Mains Test Series – 2020

COMMON TEST [TEST–14 for Batch–I] & [TEST–6 for Batch–II]

Answer Key (Paper–II)

full Syllabus

1(a) Let G be an infinite cyclic group. Prove that e is the only element in G of finite order.

Sol'n. Since G is an infinite cyclic group,

$G = (a)$ for some $a \in G$

Such that ord(a) is infinite.

Now, assume that there is an element $b \in G$

such that ord$(b) = m$ and $b \neq e$.

Since $G = (a)$

$b = a^k$ for some $k \geq 1$.

Hence, $e = b^m$

$= (a^k)^m$

$= a^{km}$

Hence, ord(a) divides km by theorem

(Let a be an element in a group G. If $a^m = e$, then ord(a) divides m.)

a contradiction since ord(a) is infinite.

Thus, e is the only element in G of finite order.

1(b) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal of R if and only if $\frac{R}{P}$ is an integral domain.

Sol^n: Given that R is the commutative ring and P is an ideal of R.

Let $\frac{R}{P}$ be an Integral domain.

we now prove that P is a prime ideal of R.

i.e $a, b \in R$ and $ab \in P \Rightarrow a \in P$ or $b \in P$

Now for any $a, b \in R$ and $ab \in P$

$\Rightarrow P+ab = P \quad (\because a \in P \Leftrightarrow P+a = P)$

$\Rightarrow (P+a) \cdot (P+b) = P+0$

$\Rightarrow (P+a) = (P+0)$ (or)

$P+b = P+0 \quad (\frac{R}{P}$ is an ID$)$

$\Rightarrow a \in P$ (or) $b \in P \quad (P+a = P \Leftrightarrow a \in P)$

$\therefore$ P is a prime ideal of R.

conversely suppose that

let P be a prime ideal of R.

we now prove that $\frac{R}{P}$ is an Integral Domain.

$P+a, P+b \in \frac{R}{P}; \; a, b \in R$

$\Rightarrow (P+a)(P+b) = P+0$

$\Rightarrow P+ab = P+0 \quad (\because S+a = S+b \Rightarrow a-b \in S)$

$\Rightarrow ab-0 \in P$

$\Rightarrow a \in P$ (or) $b \in P \quad (\because P$ is a prime ideal$)$

$\Rightarrow P+a = P+0$ (or) $P+b = P+0$

$\therefore \frac{R}{P}$ has no zero divisors.

$\Rightarrow P+a = P+0$ (or) $P+b = P+0$

$\therefore \frac{R}{P}$ has no zero divisors.

and hence $\frac{R}{P}$ is an Integral Domain.

1(c) Show that $\int_0^t \sin x\, dx = 1-\cos t$, by using Riemann integral.

Sol^n: Since $f(x)=\sin x$ is bounded and continuous on $[0,t]$, therefore, f is integrable on $[0,t]$.

Consider a partition

$P=\{0=x_0, x_1, x_2 \ldots, x_n=t\}$ of $[0,t]$ dividing it into n equal sub-intervals, each of length $\frac{t-0}{n}=\frac{t}{n}$

So that $\|P\|\to 0$ as $n\to\infty$.

Also $x_r = 0+\frac{rt}{n}=\frac{rt}{n}$ and $\delta_r=\frac{t}{n}$, $r=1,2\ldots n$.

$$\therefore \int_0^t f(x)\,dx = \lim_{\|P\|\to 0}\sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} f(x_r)\delta_r \quad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} f\left(\frac{rt}{n}\right)\cdot\frac{t}{n}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n} \frac{t}{n}\sin\frac{rt}{n}$$

$$= \lim_{n\to\infty}\frac{t}{n}\left[\sin\frac{t}{n}+\sin\frac{2t}{n}+\cdots+\sin\frac{nt}{n}\right]$$

$$= \lim_{n\to\infty}\frac{t}{n}\,\frac{\sin\left(\frac{t}{n}+\frac{n-1}{2}\cdot\frac{t}{n}\right)\sin\left(\frac{n}{2}\cdot\frac{t}{n}\right)}{\sin\frac{t}{2n}}$$

$$\left[\because \sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right]$$

$$= \lim_{n\to\infty} 2\cdot\frac{\frac{t}{2n}}{\sin\frac{t}{2n}}\cdot\sin\frac{t}{2}\left(\frac{2}{n}+\frac{n-1}{n}\right)\sin\frac{t}{2}$$

$$= \lim_{n\to\infty} 2\cdot\frac{t/2n}{\sin t/2n}\cdot\sin\frac{t}{2}\left(1+\frac{1}{n}\right)\sin\frac{t}{2} = 2\times1\times\sin\frac{t}{2}\times\sin\frac{t}{2}$$

$$= 2\sin^2\frac{t}{2} = \underline{\underline{1-\cos t}}.$$

$$\left[\because \text{as } n\to\infty, \theta=\frac{t}{2n}\to 0 \ \& \ \lim_{\theta\to 0}\frac{\theta}{\sin\theta}=1.\right]$$

1(d) Use Cauchy's theorem and/or Cauchy's integral formula to evaluate the following integrals.

(i) $\displaystyle\int_{|z-2|=2} \frac{\log(z+1)}{z-3}$ (ii) $\displaystyle\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$.

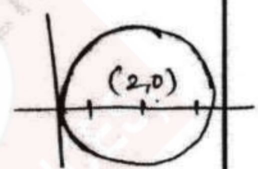
1(d) Use Cauchy's theorem and/or Cauchy's integral formula to evaluate the following integrals.

(i) $\displaystyle\int_{|z-2|=2} \frac{\log(z+1)}{z-3}\,dz$ (ii) $\displaystyle\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$

Sol'n: (i) Given that $\displaystyle\int_{|z-2|} \frac{\log(z+1)}{z-3}\,dz$

Comparing the given integral with $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz$

where $C : |z-2| = 2$
Centre $(2,0)$ and $r=2$



Since $f(z) = \log(z+1)$ is analytic in $|z-2| = 2$ and $z_0 = 3$ is a point inside $|z-2| = 2$

$\therefore$ we can apply Cauchy's integral formula

$$\int_{|z-2|=2} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \int_{|z-2|=2} \frac{\log(z+1)}{z-3} = 2\pi i \log(3+1)$$
$$= 2\pi i \log 4$$
$$= 4\pi i \log 2.$$

(ii) Given $\displaystyle\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$

$$= \frac{1}{5}\int_{|z|=5} \left[\frac{9}{z-4} - \frac{4}{z+1}\right] dz$$

$$= \frac{9}{5}\int_{|z|=5} \frac{1}{z-4}\,dz - \frac{4}{5}\int_{|z|=5} \frac{1}{z+1}\,dz$$

Since $f(z)=1$ is analytic in $|z|=5$ and $z_0 = 4$, $z_0 = -1$ are a points inside $|z|=5$

$\therefore$ By Cauchy's integral formula

$$\int_{|z|=5} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz = \frac{9}{5}(2\pi i)\,1 - \frac{4}{5}(2\pi i)(1)$$

$$= 2\pi i$$

1(e) Find an optimal solution to the following LPP be computing all basic solutions and then finding one that maximizes the objective function:

$2x_1+3x_2-x_3+4x_4=8, \quad x_1-2x_2+6x_3-7x_4=-3$

$x_1, x_2, x_3, x_4 \geq 0, \quad \text{Max } Z = 2x_1+3x_2+4x_3+7x_4.$

Sol^n: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total no. of basic solutions $^4C_2 = 6$.

The characteristics of the various basic solutions are given below.

No. of basic Sol^n	Basic variables	Non-basic variables	Values of basic variables	Is the sol^n feasible? (Are all $x_i \geq 0$?)	Value of Z	Is the sol^n optimal?
1	x_1, x_2	$x_3=0, x_4=0$	$2x_1+3x_2=8$ $x_1-2x_2=-3$ $\therefore x_1=1, x_2=2$	Yes	8	NO
2	x_1, x_3	$x_2=0, x_4=0$	$2x_1-x_3=8$ $x_1+6x_3=-3$ $\therefore x_1=-14/13, x_3=\frac{45}{13}$	NO	—	—
3	x_1, x_4	$x_2=0, x_3=0$	$2x_1+8x_4=8$ $x_1-7x_4=-3$ $\therefore x_1=22/9, x_4=7/9$	Yes	10.3	NO
4	x_2, x_3	$x_1=0, x_4=0$	$3x_2-x_3=8$ $-2x_2+6x_3=-3$ $\therefore x_2=\frac{45}{16}, x_3=\frac{7}{16}$	Yes	10.2	NO
5	x_2, x_4	$x_1=0, x_3=0$	$3x_2+4x_4=8, 2x_2+7x_4=3$ $\therefore x_2=\frac{132}{39}, x_4=-7/13$	NO	—	—
6	x_3, x_4	$x_1=0, x_2=0$	$-x_3+4x_4=8, 6x_3-7x_4=-3$ $\therefore x_3=\frac{44}{17}, x_4=45/17$	Yes	28.9	Yes

Hence the optimal basic feasible solution is $x_1=0, x_2=0, x_3=44/17, x_4=45/17$ and the maximum value of $Z=28.9$.

2(a) (i) Let G be a group of order 24. what are the possible orders for the subgroups of G.

(ii) Let $\beta = (1,2,3)(1,4,5)$. write β^{99} in cycle form.

(iii) Let $\beta = (1,5,3,2,6)(7,8,9)(4,10) \in S_{10}$. Given β^n is a 5-cycle. what can you say about n.

Sol^n: (i) write 24 as product of distinct primes.

Hence, $24 = (3)(2^3)$.

by theorem Let G be a finite group and let H be a subgroup of G. Then ord(H) divides ord(G)

Hence, we need only to find all divisors of 24.

By theorem [Let n be a +ve integer, and write $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ where p_i's are distinct prime numbers and each α_i is a +ve integer ≥ 1. Then number of all +ve divisors of n (including 1 and n) is $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$].

number of all divisors of 24 is $(1+1)(3+1) = 8$.

Hence, Possible orders for the subgroups of G are: 1, 3, 2, 4, 8, 6, 12, 24.

(ii) First, write β as disjoint cycles.

Hence $\beta = (1,4,5,2,3)$

Thus $\text{Ord}(\beta) = 5$.

Since 5 divides 100,

we have $\beta^{100} = \beta\beta^{99} = e$

Thus $\beta^{99} = \beta^{-1} = (3,2,5,4,1)$

(iii) Since β^n is a 5-cycle, we conclude that $\text{ord}(\beta^n) = 5$.

Now since β is in disjoint cycles,

we conclude that $\text{Ord}(\beta) = \text{lcm}[5,3,2] = 30$

we know that $\text{ord}(\beta^n) = 30/\gcd(n,30)$

$= 5$

Thus $\gcd(n,30) = 6$.

Thus $n = 6m$ for some $m \geq 1$

such that $\gcd(m,5) = 1$.

So, $n = 6, 12, 18, 24, 36 \ldots$ so all n such that

$\gcd(n,30) = 6$.

2(b) (i) Prove that a countable union of countable sets is countable

(ii) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

Sol^n: Consider the sets $A_i = \{a_{1i}, a_{2i}, a_{3i}, \ldots\}, i = 1, 2, 3\ldots$

Each A_i, $i = 1, 2, 3 \ldots$ is countable.

The kth element of A_i is a_{ki}.

The elements of the countable union $\bigcup_{i=1}^{\infty} A_i$ of the sets A_i's can be listed as $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}, \ldots$ (the order has been taken according to the sum $i+j=k$, $k=2,3,\ldots$ i,j being the suffices of the element $a_{ij} \in A_j$). The one-one correspondence between the elements of $\bigcup_{i=1}^{\infty} A_i$ and the set of positive integers is given by

$$\begin{array}{cccccccccc} a_{11} & a_{12} & a_{21} & a_{13} & a_{22} & a_{31} & a_{14} & a_{23} & a_{32} & a_{41} \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \ldots \end{array}$$

Hence, the set $\bigcup_{i=1}^{\infty}$ is countable.

(ii) Let $\varepsilon = \frac{1}{2}$ and δ be any +ve number such that for $n > \pi/\delta^2$

$$\left| \sqrt{\frac{n\pi}{2}} - \sqrt{\frac{(n+1)\pi}{2}} \right| < \delta$$

$\therefore$ taking $x_1 = \sqrt{\frac{n\pi}{2}}$ and $x_2 = \sqrt{\frac{(n+1)\pi}{2}}$, as any two points of the interval $[0, \infty[$

$$|f(x_1) - f(x_2)| = \left| \sin \frac{n\pi}{2} - \sin \frac{(n+1)\pi}{2} \right| = 1 > \varepsilon$$

$$|x_1 - x_2| < \delta$$

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

2(c)(i) Find zeros and discuss of singularity of the function $f(z) = \frac{(z-2)}{z^2} \sin\left(\frac{1}{z-1}\right)$

(ii) By integrating $e^{iz}/(z-ai)$, $(a>0)$, round a suitable contour prove that

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\, dx = 2\pi e^{-a}.$$

Sol^n: (i) poles of $f(z)$ are obtained by putting denominator equal to zero i.e, $z^2=0$.

This $\Rightarrow$ $z = 0, 0$

$\Rightarrow$ $z=0$ is a pole of order two

Zeros of $f(z)$ are given by

$$(z-2)\sin\left(\frac{1}{z-1}\right) = 0$$

This $\Rightarrow$ $z=2$ and $\frac{1}{z-1} = n\pi \Rightarrow z=2,\ z = \frac{1}{n\pi}+1$

Thus $z=2$ is a simple zero. The limit point of the zeros given by $z = 1+\frac{1}{n\pi}$, where $n = \pm 1, \pm 2 \ldots$ is $z=1$, therefore $z=1$ is isolated essential singularity

finally (i) $z=2$ is a simple zero.

(ii) $z=1$ is isolated essential singularity

(iii) $z=0$ is a pole of order two.

2(c)(ii) Sol^n: Consider $\int_C \frac{e^{iz}}{z-ai}\,dz = \int_C f(z)\,dz$

By residue theorem, we get

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \sum R^+ \quad \text{---} (1).$$

By Jordan's lemma $\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$

$\therefore$ from (1), we get as $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+ \quad \text{---} (2)$$

Now the only simple pole of $f(z)$ in the upper half plane is at $z = ai$

Residue (at $z = ai$) $= \lim_{z\to ai} (z-ai) f(z) = \lim_{z\to ai} e^{iz} = e^{-a}$

$\therefore$ from (2), $\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i e^{-a}$.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\cos x + i \sin x)(x+ai)}{(x-ai)(x+ai)}\,dx = 2\pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(x\cos x - a\sin x) + i(x\sin x + a\cos x)}{x^2+a^2}\,dx = 2\pi i e^{-a}$$

Equating imaginary parts on bothsides, we get

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi e^{-a}.$$

3(a) Let R be the set of all real valued continuous functions on $[0,1]$. Show that R is a commutative ring with respect to point-wise addition and point-wise multiplication. Is R an integral domain?

Sol^n: Let R be the set of all real valued continuous functions on $[0,1]$.

Let $f, g \in R$,

Define
$$(f+g)(x) = f(x) + g(x)$$
$$(f \cdot g)(x) = f(x)\, g(x). \quad \forall x \in [0,1]$$

Now it is easy to verify that R is a ring, in which the additive identity is the zero function given by
$$0(x) = 0 \quad \forall x \in [0,1]$$
and
$$(f+0)(x) = f(x) + 0(x)$$
$$= f(x) + 0$$
$$= f(x) \quad \forall x \in [0,1].$$
$\therefore f + 0 = f$. for all $f \in R$.

The additive inverse of $f \in R$ is $f_1 : [0,1] \to R$ such that
$$f_1(x) = -f(x) \quad \forall x \in [0,1].$$
$\therefore$ we have $f + f_1 = 0$.

Further $(fg)(x) = f(x) \cdot g(x) = g(x) f(x) = (gf)(x) \quad \forall x \in [0,1]$

Hence $f \cdot g = g \cdot f \quad \forall f, g \in R$.

Hence R is a commutative ring with unity, $I \in R$, where $I(x) = x$ $\forall x \in [0,1]$.

The ring R of real-valued continuous functions on $[0,1]$ is not an integral domain.

Consider,

$$f(x) = \begin{cases} x, & \text{if } x \le 0 \\ 0, & \text{if } x > 0 \end{cases}$$

and $$g(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x > 0. \end{cases}$$

Then $f \neq 0 \in R$ and $g \neq 0 \in R$

but $fg = 0$.

3(b) Show that $\int_2^\infty \frac{\cos x}{\log x}\, dx$ is conditionally convergent.

Sol'n: Let $\phi(x) = \frac{1}{\log x}$, $f(x) = \cos x$

$$\left|\int_2^x \cos x\, dx\right| = |\sin x - \sin 2| \le |\sin x| + |\sin 2| \le 2$$

So that $\int_2^x \cos x\, dx$ is bounded for all $x \ge 2$.

Also $\phi(x) = \frac{1}{\log x}$ is a monotonic decreasing function tending to 0 as $x \to \infty$,

Hence by Dirichlet's test $\int_2^\infty \frac{\cos x}{\log x}\, dx$ is convergent.

For absolute convergence consider

$$I = \int_2^\infty \left|\frac{\cos x}{\log x}\right| dx = \int_2^{3\pi/2} \frac{|\cos x|}{\log x}\, dx + \int_{3\pi/2}^{5\pi/2} \frac{|\cos x|}{\log x}\, dx + \cdots$$

$$+ \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \frac{|\cos x|}{\log x}\, dx + \cdots$$

$$\therefore I = \int_{\pi/2}^{2} \frac{|\cos x|}{\log x}\, dx + \int_2^{3\pi/2} \frac{|\cos x|}{\log x}\, dx + \cdots + \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \frac{|\cos x|}{\log x}\, dx + \cdots - \int_{\pi/2}^{2} \frac{|\cos x|}{\log x}\, dx$$

$$= \sum_{r=1}^{\infty} \int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \frac{|\cos x|}{\log x}\, dx - \int_{\pi/2}^{2} \frac{\cos x}{\log x}\, dx$$

Now, $$\int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \frac{|\cos x|}{\log x}\, dx \ge \frac{1}{\log (2r+1)\pi/2} \left| \int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \cos x\, dx \right|$$

$$= \frac{1}{\log(2r+1)\pi/2}\left|\sin(2r+1)\pi/2 - \sin(2r-1)\pi/2\right|$$

$$= \frac{|2(-1)^r|}{\log(2r+1)\pi/2} = \frac{2}{\log(2r+1)\pi/2}$$

$$\therefore \quad I \geq \sum_{r=1}^{\infty} \frac{2}{\log(2r+1)\pi/2} - \int_{\pi/2}^{2} \frac{|\cos x|}{\log x}\,dx$$

But $\sum_{x=2}^{\infty} \frac{1}{\log x}$ is divergent and $\int_{\pi/2}^{2} \frac{|\cos x|}{\log x}\,dx$ is a proper integral.

Hence $I = \int_{2}^{\infty} \frac{|\cos x|}{\log x}\,dx$ is divergent.

And so $\int_{2}^{\infty} \frac{\cos x}{\log x}\,dx$ is Conditionally convergent.

Ques: 3(c)) Solve the following linear programming problem by simplex method.

Max. $Z = -2x_1 - x_2$; subject to $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$ and $x_1, x_2 \geq 0$.

Solution:- Given; Max. $Z = -2x_1 - x_2$

subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
and $x_1, x_2 \geq 0$

By introducing slack, surplus and Artificial variable above problem converted into standard form: Max $Z = -2x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to
$$3x_1 + x_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - S_1 + 0S_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 4$$
and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

For IBFS, put $x_1 = x_2 = S_1 = 0$

$\therefore A_1 = 3$, $A_2 = 6$ and $S_2 = 4$.

$\boxed{\text{Max } Z = -9M}$

Iteration-1	$C_j \rightarrow$	-2	-1	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	A_1	A_2	b	θ
$-M$	A_1	(3)	1	0	0	1	0	3	$1 \rightarrow$
$-M$	A_2	4	3	-1	0	0	1	6	1.5
0	S_2	1	2	0	1	0	0	4	4
$Z_j = \sum C_B a_{ij}$		$-7M$	$-4M$	M	0	$-M$	$-M$		
$C_j = C_j - Z_j$		$-2+7M \uparrow$	$4M-1$	$-M$	0	0	0		

$x_1 \to$ incoming variable ; A_1 – outgoing variable.
(3) $\to$ Key element ; convert it to 1, by diving row 1 by 3; Also, make zero the same column of different rows; Omit A_1 column.
i.e $R_1 \to R_1/3$; $R_2 \to R_2 - 4R_1$; $R_3 \to R_3 - R_1$

Iteration: 2	$C_j \to$	-2	-1	0	0	-M		
C_B	Basis	x_1	x_2	S_1	S_2	A_2	b	θ
-2	x_1	1	1/3	0	0	0	1	3
-M	A_2	0	(5/3)	-1	0	1	2	1.2 →
0	S_2	0	5/3	0	1	0	3	1.8
$Z_j = \Sigma C_B a_{ij}$		-2	$-\frac{5}{3}M - \frac{2}{3}$	+M	0	-M	-2M-2	
$C_j = c_j - Z_j$		0	$\frac{5}{3}M - \frac{1}{3}$ ↑	-M	0	0		

x_2 – incoming variable , $A_2 \to$ outgoing variable
Key element (5/3). $R_2 \to \frac{3}{5}R_2$, $R_1 \to R_1 - \frac{1}{3}R_2$
A_2 Column – omitted. $R_3 \to R_3 - \frac{5}{3}R_2$

Iteration 3	$C_j \to$	-2	-1	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	b
-2	x_1	1	0	1/5	0	3/5
-1	x_2	0	1	-3/5	0	6/5
0	S_2	0	0	1	1	
$Z_j = \Sigma C_B a_{ij}$		-2	-1	+1/5	0	-12/5
$C_j = c_j - Z_j$		0	0	-1/5	0	

Since ; all $C_j \le 0$; obtimality obtained at
$x_1 = 3/5$, $x_2 = 6/5$

Max $Z = -2x_1 - x_2 = -12/5$ $\boxed{\therefore \text{Max } Z = -\frac{12}{5}}$

4(a). Find a polynomial of degree 3 irreducible over the ring of integers, Z_3, mod 3. Use it to construct a field having 27 elements.

Sol'n: $f(x) = x^3 + x^2 + 2$ is an irreducible polynomial in $Z_3[x]$.

Since $f(x)$ is of degree 3, if $f(x)$ is reducible, then one of the factors must be of degree 1. This implies that $f(x)$ has a root in Z_3. But

$$f(0) = 2$$
$$f(1) = 1$$
$$f(2) = 1$$

Let $I = \langle (x^3 + x^2 + 2) \rangle$ be an ideal of $Z_3[x]$. Since $f(x)$ is irreducible I is a maximal ideal in $Z_3[x]$ and hence $Z_3[x]/I$ is a field. Every element in $Z_3[x]/I$ can be written uniquely in the form $a_0 + a_1 x + a_2 x^2 + I$, for some $a_0, a_1, a_2 \in Z_3$. Now it is easy to see that there are 27 elements in this form.

4(b) Show that the series $\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$, is uniformly convergent on any interval, $[a,b]$, $0<a<b$, but only pointwise on $[0,b]$.

Sol^n: Let

$$f_n(x) = \frac{x}{(nx+1)\{(n-1)x+1\}}$$

$$= \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$\therefore$ n^{th} partial sum $S_n(x) = \sum_{r=1}^{n} f_r(x) = 1 - \frac{1}{nx+1}$

$\therefore$ The sum function $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Thus f is discontinuous on $[0,b]$ and therefore is not uniform on $[0,b]$, it is only pointwise.

when $x \neq 0$, let $\varepsilon > 0$ be given

$$|S_n(x) - f(x)| = \frac{1}{nx+1} < \varepsilon$$

when $n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$, but $\frac{1}{x}\left(\frac{1}{\varepsilon}-1\right)$ decreases with x, let its maximum value $\frac{1}{a}\left(\frac{1}{\varepsilon}-1\right) = m_0$ (independent of x) on $[a,b]$.

Thus for all $x \in [a,b]$, $\exists$ an integer $m\,(>m_0)$,

such that $|S_n(x) - f(x)| < \varepsilon$, for $n \geq m$

Hence, the series converges uniformly on $[a,b]$, $0<a<b$.

4(c) Show that the function f defined by
$$f(z)=\begin{cases}0 & \text{if } z=0\\ \exp(-1/z^4) & \text{if } z\neq 0\end{cases}$$ is not continuous at the origin but satisfies the C-R equations at the origin.

Sol'n: Here $u+iv = e^{-(x+iy)^{-4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$
$$= e^{\frac{-1}{(x+iy)^4}}$$

$$u+iv = e^{-\frac{1}{(x^2+y^2)^4}\left\{(x^4+y^4-6x^2y^2)-i4xy(x^2-y^2)\right\}}$$

$$\Rightarrow u = e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}}\cos 4xy(x^2-y^2)$$

and $$v = -e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}}\sin 4xy(x^2-y^2)$$

At $z=0$,
$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{e^{-x^{-4}}}{x}=0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{-e^{-y^{-4}}}{y}=0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0}{x}=0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0}{y}=0$$

Hence Cauchy - Riemann conditions are satisfied at $z=0$.

But $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \frac{e^{-z^{-4}}}{z} = \lim_{z\to 0}\frac{e^{-r^{-4}}}{re^{-i\pi/4}}$$
$$=\infty,\ \text{if } z\to 0 \text{ along } z=re^{i\pi/4}$$

showing that $f'(z)$ does not exist at $z=0$.

Hence $f(z)$ may not be analytic at $z=0$.

4(d) There are four engineers available for designing four projects. Engineer E_1 is not compitent to design project P_3. Given the time estimate required by each engineer to design a given project in the table. Find an assignment which minimise the total time.

Project

		P_1	P_2	P_3	P_4
Engineer	E_1	10	3	unsuitable	8
	E_2	4	13	1	5
	E_3	3	7	2	10
	E_4	8	6	1	9

Sol^n^:

By using Hungarian Method the optimal assignment time will be

$E_1 \to P_2$, $E_2 \to P_4$, $E_3 \to P_1$, $E_4 \to P_3$

Minimum time $= 3+5+3+1$
$= 12$

5(a) (i) Form a partial differential equation by eliminating function f from $z = y^2 + 2f(1/x + \log y)$

(ii) solve $(x-y)p + (x+y)q = 2xz$

Sol'n: (i) Given that $z = y^2 + 2f\left(1/x + \log y\right)$ —— ①

Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right]\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow -x^2\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ②}$$

and $$\frac{\partial z}{\partial y} = 2y + 2f'\left[\frac{1}{x} + \log y\right]\left(\frac{1}{y}\right)$$

$$\Rightarrow y\frac{\partial z}{\partial y} - 2y^2 = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ③}$$

from ② & ③ we get

$$-x^2\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y} - 2y^2$$

$$\Rightarrow x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2y^2$$

$$\Rightarrow x^2p + yq = 2y^2$$

which is the required partial differential equation.

5(a)(ii) Sol^n: Here the Lagrange's subsidiary equations are

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \quad \text{——} \ (1).$$

Taking the first two fractions of (1), we have.

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+(y/x)}{1-(y/x)} \quad \text{——} \ (2)$$

Let $y/x = v$ i.e. $y = xv$ —— (3)

from (3), $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$ —— (4)

using (3) and (4), (2) $\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$\Rightarrow \quad = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{1+v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1+v^2} - \frac{2v}{1+v^2}\right)dv = \frac{2dx}{x}$$

Integrating, $2\tan^{-1}v - \log(1+v^2) = 2\log x - \log c_1$

$$\log x^2 - \log(1+v^2) - \log c_1 = 2\tan^{-1}v$$

$$\log\left\{x^2(1+v^2)/c_1\right\} = 2\tan^{-1}v$$

$$\Rightarrow x^2(1+v^2) = c_1 e^{2\tan^{-1}v}$$

$$\Rightarrow x^2\left[1+\frac{y^2}{x^2}\right] = c_1 e^{2\tan^{-1}(y/x)} \quad \text{as } v = y/x \text{ by (3)}$$

$$\Rightarrow (x^2+y^2)e^{-2\tan^{-1}(y/x)} = c_1 \quad \text{——} \ (5)$$

Choosing $1, 1, -\frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{dx+dy-\frac{1}{z}dz}{(x-y)+(x+y)-\frac{1}{z}(2xz)} = \frac{dx+dy-\frac{1}{z}dz}{0}$$

$\Rightarrow dx+dy-\frac{1}{z}dz = 0$ so that $x+y-\log z = c_2$ —— (6)

from (5) & (6) the required general solution is $\phi\left(x+y-\log z, (x^2+y^2)e^{-2\tan^{-1}(y/x)}\right) = 0$.

5(b) Solve $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

Sol^n: Given equation is $(D+3D')(D-2D')z = x^2\sin(x+y)$ — ①

Its auxiliary equation is $(m+3)(m-2)=0$ so that $m=-3,2$

$\therefore$ C.F $= \phi_1(y-3x)+\phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{(D+3D')(D-2D')}x^2\sin(x+y) = \frac{1}{D+3D'}\left\{\frac{1}{D-2D'}x^2\sin(x+y)\right\}$$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx, \text{ where } c=y+2x$$

$$= \frac{1}{D+3D'}\int x^2\sin(c-x)\,dx$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \int 2x\cos(c-x)\,dx\right], \text{ integrating by parts.}$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x)+\int 2\sin(c-x)\,dx\right\}\right]$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)+2x\sin(c-x) - 2\cos(c-x)\right]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y)+2x\sin(x+y)\right], \text{ as } c=y+2x$$

$$= \int\left[(x^2-2)\cos(x+c'+3x)+2x\sin(x+c'+3x)\right]dx,$$
where $c'=y-3x$

$$= \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\frac{\sin(4x+c')}{4}dx + 2\int x\sin(4x+c')\,dx$$

(Integrating by parts 1st Integral & keeping the Second integral unchanged)

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx$$

$$= \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x\cos(4x+c')}{4} + \int\frac{\cos(4x+c')}{4}dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x),$$

$$[\because c' = y-3x]$$

$$= \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y),$$ on simplification

The required solution is $z = C.F + P.I$

$$z = \phi_1(y-3x) + \phi_2(y+2x) + \left[\left(\frac{x^2}{4}\right) - \left(\frac{13}{32}\right)\right]\sin(x+y) - \frac{3x}{8}\cos(x+y).$$

5(C) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t(sec)	0	10	20	30	40	50	60	70	85
f(cm/sec²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67.

Sol'n: Since acceleration is defined as the rate of change of velocity, we have

$$\frac{dv}{dt} = a \quad (\text{or}) \quad v = \int_0^{80} a\,dt$$

using Simpson's $\frac{1}{3}$rd rule.

we have

$$v = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

$\therefore$ The required velocity is given by

$$v = 3.0861 \text{ km/sec}$$

5(d) (i) Simplify the expression $A = xy + \bar{x}z + x\bar{y}z(xy+z)$

(ii) Simplify the Boolean expression $Y = \overline{A \cdot B} + \overline{\bar{A} + B}$ prepare truth table to show that the simplified expression is correct.

Sol^n: (i) $A = xy + \bar{x}z + x\bar{y}z(xy+z)$

$= xy + \bar{x}z + xxy\bar{y}z + x\bar{y}zz$

$= xy + \bar{x}z + x\bar{y}z$ $\quad (\because zz = z$ and $y\bar{y} = 0)$

$= xy + \bar{x} + \bar{z} + x\bar{y}z$

$= xy + \bar{z} + \bar{x} + x\bar{y}z$ (by commutative law)

$= xy + \bar{z} + \bar{x} + \bar{y}z$

$= \bar{x} + y + \bar{z} + \bar{y}$

$= 1$

(ii) $Y = \overline{A \cdot B} + \overline{\bar{A} + B}$

$= \bar{A} + \bar{B} + \bar{\bar{A}} \cdot \bar{B}$

$= \bar{A} + \bar{B}$

$= \overline{A \cdot B}$ (using Demorgan's theorem)

A	B	$A \cdot B$	$\overline{A \cdot B}$	$\bar{A}$	$\bar{A}+B$	$\overline{\bar{A}+B}$	$\overline{A \cdot B} + \overline{\bar{A}+B}$
0	0	0	1	1	1	0	1
0	1	0	1	1	1	0	1
1	0	0	1	0	0	1	1
1	1	1	0	0	1	0	0

5(e). Show that the M.I. of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is the perpendicular from the centre on the tangent.

Sol: Let the equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

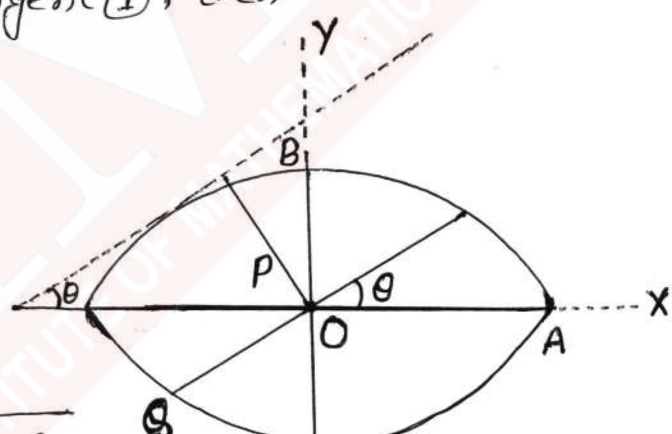
$\therefore$ equation of the tangent to the ellipse is

$$y = mx + \sqrt{(a^2m^2+b^2)}. \quad \text{—— (1)}$$

where $m=\tan\theta$, if tangent is inclined at an angle θ to the axis of x.

If p is the perpendicular from the centre (0,0) on the tangent (1), then

$$p=\frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(1+m^2)}}$$



$$=\frac{\sqrt{(a^2\tan^2\theta+b^2)}}{\sqrt{(1+\tan^2\theta)}}$$

$$=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}. \quad \text{—— (2)}$$

M.I. of the ellipse about the diameter PQ which is parallel to the tangent

$$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$$

$$=\frac{1}{4}Mb^2\cos^2\theta+\frac{1}{4}Ma^2\sin^2\theta-0$$

$$=\frac{1}{4}M(b^2\cos^2\theta+a^2\sin^2\theta)=\frac{1}{4}Mp^2, \text{ from (2)}.$$

$\therefore$ M.I. of the ellipse about the tangent

= Its M.I. about the parallel line through C.G. 'O' + M.I. of mass M at O about the tangent.

$$= \frac{1}{4}Mp^2 + Mp^2$$

$$= \frac{5}{4}Mp^2.$$

6(a) (i) Find a complete integral of $z^2 = pqxy$.

(ii) Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right) + \frac{\partial^2 z}{\partial y^2} = 0$ to Canonical form and hence solve it.

Sol'n: (i) The given equation is

$$f(x, y, z, p, q) = z^2 - pqxy = 0 \quad \text{---} (1)$$

$\therefore$ charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}}$$

$$\frac{dp}{-pqy + 2pz} = \frac{dq}{-pqx + 2qz} = \frac{dz}{-p(-qxy) - q(-pxy)} = \frac{dx}{qxy} = \frac{dy}{pxy} \quad \text{---} (2)$$

Each fraction of (2).

$$= \frac{x\,dp + p\,dx}{x(-pqy + 2pz) + pqxy} = \frac{y\,dq + q\,dy}{y(-pqx + 2qz) + pqxy}$$

$$\Rightarrow \frac{x\,dp + p\,dx}{2pxz} = \frac{y\,dq + q\,dy}{2qyz} \Rightarrow \frac{d(xp)}{xp} = \frac{d(yq)}{yq}$$

Integrating, $\log xp = \log yq + \log a^2 \Rightarrow xp = a^2 yq$ — (3)

Solving (1) & (2) for p and q, $p = (az)/x$ and $q = z/(ay)$

$\therefore dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)\,dy$

$\Rightarrow (1/z)\,dz = (a/x)\,dx + (1/ay)\,dy$

Integrating $\cdot \log z = a\log x + \frac{1}{a}\log y + \log b$

$\Rightarrow z = x^a y^{1/a} b.$

6(a)(iii) Sol^n: Re-writing the given equation $r+2s+t=0$ —①.

Comparing ① with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=1$, $S=2$, $T=1$ so that $S^2-4RT=0$.

Showing that ① is parabolic. The λ-quadrate equation reduces to $\lambda^2+2\lambda+1=0$ so that $\lambda=-1,-1$ (equal roots)

The corresponding characteristic equation is $\frac{dy}{dx}-1=0$.

Integrating $x-y=c$

Choose $u=x-y$ and $v=x+y$ —②.

where we have chosen $v=x+y$ in such a manner that u and v are independent functions as verified below.

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=1.1+1.1=2\neq 0$$

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, using ② —③

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, using ② —④

from ③ and ④ $\frac{\partial}{\partial x}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y}=-\frac{\partial}{\partial u}+\frac{\partial}{\partial v}$ —⑤

$\therefore r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$, by ③ & ⑤

$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$ —⑥

$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(-\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$, by ④ & ⑤

$=-\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$

$=\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 v}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$ —⑦.

and $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(-\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$, by (4) & (5)

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \quad \text{——} \quad (8)$$

Using (6), (7) and (8) in (1), the required Canonical form is

$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \text{(or)} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 \quad \text{——} \quad (9)$$

To find solution: Integrating (9) partially w.r.t 'v' we get

$\dfrac{\partial z}{\partial v} = \phi(u)$, ϕ being an arbitrary function —— (10)

Integrating (10) partially w.r.t 'v', $z = \int \phi(u)\,dv + \psi(u)$

$$z = v\phi(u) + \psi(u) = (x+y)\phi(x-y) + \psi(x-y)$$

which is the desired solution, ϕ, ψ being arbitrary functions.

6(b)(i) Evaluate the $1/\sqrt{14}$ (Correct to four decimal places) by Newton's iteration method.

Sol'n: Taking $N = 14$, from the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{(14x_n)}\right)$$

Since an approximate value of

$$\frac{1}{\sqrt{14}} = \frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25,$$

we take $x_0 = 0.25$

$$x_1 = \frac{1}{2}\left[x_0 + (14x_0)^{-1}\right] = \frac{1}{2}\left[0.25 + (14\times 0.25)^{-1}\right]$$

$$= 0.26785$$

$$x_2 = \frac{1}{2}\left[x_1 + (14x_1)^{-1}\right] = \frac{1}{2}\left[0.26785 + (14\times 0.26785)^{-1}\right]$$

$$= 0.2672618$$

$$x_3 = \frac{1}{2}\left[x_2 + (14x_2)^{-1}\right]$$

$$= \frac{1}{2}\left[0.2672618 + (14\times 0.2672618)^{-1}\right]$$

$$= 0.2672612$$

Since $x_2 = x_3$ upto 4 decimal places,

we take $\frac{1}{\sqrt{14}} = 0.2673$.

6(b)(ii) Solve the following equations by Gauss-Seidel method.
$83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$.

Sol'n: By Gauss Seidel method given system can be written as

$$x^{k+1} = \frac{1}{83}\left[95 - 11y^k + 4z^k\right]$$

$$y^{k+1} = \frac{1}{52}\left[104 - 7x^{k+1} - 13z^k\right]$$

$$z^{k+1} = \frac{1}{29}\left[71 - 3x^{k+1} - 8y^{k+1}\right]$$

where $k = 0, 1, 2 \ldots$

Now taking $x^{(0)} = 0$, we obtain the following iterations

Here $x^{(0)} = (\underset{x}{0}, \underset{y}{0}, \underset{z}{0})$.

First iteration:

$k = 0$: $x^{(1)} = \frac{95}{83} = 1.1445$

$y^{(1)} = \frac{1}{52}\left[104 - 7(1.1445) - 13(0)\right] = 1.84592$

$z^{(1)} = \frac{1}{29}\left[71 - 3(1.1445) - 8(1.84592)\right] = 1.82065$

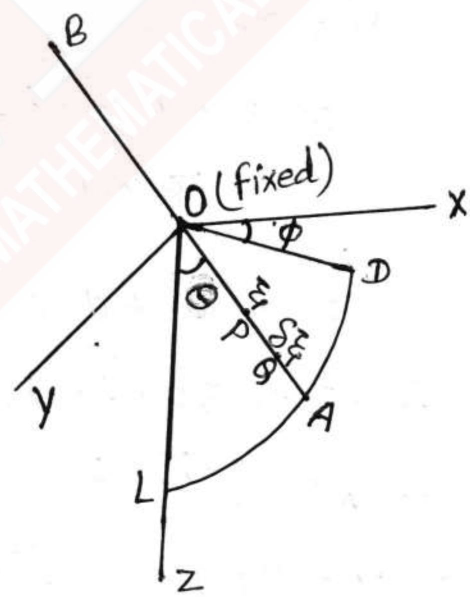
k=	x	y	z
1	0.98768	1.41188	1.95662
2	1.05176	1.36926	1.96175
3	1.0576	1.3672	1.9617
4	1.0579	1.3672	1.9617

$\therefore$ The solution is $x = 1.057$, $y = 1.3672$, $z = 1.9617$

6(c) A uniform rod, of mass $3m$ and length $2l$, has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{(2ng/l)}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is

$$\cos^{-1}\left[\sqrt{(n^2+1)}-n\right], \text{ and will then rise again.}$$

Sol: Let AB be the rod of mass $3m$ and length $2l$. The middle point O of the rod is fixed and a mass m attached at the extremity A.



Initially let the rod rest along OX in the plane of the paper. Let a line OY perpendicular to the plane of the paper and a line OZ perpendicular to OX in the plane of the paper be taken as axes of y and z respectively. At time t, let the rod turn through an angle ϕ to OX i.e., the plane

OAL containing the rod and z axis make an angle ϕ with x-z plane. And let θ be the inclination of the rod with OZ at this time t. If P is a point of the rod at a distance $OP = \xi$, from O then coordinates of P are given by

$$x_p = \xi \sin\theta \cos\phi,$$
$$y_p = \xi \sin\theta \sin\phi, \quad z_p = \xi \cos\theta.$$

$\therefore$ v_p and v_A are the velocity of the point P and A respectively, then

$$v_p^2 = \dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2$$
$$= (\xi \cos\theta \cos\phi \dot{\theta} - \xi \sin\theta \sin\phi \dot{\phi})^2 + (\xi \cos\theta \sin\phi \dot{\theta} + \xi \sin\theta \cos\phi \dot{\phi})^2 + (-\xi \sin\theta \dot{\theta})^2$$

$$= \xi(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$\therefore$ At A, $\xi = OA = l$,

$$\therefore v_A^2 = l^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

Let $PQ = \delta\xi$ be an element of the rode at P, then mass of this element, $\delta m = \frac{3m}{2l}\delta\xi$.

$\therefore$ K.E. of the element $PQ = \frac{1}{2}\delta m \cdot v_p^2$

$$= \frac{1}{2}\xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \cdot \frac{3m}{2l}\delta\xi$$

$\therefore$ K.E. of the rod $AB = \frac{3m}{4l}\int_{-l}^{l} \xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\,d\xi$

$$= \frac{1}{2}m(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\,l^2.$$

and K.E. of mass m at $A = \frac{1}{2}m v_A^2$

$$= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$\therefore$ The total kinetic energy of the system

$T =$ K.E. of the rod + K.E. of the particle

$$= ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

The work function $W = mg \cdot z_A = mgl\cos\theta$.

$\therefore$ Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e, $\frac{d}{dt}(2ml^2\dot{\theta}) - 2ml^2\dot{\phi}^2 \sin\theta\cos\theta = -mgl\sin\theta$

or $\quad 2l\ddot{\theta} - 2l\dot{\phi}^2\sin\theta\cos\theta = -g\sin\theta$ ——— (1)

And Langrange's ϕ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

i.e, $\frac{d}{dt}(2ml^2\dot{\phi}\sin^2\theta) = 0$

or $\quad \frac{d}{dt}(\dot{\phi}\sin^2\theta) = 0$ ——— (2)

Integrating (2) we get $\dot{\phi}\sin^2\theta = C$ (constant)

But initially when $\theta = (\pi/2)$

($\because$ Rod was horizontal),

$$\dot{\phi} = \sqrt{(2ng/l)}$$

$$\therefore C = \sqrt{(2ng/l)}$$

$$\therefore \dot{\phi}\sin^2\theta = \sqrt{(2ng/l)} \quad \text{——— (3)}$$

Substituting the value of $\dot{\phi}$ from (3) in (1), we get

$$2l\ddot{\theta} - 2l\frac{2ng}{l\sin^4\theta}\sin\theta\cos\theta = -g\sin\theta$$

or $\quad 2l\ddot{\theta} - 4ng\cot\theta\operatorname{cosec}^2\theta = -g\sin\theta$ ——— (4)

Multiplying both sides by $\dot{\theta}$ and integrating, we get

$$l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta + D.$$

But initially when $\theta = \pi/2$, $\dot{\theta} = 0$

$$\therefore D = 0$$

$$\therefore l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta \quad \text{——— (5)}$$

The rod will fall till $\dot{\theta} = 0$

i.e., $2ng\cot^2\theta = g\cos\theta$

or $\quad 2n\cos^2\theta - \cos\theta\sin^2\theta = 0$

or $\quad \cos\theta(2n\cos\theta - \sin^2\theta) = 0.$

$\therefore$ either $\cos\theta = 0$ i.e., $\theta = (\pi/2)$

or $2n\cos\theta - \sin^2\theta = 0$

i.e; $2n\cos\theta - (1-\cos^2\theta) = 0$

or $\cos^2\theta + 2n\cos\theta - 1 = 0$

$\therefore \cos\theta = \dfrac{-2n \pm \sqrt{(4n^2+4)}}{2}$

or $\cos\theta = -n + \sqrt{(n^2+1)}$, Leaving negative sign.

$\because$ negative value of $\cos\theta$ is inadmissible as θ can not be obtuse.

$\therefore \theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right]$.

From (4) we have

$$2l\ddot{\theta} = \frac{g(4n\cos\theta - \sin^4\theta)}{\sin^3\theta} \quad \text{———— (6)}$$

When $\cos\theta = -n + \sqrt{(n^2+1)}$,

$$\cos^2\theta = 2n^2 + 1 - 2n\sqrt{(n^2+1)}$$

$\therefore 4n\cos\theta - \sin^4\theta = 4n\cos\theta - (1-\cos^2\theta)^2$

$$= 4n\left[-n + \sqrt{(n^2+1)}\right] - \left[-2n^2 + 2n\sqrt{(n^2+1)}\right]^2$$

$$= -4n^2 + 4n\sqrt{(n^2+1)} - 4n^4 - 4n^2(n^2+1) + 8n^3\sqrt{(n^2+1)}$$

$$= -8n^2 - 8n^4 + 4n\sqrt{(n^2+1)} + 8n^3\sqrt{(n^2+1)}$$

$$= 4n\sqrt{(n^2+1)}\left[-2n\sqrt{(n^2+1)} + 1 + 2n^2\right]$$

$= 4x\sqrt{(x^2+1)}\left[-x+\sqrt{(x^2+1)}\right]^2$

which is positive.

$\because$ θ is acute angle.

$\therefore$ $\sin^3\theta$ is also positive.

$\therefore$ when $\theta = \cos^{-1}\left[\sqrt{(x^2+1)} - x\right]$, from (6), we see that $\ddot{\theta}$ is positive. Hence from this position the rod will rise again.

7(a) A taut string of length l has its ends $x=0$ and $x=l$ fixed. The midpoint is taken to a small height h and released from rest at time $t=0$. Find the displacement function $y(x,t)$.

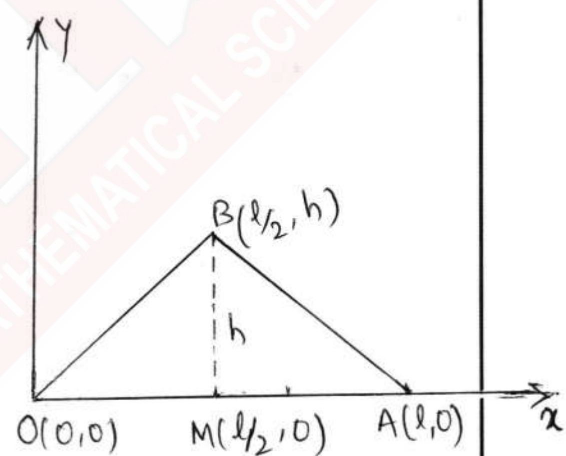
Sol'n: The displacement function $y(x,t)$ is the solution of of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\left(\frac{\partial^2 y}{\partial t^2}\right) \quad \text{—— (1)}$$

Subject to the boundary Conditions:

$y(0,t) = y(l,t) = 0$ for all $t \geq 0$.

Initial position of the string at $t=0$ is made up of two straight line segments OB & BA as shown in the figure and the string is released from rest.



The equation of OB is given by

$$y-0 = \frac{h-0}{(l/2)-0}(x-0)$$

$\Rightarrow y = \dfrac{2hx}{l}$ for $0 \leq x \leq l/2$

The equation of BA is given by

$$y-0 = \frac{h-0}{(l/2)-l}(x-l)$$

$\Rightarrow y = \{2h(l-x)\}/l$ for $l/2 \leq x \leq l$.

Hence the initial displacement is given by

$$= u(x,0) = f(x) = \begin{cases} (2hx)/l, & 0 \leq x \leq l/2 \\ \{2h(l-x)\}/l, & l/2 \leq x \leq l \end{cases} \quad \text{—— (3A)}$$

and the initial velocity $= (\partial u/\partial t)_{t=0} = 0$ —— (3B)

The solution of ① satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \text{—— (4)}$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad \text{—— (5)}$$

$$\therefore E_n = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right], \text{ using (3A)}$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4h}{l^2}\left[(x)\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$
$$+ \frac{4h}{l^2}\left[(l-x)\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$
$$+ \frac{4h}{l^2}\left(\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= (8h/n^2\pi^2)\sin(n\pi/2)$$

$$= \begin{cases}\{8(-1)^{m+1}h\}/(2m-1)^2\pi^2, & \text{if } n = 2m-1 \text{ (odd) } \& \ m = 1,2,3\ldots \\ 0, & \text{if } n = 2m \text{ (even) and } m = 1,2,3\ldots\end{cases}$$

[Note that for $n = 2m-1$, $\sin(n\pi/2) = \sin(2m-1)(\pi/2) = \sin(m\pi - \pi/2)$
$= \sin m\pi \cos\pi/2 - \cos m\pi \sin\pi/2 = (-1)^{m+1}$]

Substituting the above value of E_n in ④, the required displacement function given by $$y(x,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}.$$

7(b). (i) using Runge – Kutta method of fourth order, solve

$$\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2} \text{ with } y(0)=1 \text{ at } x=0.2, 0.4.$$

(ii) Convert 1011101.1011 to octal and then to hexadecimal.

Sol'n: (i) we have $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

To find $y(0.2)$:

Here $x_0=0$, $y_0=1$, $h=0.2$

$k_1 = hf(x_0, y_0) = (0.2)f(0,1) = 0.2$

$k_2 = hf(x_0+h/2, y_0+k_1/2) = (0.2)f(0.1, 1.1) = 0.19672$

$k_3 = hf(x_0+h/2, y_0+k_2/2) = (0.2)f(0.1, 1.09836) = 0.1967$

$k_4 = hf(x_0+h, y_0+k_3) = (0.2)f(0.2, 1.1967) = 0.1891$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.19599$

$\therefore y(0.2) = y_0+k = 1.196.$

To find $y(0.4)$:

Here $x_1=0.2$, $y_1=1.196$, $h=0.2$

$k_1 = hf(x_1, y_1) = 0.1891$

$k_2 = hf(x_1+h/2, y_1+\frac{k_1}{2}) = (0.2)f(0.3, 1.2906) = 0.1795$

$k_3 = hf(x_1+h/2, y_1+k_2/2) = (0.2)f(0.3, 1.2858) = 0.1793$

$k_4 = hf(x_1+h, y_1+k_3) = (0.2)f(0.4, 1.3753) = 0.1688$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.1792$

$\therefore y(0.4) = y_1+k = 1.196+0.1792 = 1.3752$

7(b)(ii) Sol'n: Binary number can be converted into equivalent octal number by making groups of three bits starting from LSB (least significant digit) and moving towards MSB (most significant digit) for integer part of the number and then replacing each group of three bits by its octal representation.

For fractional part, the grouping of three bits are made starting from the binary point.

$$(1011101.1011)_2 = (\underset{1}{\underline{001}}\ \underset{3}{\underline{011}}\ \underset{5}{\underline{101}} . \underset{5}{\underline{101}}\ \underset{4}{\underline{100}})_2$$

$$= (135.54)_8$$

Now, octal number can be converted to equivalent hex number by converting it to equivalent binary and then to hex number.

$$(135.54) = (001\ 011\ 101 . 101100)_2$$

$$= (0\underline{0101}\ \underline{1101} . \underline{1011})_2$$

$$= (5D.B)_{16}$$

7(c) Prove that the velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = r^{1/2}\cos(\theta/2)$ are solutions of the Laplace equation and the velocity potential $\phi_3 = (x^2-y^2) + r^{1/2}\cos(\theta/2)$ satisfies $\nabla^2\phi_3 = 0$.

Sol: The Laplace's equation in cartesian and cylindrical polar co-ordinates are given by

$$\nabla^2\phi_1 = \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0$$

and $$\nabla^2\phi_2 = \frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r} = 0$$

Here $\partial^2\phi_1/\partial x^2 = 2$ and $\partial^2\phi_1/\partial y^2 = -2$.

So $$\nabla^2\phi_1 = 2-2 = 0 \qquad \text{(1)}$$

Next,

$$\frac{\partial\phi_2}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2},$$

$$\frac{\partial^2\phi_2}{\partial r^2} = -\frac{1}{4}r^{-3/2}\cos\frac{\theta}{2},$$

$$\frac{\partial^2\phi_2}{\partial\theta^2} = -\frac{r^{1/2}}{4}\cos\frac{\theta}{2}.$$

$$\therefore \nabla^2\phi_2 = -\frac{1}{4r^{3/2}}\cos\frac{\theta}{2} - \frac{1}{4r^{3/2}}\cos\frac{\theta}{2} + \frac{1}{2r^{3/2}}\cos\frac{\theta}{2} = 0 \qquad \text{(2)}$$

① and ② show that ϕ_1 and ϕ_2 satisfy Laplaces equation.

Now,

$$\phi_3 = (x^2-y^2) + r^{1/2}\cos(\theta/2) = \phi_1 + \phi_2$$

$$\Rightarrow \nabla^2\phi_3 = \nabla^2(\phi_1+\phi_2)$$

$$= \nabla^2\phi_1 + \nabla^2\phi_2$$

$$= 0 + 0 = 0. \quad \{\text{by ① and ②}\}.$$

Hence ϕ_3 satisfies $\nabla^2\phi_1 = 0$.

8(a) A Square plate is bounded by the lines $x=0, y=0, x=10$ & $y=10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,10) = x(10-x)$ while the other three faces are kept at $0°C$. Find the steady state temperature in the plate.

Sol'n: The steady state temperature $u(x,y)$ is the solution of Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{——} \quad (1).$$

subject to boundary conditions

$u(0,y) = u(10,y) = 0, \quad 0 \le y \le 10$ —— (2)

$u(x,0) = 0, \quad 0 \le x \le 10$ —— (3a)

and $u(x,10) = 10x - x^2, \quad 0 \le x \le 10$ —— (3b)

Suppose (1) has a solution of the form

$u(x,y) = X(x)\,Y(y)$ —— (4)

Substituting this value of u in (1), we get

$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$ —— (5)

Since x and t are independent variables, each side of (5) must be equal to the constant, say μ.

Then (5) gives $X'' - \mu X = 0$ —— (6)

and $Y'' + \mu Y = 0$ —— (7)

Using (2), (4) gives $X(0)\,Y(y) = 0$ & $X(10)\,Y(y) = 0$

giving $X(0) = 0$ and $X(10) = 0$ —— (8)

where $Y(y) \ne 0$, since otherwise $u \equiv 0$ which doesnot satisfy 3(b)

we now solve (6) under B.C. (8). Three cases arise.

Case(i) Let $\mu = 0$. The solution of (6) is

$X(x) = Ax + B$ —— (9)

Using B.C. (8), we get $A = B = 0$

so that $X(x) = 0$. This leads $u \equiv 0$ which doesnot satisfy 3(b). So we reject $\mu = 0$.

Case(ii): Let $\mu = \lambda^2$, $\lambda \ne 0$. Then solution of (6) is

$X(x) = A e^{\lambda x} + B e^{-\lambda x}$ —— (10)

Using B.C (8), we get $A = B = 0$ so that

$X(x) = 0$ and hence, $u \equiv 0$.

So we reject $\mu = \lambda^2$.

Case (2): Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{--- (11)}$$

Using B.C. (8), (11) gives $0 = A$ and $0 = A\cos\lambda(10) + B\sin\lambda(10)$
$\Rightarrow \sin\lambda(10) = 0$, $B \neq 0$.

since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which doesnot satisfy 3(5).

Now, $\sin\lambda a = 0 \Rightarrow \lambda(10) = n\pi$, $n = 1, 2, 3 \ldots$
$\Rightarrow \lambda = \frac{n\pi}{10}$; $n = 1, 2, \ldots$ --- (12)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{10}\right)$ --- (13)

Using $\mu = -\lambda^2 = -\frac{n^2\pi^2}{(10)^2}$, (7) becomes $Y'' - \left(\frac{n^2\pi^2}{10^2}\right)Y = 0$ --- (14)

whose general solution is

$$Y_n(y) = C_n e^{n\pi y/10} + D_n e^{-\frac{n\pi y}{10}} \quad \text{--- (15)}$$

Using 3(9), (5) gives $0 = X(x)\,Y(0)$ so that $Y(0) = 0$, where we have taken $X(x) \neq 0$.
for otherwise we will get $u \equiv 0$ which doesnot satisfy 3(5).

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ --- (16)

Putting $y = 0$ in (15) and using (16), we have

$0 = C_n + D_n \Rightarrow D_n = -C_n$. Then (15) reduces to

$$Y_n(y) = C_n\left(e^{n\pi y/10} - e^{-n\pi y/10}\right)$$
$$= 2\sinh(n\pi y/10) \quad \text{--- (17)}$$

$\therefore u_n(x,y) = X_n(x)\,Y_n(y) = E_n \sin\left(\frac{n\pi x}{10}\right)\sinh\left(\frac{n\pi y}{10}\right)$ --- (18)

are solutions of (1), satisfying (2) and 3(a).
Here $E_n = 2B_nC_n$.

In order to satisfy condition 3(b), now consider more general solution given by

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right)\sinh\left(\frac{n\pi y}{10}\right) \quad \text{— (18)}$$

put $y=10$ in (18) and using 3(b), we get

$$u(x,10) = \underset{= f(x)}{10x - x^2} = \sum_{n=1}^{\infty} E_n \sinh(n\pi)\sin\left(\frac{n\pi x}{10}\right).$$

which is the half range fourier sine series of $f(x)$ in $(0,10)$

Hence, we have

$$E_n \sinh(n\pi) = \frac{2}{10}\int_0^{10} (10x - x^2)\sinh\left(\frac{n\pi x}{10}\right)dx$$

$$= \frac{1}{5\sinh n\pi}\left[(10x-x^2)\left(\frac{-10}{n\pi}\cos\frac{n\pi x}{10}\right) - (10-2x)\left(\frac{-100}{n^2\pi^2}\right)\sin\frac{n\pi x}{10} + 2\left(\frac{1000}{n^3\pi^3}\right)\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5\sinh n\pi}\left[\frac{-2000(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3}\right]$$

$$= \frac{400\,(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}$$

$$\therefore E_n = \begin{cases} 0, & \text{if } n = 2m \text{ and } m = 1,2,3\ldots \\ \dfrac{800\,\text{cosech}\,(2m-1)\pi}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1,\ m = 1,2\ldots \end{cases}$$

$\therefore$ from (18), we have

$$u(x,y) = \frac{800}{\pi^3}\sum_{m=1}^{\infty} \frac{\text{cosech}\,(2m-1)\pi}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{10}\sinh\frac{(2m-1)\pi y}{10}$$

which is the required temperature.

8(b) (i) Convert hexadecimal number 2647 to octal.
(ii) Convert hexadecimal number 4A.67 to binary.
(iii) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved.

Sol'n: (i) $(2647)_{16} = (0010\ 0110\ 0100\ 0111)_2$

$$= (\underset{2}{\underline{010}}\ \underset{3}{\underline{011}}\ \underset{1}{\underline{001}}\ \underset{0}{\underline{000}}\ \underset{7}{\underline{111}})_2$$

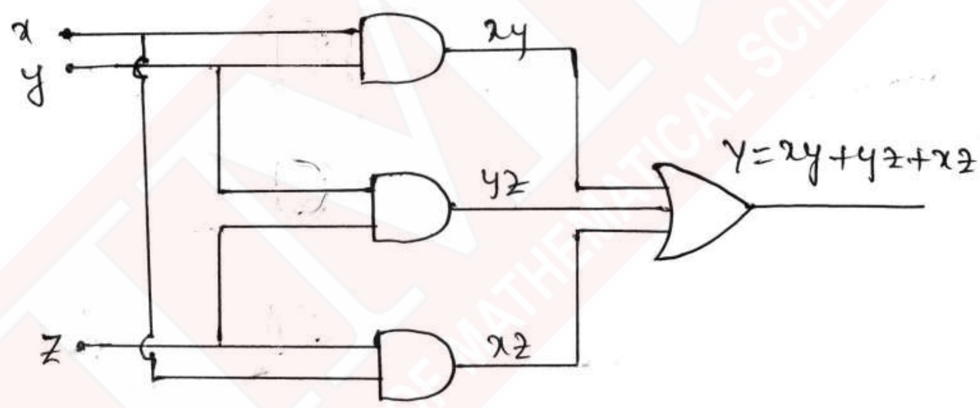
$$= (23107)_8$$

(ii) $(4A.67)_{16} = (01001010 \cdot 0110\ 0111)_2$

(iii) Let x, y, z be the responses of the three members.

x	y	z	Y	Minterms
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{x}yz$
1	0	0	0	
1	0	1	1	$x\bar{y}z$
1	1	0	1	$xy\bar{z}$
1	1	1	1	xyz

$$Y = \bar{x}yz + x\bar{y}z + xy\bar{z} + xyz$$
$$= \bar{x}yz + x\bar{y}z + xy \qquad (\because z+\bar{z}=1)$$
$$= y(x+\bar{x}z) + x\bar{y}z$$
$$= y(x+z) + x\bar{y}z$$
$$= yx + yz + x\bar{y}z$$
$$= yx + z(y+\bar{y}x)$$
$$= yx + z(y+x)$$
$$= yx + zy + zx$$



8(C). A sphere of radius R, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, show that the pressure at the surface of the sphere at time t is

$$\Pi + \frac{1}{2}\rho\left\{ \frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2 \right\}.$$

Sol: Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and equation of continuity is $x^2 v = F(t)$ so that

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

Hence $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

as ρ is constant.

Integrating w.r.t. 'x',

$$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \quad \text{——— (1)}$$

Boundary conditions are:

when $x = \infty$, $p = \Pi$, $v = 0$, ——— (2)

when $x = R$, $p = P$, $v = \dot{R}$. ——— (3)

Also $x^2 v = F(t) = R^2 \dot{R}$

$\therefore \; F'(t) = 2R(\dot{R})^2 + R^2 \ddot{R}$

Subjecting ① to the conditions ② and ③,

$$0 + 0 = -\frac{\Pi}{\rho} + C \quad \text{and}$$

$$\frac{-F'(t)}{R} + \frac{1}{2}(\dot{R})^2 = -\frac{P}{\rho} + C$$

$$= -\frac{P}{\rho} + \frac{\Pi}{\rho}$$

$$\frac{P}{\rho} = \frac{\Pi}{\rho} - \frac{1}{2}(\dot{R})^2 + \frac{1}{R}\left[2R(\dot{R})^2 + R^2\ddot{R}\right]$$

$$P = \Pi + \frac{1}{2}\rho\left[3(\dot{R})^2 + 2R\ddot{R}\right] \quad \text{④}$$

Now

$$\frac{d^2R^2}{dt^2} + (\dot{R})^2 = \frac{d}{dt}(2R\dot{R}) + \dot{R}^2$$

$$= 2\dot{R}^2 + 2R\ddot{R} + \dot{R}^2$$

Now ④ becomes

$$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \dot{R}^2\right]$$

or $$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$