IFOS - 2012

Q1 $\displaystyle\int_{2-i}^{4+i} (x+y^2-ixy)\,dz$ along AB where $A(2,-1)$, $B(4,1)$

equation of AB

$$\frac{y-(-1)}{x-2} = \frac{1+1}{4-2} = \frac{2}{2} = 1$$

$$\Rightarrow \quad y+1 = x-2$$

$$\Rightarrow \quad \boxed{y = x-3}$$

$\therefore$ Parametric eqn $\quad x = t \quad y = t-3$

$\qquad dx = dt \quad dy = dt$

$$\therefore \; dz = dx + i\,dy = dt + i\,dt = (1+i)\,dt$$

$$\int_{AB} \left[t + (t-3)^2 - i\,t(t-3)\right](1+i)\,dt \quad \text{and } 2 \le t \le 4$$

THINGS TO DO

$$= \int_2^4 (1+i)\left[(t^2-5t+9)+i(3t-t^2)\right]dt$$

$$= (1+i)\left|\left(\frac{t^3}{3}-\frac{5t^2}{2}+9t\right)+i\left(\frac{3t^2}{2}-\frac{t^3}{3}\right)\right|_2^4$$

$$=(1+i)\left[\left(\frac{64-8}{3}-\frac{5}{2}(16-4)+9(4-2)\right)+i\left(\frac{3}{2}(16-4)-\frac{1}{3}(64-8)\right)\right]$$

$$=(1+i)\left[\frac{56}{3}-30+18\right]+(1+i)i\left[18-\frac{56}{3}\right]$$

$$=(1+i)\left[\frac{20}{3}\right]+(-1+i)\left[-\frac{2}{3}\right]$$

$$=\left(\frac{20}{3}+\frac{2}{3}\right)+i\left(\frac{20}{3}-\frac{2}{3}\right)=\frac{22}{3}+6i$$

$$\therefore \int_{2-i}^{4+i}(x+y^2-ixy)\,dz=\frac{22}{3}+6i \text{ along } AB.$$

Given $u(x,y) = e^{-x}(x\cos y + y\sin y)$

$$u_x = e^{-x}\cos y - e^{-x}(x\cos y + y\sin y)$$

$$= e^{-x}[(1-x)\cos y - y\sin y]$$

$$u_{xx} = -e^{-x}[(1-x)\cos y - y\sin y] + e^{-x}(-\cos y)$$

$$= -e^{-x}[(2-x)\cos y - y\sin y]$$

$$u_y = \cancel{+} e^{-x}[-x\sin y + \sin y + y\cos y]$$

$$u_{yy} = e^{-x}[-x\cos y + \cos y + \cos y - y\sin y]$$

$$= e^{-x}[(2-x)\cos y - y\sin y]$$

$$\therefore u_{xx} + u_{yy} = -e^{-x}[(2-x)\cos y - y\sin y] + e^{-x}[(2-x)\cos y - y\sin y]$$

$$= 0$$

$\because u_{xx} + u_{yy} = 0 \quad \therefore u(x,y)$ is harmonic

Acc. to CR conditions

$$u_x = v_y = e^{-x}[(1-x)\cos y - y\sin y]$$

$$\therefore v_y = e^{-x}[(1-x)\cos y - y\sin y]$$

$$\Rightarrow V(x,y) = e^{-x} \int \left((1-x)\cos y + y \sin y\right) dy$$

$$V(x,y) = e^{-x}\left[(1-x)\sin y - y\int \sin y \, dy + \int \frac{dy}{dy}\int \sin y \, dy \, dy\right]$$

$$= e^{-x}\left[(1-x)\sin y + y\cos y - \int \cos y \, dy\right]$$

$$= e^{-x}\left[(1-x)\sin y + y\cos y - \sin y\right] + g(x)$$

$$V(x,y) = e^{-x}\left[-x\sin y + y\cos y\right]$$

$$V_x = e^{-x}\left[-\sin y\right] + e^{-x}(-1)\left[-x\sin y + y\cos y\right] + g'(x)$$

$$= e^{-x}\left[(x-1)\sin y - y\cos y\right] + g'(x)$$

Acc. to CR condition

$$V_x = -u_y$$

$$e^{-x}\left[(x-1)\sin y - y\cos y\right] + g'(x) = -e^{-x}\left[(1-x)\sin y + y\cos y\right]$$

$$\Rightarrow g'(x) = 0$$

$$\therefore g(x) = C$$

$$\Rightarrow V\cancel{(x,y) = e^{-x}\left[(1-x)\sin y\right.} +$$

$$V(x,y) = e^{-x}\left[-x\sin y + y\cos y\right] + C$$

THINGS TO DO

$u_x = \phi_1(x,y)$ and $u_y = \phi_2(x,y)$

then Using Milne's method

$$f(z) = \int \phi_1(z,0) - i\phi_2(z,0)\, dz$$

$$\phi_1(x,y) = e^{-x}[(1-x)\cos y - y\sin y]$$

$$\therefore \phi_1(z,0) = e^{-z}[(1-z)-0] = e^{-z}(1-z)$$

$$\phi_2(x,y) = e^{-x}[(1-x)\sin y + y\cos y]$$

$$\therefore \phi_2(z,0) = e^{-z}[(1-z)(0)+0] = 0$$

$$\therefore f(z) = \int e^{-z}(1-z)\,dz$$

$$= (1-z)\int e^{-z} - \int\left(\frac{d}{dz}(1-z)\int e^{-z}dz\right)dz$$

$$= (1-z)(-e^{-z}) - \int(-1)(-e^{-z})\,dz$$

$$= (z-1)e^{-z} - (-e^{-z}) + c$$

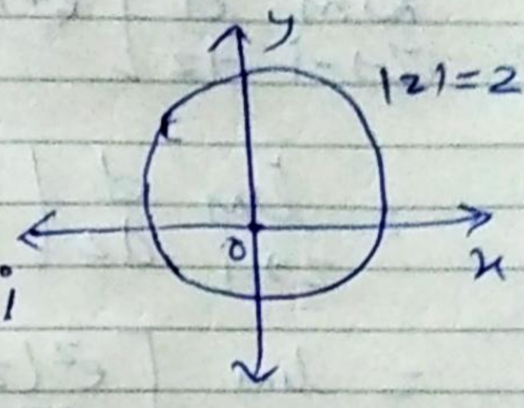
$$= e^{-z}(z-1+1) + c$$

$$= ze^{-z} + c$$

$$\therefore \boxed{f(z) = ze^{-z} + c}$$

Q3. $\int_C \frac{e^z-1}{z(z-1)(z+i)^2} dz$, C is circle $|z| = 2$.

Let $f(z) = \dfrac{e^z-1}{z(z-1)(z+i)^2}$



$f(z)$ has poles at $z = 0, 1, -i$
all lying inside C.

$\therefore$ Residue at $z = 0$ (~~Simple pole~~)(Removable singularity)

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z(e^z-1)}{z(z-1)(z+i)^2} = \frac{e^0-1}{(0-1)(0+i)^2}$$

$$= 0$$

$\therefore$ $Re(z=0) = 0$

Residue at $z = 1$ (Simple pole)

$$\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{(z-1)(e^z-1)}{z(z-1)(z+i)^2} = \frac{e-1}{1.(1+i)^2}$$

$$= \frac{e-1}{2i} \qquad \therefore Re(z=1) = \frac{e-1}{2i}$$

Residue at $z = -i$ (Pole of order 2)

$$\lim_{z \to -i} \frac{d}{dz}\left[(z+i)^2 f(z)\right] = \lim_{z \to -i}\left[\frac{d}{dz}\left(\frac{(z+i)^2\, e^z - 1}{z(z-1)(z+i)^2}\right)\right]$$

$$= \lim_{z \to -i} \frac{d}{dz}\left(\frac{e^z - 1}{z(z-1)}\right)$$

$$= \lim_{z \to -i}\left[\frac{e^z(z^2 - z) - (e^z - 1)(2z-1)}{z^2(z-1)^2}\right]$$

$$= \frac{e^{-i}(i^2 + i) - (e^{-i} - 1)(-2i-1)}{(-i)^2(-i-1)^2}$$

$$= \frac{e^{-i}(i-1) + e^{-i}(2i+1) - (2i+1)}{(-1)(1-1+2i)}$$

$$= \frac{e^{-i}(3i) - (2i+1)}{-2i}$$

Using Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\,[\text{Sum of residues}]$$

$$\therefore \int_{|z|=2} f(z)\,dz = 2\pi i\left[\text{Res}(0) + \text{Res}(1) + \text{Res}(-i)\right]$$

THINGS TO DO

$$= 2\pi i\left[0 + \frac{e-1}{2i} + \left(\frac{e^{-i}(3i) - (2i+1)}{-2i}\right)\right]$$

9

10

$$= \frac{2\pi i}{2i}\left[(e-1) + \frac{e^{-i}(3i) - 2i - 1}{-1}\right]$$

11

$$= \pi\left[(e-1) + 2i + 1 - e^{-i}(3i)\right]$$

12

$$= \pi\left[e + 2i - 3i\left(\cos(1) - i\sin(1)\right)\right]$$

1

$$= \pi\left[(e - 3\sin(1)) + i(2 - 3\cos(1))\right]$$

2

3

$$\boxed{\therefore \int_C \frac{e^z - 1}{z(z-1)(z+i)^2}\,dz = \pi\left[(e - 3\sin(1)) + i(2 - 3\cos(1))\right]}$$

4