2

Lengths of Curves

(Rectification)

§ 1. Definition.

The process of finding the length of an arc of a curve between two given points is called rectification.

§ 2. Lengths of Curves.

If s denotes the arc length of a curve measured from a *fixed point* to any point on it, then as proved in our book on Differential Calculus (§ 12, pages 199-200), we have

$$\frac{ds}{dx} = \pm\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

where +ive or –ive sign is to be taken before the radical sign according as x increases or decreases as s increases. Hence if s increases as x increases, we have

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \text{ or } ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx.$$

Integrating, we have

$$s = \int_a^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx,$$

where a is the abscissa of the fixed point from which s is measured.

Hence the arc length of the curve $\mathbf{y = f(x)}$ included between two points for which $x = a$ and $x = b$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2\right\}}\, \mathbf{dx}, \quad (\mathbf{b > a}).$$

Sometimes it is more convenient to take y as the independent variable. Then the length of the arc of the curve $\mathbf{x = f(y)}$ between $y = a$ and $y = b$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2\right\}}\, \mathbf{dy}, \quad (\mathbf{b > a}).$$

Remark. Suppose we have to find the length of the arc of a curve (whose cartesian equation is given) lying between the points (x_1, y_1) and (x_2, y_2). We can use either of the two formulae

$$s = \int_{x_1}^{x_2} \sqrt{\{1 + (dy/dx)^2\}}\, dx \quad \text{and} \quad s = \int_{y_1}^{y_2} \sqrt{\{1 + (dx/dy)^2\}}\, dy.$$

If we feel any difficulty in integration while using one of these two formulae, we must try the other formula also.

Equations of the Curve in parametric form :

If the equations of the curve be given in the parametric form $x = f(t)$, $y = \phi(t)$, then s is obviously a function of t. In this case if we measure the arc length s in the direction of t increasing, we have

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$

On integrating bewteen proper limits, the required length

$$s = \int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$

Equation of the Curve in polar form :

For the curve $r = f(\theta)$, if we measure the arc length s in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta.$$

On integrating between proper limits, the required length

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta.$$

If the equation of the curve be $\theta = f(r)$, then the required length is given by

$$s = \int_{r_1}^{r_2} \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}\, dr.$$

Equation of the Curve in pedal form :

Let $p = f(r)$ be the equation of the curve and r_1 and r_2 be the values of r at two given points of the curve. Then by differential calculus we know that

$$\frac{ds}{dr} = \frac{r}{\sqrt{(r^2 - p^2)}} \quad \text{or} \quad ds = \frac{r}{\sqrt{(r^2 - p^2)}}\, dr,$$

where s increases as r increases.

On integrating between proper limits, the required length

$$s = \int_{r_1}^{r_2} \frac{r}{\sqrt{(r^2 - p^2)}}\, dr.$$

The value of p should be put in terms of r from the equation of the curve.

Important Remark : If the curve is symmetrical about one or more lines, then find out the length of one symmetrical part and then multiply it by the number of symmetrical parts.

Examples on Rectification (Cartesian equations) :

Ex. 1. *Show that the length of the curve $y = \log \sec x$ between the points where $x = 0$ and $x = \frac{1}{3}\pi$ is $\log (2 + \sqrt{3})$.*

(Bhopal 1981; Meerut 77, 87S; Agra 82, 77)

Sol. The given curve is $y = \log \sec x$. ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x = \tan x.$$

Now $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x.$ ...(2)

If the arc length s of the given curve is measured from $x = 0$ in the direction of x increasing, we have

$$\frac{ds}{dx} = \sec x \quad \text{or} \quad ds = \sec x \, dx.$$

Therefore if s_1 denotes the arc length from $x = 0$ to $x = \frac{1}{3}\pi$, then

$$\int_0^{s_1} ds = \int_0^{\pi/3} \sec x \, dx = \Big[\log (\sec x + \tan x)\Big]_0^{\pi/3}$$

or $\quad s_1 = [\log (\sec \frac{1}{3}\pi + \tan \frac{1}{3}\pi) - \log 1] = \log (2 + \sqrt{3}).$

Ex. 2. *Find the length of the curve $y = \log [(e^x - 1)/(e^x + 1)]$ from $x = 1$ to $x = 2$.* **(Rohilkhand 1982; Bhopal 80; Meerut 86 S, 96 BP)**

Sol. The given curve is $y = \log [(e^x - 1)/(e^x + 1)]$

or $\quad y = \log (e^x - 1) - \log (e^x + 1).$ ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{(e^{2x} - 1)}.$$

Now $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left\{\frac{2e^x}{e^{2x} - 1}\right\}^2 = 1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}$

$$= \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2} = \frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}.$$

Measuring the arc length s in the direction of x increasing, we have

$\frac{ds}{dx} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, dividing the Nr. and the Dr. by e^x

or $\quad ds = \frac{e^x + e^{-x}}{e^x - e^{-x}} dx.$

$\therefore$ the required length s_1 is given by

$$s_1 = \int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

$$= \left[\log (e^x - e^{-x})\right]_1^2,$$

numerator being the diff. coeff. of denominator

$$= \log (e^2 - e^{-2}) - \log (e - e^{-1})$$

$$= \log [\{e^2 - (1/e^2)\}/\{e - (1/e)\}]$$

$$= \log (e + 1/e).$$

Ex. 3. *Find the arc length of the curve*

$$y = \tfrac{1}{2}x^2 - \tfrac{1}{4}\log x \text{ from } x = 1 \text{ to } x = 2.$$

Sol. The given curve is $y = \frac{1}{2}x^2 - \frac{1}{4}\log x$. ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = x - \frac{1}{4x} = \frac{4x^2 - 1}{4x}.$$

$\therefore$ required length of the curve

$$= \int_1^2 \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx = \int_1^2 \sqrt{\left\{1 + \frac{(4x^2 - 1)^2}{16x^2}\right\}} dx$$

$$= \int_1^2 \frac{4x^2 + 1}{4x} dx = \int_1^2 \left(x + \frac{1}{4x}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{\log x}{4}\right]_1^2 = \frac{3}{2} + \frac{1}{4}\log 2.$$

Ex. 4 (a). *Show that in the catenary $y = c \cosh (x/c)$, the length of arc from the vertex to any point is given by $s = c \sinh (x/c)$.*

(Bhopal 1983)

Sol. The given catenary is $y = c \cosh (x/c)$. ...(1)

The point $A\ (0, c)$ is the vertex of the catenary and let $P\ (x, y)$ be any point on it. We have to find the length of arc AP for which x varies from $x = 0$ to $x = x$.

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = c \cdot \frac{1}{c}\sinh\frac{x}{c} = \sinh\frac{x}{c}.$$

If s denotes the arc length of the catenary measured in the direction of x increasing, then

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \sinh^2\frac{x}{c}\right)} = \cosh\frac{x}{c}$$

or $\quad ds = \cosh (x/c)\, dx$. Integrating, we have

$$\int_0^s ds = \int_0^x \cosh\frac{x}{c} dx = \left[c \sinh\frac{x}{c}\right]_0^x = c \sinh\frac{x}{c}$$

or $\quad s = c \sinh (x/c)$, which is the required arc length.

Ex. 4 (b). *If s be the length of the arc of the catenary $y = c \cosh (x/c)$ from the vertex $(0, c)$ to the point (x, y), show that*

$$s^2 = y^2 - c^2.$$

(Indore 1972)

Sol. The given curve is $y = c \cosh (x/c)$. ...(1)

Proceeding as in Ex. 4 (a), the length s of arc extending from the vertex $(0, c)$ to any point (x, y) is given by

$$s = c \sinh (x/c). \qquad ...(2)$$

Squaring and subtracting (2) from (1), we get

$$y^2 - s^2 = c^2 \cosh^2 (x/c) - c^2 \sinh^2 (x/c) = c^2$$

or $\qquad y^2 - c^2 = s^2.$ This was to be proved.

***Ex. 5.** *If A denotes the area between the curve $y = c \cosh (x/c)$ and the two ordinates $x = x_1$ and $x = x_2$ and the x-axis and 's' stands for the length of the intervening arc, then prove that $A = cs$.* **Or**

In the catenary $y = c \cosh (x/c)$, prove that the area between the curve, the axis of x and the ordinates of two points on the curve, varies as the length of the intervening curve.

Sol. As discussed in the previous chapter, the required area

$$A = \int_{x_1}^{x_2} y\, dx = \int_{x_1}^{x_2} c \cosh \left(\frac{x}{c}\right) dx = c \int_{x_1}^{x_2} \cosh \frac{x}{c}\, dx. \qquad ...(1)$$

Also proceeding as in Ex. 4 (a), the required arc length

$$s = \int_{x_1}^{x_2} \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx = \int_{x_1}^{x_2} \sqrt{\left(1 + \sinh^2 \frac{x}{c}\right)} dx$$

$$= \int_{x_1}^{x_2} \cosh \frac{x}{c}\, dx. \qquad ...(2)$$

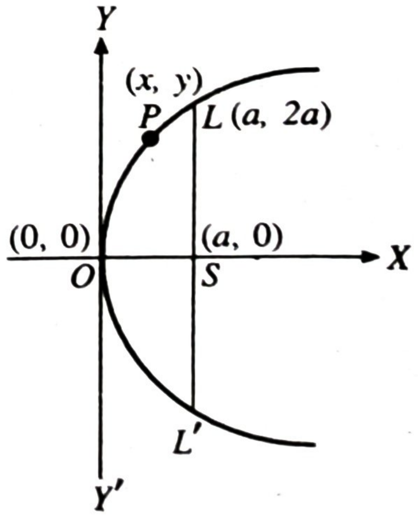
From (1) and (2), we observe that $A = cs$.

****Ex. 6 (a).** *Find the length of the arc of the parabola $y^2 = 4ax$ extending from the vertex to an extremity of the latus rectum.*

(Meerut 1976)

Sol. The given equation of parabola is

$$y^2 = 4ax. \qquad ...(1)$$



The point O (0, 0) is the vertex of the parabola and the point $L\ (a, 2a)$ is an extremity of the latus rectum LSL'. We have to find the length of arc OL. Defferentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \quad dy/dx = 2a/y$

or $\quad dx/dy = y/2a$.

Now $\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2} = \frac{1}{4a^2}(4a^2 + y^2)$. ...(2)

If 's' denotes the length of the parabola measured from the vertex O to any point $P(x, y)$ towards the point L, then s increases as y increases. Therefore ds/dy will be positive. So extracting the square root of (2) and keeping the positive sign, we have

$$\frac{ds}{dy} = \frac{1}{2a}\sqrt{(4a^2 + y^2)}, \quad \text{or} \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\, dy.$$

Let s_1 denote the arc length OL. Then

$$\int_0^{s_1} ds = \int_0^{2a} \frac{1}{2a}\sqrt{(4a^2 + y^2)}\, dy$$

or

$$s_1 = \frac{1}{2a}\left[\frac{y}{2}\sqrt{(4a^2 + y^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^{2a}$$

$$= \frac{1}{2a}[a\sqrt{(4a^2 + 4a^2)} + 2a^2\log\{2a + \sqrt{(8a^2)}\} - 0 - 2a^2\log(2a)]$$

$$= \frac{1}{2a}[2\sqrt{2}a^2 + 2a^2\log\{(2a + 2\sqrt{2}a)/2a\}]$$

$$= \frac{2a^2}{2a}[\sqrt{2} + \log(1 + \sqrt{2})] = a[\sqrt{2} + \log(1 + \sqrt{2})].$$

Ex. 6 (b). *Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.* **(Gurunanak 1975, 74; Kashmir 75)**

Sol. Here we have to find the arc length $L'OL$, which is double of the length found in Ex. 6 (a).

Ex. 7. *Find the length of an arc of the parabola measured from the vertex.*

Sol. Let the given parabola be $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \quad dy/dx = 2a/y \quad$ or $\quad dx/dy = y/2a$.

If s denotes the arc length of the parabola $y^2 = 4ax$ measured from the vertex O to any point $P(x, y)$ lying on the upper half of the parabola, then s increases as y increases.

$\therefore \quad$ the required length of arc

$$s = \int_0^y \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy \qquad \textbf{(Note)}$$

$$= \int_0^y \sqrt{\left(1 + \frac{y^2}{4a^2}\right)}\, dy = \frac{1}{2a}\int_0^y \sqrt{(y^2 + 4a^2)}\, dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{(y^2 + 4a^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(y^2 + 4a^2)}\}\right]_0^y$$

$$= \frac{1}{4a} [y\sqrt{(y^2 + 4a^2)} + 4a^2 \log \{y + \sqrt{(y^2 + 4a^2)}\} - 4a^2 \log 2a]$$

$$= \frac{1}{4a} \left[y\sqrt{(y^2 + 4a^2)} + 4a^2 \log \left\{ \frac{y + \sqrt{(y^2 + 4a^2)}}{2a} \right\} \right].$$

***Ex. 8 (a).** *Find the length of the arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of the latus rectum.*

(Kashmir 1977; Gorakhpur 71; Jiwaji 70; Meerut 85 S)

Sol. The given parabola is $x^2 = 4ay$, ...(1)
whose vertex is the point (0, 0) and whose axis is along the y-axis.

Let s_1 denote the arc length of the parabola measured from the vertex O (0, 0) to an extremity of the latus rectum $(2a, a)$.

Differentiating (1) w.r.t. x, we get

$$2x = 4a\,(dy/dx) \quad \text{or} \quad (dy/dx) = x/2a.$$

$\therefore$ the required arc length $s_1 = \displaystyle\int_0^{2a} \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx$

(Note)

$$= \int_0^{2a} \sqrt{\left\{1 + \frac{x^2}{4a^2}\right\}}\, dx = \frac{1}{2a} \int_0^{2a} \sqrt{(x^2 + 4a^2)}\, dx$$

$$= \frac{1}{2a} \left[\frac{x}{2} \sqrt{(x^2 + 4a^2)} + \frac{4a^2}{2} \log \{x + \sqrt{(x^2 + 4a^2)}\} \right]_0^{2a}$$

$= a\,[\sqrt{2} + \log (1 + \sqrt{2})]$, [proceeding as in Ex. 6 (a)].

Ex. 8 (b). *Find the length of the arc of the parabola $x^2 = 8y$ from the vertex to an extremity of the latus rectum.* **(Agra 1976)**

Sol. Proceed exactly as in Ex. 8 (a). Here $a = 2$.

Ex. 9 (a). *Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $y = 3x$.*

Sol. We have $y^2 = 4ax$ ...(1)
and $y = 3x$. ...(2)

Substituting for x in (1) from (2), we get

$$y^2 = 4a.(y/3) \quad \text{or} \quad 3y^2 - 4ay = 0 \text{ giving } y = 0,\ 4a/3.$$

$\therefore$ from (2) the corresponding values of x are 0 and $4a/9$. Hence the points of intersection of the parabola and the given line are (0, 0) and $(4a/9, 4a/3)$.

Also differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore$ $dy/dx = 2a/y$ or $dx/dy = y/2a$.

If s denotes the arc length of the parabola measured from the point (0, 0) to the point $(4a/9, 4a/3)$, then

$$s = \int_0^{4a/3} \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}\, dy = \int_0^{4a/3} \sqrt{\left[1 + \frac{y^2}{4a^2}\right]}\, dy$$

$$= \frac{1}{2a} \int_0^{4a/3} \sqrt{(y^2 + 4a^2)}\, dy$$

$$= \frac{1}{2a} \left[\frac{y}{2} \sqrt{(y^2 + 4a^2)} + \frac{4a^2}{2} \log \{y + \sqrt{(y^2 + 4a^2)}\} \right]_0^{4a/3}$$

$$= (1/2a) \left[\tfrac{1}{2} \cdot \tfrac{4}{3} a \sqrt{(\tfrac{16}{9} a^2 + 4a^2)} + 2a^2 \log \{\tfrac{4}{3} a + \sqrt{(\tfrac{16}{9} a^2 + 4a^2)}\} - 2a^2 \log (2a) \right]$$

$$= (1/2a) \left[\tfrac{2}{3} a \cdot \tfrac{2}{3} a \sqrt{(13)} + 2a^2 \log \{\tfrac{4}{3} a + \tfrac{2}{3} a \sqrt{(13)}\} - 2a^2 \log (2a) \right]$$

$$= a \left[\tfrac{2}{9} \sqrt{(13)} + \log \{\tfrac{2}{3} + \tfrac{1}{3} \sqrt{(13)}\} \right]$$

$$= a \left[\frac{2}{9} \sqrt{(13)} + \log \left\{ \frac{2 + \sqrt{(13)}}{3} \right\} \right].$$

Ex. 9 (b). *Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y = 8x$ is $a\ (\log 2 + \frac{15}{16})$.*

(Gorakhpur 1977; Jodhpur 7…)

Sol. Solving $y^2 = 4ax$ and $3y = 8x$ for y, we get

$$y^2 = 4a\,.\,(3y/8) \quad \text{or} \quad 2y^2 - 3ay = 0 \quad \text{or} \quad y = 0, 3a/2.$$

Thus the parabola and the straight line intersect at the points where $y = 0$ and $3a/2$. We need not find the x-coordinates of these points.

Also differentiating $y^2 = 4ax$, we get

$2y\,(dy/dx) = 4a$ or $(dy/dx) = (2a/y)$ or $(dx/dy) = y/2a$.

$\therefore$ if s denotes the arc length of the parabola measured in the direction of y increasing, then

$$\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \frac{1}{2a} \sqrt{(4a^2 + y^2)},$$

or

$$ds = \frac{1}{2a} \sqrt{(4a^2 + y^2)}\, dy.$$

Let s_1 denote the required arc length from $y = 0$ to $y = 3a/2$. Then

$$\int_0^{s_1} ds = \int_0^{3a/2} \frac{1}{2a} \sqrt{(4a^2 + y^2)}\, dy$$

or

$$s_1 = \frac{1}{2a} \left[\frac{y}{2} \sqrt{(4a^2 + y^2)} + \frac{4a^2}{2} \log \{y + \sqrt{(4a^2 + y^2)}\} \right]_0^{3a/2}$$

$$= \frac{1}{4a} \left[\frac{3a}{2} \sqrt{\left(\frac{25a^2}{4}\right)} + 4a^2 \log \left\{ \frac{3a}{2} + \sqrt{\left(\frac{25a^2}{4}\right)} \right\} - 4a^2 \log 2a \right]$$

$$= a\,[\tfrac{15}{16} + \log 2].$$

Ex. 10. *Find the perimeter of the curve*

$$x^2 + y^2 = a^2.$$

(Magadh 197…)

Sol. The equation of the curve is $x^2 + y^2 = a^2$. ...(1)

Here the curve is the standard equation of the circle with centre (0, 0) and radius a. Also it is symmetrical about both the axes. So the required perimeter will be four times the arc length lying in the first quadrant *i.e.*, between $x = 0$ to $x = a$.

Differentiating (1), w.r.t. x, we get $2x + 2y\,(dy/dx) = 0$

or $(dy/dx) = -(x/y) = -x/\sqrt{(a^2 - x^2)}$, from (1).

$\therefore$ the required perimeter

$= 4 \times$ {length of the arc in the first quadrant from $(0, a)$ to $(a, 0)$}

$$= 4 \int_{x=0}^{a} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= 4 \int_0^a \sqrt{\left[1 + \frac{x^2}{a^2 - x^2}\right]}\, dx, \text{ putting for } \frac{dy}{dx}$$

$$= 4a \int_0^a \frac{dx}{\sqrt{(a^2 - x^2)}} = 4a \left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$= 4a\,[\sin^{-1}(1) - \sin^{-1}(0)] = 4a\,[\tfrac{1}{2}\pi - 0] = 2a\pi$.

Ex. 11. *Find the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a).*

Sol. The given curve is $ay^2 = x^3$. ...(1)

It is symmetrical about the x-axis. We have to find the length of the arc from $x = 0$ to $x = a$ in the first quadrant.

Differentiating (1) w.r.t. x, we get $2ay\,(dy/dx) = 3x^2$.

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a\,(x^3/a)^{1/2}}, \text{ substituting for } y \text{ from (1)}$$

$= \frac{3}{2}(x^{1/2}/a^{1/2})$.

$\therefore$ required length

$$= \int_{x=0}^{a} \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_0^a \sqrt{\left\{1 + \frac{9x}{4a}\right\}}\, dx = \frac{1}{2\sqrt{a}} \int_0^a (9x + 4a)^{1/2}\, dx$$

$$= \frac{1}{2\sqrt{a}} \left[\frac{2}{3} \cdot (9x + 4a)^{3/2} \cdot \frac{1}{9}\right]_0^a = \frac{1}{27\sqrt{a}} [(13a)^{3/2} - (4a)^{3/2}]$$

$= (a/27) \cdot [13\sqrt{(13)} - 8]$.

Ex. 12. *Show that the length of the arc of the curve $x^2 = a^2(1 - e^{y/a})$ measured from the origin to the point (x, y) is*

$$a \log\{(a + x)/(a - x)\} - x.$$

Sol. The given equation of the curve is $x^2 = a^2(1 - e^{y/a})$

or $\qquad e^{y/a} = 1 - \dfrac{x^2}{a^2}$ or $\dfrac{y}{a} = \log\left(1 - \dfrac{x^2}{a^2}\right)$

or $\qquad y = a \log\left(\dfrac{a^2 - x^2}{a^2}\right)$ or $y = a \log(a^2 - x^2) - a \log a^2$ ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = a \cdot \frac{-2x}{a^2 - x^2}.$$

$\therefore$ required arc length

$$= \int_0^x \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx = \int_0^x \left[1 + \frac{4a^2x^2}{(a^2 - x^2)^2}\right]^{1/2} dx$$

$$= \int_0^x \left(\frac{a^2 + x^2}{a^2 - x^2}\right) dx = \int_0^x \left(-1 + \frac{2a^2}{a^2 - x^2}\right) dx \qquad \text{(Note)}$$

$$= \left[-x + 2a^2 \cdot \frac{1}{2a} \log \frac{a + x}{a - x}\right]_0^x = a \log \frac{a + x}{a - x} - x.$$

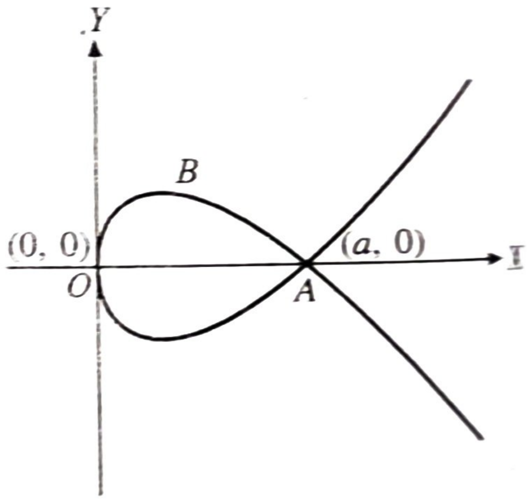
***Ex. 13.** *Prove that the length of the loop of the curve* $3ay^2 = x(x - a)^2$ *is* $4a/\sqrt{3}$.

(Meerut 1984 S, 92; Delhi 83; Rohilkhand 83; Magadh 77, 75)

Sol. The given curve is symmetrical about the x-axis.

At $y = 0$, we have $x = 0$ and $x = a$. So for the loop, x varies from 0 to a. The equation of the given curve is

$$3ay^2 = x(x - a)^2.$$



Taking logarithm, we have

$$\log 3a + 2 \log y = \log x + 2 \log (x - a).$$

Now differentiating w.r.t x, we get

$$\frac{2}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2}{x - a} = \frac{3x - a}{x(x - a)}.$$

Thus $\qquad \left(\dfrac{dy}{dx}\right)^2 = \dfrac{(3x - a)^2}{x^2 (x - a)^2} \cdot \dfrac{y^2}{4}$

$$= \frac{(3x - a)^2 x (x - a)^2}{x^2 (x - a)^2 . 12a} = \frac{(3x - a)^2}{12ax}.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{(3x - a)^2}{12ax}\right\}} = \frac{3x + a}{\sqrt{(12ax)}}.$$

$\therefore$ the required length of the whole loop

$$= 2\int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx \qquad \text{(by symmetry)}$$

$$= 2\int_0^a \frac{3x + a}{\sqrt{(12ax)}}\, dx = \frac{1}{\sqrt{(3a)}} \int_0^a (3x^{1/2} + ax^{-1/2})\, dx$$

$$= \frac{1}{\sqrt{(3a)}} \left[3 \cdot \frac{2}{3} x^{3/2} + 2ax^{1/2}\right]_0^a = \frac{1}{\sqrt{(3a)}} [4a^{3/2}] = \frac{4a}{\sqrt{3}}.$$

Ex. 14. *Find the perimeter of the loop of the curve*

$$9ay^2 = (x - 2a)(x - 5a)^2.$$

Sol. The given equation of the curve is

$$9ay^2 = (x - 2a)(x - 5a)^2. \qquad ...(1)$$

Shifting the origin to the point $(2a, 0)$, the equation (1) becomes $9ay^2 = x(x - 3a)^2$. Now this is similar to the curve of Ex. 13. (Here we have $3a$ in place of a). So to find the required length proceed exactly as in Ex. 13. The required length is

$$= 2\int_0^{3a} \frac{x + a}{\sqrt{(4ax)}}\, dx = 4a\sqrt{3}.$$

Ex. 15. *Find the perimeter of the loop of the curve*

$$3ay^2 = x^2(a - x). \qquad \textbf{(Meerut 1997)}$$

Sol. The given curve is

$$3ay^2 = x^2(a - x). \qquad ...(1)$$

Here the curve is symmetrical about the x-axis. Putting $y = 0$, we get $x = 0, x = a$. So the loop lies between $x = 0$ and $x = a$. Differentiating (1) w.r.t. x, we get

$$6ay\frac{dy}{dx} = 2ax - 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{x(2a - 3x)}{6ay}.$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2(2a - 3x)^2}{36a^2y^2} = 1 + \frac{x^2(2a - 3x)^2}{12a\,x^2(a - x)},$$

substituting for $3ay^2$ **from (1)**

$$= 1 + \frac{(2a - 3x)^2}{12a(a - x)} = \frac{12a^2 - 12ax + (2a - 3x)^2}{12a(a - x)} = \frac{(4a - 3x)^2}{12a(a - x)}.$$

$\therefore$ the required length of the loop

= twice the length of the half loop lying above the x-axis,

(by symmetry)

$$= 2\int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = 2\int_0^a \sqrt{\left\{\frac{(4a - 3x)^2}{12a(a - x)}\right\}}\, dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a \frac{(4a - 3x)}{\sqrt{(a - x)}}\, dx = \frac{1}{\sqrt{(3a)}} \int_0^a \frac{3(a - x) + a}{\sqrt{(a - x)}}\, dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a \left[\frac{3(a-x)}{\sqrt{(a-x)}} + \frac{a}{\sqrt{(a-x)}} \right] dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a [3\sqrt{(a-x)} + a(a-x)^{-1/2}] \, dx$$

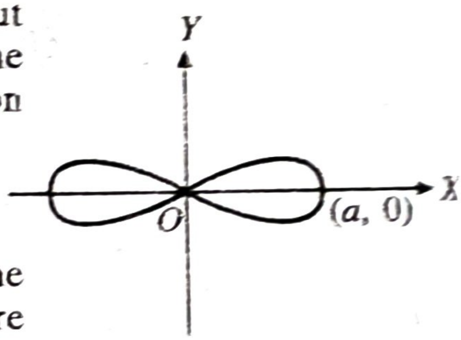
$$= \frac{1}{\sqrt{(3a)}} \left[-3 \cdot \frac{2}{3} (a-x)^{3/2} - a \cdot 2(a-x)^{1/2} \right]_0^a$$

$$= \frac{1}{\sqrt{(3a)}} \left[2a^{3/2} + 2a^{3/2} \right] = \frac{4a}{\sqrt{3}}.$$

Ex. 16. *Show that the whole length of the curve*

$$x^2 (a^2 - x^2) = 8a^2 y^2 \text{ is } \pi a \sqrt{2}.$$ **(Ranchi 1973; Meerut 98)**

Sol. The given curve is $x^2 (a^2 - x^2) = 8a^2 y^2$. ...(1)



The curve (1) is symmetrical about both the axes and it passes through the origin. Putting $y = 0$ in the given equation of the curve, we get

$$x^2 (a^2 - x^2) = 0$$

i.e., $x = 0, x = \pm a$.

So the curve passes through the points $(0, 0)$, $(a, 0)$ and $(-a, 0)$. Therefore for one loop x varies from 0 to a.

$\therefore$ the required whole length of the curve

$= 4 \times$ length of the half loop (lying above x-axis)

$$= 4 \int_0^a \sqrt{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}} \, dx. \qquad ...(2)$$

Now differentiating (1) w.r.t. x, we get

$$16a^2 y \frac{dy}{dx} = 2a^2 x - 4x^3 \quad \text{or} \quad \frac{dy}{dx} = \frac{(a^2 - 2x^2)}{8a^2 y} x.$$

$$\therefore \quad 1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{x^2 (a^2 - 2x^2)^2}{64a^4 y^2} = 1 + \frac{x^2 (a^2 - 2x^2)^2}{8a^2 x^2 (a^2 - x^2)},$$

substituting for $8a^2 y^2$ from (1)

$$= 1 + \frac{(a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)} = \frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}.$$

$\therefore$ From (2), the required whole length of the curve

$$= 4 \int_0^a \sqrt{\left\{ \frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)} \right\}} \, dx = \frac{2}{a\sqrt{2}} \int_0^a \frac{(3a^2 - 2x^2)}{\sqrt{(a^2 - x^2)}} \, dx$$

$$= \frac{\sqrt{2}}{a} \int_0^a \left[\frac{2(a^2 - x^2)}{\sqrt{(a^2 - x^2)}} + \frac{a^2}{\sqrt{(a^2 - x^2)}} \right] dx$$ **(Note)**

$$= \frac{\sqrt{2}}{a} \int_0^a 2\sqrt{(a^2 - x^2)}\, dx + \frac{\sqrt{2}}{a} \int_0^a \frac{a^2}{\sqrt{(a^2 - x^2)}}\, dx$$

$$= \frac{2\sqrt{2}}{a} \left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a + \sqrt{2}.a \left[\sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{2\sqrt{2}}{a} \left[0 + \frac{a^2}{2} \cdot \frac{\pi}{2} \right] = \sqrt{2}.a.\frac{\pi}{2} = \pi a \sqrt{2}.$$

*Ex. 17 (a). *Find the length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.*

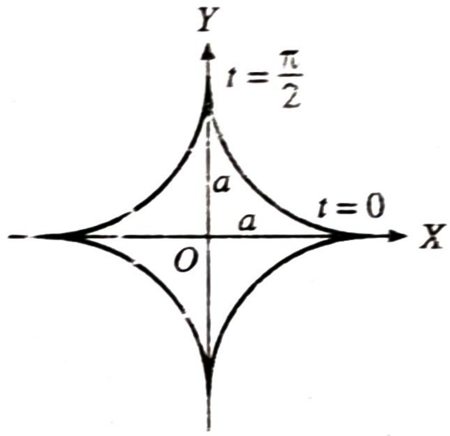
(Ranchi 1975; G.N.U. 74; Magadh 74; Agra 70; Meerut 87, 93 P. 95, 98)

Sol. The given astroid is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The curve is symmetrical in all the four quadrants. For the arc of the curve in the first quadrant x varies from 0 to a. Differentiating (1), w.r.t. x, we get

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

so that $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{1/3}$.



$\therefore$ the required whole length of the curve

= 4 × length of the curve lying in the Ist quadrant

$$= 4 \int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = 4 \int_0^a \sqrt{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)}\, dx$$

$$= 4 \int_0^a \frac{\sqrt{(x^{2/3} + y^{2/3})}}{x^{1/3}}\, dx = 4 \int_0^a \frac{\sqrt{(a^{2/3})}}{x^{1/3}}\, dx$$

$$= 4a^{1/3} \int_0^a x^{-1/3}\, dx = 4a^{1/3} \left[\frac{3}{2} x^{2/3} \right]_0^a = 6a.$$

Ex. 17 (b). *Prove that the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from $(0, a)$ to the point (x, y) is given by $s = \frac{3}{2}(ax^2)^{1/3}$.*

Sol. Proceeding as in Ex. 17 (a), the arc length s of the astroid from $x = 0$ to $x = x$ is obtained as

$$s = \int_0^x a^{1/3} x^{-1/3}\, dx = \left[a^{1/3} \frac{3}{2} x^{2/3} \right]_0^x = \frac{3}{2} (ax^2)^{1/3}.$$

Examples on rectification (parametric equations).

Ex. 18 (a). *Find the whole length of the curve (astroid)*

$x = a \cos^3 t,\ y = a \sin^3 t$. **(Gorakhpur 1977; Meerut 94 P)**

Sol. The given parametric equations of the astroid are

$$x = a \cos^3 t, y = a \sin^3 t. \qquad ...(1)$$

The shape of the curve is as shown in the figure of Ex. 17 (a).

We have $dx/dt = -3a\cos^2 t \sin t$, $dy/dt = 3a \sin^2 t \cos t$.

The astroid is symmetrical about both the axes. For the arc of the astroid lying in the first quadrant, we have $t = 0$ at the point $(a, 0)$ and $t = \pi/2$ at the point $(0, a)$.

Now $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$
$$= 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t$$
$$= 9a^2 \sin^2 t \cos^2 t\,(\sin^2 t + \cos^2 t)$$
$$= (3a \sin t \cos t)^2. \qquad ...(2)$$

If s denotes the arc length of the astroid measured from the point $t = 0$ to any point P towards the point $t = \pi/2$, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (2), we have

$$ds/dt = 3a \sin t \cos t \quad \text{or} \quad ds = 3a \sin t \cos t\, dt.$$

Let s_1 denote the arc length of the astroid lying in the first quadrant. Then

$$\int_0^{s_1} ds = \int_0^{\pi/2} 3a \sin t \cos t\, dt \quad \text{or} \quad s_1 = 3a\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3a}{2}.$$

Whole length of the curve

$= 4 \times$ length of the curve lying in the Ist quadrant $= 4.(3a/2) = 6a$.

Ex. 18 (b). *Find the length of the curve*

$$x = \frac{c^2}{a}\cos^3 t, y = \frac{c^2}{a}\sin^3 t$$

which is the evolute of an ellipse. **(Meerut 1995)**

Sol. Proceed as in Ex. 18 (a). Replace a by c^2/a.

The required length $= 6\,(c^2/a) = 6c^2/a$.

Ex. 18 (c). *Find the whole length of the curve (Hypocycloid)*

$$x = a\cos^3 t,\; y = b \sin^3 t.$$ **(Meerut 1985 P)**

Sol. The given curve is similar to that of Ex. 18 (a) *i.e.*, the curve is symmetrical about both the axes and in the first quadrant t varies from 0 to $\frac{1}{2}\pi$.

Here $dx/dt = -3a\cos^2 t \sin t$, $dy/dt = 3b \sin^2 t \cos t$.

$\therefore$ the required whole length of the curve

$= 4 \times$ length of the curve in the first quadrant

$$= 4\int_0^{\pi/2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt$$

$$= 4\int_0^{\pi/2} \sqrt{\{9a^2 \cos^4 t \sin^2 t + 9b^2 \sin^4 t \cos^2 t\}}\, dt$$

$$= 4\int_0^{\pi/2} 3 \sin t \cos t \sqrt{(a^2\cos^2 t + b^2 \sin^2 t)}\, dt.$$

Now put $a^2 \cos^2 t + b^2 \sin^2 t = z^2$, so that

$$(-2a^2 \sin t \cos t + 2b^2 \sin t \cos t)\, dt = 2z\, dz.$$

$\therefore$ $\sin t \cos t\, dt = \{z/(b^2 - a^2)\}\, dz.$

Also $z = a$ when $t = 0$ and $z = b$ when $t = \pi/2$.

Hence the required length

$$= 12 \int_a^b z \cdot \frac{z\, dz}{b^2 - a^2} = \frac{12}{b^2 - a^2} \int_a^b z^2\, dz$$

$$= \frac{12}{b^2 - a^2} \left[\frac{z^3}{3}\right]_a^b = 4 \frac{b^3 - a^3}{b^2 - a^2} = 4 \cdot \frac{(b^2 + ab + a^2)}{b + a}.$$

Ex. 18 (d). *Find the length of one quadrant of the curve*

$$(x/a)^{2/3} + (y/b)^{2/3} = 1.$$ **(Delhi 1976)**

Sol. The parametric equations of the given curve are

$$x = a \cos^3 t,\ y = b \sin^3 t.$$

Now proceed as in Ex. 18 (c).

****Ex. 19.** *Show that 8a is the length of an arch of the cycloid whose equations are*

$$x = a\,(t - \sin t), y = a\,(1 - \cos t).$$ **(Meerut 1984, 95 BP, 96)**

Sol. The given equations of the cycloid are

$$x = a\,(t - \sin t), y = a\,(1 - \cos t).$$

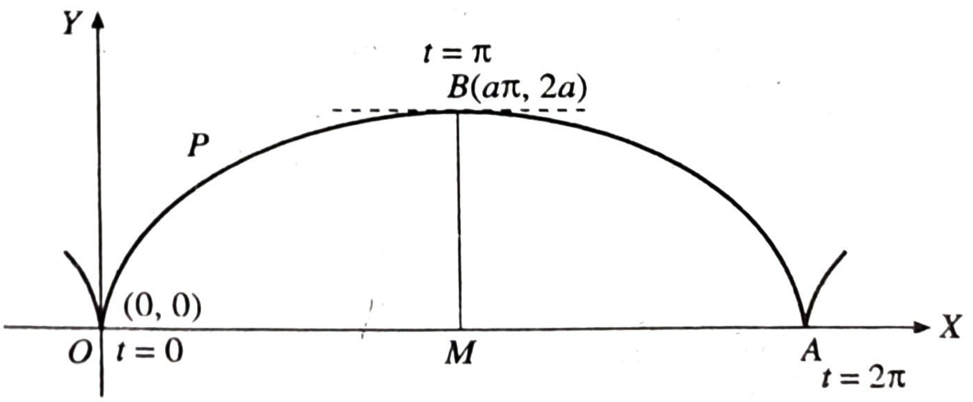
We have $dx/dt = a\,(1 - \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a\,(1 - \cos t)} = \frac{2 \sin \frac{1}{2}t \cos \frac{1}{2}t}{2 \sin^2 \frac{1}{2}t} = \cot \tfrac{1}{2}t.$$

Now $y = 0$ when $\cos t = 1$ *i.e.*, $t = 0$. At $t = 0, x = 0$, $y = 0$ and $dy/dx = \infty$. Thus the curve passes through the point $(0, 0)$ and the tangent there is perpendicular to the x-axis.

Again y is maximum when $\cos t = -1$ *i.e.*, $t = \pi$. When $t = \pi$, $x = a\pi$, $y = 2a$, $dy/dx = 0$. Thus at the point $(a\pi, 2a)$ the tangent to the curve is parallel to the x-axis.

Also in this curve y cannot be negative. Thus an arch *OBA* of the given cycloid is as shown in the figure. It is symmetrical about the line *BM* which is the axis of the cycloid.



We have $\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$

$= \{a(1 - \cos t)\}^2 + (a \sin t)^2$

$= a^2 \{(2 \sin^2 \frac{1}{2} t)^2 + (2 \sin \frac{1}{2} t \cos \frac{1}{2} t)^2\}$

$= 4a^2 \sin^2 \frac{1}{2} t(\sin^2 \frac{1}{2} t + \cos^2 \frac{1}{2} t) = 4a^2 \sin^2 \frac{1}{2} t.$ ...(1)

If s denotes the arc length of the cycloid measured from the cusp O to any point P towards the vertex B, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (1), we have $ds/dt = 2a \sin \frac{1}{2} t$, or $ds = 2a \sin \frac{1}{2} t \, dt$.

At the cusp O, $t = 0$, and at the vertex B, $t = \pi$.

Now the length of the arch $OBA = 2 \times$ length of the arc OB

$= 2 \int_0^{\pi} 2a \sin \frac{1}{2} t \, dt = 4a \left[- 2 \cos \frac{1}{2} t\right]_0^{\pi} = - 8a \left[\cos \frac{1}{2} t\right]_0^{\pi}$

$= - 8a [0 - 1] = 8a.$

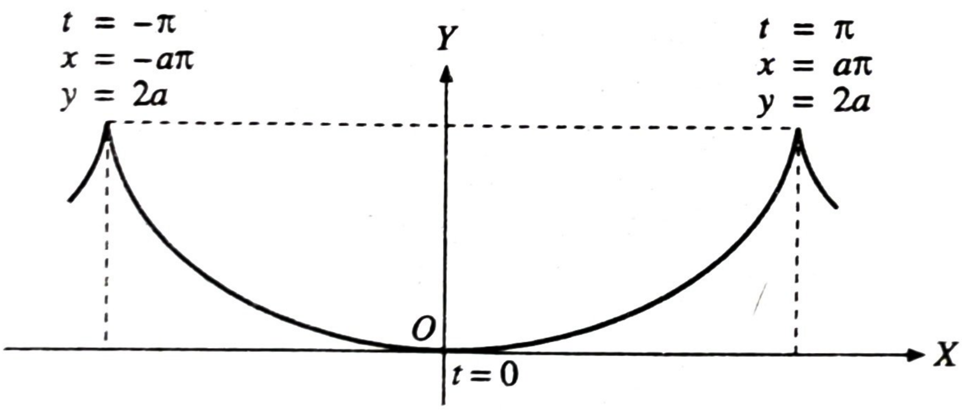
***Ex. 20.** *Rectify the curve or find the length of an arch of the curve*

$$x = a(t + \sin t), y = a(1 - \cos t).$$

(Meerut 1982 P, 84, 86, 86P, 88 S; Kanpur 77; G.N.U. 75; Indore 72)

Sol. Differentiating the given parametric equations of the cycloid w.r.t. t, we have

$$dx/dt = a(1 + \cos t), \text{ and } dy/dt = a \sin t.$$



If we measure the arc length s in the direction of t increasing, we have

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$= \sqrt{\{a^2 (1 + \cos t)^2 + a^2 \sin^2 t\}}$

$= a \{1 + \cos^2 t + 2 \cos t + \sin^2 t\}^{1/2}$

$= a\sqrt{2} (1 + \cos t)^{1/2} = a\sqrt{2} (2 \cos^2 \frac{1}{2} t)^{1/2} = 2a \cos \frac{1}{2} t.$

$\therefore \quad ds = 2a \cos \frac{1}{2} t \, dt.$

For an arch of the given cycloid lying between two successive cusps t varies from $-\pi$ to π. Also this arch is symmetrical about the y-axis and we have $t=0$ at the vertex O.

$\therefore$ the required whole length of the arch

$= 2 \times$ length of the arc from $t=0$ to $t=\pi$

$$= 2\int_0^{\pi} 2a \cos \tfrac{1}{2}t \, dt = 4a \int_0^{\pi} \cos \tfrac{1}{2}t \, dt = 4a \left[2 \sin \tfrac{1}{2}t\right]_0^{\pi}$$

$$= 4a\,[2-0] = 8a.$$

Ex. 21. *Prove that the length of an arc of the cycloid*

$$x = a\,(t + \sin t), y = a\,(1-\cos t)$$

from the vertex to the point (x, y) *is* $\sqrt{(8ay)}$.

Sol. Let s denote the arc length of the cycloid measured from the vertex to any point P (*i.e.*, from $t=0$ to $t=t$).

Then proceeding as in Ex. 20, the required arc length

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt = 2a \int_0^t \cos \tfrac{1}{2}t \, dt$$

$$= 2a \left[2 \sin \tfrac{1}{2}t\right]_0^t = 4a \sin \tfrac{1}{2}t = \sqrt{(8a^2 . 2 \sin^2 \tfrac{1}{2}t)}$$

$$= \sqrt{\{8a . a\,(1-\cos t)\}} = \sqrt{(8ay)}, \qquad [\because y = a\,(1-\cos t)].$$

Ex. 22. *Find the length of the loop of the curve*

$$x = t^2, y = t - \tfrac{1}{3}t^3.$$

Sol. Eliminating the parameter t from $x = t^2$ and $y = t - \frac{1}{3}t^3$, we get $y^2 = x\,(1-\frac{1}{3}x)^2$ as the cartesian equation of the curve and hence we observe that the curve is symmetrical about the x-axis. The loop of the curve extends from the point $(0, 0)$ to the point $(3, 0)$. Putting $y=0$ in $y = t - \frac{1}{3}t^3$, we get $t=0$ and $t=\sqrt{3}$. Therefore the arc of the upper half of the loop extends from $t=0$ to $t=\sqrt{3}$.

Now the required length of the loop

$= 2 \times$ length of the half of the loop which lies above x-axis

$$= 2\int_0^{\sqrt{3}} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{\{(2t)^2 + (1 - \tfrac{1}{3} \cdot 3t^2)^2\}}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(1 + 2t^2 + t^4)}\, dt = 2\int_0^{\sqrt{3}} (1+t^2)\, dt$$

$$= 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\,[\sqrt{3} + \sqrt{3}] = 4\sqrt{3}.$$

Ex. 23. *Find the length of the arc of the curve*

$$x = e^t \sin t, y = e^t \cos t, \text{ from } t = 0 \text{ to } t = \tfrac{1}{2}\pi.$$

(Rohilkhand 1982; Kanpur 75)

Sol. The given equations of the curve are
$$x = e^t \sin t, y = e^t \cos t.$$
Differentiating w.r.t. t, we have
$$dx/dt = e^t \cos t + e^t \sin t = e^t (\cos t + \sin t)$$
and
$$dy/dt = e^t (-\sin t) + e^t \cos t = e^t (\cos t - \sin t).$$
$\therefore$ the required arc length $= \displaystyle\int_0^{\pi/2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\right\}} dt$

$= \int_0^{\pi/2} \sqrt{\{e^{2t} (\cos t + \sin t)^2 + e^{2t} (\cos t - \sin t)^2\}}\, dt$

$= \int_0^{\pi/2} e^t \sqrt{\{2 (\cos^2 t + \sin^2 t)\}}\, dt = \sqrt{2} \int_0^{\pi/2} e^t\, dt = \sqrt{2} \left[e^t\right]_0^{\pi/2}$

$= \sqrt{2}\, [e^{\pi/2} - 1].$

Ex. 24. *Find the length of the curve* $x = e^t (\sin \frac{1}{2} t + 2 \cos \frac{1}{2} t)$, $y = e^t (\cos \frac{1}{2} t - 2 \sin \frac{1}{2} t)$ *measured from* $t = 0$ *to* $t = \pi$.

Sol. Differentiating the given parametric equations w.r.t. t, we get
$$dx/dt = e^t (\tfrac{1}{2} \cos \tfrac{1}{2} t - \sin \tfrac{1}{2} t) + e^t (\sin \tfrac{1}{2} t + 2 \cos \tfrac{1}{2} t) = \tfrac{5}{2} e^t \cos \tfrac{1}{2} t$$
and $dy/dt = e^t (-\frac{1}{2} \sin \frac{1}{2} t - \cos \frac{1}{2} t) + e^t (\cos \frac{1}{2} t - 2 \sin \frac{1}{2} t)$
$$= -\tfrac{5}{2} e^t \sin \tfrac{1}{2} t.$$
$\therefore$ the required length $= \displaystyle\int_0^{\pi} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt$

$= \int_0^{\pi} \sqrt{\{(\frac{5}{2} e^t \cos \frac{1}{2} t)^2 + (-\frac{5}{2} e^t \sin \frac{1}{2} t)^2\}}\, dt$

$= \frac{5}{2} \int_0^{\pi} e^t \sqrt{(\cos^2 \frac{1}{2} t + \sin^2 \frac{1}{2} t)}\, dt = \frac{5}{2} \int_0^{\pi} e^t\, dt = \frac{5}{2} \left[e^t\right]_0^{\pi}$

$= \frac{5}{2} [e^{\pi} - 1].$

Ex. 25. *Show that the length of the arc measured from* $t = 0$ *to any point* t, *of the curve*
$$x = a (\cos t + t \sin t), y = a (\sin t - t \cos t) \text{ is } \tfrac{1}{2} a t^3.$$ **(Agra 1979)**

Sol. Here $dx/dt = a [-\sin t + (1 . \sin t + t \cos t)] = a (t \cos t)$
and $dy/dt = a [\cos t - (1 . \cos t - t . \sin t)] = a (t \sin t)$.

$\therefore$ the required length $= \displaystyle\int_0^{t} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} dt$

$= \int_0^t at \sqrt{(\sin^2 t + \cos^2 t)}\, dt = \int_0^t at\, dt = \frac{1}{2} at^2.$

Ex. 26. *Prove that the arc of the curve given by*
$$x = a (3 \sin t - \sin^3 t), y = a \cos^3 t$$
measured from $(0, a)$ *to any point* (x, y) *is* $\frac{3}{2} a (t + \sin t \cos t)$.

Sol. Proceed exactly as in Ex. 25. Here $t = 0$ at the point $(0, a)$ and $t = t$ at the point (x, y).

***Ex. 27.** *Show that the length of an arc of the curve* $x \sin t + y \cos t = f'(t), x \cos t - y \sin t = f''(t)$ *is given by* $s = f(t) + f''(t)$, *where c is the constant of integration.*

(Kanpur 1976; Agra 75; Lucknow 71)

Sol. The given equations of the curve are

$$x \sin t + y \cos t = f'(t) \qquad \text{...(1)}$$

and

$$x \cos t - y \sin t = f''(t). \qquad \text{...(2)}$$

Multiplying (1) by $\sin t$ and (2) by $\cos t$ and adding, we get

$$x(\sin^2 t + \cos^2 t) = \sin t . f'(t) + \cos t . f''(t)$$

or

$$x = \sin t f'(t) + \cos t f''(t). \qquad \text{...(3)}$$

Again, multiplying (1) by $\cos t$ and (2) by $\sin t$ and subtracting, we get

$$y = \cos t f'(t) - \sin t f''(t). \qquad \text{...(4)}$$

Now differentiating (3) and (4) w.r.t. t, we get

$$dx/dt = \cos t f'(t) + \sin t f''(t) + \cos t f'''(t) - \sin t f''(t)$$
$$= [f'(t) + f'''(t)] \cos t$$

and

$$dy/dt = -[f'(t) + f'''(t)] \sin t.$$

Now if s be the arc length in the direction of t increasing, then

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{[\cos^2 t \{f'(t) + f'''(t)\}^2 + \sin^2 t \{f'(t) + f'''(t)\}^2]}$$
$$= [f'(t) + f'''(t)] \sqrt{(\cos^2 t + \sin^2 t)} = f'(t) + f'''(t).$$

Integrating both sides, we have

$$s = \int [f'(t) + f'''(t)] \, dt + c$$
$$= f(t) + f''(t) + c, \text{ where } c \text{ is the constant of integration.}$$

Ex. 28. *In the ellipse* $x = a \cos \phi, y = b \sin \phi$, *show that*

$$ds = a\sqrt{(1 - e^2 \cos^2 \phi)} \, d\phi,$$

and hence show that the whole length of the ellipse is

$$2\pi a \left[1 - \left(\frac{1}{2}\right)^2 \cdot \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \cdot \frac{e^6}{5} - \dots\right],$$

where e is the eccentricity of the ellipse.

Sol. The given equations of the ellipse are

$$x = a \cos \phi, y = b \sin \phi.$$

We have $\quad dx/d\phi = -a \sin \phi, dy/d\phi = b \cos \phi$.

If we measure the length s in the direction of ϕ increasing,

$$\frac{ds}{d\phi} = \sqrt{\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}}$$
$$= \sqrt{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)}$$

$$= \sqrt{\{a^2 \sin^2 \phi + a^2 (1 - e^2) \cos^2 \phi\}},$$

[$\because$ for ellipse, $b^2 = a^2 (1 - e^2)$]

$$= a \sqrt{(1 - e^2 \cos^2 \phi)}.$$

$\therefore$ $$ds = a \sqrt{(1 - e^2 \cos^2 \phi)}\, d\phi. \quad ...(1)$$

Also the ellipse is symmetrical about both the axes and in the first quadrant ϕ varies from 0 to $\frac{1}{2}\pi$. Therefore whole length of the ellipse $= 4 \times$ length of the ellipse lying in the first quadrant

$$= 4 \int_0^{\pi/2} a \sqrt{(1 - e^2 \cos^2 \phi)}\, d\phi, \quad \text{[from (1)]}$$

$$= 4a \int_0^{\pi/2} (1 - e^2 \cos^2 \phi)^{1/2}\, d\phi$$

$$= 4a \int_0^{\pi/2} \left[1 - \tfrac{1}{2} e^2 \cos^2 \phi - \frac{1}{2.4} e^4 \cos^4 \phi - \frac{1.3}{2.4.6} e^6 \cos^6 \phi - ...\right] d\phi,$$

(on expanding by binomial theorem)

$$= 4a \left[\int_0^{\pi/2} 1 . d\phi - \frac{e^2}{2} \int_0^{\pi/2} \cos^2 \phi\, d\phi - \frac{e^4}{2.4} \int_0^{\pi/2} \cos^4 \phi\, d\phi - \frac{1.3}{2..4.6} e^6 \int_0^{\pi/2} \cos^6 \phi\, d\phi - ...\right]$$

$$= 4a \left[\frac{\pi}{2} - \frac{e^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{e^4}{2.4} \frac{3.1}{4.2} \cdot \frac{\pi}{2} - \frac{1.3}{2.4.6} e^6 \cdot \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2} - ...\right]$$

$$= 2a\pi \left[1 - \frac{1}{2^2} e^2 - \frac{1^2.3}{2^2.4^2} e^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2} e^6 - ...\right]$$

$$= 2\pi a \left[1 - \left(\frac{1}{2}\right)^2 \cdot \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \cdot \frac{e^6}{5} - ...\right].$$

Ex. 29. *Show that in the epi-cycloid for which*

$$x = (a + b) \cos \theta - b \cos \{(a + b)/b\} \theta,$$
$$y = (a + b) \sin \theta - b \sin \{(a + b)/b\} \theta,$$

the length of the arc measured from the point $\theta = \pi b/a$ is

$$\{4b (a + b)/a\} \cos \{(a/2b) \theta\}.$$

Sol. Differentiating the given equations of the epicycloid w.r.t. θ, we have

$$dx/d\theta = (a + b) (- \sin \theta) - b [- \sin \{(a + b)/b\} \theta] . [(a + b)/b]$$
$$= - (a + b) [\sin \theta - \sin \{(a + b)/b\} \theta]$$

and $$dy/d\theta = (a + b) (\cos \theta) - b [\cos \{(a + b)/b\} \theta] . \{(a + b)/b\}$$
$$= (a + b) [\cos \theta - \cos \{(a + b)/b\} \theta].$$

We have

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= (a+b)^2 \left[\left\{\sin\theta - \sin\frac{a+b}{b}\theta\right\}^2 + \left\{\cos\theta - \cos\frac{a+b}{b}\theta\right\}^2\right]$$

$$= (a+b)^2 \left[\sin^2\theta + \sin^2\frac{a+b}{b}\theta - 2\sin\theta\sin\frac{a+b}{b}\theta \right.$$
$$\left. + \cos^2\theta + \cos^2\frac{a+b}{b}\theta - 2\cos\theta\cos\frac{a+b}{b}\theta\right]$$

$$= (a+b)^2 \left[1 + 1 - 2\left(\sin\theta\sin\frac{a+b}{b}\theta + \cos\theta\cos\frac{a+b}{b}\theta\right)\right]$$

$$= 2(a+b)^2\left[1 - \cos\frac{a}{b}\theta\right] = 4(a+b)^2\sin^2\frac{a}{2b}\theta. \qquad ...(1)$$

If s denotes the arc length of the epicycloid measured from the point $\theta = \pi b/a$ to the point $\theta = \theta$ in the direction of θ decreasing, then s increases as θ decreases. Therefore $ds/d\theta$ will be negative. So taking square root of both sides of (1) and keeping the negative sign, we have

$$\frac{ds}{d\theta} = -2(a+b)\sin\frac{a}{2b}\theta \qquad \textbf{(Note)}$$

or
$$ds = -2(a+b)\sin(a/2b)\,\theta\,d\theta.$$

The required length s is now given by

$$s = -\int_{b\pi/a}^{\theta} 2(a+b)\sin(a/2b)\,\theta\,d\theta$$

$$= -2(a+b)\cdot\frac{2b}{a}\left[-\cos\frac{a}{2b}\theta\right]_{b\pi/a}^{\theta}$$

$$= \frac{4b(a+b)}{a}\left[\cos\frac{a\theta}{2b} - \cos\frac{\pi}{2}\right] = \frac{4b(a+b)}{a}\cos\frac{a}{2b}\theta.$$

***Ex. 30.** *An ellipse of small eccentricity has its perimeter equal to that of a circle of radius r. Show that its area is*

$$\pi r^2\left(1 - \tfrac{3}{32}e^4\right) \textit{ nearly.}$$

Sol. The parametric equations of the ellipse are

$$x = a\cos\phi, y = b\sin\phi.$$

As proved in Ex. 28, (prove here), we have the perimeter of the ellipse

$$= 2a\pi\left[1 - \left(\frac{1}{2}\right)^2 e^2 - \left(\frac{1.3}{2.4}\right)^2\cdot\frac{e^4}{3} - ...\right]$$

$$= 2a\pi\left[1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - ...\right] = 2\pi r \qquad \text{(as given)}$$

Therefore $a = r\left[1 - \left(\tfrac{1}{4}e^2 + \tfrac{3}{64}e^4\right)\right]^{-1}$, ...(1)

retaining upto e^4 because e is small.

Now area of the ellipse

$= \pi ab = \pi a . a\sqrt{(1-e^2)}$, $\qquad [\because b^2 = a^2(1-e^2)]$

$$= \pi a^2 (1 - e^2)^{1/2} = \pi r^2 \left[1 - \left(\frac{1}{4} e^2 + \frac{3}{64} e^4\right)\right]^{-2} \cdot (1 - e^2)^{1/2},$$

from (1)

$$= \pi r^2 [1 + 2 \cdot (\tfrac{1}{4} e^2 + \tfrac{3}{64} e^4) + 3 (\tfrac{1}{4} e^2 + \tfrac{3}{64} e^4)^2 + \ldots]$$

$\times (1 - \frac{1}{2} e^2 - \frac{1}{8} e^4 - \ldots)$, expanding by binomial theorem

$$= \pi r^2 (1 + \tfrac{1}{2} e^2 + \tfrac{3}{32} e^4 + \tfrac{3}{16} e^4)(1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4),$$

retaining terms only upto e^4

$$= \pi r^2 (1 + \tfrac{1}{2} e^2 + \tfrac{9}{32} e^4)(1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4)$$

$$= \pi r^2 (1 - \tfrac{1}{2} e^2 - \tfrac{1}{8} e^4 + \tfrac{1}{2} e^2 - \tfrac{1}{4} e^4 + \tfrac{9}{32} e^4)$$

$$= \pi r^2 (1 - \tfrac{3}{32} e^4) \text{ nearly.}$$

Examples on Rectification (Polar equations).

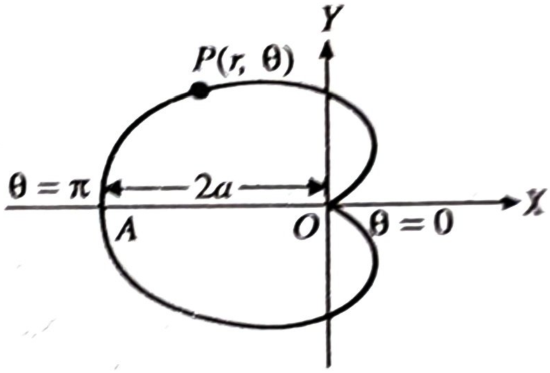
****Ex. 31.** *Find the perimeter of the cardioid $r = a(1 - \cos\theta)$.*

(Meerut 1983, 96 P, 97; Rohilkhand 77, 76; Bhopal 73; Kumayun 83; Kashmir 83)

Sol. The given curve is $r = a(1 - \cos\theta)$ ...(1)

It is symmetrical about the initial line.

We have $r = 0$ when $\cos\theta = 1$ *i.e.,* $\theta = 0$. Also r is maximum when $\cos\theta = -1$ *i.e.,* $\theta = \pi$ and then $r = 2a$. As θ increases from 0 to π, r increases from 0 to $2a$. So the curve is as shown in the figure.



By symmetry, the perimeter of the cardioid

= 2 × the arc length of the upper half of the cardioid.

Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta = a \sin\theta.$$

We have $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = a^2 (1 - \cos\theta)^2 + a^2 \sin^2\theta$

$$= a^2 (2 \sin^2 \tfrac{1}{2}\theta)^2 + a^2 (2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta)^2$$

$$= 4a^2 \sin^2 \tfrac{1}{2}\theta (\sin^2 \tfrac{1}{2}\theta + \cos^2 \tfrac{1}{2}\theta)$$

$$= 4a^2 \sin^2 \tfrac{1}{2}\theta \qquad \text{...(2)}$$

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.,* the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, then s increases as θ increases. Therefore $ds/d\theta$ will be positive.

Hence from (2), we have

$$ds/d\theta = 2a \sin \tfrac{1}{2}\theta, \quad \text{or} \quad ds = 2a \sin \tfrac{1}{2}\theta \, d\theta. \qquad ...(3)$$

At the cusp O, $\theta = 0$ and at the vertex A, $\theta = \pi$.

$\therefore$ the length of the arc $OPA = \int_0^\pi 2a \sin \tfrac{1}{2}\theta \, d\theta$

$$= 4a \left[-\cos \frac{\theta}{2}\right]_0^\pi = -4a \left[\cos \frac{\theta}{2}\right]_0^\pi = -4a(0-1) = 4a.$$

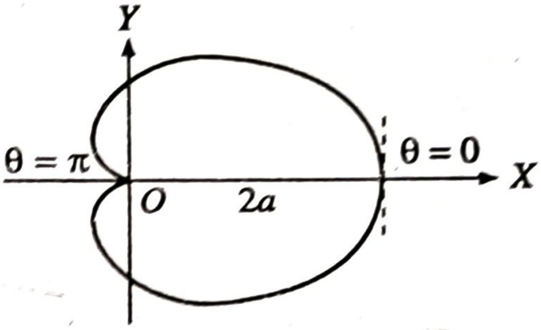
$\therefore$ the perimeter of the cardioid $= 2 \times 4a = 8a$.

****Ex. 32.** *Find the entire length of the cardioid $r = a(1 + \cos \theta)$.*

(Meerut 1983S, 87S, 94; Delhi 81)

Sol. The given curve is $r = a(1 + \cos \theta)$. ...(1)

It is symmetrical about the initial line and for the portion of the curve lying above the initial line θ varies from $\theta = 0$ to $\theta = \pi$.



Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta = -a \sin \theta.$$

If s denotes the arc length of the cardioid measured from the vertex (*i.e.*, the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2\}}$$

$$= \sqrt{\{a^2(2\cos^2 \tfrac{1}{2}\theta)^2 + a^2.(2\sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta)^2\}}$$

$$= 2a\sqrt{\{\cos^4 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta\}}$$

$$= 2a\sqrt{\{\cos^2 \tfrac{1}{2}\theta(\cos^2 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\theta)\}}$$

$$= 2a\sqrt{(\cos^2 \tfrac{1}{2}\theta)} = 2a \cos \tfrac{1}{2}\theta.$$

$$\therefore \quad ds = 2a \cos \tfrac{1}{2}\theta \, d\theta. \qquad ...(2)$$

Let s_1 denote the arc length of the upper half of the cardioid (*i.e.*, from $\theta = 0$ to $\theta = \pi$). Then

$$\int_0^{s_1} ds = \int_0^\pi 2a \cos \tfrac{1}{2}\theta \, d\theta = 2a \left[2 \sin \tfrac{1}{2}\theta\right]_0^\pi$$

or $$s_1 = 4a[\sin \tfrac{1}{2}\pi - \sin 0] = 4a(1-0) = 4a.$$

$\therefore$ by symmetry, the whole length of the cardioid $= 2 \times$ the arc length of the upper half of the cardioid $= 2.4a = 8a$.

****Ex. 33.** *Find the perimeter of the curve $r = a(1 + \cos \theta)$ and show that arc of the upper half is bisected by $\theta = \pi/3$.*

(Garhwal 1983; Berhampur 77; Rohilkhand 76; Madurai 76; Rajputana 77, 76; Calicut 75)

Sol. As proved in Ex. 32 length s_1 of the upper half of the cardioid $r = a(1+\cos\theta)$ is $4a$. (Prove it here)

$\therefore$ the perimeter of the cardioid $= 2.4a = 8a$.

Again, if s_2 denotes the arc length of the cardioid from the point $\theta = 0$ to the point $\theta = \pi/3$, then

$$s_2 = \int_0^{\pi/3} 2a \cos \tfrac{1}{2}\theta \, d\theta, \qquad \text{[refer the result (2) of Ex. 32]}$$

$$= 2a\left[2\sin\tfrac{1}{2}\theta\right]_0^{\pi/3} = 4a\left[\sin\frac{\pi}{6} - \sin 0\right] = 4a \cdot \tfrac{1}{2}$$

$= \frac{1}{2}(s_1) =$ half the arc length of the upper half of the cardioid.

***Ex. 34.** *Prove that the line $4r\cos\theta = 3a$ divides the cardioid $r = a(1+\cos\theta)$ into two parts such that lengths of the arc on either side of the line are equal.*

Sol. The given equations of the cardioid and the line are

$$r = a(1+\cos\theta) \quad \text{and} \quad 4r\cos\theta = 3a.$$

Eliminating r, we have

$$a(1+\cos\theta) = 3a/(4\cos\theta)$$

or $4\cos\theta + 4\cos^2\theta = 3$ or $4\cos^2\theta + 4\cos\theta - 3 = 0$

or $(2\cos\theta - 1)(2\cos\theta + 3) = 0$ *i.e.*, $\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{3}{2}$.

But the values of $\cos\theta$ cannot be numerically greater than 1. Therefore $\cos\theta = -\frac{3}{2}$ is inadmissible.

So we have $\cos\theta = \frac{1}{2}$ *i.e.*, $\theta = \pi/3$.

Hence the vectorial angle of the point of intersection of the cardioid with the given line is $\pi/3$. The cartesian equation of the given line is $x = 3a/4$ *i.e.*, the line is perpendicular to the x-axis. So now we have to prove that the arc of the upper half of the cardioid is bisected by $\theta = \pi/3$. This is the same as Ex. 33 and so prove it here.

***Ex. 35.** *Show that the arc of the upper half of the curve $r = a(1-\cos\theta)$ is bisected by $\theta = 2\pi/3$.*

(Ranchi 1976; Meerut 85)

Sol. As proved in Ex. 31, the arc length of the upper half of the cardioid $r = a(1-\cos\theta)$ is $4a$. (Prove it here)

Also, the arc length of the cardioid from the point $\theta = 0$ to the point $\theta = 2\pi/3$

$$= 2a\int_0^{2\pi/3} \sin\tfrac{1}{2}\theta\, d\theta, \qquad \text{[refer the result (3) of Ex. 31]}$$

$$= 2a\left[-2\cos\tfrac{1}{2}\theta\right]_0^{2\pi/3} = -4a\left[\cos\frac{\pi}{3} - \cos 0\right] = -4a\left[\tfrac{1}{2} - 1\right]$$

$= 2a = \frac{1}{2}(4a) =$ half the arc length of the upper half of the cardioid.

Hence the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$.

Ex. 36. *Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$.*

Sol. The given cardioid is $r = a(1 - \cos\theta)$. ...(1)

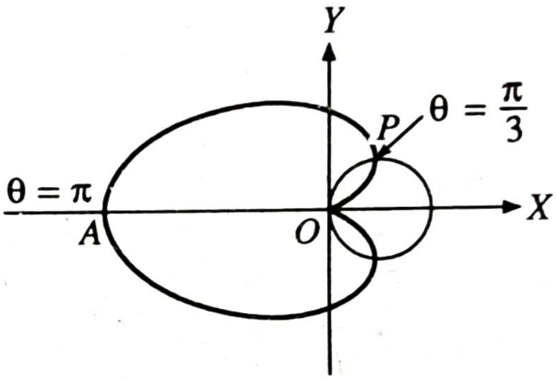
It meets the circle $r = a\cos\theta$. ...(2)

Eliminating r between (1) and (2), we have

$$a(1 - \cos\theta) = a\cos\theta \quad \text{or} \quad 1 - \cos\theta = \cos\theta$$

or $$2\cos\theta = 1 \quad \text{or} \quad \cos\theta = \tfrac{1}{2} \;\; i.e., \;\; \theta = \tfrac{1}{3}\pi.$$

Hence the vectorial angle of the point of intersection P of the cardioid $r = a(1 - \cos\theta)$ with the circle $r = a\cos\theta$ is $\pi/3$.



So for the portion of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$ (above the initial line) θ varies from $\theta = \pi/3$ to $\theta = \pi$.

Also differentiating (1), we get $dr/d\theta = -a\sin\theta$.

By symmetry, the required length of the cardioid $= 2 \times$ the arc length from $\theta = \pi/3$ to $\theta = \pi$ of the upper half of the cardioid

$$= 2\int_{\theta=\pi/3}^{\pi} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

$$= 2\int_{\pi/3}^{\pi} \sqrt{\{a^2(1 - \cos\theta)^2 + (-a\sin\theta)^2\}}\, d\theta$$

$$= 2a\int_{\pi/3}^{\pi} \sqrt{\{(2\sin^2\tfrac{1}{2}\theta)^2 + 4\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\}}\, d\theta$$

$$= 4a\int_{\pi/3}^{\pi} \sqrt{\{\sin^2\tfrac{1}{2}\theta(\sin^2\tfrac{1}{2}\theta + \cos^2\tfrac{1}{2}\theta)\}}\, d\theta$$

$$= 4a\int_{\pi/3}^{\pi} \sin\tfrac{1}{2}\theta\, d\theta = 4a\left[-2\cos\tfrac{1}{2}\theta\right]_{\pi/3}^{\pi}$$

$$= -8a[\cos\tfrac{1}{2}\pi - \cos\tfrac{1}{6}\pi] = -8a[0 - \tfrac{1}{2}\sqrt{3}] = 4a\sqrt{3}.$$

Ex. 37. *Find the length of the arc of the equiangular spiral $r = a e^{\theta\cot\alpha}$, taking $s = 0$ when $\theta = 0$.*

Sol. The given equiangular spiral is $r = ae^{\theta\cot\alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = ae^{\theta\cot\alpha}\,.\cot\alpha = r\cot\alpha.$$

If s denotes the arc length of the equiangular spiral measured from $\theta = 0$ to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$\frac{ds}{d\theta}=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}=\sqrt{(r^2+r^2\cot^2\alpha)}=r\operatorname{cosec}\alpha$$

$$=ae^{\theta\cot\alpha}\,.\operatorname{cosec}\alpha$$

or $\qquad ds=ae^{\theta\cot\alpha}\,.\operatorname{cosec}\alpha\,d\theta.$

$\therefore$ Integrating, $\int_0^s ds=a\operatorname{cosec}\alpha\int_0^\theta e^{\theta\cot\alpha}\,d\theta$

or $\quad s=a\operatorname{cosec}\alpha\cdot\dfrac{1}{\cot\alpha}\cdot\left[e^{\theta\cot\alpha}\right]_0^\theta$

$$=a\sec\alpha\,[e^{\theta\cot\alpha}-e^0]=a\sec\alpha\,[e^{\theta\cot\alpha}-1].$$

Ex. 38. *Find the length of the arc of the equiangular spiral*

$$r=ae^{\theta\cot\alpha}$$

between the points for which radii vectors are r_1 *and* r_2.

(Bhopal 1982; Gorakhpur 76)

Sol. The given curve is $r=ae^{\theta\cot\alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta=ae^{\theta\cot\alpha}\,.\cot\alpha=r\cot\alpha,\text{ from (1).}$$

$\therefore\quad d\theta/dr=1/(r\cot\alpha)$ *i.e.*, $(r\,d\theta/dr)=\tan\alpha.$ ...(2)

If s denotes the arc length of the given curve measured in the direction of r increasing, we have

$$\frac{ds}{dr}=\sqrt{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}}\qquad\textbf{(Note)}$$

$$=\sqrt{(1+\tan^2\alpha)}=\sqrt{(\sec^2\alpha)}=\sec\alpha,\qquad\text{from (2)}$$

or $\qquad ds=\sec\alpha\,dr.$

Let s_1 denote the required arc length *i.e.*, from $r=r_1$ to $r=r_2$.

Then $\qquad\int_0^{s_1}ds=\int_{r_1}^{r_2}\sec\alpha\,dr=(\sec\alpha)\left[r\right]_{r_1}^{r_2}$

or $\qquad s_1=(\sec\alpha)(r_2-r_1).$

Ex. 39 (a). *Find the length of any arc of the cissoid*

$$r=a\,(\sin^2\theta/\cos\theta).$$

(Gorakhpur 1975)

Sol. The given curve is $r=a\,(\sin^2\theta/\cos\theta)$

or $\qquad r=a\tan\theta\sin\theta.$ ...(1)

Differentiating (1) w.r.t. θ, we have

$$dr/d\theta=a\,[\tan\theta\cos\theta+\sec^2\theta\sin\theta]$$

$$=a\,[\sin\theta+\sec^2\theta\sin\theta]=a\sin\theta\,[1+\sec^2\theta].$$

We have $(ds/d\theta)^2=r^2+(dr/d\theta)^2$

$$=a^2\tan^2\theta\sin^2\theta+a^2\sin^2\theta\,(1+\sec^2\theta)^2$$

$$=a^2\sin^2\theta\,[\tan^2\theta+(1+\sec^2\theta)^2]$$

$$=a^2\sin^2\theta\,[\tan^2\theta+1+\sec^4\theta+2\sec^2\theta]$$

$$=a^2\sin^2\theta\,[\sec^2\theta+\sec^4\theta+2\sec^2\theta]$$

$$= a^2 \sin^2\theta \,.\, \sec^2\theta\,[3 + \sec^2\theta]$$
$$= a^2 \tan^2\theta\,[3 + \sec^2\theta]. \qquad ...(2)$$

If s denotes the arc length of the cissoid measured from the point $\theta = \theta_1$ in the direction of θ increasing, then

$$ds = a \tan\theta \sqrt{(3 + \sec^2\theta)}\, d\theta,$$

on taking square root of (2) and keeping the +ive sign.

Let s_1 denote the required arc length from $\theta = \theta_1$ to $\theta = \theta_2$. Then

$$\int_0^{s_1} ds = \int_{\theta=\theta_1}^{\theta_2} a \tan\theta \sqrt{(3 + \sec^2\theta)}\, d\theta$$

$$= a \int_{\theta_1}^{\theta_2} \frac{\sin\theta}{\cos\theta} \sqrt{\left(3 + \frac{1}{\cos^2\theta}\right)}\, d\theta$$

or
$$s_1 = a \int_{\theta_1}^{\theta_2} \frac{\sin\theta}{\cos^2\theta} \sqrt{(1 + 3\cos^2\theta)}\, d\theta \qquad ...(3)$$

Let us first evaluate the indefinite integral

$$I = \int \frac{\sin\theta}{\cos^2\theta} \sqrt{\{1 + 3\cos^2\theta\}}\, d\theta.$$

Put $\cos\theta = t$ so that $-\sin\theta\, d\theta = dt$.

Then
$$I = -\int \frac{\sqrt{(1 + 3t^2)}}{t^2}\, dt = -\int \frac{1 + 3t^2}{t^2\sqrt{(1 + 3t^2)}}\, dt$$

$$= -\int \frac{dt}{t^2\sqrt{(1 + 3t^2)}} - \int \frac{3\, dt}{\sqrt{(1 + 3t^2)}}.$$

To evaluate the first integral, put $t = 1/z$ so that $dt = -(1/z^2)\, dz$.

Then
$$-\int \frac{dt}{t^2\sqrt{(1 + 3t^2)}} = \int \frac{dz}{\sqrt{(1 + 3/z^2)}} = \frac{1}{2}\int \frac{2z\, dz}{\sqrt{(z^2 + 3)}}$$

$$= \sqrt{(z^2 + 3)}, \text{ by power formula}$$

$$= \sqrt{\left(\frac{1}{t^2} + 3\right)} = \sqrt{\left(\frac{1}{\cos^2\theta} + 3\right)} = \sqrt{(\sec^2\theta + 3)}.$$

Also the second integral

$$= -\int \frac{3\, dt}{\sqrt{(1 + 3t^2)}} = -\sqrt{3}\int \frac{dt}{\sqrt{(\frac{1}{3} + t^2)}}$$

$$= -\sqrt{3} \log\{t + \sqrt{(t^2 + \tfrac{1}{3})}\}$$

$$= -\sqrt{3} \log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}.$$

$\therefore$
$$I = \sqrt{(\sec^2\theta + 3)} - \sqrt{3} \log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}.$$

Hence from (3), we get the required arc length

$$s_1 = \left[a\sqrt{(\sec^2\theta + 3)} - \sqrt{3}.a \log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}\right]_{\theta_1}^{\theta_2}$$

$$= f(\theta_2) - f(\theta_1),$$

where $\quad f(\theta) = a\sqrt{(\sec^2\theta + 3)} - a\sqrt{3}\log\{\cos\theta + \sqrt{(\cos^2\theta + \frac{1}{3})}\}.$

Ex. 39 (b). *Find the length of the curve $r^{1/3} = 8\cos(\theta/3)$.*

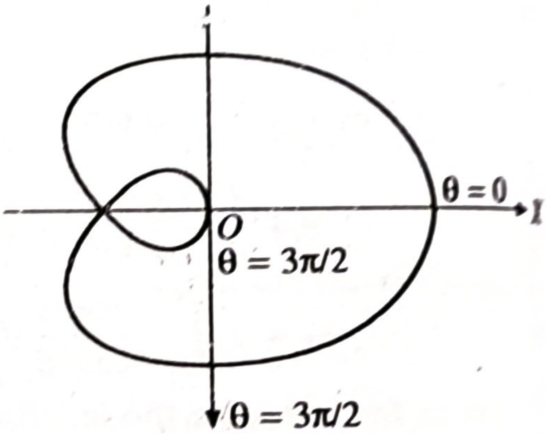
(Meerut 199...)

Sol. The given curve is $r^{1/3} = 8\cos(\theta/3)$

or $\quad r = 512\cos^3(\theta/3)$. ...(1)

The curve (1) is symmetrical about the initial line.

We have $r = 0$ when $\cos(\theta/3) = 0$ *i.e.*, $\theta/3 = \pm\pi/2$ *i.e.*, $\theta = -3\pi/2$ and $\theta = 3\pi/2$.



The entire length of the curve $= 2$. (length of the curve from $\theta = 0$ to $\theta = 3\pi/2$).

From (1),

$$\frac{dr}{d\theta} = 512 \cdot 3\left(\cos^2\frac{\theta}{3}\right)\left(-\sin\frac{\theta}{3}\right)\cdot\frac{1}{3} = -512\cos^2\frac{\theta}{3}\sin\frac{\theta}{3}.$$

Now $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2$

$$= (512)^2\cos^6\frac{\theta}{3} + (512)^2\cos^4\frac{\theta}{3}\sin^2\frac{\theta}{3}$$

$$= (512)^2\cos^4\frac{\theta}{3}\left(\cos^2\frac{\theta}{3} + \sin^2\frac{\theta}{3}\right)$$

$$= (512)^2\cos^4\frac{\theta}{3}.$$

$\therefore \quad \dfrac{ds}{d\theta} = 512\cos^2\dfrac{\theta}{3}$

or $\quad ds = 512\cos^2\dfrac{\theta}{3}\,d\theta.$

$\therefore$ the required perimeter of the curve

$$= 2\int_0^{3\pi/2} 512\cos^2\frac{\theta}{3}\,d\theta$$

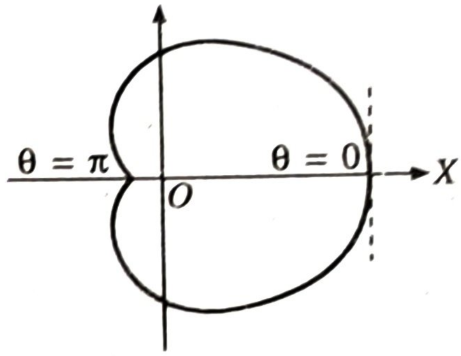
$$= 6 \cdot 512\int_0^{\pi/2}\cos^2 t\,dt, \qquad \text{putting } \frac{\theta}{3} = t \text{ so that } d\theta = 3dt$$

$$= 3072\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 768\pi.$$

Ex. 40. *Prove that the perimeter of the limacon $r = a + b\cos\theta$, if b/a be small, is approximately $2\pi a\,(1 + \frac{1}{4}b^2/a^2)$.*

Sol. The given curve is $r = a + b\cos\theta$, $(a > b)$. ...(1)

Note that b/a is given to be small so we must have $b < a$. The curve (1) is symmetrical about the initial line and for the portion of the curve lying above the initial line θ varies from $\theta = 0$ to $\theta = \pi$.



By symmetry, the perimeter of the limacon

$= 2 \times$ the arc length of the upper half of the limacon.

Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta = -\,b\sin\theta.$$

We have $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = (a + b\cos\theta)^2 + (-\,b\sin\theta)^2$

$$= a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta$$
$$= a^2 + b^2 + 2ab\cos\theta.$$

If we measure the arc length s in the direction of θ increasing,

we have $ds/d\theta = \sqrt{(a^2 + b^2 + 2ab\cos\theta)}$

or $ds = \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\,d\theta$.

The arc length of the upper half of the limacon

$$= \int_0^\pi \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\,d\theta = a\int_0^\pi \left(1 + \frac{2b}{a}\cos\theta + \frac{b^2}{a^2}\right)^{1/2} d\theta$$

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\cdot\frac{b^2}{a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2\,!}\left(4\frac{b^2}{a^2}\cos^2\theta\right)\right] d\theta,$$

expanding by binomial theorem and neglecting powers of b/a higher than two because b/a is small

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\frac{b^2}{a^2}(1 - \cos^2\theta)\right] d\theta$$

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\frac{b^2}{a^2}\sin^2\theta\right] d\theta$$

$$= a\left[\left\{\theta + \frac{b}{a}\sin\theta\right\}_0^\pi + \frac{1}{2}\frac{b^2}{a^2}\,2\int_0^{\pi/2}\sin^2\theta\,d\theta\right]$$

$$= a\left[\pi + \frac{1}{2}\frac{b^2}{a^2}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = a\pi\left[1 + \frac{b^2}{4a^2}\right].$$

$\therefore$ the perimeter of the limacon

$$= 2 \times a\pi\,[1 + (b^2/4a^2)] = 2a\pi\,[1 + (b^2/4a^2)].$$

***Ex. 41.** *Show that the whole length of the limacon*

$$r = a + b\cos\theta,\ (a > b)$$

is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.

Sol. As shown in Ex. 40, the whole length of the limacon

$$= 2\int_0^{\pi} \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\, d\theta, \text{ (prove it here).} \quad \text{...(1)}$$

Also the maximum and minimum radii vectors of the limacon are given by $\cos\theta = 1$ and $\cos\theta = -1$ and they are respectively $a + b$ and $a - b$.

Now the parametric equations of the ellipse with major axis as $(a + b)$ and minor axis as $(a - b)$ may be taken as

$$x = (a + b)\cos\phi, y = (a - b)\sin\phi. \quad \text{...(2)}$$

Differentiating (2) w.r.t. ϕ, we have

$$dx/d\phi = -(a + b)\sin\phi, dy/d\phi = (a - b)\cos\phi.$$

Now the ellipse (2) is symmetrical in all the four quadrants and for the portion of the ellipse lying in the first quadrant ϕ varies from $\phi = 0$ to $\phi = \frac{1}{2}\pi$.

By symmetry, the perimeter (whole length) of the ellipse $= 4 \times$ the arc length of the ellipse lying in the first quadrant

$$= 4\int_0^{\pi/2} \sqrt{\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}}\, d\phi$$

$$= 4\int_0^{\pi/2} \sqrt{[\{-(a + b)\sin\phi\}^2 + \{(a - b)\cos\phi\}^2]}\, d\phi$$

$$= 4\int_0^{\pi/2} \sqrt{\{a^2\sin^2\phi + b^2\sin^2\phi + 2ab\sin^2\phi + a^2\cos^2\phi + b^2\cos^2\phi - 2ab\cos^2\phi\}}\, d\phi$$

$$= 4\int_0^{\pi/2} \sqrt{\{a^2 + b^2 - 2ab(\cos^2\phi - \sin^2\phi)\}}\, d\phi$$

$$= 4\int_0^{\pi/2} \sqrt{\{a^2 + b^2 - 2ab\cos 2\phi\}}\, d\phi.$$

Now put $2\phi = t$ so that $2d\phi = dt$. Also when $\phi = 0, t = 0$ and when $\phi = \pi/2$, $t = \pi$.

Then the whole length of the ellipse

$$= 4\int_0^{\pi} \sqrt{\{a^2 + b^2 - 2ab\cos t\}}\, \tfrac{1}{2} dt$$

$$= 2\int_0^{\pi} \sqrt{\{a^2 + b^2 - 2ab\cos t\}}\, dt$$

$$= 2\int_0^{\pi} \sqrt{\{a^2 + b^2 - 2ab\cos(\pi - t)\}}\, dt,$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$$

$= 2\int_0^{\pi} \sqrt{(a^2 + b^2 + 2ab\cos t)}\, dt$

$= 2\int_0^{\pi} \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\, d\theta,$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(t)\,dt\right]$

= the whole length of the limacon, [from (1)].

Ex. 42. *Show that the length of a loop of the curve*

$$3x^2y - y^3 = (x^2 + y^2)^3 \text{ is } 2\int_0^1 \frac{dr}{\sqrt{(1 - r^6)}}.$$

Sol. Changing to polar form by putting

$$x = r\cos\theta, y = r\sin\theta,$$

the given equation of the curve becomes

$$3r^2\cos^2\theta \,.\, r\sin\theta - r^3\sin\theta = (r^2)^3$$

or $\quad r^3(3\sin\theta\cos^2\theta - \sin^3\theta) = r^6$

or $\quad 3\sin\theta(1 - \sin^2\theta) - \sin^3\theta = r^3$

or $\quad 3\sin\theta - 3\sin^3\theta - \sin^3\theta = r^3$ or $r^3 = 3\sin\theta - 4\sin^3\theta$

or $\quad r^3 = \sin 3\theta.$ ...(1)

Differentiating (1) w.r.t. θ, we have

$$3r^2(dr/d\theta) = 3\cos 3\theta \quad \text{or} \quad (dr/d\theta) = (\cos 3\theta)/r^2.$$

$$\therefore \quad r\frac{d\theta}{dr} = r \cdot \frac{r^2}{\cos 3\theta} = \frac{r^3}{\cos 3\theta} = \frac{\sin 3\theta}{\cos 3\theta} = \tan 3\theta, \qquad \text{from (1).}$$

From (1), $r = 0$ when $\sin 3\theta = 0$ *i.e.*, when $3\theta = 0, \pi$ *i.e.*, when $\theta = 0, \pi/3$. Thus two consecutive values of θ for which $r = 0$ are 0 and $\pi/3$. Therefore one loop of the curve lies between $\theta = 0$ and $\pi/3$. Also r is maximum when $\sin 3\theta = 1$ *i.e.*, $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$. Therefore half of the loop extends from $\theta = 0$ to $\theta = \pi/6$.

When $\theta = 0, r = 0$ and when $\theta = \pi/6, r = 1$.

$\therefore$ the required length of a loop

$$= 2\int_0^1 \sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}}\, dr, \qquad \text{(by symmetry)}$$

$$= 2\int_0^1 \sqrt{\{1 + \tan^2 3\theta\}}\, dr = 2\int_0^1 \sec 3\theta\, dr = 2\int_0^1 \frac{dr}{\cos 3\theta}$$

$$= 2\int_0^1 \frac{dr}{\sqrt{(1 - \sin^2 3\theta)}} = 2\int_0^1 \frac{dr}{\sqrt{(1 - r^6)}}, \qquad \text{from (1).}$$

***Ex. 43.** *If s be the length of the curve $r = a\tanh\frac{1}{2}\theta$ between the origin and $\theta = 2\pi$, and Δ be the area under the curve between the same two points, prove that*

$$\Delta = a(s - a\pi).$$

Sol. The given curve is $r = a\tanh\frac{1}{2}\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = a \cdot \tfrac{1}{2} \operatorname{sech}^2 \tfrac{1}{2}\theta.$$

We have $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = a^2 \tanh^2 \tfrac{1}{2}\theta + \dfrac{a^2}{4} \operatorname{sech}^4 \tfrac{1}{2}\theta$

$$= \tfrac{1}{4}a^2 [4 \tanh^2 \tfrac{1}{2}\theta + \operatorname{sech}^4 \tfrac{1}{2}\theta]$$

$$= \tfrac{1}{4}a^2 [4 (1 - \operatorname{sech}^2 \tfrac{1}{2}\theta) + \operatorname{sech}^4 \tfrac{1}{2}\theta]$$

$$= \tfrac{1}{4}a^2 [2 - \operatorname{sech}^2 \tfrac{1}{2}\theta]^2. \qquad \text{...(2)}$$

If we measure the arc length s in the direction of θ increasing, we have

$$ds/d\theta = \tfrac{1}{2}a (2 - \operatorname{sech}^2 \tfrac{1}{2}\theta),$$

retaining +ive sign while taking the square root of (2)

or $\quad ds = \tfrac{1}{2}a (2 - \operatorname{sech}^2 \tfrac{1}{2}\theta)\, d\theta.$

Now at the origin $r = 0$ and putting $r = 0$ in (1), we get $\theta = 0$.

$\therefore$ the arc length of the given curve between the origin ($\theta = 0$) and $\theta = 2\pi$ is given by

$$s = \tfrac{1}{2}a \int_0^{2\pi} (2 - \operatorname{sech}^2 \tfrac{1}{2}\theta)\, d\theta$$

$$= \tfrac{1}{4}a \int_0^{2\pi} 2d\theta - \tfrac{1}{2}a \int_0^{2\pi} \operatorname{sech}^2 \tfrac{1}{2}\theta\, d\theta$$

$$= \tfrac{1}{2}a \,.\, 2 \Big[\theta\Big]_0^{2\pi} - \tfrac{1}{2}a \Big[2 \tanh \tfrac{1}{2}\theta\Big]_0^{2\pi}$$

$$= 2a\pi - a \tanh \pi. \qquad \text{...(3)}$$

Also the area between the radii vectors $\theta = 0, \theta = 2\pi$ and the curve

$$= \Delta = \tfrac{1}{2}\int_0^{2\pi} r^2\, d\theta = \tfrac{1}{2}a^2 \int_0^{2\pi} \tanh^2 \tfrac{1}{2}\theta\, d\theta$$

$$= \tfrac{1}{2}a^2 \int_0^{2\pi} (1 - \operatorname{sech}^2 \tfrac{1}{2}\theta)\, d\theta = \tfrac{1}{2}a^2 \Big[\theta - 2 \tanh \tfrac{1}{2}\theta\Big]_0^{2\pi}$$

$$= \tfrac{1}{2}a^2 [2\pi - 2 \tanh \pi] = a^2 [\pi - \tanh \pi]$$

$$= a [a\pi - a \tanh \pi] = a [(2a\pi - a \tanh \pi) - a\pi]$$

$$= a (s - a\pi), \qquad \text{from (3).}$$

**Ex. 44. *Prove the formula* $s = \displaystyle\int \frac{r\, dr}{\sqrt{(r^2 - p^2)}}$.

Show that the arc of the curve $p^2 (a^4 + r^4) = a^4 r^2$ *between the limits* $r = b, r = c$ *is equal in length to the arc of the hyperbola* $xy = a^2$ *between the limits* $x = b, x = c$.

Sol. From differential calculus, we know that

$$\tan \phi = r \frac{d\theta}{dr} \text{ and } \frac{ds}{dr} = \sqrt{\left[1 + \left(r \frac{d\theta}{dr}\right)^2\right]}.$$

$$\therefore \quad \frac{ds}{dr} = \sqrt{(1 + \tan^2 \phi)} = \sqrt{(\sec^2 \phi)} = \sec \phi$$

$$= \frac{1}{\cos\phi} = \frac{1}{\sqrt{(1-\sin^2\phi)}} = \frac{1}{\sqrt{\{1-(p^2/r^2)\}}},$$

$$[\because p = r\sin\phi]$$

$$= \frac{r}{\sqrt{(r^2-p^2)}}. \qquad \text{Thus} \quad ds = \frac{r}{\sqrt{(r^2-p^2)\}}}dr.$$

Integrating between the given limits, we get

$$s = \int \frac{r}{\sqrt{(r^2-p^2)}}dr. \qquad \text{...(1)}$$

Now the given curve is $p^2(a^4+r^4) = a^4r^2$

or $\qquad p^2 = a^4r^2/(a^4+r^2)$.

We have $\quad r^2 - p^2 = r^2 - \dfrac{a^4r^2}{(a^4+r^4)} = \dfrac{r^6}{(a^4+r^4)}$. $\qquad$ ...(2)

Therefore from (1), the arc of the given curve between the limits $r = b, r = c$ is

$$= \int_b^c \frac{r\,dr}{\sqrt{(r^2-p^2)}} = \int_b^c \frac{r\,dr}{\sqrt{\{r^6/(a^4+r^4)\}}}, \qquad \text{from (2)}$$

$$= \int_b^c \frac{r\sqrt{(a^4+r^4)}}{r^3}dr = \int_b^c \frac{\sqrt{(a^4+r^4)}}{r^2}dr. \qquad \text{...(3)}$$

Also, for the hyperbola $xy = a^2$ *i.e.*, $y = a^2/x, dy/dx = -a^2/x^2$.

$\therefore$ the arc length of the hyperbola $xy = a^2$ between the limits $x = b, x = c$

$$= \int_b^c \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}dx = \int_b^c \sqrt{\left\{1 + \frac{a^4}{x^4}\right\}}dx$$

$$= \int_b^c \frac{\sqrt{(x^4+a^4)}}{x^2}dx = \int_b^c \frac{\sqrt{(r^4+a^4)}}{r^2}dr,$$

changing the variable from x to r by a property of definite integrals

$$= \int_b^c \frac{\sqrt{(a^4+r^4)}}{r^2}dr. \qquad \text{...(4)}$$

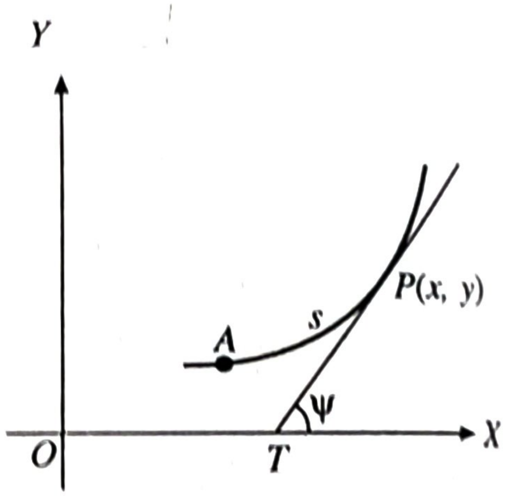
From (3) and (4) we observe that the two lengths are equal.

§ 3. Intrinsic equations.

Definition. By the *intrinsic equation* of a curve we mean a relation between s and ψ, where s is the length of the arc AP of the curve measured from a fixed point A on it to a variable point P, and ψ is the angle which the tangent to the curve at P makes with a fixed straight line usually taken as the positive direction of the axis of x. The co-ordinates s and ψ are known as **Intrinsic Co-ordinates.**

(a) To find the intrinsic equation from the cartesian equation.

Let the equation of the given curve be $y = f(x)$. Take A as the fixed point on the curve from which s is measured and take the axis of x as the fixed straight line with reference to which ψ is measured. Let $P(x, y)$ be any point on the curve and PT be the tangent at the point P to the curve. Let arc $AP = s$ and $\angle PTX = \psi$.



Now, we have

$\tan\psi = dy/dx = f'(x)$. ...(1)

Let a be the abscissa of the point A from which s is measured.

Then
$$s = \int_a^x \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_a^x \sqrt{[1 + \{f'(x)\}^2]}\, dx. \qquad ...(2)$$

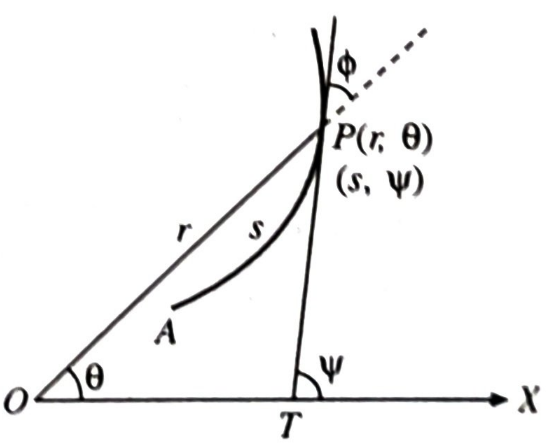
Eliminating x between (1) and (2), we obtain the required intrinsic equation.

Note. To find the intrinsic equation from the parametric equations we use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and then proceed as in case (a).

(b) Intrinsic equation from Polar equation.

Let the equation of the given curve be $r = f(\theta)$. Take A as the fixed point on the curve from which s is measured. Let P be any point (r, θ) on the curve.

Let arc $AP = s$ and $\angle PTX = \psi$, where OX is the initial line.



If ϕ is the angle between the radius vector and the tangent at P, then

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)}, \qquad ...(1)$$

and $\psi = \theta + \phi$. ...(2)

Let α be the vectorial angle of the point A. Then we have

$$s = \int_\alpha^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

$$= \int_{a}^{\theta} \sqrt{[\{f(\theta)\}^2 + \{f'(\theta)\}^2]}\, d\theta \qquad ...(3)$$

Eliminating θ and ϕ between (1), (2) and (3), we get a relation between s and ψ, which is the intrinsic equation of the curve.

(c) Intrinsic equation from Pedal Equation.

Let the pedal equation of the curve be $p = f(r)$. ...(1)

Then $$s = \int_{a}^{r} \frac{r\, dr}{\sqrt{(r^2 - p^2)}}, \qquad ...(2)$$

the arc length s being measured from the point $r = a$.

Also the radius of curvature $\rho = \dfrac{ds}{d\psi} = r\dfrac{dr}{dp}$.

Eliminating p and r between (1), (2) and (3), we obtain the required intrinsic equation.

Solved Examples

Ex. 45. *Show that the intrinsic equation of the parabola $y^2 = 4ax$ is*

$$s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi),$$

ψ being the angle between the x-axis and the tangent at the point whose arcual distance from the vertex is s.

Sol. The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \quad \tan \psi = dy/dx = 4a/2y = 2a/y.$ ...(2)

If s denotes the arc length of the parabola measured from the vertex (0, 0) in the direction of y increasing, then

$$\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \sqrt{\left\{1 + \frac{y^2}{4a^2}\right\}}, \qquad \left[\because \frac{dx}{dy} = \frac{y}{2a}\right]$$

$$= \sqrt{\left\{\frac{4a^2 + y^2}{4a^2}\right\}} = \frac{1}{2a}\sqrt{(4a^2 + y^2)}.$$

$$\therefore \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\, dy.$$

Integrating, $\int_0^s ds = \dfrac{1}{2a}\int_0^y \sqrt{(4a^2 + y^2)}\, dy$

or $$s = \frac{1}{2a}\left[\tfrac{1}{2} y \sqrt{(4a^2 + y^2)} + \tfrac{1}{2}\cdot 4a^2 \log \{y + \sqrt{(4a^2 + y^2)}\}\right]_0^y$$

$$= (1/2a)\,[\tfrac{1}{2} y \sqrt{(4a^2 + y^2)} + \tfrac{1}{2}\cdot 4a^2 \log \{y + \sqrt{(4a^2 + y^2)}\} - \tfrac{1}{2}\cdot 4a^2 \log 2a]$$

$$= \frac{1}{4a}\left[y\sqrt{(4a^2 + y^2)} + 4a^2 \log \frac{y + \sqrt{(4a^2 + y^2)}}{2a}\right] \qquad ...(3)$$

Now to obtain the intrinsic equation of the given parabola we eliminate y between (2) and (3). From (2), we have $y = 2a \cot \psi$. Putting this value of y in (3), we get

$$s = \frac{1}{4a}\left[2a \cot \psi \sqrt{(4a^2 + 4a^2 \cot^2 \psi)} + 4a^2 \log \frac{2a \cot \psi + \sqrt{(4a^2 + 4a^2 \cot^2 \psi)}}{2a}\right]$$

$$= \frac{1}{4a}[(2a \cot \psi) . 2a \sqrt{(1 + \cot^2 \psi)} + 4a^2 \log \{\cot \psi + \sqrt{(1 + \cot^2 \psi)}\}]$$

$$= a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi),$$

which is the required intrinsic equation.

Ex. 46. *Prove that the intrinsic equation of the parabola $x^2 = 4ay$ is*

$$s = a \tan \psi \sec \psi + a \log (\tan \psi + \sec \psi).$$

Sol. Proceeding exactly as in Ex. 45, we get

$$\tan \psi = dy/dx = x/2a. \quad \text{...(1)}$$

Also $s = \displaystyle\int_0^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx = \frac{1}{2a}\int_0^x \sqrt{(4a^2 + x^2)}\, dx$

$$= \frac{1}{4a}\left[x\sqrt{(4a^2 + x^2)} + 4a^2 \log \frac{x + \sqrt{(x^2 + 4a^2)}}{2a}\right], \quad \text{...(2)}$$

proceeding as in Ex. 45.

Eliminating x from (1) and (2), we get

$$s = a [\tan \psi \sec \psi + \log (\tan \psi + \sec \psi)],$$

which is the required intrinsic equation.

Ex. 47. *Find the intrinsic equation of the parabola $y^2 = 4ax$. Hence deduce the length of the arc measured from the vertex to an extremity of the latus rectum.* **(Meerut 1981S; Lucknow 72; Magadh 72)**

Sol. We have already obtained the intrinsic equation of the parabola $y^2 = 4ax$ in Ex. 45 as

$$s = a [\operatorname{cosec} \psi \cot \psi + \log (\operatorname{cosec} \psi + \cot \psi)], \quad \text{...(1)}$$

where ψ is the angle between the x-axis and the tangent at the point whose arcual distance from the vertex is s. **(Prove it here)**

Now in the intrinsic equation (1) of the parabola the arc length s has been measured from the vertex. We want to find the length of the arc from the vertex to an extremity of the latus rectum. Let this length be s_1.

At an extremity of the latus rectum, $y = 2a$. Also $\tan \psi = y/2a$. So at an extremity of the latus rectum, $\tan \psi = 2a/2a = 1$ *i.e.*, $\psi = \pi/4$. So putting $\psi = \pi/4$ in (1), we get

$s_1 = a\,[\operatorname{cosec}\tfrac{1}{4}\pi \cot\tfrac{1}{4}\pi + \log(\operatorname{cosec}\tfrac{1}{4}\pi + \cot\tfrac{1}{4}\pi)]$
$= a\,[\sqrt{2} + \log(1+\sqrt{2})]$.

Ex. 48. *Show that the intrinsic equation of the semi-cubical parabola $3ay^2 = 2x^3$ is $9s = 4a(\sec^3\psi - 1)$.* **(Meerut 1984S, 85, 88, 96 P)

Sol. The given semicubical parabola is $3ay^2 = 2x^3$. ...(1)

Differentiating (1) w.r.t. x, we get $6ay\,(dy/dx) = 6x^2$

or $$\frac{dy}{dx} = \frac{x^2}{ay} = \frac{x^2}{a\sqrt{(2x^3/3a)}} = \sqrt{\left(\frac{3x}{2a}\right)}.$$

$$\therefore \quad \tan\psi = \frac{dy}{dx} = \sqrt{\left(\frac{3x}{2a}\right)}. \qquad ...(2)$$

If s denotes the arc length of the given curve measured from the point $(0,0)$ to any point $P(x,y)$ in the direction of x increasing, then

$$s = \int_0^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx = \int_0^x \sqrt{\left(1 + \frac{3x}{2a}\right)}\,dx$$

$$= \int_0^x \left(1 + \frac{3x}{2a}\right)^{1/2} dx = \left[\frac{\{1 + (3x/2a)\}^{3/2}}{(3/2)\,.\,(3/2a)}\right]_0^x$$

$$= \frac{4a}{9}\left[\left(1 + \frac{3x}{2a}\right)^{3/2} - 1\right] \qquad ...(3)$$

Eliminating x between (2) and (3), we get

$$s = \frac{4a}{9}\,[(1 + \tan^2\psi)^{3/2} - 1] = \frac{4a}{9}(\sec^3\psi - 1),$$

which is the required intrinsic equation of the curve.

*Ex. 49. *Show that the intrinsic equation of $ay^2 = x^3$ taking its cusp as the fixed point is $27s = 8a(\sec^3\psi - 1)$.* **(Garhwal 1983)**

Sol. The given curve is $ay^2 = x^3$. ...(1)

Proceeding exactly as in Ex. 48, we get

$$\tan\psi = \frac{dy}{dx} = \frac{3}{2}\frac{1}{\sqrt{a}}x^{1/2} \qquad ...(2)$$

and $$s = \int_0^x \sqrt{\left(1 + \frac{9x}{4a}\right)}\,dx = \frac{1}{27\sqrt{a}}\,[(4a + 9x)^{3/2} - 8a^{3/2}]. \qquad ...(3)$$

Eliminating x between (2) and (3), we get

$$27s = 8a(\sec^3\psi - 1).$$

*Ex. 50. *Find the intrinsic equation of the catenary*

$y = c\cosh(x/c)$. **(Meerut 1982)**

Hence show that $c\rho = c^2 + s^2$, where ρ is the radius of curvature.

Sol. The given curve is $y = c\cosh(x/c)$. ...(1)

Differentiating (1) w.r.t. x, we get

$$dy/dx = c\sinh(x/c)\,.\,(1/c) = \sinh(x/c).$$

$\therefore \quad \tan \psi = dy/dx = \sinh (x/c)$. ...(2)

If s denotes the arc length of the catenary measured from the vertex $(0, c)$ to any point (x, y) in the direction of x increasing, then

$$s = \int_0^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int_0^x \sqrt{\left\{1 + \sinh^2 \frac{x}{c}\right\}}\, dx$$

$$= \int_0^x \cosh \frac{x}{c}\, dx = c \sinh \frac{x}{c}. \quad ...(3)$$

Eliminating x between (2) and (3), we get

$s = c \tan \psi$, which is the required intrinsic equation of the catenary.

Also $\rho = \dfrac{ds}{d\psi} = c \sec^2 \psi = c\,(1 + \tan^2 \psi) = c\left(1 + \dfrac{s^2}{c^2}\right)$

or $\quad c\rho = c^2 + s^2$.

****Ex. 51.** *Show that the intrinsic equation of the cycloid*

$$x = a\,(t + \sin t), y = a\,(1 - \cos t)$$

is $\quad s = 4a \sin \psi$.

Hence or otherwise find the length of the complete cycloid.

(Meerut 1981, 83, 83S; Allahabad 73; Agra 77)

Sol. The given equations of the cycloid are

$$x = a\,(t + \sin t), y = a\,(1 - \cos t). \quad ...(1)$$

We have $\quad dx/dt = a\,(1 + \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a\,(1 + \cos t)} = \frac{2 \sin \frac{1}{2}t \cos \frac{1}{2}t}{2 \cos^2 \frac{1}{2}t} = \tan \tfrac{1}{2}t.$$

Hence $\quad \tan \psi = dy/dx = \tan \frac{1}{2}t \quad$ or $\quad \psi = \frac{1}{2}t$. ...(2)

If s denotes the arc length of the cycloid measured from the vertex (*i.e.*, the point $t = 0$) to any point P (*i.e.*, the point 't') in the direction of t increasing, then

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt$$

$$= \int_0^t \sqrt{\{a^2 (1 + \cos t)^2 + a^2 \sin^2 t\}}\, dt$$

$$= \int_0^t \sqrt{\{2a^2 (1 + \cos t)\}}\, dt$$

$$= 2a \int_0^t \cos \tfrac{1}{2}t\, dt = 2a \left[2 \sin \tfrac{1}{2}t\right]_0^t = 4a \sin \tfrac{1}{2}t \quad ...(3)$$

Eliminating t from (2) and (3), we get

$$s = 4a \sin \psi, \quad ...(4)$$

which is the required intrinsic equation of the cycloid.

Second Part. In the intrinsic equation (4) of the cycloid the arc length s has been measured from the vertex *i.e.*, the point $\psi = 0$. At a cusp, we have $t = \pi$ and $\psi = \pi/2$. If s_1 denotes the length of the arc extending from the vertex to a cusp, then from (4), we have

$$s_1 = 4a \sin \tfrac{1}{2}\pi = 4a.$$

$\therefore$ the whole length of an arch of the cycloid $= 2 \times 4a = 8a$.

Ex. 52. *Prove that the intrinsic equation of the curve*

$$x = a(1 + \sin t), y = a(1 + \cos t) \text{ is } s + a\psi = 0.$$

Sol. The given curve is

$$x = a(1 + \sin t), y = a(1 + \cos t). \quad ...(1)$$

$\therefore$ $dx/dt = a \cos t$ and $dy/dt = -a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a \cos t} = -\tan t.$$

Hence $\tan \psi = dy/dx = -\tan t = \tan(-t)$

so that $\psi = -t$. ...(2)

Measuring the arc length s from the point $t = 0$, we have

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\right\}} dt$$

$$= \int_0^t \sqrt{(a^2 \cos^2 t + a^2 \sin^2 t)}\, dt = a \int_0^t dt$$

$$= at. \quad ...(3)$$

Eliminating t from (2) and (3), the intrinsic equation is

$$s = a(-\psi) \quad \text{or} \quad s + a\psi = 0.$$

Ex. 53. *In the four-cusped astroid $x^{2/3} + y^{2/3} = a^{2/3}$, show that*

(*i*) $s = \frac{3}{4} a \cos 2\psi$, *s being measured from the vertex;*

(Agra 1972; Meerut 84R, 92, 96 BP)

(*ii*) $s = \frac{3}{2} a \sin^2 \psi$, *s being mesured from the cusp on x-axis;*

(Meerut 1974, 96)

(*iii*) *whole length of the curve is 6a.* **(Meerut 1987)**

Sol. The parametric equations of the given curve are

$$x = a \cos^3 t, y = a \sin^3 t. \quad ...(1)$$

We have

$$dx/dt = -3a \cos^2 t \sin t, \quad \text{and} \quad dy/dt = 3a \sin^2 t \cos t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\tan t.$$

So we have $\tan \psi = dy/dx = -\tan t = \tan(-t)$.

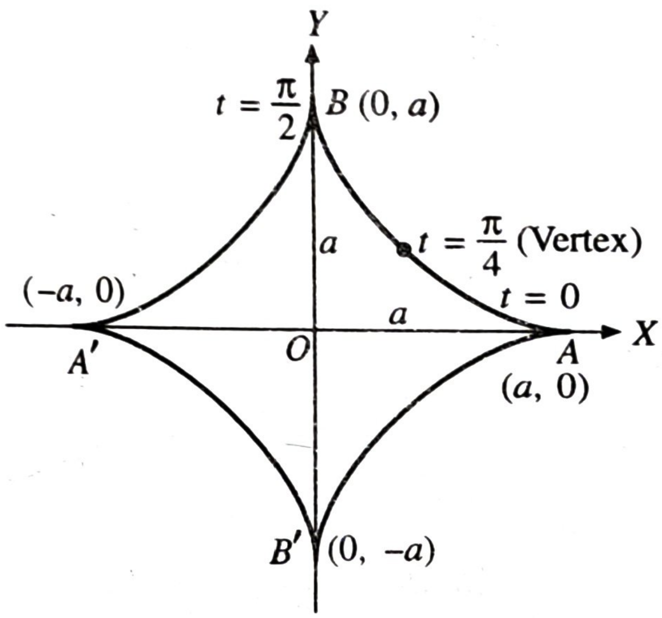
$\therefore$ $\psi = -t$. ...(2)

Now
$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= (9a^2 \cos^4 t \sin^2 t) + (9a^2 \sin^4 t \cos^2 t)$$

$$= 9a^2 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)$$

$$= 9a^2 \sin^2 t \cos^2 t. \quad ...(3)$$



(i) If s denotes the arc length of the given curve measured from the vertex (*i.e.*, the middle point of the arc in the Ist quadrant) to any point P lying towards the cusp on x-axis, then s increases as t decreases. Therefore ds/dt will be negative, so from (3), we have

$$ds/dt = -3a \sin t \cos t$$

or $$ds = -3a \sin t \cos t \, dt. \qquad \text{...(4)}$$

Now at the vertex of the given curve, we have $t = \pi/4$.

$\therefore$ from (4), the arcual distance s measured from the vertex is given by

$$s = -3a \int_{\pi/4}^{t} \sin t \cos t \, dt = -\frac{3a}{2} \int_{\pi/4}^{t} \sin 2t \, dt$$

$$= -\frac{3a}{2} \left[-\frac{\cos 2t}{2} \right]_{\pi/4}^{t}$$

$$= \tfrac{3}{4} a \cos 2t. \qquad \text{...(5)}$$

Eliminating t between (2) and (5), the required intrinsic equation of the curve is $s = \frac{3}{4} a \cos \{2(-\psi)\} = \frac{3}{4} a \cos 2\psi$.

[$\because \cos(-\theta) = \cos \theta$]

(ii) If s denotes the arc length of the given curve measured from the cusp on x-axis to any point P lying towards the second cusp on y-axis, then s increases as t increases. Therefore ds/dt will be positive. Hence from (3), we have

$$ds/dt = 3a \sin t \cos t \quad \text{or} \quad ds = 3a \sin t \cos t \, dt.$$

Also at the cusp on x-axis, we have $t = 0$.

$$\therefore \quad s = \int_0^t 3a \sin t \cos t \, dt = 3a \left[\frac{\sin^2 t}{2} \right]_0^t$$

$= \frac{3}{2} a \sin^2 t$. ...(6)

Eliminating t between (2) and (6), the required intrinsic equation of the curve is

$$s = \tfrac{3}{2} a \sin^2 (-\psi) \quad \text{or} \quad s = \tfrac{3}{2} a \sin^2 \psi.$$

(iii) The whole length of the curve is already obtained as $6a$ in Ex. 17 (a), page 89.

Ex. 54. *Find the intrinsic equation of the cardioid*

$$r = a (1 - \cos \theta).$$

(Meerut 1993, 98; Gorakhpur 77, 73; Agra 70; Indore 72, 70)

Sol. The given curve is $r = a (1 - \cos \theta)$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = a \sin \theta$.

$$\therefore \quad \tan \phi = r \frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r}{a \sin \theta} = \frac{a (1 - \cos \theta)}{a \sin \theta}$$

$$= \frac{2 \sin^2 \frac{1}{2} \theta}{2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta} = \tan \tfrac{1}{2} \theta.$$

Therefore $\phi = \frac{1}{2} \theta$, so that

$$\psi = \theta + \phi = \theta + \tfrac{1}{2} \theta = \tfrac{3}{2} \theta, \text{ giving } \theta = \tfrac{2}{3} \psi. \quad ...(2)$$

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.*, the point $\theta = 0$) to any point $P (r, \theta)$ in the direction of θ increasing, we have

$$s = \int_0^\theta \sqrt{\left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}} \, d\theta = a \int_0^\theta \sqrt{\{(1 - \cos \theta)^2 + \sin^2 \theta\}} \, d\theta$$

$$= 2a \int_0^\theta \sin \tfrac{1}{2} \theta \, d\theta = 4a \left[- \cos \tfrac{1}{2} \theta \right]_0^\theta$$

$$= 4a (1 - \cos \tfrac{1}{2} \theta) = 8a \sin^2 \tfrac{1}{4} \theta. \quad ...(3)$$

Eliminating θ between (2) and (3), we get

$$s = 8a \sin^2 \{ \tfrac{1}{4} \cdot \tfrac{2}{3} \psi \} \quad \text{or} \quad s = 8a \sin^2 \tfrac{1}{6} \psi,$$

which is the required intrinsic equation.

****Ex. 55.** *Find the intrinsic equation of the cardioid*

$$r = a (1 + \cos \theta),$$ **(Meerut 1984, 86S, 98)**

and hence, or otherwise, prove that $s^2 + 9\rho^2 = 16a^2$, *where* ρ *is the radius of curvature at any point, and* s *is the length of the arc intercepted between the vertex and the point.*

(Agra 1981; Kanpur 76, 70; Lucknow 71; Meerut 86S, 98)

Sol. The given curve is $r = a (1 + \cos \theta)$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = - a \sin \theta$.

$$\therefore \quad \tan \phi = r \frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{a (1 + \cos \theta)}{- a \sin \theta} = \frac{2 \cos^2 \frac{1}{2} \theta}{- 2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta}$$

$$= -\cot \tfrac{1}{2}\theta = \tan\left(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta\right).$$

Therefore $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$, so that

$$\psi = \theta + \phi = \theta + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta = \tfrac{1}{2}\pi + \tfrac{3}{2}\theta$$

or
$$\tfrac{1}{2}\theta = \tfrac{1}{3}\left(\psi - \tfrac{1}{2}\pi\right). \qquad ...(2)$$

If s denotes the arc length of the cardioid measured from the vertex (*i.e.*, $\theta = 0$) to any point P (*i.e.*, $\theta = \theta$) in the direction of θ increasing, then

$$s = \int_0^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta = 2a \int_0^\theta \sqrt{\{(1 + \cos\theta)^2 + \sin^2\theta\}}\, d\theta$$

$$= 2a \int_0^\theta \sqrt{\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}}\, d\theta$$

$$= 2a \int_0^\theta \sqrt{\{2(1 + \cos\theta)\}}\, d\theta = 2a \int_0^\theta \cos\tfrac{1}{2}\theta\, d\theta$$

$$= 2a\left[2\sin\tfrac{1}{2}\theta\right]_0^\theta = 4a\sin\tfrac{1}{2}\theta. \qquad ...(3)$$

Eliminating θ between (2) and (3), we get

$$s = 4a\sin\left\{\tfrac{1}{3}\left(\psi - \tfrac{1}{2}\pi\right)\right\}, \qquad ...(4)$$

which is the required intrinsic equation.

Also $\rho = \dfrac{ds}{d\psi} = \dfrac{4a}{3}\cos\frac{1}{3}\left(\psi - \frac{1}{2}\pi\right)$, from (4)

or
$$3\rho = 4a\cos\tfrac{1}{3}\left(\psi - \tfrac{1}{2}\pi\right). \qquad ...(5)$$

Squaring and adding (4) and (5), we get

$$s^2 + 9\rho^2 = (4a)^2\left\{\sin^2\tfrac{1}{3}\left(\psi - \tfrac{1}{2}\pi\right) + \cos^2\tfrac{1}{3}\left(\psi - \tfrac{1}{2}\pi\right)\right\}$$

$$= 16a^2 . 1 = 16a^2.$$

Ex. 56. *Find the intrinsic equation of $r = a\, e^{\theta\cot\alpha}$, where s is measured from the point $(a, 0)$.*

Sol. The given curve is $r = a\, e^{\theta\cot\alpha}$. ...(1)

Differentiating (1), w.r.t. θ, we have

$$(dr/d\theta) = a\cot\alpha . e^{\theta\cot\alpha} = r\cot\alpha.$$

We have $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{dr/d\theta} = \dfrac{r}{r\cot\alpha} = \tan\alpha$

or
$$\phi = \alpha \text{ so that } \psi = \theta + \phi = \theta + \alpha \quad \text{or} \quad \theta = \psi - \alpha. \qquad ...(2)$$

If we measure the arc length s from the point $\theta = 0$ to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$s = \int_0^\theta \sqrt{\{(dr/d\theta)^2 + r^2\}}\, d\theta = \int_0^\theta \sqrt{\{r^2\cot^2\alpha + r^2\}}\, d\theta$$

$$= \int_0^\theta r\sqrt{(1 + \cot^2\alpha)}\, d\theta = \operatorname{cosec}\alpha \int_0^\theta r\, d\theta$$

$$= \operatorname{cosec}\alpha \int_0^\theta a\, e^{\theta\cot\alpha}\, d\theta, \qquad \left[\because r = a\, e^{\theta\cot\alpha}\right]$$

$$= a \operatorname{cosec} \alpha \left[\frac{e^{\theta \cot \alpha}}{\cot \alpha}\right]_0^\theta = a \sec \alpha \, [e^{\theta \cot \alpha} - 1] \cdot \qquad ...(3)$$

Eliminating θ between (2) and (3), we get

$$s = a \sec \alpha \left[e^{(\psi - \alpha) \cot \alpha} - 1\right],$$

which is the required intrinsic equation.

Ex. 57. *Show that in the parabola* $\dfrac{2a}{r} = 1 + \cos \theta$,

$$\frac{ds}{d\psi} = \frac{2a}{\sin^3 \psi} \cdot$$

Hence find the length of the arc intercepted between the vertex and an extremity of the latus rectum. **(Meerut 1988 P)**

Sol. The given equation of parabola is

$$\frac{2a}{r} = 1 + \cos \theta, \qquad ...(1)$$

in which the focus O is at pole.

From (1), we have

$$r = \frac{2a}{1 + \cos \theta} = \frac{2a}{2 \cos^2 \frac{1}{2}\theta} = a \sec^2 \tfrac{1}{2}\theta.$$

$$\therefore \quad \frac{dr}{d\theta} = 2a \sec \tfrac{1}{2}\theta \,.\, (\sec \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta) \cdot \tfrac{1}{2}$$

$$= a \sec^2 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta.$$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \sec^4 \tfrac{1}{2}\theta + a^2 \sec^4 \tfrac{1}{2}\theta \tan^2 \tfrac{1}{2}\theta$$

$$= a^2 \sec^4 \tfrac{1}{2}\theta \, (1 + \tan^2 \tfrac{1}{2}\theta) = a^2 \sec^6 \tfrac{1}{2}\theta.$$

If s is the arc length of the parabola measured from the vertex A (*i.e.*, the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, then

$$\frac{ds}{d\theta} = a \sec^3 \tfrac{1}{2}\theta. \qquad ...(2)$$

We have

$$\cot \phi = \frac{1}{r} \frac{dr}{d\theta} = \frac{1}{a \sec^2 \frac{1}{2}\theta} a \sec^2 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta$$

$$= \tan \tfrac{1}{2}\theta = \cot (\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) \cdot$$

$$\therefore \quad \phi = \tfrac{1}{2}\pi - \tfrac{1}{2}\theta.$$

$$\therefore \quad \psi = \theta + \phi = \theta + \tfrac{1}{2}\pi - \tfrac{1}{2}\theta = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta. \qquad ...(3)$$

Now $\dfrac{ds}{d\psi} = \dfrac{ds}{d\theta} \cdot \dfrac{d\theta}{d\psi}$

$$= (a \sec^3 \tfrac{1}{2}\theta) \,.\, 2, \qquad \left[\because \text{ from (3)}, \frac{d\psi}{d\theta} = \tfrac{1}{2} \text{ so that } \frac{d\theta}{d\psi} = 2\right]$$

$$= 2a \sec^3 (\psi - \tfrac{1}{2}\pi) = 2a \sec^3 (\tfrac{1}{2}\pi - \psi)$$

$$= \frac{2a}{\cos^3 (\frac{1}{2}\pi - \psi)} = \frac{2a}{\sin^3 \psi}.$$

Now from $\dfrac{ds}{d\psi} = 2a \operatorname{cosec}^3 \psi$, we have

$$ds = 2a \operatorname{cosec}^3 \psi \, d\psi. \qquad ...(4)$$

At the vertex $A, \theta = 0$ and at the extremity L of latus rectum $LOL', \theta = \pi/2$. So from (3), at $A, \psi = \frac{1}{2}\pi$ and at L, $\psi = 3\pi/4$.

Integrating both sides of (4) from the point A to L, we have

$$\text{arc}\, AL = \int_{\pi/2}^{3\pi/4} 2a \operatorname{cosec}^3 \psi \, d\psi$$

$$= 2a \int_{\pi/2}^{3\pi/4} \sqrt{(1 + \cot^2 \psi)} \,.\, \operatorname{cosec}^2 \psi \, d\psi$$

$$= 2a \int_{0}^{-1} \sqrt{(1 + t^2)} \,.\, (-dt),$$

putting $\cot \psi = t$, so that $-\operatorname{cosec}^2 \psi \, d\psi = dt$

$$= 2a \int_{-1}^{0} \sqrt{(1 + t^2)} \, dt$$

$$= 2a \left[\frac{t}{2}\sqrt{(1 + t^2)} + \tfrac{1}{2} \log \{t + \sqrt{(1 + t^2)}\}\right]_{-1}^{0}$$

$$= 2a \left[0 + \tfrac{1}{2} \log 1 - \{-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2} \log (-1 + \sqrt{2})\}\right]$$

$$= 2a \left[\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2} \log (\sqrt{2} - 1)\right]$$

$$= 2a \left[\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2} \log \left(\frac{1}{\sqrt{2} - 1}\right)\right]$$

$$= 2a \left[\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2} \log (\sqrt{2} + 1)\right]$$

$$= a \left[\sqrt{2} + \log (\sqrt{2} + 1)\right].$$

***Ex. 58.** *Find the intrinsic equation of the spiral $r = a\theta$, the arc being measured from the pole.*

Sol. The given curve is $r = a\theta$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = a$.

Therefore $\tan \phi = r \dfrac{d\theta}{dr} = \dfrac{r}{dr/d\theta} = \dfrac{a\theta}{a} = \theta.$

$\therefore \quad \phi = \tan^{-1} \theta$ so that $\psi = \theta + \phi = \theta + \tan^{-1} \theta.$ ...(2)

If s denotes the arc length of the spiral measured from the pole $(0, 0)$ to any point $P(r, \theta)$, then

$$s = \int_0^{\theta} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \, d\theta = \int_0^{\theta} \sqrt{(a^2\theta^2 + a^2)} \, d\theta$$

$$= a \int_0^{\theta} \sqrt{(\theta^2 + 1)}\, d\theta$$

$$= a \left[\frac{\theta}{2} \sqrt{(\theta^2 + 1)} + \tfrac{1}{2} \log \{\theta + \sqrt{(\theta^2 + 1)}\}\right]_0^{\theta}$$

$$= \tfrac{1}{2} a \left[\theta \sqrt{(1 + \theta^2)} + \log \{\theta + \sqrt{(1 + \theta^2)}\}\right]. \qquad ...(3)$$

The required intrinsic equation is obtained by eliminating θ between (2) and (3).

Ex. 59. *Find the intrinsic equation of the equiangular spiral $p = r \sin \alpha$.*

Sol. The given pedal equation of the curve is $p = r \sin \alpha$. ...(1)

Differentiating (1) w.r.t. r, we have

$$dp/dr = \sin \alpha.$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = r \frac{dr}{dp} = \frac{r}{dp/dr} = \frac{r}{\sin \alpha} = r \operatorname{cosec} \alpha. \qquad ...(2)$$

If we measure the arc length s from the point $r = 0$ in the direction of r increasing, we have

$$s = \int_0^r \frac{r\, dr}{\sqrt{(r^2 - p^2)}} = \int_0^r \frac{r\, dr}{\sqrt{(r^2 - r^2 \sin^2 \alpha)}} = \int_0^r \sec \alpha\, dr$$

$$= \sec \alpha \int_0^r dr = \sec \alpha \left[r \right]_0^r = r \sec \alpha. \qquad ...(3)$$

Eliminating r between (2) and (3), we have

$$\frac{(ds/d\psi)}{s} = \frac{\operatorname{cosec} \alpha}{\sec \alpha} = \cot \alpha, \qquad \text{[dividing (2) by (3)]}$$

or $\quad ds/s = \cot \alpha\, d\psi.$

Integrating, $\log s = \psi \cot \alpha + \log a$, where a is constant of integration

or $\quad \log (s/a) = \psi \cot \alpha \quad$ or $\quad s = a e^{\psi \cot \alpha}$,

which is the required intrinsic eqution of the curve.

Ex. 60. *Find the intrinsic equation of the curve $p^2 = r^2 - a^2$.*

(Meerut 1993 P)

Sol. The given curve is $p^2 = r^2 - a^2$ ...(1)

Differentiating (1) w.r.t r, we have

$$2p\, (dp/dr) = 2r \quad \text{or} \quad r\, (dr/dp) = p.$$

$$\therefore \quad \rho = \frac{ds}{d\psi} = r \frac{dr}{dp} = p = \sqrt{(r^2 - a^2)}, \qquad \text{[from (1)]}. \qquad ...(2)$$

Also from the equation of the curve we have $p = 0$ for $r = a$.

If we measure the arc length s (from $r = a$) in the direction of r increasing, we have

$$s = \int_a^r \frac{r\, dr}{\sqrt{(r^2 - p^2)}} = \int_a^r \frac{r\, dr}{a}, \qquad [\because r^2 - p^2 = a^2]$$

$$= \frac{1}{a}\left[\frac{r^2}{2}\right]_a^r = \frac{1}{2a}[r^2 - a^2]$$

or $\quad 2as = r^2 - a^2 \quad$ or $\quad \sqrt{(2as)} = \sqrt{(r^2 - a^2)}$ ...(3)

Eliminating r between (2) and (3), we have

$$\frac{ds}{d\psi} = \sqrt{(2as)} \quad \text{or} \quad \frac{ds}{\sqrt{s}} = \sqrt{(2a)}\, d\psi.$$

If $s = 0$ when $\psi = 0$, then integrating, we have

$$\int_0^s \frac{ds}{\sqrt{s}} = \sqrt{(2a)} \int_0^\psi d\psi.$$

$\therefore \quad 2\sqrt{s} = \sqrt{(2a)}\,\psi \quad$ or $\quad s = \frac{1}{2} a\,\psi^2,$

which is the required intrinsic equation.

Ex. 61. *Find the intrinsic equation of the curve for which the length of the arc measured from the origin varies as the square root of the ordinate. Find also parametric equations of the curve in terms of any parameter.*

Sol. Let s denote the arc length of the curve measured from the origin to any point $P(x, y)$ such that s increases as y increases. As given $s \propto \sqrt{y}$ so that $s = \lambda\sqrt{y}$, where λ is some constant.

Choosing this constant $\lambda = \sqrt{(8a)}$ **(Note)**, we have

$s = \sqrt{(8ay)} \quad$ or $\quad s^2 = 8ay.$ ...(1)

Now differentiating (1) w.r.t. y, we have

$2s\,(ds/dy) = 8a \quad$ or $\quad ds/dy = 4a/s.$ ...(2)

Now we know that $dy/ds = \sin\psi$.

$\therefore \quad \sin\psi = dy/ds = s/4a$, [from (2)]

or $\quad s = 4a \sin\psi$, which is the required intrinsic equation.

Again from (1), we have

$y = \dfrac{s^2}{8a} = \dfrac{16a^2 \sin^2\psi}{8a} = a(1 - \cos 2\psi)$, $\quad [\because s = 4a \sin\psi]$. ...(3)

Also $\dfrac{ds}{dx} = \dfrac{ds}{d\psi} \cdot \dfrac{d\psi}{dx} = 4a\cos\psi \dfrac{d\psi}{dx}$, $\quad \left[\because \dfrac{ds}{d\psi} = 4a\cos\psi\right]$

or $\quad \dfrac{1}{\cos\psi} = 4a\cos\psi \dfrac{d\psi}{dx}$, $\quad \left[\because \dfrac{dx}{ds} = \cos\psi\right]$

or $\quad dx = 4a\cos^2\psi\, d\psi = 2a(1 + \cos 2\psi)\, d\psi.$ ...(4)

If $x = 0$ when $\psi = 0$, then integrating (4), we get

$$\int_0^x dx = 2a \int_0^\psi (1 + \cos 2\psi)\, d\psi$$

or $\quad x = 2a\left[\psi + \frac{1}{2}\sin 2\psi\right]_0^\psi \quad$ or $\quad x = a[2\psi + \sin 2\psi]$. ...(5)

So from (3) and (5), the required parametric equations of the curve are

$$x = a\,(2\psi + \sin 2\psi) \quad \text{and} \quad y = a\,(1 - \cos 2\psi),$$

which are the parametric equations of a cycloid.

Ex. 62. *Find the cartesian equation of the curve whose intrinsic equation is $s = c \tan \psi$ when it is given that at $\psi = 0$, $x = 0$ and $y = c$.*

Sol. The given intrinsic equation of the curve is

$$s = c \tan \psi. \qquad ...(1)$$

Differentiating (1) w.r.t. 'x', we have

$$ds/dx = c \sec^2 \psi \,.\, (d\psi/dx). \qquad ...(2)$$

Also $$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{(1 + \tan^2 \psi)} = \sec \psi. \qquad ...(3)$$

Equating the values of ds/dx from (2) and (3), we get

$$c \sec^2 \psi \,.\, (d\psi/dx) = \sec \psi \quad \text{or} \quad dx = c \sec \psi \, d\psi. \qquad ...(4)$$

Integrating both sides of (1), we get

$$x + A = c \log (\sec \psi + \tan \psi),$$

where A is constant of integration.

But as given $\psi = 0$ when $x = 0$ so that $A = 0$.

Therefore $x = c \log (\sec \psi + \tan \psi)$

or $$e^{x/c} = \sec \psi + \tan \psi. \qquad ...(5)$$

Now $$e^{-x/c} = \frac{1}{e^{x/c}} = \frac{1}{\sec \psi + \tan \psi} = \frac{\sec \psi - \tan \psi}{\sec^2 \psi - \tan^2 \psi}$$

$$= \sec \psi - \tan \psi. \qquad ...(6)$$

Subtracting (6) from (5), we get

$$e^{x/c} - e^{-x/c} = 2 \tan \psi$$

or $$\tan \psi = \frac{e^{x/c} - e^{-x/c}}{2} \quad \text{or} \quad \frac{dy}{dx} = \sinh \frac{x}{c} \qquad \textbf{(Note)}$$

or $$dy = \sinh (x/c)\, dx.$$

Integrating both sides, we get

$$y + B = c \cosh (x/c). \qquad ...(7)$$

But (as given) when $x = 0, y = c$, so that $B = 0$.

Therefore putting $B = 0$ in (7), we get

$$y = c \cosh (x/c),$$

which is the required cartesian equation of the given curve.

□