

Duality in Linear Programming

We were introduced to several real life problems that were formulated a L.P. models. Let us pick up one of those problems, say the diet problem with different data namely the following.

— A mother wishes her children to obtain certain amounts of nutrients from their breakfast cereals. The children have the choice of eating Krunchies or Crispies or a mixture of the two. From their breakfast, they should obtain 1 mg of vitamin 'A', 0.5 mg of vitamin 'B' and 400 calories. One ounce of Krunchies contains 0.01 mg of vitamin 'A', 1 mg of vitamin B and 110 calories. 1 ounce of crispies contains 0.25 mg of vitamin 'A', 0.5 mg of vitamin 'B' and 120 calories. One ounce of Krunchies cost 4 rupees and 1 ounce of crispies cost 5 rupees.

1 ounce = 28.35 gr

If we formulate a linear programming model for the above problem, assuming that any of the children eats x_1 ounces of Krunchies and x_2 ounces of crispies, then problem reduces to

Minimize

$$Z = 4x_1 + 5x_2$$

sub. to

$$0.01x_1 + 0.25x_2 \geq 1$$

$$1x_1 + 0.5x_2 \geq 0.5$$

$$110x_1 + 120x_2 \geq 400$$

$$x_1, x_2 \geq 0.$$

Let us consider the same problem from a different angle.

Consider a salesman, who sells nutrients in the form of vitamin tablets and calories in the form of chocolate candy. Each milligram of vitamin A tablet costs Rs. w_1, vitamin B tablet costs Rs. w_2 and the amount of chocolate candy containing one calorie costs Rs. w_3

To replace Krunchies, mother has to spend $.01w_1 + 1w_2 + 110w_3$ for which this amount should be less than 4 cents.

Similarly, to resplace crispies mother has to spend $.25w_1 + .5w_2 + 120w_3$ for which amount is less than 5 cents.

On the other hand the salesman tries to maximize his revenue which is the total cost requirement of the mother and again the mathematical formulation of the problem is given by

Maximize

$$Z = w_1 + .5w_2 + 400w_3$$

sub. to

$$0.01w_1 + w_2 + 110w_3 \leq 4$$

$$0.25w_1 + 0.5w_2 + 120w_3 \leq 5$$

$$w_1, w_2, w_3 \geq 0.$$

These two problems have different mathematical formulations, one being a minimization problem whereas other being a maximization problem, Although they are expressed in terms of same basic data with different arrangements.

(1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the duality theory.

Every linear programming problem has associated with it, another L.P.P. involving the same data and closely related optimal solutions. Such two problems are said to be duals of each other. While one of these is called the primal, the other the dual.

The importance of the duality concept is due to two main reasons.

Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it.

Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in activities being programmed.

— The notion of duality in linear programming was first introduced by von-Neumann and was later explicitly given by Gale, Kuhn and Tucker.

② <u>formulation of dual problem</u>:

Consider the following L.P.P.:

$$\max Z = c_1 x_1 + c_2 x_2 + \cdots\cdots + c_n x_n$$

sub. to the constraints

$$a_{11} x_1 + a_{12} x_2 + \cdots\cdots + a_{1n} x_n \le b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots\cdots + a_{2n} x_n \le b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots\cdots + a_{mn} x_n \le b_m$$
$$x_1, x_2, \cdots\cdots x_n \geq 0.$$

To construct the dual problem, we adopt the following guidelines:

(i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.

(ii) ($\le$) type of constraints in the primal become ($\ge$) type of constraints in the dual and vice versa

(iii) The coefficients $c_1, c_2, \cdots\cdots c_n$ in the objective function of the primal become $b_1, b_2, \cdots\cdots b_m$ in the objective function of the dual and vice versa

(iv) If the primal has 'n' variables and 'm' constraints, the dual will have 'm' variables and 'n' constraints, i.e the transpose of the body matrix of the primal problem gives the body matrix of the dual.

(v) The variables in both the primal and dual are non-negative.

Then the dual problem will be

Minimize
$$W = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$$

Sub. to constraints
$$a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \geq c_1$$
$$a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \geq c_2$$
$$\cdots\cdots\cdots$$
$$a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m \geq c_n$$
$$w_1, w_2, \ldots, w_m \geq 0.$$

→ The primal–dual relationships can be conveniently displayed as below:

		Primal variables				
		x_1	$x_2 \cdots x_n$			
Dual variables	w_1	a_{11}	$a_{12} \cdots a_{1n}$	b_1	R.H.S of primal constraints
	w_2	a_{21}	$a_{22} \cdots a_{2n}$	b_2	
	$\vdots$	$\vdots$		$\vdots$	
	w_m	a_{m1}	$a_{m2} \cdots a_{mn}$	b_m	
		c_1	$c_2 \cdots c_n$		

R.H.S of the dual constraints.

→ The information regarding the primal–dual objective, type of constraints and the sign of dual variables may be summarized in the following table:

Standard primal objective	Dual: objective function type	constraints type	Variable sign
Maximization	Minimization	$\geq$	unrestricted
Minimization	Maximization	$\leq$	unrestricted

Note:- All primal constraints are equations with non-negative RHS and all primal variables are non-negative.

problems

→ Write the dual of the following L.P.P.

Max $Z = x_1 + 2x_2$

sub. to

$$2x_1 - 3x_2 \le 3$$

$$4x_1 + x_2 \le -4$$

$$x_1, x_2 \ge 0$$

sol Dual to this L.P.P is

Min $W = 3y_1 - 4y_2$

sub. to

$$2y_1 + 4y_2 \ge 1$$

$$-3y_1 + y_2 \ge 2$$

$$y_1, y_2 \ge 0.$$

→ Write the dual of the following L.P.P

Minimize

$Z = 3x_1 - 2x_2 + 4x_3$

sub. to

$$3x_1 + 5x_2 + 4x_3 \ge 7$$

$$6x_1 + x_2 + 3x_3 \ge 4$$

$$7x_1 - 2x_2 - x_3 \le 10.$$

$$x_1 - 2x_2 + 5x_3 \ge 3$$

$$4x_1 + 7x_2 - 2x_3 \ge 2; \quad x_1, x_2, x_3 \ge 0.$$

Since the problem is of minimization type all constraints should be of ($\ge$) type. We multiply third constraint throughout by -1, so that $-7x_1 + 2x_2 + x_3 \ge -10$.

Let y_1, y_2, y_3, y_4 and y_5 be the dual variables associated with the above five constraints.

Then the dual problem is given by

Maximize

$$W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

sub. to

$$3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \le 3$$

$$5y_1 + y_2 + 2y_3 + 2y_4 + 7y_5 \le -2$$

$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \le 4$$

$$y_1, y_2, y_3, y_4, y_5 \ge 0.$$

→ Write dual for the following L.P. problems:

* Max

$$Z = 5x_1 + 3x_2$$

Sub. to

$$3x_1 + 5x_2 \le 5$$

$$x_1 + 3x_2 \le 4$$

$$x_1, x_2 \ge 0$$

* Max

$$Z = 2x_1 + 3x_2 + 5x_3$$

sub. to.

$$5x_1 + 6x_2 - x_3 \le 3$$

$$-2x_1 + x_2 + 3x_3 \le 2$$

$$x_1 + 5x_2 - 3x_3 \le 1$$

$$-3x_1 + 3x_2 - 7x_3 \le 6$$

$$x_1, x_2, x_3 \ge 0.$$

(3) Formulation of dual problem when the primal has equality constraints:

Consider the problem

Max

$$Z = c_1x_1 + c_2x_2$$

Sub. to.

$$a_{11}x_1 + a_{12}x_2 = b_1,$$

$$a_{21}x_1 + a_{22}x_2 \le b_1,$$

$$x_1, x_2 \ge 0.$$

Sol The equality constraint can be written as

$$a_{11}x_1 + a_{12}x_2 \le b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \ge b_1$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 \le b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \le -b_1$$

Now we restate the problem

Max

$Z = c_1 x_1 + c_2 x_2$

Sub. to.

$a_{11} x_1 + a_{12} x_2 \le b_1$

$-a_{11} x_1 - a_{12} x_2 \le -b_1$

$a_{21} x_1 + a_{22} x_2 \le b_2$

$x_1, x_2 \geq 0.$

Now, this primal is in the proper form i.e a maximization problem subject to all constraints '$\le$' type

$\therefore$ The dual problem is

Minimize

$W = b_1 y_1 - b_1 y_2 + b_2 y_3$

Sub. to

$a_{11} y_1 - a_{11} y_2 + a_{21} y_3 \geq c_1$

$a_{12} y_1 - a_{12} y_2 + a_{22} y_3 \geq c_2$

$y_1, y_2, y_3 \geq 0.$

$\therefore$ This can be written as

Minimize

$W = b_1 (y_1 - y_2) + b_2 y_3$

Sub. to

$a_{11}(y_1 - y_2) + a_{21} y_3 \geq c_1$

$a_{12}(y_1 - y_2) + a_{22} y_3 \geq c_1$

$y_1, y_2, y_3 \geq 0.$

The term $(y_1 - y_2)$ appears in both objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. Then the new variable $(y_1 - y_2) = u_1$ becomes unrestricted in sign being the difference of two non-negative variables

$\therefore$ The above dual problem takes the form.

Min $W = b_1 u_1 + b_2 y_2$

sub. to

$a_{11} u_1 + a_{21} y_2 \geq c_1$

$a_{12} u_1 + a_{22} y_2 \geq c_2$

u_1 unrestricted in sign,

$y_2 \geq 0$.

HW $\rightarrow$ construct the dual of the L.P.P.

Max $Z = 4x_1 + 9x_2 + 2x_3$

subject to

$2x_1 + 3x_2 + 2x_3 \leq 7$

$3x_1 - 2x_2 + 4x_3 = 5$

$x_1, x_2, x_3 \geq 0$.

HW $\rightarrow$ Obtain the dual problem of the following L.P.P.

1989

Max $Z = x_1 - 2x_2 + 3x_3$

sub. to

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$,

$x_1, x_2, x_3 \geq 0$.

$\rightarrow$ Max $Z = 3x_1 + 5x_2 + 6x_3$

sub. to

primal $\begin{cases} x_1 + x_2 \leq 5 \\ 2x_1 + x_2 - 3x_3 \leq 3 \\ 7x_2 + 2x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

$\rightarrow$ Max $Z = 3x_1 - 2x_2 + 4x_3 + x_4$

sub. to

primal $\begin{cases} 2x_1 + x_2 + 4x_3 + 3x_4 = 2 \\ 4x_1 - 8x_2 + x_3 - 2x_4 \leq 7 \\ 3x_1 + x_2 + 2x_3 + 2x_4 \leq 3 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$

(4) Dual of a problem with Unrestricted variables:

Let us now consider a case in which there is no restriction on the variables i.e when the variables involved in the problem may or may not be non-negative.

For example:

Max
$$Z = 3x_1 + 4x_2$$
sub. to

primal
$$\begin{cases} 4x_1 + 2x_2 \le 7 \\ x_1 + 3x_2 \le 5 \\ \quad x_1, x_2 \text{ unrestricted} \end{cases}$$

sol put $x_1 = x_1' - x_1''$
$x_2 = x_2' - x_2''$

so that $x_1', x_1'', x_2', x_2'' \geq 0$.

and the primal can be written as

Maximize
$$Z = 3x_1' - 3x_1'' + 4x_2' - 4x_2''$$

sub. to.
$$4x_1' - 4x_1'' + 2x_2' - 2x_2'' \le 7$$
$$x_1' - x_1'' + 3x_2' - 3x_2'' \le 5.$$
$$x_1', x_1'', x_2', x_2'' \geq 0$$

∴ Dual to this L.P.P is

Minimize $W = 7y_1 + 5y_2$

sub. to

$$4y_1 + y_2 \geq 3$$
$$-4y_1 - y_2 \geq -3$$
$$2y_1 + 3y_2 \geq 4$$
$$-2y_1 - 3y_2 \geq -4$$
$$y_1, y_2 \geq 0.$$

This can be written as

Min $W = 7y_1 + 5y_2$

sub. to $4y_1 + y_2 \geq 3$

$4y_1 + y_2 \leq 3$

$2y_1 + 3y_2 \geq 4$

$2y_1 + 3y_2 \leq 4$

$y_1, y_2 \geq 0.$

$\Rightarrow$

Min $W = 7y_1 + 5y_2$

sub. to
$4y_1 + y_2 = 3$

$2y_1 + 3y_2 = 4$

$y_1, y_2 \geq 0.$

which is the reqd dual form of given primal L.P.P.

** Write the duals of the following:

HW $\rightarrow$ Max $Z = 2x_1 + 3x_2 + 4x_3$

sub. to

$$\begin{cases} x_1 - 5x_2 + 2x_3 \leq 7 \\ 2x_1 - 3x_2 + x_3 \leq 3 \\ x_1 + 2x_2 - 4x_3 \leq 2 \\ x_1, x_2 \geq 0, \ x_3 \text{ unrestricted} \end{cases}$$

$\xrightarrow{HW}$ Max
$$Z = 3x_1 + 6x_2 + 7x_3$$
subject to
$$\begin{cases} 2x_1 + 3x_2 + 5x_3 \le 11 \\ 7x_1 + 2x_2 + 4x_3 \le 7 \\ x_1 + 5x_2 + 4x_3 \le 5 \\ x_1, x_2 \text{ unrestricted}, x_3 \geq 0 \end{cases}$$

$\xrightarrow{HW}$ Maximize
$$Z = 6x_1 - 5x_2 - 3x_3$$
sub. to
$$2x_1 - 2x_2 + 2x_3 \le 3$$
$$x_1 + 5x_2 + 4x_3 \le 4$$
$$3x_1 + x_2 - 4x_3 \le 2$$
$$x_1 \geq 0,$$
x_2, x_3 unrestricted

⑤ Dual of a problem with equality, Inequality constraints and Unrestricted variables:

For example:
$$\text{Max } Z = 8x_1 + 6x_2 + 7x_3 + 3x_4 - x_5$$
sub. to
$$4x_1 + 2x_2 + 5x_3 + 6x_4 + 2x_5 \geq 11$$
$$4x_1 + 3x_2 - 2x_3 - 5x_4 + 3x_5 \le 13$$
$$x_1, x_2, x_3 \geq 0,$$
x_4 and x_5 unrestricted.

→ Write the dual of the following linear programming in a form such that dual variables are all non-negative.

Maximize
$$Z = 3x_1 + 2x_2 - 4x_3 + x_4$$
sub. to
$$6x_1 + 4x_2 - 3x_3 + 2x_4 = 2$$
$$4x_1 - 3x_2 + 2x_3 + x_4 \le 3$$
x_1, x_2 unrestricted, $x_3, x_4 \geq 0$.

sol put $x_1 = x_1' - x_1''$

$x_2 = x_2' - x_2''$

We write the first equality constraint in the form of two inequalities of ($\le$) type.

We may rewrite the primal

Max $Z = 3x_1' - 3x_1'' + 2x_2' - 2x_2'' - 4x_3 + x_4$

sub. to

$$6x_1' - 6x_1'' + 4x_2' - 4x_2'' - 3x_3 + 2x_4 \le 2$$

$$-6x_1' + 6x_1'' - 4x_2' + 4x_2'' + 3x_3 - 2x_4 \le -2$$

$$4x_1' - 4x_1'' - 3x_2' + 3x_2'' + 2x_3 + x_4 \le 3$$

$$x_1', x_1'', x_2', x_2'', x_3, x_4 \ge 0.$$

Therefore, dual is given by

Minimize $W = 2y_1 - 2y_2 + 3y_3$

sub. to

$$6y_1 - 6y_2 + 4y_3 \ge 3$$

$$-6y_1 + 6y_2 - 4y_3 \ge -3$$

$$4y_1 - 4y_2 - 3y_3 \ge 2$$

$$-4y_1 + 4y_2 - 3y_3 \ge -2$$

$$-3y_1 + 3y_2 + 2y_3 \ge -4$$

$$2y_1 - 2y_2 + y_3 \ge 1$$

$$y_1, y_2, y_3 \ge 0.$$

(or)

Min $W = 2y_1 - 2y_2 + 3y_3$

sub. to.

$$6y_1 - 6y_2 + 4y_3 = 3, \quad 4y_1 - 4y_2 - 3y_3 = 2,$$

$$-3y_1 + 3y_2 + 2y_3 \ge -4, \quad 2y_1 - 2y_2 + y_3 \ge 1$$

$$y_1, y_2, y_3 \ge 0.$$

→ write the dual of the following primal

Max $Z = 2x_1 - x_2 + 2x_3$

sub. to

$4x_1 + 2x_2 + x_3 \le 7$

$2x_1 - x_2 + 2x_3 \ge 8$

x_1 unrestricted; $x_2, x_3 \ge 0$

→ Write the dual of the following problem in a form such that dual variables are all non-negative

max $Z = 6x_1 + 4x_2 + x_3 + 7x_4 + 5x_5$

sub. to

$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$

$2x_1 + x_2 + 3x_3 + 2x_4 + 9x_5 = 6$

$x_1, x_2, x_3, x_4 \ge 0$;

x_5 unrestricted.

(6) Dual of a problem in a standard form:

Let us now consider a general L.P.P in its standard form:

max $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$.

$$\text{subject to.}\quad \text{primal}\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ x_1, x_2, x_3, \ldots, x_n \ge 0 \end{array}\right.$$

All constraints in the primal are equality constraints,
therefore, all dual variables should be unrestricted.

Now we may write the dual as

Minimize
$$W = b_1 y_1 + b_2 y_2 + b_3 y_3 + \cdots + b_m y_m$$

Sub. to.
$$a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m \geq c_2$$
$$\cdots$$
$$\cdots$$
$$a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \geq c_n.$$

$y_1, y_2 \cdots y_m$ are all unrestricted.

2006 / 12M → Given the programme

$$\text{Max } u = 5x + 2y$$

sub. to
$$x + 3y \leq 12$$
$$3x - 4y \leq 9$$
$$7x + 8y \leq 20$$
$$x, y \geq 0.$$

Write its dual in the standard form.

2008 / 12M → Find the dual of the following LPP

$$\text{Max. } Z = 2x_1 - x_2 + x_3$$

such that

$$x_1 + x_2 - 3x_3 \le 8$$

$$4x_1 - x_2 + x_3 = 2$$

$$2x_1 + 3x_2 - x_3 \ge 5$$

$$x_1, x_2, x_3 \ge 0$$

→ Dual of the dual is the primal.

For example:

Max $Z = 6x_1 - 2x_2 + x_3$

sub. to
$4x_1 - 2x_2 - 3x_3 \le 7$
$2x_1 + x_2 - 4x_3 \ge 5$
$x_1 + x_2 + x_3 = 1$, $x_1, x_2, x_3 \ge 0$.

Sol: First, we shall write the given primal L.P.P in canonical form.

Max $Z = 6x_1 - 2x_2 + x_3$

sub. to
$4x_1 - 2x_2 - 3x_3 \le 7$
$-2x_1 - x_2 + 4x_3 \le -5$
$x_1 + x_2 + x_3 \le 1$
$-x_1 - x_2 - x_3 \le -1$; $x_1, x_2, x_3 \ge 0$.

Dual to this L.P.P is

Min $W = 7y_1 - 5y_2 + y_3 - y_4$

sub. to
$4y_1 - 2y_2 + y_3 - y_4 \ge 6$
$-2y_1 - y_2 + y_3 - y_4 \ge -2$
$-3y_1 + 4y_2 + y_3 - y_4 \ge 1$
$y_1, y_2, y_3, y_4 \ge 0$.

Let us now write this dual in a canonical form, i.e a maximization type with '$\le$' type of constraints.

∴ we have Max $-W = -7y_1 + 5y_2 - y_3 + y_4$

sub. to
$-4y_1 + 2y_2 - y_3 + y_4 \le -6$
$2y_1 + y_2 - y_3 + y_4 \le 2$
$3y_1 - 4y_2 - y_3 + y_4 \le -1$
$y_1, y_2, y_3, y_4 \ge 0$.

Duel to this L.P.P is

$-$Minimize
$$-P = -6w_1 + 2w_2 - w_3$$

Sub. to
$$-4w_1 + 2w_2 + 3w_3 \geq -7$$
$$2w_1 + w_2 - 4w_3 \geq 5$$
$$-w_1 - w_2 - w_2 \geq -1$$
$$w_1 + w_2 + w_3 \geq 1$$
$$w_1, w_2, w_3 \geq 0.$$

We can rewrite this L.P.P as

Maximize
$$P = 6w_1 - 2w_2 + w_3$$

Sub. to
$$4w_1 - 2w_2 - 3w_3 \leq 7$$
$$2w_1 + w_2 - 4w_3 \geq 5$$
$$w_1 + w_2 + w_3 = 1$$
$$w_1, w_2, w_3 \geq 0$$

Which is precisely the given primal.

HW $\Rightarrow$ verify that duel of the dual is the primal for the following L.P.P

Maximize
$$Z = 4x_1 + 3x_2$$

Sub. to.
$$5x_1 + 6x_2 = -7$$
$$x_1 + 6x_2 \leq 3$$
$$2x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0.$$

32.12. (1) DUALITY PRINCIPLE

If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and *vice-versa.*

(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal) :

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

Rule I. If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the C_j row of the optimal dual simplex table and *vice-versa.*

Rule II. If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the C_j row of the optimal dual simplex table, after deleting the constant M and *vice-versa.*

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and *vice-versa.*

Now we shall workout two examples to demonstrate the primal dual relationships.

Example 32·24. *Construct the dual of the following problem and solve both the primal and the dual :*

Maximize $Z = 2x_1 + x_2,$

subject to $\quad -x_1 + 2x_2 \le 2,\ x_1 + x_2 \le 4,\ x_1 \le 3,\ x_1, x_2 \ge 0.$

Solution *using the primal problem.* Since only two variables are involved, it is convenient to solve the problem graphically.

In the x_1, x_2-plane, the five constraints show that the point (x_1, x_2) lies within the shaded region *OABCD* of Fig. 32·12. Values of the objective function $Z = 2x_1 + x_2$ at these corners are $Z(0) = 0, Z(A) = 6, Z(B) = 7, Z(C) = 6$ and $Z(D) = 1$. Hence the optimal solution is $x_1 = 3, x_2 = 1$ and max. $(Z) = 7$.

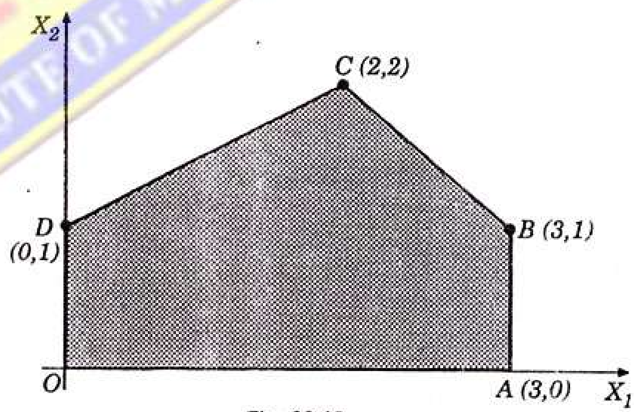

Fig. 32·12.

Solution *using the dual problem.* The dual problem of the given primal is :

Minimize $\quad W = 2y_1 + 4y_2 + 3y_3$

subject to $\quad -y_1 + y_2 + y_3 \ge 2,\ 2y_1 + y_2 \ge 1,\ y_1, y_2 \ge 0.$

Step 1. Express the problem in the standard form.

Introducing the slack and the artificial variables, the dual problem in the standard form is

Max. $W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2$

subject to $\quad -y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$

$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$

Step 2. Find an initial basic feasible solution.

Setting the non-basic variables y_1, y_2, y_3, s_1, s_2, each equal to zero, we get the initial basic feasible solution as

$y_1 = y_2 = y_3 = s_1 = s_2 = 0$ (non-basic) ; $A_1 = 2, A_2 = 1.$ (basic)

$\therefore$ Initial simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$		
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	-1	1	1	-1	0	1	0	2	$2/1$
$-M$	A_2	2	(1)	0	0	-1	0	1	1	$1/1 \leftarrow$
	Z_j	$-M$	$-2M$	$-M$	M	M	$-M$	$-M$	$-3M$	
	C_j	$M-2$	$2M-4$ $\uparrow$	$M-3$	$-M$	$-M$	0	0		

As C_j is positive under some columns, the initial solution is not optimal.

Step 3. Iterate towards an optimal solution.

(*i*) Introduce y_2 and drop A_2. Then the new simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$		
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b	θ
$-M$	A_1	-3	0	(1)	-1	1	1	-1	1	$1/1 \leftarrow$
-4	y_2	2	1	0	0	-1	0	1	1	$1/0$
	Z_j	$3M-8$	-4	$-M$	M	$4-M$	$-M$	$M-4$	$-M-4$	
	C_j	$6-3M$	0	$M-3$ $\uparrow$	$-M$	$M-4$	0	$4-2M$		

As C_j is positive under some columns, this solution is not optimal.

(*ii*) Now introduce y_3 and drop A_1. Then the revised simplex table is

	c_j	-2	-4	-3	0	0	$-M$	$-M$	
c_B	*Basis*	y_1	y_2	y_3	s_1	s_2	A_1	A_2	b
-3	y_3	-3	0	1	-1	1	1	-1	1
-4	y_2	2	1	0	0	-1	0	1	1
	Z_j	1	-4	-3	3	1	-3	-1	-7
	C_j	-3	0	0	-3	-1	$3-M$	$1-M$	

As all $C_j \le 0$, the optimal solution is attained.

Thus an optimal solution to the dual problem is

$y_1 = 0, y_2 = 1, y_3 = 1,$ Min. $W = -$ Max. $(W') = 7.$

To derive the optimal basic feasible solution to the primal problem, we note that the primal variables x_1, x_2 correspond to the artificial starting dual variables A_1, A_2 respectively. In the final simplex table of the dual problem, C_j corresponding to A_1, and A_2 are 3 and 1 respectively after ignoring M. Thus by rule II, we get opt. $x_1 = 3$ and opt. $x_2 = 1$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1 \text{ ; max. } Z = 7.$$

Obs. The validity of the duality theorem is therefore, checked since max. Z = min. $W = 7$ from both the methods.

Example 32·25. *Using duality solve the following problem :*

Minimize $Z = 0{\cdot}7x_1 + 0{\cdot}5x_2$

subject to $x_1 \geq 4,\ x_2 \geq 6,\ x_1 + 2x_2 \geq 20,\ 2x_1 + x_2 \geq 18,\ x_1, x_2 \geq 0.$ *(V.T.U., 2004)*

Sol. The dual of the given problem is Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4$,

subject to $y_1 + y_3 + 2y_4 \leq 0{\cdot}7,\ y_2 + 2y_3 + y_4 \leq 0{\cdot}5,\ y_1, y_2, y_3, y_4 \geq 0.$

Step 1. Express the problem in the standard form.

Introducing slack variables, the dual problem in the standard form becomes

Max. $W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2$,

subject to $y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0{\cdot}7$,

$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0{\cdot}5,\ y_1, y_2, y_3, y_4 \geq 0.$

Step 2. Find an initial basic feasible solution.

Setting non-basic variables y_1, y_2, y_3, y_4 each equal to zero, the basic solution is

$y_1 = y_2 = y_3 = y_4 = 0$ (non-basic) ; $s_1 = 0{\cdot}7,\ s_2 = 0{\cdot}5$ (basic)

Since the basic variables $s_1, s_2 > 0$, the initial basic solution is feasible and non-degenerate.

Initial simplex table is

	c_j	4	6	20	18	0	0		
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	0	1	2	1	0	0·7	0·7/1
0	s_2	0	1	(2)	1	0	1	0·5	0·5/2←
	Z_j	0	0	0	0	0	0	0	
	C_j	4	6	20 ↑	18	0	0		

As C_j is positive in some columns, the initial basic solution is not optimal.

Step 3. Iterate towards an optimal solution.

(*i*) Introduce y_3 and drop s_2. Then the new simplex table is

	c_j	4	6	20	18	0	0		
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b	θ
0	s_1	1	−1/2	0	(3/2)	1	−1/2	9/20	3/10←
20	y_3	0	1/2	1	$\frac{1}{2}$	0	1/2	1/4	1/2
	Z_j	0	10	20	10	0	10	5	
	C_j	4	−4	0	8 ↑	0	−10		

As C_j, is positive under some of the columns, this solution is not optimal.

(*ii*) Introduce y_4 and drop s_1. Then the revised simplex table is

	c_j	4	6	20	18	0	0	
c_B	*Basis*	y_1	y_2	y_3	y_4	s_1	s_2	b
18	y_4	2/3	−1/3	0	1	2/3	−1/3	3/10
20	y_3	−1/3	2/3	1	0	−1/3	2/3	1/10
	Z_j	16/3	22/3	20	18	16/3	22/3	74/10
	C_j	−4/3	−4/3	0	0	−16/13	−22/3	

As all $C_j \le 0$, this table gives the optimal solution.

Thus the optimal basic feasible solution is $y_1 = 0$, $y_2 = 0$, $y_3 = \cancel{20}$, $y_4 = \cancel{18}$; max. $W = 7{\cdot}4$
$\frac{1}{10}$ $\quad 3/10$

Step 4. Derive optimal solution to the primal.

We note that the primal variables x_1, x_2 correspond to the slack starting dual variables s_1, s_2 respectively. In the final simplex table of the dual problem, C_j values corresponding to s_1 and s_2 are −16/3 and −22/3 respectively.

Thus, by rule I, we conclude that opt. $x_1 = 16/3$ and opt. $x_2 = 22/3$.

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16/3,\ x_2 = 22/3\ ;\ \text{min. } Z = 7{\cdot}4.$$

> **Obs.** To check the validity of the duality theorem, the student is advised to solve the given *L.P.P.* directly by simplex method and see that min. Z = max. $W = 7{\cdot}4$.

Problems 32·7

Using duality solve the following problems (1 – 4) :

1. Minimize $Z = 2x_1 + 9x_2 + x_3$,
 subject to $\quad x_1 + 4x_2 + 2x_3 \ge 5,\ 3x_1 + x_2 + 2x_3 \ge 4$ and $x_1, x_2 \ge 0$.
2. Maximize $Z = 2x_1 + x_2$,
 subject to $\quad x_1 + 2x_2 \le 10,\ x_1 + x_2 \le 6,\ x_1 - x_2 \le 2,\ x_1 - 2x_2 \le 1,\ x_1, x_2 \ge 0$.
3. Maximize $Z = 3x_1 + 2x_2$,
 subject to $\quad x_1 + x_2 \ge 1,\ x_1 + x_2 \le 7,\ x_1 + 2x_2 \le 10,\ x_2 \le 3,\ x_1, x_2 \ge 0$. (*Madras, 1996*)
4. Maximize $Z = 3x_1 + 2x_2 + 5x_3$
 subject to $\quad x_1 + 2x_2 + x_3 \le 430,\ 3x_1 + 2x_3 \le 460,\ x_1 + 4x_2 \le 420,\ x_1, x_2, x_3 \ge 0$.

32·13. (1) DUAL SIMPLEX METHOD

... we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using *M*-method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

(2) Working procedure for dual simplex method :

Step 1. (i) Convert the problem to maximization form, if it is not so.

(ii) Convert (≥) type constraints, if any to (≤) type by multiplying such constraints by −1.

(iii) Express the problem in standard form by introducing slack variables.

Step 2. Find the initial basic solution and express this information in the form of dual simplex table.

Step 3. Test the nature of $C_j = c_j - Z_j$:

(*a*) If all $C_j \le 0$ and all $b_i \ge 0$, then optimal basic feasible solution has been attained.

(*b*) If all $C_j \le 0$ and at least one $b_i < 0$, then go to step 4.

(*c*) If any $C_j \ge 0$, the method fails.

Step 4. Mark the outgoing variable. Select the row that contains the most negative b_i. This will be the key row and the corresponding basic variable is the outgoing variable.

Step 5. Test the nature of key row elements :

(*a*) If all these elements are ≥ 0, the problem does not have a feasible solution.

(*b*) If at least one element < 0, find the ratios of the corresponding elements of C_j-row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the *incoming variable.*

Step 6. Iterate towards optimal feasible solution. Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

Example 32·26. *Using dual simplex method :*

maximize $-3x_1 - 2x_2$,

subject to $x_1 + x_2 \ge 1,\ x_1 + x_2 \le 7,\ x_1 + 2x_2 \ge 10,\ x_2 \ge 3,\ x_1 \ge 0,\ x_2 \ge 0.$

Solution consists of the following steps :

Step 1. (i) Convert the first and third constraints into (≤) type. These constraints become

$$-x_1 - x_2 \le -1,\ -x_1 - 2x_2 \le -10.$$

(ii) Express the problem in standard form

Introducing slack variables s_1, s_2, s_3, s_4 the given problem takes the form

Max. $Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

subject to $-x_1 - x_2 + s_1 = -1,\ x_1 + x_2 + s_2 = 7,\ -x_1 - 2x_2 + s_3 = -10,$

$x_2 + s_4 = 3,\ x_1, x_2, s_1, s_2, s_3, s_4 \ge 0.$

Step 2. Find the initial basic solution

Setting the decision variables x_1, x_2 each equal to zero, we get the basic solution

$$x_1 = x_2 = 0,\ s_1 = -1,\ s_2 = 7,\ s_3 = -10,\ s_4 = 3 \text{ and } Z = 0.$$

$\therefore$ Initial solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	-1	-1	1	0	0	0	-1
0	s_2	1	1	0	1	0	0	7
0	s_3	-1	(-2)	0	0	1	0	$-10 \leftarrow$
0	s_4	0	1	0	0	0	1	3
	$Z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	0
	$C_j = c_j - Z_j$	-3	-2 $\uparrow$	0	0	0	0	

Step 3. Test nature of C_j.

Since all C_j values are ≤ 0 and $b_1 = -1$, $b_3 = -10$, the initial solution is optimal but infeasible. We therefore, proceed further.

Step 4. Mark the outgoing variable.

Since b_3 is negative and numerically largest, the third row is the key row and s_3 is the outgoing variable.

Step 5. Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.

These ratios are $-3/-1 = 3$, $-2/-2 = 1$ (neglecting ratios corresponding to $+$ve or zero elements of key row). Since the smaller ratio is 1, therefore, x_2-column is the key column and (-2) is the key element.

Step 6. Iterate towards optimal feasible solution.

(*i*) Drop s_3 and introduce x_2 alongwith its associated value -2 under c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0	s_2	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
-2	x_2	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0	s_4	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1	$-2 \leftarrow$
	$Z_j = \Sigma c_B a_{ij}$	-1	-2	0	0	1	0	-10
	$C_j = c_j - Z_j$	-2 $\uparrow$	0	0	0	-1	0	

Since all C_j values are ≤ 0 and $b_4 = -2$, this solution is optimal but infeasible. We therefore proceed further.

(*ii*) *Mark the outgoing variable.*

Since b_4 is negative, the fourth row is the key row and s_4 is the outgoing variable.

(*iii*) *Calculate ratios of elements in C_j-row to the corresponding negative elements of the key row.*

This ratios is $-2/-\frac{1}{2} = 4$ (neglecting other ratios corresponding to + ve or 0 elements of key row).

$\therefore$ x_1-column is the key column and $\left(-\frac{1}{2}\right)$ is the key element.

(*iv*) *Drop s_4 and introduce x_1* with its associated value -3 under the c_B column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :

	c_j	-3	-2	0	0	0	0	
c_B	*Basis*	x_1	x_2	s_1	s_2	s_3	s_4	b
0	s_1	0	0	1	0	-1	-1	6
0	s_2	0	0	0	1	1	1	0
-2	x_2	0	1	0	0	0	1	3
-3	x_1	1	0	0	0	-10	-2	4
Z_j		-3	-2	0	0	3	4	-18
C_j		0	0	0	0	-3	-4	

Since all C_j values are ≤ 0 and all b's are ≥ 0, therefore this solution is optimal and feasible. Thus the *optimal solution* is $x_1 = 4, x_2 = 3$ and $Z_{max} = -18$.

Example 32·27. *Using dual simplex method, solve the following problem:*

Minimize $\quad Z = 2x_1 + 2x_2 + 4x_3$

subject to $\quad 2x_1 + 3x_2 + 5x_3 \ge 2,\ 3x_1 + x_2 + 7x_3 \le 3,\ x_1 + 4x_2 + 6x_3 \le 5,\ x_1, x_2, x_3 \ge 0$

Solution consists of the following steps :

Step 1. (*i*) *Convert the given problem to maximization form* by writing

Maximize $Z' = -2x_1 - 2x_2 - 4x_3$.

(*ii*) *Convert the first constraint into* ($\le$) *type.* Thus it is equivalent to

$$-2x_1 - 3x_2 - 4x_3 \le -2$$

(*iii*) *Express the problem in standard form.*

Introducing slack variables, s_1, s_2, s_3, the given problem becomes

Max. $Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $-2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$

$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$

$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$

Step 2. Find the initial basic solution.

Setting the decision variables x_1, x_2, x_3 each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0.\ s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

∴ Initial solution is given by the table below :

	c_j	−2	−2	−4	0	0	0	
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b
0	s_1	−2	(−3)	−5	1	0	0	−2 ←
0	s_2	3	1	7	0	1	0	3
0	s_3	1	4	6	0	0	1	5
	Z_j	0	0	0	0	0	0	0
	C_j	−2	−2 ↑	−4	0	0	0	

Step 3. Test nature of C_j.

Since all C_j values are ≤ 0 and $b_1 = -2$, the initial solution is optimal but infeasible.

Step 4. Mark the outgoing variable.

Since $b_1 < 0$, the first row is the key row and s_1 is the outgoing variable.

Step 5. Calculate the ratio of elements of C_j-row to the corresponding negative elements of the key row.

These ratios are $-2/-2 = 1$, $-2/-3 = 0{\cdot}67$, $-4/-5 = 0{\cdot}8$.

Since 0·67 is the smallest ratio, x_2-column is the key column and (−3) is the key element.

Step 6. Iterate towards optimal feasible solution.

Drop s_1 and introduce x_2 with its associated value −2 under c_B column. Then the revised dual simplex table is

	c_j	−2	−2	−4	0	0	0	
c_B	*Basis*	x_1	x_2	x_3	s_1	s_2	s_3	b
−2	x_2	2/3	1	5/3	−1/3	0	0	2/3
0	s_2	7/3	0	16/3	1/3	1	0	7/3
0	s_3	−5/3	0	−2/3	4/3	0	1	7/3
	Z_j	−4/3	−2	−10/3	2/3	0	0	−4/3
	C_j	−2/3	0	−2/3	−2/3	0	0	

Since all $C_j \le 0$ and all b_i's are > 0, this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 0 \text{ and max. } Z' = -4/3 \text{ i.e. min. } Z = 4/3.$$

Problems 32·8

Using dual simplex method, solve the following problems :

1. Maximize $Z = -3x_1 - x_2$
   subject to $x_1 + x_2 \geq 1,\ 2x_1 + 3x_2 \geq 2\ ;\ x_1, x_2 \geq 0.$
2. Minimize $Z = 2x_1 + x_2$,
   subject to $3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0$
3. Minimize $Z = x_1 + 2x_2 + 3x_3$,
   subject to $2x_1 - x_2 + x_3 \geq 4,\ x_1 + x_2 + 2x_3 \leq 8,\ x_2 - x_3 \geq 2;\ x_1, x_2, x_3 \geq 0.$
4. Minimize $Z = x_1 + 2x_2 + x_3 + 4x_4$
   subject to $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10,\ 3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$
   $5x_1 + 2x_2 + x_3 + 6x_4 \geq 15,\ x_1, x_2, x_3, x_4 \geq 0.$

2000 15M → Using duality or otherwise solve the linear programming problem

Minimize

$$Z = 18x_1 + 12x_2$$

subject to

$$2x_1 - 2x_2 \geq -3$$
$$3x_1 + 2x_2 \geq 3$$
$$x_1, x_2 \geq 0.$$