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2013 CSE MECH & FD

5d) Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are

$$u, v, w = \mu\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

Where μ and ϕ are functions of x, y, z, t. (10)

The D.E. of streamlines and vortex lines are respectively

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{---} (1)$$

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad \text{---} (2)$$

(1) and (2) will intersect orthogonally iff

$$u\xi + v\eta + w\zeta = 0$$

$$\mu\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

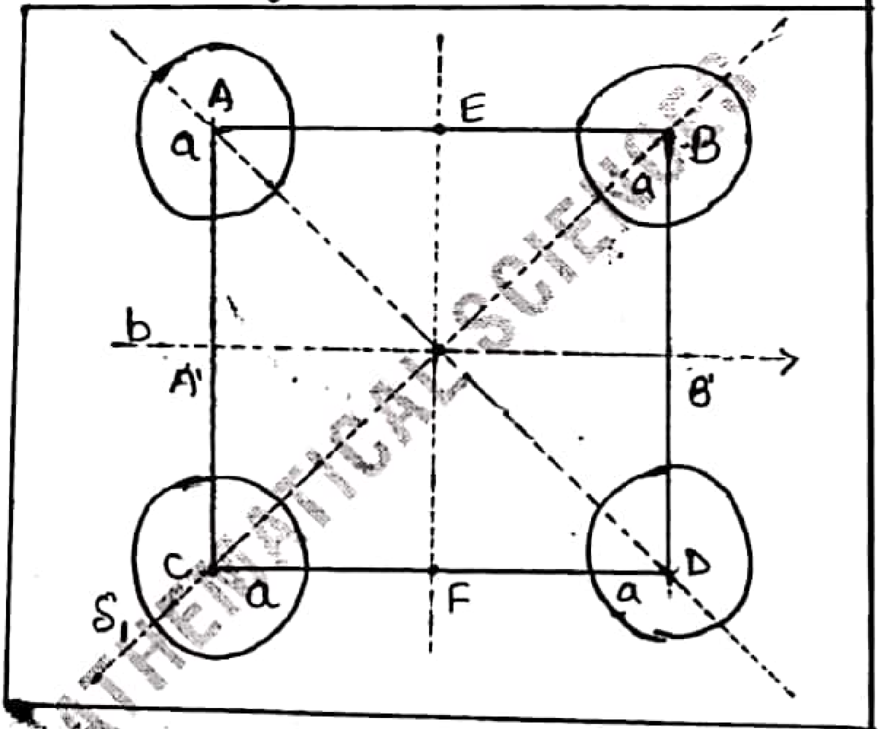
Which implies that $u\,dx + v\,dy + w\,dz$ is a perfect differential.

$$\therefore u\,dx + v\,dy + w\,dz = \mu\, d\psi$$

$$= \mu\left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial z}dx\right)$$

$$\Rightarrow u, v, w = \mu\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

Ques: 5(e) Four solid spheres A, B, C and D, each of mass 'm' and radius 'a', are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about a diagonal of the square?



Solve:-

Firstly evaluating moment of Inertia of system about AB.

Four solid sphere A, B, C and D each have mass M. and radius 'a'.

MOI of any solid sphere (S_1) about AB

= MOI of solid sphere about CD + MOI of centre C of S_1 about AB.

$$= \frac{2}{5} Ma^2 + \frac{M b^2}{4}.$$

MOI of System about AB = 4 (MOI of S_1 about AB)

$$= 4\left(\frac{2}{5} Ma^2 + \frac{M b^2}{4}\right)$$

$$= \frac{8}{5} Ma^2 + M b^2$$

Similarly, MOI of System about EF is

$$= \frac{8}{5}Ma^2 + Mb^2$$

(where EF is axis perpendicular to AB]

let; $A = B = \frac{8}{5}Ma^2 + Mb^2$

So, MOI of system about the diagonal.

$$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$$

F : Product of inertial since the system is symmetric $F = 0$, therefore

$$F = A\cos^2 45^\circ + A\sin^2 45^\circ$$

$$F = A(\cos^2 45^\circ + \sin^2 45^\circ)$$

$$F = A$$

$$\boxed{\therefore F = \frac{8}{5}Ma^2 + Mb^2}$$ required result.

Q4. Two equal rods AB and BC each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $2\pi/n$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

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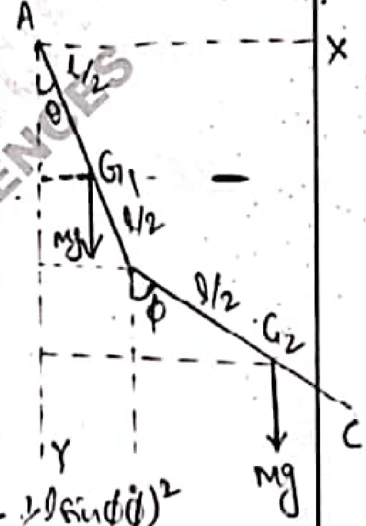
Sol'n: Let AB and BC be the rods of equal length l and mass M.
At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G G_2 of rod BC are given by



$x_{G_1} = \frac{1}{2} l\sin\theta;\ y_{G_1} = \frac{1}{2} l\cos\theta$

$x_{G_2} = l\sin\theta + \frac{1}{2} l\sin\phi,\ y_{G_2} = l\cos\theta + \frac{1}{2} l\cos\phi$

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2, then

$$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\tfrac{1}{2} l\cos\theta\dot{\theta}\right)^2 + \left(-\tfrac{1}{2} l\sin\theta\dot{\theta}\right)^2$$
$$= -\tfrac{1}{4} l^2\dot{\theta}^2$$

$$v_{G_2}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot{\theta} + \tfrac{1}{2} l\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \tfrac{1}{2} l\sin\phi\dot{\phi}\right)^2$$

$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right],\ (\because \theta, \phi \text{ are small})$$

If T be the total K.E. and W the work-function of the system, then

$$T = \text{K.E of rod AB} + \text{K.E of rod BC}$$
$$= \left[\tfrac{1}{2} M\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)^2\dot{\theta}^2 + \tfrac{1}{2} M\cdot v_{G_1}^2\right] + \left[\tfrac{1}{2} m\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)\dot{\phi}^2 + \tfrac{1}{2} M\cdot v_{G_2}^2\right]$$
$$= \tfrac{1}{2} Ml^2\left(\tfrac{4}{3}\dot{\theta}^2 + \tfrac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$$

and $W = Mg y_{G_1} + Mg y_{G_2} + c = Mg\left[\tfrac{1}{2} l\cos\theta + l\cos\theta + \tfrac{1}{2}\cos\phi\right] + c$

$$= \tfrac{1}{2} Mgl(3\cos\theta + \cos\phi)$$

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) - \dfrac{\partial T}{\partial \theta} = \dfrac{\partial W}{\partial \theta}$

i.e. $\dfrac{d}{dt}\left[\tfrac{1}{2} Ml^2\left(\tfrac{8}{3}\dot{\theta} + \dot{\phi}\right)\right] - 0 = \tfrac{1}{2} Mgl(-3\sin\theta) = -\tfrac{3}{2} Mgl\theta$ ($\because \theta$ is small)

$\Rightarrow 8\ddot{\theta} + 3\ddot{\phi} = -9c\theta$, (where $c = g/l$) —— (1).

equations (1) and (2) can be written as

$(8D^2+9c)\theta + 3D^2\phi = 0$ and $3D^2\theta + (2D^2+3c)\phi = 0$

Eliminating ϕ b/w these two equations, we get

$[(2D^2+3c)(8D^2+9c) - 9D^4] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$\theta = A\cos(nt+B)$ $\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$

substituting in (3) we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0 \quad \because \theta \neq 0.$

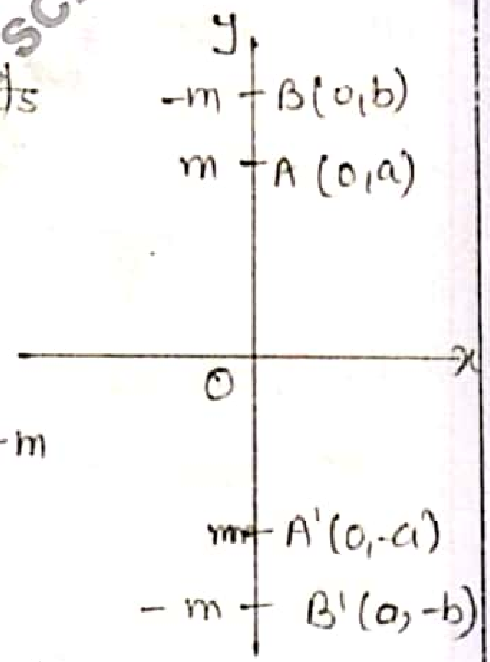
$$\therefore n^2 = \frac{42c \pm \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$$

$$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l} \quad (\because c = g/l).$$

Q (b) IAS-2013 P-II

If the fluid fills the region of space on the positive side of x-axis is a rigid boundary, and if there be a source $+m$ at the point $(0,a)$ and an equal sink at $(0,b)$ and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi \rho m^2 (a-b)^2 / ab(a+b)$, where ρ is the density of the fluid.

Sol^n The object system consists of source $+m$ at $A(0,a)$, i.e. at $z = ia$ and sink $-m$ at $z = ib$. The image system consist of source $+m$ at A' $(z = -ia)$ and sink $-m$ at B' $(z = -ib)$ w.r.t. the positive line OX which is rigid boundary.



The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary.

$$\therefore \quad w = -m\log(z-ia) + m\log(z-ib) - m\log(z+ia) + m\log(z+ib)$$

or $\quad w = -m\log(z^2+a^2) + m\log(z^2+b^2)$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

for any point on x-axis, we have $z=x$ so that

$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x=\infty$. By Bernoulli's equation for steady motion.

$$\therefore \quad \frac{p}{\rho} + \frac{1}{2}q^2 = C$$

In view of $p = p_0$, $q=0$ when $x=\infty$, we get $C = p_0/\rho$.

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundary is given by,

$$P = \int_{-\infty}^{\infty}(p_0 - p)\,dx = \int_{-\infty}^{\infty}\frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx.$$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

for any point on x- axis, we have $z = x$ so that

$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let P_0 be the pressure at $x = \infty$. By Bernoulli's equation for steady motion.

$$\therefore \quad \frac{P}{\rho} + \frac{1}{2}q^2 = C$$

In view of $P = P_0$, $q = 0$ when $x = \infty$, we get $C = P_0/\rho$.

$$\frac{P_0 - P}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundry is given by,

$$P = \int_{-\infty}^{\infty} (P_0 - P)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx.$$

$$= 4\rho m^2(a^2-b^2)^2\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

$$= 4m^2\rho\int_0^\infty\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right\}-\frac{a^2}{(x^2+a^2)^2}-\frac{b^2}{(x^2+b^2)^2}\right]dx.$$

$$= 4m^2\rho\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b}-\frac{\pi}{2a}\right\}-\frac{\pi}{4a}-\frac{\pi}{4b}\right]$$

$$= \frac{\pi\rho m^2(a-b)^2}{ab(a+b)} \quad \text{(Ans.)}$$

for $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^\infty \cos^2\theta\,d\theta\,; \quad x = a\tan\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{\pi}{2}\cdot\frac{1}{2a^3}$$

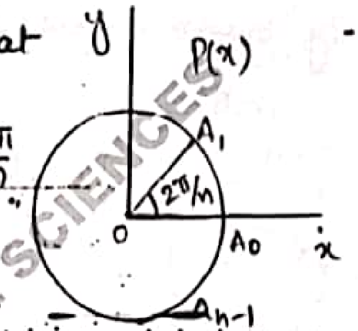
$$= \frac{\pi}{4a^3}$$

IFoS-2013 8(C) P-II

If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid.

Sol'n: from the fig., the n vortices are at $A_0, A_1, A_2 \ldots A_{n-1}$ such that

$\angle A_0OA_1 = \angle A_1OA_2 = \ldots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$



The coordinates of the points A_r are given by

$z = z_r = ae^{(2\pi/n)ir}$ where $r = 0, 1, 2 \ldots n-1$

These are n roots of the equation $z^n - a^n = 0$

$[\text{For } z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi r i}]$

Hence $z^n - a^n = (z - z_0)(z - z_1) \ldots (z - z_{n-1})$

The complex potential due to n vortices at P is given by

$$w = \frac{ik}{2\pi}\left[\log(z - z_0) + \log(z - z_1) + \ldots + \log(z - z_{n-1})\right]$$

$$= \frac{ik}{2\pi}\log(z - z_0)(z - z_1)\ldots(z - z_{n-1}) = \frac{ik}{2\pi}\log(z^n - a^n) \quad \text{——} (1)$$

For the point A_0, $z = a$ so that $r = a, \theta = 0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{ik}{2\pi}\log(z - a) = \frac{ik}{2\pi}\left[\log(z^n - a^n) - \log(z - a)\right]$$

$$\phi' + i\psi' = \frac{ik}{2\pi}\left[\log(r^n e^{in\theta} - a^n) - \log(re^{i\theta} - a)\right]$$

$$\psi' = \frac{k}{4\pi}\left[\log(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta) - \log(r^2 + a^2 - 2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1} - 2nr^{n-1}a^n\cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta} - \frac{2r - 2a\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi}\left[\frac{2nr^n a^n \sin n\theta}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} - \frac{2ra\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a} = \frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right) - \left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right] = \frac{k}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta} - \frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\lim_{x\to 0}\frac{F(x)}{G(x)} = \lim_{x\to 0}\frac{F'(x)}{G'(x)} = \lim_{x\to 0}\frac{F''(x)}{G''(x)}$ $\left[\text{form } \frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta} - \frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta \to 0$$

$$= \frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta} - \frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta \to 0$$

$$= \frac{k}{4\pi}[0+0] = 0$$

finally $\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi a}(n-1)$, $\frac{\partial\psi'}{\partial\theta} = 0$ as $r \to a, \theta \to 0$.

Consequently, the velocity q_0 of the vertex A_0 is given by

$$q_0 = \left[\left(\frac{\partial\psi'}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2} = \frac{k(n-1)}{4\pi a}$$