IFoS 2015

Q.1 Find the curvature and torsion of the curve:

$x = a\cos t,\ y = a\sin t,\ z = bt.$

Soln... The position vector $(\vec{r})$ of the curve at any point of time (t) can be given as:

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}.$$

Diffn wrt 't': $\dfrac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}$

Diffn again wrt 't': $\dfrac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j}$

Diffn again wrt 't': $\dfrac{d^3\vec{r}}{dt^3} = a\sin t\,\hat{i} - a\cos t\,\hat{j}$

(1) Curvature: $$k = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2 r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} \quad \text{--- (1)}$$

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$$\Rightarrow \frac{dr}{dt} \wedge \frac{d^2r}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\hat{i} + ab\cos t\,\hat{j} + a^2\hat{k}$$

$$\Rightarrow \left|\frac{dr}{dt} \wedge \frac{d^2r}{dt^2}\right| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$$

$$= \sqrt{a^2b^2(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^2(a^2+b^2)}$$

$$= a(a^2+b^2)^{1/2}$$

$$\Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= (a^2+b^2)^{1/2}$$

$\therefore$ from (1): $$k = \frac{\left|\frac{dr}{dt} \wedge \frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{a(a^2+b^2)^{1/2}}{(a^2+b^2)^{3/2}}$$

$$\Rightarrow \boxed{k = \frac{a}{a^2+b^2}}$$

(2) Torsion: $$\tau = \frac{\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \wedge \frac{d^2r}{dt^2}\right|^2}$$

$$\Rightarrow \left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$$= b(a^2\cos^2 t + a^2\sin^2 t)$$

$$= a^2b$$

$\Rightarrow T = \dfrac{a^2 b}{a^2(a^2+b^2)} \Rightarrow \boxed{T = \dfrac{b}{a^2+b^2}}$ Ans.

Q.2. Examine if vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F} = \text{grad}\ \phi$.

Soln... $\vec{F}$ is irrotational or not depends by curl F $(\nabla \times F) = 0$ or not.

$$\Rightarrow \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} = (3x^2z^2 - 3x^2z^2)\hat{i} - (6xyz^2 - 6xyz^2)\hat{j} + (2xz^3 - 2xz^3)\hat{k}$$

$$= \vec{0}$$

$\therefore \vec{F}$ is irrotational.

Now, $\vec{F} = \text{grad}\ \phi = \nabla\phi$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Comparing coefficients:

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad -(1); \quad \frac{\partial\phi}{\partial y} = x^2z^3 \quad -(2); \quad \frac{\partial\phi}{\partial z} = 3x^2yz^2 \quad -(3)$$

Integrating Eqn (1): $\phi = x^2yz^3 + f(y) + f(z) \quad -(4)$

Partially

Diffn (4) wrt 'y' & 'z' respectively:

$\frac{\partial \phi}{\partial y} = x^2 z^3 + f'(y)$

Comparing with eqn (2):

$f'(y) = 0$

$\Rightarrow f(y) = c_1$

$\frac{\partial \phi}{\partial z} = 3x^2 y z^2 + f'(z)$

Comparing with eqn (3):

$f'(z) = 0$

$\Rightarrow f(z) = c_2$

$\therefore$ from eqn (4): $\phi = x^2 y z^3 + c_1 + c_2$

$\Rightarrow \boxed{\phi = x^2 y z^3 + A}$ ... [whr $A = c_1 + c_2$]

Q.5 Using divergence theorem, evaluate

$\iint_S (x^3 dy\,dz + x^2 y\, dz\,dx + x^2 z\, dy\,dx)$ where S is the surface of the sphere $x^2+y^2+z^2=1$.

Soln: Acc. to divergence theorem:

$$\iint_S (F\cdot n)\,ds = \iiint_V (\nabla\cdot F)\,dv \quad \text{— (1)}$$

$\Rightarrow \iint_S (x^3 dy\,dx) + x^2 y\, dz\,dx + x^2 z\, dy\,dx = \iint_S (x^3 \hat{i} + x^2 y \hat{j} + x^2 z \hat{k})\cdot n\, ds$

$\Rightarrow \vec{F} = (x^3 \hat{i} + x^2 y \hat{j} + x^2 z \hat{k})$ ... [on comparison]

Now: $\nabla\cdot F = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$.

By (1): $\iint_S (x^3 dy\,dx + x^2 y\, dz\,dx + x^2 z\, dy\,dx) = \iiint_V 5x^2\, dV$

Converting to spherical polar co-ordinates:

$x = r\sin\theta\cos\phi$ and $dV = r^2\sin\theta \, dr\, d\theta\, d\phi$.

To cover V, the limits will be $\Rightarrow$ r from 0 to 1

θ from 0 to $\pi/2$

ϕ from 0 to 2π.

$$\therefore \iiint_V 5x^2 \, dV = 5\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=\pi/2}\int_{\phi=0}^{\phi=2\pi} (r\sin\theta\cos\phi)^2 r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$= 5\int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^4 \sin^3\theta \cos^2\phi \, dr\, d\theta\, d\phi.$$

$$= 5\int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^4 \sin^3\theta \left[\frac{1+\cos 2\phi}{2}\right] dr\, d\theta\, d\phi$$

$$= 5\int_0^1\int_0^{\pi/2} r^4 \sin^3\theta \left[\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right]_0^{2\pi} dr\, d\theta.$$

$$= 5\pi\int_0^1\int_0^{\pi/2} r^4 \sin^3\theta \, dr\, d\theta.$$

$$= 5\pi\int_0^1\int_0^{\pi/2} r^4 \left(\frac{3\sin\theta - \sin 3\theta}{4}\right) d\theta\, dr$$

$$= \frac{5\pi}{4}\int_0^1 r^4 \, dr \left[-3\cos\theta + \frac{\cos 3\theta}{3}\right]_0^{\pi/2} dr$$

$$= \frac{5\pi}{4}\cdot\frac{8}{3}\int_0^1 r^4 \, dr$$

$$= \frac{5\pi}{4}\cdot\frac{8}{3}\left[\frac{r^5}{5}\right]_0^1 = \frac{\cancel{5}\pi}{\cancel{4}}\cdot\frac{\cancel{8}^{2}}{3}\cdot\frac{1}{\cancel{5}} = \frac{2\pi}{3}.$$

Q.1 If $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \cdot \vec{F}) \cdot \hat{n}\, ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

Soln The boundary C of the surface S is the circle $x^2+y^2=a^2$, $z=0$. Suppose $x = a\cos t$, $y = a\sin t$, $z=0$ & $0 \le t \le 2\pi$ are parametric eqn's of C.

By Stokes Theorem, :

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C [y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C y\,dx + (x-2xz)\,dy + xy\,dz$$

$$= \int_C y\,dx + x\,dy \quad \ldots [\because \text{on } C,\ z=0 \ \&\ dz=0]$$

Converting to parametric form:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds = \int_C y\,dx + x\,dy$$

$$= \int_0^{2\pi} [a\sin t(-a\sin t) + a\cos t(a\cos t)]\,dt$$

$$= \int_0^{2\pi} a^2(\cos^2 t - \sin^2 t)\,dt \quad \ldots \left[\because x = a\cos t,\ dx = -a\sin t\,dt\right]$$

$$= a^2 \int_0^{2\pi} \cos 2t\,dt = a^2 \left[\frac{\sin 2t}{2}\right]_0^{2\pi} = 0.$$