UPSC 2020 Success Clap

Mechanics

(6c) Write Hamiltonian, Find the eqn of motion of a particle of mass m constrained on surface of cylinder by $x^2+y^2=R^2$, R is constant. The particle is subject to a force directed towards origin & proportional to distance r of the particle from origin given by $F=-kr$, k is const.

Note:

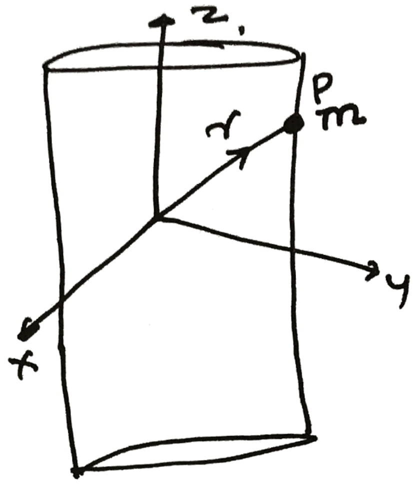
1) Same question is asked in UPSC 2006.
So solve all PYQs from 1992.
Donot stick PYQs for 10 years only

2) Checkout Video Soln of UPSC 2006 in YouTube of SuccessClap.

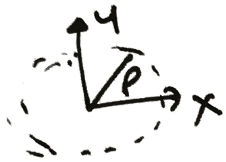
3) SAME Question is present in Question Bank of SuccessClap. So practice all Hamiltonian & Lag problems from Success Clap Question Bank.

→ Its Cylindrical Coordinate system

$v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$ (Remember Formula)



ρ is distance from origin in xy plane



$\rho = R$ is constant in our problem

$\dot{R} = 0$

$v^2 = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2$

$v^2 = R^2\dot{\theta}^2 + \dot{z}^2$

$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right)$

→ We need V (Potential)

Given $F = -kr$

But $F = -\frac{\partial V}{\partial r}$

$-\frac{\partial V}{\partial r} = -kr$

$V = \int kr\,dr = \frac{kr^2}{2}$

$V = \frac{1}{2}kr^2$

$= \frac{1}{2}k\left(x^2 + y^2 + z^2\right)$

$r^2 = x^2 + y^2 + z^2$)

But $x^2 + y^2 = R^2$ const on Cylinder

$= \frac{1}{2}k\left(R^2 + z^2\right)$

→ $L = T - V$

$= \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$

→ Canonical momenta

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \quad \Rightarrow \quad \dot\theta = \frac{p_\theta}{mR^2}$$

$$p_z = \frac{\partial L}{\partial \dot z} = m\dot z \quad \Rightarrow \quad \dot z = \frac{p_z}{m}$$

→ Hamilton $= \sum p_j \dot q_j - L$

$$= (p_\theta\dot\theta + p_z\dot z) - \left[\tfrac{1}{2}m(R^2\dot\theta^2 + \dot z^2) - \tfrac{1}{2}k(R^2+z^2)\right]$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_z^2}{m} - \left[\frac{1}{2}m\left(\frac{R^2 p_\theta^2}{m^2R^4} + \frac{p_z^2}{m^2}\right)\right] + \frac{1}{2}k(R^2+z^2)$$

$$H = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2+z^2)$$

→ Eqn of motion

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \dot p_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \dot p_z = -\frac{\partial H}{\partial z} = -kz$$

$\dot p_\theta = 0$ (Angular momentum conserved

$$\left.\begin{array}{l}\dot z = \frac{p_z}{m} \\ \dot p_z = -kz\end{array}\right] \Rightarrow \ddot z = \frac{\dot p_z}{m} = -\frac{kz}{m}$$

$$\boxed{\ddot z + \left(\frac{k}{m}\right)z = 0 \quad \text{SHM}}$$