LINEAR ALGEBRA

: CSE - 2018 :

1(a). Let A be a 3x2 matrix and B be a 2x3 matrix. Show that $C = A \cdot B$ is a singular matrix.

$\longrightarrow$ $A_{3\times 2} \cdot B_{2\times 3} = (AB)_{3\times 3}$.

Let ranks of A, B and AB be r_1, r_2 and r respectively.

Then, since A is a 3x2 matrix, $r_1 \leq 2$,
since B is a 2x3 matrix, $r_2 \leq 2$.
Since AB is a 3x3 matrix, $r \leq 3$.

Now, Since rank of $A \to r_1 \leq 2$, $\exists$ a non-singular matrix P such that

$PA = \begin{bmatrix} G \\ O \end{bmatrix}$ where G is a $r_1 \times 2$ matrix and O is a $(3-r_1) \times 2$ matrix.

Post multiplying B on both sides

$PAB = \begin{bmatrix} G \\ O \end{bmatrix} B$. —— ①

Now Rank of PAB = Rank of AB [Since multiplication with a non-singular matrix does not change the rank of a matrix]

$\therefore \rho(PAB) = r$.

But, the number of non-zero rows in $\begin{bmatrix} G \\ O \end{bmatrix} \cdot B$ cannot exceed r_1 since, there are only 'r_1' non-zero rows in $\begin{bmatrix} G \\ O \end{bmatrix}$.

$\therefore \rho(PAB) \leq r_1$. $\Rightarrow$ $r \leq r_1$. —— ②

~~Similarly~~: Now

$\rho(AB) = \rho((AB)^T) = \rho(B^T A^T) \leq \rho(B^T) = \rho(B)$ [Since $\rho(AB) \leq \rho(A)$ from ②].

$\therefore \rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$

$\therefore \rho(AB) \leq 2$.

Since AB is a 3x3 matrix and rank of AB is less than 3, therefore AB is a singular matrix.

① ⑤

1(b) Express basis vectors $e_1=(1,0)$ and $e_2=(0,1)$ as linear comb^{n}s of $\alpha_1=(2,-1)$ and $\alpha_2=(1,3)$

$\longrightarrow$ Let $e_1=(1,0) = a\alpha_1 + b\alpha_2 = a(2,-1)+b(1,3)$

$\Rightarrow (1,0) = (2a,-a)+(b,3b) = (2a+b,-a+3b)$

comparing both sides, we have $2a+b=1$

$-a+3b=0 \Rightarrow 3b=a.$

$\therefore 2a+b=1 \Rightarrow 2(3b)+b=1$

$\Rightarrow 7b=1 \Rightarrow b=\frac{1}{7}.$ $\quad 3b=a \Rightarrow \frac{3}{7}=a.$

$\therefore \boxed{e_1 = \left(\frac{3}{7}\right)\alpha_1 + \left(\frac{1}{7}\right)\alpha_2}.$

Let $e_2 = c\alpha_1 + d\alpha_2 \Rightarrow (0,1) = c(2,-1)+d(1,3)$

$\Rightarrow (0,1) = (2c+d, -c+3d)$

comparing both sides, $2c+d=0$ and $-c+3d=1$

$d=-2c \longrightarrow -c+3d=1$

$\Rightarrow -c+3(-2c)=1$

$\rightarrow -7c=1$

$\rightarrow c=-\frac{1}{7}.$

we have $d=-2c$ & $c=-\frac{1}{7}$

$\Rightarrow d=\frac{2}{7}$

$\therefore (0,1) = \frac{-1}{7}(2,-1) + \frac{2}{7}(1,3) \Rightarrow \boxed{e_2 = \frac{-1}{7}\alpha_1 + \frac{2}{7}\alpha_2}$

2(a): Show that if A and B are similar $n\times n$ matrices, then they have the same eigen values.

$\longrightarrow$ If A & B are similar, then $\exists$ a non-singular matrix P such that $B=P^{-1}AP$.

Now, the characteristic equation of B is given by

$|B-\lambda I|=0 \Rightarrow |B-\lambda I| = |P^{-1}AP-\lambda I|$ [since $B=P^{-1}AP$]

$$\begin{aligned}\Rightarrow |B-\lambda I| &= |P^{-1}AP - \lambda P^{-1}PI| = |P^{-1}AP - P^{-1}\lambda I P| \\ &= |P^{-1}(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P| \quad \left[\text{since } |P^{-1}| = \frac{1}{|P|}\right] \\ &= \frac{1}{|P|}\,|A-\lambda I|\,|P| \\ &= |A-\lambda I|\end{aligned}$$

$\therefore$ Characteristic equation of A and B are the same.

Therefore, the eigen values of A and B are same.

3(a). For the system of linear equations
$$x+3y-2z=-1$$
$$5y+3z=-8$$
$$x-2y-5z=7$$
determine which of the following statements are true & which are false.

(i) The system has no solution.
(ii) The system has a unique solution.
(iii) The system has a infinitely many solutions.

$\longrightarrow$ Given system of equations can be expressed as $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & 3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & 3 \\ 1 & -2 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

Let Augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 1 & -2 & -5 & 7 \end{array}\right]$

<u>Conditions for the three cases</u>:

(i) System has no solution if $\rho(A) \neq \rho(A|B)$
(ii) System has a unique solution if $\cancel{\rho(A) \neq 0}$ $\rho(A) = \rho(A|B)$ and is equal to the number of unknowns.
(iii) System has infinitely many solutions if $\rho(A) = \rho(A|B)$ and is less than the number of unknowns.

<u>Now</u> $[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 1 & -2 & -5 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & -5 & -3 & 8 \end{array}\right]$ $R_3 \to R_3 - R_1$

$R_3 \to R_3 + R_2$ $\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ echelon form.

$\therefore$ Echelon form of $[A|B]$ has 2 non-zero rows.
Also, A has 2 non-zero rows.
$\therefore \rho(A|B) = \rho(A) = 2 < 3 =$ No. of unknowns.

$\therefore$ The given system has infinitely many solutions
The <u>statement (iii)</u> is true and others are false.