CSE-2014

Q1 Convergence of improper integral $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$

sol. Let $g(x) = \frac{1}{x^2}$ and $f(x) = \frac{1}{x^2(1+e^{-x})}$

then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2(1+e^{-x})}{1/x^2} = \frac{1}{1+0} = 1$

As $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is finite

$\therefore \int_1^\infty f(x)$ and $\int_1^\infty g(x)$ converge or diverge together.
(Acc. to comparison test)

As $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2} dx$ is convergent

($\because \int_a^\infty \frac{1}{x^p} dx$ is convergent if $p>1$)

$\therefore \int_1^\infty f(x)$ converges. (By comparison test)

2 Find $\int_0^1 f(x)\,dx$ where

$$f(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \in (0,1) \\ 0, & x = 0 \end{cases}$$

sol. $f(x)$ is not continuos at $x = 0$ but is continuos everywhere else in $(0,1)$.
As there are only finite discontinuities hence it is Reimann integrable.

$$\int_0^1 f(x)\,dx = \lim_{h\to 0} \int_h^1 2x \sin\frac{1}{x} - \cos\frac{1}{x}\,dx$$

$$= \lim_{h\to 0} \int_h^1 d\left(x^2 \sin\frac{1}{x}\right)$$

$$= \lim_{h\to 0} \left| x^2 \sin\frac{1}{x} \right|_h^1$$

$$= \lim_{h\to 0} \left(1^2 \sin 1 - h^2 \sin\frac{1}{h}\right)$$

$$= \sin 1 - 0 = \sin 1$$

$$\therefore \boxed{\int_0^1 f(x)\,dx = \sin 1}$$

Q3 obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x,y) = \begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

also discuss continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$

sol.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(a+h, b+k) - f(a,b)}{h}$$

$$\therefore \frac{\partial f(0,0)}{\partial x} = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h(0)(3h^2 - 2(0)^2)}{h^2 + (0)^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

Also,

$$\frac{\partial f(0,y)}{\partial x} = \lim_{h \to 0} \frac{f(0+h, y) - f(0,y)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{hy(3h^2-2y^2)}{h^2+y^2} - 0}{h}$$

$$\lim_{h\to 0} \frac{y(3h^2-2y^2)}{h^2+y^2} = \frac{y(0-2y^2)}{0+y^2} = -2y$$

similarly, $\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$

$$= \lim_{k\to 0} \frac{\frac{\cancel{0}k(0^2-2k^2)}{\cancel{0}^2+k^2} - 0}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

and $\frac{\partial f}{\partial y}(x,0) = \lim_{k\to 0} \frac{f(x,0+k)-f(x,0)}{k}$

$$= \lim_{k\to 0} \frac{\frac{xk(3x^2-2k^2)}{x^2+k^2} - \frac{x.0(3x^2-0)}{x^2+0}}{k}$$

$$= \lim_{k\to 0} \frac{x\cancel{k}(3x^2-2k^2)}{\cancel{k}(x^2+k^2)} = \lim_{k\to 0} \frac{x(3x^2-2k^2)}{x^2+k^2}$$

$$= 3x$$

$\therefore$ $\boxed{f_x(0,0) = f_y(0,0) = 0}$

$\boxed{f_x(0,y) = -2y \quad \text{and} \quad f_y(x,0) = 3x}$

Now,

$$\frac{\partial f(0,0)}{\partial x\,\partial y} = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{3h - 0}{h} = 3$$

$$\frac{\partial f(0,0)}{\partial y\,\partial x} = \lim_{k\to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{-2k - 0}{k} = -2$$

$$\therefore \boxed{\frac{\partial f(0,0)}{\partial x\,\partial y} = 3 \quad \text{and} \quad \frac{\partial f(0,0)}{\partial y\,\partial x} = -2}$$

As $\frac{\partial f(0,0)}{\partial x \partial y} \neq \frac{\partial f}{\partial y \partial x}$

$\therefore$ $\frac{\partial f}{\partial x \partial y}(0,0)$ and $\frac{\partial f}{\partial y \partial x}(0,0)$ are <u>not continuos</u>

at $(0,0)$, According to the Schwartz theorem

Q. Find the minimum values of $x^2+y^2+z^2$ subject to condition $xyz = a^3$ by method of Lagrangian multiplier.

Sol. Let $f(x,y,z) = x^2+y^2+z^2$

and $g(x,y,z) = xyz - a^3$

Then Lagrangian $F(x,y,z,\lambda) = f + \lambda g$

$F(x,y,z,\lambda) = x^2+y^2+z^2+\lambda(xyz-a^3)$ where λ is multiplier

For minima, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial \lambda} = 0$

$\therefore \frac{\partial F}{\partial x} = 2x + \lambda(yz) = 0 \quad -(1)$

$\frac{\partial F}{\partial y} = 2y + \lambda(xz) = 0 \quad -(2)$

$\frac{\partial F}{\partial z} = 2z + \lambda(yx) = 0 \quad -(3)$

$\frac{\partial F}{\partial \lambda} = 0 + xyz - a^3 = 0 \quad -(4)$

Using (1), (2) and (3)

$$\lambda = \frac{-2x}{yz} = \frac{-2y}{xz} = \frac{-2z}{yx}$$

$$\therefore \frac{x^2}{xyz} = \frac{y^2}{xyz} = \frac{z^2}{xyz}$$

$$\Rightarrow x^2 = y^2 = z^2 = k \text{ (say)}$$

Using (4) we get $xyz = a^3$

$$\Rightarrow x^2 \cdot y^2 \cdot z^2 = a^6$$

$$\Rightarrow k^3 = a^6$$

$$\boxed{k = a^2}$$

$$\therefore \boxed{x^2 = y^2 = z^2 = a^2}$$

$\therefore$ Minimum value of $\boxed{f \text{ is } 3a^2}$

when $\boxed{x = y = z = a}$

By AM-GM inequality

$$\frac{x^2+y^2+z^2}{3} \geq \sqrt{x^2y^2z^2}$$

$$\therefore \quad x^2+y^2+z^2 \geq 3xyz$$

$\Rightarrow \; x^2+y^2+z^2 \geq 3a^2$ ($\therefore$ Minimum value is $3a^2$)

Hence verified.