2

Gradient, Divergence and Curl

§1. Partial Derivatives of Vectors. Suppose r is a vector depending on more than one scalar variable. Let $\mathbf{r}=\mathbf{f}(x, y, z)$ *i.e.* let $\mathbf{r}$ be a function of three scalar variables x, y and z. The partial derivative of r with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x}=\lim_{\delta x \to 0} \frac{\mathbf{f}(x+\delta x, y, z)-\mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r}/\partial x$ is nothing but the ordinary derivative of r with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial y}$ and $\dfrac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \frac{\partial^2 \mathbf{r}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If r has continuous partial derivatives of the second order at least, then, $\dfrac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\dfrac{\partial^2 \mathbf{r}}{\partial y\, \partial x}$ *i.e.* the order of differentiation is immaterial. If $\mathbf{r}=\mathbf{f}(x, y, z)$, the total differential $d\mathbf{r}$ of r is given by

$$d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial x}\, dx+\frac{\partial \mathbf{r}}{\partial y}\, dy+\frac{\partial \mathbf{r}}{\partial z}\, dz.$$

§2. The Vector Differential Operator Del. (∇). The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k} \equiv \mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ can be treated as its components along $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

§3. Gradient of a scalar Field. Definition. *Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as*

$$\nabla f=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right) f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

[Kerala 1975; Allahabad 79]

It should be noted that ∇f is a vector whose three successive components are $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$, and $\dfrac{\partial f}{\partial z}$. Thus the gradient of a scalar field defines a vector field. *If f is a scalar point function, then ∇f is a vector point function.*

§4. Formulas involving gradient.

Theorem 1. Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$\text{grad } (f+g)=\text{grad } f+\text{grad } g$$

or

$$\nabla (f+g)=\nabla f+\nabla g.$$

Proof. We have $\nabla (f+g)=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)(f+g)$

$$=\mathbf{i}\frac{\partial}{\partial x}(f+g)+\mathbf{j}\frac{\partial}{\partial y}(f+g)+\mathbf{k}\frac{\partial}{\partial z}(f+g)$$

$$=\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{i}\frac{\partial f}{\partial y}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}+\mathbf{k}\frac{\partial g}{\partial z}$$

$$=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)+\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)$$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right) f+\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right) g$$

$$=\nabla f+\nabla g=\text{grad } f+\text{grad } g.$$

Similarly, we can prove that $\nabla (f-g)=\nabla f-\nabla g$.

Theorem 2. Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that*

$$\nabla f=\mathbf{0}.$$

Proof. If $f(x, y, z)$ is constant, then

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0,\ \frac{\partial f}{\partial z}=0.$$

Therefore $\text{grad}\, f=\mathbf{i}\dfrac{\partial f}{\partial x}+\mathbf{j}\dfrac{\partial f}{\partial y}+\mathbf{k}\dfrac{\partial f}{\partial z}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$

Hence the condition is necessary.

Conversely, let $\text{grad}\, f=\mathbf{0}$. Then $\mathbf{i}\dfrac{\partial f}{\partial x}+\mathbf{j}\dfrac{\partial f}{\partial y}+\mathbf{k}\dfrac{\partial f}{\partial z}=\mathbf{0}.$

Therefore $\dfrac{\partial f}{\partial x}=0,\ \dfrac{\partial f}{\partial y}=0,\ \dfrac{\partial f}{\partial z}=0.$

$\therefore$ f must be independent of x, y and z.

$\therefore$ f must be a constant. Hence the condition is sufficient.

Theorem 3. Gradient of the product of two scalar point functions. *If f and g are two scalar point functions, then*

$$\text{grad}\,(fg)=f\,\text{grad}\, g+g\,\text{grad}\, f$$

or

$$\nabla(fg)=f\nabla g+g\nabla f.$$

[Meerut 1972; Bombay 69]

Proof. We have $\nabla(fg)=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)(fg)$

$$=\mathbf{i}\frac{\partial}{\partial x}(fg)+\mathbf{j}\frac{\partial}{\partial y}(fg)+\mathbf{k}\frac{\partial}{\partial z}(fg)$$

$$=\mathbf{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)+\mathbf{j}\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)+\mathbf{k}\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)$$

$$=f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)+g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$=f\nabla g+g\nabla f=f\,\text{grad}\, g+g\,\text{grad}\, f.$$

In particular if c is a constant, then

$$\nabla(cf)=c\nabla f+f\nabla c=c\nabla f+\mathbf{0}=c\nabla f.$$

Theorem 4. Gradient of the Quotient of two scalar functions. *If f and g are two scalar point functions, then*

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}.$$

Proof. We have $\nabla\left(\dfrac{f}{g}\right)=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)\left(\dfrac{f}{g}\right)$

$$=\mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right)+\mathbf{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right)+\mathbf{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right).$$

But $\dfrac{\partial}{\partial x}\left(\dfrac{f}{g}\right)=\dfrac{g\dfrac{\partial f}{\partial x}-f\dfrac{\partial g}{\partial x}}{g^2},\ \dfrac{\partial}{\partial y}\left(\dfrac{f}{g}\right)=\dfrac{g\dfrac{\partial f}{\partial y}-f\dfrac{\partial g}{\partial y}}{g^2},$

and $\dfrac{\partial}{\partial z}\left(\dfrac{f}{g}\right)=\dfrac{g\dfrac{\partial f}{\partial z}-f\dfrac{\partial g}{\partial z}}{g^2}.$

$$\therefore\ \nabla\left(\frac{f}{g}\right)=\frac{1}{g^2}\left\{\mathbf{i}\left(g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}\right)+\mathbf{j}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}\right)+\mathbf{k}\left(g\frac{\partial f}{\partial z}-f\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)-f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\nabla f-f\nabla g\right\}.$$

SOLVED EXAMPLES

Ex. 1. *If* $\mathbf{A}=x^2yz\,\mathbf{i}-2xz^3\,\mathbf{j}+xz^2\,\mathbf{k}$, $\mathbf{B}=2z\,\mathbf{i}+y\,\mathbf{j}-x^2\,\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})$ *at* $(1, 0, -2)$. [Kanpur 1975, 79]

Solution. We have $\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x^2yz & -2xz^3 & xz^2\\ 2z & y & -x^2\end{vmatrix}$

$$=(2x^3z^3-xyz^2)\,\mathbf{i}+(2xz^3+x^4yz)\,\mathbf{j}+(x^2y^2z+4xz^4)\,\mathbf{k}.$$

$$\therefore\ \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})=-xz^2\,\mathbf{i}+x^4z\,\mathbf{j}+2x^2yz\,\mathbf{k}.$$

Again $\dfrac{\partial^2}{\partial x\partial y}(\mathbf{A}\times\mathbf{B})=\dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\}$

$$=-z^2\,\mathbf{i}+4x^3z\,\mathbf{j}+4xyz\,\mathbf{k}. \quad \ldots(1)$$

Putting $x=1$, $y=0$ and $z=-2$ in (1), we get the required derivative at the point $(1, 0, -2)=-4\mathbf{i}-8\mathbf{j}$.

Ex. 2. *If* $f(x, y, z)=3x^2y-y^3z^2$, *find grad f at the point* $(1, -2, -1)$. [Agra 1978]

Solution. We have

$$\text{grad}\, f=\nabla f=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\mathbf{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\mathbf{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$=\mathbf{i}(6xy)+\mathbf{j}(3x^2-3y^2z^2)+\mathbf{k}(-2y^3z)$$

$$=6xy\,\mathbf{i}+(3x^2-3y^2z^2)\,\mathbf{j}-2y^3z\,\mathbf{k}.$$

Putting $x=1$, $y=-2$, $z=-1$, we get

$$\nabla f=6(1)(-2)\,\mathbf{i}+\{3(1)^2-3(-2)^2(-1)^2\}\,\mathbf{j}-2(-2)^3(-1)\,\mathbf{k}$$

$$= -12\mathbf{i}-9\mathbf{j}-16\mathbf{k}.$$

Ex. 3. *If* $r=|\mathbf{r}|$ *where* $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, *prove that*

(*i*) $\nabla f(r)=f'(r)\,\nabla r$, (*ii*) $\nabla r=\dfrac{1}{r}\,\mathbf{r}$, **[Rohilkhand 1981]**

(*iii*) $\nabla f(r)\times\mathbf{r}=\mathbf{0}$, (*iv*) $\nabla\left(\dfrac{1}{r}\right)=-\dfrac{\mathbf{r}}{r^3}$, **[Kanpur 1976]**

(*v*) $\nabla \log|\mathbf{r}|=\dfrac{\mathbf{r}}{r^2}$,

(*vi*) $\nabla r^n=nr^{n-2}\,\mathbf{r}$.

[Kanpur 1970; Rohilkhand 76; B.H.U. 70]

Solution. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

$\therefore \quad r^2=x^2+y^2+z^2.$

(i) $$\nabla f(r)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f(r)$$

$$=\mathbf{i}\frac{\partial}{\partial x}f(r)+\mathbf{j}\frac{\partial}{\partial y}f(r)+\mathbf{k}\frac{\partial}{\partial z}f(r)$$

$$=\mathbf{i}f'(r)\frac{\partial r}{\partial x}+\mathbf{j}f'(r)\frac{\partial r}{\partial y}+\mathbf{k}f'(r)\frac{\partial r}{\partial z}$$

$$=f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)=f'(r)\nabla r.$$

(ii) We have $\nabla r=\mathbf{i}\dfrac{\partial r}{\partial x}+\mathbf{j}\dfrac{\partial r}{\partial y}+\mathbf{k}\dfrac{\partial r}{\partial z}.$

Now $r^2=x^2+y^2+z^2$; $\therefore$ $2r\dfrac{\partial r}{\partial x}=2x$ *i.e.* $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

Similarly $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}$.

$$\therefore \quad \nabla r=\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}=\frac{1}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r}\mathbf{r}=\hat{\mathbf{r}}.$$

(iii) We have as in part (i), $\nabla f(r)=f'(r)\,\nabla r$.

But as in part (ii) $\nabla r=\dfrac{1}{r}\mathbf{r}$.

$$\therefore \quad \nabla f(r)=f'(r)\frac{1}{r}\mathbf{r}.$$

$$\therefore \quad \nabla f(r)\times\mathbf{r}=\left\{f'(r)\frac{1}{r}\mathbf{r}\right\}\times\mathbf{r}=\left\{\frac{1}{r}f'(r)\right\}(\mathbf{r}\times\mathbf{r})$$

$$=\mathbf{0}, \text{ since } \mathbf{r}\times\mathbf{r}=\mathbf{0}.$$

(iv) We have $\nabla\left(\dfrac{1}{r}\right)=\mathbf{i}\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right)+\mathbf{j}\dfrac{\partial}{\partial y}\left(\dfrac{1}{r}\right)+\mathbf{k}\dfrac{\partial}{\partial z}\left(\dfrac{1}{r}\right)$

$$=\mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$=-\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}\right)$$

$$=-\frac{1}{r^2}\left(\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}\right) \quad \text{[see part (ii)]}$$

$$=-\frac{1}{r^3}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-\frac{1}{r^3}\mathbf{r}.$$

(v) We have $\nabla \log|\mathbf{r}|=\nabla \log r$

$$=\mathbf{i}\frac{\partial}{\partial x}\log r+\mathbf{j}\frac{\partial}{\partial y}\log r+\mathbf{k}\frac{\partial}{\partial z}\log r$$

$$=\frac{1}{r}\frac{\partial r}{\partial x}\mathbf{i}+\frac{1}{r}\frac{\partial r}{\partial y}\mathbf{j}+\frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k}=\frac{1}{r}\left(\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}\right)$$

$$=\frac{1}{r^2}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r^2}\mathbf{r}.$$

(vi) We have $\nabla r^n=\mathbf{i}\dfrac{\partial}{\partial x}r^n+\mathbf{j}\dfrac{\partial}{\partial y}r^n+\mathbf{k}\dfrac{\partial}{\partial z}r^n$

$$=\mathbf{i}\,nr^{n-1}\frac{\partial r}{\partial x}+\mathbf{j}\,nr^{n-1}\frac{\partial r}{\partial y}+\mathbf{k}\,nr^{n-1}\frac{\partial r}{\partial z}=nr^{n-1}\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)$$

$$=nr^{n-1}\nabla r$$

$$=nr^{n-1}\frac{1}{r}\mathbf{r} \qquad \left[\because \ \nabla r=\frac{\mathbf{r}}{r} \text{ as in part (ii)}\right]$$

$$=nr^{n-2}\mathbf{r}.$$

Ex. 4. *Prove that* $f(u)\,\nabla u=\nabla\displaystyle\int f(u)\,du$.

Solution. We have $\nabla \int f(u)\,du$

$$=\Sigma\,\mathbf{i}\frac{\partial}{\partial x}\left\{\int f(u)\,du\right\} \quad \text{[by def. of gradient]}$$

$$=\Sigma\mathbf{i}\left\{\frac{d}{du}\int f(u)\,du\right\}\frac{\partial u}{\partial x}=\Sigma\mathbf{i}\,f(u)\frac{\partial u}{\partial x}=f(u)\,\Sigma\,\mathbf{i}\frac{\partial u}{\partial x}=f(u)\,\nabla u.$$

Ex. 5. *Show that*

(*i*) *grad* $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, (*ii*) *grad* $[\mathbf{r},\mathbf{a},\mathbf{b}]=\mathbf{a}\times\mathbf{b}$,

where $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.* **[Rohilkhand 1981; Bombay 70]**

Solution. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then a_1, a_2, a_3 are constants. Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$\therefore \quad \mathbf{r}\cdot\mathbf{a}=a_1x+a_2y+a_3z.$

$\therefore \quad \text{grad }(\mathbf{r}\cdot\mathbf{a})=\nabla(\mathbf{r}\cdot\mathbf{a})=\nabla(a_1x+a_2y+a_3z)$

$$=\mathbf{i}\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\mathbf{j}\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\mathbf{k}\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)$$

$$=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$$

(ii) grad $[\mathbf{r},\mathbf{a},\mathbf{b}]=\text{grad }\{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})\}$, where $\mathbf{a}\times\mathbf{b}$ is a constant vector

$$=\mathbf{a}\times\mathbf{b} \text{ as in part (i).}$$

Ex. 6. (*i*) *Interpret the symbol* $\mathbf{a}\cdot\nabla$.

(*ii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\,\phi$.

(*iii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\mathbf{a}$.

Solution. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then

$$\mathbf{a}\cdot\nabla=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)$$

$$=a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

Thus the symbol $\mathbf{a}\cdot\nabla$ stands for the operator

$$a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

(ii) $$(\mathbf{a}\cdot\nabla)\,\phi=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\cdot\nabla\phi=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=a_1\frac{\partial\phi}{\partial x}+a_2\frac{\partial\phi}{\partial y}+a_3\frac{\partial\phi}{\partial z}.$$

Hence $$(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\phi.$$

(iii) $$(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\mathbf{r}$$

$$=a_1\frac{\partial\mathbf{r}}{\partial x}+a_2\frac{\partial\mathbf{r}}{\partial y}+a_3\frac{\partial\mathbf{r}}{\partial z}.$$

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore$ $\dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \dfrac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \dfrac{\partial\mathbf{r}}{\partial z}=\mathbf{k}.$

$$\therefore\quad(\mathbf{a}\cdot\nabla)\,\mathbf{r}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$$

Exercises

1. If $\mathbf{f}=(2x^2y-x^4)\,\mathbf{i}+(e^{xy}-y\sin x)\,\mathbf{j}+x^2\cos y\,\mathbf{k}$, verify that

$$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x}=\frac{\partial^2\mathbf{f}}{\partial x\,\partial y}.$$

[Agra 1978]

2. If $\phi(x,y,z)=x^2y+y^2x+z^2$, find $\nabla\phi$ at the point (1, 1, 1). **[Agra 1979]**

Ans. $3\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

$$\left[\text{Note that } \nabla\phi=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}.\right]$$

3. Find grad f, where f is given by $f=x^3-y^3+xz^2$, at the point $(1,-1,2)$. **[Agra 1977]**

Ans. $7\mathbf{i}-3\mathbf{j}+4\mathbf{k}$.

4. If $u=x+y+z$, $v=x^2+y^2+z^2$, $w=yz+zx+xy$, prove that

$$(\text{grad } u)\cdot[(\text{grad } v)\times(\text{grad } w)]=0.$$ **[Kolhapur 1978]**

5. If $\mathbf{F}=\left(y\dfrac{\partial f}{\partial z}-z\dfrac{\partial f}{\partial y}\right)\mathbf{i}+\left(z\dfrac{\partial f}{\partial x}-x\dfrac{\partial f}{\partial z}\right)\mathbf{j}+\left(x\dfrac{\partial f}{\partial y}-y\dfrac{\partial f}{\partial x}\right)\mathbf{k}$,

prove that

(i) $\mathbf{F}=\mathbf{r}\times\nabla f$, (ii) $\mathbf{F}\cdot\mathbf{r}=0$, (iii) $\mathbf{F}\cdot\nabla f=0$.

6. If $\phi=(3r^2-4r^{1/2}+6r^{-1/3})$, show that

$$\nabla\phi=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

7. Prove that $\nabla\phi\cdot d\mathbf{r}=d\phi$.

8. ρ and p are two scalar point functions such that ρ is a function of p; show that $\nabla\rho=\dfrac{d\rho}{dp}\nabla p$.

9. Prove that $\mathbf{A}\cdot\left(\nabla\dfrac{1}{r}\right)=-\dfrac{\mathbf{A}\cdot\mathbf{r}}{r^3}$.

10. Prove that $\nabla r^{-3}=-3r^{-5}\,\mathbf{r}$. **[Agra 1974]**

§ 5. Level Surfaces. Let $f(x,y,z)$ be a scalar field over a region R. The points satisfying an equation of the type

$$f(x,y,z)=c, \text{ (arbitrary constant)}$$

constitute a family of surfaces in three dimensional space. The surfaces of this family are called *level surfaces*. Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called *iso-f surfaces*.

Theorem 1. *Let* $f(x,y,z)$ *be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof. Let (x_1,y_1,z_1) be any point of the region R. Then the level surface $f(x,y,z)=f(x_1,y_1,z_1)$ passes through this point.

Now suppose the level surfaces $f(x,y,z)=c_1$ and $f(x,y,z)=c_2$ pass through the point (x_1,y_1,z_1). Then

$$f(x_1,y_1,z_1)=c_1 \text{ and } f(x_1,y_1,z_1)=c_2.$$

Since $f(x,y,z)$ has a unique value at (x_1,y_1,z_1) therefore we have

$$c_1=c_2.$$

Hence only one level surface passes through the point (x_1,y_1,z_1).

Theorem 2. ∇f *is a vector normal to the surface* $f(x,y,z)=c$ *where c is a constant.* **[Agra 1968; Kerala 75]**

Proof. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x,y,z)$ on the level surface $f(x,y,z)=c$. Let

$$Q\,(x+\delta x,\ y+\delta y,\ z+\delta z)$$

be a neighbouring point on this surface. Then the position vector of $Q=\mathbf{r}+\delta\mathbf{r}=(x+\delta x)\,\mathbf{i}+(y+\delta y)\,\mathbf{j}+(z+\delta z)\,\mathbf{k}$.

$\therefore\quad \overrightarrow{PQ}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta\mathbf{r}=\delta x\,\mathbf{i}+\delta y\,\mathbf{j}+\delta z\,\mathbf{k}.$

As $Q\to P$, the line PQ tends to tangent at P to the level surface. Therefore $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz$$

$$=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\nabla f\cdot d\mathbf{r}.$$

Since $f(x, y, z)=\text{constant}$, therefore $df=0$.

$\therefore\quad \nabla f\cdot d\mathbf{r}=0$ so that ∇f is a vector perpendicular to $d\mathbf{r}$ and therefore to the tangent plane at P to the surface

$$f(x, y, z)=c.$$

Hence ∇f is a vector normal to the surface $f(x, y, z)=c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z)=c$ passing through that point

§ 6. Directional Derivative of a scalar point function.

[Agra 1972; Kolhapur 73; Bombay 70]

Definition. *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\displaystyle\lim_{Q\to P}\frac{f(Q)-f(P)}{PQ}$, *if it exists, is called the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of directional derivative. Let P be the point (x, y, z) and let Q be the point $(x+\delta x, y+\delta y, z+\delta z)$. Suppose $PQ=\delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If $\delta f=f(x+\delta x, y+\delta y, z+\delta z)-f(x, y, z)=f(Q)-f(P)$, then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{\mathbf{a}}$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is $\displaystyle\lim_{Q\to P}\frac{f(Q)-f(P)}{PQ}=\lim_{\delta s\to 0}\frac{\delta f}{\delta s}=\frac{df}{ds}$. It represents the rate of change of f with respect to distance at point P in the direction of unit vector $\hat{\mathbf{a}}$.

Theorem 1. *The directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by*

$$\frac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}.$$

[Allahabad 1982; Poona 70]

Proof. Let $f(x, y, z)$ define a scalar field in the region R. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ denote the position vector of any point $P(x, y, z)$ in this region. If s denotes the distance of P from some fixed point A in the direction of $\hat{\mathbf{a}}$, then δx denotes small element at P in the direction of $\hat{\mathbf{a}}$. Therefore $\dfrac{d\mathbf{r}}{ds}$ is a unit vector at P in this direction i.e.

$$\frac{d\mathbf{r}}{ds}=\hat{\mathbf{a}}.$$

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore\ \dfrac{d\mathbf{r}}{ds}=\dfrac{dx}{ds}\mathbf{i}+\dfrac{dy}{ds}\mathbf{j}+\dfrac{dz}{ds}\mathbf{k}=\hat{\mathbf{a}}.$

$$\text{Now } \nabla f\cdot\hat{\mathbf{a}}=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}\right)$$

$$=\frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds}+\frac{\partial f}{\partial z}\frac{dz}{ds}$$

$=\dfrac{df}{ds}=$ directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.

Alternative Proof. Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\hat{\mathbf{a}}$. If l, m, n are the direction cosines of the line PQ, then $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}=$ the unit vector in the direction of $PQ=\hat{\mathbf{a}}$. Further if $PQ=\delta s$, then the co-ordinates of Q are $(x+l\delta s, y+m\delta s, z+n\delta s)$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is

$$=\lim_{Q\to P}\frac{f(Q)-f(P)}{PQ}$$

$$=\lim_{\delta s\to 0}\frac{f(x+l\delta s, y+m\delta s, z+n\delta s)-f(x, y, z)}{\delta s}$$

$$=\lim_{\delta s\to 0}\frac{f(x,y,z)+\left(l\delta s\frac{\partial f}{\partial x}+m\delta s\frac{\partial f}{\partial y}+n\delta s\frac{\partial f}{\partial z}\right)+\ldots-f(x,y,z)}{\delta s}$$

on expanding by Taylor's theorem

$$=l\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z}$$

$$=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=\nabla f\cdot\hat{\mathbf{a}}.$$

Theorem 2. *If $\hat{\mathbf{n}}$ be a unit vector normal to the level surface $f(x, y, z)=c$ at a point $P(x, y, z)$ and n be the distance of P from some fixed point A in the direction of $\hat{\mathbf{n}}$ so that δn represents element of normal at P in the direction of $\hat{\mathbf{n}}$, then*

$$\text{grad } f=\frac{df}{dn}\hat{\mathbf{n}}.$$

[Agra 1971; Bombay 69]

Proof. We have grad $f=\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$.

Also grad f is a vector normal to the surface $f(x, y, z)=c$. Since $\hat{\mathbf{n}}$ is a unit vector normal to the surface $f(x, y, z)=c$, therefore let grad $f=A\hat{\mathbf{n}}$, where A is some scalar to be determined.

Now $\frac{df}{dn}$ =directional derivative of f in the direction of $\hat{\mathbf{n}}$

$$=\nabla f\cdot\hat{\mathbf{n}}$$

$$=A\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}\quad[\because\ \nabla f=\text{grad } f=A\hat{\mathbf{n}}]$$

$$=A.$$

$$\therefore\ \text{grad } f=\nabla f=\frac{df}{dn}\hat{\mathbf{n}}.$$

Note. If the vector $\hat{\mathbf{n}}$ is in the direction of f increasing, then $\frac{df}{dn}$ is positive. Therefore ∇f is a vector normal to the surface $f(x, y, z)=c$ in the direction of f increasing.

Theorem 3. *Grad f is a vector in the direction of which the maximum value of the directional derivative of f i.e. $\frac{df}{ds}$ occurs.*

[Agra 1968, 71; Bombay 69]

Proof. The directional derivative of f in the direction of $\mathbf{a}$ is given by $\frac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}$

$$=\left(\frac{df}{dn}\hat{\mathbf{n}}\right)\cdot\hat{\mathbf{a}}\qquad\left[\because\ \nabla f=\frac{df}{dn}\hat{\mathbf{n}}\right]$$

$$=\frac{df}{dn}(\hat{\mathbf{n}}\cdot\hat{\mathbf{a}})$$

$$=\frac{df}{dn}\cos\theta,\text{ where }\theta\text{ is the angle between }\hat{\mathbf{a}}\text{ and }\hat{\mathbf{n}}.$$

Now $\frac{df}{dn}$ is fixed. Therefore $\frac{df}{dn}\cos\theta$ is maximum when $\cos\theta$ is maximum *i.e.* when $\cos\theta=1$. But $\cos\theta$ will be 1 when the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is 0 *i.e.* when $\hat{\mathbf{a}}$ is along the unit normal vector $\hat{\mathbf{n}}$.

Therefore the directional derivative is maximum along the normal to the surface. Its maximum value is

$$=\frac{df}{dn}=|\text{ grad } f\,|.$$

§ 7. Tangent plane and Normal to a level surface.

To find the equations of the tangent plane and normal to the surface $f(x, y, z)=c$.

Let $f(x, y, z)=c$ be the equation of a level surface. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on this surface.

Then $\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$ is a vector along the normal to the surface at P *i.e.* ∇f is perpendicular to the tangent plane at P.

Tangent plane at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q(X, Y, Z)$ on the tangent plane at P to the surface. The vector

$$\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\mathbf{i}+(Y-y)\mathbf{j}+(Z-z)\mathbf{k}$$

lies in the tangent plane at P. Therefore it is perpendicular to the vector ∇f.

$$\therefore\quad(\mathbf{R}-\mathbf{r})\cdot\nabla f=0$$

or $$[(X-x)\mathbf{i}+(Y-y)\mathbf{j}+(Z-z)\mathbf{k}]\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)=0$$

or $$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}+(Z-z)\frac{\partial f}{\partial z}=0,\qquad\ldots(1)$$

is the equation of the tangent plane at P.

Normal at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q\ (X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \quad (\mathbf{R}-\mathbf{r})\times\nabla f=0 \qquad \text{...(2)}$$

is the vector equation of the normal at P to the given surface.

Cartesian form. The vectors

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k} \text{ and } \nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

will be parallel if

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}=p\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of **i, j, k**, we get

$$X-x=p\frac{\partial f}{\partial x},\ Y-y=p\frac{\partial f}{\partial y},\ Z-z=p\frac{\partial f}{\partial z}$$

or

$$\frac{X-x}{\dfrac{\partial f}{\partial x}}=\frac{Y-y}{\dfrac{\partial f}{\partial y}}=\frac{Z-z}{\dfrac{\partial f}{\partial z}}$$

are the equations of the normal at P.

SOLVED EXAMPLES

Ex. 1. *Find a unit normal vector to the level surface* $x^2y+2xz=4$ *at the point* $(2, -2, 3)$.

Solution. The equation of the level surface is

$$f(x, y, z)\equiv x^2y+2xz=4.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\nabla\,(x^2y+2xz)=(2xy+2z)\,\mathbf{i}+x^2\,\mathbf{j}+2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

$\therefore$ $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{|-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}|}=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{\sqrt{(4+16+16)}}=-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}.$$

The vector $-(-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k})$ *i.e.*, $\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Ex. 2. *Find the directional derivatives of a scalar point function f in the direction of coordinate axes.*

Solution. The grad f at any point (x, y, z) is the vector

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

The directional derivative of f in the direction of **i**

$$=\text{grad } f\cdot\mathbf{i}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f}{\partial x}.$$

Similarly the directional derivatives of f in the directions of **j** and **k** are $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$.

Ex. 3. *Find the directional derivative of* $f(x, y, z)=x^2yz+4xz^2$ *at the point* $(1, -2, -1)$ *in the direction of the vector* $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$. **[Allahabad 1978]**

Solution. We have $f(x, y, z)=x^2yz+4xz^2$.

$\therefore$ grad $f=(2xyz+4z^2)\,\mathbf{i}+x^2z\,\mathbf{j}+(x^2y+8xz)\,\mathbf{k}$

$=8\mathbf{i}-\mathbf{j}-10\mathbf{k}$ at the point $(1, -2, -1)$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$,

then $$\hat{\mathbf{a}}=\frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{(4+1+4)}}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}.$$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\cdot\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k})=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Ex. 4. *Find the directional derivative of*

$$f(x, y, z)=x^2-2y^2+4z^2$$

at the point $(1, 1, -1)$ *in the direction of* $2\mathbf{i}+\mathbf{j}-\mathbf{k}$. **[Agra 1979]**

Ans. $8/\sqrt{6}$.

Ex. 5. *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P\ (1, 2, 3)$ *in the direction of the line PQ where Q is the point* $(5, 0, 4)$. **[Agra 1980]**

Solution. Here grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$

$=2x\,\mathbf{i}-2y\,\mathbf{j}+4z\,\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ at the point $(1, 2, 3)$.

Also $\overrightarrow{PQ}=$ position vector of Q − position vector of P

$=(5\mathbf{i}+0\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$,

then $$\hat{\mathbf{a}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}.$$

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{\mathbf{a}}=(2\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot\left\{\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Ex. 6. *In what direction from the point* $(1, 1, -1)$ *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum ? Also find the value of this maximum directional derivative.*

Solution. We have grad $f=2x\mathbf{i}-4y\mathbf{j}+8z\mathbf{k}$

$$=2\mathbf{i}-4\mathbf{j}-8\mathbf{k} \text{ at the point } (1, 1, -1).$$

The directional derivative of f is a maximum in the direction of grad $f=2\mathbf{i}-4\mathbf{j}-8\mathbf{k}$.

The maximum value of this directional derivative

$$=|\text{ grad } f\,|=|\,2\mathbf{i}-4\mathbf{j}-8\mathbf{k}\,|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}.$$

Ex. 7. *For the function* $f=y/(x^2+y^2)$, *find the value of the directional derivative making an angle* 30° *with the positive x-axis at the point* $(0, 1)$.

Solution. We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}$

$$=\frac{-2xy}{(x^2+y^2)^2}\mathbf{i}+\frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j}=-\mathbf{j} \text{ at the point } (0, 1).$$

If $\hat{\mathbf{a}}$ is a unit vector along the line which makes an angle 30° with the positive x-axis, then

$$\hat{\mathbf{a}}=\cos 30°\,\mathbf{i}+\sin 30°\,\mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{i}+\tfrac{1}{2}\mathbf{j}.$$

$\therefore$ the required directional derivative is

$$=\text{grad } f\cdot\hat{\mathbf{a}}=(-\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)=-\tfrac{1}{2}.$$

Ex. 8. *What is the greatest rate of increase of* $u=xyz^2$ *at the point* $(1, 0, 3)$? [Agra 1968]

Solution. We have $\nabla u=yz^2\,\mathbf{i}+xz^2\,\mathbf{j}+2xyz\,\mathbf{k}$.

$\therefore$ at the point $(1, 0, 3)$, we have

$$\nabla u=0\,\mathbf{i}+9\,\mathbf{j}+0\,\mathbf{k}=9\,\mathbf{j}.$$

The greatest rate of increase of u at the point $(1, 0, 3)$

$=$ the maximum value of $\dfrac{du}{ds}$ at the point $(1, 0, 3)$

$=|\,\nabla u\,|$, at the point $(1, 0, 3)$

$=|\,9\mathbf{j}\,|=9.$

Ex. 9. *Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.*

Solution. Let $f(x, y, z)$ be a scalar point function and let $\mathbf{a}$ be a unit vector along a tangent line to the level surface $f(x, y, z)=c$.

We know that ∇f is a normal vector at any point of the surface $f(x, y, z)=c$. Therefore the vectors ∇f and $\mathbf{a}$ are perpendicular.

Now the directional derivative of f in the direction of $\mathbf{a}$

$$=\mathbf{a}\cdot\nabla f=0.$$

Ex. 10. *Find the equations of the tangent plane and normal to the surface* $2xz^2-3xy-4x=7$ *at the point* $(1, -1, 2)$.

Solution. The equation of the surface is

$$f(x, y, z)\equiv 2xz^2-3xy-4x=7.$$

We have grad $f=(2z^2-3y+4)\,\mathbf{i}-3x\,\mathbf{j}+4xz\,\mathbf{k}$

$$=7\mathbf{i}-3\mathbf{j}+8\mathbf{k}, \text{ at the point } (1, -1, 2).$$

$\therefore$ $7\mathbf{i}-3\mathbf{j}+8\mathbf{k}$ is a vector along the normal to the surface at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$ is $=\mathbf{r}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, then the vector $\mathbf{R}-\mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot\text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y+1)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$

i.e. $7(X-1)-3(Y+1)+8(Z-2)=0.$

The equations of the normal to the surface at the point $(1, -1, 2)$ are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y+1}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}}, \text{ i.e. } \frac{X-1}{7}=\frac{Y+1}{-3}=\frac{Z-2}{8}.$$

Ex. 11. *Find the equations of the tangent plane and normal to the surface* $xyz=4$ *at the point* $(1, 2, 2)$. **[Agra 1970]**

Solution. The equation of the surface is

$$f(x, y, z)\equiv xyz-4=0.$$

We have grad $f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$

$$=4\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \text{ at the point } (1, 2, 2).$$

$\therefore$ $4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a vector along the normal to the surface at the point (1, 2, 2).

The position vector of the point (1, 2, 2) is $=\mathbf{r}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, 2, 2), the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot\text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-\ )\,\mathbf{k}\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $4(X-1)+2(Y-2)+2(Z-2)=0,$

i.e. $4X+2Y+2Z=12$, i.e. $2X+Y+Z=6$.

The equations of the normal to the surface at the point (1, 2, 2) are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y-2}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}},$$

i.e. $\dfrac{X-1}{4}=\dfrac{Y-2}{2}=\dfrac{Z-2}{2}$, i.e. $\dfrac{X-1}{2}=\dfrac{Y-2}{1}=\dfrac{Z-2}{1}$.

Ex. 12. *Given the curve $x^2+y^2+z^2=1$, $x+y+z=1$ (intersection of two surfaces), find the equations of the tangent line at the point (1, 0, 0).* **[Agra 1969]**

Solution. A normal to $x^2+y^2+z^2=1$ at (1, 0, 0) is

$$\text{grad } f_1=\text{grad } (x^2+y^2+z^2)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=2\mathbf{i}.$$

A normal to $x+y+z=1$ at (1, 0, 0) is

$$\text{grad } f_2=\text{grad } (x+y+z)=1\ \mathbf{i}+1\ \mathbf{j}+1\ \mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

The tangent line at the point (1, 0, 0) is perpendicular to both these normals. Therefore it is parallel to the vector

$$(\text{grad } f_1)\times(\text{grad } f_2).$$

Now $(\text{grad } f_1)\times(\text{grad } f_2)=2\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$

$=2\mathbf{i}\times\mathbf{j}+2\mathbf{i}\times\mathbf{k}=2\mathbf{k}-2\mathbf{j}=0\mathbf{i}-2\mathbf{j}+2\mathbf{k}.$

Now to find the equations of the line through the point (1, 0, 0) and parallel to the vector $0\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

The required equations are

$$\frac{X-1}{0}=\frac{Y-0}{-2}=\frac{Z-0}{2}$$

i.e. $X=1,\ \dfrac{Y}{-1}=\dfrac{Z}{1}$.

Ex. 13. *Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z=x^2+y^2-3$ at the point (2, −1, 2).* **(Kanpur 1978, 80)**

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Solution. Angle between two surfaces at a point is the angle between the normals to the surfaces at the point.

Let $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$.

Then grad $f_1=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ and grad $f_2=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1=\text{grad } f_1$ at the point (2, −1, 2) and $\mathbf{n}_2=\text{grad } f_2$ at the point (2, −1, 2). Then

$$\mathbf{n}_1=4\mathbf{i}-2\mathbf{j}+4\mathbf{k} \text{ and } \mathbf{n}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}.$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point (2, −1, 2). If θ is the angle between these vectors then

$$\mathbf{n}_1\cdot\mathbf{n}_2=|\mathbf{n}_1|\,|\mathbf{n}_2|\cos\theta$$

or $16+4-4=\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}\cos\theta$.

$$\therefore\ \cos\theta=\frac{16}{6\sqrt{(21)}} \text{ or } \theta=\cos^{-1}\frac{8}{3\sqrt{(21)}}.$$

Exercises

1. Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point (2, 0, 0).
Ans. $4\mathbf{i}+\mathbf{j}-\mathbf{k}$, $\dfrac{1}{3\sqrt{2}}(4\mathbf{i}+\mathbf{j}-\mathbf{k})$.

2. Find the unit vector normal to the surface $x^2-y^2+z=2$ at the point (1, −1, 2).
Ans. $\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$.

3. Find the unit normal to the surface $z=x^2+y^2$ at the point (−1, −2, 5). **(Kanpur 1975, 79)**
Ans. $\left(\dfrac{1}{\sqrt{21}}\right)(2\mathbf{i}+4\mathbf{j}+\mathbf{k})$.

4. Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point (1, 1, 1). **(Allahabad 1979)**
Ans $\left(\dfrac{1}{\sqrt{11}}\right)(\mathbf{i}-3\mathbf{j}-\mathbf{k})$.

5. Find the directional derivative of $\phi=xy+yz+zx$ in the direction of vector $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ at (1, 2, 0).
Ans. 10/3.

6. Find the directional derivative of $\phi(x, y, z)=x^2yz+4xz^2$ at the point (1, −2, 1) in the direction $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$. **(Poona 1970; Allahabad 78)**
Ans. −13/3.

7. Find the directional derivative of the function
$$f=xy+yz+zx$$
in the direction of the vector $2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ at the point (3, 1, 2). **(Rohilkhand 1980, 81; Agra 75)**
Ans. 45/7.

8. Find the directional derivatives of $\phi=xyz$ at the point (2, 2, 2), in the directions

(i) i, (ii) j, (iii) i+j+k.

Ans. (i) 4, (ii) 4, (iii) $4\sqrt{3}$.

9. Find the greatest value of the directional derivative of the function $2x^2-y-z^4$ at the point $(2, -1, 1)$. **Ans.** 9.

10. Find the maximum value of the directional derivatives of $\phi=x^2yz$ at the point $(1, 4, 1)$. **(Bombay 1970)**

Ans. 9.

11. Find the equation of the tangent plane to the surface $yz-zx+xy+5=0$, at the point $(1, -1, 2)$.

Ans. $3x-3y+2z=10$.

12. Find the equations of the tangent plane and normal to the surface $x^2+y^2+z^2=25$ at the point $(4, 0, 3)$.

Ans. $4x+3z=25; \dfrac{x-4}{4}=\dfrac{y}{0}=\dfrac{z-3}{3}$.

13. Find the equations of the tangent plane and normal to the surface $z=x^2+y^2$ at the point $(2, -1, 5)$.

Ans. $4x-2y-z=5; \dfrac{x-2}{4}=\dfrac{y+1}{-2}=\dfrac{z-5}{-1}$.

14. Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$.

Ans. $\cos^{-1}\sqrt{(19/29)}$.

15. If **F** and f are point functions, show that the components of the former tangential and normal to the level surface

$f=0$ are $\dfrac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}$ and $\dfrac{(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$.

Solution. The unit normal vector to the surface $f=0$ is

$$=\frac{\nabla f}{|\nabla f|}.$$

$\therefore$ The magnitude of the component of **F** along the normal

$$=\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}.$$

$\therefore$ the component of **F** along the normal

$$=\left\{\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}\right\}\frac{\nabla f}{|\nabla f|}=\frac{(\mathbf{F}\cdot\nabla f)}{|\nabla f|^2}\nabla f=\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f.$$

Consequently the tangential component of **F** is

$$=\mathbf{F}-\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f=\frac{(\nabla f\cdot\nabla f)\,\mathbf{F}-(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$$

$$=\frac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}\quad[\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}].$$

§ 8. Divergence of a vector point function.

Definition. *Let* **V** *be any given differentiable vector point function. Then the divergence of* **V**, *written as,*

$$\nabla\cdot\mathbf{V} \text{ or } \operatorname{div}\mathbf{V},$$

is defined as $\operatorname{div}\mathbf{V}=\nabla\cdot\mathbf{V}=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)\cdot\mathbf{V}$

$$=\mathbf{i}\cdot\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{V}}{\partial z}=\Sigma\mathbf{i}\cdot\frac{\partial\mathbf{V}}{\partial x}.$$

[Meerut 1971, 72; Kerala 74; Bombay 70]

It should be noted that div **V** is a scalar quantity. Thus *the divergence of a vector point function is a scalar point function.*

Theorem. *If* $\mathbf{V}=V_1\mathbf{i}+V_2\mathbf{j}+V_3\mathbf{k}$ *is a differentiable vector point function, then* $\operatorname{div}\mathbf{V}=\dfrac{\partial V_1}{\partial x}+\dfrac{\partial V_2}{\partial y}+\dfrac{\partial V_3}{\partial z}$.

Proof. We have by definition

$$\operatorname{div}\mathbf{V}=\nabla\cdot\mathbf{V}=\mathbf{i}\cdot\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{V}}{\partial z}.$$

Now $\mathbf{V}=V_1\mathbf{i}+V_2\mathbf{j}+V_3\mathbf{k}$; $\therefore$ $\dfrac{\partial\mathbf{V}}{\partial x}=\dfrac{\partial V_1}{\partial x}\mathbf{i}+\dfrac{\partial V_2}{\partial x}\mathbf{j}+\dfrac{\partial V_3}{\partial x}\mathbf{k}$.

$$\therefore\quad \mathbf{i}\cdot\frac{\partial\mathbf{V}}{\partial x}=\mathbf{i}\cdot\left(\frac{\partial V_1}{\partial x}\mathbf{i}+\frac{\partial V_2}{\partial x}\mathbf{j}+\frac{\partial V_3}{\partial x}\mathbf{k}\right)=\frac{\partial V_1}{\partial x}.$$

Similarly $\mathbf{j}\cdot\dfrac{\partial\mathbf{V}}{\partial y}=\dfrac{\partial V_2}{\partial y}$ and $\mathbf{k}\cdot\dfrac{\partial\mathbf{V}}{\partial z}=\dfrac{\partial V_3}{\partial z}$.

Hence $\operatorname{div}\mathbf{V}=\dfrac{\partial V_1}{\partial x}+\dfrac{\partial V_2}{\partial y}+\dfrac{\partial V_3}{\partial z}$.

Solenoidal Vector. Definition. *A vector* **V** *is said to be solenoidal if* div **V** $=0$. **[Calcutta 1975]**

§ 9. Curl of a vector point function.

Definition. *Let* **f** *be any given differentiable vector point function. Then the curl or rotation of* **f**, *written as* $\nabla\times\mathbf{f}$, *curl* **f** *or rot* **f** *is defined as*

$$\text{curl }\mathbf{f}=\nabla\times\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{f}$$

$$=\mathbf{i}\times\frac{\partial\mathbf{f}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{f}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{f}}{\partial z}=\Sigma\mathbf{i}\times\frac{\partial\mathbf{f}}{\partial x}.$$

[Meerut 1971, 72; Kerala 74; Bombay 70; Delhi 64; Punjab 63]

It should be noted that curl **f** is a vector quantity. *Thus the curl of a vector point function is a vector point function.*

Theorem. *If* $\mathbf{f}=f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$ *is a differentiable vector point function, then*

$$\text{curl } \mathbf{f}=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Proof. We have by definition

$$\text{curl } \mathbf{f}=\nabla\times\mathbf{f}=\mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{f}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{f}}{\partial z}$$

$$=\mathbf{i}\times\frac{\partial}{\partial x}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})+\mathbf{j}\times\frac{\partial}{\partial y}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$

$$+\mathbf{k}\times\frac{\partial}{\partial z}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$

$$=\mathbf{i}\times\left(\frac{\partial f_1}{\partial x}\mathbf{i}+\frac{\partial f_2}{\partial x}\mathbf{j}+\frac{\partial f_3}{\partial x}\mathbf{k}\right)+\mathbf{j}\times\left(\frac{\partial f_1}{\partial y}\mathbf{i}+\frac{\partial f_2}{\partial y}\mathbf{j}+\frac{\partial f_3}{\partial y}\mathbf{k}\right)$$

$$+\mathbf{k}\times\left(\frac{\partial f_1}{\partial z}\mathbf{i}+\frac{\partial f_2}{\partial z}\mathbf{j}+\frac{\partial f_3}{\partial z}\mathbf{k}\right)$$

$$=\left(\frac{\partial f_2}{\partial x}\mathbf{k}-\frac{\partial f_3}{\partial x}\mathbf{j}\right)+\left(-\frac{\partial f_1}{\partial y}\mathbf{k}+\frac{\partial f_3}{\partial y}\mathbf{i}\right)+\left(\frac{\partial f_1}{\partial z}\mathbf{j}-\frac{\partial f_2}{\partial z}\mathbf{i}\right)$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Note. It should be noted that the expression for curl **f** can be written immediately if we treat the operator ∇ as a vector quantity. Thus

$$\text{Curl } \mathbf{f}=\nabla\times\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_2 & f_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ f_1 & f_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ f_1 & f_2\end{vmatrix}\mathbf{k}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

But we must take care that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* the functions f_1, f_2, f_3.

Irrotational vector. **Definition.** *A vector* **f** *is said to be irrotational if* $\nabla\times\mathbf{f}=0$.

§ 10. The Laplacian operator ∇^2. *The Laplacian operator* ∇^2 *is defined as* $\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

If f is a scalar point function, then

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If **f** is a vector point function, then

$$\nabla^2\mathbf{f}=\frac{\partial^2\mathbf{f}}{\partial x^2}+\frac{\partial^2\mathbf{f}}{\partial y^2}+\frac{\partial^2\mathbf{f}}{\partial z^2}.$$

It should be noted that $\nabla^2\mathbf{f}$ is also a vector quantity.

Laplace's equation. *The equation* $\nabla^2 f=0$ *is called Laplace's equation.* A function which satisfies Laplace's equation is called a **harmonic function.**

SOLVED EXAMPLES

Ex. 1. *Prove that div* $\mathbf{r}=3$.

[Agra 1978; Rohilkhand 81; Kanpur 75; Meerut 67, 71]

Solution. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

By definition, $\text{div } \mathbf{r}=\nabla\cdot\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\mathbf{r}$

$$=\mathbf{i}\cdot\frac{\partial\mathbf{r}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{r}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{r}}{\partial z}$$

$$=\mathbf{i}\cdot\mathbf{i}+\mathbf{j}\cdot\mathbf{j}+\mathbf{k}\cdot\mathbf{k}\left[\because \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i}, \frac{\partial\mathbf{r}}{\partial y}=\mathbf{j}, \frac{\partial\mathbf{r}}{\partial z}=\mathbf{k}\right]$$

$$=1+1+1=3.$$

Ex. 2. *Prove that curl* $\mathbf{r}=0$.

[Agra 1968; Kanpur 75, 79; Rohilkhand 76; Meerut 67, 71]

Solution. We have by definition

$$\text{Curl } \mathbf{r}=\nabla\times\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{r}$$

$$=\mathbf{i}\times\frac{\partial\mathbf{r}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{r}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{r}}{\partial z}.$$

Now $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i}, \frac{\partial\mathbf{r}}{\partial y}=\mathbf{j}, \frac{\partial\mathbf{r}}{\partial z}=\mathbf{k}$.

$\therefore$ $\text{Curl } \mathbf{r}=\mathbf{i}\times\mathbf{i}+\mathbf{j}\times\mathbf{j}+\mathbf{k}\times\mathbf{k}=0+0+0=0.$

Ex. 3. *If* $\mathbf{f}=x^2y\,\mathbf{i}-2xz\,\mathbf{j}+2yz\,\mathbf{k}$, *find*

(i) div **f**, *(ii) curl* **f**, *(iii) curl curl* **f**.

Solution. (i) We have

$$\text{div } \mathbf{f}=\nabla\cdot\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(x^2y\,\mathbf{i}-2xz\,\mathbf{j}+2yz\,\mathbf{k})$$

$$=\frac{\partial}{\partial x}(x^2y)+\frac{\partial}{\partial y}(-2xz)+\frac{\partial}{\partial z}(2yz)=2xy+0+2y=2y\,(x+1).$$

(ii) We have curl $\mathbf{f}=\nabla\times\mathbf{f}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & -2xz & 2yz\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(-2xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right]\mathbf{j}+\frac{\partial}{\partial x}\left[(-2xz)-\frac{\partial}{\partial y}(x^2z)\right]\mathbf{k}$$

$$=(2z+2x)\,\mathbf{i}-0\,\mathbf{j}+(-2z-x^2)\,\mathbf{k}=(2x+2z)\,\mathbf{i}-(x^2+2z)\,\mathbf{k}.$$

(iii) We have curl curl $\mathbf{f}=\nabla\times(\nabla\times\mathbf{f})$

$$=\nabla\times[(2x+2z)\,\mathbf{i}-(x^2+2z)\,\mathbf{k}]$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+2z & 0 & -x^2-2z\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(-x^2-2z)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(-x^2-2z)-\frac{\partial}{\partial z}(2x+2z)\right]\mathbf{j}+\left[0-\frac{\partial}{\partial y}(2x+2z)\right]\mathbf{k}$$

$$=0\,\mathbf{i}-(-2x-2)\,\mathbf{j}+(0-0)\,\mathbf{k}=(2x+2)\,\mathbf{j}.$$

Ex. 4. *Determine the constant a so that the vector* $\mathbf{V}=(x+3y)\,\mathbf{i}+(y-2z)\,\mathbf{j}+(x+az)\,\mathbf{k}$ *is solenoidal.* [Kanpur 1978]

Solution. A vector $\mathbf{V}$ is said to be solenoidal if div $\mathbf{V}=0$.

We have div $\mathbf{V}=\nabla\cdot\mathbf{V}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)$

$$=1+1+a=2+a.$$

Now div $\mathbf{V}=0$ if $2+a=0$ *i.e.* if $a=-2$.

Ex. 5. *Show that the vector* $\mathbf{V}=(\sin y+z)\,\mathbf{i}+(x\cos y-z)\,\mathbf{j}+(x-y)\,\mathbf{k}$ *is irrotational.*

Solution. A vector $\mathbf{V}$ is said to be irrotational if curl $\mathbf{V}=\mathbf{0}$. We have curl $\mathbf{V}=\nabla\times\mathbf{V}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \sin y+z & x\cos y-z & x-y\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]\mathbf{k}$$

$$=(-1+1)\,\mathbf{i}-(1-1)\,\mathbf{j}+(\cos y-\cos y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ $\mathbf{V}$ is irrotational.

Ex. 6. *If $\mathbf{V}$ is a constant vector, show that*

(*i*) *div* $\mathbf{V}=0$, (*ii*) *curl* $\mathbf{V}=\mathbf{0}$.

Solution. (i) We have div $\mathbf{V}=\mathbf{i}\cdot\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{V}}{\partial z}$

$$=\mathbf{i}\cdot\mathbf{0}+\mathbf{j}\cdot\mathbf{0}+\mathbf{k}\cdot\mathbf{0}=0.$$

(ii) We have curl $\mathbf{V}=\mathbf{i}\times\frac{\partial\mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{V}}{\partial z}$

$$=\mathbf{i}\times\mathbf{0}+\mathbf{j}\times\mathbf{0}+\mathbf{k}\times\mathbf{0}=\mathbf{0}.$$

Ex. 7. *If $\mathbf{a}$ is a constant vector, find*

(*i*) *div* $(\mathbf{r}\times\mathbf{a})$, **[Rohilkhand 1980, 81]**

(*ii*) *curl* $(\mathbf{r}\times\mathbf{a})$. **[Rohilkhand 1981]**

Solution. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have $\mathbf{r}\times\mathbf{a}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ a_1 & a_2 & a_3\end{vmatrix}$

$$=(a_3y-a_2z)\,\mathbf{i}+(a_1z-a_3x)\,\mathbf{j}+(a_2x-a_1y)\,\mathbf{k}.$$

(i) div $(\mathbf{r}\times\mathbf{a})=\frac{\partial}{\partial x}(a_3y-a_2z)+\frac{\partial}{\partial y}(a_1z-a_3x)+\frac{\partial}{\partial z}(a_2x-a_1y)$

$$=0+0+0=0.$$

(ii) curl $(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_3y-a_2z & a_1z-a_3x & a_2x-a_1y\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_1z-a_3x)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_3y-a_2z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(a_1z-a_3x)-\frac{\partial}{\partial y}(a_3y-a_2z)\right]\mathbf{k}$$

$$=-2a_1\mathbf{i}-2a_2\mathbf{j}-2a_3\mathbf{k}=-2\,(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=-2\mathbf{a}.$$

Ex. 8. *If* $\mathbf{V}=e^{xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, *find curl* $\mathbf{V}$. **[Meerut 1969; Agra 70]**

Solution. We have curl $\mathbf{V}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ e^{xyz} & e^{xyz} & e^{xyz}\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(e^{xyz})-\frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(e^{xyz})-\frac{\partial}{\partial x}(e^{xyz})\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(e^{xyz})-\frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$=e^{xyz}(xz-xy)\,\mathbf{i}+e^{xyz}(xy-yz)\,\mathbf{j}+e^{xyz}(yz-xz)\,\mathbf{k}.$$

Ex. 9. *Evaluate div* $\mathbf{f}$ *where*

$$\mathbf{f}=2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k}.$$ **[Kanpur 1970]**

Solution. We have

$$\text{div }\mathbf{f}=\nabla\cdot\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k})$$

$$=\frac{\partial}{\partial x}(2x^2z)+\frac{\partial}{\partial y}(-xy^2z)+\frac{\partial}{\partial z}(3y^2x)$$

$$=4xz-2xyz+0=2xz\,(2-y).$$

Ex. 10. *Show that* $\nabla^2(x/r^3)=0$.

Solution. $\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right).$

Now $\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right\}=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{x}{r}\right\}\left[\because\ r^2=x^2+y^2+z^2 \text{ gives } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$=-\frac{3}{r^4}\frac{x}{r}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}.$$

Again $\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right)=\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right\}=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}$

$$=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\}\qquad\left[\because\ \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right)=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}.$$

Similarly $\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right)=-\frac{3x}{r^5}+\frac{5xz^2}{r^7}.$

Therefore adding we get

$$\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}-\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}(x^2+y^2+z^2)=-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0.$$

Exercises

1. If $\mathbf{f}=xy^2\mathbf{i}+2x^2yz\mathbf{j}-3yz^2\mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$. What are their values at the point (1, —1, 1) ? **[Agra 1979]**

   Ans. y^2+2x^2z-6yz; $-(3z^2+2x^2y)\,\mathbf{i}+(4xyz-2xy)\,\mathbf{k}$.
   At the point $(1,-1,1)$, div $\mathbf{f}=9$ and curl $\mathbf{f}=-\mathbf{i}-2\mathbf{k}$.

2. If $\mathbf{f}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$.

   Ans. $-2x-2y-2z$; $2\,(y-z)\,\mathbf{i}+2\,(z-x)\,\mathbf{j}+2\,(x-y)\,\mathbf{k}$.

3. If $\mathbf{F}=x^2z\mathbf{i}-2y^3z^2\mathbf{j}+xy^2z\mathbf{k}$, find div $\mathbf{F}$, curl $\mathbf{F}$ at $(1,-1,1)$.

   Ans. div $\mathbf{F}=-3$, curl $\mathbf{F}=-6\mathbf{i}+2\mathbf{j}$.

4. Find div $\mathbf{f}$ and curl $\mathbf{f}$ where
   $$\mathbf{f}=\text{grad }(x^3+y^3+z^3-3xyz).$$
   Ans. div $\mathbf{f}=6\,(x+y+z)$; curl $\mathbf{f}=0$.

5. Find the divergence and curl of the vector
   $$\mathbf{f}=(x^2-y^2)\,\mathbf{i}+2xy\mathbf{j}+(y^2-xy)\,\mathbf{k}.$$ **[Agra 1977]**
   Ans. div $\mathbf{f}=4x$, curl $\mathbf{f}=(2y-x)\,\mathbf{i}+y\mathbf{j}$.

6. Given $\phi=2x^3y^2z^4$, find div (grad ϕ).

   Ans. $12xy^2z^4+4x^3z^4+24x^3y^2z^2$.

7. If $u=x^2-y^2+4z$, show that $\nabla^2u=0$.

8. If $u=3x^2y$ and $v=xz^2-2y$, then find
   $$\text{grad }[(\text{grad } u)\cdot(\text{grad } v)].$$
   Ans. $(6yz^2-4x)\,\mathbf{i}+6xz^2\,\mathbf{j}+12xyz\,\mathbf{k}$.

9. If $\mathbf{f}=(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}$, prove that
   $$\mathbf{f}\cdot\text{curl }\mathbf{f}=0.$$ **[Kanpur 1980; Agra 78, 80]**

10. If $\mathbf{f}=f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k}$, show that
    $$\nabla\cdot\mathbf{f}=\nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k},$$

$$\nabla\times\mathbf{f}=\nabla f_1\times\mathbf{i}+\nabla f_2\times\mathbf{j}+\nabla f_3\times\mathbf{k}.$$

11. Find the constants a, b, c so that the vector
$\mathbf{F}=(x+2y+az)\,\mathbf{i}+(bx-3y-z)\,\mathbf{j}+(4x+cy+2z)\,\mathbf{k}$
is irrotational.
Ans. $a=4,\ b=2,\ c=-1$.

§ 11. Important Vector Identities.

1. *Prove that div* $(\mathbf{A}+\mathbf{B})=$ *div* $\mathbf{A}+$ *div* $\mathbf{B}$

or $$\nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}.$$

Proof. We have

$$\text{div}\,(\mathbf{A}+\mathbf{B})=\nabla\cdot(\mathbf{A}+\mathbf{B})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\mathbf{A}+\mathbf{B})$$
$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\mathbf{A}+\mathbf{B})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{B})$$
$$=\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)+\mathbf{j}\cdot\left(\frac{\partial\mathbf{A}}{\partial y}+\frac{\partial\mathbf{B}}{\partial y}\right)+\mathbf{k}\cdot\left(\frac{\partial\mathbf{A}}{\partial z}+\frac{\partial\mathbf{B}}{\partial z}\right)$$
$$=\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{A}}{\partial z}\right)+\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{B}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{B}}{\partial z}\right)$$
$$=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}=\text{div}\,\mathbf{A}+\text{div}\,\mathbf{B}.$$

2. *Prove that curl* $(\mathbf{A}+\mathbf{B})=$ *curl* $\mathbf{A}+$ *curl* $\mathbf{B}$

or $$\nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}.$$

Proof. We have curl $(\mathbf{A}+\mathbf{B})=\nabla\times(\mathbf{A}+\mathbf{B})$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\mathbf{A}+\mathbf{B})=\Sigma\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})=\Sigma\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)$$
$$=\Sigma\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}+\Sigma\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}=\text{curl}\,\mathbf{A}+\text{curl}\,\mathbf{B}.$$

3. *If* $\mathbf{A}$ *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

$$\text{div}\,(\phi\mathbf{A})=(\text{grad}\,\phi)\cdot\mathbf{A}+\phi\,\text{div}\,\mathbf{A}$$

or $$\nabla\cdot(\phi\mathbf{A})=(\nabla\phi)\cdot\mathbf{A}+\phi\,(\nabla\cdot\mathbf{A}).$$

[Meerut B. Sc. Physics 1983; Venkateswara 74; Rohilkhand 80; Agra 71, 74; Bombay 70]

Proof. We have

$$\text{div}\,(\phi\mathbf{A})=\nabla\cdot(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\phi\mathbf{A})$$
$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\phi\mathbf{A})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\phi\mathbf{A})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\phi\mathbf{A})$$
$$=\Sigma\left\{\mathbf{i}\cdot\frac{\partial}{\partial x}(\phi\mathbf{A})\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$
$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\cdot\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note $\mathbf{a}\cdot(m\mathbf{b})=(m\mathbf{a})\cdot\mathbf{b}=m\,(\mathbf{a}\cdot\mathbf{b})$]

$$=\left\{\Sigma\frac{\partial\phi}{\partial x}\mathbf{i}\right\}\cdot\mathbf{A}+\phi\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A}).$$

4. *Prove that curl* $(\phi\mathbf{A})=($*grad* $\phi)\times\mathbf{A}+\phi$ *curl* $\mathbf{A}$

or $$\nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

[Agra 1968; Meerut 67, 68, 72; Bombay 68; Kanpur 76; Punjab 63]

Proof. We have

$$\text{curl}\,(\phi\mathbf{A})=\nabla\times(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\phi\mathbf{A})$$
$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\phi\mathbf{A})\right\}=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$
$$=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$
$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\times\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note that $\mathbf{a}\times(m\mathbf{b})=(m\mathbf{a})\times\mathbf{b}=m\,(\mathbf{a}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\right\}\times\mathbf{A}+\phi\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

5. *Prove that div* $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot$*curl* $\mathbf{A}-\mathbf{A}\cdot$*curl* $\mathbf{B}$

or $$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B}).$$

[Agra 1970, Punjab 66, Banaras 68, Calicut 74, Allahabad 76, 78, 79, Meerut 72]

Proof. We have

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\Sigma\left\{\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}+\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$
$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$
$$=\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\cdot\mathbf{B}\right\}-\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{A}\right)\right\}$$

[Note $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\,\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\cdot\mathbf{B}-\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\cdot\mathbf{A}\right\}=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\cdot\mathbf{A}$$
$$=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-(\text{curl}\,\mathbf{B})\cdot\mathbf{A}=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}.$$

6. *Prove that*

$$curl\ (\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\ \mathbf{A}-\mathbf{B}\ div\ \mathbf{A}-(\mathbf{A}\cdot\nabla)\ \mathbf{B}+\mathbf{A}\ div\ \mathbf{B}.$$

[Agra 1972, 74, Allahabad 77, Punjab 61]

Proof. We have curl $(\mathbf{A}\times\mathbf{B})=\nabla\times(\mathbf{A}\times\mathbf{B})$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}-(\mathbf{i}\cdot\mathbf{A})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{i}\cdot\mathbf{B})\frac{\partial\mathbf{A}}{\partial x}-\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}\right\}-\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{B}\cdot\mathbf{i})\frac{\partial\mathbf{A}}{\partial x}\right\}-\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\mathbf{A}-\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\left\{\mathbf{B}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{A}-\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\mathbf{B}$$

$$=(\text{div}\ \mathbf{B})\ \mathbf{A}-(\mathbf{A}\cdot\nabla)\ \mathbf{B}+(\mathbf{B}\cdot\nabla)\ \mathbf{A}-(\text{div}\ \mathbf{A})\ \mathbf{B}.$$

7. *Prove that*

$$grad\ (\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\ \mathbf{A}+(\mathbf{A}\cdot\nabla)\ \mathbf{B}+\mathbf{B}\times curl\ \mathbf{A}+\mathbf{A}\times curl\ \mathbf{B}.$$

[Allahabad 1980, 82, Rohilkhand 78, Punjab 67, Banaras 68]

Proof. We have

$$\text{grad}\ (\mathbf{A}\cdot\mathbf{B})=\nabla\ (\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\cdot\mathbf{B}\right)$$

$$=\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}+\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}. \quad \ldots(1)$$

Now we know that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\ \mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\ \mathbf{c}$.

$\therefore\quad (\mathbf{a}\cdot\mathbf{b})\ \mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\ \mathbf{b}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

$$\therefore\quad\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}=(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}-\mathbf{A}\times\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{i}\right)$$

$$=(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}+\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right).$$

$$\text{Thus}\ \Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\mathbf{A}\times\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=(\mathbf{A}\cdot\nabla)\ \mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B}). \quad \ldots(2)$$

$$\text{Similarly}\ \Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}=(\mathbf{B}\cdot\nabla)\ \mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}). \quad \ldots(3)$$

Putting the values from (2) and (3) in (1), we get

$$\text{grad}\ (\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\ \mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{B}\cdot\nabla)\ \mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}).$$

Note. If we put **A** in place of **B**, then

$$\text{grad}\ (\mathbf{A}\cdot\mathbf{A})=2\ (\mathbf{A}\cdot\nabla)\ \mathbf{A}+2\mathbf{A}\times(\nabla\times\mathbf{A})$$

or $$\tfrac{1}{2}\ \text{grad}\ \mathbf{A}^2=(\mathbf{A}\cdot\nabla)\ \mathbf{A}+\mathbf{A}\times\text{curl}\ \mathbf{A}.$$

8. *Prove that div grad* $\phi=\nabla^2\phi$

i.e. $$\nabla\cdot(\nabla\phi)=\nabla^2\phi.$$ **[Rohilkhand 1981; Agra 70]**

Proof. We have

$$\nabla\cdot(\nabla\phi)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero*

i.e. $$\nabla\times(\nabla\phi)=\mathbf{0},\ i.e.\ \ curl\ \ grad\ \ \phi=\mathbf{0}.$$

[Rohilkhand 1981; Agra 74; Delhi 64; Banaras 70; Meerut 72; Kerala 74; Venkateswara 74; Kanpur 70]

Proof. We have grad $\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$.

$\therefore\quad$ curl grad $\phi=\nabla\times$grad ϕ

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10. *Prove that div curl* $\mathbf{A}=0$, *i.e.*, $\nabla\cdot(\nabla\times\mathbf{A})=0$.

[Agra 1970, Kerala 74; Kolhapur 73; Bombay 68]

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

$$\text{Then curl}\ \mathbf{A}=\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

Now div curl $\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$=\frac{\partial^2 A_3}{\partial x\,\partial y}-\frac{\partial^2 A_2}{\partial x\,\partial z}+\frac{\partial^2 A_1}{\partial y\,\partial z}-\frac{\partial^2 A_3}{\partial y\,\partial x}+\frac{\partial^2 A_2}{\partial z\,\partial x}-\frac{\partial^2 A_1}{\partial z\,\partial y}$$

$=0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

11. *Prove that*

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$. **[Meerut B.Sc. Physics 1983; Allahabad 81; Agra 71]**

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then
$$\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix}$$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore\quad \nabla\times(\nabla\times\mathbf{A})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial A_3}{\partial y}-\dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z}-\dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x}-\dfrac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y\,\partial x}+\frac{\partial^2 A_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})-(\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})\right\}\mathbf{i}\right]-\nabla^2\Sigma A_1\mathbf{i}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}.$$

Solved Examples

Ex. 1. *Prove that grad $f(u)=f'(u)$ grad u.*

Solution. We have

$$\text{grad } f(u)=\mathbf{i}\frac{\partial}{\partial x}f(u)+\mathbf{j}\frac{\partial}{\partial y}f(u)+\mathbf{k}\frac{\partial}{\partial z}f(u)$$

$$=\mathbf{i}\,f'(u)\frac{\partial u}{\partial x}+\mathbf{j}\,f'(u)\frac{\partial u}{\partial y}+\mathbf{k}\,f'(u)\frac{\partial u}{\partial z}$$

$$=f'(u)\left[\mathbf{i}\frac{\partial u}{\partial x}+\mathbf{j}\frac{\partial u}{\partial y}+\mathbf{k}\frac{\partial u}{\partial z}\right]=f'(u)\text{ grad } u.$$

Ex. 2. *Taking $\mathbf{F}=x^2y\,\mathbf{i}+xz\,\mathbf{j}+2yz\,\mathbf{k}$ verify that div curl $\mathbf{F}=0$.* **[Agra 1968]**

Solution. We have
$$\text{Curl }\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & xz & 2yz\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(x^2y)\right]\mathbf{k}$$

$=(2z-x)\,\mathbf{i}-0\,\mathbf{j}+(z-x^2)\,\mathbf{k}=(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}$.

Now div curl $\mathbf{F}=$ div $[(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}]$

$$=\frac{\partial}{\partial x}(2z-x)+\frac{\partial}{\partial z}(z-x^2)=-1+1=0.$$

Ex. 3. *Find $\nabla\phi$ and $|\nabla\phi|$ when*

$$\phi=(x^2+y^2+z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}$$

Solution. Let $r^2=x^2+y^2+z^2$. Then we can write $\phi=r^2e^{-r}$.

Now $\nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$.

We have $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}=[2re^{-r}-r^2e^{-r}]\dfrac{\partial r}{\partial x}$.

But $r^2=x^2+y^2+z^2$.

Therefore $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

So $\dfrac{\partial\phi}{\partial x}=re^{-r}(2-r)\dfrac{x}{r}=(2-r)\,e^{-r}x$.

Similarly $\dfrac{\partial\phi}{\partial y}=(2-r)\,e^{-r}y$ and $\dfrac{\partial\phi}{\partial z}=(2-r)\,e^{-r}z$.

Therefore $\nabla\phi=(2-r)\,e^{-r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(2-r)\,e^{-r}\mathbf{r}$.

Also $|\nabla\phi|=|(2-r)\,e^{-r}\mathbf{r}|=(2-r)\,e^{-r}|\mathbf{r}|=(2-r)\,e^{-r}r$.

Ex. 4. *Prove that div $(r^n\mathbf{r})=(n+3)\,r^n$.* **[Meerut 1971; Rohilkhand 78; Agra 76]**

Solution. We have

$$\text{div}(\phi\mathbf{A})=\phi(\text{div }\mathbf{A})+\mathbf{A}\cdot\text{grad }\phi.$$

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=r^n$ in this identity, we get

$$\text{div}\,(r^n\,\mathbf{r})=r^n\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^n$$

$$=3r^n+\mathbf{r}\cdot(nr^{n-1}\,\text{grad}\,r)\qquad[\because\ \text{div}\,\mathbf{r}=3\ \text{and grad}\,f(u)=f'(u)\,\text{grad}\,u]$$

$$=3r^n+\mathbf{r}\cdot\left[nr^{n-1}\,\frac{1}{r}\,\mathbf{r}\right]\qquad\left[\because\ \text{grad}\,r=\hat{\mathbf{r}}=\frac{1}{r}\,\mathbf{r}\right]$$

$$=3r^n+nr^{n-2}\,(\mathbf{r}\cdot\mathbf{r})=3r^n+nr^{n-2}\,r^2=(n+3)\,r^n.$$

Ex. 5. *Prove that* $\nabla^2\,(r^n\,\mathbf{r})=n\,(n+3)\,r^{n-2}\,\mathbf{r}.$ **[Agra 1970]**

Solution. We have $\nabla^2\,(r^n\,\mathbf{r})=\nabla\,[\nabla\cdot(r^n\,\mathbf{r})]=\text{grad}\,[\text{div}\,(r^n\mathbf{r})]$

$$=\text{grad}\,[(\text{grad}\,r^n)\cdot\mathbf{r}+r^n\,\text{div}\,\mathbf{r}]$$

$$=\text{grad}\,[(nr^{n-2}\,\mathbf{r})\cdot\mathbf{r}+3r^n]=\text{grad}\,[nr^{n-2}\,r^2+3r^n]$$

$$=\text{grad}\,[nr^{n-2}\,r^2+3r^n]=\text{grad}\,[(n+3)\,r^n]$$

$$=(n+3)\,\text{grad}\,r^n=(n+3)\,nr^{n-2}\,\mathbf{r}=n\,(n+3)\,r^{n-2}\,\mathbf{r}.$$

Ex. 6. *Prove that* $\text{div}\left(\dfrac{\mathbf{r}}{r^3}\right)=0.$ **[Banaras 1970]**

Solution. We have $\text{div}\left(\dfrac{1}{r^3}\,\mathbf{r}\right)=\text{div}\,(r^{-3}\,\mathbf{r})$

$$=r^{-3}\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^{-3}=3r^{-3}+\mathbf{r}\cdot(-3r^{-4}\,\text{grad}\,r)$$

$$=3r^{-3}+\mathbf{r}\cdot\left(-3r^{-4}\,\frac{1}{r}\,\mathbf{r}\right)$$

$$=3r^{-3}-3r^{-5}\,(\mathbf{r}\cdot\mathbf{r})=3r^{-3}-3r^{-5}\,r^2=3r^{-3}-3r^{-3}=0.$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Ex. 7. *Prove that* $\text{div}\,\hat{\mathbf{r}}=2/r.$ **[Kanpur 1979]**

Solution. $\text{div}\,(\hat{\mathbf{r}})=\text{div}\left(\dfrac{1}{r}\,\mathbf{r}\right).$ Now proceed as in Ex. 4.

Alternative Method.

$$\text{div}\,\hat{\mathbf{r}}=\text{div}\left(\frac{1}{r}\,\mathbf{r}\right)=\text{div}\left[\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right]$$

$$=\text{div}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$=\left(\frac{1}{r}-\frac{x}{r^2}\,\frac{\partial r}{\partial x}\right)+\left(\frac{1}{r}-\frac{y}{r^2}\,\frac{\partial r}{\partial y}\right)+\left(\frac{1}{r}-\frac{z}{r^2}\,\frac{\partial r}{\partial z}\right).$$

Now $r^2=x^2+y^2+z^2.$ $\therefore$ $2r\,\dfrac{\partial r}{\partial x}=2x$ *i.e.* $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

Similarly $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}.$

$$\therefore\ \text{div}\,\hat{\mathbf{r}}=\frac{3}{r}-\left(\frac{x}{r^2}\,\frac{x}{r}+\frac{y}{r^2}\,\frac{y}{r}+\frac{z}{r^2}\,\frac{z}{r}\right)$$

$$=\frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}.$$

Ex. 8. *Prove that the vector* $f(r)\,\mathbf{r}$ *is irrotational.* **[Agra 1974; Kanpur 1975]**

Solution. The vector $f(r)\,\mathbf{r}$ will be irrotational if

$$\text{curl}\,[f(r)\,\mathbf{r}]=\mathbf{0}.$$

We know that $\text{Curl}\,(\phi\mathbf{A})=(\text{grad}\,\phi)\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$

Putting $\phi=f(r)$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

$$\text{Curl}\,[f(r)\,\mathbf{r}]=[\text{grad}\,f(r)]\times\mathbf{r}+f(r)\,\text{curl}\,\mathbf{r}$$

$$=[f'(r)\,\text{grad}\,r]\times\mathbf{r}+f(r)\,\mathbf{0}\qquad[\because\ \text{curl}\,\mathbf{r}=\mathbf{0}]$$

$$=\left[f'(r)\,\frac{1}{r}\,\mathbf{r}\right]\times\mathbf{r}=f'(r)\,\frac{1}{r}\,(\mathbf{r}\times\mathbf{r})=\mathbf{0},\ \text{since}\ \mathbf{r}\times\mathbf{r}=\mathbf{0}.$$

$\therefore$ The vector $f(r)\,\mathbf{r}$ is irrotational.

Ex. 9. (a) *Prove that* $\nabla^2 f(r)=f''(r)+\dfrac{2}{r}\,f'(r).$

Solution. We know that if ϕ is a scalar function then

$$\nabla^2\phi=\nabla\cdot(\nabla\phi).$$

$$\therefore\ \nabla^2 f(r)=\nabla\cdot\{\nabla\,f(r)\}=\text{div}\,\{\text{grad}\,f(r)\}$$

$$=\text{div}\,\{f'(r)\,\text{grad}\,r\}=\text{div}\left\{\frac{1}{r}\,f'(r)\,\mathbf{r}\right\}$$

$$=\frac{1}{r}\,f'(r)\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\left\{\frac{1}{r}\,f'(r)\right\}$$

$$=\frac{3}{r}\,f'(r)+\mathbf{r}\cdot\left[\frac{d}{dr}\left\{\frac{1}{r}\,f'(r)\right\}\,\text{grad}\,r\right]$$

$$=\frac{3}{r}\,f'(r)+\mathbf{r}\cdot\left[\left\{-\frac{1}{r^2}\,f'(r)+\frac{1}{r}\,f''(r)\right\}\frac{1}{r}\,\mathbf{r}\right]$$

$$=\frac{3}{r}\,f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}\,f'(r)+\frac{1}{r}\,f''(r)\right\}\right](\mathbf{r}\cdot\mathbf{r})$$

$$=\frac{3}{r}\,f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}\,f'(r)+\frac{1}{r}\,f''(r)\right\}\right]r^2$$

$$=\frac{3}{r}\,f'(r)-\frac{1}{r}\,f'(r)+f''(r)=f''(r)+\frac{2}{r}\,f'(r).$$

Ex. 9. (b) *If* $\nabla^2 f(r)=0,$ *show that*

$$f(r)=\frac{c_1}{r}+c_2,$$

where $r^2=x^2+y^2+z^2$ *and* $c_1,\,c_2$ *are arbitrary constants.* **[Bombay 1969]**

Solution. As shown in the preceding example, if

$r^2=x^2+y^2+z^2,$ then $\nabla^2 f(r)=f''(r)+\dfrac{2}{r}\,f'(r).$

$\therefore$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{2}{r}f'(r)=0 \quad \text{or} \quad \frac{f''(r)}{f'(r)}=-\frac{2}{r}.$$

Integrating with respect to r, we get

$\log f'(r)=-2\log r+\log c$, where c is a constant

$$=\log\frac{c}{r^2}.$$

$$\therefore \quad f'(r)=\frac{c}{r^2}.$$

Again integrating,

$f(r)=-\dfrac{c}{r}+c_2$ where c_2 is a constant

$=\dfrac{c_1}{r}+c_2$, replacing $-c$ by c_1.

Ex. 10. *Prove that* $\nabla^2\left(\frac{1}{r}\right)=0.$

[Agra 1976; Rohilkhand 81; Kanpur 76]

Solution. We have

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)=\text{div}\left(\text{grad}\,\frac{1}{r}\right)$$

$$=\text{div}\left(-\frac{1}{r^2}\,\text{grad}\,r\right)=\text{div}\left(-\frac{1}{r^2}\,\frac{1}{r}\,\mathbf{r}\right)=\text{div}\left(-\frac{1}{r^3}\,\mathbf{r}\right)$$

$$=\left(-\frac{1}{r^3}\right)\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\left(-\frac{1}{r^3}\right)=-\frac{3}{r^3}+\mathbf{r}\cdot\left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{grad}\,r\right]$$

$$=-\frac{3}{r^3}+\mathbf{r}\cdot\left(\frac{3}{r^4}\,\frac{1}{r}\,\mathbf{r}\right)=-\frac{3}{r^3}+\frac{3}{r^5}\,(\mathbf{r}\cdot\mathbf{r})=-\frac{3}{r^3}+\frac{3}{r^5}\,r^2=0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Ex. 11. *Prove that div grad* $r^n=n\,(n+1)\,r^{n-2}$,

i.e., $\nabla^2 r^n=n\,(n+1)\,r^{n-2}$.

[Kanpur 1978, 80; Rohilkhand 81; Agra 69; Calicut 75]

Solution. We have $\nabla^2 r^n=\nabla\cdot(\nabla r^n)=\text{div}\,(\text{grad}\,r^n)$

$$=\text{div}\,(nr^{n-1}\,\text{grad}\,r)=\text{div}\left(nr^{n-1}\,\frac{1}{r}\,\mathbf{r}\right)=\text{div}\,(nr^{n-2}\mathbf{r})$$

$$=(nr^{n-2})\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot(\text{grad}\,nr^{n-2})$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n\,(n-2)\,r^{n-3}\,\text{grad}\,r]$$

$$=3nr^{n-2}+\mathbf{r}\cdot\left[n\,(n-2)\,r^{n-3}\,\frac{1}{r}\,\mathbf{r}\right]$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n\,(n-2)\,r^{n-4}\,\mathbf{r}]=3nr^{n-2}+n\,(n-2)\,r^{n-4}\,(\mathbf{r}\cdot\mathbf{r})$$

$$=3nr^{n-2}+n\,(n-2)\,r^{n-4}\,r^2=nr^{n-2}\,(3+n-2)=n\,(n+1)\,r^{n-2}.$$

Note. If $n=-1$, then $\nabla^2(r^{-1})=(-1)(-1+1)\,r^{-3}=0$.

Ex. 12. *Prove that* $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$

[Meerut 1972; Bombay 70]

Solution. We have $\nabla^2(\phi\psi)=\nabla\cdot[\nabla(\phi\psi)]$

$$=\nabla\cdot[\phi\,(\nabla\psi)+\psi\,(\nabla\phi)]=\nabla\cdot[\phi\,(\nabla\psi)]+\nabla\cdot[\psi\,(\nabla\phi)]$$

$$=\phi\nabla\cdot(\nabla\psi)+(\nabla\psi)\cdot(\nabla\phi)+\psi\nabla\cdot(\nabla\phi)+(\nabla\psi)\cdot(\nabla\phi)$$

$$=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$$

Ex. 13. *Prove that div* $(\nabla\phi\times\nabla\psi)=0.$

Solution. We know that

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{Curl}\,\mathbf{A}-\mathbf{A}\cdot\text{Curl}\,\mathbf{B}.$$

$$\therefore\ \text{div}\,(\nabla\phi\times\nabla\psi)=(\nabla\psi)\cdot(\text{Curl}\,\nabla\phi)-(\nabla\phi)\cdot(\text{Curl}\,\nabla\psi)$$

$=(\nabla\psi)\cdot\mathbf{0}-(\nabla\phi)\cdot\mathbf{0}$ **[$\because$ curl grad $\phi=\mathbf{0}$]**

$=0.$

Ex. 14. *If* **A** *and* **B** *are irrotational, prove that* $\mathbf{A}\times\mathbf{B}$ *is solenoidal.* **[Bombay 1970; Kanpur 77, 79]**

Solution. If **A** and **B** are irrotational, then

curl $\mathbf{A}=\mathbf{0}$, curl $\mathbf{B}=\mathbf{0}$.

Now div $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}=\mathbf{B}\cdot\mathbf{0}-\mathbf{A}\cdot\mathbf{0}=0.$

Since div $(\mathbf{A}\times\mathbf{B})$ is zero, therefore $\mathbf{A}\times\mathbf{B}$ is solenoidal.

Ex. 15. *Prove that curl* $(\phi\ \textit{grad}\ \phi)=\mathbf{0}.$

Solution. We know that

$$\text{curl}\,(\phi\mathbf{A})=\text{grad}\,\phi\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$$

Putting grad ϕ in place of **A**, we get

$$\text{curl}\,(\phi\,\text{grad}\,\phi)=\text{grad}\,\phi\times\text{grad}\,\phi+\phi\,\text{curl grad}\,\phi$$

$$=\mathbf{0}+\phi\mathbf{0}.$$

Here grad $\phi\times$ grad $\phi=\mathbf{0}$, since it is the cross product of two equal vectors. Also curl grad $\phi=\mathbf{0}$.

$\therefore$ curl $(\phi$ grad $\phi)=\mathbf{0}+\mathbf{0}=\mathbf{0}$.

Ex. 16. *If f and g are two scalar point functions, prove that div* $(f\,\nabla g)=f\,\nabla^2 g+\nabla f\cdot\nabla g.$ **[Meerut 1972]**

Solution. We have $\nabla g=\dfrac{\partial g}{\partial x}\mathbf{i}+\dfrac{\partial g}{\partial y}\mathbf{j}+\dfrac{\partial g}{\partial z}\mathbf{k}.$

Therefore $f\,\nabla g=f\,\dfrac{\partial g}{\partial x}\,\mathbf{i}+f\,\dfrac{\partial g}{\partial y}\,\mathbf{j}+f\,\dfrac{\partial g}{\partial z}\,\mathbf{k}.$

$$\text{So}\quad \text{div}\,(f\,\nabla g)=\frac{\partial}{\partial x}\left(f\,\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\,\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\,\frac{\partial g}{\partial z}\right)$$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\,\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\,\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\,\frac{\partial g}{\partial z}\right)$$

$$=f\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)g$$

$$+\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right)$$

$$=f\,\nabla^2 g+\nabla f\cdot\nabla g.$$

Ex. 17. *A vector function* **f** *is the product of a scalar function and the gradient of a scalar function. Show that*

$$\mathbf{f}\cdot \textit{curl}\ \mathbf{f}=0. \qquad \textbf{[Kerala 1975]}$$

Solution. Let $\mathbf{f}=\psi$ grad ϕ, where ψ and ϕ are scalar functions. We nave curl $\mathbf{f}=$cur (ψ grad ϕ).

We know that curl $(\phi\mathbf{A})=(\text{grad }\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

$\therefore$ curl (ψ grad ϕ)=(grad ψ)$\times$(grad ϕ)+ψ (curl grad ϕ)

=(grad ψ)$\times$(grad ϕ) [$\because$ curl grad $\phi=0$]

Now $\mathbf{f}\cdot$curl $\mathbf{f}=(\psi$ grad $\phi)\cdot\{(\text{grad }\psi)\times(\text{grad }\phi)\}$

$=[\psi$ grad ϕ, grad ψ, grad $\phi]=\psi$ [grad ϕ, grad ψ, grad ϕ]

$=0$, since the value of a scalar triple product is zero if two vectors are equal.

Ex. 18. *Prove that* $\nabla\cdot\left\{r\nabla\left(\frac{1}{r^3}\right)\right\}=\frac{3}{r^4}$

or, *div* $[\mathbf{r}$ *grad* $r^{-3}]=3r^{-4}$.

Solution. We have $\nabla\left(\frac{1}{r^3}\right)=$grad r^{-3}

$$=\frac{\partial}{\partial x}(r^{-3})\,\mathbf{i}+\frac{\partial}{\partial y}(r^{-3})\,\mathbf{j}+\frac{\partial}{\partial z}(r^{-3})\,\mathbf{k}.$$

Now $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{\partial r}{\partial x}$. But $r^2=x^2+y^2+z^2$.

Therefore $2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$.

So $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{x}{r}=-3r^{-5}x$.

Similarly $\frac{\partial}{\partial y}(r^{-3})=-3r^{-5}\,y$ and $\frac{\partial}{\partial z}(r^{-3})=-3r^{-5}\,z$.

Therefore $\nabla\left(\frac{1}{r^3}\right)=-3r^{-5}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

$$\therefore\quad r\,\nabla\left(\frac{1}{r^3}\right)=-3r^{-4}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

$$\therefore\quad \nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=\frac{\partial}{\partial x}(-3r^{-4}x)+\frac{\partial}{\partial y}(-3r^{-4}y)+\frac{\partial}{\partial z}(-3r^{-4}z).$$

Now $\frac{\partial}{\partial x}(-3r^{-4}x)=12\,r^{-5}\frac{\partial r}{\partial x}x-3r^{-4}$

$$=12\,r^{-5}\frac{x}{r}x-3r^{-4}=12r^{-6}x^2-3r^{-4}.$$

Similarly $\frac{\partial}{\partial y}(-3r^{-4}y)=12r^{-6}y^2-3r^{-4}$

and $\frac{\partial}{\partial z}(-3r^{-4}z)=12r^{-6}z^2-3r^{-4}$.

Hence $\nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=12r^{-6}(x^2+y^2+z^2)-9r^{-4}$

$$=12r^{-6}r^2-9r^{-4}=12r^{-4}-9r^{-4}=3r^{-4}.$$

Ex. 19. *Prove that* $\mathbf{a}\cdot\left(\nabla\frac{1}{r}\right)=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$.

Solution. We have

$$\text{grad}\,\frac{1}{r}=-\frac{1}{r^2}\,\text{grad}\,r=-\frac{1}{r^2}\frac{1}{r}\,\mathbf{r}=-\frac{1}{r^3}\,\mathbf{r}.$$

$$\therefore\quad \mathbf{a}\cdot\left(\nabla\frac{1}{r}\right)=\mathbf{a}\cdot\left(-\frac{1}{r^3}\mathbf{r}\right)=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

Ex. 20. *Prove that*

$$\mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\frac{1}{r}\right)=\frac{3\,(\mathbf{a}\cdot\mathbf{r})\,(\mathbf{b}\cdot\mathbf{r})}{r^5}-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}$$

where **a** *and* **b** *are constant vectors.*

Solution. As shown in the last example, we have

$$\mathbf{a}\cdot\nabla\frac{1}{r}=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

$$\therefore\quad \mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\frac{1}{r}\right)=\mathbf{b}\cdot\nabla\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)=\mathbf{b}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\frac{\partial}{\partial x}(\mathbf{a}\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r})\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\left(\mathbf{a}\cdot\frac{\partial\mathbf{r}}{\partial x}\right)+3\,(\mathbf{a}\cdot\mathbf{r})\,r^{-4}\frac{\partial r}{\partial x}\right\}$$

[$\because$ **a** is a constant vector]

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{\mathbf{a}\cdot\mathbf{i}}{r^3}+\frac{3x}{r^5}(\mathbf{a}\cdot\mathbf{r})\right\}\quad\left[\because\ \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i}\text{ and }\frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\mathbf{b}\cdot\Sigma\left\{-\frac{1}{r^3}(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\,x\mathbf{i}\right\}$$

$$=\mathbf{b}\cdot\left\{-\frac{1}{r^3}\,\mathbf{a}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right\}\quad[\because\ \Sigma(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}=\mathbf{a},\text{ and }\Sigma x\mathbf{i}=\mathbf{r}]$$

$$=-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}+\frac{3\,(\mathbf{a}\cdot\mathbf{r})\,(\mathbf{b}\cdot\mathbf{r})}{r^5}.$$

Ex. 21. *Prove that div* $(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot$*curl* $\mathbf{A}$. **[Rohilkhand 1979]**

Solution. We know that

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}.$$

$$\therefore \quad \text{div}\,(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{r}$$
$$=\mathbf{r}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\mathbf{0} \qquad [\because \text{curl}\,\mathbf{r}=\mathbf{0}]$$
$$=\mathbf{r}\cdot\text{curl}\,\mathbf{A}.$$

Ex. 22. *If* $\mathbf{a}$ *is a constant vector, prove that*

$$\text{div}\,\{r^n(\mathbf{a}\times\mathbf{r})\}=0. \qquad \text{[Allahabad 1980; Rohilkhand 77]}$$

Solution. We have

$$\text{div}\,(\phi\mathbf{A})=\phi\,\text{div}\,\mathbf{A}+\mathbf{A}\cdot\text{grad}\,\phi.$$

$$\therefore \quad \text{div}\,\{r^n(\mathbf{a}\times\mathbf{r})\}=r^n\,\text{div}\,(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\text{grad}\,r^n$$
$$=r^n\,\text{div}\,(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot(nr^{n-1}\,\text{grad}\,r)$$
$$=r^n\,(\mathbf{r}\cdot\text{curl}\,\mathbf{a}-\mathbf{a}\cdot\text{curl}\,\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\left(nr^{n-1}\frac{1}{r}\,\mathbf{r}\right)$$
$$=r^n\,(\mathbf{r}\cdot\mathbf{0}-\mathbf{a}\cdot\mathbf{0})+nr^{n-2}\,(\mathbf{a}\times\mathbf{r})\cdot\mathbf{r}$$
$$[\because \text{curl of constant vector is zero and curl}\,\mathbf{r}=\mathbf{0}]$$
$$=nr^{n-2}\,[\mathbf{a},\mathbf{r},\mathbf{r}]$$

$=0$, since a scalar triple product having two equal vectors is zero.

Ex. 23. *Prove that*

$$\nabla\cdot(U\nabla V-V\nabla U)=U\nabla^2V-V\nabla^2U.$$

[Meerut 1969; Bombay 69; Agra 70]

Solution. We have $\nabla\cdot(U\nabla V-V\nabla U)$

$$=\nabla\cdot(U\nabla V)-\nabla\cdot(V\nabla U).$$

Now $\nabla\cdot(U\nabla V)=U\{\nabla\cdot(\nabla V)\}+(\nabla U)\cdot(\nabla V)$

$$=U\nabla^2V+(\nabla U)\cdot(\nabla V).$$

Interchanging U and V, we get

$$\nabla\cdot(V\nabla U)=V\nabla^2U+(\nabla V)\cdot(\nabla U).$$

$$\therefore \quad \nabla\cdot(U\nabla V-V\nabla U)$$
$$=[U\nabla^2V+(\nabla U)\cdot(\nabla V)]-[V\nabla^2U+(\nabla V)\cdot(\nabla U)]$$
$$=U\nabla^2V-V\nabla^2U.$$

Ex. 24. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors, prove that*

(*i*) *div* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=-2\mathbf{b}\cdot\mathbf{a}$, [Rohilkhand 1979]

(*ii*) *curl* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}$. [Rohilkhand 1979]

Solution. (i) We have $(\mathbf{r}\times\mathbf{a})\times\mathbf{b}=(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}$.

$$\therefore \quad \text{div}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\text{div}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$$
$$=\text{div}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-\text{div}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}] \qquad \ldots(1)$$

But $\text{div}\,(\phi\mathbf{A})=\phi\,\text{div}\,\mathbf{A}+\mathbf{A}\cdot\text{grad}\,\phi$.

Taking $\phi=\mathbf{b}\cdot\mathbf{r}$ and $\mathbf{A}=\mathbf{a}$, we get

$\text{div}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=(\mathbf{b}\cdot\mathbf{r})\,\text{div}\,\mathbf{a}+\mathbf{a}\cdot\text{grad}\,(\mathbf{b}\cdot\mathbf{r})$.

Since $\mathbf{a}$ is a constant vector, therefore $\text{div}\,\mathbf{a}=0$.

Also let $\quad \mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$.

Then $\mathbf{b}\cdot\mathbf{r}=(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$=b_1x+b_2y+b_3z$ where b_1, b_2, b_3 are constants.

$$\therefore \quad \text{grad}\,(\mathbf{b}\cdot\mathbf{r})=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}=\mathbf{b}.$$
$$\therefore \quad \text{div}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=\mathbf{a}\cdot\mathbf{b}. \qquad \ldots(2)$$

Again $\text{div}\,[(\mathbf{b}\cdot\mathbf{a})]\,\mathbf{r}=(\mathbf{b}\cdot\mathbf{a})\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,(\mathbf{b}\cdot\mathbf{a})$.

But $\text{div}\,\mathbf{r}=3$. Also $\text{grad}\,(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}$ because $\mathbf{b}\cdot\mathbf{a}$ is constant.

$$\therefore \quad \text{div}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=3\,(\mathbf{b}\cdot\mathbf{a}). \qquad \ldots(3)$$

Substituting the values from (2) and (3) in (1), we get

$$\text{div}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=(\mathbf{a}\cdot\mathbf{b})-3\,(\mathbf{b}\cdot\mathbf{a})=-2\mathbf{b}\cdot\mathbf{a}.$$

(ii) $\text{Curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$$=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-\text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}].$$

But $\quad \text{curl}\,(\phi\mathbf{A})=\text{grad}\,\phi\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}$.

$$\therefore \quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{b}\cdot\mathbf{r})\,\text{curl}\,\mathbf{a}$$
$$=\mathbf{b}\times\mathbf{a} \quad [\because \text{curl}\,\mathbf{a}=\mathbf{0} \text{ and grad}\,(\mathbf{b}\cdot\mathbf{r})=\mathbf{b}]$$

Also $\quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{a})]\times\mathbf{r}+(\mathbf{b}\cdot\mathbf{a})\,\text{curl}\,\mathbf{r}$

$$=\mathbf{0} \quad [\because \text{grad}\,(\mathbf{b}\cdot\mathbf{a})=\mathbf{0} \text{ and curl}\,\mathbf{r}=\mathbf{0}]$$

$$\therefore \quad \text{curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}-\mathbf{0}=\mathbf{b}\times\mathbf{a}.$$

Ex. 25. *If* $\mathbf{a}$ *is a constant vector, prove that*

$$\textit{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}\,(\mathbf{a}\cdot\mathbf{r}).$$

Solution. We have

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}.$$

Now $\dfrac{\partial}{\partial x}\left(\dfrac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\dfrac{3}{r^4}\dfrac{\partial r}{\partial x}\,(\mathbf{a}\times\mathbf{r})+\dfrac{1}{r^3}\left(\mathbf{a}\times\dfrac{\partial\mathbf{r}}{\partial x}\right)+\dfrac{1}{r^3}\left(\dfrac{\partial\mathbf{a}}{\partial x}\times\mathbf{r}\right) \qquad \ldots(1)$

Now $\dfrac{\partial\mathbf{a}}{\partial x}=\mathbf{0}$ because $\mathbf{a}$ is a constant vector.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\quad \therefore \dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i}$.

Further $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

$\therefore$ (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\,\frac{x}{r}\,(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,(\mathbf{a}\times\mathbf{i})$$
$$=-\frac{3x}{r^5}\,(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,(\mathbf{a}\times\mathbf{i}).$$

$$\therefore \quad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$
$$=-\frac{3x}{r^5}\Big[(\mathbf{i}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{r}\Big]+\frac{1}{r^3}\Big[(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}\Big]$$

$$= -\frac{3x}{r^5}\, x\mathbf{a} + \frac{3x}{r^5}\, a_1\mathbf{r} + \frac{1}{r^3}\,\mathbf{a} - \frac{1}{r^3}\, a_1\mathbf{i}$$

[∵ $\mathbf{i}\cdot\mathbf{r} = x$ and $\mathbf{i}\cdot\mathbf{a} = a_1$ if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$]

$$= -\frac{3x^2}{r^5}\,\mathbf{a} + \frac{3}{r^5}\, a_1 x\,\mathbf{r} + \frac{1}{r^3}\,\mathbf{a} - \frac{1}{r^3}\, a_1\mathbf{i}.$$

$$\therefore\ \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$= \left\{-\frac{3}{r^5}\,\Sigma x^2\right\}\mathbf{a} + \left\{\frac{3}{r^5}\,\Sigma\, a_1 x\right\}\mathbf{r} + \frac{3}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,\Sigma a_1\,\mathbf{i}$$

$$= -\frac{3}{r^5}\, r^2\,\mathbf{a} + \frac{3}{r^5}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r} + \frac{3}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,\mathbf{a}$$

[∵ $\Sigma x^2 = r^2$, $\Sigma a_1 x = \mathbf{r}\cdot\mathbf{a}$, $\Sigma a_1\mathbf{i} = \mathbf{a}$]

$$= -\frac{\mathbf{a}}{r^3} + \frac{3}{r^5}\,(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}.$$

Ex. 26. *Prove that div* $\left\{\frac{f(r)\,\mathbf{r}}{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$. [Agra 1971]

Solution. We have

$$\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\} = \text{div}\left\{\frac{f(r)}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\,x\right\} + \frac{\partial}{\partial y}\left\{\frac{f(r)}{r}\,y\right\} + \frac{\partial}{\partial z}\left\{\frac{f(r)}{r}\,z\right\} \qquad \ldots(1)$$

Now $\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}\,x\right\} = \frac{f(r)}{r} + x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$

$$= \frac{f(r)}{r} + x\left\{\frac{f'(r)}{r} - \frac{1}{r^2}\,f(r)\right\}\frac{x}{r} = \frac{f(r)}{r} + \frac{x^2}{r^2}\,f'(r) - \frac{x^2}{r^3}\,f(r).$$

Similarly $\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}\,y\right\} = \frac{f(r)}{r} + \frac{y^2}{r^2}\,f'(r) - \frac{y^2}{r^3}\,f(r)$

and $\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}\,z\right\} = \frac{f(r)}{r} + \frac{z^2}{r^2}\,f'(r) - \frac{z^2}{r^3}\,f(r).$

Putting these values in (1), we get

$$\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\} = \frac{3}{r}\,f(r) + \frac{r^2}{r^2}\,f'(r) - \frac{r^2}{r^3}\,f(r)$$

$$= \frac{2}{r}\,f(r) + f'(r) = \frac{1}{r^2}\left[2rf(r) + r^2 f'(r)\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right].$$

Exercises

1. Verify that curl grad $f = 0$, where
$f = x^2 y + 2xy + z^2$. [Agra 1973]

2. Prove that
curl $(\psi\nabla\phi) = \nabla\psi\times\nabla\phi = -\text{curl}\,(\phi\nabla\psi)$. [Bombay 1969]
3. Show that curl $(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a} = \mathbf{0}$, where $\mathbf{a}$ is a constant vector.
[Hint. Use identity 4. Note that $\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}$, if $\mathbf{a}$ is a constant vector.]
4. If $\mathbf{a}$ is a constant vector, then prove that
(i) $\nabla(\mathbf{a}\cdot\mathbf{u}) = (\mathbf{a}\cdot\nabla)\,\mathbf{u} + \mathbf{a}\times\text{curl}\,\mathbf{u}$,
(ii) $\nabla\cdot(\mathbf{a}\times\mathbf{u}) = -\mathbf{a}\cdot\text{curl}\,\mathbf{u}$,
(iii) $\nabla\times(\mathbf{a}\times\mathbf{u}) = \mathbf{a}\,\text{div}\,\mathbf{u} - (\mathbf{a}\cdot\nabla)\,\mathbf{u}$.
5. Prove that $\mathbf{a}\cdot\{\nabla(\mathbf{v}\cdot\mathbf{a}) - \nabla\times(\mathbf{v}\times\mathbf{a})\} = \text{div}\,\mathbf{v}$, where $\mathbf{a}$ is a constant *unit* vector.
6. Given that $\rho\mathbf{F} = \nabla p$ where ρ, p, $\mathbf{F}$ are point functions, prove that $\mathbf{F}\cdot\text{curl}\,\mathbf{F} = 0$. [Kerala 1975]
7. Show that curl $\mathbf{a}\phi(r) = \frac{1}{r}\,\phi'(r)\,\mathbf{r}\times\mathbf{a}$, where $\mathbf{a}$ is a constant vector.
8. Prove that curl $(\mathbf{a}\times\mathbf{r})\,r^n = (n+2)\,r^n\,\mathbf{a} - n\,r^{n-2}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}$. [Rohilkhand 1977]
9. Prove that curl grad $r^n = \mathbf{0}$.
10. If $\mathbf{r}$ is the position vector of the point (x, y, z) show that curl $(r^n\mathbf{r}) = \mathbf{0}$, where r is the module of $\mathbf{r}$. [Kanpur 1978]
11. Prove that $r^n\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n+3=0$. [Agra 1976; Rohilkhand 78]
12. If $\mathbf{u} = (1/r)\,\mathbf{r}$, show that $\nabla\times\mathbf{u} = \mathbf{0}$. [Kanpur 1979]
13. If $\nabla^2 f(r) = 0$, show that $f(r) = c_1\log r + c_2$ where $r^2 = x^2 + y^2$ and c_1, c_2 are arbitrary constants. [Poona 1970]
[Hint. First show that if $r^2 = x^2 + y^2$, then
$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(r) = \frac{f'(r)}{r} + f''(r).\Big]$$
14. If $\mathbf{u} = (1/r)\,\mathbf{r}$ find grad (div $\mathbf{u}$). [Kanpur 1976]
Ans. $(-2/r^3)\,\mathbf{r}$.
15. Prove that $\frac{1}{2}\nabla\mathbf{a}^2 = (\mathbf{a}\cdot\nabla)\,\mathbf{a} + \mathbf{a}\times\text{curl}\,\mathbf{a}$.
16. If $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that
$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r}) = \mathbf{a}\cdot\mathbf{b}$.
17. Prove that $\nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right] = 2r^{-4}$.
18. Evaluate div $\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}$, where $\mathbf{a}$ is a constant vector. [Kanpur 1976]
Ans. $2a^2$.

§ 12. Invariance.

Theorem 1. *Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator ∇ remains invariant.*

Proof. Let O be the fixed origin. Let Ox, Oy, Oz be one system of rectangular axes and Ox', Oy', Oz' be the other system of rectangular axes. Take **i, j, k** as unit vectors along Ox, Oy, Oz and **i', j', k'** as unit vectors along Ox', Oy', Oz'. Let P be any point in space whose co-ordinates are (x, y, z) or (x', y', z') with respect to the two systems of axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the lines Ox', Oy', Oz' with respect to the co-ordinate axes Ox, Oy, Oz.

The scheme of transformation will be as follows ;

$$\left.\begin{aligned} x'&=l_1x+m_1y+n_1z \\ y'&=l_2x+m_2y+n_2z \\ z'&=l_3x+m_3y+n_3z \end{aligned}\right\} \quad \ldots(1)$$

Also we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where **i, j, k** are unit vectors along co-ordinate axes. Therefore

$$\left.\begin{aligned} \mathbf{i}'&=l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k} \\ \mathbf{j}'&=l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k} \\ \mathbf{k}'&=l_3\mathbf{i}+m_3\mathbf{j}+n_3\mathbf{k} \end{aligned}\right\} \quad \ldots(2)$$

If V is any function (vector or scalar) of x, y, z, then

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial V}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial V}{\partial z'}\frac{\partial z'}{\partial x}.$$

$$\therefore \quad \frac{\partial}{\partial x}\equiv\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}+\frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned} \therefore\quad \frac{\partial}{\partial x}&\equiv l_1\frac{\partial}{\partial x'}+l_2\frac{\partial}{\partial y'}+l_3\frac{\partial}{\partial z'} \\ \text{Similarly}\quad \frac{\partial}{\partial y}&\equiv m_1\frac{\partial}{\partial x'}+m_2\frac{\partial}{\partial y'}+m_3\frac{\partial}{\partial z'} \\ \frac{\partial}{\partial z}&\equiv n_1\frac{\partial}{\partial x'}+n_2\frac{\partial}{\partial y'}+n_3\frac{\partial}{\partial z'} \end{aligned}\right\} \quad \ldots(3)$$

Multiplying the equations (3) by **i, j, k** respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\equiv\mathbf{i}'\frac{\partial}{\partial x'}+\mathbf{j}'\frac{\partial}{\partial y'}+\mathbf{k}'\frac{\partial}{\partial z'}.$$

Theorem 2. *If $\phi(x, y, z)$ is a scalar invariant with respect to a rotation of axes, then grad ϕ is a vector invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorem 1 and obtain the equations (1) and (2).

Now suppose the function $\phi(x, y, z)$ becomes $\phi'(x', y', z')$ after rotation of axes. Then by hypothesis $\phi(x, y, z)=\phi'(x', y', z')$.

By chain rule of differentiation, we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned} \therefore\quad \frac{\partial\phi}{\partial x}&=l_1\frac{\partial\phi'}{\partial x'}+l_2\frac{\partial\phi'}{\partial y'}+l_3\frac{\partial\phi'}{\partial z'} \\ \text{Similarly}\quad \frac{\partial\phi}{\partial y}&=m_1\frac{\partial\phi'}{\partial x'}+m_2\frac{\partial\phi'}{\partial y'}+m_3\frac{\partial\phi'}{\partial z'} \\ \text{and}\quad \frac{\partial\phi}{\partial z}&=n_1\frac{\partial\phi'}{\partial x'}+n_2\frac{\partial\phi'}{\partial y'}+n_3\frac{\partial\phi'}{\partial z'} \end{aligned}\right\} \quad \ldots(3)$$

Multiplying these equations by **i, j, k** respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}=\mathbf{i}'\frac{\partial\phi'}{\partial x'}+\mathbf{j}'\frac{\partial\phi'}{\partial y'}+\mathbf{k}'\frac{\partial\phi'}{\partial z'}$$

or $\qquad$ grad ϕ = grad ϕ'.

Theorem 3. *If $\mathbf{V}(x, y, z)$ is a vector function invariant with respect to a rotation of axes, then div $\mathbf{V}$ is a scalar invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorems 1 and 2.

Now suppose the function $\mathbf{V}(x, y, z)$ becomes $\mathbf{V}'(x', y', z')$ after rotation of axes. Then by hypothesis

$$\mathbf{V}(x, y, z)=\mathbf{V}'(x', y', z').$$

By chain rule of differentiation, we have

$$\frac{\partial\mathbf{V}}{\partial x}=\frac{\partial\mathbf{V}'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned}
\therefore\quad & \frac{\partial \mathbf{V}}{\partial x}=l_1\frac{\partial \mathbf{V}'}{\partial x'}+l_2\frac{\partial \mathbf{V}'}{\partial y'}+l_3\frac{\partial \mathbf{V}'}{\partial z'}\\
\text{Similarly}\quad & \frac{\partial \mathbf{V}}{\partial y}=m_1\frac{\partial \mathbf{V}'}{\partial x'}+m_2\frac{\partial \mathbf{V}'}{\partial y'}+m_3\frac{\partial \mathbf{V}'}{\partial z'}\\
\text{and}\quad & \frac{\partial \mathbf{V}}{\partial z}=n_1\frac{\partial \mathbf{V}'}{\partial x'}+n_2\frac{\partial \mathbf{V}'}{\partial y'}+n_3\frac{\partial \mathbf{V}'}{\partial z'}
\end{aligned}\right\}\quad \ldots(3)$$

Taking dot product of these three equations by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{div } \mathbf{V}=\text{div } \mathbf{V}'.$$

Theorem 4. *If* $\mathbf{V}(x, y, z)$ *is a vector function invariant under a rotation of axes, then curl* $\mathbf{V}$ *is a vector invariant under this rotation.* **[Punjab 1966]**

Proof. Proceed exactly in the same manner as in theorem 3.

In place of taking dot product of equations (3), take cross product. We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

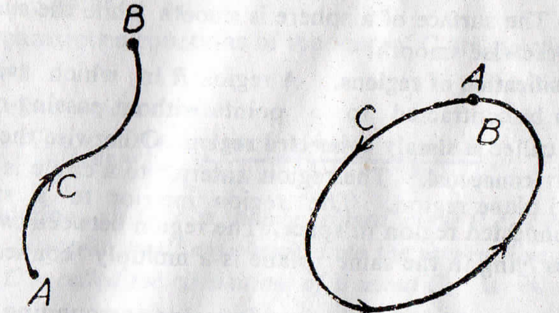
or $$\text{curl } \mathbf{V}=\text{curl } \mathbf{V}'.$$

3

Green's, Gauss's and Stoke's Theorems

§ 1. Some preliminary concepts.

Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction*; the opposite direction along C is then called the *negative direction*. Suppose A is the initial point and B the terminal point



Oriented closed curve

of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve*.

Smooth curve. Let $\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t=t_1$ and $t=t_2$ respectively. We know that $\frac{d\mathbf{r}}{dt}$ is a tangent vector to this curve at the point $\mathbf{r}$. Suppose the function $\mathbf{r}(t)$ is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a smooth curve.

A curve C is said to be piecewise smooth if it is composed of a finite number of smooth curves. The curve C in the adjoining figure