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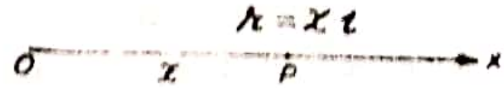
Rectilinear Motion. (S.H.M.)

§ 1. **Introduction.** When a point (or particle) moves along a straight line, its motion is said to be a **rectilinear motion.** Hence in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

§ 2. **Velocity and acceleration.**



Suppose a particle moves along a straight line OX where O is a fixed point on the line. Let P be the position of the particle at time t, where $OP = x$. If $\mathbf{r}$ denotes the position vector of P and $\mathbf{i}$ denotes the unit vector along OX, then $\mathbf{r} = \vec{OP} = x\,\mathbf{i}$.

Let $\mathbf{v}$ be the velocity vector of the particle at P. Then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\,\mathbf{i}) = \frac{dx}{dt}\,\mathbf{i} + x\,\frac{d\mathbf{i}}{dt} = \frac{dx}{dt}\,\mathbf{i},$$

because $\mathbf{i}$ is a constant vector. Obviously the vector $\mathbf{v}$ is collinear with the vector $\mathbf{i}$. Thus for a particle moving along a straight line the direction of velocity is always along the line itself. If at P the particle be moving in the direction of x increasing (*i.e.*, in the direction OX) and if the magnitude of its velocity *i.e.*, its *speed* be v, we have

$$\mathbf{v} = v\,\mathbf{i} = \frac{dx}{dt}\,\mathbf{i}. \quad \text{Therefore } \frac{dx}{dt} = v.$$

On the other hand if at P the particle be moving in the direction of x decreasing (*i.e.*, in the direction XO) and if the magnitude of its velocity be v, we have

$$\mathbf{v} = -v\,\mathbf{i} = \frac{dx}{dt}\,\mathbf{i}. \quad \text{Therefore, } \frac{dx}{dt} = -v.$$

Remember. *In the case of a rectilinear motion the velocity of a particle at time t is dx/dt along the line itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.*

Now let **a** be the acceleration vector of the particle at P. Then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\,\mathbf{i}\right)=\frac{d^2x}{dt^2}\,\mathbf{i}.$$

Thus the vector **a** is collinear with **i** *i.e.*, the direction of acceleration is always along the line itself. If at P the acceleration be acting in the direction of x increasing and if its magnitude be f, we have $\mathbf{a}=f\,\mathbf{i}=\frac{d^2x}{dt^2}\,\mathbf{i}$. Therefore $\frac{d^2x}{dt^2}=f$. On the other hand if at P the acceleration be acting in the direction of x decreasing and if its magnitude be f, we have

$$\mathbf{a}=-f\,\mathbf{i}=\frac{d^2x}{dt^2}\,\mathbf{i}\ ;\ \text{therefore}\ \frac{d^2x}{dt^2}=-f.$$

Remember. *In the case of a rectilinear motion the acceleration of a particle at time t is d^2x/dt^2 along the line itself and is taken with positive or negative sign according as it acts in the direction of x increasing or x decreasing.*

Since the acceleration is produced by the force, therefore while considering the sign of d^2x/dt^2 we must notice the direction of the acting force and not the direction in which the particle is moving. For example if the direction of the acting force is that of x increasing, then d^2x/dt^2 must be taken with positive sign whether the particle is moving in the direction of x increasing or in the direction of x decreasing.

Other Expressions for acceleration :

Let $v=\frac{dx}{dt}$. We can then write

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\,\frac{dv}{dx}.$$

Thus $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\,\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them may be used to suit the convenience in working out the problems.

Note. Often we denote dx/dt by $\dot{x}$ and d^2x/dt^2 by $\ddot{x}$.

Illustrative Examples :

Ex. 1. *If at time t the displacement x of a particle moving away from the origin is given by $x=a\sin t+b\cos t$, find the velocity and acceleration of the particle.* **[Meerut 1977]**

Sol. Given that $x=a\sin t+b\cos t$.

Differentiating w.r.t. 't', we have the velocity $v=dx/dt=a\cos t-b\sin t$.

Differentiating again, we have
the acceleration $=dv/dt=-a \sin t-b \cos t=-x$.

Ex. 2. *A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that*

$$v^2=nfs/(n-1).$$ **[Meerut 1981, 84 P, 85 S]**

Sol. Here, distance $s \propto t^n$.

$$\therefore \quad \text{let } s=k\,t^n, \quad \ldots(1)$$

where k is a constant of proportionality.

Differentiating (1), w.r.t. 't', we have
the velocity $v=ds/dt=knt^{n-1}$. ...(2)

Again differentiating (2),
the acceleration $f=dv/dt=kn\,(n-1)\,t^{n-2}$. ...(3)

$$\therefore \quad v^2=(knt^{n-1})^2=k^2n^2t^{2n-2}$$

$$=\frac{n.\{kn(n-1)\,t^{n-2}\}.kt^n}{(n-1)}$$

$$=\frac{nfs}{(n-1)}, \text{ substituting from (1) and (3).}$$

Ex. 3. *A particle moves along a straight line such that its displacement x, from a point on the line at time t, is given by*

$$x=t^3-9t^2+24t+6.$$

Determine (i) the instant when the acceleration becomes zero, (ii) the position of the particle at that instant and (iii) the velocity of the particle, then. **[Meerut 1971]**

Sol. Here, $x=t^3-9t^2+24t+6$.

$\therefore$ the velocity $v=dx/dt=3t^2-18t+24$

and the acceleration $f=d^2x/dt^2=6t-18$.

(i) Now the acceleration $=0$, when $6t-18=0$ or $t=3$.

Thus the acceleration is zero when $t=3$ seconds.

(ii) When $t=3$, position of the particle is given by

$$x=3^3-9.3^2+24.3+6=24 \text{ units.}$$

(iii) When $t=3$, the velocity $v=3\cdot3^2-18\cdot3+24=-3$ units.

Thus when $t=3$, the velocity of the particle is 3 units in the direction of x decreasing.

Ex. 4. *A particle moves along a straight line and its distance from a fixed point on the line is given by $x=a \cos(\mu t+\epsilon)$. Show that its acceleration varies as the distance from the origin and is directed towards the origin.*

Sol. We have $x=a \cos(\mu t+\epsilon)$. ...(1)

Differentiating w.r.t. t, we get

$dx/dt=-a\mu \sin(\mu t+\epsilon)$, from (1)

and $d^2x/dt^2=-a\mu^2 \cos(\mu t+\epsilon)=-\mu^2 x$.

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the negative sense of x-axis *i.e.*, towards the origin.

Ex. 5. *A particle moves along a straight line such that its distance x from a fixed point on it and the velocity v there are related by $v^2=\mu(a^2-x^2)$. Prove that the acceleration varies as the distance of the particle from the origin and is directed towards the origin.* **[Agra 1975]**

Sol. We have $v^2=\mu(a^2-x^2)$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2v\frac{dv}{dx}=\mu(-2x). \quad \therefore \quad \frac{d^2x}{dt^2}=v\frac{dv}{dx}=-\mu x.$$

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the direction of x decreasing *i.e.*, towards the origin.

Ex. 6. *The velocity of a particle moving along a straight line, when at a distance x from the origin (centre of force) varies as $\sqrt{\{(a^2-x^2)/x^2\}}$. Find the law of acceleration.* **[Agra 1979]**

Sol. Let v be the velocity of the particle when it is at a distance x from the origin. Then according to the question, we have

$v=\mu\sqrt{\{(a^2-x^2)/x^2\}}$, where μ is a constant.

$\therefore \quad v^2=\mu^2(a^2-x^2)/x^2=\mu^2(a^2/x^2-1)$.

Differentiating w.r.t. x, we get

$$2v\frac{dv}{dx}=\mu^2\left(-\frac{2a^2}{x^3}\right). \quad \therefore \quad v\frac{dv}{dx}=\frac{d^2x}{dt^2}=-\frac{\mu^2 a^2}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance from the origin and is directed towards the centre of force.

Ex. 7. *The law of motion in a straight line being given by $s=\frac{1}{2}vt$, prove that the acceleration is constant.* **[Meerut 1979]**

Sol. We have $s=\frac{1}{2}vt=\frac{1}{2}\frac{ds}{dt}t. \quad \left[\because \quad v=\frac{ds}{dt}\right]$

Differentiating w.r.t., 't' we get

$$\frac{ds}{dt}=\frac{1}{2}\frac{d^2s}{dt^2}t+\frac{1}{2}\frac{ds}{dt} \quad \text{or} \quad \frac{1}{2}\frac{ds}{dt}=\frac{1}{2}\frac{d^2s}{dt^2}t$$

or $$\frac{ds}{dt}=\frac{d^2s}{dt^2}t.$$

Differentiating again w.r.t. t, we get

$$\frac{d^2s}{dt^2}=\frac{d^2s}{dt^2}+\frac{d^3s}{dt^3}\,t \quad \text{or} \quad \frac{d^3s}{dt^3}\,t=0 \quad \text{or} \quad \frac{d^3s}{dt^3}=0$$

because $t \neq 0$.

Now $\dfrac{d^3s}{dt^3}=0 \Rightarrow \dfrac{d}{dt}\left(\dfrac{d^2s}{dt^2}\right)=0 \Rightarrow \dfrac{d^2s}{dt^2}=$ constant.

Hence the acceleration is constant.

Ex. 8. *A point moves in a straight line so that its distance from a fixed point in that line is the square root of the quadratic function of the time; prove that its acceleration varies inversely as the cube of the distance from the fixed point.*

Sol. At any time t, let x be the distance of the particle from a fixed point on the line. Then according to the question, we have

$$x=\sqrt{(at^2+2bt+c)}, \text{ where } a, b, c \text{ are constants.}$$

$$\therefore \quad x^2=at^2+2bt+c. \qquad \text{...(1)}$$

Differentiating w.r.t. t, we get

$$2x\,\frac{dx}{dt}=2at+2b$$

or
$$\frac{dx}{dt}=\frac{at+b}{x}. \qquad \text{...(2)}$$

Differentiating again w.r.t. 't', we have

$$\frac{d^2x}{dt^2}=\frac{ax-(at+b)\,(dx/dt)}{x^2}=\frac{ax-(at+b)\,\{(at+b)/x\}}{x^2}, \text{ [from (2)]}$$

$$=\frac{ax^2-(at+b)^2}{x^3}=\frac{a\,(at^2+2bt+c)-(a^2t^2+2abt+b^2)}{x^3}$$

$$=\frac{ac-b^2}{x^3}=(\text{some constant}).\frac{1}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance x from the fixed point.

Ex. 9. *If a point moves in a straight line in such a manner that its retardation is proportional to its speed, prove that the space described in any time is proportional to the speed destroyed in that time.*

Sol. Here it is given that the retardation $\propto$ speed.

$$\therefore \quad -\frac{dv}{dt}=kv, \quad \text{where } k \text{ is a constant of proportionality}$$

or
$$-v\,\frac{dv}{dx}=kv \quad \text{or} \quad dx=-\frac{1}{k}\,dv.$$

Integrating, $x=-(v/k)+A$,

where A is constant of integration.

Suppose the particle starts from the origin with velocity u. Then $v=u$, $x=0$.

$$\therefore \quad 0=-\frac{u}{k}+A \quad \text{or} \quad A=\frac{u}{k}.$$

$$\therefore \quad x=-\frac{v}{k}+\frac{u}{k}=\frac{1}{k}(u-v) \qquad ...(1)$$

or $$(u-v)=kx.$$

Now the space described in time t is x and the speed destroyed in time $t=u-v$. Hence from (1), we conclude that the space described in any time is proportional to the speed destroyed in that time.

Ex. 10. *Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.*

Sol. If x is the distance of the particle from the fixed point O at any time t, then its speed v at that time is given by $v=kx^n$, where k is a constant and n is not less than 1.

Since the particle is moving towards the fixed point *i.e.*, in the direction of x decreasing, therefore

$$dx/dt=-v$$

or $$dx/dt=-kx^n. \qquad ...(1)$$

Case I. If $n=1$, then from (1), we have

$$dx/dt=-kx$$

or $$dt=-\frac{1}{k}\frac{dx}{x}.$$

Integrating, $t=-(1/k)\log x+A$, where A is a constant.

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=-(1/k)\log 0+A=\infty$$

i.e., the particle will never reach the fixed point O.

Case II. If $n>1$, then from (1), we have

$$dt=-\frac{1}{k}x^{-n}\,dx.$$

Integrating, $t=-\dfrac{1}{k}\dfrac{x^{-n+1}}{-n+1}+B$, where B is a constant

or $$t=\frac{1}{k(n-1)x^{n-1}}+B.$$

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=\infty+B=\infty$$

i.e., the particle will never reach the fixed point O.

Hence if $n\geqslant 1$, the particle will never reach the fixed point it is approaching.

Ex. 11. *The velocity of a particle moving along a straight line is given by the relation* $v^2=ax^2+2bx+c$. *Prove that the acceleration varies as the distance from a fixed point in the line.*

Sol. Here given that $v^2=ax^2+2bx+c$.

Differentiating w.r.t. 'x', we have

$$2v\frac{dv}{dx}=2ax+2b$$

or
$$f=v\frac{dv}{dx}=ax+b=a\left(x+\frac{b}{a}\right).$$

Let P be the position of the particle at time t.

If $x=-(b/a)$ is the fixed point O', then the distance of the particle at time t from O'

$$=O'P=x-\left(-\frac{b}{a}\right)=x+\frac{b}{a}.$$

$$\therefore\quad f=a.O'P \quad \text{or} \quad f \propto O'P.$$

Hence the acceleration varies as the distance from a fixed point $x=-(b/a)$ in the line.

Ex. 12. *If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by* $f^3\,(d^2t/dv^2)$, *f being the acceleration.* **[Meerut 80. 83]**

Sol. Let f be the acceleration at time t. Then $f=dv/dt$.

Now the rate of decrease of acceleration $=-df/dt$

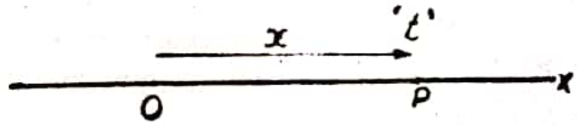
$$=-\frac{d}{dt}\left(\frac{dv}{dt}\right)=-\frac{d}{dt}\left(\frac{dt}{dv}\right)^{-1},$$

regarding t as a function of v

$$=-\left\{\frac{d}{dv}\left(\frac{dt}{dv}\right)^{-1}\right\}\cdot\frac{dv}{dt}=\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\cdot\frac{dv}{dt}$$

$$=\left(\frac{dv}{dt}\right)^2\cdot\frac{dv}{dt}\cdot\frac{d^2t}{dv^2}=\left(\frac{dv}{dt}\right)^3\cdot\frac{d^2t}{dv^2}=f^3\frac{d^2t}{dv^2}.$$

§ 2. Motion under constant acceleration. *A particle moves in a straight line with a constant acceleration f, the initial velocity being u, to discuss the motion.* **[Meerut 78]**



Suppose a particle moves in a straight line OX starting from O with velocity u. Take O as origin. Let P be the position of the particle at any time t, where $OP=x$. The acceleration of P is constant and is f. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=f. \qquad \ldots(1)$$

If v is the velocity of the particle at any time t, then $v=dx/dt$. So integrating (1) w.r.t. t, we get

$v=dx/dt=ft+A$, where A is constant of integration.

But initially at O, $v=u$ and $t=0$; therefore $A=u$. Thus we have

$$v=\frac{dx}{dt}=u+ft. \qquad \text{...(2)}$$

The equation (2) gives the velocity v of the particle at any time t.

Now integrating (2) w.r.t. 't', we get

$x=ut+\frac{1}{2}ft^2+B$, where B is a constant.

But at O, $t=0$ and $x=0$; therefore $B=0$. Thus we have

$$x=ut+\tfrac{1}{2}ft^2. \qquad \text{...(3)}$$

The equation (3) gives the position of the particle at any time t.

The equation of motion (1) can also be written as

$$v\frac{dv}{dx}=f \quad \text{or} \quad 2v\frac{dv}{dx}=2f.$$

Integrating it w.r.t. x, we get

$v^2=2fx+C$. But at O, $x=0$ and $v=u$; therefore $C=u^2$.

Hence we have

$$v^2=u^2+2fx. \qquad \text{...(4)}$$

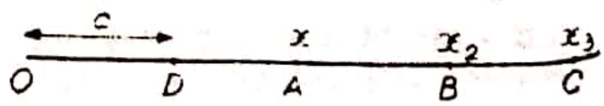
Thus in equations (2), (3) and (4) we have obtained the three well known formulae of rectiliner motion with constant acceleration.

Illustrative Fxamples

Ex. 13. *A particle moves in a straight line with constant acceleration and its distances from the origin O on the line (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively. Show that if t_1, t_2, t_3 form an A. P. whose common difference is d and x_1, x_2, x_3 are in G. P., then the acceleration is*

$$(\sqrt{x_1}-\sqrt{x_3})^2/d^2.$$

Sol. Let O be the origin and D the point of start *i.e.*, the position at $t=0$.



Let $OD=c$. Suppose u is the initial velocity and f the constant acceleration. Let A, B, C be the positions of the particle at times t_1, t_2, t_3 respectively and let $OA=x_1$, $OB=x_2$ and $OC=x_3$. Then

$$x_1-c=ut_1+\tfrac{1}{2}ft_1^2,\ x_2-c=ut_2+\tfrac{1}{2}ft_2^2,\ x_3-c=ut_3+\tfrac{1}{2}ft_3^2.$$

These equations give

$$x_1+x_3-2x_2=u\,(t_1+t_3-2t_2)+\tfrac{1}{2}f\,(t_1^2+t_3^2-2t_2^2). \qquad \ldots(1)$$

But x_1, x_2, x_3 are in G.P., so that $x_2=\sqrt{(x_1\,x_3)}$. Also t_1, t_2, t_3 are in A.P. whose common difference is d. Therefore $t_1+t_3=2\,t_2$ and $t_3-t_1=2d$. Putting these values in (1), we get

$$x_1+x_3-2\sqrt{(x_1\,x_3)}=u.0+\tfrac{1}{2}f\left[t_1^2+t_3^2-2\left(\frac{t_1+t_3}{2}\right)^2\right].$$

$$\therefore \quad (\sqrt{x_1}-\sqrt{x_3})^2=\tfrac{1}{4}f\,[2t_1^2+2t_3^2-(t_1^2+t_3^2+2t_1t_3)]$$

$$=\tfrac{1}{4}f\,(t_3-t_1)^2=\tfrac{1}{4}f\,(2d)^2=fd^2.$$

Hence $\quad f=(\sqrt{x_1}-\sqrt{x_3})^2/d^2.$

Ex. 14. *Two cars start off to race with velocities u and u' and travel in a straight line with uniform accelerations f and f' respectively. If the race ends in a dead heat, prove that the length of the course is*

$$\{2\,(u-u')\,(uf'-u'f)\}/(f-f')^2.$$

Sol. Let s be the length of the course. By dead heat we mean that each car moves the distance s in the same time, say t. Then considering the motion of the first car we have $s=ut+\frac{1}{2}ft^2$; and considering the motion of the second car, we have $s=u't+\frac{1}{2}f't^2$. These equations can be written as

$$\tfrac{1}{2}ft^2+ut-s=0, \qquad \ldots(1)$$

and

$$\tfrac{1}{2}f't^2+u't-s=0. \qquad \ldots(2)$$

By the method of cross multiplication, we get from (1) and (2)

$$\frac{t^2}{\begin{vmatrix} u & -s \\ u' & -s \end{vmatrix}}=\frac{t}{\begin{vmatrix} -s & \frac{1}{2}f \\ -s & \frac{1}{2}f' \end{vmatrix}}=\frac{1}{\begin{vmatrix} \frac{1}{2}f & u \\ \frac{1}{2}f' & u' \end{vmatrix}}$$

or

$$\frac{t^2}{(u'-u)\,s}=\frac{t}{\frac{1}{2}s\,(f-f')}=\frac{1}{\frac{1}{2}\,(fu'-f'u)}.$$

Eliminating t, we have

$$\frac{(u'-u)\,s}{\frac{1}{2}\,(fu'-f'u)}=\left[\frac{\frac{1}{2}s\,(f-f')}{\frac{1}{2}\,(fu'-f'u)}\right]^2=\frac{s^2\,(f-f')^2}{(fu'-f'u)^2}.$$

Since $s\neq 0$, therefore $s=\{2\,(u'-u)\,(fu'-f'u)\}/(f-f')^2$

$=\{2\,(u-u')\,(uf'-u'f)\}/(f-f')^2.$

Ex. 15. *Two particles P and Q move in a straight line AB. The particle P starts from A in the direction AB with velocity u and constant acceleration f, and at the same time Q starts from B in the direction BA with velocity u_1 and constant acceleration f_1; if they pass one another at the middle point of AB and arrive at the other ends of AB with equal velocities, prove that*

$$(u+u_1)(f-f_1)=8(fu_1-f_1u).$$

Sol. Let $AB=2s$. Let v be the velocity of either particle after moving the distance $AB=2s$. Then

$$v^2=u^2+2f(2s)=u_1^2+2f_1(2s).$$

$$\therefore \quad s=\frac{u^2-u_1^2}{4(f_1-f)}.$$

Now let t be the time taken by each particle to reach the middle point of AB. Then each particle moves distance s in time t. Therefore

$$s=ut+\tfrac{1}{2}ft^2=u_1t+\tfrac{1}{2}f_1t^2. \qquad \ldots(1)$$

Since $t\neq0$, therefore from (1), we have $u+\frac{1}{2}ft=u_1+\frac{1}{2}f_1t$

or $$t=2(u-u_1)/(f_1-f).$$

Now considering the motion of the particle P to cover the first half of the journey AB and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$\frac{u^2-u_1^2}{4(f_1-f)}=u.\frac{2(u-u_1)}{(f_1-f)}+\tfrac{1}{2}f\,\frac{4(u-u_1)^2}{(f_1-f)^2}$$

or $$(u+u_1)(f_1-f)=8u(f_1-f)+8f(u-u_1) \qquad [\because \; u-u_1\neq0]$$

or $$(u+u_1)(f_1-f)=8(uf_1-fu_1)$$

or $$(u+u_1)(f-f_1)=8(fu_1-f_1u).$$

Ex. 16. *A train travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of journey it moves with constant acceleration f and in the second part with constant retardation f′. Show that if s is the distance between the two stations, then*

$$t=\sqrt{[2s(1/f+1/f')]}.$$

Sol. Let v be the velocity at the end of the first part of the motion, or say in the beginning of the second part of the motion and t_1 and t_2 be the times for the two motions respectively. Then $t=t_1+t_2$.

Let x be the distance described in the first part. Then the distance described in the second part is $s-x$. Considering the first part of the motion with constant acceleration f, we have

$$\left.\begin{aligned} &v=0+ft_1=ft_1,\\ \text{and} \quad &v^2=0+2fx=2fx. \end{aligned}\right\} \qquad \ldots(1)$$

Again considering the second part of the motion with constant retardation f', we have

$$\left.\begin{aligned} &0=v-f't_2 \text{ i.e., } v=f't_2,\\ \text{and} \quad &0=v^2-2f'(s-x) \text{ i.e., } v^2=2f'(s-x). \end{aligned}\right\} \qquad \ldots(2)$$

From (1) and (2), we have

$$(s-x)+x=\frac{v^2}{2f'}+\frac{v^2}{2f}, \text{ or } s=\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{f'}\right). \qquad \ldots(3)$$

Also $t_1+t_2=v/f+v/f'=v\ (1/f+1/f')$. ...(4)

Substituting the value of v from (3) in (4), we get

$$t=t_1+t_2=\sqrt{\left\{\frac{2s}{(1/f+1/f')}\right\}}\cdot\left(\frac{1}{f}+\frac{1}{f'}\right)=\sqrt{\left[2s\left(\frac{1}{f}+\frac{1}{f'}\right)\right]}.$$

Ex. 17. *A point moving in a straight line with uniform acceleration describes distances a, b feet in successive intervals of t_1, t_2 seconds. Prove that the acceleration is* $2\ (t_1b-t_2a)/[t_1t_2\ (t_1+t_2)]$.

[Kanpur 1981; Meerut 69, 84S]

Sol. Let u be the initial velocity and f be the uniform acceleration of the particle. Then from $s=ut+\frac{1}{2}ft^2$, we have

$$a=ut_1+\tfrac{1}{2}ft_1^2 \quad ...(1)$$

and

$$a+b=u\ (t_1+t_2)+\tfrac{1}{2}f\ (t_1+t_2)^2. \quad ...(2)$$

Subtracting (1) from (2), we have

$$b=ut_2+\tfrac{1}{2}f\ (t_2^2+2t_1t_2). \quad ...(3)$$

Multiplying (3) by t_1 and (1) by t_2 and subtracting, we have

$$bt_1-at_2=\tfrac{1}{2}f\ (t_2^2+2t_1t_2)\ t_1-\tfrac{1}{2}f\ t_1^2t_2$$

$$=\tfrac{1}{2}f\ (t_2^2t_1+t_1^2t_2)=\tfrac{1}{2}f\ t_1t_2\ (t_2+t_1).$$

$$\therefore \quad f=\frac{2\ (bt_1-at_2)}{t_1t_2\ (t_1+t_2)}.$$

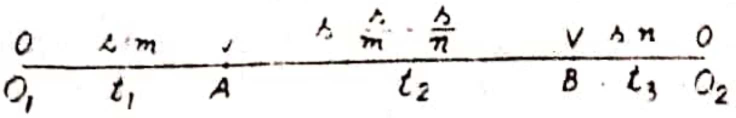
Ex. 18. *For $1/m$ of the distance between two stations a train is uniformly accelerated and for $1/n$ of the distance it is uniformly retarded ; it starts from rest at one station and comes to rest at the other. Prove that the ratio of its greatest velocity to its average velocity is* $\left(1+\frac{1}{m}+\frac{1}{n}\right):1$.

[Meerut 1977]

Sol. Let O_1 and O_2 be two stations at a distance s apart and A and B two points between O_1 and O_2 such that

$$O_1A=s/m \qquad \text{and} \qquad BO_2=s/n.$$

$$\therefore \quad AB=s-s/m-s/n.$$



The train starts at rest from O_1 and moves with uniform acceleration f from O_1 to A. Let V be its velocity at the point A. It moves with constant velocity V from A to B and then moves with uniform retardation f' from B to O_2. The velocity at the station O_2 is zero.

Let t_1, t_2, t_3 be the times taken to travel the distances O_1A, AB and BO_2 respectively.

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Now the greatest velocity of the train during its journey from O_1 to $O_2 = V$

and the average velocity of the train $= s/(t_1+t_2+t_3)$.

$\therefore$ the required ratio $= \dfrac{\text{greatest velocity}}{\text{average velocity}} = \dfrac{V}{s/(t_1+t_2+t_3)}$

$$= \frac{V(t_1+t_2+t_3)}{s}. \qquad \ldots(1)$$

For motion from O_1 to A, using the formula $v=u+ft$, we have

$V=0+ft_1$. $\qquad \therefore \quad f=\dfrac{V}{t_1}$.

Now using the formula $s=ut+\frac{1}{2}ft^2$ for the same motion, we have

$$\frac{s}{m}=0+\tfrac{1}{2}\,\frac{V}{t_1}\,t_1^2$$

or
$$t_1=\frac{2s}{Vm}. \qquad \ldots(2)$$

For motion from A to B, $AB=V.t_2$.

$$\therefore \quad t_2=\frac{AB}{V}=\frac{s-s/m-s/n}{V}. \qquad \ldots(3)$$

For motion from B to O_2, using the formula $v=u+ft$, we have

$0=V-f't_3$. $\qquad \therefore \quad f'=V/t_3$.

Using the formula $s=ut+\frac{1}{2}ft^2$ for the same motion, we have

$$\frac{s}{n}=Vt_3-\tfrac{1}{2}.\frac{V}{t_3}.t_3^2=\frac{Vt_3}{2}$$

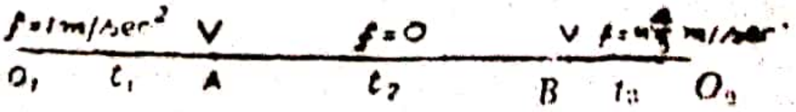
or
$$t_3=\frac{2s}{Vn}. \qquad \ldots(4)$$

Substituting from (2), (3) and (4) in (1), the required ratio

$$=\frac{V\left\{\frac{2s}{Vm}+\frac{1}{V}\left(s-\frac{s}{m}-\frac{s}{n}\right)+\frac{2s}{Vn}\right\}}{s}=\frac{\frac{1}{m}+\frac{1}{n}+1}{1}.$$

Ex. 19. *The greatest possible acceleration of a train is 1 m/sec² and the greatest possible retardation is $\frac{4}{3}$ m/sec². Find the least time taken to run between two stations 12 km. appart if the maximum speed is 22 m/sec.* **[Meerut 1972, 76, 88S]**

Sol. Let a train start from the station O_1 and move with uniform acceleration 1 m/sec² upto A for time t_1 seconds.



Let the velocity of the train at A be $V=22$ m/sec. Then the train moves with constant velocity V from A to B for time t_2 seconds. In the last the train moves from B to the second station O_2 under constant retardation $\frac{4}{3}$ m/sec.^{2} for time t_3 seconds. Thus the least time to travel between the two stations O_1 and O_2 is $(t_1+t_2+t_3)$ seconds.

Also $O_1O_2=12$ km.$=12000$ meters.

Now using the formula $v=u+ft$ for the parts O_1A and BO_2 of the journey, we have

$$V=22=0+1.\ t_1 \text{ so that } t_1=22,$$

$$\text{and}\quad 0=22-\frac{4}{3}\,t_3 \text{ so that } t_3=\frac{33}{2}.$$

Now $O_1A=$(Average velocity from O_1 to A)$\times t_1$

$$=\frac{0+22}{2}\times 22=242 \text{ meters,}$$

$$\text{and } BO_2=\frac{22+0}{2}\times\frac{33}{2}=\frac{363}{2} \text{ meters.}$$

$$\therefore\quad AB=O_1O_2-O_1A-BO_2=12000-242-\frac{363}{2}$$

$$=\frac{23153}{2} \text{ meters.}$$

$$\therefore\quad t_2=\frac{AB}{V}=\frac{23153}{2\times 22}=\frac{23153}{44} \text{ seconds.}$$

$\therefore$ the required time$=(t_1+t_2+t_3)$ seconds

$$=\left(22+\frac{33}{2}+\frac{23153}{44}\right) \text{ seconds}=\frac{24847}{44} \text{ seconds}$$

$=9$ minutes 25 seconds approximately.

Ex. 20. *Two points move in the same straight line starting at the same moment from the same point in the same direction. The first moves with constant velocity u and the second with constant acceleration f (its initial velocity being zero). Show that the greatest distance between the points before the second catches first is $u^2/2f$ at the end of the time u/f from the first.*

Sol. If s_1 and s_2 are the distances moved by the two particles in time t, then

$$s_1=ut \text{ and } s_2=0+\tfrac{1}{2}ft^2.$$

$\therefore$ the distance s between the two particles at time t is given by

$$s=s_1-s_2=ut-\tfrac{1}{2}ft^2=\frac{f}{2}\left(\frac{2u}{f}t-t^2\right)$$

or $$s=\frac{f}{2}\left[\frac{u^2}{f^2}-\left(t-\frac{u}{f}\right)^2\right].\quad \ldots(1)$$

Now s is greatest if $(t-u/f)^2=0$ *i.e.*, if $t=u/f$.

Also the greatest value of $s=\frac{f}{2}\cdot\frac{u^2}{f^2}=\frac{u^2}{2f}$.

Ex. 21. *The speed of a train increases at constant rate α from zero to v, then remains constant for an interval and finally decreases to zero at a constant rate β. If l be the total distance described, prove that the total time occupied is $(l/v)+(v/2)(1/\alpha+1/\beta)$. Also find the least value of time when $\alpha=\beta$.* **[Allahabad 1975]**

Sol. Let t_1, t_2, t_3 be the times taken to cover the distances x, y, z of the first, second and last phase of the journey. Whole distance $l=x+y+z$.

Equations for the first and last part of the journey are

$$\left.\begin{array}{l} v^2=2\alpha x, \\ \text{and } v=\alpha t_1 \end{array}\right\}\ \ldots(1); \qquad \left.\begin{array}{l} v^2=2\beta z. \\ \text{and } v=\beta t_3 \end{array}\right\}.\quad (2)$$

From (1), on eliminating α, we have $x=\frac{1}{2}vt_1$; and from (2), on eliminating β, we have $z=\frac{1}{2}vt_3$.

Also considering the motion for the middle part of the journey, we have $y=vt_2$.

Thus $x+y+z=v\left(\frac{1}{2}t_1+t_2+\frac{1}{2}t_3\right)$

i.e., $$l=v\left[(t_1+t_2+t_3)-\tfrac{1}{2}(t_1+t_3)\right]$$

or $$\frac{l}{v}=(t_1+t_2+t_3)-\tfrac{1}{2}(t_1+t_3).$$

$\therefore$ the total time occupied *i.e.*, $t_1+t_2+t_3=(l/v)+\frac{1}{2}(t_1+t_3)$

$$=\frac{l}{v}+\frac{1}{2}\left(\frac{v}{\alpha}+\frac{v}{\beta}\right), \quad \text{[from (1) and (2)]}$$

$$=\frac{l}{v}+\frac{1}{2}v\left(\frac{1}{\alpha}+\frac{1}{\beta}\right). \quad \ldots(3)$$

Let t denote the total time occupied when $\alpha=\beta$.

Then putting $\alpha=\beta$ in the above result (3), we have $t=\frac{l}{v}+\frac{v}{\alpha}$. Therefore $\frac{dt}{dv}=-\frac{l}{v^2}+\frac{1}{\alpha}$.

For least value of t, we have $dt/dv=0$, *i.e.*, $-\frac{l}{v^2}+\frac{1}{\alpha}=0$

i.e., $\frac{l}{v}=\frac{v}{\alpha}$ *i.e.*, $v=\sqrt{(l\alpha)}$.

Also then the time $=2\left(\frac{l}{v}\right)=\frac{2l}{\sqrt{(l\alpha)}}=2\sqrt{(l/\alpha)}$. This time is least because $d^2t/dv^2=2l/v^3$ which is positive for $v=\sqrt{(l\alpha)}$.

Ex. 22. *A lift ascends with constant acceleration f, then with constant velocity and finally stops under constant retardation f. If the total distance ascended is s and the total time occupied is t, show that the time during which the lift is ascending with constant velocity is* $\sqrt{\{t^2-(4s/f)\}}$.

Sol. Let v be the maximum velocity produced during the ascent. Since this velocity is produced under a constant acceleration f during the first part of the ascent and destroyed under the same retardation f during the last part of the ascent, therefore, the distances as well as the times for these two ascents are equal. Let x be the distance and t_1 the time for each of these two parts. We have then

$$\left.\begin{aligned} v^2&=2fx, \\ \text{and} \quad v&=ft_1 \end{aligned}\right\} \qquad \ldots(1)$$

for the first and last part of the motion.

Also considering the middle part of the motion, we have

$$v\,(t-2t_1)=s-2x.$$

From (1) and (2), on eliminating v and x, we have

$$ft_1\,(t-2t_1)=s-\frac{v^2}{f}=s-\frac{f^2t_1^2}{f}=s-ft_1^2.$$

$$\therefore \quad ft_1^2-ftt_1+s=0.$$

Solving this as a quadratic in t_1, we get

$$t_1=\frac{ft\pm\sqrt{(f^2t^2-4fs)}}{2f}$$

or $$2t_1=t\pm\sqrt{\left(t^2-\frac{4s}{f}\right)} \quad \text{or} \quad t-2t_1=\sqrt{\left(t^2-\frac{4s}{f}\right)}.$$

This gives the time of ascent with constant velocity.

Ex. 23. *Prove that the shortest time from rest to rest in which a steady load of P tons can lift a weight of W tons through a vertical distance h feet is* $\sqrt{\{(2h/g).P/(P-W)\}}$ *seconds.*

Sol. The time will be shortest if the load acts continuously during the first part of the ascent. Let f be the acceleration during the first part of the ascent. Then by Newton's second law of motion, f is given by

$$P-W=(W/g)\,f. \qquad \ldots(1)$$

During the second part of the ascent, P ceases to act and W then moves only under gravity. Therefore the retardation is g.

Let x and y be the distances and t_1, t_2 the corresponding times for the two parts in the ascent.

If v be the velocity at the end of the first part of the ascent or at the beginning of the second part of the ascent, we have then

$$\left.\begin{aligned} v^2&=2fx, \\ v&=ft_1 \end{aligned}\right\} \qquad \dots(2)$$

[Equations for the first part of the ascent]

and

$$\left.\begin{aligned} v^2&=2gy \\ v&=gt_2 \end{aligned}\right\} \qquad \dots(3)$$

[Equations for the second part of the ascent]

Also $x+y=h$ (given)

From (2) and (3), we get

$$\frac{v^2}{2f}+\frac{v^2}{2g}=x+y$$

i.e.,

$$\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h. \qquad \dots(4)$$

Also

$$\frac{v}{f}+\frac{v}{g}=t_1+t_2. \qquad \dots(5)$$

Now the total time of ascent

$$=t_1+t_2=\left(\frac{1}{f}+\frac{1}{g}\right)v \qquad \text{[from (5)]}$$

$$=\left(\frac{1}{f}+\frac{1}{g}\right)\sqrt{\left[2h\Big/\left(\frac{1}{f}+\frac{1}{g}\right)\right]} \qquad \text{[from (4)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}=\sqrt{\left[\frac{2h}{g}\left(\frac{g}{f}+1\right)\right]}$$

$$=\sqrt{\left[\frac{2h}{g}\left(\frac{W}{P-W}+1\right)\right]} \qquad \text{[from (1)]}$$

$$=\sqrt{\left[\frac{2h}{g}\cdot\frac{P}{P-W}\right]}.$$

Ex. 24. *Prove that the mean kinetic energy of a particle of mass m moving under a constant force, in any interval of time is* $\frac{1}{6}m\,(u_1^2+u_1u_2+u_2^2)$, *where* u_1 *and* u_2 *are the initial and final velocities.*

Sol. Let the interval of time during which the particle moves be T. If the particle moves under a constant acceleration f and v be its velocity at any time t, we have $v=u_1+ft$.

Now the mean kinetic energy of the particle during the time T

$$=\frac{1}{T}\int_0^T \tfrac{1}{2}mv^2\,dt=\frac{m}{2T}\int_0^T (u_1+ft)^2\,dt=\frac{m}{2fT}\cdot\frac{1}{3}\Big[(u_1+ft)^3\Big]_0^T$$

$$=\frac{m}{6fT}\Big[(u_1+fT)^3-u_1^3\Big]=\frac{m}{6\,(u_2-u_1)}\,(u_2^3-u_1^3)$$

$[\because\ u_2=u_1+fT$ and so $u_2-u_1=fT]$

$$=\tfrac{1}{6}m\,(u_1^2+u_1u_2+u_2^2).$$

Ex. 25. *A bullet fired into a target loses half its velocity after penetrating 3 cm. How much further will it penetrate ?*

[Meerut 1972, 76, 79S, 83, 86P, 88]

Sol. If u cm./sec. is the initial velocity of the bullet then its velocity after penetrating 3 cm. will be $\frac{1}{2}u$ cm./sec. Let f cm./sec^2. be the retardation of the bullet.

Then from $v^2=u^2+2fs$, we have

$$(u/2)^2=u^2-2.f\,3 \text{ giving } f=u^2/8.$$

If the bullet penetrates further by a cm, then from $v^2=u^2+2fs$, we have

$$0=(u/2)^2-2.(u^2/8).a.$$

$$\therefore \quad a=1 \text{ cm.}$$

Ex. 26. *A load W is to be raised by a rope, from rest to rest, through a height h; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is* $[2nh/(n-1)\,g]^{1/2}$. **[Meerut 1986]**

Sol. Obviously the time for ascent is least when the acceleration of the load is greatest. If m is the mass of the load, then $W=mg$ or $m=W/g$. Let f be the greatest acceleration of the load in the upward direction. Since the rope can bear the greatest tension nW, therefore when f is the greatest acceleration of the load, then the tension T in the rope is nW.

$\therefore$ by Newton's second law of motion $P=mf$, we have

$$T-W=nW-W=mf \text{ or } f=(n-1)\,(W/m)=(n-1)\,g. \quad \ldots(1)$$

Let the load W move upwards upto the height h_1 under the acceleration f. After that the tension in the rope ceases to act and therefore above the height h_1 the load will move under gravity which acts vertically downwards. If the load comes to rest after moving through a subsequent height h_2 above the height h_1, then according to the question

$$h_1+h_2=h. \quad \ldots(2)$$

If V is the maximum velocity of the load acquired at the end of the first part and t_1, t_2 are the times taken for describing the heights h_1 and h_2 respectively, then from $v=u+ft$, we have

$$V=0+ft_1 \quad \text{and} \quad 0=V-gt_2.$$

$$\therefore \quad t_1=V/f \quad \text{and} \quad t_2=V/g.$$

Also from $v^2=u^2+2fs$, we have

$$V^2=0+2fh_1 \quad \text{and} \quad 0=V^2-2gh_2.$$

$$\text{and} \quad h_2=\frac{V^2}{2g}.$$

Now from $h_1+h_2=h$, we have

$$\frac{V^2}{2f}+\frac{V^2}{2g}=h \text{ or } \frac{V^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h. \quad \ldots(3)$$

$$\therefore \quad V=\sqrt{\{2h/(1/f+1/g)\}}.$$

$\therefore$ the least time of ascent

$$=t_1+t_2=\frac{V}{f}+\frac{V}{g}=V\left(\frac{1}{f}+\frac{1}{g}\right)$$

$$=\sqrt{\left\{\frac{2h}{(1/f+1/g)}\right\}}.\left(\frac{1}{f}+\frac{1}{g}\right) \quad \text{[substituting for } V \text{ from (3)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}$$

$$=\sqrt{\left[2h\left\{\frac{1}{(n-1)\,g}+\frac{1}{g}\right\}\right]} \quad \text{[substituting for } f \text{ from (1)]}$$

$$=\left[\frac{2nh}{(n-1)g}\right]^{1/2}$$

§ 3. **Newton's Laws of Motion.** **[Allahabad 1979; Meerut 81]**

The Newton's laws of motion are as follows.

Law 1. *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled by some external force or forces to change its state.*

Law 2. *The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.*

Law 3. *To every action there is an equal and opposite reaction.*

§ 4. **Equation of motion** *of a particle moving in a straight line as deduced from the Newton's second law of motion.*

Let v be the velocity at time t of a particle of mass m moving in a straight line under the action of the impressed force P. Since from Newton's second law of motion the rate of change of momentum is proportional to the impressed force, therefore

$$P \propto \frac{d}{dt}(mv), \quad [\because \text{ by def.. momentum}=\text{mass}\times\text{velocity}]$$

or $\quad P=k\dfrac{d}{dt}(mv)$, where k is some constant

or $\quad P=km\dfrac{dv}{dt}$ provided m is constant

or $\quad P=kmf. \quad \ldots(1)$

$[\because f=\text{acceleration}=dv/dt]$.

Let us suppose that a unit force is that which produces a unit acceleration in a particle of unit mass. Then

$$P=1, \text{ when } m=1 \text{ and } f=1.$$

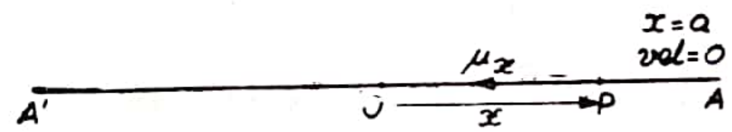
$\therefore$ from (1), we have $k=1$.

Hence we have, $P=mf$, which is the required equation of motion of the particle.

§ 5. **Simple Harmonic Motion. (S.H.M.) Definition.** *The kind of motion, in which a particle moves in a straight line in such a way that its acceleration is always directed towards a fixed point on the line (called the centre of force) and varies as the distance of the particle from the fixed point, is called simple harmonic motion.*

[Meerut 1976, 78, 79, 81, 82, 85, 86; Kanpur 76, 77; Lucknow 80; Agra 77]

Let O be the centre of force taken as origin. Suppose the particle starts from rest from the point A where $OA=a$. It begins to move towards the centre of attraction O. Let P be the position of the particle after time t, where $OP=x$. By the definition of S.H.M. the magnitude of acceleration at P is proportional to x.



Let it be μx, where μ is a constant called, the **intensity of force.** Also on account of a centre of attraction at O, the acceleration of P is towards O *i.e.*, in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\mu x, \qquad \ldots(1)$$

where the negative sign has been taken because the **force acting on P is towards O** *i.e.*, in the direction of x decreasing. The equation (1) gives the acceleration of the particle at any position.

Multiplying both sides of (1) by $2dx/dt$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}=-2\mu x\frac{dx}{dt}.$$

Integrating with respect to t, we get

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+C,$$

where C is a constant of integration and v is the velocity at P.

Initially at the point A, $x=a$ and $v=0$; therefore $C=\mu a^2$.

Thus, we have

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+\mu a^2$$

or

$$v^2=\mu(a^2-x^2). \qquad \ldots(2)$$

The equation (2) gives the velocity at any point P. From (2) we observe that v^2 is maximum when $x^2=0$ or $x=0$. **Thus in a S.H.M. the velocity is maximum at the centre of force O.** Let this maximum velocity be v_1. Then at O, $x=0$, $v=v_1$. So from (2) we get $v_1^2=\mu a^2$ or $v_1=a\sqrt{\mu}$.

Also from (2) we observe that $v=0$ when $x^2=a^2$ i.e., $x=\pm a$. **Thus in a S.H.M. the velocity is zero at points equidistant from the centre of force.**

Now from (2), on taking square root, we get $dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$, where the $-$ive sign has been taken because at P the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}=dt \qquad \ldots(3)$$

Integrating both sides, we get

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t+D, \text{ where } D \text{ is a constant.}$$

But initially at A, $x=a$ and $t=0$; therefore $D=0$.

Thus we have

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t \text{ or } \mathbf{x=a\cos(\sqrt{\mu}t)}. \qquad \ldots(4)$$

[Lucknow 1978]

The equation (4) gives a relation between x and t, where t is the time measured from A. If t_1 be the time from A to O, then at O, we have $t=t_1$ and $x=0$. So from (4), we get $t_1=\dfrac{1}{\sqrt{\mu}}\cos^{-1}0$

$=\dfrac{1}{\sqrt{\mu}}\dfrac{\pi}{2}=\dfrac{\pi}{2\sqrt{\mu}}$, which is independent of the initial displacement a of the particle. *Thus in a S.H.M. the time of descent to the centre of force is independent of the initial displacement of the particle.*

[Meerut 1984, 85]

Note. The time of descent t_1 from A to O can also be found from (3) with the help of the definite integrals $-\dfrac{1}{\sqrt{\mu}}\displaystyle\int_a^0\frac{dx}{\sqrt{(a^2-x^2)}}=\int_0^{t_1}dt.$

For fixing the limits of integration, we observe that at A, $x=a$ and $t=0$ while at O, $x=0$ and $t=t_1$.

Nature of Motion. The particle starts from rest at A where its acceleration is maximum and is μa towards O. It begins to move towards the centre of attraction O and as it approaches the centre of force O, its velocity goes on increasing. When the particle reaches O its acceleration is zero and its velocity is maximum and is $a\sqrt{\mu}$ in the direction OA'. Due to this velocity gained at O the particle moves towards the left of O. But on account of the centre of attraction at O a force begins to act upon the particle against its direction of motion. So its velocity goes on decreasing and it comes to instantaneous rest at A' where $OA'=OA$. The rest at A' is only instantaneous. The particle at once begins to move towards the centre of attraction O and retracing its path it again comes to instantaneous rest at A. Thus the motion of the particle is **oscillatory** and it continues to oscillate between A and A'. To start from A and to come back to A is called one *complete oscillation.*

Few Important Definitions :

1. **Amplitude.** *In a S.H.M. the distance from the centre of force of the position of maximum displacement is called the amplitude of the motion.* Thus the amplitude is the distance of a position of instantaneous rest from the centre of force. In the formulae (2) and (4) of this article the amplitude is a.

2. **Time period.** [**Kanpur 1977**]. *In a S.H.M. the time taken to make one complete oscillation is called time period or periodic time.* Thus if T is the time period of the S.H.M., then

$T=4.$ (time from A to O)$=4.\dfrac{\pi}{2\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}$, which is **independent of the amplitude** a. [**Meerut 1989**]

3. **Frequency.** *The number of complete oscillations in one second is called the frequency of the motion.* Since the time taken to make one complete oscillation is $\dfrac{2\pi}{\sqrt{\mu}}$ seconds, therefore if n is the frequency, then $n.\dfrac{2\pi}{\sqrt{\mu}}=1$ or $n=\dfrac{\sqrt{\mu}}{2\pi}$.

Thus the frequency is the reciprocal of the periodic time.

Important Remark 1. In a S.H.M. if the centre of force is not at origin but is at the point $x=b$, then the equation of motion is $d^2x/dt^2=-\mu\,(x-b)$. Similarly $d^2x/dt^2=-\mu\,(x+b)$ is the equation of a S.H.M. in which the centre of force is at the point $x=-b$.

Important Remark 2. In the above article when after instantaneous rest at A' the particle begins to move towards A, we have

from (2) $$\frac{dx}{dt}=+\sqrt{\mu}\sqrt{(a^2-x^2)},$$

where the $+$ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have $\dfrac{dx}{\sqrt{(a^2-x^2)}}=\sqrt{\mu}\,dt$.

Integrating, we get $-\cos^{-1}(x/a)=\sqrt{\mu}\,t+B$. Now the time from A to A' is $\pi/\sqrt{\mu}$. Therefore at A', we have $t=\pi/\sqrt{\mu}$ and

$x=-a$. These give $-\cos^{-1}(-a/a)=\sqrt{\mu}.(\pi/\sqrt{\mu})+B$ or $-\cos^{-1}(-1)=\pi+B$ or $-\pi=\pi+B$ or $B=-2\pi$. Thus we have $-\cos^{-1}(x/a)=\sqrt{\mu}\,t-2\pi$ or $\cos^{-1}(x/a)=2\pi-\sqrt{\mu}t$ or $x=a\cos(2\pi-\sqrt{\mu}t)$ or $x=a\cos\sqrt{\mu}t$. Thus in S.H.M. the equation $x=a\cos\sqrt{\mu}t$ is valid throughout the entire motion from A to A' and back from A' to A.

4. Phase and Epoch. From equation (1), we have

$$\frac{d^2x}{dt^2}+\mu x=0,$$

which is a linear differential equation with constant coefficients and its general solution is given by

$$x=a\cos(\sqrt{\mu}t+\epsilon). \qquad \ldots(5)$$

The constant ϵ is called the **starting phase** or the **epoch** of the motion and the quantity $\sqrt{\mu}t+\epsilon$ is called the **argument** of the motion.

The **phase** *at any time t of a S.H.M. is the time that has elapsed since the particle passed through its extreme position in the positive direction.*

From (5), x is maximum when $\cos(\sqrt{\mu}t+\epsilon)$ is maximum *i.e.*, when $\cos(\sqrt{\mu}t+\epsilon)=1$.

Therefore if t_1 is the time of reaching the extreme position in the positive direction, then

$$\cos(\sqrt{\mu}t_1+\epsilon)=1$$

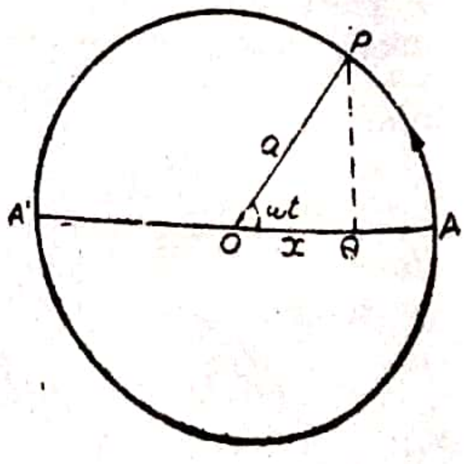
or $$\sqrt{\mu}t_1+\epsilon=0 \quad \text{or} \quad t_1=-\frac{\epsilon}{\sqrt{\mu}}$$

$\therefore$ **the phase at time** $t=t-t_1=t+\dfrac{\epsilon}{\sqrt{\mu}}$.

5. Periodic Motion. *A point is said to have a* **periodic motion** *when it moves in such a manner that after a certain fixed interval of time called periodic time it acquires the same position and moves with the same velocity in the same direction. Thus S.H.M. is a periodic motion.*

§ 6. Geometrical representation of S.H.M. [Lucknow 1975]

Let a particle move with a uniform angular velocity ω, round the circumference of a circle of radius a. Suppose AA' is a fixed diameter of the circle. If the particle starts from A and P is its position at time t, then $\angle AOP=\omega t$.



Draw PQ perpendicular to the diameter AA'.

If $OQ=x$, then

$$x=a \cos \omega t. \qquad ..(1)$$

As the particle P moves round the circumference, the foot Q of the perpendicular on the diameter AA' oscillates on AA' from A to A' and from A' to A back. Thus the motion of the point Q is periodic.

From (1), we have

$$\frac{dx}{dt}=-a\omega \sin \omega t \qquad ...(2)$$

and

$$\frac{d^2x}{dt^2}=-a\omega^2 \cos \omega t=-\omega^2 x. \qquad ...(3)$$

The equations (2) and (3) give the velocity and acceleration of Q at any time t.

The equation (3) shows that Q executes a simple harmonic motion with centre at the origin O. From equation (1), we see that the amplitude of this S.H.M. is a because the maximum value of x is a.

The periodic time of Q= The time required by P to turn through an angle 2π with a uniform angular velocity ω

$$=\frac{2\pi}{\omega}.$$

Thus if a particle describes a circle with constant angular velocity, the foot of the perpendicular from it on any diameter executes a S.H.M.

§ 7. **Important results about S.H.M.**

We summarize the important relations of a S.H.M. as follows : (Remember them).

(i) Referred to the centre as origin the equation of S.H.M. is $\ddot{x}=-\mu x$,

or the equation $\ddot{x}=-\mu x$ represents a S.H.M. with centre at the origin.

(ii) The velocity v at a distance x from the centre and the distance x from the centre at time t are respectively given by

$$v^2=\mu (a^2-x^2) \text{ and } x=a \cos \sqrt{\mu} t,$$

where a is the amplitude and the time t has been measured from the extreme position in the positive direction.

(iii) Maximum acceleration $=\mu a$, (at extreme points)

(iv) Maximum velocity $=\sqrt{\mu} a$, (at the centre)

(v) Periodic time $T=\dfrac{2\pi}{\sqrt{\mu}}$.

$\left(T=\dfrac{2\pi}{\sqrt{K}}\right)$

(vi) *Frequency* $n=\frac{1}{T}=\frac{\sqrt{\mu}}{2\pi}$.

Illustrative Examples.

Ex. 27. *The maximum velocity of a body moving with S.H.M. is 2 ft./sec. and its period is $\frac{1}{5}$ sec. What is its amplitude ?*

Sol. Let the amplitude be a ft. Then the maximum velocity $=a\sqrt{\mu}$ ft./sec. $=2$ ft./sec. (given).

$$\therefore \quad a\sqrt{\mu}=2. \quad \ldots(1)$$

Also the time period $T=2\pi/\sqrt{\mu}$ seconds $=\frac{1}{5}$ seconds (given)

$$\therefore \quad \frac{2\pi}{\sqrt{\mu}}=\frac{1}{5}. \quad \ldots(2)$$

Multiplying (1) and (2) to eliminate μ, we have

$$2\pi a=\frac{2}{5}. \qquad \therefore \quad a=\frac{1}{5\pi}.$$

$\therefore$ the required amplitude $=\frac{1}{5\pi}$ ft. $=\cdot064$ ft. nearly.

Ex. 28. *At what distance from the centre the velocity in a S.H.M. will be half of the maximum ?*

Sol. Take the centre of the motion as origin. Let a be the amplitude. In a S.H.M., the velocity v of the particle at a distance x from the centre is given by

$$v^2=\mu\ (a^2-x^2). \quad \ldots(1)$$

From (1), v is max. when $x=0$. Therefore max velocity $=\sqrt{\mu}a$.

Let x_1 be the distance from the centre of the point where the velocity is half of the maximum *i.e.*, where the velocity is $\frac{1}{2}a\sqrt{\mu}$. Then putting $x=x_1$ and $v=\frac{1}{2}a\sqrt{\mu}$ in (1), we get

$$\tfrac{1}{4}\ a^2\mu=\mu\cdot(a^2-x_1^2), \text{ or } \tfrac{1}{4}\ a^2=a^2-x_1^2$$

or $$x_1^2=\frac{3a^2}{4} \text{ or } x_1=\pm a\sqrt{3}/2.$$

Thus there are two points, each at a distance $a\sqrt{3}/2$ from the centre, where the velocity is half of the maximum.

Ex. 29. *A particle moves in a straight line and its velocity at a distance x from the origin is $k\sqrt{(a^2-x^2)}$, where a and k are constants. Prove that the motion is simple harmonic and find the amplitude and the periodic time of the motion.*

Sol. We know that in a rectilinear motion the expression for velocity at a distance x from the origin is dx/dt. So according to the question, we have

$$\ldots(1)$$

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=k^2\left(-2x\frac{dx}{dt}\right).$$

$\therefore\quad \dfrac{d^2x}{dt^2}=-k^2x$, which is the equation of a S. H. M. with centre at the origin and $\mu=k^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{k^2}=2\pi/k$.

Now to find the amplitude we are to find the distance from the centre of a point where the velocity is zero. So putting $dx/dt=0$ in (1), we get $0=k^2(a^2-x^2)$ or $x=\pm a$. Since here the centre is at origin, therefore the amplitude $=a$.

Ex. 30. *Show that if the displacement of a particle in a straight line is expressed by the equation $x=a\cos nt+b\sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{(a^2+b^2)}$ and period is $2\pi/n$.* **[Meerut 1977]**

Sol. Given $x=a\cos nt+b\sin nt$. ...(1)

$\therefore\quad dx/dt=-an\sin nt+bn\cos nt,$ ...(2)

and $\quad d^2x/dt^2=-an^2\cos nt-bn^2\sin nt=-n^2(a\cos nt+b\sin nt)$
$=-n^2x$, from (1).

Now $d^2x/dt^2=-n^2x$ is the equation of a S. H. M. with centre at the origin and $\mu=n^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{n^2}=2\pi/n$. Also the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is at origin, therefore the amplitude is the value of x when $dx/dt=0$. Putting $dx/dt=0$ in (2), we get

$0=-an\sin nt+bn\cos nt\quad$ or $\quad\tan nt=b/a$.

$\therefore\quad \sin nt=b/\sqrt{(a^2+b^2)}\quad$ and $\quad\cos nt=a/\sqrt{(a^2+b^2)}$.

Substituting these in (1), we have

$$\text{the amplitude}=a\,\frac{a}{\sqrt{(a^2+b^2)}}+b\cdot\frac{b}{\sqrt{(a^2+b^2)}}=\frac{a^2+b^2}{\sqrt{(a^2+b^2)}}$$
$$=\sqrt{(a^2+b^2)}.$$

Ex. 31. *The speed v of a particle moving along the axis of x is given by the relation $v^2=n^2(8bx-x^2-12b^2)$. Show that the motion is simple harmonic with its centre at $x=4b$, and amplitude $=2b$.*

Sol. Given $v^2=(dx/dt)^2=n^2(8bx-x^2-12b^2)$. ...(1)

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=n^2(8b-2x)\frac{dx}{dt}.$$

$\therefore \quad \frac{d^2x}{dt^2}=n^2(4b-x)=-n^2(x-4b)$, which is the equation of a S.H.M. with centre at the point $x-4b=0$ *i.e.*, at the point $x=4b$. [Note that centre is the point where the acceleration d^2x/dt^2 is zero.]

Now $v=0$ where $8bx-x^2-12b^2=0$ *i.e.*, $x^2-8bx+12b^2=0$ *i.e.*, $(x-6b)(x-2b)=0$ *i.e.*, $x=6b$ or $2b$. Thus the positions of instantaneous rest are given by $x=2b$ and $x=6b$. The distance of any of these two positions from the centre $x=4b$ is the amplitude.

Hence the amplitude is the distance of the point $x=6b$ from the point $x=4b$. Thus the amplitude$=6b-4b=2b$.

Ex. 32. *The speed v of the point P which moves in a line is given by the relation $v^2=a+2bx-cx^2$, where x is the distance of the point P from a fixed point on the path, and a, b, c are constants. Show that the motion is simple harmonic if c is positive ; determine the period and the amplitude of the motion.* **[Kanpur 1979]**

Sol. Here given that, $v^2=a+2bx-cx^2$. ...(1)

Differentiating both sides of (1) w.r.t. x, we have

$$2v\frac{dv}{dx}=2b-2cx.$$

$$\therefore \quad \frac{d^2x}{dt^2}=v\frac{dv}{dx}=-c\left(x-\frac{b}{c}\right). \quad ...(2)$$

Since c is positive, therefore the equation (2) represents a S. H. M. with the centre of force at the point $x=b/c$.

Hence the relation (1) represents a S. H. M. of period

$$T=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{c}},$$ because in the equation (2), $\mu=c$.

To determine the amplitude, putting $v=0$ in (1), we have

$$a+2bx-cx^2=0$$

or $$cx^2-2bx-a=0.$$

$$\therefore \quad x=\frac{b\pm\sqrt{(b^2+ac)}}{c}$$

$\therefore$ the distances of the two positions of instantaneous rest A and A' from the fixed point O are given by

$$OA=\frac{b+\sqrt{(b^2+ac)}}{c} \quad \text{and} \quad OA'=\frac{b-\sqrt{(b^2+ac)}}{c}$$

The distance of any of these two positions from the centre $x=(b/c)$ is the amplitude of the motion.

$$\therefore \quad \text{the amplitude}=\frac{b+\sqrt{(b^2+ac)}}{c}-\frac{b}{c}=\frac{\sqrt{(b^2+ac)}}{c}.$$

Ex. 33. *In a S. H. M. of period $2\pi/\omega$ if the initial displacement be x_0 and the initial velocity u_0, prove that*

(i) *amplitude* $=\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$

(ii) *position at time* $t=\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}\cdot\cos\left\{\omega t-\tan^{-1}\left(\dfrac{u_0}{\omega x_0}\right)\right\}$,

and (iii) *time to the position of rest* $=\dfrac{1}{\omega}\tan^{-1}\left(\dfrac{u_0}{\omega x_0}\right)$.

Sol. We know that in a S. H. M. the time period $=2\pi/\sqrt{(\mu)}$.

Since here the time period is $2\pi/\omega$, therefore $2\pi/\sqrt{(\mu)}=2\pi/\omega$ *i.e.*, $\mu=\omega^2$.

Now taking the centre of the motion as origin, the equation of the given S. H. M. is

$$\frac{d^2x}{dt^2}=-\omega^2 x. \quad \text{...(1)}$$

Multiplying (1) by 2 (dx/dt) and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+A, \text{ where } A \text{ is a constant.}$$

But initially at $x=x_0$, the velocity $\dfrac{dx}{dt}=u_0$.

Therefore $\quad u_0^2=-\omega^2x_0^2+A \quad$ or $\quad A=u_0^2+\omega^2x_0^2$.

Thus we have

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+u_0^2+\omega^2x_0^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}-x^2\right) \quad \text{...(2)}$$

(i) Now the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is origin, therefore the amplitude is the value of x when velocity is zero.

Putting $\dfrac{dx}{dt}=0$ in (2), we get $x=\pm\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$.

Here the required amplitude is $\sqrt{\left(x_0^2+\dfrac{u_0^2}{\omega^2}\right)}$.

(ii) Assuming that the particle is moving in the direction of x increasing, we have from (2)

$$\frac{dx}{dt}=\omega\sqrt{\left\{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)-x^2\right\}}$$

or
$$dt=\frac{1}{\omega}\frac{dx}{\sqrt{\{(x_0^2+u_0^2/\omega^2)-x^2\}}}.$$

Integrating, $\quad t=-\dfrac{1}{\omega}\cos^{-1}\left\{\dfrac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+B,$

where B is a constant.

But initially, when $t=0$, $x=x_0$.

$$\therefore \quad B=\frac{1}{\omega}\cos^{-1}\left\{\frac{x_0}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

$$\therefore \quad t=-\frac{1}{\omega}\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$$

or $$\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=-\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or $$\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}=\cos\left[-\left\{\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right\}\right]$$

$$=\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right)$$

or $$x=\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right), \quad \ldots(3)$$

which gives the position of the particle at time t.

(iii) Substituting the value of x from (3) in (2), we get

$$\left(\frac{dx}{dt}\right)^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right).\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}.$$

Putting $\frac{dx}{d}=0$, we get

$$0=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right)\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or $$\sin\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}=0$$

or $$\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)=0 \quad \text{or} \quad t=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

Hence the time of the position of rest $=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$.

Ex. 34. *Show that in a simple harmonic motion of amplitude a and period 'T', the velocity v at a distance x from the centre is given by the relation $v^2T^2=4\pi^2(a^2-x^2)$.*

Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}a$ from the centre; the period remaining unaltered.

Sol. Let the equation of S. H. M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$. $\quad \ldots(1)$

Let a be the amplitude. Then the velocity v at a distance x from the centre is given by

$$v^2=\mu(a^2-x^2). \quad \ldots(2)$$

From (1), $\mu=4\pi^2/T^2$. Putting this value of μ in (2), we have

$$v^2=\frac{4\pi^2}{T^2}(a^2-x^2) \quad \text{or} \quad v^2T^2=4\pi^2(a^2-x^2). \qquad ...(3)$$

Let v_1 be the velocity at a distance $\frac{1}{2}a$ from the centre. Then putting $x=\frac{1}{2}a$ and $v=v_1$ in (3), we get

$$v_1^2T^2=4\pi^2(a^2-\tfrac{1}{4}a^2)=3\pi^2a^2. \qquad ...(4)$$

Let a_1 be the new amplitude when the velocity at the point $x=\frac{1}{2}a$ is doubled *i.e.*, when the velocity at the point $x=\frac{1}{2}a$ is any how made $2v_1$. Since the period remains unchanged, therefore putting $v=2v_1$, $a=a_1$ and $x=\frac{1}{2}a$ in (3), we get

$$4v_1^2T^2=4\pi^2(a_1^2-\tfrac{1}{4}a^2)$$

or $4\times 3\pi^2a^2=4\pi^2(a_1^2-\frac{1}{4}a^2)$ $\qquad$ [$\because$ from (4), $v_1^2T^2=3\pi^2a^2$]

or $a_1^2=3a^2+\frac{1}{4}a^2=13a^2/4$. Hence the new amplitude $a_1=(a\sqrt{13})/2$.

Ex. 35. *Show that the particle executing S.H.M. requires one sixth of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.*

(Kanpur 1973)

Sol. Let the equation of S.H.M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$.

Let a be the amplitude of the motion. Then

$$(dx/dt)^2=\mu(a^2-x^2).$$

Suppose the particle is moving from the position of maximum displacement $x=a$ in the direction of x decreasing. Then

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}\sqrt{(a^2-x^2)}}. \qquad ...(1)$$

Let t_1 be the time from the maximum displacement $x=a$ to the point $x=\frac{1}{2}a$. Then integrating (1), we get

$$\int_0^{t_1} dt=-\frac{1}{\sqrt{\mu}}\int_a^{a/2}\frac{dx}{\sqrt{(a^2-x^2)}}.$$

$$\therefore \quad t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^{a/2}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{1}{2}-\cos^{-1}1\right],$$

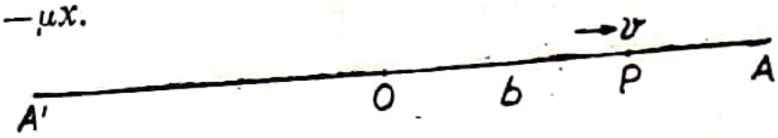
$$=\frac{1}{\sqrt{\mu}}\left[\frac{\pi}{3}-0\right]=\frac{1}{\sqrt{\mu}}\cdot\frac{\pi}{3}=\frac{1}{6}\left(\frac{2\pi}{\sqrt{\mu}}\right)=\frac{1}{6}. \quad \text{(time period } T).$$

Ex. 36. *A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP=b$ with velocity v in the direction OP; prove that the time which elapses before it returns to P is*

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

[Lucknow 1979; Meerut 72, 83, 87, 90; Kanpur 74]

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$.



The time period $T=2\pi/\sqrt{\mu}$.

Let the amplitude be a. Then $(dx/dt)^2=\mu\,(a^2-x^2)$. ...(1)

When the particle passes through P its velocity is given to be v in the direction OP. Also $OP=b$. So putting $x=b$ and $dx/dt=v$ in (1), we get

$$v^2=\mu\,(a^2-b^2). \quad ...(2)$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

$\therefore$ the required time $=2$. time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}.$$

Let t_1 be the time from A to P. Then at A, $t=0$, $x=a$ and at P, $t=t_1$ and $x=b$. Therefore integrating (3), we get

$$\int_0^{t_1} dt=\frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{(a^2-x^2)}}; \quad \text{or} \quad t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a}-\cos^{-1}1\right]=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}.$$

Hence the required time $=2t_1=\dfrac{2}{\sqrt{\mu}}\cos^{-1}\dfrac{b}{a}$

$$=\frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{(a^2-b^2)}}{b}\right\}=\frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

$$\left[\because \text{ from (2), } \sqrt{(a^2-b^2)}=\frac{v}{\sqrt{\mu}}\right]$$

$$=\frac{2}{2\pi/T}\tan^{-1}\left\{\frac{v}{b(2\pi/T)}\right\} \quad [\because T=2\pi/\sqrt{\mu} \text{ so that } \sqrt{\mu}=2\pi/T]$$

$$=\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

Ex. 37. *A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is*

$$2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}.$$

[Meerut 1977 ; Kanpur 84]

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$. Then the time period $T=2\pi/\sqrt{\mu}$.

If a be the amplitude of the motion, we have

$$v^2=\mu\ (a^2-x^2),$$

where v is the velocity at a distance x from the centre.

But when $x=x_1$, $v=v_1$ and when $x=x_2$, $v=v_2$.

Therefore from (1), we have

$$v_1^2=\mu\ (a^2-x_1^2) \text{ and } v_2^2=\mu\ (a^2-x_2^2).$$

These give $v_2^2-v_1^2=\mu\ \{(a^2-x_2^2)-(a^2-x_1^2)\}=\mu\ (x_1^2-x_2^2)$

i.e.,
$$\mu=(v_2^2-v_1^2)/(x_1^2-x_2^2).$$

Hence the time period $T=2\pi/\sqrt{\mu}=2\pi\sqrt{\left(\dfrac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$.

Ex. 38. *A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a complete oscillation is*

$$2\pi\Big/\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right).$$

Sol. Take the middle point of the path as origin. Let the equation of the S.H.M. be $d^2x/dt^2=-\mu x$. Then the time period $T=2\pi/\sqrt{\mu}$.

Let a be the amplitude of the motion. If the time t be measured from the position of instantaneous rest $x=a$, we have

$$x=a\cos\sqrt{\mu}t,$$

where x is the distance of the particle from the centre at time t.

Let x_1, x_2, x_3 be the distances of the particle from the centre at the ends of t_1^{th}, $(t_1+1)^{th}$ and $(t_1+2)^{th}$ seconds. Then from (1),

$$x_1=a\cos\sqrt{\mu}t_1, \qquad \ldots(2)$$

$$x_2=a\cos\sqrt{\mu}(t_1+1), \qquad \ldots(3)$$

and
$$x_3=a\cos\sqrt{\mu}\ (t_1+2). \qquad \ldots(4)$$

$\therefore\quad x_1+x_3=a\ [\cos\sqrt{\mu}t_1+\cos\sqrt{\mu}(t_1+2)]$

$\qquad =2a\cos\sqrt{\mu}(t_1+1)\cos\sqrt{\mu}=2x_2\cos\sqrt{\mu}$, [from (3)].

$\therefore\quad \cos\sqrt{\mu}=(x_1+x_3)/2x_2 \quad$ or $\quad \sqrt{\mu}=\cos^{-1}\{(x_1+x_3)/2x_2\}$.

Hence the time period $T=\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{2\pi}{\cos^{-1}\{(x_1+x_3)/2x_2\}}$.

Ex. 39 (a). *At the ends of three successive seconds the distances of a point moving with S.H.M. from the mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $2\pi/\theta$ where $\cos\theta=3/5$.*

[Meerut 1969, 72, 90P]

Sol. Proceed as in Ex. 38.

Ex. 39 (b). *At the end of three successive seconds, the distances of a point moving with simple harmonic motion from its mean position measured in the same direction are 1, 3 and 4. Show that the period of complete oscillation is*

$$\frac{2\pi}{\cos^{-1}(5/6)}.$$

[Meerut 1987P, 88P]

Ex. 40. *A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b-a)/v$.*

Sol. In the figure, A and B are the positions of instantaneous rest in a S.H.M. Let C be the middle point of AB. Then C is the centre of the motion. Also it is given that $OA=a$, $OB=b$.

The amplitude of the motion $=\frac{1}{2}AB=\frac{1}{2}(OB-OA)=\frac{1}{2}(b-a)$.

Now in a S.H.M. the velocity at the centre $=(\sqrt{\mu})\times$ amplitude. Since in this case the velocity at the centre is given to be v,

therefore $\quad v=\frac{1}{2}(b-a).\sqrt{\mu}\quad$ or $\quad\sqrt{\mu}=2v/(b-a)$.

Hence time period $T=2\pi/\sqrt{\mu}=2\pi\,[(b-a)/2v]=\pi(b-a)/v$.

Ex. 41. *A point executes S.H.M. such that in two of its positions velocities are u, v and the two corresponding accelerations are α, β; show that the distance between the two positions is $(v^2-u^2)/(\alpha+\beta)$ and the amplitude of the motion is*

$$\frac{\{(v^2-u^2)(\alpha^2v^2-\beta^2u^2)\}^{1/2}}{\alpha^2-\beta^2}.$$

[Meerut 1990S; Allahabad 77]

Sol. Let the equation of the S.H.M. with centre as origin be $d^2x/dt^2=-\mu x$.

If a be the amplitude of the motion, we have

$$(dx/dt)^2=\mu\,(a^2-x^2),$$

where dx/dt is the velocity at a distance x from the centre.

Let x_1 and x_2 be the distances from the centre of the two positions where u and v are the velocities and α and β are the accelerations respectively. Then

$$\alpha=\mu x_1, \qquad ...(1)$$
$$\beta=\mu x_2, \qquad ...(2)$$
$$u^2=\mu\,(a^2-x_1^2), \qquad ...(3)$$
and
$$v^2=\mu\,(a^2-x_2^2). \qquad ...(4)$$

Adding (1) and (2), we get $\alpha+\beta=\mu\,(x_1+x_2)$. ...(5)

Also subtracting (3) from (4), we get

$$v^2-u^2=\mu\,(x_1^2-x_2^2)=\mu\,(x_1-x_2)\,(x_1+x_2)=(\alpha+\beta)\,(x_1-x_2).$$
[from (5)]

$\therefore$ $(x_1-x_2)=(v^2-u^2)/(\alpha+\beta)$. This gives the distance between the two positions.

Now to get the amplitude a it is obvious that we have to eliminate x_1, x_2 and μ from the equations (1), (2), (3) and (4). Substituting for x_1 and x_2 from (1) and (2) in (3) and (4), we have

$$u^2=\mu\left(a^2-\frac{\alpha^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2-u^2\mu-\alpha^2=0 \qquad ...(6)$$

and $$v^2=\mu\left(a^2-\frac{\beta^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2-v^2\mu-\beta^2=0. \qquad ...(7)$$

By the method of cross multiplication, we have from (6) and (7),

$$\frac{\mu^2}{u^2\beta^2-v^2\alpha^2}=\frac{\mu}{-a^2\alpha^2+a^2\beta^2}=\frac{1}{a^2u^2-a^2v^2}.$$

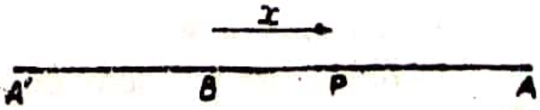
Equating the two values of μ^2 found from the above equations, we get

$$\frac{\alpha^2v^2-u^2\beta^2}{a^2\,(v^2-u^2)}=\left[\frac{a^2\,(\alpha^2-\beta^2)}{a^2\,(v^2-u^2)}\right]^2, \text{ or } \frac{\alpha^2v^2-u^2\beta^2}{a^2\,(v^2-u^2)}=\frac{(\alpha^2-\beta^2)^2}{(v^2-u^2)^2}.$$

$$\therefore \quad a^2=\frac{(\alpha^2v^2-\beta^2u^2)\,(v^2-u^2)}{(\alpha^2-\beta^2)^2} \text{ or } a=\frac{\{(v^2-u^2)\,(\alpha^2v^2-\beta^2u^2)\}^{1/2}}{(\alpha^2-\beta^2)}.$$

Ex. 42. *A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is displaced slightly towards one of them, show that the time of a small oscillation is $2\pi/\sqrt{(\mu+\mu')}$.* **[Agra 1980, 86; Rohilkhand 88]**

Sol. Suppose A and A' are the two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at B under the attraction of these two centres. If $AB=a$ and $A'B=a'$, the forces of attraction at B due to the centres A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions. As these two forces balance, we have



$$m\mu a=m\mu' a'. \qquad ..(1)$$

Now suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP=x$.

The attraction at P due to the centre A is $m\mu.AP$ or $m\mu(a-x)$ in the direction PA *i.e.*, in the direction of x increasing. Also the attraction at P due to the centre A' is $m\mu'.A'P$ or $m\mu'(a'+x)$ in the direction PA' *i.e.*, in the direction of x decreasing. Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\,(d^2x/dt^2)=m\mu\,(a-x)-m\mu'\,(a'+x), \qquad \ldots(2)$$

where the force in the direction of x increasing has been taken with +ive sign and the force in the direction of x decreasing has been taken with −ive sign.

Simplifying the equation (2), we get

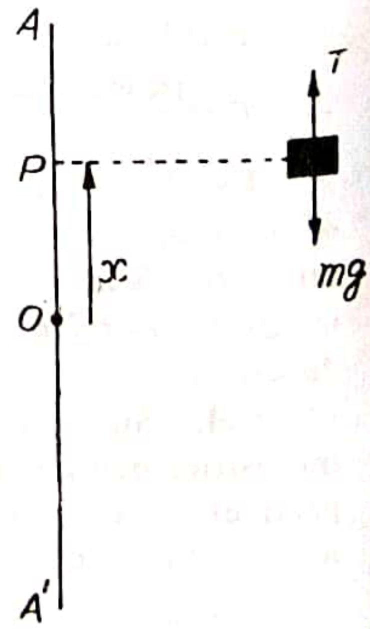
$$m\,(d^2x/dt^2)=m\,(\mu a-\mu x-\mu' a'-\mu' x)$$

or $$d^2x/dt^2=-(\mu+\mu')\,x. \qquad [\because \text{ by (1), } m\mu a=m\mu' a']$$

This is the equation of a S. H. M. with centre at the origin. Hence the motion of the particle is simple harmonic with centre at B and its time period is $2\pi/\sqrt{(\mu+\mu')}$.

Ex. 43. *A body is attached to one end of an inelastic string, and the other end moves in a vertical line with S.H.M. of amplitude a, making n oscillations per second. Show that the string will not remain tight during the motion unless* $n^2<g/(4\pi^2 a)$

[Meerut 1970, 80, 86 P, 88; Agra 75]



Sol. Suppose the string remains tight during the motion so that the body also moves in an identical S.H.M. Let m be the mass of the body.

Let the body move in S.H.M. between A and A' and suppose O is the centre of the motion, where $OA=a$.

Since the body makes n oscillations per second, therefore its time period $\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{1}{n}$.

This gives $\mu=4\pi^2 n^2$.

At time t, let the body be in a position P, where $OP=x$. The impressed force acting on the body is $T-mg$ along OP. Here T is the tension of the string. By Newton's law, the equation of motion of the body is $$m\,(d^2x/dt^2)=T-mg.$$

$$\therefore\quad T=mg+m\,(d^2x/dt^2).$$

Obviously T is least when d^2x/dt^2 is least. But the least value of d^2x/dt^2 is $-\mu a$. Hence least $T=mg-m\mu a$.

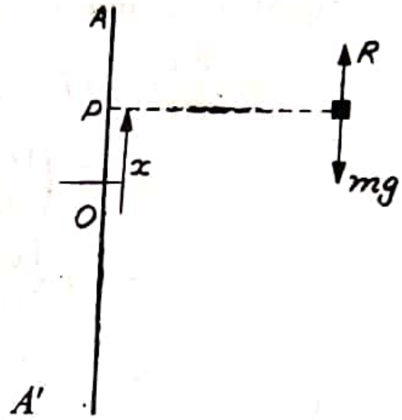
The string will remain tight if this least tension is positive *i.e.*, if $m\mu a<mg$

i.e., if $m4\pi^2n^2a<mg$ $\quad [\because \ \mu=4\pi^2n^2]$

i.e., if $n^2<g/(4\pi^2a)$. Hence the result.

Ex. 44. *A horizontal shelf is moved up and down with S. H. M. of period $\frac{1}{2}$ sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off?* **[Lucknow 1979]**

Sol. Let m be the mass of the body placed on the shelf. Suppose along with the shelf, the body moves in an identical S. H. M. between A and A'. Let O be the centre of the motion so that $OA=a$ is the amplitude.



The time period $2\pi/\sqrt{\mu}=\frac{1}{2}$; (given)

$\therefore \quad \mu=16\pi^2.$

Let P be the position of the body at time t, where $OP=x$. The impressed force acting on the body is $R-mg$ along OP. Here R is the reaction of the shelf. By Newton's law the equation of motion of the body is

$$m(d^2x/dt^2)=R-mg.$$

$$R=mg+m\,(d^2x/dt^2).$$

Obviously R is least when d^2x/dt^2 is least and the least value of d^2x/dt^2 is $-\mu a$. Hence least $R=mg-m\mu a$.

The body will not be jerked off if this least value of R remains non-negative *i.e.*, if $m\mu a\leqslant mg$

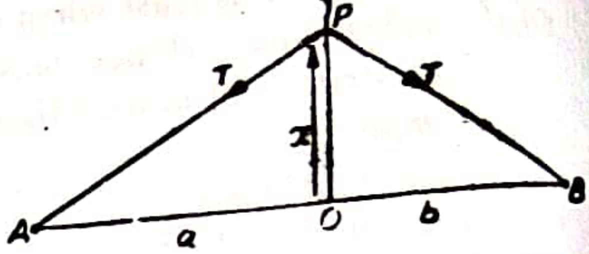
i.e., if $m16\pi^2a\leqslant mg$ $\quad [\because \ \mu=16\pi^2]$

i.e., if $a \leqslant g/(16\pi^2)$. Hence the greatest admissible value of the amplitude $a=g/(16\pi^2)$.

Ex. 45. *A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distance of the particle from the two ends, prove that the period of small transverse oscillation of mass m is*

$$\frac{2\pi}{\sqrt{\left\{\dfrac{T\,(a+b)}{mab}\right\}}}.$$

Sol. Let a light wire be stretched tightly between the fixed points A and B with a tension T. Let a particle of mass m be attached at the point O of the wire where $AO=a$ and $OB=b$.



Let the particle be displaced slightly perpendicular to AB (*i.e.*, in the transverse direction) and then let go. Let P be the position of the particle at any time t, where $OP=x$. Since the displacement is small, therefore the tension in the string in any displaced position can be taken as T which is the tension in the string in the original position. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=-(T\cos\angle OPA+T\cos\angle OPB)$$

$$=-T\left(\frac{OP}{AP}+\frac{OP}{BP}\right)=-T\left(\frac{x}{\sqrt{(a^2+x^2)}}+\frac{x}{\sqrt{(b^2+x^2)}}\right)$$

$$=-T\left\{\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2}+\frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}\right\}$$

$$=-T\left[\frac{x}{a}\left(1-\tfrac{1}{2}.\frac{x^2}{a^2}+\dots\right)+\frac{x}{b}\left(1-\tfrac{1}{2}.\frac{x^2}{b^2}+\dots\right)\right]$$

$$=-T\left(\frac{x}{a}+\frac{x}{b}\right),\text{ neglecting higher powers of } x/a \text{ and } x/b$$

which are very small

$$=-T\left(\frac{a+b}{ab}\right)x.$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{T(a+b)}{mab}x=-\mu x,\text{ where }\mu=\frac{T(a+b)}{mab}.$$

This is the standard equation of a S. H. M. with centre at the origin. The time period

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\Big/\sqrt{\left\{\frac{T(a+b)}{mab}\right\}}=2\pi\sqrt{\left\{\frac{mab}{T(a+b)}\right\}}.$$

Ex. 46. *If in a S. H. M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation*

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a)=\begin{vmatrix} u & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1\end{vmatrix}.$$

[Kanpur 1980, 85, 88; Meerut 84]

Sol. Let O and O' be the centre of force and the fixed point respectively on the line of motion and let

$OO'=l$. Let u, v, w be the velocities of the particle at P, Q, R respectively where

$$O'P=a,\ O'Q=b,\ O'R=c.$$

For a S.H.M. of amplitude A, the velocity V at a distance x from the centre of force is given by

$$V^2=\mu\,(A^2=x^2). \quad \ldots(1)$$

At P, $x=OP=l+a$, $V=u$

at Q, $x=OQ=l+b$; $V=v$

and at R, $x=OR=l+c$, $V=w$.

$\therefore$ from (1), we have

$$u^2=\mu\,\{A^2-(l+a)^2\}$$

or
$$\frac{u^2}{\mu}=A^2-l^2-a^2-2al$$

or
$$\left(\frac{u^2}{\mu}+a^2\right)+2l.a+(l^2-A^2)=0. \quad \ldots(2)$$

Similarly,

$$\left(\frac{v^2}{u}+b^2\right)+2l.b+(l^2-A^2)=0, \quad \ldots(3)$$

and
$$\left(\frac{w^2}{\mu}+c^2\right)+2l.c+(l^2-A^2)=0. \quad \ldots(4)$$

Eliminating $2l$ and (l^2-A^2) from (2), (3) and (4), we have

$$\begin{vmatrix} \frac{u^2}{\mu}+a^2 & a & 1 \\ \frac{v^2}{\mu}+b^2 & b & 1 \\ \frac{w^2}{\mu}+c^2 & c & 1 \end{vmatrix}=0$$

or
$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix}+\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}=0$$

$$\text{or} \quad -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu} \begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

$$\text{or} \quad \mu \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{or} \quad \mu\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \qquad \ldots(5)$$

But the time period $T=\dfrac{2\pi}{\sqrt{\mu}}$, so that $\mu=\dfrac{4\pi^2}{T^2}$.

Hence from (5), we have

$$\frac{4\pi^2}{T^2}\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

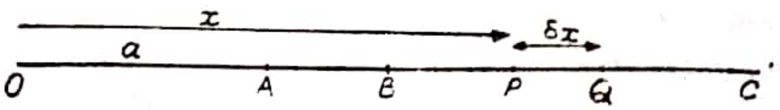
§ 8. Hooke's Law :

Statement. *The tension of an elastic string is proportional to the extension of the string beyond its natural length.*

If x is the stretched length of a string of natural length l, then by Hooke's law the tension T in the string is given by $T=\lambda.\dfrac{x-l}{l}$, where λ is called the **modulus of elasticity** of the string. **Remember** that the direction of the tension is always opposite to the extension.

Theorem. *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.* **[Kanpur 1977]**

Proof. Let $OA=a$ be the natural length of a string whose one end is fixed at O. Let the string be stretched beyond its natural



length. Let B and C be the two positions of the free end A of the string during its any extension and let $OB=b$ and $OC=c$.

Then by Hooke's law,

the tension at $B=T_B=\lambda\dfrac{b-a}{a}$, ...(1)

and the tension at $C=T_C=\lambda\dfrac{c-a}{a}$, ...(2)

where λ is the modulus of elasticity of the string.

Now we find the work done against the tension in stretching the string from B to C.

Let P be any position of the free end of the string during its extension from B to C and let $OP=x$.

Then the tension at $P=T_P=\lambda.\ \dfrac{x-a}{a}$.

Now suppose the free end of the string is slightly stsetched from P to Q, where $PQ=\delta x$. Then the work done against the tension in stretching the string from P to Q

$$=T_P\ \delta x=\lambda\,\frac{(x-a)}{a}.\delta x.$$

$\therefore$ the work done against the tension in stretching the string from B to C

$$=\int_b^c \frac{\lambda}{a}(x-a)\,dx=\frac{\lambda}{2a}\Big[(x-a)^2\Big]_b^c$$

$$=\frac{\lambda}{2a}[(c-a)^2-(b-a)^2]=\frac{\lambda}{2a}[\{(c-a)-(b-a)\}\{(c-a)+(b-a)\}]$$

$$=(c-b).\frac{1}{2}\left[\frac{\lambda}{a}(c-a)+\frac{\lambda}{a}(b-a)\right]$$

$$=(c-b).\tfrac{1}{2}[T_C+T_B], \qquad \text{[from (1) and (2)]}$$

$=BC\times$(mean of the tension at B and C).

Hence, the work done against the tension in stretching the string is equal to the product of the extension and the mean of the initial and final tensions.

Now we shall discuss a few simple and interesting cases of simle harmonic motion.

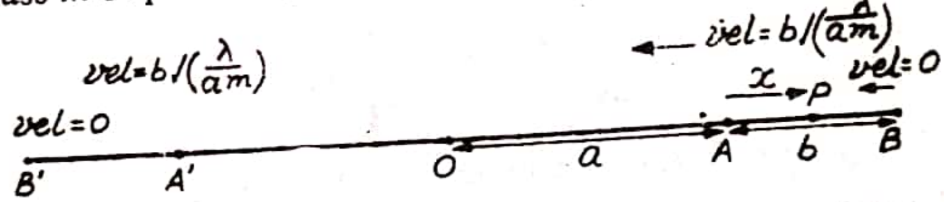
§ 9. Particle attached to one end of a horizontal elastic string.

A particle of mass m is attached to one end of a horizontal elastic string whose other end is fixed to a point on a smooth hori-

zontal table. The particle is pulled to any distance in the direction of the string and then let go; to discuss the motion.

[Lucknow 1977; Allahabad 76]

Let a string OA of natural length a lie on a smooth horizontal table. The end O of the string is attached to a fixed point of the table and a particle of mass m is attached to the other end A. The mass m is pulled upto B, where $AB=b$, and then let go.



Let P be the position of the particle after time t, where $AP=x$. The table being smooth, the only horizontal force acting on the particle at P is the tension T in the string OP. Since the direction of tension is always opposite to the extension, therefore, the force T acts in the direction PA *i.e.*, in the direction of x decreasing. Also by Hooke's law $T=\lambda\,(x/a)$. Hence the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\lambda\frac{x}{a} \text{ or } \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \qquad \ldots(1)$$

The equation (1) shows that the motion of the particle is simple harmonic with centre at the origin A. The equation of motion (1) holds good so long as the string is stretched. Since the string becomes slack just as the particle reaches A, therefore the equation (1) holds good for the motion of the particle from B to A.

Multiplying (1) by $2\,(dx/dt)$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{am}x^2+C, \text{ where } C \text{ is a constant.}$$

At the point B, $x=b$ and $dx/dt=0$; $\therefore \quad C=(\lambda/am)\,b^2$.

Thus we have $\left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}(b^2-x^2)$. $\qquad \ldots(2)$

This equation gives velocity in any position from B to A. Putting $x=0$ in (2), we have the velocity at $A=\sqrt{(\lambda/am)}\,b$, in the direction AO.

The time from B to A is $\frac{1}{4}$ of the complete time period of a S.H.M. whose equation is (1).

Character of the motion. The motion from B to A is simple harmonic. When the particle reaches A, the string becomes slack and the simple harmonic motion ceases. But due to the velocity gained at A the particle continues to move to the left of A. So long as the string is loose there is no force on the particle to change its velocity because the only force here is that of tension and the tension is zero so long as the string is loose. Thus the particle moves from A to A' with uniform velocity $\sqrt{(\lambda/am)}\, b$ gained by it at A. Here A' is a point on the other side of O such that $OA'=OA$. When the particle passes A' the string again becomes tight and begins to extend. The tension again comes into picture and the particle begins to move in S. H. M. But now the force of tension acts against the direction of motion of the particle. So the velocity of the particle starts decreasing and the particle comes to instantaneous rest at B', where $A'B'=AB$. The time from A' to B' is the same as that from B to A. At B' the particle at once begins to move towards A' because of the tension which attracts it towards A'. Retracing its path the particle again comes to instantaneous rest at B and thus it continues to oscillate between B and B'.

During one complete oscillation the particle covers the distance between A and B and also that between A' and B' twice while moving in S. H. M. Also it covers the distance between A and A' twice with uniform velocity $\sqrt{(\lambda/am)}\, b$. Hence the total time for one complete oscillation

$=$the complete time period of a S.H.M. whose equation is (1)
$+$the time taken to cover the distance $4a$ with uniform velocity $\sqrt{(\lambda/am)}\, b$

$$=\frac{2\pi}{\sqrt{(\lambda/am)}}+\frac{4a}{\sqrt{(\lambda/am)}\, b}=2\pi\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{4a}{b}\sqrt{\left(\frac{am}{\lambda}\right)}$$

$$=2\left(\pi+\frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Illustrative Examples :

Ex. 47. *One end of an elastic string (modulus of elasticity λ) whose natural length is a, is fixed to a point on a smooth horizontal table and the other is tied to a particle of mass m. which is lying on the table. The particle is pulled to a distance from the point of attachment of the string equal to twice its natural length and then let go. Show that the time of a complete oscillation is*

$$2(\pi+2)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

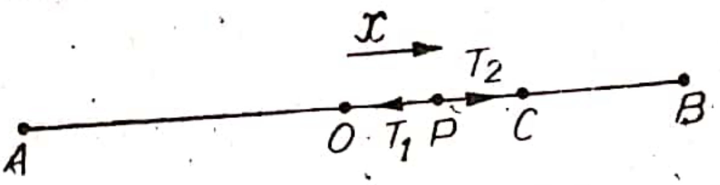
[Lucknow 1981]

Sol. Proceed exactly in the same way as in § 9. Here, the particle is pulled to a distance from the point of attachment of the string equal to twice its natural length. Therefore initially the increase b in the length of the string is equal to $2a-a$ i.e., a.

Now proceed as in § 9, taking $b=a$.

Ex. 48. *A light elastic string whose modulus of elasticity is λ is stretched to double its length and is tied to two fixed points distant $2a$ apart. A particle of mass m tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed points and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.*

Sol. Let an elastic string of natural length a be stretched between two fixed points A and B distant $2a$ apart, O being the middle point of AB. We have, $OA=OB=a$.



Natural length of the portions OA and OB each is $a/2$ (since the string is stretched to double its length). A particle of mass m attached to the middle point O is displaced towards B upto a point C, where $OC=a/2$ and then let go. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing]. At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

[Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA].

By Hooke's law, $T_1=\lambda\dfrac{a+x-\frac{1}{2}a}{a/2}$ and $T_2=\lambda\dfrac{a-x-\frac{1}{2}a}{a/2}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\lambda\frac{a-x-a/2}{a/2}-\lambda\frac{a+x-a/2}{a/2}=-\frac{4\lambda x}{a}.$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{4\lambda}{am}x. \qquad \text{...(1)}$$

Thus the motion is S.H.M. with centre at the origin O. Since we have displaced the particle towards B only upto the point C so that the portion BC of the string is just in its natural length, therefore during the entire motion of the particle both the portions of the string remain taut and so the entire motion of the particle is governed by the above equation. Thus the particle makes oscillations in S.H.M. about O and the time period of one complete oscillation

= the time period of S.H.M. whose equation is (1)

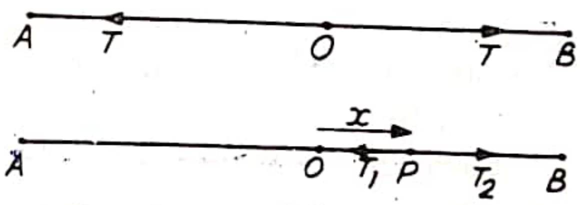
$$=2\pi\Big/\sqrt{\left(\frac{4\lambda}{am}\right)}=\pi\sqrt{(am/\lambda)}.$$

The amplitude (*i.e.*, the maximum displacement from the centre) of this S.H.M. is $a/2$.

$\therefore$ the maximum velocity $=(\sqrt{\mu})\times$ amplitude

$$=\sqrt{(4\lambda/am)}\,.\,(a/2)=\sqrt{(a\lambda/m)}.$$

Ex. 49. *A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T; show that the time of an oscillation is* $2\pi\,\{mll'/T\,(l+l')\}^{1/2}$, *where l. l' are the extensions of the strings beyond their natural legths.*



Sol. A particle of mass m rests at O being pulled by two horizontal strings AO and BO whose other ends are connected to two fixed points A and B. Let a, a' be the natural lengths of the strings AO and BO whose extensions beyond their natural lengths are l and l' respectively. Let λ and λ' be the respective modulii of elasticity of the two strings AO and BO. At O the particle is in equilibrium under the tensions of the two strings. Therefore

$$\frac{\lambda l}{a}=\frac{\lambda' l'}{a'}=T \text{ (given)}. \qquad \ldots(1)$$

From (1), we have $\dfrac{T}{l}=\dfrac{\lambda}{a}$ and $\dfrac{T}{l'}=\dfrac{\lambda'}{a'}$ ...(2)

Now suppose the particle is slightly pulled towards B and then let go. It beings to move towards O. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing.]

At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA, *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB, *i.e.*, in the direction of x increasing. [Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA.]

By Hooke's law, $T_1=\lambda\dfrac{(l-x)}{a}$ and $T_2=\lambda'\dfrac{(l'-x)}{a'}$.

Hence by Newton's second law of motion $(P=mf)$, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda'(l'-x)}{a'}-\frac{\lambda(l+x)}{a}$$

$$=\frac{\lambda' x}{a'}-\frac{\lambda x}{a}, \qquad \left[\because \text{ by (1) } \frac{\lambda l'}{a'}=\frac{\lambda l}{a}\right]$$

$$=-x\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right).$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{x}{m}\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right)=-\frac{x}{m}\left(\frac{T}{l'}+\frac{T}{l}\right), \qquad \text{from (2)}$$

$$=-\frac{T(l+l')}{mll'}x, \qquad \ldots(3)$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and so the entire motion of the particle is governed by the equation (3). Thus the particle makes small oscillations in S.H.M. about O and the time period of one complete oscillation

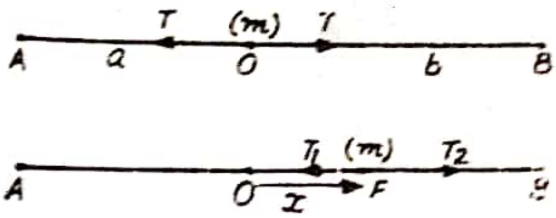
$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\{T(l+l')/mll'\}}}=2\pi\left[\frac{mll'}{T(l+l')}\right]^{1/2}$$

Remark. In order that the entire motion of the particle should remain simple harmonic with centre at O, the particle must be pulled towards B only upto that distance which does not allow the string OB to become slack.

Ex. 50. *Two light elastic strings are fastened to a particle of mass m and their other ends to fixed points so that the strings are taut. The modulus of each is λ, the tension T, and length a and b. Show that the period of an oscillation along the line of the strings is*

$$2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

[Meerut 1981, 84, 85]



Sol. Let the two light elastic strings be fastened to a particle of mass m at O and their other ends be attached to two fixed points A and B so that the strings are taut and $OA=a$, $OB=b$. If l and l' are the natural lengths of the strings OA and OB respectively, then in the position of equilibrium of the particle at O,

tension in the string OA=tension in the string $OB=T$; (as given).

Applying Hooke's law, we have

$$T=\lambda\frac{a-l}{l}=\lambda\frac{b-l'}{l'}. \quad \text{...(1)}$$

From $T=\lambda\dfrac{a-l}{l}$, we have $Tl=\lambda a-\lambda l$

i.e., $$l(T+\lambda)=\lambda a$$

i.e., $$\frac{\lambda}{l}=\frac{T+\lambda}{a}. \quad \text{...(2)}$$

Similarly $$\frac{\lambda}{l'}=\frac{T+\lambda}{b}. \quad \text{...(3)}$$

Now suppose the particle is slightly pulled towards B and then let go. It begins to move towards O. Let P be the position of the particle after any time t, where $OP=x$. The direction OP is that of x increasing and the direction PO is that of x decreasing.

At P there are two horizontal forces acting on the particle.

(i) The tension T_1 in the string AP acting in the direction PA i.e., in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB i.e., in the direction of x increasing.

By Hooke's law, $T_1=\lambda\dfrac{a+x-l}{l}$, $T_2=\lambda\dfrac{b-x-l'}{l'}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda(b-x-l')}{l'}-\frac{\lambda(a+x-l)}{l}$$

$$=-\frac{\lambda}{l'}x-\frac{\lambda}{l}x, \quad \left[\because \text{ from (1), } \frac{\lambda(b-l')}{l'}=\frac{\lambda(a-l)}{l}\right]$$

$$=-\left[\frac{T+\lambda}{b}+\frac{T+\lambda}{a}\right]x. \quad \text{[from (2) and (3)]}$$

$$= -\frac{(T+\lambda)(a+b)}{ab}\, x.$$

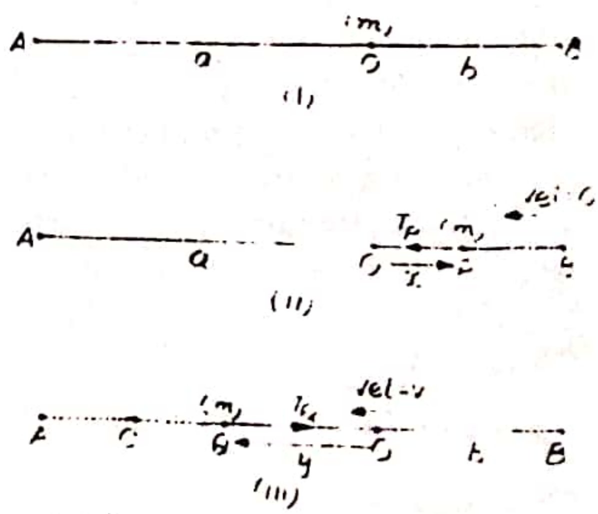
$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{(T+\lambda)(a+b)}{mab}\, x, \qquad \ldots(4)$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and the entire motion of the particle is governed by the equation (4). Thus the particle makes small oscillations in S. H. M. about O and the time period of one complete oscillation

$$= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\{(T+\lambda)(a+b)/mab\}}} = 2\pi \left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

Ex. 51. *An elastic string of natural length $(a+b)$ where $a > b$ and modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to and fro through a distance $\frac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}$ in a periodic time $\pi(\sqrt{a}+\sqrt{b})\sqrt{(m/\lambda)}$.*



Sol. Let AB be an elastic string of natural length $a+b$ attached to two fixed points A and B distant $a+b$ apart. Let a particle of mass m be attached to the point O of the string such that $OA = a$, $OB = b$ and $a > b$.

When the particle is held at B, the portion AO of the string is stretched while the portion OB is slack and so when the particle is released from B, it moves towards O starting from rest at B.

If P is the position of the particle between O and B, [see fig. (ii)], at any time t after its release from B and $OP = x$, then the tension in the string AP is $T_P = \lambda \frac{x}{a}$ acting towards O and the tension in the string PB is zero because it is slack.

$\therefore$ the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-T_P=-\frac{\lambda}{a}x$$

or

$$\frac{d^2x}{dt^2}=-\frac{\lambda}{am}x, \qquad \text{...(1)}$$

which represents a S. H. M. with centre at O and amplitude OB.

If t_1 be the time from B to O, then

$t_1=\frac{1}{4}\times$ time period of the S. H. M. represented by (1)

$$=\frac{1}{4}\cdot\frac{2\pi}{\sqrt{(\lambda/am)}}=\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}. \qquad \text{...(2)}$$

Now multiplying both sides of (1) by $2(dx/dt)$ and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{am}.x^2+k, \text{ where } k \text{ is a constant.}$$

But at the point B, $x=OB$ and $dx/dt=0$.

$$\therefore \quad 0=-\frac{\lambda}{am}b^2+k \quad \text{or} \quad k=\frac{\lambda b^2}{am}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}(b^2-x^2). \qquad \text{...(3)}$$

If V is the velocity of the particle at O, where $x=0$, then from (3), we have

$$V^2=\frac{\lambda}{am}.b^2 \quad \text{or} \quad V=\sqrt{\left(\frac{\lambda}{am}\right)}.b. \qquad \text{...(4)}$$

At the point O, the tension in either of the two portions of the string is zero and the velocity of the particle is V to the left of O, due to which the particle moves towards the left of O. As the particle moves to the left of O, the string OA becomes slack and the string OB is stretched.

If Q is the position of the particle between O and A, [see fig. (iii)], at any time t, since it starts moving from O to the left of it and $OQ=y$, then the tension in the string QB is $T_Q=\lambda\frac{y}{b}$ acting towards O and the tension in the string $QA=0$ because it is slack.

$\therefore$ The equation of motion of the particle at Q is

$$m\frac{d^2y}{dt^2}=-T_Q=-\frac{\lambda y}{b}$$

or

$$\frac{d^2y}{dt^2}=-\frac{\lambda}{bm}y. \qquad \text{...(4)}$$

Multiplying both sides of (4) by $2(dy/dt)$ and then integrating, we have

$$\left(\frac{dy}{dt}\right)^2=-\frac{\lambda}{bm}y^2+D, \text{ where } D \text{ is a constant.}$$

But at O, $y=0$ and $\left(\frac{dy}{dt}\right)^2=V^2=\frac{\lambda}{am}.b^2$.

$$\therefore \quad \frac{\lambda}{am}.b^2=-\frac{\lambda}{bm}.0+D \quad \text{or} \quad D=\frac{\lambda}{am}b^2.$$

$$\therefore \quad \left(\frac{dy}{dt}\right)^2=\frac{\lambda}{m}\left(\frac{b^2}{a}-\frac{1}{b}y^2\right)$$

or

$$\left(\frac{dy}{dt}\right)^2=\frac{\lambda}{bm}\left(\frac{b^3}{a}-y^2\right). \qquad \ldots(5)$$

If the particle comes to instantaneous rest at the point C between O and A such that $OC=c$, then at C, $y=c$ and $dy/dt=0$.

$\therefore$ from (5), we have

$$0=\frac{\lambda}{bm}\left(\frac{b^3}{a}-c^2\right) \quad \text{or} \quad c=b\sqrt{\left(\frac{b}{a}\right)}.$$

From C the particle retraces its path and comes to instantaneous rest at B.

The particle thus oscillates to and fro through a distance

$$BC=BO+OC=b+c=b+b\sqrt{\left(\frac{b}{a}\right)}=\frac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}.$$

The equation (4) represents a S. H. M. with centre at O, amplitude OC and time period $T'=2\pi\Big/\sqrt{\left(\frac{\lambda}{bm}\right)}=2\pi\sqrt{\left(\frac{bm}{\lambda}\right)}$.

If t_2 be the time from O to C, we have

$$t_2=\tfrac{1}{4}.(T')=\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}.$$

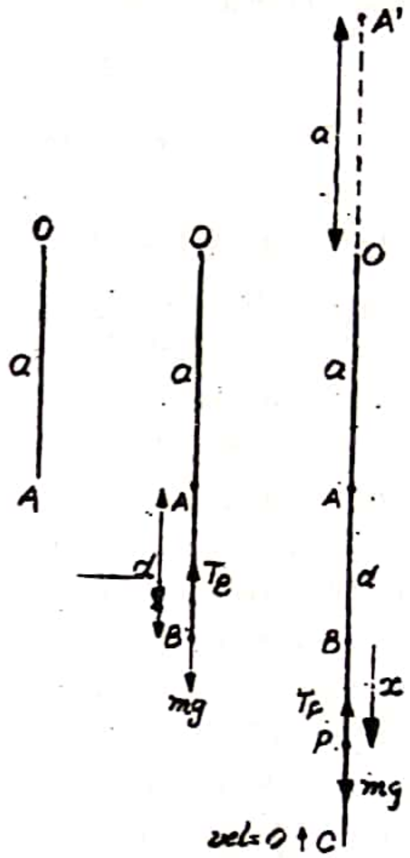
Hence the required periodic time for making a complete oscillation between B and C

$$=2.(\text{time from } B \text{ to } C)=2(t_1+t_2)$$

$$=2\left[\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}\right]=\pi(\sqrt{a}+\sqrt{b})\sqrt{\left(\frac{m}{\lambda}\right)}.$$

§ 10. **Particle suspended by an elastic string.** *A particle of mass m is suspended from a fixed point by a light elastic string of natural length a and modulus of elasticity λ. The particle is pulled down a little in the line of the string and released; to discuss the motion.*

(Meerut 1988 S)



Let one end of the string OA of natural length a be attached to the fixed point O and a particle of mass m be attached to the other end A. Due to the weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB=d$, then the tension T_B in the string will balance the weight of the particle

i.e., $$mg=T_B$$

or $$mg=\lambda\frac{AB}{OA}=\lambda\frac{d}{a}. \quad ...(1)$$

The particle is pulled down to a point C such that $BC=c$ and then released. At the point C, the tension in the string is greater than the weight of the particle and so the particle starts moving vertically upwards with velocity zero at C. Let P be the position of the particle at any time t, where $BP=x$. The tension in the string when the particle is at P is $T_P=\lambda\dfrac{d+x}{a}$, acting vertically upwards.

The resultant force acting on the particle at P in the vertically upwards direction $=T_P-mg=\lambda\left(\dfrac{d+x}{a}\right)-mg=\dfrac{\lambda d}{a}+\dfrac{\lambda x}{a}-mg$

$$=\frac{\lambda x}{a}, \left[\because \frac{\lambda d}{a}=mg, \text{ from (1)}\right].$$

Also the acceleration of the particle at P is d^2x/dt^2 in the direction of x increasing *i.e.,* in the vertically downwards direction.

$\therefore$ by Newton's law, the equation of motion of P is given by

$$m\frac{d^2x}{dt^2}=-\frac{\lambda x}{a} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \quad ...(2)$$

This equation holds good so long as the tension operates *i.e.*, when the string is extended beyond its natural length.

Equation (2) is the standard equation of a S.H.M. with centre at the origin B and the amplitude of the motion is $BC=e$.

The periodic time T of the S.H.M. represented by the equation (2) is given by

$$T=2\pi \Big/ \sqrt{\left(\frac{\lambda}{am}\right)}=2\pi \sqrt{\left(\frac{am}{\lambda}\right)}. \qquad \ldots(3)$$

The motion of the particle remains simple harmonic as long as there is tension in the string *i.e.*, as long the particle remains in the region from C to A.

In case the string becomes slack during the motion of the particle, the particle will begin to move freely under gravity.

Now there are two cases.

Case I. If BC$\leqslant$AB *i.e.*, **c$\leqslant$d.** In this case the particle will not rise above A and it will come to instantaneous rest before or just reaching A. The whole motion will be S.H.M. with centre at B, amplitude BC and period T given by (3).

Case II. If BC$>$AB *i.e.*, **c$>$d.** In this case the particle will rise above A, and the motion will be simple harmonic upto A and above A the particle will move freely under gravity.

Multiplying both sides of (2) by 2 (dx/dt) and then integrating, we have $\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{am}x^2+k$. where k is a constant.

But at C, $x=BC=c$ and $dx/dt=0$.

$$\therefore \quad 0=-\frac{\lambda}{am}c^2+k \quad \text{or} \quad k=\frac{\lambda}{am}c^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}(c^2-x^2). \qquad \ldots(4)$$

Now if V is the velocity of the particle at A, where $x=-BA=-d$, then, from (4), we have

$$V^2=\frac{\lambda}{am}(c^2-d^2) \quad \text{or} \quad V=\sqrt{\left[\frac{\lambda}{am}(c^2-d^2)\right]}, \qquad \ldots(5)$$

the direction of V being vertically upwards.

If h is the height to which the particle rises above A, then

$$h=\frac{V^2}{2g}=\frac{\lambda(c^2-d^2)}{2amg}, \qquad \ldots(6)$$

provided $h \leqslant 2a$.

Also in this case the maximum height attained by the particle during its entire motion

$$=CB+BA+h,$$
$$=c+d+h. \qquad \ldots(7)$$

If $h \leqslant 2a$ *i.e.*, if $h \leqslant AA'$, then the particle, after coming to instantaneous rest, will retrace its path *i.e.*, it will fall freely under gravity upto A and below A it will move in S.H.M. till it comes to instantaneous rest at C.

If $h=2a=AA'$, then the particle will just come to rest at A' and will then move downwards, retracing its path.

In this case the maximum height attained by the particle

$$=c+d+2a. \qquad \ldots(8)$$

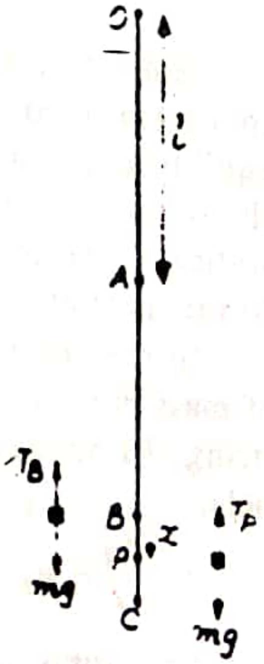
If $h>2a$ *i.e.*, if $h>AA'$, then the particle will rise above A' also and so the string will again become stretched and the particle will again begin to move in simple harmonic motion. After coming to instantaneous rest the particle will retrace its path.

Illustrative Examples

Ex. 52 (a). *An elastic string without weight of which the unstretched length is l and modulus of elasticity is the weight of n oz. is suspended by one end and a mass m oz. is attached to the other end. Show that the time of a small vertical oscillation is*

$$2\pi\sqrt{(ml/ng)}. \qquad \textbf{[Meerut 1971, 76, 78, 79]}$$

Sol. $OA=l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a particle of mass m oz. attached to the other end of the string. Considering the equilibrium of the particle at B, we have mg=the tension T_B in the string OB.



$$\therefore \quad mg=ng\,\frac{AB}{l}, \qquad \ldots(1)$$

because modulus of elasticity of the string is given to be ng.

Now suppose the particle is pulled slightly upto C (so that $BC<AB$), and then let go. It starts moving vertically upwards with velocity zero at C. Let P be its position at any point t, where $BP=x$. The direction BP is that of x increasing and the direction PB is that of x decreasing. At P there are two forces acting on the particle :

(i) The weight mg acting vertically downwards *i.e.*, in the direction of x increasing.

and (ii) the tension $T_P=ng\,\dfrac{AB+x}{l}$ in the string OP,

cally upwards *i.e.*, in the direction of x decreasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-ng\frac{AB+x}{l}=mg-ng\frac{AB}{l}-ng\frac{x}{l}$$

$$=-ng\frac{x}{l}, \qquad \left[\because \text{ from (1), } mg=ng\frac{AB}{l}\right].$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{ng}{lm}x, \qquad \ldots(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC.

Since $BC<AB$, therefore during the entire motion of the particle the string will not become slack.

Thus the entire motion of the particle is governed by the equation (2) and the particle will make oscillations in simple harmonic motion about the centre B.

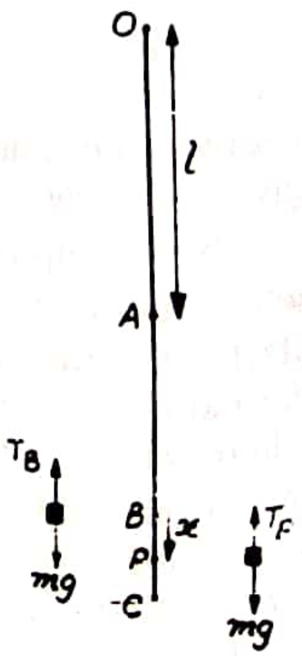
The time of one oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(ng/lm)}}=2\pi\sqrt{\left(\frac{lm}{ng}\right)}.$$

Ex. 52 (b). *A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that*

$$g(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$ **[Rohilkhand 1985]**

Sol. One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m. Let $AB=d$.



In the equilibrium position of the particle of mass m at B, the tension $T_B=\lambda(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l}=mg \text{ or } \frac{\lambda}{lm}=\frac{g}{d}. \qquad \ldots(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP=x$. When the particle is at P, the tension T_P in the string P is $\lambda\frac{d+x}{l}$, acting

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\frac{\lambda(d+x)}{l}+mg,$$

[**Note** that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards *i.e.*, in the direction of x increasing.]

or
$$m\frac{d^2x}{dt^2}=-\frac{\lambda d}{l}-\frac{\lambda x}{l}+mg$$

$$=-\frac{\lambda x}{l}, \qquad \left[\because \frac{\lambda d}{l}=mg\right].$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x, \text{ [from (1)].}$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\frac{2\pi}{\sqrt{(g/d)}}$ *i.e.*, $2\pi\sqrt{\left(\frac{d}{g}\right)}$.

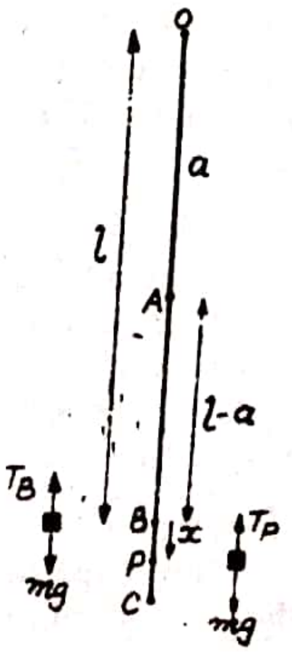
But according to the question, the period is t_1 when $d=c_1$ and the period is t_2 when $d=c_2$.

$$\therefore \quad t_1=2\pi\sqrt{(c_1/g)} \text{ and } t_2=2\pi\sqrt{(c_2/g)},$$

so that
$$t_1^2-t_2^2=(4\pi^2/g)(c_1-c_2)$$

or
$$g(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$

Ex. 53. *A mass m hangs from a light spring and is given a small vertical displacement. If l is the length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is* $l-(g/4\pi^2n^2)$.



Sol. Let $OA=a$ be the natural length of the spring which extends to a length $OB=l$ when a particle of mass m hangs in equilibrium. In the position of equilibrium of the particle at B, the tension T_B in the spring is $\lambda\{(l-a)/a\}$ and it balances the weight mg of the particle.

$$\therefore \quad \lambda[(l-a)/a]=mg. \qquad \ldots(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. It moves towards B starting at rest from C. Let P be the position of the particle after any time t, where $BP=x$. When the particle is at P, the tension T_P in the spring OP is $\lambda\,\dfrac{l+x-a}{a}$, acting vertically upwards *i.e.*, in the direction of x decreasing.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\,\frac{l+x-a}{a}=mg-\lambda\,\frac{l-a}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \text{ [from (1)].}$$

$$\therefore\ \frac{d^2x}{dt^2}=-\frac{\lambda}{am}\,x=-\frac{g}{l-a}\,x, \left[\because\ \text{ from (1), } \frac{\lambda}{am}=\frac{g}{l-a}\right].$$

Hence the motion of the particle is simple harmonic with centre at the origin B and the time period T (*i.e.*, the time for one complete oscillation)$=2\pi\sqrt{\left(\dfrac{l-a}{g}\right)}$ seconds.

Since n is given to be the number of oscillations per second, therefore $n.T=1$ or $n^2T^2=1$

or $$n^2\,\frac{4\pi^2\,(l-a)}{g}=1 \quad\text{or}\quad l-a=\frac{g}{4\pi^2n^2}$$

or $$a=l-\frac{g}{4\pi^2n^2}.$$

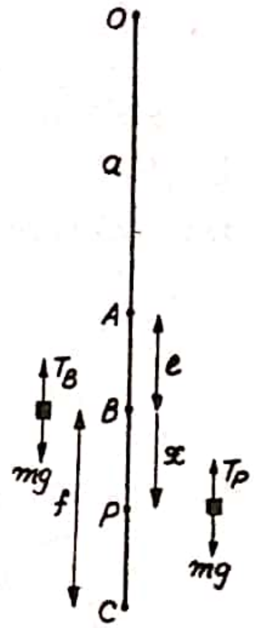
This gives the natural length a of the spring.

Ex. 54. *A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f and then let go: determine the height to which it will arise if $f^2-e^2=4ae$, e being the unstretched length of the string.*

Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. It is given that $AB=e$. In the position of equilibrium of the particle at B, the tension T_B in the string OB is $\lambda(e/a)$ and it balances the weight mg of the particle.

$$\therefore\quad mg=\lambda\,(e/a). \qquad \ldots(1)$$

Now suppose the particle is pulled down to a point C, such that $BC=f$, and then let go. It moves towards B starting with



velocity zero at C. Let P be the position of the particle after any time t, where $BP=x$. Note that we have taken B as the origin. When the particle is at P, there are two forces acting upon it :

(i) the tension $T_P=\lambda \dfrac{OP-OA}{OA}=\lambda \dfrac{e+x}{a}$

in the string OP, acting vertically upwards *i.e.*, in the direction of x decreasing, and (ii) the weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{e+x}{a}=mg-\frac{\lambda e}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \qquad \left[\because \text{ from (1), } mg=\frac{\lambda e}{a}\right].$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{g}{e}x, \quad \left[\because \text{ from (1), } \frac{\lambda}{am}=\frac{g}{e}\right].$$

Thus the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\frac{g}{e}x, \qquad \text{...(2)}$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC. The equation (2) governs the motion of the particle so long as the string does not become slack.

Since $f^2-e^2=4ae=+\text{ive}$, therefore $f>e$ *i.e.*, $BC>AB$. So when the particle, while moving in simple harmonic motion, reaches the point A, its velocity is not zero. But at A the string becomes slack and so above A the particle will move freely under gravity.

Let us first find the velocity at A for the S.H.M. given by (2). Multiplying both sides of (2) by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{e}x^2+k, \text{ where } k \text{ is a constant.}$$

But at C, $x=BC=f$ and $\left(\dfrac{dx}{dt}\right)=0$. Therefore $0=-\left(\dfrac{g}{e}\right)f^2+k$

or $$k=\left(\frac{g}{e}\right)f^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=-\frac{g}{e}x^2+\frac{g}{e}f^2=\frac{g}{e}(f^2-x^2). \qquad \ldots(3)$$

The equation (3) gives the velocity of the particle at any point from C to A. Let v_1 be the velocity of the particle at A. Then at A, $x=-e$ and $\left(\frac{dx}{dt}\right)^2=v_1^2$. Therefore, from (3), we have

$$v_1^2=\frac{g}{e}(f^2-e^2)=\frac{g}{e}\,4ae \qquad [\because\ f^2-e^2=4ae]$$

$=4ag$, the direction of v_1 being vertically upwards.

Above A the motion of the particle is freely under gravity. If the particle rises to a height h above A, we have

$0=v_1^2-2gh,$ [using the formula $v^2=u^2+2fs$]

$=4ag-2gh,$ [$\because\ v_1^2=4ag$].

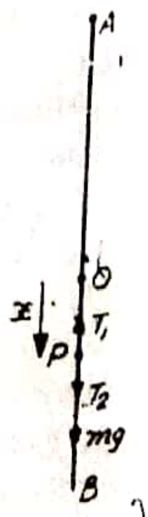
$\therefore\ 2gh=4ag$ or $h=2a$.

Hence the total height to which the particle rises above C

$$=CB+BA+h=f+e+2a.$$

Ex. 55. *A heavy particle is attached to one point of a uniform elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is* $2\pi\sqrt{(mh/2\lambda)}$, *where* λ *is the coefficient of elasticity of the string and* h *the harmonic mean of the unstretched lengths of the two parts of the string.*

Sol. Let a particle of mass m be attached to a point O of a string whose ends have been fastened to two fixed points A and B in a vertical line. The string is taut and the particle is in equilibrium at O. Let $OA=a$ and $OB=b$. Also let a_1 and b_1 be the natural lengths of the stretched portions OA and OB of the string.



Considering the equilibrium of the particle at O, we have the resultant upward force = the resultant downward force

ension in OA = the tension in OB + the weight of the particle

$$\lambda\frac{(a-a_1)}{a_1}=\lambda\frac{(b-b_1)}{b_1}+mg. \qquad \ldots(1)$$

Now suppose the particle is slightly displaced towards B and then let go. During this slight displacement of the particle both the portions of the string remain taut. Let P be the position of the particle after any time t, where $OP=x$.

When the particle is at P, there are three forces acting upon it :

(i) The tension $T_1=\lambda\,\dfrac{a+x-a_1}{a_1}$ in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension $T_2=\lambda\,\dfrac{b-x-b_1}{b_1}$ in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

(iii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=-\lambda\,\frac{a+x-a_1}{a_1}+\lambda\,\frac{b-x-b_1}{b_1}+mg$$

$$=-\lambda\,\frac{a-a_1}{a_1}+\lambda\,\frac{b-b_1}{b_1}+mg-\frac{\lambda x}{a_1}-\frac{\lambda x}{b_1}$$

$$=-\lambda\left(\frac{1}{a_1}+\frac{1}{b_1}\right)x \qquad [\text{by } (1)]$$

$$=-\lambda\left(\frac{a_1+b_1}{a_1b_1}\right)x.$$

$\therefore\ \dfrac{d^2x}{dt^2}=-\dfrac{\lambda}{m}\,\dfrac{(a_1+b_1)}{a_1b_1}\,x$, which is the equation of motion of a S.H.M. with centre at the origin O. This equation of motion holds good so long as both the portions of the string remain taut. But the initial displacement given to the particle below O being small, both the portions of the string must remain taut for ever. Hence this equation governs the entire motion of the particle. Thus the entire motion of the particle is simple harmonic about the centre O and the time period of one complete oscillation

$$=2\pi\sqrt{\left\{\frac{m\,a_1b_1}{\lambda\,(a_1+b_1)}\right\}}=\pi\sqrt{\left\{\frac{m\,(2a_1b_1)}{2\lambda\,(a_1+b_1)}\right\}}=2\pi\sqrt{\left(\frac{mh}{2\lambda}\right)},$$

where $h=\dfrac{2a_1b_1}{a_1+b_1}$ is the harmonic mean between a_1 and b_1.

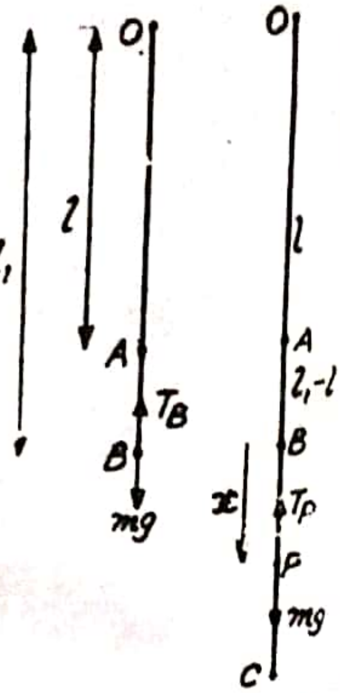
Ex. 56. *A light elastic string of natural length l has one extremity fixed at a point O and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1. Show that if the stone be dropped from rest at O, it will come to instantaneous rest at a depth $\sqrt{(l_1^2-l^2)}$ below the equilibrium position.* **[Kanpur 1978; Meerut 80, 84 P, 88 P, Allahabad 75]**

Sol. $OA=l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a stone of mass m

attached to the other end of the string and $OB=l_1$. When the stone rests at B, the tension T_B of the string balances the weight the stone. Therefore

$$T_B=\frac{\lambda\,(l_1-l)}{l}=mg, \qquad \ldots(1)$$

where λ is the modulus of elasticity of the string.



Now the stone is dropped from O. It falls the distance OA $(=l)$ freely under gravity. If v_1 be the velocity gained by the stone at A, we have $v_1=\sqrt{(2gl)}$ downwards. When the stone falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B, the force of tension acting vertically upwards remains less than the weight of the stone acting vertically downwards. Therefore during the fall from A to B the velocity of the stone goes on increasing. When the stone begins to fall below B, its velocity goes on decreasing because now the force of tension exceeds the weight of the stone. Let the stone come to instantaneous rest at C, where $BC=a$.

During the motion of the stone below A, let P be its position after any time t, where $BP=x$. [Note that we have taken the position of equilibrium B of the stone as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing].

When the stone is at P, there are two forces acting upon it :

(i) The tension $T_P=\lambda\dfrac{(l_1+x)-l}{l}$ in the string OP acting in the direction OP *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the stone acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion $(P=mf)$, the equation of motion of the stone at P, is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{(l_1+x)-l}{l}=mg-\lambda\frac{(l_1-l)}{l}-\frac{\lambda x}{l}$$

$$=-\frac{\lambda x}{l}, \qquad \text{[from (1)]}.$$

[Note that the force acting in the direction of x increasing has been taken with + ive sign and that in the direction of x decreasing with − ive sign].

Thus $\dfrac{d^2x}{dt^2}=-\dfrac{\lambda}{lm}x,$ ...(2)

which is the equation of a S.H.M. with centre at the origin B. The equation (2) holds good so long as the string is stretched *i.e.*, for the motion of the stone between A and C.

Multiplying (2) by 2 (dx/dt) and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{lm}x^2+D, \text{ where } D \text{ is a constant.}$$

At A, $x=-(l_1-l)$ and $dx/dt=\sqrt{(2gl)}$;

$$\therefore\quad 2gl=-\frac{\lambda}{lm}(l_1-l)^2+D \quad\text{or}\quad D=2gl+\frac{\lambda}{lm}(l_1-l)^2.$$

Thus, we have $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{\lambda}{lm}x^2+2gl+\dfrac{\lambda}{lm}(l_1-l)^2.$ ... (3)

The equation (3) gives velocity of the stone at any point between A and C. At C, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=-\frac{\lambda}{lm}a^2+2gl+\frac{\lambda}{lm}(l_1-l)^2$$

or
$$-\frac{g}{(l_1-l)}a^2+2gl+\frac{g}{(l_1-l)}(l_1-l)^2=0$$

$$\left[\because \text{ from (1), } \frac{\lambda}{lm}=\frac{g}{l_1-l}\right]$$

or
$$\frac{a^2}{l_1-l}=2l+l_1-l=l_1+l$$

or
$$a^2=(l_1-l)(l_1+l)=l_1^2-l^2.$$

$$\therefore\ a=\sqrt{(l_1^2-l^2)}.$$

Ex. 57. *A light elastic string whose natural length is a has one end fixed to a point O, and to the other end is attached a weight which in equilibrium would produce an extension e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae+e^2)}$ below the position of equilibrium.*

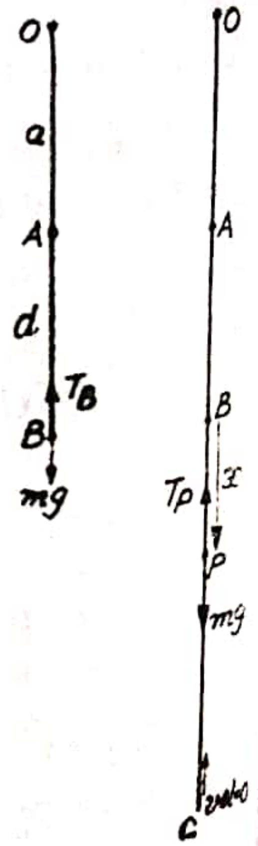
Sol. Proceed as in the preceding example 56. Take $l=a$, $l_1-l=e$ or $l_1=e+a$. Then the required distance $=\sqrt{(l_1^2-l^2)}$ $=\sqrt{[(e+a)^2-a^2]}=\sqrt{(2ae+e^2)}$.

Ex. 58. *A light elastic string of natural length a has one extremity fixed at a point O and the other attached to a body of mass m. The equilibrium length of the string with the body attached* *is $a \sec \theta$. Show that if the body be dropped from rest at O it will come to instantaneous rest at a depth $a \tan \theta$ below the position of equilibrium.*

Sol. Proceed as in Example 56. Take $l=a$ and $l_1=a \sec \theta$. We have then, the required depth below the equilibrium position $=\sqrt{(a^2 \sec^2 \theta-a^2)}=a\sqrt{(\sec^2 \theta-1)}=a \tan \theta$.

Ex. 59. *A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time* $\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, *where a is the natural length of the string.*

[Lucknow 1976; Kanpur 83; Agra 80; Meerut 88]



Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B=\lambda \frac{OB-OA}{OA}=\lambda \frac{d}{a},$$

where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B. we have

$$mg=T_B=\lambda \frac{d}{a}=mg \frac{d}{a}. \quad \left[\because \ \lambda=mg, \text{ as given}\right]$$

$$\therefore \quad d=a.$$

Now the particle is pulled down to a point C such that $OC=4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP=x$. [**Note** that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda \frac{a+x}{a}=\frac{mg}{a}(a+x)$ in the string OP acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\frac{mg}{a}(a+x)=-\frac{mgx}{a}.$$

Thus $$\frac{d^2x}{dt^2}=-\frac{g}{a}x, \qquad \text{...(1)}$$

which is the equation of a S.H.M. with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+k$, where k is a constant.

At the point C, $x=BC=2a$, and the velocity $dx/dt=0$;—

$$\therefore \quad k=\frac{g}{a}\cdot 4a^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{a}(4a^2-x^2). \qquad \text{...(2)}$$

Taking square root of (2), we have

$$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\sqrt{(4a^2-x^2)},$$

the —ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variabes, we have

$$dt=-\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{(4a^2-x^2)}}. \qquad \text{...(3)}$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1}dt=-\sqrt{\left(\frac{a}{g}\right)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2-x^2)}}$$

or $$t_1=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}(-\tfrac{1}{2})-\cos^{-1}(1)\right]=\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}.$$

Let v_1 be the velocity of the particle at A. Then at A

$$x=-a \text{ and } (dx/dt)^2=v_1^2.$$

So from (2), we have $v_1^2=(g/a)(4a^2-a^2)$

or $v_1=\sqrt{(3ag)}$, the direction of v_1 being vertically upwards.

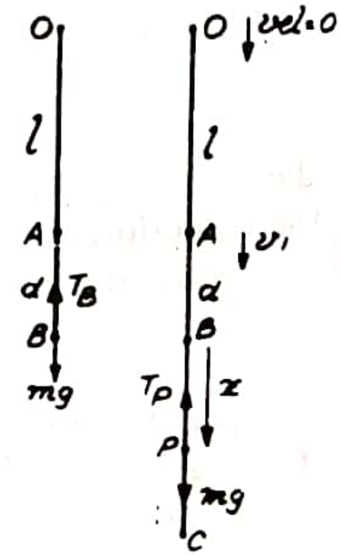
Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

$$0=\sqrt{(3ag)}-gt_2, \text{ so that } t_2=\sqrt{\left(\frac{3a}{g}\right)}.$$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to $C=2\ (t_1+t_2)$

$$=2\left[\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}+\sqrt{\left(\frac{3a}{g}\right)}\right]=\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right].$$

Ex. 60. *A heavy particle of mass m is attached to one end of an elastic string of natural length l, whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is $l\ \cot^2(\theta/2)$, the modulus of elasticity of the string is $\frac{1}{2}mg\ \tan^2\theta$.* **[Meerut 1988]**



Sol. Let $OA=l$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and let $AB=d$. In the equilibrium position at B, the tension T_B in the string OB balances the weight mg of the particle. Therefore,

$$T_B=\lambda\frac{d}{l}=mg, \qquad \ldots(1)$$

where λ is the modulus of elasticity of the string. Now the particle is dropped at rest from O. It falls the distance OA freely under gravity. If v_1 be the velocity gained by it at A, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When the particle falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B the force of tension acting vertically upwards remains less than the weight of the particle acting vertically downwards. Therefore during the fall from A to B the velocity of the particle goes on increasing. When the particle begins to fall below B, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at C, where $OC=l\ \cot^2\frac{1}{2}\theta$, as given.

During the motion of the particle below A, let P be its position after any time t, where $BP=x$. [**Note** that we have taken the position of equilibrium B of the particle as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\dfrac{d+x}{l}$ in the string OP, acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{d+x}{l}$$

$$=mg-\frac{\lambda d}{l}-\frac{\lambda x}{l}=-\frac{\lambda x}{l}, \text{ by (1).}$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x. \qquad \ldots$$

$$\left[\because \text{ from (1), } \frac{\lambda}{lm}=\frac{g}{d}\right]$$

The equation (2) represents a S. H. M. with centre at the point B and amplitude BC. Hence the motion of the particle below A is simple harmonic.

Multiplying (2) by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$\left(\dfrac{dx}{dt}\right)^2=-\dfrac{g}{d}\cdot x^2+D$, where D is a constant.

At the point A, $x=-d$ and the velocity$=dx/dt=\sqrt{(2gl)}$.

$$\therefore\quad D=2gl+gd.$$

$$\therefore\quad \text{we have, (velocity)}^2=\left(\frac{dx}{dt}\right)^2=-\frac{g}{d}x^2+2gl+gd. \qquad \ldots(3)$$

The above equation (3) gives the velocity of the particle at any point between A and C. At C, $x=BC=OC-OB=l\cot^2\frac{1}{2}\theta-(l+d)$ and $dx/dt=0$. Therefore (3) gives

$$0=-\frac{g}{d}\left[(l\cot^2\tfrac{1}{2}\theta-l)-d\right]^2+2gl+gd$$

$$=-\frac{g}{d}\left[(l\cot^2\tfrac{1}{2}\theta-l)^2+d^2-2ld\,(\cot^2\tfrac{1}{2}\theta-1)\right]+2gl+gd$$

$$=-\left[\frac{g}{d}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right]$$

$$=-\left[\frac{\lambda}{ml}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right], \left[\because \frac{g}{d}=\frac{\lambda}{ml} \text{ by (1)}\right].$$

$$\lambda=\frac{2mgl^2\cot^2\frac{1}{2}\theta}{(l\cot^2\frac{1}{2}\theta-l)^2}=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cot^2\frac{1}{2}\theta-1)^2}$$

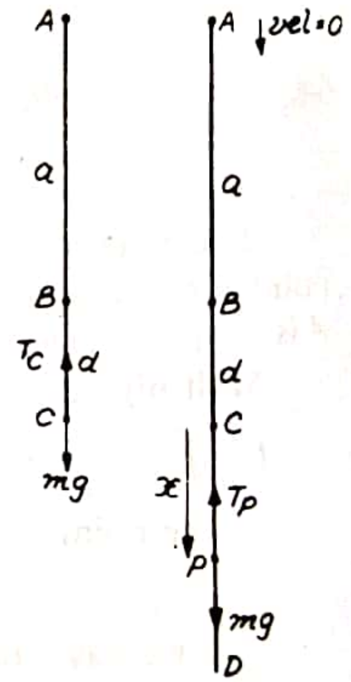
$$=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)^2}.\sin^4\tfrac{1}{2}\theta$$

$$=\tfrac{1}{2}\,\frac{mg.4\cos^2\frac{1}{2}\theta\sin^2\frac{1}{2}\theta}{\cos^2\theta}=\tfrac{1}{2}mg.\frac{\sin^2\theta}{\cos^2\theta}=\tfrac{1}{2}mg\tan^2\theta.$$

Ex. 61. *One end of a light elastic string of natural length a and modulus of elasticity 2mg is attached to a fixed point A and the other end to a particle of mass m. The particle initially held at rest at A, is let fall. Show that the greatest extension of the string is* $\frac{1}{2}a\,(1+\sqrt{5})$ *during the motion and show that the particle will reach back A again after a time* $(\pi+2-\tan^{-1}2)\sqrt{(2a/g)}$.

Sol. $AB=a$ is the natural length of an elastic string whose one end is fixed at A. Let C be the position of equilibrium of a particle of mass m attached to the other end of the string and let $BC=d$. In the position of equilibrium of the particle at C, the tension $T_C=\lambda\,\frac{d}{a}=2mg\,\frac{d}{a}$ in the string AC balances the weight mg of the particle.

$\therefore\quad mg=2mg\,(d/a)$ or $d=a/2.\qquad\ldots(1)$



Now the particle is dropped at rest from A. It falls the distance AB freely under gravity. If v_1 be the velocity gained at B, we have $v_1=\sqrt{(2ga)}$ in the downward direction. When the particle falls below B, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from B to C the velocity of the particle goes on increasing as the tension remains less than the weight of the particle and when the particle begins to fall below C, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at D.

During the motion of the particle below B, let P be its position after any time t, where $CP=x$. If T_P be the tension in the string AP, we have $T_P=\lambda\,\frac{d+x}{a}=2mg\,\frac{\frac{1}{2}a+x}{a}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-T_P=mg-2mg\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}x.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{2g}{a}x, \qquad \ldots(2)$$

which is the equation of a S. H. M. with centre at the point C and amplitude CD.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}.x^2+k$, where k is a constant.

At the point B, the velocity

$$=dx/dt=\sqrt{(2ga)} \text{ and } x=-d=-\frac{a}{2}.$$

$$\therefore \quad k=2ga+\frac{2g}{a}.\frac{a^2}{4}=2ga+\frac{2ga}{4}=\frac{5ag}{2}.$$

$$\therefore \quad \text{We have } \left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}x^2+\frac{5ag}{2}. \qquad \ldots(3)$$

The equation (3) gives the velocity of the particle at any point between B and D. At D, $x=CD$ and $dx/dt=0$. So putting $dx/dt=0$ in (3), we have

$$0=-\frac{2g}{a}x^2+\frac{5ag}{2} \quad \text{or} \quad x^2=\frac{5a^2}{4}$$

or $$x=\frac{a}{2}\sqrt{5}=CD.$$

$\therefore$ the greatest extension of the string

$$=BC+CD=\tfrac{1}{2}a+\tfrac{1}{2}a\sqrt{5}=\tfrac{1}{2}a\,(1+\sqrt{5}).$$

Now from (3), we have $\left(\frac{dx}{dt}\right)^2=\frac{2g}{a}\left[\frac{5}{4}a^2-x^2\right].$

$\therefore \quad \frac{dx}{dt}=\sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left[\frac{5}{4}a^2-x^2\right]}$, the +ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have $dt=\sqrt{\left(\frac{a}{2g}\right)}\frac{dx}{\sqrt{\left[\frac{5}{4}a^2-x^2\right]}}$

If t_1 is the time from B to D, then

$$\int_0^{t_1} dt=\sqrt{\left(\frac{a}{2g}\right)}\int_{-a/2}^{(a\sqrt{5})/2}\frac{dx}{\sqrt{\left[\frac{5}{4}a^2-x^2\right]}}$$

or $$t_1=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}\left\{\frac{x}{(a\sqrt{5})/2}\right\}\right]_{-a/2}^{(a\sqrt{5})/2}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}1+\sin^{-1}\frac{1}{\sqrt{5}}\right]=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\tan^{-1}\tfrac{1}{2}\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\cot^{-1} 2\right)=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1} 2\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\,(\pi-\tan^{-1} 2).$$

And if t_2 is the time from A to B, (while falling freely under gravity), then

$$a=0.t_2+\frac{1}{2}\,g t_2^2 \quad \text{or} \quad t_2=\sqrt{\left(\frac{2a}{g}\right)}.$$

$\therefore$ the total time to return back to $A=2$ (time from A to D)

$$=2\,(t_2+t_1)=2\left[\sqrt{\left(\frac{a}{2g}\right)}\,(\pi-\tan^{-1} 2)+\sqrt{\left(\frac{2a}{g}\right)}\right]$$

$$=\sqrt{\left(\frac{2a}{g}\right)}\,[\pi-\tan^{-1} 2+2].$$

This proves the required result.

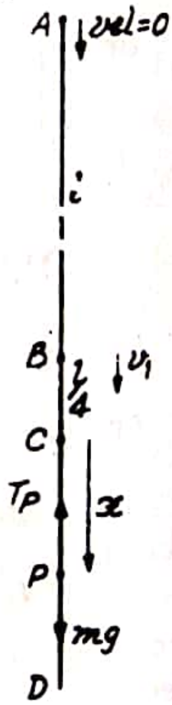
Ex. 62. *A light elastic string AB of length l is fixed at A and is such that if a weight w be attached to B, the string will be stretched to a length $2l$. If a weight $\frac{1}{4}w$ be attached to B and let fall from the level of A prove that (i) the amplitude of the S.H.M. that ensues is $3l/4$; (ii) the distance through which it falls is $2l$; and (iii) the period of oscillation is*

$$\sqrt{\left(\frac{l}{4g}\right)}\,(4\sqrt{2}+\pi+2\sin^{-1}\tfrac{1}{3}).$$

[Meerut 1981S, 88S]

Sol. $AB=l$ is the natural length of an elastic string whose one end is fixed at A. Let λ be the modulus of elasticity of the string. If a weight w be attached to the other end of the string, it extends the string to a length $2l$ while hanging in equilibrium. Therefore

$$w=\lambda\,\frac{2l-l}{l}=\lambda. \qquad \ldots(1)$$



Now in the actual problem a particle of weight $\frac{1}{4}w$ or mass $\frac{1}{4}(w/g)$ is attached to the free end of the string. Let C be the position of equilibrium of this weight $\frac{1}{4}w$. Then considering the equilibrium of this weight at C, we have

$$\tfrac{1}{4}w=\lambda\,\frac{BC}{l}=w\,\frac{BC}{l} \qquad [\because \text{ by (1)}, \lambda=w]$$

$\therefore\ BC=\frac{1}{4}l.$

Now the weight $\frac{1}{4}w$ is dropped from A. It falls the distance $AB\,(=l)$ freely under gravity. If v_1 be the velocity gained by this weight at B, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When this weight falls below B, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the weight continues increasing upto C, after which it starts decreasing. Suppose the weight comes to instantaneous rest at D, where $CD=a$.

During the motion of the weight below B, let P be its position after any time t, where $CP=x$. [Note that we have taken C as origin and CP is the direction of x increasing]. If T_P be the tension in the string AP, we have $T_P=w\dfrac{\frac{1}{4}l+x}{l}$ acting vertically upwards.

The equation of motion of this weight $w/4$ at P is

$$\frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=\frac{1}{4}w-w\frac{\frac{1}{4}l+x}{l}=\frac{1}{4}w-\frac{1}{4}w-w\frac{x}{l}$$

or $$\frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=-w\frac{x}{l} \quad \text{or} \quad \frac{d^2x}{dt^2}=-\frac{4g}{l}x, \qquad \ldots(2)$$

which is the equation of a S.H.M. with centre at the origin C, and amplitude CD $(=a)$. The equation (2) holds good so long as the string is stretched *i.e.*, for the motion of the weight from B to D.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+k, \text{ where } k \text{ is a constant.}$$

At B, $x=-\frac{1}{4}l$ and $dx/dt=\sqrt{(2gl)}$;

$$\therefore \quad 2gl=-\frac{4g}{l}\cdot\frac{1}{16}l^2+k \quad \text{or} \quad k=\frac{9}{4}gl.$$

Thus, we have $$\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+\frac{9}{4}gl=\frac{4g}{l}\left(\frac{9}{16}l^2-x^2\right) \qquad \ldots(3)$$

The equation (3) gives velocity at any point between B and D. At D, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=\frac{4g}{l}\left(\frac{9}{16}l^2-a^2\right) \text{ or } a=\frac{3}{4}l.$$

Hence the amplitude a of the S.H.M. that ensues is $\frac{3}{4}l$.

Also the total distance through which the weight falls $=AB+BC+CD=l+\frac{1}{4}l+\frac{3}{4}l=2l.$

Now let t_1 be the time taken by the weight to fall freely under gravity from A to B.

Then using the formula $v=u+ft$, we get

$$\sqrt{(2gl)}=0+gt_1 \quad \text{or} \quad t_1=\sqrt{(2l/g)}.$$

Again let t_2 be the time taken by the weight to fall from B to D while moving in S.H.M. From (3), on taking square root, we get $\quad \dfrac{dx}{dt}=+\sqrt{\left(\dfrac{4g}{l}\right)}\sqrt{\left(\dfrac{9}{16}\, l^2-x^2\right)},$

where the +ive sign has been taken because the weight is moving in the direction of x increasing. Separating the variables, we get

$$\sqrt{\left(\frac{l}{4g}\right)}\cdot\frac{dx}{\sqrt{\left(\frac{9l^2}{16}-x^2\right)}}=dt.$$

Integrating from B to D, we get

$$\int_0^{t_2} dt=\sqrt{\left(\frac{l}{4g}\right)}\int_{-l/4}^{3l/4}\frac{dx}{\sqrt{\left(\frac{9}{16}\, l^2-x^2\right)}}.$$

$$\therefore\quad t_2=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}\frac{x}{\frac{3}{4}l}\right]_{-l/4}^{3l/4}=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}1-\sin^{-1}\left(-\tfrac{1}{3}\right)\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right].$$

Hence the total time taken to fall from A to $D=t_1+t_2$

$$=\sqrt{\left(\frac{2l}{g}\right)}+\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{\pi}{2}+\sin^{-1}\tfrac{1}{3}+2\sqrt{2}\right].$$

Now after instantaneous rest at D, the weight begins to move upwards. From D to B it moves in S.H.M. whose equation is (2). At B the string becomes slack and S.H.M. ceases. The velocity of the weight at B is $\sqrt{(2gl)}$ upwards. Above B the weight rises freely under gravity and comes to instantaneous rest at A. Thus it oscillates again and again between A and D.

The time period of one complete oscillation = 2. time from A to $D=2.(t_1+t_2)=\sqrt{\left(\dfrac{l}{4g}\right)}\left\{\pi+4\sqrt{2}+2\sin^{-1}\tfrac{1}{3}\right\}.$

Ex. 63. *A heavy particle of mass m is attached to one end of an elastic string of natural length l ft., whose modulus of elasticity is equal to the weight of the particle and the other end is fixed at O. The particle is let fall from O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\, l$ ft. Show that this depth is attained in time*

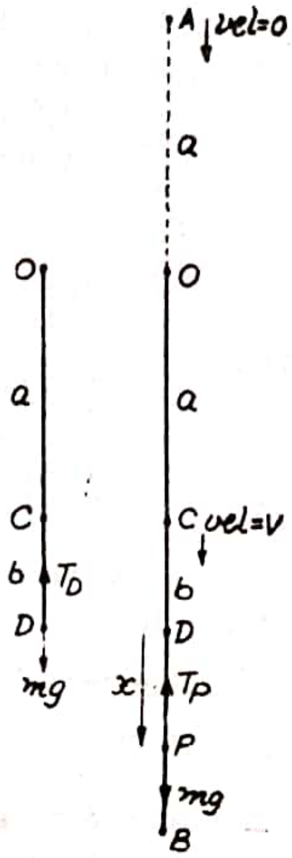
$[\sqrt{2}+\pi-\cos^{-1}(1/\sqrt{3})]\sqrt{(l/g)}$ *seconds.* **[Lucknow 1980]**

Sol. Proceed as in the preceding example.

Ex. 64. *A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity 2mg, whose other end is fixed at O. The particle is let fall from A, when A is* *vertically above O and OA=a. Show that its velocity will be zero at B, where OB=3a.* **[Meerut 77, 83)**

Calculate also the time from A to B.

Sol. Let $OC=a$, be the natural length of an elastic string suspended from the fixed point O. The modulus of elasticity λ of the string is given to be equal to $2mg$, where m is the mass of the particle attached to the other end of the string.



If D is the position of equilibrium of the particle such that $CD=b$, then at D the tension T_D in the string OD balances the weight of the particle.

$$\therefore \qquad mg=T_D=\lambda\,\frac{b}{a}=2mg.\frac{b}{a}$$

or $$b=a/2.$$

The particle is let fall from A where $OA=a$. Then the motion from A to C will be freely under gravity.

If V is the velocity of the particle gained at the point C, then

$$V^2=0+2g.2a \quad \text{or} \quad V=2\sqrt{(ag)}, \qquad \ldots(1)$$

in the downward direction.

As the particle moves below C, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the particle continues increasing upto D after which it starts decreasing. Suppose that the particle comes to instantaneous rest at B. During the motion below C, let P be the position of the particle at any time t, where $DP=x$. If T_P is the tension in the string OP, we have

$$T_P=\lambda\,\frac{b+x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=mg-T_P=mg-\lambda.\frac{b+x}{a}$$

$$=mg-2mg.\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}\,x$$

or $$\frac{d^2x}{dt^2}=-\frac{2g}{a}\,x, \qquad \ldots(2)$$

which represents a S. H. M. with centre at D and holds good for the motion from C to B.

Multiplying both sides of (2) by 2 (dx/dt) and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}x^2+k,\quad \text{where } k \text{ is a constant.}$$

But at C, $x=-DC=-b=-a/2$ and $(dx/dt)^2=V^2=4ag.$

$$4ag=-\frac{2g}{a}\cdot\frac{a^2}{4}+k \qquad \text{or} \qquad k=\tfrac{9}{2}ag.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}x^2+\frac{9}{2}ag$$

or
$$\left(\frac{dx}{dt}\right)^2=\frac{2g}{a}\left(\frac{9}{4}a^2-x^2\right) \qquad \ldots(3)$$

If the particle comes to instantaneous rest at B where $DB=x_1$, (say), then

at B, $x=x_1$ and $dx/dt=0$. Therefore from (3), we have

$$0=\frac{2g}{a}\left(\frac{9}{4}a^2-x_1^2\right),\quad \text{giving } x_1=\frac{3}{2}a.$$

Now $OB=OC+CD+DB=a+\frac{1}{2}a+\frac{3}{2}a=3a$,
which proves the first part of the question.

To find the time from A to B.

If t_1 is the time from A to C, then from $s=ut+\frac{1}{2}ft^2$,

$$2a=0+\tfrac{1}{2}gt_1^2. \qquad \therefore \quad t_1=2\sqrt{(a/g)}. \qquad \ldots(4)$$

Now from (3), we have

$$\frac{dx}{dt}=\sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left(\frac{9}{4}a^2-x^2\right)};$$

the +ive sign has been taken because the particle is moving in the direction of x increasing

or
$$dt=\sqrt{\left(\frac{a}{2g}\right)}\cdot\frac{dx}{\sqrt{(\frac{9}{4}a^2-x^2)}}.$$

Integrating from C to B, the time t_2 from C to B is given by

$$\therefore \quad t_2=\sqrt{\left(\frac{a}{2g}\right)}\int_{x=-a/2}^{3a/2}\frac{dx}{\sqrt{(\frac{9}{4}a^2-x^2)}}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\sin^{-1}\left(\frac{x}{3a/2}\right)\right]_{-a/2}^{3a/2}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\sin^{-1}1-\sin^{-1}\left(-\frac{1}{3}\right)\right]$$

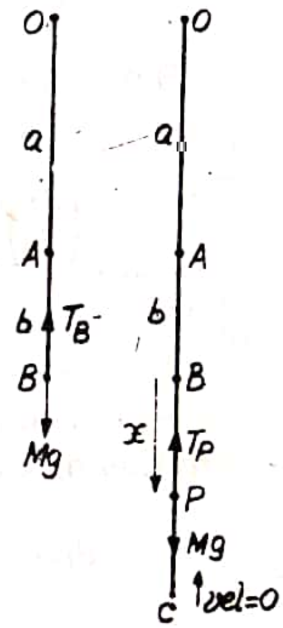
$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{3}\right)\right].$$

$\therefore$ the time from A to $B=t_1+t_2$

$$=2\sqrt{(a/g)}+\sqrt{(a/2g)}.[\pi/2+\sin^{-1}(1/3)]$$
$$=\tfrac{1}{2}\sqrt{(a/2g)}\,[4\sqrt{2}+\pi+2\sin^{-1}(1/3)].$$

Ex. 65. *Two bodies of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off; show that the distance of M from the upper end of the string at time t is $a+b+c\cos\{\sqrt{(g/b)}\,t\}$, where a is the unstretched length of the string, b and c the distances by which it would be stretched when supporting M and M' respectively.*

[Lucknow 1978]



Sol. Let $OA=a$ be the natural length of an elastic string suspended from the fixed point O. If B is the position of equilibrium of the particle of mass M attached to the lower end of the string and $AB=b$, then

$$Mg=\lambda\frac{AB}{a}=\lambda\frac{b}{a}. \quad ...(1)$$

Similarly $M'g=\lambda\dfrac{c}{a}$. $\quad ...(2)$

Adding (1) and (2), we have

$$(M+M')\,g=\lambda\frac{b+c}{a}.$$

Thus the string will be stretched by the distance $b+c$ when supporting both the masses M and M' at the lower end. Let OC be the stretched length of the string when both the masses M and M' are attached to its lower end. Then

$$AC=b+c \text{ and so } BC=AC-AB=b+c-b=c.$$

Now when M' falls off at C, the mass M will begin to move towards B starting with velocity zero at C. Let P be the position of the particle of mass M at any time t, where $BP=x$.

If T_P be the tension in the string OP, then

$$T_P=\lambda\frac{b+x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ the equation of motion of the particle of mass M at P is

$$M\frac{d^2x}{dt^2}=Mg-T_P=Mg-\lambda\frac{b+x}{a}$$

$$=Mg-\lambda\frac{b}{a}-\frac{\lambda x}{a}$$

$$=Mg-Mg-\frac{Mg}{b}x, \quad \left[\because \text{ from (1), } Mg=\frac{\lambda b}{a}\right]$$

$$=-\frac{Mg}{b}x.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{b}x, \qquad \ldots(3)$$

which represents a S. H. M. with centre at B and amplitude BC.

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{b}x^2+k, \quad \text{where } k \text{ is a constant.}$$

But at the point C, $x=BC=c$ and $dx/dt=0$.

$$\therefore \quad 0=-(g/b)c^2+k \quad \text{or} \quad k=(g/b)c^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{b}(c^2-x^2)$$

or $$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{b}\right)}\sqrt{\left(c^2-x^2\right)},$$

the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

$$\therefore \quad dt=-\sqrt{\left(\frac{b}{g}\right)}\frac{dx}{\sqrt{(c^2-x^2)}}, \text{ separating the variables.}$$

Integrating, $t=\sqrt{(b/g)}\cos^{-1}(x/c)+D$, where D is a constant.

But at C, $t=0$ and $x=c$; $\quad \therefore \quad D=0$.

$$\therefore \quad t=\sqrt{(b/g)}\cos^{-1}(x/c)$$

or $$x=BP=c\cos\{\sqrt{(g/b)}\,t\}.$$

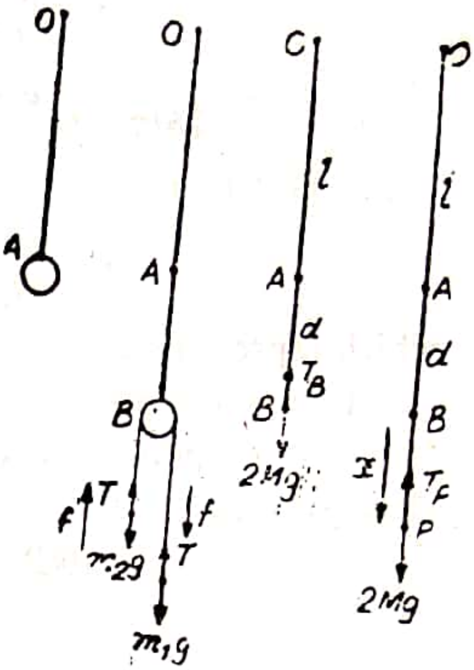
$\therefore$ the required distance of the particle of mass M at time t from the point O

$$=OP=OA+AB+BP=a+b+c\cos\{\sqrt{(g/b)}\,t\}.$$

Ex. 66. *A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity mg. If masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point of suspension is $l\{1+(2M/m)\}$, where M is the harmonic mean between m_1 and m_2.*

[Meerut 1981, 84, 85 S]

Sol. Let a smooth **light** pulley be suspended from a fixed point O by a spring OA of natural length l and modulus of elasticity $\lambda=ng$. Let B be the position of equilibrium of the pulley when masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley. Let T be the tension in the inextensible string passing round the pulley. Let us first find the value of T.



Let f be the common acceleration of the particles m_1, m_2 which hang at the ends of a light inextensible string passing round the pulley. If $m_1>m_2$, then the equations of motion of m_1, m_2 are

$$m_1g-T=m_1f \quad \text{and} \quad T-m_2g=m_2f.$$

Solving, we get $T=\dfrac{2m_1m_2}{(m_1+m_2)}\,g=Mg,$

where $M=\dfrac{2m_1m_2}{m_1+m_2}=$the harmonic mean between m_1 and m_2.

Now the pressure on the pulley$=2T=2Mg$ and therefore the pulley, which itself is light, behaves like a particle of mass $2M$.

Now the problem reduces to the vertical motion of a mass $2M$ attached to the end A of the string OA whose other end is fixed at O. If B is the equilibrium position of the mass $2M$ and $AB=d$, then the tension T_B in the spring OB is $\lambda(d/l)$, acting vertically upwards.

For equilibrium of the pulley of mass $2M$ at the point B,

we have $$2Mg=T_B=\lambda\,\frac{d}{l}=ng\,\frac{d}{l}$$

or $$d=\frac{2Ml}{n}. \qquad \ldots(1)$$

Now let the particle of mass $2M$ be slightly pulled down and then let go. If P is the position of this particle at time t such that $BP=x$, then the tension in the spring OP

$$=T_P=\lambda\,\frac{d+x}{l}=ng.\,\frac{d+x}{l}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the pulley is given by

$$2M.\frac{d^2x}{dt^2}=2Mg-T_P$$

$$=2Mg-ng\frac{d+x}{l}=2Mg-ng\frac{d}{l}-\frac{ng}{l}x=-\frac{ng}{l}x,$$

[by (1)]

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{ng}{2Ml}x,$$

which represents a simple harmonic motion about the centre B.

Hence the pulley executes simple harmonic motion with centre at the point B whose depth below the point of suspension O is given by

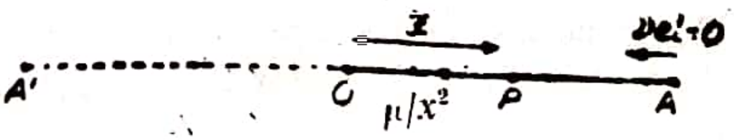
$$OB=OA+AB=l+d$$
$$=l+\frac{2Ml}{n}=l\left(1+\frac{2M}{n}\right)$$

§ 11. Motion under inverse square law.

A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance from the fixed point. If the particle was initially at rest, to investigate the motion.

[Lucknow 1977, Meerut 83S, 84, 86, 87S]

Let a particle start from rest from a point A such that $OA=a$, where O is the fixed point (*i.e.*, the centre of force) on the line and is taken as origin. Let P be the position of the particle at any time t, such that $OP=x$. Then the acceleration at $P=\mu/x^2$, towards O, where μ is a constant.



$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \qquad ...(1)$$

[$-$ive sign has been taken because d^2x/dt^2 is positive in the direction of x increasing while here μ/x^2 acts in the direction of x decreasing].

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2\mu}{x}+A$, where A is constant of integration.

But at A, $x=OA=a$ and $dx/dt=0$.

$$\therefore \quad 0=\frac{2\mu}{a}+A \quad \text{or} \quad A=-\frac{2\mu}{a}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right), \qquad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force O.

From (2), we have on taking square root

$$\frac{dx}{dt}=-\sqrt{\left(\frac{2\mu}{a}\right)}\cdot\sqrt{\left(\frac{a-x}{x}\right)}.$$

[Here $-$ive sign is taken since the particle is moving in the direction of x decreasing].

Separating the variables, we get

$$dt=-\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\sqrt{\left(\frac{x}{a-x}\right)}\,dx.$$

Integrating, $t=-\sqrt{\left(\frac{a}{2\mu}\right)}\int\sqrt{\left(\frac{x}{a-x}\right)}\,dx+B$, where B is constant of integration.

Putting $x=a\cos^2\theta$, so that $dx=-2a\cos\theta\sin\theta\,d\theta$, we have

$$t=\sqrt{\left(\frac{a}{2\mu}\right)}\int\sqrt{\left(\frac{a\cos^2\theta}{a-a\cos^2\theta}\right)}.2a\sin\theta\cos\theta\,d\theta+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int 2\cos^2\theta\,d\theta+B=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int(1+\cos 2\theta)\,d\theta+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left(\theta+\frac{\sin 2\theta}{2}\right)+B=a\sqrt{\left(\frac{a}{2\mu}\right)}(\theta+\sin\theta\cos\theta)+B$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}[\theta+\sqrt{(1-\cos^2\theta)}.\cos\theta]+B.$$

But $x=a\cos^2\theta$ means $\cos\theta=\sqrt{(x/a)}$ and $\theta=\cos^{-1}\sqrt{(x/a)}$.

$$\therefore \quad t=a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right]+B.$$

But initially at A, $t=0$ and $x=OA=a$.

$$\therefore \quad 0=a\sqrt{\left(\frac{a}{2\mu}\right)}[0+0]+B \quad \text{or} \quad B=0.$$

$$\therefore \quad t=a\sqrt{\left(\frac{a}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right], \qquad \ldots(3)$$

which gives the time from the initial position A to any point distant x from the centre of force.

Putting $x=0$ in (3), the time t_1 taken by the particle from A to O is given by

$$t_1 = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \left[\frac{\pi}{2} + 0\right] = \frac{\pi}{2}\sqrt{\left(\frac{a^3}{2\mu}\right)}. \quad \ldots(4)$$

Putting $x=0$ in (2), we see that the velocity at O is infinite and therefore the particle moves to the left of O. But the acceleration on the particle is towards O, so the particle moves to the left of O under retardation which is inversely proportional to the square of the distance from O. The particle will come to instantaneous rest at A', where $OA' = OA = a$, and then retrace its path. Thus, the particle will oscillate between A and A'.

Time of one complete oscillation $= 4 \times$ (time from A to O)

$$= 4t_1 = 2\pi\sqrt{(a^3/2\mu)}.$$

§ 12. Motion of a particle under the attraction of the earth. Newton's law of gravitation. When a particle moves under the attraction of the earth, the acceleration acting on it towards the centre of the earth will be as follows :

1. When the particle moves (upwards or downwards) outside the surface of the earth, the acceleration varies inversely as the square of the distance of the particle from the centre of the earth.
2. When the particle moves inside the earth through a hole made in the earth, the acceleration varies directly as the distance of the particle from the centre of the earth.
3. The value of the acceleration at the surface of the earth is g.

Illustrative Examples :

Ex. 67. *Show that the time occupied by a body, under the acceleration K/x^2 towards the origin, to fall from rest at distance a to distance x from the attracting centre can be put in the form*

$$\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right].$$

Prove also that the time occupied from $x=3a/4$ to $a/4$ is one-third of the whole time of descent from a to 0.

Sol. For the first part see equation (3) of § 11. (Deduce this equation here).

Thus the time t measured from the initial position $x=a$ to any point at a distance x from the centre O is given by

$$t = \sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right]. \quad \ldots(1)$$

Note that here $\mu = K$.

Let t_1 be the whole time of descent from $x=a$ to $x=0$. Then at O, $x=0$, $t=t_1$. Putting these values in the relation (1) connecting x and t, we have

$$t_1=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1} 0+0\right]=\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}. \quad ...(2)$$

Now let t_2 be the time from $x=a$ to $x=3a/4$. Then putting $x=3a/4$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{\left(\frac{3}{4}\cdot\frac{1}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right].$$

Again let t_3 be the time from $x=a$ to $x=a/4$. Then putting $x=a/4$ and $t=t_3$ in (1), we get

$$t_3=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\tfrac{1}{2}+\sqrt{\left(\frac{1}{4}\cdot\frac{3}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right].$$

Therefore if t_4 be the time from $x=3a/4$ to $x=a/4$, we have

$$t_4=t_3-t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{\pi}{6}\sqrt{\left(\frac{a^3}{2K}\right)}$$

$$=\tfrac{1}{3}\left[\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}\right]=\tfrac{1}{3}\,t_1, \text{ from (2).}$$

Hence the time from $x=3a/4$ to $x=a/4$ is one-third of the whole time of descent from $x=a$ to $x=0$.

Note. To find the time from $x=3a/4$ to $x=a/4$, we have first found the times from $x=a$ to $x=3a/4$ and from $x=a$ to $x=a/4$ because in the relation (1) connecting x and t the time t has been measured from the point $x=a$.

Ex. 68. *Show that the time of descent to the centre of force, varying inversely as the square of the distance from the centre, through first half of its initial distance is to that through the last half as* $(\pi+2):(\pi-2)$.

[**Lucknow 1975; Meerut 83P; Rohilkhand 87**]

Sol. Let the particle start from rest from the point A at a distance a from the centre of force O. If x is the distance of the particle from the centre of force at time t, then the equation of motion of the particle at time t is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

Now proceeding as in § 11, page 126, we find that the time t measured from the initial position $x=a$ to any point distant x from the centre O is given by the equation

$$t=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right] \quad ...(1)$$

[Give the complete proof for deducing this equation here].

Now let B be the middle point of OA. Then at B, $x=a/2$.

Let t_1 be the time from A to B *i.e.*, the time to cover the first half of the initial displacement. Then at B, $x=a/2$ and $t=t_1$. So putting $x=a/2$ and $t=t_1$ in (1), we get

$$t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tfrac{1}{2}\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4}+\tfrac{1}{2}\right].$$

Again let t_2 be the time from A to O. Then at O, $x=0$ and $t=t_2$. So putting $x=0$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}0+0\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\frac{\pi}{2}.$$

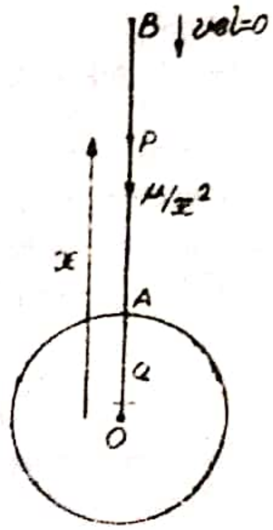
Now if t_3 be the time from B to O (*i.e.*, the time to cover the last half of the initial displacement), then

$$t_3=t_2-t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\left[\frac{\pi}{4}-\tfrac{1}{2}\right].$$

We have $\dfrac{t_1}{t_3}=\dfrac{\frac{1}{4}\pi+\frac{1}{2}}{\frac{1}{4}\pi-\frac{1}{2}}=\dfrac{\pi+2}{\pi-2}$, which proves the required result.

Ex. 69. *If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is* $\sqrt{\left(\dfrac{a+h}{2g}\right)}\left[\sqrt{\left(\dfrac{h}{a}\right)}+\dfrac{a+h}{a}\sin^{-1}\sqrt{\left(\dfrac{h}{a+h}\right)}\right]$, *where a is the radius of the earth.* **[Meerut 1981, 84, 85, 85S, 90; Lucknow 79; Kanpur 74]**

Sol. Let O be the centre of the earth taken as origin. Let OB be the vertical line through O which meets the surface of the earth at A and let $AB=h$; $OA=a$ is the radius of the earth.



A particle falls from rest from B towards the surface of the earth. Let P be the position of the particle at any time t. where $OP=x$. [**Note** that O is the origin and OP is the direction of x increasing]. According to the Newton's law of gravitation the acceleration of the particle at P is μ/x^2 directed towards O *i.e.*, in the direction of x decreasing. Hence the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \qquad \ldots(1)$$

The equation (1) holds good for the motion of the particle from B to A. At A (*i.e.*, on the surface of the earth $x=a$ and $d^2x/dt^2=-g$. Therefore $-g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Integrating, we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C. \quad \text{At } B,\ x=OB=a+h,\ \frac{dx}{dt}=0.$$

$$\therefore \quad 0=\frac{2a^2g}{a+h}+C \quad \text{or} \quad C=-\frac{2a^2g}{a+h}.$$

Thus, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}-\frac{2a^2g}{a+h}=2a^2g\left(\frac{1}{x}-\frac{1}{a+h}\right).$$

For the sake of convenience let us put $a+h=b$. Then

$$\left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right)=\frac{2a^2g}{b}\left(\frac{b-x}{x}\right). \quad \ldots(2)$$

The equation (2) gives velocity at any point from B to A.

From (2) on taking square root, we get

$$\frac{dx}{dt}=-a\sqrt{\left(\frac{2g}{b}\right)}\sqrt{\left(\frac{b-x}{x}\right)},$$

where the negative sign has been taken because the particle is moving in the direction of x decreasing.

$$\therefore \quad dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\sqrt{\left(\frac{x}{b-x}\right)}\,dx. \quad \ldots(3)$$

Let t_1 be the time from B to A. Then integrating (3) from B to A, we get

$$\int_0^{t_1} dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a \sqrt{\left(\frac{x}{b-x}\right)}\,dx.$$

$$\therefore \quad t_1=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a \sqrt{\left(\frac{x}{b-x}\right)}\,dx.$$

Put $x=b\cos^2\theta$; so that $dx=-2b\cos\theta\sin\theta\,d\theta$.

$$\therefore \quad t_1=\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_0^{\cos^{-1}\sqrt{(a/b)}}\frac{\cos\theta}{\sin\theta}\,2b\cos\theta\sin\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}} 2\cos^2\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\ldots}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\sin\theta\cos\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\cos\theta\sqrt{(1-\cos^2\theta)}\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\left[\frac{b}{a}\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{b}{a}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\cos^{-1}\sqrt{\left(\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(1-\frac{a}{a+h}\right)}\right]$$

[replacing b by $a+h$]

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(1-\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(\frac{h}{a+h}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}+\sqrt{\left(\frac{h}{a}\right)}\right].$$

Ex. 70. *A particle falls towards the earth form infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.*

[Kanpur 1975; Agra 87]

Sol. Let a be the radius of the earth and O be the centre of the earth taken as origin. Let the vertical line through O meet the earth's surface at A. [Draw figure as in Ex. 69].

A particle falls from rest from infinity towards the earth. Let P be the position of the particle at any time t, where $OP=x$. [**Note** that O is the origin and OP is the direction of x increasing.] According to Newton's law of gravitation the acceleration of the particle at P is μ/x^2 towards O *i.e.*, in the direction of x decreasing. Hence the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \qquad ...(1)$$

The equation (1) holds good for the motion of the particle upto A. At A (*i.e.*, on the surface of the earth),

$$x=a \text{ and } \frac{d^2x}{dt^2}=-g$$

$\therefore \quad -g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Multiplying both sides by 2 (dx/dt) and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C.$$

But initially when $x=\infty$, the velocity $dx/dt=0$. Therefore $C=0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x}. \qquad \ldots(2)$$

Putting $x=a$ in (2), the velocity V at the earth's surface is given by

$$V^2 = 2a^2 g/a = 2ag \quad \text{or} \quad V = \sqrt{(2ag)}. \qquad \ldots(3)$$

If v_1 is the velocity acquired by the particle in falling a distance equal to the earth's radius a with constant acceleration g, then

$$v_1^2 = 0 + 2ag \quad \text{or} \quad v_1 = \sqrt{(2ag)}. \qquad \ldots(4)$$

From (3) and (4), we have $V = v_1$, which proves the required result.

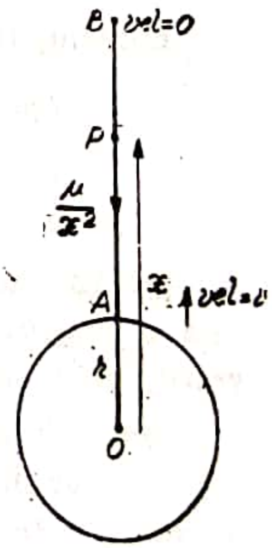
Ex. 71. *If h be the height due to the velocity v at the earth's surface supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that $\frac{1}{h} - \frac{1}{H} = \frac{1}{r}$, where r is the radius of the earth.*

[Kanpur 1978; Meerut 82, 85P; Rohilkhand 85]

Sol. If h is the height of the particle due to the velocity v at the earth's surface, supposing its attraction constant (*i.e.*, taking the acceleration due to gravity as constant and equal to g), then from the formula $v^2 = u^2 + 2fs$, we have

$$0^2 = v^2 - 2gh.$$

$$\therefore \quad v^2 = 2gh. \qquad \ldots(1)$$



When the variation of gravity is taken into account, let P be the position of the particle at any time t measured from the instant the particle is projected vertically upwards from the earth's surface with velocity v, and let $OP = x$.

The acceleration of the particle at P is μ/x^2 directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \ldots(2)$$

[Here the $-$ive sign is taken since the acceleration acts in the direction of x decreasing.]

Buf at A *i.e.*, on the surface of the earth,

$$x = OA = r \text{ and } \frac{d^2x}{dt^2} = -g.$$

$\therefore$ from (2), we have $-g=-\mu/r^2$ or $\mu=gr^2$.

Substituting in (2), we have

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \qquad \ldots(3)$$

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+A$, where A is a constant of integration.

But at the point A, $x=OA=r$ and $dx/dt=v$, which is the velocity of projection at A.

$$\therefore \quad v^2=\frac{2gr^2}{r}+A \quad \text{or} \quad A=v^2-2gr.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+v^2-2gr. \qquad \ldots(4)$$

Suppose the particle in this case rises upto the point B, where $AB=H$. Then at the point B, $x=OB=OA+AB=r+H$ and $dx/dt=0$.

$\therefore$ from (4), we have $0=\dfrac{2gr^2}{r+H}+v^2-2gr$

or
$$v^2=\frac{2gr^2}{r+H}+2gr=\frac{2grH}{r+H} \qquad \ldots(5)$$

Equating the values of v^2 from (1) and (5), we have

$$2gh=\frac{2grH}{r+H} \quad \text{or} \quad \frac{1}{h}=\frac{r+H}{rH}$$

or
$$\frac{1}{h}=\frac{1}{H}+\frac{1}{r} \quad \text{or} \quad \frac{1}{h}-\frac{1}{H}=\frac{1}{r}.$$

Ex. 72. *A particle is shot upwards from the earth's surface with a velocity of one mile per second. Considering variations in gravity, find roughly in miles the greatest height attained.*

Sol. [Refer fig. of Ex. 71].

Let r be the radius of the earth. Suppose the particle is projected vertically upwards from the surface of the earth with velocity u and it rises to a height H above the surface of the earth. Let P be the position of the particle at any time t and x the distance of P from the centre of the earth. Since P is outside the surface of the earth, therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

But on the surface of the earth, $x=r$ and $d^2x/dt^2=-g$. Therefore $-g=-(\mu/r^2)$ or $\mu=gr^2$.

$\therefore$ the equation of motion of P becomes

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \quad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+C$, where C is constant of integration.

When $x=r$, $dx/dt=u$. Therefore $u^2=2gr+C$ or $C=u^2-2gr$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+u^2-2gr. \quad \ldots(2)$$

Since the particle rises to a height H above the surface of the earth, therefore $dx/dt=0$ when $x=r+H$.

Putting these values in (2), we get

$$0=\frac{2gr^2}{r+H}+u^2-2gr$$

or $\quad 0=2gr^2+u^2(r+H)-2gr(r+H)$

or $\quad u^2r+u^2H-2grH=0$

or $\quad H(2gr-u^2)=u^2r.$

$$H=\frac{u^2r}{(2gr-u^2)}.$$

But according to the question, $u=1$ mile/second. Also $r=$ the radius of the earth $=4000$ miles, and

$$g=32 \text{ ft./second}^2=\frac{32}{3\times 1760} \text{ miles/sec}^2.$$

Hence, $H=\dfrac{4000}{\dfrac{2\times 32\times 400}{3\times 1760}-1}$ miles $=\dfrac{1}{\dfrac{2}{165}-\dfrac{1}{4000}}$ miles

$=\dfrac{165}{2}\left[1-\dfrac{165}{8000}\right]^{-1}$ miles $=\dfrac{165}{2}\left[1+\dfrac{165}{8000}\right]$ miles approximately,

[expanding by binomial theorem and neglecting higher powers]

$=\left[\dfrac{165}{2}+\dfrac{(165)^2}{16000}\right]$ miles $=82\cdot5$ miles $+1\cdot5$ miles nearly

$=84$ miles approximately.

Remark. If the particle is projected from the surface of the earth with a velocity 1 kilometre per second, then for the calculation work we shall take $r=6380$ km. and $g=9\cdot8$ metre/sec^2 $=10^{-3}\times 9\cdot8$ km./sec^2. The answer in this case is $51\cdot43$ km approximately.

Ex. 73. *A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is*

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right],$$

where a is the radius of the earth.

[Meerut 1979, 86S, 88P; Kanpur 76, 87; Agra 84, 85, 88; Rohilkhand 88]

Sol. [Refer fig. of Ex. 71]

Let O be the centre of the earth and A the point of projection on the earth's surface.

If P is the position of the particle at any time t, such that $OP=x$, then the acceleration at $P=\mu/x^2$ directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \qquad \ldots(1)$$

But at the point A, on the surface of the earth, $x=a$ and $d^2x/dt^2=-g$.

$$\therefore \quad -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Multiplying by 2 (dx/dt) and intergating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C, \text{ where } C \text{ is a constant.}$$

But when $x\to\infty$, $dx/dt\to0$. $\therefore$ $C=0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x} \quad \text{or} \quad \frac{dx}{dt}=\frac{a\sqrt{(2g)}}{\sqrt{x}} \qquad \ldots(2)$$

[Here +ive sign is taken because the particle is moving in the direction of x increasing.]

Separating the variables, we have

$$dt=\frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx.$$

Integrating between the limits $x=a$ to $x=a+h$, the required time t to reach a height h is given by

$$t=\frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx=\frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$=\frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2}-a^{3/2}\right]=\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

Ex. 74. *Calculate in miles per second the least velocity which will carry the particle from earth's surface to infinity.* [Agra 1977]

Sol. The least velocity of projection from the earth's surface to carry the particle to infinity is that for which the velocity of the particle tends to zero as the distance of the particle from the earth's surface tends to infinity. Now proceed an in Ex. 73.

The velocity at a distance x from the centre of the earth is given by $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}$.

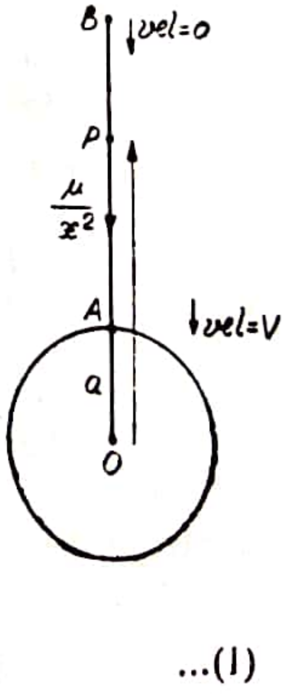
$\therefore$ putting $x=a$, the least velocity V at the earth's surface which will carry the particle to infinity is given by $V=\sqrt{(2ag)}$.

But $a=4000$ miles$=4000\times3\times1760$ ft. and $g=32$ ft/sec^2.

$$\therefore \quad V=\sqrt{[2\times4000\times3\times1760\times32]} \text{ ft./sec.}$$
$$=8\times200\times4\sqrt{(33)} \text{ ft./sec.}$$
$$=\frac{8\times200\times4\times\sqrt{33}}{3\times1760} \text{ miles/sec.}$$
$$=7 \text{ miles/sec. approximately.}$$

Ex. 75. *Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b $(b>a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga\,(3b-2a)/b]}$ and the time to travel from the surface to the centre of the earth is $\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.* **[I.F.S. 1976; Meerut 81S; 83S; Agra 84, 86]**

Sol. Let the particle fall from rest from the point B such that $OB=b$, where O is the centre of the earth. Let P be the position of the particle at any time t measured from the instant it starts falling from B and let $OP=x$.



Acceleration at $P=\mu/x^2$ towards O. The equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2},$$

which holds good for the motion from B to A *i.e.*, outside the surface of the earth.

But at the point A (on one earth's surface) $x=a$ and $d^2x/dt^2=-g$.

$$\therefore \quad -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+A$, where A is a constant.

But at B, $x=OB=b$ and $dx/dt=0$.

$$\therefore \quad 0=\frac{2a^2g}{b}+A \text{ or } A=-\frac{2a^2g}{b}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right). \qquad \ldots(2)$$

If V is the velocity of the particle at the point A, then at A, $x=OA=a$ and $(dx/dt)^2=V^2$.

$$\therefore \quad V^2=2a^2g\left(\frac{1}{a}-\frac{1}{b}\right). \qquad \ldots(3)$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x<a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is a constant.

The equation of motion (inside the earth) is $\dfrac{d^2x}{dt^2}=-\lambda x$, which holds good for the motion from A to O.

At A, $x=a$ and $d^2x/dt^2=-g$. $\therefore$ $\lambda=g/a$.

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{a}x.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+B, \text{ where } B \text{ is a constant.} \qquad \ldots(4)$$

But at A, $x=OA=a$ and $\left(\dfrac{dx}{dt}\right)^2=V^2=2a^2g\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$, from (3).

$$\therefore \quad 2a^2g\left(\frac{1}{a}-\frac{1}{b}\right)=-\frac{g}{a}a^2+B \quad \text{or} \quad B=ag\left(\frac{3b-2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2=ag\left(\frac{3b-2a}{b}\right)-\frac{g}{a}x^2. \qquad \ldots(5)$$

Putting $x=0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{[ga(3b-2a)/b]}$.

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2=\frac{g}{a}\left[a^2\frac{(3b-2a)}{b}-x^2\right]$$

$$=\frac{g}{a}(c^2-x^2), \text{ where } c^2=\frac{a^2}{b}(3b-2a).$$

$$\therefore \quad \frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\cdot\sqrt{(c^2-x^2)},$$ the $-$ive sign being taken because the particle is moving in the direction of x decreasing

or $$dt=-\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{dx}{\sqrt{(c^2-x^2)}},$$ separating the variables.

Integrating from A to O, the required time t is given by

$$t=-\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2-x^2)}}$$

$$=\sqrt{\left(\frac{a}{g}\right)}\int_{0}^{a}\frac{dx}{\sqrt{(c^2-x^2)}}=\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\frac{x}{c}\right]_0^a$$

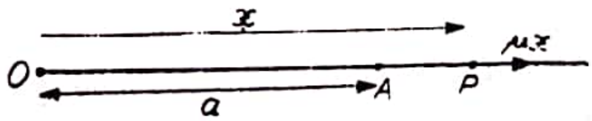
$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left(\frac{a}{c}\right)=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{a\sqrt{\{(3b-2a)/b\}}}\right],$$

substituting for c

$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left(\frac{b}{3b-2a}\right)}$$

§ 13. *A particle moves under an acceleration varying as the distance and directed away from a fixed point, to investigate the motion.*

Sol. Let O be the fixed point and x the distance of the particle from O, at any time t. Then the acceleration of the particle at this point is μx in the direction of x increasing.



$\therefore$ the equation of motion of the particle is $\dfrac{d^2x}{dt^2}=\mu x$, ...(1)

where the $+$ive sign has been taken since the acceleration acts in the direction of x increasing.

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$(dx/dt)^2=\mu x^2+A$, where A is a constant.

Suppose the particle starts from rest at a distance a from O, *i.e.*, $dx/dt=0$ at $x=a$. Then $0=\mu a^2+A$, or $A=-\mu a^2$.

$$\therefore \quad (dx/dt)^2=\mu\,(x^2-a^2), \qquad ...(2)$$

which gives the velocity at any distance x from O.

From (2), on extracting square root, we have

$$dx/dt=\sqrt{\mu}\sqrt{(x^2-a^2)}$$

[$+$ive sign being taken because the particle moves in the direction of x increasing].

or $$dt = \frac{1}{\sqrt{\mu}} \cdot \frac{dx}{\sqrt{(x^2 - a^2)}}.$$

Integrating, $t = \dfrac{1}{\sqrt{\mu}} \cosh^{-1} \dfrac{x}{a} + B.$

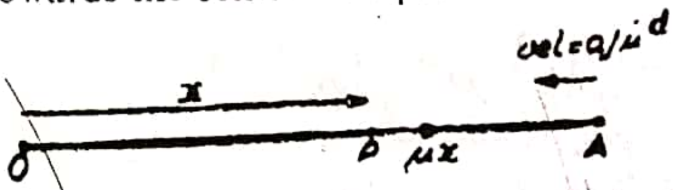
But when $t=0$, $x=a$. $\therefore$ $B=0$.

$$\therefore \quad t = \frac{1}{\sqrt{\mu}} \cosh^{-1} \frac{x}{a} \quad \text{or} \quad x = a \cosh(\sqrt{\mu}\, t), \qquad \ldots(3)$$

which gives the position of the particle at time t.

Ex. 76. *If a particle is projected towards the centre of repulsion, varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$; prove that the particle will approach the centre but will never reach it.* **[Lucknow 1978; Alld. 80; Agra 84]**

Sol. Let the particle be projected from the point A with velocity $a\sqrt{\mu}$ towards the centre of repulsion O and let $OA = a$.



If P is the position of the particle at time t such that $OP = x$, then at P, the acceleration on the particle is μx in the direction PA.

$\therefore$ the equation of motion of the particle is

$\dfrac{d^2x}{dt^2} = \mu x.$ [+ive sign is taken because the acceleration is in the direction of x increasing].

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we have

$(dx/dt)^2 = \mu x^2 + C$, where C is a constant.

But at A, $x = a$ and $(dx/dt)^2 = a^2\mu$. $\therefore$ $C = 0$.

$\therefore$ $(dx/dt)^2 = \mu x^2$ or $dx/dt = -\sqrt{\mu}\, x$. ...(1)

[$-$ive sign is taken because the particle is moving in the direction of x decreasing].

The equation (1) shows that the velocity of the particle will be zero when $x=0$ and not before it and so the particle will approach the centre O.

From (1), we have $dt = -\dfrac{1}{\sqrt{\mu}} \dfrac{dx}{x}.$

Integrating between the limits $x=a$ to $x=0$, the time t_1 from A to O is given by

$$t_1 = -\frac{1}{\sqrt{\mu}} \int_a^0 \frac{dx}{x} = \frac{1}{\sqrt{\mu}} \Big[\log x\Big]_0^a = \frac{1}{\sqrt{\mu}} (\log a - \log 0)$$
$$= \infty. \qquad [\because \log 0 = -\infty]$$

Hence the particle will take an infinite time to reach the centre O or in other words it will never reach the centre O.

§ **14.** *A particle moves in such a way that its acceleration varies inversely as the cube of the distance from a fixed point and is directed towards the fixed point; discuss the motion.*

[Lucknow 1976; Agra 79]

Let O be the fixed point and x the distance of the particle from O, at any time t. Then the equation of motion of the particle is
$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^3}.$$

[The $-$ive sign has been taken because the force is given to be attractive.]

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have
$$\left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}+A.$$

Suppose the particle starts from rest at a distance a from O, i.e., $dx/dt=0$ at $x=a$.

Then $0=\frac{\mu}{a^2}+A$ or $A=-\frac{\mu}{a^2}$.
$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right), \qquad \text{...(2)}$$
which gives the velocity at any distance x from the centre of force O.

From (2), we have $\frac{dx}{dt}=-\frac{\sqrt{\mu}}{a}\frac{\sqrt{(a^2-x^2)}}{x}$

[the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.]

or $$dt=-\frac{a}{\sqrt{\mu}}\cdot\frac{x\,dx}{\sqrt{(a^2-x^2)}}, \text{ separating the variables}$$
$$=\frac{a}{2\sqrt{\mu}}\cdot(a^2-x^2)^{-1/2}\,(-2x)\,dx.$$

Integrating, $t=\frac{a}{\sqrt{\mu}}\cdot\sqrt{(a^2-x^2)}+B.$

But initially when $t=0$, $x=a$. $\therefore$ $B=0$.
$$\therefore \quad t=\frac{a}{\sqrt{\mu}}\sqrt{(a^2-x^2)}, \qquad \text{...(3)}$$
which gives the position of the particle at any time t.

Ex. 77. *A particle moves in a straight line towards a centre of force $\mu/(\text{distance})^3$ starting from rest at a distance a from the* *centre of force; show that the time of reaching a point distant b from the centre of force is a* $\sqrt{\left(\frac{a^2-b^2}{\mu}\right)}$, *and that its velocity then is* $\sqrt{[\mu(a^2-b^2)]}/ab$. *Also show that the time to reach the centre is* $a^2/\sqrt{\mu}$. **(Kanpur 1987)**

Sol. Let the particle start at rest from A and at time t let it be at P, where $OP=x$; O being the centre of force.

Given that the acceleration at P is μ/x^3 towards O, we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^3}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}+C.$

When $x=a$, $dx/dt=0$, so that $C=-\mu/a^2$.

Hence $\left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right)=\mu\left(\frac{a^2-x^2}{a^2x^2}\right)$.

$$\therefore \quad \frac{dx}{dt}=-\frac{\sqrt{[\mu(a^2-x^2)]}}{ax}, \qquad \ldots(2)$$

the negative sign being taken b cause the particle is moving towards x decreasing.

Putting $x=b$ in (2), the velocity at $x=b$ is $\sqrt{[\mu(a^2-b^2)]}/ab$. in magnitude. This proves the second result.

If t_1 is the time from $x=a$ to $x=b$, then integrating (2) after seperating the variables, we get

$$t_1=-\frac{a}{\sqrt{\mu}}\int_a^b \frac{x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\int_a^b \frac{-2x}{\sqrt{(a^2-x^2)}}\,dx$$

$$=\frac{a}{2\sqrt{\mu}}\Big[2\sqrt{(a^2-x^2)}\Big]_a^b=\frac{a\sqrt{(a^2-b^2)}}{\sqrt{\mu}}.$$

This proves the first result.

And if T be the time to reach the centre O, where $x=0$, then

$$T=\frac{a}{2\sqrt{\mu}}\int_a^0 \frac{-2x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\Big[2\sqrt{(a^2-x^2)}\Big]_a^0=\frac{a^2}{\sqrt{\mu}}$$

§ 15. Motion under miscellaneous laws of forces.

Now we shall give a few examples in which the particle moves under different laws of acceleration.

Ex. 78. *A particle whose mass is m is acted upon by a force* $m\mu\left[x+\frac{a^4}{x^3}\right]$ *towards origin; if it starts from rest at a distance a, show that it will arrive at origin in time* $\pi/(4\sqrt{\mu})$.

[Lucknow 1981; Meerut 87; Kanpur 84; Agra 77, 85]

Sol. Given $\dfrac{d^2x}{dt^2}=-\mu\left[x+\dfrac{a^4}{x^3}\right]$, ...(1)

the $-$ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2=\mu\left[-x^2+\frac{a^4}{x^2}\right]+C$$

When $x=a$, $dx/dt=0$, so that $C=0$.

$$\therefore\quad \left(\frac{dx}{dt}\right)^2=\mu\left[\frac{a^4-x^4}{x^2}\right]$$

or
$$\frac{dx}{dt}=-\frac{\sqrt{\mu}\sqrt{(a^4-x^4)}}{x},$$

the $-$ive sign is taken because the particle is moving in the direction of x decreasing. ...(2)

If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1=-\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4-x^4)}}\,dx=\frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4-x^4)}}$$

Put $x^2=a^2\sin\theta$ so that $2x\,dx=a^2\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore\quad t_1=\frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}=\frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta=\frac{1}{2\sqrt{\mu}}\Big[\,\theta\,\Big]_0^{\pi/2}$$

$$=\frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2}=\frac{\pi}{4\sqrt{\mu}}$$

Ex. 79. *A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to* $\mu/x^2-\lambda/x^3$ *at a distance* x *from the given point; the particle starts from rest at a distance* a*: show that it oscillates between this distance and the distance* $\dfrac{\lambda c}{(2\mu a-\lambda)}$ *and the periodic time is* $\dfrac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$.

Sol. Let O be the fixed point taken as origin and A the starting point such that $OA=a$. At any time t let P be the position of the particle, where $OP=x$. Equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right).\ \text{[given]} \qquad ...(1)$$

Integrating, we get $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2\mu}{x}-\dfrac{\lambda}{x^2}+C.$

When $x=a$, $\dfrac{dx}{dt}=0$, so that $C=-\dfrac{2\mu}{a}+\dfrac{\lambda}{a^2}$.

$$\therefore\quad \left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)-\lambda\left(\frac{1}{x^2}-\frac{1}{a^2}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(2\mu-\frac{\lambda}{x}-\frac{\lambda}{a}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{a}-\frac{\lambda}{x}\right)$$

$$=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\lambda a}-\frac{1}{x}\right). \quad \ldots(2)$$

The particle comes to rest where $dx/dt=0$, *i.e.*, where

$$\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\mu a}-\frac{1}{x}\right)=0.$$

One solution of this equation is $\frac{1}{x}-\frac{1}{a}=0$ *i.e.*, $x=a$, which gives the initial position. Another solution is $\frac{2a\mu-\lambda}{\lambda a}-\frac{1}{x}=0$ *i.e.*, $x=\frac{\lambda a}{(2a\mu-\lambda)}$ which gives the other position of instantaneous rest.

Hence the particle oscillates between $x=a$ and $x=\frac{\lambda a}{(2a\mu-\lambda)}$.

This proves one result. To prove the other result, put $\frac{\lambda a}{2a\mu-\lambda}=b$, so that the equation (2) becomes

$$\left(\frac{dx}{dt}\right)^2=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{x}\right)=\frac{\lambda}{ab}\frac{(a-x)(x-b)}{x^2}$$

or
$$\frac{dx}{dt}=-\sqrt{\left(\frac{\lambda}{ab}\right)}\cdot\frac{\sqrt{\{(a-x)(x-b)\}}}{x}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or
$$dt=-\sqrt{\left(\frac{ab}{\lambda}\right)}\cdot\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}.$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from one position of rest to the other position of rest is given by

$$t_1=-\sqrt{\left(\frac{ab}{\lambda}\right)}\int_a^b\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{[-ab-\{x^2-(a+b)x\}]}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{[\frac{1}{4}(a-b)^2-\{x-\frac{1}{2}(a+b)\}^2]}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{\{\frac{1}{2}(a+b)+y\}\,dy}{\sqrt{[\frac{1}{4}(a-b)^2-y^2]}},$$

putting $x-\frac{1}{2}(a+b)=y$ so that $dx=dy$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{\frac{1}{2}(a+b)}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}dy$$

$$+\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{y}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}dy$$

$$=2\sqrt{\left(\frac{ab}{\lambda}\right)}\cdot\frac{1}{2}(a+b)\int_0^{(a-b)/2}\frac{y}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}},$$

the second integral vanishes because the integrand is an odd function of y

$$=(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1}\left\{\frac{y}{\frac{1}{2}(a-b)}\right\}\right]_0^{(a-b)/2}$$

$$=(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1}1-\sin^{-1}0\right]=\frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}.$$

$\therefore$ the periodic time of one complete oscillation

$$=2t_1=2.\frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}$$

$$=\pi\left(a+\frac{\lambda a}{2a\mu-\lambda}\right)\sqrt{\left\{\frac{a}{\lambda}\cdot\frac{\lambda a}{2a\mu-\lambda}\right\}}$$

$$=\pi\frac{2a^2\mu}{(2a\mu-\lambda)}\cdot\frac{a}{\sqrt{(2a\mu-\lambda)}}=\frac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}.$$

Remark. To evaluate the integral giving the time t_1, we can also make the substitution $x=a\cos^2\theta+b\sin^2\theta$, so that $dx=-2(a-b)\sin\theta\cos\theta\,d\theta$. Also $\theta=0$ when $x=a$ and $\theta=\pi/2$ when $x=b$.

Ex. 80. *A particle moves in a straight line under a force to a point in it, varying as (distance)$^{-4/3}$. Show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest at a distance a to a distance a/8.* **[Kanpur 1977]**

Sol. If x is the distance of the particle from the fixed point at time t, then the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\mu x^{-4/3}. \qquad ...(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{6\mu}{x^{1/3}}+A. \qquad ...(2)$$

If the particle falls from rest at infinity, *i.e.*, $dx/dt=0$ when $x=\infty$, we have from (2), $A=0$.

$$\therefore \quad (dx/dt)^2=6\mu/x^{1/3}. \qquad ...(3)$$

If v_1 is the velocity of the particle at $x=a$, then

$$v_1^2=6\mu/a^{1/3}.$$

Again if the particle falls from rest at a distance a, *i.e.*, if $dx/dt=0$ when $x=a$, we have, from (2)

$$0=\frac{6\mu}{a^{1/3}}+A \quad \text{or} \quad A=-\frac{6\mu}{a^{1/3}}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=6\mu\left(\frac{1}{x^{1/3}}-\frac{1}{a^{1/3}}\right).$$

If in this case v_2 is the velocity of the particle at $x=a/8$, then

$$v_2^2=6\mu\left[\left(\frac{8}{a}\right)^{1/3}-\frac{1}{a^{1/3}}\right]=6\mu\left(\frac{2}{a^{1/3}}-\frac{1}{a^{1/3}}\right)=\frac{6\mu}{a^{1/3}}. \quad \text{...(4)}$$

From (3) and (4), we observe that $v_1=v_2$, which proves the required result.

Ex. 81. *Find the time of descent to the centre of force, when the force varies as (distance)$^{-5/3}$, and show that the velocity at the centre is infinite.*

Sol. Let O be the centre of force taken as the origin. Suppose a particle starts at rest from A, where $OA=a$. The particle moves towards O on account of a centre of attraction at O. Let P be the position of the particle at any time t, where $OP=x$. The acceleration of the particle at P is $\mu x^{-5/3}$ directed towards O. Therefore the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\mu x^{-5/3}. \quad \text{...(1)}$$

Multiplying both sides of (1) by 2 (dx/dt) and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{2\mu x^{-2/3}}{-2/3}+k=\frac{3\mu}{x^{2/3}}+k, \text{ where } k \text{ is a constant.}$$

At A, $x=a$ and $dx/dt=0$, so that $(3\mu/a^{2/3})+k=0$

or $\qquad k=-3\mu/a^{2/3}$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{3\mu}{x^{2/3}}-\frac{3\mu}{a^{2/3}}=\frac{3\mu\,(a^{2/3}-x^{2/3})}{a^{2/3}\,x^{2/3}}, \quad \text{...(2)}$$

which gives the velocity of the particle at any distance x from the centre of force O. Putting $x=0$ in (2), we see that at O, $(dx/dt)^2=\infty$. Therefore the velocity of the particle at the centre is infinite.

Taking square root of (2), we get

$$\frac{dx}{dt}=-\sqrt{(3\mu)}\sqrt{\left(\frac{a^{2/3}-x^{2/3}}{a^{2/3}\,x^{2/3}}\right)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt=-\frac{a^{1/3}}{\sqrt{(3\mu)}}\,\frac{x^{1/3}}{\sqrt{(a^{2/3}-x^{2/3})}}\,dx. \quad \text{...(3)}$$

Let t_1 be the time from A to O. Then at A, $t=0$ and $x=a$ while at O, $t=t_1$ and $x=0$. So integrating (3) from A to O, we have
$$\int_0^{t_1} dt=-\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_a^0 \frac{x^{1/3}}{\sqrt{(a^{2/3}-x^{2/3})}}\,dx$$
$$=\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^a \frac{x^{1/3}\,dx}{\sqrt{(a^{2/3}-x^{2/3})}}.$$

Putting $x=a\sin^3\theta$, so that $dx=3a\sin^2\theta\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.
$$\therefore\quad t_1=\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\frac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}\,3a\sin^2\theta\cos\theta\,d\theta$$
$$=\frac{3a^{4/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\sin^3\theta\,d\theta=\frac{3a^{4/3}}{\sqrt{(3\mu)}}\cdot\frac{2}{3.1}=\frac{2a^{4/3}}{\sqrt{(3\mu)}}.$$

Hence the time of descent to the centre of force is $2a^{4/3}/\sqrt{(3\mu)}$.

Ex. 82. *A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\sqrt{(\pi/2\mu)}$.*

[Meerut 1980, 84, 85, 88P]

Sol. If x is the distance of the particle from the centre of force at time t, then the equation of motion is
$$\frac{d^2x}{dt^2}=-\frac{\mu}{x}.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have $(dx/dt)^2=-2\mu\log x+A$, where A is a constant.

But initially at $x=a$, $dx/dt=0$.

$\therefore\quad 0=-2\mu\log a+A$ or $A=2\mu\log a$.

$\therefore\quad (dx/dt)^2=2\mu(\log a-\log x)=2\mu\log(a/x)$

or
$$dx/dt=-\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}},$$
where the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

Separating the variables, we have
$$dt=-\frac{1}{\sqrt{(2\mu)}}\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Integrating from $x=a$ to $x=0$, the required time t_1 to reach the centre is given by
$$t_1=-\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Put $\log\left(\frac{a}{x}\right)=u^2$ i.e., $x=ae^{-u^2}$, so that $dx=-2ae^{-u^2}u\,du$.

When $x=a$, $u=0$ and when $x\to0$, $u\to\infty$.

$\therefore\ t_1=\dfrac{2}{\sqrt{(2\mu)}}\displaystyle\int_0^\infty e^{-u^2}du$. But $\displaystyle\int_0^\infty e^{-u^2}du=\frac{\sqrt{\pi}}{2}$ **(Remember)**

$$\therefore \quad t_1 = \frac{2a}{\sqrt{(2\mu)}} \cdot \frac{\sqrt{\pi}}{2} = a\sqrt{\left(\frac{\pi}{2\mu}\right)}.$$

Ex. 83. *A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to* $\mu\,(a^5/x^2)^{1/3}$ *when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity* $a\sqrt{(6\mu)}$ *after time* $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.

[Agra 1980, 84; Meerut 86, 87P, 90S]

Sol. Take the centre of force O as origin. Suppose a particle starts from rest at A, where $OA=a$. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where $OP=x$. The acceleration of the particle at P is $\mu a^{5/3}\,x^{-2/3}$ directed towards O. Therefore the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu a^{5/3}\,x^{-2/3}. \quad \text{...(1)}$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3}\,x^{1/3}}{1/3} + k = -6\mu a^{5/3}\,x^{1/3} + k,$$

where k is a constant.

At A, $x=a$ and $dx/dt=0$, so that

$$-6\mu a^{5/3}\,a^{1/3} + k = 0 \quad \text{or} \quad k = 6\mu a^2.$$

$$\therefore \quad (dx/dt)^2 = -6\mu a^{5/3}\,x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}\,(a^{1/3} - x^{1/3}), \quad \text{...(2)}$$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x=0$ and $(dx/dt)^2 = v_1^2$. So from (2), we have

$$v_1^2 = 6\mu a^{5/3}\,(a^{1/3} - 0) = 6\mu a^2 \quad \text{or} \quad v_1 = a\sqrt{(6\mu)}.$$

Now taking square root of (2), we get

$$dx/dt = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3} - x^{1/3})},$$

where the $-$ive sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \cdot \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}. \quad \text{...(3)}$$

Let t_1 be the time from A to O. Then integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \int_a^0 \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

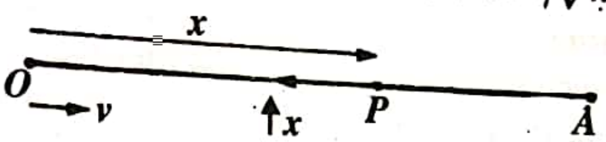
$$= \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^a \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

Put $x=a \sin^6 \theta$, so that $dx=6a \sin^5 \theta \cos \theta \, d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore \quad t_1=\frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^{\pi/2} \frac{6a \sin^5 \theta \cos \theta \, d\theta}{a^{1/6} \cos \theta}$$

$$=\sqrt{\left(\frac{6}{\mu}\right)} \int_0^{\pi/2} \sin^5 \theta \, d\theta=\sqrt{\left(\frac{6}{\mu}\right)} \cdot \frac{4.2}{5.3.1}=\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}.$$

Ex. 84. *A particle starts with a given velocity v and moves under a retardation equal to k times the space described. Show that the distance traversed before it comes to rest is* $v/\sqrt{k}$.



Sol. Suppose the particle starts from O with velocity v and moves in the straight line OA. Let P be the position of the particle after any time t, where $OP=x$. Then the retardation of the particle at P is kx *i.e.*, the acceleration of the particle at P is kx and is directed towards O *i.e.*, in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$d^2x/dt^2=-kx. \qquad \text{...(1)}$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have $(dx/dt)^2=-kx^2+C$, where C is a constant.

At O, $x=0$ and $dx/dt=v$, so that $v^2=C$.

$$\therefore \quad (dx/dt)^2=v^2-kx^2, \qquad \text{...(2)}$$

which gives the velocity of the particle at a distance x from O

From (2), $dx/dt=0$ when $v^2-kx^2=0$ *i.e.*, when $x=v/\sqrt{k}$.

Hence the distance traversed before the particle comes to rest is $v/\sqrt{k}$.

Ex. 85. *Assuming that at a distance x from a centre of force, the speed v of a particle, moving in a straight line is given by the equation* $x=ae^{bv^2}$, *where a and b are constants. Find the law and the nature of the force.*

Sol. Given, $x=ae^{bv^2}$. Therefore $e^{bv^2}=x/a$

or $$bv^2=\log (x/a)=\log x-\log a. \qquad \text{...(1)}$$

Differentiating both sides of (1) w.r.t. x, we get

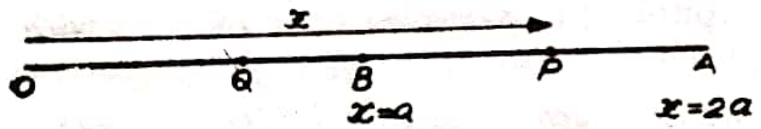
$$2bv \frac{dv}{dx}=\frac{1}{x} \quad \text{or} \quad v\frac{dv}{dx}=\frac{1}{2b}\cdot\frac{1}{x}.$$

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=\frac{1}{2b}\cdot\frac{1}{x}. \qquad \left[\text{Note that } v\frac{dv}{dx}=\frac{d^2x}{dt^2}\right]$$

Hence the acceleration varies inversely as the distance of the particle from the centre of force. Also the force is repulsive or attractive according as b is positive or negative.

Ex. 86. *A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by the formula $m\mu a^2/x^2$ for values of $x \geqslant a$, and by the formula $m\mu x/a$ for $x \leqslant a$, where x is the distance from a fixed origin in the line. If the particle starts at a distance 2a from the origin, prove that it will reach the origin with velocity $(2\mu a)^{1/2}$. Prove further that the time taken to reach the origin is $(1+\frac{3}{4}\pi)\sqrt{(a/\mu)}$.* **[Lucknow 1981]**



Sol. Let O be the origin and A the point from which the particle starts. We have $OA=2a$ and let $OB=a$, so that B is the middle point of OA.

Motion from A to B. The particle starts from rest at A and it moves towards B. Let P be its position at any time t, where $OP=x$. According to the question the acceleration of P is $\mu a^2/x^2$ and is directed towards O *i.e.* in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu a^2}{x^2}. \qquad ...(1)$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2\mu a^2}{x}+C.$$

When $x=2a$, $dx/dt=0$, so that $C=-2\mu a^2/2a$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{2\mu a^2}{x}-\frac{2\mu a^2}{2a}=2a^2\mu\left[\frac{1}{x}-\frac{1}{2a}\right]=a\mu\,\frac{2a-x}{x}, \qquad ...(2)$$

which gives the velocity of the particle at any position between A and B. Suppose the particle reaches B with the velocity v_1. Then at B, $x=a$ and $(dx/dt)^2=v_1^2$. So from (2), we get

$$v_1^2=a\mu\,\frac{2a-a}{a}=a\mu \text{ or } v_1=\sqrt{(a\mu)}, \text{ its direction being towards}$$

the origin O.

Now taking square root of (2), we get

$$\frac{dx}{dt}=-\sqrt{(a\mu)}\sqrt{\left(\frac{2a-x}{x}\right)}, \text{ where the } -\text{ive sign has been taken}$$

Separating the variables, we get

$$dt=-\frac{1}{\sqrt{(a\mu)}}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx. \qquad \ldots(3)$$

Let t_1 be the time from A to B. Then at A, $x=2a$ and $t=0$, while at B, $x=a$ and $t=t_1$. So integrating (3) from A to B, we get

$$\int_0^{t_1} dt=-\frac{1}{\sqrt{(a\mu)}}\int_{2a}^{a}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx.$$

Put $x=2a\cos^2\theta$, so that $dx=-4a\cos\theta\sin\theta\,d\theta$. When $x=2a$, $\theta=0$ and when $x=a$, $\theta=\pi/4$.

$$\therefore \quad t_1=-\frac{1}{\sqrt{(a\mu)}}\int_0^{\pi/4}\frac{\cos\theta}{\sin\theta}\cdot(-4a\cos\theta\sin\theta)\,d\theta$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}2\cos^2\theta\,d\theta=2\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta$$

$$=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right].$$

Motion from B to O. Now the particle starts from B towards O with velocity $\sqrt{(a\mu)}$ gained b it during its motion from A to B. Let Q be its position after time t since it starts from B and let $OQ=x$. Now according to the question the acceleration of Q is $\mu x/a$ directed towards O. Therefore the equation of motion of Q is

$$\frac{d^2x}{dt^2}=-\frac{\mu x}{a}. \qquad \ldots(4)$$

Multiplying both sides of (4) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+D.$$

At B, $x=a$ and $(dx/dt)^2=v_1^2=a\mu$, so that $a\mu=-a\mu+D$

or $\quad D=2a\mu$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+2a\mu=\frac{\mu}{a}(2a^2-x^2), \qquad \ldots(5)$$

which gives the velocity of the particle at any position between B and O. Let v_2 be the velocity of the particle at O. Then putting $x=0$ and $(dx/dt)^2=v_2^2$ in (5), we get

$$v_2^2=\frac{\mu}{a}(2a^2-0)=2a\mu \quad \text{or} \quad v_2=\sqrt{(2a\mu)}.$$

Hence the particle reaches the origin with the velocity $\sqrt{(2a\mu)}$.

Now taking square root of (5), we get

$\dfrac{dx}{dt}=-\sqrt{\left(\dfrac{\mu}{a}\right)}\sqrt{(2a^2-x^2)}$, where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt=-\sqrt{\left(\frac{a}{\mu}\right)}\frac{dx}{\sqrt{(2a^2-x^2)}}. \qquad ...(6)$$

Let t_2 be the time from B to O. Then at B, $t=0$ and $x=a$ while at O, $x=0$ and $t=t_2$. So integrating (6) from B to O, we get

$$\int_0^{t_2} dt=-\sqrt{\left(\frac{a}{\mu}\right)}\int_a^0 \frac{dx}{\sqrt{(2a^2-x^2)}}$$

i.e.,

$$t_2=\sqrt{\left(\frac{a}{\mu}\right)}\left[\cos^{-1}\frac{x}{a\sqrt{2}}\right]_a^0$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}.$$

Hence the whole time taken to reach the origin $O=t_1+t_2$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right]+\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{3\pi}{4}+1\right].$$

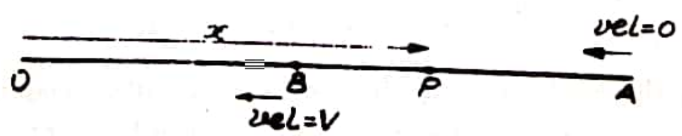
Ex. 87. *A particle moves along the axis of x starting from rest at $x=a$. For an interval t_1 from the beginning of the motion the acceleration is $-\mu x$, for a subsequent time t_2 the acceleration is μx, and at the end of this interval the particle is at the origin; prove that*

$$\tan(\sqrt{\mu}t_1).\tanh(\sqrt{\mu}t_2)=1.$$

[I.F.S. 1976; Meerut 82S, 90P]

Sol. Let the particle moving along the axis of x start from rest at A such that $OA=a$.

Let $-\mu x$ be the acceleration for an interval t_1 from A to B and μx that for an interval t_2 from B to O, where $OB=b$.



For motion from A to B, the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x. \qquad ...(1)$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$(dx/dt)^2=-\mu x^2+A$, where A is a constant.

But at $x=a$, $dx/dt=0$. $\therefore$ $0=-\mu a^2+A$ or $A=\mu a^2$.

$$\therefore \ (dx/dt)^2=\mu(a^2-x^2) \qquad ...(2)$$

or $$dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or $$dt=\frac{1}{\sqrt{\mu}}\cdot\frac{dx}{\sqrt{(a^2-x^2)}},\quad \text{[separating the variables].}$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from A to B is given by

$$t_1=-\frac{1}{\sqrt{\mu}}\int_{x=a}^{b}\frac{dx}{\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_{x=a}^{b}=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}.$$

$\therefore$ $\cos(\sqrt{\mu}t_1)=b/a$ and $\sin(\sqrt{\mu}t_1)=\sqrt{[1-\cos^2(\sqrt{\mu}t_1)]}$

$$=\sqrt{\left(1-\frac{b^2}{a^2}\right)}=\frac{\sqrt{(a^2-b^2)}}{a}.$$

Dividing, $$\tan(\sqrt{\mu}t_1)=\frac{\sqrt{(a^2-b^2)}}{b}. \quad \ldots(3)$$

If V is the velocity at B where $x=b$, then from (2),

$$V^2=\mu\,(a^2-b^2). \quad \ldots(4)$$

For motion from B to O, the velocity at B is V and the particle moves towards O under the acceleration μx.

$\therefore$ the equation of motion is $$\frac{d^2x}{dt^2}=\mu x. \quad \ldots(5)$$

Integrating, $(dx/dt)^2=\mu x^2+B$, where B is a constant.

But at the point B, $x=b$ and $(dx/dt)^2=V^2=\mu\,(a^2-b^2)$.

$\therefore$ $\mu\,(a^2-b^2)=\mu b^2+B$ or $B=\mu\,(a^2-2b^2)$.

$\therefore$ $$\left(\frac{dx}{dt}\right)^2=\mu\,[x^2+(a^2-2b^2)] \text{ or } \frac{dx}{dt}=-\sqrt{\mu}\sqrt{[x^2+(a^2-2b^2)]}$$

or $$dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$

Integrating between the limits $x=b$ to $x=0$, the time t_2 from B to O is given by

$$t_2=-\frac{1}{\sqrt{\mu}}\int_{x=b}^{0}\frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$
$$=-\frac{1}{\sqrt{\mu}}\left[\sinh^{-1}\frac{x}{\sqrt{(a^2-2b^2)}}\right]_b^0=\frac{1}{\sqrt{\mu}}\sinh^{-1}\frac{b}{\sqrt{(a^2-2b^2)}}$$

$\therefore$ $$\sinh(\sqrt{\mu}t_2)=\frac{b}{\sqrt{(a^2-2b^2)}} \text{ so that}$$

$$\cosh(\sqrt{\mu}t_2)=\sqrt{\{1+\sinh^2(\sqrt{\mu}t_2)\}}$$
$$=\sqrt{\left(1+\frac{b^2}{a^2-2b^2}\right)}=\sqrt{\left(\frac{a^2-b^2}{a^2-2b^2}\right)}.$$

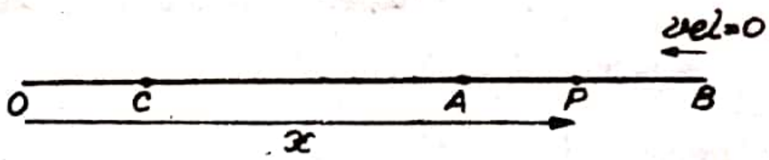
Dividing, $$\tanh(\sqrt{\mu}t_2)=\frac{b}{\sqrt{(a^2-b^2)}}. \quad \ldots(6)$$

Multiplying (3) and (6), we have

$$\tan(\sqrt{\mu}t_1).\tanh(\sqrt{\mu}t_2)=1.$$

Ex. 88. *A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left[1-\frac{a}{x}\right]$ towards the point when $x>a$ but $\mu\left[\frac{a^2}{x^2}-\frac{a}{x}\right]$ from the same point when $x<a$; prove that particle will oscillate through a space $\left[\frac{b^2-a^2}{b}\right]$.*

Sol. Let the particle start from rest at B, where $OB=b$, and move towards the centre of force. Let $OA=a$.



Motion from B to A *i.e.*, when $x>a$.

Since the law of force, when $x>a$, is $\mu(1-a/x)$ towards O, therefore the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu\left(1-\frac{a}{x}\right).$$

Multiplying both sides by $2(dx/dt)$ and integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=-2\mu(x-a\log x)+C$, where C is a constant.

But at B, $x=OB=b$ and $dx/dt=0$. $\therefore$ $C=2\mu(b-a\log b)$.

$$\therefore\quad \left(\frac{dx}{dt}\right)^2=2\mu(b-a\log b-x+a\log x). \quad ...(1)$$

If V is the velocity at the point A where $x=OA=a$, then from (1), we have $\quad V^2=2\mu(b-a-a\log b+a\log a). \quad ...(2)$

Motion from A towards O *i.e.*, when $x<a$.

The velocity of the particle at A is V and it moves towards O under the law of force $\mu\left(\frac{a^2}{x^2}-\frac{a}{x}\right)$ at the distance x from the fixed point O.

$\therefore$ the equation of motion is $\frac{d^2x}{dt^2}=\mu\left[\frac{a^2}{x^2}-\frac{a}{x}\right]$.

Multiplying both sides by $2(dx/dt)$ and integrating, we have

$$\left(\frac{dx}{dt}\right)^2=2\mu\left(-\frac{a^2}{x}-a\log x\right)+D,$$ where D is a constant.

But at the point A,

$x=a$ and $(dx/dt)^2=V^2=2\mu(b-a-a\log b+a\log a)$.

$\therefore\quad D=2\mu\,(b-a-a\log b+a\log a)+2\mu\,(a+a\log a)$

$\quad=2\mu\,(b-a\log b+2a\log a)=2\mu\,\{b+a\log(a^2/b)\}.$

$$\therefore\quad \left(\frac{dx}{dt}\right)^2=-2\mu\left(\frac{a^2}{x}+a\log x\right)+2\mu\left\{b+a\log\left(\frac{a^2}{b}\right)\right\}. \qquad \ldots(3)$$

If the particle comes to rest at the point C, where $x=c$, then putting $x=c$ and $dx/dt=0$ in (3), we get

$$2\mu\left(\frac{a^2}{c}+a\log c\right)=2\mu\left\{b+a\log\left(\frac{a^2}{b}\right)\right\}$$

or
$$\frac{a^2}{c}+a\log c=\frac{a^2}{(a^2/b)}+a\log\left(\frac{a^2}{b}\right).$$

$$\therefore\quad c=a^2/b \text{ i.e., } OC=a^2/b.$$

Since B and C are the positions of instantaneous rest of the particle, therefore the particle oscillates through the space BC.

We have $BC=OB-OC=b-\dfrac{a^2}{b}=\dfrac{b^2-a^2}{b}$,

which proves the required result.