IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2021

Mains

MATHEMATICS

Optional Paper-II

Solutions

Section A

1(a) Let $m_1, m_2 \cdots m_k$ be positive integers and $d>0$ the greatest common divisor of $m_1, m_2 \cdots m_n$. Show that there exist integers $x_1, x_2, \cdots x_k$ such that

$$d = x_1 m_1 + x_2 m_2 + \cdots + x_k m_k.$$

Sol Definition (for student understanding)

An integer $d>0$ is called greatest common divisor (gcd) of two integers a, b (non zero) if (i) $d|a$, $d|b$
(ii) if $c|a$, $c|b$ then $c|d$.

we write $d = \gcd(a,b)$
or $d = (a,b)$

Let $S = \{au+bv \mid u, v \text{ are integers and } au+bv>0\}$.

if $a>0$ then $a = a\cdot 1 + b(0) > 0$
$\Rightarrow a \in S$.

if $a<0$ then $-a = a(-1)+b\cdot(0) > 0$
$\Rightarrow -a \in S$.

Sly if $b>0$ then $b \in S$
if $b<0$ then $-b \in S$.

$\therefore S \neq \phi$ and S contains +ve integers.

By well-ordering principle, S has a least element say 'd'.

Now we have
$d \in S$ s.t $d = ax + by$ — (1) for some integers x, y.
Also $d > 0$.

<u>To prove that d is g.c.d. of a and b</u>

Let $a = dq + r$; $0 \le r < d$. — (2)
if $r \neq 0$ then $r = a - dq$
$= a - (ax+by)q$ (by (1))
$= a(1-xq) + b(-yq) > 0$
$\therefore r > 0$ ($\because 1-xq, -yq$ are integers)
$\therefore r \in S$
if $r < d$ and $r \in S$
which is the contradiction to the fact that 'd' is least element of S.

$\therefore \boxed{r = 0}$

$\therefore a = dq \Rightarrow \frac{a}{d} = q \Rightarrow d|a$
$\text{sly } d|b$

suppose $c|a, c|b \Rightarrow c|ax+by$
$\Rightarrow c|d$.

$\therefore d|a, d|b$
also if $c|a, c|b \Rightarrow c|d$.
$\therefore d$ is gcd of 'a' and 'b' (by definition)
if possible d' is also gcd of a and b.

Then $d'|a, d'|b \Rightarrow d|d'$. —— (3)

Sly $d|a, d|b \Rightarrow d'|d$ —— (4).

$\therefore$ from (3) & (4), $d = d'$.

$\therefore$ d is uniquely determined by a & b.

The above can be extended to more than two integers.

if $d = \gcd(m_1, m_2, \ldots m_k)$ there exist integers $x_1, x_2, \ldots x_k$ such that

$$d = m_1x_1 + m_2x_2 + \cdots + m_kx_k$$

$$\Rightarrow \boxed{d = x_1m_1 + x_2m_2 + \cdots + x_km_k}.$$

1.(b) → Test the uniform convergence of the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \cdots\cdots$$

on $[0,1]$.

Soln: Here $S_n(x)$ = nth partial sum

$$= f_1(x) + f_2(x) + \cdots\cdots + f_n(x)$$

$$= x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots\cdots + \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4\left[1 - \{1/(1+x^4)^n\}\right]}{1 - \{1/(1+x^4)\}}$$

$$= 1 + x^4 - \frac{1}{(1+x^4)^{n-1}}$$

$\therefore$ sum function $S(x) = \lim\limits_{n\to\infty} S_n(x)$

$$= \begin{cases} 1+x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Since the sum function $S(x)$ is discontinuous at $x = 0 \in [0,1]$. the given series is not uniform convergent on $[0,1]$.

1.(c) → If a function f is monotonic in the interval $[a,b]$, then prove that f is Riemann integrable in $[a,b]$.

Soln: Let f be monotonically increasing on $[a,b]$, then

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in [a,b]$$

$\Rightarrow f$ is bounded on $[a,b]$ and $\inf f = f(a)$ and $\sup f = f(b)$

Let $\epsilon > 0$ be given and

$$P = \{a = x_0, x_1, x_2, \cdots, x_n = b\}$$

be a partition $[a,b]$ such that

$$\delta_r < \frac{\epsilon}{f(b)-f(a)+1} \text{ for } r = 1,2,\cdots n.$$

Let m_r and M_r be the infimum and supremum of f on

$$I_r = [x_{r-1}, x_r].$$

Since f is monotonically increasing,

$$m_r = f(x_{r-1}) \text{ and } M_r = f(x_r)$$

Now
$$U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$
$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\delta_r$$
$$< \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\epsilon}{f(b)-f(a)+1}$$

$$= \frac{\epsilon}{f(b)-f(a)+1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$

$$= \frac{\epsilon}{f(b)-f(a)+1} [f(x_n) - f(x_0)]$$

$$= \frac{\epsilon}{f(b)-f(a)+1} [f(b) - f(a)]$$

$$= \frac{f(b)-f(a)}{f(b)-f(a)+1} \epsilon < \epsilon$$

$\therefore$ for each $\epsilon > 0$, $\exists$ a partition P such that

$$U(P,f) - L(P,f) < \epsilon$$

$\Rightarrow f$ is integrable on $[a,b]$.

Similarly, when f is monotonically decreasing on $[a,b]$, we can prove that f is integrable on $[a,b]$.

Hence, f is monotonic on $[a,b]$

$\Rightarrow f$ is integrable on $[a,b]$.

1(d) Let $c : [0,1] \to \mathbb{C}$ be the curve, where $c(t) = 1e^{4\pi i t}$, $0 \le t \le 1$,
Evaluate the contour integral

$$\int_C \frac{dz}{2z^2-5z+2}.$$

Sol Let $z(t) = e^{4\pi i t}$, $0 \le t \le 1$

then $|z(t)| = 1$.

$\therefore$ $C : |z-0| = 1$.

$\therefore$ C is a circle of unit radius and center is origin $(0,0)$ O.

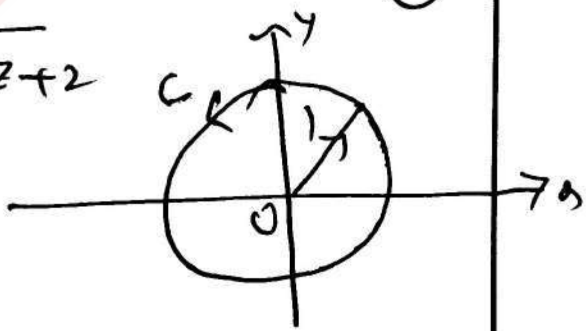
$\therefore$ let us find

$$\int_{C : |z-0|=1} \frac{1}{2z^2-5z+2}\,dz = \int_{C:|z|=1} f(z)\,dz \text{ say} \quad \text{---}(1)$$

where $f(z) = \dfrac{1}{2z^2-5z+2}$

$$= \frac{1}{(2z-1)(z-2)}$$

$$= \frac{\frac{1}{2}}{(z-\frac{1}{2})(z-2)}.$$



clearly $z = \frac{1}{2}$ lies in $|z| = 1$.

$$\therefore \int_{C:|z-0|=1} f(z)\,dz = \int_C \frac{\frac{1}{2(z-2)}}{(z-\frac{1}{2})}\,dz$$

$$= \int_C \frac{f_1(z)}{(z-\frac{1}{2})}\,dz \text{ say} \quad \text{---}(2)$$

where $f_1(z) = \frac{1}{2(z-2)}$.

$\therefore$ By cauchy's 1st integral formula:

$$\int_{C'} \frac{f_1(z)}{(z-\frac{1}{2})^1} dz = 2\pi i f_1(\tfrac{1}{2})$$

$$\Rightarrow \int_C \frac{\frac{1}{2(z-2)}}{(z-\frac{1}{2})} dz = 2\pi i \left[\frac{1}{2(\frac{1}{2}-2)}\right]$$

$$= \frac{\pi i}{\frac{1-4}{2}} = \frac{2\pi i}{-3}$$

$$= \frac{-2\pi i}{3}.$$

$$\therefore \boxed{\int_C f(z)\,dz = \frac{-2\pi i}{3}}$$

Since $c(t) = e^{4\pi i t}$, $0 \le t \le 1$.

clearly 'c' winds up two times about the origin with unit radius.

$$\therefore \int_C f(z)\,dz = \frac{-4\pi i}{3}.$$

1(e)→ A department of a company has five employees with five jobs to be performed. The time (in hours) that each man takes to perform each job is given in the effectiveness matrix. Assign all the jobs to these five employees to minimize the total processing time.

Employees

Jobs	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

Soln:

Subtracting the minimum element of each row from all elements of that row we get

5	0	8	10	11
0	6	15	10	3
8	5	0	0	0
0	4	2	0	5
3	5	6	0	8

Subtracting the minimum element of each column from elements of that column, and cover all the zeros by minimum no. of horizontal and vertical lines.

5	0	8	10	4
0	6	15	10	3
8	5	0	0	0
0	4	2	0	5
3	5	6	0	8

see that we can cover all the zeros by 4 lines only.

so, $r = 4 < 5 = n$

Now,

2 is the least uncovered element. subtract 2 from all the uncovered elements and add 2 to the elements at intersection of the covering lines namely 8, 5 and 0 at positions (3,1), (3,2) and (3, 4) respectively.

and leave other covered elements unchanged. The reduced cost-matrix so obtained is:

5	0	6	10	9
0	6	13	10	1
10	7	0	2	0
0	4	0	0	3
3	5	4	0	6

In the above table, cover the zeros by

minimum no. of horizontal and vertical lines. we require exactly 5 lines to cover all the zeros.

As $r = 5 = n$; optimal assignment can be made at this stage.

For making assignments, proceed as follows.

	I	II	III	IV	V
A	5	0	6	10	9
B	0	6	13	10	1
C	10	7	~~0~~	2	0
D	~~0~~	4	0	~~0~~	3
E	3	5	4	0	6

- 1^{st}, 2^{nd} and 5^{th} rows have only one zero in positions (1,2), (2,1) and (5,4) respectively. So encircle this zero and cross all other zeros in its column.
- 5^{th} column contains only one zero in position (3,5). so encircle it and cross all the zeros in its row.
- There is only one zero in 4 row, so encircle it.

Now, since each row and each

column has a single encircled zero.
i.e, each row and each column has one and only one assignment.

So, an optimal assignment is reached.

∴ The optimal assignment is

$A \to \text{II}$, $B \to \text{I}$, $C \to \text{V}$

$D \to \text{III}$, $E \to \text{IV}$.

The minimum assignment cost is

$c_{12} + c_{21} + c_{35} + c_{43} + c_{54}$

$= 5 + 3 + 2 + 9 + 4$

$= 23$

2(a) Find the maximum and minimum values of $f(x)=x^3-9x^2+26x-24$ for $0\le x\le 1$.

Sol Given that $f(x)=x^3-9x^2+26x-24$ $\forall x\in[0,1]$ — (1)

$\therefore f'(x)=3x^2-18x+26$ — (2)

if $f'(x)=0$ then $3x^2-18x+26=0$

we have
$$x=\frac{18\pm\sqrt{324-4(3)(26)}}{6}$$
$$\Rightarrow x=\frac{18\pm\sqrt{12}}{6}$$
$$\Rightarrow x=\frac{9\pm\sqrt{3}}{3}$$
$$\Rightarrow x=\frac{9+\sqrt{3}}{3},\ \frac{9-\sqrt{3}}{3}.$$

These are the critical points of the $f(x)$ which $\notin[0,1]$

Thus $f(x)$ is monoton on $[0,1]$.

Now let us check whether it is monotonic increasing or decreasing:-

we have
$$f'(0)=3(0)^2-18(0)+26=26>0$$
and $f'(1)=3(1)-18(1)+26=11>0$.

$\therefore f'(x)>0$ $\forall x\in[0,1]$.

$\therefore$

$\therefore f(x)$ is increasing on $[0,1]$

$\Rightarrow f(x)$ is Monotonically increasing on $[0,1]$.

$\therefore f_{max} = f(x)\Big|_{x=1} = 1-9+26-24$
$= -6.$

$\therefore \boxed{f_{max} = -6 \text{ at } x=1}$

$f_{min} = f(x)\Big|_{x=0} = 0-0+0-24$
$= -24.$

$\therefore f_{min}(0) = -24 \;;\; f_{max}(1) = -6.$

2(c) Find the Laurent series expansion of $f(z) = \dfrac{z^2-z+1}{z(z^2-3z+2)}$ in powers of $(z+1)$ in the region $|z+1|>3$.

Sol Let $f(z) = \dfrac{z^2-z+1}{z(z^2-3z+2)}$.

Let $z+1=u$. then $\boxed{z = u-1}$

$$\therefore f(u) = \frac{(u-1)^2-(u-1)+1}{(u-1)\left[(u-1)^2-3(u-1)+2\right]}$$

$$\Rightarrow f(u) = \frac{u^2+1-2u-u+1+1}{(u-1)\left[u^2+1-2u-3u+3+2\right]}$$

$$\Rightarrow f(u) = \frac{u^2-3u+3}{(u-1)(u^2-5u+6)}$$

$$f(u) = \frac{u^2-3u+3}{(u-1)(u+2)(u-3)}$$

6

-2 -3

$$= \frac{A}{u-1} + \frac{B}{u-2} + \frac{C}{u-3} \quad \text{--- (1)}$$

$$\Rightarrow u^2-3u+3 = A(u-2)(u-3)+B(u-1)(u-3) + C(u-1)(u-2)$$

if $u=2$ then

$4-6+3 = B(1)(-1) \Rightarrow \boxed{B=-1}$

if $u=1$ then $1-3+3 = A(-1)(-2)$

$\Rightarrow \boxed{A = 1/2}$

if $u=3$ then $9-9+3 = C(2)(1) \Rightarrow \boxed{C=3/2}$

$\therefore$ (1) $\equiv$

$$f(u) = \frac{\frac{1}{2}}{u-1} + \frac{-1}{u-2} + \frac{\frac{3}{2}}{u-3} \quad \text{——— (2)}$$

<u>corr. to $\frac{1}{u-1}$:</u>

$$\frac{1}{u-1} = \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} = \frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+\cdots\right)$$

is valid for $\left|\frac{1}{u}\right|<1$ and $|u|>0$

$$\therefore \frac{1}{u-1} = \frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots$$ is valid for $|u|>1$ ——— (3)

<u>corr. to $\frac{1}{u-2}$:</u>

$$\therefore \frac{1}{u-2} = \frac{1}{u}\left(1-\frac{2}{u}\right)^{-1} = \frac{1}{u}\left(1+\frac{2}{u}+\frac{4}{u^2}+\cdots\right)$$

is valid for $\left|\frac{2}{u}\right|<1$ and $|u|>0$

$$\therefore \frac{1}{u-2} = \frac{1}{u}+\frac{2}{u^2}+\frac{4}{u^3}+\cdots$$ is valid for $|u|>2$. ——— (4)

<u>corr to $\frac{1}{u-3}$:</u>

$$\therefore \frac{1}{u-3} = \frac{1}{u}\left(1-\frac{3}{u}\right)^{-1} = \frac{1}{u}\left(1+\frac{3}{u}+\frac{9}{u^2}+\cdots\right)$$

is valid for $|u|>0$ and $\left|\frac{3}{u}\right|<1$

$$\therefore \frac{1}{u-3} = \frac{1}{u}+\frac{3}{u^2}+\frac{9}{u^3}+\cdots$$

is valid for $|u|>3$. ——— (5)

$\therefore$ from ②,

$$f(u) = \frac{1}{2}\left(\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right)$$
$$-\left(\frac{1}{u}+\frac{2}{u^2}+\frac{4}{u^3}+\cdots\right)$$
$$+\frac{3}{2}\left(\frac{1}{u}+\frac{3}{u^2}+\frac{9}{u^3}+\cdots\right) \text{ is valid}$$

for $|u|>3$.

$$\Rightarrow f(u) = \left(\tfrac{1}{2}-1+\tfrac{3}{2}\right)\tfrac{1}{u}+\left(\tfrac{1}{2}-2+\tfrac{9}{2}\right)\tfrac{1}{u^2}$$
$$+\left(\tfrac{1}{2}-4+\tfrac{27}{2}\right)\tfrac{1}{u^3}+\cdots$$

is valid for $|u|>3$.

$$\Rightarrow f(u) = \frac{1}{u}+\frac{3}{u^2}+\frac{11}{u^3}+\cdots$$

is valid for $|u|>3$.

$$\therefore f(z) = \frac{1}{z+1}+\frac{3}{(z+1)^2}+\frac{11}{(z+1)^3}+\cdots$$

is valid for $|z+1|>3$

3(c) Let f be an entire function whose taylor series expansion with centre $z=0$ has infinitely many terms.
Show that $z=0$ is an essential singularity of $f(1/z)$.

Sol Let $f(z)$ be an entire function whose Taylor series expansion with centre $\boxed{z=0}$ has infinitely many terms.

Clearly $f(z)=e^{z}$ is an entire function and its taylor's series expansion with centre $z=0$ has infinitely many terms

i.e $f(z)=e^{z}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$

we have $f(1/z)=e^{1/z}$

$\Rightarrow f(1/z)=1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\cdot\frac{1}{z^3}+\cdots$

clearly the principal part of Laurent series contains infinitely many terms.

$\therefore z=0$ is an essential singular point

3.(b) Find the stationary values of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$. Interpret the result geometrically.

Sol^n: Let $u = x^2+y^2+z^2$ ——— ①

Given $ax^2+by^2+cz^2=1$ ——— ②

and $lx+my+nz=0$ ——— ③

From ① we get

$$du = 2x\,dx + 2y\,dy + 2z\,dz = 0 \quad \text{——— ④}$$

from ② and ③ we have

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0 \quad \text{——— ⑤}$$

and $l\,dx + m\,dy + n\,dz = 0$ ——— ⑥

Multiplying ④, ⑤ and ⑥ by $1, \lambda_1, \lambda_2$ and adding we get

$$(x\,dx + y\,dy + z\,dz) + \lambda_1(ax\,dx + by\,dy + cz\,dz) + \lambda_2(l\,dx + m\,dy + n\,dz) = 0$$

or $$(x + a\lambda_1 x + l\lambda_2)dx + (y + b\lambda_1 y + m\lambda_2)dy + (z + c\lambda_1 z + n\lambda_2)dz = 0$$

Equating to zero the coefficients of dx, dy and dz we get

$$\left.\begin{aligned} x + a\lambda_1 x + l\lambda_2 &= 0 \\ y + b\lambda_1 y + m\lambda_2 &= 0 \\ z + c\lambda_1 z + n\lambda_2 &= 0 \end{aligned}\right\} \text{——— ⑦}$$

Multiplying these by x, y, z and adding, we get

$$(x^2+y^2+z^2)+\lambda_1(ax^2+by^2+cz^2)+\lambda_2(lx+my+nz)=0$$

or $u+\lambda_1(1)+\lambda_2(0)=0$, from ①, ② and ③

or $u+\lambda_1=0$ or $\lambda_1=-u$

$\therefore$ from ⑦ we have

$$x-axu+l\lambda_2=0,\quad y-byu+m\lambda_2=0,$$
$$z-czu+n\lambda_2=0$$

or $x(1-au)=-l\lambda_2$, $y(1-bu)=-m\lambda_2$,

$$z(1-cu)=-n\lambda_2$$

or $x=\dfrac{l\lambda_2}{au-1}$, $y=\dfrac{m\lambda_2}{bu-1}$, $z=\dfrac{n\lambda_2}{cu-1}$

Substituting these values in ③, we get

$$\frac{l^2}{(au-1)}+\frac{m^2}{(bu-1)}+\frac{n^2}{(cu-1)}=0$$

which gives the stationary values of u, i.e., $x^2+y^2+z^2$.

3.(c) Convert the following LPP into dual LPP:

Minimize $z = x_1 - 3x_2 - 2x_3$

Subject to

$$3x_1 - x_2 + 2x_3 \leq 7$$
$$2x_1 - 4x_2 \geq 12$$
$$-4x_1 + 3x_2 + 8x_3 = 10$$

where $x_1, x_2 \geq 0$ and x_3 is unrestricted in sign.

Solⁿ: Given that

Minimize $z = x_1 - 3x_2 - 2x_3$

Subject to

$$3x_1 - x_2 + 2x_3 \leq 7 \quad \text{——} ①$$
$$2x_1 - 4x_2 \geq 12 \quad \text{——} ②$$
$$-4x_1 + 3x_2 + 8x_3 = 10 \quad \text{——} ③$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign

Since, the constraints ① in ($\leq$) type and the problem is of minimization, all the constraints should be of ($\geq$) type.

we multiply constraints ① by (−1)

So that $\quad -3x_1 + x_2 - 2x_3 \geq -7$

and the constraints with equality sign can be written as

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$\Rightarrow \quad 4x_1 - 3x_2 - 8x_3 \geq -10$$
$$-4x_1 + 3x_2 + 8x_3 \geq 10$$

Put $\quad x_3 = x_3' - x_3''$ so that $x_3', x_3'' \geq 0$

and the primal can be written as

$$\text{Min } Z = x_1 - 3x_2 - 2x_3' + 2x_3''$$

Subject to

$$-3x_1 + x_2 - 2x_3' + 2x_3'' \geq -7$$
$$2x_1 - 4x_2 \geq 12$$
$$4x_1 - 3x_2 - 8x_3' + 8x_3'' \geq -10$$
$$-4x_1 + 3x_2 + 8x_3' - 8x_3'' \geq 10$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

$\therefore$ Dual to this LPP is:

$$\text{Maximize } W = -7y_1 + 12y_2 - 10y_3 + 10y_4$$

Subject to

$$-3y_1 + 2y_2 + 4y_3 - 4y_4 \leq 1$$
$$y_1 - 4y_2 - 3y_3 + 3y_4 \leq -3$$
$$-2y_1 + 0y_2 - 8y_3 + 8y_4 \leq -2$$
$$2y_1 + 0y_2 + 8y_3 - 8y_4 \leq 2$$
$$y_1, y_2, y_3, y_4 \geq 0$$

This can also be written as

$$\text{Max } W = -7y_1 + 12y_2 - 10(y_3 - y_4)$$

Subject to

$$-3y_1 + 2y_2 + 4(y_3 - y_4) \leq 1$$
$$y_1 - 4y_2 - 3(y_3 - y_4) \leq -3$$
$$-2y_1 - 8(y_3 - y_4) \leq -2$$
$$2y_1 + 8(y_3 - y_4) \leq 2$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

The term (y_3-y_4) in both objective function and constraints of the dual.

This is always whenever there is equality constraints in the primal. then, the new variable $(y_3-y_4)=u_1$ becomes unrestricted in sign, being the difference of two non-negative variable.

$\therefore$ The above dual problem takes the form

$$\text{Min } W = -7y_1+12y_2+10u_1$$

subject to

$$-3y_1+2y_2+4u_1 \leq 1$$

$$y_1-4y_2-3u_1 \leq -3$$

$$-2y_1-8u_1 \leq -2$$

$$2y_1+8u_1 \leq 2$$

$y_1, y_2 \geq 0$ and u_1 is unrestricted in sign.

It can be written as

$$\text{Max } Z = -7y_1+12y_2+10u_1$$

Subject to

$$-3y_1+2y_2+4u_1 \leq 1$$

$$y_1-4y_2-3u_1 \leq -3$$

$$-2y_1+8u_1 = -2$$

$y_1, y_2 \geq 0$ and u_1 is unrestricted.

4(a) Show that there are infinitely many subgroups of the additive group Q of rational numbers.

Sol Let $Q = \left\{ \frac{p}{q} / p, q \in I ; q \neq 0 \right\}$ be a group w.r.t +n.

Let $H = \langle \frac{a}{b} \rangle$: $\frac{a}{b} \in Q$, $a, b \in I$, $b \neq 0$.

be a cyclic subgroup of Q.

clearly $H = \langle \frac{a}{b} \rangle = \left\{ \frac{a}{b} x / x \in Q \right\}$

clearly it is an infinite cyclic subgroup of Q.

We know that any infinite cyclic group is isomorphic to the additive group Z of all integers.

clearly $Z \cong H$.

Also $H \cong Z$.

Here $H = \langle \frac{a}{b} \rangle$,

Since Z has infinitely many subgroups.

$\therefore$ H has infinitely many subgroups.

$\therefore$ An additive group Q of rational numbers has infinitely many subgroups.

4.(b) → Using contour integration, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x(x^2+a^2)}, \; a > 0.$$

Solⁿ: consider the integral

$$\int_C \frac{e^{iz}\, dz}{z(z^2+a^2)} \int_C f(z)\, dz,$$ taken round

the contour C consisting of

(i) the real axis from ρ to R,

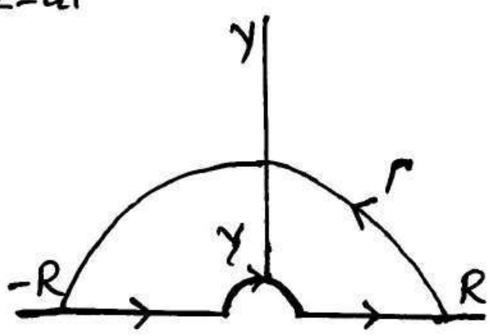
(ii) a large semi-circle Γ in the upper half plane given by $|z| = R$,

(iii) the real axis $-R$ to $-\rho$, and

(iv) a small semi-circle γ given by $|z| = \rho$.

Now $f(z)$ has simple poles at $z = 0, \pm ai$ of which only $z = ai$ lies within C.

The residue of $f(z)$ at $z = ai$ is given by



$$\lim_{z \to ai} (z - ai) f(z)$$

$$= \lim_{z \to ai} \frac{e^{iz}}{z(z+ai)}$$

$$= \frac{e^{-a}}{-2a^2}$$

∴ By residue theorem, we get

$$\int_C f(z)dz = \int_{\mu}^{R} f(z)dz + \int_{\Gamma} f(z)dz + \int_{-R}^{-p} f(x)dx + \int_{\gamma} f(z)dz$$

$$= 2\pi i \frac{e^{-a}}{-2a^2} = \frac{-\pi i}{a^2} e^{-a} \quad \text{——— (1)}$$

By jordan's lemma,

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$$

Also since $\lim\limits_{z\to 0} zf(z) = \lim\limits_{z\to 0} \frac{e^{iz}}{z^2+a^2} = \frac{1}{a^2}$

Hence $\int_{\gamma} f(z)\,dz = \frac{i}{a^2}(0-\pi) = -\frac{\pi i}{a^2}$

Hence as $p\to 0$ and $R\to\infty$, we get from (1)

$$\int_0^{\infty} f(x)dx + \int_{-\infty}^{0} f(x)dx - \frac{\pi i}{a^2} = -\frac{\pi i}{a^2} e^{-a}$$

or $$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(a^2+x^2)}dx = \frac{\pi i}{a^2}(1-e^{-a}).$$

Equating imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x(x^2+a^2)} = \frac{\pi}{a^2}(1-e^{-a}).$$

4(c) Solve the following linear programming problem using Big-M method:

Maximize $Z = 4x_1 + 5x_2 + 2x_3$

subject to

$2x_1 + x_2 + x_3 \geq 10$

$x_1 + 3x_2 + x_3 \leq 12$

$x_1 + x_2 + x_3 = 6.$

$x_1, x_2, x_3 \geq 0.$

Sol^n:

The objective function of the given LPP is of maximization type.

Now we write the given LPP in standard form

Maximize $Z = 4x_1 + 5x_2 + 2x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

subject to

$2x_1 + x_2 + x_3 - S_1 + 0S_2 + A_1 + 0A_2 = 10$

$x_1 + 3x_2 + x_3 + 0S_1 + S_2 + 0A_1 + 0A_2 = 12$

$x_1 + x_2 + x_3 + 0S_1 + 0S_2 + 0A_1 + A_2 = 6$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0.$

where S_1 is the surplus variable

S_2 is the slack variable

A_1, A_2 are the artificial variables.

Now the surplus variable S_1 is not a basic variable since its value is -10. As negative quantities are not feasible, $\therefore$ S_1 must be prevented from appearing in the initial solution. This is done by taking $s_1 = 0$.

By setting other non-basic variables $x_1 = x_2 = x_3 = 0$.

we obtain the IBFS as

$x_1 = x_2 = x_3 = s_1 = 0$, $A_1 = 10$, $A_2 = 6$, $S_2 = 12$

Thus the initial simplex table is:

	c_j	4	5	2	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	$\boxed{2}$	1	1	-1	0	1	0	10	5 →
0	s_2	1	3	1	0	1	0	0	12	12
$-M$	A_2	1	1	1	0	0	0	1	6	6
$Z_j = \Sigma C_B a_{ij}$		$-3M$	$-2M$	$-2M$	M	0	$-M$	$-M$	$-16M$	
$C_j = C_j - Z_j$		$3M+4$ ↑	$2M+5$	$2M+2$	$-M$	0	0	0		

From the above table, the variable x_1 is entering variable, A_1 is the outgoing variable and omit column for this variable in the next simplex table.

Here $\boxed{2}$ is the key element and convert it into unity and all other elements in its column to zero.

Then the new simplex table is:

	c_j	4	5	2	0	0	$-M$		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	A_2	b	θ
4	x_1	1	$1/2$	$1/2$	$-1/2$	0	0	5	10
0	s_2	0	$5/2$	$1/2$	$1/2$	1	0	7	2.8
$-M$	A_2	0	$\boxed{1/2}$	$1/2$	$1/2$	0	1	1	$2 \rightarrow$
$Z_j = \sum c_B a_{ij}$		4	$2-\frac{M}{2}$	$2-\frac{M}{2}$	$-2-\frac{M}{2}$	0	$-M$	$20-M$	
$C_j = c_j - Z_j$		0	$3+\frac{M}{2}$ $\uparrow$	$\frac{M}{2}$	$2+\frac{M}{2}$	0	0		

From the above table, x_2 is the entering variable, A_2 is the outgoing variable and omit its column in the next simplex table. Here $\boxed{1/2}$ is the key element and make it unity and all other elements in its column equal to zero.

Then revised simplex table is

	c_j	4	5	2	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	b
4	x_1	1	0	0	-1	0	4
0	s_2	0	0	-2	-2	1	2
5	x_2	0	1	1	1	0	2
$Z_j = \sum c_B a_{ij}$		4	5	5	1	0	26
$C_j = c_j - Z_j$		0	0	-3	-1	0	

From the above table, all C_j's ≤ 0.
There remains no artificial variable in the basis.
$\therefore$ The solution is an optimal BFS to the problem and is given by $x_1 = 4$, $x_2 = 2$, $x_3 = 0$

$\therefore \text{Max } Z = 26$

SECTION-B

5(a) Obtain the partial differential equation by eliminating arbitrary function f from the equation $f(x+y+z,\ x^2+y^2+z^2)=0$.

Sol: Given that $f(x+y+z,\ x^2+y^2+z^2)=0$ —— ①

Let $u = x+y+z$, $v = x^2+y^2+z^2$ —— ②

Then ① becomes $\phi(u,v)=0$ —— ③

Differentiating (3) w.r.t x partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{—— ④}$$

From ②,

$$\left.\begin{aligned}&\frac{\partial u}{\partial x}=1,\quad \frac{\partial u}{\partial z}=1,\quad \frac{\partial v}{\partial x}=2x,\quad \frac{\partial v}{\partial z}=2z\\ &\frac{\partial u}{\partial y}=1,\quad \frac{\partial v}{\partial y}=2y\end{aligned}\right\} \text{—— ⑤}$$

from ④ & ⑤,

$$\frac{\partial\phi}{\partial u}(1+p)+2\frac{\partial\phi}{\partial v}(x+pz)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = -\frac{2(x+pz)}{1+p} \quad \text{—— ⑥}$$

Again differentiating ③ w.r.t y partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}(1+q)+2\frac{\partial\phi}{\partial v}(y+zq)=0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{-2(y+zq)}{1+q}. \quad \text{—— ⑦}$$

$\therefore$ From (6) & (7) by eliminating ϕ,
we obtain,

$$\frac{(x+pz)}{1+p} = \frac{(y+qz)}{1+q}$$

$$\Rightarrow (1+q)(x+pz) = (1+p)(y+qz)$$

$$\Rightarrow x+pz+qx+\cancel{pqz} = y+qz+py+pqz$$

$$\Rightarrow x-y = p(y-z)+q(z-x)$$

$$\Rightarrow (y-z)p+(z-x)q = x-y.$$

which is the desired partial differential equation of first order.

5.(b) Find a positive root of the equation $3x = 1+\cos x$ by a numerical technique using initial values $0, \frac{\pi}{2}$; and further improve the result using Newton-Raphson method correct to 8 significant figures.

Sol^n: Given, the equation

$$3x = 1+\cos x$$

$$\Rightarrow f(x) = 3x - \cos x - 1$$

$$f'(x) = 3+\sin x$$

and also given the initial values are 0 and $\pi/2$.

Clearly the function $f(x)$ and $f'(x)$ are continous on $[0, \pi/2]$.

We have, $f(0) = 3(0) - \cos(0) - 1 = -2$

and $f(\pi/2) = 3\pi/2 - \cos\left(\frac{\pi}{2}\right) - 1 = 3.71238898$

$\because f(0)f(\pi/2) < 0$

$\therefore$ The root of the given equation lies between 0 and $\pi/2$.

Let $x_0 = 0.7$ be the initial approximation to the root.

We have, $f(x_0) = 0.3351578127$

$$f'(x_0) = 3.644217687$$

Now, using the Newton-Raphson formula for performing the iterations

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = 0.7 - \frac{0.3351578127}{3.644217687}$$

$$\Rightarrow x_1 = 0.7 - 0.09196975633$$

$$\Rightarrow \boxed{x_1 = 0.6080302437}$$

Now, $f(x_1) = 0.003315894788$

$f'(x_1) = 3.571251843$

Now, finding the second approximation of the root also by Newton Raphson formula.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_2 = 0.6080302437 - \frac{0.003315894788}{3.571251843}$$

$$\Rightarrow x_2 = 0.6080302437 - 0.0009284964863$$

$$\boxed{x_2 = 0.6071017472}$$

then $f(x_2) = 0.00000035386538$

$f'(x_2) = 3.57048951$

Now, finding the third approximation of the root

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\Rightarrow x_3 = 0.6071017472 - \frac{0.00000035386538}{3.57048951}$$

$$\Rightarrow x_3 = 0.6071017472 - 0.009910836568$$

$$\boxed{x_3 = 0.6071016481}$$

then, $f(x_3) = -0.00000000000036251$

$f'(x_3) = 3.570489429$

Now, finding the fourth approximation of the root

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\Rightarrow x_4 = 0.6071016481 + \frac{0.000000000036251}{3.570489429}$$

$$\Rightarrow x_4 = 0.6071016481 + 0.0000000000101529 5$$

$$\boxed{x_4 = 0.6071016481}$$

Since x_3 and x_4 are same upto eight significant figures.

$\therefore$ we stop the iteration here.

Hence the root of $f(x)$ is 0.6071016481

5(c)(ii) Convert $(3798.3875)_{10}$ into octal and hexadecimal equivalents.

Sol^n: Decimal to octal conversion.

Integer part:

$$\begin{array}{r|l}
8 & 3798 \\ \hline
8 & 474 - 6 \\ \hline
8 & 59 - 2 \\ \hline
8 & 7 - 3 \\ \hline
 & 0 - 7
\end{array}$$

$\therefore (3798)_{10} = (7326)_8$

fractional part:

Fraction	fraction × 8	Remainder new fraction	Integer
0.3875	3.1	0.1	3
0.1	0.8	0.8	0
0.8	6.4	0.4	6
0.4	3.2	0.2	3
⋮	⋮	⋮	⋮

Thus $(0.3875)_{10} \approx (0.3063)_8$

$\therefore (3798.3875)_{10} = (7326.3063)_8$.

Decimal to hexadecimal conversion.

Integral part:

$$\begin{array}{r|l} 16 & 3798 \\ \hline 16 & 237 - 6 \\ \hline 16 & 14 - D \\ \hline & 0 - E \end{array} \uparrow$$

$\therefore (3798)_{10} = (ED6)_{16}$

fractional part:

Fraction	fraction×16	Remainder new fraction	Integer
0.3875	6.2	0.2	6
0.2	3.2	0.2	3
0.2	3.2	0.2	3

Thus $(0.3875)_{10} = (0.633)_{16}$

$\therefore (3798.3875)_{10} = (ED6.633)_{16}$.

(or)

$(3798.3875)_{10} = (7326.3063)_8$

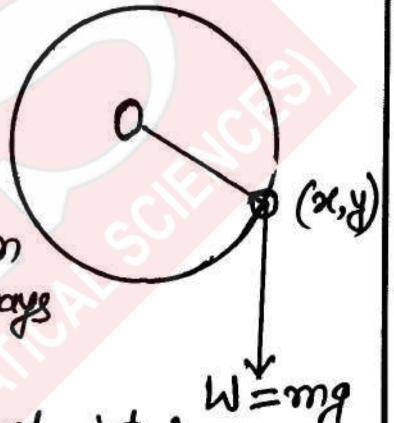
$= (111011\,010\,110 . 011000110011)_2$

$= (ED6.633)_{16}$

5.(d) A particle is constrained to move along a circle lying in the vertical xy-plane. with the help of the D'Alembert's principle, show that its equation of motion is $\ddot{x}y - \ddot{y}x - gx = 0$, where g is the acceleration due to gravity.

Solⁿ: consider a particle of mass m be moving along a circle of radius r in xy-plane.

Let (x,y) be the position of the particle at any instant t with respect to the fixed point O.



The constraint on the motion of the particle is that the position co-ordinates of the particle always lie on the circle.

Hence the equation of the constraint is:

$$x^2+y^2=r^2 \quad \text{——— (1)}$$

$$\Rightarrow 2x\delta x + 2y\delta y = 0$$

$$\text{or} \quad \delta x = -\frac{y}{x}\delta y \quad \text{——— (2)}$$

where δx and δy are displacement in x and y respectively. Now from D'Alembert's principle, we have

$$(F - m\ddot{r})\delta r = 0.$$

In terms of components we have

$$(F_x - m\ddot{x})\delta x + (F_y - m\ddot{y})\delta y = 0 \quad \text{——— (3)}$$

However, the only force acting on the particle at any instant t is its weight mg in the downward direction. Resolving the force horizontally and vertically, we have $F_x = 0$ and $F_y = -mg$. Therefore equation (3) becomes

$$-m\ddot{x}\delta x - (mg + m\ddot{y})\delta y = 0.$$

on using (2) we have

$$m(-\ddot{x}y + \ddot{y}x + gx)\delta x = 0.$$

for $\delta x \neq 0$, and $m \neq 0$ we have

$$\ddot{x}y - \ddot{y}x - gx = 0, \quad \text{——— (4)}$$

which is the required equation of motion.

5.(e) What arrangements of sources and sinks can have the velocity potential $w = \log_e\left(z - \frac{a^2}{z}\right)$? Draw the corresponding sketch of the stream-lines and prove that two of them subdivide into the circle $r=a$ and the axis of y.

Sol^n: The complex potential is given by

$$w = \log_e\left(z - \frac{a^2}{z}\right) = \log_e\left\{\frac{(z-a)(z+a)}{z}\right\}$$

or $$w = \log_e(z-a) + \log_e(z+a) - \log_e z, \quad \text{—— (1)}$$

which shows that there are two sinks of unit strength at distance $z=a$ and $z=-a$ and a sources of unit strength at an origin.

The relation (1) can be expressed as

$$\phi + i\psi = \log\{(x-a)+iy\} + \log\{(x+a)+iy\} - \log(x+iy).$$

Equating the imaginary parts, we have

$$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

or $$\psi = \tan^{-1}\left(\frac{\frac{y}{(x-a)} + \frac{y}{(x+a)}}{1 - \frac{y}{(x-a)}\frac{y}{(x+a)}}\right) - \tan^{-1}\frac{y}{x}$$

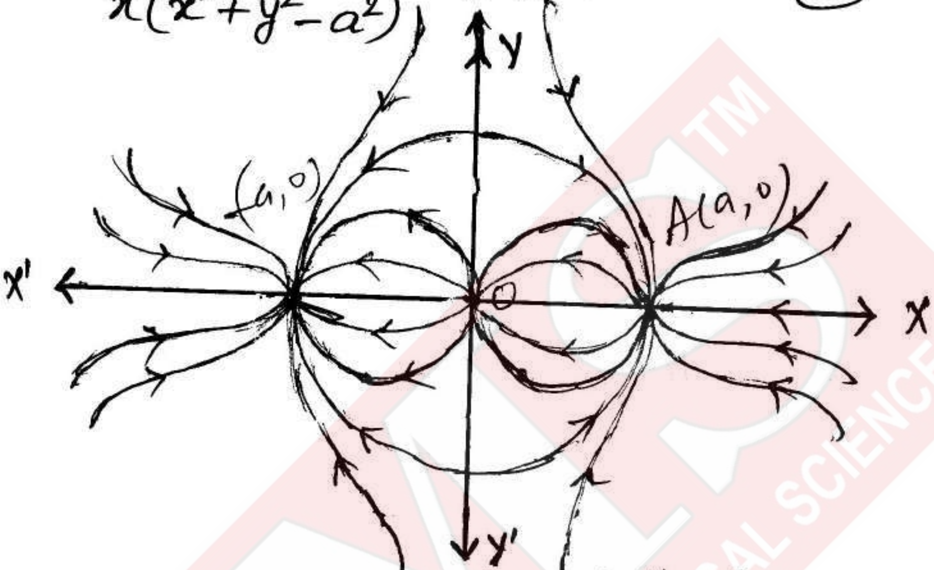
or $$\psi = \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right) - \tan^{-1}\frac{y}{x}$$

or $$\psi = \tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2-a^2} - \frac{y}{x}}{1 + \frac{2xy}{x^2-y^2-a^2}\frac{y}{x}}\right)$$

or $\psi = \tan^{-1} \frac{y}{x} \frac{(x^2+y^2+a^2)}{(x^2+y^2-a^2)}$.

The streamlines are given by ψ = constant.

or $\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = K$. ——— (2)



I. If the constant K is infinite then

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \infty$$

$\Rightarrow x(x^2+y^2-a^2) = 0 \Rightarrow x = 0$ and $x^2+y^2 = a^2$.

Thus $x = 0$ shows that y-axis is a streamline and the equation $x^2+y^2 = a^2$ (i.e., $r = a$) shows that the circle is a streamline with centre as origin.

II. If the constant K is zero, then $y = 0$ implies that axis of x is a streamline, Therefore the rough sketch of the lines is with a source of unit strength at origin O(0,0) and two sinks of unit strength at A(a,0) and B(-a,0).

6.(a) Solve the wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0<x<L, \quad t>0$$

subject to the conditions

$$u(0,t)=0, \quad u(L,t)=0$$
$$u(x,0)=\frac{1}{4}x(L-x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0.$$

Soln: Consider the following wave equation with boundary conditions:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0<x<L, \; t>0 \qquad \text{--- (1)}$$

$$u(0,t)=0, \quad u(L,t)=0, \quad t>0$$
$$u(x,0)=\frac{1}{4}x(L-x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0, \quad 0<x<L$$

Let $u(x,t)=X(x).T(t)$ be the solution to the equation (1) Then

$$\frac{\partial u}{\partial x}=X'(x).T(t), \quad \frac{\partial u}{\partial t}=X(x).T'(t)$$

$$\frac{\partial^2 u}{\partial x^2}=X''(x)T(t), \quad \frac{\partial^2 u}{\partial t^2}=X(x).T''(t)$$

Substitute these expressions in (1) gives

$$a^2 X''(x)\,T(t) = X(x)\,T''(t)$$

$$\Rightarrow \frac{X''}{X}=\frac{T''}{a^2 T}=-\lambda \text{ (say)}$$

$$\Rightarrow X''+\lambda X=0 \text{ and } T''+\lambda a^2 T=0$$

These differential equations yield different solutions corresponding to different values of λ.

For $\lambda = 0$: $X = c_1 x + c_2$ and $T = c_3 t + c_4$

So that the solution to ① is

$$u(x,t) = (c_1 x + c_2)(c_3 t + c_4)$$

The boundary condition leads to the trivial solution $u=0$.

For $\lambda = -\alpha^2 (\alpha > 0)$:

$$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x,$$

$$T = c_3 \cosh a\alpha t + c_4 \sinh a\alpha t$$

So that the solution to ① is

$$u(x,t) = (c_1 \cosh \alpha x + c_2 \sinh \alpha x)\cdot(c_3 \cosh a\alpha t + c_4 \sinh a\alpha t)$$

The boundary conditions lead to the trivial solution $u=0$

For $\lambda = \alpha^2 (\alpha > 0)$:

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x, \quad T = c_3 \cos a\alpha t + c_4 \sin a\alpha t$$

So that the solution to ① is

$$u(x,t) = (c_1 \cos \alpha x + c_2 \sin \alpha x)(c_3 \cos a\alpha t + c_4 \sin a\alpha t)$$

use the boundary conditions $u(0,t)=0$ and $u(L,t)=0$, here lead to $c_1 = 0$ and $\alpha = \frac{n\pi}{L}$.

Therefore, the solution to ① can be written as

$$u(x,t) = \left(c_3 \cos \frac{a\pi n}{L} t + c_4 \sin \frac{a\pi n}{L} t\right)\cdot \sin \frac{n\pi}{L} x$$

$$u_n(x,t) = \left(A_n \cos \frac{a\pi n}{L} t + B_n \sin \frac{a\pi n}{L} t\right)\cdot \sin \frac{n\pi}{L} x$$

Apply superposition principle, the solution can be written as

$$u(x,t)=\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi}{L}t+B_n\sin\frac{an\pi}{L}t\right)\cdot\sin\frac{n\pi}{L}x \quad \text{—— (2)}$$

Thus, the solution given above by (2) satisfies the conditions,

$$u(0,t)=0 \text{ and } u(L,t)=0$$

$$u(x,0)=\frac{1}{4}x(L-x)$$

However, this has also to satisfy the initial condition

$$u(x,0)=\frac{1}{4}x(L-x)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x \quad \text{—— (3)}$$

Therefore, on taking $t=0$ in (2), gives the series on the right being the half range sine series, A_n can be evaluated by L

$$A_n=\frac{2}{L}\int_0^L\frac{1}{4}x(L-x)\cdot\sin\frac{n\pi}{L}x\,dx$$

$$=\frac{1}{2L}\int_0^L Lx\sin\left(\frac{n\pi}{L}x\right)dx-\frac{1}{2L}\int_0^L x^2\sin\left(\frac{n\pi}{L}x\right)dx$$

$$=\frac{1}{2L}\left[\frac{-L^2x}{n\pi}\cos\frac{n\pi}{L}x+\frac{L^3}{n^2\pi^2}\sin\frac{n\pi}{L}x\right]_0^L$$

$$+\frac{1}{2L}\left[\frac{x^2L}{n\pi}\cos\frac{n\pi}{L}x\right]_0^L-\frac{2L}{n\pi}\int_0^L x\cos\frac{n\pi}{L}x\,dx$$

$$=-\frac{L^2}{2n\pi}\cos n\pi+\frac{L^2}{2n\pi}\cos n\pi-\frac{1}{n\pi}\left[\frac{xL}{n\pi}\sin\frac{n\pi}{L}x+\frac{L^2}{n^2\pi^2}\cos\frac{n\pi}{L}x\right]_0^L$$

$$=-\frac{L^2}{n^3\pi^3}\cdot\left[\cos n\pi-1\right]$$

$$= \frac{L^2(1-\cos n\pi)}{n^3\pi^3}$$

$$= \frac{L^2(1-(-1)^n)}{n^3\pi^3} \quad \text{------} \quad (4)$$

Next we use the condition $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ in (2)

leads to

$$0 = \sum_{n=1}^{\infty} B_n \cdot \frac{a\pi n}{L} \cdot \sin \frac{n\pi}{L} x$$

$$\Rightarrow B_n = 0 \quad \text{------} \quad (5)$$

Therefore, the solution to the equation (1) satisfying the given boundary conditions is

$$\boxed{u(x,t) = \frac{L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3} \sin\left(\frac{n\pi}{L} x\right) \cos\left(\frac{a\pi n}{L} t\right)}$$

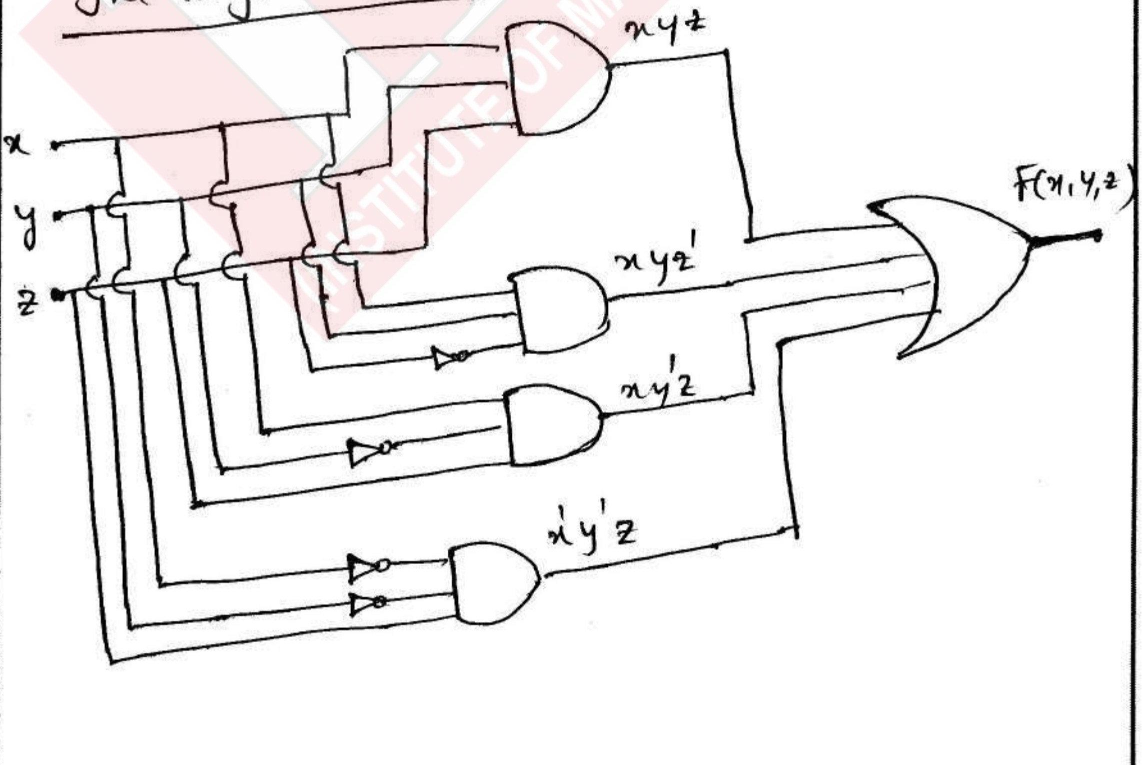
6(b) Obtain the Boo function $f(x,y,z)$ based on the table given below. Then simplify $f(x,y,z)$ and draw the corresponding GATE network:

x	y	z	f(x,y,z)	
1	1	1	1	xyz
1	1	0	1	xyz'
1	0	1	1	$xy'z$
1	0	0	0	
0	1	1	1	$x'y'z$
0	1	0	0	
0	0	1	0	
0	0	0	0	

Since Minterms are xyz, xyz', $xy'z$, $x'y'z$

So, $f(x,y,z) = xyz + xyz' + xy'z + x'y'z$

The logic circuit is:



Reducing

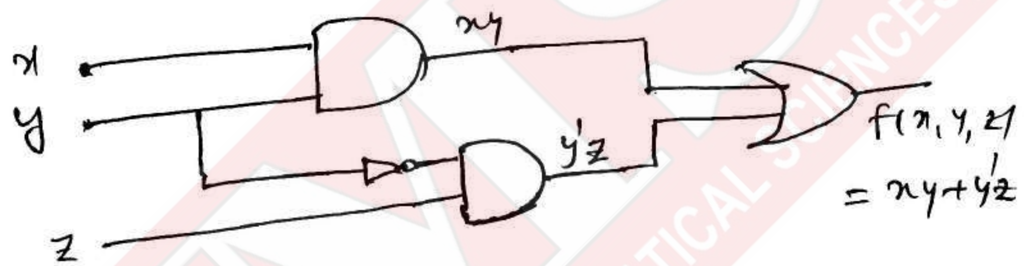
$f(x,y,z) = xyz + xyz' + xy'z + x'y'z$

$= xy(z+z') + y'z(x+x')$

$= xy(1) + y'z(1) \quad \left(\because x+x'=1,\ z+z'=1\right)$

$= xy + y'z$

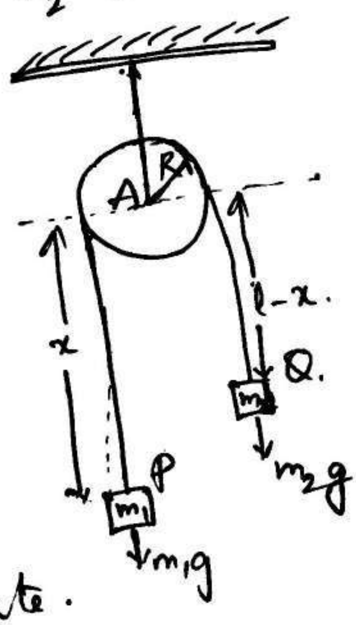
Simple circuit is:



6(c) → obtain the Lagrangian equation for the motion of a system of two particles of unequal masses connected by an inextensible string passing over a small smooth pulley.

Solⁿ: Let the system of two particles of unequal masses m_1 and m_2 connected by an inextensible string passing over a small smooth pulley of radius R, i.e., both the ends of the string are attached the masses m_1 and m_2 respectively.

Let the length of the string between m_1 and m_2 be l.



Then we have from figure that $PA = x$ and $QA = l - x$.

The system has only one degree of freedom and x is the only generalized co-ordinate.

$\therefore$ The instantaneous configuration is specified by $q = x$.

Assuming that the cord doesn't slip, the angular velocity of the pulley is $\frac{\dot{x}}{R}$.

Hence the K.E of the system is given by

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{R^2}$$

The potential energy of the system is

$$V = -m_1 g x - m_2 g(l - x)$$

And the Lagrangian is

$$L = T - V = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2 gl$$

$$\frac{\partial L}{\partial \dot{x}} = \left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x} \quad ; \quad \frac{\partial L}{\partial x} = (m_1 - m_2)g$$

Lagrange's equation of motion

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ becomes

$$\frac{d}{dt}\left[\left(m_1 + m_2 + \frac{I}{R^2}\right)\dot{x}\right] - g(m_1 - m_2) = 0$$

$$\Rightarrow \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g$$

$$\Rightarrow \ddot{x} = \frac{(m_1 - m_2)g}{\left(m_1 + m_2 + \frac{I}{R^2}\right)}.$$

7(a) → Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

Solution:

The given equation can be re-written as

$(D-D')(D+D'-3)z = xy + e^{x+2y}$

$\therefore$ C.F. $= \phi_1(y+x) + e^{3x}\phi_2(y-x)$,

ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to xy

$$= \frac{1}{(D-D')(D+D'-3)}xy = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)^{-1}xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\right]xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left(1+\frac{D+D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{D'}{D}+\frac{D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D+D'-3)(D-D')}e^{x+2y} = \frac{1}{(D+D'-3)}\cdot\frac{1}{(1-2)}\cdot e^{x+2y}$$

$$= -\frac{1}{(D+D'-3)} e^{1x+2y} \cdot 1$$

$$= -e^{1.x+2y} \frac{1}{(D+1)+(D'+2)-3} \cdot 1$$

$$= -e^{x+2y} \cdot \frac{1}{D+D'} \cdot 1$$

$$= -e^{x+2y} \frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1} \cdot 1$$

$$= -e^{x+2y} \frac{1}{D}(1+\cdots) \cdot 1$$

$$= -x e^{x+2y}$$

Hence,
the required general solution is

$$z = C.F + P.I.$$

i.e., $$\boxed{z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - (1/6)x^2y - (1/6)xy - (1/9)x^2 - (1/18)x^3 - (2/27)x - xe^{x+2y}.}$$

Hence the result.

7(b) → Solve the system of equations

$$3x_1+9x_2-2x_3=11$$
$$4x_1+2x_2+13x_3=24$$
$$4x_1-2x_2+x_3=-8$$

correct upto 4 significant figures by using Gauss-Seidel method after verifying whether the method is applicable in your transformed form of the system.

Sol^n: Here we use pivoting By taking initially
Let $X^{(0)}=(x_1^{(0)},x_2^{(0)},x_3^{(0)})=(0,0,0)$, be an initial approximation.

$$4x_1-2x_2+x_3=-8$$
$$3x_1+9x_2-2x_3=11$$
$$4x_1+2x_2+13x_3=24$$

Let us Apply Gauss seidel method by using partial pivoting:

$$x_1^{K+1}=\frac{1}{4}(-8+2x_2^{K}-x_3^{K})$$

and $x_2^{K+1}=\frac{1}{9}(11-3x_1^{K+1}+2x_3^{K})$

$$x_3^{K+1}=\frac{1}{13}(24-2x_2^{K+1}-4x_1^{K+1})$$

First Iteration $K=0$

$$x_1^{(1)}=\frac{1}{4}(-8)=-2$$

$$x_2^{(1)}=\frac{1}{9}(11+6)=1.8888$$

$$x_3^{(1)}=\frac{1}{13}(24-2(1.8888)-4(-2))$$
$$=\frac{1}{13}[32-3.7776]$$
$$=2.1709$$

2nd Iteration $K=1$

$$x_1^{(2)} = \frac{1}{4}\left(-8+2(1.8888)-2.1709\right) = -1.5983$$

$$x_2^{(2)} = \frac{1}{9}\left[11-3(-1.5983)+2(2.1709)\right] = 2.2324$$

$$x_3^{(2)} = \frac{1}{13}\left[24-2(2.2324)-4(-1.5983)\right] = 1.9944$$

3rd Iteration $K=2$

$$x_1^{(3)} = \frac{1}{4}\left(-8+2(2.2324)-1.9944\right) = -1.3824$$

$$x_2^{(3)} = \frac{1}{9}\left[11-3(-1.3824)+2(1.9944)\right] = 2.1262$$

$$x_3^{(3)} = \frac{1}{13}\left[24-2(2.1262)-4(-1.3824)\right] = 1.9444$$

4th Iteration $K=3$

$$x_1^{(4)} = \frac{1}{4}\left[-8+2(2.1262)-1.9944\right] = -1.4355$$

$$x_2^{(4)} = \frac{1}{9}\left[11-3(-1.4355)+2(1.9444)\right] = 2.1328$$

$$x_3^{(4)} = \frac{1}{13}\left[24-2(2.1328)-4(-1.4355)\right] = 1.9597$$

NOTE:

Advise to the student continuing in this way upto the consecutive iterations having same four significant figures.

Let us try to Apply the Gauss Seidel Method withought using pivoting.

$$x_1^{(K+1)} = \frac{1}{3}\left(11-9x_2^{(K)}+2x_3^{(K)}\right)$$

$$x_2^{(K+1)} = \frac{1}{2}\left(24-4x_1^{(K+1)}-13x_3^{(K)}\right)$$

$$x_3^{(k+1)} = \left(-8 + 2x_2^{(k+1)} - 4x_1^{(k+1)}\right)$$

let $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0,0,0)$ be an initial approximation.

1st Iteration: $x_1^{(1)} = \frac{1}{3}(11) = 3.6666$

$$x_2^{(1)} = \frac{1}{2}[24 - 4(3.6666)] = 4.6668$$

$$x_3^{(1)} = -8 + 2(4.6668) - 4(3.6666) = -13.3328$$

2nd Iteration:

$$x_1^{(2)} = \frac{1}{3}[11 - 9(4.6668) + 2\cdot(-13.3328)] = -19.2222$$

$$x_2^{(2)} = \frac{1}{2}[24 - 4(-19.2222) - 13(-13.3328)]$$

$$= 137.1076$$

$$x_3^{(2)} = [-8 + 2(137.1076) - 4(-19.2222)]$$

$$= 343.104$$

etc.

Clearly from the above, we observe that withought pivoting between iterations the approximate values diverging largely. but this is not the case by applying the pivoting. It means that the Gauss Seidel Method is strictly applicable by using pivotal process.

7.(C) → Show that $\vec{q} = \dfrac{\lambda(-y\hat{i}+x\hat{j})}{x^2+y^2}$, ($\lambda$ = constant) is a possible incompressible fluid motion. Determine the streamlines. Is the kind of the motion potential? If yes, then find the velcity potential.

Solⁿ: We know that, $\nabla \cdot q = 0$ ——— ①

or $\lambda\left\{-\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right)+\dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right)\right\}=0,$

or $\lambda\left\{\dfrac{2xy}{(x^2+y^2)^2}-\dfrac{2xy}{(x^2+y^2)^2}\right\}=0,$

which is evident. Thus the equation of continuity for an incompressible fluid is satisfied and hence it is a possible motion for an incompressible fluid.

The equation of the streamlines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

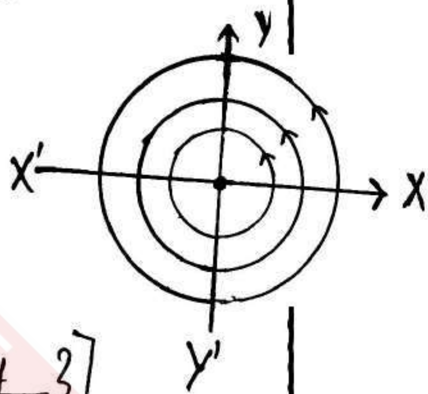
or $\dfrac{dx}{-\lambda y/(x^2+y^2)}=\dfrac{dy}{\lambda x/(x^2+y^2)}=\dfrac{dz}{0}$

or $x\,dx+y\,dy=0,\ dz=0$

By integrating, we have

$$x^2+y^2=\text{constant}, \text{ ——— } ②$$
$$z=\text{constant} \text{ ——— } ③$$

Thus the streamlines are circles whose centres are on z-axis, their planes being perpendicular to the axis.



Again $\nabla\times q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{\lambda y}{(x^2+y^2)} & \frac{\lambda x}{(x^2+y^2)} & 0 \end{vmatrix}$

or $\nabla\times q = k\left[\frac{\partial}{\partial x}\left\{\frac{\lambda x}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{\lambda y}{x^2+y^2}\right\}\right]$

or $\nabla\times q = k\lambda\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0.$

Thus the flow is of potential kind, so we can determine $\phi(x,y,z)$ such that

$$q = -\nabla\phi$$

or $\frac{\partial\phi}{\partial x} = -u = \frac{\lambda y}{x^2+y^2}$, ——— (4)

$\frac{\partial\phi}{\partial y} = -v = -\frac{\lambda x}{x^2+y^2}$, ——— (5)

$\frac{\partial\phi}{\partial z} = -w = 0$, ——— (6)

which shows that ϕ is independent of z, hence $\phi = \phi(x,y)$.

Integrating the relation (4), we have

$$\phi(x,y) = \lambda\tan^{-1}(x/y) + f(y)$$

or $\frac{\partial\phi}{\partial y} = f'(y) - \lambda x/(x^2+y^2)$,

using the relation (5), we get

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}.$$

Therefore $\phi(x,y) = \lambda\tan^{-1}(x/y)$.

8(a) Find a complete integral of the partial differential equation $p=(z+qy)^2$ by using Charpit's method.

Sol^n: Here given equation is

$$f(x,y,z,p,q) = (z+qy)^2 - p \quad \text{——} (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\frac{dp}{2p(z+qy)} = \frac{dq}{4q(z+qy)} = \frac{dz}{p-2qy(z+qy)} = \frac{dx}{1} = \frac{dy}{-2y(z+qy)}$$

Taking the first and fifth fractions,

$$\frac{dp}{2p(z+qy)} = \frac{dy}{-2y(z+qy)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dy}{-y}$$

$$\Rightarrow \log p = \log a - \log y$$

$$\Rightarrow \boxed{p = a/y} \quad \text{——} (2)$$

Substituting (2) in (1), we have

$(z+qy)^2 = \frac{a}{y}$

$\Rightarrow z+qy = \sqrt{a}/\sqrt{y}$

$\Rightarrow qy = \frac{\sqrt{a}}{\sqrt{y}} - z$

$\Rightarrow \boxed{q = \frac{\sqrt{a}}{y\sqrt{y}} - \frac{z}{y}}$ —— (3)

$\therefore dz = p\,dx + q\,dy$

$dz = \frac{a}{y}dx + \left(\frac{\sqrt{a}}{y\sqrt{y}} - \frac{z}{y}\right)dy$

$\Rightarrow y\,dz = a\,dx + \sqrt{a}\,y^{-1/2}dy - z\,dy$

$\Rightarrow y\,dz + z\,dy = a\,dx + \sqrt{a}\,y^{-1/2}dy$

$\Rightarrow d(yz) = a\,dx + \sqrt{a}\,y^{-1/2}dy$

Integrating,

$yz = ax + 2\sqrt{a}\sqrt{y} + b$,

i.e., $yz = ax + 2\sqrt{ay} + b$.

a, b being arbitrary constants.

8.(b) Derive Newton's backward difference interpolation formula and also do error analysis.

Soln: Let $y=f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots y_n$ corresponding to the values $x_0, x_1, x_2, \ldots x_n$ of the independent variable x. Let the values of x be equally spaced with h as the interval of differencing,

i.e., Let $x_r = x_0 + rh$, $r=0,1,2,\ldots n$.

Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, x_2 \ldots x_n$ i.e., $\phi(x)$ represents $y=f(x)$ such that $f(x_r) = \phi(x_r)$, $r=0,1,2,\ldots n$.

we may write $\phi(x)$ as

$$f(x) \approx \phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + \cdots + a_n(x-x_n)(x-x_{n-1})\cdots\cdots(x-x_1) \quad \text{——— (1)}$$

Putting $x = x_n$ in (1) we get

$$f(x_n) \approx \phi(x_n) = a_0$$

$$\Rightarrow y_n = a_0$$

Putting $x = x_{n-1}$ in (1) we get

$$f(x_{n-1}) \approx \phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow y_{n-1} = y_n + a_1(-h)$$

$$\Rightarrow a_1 h = y_n - y_{n-1} = \nabla y_n$$

$$\Rightarrow a_1 = \nabla y_n / 1!\,h$$

Putting $x = x_{n-2}$ in ① we get

$$f(x_{n-2}) \approx \phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow y_{n-2} = y_n + \left(\frac{y_n - y_{n-1}}{h}\right)(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow y_{n-2} = y_n - 2y_n + 2y_{n-1} + (2h^2)a_2$$

$$\Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!\,h^2}$$

Similarly putting $x = x_{n-3}$, $x = x_{n-4}, \ldots\ldots,$ we get

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}, \quad a_4 = \frac{\nabla^4 y_n}{4!\,h^4}, \ldots\ldots a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

Substituting these values in ①

$$f(x) \approx \phi(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x - x_n)(x - x_{n-1}) + \frac{\nabla^3 y_n}{3!\,h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2})$$
$$+ \cdots\cdots + \frac{\nabla^n y_n}{n!\,h^n}(x - x_n)(x - x_{n-1})\cdots\cdots(x - x_1) \quad \text{——— ②}$$

writing $u = \frac{x - x_n}{h}$ we get

$$x - x_n = uh$$

$$\therefore x - x_{n-1} = x - x_n + x_n - x_{n-1} = (uh) + h = (u+1)h$$

$$\Rightarrow x - x_{n-2} = (u+2)h, \ldots, (x - x_1) = u + n - 1)h$$

∴ The equation (2) may be written as

$$f(x) \approx \phi(x) = y_n + \frac{u \nabla y_n}{1!} + \frac{u(u+1)}{2!} \nabla^2 y_n + \frac{u(u+1)(u+2)}{3!} \nabla^3 y_n + \cdots\cdots + \frac{u(u+1)(u+2)\cdots(u+n-1)}{n!} \nabla^n y_n$$

The error in the Newton's backward interpolation formula as

$$R(x) = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1)\cdots\cdots(u+n)$$

where $uh = x - x_n$.

8.(c) → Show that for the complex potential $\tan^{-1}z$, the streamlines and equipotential curves are circles. Find the velocity at any point and check the singularities at $z=\pm i$.

Sol^n: The complex potential is given by

$$w = \phi + i\psi = \tan^{-1}z, \quad \text{---(1)}$$

Also $$\bar{w} = \phi - i\psi = \tan^{-1}\bar{z}, \quad \text{---(2)}$$

By subtracting (1) and (2), we have

$$2i\psi = \tan^{-1}z - \tan^{-1}\bar{z}$$

$$= \tan^{-1}\frac{z-\bar{z}}{1+z\bar{z}}$$

or $$\tan 2i\psi = \frac{2iy}{1+x^2+y^2}$$

$$\Rightarrow x^2+y^2+1 = 2y\coth 2\psi.$$

The streamlines $\psi=$ constant represent the circles.

$$x^2+y^2+1 = 2y\coth 2\psi. \quad \text{---(3)}$$

Similarly, by adding (1) and (2), we have

$$2\phi = \tan^{-1}z + \tan^{-1}\bar{z} = \tan^{-1}\frac{z-\bar{z}}{1-z\bar{z}}$$

or $$1-x^2-y^2 = 2x\cot 2\phi. \quad \text{---(4)}$$

The equi-potentials $\phi=$ const. also represent circles which are orthogonal to the streamlines $\psi=$ const. and form a co-axial system with limit points at $z=\pm i$. The velocity component (u,v) are given by

$$\frac{dw}{dz} = -u+iv = \frac{1}{z^2+1} \quad \text{——— (5)}$$

The denominator vanishes at $z=\pm i$, therefore, it represents the singularities at these points.

At $z=+i$, substitute $z=i+z_1$, where $|z_1|$ is very small

$$-u+iv = \frac{dw}{dz} = \frac{dw}{dz_1} = \frac{1}{1+(-1+2iz_1)} = \frac{1}{2iz_1}$$

By integrating, we have

$$w = -\frac{1}{2} i \log z_1$$

$\Rightarrow$ that the singularity at $z=i$ is a vortex of strength $K=-\frac{1}{2}$ with circulation $-\pi K$,

Similarly, the singularity at $z=-i$ is a vortex of strength $K=\frac{1}{2}$ with circulation πK.