Previous Years' Paper (Solved)

IFS Mathematics Main Examination, 2016

PAPER-I

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos.* ***1*** *and* ***5*** *are compulsory. Out of the remaining six questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections* ***A*** *and* ***B****. Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Let T : $\mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$.

Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (*i.e.*, (1, 0, 0), (0, 1, 0), etc.) Examine if T is a linear map.

(*b*) Show that $\dfrac{x}{(1+x)} < \log(1 + x) < x$ for $x > 0$.

(*c*) Examine if the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at (0, 0). Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin.

(*d*) If the point (2, 3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord.

(*e*) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$.

2. (*a*) After changing the order of integration of

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx \, dx \, dy,$$

show that $\displaystyle\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$.

(*b*) A perpendicular is drawn from the centre of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

(*c*) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve.

(*d*) Let T be a linear map such that $T : V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis B_1[(1, 1, 0) (1, 0, 1) (0, 1, 1)] and B_2[(1, 1) (1, –1)]. Obtain the matrix T_1 relative to B_1 and B_2.

3. (*a*) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ = I. Hence find A^{-1}.

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(*b*) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point $(1, 0, 0)$.

(*c*) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$.

(*d*) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle.

4. (*a*) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature.

(*b*) Show that the integral $\int_0^\infty e^{-x}x^{\alpha-1}dx,\ \alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$.

(*c*) Prove that $\boxed{2z} = \frac{2^{2z-1}}{\sqrt{\pi}}\boxed{z}\boxed{z+\frac{1}{2}}$.

(*d*) A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C.

SECTION–B

5. (*a*) Obtain the curve which passes through (1, 2) and has a slope $= \frac{-2xy}{x^2+1}$. Obtain one asymptote to the curve.

(*b*) Solve the dE to get the particular integral of $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y = x^2\cos x$.

(*c*) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings.

(*d*) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot\beta = 2\tan(\alpha - \beta)$.

(*e*) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \overline{F}.dS$ where S is the surface bounding the volume E and $\overline{F} = (zx\sin yz + x^3)\hat{i} + \cos yz\,\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}$.

6. (*a*) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet?

(*b*) Using the method of variation of parameters, solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y = x^2e^x.$$

(*c*) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line.

(*d*) Evaluate $\iint_S(\nabla\times\overline{f}).\hat{n}\,dS$ for $\overline{f} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane.

7. (*a*) State Stokes' theorem. Verify the Stokes' theorem for the function $\overline{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one.

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(*b*) A uniform rod of weight W is resting against an equally rough horizon and a well, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W \tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α?

(*c*) Obtain the singular solution of the differential equation:

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2,\ p = \frac{dy}{dx}.$$

8. (*a*) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio 3 : 2.

(*b*) Solve the differential equation

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x).$$

(*c*) Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$.

(*d*) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary.

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5. (a) Given, $$\frac{dy}{dx} = -\frac{2xy}{x^2+1}$$

and (xy) pass through (1, 2) separate variables

$$\frac{dy}{y} = -\frac{2x}{x^2+1}dx$$

Integrate on both sides

$$\int\frac{dy}{y} = -\int\frac{2x}{x^2+1}dx+c$$

Put, $x^2 + 1 = t$

$2xdx = dt$

$\therefore$ $$\log y = -\int\frac{dt}{t}+c$$

$\log y = -\log t + c$

$\log y = -\log(x^2 + 1) + c$

Put, $x = 1, y = 2$

$\log 2 = -\log(2) + c$

$\Rightarrow$ $c = 2\log 2 = \log 4$

$\therefore$ $\log y = -\log(x^2 + 1) + \log 4$

$\Rightarrow$ $$y = \frac{4}{x^2+1}$$

5. (b) Hints: The given differential equation is

$$\frac{d^2y}{dx^4}+2.\frac{d^2y}{dx^2}+y = x^2.\cos x$$

$(D^2 + 1)^2y = x^2.\cos x$

The auxiliary equation is,

$(m^2 + 1)^2 = 0$

Solving for m, we get

$(m^2 + 1)(m^2 + 1) = 0$

i.e., $m^2 = -1 = i^2, i^2$

Therefore, $m = \pm i, \pm i$

The roots are pair of complex conjugates

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The complementary function is

C.F. = (A + Bx) cos$x$ + (C + Dx) sin$x$

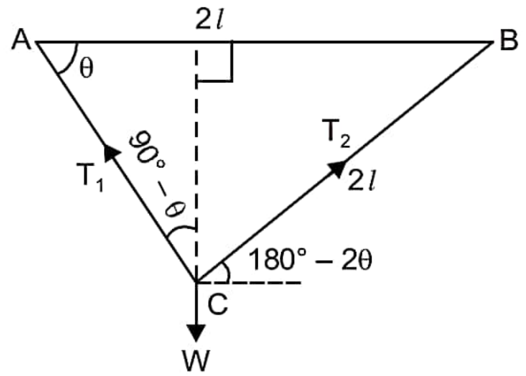
Now, we have to the particular integral

$$\text{P.I.} = \frac{x^2.\cos x}{(D^2+1)^2}$$

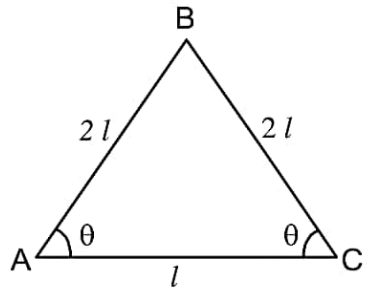
General Solution = C.F. + P.I.

$$= (A + Bx)\cos x + (C + Dx)\sin x + \frac{x^2.\cos x}{(D^2+1)^2}$$

5. (*c*) **Hint:** Let, tension in two string AC of length l and BC of length $2l$ are T_1 and T_2 respectively.



From ΔABC,



$$\angle A = \angle C = \theta \text{ (say)}$$

$$T_1\sin(90 - \theta) = T_2\cos(180° - 2\theta)$$

$$T_1\cos\theta = T_2\cos 2\theta \qquad ...(i)$$

and $T_1\cos(90° - \theta) + T_2\sin(180° - 2\theta) = W$

$$T_1\sin\theta + T_2\sin 2\theta = W \qquad ...(ii)$$

From (*i*) and (*ii*), we can get the required values of Tension in two strings.

5. (*d*) Suppose the particle strike the inclined plane at A. Let OA = R. Let T be the time of flight from O to A. As shown in the figure, the components of initial velocity of the particle along and perpendicular to the inclined plane are $u\cos(\alpha - \beta)$ and $u\sin(\alpha - \beta)$ respectively. Again, the component of g

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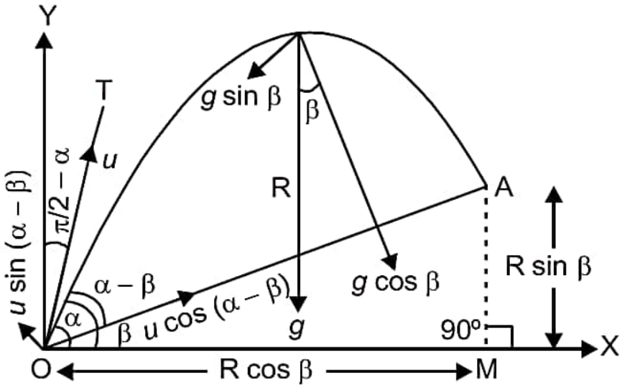

along the inclined is g sin β (down the plene) and the component of g perpendicular to the inclined plane is g sin β (along the downward normal to the plane OA).

Let time taken from O to A be T.

While moving from O to A, the displacement of the particle perpendicular to OA is zero. So, considering motion of the particle from O to A perpendicular to OA and using the formula "$s = ut + (1/2) ft^2$",

We have $\quad s = u.t + \dfrac{1}{2}a.t^2$

$0 = u \sin(\alpha - \beta).T - (1/2)g \cos\beta.T^2$ or $T\{g \cos\beta.T - 2u\sin(\alpha - \beta)\} = 0$

Since T = 0 gives time from O to O, hence time from O to A is given by

∴ T = time of flight up the inclined plane

$$= \frac{2u\sin(\alpha-\beta)}{g\cos\theta} \quad ...(i)$$

Since the particle strikes the plane OA at right angles at A, hence the direction of velocity of the particle at A is perpendicular to OA and so the component of velocity of the particle at A along OA is zero.

So, considering the motion of the particle from O to A along OA and using the formula.

$$V = u + a.t$$

$$O = u\cos(\alpha - \beta) - g\sin\beta.T$$

$$T = \frac{u}{g}.\frac{\cos(\alpha-\beta)}{\sin\beta} \quad ...(ii)$$

From (i) and (ii), we have

$$\frac{2u}{g}.\frac{\sin(\alpha-\beta)}{\cos\beta} = \frac{u}{g}.\frac{\cos(\alpha-\beta)}{\sin\beta}$$

$$2\tan(\alpha - \beta) = \cot\beta$$

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5. (e) Given that $\vec{F} = (zx\sin yz + x^3)\hat{i} + \cos yz\,\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}$

$$\text{div F} = \frac{\partial}{\partial x}(xz\sin(yz) + x^3) + \frac{\partial}{\partial y}(\cos(yz)) + \frac{\partial}{\partial z}\left(3zy^2 - e^{x^2+y^2}\right)$$

$$= (z \sin (yz) + 3x^2) + (- z \sin (yz)) + (3y^2) = 3x^2 + 3y^2$$

Thus, we have from the divergence theorem

$$\iint_S \text{F}.d\text{S} = \iiint_E \text{div F } d\text{V}$$

$$= \iint_D \int_0^{4-x^2-y^2} (3x^2 + 3y^2) dz\, d\text{A}$$

where D is the disk $x^2 + y^2 \leq 4$ in the xy-plane. Thus, we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\iint_S \text{F}.d\text{S} = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} (3r^2) r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (12r^3 - 3r^5) dr\, d\theta$$

$$= \int_0^{2\pi}\left[3r^4 - \frac{1}{2}r^6\right]_0^2 d\theta$$

$$= \int_0^{2\pi}(48 - 32)\, d\theta = 32\pi.$$

6. (a) Let u be the initial velocity of projection. Since the greatest height is 40 m, we have

$$0 = u^2 - 2g.40$$

$$\therefore \quad u = \sqrt{2g \times 40} = 28 \text{ m}$$

Let T be the time after the first stone starts before the two stones meet.

Then the distance traversed by the first stone in time T = distance traversed by the second stone in time (T – 2).

$$\therefore \quad 28\text{T} - \frac{1}{2}g\text{T}^2 = 28(\text{T}-2) - \frac{1}{2}g(\text{T}-2)^2$$

$$= 28\text{T} - 56 - \frac{1}{2}g(\text{T}^2 - 4\text{T} + 4)$$

$$\therefore \quad 56 = \frac{1}{2}g(4\text{T} - 4) = 4.9(4\text{T} - 4)$$

$$\therefore \quad \text{T} = 3\frac{6}{7} \text{ seconds.}$$

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Also the height at which they meet

$$= 28\times\frac{27}{7}-\frac{1}{2}\times 9.8\times\left(\frac{27}{7}\right)^2$$

$$= 108 - 72.9 = 35.1 \text{ m}$$

The first stone will be coming down and the second stone going upwards.

6. (*b*) **Hints:** Let, $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}$$

and $$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Now, $x^2.\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y = 0$

$x^2.m(m-1).x^{m-2} + x.mx^{m-1} - x^m = 0$

$x^m\{m(m-1) + m - 1\} = 0$

$x^m\{m^2 - 1\} = 0$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

The general solution is then

$$y = c_1e^{-x} + c_2.e^x$$

6. (*c*) Given Data:

Diameter of pipe :

$$d = 80 \text{ mm} = 0.08 \text{ m}$$

Gauge pressure of water:

$$p = 60 \text{ kPa} = 60 \times 10^3 \text{ pa or N/m}^2$$

Mean velocity of water:

$$V = 2 \text{ m/s}$$

Datum head:

$$z = 7 \text{ m}$$

According to Bernoulli's equation:

Total head of water :

$$H = \frac{p}{\rho g}+\frac{V^2}{2g}+z$$

$$= \frac{60\times10^3}{1000\times9.81}+\frac{(2)^2}{2\times9.81} + 7$$

$$= 6.11 + 0.20 + 7$$

$$= 13.31 \text{ m of water}$$

6. (*d*) $$\int_C F\cdot dr = \oint_C (F_x dx+F_y dy+F_z dz)$$

$$= \oint_C \left\{(2x-y)dx - yz^2dy - y^2z\,dz\right\}$$

But the boundary C of S is a circle in the xy-plane of radius unity and centre at (0,0,0); Hence the parametric equations of C are $x = \cos\theta$, $y = \sin\theta$, $z = 0$ where θ varies from 0 to 2π.

Thus,

$$\int_C F\cdot dr = \int_{\theta=0}^{2\pi}\{(2\cos\theta - \sin\theta)(-\sin\theta\, d\theta) - 0 - 0\}$$

$$= \int_0^{2\pi}(2\cos\theta - \sin\theta)\sin\theta\, d\theta$$

$$= \int_0^{2\pi}(\sin 2\theta - \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi}\left\{\sin 2\theta - \frac{1-\cos 2\theta}{2}\right\}d\theta$$

$$= -\left[\frac{\cos 2\theta}{2} - \frac{\theta}{2} + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \pi$$

Further $\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2 z \end{vmatrix} = k$

Hence, $\iint_S (\nabla\times A).ds = \iint_S k.ds = \iint_R dx\,dy$

where R is the projection of S on xy-plane and $k.ds = dx\,dy$ = projection of ds on xy-plane.

Thus, R is $x^2 + y^2 = 1$

$$\therefore \iint_R dx\,dy = 4\int_0^1\int_0^1 \sqrt{(1-x^2)}\,dx\,dy$$

$$= 4\int_0^1 \sqrt{(1-x^2)}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= 4\left[\frac{\pi}{4}\right] = \pi$$

Thus, from above, we have

$\int_C A.dr = \iint_S (\nabla\times A).ds$ and hence Stoke's Theorem is verified.

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