4(a) To show that the bilinear transformation

$$w = e^{i\theta_0}\left(\frac{z-z_0}{z-\bar{z}_0}\right)$$

maps the upper half of z-plane into the interior of unit circle in w-plane.

Let, $w = \dfrac{az+b}{cz+d}$, such that $ad-bc \neq 0$ — (1)

be any bilinear mapping which maps upper half plane $I(z) = y \geq 0$ into the unit circle $|w| \leq 1$.

Now,

$$w = \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)} \quad \text{— (2)}$$

This implies that $c \neq 0$, otherwise the points at ∞ in the two planes would correspond.

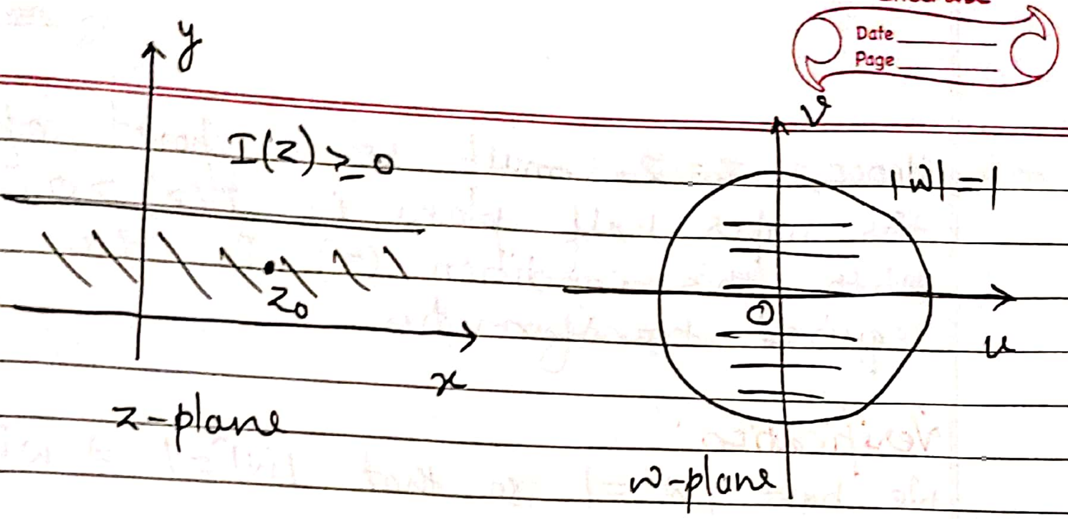
If $w=0$, then (2) $\Rightarrow$ $z = -\dfrac{b}{a}$

If $w=\infty$, then (2) $\Rightarrow$ $z = -\dfrac{d}{c}$

($\because$ denominator $= 0$)

Also $w=0$, $w=\infty$ are inverse points of $|w|=1$.

Now, transformation (2) transforms a straight line of z-plane into a circle and points which are symmetrical about the line transform into inverse points of the circle of w-plane.



Hence, the points $z, \bar{z}$ symmetrical about the real axis i.e. $I(z) = 0$ will correspond to $w = 0$, $w = \infty$ respectively.

Hence, we may take, $\frac{-b}{a} = z_0$, $\frac{-d}{c} = \bar{z}_0$

Then (2) becomes

$$w = \frac{a}{c}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$$

The point $z = 0$ on the boundary of the ~~up~~ half plane $I(z) \geq 0$ must correspond to a point on the boundary of the circle $|w| = 1$, so that

$$1 = |w| = \left|\frac{a}{c}\right| \left|\frac{0 - z_0}{0 - \bar{z}_0}\right| = \left|\frac{a}{c}\right|$$

or $\left|\frac{a}{c}\right| = 1$ $\Rightarrow \frac{a}{c} = e^{i\theta_0}$, where θ_0 is real.

$$\therefore \quad w = e^{i\theta_0}\left(\frac{z - z_0}{z - \bar{z}_0}\right) \qquad (3)$$

Evidently, $z = z_0$ gives $w = 0$. But $w = 0$ is an interior point of the circle $|w| = 1$.

Hence, $z = z_0$ must be a point of the upper half plane i.e. $I(z) > 0$. With this condition (3) is the required transformation.

Verification:

We have $|w| = 1$ so that $|w|^2 = 1 \Rightarrow w\bar{w} = 1$.

Consider,

$$w\bar{w} - 1 = e^{i\theta}\left(\frac{z - z_0}{z - \bar{z}_0}\right) e^{-i\theta}\left(\frac{\bar{z} - \bar{z}_0}{\bar{z} - z_0}\right) - 1$$

$$= \frac{(z - z_0)}{(z - \bar{z}_0)}\left(\frac{\bar{z} - \bar{z}_0}{\bar{z} - z_0}\right) - 1$$

$$= \frac{(z\bar{z} - z_0\bar{z} - z\bar{z}_0 + z_0\bar{z}_0) - (z\bar{z} - \bar{z}_0\bar{z} - zz_0 + \bar{z}_0 z_0)}{(z - \bar{z}_0)(\bar{z} - z_0)}$$

$$= \frac{z(z_0 - \bar{z}_0) - \bar{z}(z_0 - \bar{z}_0)}{|z - \bar{z}_0|^2}$$

$$= \frac{(z_0 - \bar{z}_0)(z - \bar{z})}{|z - \bar{z}_0|^2} = \frac{-4\, I(z_0)\, I(z)}{|z - \bar{z}_0|^2}$$

Finally, $$\frac{4\, I(z_0)\, I(z)}{|z - \bar{z}_0|^2} = 1 - |w|^2 \qquad (4)$$

Also, we have proved that $I(z_0) > 0$.

From (4) it is clear that

i) $I(z) = 0$ corresponds to $|w| = 1$

ii) $I(z) > 0$ corresponds to $1 - |w|^2 > 0$ i.e. $|w| < 1$

Hence $I(z) \geq 0$ corresponds to $|w| \leq 1$.

Finally, we have to find a bilinear transformation such that

$z = i$ is mapped into $w = 0$ and

$z = \infty$ is mapped into $w = -1$.

Here, $$w = e^{i\theta}\left(\frac{z - z_0}{z - \bar{z}_0}\right)$$

$$\therefore \quad 0 = e^{i\theta}\left(\frac{i - z_0}{i - \bar{z}_0}\right) \Rightarrow z_0 = i$$

$$\therefore \quad w = e^{i\theta}\left(\frac{z - i}{z + i}\right)$$

Applying second condition,

$$-1 = e^{i\theta}\left(\frac{\infty - i}{\infty + i}\right) = e^{i\theta}(1)$$

$$\Rightarrow -1 = \cos\theta + i\sin\theta \quad \text{i.e. } \theta = \pi$$

$$\left(\frac{\infty - i}{\infty + i} = \lim_{n \to \infty} \frac{n - i}{n + i} \quad \left(\frac{\infty}{\infty} \text{ form}\right)\right.$$

$$\left. = \lim_{n \to \infty} \frac{1 - 0}{1 + 0} = 1 \right)$$

Hence, require bilinear transformation is

$$w = -\left(\frac{z - i}{z + i}\right)$$

4(b) Let K be a finite field. Show that the number of elements in K is p^n, where p is a prime, which is characteristic of K and $n \geq 1$ is an integer. Also prove that $\frac{\mathbb{Z}_3(X)}{(x^2+1)}$ is a field.
How many elements does this field have? (15)

Sol: Since $pa = 0$ for all non zero a in K

$\therefore o(a) \mid p \Rightarrow o(a) = p \;\forall\; 0 \neq a \in K$

$\Rightarrow \langle K, + \rangle$ is a finite p-group.
i.e. every element has order p under addition operation.

Now, we prove that, $o(K) = p^n$

Let q be a prime dividing $o(K)$.
By Cauchy's theorem $\exists\, x \in K$
s.t. $o(x) = q$.

But $o(x) = p^r$ as K is a p-group.

$\therefore q = p^r \Rightarrow q = p$.

So, p is the only prime dividing $o(K)$.

$\therefore o(K) = p^n$.

Now, we prove that the quotient ring $\frac{\mathbb{Z}_3(x)}{(x^2+1)}$ is a field.

Let $f(x) = x^2+1$
We claim that polynomial $f(x)$ is irreducible over $\mathbb{Z}_3$

To see it, note that $f(x)$ is a quadratic polynomial.
So, $f(x)$ is irreducible over $\mathbb{Z}_3$ if it does not have a root in $\mathbb{Z}_3$.

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 2^2+1 = 2 \text{ in } \mathbb{Z}_3$$

Hence $f(x)$ is irreducible over $\mathbb{Z}_3$.
It implies that the quotient ring

$$\frac{\mathbb{Z}_3(x)}{\langle x^2+1 \rangle} \text{ is a field.}$$

(... nal

(Theorem: $\dfrac{F[x]}{\langle p(x) \rangle}$ is a field iff $p(x)$ is irreducible)

Since, x^2+1 is quadratic, the extension-degree of $\mathbb{Z}_3(x)/(x^2+1)$ over $\mathbb{Z}_3$ is 2.

Hence, the number of elements in the field is $3^2 = 9$.

Remark: You can construct a field of order p^n for any prime p and $n \geq 1$.

Just take $\mathbb{Z}_p[x]$ and form the quotient ring $\dfrac{\mathbb{Z}_p[x]}{\langle p(x) \rangle}$, where $p(x)$ is an irreducible polynomial of degree n.

4(c) Vogel's Approximation Method (VAM) to find the minimum transportation cost

Destinations

	D_1	D_2	D_3	D_4	Availability
S_1	9	16	15	9	15
S_2	2	1	3	5	25
S_3	6	4	7	3	20
Demand	10	15	25	10	60

Source

(15M)

Here, we note that
total demand = total Availability,
hence Transportation problem is balanced.

~~of two min values~~,
Row-Difference/Penalty

9	16	$\boxed{5}$ 15	$\boxed{10}$ 9	~~15~~ 5	0	6	1	1
$\boxed{10}$ 2	1	$\boxed{15}$ 3	5	~~25~~ ~~15~~ 0	1	2	2	–
6	$\boxed{15}$ 4	$\boxed{5}$ 7	3	~~20~~ 5	1	1	3	3
~~10~~ 0	~~15~~ 0	~~25~~ 10	~~10~~ 0					

Column Diff

4	3	4	2
–	3	4	2
–	3	4	–
–	12	8	–

We choose highest Penalty Cost from Row/Column for making Allocation

Total transportation cost

$= 5\times15 + 10\times9 + 10\times2 + 15\times3 + 15\times4 + 5\times7$

$= 75 + 90 + 20 + 45 + 60 + 35$

$= 325.$

No. of Allocation = 6 = 3 (Row) + Column (4) − 1
(Non-degenerate Solution).