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e) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $f(r)$ is differentiable, show that

$$\operatorname{div}\left(f(r)\,\vec{r}\right) = r f'(r) + 3 f(r)$$

Hence or otherwise show that:

$$\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0$$

We know that

$$\operatorname{div}(\phi A) = (\operatorname{grad}\phi)\cdot A + \phi \operatorname{div}(A)$$

$$\therefore\ \nabla\cdot\left(f(r)\,\vec{r}\right) = \left[\nabla f(r)\right]\cdot\vec{r} + f(r)\,\nabla\cdot\vec{r}$$

$$= \left[f'(r)\,\nabla r\right]\cdot\vec{r} + f(r)\left[1+1+1\right]$$

$$= \left[f'(r)\,\frac{\vec{r}}{r}\right]\cdot\vec{r} + 3 f(r)$$

$$= r f'(r) + 3 f(r) \qquad \left[\because \vec{r}\cdot\vec{r} = r^2,\ r = |\vec{r}|\right]$$

Now, taking $f(r) = \frac{1}{r^3}$

$$\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = r\left(\frac{-3}{r^4}\right) + 3\cdot\frac{1}{r^3}$$

$$= \frac{-3}{r^3} + \frac{3}{r^3} = 0.$$

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6(c) Show that $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2 k$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1,-2,1)$ to $(3,1,4)$.

$$\text{Curl}(F) = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= i(0-0) + j(3z^2 - 3z^2) + k(2x-2x)$$
$$= 0$$

Hence, $\vec{F}$ is a conservative force.

Let ϕ be the scalar potential

$$\nabla\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$$
$$= (2xy+z^3)i + x^2 j + 3xz^2 k$$

$$\Rightarrow \quad \phi(x,y,z) = x^2 y + xz^3$$

Work done in moving particle from $(1,-2,1)$ to $(3,1,4)$

$$= \phi(3,1,4) - \phi(1,-2,1)$$
$$= (9\cdot 1 + 3\cdot 64) - (1(-2) + 1\cdot 1)$$
$$= 201 + 1 = 202$$

which expresses $\dot{\mathbf{t}}$, $\dot{\mathbf{n}}$ and $\dot{\mathbf{b}}$ in terms of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ is *skew-symmetric*, i.e., it is equal to the negative of its transpose. This helps when trying to remember the equations. (The 'reason' for this skew-symmetry can be seen in Exercise 2.3.6.)

Here is a simple application of Frenet–Serret:

Proposition 2.3.5

Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Then, γ is a parametrization of (part of) a circle.

Proof

This result is actually an immediate consequence of Example 2.2.7 and Proposition 2.3.3, but the following proof is instructive and gives more information, namely the centre and radius of the circle and the plane in which it lies.

By the proof of Proposition 2.3.3, the binormal $\mathbf{b}$ is a constant vector and γ is contained in a plane Π, say, perpendicular to $\mathbf{b}$. Now

$$\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\mathbf{n}\right) = \mathbf{t} + \frac{1}{\kappa}\dot{\mathbf{n}} = \mathbf{0},$$

using the fact that the curvature κ is constant and the Frenet–Serret equation

$$\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b} = -\kappa\mathbf{t} \quad (\text{since } \tau = 0)$$

(the reason for considering $\gamma + \frac{1}{\kappa}\mathbf{n}$ can be found in Exercise 2.2.6). Hence, $\gamma + \frac{1}{\kappa}\mathbf{n}$ is a constant vector, say $\mathbf{a}$, and we have

$$\parallel \gamma - \mathbf{a} \parallel = \parallel -\frac{1}{\kappa}\mathbf{n} \parallel = \frac{1}{\kappa}.$$

This shows that γ lies on the sphere $\mathcal{S}$, say, with centre $\mathbf{a}$ and radius $1/\kappa$. The intersection of Π and $\mathcal{S}$ is a circle, say $\mathcal{C}$, and we have shown that γ is a parametrization of part of $\mathcal{C}$. If r is the radius of $\mathcal{C}$, we have $\kappa = 1/r$ so $r = 1/\kappa$ is also the radius of $\mathcal{S}$. It follows that $\mathcal{C}$ is a *great circle* on $\mathcal{S}$, i.e., that Π passes through the centre $\mathbf{a}$ of $\mathcal{S}$. Thus, $\mathbf{a}$ is the centre of $\mathcal{C}$ and the equation of Π is $\mathbf{v} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. □

We conclude this chapter with the analogue of

5.5. **Show that for a curve lying on a sphere of radius a and such that the torsion τ is never 0, the following equation is satisfied**

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = a^2$$

Let $\mathbf{x} = \mathbf{x}(s)$ lie on the sphere with center $\mathbf{y}_0$ and radius a. Then for all s,

$$(\mathbf{x}(s) - \mathbf{y}_0) \cdot (\mathbf{x}(s) - \mathbf{y}_0) = a^2$$

Differentiating, $\qquad 2(\mathbf{x} - \mathbf{y}_0) \cdot \dot{\mathbf{x}} = 0 \quad \text{or} \quad (\mathbf{x} - \mathbf{y}_0) \cdot \mathbf{t} = 0$

Differentiating again,

$$(\mathbf{x} - \mathbf{y}_0) \cdot \dot{\mathbf{t}} + \dot{\mathbf{x}} \cdot \mathbf{t} = 0 \quad \text{or} \quad \kappa(\mathbf{x} - \mathbf{y}_0) \cdot \mathbf{n} + 1 = 0$$

Note it follows that $\kappa \neq 0$ and $(\mathbf{x} - \mathbf{y}_0) \cdot \mathbf{n} = -1/\kappa$. Finally, differentiating again,

$$\dot{\mathbf{x}} \cdot \mathbf{n} + (\mathbf{x} - \mathbf{y}_0) \cdot \dot{\mathbf{n}} = \dot{\kappa}/\kappa^2 \quad \text{or} \quad (\mathbf{x} - \mathbf{y}_0) \cdot (-\kappa\mathbf{t} + \tau\mathbf{b}) = \dot{\kappa}/\kappa^2$$

Using $(\mathbf{x} - \mathbf{y}_0) \cdot \mathbf{t} = 0$, we have, where $\tau \neq 0$, $(\mathbf{x} - \mathbf{y}_0) \cdot \mathbf{b} = \dot{\kappa}/\kappa^2\tau$. Thus the components of $\mathbf{x} - \mathbf{y}_0$ with respect to $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are $0, -1/\kappa, \dot{\kappa}/\kappa^2\tau$. Hence

$$\mathbf{x} - \mathbf{y}_0 = \frac{-1}{\kappa}\mathbf{n} + \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}$$

But on the sphere,

$$(\mathbf{x} - \mathbf{y}_0) \cdot (\mathbf{x} - \mathbf{y}_0) = \left(-\frac{1}{\kappa}\mathbf{n} + \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}\right) \cdot \left(-\frac{1}{\kappa}\mathbf{n} + \frac{\dot{\kappa}}{\kappa^2\tau}\mathbf{b}\right) = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = a^2$$