Set - IV

Limits and Continuity

Real valued functions:

Constant function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = K$, $(K \in \mathbb{R})$ is called a constant function.

Range of $f = \{K\}$ = a singleton set.

Identity function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is called the identity function.

Range of $f = \mathbb{R}$ = Domain of f.

polynomial function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$, where $a_0, a_1, a_2, \ldots a_n \in \mathbb{R}$, $n \in N$ and $a_n \neq 0$ is called a polynomial function of n^{th} degree.

If $a_0 = a_1 = \cdots\cdots = a_n = 0$ then $f(x) = 0$ $\forall x \in \mathbb{R}$.

In this case we say that f is a zero polynomial function.

Rational function: If f, g are two polynomial functions and $A = \{x / x \in \mathbb{R}, g(x) \neq 0\}$ then the function $h: A \to \mathbb{R}$ defined by $h(x) = \frac{f(x)}{g(x)}$ is called a rational function.

Ex: $h(x) = \frac{1}{x}$ is a rational function with domain $\mathbb{R} - \{0\}$.

power function: A function $f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = x^n$ where $n \in \mathbb{R}$ is called power function.

Ex: $f(x) = \sqrt{x}$ is the square root function defined from $[0, \infty)$ to $\mathbb{R}$.

→ **Absolute value function (or) Mod function:**

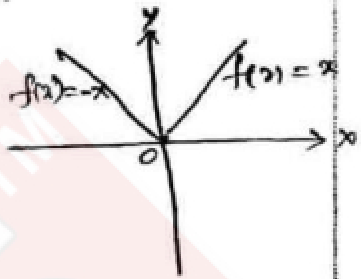
The function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x \text{ if } x \geq 0$$
$$= -x \text{ if } x < 0$$

is called mod function.

It is denoted by $f(x) = |x|$.

Range of $f = [0, \infty)$.

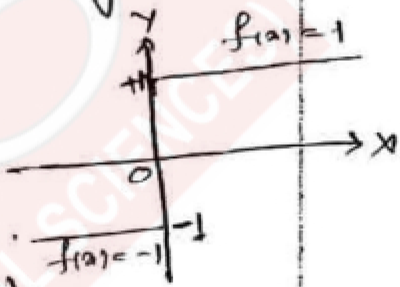


→ **Signature function:**

The function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1 \;;\; x > 0$$
$$= 0 \;;\; x = 0$$
$$= -1 \;;\; x < 0$$

is called signature function.



It is denoted by $f(x) = \operatorname{sgn}(x)$.

i.e, $\operatorname{sgn}(x) = 1$ if $x > 0$
$= 0$ if $x = 0$
$= -1$ if $x < 0$

i.e, $\operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Range of $\operatorname{sgn}(x) = \{-1, 0, 1\}$

→ **Integral part function or step function or greatest integer function:**

The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$ = integral part of x, is called step function.

i.e, $f(x) = [x]$ is a greatest integer $\leq x$, is called the greatest integer function.

i.e, for every $x \in \mathbb{R}$, $\exists$ unique $n \in \mathbb{Z}$ such that $n \le x < n+1$ and $[x] = n$.

The range of the step function $= \mathbb{Z}$.

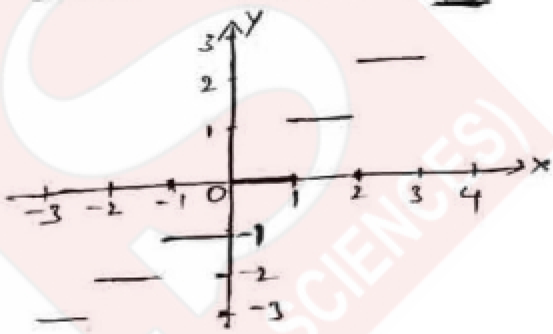
Ex: $x = 2.5$; $[x] = 2$ since $2 < x < 3$. i.e, $2 < x < 2+1$

$x = 0.1$; $[x] = 0$ since $0 < x < 1$

$x = 0$; $[x] = 0$ since $0 \le x < 1$.

$x = -2.5$; $[x] = -3$ since $-3 < x < -2$

$x = 1.5$; $[x] = 1$



→ **Exponential function:**

The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is called exponential function.

The range of exponential function $= \mathbb{R}^+$.

The range of exponential function is also

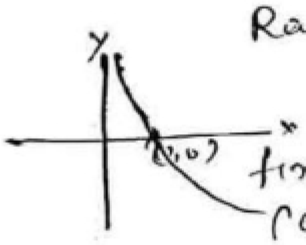
→ If $a \in \mathbb{R}^+ - \{1\}$ then $f(x) = a^x$ from $\mathbb{R} \to \mathbb{R}^+$ is also called exponential function.

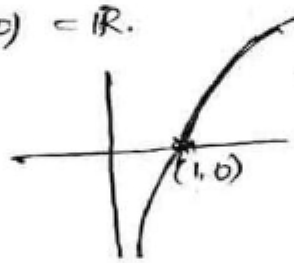
→ **Logarithmic function:**

The exponential function $f: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$ is both 1-1 and onto. The inverse function of this exponential function is called Logarithmic function.

$f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \log x$ is the natural logarithmic function.

Range $= (-\infty, \infty) = \mathbb{R}$.





→ Trigonometric functions:

— The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is called sine function.

$$\text{Range } f = [-1, 1]$$

— The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$ is called cosine function.

$$\text{Range } f = [-1, 1]$$

— If $A = \{x \in \mathbb{R} / x = n\pi + \frac{\pi}{2} \text{ (or) } (2n+1)\frac{\pi}{2} ; n \in \mathbb{Z}\}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin x}{\cos x} = \tan x$$

is called tangent function.

Domain $f = \mathbb{R} - A$ (odd values)

Range $f = \mathbb{R}$.

— If $A = \{x \in \mathbb{R} / x = n\pi : n \in \mathbb{Z}\}$ then the function $f: \mathbb{R} - A \to \mathbb{R}$ defined by $f(x) = \frac{\cos x}{\sin x} = \cot x$, is called cotangent function.

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R}$.

— If $A = \{x \in \mathbb{R} / x = (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$. defined by $f(x) = \frac{1}{\cos x} = \sec x$.

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R} - (-1, 1)$.

— If $A = \{x \in \mathbb{R} / x = n\pi : n \in \mathbb{Z}\}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sin x} = \operatorname{cosec} x$. is called cosecant function.

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R} - (-1, 1)$.

Boundedness of a function:

A function f is said to be bounded if its range is bounded. Otherwise it is unbounded

i.e, A function f is said to be bounded on a domain D if there exist two real numbers h, k such that $h \leq f(x) \leq k \ \forall x \in D$.

where h is called a lower bound of f.

k is called an upperbound of f.

(or)

A function f is said to be bounded on a domain D if there exist a +ve real number M (i.e, $M > 0$) such that $|f(x)| \leq M \ \forall x \in D$.

Ex: $f(x) = \sin x$, $f(x) = \cos x$ are bounded functions on $\mathbb{R}$

But $f(x) = \tan x$ is not bounded on $\mathbb{R}$.

Cluster point of a set or Limit point of a set:

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$ there exists atleast one point $x \in A$, $x \neq c$, such that $|x - c| < \delta$.

i.e, $0 < |x - c| < \delta$.

(or)

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if every δ-nbd of c contains atleast one point of A. other than c.

i.e, $\delta > 0$, $(c - \delta, c + \delta)$ contains atleast one point of the set A other than c.

(or)

A point $c \in \mathbb{R}$ is a cluster point of A if every

nbd of C contains infinitely many points of A
i.e, $\delta > 0$, $(c-\delta, c+\delta)$ contains infinitely many points of A.

Ex: (1) For the open interval $A_1 = (0,1)$, every point of the closed interval $[0,1]$ is a cluster point of A_1.
- The points 0 & 1 are cluster points of A_1, but donot belong to A_1.
- All the points of A_1 are cluster points of A_1

(2) A finite set has no cluster points.
(3) The infinite set N has no cluster points.
(4) The set $A_4 = \{\frac{1}{n} / n \in N\}$ has only the point '0' as a cluster point.
None of the points in A_4 is a cluster point of A_4.

Note: A cluster point of the set A may (or) may not belong to the set A.

Limit of a function:

Let $A \subseteq \mathbb{R}$ and let C be a cluster point of A. For a function $f: A \to R$, a real number L is said to be a limit of f at c, if given any $\epsilon > 0$, there exists a $\delta > 0$ (depending on ϵ, i.e, $\delta(\epsilon)$) such that if $x \in A$ and $0 < |x-c| < \delta$ then $|f(x) - L| < \epsilon$.

$|f(x) - L| < \epsilon$ whenever $0 < |x-c| < \delta$.
i.e, $|f(x) - L| < \epsilon$ $\forall x \in (c-\delta, c+\delta)$; $x \neq c$
i.e, $f(x) \in (L-\epsilon, L+\epsilon)$

Note: (1) If L is a limit of f at 'c' then we also say that f cgs to L at c.

we write $\lim_{x\to c} f(x) = L$ (or) $\lim_{x\to c} f = L$

we also say that $f(x)$ approaches L as x approaches c.

i.e, $f(x) \to L$ as $x \to c$.

(2) If the limit of 'f' at 'c' doesnot exist, we say that f diverges at c.

(3) If 'c' is not a cluster point of A then 'f' doesnot discuss the limit of a function at 'c'.

(4) The function f may (or) may not be defined at the limit points.

Ex: If $A = (0,1)$ and if $f: A \to \mathbb{R}$ and $f(x) = \frac{1}{x(1-x)}$ then

1 is a cluster point of A.

but f is not defined at 1.

similarly at '0'.

(5) In order to prove that $\lim_{x\to c} f(x) \neq L$, we have to show that for an $\epsilon > 0$, and any $\delta > 0$ there is $x \in A$, $0 < |x-c| < \delta \Rightarrow |f(x) - L| \geq \epsilon$.

(6) If $f: A \to \mathbb{R}$ and if c is a cluster point of A then f can have only one limit at 'c'.

Sequential Criterion:

Let $f: A \to \mathbb{R}$ and let 'c' be a cluster point of A then the following are equivalent.

(i) $\lim_{x\to c} f(x) = L$

(ii) for every (x_n) in A that cgs to c such that

$x_n \neq c$ $\forall n \in \mathbb{N}$, the sequence $(f(x_n))$ cgs to L.

→ Use the $\epsilon-\delta$ definition of limit, to show that

$\underset{x\to c}{Lt} \frac{1}{x} = \frac{1}{c}$, $c>0$.

Sol: Let $f(x) = \frac{1}{x}$; $x>0$

and let $c>0$.

To show that $\underset{x\to c}{Lt} f(x) = \frac{1}{c}$.

For this we are enough to show that for any $\epsilon>0$, $\exists$ a $\delta>0$ (depends on ϵ) such that

$\left|f(x)-\frac{1}{c}\right| < \epsilon$ whenever $0<|x-c|<\delta$.

Now we have $\left|f(x)-\frac{1}{c}\right| = \left|\frac{1}{x}-\frac{1}{c}\right|$

$= \left|\frac{c-x}{xc}\right|$

$= \frac{|x-c|}{|cx|}$ ——— (1)

for $x \to c$,

By taking x sufficiently close to c

we have $0<|x-c|<\frac{1}{2}c$. $(\because \delta \leq \frac{1}{2}c)$

$\Rightarrow |x-c|>0$ and $|x-c|<\frac{1}{2}c$.

$x\neq c$ and $-\frac{1}{2}c < x-c < \frac{1}{2}c$.

$x\neq c$ and $\frac{c}{2} < x < \frac{3}{2}c$.

$x\neq c$ and $\frac{c}{2} < x$.

$x\neq c$ and $xc > \frac{c^2}{2}$

and $|xc| > \frac{c^2}{2}$

and $\frac{1}{|xc|} < \frac{2}{c^2}$

$\therefore$ (1) $\equiv$ $\left|f(x)-\frac{1}{c}\right| < \frac{2}{c^2}|x-c|$

$<\epsilon$ whenever $|x-c| < \frac{c^2}{2}\epsilon$.

Choosing $\delta = \min\{\frac{1}{2}c, \frac{c^2}{2}\epsilon\}$

$\therefore \left|f(x) - \frac{1}{c}\right| < \epsilon$ whenever $0 < |x-c| < \delta$.

$\therefore f(x) \to \frac{1}{c}$ as $x \to c$.

i.e. $\underset{x\to c}{Lt} f(x) = \frac{1}{c}$; $c > 0$.

→ Use either the ϵ-δ definition of limit (or) the sequential criterion for ~~limits~~ limits to establish the following limits.

(1) $\underset{x\to 2}{Lt} \frac{1}{1-x} = -1$ (2) $\underset{x\to 1}{Lt} \frac{x}{1+x} = \frac{1}{2}$

(3) $\underset{x\to 0}{Lt} \frac{x^2}{|x|} = 0$ (4) $\underset{x\to 1}{Lt} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

Solⁿ:

(i) ϵ-δ method

Let $f(x) = \frac{1}{1-x}$, then we prove that $\underset{x\to 2}{Lt} f(x) = -1$.

For this we are enough to prove that for each $\epsilon > 0$ $\exists$ a $\delta > 0$ such that $|f(x) - (-1)| < \epsilon$ whenever $0 < |x-2| < \delta$.

we have

$$
\begin{aligned}
|f(x) - (-1)| &= \left|\frac{1}{1-x} - (-1)\right| \\
&= \left|\frac{1}{1-x} + 1\right| \\
&= \left|\frac{2-x}{1-x}\right| \\
&= \left|\frac{x-2}{x-1}\right|
\end{aligned}
$$

$\therefore |f(x) - (-1)| = \left|\frac{x-2}{x-1}\right|$ ——— (1)

for $x \to 2$,

By taking x sufficiently close to 2.

we have $0 < |x-2| < 1 \quad (\because 0 < \delta \le 1)$

$\Rightarrow |x-2| > 0$ and $|x-2| < 1$

$\Rightarrow x \neq 2$ and $-1 < x-2 < 1$

$\Rightarrow x \neq 2$ and $2-1 < x < 2+1$

i.e., $1 < x < 3$

Since $x > 1$

$\Rightarrow x-1 > 0 \Rightarrow \frac{1}{x-1} > 0$

$\Rightarrow \left|\frac{1}{x-1}\right| > 0$

$\Rightarrow 0 < \left|\frac{1}{x-1}\right| \le 1$

$\Rightarrow \frac{1}{|x-1|} \le 1$

$\therefore$ (1) $\equiv |f(x) - (-1)| \le 1 \cdot |x-2|$

$< \epsilon$ whenever $|x-2| < \frac{\epsilon}{1}$

choosing $\delta = \min\left\{1, \frac{\epsilon}{1}\right\}$

$\therefore |f(x) - (-1)| < \epsilon$ whenever $0 < |x-2| < \delta$.

$\therefore f(x) \to -1$ as $x \to 2$

$\therefore \lim_{x \to 2} f(x) = -1$

Sequential Method:

Let $f(x) = \frac{1}{1-x}$; $c = 2$

Take $x_n = \frac{2n}{n+1} \quad \forall n \in N$

$\therefore \lim_{n \to \infty} x_n = 2$

$\therefore f(x_n) = \frac{1}{1-x_n}$

$= \frac{1}{1-\frac{2n}{n+1}} = \frac{n+1}{1-n}$

$$\underset{n\to\infty}{Lt} f(x_n) = \underset{n\to\infty}{Lt}\left(\frac{1+n}{1-n}\right)$$

$$= \underset{n\to\infty}{Lt}\left(\frac{\frac{1}{n}+1}{\frac{1}{n}-1}\right)$$

$$= \frac{0+1}{0-1} = -1$$

$\therefore$ The sequence $(f(x_n))$ cgs to $-1 = L$

$$\therefore \underset{x\to 2}{Lt} f(x) = L$$

$$\Rightarrow \underset{x\to 2}{Lt} f(x) = -1$$

(3) By ϵ-δ method:

$$\left|\frac{x^2}{|x|} - 0\right| = \frac{|x^2|}{|x|}$$

$$= \frac{|x|^2}{|x|}$$

$$= |x| < \epsilon \text{ (say)}$$

$\therefore \left|\frac{x^2}{|x|} - 0\right| < \epsilon$ whenever $|x| < \delta = \epsilon$.

$\therefore \forall \epsilon > 0, \exists \delta = \epsilon > 0$ such that

$\left|\frac{x^2}{|x|} - 0\right| < \epsilon$ whenever $|x - 0| < \delta$

$$\therefore \underset{x\to 0}{Lt} \frac{x^2}{|x|} = 0$$

Divergence Criteria:

$A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

(a) If $L \in \mathbb{R}$ then f doesnot have limit 'L' at 'c' iff there exists a sequence (x_n) in A with $x_n \neq c$ $\forall n \in \mathbb{N}$ such that the sequence (x_n) cgs to 'c' but the sequence $(f(x_n))$ doesnot converge to L

(b) The function f doesnot have a limit at 'c' iff there exists a sequence (x_n) in A with $x_n \neq c$ $\forall n \in N$ such that the sequence (x_n) cgs to 'c' but the sequence $(f(x_n))$ doesnot converge in $\mathbb{R}$.

Ex: $\lim_{x\to 0} \frac{1}{x}$ doesnot exist in $\mathbb{R}$.

Sol^n: Let $f(x) = \frac{1}{x}$; $c=0$

Let $x_n = \frac{1}{n}$ $\forall n$

then $\lim_{n\to\infty} x_n = 0 = c$

$\therefore$ (x_n) cgs to '0'.

Now $f(x_n) = \frac{1}{x_n} = \frac{1}{1/n} = n$ $\forall n$

$\therefore \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = +\infty$

$\therefore (f(x_n))$ is not cgt in $\mathbb{R}$.

$\therefore \lim_{x\to 0} f(x)$ doesnot exist in $\mathbb{R}$.

→ Show that the following limits do not exist.

(a) $\lim_{x\to 0} \frac{1}{x^2}$ $(x>0)$ (b) $\lim_{x\to 0} \frac{1}{\sqrt{x}}$ $(x>0)$

(c) $\lim_{x\to 0} \operatorname{sgn}(x)$ (d) $\lim_{x\to 0}(x + \operatorname{sgn}(x))$ (e) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

(f) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$.

Sol^n: (c) $\lim_{x\to 0} \operatorname{sgn}(x)$.

Let $f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

Now $\operatorname{sgn}(x) = \frac{|x|}{x}$ if $x \neq 0$.

Now we have to show that $sgn(x)$ doesnot have a limit at $x=0$.

Let $x_n = \frac{(-1)^n}{n}$ $\forall n$ then $\lim_{n\to\infty} x_n = 0$

$\therefore (x_n)$ cgs to '0'

Now $sgn(x_n) = \frac{(-1)^n/n}{|(-1)^n/n|} = (-1)^n$ $\forall n$

$\therefore \lim_{n\to\infty} sgn(x_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ +1 & \text{if } n \text{ is even.} \end{cases}$

$\therefore$ $sgn(x_n)$ doesnot convergent.

$\therefore \lim_{x\to 0} sgn(x)$ doesnot exist

(e) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Let $f(x) = \sin(1/x)$; $c=0$

By introducing two sequences (x_n) & (y_n).

Let $x_n = \frac{1}{n\pi}$ $\forall n$

Then $\lim_{n\to\infty} x_n = 0$

Now $f(x_n) = \sin(n\pi)$
$= 0$ $\forall n$

$\therefore \lim_{n\to\infty} f(x_n) = 0$

and let $y_n = \frac{1}{\frac{1}{2}\pi + 2n\pi}$

then $\lim_{n\to\infty} y_n = 0$

Now $f(y_n) = \sin\left(\frac{1}{\frac{1}{\frac{1}{2}\pi+2n\pi}}\right)$

$= \sin\left(\frac{1}{2}\pi + 2n\pi\right)$

$= 1$ $\forall n$

$\therefore \lim_{n\to\infty} f(y_n) = 1$

$\therefore \lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

$\therefore \lim_{n\to\infty} f(x)$ doesnot exist.

(f) $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$.

Let $f(x) = \sin\left(\frac{1}{x^2}\right)$; $c=0$

let $x_n = \frac{1}{\sqrt{n\pi}}$

Algebra of limits:

Let $A \subseteq \mathbb{R}$. Let f & g be two functions on A to $\mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Further let $b \in \mathbb{R}$ if $\underset{x\to c}{Lt} f = L$ and $\underset{x\to c}{Lt}\, g = M$.

then (i) $\underset{x\to c}{Lt}(f \pm g) = L \pm M$

(ii) $\underset{x\to c}{Lt}(fg) = LM$

(iii) $\underset{x\to c}{Lt}(bf) = bL$ (iv) $\underset{x\to c}{Lt}\left(\frac{f}{g}\right) = \frac{L}{M}$ provided $M \neq 0$.

Theorem Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

If $a \leq f(x) \leq b$ $\forall x \in A$; $x \neq c$

and $\underset{x\to c}{Lt} f(x)$ exists.

then $a \leq \underset{x\to c}{Lt} f(x) \leq b$.

Squeeze theorem:

Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$; $x \neq c$.

and if $\underset{x\to c}{Lt} f(x) = L = \underset{x\to c}{Lt} h(x)$

then $\underset{x\to c}{Lt} g(x) = L$

Note: If $x \in \mathbb{R}$, $x \geq 0$ then

we have (i) $-x \leq S(x) \leq x$

(ii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$

(iii) $x - \frac{1}{6}x^3 \leq S(x) \leq x$

(iv) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Here $\sin x = S(x)$ & $\cos x = C(x)$

Sol: Let $-1 \le C(t) \le 1 \quad \forall t \in \mathbb{R}$

if $x \geq 0$ then

$$-\int_{t=0}^{x} dt \le \int_0^x C(t)\,dt \le \int_0^x dt$$

$$\Rightarrow \boxed{-x \le S(x) \le x}$$

Integrating, we get

$$-\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$$

$$\Rightarrow -\frac{x^2}{2} \le \cos x - 1 \le \frac{x^2}{2}$$

$$\Rightarrow 1 - \frac{x^2}{2} \le \cos x \le 1 + \frac{x^2}{2}.$$

$$\Rightarrow \boxed{1 - \frac{x^2}{2} \le \cos x \le 1} \quad (\because \text{ range of } \cos x \le 1)$$

and so on.

Problems:

→ $\underset{x\to 0}{Lt}\, x^{3/2} = 0$; $(x>0)$

Sol: Let $g(x) = x^{3/2}$; $x>0$

we have $x < x^{1/2} \le 1$ for $0 < x \le 1$

$\Rightarrow x^2 < x^{3/2} \le x$ for $0 < x \le 1$.

is of the form $f(x) \le g(x) \le h(x)$.

where $f(x) = x^2$, $g(x) = x^{3/2}$; $h(x) = x$

$\therefore \underset{x\to 0}{Lt}\, f(x) = 0 = \underset{x\to 0}{Lt}\, h(x)$

$\therefore$ By squeeze theorem

$$\underset{x\to 0}{Lt}\, g(x) = 0$$

→ $\underset{x\to 0}{Lt}\, \sin x = 0$

Since $-x \le \sin x \le x \quad \forall x \geq 0$.

is of the form $f(x) = -x$; $g(x) = \sin x$; $h(x) = x$.

$\therefore \underset{x\to 0}{Lt}\, f(x) = 0 = \underset{x\to 0}{Lt}\, h(x)$

$\therefore$ By squeeze theorem $\underset{x\to 0}{Lt}\, g(x) = 0$.

$\rightarrow \underset{x\to 0}{Lt}\, \cos x = 1$

Solⁿ: Since $1-\frac{x^2}{2} \le \cos x \le 1$

$\rightarrow \underset{x\to 0}{Lt}\left(\frac{\cos x - 1}{x}\right) = 0$

Solⁿ: Since $1-\frac{x^2}{2} \le \cos x \le 1 \quad \forall x>0$

$$\Rightarrow -\frac{x^2}{2} \le \cos x - 1 \le 0$$

$$\Rightarrow -\frac{x}{2} \le \frac{\cos x - 1}{x} \le 0.$$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = -\frac{x}{2}$; $g(x) = \frac{\cos x - 1}{x}$

and $h(x) = 0$.

$\therefore \underset{x\to 0}{Lt}\, f(x) = 0 = \underset{x\to 0}{Lt}\, h(x)$

$\therefore$ By squeeze theorem

$\underset{x\to 0}{Lt}\, g(x) = 0$

$\rightarrow \underset{x\to 0}{Lt}\, \frac{\sin x}{x} = 1$

Solⁿ: Since $x - \frac{x^3}{6} \le \sin x \le x \quad \forall x>0$

$$\Rightarrow 1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \forall x>0$$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = 1-\frac{x^2}{6}$; $g(x) = \frac{\sin x}{x}$

and $h(x) = 1$

$\therefore \underset{x\to 0}{Lt}\, f(x) = 1 = \underset{x\to 0}{Lt}\, h(x)$

$\therefore$ By squeeze theorem

$\underset{x\to 0}{Lt}\, g(x) = 1$

$\rightarrow \underset{x\to 0}{Lt}\left(x \sin \frac{1}{x}\right) = ?$

Solⁿ: Let $f(x) = x \sin \frac{1}{x}$.

Since $-1 \le \sin\frac{1}{x} \le 1$; $x \neq 0$

$\Rightarrow -x \le x \sin\frac{1}{x} \le x$; $x > 0$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = -x$; $g(x) = x\sin\frac{1}{x}$ and $h(x) = x$

$\therefore \underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} h(x) = 0$

$\therefore$ By squeeze theorem,

$\underset{x\to 0}{Lt}\, g(x) = 0$

H.W → $\underset{x\to 0}{Lt}\, x\sin\left(\frac{1}{x^2}\right) = ?$; $x \neq 0$.

→ $\underset{x\to 0}{Lt}\, \operatorname{sgn}\left(\sin\frac{1}{x}\right) = ?$

Sol^n: Let $f(x) = \operatorname{sgn}\left(\sin\frac{1}{x}\right)$; $x \neq 0$

$$= \frac{\sin\frac{1}{x}}{\left|\sin\frac{1}{x}\right|} \quad ; x \neq 0$$

Since $\underset{x\to 0}{Lt}\sin\frac{1}{x}$ doesnot exist.

$\therefore \underset{x\to 0}{Lt}\, \operatorname{sgn}\left(\sin\frac{1}{x}\right)$ doesnot exist.

One-Sided Limits:

→ Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$

if $c \in \mathbb{R}$ is a cluster point of the set

$A \cap (c, \infty) = \{x \in A / x > c\}$

then we say that $L \in \mathbb{R}$ is a right-hand limit of 'f' at c if given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$.

i.e, $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

— The right-hand limit (RHL) is denoted by

$\underset{x\to c^+}{Lt} f(x)$ (or) $\underset{x\to c^+}{Lt} f$

(ii) If $c \in \mathbb{R}$ is a cluster point of the set

$$A \cap (-\infty, c) = \{x \in A \mid x < c\},$$

then we say that $L \in \mathbb{R}$ is a left-hand limit of 'f' at 'c'.

if given any $\epsilon > 0$, $\exists$ a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$

i.e., $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta$.

– The left-hand limit (LHL) is denoted by

$$\underset{x \to c-}{Lt} f(x) \quad (\text{or}) \quad \underset{x \to c-}{Lt} f.$$

<u>Existence of a limit</u>

$$\underset{x \to c}{Lt} f(x) = L \iff \underset{x \to c+}{Lt} f(x) = L = \underset{x \to c-}{Lt} f(x)$$ ✓

→ <u>Sequential Criteria</u>:

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$ then the following statements are equivalent.

(i) $\underset{x \to c+}{Lt} f(x) = L$

(ii) for every sequence (x_n) that cgs to 'c' such that $x_n \in A$ and $x_n > c$ $\forall n \in \mathbb{N}$, the sequence $(f(x_n))$ cgs to L.

$\boxed{\text{In this way for left-hand limit.}}$

<u>Examples</u>:

→ $\underset{x \to 0}{Lt} \, Sgn(x) = ?$

<u>Sol</u>: Let $f(x) = Sgn(x)$; $x \neq 0$

$$= \frac{x}{|x|}, \; x \neq 0$$

$$= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Now $\underset{x\to 0+}{Lt} f(x) = 1$ & $\underset{x\to 0-}{Lt} f(x) = -1$

$\therefore \underset{x\to 0+}{Lt} f(x) \neq \underset{x\to 0-}{Lt} f(x)$

$\therefore \underset{x\to 0}{Lt} f(x)$ doesnot exist.

→ $\underset{x\to 0}{Lt} \sin\left(\frac{1}{x}\right) = ?$

Let $f(x) = \sin\left(\frac{1}{x}\right)$; $x \neq 0$ (i.e., $x<0$ or $x>0$)

LHL: $\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{Lt} \sin\frac{1}{x}$.

now $x \to 0+$, (i.e $x>0$) $\sin\frac{1}{x}$ is finite and oscillates between -1 & 1

$\therefore$ It doesnot tend to any unique number.

$\therefore \underset{x\to 0+}{Lt} \sin\frac{1}{x}$ doesnot exist.

Similarly, $\underset{x\to 0-}{Lt} \sin\frac{1}{x}$ doesnot exist.

$\therefore \underset{x\to 0}{Lt} \sin\frac{1}{x}$ doesnot exist.

→ $\underset{x\to 0}{Lt} x\sin\frac{1}{x} = ?$

Soln: Let $f(x) = x\sin\frac{1}{x}$; $x \neq 0$

LHL: $\underset{x\to 0-}{Lt} f(x) = \underset{x\to 0-}{Lt} x\sin\frac{1}{x}$

$= 0 \times$ [finite number between -1 & $+1$]

$= 0$

RHL: $\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{Lt} x\sin\frac{1}{x}$

$= 0 \times$ (fininte number b/w -1 & $+1$)

$= 0$

$\therefore \underset{x\to 0+}{Lt} f(x) = \underset{x\to 0-}{Lt} f(x) = 0$

$\therefore \underset{x\to 0}{Lt} f(x) = 0$

→ Limits at infinity and Infinite limits :

(i) $\underset{x\to\infty}{Lt}\ f(x) = L$.
A function $f(x)$ is said to tend to 'L' as $x\to\infty$, if given any $\epsilon>0$ (however small) $\exists$ a +ve number K (depends on ϵ) such that $x\geq K \Rightarrow |f(x)-L|<\epsilon$.

(ii) $\underset{x\to-\infty}{Lt}\ f(x) = L$
A function $f(x)$ is said to tend to L as $x\to-\infty$, if given any $\epsilon>0$ (however small), $\exists$ a +ve number K (depends on ϵ) such that $x\leq -K \Rightarrow |f(x)-L|<\epsilon$

(iii) $\underset{x\to c}{Lt}\ f(x) = +\infty$
A function $f(x)$ is said to tend to ∞ as $x\to c$ if given any $K>0$ (however large) $\exists$ a +ve number 'δ' such that $0<|x-c|<\delta \Rightarrow f(x)>K$.

(iv) $\underset{x\to c}{Lt}\ f(x) = -\infty$
A function $f(x)$ is said to tend to $-\infty$ as $x\to c$, if given $K>0$ (however large), $\exists$ a $\delta>0$ such that $0<|x-c|<\delta \Rightarrow f(x)<-K$

(v) $\underset{x\to\infty}{Lt}\ f(x) = +\infty$
A function $f(x)$ is said to tend to $+\infty$ as $x\to\infty$ if any $K>0$ (however large), $\exists$ a number $K'>0$ such that $x\geq K' \Rightarrow f(x)>K$.

(vi) $\underset{x\to\infty}{Lt}\ f(x) = -\infty$.
A function $f(x)$ is said to tend to $-\infty$ as $x\to\infty$, if given $K>0$ (however large), $\exists$ a number $K'>0$ such that $x\geq K' \Rightarrow f(x)<-K$.

(vii) $\underset{x\to-\infty}{Lt}\ f(x) = \infty$.
A function $f(x)$ is said to tend to ∞ as $x\to-\infty$ if given $K>0$ (however large), $\exists$ a number $K'>0$ (depends on K) such that $x<-K' \Rightarrow f(x)>K$.

(viii) $\underset{x\to-\infty}{Lt}\ f(x) = -\infty$.
A function $f(x)$ is said to tend to $-\infty$ as $x\to-\infty$ if given any $K>0$ (however large), $\exists$ $K'>0$ (depends on K) such that $x<-K' \Rightarrow f(x)<-K$

Continuous functions:

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$ be a cluster point of A then we say that 'f' is continuous at 'c' if $\underset{x \to c}{Lt} f(x) = f(c)$

$$\text{(or)} \quad \underset{x \to c-}{Lt} f(x) = \underset{x \to c+}{Lt} f(x) = f(c)$$

(or)

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ and $c \in A$ be a cluster point of A then we say that f is continuous at 'c' if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depending on ϵ) such that if $x \in A$ satisfying $|x - c| < \delta$

then $|f(x) - f(c)| < \epsilon$

i.e., $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

i.e, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$

$\forall x \in (c - \delta, c + \delta)$

Continuous from the left at a point:

A function f is continuous from the left (or left continuous) at the point $x = c$ if $\underset{x \to c-}{Lt} f(x) = f(c)$.

(or)

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, $c \in A$ is a cluster point of $A \cap (-\infty, c]$ then we say that f is left continuous at 'c', if given any $\epsilon > 0$ (however small), $\exists$ a $\delta > 0$ (depends on ϵ) such that

$$c - \delta < x \le c \Rightarrow |f(x) - f(c)| < \epsilon.$$

Continuity from the right at a point:

A function f is continuous from the right (or) right continuous at the point $x = c$ if $\underset{x \to c+}{Lt} f(x) = f(c)$

(or)

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, $c \in A$ is a cluster point of $A \cap [c, \infty) = \{x \in A / x \ge c\}$ then we say that f is right continuous at 'c', if given any $\epsilon > 0$ (however small) $\exists$ a $\delta > 0$ (depends on ϵ) such that

$$c \le x < c + \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Discontinuity:

If f is not continuous at 'c' then f is said to be discontinuous at 'c'

i.e, $\underset{x \to c}{Lt} f(x) \ne f(c)$

(or)

$A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, $c \in A$ is a cluster point of A then f is not continuous at 'c', if $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t x is any point of A satisfying $|x - c| < \delta \Rightarrow |f(x) - f(c)| \geq \epsilon$.

<u>Note</u>:- ① If c is a cluster point of A then the following three conditions must hold for f to be continuous at 'c'.

(i) f should be defined at 'c'. (ie f(c) exists).

(ii). $\underset{x \to c}{Lt} f(x)$ exists and

(iii) $f(c)$ & $\underset{x \to c}{Lt} f(x)$ are equal.

② f is discontinuous at $x = c$ because of any one of the following reason:

(i) f is not defined at 'c'

(ii) $\underset{x \to c}{Lt} f(x)$ does not exist

i.e $\underset{x \to c-}{Lt} f(x) \neq \underset{x \to c+}{Lt} f(x)$

(or)

one of the limit does not exist or both of the limits donot exist

and

(iii) $\underset{x \to c}{Lt} f(x)$ & $f(c)$ exist but are not equal.

(12)

$\Rightarrow$ <u>Sequential criterion for continuity</u>:

A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ iff for every sequence (x_n) in A that cgs to 'c', the sequence $(f(x_n))$ cgs to $f(c)$.

$\Rightarrow$ <u>Discontinuity criterion</u>:-

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at 'c' iff there exists a sequence (x_n) in A that cgs to 'c', but the sequence $(f(x_n))$ does not converge to $f(c)$.

$\rightarrow$ Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $B \subseteq A$, we say that f is continuous on the set B if f is continuous at every point of B.

* Continuity in an open interval :-

A function f is said to be continuous in an open interval (a,b), if it is continuous at every point of (a,b).

i.e $\lim_{x\to c} f(x) = f(c)$, $\forall c\in(a,b)$

* Continuity in a closed interval :-

A function f is said to be conti. in a closed interval $[a,b]$ if it is

(i) right conti at 'a'
i.e $\lim_{x\to a+} f(x) = f(a)$

(ii) left conti at 'b'
i.e $\lim_{x\to b-} f(x) = f(b)$

(iii) conti in (a,b)
i.e $\lim_{x\to c} f(x) = f(c)$; $c\in(a,b)$.

→ A function which is not continuous even at a single point of an interval is said to be discontinuous in that interval.

* Types of Discontinuity :

① Removable discontinuity :-

If $\lim_{x\to c} f(x)$ exists but is not equal to $f(c)$ then f is said to be removable discontinuity at 'c'

i.e $\left(\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)\right) \neq f(c)$

Ex :- ①

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x\neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

Sol

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x}$

$= 1.$

at $x=0$,

$f(0) = 2$

$\therefore \lim_{x\to 0} f(x) \neq f(0).$

Ex :- ②

$$f(x) = \begin{cases} x^2-2 & \text{if } x>2 \\ 4-x & \text{if } x<2 \\ 1 & \text{if } x=2 \end{cases}$$

Sol LHL

$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (4-x)$

$= 4-2$

$= 2$

RHL

$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} x^2-2$

$= 4-2$

$= 2$

at $x=2$

$f(2) = 1$

$\left(\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x)\right) \neq f(2)$

i.e $\lim_{x\to 2} f(x) \neq f(2)$

(2) Discontinuity of first kind (or) jump discontinuity (or) Ordinary discontinuity :-

If $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ both exist but are not equal and $f(c)$ exists, it is equal to the either (or) neither of $\lim_{x\to c-} f(x)$ (or) $\lim_{x\to c+} f(x)$ then f is called discontinuity of first kind.

→ f is said to be discontinuity of first kind from the left at 'c' if $\lim_{x\to c-} f(x)$ exists but it is not equal to $f(c)$.

→ f is said to be discontinuity of the first kind from right at 'c' if $\lim_{x\to c+} f(x)$ exist but is not equal to $f(c)$.

Ex:- $f(x) = \begin{cases} x^2-2 & \text{if } x>2 \\ 3-x & \text{if } x<2 \\ 1 & \text{if } x=2 \end{cases}$

(3) Disconti. of second kind :-

→ If $\lim_{x\to c-} f(x)$ & $\lim_{x\to c+} f(x)$ both donot exist then f is called disconti. of 2^{nd} kind at c.

→ f is said to be a disconti. of the second kind from the left at 'c' if $\lim_{x\to c-} f(x)$ does not exist.

→ f is said to be a disconti of the second kind from the right at 'c' if $\lim_{x\to c+} f(x)$ does not exist.

Ex:- $f(x) = \begin{cases} \sin(1/x) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

Sol LHL

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \sin(1/x)$

$= l \quad (\because -1\le l\le 1)$

$\therefore l$ is finite number but it is not fixed because l rotates with -1 to 1.

$\therefore \lim_{x\to 0-} f(x)$ does not exist.

Hly RHL does not exist

(4) **Mixed discontinuity:-**

If a function f has discontinuity of the second kind on one side of 'c' and other side a discontinuity of first kind (or) may be continuous then f is called a mixed discontinuity at 'c'

(or)

If one of the limits $\lim_{x\to c-} f(x)$ & $\lim_{x\to c+} f(x)$ exist but not the other then f is called mixed discontinuous at c.

i.e $\lim_{x\to c-} f(x)$ does not exist and $\lim_{x\to c+} f(x)$ exists and may (or) may not equal to $f(c)$

(or)

$\lim_{x\to c+} f(x)$ does not exist and $\lim_{x\to c-} f(x)$ exists and may (or) may not equal to $f(c)$.

Ex:- $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x>0 \\ 2-x & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases}$

Sol:

(5). **Infinite disconti:-**

If one (or) both limits $\lim_{x\to c-} f(x)$ & $\lim_{x\to c+} f(x)$ are $-\infty$ (or) $+\infty$ then f is called infinite discontinuity at 'c'.

Ex:- $f(x) = \begin{cases} \frac{1}{x-2} & ; x\neq 2 \\ 0 & ; x=2. \end{cases}$

Sol:

Algebra of continuous functions:-

If $f(x)$ & $g(x)$ are continuous functions at $x=c$ then $\lim_{x\to c} f(x) = f(c)$ & $\lim_{x\to c} g(x) = g(c)$.

(i) $\lim_{x\to c} (f\pm g)(x) = \lim_{x\to c} (f(x)\pm g(x))$

$= \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$

$= f(c) \pm g(c)$.

$= (f\pm g)(c)$

(ii) $\lim_{x\to c} (f\cdot g)(x) = \lim_{x\to c} (f(x)\cdot g(x))$

$= \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

$= f(c)\cdot g(c)$

$= (fg)(c)$

(iii) $\lim_{x\to c} (af)(x) = \lim_{x\to c} [a f(x)]$

$= a \lim_{x\to c} f(x)$

$= a f(c) = (af)(c)$; $a\in\mathbb{R}$.

(iv) $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \lim_{x\to c}\frac{f(x)}{g(x)}$

$= \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

$= \frac{f(c)}{g(c)}$

$= \left(\frac{f}{g}\right)(c)$

provided $g\neq 0$.

problems

Using $\epsilon-\delta$ definition, prove that

(i) $f(x) = 3x+1$ is continuous at $x=2$

(ii) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x\neq 2 \\ 4 & \text{if } x=2\end{cases}$

sol (i) $f(x) = 3x+1$;

at $x=2$

$f(2) = 3(2)+1$
$= 7.$

Let $\epsilon>0$ be given,

we have

$|f(x)-f(2)| = |3x+1-7|$

$= |3x-6|$

$= 3|x-2| < \epsilon$

whenever $|x-2|<\frac{\epsilon}{3}$

If we choose $\delta=\frac{\epsilon}{3}$, then

$|f(x)-f(2)|<\epsilon$ whenever $|x-2|<\delta$

$\therefore f(x)$ is conti at $x=2$.

(ii) $f(x) = \frac{x^2-4}{x-2}$; $x\neq 2$

at $x=2$; $f(2)=4$.

(14)

Let $\epsilon>0$ be given,

Now we have

$|f(x)-f(2)| = \left|\frac{x^2-4}{x-2}-4\right|$

$= \left|\frac{x^2-4-4x+8}{x-2}\right|$

$= \left|\frac{x^2-4x+4}{x-2}\right|$

$= \left|\frac{(x-2)^2}{x-2}\right|$

$= |x-2| < \epsilon$

whenever $|x-2|<\frac{\epsilon}{1}$

choosing $\delta=\frac{\epsilon}{1}$.

$\therefore |f(x)-f(2)|<\epsilon$ whenever $|x-2|<\delta$

$\therefore f(x)$ is conti at $x=2$

→ The constant function $f(x)=k$ is conti on $\mathbb{R}$

→ $g(x)=x$ is conti. on $\mathbb{R}$

→ $h(x)=x^2$ is conti on $\mathbb{R}$

→ $\phi(x)=\frac{1}{x}$ is conti on $A=\{x\in\mathbb{R}/x\neq 0\}$

sol^n Let $x=c\in\mathbb{R}$ $(c\neq 0)$

then $\phi(c)=\frac{1}{c}$

and $\lim_{x\to c}\phi(x) = \frac{1}{c}$.

$\therefore \lim_{x\to c}\phi(x) = \phi(c)$.

$\therefore \phi(x)$ is conti at $x=c\in\mathbb{R}$ $(c\neq 0)$

→ $\phi(x) = \frac{1}{x}$ is not conti at $x=0$

because ϕ is not defined at $x=0$ and $\lim_{x\to 0}\phi$ does not exist

→ The signum function sgn is not conti. at $x=0$

Sol. Let $f(x) = sgn(x)$

$$= \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ +1 & \text{if } x>0. \end{cases}$$

$\lim_{x\to 0-} f(x) = -1$ & $\lim_{x\to 0+} f(x) = 1$

$\therefore \lim_{x\to 0-} f(x) \neq \lim_{x\to 0+} f(x)$

$\therefore \lim_{x\to 0} f(x)$ does not exist.

$\therefore f(x)$ is not cont at $x=0$.

→ Let $A = \mathbb{R}$ and let $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

which is known as Dirichlet's function.

→ S.T that the Dirichlet's function is not continuous any point of $\mathbb{R}$.

Sol Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Let $x = c \in \mathbb{R}$ then c is either rational or irrational number

If c is a rational number:

$\therefore f(c) = 1$

Let (x_n) be sequence of irrational numbers that converges to 'c'

Since $f(x_n) = 0 \ \forall n$ ($\because x_n$ is irrational)

$\therefore \lim_{n\to\infty} f(x_n) = 0 \neq f(c)$

$\therefore (f(x_n))$ does not converge to $f(c)$.

$\therefore f(x)$ is not continuous at the rational number 'c'.

If c is an irrational number:

$f(c) = 0$

Let (x_n) be a sequence of rational numbers that converges to 'c'

Since $f(x_n) = 1 \ \forall n$.

$\therefore \lim_{n\to\infty} f(x_n) = 1 \neq f(c)$

$\therefore (f(x_n))$ does not converge to $f(c)$.

$\therefore f(x)$ is not conti. at the irrational number 'c'.

(15)

P-II 2006 → Prove that the function f defined by

$$f(x)=\begin{cases}1, & \text{when } x \text{ is rational}\\ -1 & \text{when } x \text{ is irrational}\end{cases}$$

is nowhere continuous.

(1) → prove that the function f defined by

$$f(x)=\begin{cases}1/2, & \text{if } x \text{ is rational}\\ 1/3 & \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous everywhere.

→ Define $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}2x, & \text{for } x \text{ rational}\\ x+3, & \text{for } x \text{ irrational.}\end{cases}$$

Find all points at which g is continuous.

Sol Given that

$$g(x)=\begin{cases}2x, & \text{for } x \text{ rational}\\ x+3 & \text{for } x \text{ irrational}\end{cases}$$

Let x be any real number,

For each $n\in\mathbb{N}$, ∃ a rational number 'a_n' and an irrational number 'b_n' such that

$x-1/n < a_n < x+1/n$ and

$x-1/n < b_n < x+1/n$ ∀n.

$\Rightarrow |a_n-x|<\frac{1}{n}$ and $|b_n-x|<\frac{1}{n}$ ∀n∈N.

$\Rightarrow \lim_{n\to\infty} a_n = x$ & $\lim_{n\to\infty} b_n = x$.

$\Rightarrow \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$ ——— (1).

If g is continuous at 'x' then we must have

$$\lim_{n\to\infty} g(a_n) = g(x) = \lim_{n\to\infty} g(b_n)$$

But $g(a_n) = 2a_n$ &

$g(b_n) = b_n+3$.

$\therefore \lim_{n\to\infty} 2a_n = g(x) = \lim_{n\to\infty}(b_n+3)$

$\Rightarrow 2\lim_{n\to\infty} a_n = g(x) = \lim_{n\to\infty} b_n+3$

$\Rightarrow 2x = g(x) = x+3$ (by (1)).

$\Rightarrow 2x = x+3$

$\Rightarrow \boxed{x=3}$

∴ 3 is the only possible point of continuous and discontinuity at every other point.

Now we show that g is continuous at x=3.

at x=3, $g(3) = 6$.

Let ε>0 be given,

For a rational number 'x', we have

$|g(x)-g(3)| = |2x-6|$

$= 2|x-3|$ → (2)

For an irrational number 'x' we have

$|g(x)-g(3)| = |x+3-6|$

$= |x-3|$ → (3)

from ②,

$|g(x)-g(3)| = 2|x-3| < \epsilon$

whenever $|x-3| < \frac{\epsilon}{2}$

choosing $\delta = \frac{\epsilon}{2}$

$\therefore |g(x)-g(3)| < \epsilon$ whenever $|x-3| < \delta$

from ①,

$|g(x)-g(3)| = 1\cdot|x-3| < \epsilon$

whenever $|x-3| < \frac{\epsilon}{1}$.

choosing $\delta = \frac{\epsilon}{1}$

$\therefore |g(x)-g(3)| < \epsilon$ whenever $|x-3| < \delta$

$\therefore g(x)$ is continuous at $x=3$.

P-II 2001 → Let f be defined on $\mathbb{R}$ by setting $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$

show that f is continuous at $x = \frac{1}{2}$ but discontinuous at every other point.

IAS → show that the function f defined by

$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

is continuous only at $x=0$

P-II 2004 → S.T the function $f(x)$ defined on $\mathbb{R}$ by:

$f(x) = \begin{cases} x & \text{when } x \text{ is rational} \\ -x & \text{when } x \text{ is irrational} \end{cases}$

is continuous only at $x=0$.

* Examine the continuity of the following functions at the indicated point.

(i) $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$

(ii) $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$

(iii) $f(x) = \begin{cases} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} & \text{if } x \neq 0 \\ 1, & \text{if } x=0 \end{cases}$ at $x=0$.

(iv) $f(x) = \begin{cases} (x-a)\dfrac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1} & \text{if } x \neq a \\ 0, & \text{if } x=a \end{cases}$ at $x=a$.

(v) $f(x) = \begin{cases} \dfrac{x\, e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$ at $x=0$

Sol

(i) Since

$x \to 0^- \Rightarrow \frac{1}{x} \to -\infty$

and $x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty$.

LHL

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1}$

$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1}$

$= -1.$

RHL

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1}$

$$= \underset{x\to 0+}{Lt}\left[\frac{1-e^{-1/x}}{1+e^{-1/x}}\right]$$

$$= \frac{1-e^{-\infty}}{1+e^{-\infty}}$$

$$= \frac{1-0}{1+0} = 1.$$

$\therefore \underset{x\to 0+}{Lt} f(x) \neq \underset{x\to 0-}{Lt} f(x)$

$\therefore \underset{x\to 0}{Lt} f(x)$ does not exist

(iv) Since $x \to a- \Rightarrow (x-a) \to 0^-$

$\Rightarrow \frac{1}{x-a} \to -\infty$

and $x \to a+ \Rightarrow (x-a) \to 0+$

$\Rightarrow \frac{1}{x-a} \to +\infty.$

LHL

$$\underset{x\to a-}{Lt} f(x) = \underset{x\to a-}{Lt}(x-a)\left[\frac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}\right]$$

$$= 0\times\left[\frac{e^{-\infty}-1}{e^{-\infty}+1}\right]$$

$$= 0\left[\frac{0-1}{0+1}\right] = 0\times(-1)$$

$$= 0.$$

RHL

$$\underset{x\to a+}{Lt} f(x) = \underset{x\to a+}{Lt}(x-a)\left[\frac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}\right]$$

$$= \underset{x\to a+}{Lt}(x-a)\left[\frac{1-e^{-\frac{1}{x-a}}}{1+e^{-\frac{1}{x-a}}}\right]$$

$$= 0\times\left[\frac{1-e^{-\infty}}{1+e^{-\infty}}\right]$$

$$= 0\times\left[\frac{1-0}{1+0}\right]$$

$$= 0\times 1$$

$$= 0.$$

(16)

at $x=a$,

$f(a) = 0.$

$$\therefore \left(\underset{x\to a-}{Lt} f(x) = \underset{x\to a+}{Lt} f(x)\right) = f(a)$$

$$\Rightarrow \underset{x\to a}{Lt} f(x) = f(a).$$

$\therefore f(x)$ is conti at $x=a$.

→ Discuss the continuity of the following functions at $x=0$

(i) $f(x) = \begin{cases} 2^{1/x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(ii) $f(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(iii) $f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(iv) $f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(v) $f(x) = \begin{cases} \cos 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(vi) $f(x) = \begin{cases} x \cos 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(vii) $f(x) = \begin{cases} x^2 \cos 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

(viii) $f(x) = \begin{cases} \cos x & \text{if } x \geq 0 \\ -\cos x & \text{if } x < 0 \end{cases}$

Sol (i) Since $x \to 0+$
$\Rightarrow \frac{1}{x} \to +\infty$

$x \to 0-$
$\Rightarrow \frac{1}{x} \to -\infty.$

LHL

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} 2^{1/x}$$

$$= 2^{-\infty}$$

$$= \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0.$$

RHL

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} 2^{1/x}$$

$$= 2^{\infty} = \infty.$$

$\therefore \lim_{x\to 0+} f(x)$ does not exist

$\therefore$ f is disconti at $x=0$

(ii) Since $x \to 0- \Rightarrow \frac{1}{x} \to -\infty$

$x \to 0+ \Rightarrow \frac{1}{x} \to +\infty.$

LHL $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \sin \frac{1}{x}$

$$= l \quad (\because -1 \le l \le 1)$$

Since l is finite number but it is not fixed number because l rotates with -1 to 1

$\therefore \lim_{x\to 0-} f(x)$ does not exist

Sly RHL does not exist.

$\therefore$ f is disconti at $x=0$.

(iii) Since $x \to 0+ \Rightarrow \frac{1}{x} \to \infty$

$x \to 0- \Rightarrow \frac{1}{x} \to -\infty.$

LHL

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x \sin \frac{1}{x}$$

$$= 0 \times l \quad (\because -1 \le l \le 1)$$

$$= 0.$$

RHL $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x \sin \frac{1}{x}$

$$= 0 \times l \quad (\because -1 \le l \le 1).$$

$$= 0.$$

at $x=0$
$f(0) = 0.$

$\therefore \left[\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)\right] = f(0)$

$\Rightarrow \lim_{x\to 0} f(x) = f(0).$

$\therefore$ f is conti at $x=0$.

(v) Since $x \to 0- \Rightarrow \frac{1}{x} \to -\infty$

$x \to 0+ \Rightarrow \frac{1}{x} \to +\infty.$

LHL $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \cos \frac{1}{x}$

$$= l$$

Since l is finite number but is not fixed because l rotates with -1 to $+1$.

$\therefore \lim_{x\to 0-} f(x)$ does not exist.

Sly RHL does not exist

$\therefore$ f is not continuous at $x=0$.

→ Discuss the continuity of the following functions at $x=a$.

(i) $f(x)=\begin{cases}(x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x\neq a\\ 0 & \text{if } x=a.\end{cases}$

(ii) $f(x)=\begin{cases}(x-a)\cos\left(\frac{1}{x-a}\right) & \text{if } x\neq a\\ 0 & \text{if } x=a.\end{cases}$

(iii) $f(x)=\begin{cases}\frac{1}{x-a}\operatorname{cosec}\left(\frac{1}{x-a}\right) & \text{if } x\neq a\\ 0 & \text{if } x=a\end{cases}$

→ Examine the discontinuity of the following functions at the indicated point.

(i) $f(x)=\begin{cases}\frac{|x|}{x}, & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}$ at $x=0$

(ii) $f(x)=|x|+|x-1|$ at $x=0$ and $x=1$.

(iii) $f(x)=\begin{cases}\frac{x-|x|}{x}, & \text{if } x\neq 0\\ 1 & \text{if } x=0;\end{cases}$

(iv) $f(x)=\begin{cases}\frac{|x-2|}{x-2} & \text{if } x\neq 2\\ -1 & \text{if } x=2\end{cases}$; at $x=2$

(v) $f(x)=\begin{cases}\frac{x}{|x|+x^2} & \text{if } x\neq 0\\ 0 & \text{if } x=0;\end{cases}$ at $x=0$.

(vi) $f(x)=\begin{cases}\frac{|x|}{x^2+2x} & \text{if } x\neq 0\\ 0 & \text{if } x=0,\end{cases}$ at $x=0$

(vii) $f(x)=\begin{cases}\frac{2|x|+x^2}{x}, & \text{if } x\neq 0,\\ 2 & \text{if } x=0,\end{cases}$ at $x=0$

Sol. (i)

Since $|x| = x$ if $x\geq 0$
$= -x$ if $x<0$

LHL

$\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt}\frac{|x|}{x} = \underset{x\to 0^-}{Lt}\frac{(-x)}{x}$
$= \underset{x\to 0^-}{Lt}(-1)$
$=-1.$

RHL

$\underset{x\to 0^+}{Lt} f(x) = +1.$

at $x=0$, $f(0)=1.$

$\therefore \underset{x\to 0^-}{Lt} f(x) \neq \underset{x\to 0^+}{Lt} f(x).$

$\therefore f(x)$ is not conti at $x=0$.

(ii) $f(x) = |x|+|x-1|.$

$x<0$, $0\leq x\leq 1$, $1<x$.

if $x<0$ then $|x|=-x$ & $|x-1| = -(x-1) = 1-x.$

$\therefore f(x) = |x|+|x-1|$
$= -x+1-x$
$= 1-2x$

If $0 \le x \le 1$ then $|x| = x$

and $|x-1| = -(x-1) = 1-x$.

$\therefore f(x) = |x| + |x-1|$

$= x + 1 - x$

$= 1.$

If $x > 1$ then $|x| = x$ and

$|x-1| = x-1$.

$\therefore f(x) = x + x - 1$

$= 2x - 1.$

$\therefore f(x) = \begin{cases} 1-2x & \text{if } x<0 \\ 1 & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } x>1 \end{cases}$

Continuity at $x=0$:-

at $x=0$, $f(0) = 1$.

LHL $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1-2x)$

$= 1 - 2(0) = 1.$

RHL $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1)$

$= 1.$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0).$

$\therefore$ f is conti at $x=0$.

Continuity at $x=1$:

At $x=1$, $f(1) = 1$.

LHL $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1)$

$= 1$

RHL $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1)$

$= 2(1) - 1$

$= 1.$

$\therefore \left(\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)\right) = f(1)$

$\Rightarrow \lim_{x\to 1} f(x) = f(1).$

$\therefore$ f is conti. at $x=1$.

(iv)

at $x=2$ $f(2) = -1$.

LHL $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{|x-2|}{x-2}$

$= \lim_{x\to 2^-} \frac{-(x-2)}{x-2}$

$\left(\because x \to 2^- \text{ then } x<2 \text{ i.e. } (x-2)<0\right)$

$= \lim_{x\to 2^-} (-1)$

$= -1.$

RHL $\lim_{x\to 2^+} f(x) = +1.$

$\therefore \lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} f(x).$

$\therefore \lim_{x\to 2} f(x)$ does not exist.

$\therefore$ f is not conti. at $x=2$.

Prob (IFoS 2008) Let $f: \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c & \text{if } x=0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{b\,x^{3/2}}, & \text{if } x>0. \end{cases}$$

Determine the values of a, b, c for which the function is continuous at $x=0$.

Sol at $x=0$,

$f(0)=c$.

LHL

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\sin(a+1)x+\sin x}{x}$

$= \lim_{x\to 0-}\left[\frac{\sin(a+1)x}{x}+\frac{\sin x}{x}\right]$

$= \lim_{x\to 0-}\left[(a+1)\frac{\sin(a+1)x}{(a+1)x}\right] + \lim_{x\to 0^-}\frac{\sin x}{x}$

$= (a+1)\lim_{(a+1)x\to 0^-}\frac{\sin(a+1)x}{(a+1)x} + \lim_{x\to 0^-}\frac{\sin x}{x}$

$(\because x\to 0^- \Rightarrow (a+1)x\to 0^-)$

$= (a+1)(1)+(1)$ $\left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right)$

$= a+2$.

RHL

$\lim_{x\to 0+} f(x) = \lim \frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}}$

$= \lim_{x\to 0+} \frac{x^{1/2}(1+bx)^{1/2}-x^{1/2}}{bx^{3/2}}$

$= \lim_{x\to 0+}\frac{(1+bx)^{1/2}-1}{bx}$.

$= \lim_{x\to 0+}\frac{(\sqrt{1+bx}-1)}{bx}\times\frac{(\sqrt{1+bx}+1)}{(\sqrt{1+bx}+1)}$

$= \lim_{x\to 0+}\frac{1+bx-1}{bx(\sqrt{1+bx}+1)}$

$= \lim_{x\to 0+}\frac{1}{(\sqrt{1+bx}+1)}$

$= \frac{1}{\sqrt{1+b(0)}+1}$ $(\because b\neq 0)$

$= \frac{1}{2}$ This is the independent of b

so that b can have any non-zero real value.

Since f is conti at $x=0$.

we have,

$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$

$\Rightarrow a+2 = \frac{1}{2} = c$

$\Rightarrow a+2=\frac{1}{2}$ & $c=\frac{1}{2}$

$\Rightarrow a=-\frac{3}{2}$ & $c=\frac{1}{2}$

$\therefore a=-\frac{3}{2}$, $b\neq 0$ & $c=\frac{1}{2}$.

$\Longrightarrow$ Discuss the continuity of the function $f(x)=[x]$ at the points $\frac{1}{2}$ & 1, where $[x]$ denotes the greatest integer $\leq x$.

Sol $f(x)=[x]$.

conti at $x=\frac{1}{2}$:-

at $x=\frac{1}{2}$, $f(\frac{1}{2})=[\frac{1}{2}]=0$

LHL

$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-}[x]$

$= 0$ $\left(\because x\to \frac{1}{2}^- \Rightarrow x<\frac{1}{2}=.5 \quad x=.4, .49, .499, .4995\right)$

RHL $\lim_{x\to \frac{1}{2}+} f(x) = \lim_{x\to\frac{1}{2}+}[x] = 0$.

$\therefore \left(\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x)\right) = f(\tfrac{1}{2})$

$\therefore f(x)$ is conti at $x=\frac{1}{2}$

conti. at $x=1$:

$f(1) = [1] = 1.$

LHL $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} [x]$

$= 0 \quad \left(\because x\to 1- \Rightarrow x<1,\ x = .6, .7, .8, .9 \ldots\right)$

RHL $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} [x]$

$= 1. \quad \left(\because x\to 1+ \Rightarrow x>1,\ x = 1.1, 1.2, 1.3 \ldots\right)$

$\therefore \lim_{x\to 1-} f(x) \neq \lim_{x\to 1+} f(x)$

$\therefore f(x)$ is not conti at $x=1$

→ Discuss the continuity of f at $x=1$, where $f(x) = [1-x] + [x-1]$.

Sol $f(1) = [1-1] + [1-1]$
$= [0] + [0]$
$= 0.$

LHL $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} [1-x] + [x-1]$

$= \lim_{x\to 1-} [1-x] + \lim_{x\to 1+} [x-1]$

$= 0 + (-1) = -1.$

$\left(\because x\to 1- \Rightarrow x<1 \Rightarrow x = .5, .6, .7, .8, .9\right)$

RHL

$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} ([1-x] + [x-1])$

$= \lim_{x\to 1+} [1-x] + \lim_{x\to 1+} [x-1]$

$= -1 + 0.$

$= -1.$

$\therefore \left(\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x)\right) \neq f(1)$

$\therefore f$ is not conti at $x=1$.

→ Show that the function f defined by

$$f(x) = \begin{cases} [x-1] + |x-1| & \text{if } x\neq 1 \\ 0 & \text{if } x=1 \end{cases}$$

is discontinuity at $x=1$.

→ Examine the continuity of f at $x=2$, where

$$f(x) = \begin{cases} x-[x], & \text{if } x<2 \\ 1 & \text{if } x=2 \\ 3x-5 & \text{if } x>2. \end{cases}$$

→ Determine the points of continuity of the following functions:

(i) $f(x) = [x]$, (ii) $f(x) = x[x]$

(iii) $h(x) = x - [x]$ (iv) $k(x) = \left[\frac{1}{x}\right]$, $x \neq 0$

Sol (i) $f(x) = [x]$; $x \in \mathbb{R}$

(a) Let $x = c \in \mathbb{Z}$ (i.e. integral values)

then $f(c) = [c] = c$

LHL

$\underset{x \to c-}{Lt} f(x) = \underset{x \to c-}{Lt} [x]$

putting $x = c - h$ ($\because h > 0$, $h \to 0$)

$\therefore \underset{x \to c-}{Lt} f(x) = \underset{h \to 0}{Lt} [c-h]$

$= \underset{h \to 0}{Lt} (c-1)$ ($\because c-1 < c-h < c$)

$= c-1$

$[2-h] = 1 = 2-1$

RHL

$\underset{x \to c+}{Lt} f(x) = \underset{x \to c+}{Lt} [x]$

putting $x = c+h$ ($\because h > 0$)

$\therefore \underset{x \to c+}{Lt} f(x) = \underset{h \to 0}{Lt} [c+h]$

$= \underset{h \to 0}{Lt} (c)$ ($\because c < c+h < c+1$).

$= c$

$\therefore \underset{x \to c-}{Lt} f(x) \neq \underset{x \to c+}{Lt} f(x)$

$\therefore f(x)$ is not continuous at $x = c \in \mathbb{Z}$.

(b) Let $x = c \in \mathbb{R} - \mathbb{Z}$.

i.e. $x =$ non-integral value

If n is the greatest integer less than 'c' then $[c] = n$.

where $n < c < n+1$.

Now $f(c) = [c] = n$.

LHL

$\underset{x \to c-}{Lt} f(x) = \underset{x \to c-}{Lt} [x]$

$= \underset{h \to 0}{Lt} [c-h]$ (put $x = c-h$, $h > 0$)

$= \underset{h \to 0}{Lt} (n)$ ($\because n < (c-h) < n+1$)

$= n$

$x \to c-$
$x \to c = 2.5$
$x = 2.1$ Lt $h \to$
$[x] = 2 = n$

RHL

$\underset{x \to c+}{Lt} f(x) = \underset{x \to c+}{Lt} [x]$

$= \underset{h \to 0}{Lt} [c+h]$ (put $x = c+h$, $h > 0$)

$= \underset{h \to 0}{Lt} (n)$ ($\because n < (c+h) < n+1$)

$= n$.

$\therefore \left(\underset{x \to c-}{Lt} f(x) = \underset{x \to c+}{Lt} f(x)\right) = f(c)$.

$\therefore f$ is cont. at $x = c \in \mathbb{R} - \mathbb{Z}$

i.e. f is cont. at non-integral values.

(iv) $k(x) = \left[\frac{1}{x}\right]$; $(x \neq 0)$, $x \in \mathbb{R}$

(a) Let $x = c$ be an integer (except 0 & ± 1).

If n is the greatest integer less than $\frac{1}{c}$ then $\left[\frac{1}{c}\right]=n$.

where $n \le \frac{1}{c} < n+1$

NOW

LHL

$$\lim_{x\to c-} f(x) = \lim_{x\to c-}\left[\frac{1}{x}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{c-h}\right] \quad (\text{put } x=c-h;\ h>0)$$

$$= \lim_{h\to 0}(n) \quad \left(\because n < \frac{1}{c-h} < n+1\right)$$

$$= n.$$

RHL

$$\lim_{x\to c+} f(x) = \lim_{x\to c+}\left[\frac{1}{x}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{c+h}\right] \quad (\text{put } x=c+h;\ h>0)$$

$$= \lim_{h\to 0}(n) \quad \left(\because n < \frac{1}{c+h} < n+1\right)$$

$$= n.$$

$$\therefore \left(\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x)\right) = f(c).$$

$\therefore f(x)$ is conti at $x=c\in\mathbb{Z}$. except 0 & ± 1

(b) Let $x=c\in\mathbb{R}-\mathbb{Z}$

then ① $\frac{1}{x}=\frac{1}{c}$ is an integer if $x=c=\pm\frac{1}{2}, \pm\frac{1}{3}$

② $\frac{1}{x}=\frac{1}{c}$ is not an integer if $x=c \ne \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

① if $\frac{1}{x}=\frac{1}{c}$ is an integer then $\left[\frac{1}{c}\right]=\frac{1}{c}$.

LHL

$$\lim_{x\to c-} f(x) = \lim_{x\to c-}\left[\frac{1}{x}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{c-h}\right] \quad (\text{put } x=c-h;\ h>0).$$

$$= \lim_{h\to 0}\left(\frac{1}{c}-1\right) \quad \left(\because \frac{1}{c}-1 < \frac{1}{c-h} < \frac{1}{c}\right)$$

$$= \frac{1}{c}-1$$

$x \to c-$

$x \to c = \frac{1}{2}-$

$\frac{1}{x} \to \frac{1}{c} = 2-$

$\frac{1}{x} = 1.6, 1.8, 1.9, 1.9$

$\left[\frac{1}{x}\right] = 1 = 2-1$

$\left(\frac{1}{c}-1\right)$.

RHL

$$\lim_{x\to c+} f(x) = \lim_{x\to c+}\left[\frac{1}{x}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{c+h}\right] \quad (\text{put } x=c+h;\ h>0).$$

$$= \lim_{h\to 0}\left(\frac{1}{c}\right) \quad \left(\frac{1}{c} < \frac{1}{c+h} < \frac{1}{c}+1\right)$$

$$= \frac{1}{c}.$$

$$\therefore \lim_{x\to c+} f(x) \ne \lim_{x\to c-} f(x)$$

$\therefore f$ is not conti. at $x=\pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

(2) if $\frac{1}{x} = \frac{1}{c}$ is not an integer if

$x \neq \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

Let n be the greatest integer less than $\frac{1}{c}$ then $[\frac{1}{c}] = n$. where $n < \frac{1}{c} < n+1$.

LHL $\underset{x\to c-}{Lt} f(x) = \underset{x\to c-}{Lt} [\frac{1}{x}]$

$= \underset{h\to 0}{Lt} [\frac{1}{c-h}]$ (Put $x = c-h$, $h>0$)

$= \underset{h\to 0}{Lt} (n)$ $(\because n < \frac{1}{c-h} < n+1)$.

$= n$.

RHL $\underset{x\to c+}{Lt} f(x) = \underset{x\to c+}{Lt} [\frac{1}{x}]$

$= \underset{h\to 0}{Lt} \left[\frac{1}{c+h}\right]$ put $x = c+h$, $h>0$

$= \underset{h\to 0}{Lt} (n)$

$= n$

$\therefore \underset{x\to c-}{Lt} f(x) = \underset{x\to c+}{Lt} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x \neq \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

Imp → Show that the absolute function $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$

Sol Given that $f(x) = |x|$ $\forall x \in \mathbb{R}$

Let $x = c \in \mathbb{R}$ then $f(c) = |c|$

Now we will show that

$\underset{x\to c}{Lt} f(x) = f(c)$

i.e $f(x) \to f(c)$ as $x \to c$.

Let $\epsilon > 0$ be given.

Now we have

$|f(x) - f(c)| = ||x| - |c||$

$\leq |x - c|$

$< \epsilon$ whenever $|x-c| < \frac{\epsilon}{1}$.

choosing $\delta = \frac{\epsilon}{1}$.

$\therefore |f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$.

i.e $f(x) \to f(c)$ as $x \to c$.

$\underset{x\to c}{Lt} f(x) = f(c)$.

$\therefore f(x)$ is continuous at $x = c$

(1) Imp → Let $K > 0$ and let $f: \mathbb{R} \to \mathbb{R}$ satisfy the condition

$|f(x) - f(y)| \leq K|x - y|$ $\forall x, y \in \mathbb{R}$.

Show that f is continuous at every point $c \in \mathbb{R}$.

Sol $K > 0$, $f: \mathbb{R} \to \mathbb{R}$ satisfies the condition

$|f(x) - f(y)| \leq K|x - y|$ $\forall x, y \in \mathbb{R}$ —— (1)

Now we shall show that

$\underset{x\to c}{Lt} f(x) = f(c)$.

i.e $f(x) \to f(c)$ as $x \to c$.

For this we are enough to show that,

given any $\epsilon > 0$ (however small), $\exists$ a $\delta > 0$ (depends on ϵ) s.t

$|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$

Now from (1),

we have

$|f(x) - f(y)| \leq K|x-y|$
$\forall x, y \in (c-\delta, c+\delta)$

Taking $x=x$, $y=c$, we get

$|f(x) - f(c)| \leq K|x-c|$
$< \epsilon$ whenever $|x-c| < \frac{\epsilon}{K}$

choosing $\delta = \frac{\epsilon}{K}$

$\therefore |f(x) - f(c)| < \epsilon$ whenever $|x-c|<\delta$

i.e $f(x) \to f(c)$ as $x \to c$

i.e $\lim_{x\to c} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x = c \in \mathbb{R}$.

→ Let g be defined on $\mathbb{R}$ by $g(x) = \begin{cases} 0 & \text{for } x=1 \\ 2 & \text{for } x \neq 1 \end{cases}$

and let $f(x) = x+1 \ \forall x \in \mathbb{R}$.

Show that $\lim_{x\to 0} (g\circ f)(x) \neq (g\circ f)(0)$.

Sol $g(x) = \begin{cases} 2 & \text{for } x \neq 1 \\ 0 & \text{for } x=1. \end{cases}$

and $f(x) = x+1 \ \forall x \in \mathbb{R}$.

Now $f(x) = x+1 \ \forall x \in \mathbb{R}$.

Now $(g\circ f)(x) = g(f(x))$

$= g(x+1)$

$= \begin{cases} 2 & \text{for } x+1 \neq 1 \\ 0 & \text{for } x+1 = 1 \end{cases}$

$= \begin{cases} 2 & \text{for } x \neq 0 \\ 0 & \text{for } x=0. \end{cases}$

Now $\lim_{x\to 0-} (g\circ f)(x) = \lim_{x\to 0-} (2)$

$= 2$

and $\lim_{x\to 0+} (g\circ f)(x) = \lim_{x\to 0+} (2)$

$= 2$

$\therefore \left(\lim_{x\to 0-} (g\circ f)(x) = \lim_{x\to 0+} (g\circ f)(x)\right) = 2$

$\Rightarrow \lim_{x\to 0} (g\circ f)(x) = 2$ ——(1)

But at $x=0$,

$(g\circ f)(0) = 0$. ——(2)

from (1) & (2)

$\lim_{x\to 0} (g\circ f)(x) \neq (g\circ f)(0)$.

→ **Def:** A function $f: \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x+y) = f(x) + f(y) \ \forall x, y \in \mathbb{R}$

Prove that if f is continuous at some x_0, then it is continuous at every point of $\mathbb{R}$.

Sol Continuity at the point x_0:

$\lim_{x\to x_0^-} f(x) = \lim_{h\to 0} f(x_0 - h)$ | putting $x = x_0 - h$; $h>0$

$= \lim_{h\to 0} [f(x_0) + f(-h)]$ | $\because f(x+y) = f(x)+f(y)$

$= \lim_{h\to 0} f(x_0) + \lim_{h\to 0} f(-h)$

$= f(x_0) + \lim_{h\to 0} f(-h)$.

$\therefore \boxed{\lim_{x\to x_0^-} f(x) = f(x_0) + \lim_{h\to 0} f(-h)}$

Sly $\lim_{x \to x_0^+} f(x) = \lim_{h \to 0} f(x_0+h)$ | putting $x = x_0+h$, $h>0$

$= \lim_{h \to 0} [f(x_0) + f(h)]$ | $\because f(x+y) = f(x)+f(y)$

$= \lim_{h \to 0} f(x_0) + \lim_{h \to 0} f(h)$

$= f(x_0) + f(h)$.

$\therefore \boxed{\lim_{x \to x_0^+} f(x) = f(x_0) + f(h)}$

Since f is continuous at $x = x_0$

$\therefore \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$

$\Rightarrow f(x_0) + \lim_{h \to 0} f(-h) = f(x_0) + \lim_{h \to 0} f(h) = f(x_0)$.

$\Rightarrow \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 0$ $\longrightarrow (1)$

Let c be any real number

then $\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c-h)$ | putting $x = c-h$, $h>0$

$= \lim_{h \to 0} (f(c) + f(-h))$ | $\because f(x+y) = f(x)+f(y)$.

$= \lim_{h \to 0} f(c) + \lim_{h \to 0} f(-h)$

$= f(c) + 0$ (from (1)).

$\therefore \boxed{\lim_{x \to c^-} f(x) = f(c)}$.

Also

$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h)$. | putting $x = c+h$, $h>0$

$= \lim_{h \to 0} [f(c) + f(h)]$

$= \lim_{h \to 0} f(c) + \lim_{h \to 0} f(h)$

$= f(c) + 0$ (from (1))

$\therefore \boxed{\lim_{x \to c^+} f(x) = f(c)}$.

$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$.

$\therefore f(x)$ is continuous at $x = c \in \mathbb{R}$

Theorem $\rightarrow$ If a function f is continuous at 'c' then $|f|$ is also continuous at 'c'.

Proof Since f is continuous at $x = c$.

$\therefore$ Given $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t.

$|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$ $\longrightarrow (1)$

Now we have

$\big||f(x)| - |f(c)|\big| \le |f(x) - f(c)| < \epsilon$

whenever $|x-c| < \delta$ (from (1))

$\therefore \big||f(x)| - |f(c)|\big| < \epsilon$ whenever $|x-c| < \delta$

$\therefore |f|$ is continuous at 'c'.

Note:- The converse of the above theorem need not be true.

i.e If $|f|$ is continuous at 'c' then f need not be conti at 'c'.

For example

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Sol

$|f|(x) = |f(x)| = 1 \quad \forall x \in \mathbb{R}.$

$\lim_{x \to 0} |f|(x) = \lim_{x \to 0} (1) = 1.$

at $x = 0$,
$|f|(0) = |f(0)| = 1.$

$\therefore \lim_{x \to 0} |f|(x) = |f|(0).$

$\therefore |f|$ is conti at $x=0$.

Now $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-1) = -1$ &

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (+1) = +1.$

$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x)$

$\therefore f(x)$ is not at $x=0$.

Theorem ⇒ (4) Let $f, g : \mathbb{R} \to \mathbb{R}$ continuous at 'c' and let $h(x) = \text{Max}\{f(x), g(x)\}$

i.e, $\sup\{f(x), g(x)\}$ for $x \in \mathbb{R}$

Show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)| \quad \forall x \in \mathbb{R}.$

Use this to show that h is continuous at 'c'.

proof: Since $f : \mathbb{R} \to \mathbb{R}$ & $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions at 'c'

$\therefore \lim_{x \to c} f(x) = f(c)$ &

$\lim_{x \to c} g(x) = g(c)$ ——— (1)

and since

$h(x) = \sup\{f(x), g(x)\}$; $x \in \mathbb{R}$

ie $h(x) = \text{Max}\{f(x), g(x)\}$; $x \in \mathbb{R}$

(i)
$$h(x) = \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) \leq g(x). \end{cases}$$ ——— (A)

Now since, $\forall x \in \mathbb{R}$,

$\frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$

$$= \begin{cases} \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}[f(x) - g(x)], & \text{if } f(x) \geq g(x) \\ \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}[-(f(x) - g(x))] & \text{if } f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) \leq g(x). \end{cases}$$ ——— (B)

$= h(x)$ (by (A)).

(ii) To show h is conti at 'c':

Since $f, \& g$ are continuous at 'c'

$\therefore f+g$ is also conti at 'c'

$\Rightarrow \frac{1}{2}(f+g)$ is also conti at 'c' ——— (C)

also $(f-g)$ is conti at 'c'

$\Rightarrow |f-g|$ is also continuous at 'c'.

$\Rightarrow \frac{1}{2}|f-g|$ is also conti at 'c' ——— (D) from (C) & (D)

$\frac{1}{2}(f+g)+\frac{1}{2}|f-g|$ is also continuous at 'c'.

$\therefore h(x)=\frac{1}{2}\left(f(x)+g(x)\right)+\frac{1}{2}|f(x)-g(x)|$ is continuous at $x=c$

(or)

$$\lim_{x\to c} h(x) = \lim_{x\to c} \frac{1}{2}\left[\left[f(x)+g(x)\right]+\frac{1}{2}\left|f(x)-g(x)\right|\right]$$

$$= \frac{1}{2}\left[\lim_{x\to c} f(x)+\lim_{x\to c} g(x)\right]+\frac{1}{2}\left[\left|\lim_{x\to c} f(x)+\lim_{x\to c} g(x)\right|\right]$$

$$= \frac{1}{2}\left[f(c)+g(c)\right]+\frac{1}{2}\left[\left|f(c)-g(c)\right|\right] \text{ by (1)}$$

$$= h(c)$$

$\therefore \lim_{x\to c} h(x) = h(c)$.

$\therefore h(x)$ is continuous at 'c'.

(1) H.W → Let $f, g: \mathbb{R}\to\mathbb{R}$ be continuous at 'c' and let

$h(x)=\inf\{f(x), g(x)\}$

i.e., $h(x)=\min\{f(x), g(x)\}$ for $x\in\mathbb{R}$

Show that $h(x)=\frac{1}{2}\left(f(x)+g(x)\right)-\frac{1}{2}|f(x)-g(x)|$ $\forall x\in\mathbb{R}$

Use this to show that h is continuous at 'c'.

Theorem → A function f is continuous at 'c' iff for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x_1)-f(x_2)|<\epsilon$ whenever $x_1, x_2\in(c-\delta, c+\delta)$.

proof: (i) Let f be a continuous function at 'c'. then for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x)-f(c)|<\frac{\epsilon}{2}$ whenever $|x-c|<\delta$.

$\Rightarrow |f(x)-f(c)|<\epsilon/2$ whenever $-\delta<x-c<\delta$.

$\Rightarrow |f(x)-f(c)|<\epsilon/2$ whenever $c-\delta<x<c+\delta$.

$\Rightarrow |f(x)-f(c)|<\epsilon/2$ whenever $x\in(c-\delta, c+\delta)$

Now for $x_1, x_2\in(c-\delta, c+\delta)$

$|f(x_1)-f(c)|<\epsilon/2$ & $|f(x_2)-f(c)|<\epsilon/2$ — (1)

Now we have

$$|f(x_1)-f(x_2)| = |f(x_1)-f(c)+f(c)-f(x_2)|$$
$$\leq |f(x_1)-f(c)|+|f(x_2)-f(c)|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \quad (\text{by } ①)$$

$\therefore |f(x_1)-f(x_2)| < \epsilon$ whenever $x_1, x_2 \in (c-\delta, c+\delta)$

(ii) Conversely suppose that for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x_1)-f(x_2)| < \epsilon$ whenever $x_1, x_2 \in (c-\delta, c+\delta)$.

Taking $x_1 = x$ & $x_2 = c$

we have $|f(x)-f(c)| < \epsilon$ whenever $x \in (c-\delta, c+\delta)$

$\therefore f$ is continuous at $x = c$

Theorem → If a function f is continuous at 'c' then it is bounded in some nbd of 'c'.

proof: Since f is continuous at 'c'.

$\therefore$ Given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta$; $x \in D_f$.

$\Rightarrow f(c)-\epsilon < f(x) < f(c)+\epsilon$ whenever $c-\delta < x < c+\delta$; $x \in D_f$

Let $M = \max\{|f(c)-\epsilon|, |f(c)+\epsilon|\}$

then $-M \leq f(x) \leq M$ whenever $x \in (c-\delta, c+\delta) \cap D_f$

$\Rightarrow |f(x)| \leq M$ whenever $x \in (c-\delta, c+\delta) \cap D_f$

$\therefore f$ is bounded in some nbd of 'c'.

Ex: $f(x) = \sin x$ is continuous for all $x \in \mathbb{R}$ and the range of $\sin x$ is $[-1, 1]$

$\therefore -1 \leq \sin x \leq 1 \ \forall x \in \mathbb{R}$.

$\inf = -1$ & $\sup = 1$

$\therefore f$ is bdd. (for each nbd of x)

2008 12M (2(a)(i))

If f is a continuous function of x satisfying the functional equation

$$f(x+y) = f(x) + f(y)$$

show that $f(x) = ax$, where 'a' is a constant. $\forall x \in \mathbb{R}$.

Sol^n: Given that 'f' is continuous and $f(x+y) = f(x) + f(y)$ —— (1)

Taking $x = 0 = y$ in (1)

(1) $\equiv$ $f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow f(0) + 0 = f(0) + f(0)$

$\Rightarrow f(0) = 0$

Taking $y = -x$.

(1) $\equiv$ $f(x+(-x)) = f(x) + f(-x)$

$\Rightarrow f(0) = f(x) + f(-x)$

$\Rightarrow 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x)$

If x be a +ve integer,

we have

$$f(x) = f(1+1+1+\cdots+1)$$
$$= f(1) + f(1) + f(1) + \cdots + f(1)$$
$$= x f(1)$$
$$= ax, \text{ say}$$

where $f(1) = a$

Now. let x be a −ve integer.

we write $x = -y$ so that y is +ve integer.

we have

$$f(x) = f(-y)$$
$$= -f(y) \quad [\because f(-y) = -f(y)]$$
$$= -ay$$
$$= a(-y)$$
$$= ax.$$

Again let $x = \frac{p}{q}$ be a rational number; q being +ve.

we have

$$f(p) = f\left(\frac{p}{q} \cdot q\right)$$
$$= f\left(\frac{p}{q} + \frac{p}{q} + \cdots \text{ q times}\right)$$
$$= f(p/q) + f(p/q) + \cdots \text{ q times}$$
$$= q f(p/q)$$

$\therefore f(p) = q f(p/q)$

$\Rightarrow ap = q f(p/q) \quad (\because f(p) = ap)$

$\Rightarrow f(p/q) = a \cdot \frac{p}{q}$

$\Rightarrow f(x) = ax. \quad (\because \frac{p}{q} = x)$

Now, suppose that x is any real number. Let $\{x_n\}$ be a sequence of rational numbers such that $\lim x_n = x$.

we have, x_n, being rational

$$f(x_n) = ax_n \quad \text{——— (2)}$$

Let $n \to \infty$.

As f is a continuous function.

we obtain from (2)

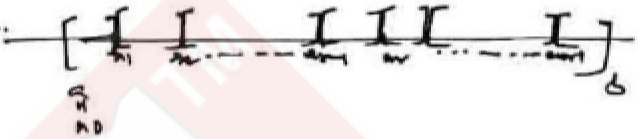
$$f(x) = ax \quad \forall x.$$

Hence the result

* Partition of a closed interval :—

Let $[a,b]$ be a closed interval.

If $a = x_0 < x_1 < x_2 < \cdots x_{r-1} < x_r < x_{r+1} < \cdots < x_n = b$ where $r = 1, 2, \ldots$



then the finite set

$P = \{x_0, x_1, x_2, \ldots x_{r-1}, x_r, \ldots x_n\}$ is called a partition of $[a,b]$

— The $(n+1)$ points $x_0, x_1, \ldots x_n$ are called partition points of the set 'P'.

— The closed intervals $[x_0, x_1]$, $[x_1, x_2], \ldots [x_{r-1}, x_r], \ldots [x_{n-1}, x_n]$ are called the n subinterval of the closed interval $[a,b]$.

The r^{th} subinterval $[x_{r-1}, x_r]$ is denoted by I_r and its length $x_r - x_{r-1}$ is denoted by δ_r.

It is denoted
i.e $\delta_r = x_r - x_{r-1}$.

$\Rightarrow$ If f is contf on $[a,b]$ then given $\epsilon > 0$ (however small), the closed interval $[a,b]$ can be divided into a finite number of subintervals. in each of which the oscillation of f is less than ϵ
i.e $|f(x_1) - f(x_2)| < \epsilon$ for any two points x_1 & x_2 in the same subinterval.

Theorem: If f is continuous on $[a,b]$ then f is bdd on that interval.

proof Since f is continuous on $[a,b]$

$\therefore$ Given $\epsilon > 0$ (however small), $[a,b]$ can be divided into finite number of subintervals in each of which the oscillation of f is less than ϵ

i.e Let $[a=a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n=b]$

s.t $|f(x_1) - f(x_2)| < \epsilon$ ——— (1)

for any two points x_1, x_2 belonging to the same subinterval.

Let x be any point of the first subinterval $[a, a_1]$ then by (1),

$|f(x) - f(a)| < \epsilon$

$\therefore |f(x)| = |f(x) - f(a) + f(a)|$
$\leq |f(x) - f(a)| + |f(a)|$
$< \epsilon + |f(a)|$.

In particular $x = a_1$

$|f(a_1)| < \epsilon + |f(a)|$. ——— (2)

Let $x \in [a_1, a_2]$ then by (1)

$|f(x) - f(a_1)| < \epsilon$

$\therefore |f(x)| = |f(x) - f(a_1) + f(a_1)|$
$\leq |f(x) - f(a_1)| + |f(a_1)|$
$< \epsilon + |f(a_1)|$
$< \epsilon + \epsilon + |f(a)|$ (by (2))
$= 2\epsilon + |f(a)|$

$\therefore |f(x)| < 2\epsilon + |f(a)|$.

In particular $x = a_2$,

$|f(a_2)| < 2\epsilon + |f(a)|$. ——— (3)

proceeding, similarly, we have

$|f(x)| < n\epsilon + |f(a)|$

$\forall x \in [a_{n-1}, a_n]$.

$\therefore$ This inequality is satisfied over the whole interval $[a,b]$

$\therefore f$ is bdd on $[a,b]$.

Note:- The converse of the above theorem need not be true

ie If f is bdd on $[a,b]$ then f need not continuous on $[a,b]$

Ex:-

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \forall x \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$$

Sol $f(x) = \sin \frac{1}{x}$

$\Rightarrow f(-2/\pi) = \sin\left(\frac{-\pi}{2}\right) = -1$

& $f\left(\frac{2}{\pi}\right) = \sin(\pi/2) = 1$.

$\therefore -1 \leq f(x) \leq 1 \quad \forall x \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$;

$\therefore f$ is bdd on $\left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$.

but is not continuous at $x=0$.

Because:

at $x=0$; $f(0)=0$.

Now since $-1 \le \sin\frac{1}{x} \le 1$ $\forall x \in [\frac{-1}{\pi}, \frac{1}{\pi}]$, $x \neq 0$.

is of the form

$g(x) \le f(x) \le h(x)$

where

$g(x)=-1$, $f(x)=\sin\frac{1}{x}$ &

$h(x)=+1$

with $\underset{x\to0}{Lt}\, g(x) \neq \underset{x\to0}{Lt}\, h(x)$

$\therefore$ By squeeze ~~theorem~~

$\underset{x\to0}{Lt}\, f(x)$ does not exist.

$\therefore f(x)$ is not conti. at $x=0$

$\therefore f(x)$ is not conti on $[\frac{-1}{\pi}, \frac{1}{\pi}]$

Note:- If f is conti. on (a,b) then f need not be bdd in that interval.

Ex:- $f(x)=\frac{1}{x}$ $\forall x \in (0,1)$.

since f is conti. on $(0,1)$

but is not bdd on $(0,1)$

Because: $\because x>0$

$\Rightarrow \frac{1}{x}>0$

$\Rightarrow 0<\frac{1}{x}<\infty$ $\forall x \in (0,1)$

$\Rightarrow 0<\frac{1}{x}$ $\forall x \in (0,1)$

$\Rightarrow 0<f(x)$ $\forall x \in (0,1)$

$\therefore f$ is not bdd on $(0,1)$.

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Theorem: If f is continuous on $[a,b]$ then f attains its bounds.

(or)

If f is conti on $[a,b]$ then f attains its supremum & infimum at least once in $[a,b]$.

Proof

Let f be conti. on $[a,b]$

then f is bdd on $[a,b]$.

$\therefore$ sup f & inf on $[a,b]$ exist.

Let $M =$ sup of f (L.u.b) &

$m =$ inf of f (g.l.b) on $[a,b]$

$\therefore f(x) \le M$ & $f(x) \ge m$ $\forall x \in [a,b]$

now we have to show that f attains its sup & inf at least once in $[a,b]$.

i.e $\exists\, \alpha, \beta \in [a,b]$ such that $f(\alpha)=M$ & $f(\beta)=m$.

now if possible suppose that f doesn't attain M on $[a,b]$.

$\therefore f(x) \neq M$ $\forall x \in [a,b]$.

$M-f(x) \neq 0$ $\forall x \in [a,b]$.

since M is constant, it is continuous for all x, it is continuous on $[a,b]$.

and f is continuous on $[a,b]$

$\therefore M-f(x)$ is also conti on $[a,b]$

$\Rightarrow \frac{1}{M-f(x)}$ is also conti on $[a,b]$ $(\because M-f(x) \neq 0)$

$\Rightarrow \frac{1}{M-f(x)}$ is bdd on $[a,b]$.

$\therefore \exists$ +ve real number K (i.e $K>0$)

s.t $\frac{1}{M-f(x)} \le K \;\forall x\in[a,b]$

$\Rightarrow M-f(x) \ge \frac{1}{K} \;\forall x\in[a,b]$

$\Rightarrow M-\frac{1}{K} \ge f(x) \;\forall x\in[a,b]$

$\Rightarrow f(x) \le M-\frac{1}{K} \;\forall x\in[a,b]$

$< M \;\forall x\in[a,b]$.

$\Rightarrow M-\frac{1}{K}$ is an upper bound of 'f' on $[a,b]$

and this upper bound less than sup of f on $[a,b]$.

$\therefore$ which is contradiction to the hypothesis that M is sup (lub) of f on $[a,b]$

$\therefore \exists\, \alpha\in[a,b]$ s.t $f(\alpha)=M$

$\therefore f$ attains its sup atleast once on $[a,b]$

sly, f attains its inf atleast once on $[a,b]$

Note 1:- The above theorem is not true if the interval is not closed

Ex:- $f(x)=x \;\forall x\in(0,1]$

f is conti. on $(0,1]$ and is bdd on $(0,1]$

because $\underset{\text{inf}}{f(0)=0}$ $\underset{\text{sup}}{f(1)=1}$

$\therefore 0<f(x)\le 1 \;\forall x\in(0,1]$

Clearly f attains sup but not attain inf on $(0,1]$

sly f on $[0,1)$ attains the infimum but not the sup

sly f on $(0,1)$ doesnot attain inf & sup

Note (2) The converse of above theorem need not be true

Ex:- $f(x)=\begin{cases}\sin\left(\frac{1}{x}\right); & x\neq 0\\ 0; & x=0\end{cases}$

$\forall x\in\left[\frac{-2}{\pi},\frac{2}{\pi}\right]$.

Theorem

If f is continuous on $[a,b]$ then f is bdd and attains its bounds atleast once on $[a,b]$

proof: Above two theorems proofs combined.

Sign preservation theorem:

If f is continuous on $[a,b]$ and $a<c<b$ such that $f(c) \neq 0$ then $\exists$ a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$ $\forall x \in (c-\delta, c+\delta)$.

Theorem If a function f is continuous on $[a,b]$ and $f(a)$ & $f(b)$ are of opposite signs then $\exists$ atleast one point $c \in (a,b)$ such that $f(c)=0$

Intermediate value theorem:

If f is continuous on $[a,b]$ and $f(a) \neq f(b)$ then f assumes every value between $f(a)$ & $f(b)$ atleast once.

Uniform Continuity:-

w.k.T a function f is continuous at a point x_0 of an interval I, if given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x)-f(x_0)| < \epsilon$ whenever $|x-x_0| < \delta$.

Here δ depends, in general, not only on ϵ but also on the point x_0 at which the continuity of 'f' is considered.

i.e, $\delta = \delta(\epsilon, x_0)$.

For example:

$f(x) = x^2 \quad \forall x \in \mathbb{R}$.

Let $\epsilon = 1/4$ & $x_0 = 0$

then $|f(x)-f(x_0)| = |x^2 - 0|$

$= |x^2|$

$= |x|^2 < \epsilon$ whenever $|x| < \frac{\sqrt{\epsilon}}{1}$.

Since $\epsilon = \frac{1}{4}$

$\therefore |f(x) - f(x_0)| < \frac{1}{4}$ whenever $|x| < \frac{1}{2}$

Taking $\delta = \frac{1}{2}$

$\therefore |f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

$\therefore \delta = \frac{1}{2}$ works at $x_0 = 0$ corresponding to $\epsilon = \frac{1}{4}$

Now let $\epsilon = \frac{1}{4}$ and $x_0 = 1$, then $\delta = \frac{1}{2}$ doesnot work.

<u>because</u>: Let $x = 1.4$ then

$|x - x_0| = |1.4 - 1| = 0.4 < \frac{1}{2}$

But $|f(x) - f(x_0)| = |1.96 - 1|$
$= 0.96 \nless \frac{1}{4} (= \epsilon)$

$\therefore \epsilon > 0$, the same value of δ doesnot work for different points of the Interval.

$\therefore$ If a continuous function f is such that given $\epsilon > 0$, we can find a uniform $\delta > 0$ which depends only on 'ϵ' and not on the point x_0 at which the continuity is considered, then we say that f is uniformly continuous.

<u>Defn</u>: A function defined on an interval I is said to be uniformly continuous on I, if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depends on ϵ only) such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where $x_1, x_2 \in I$.

<u>Note</u>: (i) Uniform continuity of a function is a global property we talk of uniform continuity on set and not at a point.

[2]. Continuity on the otherhand is a local property.

[3]. A function f is not uniformly continuous on I if $\exists$ some $\epsilon > 0$ for which no $\delta > 0$ works.

i.e, for any $\delta > 0$, $\exists\, x_1, x_2 \in I$ such that $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \epsilon$.

Problems

→ Every Constant function is uniformly continuous on $\mathbb{R}$.

Soln: Let $f(x) = c \ (\in \mathbb{R})$ constant function.

Given $\epsilon > 0$,

Now choosing $\delta > 0$ such that

$|x_1 - x_2| < \delta$; $x_1, x_2 \in \mathbb{R}$.

$\Rightarrow |f(x_1) - f(x_2)| = |c - c|$

$= 0 < \epsilon$

$\therefore f(x) = c \ (\in \mathbb{R})$ is uniformly continuous on $\mathbb{R}$.

→ The identity function $f(x) = x \ \forall x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Soln: Given $f(x) = x \ \forall x \in \mathbb{R}$

Let $\epsilon > 0$ be given.

Let $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$.

Now we have

$|f(x_1) - f(x_2)| = |x_1 - x_2|$

$< \epsilon$ whenever $|x_1 - x_2| < \frac{\epsilon}{1}$

Choosing $\delta = \frac{\epsilon}{1}$

$\therefore |f(x_1) - f(x_2)| < \epsilon$.

$\therefore f(x) = x$ is uniformly continuous on $\mathbb{R}$.

→ S.T $f(x) = x^2$ is uniformly continuous on $[-1,1]$

Sol^n: Let $\epsilon > 0$ be given and let $x_1, x_2 \in [-1,1]$

$\Rightarrow x_1 \in [-1,1]$ & $x_2 \in [-1,1]$.

$\Rightarrow -1 \le x_1 \le 1$ & $-1 \le x_2 \le 1$

$\Rightarrow |x_1| \le 1$ & $|x_2| \le 1$.

Now we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| \\ &= |(x_1 - x_2)(x_1 + x_2)| \\ &= |x_1 - x_2|\,|x_1 + x_2| \\ &\le (|x_1| + |x_2|)(|x_1 - x_2|) \\ &< (1+1)|x_1 - x_2| \\ &= 2|x_1 - x_2| \\ &< \epsilon \text{ whenever } |x_1 - x_2| < \frac{\epsilon}{2} \end{aligned}$$

Choosing $\delta = \epsilon/2$

$\therefore |f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$.

$\therefore f$ is uniformly continuous on $[-1, 1]$.

→ S.T $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, 2]$

Sol^n Let $\epsilon > 0$ be given,

Let $x_1, x_2 \in [0, 2]$

$\Rightarrow 0 \le x_1 \le 2$ & $0 \le x_2 \le 2$ —— (1)

we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left| \frac{x_1}{x_1+1} - \frac{x_2}{x_2+1} \right| \\ &= \left| \frac{x_1 - x_2}{(x_1+1)(x_2+1)} \right| \\ &= \frac{|x_1 - x_2|}{|x_1+1|\,|x_2+1|} \quad \text{—— (2)} \end{aligned}$$

① $\equiv$ $1 < x_1+1 \leq 3$ & $1 \leq x_2+1 \leq 3$.

$\Rightarrow$ $|x_1+1| \geq 1$ & $|x_2+1| \geq 1$

$\Rightarrow$ $\frac{1}{|x_1+1|} \leq 1$ & $\frac{1}{|x_2+1|} \leq 1$

$\therefore$ ② $\equiv$ $|f(x_1)-f(x_2)| \leq (1)(1)|x_1-x_2|$

$< \epsilon$ whenever $|x_1-x_2| < \frac{\epsilon}{1}$

Choosing $\delta = \frac{\epsilon}{1}$.

$\therefore$ $|f(x_1)-f(x_2)| < \epsilon$ whenever $|x_1-x_2| < \delta$.

$\therefore$ f is uniformly continuous on $[0,2]$

$\rightarrow$ S.T $f(x) = \frac{2x}{2x-1}$ is uniformly continuous on $[1,\infty)$

Sol^n: Let $\epsilon > 0$, be given.

Let $x_1, x_2 \in [1,\infty)$

then $x_1 \geq 1$ & $x_2 \geq 1$ ——— ①

we have

$$|f(x_1)-f(x_2)| = \left|\frac{2x_1}{2x_1-1} - \frac{2x_2}{2x_2-1}\right|$$

$$= \frac{2|x_1-x_2|}{|2x_1-1||2x_2-1|} \quad \text{——— ②}$$

① $\equiv$ $2x_1-1 \geq 1$ & $2x_2-1 \geq 1$

$\Rightarrow$ $|2x_1-1| \geq 1$ & $|2x_2-1| \geq 1$

$\Rightarrow$ $\frac{1}{|2x_1-1|} \leq 1$ & $\frac{1}{|2x_2-1|} \leq 1$.

$\therefore$ ② $\equiv$

we have

$|f(x_1)-f(x_2)| \leq (1)(1)(2)|x_1-x_2|$

$< \epsilon$ whenever $|x_1-x_2| < \epsilon/2$.

Choosing $\delta = \epsilon/2$.

$\therefore$ $|f(x_1)-f(x_2)| < \epsilon$ whenever $|x_1-x_2| < \delta$.

$\therefore$ f is uniformly continuous on $[1,\infty)$

Note: Every uniformly continuous is always continuous but not converse.
i.e, every continuous function need not be uniformly continuous.

for example:
$f(x)=x^2$ is continuous on $\mathbb{R}$, but not uniformly continuous on $\mathbb{R}$.

because:
Let $\epsilon>0$ be given
Now we shall show that for each $\delta>0$,
$\exists\, x_1, x_2 \in \mathbb{R}$ such that $|x_1-x_2|<\delta$
$\Rightarrow |f(x_1)-f(x_2)| \geq \epsilon$.

Taking $x_2 = x_1+\frac{\delta}{2}$

$\therefore |x_1-x_2| = |x_1-x_1-\frac{\delta}{2}|$
$= \frac{\delta}{2} < \delta$.

Now $|f(x_1)-f(x_2)| = |x_1^2-x_2^2|$
$= |x_1-x_2|\,|x_1+x_2|$
$= \frac{\delta}{2}\,|x_1+x_1+\frac{\delta}{2}|$
$= \left(\frac{\delta}{2}\right)|2x_1+\frac{\delta}{2}|$
$= x_1\delta+\frac{\delta^2}{4}$ $(\because x_1>0)$
$> \epsilon$.

Since $\frac{\delta^2}{4}>0$ and $x_1\delta<\epsilon\ \forall x_1 \in \mathbb{R}$; $x_1>0$
it is impossible, δ depends on ϵ & x_1
$\therefore$ The given function is not uniformly continuous on $\mathbb{R}$.

→ If a function f is continuous on $[a,b]$ then it is uniformly continuous on $[a,b]$.
$f(x) = x^2+5x+3\ \forall x \in [0,4]$