STABLE, UNSTABLE AND NEUTRAL EQUILIBRIUM

CHAPTER 12

12.1. EQUILIBRIUM OF BODIES

If a heavy body of weight W is suspended freely from a point by means of a string, the forces acting on it are

(*i*) weight of the body acting vertically downwards through its centre of gravity.

(*ii*) tension in the string.

The body will be in equilibrium under the action of only these two forces which must be equal in magnitude but opposite in direction along the same line of action. Since the weight W is acting vertically downwards, therefore the tension T in the string must also act vertically upwards through the centre of gravity of the body.

Similarly, if a heavy body rests in equilibrium on a horizontal plane at a point under the action of gravity alone, the normal reaction through the point of contact must pass through the centre of gravity of the body.

12.2. POSITION OF EQUILIBRIUM

Consider that one point of a rigid body is fixed, then the forces acting on the body are its weight and the reaction at the fixed point. For equilibrium, they should be equal and opposite and have the same line of action. Now there are three possibilities.

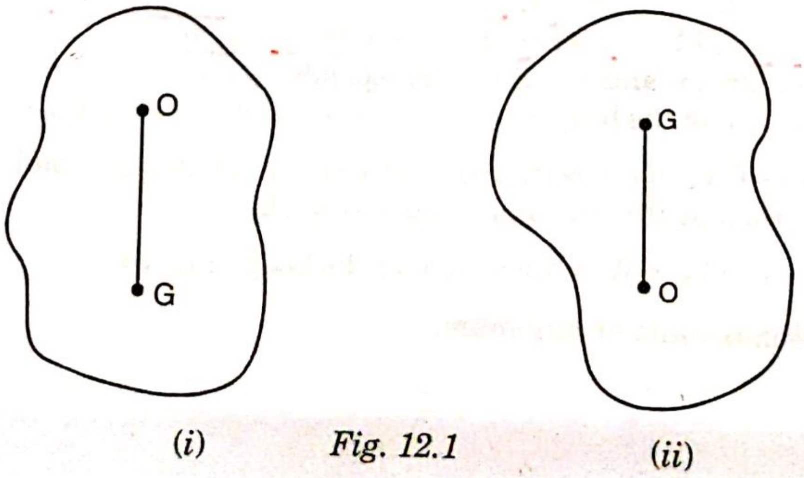

Fig. 12.1

In first case, the C.G., (G) of the body is below the fixed point O as shown in fig. 12.1(*i*). In this case, if the body is displaced from its position of equilibrium, it will tend to come back to its original position.

In the second case, the C.G., (G) of the body is above the fixed point O as shown in fig. 12.1(*ii*). In this case, if the body is displaced from its position of equilibrium, it will not tend to return to its position of equilibrium.

In the third case, the fixed point is at the C.G. and the body will remain in equilibrium at any position if displaced.

These three bodies are said to be in stable, unstable and neutral equilibrium respectively.

If a body (or a system of bodies) is placed with its base in contact with a horizontal plane, it will remain in equilibrium position, or fall according as the vertical line drawn through the centre of gravity of the body meets the plane within or outside the base.

Hence when a body is slightly displaced from the position of equilibrium, there are three states of equilibrium of a body, which are as follows :

(*i*) Stable equilibrium. *A body is said to be in stable equilibrium if, when slightly displaced, it tends to return to its original position of equilibrium.*

e.g., a cone resting with its flat circular base in contact with a horizontal plane would return to its position of equilibrium if slightly displaced. This is the position of stable equilibrium.

(*ii*) Unstable equilibrium. *A body is said to be in unstable equilibrium if, when slightly displaced, it tends to recede still further from its original position.*

A cone resting with its vertex in contact with plane is in unstable equilibrium.

Note. The bodies which have small base are unstable.

(*iii*) Neutral equilibrium. *A body is said to be in neutral equilibrium, if slightly displaced, it remains in equilibrium in any position.*

If the cone is placed with its slant side in contact with the plane, it will remain in equilibrium in any position.

12.3. CONDITIONS OF STABILITY OF EQUILIBRIUM

A body is in a stable equilibrium when the C.G. is in the lowest position it can take up and is in an unstable position of equilibrium when the C.G. is in the highest position it can take up. Hence the position of equilibrium and stability corresponds to the maximum and minimum values of the height of the C.G. of the body above a fixed point or a fixed horizontal plane.

If z be the height of the C.G. of the body above a fixed horizontal plane and dz the virtual displacement of the C.G. , then by the principle of virtual work,

$$-\,W dz = 0, \text{ where W is the weight of the body} \quad i.e., \; dz = 0$$

Therefore z is either maximum or minimum.

Hence to determine the position of equilibrium and stability of a system, we use the following working rule.

(i) Let the height z of the C.G. of the system above a fixed point or fixed horizontal plane be expressed as a function of some variable θ say, $z = f(\theta)$

(ii) Solve the equation $\frac{dz}{d\theta} = 0$. The values of θ so obtained will give the position of equilibrium as under :

(a) If $\frac{d^2z}{d\theta^2}$ is *positive* for any value of θ, then the equilibrium is *stable*.

(b) If $\frac{d^2z}{d\theta^2}$ is *negative*, then the equilibrium is *unstable* for that value of θ.

Remark :

If $z = f(\theta)$ be the depth of the C.G. of the body below a fixed horizontal plane, then the conditions for the stability and unstability of the equilibrium are reversed. The body is in stable position of equilibrium when the depth of C.G. is greatest. In this case the positions of equilibrium are as under :

(i) If $\frac{d^2z}{d\theta^2}$ is *positive*, then the equilibrium is *unstable*.

(ii) If $\frac{d^2z}{d\theta^2}$ is *negative*, then the equilibrium is *stable*.

12.4. THEOREM

A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively and the straight line joining the centres of the sphere being vertical. If the first body be slightly displaced, to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent sliding. [*M.D.U. 2016*]

Proof. Let O, O_1 be the centres of spherical surfaces of the lower body and upper body respectively. In the position of rest, let G_1 be the position of the C.G. of the upper body. The section of the bodies by vertical plane through G_1 is shown in fig. 12.2. Let $A_1G_1 = h$ where A_1 is the point of contact.

Let the upper body be slightly displaced by rolling so that O_2 is the new centre of the upper body. Let the new positions of A_1 be C and G_1 be G_2 so that $A_1G_1 = CG_2 = h$

Let A_2 be new point of contact of the two surfaces.

Let $\angle A_1OA_2 = \theta$

$\angle CO_2A_2 = \phi$

$\therefore$ $\angle G_2O_2H = \phi + \theta$

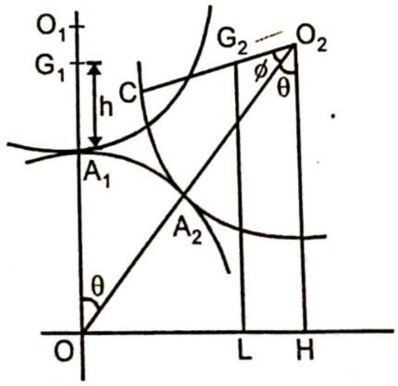

Fig. 12.2

Since the upper body has rolled on the lower body and there is no slipping, therefore

$$\text{arc } A_1A_2 = \text{arc } CA_2$$

or
$$R\theta = r\phi \qquad ...(1)$$

In order to find the nature of equilibrium, we have to find the height of C.G. in the new position above O.

Let
$$z = G_2L = O_2H - O_2G_2 \cos(\theta + \phi)$$
$$= OO_2 \cos\theta - [CO_2 - CG_2] \cos(\theta + \phi)$$
$$= (R + r)\cos\theta - (r - h)\cos(\theta + \phi)$$
$$= (R + r)\cos\theta - (r - h)\cos\left(\theta + \frac{R\theta}{r}\right) \qquad [Using\ (1)]$$

$$\therefore \quad z = (R + r)\cos\theta - (r - h)\cos\left[\frac{\theta(r + R)}{r}\right]$$

Differentiating w.r.t. θ,

$$\frac{dz}{d\theta} = -(R + r)\sin\theta + (r - h)\left(\frac{r + R}{r}\right)\sin\left[\frac{\theta(r + R)}{r}\right] = 0$$

For equilibrium, $\frac{dz}{d\theta} = 0$

$$\therefore \quad -(R + r)\sin\theta + (r - h)\left(\frac{r + R}{r}\right)\sin\left[\frac{\theta(r + R)}{r}\right] = 0$$

or
$$\sin\theta = \frac{r - h}{r}\sin\theta\left(\frac{r + R}{r}\right)$$

This is satisfied for $\theta = 0$

Also,
$$\frac{d^2z}{d\theta^2} = -(R + r)\cos\theta + (r - h)\left(\frac{r + R}{r}\right)^2\cos\left[\frac{\theta(r + R)}{r}\right]$$

$$\left(\frac{d^2z}{d\theta^2}\right)_{\theta = 0} = -(R + r) + (r - h)\left(\frac{r + R}{r}\right)^2$$

$$= \left(\frac{R + r}{r}\right)^2\left[(r - h) - \frac{r^2}{R + r}\right] = \left(\frac{R + r}{r}\right)^2\left[r - \frac{r^2}{R + r} - h\right]$$

$$= \left(\frac{R + r}{r}\right)^2\left[\frac{rR + r^2 - r^2}{R + r} - h\right]$$

$$= \left(\frac{R + r}{r}\right)^2\left[\frac{rR}{R + r} - h\right]$$

The equilibrium is stable if $\frac{d^2z}{d\theta^2} > 0$

This is possible if $$\frac{rR}{R+r} - h > 0$$

or if $$\frac{rR}{R+r} > h$$

or if $$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

The equilibrium is unstable if $$\frac{d^2z}{d\theta^2} < 0$$

This is possible if $$\frac{rR}{R+r} < h$$

or if $$\frac{1}{h} < \frac{1}{r} + \frac{1}{R}$$

Third case arises when $\frac{1}{h} = \frac{1}{r} + \frac{1}{R}$ *i.e.*, $h = \frac{rR}{R+r}$

Here $$\frac{d^2z}{d\theta^2} = 0$$

$$\frac{d^3z}{d\theta^3} = (R+r)\sin\theta - (r-h)\left(\frac{r+R}{r}\right)^3 \sin\left(\frac{r+R}{r}\theta\right)$$

$$\frac{d^4z}{d\theta^4} = (R+r)\cos\theta - (r-h)\left(\frac{r+R}{r}\right)^4 \cos\left(\frac{r+R}{r}\theta\right)$$

At $\theta = 0$, $\frac{d^3z}{d\theta^3} = 0$

$$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R+r) - (r-h)\left(\frac{r+R}{r}\right)^4$$

$$= (R+r) - \left[r - \frac{rR}{R+r}\right]\left[\frac{R+r}{r}\right]^4$$

$$= (R+r) - \left[\frac{rR + r^2 - rR}{R+r}\right]\left[\frac{R+r}{r}\right]^4$$

$$= (R+r) - \left[\frac{r^2}{R+r}\right]\left[\frac{R+r}{r}\right]^4 = (R+r) - \frac{(R+r)^3}{r^2}$$

$$= (R+r)\left[1 - \frac{(R+r)^2}{r^2}\right] = (R+r)\left[\frac{r^2 - (R+r)^2}{(R+r)^2}\right]$$

$$= (R+r)\left[\frac{r^2 - R^2 - r^2 - 2rR}{(R+r)^2}\right] = -\frac{(R^2+2rR)}{(R+r)} = -\text{ve}$$

$\therefore$ z is maximum and hence equilibrium is unstable.

Thus equilibrium is $\boxed{\text{stable if } \frac{1}{h} > \frac{1}{r} + \frac{1}{R}}$ and $\boxed{\text{unstable if } \frac{1}{h} \le \frac{1}{r} + \frac{1}{R}}$.

Cor. 1. The upper body has a plane face in contact with the lower body of radius R.

Here r is infinite and hence $\frac{1}{r} = 0$

$\therefore$ For stable equilibrium $\frac{1}{h} > \frac{1}{R}$ *i.e.,* $R > h$

and for unstable equilibrium $\frac{1}{h} \le \frac{1}{R}$ *i.e.,* $R \le h$

Thus the equilibrium is stable if the radius of the lower body R is greater than the distance of the centre of gravity of the body from its plane face; otherwise the equilibrium is unstable.

Cor. 2. The lower body is plane so that $R \to \infty$ *i.e.,* $\frac{1}{R} \to 0$

For stable equilibrium $\frac{1}{h} > \frac{1}{r}$ *i.e.,* $h < r$

Thus, if a body of spherical base is placed on a horizontal table, it is in stable equilibrium if the distance of its C.G. from the point of contact is less than the radius of the spherical surface.

12.5. A BODY RESTING INSIDE OTHER FIXED CONCAVE BODY

Let O be the centre and R be the radius of the spherical surface of the lower body and O_1 be the centre of the upper body such that G_1 is its C.G. Let A_1 be the point of contact of the two bodies.

Let A_2 be the new point of contact after displacement (by rolling) so that A_1 moves to C, O_1 moves to O_2 and G_1 moves to G_2 in the displaced position.

Let $A_1G_1 = CG_2 = h$

$\angle A_1OA_2 = \theta = \angle A_2O_2N$

$\angle CO_2A_2 = \phi$

$\therefore$ $\angle CO_2N = \phi - \theta$

Also $OA_1 = OA_2 = R$

$O_1A_1 = O_2A_2 = r$

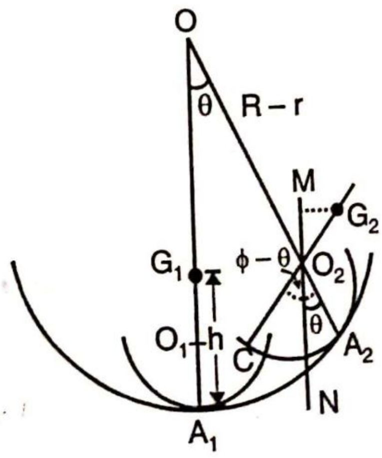

Fig. 12.3

Since the upper body has rolled on the lower body and there is no slipping, therefore

$$\text{arc } A_1A_2 = \text{arc } CA_2$$

or

$$R\theta = r\phi \qquad ...(1)$$

In order to find the nature of equilibrium, we have to find height of C.G. in the new position.

If z denotes the depth of G_2 below O, then

$$\begin{aligned} z &= OO_2 \cos\theta - O_2M \\ &= OO_2 \cos\theta - O_2G_2 \cos(\phi - \theta) \\ &= OO_2 \cos\theta - (CG_2 - CO_2)\cos(\phi - \theta) \\ &= OO_2 \cos\theta - (A_1G_1 - r)\cos(\phi - \theta) \\ &= (R - r)\cos\theta - (h - r)\cos(\phi - \theta) \\ &= (R - r)\cos\theta - (h - r)\cos\left(\frac{R\theta}{r} - \theta\right) \qquad \text{[Using (1)]} \\ &= (R - r)\cos\theta - (h - r)\cos\left(\frac{R - r}{r}\right)\theta \end{aligned}$$

Differentiating w.r.t. θ, we have

$$\frac{dz}{d\theta} = -(R - r)\sin\theta + (h - r)\left(\frac{R - r}{r}\right)\sin\frac{(R - r)\theta}{r}$$

$$\frac{d^2z}{d\theta^2} = -(R - r)\cos\theta + (h - r)\left(\frac{R - r}{r}\right)^2\cos\frac{(R - r)\theta}{r}$$

$$\frac{d^3z}{d\theta^3} = (R - r)\sin\theta - (h - r)\left(\frac{R - r}{r}\right)^3\sin\frac{(R - r)\theta}{r}$$

$$\frac{d^4z}{d\theta^4} = (R - r)\cos\theta - (h - r)\left(\frac{R - r}{r}\right)^4\cos\frac{(R - r)\theta}{r}$$

Here z stands for depth of C.G., therefore the conditions of equilibrium will be reversed.

For the position of equilibrium, $\dfrac{dz}{d\theta} = 0$

$$\therefore \qquad -(R - r)\sin\theta + (h - r)\left(\frac{R - r}{r}\right)\sin\left(\frac{R - r}{r}\right)\theta = 0$$

This is satisfied for $\theta = 0$

$$\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} = -(R - r) + (h - r)\left(\frac{R - r}{r}\right)^2$$

$$= \left(\frac{R-r}{r}\right)^2 \left[(h-r) - \frac{r^2}{R-r}\right]$$

$$= \left(\frac{R-r}{r}\right)^2 \left[h - \left(r + \frac{r^2}{R-r}\right)\right] = \left(\frac{R-r}{r}\right)^2 \left[h - \left(\frac{rR}{R-r}\right)\right]$$

Case I. $\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} > 0$ if $h - \frac{rR}{R-r} > 0$

or if $$h > \frac{rR}{R-r}$$

or if $$\frac{1}{h} < \frac{R-r}{rR}$$

or if $$\frac{1}{h} < \frac{1}{r} - \frac{1}{R}$$

$\therefore$ Equilibrium is unstable if $\frac{1}{h} < \frac{1}{r} - \frac{1}{R}$

Case II. $\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} < 0$ if $h - \frac{rR}{R-r} < 0$

or if $$\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$$

$\therefore$ Equilibrium is stable if $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$.

Case III. $\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} = 0$ if $\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$.

Here $$\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0} = 0$$

$$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R-r) - (h-r)\left(\frac{R-r}{r}\right)^4$$

$$= (R-r)\left[1 - \frac{(R-r)^3}{r^4}(h-r)\right]$$

$$= (R-r)\left[1 - \frac{(R-r)^3}{r^4}\left(\frac{Rr}{R-r} - r\right)\right] \qquad \left[\because h = \frac{Rr}{R-r}\right]$$

$$= (R - r)\left[1 - \frac{(R-r)^3}{r^4}\cdot\left(\frac{Rr - rR + r^2}{R - r}\right)\right]$$

$$= (R - r)\left[1 - \frac{(R-r)^2}{r^2}\right] = (R - r)\left[1 - \left(\frac{R}{r} - 1\right)^2\right]$$

This will be positive if $1 > \left(\frac{R}{r} - 1\right)^2$ *i.e.,* $R < 2r$

and negative if $1 < \left(\frac{R}{r} - 1\right)^2$ *i.e.,* $R > 2r$

Thus at $h = \frac{Rr}{R - r}$, the equilibrium will be stable if $R > 2r$ and unstable if $R < 2r$.

Working rule. If a body rests inside other fixed concave body, then

(*i*) $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$ $\Rightarrow$ stable equilibrium

(*ii*) $\frac{1}{h} < \frac{1}{r} - \frac{1}{R}$ $\Rightarrow$ unstable equilibrium

(*iii*) $\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$, $R > 2r$ $\Rightarrow$ stable equilibrium

$\frac{1}{h} = \frac{1}{r} - \frac{1}{R}$, $R < 2r$ $\Rightarrow$ unstable equilibrium.

SOLVED EXAMPLES

Example 1.

A uniform beam, of thickness 2b, rests symmetrically on a perfectly rough horizontal cylinder of radius a. Show that equilibrium of the beam will be stable or unstable according as b is less or greater than a. [*M.D.U. 2018*]

Solution. Here A is the point of contact and G is the C.G. of the upper body.

$\therefore$ Height of C.G. of upper body from point of contact = AG = b

r = radius of upper body (flat surface) = ∞

R = radius of lower body = a

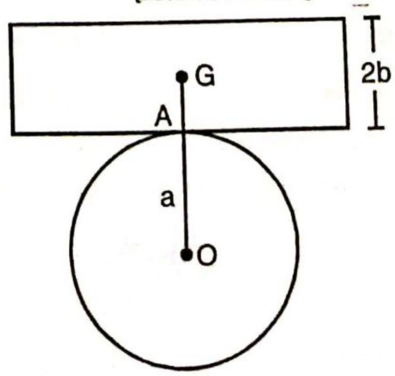

Fig. 12.4

The equilibrium is stable if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e., $\frac{1}{b} > \frac{1}{\infty} + \frac{1}{a}$ *i.e.*, $\frac{1}{b} > \frac{1}{a}$ *i.e.*, $b < a$

Also the equilibrium is unstable if $\frac{1}{h} < \frac{1}{r} + \frac{1}{R}$ *i.e.*, $b > a$

Hence the equilibrium is stable or unstable according as $b < a$ or $b > a$.

Example 2.

A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of cube be $\frac{\pi r}{2}$, show that the cube can rock through a right angle without falling.

[*M.D.U. 2017, 13, 09; C.D.L.U. 2013; K.U. 2010, 06*]

Solution. Let O be the centre of sphere and A be the point of contact of sphere with cube such that $OA = r$.

If G be the C.G. of the cube, then

$$AG = \frac{\text{side of cube}}{2} = \frac{\pi r}{4}$$

$$\therefore \quad h = \frac{\pi r}{4}$$

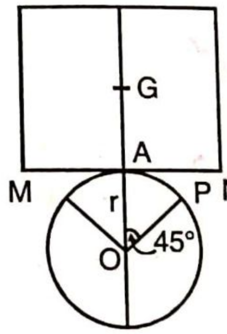

Fig. 12.5

The upper surface has a plane face in contact with the lower body and so radius of upper surface = ∞.

Here it is given that, radius of lower surface = r

The equilibrium is stable if $\frac{1}{h} > \frac{1}{R} + \frac{1}{r'}$ [*Here* $r' = \infty$, $R = r$]

or if $\frac{4}{\pi r} > \frac{1}{r} + \frac{1}{\infty}$

or if $\frac{4}{\pi} > 1$

or if $4 > \pi$, which is true.

Hence the equilibrium is stable.

Further the cube will not fall down till the end N of the lowest edge comes in contact with the sphere and then distance AN = arc AP.

or $\frac{\pi r}{4} = r\theta$ where $\angle AOP = \theta$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

This shows that the angle through which the cube can rock (turn) without falling on R.H.S. is $\frac{\pi}{4}$. Similarly, the cube can turn through an angle of $\frac{\pi}{4}$ on the other side *i.e.*, L.H.S.

Hence the total angle through which the cube can swing without falling in $\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$.

Example 3.

A heavy hemispherical shell of radius r has a particle attached to a point on the rim and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $\frac{R}{r} > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle. [K.U. 2012, 09; C.D.L.U. 2012]

Solution. Suppose that a weight W is attached at A lying on the rim of the hemisphere. Let G_1 be the C.G. of the hemisphere such that $CG_1 = \frac{r}{2}$.

Let G be the C.G. of the combined body (hemisphere and weight W attached at A), then G lies on the line AG_1. Let N be the point of contact of the two surfaces such that $NG = h$.

The equilibrium is stable, if $\frac{1}{h} > \frac{1}{r}+\frac{1}{R}$

or if $$\frac{1}{h} > \frac{R+r}{Rr}$$

or if $$h < \frac{Rr}{R+r} \qquad ...(1)$$

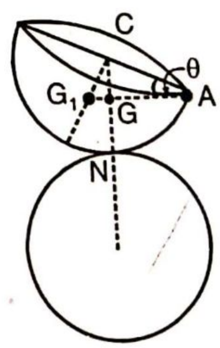

Fig. 12.6

Result (1) will be true if h is maximum *i.e.*, NG is maximum.

Let $\angle CAG_1 = \theta$

In right angled ΔACG_1,

$$\tan \angle CAG_1 = \frac{CG_1}{AC}$$

$$\therefore \quad \tan\theta = \frac{\frac{r}{2}}{r} = \frac{1}{2}$$

Here $$\sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}$$

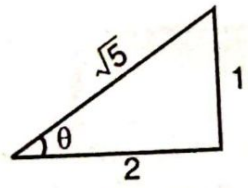

Fig. 12.7

For NG to be maximum, CG must be minimum.

This is possible if $CG \perp AG_1$. In this case

$$CG = AC\sin\theta = r\sin\theta = \frac{r}{\sqrt{5}}$$

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$\therefore$ Minimum value of CG is $\frac{r}{\sqrt{5}}$

$\Rightarrow$ Maximum value of NG = NC – minimum value of CG

$$= r - \frac{r}{\sqrt{5}}$$

$$\therefore \quad h_{max.} = r - \frac{r}{\sqrt{5}}$$

The equilibrium is stable, if $h < \frac{Rr}{R+r}$

or if $$h_{max.} < \frac{Rr}{R+r}$$

or if $$r - \frac{r}{\sqrt{5}} < \frac{Rr}{R+r}$$

or if $$1 - \frac{1}{\sqrt{5}} < \frac{R}{R+r}$$

or if $$-\frac{1}{\sqrt{5}} < \frac{R}{R+r} - 1$$

or if $$-\frac{1}{\sqrt{5}} < -\frac{r}{R+r}$$

or if $$\frac{1}{\sqrt{5}} > \frac{r}{R+r}$$

or if $$\sqrt{5} < \frac{R+r}{r}$$

or if $$\sqrt{5} < \frac{R}{r} + 1$$

or if $$\sqrt{5} - 1 < \frac{R}{r}$$

or if $$\frac{R}{r} > \sqrt{5} - 1$$

Since the equation (1) and h are free from weight W attached at A, hence equilibrium will be stable whatever be the weight of the particle at A and for stable equilibrium $\frac{R}{r} > \sqrt{5} - 1$.

Example 4.

A body consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table, the hemisphere being in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

[M.D.U. 2014, 10, 08, 06, 04; K.U. 2013, 10]

Solution. Here we have to find out the C.G. of the whole body from the point of contact A as the combined body rests on the table

$$w_1 = \text{weight of hemisphere}$$
$$= \frac{2}{3}\pi a^3 \rho g$$
$$x_1 = \text{distance of the C.G.* of hemisphere from the table}$$
$$= AG_1$$
$$= a - \frac{3a}{8} = \frac{5a}{8}$$
$$w_2 = \text{weight of the cone}$$
$$= \frac{1}{3}\pi a^2 H\rho g, \text{ where H is the height of cone.}$$
$$x_2 = \text{distance of C.G. of cone from the table}$$
$$= AO + OG_2 = a + \frac{H}{4}$$

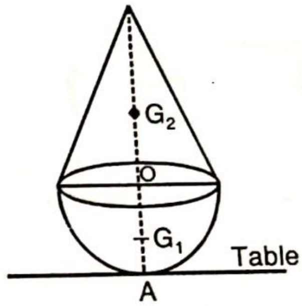

Fig. 12.8

If h is the distance of C.G. of the combined body from the horizontal plane, then

$$h = \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$$
$$= \frac{\frac{2}{3}\pi a^3 \rho g . \frac{5a}{8} + \frac{1}{3}\pi a^2 H\rho g\left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H\rho g}$$

$\therefore$
$$h = \frac{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)}{2a + H}$$
$$R = \text{radius of lower surface} = \infty$$
$$r = \text{radius of upper surface} = a$$

For stable equilibrium $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e.,
$$\frac{2a + H}{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)} > \frac{1}{a} + \frac{1}{\infty}$$

or
$$\frac{2a + H}{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)} > \frac{1}{a}$$

* C.G. of a solid hemisphere is a point on its axis at a distance $\frac{3a}{8}$ from the centre of base, where a is radius of sphere.

or
$$2a^2 + Ha > \frac{5}{4}a^2 + Ha + \frac{H^2}{4}$$

or
$$\frac{3a^2}{4} > \frac{H^2}{4} \quad i.e., \quad H < \sqrt{3}\, a$$

Hence the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

Example 5.

A sphere of weight W and radius a lies within a fixed spherical shell of radius b and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$ and that if $\frac{W}{w}$ be equal to this ratio, then the equilibrium is essentially stable.

Solution. The combined body consists of sphere of weight W and weight w attached at A′, the vertical end of the diameter of sphere.

Let h be the height of C.G. of the combined body from the point of contact O of sphere and shell, then

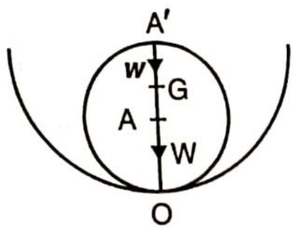

Fig. 12.9

$$h = \frac{w.(OA') + W(OA)}{w + W}$$
$$= \frac{w.2a + W.a}{w + W} = \frac{(2w + W)a}{w + W}$$
$$r = \text{radius of sphere} = a$$
$$R = \text{radius of shell} = b$$

Case I. The equilibrium will be stable if $\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$

or
$$\frac{w + W}{(2w + W)\, a} > \frac{1}{a} - \frac{1}{b}$$

or
$$\frac{w + W}{a(2w + W)} > \frac{b - a}{ab}$$

or
$$(w + W)\, b > (b - a)(2w + W)$$

or
$$bw + bW > 2bw + bW - 2aW - aW$$

or
$$aW > (b - 2a)\, w$$

or
$$\frac{W}{w} > \frac{b - 2a}{a}$$

Case II. $\frac{W}{w} = \frac{b - 2a}{a}$

Here $\frac{W}{w} > 0$

$$\therefore \qquad \frac{b-2a}{a} > 0$$

$$\Rightarrow \qquad b - 2a > 0 \Rightarrow b > 2a$$

$$\Rightarrow \qquad R > 2r$$

$\therefore$ Equilibrium is essentially stable.

Thus the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$ and essentially stable if $\frac{W}{w} = \frac{b-2a}{a}$.

Example 6.

A heavy uniform beam rests between two smooth planes, each inclined at an angle $\frac{\pi}{4}$ to the horizontal, so that the beam is in a vertical plane and perpendicular to the line of intersection of the planes. Find the equilibrium position of the beam and test its stability. *[K.U. 2014; M.D.U. 2012]*

Solution. Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB, each inclined at an angle $\frac{\pi}{4}$ to the horizontal. Let the beam be inclined at an angle θ to the horizontal in one of the equilibrium position.

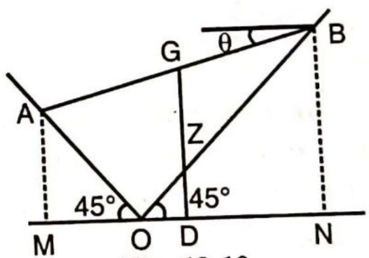

Fig. 12.10

Let G be the C.G. of the beam AB and Z be its height above the horizontal plane through O. Draw AM, GD and BN perpendicular on the horizontal line MN. Then

$$Z = GD = \frac{1}{2}[AM + BN]$$

[$\because$ G is mid-point of AB]

$$= \frac{1}{2}(OA \sin 45° + OB \sin 45°)$$

$$= \frac{1}{2\sqrt{2}}(OA + OB) \qquad ...(1)$$

Now $\angle OAB = 45° + \theta$; $\angle OBA = 45° - \theta$ *[$\because \angle AOB = 90°$]*

$\therefore$ By sine formula,

$$\frac{OA}{\sin(45° - \theta)} = \frac{OB}{\sin(45° + \theta)} = \frac{AB}{\sin 90°}$$

or $$\frac{OA}{\sin(45° - \theta)} = \frac{OB}{\sin(45° + \theta)} = 2a$$

$$\therefore \qquad OA = 2a \sin(45° - \theta)$$

$$OB = 2a \sin(45° + \theta)$$

$\therefore$ From (1), $Z = \frac{1}{2\sqrt{2}}[2a \sin(45° - \theta) + 2a \sin(45° + \theta)]$

$$= \frac{a}{\sqrt{2}}[2\sin 45° \cos\theta] = a\cos\theta$$

$$\therefore \quad \frac{dZ}{d\theta} = -a\sin\theta \quad \text{and} \quad \frac{d^2Z}{d\theta^2} = -a\cos\theta$$

For equilibrium of the beam, we have

$$\frac{dZ}{d\theta} = 0 \quad i.e., \quad \sin\theta = 0 \Rightarrow \theta = 0$$

Thus the beam rests in a horizontal position between the two inclined planes.

When $\theta = 0$, $\frac{d^2Z}{d\theta^2} = -a\cos\theta = -a$, which is negative.

$\therefore$ Z is maximum when $\theta = 0$

Hence the equilibrium is unstable.

Example 7.

Two equal particles are connected by a light string which is slung over the top of a smooth vertical circle. Verify that the position of equilibrium is unstable (It may be supposed that both particles rest on the circle so that the length of the string is less than one-half of the circumference of the circle). [*M.D.U. 2011, 05*]

Solution. Let the two equal particles P and Q each of weight W be connected by a light string of length $2a\alpha$, which rests over the top of a smooth vertical circle of radius a.

Since the length of the string is less than one half of the circumference of the circle, we suppose $\angle POQ = 2\alpha$ (less than π)

Let $\angle POR = \theta$, so that

$\angle QOR = 2\alpha - \theta$

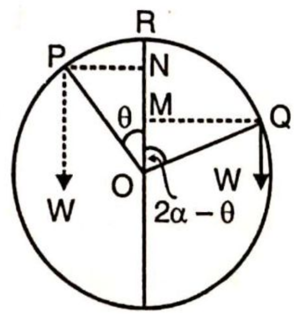

Fig. 12.11

The height of C.G. of the particle P above O

$$= ON = a\cos\theta.$$

The height of C.G. of the particle Q above O

$$= OM = a\cos(2\alpha - \theta)$$

The height of the C.G. of the two particles above O is given by

$$z = \frac{W(ON) + W(OM)}{W + W}$$

$$= \frac{1}{2}[a\cos\theta + a\cos(2\alpha - \theta)]$$

$$= \frac{a}{2}[2 \cos \alpha \cos (\alpha - \theta)] = a \cos \alpha \cos (\alpha - \theta)$$

$$\therefore \qquad \frac{dz}{d\theta} = a \cos \alpha [- \sin (\alpha - \theta)] (-1) = r \cos \alpha \sin (\alpha - \theta)$$

and
$$\frac{d^2z}{d\theta^2} = - r \cos \alpha \cos (\alpha - \theta)$$

For position of equilibrium, $\frac{dz}{d\theta} = 0$

i.e., $\qquad r \cos \alpha \sin (\alpha - \theta) = 0$

or $\qquad \sin (\alpha - \theta) = 0 \Rightarrow \alpha - \theta = 0 \Rightarrow \theta = \alpha$

and $\qquad \left(\frac{d^2z}{d\theta^2}\right)_{\text{at } \theta = \alpha} = - r \cos \alpha$, which is negative.

$\therefore$ z is maximum for $\theta = \alpha$.

Hence the equilibrium is unstable.

Example 8.

A heavy uniform rod of length 2a, rests partly within and partly without a fixed smooth hemispherical bowl of radius r, the rim of the bowl is horizontal and one point of the rod is in contact with the rim. If θ be the inclination of the rod to the horizontal, show that 2r cos 2θ = a cos θ. Also show that the equilibrium of the rod is stable.

[K.U. 2016, 07]

Solution. Let AB be the rod of length $2a$ and weight W.

Let D be the point of contact of the rod with the rim of the hemispherical bowl of radius r and centre O. The rod is in equilibrium under the action of following three forces :

(*i*) The reaction R at A acting along the normal AO to the bowl.

(*ii*) The reaction S at D acting perpendicular to the rod.

(*iii*) W, the weight of the rod acting vertically downwards through its middle point G.

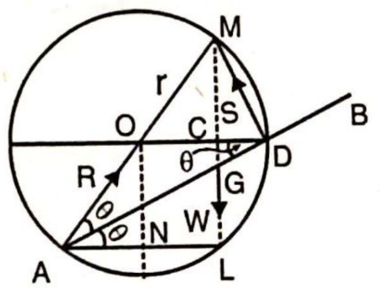

Fig. 12.12

Since OD is parallel to AL

$\therefore \qquad \angle ODA = \angle DAL = \theta$

Also, $\qquad OD = OA$

$\therefore \qquad \angle OAD = \theta$ and $\angle MAL = 2\theta$

For equilibrium, the three forces R, S and W must meet in a point M (say) and the point M lies on the sphere of which bowl is a part. Join M, G and produce it so that it meets the bowl at L. By geometry, $\angle ALM = 90°$ and AL is a horizontal line. Let the rod be inclined at an angle θ with AL.

Let z be the depth of the C.G. of the rod below the fixed horizontal line OD.

$$\therefore \quad z = CG = CL - GL = ON - GL$$
$$= OA \sin 2\theta - AG \sin \theta$$
$$= r \sin 2\theta - a \sin \theta$$

$$\therefore \quad \frac{dz}{d\theta} = 2r \cos 2\theta - a \cos \theta$$

For the equilibrium, $\frac{dz}{d\theta} = 0$, which gives

$$2r \cos 2\theta - a \cos \theta = 0 \quad \Rightarrow \quad 2r \cos 2\theta = a \cos \theta$$

which gives the position of equilibrium.

Also, $$\frac{d^2z}{d\theta^2} = -4r \sin 2\theta + a \sin \theta$$
$$= -2(2r \sin 2\theta) + a \sin \theta$$
$$= -2(AM \sin 2\theta) + AG \sin \theta$$
$$= -2ML + GL = \text{negative} \qquad [\because ML > GL]$$

$\therefore$ z, the depth of centre of gravity of the rod below a fixed horizontal line is maximum.

Hence the equilibrium is stable.

Example 9.

A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and show that it is unstable.

[K.U. 2017, 15, 11: M.D.U. 2015, 14, 11, 06]

Solution. Let a uniform rod AB of length $2a$ with its C.G. at G be inclined at an angle θ with the vertical, the lower end A resting against a vertical wall. Let P be the peg at a distance b (= PM) from the wall. Let z be the height of G above the peg. Then

$$z = GD = AL - AM$$
$$= AG \cos \theta - b \cot \theta \qquad [\because AM = PM \cot \theta = b \cot \theta]$$
$$= a \cos \theta - b \cot \theta$$

$$\therefore \quad \frac{dz}{d\theta} = -a \sin \theta + b \operatorname{cosec}^2 \theta$$

and $$\frac{d^2z}{d\theta^2} = -a\cos\theta - 2b\,\mathrm{cosec}^2\theta\cot\theta$$

For equilibrium, $\frac{dz}{d\theta} = 0$, which gives

$$-a\sin\theta + b\,\mathrm{cosec}^2\theta = 0$$

or $$a\sin\theta = b\,\mathrm{cosec}^2\theta = \frac{b}{\sin^2\theta}$$

or $$\sin^3\theta = \frac{b}{a}$$

or $$\theta = \left(\sin^{-1}\frac{b}{a}\right)^{1/3} \quad ...(1)$$

which gives the position of equilibrium.

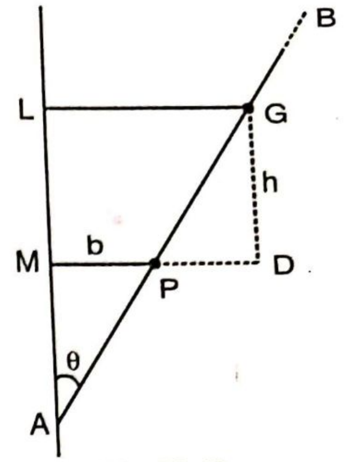

Fig. 12.13

Also, $$\frac{d^2z}{d\theta^2} = -\cos\theta\left[a + 2b\,\mathrm{cosec}^2\theta.\frac{1}{\sin\theta}\right]$$

$$= -\cos\theta\left[a + \frac{2b}{\sin^3\theta}\right]$$

$$= -\cos\theta\left[a + 2b\,.\frac{a}{b}\right] \quad \text{[From (1)]}$$

$$= -3a\cos\theta, \text{ which is negative}$$

$\therefore$ z is maximum when $\theta = \left(\sin^{-1}\frac{b}{a}\right)^{1/3}$

Hence equilibrium is unstable.

Example 10.

A heavy body, the section of which is a cycloid, rests on a rough horizontal plane and has its C.G. at the centre of curvature of the curve at the point of contact. Show that the equilibrium is unstable. [*C.D.L.U. 2016; K.U. 2015, 08; M.D.U. 2010, 05*]

Solution. The equation of cycloid is $s = 4a\sin\psi$

Now, $$\rho = \frac{ds}{d\psi} = 4a\cos\psi \quad ...(1)$$

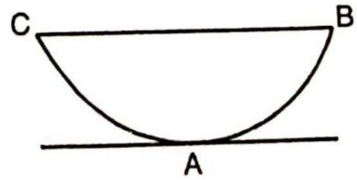

Fig. 12.14

At the point of contact $\psi = 0$

$\therefore$ From (1), $\rho = 4a$

At the point of contact, $\rho_1 = \infty$, as lower surface is a plane

Also $\rho_2 = \rho = 4a$

It is given that C.G. lies at the centre of curvature of the curve at the point of contact

$\therefore \quad h = \rho_2 = 4a$

Now, the equilibrium is stable or unstable according as

$$h < \text{or} > \frac{\rho_1\rho_2 \cos\alpha}{\rho_1 + \rho_2}$$

or
$$\frac{1}{h} > \text{or} < \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

or
$$\frac{1}{4a} > \text{or} < \frac{1}{\infty} + \frac{1}{4a} \qquad [\because h = \rho_2 = 4a, \rho_1 = \infty]$$

or
$$\frac{1}{4a} > \text{or} < 0 + \frac{1}{4a}, \text{ which is impossible.}$$

Therefore it does not give any result. In such case the equilibrium is stable or unstable if

$$\frac{d}{ds}\left(\frac{1}{\rho_1}\right) + \frac{d}{ds}\left(\frac{1}{\rho_2}\right) < \text{or} > 0$$

Now,
$$\frac{d}{ds}\left(\frac{1}{\rho_1}\right) + \frac{d}{ds}\left(\frac{1}{\rho_2}\right) = \frac{d}{ds}(0) + \frac{d}{ds}\left(\frac{1}{4a}\right) = 0 + 0$$

$\therefore$ This test fails.

Now, we find the nature of equilibrium with the help of the following formula

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho_1}\right) + \frac{d^2}{ds^2}\left(\frac{1}{\rho_2}\right) + \frac{(\rho_1 + \rho_2)(\rho_1 + 2\rho_2)}{\rho_1^3 \rho_2^2}$$

Putting the values of ρ_1 and ρ_2 in the above, we get a positive quantity.
Hence, the equilibrium is unstable.

EXERCISE 12.1

1. A hemisphere rests in equilibrium on a sphere of equal radius. Show that the equilibrium is unstable when the curved surface rests on the sphere and stable when the flat surface of the hemisphere rests on the sphere. *[M.D.U. 2014, 11]*
2. A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable. *[M.D.U. 2015]*
3. A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with

the sphere. Show that , if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin \alpha$ be less than $\frac{3r}{a}$, where a is one of the equal sides of the triangle.

4. A thin hemispherical bowl of radius b and weight W rests in equilibrium on the highest point of a fixed sphere of radius a which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w. Show that the equilibrium is unstable unless $w > W\frac{a-b}{2b}$.

5. A solid homogeneous hemisphere of radius a has a solid right cone of the same substance constructed on its base; the hemisphere rests on the convex sides of a fixed sphere of radius b, the axis of the cone being vertical. Show that the greatest height of the cone consistent with stability for a small rolling displacement is $\frac{a}{a+b}[\sqrt{(3b+a)(b-a)}-2a]$.

[M.D.U. 2015, 07]

6. A lamina in the form of a cycloid, whose generating circle is of radius a, rests on the top of another cycloid whose generating circle is of radius b, their vertices being in contact and their axes vertical. If h be the height of C.G. of the upper cycloid above its vertex, show that the equilibrium is stable only if $h < \frac{4ab}{a+b}$ and is unstable if $h \geq \frac{4ab}{a+b}$.

[K.U. 2005]

7. A solid ellipsoid whose axes are of lengths $2a$, $2b$, $2c$ with the axes vertical lies on a rough horizontal plane. The centre of gravity is on the vertical axis at a distance h from the bottom vertex. Show that the equilibrium is stable if $h < \frac{a^2}{c}$ and $\frac{b^2}{c}$ both.

[M.D.U. 2013]

8. Two equal uniform rods are firmly jointed at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in stable or unstable equilibrium according as the length of the rod is greater than or less than $4r \operatorname{cosec} \alpha$.

[M.D.U. 2012, 09]

9. A square lamina rests with its plane perpendicular to a smooth wall; one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.

[K.U. 2014; 04; C.D.L.U. 2013; M.D.U. 2008, 07]

10. A uniform beam of length $2a$ rests with its ends on two smooth inclined planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\alpha > \beta$), show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by $\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$ and show that the beam is in unstable equilibrium.

[M.D.U. 2017]

11. A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse. If the length of beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.

12. One end A of a uniform rod AB of weight W and length l is smoothly hinged at a fixed point while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A and carries a weight $\frac{W}{4}$. If $l < a < 2l$, show that the system is in stable equilibrium when AB is vertically upwards and that there is also a configuration of equilibrium in which the rod is at a certain angle to the vertical.

13. A uniform rod of length $2l$, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium is given by $\cos^2\theta = \frac{2a}{l}$ and that if these positions exist they are stable.

[*M.D.U. 2013, 08*]

14. A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus and rests with one end on the parabola. Prove that the rod will be in equilibrium, if it makes with the vertical, an angle θ, given by the equation $\cos^4\frac{\theta}{2} = \frac{a}{2c}$ where $4a$ is the latus rectum of the parabola and $2c$ the length of the rod.

[*M.D.U. 2007*]

ANSWERS

2. $\frac{a}{2}$

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