2016

Ques)

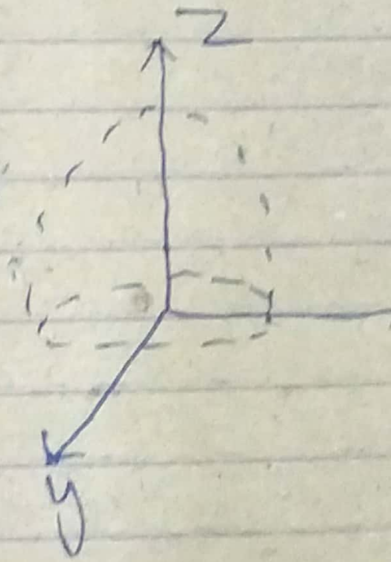
If E be the solid bounded by the xy plane & the paraboloid $z = 4 - x^2 - y^2$ then evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where S is the surface bounding the volume E & $\vec{F} = (zx \sin yz + x^3)\hat{i} + \cos(yz)\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}$

Ans We know as per ~~Stokes~~ Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint (\nabla \cdot F)\, dV$$

$$\nabla \cdot F = z(\sin yz) + 3x^2 - \sin(yz) \cdot z + 3y^2$$

$$= 3(x^2 + y^2)$$



Now, $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint (\nabla \cdot F)\, dV$

$$= \iiint_v 3(x^2 + y^2)\, dx\, dy\, dz$$

$z = 4 - x^2 - y^2$

Taking $x = r\cos\theta$
$y = r\sin\theta$

Limits:

$z = 0 \to 4 - r^2$
$r = 0 \to 2$
$\theta = 0 \to 2\pi$

$$\therefore \iiint_V 3(x^2+y^2)\, dx\, dy\, dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{4-r^2} 3(r^2)\, r\, dr\, d\theta\, dz$$

$$3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3(4-r^2)\, dr\, d\theta$$

$$3\int_{\theta=0}^{2\pi}\left[\frac{4r^4}{4} - \frac{r^6}{6}\right]_0^2 d\theta$$

$$3\int_{\theta=0}^{2\pi}\left(2^4 - \frac{2^6}{6}\right) d\theta$$

$$= \cancel{3 \cdot 2^4\left(1 - \frac{8}{7}\right) \cdot 2\pi} \Rightarrow 3 \cdot 2^4 \cdot \frac{1}{3} \cdot 2\pi$$

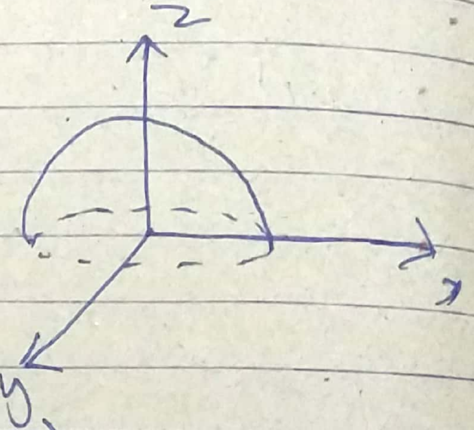
$$= 32\pi$$ //

Ans 2

Evaluate $\iint_S (\nabla \times \vec{f}) \cdot \hat{n} \, dS$ for

$\vec{f} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z k$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by its projection on the xy plane.

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$



$$= \hat{i}(-2yz+2yz) - \hat{j}(0-0) + k(1)$$

$$= k$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|}, \quad S = x^2+y^2+z^2-1$$

$$= \frac{2x(\hat{i}) + 2y(\hat{j}) + 2z(k)}{2\cdot\sqrt{x^2+y^2+z^2}}$$

$$= x\hat{i} + y\hat{j} + zk$$

$(\nabla \times F) \cdot \hat{n} = z$

Taking projection of the surface S on the xy plane as R,

$$ds = \frac{dx\,dy}{|\hat{n} \cdot k|} = \frac{dx\,dy}{z}$$

$$\therefore \iint_S (\nabla \times F) \cdot \hat{n}\, ds = \iint_R (\nabla \times F) \cdot \hat{n} \frac{dx\,dy}{z}$$

$$= \iint_R z \frac{dx\,dy}{z} = \iint_R dx\,dy$$

R is the circle $x^2+y^2=1$.

$$\therefore \iint_R dx\,dy = \pi(1)^2$$
$$= \pi$$

By Stokes

We know that as per the Stokes theorem,

$$\oint_C F \cdot dr = \iint_S (\nabla \times f) \cdot \hat{n}\, ds$$

Here, C is the boundary of the surface S.

$C: x^2+y^2=1$

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$$F \cdot dr = (2x-y)\,dx - (yz^2)\,dy - (y^2 z)\,dz$$

$$\int_C F \cdot dr = \int_C (2x-y)\,dx - (yz^2)\,dy - (y^2 z)\,dz$$

Taking $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

$z = 0$; $dz = 0$ in C.

$$\int_C F \cdot dr = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

$$= \int_{\theta=0}^{2\pi} -\sin(2\theta) + \sin^2\theta \quad d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) \right]_{\theta=0}^{2\pi}$$

$$= \left[\frac{1}{2} + \frac{1}{2}(2\pi) - \frac{1}{2} - 0 \right]$$

$$= \pi \quad //$$

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Ans 3

State Stoke's Theorem. Verify the stokes theorem for the function $f = xi + zj + 2yk$, where c is the curve obtained by the intersection of the plane $z = x$ & cylinder $x^2 + y^2 = 1$ & S is the surface inside the intersected curve

Stoke's theorem gives us the relation between a line integral & a surface integral.

As per Stoke's theorem,

$$\oint_c \vec{F} \cdot dr = \iint_s (\nabla \times f) \cdot \hat{n} \, ds$$

where S is a surface
$\hat{n}$ is the normal to the surface
c is the boundary of the surface

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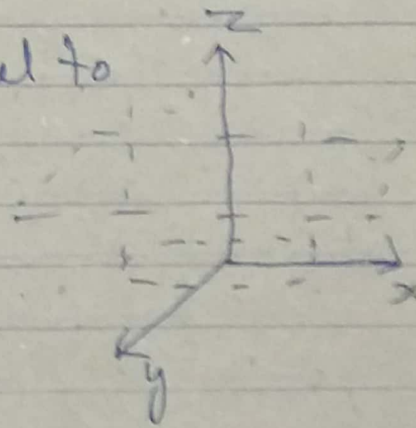
We have
$F = x\hat{i} + z\hat{j} + 2y\hat{k}$

Let C be the intersection of the plane $x-z=0$ & cylinder $x^2+y^2=1$ & S be the surface on it.

$\because$ $x-z=0$ is a plane parallel to the y axis, C will be a circle, i.e.,
$C: x^2+y^2=1$



Now,
$$\int_C F\cdot dr = \iint (\nabla \times F)\cdot \hat{n}\, ds$$

$$F\cdot dr = x\,dx + z\,dy + 2y\,dz$$

$$\int_C F\cdot dr = \int_C x\,dx + z\,dy + 2y\,dz$$

On C, $x = z$
Taking $x = 1\cdot\cos\theta$
$y = 1\cdot\sin\theta$

$$\int_C F\cdot dr = \int_{\theta=0}^{2\pi} \cos\theta(-\sin\theta) + \cos\theta(\cos\theta) + 2\sin\theta(-\sin\theta)$$

$$= \int_{\theta=0}^{2\pi} \left(-\frac{\sin 2\theta}{2} + \cos^2\theta - 2\sin^2\theta\right) d\theta$$

$$= \left[\frac{\cos 2\theta}{2 \cdot 2} + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{2}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_{\theta=}$$

$$= \left[\frac{1}{4} + \frac{1}{2}(2\pi) - (2\pi) - \frac{1}{4} - 0 + 0\right]$$

$$= -\pi$$

Now,

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & z & 2y \end{vmatrix}$$

$$= i(2-1) - j(0) + k(0)$$

$$= i$$

$\hat{n}$ will be the normal to the plane $x - z = 0$

$$\hat{n} = \frac{i - k}{\sqrt{2}}$$

$$(\nabla \times f) \cdot \hat{n} = \frac{1}{\sqrt{2}}$$

Taking projection of the surface S

on the xy plane as R,

$$ds = \frac{dx\,dy}{|\hat{n}\cdot k|} = \underline{-dx\,dy\,.\sqrt{2}}$$

$\therefore$,

$$\iint_S (\nabla\times F)\cdot\hat{n}\,ds = \iint_R -(\nabla\times F)\cdot\hat{n}\,dx\,dy\sqrt{2}$$

$$= -\iint_R \frac{1}{\sqrt{2}}\cdot\sqrt{2}\,dx\,dy = -\iint_R dx\,dy$$

R is the circle $x^2+y^2=1$.

$$\therefore -\iint_R dx\,dy = -\pi(1)^2$$

$$= -\pi.$$

Hence Stokes theorem has been verified.

April May

Ques 4

Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ iff either $\vec{b} = \vec{0}$ or $\vec{c}$ is collinear with $\vec{a}$ or $\vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{c}$.

Ans 4

Sufficient Condition

We have,

i) $\vec{b} = 0$

Now,

$\cancel{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}}$

$\vec{a} \times (\vec{b} \times \vec{c}) = 0$

$(\vec{a} \times \vec{b}) \times \vec{c} = 0$

Hence, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$

ii) $\vec{c} = k\vec{a}$ ($\vec{c}, \vec{a}$ are collinear)

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times (\vec{b} \times k\vec{a})$

$= k(\vec{a} \times (\vec{b} \times \vec{a}))$

$= k(-(\vec{b} \cdot \vec{a})\vec{a} + (\vec{a} \cdot \vec{a})\vec{b})$

$= k(-\vec{a}(\vec{a} \cdot \vec{b}) + a^2\vec{b})$

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Also, $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\times\vec{b})\times k\vec{a}$

$$= k\left((\vec{a}\cdot\vec{a})\vec{b} - (\vec{b}\cdot\vec{a})\vec{a}\right)$$

$$= k\left(a^2\vec{b} - \vec{a}(\vec{b}\cdot\vec{a})\right)$$

Clearly $\vec{a}\times(\vec{b}\times\vec{c}) = \cancel{\vec{a}\times(\vec{b}\times\vec{c})}$ $(\vec{a}\times\vec{b})\times\vec{c}$

(iii) $\vec{b}$ is perpendicular to both $\vec{a}$ & $\vec{c}$

i.e, $\vec{b}\cdot\vec{a}=0$, $\vec{b}\cdot\vec{c}=0$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \left(-(\vec{b}\cdot\vec{a})\vec{c} + (\vec{c}\cdot\vec{a})\vec{b}\right)$$

$$= \left(-0 + \vec{b}(\vec{a}\cdot\vec{c})\right)$$

$$= \vec{b}(\vec{a}\cdot\vec{c})$$

Also, $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

$$= (\vec{a}\cdot\vec{c})\vec{b} - 0$$

$$= \vec{b}(\vec{a}\cdot\vec{c})$$

Hence, $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$

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Necessary condition

We have,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$

$$-(\vec{b}\cdot\vec{a})\vec{c} + (\vec{c}\cdot\vec{a})\vec{b} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} = 0$$

$$(\vec{a} \times \vec{c})\cdot\vec{b} = 0$$

Its clear from above that either

i) $\vec{b} = 0 \Rightarrow (\vec{a} \times \vec{c})\cdot\vec{b} = 0$

or

ii) $\vec{c} = k\vec{a} \Rightarrow (\vec{a} \times k\vec{a})\cdot\vec{b} = 0\cdot\vec{b}$

$= 0$

or

iii) $\vec{b}\cdot\vec{a} = 0$, $\vec{b}\cdot\vec{c} = 0$

$\Rightarrow (\vec{a} \times \vec{c})\cdot\vec{b} = k\vec{b} \times \vec{b}$

$= 0$

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