MAINS TEST SERIES-2021
TEST- 9
FULL SYLLABUS (PAPER- I)
ANSWER KEY

1(a) Suppose U and W are distinct four dimentional subspaces of a vector space 'V', where dim V = 6. Find the possible dimension of subspace $U \cap W$?

Sol: Given; U and W are distinct Subspaces; such that

$$\left.\begin{array}{l} \dim(U) = \dim(W) = 4 \\ \dim(V) = 6 \end{array}\right\} \text{ of a vector space } V. \quad \text{——(1)}$$

we know that

$$U, W \subseteq U+W \subseteq V \quad \text{——(2)}$$

Using formula.

$$\dim(U) + \dim(W) - \dim(U\cap W) = \dim(U+W)$$

$$\dim(U\cap W) = -\dim(U+W) + \dim(U) + \dim(W)$$

$$\dim(U\cap W) = -\dim(U+W) + 4 + 4$$

$$\dim(U\cap W) = 8 - \dim(U+W) \quad \text{——(3)}$$

from (2)

$$\dim(U) = 4 \leq \dim(U+W) \leq 6 \; (=\dim(V)) \quad \text{——(4)}$$

Using (3) and (4) we get

$$2 \leq \dim(U\cap W) \leq 4$$

Since; U, W are distinct

$$U\cap W \neq U \text{ or } W \text{ or } \dim(U\cap W) \neq 4$$

Hence; $\dim(U\cap W) = \underline{\underline{2 \text{ or } 3}}$

1.(b) Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix $\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

Sol^n: Let $A = \begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

then $A+A^{\theta}$ is Hermitian and

$A-A^{\theta}$ is skew-Hermitian.

$\therefore \frac{1}{2}(A+A^{\theta})$ is Hermitian and

$\frac{1}{2}(A-A^{\theta})$ is skew-Hermitian.

Given that $A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$

$= P+Q$ (say)

where P is Hermitian and

Q is skew-Hermitian.

To find P and Q:

Now $A^{\theta} = (\bar{A})^T$

$$= \begin{bmatrix} -2i & 3 & -1 \\ 1 & 2-3i & 2 \\ +i+1 & 4 & -5i \end{bmatrix}^T$$

$$= \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}$$

we have

$$P = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} + \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix} \right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & i \\ 4 & 4 & 6 \\ -i & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$$

now we have

$$Q = \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} - \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix} \right)$$

$$= \frac{1}{2}\begin{bmatrix} 4i & 2 & -i-2 \\ -2 & 6i & -2 \\ -i+2 & 2 & 10i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 1 & -\frac{i}{2}-1 \\ -1 & 3i & -1 \\ -\frac{i}{2}+1 & 1 & 5i \end{bmatrix}$$

$\therefore$ The required Hermitian and Skew-Hermitian matrices are $\begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$ and

$\begin{bmatrix} 2i & 1 & -\frac{i}{2}-1 \\ -1 & 3i & -1 \\ -\frac{i}{2}+1 & 1 & 5i \end{bmatrix}$ respectively.

1.(c) Let $f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 1 & ; \text{if } (x,y) = (0,0) \end{cases}$

Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$.

Solution:-

Given; $f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 1 & ; \text{if } (x,y) = (0,0) \end{cases}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\frac{(h+0)^2}{h^2+0^2} - 1\right]}{h}$$

$$= \lim_{h\to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\left[\frac{k^2}{k^2} - 1\right]}{k}$$

$$= \lim_{k\to 0} \frac{1-1}{k} = 0$$

Therefore; $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$.

To check continuity

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$$

Assuming (x,y) approaches $(0,0)$ along $y=mx$

$y = mx$

$$= \underset{x \to 0}{\mathrm{lt}} \frac{(x+mx)^2}{x^2+m^2x^2}$$

$$= \underset{x \to 0}{\mathrm{lt}} \frac{\cancel{x^2}(1+m)^2}{\cancel{x^2}(1+m^2)}$$

$$\boxed{\underset{(x,y)\to(0,0)}{\mathrm{lt}} f(x,y) = \frac{(1+m)^2}{1+m^2}.}$$

Limit depends on 'm' thus limit does not exist and $f(x,y)$ is not continuous at $(0,0)$.

1.(d) Evaluate $\displaystyle\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$

Sol^n $\displaystyle I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx \quad — (1)$

We know $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4(\pi/2-x)+\cos^4(\pi/2-x)}\,dx$$

$$I = \int_0^{\pi/2} \frac{(\pi/2-x)\cos x\sin x}{\cos^4 x+\sin^4 x}\,dx \quad — (2)$$

Adding eq^n (1) and (2), we get -

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}\,dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{1-\frac{\sin^2 2x}{2}}\,dx \quad \left[\begin{array}{l}\because \sin 2x = 2\sin x\cos x \\ \& \sin^2 x+\cos^2 x=1\end{array}\right.$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{2\sin x\cos x}{1+\cos^2 2x}\,dx$$

$$\Rightarrow I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin 2x}{1+\cos^2 2x}\,dx$$

Let $\cos 2x = t \Rightarrow -2\sin 2x\,dx = dt$

at $x = 0 \Rightarrow t = 1$

$x = \pi/2 \Rightarrow t = -1$

$$\Rightarrow I = \frac{\pi}{4}\int_1^{-1} \frac{(-\frac{1}{2})dt}{1+t^2}$$

$$\Rightarrow I = \frac{\pi}{8}\int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{8}\left[\tan^{-1}t\right]_{-1}^{1}$$

$$\Rightarrow I = \frac{\pi}{8}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow \boxed{I = \frac{\pi^2}{16}}$$

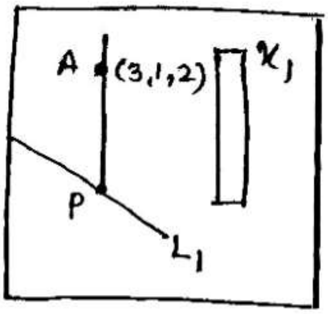
$$\Rightarrow \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx = \frac{\pi^2}{16}$$

Ans

1.(e) Find the equations of the straight line through the point $(3,1,2)$ to intersect the straight line $x+4=y+1=2(z-2)$ and parallel to the plane $4x+y+5z=0$.

Solution:-

Given, point $(3,1,2)$ to intersect the straight line $x+4=y+1=2(z-2)$.



$\therefore L_1 \Rightarrow \frac{x+4}{2}=\frac{y+1}{2}=\frac{z-2}{1}$

and the general point on L_1, let P be the point

$$\frac{x+4}{2}=\frac{y+1}{2}=\frac{z-2}{1}=r$$

$$P\,(2r-4,\ 2r-1,\ r+2)$$

Let, straight line through $(3,1,2)$ meet the given line 'p', therefore Direction ratio's of line AP to be determined by.

$AP\,(2r-4-3,\ 2r-1-1,\ r+2-2] \Rightarrow AP(2r-7, 2r-2, r)$

The line is parallel to the plane $4x+y+5z=0$

$\therefore$ Dr's of normal to the plane is $\perp^r$ to Dr's of determined line thus

$$(2r-7)4+2r-2+5r=0 \Rightarrow 15r=30 \Rightarrow \boxed{r=2}$$

$\therefore$ Dr's of $AP=(-3,2,2)$

And Point $P=(0,3,4)$

Therefore; straight line AP, which intersect $(3,1,2)$.

$$\left[\frac{x-3}{-3}=\frac{y-1}{2}=\frac{z-2}{2}\right]$$

2(a) (i) Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$.
Let $W=\{A\in V \mid \text{Trace } A=0\}$. Show that W is a subspace of V. Find a basis of W and dimention of W.

(ii) Find the dimention and a basis of the solution space W of the system

$$x+2y+2z-s+3t=0,\quad x+2y+3z+s+t=0,$$
$$3x+6y+8z+s+5t=0.$$

Sol^n: (ii) we write the single matrix equation is $AX=0$ —— (1)

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A=\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}$

$$\sim\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix}\quad \begin{array}{l} R_2\to R_2-R_1 \\ R_3\to R_3-3R_1 \end{array}$$

$$\sim\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\quad R_3\to R_3-2R_2$$

Clearly it is in echelon form

Again we write a single matrix equation by using above, we get

$$x + 2y + 2z - s + 3t = 0$$

$$z + 2s - 2t = 0$$

$$\Rightarrow \boxed{z = 2t - 2s} \text{ and}$$

$$\boxed{x = -2y + 5s - 7t}$$

Clearly y, s, t are free variables.

$\therefore$ dimension of $W = 3$

(i) Let $y = 1, s = 0, t = 0$ to obtain the solution

$$v_1 = (-2, 1, 0, 0, 0)$$

(ii) Let $y = 0, s = 1, t = 0$ to obtain the solution

$$v_2 = (5, 0, -2, 1, 0)$$

(iii) Let $y = 0, s = 0, t = 1$ to obtain the solution

$$v_3 = (-7, 0, 2, 0, 1)$$

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space W.

2(a)(i) Sol^n: Given, V is a vector space of 2×2 matrices over R.

i.e, $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\}$

and W is a subset of V such that Trace(A)=0 when $A \in W$.

Clearly $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

i.e, W is not empty.

Now let $A_1, A_2 \in W$

then, Trace $(A_1) = 0$

and Trace $(A_2) = 0$

then $tr(xA_1 + yA_2) = x\,Tr(A_1) + y\,Tr(A_2)$

$= x.0 + y.0 = 0$

$\Rightarrow xA_1 + yA_2 = 0$

$\Rightarrow xA_1 + yA_2 \in W$

i.e. W is a subspace of

If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in W$

then $x + w = 0$

i.e, it can have at maximum three free variables.

Hence, dimention of $W = 4 - 1 = 3$

and the basis of W are $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

i.e. $W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

2(b)i) For all real numbers, $f(x)$ is given as

$$f(x) = \begin{cases} e^x + a\sin x & \text{if } x<0 \\ b(x-1)^2 + x-2 & \text{if } x \geq 0 \end{cases}$$, find values of a and b for which f is differentiable at $x=0$.

Sol'n: Since f is differentiable at $x=0$

hence f must be continuous at $x=0$.

Now $\underset{x\to 0-}{Lt} f(x) = \underset{x\to 0-}{Lt} e^x + a\sin x = 1$

$$\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{Lt} b(x-1)^2 + x-2 = b-2$$

Also $f(0) = b-2$

Since f is continuous at $x=0$

$$\therefore \underset{x\to 0-}{Lt} f(x) = \underset{x\to 0+}{Lt} f(x) = f(0)$$

$$\Rightarrow 1 = b-2$$

$$\Rightarrow b-2 = 1 \Rightarrow \boxed{b=-3}$$

Now $Lf'(0) = \underset{x\to 0-}{Lt} \dfrac{f(x)-f(0)}{x-0}$

$$= \underset{x\to 0-}{Lt} \frac{e^x + a\sin x - (b-2)}{x}$$

$$= \underset{x\to 0-}{Lt} \frac{e^x + a\sin x - 1}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{x\to 0-}{Lt} e^x + a\cos x \quad \text{(by using L'Hospitals rule)}$$

$$= 1+a$$

$$Rf'(0) = \lim_{x \to 0+} \frac{b(x-1)^2 + x - 2 - (b-2)}{x}$$

$$= \lim_{x \to 0+} \frac{3(x-1)^2 + x - 2 - 1}{x}$$

$$= \lim_{x \to 0+} \frac{3(x-1)^2 + x - 3}{x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0+} 6(x-1) + 1 \quad (\text{by L'Hospital's rule})$$

$$= -5$$

Since f is differentiable

$\therefore Lf'(0) = Rf'(0)$

$1 + a = -5$

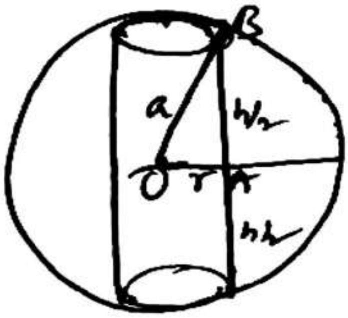
$\Rightarrow \boxed{a = -6}$.

$\therefore \boxed{a = -6}$ and $\boxed{b = 3}$

2(b)(ii) Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

Sol: Let O be the centre of the sphere of radius 'a'.

Let 'h' be the height and 'r' be the radius of the cylinder.



Then $OA = r = \sqrt{a^2 - h^2/4}$

The volume V of the cylinder $= \pi r^2 h$

$$= \pi\left(a^2 - \frac{h^2}{4}\right)h$$

$$\therefore \frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right)$$

for V to be maximum or minimum, we must have

$$\frac{dV}{dh} = 0.$$

i.e, $\pi\left(a^2 - \frac{3}{4}h^2\right) = 0 \Rightarrow h = \frac{2a}{\sqrt{3}}$

Also, $\frac{d^2V}{dh^2} = -\frac{6}{4}h = -\frac{3}{2}h < 0$ at $h = \frac{2a}{\sqrt{3}}$.

Hence V is maximum when $\boxed{h = \frac{2a}{\sqrt{3}}}$.

and maximum value $= \pi\left(a^2 - \frac{h^2}{4}\right)h$

$$= \pi\left[a^2 - \frac{1}{4}\left(\frac{2a}{\sqrt{3}}\right)^2\right]\frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}}.$$

2.(c) A sphere S has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of diameter. find the eqn of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle

sol^n:- Equation of the sphere S on joining the two given points $(0,1,0)$, $(3,-5,2)$ is given by

$$S \equiv (x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$$

$$\Rightarrow x^2+y^2+z^2-3x+4y-2z-5=0$$

Equation of any sphere having the intersection of the sphere S with the plane $P \equiv 5x-2y+4z+7=0$ is $S+\lambda P=0$

i.e, $(x^2+y^2+z^2-3x+4y-2z-5)+\lambda(5x-2y+4z+7)=0$ —— (1).

$$\Rightarrow x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z$$
$$+(-5+7\lambda)=0.$$

Its centre is $\left(\frac{3-5\lambda}{2}, \frac{2\lambda-4}{2}, \frac{2-4\lambda}{2}\right)$. —— (2)

Now if the given circle (which is the section of the sphere $x^2+y^2+z^2-3x+4y-2z-5$ by the plane $5x-2y+4z+7=0$) is a great circle of the sphere (1), the centre of the sphere (1) must lie on the plane

$5x-2y+4z+7=0.$

$\Rightarrow 5\left(\frac{3-5\lambda}{2}\right)-2\left(\frac{2\lambda-4}{2}\right)+4\left(\frac{2-4\lambda}{2}\right)+7=0$

$\Rightarrow 15-25\lambda-4\lambda+8+8-16\lambda+14=0$

$\Rightarrow 45-45\lambda=0$

$\Rightarrow \boxed{\lambda=1}.$

$\therefore$ from (1), we get

$x^2+y^2+z^2-3x+4y-2z-5+(5x-2y+4z+7)=0$

$\Rightarrow x^2+y^2+z^2+2x+2y+2z+2=0$

which is the required equation of the sphere

3.(a)(ii) → Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\}$. Hence find a basis for the subspace.

Sol.

Let $\mathbb{R}^4 = \{(x_1,x_2,x_3,x_4) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\}$ be the given vector space over the field $\mathbb{R}$.

Let W be the subspace of $\mathbb{R}^4$ spanned by S where

$$S=\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\} \subseteq$$

Let us construct a matrix A whose rows are the given vectors of S and convert it into the echelon form.

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&2&0&1\\0&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&0&1\\0&0&0&1\end{bmatrix} \quad R_3 \to R_3-R_1$$

$$\sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{bmatrix} \quad R_3 \to R_3-2R_2$$

$$\sim \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}. \quad R_4 \to R_4-R_3$$

Clearly which is in echelon form and the number of non-zero rows are equal to 3. Corresponding these rows the vectors of S $(1,0,0,0),(0,1,0,0),(1,2,0,1)$ form a basis of W.

i.e $S'=\{(1,0,0,0),(0,1,0,0),(1,2,0,1)\}$ is a maximum number of linearly independent subset of W.

and it forms a basis of W.

3.(b)i) If $z = xf(y/x) + g(y/x)$, show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

Sol^n : Given $z = xf(y/x) + g(y/x)$ —— ①

Differentiating ① w.r.t x, partially we get

$$\frac{\partial z}{\partial x} = f(y/x) + xf'(y/x)\left(\frac{-y}{x^2}\right) + g'(y/x)(-y/x^2)$$

$$= f(y/x) - \frac{y}{x} f'(y/x) - y/x^2\, g'(y/x)$$

Again Differentiating w.r.t x, partially

$$\frac{\partial^2 z}{\partial x^2} = f'(y/x)(-y/x^2) - \left[-y/x^2 f'(y/x) + y/x\, f''(y/x)(-y/x^2)\right]$$

$$-\left[y/x^2\, g''(y/x)(-y/x^2) + (-2y/x^3)\, g'(y/x)\right]$$

$$= -y/x^2 f'(y/x) + y/x^2 f'(y/x) + y^2/x^3 f''(y/x)$$

$$+ \frac{y^2}{x^4} g''(y/x) + 2y/x^3\, g'(y/x)$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} = \frac{y^2}{x} f''(y/x) + y^2/x^2\, g''(y/x) + \frac{2y}{x} g'(y/x) \text{ —— ②}$$

Differentiating ① w.r.t y, partially we get

$$\frac{\partial z}{\partial y} = xf'(y/x)(1/x) + g'(y/x)(1/x)$$

$$= f'(y/x) + g'(y/x)(1/x) \text{ —— ③}$$

Again Differentiating w.r.t y we get

$$\frac{\partial^2 z}{\partial y^2} = f''(y/x)(1/x) + g''(y/x)(1/x^2)$$

$$\Rightarrow y^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} f''(y/x) + y^2/x^2\, g''(y/x) \text{ —— ④}$$

Differentiating (3) w.r.t x, Partially, we get

$$\frac{\partial^2 z}{\partial x \partial y} = f''(y/x)\left(-y/x^2\right) - \frac{1}{x^2} g'(y/x) + \frac{1}{x} g''(y/x)\left(-y/x^2\right)$$

$$= -\frac{y}{x^2} f''(y/x) - \frac{1}{x^2} g'(y/x) - \frac{y}{x^3} g''(y/x)$$

$$\Rightarrow 2xy \frac{\partial^2 z}{\partial x \partial y} = 2xy\left[-\frac{y}{x^2} f''(y/x) - \frac{1}{x^2} g'(y/x) - \frac{y}{x^3} g''(y/x)\right]$$

$$= -\frac{2y^2}{x} f''(y/x) - \frac{2y}{x} g'(y/x) - \frac{2y^2}{x^2} g''(y/x) \quad \text{--- (5)}$$

from (2), (4) & (5)

we have

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

$$= \frac{y^2}{x} \cancel{f''(y/x)} + \frac{y^2}{x^2} \cancel{g''(y/x)} + \cancel{\frac{2y}{x} g'(y/x)}$$

$$- \cancel{\frac{2y^2}{x} f''(y/x)} - \cancel{\frac{2y}{x} g'(y/x)} - \cancel{\frac{2y^2}{x^2} g''(y/x)}$$

$$+ \cancel{\frac{y^2}{x} f''(y/x)} + \cancel{\frac{y^2}{x^2} g''(y/x)}$$

$$= 0$$

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$

3(b)(ii) Find constant a and b for which

$$F(a,b) = \int_0^\pi \{\sin x - (ax^2+bx)\}^2 dx$$ is

a minimum.

Sol^n $F(a,b) = \int_0^\pi \{\sin^2 x + (ax^2+bx)^2 - 2\sin x(ax^2+bx)\} dx$

$$= \int_0^\pi \frac{1-\cos 2x}{2} dx + \int_0^\pi (ax^2+bx)^2 dx - 2\int_0^\pi \sin x(ax^2+bx) dx$$

(I_1) (I_2) (I_3)

$I_1 = \pi/2$

$$I_2 = \frac{a^2\pi^5}{5} + \frac{b^2\pi^3}{3} + \frac{ab\pi^4}{2}$$

$I_3 = (a\pi^2 + b\pi - 4a)$

↳ solving 'by parts'

$$F(a,b) = \frac{\pi}{2} + \frac{a^2\pi^5}{5} + \frac{b^2\pi^3}{3} + \frac{ab\pi^4}{2} - 2a\pi^2 - 2b\pi + 8a$$

For minimum $\frac{\partial f}{\partial a} = 0$, $\frac{\partial f}{\partial b} = 0$

we get $\Rightarrow 5\pi^4 b + 4a\pi^5 - 20\pi^2 + 80 = 0$ — ①

$3a\pi^4 + 4b\pi^3 = 12\pi$ — ②

Solving eq^n ① and ② we get –

$$a = \frac{(20\pi^2 - 320)}{\pi^5}, \quad b = \frac{(240 - 12\pi^2)}{\pi^4}$$

4(a)(i) Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1. If $B = bA$ is orthogonal determine the values of a and b.

Sol'n: Given that $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$

$$B = bA = \begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix}$$

As matrix B is orthogonal matrix.

$$B \cdot B^T = I$$

$$\begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix}\begin{bmatrix} ab & b & b \\ b & ab & b \\ b & b & ab \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a^2b^2+2b^2 & 2ab^2+b^2 & 2ab^2+b^2 \\ 2ab^2+b^2 & a^2b^2+2b^2 & 2ab^2+b^2 \\ 2ab^2+b^2 & 2ab^2+b^2 & a^2b^2+2b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow a^2b^2 + 2b^2 = 1$ and $2ab^2 + b^2 = 0$

from $(2a+1)b^2 = 0$

$b \neq 0$

$\therefore 2a + 1 = 0$

$\Rightarrow a = -\frac{1}{2}$

Again, $b^2(a^2+2) = 1$

$$b^2\left(\left(-\tfrac{1}{2}\right)^2+2\right)=1$$

$$b^2\left(\tfrac{9}{4}\right)=1$$

$$b=\sqrt{\tfrac{4}{9}}$$

$$b=\pm\tfrac{2}{3}$$

$$\therefore a=-\tfrac{1}{2},\quad b=\pm\tfrac{2}{3}$$

4(a)(ii) If $M_2(\mathbb{R})$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomial of degree at most 2, then find the matrix representation of $T: M_2(\mathbb{R}) \longrightarrow P_2(x)$ such that

$$T = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$$

with respect to the standard bases of $M_2(\mathbb{R})$ and $P_2(x)$. Further find the null space of T.

Solution:- Given that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2 \quad \text{—} ①$$

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$T = \{1, x, x^2\}$ be the standard bases of $M_2(\mathbb{R})$ and $P_2(x)$.

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1+x+0x^2 = 1(1)+1(x)+0(x^2)$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0+0x+x^2 = 0(1)+0(x)+1(x^2)$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 1+0x+1x^2 = 1(1)+0(x)+(1)x^2$$

$$T\left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0-1x+0x^2 = 0(1)+(-1)x+0(x^2)$$

∴ The required Matrix of linear transformation.

$$[T: M_2, P_2] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

If $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0 = 0+0x+0x^2$

Then; $(a+c)+(a-b)x+(b+c)x^2 = 0+0x+0x^2$

$$\left.\begin{matrix} a+c=0 \\ a-d=0 \end{matrix}\right] \Rightarrow \quad c+d=0 \quad \text{—} ③$$

$a-d=0$ — ①

$b+c=0$ — ②

③ − ② ⇒ $b-d=0$ ⇒ $\boxed{b=d}$

① ≡ $\boxed{a=d}$

3 ≡ $\boxed{c=-d}$

∴ The null space of T

$$N(T) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ \begin{matrix} b=d \\ a=d \\ c=-d \end{matrix} \right\} = \left\{ \begin{bmatrix} d & d \\ -d & d \end{bmatrix} \Big/ d \in \mathbb{R} \right\}$$

$$\boxed{\therefore N(T) = \left\{ \begin{bmatrix} d & d \\ -d & d \end{bmatrix} \Big/ d \in \mathbb{R} \right\}}$$

4.(b) Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ from the origin, by the method of Lagrange's multipliers.

Solution:-

Let (x,y,z) be the point on line formed by intersection of $\pi_1: 3x+4y+5z=7$ & $\pi_2 = x-z=9$

Distance from origin be $d^2 = x^2+y^2+z^2$

Using Lagrange's method to evaluate minimum value of $x^2+y^2+z^2$ subject to π_1 & π_2

$$F(x,y,z) = x^2+y^2+z^2+\lambda_1(3x+4y+5z-7)+\lambda_2(x-z-9)$$

$$\frac{\partial f}{\partial x} = 2x+\lambda_1 3+\lambda_2 = 0 \quad \text{— (1)}$$

$$\frac{\partial f}{\partial y} = 2y+4\lambda_1 = 0 \quad \text{— (2)} \Rightarrow \boxed{\lambda_1 = -\frac{y}{2}}$$

$$\frac{\partial f}{\partial z} = 2z+\lambda_1(5)-\lambda_2 = 0 \quad \text{— (3)}$$

Adding (1) and (3), we get

$$2(x+z)+8\lambda_1 = 0$$

$$\lambda_1 = -\frac{\cancel{2}(x+z)}{\cancel{8}4}$$

$$-\frac{y}{2} = -\frac{(x+z)}{4} \Rightarrow \boxed{2y = x+z} \quad \text{— (4)}$$

Solving (4) & π_2

$$x = \frac{2y+9}{2} \; ; \; z = \frac{2y-9}{2}$$

Put these in π_1.

4.(c) (i) The plane $x-2y+3z=0$ is rotated through a right-angle about its line of intersection with the plane $2x+3y-4z-5=0$, find the equation of the plane in its new position.

Sol'n: The equation of any plane through the line of intersection of the plane $x-2y+3z=0$ and $2x+3y-4z-5=0$ is

$$(x-2y+3z)+\lambda\,(2x+3y-4z-5)=0$$

$$\Rightarrow (1+2\lambda)x+(3\lambda-2)y+(3-4\lambda)z-5\lambda=0 \quad \text{— (1)}$$

It is given that the angle between the plane $x-2y+3z=0$ and (1) is a right angle, so the angle between their normals is a right angle. Also the d.r's of their normals are $1,-2,3$ and $1+2\lambda,\ 3\lambda-2,\ 3-4\lambda$.

$$\therefore\ 1(1+2\lambda)-2(3\lambda-2)+3(3-4\lambda)=0$$

$$\Rightarrow 1+2\lambda-6\lambda+4+9-12\lambda=0$$

$$\Rightarrow 16\lambda=14 \Rightarrow \lambda=7/8.$$

$\therefore$ from (1) the required equation is

$$\left[1+2\left(7/8\right)\right]x+\left[3\left(7/8\right)-2\right]y+\left[3-4\left(7/8\right)\right]z-5\left(7/8\right)=0$$

$$\Rightarrow 22x+5y-4z-35=0.$$

5.(a) Solve: $\frac{dy}{dx}=\frac{y^2(x-y)}{3xy^2-x^2y-4y^3}, y(0)=1$.

SOLUTION

Given equation is

$$\frac{dy}{dx} = \frac{y^2(x-y)}{3xy^2-x^2y-4y^3}$$

$y^2(x - y)\,dx - (3xy^2 - x^2y - 4y^3)\,dy = 0$

We need to check weather above equation is in exact form.

$$Mdx + Ndy = 0$$

$\therefore$ Let

$$M = y^2(x - y)$$

$$N = -(3xy^2 - x^2y - 4y^3)$$

$$\frac{\partial M}{\partial y} = 2yx - 3y^2$$

$$\frac{\partial N}{\partial x} = -3y^2 + 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \text{ exact form}$$

$\therefore$ General solution $\Rightarrow \int Mdx + \int Ndy = c$

y constant y terms not containing X

$$\int y^2(x-y)dx - \int\left(3xy^2 - x^2y - 4y^3\right)dy$$

Required solutions

$$\boxed{y^2\frac{x^2}{2} - y^3 + y^4 = c}$$

5(b) Find (i) $L\{F(t)\}$ and (ii) $L\{F'(t)\}$ for the function given by $F(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1 \end{cases}$

Sol^n: Given that $F(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ t, & t > 1. \end{cases}$ —— (1)

(i) By definition of Laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = \int_0^1 e^{-st} F(t)\,dt + \int_1^\infty e^{-st} F(t)\,dt$$

$$= \int_0^1 e^{-st}(2t)\,dt + \int_1^\infty e^{-st}\, t\,dt, \text{ using (1)}$$

$$= \left[(2t)\left(-\frac{1}{s}e^{-st}\right)\right]_0^1 - \int_0^1 2\left(-\frac{1}{s}e^{-st}\right)dt + \left[(t)\left(-\frac{1}{s}e^{-st}\right)\right]_1^\infty - \int_1^\infty (1)\left(-\frac{1}{s}e^{-st}\right)dt$$

(on integration by parts)

$$= -\frac{2}{s}e^{-s} + \frac{2}{s}\left[-\frac{1}{s}e^{-st}\right]_0^1 - \frac{1}{s}\lim_{t\to\infty} te^{-st} + \frac{1}{s}e^{-s} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_1^\infty$$

$$= -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^2} - \frac{1}{s}\lim_{t\to\infty}\left(\frac{t}{e^{st}}\right) + \frac{1}{s}e^{-s}$$

$$-\frac{1}{s^2}\lim_{t\to\infty} e^{-st} + \frac{1}{s^2}e^{-s}$$

$$= -\frac{1}{s}e^{-s} + \frac{2}{s^2} - \frac{1}{s^2}e^{-s} \quad \left[\because \text{ If } s>0, \text{ then } \lim_{t\to\infty} e^{-st} = 0 \right.$$

$$\left. \text{and } \lim_{t\to\infty}\frac{t}{e^{st}} = \lim_{t\to\infty}\frac{1}{se^{st}} = 0, \text{ by L Hospital's rule}\right]$$

$$\therefore L\{F(t)\} = \frac{2}{s^2} - \left(\frac{1}{s} + \frac{2}{s^2}\right)e^{-s}, \; s>0 \quad \text{—— (2)}$$

Finding $L\{F'(t)\}$: from ①, we have

$$F'(t) = \begin{cases} 2, & 0<t<1 \\ 1, & t>1 \end{cases}$$

Using the definition of Laplace transform, we get

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)dt = \int_0^1 e^{-st}F'(t)dt + \int_1^\infty e^{-st}F'(t)dt$$

$$= \int_0^1 e^{-st}.2\,dt + \int_1^\infty e^{-st}.1\,dt, \text{ using } ⑤$$

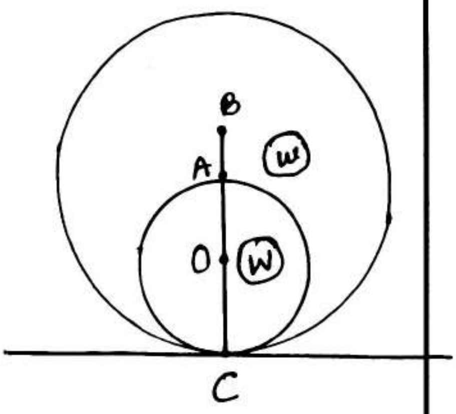
$$= 2\left[-\frac{e^{-st}}{s}\right]_0^1 + \left[\frac{-e^{-st}}{s}\right]_1^\infty$$

$$= -\frac{2}{s}e^{-s} + \frac{2}{s} - 0 + \frac{e^{-s}}{s}, \; s>0$$

$$= \frac{2}{s} - \frac{1}{s}\cdot e^{-s} = \frac{1}{s}(2-e^{-s}), \; s>0.$$

5.(c) A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$

Sol: C is the point of contact of the sphere and the spherical shell, O is the centre of the sphere, CA is the vertical diameter of the sphere and B is the centre of the spherical shell.



We have $OC = a$ and $BC = b$.

The weight W of the sphere acts at O and a particle of weight w is attached to A If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then

$$h = \frac{W\cdot a + w\cdot 2a}{W+w} = \frac{W+2w}{W+w}a$$

Here $\rho_1 = a$ and $\rho_2 = -b$

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

i.e. $$\frac{1}{h} > \frac{1}{a} - \frac{1}{b}$$

i.e. $$\frac{W+w}{a(W+2w)} > \frac{b-a}{ab}$$

i.e. $(W+w)ab > a(b-a)(W+2w)$

i.e. $(W+w)b > (b-a)(W+2w)$

i.e. $W\{b-(b-a)\} > w\{2(b-a)-b\}$

i.e. $Wa > w(b-2a)$

i.e. $\dfrac{W}{w} > \dfrac{b-2a}{a}$

5(d) → A particle of mass 'm', is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time 't'.

Solution:- Let a particle of mass 'm' falling under gravity be at a distance 'x' from the starting point, after time 't'. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e., in the direction of 'x' decreasing. The weight 'mg' of the particle acts vertically downwards i.e., in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

or $\frac{dv}{dt} = g - \frac{\mu}{m}v$, $\left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$

or $dt = \frac{dv}{g-(\mu/m)v}$

Integrating, we have

$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A$; where A = constant

But initially, when $t=0$, $v=0$;

$\therefore \boxed{A = (m/\mu)\log g}$

$$\therefore \quad t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + \frac{m}{\mu}\log g$$

$$t = -\frac{m}{\mu}\log\left\{\frac{g - (\mu/m)v}{g}\right\}$$

$$-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm}\cdot v\right)$$

$$1 - \frac{\mu}{gm}\cdot v = e^{-\frac{\mu t}{m}}$$

$$\Rightarrow \quad v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)$$

$$dx = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)dt$$

Integrating, we have.

$$\boxed{x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B} \quad \text{——— (1)}$$

B = constant

But initially, when $t=0$, $x=0$

$$\therefore \quad \boxed{0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B} \quad \text{——— (2)}$$

Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right]$$

$$\therefore \boxed{x = \frac{gm^2}{\mu^2}\left[e^{-\mu t/m} - 1 + \frac{\mu t}{m}\right]}$$

which is required result

5(e) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$.

Sol^n: The field F will be conservative if

$$\nabla \times F = 0$$

we have

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= i(0-0) - j(3z^2 - 3z^2) + k(2x - 2x)$$

$$= 0$$

$\therefore$ F is conservative force field.

Let $F = \nabla\phi$

$$\Rightarrow (2xy+z^3)i + x^2 j + 3xz^2 k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3$$

Integrating partially w.r.t x

$$\phi = x^2y + z^3x + f_1(y,z) \quad \text{—①}$$

where $f_1(y,z)$ is arbitrary function.

$\frac{\partial\phi}{\partial y} = x^2$ Integrating partially w.r.t y

$$\phi = x^2y + f_2(x,z) \quad \text{—②}$$

$\frac{\partial\phi}{\partial z} = 3xz^2$ Integrating partially w.r.t z

$$\phi = xz^3 + f_3(x,y) \quad \text{—③}$$

Eqⁿ ①, ②, ③ each represents ϕ.
These agree if we choose
$f_1(y,z) = 0$, $f_2(x,z) = z^3 x$, $f_3(x,y) = x^2 y$

$\therefore \phi = x^2 y + xz^3$ to which may be added any constant.

$$\therefore \phi = x^2 y + xz^3 + C$$

$$\text{workdone} = \int_{(1,-2,1)}^{(3,1,4)} F \cdot dr$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi$$

$$= [\phi]_{(1,-2,1)}^{(3,1,4)}$$

$$= [x^2 y + xz^3]_{(1,-2,1)}^{(3,1,4)}$$

$$= 202$$

6(a) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, where λ is a parameter

Sol: The given family of curves is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \text{ where } \lambda \text{ as a parameter} \quad \text{---(1)}$$

Differentiating (1) w.r.t. x, we get

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow x(b^2+\lambda)+y(a^2+\lambda)\frac{dy}{dx}=0$$

$$\Rightarrow \lambda\left(x+y\frac{dy}{dx}\right)=-\left(b^2x+a^2y\frac{dy}{dx}\right)$$

$$\Rightarrow \lambda=-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$\therefore a^2+\lambda=a^2-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$=\frac{(a^2-b^2)x}{x+y\frac{dy}{dx}}$$

and $$b^2+\lambda=b^2-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$=\frac{-(a^2-b^2)y\frac{dy}{dx}}{\left(x+y\frac{dy}{dx}\right)}$$

putting the above values of $a^2+\lambda$ and $b^2+\lambda$ in ①, we have

$$\frac{x^2\left\{x+y\frac{dy}{dx}\right\}}{(a^2-b^2)x} - \frac{y^2\left\{x+y\frac{dy}{dx}\right\}}{(a^2-b^2)\,y\frac{dy}{dx}} = 1$$

$$\Rightarrow \left\{x+y\frac{dy}{dx}\right\}\left\{x-y\frac{dx}{dy}\right\} = a^2-b^2 \quad \text{②}$$

which is the differential equation of the given family of curves ①

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ②, the differential equation of the required orthogonal trajectories is

$$\left\{x+y\left(-\frac{dx}{dy}\right)\right\}\left[x-y\left\{-\frac{dy}{dx}\right\}\right] = a^2-b^2$$

$$\Rightarrow \left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right) = a^2-b^2$$

which is same as the differential equation ② of the given family of curves ①.

Hence the system of given curves ① is self orthogonal, i.e, each member of the given family of curves intersects its own members orthogonally.

6(b) Solve $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$

$y(0)=0$ and $y'(0)=2$.

Sol^n: Given $[(1+2x)^2D^2-6(1+2x)D+16]y=8(1+2x)^2$ —— (1)

Let $(1+2x)=e^z$

$\log(1+2x)=z$

Also let $D_1\equiv d/dz$ —— (2)

Then $(1+2x)D=2D_1$, $(1+2x)^2D^2=2^2D_1(D_1-1)$

and so (1) becomes

$[2^2D_1(D_1-1)-6.2D_1+16]y=8e^{2z}$

$\Rightarrow(D_1-2)^2y=2e^{2z}$ —— (3)

Its auxiliary equation is $(D_1-2)^2=0$

$\Rightarrow D_1=2,2$

$\therefore$ C.F $=(C_1+C_2z)e^{2z}=(C_1+C_2z)(e^z)^2$

$=[C_1+C_2\log(1+2x)](1+2x)^2$

where C_1 and C_2 are arbitrary constants.

$P.I=\frac{1}{(D_1-2)^2}2e^{2z}=2\frac{z^2}{2!}e^{2z}$, as $\frac{1}{(D_1-a)^n}e^{az}=\frac{z^n}{n!}e^{az}$

$=z^2(e^z)^2=[\log(1+2x)]^2(1+2x)^2$, using (2)

$\therefore$ solution is $y=[C_1+C_2\log(1+2x)](1+2x)^2+[\log(1+2x)]^2(1+2x)^2$

$y=(1+2x)^2[C_1+C_2\log(1+x)+\{\log(1+2x)^2\}^2]$ —— (4)

$y(x)=(1+2x)^2[C_1+C_2\log(1+2x)+\{\log(1+2x)\}^2]$ —— (5)

Differentiating bothsides of (5) w.r.t x we have

$$y'(x) = (2\times 2)\,(1+2x)^2\left[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2\right] + (1+2x)^2\left[\frac{2c_2}{1+2x} + \frac{2\log(1+2x)}{1+2x}\times 2\right] \quad \text{—} (6)$$

$$y'(x) = (2\times 2)\,(1+2x)^2\left[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2\right] + (1+2x)^2\left[\frac{2c_2}{1+2x} + \frac{2\log(1+2x)}{(1+2x)}\times 2\right] \quad \text{—} (6)$$

putting $x=0$ in (5) & noting that $y(0)=0$ (given), we get

$$0 = 1\left[c_1 + (c_2\times 0) + 0^2\right] \quad \text{so that } c_1 = 0$$

putting $x=0$ in (6) and noting that $y'(0)=2$ (given)

we get $$2 = 4\left[c_1 + (c_2\times 0) + 0^2\right] + 1\left[2c_2 + (2\times 0\times 2)\right]$$

so that $c_2 = 1$, as $c_1 = 0$

putting the above values of c_1 and c_2 in (5), the required solution is

$$y(x) = (1+2x)^2 \log(1+2x)\left[1+\log(1+2x)\right].$$

6(c) Solve the differential equation $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$, where $p=dy/dx$, by reducing it to Clairaut's form using suitable substitution.

Sol^n: Given that

$$(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$$

put $x+y=u$, $x^2+y^2=v$.

$$\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx} \; ; \; 2x+2y\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow 1+p=\frac{du}{dx} \; ; \; 2x+2yp=\frac{dv}{dx}$$

$$\therefore \frac{dv}{du}=\frac{2(x+yp)}{1+p}$$

$$\Rightarrow P=\frac{2(x+yp)}{1+p} \; ; \text{ where } P=\frac{dv}{du}, \; p=\frac{dy}{dx}$$

$$\Rightarrow P(1+p)=2(x+yp)$$

$$\Rightarrow P+Pp=2x+2yp \Rightarrow P-2x=p(2y-P)$$

$$\Rightarrow p=\frac{P-2x}{2y-P} \quad \text{——— ①}$$

using ① the given equation becomes

$$(x^2+y^2)\left[1+\frac{P-2x}{2y-P}\right]^2-2(x+y)\left(1+\frac{P-2x}{2y-P}\right)\left(x+y\,\frac{P-2x}{2y-P}\right)+\left[x+y\left(\frac{P-2x}{2y-P}\right)\right]^2=0$$

$$\Rightarrow (x^2+y^2)\left[\frac{2y-2x}{2y-P}\right]^2-2(x+y)\left[\frac{2y-2x}{2y-P}\right]P\,\frac{(y-x)}{2y-P}+P^2\left(\frac{y-x}{2y-P}\right)^2=0$$

$$\Rightarrow (x^2+y^2)\,4(y-x)^2-4(x+y)(y-x)^2P+P^2(y-x)^2=0$$

$$\Rightarrow 4(x^2+y^2)-4P(x+y)+P^2=0$$

$$\Rightarrow 4v-4Pu+P^2=0$$

$$\Rightarrow v=Pu-\frac{P^2}{4}$$

which is of Clairaut's form and its solution is

$$v=uc-c^2/4$$

i.e, $x^2+y^2=(x+y)c-c^2/4$

6.(d) Use Laplace transform method to solve the following initial value problem:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t,\ x(0) = 2 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = -1$$

SOLUTION

Given $x'' - 2x' + x = e^t$

Applying Laplace transform on both sides

$$s^2\mathcal{L}(x) - sx(0) - x'(0) - 2s\,\mathcal{L}(x) + 2x(0) + \mathcal{L}(x) = \frac{1}{s-1}$$

(Given $x(0) = 2$ and $x'(0) = -1$)

$$(s^2 - 2s + 1)\,\mathcal{L}(x) - 2s + 1 + 4 = \frac{1}{s-1}$$

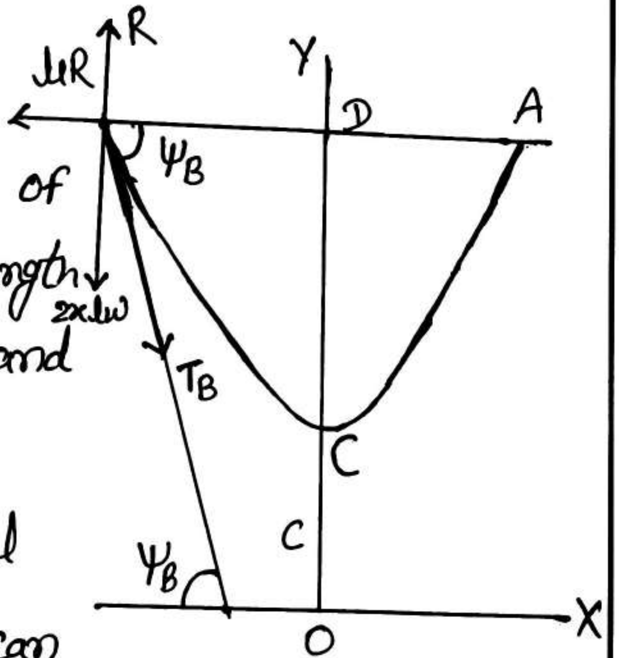
$$\mathcal{L}(x) = \frac{1}{(s-1)(s-1)^2} + \frac{2s-5}{(s-1)^2}$$

$$\mathcal{L}(x) = \frac{1}{(s-1)^3} + \frac{2}{(s-1)} - \frac{3}{(s-1)^2}$$

$$X(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^3} + \frac{2}{s-1} - \frac{3}{(s-1)^2}\right]$$

$$= e^t\left(\frac{t^2}{2} - 3t + 2\right)$$

7.(a) A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\{\lambda+\sqrt{(1+\lambda^2)}\}$, where $1/\lambda = \mu(2n+1)$ and μ is the coefficient of friction.



Sol: Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A. Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A.

In this position of limiting equilibrium the forces acting on the ring are:

(i) the weight $2nlw$ of the ring acting vertically downwards,

(ii) the normal reaction R of the rod,

(iii) the force of limiting friction μR of the rod acting in the direction AB, and

(iv) the tension T_B in the string at B acting along the tangent to the string at B.

For the equilibrium of the ring at B. resolving the forces acting on it horizontally and vertically, we have

$$\mu R = T_B \cos \psi_B \quad \text{——— (1)}$$

and $$R = 2nlw + T_B \sin \psi_B, \quad \text{——— (2)}$$

where ψ_B is the angle of inclination of the tangent at B to the horizontal.

Let C be the lowest point of the catenary formed by the chain, OX be the

directrix and $OC = c$ be the parameter.
we have arc $CB = s_B = l$. By the formula $T\cos\psi = wc$, we have $T_B \cos\psi_B = wc$. Also by the formula $T\sin\psi = ws$, we have

$$T_B \sin\psi_B = ws_B = wl.$$

Putting these values in ① and ②, we have

$$\mu R = wc \quad \text{and} \quad R = 2nlw + wl$$
$$= (2n+1)wl.$$

$$\therefore \mu(2n+1)wl = wc$$

or $\mu(2n+1)l = c$.

But it is given that

$$\mu(2n+1) = 1/\lambda$$

$$\therefore l/\lambda = c \quad \text{——— ③}$$

Using the formula $s = c\tan\psi$ for the point B, we have

$$l = c\tan\psi_B$$

$$\therefore \tan\psi_B = l/c = \lambda \quad \text{——— ④}$$

Now the required greatest possible distance of the ring from A

$$= AB = 2DB = 2x_B$$

$$= 2c\log(\sec\psi_B + \tan\psi_B)$$

$$[\because x = c\log(\sec\psi + \tan\psi)]$$

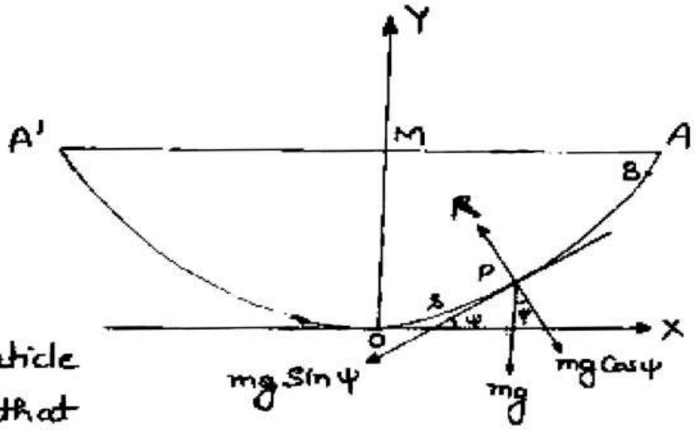
$$= 2c\log\left[\tan\psi_B + \sqrt{(1+\tan^2\psi_B)}\right]$$

$$= \frac{2l}{\lambda}\log\left[\lambda + \sqrt{(1+\lambda)^2}\right]$$

[$\because$ from 3, $c = l/\lambda$ and
from 4, $\tan\psi_B = \lambda$]

7.(b) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.

Sol^n Let a Particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc OP = s then the equations of motion of the particle are



$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{—— (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{—— (2)}$$

for the cycloid, $s = 4a\sin\psi$ —— (3)

from equ^n (1) and (3) we have –

$$\frac{d^2s}{dt^2} = \frac{-g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and integrating we have, $$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{-g}{4a}s^2 + A.$$

But initially at the cusp A,
$$s = 4a \text{ and } \left(\frac{ds}{dt}\right)^2 = V^2$$

$$\therefore V^2 = -(g/4a)\cdot 16a^2 + A \quad \text{or} \quad A = V^2 + 4ag.$$

$$\therefore v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2 = \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag) - s^2\right]$$

or, $\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}(V^2+4ag)-s^2\right]}$

(-ve sign is taken because the particle is moving in the direction of s decreasing)

or $dt = -2\sqrt{(a/g)}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$

Integrating, the time t_1, from the cusp A to the vertex O is given by -

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$= 2\sqrt{(a/g)}\int_{0}^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$= 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{V^2+4ag}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)}\sin^{-1}\left\{\frac{2\sqrt{ag}}{\sqrt{V^2+4ag}}\right\}$$

$$= 2\sqrt{(a/g)}\cdot\theta \quad \text{—— (4)}$$

Where $\theta = \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$

We have $\sin\theta = \frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}$

$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{\left[1-\frac{4ag}{V^2+4ag}\right]} = \frac{V}{\sqrt{V^2+4ag}}$

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{V} = \frac{\sqrt{(4ag)}}{V}$

or. $\theta = \tan^{-1}\left[\sqrt{(4ag)}/V\right]$

$\therefore$ from (4), the time of reaching the vertex is $= 2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/V\right]$.

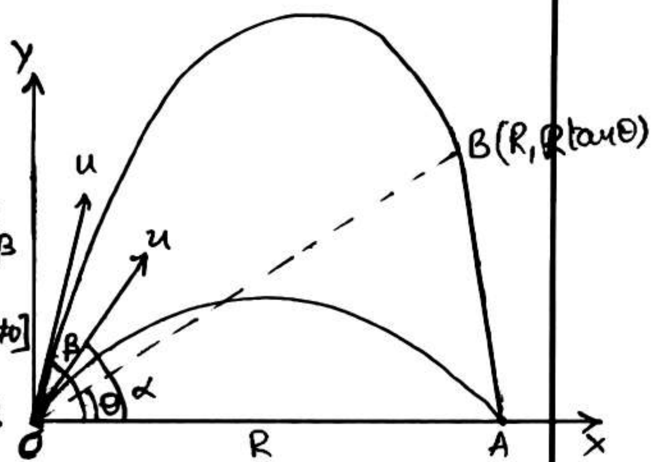
7.(C) A shot fired at an elevation α is observed to strike the foot of a tower which raise above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is $\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha \cos\theta)\right]$.

Sol^n: Let AB be the tower and O the point of projection.
It is given that $\angle AOB = \theta$.
Let u be the velocity of projection of the shot. when the shot is fired at an elevation α from O, it strikes the foot A of the tower AB, Let $OA = R$.

Then $\quad R = \dfrac{u^2 \sin 2\alpha}{g}$.

Referred to the horizontal & vertical lines OX & OY lying in the plane of motion as the coordinate axes, the coordinates of the top B of the tower are $(R, R\tan\theta)$.
If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y = x\tan\beta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\beta}$$



$$\therefore R\tan\theta = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{R}{u^2\cos^2\beta} \quad [\because R \neq 0]$$

Substituting the value of R from ①, we get

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

$$\Rightarrow \tan\theta = \tan\beta - \frac{\sin 2\alpha}{2\cos^2\beta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{\sin 2\alpha}{2\cos^2\beta}$$

Multiplying both sides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta = 2\sin\beta\cos\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow (1+\cos 2\beta)\sin\theta = \sin 2\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin 2\beta\cos\theta - \cos 2\beta\sin\theta = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin(2\beta-\theta) = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow 2\beta-\theta = \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow 2\beta = \theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow \beta = \frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)\right]$$

8(a) Find the angle b/w the tangents to the curve $\vec{r} = t^2\hat{i} - 2t\hat{j} + t^3\hat{k}$ at the points $t=1$ and $t=2$.

Sol'n: Given that $\vec{r} = t^2\hat{i} - 2t\hat{j} + t^3\hat{k}$

Tangent vector is given by

$$\frac{d\vec{r}}{dt} = 2t\hat{i} - 2\hat{j} + 3t^2\hat{k}$$

At $t=1$, $\frac{d\vec{r}}{dt} = 2\hat{i} - 2\hat{j} + 3\hat{k} = T_1$ (say)

$t=2$, $\frac{d\vec{r}}{dt} = 4\hat{i} - 2\hat{j} + 12\hat{k} = T_2$ (say)

Angle b/w the tangents T_1 and T_2 is given by

$$\cos\theta = \frac{T_1 \cdot T_2}{|T_1||T_2|}$$

$$= \frac{(2\hat{i} - 2\hat{j} + 3\hat{k})(4\hat{i} - 2\hat{j} + 12\hat{k})}{\sqrt{4+4+9}\sqrt{16+4+144}}$$

$$= \frac{(8+4+36)}{\sqrt{17}\sqrt{164}}$$

$$= \frac{48}{\sqrt{17}\sqrt{164}}$$

$$\cos\theta = \frac{48}{2\sqrt{41}\sqrt{17}}$$

$$\theta = \cos^{-1}\left(\frac{24}{\sqrt{17}\sqrt{41}}\right)$$

8.(b)(i) Find f(r) such that $\nabla f = \frac{\vec{r}}{r^5}$ and f(1) = 0.

SOLUTION

We know that,

$$\nabla f = f'(r)\nabla r = f'(r)\frac{\vec{r}}{r} \qquad \left(\because \nabla r = \frac{\vec{r}}{r}\right)$$

We have, $$\nabla f = \frac{\vec{r}}{r^5}$$

$$\therefore \quad f'(r)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^5}$$

$$\Rightarrow \quad \vec{r}\left[\frac{f'(r)}{r} - \frac{1}{r^5}\right] = 0$$

Since, $\vec{r} \neq 0$

$$\therefore \quad f'(r) = \frac{1}{r^4}$$

Integrating we get,

$$f(r) = \frac{-1}{3r^3} + C$$

$$f(1) = 0 \Rightarrow 0 = \frac{-1}{3.1} + c \Rightarrow c = \frac{1}{3}$$

$$\therefore \quad f(r) = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$$

8(b)(ii) Find the curvature and torsion at any point of the curve $x = a\cos 2t$, $y = a\sin 2t$, $z = 2a\sin t$.

Sol^n: The position vector for any point on the curve is

$$\vec{r} = a\cos 2t\,\hat{i} + a\sin 2t\,\hat{j} + 2a\sin t\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2a\sin 2t\,\hat{i} + 2a\cos 2t\,\hat{j} + 2a\cos t\,\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 4a\cos 2t\,\hat{i} - 4a\sin 2t\,\hat{j} - 2a\sin t\,\hat{k}$$

and $$\frac{d^3\vec{r}}{dt^3} = -8a\sin 2t\,\hat{i} - 8a\cos 2t\,\hat{j} - 2a\cos t\,\hat{k}$$

Now $$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ 2a\sin 2t & 2a\cos 2t & 2a\cos t \\ 4a\cos 2t & -4a\sin 2t & -2a\sin t \end{vmatrix}$$

$$= i(-4a^2\sin t\cos 2t + 8a^2\cos t\sin 2t)$$
$$+ j(8a^2\cos t\cos 2t + 4a^2\sin t\sin 2t)$$
$$+ k(-8a^2\sin^2 2t - 8a^2\cos^2 2t)$$

$$= ia^2(-4\sin t\cos 2t + 8\cos t\sin 2t) +$$
$$ja^2(8\cos t\cos 2t + 4\sin t\sin 2t) - 8a^2 k$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = a^2\sqrt{(-4\sin t\cos 2t + 8\cos t\sin 2t)^2 + (8\cos t\cos 2t + 4\sin t\sin 2t)^2 + (8)^2}$$

$$= a^2\sqrt{16\sin^2 t(\cos^2 2t + \sin^2 2t) + 64\cos^2 t(\sin^2 2t + \cos^2 2t) + 64}$$

$$= a^2\sqrt{16\sin^2 t(1) + 64\cos^2 t(1) + 64}$$

$$= a^2\sqrt{16(\sin^2 t+\cos^2 t)+48\cos^2 t+64}$$

$$= a^2\sqrt{48\cos^2 t+80}$$

$$= 4a^2\sqrt{3\cos^2 t+5}$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^3\vec{r}}{dt^3}$$

$$= (-8a\sin 2t)\left[-4a^2\sin t\cos 2t+8a^2\cos t\sin 2t\right]$$

$$-8a\cos 2t\left[8a^2\cos t\cos 2t+4a^2\sin t\sin 2t\right]$$

$$-2a\cos t\left[-8a^2\right]$$

$$= a^3\left[-64\cos t\,(\sin^2 2t+\cos^2 2t)+16\cos t\right]$$

$$= a^3\left[-64\cos t+16\cos t\right] = -48a^3\cos t$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4a^2\sin^2 2t+4a^2\cos^2 2t+4a^2\cos^2 t} = 2a\sqrt{1+\cos^2 t}$$

$$\therefore \text{Curvature}(\kappa) = \left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| \Big/ \left|\frac{d\vec{r}}{dt}\right|^3$$

$$= \frac{4a^2\sqrt{3\cos^2 t+5}}{2a\sqrt{1+\cos^2 t}} = 2a\frac{\sqrt{3\cos^2 t+5}}{\sqrt{1+\cos^2 t}}$$

$$\text{Torsion}(\tau) = \frac{\left[\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|^2}$$

$$= \frac{-48a^3\cos t}{\left[4a^2\sqrt{3\cos^2 t+5}\right]^2} = \frac{-48a^3\cos t}{16a^4(3\cos^2 t+5)}$$

$$= \frac{-3\cos t}{3\cos^2 t+5}$$

8(c) Evaluate the integral : $\iint_S \vec{F}\cdot\hat{n}\,ds$; where $\vec{F} = 3xy^2\hat{i} + (yx^2 - y^3)\hat{j} + 3zx^2\hat{k}$ and 'S' is a surface of cylinder $y^2+z^2 \leq 4$; $-3 \leq x \leq 3$; using divergence theorem.

Sol- Using Gauss divergence theorem.

$$\iiint \nabla\cdot f\,ds = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Now, $\vec{F} = 3xy^2\hat{i} + (yx^2-y^3)\hat{j} + 3zx^2\hat{k}$

$$\nabla\cdot\vec{F} = \cancel{3y^2} + x^2 \cancel{-3y^2} + 3x^2 = x^2+3x^2 = 4x^2$$

$$\therefore \iint F\cdot\hat{n}\,ds = \iiint \nabla\cdot F\,ds$$

$$I = \iiint 4x^2\,dx\,dy\,dz$$

converting to cylinder co-ordinates.

$y = r\cos\theta$; $z = r\sin\theta$; $x = x$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{-3}^{3} 4x^2\,dx\,dr\,d\theta\,[r]$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left[\frac{4x^3}{3}\right]_{-3}^{3} = 72\int_0^{2\pi}\int_0^2 r\,dr\,d\theta$$

$$I = 72\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta = 144\int_0^{2\pi} d\theta$$

$$\boxed{I = 288\pi}$$

8(d) $\rightarrow$ Verify Stoke's theorem for the vector $A = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$, where 'S' is the surface of the paraboloid ; $2z = x^2+y^2$ bounded by $z = 2$ and C is its boundary.

Solution:-
The boundary C of the surface 'S' is the circle in the plane $z=2$, whose equations are $x^2+y^2=4$, $z=2$.
The radius of this circle is 2 and centre $(0,0,2)$.
Suppose $x = 2\cos t$, $y = 2\sin t$, $z=2$; $0 \le t < 2\pi$ are parametric equations of 'C'. By Stoke's theorem

$$\oint_C A\cdot dr = \iint_S (\text{Curl } A)\cdot n \, dS .$$

when 'n' is a unit vector along outward drawn normal to the surface S.

We have;

$$\oint_C A\cdot dr = \oint_C (3y\hat{i} - xz\hat{j} + yz^2\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (3y\,dx - xz\,dy + yz^2 dz)$$

$$= \oint_C (3y\,dx - 2x\,dy),$$

Since, on C; $z=2$ and $dz=0$

$$= \int_{2\pi}^{0} \left(3y\frac{dx}{dt} - 2x\frac{dy}{dt}\right) dt$$

[Note that here the surface 'S' lies below the curve 'C' and so direction of 'C' is positive if 'C' is traversed in clockwise sense].

$$= -\int_0^{2\pi} [3.2\sin t.(-2\sin t) - 2.2\cos t.2\cos t]\,dt$$

$$= -\int_0^{2\pi} [-12\sin^2 t - 8\cos^2 t]\,dt$$

$$= 4\left[12\int_0^{\pi/2}\sin^2 t\,dt + 8\int_0^{\pi/2}\cos^2 t\,dt\right]$$

$$= 4\left[12.\frac{1}{2}.\frac{\pi}{2} + 8.\frac{1}{2}.\frac{\pi}{2}\right] = \frac{\cancel{4}}{\cancel{4}}[12\pi + 8\pi]$$

$$= 20\pi. \quad \text{———} \quad (1)$$

Let, S_1 be the plane region bounded by the circle 'C'. If S' is the surface consisting of the surface S and S_1, then S' is a closed surface.

Let, V be the volume bounded by S'.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } A).n\,ds = \iiint_V \text{div curl } A\,dV$$

$$= 0; \ [\text{since div curl } A = 0].$$

$$\therefore \iint_S (\text{curl } A).n\,ds + \iint_{S_1} (\text{curl } A).n\,ds = 0$$

$[\because S'$ consists of S and $S_1]$

$$\iint_S (\text{curl } A).n\,ds = -\iint_{S_1} (\text{curl } A).n\,ds$$

$$= -\iint_{S_1} (\text{curl } A).K\,ds$$

$[\because$ on S_1, $n = K]$

Now; $\text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & -xz & yz^2 \end{vmatrix}$

$$\text{Curl } A = i\left[\frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)\right] - j\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial z}(3y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(3y)\right]$$

$$\boxed{\text{Curl } A = (z^2+x)\hat{i} - (z+3)\hat{k}}$$

$\therefore \iint_S (\text{curl } A)\cdot n \, ds = -\iint_{S_1} [(z^2+x)\,\hat{i} - (z+3)\hat{k}]\cdot\hat{k}\, ds$

$= \iint_{S_1} (z+3)\, ds = \iint_{S_1} 5\, ds \quad [\because \text{ on } S_1,\ z=2]$

$= 5S_1$, where S_1 is the area of a circle of radius 2.

$= 5\cdot\pi\cdot(2)^2 = 20\pi.$

$$\boxed{\therefore \oint_c A\cdot dr = \iint_S (\text{curl } A)\cdot n\, ds = 20\pi}$$

Hence proved.