(26) Prove $f(x) = \sin x^2$ is Not uniformly continous on $[0, \infty[$

Success Clap

This problem is present in Success Clap Question Bank (Qn-99)

Method - 1

Given in all books

Let $x_1 = \sqrt{\frac{n\pi}{2}}$ $\quad x_2 = \sqrt{(n+1)\frac{\pi}{2}}$

$$|f(x_2) - f(x_1)| = |\sin x_2^2 - \sin x_1^2|$$

$$= \left|\sin(n+1)\frac{\pi}{2} - \sin\frac{n\pi}{2}\right|$$

$$= |0 - (\pm 1)| = 1 \text{ if } n \text{ is odd}$$

$$= |1 \pm 1 - 0| = 1 \quad \text{if } n \text{ is even}$$

If I take $\varepsilon = \frac{1}{2}$ which is < 1
ie $\varepsilon < 1$

we have $|f(x_2) - f(x_1)| = 1 > \varepsilon$

& $|x_2 - x_1| = \left|\frac{x_2^2 - x_1^2}{x_2 + x_1}\right| = \frac{\pi/2}{\sqrt{n+\frac{1}{2}\pi} + \sqrt{\frac{n\pi}{2}}}$

$$< \frac{\pi}{2\left\{2\sqrt{\frac{n\pi}{2}}\right\}} < \frac{\pi}{\sqrt{n\pi}} = \sqrt{\frac{\pi}{n}} < \varepsilon$$

So we got $|f(x_2) - f(x_1)| > \varepsilon$

when $|x_2 - x_1| < \varepsilon$

So Not uniform convergence

BUT BUT

If we could not remember

$x_1 = \sqrt{\frac{n\pi}{2}}$ $x_2 = \sqrt{(n+1)\frac{\pi}{2}}$

Then what??

Solve by basic

Let given $\varepsilon > 0$ (Arbit

$$|f(x_2) - f(x_1)| = |\sin x_2^2 - \sin x_1^2|$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$|f(x_2)-f(x_1)| = \left|2\cos\frac{x_1^2+x_2^2}{2}\sin\frac{x_2^2-x_1^2}{2}\right|$$

$$|\sin\theta| \leq \theta$$

$$|\cos\theta| \leq 1$$

$$|ab| < |a||b|$$

$$|f(x_2)-f(x_1)| \leq 2\left|\cos\frac{x_1^2+x_2^2}{2}\right|\left|\sin\left(\frac{x_2^2-x_1^2}{2}\right)\right|$$

$$\leq 2\times 1\times \frac{x_2^2-x_1^2}{2}$$

$$\leq |(x_2+x_1)(x_2-x_1)| \leq |x_1+x_2|\,|x_2-x_1|$$

My aim:

Given $\varepsilon > 0$, get (x_1, x_2) s.t

$|f(x_2)-f(x_1)| < \varepsilon$

s.t $|x_2-x_1| < \delta$

in the given Range (Note)

observe : inverval is $[0, \infty[$

$\downarrow$

Let us say interval is fixed at right side

ie $[0, \lambda]$ (say)

Then x_1 can have max λ value
x_2 can have max λ value

$x_1 + x_2$ can have max 2λ value

Then $|x_1 + x_2| \leq 2\lambda$

$$|f(x_2) - f(x_1)| \leq |x_1 + x_2| |x_2 - x_1|$$
$$\leq 2\lambda |x_2 - x_1|$$

Given $\varepsilon > 0$
what I do is, I will reduce my x_1, x_2 value such that

$$|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$$

I will choose x_1, x_2, such that $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

Now when $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

we get
$$|f(x_2) - f(x_1)| < 2\lambda |x_2 - x_1|$$
$$< 2\lambda \cdot \frac{\varepsilon}{2\lambda} = \varepsilon$$

So if $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

we get $|f(x_2) - f(x_1)| < \varepsilon$

Hence we get x_1, x_2 such that $|x_2 - x_1| < \delta$

where $\delta = \frac{\varepsilon}{2\lambda}$

and satisfy codetion for uniform continuity

So we found that if

the internal is $[0, \lambda[$

if is Uniform Continous

Come to our problem

$[0, \infty[$

Given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{2\lambda}$

Here (I donot know) what is that λ, because it extends to ∞, but No definite boundary,

So I could not choose δ

Given $\varepsilon > 0$, I cannot get δ

So Not Uniform Continous