2015 - PDE IFoS

Q.1.

$$u_{xx} - 3u_{xy} + u_{yy} = \sin(x-2y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = \sin(x-2y)$$

$$(D^2 - 3DD' + D'^2)u = \sin(x-2y)$$

Aux eqn $(m^2 - 3m + 1) = 0$

$$m = \frac{3 \pm \sqrt{5}}{2}$$

$$m = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$$

C.F $z = \phi_1\left(y + \frac{3+\sqrt{5}}{2}x\right) + \phi_2\left(y + \frac{3-\sqrt{5}}{2}x\right)$

P.I $= \frac{1}{(D^2 - 3DD' + D'^2)} \sin(x-2y) \rightarrow \phi(ax+by) \Rightarrow a = 1, b = -2$

$$= \frac{1}{-(1)^2 - 3(-(1)(-2)) + (-(-2)^2)} \sin(x-2y)$$

$$\left[\frac{1}{f(D,D')} \sin(ax+by)\right] \quad D^2$$

$$= \frac{1}{-1-6-4} \sin(x-2y)$$

$$= \frac{-1}{11} \sin(x-2y)$$

gen sol^n = C.F + P.I

$$= \phi_1\left(y + \left(\frac{3+\sqrt{5}}{2}\right)x\right) + \phi_2\left(y + \left(\frac{3-\sqrt{5}}{2}\right)x\right) - \frac{1}{11}\sin(x-2y)$$

Ans

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3 $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $0<x<1$ $t>0$ —(i)

B.C $u(0,t) = u(1,t) = 0$

I.C $u(x,0) = \sin \pi x$

let $u(x,t) = X(x)\,T(t)$ be sol^n of eq (i)

$u(0,t)=0 \Rightarrow \boxed{X(0)=0}$ —(ii)

$u(1,t)=0 \Rightarrow \boxed{X(1)=0}$ —(iii)

$\frac{\partial u}{\partial x} = X'(x)\,T(t)$

$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$

$\frac{\partial u}{\partial t} = X(x)\,T'(t)$

$\Rightarrow X(x)T'(t) = X''(x)T(t)$

$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \mu$ —(iv)

$\Rightarrow X''(x) - \mu X(x) = 0$

$T'(t) - \mu T(t) = 0$

Case i) $\mu = 0$

$\Rightarrow X(x) = Ax + B$

$\Rightarrow X(0) = 0 \Rightarrow B = 0$

$\Rightarrow X(1) = 0 \Rightarrow A = 0$

$\Rightarrow X(x) = 0 \Rightarrow u(x,t) = 0$ which doesn't satisfy initial condition. $\therefore$ rejecting $\mu = 0$

Case ii) $\mu = \lambda^2$

$X''(x) - \lambda^2 X(x) = 0$

$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

$\Rightarrow X(0) = 0 \Rightarrow C_1 + C_2 = 0$

$X(1) = 0 \Rightarrow C_1 = C_2 = 0$

$\Rightarrow X(x) = 0 \Rightarrow u(x,t) = 0$ $\therefore$ rejecting $\mu = \lambda^2$

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Case iii) $\mu = -\lambda^2$

$X''(x) + \lambda^2 X(x) = 0$

$\Rightarrow X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$

$X(0) = C_1 = 0$

$X(1) = 0 \Rightarrow 0 = C_2 \sin \lambda$

taking $C_2 \neq 0$ $\quad$ & $\sin \lambda = 0$

$\Rightarrow \lambda = n\pi$

$\Rightarrow \quad X(x) = \cancel{C_1 \cos n\pi x} + C_2 \sin n\pi x$

$T'(t) + \lambda^2 T(t) = 0$

$\Rightarrow T(t) = C_3 e^{-\lambda^2 t}$

$\Rightarrow U(x,t) = \left[\cancel{C_1 C_3 \cos n\pi x} + C_2 C_3 \sin n\pi x\right] e^{-\lambda^2 t}$

$U_n(x,t) = \left[\cancel{E_n \cos n\pi x} + F_n \sin n\pi x\right] e^{-\lambda^2 t}$

$\begin{bmatrix} E_n = C_1 C_3 \\ F_n = C_2 C_3 \end{bmatrix}$

generalized soln

$U(x,t) = \sum_{n=1}^{\infty} \left(\cancel{E_n \cos n\pi x} + F_n \sin n\pi x\right) e^{\lambda^2 t}$

Now, using initial con

$\cancel{U(x,0)} = \cancel{\sin \pi x} = \cancel{\sum_{n=1}^{\infty} E_n \cos n\pi x}$

$U(x,0) = \sin \pi x = \sum_{n=1}^{\infty} F_n \sin n\pi x$

$\sin \pi x = F_1 \sin \pi x + F_2 \sin 2\pi x + \cdots$

$\Rightarrow F_1 = 1 \;\; \& \;\; F_n = 0 \;\; \forall n \geq 2$

$\Rightarrow U(x,t) = \sin (n\pi x) e^{-\lambda^2 t}$ $\quad [\because \lambda = n\pi]$

$\boxed{U(x,t) = \sin(\pi x) e^{-\pi^2 t}}$

Ans

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—(1)}$$

$\underline{B.C}$ $u(0,t) = 0$

$u(l,t) = 0$

$\underline{I.C}$ $u(x,0) = \begin{cases} \frac{2x}{l} & 0 < x < \frac{l}{2} \\ \frac{2}{l}(l-x) & \frac{l}{2} \le x < l \end{cases}$ & $u_t(x,0) = 0$

let $U(x,t) = X(x)\, T(t)$ —(ii)

using $\underline{B.C}$ $u(0,t) = \boxed{0 = X(0)}$

$u(l,t) = \boxed{0 = X(l)}$

diff (ii) partially w.r.t. x & t

$$\frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t) \quad \& \quad \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

$$\Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} = k$$

$\Rightarrow$ $X''(x) - k(X(x)) = 0$ | $T''(t) - kc^2\,T(t) = 0$

$\underline{\text{Case (i)}}$ $k = 0 \to$ rejecting

(ii) $k = \lambda^2 \to$ rejecting

(iii) $k = -\lambda^2$

$\Rightarrow X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$

$X(0) = 0 = c_1$

$X(l) = 0 = c_2 \sin \lambda l = 0$

taking $c_2 \neq 0$ & $\sin \lambda l = 0$

$\rightarrow$ $\lambda l = n\pi$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

$\Rightarrow X(x) = C_2 \sin \frac{n\pi x}{l}$

$T(t) = C_3 \cos(\lambda c t) + C_4 \sin(\lambda c t)$

$\Rightarrow u_n(x,t) = \left[E_n \cos(\lambda c t) + F_n \sin \lambda c t\right] \sin \frac{n\pi x}{l}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left[E_n \cos(\lambda c t) + F_n \sin(\lambda c t)\right] \sin \frac{n\pi x}{l} \quad \left[\begin{array}{l} C_2 C_3 = E_n \\ C_2 C_4 = F_n \end{array}\right]$

Using initial condⁿ

$u(x,0) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{l}$

$\Rightarrow E_n = \frac{2}{l} \int_0^l u(x,0) \sin \frac{n\pi x}{l}$

$E_n = \frac{2}{l}\left[\int_0^{l/2} \frac{2x}{l} \sin \frac{n\pi x}{l} dx + \int_{l/2}^{l} \frac{2}{l}(l-x) \sin \frac{n\pi x}{l} dx\right]$

$\Rightarrow \frac{4}{l^2}\left[\int_0^{l/2} x \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} (l-x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$

$E_n = \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 2m \quad m \in N \\ \pm \frac{8}{(2m-1)^2} & n = (2m-1) \end{cases}$

$\Rightarrow$ ~~$u(x,t) =$~~

to find F_n diff. partially u w.r.t. t

$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \left[-E_n \sin(\lambda c t)\lambda c + \lambda c F_n \cos(\lambda c t)\right] \sin \frac{n\pi x}{l}$

Using i.c. $u_t(x,0) = 0 = F_n$

$\boxed{F_n = 0}$

$\Rightarrow$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{\pm 8}{(2m-1)^2} \cos\left(\frac{(2m-1)\pi c t}{l}\right) \sin \frac{(2m-1)\pi x}{l}}$$ Ans