Mains Test Series – 2019

Test-1 (Paper-I)

Answer Key

Linear Algebra, Calculus & 3D

SECTION – A

Q.1. (a) Determine the angle between the vectors $\vec{u} = (1,0,0)$ and $\vec{v} = (1,0,1)$ in $\mathbb{R}^3$.

Solution:

The given vectors are $\vec{u} = (1,0,0)$ and $\vec{v} = (1,0,1)$ in $\mathbb{R}^3$.

To find angle between two vectors $\vec{u}, \vec{v}$, we proceed as follows →

$$\vec{u}\cdot\vec{v} = (1,0,0)\cdot(1,0,1) = 1.$$

$$\|\vec{u}\| = \sqrt{1^2+0^2+0^2} = 1$$

$$\|\vec{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$$\therefore \cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$\Rightarrow \boxed{\theta = 45^\circ}$ Thus, the angle between the given vectors $\vec{u} = (1,0,0)$ and $\vec{v} = (1,0,1)$ is $45°$ (or $\frac{\pi^c}{4}$)

Q. 1. (b) The transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ -x \end{bmatrix}$ defines a reflection in the line $y = -x$. Show that T is linear transformation. Determine the standard matrix of this transformation. Find the image of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Solution:

Let $V_2(F)$ be the given vector space.

Let $T: V_2(F) \to V_2(F)$ be the given transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

Let $\alpha = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\beta = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ be two vectors of $V_2(F)$

For $a, b \in F$,

$$T(a\alpha + b\beta) = T\left(a\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + b\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} ax_1 \\ ay_1 \end{bmatrix} + \begin{bmatrix} bx_2 \\ by_2 \end{bmatrix}\right)$$

$$= T\left(\begin{bmatrix} ax_1+bx_2 \\ ay_1+by_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -(ay_1+by_2) \\ -(ax_1+bx_2) \end{bmatrix}$$

$$= \begin{bmatrix} -ay_1-by_2 \\ -ax_1-bx_2 \end{bmatrix}$$

$$= a\begin{bmatrix} -y_1 \\ -x_1 \end{bmatrix} + b\begin{bmatrix} -y_2 \\ -x_2 \end{bmatrix}$$

$$= aT\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + bT\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$\therefore T(a\alpha+b\beta) = aT(\alpha) + bT(\beta) \quad \forall a, b \in F,$
$\alpha, \beta \in V_2(F)$.

$\therefore$ T is a linear transformation from $V_2(F)$ to $V_2(F)$.

Now, we have,
standard basis in $V_2(F)$ as $B = \{(1,0), (0,1)\}$

$\therefore T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0\cdot\alpha_1 + (-1)\cdot\alpha_2$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = (-1)\cdot\alpha_1 + 0\cdot\alpha_2$

where $\alpha_1 = (1,0)$ and $\alpha_2 = (0,1)$.

$\therefore$ The matrix of a linear transformation T w.r.t. standard basis is $[T:B] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Further, the given transformation can be written as

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Now, applying the transformation to the point $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, we get,

$$T\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$= \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ which is the required image of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Q.1.(c) If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that:

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u.$$

<u>Solution:</u>

We have, $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$ which is not a homogeneous function.

However, we write $\sin u = \dfrac{x^2+y^2}{x+y}$ $(= z)$ say ——— (1)

$$\Rightarrow z = \frac{x^2+y^2}{x+y} = x\left[\frac{1+(y/x)^2}{1+(y/x)}\right] = x \cdot f(y/x)$$

$\Rightarrow$ z is a homogeneous function of x, y of degree 1.

∴ By Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z \quad \text{—— ②}$$

But from ①,

$$\frac{\partial z}{\partial x}=\cos u\cdot\frac{\partial u}{\partial x}, \quad \frac{\partial z}{\partial y}=\cos u\cdot\frac{\partial u}{\partial y} \quad \text{—— ③}$$

Putting ③, ① in ②, we have,

$$x\cdot\cos u\frac{\partial u}{\partial x}+y\cdot\cos u\frac{\partial u}{\partial y}=\sin u$$

$$\Rightarrow \cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u \quad \text{—— ④}$$

Partially differentiating ④ w.r.t. x and multiplying by x on both sides, we get,

$$x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=x\cdot\sec^2 u\frac{\partial u}{\partial x} \quad \text{— ⑤}$$

Similarly, partially differentiating ④ w.r.t. y and multiplying by y on both sides, we get,

$$xy\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=y\cdot\sec^2 u\frac{\partial u}{\partial y} \quad \text{— ⑥}$$

Adding ⑤ and ⑥, we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

Using ④, we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\tan u=\sec^2 u\cdot\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan u(\sec^2 u-1)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan^3 u$$

$\left[\because \sec^2 u-1=\tan^2 u\right]$

Hence Proved.

Q.1.(d). Show that:

$$\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

Solution:

Let $I=\int_0^{\pi/2}\frac{\sin^2 x\,dx}{\sin x+\cos x}$ —— ①

$$\therefore I=\int_0^{\pi/2}\frac{\sin^2(\pi/2-x)\,dx}{\sin(\pi/2-x)+\cos(\pi/2-x)}$$

$\therefore I=\int_0^{\pi/2}\frac{\cos^2 x\,dx}{\cos x+\sin x}$ —— ②

On adding ① and ②, we get,

$$2I=\int_0^{\pi/2}\frac{dx}{\sin x+\cos x}=\int_{t=0}^{1}\frac{2\,dt}{(1+t^2)\left[\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right]}$$

where $t = \tan x/2$

$$= \int_0^1 \frac{dt}{1+2t-t^2} = 2\int_0^1 \frac{dt}{2-(t-1)^2} = \frac{2 \cdot 1}{2\sqrt{2}} \left| \log \frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)} \right|_0^1$$

$$= \frac{1}{\sqrt{2}} \left[0 - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right] = -\frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{(\sqrt{2}+1)^2}$$

$$= -\frac{1}{\sqrt{2}} \cdot \log \frac{1}{(\sqrt{2}+1)^2} = -\frac{1}{\sqrt{2}} \log (\sqrt{2}+1)^{-2}$$

$$\Rightarrow 2I = \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

Hence Proved.

Q.1.(e) P is a point on the plane $lx+my+nz = p$. A point Q is taken to be on the line OP such that $OP \cdot OQ = p^2$, prove that the locus of Q is $p(lx+my+nz) = x^2+y^2+z^2$.

Solution:

Let $Q = (\alpha, \beta, \gamma)$ and $OQ = R$, then the d.r.'s of the line OQ are $\alpha-0, \beta-0, \gamma-0$

i.e. α, β, γ.

$\therefore$ The d.c.'s of OQ are $\frac{\alpha}{R}, \frac{\beta}{R}, \frac{\gamma}{R}$ where

$R = OQ = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ —— (1)

$\therefore$ Equations of line OQ are

$$\frac{x-0}{\alpha/R} = \frac{y-0}{\beta/R} = \frac{z-0}{\gamma/R} = r \text{ (say)}$$

where r is the distance of any point from $(0,0,0)$.

Let $OP = r$, then the co-ordinates of P are

$\left(\frac{\alpha r}{R}, \frac{\beta r}{R}, \frac{\gamma r}{R}\right)$.

But, it is given that P is a point on the plane

$lx + my + nz = p$

$\therefore l \cdot \frac{\alpha r}{R} + m \cdot \frac{\beta r}{R} + n \cdot \frac{\gamma r}{R} = p$

$\Rightarrow \frac{r}{R}(l\alpha + m\beta + n\gamma) = p$ —— (2)

Also, $OP \cdot OQ = p^2$ is given.

i.e., $r \cdot R = p^2$ $\quad (\because OP = r, OQ = R)$

$\Rightarrow r = p^2/R$

$\therefore$ (2) $\equiv$ $p(l\alpha + m\beta + n\gamma) = R^2$

$\Rightarrow p(l\alpha + m\beta + n\gamma) = \alpha^2 + \beta^2 + \gamma^2 \rightarrow$ from (1)

$\therefore$ The locus of $Q(\alpha, \beta, \gamma)$ is

$$p(lx + my + nz) = x^2 + y^2 + z^2.$$

Hence, proved.

Q.2.(a) (i) Find the kernel and range of the linear operator $T(x,y,z) = (x,y,0)$ and describe transformation geometrically.

(ii) If α, β are any scalars, then prove that $A^2 - (\alpha+\beta)A + \alpha\beta I = (A-\alpha I)(A-\beta I)$, where A is any square matrix of order n and $I = I_n$.

Solution:

(i) Given $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x,y,z) = (x,y,0)$$

Also, the kernel and range will both be subspaces of $\mathbb{R}^3$.

Kernel:

Ker (T) is the subset that is mapped into $(0,0,0)$

i.e. $\text{Ker } T = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \,/\, T(x_1,x_2,x_3) = (0,0,0)\}$

i.e. $T(x,y,z) = (0,0,0)$

$\Rightarrow (x,y,0) = (0,0,0)$ if $x=0, y=0$.

$\therefore \text{Ker } T = \{(0,0,z) \,/\, z \in \mathbb{R}\}$

Geometrically, Ker(T) is the set of all vectors that lie on the z-axis.

Range:

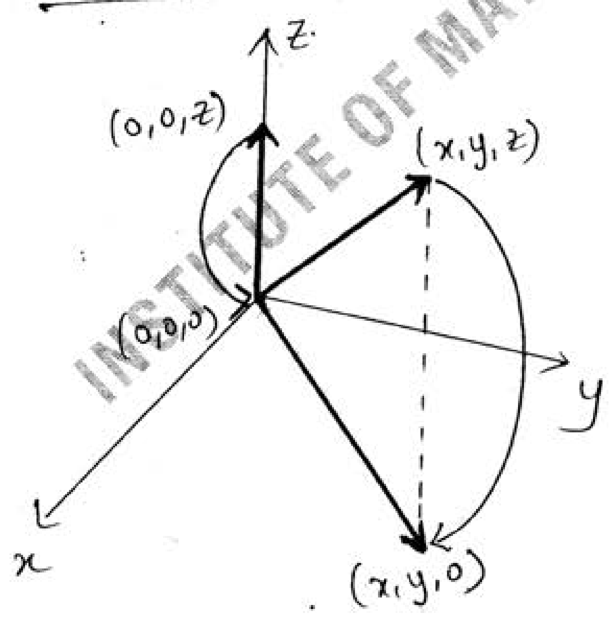
Range space of $T = \{ \beta \in \mathbb{R}^3 / T(\alpha) = \beta$ for $\alpha \in \mathbb{R}^3 \}$

i.e. The range space consists of all vectors of the type $(x, y, 0)$ for all $(x, y, z) \in \mathbb{R}^3$.

i.e. $\text{Range}(T) = \{ (x, y, 0) \in \mathbb{R}^3 / T(x, y, z) = (x, y, 0) \ \forall (x, y, z) \in \mathbb{R}^3 \}$

Geometrically, Range (T) is the set of all vectors that lie in the xy-plane.

Geometric interpretation of the transformation:



T projects the vector (x, y, z) into the vector $(x, y, 0)$ in the xy-plane & also T projects all vectors onto xy plane.

[T is an example of projection operator.]

(ii) Given A is any square matrix of order n and $I = I_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$, $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$

then,

$$A^2 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} \Sigma a_{1i} a_{i1} & \Sigma a_{1i} a_{i2} & \cdots & \Sigma a_{1i} a_{in} \\ \Sigma a_{2i} a_{i1} & \Sigma a_{2i} a_{i2} & \cdots & \Sigma a_{2i} a_{in} \\ \vdots \\ \Sigma a_{ni} a_{i1} & \Sigma a_{ni} a_{i2} & \cdots & \Sigma a_{ni} a_{in} \end{bmatrix}$$

Now, for any α, β as scalars, consider L.H.S.

i.e. $A^2 - (\alpha+\beta)A + \alpha\beta I$

$$= \begin{bmatrix} \Sigma a_{1i} a_{i1} & \Sigma a_{1i} a_{i2} & \cdots & \Sigma a_{1i} a_{in} \\ \Sigma a_{2i} a_{i1} & \Sigma a_{2i} a_{i2} & \cdots & \Sigma a_{2i} a_{in} \\ \vdots \\ \Sigma a_{ni} a_{i1} & \Sigma a_{ni} a_{i2} & \cdots & \Sigma a_{ni} a_{in} \end{bmatrix} -$$

$$\begin{bmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} & \cdots & (\alpha+\beta)a_{1n} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} & \cdots & (\alpha+\beta)a_{2n} \\ \vdots & & & \\ (\alpha+\beta)a_{n1} & (\alpha+\beta)a_{n2} & \cdots & (\alpha+\beta)a_{nn} \end{bmatrix} +$$

$$\begin{bmatrix} \alpha\beta & 0 & \cdots & 0 \\ 0 & \alpha\beta & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \alpha\beta \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma a_{1i}a_{i1} - (\alpha+\beta)a_{11} + \alpha\beta & \Sigma a_{1i}a_{i2} - (\alpha+\beta)a_{12} & \cdots \Sigma a_{1i}a_{in} - (\alpha+\beta)a_{1n} \\ \Sigma a_{2i}a_{i1} - (\alpha+\beta)a_{21} & \Sigma a_{2i}a_{i2} - (\alpha+\beta)a_{22} + \alpha\beta & \cdots \Sigma a_{2i}a_{in} - (\alpha+\beta)a_{2n} \\ \vdots & & \\ \Sigma a_{ni}a_{i1} - (\alpha+\beta)a_{n1} & \Sigma a_{ni}a_{i2} - (\alpha+\beta)a_{n2} & \cdots \Sigma a_{ni}a_{in} - (\alpha+\beta)a_{nn} + \alpha\beta \end{bmatrix} \quad \text{——— (1)}$$

Now,
consider R.H.S,
i.e. $(A - \alpha I)(A - \beta I)$

$$= \begin{bmatrix} a_{11}-\alpha & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\alpha & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\alpha \end{bmatrix} \begin{bmatrix} a_{11}-\beta & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\beta & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\beta \end{bmatrix}$$

$$= \begin{bmatrix} \sum a_{1i} a_{i1} - (\alpha+\beta)a_{11} + \alpha\beta & \sum a_{1i} a_{i2} - (\alpha+\beta)a_{12} & \cdots & \sum a_{1i} a_{in} - (\alpha+\beta)a_{1n} \\ \sum a_{2i} a_{i1} - (\alpha+\beta)a_{21} & \sum a_{2i} a_{i2} - (\alpha+\beta)a_{22} + \alpha\beta & \cdots & \sum a_{2i} a_{in} - (\alpha+\beta)a_{2n} \\ \vdots & & & \\ \sum a_{ni} a_{i1} - (\alpha+\beta)a_{n1} & \sum a_{ni} a_{i2} - (\alpha+\beta)a_{n2} & \cdots & \sum a_{ni} a_{in} - (\alpha+\beta)a_{nn} + \alpha\beta \end{bmatrix} \quad \text{--- (2)}$$

From (1) & (2), we conclude that,

$L.H.S = R.H.S$

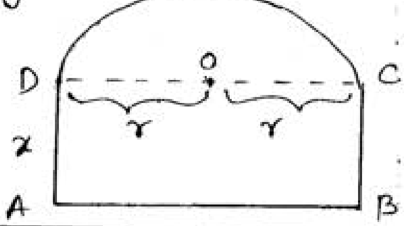
i.e. $A^2 - (\alpha+\beta)A + \alpha\beta I = (A-\alpha I)(A-\beta I)$

for any square matrix A of order n and $I = I_n$ where α, β are any scalars.

Hence, Proved.

Q.2.(b) A window has the form of a rectangle surmounted by a semi-circle. If the perimeter is 40 ft., find its dimensions so that the greatest amount of light may be admitted.

Solution:



Perimeter of the window when the

width of window is 'x' and '$2r$' is the length.

$\Rightarrow$ Perimeter $= 40$

$\Rightarrow 2x + 2r + \pi r = 40$

$\Rightarrow 2x = 40 - r(\pi + 2)$ —— (1)

For admitting the greatest amount of light through the opening, the area of the window must be maximum.

Let A = sum of areas of rectangle and semi-circle.

$\therefore A = 2r.x + \frac{1}{2}(\pi r^2)$

$= r\left[40 - r(\pi+2)\right] + \frac{1}{2}\pi r^2$ ... from (1).

$\therefore A = 40r - \left(\frac{1}{2}\pi + 2\right)r^2$

For evaluating either minimum or maximum, we consider $\frac{dA}{dr} = 0$.

$\therefore 40 - (\pi + 4)r = 0$

$\Rightarrow \boxed{r = \frac{40}{\pi + 4}}$

Now, consider $\frac{d^2A}{dr^2} = -(\pi + 4)$

i.e. $\frac{d^2A}{dr^2}$ is $-ve \Rightarrow A$ is maximum.

$\therefore$ from (1), $40 = (\pi + 2)\left(\frac{40}{\pi+4}\right) + 2x$

$\Rightarrow 40 = \frac{40(\pi+2) + 2x(\pi+4)}{\pi + 4}$

$\therefore\ 40(\pi+4) - 40(\pi+2) = 2x(\pi+4)$

$\Rightarrow \quad 40 = x(\pi+4)$

$\Rightarrow \quad \boxed{x = \dfrac{40}{\pi+4}}$

$\therefore$ Length of rectangle $= 2r$
$= 2\cdot\dfrac{40}{\pi+4}$

$= \dfrac{80}{\pi+4}$ ft.

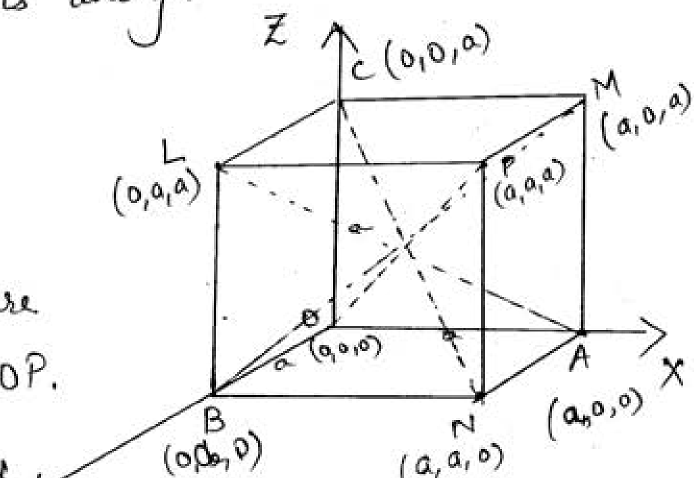
and Breadth of rectangle $= \dfrac{40}{\pi+4}$ ft.

Q.2.(c) (i) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube; prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$.

(ii) Find the equation of the planes through the intersection of the planes $x+3y+6=0$; $3x-y-4z=0$ and whose perpendicular distance from the origin is unity.

Solution:

(i) From the figure alongside, the four diagonals are AL, BM, CN and OP.



The d.c.'s of the diagonal OP are proportional to $a-0, a-0, a-0$ [using $x_2-x_1, y_2-y_1, z_2-z_1$]

i.e. a, a, a or $1, 1, 1$.

$\therefore$ Actual d.c.'s of OP are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

Similarly,

the d.c.'s of diagonal AL are $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$;

the d.c.'s of diagonal BM are $1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}$;

the d.c.'s of diagonal CN are $1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}$.

Let l, m, n be the d.c.'s of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals OP, AL, BM and CN respectively (say).

$$\therefore \cos\alpha = \frac{1}{\sqrt{3}}\cdot l + \frac{1}{\sqrt{3}}\cdot m + \frac{1}{\sqrt{3}}\cdot n$$

$$\therefore \cos\alpha = \frac{1}{\sqrt{3}}(l+m+n).$$

Similarly,

$$\cos\beta = \frac{1}{\sqrt{3}}(-l+m+n); \quad \cos\gamma = \frac{1}{\sqrt{3}}(l-m+n);$$

$$\cos\delta = \frac{1}{\sqrt{3}}(l+m-n).$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}(l+m+n)^2 + \frac{1}{3}(-l+m+n)^2 + \frac{1}{3}(l-m+n)^2 + \frac{1}{3}(l+m-n)^2$$

$$= \frac{1}{3}\Big[(l^2+m^2+n^2+2lm+2mn+2ln)+(l^2+m^2+n^2-2lm+2mn-2ln)+(l^2+m^2+n^2-2ln-2mn+2ln)+(l^2+m^2+n^2+2lm-2mn-2ln)\Big]$$

$$= \frac{1}{3}\left[4(l^2+m^2+n^2)\right]$$

$$\left[\because l^2+m^2+n^2=1\right]$$

$$= \frac{4}{3}$$

Hence Proved.

(ii)

The equation of any plane through the intersection of the given planes is

$$(x+3y+6)+\lambda(3x-y-4z)=0$$

i.e. $(1+3\lambda)x+(3-\lambda)y-4\lambda z+6=0$ —— (1)

The length of perpendicular from the origin i.e. $(0,0,0)$ to Eqn (1) is given as unity, so we have

$$1 = \frac{6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}$$

$$\Rightarrow (1+3\lambda)^2+(3-\lambda)^2+16\lambda^2=36$$

$$\Rightarrow 26\lambda^2+10=36 \quad \Rightarrow 26\lambda^2=26$$

$$\Rightarrow \lambda^2=1. \quad \Rightarrow \lambda=\pm 1.$$

Substituting these values of λ in (1), we have the required equations as

$$2x+y-2z+3=0 \quad \text{and} \quad x-2y-2z-3=0.$$

Q.3. (a). Consider the linear transformation $T(x,y)=(3x+4y,\ 5x+7y)$ of $\mathbb{R}^2 \to \mathbb{R}^2$.

(i) Prove that T is invertible and find the inverse of T.

(ii) Determine the pre-image of the vector (1, 2).

<u>Solution :</u>

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x,y) = (3x+4y,\ 5x+7y)$$

(i)

If $T(x,y) = (0,0)$ then

$$(3x+4y,\ 5x+7y) = (0,0)$$

$\Rightarrow$ $3x+4y=0$ and $5x+7y=0$

On solving these two equations, we get

$\underline{x=0}$ and $\underline{y=0}$.

$\therefore$ $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Hence, T is non-singular and therefore T^{-1} exists.

i.e. T is invertible.

Now,

Let $T(x,y) = (a,b)$

then $T^{-1}(a,b) = (x,y)$.

$\because T(x,y) = (a,b)$

$\Rightarrow (3x+4y, 5x+7y) = (a,b)$

$\Rightarrow 3x+4y = a$ and $5x+7y = b$

$\Rightarrow x = \frac{a-4y}{3}$; $5\left(\frac{a-4y}{3}\right) + 7y = b$

$\Rightarrow 5a - 20y + 21y = 3b$

$\Rightarrow \underline{y = -5a+3b}$

$\Rightarrow x = \frac{a-4(-5a+3b)}{3}$

$\Rightarrow x = \frac{21a-12b}{3}$

$\Rightarrow \underline{x = 7a-4b}$

$\therefore T^{-1}(a,b) = (7a-4b, -5a+3b)$ —— (A)

(ii) The pre-image of vector $\binom{1}{2}$ is given by (A) as follows,

$T^{-1}(1,2) = (7(1)-4(2), -5(1)+3(2))$

$\therefore \boxed{T^{-1}(1,2) = (-1,1)}$

Hence, the result.

Q.3.(b). Show that the following function is discontinuous at (0,0):

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & : x \neq y \\ 0 & : x = y \end{cases}$$

Solution:

Let $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & : x \neq y \\ 0 & : x = y \end{cases}$

Let us approach (0,0) along the path $y = x - mx^3$, then we have,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + (x-mx^3)^3}{x-(x-mx^3)}$$

$$= \lim_{x\to 0} \frac{x^3 + x^3 - m^3x^9 - 3x^2mx^3 + 3xm^2x^6}{mx^3}$$

$$= \lim_{x\to 0} \frac{2x^3 - m^3x^9 - 3x^5m - 3x^7m^2}{mx^3}$$

$$= \lim_{x\to 0} \frac{2 - m^3x^6 - 3x^2m - 3x^4m^2}{m}$$

$$= \frac{2}{m}$$

which is different for different values of m.

Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore$ The given function is discontinuous at (0,0). //

Q.3. (c) Evaluate $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

<u>Solution:</u>

The given limit is 1^∞ form.

Let $y=\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ so that

$$\log y = \tan\left(\frac{\pi x}{2a}\right)\cdot\log\left(2-\frac{x}{a}\right)$$

$$\therefore \lim_{x\to a}\log y = \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to a}\frac{\frac{1}{\left(2-\frac{x}{a}\right)}\cdot\cancel{-}\frac{1}{\cancel{a}}}{\cancel{-}\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2\cancel{a}}}$$

$$\therefore \lim_{x\to a}\log y = \frac{2}{\pi}$$

$$\therefore \log\lim_{x\to a} y = 2/\pi$$

$$\Rightarrow \lim_{x\to a} y = e^{2/\pi}$$

Hence, $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)} = e^{2/\pi}$

Q.3.(d) (i) Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1,1,-1)$ and passes through the origin

(ii) Prove that the condition that the plane $ux+vy+wz=0$ may cut the cone $ax^2+by^2+cz^2=0$ in perpendicular generators if $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$.

Solution:

(i) Equation of the tangent plane to the given sphere $x^2+y^2+z^2-x+3y+2z-3=0$ — ① at $(1,1,-1)$ is

$$x\cdot 1+y\cdot 1+z\cdot(-1)-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+(z-1)-3=0$$

i.e. $x+5y-6=0$ — ②

$\left[\begin{array}{l} \because \text{ The equation of tangent plane to the sphere } x^2+y^2+z^2+2ux+2vy+2wz+d=0 \\ \text{at the point } (\alpha,\beta,\gamma) \text{ is given by} \\ \quad x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+ \\ \quad w(z+\gamma)+d=0 \end{array}\right]$

The required sphere is the sphere through the point circle (or a circle of zero radius) of intersection of the sphere and the plane and also passing through the point (0,0,0).

The equation of any sphere through the circle of intersection (point circle) is $(x^2+y^2+z^2-x+3y+2z-3)+\lambda(x+5y-6)=0$ — (3)

If it passes through (0,0,0) then,

$$-3-6\lambda=0$$

$$\Rightarrow \lambda=-1/2$$

Substituting this value of λ in (3), the required equation is

$$(x^2+y^2+z^2-x+3y+2z-3)+(-1/2)(x+5y-6)=0$$

$$\Rightarrow 2(x^2+y^2+z^2-x+3y+2z-3)-(x+5y-6)=0$$

$$\Rightarrow 2(x^2+y^2+z^2)-3x+y+4z=0$$

(ii)

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines in which the plane $ux+vy+wz=0$ meets the cone $ax^2+by^2+cz^2=0$, then we have,

$ul + vm + wn = 0$ —— (1) and

$al^2 + bm^2 + cn^2 = 0$ —— (2)

Eliminating 'n' between (1) & (2), we get,

$$al^2 + bm^2 + c\left[-\frac{(ul+vm)}{w}\right]^2 = 0$$

$$\Rightarrow (aw^2 + cu^2)l^2 + 2cuvlm + (bw^2 + cv^2)m^2 = 0$$

$$\Rightarrow (aw^2 + cu^2)\left(\frac{l^2}{m^2}\right) + 2cuv\left(\frac{l}{m}\right) + (bw^2 + cv^2) = 0$$

If l_1/m_1 and l_2/m_2 are its roots, then we have,

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{bw^2 + cv^2}{aw^2 + cu^2} \quad \left(\frac{c}{a}\right)$$

$$\Rightarrow \frac{l_1 \cdot l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{cu^2 + aw^2} = \frac{n_1 n_2}{av^2 + bu^2} \quad \left(\text{by symmetry}\right)$$

If the lines are perpendicular then,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow (bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2) = 0$$

$$\Rightarrow (b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$$

Hence, proved.

Q. 4. (a) (i) Solve the following homogeneous system of linear equations. Interpret the set of solutions as a subspace. Sketch the subspace of solutions.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$x_1 + x_2 + 4x_3 = 0.$$

(ii) In any vector space V,

(A) $\alpha\vec{0} = \vec{0}$ for every scalar α.

(B) $0.\vec{u} = \vec{0}$ for every $u \in V$

(C) $(-1).\vec{u} = -\vec{u}$ for every $u \in V$.

Solution:

(i) The given system of homogeneous linear equations are

$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$x_1 + x_2 + 4x_3 = 0$$

Now, we write a single augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$R_2 \to (-1) \times R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$R_3 \to R_3 + R_2, \quad R_1 \to R_1 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore\ \rho(A) = \rho(A|B) = 2 <$ no. of unknowns x_1, x_2, x_3.

$\therefore$ The given system has a non-zero solution.

The echelon matrix equation is

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} x_1 \quad + 5x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -5x_3 \\ x_2 &= x_3 \end{aligned}$$

$\therefore$ The solutions are vectors of the form

$(x_1, x_2, x_3) = (-5x_3, x_3, x_3)$.

i.e. Solution Set $= \{(-5, 1, 1)x_3 / x_3 \in \mathbb{R}\}$

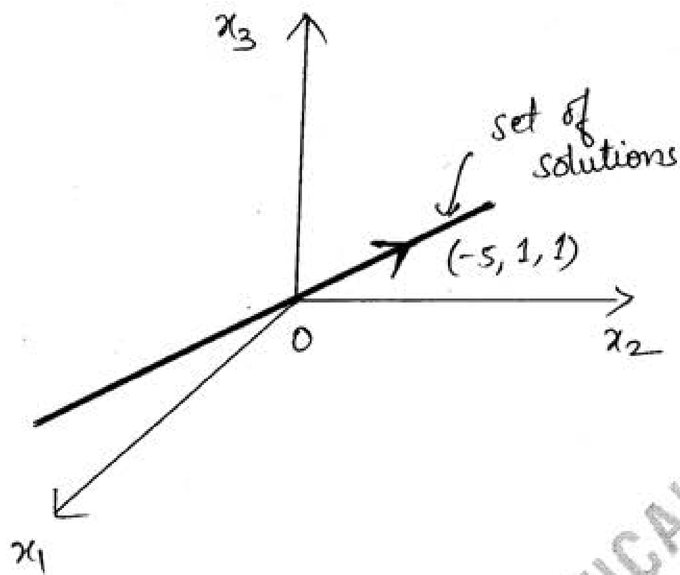
__Interpretation :__

These vectors form a one-dimensional subspace of $\mathbb{R}^3$ with basis $(-5, 1, 1)$.

Also, this subspace is the kernel of the transformation defined by the matrix of co-efficients of the system,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}.$$

Sketch of subspace of solutions :-



(ii)

(A) $\alpha\vec{0} = \alpha(\vec{0}+\vec{0})$ by Additive identity

$= \alpha\vec{0} + \alpha\vec{0}$ by Distributivity

Adding $-(\alpha\vec{0})$ to both sides, we get,

$\vec{0} = -(\alpha\vec{0}) + (\alpha\vec{0} + \alpha\vec{0})$

$= \left[-(\alpha\vec{0}) + (\alpha\vec{0})\right] + \alpha\vec{0}$ — by Associativity

$= \vec{0} + \alpha\vec{0}$ by Additive inverse

$\therefore\ \vec{0} = \alpha\vec{0}$ by Additive identity

(B) $0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u}$ by Distributivity

Adding $-(0\vec{u})$ to both sides, we get,

$\vec{0} = -(0\vec{u}) + (0\vec{u} + 0\vec{u}) = \left[-(0\vec{u}) + (0\vec{u})\right] + 0\vec{u}$

by Associativity

$= \vec{0} + 0\vec{u}$ by Additive inverse

$\therefore \vec{0} = 0\vec{u}$ by Additive identity

(C) $(-1)\vec{u} + \vec{u} = (-1)\cdot\vec{u} + 1\cdot\vec{u}$

— by Existence of multiplicative identity

$= (-1+1)\cdot\vec{u}$ by Distributivity

$= 0\cdot\vec{u}$ by Additive inverse

$\therefore (-1)\vec{u} + \vec{u} = \vec{0}$ by (B).

So, by additive inverse and identity along with the uniqueness of the negative, $(-1)\vec{u}$ is the negative of $\vec{u}$.

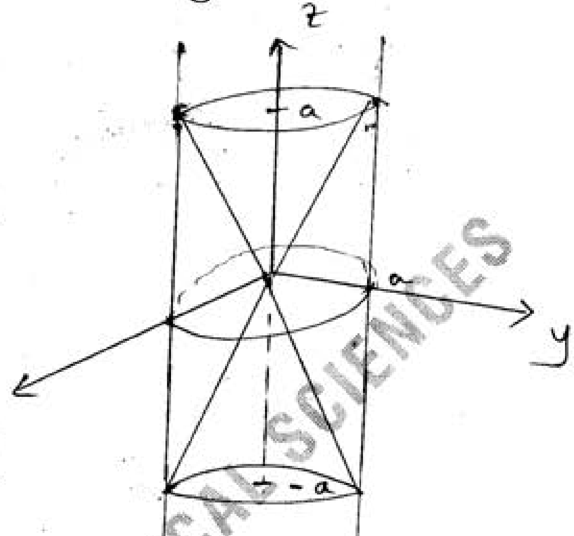
i.e. $(-1)\vec{u} = -\vec{u}$

Hence, proved.

Q.4.(b). Find the volume bounded by the cylinder $x^2+y^2=a^2$ and the cone $x^2+y^2=z^2$.

Solution:

Volume bounded by cylinder $x^2+y^2=a^2$ and the cone $x^2+y^2=z^2$ is given by



$$V = 8\times\int_0^a dx\int_0^{\sqrt{a^2-x^2}} dy\int_0^{\sqrt{x^2+y^2}} dz$$

$$= 8\times\int_0^a dx\int_0^{\sqrt{a^2-x^2}}\left[z\right]_0^{\sqrt{x^2+y^2}}$$

$$= 8\times\int_0^a dx\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy$$

$$= 8\times\int_0^a dx\left[\frac{y}{2}\sqrt{x^2+y^2}+\frac{x^2}{2}\log\left|y+\sqrt{x^2+y^2}\right|\right]_0^{\sqrt{a^2-x^2}}$$

$$= 8\times\int_0^a dx\left[\frac{\sqrt{a^2-x^2}}{2}\cdot a+\frac{x^2}{2}\log\left|a+\sqrt{a^2-x^2}\right|-\frac{x^2}{2}\log x\right]$$

$$= 8\int_0^a \frac{a}{2}\sqrt{a^2-x^2} + \frac{x^2}{2}\log\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| dx$$

Let $x = a\sin\theta$

$$= 8\times\int_0^{\pi/2} \frac{a}{2}\sqrt{a^2-a^2\sin^2\theta} + \frac{a^2\sin^2\theta}{2}\log\left|\frac{a+a\cos\theta}{a\sin\theta}\right|$$

$$a\cos\theta\, d\theta$$

$$= 8\cdot\int_0^{\pi/2}\left[\frac{a^2}{2}\cos\theta + \frac{a^2\sin^2\theta}{2}\log(\cot\theta/2)\right]\cdot a\cos\theta\, d\theta$$

$$= 8\cdot\int_0^{\pi/2}\frac{a^3\cos^2\theta}{2}\,d\theta + 8\int_0^{\pi/2}\frac{a^3\sin^2\theta\cos\theta}{2}\log(\cot\theta/2)\,d\theta$$

$$= \pi a^3 + \frac{1}{3}\pi a^3 \qquad \text{(On Integrating)}$$

$$\therefore \boxed{V = \frac{4}{3}\pi a^3}$$

Alternatively, Volume of cylinder between $z = \pm a$ is $(\pi a^2)\times 2a = 2\pi a^3$; Volume of each half cone $= \frac{1}{3}\pi a^2 \times a = \frac{1}{3}\pi a^3 \Rightarrow$ Volume of 2 cones $= \frac{2}{3}\pi a^3$.

$\therefore$ Required volume $= 2\pi a^3 - \frac{2}{3}\pi a^3 = \frac{4}{3}\pi a^3$

Hence, the result.

Q.4.(c) Prove that the tangent planes to the hyperboloid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1$ which are parallel to tangent planes to the cone $\frac{b^2c^2x^2}{c^2-b^2}+\frac{c^2a^2y^2}{c^2-a^2}+\frac{a^2b^2z^2}{a^2+b^2}=0$ cut the surface in perpendicular generators.

Solution:

Since, the equation of cone reciprocal to the cone $ax^2+by^2+cz^2=0$ is $\left(\frac{x^2}{a}\right)+\left(\frac{y^2}{b}\right)+\left(\frac{z^2}{c}\right)=0$

$\therefore$ The equation of the cone reciprocal to the given cone is

$$\frac{c^2-b^2}{b^2c^2}\cdot x^2+\frac{c^2-a^2}{c^2a^2}\cdot y^2+\frac{a^2+b^2}{a^2b^2}\cdot z^2=0 \quad \text{—} (1)$$

Let $lx+my+nz=0$ be a tangent plane to the given cone so that by definition, its normal with d.r.'s l, m, n is a generator of its reciprocal cone (1).

Also, $$\frac{c^2-b^2}{b^2c^2}\cdot l^2+\frac{c^2-a^2}{c^2a^2}m^2+\frac{a^2+b^2}{a^2b^2}\cdot n^2=0 \quad \text{—} (2)$$

Let any plane parallel to the tangent plane to the given cone be $lx+my+nz=p$ — (3)

If it is a tangent plane to the given hyperboloid, then $p^2 = a^2l^2 + b^2m^2 - c^2n^2$ —— (4)

Also, if it is a tangent plane at the point (x_1, y_1, z_1) then its equation is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \frac{zz_1}{c^2} = 1 \quad \text{—— (5)}$$

Comparing (3) & (5), we get,

$$\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{z_1/c^2}{n} = 1/p$$

$$\Rightarrow \frac{x_1}{la^2} = \frac{y_1}{mb^2} = \frac{z_1}{-nc^2} = \frac{1}{p} \quad \text{—— (6)}$$

Also, the plane (3) cuts the given hyperboloid in perpendicular generators if (x_1, y_1, z_1) lies on the director sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 - c^2$$

$$\therefore x_1^2 + y_1^2 + z_1^2 = a^2 + b^2 - c^2$$

$$\Rightarrow \left(\frac{a^2l}{p}\right)^2 + \left(\frac{b^2m}{p}\right)^2 + \left(-\frac{c^2n}{p}\right)^2 = a^2 + b^2 - c^2 \quad \text{— from (6)}$$

$$\Rightarrow a^4l^2 + b^4m^2 + c^4n^2 = (a^2 + b^2 - c^2)p^2$$

$$= (a^2 + b^2 - c^2)(a^2l^2 + b^2m^2 - c^2n^2) \quad \text{— from (4)}$$

$$\Rightarrow a^2l^2(b^2 - c^2) + b^2m^2(a^2 - c^2) - c^2n^2(a^2 + b^2) = 0$$

$$\Rightarrow \frac{l^2(c^2-b^2)}{b^2c^2} + \frac{m^2(c^2-a^2)}{c^2a^2} + \frac{n^2(a^2+b^2)}{a^2b^2} = 0$$

[by dividing each term by $-a^2b^2c^2$]

This is true by virtue of Equation (2).

Hence, the result.

SECTION – B

Q.5. (a) Show that the function $h(x) = 4x^2+3x-7$ lies in the space span $\{f, g\}$ generated by $f(x) = 2x^2-5$ and $g(x) = x+1$.

Solution:

To show that the function $h(x) = 4x^2+3x-7$ lies in the space span $\{f, g\}$, we express $h(x)$ as a linear combination of $f(x)$ and $g(x)$.

Let $a, b \in F$ such that

$$h(x) = a\,f(x) + b\,g(x)$$

$$\Rightarrow (4x^2+3x-7) = a(2x^2-5) + b(x+1)$$

$$\Rightarrow (4x^2+3x-7) = 2ax^2 + bx - 5a + b.$$

Two polynomials can only be equal for all values of x if the corresponding co-efficients are equal.

$\therefore$ By comparing the co-efficients of x^2, x and the constant terms on both the sides, we have,

$$2a = 4 \; ; \quad b = 3 \; ; \quad -5a + b = -7$$

$$\Rightarrow a = 2 \; ; \quad b = 3.$$

$$\therefore h(x) = 2f(x) + 3g(x)$$

$\therefore$ The function $h(x) = 4x^2 + 3x - 7$ lies in the space generated by $f(x) = 2x^2 - 5$ and $g(x) = x + 1$.

Hence, the result.

Q.5.(b). Show that the set $\{(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)\}$ is linearly independent in $\mathbb{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbb{R}^4$. Construct an orthonormal basis for V.

Solution:

To show that the given set $S = \{(1,2,0,3), (4,0,5,8), (8,1,5,6)\}$ is L.I., we proceed further as follows:-

Let $a, b, c \in F$ such that

$$a(1,2,0,3) + b(4,0,5,8) + c(8,1,5,6) = (0,0,0,0)$$

$$\Rightarrow (a + 4b + 8c, \; 2a + c, \; 5b + 5c, \; 3a + 8b + 6c) = (0,0,0,0)$$

$\Rightarrow a+4b+8c=0;\quad 2a+c=0;\quad 5b+5c=0;$

$3a+8b+6c=0.$

$\Rightarrow 2a+c=0 \Rightarrow a=-\frac{c}{2}.$

$5b+5c=0 \Rightarrow b=-c$

$a+4b+8c=0 \Rightarrow -\frac{c}{2}-4c+8c=0$

$\Rightarrow -c-8c+16c=0$

$\Rightarrow 7c=0$

$\Rightarrow \underline{c=0}.$

$\therefore \underline{a=0}$ and $\underline{b=0}$.

$\therefore$ The set S is L.I. where $v_1=(1,2,0,3)$, $v_2=(4,0,5,8)$, $v_3=(8,1,5,6)$.

Now, using Gram-Schmidth process to construct an orthogonal set (u_1, u_2, u_3) from these vectors, we proceed ahead.

Let $u_1 = v_1 = (1,2,0,3)$.

Let $u_2 = v_2 - \text{proj}_{u_1} v_2$

$$= v_2 - \frac{(v_2\cdot u_1)}{(u_1\cdot u_1)}\cdot u_1$$

$$= (4,0,5,8) - \frac{(4,0,5,8)\cdot(1,2,0,3)}{(1,2,0,3)\cdot(1,2,0,3)}(1,2,0,3)$$

$$= (4,0,5,8) - 2(1,2,0,3)$$

$\therefore\ u_2 = (2,-4,5,2)$

Let $u_3 = v_3 - \text{proj}_{u_1} v_3 - \text{proj}_{u_2} v_3$

$$= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= (8,1,5,6) - \frac{(8,1,5,6)\cdot(1,2,0,3)}{(1,2,0,3)\cdot(1,2,0,3)}(1,2,0,3)$$

$$- \frac{(8,1,5,6)\cdot(2,-4,5,2)}{(2,-4,5,2)\cdot(2,-4,5,2)}(2,-4,5,2)$$

$$= (8,1,5,6) - 2(1,2,0,3) - 1(2,-4,5,2)$$

$\therefore\ u_3 = (4,1,0,-2)$

The set $\{(1,2,0,3), (2,-4,5,2), (4,1,0,-2)\}$ is an orthogonal basis for V.

Now, we compute the norm of each vector and then normalize the vectors to get an orthonormal basis.

$\|(1,2,0,3)\| = \sqrt{1^2+2^2+0^2+3^2} = \sqrt{14}$

$\|(2,-4,5,2)\| = \sqrt{2^2+(-4)^2+5^2+2^2} = 7$

$\|(4,1,0,-2)\| = \sqrt{4^2+1^2+0^2+(-2)^2} = \sqrt{21}$

$\therefore$ The set of normalized vectors using the values of the norms which provide the orthonormal basis for V is

$$\left\{\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, 0, \frac{3}{\sqrt{14}}\right), \left(\frac{2}{7}, -\frac{4}{7}, \frac{5}{7}, \frac{2}{7}\right), \left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, 0, \frac{-2}{\sqrt{21}}\right)\right\}$$

Hence, the result.

Q.5.(c). Applying Lagrange's mean value theorem to the function defined by $f(x) = \log(1+x)$ for all $x \geq 0$, show that $0 < [\log(1+x)]^{-1} - x^{-1} < 1$ whenever $x > 0$.

Solution:

Let $f(x) = \log(1+x)$ in $[0, x]$ so that

$$f'(x) = \frac{1}{1+x}$$

Then f is continuous in $[0, x]$ and derivable in $[0, x]$. So, by Lagrange's Mean Value Theorem there exists θ, $0 < \theta < 1$, such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\theta x) \qquad \left[\because (0, x) = (0x, 1x) \text{ open interval. } c = \theta x \right.$$

$\Rightarrow \log(1+x) = \dfrac{x}{1+\theta x}$ —— (1)

$[\because f(0)=0]$

Now, $0<\theta<1$ and $x>0$

$\Rightarrow \theta x < x$

$\Rightarrow 1+\theta x < 1+x$

$\Rightarrow \dfrac{1}{1+\theta x} > \dfrac{1}{1+x}$

$\Rightarrow \dfrac{x}{1+\theta x} > \dfrac{x}{1+x}$

i.e. $\dfrac{x}{1+x} < \dfrac{x}{1+\theta x}$ —— (2)

Again, $0<\theta<1$ and $x>0$

$\Rightarrow 1 < 1+\theta x$

$\Rightarrow \dfrac{1}{1+\theta x} < 1$

$\Rightarrow \dfrac{x}{1+\theta x} < x$ —— (3)

from (2) and (3), we obtain,

$\dfrac{x}{1+x} < \dfrac{x}{1+\theta x} < x$ —— (4)

from (1) and (4), we obtain,

$\dfrac{x}{1+x} < \log(1+x) < x$

$\Rightarrow \dfrac{1+x}{x} > \dfrac{1}{\log(1+x)} > \dfrac{1}{x}$

$\Rightarrow 1+\frac{1}{x} > \frac{1}{\log(1+x)} > \frac{1}{x}$

$\Rightarrow 1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 0$

Hence, $0 < [\log(1+x)]^{-1} - x^{-1} < 1$, $x > 0$.

Hence, the result.

Q. 5. (d) A sphere of constant radius $2k$ passes through origin and meets the axes in A, B and C. Find the locus of the centroid of the tetrahedron OABC.

Solution:

Since, the sphere meets the axes in A, B and C, it passes through the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ respectively (say).

Also, it passes through the origin i.e. $(0,0,0)$.

Let the equation of sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

Since, it passes through $(0,0,0)$ we have $d=0$ —— (2)

Now, (1) passes through $(a,0,0)$ then we have

$a^2+2ua+d=0 \Rightarrow a^2+2ua=0$

$\Rightarrow u = -\frac{1}{2}a$ (as $a \neq 0$).

Similarly, as ① passes through $(0,b,0)$ and $(0,0,c)$, we get, $v = -\frac{1}{2}b$ and $w = -\frac{1}{2}c$.

$\therefore$ from ①, the equation of sphere is

$$x^2+y^2+z^2-ax-by-cz=0 \quad \text{—— ③}$$

It's radius $= \sqrt{\left(\frac{1}{2}a\right)^2+\left(\frac{1}{2}b\right)^2+\left(\frac{1}{2}c\right)^2}$

$$= \frac{1}{2}\sqrt{a^2+b^2+c^2}$$

i.e. $\frac{1}{2}\sqrt{a^2+b^2+c^2} = 2k$ (given)

$\Rightarrow \quad a^2+b^2+c^2 = 16k^2$ —— ④

If (x_1, y_1, z_1) be the centroid of the tetrahedron OABC, then

$$x_1 = \frac{1}{4}\left[0+a+0+0\right] = \frac{1}{4}a \text{ or } a = 4x_1$$

Similarly, $b = 4y_1$ and $c = 4z_1$

Substituting these values of a, b, c in ④, we get,

$$(4x_1)^2+(4y_1)^2+(4z_1)^2 = 16k^2$$

$$\Rightarrow x_1^2+y_1^2+z_1^2 = k^2$$

$\therefore$ The required locus of (x_1, y_1, z_1) is

$$\underline{x^2+y^2+z^2 = K^2}.$$

Hence, the result.

Q. 5. (c). Find the equations of the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$.

Solution:

Any plane through the given lines is

$(x+9y-3z) + \lambda(3x-3y+6z-5) = 0$

$\Rightarrow (1+3\lambda)x + (9-3\lambda)y + (6\lambda-3)z = 5\lambda$ — (1)

If this plane touches the given hyperboloid

$2x^2 - 6y^2 + 3z^2 = 5$ or $\frac{x^2}{5/2} + \frac{y^2}{-5/6} + \frac{z^2}{5/3} = 1$.

then, $a^2l^2 + b^2m^2 + c^2n^2 = p^2$.

$\Rightarrow \frac{5}{2}(1+3\lambda)^2 + \left(-\frac{5}{6}\right)(9-3\lambda)^2 + \left(\frac{5}{3}\right)(6\lambda-3)^2 = (5\lambda)^2$

$\Rightarrow 15(9\lambda^2+6\lambda+1) - 5(9\lambda^2-54\lambda+81) + 10(36\lambda^2-36\lambda+9) = 150\lambda^2$

$\Rightarrow 300\lambda^2 - 300 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$.

$\therefore$ from (1), the required tangent planes are

$(1 \pm 3)x + (9 \mp 3)y + (\pm 6 - 3)z = \pm 5$

$\Rightarrow 4x + 6y + 3z = 5$ and $2x - 12y + 9z = 5$.

Hence, the result.

Q. 6. (a) (i) A square matrix A is said to be involutory if $A^2 = I$. Prove that the matrices $\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$ are involutary for all scalars α. Determine all 2×2 involutary matrices

(ii) Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--- (i)}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--- (ii)}$$

From (i) & (ii), we conclude that A and B are involutary matrices for all scalars a.

Now, consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that M is an involutary matrix.

i.e. $M^2 = I$.

$$M^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

— [$\because$ M is an involutory matrix]

$\Rightarrow a^2+bc = 1$ —— (1)

$bc+d^2 = 1$ —— (2)

$b(a+d) = 0$ —— (3)

$c(a+d) = 0$ —— (4)

From (1) − (2), we have, $a^2-d^2 = 0 \Rightarrow a = \pm d$.

$\left.\begin{array}{l} (3) \Rightarrow b=0 \text{ or } a=-d \\ (4) \Rightarrow c=0 \text{ or } a=-d. \end{array}\right\}$ —— (5)

Considering following cases for enumerating the required matrices:-

Case(i) :- $a=d \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

[$\because b=0=c$ from (5)]

$$\therefore M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{—— (*)}$$

Case (ii) :- $a=-d \Rightarrow bc = 1-a^2$ $(b, c \neq 0)$ [from (5)]

Let $a = \alpha$, $b = \beta \Rightarrow d = -\alpha$, $c = \frac{1-\alpha^2}{\beta}$

$$\therefore M_3 = \begin{bmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{bmatrix} \quad \text{—— (**)}.$$

from (*) & (**), we conclude that M_1, M_2, M_3 are all 2×2 involutory matrices where M_3 includes Pauli's matrices $\sigma_1, \sigma_2, \sigma_3$.

(ii) The given matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$\sim \underset{R_4 \to R_4 - R_1}{\overset{R_2 \to R_2 + R_1,\ R_3 \to R_3 - 2R_1,}{}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \overset{R_2 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$$\overset{R_2 \to R_2 \times (-1)}{\sim} \begin{bmatrix} 1 & +2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix} \overset{R_3 \to R_3 + 3R_2,\ R_4 \to R_4 - 5R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix} \overset{R_4 \to R_4 + 9R_3}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_3 \to R_3 \times (-1)}{\sim} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_2 \to R_2 + 2R_3,\ R_1 \to R_1 + R_3}{\sim} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly which is in row-reduced echelon form.
The number of non-zero rows in the echelon matrix is 3.

$\therefore \boxed{\text{Rank of } A = 3.}$

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Q.6. (b) Let

$$\left\{ v_1 = \begin{bmatrix} i \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ i \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ i \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ i \end{bmatrix} \right\}$$

be a basis of $\mathbb{C}^4$ and let $T \in L(\mathbb{C}^4)$:

$Tv_1 = iv_1 + v_2 - iv_3 + v_4$,

$Tv_2 = v_1 + iv_2 + v_3 - iv_4$,

$Tv_3 = -iv_1 + v_2 + iv_3 + v_4$,

$Tv_4 = v_1 - iv_2 + v_3 + iv_4$.

What is the matrix of T^* with respect to the basis $\{v_1, v_2, v_3, v_4\}$?

Solution:

The matrix of T relative to the basis $\{v_1, v_2, v_3, v_4\}$ is $[T] = [Tv_1 \; Tv_2 \; Tv_3 \; Tv_4]$

$$\therefore [T] = \begin{bmatrix} i & 1 & -i & 1 \\ 1 & i & 1 & -i \\ -i & 1 & i & 1 \\ 1 & -i & 1 & i \end{bmatrix} \quad \text{--- (1)}$$

Now, by definition of T^*

i.e. Let V be an inner product space and $T: U \to V$ be a linear transformation. If there exists a unique linear operator T^* on V such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall\, x, y \in V,$$

then T^* is called the <u>adjoint</u> of the operator T.

Also, we know that, if the matrix of T with respect to the basis set B is $[M]$, then

$$\boxed{[M^*] = [M]^*}$$

where $[M^*]$ represents adjoint of the operator T;
$[M]^*$ represents complex conjugate transpose matrix.

$$\therefore [T^*] = [T]^*$$

$$= \begin{bmatrix} -i & 1 & i & 1 \\ 1 & -i & 1 & i \\ i & 1 & -i & 1 \\ 1 & i & 1 & -i \end{bmatrix}$$

— It is obtained by interchanging rows and columns of $[T]$ wherein the sign of the imaginary part of each element is reversed (or negated)

$$\therefore \boxed{[T^*] = \begin{bmatrix} -i & 1 & i & 1 \\ 1 & -i & 1 & i \\ i & 1 & -i & 1 \\ 1 & i & 1 & -i \end{bmatrix}}$$

is the required matrix.

Hence, the result.

Q.6.(c) Consider the linear operator $T(x,y) = (2x, x+y)$ on $\mathbb{R}^2$. Find the matrix of T with respect to the standard basis $B = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$. Use the transformation $A' = P^{-1}AP$ to determine the matrix A' with respect to the basis $B' = \{(-2,3),(1,-1)\}$.

Solution:

Given $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x,y) = (2x, x+y)$$

Also, the standard basis of $\mathbb{R}^2$ is $B = \{(1,0),(0,1)\}$.

Now,

$$T(1,0) = (2,1) = 2(1,0) + 1(0,1)$$

$$T(0,1) = (0,1) = 0(1,0) + 1(0,1).$$

$\therefore$ The matrix of T relative to the standard bases B is $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ i.e. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Now, we find the transition matrix P from B' to B.

$\therefore$ We express the vectors of B' in terms of those of B.

i.e. $(-2,3) = -2(1,0) + 3(0,1)$

$(1,-1) = 1(1,0) + (-1)(0,1)$

$\therefore$ The transition matrix $P = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$

$$\therefore A' = P^{-1}AP = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} -3 & 2 \\ -10 & 6 \end{bmatrix}$$

Hence, the result.

Q.7. (a) The function f defined by

$$f(x) = \begin{cases} x^2+3x+a & : x \leq 1 \\ bx+2 & : x > 1 \end{cases}$$

is given to be derivable for every x. Find a and b.

Solution:

Since $f(x)$ is derivable for every x, it is derivable and also continuous at $x = 1$.

Now, considering the continuity at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-} x^2+3x+a = \lim_{x\to 1^+} bx+2 = x^2+3x+a.$$

$\Rightarrow 1+3+a = b+2$

$\Rightarrow a-b+2=0$ i.e. $a-b=-2$ —— ①.

Now, considering differentiability at $x=1$.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1^-} \frac{(x^2+3x+a)-(a+4)}{x-1} = \lim_{x\to 1^+} \frac{(bx+2)-(b+2)}{x-1}$$

$$\Rightarrow \lim_{x\to 1^-} \frac{x^2+3x-4}{x-1} = \lim_{x\to 1^+} \frac{bx-b}{x-1}$$

$$\Rightarrow \lim_{x\to 1^-} \frac{(x+4)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x\to 1^+} \frac{b\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$\Rightarrow \lim_{x\to 1^-} (x+4) = \lim_{x\to 1^+} b$$

$\Rightarrow \boxed{5 = b}$ $\longrightarrow$ on taking limits,

Using $b=5$ in ①, we have,

$a-5=-2 \Rightarrow \boxed{a=3}$

Hence, the values of 'a' and 'b' are 3 and 5 respectively.

Q.7. (b) Determine the values of p and q for which $\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

Solution:

Let $\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3} = 1$ —— (1)

The L.H.S. of (1) is of the form $\frac{0}{0}$ and so, by using L Hospital's rule, we have,

$$\lim_{x\to 0} \frac{(1+p\cos x)-px\sin x-q\cos x}{3x^2} = 1$$

$$\therefore \lim_{x\to 0} \frac{1+(p-q)\cos x-px\sin x}{3x^2} = 1$$

Here, L.H.S. is of the form $\frac{1+p-q}{0}$, so to get the required limit, we take $1+p-q=0$ —— (2)

Thus, $\lim_{x\to 0} \frac{1+(p-q)\cos x-px\sin x}{3x^2} = 1$

(L.H.S. is of the form $\frac{0}{0}$)

$$\therefore \lim_{x\to 0} \frac{-(p-q)\sin x-p\sin x-px\cos x}{6x} = 1$$

(Still, L.H.S. is of the form $\frac{0}{0}$)

$$\therefore \lim_{x\to 0} \frac{-(p-q)\cos x-2p\cos x+px\sin x}{6} = 1$$

$\therefore \quad -\dfrac{(p-q)-2p}{6} = 1$

$\Rightarrow \quad -\dfrac{3p+q}{6} = 1.$

$\therefore \quad -3p+q = 6 \quad \text{——} \ (3)$

Thus, from (2) & (3), we have

$1+p-q=0$

$-6-3p+q=0.$

On solving these two equations for p and q, we obtain,

$\boxed{p = \dfrac{-5}{2}}$, $\boxed{q = -\dfrac{3}{2}}$

Hence, the result.

Q.7.(c) The temperature T at any point (x,y,z) in space is $T = 400xyz^2$. Find the highest temperature on the unit sphere $x^2+y^2+z^2=1$.

Solution:

The problem is to maximize $T(x,y,z) = 400xyz^2$ which is subject to the constraint $g(x,y,z) = x^2+y^2+z^2-1=0$.

The Lagrange's equations $T_x = \lambda g_x$, $T_y = \lambda g_y$, $T_z = \lambda g_z$ become

$$400yz^2 = \lambda 2x \quad \text{——} \ (1)$$

$$400xz^2 = \lambda 2y \quad \text{——} \ (2)$$

$$800xyz = \lambda 2z \quad \text{——} \ (3)$$

Eliminating λ from these equations, we obtain

$$\frac{200\,yz^2}{x} = \frac{200\,xz^2}{y} = \frac{400\,xy}{1.}$$

Also, $z \neq 0$ otherwise from Lagrange's equations it would follow that $x = y = z = 0$ which is impossible for the given constraint.

On solving (1) & (2) ; (2) & (3), we have

$$(1)\times x - (2)\times y \Rightarrow 2\lambda(x^2 - y^2) = 0 \Rightarrow x^2 = y^2 \quad (\text{i.e. } x = \pm y) \quad \text{---(4)}$$

$$(2)\times 2y - (3)\times 2 \Rightarrow 2\lambda(2y^2 - z^2) = 0 \Rightarrow z^2 = 2y^2 \quad (\text{i.e. } z = \pm\sqrt{2}y) \quad \text{---(5)}$$

$\therefore$ Stationary points are $(\pm y, y, \pm\sqrt{2}y)$

$\because$ they lie on sphere, it must satisfy its equation.

$$\therefore\ x^2 + y^2 + 2y^2 = 1$$

$$\Rightarrow y^2 = 1/4 \Rightarrow y = \pm 1/2 \Rightarrow x = \pm 1/2,$$

$$z = \pm\frac{\sqrt{2}}{2} = \pm\frac{1}{\sqrt{2}}.$$

$\therefore$ Stationary points are $\left(\pm 1/2, \pm 1/2, \pm 1/\sqrt{2}\right)$.

The maximal value of the temperature $T(x,y,z)$ on the sphere $x^2+y^2+z^2 = 1$ is therefore

$$T\left(1/2, 1/2, 1/\sqrt{2}\right) = 400\left(1/2\right)\left(1/2\right)\left(1/\sqrt{2}\right)^2 = \underline{\underline{50 \text{ units}}}.$$

Hence, the result.

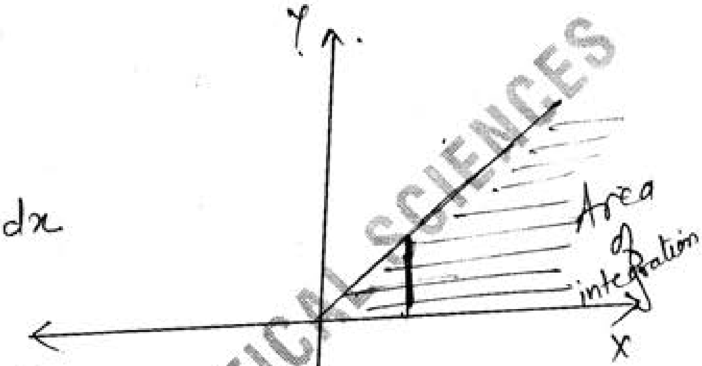
Q.7.(d). Evaluate the following integral

$$\int_0^\infty \int_0^x x\cdot e^{-x^2/y}\,dx\,dy$$ by changing the order of integration.

Solution:

Given,

$$I = \int_{x=0}^{x=\infty}\int_{y=0}^{y=x} x\cdot e^{-x^2/y}\,dy\,dx$$



By changing the order of integration,

Old limits	New limits
y : 0 to x	x : y to ∞
x : 0 to ∞	y : 0 to ∞

$$I = \int_{y=0}^{y=\infty}\left(\int_{x=y}^{x=\infty} x\cdot e^{-x^2/y}\,dx\right)dy$$

$$\therefore I = \int_{y=0}^{y=\infty}\left[\int_{x=y}^{x=\infty} x\cdot e^{-x^2/y}\,dx\right]dy$$ Put $-x^2=t$, $-2x\,dx=dt$

$$= \int_{y=0}^{y=\infty}\left[\int_{t=-\infty}^{t=y^2} e^{t/y}\left(\frac{dt}{-2}\right)\right]dy = \int_{y=0}^{y=\infty}\left[-\frac{y}{2}e^{-x^2/y}\right]_{y}^{\infty}dy$$

$$= \int_0^\infty \frac{y}{2}\left(e^{-y}\right) dy$$

$$= \frac{1}{2}\left[y(-e^{-y})\Big|_0^\infty - \int_0^\infty (1)(-e^{-y})dy\right]$$

$$= \frac{1}{2}\left[-y.e^{-y} - e^{-y}\right]\Big|_0^\infty$$

$$= -\frac{1}{2}\left[(y+1)e^{-y}\right]_0^\infty$$

$$= -\frac{1}{2}\left[0-1\right]$$

$$= \frac{1}{2}$$

$$\therefore \boxed{I = \int_0^\infty \int_0^x x.e^{-x^2/y}\,dx\,dy = \frac{1}{2}}$$

Q. 8. (a). Show that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to z-axis meets the plane $z = 0$ in parabolas.

Solution:

The d.c.'s of the z-axis are 0, 0, 1.

∴ The d.r.'s of the line perpendicular to z-axis are $l, m, 0$.

Let $P(\alpha, \beta, \gamma)$ be a point on the enveloping cylinder, then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{0} = r \text{ (say)}$$

Any point on it is $(\alpha + lr, \beta + mr, \gamma)$

If this point lies on the given conicoid, we get,

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma)^2 = 1.$$

$$r^2(al^2 + bm^2) + 2r(a\alpha l + b\beta m) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad \text{--- (1)}$$

Since this generator is tangent to the given conicoid so the two values of r obtained from (1) must be equal and the condition for the same is

$$(a\alpha l + b\beta m)^2 = (al^2+bm^2)(a\alpha^2+b\beta^2+(\gamma^2-1)$$

$\therefore$ The equation of the enveloping cylinder of the given conicoid i.e. the locus of $P(\alpha, \beta, \gamma)$ is

$$(alx + bmy)^2 = (al^2+bm^2)(ax^2+by^2+(z^2-1)$$

It's section by the plane $z=0$ is

$$(alx+bmy)^2 = (al^2+bm^2)(ax^2+by^2-1), \ z=0.$$

$$\Rightarrow a^2\cancel{l^2}x^2 + b^2m^2y^2 + 2ablmxy = a^2\cancel{l^2}x^2 + abl^2y^2 - al^2 + abm^2x^2 + b^2m^2y^2 - bm^2 \ ; \ z=0$$

$$\Rightarrow ab(m^2x^2 + l^2y^2 - 2lmxy) = al^2+bm^2 ; z=0$$

$$\Rightarrow ab(mx-ly)^2 = al^2+bm^2 \ ; \ z=0.$$

which represents a parabola as the second degree terms forms a perfect square.

Hence, the result.

Q.8.(b) Find the surface generated by a line which intersects the lines $y=a=z$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0$.

Solution:

Any line that intersects the given line is given by the planes

$(y-a)+k_1(z-a)=0$ and $(x+3z-a)+k_2(y+z-a)=0$ ——— ①

If λ, μ, ν are d.r.'s of this line, then

$\mu+k_1\nu=0$ and $\lambda+k_2\mu+(3+k_2)\nu=0$

Solving these, we get,

$$\frac{\lambda}{(3+k_2)-k_1k_2}=\frac{\mu}{k_1}=\frac{\nu}{-1}$$

$\therefore$ The d.r.'s of the line intersecting the given lines are $3+k_2-k_2k_1, k_1, -1$.

If this line is parallel to the plane $x+y=0$, then this line is perpendicular to the normal to this plane.

We have, $(3+k_2-k_1k_2).1+k_1.1=0$

$$\Rightarrow 3+k_1+k_2-k_1k_2=0 \quad \text{——— ②}$$

Also, from ①, we get,

$$k_1=\frac{a-y}{z-a}, \quad k_2=\frac{a-x-3z}{y+z-a}$$

$\therefore$ from ②, the required locus is

$$3+\frac{a-y}{z-a}+\frac{a-x-3z}{y+z-a}-\left\{\frac{a-y}{z-a}\right\}\cdot\left\{\frac{a-x-3z}{y+z-a}\right\}=0$$

$\Rightarrow 3(z-a)(y+z-a)+(a-y)(y+z-a)+(a-x-3z)$
$(z-a)-(a-y)(a-x-3z)=0$

$\Rightarrow (y+z-a)\left[3z-3a+a-y\right]=(a-x-3z)\left[a-y-z+a\right]$

$\Rightarrow (y+z-a)(-2a-y+3z)=(a-x-3z)(2a-y-z)$

$\Rightarrow -2ay-y^2+3yz-2az-yz+3z^2+2a^2+ay$
$-3az=2a^2-ay-az-2ax+xy+xz-6az+3yz$
$+3z^2$

$\Rightarrow -y^2-5az-yz+az+2ax-xy-xz+6az=0$

$\Rightarrow -y^2+2az+2ax-xy-xz-yz=0$

$\Rightarrow y^2+yz+xy+xz-2az-2ax=0$

$\Rightarrow \boxed{y^2+yz+xy+xz=2a(x+z)}$

which is the required equation of the surface.

Hence, the result.

Q. 8. (c). Show that the surface generally represented by the equation

$x^2+y^2+z^2-yz-xz-xy-3x-6y-9z+21=0$

is a paraboloid of revolution, the coordinates of the focus being $(1,2,3)$ and the equations to axis are $x=y-1=z-2$.

Solution :

Here, the discriminating cube is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1/2 & -1/2 \\ -1/2 & 1-\lambda & -1/2 \\ -1/2 & -1/2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - 1/4\right] + \tfrac{1}{2}\left[-1/2(1-\lambda) - 1/4\right]$$

$$-\frac{1}{2}\left[1/4 + 1/2(1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[8(1-\lambda)^2 - 2\right] - 2\left[2(1-\lambda)+1\right] = 0$$

$$\Rightarrow 8\lambda^2 - 16\lambda + \cancel{6} - 8\lambda^3 + 16\lambda^2 - 6\lambda - \cancel{6} + 4\lambda = 0$$

$$\Rightarrow -8\lambda^3 + 24\lambda^2 - 18\lambda = 0$$

$$\Rightarrow 4\lambda^3 - 12\lambda^2 + 9\lambda = 0$$

$$\Rightarrow \lambda\left[4\lambda^2 - 12\lambda + 9\right] = 0$$

$$\Rightarrow \lambda(2\lambda - 3)^2 = 0$$

$$\Rightarrow \lambda = 0, 3/2, 3/2.$$

As, two roots of the discriminating cube are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

The d.r.'s of the axis are given by

$al + hm + gn = 0,\ hl + bm + fn = 0,\ gl + fm + cn = 0.$

$\Rightarrow l - \frac{1}{2}m - \frac{1}{2}n = 0\ ;\ -\frac{1}{2}l + m - \frac{1}{2}n = 0\ ;\ -\frac{1}{2}l - \frac{1}{2}m + n = 0.$

$\Rightarrow 2l - m - n = 0\ ;\ -l + 2m - n = 0\ ;\ -l - m + 2n = 0.$

These give $l = m = n = \frac{1}{\sqrt{3}}$.

Now,
$k = ul + vm + wn$

$\Rightarrow k = \left(-\frac{3}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + (-3)\left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{9}{2}\right)\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow k = -3\sqrt{3}\ \ (\neq 0).$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

$\Rightarrow \left(\frac{3}{2}\right)x^2 + \left(\frac{3}{2}\right)y^2 + 2(-3\sqrt{3})z = 0$

$\Rightarrow \underline{x^2 + y^2 = 4\sqrt{3}\,z}$ which represents a <u>paraboloid of revolution</u>.

Also, the co-ordinates of the vertex of the paraboloid are obtained by solving any two of the three equations (I) with the equation (II) (which are mentioned as follows):

$$\frac{\left(\frac{\partial f}{\partial x}\right)}{l} = \frac{\left(\frac{\partial f}{\partial y}\right)}{m} = \frac{\left(\frac{\partial f}{\partial z}\right)}{n} = 2k$$

$$\Rightarrow \frac{2x-y-z-3}{\frac{1}{\sqrt{3}}} = \frac{2y-z-x-6}{\frac{1}{\sqrt{3}}} = \frac{2z-y-x-9}{\frac{1}{\sqrt{3}}} = -6\sqrt{3}$$

$\Rightarrow 2x-y-z-3 = 2y-z-x-6 = 2z-y-x-9 = -6$

$\Rightarrow 2x-y-z+3=0;\quad x-2y+z=0;$
$x+y-2z+3=0$ —— (I)

and $K(lx+my+nz)+ux+vy+wz+d=0$

$\Rightarrow -3\sqrt{3}\left(\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z\right)+\left(\frac{-3}{2}\right)x+$

$(-3)y+\left(\frac{-9}{2}\right)z+21=0$ —— (II)

$\Rightarrow 3x+4y+5z-14=0$

Now,
solving $2x-y-z+3=0$, $x-2y+z=0$,
$3x+4y+5z-14=0$;

we get $x=0$, $y=1$, $z=2$.

$\therefore$ The required vertex is $(0,1,2)$.

$\therefore$ Equations of the axis are $\frac{x-0}{1} = \frac{y-1}{1} = \frac{z-2}{1}$

$\Rightarrow$ $x = y-1 = z-2$.

Also, the focus will be a point on the axis whose actual d.c.'s are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and will be at a distance $\left(\frac{1}{4}\right)4\sqrt{3}$ i.e. $\sqrt{3}$ from the vertex $(0, 1, 2)$.

$\therefore$ Co-ordinates of the focus are given by

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = \sqrt{3}$$

$\Rightarrow$ $x = 1, y = 2, z = 3$.

$\therefore$ The required focus is $(1, 2, 3)$.

Hence, the result.