Mains Test Series - 2019

TEST-17 (Paper-I Full Syllabus)

Section-A

Ques:1(a) Find the characteristics polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. Hence find A^{-1} and A^6.

Solution:- Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}_{2\times 2}$, $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then; $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$

$$|A-\lambda I| = (1-\lambda)(3-\lambda)+1$$

$\therefore$ Characteristic polynomial of A is

$$|A-\lambda I| = 0$$

i.e $(1-\lambda)(3-\lambda)+1=0$

$\Rightarrow 3-\lambda-3\lambda+\lambda^2+1=0$

$\Rightarrow \lambda^2-4\lambda+4=0$ ——— (1)

which is required characteristic polynomial equation of A.

We know that every square matrix satisfies its own characteristic equation.

$\therefore$ ① $\equiv A^2 - 4A + 4I = 0$ ——— ②.

Since $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \Rightarrow |A| = 3+1 = 4 \neq 0$

$\therefore$ A^{-1} exists.

② $\equiv A - 4I + 4A^{-1} = 0$ [$\because$ ② is multiplied by A^{-1}]

$\Rightarrow 4A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$\Rightarrow 4A^{-1} = \begin{bmatrix} 3 & -1 \\ +1 & 1 \end{bmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}}$

Again from ②, we have.

$A^6 - 4A + 4A = 0$ [$\because$ ② multiplied by A^4]

$\therefore A^6 = 4A^5 - 4A^4$.

$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}$

$A^4 = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix} = \begin{bmatrix} -16 & 32 \\ -32 & 48 \end{bmatrix}$

$A^5 = A^4 . A = \begin{bmatrix} -16 & 32 \\ -32 & 48 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -48 & 80 \\ -80 & 112 \end{bmatrix}$

$\therefore A^6 = 4\begin{bmatrix} -48 & 80 \\ -80 & 112 \end{bmatrix} - 4\begin{bmatrix} -16 & 32 \\ -32 & 48 \end{bmatrix}$

$A^6 = \begin{bmatrix} -192 & 320 \\ -320 & 448 \end{bmatrix} + \begin{bmatrix} 64 & -128 \\ 128 & -192 \end{bmatrix}$

$\boxed{A^6 = \begin{bmatrix} -128 & 192 \\ -192 & 256 \end{bmatrix}}$ which is required solution.

Ques: 1(b)) Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by $(2x_1+x_2+x_3, x_1+x_2, x_1+x_3, 3x_1+x_2+2x_3)$ for each $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Determine a basis for Null space of T. What is the dimension of the Range space of T.

Solution:-

Let, $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by -

$$T(x_1, x_2, x_3) = (2x_1+x_2+x_3, x_1+x_2, x_1+x_3, 3x_1+x_2+2x_3)$$

where for each $(x_1, x_2, x_3) \in \mathbb{R}^3$ — (1)

The null space of T

$$N(T) = \{ \alpha \in \mathbb{R}^3 / T(\alpha) = \hat{0} \text{ in } \mathbb{R}^4 \}.$$

where $\hat{0} = \{0, 0, 0, 0\}$

Let $\alpha \in N(T)$, then $T(\alpha) = \hat{0}$

$$\Rightarrow T(x_1, x_2, x_3) = (0, 0, 0, 0)$$

$$\Rightarrow (2x_1+x_2+x_3, x_1+x_2, x_1+x_3, 3x_1+x_2+2x_3) = (0,0,0,0)$$

$$\Rightarrow 2x_1 + x_2 + x_3 = 0$$

$x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$

$x_1 + x_3 = 0 \Rightarrow x_3 = -x_1$

i.e $x_2 = x_3 = -x_1$

$3x_1 + x_2 + 2x_3 = 0$

$$\therefore N(T) = \{ (x_1, -x_1, -x_1) / x_1 \in \mathbb{R} \}$$

To find the basis for $N(T)$

$\exists$ a finite subset of 'S', such that

i) S is Linearly Independent (L.I)

ii) $L(S) = N(T)$

Now, let $\alpha = (x_1, -x_1, -x_1) \in N(T)$; $x_1 \in \mathbb{R}$

$$\alpha = x_1(1,-1,-1) \in L(S)$$

where $S = \{1,-1,-1\} \subseteq N(T)$

$\therefore \quad N(T) \subseteq L(S)$ —— ①

Since; $\quad S \subseteq N(T)$

$\Rightarrow \quad L(S) \subseteq N(T)$ —— ②

$\therefore$ from ① and ②, we have

$$\boxed{L(S) = N(T)}$$

Since, the singleton non-zero vector of 'S' is linearly independent

$\therefore$ S forms a basis of $N(T)$ and the number of elements $= 1$

$$\therefore \boxed{\dim(N(T)) = 1}$$

We know that $\dim(R(T)) + \dim(N(T)) = \dim(\mathbb{R}^3)$

$\Rightarrow \quad \dim(R(T)) + 1 = 3$

$\Rightarrow \quad \boxed{\dim[R(T)] = 2}$

i.e dimension of Range space of $T = 2$.

Ques: 1. c) Let $f(x,y) = xy/\sqrt{x^2+y^2}$, $(x,y) \neq (0,0)$ $f(0,0) = 0$. Show that 'f' is continuous at the origin (0,0). possesses partial derivatives thereat but is not differentiable at origin.

Solution:- Given that

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} \quad \text{if } (x,y) \neq 0$$

$$f(0,0) = 0$$

Now; $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

and $f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0.$$

$\therefore$ f possesses partial derivative at (0,0).
Now, we prove that 'f' is not differentiable at (0,0).

We have $f(0+h, 0+k) = \dfrac{hk}{\sqrt{h^2+k^2}}$

$$\therefore f(0+h,0+k) - f(0,0) = 0\cdot h + 0\cdot k + \sqrt{h^2+k^2}\,\frac{hk}{h^2+k^2}$$

So that $A=0$, $B=0$ and

$$\phi(h,k) = \frac{hk}{h^2+k^2}$$

i.e. A and B are independent of h & k.
If we put $k = mh$, Then we have.-

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$

$$= \lim_{h\to 0} \frac{mh^2}{h^2+m^2h^2} = \lim_{h\to 0} \frac{\cancel{h^2}\cdot m}{\cancel{h^2}(1+m^2)}$$

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \frac{m}{1+m^2}$$

This limit does not exist since it depends upon 'm'

$\therefore$ lt $\phi(h,k) \neq 0$ as $(h,k) \longrightarrow (0,0)$.

It follows that the given function is not differentiable at $(0,0)$.

Ques: 1(d) Evaluate $\int_0^1 (x\log x)^3 dx$.

Solution:- Given $I = \int_0^1 (x\log x)^3 dx$.

$$I = \int_0^1 \underset{\text{II}}{x^3} \cdot \underset{\text{I}}{(\log x)^3} dx.$$

Integration by parts.

$$I = \left[(\log x)^3 \cdot \frac{x^4}{4} - \int 3(\log x)^2 \cdot \frac{1}{x} \cdot \frac{x^4}{4} dx\right]_0^1$$

$$I = \left[\frac{x^4(\log x)^3}{4} - \frac{3}{4}\left\{(\log x)^2 \cdot \frac{x^4}{4} - 2\int \ln x \cdot \frac{x^3}{4} dx\right\}\right]_0^1$$

$$I = \left[\frac{x^4(\ln x)^3}{4} - \frac{3}{16}(\ln x)^2 x^4 + \frac{3}{8}\left\{(\ln x)\frac{x^4}{4} - \int \frac{x^3}{4} dx\right\}\right]_0^1$$

$$I = \left[x^4\left\{\frac{(\ln x)^3}{4} - \frac{3}{16}(\ln x)^2 + \frac{3}{32}(\ln x) - \frac{3}{128}\right\}\right]_0^1$$

put the limits

$$I = \left[1 \cdot \left[0 - 0 + 0 - \frac{3}{128}\right] - 0[0]\right]$$

$$I = 1 \times -\frac{3}{128} = -\frac{3}{128}.$$

$$\therefore \boxed{I = \int_0^1 (x\log x)^3 dx = \frac{-3}{128}}$$

Alternate Method

$$I = \int_0^1 (x\log x)^3 dx = \int_0^1 x^3.(\log x)^3 dx.$$

put $\log x = u \Rightarrow x = e^u$

$\frac{1}{x}dx = du \Rightarrow dx = x\,du.$

at $x = 1$; $u = 0$

$x = 0$; $u = \infty$

$$I = \int_{-\infty}^{0} e^{4t}.\, t^3 dt. \text{ [using integration by parts]}$$

$$I = \left[t^3.\frac{e^{4t}}{4}\right]_{-\infty}^{0} - \int_{-\infty}^{0} 3t^2 \frac{e^{4t}}{4} = \frac{-3}{4}\left[\left[t^2\frac{e^{4t}}{4}\right]_{-\infty}^{0} - \int_{-\infty}^{0}\frac{2t\,e^{4t}}{4}dt\right]$$

$$I = \frac{3}{8}\left[t\frac{e^{4t}}{4}\right]_{-\infty}^{0} - \int_{-\infty}^{0}\frac{e^{4t}}{4}dt.$$

$$I = -\frac{3}{8}\left[\frac{e^{4t}}{16}\right]_{-\infty}^{0} = -\frac{3}{8}\left[\frac{1}{16}\right]$$

$$\boxed{I = \frac{-3}{128}}$$

$$\therefore \boxed{I = \int_0^1 (x\log x)^3 dx = \frac{-3}{128}}$$

Q.1 (e) Find the Equation of the plane which passes through the points $(0,1,1)$ and $(2,0,-1)$ and is parallel to the line joining the points $(-1,1,-2)$, $(3,-2,4)$. Find also the distance b/w the line and the plane.

Solution: Equation of plane through $(0,1,1)$ is

$$a(x-0)+b(y-1)+c(z-1)=0$$

$$\Rightarrow ax+by+cz-b-c=0 \longrightarrow (1)$$

this also passes through $(2,0,-1)$, then

$$a(2)+b(0)+c(-1)-b-c=0$$

$$\Rightarrow 2a-b-2c=0 \longrightarrow (2)$$

Given plane is parallel to line joining $(-1,1,-2)$ and $(3,-2,4)$

Now dr's of line joining $(-1,1,-2)$ and $(3,-2,4)$
$(4,-3,6)$

as plane is parallel to this line, its normal will be perpendicular to it, Then

$$4a-3b+6c=0 \longrightarrow (3)$$

from (2) and (3)

$$\frac{a}{-6-6}=\frac{b}{-8-12}=\frac{c}{-6+4}$$

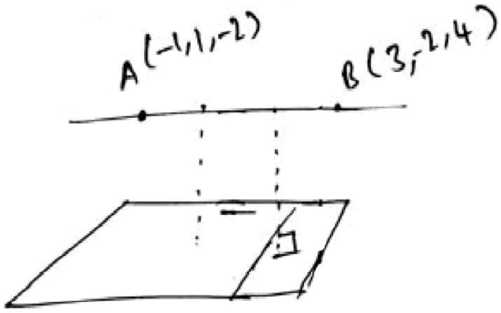
(or) $$\frac{a}{-12}=\frac{b}{-20}=\frac{c}{-2}$$

(or) $$\frac{a}{6}=\frac{b}{10}=\frac{c}{1}$$

Thus, Equation of plane is

$$P \equiv 6x+10y+z-11=0$$

and Equation of line is



$$\frac{x+1}{4} = \frac{y-1}{-3} = \frac{z+2}{6}$$

from figure, let M be the foot of the perpendicular from point B $(3,-2,4)$ on the plane p.

Equation of BM $= \dfrac{x-3}{\frac{6}{\sqrt{137}}} = \dfrac{y}{\frac{10}{\sqrt{137}}} = \dfrac{z-4}{\frac{1}{\sqrt{137}}} = r.$

where $r = |BM|$

then $\left(3+\frac{6r}{\sqrt{137}}, -2+\frac{10r}{\sqrt{137}}, 4+\frac{r}{\sqrt{137}}\right)$ lies on the plane p.

thus $6\left(3+\frac{6r}{\sqrt{137}}\right)+10\left(-2+\frac{10r}{\sqrt{137}}\right)+\left(4+\frac{r}{\sqrt{137}}\right)-11=0$

(or) $\dfrac{137r}{\sqrt{137}}+18-20+4-11=0$

$\Rightarrow \sqrt{137}\, r = 9$

(or) $r = \dfrac{9}{\sqrt{137}}.$

Ques: 2(a)) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

Solution:- Given set of system is -

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

Representing the given equation in form of

$$[AX] = [B]$$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Hence, the augumented matrix $[A:B]$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right] \approx \left[\begin{array}{ccc:c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{ccc:c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & (c-a)+2(b-2a) \end{array}\right] R_3 \to R_3 + 2R_2.$$

$(c-a) + 2(b-2a) = 0$ [For the system to have more than one solution].

It should not have a unique solution as $\rho(A) = \rho(A:B) <$ No. of rows.

Hence; the system has more than one solution at

$$(c-a) + 2(b-2a) = 0$$

$$c - a + 2b - 4a = 0$$

$$\boxed{c + 2b = 5a}$$

which is required condition on a, b and c to have solution for given system.

Ques: 2(b)(i) Consider the linear mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = (3x+4y, 2x-5y)$$

Find the matrix A relative to the basis $\{(1,0),(0,1)\}$ and the matrix B relative to the basis $\{(1,2),(2,3)\}$

Solution:- Given, linear mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ by.

$$f(x,y) = (3x+4y, 2x-5y) \text{ ——— } (1)$$

To find the matrix 'A' relative to basis $\{(1,0),(0,1)\}$

Consider $B_1 = \{(1,0),(0,1)\} = \{\alpha_1, \alpha_2\}$

$(a,b) \in \mathbb{R}^2 \Rightarrow (a,b) = a(1,0) + b(0,1)$ ——— (2)

$f(\alpha_1) = (3,2) \longrightarrow$ [using equation (1)]

$f(\alpha_1) = 3(1,0) + 2(0,1) \longrightarrow$ [using equation (2)] — (3)

Similarly ; $f(\alpha_2) = (4,-5)$

$f(\alpha_2) = 4(1,0) - 5(0,1)$ ——— (4)

from (3) and (4) matrix A, relative to B_1 is

$$A = [f : B_1] = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} \text{ ——— } (5)$$

Now to find matrix B relative to basis $\{(1,2),(2,3)\}$

Consider; $B_2 = \{(1,2),(2,3)\} = \{\beta_1, \beta_2\}$

where; $\beta_1 = (1,2)$ and $\beta_2 = (2,3)$.

$(a,b) \in \mathbb{R}^2 \quad (a,b) = x\beta_1 + y\beta_2$.

$\Rightarrow (a,b) = x(1,2) + y(2,3)$

$(a,b) = (x+2y, 2x+3y)$

$x + 2y = a$ ——— (6)

$2x + 3y = b$ ——— (7)

On solving (6) and (7), we get

$y = 2a-b$ and $x = -3a+2b$

$\Rightarrow (a,b) = (-3a+2b)(1,2) + (2a-b)(2,3)$ —— (8)

Now, $f(\beta_1) = (11,-8)$ —— using eq^n (1)

$f(\beta_1) = (-49)(1,2) + (30)(2,3)$ —— (9)

using (8)

$f(\beta_2) = (18,-11)$

$f(\beta_2) = (-76)(1,2) + 47(2,3)$ —— (10)

from (9) and (10)

matrix 'B' relative to Basis B_2 is given by —

$$B = [f : B_2] = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

$$\therefore \boxed{A = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix} \text{ and } B = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}}$$

are required solution.

Ques: 2(b)(ii) Examine the correctness or otherwise of the statements:

(A) The division law is not valid in matrix algebra.

(B) If A and B are square matrices each of order 'n' and 'I' is the corresponding unit matrix, the equation $AB - BA = I$ can never hold.

Solution:-

(A) True

Justification

The division law states that

for $A, B \in M_n(F)$,

$\exists$ $Q, R \in M_n(F)$

s.t. $A = Q \cdot B + R$ where $0 \leq \rho(R) < n$

$\Rightarrow B = \dfrac{A - R}{Q}$

which is absurd.

[$\because$ Division of Matrices is not defined)

Hence, division law is not valid in Matrix Algebra.

(B) True:

Justification:

Let us suppose that the given equation

$AB - BA = I$ holds.

$\Rightarrow$ trace $(AB - BA) =$ trace I

$\Rightarrow$ trace(AB) $-$ trace(BA) $=$ trace I

$\Rightarrow \boxed{0 = n}$

Which is contradiction to hypothesis that I is a unit matrix of order 'n' that is due to our wrong assumption.

Hence; $AB - BA = I$ can never hold.

Ques: 2 (c)> Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ from the origin, by the method of Lagrange's multipliers.

Solution:-

Let (x,y,z) be the point on line formed by intersection of $\pi_1 : 3x+4y+5z=7$ & $\pi_2 : x-z=9$

Distance from origin be $d^2 = x^2+y^2+z^2$

Using Lagrange's method to evaluate minimum value of $x^2+y^2+z^2$ subject to π_1 & π_2

$$F(x,y,z) = x^2+y^2+z^2+\lambda_1(3x+4y+5z-7)+\lambda_2(x-z-9)$$

$$\frac{\partial f}{\partial x} = 2x+\lambda_1 3+\lambda_2 = 0 \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial y} = 2y+4\lambda_1 = 0 \quad \text{—— (2)} \Rightarrow \boxed{\lambda_1 = -\frac{y}{2}}$$

$$\frac{\partial f}{\partial z} = 2z+\lambda_1(5)-\lambda_2 = 0 \quad \text{—— (3)}$$

Adding (1) and (3), we get

$$2(x+z)+8\lambda_1 = 0$$

$$\lambda_1 = -\frac{\cancel{2}(x+z)}{\cancel{8}\,4}$$

$$-\frac{y}{2} = -\frac{(x+z)}{4} \Rightarrow \boxed{2y = x+z} \quad \text{—— (4)}$$

Solving (4) & π_2

$$x = \frac{2y+9}{2} \; ; \; z = \frac{2y-9}{2}$$

Put these in π_1.

Ques: 2(d)) A line with direction ratios 2, 7, −5 is drawn to intersect the lines

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{4} \text{ and } \frac{x-11}{3} = \frac{y-5}{1} = \frac{z}{1}.$$

Find the co-ordinates of the points of intersection and the length intercepted on it?

Solution:- Given lines are

$$\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{4} = r \text{ (say)} \quad \text{——— } (1)$$

$$\frac{x-11}{3} = \frac{y-5}{1} = \frac{z}{1} = r' \text{ (say)} \quad \text{——— } (2)$$

Any point on (1) is $P(3r, 2r+1, 4r+2)$ and

any point on (2) is $P'(3r'+11, r'+5, r')$ ——— (3)

∴ The direction ratios of the line PP' are

$$[3r-(3r'+11),\ 2r+1-(r'+5),\ 4r+2-r']$$

or $[3r-3r'-11,\ 2r-r'-4,\ 4r-r'+2]$

But the direction ratios of PP' are given to be $2, +7, -5$, so we get

$$\frac{3r-3r'-11}{2} = \frac{2r-r'-4}{7} = \frac{4r-r'+2}{-5}$$

which gives; $7(3r-3r'-11) = 2(2r-r'-4)$

and $-5(3r-3r'-11) = 2(4r-r'+2)$

(or) $17r - 19r' - 69 = 0$

and $-23r + 17r' + 51 = 0$

$$\frac{r}{-19(51)+69(17)} = \frac{r'}{17(51)-69(23)} = \frac{1}{289-23(19)}$$

$$\Rightarrow \quad \frac{r}{204} = \frac{r'}{720} = \frac{1}{-148}$$

$$\Rightarrow \quad r = \frac{204}{-148} = -\frac{51}{37}$$

$$r' = \frac{720}{-148} = -\frac{180}{37}$$

Substituting the values in equation (3), we get the co-ordinates of required points of intersection P and P'

$$\boxed{P\left(\frac{-153}{37}, \frac{-65}{37}, \frac{-130}{37}\right) \text{ and } P'\left(\frac{597}{37}, \frac{365}{37}, \frac{180}{37}\right).}$$

Required Solution

Q.3 (a)(i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let 'W' be the set consisting of all matrices with zero determinant. Is 'W' a subspace of V? Justify your answer.

Sol:- let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

& $W = \left\{ A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \middle| \det A = 0 \right\} \subseteq V$

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\text{Det } A = 0$ & $\text{Det } B = 0$

Clearly; $A, B \in W$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(A+B) = 1 \neq 0$

$\Rightarrow A+B \notin W.$

Hence, W is not the subspace of V.

Q.3 (a)(ii) Find the dimension and a basis for the space 'W' of all solutions of the following homogeneous system using matrix notation:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$$

Sol^n Given homogeneous system is:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{--- (1)}$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0 \quad \text{--- (2)}$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0 \quad \text{--- (3)}$$

Co-efficient matrix of given system is —

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, corresponding homogeneous system is

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{--- (4)}$$
$$2x_3 + 5x_4 + x_5 = 0 \quad \text{--- (5)}$$

Here, 3 variables x_2, x_4, x_5, and values of x_1 and x_3 depend upon x_2, x_4, x_5

from (5) $\quad x_3 = -\dfrac{(5x_4 + x_5)}{2} \quad \text{——— (6)}$

from (4) $\quad x_1 = -2x_2 - 3x_3 + 2x_4 - 4x_5$

$$x_1 = (-2x_2) + \frac{19x_4 - 5x_5}{2}$$

$$\Rightarrow x_1 = \frac{-4x_2 + 19x_4 - 5x_5}{2} \quad \text{— (7)}$$

Now, put $x_2 = 1$, $x_4 = 0$, $x_5 = 0$

$$\Rightarrow x_3 = 0, \; x_1 = -2$$

Then $(x_1, x_2, x_3, x_4, x_5) = (-2, 1, 0, 0, 0)$ — (8)

put $x_2 = 0$, $x_4 = 1$, $x_5 = 0$

$$\Rightarrow x_3 = -\frac{5}{2}, \; x_1 = \frac{19}{2}$$

then $(x_1, x_2, x_3, x_4, x_5) = (19/2, 0, -\frac{5}{2}, 1, 0)$ — (9)

Put $x_2 = 0$, $x_4 = 0$, $x_5 = 1$

$$\Rightarrow x_3 = -\frac{1}{2}; \; x_1 = -\frac{5}{2}$$

then $(x_1, x_2, x_3, x_4, x_5) = (-\frac{5}{2}, 0, -\frac{1}{2}, 0, 1)$ — (10)

Hence, from (8), (9) & (10)

$$B = \left\{(-2, 1, 0, 0, 0), \left(\frac{19}{2}, 0, \frac{-5}{2}, 1, 0\right), \left(\frac{-5}{2}, 0, \frac{-1}{2}, 0, 1\right)\right\}$$

is the basis for the space W.

& $\boxed{\text{Dim } W = 3}$

Ques: 3(b) (i) Find $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$.

Solution:- Given, we have.

$$y = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}$$

$$\Rightarrow \log_e y = \underset{x\to 0}{\mathrm{lt}} \log\left(\frac{\tan x}{x}\right)^{1/x}.$$

$$\Rightarrow \log_e y = \underset{x\to 0}{\mathrm{lt}} \frac{1}{x}\log\left(\frac{\tan x}{x}\right)$$

$$\Rightarrow \log_e y = \underset{x\to 0}{\mathrm{lt}} \frac{\log\tan x - \log x}{x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$\log_e y = \underset{x\to 0}{\mathrm{lt}}\left(\frac{\sec^2 x}{\tan x} - \frac{1}{x}\right]$$

$$\log_e y = \underset{x\to 0}{\mathrm{lt}}\left[\frac{x - \sin x\cos x}{x\cdot\sin x\cos x}\right]$$

$$\log_e y = \underset{x\to 0}{\mathrm{lt}}\left[\frac{2x - \sin 2x}{x\sin 2x}\right] \quad \left[\frac{0}{0}\text{ form}\right].$$

$$\log_e y = \underset{x\to 0}{\mathrm{lt}} \frac{2 - 2\cos 2x}{\sin 2x + 2x\cos 2x} \quad \left[\frac{0}{0}\text{ form}\right]$$

$$\log_e y = \underset{x\to 0}{\mathrm{lt}} \frac{4\sin 2x}{4\cos 2x - 4x\sin 2x}$$

$$\log_e y = 0$$

$$\boxed{y = e^0 = 1}$$ Required result.

Ques: 3(b) ii) If $v = At^{-1/2} \cdot e^{-x^2/4a^2t}$, prove that

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}.$$

Solution:- Given;

$$V = At^{-1/2} \cdot e^{-x^2/4a^2t}$$

$$\Rightarrow \frac{\partial v}{\partial x} = At^{-1/2} \cdot e^{-x^2/4a^2t}\left(\frac{-2x}{4a^2t}\right)$$

$$\Rightarrow \boxed{\frac{\partial v}{\partial x} = \frac{-x}{2a^2t} \cdot v}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2t}\left[v + x\frac{\partial v}{\partial x}\right]$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2t}\left[v + x \cdot \left[\frac{-x}{2a^2t} \cdot v\right]\right]$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{v}{4a^4t^2}\left(-2a^2t + x^2\right) \quad \text{—— (1)}$$

Again; $\frac{\partial v}{\partial t} = At^{-1/2} \cdot e^{-x^2/4a^2t}\left(\frac{x^2}{4a^2t^2}\right) - A \cdot \frac{1}{2} \cdot t^{-3/2} \cdot e^{-x^2/4a^2t}$

$$\frac{\partial v}{\partial t} = At^{-1/2} \cdot e^{-x^2/4a^2t}\left[\frac{x^2}{4a^2t} - \frac{1}{2t}\right].$$

$$\frac{\partial v}{\partial t} = \frac{v}{4a^2t}\left[x^2 - 2a^2t\right]. \quad \text{—— (2)}$$

$$\therefore \boxed{\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}}$$ (from (1) & (2))

Hence proved.

Ques: 3(c)(i) Find the surface generated by a line which intersects the lines $y = a = z$, $x+3z = a = y+z$ and parallel to plane $x+y=0$.

Solution:-

Given; lines are $y = a = z$

$$x + 3z = a = y + z$$

parallel to the plane $x+y=0$

Equation of the required line :-

$$(y-a) + \lambda(z-a) = 0 \quad \text{and}$$

$$(x+3z-a) + \mu(y+z-a) = 0$$

$$\Rightarrow y + \lambda z - a - \lambda a = 0 \quad \text{--- (1)}$$

$$x + \mu y + (3+\mu)z - a(1+\mu) = 0 \quad \text{--- (2)}$$

parallel to the plane $x+y=0$ --- (3)

Then, ratios of d.r's should be same.

$$\frac{x}{3+\mu-\lambda\mu} = \frac{y}{\lambda} = \frac{z}{-1}$$

So Dr's (x, y, z) are in ratio of $((3+\mu-\lambda\mu), \lambda, -1)$

As, it's parallel to $x+y=0$

$$\therefore 3+\mu-\lambda\mu+\lambda = 0 \quad \text{--- (4)}$$

from (1) ; $\lambda = \dfrac{a-y}{z-a}$; $\mu = \dfrac{a-x-3z}{y+z-a}$

Substituting λ and μ in (4), we get

$$\boxed{(y+z)(x+y) = 2a(x+z)}$$

Ques: 3(c) ii) Show that the length of the shortest distance between the line $z = x\tan\alpha$, $y=0$ and any tangent to the ellipse

$$x^2\sin^2\alpha + y^2 = a^2,\ z=0 \text{ is constant.}$$

Solution:-

Given; $z = x\tan\alpha$; $y=0$

given tangent to the ellipse -

$$x^2\sin^2\alpha + y^2 = a^2 \ ; \ z=0$$

$$x x_1 \sin^2\alpha + y y_1 = a^2,\ z=0 \quad \text{--- (1)}$$

Equation of ellipse

$$\frac{x^2\sin\alpha}{a^2} + \frac{y^2}{a^2} = 1,\ z=0$$

Any general point on the ellipse

$$x = \frac{a\sin\theta}{\sin\alpha} \ ; \ y = a\cos\theta.$$

⇒ Equation of tangent to the plane

$$= \frac{\sin^{\cancel{2}}\alpha}{a^{\cancel{2}}}\left(x\cdot\frac{\cancel{a}\sin\theta}{\cancel{\sin\alpha}}\right) + \frac{1}{a^{\cancel{2}}}\left(y\cdot\cancel{a}\cos\theta\right) = 1$$

$$\Rightarrow \sin\alpha\cdot\sin\theta\cdot\frac{x}{a} + \frac{y}{a}\cdot\cos\theta = 1. \ ; \ z=0$$

$$\Rightarrow x\sin\alpha\sin\theta + y\cos\theta = a \quad ; \ z=0$$

$$\Rightarrow \frac{x\sin\theta\cdot\sin\alpha}{1} = \frac{-a+y\cos\theta}{-1} = \frac{z-0}{0}$$

$$\Rightarrow \frac{x-0}{\frac{1}{\sin\theta\cdot\sin\alpha}} = \frac{y-a/\cos\theta}{\frac{-1}{\cos\theta}} = \frac{z-0}{0} \quad \text{--- (2)}$$

from (1) and (2)

$$\frac{l}{\frac{-\sin^2\alpha\tan\alpha}{y_1}} = \frac{m}{\frac{\tan\alpha}{x_1}} = \frac{n}{\frac{-\sin^2\alpha}{y_1}} \quad \text{--- (3)}$$

shortest distance between two lines is $= \frac{1}{\sqrt{l^2+m^2+n^2}} \begin{bmatrix} 0 & a^2/y_1 & 0 \\ 1 & 0 & \tan\alpha \\ \frac{+1}{x_1} & \frac{-\sin^2\alpha}{y_1} & 0 \end{bmatrix}$

$$\Rightarrow \frac{a^2}{y_1}\left(-\frac{\tan\alpha}{x_1}\right) \Big/ \sqrt{l^2+m^2+n^2}$$

$$\Rightarrow -\frac{a^2\tan\alpha}{x_1 y_1} \Big/ \sqrt{\frac{\sin^4\alpha\tan^2\alpha}{y_1^2} + \frac{\sin^4\alpha}{y_1^2} + \frac{\tan^2\alpha}{x_1^2}}$$

$$\Rightarrow \frac{-a^2\tan\alpha}{\cancel{x_1 y_1}} \Big/ \frac{1}{\cancel{y_1 x_1}} \sqrt{x_1^2(\sin^4\alpha\tan^2\alpha) + x_1^2\sin^4\alpha + y_1^2\tan^2\alpha}$$

$$\Rightarrow \frac{-a^2\cancel{\tan\alpha}}{\cancel{\tan\alpha}\sqrt{x_1^2\sin^4\alpha + x_1^2\cos^2\alpha\sin^2\alpha + y_1^2}}$$

By solving. $\Rightarrow \dfrac{-a^2}{\sqrt{x_1^2\sin^2\alpha + y_1^2}}$ $\quad [\because \cos^2\alpha + \sin^2\alpha = 1]$

$\Rightarrow \dfrac{-a^{\cancel{2}}}{\cancel{a}}$ $\quad [\because x_1^2\sin^2\alpha + y_1^2 = a^2]$

$\Rightarrow -a$ [-ve sign due to direction.

$\boxed{S.D = a}$ which is constant

Hence proved.

Ques:-4(a)) Examine whether the Matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix P such that $P^{-1}AP$ is a diagnol matrix.

Solution: Given; Matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

To get the eigen values – By characteristics equation

$|A - \lambda I| = 0$ where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\therefore A - \lambda I = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = -(2+\lambda)(-\lambda(1-\lambda)-12) - 2(-2\lambda - 6) - 3[-4 + (1-\lambda)] = 0$$

After solving –

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$(\lambda - 5)(\lambda+3)^2 = 0$$

$$\boxed{\therefore \lambda = +5, -3, -3}$$

$\therefore$ Eigen values of Matrix A is $5, -3, -3$.

Eigen vector at $\lambda = 5$:-

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{2}{7} R_1$ and $R_3 \to R_3 - \frac{1}{7} R_1$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & -\frac{16}{7} & -\frac{32}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to -\frac{7}{24} R_2$; $R_3 \to -\frac{7}{16} R_3$; $R_3 \to R_3 - R_2$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $x_3 = k$; $x_2 + 2x_3 = 0 \Rightarrow x_2 = -2k$

$-7x_1 + 2(-2k) + (-3k) = 0$

$-7x_1 - 7k = 0 \Rightarrow \boxed{x_1 = -k}$

So eigen vectors $X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ for $x = 5$

For $x = -3$; Eigen vector is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_3 = k_1$; $x_2 = k_2$

$x_1 + 2k_2 - 3k_3 = 0$

$\boxed{x_1 = 3k_3 - 2k_2}$

$\therefore \quad X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

and we have $x_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

So; $P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow |P| = -1(1) + 2(-2) + 3(-1)$
$= -1 - 4 - 3 = -8$

$$P^{-1} = \frac{1}{|P|} \begin{bmatrix} +1 & +2 & -3 \\ +2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{8} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

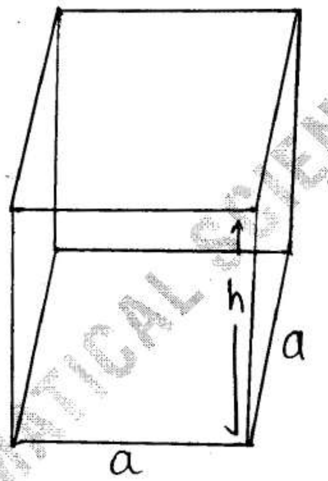
$$P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$\therefore$ Hence, A is diagonalizable.

Ques: 4(b)(i) An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.

Solution :-

Let 'a' be the length of the sides of the square base and 'h' be the height of the open tank.



Total surface area of open tank $= a^2 + 4(ah)$

Volume is fixed: $V = a^2h$ = constant

The aim is to minimize $S = a^2 + 4ah$.

subject to $V = a^2h \Rightarrow h = \frac{V}{a^2}$

$$\Rightarrow S = a^2 + 4\cancel{a}\cdot\frac{V}{a^{\cancel{2}}} \Rightarrow S = a^2 + \frac{4}{a}V$$

$$\frac{dS}{da} = 2a - \frac{4}{a^2}V$$

$$\frac{dS}{da} = 0 \quad \text{i.e} \quad 2a - \frac{4}{a^2}V = 0$$

$$2a = \frac{4}{a^2}V$$

$$\cancel{2}a^3 = \cancel{4}V$$

$$\boxed{a^3 = 2V}$$

Also $V = a^2 h.$

$$\Rightarrow a^3 = 2(a^2 h)$$

$$\Rightarrow \frac{a^{\cancel{3}}}{\cancel{a^2}} = 2h$$

$$\Rightarrow \boxed{\frac{h}{a} = \frac{1}{2}}$$

Hence, the ratio of depth (height) to width $= 1:2$

Q.4 (b)(ii) Examine, if the improper integral $\int_0^3 \frac{2x}{(1-x^2)^{2/3}}$ exists?

Sol: let $I = \int_0^3 \frac{2x}{(1-x^2)^{2/3}} dx.$

The only point of discontinuity shall be '1', which belongs to (0,3)

$$I = \lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{2x}{(1-x^2)^{2/3}} dx + \lim_{\epsilon \to 0^+} \int_{1+\epsilon}^3 \frac{2x}{(1-x^2)^{2/3}} dx.$$

$$= \lim_{\epsilon \to 0} \left[-3(1-x^2)^{1/3}\right]_0^{1-\epsilon} + \lim_{\epsilon \to 0^+} \left[-3(1-x^2)^{1/3}\right]_{1+\epsilon}^3$$

$$= \lim_{\epsilon \to 0^+} \left[-3(1-(1-\epsilon^2))^{1/3} + (-3(1-3^2)^{1/3} + 3(1-(1+\epsilon)^2)\right)^{1/3}$$

$$= -3[1-1)^{1/3} + (-3(-8)^{1/3} + 3(1-1)^{1/3}$$

$$= -3[0] + (-3((-2)^{3 \times 1/3}) + 3(0)$$

$$= 0 - 3 \times -2 + 0$$

$$= 0 + 6 + 0 = 6.$$

which is finite.

Hence, I exists.

Ques:-4 (c) A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left[\frac{b}{c}+\frac{c}{b}\right]+zx\left[\frac{c}{a}+\frac{a}{c}\right]+xy\left[\frac{a}{b}+\frac{b}{a}\right]=0$$

Solution:-

Given; plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ —— (1)

Any parallel to above plane can be written

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=d \quad \text{—— (2)}$$

It meets axes in

A $(ad, 0, 0)$; B $(0, bd, 0)$; C $(0, 0, cd)$

Circle ABC constructed when sphere OABC is cut by the plane (2).

Sphere OABC can be given as,

$$x^2+y^2+z^2-adx-bdy-cdz=0 \quad \text{—— (3)}$$

sphere OABC and plane (2), gives circle ABC thus, to construct cone in which circle lies can be attained by replacing 'd' from (2) and (3)

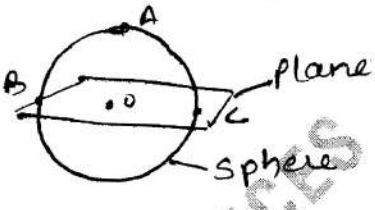
$$x^2+y^2+z^2-a\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)x-b\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)y-c\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)z=0$$

By solving →

$$-ax\left(\frac{y}{b}+\frac{z}{c}\right)-by\left(\frac{x}{a}+\frac{z}{c}\right)-cz\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

So; Since;



$\lim_{n\to\infty} M_n \neq 0$;

therefore $f_n(x)$ is not uniform convergence.

Section-B

Ques: 5(a) Solve the Ordinary differential Equation.

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2} \sin 6x + \sin^2 3x; \ 0<x<\pi/2$$

Solution:-

The given differential equation is

$$\cos 3x \frac{dy}{dx} - 3y \sin 3x = \frac{1}{2} \sin 6x + \sin^2 3x$$

Dividing by $\cos 3x$, we get

$$\frac{dy}{dx} - (3\tan 3x) y = \frac{1}{2} \frac{2 \sin 3x \cancel{\cos 3x}}{\cancel{\cos 3x}} + \frac{\sin^2 3x}{\cos 3x}$$

$$\Rightarrow \frac{dy}{dx} - (3\tan 3x) y = \sin 3x + \tan 3x \sin x.$$

which is a linear equations.

$$I.F = e^{-\int 3\tan 3x \, dx} = e^{\log \cos 3x}.$$

$$\boxed{I.F = \cos 3x}$$

$$y(I.F) = \int I.F \left(\sin 3x + \tan 3x . \sin x \right) dx + c$$

$$y \cos 3x = \int (\cos 3x \sin 3x + \tan 3x . \sin x . \cos x) dx + c$$

$$= \frac{1}{2} \int \sin 6x \, dx + \int \sin^2 3x \, dx + c.$$

$$= \frac{1}{2} \left[-\frac{\cos 6x}{6} \right] + \frac{1}{2} \int (1 + \cos 6x) dx + c$$

$$= -\frac{\cos 6x}{12} + \frac{1}{2} \left[x + \frac{\sin 6x}{6} \right] + c$$

$$\therefore \boxed{y \cos 3x = \frac{x}{2} + \frac{\sin 6x - \cos 6x}{12} + c}$$ required solution.

5(b). Obtain the equation of the orthogonal trajectory of the family of curves represented by $r^n = a \sin n\theta$, (r, θ) being the plane polar coordinates.

SOLUTION

Given $$r^n = a \sin n\theta \qquad .....(1)$$

applying log on both sides

$$n \log r = \log a + \log \sin n\theta$$

differenting w.r.t. θ.

$$\frac{n}{r}\frac{dr}{d\theta} = n \cot n\theta$$

for finding orthogonal trajectories replace

$\dfrac{dr}{d\theta}$ by $\dfrac{-r^2 d\theta}{dr}$

$$\therefore \qquad \frac{n}{r}\left(\frac{-r^2 d\theta}{dr}\right) = n \cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = \frac{-dr}{r}$$

integerating both sides

$$\frac{\log(\sec n\theta)}{n} = -\log r + c$$

∴ Simplifying

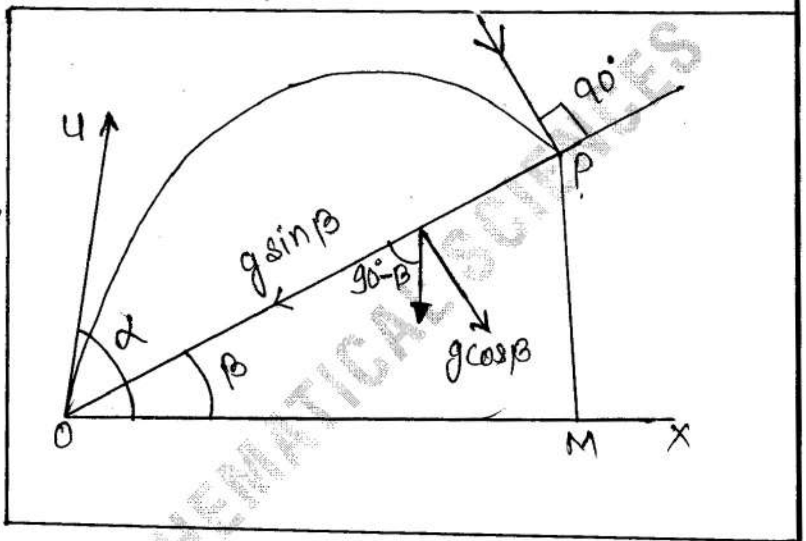
$\boxed{r^n = K \cos n\theta}$ K being arbitrary constant

Ques: 5(c). A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is 'β'. Show that it will strike the plane at right angle if

$$\boxed{\cot\beta = 2\tan(\alpha-\beta)}\ ?$$

Solution:-

Let 'O' be the point of projection, 'u' the velocity of projection and 'P' the point where the particle strikes the plane.



Let T be the time of flight from 'O' to 'P'. Then by usual formula for time

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \quad \text{—— ①}$$

Since, in this question the particle strikes the inclined plane at right angles at P, therefore the direction of the velocity of the particle at 'P' is perpendicular to the inclined plane. Consequently the resolved part of the velocity of the particle at 'O' along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration 'g' along the inclined plane is '$g\sin\beta$' downwards. So, considering the motion of the particle from O to P along the inclined plane and using the formula —

$v = u + ft$, we have.

$$0 = u\cos(\alpha-\beta) - g\sin\beta\, T$$

$$T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{——— (2)}$$

Equating the values of T from (1) and (2), we have

$$\frac{2\cancel{u}\sin(\alpha-\beta)}{\cancel{g}\cos\beta} = \frac{\cancel{u}\cos(\alpha-\beta)}{\cancel{g}\sin\beta}$$

$$\Rightarrow 2\,\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{\cos\beta}{\sin\beta}$$

$$\Rightarrow \boxed{2\tan(\alpha-\beta) = \cot\beta}$$

Hence proved.

Ques: 5(d) Apply Green's theorem to evaluate the line integral $\oint_C (4x-2y)dx+(2x-4y)dy$ where 'C' is the circle $(x-2)^2+(y-2)^2=4$.

Solution:-

By Green's theorem in the plane we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C M\,dx+N\,dy.$$

Here $M = 4x-2y$ $N = 2x-4y$.

and c is the circle

$$(x-2)^2+(y-2)^2=4 \;;\; r=2$$

$$\frac{\partial M}{\partial y}=-2 \;;\; \frac{\partial N}{\partial x}=2$$

$$\therefore \oint_C M\,dx+N\,dy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \iint (2-(-2))\,dx\,dy$$

$$= 4\iint dx\,dy.$$

$= 4$ (Area of the circle of Radius 2 with centre $(2,2)$]

$$= 4\,(\pi(2)^2)$$

$$= 16\pi.$$

$$\therefore \boxed{\oint_C M\,dx+N\,dy = 16\pi \cdot \text{sq. units}}$$ required result

Q.5 (e) Find f(r) such that $\nabla f = \frac{\vec{r}}{r^5}$ and f(1) = 0.

SOLUTION

We know that,

$$\nabla f = f'(r)\nabla r = f'(r)\frac{\vec{r}}{r} \qquad \left(\because \nabla r = \frac{\vec{r}}{r}\right)$$

We have, $$\nabla f = \frac{\vec{r}}{r^5}$$

$$\therefore \quad f'(r)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^5}$$

$$\Rightarrow \quad \vec{r}\left[\frac{f'(r)}{r} - \frac{1}{r^5}\right] = 0$$

Since, $\vec{r} \neq 0$

$$\therefore \quad f'(r) = \frac{1}{r^4}$$

Integrating we get,

$$f(r) = \frac{-1}{3r^3} + C$$

$$f(1) = 0 \Rightarrow 0 = \frac{-1}{3.1} + c \Rightarrow c = \frac{1}{3}$$

$$\therefore \quad f(r) = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$$

Q.6 (a) Find the Wronskian of the set of functions

$\left\{3x^3, |3x^3|\right\}$

on the interval [–1, 1] and determine whether the set is linearly dependent on [–1, 1].

SOLUTION

Given functins are $\left\{3x_1^3 \left|3x^3\right|\right\}$

Let $f(x) = 3x^3$

$g(x) = (3x^3)$

$$\Rightarrow \quad f(x) = \begin{cases} 3x^3 & \text{for } x \in [-1, 0] \\ 3x^3 & \text{for } x \in [0, 1] \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} -3x^3 & \text{for } x \in [-1, 0] \\ 3x^3 & \text{for } x \in [0, 1] \end{cases}$$

wronskian of

$$f(x),\ g(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

for $x \in [-1, 0]$ $\quad w = \begin{vmatrix} 3x^3 & -3x^3 \\ 9x^2 & -9x^2 \end{vmatrix} = 0$

for $x \in [0, 1]$ $\quad w = \begin{vmatrix} 3x^3 & 3x^3 \\ 9x^2 & 9x^2 \end{vmatrix} = 0$

w = 0 for $x \in [-1, 1]$ the given functions are linearly dependent on [-1, 1]

Ques: 6 (b) Solve the following differential equation:

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log_e x).$$

Solution:-

Given; $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log_e x)$

which can be rewritten as-

$$(x^3D^3+3x^2D^2+xD+8)y=65\cos(\log x) \quad \text{—— (1)}$$

Let $x=e^z$

$\Rightarrow \log x = z$

and let $D_1=\frac{d}{dz}$

Then; $\boxed{xD=D_1}$ and $x^2D^2=D_1(D_1-1)$

$x^3D^3=D_1(D_1-1)(D_1-2)$

$\therefore$ (1) $\equiv$ we have

$$(D_1(D_1-1)(D_1-2)+3(D_1-1)D_1+D_1+8)y=65\cos z$$

$$\Rightarrow (D_1^3+8)y=65\cos z \quad \text{—— (2)}$$

Auxillary equation of (2) is.

$D_1^3+8=0$

$\Rightarrow (D_1+2)(D_1^2-2D_1+4)=0$

$\Rightarrow D_1=-2,\ 1\pm\sqrt{3}i$

$$\therefore C\cdot F=y_c=C_1e^{-2z}+e^z(C_2\cos\sqrt{3}z+C_3\sin\sqrt{3}z)$$

$$P.I. = \frac{1}{(D_1^3+8)}(65\cos z)$$

$$P.I. = \frac{65}{-D_1+8}\cos z = \frac{(D_1+8)65}{-D_1^2+64}\cos z.$$

$$P.I = \frac{65}{65}(D_1+8)\cos z$$

$$P.I. = (D_1+8)\cos z. = -\sin z + 8\cos z.$$

$$\therefore\ y = C.F + P.I = y_c + y_p.$$

$$y = c_1 e^{-2z} + e^{z}(c_2\cos\sqrt{3}z + c_3\sin\sqrt{3}z) - \sin z + 8\cos z$$

$$y = c_1 x^{-2} + x(c_2\cos\sqrt{3}z + c_3\sin\sqrt{3}z) - \sin z + 8\cos z$$

put $z = \log x$.

$$\boxed{y = c_1 x^{-2} + x(c_2\cos\sqrt{3}(\log x) + c_3\sin\sqrt{3}(\log x)) - \sin\log x + 8\cos\log x}$$

Required solution.

Ques: 6(c) Solve the differential equation
$$x^2y'' - 4xy' + 6y = x^4\sec^2 x.$$
by variation of parameter.

Solution:-

The given equation is
$$x^2y'' - 4xy' + 6y = x^4\sec^2 x \quad \text{——} (1)$$
Re-writing the given equation, we have
$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x^2\sec^2 x \quad \text{——} (2)$$
Comparing (2) with $y'' + Py' + Qy = R$, we have.
$$P = -\frac{4}{x}, \quad Q = \frac{6}{x^2}, \quad R = x^2\sec^2 x$$
Consider, $y'' - \left(\frac{4}{x}\right)y' + \left(6/x^2\right)y = 0$
$$\Rightarrow (x^2D^2 - 4xD + 6)y = 0 \quad \text{——} (3)$$
where $D = \frac{d}{dx}$

In order to apply the method of variation of parameters, we shall reduce (3) into linear, differential equation with constant co-efficients.

Let $x = e^z$ i.e $\log x = z$ and

let $D_1 = d/dz$ —— (4)

Then, $xD = D_1$, $x^2D^2 = D_1(D_1-1)$ and so on,

Eq^n (3) reduces to
$$\{D_1(D_1-1) - 4D_1 + 6\}y = 0$$

$\Rightarrow (D_1^2 - 5D_1 + 6)y = 0$, whose auxillary equation is $D_1^2 - 5D_1 + 6 = 0 \Rightarrow (D_1-2)(D_1-3) = 0$

giving $D_1 = 2, 3$.

$\therefore$ C.F of (1) $= C_1 e^{2z} + C_2 e^{3z}$.

$$CF = c_1(e^z)^2 + c_2(e^z)^3$$

$$\boxed{C.F = C_1 x^2 + C_2 x^3} \quad \text{—— (5)}$$

Let $u = x^2$ and $v = x^3$.

Also, here $R = x^2 \sec^2 x$ —— (6)

Here; $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$

$$W = 3x^4 - 2x^4 = x^4 \neq 0$$

Hence, P.I of (1) $= u \cdot f(x) + v \cdot g(x)$. —— (7)

where; $f(x) = -\int \frac{vR}{W} dx$

$$f(x) = -\int \frac{x^3 \cdot x^2 \cdot \sec^2 x}{x^4} dx$$

$$f(x) = -\int x \cdot \sec^2 x \, dx.$$

$$f(x) = -\left[x \tan x - \int \tan x \, dx\right]$$

$$\boxed{f(x) = -\left[x \tan x + \log \cos x\right]} \quad \text{—— (8)}$$

and $g(x) = \int \frac{uR}{W} dx$

$$g(x) = \int \frac{x^2 \cdot x^2 \cdot \sec^2 x}{x^4}\, dx$$

$$g(x) = \int \sec^2 x\, dx$$

$$\boxed{g(x) = \tan x} \quad \text{—— (9)}$$

Using (6), (8) and (9), (7) reduces to –

$$\text{P.I. of (1)} = -x^2 [x\tan x + \log\cos x] + x^3 \tan x$$

$$\text{P.I} = -x^2 \log\cos x + x^3 (\tan x - \tan x)$$

$$\boxed{\text{P.I of (1)} = -x^2 \log\cos x.} \quad \text{—— (10)}$$

Hence, the required general solution is

$$y = y_c + y_p = C.F + P.I$$

$$\boxed{y = c_1 x^2 + c_2 x^3 - x^2 \log\cos x}$$ where, c_1, c_2 are arbitrary constants

Ques: 6(d)> Solve the initial value problem

$$\frac{d^2y}{dt^2}+y = 8e^{-2t}\sin t \; ; \; y(0)=0, \; y'(0)=0$$

by using Laplace transform.

Solution:- Given that $\frac{d^2y}{dt^2}+y = 8e^{-2t}\sin t$

and $y(0)=0$, $y'(0)=0$

$$\frac{d^2y}{dt^2} = Lf''(t) = p^2 f(p).$$

$$\therefore \; Lf''(t) + Lf(t) = 8\, L e^{-2t}\sin t$$

$$p^2 f(p) + f(p) = 8 \cdot \frac{1}{((p+2)^2+1)}$$

$\because$ $Lf'(t) = p\,Lf(t) - f(0)$

$$Lf''(t) = p\,L[f'(t)] - f'(0)$$

$$Lf''(t) = p(\,p f(p) - f(0)) - f'(0)$$

$$Lf''(t) = p^2 f(p) - \underbrace{p f(0) - f'(0)}_{=0 \text{ (given)}}$$

$$\boxed{Lf''(t) = p^2 f(p)}$$

$$f(p)[\,p^2+1\,] = \frac{8\cdot 1}{(p^2+4p+5)}$$

$$f(p) = \frac{8}{(p^2+1)(p^2+4p+5)}$$

$$f(p) = \frac{\cancel{8}\,2}{\cancel{4}(p+1)}\left[\frac{1}{p^2+1} - \frac{1}{p^2+4p+5}\right]$$

$$f(p) = \frac{2}{p+1}\left[\frac{1}{p^2+1} - \frac{1}{p^2+4p+5}\right]$$

$$f(p) = \frac{2}{(p+1)(p^2+1)} - \frac{2}{(p+1)(p^2+4p+5)}$$

Taking Laplace Inverse.

$$f(t) = 2L^{-1}\left[\frac{1}{(p+1)}\cdot\frac{1}{(p^2+1)}\right] - 2L^{-1}\left\{\frac{1}{p+1}\cdot\frac{1}{(p^2+4p+5)}\right]$$

Ist part:

$$L^{-1}\left[\underset{f(p)}{\frac{1}{(p+1)}}\;\underset{g(p)}{\frac{1}{(p^2+1)}}\right] = \int_0^t e^{-x}\sin(t-x)\,dx.$$

$$\therefore L^{-1}\left[\frac{1}{(p+1)(p^2+1)}\right] = \frac{e^{-t}+\sin t - \cos t}{2}$$

IInd part:

$$L^{-1}\left[\underset{f(p)}{\frac{1}{(p+1)}}\;\underset{g(p)}{\frac{1}{(p^2+4p+5)}}\right] = \frac{e^{+t}-\sin t-\cos t}{2}$$

Now, substitute the values in the f(t) equation

$$f(t) = \cancel{2}\left[\frac{e^{-t}+\sin t-\cos t}{\cancel{2}}\right] - \cancel{2}\left[\frac{e^{+t}-\sin t-\cos t}{\cancel{2}}\right]$$

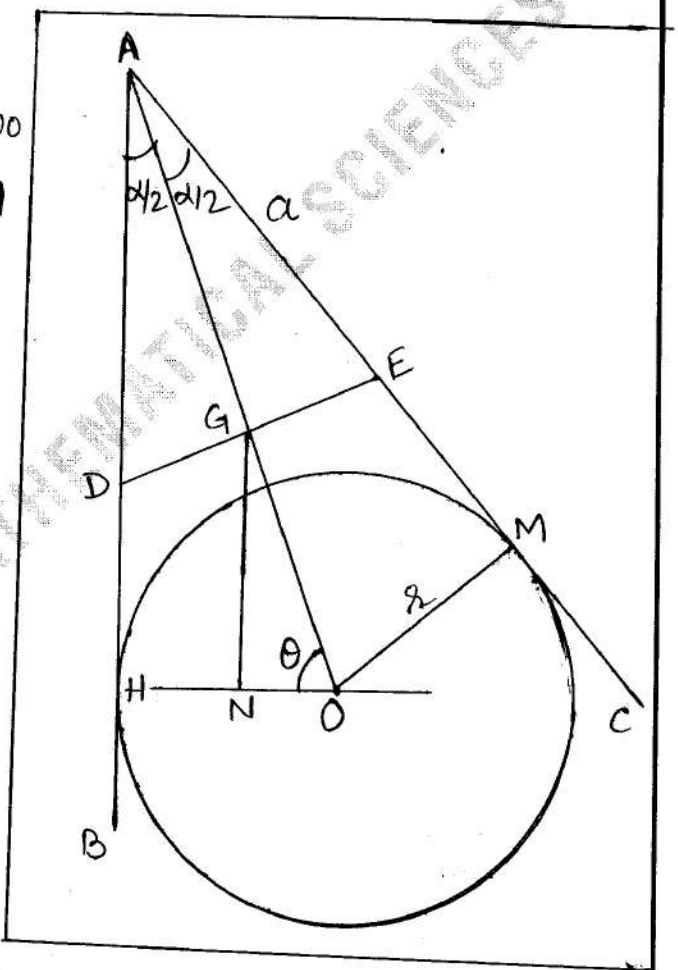
$$f(t) = e^{-t}+\sin t-\cancel{\cos t} - e^{t}+\sin t+\cancel{\cos t}$$

$$\boxed{f(t) = e^{-t}-e^{t}+2\sin t.}$$

Hence the result

Ques: 7(a) Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius 'r'. Show that they are in a stable or unstable equilibrium according as the length of rod is $>$ or $<$ $4r\operatorname{cosec}\alpha$.

Solution:-



Let AB and AC be two rods jointed at A and placed in a vertical plane on a smooth sphere of centre 'O' and radius 'r'. We have $\angle BAC = \alpha$.

Since, the rods are tangential to the sphere, therefore

$\angle BAO = \angle CAO = \frac{1}{2}\alpha$.

Suppose; $AB = AC = 2a$.

If D and E are the middle points of the rods AB and AC, then the combined C.G. of the rods is at the middle point G of ED which must be on AO. Suppose the rod AC touches the sphere at M. We have,

$OM = r$, $AE = a$, $\angle AMO = 90°$, $\angle AGE = 90°$

Suppose AO makes an angle θ with the horizontal line OH through the fixed point 'O'. Let z be the height of the C.G. of the system above the horizontal through 'O'. Then,

$$z = GN = OG \sin\theta = (AO - AG)\sin\theta$$

$$= \left(r \operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\sin\theta$$

$$\therefore \frac{dz}{d\theta} = \left(r \operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\cos\theta.$$

for equilibrium of the rods, we must have

$$\frac{dz}{d\theta} = 0 \quad \text{i.e} \left(r \operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\cos\theta = 0$$

$$\text{i.e. } \cos\theta = 0 \quad \text{i.e} \quad \theta = \frac{\pi}{2}$$

Thus, in the position of equilibrium of rods, the line AO must be vertical.

Also; $$\frac{d^2z}{d\theta^2} = -\left(r \operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\sin\theta$$

$$\frac{d^2z}{d\theta^2} = -r\operatorname{cosec}\tfrac{1}{2}\alpha + a\cos\tfrac{1}{2}\alpha \quad \left[\because \sin\pi/2 = 1 \text{ as } \theta = \pi/2\right]$$

The equilibrium will be stable or unstable according as the height 'z' of the C.G. of the system is minimum or maximum in the position of equilibrium,

i.e. according as $d^2z/d\theta^2$ is positive or negative at $\theta = \pi/2$

i.e according as $a\cos\frac{1}{2}\alpha >$ or $< r\operatorname{cosec}\frac{1}{2}\alpha$

i.e according as $2a >$ or $< \dfrac{2r}{\sin\frac{1}{2}\alpha \cdot \cos\frac{1}{2}\alpha}$

$\Rightarrow$ according as $2a >$ or $< \frac{4r}{\sin\alpha}$

$\Rightarrow$ according as $2a >$ or $< 4r\,\text{cosec}\,\alpha$.

$\left[\because \text{cosec}\,\theta = \frac{1}{\sin\theta}\right]$

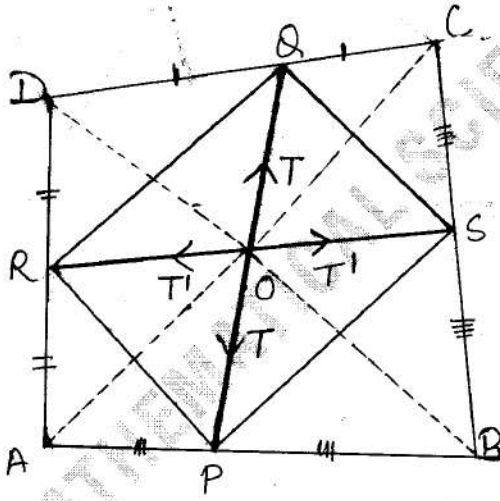
$\Rightarrow$ $\boxed{\text{Hence; } 2a > \text{or} < 4r\,\text{cosec}\,\alpha}$

Hence proved.

Ques:-7(b)) The middle points ~~of~~ the opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T' be the tensions in these rods, prove that

$$\frac{T}{l}+\frac{T'}{l'}=0$$

Solution:-



ABCD - framework in the form of a quadrilateral formed by four light rods.

The middle points P & Q of the rods AB and CD are joined by a string in a state of tension T and the middle points of R and S of rods AD and BC are joined by a light string with tension T'. and the strings PQ of length l and RS of length l'.

Since, P,Q,R,S are the middle points of sides ABCD, therefore PSQR is a parallelogram, and PQ & RS are diagonals bisects each other at 'O'.

Now, give the system a small displacement in which PQ changes to $PQ+\delta(PQ)$ and RS changes to $RS+\delta(RS)$. The lengths of rods AB, BC, CD & DA do not change.

The equation of virtual work is —

$$-T\,\delta(PQ) - T'\,\delta(RS) = 0$$

$$T\,\delta(PQ) = -T'\,\delta(RS)$$

$$\frac{\delta(PQ)}{\delta(RS)} = -\frac{T'}{T}$$

Also; $\dfrac{\delta(PQ)}{\delta(RS)} = \dfrac{RS}{PQ}$

$$\therefore\ -\frac{T'}{T} = \frac{RS}{PQ}$$

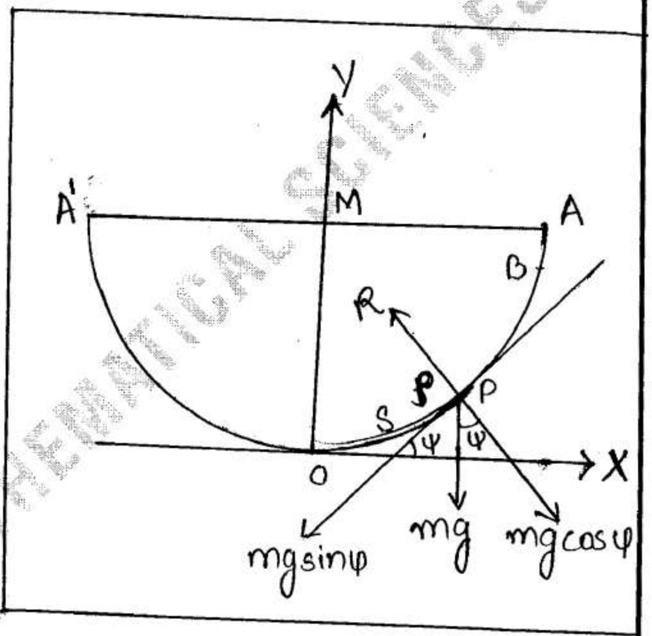
or $\dfrac{T}{PQ} + \dfrac{T'}{RS} = 0$

and as length of $PQ = l$ and $RS = l'$

$\boxed{\dfrac{T}{l} + \dfrac{T'}{l'} = 0}$ [$\because$ in the position of equilibrium].

Hence the result.

Ques: 7(c)) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc. Show that the time of reaching the vertex is $2\sqrt{\frac{a}{g}}\cot^{-1}\left(\frac{V}{2\sqrt{ag}}\right)$, where a is the radius of the generating circle?



Solve:

Let a particle be projected with velocity v from the cusp A of a smooth inverted cycloid down the arc.

If P be the position of the particle at time t such that the tangent at 'P' is inclined at an angle ψ to the horizontal and arc. OP = S. Then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{---(1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{---(2)}$$

For the cycloid, $s = 4a\sin\psi$ ---(3)

from (1) and (3); we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$

Multiplying both sides by $2(ds/dt)$ and integrating we have–

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

But initially at the cusp A,

$$s = 4a \quad \text{and} \quad \left(\frac{ds}{dt}\right)^2 = V^2$$

$$\therefore \; V^2 = -\left(g/4a\right)\cdot 16a^2 + A$$

or $\quad A = V^2 + 4ag$

$$\therefore \; v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2$$

$$= \frac{g}{4a}\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]$$

or $\quad \dfrac{ds}{dt} = -\dfrac{1}{2}\sqrt{g/a}\sqrt{\left[\left(\dfrac{4a}{g}\right)(V^2 + 4ag) - s^2\right]}$

(-ve sign is taken because the particle is moving in the direction of s decreasing).

or $\quad dt = -2\sqrt{\dfrac{a}{g}}\dfrac{ds}{\sqrt{\left[\left(\dfrac{4a}{g}\right)(V^2 + 4ag) - s^2\right]}}$

Integrating, the time t_1, from the cusp A to the vertex 'O' is given by -

$$t_1 = -2\sqrt{a/g}\int_{s=4a}^{0}\frac{ds}{\sqrt{\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]}}$$

$$t_1 = 2\sqrt{\frac{a}{g}}\int_{s=0}^{4a}\frac{ds}{\sqrt{\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]}}$$

$$t_1 = 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{a/g}\sqrt{(v^2+4ag)}}\right]_0^{4a}$$

$$t_1 = 2\sqrt{(a/g)}\ \sin^{-1}\left\{\frac{2\sqrt{ag}}{\sqrt{v^2+4ag}}\right\}.$$

$$t_1 = 2\sqrt{(a/g)}\cdot\theta \quad \text{——} (4)$$

where; $\theta = \sin^{-1}\left[\frac{2\sqrt{ag}}{\sqrt{v^2+4ag}}\right]$

we have; $\boxed{\sin\theta = \frac{2\sqrt{ag}}{\sqrt{v^2+4ag}}}$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{\left[1-\frac{4ag}{v^2+4ag}\right]}$$

$$\boxed{\cos\theta = \frac{v}{\sqrt{(v^2+4ag)}}}$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{ag}}{v} = \frac{\sqrt{4ag}}{v}$$

or $\boxed{\theta = \tan^{-1}\left[\frac{\sqrt{4ag}}{v}\right]}$

$\therefore$ from (4), the time of reaching the vertex is

$$= 2\sqrt{(a/g)}\cdot\tan^{-1}\left[\sqrt{4ag}/v\right]$$

$$t_1 = 2\sqrt{\frac{a}{g}}\cdot\tan^{-1}\left[\sqrt{4ag}/v\right]$$

$$\boxed{t_1 = 2\sqrt{\frac{a}{g}}\ \cot^{-1}\left[\frac{v}{2\sqrt{ag}}\right]}$$ required solution

Ques: 8(a) (i) Show that

$$\text{curl}\left[k \times \text{grad}\,\frac{1}{r}\right] + \text{grad}\left[k \cdot \text{grad}\,\frac{1}{r}\right] = 0$$

where 'r' is the distance from the origin and k is the unit vector in the direction of OZ.

Solution:-

To prove;

$$\text{curl}\left[k \times \text{grad}\,\frac{1}{r}\right] + \text{grad}\left[k \cdot \text{grad}\,\frac{1}{r}\right] = 0$$

Which can be written as.

$$\nabla \times \left[k \times \text{grad}\,\frac{1}{r}\right] + \nabla \cdot \left[k \cdot \text{grad}\,\frac{1}{r}\right] = 0$$

$$\text{grad}\,\frac{1}{r} = \nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$$

So $\hat{k} \times \frac{-\vec{r}}{r^3} = \frac{\vec{r} \times \vec{k}}{r^3}$; $\hat{k}\cdot\left(\frac{-\vec{r}}{r^3}\right) = \frac{-\vec{r}\cdot\hat{k}}{r^3}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $\hat{k}$ = unit vector in direction of OZ.

so; $\hat{k} \times \nabla\left(\frac{1}{r}\right) = \frac{\vec{r} \times \vec{k}}{r^3} = \frac{(x\hat{i}+y\hat{j}+z\hat{k}) \times \hat{k}}{r^3} = \frac{-x\hat{j}+y\hat{i}}{r^3}$

$$\hat{k}\cdot\nabla\left(\frac{1}{r}\right) = \frac{-(x\hat{i}+y\hat{j}+z\hat{k})\cdot\hat{k}}{r^3} = \frac{-z}{r^3}$$

$$\Rightarrow \nabla \times \left[\frac{-1}{r^3}(x\hat{j} - y\hat{i})\right] + \nabla\cdot\left(\frac{-z}{r^3}\right)$$

$$= \hat{i}\left(\frac{-3zx}{r^5} + \frac{3zx}{r^5}\right] + j\left[\frac{-3zy}{r^5} + \frac{3zy}{r^5}\right] + \hat{k}\left[\frac{-3r^2 + 3(x^2+y^2+z^2)}{r^5}\right]$$

$$\Rightarrow \hat{i}(0) + \hat{j}(0) + \hat{k}\left[\frac{-3r^2 + 3(r^2)}{r^5}\right] \quad \left[\because r^2 = x^2+y^2+z^2\right]$$

$$\Rightarrow 0 + 0 + \hat{k}(0) = 0.$$ Hence proved

Ques: 8(a)(ii)} Find the values of constants a, b and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at $(1,-1,1)$ is in the direction parallel to y axis and has magnitude 6?

Solution:- Given; $f = axy^2 + byz + cx^2z^2$.

$$\nabla f = (ay^2 + 2cxz)\hat{i} + (2axy + bz)\hat{j} + (by + 2cx^2z)\hat{k}$$

$$\nabla f\Big|_{(1,-1,1)} = (a + 2c)\hat{i} + (-2a + b)\hat{j} + (-b + 2c)\hat{k}$$

given; $\nabla f\Big|_{(1,-1,1)} = 6\hat{j}$ $\left[\because \text{Magnitude} = 6 \text{ in direction parallel to y axis}\right]$

from above two equations of $\nabla f\Big|_{(1,-1,1)}$ we get

$$a + 2c = 0 \Rightarrow 2c = -a$$
$$-2a + b = 6 \qquad 2c = b$$
$$-b + 2c = 0 \qquad \text{i.e } \boxed{a = -b}$$

$$-2\times(-b) + b = 6$$
$$+2b + b = 6 \Rightarrow 3b = 6 \Rightarrow \boxed{b = 2}$$

Hence; $\boxed{a = -2}$ & $2c = 2 \Rightarrow \boxed{c = 1}$

$\therefore$ $\boxed{\text{values of } (a, b, c) \text{ are } (-2, 2, 1).}$

Ques: 8(b)(i) Prove that $\nabla^2 r^n = n(n+1) r^{n-2}$ and that $r^n \cdot \vec{r}$ is irrotational, where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

Solution:-

Consider; $\nabla^2 r^n = \nabla(\nabla r^n)$

$$\nabla r^n = \Sigma i \frac{\partial}{\partial x} r^n = \Sigma \hat{i}\, n r^{n-1} \cdot \frac{x}{r}$$

$$= \Sigma \hat{i}\, n r^{n-2} \cdot x$$

$$= n \cdot r^{n-2} \cdot \vec{r} \quad [\text{where } \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}]$$

$$\nabla(\nabla r^n) = \nabla(n \cdot r^{n-2} \cdot \vec{r})$$

$$= \Sigma i \frac{\partial}{\partial x}(n r^{n-2} \cdot \vec{r})$$

$$= n \Sigma i \left[(n-2) r^{n-3} \cdot \frac{x}{r} \cdot \vec{r} + r^{n-2}\hat{i}\right)$$

$$= n \left[\Sigma i \left[(n-2) r^{n-3} \cdot \frac{\vec{r}\cdot\vec{r}}{r} + r^{n-2}\hat{i}\right]\right.$$

$$= n \left[\Sigma i \left[(n-2) r^{n-3} \cdot \frac{r^2}{r}\right] + r^{n-2}\hat{i}\right] \quad [\because \vec{r}\cdot\vec{r} = r^2]$$

$$= n\left[(n-2) r^{n-2} + 3 r^{n-2}\right] \quad \left[\begin{array}{l}\because \hat{i} = \nabla\cdot\vec{r} \\ = 3\end{array}\right]$$

$$= n((n-2+3) r^{n-2}]$$

$$= n(n+1) r^{n-2}.$$

$$\therefore \boxed{\nabla^2 r^n = n(n+1) r^{n-2}}$$

Hence proved.

for $r^n \cdot \vec{r}$ to be irrotational.

$$\nabla \times (r^n \cdot \vec{r}) = 0 \quad \text{or} \quad \text{curl}\,(r^n \cdot \vec{r}) = 0$$

$$\nabla \times (r^n \vec{r}) = \nabla \times \vec{r} \cdot r^n + \nabla r^n \times \vec{r}$$

$[\because \nabla \times (\phi A) = (\nabla \times A) \cdot \phi + \nabla \phi \times \vec{A}]$

$$= \underline{\nabla \times \vec{r}} \cdot r^n + \cancel{n} r^{n-1} \cdot \frac{\vec{r}}{\cancel{r}} \times \vec{r} \quad [\because \nabla \times \vec{r} = 0]$$

$$= 0 + r^{n-1} (\vec{r} \times \vec{r}) \quad [\because \vec{r} \times \vec{r} = 0]$$

$$= 0 + 0$$

$$= 0.$$

$$\boxed{\therefore \nabla \times (r^n \cdot \vec{r}) = \text{curl}\,(r^n \cdot \vec{r}) = 0}$$

Hence $r^n \cdot \vec{r}$ is irrotational

Ques:- 8(b) ii) Find the value of a if $A = a\hat{i} + \hat{j} + \sqrt{5}\hat{k}$ subtends an angle of $60°$ with $4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$.

Solution:- Given; $A = a\hat{i} + \hat{j} + \sqrt{5}\hat{k}$, $B = 4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$

Angle between A and B, $\theta = 60°$

$$A = |A| = \sqrt{a^2 + 1^2 + (\sqrt{5})^2} = \sqrt{a^2+6} \quad \text{--- (1)}$$

$$B = |B| = \sqrt{4^2 + (5)^2 + (\sqrt{5})^2} = \sqrt{16+25+5}$$

$$|B| = \sqrt{46} \quad \text{--- (2)}$$

Also; $A \cdot B = (a\hat{i} + \hat{j} + \sqrt{5}\hat{k})(4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k})$

$$A \cdot B = 4a - 5 + 5 = 4a \quad \text{--- (3)}$$

we have; $A \cdot B = AB\cos\theta$

Substituting values from (1), (2) and (3),

$$4a = \sqrt{a^2+6} \cdot \sqrt{46}\cos 60°$$

$$\Rightarrow 4a = \sqrt{a^2+6}\,\sqrt{46} \cdot \frac{1}{2}$$

$$\Rightarrow 8a = \sqrt{46(a^2+6)}$$

Squaring both sides, we get.

$$\Rightarrow 64a^2 = 46(a^2+6)$$

$$\Rightarrow 64a^2 - 46a^2 = 276$$

$$\Rightarrow 18a^2 = 276$$

$$a^2 = \frac{46}{3}$$

$$\Rightarrow \boxed{a = \sqrt{\frac{46}{3}}}$$ required result

Q.8 (c) For two vectors $\vec{a}$ and $\vec{b}$ given respectively by $\vec{a}=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$ and $\vec{b}=\sin t\hat{i}-\cos t\hat{j}$

Determine: $(i)\dfrac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$ and $(ii)\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)$.

SOLUTION

Given vector are

$$\vec{a} = 5t^2\hat{i}+t\hat{j}-t^3\hat{k}$$

$$\vec{b} = \sin\hat{i}-\cos t\hat{j}$$

(i) $$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right) = \frac{d}{dt}\left(5t^2\sin t - t\cos t\right)$$

$$= 10\text{ t sint} + 5t^2\text{ cost} - \text{cost} + \text{t sint}$$

$$= 11\text{ t sint} + (5t^2 - 1)\text{ cost}$$

(ii) $\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)$

$$\because \qquad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}\left(-t^3\cos t\right)-\hat{j}\left(+t^3\sin t\right)+\hat{k}\left(-5t^2\cos t - t\sin t\right)$$

$$= -t^3\cos\hat{i}-t^3\sin\hat{j}-\left(5t\cos t+t\sin t\right)\hat{k}$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right) = \left(-3t^2\cos t+t^3\sin t\right)\hat{i}-\left(3t^2\sin t+t^3\cos t\right)\hat{j}-\left(10\cos t-5t^2\sin t+\sin t+t\cos t\right)\hat{k}$$

$$= \left(t^3\sin t-3t^2\cos t\right)\hat{i}-\left(t^3\cos t+3t^2\sin t\right)\hat{j}-\left\{11t\cos t+\left(1-5t^2\right)\sin t\right\}\hat{k}$$

is the required

Ques: 8(d)) State Stoke's theorem. Verify the Stoke's theorem for the function $\vec{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where C is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one?

Solution:-

Stoke's theorem:. If 'S' is an open, two-sided surface bounded by a closed, non-intersecting curve 'C' (Simple close curve), then if 'A' has continuous derivatives.

$$\boxed{\oint_C A \cdot dx = \iint_S (\nabla \times A) \cdot n\, ds = \iint_S (\nabla \times A) \cdot dS}$$

where C is traversed in the positive direction. The direction of C is called positive, if an observer, walking on the boundary of S in the direction, with his head pointing in the direction of the positive normal to 'S', has the surface on his left.

(ii) Given; $\vec{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where 'C' is the curved obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one.

$$\therefore \boxed{\oint_C F \cdot dx = \iint_S (\nabla \times F) \cdot n\, ds}$$ By Stoke's Theorem.

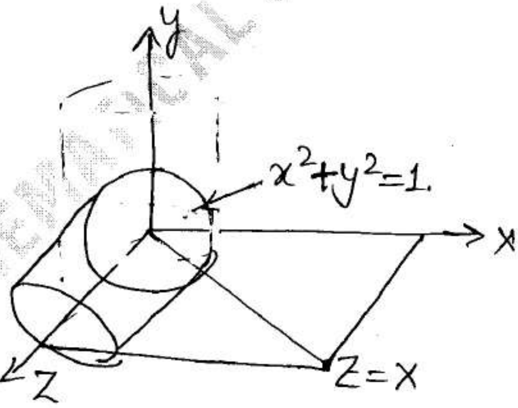
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & z & 2y \end{vmatrix}$$

$$= i\left[2-1\right] - j(0-0) + \hat{k}(0-0)$$

$$\boxed{\nabla \times \vec{F} = \hat{i}}$$

The region S in the area enclosed within $x^2+y^2=1$ of the plane $z=x$.

Taking the projection of the area $d\vec{s}$ on to the x, y plane i.e $z=0$ we have



$$I_1 = \iint_S (\nabla \times \vec{F})\,\hat{n}\,ds$$

$ds = dx\,dy$

$n = \hat{k}$

$$= \iint_{R:\, x^2+y^2 \le 1;\, z=0} \hat{i} \cdot \hat{k}\, dx\, dy.$$

$$I_1 = 0$$

Similarly, for L.H.S.

$$I_2 = \oint_C \vec{F} \cdot d\vec{r} = \int x\,dx + z\,dy + 2y\,dz$$

$$I_2 = \oint_C x\,dx. \qquad [\because z=0,\ dz=0]$$

$$I_2 = \int_{x=-1}^{1} x\,dx \qquad \left[\because x^2+y^2=1 \\ -1 \le x \le 1\right]$$

$$I_2 = \left[\frac{x^2}{2}\right]_{-1}^{1}$$

$$\boxed{I_2 = \left[\frac{1}{2} - \frac{1}{2}\right] = 0}$$

As; $\boxed{I_1 = I_2}$ stroke's theorem verified.

$$\therefore \boxed{\oint_C F\cdot dr = \iint_S (\nabla \times F)\,\hat{n}\,dS = 0}$$ Hence the result