Mains Test Series - 2018

Test - 13, Paper - I

Answer Key

1(b) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$. Find the minimal polynomial for T.

Sol'n: The characteristic equation of T is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)\left[(1-\lambda)(4-\lambda)+2\right]=0$

$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-3)=0$

$\Rightarrow \lambda = 2,2,3$

Hence the characteristic values of T are 2, 2, 3.

The characteristic vector corresponding to $\lambda=2$ is given by $(A-2I)X=0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$\Rightarrow x_2 = 0,\ x_3 = 0$ and x_1 can be given any value.

we take $x_1 = 1,\ x_2 = 0,\ x_3 = 0$.

clearly, there is only one LI vector corresponding to the characteristic value 2.

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus the geometric multiplicity of the eigen value 2 is one while its algebraic multiplicity is 2. Since the geometric multiplicity of this eigen value is not equal to its algebraic multiplicity therefore A is not similar to a diagonal matrix.

i.e. T is not diagonalizable.

we know that the minimal polynomial for T divides its characteristic polynomial. Thus the possible minimal polynomials for T can be either.

$$P(\lambda) = (3-\lambda)(\lambda-2) \text{ (or) } (\lambda-2)^2(3-\lambda)$$

Let us take $P(\lambda) = (3-\lambda)(\lambda-2)$

we have $P(A) = (3I - A)(A - 2I) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

$$P(A) = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

This shows that $P(\lambda) = (3-\lambda)(\lambda-2)$ is not the minimal polynomial for T.

Hence the minimal polynomial for T is.

$$P(\lambda) = (3-\lambda)(\lambda-2)^2$$

which is same as the characteristic polynomial of T.

1(C). If $V = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$, find the value $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$ when $x=0,\ y=1,\ z=2$.

Sol'n: Given

$$V = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow e^v = \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow \sin^{-1}e^v = \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}} \ (=u) \text{ say} \quad \text{——— (1)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu \quad \text{——— (2).}$$

where $n = 1-\frac{2}{3} = \frac{1}{3}$

But from (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-e^{2v}}}e^v\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-e^{2v}}}e^v\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial z} = \frac{1}{\sqrt{1-e^{2v}}}e^v\frac{\partial v}{\partial z}$$

$\therefore$ from (2)

$$\frac{e^v}{\sqrt{1-e^{2v}}}\left[x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right] = \frac{1}{3}(\sin^{-1}e^v)$$

$$\Rightarrow x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z} = \frac{1}{3}(\sin^{-1}e^v)\frac{\sqrt{1-e^{2v}}}{e^v} \quad \text{——— (3)}$$

when $(x,y,z) = (0,1,2)$

$$v = \log_e \sin\left\{\frac{\pi(1)^{1/2}}{2(4+4)^{1/3}}\right\}$$

$$= \log_e \sin\left[\frac{\pi}{2(8)^{1/3}}\right]$$

$$= \log_e \sin\left(\pi/4\right)$$

$$v = \log_e\left(1/\sqrt{2}\right)$$

$$\Rightarrow e^v = 1/\sqrt{2}$$

and $u = \sin^{-1} e^v = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$

$\therefore$ from (3)

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3}\left(\pi/4\right)\frac{\sqrt{1-1/2}}{\frac{1}{\sqrt{2}}}$$

$$= \pi/12\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right)$$

$$= \pi/12$$

1(d) Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Sol: Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx$ —— (1)

then $I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x+\sin x}\,dx$ —— (2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}\,dx = \int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{dx}{\sin\left(\frac{\pi}{4}+x\right)}$$

$$= \frac{2}{\sqrt{2}}\int_0^{\pi/4} \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\left[\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left\{\operatorname{cosec}\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right\}\right]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{\operatorname{cosec}\frac{\pi}{2}-\cot\frac{\pi}{2}}{\operatorname{cosec}\frac{\pi}{4}-\cot\frac{\pi}{4}}\right)$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{1-0}{\sqrt{2}-1}\right)$$

$$I = \frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

$$\therefore \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log(1+\sqrt{2})$$

1(e) The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation of the plane in its new position is $lx+my\pm z\sqrt{l^2+m^2}\tan\alpha=0$

Soln:

The equation of the plane through the line of intersection of the planes

$lx+my=0$ — (1) and $z=0$ — (2)

is $lx+my+\lambda z=0$ — (3)

According to the given condition, plane (3) makes an angle α with the plane (1)

$$\therefore \cos\alpha=\frac{l^2+m^2}{\sqrt{l^2+m^2}\sqrt{(l^2+m^2+\lambda^2)}}$$

$$\Rightarrow \cos^2\alpha=\frac{l^2+m^2}{l^2+m^2+\lambda^2}$$

$$\Rightarrow \lambda=\pm\sqrt{l^2+m^2}\tan\alpha$$

Hence the required plane is

$$(lx+my)\pm z\sqrt{(l^2+m^2)}\tan\alpha=0$$

2(a)(i) verify that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix? State the theorem that applies here.

Sol'n: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}_{3\times3}$ be a given matrix.

Then its characteristic equation is

$$|A-\lambda I|=0; \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)\left[(1-\lambda)(-1-\lambda)-4\right]=0$$

$$\Rightarrow (1+\lambda)\left[1-\lambda^2+4\right]=0$$

$$\Rightarrow (1+\lambda)(5-\lambda^2)=0$$

$$\Rightarrow 5-\lambda^2+5\lambda-\lambda^3=0$$

$$\Rightarrow \lambda^3+\lambda^2-5\lambda-5=0 \quad \text{——— (1)}.$$

Now we have

$$A^2 = A\cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$A^3 = A^2\cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Let us consider

$$A^3+A^2-5A-5I=\begin{bmatrix}5&10&0\\10&-5&0\\0&0&-1\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&1\end{bmatrix}-5\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}-5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\therefore$ The matrix A satisfies the characteristic equation

$$\lambda^3+\lambda^2-5\lambda-5=0$$

It is true for every square matrix.
According to "Cayley - Hamilton" theorem, Every square matrix satisfies its own characteristic equation.

2(a)(ii) $\to$ Let $V=R^4(R)$ and $W=\{(a,b,c,d)\in R^4 : a=b+c,\ c=b+d\}$.
Find a basis and the dimension of W.

Sol'n: Let $\alpha_1=(1,1,0,-1)$ and $\alpha_2=(0,1,-1,-2)$

then $\alpha_1,\alpha_2\in W$ and are L.I

Since $x\alpha_1+y\alpha_2=0$ where $x,y\in\mathbb{R}$

$\Rightarrow x(1,1,0,-1)+y(0,1,-1,-2)=0$

$\Rightarrow (x,\ x+y,\ -y,\ -x-2y)=(0,0,0,0)$

$\Rightarrow x=0=y$.

To show W is spanned by α_1 & α_2

Let $(a,b,c,d)\in W$ then $a=b+c$ & $c=b+d$ — ①

Since $a(1,1,0,-1)-c(0,1,-1,-2)$
$=(a,\ a-c,\ c,\ -a+2c)$
$=(a,b,c,d)$ by ①

W is spanned by $\{\alpha_1,\alpha_2\}$

$\therefore \{\alpha_1,\alpha_2\}$ is a basis of W and $\dim W=2$.

2(c) prove that the enveloping cylinder of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ whose generators are parallel to the line $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c}$ meet the plane $z=0$ in circles.

Sol: The given ellipsoid is $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— (1)

and the given lines are

$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c} \quad \text{—— (2)}$$

The equation of enveloping cylinder is

$$SS_1 = T^2$$

i.e., $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left[\frac{1}{a^2}(0) + \frac{1}{b^2}(\pm\sqrt{a^2-b^2})^2 + \frac{1}{c^2}(c)^2\right]$

$$= \left[\frac{1}{a^2}(0)(x) + \frac{1}{b^2}(\pm\sqrt{a^2-b^2})y + \frac{1}{c^2}(c)z\right]^2$$

Here $S_1 = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}$, $l=0, m=\pm\sqrt{a^2-b^2}, c=c$

$$\Rightarrow \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} - 1\right)\left(\frac{a^2-b^2}{b^2} + 1\right) = \left(\pm\frac{\sqrt{a^2-b^2}}{b^2}y + \frac{z}{c}\right)^2$$

This meets the plane $z=0$ —— (1)

$$(1) \equiv \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{a^2-b^2}{b^2} + 1\right) = \left(\pm\frac{\sqrt{a^2-b^2}}{b^2}y + 0\right)^2$$

$$\Rightarrow \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\frac{a^2}{b^2} = \frac{(a^2-b^2)y^2}{b^4}$$

$$\Rightarrow x^2 + \frac{a^2y^2}{b^2} - a^2 = \frac{a^2}{b^2}y^2 - y^2$$

$$\Rightarrow x^2 + y^2 = a^2$$

which is a circle.

3(a) Let T be the linear operator on R^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what conditions $a, b,$ and c is T diagonalizable?

Sol^n:- Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be the given linear operator represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Now we have $|A - \lambda I| = 0$

3(a)

$$\Rightarrow \begin{vmatrix} 0-\lambda & 0 & 0 & 0 \\ a & 0-\lambda & 0 & 0 \\ 0 & b & 0-\lambda & 0 \\ 0 & 0 & c & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0-\lambda)^4 = 0$$

$$\Rightarrow \lambda = 0, 0, 0, 0$$

$\therefore$ 0 is the only characteristic value of A.

Let W_0 be the corresponding eigenspace

then $W_0 = \{v \in V \,/\, T(v) = 0\} = \ker T$.

T is diagonalizable $\Leftrightarrow \dim W_0 = 4$.

$\Leftrightarrow \dim(\ker T) = 4$

$\Leftrightarrow$ nullity of $T = 4$.

$\therefore$ we know that

rank$T$ + nullity T = $\dim(\mathbb{R}^4) = 4$

$\therefore$ It follows that rank $T = 0$

i.e, dimension of range space $= 0$

i.e, Range space of $T = \{0\}$

$\Rightarrow T = 0$

$\Rightarrow a = b = c = 0$

Hence T is diagonalizable if $a = b = c = 0$

 7

3(b) → Obtain the volume bounded by the elliptic paraboloids given by the equations $z = x^2+9y^2$ and $z = 18-x^2-9y^2$.

Sol: The elliptic paraboloids intersect on the elliptic cylinder

$$x^2+9y^2 = 18-x^2-9y^2$$

$$\Rightarrow x^2+9y^2 = 9.$$

The volume projects into the region R (in the xy-plane) that is enclosed by the elliptic $x^2+9y^2=9$.

In the double integral w.r.t. x and y over R, if we integrate w.r.t x, holding y fixed, x varies from $-\sqrt{9-9y^2}$ to $\sqrt{9-9y^2}$. Then y varies from -1 to 1.

Thus we have

$$V = \int_{y=-1}^{1} \int_{x=-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} \int_{z=x^2+9y^2}^{18-x^2-9y^2} dz\,dx\,dy$$

$$= \int_{-1}^{1} \int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} \left(18-2x^2-18y^2\right) dx\,dy$$

$$= 2\int_{-1}^{1} \int_{0}^{\sqrt{9-9y^2}} \left[18(1-y^2)-2x^2\right] dx\,dy$$

$$= 2\int_{-1}^{1}\left(18(1-y^2)x - \frac{2}{3}x^3\right)_0^{3\sqrt{1-y^2}} dy$$

$$= 2\int_{-1}^{1} 36(1-y^2)^{3/2}\, dy$$

$$= 72\times 2\int_0^1 (1-y^2)^{3/2} dy$$

put $y = \sin\theta$
$dy = \cos\theta\, d\theta$

$$= 72\times 2\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 144\times\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= 27\pi$$

3(c) Prove that the shortest distance between generators of the same system drawn at the ends of diameters of the principal elliptic section of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ lie on the surfaces whose equations are $\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}$.

Sol Let any point on the elliptic section $P(a\cos\alpha, b\sin\alpha, 0)$

equation through generator

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z-0}{c} \quad \text{--- (i)}$$

Other extremity of diameter is obtained by putting $\alpha+\pi$ for α in P. $\therefore Q(-a\cos\alpha, -b\sin\alpha, 0)$

$\therefore$ generator passing through Q

$$\frac{x+a\cos\alpha}{-a\sin\alpha}=\frac{y+b\sin\alpha}{b\cos\alpha}=\frac{z-0}{c} \quad \text{--- (ii)}$$

l, m, n be the direction cosines of shortest distance (S.D)

$$la\cos\alpha - mb\cos\alpha + nc = 0$$
$$-al\sin\alpha + mb\cos\alpha + nc = 0$$

$$\frac{l}{2bc\cos\alpha}=\frac{m}{-2ac\sin\alpha}=\frac{n}{0}.$$

$\therefore$ equation of plane through generator (i) and S.D $\begin{vmatrix} x-a\cos\alpha & y-b\sin\alpha & z-0 \\ a\sin\alpha & -b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix}=0$

equation of plane through generator (ii) and S.D

$$\begin{vmatrix} x+a\cos\alpha & y+b\sin\alpha & z-0 \\ -a\sin\alpha & b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix} = 0$$

expanding above determinants.

$$z(a^2\sin^2\alpha+b^2\cos^2\alpha) - c[a\sin\alpha(x-a\cos\alpha) - b\cos\alpha(y-b\sin\alpha)] = 0 \quad \text{---(iii)}$$

$$-z(a^2\sin^2\alpha+b^2\cos^2\alpha) - c[a\sin\alpha(x+a\cos\alpha) - b\cos\alpha(y+b\sin\alpha)] = 0 \quad \text{---(iv)}$$

eliminating α.

(iii)+(iv) $\Rightarrow$ $-2acx\sin\alpha + 2bcy\cos\alpha = 0 \Rightarrow \tan\alpha = (by)/ax$

$$2z(a^2\sin^2\alpha + b^2\cdot\cos^2\alpha) + 2c(a^2-b^2)\sin\alpha\cos\alpha = 0$$

$$2z(a^2\tan^2\alpha + b^2) + 2c(a^2-b^2)\tan\alpha = 0$$

$$2z(a^2(by/ax)^2 + b^2) + 2c(a^2-b^2)(by/ax) = 0$$

$$\frac{b^2z(x^2+y^2)}{x^2} + \frac{c(a^2-b^2)by}{ax} = 0$$

$$\boxed{\frac{cxy}{x^2+y^2} = \frac{-abz}{a^2-b^2}}$$

Similarly for the other system $\dfrac{cxy}{x^2+y^2} = \dfrac{abz}{a^2-b^2}$

$\therefore$ Required S.D locus $\boxed{\dfrac{cxy}{x^2+y^2} = \pm\dfrac{abz}{a^2-b^2}}$

4(a) Find the condition on a, b and c so that the following system in unknowns x, y, and z has a solution?

$$x+2y-3z=a$$
$$2x+6y-11z=b$$
$$x-2y+7z=c.$$

Sol: The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -5 \\ 0 & -4 & +10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c-a \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c+2b-5a \end{bmatrix}$$

The system will have no solution if $c+2b-5a \neq 0$.

Thus the system will have atleast one solution if $c+2b-5a=0$ i.e. $5a=2b+c$.
which is the required condition.

Note: In this case the system will have infinitely many solutions. In other words, the system cannot have a unique solution.

4(c)(ii) Given $w = (x, y)$ with $x = u+v$, $y = u-v$

Show that $\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$.

Sol Since $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$ &

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$ & $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$ — (1)

Again partially differentiating (1) w.r.t v & u respectively, we get

$$\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial y}\right) \text{ \&}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 w}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial v} + \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\cdot\frac{\partial y}{\partial v} \text{ \&}$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial u} - \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial u}$$

$$\Rightarrow \frac{\partial^2 w}{\partial v \partial u} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

4(c)(i) Show that the set $S = \left\{ 1 + \frac{(-1)^n}{2^n} : n \text{ is a positive integer} \right\}$ is bounded. Show that 1 is a limit point of S. Are there any other limit points of S?

Sol'n: The given set S is the sequence $\langle S_n \rangle$

where $S_n = 1 + \frac{(-1)^n}{2^n}$, $n \in N$.

The subsequence $\langle S_{2n-1} \rangle$ of the sequence $\langle S_n \rangle$ is the sequence.

$$\left\langle 1 - \frac{1}{2}, 1 - \frac{1}{2^3}, 1 - \frac{1}{2^5}, \ldots \right\rangle$$

which is monotonic increasing and has each term less than 1.

$\therefore$ $1 - \frac{1}{2} = \frac{1}{2}$ is a lower bound for $\langle S_{2n-1} \rangle$ and 1 is an upper bound for $\langle S_{2n-1} \rangle$.

Again the subsequence $\langle S_{2n} \rangle$ of the sequence.

$\langle s_n \rangle$ is the sequence

$\langle 1+\frac{1}{2^2}, 1+\frac{1}{2^4}, 1+\frac{1}{2^6}, \ldots \rangle$

which is monotonic decreasing and has each term greater than 1.

$\therefore 1+\frac{1}{2^2} = \frac{5}{4}$ is an upper bound for $\langle s_{2n} \rangle$ and 1 is a lower bound for it.

Hence for the sequence $\langle s_n \rangle$, we have

$s_n \leq \frac{5}{4}$ $\forall n \in N$ and $s_n \geq \frac{1}{2}$ $\forall n \in N$

$\therefore$ The set S is bounded, $\frac{5}{4}$ is an upper bound for S and $\frac{1}{2}$ is a lower bound for S.

Infact we have $\sup S = \frac{5}{4}$ and $\inf S = \frac{1}{2}$.

we have $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left\{1+\frac{(-1)^n}{2^n}\right\} = 1$

$\therefore$ the sequence $\langle s_n \rangle$ converges to 1.

Now we shall show that 1 is a limit point of the set S.

Take any positive real number ϵ. Since the sequence $\langle s_n \rangle$ converges to 1.

$\therefore$ for given $\epsilon > 0$, $\exists$ $m > 0$ such that

$|s_n - 1| < \epsilon$ $\forall n \geq m$

Thus for every $\epsilon > 0$, $(1-\epsilon, 1+\epsilon)$ contains infinite terms of the sequence $\langle s_n \rangle$

i.e, infinite distinct points of the set S.

Hence 1 is a limit point of the set S.

Now we shall show that 1 is the only limit point of the set S i.e., if l is any limit point of the set S. we must have $l=1$.

Let $\epsilon>0$ be arbitrary. Since the sequence $\langle s_n \rangle$ converges to 1, therefore for given real number $\frac{\epsilon}{2}>0$.

$\exists$ $p>0$ such that $|s_n-1|<\frac{\epsilon}{2}$ $\forall n\geq p$ — (1)

Since l is a limit point of the S, therefore

$\therefore$ $(l-\epsilon/2, l+\epsilon/2)$ contains infinite distinct points of the set S.

i.e., infinite terms of the sequence $\langle s_n \rangle$.

So there exist a +ve integer $q>p$ such that

$l-\epsilon/2 < s_q < l+\epsilon/2$

$\Rightarrow |s_q-l|<\epsilon/2$ — (2)

putting $n=q$ in (1), we have $|s_q-1|<\epsilon/2$ — (3)

Now $|1-l| = |(s_q-l)+(1-s_q)|$

$\leq |s_q-l|+|1-s_q|$ $(\because |x+y|\leq |x|+|y|)$

$= |s_q-l|+|s_q-1|$ $(\because |x|=|-x|)$

$< \epsilon/2+\epsilon/2$ from (2) & (3)

i.e, $|1-l|<\epsilon$

Since ϵ is arbitrary.

Hence we must have $|1-l|=0$

i.e, $1-l=0$

$\Rightarrow l=1$

Hence 1 is the only limit point of the given set S. i.e. $D(S)=\{1\}$

4(d) → Two perpendicular tangent planes to the paraboloid $x^2/a + y^2/b = 2z$ intersect in a line lying on the plane $x=0$. Prove that the line touches the parabola $x=0$, $y^2=(a+b)(2z+a)$.

Sol'n: Let the line of intersection of the two tangent planes be $my+nz=\lambda,\ x=0$ —— ①

Since this lies on the plane $x=0$ (given)

∴ Equation of the plane through the line ① is

$(my+nz-\lambda)+kx=0 \Rightarrow kx+my+nz=\lambda$ —— ②

If the plane ② touches the paraboloid, then

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{2\lambda n}{c}=0$$

$$\Rightarrow ak^2+bm^2+2\lambda n=0 \quad \text{—— ③}$$

This being a quadratic in k, gives two values of k say k_1, k_2 such that

$$k_1k_2=(bm^2+2\lambda n)/a \quad \text{—— ④}$$

Also from ② the direction ratios of the normals to the two tangent planes whose line of intersection is ② are k_1, m, n and k_2, m, n.

Also as these two tangent planes are ⊥lar, so are their normals and consequently we have

$k_1k_2+m.m+n.n=0$

$\Rightarrow [(bm^2+2\lambda n)/a]+m^2+n^2=0$, from ④

$\Rightarrow (a+b)m^2+an^2+2\lambda n=0$ —— ⑤

Now we are to prove that the line ① touches a parabola, so we are to find the envelope of ① which satisfies the condition ⑤

Eliminating λ between ① and ⑤, the equations of the line of intersection of two tangent planes is

$$(a+b)m^2 + an^2 + 2(my+nz)n = 0, \ x=0$$

$$\Rightarrow (a+b)(m/n)^2 + 2y(m/n) + (a+2z) = 0, \ x=0$$

It is quadratic in (m/n) so its envelope is given by

$$B^2 - 4AC = 0, \ x=0$$

$$\Rightarrow (2y)^2 - 4(a+b)(a+2z) = 0, \ x=0$$

$$\Rightarrow y^2 = (a+b)(a+2z), \ x=0$$ Hence proved.

5(a) Find the orthogonal trajectories of family of curves $r^2 = a^2 \cos 2\theta$.

Sol'n: Given equation of family of curves is $r^2 = a^2 \cos 2\theta$ —— (1) with 'a' as parameter.

From (1),

$2\log r = \log a^2 + \log \cos 2\theta$ —— (2)

Differentiating (2) w.r.t. θ, we get

$\frac{2}{r}\frac{dr}{d\theta} = \frac{1}{\cos 2\theta}(-2)\sin 2\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta$ —— (3)

which is the differential equation of the given family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), the differential equation of the required orthogonal trajectories is

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan 2\theta$

$\Rightarrow r\frac{d\theta}{dr} = \tan 2\theta$

$\Rightarrow \frac{dr}{r} = \cot 2\theta \, d\theta$

Integrating,

$\log r = \frac{1}{2}\log \sin 2\theta + \log c$

$\Rightarrow 2\log r = \log \sin 2\theta + 2\log c$

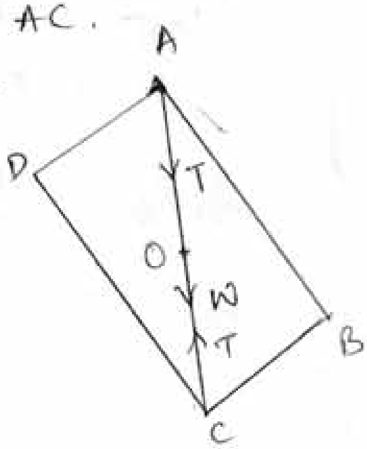
$\Rightarrow \boxed{r^2 = c^2 \sin 2\theta}$

which is the required equation of orthogonal trajectories, c being parameter.

5(b) Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.

Sol^n: ABCD is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC.

Let T be the tension in the string AC.



The total weight W of all the four rods AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC = 2x$

Give the system a small displacement in which x changes to $x + \delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes. we have $AO = x$.

By the principle of virtual work, we have

$$-T\delta(AC) + W\delta(AO) = 0$$

$$\Rightarrow -T\delta(2x) + W\delta(x) = 0$$

$$\Rightarrow -2T\delta x + W\delta x = 0$$

$$\Rightarrow [-2T + W]\delta x = 0 \qquad [\because \delta x \neq 0]$$

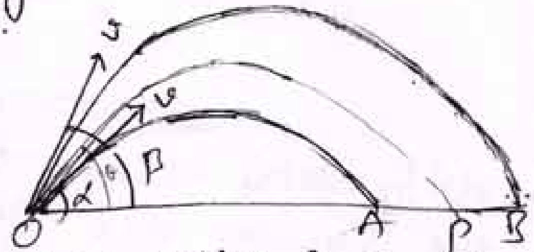
$$\Rightarrow -2T + W = 0.$$

$$\Rightarrow T = \tfrac{1}{2}W = \tfrac{1}{2}\text{ (total weight of all the four rods).}$$

5(c) A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is $\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}$

Sol^n:- Let O be the point of projection and v be the velocity of projection in all the cases. Let P be the point in the horizontal plane through O required to be hit from O. Let θ be the correct angle of projection to hit P from O.

Then OP = the range for the angle of projection $\theta = \frac{v^2\sin 2\theta}{g}$.



When the angle of projection is α, the particle falls at A and when the angle of projection is β, it falls at B.

We have $OA = \frac{v^2\sin 2\alpha}{g}$ and $OB = \frac{v^2\sin 2\beta}{g}$

According to the question

$AP = OP - OA = a$ and $PB = OB - OP = b$

$\therefore a = \frac{v^2 \sin 2\theta}{g} - \frac{v^2 \sin 2\alpha}{g}$

$= \frac{v^2}{g}(\sin 2\theta - \sin 2\alpha)$ —— (1)

and $b = \frac{v^2 \sin 2\beta}{g} - \frac{v^2 \sin 2\theta}{g}$

$= \frac{v^2}{g}(\sin 2\beta - \sin 2\theta)$ —— (2)

Dividing (1) by (2), we get

$\frac{a}{b} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$

$\Rightarrow a \sin 2\beta - a \sin 2\theta = b \sin 2\theta - b \sin 2\alpha$

$\Rightarrow (a+b) \sin 2\theta = a \sin 2\beta + b \sin 2\alpha$

$\Rightarrow \sin 2\theta = \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}$

$\Rightarrow \theta = \frac{1}{2} \sin^{-1}\left(\frac{a \sin 2\beta + b \sin 2\alpha}{a+b}\right)$

5(d) → verify Stoke's theorem for $\vec{F} = -y^3\hat{i} + x^3\hat{j}$, where S is the circular disc $x^2+y^2 \le 1$, $z=0$.

Sol'n: The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x = \cos t$, $y = \sin t$, $z=0$, $0 \le t < 2\pi$. are parametric equations of C.

Then $\oint_C F \cdot dr = \oint_C (-y^3\hat{i} + x^3\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

15

$$= \oint_C \left(-y^3 dx + x^3 dy\right)$$

$$= \int_{t=0}^{2\pi} \left(-y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt}\right) dt$$

$$= \int_0^{2\pi} \left[-\sin^3 t(-\sin t) + \cos^3 t\,(\cos t)\right] dt$$

$$= \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\left\{\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\} = \frac{3\pi}{2}.$$

Also $\nabla \times \underline{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = (3x^2 + 3y^2)\hat{k}$.

Here $n = \hat{k}$ because the surface S is the xy-plane.

$$(\nabla \times F)\cdot \hat{n} = (3x^2 + 3y^2)\,\hat{k}\cdot\hat{k} = 3(x^2+y^2)$$

$$\therefore \iint_S (\nabla \times F)\cdot \vec{n}\, ds = 3\iint_S (x^2+y^2)\, ds$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\, r\, d\theta\, dr$$

by Changing to polars

$$= \frac{3}{4}\int_0^{2\pi} d\theta$$

$$= \frac{3}{4}(2\pi) = \frac{3\pi}{2}.$$

Hence the theorem is verified.

5(e) → Given the space curve $x=t,\ y=t^2,\ z=\frac{2}{3}t^3$, find (i) the curvature κ, (ii) the torsion τ.

Sol'n: Given that $\vec{r} = ti + t^2 j + \frac{2}{3}t^3 k$

$$\Rightarrow \frac{d\vec{r}}{dt} = i + 2tj + 2t^2 k$$

$$\frac{d^2\vec{r}}{dt^2} = 0i + 2j + 4tk$$

and $\frac{d^3\vec{r}}{dt^3} = 0i + 0j + 4k$

Now $$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}$$

$$= i(8t^2 - 4t^2) + j(0 - 4t) + k(2-0)$$

$$= 4t^2 i - 4tj + 2k$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{16t^4 + 16t^2 + 4}$$

$$= 2\sqrt{4t^4 + 4t^2 + 1}$$

$$= 2\sqrt{(2t^2+1)^2}$$

$$= 2(2t^2+1)$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}$$

$$= (4t^2 i - 4tj + 2k)\cdot(4k)$$

$$= 8$$

$$\therefore \text{ Curvature } (K) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$$

$$= \frac{2(2t^2+1)}{\left[\sqrt{1+4t+4t^4}\right]^3}$$

$$= \frac{2(2t^2+1)}{(2t^2+1)^3} = \frac{2}{2t^2+1}$$

$$\text{Torsion } (\tau) = \left[\frac{d\vec{r}}{dt} \;\; \frac{d^2\vec{r}}{dt^2} \;\; \frac{d^3\vec{r}}{dt^3}\right] \Big/ \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2$$

$$= \frac{8}{4(2t^2+1)^2}$$

$$= \frac{2}{(2t^2+1)^2}$$

$$\therefore K = \tau = \frac{2}{(2t^2+1)^2}$$

$\therefore$ For the space curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$ the curvature (K) and torsion (τ) are same at every point.

6(a) Solve $(x^2-4)p^2-2pxy-x^2=0$ and examine for singular solutions and extraneous loci.

Sol^n: The given equation is

$$(x^2-4)p^2-2xyp-x^2=0 \quad \text{--- (1)}$$

solving for y,

$$2y = xp-\frac{4}{x}p-\frac{x}{p} \quad \text{--- (2)}$$

Differentiating (2) partially w.r.t. x, we have

$$2p = p+x\frac{dp}{dx}-\frac{4}{x}\frac{dp}{dx}+\frac{4p}{x^2}-\frac{1}{p}+\frac{x}{p^2}\frac{dp}{dx}$$

$$\Rightarrow (p^2x^2-4p^2+x^2)\left(p-x\frac{dp}{dx}\right)=0$$

from $p-x\frac{dp}{dx}=0$ (omitting the 2nd factor)

$$\Rightarrow \frac{dx}{x}-\frac{dp}{p}=0$$

$$\Rightarrow p=xc$$

putting $p=xc$ in (1), we get

$$x^2c^2(x^2-4)-2xy(xc)-x^2=0$$

$$\Rightarrow x^2[c^2(x^2-4)-2yc-1]=0$$

$$\Rightarrow (x^2-4)c^2-2cy-1=0$$

which is the general solution of (1)

The c-discriminant relation is

$$(-2y)^2-4(x^2-4)(-1)=0$$

$$\Rightarrow y^2+x^2-4=0$$

and p-disc. relation is $x^2(x^2+y^2-4)=0$.

Now $x^2+y^2=4$ occurs once in both the discriminant relations and satisfies the given differential equation and therefore it is a singular solution. Also $x=0$ occurs twice in the p-disc. relation doesnot occur in the c-disc. relation. and doesnot satisfy the differential equation. $\therefore x=0$ is a tac locus.

6(b) Apply the method of variation of parameters to solve $x^2y_2+3xy_1+y = 1/(1-x)^2$

sol'n: Given that $x^2y_2+3xy_1+y=\frac{1}{(1-x)^2}$ — ①

① can be written as $y_2+\frac{3}{x}y_1+\frac{1}{x^2}y = x^{-2}(1-x)^{-2}$ — ②

Consider $y_2+\frac{3}{x}y_1+\frac{1}{x^2}y=0 \Rightarrow x^2y_2+3xy_1+y=0$

$\Rightarrow (x^2D^2+3xD+1)y=0$ — ③

Let $x=e^z$, $\log x = z$ and $D_1=\frac{d}{dz}$

Then $xD=D_1$ and $x^2D^2=D_1(D_1-1)$ and therefore ③ becomes

$\{D_1(D_1-1)+3D_1+1\}y=0$

$\Rightarrow (D_1+1)^2y=0 \Rightarrow D_1=-1,-1$

$\therefore$ C.F of ② is $=(c_1+c_2z)e^{-z}$

$=(C_1+c_2\log x)\frac{1}{x}$ — ④

Let $u=x^{-1}$, $v=x^{-1}\log x$ and $R=x^{-2}(1-x)^{-2}$

$W=\begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x^{-1} & x^{-1}\log x \\ -x^{-2} & x^{-2}-x^{-2}\log x \end{vmatrix} = x^{-3}\neq 0$

$\therefore$ P.I of ② $= uf(x)+vg(x)$

where $f(x) = -\int \frac{vR}{W}dx = -\int \frac{x^{-1}\log x . x^{-2}(1-x)^{-2}}{x^{-3}}dx$

$= -\int (1-x)^{-2}\log x\, dx$

$= -\left[\frac{1}{1-x}\log x - \int\frac{1}{x(1-x)}dx\right]$

$= -(1-x)^{-1}\log x + \log x - \log(1-x)$

and $g(x) = \int\frac{uR}{W}dx = \int\frac{x^{-1}x^{-2}(1-x)^{-2}}{x^{-3}}dx = (1-x)^{-1}$

$\therefore$ P.I of ② is $= x^{-1}\left\{-(1-x)^{-1}\log x+\log x-\log(1-x)\right\}$
$+x^{-1}\log x.(1-x)^{-1}$

$= x^{-1}\{\log x - \log(1-x)\} = x^{-1}\log\left\{\frac{x}{1-x}\right\}$

Hence the general solution of ② is $y = \text{C.F}+\text{PI}$

i.e. $y = c_1x^{-1}+c_2x^{-1}\log x + x^{-1}\log\left\{\frac{x}{1-x}\right\}$

$= x^{-1}\left\{C_1+c_2\log x+\log\left\{\frac{x}{1-x}\right\}\right\}$

6(c) A uniform rod AB of length 2a movable about a hinge at A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, Prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod.

Solⁿ: Let a uniform rod AB of length 2a movable about the hinge at the end A rest with a smooth vertical wall CD.

Let W be the weight of the rod and G its middle point.

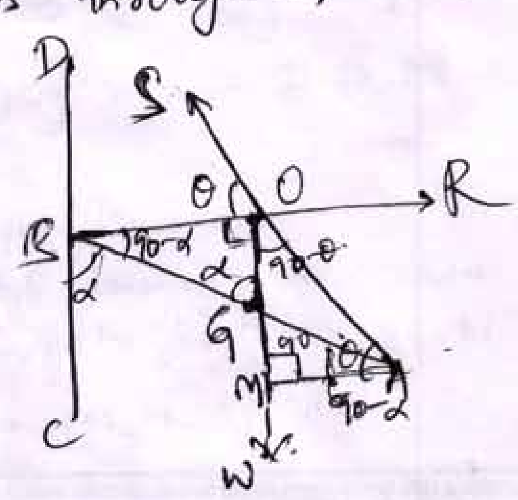
The rod is in equilibrium under the action of the following three forces only.

(i) R, the reaction of the wall at B acting at right angles to the wall.

(ii) S, the reaction of the hinge at A. and

(iii) W, the weight of the rod acting vertically downwards at its middle point G.

Since the force R and the line of action of W meet at O, therefore the reaction S of the hinge at A must also pass through O, as shown in the figure.



Let the rod AB and the reaction S make angles α and θ ~~respectively~~ with the vertical and horizontal respectively.

i.e, $\angle ABC = \alpha$, and $\angle OAM = \theta$. and $\angle ABO = 90-\alpha$.

$\therefore \angle OGB = \alpha$ and $\angle AOM = 90-\theta$.

In ΔOAB, by the trigonometrical theorem, we have

$$(AG+BG)\cot OGB = AG\cot AOG - BG\cot BOG.$$

$$(a+a)\cot\alpha = a\cot(90-\theta) - a\cot 90.$$

$$2a\cot\alpha = a\tan\theta - 0$$

$$\tan\theta = 2\cot\alpha. \quad \text{---} (1)$$

$\therefore$ the reaction S at the hinge makes an angle $\theta = \tan^{-1}(2\cot\alpha)$ with the horizontal.

Now by Lami's theorem at the point O, we have

$$\frac{S}{\sin 90} = \frac{W}{\sin(180-\theta)} = \frac{R}{\sin(90+\theta)}$$

$$\therefore S = \frac{W}{\sin\theta} = W\operatorname{cosec}\theta = W\sqrt{1+\cot^2\theta}$$

$$= W\sqrt{\left(1+\frac{1}{4}\tan^2\alpha\right)}$$

$$= \frac{W}{2}\sqrt{4+\tan^2\alpha}.$$

6(d)(i) what is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ at $(-1,2,1)$?

Sol'n: we have $\phi = xy^2 + yz^3$

$$\Rightarrow \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= y^2 i + (2xy + z^3)j + 3yz^2 k$$

$$\nabla\phi\Big|_{(2,-1,1)} = i - 3j - 3k \quad \text{--- (1)}$$

and also $\nabla(x\log z - y^2 + 4) = i(\log z) - 2yj + \frac{x}{z}$

$$\nabla(x\log z - y^2 + 4)\Big|_{(1,-2,1)} = -4j - k \quad \text{--- (2)}$$

But (1) is normal to surface $x\log z - y^2 = -4$

$$\therefore \hat{a} = \frac{\nabla(x\log z - y^2 + 4)}{|\nabla(x\log z - y^2 + 4)|}$$

$$= \frac{-4j - k}{|-4j - k|} = \frac{+4j - k}{\sqrt{16+1}} = \frac{-4j - k}{\sqrt{17}}$$

$\therefore$ the directional derivative of ϕ at $(2,-1,1)$ in the direction of normal to the surface

$$= \nabla\phi \cdot \hat{a} = (i - 3j + 3k)\cdot\left(\frac{4j - k}{\sqrt{17}}\right)$$

$$= \frac{15}{\sqrt{17}}$$

7(a) Solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$.

Sol'n: Given $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

The auxiliary equation is $D^4+D^2+1=0$

$$\Rightarrow (D^2+1)^2 - D^2 = 0$$

$$\Rightarrow (D^2+D+1)(D^2-D+1)=0$$

$$\Rightarrow D^2+D+1=0 \text{ (or) } D^2-D+1=0$$

$$\Rightarrow D = \frac{-1\pm i\sqrt{3}}{2},\ \frac{1\pm i\sqrt{3}}{2}$$

$$\therefore \text{C.F} = e^{-x/2}\left[c_1\cos\left(\frac{\sqrt{3}x}{2}\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right)+c_4\sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

$$\text{P.I} = \frac{1}{D^4+D^2+1}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})(D^2-2D+\frac{7}{4})}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{-\frac{3}{4}-2D+\frac{7}{4}}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{(1-2D)}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{(1+2D)}{1-4D^2}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D^2+3/4)}\frac{1}{4}(1+2D)\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{e^{-x/2}}{4}\frac{1}{(D^2+3/4)}\left(\cos\frac{\sqrt{3}}{2}x - \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{1}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2}\cos\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2}\sin\frac{\sqrt{3}}{2}x\right]$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{x}{2\left(\sqrt{3}/2\right)}\sin\left(\frac{x\sqrt{3}}{2}\right) - \frac{\sqrt{3}x}{2\left(\sqrt{3}/2\right)}\left(-\cos\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{x}{4\sqrt{3}}e^{-x/2}\left[\sin\left(\frac{\sqrt{3}}{2}x\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$\therefore y = C.F + PI$$

$$= e^{-x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right) + c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$+ e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right) + c_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$+ \frac{x}{4\sqrt{3}}e^{-x/2}\left[\sin\left(\frac{\sqrt{3}}{2}x\right) + \sqrt{3}\cos\left(\frac{\sqrt{3}}{2}x\right)\right]$$

7(b) Solve $x(1-x^2)dy + (2x^2y - y - ax^3)dx = 0$.

Sol'n: Rewriting the given equation, we have

$$x(1-x^2)\frac{dy}{dx} + y(2x^2-1) = ax^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}y = \frac{ax^2}{1-x^2} \quad \text{——} \quad (1)$$

Comparing (1) with $dy/dx + Py = Q$, we have

$$P = \frac{2x^2-1}{x(1-x^2)} = -\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)} \text{ and}$$

$$Q = \frac{ax^2}{1-x^2} \quad \text{——} \quad (2)$$

$$\int P\,dx = -\int\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}\right]dx$$

$$= -\left[\log x + \frac{1}{2}\times(\log(x+1) + \frac{1}{2}\times\log(x-1)\right]$$

$$= -\left[\log x + \frac{1}{2}\log(x^2-1)\right]$$

$$= -\log\left[x(x^2-1)^{1/2}\right] = \log\left[x(x^2-1)^{1/2}\right]^{-1}$$

$\therefore$ Integrating factor $= e^{\int P dx} = e^{\log\{x(x^2-1)^{1/2}\}^{-1}}$

$$= \{x(x^2-1)^{1/2}\}^{-1} = \frac{1}{\{x(x^2-1)^{1/2}\}}$$

Solution of (1) is

$y(\text{I.F}) = \int Q(\text{I.F})dx + c$, c being an arbitrary constant.

$$\Rightarrow \frac{y}{x(x^2-1)^{1/2}} = \int \frac{ax^2}{1-x^2}\times\frac{1}{x(x^2-1)^{1/2}}dx + C = C - a\int\frac{x\,dx}{(x^2-1)^{3/2}}$$

$$\frac{y}{x(x^2-1)^{1/2}} = C - \frac{a}{2}\int\frac{dt}{t^{3/2}}, \text{ Putting } x^2-1 = t \;\&\; 2x\,dx = dt$$

$$\Rightarrow \frac{y}{x(x^2-1)^{1/2}} = C - \frac{a}{2}\left[\frac{t^{-1/2}}{-1/2}\right] = C + \frac{a}{\sqrt{t}} = C + \frac{a}{(x^2-1)^{1/2}}$$

$$\Rightarrow y = ax + Cx(x^2-1)^{1/2}$$

7(c) A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$, $a>b$ and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu/(a+b)}$. Show that the equation of its path is $r = a + b\cos\theta$.

Solⁿ Here the central acceleration.

$$P = \mu\{3au^4 - 2(a^2-b^2)u^5\}$$

∴ the diffⁿ equation of the path is -

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\{3au^4 - 2(a^2-b^2)u^5\}$$

or $$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\{3au^2 - 2(a^2-b^2)u^3\}$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[au^3 - 2(a^2-b^2)\frac{u^4}{4}\right] + A$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left[2au^3 - (a^2-b^2)u^4\right] + A$$

where A is constant. ——— (1)

But initially at an apse $r = a+b$, $u = \frac{1}{(a+b)}$

$\frac{du}{d\theta} = 0$ and $v = \sqrt{\frac{\mu}{(a+b)}}$

∴ from (1) we have

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right] + A$$

$\therefore\ h^2 = \mu$ and $A = 0$

Substituting the values of h^2 and A in (1) we have,

$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left\{2au^3 - (a^2-b^2)u^4\right\}$$

or $$\left(\frac{du}{d\theta}\right)^2 = -u^2 + 2au^3 - (a^2-b^2)u^4 \quad \text{—— (2)}$$

But, $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = \frac{-1}{r^2}\frac{dr}{d\theta}$

Substituting in (2), we have,

$$\left(\frac{-1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2 + 2ar - (a^2-b^2)\right]$$

or $$\left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2$$
$$= b^2 - (r^2 - 2ar + a^2)$$
$$= b^2 - (r-a)^2$$

$\therefore\ \frac{dr}{d\theta} = \sqrt{\{b^2-(r-a)^2\}}$ or $d\theta = \frac{dr}{\sqrt{\{b^2-(r-a)^2\}}}$

Integrating $\theta + B = \sin^{-1}\left(\frac{r-a}{b}\right)$ —— (3)

But initially when $r = a+b$, let us take $\theta = 0$.

Then from (3), $B = \sin^{-1}(1) = \pi/2$

Substituting in (3) we have,

$$\theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r-a}{b}\right), \text{ or } r - a = b\sin\left(\frac{\pi}{2}+\theta\right)$$

$r = a + b\cos\theta$. which is the required equation of path.

7(d) (i) show that $r^n\vec{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n=-3$ ($\vec{r}$ is position vector of a point)

(ii) Find the value of a, b and c such that

$F=(3x-4y+az)\hat{i}+(cx+5y-2z)\hat{j}+(x-by+7z)\hat{k}$ is irrotational.

Sol^n: (i) Let $F=r^n\vec{r}$

The vector F is irrotational if Curl F = 0 putting $\phi=r^n$ and $A=\vec{r}$

and we know that $\text{curl}(\phi A)=\nabla\phi\times A+\phi\,\text{curl}\,A$.

$$\therefore \text{Curl}(r^n\vec{r}) = \nabla r^n\times\vec{r}+r^n\,\text{curl}\,\vec{r}$$

$$=(nr^{n-1}\nabla r)\times\vec{r}+r^n(0) \quad (\because \nabla f(r)=f'(r)\nabla r)$$

$$=\left(nr^{n-1}\frac{1}{r}\vec{r}\right)\times\vec{r} \quad \left(\because \nabla r=\frac{1}{r}\vec{r}\right)$$

$$= nr^{n-2}(\vec{r}\times\vec{r})=0$$

The vector F is Solenoidal if $\text{div}F=0$

we know that $\text{div}(\phi A)=\phi(\text{div}A)+A.(\text{grad}\phi)$

$$\Rightarrow \text{div}(r^n\vec{r})=r^n\text{div}\vec{r}+\vec{r}.\text{grad}\,r^n$$

$$=3r^n+\vec{r}.(nr^{n-1}\text{grad}\,r)$$

$$=3r^n+\vec{r}.\left(nr^{n-1}.\frac{1}{r}\vec{r}\right)$$

$$(\because \text{div}\vec{r}=3 \text{ \& } \nabla f(r)=f'(r)\nabla r)$$

$$=3r^n+\vec{r}.\left(nr^{n-1}.\frac{1}{r}\vec{r}\right)$$

$$=3r^n+nr^{n-2}(\vec{r}.\vec{r})$$

$$=3r^n+nr^n$$

$$\text{div}(r^n\vec{r})=r^n(n+3)$$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if $(n+3)r^n=0$

i.e. $n+3=0$

$\Rightarrow \underline{n=-3}$

(ii) For an irrotational vector F, Curl F = 0

$\therefore$ Curl F $=\nabla\times F$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x-4y+az) & (cx+5y-2z) & (x-by+7z) \end{vmatrix} = 0.$$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(x-by+7z) - \frac{\partial}{\partial z}(cx+5y-2z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(3x-4y+az) - \frac{\partial}{\partial x}(x-by+7z)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(cx+5y-2z) - \frac{\partial}{\partial y}(3x-4y+az)\right] = 0$$

$$\Rightarrow \hat{i}(-b+2) + \hat{j}(a-1) + \hat{k}(c+4) = 0$$

As $\hat{i}, \hat{j}$ & $\hat{k}$ orthogonal and independent vectors, the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$ should be zero separately, therefore

$$(-b+2)=0, \ (a-1)=0 \text{ and } (c+4)=0$$

ie $b=2, a=1, c=-4$

Thus for the given vector F to be irrotational.

$$a=1, \ b=2 \text{ and } c=-4$$

8.(a) By using Laplace transform method, solve $(D^2+m^2)x = a\cos nt$, $t>0$ if $x = Dx = 0$ when $t=0$.

Sol'n: Given $y''+m^2y = a\cos nt$

Applying Laplace transform on both sides

$$s^2L(y) - sy(0) - y'(0) + m^2L(y) = aL(\cos nt)$$

$$(s^2+m^2)L(y) = a \cdot \frac{s}{s^2+n^2}$$

$$L(y) = a \cdot \frac{s}{(s^2+m^2)(s^2+n^2)}$$

Taking partial fractions.

$$\frac{s}{(s^2+m^2)(s^2+n^2)} = \frac{As+B}{s^2+n^2} + \frac{Cs+D}{s^2+m^2}$$

$$s = (As+B)(s^2+m^2) + (Cs+D)(s^2+n^2)$$

$$s = (A+C)s^3 + (B+D)s^2 + (Am^2+Cn^2)s + Bm^2+Dn^2$$

$$A+C=0, \quad B+D=0, \quad Am^2+Cn^2=1, \quad Bm^2+Cn^2=0$$

$$A = \frac{1}{m^2-n^2}, \quad C = \frac{1}{n^2-m^2}$$

$$\therefore L(y) = a\left[\frac{1}{(m^2-n^2)}\left[\frac{s}{s^2+n^2}\right] + \frac{1}{(n^2-m^2)}\left[\frac{s}{s^2+m^2}\right]\right].$$

Taking inverse Laplace Transform

$$y = \frac{a}{m^2-n^2}\left[\cos nt - \cos mt\right]$$ Required solution.

8(b) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest starting at rest from the cusp. prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Soln: let a particle starts from rest from the cusp A of the cycloid. proceeding as in the ~~last example~~ the velocity v of the particle at any point P at time 't' is given by

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\left(16a^2 - s^2\right)$$

(or) $\frac{ds}{dt} = -\frac{1}{2}\sqrt{\frac{g}{a}}\sqrt{16a^2-s^2}$ the -ve sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore dt = -2\sqrt{a/g}\,\frac{ds}{\sqrt{16a^2-s^2}} \longrightarrow (1)$$

The vertical height of the cycloid is 2a. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$ putting $y=a$ in the equation $s^2 = 8ay$ we get $s^2 = 8a^2$ (or) $s = 2\sqrt{2}a$.

∴ Integrating ① from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of vertical height of the cycloid is given by

$$t_1 = -2\sqrt{a/g}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{16a^2-s^2}} = 2\sqrt{\frac{a}{g}}\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{4a}^{2\sqrt{2}a}$$

$$= 2\sqrt{\frac{a}{g}}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a} - \cos^{-1}1\right]$$

$$= 2\sqrt{\frac{a}{g}}\left[\cos^{-1}\frac{1}{\sqrt{2}} - \cos^{-1}1\right]$$

$$= 2\sqrt{\frac{a}{g}}\left[\frac{\pi}{4}-0\right] = \frac{\pi}{2}\sqrt{a/g}.$$

As integrating ① from $s=2\sqrt{2}a$ to $s=0$ the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2 = -2\sqrt{\frac{a}{g}}\int_{s=2\sqrt{2}a}^{0}\sqrt{16a^2-s^2}\,ds$$

$$= 2\sqrt{\frac{a}{g}}\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0} = 2\sqrt{\frac{a}{g}}\left[\cos^{-1}0 - \cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{\frac{a}{g}}\left[\left(\frac{\pi}{2}-\frac{\pi}{4}\right)\right] = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

Hence $t_1 = t_2$ i.e the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

8(c) A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t=0$ and $t = \frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.

Sol'n: Let $\vec{r}$ be the position vector of the particle at time t.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\,\hat{k}$

If $\vec{v}$ is the velocity of the particle at time t and $\vec{a}$ its acceleration at that time then,

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$$

Magnitude of the velocity at time

$t = |\vec{v}|$

$= \sqrt{16\sin^2 t + 16\cos^2 t + 36}$

$= \sqrt{52}$

$= 2\sqrt{13}$

Magnitude of acceleration

$|\vec{a}| = \sqrt{16\cos^2 t + 16\sin^2 t}$

$= \sqrt{16} = 4$

At $t=0$, $\vec{v} = 4\hat{j} + 6\hat{k}$, $\vec{a} = -4\hat{i}$

At $t = \frac{1}{2}\pi$, $\vec{v} = -4\hat{i} + 6\hat{k}$, $\vec{a} = -4\hat{j}$

8(d). If $A = 2yz\,i - (x+2y-2)\,j + (x^2+z)\,k$, evaluate $\iint_S (\nabla\times A)\cdot n\, ds$ over the surface of intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant.

Ans: $\dfrac{-a^2}{12}[3\pi+8a]$