Test series – 2018

Answer Key – Test – 3 (Paper-I)

ODE, Statics & Dynamics and Vector Analysis

1(a) (i) Find $L^{-1}\left\{\log\frac{p+3}{p+2}\right\}$, (ii) Find $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$

Sol'n: (i) Let $f(p) = \log\frac{p+3}{p+2}$

$$= \log(p+3) - \log(p+2)$$

$$\therefore f'(p) = \frac{1}{p+3} - \frac{1}{p+2}$$

$$\therefore L^{-1}\{f^{-1}(p)\} = e^{-3t} - e^{-2t}$$

$$\Rightarrow (-1)t\,L^{-1}\{f(p)\} = e^{-3t} - e^{-2t}$$

$$\Rightarrow L^{-1}\log\left\{\frac{p+3}{p+2}\right\} = \frac{1}{t}\left(e^{-2t} - e^{-3t}\right)$$

(ii) Let $f(p) = \log\left(1-\frac{1}{p^2}\right)$

$$= -\log\frac{p^2-1}{p^2} = -2\log p + \log(p^2-1)$$

$$\therefore f'(p) = -2\left(\frac{1}{p} - \frac{p}{p^2-1}\right)$$

$$\therefore L^{-1}\{f'(p)\} = -2(1-\cosh t)$$

$$\Rightarrow -t\,L^{-1}\{f(p)\} = -2(1-\cosh t)$$

$$\Rightarrow L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\} = \frac{2}{t}(1-\cosh t)$$

1(b) Find the orthogonal trajectories of the family of cardioids $r = a(1+\cos\theta)$.

Sol^n: The given family of cardioids in

$$r = a(1+\cos\theta) \quad \text{--- (1)}$$

Taking logarithm of bothsides of (1) we get

$$\log r = \log a + \log(1+\cos\theta) \quad \text{--- (2)}$$

Differentiating (2) w.r.t 'θ' we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1+\cos\theta} \quad \text{--- (3)}$$

Since (3) is free from parameter a, hence (3) is the differential equation of the given family (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{\sin\theta}{1+\cos\theta}$$

$$= -\frac{2\sin\theta/_2\cos\theta/_2}{2\cos^2\theta/_2}$$

$$\cancel{-}r\frac{d\theta}{dr} = \cancel{-}\tan\theta/_2$$

$$\frac{dr}{r} = \cot\theta/_2\, d\theta \text{ (On seperating variables)}$$

Integrating,

$$\log r = 2\log\sin\theta/_2 + \log c.$$

$$\Rightarrow \log r = \log\sin^2\theta/_2 + \log c$$

$$\Rightarrow r = c\sin^2\theta/_2$$

$$\Rightarrow r = \frac{c}{2}(1-\cos\theta)$$

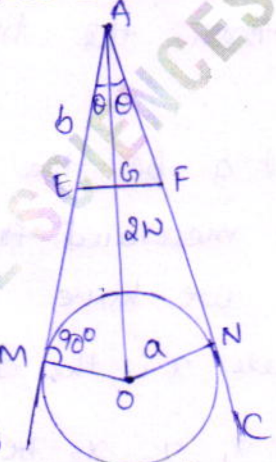
$$\Rightarrow \boxed{r = b(1-\cos\theta)}$$

where $b = \frac{c}{2}$ is arbitrary constant.

which is the required equation of orthogonal trajectories, b being parameter.

1(c) Two equal rods, AB and AC, each of length 2b are freely joined at A and rest on a smooth vertical circle of radius a. If 2θ be the angle between the rods then find the relation between b, a and θ using the principle of virtual work.

Sol^n: Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC.



If E and F are the middle points of AB and AC, then the total weight 2W of the two rods can be taken acting at G, the middle point of EF. The line AO is vertical. we have

$\angle BAO = \angle CAO = \theta$

Also $AB = 2b$, $AE = b$. If the rod AB touches the circle at M, then $\angle OMA = 90°$ and OM = the radius of the circle = a

Give the rods a small symmetrical displacement in which $\theta \to \theta + \delta\theta$. The point 'O' remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains $90°$. we have, the height of G above the fixed point O.

$$= OG = OA - GA = OM\operatorname{cosec}\theta - AE\cos\theta$$

$$= a\operatorname{cosec}\theta - b\cos\theta$$

The equation of virtual work is

$$-2W\delta(OG) = 0$$

$$\Rightarrow \delta(OG) = 0$$

$$\Rightarrow \delta(a\operatorname{cosec}\theta - b\cos\theta) = 0 \qquad [\because \delta\theta \neq 0]$$

$$\Rightarrow -a\operatorname{cosec}\theta\cot\theta + b\sin\theta = 0$$

$$\Rightarrow a\operatorname{cosec}\theta\cot\theta = b\sin\theta$$

$$\Rightarrow a\cos\theta = b\sin^3\theta$$

1(d). A particle is moving with S.H.M and while making an excursion from one position of rest to other, its distances from the middle point of its path at three consecutive seconds are observe to be x_1, x_2, x_3; Prove that the time of a complete oscillations is $2\pi / \cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right)$.

Sol'n: Take the middle point of the path as origin.
Let the equation of the S.H.M be $d^2x/dt^2 = -\mu x$.
Then the time period

$$T = 2\pi/\sqrt{\mu}$$

Let a be the amplitude of the motion. If the time t be measured from the position of instantaneous rest $x=a$, we have $x = a\cos\sqrt{\mu}\,t$, ——— (1)

where x is the distance of the particle from the centre at time t.

Let x_1, x_2, x_3 be the distances of the particle from the Centre at the ends of $t_1, (t_1+1)$ & (t_1+2) seconds. Then from (1)

$$x_1 = a\cos\sqrt{\mu}\,t_1 \quad \text{——— (2)}$$

$$x_2 = a\cos\sqrt{\mu}\,(t_1+1) \quad \text{——— (3)}$$

and $$x_3 = a\cos\sqrt{\mu}\,(t_1+2) \quad \text{——— (4)}$$

$$\therefore x_1+x_3 = a\left[\cos\sqrt{\mu}\,t_1 + \cos\sqrt{\mu}\,(t_1+2)\right]$$

$$= 2a\cos\sqrt{\mu}\,(t_1+1)\cos\sqrt{\mu} = 2x_2\cos\sqrt{\mu}, \quad [\text{from (3)}]$$

$$\therefore \cos\sqrt{\mu} = (x_1+x_3)/2x_2$$

$$\Rightarrow \sqrt{\mu} = \cos^{-1}\left\{\frac{x_1+x_3}{2x_2}\right\}$$

Hence the time period $T = \frac{2\pi}{\sqrt{\mu}}$

$$= \frac{2\pi}{\cos^{-1}\{(x_1+x_3)/2x_2\}}$$

1(c), The position of a point at time t is given by the formulas $x = e^t \cos t$, $y = e^t \sin t$

(i) Show that $\vec{a} = 2\vec{v} - 2\vec{r}$

(ii) Show that the angle between the radius vector r and the acceleration vector a is constant, and find this angle

Soln:
$$\vec{r} = e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = e^t(\cos t - \sin t)\hat{i} + e^t(\sin t + \cos t)\hat{j}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^t\left[\cancel{\cos t} - \sin t - \sin t - \cancel{\cos t}\right]\hat{i} + e^t(\cancel{\sin t} + \cos t + \cos t - \cancel{\sin t})\hat{j}$$

$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j} \quad \text{—— (1)}$$

(i) $$2\vec{v} - 2\vec{r} = 2\left[e^t(\cos t - \cancel{\sin t})\hat{i} + \cancel{e^t}(\cancel{\sin t} + \cos t)\hat{j}\right] - 2\left[e^t \cos t\,\hat{i} + e^t \sin t\,\hat{j}\right]$$

$$= -2e^t \sin t\,\hat{i} + 2e^t \cos t\,\hat{j}$$

$$= \vec{a} \quad (\because \text{from (1)})$$

$\therefore 2\vec{v} - 2\vec{r} = \vec{a}$

(ii) Let the angle between $\vec{r}$ and $\vec{a}$ be θ

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{r}}{|\vec{a}||\vec{r}|} = \frac{-2e^t \sin t \cos t + 2e^t \sin t \cos t}{|\vec{a}||\vec{r}|}$$

$$\cos\theta = 0$$

$\Rightarrow \theta = \pi/2$.

$\vec{a}$ and $\vec{r}$ are at right angle to each other.

2(a)) Solve and examine for singular solution of the equation

$$(1+p)^3 = \left(\frac{27}{8a}\right)(x+y)(1-p)^3$$

Hint: put $x+y=u$, $x-y=v$.

Then the general solution is

$$(x-y+c)^3 = a(x+y)^2$$

$x+y=0$ is a singular solution.

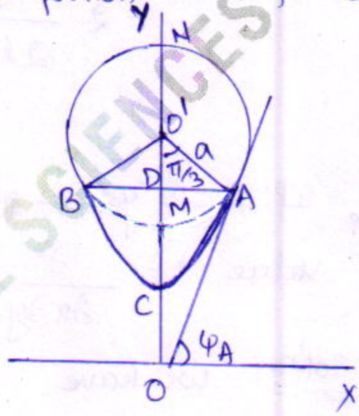
2(b). Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with the two thirds of the circumference of the pulley.

Sol^n: Let ANBMA be the circular pulley of radius 'a' and ANBCA the endless chain hanging over it.

Since the chain is in contact with the $2/3^{rd}$ of the circumference of the pulley, hence the length of this portion ANB of the chain

$= \frac{2}{3}$ (circumference of the pulley)

$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a$



Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e. the vertex, CO'N the axis and OX the directrix of this catenary.

Let $OC = c$ = the parameter of the catenary

The tangent at A will be ⊥lar to the radius O'A.

∴ If the tangent at A is inclined at an angle φ_A to the horizontal, then

$$\varphi_A = \angle AO'D = \tfrac{1}{2}(\angle AO'B) = \tfrac{1}{2}\left(\tfrac{1}{3}\cdot 2\pi\right) = \tfrac{1}{3}\pi.$$

From the triangle AO'D, we have

$$DA = O'A \sin \tfrac{1}{3}\pi = a\sqrt{3}/2$$

∴ from $x = c\log(\tan\varphi + \sec\varphi)$, for the point A, we have

$$x = DA = c\log(\tan\varphi_A + \sec\varphi_A)$$

$$\Rightarrow a\frac{\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right) = c\log(\sqrt{3}+2)$$

$$\therefore c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $S = c\tan\psi$, applied for the point A, we have

$$\text{arc } CA = c\tan\psi_A = c\tan\tfrac{1}{3}\pi = c\sqrt{3} = \frac{3a}{2\log(2+\sqrt{3})}$$

Hence the total length of the chain

$= \text{arc } ABC +$ length of the chain in contact with the pulley

$$= 2(\text{arc } CA) + \tfrac{4}{3}\pi a$$

$$= 2\,\frac{3a}{2\log(2+\sqrt{3})} + \tfrac{4}{3}\pi a = a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$$

2(c) If $A = x^2yz\,i - 2xz^3\,j + xz^2\,k$, $B = 2z\,i + y\,j - x^2\,k$, find the value of $\frac{\partial^2}{\partial x\,\partial y}(A\times B)$ at $(1,0,-2)$

Sol^n: we have $A\times B = \begin{vmatrix} i & j & k \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= (2x^3z^3 - xyz^2)\,i + (2xz^3 + x^4yz)\,j + (x^2y^2z + 4xz^4)\,k$$

$$\therefore \frac{\partial}{\partial y}(A\times B) = -xz^2\,i + x^4z\,j + 2x^2yz\,k$$

Again $\frac{\partial^2}{\partial x\,\partial y}(A\times B) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}(A\times B)\right\}$

$$= -z^2\,i + 4x^3z\,j + 4xyz\,k$$

Putting $x=1$, $y=0$ and $z=-2$ in ①, we get the required derivative at the point $(1,0,-2) = -4i - 8j$.

2(d) If $F = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)i + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)j + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)k$

Prove that (i) $F = r\times\nabla f$, (ii) $F\cdot r = 0$, (iii) $F\cdot\nabla f = 0$.

Sol^n: we have $\nabla f = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k$

and $r = xi + yj + zk$.

(i)

$$r \times \nabla f = \begin{vmatrix} i & j & k \\ x & y & z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)i + \left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)j + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)k$$

$$= F$$

(ii) $F \cdot r = (r \times \nabla f) \cdot r$

$= 0$, because the value of a scalar triple product having two vectors equal is zero.

(iii) $F \cdot \nabla f = (r \times \nabla f) \cdot \nabla f = [r, \nabla f, \nabla f]$

$= 0$, because the value of a scalar triple product having two vectors equal is zero.

3(a) solve $x\left(d^2y/dx^2\right) - \left(dy/dx\right) + (1-x)y = x^2 e^{-x}$

Sol'n: Rewriting $\left(\frac{d^2y}{dx^2}\right) - \left(\frac{1}{x}\right)\left(\frac{dy}{dx}\right) + \left(\frac{1}{x} - 1\right)y = xe^{-x}$ — ①

Comparing ① with $y'' + Py' + Qy = R$,

$P = -\frac{1}{x}$, $Q = \frac{1}{x} - 1$, $R = xe^{-x}$ — ②

Here $1 + P + Q = 0$ showing that

$y = u = e^{x}$ — ③

is a part of C.F of the given equation

Let the required general solution be $y = uv$ — ④

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-\frac{1}{x} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{xe^{-x}}{e^x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(2-\frac{1}{x}\right)\frac{dv}{dx} = xe^{-2x} \quad \text{--- (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$

Then (5) becomes $\frac{dq}{dx} + \left(2-\frac{1}{x}\right)q = xe^{-2x}$, which is linear in q and x.

Its I.F $= e^{\int(2-\frac{1}{x})dx} = e^{2x-\log x} = e^{2x}\cdot e^{-\log x} = e^{2x}\cdot x^{-1}$

$\therefore q = xe^{-2x}(x+c_1)$

$\Rightarrow \frac{dv}{dx} = (x^2+c_1x)e^{-2x}$, by (6)

$\therefore \int dv = \int (x^2+c_1x)e^{-2x}dx + c_2$

$\Rightarrow v = c_2 + (x^2+c_1x)\left(-\frac{1}{2}\right)e^{-2x} - \int(2x+c_1)\left(-\frac{1}{2}\right)e^{-2x}dx$, integrating by parts

$= c_2 - \frac{1}{2}(x^2+c_1x)e^{-2x} + \frac{1}{2}\int(2x+c_1)e^{-2x}dx$

$= c_2 - \frac{1}{2}(x^2+c_1x)e^{-2x} + \frac{1}{2}\left[(2x+c_1)\left(\frac{e^{-2x}}{-2}\right) - \int(2)\left(\frac{e^{-2x}}{-2}\right)dx\right]$

$= c_2 - \left(\frac{1}{2}\right)(x^2+c_1x)e^{-2x} - \frac{1}{4}(2x+c_1)e^{-2x} - \left(\frac{1}{4}\right)e^{-2x}$

$= c_2 - \left(\frac{1}{4}\right)e^{-2x}(2x^2+2c_1x+2x+c_1+1)$

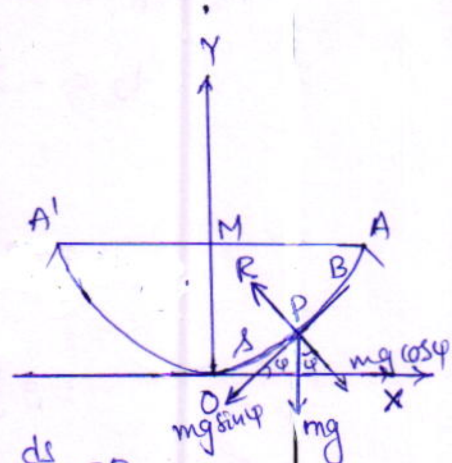
$\Rightarrow v = c_2 - \frac{1}{4}e^{-2x}(2x^2+2x+1) - \left(\frac{1}{4}\right)e^{-2x}(2x+1)c_1$ — (7)

from (3), (4) & (7), the required general solution is $y = uv = e^x\left[c_2 - \left(\frac{1}{4}\right)e^{-2x}(2x^2+2x+1) - \left(\frac{1}{4}\right)e^{-2x}(2x+1)c_1\right]$

$\Rightarrow y = c_1'(2x+1)e^{-x} + c_2e^x - \left(\frac{1}{4}\right)(2x^2+2x+1)e^{-x}$, where $c_1' = -\left(\frac{1}{4}\right)c_1$

3(b) A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards.. Prove that when it has fallen through half the distance measured along the arc to the vertex, 2/3rds of the time of descent will have elapsed.

Soln: Let a particle of mass m start from rest from the cusp A of the cycloid. If P is the position of the particle after time t such that arc OP = s, the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{——— (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{——— (2)}$$

For the cycloid, $s = 4a\sin\psi$ ——— (3)

from (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$

multiplying bothsides by $2(ds/dt)$ and then integrating we have

$$\left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

Initially at the cusp A, $s = 4a$ and $\frac{ds}{dt} = 0$

$$\therefore A = \frac{g}{4a}\cdot(4a)^2 = 4ag$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + 4ag = \frac{g}{4a}(16a^2 - s^2) \quad \text{——— (4)}$$

$$\Rightarrow \frac{ds}{dt} = -\frac{1}{2}\sqrt{g/a}\cdot\sqrt{(16a^2 - s^2)}$$

the -ve sign is taken because the particle is moving in the direction of s decreasing.

Separating the variables, we have

$$dt = -2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(16a^2 - s^2)}} \quad \text{——— (5)}$$

If t_1 is the time from the cusp A (i.e. $s=4a$) to the vertex O (i.e. $s=0$), then integrating ⑤.

$$t_1 = -2\sqrt{(a/g)}\int_{4a}^{0} \frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$= -2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2a}$$

$$= 2\sqrt{(a/g)}\ \frac{\pi}{2} = \pi\sqrt{(a/g)}$$

Again if t_2 is the time taken to move from the cusp A (i.e. $s=4a$) to half the distance along the arc to the vertex i.e. to $s=2a$, then integrating ⑤

$$t_2 = -2\sqrt{(a/g)}\int_{s=4a}^{2a} \frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$= 2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2a}$$

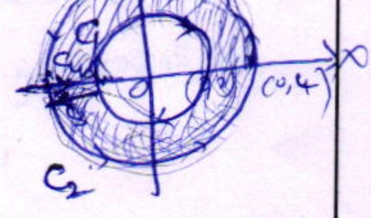
$$= 2\sqrt{(a/g)}\left[\cos^{-1}\left(\tfrac{1}{2}\right)-\cos^{-1}(1)\right]$$

$$= 2\sqrt{(a/g)}\cdot\frac{\pi}{3} = \frac{2}{3}t_1$$

3(c)(i) verify Green's theorem in the plane for $\oint_C x^2y\,dx + (y^3-xy^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2+y^2=4$, $x^2+y^2=16$

Sol: By Green's theorem, we have

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy. \quad \text{—①}$$



The boundary of the curve C is given by
$C: C_1+C_2+C_3+C_4$ and R is the region bounded by the circles $x^2+y^2=4$ and $x^2+y^2=16$.

$$\therefore L.H.S = \int_C Mdx+Ndy = \int_{C_1+C_2+C_3+C_4} Mdx+Ndy \quad \text{--- (2)}$$

Here note that along C_3 & C_4

$$\int_{C_3+C_4} Mdx+Ndy = 0. \quad \left[\because C_3 \,\&\, C_4 \text{ are in opposite directions, i.e. } \int_{C_3} = -\int_{C_4}\right]$$

$\therefore$ from (1), we have

$$\int_C Mdx+Ndy = \int_{C_1+C_2} Mdx+Ndy$$

$$= \int_{C_1} Mdx+Ndy + \int_{C_2} Mdx+Ndy$$

Let $\int_{C_1} Mdx+Ndy = \int_{C_1} x^2y\,dx + (y^3-xy^2)\,dy$

putting $x = 2\cos\theta$; $y = 2\sin\theta$
$\Rightarrow dx = -2\sin\theta\,d\theta$ & $dy = 2\cos\theta\,d\theta$.

$$\therefore \int_{C_1} Mdx+Ndy = \int_{2\pi}^{0} 4\cos^2\theta\cdot(+2\sin\theta)(-2\sin\theta)\,d\theta + \int_{2\pi}^{0}(8\sin^3\theta - 2\cos\theta\cdot 4\sin^2\theta)\,2\cos\theta\,d\theta$$

$$= \int_{2\pi}^{0} -32\cos^2\theta\sin^2\theta\,d\theta + \int_{2\pi}^{0} 16\sin^3\theta\cos\theta\,d\theta$$

$$= \int_0^{2\pi} 8\sin^2 2\theta\,d\theta + 16\left[\frac{\sin^4\theta}{\theta}\right]_{2\pi}^{0}$$

$$= 8\left[\frac{1-\cos 4\theta}{2}\right]_0^{2\pi} = 8\pi.$$

$\int_{C_2} Mdx + Ndy = \int_0^{2\pi} -256\cos^2\theta\sin^2\theta\, d\theta + \int_0^{2\pi} (64\sin^3\theta - 4\cos\theta \cdot 16\sin^2\theta)\, 4\cos\theta d\theta$

by putting $x = 4\cos\theta$; $dx = -4\sin\theta d\theta$
$y = 4\sin\theta$; $dy = 4\cos\theta d\theta$

$= -64\int_0^{2\pi} \sin^2 2\theta\, d\theta + 256\int_0^{2\pi} \sin^3\theta\cos\theta\, d\theta - 64\int_0^{2\pi}$

$= -128\int_0^{2\pi} \sin^2 2\theta + \left[\frac{\sin^4\theta}{4}\right]_0^{2\pi}(256)$

$= -128\int_0^{2\pi} \left[\frac{1-\cos 4\theta}{2}\right]$

$= -\frac{128}{2}[\theta]_0^{2\pi}$

$= -128\pi$

$\therefore$ from (2)

$\oint_C Mdx + Ndy = 8\pi - 128\pi = -120\pi$

$M = x^2y \quad N = y^3 - xy^2$

$\Rightarrow \frac{\partial M}{\partial y} = x^2 \quad \frac{\partial N}{\partial x} = -y^2$

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint -y^2 - x^2\, dxdy$

$= -\iint (x^2 + y^2)\, dxdy$

$x = r\cos\theta$; $y = r\sin\theta$
$dx\,dy = r\,d\theta\,dr$

$$= -\int_0^{2\pi}\int_{r=2}^{4} r^2 \cdot r\,d\theta\,dr$$

$$= -\int_0^{2\pi}\left[\frac{r^4}{4}\right]_2^4 d\theta$$

$$= -\int_0^{2\pi}\left[\frac{256-16}{4}\right] d\theta$$

$$= -\int_0^{2\pi} 60\,d\theta$$

$$= -60\,[\theta]_0^{2\pi}$$

$$= -60\,(2\pi)$$

$$= -120\pi.$$

Hence Green's theorem is verified.

3(c)(ii) If $E = r\vec{r}$, is there a function ϕ such that $E = -\nabla\phi$? If so, find it. Evaluate $\oint_C E\cdot dr$ if C is a simple closed curve.

Sol^n: $E = r\vec{r}$ and $E = -\nabla\phi$, $r^2 = x^2+y^2+z^2$

$$\therefore r(x\hat{i}+y\hat{j}+z\hat{k}) = -\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$\Rightarrow r(x) = -\frac{\partial\phi}{\partial x}$$

i.e. $x\sqrt{x^2+y^2+z^2} = -\frac{\partial\phi}{\partial x}$

$$\Rightarrow \phi = -\frac{1}{3}(x^2+y^2+z^2)^{3/2} + f_1(y,z)$$

Illy $\phi = -\frac{1}{3}(x^2+y^2+z^2)^{3/2} + f_2(x,z)$ —②; $\phi = -\frac{1}{3}(x^2+y^2+z^2)^{3/2} + f_3(x,y)$ —③

①, ②, ③ each represents ϕ. These agree if we choose

$$f_1(y,z) = f_2(x,z) = f_3(x,y) = c.$$

$$\therefore \phi = -\frac{1}{3}(x^2+y^2+z^2)^{3/2} + c$$

$$= -\frac{1}{3}r^3 + c.$$

(iii) $\oint_C E \cdot dr = 0$ (Try yourself.)

4(a) (i) The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, find the number of bacteria which will be present at the end of 5 hours.

(ii) Solve $(D^2+1)y = x^2 \sin 2x$.

Sol'n: (i) Suppose that the number of bacteria is x_0 when $t=0$, and it is x at time t (in hrs).

Then given that

$$\frac{dx}{dt} \propto x$$

$$\Rightarrow \frac{dx}{dt} = kx \quad \text{——— (1)}, \text{ where } k = \text{Constant of proportionality.}$$

$$(1) \Rightarrow \frac{dx}{x} = k\,dt = \int \frac{dx}{x} = k\int dt$$

$$\Rightarrow \log x - \log c = kt$$

$$\therefore \log(x/c) = kt \text{ so that } x = ce^{kt} \quad \text{——— (2)}$$

By our assumption, when $t=0$, $x = x_0$ so that

$$(2) \Rightarrow x_0 = c \text{ and so } (2) \Rightarrow x = x_0 e^{kt} \quad \text{——— (3)}$$

Given $x = 3x_0$ when $t=1$, so (3) yields

$$3x_0 = x_0 e^k$$

$$\Rightarrow e^k = 3 \quad \text{——— (4)}$$

let $x = x'$ when $t=5$, then (3) yields

$$x' = x_0 e^{5k} = x_0 (e^k)^5 = x_0 \cdot 3^5. \text{ by (4)}$$

Hence, the bateria is expected to grow 3^5 times at the end of 5hrs.

(ii) The auxiliary equation is $m^2+1=0$

$$\therefore m = \pm i = 0 \pm i$$

$\therefore$ C.F $= e^{0x}(C_1\cos x + C_2 \sin x) = C_1 \cos x + C_2 \sin x$

And P.I $= \dfrac{1}{(D^2+1)} x^2 \sin 2x$

$= \text{Imaginary part of } \dfrac{1}{(D^2+1)} x^2 e^{2ix}$

$= \text{I.P of } e^{2ix} \dfrac{1}{\{(D+2i)^2+1\}} x^2$

$= \text{I.P. of } e^{2ix} \dfrac{1}{(D^2+4iD-3)} x^2 \quad [\because i^2=-1]$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1-\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)\right]^{-1} x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1+\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)+\left(\dfrac{4iD}{3}+\dfrac{D^2}{3}\right)^2+\cdots\right] x^2$

$= \text{I.P. of } \dfrac{e^{2ix}}{-3}\left[1+\dfrac{4iD}{3}+\dfrac{D^2}{3}-\dfrac{16D^2}{9}+\cdots\right] x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[1+\dfrac{4Di}{3}-\dfrac{13D^2}{9}+\cdots\right] x^2$

$= \text{I.P of } \dfrac{e^{2ix}}{-3}\left[x^2+\dfrac{4i}{3}.2x-\dfrac{13}{9}.2\right]$

$= \text{I.P of } -\frac{1}{3}(\cos 2x + i \sin 2x)\{x^2+(8/3)ix-(26/9)\}$

$= -\frac{1}{3}.(8/3\, x\cos 2x) - \frac{1}{3}\{x^2-(26/9)\}\sin 2x.$

$= -(1/27)[24x\cos 2x + (9x^2-26)\sin 2x]$

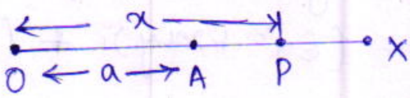
Hence the Complete solution is $y =$ C.F + P.I

$\Rightarrow y = C_1\cos x + C_2 \sin x - \frac{1}{27}[24x\cos 2x - (9x^2-26)\sin 2x]$

4(b) A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\left(\sqrt{\pi/2\mu}\right)$.

Sol^n: Let O be the centre of force and A be the initial position such that OA = a. Let P be its position at any time t such that OP = x. Then its equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x}$$



$$v\left(\frac{dv}{dx}\right) = -\frac{\mu}{x}$$

$$\Rightarrow 2v\,dv = \left(\frac{-2\mu}{x}\right)dx$$

Integrating, $v^2 = -2\mu\log x + A$, A is constant.

Initially at A, $x = a$, $v = 0$, so ① gives $A = 2\mu\log a$

$\therefore$ ① gives

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\,(\log a - \log x) = 2\mu\log(a/x)$$

$$\Rightarrow \frac{dx}{dt} = -(2\mu)^{1/2}\{\log(a/x)\}^{1/2}$$

[-ve sign is taken because as t increases, here x decreases]

Separating the variables,

$$dt = -\frac{1}{(2\mu)^{1/2}}\cdot\frac{dx}{\{\log(a/x)\}^{1/2}}$$

Integrating between the limits $x = a$ to $x = 0$, the time t_1 from A to O is given by

$$\int_0^{t_1} dt = -\frac{1}{(2\mu)^{1/2}}\int_a^0 \frac{dx}{\{\log(a/x)\}^{1/2}}.$$

$$= -\frac{1}{(2\mu)^{1/2}}\int_0^\infty \frac{-2az^2}{z}\,dz \qquad \left[\text{putting } \log(a/x) = z^2 \text{ (or) } x = ae^{-z^2} \text{ and } dx = -2aze^{-z^2}dz\right]$$

$$\Rightarrow t_1 = \frac{2a}{(2\mu)^{1/2}}\int_0^\infty e^{-z^2}dz = \frac{2a}{(2\mu)^{1/2}}\frac{\sqrt{\pi}}{2}, \text{ as } \int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2}$$

Thus, the required time $= t_1 = a\left(\frac{\pi}{2\mu}\right)^{1/2}$.

4(c) Use divergence theorem to evaluate the surface integral $\iint_S F.n\,ds$ where S is the surface of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=4$ and the plane $z=4$, and $F=(6x^2+2xy)i+(2y+x^2z)j+4x^2y^3k$.

Ans: $128+8\pi$.

Try yourself.

5(d) A rocket leaves the point $(1,-2,3)$ at time $t=0$ and travels with constant speed 1 unit in a straight line toward the point $(3,0,0)$. ~~Find as~~ functions of t, the
(i) position vector R (ii) velocity v
(iii) unit tangent vector T, (iv) acceleration $\bar{a}$
(v) curvature k.

Sol^n: (i) $R(t) = \left[1+\left(\frac{2}{\sqrt{17}}\right)t\right]\hat{i} + \left[-2+\left(2/\sqrt{17}\right)t\right]\hat{j} + \left[3-\left(3/\sqrt{17}\right)t\right]\hat{k}$

(ii) $v(t) = \frac{1}{\sqrt{17}}(2i+2j-3k)$

(iii) $T(t) = \frac{1}{\sqrt{17}}(2i+2j-3k)$.

(iv) $a(t) = 0$

(v) $k = 0$

5(e) Find the workdone by the force $f = -4xy\hat{i} + 8y\hat{j} + 2k$ as the point of application moves along the parabola $y=x^2$, $z=1$ from $A(0,0,1)$ to $B(2,4,1)$

Sol^n: Ans 48.

5(a) solve $(2x+y-3)dy = (x+2y-3)dx$.

sol'n: the given equation can be written as.

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

Here $\frac{a}{a_1} \neq \frac{b}{b_1}$; therefore putting $x = X+h$, $y = Y+k$, so that $\frac{dy}{dx} = \frac{dY}{dX}$ — ①

the above equation reduces to

$$\frac{dY}{dX} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)} \quad \text{— ②}$$

Choose h, k so that $h+2k-3=0$ and $2h+k-3=0$ — ③

solving ③ we get $h=1$, $k=1$ so that from ①, we have

$$X = x-1, \quad Y = y-1 \quad \text{— ④}$$

using ③ in ②, we get

$$\frac{dY}{dX} = \frac{X+2Y}{2X+Y} = \frac{1+(2Y/X)}{1+(Y/X)} \quad \text{— ⑤}$$

Take $\frac{Y}{X} = v$, i.e. $Y = vX$, so $\frac{dY}{dX} = v + X\frac{dv}{dX}$ — ⑥

from ⑤ and ⑥,

$$v + X\frac{dv}{dX} = \frac{1+2v}{2+v} \implies X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}$$

$$\implies \frac{dX}{X} = \frac{(2+v)dv}{(1-v)(1+v)}$$

$$= \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv$$

[Resolving into partial fractions]

Integrating $\log X + \log c = \frac{1}{2}[\log(1+v) - 3\log(1-v)]$

$$\implies 2\log(cX) = \log\frac{1+v}{(1+v)^3}$$

$$\implies X^2c^2 = \frac{1+v}{(1-v)^3}$$

$$\implies X^2c^2\left(1-\frac{Y}{X}\right)^3 = 1+\frac{Y}{X}, \quad \text{as } v = \frac{Y}{X}$$

$\Rightarrow c^2(X-Y)^3 = X+Y$

$\Rightarrow c^2\{x-1-(y-1)\}^3 = x-1+y-1$ by (4).

$\Rightarrow c'(x-y)^3 = x+y-2$, taking $c'=c^2$

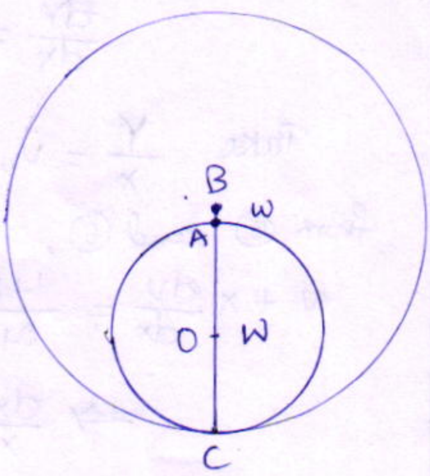
5(b) → A Sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$.

Sol^n: C is the point of contact of the sphere and the spherical shell, O is the centre of the sphere, CA is the vertical diameter of the sphere and B is the centre of the spherical shell. we have

$OC = a$ & $BC = b$

the weight W of the sphere acts at O and a particle of weight w is attached to A.

If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then



$$h = \frac{W.a + w.2a}{W+w} = \frac{W+2w}{W+w}a$$

Here $\rho_1 = a$ and $\rho_2 = -b$

If equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e. } \frac{1}{h} > \frac{1}{a} - \frac{1}{b} \Rightarrow \frac{W+w}{a(W+2w)} > \frac{b-a}{ab}$$

$\Rightarrow (W+w)ab > a(b-a)(W+2w)$

$\Rightarrow (W+w)b > (b-a)(W+2w)$

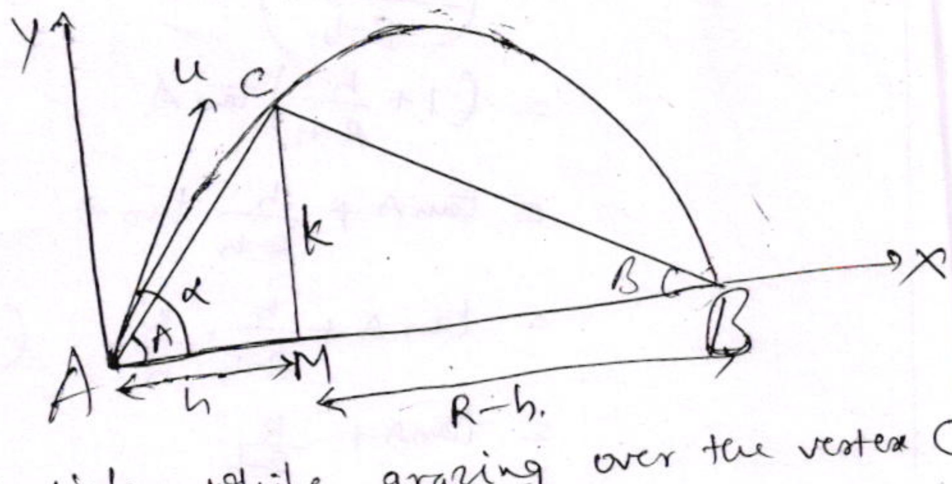
$\Rightarrow W\{b-(b-a)\} > w\{2(b-a)-b\}$

$\Rightarrow Wa > w(b-2a)$

$\Rightarrow \frac{W}{w} > \frac{b-2a}{a}$

5(c) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection. Prove that $\tan\alpha = \tan A + \tan B$.

Sol^n: Let A be the point of projection, u the velocity of projection and α the angle of projection



The particle while grazing over the vertex C falls at the point B. If AB = R, Then

$$R = \frac{2u^2 \sin\alpha\cos\alpha}{g} \quad \text{--- (1)}$$

Take the horizontal line AB as the x-axis and the vertical line AY as the y-axis. Let the co-ordinates of the vertex C be (h, k). Then the point (h, k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore k = h\tan\alpha - \frac{1}{2}g\frac{h^2}{u^2\cos^2\alpha}$$

$$= h\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$$= h\tan\alpha\left[1 - \frac{h}{R}\right] \quad (\text{by } 1)$$

$$\therefore \frac{k}{h} = \tan\alpha\left(\frac{R-h}{R}\right)$$

$$\Rightarrow \tan A = \tan\alpha\left(\frac{R-h}{R}\right) \quad \left(\because \text{from } \Delta CAM,\ \tan A = \frac{k}{h}\right)$$

$$\Rightarrow \tan\alpha = \left(\frac{R}{R-h}\right)\tan A$$

$$= \left(\frac{R-h+h}{R-h}\right)\tan A$$

$$= \left(1 + \frac{h}{R-h}\right)\tan A$$

$$= \tan A + \frac{h}{R-h}\tan A.$$

$$= \tan A + \frac{h}{R-h}\cdot\frac{k}{h} \quad \left(\because \tan A = \frac{k}{h}\right)$$

$$= \tan A + \frac{k}{R-h}$$

But from the ΔCMB, $\tan B = k/R-h$

$$\therefore \boxed{\tan\alpha = \tan A + \tan B}$$

6(a) (i) Solve $x(1-x^2)dy+(2x^2y-y)dx=ax^3dx$

(ii) Solve $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$.

Sol^n: (i) the given equation can be written as.

$$\frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}\cdot y=\frac{ax^2}{1-x^2}, \text{ which is linear.}$$

Here $P=\dfrac{2x^2-1}{x(1-x^2)}=\dfrac{2x^2-1}{x(1-x)\cdot(1+x)}$

$$=-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}, \text{ by partial fractions}$$

$$Q=\frac{ax^2}{1-x^2}$$

we have

$$\int P\,dx=-\int\left[\frac{1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}\right]dx$$

$$=-\left[\log x+\tfrac{1}{2}\log(x-1)+\tfrac{1}{2}\log(x+1)\right]$$

$$=-\left[\log x+\tfrac{1}{2}\log(x^2-1)\right]$$

$$=-\log\left\{x\sqrt{(x^2-1)}\right\}=\log\left[\frac{1}{x\sqrt{(x^2-1)}}\right]$$

$$\therefore \text{I.F.}=e^{\int P\,dx}=\frac{1}{x\sqrt{x^2-1}}$$

$\therefore$ the solution is

$$y(\text{IF})=\int[Q\cdot(\text{I.F})]\,dx+c$$

i.e. $$y\cdot\frac{1}{x\sqrt{(x^2-1)}}=\int\frac{ax^2}{(1-x^2)}\cdot\frac{1}{x\sqrt{(x^2-1)}}\,dx+c$$

$$=-\tfrac{1}{2}a\int(x^2-1)^{-3/2}(2x)\,dx+c$$

$$=-\tfrac{1}{2}a\,\frac{(x^2-1)^{-1/2}}{-1/2}+c, \text{ by power formula}$$

$$=\frac{a}{\sqrt{(x^2-1)}}+c$$

Hence the required solution is

$$y=ax+cx\sqrt{(x^2-1)}$$

(ii) The given equation can be written as

$$(xy+2x^2y^2)y\,dx+(xy-x^2y^2)x\,dy.$$

Here $Mx-Ny=(xy+2x^2y^2)xy-(xy-x^2y^2)xy$

$$=x^2y^2+2x^3y^3-x^2y^2+x^3y^3=3x^3y^3\neq 0.$$

$\therefore$ the Integrating factor $=\dfrac{1}{Mx-Ny}=\dfrac{1}{3x^3y^3}$

Multiplying the given equation by the I.F $\frac{1}{3x^3y^3}$, we get

$$\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0 \quad \text{①}$$

Now the eqⁿ ① is of the form $M\,dx+N\,dy=0$

$\therefore M=\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)$ and $N=\frac{1}{3}\left(\frac{1}{xy^2}-\frac{1}{y}\right)$

we have

$\dfrac{\partial M}{\partial y}=\dfrac{1}{3x^2}\left(-\dfrac{1}{y^2}\right)$ and $\dfrac{\partial N}{\partial x}=\dfrac{1}{3y^2}\left(-\dfrac{1}{x^2}\right)$, Thus $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

and hence equation ① is exact.

Now $\int M\,dx=\int\frac{1}{3}\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx=\frac{1}{3}\left(-\frac{1}{xy}+2\log x\right)$ —— ②

and $\int N\,dy=\int-\frac{1}{3}\frac{1}{y}\,dy=-\frac{1}{3}\log y.$ —— ③

the solution is given by

② + ③ = c (an arbitrary constant).

$$-\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=\frac{1}{3}\log c$$

$$\Rightarrow -\frac{1}{xy}+\log\frac{x^2}{y}=\log c$$

$$\Rightarrow \log\frac{x^2}{cy}=\frac{1}{xy}$$

$$\Rightarrow \frac{x^2}{cy}=e^{1/xy}$$

$$\Rightarrow x^2=cye^{1/xy}$$

6(b) Solve $ap^2 + py - x = 0$

Sol^n: Solving the given differential equation for x, we get

Differentiating ① w.r.t y and writing $1/p$ for dx/dy,

we get $\frac{1}{p} = p + y\frac{dp}{dy} + 2ap\frac{dp}{dy}$

$$\Rightarrow \frac{1-p^2}{p} = y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

$$\Rightarrow \frac{1-p^2}{p}\frac{dy}{dp} - y = 2ap, \text{ (multiplying bothsides by } dy/dp)$$

$$\Rightarrow \frac{dy}{dp} - \frac{1}{p^2-1}y = -\frac{2ap^2}{p^2-1}$$

which is a linear differential equation

Here the $I.F = e^{\int\{p/(p^2-1)\}dp} = e^{\frac{1}{2}\log(p^2-1)} = (p^2-1)^{\frac{1}{2}}$

$\therefore$ the solution of ② is

$$y(p^2-1)^{1/2} = \int \frac{-2ap^2}{p^2-1}(p^2-1)^{1/2}dp + c$$

$$= -2a\int \frac{(p^2-1)+1}{\sqrt{(p^2-1)}}dp + c$$

$$= -2a\int\left[\sqrt{(p^2-1)} + \frac{1}{\sqrt{(p^2-1)}}\right]dp + c$$

$$= -2a\left[\tfrac{1}{2}p\sqrt{(p^2-1)} - \tfrac{1}{2}\cosh^{-1}p + \cosh^{-1}p\right] + c$$

$$= -ap\sqrt{(p^2-1)} - a\cosh^{-1}p + c$$

$$y = \frac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap$$

Substituting this value of y in ①, we get

$$x = p\left[\frac{c-a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap\right] + ap^2$$

$$\Rightarrow x = \frac{p(c-a\cosh^{-1}p)}{\sqrt{(p^2-1)}} \quad \text{—④}$$

The eqn's ③ & ④ Constitute the parametric equations of the required solution.

6(c) Apply the method of variation of parameters to solve the equation $(x+2)y_2 - (2x+5)y_1 + 2y = (x+1)e^x$.

Sol^n: putting the given eq^n in standard form

$$y_2 + Py_1 + Qy = R, \text{ we get}$$

$$y_2 - \frac{2x+5}{x+2}y_1 + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x \quad \text{--- (1)}$$

Consider $y_2 - \frac{2x+5}{x+2}y_1 + \frac{2}{x+2}y = 0$ --- (2)

Comparing (2) with $y_2 + Py_1 + Qy = R$, we have

$$P = -(2x+5)/(x+2), \quad Q = 2/(x+2) \text{ and } R = 0$$

Here $2^2 + 2P + Q = 4 - \frac{2(2x+5)}{x+2} + \frac{2}{x+2} = 0$

Hence $u = e^{2x}$ is an integral of (2), Let the Complete solution of (1) be $y = uv$. then (2) reduces to

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx} + \left[-\frac{2x+5}{x+2} + \frac{1}{e^{2x}}\cdot 2e^{2x}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{2x+3}{2x+2}\frac{dv}{dx} = 0 \quad \text{--- (3)}.$$

Putting $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$, (3) becomes

$$\frac{dq}{dx} + \left[2 - \frac{1}{x+2}\right]q = 0.$$

$$\Rightarrow \frac{dq}{q} + \left[2 - \frac{1}{x+2}\right]dx = 0$$

Integrating $\log q - \log a' - \log(x+2) = -2x$

$\Rightarrow q = a'(x+2)e^{-2x}$ (or) $\frac{dv}{dx} = a'(x+2)e^{-2x}$

Integrating by Chain rule of integration by parts, we have

$$\Rightarrow v = a'\left[(x+2)\left(-\tfrac{1}{2}e^{-2x}\right) - (1)\left(\tfrac{1}{4}e^{-2x}\right)\right] + b$$

$$\Rightarrow v = -(a'/4)e^{-2x}(2x+4+1)+b = a(2x+5)+b$$
where $a = -a'/4$

Hence the solution of ② is

$$y = uv = e^{2x}\left[a(2x+5)e^{-2x}+b\right] = a(2x+5)+be^{2x}$$

Thus $a(2x+5)+be^{2x}$ is C.F of ①, a & b being arbitrary constants.

Let $y = A(2x+5)+Be^{2x}$ —— ④

be complete solution of ①, Then A and B are functions of x which are so chosen that ① will be satisfied.

Differentiating ④, we get

$$y_1 = A_1(2x+5)+2A+B_1e^{2x}+2Be^{2x} \quad \text{—— ⑤}$$

Choose A and B such that $A_1(2x+5)+B_1e^{2x}=0$ —— ⑥

then ④ reduces to $y_1 = 2A+2Be^{2x}$ —— ⑦

Differentiating ⑥, $y_2 = 2A_1+2B_1e^{2x}+4Be^{2x}$ —— ⑧

using ④, ⑦ and ⑧, ① reduces to

$$2A_1+2B_1e^{2x} = \left[(x+1)/(x+2)\right]e^{x} \quad \text{—— ⑨}$$

Multiplying ⑥ by 2 & substracting it from ⑨, we get

$$A_1(-4x-8) = \frac{x+1}{x+2}e^{x} \Rightarrow A_1 = \frac{dA}{dx} = -\frac{(x+1)}{4(x+2)^2}e^{x} \quad \text{—— ⑩}$$

Integrating, $A = -\frac{1}{4}\int \frac{x+1}{(x+2)^2}e^{x}\,dx + C_1$

$$\Rightarrow A = C_1 - \frac{1}{4}\int \frac{(x+2)-1}{(x+2)^2}e^{x}\,dx = C_1 - \frac{1}{4}\int e^{x}\left[(x+2)^{-1}-(x+2)^{-2}\right]dx$$

$$\Rightarrow A = C_1 - \frac{1}{4}e^{x}(x+2)^{-1} \quad \left[\because \int e^{x}[f(x)+f'(x)]dx = e^{x}f(x)\right] \quad \text{—— ⑪}$$

using ⑩, ⑥ gives $B_1 = \frac{dB}{dx} = \frac{(2x+5)(x+1)}{4(x+2)^2}e^{-x}$

$$\therefore \frac{dB}{dx} = \frac{(2x^2+7x+5)e^{-x}}{4(x+2)^2} = \frac{2(x+2)^2-(x+3)}{4(x+2)^2}e^{-x}$$

$\Rightarrow \frac{dB}{dx} = \frac{1}{2}e^{-x} - \frac{(x+2)+1}{4(x+2)^2}e^{-x} = \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}\left[-(x+2)^{-1} - (x+2)^{-2}\right]$

Integrating $B = c_2 - \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}(x+2)^{-1}$ —— (12)

[using $\int e^{ax}[af(x) + f'(x)]dx = e^{ax}f(x)$ for $a=-1$]

Using (11) and (12) in (4), the required solution is

$y = \left[c_1 - \frac{1}{4}e^{x}(x+2)^{-1}\right](2x+5) + \left[c_2 - \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}(x+2)^{-1}\right]e^{2x}$

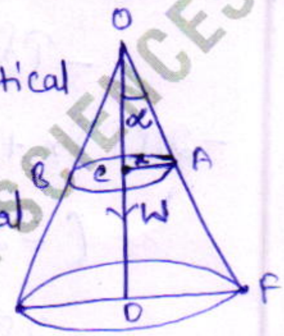
$\Rightarrow y = c_1(2x+5) + c_2 e^{2x} + \frac{1}{4}e^{x}\left[\frac{1}{x+2} - 2 - \frac{2x+5}{x+2}\right]$

$\Rightarrow y = c_1(2x+5) + c_2e^{2x} - e^{x}$

(6d) By using Laplace transformation solve $(D^2+m^2)x = a\sin nt$, $t>0$, when x, Dx equal to x_0 and x_1, when $t=0$, $n \neq m$.

Ans: $x_0\cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}\left(\sin nt - \frac{n}{m}\sin mt\right)$

7(a) A heavy elastic string, whose Natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semivertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is $a\left(1+\frac{W}{2\lambda\pi}\cot\alpha\right)$.



Sol^n:- OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical. A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x. The weight W of the string acts at its centre of gravity C. Let P be the tension in this string.

Give the string a small displacement in which x changes to $x+\delta x$. The point O remains fixed, the point C is slightly displaced, $\angle\alpha$ is fixed and the length of the string slightly changes.

We have the length of the string AB in the form of a circle of radius $x = 2\pi x$ and so the work done by the tension P of this string is $-P\delta(2\pi x)$

Also the depth of the point of application C of the weight W below the fixed point O

$= OC = AC\cot\alpha = x\cot\alpha$.

and so the work done by the weight W during this small displacement.

$$= W\delta(x\cot\alpha).$$

Since the reactions at the various points of contact do no work, we have, by the principle of virtual work.

$$-T\delta(2\pi x) + W\delta(x\cot\alpha) = 0$$

or $\quad -2\pi T\,\delta x + W\cot\alpha\,\delta x = 0$ (or)

$$(-2\pi T + W\cot\alpha)\,\delta x = 0$$

$$-2\pi T + W\cot\alpha = 0 \quad [\because \delta x \neq 0]$$

$$T = (W\cot\alpha)/2\pi.$$

By Hooke's law the tension T in the elastic string AB is given by

$$T = \lambda\,\frac{2\pi x - 2\pi a}{2\pi a} = \lambda\,\frac{x-a}{a}.$$

Equating the two values of T, we get

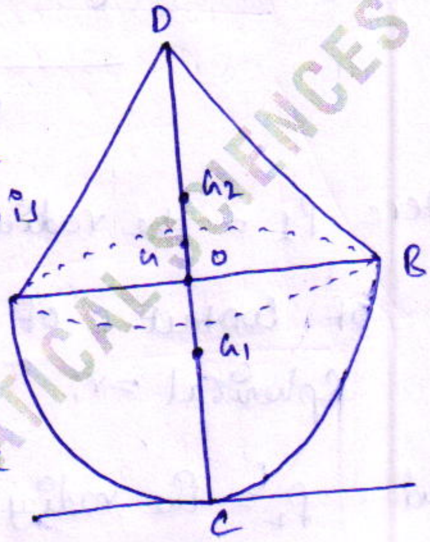
$$\frac{W\cot\alpha}{2\pi} = \lambda\,\frac{x-a}{a}$$

$$x - a = \frac{a}{2\pi\lambda}\,W\cot\alpha$$

$$x = a\left(1 + \frac{W\cot\alpha}{2\pi\lambda}\right).$$

which gives the radius of the string in equilibrium.

7(b) → A body, consisting of a cone and a hemisphere on the same base rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable is $\sqrt{3}$ times the radius of the hemisphere.



Sol^n AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium.

The hemisphere touches the table at C.

Let H be the height OD of the cone and r be the radius OA or OC of the hemisphere. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then

$$OG_1 = 3r/8 \text{ and } OG_2 = H/4$$

If h be the height of the centre of gravity (G) of the combined body composed of the hemisphere and the cone above the point of contact C, then using the formula $x = \dfrac{W_1 x_1 + W_2 x_2}{W_1 + W_2}$,

we have $h = \dfrac{\frac{1}{3}\pi r^2 H \cdot CG_2 + \frac{2}{3}\pi r^3 \cdot CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$

$$= \frac{\frac{1}{3}\pi r^2 H\left(r+\frac{1}{4}H\right)+\frac{2}{3}\pi r^3\cdot\frac{5}{8}r}{\frac{1}{3}\pi r^2 H+\frac{2}{3}\pi r^3}$$

$$= \frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of curvature at the point of contact c of the upper body which is spherical $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$

$\therefore$ the equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2}$ i.e $\frac{1}{h} > \frac{1}{r}+\frac{1}{\infty}$ i.e $\frac{1}{h} > \frac{1}{r}$

i.e $\frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r} < r$ i.e $Hr+\frac{1}{4}H^2+\frac{5}{4}r^2 < Hr+2r^2$

i.e $\frac{1}{4}H^2 < \frac{3}{4}r^2$ i.e $H^2 < 3r^2$ i.e $H < r\sqrt{3}$

Hence the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

7(c) → A particle moves with a central acceleration $\mu\left(r + a^4/r^3\right)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. prove that it describes the curve $r^2(2 + \cos\sqrt{3}\theta) = 3a^2$.

Sol^n: Here, the central acceleration

$$P = \mu\left(r + \frac{a^4}{r^3}\right) = \mu\left\{\frac{1}{u} + a^4u^3\right\}$$

where $u = 1/r$

∴ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left[\frac{1}{u} + a^4u^3\right]$$

(or) $$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\left[\frac{1}{u^3} + a^4u\right].$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating w.r.t to θ we have

$$h^2\left[2\cdot\frac{u^2}{2} + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(\frac{-1}{2u^2} + \frac{a^4u^2}{2}\right) + A$$

(or) $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + A$$ → eq^n (1)

where A is constant.

Now initially the particle has been projected from an apse (say the point A) at a distance 'a' with velocity $2\sqrt{\mu}a$. Therefore, when $r = a$ i.e

$u = 1/a$, $\frac{du}{d\theta} = 0$ (at an apse) and $v = 2\sqrt{\mu a}$.

$\therefore$ from (1), we have

$$\underset{(i)}{4\mu a^2} = h^2 \underset{(ii)}{\left[\frac{1}{a^2}\right]} = \mu^2 \underset{(iii)}{\left(-a^2 + a^4 \cdot \frac{1}{a^2}\right)} + A$$

from (i) and (ii), we have $h^2 = 4\mu a^4$ and.
from (i) and (iii), we have

$$4\mu a^2 = 0 + A \quad \text{i.e,} \quad A = 4\mu a^2.$$

Substituting the values of h^2 and A in (1), we have

$$4\mu a^4 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4 u^2\right) + 4\mu a^2$$

(or) $$4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$$

(or) $$4a^4u^2\left(\frac{du}{d\theta}\right)^2 = \left(-1 - 3a^4u^4 + 4a^2u^2\right) \rightarrow (2)$$

(or) $$2a^2u\frac{du}{d\theta} = \sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}$$ [taking square root]

(or) $$d\theta = \frac{2a^2u\,du}{\sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\sqrt{\left[-\frac{1}{3} - \left(a^4u^4 - \frac{4}{3}a^2u^2\right)\right]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\cdot\sqrt{\left[-\frac{1}{3}-(a^2u^2-\frac{2}{3})^2+\frac{4}{9}\right]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\sqrt{\left[(\frac{1}{3})^2-(a^2u^2-\frac{2}{3})^2\right]}}$$

(or) $$\sqrt{3}\,d\theta = \frac{2a^2u\,du}{\sqrt{(\frac{1}{3})^2-(a^2u^2-\frac{2}{3})^2}}$$

Substituting $a^2u^2-\frac{2}{3}=z$, so that $2a^2u\,du=dz$,

we have $$\sqrt{3}\,d\theta = \frac{dz}{\sqrt{\left[(\frac{1}{3})^2-z^2\right]}}$$

Integrating, $\sqrt{3}\theta + B = \sin^{-1}(3z)$ where B is a constant

(or) $\sqrt{3}\theta + B = \sin^{-1}(3a^2u^2-2)$. $\longrightarrow$ (3)

Now take the apse-line OA as the initial line.

Then initially

$r=a$, $u=1/a$ and $\theta=0$

$\therefore$ from (3), $0+B=\sin^{-1}(1)$ (or) $B=\frac{1}{2}\pi$.

putting $B=\frac{1}{2}\pi$ in (3), we have

$$\sqrt{3}\theta+\frac{1}{2}\pi = \sin^{-1}(3a^2u^2-2)$$

(or) $3a^2u^2 - 2 = \sin\left(\frac{1}{2}\pi + \sqrt{3}\theta\right) = \cos(\sqrt{3}\theta)$

(or) $\frac{3a^2}{r^2} - 2 = \cos(\sqrt{3}\theta)$ (or)

$$3a^2 - 2r^2 = r^2\cos(\sqrt{3}\theta)$$

$$\therefore 3a^2 = r^2\left[2 + \cos(\sqrt{3}\theta)\right],$$

which is the equation of the required curve.

8(a) (i) If F is a conservative field, Prove that curl $F = \nabla \times F = 0$ (i.e. F is irrotational)

(ii) Conversely, if $\nabla \times F = 0$, (i.e. F is irrotational). Prove that F is Conservative.

sol^n: (i) If F is a conservative field

$F = \nabla\phi$, where ϕ is single-valued and has continuous partial derivatives.

Now Curl $F = \nabla \times (\nabla\phi) = \nabla \times \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\hat{i} + \left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\hat{j} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\hat{k}$$

$$= \left(\frac{\partial^2\phi}{\partial y \partial z} - \frac{\partial^2\phi}{\partial z \partial y}\right)\hat{i} + \left(\frac{\partial^2\phi}{\partial z \partial x} - \frac{\partial^2\phi}{\partial x \partial z}\right)j + \left(\frac{\partial^2\phi}{\partial x \partial y} - \frac{\partial^2\phi}{\partial y \partial x}\right)k$$

$$= 0$$

Provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

(ii) If $\nabla \times F = 0$, then $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = 0$ and thus

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

we must prove that $F = \nabla\phi$ follows as a consequence of this.

The workdone in moving a particle from (x_1, y_1, z_1) to (x, y, z) in the force field F is

$$\int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

where C is a path joining (x_1, y_1, z_1) and (x, y, z). Let us choose as a particular path the straight line segments from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z_1) to (x, y, z) and call $\phi(x, y, z)$ the workdone along this particular path.

Then

$$\phi(x,y,z) = \int_{x_1}^{x} F_1(x, y_1, z_1)\,dx + \int_{y_1}^{y} F_2(x, y, z_1)\,dy + \int_{z_1}^{z} F_3(x,y,z)\,dz$$

It follows that $\frac{\partial\phi}{\partial z} = F_3(x,y,z)$

$$\frac{\partial\phi}{\partial y} = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + F_2(x,y,z)\Big|_{z_1}^{z} = F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1)$$

$$= \underline{F_2(x,y,z)}.$$

$$\frac{\partial\phi}{\partial x} = F_1(x, y_1, z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1)\Big|_{y_1}^{y} + F_1(x,y,z)\Big|_{z_1}^{z}$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1) - F(x,y_1,z_1) + F_1(x,y,z) - F(x,y,z_1)$$

$$= F_1(x,y,z)$$

Then $\overline{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \nabla\phi$

Thus a necessary and sufficient condition that a field F be conservative is that $\underline{\text{curl}\,F = \nabla\times F = 0}$.

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8(b) (i) Find the angle of intersection at $(4,-3,2)$ of spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$.

Sol'n: Let $f_1 = x^2+y^2+z^2-29$ and $f_2 = x^2+y^2+z^2+4x-6y-8z-47$.

Then $\text{grad } f_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

and $\text{grad } f_2 = (2x+4)i+(2y-6)j+(2z-8)k$

Let $n_1 = \text{grad } f_1$ at the point $(4,-3,2)$

and $n_2 = \text{grad } f_2$ at the point $(4,-3,2)$. then

$$n_1 = 8i-6j+4k = 2(4i-3j+2k)$$

$$\& \; n_2 = 12i-12j-4k = 4(3i-3j-k)$$

The vectors n_1 and n_2 are along normals to the two spheres at the point $(4,-3,2)$ and the angle θ between these two vectors is the angle of intersection of the two spheres at the point $(4,-3,2)$.

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

$$= \frac{8(12+9-2)}{2\sqrt{16+9+4}\cdot 4\sqrt{9+9+1}}$$

$$= \frac{19}{\sqrt{29}\cdot\sqrt{19}}$$

$$\theta = \cos^{-1}\sqrt{\frac{19}{29}}$$

8(b)(iii) In what direction from the point $(1,3,2)$ is the directional derivative of $\phi = 2xz - y^2$ a maximum? What is the magnitude of this maximum.

Sol^n: We know that the directional derivative of $\phi = 2xz - y^2$ at the point (x, y, z) is maximum in the direction of the normal to the surface ϕ = constant

i.e, in the direction of the vector grad ϕ.

Now $\text{grad}\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$

(2) $= 2zi + (-2y)j + 2xk$

$\text{grad}\phi\Big|_{(1,3,2)} = 2(2)i - 2(3)j + 2(1)k$

$= 4i - 6j + 2k$

Hence the directional derivative of ϕ at the point $(1,3,2)$ will be maximum in the direction of the vector $4i - 6j + 2k$.

Also the magnitude of this maximum directional derivative

$= $ modulus of grad ϕ at $(1,3,2)$

$= |4i - 6j + 2k|$

$= \sqrt{16+36+4}$

$= \sqrt{56}$

$= 2\sqrt{14}$

8(c) Prove that $\text{curl}[r^n(a\times r)] = (n+2)r^n a - nr^{n-2}(r\cdot a)r$, where a is a constant vector.

Sol'n: we know that

$$\text{curl}(\phi\vec{A}) = (\text{grad}\,\phi)\vec{A} + \phi\,\text{curl}\vec{A}$$

putting $\phi = r^n$ and $\vec{A} = \vec{a}\times\vec{r}$

$$\therefore \text{curl}[r^n(\vec{a}\times\vec{r})] = \nabla r^n \times (\vec{a}\times\vec{r}) + r^n\,\text{curl}(\vec{a}\times\vec{r}) \quad \text{--- (1)}$$

Now $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\left(\frac{1}{r}\right)\vec{r} = nr^{n-2}\vec{r}$.

$$\therefore \nabla r^n \times (\vec{a}\times\vec{r}) = (nr^{n-2}\vec{r})\times(\vec{a}\times\vec{r})$$

$$= nr^{n-2}\{\vec{r}\times(\vec{a}\times\vec{r})\}$$

$$= nr^{n-2}[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}]$$

$$= nr^{n-2}[r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}]$$

$$= nr^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r} \quad \text{--- (2)}.$$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

where the scalars a_1, a_2, a_3 are all constants.

then $$\vec{a}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i(a_2 z - a_3 y) + j(a_3 x - a_1 z) + k(a_1 y - a_2 x)$$

$$\therefore \text{curl}(\vec{a}\times\vec{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1+a_1)\hat{i} + (a_2+a_2)\hat{j} + (a_3+a_3)\hat{k} = 2(a_1 i + a_2 j + a_3 \hat{k})$$
$$= 2\vec{a} \quad \text{--- (3)}$$

Substituting (2) & (3) in (1), we get

$$\text{curl}[r^n(\vec{a}\times\vec{r})] = nr^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r} + r^n(2\vec{a})$$
$$= (n+2)r^n - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

8(d) By converting into a line integral evaluate $\iint_S (\nabla\times F)\cdot n\,ds$ where $F=(x^2+y-4)i+3xyj+(2xy+z^2)k$ and S is the surface of the paraboloid $z=4-(x^2+y^2)$ above the xy-plane.

Sol^n: The xy-plane cuts the surface S of the paraboloid $z=4-(x^2+y^2)$ in the circle C whose equations are $x^2+y^2=4$, $z=0$. Thus the boundary of the surface S is the circle C and the surface S lies above the circle C.

Let the parametric equations of the curve C be $x=2\cos t$, $y=2\sin t$, $z=0$, $0\le t<2\pi$.

By Stokes theorem, we have

$$\iint_S (\nabla\times F)\cdot n\,ds=\oint_C F\cdot dr$$

$$=\int_C\left[(x^2+y-4)\hat{i}+3xy\hat{j}+(2xy+z^2)\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$=\int_C\left[(x^2+y-4)dx+3xy\,dy+(2xz+z^2)dz\right]$$

$$=\int_C\left[(x^2+y-4)dx+3xy\,dy\right],\ \text{since on } C,\ z=0 \text{ and } dz=0.$$

$$=\int_{t=0}^{2\pi}\left[(x^2+y-4)\frac{dx}{dt}+3xy\frac{dy}{dt}\right]dt.$$

$$=\int_{t=0}^{2\pi}\left[(4\cos^2t+2\sin t-4)(-2\sin t)+3.2\cos t.2\sin t.2\cos t\right]dt$$

$$=-8\int_0^{2\pi}\cos^2t\sin t\,dt-4\int_0^{2\pi}\sin^2t\,dt+8\int_0^{2\pi}\sin t\,dt+24\int_0^{2\pi}\cos^2t\sin t\,dt$$

$$=8\left[\frac{\cos^3t}{3}\right]_0^{2\pi}-4.2.2\int_0^{\pi/2}\sin^2t\,dt+8\left[-\cos t\right]_0^{2\pi}-24\left[\frac{\cos^3t}{3}\right]_0^{2\pi}$$

$$=8.0-16.\tfrac{1}{2}.\tfrac{\pi}{2}+8.0-\tfrac{24}{3}.0=-4\pi$$