EXADEMY MEGA TEST

PAPER II

There are 8 questions divided in two sections.

Candidate has to attempt FIVE questions in all.

Q1 and Q5 are compulsory and out of the remaining, THREE are to be attempted choosing at least ONE question from each section

Total Marks : 250

SECTION - I

Q1. (i) Show that a subgroup H of G is normal iff $Ha \neq Hb \rightarrow aH \neq bH$.

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(ii) For the real numbers x, a, $\varepsilon > 0$, show that

I. $|x| < \varepsilon \leftrightarrow -\varepsilon < x < \varepsilon$,
II. $|x - a| < \varepsilon \leftrightarrow a - \varepsilon < x < a + \varepsilon$

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(iii) Find a complete integral of $px + qy = pq$

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(iv) Suppose $z1 = 2 + I$, $z2 = 3 - 2i$ and $z3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Evaluate each of the following:

I. $|3z_1 - 4z_2|$
II. $z_1^3 - 3z_1^3 + 4z_1 - 8$
III. $(\overline{Z3})^4$
IV. $\left|\frac{2z2+z1-5-i}{2z1-z2+3-i}\right|^2$

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(v) Use Newton's method to find a root of the equation $x^3 - 3x - 5 = 0$.

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Q2. (i) Find the complete integrals of the following equations:

I. $q = (z + px)^2$
II. $p = (z + qy)^2$

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(ii) Prove that the area of a parallelogram having sides z_1 and z_2 is $|z_1 \times z_2|$.

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(iii) a. What are the invariant points in a mapping?

b. Which are the invariant points of the mapping $w = z^2$?

5+5

(iv) Suppose $z_1 = 3 - 4i$ and $z_2 = -4 + 3i$. Find

1. $z_1.z_2$
2. $|z_1 \times z_2|$

Also find the acute angles between the two vectors.

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Q3. (i) Find a complete integral of $(p^2 + q^2)x = pz$

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(ii) If $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ and if $\lambda_k \geq 0$, $|z_k| < 1$ for $k = 1, 2, \ldots n$, then show that

$|\lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_n z_n| < 1$

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(iii) Find z which satisfies $|z| - z = 1 + 2i$

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(iv) Show that every complex number z whose absolute value is 1, can be expressed in the form $Z = (1 + it) / (1 - it)$

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Q4. (i) If L, M N are three subspaces of a vector space V, such that $M \subseteq L$ then show that

$L \cap (M + N) = (L \cap M) + (L \cap N) = M + (L \cap N)$.

Also give an example, where the result fails to hold when $M \nsubseteq L$.

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(ii) Find the complete integral of the partial differential equation $(p^2 + q^2)x = pz$ and deduce the solution which passes through the curves $x = 0$, $z^2 = 4y$.

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(iii) Express each equation in terms of conjugate coordinates:

1. $2x + y = 5$
2. $x^2 + y^2 = 36$

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(iv) Suppose n = 2, 3, 4, …. Prove that

I. $\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \dots + \cos\frac{2(n-1)\pi}{n} = -1$

II. $\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \dots + \sin\frac{2(n-1)\pi}{n} = 0$

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(v) Find all the 5^{th} roots of unity

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SECTION - II

Q5. (i) Show that a set S of real numbers is bounded if there exists a real number $G > 0$, such that $|x| \leq G$, for all $x \in S$.

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(ii) Apply Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e (1 + e^x)$ to the terms containing x^4.

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(iii) The equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between 1.4 and 1.5. Find the root to four decimals by false position method.

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(iv) Which are the fixed points under the mapping $w = az + b$, where a and b are complex constants.

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Q6. (i) Solve $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$

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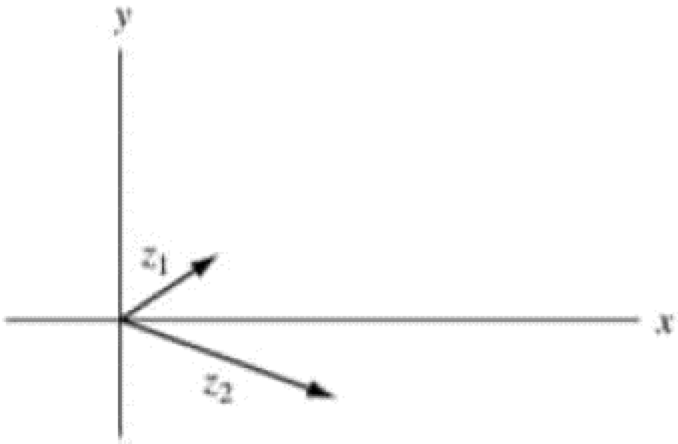
(ii) Suppose z1 and z2 are two given complex numbers (vectors) as in Fig. Construct graphically

1. $3z_1 - 2z_2$
2. $\frac{1}{2} z_2 + 5/3\, z_1$



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(iii) Show that in the motion of a fluid on two dimensions if the coordinates (x, y) of an element at any time may be expressed in terms of the initial coordinates (a, b) and the time, the motion is irrotational, if

$\frac{\partial(x',x)}{\partial(a,b)} = \frac{\partial(y',y)}{\partial(a,b)} = 0.$ Here x' = dx/dt and y' = dy/dt.

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(iv) Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$

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Q7. (i) Let F be the field of quotients of an integral domain R. Define

$\Theta : R[x] \rightarrow r[x]$ s.t.,

$\Theta(a_0 + a_1x + \ldots + a_nx^n) = a_0x^n + a_1x^{n1-1} + \ldots + a_{n-1}x+a_n.$

Then show that

1. $\Theta(f(x)) = x^n f(1/x)$ in F[x]
2. Θ is 1 – 1, onto
3. $\Theta(fg) = \Theta(f)\,\Theta(g)$.

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(ii) Suppose p + qi is a root of $a_0z^n + a_1z^{n-1} + \ldots + a_n = 0$, where $a_0 \neq 0$, $a_1, \ldots, a_n$, p and q are real. Prove that p – qi is also a root.

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(iii) Prove

I. $\overline{z1 + z2} = \overline{z1} + \overline{z2}$

II. |z1 z2| = |z1| |z2

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(iv) Find the images of the following circles under the transformation w = 1/z.

1. |z – 1| = 1
2. |z + 1| = 1
3. |z – 3i| = 3
4. |z + 2i| = 2

8

(v) State whether the following sets are open, closed and arc-wise connected and whether they are regions: |z| ≤ R, 0 < |z| ≤ R, |z| < R, |z| > R, Re z > 0, Re z ≠ 0.

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Q8. (i) Show that

$\lim\limits_{n\to\infty} \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n = 0$, where |x| < 1 and m is any real number.

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(ii) Show that the area of the triangle whose vertices are z1, z2, z3 is $\frac{1}{4i}\sum \frac{((z2-z3)|z1|^2}{z1}$

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(iii) Identify the following sets in the complex plane.

1. |z| < 1
2. |z| > 1
3. |z – 2| < 3
4. |z – 2| > 3
5. Re Z > 0
6. Im z > 0
7. |z – 1| < |z + 1|
8. |z – 1| ≥ |z + 1|

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(iv) Prove

I. $|z1 + z2| \leq |z1| + |z2|$
II. $|z1 + z2 + z3| \leq |z1| + |z2| + |z3|$
III. $|z1 - z2| \geq |z1| - |z2|$

And give a graphical interpretation.

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