IAS MATHEMATICS (OPT.)

PAPER - I : LINEAR ALGEBRA (2007 to 2000)

IAS-2007

12/7/2007 → (1) Let S be the vector space of all polynomials $p(x)$, with real coefficients, of degree less than or equal to two considered over the real field $\mathbb{R}$, such that $p(0)=0$ and $p(1)=0$. Determine a basis for 'S' and hence its dimension.

1(a).

Sol Let $S = \left\{ p(x) \,/\, \begin{matrix} p(x) = a_0 + a_1 x + a_2 x^2;\ a_0, a_1, a_2 \in \mathbb{R}; \\ p(0) = 0 \text{ and } p(1) = 0 \end{matrix} \right\}$.

be the given vectorspace of all polynomials with real coefficients, of degree less than or equal to two over the field $\mathbb{R}$ such that $p(0)=0$ and $p(1)=1$.

TO determine a basis for S:-.

given that $p(0)=0 \Rightarrow a_0 + a_1(0) + a_2(0)^2 = 0$

$\Rightarrow \boxed{a_0 = 0}$

and $p(1)=0 \Rightarrow a_0 + a_1(1) + a_2(1)^2 = 0$

$\Rightarrow a_0 + a_1 + a_2 = 0$

$\Rightarrow 0 + a_1 + a_2 = 0$

$\Rightarrow \boxed{a_2 = -a_1}$

∴ The given vector space S becomes

$S = \{ +a_1 x - a_1 x^2 \,/\, a_1 \in \mathbb{R} \}$

TO find a basis for 'S', ∃ a finite subset S' of S s.t (i) S' is L.I

(ii) $L(S') = S$.

Let $p_1(x) = a_1 x - a_1 x^2 \in S$; $a_1 \in \mathbb{R}$

then $p_1(x) = a_1(x) + (-a_1)x^2$

$\in L(S')$

where $S' = \{x, x^2\} \subseteq S$.

$\therefore S \subseteq L(S')$ —— (i)

$\because S' \subseteq S$

$\Rightarrow L(S') \subseteq S$ —— (ii)

From (i) and (ii), we have

$\boxed{L(S') = S.}$

i.e. S' spans S.

Let $a, b \in \mathbb{R}$ s.t. $a(x) + b(x^2) = 0$

$\Rightarrow a(x) + bx^2 = 0(x) + 0(x^2)$.

$\Rightarrow a = b = 0$.

$\therefore S'$ is LI subset of S.

$\therefore S'$ forms basis of S and

the number of elements $= 2$.

$\therefore \boxed{\dim(S) = 2}$.

1(b).

12M 2007 → (2) Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3,\ x_1 + x_2,\ x_1 + x_3,\ 3x_1 + x_2 + 2x_3)$$

for each $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Determine a basis for the null space of T.
What is the dimension of the Range space of T.

Sol. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3,\ x_1 + x_2,\ x_1 + x_3,\ 3x_1 + x_2 + 2x_3)$$
$(x_1, x_2, x_3) \in \mathbb{R}^3$ ——— (1).

The null space of T $N(T) = \{ \alpha \in \mathbb{R}^3 \mid T(\alpha) = \hat{0} \text{ in } \mathbb{R}^4 \}$. ——— (2)

Let $\alpha \in N(T)$ then $T(\alpha) = \hat{0}$

$$\Rightarrow T(x_1, x_2, x_3) = (0, 0, 0, 0)$$

$$\Rightarrow (2x_1 + x_2 + x_3,\ x_1 + x_2,\ x_1 + x_3,\ 3x_1 + x_2 + 2x_3) = (0,0,0,0)$$

$$\Rightarrow 2x_1 + x_2 + x_3 = 0 \quad (i)$$
$$x_1 + x_2 = 0 \quad (ii) \Rightarrow x_2 = -x_1$$
$$x_1 + x_3 = 0 \quad (iii) \Rightarrow x_3 = -x_1.$$
$$3x_1 + x_2 + 2x_3 = 0 \quad (iv)$$

$\therefore N(T) = \{ (x_1, -x_1, -x_1) \mid x_1 \in \mathbb{R} \}$.

To find a basis for $N(T)$,
$\exists$ a finite subset of 'S' such that
(i) S is L.I
(ii) $L(S) = N(T)$.

Now let $\alpha = (x_1, -x_1, -x_1) \in N(T)$; $x_1 \in \mathbb{R}$
then $\alpha = x_1(1, -1, -1)$
$\in L(S)$ where $S = \{1, -1, 1\} \subseteq N(T)$

$\therefore N(T) \subseteq L(S)$ —— ①

Since $S \subseteq N(T)$

$\Rightarrow L(S) \subseteq N(T)$ —— ②

$\therefore$ from ① and ②, we have

$$\boxed{L(S) = N(T)}$$

Since the singleton non-zero vector of 'S' is linearly independent.

$\therefore$ S forms a basis of $N(T)$ and the number of elements $= 1$.

$\dim(N(T)) = 1$.

We know that $\dim(R(T)) + \dim(N(T)) = \dim(\mathbb{R}^3)$

$\Rightarrow \dim(R(T)) + 1 = 3$.

$\Rightarrow \dim(R(T)) = 3 - 1$

$\Rightarrow \boxed{\dim R(T) = 2}$.

i.e dimension of range space of T $= 2$

20M (3) Let W be the set of all 3×3 symmetric matrices over ℝ. Does it form a subspace of the vector space of the 3×3 matrices over ℝ? In case it does, construct a basis for this space and determine its dimension.

2007
2(a).

Sol Let $V = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}_{3\times 3} \middle| \begin{matrix} x_1, x_2, x_3 \\ y_1, y_2, y_3 \\ z_1, z_2, z_3 \in \mathbb{R} \end{matrix} \right\}$ be the vectorspace of all 3×3 matrices over ℝ.

Let $W = \left\{ \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times 3} \middle| a, b, c, h, g, f \in \mathbb{R} \right\}$ be the set of all 3×3 symmetric matrices over ℝ and $W \subseteq V$.

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in W \quad \therefore W \neq \phi.$

$\therefore$ W is non-empty subset of V.

Let $A = \begin{bmatrix} a_1 & h_1 & g_1 \\ h_1 & b_1 & f_1 \\ g_1 & f_1 & c_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & h_2 & g_2 \\ h_2 & b_2 & f_2 \\ g_2 & f_2 & c_2 \end{bmatrix}$ be two matrices in W and Let $x, y \in \mathbb{R}$

then we have

$$xA + yB = \begin{bmatrix} xa_1 + ya_2 & xh_1 + yh_2 & xg_1 + yg_2 \\ xh_1 + yh_2 & xb_1 + yb_2 & xf_1 + yf_2 \\ xg_1 + yg_2 & xf_1 + yf_2 & xc_1 + yc_2 \end{bmatrix}$$

$= C_{3\times 3} \in W$ ($\because C_{3\times 3}$ is symmetric matrix)

$\therefore$ W forms subspace of V over the field ℝ.

To find a basis for W:-

– For this, ∃ a finite subset 'S' of W

s.t (i) S is L.I

(ii) $L(S) = W$.

Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \in W$; $a, b, c, f, h, g \in \mathbb{R}$

then $A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$+ b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\in L(S)$

where $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset W$

$\therefore W \subseteq L(S)$ — (i)

since $S \subseteq W$

$\Rightarrow L(S) \subseteq W$ — (ii)

∴ from (i) & (ii), we have

$L(S) = W$.

i.e. S spans W.

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Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{R}$ s.t

$$\lambda_1\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\lambda_2\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+\lambda_3\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+\lambda_4\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}+\lambda_5\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+\lambda_6\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\lambda_1&\lambda_2&\lambda_3\\\lambda_2&\lambda_4&\lambda_5\\\lambda_3&\lambda_5&\lambda_6\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow \lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6=0.$

$\therefore$ S is linearly independent subset of W.

$\therefore$ S forms a basis of W and

~~the~~ number of elements ~~of~~ in S = 6.

$\therefore$ $\boxed{\dim W = 6}$.

20M 2007 2(b). Consider the vectorspace $X := \{P(x) \mid P(x)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$, over the real field $\mathbb{R}$. Define the map $D : X \to X$ by

$$D(p(x)) := p_1 + 2p_2x + 3p_3x^2$$

where $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the basis $\{1, x, x^2, x^3\}$ for X.

Sol'n: Let $p(x), q(x) \in X$, $a, b \in \mathbb{R}$

Given map $D : X \to X$ defined by

$$D(p(x)) = p_1 + 2p_2x + 3p_3x^2$$

where $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$

i.e. $D(p_0 + p_1x + p_2x^2 + p_3x^3) = p_1 + 2p_2x + 3p_3x^2$ —— (1)

Now
$$D[a\,p(x) + b\,q(x)] = D[a(p_0 + p_1x + p_2x^2 + p_3x^3) + b(q_0 + q_1x + q_2x^2 + q_3x^3)]$$

$$= D[(ap_0 + bq_0) + (ap_1 + bq_1)x + (ap_2 + bq_2)x^2 + (ap_3 + bq_3)x^3]$$

$$= (ap_1 + bq_1) + 2(ap_2 + bq_2)x + 3(ap_3 + bq_3)x^2$$

$$= a(p_1 + 2p_2x + 3p_3x^2) + b(q_1 + 2q_2x + 3q_3x^2)$$

$$= a\,D(p(x)) + b\,D(q(x))$$

$\therefore D : X \to X$ is a linear transformation.

Now from ①,

$D(1) = 0 = 0 + 0x + 0x^2 + 0x^3$

$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0 + 2.x + 0x^2 + 0x^3$

$D(x^3) = 3x^2 = 0 + 0.x + 0.x^2 + 3x^2$

Hence the matrix representation of D w.r.to the ordered basis $\{1, x, x^2, x^3\}$

is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

IAS-2006

12M 2006 → (i)

1(a). Let V be the vector space of all 2×2 matrices over the field F. prove that V has dimension 4 by exhibiting a basis for V.

Sol Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \middle| \; a, b, c, d \in F \right\}$ be the given vector space over the field F.

To find a basis for V, ∃ a finite subset S of V such that (i) S is L.I (ii) $L(S) = V$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$; $a, b, c, d \in F$

then $A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\in L(S)$

where $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq V$

$\therefore V \subseteq L(S)$ ——— (i)

Since $S \subseteq V$

$\Longrightarrow L(S) \subseteq V$ ——— (ii)

from (i) and (ii), we have

$L(S) = V$.

i.e S spans V.

Let $x, y, z, t \in F$ s.t

$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ——— (1).

$\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x = y = z = t = 0$.

$\therefore$ S is linearly independent subset of V.

S forms a basis for V.
and the number of elements in S = 4.

$\boxed{\dim V = 4}$.

2006
12m.

1(b). State Cayley Hamilton theorem and using it, find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Sol'n : Statement : Every square matrix satisfies its characteristic equation.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= 4 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda - 2$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

The given matrix A satisfies the characteristic equation

$$\therefore A^2 - 5A - 2I = 0 \quad \text{——— (1)}.$$

Now multiplying (1) by A^{-1}, we get

$$A - 5I - 2A^{-1} = 0$$

$$\Rightarrow 2A^{-1} = A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{2}[A - 5I] \quad \text{——— (2)}$$

$$A-5I = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

$\therefore$ (2) $\equiv$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

2(a).

15M 2006 (3) If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y)=(2x-3y, x+y)$. compute the matrix of T relative to the basis $B=\{(1,2),(2,3)\}$.

Sol Given that $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by

$$T(x,y) = (2x-3y, x+y) \quad \text{(1)}.$$

To find matrix of T w.r.t the basis $B=\{(1,2),(2,3)\}$ of $\mathbb{R}^3$.

Now from (1), we have

$$T(1,2) = (2(1)-3(2), 1+2)$$

$$\therefore \boxed{T(1,2) = (-4,3)} \quad \text{(A)}$$

$$T(2,3) = (4-9, 2+3)$$

$$\therefore \boxed{T(2,3) = (-5,5)} \quad \text{(B)}$$

Since $B=\{(1,2),(2,3)\}$ is a basis of $\mathbb{R}^3$.

let $\alpha=(x,y)\in\mathbb{R}^3$;

$$(x,y) = a(1,2)+b(2,3). \quad \text{(2)}$$

$\Rightarrow \quad a + 2b = x \quad \text{---(i)}$

$2a + 3b = y \quad \text{---(ii)}$

$2\times(i) - (ii) \equiv 4b - 3b = 2x - y$

$\Rightarrow \boxed{b = 2x - y}$

$(i) \equiv a = x - 2b$

$= x - 2(2x - y)$

$\boxed{a = -3x + 2y}$

$\therefore$ ② $\equiv (x, y) = (-3x + 2y)(1, 2) + (2x - y)(2, 3)$.

Now from Ⓐ and Ⓑ, we have

$T(1,2) = (-4, 3) = 18(1,2) + (-11)(2,3)$.

$T(2,3) = (-5, 5) = -5(1,2) + (-20)(2,3)$.

Now the matrix of the linear transformation T is $[T : B] = \begin{bmatrix} 18 & -5 \\ -11 & -20 \end{bmatrix}$.

15M 2006 2(b). (4) Using elementary row-operations, find the rank of the matrix $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Sol

Let $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Let us convert it into echelon form by using elementary row-operation.

$$A \sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & 5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & 5 \\ 0 & 2 & 2 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to R_4 + 2R_3$$

Clearly which is echelon form and the number of non-zero rows of echelon form is the rank of A $\therefore \boxed{\text{rank of } A = 4}$.

15M 2006 2(c). (5) Investigate for what values of λ and μ the equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions.

Sol^n: Write the matrix equation of the given system.

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$R_3 \to$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad R_3 \to R_3 - R_2$$

If $\lambda = 3$ & $\mu \neq 10$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A/B) \neq \rho(A)$

$\therefore$ The given equations have no solutions

If $\lambda \neq 3$ and μ = any value then

$\rho(A/B) = \rho(A) = 3$ = the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

If $\lambda = 3$ and $\mu = 10$ then

$\rho(A/B) = \rho(A) = 2 <$ The number of unknown variables

$\therefore$ The given equations are consistent and have infinite solutions.

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IAS-2005

1(a).

12M 1(b) 2005 Find the values of K for which the vectors $(1,1,1,1)$, $(1,3,-2,K)$, $(2, 2K-2, -K-2, 3K-1)$ and $(3, K+2, -3, 2K+1)$ are linearly independent in $\mathbb{R}^4$.

Sol^n: form the matrix A whose rows are given vectors

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & K \\ 2 & 2K-2 & -K-2 & 3K-1 \\ 3 & K+2 & -3 & 2K+1 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & K \\ 2 & 2K-2 & -K-2 & 3K-1 \\ 3 & K+2 & -3 & 2K+1 \end{vmatrix} \neq 0$$

$R_2 \to R_2 - R_1$
$R_3 - R_3 \to 2R_1$
$R_4 \to R_4 - 3R_1$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & K-1 \\ 0 & 2K-4 & -K-4 & 3K-3 \\ 0 & K-1 & -6 & 2K-2 \end{vmatrix} \neq 0$$

$$\Rightarrow 1 \begin{vmatrix} 2 & -3 & K-1 \\ 2K-4 & -K-4 & 3K-3 \\ K-1 & -6 & 2K-2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} 2 & -3 & K-1 \\ 2K-4 & -K-4 & 3(K-1) \\ K-1 & -6 & 2(K-1) \end{vmatrix} \neq 0$$

$$\Rightarrow (K-1) \begin{vmatrix} 2 & -3 & 1 \\ 2K-4 & -K-4 & 3 \\ K-1 & -6 & 2 \end{vmatrix} \neq 0$$

(2)

$$\Rightarrow (k-1)\begin{vmatrix} 2 & -3 & 1 \\ 2k-10 & -k+5 & 0 \\ k-5 & 0 & 0 \end{vmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\Rightarrow (k-1)(k-5)\begin{vmatrix} 2 & -3 & 1 \\ 2 & -1 & 0 \\ k-5 & 0 & 0 \end{vmatrix} \neq 0$$

$$\Rightarrow (k-1)(k-5)(k-5)(0+1) \neq 0$$

$$\Rightarrow (k-1)(k-5)^2 \neq 0$$

$$\Rightarrow k \neq 1, 5.$$

∴ The given vectors are linearly independent if $k \neq 1, 5$.

i.e i.e $k \in \mathbb{R} - \{1, 5\}$.

12M 2005 → **1(b).** (2) Let V be the vector space of polynomials in x of degree $\leq n$ over $\mathbb{R}$. Prove that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x.

Sol^n: Let $V = \{p(x) \,/\, p(x)$ is a polynomial in x of degree $\leq n$ over $\mathbb{R}\}$.

Let $S = \{1, x, x^2, \ldots\ldots x^n\} \subseteq V$

To show that 'S' is a basis of V :—

(i) To prove S is LI :-

Let $a_0, a_1, a_2, \ldots\ldots a_n \in F$ then

$a_0(1) + a_1(x) + a_2x^2 + \ldots\ldots + a_n(x^n) = 0$ (zero polynomial)

$\Rightarrow a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n = 0 + 0x + \ldots + 0x^n$.

$\Rightarrow a_1 = a_2 = \ldots\ldots = a_n = 0$.

$\therefore$ S is LI.

(ii) To prove $L(S) = V$:

we know that $L(S) \subseteq V$ —(i)

Let $p(x)$ be any polynomial of degree $\leq n$ over $\mathbb{R}$ i.e $p(x) \in V$.

then $p(x) = b_0 + b_1x + b_2x^2 + \ldots\ldots + b_nx^n$ where $b_0, b_1, \ldots b_n \in \mathbb{R}$.

then $p(x) = b_0(1) + b_1(x) + b_2(x^2) + \ldots\ldots + b_n(x^n)$.
= linear combination of elements of S.
= $L(S)$

$\therefore p(x) \in L(S)$

$V \subseteq L(S)$ —(ii)

$\therefore$ from (i) & (ii), we have
S is a basis of V.

Let $F[x]$ be the vectorspace of all polynomials over the field $\mathbb{R}$.

Let $S' = \{1, x, x^2, \ldots\ldots x^n, \ldots\ldots\} \subseteq F[x]$.

Since S is a finite subset of S' having 'n' vectors and S is LI.

$\therefore$ Every finite subset of S' is LI.

$\therefore S'$ is LI.

(ii) To prove $L(S') = F[x]$:–

we know that $L(S') \subseteq F[x]$ —— (i)

Let $f(x) \in F[x]$

i.e $f(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$ be

any polynomial of degree 'm' in $F[x]$.

$= b_0(1) + b_1(x) + b_2(x^2) + \ldots b_m(x^m) + 0x^{m+1} + 0x^{m+2} + \ldots$

= linear combination of S'.

$\in L(S')$.

$\therefore f(x) \in L(S')$.

$\therefore F[x] \subseteq L(S')$ —— (ii)

from (i) & (ii), we have

$L(S') = F[x]$.

$\therefore S'$ is a basis of $F[x]$.

i.e S' extends to form a basis of all polynomials fun. vectorspace

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15M 2005

2(a). Let T be a linear transformation on $\mathbb{R}^3$, whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

find the matrix of T relative to the basis $B = \{(1,1,1),(1,1,0),(0,1,1)\}$.

Sol) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Let $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ be the standard basis of $\mathbb{R}^3$. then $T(1,0,0) = (2,1,3)$

$T(0,1,0) = (1,2,3)$

$T(0,0,1) = (1,2,4)$.

Since S is a basis of $\mathbb{R}^3$

$\therefore \alpha = (x,y,z) \in \mathbb{R}^3$; $x,y,z \in \mathbb{R}$

$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$= x(2,1,3) + y(1,2,3) + z(1,2,4)$

$T(x,y,z) = (2x+y+z,\ x+2y+2z,\ 3x+3y+4z)$ ...(1)

which is the required explicitly condition of L.T.

To find the matrix of T relative to the basis $B = \{(1,1,1),(1,1,0),(0,1,1)\}$.

from (1),

$T(1,1,1) = (4,5,10)$

$T(1,1,0) = (3,3,6)$ ——— (A).

$T(0,1,1) = (2,4,7)$

Since $B = \{(1,1,1),(1,1,0),(0,1,1)\}$ be a basis of $\mathbb{R}^3$.

Let $(a,b,c) \in \mathbb{R}^3$, $a, b, c \in \mathbb{R}$.

$$(a,b,c) = x(1,1,1) + y(1,1,0) + z(0,1,1) \quad \text{--- (B)}$$

$x, y, z \in \mathbb{R}$.

$$\Rightarrow x + y = a$$

$$x + y + z = b \Rightarrow \boxed{z = b - a}$$

$$x + z = c. \Rightarrow x = c - z$$

$$\Rightarrow x = c - (b-a)$$

$$\Rightarrow \boxed{x = a - b + c}$$

and $y = a - x$

$$\Rightarrow y = a - (a - b + c)$$

$$\Rightarrow \boxed{y = b - c}$$

$\therefore$ (B) $\equiv$

$$(a,b,c) = (a-b+c)(1,1,1) + (b-c)(1,1,0) + (b-a)(0,1,1).$$

Now from (A),

$$T(1,1,1) = (4,5,10) = 9(1,1,1) + (-5)(1,1,0) + 1(0,1,1)$$

$$T(1,1,0) = (3,3,6) = 6(1,1,1) + (-3)(1,1,0) + (-3)(0,1,1)$$

$$T(0,1,1) = (2,4,7) = 5(1,1,1) + (-2)(1,1,0) + 2(0,1,1).$$

$\therefore$ The matrix of linear transformation

$T: \mathbb{R}^2 \to \mathbb{R}^3$ is

$$[T:B] = \begin{bmatrix} 9 & 6 & 5 \\ -5 & -3 & -3 \\ 1 & -3 & 2 \end{bmatrix}.$$

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15M(4) **2(b).** Find the inverse of the matrix given below using Elementary row operations only. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol'n: Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ then

$$A = I_3 A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\sim \begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_2 \to R_2 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A \qquad R_3 \to \tfrac{1}{3} R_3$$

$$I_3 = BA$$

where $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\therefore A^{-1} = B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

2000

15M (5) If S is skew-Hermitian matrix, then show that

2(c). $A = (I+S)(I-S)^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided -1 is not an eigen value of A.

Sol'n: Given that 'S' is a skew-Hermitian matrix

$$\therefore S^{\theta} = -S \quad \text{——} \quad (1)$$

we know that the eigen values of a skew-Hermitian matrix 'S' are either purely imaginary (or) zero

$\therefore$ Neither 1 nor -1 is a root of the equation

$$|S - \lambda I| = 0$$

$$\Rightarrow |S-I| \neq 0 \text{ and } |S+I| \neq 0$$

$$\Rightarrow |I-S| \neq 0 \text{ and } |I+S| \neq 0$$

$$(\because |-A| \neq 0 \Rightarrow |A| \neq 0)$$

$\therefore$ $I-S$ and $I+S$ are both non-singular matrices.

Now given that $A = (I+S)(I-S)^{-1}$

consider $A^{\theta} = [(I+S)(I-S)^{-1}]^{\theta}$

$$= [(I-S)^{-1}]^{\theta} (I+S)^{\theta}$$

$$= [(I-S)^{\theta}]^{-1} (I^{\theta} + S^{\theta})$$

$$= (I^{\theta} - S^{\theta})^{-1} (I-S) \quad (\text{by } (1))$$

$$= (I+S)^{-1} (I-S) \quad (\text{by } (1))$$

$$\therefore A^{\theta} A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$$

$$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$$

$$[\because (I-S)(I+S) = (I+S)(I-S)]$$

$$= I \cdot I$$

$$= I$$

$$\therefore A^{\theta} A = I$$

A is a unitary matrix

IAS-2004

1(a).

12M 2004 (1) Let S be space generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,-2)\}$. what is the dimension of the space 'S'? find a basis for S.

Sol Given that S is a space generated by the set $S' = \{(0,2,6),(3,1,6),(4,-2,-2)\} \subseteq S$.

i.e $(S') = S$

Let us construct a matrix A whose rows are the given vectors of S' and convert it into echelon form.

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 1 & 6 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 4 & -2 & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & \frac{-10}{3} & -10 \end{bmatrix} \quad R_3 \to R_3 - \frac{4}{3}R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + \frac{5}{3}R_2$$

Clearly which is in echelon form and the number of non-zero rows of echelon form = 2.

corresponding to these non-zero rows the vectors $(0,2,6),(3,1,6)$ form a basis of S. and $\boxed{\dim(S)=2}$.

12M. 2004

1(b). Show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel? Find a basis for the kernel.

Sol^n: To show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation.

where $f(x,y,z) = 3x + y - z$ —— (1).

Let $\alpha, \beta \in \mathbb{R}^3$ s.t $\alpha = (x, y, z)$

$\beta = (x_1, y_1, z_1) \in \mathbb{R}^3$

let $a, b \in \mathbb{R}$ then $a\alpha + b\beta =$

$(ax + bx_1, ay + by_1, az + bz_1) \in \mathbb{R}^3$

$\therefore f(a\alpha + b\beta) = f(ax + bx_1, ay + by_1, az + bz_1)$

$= 3(ax + bx_1) + (ay + by_1) - (az + bz_1)$ (by defn).

$= a(3x + y - z) + b(3x_1 + y_1 - z_1)$

$f(a\alpha + b\beta) = a f(\alpha) + b f(\beta)$.

$\therefore f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation

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Now the kernel of f = ker f

$= K = \{\alpha \in \mathbb{R}^3 / f(\alpha) = 0 \in \mathbb{R}\}$

Let $\alpha = (x, y, z) \in K$.

then $f(\alpha) = 0$

$\Rightarrow f(x, y, z) = 0$

$\Rightarrow 3x + y - z = 0$

$\Rightarrow \boxed{z = 3x + y}$

$\therefore K = \{(x, y, 3x+y) / x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

To find a basis of K i.e.

a finite subset 'S' of K s.t. (i) S is LI

(ii) $L(S) = K$

Let $\beta = (x, y, 3x+y) \in K$.

then $\beta = x(1,0,3) + y(0,1,1)$

$\in L(S)$ where $S = \{(1,0,3), (0,1,1)\} \subseteq K$

$\therefore K \subseteq L(S)$ ——(i)

Since $S \subseteq K$

$\Rightarrow L(S) \subseteq K$ ——(ii)

$\therefore$ from (i) & (ii), we have

$L(S) = K$.

Since no vector of S is a scalar multiple of other

$\therefore$ S is LI.

$\therefore$ S forms a basis of K and

the number of elements of S = 2.

$\boxed{\dim(K) = 2}$

15M(4) 2004

2(b). verify whether the following system of equations is consistent:

$$x+3z=5$$
$$-2x+5y-z=0$$
$$-x+4y+z=4.$$

Sol The given system is

$$\left.\begin{array}{l} x+3z=5 \\ -2x+5y-z=0 \\ -x+4y+z=4 \end{array}\right\} \text{ (A)}$$

We write the given system in matrix equation is $AX=B$

i.e
$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$$

where $A=\begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B=\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$.

Now the augmented matrix

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ -2 & 5 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Let us convert into echelon form by using elementary row transformations

$$[A|B]\sim\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & 4 & 4 & 9 \end{array}\right] \quad \begin{array}{l} R_2\to R_2+2R_1 \\ R_3\to R_3+R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - \frac{4}{5}R_2$$

clearly which is in echelon form

$\therefore\ \rho(A) = 2$ & $\rho(A|B) = 3$.

$\therefore\ \rho(A) \neq \rho(A|B)$.

$\therefore$ The given system of equations is not consistent.

157 → (5) find the characteristic polynomial of
2004
2(c). the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$. Hence, find A^{-1} and A^6.

Sol Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}_{2\times 2}$, $I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

λ is a scalar

then $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda) + 1$

$\therefore$ Characteristic polynomial of A

is $|A - \lambda I| = 0$

i.e $(1-\lambda)(3-\lambda) + 1 = 0$.

$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$ — (1).

which is the required characteristic polynomial equation of A.

We know that every square matrix satisfies its own characteristic equation.

$\therefore$ (1) $\equiv A^2 - 4A + 4I = 0$. — (2).

Since $|A| = 3 + 1$
$= 4$
$\neq 0$.

$\therefore A^{-1}$ exists.

$$(2) \equiv A - 4I + 4A^{-1} = 0.$$

$$\Rightarrow 4A^{-1} = 4I - A$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\therefore 4A^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.}$$

Again from (2), we have

$$A^6 - 4A^4 + 4A^3 = 0.$$

$$\Rightarrow A^6 = 4A^4 + 4A^3. \quad \text{--- (3)}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}$$

$$A^6 = A^3 A^3 = \begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}\begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 160 & 192 \\ -192 & 256 \end{bmatrix}$$

IAS-2003

1(a). 12M Let S be any non-empty subset of a vector space V over the field F.

Show that the set $\{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : a_1, a_2, \ldots a_n \in F, \alpha_1, \alpha_2, \ldots, \alpha_n \in S, n \in N\}$ is the subspace generated by S.

Proof: Given that V(F) is a vectorspace and $S \subseteq V$

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and $L(S) = \left\{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \,\middle|\, \begin{matrix}\alpha_1, \alpha_2 \ldots \alpha_n \in S \\ a_1, a_2 \ldots a_n \in F\end{matrix}\right\} \subseteq V$

Now $\forall\, a, b \in F$; $\alpha, \beta \in L(S)$

choose $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$

where a's, b's $\in F$ and α's $\in S$

$\Rightarrow a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$

$= (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bb_n)\alpha_n$

$\in L(S)$

$(\because aa_1 + bb_1, aa_2 + bb_2, \ldots aa_n + bb_n \in F)$

$\therefore$ L(S) is a subspace of V(F).

Let $\alpha_i \in S$, $i = 1, 2, \ldots n$

then $\alpha_i = 1\alpha_i$

$=$ linear Combination of α_i

$\in L(S)$

$\therefore \alpha_i \in L(S)$

$\therefore S \subseteq L(S)$.

Now let W be any subspace of V(F) containing S. $\therefore S \subseteq W$

if $\alpha \in L(S)$ then $\alpha =$ the linear Combination of a finite no. of elements of S

$\in W$ $(\because S \subseteq W)$

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$\therefore$ If $\alpha \in L(S)$ then $\alpha \in W$

$$\therefore L(S) \subseteq W$$

$$\therefore S \subseteq L(S) \subseteq W \subseteq V$$

$\therefore L(S)$ is the smallest subspace of V containing S.

i.e, $L(S) = \{S\}$

P.I

12M 2003 1(b). If $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$, then find the matrix represented by: $2A^{10}-10A^9+14A^8-6A^7-3A^6+15A^5-21A^4+9A^3+A-I$.

Sol^n: Given that

$$A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$$

Now
$$A^2=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$$
$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$$

$$A^3=A^2A$$
$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$$
$$=\begin{bmatrix}14&13&13\\0&1&0\\13&13&14\end{bmatrix}$$

$$A^4=A^2A^2$$
$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$$
$$=\begin{bmatrix}41&40&40\\0&1&0\\40&40&41\end{bmatrix}$$

$$A^5 = A^4 A$$

$$= \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix}$$

$$A^6 = A^5 A = \begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 364 & 365 \end{bmatrix}$$

$$A^7 = A^6 A = \begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 364 & 365 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix}$$

$$A^8 = A^7 A = \begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix}$$

$$A^9 = A^8 A = \begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix}$$

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$$A^{10} = A^9 A = \begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29525 & 29524 & 29524 \\ 0 & 1 & 0 \\ 29524 & 29524 & 29525 \end{bmatrix}$$

$$2A^{10} - 10A^9 + 14A^8 - 6A^7 - 3A^6 + 15A^5 - 21A^4 + 9A^3 + A - I$$

$$= 2\begin{bmatrix} 29525 & 29524 & 29524 \\ 0 & 1 & 0 \\ 29524 & 29524 & 29525 \end{bmatrix} - 10\begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix}$$

$$+14\begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix} - 6\begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix}$$

$$-3\begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 365 & 365 \end{bmatrix} + 15\begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix} - 21\begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

$$+9\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

2003 15M (2) 2(a). Prove that the eigen vectors Corresponding to distinct eigen values of a Square matrix are linearly independent.

Proof: Let $x_1, x_2, x_3 \ldots x_m$ be the Characteristic Vectors of a matrix A Corresponding to distinct characteristic values $\lambda_1, \lambda_2, \ldots \lambda_m$.

Then $Ax_i = \lambda_i x_i$; $i = 1, 2, \ldots m$. —— (1)

To Prove that the vectors $x_1, x_2, \ldots x_m$ are linearly dependent.

then we Can choose $r\,(1 \leq r < m)$ Such that $x_1, x_2, \ldots x_r$ are linearly independent and $x_1, x_2, \ldots x_r, x_{r+1}$ are linearly dependent.

$\therefore$ we Can choose the Scalars $k_1, k_2, \ldots k_{r+1}$ not all zeros Such that

$$k_1x_1 + k_2x_2 + \ldots + k_rx_r + k_{r+1}x_{r+1} = 0 \quad \text{—— (2)}$$

$$\Rightarrow A(k_1x_1 + k_2x_2 + \ldots + k_rx_r + k_{r+1}x_{r+1}) = A(0)$$

$$\Rightarrow k_1(Ax_1) + k_2(Ax_2) + \ldots + k_r(Ax_r) + k_{r+1}(Ax_{r+1}) = 0$$

$\Rightarrow k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) + \cdots + k_r(\lambda_r x_r) + k_{r+1}(\lambda_{r+1} x_{r+1}) = 0$ —— ③

(by using ①)

Now ③ $- \lambda_{r+1}$ ② $\equiv$

$k_1(\lambda_1 - \lambda_{r+1})x_1 + \cdots k_r(\lambda_r - \lambda_{r+1})x_r = 0$ —— ④

Since $x_1, x_2, \cdots x_r$ are linearly independent and $\lambda_1, \lambda_2, \cdots \lambda_r, \lambda_{r+1}$ are distinct

$\therefore k_1 = 0,\ k_2 = 0, \cdots k_r = 0$

Putting $k_1 = 0, k_2 \cdots k_r = 0$ in ②.

we get $k_{r+1} x_{r+1} = 0$

$\Rightarrow k_{r+1} = 0 \ (\because x_{r+1} \neq 0)$

$\therefore$ from ⑤, $k_1 = 0, k_2 = 0, \cdots k_r = 0, k_{r+1} = 0$

$\therefore$ which is Contradiction to our assumption that the Scalars.

$k_1, k_2, \cdots k_r, k_{r+1}$ are not all zeros.

$\therefore$ our assumption that $x_1, x_2 \cdots x_m$ are linearly dependent is wrong.

$\therefore x_1, x_2, x_3 \cdots x_m$ are linearly independent.

$\therefore x_1, x_2, \cdots x_m$ which corresponds to distinct characteristic roots of A are linearly independent.

IAS-2002

1(a). Show that the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(a, b, c) = (a-b, b-c, a+c)$ is linear and non-singular.

Sol Given that $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t

$$T(a, b, c) = (a-b, b-c, a+c) \quad \text{--- (1)}.$$

$(a, b, c) \in \mathbb{R}^3$.

Let $\alpha = (a, b, c)$

$\beta = (a_1, b_1, c_1) \in \mathbb{R}^3$; $x, y \in \mathbb{R}$

then $x\alpha + y\beta = x(a, b, c) + y(a_1, b_1, c_1)$

$= (xa + ya_1, bx + yb_1, cx + yc_1)$

Now we have

$T(x\alpha + y\beta) = T(xa + ya_1, bx + yb_1, cx + yc_1)$

$= (xa + ya_1 - bx - yb_1, bx + yb - cx - yc_1, xa + ya_1 + cx + yc_1)$

$= (x(a-b) + y(a_1 - b_1), x(b-c) + y(b_1 - c_1), x(a+c) + y(a_1 + c_1))$

$= x(a-b, b-c, a+c) + y(a_1 - b_1, b_1 - c_1, a_1 + c_1)$

$T(x\alpha + y\beta) = xT(\alpha) + yT(\beta)$.

$\therefore$ T is a linear transformation.

<u>To show that T is non-singular:</u>–

The null space $N(T) = \{\alpha \in \mathbb{R}^3 / T(\alpha) = \bar{0} \text{ in } \mathbb{R}^3\} \subseteq \mathbb{R}^3$.

Let $\alpha \in N(T)$ then $T(\alpha) = \bar{0}$

$\Rightarrow (a-b, b-c, a+c) = (0, 0, 0)$

$\Rightarrow a-b=0$ (i)

$b-c=0$ (ii)

$a+c=0$ (iii)

(i) + (ii) $\equiv a-c=0$

$a+c=0.$

$\Rightarrow a=0=b=c.$

$\therefore N(T)=\{(0,0,0)\}\subseteq \mathbb{R}^3.$

$\therefore N(T)$ contains only zero vector

$T:\mathbb{R}^3\to\mathbb{R}^3$ is non-singular.

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IAS 2002

15M 2002

2(b). Let A be a real 3×3 symmetric matrix with eigen values 0,0,5. If the corresponding eigen vectors are (2,0,1), (2,1,1) and (1,0,−2) then find the matrix A.

Sol^n: Given that the matrix A is a real 3×3 symmetric matrix with eigen values 0, 0, 5 and the corresponding eigen vectors (2, 0, 1), (2, 1, 1) and (1, 0, −2).

Let $x_1 = (2,0,1)^T$; $x_2 = (2,1,1)^T$; $x_3 = (1,0,-2)^T$

Now let us normalize the vectors x_1 x_2 & x_3 &

for this

$\|x_1\| = \sqrt{5}$; $\|x_2\| = \sqrt{6}$; $\|x_3\| = \sqrt{5}$

$$\therefore \hat{x}_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{5}}(2,0,1)^T$$

$$= \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)^T$$

$$\hat{x}_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{6}}(2,1,1)^T$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$$

$$\hat{x}_3 = \frac{x_3}{\|x_3\|} = \frac{1}{\sqrt{5}}(1,0,-2)^T$$

$$= \left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right)^T$$

Let $P = \left[\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3\right]$

$$= \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{6} & 1/\sqrt{5} \\ 0 & 1/\sqrt{6} & 0 \\ 1/\sqrt{5} & 1/\sqrt{6} & -2/\sqrt{5} \end{bmatrix}$$

which is the required orthogonal matrix.

Since P is orthogonal.

$\therefore P^T P = I$

P is non-singular

$\therefore P^T = P^{-1}$

$$\therefore P^{-1} = P^T = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

Since every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

$\therefore P^{-1}AP = D = \text{dia}(0, 0, 5)$

$\Rightarrow AP = PD$

$\Rightarrow A = PDP^{-1}$

$\Rightarrow A = PDP^T$

$$\Rightarrow A = \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{6} & 1/\sqrt{5} \\ 0 & 1/\sqrt{6} & 0 \\ 1/\sqrt{5} & 1/\sqrt{6} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 5/\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & -10/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$ which is the required symmetric matrix

IAS-2001

1(a).

IAS- 2001

12M 2001 (i) Show that the vectors $(1,0,-1)$, $(0,-3,2)$ and $(1,2,1)$ form a basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

Sol Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be the given vector space.

Let $S = \{(1,0,-1), (0,-3,2), (1,2,1)\} \subseteq \mathbb{R}^3$.

We know that $\dim(\mathbb{R}^3) = 3$.

Now we construct a matrix A whose rows are given vectors of S

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Now we have

$$|A| = 1(-7) - 0(-2) - 1(3)$$
$$= -7 - 3$$
$$= -10$$
$$\neq 0$$

$\therefore$ The given vectors of S are linearly independent and number of linearly independent vectors $= 3$.

$\therefore$ S forms a basis of $\mathbb{R}^3(\mathbb{R})$.

2001 12M (2)

1(b). If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

Sol'n: Since 'λ' is a characteristic root of a non-singular matrix, therefore $\lambda \neq 0$. Also λ is a characteristic root of A implies there exists a non-zero vector x such that $AX = \lambda x$

$$\Rightarrow (\text{Adj } A)(Ax) = (\text{Adj } A)(\lambda x)$$

$$\Rightarrow [(\text{Adj } A)A]x = \lambda(\text{Adj } A)x$$

$$\Rightarrow |A| I x = \lambda(\text{Adj } A)x \quad (\because (\text{Adj} A)A = |A| I)$$

$$\Rightarrow \frac{|A|}{\lambda} x = (\text{Adj } A)x$$

$$\Rightarrow (\text{Adj } A)x = \frac{|A|}{\lambda} x$$

$\Rightarrow \frac{|A|}{\lambda}$ is a characteristic root of the matrix Adj A.

2001
15m (3)

2(a). If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, determine A^{50}.

Sol'n: If $n = 3$ then

$$A^3 = A + A^2 - I \quad \text{—— (1)}$$

Since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now $A + A^2 - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Suppose $A^n = A^{n-2} + A^2 - I$ is true for $n=k$

$\therefore A^k = A^{k-2} + A^2 - I$

Now for $n = k+1$;

$$A^{k+1} = A \cdot A^k$$
$$= A \cdot [A^{k-2} + A^2 - I]$$
$$= A^{k-1} + A^3 - A$$
$$= A^{k-1} + A + A^2 - I - A \quad (\text{from } 1)$$
$$= A^{k-1} + A^2 - I$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = k+1$

$\therefore$ By mathematical induction, it is true for every integer $n \geq 3$.

$$A^n = A^{n-2} + A^2 - I \quad \text{——} (2)$$

Now $A^3 = A + A^2 - I$

$$\boxed{A^4 = 2A^2 - I}$$

$$A^6 = A^4 + A^2 - I$$

$$\boxed{A^6 = 3A^2 - 2I}$$

$$A^8 = A^6 + A^2 - I$$

$$\boxed{A^8 = 4A^2 - 3I}$$

$$A^{10} = A^8 + A^2 - I$$

$$\boxed{A^{10} = 5A^2 - 4I}$$

Similarly $A^{12} = 6A^2 - 5I$

$\left(\text{i.e. } A^{12} = \frac{12}{2} A^2 - \left(\frac{12}{2} - 1\right) I\right)$

$\vdots$

$A^{50} = 25A^2 - 24I$ $\quad \left(\text{i.e. } A^{50} = \frac{50}{2} A^2 - \left(\frac{50}{2} - 1\right) I\right)$

$$A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

15M(4) 2001 **2(c).** Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}.$$

Sol^n: The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\left[(8+\lambda)^2 - 1\right] + 4\left[4 + 4(8+\lambda)\right] - 4\left[-4 - 4(8+\lambda)\right]$$

$$\Rightarrow 729 + 81\lambda - 9\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow -\lambda^3 - 9\lambda^2 + 81\lambda + 729 = 0$$

$$\Rightarrow (\lambda - 9)(-\lambda^2 - 18\lambda - 81) = 0$$

$$\Rightarrow (\lambda - 9)(\lambda^2 + 18\lambda + 81) = 0$$

$$\Rightarrow (\lambda - 9)(\lambda + 9)^2 = 0$$

$$\Rightarrow \boxed{\lambda = 9, -9, -9.}$$

IAS-2000

1(a).

12-1(3) 2000

Let V be a vectorspace over $\mathbb{R}$ and $T = \{(x, y) / x, y \in V\}$.

Define addition in T componentwise and scalar multiplication by a complex number $\alpha + i\beta$ by $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall\ \alpha, \beta \in \mathbb{R}$

Show that T is a vectorspace over C.

Sol: Given that V is a vectorspace over the field $\mathbb{R}$.

$T = \{(x, y) / x, y \in V\}$

Define addition in T componentwise

(1) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ $\forall (x, y), (x_1, y_1) \in T$; $x, y; x_1, y_1 \in V$.

and scalar multiplication

(2) $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall\ \alpha, \beta \in \mathbb{R}$

To show that T is a vectorspace over the field C.

Internal composition in T:-

by (1) $(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ $\forall (x, y), (x_1, y_1) \in T$; $x, y; x_1, y_1 \in V$.

$\in T$ $(\because x + x_1, y + y_1 \in V)$

$\therefore$ Internal composition in T is satisfied.

External composition in T over C:-

by (2) $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall\ \alpha, \beta \in \mathbb{R}$

$\in T$ $(\because \alpha x - \beta y, \beta x + \alpha y \in V$ (vector space)$)$

$\therefore$ External composition in T over C is satisfied.

(I) To show that $(T, +)$ is an abelian group:

(i) By internal composition in T,
T is closed.

(ii) $\forall (x, y), (x_1, y_1), (x_2, y_2) \in T$;

$$[(x, y) + (x_1, y_1)] + (x_2, y_2) = (x+x_1, y+y_1) + (x_2, y_2) \quad (\text{by } ①)$$
$$= ((x+x_1)+x_2, (y+y_1)+y_2)$$
$$= (x+(x_1+x_2), y+(y_1+y_2)) \quad (\text{by associative prop in } V)$$
$$= (x, y) + (x_1+x_2, y_1+y_2)$$
$$= (x, y) + [(x_1, y_1) + (x_2, y_2)] \quad (\text{by } ①)$$

$\therefore$ associative property in V is satisfied.

(iii) Existence of right identity in T:

$\forall (x, y) \in T, \exists (0, 0) \in T$; $0 \in V$ s.t

$$(x, y) + (0, 0) = (x+0, y+0) \quad (\text{by } ①)$$
$$= (x, y) \quad (\text{by identity in } V).$$

$\therefore (0, 0)$ is a right identity in T

(iv) Existence of right inverse in T:

for each $(x, y) \in T$, $\exists (-x, -y) \in T$; $-x, -y \in V$

s.t $(x, y) + (-x, -y) = (x+(-x), y+(-y))$

$$= (0, 0) \quad (\text{by inverse prop. of } V)$$

identity in T

$\therefore (-x, -y)$ is the right inverse of (x, y) in T.

(v) commutative property for T

$\forall (x, y) + (x_1, y_1) \in T$;

$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$ (by ①)

$= (x_1 + x, y_1 + y)$ (by comm. for $\mathbb{R}$)

$= (x_1, y_1) + (x, y)$ (by ①)

$\therefore$ Commutative prop for T is satisfied.

(II) Let $X = (x, y)$, $Y = (x_1, y_1) \in T$

$a = \alpha + i\beta$; $b \in \alpha_1 + i\beta_1 \in \mathbb{C}$; $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{R}$.

then we have

(i) $a(X + Y) = aX + aY$

(ii) $(a+b)X = aX + bX$

(iii) $(ab)X = a(bX)$

(iv) $1X = X$; $1 = 1 + i0 \in \mathbb{C}$ (identity in $\mathbb{C}$).

15M 2000 (i)

2(a). Prove that a system $Ax=B$ of 'n' non-homogeneous equations 'n' unknowns has a unique solution provided the coefficient matrix is non-singular.

Sol: Since A is a non-singular matrix of order 'n'.

$\therefore |A| \neq 0.$

$\Rightarrow \rho(A) = n$ and $\rho(A|B) = n$

$\therefore \rho(A) = \rho(A|B)$

$\Rightarrow Ax = B$ is consistent and it has a solution.

Also A^{-1} exists $(\because |A| \neq 0)$

$\therefore A^{-1}(Ax) = A^{-1}B$

$\Rightarrow (A^{-1}A)x = A^{-1}B$

$\Rightarrow Ix = A^{-1}B$

$\Rightarrow x = A^{-1}B$ is a solution of $Ax=B$.

If possible let x_1 & x_2 be two solutions of $Ax=B$.

$\therefore Ax_1 = B$ and $Ax_2 = B$

$\Rightarrow Ax_1 = Ax_2$

$\Rightarrow A^{-1}(Ax_1) = A^{-1}(Ax_2)$

$\Rightarrow (A^{-1}A)x_1 = (A^{-1}A)x_2$

$\Rightarrow Ix_1 = Ix_2$

$\Rightarrow x_1 = x_2$

$\therefore$ The solution $x = A^{-1}B$ of $Ax=B$ is unique.

1(a) 2000 2(c). Prove that two similar matrices have the same characteristic roots. Is its converse true? Justify your claim.

Sol: Let A and B be similar matrices.

Then $\exists$ an invertible matrix P such that

$$B = P^{-1}AP.$$

$$\therefore B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}AP - P^{-1}\lambda P$$

$$= P^{-1}AP - P^{-1}\lambda(IP)$$

$$= P^{-1}AP - P^{-1}(\lambda I)P$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}|\,|A - \lambda I|\,|P|$$

$$= |A - \lambda I|\,|P^{-1}P|$$

$$= |A - \lambda I|$$

$$\therefore |B - \lambda I| = |A - \lambda I|$$

$\therefore$ A and B have the same characteristic polynomial and hence same characteristic roots.

The converse of the above need not be true i.e., if the two matrices have the same characteristic roots then it is not necessary that they are similar.

for example:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$$

have the same characteristic roots but are not similar.