A particle moves with a central acceleration which varies inversely as the cube of the distance. If it is projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then find the equation to the path. 10

"a" from the origin ...

PYQ 2016

Ans) Let the central acc. be P.
Let ~~a~~ r be the distance.

$$P \propto \frac{1}{r^3}$$

$$P = \frac{\mu}{r^3}$$, μ is ~~a~~ constant of proportionality

Let the velocity of projn be V.

Given that

$$V = V_{circle} \cdot \sqrt{2}$$

where

$$\frac{(V_{circle})^2}{r} = P$$

$$\frac{(V_{circle})^2}{a} = \frac{\mu}{a^3}$$

$$V_{circle} = \frac{\sqrt{\mu}}{a}$$

Now, we know that,

$$\frac{P}{h^2 u^2} = \left(\frac{d^2 u}{d\theta^2}\right) + u$$

Scanned with CamScanner

$$\frac{P}{u^2} = h^2\left(\left(\frac{d^2u}{d\theta^2}\right)+u\right)$$

$$\frac{\mu \cdot u^3}{u^2} = h^2\left(\frac{d^2u}{d\theta^2}+u\right) \quad \left[\because r=\frac{1}{u}\right]$$

$$\mu \cdot u = h^2\left(\frac{d^2u}{d\theta^2}+u\right)$$

Multiplying by $2\left(\frac{du}{d\theta}\right)$ & integrating,

$$2u\cdot\left(\frac{du}{d\theta}\right)\cdot\mu = h^2\left(2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta}\right)$$

$$v^2 = u^2\mu + A = h^2\left(\left(\frac{du}{d\theta}\right)^2+u^2\right)$$

Now, $v^2 = \mu\cdot u^2 + A$

When $r=\frac{1}{u} = a$, $\frac{du}{d\theta} = 0$

$$\left(\frac{\sqrt{\mu}}{a}\cdot\sqrt{2}\right)^2 = \mu\cdot\frac{1}{a^2} + A$$

$$\frac{2\mu}{a^2} = \frac{\mu}{a^2} + A$$

$$A = \frac{\mu}{a^2}$$

Also, $\frac{2\mu}{a^2} = h^2\left(0+\frac{1}{a^2}\right)$

$$h^2 = 2\mu$$

Putting these values back:

$$m \cdot u^2 + \frac{m}{a^2} = 2m\left(\left(\frac{du}{d\theta}\right)^2 + u^2\right)$$

$$u^2 + \frac{1}{a^2} = 2\left(\frac{du}{d\theta}\right)^2 + 2u^2$$

$$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} - u^2$$

$$2\left(\frac{du}{d\theta}\right)^2 = \frac{1 - a^2u^2}{a^2}$$

$$\sqrt{2}\left(\frac{du}{d\theta}\right) = \frac{\sqrt{1 - a^2u^2}}{a}$$

$$\frac{a\sqrt{2} \cdot du}{\sqrt{1 - a^2u^2}} = d\theta$$

$$\frac{\sqrt{2}\, du}{\sqrt{\left(\frac{1}{a}\right)^2 - u^2}} = d\theta$$

Integrating:

$$\sqrt{2} \cdot \sin^{-1}(u \cdot a) = \theta + B$$

Now, when $r = \frac{1}{u} = a$, $\theta = 0$

$$\sqrt{2} \sin^{-1}(1) = B$$

$$B = \frac{\pi}{\sqrt{2}}$$

Scanned with CamScanner

$$\sqrt{2}\,\sin^{-1}(v.a) = \theta + \frac{\pi}{\sqrt{2}}$$

$$\sin^{-1}(av) = \left(\frac{\theta}{\sqrt{2}} + \frac{\pi}{2}\right)$$

$$av = \cos\left(\frac{\theta}{\sqrt{2}}\right)$$

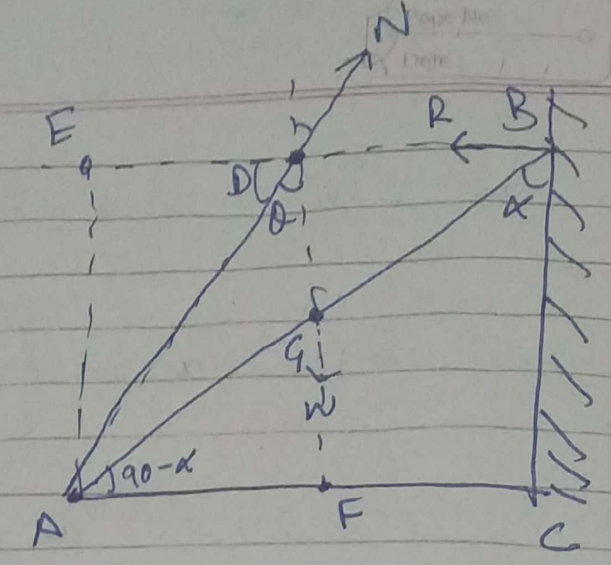
$$v = a\sec\left(\frac{\theta}{\sqrt{2}}\right) \;//.$$

A uniform rod AB of length $2a$ movable about a hinge at A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, prove that the magnitude of reaction of the hinge is

$$\frac{1}{2} W\sqrt{4+\tan^2 \alpha}$$

where W is the weight of the rod. 15

Ans 2



We have,

AB a rod hinged at A.
G is the COM of the rod.

$AG = GB = a$

Since, the system is in equilibrium under the action of 3 forces, the forces must be concurrent. They meet at D.

→ W is the weight of the rod downwards

→ N is the reaction at the hinge at A.

→ R is the reaction at the wall at B.

Let $\angle ADG = \theta$

Applying Lami's at D;

$$\frac{R}{\sin(\pi-\theta)} = \frac{W}{\sin(90+\theta)} = \frac{N}{\sin(90^\circ)}$$

$$\frac{R}{\sin\theta} = \frac{W}{\cos\theta} = N$$

$$N = \frac{W}{\cos\theta} \quad -(1)$$

~~Taking moments around B to eliminate R.~~

In ΔADE,

$$\tan(90-\theta) = \frac{AE}{DE}$$

$$\cot\theta = \frac{BC}{AF} \quad -(2)$$

In ΔABC,

$$\cos(90-\alpha) = \frac{AC}{AB} = \frac{AC}{2a}$$

$$\sin\alpha = \frac{AC}{2a} \quad -(3)$$

In ΔAGF,

$$\cos(90-\alpha) = \frac{AF}{AG}$$

$$\sin\alpha = \frac{AF}{a} \quad -(4)$$

from (3) & (4),

$$AF = \frac{AC}{2} \quad -(5)$$

In ΔABC,

$$\sin(90-\alpha) = \frac{BC}{AB}$$

$$\cos\alpha = \frac{BC}{2a} \quad -(6)$$

Scanned with CamScanner

Ans 3

$$\tan(90-\alpha) = \frac{BC}{AC}$$

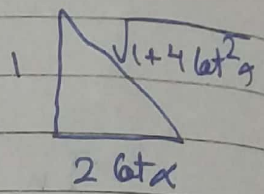
$$\cot\alpha \cdot (AC) = BC \quad - (7)$$

Using (5) & (7) in (2)

$$\cot\theta = \frac{(AC)\cot\alpha}{AC} \cdot 2$$

$$\cot\theta = 2\cot\alpha \quad - (8)$$

Putting (8) in (1)



$$N = \frac{W}{\cos\theta}$$

$$= \frac{W}{2\cot\alpha}\sqrt{1+4\cot^2\alpha}$$

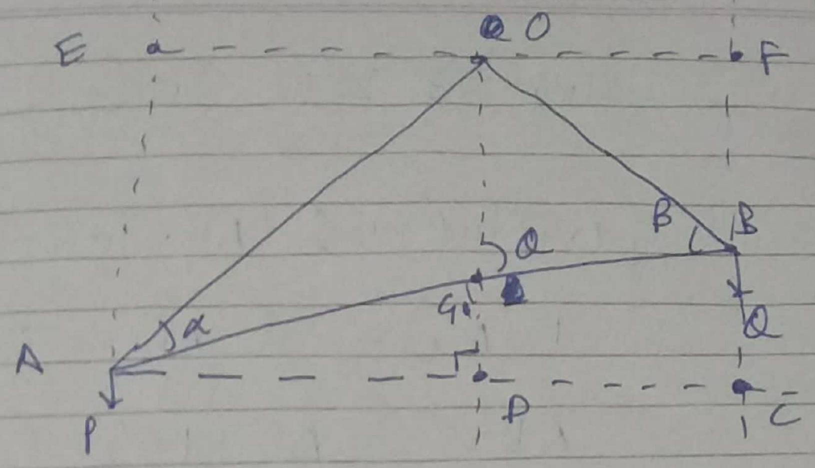
$$= \frac{W}{2}\sqrt{4+\tan^2\alpha} \quad //$$

Scanned with CamScanner

Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If the strings OA and OB make angles α and β with the rod AB, show that the angle θ which the rod makes with the vertical is given by

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}$$ 15

Ans 3



Applying the m, n Cot rule to the triangle ABO,

$$(AG+GB)\cot\theta = (GB)\cot\alpha - (AG)\cot\beta$$

$$(AB)\cot\theta = (GB)\cot\alpha - (AG)\cot\beta \quad — (1)$$

Since the system is in equilibrium, taking the moments around O,

$$P(OE) = Q(OF)$$

$$P(AD) = Q(DC) \quad — (2)$$

Now, in ΔADG,

$$\sin\theta = \frac{AD}{AG}$$

$$AD = AG\sin\theta \quad — (3)$$

In ΔABC,

$$\sin\theta = \frac{AC}{AB}$$

$$AC = AB\sin\theta$$

$$AD + DC = AB\sin\theta \quad — (4)$$

Using (3) & (4) in (2),

$$P(AG\sin\theta) = Q(AB\sin\theta - AG\sin\theta)$$

$$P(AG) = Q(AB) - Q(AG)$$

$$AB = \frac{AG(P+Q)}{Q} \quad -(5)$$

Also,

$$AG + GB = AB$$

$$GB = AB - AG$$

$$= AG\left(\frac{P}{Q}\right) \quad -(6)$$

Using (5) & (6) in (1),

$$AG\frac{(P+Q)}{Q}\cot\theta = AG\left(\frac{P}{Q}\right)\cot\alpha - AG\cot\beta$$

$$\frac{(P+Q)}{Q}\cot\theta = \frac{P}{Q}\cot\alpha - \cot\beta$$

$$\frac{(P+Q)}{Q}\cot\theta = \frac{(P\cot\alpha - Q\cot\beta)}{Q}$$

$$\cot\theta = \frac{P\cot\alpha - Q\cot\beta}{(P+Q)}$$

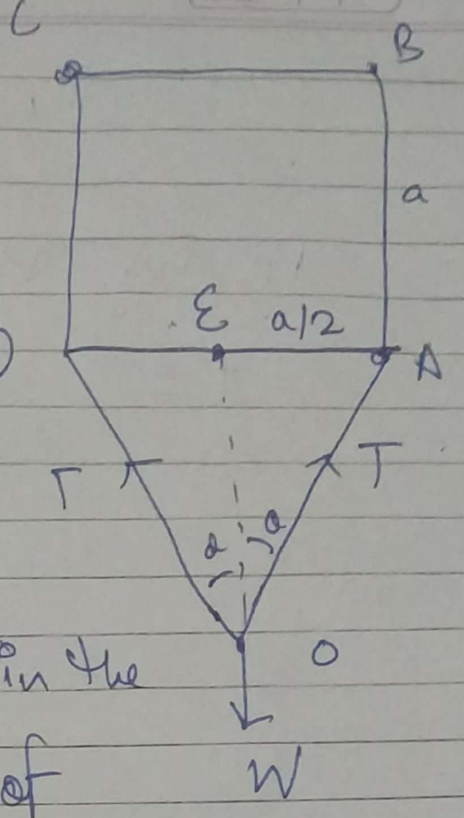
$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta} \quad //$$

A square $ABCD$, the length of whose sides is a, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l(> 4a)$ passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that the tension of the string is $\dfrac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}$. 20

Ans 4

ABCD is a square of edge length 'a'.

OABCDO is a string of length l passing over the pegs at A, B, C, D.



T is the tension in the string.
W is the weight of a ring at O.

Resolving forces at O,

$$2T\cos\theta = W$$
$$T = \frac{W}{2\cos\theta} \quad \text{—} (1)$$

Consider ΔOAE,

$$OA = \frac{l-3a}{2}$$

$$AE = a/2$$

$$OE = \sqrt{\left(\frac{l-3a}{2}\right)^2 - \left(\frac{a}{2}\right)^2}$$

$$\cos\theta = \frac{OE}{OA}$$
$$= \frac{\sqrt{\dfrac{l^2+9a^2-6al-a^2}{4}}}{\dfrac{l-3a}{2}}$$

$$= \frac{\sqrt{l^2 + 8a^2 - 6al}}{l - 3a}$$

Putting this in (1),

$$T = \frac{W(l-3a)}{2\sqrt{l^2+8a^2-6al}}$$

//.

A particle moves in a straight line. Its acceleration is directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then find the time, the particle will arrive at O. 15

Ans 6

$$\left(\frac{d^2x}{dt^2}\right) = -\mu\left(\frac{a^5}{x^2}\right)^{1/3}$$

(-ve because x is decreasing)

multiplying by $2\left(\frac{dx}{dt}\right)$ & integrating,

$$2\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) = -\mu\left(\frac{a^5}{x^2}\right)^{1/3} \cdot 2\left(\frac{dx}{dt}\right)$$

$$\left(\frac{dx}{dt}\right)^2 = -\mu \cdot a^{5/3}\, x^{1/3} \cdot 3 \cdot 2 + A$$

$$\left(\frac{dx}{dt}\right)^2 = -6\mu a^{5/3} \cdot x^{1/3} + A$$

Now, its given,

at $t=0$,

$x = a$

$\left(\frac{dx}{dt}\right) = 0$

$0 = -6\mu a^2 + A$

$A = 6\mu a^2$

$$\left(\frac{dx}{dt}\right)^2 = 6\mu a^2 - 6\mu a^{5/3} \cdot x^{1/3}$$

$$\left(\frac{dx}{dt}\right) = \pm\sqrt{6\mu a^2 - 6\mu a^{5/3} \cdot x^{1/3}}$$

Since x is decreasing,

$$\frac{dx}{dt} = -\sqrt{6\mu a^2 - 6\mu a^{5/3} \cdot x^{1/3}}$$

$$\frac{dx}{\sqrt{6\mu a^2}\sqrt{1 - \frac{x^{1/3}}{a^{1/3}}}} = -dt$$

Integrating

$$\int_{x=a}^{0} \frac{dx}{\sqrt{6\mu a^2}\sqrt{1-\left(\frac{x}{a}\right)^{1/3}}} = \int_{t=0}^{t_1} -dt$$

$$\frac{1}{\sqrt{6\mu a^2}} \int_{x=a}^{0} \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^{1/3}}} = -t_1$$

Let $\left(\frac{x}{a}\right)^{1/6} = \sin\theta$

$$x^{1/6} = a^{1/6} \sin\theta$$

$$dx = 6\cos\theta \cdot a^{1/6} \cdot x^{5/6}\, d\theta$$

$$dx = 6a^{1/6} \cos\theta\, a^{5/6} \sin^5\theta\, d\theta$$

$$dx = 6a \cos\theta \cdot \sin^5\theta\, d\theta$$

$$\frac{1}{\sqrt{6\mu a^2}} \int_{\pi/2}^{0} \frac{6a \cdot \cos\theta \sin^5\theta}{\cos\theta}\, d\theta = -t_1$$

$$\frac{\sqrt{6}}{\sqrt{\mu}} \int_{0}^{\pi/2} \sin^5\theta\, d\theta = +t_1$$

$$t_1 = \frac{\sqrt{6}}{\sqrt{\mu}} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}\sqrt{\frac{6}{\mu}} \quad //$$