CSE - 2018 Complex Analysis

1 (c) (10M) $\rightarrow$ $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$

$$\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6(x-1)$$

$$\frac{\partial u}{\partial y} = -6xy + 6y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6(x-1).$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x-1) - 6(x-1) = 0$$

$\therefore$ $u(x,y)$ is a harmonic function.

Let $f(z) = u(x,y) + i\,v(x,y)$ such that $f(z)$ be an analytic function then, by C-R conditions we get.

$$\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 = \frac{\partial v}{\partial y}$$

Integrating w.r.t y we get, $v = 3(x-1)^2 y - y^3 + f_1(x)$

Again $\frac{\partial u}{\partial y} = -6xy + 6y = -\frac{\partial v}{\partial x} = -[6(x-1)y + f_1'(x)]$

$$\Rightarrow f_1'(x) = 0 \Rightarrow f_1(x) = K \text{ (constant)}$$

So we get $\boxed{v = 3(x-1)^2 y - y^3 + K}$ $\Rightarrow$ harmonic conjugate of $u(x,y)$.

Now to find analytic function $f(z)$ in terms of z we will use Milne thompson's method,

Let $f(z) = u(x,y) + i\,v(x,y)$ where $z = x+iy$ & $\bar{z} = x-iy$.

Putting $z = \bar{z}$ we get, $\boxed{x = z \text{ \& } y = 0}$

$\Rightarrow f(z) = u(z,0) + i\,v(z,0)$

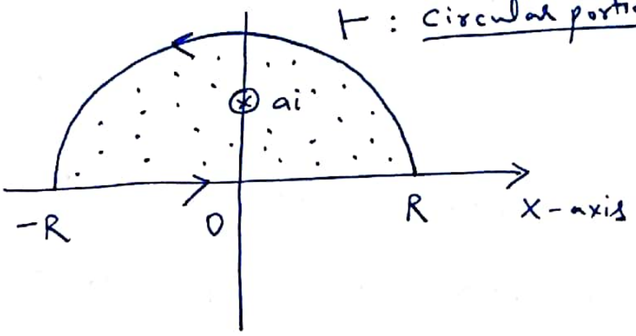
$\Rightarrow f(z) = (z-1)^3 - 3z(0)^2 + 3(0)^2 + i\left[3(z-1)^2(0) - (0)^3 + k\right]$

$$\Rightarrow \boxed{f(z) = (z-1)^3 + ki}$$

3(b) (15 M) $\rightarrow$ $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a > 0.$

Let $f(z) = \dfrac{1}{(z^2+a^2)^2}$ and $\displaystyle\int_C f(z)\,dz$ be the integration of $f(z)$ along the contour C where C is a semi-circle of radius R with centre as the origin and above x-axis.

Γ : Circular portion



$$\int_C f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{X\text{-axis}} f(z)\,dz$$

$f(z) = \dfrac{1}{(z^2+a^2)^2} \Rightarrow$ Poles $\equiv \pm ai$

In the given region only $+ai$ will be contained.

Residue at $z = ai$ [double pole] is $\dfrac{1}{1!} \displaystyle\lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 f(z)\right]$

$$\text{Residue at } (z = ai) = \lim_{z\to ai} \frac{-2}{(z+ai)^3} = \frac{-2}{(2ai)^3} = \frac{-i}{4a^3}.$$

$$\therefore \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_\Gamma f(z)\,dz = 2\pi i\left[\text{Residue at } z = ai\right].$$

Now $\displaystyle\left|\int_\Gamma \frac{dz}{(z^2+a^2)^2}\right| \le \int_\Gamma \frac{|dz|}{|z^2+a^2|^2} \le \int \frac{R\,d\theta}{(|z|^2-a^2)^2} = \int_0^\pi \frac{R\,d\theta}{(R^2-a^2)^2}$

Now as $R \to \infty$ $\dfrac{R\pi}{(R^2-a^2)^2} \to 0 \Rightarrow \left|\displaystyle\int_\Gamma \frac{dz}{(z^2+a^2)^2}\,dz\right| \to 0$

$$\Rightarrow \int_\Gamma \frac{dz}{z^2+a^2}\,dz = 0.$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = 2\pi i \left(\frac{-i}{4a^3}\right) = \frac{\pi}{2a^3}.$$

As $\frac{1}{(x^2+a^2)^2}$ is an even function we can write

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}\,dx = \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

$$\Rightarrow \boxed{\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.}$$

4 (b) (15 M) $\rightarrow$ $f(z) = \dfrac{1}{(z^2+2)(z+2)} = \dfrac{1}{5}\left[\dfrac{1}{(z+2)} - \dfrac{z-2}{(z^2+1)}\right]$

(i) $|z| < 1$

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - (z-2)(1+z^2)^{-1}\right].$$

$$f(z) = \frac{1}{10}\left[1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 \cdots\cdots\right] - \frac{1}{5}(z-2)\left(1 - z^2 + z^4 - z^6 + \cdots\cdots\right)$$

$$= \frac{1}{10}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n - \frac{z-2}{5}\sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Clearly it represents the taylor series expansion of $f(z)$.

(ii) $1 < |z| < 2$

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} - (z-2)\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}\right].$$

$$= \frac{1}{5}\left[\frac{1}{2}\left\{1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 \cdots\cdots\right\} - \frac{z-2}{z^2}\left\{1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 \cdots\cdots\right\}\right]$$

$$f(z) = \frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n(-1)^n - \frac{z-2}{5z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n .$$

Case (iii) $|z| > 2$.

$$f(z) = \frac{1}{5}\left[\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}\right].$$

$$= \frac{1}{5z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2\cdots\cdots\right) - \frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2\cdots\cdots\right)$$

$$= \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{z-2}{5z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n .$$