IMS
(INSTITUTE OF MATHEMATICAL SCIENCES)

IAS MATHEMATICS (OPT.)-2012

PAPER - I : SOLUTIONS

2012

1 (b) Let p and q be positive real numbers such that $1/p + 1/q = 1$. Show that for real numbers $a, b \geq 0$

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Sol:- If each $a, b > 0$ and each $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$; then.

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab. \quad \text{——} (1)$$

where the equality occurs only when $a^p = b^q$ from the following way-

let $f(\alpha) = \alpha^x (1-\alpha)^{1-x}$; where $x \in (0,1)$ and $\alpha \in [0,1]$.

Since $f(\alpha)$ is continuous on $[0,1]$ and.

$$f'(\alpha) = \alpha^x (1-\alpha)^{1-x} \left[\frac{x}{\alpha} - \left[\frac{1-x}{1-\alpha}\right]\right] = \alpha^x (1-\alpha)^{1-x} \left[\frac{x-\alpha}{\alpha(1-\alpha)}\right]$$

gives: $f'(\alpha) > 0$ or < 0 on $(0,1)$ according as $\alpha < x$, or $> x$, therefore; $\alpha^x (1-\alpha)^{1-x}$ is increasing or decreasing according as $\alpha < x$, or $> x$;

i.e $x^x (1-x)^{1-x} > \alpha^x (1-\alpha)^{1-x}$ for $\alpha < x$ and $\alpha > x$ when $x, \alpha \in (0,1)$. for $x = \alpha$

$x^x (1-x)^{1-x} = \alpha^x (1-\alpha)^{1-x}$. — Hence the result —(2)

The ~~above~~ (1) result is equivalent to that of above (2) preceeding, for

$$x = \frac{1}{p} \; ; \quad \alpha = \frac{qa^p}{qa^p + pb^q}$$

On replacing x by $x_1/(x_1+x_2)$ and α by $a_1(a_1+a_2)$
$\forall\ x_1, x_2, a_1$ and $a_2 \in R^+$, we get

$$\frac{x_1^{x_1/(x_1+x_2)} \cdot x_2^{x_2/(x_1+x_2)}}{(x_1+x_2)} \geq \frac{a_1^{x_1/(x_1+x_2)} \cdot a_2^{x_2/(x_1+x_2)}}{(a_1+a_2)}$$

i.e. $$\left(\frac{x_1}{a_1}\right)^{x_1}\left(\frac{x_2}{a_2}\right)^{x_2} \geq \left(\frac{x_1+x_2}{a_1+a_2}\right)^{x_1+x_2} \quad \text{—— (3)}$$

where the equality occurs only when

$\frac{x_1}{x_1+x_2} = \frac{a_1}{a_1+a_2}$; i.e. only when $\frac{x_1}{a_1} = \frac{x_2}{a_2}$

An obvious extension of (3) $\forall\ x_1, \cdots x_n$ and $a_1 \cdots a_n \in R^+$

is $$\left(\frac{x_1}{a_1}\right)^{x_1} \cdots \left(\frac{x_n}{a_n}\right)^{x_n} \geq \left(\frac{x_1+x_2+\cdots+x_n}{a_1+a_2+\cdots+a_n}\right)^{x_1+x_2+\cdots+x_n}$$

where the equality holds only when $\frac{x_1}{a_1} = \cdots\cdots = \frac{x_n}{a_n}$.

Replacing a_1 by $\frac{x_1^2}{a_1}, \ldots, a_n$ by $\frac{x_n^2}{a_n}$ $\forall \left.\begin{matrix} x_1 \cdots x_n \\ a_1 \cdots a_n \end{matrix}\right\} \in R^+$

we get;

$$\left(\frac{a_1}{x_1}\right)^{x_1} \cdots \left(\frac{a_n}{x_n}\right)^{x_n} \geq \left(\frac{x_1+\cdots+x_n}{x_1^2/a_1+\cdots+x_n^2/a_n}\right)^{x_1+x_2+\cdots+x_n}$$

~~On combining the preceeding two equalities~~,
where the equality occurs only, when

$\frac{x_1}{\frac{x_1^2}{a_1}} = \cdots\cdots = \frac{x_n}{x_n^2/a_n}$; i.e. only when $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$

On combining the preceeding two inequalities.
$\forall\ x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in R^+$. we get

$$\left[\frac{x_1^2/a_1 + \cdots + x_n^2/a_n}{x_1 + x_2 + \cdots + x_n}\right]^{x_1 + \cdots + x_n} \geq \left(\frac{x_1}{a_1}\right)^{x_1} \cdots \left(\frac{x_n}{a_n}\right)^{x_n} \geq \left(\frac{x_1 + x_2 + \cdots + x_n}{a_1 + a_2 + \cdots + a_n}\right)^{x_1 + x_2 + \cdots + x_n} \quad \text{--- (4)}$$

where the equality occurs only when

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$$

Replacement of a_1 by $x_1/a_1, \ldots, a_n$ by x_n/a_n in (4), it equivalently gives Roger's form of the generalised law of Means :

Ques) 1(c) Prove or disprove the following statement if $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for $\mathbb{R}^5$ and V is a two-dimensional subspace of $\mathbb{R}^5$, then V has a basis made of just two members of B.

Sol: Given that $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for $\mathbb{R}^5 = \{(a,b,c,d,e) / a,b,c,d,e \in \mathbb{R}\}$

Let $S = \{b_1, b_2, b_3, b_4, b_5\}$ be a basis of $\mathbb{R}^5$ where $b_1 = (1,0,0,0,0)$, $b_2 = (0,1,0,0,0)$, $b_3 = (0,0,1,0,0)$; $b_4 = (0,0,0,1,0)$, $b_5 = (0,0,0,0,1)$.

Let the two elements of V be a

$\alpha = (1,0,0,1,1)$ and $(1,0,1,1,0) = \beta$.

clearly these vectors are linearly independent and V is a two dimensional vectorspace.

$\therefore$ span $V = \{x(1,0,0,1,1) + y(1,0,1,1,0) / x,y \in \mathbb{R}\}$.

$= \{(x+y, 0, y, x+y, x) / x,y \in \mathbb{R}\}$.

clearly V is not forming by taking any two vectors of b_1, b_2, b_3, b_4, b_5.

as there are ten possibilities of the span of any two vectors of B i.e.

~~2011~~

~~2010~~

~~(i)~~(a)

$b_1+b_2,\ b_2+b_3,\ b_3+b_4,\ b_4+b_5,\ b_5+b_1,\ b_1+b_3,$
$b_3+b_5,\ b_2+b_4,\ b_2+b_5,\ b_1+b_4$

then,

$\text{span}(b_1+b_2) = \{x(1,0,0,0,0)+y(0,1,0,0,0)\} = \{(x,y,0,0,0)\}$

$\text{span}(b_2+b_3) = \{(0,x,y,0,0)\}$ $\forall\ x,y\in\mathbb{R}$

$\text{span}(b_3+b_4) = \{(0,0,x,y,0)\}$

$\text{span}(b_4+b_5) = \{(0,0,0,x,y)\}$

$\text{span}(b_5+b_1) = \{(x,0,0,0,y)\}$

$\text{span}(b_1+b_3) = \{(x,0,y,0,0)\}$

$\text{span}(b_3+b_5) = \{(0,0,x,0,y)\}$ $\forall\ x,y\in\mathbb{R}$

$\text{span}(b_2+b_4) = \{(0,x,0,y,0)\}$

$\text{span}(b_2+b_5) = \{(0,x,0,0,y)\}$

$\text{span}(b_1+b_4) = \{(x,0,0,y,0)\}$

Here, span of any two vector of B is not contained in the span of V i.e.

$\text{span}(b_i+b_j) \not\subseteq \text{span}\ V$ $\forall\ i,j=1,2,3,4,5,$ $i\neq j.$

Neither, $\text{span}(V) \not\subseteq \text{span}(b_i+b_j)$.

thus, V ~~has~~ does not have a basis made of just two members of B.

Que) 1(d) IAS -2012

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\alpha, \beta, \gamma) = (\alpha + 2\beta - 3\gamma, 2\alpha + 5\beta - 4\gamma, \alpha + 4\beta + \gamma)$. —(1)

Find a basis and the dimension of the image of T and the kernel of T.

Sol: Let, $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$ (domain).

Now take transformation on 'S'.

$$\left.\begin{aligned} T(1,0,0) &= (1,2,1) \\ T(0,1,0) &= (2,5,4) \\ T(0,0,1) &= (-3,-4,1) \end{aligned}\right\} \text{ using equation (1)}$$

Now, consider $S_1 = \{(1,2,1), (2,5,4), (-3,-4,1)\} \subseteq \mathbb{R}^3$ (co-domain)

Now, to check whether S_1 is L.I or not

let us consider a matrix 'A' whose rows are vectors of S_1.

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ -3 & -4 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2 \qquad A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, this is echelon form of matrix A and here 2 non-zero rows

$\therefore \rho(A) =$ no. of non-zero rows $= 2$

$\rho(A) = \dim[R(T)] \Rightarrow \text{Dim } R(T) = 2.$

& $\text{Basis}_1 = \{(1,2,1),(0,1,2)\}$ forms the basis of $R(T)$.

Now, to find Basis and dimension of Nullspace of T:

$N(T) = \{(a,b,c) \mid T(a,b,c) = \hat{0}\}$.

$T(a,b,c) = 0$

using equation ①

$\Rightarrow (a+2b-3c,\ 2a+5b-4c,\ a+4b+c) = (0,0,0)$

$\Rightarrow a+2b-3c = 0$ —(i)

$2a+5b-4c = 0$ —(ii)

$a+4b+c = 0$ —(iii)

Its Matrix form [Co-efficient Matrix]

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 4 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$B \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} R_3 \to R_3 - 2R_2 \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim B$$

Corresponding to the system is

$a+2b-3c = 0$ —(iv)

$b+2c = 0$ —(v)

$\Rightarrow$ from (v) $\Rightarrow b = -2c$

from (iv) $\Rightarrow a = -7c$

Here, only one free variable 'c' and a&b are dependent on 'c'

Hence; $\dim(N(T))$ = No. of free variables = 1

$\therefore \boxed{\dim[N(T)] = 1}$

Choose; $c = 1$; and $a = -7$, $b = -2$.

$\Rightarrow (a,b,c) = (-7,-2,1)$

Thus; $\overline{\text{Basis}_2} = \{(-7,-2,1)\}$ is Basis for $N(T)$.

Qu. 2(c)(i) IFoS 2012

Let V be the vector space of all 2×2 matrices over the field of real numbers. Let 'W' be the set consisting of all matrices with zero determinant. Is 'W' a subspace of V? Justify your answer.

Sol:- let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \middle| a, b, c, d \in \mathbb{R} \right\}$

& $W = \left\{ A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \middle| \det A = 0 \right\} \subseteq V$

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ & $\det B = 0$

Clearly; $A, B \in W$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A+B) = 1 \neq 0$

$\Rightarrow A+B \notin W$.

Hence, W is not the subspace of V.

Ques) 2c(ii) RAS -2012

Find the dimension and a basis for the space 'W' of all solutions of the following homogeneous system using matrix notation:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$$

Sol: Given homogeneous system is:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{--- (1)}$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0 \quad \text{--- (2)}$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0 \quad \text{--- (3)}$$

Co-efficient matrix of given system is —

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, corresponding homogeneous system is

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{--- (4)}$$
$$2x_3 + 5x_4 + x_5 = 0 \quad \text{--- (5)}$$

Here, 3 variables x_2, x_4, x_5, and values of x_1 and x_3 depend upon x_2, x_4, x_5

from (5) $\quad x_3 = -\dfrac{(5x_4 + x_5)}{2} \quad \text{--- (6)}$

from (4) $\quad x_1 = -2x_2 - 3x_3 + 2x_4 - 4x_5$

$$x_1 = (-2x_2) + \frac{19x_4 - 5x_5}{2}$$

$$\Rightarrow x_1 = \frac{-4x_2 + 19x_4 - 5x_5}{2} \quad \text{--- (7)}$$

Now, put $x_2 = 1$, $x_4 = 0$, $x_5 = 0$

$$\Rightarrow x_3 = 0, \; x_1 = -2$$

Then $(x_1, x_2, x_3, x_4, x_5) = (-2, 1, 0, 0, 0)$ --- (8)

put $x_2 = 0$, $x_4 = 1$, $x_5 = 0$

$$\Rightarrow x_3 = -\frac{5}{2}, \; x_1 = \frac{19}{2}$$

then $(x_1, x_2, x_3, x_4, x_5) = (19/2, 0, -\frac{5}{2}, 1, 0)$ --- (9)

Put $x_2 = 0$, $x_4 = 0$, $x_5 = 1$

$$\Rightarrow x_3 = -\frac{1}{2}; \; x_1 = -\frac{5}{2}$$

then $(x_1, x_2, x_3, x_4, x_5) = (-\frac{5}{2}, 0, -\frac{1}{2}, 0, 1)$ --- (10)

Hence, from (8), (9) & (10)

$$B = \left\{(-2, 1, 0, 0, 0), \left(\frac{19}{2}, 0, \frac{-5}{2}, 1, 0\right), \left(\frac{-5}{2}, 0, \frac{-1}{2}, 0, 1\right)\right\}$$

is the basis for the space W.

& $\boxed{\text{Dim } W = 3}$

2(b)(i) IAS 2012

Q. Consider the linear Mapping $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,y) = (3x+4y, 2x-5y)$; find the matrix 'A' relative to the basis $\{(1,0),(0,1)\}$ and the matrix 'B' relative to the basis $\{(1,2),(3,4)\}$

Sol:- Given: linear mapping $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$f(x,y) = (3x+4y, 2x-5y) \quad \text{---} (1)$$

To find the matrix 'A' relative to basis $\{(1,0),(0,1)\}$

Consider $B_1 = \{(1,0),(0,1)\} = \{\alpha_1, \alpha_2\}$.

$$(a,b) \in \mathbb{R}^2 \Rightarrow (a,b) = a(1,0) + b(0,1) \quad \text{---} (2)$$

$$f(\alpha_1) = (3,2) \quad \text{--- (using eq (1))}$$

$$f(\alpha_1) = 3(1,0) + 2(0,1) \quad \text{(using eq (2))} \quad \text{---} (3)$$

Similarly; $f(\alpha_2) = (4,-5)$

$$f(\alpha_2) = 4(1,0) + (-5)(0,1) \quad \text{---} (4)$$

from (3) & (4) matrix 'A', relative to B_1 is

$$A = [f : B_1] = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} \quad \text{---} (5)$$

Now, to find matrix B relative to the basis $\{(1,2),(2,3)\}$

consider $B_2 = \{(1,2),(2,3)\} = \{\beta_1, \beta_2\}$

where; $\beta_1 = (1,2)$ & $\beta_2 = (2,3)$

$(a,b) \in \mathbb{R}^2 \quad (a,b) = x\beta_1 + y\beta_2$

$\Rightarrow (a,b) = x(1,2) + y(2,3)$

$(a,b) = (x+2y, 2x+3y)$

$x + 2y = a$ —— (6)

$2x + 3y = b$ —— (7)

On solving (6) & (7), we get

$y = 2a - b$ & $x = -3a + 2b$

$\Rightarrow (a,b) = (-3a+2b)(1,2) + (2a-b)(2,3)$ —— (8)

Now, $f(\beta_1) = (11,-8)$ using eq (1)

$f(\beta_1) = (-49)(1,2) + 30(2,3)$ —— (9) using (8)

$f(\beta_2) = (18,-11)$ — using eq (1)

$f(\beta_2) = (-76)(1,2) + 47(2,3)$ —— (10)

from (9) & (10),

matrix 'B' relative to basis B_2

is $B = [f : B_2] = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$

Q. If λ is a characteristic root of a non-singular matrix 'A', then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

2 (b)(ii) IAS-2017

Sol: 'A' be a non-singular matrix,

let λ be an eigen value of A

$\Rightarrow \quad AX = \lambda X \quad \text{——} \; (1)$

$\because \quad A^{-1} = \frac{\text{Adj } A}{|A|}$

$\Rightarrow \frac{(\text{Adj } A) \cdot A}{|A|} = A^{-1}A$

$(\text{Adj } A) A = |A| I \quad \text{——} \; (2)$

$|(\text{Adj } A) A| X = |A| I X$

$\text{Adj } A \cdot (AX) = |A| X$

$\text{Adj } A \cdot (\lambda X) = |A| X$

$(\text{Adj } A) X = \frac{|A|}{\lambda} \cdot X$

[By associativity of matrix multiplication]

$\Rightarrow \frac{|A|}{\lambda}$ is a characteristic root of Adj 'A'.

2(c) IAS —2012 → Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian Matrix. Find a non-singular Matrix P such that $D = P^T H \bar{P}$ is diagonal. P-II

Sol Let us construct a block Matrix

$$[H \mid I] = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 1 & i & i & 1 & 0 \\ 0 & -i & -3 & -(2-i) & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + iR_1 \\ R_3 \to R_3 - (2-i)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & -(2i-1) \\ 0 & -i & -3 & -(2-i) & 2+i & -4 \end{array}\right] \quad \begin{array}{l} C_2 \to C_2 - iC_1 \\ C_3 \to C_3 - (2+i)C_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & -(2i-1) \\ 0 & 0 & -4 & -3+i & 4i+1 & -2+i \end{array}\right] \quad R_3 \to R_3 + iR_2$$

$$C_3 \to C_3 - iC_1$$

$$[H|I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -3-i \\ 0 & 1 & 0 & i & 2 & -4i+1 \\ 0 & 0 & -4 & -3+i & 4i+1 & 2 \end{array}\right]$$

$$P = \begin{bmatrix} 1 & -i & -3-i \\ i & 2 & -4i+1 \\ -3+i & 4i+1 & 2 \end{bmatrix}$$

and $$P^T H \bar{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = D$$

5(a). Solve $\dfrac{dy}{dx}=\dfrac{2xy\,e^{(x/y)^2}}{y^2(1+e^{(x/y)^2})+2x^2e^{(x/y)^2}}$

SOLUTION

Clearly given equation is homogenous

Let $\dfrac{y}{x}=t \Rightarrow \dfrac{dy}{dx}=t+x\dfrac{dt}{dx}$

$$\therefore \qquad t+x\frac{dt}{dx} = \frac{2t\,e^{1/t^2}}{t^2\left(1+e^{1/t^2}\right)+2e^{1/t^2}}$$

$$\frac{x\,dt}{dx} = \frac{2te^{1/t^2}}{t^2\left(1+e^{1/t^2}\right)+2e^{1/t^2}}-t$$

$$\frac{x\,dt}{dx} = \frac{-t^3\left(1+e^{1/t^2}\right)}{t^2\left(1+e^{1/t^2}\right)+2e^{1/t^2}}$$

$$-\frac{dx}{x} = \frac{\left(1+e^{1/t^2}\right)t^2+2e^{1/t^2}}{t^3\left(1+e^{1/t^2}\right)}\cdot dt$$

$$-\frac{dx}{x} = \frac{dt}{t}+\frac{d\left(1+e^{1/t^2}\right)}{\left(1+e^{1/t^2}\right)}$$

integrating both sides

$$-\ell n\,x = \ell n\,t-\ell n\left(1+e^{1/t^2}\right)C$$

$$\ell n\,tx = \ell n\left(1+e^{1/t^2}\right)C$$

$$\boxed{\therefore y=C\left(1+e^{(x/y)^2}\right)}$$

Required solution.

5(b). Find the orthogonal trajectories of the family of curves $x^2 + y^2 = ax.$

SOLUTION

Goven equation $x^2 + y^2 = ax$

differenting w.r.t to x

$$2x + 2y\frac{dy}{dx} = a$$

eliminating the arbitary constant a

$$\therefore \quad \left(x^2 + y^2\right) = \left(2x + xy\frac{dy}{dx}\right)x$$

for othogonal trajections replace

$$\frac{dy}{dx} \text{ by } -\frac{1}{dy/dx}$$

$$x^2 + y^2 = 2x^2 + 2xy\frac{(-1)}{dy/dy}$$

$$2xy\frac{dx}{dy} = (x^2 - y^2)$$

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

Take common x^2 from N^r and D^r

$$\frac{dy}{dx} = \frac{2(y/x)}{1-(y/x)^2}$$

Clearly given equation is homogenous, then

put $\frac{y}{x} = t \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(tn) = t + x\frac{dt}{dx}$

$$\therefore \quad t + x\frac{dt}{dx} = \frac{2t}{1-t^2}$$

$$x\frac{dt}{dx} = \frac{t+t^3}{1-t^2}$$

$$\frac{dx}{x} = \frac{\left(1-t^2\right)dt}{t+t^3}$$

$$\frac{dx}{x} = \left[\frac{1}{t} - \frac{2t}{1+t^2}\right]dt$$

$$\ell nx = \ell n\,t - \ell n(1+t^2) + C$$

$$x = \frac{Ct}{1+t^2}$$

Put $\quad t = y/x$

$$\therefore \quad \boxed{x^2 + y^2 = cy}$$

5(c). Using Laplace transforms, solve the intial value problem

$$y'' + 2y' + y = e^{-t}, y(0) = -1, y'(0) = 1$$

SOLUTION

Given equation $y'' + 2y' + y = e^{-t}$

Applying Laplace transform as both sides

$$\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(e^{-t})$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 2s\mathcal{L}(y) - 2y(0) + \mathcal{L}(y) = \frac{1}{s+1}$$

(given, $y(0) = -1$, $y'(0) = 1$)

$$\left(s^2 + 2s + 1\right) \mathcal{L}(y) + s - 1 + 2 = \frac{1}{s+1}$$

$$(s+1)^2 \mathcal{L}(y) = \frac{1}{s+1} - (s+1)$$

$$\mathcal{L}(y) = \frac{1}{(s+1)^3} - \frac{1}{(s+1)}$$

Applying inverse transform

$$y(t) = e^{-t}\frac{t^2}{2} - e^{-t}$$

$$\boxed{y(t) = e^{-t}\left(\frac{t^2}{2} - 1\right)}$$

5(d) IAS-2012 P-I

A particle whose mass is m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^3}\right]$ towards origin, if it starts from rest at a distance a. show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$

Sol^n: Given $\frac{d^2x}{dt^2} = -\mu\left[x+\frac{a^4}{x^3}\right]$, ——— (i)

the $-$ve sign being taken because the force is attractive

multiply on bothside by $2(dx/dt)$ and integrating we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2+\frac{a^4}{x^2}\right]+C$$

when $x=a$, $dx/dt = 0$, so that $C=0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4-x^4}{x^2}\right]$$

$$\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4-x^4)}}{x} \quad ——— (2)$$

the $-$ve sign is taken because the particle is moving in the direction of x decreasing.

If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{a^4-x^4}}dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{a^4-x^4}}$$

Put $x^2 = a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$. when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$

$$\therefore t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta$$

$$= \frac{1}{2\sqrt{\mu}}\left[\theta\right]_0^{\pi/2} = \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}$$

5.(e) If $\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$

$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$

Find the value of $\frac{\partial^2}{\partial x \partial y}\left(\vec{A}\times\vec{B}\right)$ at (1, 0, –2).

SOLUTION

Given $\vec{A} = x^2yz\vec{i} - 2xz^3\vec{j} + xz^2\vec{k}$

$\vec{B} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$

Then the value of $\frac{\partial^2}{\partial n \partial y}(\vec{A}\times\vec{B}) = ?$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \left(2x^3z^3 - xyz^2\right)\hat{i} + \hat{j}\left(x^4yz + 2xz^3\right) + \hat{k}\left(x^2y^2z + 4xz^4\right) \quad .....(1)$$

Partially differentiating (1) w.r.t. 'y'

$$\Rightarrow \frac{\partial(\vec{A}\times\vec{B})}{\partial y} = -xz^2\,\hat{i} + x^4z\,\hat{j} + 2x^2zy\,\hat{k} \quad .....(2)$$

Now partially differentiating (2) w.r.t.'x'

$$\frac{\partial^2(\vec{A}\times\vec{B})}{\partial x\,\partial y} = -z^2\,\hat{i} + 4x^3z\,\hat{j} + 4xyz\,\hat{k}$$

At (1, 0, –2)

$$\frac{\partial^2(\vec{A}\times\vec{B})}{\partial x\,\partial y} = -(-2)^2\,\hat{i} + 4\times 1\times(-2)\,\hat{j} + 0$$

$$\frac{\partial^2(\vec{A}\times\vec{B})}{\partial x\,\partial y} = -4\hat{i} - 8\,\hat{j}$$

is the required answer

6(a). Show that the differential equation

$$(2xy\log y)dx+\left(x^2+y^2\sqrt{y^2+1}\right)dy=0$$

is not exact. Find an integrating factor and hence, the solution of the equation.

SOLUTION

Given equation is

$$\left(2xy\log y\right)dx+\left(x^2+y^2\sqrt{1+y^2}\right)dy=0$$

comparing with standard exact equation Mdx + Ndy = 0

$\therefore$ $$M = 2xy\log y$$

$$N = x^2+y^2\sqrt{1+y^2}$$

$$\frac{\partial M}{\partial y} = 2x\log y + 2x.$$

$$\frac{\partial N}{\partial x} = 2x$$

$$\sin u\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

$\therefore$ Given equation is not exact

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{-2x\log y}{2xy\log y}=\frac{-1}{y}=f(y)$$

$\therefore$ $$I.F. = e^{-\int\frac{1}{y}dy} = e^{-\log y}=e^{\log\frac{1}{y}}$$

$$I.F. = \frac{1}{y}$$

Multiply by integrating factors we have

$$\left(2x\log y\right)dx+\left(\frac{x^2}{y}+y\sqrt{1+y^2}\right)dy=0$$

this is exact differential equation.

$\therefore$ $$\int\left(2x\log y\right)dx+\int y\sqrt{y^2+1}\,dy=C$$

$$x^2\log y+\frac{\left(y^2+1\right)^{3/2}}{3}=C$$

$\therefore$ Solution of given equation

$$\boxed{\left[x^2\log y+\frac{\left(y^2+1\right)^{3/2}}{3}\right]=C}$$

6(b). Find the general solution of the equation

$y''' - y'' = 12x^2 + 6x.$

SOLUTION

Given equation $y'' - y' = 12x^2 + 6x$

$\Rightarrow \quad (D^3 - D^2)y = 12x^2 + 6x$

Homogenous equation of given problem is

$$D^3 - D^2 = 0$$

Auxillary equation of given equation $m^3 - m^2 = 0$

$$m = 0, 0, 1$$

$\therefore$ Complementary fuction

$$CF = C_1 + C_2\, x + C_3 e^x$$

Particular integral

$$P.I = \frac{\left(12x^2 + 6x\right)}{D^3 - D^2}$$

$$= \frac{-1}{D^2}\left[1 - D\right]^{-1}\left(12x^2 + 6x\right)$$

$$= \frac{-1}{D^2}\left[1 + D + D^2 .....\right]\left(12x^2 + 6x\right)$$

$$= \frac{-1}{D^2}\left[12x^2 + 6x + 24x + 6 + 24\right]$$

$$\boxed{y_p = -x^4 - 5x^3 - 15x^2}$$

$\therefore$ General solution $= C_1 + C_2 x + C_3\, e^x - x^4 - 5x^3 - 15x^2$

6(c). Solve the ordinary differential equation

$$x(x-1)y''-(2x-1)y'+2y=x^2(2x-3)$$

SOLUTION

Given $y''-\frac{(2x-1)}{x(x-1)}y'+\frac{2y}{x(x-1)}=\frac{x(2x-3)}{(x-1)}$

comparing with $y''+Py'+Qy=R$, we have

$$P=\frac{-(2x-1)}{x(x-1)}$$

$$Q=\frac{2}{x(x-1)}$$

$$R=\frac{x(2x-3)}{(x-1)}$$

Clearly $2+2Px+Qx^2=0$

$u=x^2$ is the part of complementary function when the part of complementary function is know.

Let $y=uv$

when $u=x^2$

$\therefore$ new equation is v.

$$\frac{d^2v}{dx^2}+\left(P+\frac{2u'}{u}\right)\frac{dv}{dx}=\frac{R}{u}$$

$$\frac{d^2v}{dx^2}+\left[\frac{-(2x-1)}{x(x-1)}+\frac{4}{x}\right]\frac{dv}{dx}=\frac{(2x-3)}{x(x-1)}$$

$$\frac{d^2v}{dx^2}+\frac{(2x-3)}{x(x-1)}\frac{dv}{dx}=\frac{2x-3}{x(x-1)}$$

Let $\frac{dv}{dx}=t$

$$\therefore \quad \frac{dt}{dx}+\frac{2x-3}{x(x-1)}t=\frac{2x-3}{x(x-1)}$$

$$\frac{dt}{1-t}=\left[\frac{2x-3}{x(x-1)}\right]dx$$

integrating on bothsides

$$-\ell n(1-t)=\int\left[\frac{3}{x}+\frac{-1}{x-1}\right]dx+c$$

$$-\ell n(1-t)=3\ \ell nx-\ell n\ (x-1)+c$$

$$(1-t)=c\frac{(x-1)}{x^3}$$

$$\frac{dv}{dx}=1-\frac{c}{x^2}+\frac{c}{x^3}$$

$$v=x+\frac{c_1}{x}-\frac{c_1}{2x^2}+c_2$$

$\therefore$ genral solution $y=uv$

$$y=x^3+c_1x-\frac{c_1}{2}+c_2x^2$$

$$\boxed{y=c_1\left(\frac{x-1}{2}\right)+c_2x^2+x^3}$$

8.(a) Derive the Frenet-Serret formulae.

Define the curvature and torsion for a space curve. Compute them for the space curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

Show that the curvature and torsion are equal for this curve.

SOLUTION

8(a) IAS 2012 P-I

* Serret – Frenet formulae :–

The set of relations involving the derivatives of the fundamental vectors T, N, B is known collectively as the Serret-Frenet formulae given by

(1) $\frac{dT}{ds} = kN$; (2) $\frac{dB}{ds} = -\tau N$ (3) $\frac{dN}{ds} = \tau B - kT$.

where τ is a scalar called the torsion.

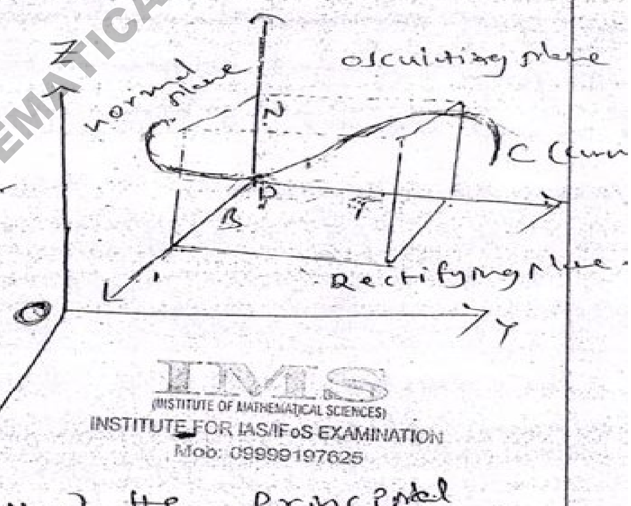
The quantity $\sigma = \frac{1}{\tau}$ is called the radius of the torsion.

* Principal Normal vector :–

Any line perpendicular to the tangent to a curve at a point 'P' is called a normal line at 'P'.

The normal line lying in the osculating plane is called the principal normal at 'P'.

– The unit principal normal is denoted by N.



– The osculating plane to a curve at a point is the plane containing the tangent and principal normal at 'P'.

Normal plane :–

The plane through the point 'P' perpendicular to the tangent at 'P' is called the normal plane at 'P'

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Rectifying plane :-

The plane through the point 'p' perpendicular to the principal normal is called rectifying plane.

Binormal:

Let T be the unit tangent vector, N be the unit principal normal vector to the curve at a point 'p'.
If B is the unit vector perpendicular to both T and N such that T, N, B form a right handed system then B is called binormal vector to the curve at 'p'.

Thus the binormal is the perpendicular to the osculating plane.

* Right-handed system of T, N, B :-

The vectors T, N, B form a right handed system of unit vectors

$\therefore T\cdot T = 1,\ N\cdot N = 1,\ B\cdot B = 1;\ T\cdot N = N\cdot B = B\cdot T = 0.$

and $T\times N = B,\ N\times B = T,\ B\times T = N$;

$T\times T = N\times N = B\times B = 0.$

Proof of Serret-Frenet formulae:

(1) $\frac{dT}{ds} = kN.$

Let $\vec{r}(t)$ be the position vector of the point 'p' on the curve $(\bar{r}(t) = \bar{r}(t))$, then the unit vector T at p is given by $\frac{d\bar{r}}{ds} = T$.

Since $|T| = 1$ i.e T is of constant magnitude

we have $T\cdot\frac{dT}{ds} = 0.$

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(28)

$\therefore \frac{dT}{ds}$ is perpendicular to T.

But we know that $\frac{dT}{ds}$ lies in the osculating plane.

$\therefore \frac{dT}{ds}$ is parallel to N. $\Rightarrow \frac{dT}{ds} = \pm kN$ for some scalar k.
By convention, we take +ve sign,

$\Longrightarrow \frac{dT}{ds} = kN$. for some scalar k. —— (i)

Curvature:- If T is the unit tangent vector to the curve $\vec{r}(s)$ at a point P then the rate of change of T w.r.t 's' is called curvature of the curve at 'P'. It is denoted by k.

The reciprocal of k is called radius of curvature of the curve at 'P'. It is denoted by ρ i.e. $\rho = \frac{1}{k}$.

Note: $\left|\frac{dT}{ds}\right| = k$. (from (i), $|N|=1$).

(2) $\frac{dB}{ds} = -\tau N$.

since $|B| = 1$
i.e. B is of constant magnitude.

$\therefore B \cdot \frac{dB}{ds} = 0$.

$\therefore \frac{dB}{ds}$ is perpendicular to B

we know that $\frac{dB}{ds}$ lies in the osculating plane

Now we have

$B \cdot T = 0$

$\Longrightarrow B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T = 0$ (by diff w.r.t s).

$\Longrightarrow B \cdot (kN) + \frac{dB}{ds} \cdot T = 0$. ($\because \frac{dT}{ds} = kN$).

$$\Rightarrow \frac{dB}{ds}\cdot T + (B\cdot N)k = 0.$$

$$\Rightarrow \frac{dB}{ds}\cdot T = 0. \quad (\because B\cdot N = 0)$$

$$\Rightarrow T\cdot\frac{dB}{ds} = 0.$$

$$\Rightarrow \frac{dB}{ds} \text{ is } \perp \text{ to } T.$$

Since $\frac{dB}{ds}$ lies in the osculating plane

so it must be parallel to N.

$$\therefore \frac{dB}{ds} = \pm\tau N.$$

By convention, $\frac{dB}{ds} = -\tau N.$ — (ii)

Torsion:- If B is the binormal vector to the curve $\vec{r}(s)$ at a point 'p' then the rate of change of B w.r.t. 's' is called torsion of the curve at 'p'. It is denoted by τ.

— The reciprocal of τ is called the radius of torsion and is denoted by σ

i.e. $\sigma = \frac{1}{\tau}$.

Note:- $\left|\frac{dB}{ds}\right| = \tau$ (from (ii), $|N|=1$).

(3) $\frac{dN}{ds} = \tau B - kT$

Now we have $B\times T = N$.

$$\Rightarrow B\times\frac{dT}{ds} + \frac{dB}{ds}\times T = \frac{dN}{ds} \quad (\text{by diff. w.r.t } s)$$

$$\Rightarrow B\times(kN) + (-\tau N)\times T = \frac{dN}{ds}$$

$$\Rightarrow \frac{dN}{ds} = k(B\times N) - \tau(N\times T)$$

$$= k(-T) - \tau(-B)$$

$$= \tau B - kT.$$

Hence the

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Part-2

Given curve is

$$x = t,\ y = t^2,\ z = \frac{2}{3}t^3$$

$$\therefore \quad \vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$$

$$\because \quad \frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 2t^2\hat{k} \qquad .....(1)$$

$$\because \quad \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt} = \sqrt{1+4t^2+4t^4}$$

$$\frac{d^3\vec{r}}{dt^3} = 4\hat{k}$$

$$\because \quad k = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$$

$$\because \quad \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}$$

$$= \hat{i}(8t^2 - 4t^2) - \hat{j}(4t) + \hat{k}(2)$$

$$= (4t^2)\hat{i} - (4t)\hat{j} + 2\hat{k}$$

$$\boxed{k = \frac{2\sqrt{1+4t^2+4t^4}}{\left(\sqrt{1+4t^2+4t^4}\right)^3} = \frac{2}{1+4t^2+4t^4}}$$

$$T = \text{Torsion} = \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$$

$$= \frac{\begin{vmatrix} 1 & 2t & 2t^2 \\ 0 & 2 & 4t \\ 0 & 0 & 4 \end{vmatrix}}{4(1+4t+4t^4)}$$

$$= \frac{4(2)}{4(1+4t^2+4t^4)}$$

$$\boxed{T = \frac{2}{1+4t^2+4t^4}}$$

$$\boxed{K = T}$$

8(b). Verify Green's theorem in the plane for

$$\oint_C \left\{\left\{xy+y^2\right\}dx+x^2dy\right\}$$

where C is the closed curve of the region bounded by y = x and y = x^2.

SOLUTION

Given integral is

$$\oint_C \left[\left(xy+y^2\right)dx+x^2dy\right]$$

Comparing above integral with

$$\oint_C \left(P\,dx+Q\,dy\right)$$

we get $P = xy + y^2$

$Q = x^2$

According to Green's theorem

$$\oint_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$\text{L.H.S.} = \oint_C P\,dx+Q\,dy$$

$$= \oint_C \left(dy+y^2\right)dx+x^2dy \qquad .....(2)$$

where C(curve) is region bounded by

$$\text{Let}\quad \left.\begin{aligned} C_2 &: y=x \\ C_1 &: y=x^2\end{aligned}\right\} \qquad .....(1)$$

Along the curve C_1 : $y = x^2$

$dy = 2x\,dx$

$$(2)\Rightarrow \text{L.H.S.}\int_{C_1}\left(x\cdot x^2+x^4\right)dx+x^2\left(2x\,dx\right)$$

$$= \int_0^1\left(x^4+x^3+2x^3\right)dx$$

$$= \int_0^1\left(x^4+3x^3\right)dx$$

$$= \left.\frac{x^5}{5}+\frac{3x^4}{4}\right|_0^1=\frac{1}{5}+\frac{3}{4}=\frac{19}{20}$$

Along the curve C_2

$$= \int_C\left(xy+y^2\right)dx+x^2dy$$

Now put y = x

$$= \int_{C_2}\left(x^2+x^2\right)dx+x^2dx$$

$$= \int_1^0 3x^2 dx = 3\left.\frac{x^3}{3}\right|_1^0 = -1$$

$$\therefore \qquad \int_C P\,dx + Q\,dy = +\frac{19}{20} - 1 = \frac{-1}{20}$$

$$\text{R.H.S.} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x} \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}\left(xy + y^2\right)\right] dy \cdot dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (2x - x - 2y)\,dy \cdot dx$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\,dy \cdot dx$$

$$= \int_{x=0}^{1} xy - y^2\Big|_{x^2}^{x} \cdot dx$$

$$= \int_{x=0}^{1} \left(-x^3 + x^4 + x^2 - x^2\right) dx = \int_{x=0}^{1} x^4 - x^3 dx$$

$$= \left.\frac{-x^4}{4} + \frac{x^5}{5}\right|_0^1$$

$$= \frac{-1}{4} + \frac{1}{5} = \frac{-1}{20}$$

Hence L.H.S. = R.H.S.

Green's thorem verified

8.(c) If $\vec{F} = y\vec{i} + (x-2xz)\vec{j} - xy\vec{k}$, evaluate

$$\iint_S \left(\vec{\nabla}\times\vec{F}\right).\vec{n}\,d\,\vec{s}$$

where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the $xy-$plane.

SOLUTION

$$\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$$

$$I = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\vec{s}$$

where S : $x^2 + y^2 + z^2 = a^2$ above xy plane

$$\because \qquad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial(-xy)}{\partial y} - \frac{\partial}{\partial z}(x-2xz)\right) - \hat{j}\left(\frac{-\partial}{\partial z}(y) + \frac{\partial}{\partial x}(-xy)\right) + +\hat{k}\left(\frac{\partial}{\partial x}(x-2xz) - \frac{\partial}{\partial y}(y)\right)$$

$$= (-x+2x)\hat{i} - \hat{j}(-y) + \hat{k}(1-2z-1)$$

$$= x\hat{i} + y\hat{j} - 2x\hat{k}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$\therefore \qquad \left(\vec{\nabla}\times\vec{F}\right) = \left(\frac{x^2+y^2-2z^2}{a}\right)$$

$$dS = \frac{dx\,dy}{|\hat{n}-\hat{k}|} = \frac{dx\,dy}{\dfrac{\left|\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\left(-\hat{k}\right)\right|}{a}} = \frac{dx\,dy}{\left|\left(\dfrac{-z}{a}\right)\right|} = \frac{dx\,dy}{\dfrac{z}{a}}$$

$$\therefore \qquad I = \int\left(\frac{x^2+y^2-2z^2}{a}\right)\frac{dx\,dy}{z/a} \qquad \because z\,7x^2 - y^2$$

$$= \iint \frac{x^2+y^2-2\left(x^2-y^2\right)}{\sqrt{a^2-x^2-y^2}}dx\,dy = \iint \frac{3\left(x^2+y^2\right)-2a^2}{\sqrt{a^2-x^2-y^2}}dx\,dy$$

In xy-plane

$$x^2 + y^2 = r^2 \ \& \ z = 0$$

Let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$\Rightarrow \quad I = \iint \frac{3r^2 - 2z^2}{a^2 - r^2} dx\,dy$$

$$= \iint \frac{3r^2 - 2z^2}{a^2 - r^2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} r\,dr \right) d\theta \frac{\pi}{2} \times \int_{r=0}^{a} \frac{3r^2 - 2a^2 r}{\sqrt{a^2 - r^2}}$$

Let $r = a\sin t$

$dr = a\cos t\,dt$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{3a^2 \sin t - 2a^2}{a\cos t} \times a\sin t \times a\cos t\,dt$$

$$= \frac{\pi}{2} - \int_0^{\pi/2} \left[3a^3 \left(1 - \cos^2 t\right) - 2a^3 \right] d(\cot)$$

$$= \frac{-\pi}{2} \int_0^{\pi/2} \left[3a^3 \cot - 3a^3 \frac{\cos^3 t}{3} - 2a^3 \cot \right]_0^{\pi/2}$$

$$= \frac{-\pi}{2} \left[a^3 \cot - a^3 \cos^3 t \right] = 0$$

Hence value of integral

$$\boxed{\iint_S \left(\vec{\nabla} \times \vec{F} \right) \vec{n}\,ds = 0}$$

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