2018

1. (5 b) Let $\bar{r}_1 = 3t\hat{i} + 3t^2\hat{j} + 3t^3\hat{k}$

Tangent $\Rightarrow \bar{T} = \frac{d\bar{r}_1}{dt} = 3\hat{i} + 6t\hat{j} + 9t^2\hat{k}$

For the line $y = z - x = 0 \Rightarrow x = z, \; y = 0$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{1} \Rightarrow \bar{r}_2 = \hat{i} + \hat{k}$$

Angle b/w $\bar{r}_2$ and $\bar{T} = \frac{\bar{r}_2 \cdot \bar{T}}{|\bar{r}_2| \cdot |\bar{T}|} = \frac{(3\hat{i} + 6t\hat{j} + 9t^2\hat{k})(\hat{i} + \hat{k})}{\sqrt{9 + 36t^2 + 36t^4} \cdot \sqrt{1+1}}$

$$\cos\theta = \frac{3 + 9t^2}{3\sqrt{1 + 4t^2 + 4t^4} \cdot \sqrt{2}} = \frac{1 + 3t^2}{(1 + 2t^2)\sqrt{2}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left[\frac{1 + 3t^2}{\sqrt{2}(1 + 2t^2)}\right]}$$

2. (6d) $\iint_S \left[(x+z)\,dy + (y+z)\,dz\,dx + (y+x)\,dx\,dy\right]$

$$\bar{F} = (x^2 + z)\hat{i} + (y + z)\hat{j} + (x + y)\hat{j}$$

By ~~Gauss~~ Divergence Theorem, $\iint \bar{F} \cdot \hat{n}\, dS = \iiint (\nabla \cdot \bar{F})\, dV$

$\nabla \cdot \bar{F} = 2$

$$\Rightarrow I = \iiint 2\,dV = 2 \times \frac{4}{3}\pi a^3 = \frac{8}{3}\pi a^3$$

3. (7b) $\bar{r} = a(u - \sin u)\hat{i} + a(1-\cos u)\hat{j} + bu\hat{k}$

$$\frac{d\bar{r}}{du} = a(1-\cos u)\hat{i} + a\sin u\hat{j} + b\hat{k}$$

$$\frac{d^2\bar{r}}{du^2} = a\sin u\hat{i} + a\cos u\hat{j}$$

$$\frac{d^3\bar{r}}{du^3} = a\cos u\hat{i} - a\sin u\hat{j}$$

Curvature $k = \dfrac{\left|\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right|}{\left|\frac{dr}{du}\right|^3}$ $\qquad \tau = \dfrac{\left[\frac{d\bar{r}}{du}\ \frac{d^2\bar{r}}{du^2}\ \frac{d^3\bar{r}}{du^3}\right]}{\left|\frac{d\bar{r}}{du} \times \frac{d^2\bar{r}}{du^2}\right|}$

$$\frac{dr}{du} \times \frac{d^2r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(1-\cos u) & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix} = -ab\cos u\hat{i} + ab\sin u\hat{j} + a^2(\cos u - 1)\hat{k}$$

$$\left|\frac{dr}{du} \times \frac{d^2r}{du^2}\right| = \sqrt{(ab)^2\cos^2 u + (ab)^2\sin^2 u + a^4(\cos u - 1)^2}$$

$$= a\sqrt{b^2 + a^2(\cos u - 1)^2}$$

$$\left|\frac{dr}{du}\right| = \sqrt{a^2(1-\cos u)^2 + a^2\sin^2 u + b^2} = \sqrt{b^2 + 2a^2(1-\cos u)}$$

$$\left[\frac{dr}{du}\ \frac{d^2r}{du^2}\ \frac{d^3r}{du^3}\right] = (a\cos u\hat{i} - a\sin u\hat{j}) \cdot \left[-ab\cos u\hat{i} + ab\sin u\hat{j} + a^2(\cos u - 1)\hat{k}\right]$$

$$= -a^2b\cos^2 u - a^2b\sin^2 u = -a^2b$$

$$\Rightarrow k = \frac{a\sqrt{b^2 + a^2(\cos u - 1)^2}}{\left[\sqrt{b^2 + 2a^2(1-\cos u)}\right]^3}, \quad \tau = \frac{-a^{\cancel{2}}b}{\cancel{a}\sqrt{b^2 + a^2(1-\cos u)^2}}$$

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4.(2a) $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\text{Curl } \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \hat{j}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\text{Curl }(\text{Curl } v) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 v_2}{\partial x \partial y} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 v_3}{\partial x \partial z}\right) + \hat{j}\left(\frac{\partial^2 v_3}{\partial y \partial z} - \frac{\partial^2 v_2}{\partial z^2} - \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_1}{\partial x \partial y}\right)$$

$$+ \hat{k}\left[\frac{\partial^2 v_1}{\partial x \partial z} - \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial y^2} + \frac{\partial^2 v_2}{\partial y \partial z}\right]$$

$$= \hat{i}\left(\frac{\partial^2 v_2}{\partial x \partial y} + \frac{\partial^2 v_3}{\partial x \partial z} + \underbrace{\frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2}} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2}\right) + \hat{j}\left[\frac{\partial^2 v_1}{\partial x \partial y} + \frac{\partial^2 v_3}{\partial y \partial z} + \underbrace{\frac{\partial^2 v_2}{\partial y^2} - \frac{\partial^2 v_2}{\partial y^2}} - \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial z^2}\right]$$

$$+ \hat{k}\left(\frac{\partial^2 v_1}{\partial x \partial z} + \frac{\partial^2 v_2}{\partial y \partial z} + \underbrace{\frac{\partial^2 v_3}{\partial z^2} - \frac{\partial^2 v_3}{\partial z^2}} - \frac{\partial^2 v_3}{\partial x^2} - \frac{\partial^2 v_3}{\partial y^2}\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right) - \hat{i}\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)$$

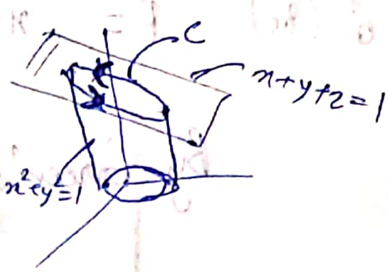
$$+ \hat{j}\frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right) - \hat{j}\left(\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}+\frac{\partial^2 v_2}{\partial z^2}\right)$$

$$+ \hat{k}\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right) - \hat{k}\left(\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial y^2}+\frac{\partial^2 v_3}{\partial z^2}\right)$$

$$= \sum \left\{ \frac{\partial}{\partial x}(\nabla\cdot V) - \nabla^2 v_1 \right\}\hat{i}$$

$$= \nabla\cdot(\nabla V) - \nabla^2 V. \quad = \underline{\underline{\text{grad}(\nabla\cdot V) - \nabla^2 V}}$$

5. (8b) $\int_C -y^3 dx + x^3 dy + z^3 dz = \oint_C \bar{F} \cdot d\bar{r}$



$\bar{F} = -y^3\hat{i} + x^3\hat{j} + z^3\hat{k}$

By Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, dS$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & z^3 \end{vmatrix} = 3(x^2+y^2)\hat{k}$$

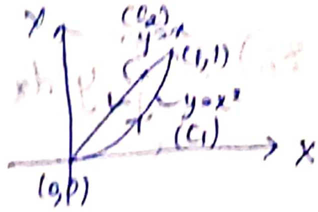
$\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ $\qquad (\nabla\times\bar{F})\cdot\hat{n} = \sqrt{3}(x^2+y^2)$

$$I = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, dS = \iint \sqrt{3}(x^2+y^2)\, \frac{dx\,dy}{1/\sqrt{3}} = 3\iint_{x^2+y^2\le 1} (x^2+y^2)\, dx\,dy$$

$x = r\cos\theta, \quad y = r\sin\theta$

$$I = 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3\, dr\, d\theta = \frac{3}{4}\cdot[2\pi] = \frac{3}{2}\pi$$

6. (8c) $\bar{F} = xy^2\,\hat{i} + (y+x)\,\hat{j}$



By Green's Theorem, $\iint_S (\nabla \times \bar{F})\cdot\hat{k} = \oint_C \bar{F}\cdot d\bar{r}$

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_{C_1} \bar{F}\cdot d\bar{r} + \oint_{C_2} \bar{F}\cdot d\bar{r}$$

Along C_1: $\oint_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} \left[xy^2\,dx + (y+x)\,dy\right]$

Putting $y = x^2$, $dy = 2x\,dx$

$$= \int_0^1 (x^5 + 2x^2 + 2x^3)\,dx = \frac{1}{6} + \frac{2}{3} + \frac{1}{2} = \frac{4}{3}$$

Along C_2: $\oint_{C_2} \bar{F}\cdot d\bar{r} = \int_{C_2} \left[xy^2\,dx + (y+x)\,dy\right]$

Putting $y = x$, $dy = dx$

$$= \int_{x=1}^{0} (x^3 + 2x)\,dx = -1 - \frac{1}{4} = -\frac{5}{4}$$

$$\Rightarrow I = \oint_C \bar{F}\cdot d\bar{r} = \frac{4}{3} - \frac{5}{4} = \underline{\underline{\frac{1}{12}}}.$$

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