§ 3·5. To obtain B.F.S. from F.S.

Theorem. *If a L.P.P.*

$$\text{Max. } Z = \mathbf{c}\,\mathbf{x},\ s.t.\ A\,\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0,$$

where A is $m \times (m+n)$ matrix of coefficients given by $A = (\alpha_1, \alpha_2, \ldots, \alpha_{m+n})$, has at least one feasible solution. then it has at least one basic feasible solution also.*

[Meerut 95 (BP), 97 (BP), 98 (OLD); Raj. 87]

Proof. Let $\mathbf{x} = [x_1, x_2, \ldots, x_{m+n}]$ be a feasible solution of the given L.P.P.

Suppose that k components (variables) in $\mathbf{x}$ are non-zero and remaining $(m+n-k)$ components are zero. We can assume these non-zero components as the first k components of x.

i.e. $$\mathbf{x} = [x_1, x_2, \ldots, x_k, \underbrace{0, 0, \ldots, 0}_{(m+n-k)}]$$

---

* In L.P.P. any solution which satisfies $x \geq 0$ will be termed as feasible solution.

$$\therefore \quad \sum_{i=1}^{k} x_i \alpha_i = b. \qquad \dots (1)$$

Now there are two possibilities.

1. If the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are L.I. If the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are L.I., then by defination x is a B.F.S. Hence the theorem is true in this case. This solution is degenerate if $k < m$ and non-degenerate if $k = m$.

2. If the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are L.D. This is the case when $k > m$. In this case we can reduce the number of non-zero variables step by step, so that the feasible solution become B.F.S.

Proceeding as in § 3·4, can reduce the F.S.

$$\mathbf{x}' = [x_1, x_2, \dots, x_k, \underbrace{0, 0, \dots, 0}_{(m+n-K)}]$$

to a new basic feasible solution.

$$\mathbf{x}' = \left[x_1 - \frac{\lambda_1}{y}, x_2 - \frac{\lambda_2}{\nu}, \dots, x_k - \frac{\lambda_k}{\nu}, \underbrace{0, 0, \dots, 0}_{(m+n-k)}\right]$$

which cannot contain more than $(k-1)$ non-zero components. Thus we have derived a new F.S. from the given F.S. containing less number of non-zero (positive) variables.

If the column vectors associated to non-zero variable in this new sol^n are L.I., then this sol^n is B.F.S. and hence the theorem is true. But if these associated vectors are not L.I. then this solution is not B.S.F In this case we repeat the above prodess again and get another B.F.S. of the given system containing not more than $(k-2)$ non-zero variables. Continueing in this way for finite number of times we will certainly get an optimal B.F.S. of the system. Hence the theorem is true.

Note. The two theorems 3·4 and 3·5 in combined form may be stated *as follows.*

If the L.P.P. Max. $Z = \mathbf{cx}$, s.t. $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq 0$ has feasible solution, then at least one of the B.F. solution will be optimal.

Illustrative Examples

Ex. 1. *If $x_1 = 2, x_2 = 3, x_3 = 1$, be a feasible solution of the L.P.P.*

$$\begin{aligned} \text{Max.} \quad & Z = x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + 4x_3 = 11 \\ & 3x_1 + x_2 + 5x_3 = 14 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

then find a B.F.S. [Meerut 89, 94 (P), 95]

Sol. The given L.P.P. can be written as

Max. $Z = x_1 + 2x_2 + 4x_3$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = b \qquad \ldots (1)$$

$$x_1, x_2, x_3 \geq 0$$

where $\alpha_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, b = \begin{bmatrix} 11 \\ 14 \end{bmatrix}$

Here m (number of constraints) = 2, so the B.F.S. of this L.P.P. can not have more that 2 non-zero variables.

$\therefore$ The given feasible solution $x_1 = 2, x_2 = 3, x_3 = 1$ is not B.F.S. In order to reduce the F.S. to a B.S.F. we have to make at least one variable zero. For this we proceed as follows.

Since the vectors $\alpha_1, \alpha_2, \alpha_3$, associated with the non-zero variables are L.D., therefore one of these vectors may be expressed as a L.C. of the remaining two.

Let $$\alpha_1 = \lambda_2 \alpha_2 + \lambda_3 \alpha_3$$

or $$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \lambda_2 + 4\lambda_3 \\ \lambda_2 + 5\lambda_3 \end{bmatrix}$$

$\therefore \quad \lambda_2 + 4\lambda_3 = 2$ and $\lambda_2 + 5\lambda_3 = 3$.

Solving, we have $\lambda_2 = -2$ and $\lambda_3 = 1$

$$\therefore \alpha_1 = -2\alpha_2 + 1\alpha_3$$

or $$\alpha_1 + 2\alpha_2 - \alpha_3 = 0 \qquad \ldots (2)$$

or $$\sum_{i=1}^{3} \lambda_i x_i = 0, \text{ where } \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1$$

Now $$v = \underset{1 \leq i \leq 3}{Max} \left(\frac{\lambda_i}{x_i}\right)$$

$$= \text{Max.} \left(\frac{\lambda_1}{x_1}, \frac{\lambda_2}{x_2}, \frac{\lambda_3}{x_3}\right)$$

$$= \text{Max.} \left(\frac{1}{2}, \frac{2}{3}, \frac{-1}{1}\right) = \frac{2}{3} = \frac{\lambda_2}{x_2}.$$

$\therefore$ The variables x_2 should be zero or the vector α_2 should be eliminated.

Substituting the given feasible solution $x_1 = 2, x_2 = 3, x_3 = 1$ in (1), we have

Substituting the value of α_2 from (2) in (3), we have

$$2\alpha_1 + 3\left(\frac{\alpha_3 - \alpha_1}{2}\right) + \alpha_3 = b$$

or $$\frac{1}{2}\alpha_1 + \frac{5}{2}\alpha_3 = b$$

or $$\frac{1}{2}\alpha_1 + 0.\alpha_2 + \frac{5}{2}\alpha_3 = b.$$

$\therefore$ The new feasible solution is $x_1 = 1/2, x_2 = 0, x_3 = 5/2$.

Here the vector $\alpha_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, associated to non-zero varibles

are L.I. Hence the F.S. $x_1 = 1/2, x_2 = 0, x_3 = 5/2$ is B.F.S.

Note 1. Another B.F.S. of the L.P.P. is $x_1 = 3, x_2 = 5, x_3 = 0$.

2. The new values of the variables can also be found by using the formula.

$$x_i' = x_i - \frac{\lambda_i}{v}.$$

Ex. 2. *If $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 3$ be a F.S. to the set of equations*

$$\left.\begin{aligned} 5x_1 - 4x_2 + 3x_3 + x_4 &= 3 \\ 2x_1 + x_2 + 5x_3 - 3x_4 &= 0 \\ x_1 + 6x_2 - 4x_3 + 2x_4 &= 15 \end{aligned}\right\}$$

then find a B. F. S. **[Meerut 87]**

Sol. The given set of equation can be written as

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4 = b \qquad \ldots(1)$$

where $\alpha_1 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

$$b = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix}.$$

Here $m = 3$, so the B.F.S. of the set of equations can not have more than 3 non-zero variables.

$\therefore$ The given F.S. $x_1 = 1, x_2 = 2, x_3 = 1, x_4 = 3$ is not B.F.S. In order to reduce this F.S. to B.F.S. we have to make at least one variable zero. For this we proceed follows.

Let $\alpha_1 = \lambda_2 \alpha_2 + \lambda_3 \alpha_3 + \lambda_4 \alpha_4$

or
$$\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix} + \lambda_4 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4\lambda_2 + 3\lambda_3 + \lambda_4 \\ \lambda_2 + 5\lambda_3 - 3\lambda_4 \\ 6\lambda_2 - 4\lambda_3 + 2\lambda_4 \end{bmatrix}$$

$$-4\lambda_2 + 3\lambda_3 + \lambda_4 = 5$$
$$\lambda_2 + 5\lambda_3 - 3\lambda_4 = 2$$
$$6\lambda_2 - 4\lambda_3 + 2\lambda_4 = 1.$$

Solving, we have $\lambda_2 = \frac{22}{43}, \lambda_3 = \frac{139}{86}, \lambda_4 = \frac{189}{86}$.

$$\alpha_1 = \frac{22}{43}\alpha_2 + \frac{139}{86}\alpha_3 + \frac{189}{86}\alpha_4$$

or
$$86\alpha_1 - 44\alpha_2 - 139\alpha_3 - 189\alpha_4 = 0 \qquad \ldots (2)$$

or $\sum_{i=1}^{4} \lambda_i \alpha_i = 0$, where $\lambda_1 = 86$, $\lambda_2 = -44$, $\lambda_3 = -139$, $\lambda_4 = -189$

Now $v = \underset{1 \le i \le 4}{Max} \left(\frac{\lambda_i}{x_i}\right)$

$$= \text{Max.} \left(\frac{\lambda_1}{x_1}, \frac{\lambda_2}{x_2}, \frac{\lambda_3}{x_3}, \frac{\lambda_4}{x_4}\right)$$

$$= \text{Max.} \left(\frac{86}{1}, \frac{-44}{2}, \frac{-139}{1}, \frac{-189}{3}\right) = \frac{86}{1} = \frac{\lambda_1}{x_1}.$$

$\therefore$ The variable x_1 should be zero or the vector α_1 should be eliminated.

Substituting he given F.S. (1), we have

$$\alpha_1 + 2\alpha_2 + \alpha_3 + 3\alpha_4 = b \qquad \ldots (3)$$

substituting the value of α_1 from (2) in (3), we have

$$\frac{44\alpha_2 + 139\alpha_3 + 189\alpha_4}{86} + 2\alpha_2 + \alpha_3 + 3\alpha_4 = b$$

or
$$0.\alpha_1 + \frac{216}{86}\alpha_2 + \frac{225}{86}\alpha_3 + \frac{447}{86}\alpha_4 = b$$

$\therefore$The new F.S. is $x_1 = 0, x_2 = \frac{216}{86}, x_3 = \frac{225}{86}, x_4 = \frac{447}{86}$. Here the vectors $\alpha_2 = \begin{bmatrix} -4 \\ 1 \\ 6 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, associated to non-zero variables are L.I. Hence this F.S. is B.S.F.

Note. Another basic feasible solution of the set of equations is

$$x_1 = \frac{25}{139}, \quad x_2 = \frac{234}{139}, \quad x_3 = 0, \quad x_4 = \frac{228}{139}.$$

Step 2. Draw the *minimum* possible number of horizontal and vertical lines to cover all the zeros of the starting cost matrix. With this operation we may face the following two situations:

(i) The number of lines (N) so drawn may be equal to the order (m) of the cost matrix. In this situation an *optimum assignment* has been reached.

(ii) The number of lines so drawn is less than the order of the cost matrix ($N < m$).

When situation (i) occurs to find the optimum assignment we follow the next step.

Step 3. Write the matrix consisting of only zero elements by performing operations in *Step* 1 and *Step* 2. Then starting with the first row, examine all the rows which contain *only one zero* (row operation). Mark this zero under rectangle ▭, where an assignment will be made.

Apply similar procedure to the column (column operation) only after all the rows have been completely examined. As there is no *unmarked zero* is left we immediately have an optimum solution.

But whenever the situation (ii) occurs, follow the following step.

Step 3A. Search the *smallest element* after performing *Step* 1 and *Step* 2 among the *uncovered* element left after drawing the lines. Subtract this smallest element from all uncovered elements and at the same time add this smallest element to the elements lying at the point of intersection of horizontal and vertical lines. *Do not touch* the elements through which only one line passes. This operation will create a new MODIFIED MATRIX with more zeros.

Go to *Step* 2 with this modified matrix. If a complete assignment be not still available then repeat *Step* 3A and *Step* 2 successively and finally apply *Step* 3.

Step 4. Repeat *row and column* operations of *Step* 3 until one of the following two cases arise.

Case 1. There will be no unmarked zero left.

Case 2. There lie more than one unmarked zero in one row or column.

For Case 1 process stops and we have exactly one *marked* zero in each row and each column.

For Case 2 mark under ▭ one of the unmarked zeros arbitrarily and ignore the remaining zeros in that row or column. Repeat the process until no unmarked zero is left.

The procedure described above is called *Hungarian method* as it is based on the works of two famous Hungarian mathematicians König and Egerváry.

Example 7.2.1. *Find the optimal assignment and minimum cost for the following assignment problem.*

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

[KU(H) 1994, BESU(ME) 2005]

Solution: The minimum elements of the rows A, B, C, D are respectively (10, 5, 11, 8). We subtract these elements from all the elements of the respective rows yielding *Table* 1.

Table 1

	1	2	3	4
A	0	2	9	1
B	0	5	2	3
C	1	3	2	0
D	0	7	3	1.

Table 1 shows the matrix has at least one zero in every row. Now the minimum elements of the columns 1, 2, 3, 4 respectively (0, 2, 2, 0). We subtract these elements from all the elements of the respective columns giving the *starting cost matrix* in *Table* 2.

Table 2

	1	2	3	4
A	0	0	7	1
B	0	3	0	3
C		1	0	0
D	0	5		1

Now we draw the minimum number of horizontal and vertical lines to cover all the zeros of the starting matrix. The number of lines $N = 4 =$ order of the cost matrix so that a complete assignment will be possible from this matrix. Now follow *Step* 3 of Hungarian method.

Table 3

	1	2	3	4
A	$\cancel{0}$	$\boxed{0}$		
B	$\cancel{0}$		$\boxed{0}$	
C			$\cancel{0}$	$\boxed{0}$
D	$\boxed{0}$			

Assignments are made in the cell (4, 1), (1, 2), (3, 4) and (2, 3) for the fourth, first, third and second rows respectively, which contains only one zero in each. Cross of zeros lying on the corresponding columns marked zeros. Hence, the optimal assignment schedule will be A ⟶ 2, B ⟶ 3, C ⟶ 4, D ⟶ 1 (represents a one-one mapping) and minimum cost $= 12 + 7 + 11 + 8 = 38$.

Example 7.2.2. *Solve the following assignment problem whose cost matrix is given below.*

	a	b	c	d
1	18	26	17	11
2	13	28	14	26
3	38	19	18	15
4	19	26	24	10

[CU(H) 1987, 2003]

Solution: Using the same procedure as described in details in Ex. 7.2.1 the starting cost matrix of the problem is given by:

Table 1

	a	b	c	d
1	7	11	5	0
2	0	11	0	13
3	23	0	2	0
4	9	12	12	0

Draw minimum number of horizontal and vertical lines to cover all zeros of *Table* 1, which shows that number of lines $N = 3 < 4 = m$, order of matrix. So we follow *Step* 3A of the method. The smallest element among the uncovered elements by the lines is 2. We subtract 2 from each uncovered element and add 2 to the element at the point of intersection of horizontal and vertical lines but untouch those elements through which the lines pass. This will create a modified cost matrix given in *Table* 2.

Table 2

a	b	c	d
5	11	3	0
0	13	0	15
21	0	0	0
7	12	10	0

Again, $N = 3 < 4 = m$, order of matrix. Now smallest element among uncovered elements $= 3$. Perform the same operation which will give a second modified cost matrix given in *Table* 3.

Table 3

	a	*b*	*c*	*d*
1	2	8	0	0
2	0	13	0	18
3	22	0	0	3
4	4	9	7	0

Table 4

	a	*b*	*c*	*d*
1			$\boxed{0}$	$\cancel{0}$
2	$\boxed{0}$		$\cancel{0}$	
3		$\boxed{0}$	$\cancel{0}$	
4				$\boxed{0}$

Now $N = 4 =$ order of matrix, an optimum assignment has been reached. To find the assignment follow the following.

Single zero row in cell (4,4) assigns first zero in that row under ▭ and cross of all zero in the corresponding column. Next single zero in cell (2,1) assigns second assignment thereunder ▭ and cross of zero in the corresponding row. Proceeding in this manner, we get the following optimal assignment schedule:

$$(1 \longrightarrow c,\ 2 \longrightarrow a,\ 3 \longrightarrow b,\ 4 \longrightarrow d).$$

Minimum cost = 17 + 13 + 19 + 10 = 59.

Example 7.2.3. *Solve the following assignment problem whose cost matrix is given below.*

	I	II	III	IV	V
A	6	5	8	11	16
B	1	13	16	1	13
C	16	11	8	8	8
D	9	14	12	10	16
E	10	13	11	8	16

[BU(H) 1996]

Solution: Using the same procedure, we get the following starting cost matrix.

The minimum lines to cover all zeros of starting matrix are 4, which is less than the order 5 of the matrix. So we search for the smallest element among the uncovered element, which is 3 and perform *Step* 3A to get *Table* 2.

Table 1

	I	II	III	IV	V
A	1	0	3	6	11
B	0	12	15	0	9
C	0	3	0	0	0
D	0	5	3		7
E	2	5	3	0	8

Table 2

	I	II	III	IV	V
A	4	0	3	0	11
B	0	9	12	0	6
C	11	3	0	3	0
D	0	2	0	1	4
E	2	2	0	0	5

Now the minimum number of lines is equal to the order of the matrix so that an optimum solution has been reached. To find the optimum assignment consider *Table* 3 and *Table* 4.

Table 3

	I	II	III	IV	V
A		$\boxed{0}$			
B	$\boxed{0}$			$\cancel{0}$	
C			$\cancel{0}$		$\boxed{0}$
D	$\cancel{0}$		$\boxed{0}$		
E			$\cancel{0}$	$\boxed{0}$	

Table 4

	I	II	III	IV	V
A		$\boxed{0}$			
B	$\cancel{0}$			$\boxed{0}$	
C			$\cancel{0}$		$\boxed{0}$
D	$\boxed{0}$		$\cancel{0}$		
E			$\boxed{0}$	$\cancel{0}$	

Single zero row in cell (1,2) in both tables assigns first in that cell under ▭. Then there is no other single zero in the row of both tables. Now examine single column which is the fifth column and assigns there. Cross off all zeros in that column and row.

Next we observe that there lie more than one unmarked zero in row as well as column. In *Table* 3 mark zero in (2, 1) cell and cross off zero in that particular row and column but in *Table* 4 mark zero in (2, 4) cell and similarly cross off all other zeros in that row and column. This will be done arbitrarily as mentioned in *Step* 4 and we get the following two alternate optimal assignments.

(i) A ⟶ II, B ⟶ I, C ⟶ V, D ⟶ III, E ⟶ IV

or, (ii) A ⟶ II, B ⟶ IV, C ⟶ V, D ⟶ I, E ⟶ III

and Minimum cost = 34.

7.2.4 Different Forms of An Assignment Problem

(A) Unbalanced Assignment Problem

When the number of facilities is not equal to the number of jobs (i.e., when the matrix is rectangular) such a problem is called an *unbalanced assignment problem*. Since the Hungarian method requires a square matrix, fictitious facilities or jobs may be added and zero costs be assigned to the corresponding cells of the matrix. These cells are then treated with the same way as the real cost cells during the solution procedure.

Example 7.2.4. *A manufacturer of electronic equipment has just received a sizeable contract and plans to subcontract a part of the job. He has solicited bids for 6 subcontracts from 3 firms. Each job is sufficiently large and any firm can take only one job. The table below shows the bids as well as the cost estimates (in lakh of rupees) for doing the job internally. Not more than three jobs can be performed internally.*

Find the optimal assignment that will result in minimum total cost. [IAS(Main) 2002]

Job / Firm	J_1	J_2	J_3	J_4	J_5	J_6
F_1	44	67	41	53	48	64
F_2	46	69	40	45	45	68
F_3	43	73	37	51	44	62
Internal	50	65	35	50	46	63

Solution: The problem involves a rectangular cost matrix of order 4 × 6. As three jobs can be performed internally we convert it into a square matrix of order 6 × 6 as follows.

Now using row and then column reduction, we get *Table* 2.

Table 1

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	44	67	41	53	48	64
F_2	46	69	40	45	45	68
F_3	43	73	37	51	44	62
I_1	50	65	35	50	46	63
I_2	50	65	35	50	46	63
I_3	50	65	35	50	46	63

Table 2

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	0	0	0	7	3	0
F_2	3	3	0	0	0	5
F_3	3	10	0	9	2	2
I_1	12	4	0	10	6	5
I_2	12	4	0	10	6	5
I_3	12	4	0	10	6	5

The minimum number of lines covering all zeros is only 3(<6). Hence, optimal assignment cannot be made in the current feasible solution. Proceeding similarly as in previous examples, we get the following.

Table 3

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	0	0	1	7	2	0
F_2	3	3	1	0	0	5
F_3	1	8	0	7	0	0
I_1	10	2	0	8	4	3
I_2	10	2	0	8	4	3
I_3	10	2	0	8	4	3

min line $N = 4 < 6$

Table 4

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	0	0	1	7	2	0
F_2	3	1	1	0	0	5
F_3	1	8	1	7	0	0
I_1	8	0	0	6	2	1
I_2	8	0	0	6	2	1
I_3	8	0	0	6	2	1

min line $N = 5 < 6$

Table 5

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	0		5	7	3	0
F_2	3	1	5	0	0	5
F_3	1	0	3	7	0	0
I_1	7	2	0	5		0
I_2	7	2	0	5		0
I_3	7	2	0	5		0

Number of lines $= N = 6$, current feasible solution becomes optimum. To find optimal assignment we construct as shown in the adjoining *Table* 6.

Table 6

	J_1	J_2	J_3	J_4	J_5	J_6
F_1	$\boxed{0}$					$\cancel{0}$
F_2				$\boxed{0}$	$\cancel{0}$	
F_3					$\boxed{0}$	
I_1		$\boxed{0}$	$\cancel{0}$			$\cancel{0}$
I_2		$\cancel{0}$	$\boxed{0}$			$\cancel{0}$
I_3		$\cancel{0}$	$\cancel{0}$			$\boxed{0}$

Optimal assignment schedule: $F_1 \longrightarrow J_1$, $F_2 \longrightarrow J_4$, $F_3 \longrightarrow J_5$, $I_1 \longrightarrow J_2$, $I_2 \longrightarrow J_3$, $I_3 \longrightarrow J_6$.

Associated minimum cost = ₹296 lac.

(B) Maximisation Problem

Sometimes the assignment problem may be a maximisation problem (with a profit matrix). Before the Hungarian method be applied the problem has to be changed to minimisation. This conversion may be done in either of the following two ways:

(i) The largest element of the profit matrix is selected and then a new *cost matrix* is formed whose elements are obtained by subtracting all the elements from the largest element.

(ii) By multiplying the matrix elements by (−1).

The Hungarian method can then be applied to this equivalent minimisation problem to obtain the optimal assignment.

Example 7.2.5. *The captain of Indian cricket team has to allot five middle order batsmen. The average runs scored by each batsman at these positions are as follows.*

(i) *Find the assignment of batsmen to position, which will give maximum number of runs.*

(ii) *If another batsman B_6 with the following average runs in batting position as given below joins the team, should he be included in the team? If so, who will be replaced by him?*

Batsman	*Batting position*				
	P_1	P_2	P_3	P_4	P_5
B_1	40	40	35	25	50
B_2	42	30	16	25	27
B_3	50	48	40	60	50
B_4	20	19	20	18	25
B_5	58	60	59	55	53

Batsman	*Batting position*				
	P_1	P_2	P_3	P_4	P_5
B_6	54	52	38	50	49

[JNU(Comp) 1999

Solution: (i) The largest element is 60, the equivalent minimisation problem is:

Performing the row and the column reduction, we get *Table 2*.

Table 1

	P_1	P_2	P_3	P_4	P_5
B_1	20	20	25	35	10
B_2	18	30	44	35	33
B_3	10	12	20	0	10
B_4	40	41	40	42	35
B_5	2	0	1	5	7

Table 2

	P_1	P_2	P_3	P_4	P_5
B_1	10	10	14	25	0
B_2	0	12	26	17	15
B_3	10	12	19	0	10
B_4	5	6	4	7	0
B_5	2	0	0	5	[illegible]

Table 3

	P_1	P_2	P_3	P_4	P_5
B_1	0	6	10	21	0
B_2	0	12	25	17	19
B_3	10	12	19	0	14
B_4	[illegible]	2	0	[illegible]	0
B_5	2	0	0	[illegible]	[illegible]

Number of lines $N = 4 < 5$, current feasible solution is not optimum.

Minimum number of lines $N = 5$. Therefore, optimum solution has been reached and we get *Table 4* for optimum assignment.

Table 4

	P_1	P_2	P_3	P_4	P_5
B_1					$\boxed{0}$
B_2	$\boxed{0}$				
B_3				$\boxed{0}$	
B_4			$\boxed{0}$		$\cancel{0}$
B_5		$\boxed{0}$	$\cancel{0}$		

Optimum assignment policy: Batsman ⟶ Position ⟶ No. of runs

Batsman	Position	No. of runs
B_1	P_5	50
B_2	P_1	42
B_3	P_4	60
B_4	P_3	20
B_5	P_2	60
		$\overline{232}$

(ii) Assignment matrix if batsman B_6 joins the team is:

	P_1	P_2	P_3	P_4	P_5
B_1	40	40	35	25	50
B_2	42	30	16	25	27
B_3	50	48	40	60	50
B_4	20	19	20	18	25
B_5	58	60	59	55	53
B_6	45	52	38	50	49

This is an unbalanced assignment problem. First making it balanced by adding a dummy column P_D and then transforming it into a minimisation problem by usual method, we get *Table* 5.

Row reduction and then column reduction will give *Table* 6.

Table 5

	P_1	P_2	P_3	P_4	P_5	P_D
B_1	20	20	25	35	10	60
B_2	18	30	44	35	33	60
B_3	10	12	20	0	10	60
B_4	40	41	40	42	35	60
B_5	2	0	1	5	7	60
B_6	15	8	22	10	11	60

Table 6

	P_1	P_2	P_3	P_4	P_5	P_D
B_1	10	10	14	25	0	25
B_2	0	12	25	17	15	17
B_3	10	12	19	0	10	35
B_4	5	6	4	7	0	0
B_5	2	0	0	5	7	35
B_6	7	0	13	2	3	7

Minimum number of lines $N = 6$, current solution becomes optimal and is given in *Table* 7.

Table 7

	P_1	P_2	P_3	P_4	P_5	P_D
B_1					$\boxed{0}$	
B_2	$\boxed{0}$					
B_3				$\boxed{0}$		
B_4					$\cancel{0}$	$\boxed{0}$
B_5		$\cancel{0}$	$\boxed{0}$			
B_6		$\boxed{0}$				

Optimal assignment policy after the inclusion of batsman B_6 in Indian team:

Batsman ⟶	Batting position ⟶	No. of runs
B_1	P_5	50
B_2	P_1	42
B_3	P_4	60
B_4	P_D	0
B_5	P_3	59
B_6	P_2	52
		263

Conclusion. Since the number of runs increases, batsman B_6 should be included in the team. Obviously, in place of B_4 batsman B_6 enters into the Indian team.

(C) Impossible Assignment

Sometime due to certain restrictions we do not permit the assignment of a particular facility to a particular job. To tackle such problems we assign a very large cost to the corresponding cell. Such job will then be automatically excluded from further consideration.

Example 7.2.6. *Consider the problem of assigning the operators to the machines. The assignment cost in rupees are given in the table. Operator O_2 cannot be assigned to machine M_2 and operator O_5 cannot be assigned to machine M_4. Find the optimal cost of assignment.* [BESU(BE) 2010]

	M_1	M_2	M_3	M_4	M_5
O_1	8	4	2	6	1
O_2	0	–	5	5	4
O_3	3	8	9	2	6
O_4	4	3	1	0	3
O_5	9	5	8	–	5

Solution: The problem indicates that two assignments are impossible. Therefore, we put a very large cost 10 in the cells (2, 2) and (5, 4), then proceed as usual. Performing first row reduction and then column reduction, we get *Table* 1.

Minimum number of lines $N = 5$, which is equal to order of the matrix. Current basic solution is therefore optimal and is given in *Table* 2.

Table 1

	M_1	M_2	M_3	M_4	M_5
O_1	7	3	0	5	0
O_2	0	10	4	5	4
O_3	1	6	6	0	4
O_4	[illegible]	3	0	0	3
O_5	[illegible]	0	2	[illegible]	0

Table 2

	M_1	M_2	M_3	M_4	M_5
O_1			⊠		$\boxed{0}$
O_2	$\boxed{0}$				
O_3				$\boxed{0}$	
O_4			$\boxed{0}$	⊠	
O_5		$\boxed{0}$			⊠

Optimal assignment: $O_1 \longrightarrow M_5$, $O_2 \longrightarrow M_1$, $O_3 \longrightarrow M_4$, $O_4 \longrightarrow M_3$, $O_5 \longrightarrow M_2$.

Minimum cost = 1 + 0 + 2 + 1 + 5 = ₹9.

(D) Negative Cost

If the cost matrix contains some negative cost, then we add to each element of the rows and columns a quantity sufficient to make all the cell ...

Example 7.2.7. *Solve the following assignment problem whose cost matrix is given.*

[JNU(Comp) 1992, IAS 1990]

	J_1	J_2	J_3	J_4
M_1	2	−1	−1	−2
M_2	1	0	−2	−1
M_3	1	−1	−2	0
M_4	2	2	1	1

Solution: First of all we make all the cell elements non-negative by adding 3 to each element of the matrix and get *Table* 1.

Table 1

	J_1	J_2	J_3	J_4
M_1	5	2	2	1
M_1	4	3	1	2
M_1	4	2	1	3
M_1	5	5	4	4

Now row reduction and column reduction will give *Table* 2.

Table 2

	J_1	J_2	J_3	J_4
M_1	3	0	1	0
M_2	2	1	0	1
M_3	2	0	0	2
M_4	0	0	0	0

Minimum number of lines $N = 4$, which is equal to the order of the cost matrix. Hence, an optimal solution has been reached and is given by *Table* 3.

Table 3

	J_1	J_2	J_3	J_4
M_1		$\cancel{0}$		$\boxed{0}$
M_2			$\boxed{0}$	
M_3		$\boxed{0}$	$\cancel{0}$	
M_4	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Optimal assignment: $M_1 \longrightarrow J_4$, $M_2 \longrightarrow J_3$, $M_3 \longrightarrow J_2$, $M_4 \longrightarrow J_1$.

Minimum cost $= (-2)+(-2)+(-1)+2 = -3$.

(E) Alternate Optimal Solution

Sometimes, it is possible to have two or more ways to strike off all zero elements in the reduced matrix for a given problem. In such cases, there will be alternate optimal solutions. How to find alternate solution that is already mentioned in computational procedures. In fact, an alternate optimal solutions offer a great flexibility to the management since it can select the one which is most suitable to its requirement.

Ex. 7.2.3 illustrate an alternate optimal solution of a given problem.

7.3 TRAVELLING SALESMAN PROBLEM

Suppose, a salesman wants to visit a certain number of cities from his headquarters. The distances (or cost or time) of journey between every pair of cities be denoted by c_{ij}, distance from city i to city j is assumed to be known. The problem is: Salesman starting from its home city visiting each city only once and return to his home city in the shortest possible distance (or at the least cost or in the least time).

Given n cities and distance c_{ij}, the salesman starts from city 1, then any permutation of $2, 3, \ldots, n$ represents the number of possible ways for his tour. So, there are $(n-1)!$ possible ways for his tour. The problem is to select the optimal route that could achieve the salesman's objective.

$$\text{Let } x_{ij} = \begin{cases} 1, & \text{if the salesman goes directly from city } i \text{ to city } j \\ 0, & \text{otherwise.} \end{cases}$$

Here $\min\{M_{1j}, M_{4j}\} = 7$ for M_{42}, do J_2 last
$= 8$ for M_{13} and M_{44}, schedule J_3 first and J_4 next to the last
$= 10$ for M_{11} and M_{45}.

The optimum sequence becomes:

J_3	J_1	J_5	J_4	J_2

7.5 EXERCISE 7

1. In a (3×3) transportation problem let x_{ij} be the amount shipped from source i to destination j and c_{ij}, the corresponding per unit transportation cost. The supplies at sources 1, 2 and 3 are 15, 30 and 85 units and the demands at destinations 1, 2 and 3 are 20, 30 and 80 units. Assuming that the starting solution obtained by the north-west corner method gives the optimal basic solution to the problem, let the associated values of the multipliers for sources 1, 2 and 3 be 2, 3 and 5 and those of destinations 1, 2 and 3 be 2, 5 and 10.
   (a) Find the total optimal transportation cost.
   (b) What are the smallest values of c_{ij} for the non-basic variables that will keep the solution optimal?

2. Solve the adjoining transportation model by using north-west corner method and Volgel's approximation method (VAM) to obtain the starting solution. Compare the computations:

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

3. Find the starting solution in the adjoining transportation problem by the (a) North-west corner method (b) Matrix-minima method (c) VAM method. Obtain the optimal solution by using the best starting solution.

10	20	5	7	10
13	9	12	8	20
4	15	7	19	30
14	7	1	0	40
3	12	5	19	50
60	60	20	10	

4. For the balanced transportation problem, prove that set of optimal solutions is unaffected if the cost matrix (c_{ij}) is replaced by the cost matrix (b_{ij}), where $b_{ij} = c_{ij} + p_i + q_j$ and p_i, q_j are real numbers. What would happen to the set of optimum solutions if the matrix (c_{ij}) is replaced by the cost matrix (αc_{ij}), where α is a given positive number?

5. In the balanced transportation problem, if all a_i, b_j are integers and if b_j is replaced by $b_j + \frac{1}{n}$ for all $j = 1, 2, \ldots, n$ and a_m by $a_m + 1$, then prove that the new balanced transportation problem is never degenerate for a basic feasible solution.

6. Find the optimal (minimum) solution of the following transportation problem:

(a)

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	3	4	6	8	8	20
O_2	2	10	0	5	8	30
O_3	7	11	20	40	3	15
O_4	1	0	9	14	16	13
b_j	40	6	8	18	6	

[KU(MSc) 1991, BESU(BE) 2000]

(b)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
b_j	10	25	18	7	

[JU(MSc) 1983, CU(H) 1991]

(c)

	a	b	c	d	e	f	a_i
A	5	3	7	3	8	5	3
B	5	6	12	5	7	11	4
C	2	1	3	4	8	2	2
D	9	6	10	5	10	9	8
b_j	3	3	6	2	1	2	

[CU(H) 2001, MCA(WBUT) 2005]

(d)

	D_1	D_2	D_3	a_i
O_1	0	2	1	5
O_2	2	1	5	10
O_3	2	4	3	5
b_j	5	5	10	

[CU(H) 1993]

(e)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

[Delhi(MSc) 1968, JNU(Comp Sc) 2001]

(f)

	D_1	D_2	D_3	D_4	a_i
O_1	8	10	7	6	50
O_2	12	9	4	7	40
O_3	9	11	10	8	30
b_j	25	32	40	33	

[IGNOU(MBA) Dec 2006, Delhi(MSc) 1975]

(g)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	11	3	11	2
O_4	6	8	11	2	2	10	9
b_j	4	4	6	2	4	2	

[IAS(Main) 1995, Meerut(MSc) 2000, 1995]

(h)

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	4	7	3	8	2	4
S_2	1	4	7	3	8	7
S_3	7	2	4	7	7	9
S_4	4	7	2	4	7	2
b_j	8	3	7	2	2	

(i)

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	2	11	10	3	7	4
S_2	1	4	7	2	1	8
S_3	3	9	4	8	12	9
b_j	3	3	4	5	6	

7. Is $x_{13} = 50, x_{14} = 20, x_{21} = 55, x_{31} = 30, x_{32} = 35, x_{34} = 25$ an optimal solution of the adjoining transportation problem?
If not, modify it to obtain a better solution.

[IISc (1991)]

	D_1	D_2	D_3	D_4	a_i
S_1	6	1	9	3	70
S_2	11	5	2	8	55
S_3	10	12	4	7	90
b_j	85	35	50	45	

8. Find the minimum cost of the adjoining unbalanced transportation problem.

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	1	5	0	13
O_3	3	8	6	12
b_j	8	5	4	

[CU(H) 1982]

9. Solve the adjoining unbalanced transportation problem.

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	1	3	0	5	1	20
O_2	−6	5	10	3	−5	34
O_3	2	−4	3	2	0	11
b_j	10	8	15	12	8	

10. Define an assignment problem. Let $C = (c_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices, where $b_{ij} = c_{ij} - d_i - e_j$, d_i, e_j are arbitrary constants. Then show that an optimal solution to the assignment problem with the cost matrix B is also an optimal solution to the assignment problem with the cost matrix C, and vice versa.

11. *If P is a non-empty subset of the points of matrix* $C = (c_{ij})$, *then the maximum number of independent points that can be selected in P is equal to the minimum number of lines covering all the elements of P.* — Prove it.

12. There are four men and each of them has to perform one of the four tasks. The men differ in their efficiency and ability to complete the tasks. The estimate of the time required by each person to complete each task as shown in the adjoining. Assign a task to each man so as to *minimise the total time* spent on the four assignments.

		Man			
		A	B	C	D
Task	a	18	26	17	11
	b	14	28	14	26
	c	38	19	18	15
	d	19	26	24	10

13. A machine shop purchased a drilling machine and two lathes of different capacities. The positioning of the machines among four possible locations on the shop-floor is important from the standpoint of materials handling. Given the *cost estimate per unit of time* of materials handling as shown in the adjoining table. Determine the optimal location of the three machines.

[CU(H) 2005, MCA(WBUT) 2009]

		Location			
		1	2	3	4
Machine	L_1	12	9	12	9
	D	15	unsuitable	13	20
	L_2	4	8	10	6

14. There are five pumps available for developing five wells. The efficiency of each pump in producing the maximum yield in each well is shown in the adjoining table. In what way should the pumps be assigned so as to *maximise* the overall efficiency? [IAS(Main) 1993]

		Efficiency well				
		W_1	W_2	W_3	W_4	W_5
Pump	P_1	45	40	65	30	55
	P_2	50	30	25	60	30
	P_3	25	20	15	20	40
	P_4	35	25	30	25	20
	P_5	80	60	60	70	50.

15. There are four engineers available for designing four projects. Engineer E_1 is not competent to design project P_3. Given the time estimate required by each engineer to design a given project in the adjoining table. Find an assignment which *minimise the total time.*

		Project			
		P_1	P_2	P_3	P_4
Engineer	E_1	10	3	unsuitable	8
	E_2	4	13	1	5
	E_3	3	7	2	10
	E_4	8	6	1	9

16. A construction company has to move four large cranes from old construction site to new construction site. The distance in kilometres between the old and new locations are as given in the adjoining table. The crane at O_3 cannot be used at N_2 but all the cranes can work equally well at any of the other new sites. Determine a plan for moving the cranes that will *minimise* the total distance involved in the move.

		New cons. sites			
		N_1	N_2	N_3	N_4
Old cons. sites	O_1	15	20	13	40
	O_2	38	42	15	20
	O_3	25	17	30	18
	O_4	18	30	40	35

17. (a) Consider the problem of assigning four operations to four machines. The assignment costs in rupees are given below. Operator O_1 cannot be assigned to machine M_3. Also O_3 cannot be assigned to M_4. Find the optimal assignment.

		Machine			
		M_1	M_2	M_3	M_4
Operator	O_1	5	5	–	2
	O_2	7	4	2	3
	O_3	9	3	5	–
	O_4	7	2	6	7

(b) Suppose that in problem (a) a fifth machine M_5 is made available, its respective assignment costs to the four operators are 2, 1, 2 and 8. The new machine M_5 replaces an existing one if the replacement can be justified economically. Reformulate the problem as an assignment model and find the optimal solution.

18. Solve the following assignment models

(a)

	A	*B*	*C*	*D*	*E*
a	3	8	2	10	3
b	8	7	2	9	7
c	6	4	2	7	5
d	8	4	2	3	5
e	9	10	6	9	10

(b)

	I	II	III	IV	V
1	3	9	2	3	7
2	6	1	5	6	6
3	9	1	7	10	3
4	2	5	4	2	1
5	9	6	2	4	6

[CU(H) 1987, CU(P) 2000]

(c)

	a	*b*	*c*	*d*
1	18	26	17	11
2	13	28	14	26
3	38	19	18	15
4	19	26	24	10

[CU(H) 1987, 2002]

(d)

	I	II	III	IV	V
A	1	3	2	3	6
B	2	4	3	1	5
C	5	6	3	4	6
D	3	1	4	2	2
E	1	5	6	5	4