Sequences

Set - II (1)

Sequence :- A function whose domain is the set N of all natural numbers and the range is a subset of real numbers is called a sequence. (or) Real sequence.

The sequence is denoted by $x : N \to \mathbb{R}$. (or) $S : N \to \mathbb{R}$.

— A set of numbers which are 1-1 correspondence with natural numbers is called a sequence.

Note

— The domain for a sequence is always natural numbers.

— A sequence is specified by the values $x(n)$ (or) x_n $\forall n \in N$

— A sequence may be denoted by $\{x_n : n \in N\}$ (or) $(x_n : n \in N)$ (or) x (or) $\{x_1, x_2, \ldots x_n, \ldots\}$

The values $x_1, x_2, \ldots x_n, \ldots$ are called first, second, third.... terms of the sequence.

— The m^{th} & n^{th} terms. x_m & x_n for $m \neq n$ are treated as distinct terms even if $x_m = x_n$.

i.e, the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occuring at different positions are treated as distinct terms even if they have the same value.

Range of a sequence :-

The set of all distinct terms in a sequence is called its range.

Note:- In a sequence $\{x_n : n \in N\}$ and N is infinite, the number of terms in a sequence is always infinite.

— The range of a sequence may be finite set.

Ex: If $x_n = (-1)^n$; $n \in N$ then $\{x_n\} = \{-1, +1, -1, +1, \ldots\}$

$\therefore x_1 = -1, \quad x_2 = +1, \quad x_3 = -1, \quad x_4 = +1, \ldots\ldots$

$\therefore$ There are only two distinct elements.

$\therefore$ The range of a sequence $\{x_n\} = \{-1, +1\}$.

which is finite.

Ex: $\{x_n\} = \{\frac{1}{n}\}_{n\in N}$

$= \{1, \frac{1}{2}, \ldots\}$

All the elements of the sequence are distinct.

$\therefore$ The range of a sequence is infinite.

Constant sequence:- A sequence $\{x_n\}$ is defined by $x_n = c \in R \ \forall n \in N$ is called constant sequence.

i.e, $\{x_n\} = \{c, c, c, \ldots\ldots\}$ is constant sequence with a range $= \{c\}$

which is a singleton set.

Ex $\{x_n\}_{\forall n\in N} = \{1\}$

Problems:-

→ The sequence $\{x_n\}$ is defined by the following formulas for the n^{th} term. Write the first five terms in each case.

(a) $x_n = 1 + (-1)^n$ (b) $x_n = \frac{(-1)^n}{n}$ (c) $x_n = \frac{1}{n(n+1)}$ (d) $x_n = \frac{1}{n^2+2}$

→ The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the n^{th} term x_n.

(a) $5, 7, 9, 11, \ldots$ (b) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$

(c) $1/2, 2/3, 3/4, 4/5, \ldots$ (d) $1, 4, 9, 16, \ldots$

Sol: Let $x = (5, 7, 9, 11, \ldots) = (2n+3 \mid n \in N)$.

→ Sum of sequences:- If $x = (x_n)$ and $y = (y_n)$ are two sequences in $\mathbb{R}$ then $x+y = (x_n+y_n)$ in $\mathbb{R}$ is called sum of two sequences. (2)

→ Difference of sequences :- If $x = (x_n)$ and $y = (y_n)$ are two sequences in $\mathbb{R}$ then $x-y = (x_n - y_n)$ in $\mathbb{R}$ is called difference of two sequences.

→ Product of sequences :- If $x = (x_n)$ and $y = (y_n)$ are two sequences then $xy = (x_n y_n)$ in $\mathbb{R}$ is called product of two sequences.

→ Quotient: If $x = (x_n)$, $y = (y_n)$ are two sequences in $\mathbb{R}$ then $\frac{x}{y} = \left(\frac{x_n}{y_n}\right)$ $(y_n \neq 0)$ is called quotient.

Bounds of a sequence :- If the range of a sequence is bdd below, then the sequence is said to be bdd below.

i.e, A sequence $\{x_n\}$ is said to be bdd below if $\exists\, k \in \mathbb{R}$ s.t $x_n \geq k\ \forall n \in N$

→ If the range of a sequence is bdd above then the sequence is said to be bdd above.

i.e, A sequence $\{x_n\}$ is said to be bdd above if $\exists\, K \in \mathbb{R}$ s.t $x_n \leq K\ \forall n \in N$

→ If the range of a sequence is bdd, the sequence is said to be bdd.

i.e, A sequence $\{x_n\}$ is bdd, if $\exists$ two real numbers k, K s.t $k \leq x_n \leq K\ \forall n \in N$.

→ A sequence is said to be unbdd if it is not bdd.

→ If k is a lowerbound of the sequence $\{x_n\}$, every real number less than k is also lower bound of seq $\{x_n\}$. The greatest of all lower bounds is called glb (or) inf of $\{x_n\}$.

→ If K is an upperbound of the seq $\{x_n\}$, every real number greater than K is also an upperbound of a seq $\{x_n\}$. The least of all upper bounds is called lub (or) sup of $\{x_n\}$.

Examples of bounds of a sequence:-

→ (1) $x = (x_n)$ or $\{x_n\}$ where $x_n = n \; \forall n \in N$

$\{x_n\} = \{n / n \in N\} = \{1, 2, 3, \ldots\}$ is not bdd sequence.

Since L.B = 1 ; U.B is not defined

∴ sequence $\{x_n\}$ is bdd below.

→ (2) $x = \{-n / n \in N\}$.

→ (3) $x = \{(-1)^n / n \in N\} = \{-1, +1, -1, +1, \ldots\} = \{-1, +1\}$ is bdd seq.

L.B = −1, U.B = +1

→ (4) $x = \{(-1)^n n / n \in N\}$

→ (5) $x = \{\sqrt{n} / n \in N\}$.

→ Theorem:- A sequence $\{x_n\}$ is bounded if and only if ∃ a real number M (i.e, M>0) s.t $|x_n| \le M \; \forall n \in N$.

N.C: Let $\{x_n\}$ be a bdd seq.

∴ By defn ∃ two real numbers h, k st

$h \le x_n \le k \quad \forall n \in N$ ——(1)

Let $M = \max\{|h|, |k|\}$

$\Rightarrow |h| \le M \,,\; |k| \le M$

$\Rightarrow -M \le h \le M$ ——(2), $-M \le k \le M$ ——(3)

from (1), (2) & (3)

$-M \le h \le x_n \le k \le M \quad \forall n \in N$

$\Rightarrow -M \le x_n \le M \quad \forall n \in N$

$\Rightarrow |x_n| \le M \quad \forall n \in N$.

S.C: $|x_n| \le M \quad \forall n \in N$

$\Rightarrow -M \le x_n \le M \quad \forall n \in N$

∴ $\{x_n\}$ is bdd.

* Limit of a sequence:- Let $x = (x_n)$ be a sequence and $x \in \mathbb{R}$, the real number x is said to be the limit of the sequence $\{x_n\}$ if to each $\epsilon > 0$ (however small) ∃ $k \in N$. (k depending on ε; i.e, k(ε)) s.t $|x_n - x| < \epsilon \; \forall n \ge k(\epsilon)$

i.e, If x is the limit of (x_n), then we write $x_n \to x$ as $n \to \infty$ (or) $\lim_{n\to\infty} x_n = x$.

Note: $|x_n - x| < \epsilon \;\; \forall n \geq k$

$\Rightarrow -\epsilon < x_n - x < \epsilon \;\; \forall n \geq k$

$\Rightarrow x - \epsilon < x_n < x + \epsilon \;\; \forall n \geq k$

$\Rightarrow x_n \in (x-\epsilon, x+\epsilon) \;\; \forall n \geq k.$

* Cgt of a sequence:— Let (x_n) be a sequence if $\lim\limits_{n\to\infty} x_n = x$. then the sequence (x_n) is said to be cgs to x.

If a sequence (x_n) has a limit then the sequence (x_n) is called cgt sequence.

(or)

→ A sequence (x_n) is said to be cgs to x, if for given $\epsilon > 0$ (however small), $\exists$ a +ve integer k (k depending on ϵ, i.e, $k(\epsilon)$) s.t $|x_n - x| < \epsilon \;\; \forall n \geq k$

Here the real number x is limit of the sequence (x_n)

* Divergence of a sequences:— Let (x_n) be a sequence, if $\lim\limits_{n\to\infty} x_n = +\infty$ (or) $-\infty$. then the sequence (x_n) is called dgt sequence.

(or)

→ If a sequence has no limit then the sequence is called dgt sequence.

(or)

(i) A sequence (x_n) is said to dgs to $+\infty$. If given any +ve real number k (however large) $k>0$. $\exists$ a +ve integer m (depending on k) s.t $x_n > k \;\; \forall n \geq m$.

i.e, $\lim\limits_{n\to\infty} x_n = +\infty$ (or) $x_n \to \infty$ as $n \to \infty$.

(ii) A sequence (x_n) is said to dgs to $-\infty$. If given any +ve real number k (however large) $\exists$ a +ve integer m (depending on k) s.t $x_n < -k \;\; \forall n \geq m$.

i.e, $\lim_{n\to\infty} x_n = -\infty$

i.e, $x_n \to -\infty$ as $n \to \infty$

Oscillatory sequences:- If a sequence (x_n) neither cgs to a finite number nor diverges to $+\infty$ (or) $-\infty$, then the sequence (x_n) is called an oscillatory sequence.

→ If the oscillatory sequence is bdd then the sequence is called finite oscillatory sequence.

→ If the oscillatory sequence is unbdd then the sequence is called an infinite oscillatory seq.

Ex: (1) $(x_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$

$U.B = 1; \quad L.B = 0$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore (x_n)$ is cgt.

(2) $(x_n) = \frac{1}{3^n}$ (3) $(x_n) = n^2$ (4) $(x_n) = -n$

(5) $(x_n) = (-1)^n = (-1, +1, -1, +1, \ldots)$

$L.B = -1 \;;\; U.B = +1$

$\lim_{n\to\infty} x_n = -1$ if n is odd

$= +1$ if n is even

$\therefore (x_n)$ is neither cgt nor dgt.

$\therefore$ It is oscillatory sequence. and it is bdd seq.

$\therefore$ finite oscillatory sequence

(6) $(x_n) = \left((-1)^n \cdot n\right) = (-1, +2, -3, +4, \ldots)$

$U.B$ = not defined ; $L.B$ = not defined.

$\lim_{n\to\infty} x_n = +\infty$ if n is even

$-\infty$ if n is odd.

$\therefore$ It is neither cgt nor dgt.

$\therefore$ It is oscillatory sequence and it is unbdd.

$\therefore$ It is infinite oscillatory sequence.

* **Null sequence:-** A sequence (x_n) is said to be a null sequence if it cgs to zero. i.e, $\lim_{n\to\infty} x_n = 0$. Ex: The sequences $\left(\frac{1}{n}\right), \left(\frac{1}{n+1}\right), \left(\frac{1}{2n}\right)$ are null sequences.

→ **Theorem** :- Every cgt sequence has unique limit. (4)

i.e, A sequence cannot converge to more than one limit.

Proof : If possible let a sequence (a_n) converge to two distinct limits x' & x''.

since $x' \neq x'' \Rightarrow |x' - x''| > 0$.

let $\epsilon = \frac{1}{2}|x' - x''|$

since the sequence (a_n) cgs to x'.

$\therefore$ Given $\epsilon > 0$, $\exists$ a +ve integer k' (depending on ϵ)

s.t $|a_n - x'| < \epsilon/2 \quad \forall n \geq k'$.

and also the sequence (a_n) cgs to x''.

$\therefore$ Given $\epsilon > 0$, $\exists$ a +ve integer k'' (depending on ϵ)

s.t $|a_n - x''| < \epsilon/2 \quad \forall n \geq k''$.

let $k = \max\{k', k''\}$.

then $|a_n - x'| < \epsilon/2$ & $|a_n - x''| < \epsilon/2 \quad \forall n \geq k$

now $|x' - x''| = |x' - a_n + a_n - x''|$

$\leq |x' - a_n| + |a_n - x''|$

$< \epsilon/2 + \epsilon/2 = \epsilon$

$\therefore |x' - x''| < \epsilon \quad \forall n \geq k$.

which is a contradiction to $\epsilon = \frac{1}{2}|x' - x''|$

$\therefore$ our assumption that a sequence cgs to two distinct limits x', x'' is wrong.

$\therefore x' = x''$.

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* **Theorem** : Every cgt sequence is bdd.

Pf: Let $X = (x_n)$ be a cgt sequence.

It cgs to x (say)

$\therefore$ Given $\epsilon > 0$, $\exists$ a natural number K s.t $|x_n - x| < \epsilon$ $\forall n \geq K$.

we have $|x_n| = |x_n - x + x| \quad \forall n \geq k$

$\leq |x_n - x| + |x|$

$< \epsilon + |x|$.

Let $M = \sup\{|x_1|, |x_2|, \dots |x_{k-1}|, \epsilon + |x|\}$

$\therefore |x_n| \leq M \;\; \forall n \in \mathbb{N}$

$\therefore (x_n)$ is bdd.

<u>Note</u>: The converse of above theorem need not be true.

i.e, Every bdd sequence need not be cgt.

<u>Ex</u>: $(x_n) = ((-1)^n)$ is bdd.

but $\lim\limits_{n\to\infty} x_n = -1$ if n is odd

$= +1$ if n is even.

$\therefore$ It is an oscillatory sequence.

→ Use the defn of the limit of a sequence to S.T

(1) $\lim \left(\frac{1}{n}\right) = 0$.

<u>Sol</u>: Given $\epsilon > 0$

we have $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$ —(1)

for given $\epsilon > 0$, by Archemedian property $\exists k \in \mathbb{Z}^+$

s.t $k\epsilon > 1$

$\Rightarrow \frac{1}{k} < \epsilon$ —(2)

Now we have, $\forall n \geq k \Rightarrow \frac{1}{n} \leq \frac{1}{k}$

$\Rightarrow \frac{1}{n} \leq \frac{1}{k} < \epsilon$ (by (2)) —(3)

$\therefore$ (1) $\equiv \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$ (by (3)) $\forall n \geq k$.

$\therefore \left|\frac{1}{n} - 0\right| < \epsilon \quad \forall n \geq k$.

i.e, $\frac{1}{n} \to 0$ as $n \to \infty$. i.e, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

(2) $\lim \frac{1}{n^2+1} = 0$

(3) $\lim \left(\frac{n}{n^2+1}\right) = 0$

(4) $\lim \frac{2n}{n+1} = 2$

(5) $\lim \frac{3n+1}{2n+5} = 3/2$

(6) $\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$

(7) $\lim \frac{1}{\sqrt{n+7}} = 0$

(8) $\lim \frac{2n}{n+2} = 2$

(9) $\lim \frac{\sqrt{n}}{n+1} = 0$

(10) $\lim \frac{(-1)^n \cdot n}{n^2+1} = 0$

(11) for any $b \in \mathbb{R}$ P.T $\lim \left(\frac{b}{n}\right) = 0$

(12) $\lim \left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$

→ **Theorem:** Let (x_n) be a sequence of real numbers (5) and let $x \in \mathbb{R}$. If (a_n) is a sequence of +ve real numbers with $\lim_{n\to\infty} a_n = 0$ and if for some constant $c>0$ and some $m \in \mathbb{N}$. we have $|x_n - x| \leq c\,a_n \;\forall n \geq m$ then it follows that $\lim x_n = x$.

proof: Since $\lim_{n\to\infty} a_n = 0$

i.e, $a_n \to 0$ as $n \to \infty$

Given $\epsilon>0$, $\Rightarrow \frac{\epsilon}{c} > 0$, $\exists$ $K \in \mathbb{Z}^+$ $(K = K(\epsilon/c))$

s.t $n \geq K \Rightarrow |a_n - 0| < \epsilon/c$

$\Rightarrow |a_n| < \epsilon/c$

$\Rightarrow a_n < \epsilon/c$ — (1)

Since $|x_n - x| \leq c\,a_n \;\forall n \geq m$

i.e, $n \geq m \Rightarrow |x_n - x| \leq c\,a_n < c(\epsilon/c)$

$\Rightarrow |x_n - x| < \epsilon$ (from (1))

$\therefore$ we have $n \geq m \Rightarrow |x_n - x| < \epsilon$.

i.e, $x_n \to x$ as $n \to \infty$

$\therefore \lim_{n\to\infty} x_n = x$.

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corresponding problems M.G No. 203.

Bernoulli's Inequality:- If $x > -1$ then $(1+x)^n \geq 1+nx \;\forall n \in \mathbb{N}$.

Theorem:

(a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that cgs to x & y respectively and $c \in \mathbb{R}$ then the sequences $X+Y$, $X-Y$, $X\cdot Y$ and cX converge to $x+y$, $x-y$, xy and cx respectively.

(b) If $X = (x_n)$ cgs to x and $Z = (z_n)$ is sequence of non-zero real numbers that cgs to z and if $z \neq 0$ then the quotient sequence $\frac{X}{Z}$ cgs to $\frac{x}{z}$.

Note (1) If $A=(a_n)$, $B=(b_n)$, ... $z=(z_n)$ are cgt sequences then $A+B+\dots+Z=(a_n+b_n+\dots+z_n)$ is also cgt.
and $Lt(a_n+b_n+\dots+z_n) = Lt\, a_n + Lt\, b_n + \dots + Lt\, z_n$.

(2) $A\cdot B\cdot \dots \cdot Z = (a_n\, b_n \cdots z_n)$ is cgt sequence.
and $Lt(a_n b_n \cdots z_n) = Lt\, a_n \cdot Lt\, b_n \cdots Lt\, z_n$

(3.) If $k\in N$ and if $A=(a_n)$ is a cgt sequence then
$$Lt\, a_n^k = (Lt(a_n))^k.$$

Theorem: If $x=(x_n)$ is cgt to x and if $x_n \geq 0\ \forall n\in N$
then $x = Lt\, x_n \geq 0$ (i.e., $x\geq 0$.)

proof: If possible suppose that $x<0$
Since the sequence (x_n) cgt to 'x'.
$\therefore \exists\, k\in I^+$ s.t $|x_n - x| < \epsilon \quad \forall n\geq k$.
$$\Rightarrow x-\epsilon < x_n < x+\epsilon \quad \forall n\geq k.$$
Taking $\epsilon = -x/2 > 0$ ($\because x<0$).
$$\therefore x+\frac{x}{2} < x_n < x-\frac{x}{2} \quad \forall n\geq k.$$
$$x_n < x/2 < 0 \quad \forall n\geq k.$$
$\Rightarrow x_n < 0 \quad \forall n\geq k$

But which is contradiction to the hypothesis that $x_n \geq 0\ \forall n\in N$.
$\therefore$ Our supposition that $x<0$ is wrong.
$\therefore x\geq 0$.

Theorem If $x=(x_n)$ and $y=(y_n)$ are cgt and if $x_n \leq y_n\ \forall n\in N$ then $Lt\, x_n \leq Lt\, y_n$.

proof: Since (x_n) & (y_n) cgt sequences and converge to x & y (say).
$\therefore Lt\, x_n = x$; $Lt\, y_n = y$.
Let $z_n = y_n - x_n$; then $z_n \geq 0\ \forall n$ ($\because y_n \geq x_n$).
NOW $\underset{n\to\infty}{Lt} z_n = Lt\, y_n - Lt\, x_n = y - x$. $\Rightarrow y-x\geq 0 \Rightarrow y\geq x$
$\Rightarrow x\leq y \Rightarrow Lt\, x_n \leq Lt\, y_n$.

Theorem:- If $x=(x_n)$ is a convergent sequence and if $a \le x_n \le b \ \forall n \in N$ then $a \le Lt\, x_n \le b$.

Proof:- Let $y_n = b - x_n$ then

$$y_n \ge 0 \ \forall n \ (\because b \ge x_n)$$

$$\therefore Lt\, y_n = Lt(b - x_n)$$
$$= b - Lt\, x_n$$
$$\Rightarrow b - Lt\, x_n \ge 0 \quad (\because y_n \ge 0)$$
$$\Rightarrow b \ge Lt\, x_n$$
$$\Rightarrow Lt\, x_n \le b$$

Similarly $Lt\, x_n \ge a$ (Let $y_n = x_n - a$)

$$\therefore a \le Lt\, x_n \le b$$

Squeeze Theorem:-

Suppose that $X=(x_n)$, $Y=(y_n)$ and $Z=(z_n)$ are sequences of real numbers such that $x_n \le y_n \le z_n \ \forall n \in N$ and that $Lt\, x_n = Lt\, z_n$ then $Y=(y_n)$ is convergent and

$$Lt(x_n) = Lt(y_n) = Lt(z_n).$$

Proof:- Let $Lt(x_n) = Lt\, z_n = \omega$

i.e. the sequences (x_n) & (z_n) are convergent to ω.

$\therefore$ Given $\epsilon > 0$, $\exists\, K \in I^+$ such that

$n \ge K \Rightarrow |x_n - \omega| < \epsilon$; $|z_n - \omega| < \epsilon$

$n \ge K \Rightarrow \omega - \epsilon < x_n < \omega + \epsilon$

and $\omega - \epsilon < z_n < \omega + \epsilon$

since $x_n \le y_n \le z_n \ \forall n \in N$.

$\therefore$ we have $\omega - \epsilon < x_n \le y_n \le z_n < \omega + \epsilon \ \forall n \ge K$.

$$\Rightarrow \omega - \epsilon < y_n < \omega + \epsilon \quad \forall n \ge K.$$
$$\Rightarrow |y_n - \omega| < \epsilon \quad \forall n \ge K.$$
$$\Rightarrow \lim_{n\to\infty} y_n = \omega$$

$\therefore (y_n)$ converges to ω.

and also $Lt\, x_n = Lt\, y_n = Lt\, z_n$.

Theorem:- Let the sequence $x=(x_n)$ converge to x then the sequence $(|x_n|)$ of absolute values converges to $|x|$. i.e. if $Lt\, x_n = x$ then $Lt(|x_n|) = |x|$.

Proof:- Since $x=(x_n)$ is convergent to x.

$\therefore$ Given $\epsilon > 0$, $\exists\, K \in I^+$ such that

$|x_n - x| < \epsilon \ \forall n \ge K$.

Now we have

$$\big||x_n| - |x|\big| \le |x_n - x| < \epsilon \quad \forall n \ge K.$$
$$\therefore \big||x_n| - |x|\big| < \epsilon \quad \forall n \ge K$$

$\therefore |x_n|$ converges to $|x|$.

Converse part

Ex:- $(x_n) = ((-1)^n) \ \forall n \in N$.

Theorem:- Let (x_n) be a sequence of +ve real numbers such that $L = Lt\left(\frac{x_{n+1}}{x_n}\right)$ exists.

If $L < 1$ then (x_n) converges and

$$\lim_{n\to\infty}(x_n) = 0$$

Proof:- Since $x_n > 0 \ \forall n$

$\Rightarrow x_{n+1} > 0 \ \forall n$

$\therefore \frac{x_{n+1}}{x_n} > 0 \ \forall n$

$\therefore \underset{n\to\infty}{Lt}\left(\frac{x_{n+1}}{x_n}\right) = L \geq 0$

i.e. $L \geq 0$.

Since $\underset{n\to\infty}{Lt} \frac{x_{n+1}}{x_n} = L$

$\therefore$ Given $\epsilon > 0$, $\exists$ $K \in N$ such that

$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon \ \forall n \geq K.$

$\Rightarrow -\epsilon < \frac{x_{n+1}}{x_n} - L < \epsilon \ \forall n \geq K$

$\Rightarrow L-\epsilon < \frac{x_{n+1}}{x_n} < L+\epsilon \ \forall n \geq K$ —— ①

Now replacing n by $K, K+1, K+2, \ldots n-1$ in ① we get

$L-\epsilon < \frac{x_{K+1}}{x_K} < L+\epsilon$

$L-\epsilon < \frac{x_{K+2}}{x_{K+1}} < L+\epsilon$

$L-\epsilon < \frac{x_{K+3}}{x_{K+2}} < L+\epsilon$

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$L-\epsilon < \frac{x_n}{x_{n-1}} < L+\epsilon$

Now multiplying the above $(n-K)$ inequalities, we have

$(L-\epsilon)^{n-K} < \frac{x_n}{x_K} < (L+\epsilon)^{n-K}$ —— ②

Since $L < 1$

choosing $\epsilon > 0$ such that $L+\epsilon < 1$

Since $L \geq 0$

$\therefore 0 \leq L < L+\epsilon < 1$

$\Rightarrow 0 < L+\epsilon < 1$ —— ③

$\therefore$ from ②, we have

$\frac{x_n}{x_K} < (L+\epsilon)^{n-K}$

$\Rightarrow x_n < x_K (L+\epsilon)^{n-K}$

$\Rightarrow x_n < x_K (L+\epsilon)^n \frac{1}{(L+\epsilon)^K}$

since $x_n > 0 \ \forall n$

$\therefore 0 < x_n < x_K (L+\epsilon)^n \frac{1}{(L+\epsilon)^K}$ —— ④

Let $m = x_K \frac{1}{(L+\epsilon)^K} > 0$

$\therefore 0 < x_n < m(L+\epsilon)^n$ (by ④) —— ⑤

Since $0 < L+\epsilon < 1$ (by ③)

$\therefore \underset{n\to\infty}{Lt} (L+\epsilon)^n = 0$

Since the equ'n ⑤ is of the form $y_n < x_n < z_n \ \forall n$

with $Lt\, y_n = Lt\, z_n = 0$.

$\therefore$ By squeeze theorem, $Lt\, x_n = 0$

and (x_n) is convergent to zero.

$\frac{x_n}{x_K} = x_{n-K}$

$70 \geq 50$

$n \geq K$.

$n = 50, 50+1, 50+2, \ldots 69$

$K, K+1, K+2, \ldots n-1$

$69 \rightarrow 50 = 20$ terms

$n-K = 20$

Problems

→ Apply above theorem (i.e. Let (x_n) be a sequence of +ve real numbers such that $L = Lt\left(\frac{x_{n+1}}{x_n}\right)$ exists. If $L<1$, then (x_n) converges and $\lim_{n\to\infty}(x_n)=0$) to the following sequences, where a, b satisfy $0<a<1$, $b>1$.

(a) $\left(\frac{n}{b^n}\right)$, (b) $\left(\frac{2^{3n}}{3^{2n}}\right)$, (c) $\left(\frac{b^n}{2^n}\right)$

Sol'n:- a) $\left(\frac{n}{b^n}\right)$

Let $x_n = \frac{n}{b^n}$ then $x_{n+1} = \frac{n+1}{b^{n+1}}$

Now $\frac{x_{n+1}}{x_n} = \frac{n+1}{b^{n+1}} \times \frac{b^n}{n}$

$= \frac{n+1}{bn}$

$= \frac{1+1/n}{b}$

$Lt\left(\frac{x_{n+1}}{x_n}\right) = Lt\left(\frac{1+1/n}{b}\right)$

$= \frac{1+0}{b} = 1/b < 1$

$(\because b>1)$

$\therefore Lt\,\frac{x_{n+1}}{x_n} = 1/b < 1$ (i.e. $L<1$)

$\therefore (x_n)$ Converges & $Lt\, x_n = 0$.

(b) $\left(\frac{2^{3n}}{3^{2n}}\right)$

Let $x_n = \frac{2^{3n}}{3^{2n}}$ then $x_{n+1} = \frac{2^{3n+3}}{3^{2n+2}}$

Now $\lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = Lt\left[\frac{2^{3n+3}}{3^{2n+2}} \times \frac{3^{2n}}{2^{3n}}\right]$

$= \lim_{n\to\infty}\left(\frac{2^3}{3^2}\right) = 8/9 < 1$

$\therefore (x_n)$ is convergent & $Lt\, x_n = 0$.

(c) $\left(\frac{b^n}{2^n}\right)$

Let $x_n = \frac{b^n}{2^n}$ then $x_{n+1} = \frac{b^{n+1}}{2^{n+1}}$

Now $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{2^{n+1}} \cdot \frac{2^n}{b^n}$

$= b/2$

$\therefore Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2$

if $1<b<2$ then $Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2 < 1$.

$\therefore (x_n)$ is convergent & $Lt\, x_n = 0$.

If $b>2$ then $Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2 > 1$.

$\therefore (x_n)$ is not convergent & $Lt\, x_n \neq 0$.

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* Cauchy's First theorem on limits :-

If $\{a_n\}$ converges to l then the sequence $\{x_n\}$ where $x_n = \frac{a_1+a_2+\cdots+a_n}{n}$ also converges to l. (or)

$\lim_{n\to\infty} a_n = l \Rightarrow \lim_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n} = l$.

Proof :- Let $b_n = a_n - l$ then

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n - l$

$= l - l$

$= 0$

$\therefore \lim_{n\to\infty} b_n = 0$

i.e. $b_n \to 0$ as $n \to \infty$ ——— ①.

Since $a_n = b_n + l \ \forall n$

$$\therefore x_n = \frac{(b_1+l)+(b_2+l)+\cdots+(b_n+l)}{n}$$

$$= \frac{(b_1+b_2+\cdots+b_n)+nl}{n}$$

$$= \frac{b_1+b_2+\cdots+b_n}{n} + l$$

$\therefore$ In order to prove that $x_n \to l$,

For this we are enough to show that

$$\frac{b_1+b_2+\cdots+b_n}{n} \to 0. \quad \text{——— ②}$$

From ①, $\lim b_n = 0$ i.e. $b_n \to 0$ as $n \to \infty$.

$\therefore$ Given $\epsilon > 0$, $\exists$ $m \in I^+$ such that

$|b_n - 0| < \epsilon/2 \ \forall n \geq m$.

$\Rightarrow |b_n| < \epsilon/2 \ \forall n \geq m$, ——— ③

Also the sequence $\{b_n\}$ is convergent

$\therefore \{b_n\}$ is bounded.

$\therefore \exists M > 0$ such that $|b_n| \leq M \ \forall n$ ——— ④

Now let us prove ②

We have

$$\left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| = \left|\frac{b_1+b_2+\cdots+b_n}{n}\right|$$

$$= \frac{|b_1+b_2+\cdots+b_n|}{|n|}$$

$$= \frac{1}{n}\left[|b_1+b_2+\cdots+b_m+b_{m+1}+b_{m+2}+\cdots+b_n|\right]$$

$(\because n \in N \Rightarrow |n| = n)$

$$\leq \frac{1}{n}\left[(|b_1|+|b_2|+\cdots+|b_m|)+(|b_{m+1}|+|b_{m+2}|+\cdots+|b_n|)\right]$$

$$< \frac{1}{n}\left[(M+M+\cdots+M(m\text{ times}))+(\epsilon/2+\epsilon/2+\cdots+\epsilon/2(n-m\text{ times}))\right]. \ \forall n \geq m.$$

(using ③ & ④)

$$\Rightarrow \left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \frac{1}{n}\left[mM + (n-m)\frac{\epsilon}{2}\right] \ \forall n \geq m$$

$$\Rightarrow \left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \frac{mM}{n} + \frac{(n-m)\epsilon}{2n} \ \forall n \geq m$$

$$< \frac{mM}{n} + \frac{\epsilon}{2}$$

$\left(\because \frac{n-m}{n} = 1 - \frac{m}{n} < 1\right) \forall n \geq m$ ——— ⑤

Now $\frac{mM}{n} < \epsilon/2$ if $\frac{n}{mM} > \frac{2}{\epsilon}$

$\Rightarrow$ if $n > \frac{2mM}{\epsilon}$

If P is a natural number $> \frac{2mM}{\epsilon}$

then $n \geq P$.

Let $q = \text{Max}\{P, m\}$.

$\therefore$ From ⑤,

$$\left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \ \forall n \geq q$$

$$= \epsilon$$

$$\therefore \left|\frac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \epsilon \ \forall n \geq q.$$

$\therefore \dfrac{b_1+b_2+\cdots+b_n}{n} \to 0$ as $n\to\infty$

$\therefore x_n \to l$ as $n\to\infty$

i.e. $\underset{n\to\infty}{Lt}\, x_n = l$

Hence the theorem.

Note:– The converse of the above theorem need not be true.

i.e. If the sequence $\{x_n\}$ converges to l then the sequence $\{a_n\}$ need not be converge to l.

where $x_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$

Ex:– Let $\{a_n\} = \{(-1)^n\}$

$= \{-1, +1, -1, +1, \cdots\}$

then $x_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$

if n is even i.e, n=1 to 6 then $x_n = \dfrac{-1+1-1+1-1+1}{n} = 0$

$= 0$ if n is even

$= -\dfrac{1}{n}$ if n is odd.

$\therefore \underset{n\to\infty}{Lt}\, x_n = 0$ i.e. the sequence $\{x_n\}$ Convergent to 0.

But $\{a_n\}$ is not Convergent.

because $\underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt}(-1)^n = +1$ if n is even.

$= -1$ if n is odd.

$\therefore \{a_n\}$ is oscillatory sequence.

$\therefore$ It is not convergent.

Problems:–

→ show that $\underset{n\to\infty}{Lt}\, \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) = 0$.

Sol'n:– Let $a_n = 1/n$ then $\underset{n\to\infty}{Lt}\, \frac{1}{n} = 0$

$\therefore$ By Cauchy's first theorem on limits,

$$\underset{n\to\infty}{Lt}\, \frac{a_1+a_2+\cdots+a_n}{n} = 0$$

$$\Rightarrow \underset{n\to\infty}{Lt}\, \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$$

→ show that

$$\underset{n\to\infty}{Lt}\, \frac{1}{n}\left(\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots+\frac{(n+1)}{n}\right) = 1$$

Let $a_n = \dfrac{n+1}{n}$ then $\underset{n\to\infty}{Lt}\, a_n = Lt\left(1+\frac{1}{n}\right)$

$= 1$

$\therefore$ By Cauchy's first theorem on limits.

$$\underset{n\to\infty}{Lt}\, \frac{a_1+a_2+\cdots+a_n}{n} = 1$$

$$\Rightarrow \underset{n\to\infty}{Lt}\, \frac{\frac{2}{1}+\frac{3}{2}+\cdots+\left(\frac{n+1}{n}\right)}{n} = 1$$

→ show that

$$\underset{n\to\infty}{Lt}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right] = 1$$

L.H.S. $\underset{n\to\infty}{Lt}\left[\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right]$

$$= \underset{n\to\infty}{Lt}\, \frac{1}{n}\left[\frac{1}{\sqrt{1+1/n^2}}+\frac{1}{\sqrt{1+2/n^2}}+\cdots+\frac{1}{\sqrt{1+1/n}}\right]$$

$$= \underset{n\to\infty}{Lt}\, \frac{\left[\frac{1}{\sqrt{1+1/n^2}}+\frac{1}{\sqrt{1+2/n^2}}+\cdots+\frac{1}{\sqrt{1+1/n}}\right]}{n} \quad \text{——— (i)}$$

Let $a_n = \frac{1}{\sqrt{1+1/n}}$ then $\lim_{n\to\infty} a_n = 1$

$\therefore$ By Cauchy's first theorem on limits.

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{1+1/n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \cdots + \frac{1}{\sqrt{1+1/n^2}}\right]}{n} = 1$$

i.e. $\lim \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} = 1$

$\rightarrow$ show that

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}\right] = \infty$$

L.H.S. $\lim \frac{1}{n}\left[\frac{n}{\sqrt{n}} + \frac{n}{\sqrt{n+1}} + \cdots + \frac{n}{\sqrt{2n}}\right]$ — (i)

Let $a_n = \frac{n}{\sqrt{2n}}$ then $a_n = \frac{1}{\sqrt{2}}\sqrt{n}$

$\therefore \lim_{n\to\infty} a_n = \infty$

$\therefore$ By Cauchy's first theorem on limits

$$\lim \frac{\left[\frac{n}{\sqrt{n}} + \frac{n}{\sqrt{n+1}} + \cdots + \frac{n}{\sqrt{2n}}\right]}{n} = \infty$$

i.e. $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{2n}}\right] = \infty$

H.W. $\rightarrow$ show that

$$\lim_{n\to\infty}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right] = 0.$$

$\rightarrow$ **theorem :-**

If $\{a_n\}$ is a sequence of +ve terms for all n and $\lim_{n\to\infty} a_n = l$ then

$$\lim_{n\to\infty} (a_1 . a_2 \cdots a_n)^{1/n} = l$$

Proof :- Let $b_n = \log a_n \ \forall n \ (\because a_n > 0)$ — (1)

Since $\lim_{n\to\infty} a_n = l$

$\therefore \lim_{n\to\infty} b_n = \log l$ ($\because l>0$ because $a_n>0 \ \forall n$)

$\therefore$ By Cauchy's first theorem on limits.

$$\lim_{n\to\infty} \frac{b_1+b_2+\cdots+b_n}{n} = \log l$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n} = \log l \quad (\text{by } (1))$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log[a_1 . a_2 \cdots a_n]}{n} = \log l$$

$$\Rightarrow \lim_{n\to\infty} \log(a_1 \cdots a_n)^{1/n} = \log l.$$

$$\Rightarrow \log\left[\lim_{n\to\infty}(a_1 . a_2 \cdots a_n)^{1/n}\right] = \log l$$

$$\Rightarrow \lim_{n\to\infty}(a_1 . a_2 \cdots a_n)^{1/n} = l$$

Theorem If (a_n) is a sequence such that $a_n > 0 \ \forall n$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ then $\lim_{n\to\infty} (a_n)^{1/n} = l$.

Proof :- Let us define the sequence $\{b_n\}$ such that

$b_1 = a_1,\ b_2 = \frac{a_2}{a_1},\ b_3 = \frac{a_3}{a_2}, \cdots b_n = \frac{a_n}{a_{n-1}} \cdots$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = l$

$\Rightarrow \underset{n\to\infty}{Lt} \frac{a_n}{a_{n-1}} = l \Rightarrow \underset{n\to\infty}{Lt}\, b_n = l$ —— (1)

Since $a_n > 0 \ \forall n$

$\therefore b_n > 0 \ \forall n$

Now we have a sequence $\{b_n\}$ such that $b_n > 0 \ \forall n$ and $\underset{n\to\infty}{Lt}\, b_n = l$

$\therefore \underset{n\to\infty}{Lt} (b_1 . b_2 \cdots b_n)^{1/n} = l$

(By previous theorem)

$\Rightarrow \underset{n\to\infty}{Lt} (a_n)^{1/n} = l$

Note :- The converse of above theorem need not be true.

Ex:- Let $a_n = 2^{-n+(-1)^n}$

then $a_n^{1/n} = 2^{-1+\frac{(-1)^n}{n}}$

$\therefore \underset{n\to\infty}{Lt}\, a_n^{1/n} = 2^{-1}$ $\quad \underset{n\to\infty}{Lt} \frac{(-1)^n}{n} = 0$

$= 1/2$

But $\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n+(-1)^n}}$

$= 2^{-(n+1)+(-1)^{n+1}} 2^{n-(-1)^n}$

$= 2^{-1+(-1)^{n+1}-(-1)^n}$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = 2^{-1-1-1} = 2^{-3}$ if n is even.

$= 2^{-1+1+1} = 2^1$ if n is odd

$\therefore \underset{}{Lt} \frac{a_{n+1}}{a_n} = 1/8$ if n is even.

$= 2$ if n is odd.

$\therefore \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n}$ does not exist.

Note :- The above theorem known as Cauchy's second theorem on limits.

Problems:

→ Show that $\{n^{1/n}\}$ converges to 1.

Sol'n: Let $a_n = n$ then

$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{n}$

$= (1 + 1/n)$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = 1$

$\therefore$ By Cauchy's second theorem on limits.

$\underset{n\to\infty}{Lt}\, a_n^{1/n} = 1$

→ Show that

$\frac{1}{n}\left(1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}\right) = 1$

First theorem.

→ Find $\underset{n\to\infty}{Lt} (n!)^{1/n}$

Let $a_n = n!$ then $a_{n+1} = (n+1)!$

Now $\frac{a_{n+1}}{a_n} = n+1$

$\therefore \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \infty$

$\therefore$ By Cauchy's second theorem on limits

$\underset{n\to\infty}{Lt} (a_n)^{1/n} = \infty$

i.e. $\underset{n\to\infty}{Lt} (n!)^{1/n} = \infty$

→ If $x_n = \frac{n!}{n^n}$ then $\lim_{n\to\infty} x_n = 0$.

Sol'n :– Now $x_n = \frac{n!}{n^n}$

$$\Rightarrow x_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Now $\frac{x_n}{x_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{e} < 1$$

$$\therefore \lim x_n = 0$$

→ If $x_n = \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$

then show that $\lim_{n\to\infty} x_n = e$

Let $a_n = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n$

then $a_{n+1} = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)^3 \cdots \cdots \left(\frac{n+1}{n}\right)^n \left(\frac{n+2}{n+1}\right)^{n+1}$

Now $\frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{n+1}$

$$= \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\therefore \lim \frac{a_{n+1}}{a_n} = e$$

∴ Cauchy's second theorem on limits.

$$\lim_{n\to\infty} (a_n)^{1/n} = e$$

i.e. $\lim x_n = e$

H.W → Prove that $\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} = e$

(Cauchy's second theorem)

H.W → Show that $\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = 1/e$

(Cauchy's second theorem)

sol'n – $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{1/n}$.

* Monotonic Sequences :-

→ A sequence (x_n) is said to be monotonically increasing if $x_{n+1} \geq x_n \ \forall n \in N$; i.e. $x_n \leq x_{n+1} \ \forall n$

i.e. $x_1 \leq x_2 \leq x_3 \cdots \leq x_n \leq x_{n+1} \cdots$

→ A sequence (x_n) is said to be monotonically decreasing if $x_{n+1} \leq x_n \ \forall n \in N$ i.e. $x_n \geq x_{n+1} \forall n$

i.e. $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$

→ A sequence (x_n) is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

→ A sequence is said to be strictly monotonically increasing if $x_n < x_{n+1} \ \forall n$.

→ A sequence (x_n) is said to be strictly monotonically decreasing if $x_n > x_{n+1} \ \forall n$.

→ A sequence (x_n) is said to be strictly monotonic if it is either strictly increasing or strictly decreasing.

Ex-① :-

$(1,2,3,4,\cdots n,\cdots)$, $(1,2,2,3,3,3,4,4,4\cdots)$

$(a, a^2, a^3, \cdots a^n, \cdots)$ if $a>1$ are increasing

$\left(1-\frac{1}{n} : n\in N\right) = (1, 1-\frac{1}{2}, 1-\frac{1}{3}, \cdots)$ sequences

②. $(1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots)$, $(1, \frac{1}{2}, \frac{1}{2^2} \cdots \frac{1}{2^{n-1}}, \cdots)$

$(b, b^2, b^3 \cdots b^n, \cdots)$ if $0<b<1$.

$\left(1+\frac{1}{n} : n\in N\right) = (1+1, 1+\frac{1}{2}, 1+\frac{1}{3} \cdots)$ are decreasing sequences.

③ $(+1, -1, +1, \cdots (-1)^{n+1}, \cdots)$;

$(-1, +2, -3, \cdots (-1)^n n, \cdots)$ are not monotonic sequences. Because which are neither increasing nor decreasing.

Theorem : If $X = (x_n)$ is a bounded increasing sequence, then

$\lim_{n\to\infty} (x_n) = \sup\{x_n : n\in N\}$. Further if (x_n) is unbounded increasing sequence then $\lim_{n\to\infty} x_n = \infty$.

(OR)

Every monotonically increasing sequence which is bounded above converge to its least upperbound.

Further every monotonically increasing sequence which is not bounded above diverges to ∞.

Proof :- Case 1:

Let $X = (x_n)$ be a bounded increasing sequence.

and let $X = (x_n)$ be a monotonically increasing which is bounded above.

To prove that the (x_n) converges to its least upper bound.

Since (x_n) is monotonically increasing sequence which is bounded above.

Let 'x' be the least upper bound of the sequence (x_n).

i.e. $x_n \leq x \ \forall n$.

If $\epsilon > 0$ is given then $x-\epsilon$ is not an upper bound of the sequence (x_n)

$\therefore \exists$ at least one term of the sequence (x_n) is x_m in the interval $(x-\epsilon, x]$.

$\Rightarrow x_m \in (x-\epsilon, x]$

$\Rightarrow x-\epsilon < x_m \leq x < x+\epsilon$ —①

Since (x_n) is monotonically increasing sequence.

$\therefore x_n \leq x_{n+1} \ \forall n \in N$.

$$① \equiv x-\epsilon < x_m \leq x_{m+1} \leq x_{m+2} \leq \cdots \leq x < x+\epsilon$$

$\therefore x-\epsilon < x_n < x+\epsilon \ \forall n \geq m$

$\Rightarrow -\epsilon < x_n - x < \epsilon \quad \forall n \geq m$

$\Rightarrow |x_n - x| < \epsilon \quad \forall n \geq m$.

i.e. $\lim\limits_{n\to\infty} x_n = x$

i.e. the sequence (x_n) converges to x.

Case 2 :- Let the sequence (x_n) be unbounded $\uparrow$ and let (x_n) be M$\uparrow$ and which is not bounded above.

To Prove that it diverges to ∞.

Since (x_n) is M$\uparrow$ and which is not bounded above.

Now $\exists$ at least one term of the sequence (x_n) is x_m such that $x_m > K$, $K > 0$ (however large) —①

Since the sequence (x_n) is M$\uparrow$ sequence.

$\therefore x_n \leq x_{n+1} \ \forall x_n$

$① \equiv K < x_m \leq x_{m+1} \leq \cdots$

$\Rightarrow K < x_n \ \forall n \geq m$.

$\therefore (x_n)$ diverges to $+\infty$

i.e. $\lim x_n = \infty$

Theorem :- If $Y = (y_n)$ is a bounded decreasing sequence then

$$\lim (y_n) = \inf \{y_n : n \in N\}$$

further if (y_n) is an unbounded decreasing then $\lim y_n = -\infty$.

(OR)

Every monotonically decreasing sequence, which is bounded below converges to its greatest lower bound.

further, Every monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Monotone Convergence Theorem

A monotone sequence of real numbers convergent iff it is bounded.

Necessary condition :- Let the monotone

sequence (x_n) be convergent sequence.
Then we have to prove that (x_n) is bounded.

Sufficient Condition :-

Let the sequence (x_n) be monotone bounded sequence.
then we have to Prove that the sequence (x_n) is Convergent.

Since (x_n) is monotone bounded sequence.

$\therefore$ it is either $M\uparrow$ Sequence or $M\downarrow$ Sequence.

Also it is bounded above as well as bounded below.

(i) suppose that the sequence (x_n) is bounded $M\uparrow$ sequence then (x_n) is bounded above.

(ii) suppose that the sequence (x_n) is bounded $M\downarrow$ sequence, then (x_n) is bounded below.

* Limit Points of a Sequence:

→ A real number l is said to be limit point of a sequence (x_n) if every neighbourhood of l contains infinitely many terms of the sequence.

i.e. $l \in \mathbb{R}$ is limit point of the sequence $(x_n) \Leftrightarrow$ every neighbourhood of l contains infinitely many terms of the sequence.

$\Leftrightarrow \forall\ \epsilon>0,\ x_n \in (l-\epsilon, l+\epsilon)$ for infinitely many values of n.

$\Leftrightarrow \forall\ \epsilon>0,\ |x_n - l| < \epsilon$ for infinitely many values of n.

Ex:- (1) $(x_n) = ((-1)^n)$

$= (-1, +1, -1, +1, -1, \ldots)$

has two limit points -1 & $+1$.

Let $x_n = (-1)^n\ \forall\ n$

then $x_n = -1$ if n is odd.

and $x_n = +1$ if n is even.

$\therefore$ Every neighbourhood of -1 Contains all the odd terms of the sequence (x_n).

$\therefore -1$ is a limit point.

Similarly every neighbourhood of $+1$ Contains all even terms of the sequence (x_n).

$\therefore +1$ is a limit point.

Ex:(2). $(x_n) = \left(\frac{1}{n}\right)$

$= \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ has a limit point '0'.

Because the neighbourhood of 0 Contains infinitely many terms of the sequence.

(3) The constant sequence (x_n), where $x_n = 1 \ \forall \ n \in N$ has the only limit point 1.

<u>Note</u> :- (1) A limit point of a sequence is also called cluster point (or) an accumulation point (or) condensation point of the sequence.

(2) Limit point of a sequence is different from limit of a sequence.

i.e. if $l \in \mathbb{R}$ is the limit of a sequence (x_n) then for $\epsilon > 0$, $\exists \ m \in N$ such that $|x_n - l| < \epsilon \ \forall n \geq m$.

$\Leftrightarrow x_n \in (l-\epsilon, l+\epsilon) \ \forall n \geq m$.

i.e. Every neighbourhood of l contains all except a finite number of terms of the sequence.

<u>where as</u> if $l \in \mathbb{R}$ is a limit point of the sequence (x_n) then every neighbourhood of l contains infinitely many terms of the sequence (x_n) doesnot exclude the possibility of an infinite number of terms of the sequence lying outside that neighbourhood.

Hence limit of a sequence is a limit point of the sequence, but a limit point of a sequence need not be the limit of the sequence.

(3) If $x_n = l$ for infinitely many values of n then l is a limit point of (x_n).

(4) If for $\epsilon > 0$, $x_n \in (l-\epsilon, l+\epsilon)$ for finitely many values of n then l is not a limit point of the sequence (x_n).

(5) Limit point of a sequence need not be a term of the sequence.

→ <u>Bolzano – Weierstrass Theorem for Sequences</u>:

Every bounded sequence has at least one limit point.

<u>Cauchy's General principle of Convergence</u> :—

A necessary and sufficient condition for the convergence of a sequence (x_n) is that, for each $\epsilon > 0$, $\exists \ m \in I^+$ such that $|x_{n+p} - x_n| < \epsilon \ \forall n \geq m$ and $p \geq 1$.

<u>Necessay Condition</u> :-

Let the sequence (x_n) be convergent and let it be convergent to l.

$\therefore \lim\limits_{n\to\infty} x_n = l$

i.e. $x_n \to l$ as $n \to \infty$

$\therefore$ Given $\epsilon>0$, $\exists\ m\in I^+$ such that

$|x_n-l| < \epsilon/2 \ \forall\ n\geq m$. ——(1)

Since $p\geq 1 \Rightarrow n+p\geq n+1>n\geq m$.

$\therefore |x_{m+p}-l|<\epsilon/2\ \forall\ n\geq m\ \&\ p\geq 1$ ——(2)

Now we have

$$|x_{n+p}-x_n| = |x_{n+p}-l+l-x_n|$$
$$\leq |x_{n+p}-l|+|x_n-l|$$
$$< \epsilon/2+\epsilon/2\ \forall\ n\geq m \text{ and } p\geq 1.$$

$\Rightarrow |x_{n+p}-x_n|<\epsilon\ \forall\ n\geq m$ and $p\geq 1$.

Sufficient Condition :-

Given that for each $\epsilon>0$, $\exists\ m\in I^+$ such that $|x_{n+p}-x_n|<\epsilon$ $\forall\ n\geq m$ and $p\geq 1$

Inparticular $n=m$.

$\therefore |x_{m+p}-x_m|<\epsilon\ \forall\ p\geq 1$.

$\Rightarrow -\epsilon< x_{m+p}-x_m<\epsilon\ \ \forall\ p\geq 1$.

$\Rightarrow x_m-\epsilon<x_{m+p}<x_m+\epsilon\ \ \forall\ p\geq 1$

$\left[\{x_n\}=\left\{\frac{1}{n}\right\}\right.$

$x_m=\frac{1}{m}$

$x_{m+1}=\frac{1}{m+1}$

$\frac{1}{m}-\epsilon<x_{m+1}<\frac{1}{m}+\epsilon$

$\frac{1}{2}-\epsilon<\frac{1}{2+1}<\frac{1}{2}+\epsilon$

$\left.0<0.33<0.6\right]$

Let $h=\text{Min}\{x_1, x_2, \ldots x_{m-1}, x_m-\epsilon\}$.

$K=\text{Max}\{x_1, x_2, \ldots x_{m-1}, x_m+\epsilon\}$

$\therefore h\leq x_n\leq K\ \forall\ n$.

$\therefore (x_n)$ is bounded.

$\therefore$ By Bolzano – weierstrass theorem every bounded sequence has at least one limit point.

i.e. the sequence (x_n) has a limit point say l.

we shall show that the sequence (x_n) converges to l.

i.e. $\lim x_n=l$ ——(3)

Given that, for each $\epsilon>0$, $\exists\ m\in I^+$ such that

$|x_{n+p}-x_n|<\epsilon/3\ \forall\ n\geq m\ \&\ p\geq 1$. ——(4)

Inparticular $n=m$

$\therefore |x_{m+p}-x_m|<\epsilon/3\ \forall\ p\geq 1$ ——(5)

Since l is a limit point.

$\therefore \exists\ m_1>m$ such that

$|x_{m_1}-l|<\epsilon/3$ ——(6)

Since $m_1>m$

$\therefore$ from (5),

$|x_{m_1}-x_m|<\epsilon/3$ ——(7)

Now we have

$$|x_{m+p}-l| = |x_{m+p}-x_m+x_m-x_{m_1}+x_{m_1}-l|$$
$$\leq |x_{m+p}-x_m|+|x_m-x_{m_1}|+|x_{m_1}-l|$$
$$<\epsilon/3+\epsilon/3+\epsilon/3\ \forall\ p\geq 1.$$
$$=\epsilon$$

$\therefore |x_{m+p}-l|<\epsilon\ \forall\ p\geq 1$

$\Rightarrow |x_n-l|<\epsilon\ \forall\ n\geq m$.

$\therefore (x_n)$ is convergent to l.

Cauchy Sequence :-

A sequence (x_n) is said to be Cauchy sequence (or) fundamental sequence.

if for each $\epsilon>0$, $\exists\, m \in I^+$ such that

$$|x_{m+p} - x_n| < \epsilon \;\forall\, n \geq m \text{ and } p \geq 1.$$

(or) $|s_p - s_q| < \epsilon \;\forall\, p, q \geq m.$

Theorem → Every Cauchy's sequence is bounded.

Theorem → If $X = (x_n)$ is a convergent sequence of real numbers then X is a Cauchy sequence.

Note: A sequence cannot converge if for each $\epsilon>0$, $\exists\, m \in I^+$ such that

$$|x_{m+p} - x_m| \nless \epsilon \;\forall\, n \geq m \;\&\; p \geq 1.$$

Problems

→ Use the definition of the limit of a sequence to establish the following limits.

(1) $\underset{}{Lt}\left(\frac{3n+1}{2n+5}\right) = 3/2$

Sol^n :- Let $\epsilon>0$ be given

Now $\left|\frac{3n+1}{2n+5} - 3/2\right| = \left|\frac{-13}{4n+10}\right|$

$$= \frac{13}{4n+10} < \frac{13}{4n}$$
$$< \epsilon$$

if $\frac{4}{13}n > \frac{1}{\epsilon}$

i.e. $n > \frac{13}{4\epsilon}$.

Now if m is a +ve integer $> \frac{13}{4\epsilon}$.

then $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon \;\forall\; n \geq m$.

$\therefore \frac{3n+1}{2n+5} \longrightarrow 3/2$ as $n \longrightarrow \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{3n+1}{2n+5} = 3/2$

(or)

Now we have

$$\left|\frac{3n+1}{2n+5} - 3/2\right| = \left|\frac{-13}{4n+10}\right|$$

$$= \frac{13}{4n+10} < \frac{13}{4n} \quad \text{——①}$$

For $\epsilon>0$, by Archemedian property $\exists\; K \in I^+$ such that $K\epsilon > \frac{13}{4}$.

$$\Rightarrow \frac{1}{K} < \frac{4}{13}\epsilon \quad \text{——②}$$

Now we have

$n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K}$

$\Rightarrow \frac{1}{n} \leq \frac{1}{K} < \frac{4}{13}\epsilon$ (by ②) ——③

$\therefore$ ① $\equiv \left|\frac{3n+1}{2n+5} - 3/2\right| < \frac{13}{4}\left(\frac{4}{13}\right)\epsilon \;\forall\; n \geq K$ (by ③)

$\therefore \left|\frac{3n+1}{2n+5} - 3/2\right| < \epsilon \;\forall\; n \geq K$.

i.e. $\frac{3n+1}{2n+5} \longrightarrow 3/2$ as $n \longrightarrow \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{3n+1}{2n+5} = 3/2$

(2) $\underset{n\to\infty}{Lt} \frac{1}{\sqrt{n+7}} = 0$

(3) $\underset{n\to\infty}{Lt} \left[(-1)^n \cdot \frac{n}{n^2+1}\right] = 0$

(4) $\underset{n\to\infty}{Lt} \frac{1}{3^n} = 0$

Sol^n :- Let $\epsilon>0$ be given

Now we have

$\left|\frac{1}{3^n} - 0\right| = \left|\frac{1}{3^n}\right|$

$= \frac{1}{3^n} < \epsilon$. if $3^n > \frac{1}{\epsilon}$

i.e. if $n\log 3 > \log\left(\frac{1}{\epsilon}\right)$

i.e. if $n > \frac{\log(1/\epsilon)}{\log 3}$ ($\because \log 3 > 0$)

Now if m is +ve integer $> \frac{\log(1/\epsilon)}{\log 3}$

then $\left|\frac{1}{3^n} - 0\right| < \epsilon \;\forall\; n \geq m$

i.e. $\frac{1}{3^n} \longrightarrow 0$ as $n \longrightarrow \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{1}{3^n} = 0$

Imp. If $x_n = 1+\frac{(-1)^n}{2n}$, find the least +ve integer m such that

$|x_n - 1| < \frac{1}{10^3} \quad \forall\ n > m.$

Sol'n :- Now $|x_n - 1| = \left|1+\frac{(-1)^n}{2n} - 1\right|$

$= \left|\frac{(-1)^n}{2n}\right|$

$= \frac{1}{2n}$ — ①

Since $|x_n - 1| < \frac{1}{10^3}$

$\Rightarrow \frac{1}{2n} < \frac{1}{10^3}$ (by ①)

$\Rightarrow 2n > 10^3$

$\Rightarrow n > 500$

$\therefore$ Taking $m = 500$,

we have

$|x_n - 1| < \frac{1}{10^3} \quad \forall\ n > m$ where $m = 500$

H.W. If $x_n = 2+\frac{(-1)^n}{n^2}$, find the least +ve integer m such that $|x_n - 2| < \frac{1}{10^4} \quad \forall\ n > m.$

Theorem Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$, if (a_n) is a sequence of +ve real numbers with $\lim\limits_{n\to\infty} (a_n) = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$. we have $|x_n - x| \le c a_n \quad \forall\ n \ge m$. then it follows that

$\lim x_n = x.$

→ Bernoulli's Inequality :-

If $x > -1$ then $(1+x)^n \ge 1+nx \ \forall n \in \mathbb{N}$

Problems

① If $a > 0$ then $\lim \left(\frac{1}{1+na}\right) = 0$

Sol'n : Since $a > 0$

$\Rightarrow na > 0 \ \forall n \in \mathbb{N}$

$\Rightarrow 0 < na < 1+na \ \forall n \in \mathbb{N}$

$\Rightarrow 0 < \frac{1}{1+na} < \frac{1}{na} \ \forall n \in \mathbb{N}$

Now we have

$\left|\frac{1}{1+na} - 0\right| = \left|\frac{1}{1+na}\right|$

$= \frac{1}{1+na} < \frac{1}{na} \ \forall n \in \mathbb{N}$

$= \left(\frac{1}{a}\right)\left(\frac{1}{n}\right) \ \forall n \in \mathbb{N}.$

$\therefore \left|\frac{1}{1+na} - 0\right| < \left(\frac{1}{a}\right)\left(\frac{1}{n}\right) \ \forall n \in \mathbb{N}$ — ①

Since $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$

and $a > 0$

$\Rightarrow \frac{1}{a} > 0.$

$\therefore \lim\limits_{n\to\infty} \left(\frac{1}{1+na}\right) = 0.$

② If $0 < b < 1$ then $\lim (b^n) = 0$

Sol'n :- Since $0 < b < 1$

Take $b = \frac{1}{1+a}$

where $a = \left(\frac{1}{b}\right) - 1$

$\Rightarrow a > 0$

By Bernoulli's inequality,

we have $(1+a)^n \ge 1+na \ \forall n.$

$\Rightarrow \frac{1}{(1+a)^n} \le \frac{1}{1+na} \ \forall n \in \mathbb{N}$ — ①

Now, $0 < b^n = \frac{1}{(1+a)^n}$

$\leq \frac{1}{1+na}$ (by ①)

$< \frac{1}{na} \; \forall n \in N$ ——— ②

Now we have

$|b^n - 0| = \left| \frac{1}{(1+a)^n} - 0 \right|$

$= \frac{1}{1+na}$

$< \frac{1}{na}$ (by ②). $\forall n \in N$

$= \left(\frac{1}{a}\right)\left(\frac{1}{n}\right)$

Since $\underset{n\to\infty}{Lt} \left(\frac{1}{n}\right) = 0$

and $a > 0$

$\Rightarrow 1/a > 0$.

$\therefore \underset{n\to\infty}{Lt} b^n = 0$.

→ If $c>0$ then $\lim_{n\to\infty}(c^{1/n})=1$

Sol'n :- Case(i):

Let $c=1$ then

$(c^{1/n})=(1,1,1,\ldots)$

$\therefore \lim_{n\to\infty}(c^{1/n})=1.$

Case(ii): Let $c>1$;

then $c^{1/n}=1+d_n$ for some $d_n>0$

$\Rightarrow \boxed{c^{1/n}-1=d_n}$

and $c=(1+d_n)^n \geq (1+nd_n)\,\forall n$

(by Bernoullis inequality)

$\Rightarrow c\geq 1+nd_n \;\forall n$

$\Rightarrow \frac{c-1}{n}\geq d_n \;\forall n$ ——(1)

Now we have,

$|c^{1/n}-1| = d_n$

$\leq \frac{c-1}{n}\;\forall n$ (by (1))

$=(c-1)\left(\frac{1}{n}\right)$ ——(2)

since $\lim(1/n)=0$

and $(c-1)>0$ $(\because c>1)$

$\therefore \lim_{n\to\infty}(c^{1/n})=1$

Case(iii):

Let $0<c<1$

then $c^{1/n}=\frac{1}{1+h_n}$ for some $h_n>0$

$\Rightarrow c=\frac{1}{(1+h_n)^n}$ ——(3)

By Bernoulli's inequality,

$(1+h_n)^n \geq 1+nh_n \;\forall n\in N$

Rough Idea

If $c=2$ then $c^{1/n}=2^{1/n}$

$=2^1, 2^{1/2}, 2^{1/3}, \ldots$

$=2, \sqrt{2}, \sqrt[3]{2}, \ldots$

$=2, 1.414, \ldots$

$=1+1, 1+0.414, \ldots \longrightarrow 1$

$=1+d_n;\; d_n>0$

If $c=3$ then $c^{1/n}=3^{1/n}$

$=3^1, 3^{1/2}, 3^{1/3}, \ldots$

$=3, \sqrt{3}, \sqrt[3]{3}, \ldots$

$=3, 1.732, \ldots \longrightarrow 1$

$=1+2, 1+0.732, \ldots$

$=1+d_n;\; d_n>0.$

Rough Idea

$c=0.5$

$=1/2$

$c^{1/n}=(1/2)^{1/n}$

$=(1/2)^1, (1/2)^{1/2}, (1/2)^{1/3}, \ldots$

$=\frac{1}{1+1}, \frac{1}{1+0.414}, \ldots \longrightarrow 1$

$=\frac{1}{1+h_n};\; h_n>0.$

$\therefore \dfrac{1}{1+h_n} \le \dfrac{1}{1+nh_n} \quad \forall n$ — (4)

From (3) & (4)

$$c = \frac{1}{(1+h_n)^n} \le \frac{1}{1+nh_n} \quad \forall n$$

$$\Rightarrow c \le \frac{1}{1+nh_n} < \frac{1}{nh_n} \quad \forall n \quad \text{— (5)}$$

Now we have,

$$0 < c < \frac{1}{nh_n} \quad \forall n$$

$$\Rightarrow \quad 0 < ch_n < \frac{1}{n} \quad \forall n$$

$$\Rightarrow \quad 0 < h_n < \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \quad \forall n \quad \text{— (6)}$$

Now we have,

$$|c^{x_n} - 1| = \left|\frac{1}{1+h_n} - 1\right|$$

$$= \left|\frac{-h_n}{1+h_n}\right|$$

$$= \frac{h_n}{1+h_n} < h_n \quad \forall n$$

$$< \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \quad \forall n \quad (\text{by } (6))$$

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

& $c > 0$

$\Rightarrow \frac{1}{c} > 0$

$\therefore \lim\limits_{n\to\infty} c^{x_n} = 1.$

2003 → Let 'a' be a +ve real number (i.e $a>0$) and $\{x_n\}$ a sequence of rational numbers such that $\lim\limits_{n\to\infty} x_n = 0$.

Show that

$$\lim_{n\to\infty} a^{x_n} = 1.$$

Sol^n: Given that $\{x_n\}$ a sequence of rational numbers such that $\lim\limits_{n\to\infty} x_n = 0$

Let the sequence

$$\{x_n\} = \left\{\frac{1}{n}\right\}$$

then we show that

$$\lim_{n\to\infty} a^{x_n} = 1.$$

(proceed as in the above problem.)

→ Prove that the sequences whose n^{th} terms are given below, are monotonic. Find out whether they are increasing (or) decreasing.

(i) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\cdots + \frac{1}{2^n}$.

Sol^n: Let $x_n = \frac{1}{2} + \frac{1}{2^2} + \cdots\cdots + \frac{1}{2^n}$

and $x_{n+1} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$

Now $x_{n+1}-x_n = \frac{1}{2^{n+1}} > 0 \; \forall n$

$\Rightarrow x_{n+1} > x_n \; \forall n$

$\Rightarrow x_n < x_{n+1} \; \forall n$

$\therefore (x_n)$ is an increasing sequence.

$\therefore (x_n)$ is monotonic sequence.

(ii) $\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{2n-1}$

(iii) $\frac{3n+7}{4n+8}$ (iv) $\frac{2n+7}{3n+8}$

(v) $1+\frac{1}{1!}+\frac{1}{2!}+\cdots\cdots+\frac{1}{(n-1)!}$

(vi) $\frac{1}{2n+5}$ (vii) $\frac{-1}{2n+1}$

(viii) $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}$ (ix) $\left(1+\frac{1}{n}\right)^n$

Sol^n: Let $x_n = \left(1+\frac{1}{n}\right)^n$

$= n_{c_0}(1)^n\left(\frac{1}{n}\right)^0 + n_{c_1}(1)^{n-1}\left(\frac{1}{n}\right)^1 + n_{c_2}(1)^{n-2}\left(\frac{1}{n}\right)^2+\cdots+ n_{c_n}(1)^{n-n}\left(\frac{1}{n}\right)^n.$

$= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right)+\cdots\cdots+\frac{n(n-1)(n-2)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$

$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left[1-\frac{(n-1)}{n}\right]$ — (1)

Now $x_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\cdots\left(1-\frac{n-1}{n}\right)\left(1-\frac{n}{n+1}\right)$ — (2)

Now we have,

$\frac{1}{n+1} < \frac{1}{n} \; \forall n \in N.$

$\Rightarrow \frac{K}{n+1} < \frac{K}{n} \; \forall n \in N.$

$\Rightarrow \frac{-K}{n+1} > -\frac{K}{n} \; \forall n \in N$

$\Rightarrow 1-\frac{K}{n+1} > 1-\frac{K}{n}$; $K=1,2\cdots n$

From (1) & (2) we have

$x_{n+1} > x_n \; \forall n$

$\therefore (x_n)$ is M $\uparrow$

$\therefore (x_n)$ is monotonic sequence.

(x) $a_1=1$ and $a_n=\sqrt{2+a_{n-1}} \; \forall n \geq 2$.

Sol^n: Given that $a_1=1$ & $a_n=\sqrt{2+a_{n-1}} \; \forall n \geq 2$

$a_2 = \sqrt{2+a_1}$

$= \sqrt{2+1}$

$= \sqrt{3} > 1 = a_1$

$\therefore a_2 > a_1.$

Similarly $a_3 > a_2$

Now suppose, $a_n > a_{n-1}$ for some 'n', $n>1$

$\Rightarrow 2+a_n > 2+a_{n-1}$

$\Rightarrow \sqrt{2+a_n} > \sqrt{2+a_{n-1}}$

$\Rightarrow a_{n+1} > a_n$

$\therefore$ By mathematical induction

$a_n < a_{n+1} \; \forall n$

$\therefore (a_n)$ is an increasing sequence.

$\therefore (a_n)$ is monotonic sequence.

→ Show that the sequence (x_n) defined by $x_n = \left(1+\frac{1}{n}\right)^n$ is cgt.

Sol^n: Given that $x_n = \left(1+\frac{1}{n}\right)^n \; \forall n \in N$

$\Rightarrow x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

and $x_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$

$\therefore x_n \leq x_{n+1} \; \forall n$

$\therefore (x_n)$ is an increasing sequence.

$\therefore (x_n)$ is monotonic sequence.

Since $x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$

$= 1+1+\frac{1}{2\cdot1}+\frac{1}{3\cdot2\cdot1}+\cdots+\frac{1}{n(n-1)\cdots3\cdot2\cdot1}$

$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}$

$= 1+\frac{\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$ (By G.P i.e. $\frac{a(1-r^n)}{1-r}$)

$= 1+2\left(1-\frac{1}{2^n}\right)$

$= 3-\frac{1}{2^{n-1}}$

$< 3 \; \forall n$

$\therefore x_n < 3 \; \forall n$

$\therefore (x_n)$ is bdd above.

Since (x_n) is monotonically increasing & bdd above.

$\therefore (x_n)$ is cgt.

Note: Clearly $2 \leq x_n \; \forall n$

$\therefore 2 \leq x_n < 3 \; \forall n$

$\Rightarrow 2 \leq \lim_{n\to\infty} x_n < 3$

$\Rightarrow 2 \leq \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n < 3$

$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

→ Discuss the convergence of the sequence (x_n) where

(i) $x_n = \frac{n+1}{n}$ (ii) $x_n = \frac{n}{n^2+1}$

(iii) $x_n = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{3^n}$.

→ P.T the sequence $\left\{\frac{2n-7}{3n+2}\right\}$,

(i) is monotonically increasing

(ii) is bounded

(iii) tends to the limit $\frac{2}{3}$.

Sol^n: Let $x_n = \frac{2n-7}{3n+2}\ \forall n$

$x_{n+1} = \frac{2n-5}{3n+5}$

$\therefore x_{n+1}-x_n = \frac{25}{(3n+5)(3n+2)}$

$>0\ \forall n$

$\therefore x_{n+1} > x_n\ \forall n$

$\therefore (x_n)$ is monotonically increasing.

(ii) The given sequence is

$\left\{-1, -\frac{3}{8}, -\frac{1}{11}, \frac{1}{14}, \frac{3}{17}, \cdots\right\}_2$ approach to 1

Clearly $x_n \geq -1\ \forall n$

and also $1-x_n = 1-\frac{2n-7}{3n+2}$

$= \frac{n+9}{3n+2} > 0\ \forall n$

$\therefore 1-x_n > 0\ \forall n$

$\Rightarrow 1 > x_n\ \forall n$

$\Rightarrow x_n < 1\ \forall n$

$\Rightarrow -1 \leq x_n < 1\ \forall n$

$\therefore (x_n)$ is bounded.

(iii) Since (x_n) is M ↑ and bdd above.

$\therefore$ It cgs.

Now $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{2n-7}{3n+2}$

$= \lim\limits_{n\to\infty} \frac{2-\frac{7}{n}}{3+\frac{2}{n}}$

$= \frac{2}{3}$.

$\therefore (x_n)$ cgs to $2/3$.

→ H.W. P.T the sequence whose n^{th} term is $\frac{3n+4}{2n+1}$

(i) is monotonically decreasing

(ii) is bdd and

(iii) cgs to $3/2$.

1999 P-II → Show that the sequence (x_n) defined by $x_{n+1} = \sqrt{3x_n}$, $x_1 = 1$ cgs to 3.

Sol^n: Given that $x_1 = 1$, $x_{n+1} = \sqrt{3x_n}\ \forall n$

Now $x_2 = \sqrt{3x_1}$
$= \sqrt{3(1)}$
$= \sqrt{3} > 1 = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$

Now suppose $x_{n+1} > x_n$

$\Rightarrow 3x_{n+1} > 3x_n$

$\Rightarrow \sqrt{3x_{n+1}} > \sqrt{3x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$.

$\therefore$ By mathematical induction

$x_{n+1} > x_n \ \forall n$

$\therefore (x_n)$ is monotonically increasing.

Now
$x_1 = 1 < 3$
$x_2 = \sqrt{3x_1}$
$= \sqrt{3} < 3$

$x_3 = \sqrt{3x_2}$
$= \sqrt{3 \cdot \sqrt{3}} < 3$

Suppose $x_n < 3$

then $x_{n+1} = \sqrt{3x_n}$
$< \sqrt{3 \cdot 3} = 3$

$\therefore x_{n+1} < 3$.

$\therefore$ By mathematical induction $x_n < 3 \ \forall n$.

$\therefore (x_n)$ is bdd above by 3.

Since (x_n) is $\uparrow$ and bdd above

$\therefore$ It is cgt.

Now let $\lim_{n\to\infty} x_n = l$

then $\lim_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{3x_n} \ \forall n$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{3 \lim_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{3l}$

$\Rightarrow l^2 - 3l = 0$

$\Rightarrow l(l-3) = 0$

$\Rightarrow l = 0, \ l = 3$.

But $l \neq 0$, Since $x_n \geq 1 \ \forall n$

$\therefore \lim_{n\to\infty} x_n = 3$

H.W $\rightarrow$ Show that the sequence (x_n), where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}} \ \forall n \geq 2$. is cgt and cgs to 2.

$\rightarrow$ P.T the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{7 + x_n}$ cgs to the +ve root of the equation $x^2 - x - 7 = 0$

Sol^n: Given that

$x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{7+x_n}$

$x_2 = \sqrt{7+x_1}$

$= \sqrt{7+\sqrt{7}} > \sqrt{7} = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$

Suppose $x_{n+1} > x_n$ for some 'n'.

$\Rightarrow 7+x_{n+1} > 7+x_n$

$\Rightarrow \sqrt{7+x_{n+1}} > \sqrt{7+x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$

$\therefore$ By mathematical induction

$x_{n+1} > x_n \ \forall n.$

$\therefore (x_n)$ is M $\uparrow$

Now $x_1 = \sqrt{7} < 7$

$x_2 = \sqrt{7+\sqrt{7}} < 7$

Similarly $x_3 < 7$.

Suppose $x_n < 7$

$\Rightarrow 7+x_n < 14$

$\Rightarrow \sqrt{7+x_n} < \sqrt{14}$

$\Rightarrow x_{n+1} < \sqrt{14} < \sqrt{49} = 7$

$\Rightarrow x_{n+1} < 7$

$\therefore$ By mathematical induction

$x_n < 7 \ \forall n.$

$\therefore (x_n)$ is bdd above.

Since (x_n) is M $\uparrow$ & bdd above.

$\therefore$ It is cgt.

Let $\lim_{n\to\infty} x_n = l$ &

$\lim_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{7+x_n}$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{7+\lim_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{7+l}$

$\Rightarrow l^2 - l - 7 = 0$

$\Rightarrow l = \dfrac{1\pm\sqrt{29}}{2}$

But $l = \dfrac{1-\sqrt{29}}{2} < 0$ where as $x_n > 0 \ \forall n$.

$\therefore l \neq \dfrac{1-\sqrt{29}}{2}$.

$\therefore x_n$ cgs to $\dfrac{1+\sqrt{29}}{2}$.

which is +ve root of the equation $x^2 - x - 7 = 0$

H.W. P.T the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2+x_n}$ cgs to the +ve root of the equation

$x^2 - x - 2 = 0$

→ Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ $\forall n$

Show that (x_n) is bdd and monotone, find the limit.

Sol^n: Given that

$x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$

$x_2 = \frac{1}{2}x_1 + 2$

$= \frac{1}{2}(8) + 2 = 6 < 8 = x_1$

$\therefore x_2 < x_1$

Similarly $x_3 < x_2$

Now Suppose $x_{n+1} < x_n$ for some 'n'.

$\Rightarrow \frac{1}{2}x_{n+1} < \frac{1}{2}x_n$

$\Rightarrow \frac{1}{2}x_{n+1} + 2 < \frac{1}{2}x_n + 2$

$\Rightarrow x_{n+2} < x_{n+1}$

$\therefore$ By mathematical induction

$x_{n+1} < x_n$ $\forall n$.

$\therefore (x_n)$ is Monotonically decreasing.

But w.k.T every decreasing sequence is always bdd below.

Since $x_n > x_{n+1}$ $\forall n \in N$.

$\Rightarrow x_n > \frac{1}{2}x_n + 2$ $\forall n$

$\Rightarrow 2x_n > x_n + 4$ $\forall n$

$\Rightarrow x_n > 4$ $\forall n$

$\therefore (x_n)$ is bdd below

$\therefore (x_n)$ is bdd.

Since (x_n) is M↓ and bdd below.

$\therefore$ It is cgt.

Let $\lim\limits_{n\to\infty} x_n = l$ & $\lim\limits_{n\to\infty} x_{n+1} = l$

Since $x_{n+1} = \frac{x_n}{2} + 2$ $\forall n$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = \frac{1}{2}\lim\limits_{n\to\infty} x_n + 2$

$\Rightarrow l = \frac{1}{2}l + 2$

$\Rightarrow l = 4$

$\therefore \lim\limits_{n\to\infty} x_n = 4$.

H.W → Let $Y = (y_n)$ be defined inductively by $y_1 = 1$;

$y_{n+1} = \frac{1}{4}(2y_n + 3)$ $\forall n \geq 1$.

Show that $\lim Y = 3/2$

H.W. → Let $Z = (z_n)$ be the sequence of real numbers defined by $z_1 = 1$, $z_{n+1} = \sqrt{2z_n}$ for $n \in N$ then Show that

$\lim\limits_{n\to\infty}(z_n) = 2$

IAS 2017 Similar Q

→ Let $y_1 = \sqrt{p}$, $p > 0$, $y_{n+1} = \sqrt{p + y_n}$ for $n \in N$

Show that (y_n) cgs and find limit.

Solⁿ: $y_1 = \sqrt{p}$; $p > 0$ &

$y_{n+1} = \sqrt{p + y_n}$ $\forall n$

Now $y_2 = \sqrt{p + y_1}$

$= \sqrt{p+p} > \sqrt{p} = y_1$

$\therefore y_2 > y_1$

Similarly $y_3 > y_2$

Now Suppose, $y_{n+1} > y_n$

$\Rightarrow p + y_{n+1} > p + y_n$; $p > 0$

$\Rightarrow \sqrt{p + y_{n+1}} > \sqrt{p + y_n}$

$\Rightarrow y_{n+2} > y_{n+1}$

$\therefore$ By mathematical induction

$y_{n+1} > y_n$ $\forall n$

$\therefore (y_n)$ is M ↑.

Since $y_n < y_{n+1}$ $\forall n$

$\Rightarrow y_n < \sqrt{p + y_n}$ $\forall n$

$\Rightarrow y_n^2 < p + y_n$ $\forall n$

$\Rightarrow y_n^2 - y_n - p < 0$ $\forall n$

$\Rightarrow \left[y_n - \frac{1+\sqrt{1+4p}}{2}\right]\left[y_n - \frac{1-\sqrt{1+4p}}{2}\right] < 0$

$\Rightarrow \left[y_n - \frac{1+\sqrt{1+4p}}{2}\right] < 0$ $\left(\because \left(y_n - \frac{1-\sqrt{1+4p}}{2}\right) > 0\right)$

$\Rightarrow y_n < \frac{1+\sqrt{1+4p}}{2}$ $\forall n$

$< \frac{1+1+\sqrt{1+4p}}{2}$

$< 1 + \frac{\sqrt{1+4p}}{2}$

$< 1 + \frac{\sqrt{4p}}{2}$

$< 1 + \sqrt{p}$

$\therefore (y_n)$ is bdd.

Since (y_n) is M ↑ & bdd above.

$\therefore$ It is cgt.

<u>To find limit of (y_n):</u>

Let $\lim_{n\to\infty} y_n = l$ & $\lim_{n\to\infty} y_{n+1} = l$

Now $y_{n+1} = \sqrt{p + y_n}$; $p > 0$

$\Rightarrow \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \sqrt{p + y_n}$

$\Rightarrow l = \sqrt{p + l}$

$\Rightarrow l^2 - l - p = 0$

$\Rightarrow l = \frac{1 \pm \sqrt{1+4p}}{2}$

$\therefore l = \frac{1+\sqrt{1+4p}}{2}$ $\left(\because l = \frac{1-\sqrt{1+4p}}{2} < 0 \text{ But } y_n > 0\right)$

$\therefore \lim_{n\to\infty} y_n = \frac{1}{2}\left(1 + \sqrt{1+4p}\right)$.

→ Let $y_n = \sqrt{n+1} - \sqrt{n}$ $\forall n \in N$

Show that (y_n) and $(\sqrt{n}\,y_n)$ converge find their limits.

Sol^n: (i) $y_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\leq \frac{1}{\sqrt{n}+\sqrt{n}} \quad (\because \sqrt{n+1} \geq \sqrt{n})$$

$\therefore y_n \leq \frac{1}{2\sqrt{n}}$ —— (A)

Since $0 < y_n$ $\forall n$ —— (B)

from (A) & (B),

we have

$$0 < y_n \leq \frac{1}{2\sqrt{n}}$$

which is in the form of $x_n < y_n \leq z_n$ $\forall n$

with $\lim\limits_{n\to\infty} x_n = 0 = \lim\limits_{n\to\infty} z_n$

$\therefore$ By squeeze theorem

$\lim y_n = 0$

$\therefore (y_n)$ cgs to zero.

(ii) $\sqrt{n}\,y_n = \sqrt{n}\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right)$

$$\leq \sqrt{n}\left(\frac{1}{2\sqrt{n}}\right) \quad \text{(By (A))}$$

$$= \frac{1}{2}$$

$\therefore \sqrt{n}\,y_n \leq \frac{1}{2}$ —— (C)

Now since

$\sqrt{n+1} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$

$\Rightarrow 2\sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$

$\Rightarrow \frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$\Rightarrow \frac{\sqrt{n}}{2\sqrt{n+1}} < \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}}$

$\Rightarrow \frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}\,y_n.$ —— (D)

from (C) & (D), we have

$$\frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}\,y_n \leq \frac{1}{2}$$

of the form

$x_n < v_n \leq z_n.$

with $\lim x_n = \lim z_n = 1/2$

$\therefore$ By sqeeze theorem

$$\lim_{n\to\infty}(\sqrt{n}\,y_n) = 1/2$$

→ Establish the convergence of the sequence (y_n), where $y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots\cdots + \frac{1}{2n}$; $n \in N$.

Sol^n: $y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}$

Now $y_{n+2} - y_{n+1} = \left[\frac{1}{n+2} + \frac{1}{n+3} + \cdots\cdots + \frac{1}{2n+1} + \frac{1}{2n+2}\right] - \left[\frac{1}{n+1} + \cancel{\frac{1}{n+2}} + \cdots\cdots + \cancel{\frac{1}{2n}}\right]$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0 \ \forall n$$

$\therefore y_{n+1} > y_n \ \forall n$

$\therefore (y_n)$ is M $\uparrow$.

Now $y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \cdots + \frac{1}{n+n}$

$< \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$

$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

$= \frac{n}{n}$

$= 1$

$\therefore y_n < 1 \ \forall n \in N$.

$\therefore (y_n)$ is bdd above.

$\therefore (y_n)$ is cgt

→ If (b_n) is bdd sequence and $\underset{n\to\infty}{Lt} (a_n) = 0$ then show that $\underset{n\to\infty}{Lt} (a_n b_n) = 0$.

Sol^n^: Since (b_n) is bdd sequence.

$\therefore \exists M > 0$ such that

$|b_n| \le M \ \forall n$ ——— (1)

and since $\underset{n\to\infty}{Lt} a_n = 0$

i.e, $a_n \to 0$ as $n \to \infty$

$\therefore$ Given $\epsilon > 0 \ \exists k \in I^+$

such that $|a_n - 0| < \frac{\epsilon}{M}$, $(M > 0)$ $\forall n \ge k$. ——— (2)

Now we have

$|a_n b_n - 0| = |a_n b_n|$

$= |a_n| |b_n|$

$< \frac{\epsilon}{M} \cdot M$. $\forall n \ge k$

$= \epsilon$

$\therefore |a_n b_n - 0| < \epsilon \ \forall n \ge k$.

$\therefore \underset{n\to\infty}{Lt} a_n b_n = 0$

(10) Set-II Real Analysis related

EQUIVALENT SETS

A set 'A' is called equivalent to a set B, written $A \sim B$, if $\exists$ a function $f: A \to B$ which is one-one and onto.

The function 'f' is then said to define a one-to-one correspondence between the sets A and B.

— A set is finite iff it is empty or equivalent to $\{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$:
Otherwise it is said to be infinite.

Clearly two finite sets are equivalent iff they contain the same number of elements.
Hence, for finite sets, equivalence corresponds to the usual meaning of two sets containing the same number of elements.

For example:

Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and
$E = \{2, 4, 6, 8, \ldots\}$.

Then the function $f: \mathbb{N} \to E$ defined $f(x) = 2x$ is both one-one and onto.

Hence the open interval $(-1, 1)$ is equivalent to $\mathbb{R}$ (the set of real numbers).

Observe that an infinite set can be equivalent to a proper subset of itself.
This property is true of infinite sets generally.

→ The relation in any collection of sets defined by $A \sim B$ is an equivalence relation.

Sol^n: (i) '$\sim$' is reflexive:

For any set A, the identity function $I_A : A \to A$ is one-one and onto.

$\therefore$ $A \sim A$ for any set A.

$\Rightarrow$ '$\sim$' is reflexive.

(ii) '$\sim$' is symmetric:

If $A \sim B$, then $\exists$ $f : A \to B$ which is one-one and onto.

But then f has an inverse $f^{-1} : B \to A$ which is also one-one and onto.

Hence $A \sim B \Rightarrow B \sim A$.

$\Rightarrow$ '$\sim$' is symmetric.

(iii) '$\sim$' transitive:

If $A \sim B$ and $B \sim C$, then there exist functions $f : A \to B$ and $g : B \to C$ which are one-one and onto.

Thus the composition function $g \circ f : A \to C$ is also one-one and onto.

Hence $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

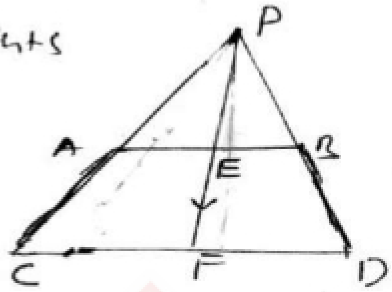
$\therefore$ '$\sim$' is transitive.

$\therefore$ $\sim$ is an equivalence relation.

Note: The advantage of using the concept of one-one correspondence is that it helps in studying the countability of infinite sets.

→ Now, consider any two line segments AB and CD.

Let M denote the set of points on AB and N denote the set of points on CD.



Let us check whether M and N are equivalent.

Join CA and DB to meet in the point p.

Let a line through 'P' meet AB in E and CD in F.

Define $f: M \to N$ as $f(x) = y$

where x is any point on AB and y is any point on CD.

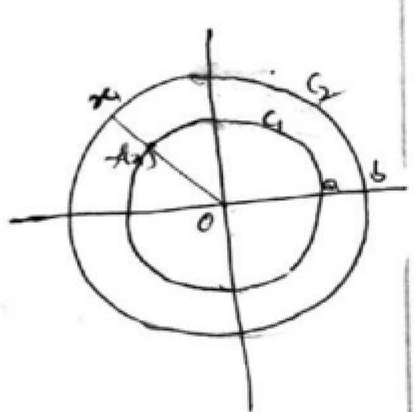
The construction shows that f is a one-one correspondence.

Thus M and N are equivalent sets.

→ Consider the sets concentric circles

$C_1 = \{(x,y) \mid x^2+y^2 = a^2\}$

$C_2 = \{(x,y) \mid x^2+y^2 = b^2\}$ where $0<a<b$.



Let us check whether C_1 and C_2 are equivalent.

Let $x \in C_2$.

consider the function $f: C_2 \to C_1$,

where $f(x)$ is the point of intersection of the radius from the centre of C_2 (and C_1) to x and C_1 as shown in the adjacent diagram.

Note that f is both one-one and onto.

Thus f defines a one-to-one correspondence between C_1 and C_2.

* DENUMERABLE (OR ENUMERABLE) SET :-

— Let N be the set of positive integers $\{1, 2, 3, \ldots\}$

A set 'x' is called denumerable and is said to have cardinality iff it is equivalent to N.

(or)

A set is said to be denumerable if it is equivalent to 'N' (the set of natural numbers).

* Countable set :—

A set is said to be countable iff it is a finite or denumerable. Otherwise it is said to be uncountable.

For example: ① The set of terms in any infinite sequence $a_1, a_2, a_3, \ldots a_n \ldots$ of distinct terms is denumerable

i.e the sequence $\{a_n / n \in N\}$

$= \{a_1, a_2, \ldots a_n, \ldots\}$ of distinct terms is denumerable,

for a sequence is essentially a function $f(n) = a_n$ whose domain is N and the range subset of real number set R are distinct, the function is one-one and onto.

Accordingly, each of the following sets is denumerable:

$\{\frac{1}{n} / n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\{1, -2, 3, -4 \ldots\} = \{(-1)^{n+1} n / n \in N\}$

$\{\langle n^2, n^3 \rangle\} = \{\langle 1, 1 \rangle, \langle 4, 8 \rangle, \langle 9, 27 \rangle, \ldots \langle n^2, n^3 \rangle, \ldots\}$.

2

Countability of Sets and The Real Number System

2·1. INITIAL SEGMENT OF N

Let $m \in N$. Then the subset

$$N_m = \{n : n \in N \text{ and } n < m\}$$
$$= \{1, 2, 3, \ldots\ldots, m\}$$

of N is called an initial segment of N determined by the natural number m.

e.g. $\{1, 2, 3, 4, 5\} = N_5$

is the initial segment of N determined by 5.

Thus an initial segment of N determined by $m \in N$ contains all natural numbers from 1 to m.

2·2. EQUIVALENT SETS (*M.D.U. 1991*)

Two sets A and B are said to be equivalent if $\exists$ a bijection *i.e.* a one-one and onto function, from A to B.

A is equivalent to B is written as $A \sim B$.

For example, the sets N and $E = \{2, 4, 6, \ldots\ldots\}$ of all even natural numbers are equivalent because the function

$f : N \to E$ defined by $f(n) = 2n$, $n \in N$ is one-one from N onto E.

2·3. THEOREM

The relation '$\sim$' is an equivalence relation.

Proof. '$\sim$' will be an equivalence relation if it is :

(*i*) reflexive (*ii*) symmetric and

(*iii*) transitive.

(*i*) *'$\sim$' is reflexive*

For any set A, the identity function $I_A : A \to A$ is one-one and onto.

$\therefore$ $A \sim A$ for any set A

$\Rightarrow$ '$\sim$' is reflexive.

(*ii*) '$\sim$' *is symmetric.*

Let $A \sim B$, then $\exists$ a function $f : A \to B$ which is one-one and onto. Its inverse function $f^{-1} : B \to A$ is also one-one and onto.

$\therefore \qquad B \sim A$

Since $\qquad A \sim B \quad \Rightarrow \quad B \sim A$

'$\sim$' is symmetric.

(*iii*) '$\sim$' *is transitive.*

Let $A \sim B$ and $B \sim C$, then $\exists$ one-one and onto functions $f : A \to B$ and $g : B \to C$.

Their composite function $g \circ f : A \to C$ is also one-one and onto.

$\therefore \qquad A \sim C$

Since $\qquad A \sim B$ and $B \sim C \quad \Rightarrow \quad A \sim C$

'$\sim$' is transitive.

From (*i*), (*ii*) and (*iii*), '$\sim$' is an equivalence relation.

Note. For $n, m \in N$, $N_m \sim N_n \Leftrightarrow m = n$.

2·4. FINITE SET

A set which is either empty or equivalent to a subset N_m of N is said to be a finite set.

Thus A is finite if $A = \phi$ or $A \sim N_m$ for some natural number m.

Note. If $A \sim N_m$, then m is called the cardinal number of A.

2·5. INFINITE SET

A set which is not finite is called an infinite set.

Thus A is an infinite set if $A \neq \phi$ and A is not equivalent to N_m for any $m \in N$.

2·6. DENUMERABLE (OR ENUMERABLE) SET

(*M.D.U. 1983, 90*)

A set is said to be denumerable if it is equivalent to N, the set of all natural numbers. Thus A is denumerable $\Rightarrow A \sim N$.

2·7. COUNTABLE SET (*M.D.U. 1983, 91*)

A set A is said to be countable if either A is finite or A is denumerable *i.e.* if either A is finite or $A \sim N$, the set of all natural numbers.

2·8. UNCOUNTABLE SET

A set which is not countable is said to be an uncountable set.

Thus a set A is uncountable if A is not finite and A is not equivalent to N, the set of all natural numbers.

2·9. THEOREM

A is finite and $B \subset A \Rightarrow$ B is finite (Every subset of a finite set is finite) *(M.D.U. 1991)*

(*i*) If $B = \phi$, then B is finite [By def.]

(*ii*) If $B = A$, then B is finite because A is finite.

(*iii*) Suppose $B \neq \phi$ and $B \neq A$.

Since A is finite, there exists $m \in N$ such that $A \sim N_m$.

Since $B \subset A$, B has k elements where $k < m$ *i.e.* $B \sim N_k$

$\therefore$ B is finite.

2·10. THEOREM

A is infinite and $B \supset A \Rightarrow$ B is infinite (Every super-set of an infinite set is infinite)

Proof. Let B be finite, then $A \subset B$ and B is finite

$\Rightarrow$ A is finite [Th. 2·9]

which is a contradiction. Hence B is infinite.

2·11. THEOREM

If A and B are finite sets, then $A \cap B$ is also a finite set

Proof. $A \cap B \subset A$ and A is finite.

$\therefore$ $A \cap B$ is a finite set.

2·12. THEOREM

If A and B are finite sets, then $A \cup B$ is also a finite set

Proof. (*i*) If $A = \phi = B$, then $A \cup B = \phi$ is finite.

(*ii*) If $A = \phi$ or $B = \phi$, then $A \cup B$ is either B or A, both of which are finite

$\therefore$ $A \cup B$ is finite.

(*iii*) If $A \neq \phi$, $B \neq \phi$, since A and B are finite, there exist natural numbers n and m such that $A \sim N_n$ and $B \sim N_m$.

If $A \cap B = \phi$, then $A \cup B$ has $m+n$ elements and $A \cup B \sim N_{n+m}$

$\Rightarrow$ $A \cup B$ is finite.

If $A \cap B \neq \phi$, there exists a natural number $k < \min \{n, m\}$ such that $A \cap B \sim N_k$.

Now $A \cup B$ has $n+m-k$ elements.

$\therefore$ $A \cup B \sim N_{n+m-k}$ $\Rightarrow$ $A \cup B$ is finite.

2.13. THEOREM

Every subset of a countable set is countable. *(M.D.U. 1982, 83 S ; Gorakhpur 1985)*

Proof. Let A be a countable set. Then A is either finite or denumerable.

Case I. When A is finite.

Since every subset of a finite set is finite, every subset of A is finite and hence countable.

Case II. When A is denumerable.

Here $A \sim N$, the set of natural numbers.

Let $A = \{a_1, a_2, a_3, \ldots\ldots\}$ and let $B \subset A$.

Sub-case 1. If B is finite, then B is countable. (By def.)

Sub-case 2. If B is infinite, let n_1 be the least +ve integer s.t. $a_{n_1} \in B$.

Since B is infinite, $B \neq \{a_{n_1}\}$. Let n_2 be the least +ve integer s.t. $n_2 > n_1$ and $a_{n_2} \in B$.

Since B is infinite, $B \neq \{a_{n_1}, a_{n_2}\}$.

Continuing like this, $B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\ldots\ldots\ldots\}$ where $n_1 < n_2 < n_3 < \ldots\ldots\ldots$

Define $f : N \to B$ by

$$f(k) = a_{n_k} \quad \forall k \in N$$

Then f is one-one and onto.

$\therefore \quad B \sim N$

$\Rightarrow$ B is denumerable

$\Rightarrow$ B is countable.

Cor. 1. Every infinite subset of a denumerable set is denumerable.

[See Case II, Sub-case 2].

Cor 2. If A and B are countable sets, then $A \cap B$ is also a countable set.

$A \cap B \subset A$ and A is countable

$\Rightarrow A \cap B$ is countable.

Cor. 3. Every super-set of an uncountable set is uncountable.

Let A be an uncountable set and let B be any super-set of A.

Suppose B is countable. Then A being a subset of a countable set must be countable, which is a contradiction.

Hence B is uncountable.

2·14. THEOREM

Every infinite set has a countable subset.

(*K.U. 1981 S, Gorakhpur 1981*)

Proof. Let A be an infinite set. Let $a_1 \in A$.

Since A is infinite, $A \neq \{a_1\}$

$\Rightarrow \exists\ a_2 \neq a_1$ s.t. $a_2 \in A$.

Since A is infinite, $A \neq \{a_1, a_2\}$.

$\Rightarrow \quad \exists\ a_3 \neq a_2 \neq a_1$ s.t. $a_3 \in A$.

Continuing like this **as long as we please,** we can have a proper subset $B = \{a_1, a_2, a_3, \ldots\ldots\}$ of A.

If B is finite, B is countable. (By def.)

If B is infinite, define $f : N \to B$ by

$$f(k) = a_k \ \forall\ k \in N$$

Then f is one-one and onto.

$\therefore \qquad B \sim N$

$\Rightarrow$ B is denumerable

$\Rightarrow$ B is countable.

2·15. THEOREM

A is countable, B is countable ⇒ A∪B is countable.

(*M.D.U. 1981*)

Proof. Case I. If A and B are both finite, then so is A∪B.

$\Rightarrow$ A∪B is countable.

Case II. If one of A and B is finite and the other is denumerable.

Let us assume that A is finite and B is denumerable. Then we can write

$A = \{a_1, a_2, a_3, \ldots\ldots, a_m\} \qquad (A \sim N_m)$

$B = \{b_1, b_2, b_3, \ldots\ldots\}$

Let $C = B - A$, then $C \in B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable, so we can express C as

$$C = \{c_1, c_2, c_3, \ldots\ldots\}$$

Clearly, $A \cup B = A \cup C = \{a_1, a_2, \ldots\ldots, a_m, c_1, c_2, \ldots\ldots\}$

and $\qquad A \cap C = \phi$

Define $f : N \to A \cup C$ by

$$f(k) = \begin{cases} a_k, \text{ if } k = 1, 2, \ldots\ldots, m \\ c_{k-m}, \text{ if } k \geq m+1 \end{cases}$$

Then f is one-one and onto.

$\therefore \quad A \cup C \sim N \Rightarrow A \cup B \sim N$

$\Rightarrow$ A∪B is denumerable

$\Rightarrow$ A∪B is countable.

Case III. If A and B are both denumerable sets, we can write.

$$A=\{a_1, a_2, a_3, \ldots\ldots\}$$
$$B=\{b_1, b_2, b_3, \ldots\ldots\}$$

Let $C=B-A$, then $C\subset B$ and $A\cup B=A\cup C$.

If C is finite, then $A\cup C$ is countable. (By case II)

If C is infinite, then C is denumerable and we can write

$$C=\{c_1, c_2, c_3, \ldots\ldots\}$$

Clearly, $A\cup C=\{a_1, c_1, a_2, c_2, a_3, c_3, \ldots\ldots\}$

Define $f: N\to A\cup C$ by

$$f(n)=\begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ c_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then f is one-one and onto.

$\therefore$ $A\cup C\sim N$

$\Rightarrow$ $A\cup B\sim N$

$\Rightarrow$ $A\cup B$ is denumerable

$\Rightarrow$ $A\cup B$ is countable.

2·16. THEOREM

The union of a denumerable collection of denumerable sets is denumerable (*K.U. 1980 S ; M.D.U. 1983, 91*)

Proof. Let $\{A_i\}_{i\in N}$ be a denumerable collection of denumerable sets.

Since each A_i is denumerable, we have

$$A_1=\{a_{11}, a_{12}, a_{13}, \ldots\ldots a_{1n}, \ldots\ldots\}$$
$$A_2=\{a_{21}, a_{22}, a_{23}, \ldots\ldots a_{2n}, \ldots\ldots\}$$
$$A_3=\{a_{31}, a_{32}, a_{33}, \ldots\ldots a_{3n}, \ldots\ldots\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Then a_{ij} is the jth element of A_i.

Let us list the elements of $\bigcup_{i\in N} A_i$, as follows :

$$a_{11}$$
$$a_{21}, a_{12}$$
$$a_{31}, a_{22}, a_{13}$$
$$a_{41}, a_{32}, a_{23}, a_{14}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

From the above scheme it is evident that a_{pq} is the qth element of $(p+q-1)$th row. Thus all the elements of $\bigcup_{i\in N} A_i$ have been arranged in an infinite sequence as

$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}\ldots\ldots\}$

In fact, the map $f : \bigcup\limits_{i \in N} A_i \to N$ defined by

$$f(a_{pq}) = \frac{(p+q-2)(p+q-1)}{2} + q$$

gives an enumeration of $\bigcup\limits_{i \in N} A_i$

$$\therefore \qquad \bigcup_{i \in N} A_i \sim N$$

Hence $\bigcup\limits_{i \in N} A_i$ is denumerable.

2·17. THEOREM

The set of real numbers x such that $0 < x < 1$ is not countable *(M.D.U. 1984 ; K.U. 1980, 82 S, 84)*

Or

The unit interval [0, 1] is not countable.

Proof. Let us assume that [0, 1] is countable.

$\Rightarrow$ either [0, 1] is finite or denumerable.

Since every interval is an infinite set, [0, 1] is denumerable.

$\Rightarrow$ There is an enumeration $x_1, x_2, x_3\ldots\ldots$ of real numbers in [0, 1].

Expanding each x_i in the form of an infinite decimal, we have

$$x_1 = 0\,.\,a_{11}\,a_{12}\,a_{13}\,a_{14}\ldots\ldots a_{1n}\ldots\ldots$$
$$x_2 = 0\,.\,a_{21}\,a_{22}\,a_{23}\,a_{24}\ldots\ldots a_{2n}\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_n = 0\,.\,a_{n1}\,a_{n2}\,a_{n3}\,a_{n4}\ldots\ldots a_{nn}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where each $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Consider the number b with decimal representation

$$b = 0\,.\,b_1\,b_2\,b_3\ldots\ldots b_n\ldots\ldots$$

where b_1 is any integer from 1 to 8 s.t. $b_1 \neq a_{11}$

b_2 is any integer from 1 to 8 s.t. $b_2 \neq a_{22}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

b_n is any integer from 1 to 8 s.t. $b_n \neq a_{nn}$ and so on.

Clearly, $b \in [0, 1]$ and $b \neq x_n\ \forall n$ since the decimal representation of b is different from the decimal representation of x_n as $b_n \neq a_{nn}$. Thus b escapes enumeration and we arrive at a contradiction.

Hence [0, 1] is not countable.

2·18. THEOREM

The set of real numbers is not countable.

(*M.D.U. 1990 ; Gorakhpur 1984*)

Proof. We know that every subset of a countable set is countable.

If R were countable, then [0, 1] which is a subset of R must also be countable.

But the unit interval [0, 1] is not countable. [Theorem 2·17]

Hence R is not countable.

2·19. THEOREM

The set of all rational numbers is countable.

(*M.D.U. 1938 S, 84 S, 86 ; K.U. 1982 ; Gorakhpur 1981, 83, 85, 90*)

Proof. Consider the sets

$$A_1=\left\{\frac{0}{1}, \frac{-1}{1}, \frac{1}{1}, \frac{-2}{1}, \frac{2}{1}, \ldots\ldots\right\} \text{(Common denom. 1)}$$

$$A_2=\left\{\frac{0}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-2}{2}, \frac{2}{2}, \ldots\ldots\right\} \text{(Common denom. 2)}$$

....

$$A_n=\left\{\frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n}, \ldots\ldots\right\} \text{(Common denom. } n\text{]}$$

....

Clearly, the set of rational numbers $Q=\bigcup_{i\in N} A_i$

Consider a mapping $f: N\rightarrow A_n$ defined by

$$f(r)=\begin{cases} \dfrac{r-1}{2n} & \text{if } r \text{ is odd} \\ \dfrac{-r}{2n} & \text{if } r \text{ is even} \end{cases}$$

f is one-one and onto. $\therefore A_n \sim N$

i.e. A_n is denumerable. $\Rightarrow A_n$ is countable.

Since $Q=\bigcup_{i\in N} A_i$ is the union of a countable collection of countable sets.

$\therefore$ Q is countable.

2·20. THEOREM

The set of all positive rational numbers is countable.

Proof. Let Q^+ denote the set of positive rational numbers ; then $Q^+ \subset Q$.

Since every subset of a countable set is countable, and Q is countable.

$\therefore Q^+$ is countable.

2·21. THEOREM

The set of irrational numbers is uncountable.

(*Gorakhpur 1986*)

Proof. Suppose the set of irrational numbers is countable. We know that the set of rational numbers is countable. Since R, the set of real numbers is the union of the set of rational numbers and the set of irrational numbers, therefore, R is countable. But R is not countable. We are, thus, led to a contradiction.

Hence the set of irrational numbers is uncountable.

2·22. THEOREM

A finite set is not equivalent to any of its proper subsets.

Proof. Let A be a finite set.

If $A=\phi$, then A has no proper subset and we have nothing to prove.

If $A\neq\phi$ then $A\sim N_m$ for some $m\in N$

Let B be a proper subset of A, then B has k elements, where $k<m$ *i.e.* $B\sim N_k$

Since A and B do not have same number of elements, A cannot be equivalent to B.

2·23. THEOREM

Every infinite set is equivalent to a proper subset of itself.

Proof. Let A be an infinite set. Since every infinite set contains a denumerable subset. [See Th. 2·14]

Let $B=\{a_1, a_2, a_3, \ldots\ldots\}$ be a denumerable subset of A.

Let $C=A-B$, then $A=B\cup C$

Let $P=A-\{a_1\}$ be a proper subset of A

Consider the mapping $f: A\to P$

defined by $\quad f(a_i)=a_{i+1} \quad$ for $a_i\in B$

and $\quad f(a)=a \quad$ for $a\in C$

Then f is one-one and onto. Hence $A\sim P$.

Note. If A is a denumerable set, then $A\sim N$ and we can write A as the indexed set $\{a_i : i\in N\}$, where $a_i\neq a_j$ for $i\neq j$. The process of writing a denumerable set in this form is called **enumeration.**

2·24. THEOREM

The union of a finite set and a countable set is a countable set.

Proof. Let A be a finite set and B be a countable set.

If B is finite then $A\cup B$ is a finite set and hence countable.

If B is denumerable then there are two possibilities :

(*i*) $A\cap B=\phi$ and (*ii*) $A\cap B\neq\phi$

Case (*i*) When $A\cap B=\phi$

Let $A=\{a_1, a_2, \ldots\ldots, a_p\}$

and $B=\{b_1, b_2, \ldots\ldots, b_n, \ldots\ldots\}$

Then $A\cup B=\{a_1, a_2, \ldots\ldots, a_p, b_1, b_2, \ldots\ldots, b_n, \ldots\ldots\}$

Define a function $f: N\rightarrow A\cup B$ by

$$f(n)=\begin{cases} a_n & \text{if } 1\leqslant n<p \\ b_{n-p} & \text{if } n\geqslant p+1 \end{cases}$$

Clearly f is one-one and onto.

$\therefore$ $A\cup B\sim N$. Hence $A\cup B$ is denumerable and so countable.

Case (*ii*) When $A\cap B\neq\phi$

Let $C=B-A$, then $C\subset B$

Since A is finite, C is infinite.

C being an infinite subset of a denumerable set is denumerable.

Clearly $A\cup B=A\cup C=\{a_1, a_2, \ldots\ldots, a_p, c_1, c_2, \ldots\ldots\}$

and $A\cap C=\phi$

$\therefore$ By case (*i*), $A\cup C$ is countable. Hence $A\cup B$ is countable.

2·25. THEOREM

The set $N\times N$ is countable

Proof. Consider the sets

$$A_1=\{(1, 1), (1, 2), (1, 3), \ldots\ldots\}$$
$$A_2=\{(2, 1), (2, 2), (2, 3), \ldots\ldots\}$$
$$A_3=\{(3, 1), (3, 2), (3, 3), \ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_n=\{(n, 1), (n, 2), (n, 3), \ldots\ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly $N\times N=\bigcup\limits_{n\in N} A_n$

Also the function $f : A_n\rightarrow N$ defined by

$f(n, i)=i$ is one-one and onto.

$\therefore$ A_n is denumerable. Since $N\times N$ is a denumerable collection of denumerable sets, it is denumerable and hence countable.

Corollary 1. The set of all positive rational numbers is countable.

Proof. $Q^+=\left\{\dfrac{p}{q} : p, q \text{ are co-prime positive integers}\right\}$

Let $A=\{(p, q) ; p, q \text{ are co-prime positive integers}\}$

Clearly the elements of Q^+ and A are in one-one correspondence and therefore Q^+ is countable iff A is countable. Since $A \subset N \times N$ and $N \times N$ is countable, therefore, A is countable. Hence Q^+ is countable.

Note 1. The set Q^+ is denumerable can also be proved as under.

Consider the sets

$$A_1=\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\ldots\right\}$$

$$A_2=\left\{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \ldots\ldots\right\}$$

$$A_3=\left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_n=\left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Clearly $Q^+ = \bigcup_{n \in N} A_n$

Also, the function $f : A_n \to N$ defined by

$$f\left(\frac{i}{n}\right)=i \text{ is one-one and onto.}$$

$\therefore$ A_n is denumerable. Since Q^+ is a denumerable collection of denumerable sets, it is denumerable and hence countable.

Note 2. $Q=Q^- \cup \{0\} \cup Q^+$ is denumerable, since Q^+ and Q^- are in one-one correspondence.

No need of theorem proofs.

Note: (1) The set of limit points of a bounded sequence is bounded

(2) The bounds of ~~the~~ set of all limit points of a bounded sequence are the same as the bounds of the sequence.

(3) The set of all limit points of unbounded sequence may or may not be bounded.

For example:

(i) The sequence $\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \ldots\ldots\}$ is unbounded but the set of its limit points $\{0\}$, which is bounded.

(ii) $\{2, 1+\frac{1}{2}, 2+\frac{1}{2}, 1+\frac{1}{3}, 2+\frac{1}{3}, 3+\frac{1}{3}, 1+\frac{1}{4}, 2+\frac{1}{4}, 3+\frac{1}{4}, 4+\frac{1}{4} \ldots\ldots\}$ is unbounded and the set of its limit points is N, which is unbounded.

(4) Every bounded sequence has the greatest and least limit points.

4·27. Limit Superior and Limit Inferior of a Sepuence

Let $<a_n>$ be a bounded sequence, then the sequence has the least and the greatest limit points. (See Theorem 5 with 4·26)

The least limit point of $<a_n>$ is called the *limit inferior* (or *inferior limit* or *lower limit*) of $<a_n>$ and is denoted by

$$\lim_{n\to\infty} \inf. \ a_n \quad \text{or} \quad \underline{\lim}_{n\to\infty} a_n.$$

The greatest limit point of $<a_n>$ is called the *limit superior* (or *superior limit* or *upper limit*) of $<a_n>$ and is denoted by

$$\lim_{n\to\infty} \sup. \ a_n \quad \text{or} \quad \overline{\lim}_{n\to\infty} a_n.$$

Note 1. If $<a_n>$ is unbounded above, we write

$$\lim_{n\to\infty} \sup. \ a_n = \infty$$

If $<a_n>$ is unbounded below, we write

$$\lim_{n\to\infty} \inf. \ a_n = -\infty.$$

Note 2. Since the greatest limit point of a sequence $\{a_n\} \geqslant$ the least limit point

$$\therefore \lim_{n\to\infty} \sup. \ a_n \geqslant \lim_{n\to\infty} \inf. \ a_n.$$

Examples. (*i*) For the sequence $<a_n>$ defined by $a_n = (-1)^n$ $\forall n$, the only limit points are -1 and 1.

The set of limit points $= \{-1, 1\}$ which is bounded.

$$\therefore \lim_{n\to\infty} \inf. \ a_n = -1 \quad \text{and} \quad \lim_{n\to\infty} \sup. \ a_n = 1.$$

(*ii*) For the sequence $<a_n>$ defined by $a_n = \dfrac{1}{n}$ $\forall n$, the only limit point is 0.

The set of limit points $= \{0\}$ which is bounded.

$$\therefore \lim_{n\to\infty} \inf. \ a_n = 0 = \lim_{n\to\infty} \sup. \ a_n.$$

(*iii*) For the constant sequence $a_n = k \ \forall n$,

$$\lim_{n\to\infty} \inf. \ a_n = k = \lim_{n\to\infty} \sup \ a_n.$$

(*iv*) If $a_n = \begin{cases} 2 & \text{when } n \text{ is odd} \\ -n & \text{when } n \text{ is even} \end{cases}$

then 2 is a limit point of $<a_n>$ which is unbounded below.

$$\therefore \lim_{n\to\infty} \inf. \ a_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \sup. \ a_n = 2.$$

(v) For the sequence $a_n = (-1)^n n \ \forall\ n \in \mathbb{N}$,

$$\lim_{n\to\infty} \inf. a_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \sup. a_n = \infty.$$

4·28. Theorem 1

A real number u is the limit superior of a bounded sequence $<a_n>$ if and only if the following two conditions are statisfied :

(i) for each $\varepsilon>0$, $a_n > u-\varepsilon$ for infinity many values of n

(ii) for each $\varepsilon>0$, $a_n < u+\varepsilon$ for all except finitely many values of n.

Proof. Necessity

Let u be the limit superior of a bounded sequence $<a_n>$ and let $\varepsilon>0$ be given.

Since u is a limit point of $<a_n>$, we have

$$u-\varepsilon < a_n < u+\varepsilon$$

for infinitely many values of n.

In particular $a_n > u-\varepsilon$ for infinitely many values of n.

Again, since u is the greatest limit point, $u+\varepsilon$ is not a limit point and, therefore, $a_n \geqslant u+\varepsilon$ for only finitely many values of n. (If for some $\varepsilon>0$, $a_n \geqslant u+\varepsilon$ for infinitely many values of n, then $<a_n>$ will have a limit point $p \geqslant u+\varepsilon$).

$\therefore$ $a_n < u+\varepsilon$ for all except finitely many values of n.

Sufficiency

Let us assume that u satisfies both the conditions.

Given any $\varepsilon>0$, $u-\varepsilon<a_n$ for infinitely many values of n and $a_n<u+\varepsilon$ for all except finitely many values of n.

$\Rightarrow$ $u-\varepsilon<a_n<u+\varepsilon$ for infinitely many values of n.

$\therefore$ u is a limit point of $<a_n>$.

Now we shall show that no number greater than u can be a limit point of $<a_n>$.

Let u' be any number greater than u. Let p and q be two numbers such that

$$u<p<u'<q.$$

By the second condition, for each $\varepsilon>0$, $a_n<u+\varepsilon$ for all except finitely many value of n.

Choosing $\varepsilon=p-u>0$, we have $a_n<p$ for all except finitely many values of n and therefore, (p, q) is a nbd of u' containing a_n for finitely many values of n. This implies that u' is not a limit point of $<a_n>$ so that u is the greatest limit point of $<a_n>$.

Hence u is the limit superior of the sequence $<a_n>$.

Theorem 2. A real number l is the limit inferior of a bounded sequence $<a_n>$ if and only if the following two conditions are satisfied.

(*i*) for each $\varepsilon>0$, $a_n<l+\varepsilon$ for infinitely many values of n.

(*ii*) for each $\varepsilon>0$, $a_n>l-\varepsilon$ for all except finitely many values of n.

Proof. Necessity.

Let l be the limit inferior of a bounded sequence $<a_n>$ and let $\varepsilon>0$ be given.

Since l is a limit point of $<a_n>$, we have

$$l-\varepsilon<a_n<l+\varepsilon$$

for infinitely many values of n.

In particular, $a_n<l+\varepsilon$ for infinitely many values of n.

Again, since l is the least limit point, $l-\varepsilon$ is not a limit point and, therefore, $a_n\leqslant l-\varepsilon$ for only finitely many values of n. (If for some $\varepsilon>0$, $a_n\leqslant l-\varepsilon$ for infinitely many values of n, then $<a_n>$ will have a limit point $p\leqslant l-\varepsilon$).

$\therefore$ $a_n>l-\varepsilon$ for all except finitely many values of n.

Sufficiency

Let us assume that l satisfies both the conditions.

Given any $\varepsilon>0$, $a_n<l+\varepsilon$ for infinitely many values of n and $a_n>l-\varepsilon$ for all except finitely many value of n.

$\Rightarrow$ $l-\varepsilon<a_n<l+\varepsilon$ for infinitely many values of n.

$\therefore$ l is a limit point of $<a_n>$.

Now we shall show that no number less than l can be a limit point of $<a_n>$.

Let l' be any number less than l. Let p and q be two numbers such that

$$p<l'<q<l.$$

By the second condition, for each $\varepsilon>0$, $a_n>l-\varepsilon$ for all except finitely many values of n.

Choosing $\varepsilon=l-q>0$, we have $a_n>q$ for all except finitely many values of n and, therefore, (p, q) is a nbd of l' containing a_n for finitely many values of n. This implies that l' is not a limit point of $<a_n>$ so that l is the least limit point of $<a_n>$. Hence l is the limit inferior of the sequence $<a_n>$.

Theorem 3. A sequence $<a_n>$ converges to l if and only if

$$\lim_{n\to\infty} \sup. a_n = \lim_{n\to\infty} \inf. a_n = l. \qquad \textit{(M.D.U. 1992)}$$

Proof. Let the sequence $<a_n>$ converge to l.

Then given $\varepsilon>0$, $\exists$ a positive integer m, such that

$$|a_n-l|<\varepsilon \;\forall\; n\geqslant m$$

$$\Rightarrow \qquad l-\varepsilon<a_n<l+\varepsilon \;\forall\; n\geqslant m$$

Since the nbd $(l-\varepsilon, l+\varepsilon)$ of l contains a_n for infinitely many values of n and since ε is arbitrary, therefore, every nbd of l contains infinitely terms of the sequence $<a_n>$.

$\therefore$ l is a limit point of $<a_n>$.

Now we shall show that l is the only limit point of the sequence $<a_n>$.

Let l' be any number other than l. Two cases arise :

(i) $l<l'$ (ii) $l'<l$

Suppose $l<l'$. Let p, q, r be three numbers such that

$$p<l<q<l'<r$$

Since $a_n \to l$, therefore, every nbd of l contains a_n for all except finitely many values of n. In particular, $a_n \in (p, q)$ for all except finitely many values of n.

$\Rightarrow$ The nbd (q, r) of l' contains a_n for atmost finitely many values of n.

$\Rightarrow$ l' cannot be a limit point of $<a_n>$.

Similarly, when $l'<l$, l' is not a limit point of $<a_n>$.

Thus l is the only limit point of $<a_n>$.

Hence $\lim\limits_{n\to\infty} \sup. a_n = \lim\limits_{n\to\infty} \inf. a_n = l$.

Conversely, suppose that

$$\lim_{n\to\infty} \sup. a_n = \lim_{n\to\infty} \inf. a_n = l$$

Let $\varepsilon>0$ be given.

Since l is a limit superior of $<a_n>$, therefore, $a_n<l+\varepsilon$ for all except finitely many values of n.

$\Rightarrow$ $\exists$ a positive integer m, such that

$$a_n<l+\varepsilon \quad \forall n \geqslant m_1 \qquad \ldots(1)$$

Again, since l is a limit inferior of $<a_n>$, therefore, $a_n>l-\varepsilon$ for all except finitely many values of n.

$\Rightarrow$ $\exists$ a positive integer m_2 such that

$$a_n>l-\varepsilon \ \forall \ n \geqslant m_2 \qquad \ldots(2)$$

Let $m=\max. \{m_1, m_2\}$, then from (1) and (2), we have

$$l-\varepsilon<a_n<l+\varepsilon \ \forall \ n \geqslant m$$

$$\Rightarrow \quad |a_n-l|<\varepsilon \quad \forall \ n \geqslant m$$

$$\Rightarrow \quad a_n \to l.$$

4·29. Limit Superior and Limit Inferior of a Bounded Sequence (*Second Definition*)

Let $<a_n>$ be a bounded sequence. Let $<a_n>$ be bounded above by K. Then, for each $n \in \mathbb{N}$, the set $S_n=\{a_n, a_{n+1}, \ldots\ldots\}$ is bounded above by K. By completeness axiom, S_n has the l.u.b.

(or supermum) M_n (say). Clearly $M_n > M_{n+1}$ $\forall$ $n \in N$. Thus, the sequence $<M_n>$ being a decreasing sequence is either convergent or diverges to $-\infty$. If $<M_n>$ is convergent, then $\lim_{n\to\infty} M_n$ is called the limit superior of $<a_n>$. If $<M_n>$ is divergent, then the limit superior of $<a_n>$ is taken as $-\infty$.

$$\text{Thus } \lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} M_n$$

$$= \lim_{n\to\infty} \sup S_n$$

$$= \lim_{n\to\infty} \sup \{a_n, a_{n+1}, \ldots\ldots\}$$

Let $<a_n>$ be a bounded sequence. Let $<a_n>$ be bounded below by k. Then, for each $n \in N$, the set $S_n = \{a_n, a_{n+1}, \ldots\ldots\}$ is bounded below by k. By completeness axiom, S_n has the g.l.b. (or infinium) m_n (say). Clearly $m_n \leq m_{n+1}$ $\forall n \in N$. Thus, the sequence $<m_n>$ being an increasing sequence is either convergent or diverges to ∞. If $<m_n>$ is convergent, then $\lim_{n\to\infty} m_n$ is called the limit inferior of $<a_n>$. If $<m_n>$ is divergent, then the limit inferior of $<a_n>$ is taken as ∞.

$$\text{Thus } \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} m_n$$

$$= \lim_{n\to\infty} \inf. S_n$$

$$= \lim_{n\to\infty} \inf. \{a_n, a_{n+1} \ldots\ldots\}$$

4·30. Theorem

(*i*) If a sequence $<a_n>$ is such that

$$\overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n = \infty$$

then $<a_n>$ diverges to ∞.

(*ii*) If a sequence $<a_n>$ is such that

$$\overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n = -\infty$$

then $<a_n>$ diverges to $-\infty$.

Proof. (*i*) Let $M_n = \sup \{a_n, a_{n+1}, \ldots\ldots\}$

and $\qquad m_n = \inf. \{a_n, a_{n+1}, \ldots\ldots\}$ $\forall$ $n \in N$.

Then $\qquad m_n \leq a_n \leq M_n$ $\forall n$ $\qquad \ldots(1)$

Since $\qquad \lim_{n\to\infty} m_n = \underline{\lim_{n\to\infty}}\, a_n = \infty$

and $\qquad \lim_{n\to\infty} M_n = \overline{\lim_{n\to\infty}}\, a_n = \infty$

$\therefore$ From (1), $\lim_{n\to\infty} a_n = \infty$

$\Rightarrow$ The sequence $<a_n>$ diverges to ∞.

(*ii*) Please try yourself.

ILLUSTRATIVE EXAMPLES

Example 1. *Give examples of sequences having*

(*i*) *no cluster point*

(*ii*) *one cluster point*

(*iii*) *two cluster points*

(*iv*) *infinitely many cluster points.*

Sol. (*i*) The sequence $<n>$ has no cluster point.

(*ii*) The sequence $<\frac{1}{n}>$ has one cluster point, namely 0.

(*iii*) The sequence $<(-1)^n>$ has two cluster points, namely -1 and 1.

(*iv*) The sequence $<2,\ 1+\frac{1}{2},\ 2+\frac{1}{2},\ 1+\frac{1}{3},\ 2+\frac{1}{3},\ 3+\frac{1}{3},\ 1+\frac{1}{4},\ 2+\frac{1}{4},\ 3+\frac{1}{4},\ 4+\frac{1}{4},\ \ldots\ldots>$ has infinitely many cluster points. Every natural number is a cluster point.

Example 2. *Examine the sequences whose nth terms are given below, for cluster points :*

(*i*) $(-1)^n$ (*ii*) 5

(*iii*) $\frac{(-1)^n}{n}$ (*iv*) n

(*v*) $(-1)^n\left(1+\frac{1}{n}\right)$ (*vi*) $\left(1+\frac{1}{n}\right)^{n+1}$

(*vii*) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots+\frac{1}{n!}$

Sol. (*i*) Here $a_n=(-1)^n=\begin{cases}-1 & \text{when } n \text{ is odd}\\ 1 & \text{when } n \text{ is even}\end{cases}$

$\therefore$ The sequence $<a_n>$ has two cluster points, namely -1 and 1.

(*ii*) Here $a_n=5$ is a constant sequence converging to 5.

$\therefore$ The sequence $<a_n>$ has only one cluster point, namely 5.

(*iii*) Here $a_n=\frac{(-1)^n}{n}=\begin{cases}-\frac{1}{n} & \text{when } n \text{ is odd.}\\ \frac{1}{n} & \text{when } n \text{ is even}\end{cases}$

$\lim\limits_{n\to\infty} a_n=0$ so that the sequence $<a_n>$ converges to 0.

$\therefore$ The sequence $<a_n>$ has only one cluster point, 0.

(*iv*) For any $l\in R$, the nbd $\left(l-\frac{1}{4},\ l+\frac{1}{4}\right)$ of l contains at most one term of the sequence $<n>$.

$\therefore$ $l\in R$ is not a limit point of $<n>$.

$\Rightarrow$ The sequence $<n>$ has no limit point.

(v) Here $a_n = (-1)^n \left(1+\frac{1}{n}\right)$

$$= \begin{cases} -\left(1+\frac{1}{n}\right) & \text{when } n \text{ is odd} \\ \left(1+\frac{1}{n}\right) & \text{when } n \text{ is even} \end{cases}$$

The sequence $\langle a_n \rangle$ has two cluster points, -1 and 1.

(vi) Here $a_n = \left(1+\frac{1}{n}\right)^{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right) = e \times 1 = e$$

$\Rightarrow$ The sequence $\langle a_n \rangle$ converges to e.

$\Rightarrow$ The sequence has only one cluster point e.

(vii) Here $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$

The sequence $\langle a_n \rangle$ converges to e.

$\therefore$ The sequence has only one cluster point e.

Example 3. *Find the limit superior and limit inferior of each of the following sequences :*

(i) $\langle 1, 3, 5, 1, 3, 5, 1, 3, 5, \ldots\ldots \rangle$

(ii) $\langle 1, 5, 17, 19, 1, 5, 17, 19, 1, 5, 17, 19, \ldots\ldots \rangle$

(iii) $\langle a_n \rangle$ *where* $a_n = \sin \frac{n\pi}{3}$

(iv) $\langle a_n \rangle$ *where* $a_n = (-2)^{-n} \left(1+\frac{1}{n}\right)$

(v) $\langle a_n \rangle$ *where* $a_n = (-10)^n \left(1+\frac{1}{n}\right)^2$

(vi) $\langle a_n \rangle$ *where* $a_n = (-1)^n \left(1-\frac{1}{n}\right)$

(vii) $\langle a_n \rangle$ *where* $a_n = \left(1+\frac{1}{n}\right)^{n+1}$

(viii) $\langle a_n \rangle$ *where* $a_n = (-1)^n (2^n + 3^n)$.

Sol. (i) Here $a_n = \begin{cases} 1 & \text{if } n = 3m-2 \\ 3 & \text{if } n = 3m-1, \quad m \in \mathbb{N} \\ 5 & \text{if } n = 3m \end{cases}$

The set of cluster points of $\langle a_n \rangle$ is $E = \{1, 3, 5\}$.

$$\therefore \quad \overline{\lim_{n\to\infty}}\ a_n = \max.\ \{1, 3, 5\} = 5$$

$$\underline{\lim_{n\to\infty}}\ a_n = \min.\ \{1, 3, 5\} = 1.$$

(*ii*) Please try yourself. [Ans. $\overline{\lim_{n\to\infty}}\ a_n = 19$, $\underline{\lim_{n\to\infty}}\ a_n = 1$]

(*iii*) Here $a_n = \sin \dfrac{n\pi}{3}$

$$= \begin{cases} 0 & \text{if } n = 3m \\ \dfrac{\sqrt{3}}{2} & \text{if } n = 6m-5 \text{ or } 6m-4 \\ -\dfrac{\sqrt{3}}{2} & \text{if } n = 6m-2 \text{ or } 6m-1 \end{cases}$$

where $m \in \mathbf{N}$

The set of cluster points of $\langle a_n \rangle$ is

$$E = \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\}$$

$$\therefore \quad \overline{\lim_{n\to\infty}}\ a_n = \max.\ \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\} = \frac{\sqrt{3}}{2}$$

$$\underline{\lim_{n\to\infty}}\ a_n = \min.\ \left\{0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}\right\} = -\frac{\sqrt{3}}{2}.$$

(*iv*) Here $a_n = (-2)^{-n}\left(1+\dfrac{1}{n}\right)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-2)^{-n}\left(1+\frac{1}{n}\right) = 0 \times 1 = 0$$

$\Rightarrow$ The sequence $\langle a_n \rangle$ converges to 0.

$\Rightarrow$ The set of cluster points of $\langle a_n \rangle$ is

$$E = \{0\}$$

$$\therefore \quad \overline{\lim_{n\to\infty}}\ a_n = 0 = \underline{\lim_{n\to\infty}}\ a_n.$$

(*v*) Here $a_n = (-10)^n\left(1+\dfrac{1}{n}\right)^2$

$$= \begin{cases} -(10)^n\left(1+\dfrac{1}{n}\right)^2 & \text{when } n \text{ is odd} \\ (10)^n\left(1+\dfrac{1}{n}\right)^2 & \text{when } n \text{ is even} \end{cases}$$

Since $\lim_{n\to\infty} (10)^n\left(1+\frac{1}{n}\right)^2=\infty$, the set of cluster points of $<a_n>$ is

$$E=\{-\infty, \infty\}$$

$\therefore \overline{\lim_{n\to\infty}} a_n=\infty$ and $\underline{\lim_{n\to\infty}} a_n=-\infty$.

(*vi*) Please try yourself. [Ans. $\overline{\lim_{n\to\infty}} a_n=1$, $\underline{\lim_{n\to\infty}} a_n=-1$]

(*vii*) Please try yourself. [Ans. $\overline{\lim_{n\to\infty}} a_n=e=\underline{\lim_{n\to\infty}} a_n$]

(*viii*) Here $a_n=(-1)^n (2^n+3^n)$

$$=\begin{cases}-(2^n+3^n) & \text{when } n \text{ is odd}\\ (2^n+3^n) & \text{when } n \text{ is even}\end{cases}$$

Since $\lim_{n\to\infty} (2^n+3^n)=\infty$, the set of cluster points of $<a_n>$ is

$$E=\{-\infty, \infty\}$$

$\therefore \overline{\lim_{n\to\infty}} a_n=\infty$ and $\underline{\lim_{n\to\infty}} a_n=-\infty$.

Note. E is the set of all the cluster points of $<a_n>$, including $+\infty$ and $-\infty$.

Example 4. *For a sequence $<a_n>$, $\underline{\lim_{n\to\infty}} a_n \leqslant \overline{\lim_{n\to\infty}} a_n$.*

Sol. If $<a_n>$ is unbounded, then either

$$\overline{\lim_{n\to\infty}} a_n=\infty \quad \text{or} \quad \underline{\lim_{n\to\infty}} a_n=-\infty$$

and hence there is nothing to prove.

Now, let $<a_n>$ be a bounded sequence.

Let $m_n=\text{g.l.b. }\{a_n, a_{n+1}, \ldots\ldots\}$

and $M_n=\text{l.u.b. }\{a_n, a_{n+1}, \ldots\ldots\} \quad \forall\, n\in N$

Then $m_n \leqslant M_n$

$\Rightarrow \lim_{n\to\infty} m_n \leqslant \lim_{n\to\infty} M_n$

$\Rightarrow \underline{\lim_{n\to\infty}} a_n \leqslant \overline{\lim_{n\to\infty}} a_n$.

Example 5. *If $<a_n>$ and $<b_n>$ are bounded sequences such that $a_n \leqslant b_n \ \forall\, n\in N$, then*

(*i*) $\underline{\lim_{n\to\infty}} a_n \leqslant \underline{\lim_{n\to\infty}} b_n$

(*ii*) $\overline{\lim_{n\to\infty}} a_n \leqslant \overline{\lim_{n\to\infty}} b_n$.

Sol. Let $M_n = \text{l.u.b.}\{a_n, a_{n+1}, \ldots\ldots\}$
$M_n' = \text{l.u.b.}\{b_n, b_{n+1}, \ldots\ldots\}$
$m_n = \text{g.l.b.}\{a_n, a_{n+1}, \ldots\ldots\}$
and $m_n' = \text{g.l.b.}\{b_n, b_{n+1}, \ldots\ldots\}$

Since $a_n \leqslant b_n \quad \forall\ n \in N$

$\therefore \quad M_n \leqslant M_n'$ and $m_n \leqslant m_n' \quad \forall\ n$

$\Rightarrow \lim\limits_{n\to\infty} M_n \leqslant \lim\limits_{n\to\infty} M_n'$ and $\lim\limits_{n\to\infty} m_n \leqslant \lim\limits_{n\to\infty} m_n'$

$\Rightarrow \overline{\lim\limits_{n\to\infty}}\, a_n \leqslant \overline{\lim\limits_{n\to\infty}}\, b_n$ and $\underline{\lim\limits_{n\to\infty}}\, a_n \leqslant \underline{\lim\limits_{n\to\infty}}\, b_n.$

Example 6. *If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded sequences, then show that*

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(i) $\overline{\lim\limits_{n\to\infty}}\,(a_n+b_n) \leqslant \overline{\lim\limits_{n\to\infty}}\, a_n + \overline{\lim\limits_{n\to\infty}}\, b_n$

(ii) $\underline{\lim\limits_{n\to\infty}}\,(a_n+b_n) \geqslant \underline{\lim\limits_{n\to\infty}}\, a_n + \underline{\lim\limits_{n\to\infty}}\, b_n.$

Sol. Let $M_n = \sup\{a_n, a_{n+1}, \ldots\ldots\}$
$M_n' = \sup\{b_n, b_{n+1}, \ldots\ldots\}$
$m_n = \inf\{a_n, a_{n+1}, \ldots\ldots\}$
and $m_n' = \inf\{b_n, b_{n+1}, \ldots\ldots\}$

Since $\langle a_n \rangle$ and $\langle b_n \rangle$ are bounded, so is $\langle a_n + b_n \rangle$

and $\sup\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\ldots\}$
$\leqslant \sup\{a_n, a_{n+1}, \ldots\ldots\} + \sup\{b_n, b_{n+1}, \ldots\ldots\}$
$= M_n + M_n'$
$\inf\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\ldots\}$
$\geqslant \inf\{a_n, a_{n+1}, \ldots\ldots\} + \inf\{b_n, b_{n+1}, \ldots\ldots\}$
$= m_n + m_n'.$

(i)
$$\overline{\lim_{n\to\infty}}\,(a_n+b_n) = \lim_{n\to\infty} \sup\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\ldots\}$$
$$\leqslant \lim_{n\to\infty} (M_n + M_n')$$
$$= \lim_{n\to\infty} M_n + \lim_{n\to\infty} M_n'$$
$$= \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n$$

$$\therefore \quad \overline{\lim_{n\to\infty}}\,(a_n+b_n) \leqslant \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n.$$

(ii)
$$\underline{\lim_{n\to\infty}}\,(a_n+b_n) = \lim_{n\to\infty} \inf\{a_n+b_n, a_{n+1}+b_{n+1}, \ldots\ldots\}$$
$$\geqslant \lim_{n\to\infty} (m_n + m_n')$$

$$= \lim_{n\to\infty} m_n + \lim_{n\to\infty} m_n'$$

$$= \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n$$

$$\therefore \quad \underline{\lim}_{n\to\infty} (a_n+b_n) \geqslant \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n.$$

Note 1. Since $\underline{\lim}_{n\to\infty} (a_n+b_n) \leqslant \overline{\lim}_{n\to\infty} (a_n+b_n)$

we have, by combining the above two parts,

$$\underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n \leqslant \underline{\lim}_{n\to\infty} (a_n+b_n) \leqslant \overline{\lim}_{n\to\infty} (a_n+b_n)$$

$$\leqslant \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n.$$

Note 2. In certain cases, strict inequalities may hold.

For example, consider $a_n=(-1)^n$ and $b_n=(-1)^{n+1}$

then $\overline{\lim}_{n\to\infty} a_n=1, \qquad \overline{\lim}_{n\to\infty} b_n=1$

Since $a_n+b_n=0 \quad \forall\ n$

$$\overline{\lim}_{n\to\infty} (a_n+b_n)=0$$

Clearly $\overline{\lim}_{n\to\infty} (a_n+b_n) \neq \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

rather $\overline{\lim}_{n\to\infty} (a_n+b_n) < \overline{\lim}_{n\to\infty} a_n + \overline{\lim}_{n\to\infty} b_n$

Similarly $\underline{\lim}_{n\to\infty} (a_n+b_n) \neq \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n$

rather $\underline{\lim}_{n\to\infty} (a_n+b_n) > \underline{\lim}_{n\to\infty} a_n + \underline{\lim}_{n\to\infty} b_n.$

Example 7. *If $\langle a_n \rangle$ is a bounded sequence, show that*

(i) $\overline{\lim}_{n\to\infty} (-a_n) = -\underline{\lim}_{n\to\infty} a_n$

(ii) $\underline{\lim}_{n\to\infty} (-a_n) = -\overline{\lim}_{n\to\infty} a_n$

(iii) $\overline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \overline{\lim}_{n\to\infty} a_n, \qquad \lambda>0$

(iv) $\underline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \underline{\lim}_{n\to\infty} a_n, \qquad \lambda>0$

(v) $\overline{\lim}_{n\to\infty} (\lambda a_n) = \lambda\, \underline{\lim}_{n\to\infty} a_n, \qquad \lambda<0.$

Sol. Since $\langle a_n \rangle$ is a bounded sequence, so are $\langle -a_n \rangle$ and $\langle \lambda a_n \rangle$.

(i) $\overline{\lim_{n\to\infty}}(-a_n) = \lim_{n\to\infty} \sup\{-a_n, -a_{n+1}, \ldots\ldots\}$

$$= \lim_{n\to\infty} -\inf\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= -\lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= -\underline{\lim_{n\to\infty}}\ a_n.$$

(ii) $\underline{\lim_{n\to\infty}}(-a_n) = \lim_{n\to\infty} \inf\{-a_n, -a_{n+1}, \ldots\ldots\}$

$$= \lim_{n\to\infty} -\sup\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= -\lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= -\overline{\lim_{n\to\infty}}\ a_n.$$

(iii) $\overline{\lim_{n\to\infty}}(\lambda a_n) = \lim_{n\to\infty} \sup\{\lambda a_n, \lambda a_{n+1}, \ldots\ldots\}$

$$= \lim_{n\to\infty} \lambda \sup\{a_n, a_{n+1}, \ldots\ldots\} \qquad (\because \lambda > 0)$$

$$= \lambda \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= \lambda\ \overline{\lim_{n\to\infty}}\ a_n.$$

(iv) Please try yourself.

(v) $\overline{\lim_{n\to\infty}}(\lambda a_n) = \lim_{n\to\infty} \sup\{\lambda a_n, \lambda a_{n+1}, \ldots\ldots\}$

$$= \lim_{n\to\infty} \lambda \inf\{a_n, a_{n+1}, \ldots\ldots\} \qquad (\because \lambda < 0)$$

$$= \lambda \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\ldots\}$$

$$= \lambda\ \underline{\lim_{n\to\infty}}\ a_n.$$

4·31. Subsequences

In a sequence $<a_n>$ if we keep only the terms whose suffixes are $n_1, n_2, n_3, \ldots$ maintaining the same order as in the sequence, we get another sequence $<a_{n_k}>$. It is apt to call $<a_{n_k}>$ a subsequence of $<a_n>$. In the subsequence $<a_{n_k}>$, the suffixes $n_1, n_2, n_3, \ldots\ldots$ form a strictly increasing sequence of positive integers. Sometimes it is convenient to determine the nature of a sequence by using its subsequences.

Def. *Let $<a_n>$ be a given sequence. If $<n_k>$ is a strictly increasing sequence of natural numbers (i.e., $n_1 < n_2 < n_3 < \ldots\ldots$), then $<a_{n_k}>$ is called a subsequence of $<a_n>$.*

For example (i) $<a_{2n}>$, $<a_{2n-1}>$, $<a_n^2>$ are all subsequences of $<a_n>$.

(ii) The sequences $<2, 4, 6, \ldots\ldots>$, $<1, 3, 5, \ldots\ldots>$, $<1, 4, 9, 16, \ldots\ldots>$ are all subsequences of the sequence $<n>$.

Note 1. The terms of a subsequence occur in the same order in which they occur in the original sequence.

Note 2. If $<u_n>$ is a subsequence of $<a_n>$ and $<v_n>$ is a subsequence of $<u_n>$, then $<v_n>$ is a subsequence of $<a_n>$.

Note 3. Every sequence is a subsequence of itself.

Note 4. The interval in the various terms of a subsequence need not be regular.

Note 5. Given a term a_m of the sequence $<a_n>$, there is a term of the subsequence following it.

4·32. Theorem

1. If a sequence $<a_n>$ converges to l, then every subsequence of $<a_n>$ also converges to l.

Proof. Let $<a_{n_k}>$ be a subsequence of $<a_n>$.

Since $<a_n>$ converges to l.

$\therefore$ Given $\epsilon>0$, $\exists$ a positive integer m such that

$$|a_n-l|<\epsilon \qquad \forall\ n\geq m \qquad \ldots(1)$$

We can find a natural number $n_{k_0}\geq m$

If $n_k\geq n_{k_0}$, then $n_k\geq m$.

$\therefore$ From (1), we have

$$|a_{n_k}-l|<\epsilon \qquad \forall\ n_k\geq m$$

$\Rightarrow$ $<a_{n_k}>$ converges to l.

Note 1. *The converse of the above theorem is not true.*

That is if a subsequence or even if infinitely many subsequences of a given sequence converge, the original sequence may not converge.

For example, let $a_n=(-1)^n$.

The sequence $<a_n>$ does not converge. However, the two subsequences.

$$<a_1, a_3, a_5, \ldots\ldots>=<a_{2n-1}>$$

and $$<a_2, a_4, a_6, \ldots\ldots>=<a_{2n}>$$

converge to -1 and 1 respectively.

Note 2. If all subsequences of a sequence $<a_n>$ converge to the same limit l, only then $<a_n>$ converges to l.

To prove that a given sequence is not convergent, it is sufficient to show that two of its subsequences converge to different limits. (See example with Note 1).

Theorem 2. If the subsequences $<a_{2n-1}>$ and $<a_{2n}>$ of a sequence $<a_n>$ converge to the same limit l, then the sequence $<a_n>$ converges to l.

Proof. Let $\varepsilon>0$ be given

$<a_{2n-1}>$ converges to l

$\Rightarrow$ For $\varepsilon>0$, $\exists$ a positive integer m_1 such that

$$|a_{2n-1}-l|<\varepsilon \quad \forall\ n\geqslant m_1$$

$<a_{2n}>$ converges to l

$\Rightarrow$ For $\varepsilon>0$, $\exists$ a positive integer m_2 such that

$$|a_{2n}-l|<\varepsilon \quad \forall\ n\geqslant m_2$$

Let $m=\max.\{m_1, m_2\}$, then

$$|a_{2n-1}-l|<\varepsilon \quad \text{and} \quad |a_{2n}-l|<\varepsilon \quad \forall\ n\geqslant m$$

$\Rightarrow$ $|a_n-l|<\varepsilon \quad \forall\ n\geqslant m$

$\Rightarrow$ $<a_n>$ converges to l.

Theorem 3. (*i*) If a sequence $<a_n>$ diverges to ∞, then every subsequence of $<a_n>$ also diverges to ∞.

(*ii*) If a sequence $<a_n>$ diverges to $-\infty$, then every subsequence of $<a_n>$ also diverges to $-\infty$.

Proof. (*i*) Let $<a_{n_k}>$ be a subsequence of $<a_n>$.

Since $<a_n>$ diverges to ∞.

$\therefore$ For every positive real number K, however large, $\exists$ a positive integer m such that

$$a_n>K \quad \forall\ n\geqslant m \qquad \ldots(1)$$

We can find a natural number $n_{k_0}\geqslant m$.

If $n_k\geqslant n_{k_0}$, then $n_k\geqslant m$.

$\therefore$ From (1), we have

$$a_{n_k}>K \quad \forall\ n_k\geqslant m$$

$\Rightarrow$ $<a_{n_k}>$ diverges to ∞.

(*ii*) Please try yourself.

Note 1. *The converse of the above theorem is not true.*

That is if a subsequence of a given sequence diverges to $+\infty$ (or $-\infty$), the sequence need not diverges to $+\infty$ (or $-\infty$).

For example, let $a_n=(-1)^n\, n=\begin{cases}-n & \text{if } n \text{ odd}\\ \ \ n & \text{if } n \text{ even}\end{cases}$

Then the subsequence $<a_{2n-1}>$ diverges to $-\infty$ and the subsequence $<a_{2n}>$ diverges to $+\infty$, but the sequence $<a_n>$ does not diverge either to $+\infty$ or $-\infty$. $<a_n>$ is an oscillatory sequence.

Note 2. If all the subsequences of a sequence $<a_n>$ diverge to ∞ (or $-\infty$), only then the sequence $<a_n>$ diverges to ∞ (or $-\infty$).

4·33. Peak Point of a Sequence

Def. A natural number m is called a **peak point** of the sequence $<a_n>$ if

$$a_n < a_m \quad \forall \quad n > m.$$

For example (*i*) The sequence $<(-1)^n>$ has no peak point.

(*ii*) If $a_n = \dfrac{1}{n}$ when $n \leqslant 5$

$= -n$ when $n > 5$

then 1, 2, 3, 4, 5 are 5 peak points.

(*iii*) If $a_n = 1$ when $n = 1, 2, \ldots\ldots, m$

$= -1$ when $n > m$

then m is the only peak point.

(*iv*) If $a_n = \dfrac{1}{n}$, then every natural number is a peak point.

For, let m be any natural number, then for $n > m$, we have

$$\frac{1}{n} < \frac{1}{m}$$

i.e. $\quad a_n < a_m \quad \forall \quad n > m$

Thus a sequence may have no peak point, a finite number of peak points or an infinite number of peak points.

Note. For a strictly monotonically decreasing sequence, every natural number is a peak point.

4·34. Theorem

1. Every sequence contains a monotonic subsequence.

Proof. Let $<a_n>$ be any sequence. Then three cases arise according as it has no peak point, finitely many peak points or infinitely many peak points.

Case (*i*) The sequence has no peak point.

Since 1 is not a peak point, $\exists$ a natural number $n_2 > 1$ such that $a_{n_2} > a_1$.

Again, since n_2 is not a peak point, $\exists$ a natural number $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.

Repeating the above argument, we get a subsequence

$$<a_{n_k}> \text{ so that } a_{n_1} < a_{n_2} < a_{n_3} < \ldots\ldots$$

where $\quad n_1 = 1$.

Thus the sequence $<a_n>$ contains a monotonically increasing subsequence $<a_{n_k}>$.

Case (*ii*) The sequence has a finite number of peak points.

Let m be the largest peak point. Let n_1 be a natural number such that $n_1 > m$, then n_1 is not a peak point.

$\therefore$ $\exists$ a natural number $n_2 > n_1$ such that

$$a_{n_2} > a_{n_1}$$

Again n_2 is not a peak point.

$\therefore$ $\exists$ a natural number $n_3 > n_2$ such that

$$a_{n_3} > a_{n_2}$$

Repeating the above argument, we get a subsequence $<a_{n_k}>$ so that $a_{n_1} < a_{n_2} < a_{n_3} < \ldots\ldots$

Thus the sequence $<a_n>$ contains a monotonically increasing sequence $<a_{n_k}>$.

Case (*iii*) **The sequence has an infinite number of peak points.**

Let the peak points be $n_1, n_2, n_3, \ldots\ldots$ such that

$$n_1 < n_2 < n_3 < \ldots\ldots$$

$\because$ n_1 is a peak point and $n_2 > n_1$, therefore, $a_{n_2} < a_{n_1}$.

$\because$ n_2 is a peak point and $n_3 > n_2$, therefore, $a_{n_3} < a_{n_2}$.

Repeating the above argument, we get a subsequence $<a_{n_k}>$ so that $a_{n_1} > a_{n_2} > a_{n_3} > \ldots\ldots$

Thus the sequence $<a_n>$ contains a monotonically decreasing sequence $<a_{n_k}>$.

Theorem 2. **Every bounded sequence in R contains a convergent subsequence.** (*M.D.U. 1991*)

Proof. Let $<a_n>$ be a bounded sequence.

Since every sequence has a monotonic subsequence, therefore, $<a_n>$ has a monotonic subsequence, say $<a_{n_k}>$.

Since $<a_n>$ is bounded, therefore, the subsequence $<a_{n_k}>$ is also bounded.

Now $<a_{n_k}>$ is a bounded monotonic sequence, therefore, $<a_{n_k}>$ is convergent.

Hence $<a_n>$ has a convergent subsequence.

4·35. Subsequential Limit

Def. Let $<a_n>$ be a sequence. A real number l is called a *subsequential limit* of the sequence $<a_n>$ if there exists a subsequence of $<a_n>$ converging to l.

A subsequential limit of a sequence $<a_n>$ is also a *cluster point* (or *limit point*) of the sequence $<a_n>$.

The following theorem establishes the equivalence between the cluster points of a sequence and its subsequential limits.

4·36. Theorem

A real number l is a subsequential limit of the sequence $<a_n>$ if and only if each neighbourhood $(l-\varepsilon, l+\varepsilon)$, $\varepsilon>0$, of l contains infinitely many terms of $<a_n>$ (i.e., if and only if l is a cluster point of $<a_n>$).

Proof. Let l be a subsequential limit of $<a_n>$, then $\exists$ a subsequence $<a_{n_k}>$ of $<a_n>$ converging to l.

$\therefore$ Given $\varepsilon>0$, $\exists$ a positive integer k_0 such that

$$|a_{n_k}-l|<\varepsilon \quad \forall\ k\geqslant k_0$$

$$\Rightarrow \qquad a_{n_k}\in(l-\varepsilon, l+\varepsilon) \quad \forall\ k\geqslant k_0$$

$\Rightarrow$ Infinitely many terms of the subsequence $<a_{n_k}>$ and hence of the sequence $<a_n>$ lie in $(l-\varepsilon, l+\varepsilon)$.

Conversely. Let each nbd. $(l-\varepsilon, l+\varepsilon)$ of l contain infinitely many terms of $<a_n>$.

Then $a_n\in(l-\varepsilon, l+\varepsilon)$ for infinitely many values of n.

In particular, $a_n=\left(l-\dfrac{1}{n},\ l+\dfrac{1}{n}\right)=\mathrm{I}_n$ for infinitely many values of n. $\left|\ \text{Taking } \varepsilon=\dfrac{1}{n}\right.$

Choose $a_{n_1}\in(l-1,\ l+1)=\mathrm{I}_1$. Then $\exists\ n_2>n_1$ such that $a_{n_2}\in\left(l-\frac{1}{2}, l+\frac{1}{2}\right)=\mathrm{I}_2$.

Continuing like this, $\exists$ a natural number n_{k_0} such that

$$n_{k_0}>\ldots\ldots>n_2>n_1 \text{ and } a_{n_{k_0}}\in\left(l-\frac{1}{k_0}, l+\frac{1}{k_0}\right)=\mathrm{I}_{k_0}$$

Again, continuing like this, we get a subsequence $<a_{n_k}>$ of $<a_n>$.

Now for all $n_k\geqslant n_{k_0}$, we have $k\geqslant k_0$

$$\therefore \qquad \frac{1}{k}\leqslant\frac{1}{k_0} \quad \text{and} \quad -\frac{1}{k}\geqslant-\frac{1}{k_0}$$

$$\Rightarrow \qquad l+\frac{1}{k}\leqslant l+\frac{1}{k_0} \quad \text{and} \quad l-\frac{1}{k}\geqslant l-\frac{1}{k_0}$$

$$\Rightarrow \qquad \left(l-\frac{1}{k},\ l+\frac{1}{k}\right)\subset\left(l-\frac{1}{k_0},\ l+\frac{1}{k_0}\right)$$

$$\Rightarrow \qquad \mathrm{I}_k\subset\mathrm{I}_{k_0} \quad \forall\ k\geqslant k_0 \quad \text{i.e.} \quad \forall\ n_k\geqslant n_{k_0}$$

$$\Rightarrow \qquad \forall\ n_k\geqslant n_{k_0},\ a_{n_k}\in\mathrm{I}_k \ \Rightarrow\ a_{n_k}\in\mathrm{I}_{k_0}$$

$\Rightarrow \quad a_{n_k} \in \left(l - \frac{1}{k_0},\ l + \frac{1}{k_0} \right) \ \forall \ n_k \geq n_{k_0}$

$\Rightarrow \quad \left| a_{n_k} - l \right| < \frac{1}{k_0} = \varepsilon \ \forall \ n_k \geq n_{k_0}$

$\Rightarrow \quad < a_{n_k} >$ converges to l.

$\Rightarrow \quad l$ is a subsequential limit of the sequence $<a_n>$.