Lagrangian Dynamics

1. Terms
   - Scleronomic System ≡ independent of time
   - Rheonomic System ≡ t enters explicitly (moving coord.)
     - L: A particle on a wire rotating with constant ω
   - Holonomic, Non-holonomic system
     - L non-integratable equations (Sphere rolling on a sphere).
   - Degrees of freedom ≡ Rigid body → 6.

* Finding motion ⇒ Lagrangian or

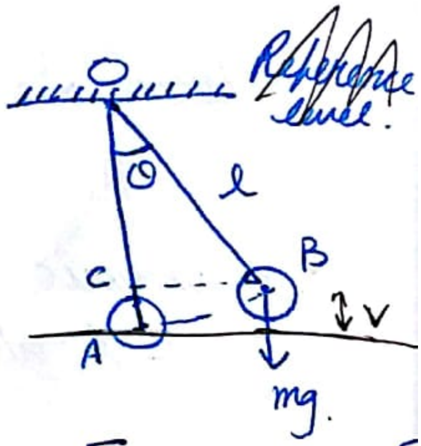
Simply

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \frac{\partial W}{\partial q_\alpha}}$$

$T \equiv$ total K.E $\qquad W \equiv$ Work function (generali...)

* Lagrangian

Ex. Simple Pendulum (massless rod).



Let generalised coordinate be angle θ made by OB with OA, l = length

① Find $V = mgl - mgl\cos\theta$.

(KE) $T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$ $[l\cos\theta, l\sin\theta]$

② $L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl + mgl\cos\theta$.

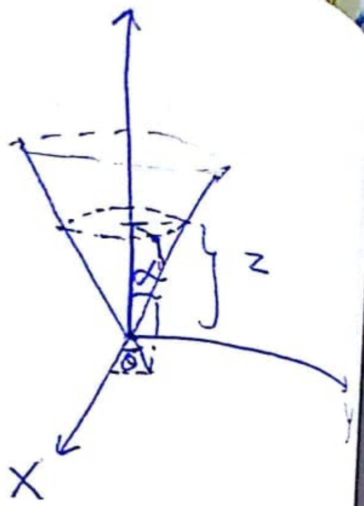
③ Force is gravity and hence conservative

$$\text{Lagrangian} \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$ not $\dot{\theta}$ term

$$\Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\right) - \left(-mgl\sin\theta\right) = 0$$

$$l^{\cancel{2}}\ddot{\theta} = -g\cancel{l}\sin\theta \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

Lagrangian on a Cone



$$x^2+y^2 = z^2 \cancel{\cot^2\alpha}.\tan^2\alpha.$$

~~R cos~~

$$(z\tan\alpha\cos\theta,\ z\tan\alpha\sin\theta,\ z)$$

$$(\dot z\tan\alpha\cos\theta - z\tan\alpha\sin\theta\dot\theta,\ \dot z\tan\alpha\sin\theta + z\tan\alpha\cos\theta\dot\theta,\ \dot z)$$

$$v^2 \equiv \dot z^2\tan^2\alpha. + z^2\tan^2\alpha\dot\theta^2 + \dot z^2.$$

$$T = \frac{1}{2}mv^2 \qquad V = mgz.$$

$$L \equiv \frac{1}{2}m\left(\dot z^2\sec^2\alpha + z^2\tan^2\alpha\dot\theta^2 + \dot z^2\right) - mgz.$$

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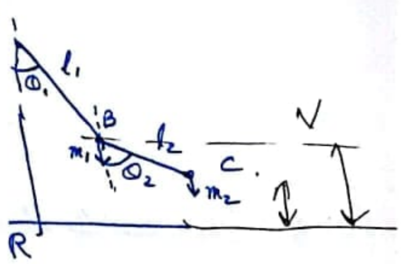
① Double pendulum in a vertical plane



$B \equiv (x_1, y_1) = (l_1 \sin\theta_1, l_1 \cos\theta_1)$

$C \equiv (x_2, y_2) = (l_1 \sin\theta_1 + l_2 \sin\theta_2, l_1 \cos\theta_1 + l_2 \cos\theta_2)$

$\dot{x}_1, \dot{y}_1 = (l_1 \cos\theta_1 \dot{\theta}_1, -l_1 \sin\theta_1 \dot{\theta}_1)$

$\dot{x}_2, \dot{y}_2 \equiv (l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2, -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2)$

$$\underset{(T)}{KE} = \frac{1}{2} m_1 (l_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \cos\theta_1 \cos\theta_2 \dot{\theta}_1 \dot{\theta}_2 + 2 l_1 l_2 \sin\theta_1 \sin\theta_2 \dot{\theta}_1 \dot{\theta}_2\right)$$

Taking reference zero PE at $l_1 + l_2$ below A at R.

$$V = m_1 g (l_1 + l_2 - l_1 \cos\theta_1) + m_2 g (l_1 + l_2 - l_1 \cos\theta_1 - l_2 \cos\theta_2)$$

$L = T - V$

2 parameters $\theta_1, \theta_2 \Rightarrow$ 2 Lagrangian equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0.$$

$$\cancel{\frac{d}{dt}\left(m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + 2 l_1 l_2 \cos(\theta_1 - \theta_2)\right) - \left[m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1\right]}$$

$$\frac{d}{dt}\left[m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

$$\boxed{m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)}$$

$$-\frac{\partial L}{\partial \theta_1} = + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1$$

$$\boxed{(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1}$$

Reference line?

B

g.

$[l \sin\theta]$

term

(2) A particle moves in a plane

L attraction towards origin $\propto$ distance from it

L Force $\perp$ to radius $\propto \frac{1}{\text{distance from origin}}$

$\circlearrowleft$ anti-clockwise

(r, θ) be coordinates of mass m at instant t.

Kinetic Energy $\equiv T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

Force on particle $F = -kr$.

$$F = -\frac{dV}{dr} \Rightarrow +\frac{dV}{dr} = +kr$$

$$\Rightarrow V = \frac{kr^2}{2} \cancel{+A}.$$

at $r = 0$ (origin)
$V = 0 \Rightarrow A = 0$

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) - \frac{kr^2}{2}$$

Lagranges Equations $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = Q_r$$

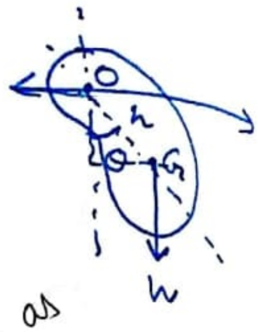
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta$$

} where Q_r, Q_θ are non-conservative

Since, F is conservative (as integrable), so $Q_r = 0$.

Q E Lagrange of Compound Pendulum

Let moment of inertia about O be Mk^2



$V = -Mgh\cos\theta$ Or $(Mgh - Mgh\cos\theta)$

Better as follow same technique as before

$T = \frac{1}{2} Mk^2 \dot{\theta}^2.$

$L = \frac{1}{2} Mk^2 \dot{\theta}^2 + Mgh\cos\theta.$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(Mk^2\dot{\theta}\right) + Mgh\sin\theta.$$

$$0 = Mk^2\ddot{\theta} - Mgh\sin\theta.$$

$$\ddot{\theta} = -\frac{gh}{k^2}\sin\theta \Rightarrow \boxed{D^2\theta = -\frac{gh}{k^2}\sin\theta}$$ Ans

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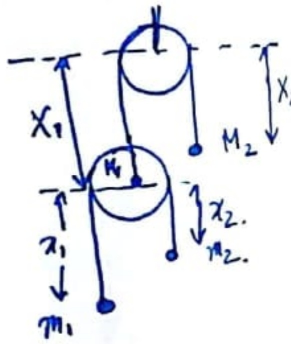
Q E



$$X_1 + X_2 = k \Rightarrow \dot{X}_1 + \dot{X}_2 = 0$$

$$x_1 + x_2 = l \Rightarrow \dot{x}_1 + \dot{x}_2 = 0$$

$$\cancel{T = \frac{1}{2} M_2 \dot{X}_2^2 + \frac{1}{2} M_1 \dot{X}_1^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \dot{X}_1^2 + \frac{1}{2} m_2 \dot{X}_1^2}.$$

$$V = -M_1 g X_1 - M_2 g X_2 - m_1 g (X_1 + x_1) - m_2 g (X_1 + x_2)$$

$$= -M_1 g X_1 - M_2 g (k - X_1) - m_1 g (X_1 + x_1) - m_2 g (X_1 + l - x_1).$$

$$T = \frac{1}{2} M_2 \dot{X}_1^2 + \frac{1}{2} M_1 \dot{X}_1^2 + \frac{1}{2} m_1 (\dot{X}_1 + \dot{x}_1)^2 + \frac{1}{2} m_2 (\dot{X}_1 - \dot{x}_1)^2.$$

$L = T - V.$

Lagrangian $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1}$

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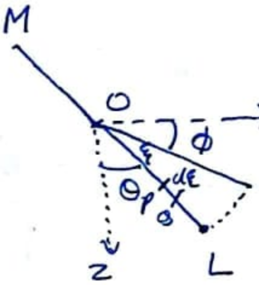
[8] A rod $(3m, 2l)$ with middle pt fixed and mass m at one extremity. Rod in horizontal position is set rotating about vertical axis with $\omega = \sqrt{\frac{2ng}{l}}$. Show heavy end of rod falls till inclination to vertical is $\cos^{-1}\{\sqrt{n^2+1} - n\}$.

M

Any point $P \Rightarrow$

$x = \xi \sin\theta \cos\phi \quad y = \xi \sin\theta \sin\phi \quad z = \xi \cos\phi$



$\dot{x} = \xi \cos\theta \cos\phi\, \dot{\theta} - \xi \sin\theta \sin\phi\, \dot{\phi}$

$\dot{y} = \xi \cos\theta \sin\phi\, \dot{\theta} + \xi \sin\theta \cos\phi\, \dot{\phi}$

$\dot{z} = -\xi \sin\theta\, \dot{\theta}$

$v_P^2 = \xi^2 \left[\dot{\theta}^2 \cos^2\theta + \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \sin^2\theta\right]$

$= \xi^2 \left[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right]$

mass $(PQ) = \frac{3m}{2l}\, d\xi = dm.$

KE of rod $= \frac{3m}{2l}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) \int_{-l}^{l} \xi^2\, d\xi = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$

KE of $m = \frac{1}{2} m\, l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$

$\rightarrow$ Work function for rod $\equiv \boxed{W = mgl\cos\theta + C.}$ $*$

① Lagrange $= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi} \Rightarrow \dot{\phi} \sin^2\theta = K.$

Initially $\Rightarrow \theta = \frac{\pi}{2}, \dot{\phi} = \sqrt{\frac{2ng}{l}} \Rightarrow K = \sqrt{\frac{2ng}{l}}$

$$\boxed{\dot{\phi} \sin^2\theta = \sqrt{\frac{2ng}{l}}}$$

② $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta} \Rightarrow 2l\ddot{\theta} - 2l\dot{\phi}^2 \sin\theta \cos\theta = -g\sin\theta.$

$2l\ddot{\theta} + \cancel{2l}\, 4ng \cot\theta \csc^2\theta = -g\sin\theta.$

$2l\dot{\theta}^2 + 4ng\cot^2\theta = 2g\cos\theta + k \Rightarrow \theta = \frac{\pi}{2}, \dot{\theta} = 0 \Rightarrow k = 0$

Rod falls till $\dot{\theta} = 0 \quad 2n\cot^2\theta = \cos\theta \Rightarrow 2n\cos^2\theta - \cos\theta\sin^2\theta = 0$

$\cos\theta(2n\cos\theta - \sin^2\theta) = 0 \qquad \theta = \frac{\pi}{2} \Rightarrow$ initial

$\cos^2\theta = 2n\cos\theta - 1 \Rightarrow \cos\theta = -n \pm \sqrt{n^2+1}$ Ans

(9) Particle m in a conservative force field. (In Cylindrical coor

E Find • Lagrangian • Equations of motion.

$\Rightarrow$ Let at point $P(\rho, \theta, z)$ $\vec{P} \equiv \rho\cos\theta\,\hat{i} + \rho\sin\theta\,\hat{j} + z\hat{k}$

$$v = \dot{\rho}\cos\theta\,\hat{i} + \dot{\rho}\sin\theta\,\hat{j} - \rho\sin\theta\,\dot{\theta}\,\hat{i} + \rho\cos\theta\,\dot{\theta}\,\hat{j} + \dot{z}\hat{k}$$

Unit vector $\rightarrow$ $\hat{\rho}_1 = \dfrac{\partial\vec{P}/\partial\rho}{\left|\partial\vec{P}/\partial\rho\right|} = \dfrac{\cos\theta\,\hat{i} + \sin\theta\,\hat{j}}{1}$ (along $\vec{\rho}$ in

NOT NEEDED (Just compute V^2)

$$\hat{\theta} = \frac{\partial\bar{P}/\partial\theta}{\left|\partial P/\partial\theta\right|} = \frac{\cancel{\rho\dot{\theta}}(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})}{\cancel{\rho\dot{\theta}}}$$

$$v = \dot{\rho}\,\hat{\rho}_1 + \rho\dot{\theta}\,\hat{\theta} + \dot{z}\hat{k}$$

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) \qquad V = V(\rho, \theta, z)$$

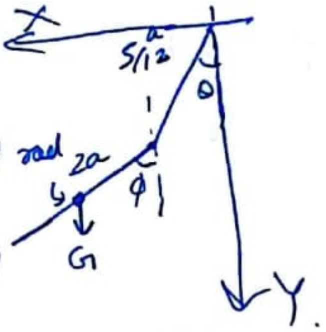
• Lagrangian $= T - V = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) - V(\rho, \theta, z)$

$$\Rightarrow \frac{d}{dt}\left(m\dot{\rho}\right) - \left(m\rho\dot{\theta}^2 - \frac{\partial V}{\partial\rho}\right) = 0.$$

$$\boxed{m\left(\ddot{\rho} - \rho\dot{\theta}^2\right) = -\frac{\partial V}{\partial\rho}}$$

$$\Rightarrow \frac{d}{dt}\left(m\rho^2\dot{\theta}\right) - \left(-\frac{\partial V}{\partial\theta}\right) = 0 \Rightarrow \boxed{m\rho^2\ddot{\theta} = -\frac{\partial V}{\partial\theta}}$$

(8) M/ Find position and period of oscillations ⇒



$x_G = \frac{5a}{12}\sin\theta + a\sin\phi \Rightarrow \frac{5a}{12}\cos\theta\dot\theta + a\cos\phi\dot\phi\ (\dot x)$ and

$y_G = \frac{5a}{12}\cos\theta + a\cos\phi \Rightarrow -\frac{5a}{12}\sin\theta\dot\theta - a\sin\phi\dot\phi\ (\dot y)$

$V_G^2 = \left(\frac{5a}{12}\right)^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5a^2}{6}(\cos\theta\cos\phi + \sin\theta\sin\phi)\dot\theta\dot\phi$

$\hookrightarrow \cos(\theta+\phi) = 1$ θ, ϕ small.

$= \left(\frac{5a}{12}\right)^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5a^2}{6}\dot\theta\dot\phi$

$T = \frac{1}{2}m\left[\left(\frac{5a}{12}\right)^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5a^2}{6}\dot\theta\dot\phi + \frac{a^2}{3}\dot\phi^2\right]$

$W = mg\left(\frac{5}{12}a\cos\theta + a\cos\phi\right) + K.$

Lagrange θ-equation ⇒.

$\frac{d}{dt}\left(\frac{1}{2}m\left(\left(\frac{5a}{12}\right)^2\dot\theta\times 2 + \frac{5a^2}{6}\dot\phi\right)\right) - 0 = -\frac{5m}{12}ga\sin\theta$

$\frac{1}{2}m\left(\frac{25a^2}{144}\times 2\ddot\theta + \frac{5a^2}{6}\ddot\phi\right) = -\frac{5mga}{12}\theta$ (As θ is small).

$\frac{\cancel{50}^{10}\cancel{a^2}}{\cancel{144}_{24}}D^2\theta + \frac{\cancel{5}\cancel{a^2}}{\cancel{6}}D^2\phi = -\frac{\cancel{5}g\cancel{a}}{\cancel{6}}\theta.$

$10D^2\theta + 24D^2\phi = -24\frac{g}{a}\theta$ ⇒ $\boxed{\left(5D^2 + 12\frac{g}{a}\right)\theta + 12D^2\phi = 0}$

$L \times 16D^2 + 12\frac{g}{a}$

ϕ equation ⇒.

$\frac{d}{dt}\left(\frac{1}{2}\cancel{m}\times\left(\frac{8a^2}{3}\dot\phi + \frac{5a^2}{6}\dot\theta + \cancel{\frac{2a^2}{3}\dot\phi}\right)\right) = -\cancel{m}ga\sin\phi$

$16a^2\ddot\phi + 5a^2\ddot\theta = -\frac{2ag\sin\phi\times 6}{a} = -\frac{12g\phi}{a}$

$\Rightarrow \boxed{\left(16D^2 + 12\frac{g}{a}\right)\phi + 5D^2\theta = 0} \times 12D^2.$

$\frac{12g}{a} = c$.

① − ②

$$\left[\left(5D^2 + 12\frac{g}{a}\right)\left(16D^2 + 12\frac{g}{a}\right) - 60D^4\right]\theta = 0.$$

$$\left(20D^4 + 21cD^2 + c^2\right)\theta = 0.$$

Let Solution be $\boxed{\theta = A\cos(pt + B)}$

$D\theta = -Ap\sin(pt+B)$ $\quad D^2\theta = -Ap^2\cos(pt+B)$

$D^3\theta = Ap^3\sin(pt+B)$ $\quad D^4\theta = Ap^4\cos(pt+B) = p^4\theta.$

$$20p^4\theta + c21(-p^2\theta) + c^2\theta = 0 \Rightarrow 20p^4 - 21cp^2 + c^2 = 0.$$

$$c\left(\frac{21 \pm \sqrt{441-80}}{40}\right) = c\left(\frac{21 \pm 19}{40}\right) = c\left(1, \frac{1}{20}\right) = c, \frac{c}{20}.$$

$$p^2 = \frac{12g}{a}, \frac{12g}{20a} \Rightarrow p = \sqrt{\frac{12g}{a}}, \sqrt{\frac{3g}{5a}}$$

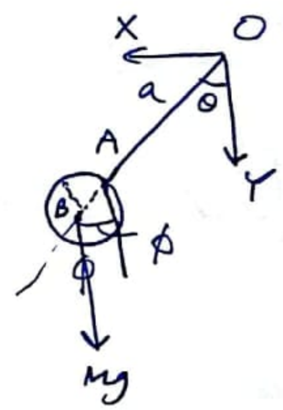
Time period $\equiv \frac{2\pi}{p_1}, \frac{2\pi}{p_2}$

- When $\cos\theta \approx \left(1 - \frac{\theta^2}{2}\right)$ and $\sin\theta \approx \theta$.

- $p = \sqrt{\frac{g}{l}}$; Period $= \frac{2\pi}{p}$
- No differential element unless 3-D motion
- Use Work function ≡ Taken +ve (intuition says −ve) (As per coordinates)

8

A solid sphere (M, b) is freely hinged to a rod (nM, a). Find periods of small oscillation.



$T \Rightarrow$

Rod $\equiv \frac{1}{2} nM \frac{a^2}{3} \dot{\theta}^2$ $\leftarrow$ $\frac{1}{2} I_p \omega^2$

Same as $\frac{1}{2} I\omega^2 + \frac{1}{2} M v_G^2$.

Sphere

~~$B = (a\sin\theta + b\sin\phi, a\cos\theta + b$~~.

$B = (a\sin\theta + b\sin\phi, a\cos\theta + b\cos\phi)$

$v_B^2 = a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}$

$$T = \frac{1}{2} M \left[\frac{2b^2}{5} \dot{\phi}^2 + a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi} \right] + \frac{1}{2} nM \frac{a^2}{3} \dot{\theta}^2$$

$$W = nMg \frac{a}{2} \cos\theta + Mg(a\cos\theta + b\cos\phi)$$

$\rightarrow$ Lagrange $\Rightarrow \theta$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

$$\frac{d}{dt}\left[Ma^2\dot{\theta} + Mab\dot{\phi} + nM\frac{a^2}{3}\dot{\theta} \right] - [0] = -\frac{nMga\sin\theta}{2} - Mga\sin\theta.$$

$$\ddot{\theta}\left(Ma^2 + \frac{na^2}{3}M\right) + \ddot{\phi}\, Mab = -Mga\sin\theta\left(\frac{2+n}{2}\right)$$

$$\ddot{\theta}\, 2a(n+3) + 6b\ddot{\phi} = -3(n+2)g \cancel{\sin}\,\theta$$

(Approx for small)

· Lagrange ϕ equation $\Rightarrow$

$$7b\ddot{\phi} + 5a\ddot{\theta} = -5g\phi$$

Eliminate ϕ between both $\Rightarrow$

$$\left[2ab(7n+6)D^4 + \{10(n+3)+21b(n+2)\}gD^2 + 15(n+2)g^2\right]\theta = 0$$

Put $\theta = A\cos(pt+B)$

We have $\Rightarrow$

$$2ab(7n+6)p^4 - \{10(n+3)a + 21b(n+2)\}p^2 g + 15(n+2)g^2 = 0$$

gives solution

Hamiltonian

- $H = \sum p_k \dot{q}_k - L.$

  where $p_k = \frac{\partial L}{\partial \dot{q}_k}$

- If H is independent of $t \Rightarrow H = T + V.$
  (However remove $\dot{q}_k$ with p_k)

- Hamiltonian equations

$$\rightarrow \boxed{\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial q_k} \\ \dot{q}_k &= \frac{\partial H}{\partial p_k}. \end{aligned}}$$

- L is a function of $q, \dot{q}$
  H is a function of $q, p.$

$$H = \sum p \dot{q} - L., \quad p = \frac{\partial L}{\partial \dot{q}}$$

- If H independent of $t \Rightarrow H = T + V$

Examples

① 1-D spring

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

To find H from this we wish to remove $\dot{x}$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \Rightarrow \boxed{\dot{x} = \frac{p}{m}}$$

generalised momentum

$$\boxed{H = T + V = \frac{1}{2\cancel{m}}\cdot\frac{p_x^2}{m} + \frac{1}{2}kx^2}$$

Equations of motion

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx \Rightarrow \dot{p}_x = -kx$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \Rightarrow m\dot{x} = p.$$

$$\dot{p} = m\ddot{x} = -kx$$

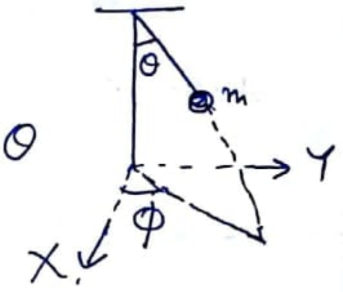
$$\Rightarrow \boxed{m\ddot{x} + kx = 0}$$

OR

$$H = \sum p_\alpha \dot{q}_\alpha - L. = p \times \frac{p}{m} - \left[\frac{1}{2}\cancel{m}\cdot\frac{p^2}{m^{\cancel{2}}} - \frac{1}{2}kx^2\right]$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2}$$

② Spherical Pendulum



$x = l\sin\theta\cos\varphi \;;\; y = l\sin\theta\sin\varphi \;;\; z = l\cos\theta$

$$T = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right)$$

$$V = -mgl\cos\theta.$$

$$L = T - V = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + mgl\cos\theta$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2}$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta \Rightarrow \dot{\phi} = \frac{P_\phi}{ml^2\sin^2\theta}$$

$$H = T + V = \frac{1}{2} ml^2 \left(\frac{P_\theta^2}{m^2l^4} + \frac{P_\phi^2}{m^2l^4\sin^2\theta}\right) - mgl\cos\theta$$

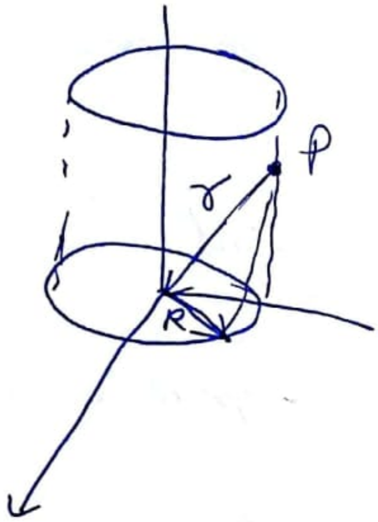
$$H = \frac{P_\theta^2}{2ml^2} + \frac{P_\phi^2}{2ml^2\sin^2\theta} - mgl\cos\theta.$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = \frac{P_\phi^2\cos\theta}{ml^2\sin^3\theta} - mgl\sin\theta.$$

$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0.$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{ml^2} \;;\; \dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{ml^2\sin^2\theta}.$$

(3) Particle on cylindrical surface $x^2+y^2=R^2$; subject to $\vec{F}=-k\vec{r}$



$F = -k(x\hat{i}+y\hat{j}+z\hat{k})$

$P = \{R\cos\theta, R\sin\theta, z\}$

$r^2 = R^2+z^2$

$$F = -\frac{\partial V}{\partial r} \Rightarrow V = \frac{1}{2}kr^2 = \frac{1}{2}k(R^2+z^2)$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2)$$

$$\boxed{L = \frac{1}{2}m(R^2\dot{\theta}^2+\dot{z}^2) - \frac{1}{2}k(R^2+z^2)}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad ; \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$H = T+V$$

$$= \frac{1}{2}m\left(\frac{R^2 p_\theta^2}{m^2R^4} + \frac{p_z^2}{m^2}\right) + \frac{1}{2}k(R^2+z^2).$$

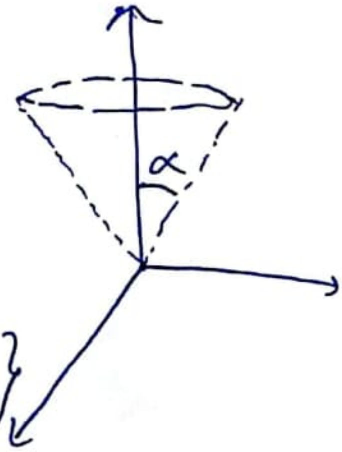
$$= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2 + \frac{1}{2}kR^2$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad , \quad \dot{P}_z = -\frac{\partial H}{\partial z} = -kz$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mR^2} \quad , \quad \dot{z} = \frac{\partial H}{\partial P_z} = \frac{P_z}{m}$$

④ Hamiltonian for a cone $x^2+y^2=z^2\tan^2\alpha$.

Particle of mass m.



$$P \equiv \{\rho\cos\phi, \rho\sin\phi, \rho\cot\alpha\}$$

$$\dot{P} = \{\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi}, \dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi}, \dot{\rho}\cot\alpha\}$$

$$v_p^2 = (\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{\rho}^2\cot^2\alpha)$$

$$T = \frac{1}{2}m(\dot{\rho}^2\operatorname{cosec}^2\alpha + \rho^2\dot{\phi}^2) \quad -(1)$$

$$V = +mgz = mg\rho\cot\alpha. \quad \text{(As above origin)} -(2)$$

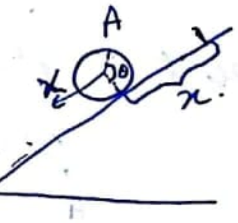
$$L = \frac{1}{2}m(\dot{\rho}^2\operatorname{cosec}^2\alpha + \rho^2\dot{\phi}^2) - mg\rho\cot\alpha. \quad -(1),(2)$$

$$P_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho}\operatorname{cosec}^2\alpha \quad ; \quad P_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi}$$

$$H = T+V = \frac{1}{2}m\left(\frac{P_\rho^2}{m^2\operatorname{cosec}^2\alpha} + \frac{\cancel{\rho^2}\cdot P_\phi^2}{m^2\rho^2}\right) + mg\rho\cot\alpha$$

$$H = \frac{P_\rho^2}{2m\operatorname{cosec}^2\alpha} + \frac{P_\phi^2}{2m\rho^2} + mg\rho\cot\alpha$$

(5) Sphere rolls down a rough inclined plane: if x is distrance of pt. of contact from a fixed point on plane, find acceleration.



$$\cdot\ T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\, m \cdot \frac{2}{5} a^2 \dot{\theta}^2$$

$$a\theta = x \Rightarrow \dot{\theta} = \frac{\dot{x}}{a}$$

$$\rightarrow T = \frac{1}{2} m\dot{x}^2 + \frac{m\cancel{a^2}}{5}\frac{\dot{x}^2}{\cancel{a^2}} = \frac{7}{10} m\dot{x}^2$$

$$V = -mgx\sin\alpha$$

$$\therefore L = T - V = \frac{7}{10} m\dot{x}^2 + mgx\sin\alpha$$

$$P_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5} m\dot{x} \Rightarrow \boxed{\dot{x} = \frac{5P_x}{7m}}$$

$$H = \sum p_i \dot{q}_i - L = \overset{P_x \dot{x}}{\cancel{\frac{7}{5} m\dot{x}^2}} - \frac{7}{10} m\dot{x}^2 - mgx\sin\alpha$$

$$= \frac{5P_x^2}{7m} - \frac{\cancel{7}}{\cancel{10}_2}\, m \cdot \frac{\cancel{25}^5 P_x^2}{\cancel{49}_7\, m} - mgx\sin\alpha$$

$$= \boxed{\frac{5P_x^2}{14m} - mgx\sin\alpha}$$ Ans

$$\dot{P}_x = -\frac{\partial H}{\partial x} = mg\sin\alpha$$

$$\dot{x} = \frac{\cancel{10}^5 P_x}{\cancel{14}_7\, m} \Rightarrow \boxed{\ddot{x} = \frac{5}{7} g\sin\alpha}$$ Ans

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(9) Particle m moves under gravity on smooth sphere r.
Find equations of motion in cartesian coordinates

→ Constraint ≡ $x^2+y^2+z^2=r^2$.

$$\underbrace{x}_{A}dx+\underbrace{y}_{B}dy+\underbrace{z}_{C}dz=0$$

$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ $\quad V=mgz$ $\quad L=T-V$

Lagrange's equations of motion:

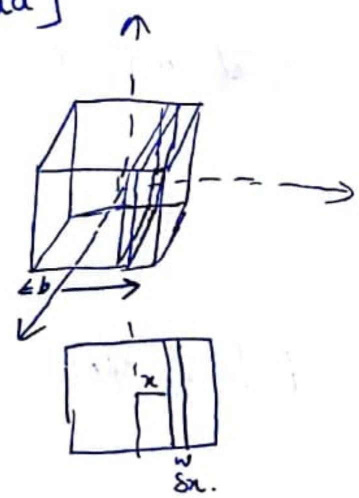
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=A\lambda=\lambda x.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=\lambda y.$$

$\vdots$

Moment of Inertia — 9:0 am

- Rectangular Parallelopiped $(2a, 2b, 2c)$ [Solid]

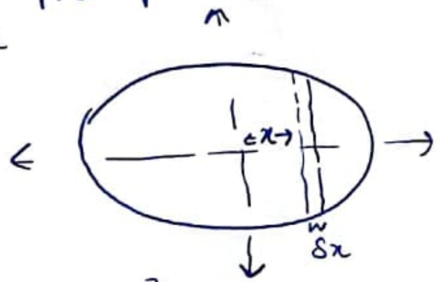


Mass of unit rectangle $= \dfrac{M}{8abc} 2a.2c.\delta x.$

$$MI = \int_{-b}^{b} \frac{4ac\,M}{8abc}\left(\frac{a^2+c^2}{3}\right)\delta x$$

$$= \boxed{M\left(\frac{a^2+c^2}{3}\right)} \text{ Ans}$$

- Elliptic Disc. $\quad y^2 = \dfrac{b^2(a^2-x^2)}{a^2} \qquad (M = \rho\,\pi ab).$



$$MI = \int_{-a}^{a} \rho.2y\,\delta x.\frac{y^2}{3}$$

$$= \int_{-a}^{a} \frac{M}{\pi ab}.\frac{2y^3}{3}\,dx = \int_{-a}^{a} \frac{M}{\pi ab}.\frac{2}{3}\left(\frac{b}{a}\right)^3 (a^2-x^2)^{3/2}\,dx.$$

$x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

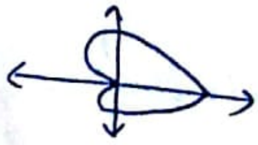
$$= \frac{2}{3}.\frac{M}{\pi \cancel{a}b}.\frac{b^3}{\cancel{a^3}}.2\int_0^{\pi/2} a^3\cos^3\theta.\,a\cos\theta\,d\theta$$

$$= \frac{3\cancel{\pi}}{16}.\frac{4}{3\cancel{\pi}}.\frac{b^{\cancel{3}}M}{\cancel{b}} = \boxed{\frac{Mb^2}{4}} \text{ Ans}$$

$$\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

$$\boxed{M = \rho\pi ab}$$

- $r = a(1+\cos\theta)$



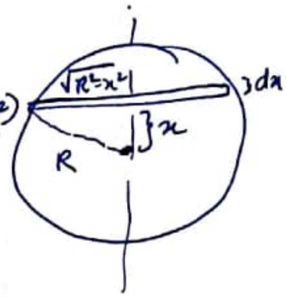
$$\int_0^{\pi}\int_0^{a(1+\cos\theta)}$$

- **M I solid sphere**

ALTER : Take a disc

MI of disk about axis $\equiv m_0 \dfrac{(R^2-x^2)}{2}$

$= \dfrac{M}{\frac{4}{3}\pi R^3} \cdot \pi(R^2-x^2).\delta x \dfrac{(R^2-x^2)}{2}$



MI of sphere $\Rightarrow \displaystyle\int_{-R}^{R} \frac{3M}{4\pi R^3} \frac{\pi}{2} (R^2-x^2)^2 \delta x$

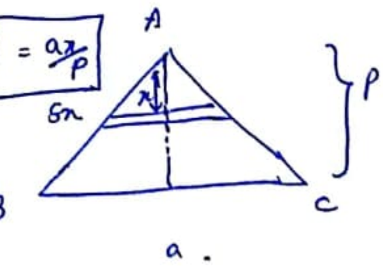
$= \dfrac{3M}{8R^3} \cdot 2 \displaystyle\int_0^R R^4+x^4-2R^2x^2 \, dx$

$= \dfrac{3M}{4R^3}\left[R^5+\dfrac{R^5}{5}-\dfrac{2}{3}R^5\right] = \dfrac{3M}{4R^3}\left[\dfrac{15+3-10}{15}R^5\right]$

$= \boxed{\dfrac{2MR^2}{5}}$ Ans

- **Δ lamina** about a side (BC)

$\rho_0 = \dfrac{M}{\frac{1}{2}ap}$; $\qquad \dfrac{x}{p} = \dfrac{l}{a} \Rightarrow \boxed{l = \dfrac{ax}{p}}$

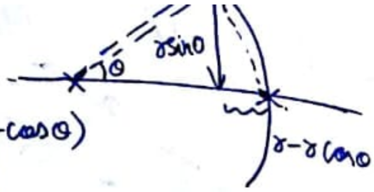


Mass of strip $\equiv \dfrac{2M}{\cancel{a}p} \dfrac{\cancel{a}x}{p} \delta x$.

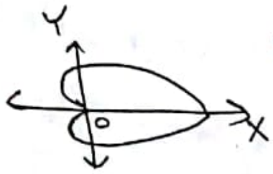
MI of strip about BC $\equiv \left(\dfrac{2M}{p^2} x\,\delta x\right)(p-x)^2$.

Total MI $\equiv \dfrac{2M}{p^2}\displaystyle\int_0^p x(p-x)^2\,dx = \boxed{\dfrac{1}{6}Mp^2}$ Ans

Mass of Ellipsoid $\equiv \dfrac{4}{3}\pi\rho abc$

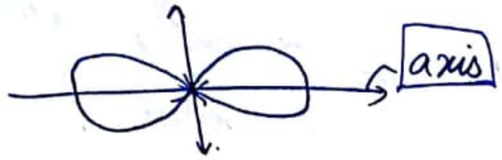


- $r = a(1+\cos\theta)$



$$\int_0^{\pi}\int_0^{a(1+\cos\theta)}$$

- $r^2 = a^2\cos 2\theta$



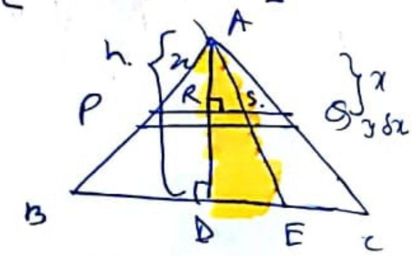
$$r = 2\int_0^{a\sqrt{\cos 2\theta}}\int_{-\pi/4}^{\pi/4}$$ ⇐ For integration (IMP)

- Find AE
- Find AS, PQ.

Assume AD given

- MI of triangle ABC about ⊥r through A [Use a, b, c, h]

AD ⊥ BC, AE is median. $\boxed{AD = h}$ altitude



$$\Delta ARS \sim \Delta ADE \Rightarrow \frac{x}{AD} = \frac{AS}{AE}$$

$$AE^2 = AD^2 + (BE - DB)^2 \Rightarrow (AD^2 + BD^2) + BE^2 - 2BE.BD$$

$$= c^2 + \frac{a^2}{4} - 2\cdot\frac{BC}{2}\cdot AB\cos B$$

$$= c^2 + \frac{a^2}{4} - \cancel{2}\frac{a}{\cancel{2}}\cdot \cancel{c}\,\frac{a^2+c^2-b^2}{\cancel{2ac}}$$

$$(AE)^2 = \frac{2b^2 + 2c^2 - a^2}{4} = r^2.$$

$$\boxed{\rho = \frac{M}{\frac{1}{2}ah}}$$

$$\frac{x}{h} = \frac{AS}{r} \Rightarrow AS = \frac{xr}{h}$$

$\boxed{S \text{ is mid-point of } PQ}$

$$PQ \Rightarrow \frac{x}{h} = \frac{PQ}{a} \Rightarrow PQ = \frac{ax}{h}$$

$$MI \Rightarrow \int_0^h \underbrace{\frac{1}{3}\overbrace{\rho\left(\frac{ax}{h}\right)\delta x}^{M}\left(\frac{ax}{2h}\right)^2}_{\text{(MI PQ about S)}} + \underbrace{\overbrace{\rho\left(\frac{ax}{h}\right)\delta x}^{M}\left(\frac{xr}{h}\right)^2}_{\text{(MI about A)}}$$

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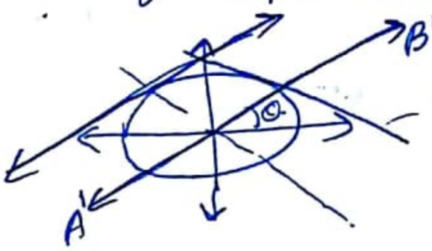
* MI of ellipse about a tangent (p is $\perp$r from O on tangent)



$y = mx + \sqrt{a^2m^2+b^2}$

$y = \tan\theta\, x + \sqrt{a^2\tan^2\theta + b^2}$

A' B' be tangent parallel through O

$\boxed{y = x\tan\theta}$

About a diameter ∥ to given tangent

MUG IT!

$$\boxed{MI_{(A'B')} = \left(\frac{Mb^2}{4}\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta\right)}$$

$$p = \frac{\sqrt{a^2\tan^2\theta + b^2}}{(1+\tan^2\theta)}$$

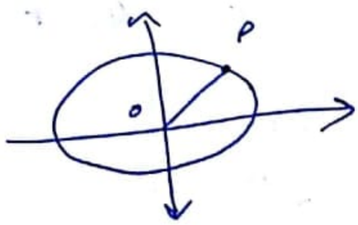
$$\boxed{p^2 = a^2\sin^2\theta + b^2\cos^2\theta}$$

$$MI_{(\text{tangent})} = \frac{Mb^2}{4}\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta + Mp^2$$

(tangent) dist p

$$= \frac{M}{4}\left\{(b^2\cos^2\theta + a^2\sin^2\theta) + p^2\right\} = \boxed{\frac{5M}{4}p^2} \text{ Ans}$$

* MI about a diameter of length r.



$$OP = \frac{r}{2} \Rightarrow P = \left(\frac{r\cos\theta}{2}, \frac{r\sin\theta}{2}\right)$$

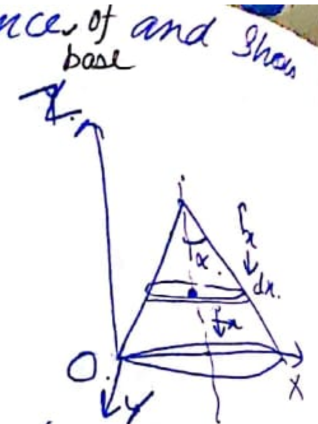
$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 4$$

$$b^2\cos^2\theta + a^2\sin^2\theta = \frac{4a^2b^2}{r^2}$$

$$MI = \frac{Mb^2}{4}\cdot\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta = \frac{M}{\cancel{4}}\left(\frac{\cancel{4}a^2b^2}{r^2}\right)$$

$$\left(\frac{M}{4}\frac{a^2b^2}{r^2}\right) \leftarrow \text{Answer}$$

* Principal Axes of Cone at point on circumference of base and Show one of them is through CG. if $\alpha = \tan^{-1}(1/2)$



MI about OX ⇒ (A)

$$\int_0^h \rho\pi(x\tan\alpha)^2 dx \left[\frac{(x\tan\alpha)^2}{4} + (h-x)^2\right] dx. \qquad r = (\cancel{h-}x)\tan\alpha.$$

$$\int_0^h \rho\pi\tan^2\alpha \left[x^4 \cdot \frac{\tan^2\alpha}{4} + h^2x^2 + x^4 - 2hx^3\right] dx. \qquad \tan\alpha = \frac{a}{h}.$$

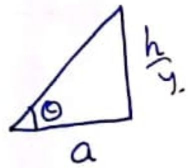
$$\rho\pi\,\cancel{\tan^2\alpha}\,\frac{a^2}{h^2}\left[\frac{h^5\,\cancel{\tan^2\alpha}\,a^2}{20\,h^2} + \frac{h^5}{3} + \frac{h^5}{5} - \frac{h^5}{2}\right]$$

$$\rho\pi a^2\left[\frac{ha^2}{20} + \frac{h^3}{30}\right] \Rightarrow \underbrace{\frac{1}{3}\rho\pi a^2 h}\; M\left[\frac{3a^2}{20} + \frac{2h^2}{20}\right] \text{ Ans}$$

B = about Y ⇒ $\boxed{\frac{M}{20}(3a^2+2h^2) + Ma^2}$

C = about Z ⇒ $\boxed{\frac{13}{10}Ma^2}$

$CG \equiv [a, 0, h/4]$ < $D = 0$, $E = Mah/4$, $F = 0$



Y-axis is principal axis as $\boxed{D, F = 0}$

(*) ⇒ $\tan 2\theta = \frac{2E}{C-A} = \frac{10ah}{23a^2 - 2h^2}$ (3)

(*) passes through CG. $\boxed{\tan\theta = \frac{h/4}{a} = h/4a}$ (1)

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ ⇒ $\boxed{h = 2a}$

$$\frac{2 \cdot \frac{h}{4a}}{1 - \frac{h^2}{16a^2}} = \boxed{\frac{8ah}{16a^2 - h^2}} \;(2)$$

Put (1) here ↗ So $\boxed{\tan\alpha = \frac{1}{2}}$

(2) = (3) gives ↑

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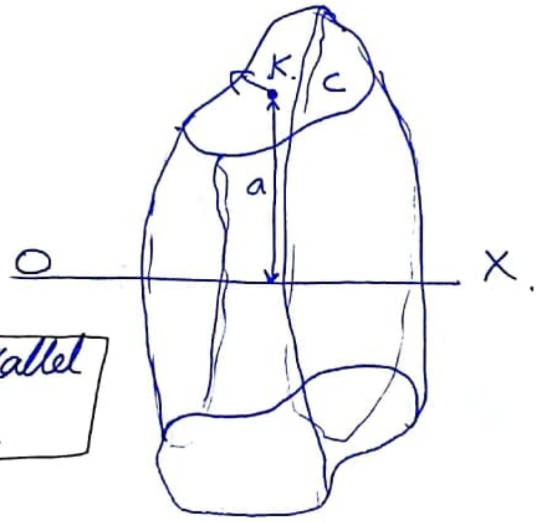
* Revolution of an object about an axis.

M.I of body of revolution

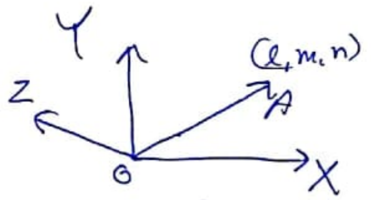
↳ $\boxed{M\left[a^2 + 3K^2\right]}$

about a line parallel to XO through C.



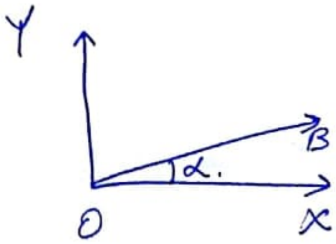
* Moment of inertia about axes.

A,	B,	C,	D,	E,	F
X	Y	Z	(yz)	(zx)	(xy).
$\Sigma m(y^2+z^2)$	- - -		$\Sigma m yz$	- - -	- -



• About any line OA $\qquad \dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$

$$\boxed{Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm}$$



• Given about OX, OY

To find about OB

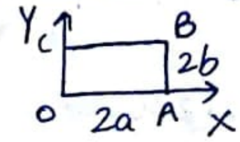
$$\boxed{A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha}$$

* Finding product of inertia.

- $CG \equiv (l, m, n)$.
  - If 2 are 0 $\Rightarrow D, E, F = 0$.
  - If only 1 is 0 $(n = 0) \Rightarrow \boxed{F = (lm)M}$

    Rest are 0.

* MI about a point P (dist r from CG) $=$ MI at CG $+ Mr^2$.

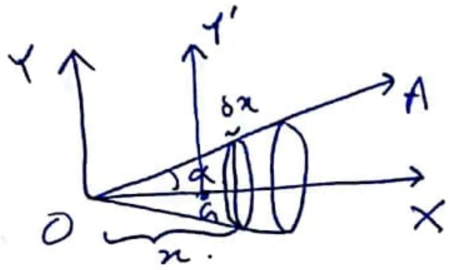
- 

  $F = Mab$

  (Product of inertia)

- MI of solid cone (h, a) about slant side

- We need about OA



$$\boxed{A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha}$$

$$\boxed{\tan\alpha = a/h}$$

- About OX (A).

At distance x from O $\Rightarrow R = x\tan\alpha$.

$$\boxed{\rho = \frac{3M}{\pi a^2 h}}$$

$$A = \int_0^h \frac{3M}{\pi a^2 h} \pi (x\tan\alpha)^2 \delta x \frac{(x\tan\alpha)^2}{2}$$

$$A = \boxed{\frac{3M}{\pi a^2 h} \frac{\pi \tan^4\alpha \cdot h^5}{2 \cdot 5}}$$

- About OY (Just add mx^2 term to prev)

$$B = \boxed{\rho \frac{\pi \tan^2\alpha \cdot h^5}{5}\left(1 + \frac{\tan^2\alpha}{4}\right)}$$

- $F = 0$

Just Simplify.

- MI about OY' at G = MI at OY $\underbrace{- M(OG)^2}$.

$-$ sign when finding at G.

• MI of thin homogenous ellipsoid shell $\{a, b, c, M\}$

- MI of solid about OX $\equiv M_0 \frac{(b^2+c^2)}{5}$

$\left(\frac{4}{3}\pi abc\rho_0\right)$

- $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$ where a', b', c' is concentric.

$b = \frac{b'}{a'} a = pa$; $c = \frac{c'}{a'} a = qa$

- MI of shell $\equiv d\left\{\frac{4}{3}\pi abc\rho_0 \frac{(b^2+c^2)}{5}\right\}$

$\equiv d\left\{\frac{4}{3}\pi pq\rho_0 a^5 \frac{(p^2+q^2)}{5}\right\}$

$= \boxed{\frac{4}{3}pq\rho_0\pi(p^2+q^2)a^4 \underbrace{da}_{\text{To find } da}}$

- Mass of shell $= d\left\{\frac{4}{3}\pi abc\rho_0\right\} = d\left\{\frac{4}{3}\pi\rho_0 pq a^3\right\}$

$\boxed{M = \cancel{3}\cdot\frac{4}{\cancel{3}}\pi\rho_0 pq a^2 da}$

$MI = \frac{\cancel{4}}{3}\cancel{\rho_0 pq\pi}(p^2+q^2)a^{\cancel{4}} \frac{M}{\cancel{4\pi\rho_0 pq a^2}}$

$= \frac{M}{3}(p^2a^2+q^2a^2) = \boxed{\frac{M}{3}(b^2+c^2)}$ Ans

As SHELL → So, no integration

MI of heterogenous ellipsoid about major axis, layers of uniform density being similar ellipsoids with density varying along major axis as distance from centre.

→ $\rho_0 = \lambda a$ (given). $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

$$\text{Mass} = \frac{4}{3}\pi\, a\, pa\, qa \cdot \rho_0 = \frac{4}{3}\pi \rho_0 pq\, a^3.$$

$$MI = \frac{4}{3}\pi pq\, a^3 \rho_0 \left(\frac{p^2+q^2}{5}\right)a^2 = \frac{4}{15}\pi pq \rho_0 a^5 (p^2+q^2)$$

For shell, (ρ_0 value not during differentiation but during integration for both MI, Mass)

$$MI \equiv d\left\{\frac{4}{15}\pi pq \rho_0 a^5 (p^2+q^2)\right\} = \frac{4}{3}\pi pq \rho_0 (p^2+q^2) a^4 da$$

$$\text{Mass} = d\left\{\frac{4}{3}\pi \rho_0 pq\, a^3\right\} = \frac{4}{3}\pi \rho_0 pq\, a^2 da$$

Overall, [Here put ρ_0 value]

$$MI = \frac{4}{3}\pi pq \rho_0 (p^2+q^2) \int_0^a \lambda a^5 da = \frac{\lambda a^6}{6}.$$

$$= \boxed{\frac{2}{9}\lambda pq (p^2+q^2) a^6}$$

$$\text{Mass} = \cancel{\pi\rho_0}\ \boxed{\pi \lambda pq\, a^4}$$

$$\text{So, } MI \equiv \frac{2}{9} M (b^2+c^2)$$

*

- Momental Ellipsoid

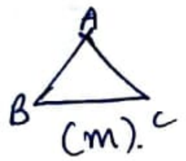
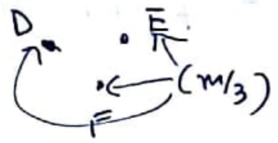
$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy = \text{constant}$$

- Principal axes
  Through which $D, E, F = 0$

- Momental Ellipsoid at $P(p,q,r)$ given at CG. $[A, B, C, D, E, F]$

  $A' = A + M(q^2+r^2) \dots$

  $D' = D + Mqr$

-  $\equiv$  [All moments and inertia products are same]

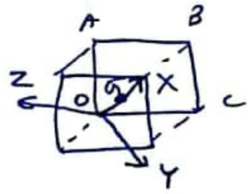
- Equimomental Bodies
  Same CG (+) Same moments, products about any one axes [Just show for 1 random]

  (+) Same Mass

Product inertia for $(x_1,y_1), (x_2,y_2) \equiv \sum m_i x_1 y_1 + m_2 x_2 y_2$

7) Momental Ellipsoid at corner of a cube of side 2a is $(2x^2 + 11(y^2+z^2) = c)$

- To eliminate product inertia, take OX through CG



MUG :

For any cube (2a), MI through any axis through CG. is $\boxed{\frac{2}{3}Ma^2} = A$. MUG IT!

IIly MI through axes at G II to OY, OZ also $\frac{2}{3}Ma^2$

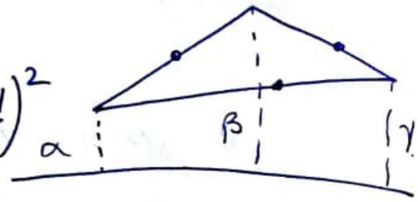
$\left[\text{Diagonal of cube} \equiv (2\sqrt{3})(a^2)^{1/2}\right] = \underline{2\sqrt{3}\,a}$

$B = \frac{2}{3}Ma^2 + M\,3a^2 = \frac{11}{3}Ma^2 = C.$

D, E, F = 0

Hence proved

- MI of Δ lamina, given distances α, β, γ of vertices from a fixed line.



$$MI = \frac{1}{3} m \left(\frac{\alpha+\beta}{2}\right)^2 + \frac{1}{3} m \left(\frac{\beta+\gamma}{2}\right)^2 + \frac{1}{3} m \left(\frac{\alpha+\gamma}{2}\right)^2$$

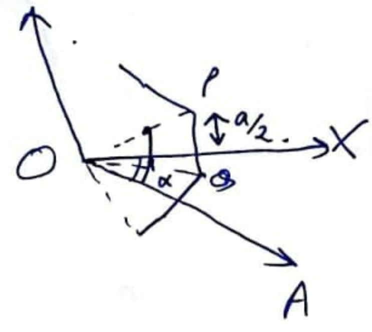
$$= \boxed{\frac{1}{6} m \left(\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta\right)}$$

Remember as a result.

Use whenever to show Something is equi-momental to a Δ lamina.

= Same as lamina.

· MI of regular polygon (n sides) about any straight line through centre.



- To find about OA
Divide into n isosceles Δ
mass ΔOPQ = M/n, $\angle POQ = \frac{2\pi}{n}$
- Consider weights placed at mid-points
↳ MI ΔOPQ about OX $= \frac{M}{3n}\left[\frac{a^2}{16}+\frac{a^2}{16}\right]$
$= \frac{M}{24n}a^2$

↳ MI ΔOPQ about OY $= \frac{M}{3n}\left[\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2 + 2\left(\frac{a}{4}\cot\frac{\pi}{n}\right)^2\right]$
$= \frac{Ma^2}{8n}\cot^2\left(\frac{\pi}{n}\right)$

↳ About OA $\equiv \frac{M}{24n}a^2\cos^2\alpha + \frac{Ma^2}{8n}\cot^2\left(\frac{\pi}{n}\right)\sin^2\alpha$.

||ly taking all Δs ⇒

$$MI = \frac{M}{24n}a^2\left[\cos^2\alpha + \cos^2\left(\alpha+\frac{2\pi}{n}\right) + \cos^2\left(\alpha+\frac{4\pi}{n}\right) + \dots n\right]$$
$$+ \frac{Ma^2}{8n}\cot^2\frac{\pi}{n}\left[\sin^2\alpha + \sin^2\left(\alpha+\frac{2\pi}{n}\right) + \dots\right]$$

$$= \frac{M}{24n}a^2\cdot\frac{1}{2}\left[1+\cos 2\alpha + 1 + \cos\left(2\alpha+\frac{4\pi}{n}\right)+\dots\right] +$$
$$\frac{Ma^2}{8n}\cdot\cot^2\left(\frac{\pi}{n}\right)\cdot\frac{1}{2}\left[1-\cos 2\alpha + 1 - \cos\left(2\alpha+\frac{4\pi}{n}\right)+\dots\right]$$
↳ Only this remains

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$$= \frac{Ma^2 \cdot n}{48n} + \frac{Ma^2}{8n}\cot^2\left(\frac{\pi}{n}\right)\cdot\frac{1}{2}n \cdot + \frac{Ma^2}{48n}\left[\cos 2\alpha + \overset{\nearrow 0}{\cos\left(2\alpha + \frac{4\pi}{n}\right)} + \dots\right]$$

IMPORTANT →

$$-\frac{Ma^2}{16n}\cot^2\left(\frac{\pi}{n}\right)\left[\cos 2\alpha + \overset{\nearrow 0}{\cos\left(2\alpha + \frac{4\pi}{n}\right)} + \dots\right]$$

$$= Ma^2\left[\frac{1}{48} + \frac{1}{16}\cot^2\left(\frac{\pi}{n}\right)\right] = \frac{Ma^2}{48}\left\{\frac{\sin^2(\pi/n) + 3\cos^2(\pi/n)}{\sin^2(\pi/n)}\right\}$$

$$= \frac{Ma^2}{24}\left\{\frac{1 + 2\cos^2 \pi/n}{1 - \cos(2\pi/n)}\right\} = \boxed{\frac{Ma^2}{24}\left\{\frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}\right\}} \text{ Ans}$$

Only 1st terms after changing $\cos^2/\sin^2$ to $\cos 2\theta/\sin 2\theta$ form

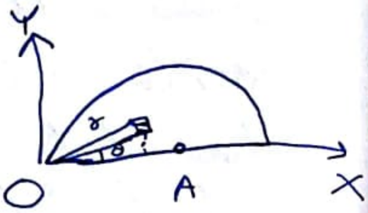
* Principal Axes. ($D, E, F = 0$)

. OX, OY, OZ be normal axes

. A, B are known $\Rightarrow$ Angle with $OX \equiv \boxed{\tan 2\theta = \frac{2F}{B-A}}$

. Other axis is $\perp$r $\left(\frac{\pi}{2}+\theta\right)$

· Find principal axis at an extremity of a boundary diameter of a semi-circular lamina.



$$A = \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\,\delta\theta\,\delta r\,(r\sin\theta)^2.$$

$$= \rho \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^{2a\cos\theta} \sin^2\theta\, d\theta$$

$$= \frac{1}{3}\pi\rho a^4$$

Let, $\boxed{r = 2a\cos\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$

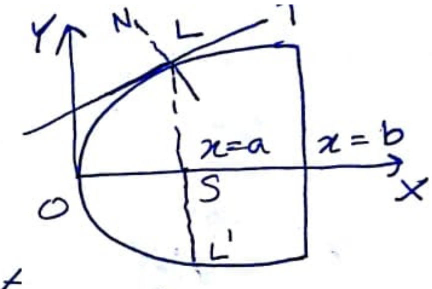
Element $r\,\delta r\,\delta\theta$

$$B = \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\,\delta\theta\,\delta r\,(r^2\cos^2\theta) = \frac{5}{3}\pi\rho a^4$$

$$F = \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\,\delta\theta\,\delta r\,(r\sin\theta)(r\cos\theta) = \frac{2}{3}\rho a^4$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} = \frac{1}{2}\tan^{-1}\left(\frac{8}{3\pi}\right)$$ Ans

- A uniform lamina of latus rectum $4a$ and a double ordinate at $\boxed{x=b}$. Show that two principal axes at end of latus rectum are tangent and normal there when ⊛



$y^2 = 4ax.$

$\Rightarrow$ At L $(x=a,\ y=2a)$ $\qquad \dfrac{dy}{dx} = \sqrt{\dfrac{a}{x}}$

Tangent at $L \Rightarrow$ slope $= 1$, $y-2a = x-a \Rightarrow \boxed{y-x=a}$

Normal at $L \Rightarrow$ Slope $= -1$, $\boxed{y+x=3a}$

At any point (x_0, y_0) consider $\delta x \delta y$

Product of inertia $= \displaystyle\int_0^b \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \frac{(y+x-3a)}{\sqrt{2}} \frac{(y-x-a)}{\sqrt{2}}\, dx\, dy$
(Should be 0)

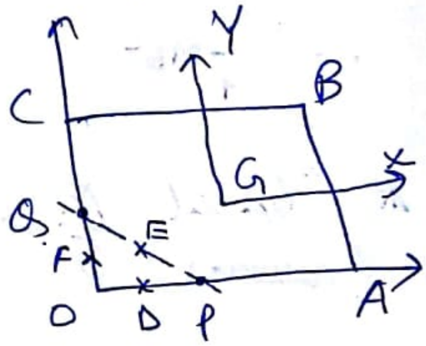
$$= \frac{4}{21}\sqrt{a}\, b\sqrt{b}\,(14ab + 21a^2 - 3b^2)$$

For it to be $0 \Rightarrow$

$21a^2 + 14ab - 3b^2 = 0$

$\Rightarrow b = \dfrac{14a \pm \sqrt{196a^2 + 12 \times 21a^2}}{6} \Rightarrow \boxed{b = \dfrac{a}{3}(7+4\sqrt{7})\ \text{Ans}}$ ⊛

* A square lamina of $2c$, corner cut off by $\frac{x}{a}+\frac{y}{b}=2$. Find inclination of principal axis at centre to X-axis.



ΔOPQ is cut-off

$\Delta DPQ \equiv DEF$

- $m(OPQ) = m$.

$F \equiv (0,b) - \frac{m}{3}$ $\quad D \equiv (a,0) - \frac{m}{3}$ $\quad E \equiv (a,b) - \frac{m}{3}$

- Consider $GX, GY \equiv$

$D' \equiv (a-c, -c)$, $E' \equiv (a-c, b-c)$, $F' \equiv (-c, b-c)$

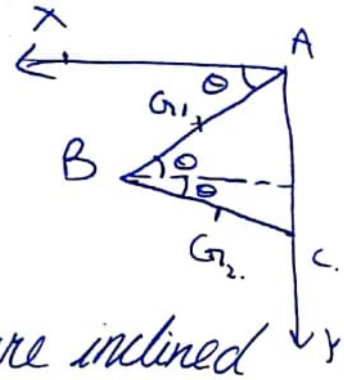
MI of new = MI square - MI cut off.

$A \equiv (GX) \rightarrow \frac{1}{3}Mc^2 - \frac{1}{3}m\left[c^2 + (b-c)^2 + (b-c)^2\right]$

$B \equiv (GY) \rightarrow \frac{1}{3}Mc^2 - \frac{1}{3}m\left[c^2 + (c-a)^2 + (c-a)^2\right]$

* Conservation of Energy

(1) Like rods AB, BC (2a) are freely jointed at B; C can move freely on a vertical axis.
Initially C = A (coincident).
Find ω of either rod when rods are inclined at θ to horizontal.



$G_1 \equiv (a\cos\theta, a\sin\theta), G_2 \equiv (a\cos\theta, 3a\sin\theta)$

→ Energy equation.

$$\rightarrow \frac{1}{2} m \left[\frac{a^2}{3}\dot{\theta}^2 + (-a\sin\theta\dot{\theta})^2 + (a\cos\theta\dot{\theta})^2\right] + \frac{1}{2} m \left[\frac{a^2}{3}\dot{\theta}^2 + (a\sin\theta\dot{\theta})^2 + (3a\cos\theta\dot{\theta})^2\right]$$

$$= mga\sin\theta + mg3a\sin\theta.$$

$$\rightarrow \frac{4}{3}a^2\dot{\theta}^2 + \frac{4}{3}a^2\dot{\theta}^2 + 8a^2\cos^2\theta\dot{\theta}^2 = 8ag\sin\theta.$$

$$\rightarrow \boxed{\dot{\theta}^2 = \frac{3g\sin\theta}{a(1+3\cos^2\theta)}} \quad \text{Ans}$$

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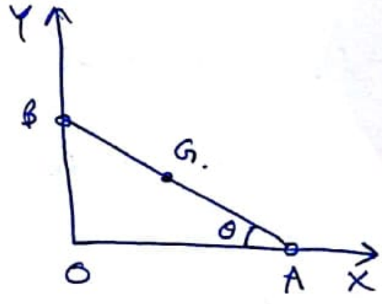
(2) A rod of length $2a$ has two small rings at its ends which can slide on OX, OY. Rod starts at α with $\omega = \sqrt{3g(1-\sin\alpha)/2a}$ and moves down. Find taken time to strike horizontal.



$G \equiv (a\cos\theta, a\sin\theta)$

$T = \frac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right]$

$V = mga\sin\theta.$

Using COE,

$$\cancel{m}ga(\sin\alpha - \sin\theta) = \frac{(-1)}{\cancel{2}}\cancel{m}.\frac{\cancel{4}}{\cancel{3}}a^{\cancel{2}}.\frac{\cancel{3}g(1-\sin\alpha)}{\cancel{2a}} + \frac{1}{\cancel{2}}\cancel{m}.\frac{4}{3}a^2\dot{\theta}^2$$

$$\cancel{ga\sin\alpha + \frac{ga}{2}(\sin\alpha - 1) - ga\sin\theta = \frac{2}{3}a^2\dot{\theta}^2}$$

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\sin\theta) \Rightarrow \dot{\theta} = \sqrt{\frac{3g}{2a}}\sqrt{1-\sin\theta}$$

$$\Rightarrow \int \frac{d\theta}{\sqrt{1-\sin\theta}}\cdot\sqrt{\frac{3g}{2a}} = \int dt.$$

$$\boxed{\frac{1}{\sqrt{1-\sin\theta}} = \frac{1}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}} = \frac{1}{\sqrt{2}\left(\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)} = \frac{1}{\sqrt{2}}\operatorname{cosec}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

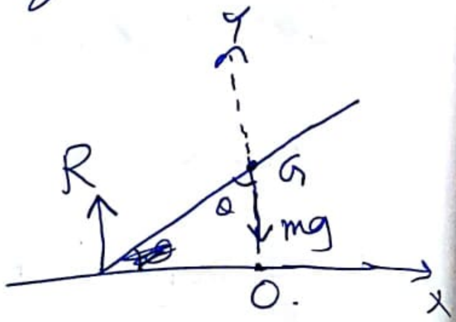
$$t = \frac{1}{\sqrt{\frac{3g}{2a}}} \times \frac{1}{\sqrt{2}}.2\left[\log\left[\tan\left(\frac{\pi}{8}-\frac{\theta}{4}\right)\right]\right]_0^\alpha$$

$$= 2\sqrt{\frac{a}{3g}}\log\left[\frac{\tan\pi/8}{\tan(\pi/8 - \alpha/4)}\right] \quad \text{Ans}$$

③ A uniform rod of 2a is place on a smooth table at α. Find its ω when inclined at θ and Reaction of table.

- No external force.

$G \equiv (0, a\cos\theta)$



COE $\Rightarrow$

$$\frac{1}{2} m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\sin^2\dot{\theta}^2\right] = mga\cos\alpha - mga\cos\theta.$$

$$\Rightarrow \quad \dot{\theta} = \sqrt{\frac{6g}{a}}\sqrt{\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}}.$$

Find Rxn $\Rightarrow$ Take torque about G.

$$Ra\sin\theta = \underbrace{\frac{ma^2}{3}\ddot{\theta}}_{(I\alpha)} \rightarrow R = \frac{m}{3}a\,\mathrm{cosec}\,\theta\,\ddot{\theta}$$

Differentiate $\dot{\theta}^2$

$$2\dot{\cancel{\theta}}\ddot{\theta} = \frac{d}{d\theta}\left(\cancel{\sqrt{\frac{6g}{a}}}\;\frac{d}{d\theta}\left(\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right)\right) \Leftarrow \dot{\theta} \text{ term comes out.}$$

$$= (\quad)\,\dot{\cancel{\theta}} \quad \Leftarrow \text{Gives answer easily}$$