18 Integration on $\mathbf{R}^3$

(Line, Surface and Volume Integrals)

This chapter deals mainly with three types of integrals in a three-dimensional space, viz.,

(*i*) Line integrals,

(*ii*) Surface integrals,

(*iii*) Volume (triple) integrals.

The discussion starts with the consideration of rectifiable curves and rectification, and goes on to consider the line integrals, surface areas and the surface integrals, the volumes and the volume (triple) integrals in that order. Stokes' theorem, which connects a line integral with a double integral, and Gauss's theorem, which establishes a relation between a surface integral and a volume integral, have also been considered.

1. RECTIFIABLE CURVES

We are now in a position to consider some suitable definitions of the length of the curve (rectification) and to discuss the conditions under which a curve is rectifiable. *We shall be concerned with continuous curves only* in the present discussion. The reader will see the importance of the functions of bounded variation in such a discussion.

A *curve* in space is defined to be a vector-valued function γ with domain as a subset of $\mathbf{R}$ and range a subset of $\mathbf{R}^3$. The curve is *continuous* if γ is continuous, and is called a *Jordan arc* if γ is one-to-one.

1.1 Length of a Curve

Let $\gamma = (X, Y, Z)$, where X, Y, Z are three real *single-valued, continuous* functions of t defined on an interval $[a, b]$, be a *continuous curve* in $\mathbf{R}^3$, so that $\gamma(t) = [X(t), Y(t), Z(t)]$ is a point on the curve* corresponding to $t \in [a, b]$.

Corresponding to any partition

$$P = \{a = t_0, t_1, t_2, \ldots, t_n = b\}$$

of $[a, b]$, we get an ordered set of points

$$\{\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_n)\}$$

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* $x = X(t)$, $y = Y(t)$, $z = Z(t)$, $t \in [a, b]$

may be thought of as a (parametric) representation of the curve γ in $\mathbf{R}^3$.

on the curve γ. This set of points may be thought of as a polygon inscribed in the curve.

$|\gamma(t_i) - \gamma(t_{i-1})|$ represents the distance between the points $\gamma(t_{i-1})$ and $\gamma(t_i)$ and so the sum

$$s(P, a, b) = \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})|$$

is the length of the polygon inscribed in the curve with vertices at the points $\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_n)$ in this order. The length of the polygon clearly depends upon the particular partition of $[a, b]$. Let us consider the set of these sums for all possible partitions of $[a, b]$.

As the mesh of the partition tends to zero (or $n \to \infty$) these polygons approach the curve more and more closely. Accordingly,

The curve is said to be *rectifiable* if this set of sums is bounded and then, the supremum of the set is known as the *length of the curve.*

Thus a continuous curve γ is *rectifiable* when the supremum of the sums $\sum |\gamma(t_i) - \gamma(t_{i-1})|$, taken over all partitions of $[a, b]$, exists, *i.e.*, when γ is of bounded variation over $[a, b]$. The length, $s(a, b)$ of the curve, from $t = a$ to $t = b$ is then the total variation of γ over $[a, b]$. Thus

$$s(a, b) = V(\gamma, a, b)$$

The length of the curve over the subinterval $[a, t]$ is generally denoted by $s(t)$, where

$$s(t) = s(a, t) = V(\gamma, a, t) = v_\gamma(t)$$

is a function of t.

Note: $\gamma = (\phi, \Psi, \theta)$ being a vector-valued function, the meaning of $|\gamma(t_i) - \gamma(t_{i-1})|$ or $V(\gamma, a, b)$ should be clearly understood. Here,

$$|\gamma(t_i) - \gamma(t_{i-1})| = \sqrt{\{[\phi(t_i) - \phi(t_{i-1})]^2 + [\Psi(t_i) - \Psi(t_{i-1})]^2 + [\theta(t_i) - \theta(t_{i-1})]^2\}}.$$

1.2 Properties of Rectifiable Curves

We know that a rectifiable continuous curve in space is a continuous vector-valued function (domain **R** and range $\mathbf{R}^3$) of bounded variation. The following properties of such curves are the direct consequences of the vector-valued functions of bounded variation.

1. Since for a vector-valued function, $\gamma = (X, Y, Z)$, of bounded variation,

$$V(X, a, b) \le V(\gamma, a, b) \le V(X, a, b) + V(Y, a, b) + V(Z, a, b)$$

we at once deduce that

$$V(X, a, b) \le s(a, b);\ V(Y, a, b) \le s(a, b);\ V(Z, a, b) \le s(a, b)$$

and

$$s(a, b) \le V(X, a, b) + V(Y, a, b) + V(Z, a, b).$$

2. A curve $\gamma = (X, Y, Z)$ is rectifiable if X, Y, Z are derivable and the derivatives are bounded. Under the given hypothesis, γ' exists and is bounded and therefore γ is of bounded variation. Consequently, γ is rectifiable.

3. Since a function of bounded variation over $[a, b]$ is of bounded variation over each sub-interval of $[a, b]$, an arc (part) of a rectifiable curve corresponding to any of its sub-intervals is also rectifiable.
4. Since for a function of bounded variation,

$$V(\gamma, a, b) = V(\gamma, a, c) + V(\gamma, c, b), \quad a \le c \le b$$

it follows that

$$s(a, b) = s(a, c) + s(c, b)$$

5. The arc $s(t)$ of a rectifiable curve is a monotone increasing function of t.

Since the total variation function $v_\gamma(t)$ of a function of bounded variation is monotone increasing over $[a, b]$, the arc $s(t)$ is also monotone increasing on $[a, b]$.

Again since $v_\gamma(t)$ is a *strictly* monotone increasing function over $[a, b]$ unless X, Y, Z (or equivalently γ) are constant functions over some sub-interval of $[a, b]$, it follows that the arc $s(t)$ is also a strictly monotone increasing function unless all the three functions X, Y, Z (or γ) are constant over that sub-interval.
6. The arc $s(t)$ is continuous over $[a, b]$.

Since we are dealing with continuous curves, X, Y, Z and therefore γ are all continuous. Also, since the total variation function $v_\gamma(t)$ of a continuous function is continuous, the arc $s(t)$ is continuous over $[a, b]$.

Remark: If γ (or equivalently, X, Y, Z) is not constant over any sub-interval of $[a, b]$, $s(t)$ is a strictly monotone and increasing continuous function of t on $[a, b]$, therefore the function s is invertible and s^{-1} is continuous on $[a, b]$.

Thus t can be regarded as a strictly monotone increasing function of s over $[s(a), s(b)]$, so that the arc length s may be used as a parameter in place of t.

1.3 The Riemann Integral for Length of a Curve

In certain cases, the length of a curve can be found with the help of a Riemann integral. We shall prove this for *smooth* curves, *i.e.*, for curves which have no multiple points and which are continuously differentiable. Thus if $\gamma = (X, Y, Z)$ is a continuously differentiable curve on $[a, b]$, then X', Y', Z' all exist, are continuous and do not vanish simultaneously on $[a, b]$. The last condition is equivalent to saying that $(X'^2 + Y'^2 + Z'^2)$ does not vanish for any value of t on $[a, b]$.

Theorem 1. *If γ is a smooth curve in $\mathbb{R}^3$ such that γ' exists and is continuous on $[a, b]$, then γ is rectifiable and has a length*

$$\int_a^b |\gamma'(t)|\, dt$$

Since the function γ has a bounded derivative, it is of bounded variation on $[a, b]$ and is therefore rectifiable on $[a, b]$. We have to prove that,

$$\int_a^b |\gamma'(t)|\, dt = V(\gamma, a, b)$$

For any partition $\{a = t_0, t_1, t_2, \ldots, t_n = b\}$ of $[a, b]$, we have

$$\Sigma|\gamma(t_i) - \gamma(t_{i-1})| = \Sigma\left|\int_{t_{i-1}}^{t_i} \gamma'(u)\, du\right| \leq \Sigma\int_{t_{i-1}}^{t_i} |\gamma'(u)|\, du = \int_a^b |\gamma'(t)|\, dt$$

$$\Rightarrow \qquad V(\gamma, a, b) \leq \int_a^b |\gamma'(t)|\, dt \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given.

Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|\gamma'(v) - \gamma'(u)| < \varepsilon, \text{ if } |v - u| < \delta$$

Let $P = \{a = t_0, t_1, \ldots\ldots, t_n = b\}$ be a partition of $[a, b]$ with mesh $\mu(P) < \delta$.

Now for $t_{i-1} \leq u \leq t_i$,

$$|\gamma'(t_i) - \gamma'(u)| < \varepsilon$$

or

$$|\gamma'(u)| - \varepsilon < |\gamma'(t_i)|$$

so that, integrating from t_{i-1} to t_i, we get

$$\int_{t_{i-1}}^{t_i} |\gamma'(u)|\, du - \varepsilon\, \Delta t_i < |\gamma'(t_i)|\, \Delta t_i - |\gamma'(t_i)\, \Delta t_i|$$

$$= \left|\int_{t_{i-1}}^{t_i} [\gamma'(u) + \gamma'(t_i) - \gamma'(u)]\, du\right|$$

$$\leq \left|\int_{t_{i-1}}^{t_i} \gamma'(u)\, du\right| + \left|\int_{t_{i-1}}^{t_i} [\gamma'(t_i) - \gamma'(u)]\, du\right|$$

$$\leq |\gamma(t_i) - \gamma(t_{i-1})| + \int_{t_{i-1}}^{t_i} |\gamma'(t_i) - \gamma'(u)|\, du$$

$$\Rightarrow \qquad \int_{t_{i-1}}^{t_i} |\gamma'(u)|\, du < |\gamma(t_i) - \gamma(t_{i-1})| + 2\varepsilon\, \Delta t_i$$

Writing these inequalities for $i = 1, 2, \ldots, n$, and adding, we get

$$\int_a^b |\gamma'(t)|\, dt < \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| + 2\varepsilon(b - a)$$

$$\leq V(\gamma, a, b) + 2\varepsilon(b - a)$$

But since ε is arbitrary,

$$\int_a^b |\gamma'(t)\, dt \leq V(\gamma, a, b) \qquad \ldots($$

(1) and (2) imply

$$\int_a^b |\gamma'(t)|\, dt = V(\gamma, a, b)$$

i.e., the length of the curve is $\int_a^b |\gamma'(t)|\, dt$.

Remark: The theorem may be stated as follows:

If $\gamma = (X, Y, Z)$ is a curve such that X', Y', Z' exist, and are continuous on $[a, b]$ and do not vanish simultaneously for any $t \in [a, b]$, then the curve is rectifiable and has a length $\int_a^b |\gamma'(t)|\, dt$.

Note: Since γ is a vector-valued function,

$$|\gamma'(t)| = \sqrt{X'^2(t) + Y'^2(t) + Z'^2(t)}\,.$$

Example 1. Find the length of the curve

$$x = at^2,\ y = 2at,\ z = at,\ 0 \le t \le 1$$

■
$$s(0, 1) = \int_0^1 \sqrt{(2at)^2 + (2a)^2 + a^2}\, dt = a \int_0^1 \sqrt{5 + 4t^2}\, dt$$

Put $2t = \sqrt{5} \sinh u$.

$$s(0, 1) = \frac{5a}{4} \int_0^{\log\sqrt{5}} (1 + \cosh 2u)\, du = \frac{5a}{4}\left[\log \sqrt{5} + \frac{6}{5}\right] = \frac{a}{8}[5 \log 5 + 12].$$

Ex. Show that the length of the curve

1. $x = a \cos \theta,\ y = a \sin \theta,\ z = a\theta,\ 0 \le \theta \le 2\pi$ is $2\sqrt{2}a\pi$.
2. $x = 2t - 1,\ y = t + 1,\ z = t - 2,\ 0 \le t \le 3$ is $3\sqrt{6}$.
3. $x = a(\theta - \sin \theta),\ y = a(1 - \cos \theta),\ z = a\theta,\ -\pi \le \theta \le \pi$ is
$$\int_0^{\pi} 2a\sqrt{3 - 2 \cos \theta}\, d\theta\,.$$
4. $x = 2t,\ y = 2t + 1,\ z = t^2 + t$ between the points (0, 1, 0) and (2, 3, 2) is
$$3\left(\sqrt{17} - 1\right)/4 + \log\left\{\left(13 + 3\sqrt{17}\right)/8\right\}.$$

2. LINE INTEGRALS

We have considered line integrals of functions along plane curves. In this section we shall consider line integrals along space curves. Most of the facts of the theory of plane curves are automatically generalized to the case of space curves.

Definition. Let $x = X(t)$, $y = Y(t)$, $z = Z(t)$, $a \le t \le b$ be a curve C in a space of three dimensions.

Let a bounded vector-valued function $F = (f, g, h)$ be defined at every point of the curve C, so that f, g, h are all bounded and defined at every point (x, y, z) of the curve.

Let $\{a = t_0, t_1 ..., t_n = b\}$ be any partition of $[a, b]$, and ξ_i any point of Δt_i. Let (x_i, y_i, z_i) be a point of the curve corresponding to $t = t_i$, where

$$x_i = X(t_i), \quad y_i = Y(t_i), \quad z_i = Z(t_i)$$

Form the sum

$$\sum_{i=1}^{n} [f(X(\xi_i), Y(\xi_i), Z(\xi_i))\, \Delta x_i + g(X(\xi_i), Y(\xi_i), Z(\xi_i))\, \Delta y_i$$
$$+ h(X(\xi_i), Y(\xi_i), Z(\xi_i))\, \Delta z_i]$$

If, as the norm of the partition tends to zero, the sum tends to a finite limit, independent of the choice of ξ_i in Δt_i, the line integral of F along C exists and is denoted by $\int_C F\, dt$ or

$$\left.\begin{aligned} &\int_C f(X, Y, Z)\, dx + g(X, Y, Z)\, dy + h(X, Y, Z)\, dz \\ \text{or} \quad &\int_C f\, dx + g\, dy + h\, dz \end{aligned}\right\} \qquad ...(1)$$

and call it the *line integral* of $F = (f, g, h)$ along C.

2.1 A Sufficient Condition for Existence

Applying arguments completely similar to those used for the case of a line integral along plane curves (§ 1.4, Ch. 16), it can be proved that if

(*i*) X, Y, Z possess continuous derivatives in $[a, b]$, and

(*ii*) $F = (f, g, h)$ is continuous at every point of C,

then the integral $\int_C f\, dx + g\, dy + h\, dz$ exists and is equal to the ordinary integral

$$\int_a^b \{f(X, Y, Z)X' + g(X, Y, Z)Y' + h(X, Y, Z)Z'\}\, dt\,.$$

Note: For evaluating a line integral, the parametric equation of the curve should be known.

Remark: *Vectorial formulation.* Let $\mathbf{r}$ be the position vector of a point (x, y, z) on the curve, so that

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$
$$= \mathbf{i}X(t) + \mathbf{j}Y(t) + \mathbf{k}Z(t), \quad a \le t \le b$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the coordinate axes.

Let a vector function be represented as

$$\mathbf{F}(x, y, z) = \mathbf{i}P(x, y, z) + \mathbf{j}Q(x, y, z) + \mathbf{k}R(x, y, z)$$

$$\therefore \qquad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}$$

$$\Rightarrow \qquad \int_a^b \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_\alpha^\beta \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt$$

Thus the line integral can be expressed as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt$$

2.2 Line Integral with Respect to Arc Length

We have seen that the line integral, $\int f\,dx + g\,dy + h\,dz$ of a function $F = (f, g, h)$ along a curve C,

$$x = X(t), \quad y = Y(t), \quad z = Z(t), \quad a \le t \le b$$

can be expressed as an ordinary integral

$$\int_a^b \{f(X, Y, Z)X' + g(X, Y, Z)Y' + h(X, Y, Z)Z'\}\,dt$$

If the curve is *smooth*, it can be represented as

$$x = \theta(s), \quad y = \phi(s), \quad z = \Psi(s)$$

where the arc length s varies from 0 to l (l being the length of the curve as t varies from a to b).

Considering the partitions $\{0 = s_0, s_1 \ldots, s_n = l\}$ and proceeding as in § 2.1 (with parameter s in place of t) we deduce that the line integral

$$\int_C f\,dx + g\,dy + h\,dz \text{ reduces to}$$

$$\int_0^l \left\{f(\theta, \phi, \Psi)\frac{dx}{ds} + g(\theta, \phi, \Psi)\frac{dy}{ds} + h(\theta, \phi, \Psi)\frac{dz}{ds}\right\} ds$$

which is equivalent to the ordinary integral, or

$$\int_a^b \left(f\frac{dx}{ds} + g\frac{dy}{ds} + h\frac{dz}{ds}\right)\frac{ds}{dt}\,dt$$

where
$$\frac{ds}{dt} = \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 + \left(\frac{dZ}{dt}\right)^2}.$$

Example 2. Prove that

$$\int \frac{x^2 + y^2}{p}\,ds = \frac{\pi ab}{4}[4 + (a^2 + b^2)(a^{-2} + b^{-2})]$$

when the integral is taken round the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and p is the length of the perpendicular from the centre to the tangent.

▪ The parametric equation of the ellipse is

$$x = a\cos\phi, \quad y = b\sin\phi$$

$$p = \frac{y - x(dy/dx)}{\sqrt{1+(dy/dx)^2}} = \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}$$

$$\frac{ds}{d\phi} = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}$$

$$\therefore \quad \int_C \frac{x^2+y^2}{p}\,ds = \int_0^{2\pi} \frac{x^2+y^2}{p}\frac{ds}{d\phi}\,d\phi$$

$$= \frac{1}{ab}\int_0^{2\pi}(a^2\cos^2\phi + b^2\sin^2\phi)(a^2\sin^2\phi + b^2\cos^2\phi)\,d\phi$$

$$= \frac{4}{ab}\int_0^{\pi/2}[(a^4+b^4)\cos^2\phi\sin^2\phi + a^2b^2(\cos^4\phi + \sin^4\phi)]\,d\phi$$

$$= \frac{\pi}{4ab}[a^4+b^4+6a^2b^2] = \frac{\pi ab}{4}[(a^2+b^2)(a^{-2}+b^{-2})+4]\ .$$

Example 3. Show that

$$\int_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz = -2\pi ab^2$$

where the curve C is the part for which $z \geq 0$ of the intersection of the surfaces

$$x^2+y^2+z^2 = 2ax, \quad x^2+y^2 = 2bx, \quad a > b > 0$$

▪ The curve begins at the origin and runs at first in the positive octant.
Curve C is the intersection of the two surfaces,

$$(x-b)^2 + y^2 = b^2$$
$$z^2 = 2(a-b)x$$

So, the parametric equation of C is

$$x = b(1+\cos\theta) = 2b\cos^2\frac{\theta}{2}$$

$$y = b\sin\theta = 2b\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$z = \sqrt{2(a-b)b(1+\cos\theta)}$$
$$= 2\sqrt{b(a-b)}\cos\frac{\theta}{2}$$

and θ varies from π to $-\pi$.

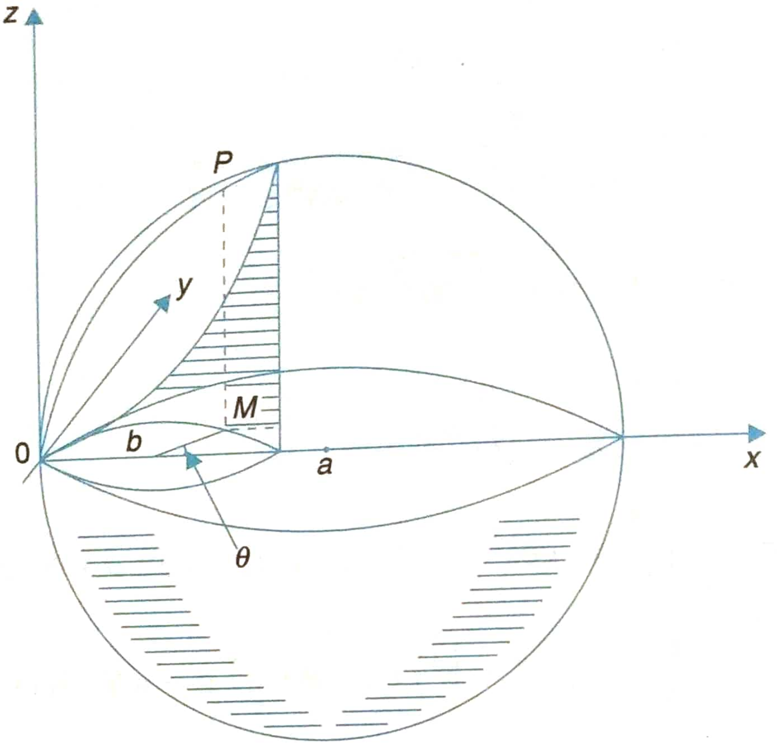

Fig. 1

$$\therefore \quad y^2 + z^2 = -4b^2 \cos^4 \frac{\theta}{2} + 4ab \cos^2 \frac{\theta}{2}$$

$$z^2 + x^2 = 4ab \cos^2 \frac{\theta}{2} - 4b^2 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}$$

$$x^2 + y^2 = 4b^2 \cos^2 \frac{\theta}{2}$$

$$\int_C (y^2 + z^2)\, dx + (z^2 + x^2)\, dy + (x^2 + y^2)\, dz$$

$$= +4b^2 \int_{-\pi}^{\pi} \left(b \cos^4 \frac{\theta}{2} - a \cos^2 \frac{\theta}{2}\right) \sin\theta\, d\theta$$

$$- 4b^2 \int_{-\pi}^{\pi} \left(a \cos^2 \frac{\theta}{2} - b \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}\right) \cos\theta\, d\theta$$

$$+ 4b^2 \sqrt{b(a-b)} \int_{-\pi}^{\pi} \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2}\, d\theta .$$

As the first and the third integral on the right vanish,

$$= -8b^2 \int_0^{\pi} \left(a \cos^2 \frac{\theta}{2} - b \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}\right) \cos\theta\, d\theta$$

$$= -8b^2 \int_0^{\pi} a \cos^2 \frac{\theta}{2} \left(2 \cos^2 \frac{\theta}{2} - 1\right) d\theta$$

$$= -16b^2 \int_0^{\pi/2} a \cos^2 t (2\cos^2 t - 1)\, dt, \quad t = \frac{\theta}{2} = -2\pi ab^2 .$$

EXERCISE

1. Evaluate the integrals:

(i) $\int \frac{ds}{x-y}$, along the line $2y = x - 4$ between the points $(0, -2)$ and $(4, 0)$.

(ii) $\int y\, ds$, along the arc of the parabola $y^2 = 2px$ cut off by $x^2 = 2py$.

(iii) $\int xy\, ds$, along the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant.

(iv) $\int x\sqrt{x^2 - y^2}\, ds$, along the half-lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, $(x \geq 0)$.

[*Hint*: $x = a \cos^{1/2} 2\theta \cos\theta$, $y = a \cos^{1/2} 2\theta \sin\theta$, $-\pi/4 \leq \theta \leq \pi/4$.]

2. Show that

$$\int_\Gamma xyz\, ds = \frac{\sqrt{3}}{32} R^4$$

where Γ is the quarter circle of the circle $x^2 + y^2 + z^2 = R^2$, $x^2 + y^2 = R^2/4$ lying in the first octant.

3. Show that

$$\int_r (x + y)\, ds = \sqrt{2}a^2$$

where Γ is the quarter $x^2 + y^2 + z^2 = a^2$, $y = x$ lying in the first octant.

4. Evaluate the following integrals along segment of straight lines joining the given points.

(i) $\int x\, dx + y\, dy + (x + y - 1)\, dz$, $(1, 1, 1)$ and $(2, 3, 4)$.

(ii) $\int \frac{x\, dx + y\, dy + z\, dz}{\sqrt{(x^2 + y^2 + z^2 - x - y - 2z)}}$, $(1, 1, 1)$ and $(4, 4, 4)$.

5. Find the line integral

$$\int_C (y + z)\, dx + (z + x)\, dy + (x + y)\, dz$$

where C is the circle $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$.

6. Evaluate:

$$\int_C x^2 y^3\, dx + dy + z\, dz,$$

where C is the circle $x^2 + y^2 = R^2$, $z = 0$.

7. Show that

$$\int_\Gamma yz\, dx + zx\, dy + xy\, dz = 0$$

where Γ is the arc of the curve $x = b \cot t$, $y = b \sin t$, $z = at/2\pi$, from the point it intersects $z = 0$ to the point it intersects $z = a$.

8. Show that

$$\int_C y^2\, dx + z^2\, dy + x^2\, dz = -\frac{\pi a^3}{4}$$

where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$, $(a > 0, z \geq 0)$, integrated anticlockwise when viewed from the origin.

ANSWERS

1. (i) $\sqrt{5}\log 2$ (ii) $(5\sqrt{5}-1)\dfrac{p^2}{3}$ (iii) $\dfrac{ab(a^2+ab+b^2)}{3(a+b)}$ (iv) $\dfrac{2\sqrt{2a^3}}{3}$

4. (i) 13 (ii) $3\sqrt{3}$ 5. 0 6. $\dfrac{-\pi R^6}{8}$

3. SURFACES

A curve in R^3 is a vector valued function whose domain is a subset of R and range a subset of R^3. This idea is extended to define a surface in R^3.

Definition. A surface in R^3 is a vector-valued function with domain a subset of R^2 and range a subset of R^3.

Very often we do not make a distinction between a surface and its range set, and take the range of the surface as the surface itself. *Thus if X, Y, Z are three real-valued functions defined on a domain $E \subset R^2$, where*

$$x = X(u, v),\ y = Y(u, v),\ z = Z(u, v),\ (u, v) \in E,$$

then the set

$$\{(X(u, v), Y(u, v), Z(u, v) : (u, v) \in E\} \qquad ...(1)$$

is a (parametric representation of) surface in R^3.

Thus while a curve requires one parameter, the surface requires two parameters for its representation.

A surface of the form

$$z = \Psi(x, y) \quad x \in R,\ y \in R \qquad ...(2)$$

which is met in not more than one point by any line parallel in the axis of z, is known as quadratic or regular with respect to z-axis. Such a surface is projectible in a one-to-one manner on the xy-plane.

$x = \theta(y, z)$ and $y = \phi(z, x)$ are surfaces quadratic (regular) with respect to x-axis and y-axis, respectively.

A surface which can be divided by a finite number of smooth curves into a finite number of portions each of which is quadratic (regular) with respect to the axes, is called a *piecewise quadratic surface.*

A surface is said to be smooth if X, Y, Z possess continuous first order partial derivatives at each point of E and

$$\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}$$

do not all vanish simultaneously at any point.

3.1 The Surface Area

Let it be required to compute the area S of a surface bounded by a curve C. The surface being defined by the equation

$$z = \Psi(x, y)$$

where the function Ψ is continuous and possesses continuous partial derivatives. Denote the projection of C on the xy-plane bounded by Γ, and let D be the domain on the xy-plane bounded by Γ. Let σ denote the area of D.

Let an arbitrary partition P of D give rise to sub-domains of areas $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$. In each sub-domain take a point $P_i(\xi_i, \eta_i)$. To the point P_i, there will correspond, on the surface, a point $Q_i(\xi_i, \eta_i, \Psi(\xi_i, \eta_i))$. Through Q_i draw a tangent plane to the surface, and on this plane pick out a sub-domain ΔS_i which projects onto $\Delta\sigma_i$ on the xy-plane.

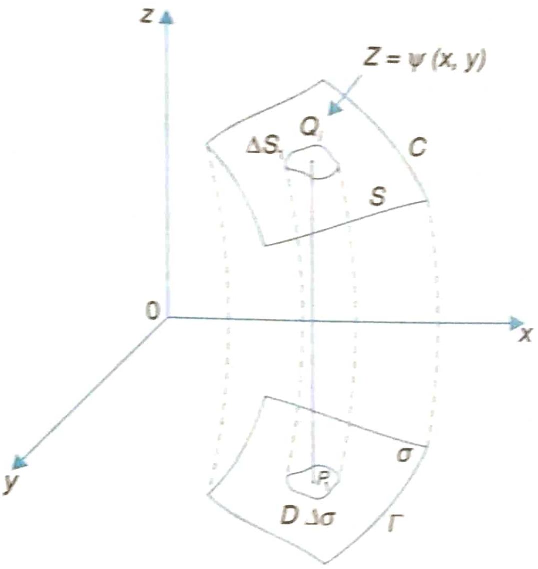

Fig. 2

Consider the sum $\sum_{i=1}^{n} \Delta S_i$.

If the limit of this sum exists, when the norm $\mu(P)$ of the partition (the greatest of the diameters of the sum-domains) approaches zero, the surface is said to be *Squarable* and its area S is given by

$$S = \lim_{\mu(P)\to 0} \sum_{i=1}^{n} \Delta S_i$$

Let us now calculate the value of S.

If γ_i denote the angle between the tangent plane (at Q_i) and the xy-plane, we know from analytical geometry that

$$\Delta\sigma_i = \Delta S_i \cos\gamma_i$$

and

$$\cos\gamma_i = \frac{1}{\sqrt{1 + \Psi_x^2(\xi_i, \eta_i) + \Psi_y^2(\xi_i, \eta_i)}}$$

$$\therefore \quad \Delta S_i = \sqrt{1 + \Psi_x^2(\xi_i, \eta_i) + \Psi_y^2(\xi_i, \eta_i)}\ \Delta\sigma_i$$

$$\therefore \quad S = \lim_{\mu(P)\to 0} \sum_i \Delta S_i$$

$$= \lim_{\mu(P)\to 0} \sum_i \sqrt{1 + \Psi_x^2(\xi_i, \eta_i) + \Psi_y^2(\xi_i, \eta_i)}\ \Delta\sigma_i$$

$$= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\ dy \qquad \ldots(1)$$

a formula to compute the area of the surface $z = \Psi(x, y)$.

If the equation of the surface is given in the form

$$x = \theta(y, z), \quad \text{or} \quad y = \phi(z, x)$$

then the corresponding formulas for calculating the surface area are of the form

$$S = \iint_{D_1} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\ dy\ dz \qquad \ldots(2)$$

$$S = \iint_{D_2} \sqrt{1 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\ dz\ dx \qquad \ldots(3)$$

where D_1, D_2 are the domains in the *yz*-plane and *zx*-plane in which the given surface is projected.

3.2 The formulas obtained in the above section enable us to find the areas of smooth surfaces (of the form $z = \Psi(x, y)$) which are projectable in a one-to-one manner on the coordinate planes.

However, they help us to deduce a formula to find the area of a smooth surface represented parametrically which is not necessarily projectable in a one-to-one manner on one of the coordinate planes.

To find the area of a smooth surface represented parametrically as

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v), \quad (u, v) \in E$$

we have only to change the variables in formula (1) § 3.1, where the equation of surface is $z = \Psi(x, y)$.

Put
$$x = X(u, v),\ y = Y(u, v) \text{ so that}$$
$$z = \Psi[X(u, v), Y(u, v)] = Z(u, v)$$

Also (by § 8 Ch. 11),

$$\frac{\partial z}{\partial x} = -\frac{\partial(Y, Z)}{\partial(u, v)} \Big/ \frac{\partial(X, Y)}{\partial(u, v)}; \quad \frac{\partial z}{\partial y} = -\frac{\partial(Z, X)}{\partial(u, v)} \Big/ \frac{\partial(X, Y)}{\partial(u, v)}$$

and then

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\left[\frac{\partial(X, Y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Y, Z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Z, X)}{\partial(u, v)}\right]^2} \Big/ \frac{\partial(X, Y)}{\partial(u, v)}$$

and the Jacobian of transformation $J = \dfrac{\partial(X, Y)}{\partial(u, v)}$.

Thus formula (1) yields

$$S = \iint_D \sqrt{\left[\frac{\partial(X, Y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Y, Z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Z, X)}{\partial(u, v)}\right]^2}\, du\, dv \qquad \ldots(4)$$

which is the required formula.

It can be easily verified that

$$\left.\begin{aligned}
&\left[\frac{\partial(X, Y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Y, Z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(Z, X)}{\partial(u, v)}\right]^2 = AC - B^2\\
&\text{where}\quad \left(\frac{\partial X}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 + \left(\frac{\partial Z}{\partial u}\right)^2 = A\\
&\frac{\partial X}{\partial u}\frac{\partial X}{\partial v} + \frac{\partial Y}{\partial u}\frac{\partial Y}{\partial v} + \frac{\partial Z}{\partial u}\frac{\partial Z}{\partial v} = B\\
&\left(\frac{\partial X}{\partial v}\right)^2 + \left(\frac{\partial Y}{\partial v}\right)^2 + \left(\frac{\partial Z}{\partial v}\right)^2 = C
\end{aligned}\right\} \qquad \ldots(5)$$

Thus (4) may be expressed as

$$S = \iint_D \sqrt{AC - B^2}\, du\, dv. \qquad \ldots(6)$$

Note: The following results of differential geometry are very useful.

1. (*a*) For the surface $x = X(u, v)$, $y = Y(u, v)$, $z = Z(u, v)$ the direction cosines of the normal to the surface at any point (x, y, z) are

$$\frac{\partial(y, z)}{\partial(u, v)}\bigg/\sqrt{K},\quad \frac{\partial(z, x)}{\partial(u, v)}\bigg/\sqrt{K},\quad \frac{\partial(x, y)}{\partial(u, v)}\bigg/\sqrt{K}$$

where
$$K = \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2$$
$$= AC - B^2 \quad [\text{ref. (5) § 3.2}]$$

(*b*) The elementary surface area dS is given by

$$dS = \sqrt{K}\, du\, dv.$$

2. (*a*) For the surface $z = \Psi(x, y)$, the direction cosines of the normal at any point (x, y, z) of the surface area are

$$-\frac{\partial z}{\partial x}\bigg/\sqrt{K},\quad -\frac{\partial z}{\partial y}\bigg/\sqrt{K},\quad 1/\sqrt{K}.$$

where $$K = 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

(*b*) The elementary surface area dS is given by

$$dS = \sqrt{K}\, dx\, dy = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy.$$

Example 4. Compute the surface area S of the sphere

$$x^2 + y^2 + z^2 = a^2$$

■ The surface area of the sphere is twice the surface area of the upper half-sphere, $z = \sqrt{a^2 - x^2 - y^2}$. Now

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

so that

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}$$

The domain of integration is the circle $x^2 + y^2 = a^2$ on the xy-plane.

Thus by formula (1), we have

$$\tfrac{1}{2} S = \int_{-a}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy$$

On passing to polars, we have

$$S = 2\int_0^{2\pi}\left[\int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\right] d\theta = 4\pi a^2.$$

Example 5. Find the area of that part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$.

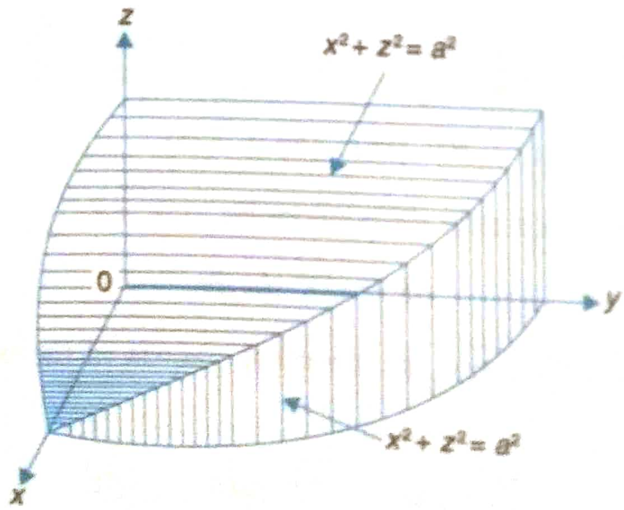

Fig. 3

The figure shows 1/8th of the desired surface.
The equation of the surface has the form

$$y = \sqrt{a^2 - x^2}$$

so that

$$\frac{\partial y}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2}}, \quad \frac{\partial y}{\partial z} = 0$$

$$\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} = \frac{a}{\sqrt{a^2 - x^2}}$$

The domain of integration is a quarter circle $x^2 + z^2 = a^2, x \geq 0, z \geq 0$, on the xz-plane. Thus by formula (3), we have

$$\frac{1}{8} S = \int_0^a dx \int_0^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}} dz = a^2$$

$$\therefore \qquad S = 8a^2.$$

Example 6. The x and y coordinates of a point on the paraboloid $2z = x^2/a + y^2/b$ are expressed in the form

$$x = a \tan\theta \cos\phi, \; y = b \tan\theta \sin\phi$$

where θ is the angle of inclination of the normal at any point to the axis of z. Show that the area of the cap of the surface cut off by the curve $\theta = \lambda$ is $2\pi ab(\sec^3 \lambda - 1)/3$.

▪ Here

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

so that by formula (1), surface area S is given bys

$$S = \iint_D \sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}} \, dx \, dy$$

Let us put $x = a \tan\theta \cos\phi$, $y = b \tan\theta \sin\phi$.

$$\therefore \qquad \sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}} = \sec\theta$$

and

$$J = \frac{\partial(x, y)}{\partial(\theta, \phi)} = ab \sec^2\theta \tan\theta$$

$$\therefore \qquad S = \int_0^{2\pi} d\phi \int_0^{\lambda} \sec\theta \, ab \sec^2\theta \tan\theta \, d\theta$$

$$= 2\pi \int_0^{\lambda} ab \sec^3 \theta \tan \theta \, d\theta = \frac{2\pi ab}{3} (\sec^3 \lambda - 1).$$

Example 7. A surface is given by the equations

$$x = c \sin u, \quad y = c \cos v, \quad z = c(\cos u + \cos v)$$

prove that its area bounded by

$$u = 0, \quad u = \pi/2; \quad v = 0, \quad v = \pi/2$$

is

$$\frac{\pi c^2}{2} \left[1 - \sum_{n=1}^{\infty} (-1)^n \frac{p_{2n}^2}{2n-1} \right]$$

where
$$p_{2n} = \frac{(2n-1)(2n-3)\ldots 3.1}{2n(2n-2)\ldots 4.2}$$

■ Here

$$A = c^2 \cos^2 u + c^2 \sin^2 u = c^2$$
$$B = c^2 \sin u \sin v$$
$$C = c^2 \sin^2 v + c^2 \sin^2 v = 2c^2 \sin^2 v$$

$$\therefore \quad S = \iint_D c^2 \sqrt{2 \sin^2 v - \sin^2 u \sin^2 v} \, du \, dv$$

$$= c^2 \int_0^{\pi/2} \sin v \, dv \int_0^{\pi/2} \sqrt{1 + \cos^2 u} \, du = c^2 \int_0^{\pi/2} (1 + \cos^2 u)^{1/2} \, du$$

$$= c^2 \int_0^{\pi/2} \left[1 + \frac{1}{2} \cos^2 u + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} (\cos^2 u)^2 + \ldots + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right) \ldots \left(\frac{1}{2} - n + 1\right)}{n!} (\cos^2 u)^2 + \ldots \right] du$$

$$= c^2 \int_0^{\pi/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \frac{(2n-1)(2n-3)\ldots 3.1}{2n(2n-2)\ldots 4.2} \cos^{2n} u \right] du$$

$$= \frac{\pi}{2} c^2 \left[1 - \sum_{n=1}^{\infty} \frac{(-1)^n p_{2n}^2}{2n-1} \right].$$

EXERCISE

1. Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$.
2. Find the area of the part of the spherical surface

$$x^2 + y^2 + z^2 = 4a^2$$

enclosed by the cylinder

$$(x^2 + y^2)^2 = 2a^2(2x^2 + y^2).$$

3. Find the areas of the indicated parts of the given surfaces.
   (i) The part of $z^2 = x^2 + y^2$ cut off by the cylinder $z^2 = 2py$.
   [*Hint*: Take projection on xy-plane.]
   (ii) The part of $y^2 + z^2 = x^2$ inside the cylinder $x^2 + y^2 = a^2$.
   [*Hint*: Projection on yz-plane.]
   (iii) The part of $y^2 + z^2 = x^2$ cut off by the cylinder $x^2 - y^2 = a^2$ and the planes $y = b$, $y = -b$.
   (iv) The part of $z^2 = 4x$ cut off by the cylinder $y^2 = 4x$ and the plane $x = 1$.
   (v) The part of $z = xy$ cut off by the cylinder $x^2 + y^2 = a^2$.
   (vi) The part of $2z = x^2 + y^2$ cut off by the cylinder $x^2 + y^2 = 1$.
   (vii) The part of the cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 2x$.
   (viii) The part of $x^2 = y^2 + z^2$ between the cylinder $y^2 = z$ and the plane $y = z - 2$.
   (ix) The part of the cone $x^2 = y^2 + z^2$ inside the sphere $x^2 + y^2 + z^2 = 2z$.
   (x) The part of $y^2 + z^2 = 2z$ cut off by the cone $x^2 = y^2 + z^2$.
4. Find the area of the surface of the cylinder $x^2 + y^2 = 4a^2$ above the xy-plane and bounded by the planes $y = 0$, $z = a$ and $y = z$.
5. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ above $z = 0$ plane, cut off by the vertical cylinder erected on one loop of the curve whose equation in polars is $r = a \cos 2\theta$.
6. Show that the area of the surface of the paraboloid $\frac{x^2}{a} + \frac{y^2}{b} = 2z$ inside the cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = k$ is $\frac{2}{3}\pi\{(1 + k)^{3/2} - 1\}ab$.
7. Show that the surface area of the sphere $x^2 + y^2 + z^2 = 1$ that lies inside the cylinder $2x^2(x^2 + y^2) = 3(x^2 - y^2)$ is $2\pi - 4\sqrt{2}\{\sqrt{3}\log(\sqrt{3} + \sqrt{2}) - 2\log(1 + \sqrt{2})\}$.
8. Calculate the area of the spherical surface given by
$$x = a\cos\theta\cos\phi,\quad y = a\cos\theta\sin\phi,\quad z = a\sin\theta$$
where $0 < \phi < 2\pi$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
9. Find the surface area of the torus
$$x = (r - \cos v)\cos u,\quad y = (r - \cos v)\sin u,\quad z = \sin v,$$
where $-\pi \le u, v \le \pi$, $r > 1$.
10. Compute the area of the part of the earth's surface (considering it as a sphere of radius R, km) contained between the meridians $\phi = 30^\circ$, $\phi = 60^\circ$ and parallels $\theta = 45^\circ$, $\theta = 60^\circ$.

ANSWERS

2. $16(\pi - \sqrt{2})a^2$
3. (i) $2\sqrt{2}\pi p^2$ (ii) $2\pi a^2$
   (iii) $8\sqrt{2}ab$ (iv) $16(\sqrt{8} - 1)/3$
   (v) $\frac{2\pi}{3}\{(1 + a^2)^{3/2} - 1\}$ (vi) $2\pi(\sqrt{8} - 1)/3$

(vii) $2\pi\sqrt{2}$ (viii) $9\sqrt{2}$

(ix) $\pi\sqrt{2}$ (x) 16.

4. $\frac{a^2}{3}(\pi + 6\sqrt{3} - 12)$

5. $\frac{1}{2}(\pi - 2)a^2$

8. $4\pi a^2$

9. $4\pi^2 r$

10. $\frac{\pi R^2}{12}(\sqrt{3} - \sqrt{2})$

4. SURFACE INTEGRALS

In many physical problems we encounter functions defined on various surfaces. For example, density of a charge distribution over the surface of a conductor, intensity of illumination of a surface, velocity of the particles of a fluid passing through a surface, and the like. This section is devoted to studying integrals of functions defined on surfaces, the so-called surface integrals. The theory is in many respects analogous to the theory of line integrals presented in earlier sections.

4.1 Surface Integrals of the First Type (*Definition*)

Let S be a smooth (or piecewise smooth) surface bounded by a smooth (or piecewise smooth) contour C. Let a bounded function f be defined at all points (x, y, z) of the surface.

Let the surface be partitioned into sub-surfaces of areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$ by means of smooth (or piecewise smooth) curves. Let $P(\xi_i, \eta_i, \zeta_i)$ be a point of ΔS_i. For the sum

$$\sum_i f(\xi_i, \eta_i, \zeta_i)\, \Delta S_i \,. \qquad \ldots(1)$$

If the sum tends to a finite limit as the norm of the partition (the maximal of the diameters of the sub-areas ΔS_i) tends to zero, and for all positions of (ξ_i, η_i, ζ_i) in ΔS_i the limit is called the *surface integral of the first type* of the function f over the surface S and is denoted by the symbol

$$\iint_S f(x, y, z)\, dS \qquad \ldots(2)$$

Note: Let a plane surface D bounded by contour Γ be the projection of the surface S on the xy-plane and $\Delta\sigma_i$ the corresponding sub-areas of D.

Then

$$\Delta\sigma_i = \Delta S_i \cos\gamma_i$$

where γ_i is the angle of inclination of the normal at (ξ_i, η_i, ζ_i) to the surface ΔS_i with the z-axis. Sum (1) then becomes

$$\sum_i f(\xi_i, \eta_i, \zeta_i)\, \Delta S_i = \sum_i f(\xi_i, \eta_i, \zeta_i)\, \frac{\Delta\sigma_i}{\cos\gamma_i}$$

which, in the limit, yields

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, z)\, \frac{dx\, dy}{\cos\gamma} \qquad \text{...(3)}$$

where z is on the right hand side, is expressed in terms of x, y with the help of the equation of the surface.

Relation (3) expresses a surface integral in terms of a double integral.

4.2 Reducing a Surface Integral of First Type to a Double Integral

(*i*) If the surface is represented by $z = \Psi(x, y)$, we know

$$\Delta S_i = \sqrt{1 + \Psi_x^2 + \Psi_y^2}\; \Delta x_i\, \Delta y_i$$

so that in the limit, (1) takes the form

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx\, dy \qquad \text{...(4)}$$

where z, in the double integral, is expressed in terms of x, y and D is the projection of S on the xy-plane.

If the surface is represented by equation of the form

$$x = \theta(y, z) \quad \text{or} \quad y = \phi(z, x)$$

then interchanging the roles of the variables x, y, z, the corresponding formulas are of the form

$$\iint_S f(x, y, z)\, dS = \iint_{D_1} f(\theta(y, z), y, z) \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\; dy\, dz \qquad \text{...(5)}$$

$$\iint_S f(x, y, z)\, dS = \iint_{D_2} f(x, \phi(z, x), z) \sqrt{1 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\; dz\, dx \qquad \text{...(6)}$$

where D_1 and D_2 are respectively the projection of S on the yz-plane and zx-plane.

Note: If the surface S is composed of several parts, each of which can be represented by an equation of the form

$$x = \theta(y, z), \quad y = \phi(z, x) \quad \text{or} \quad z = \Psi(x, y)$$

then, since by definition, the surface integral over S is equal to the sum of the integrals over the parts of S, applying formulas (4), (5) or (6) to these integrals separately, we reduce the integral over S to the sum of the double integrals.

(*ii*) In case a smooth surface S is represented by a parametric equation of the form

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v)$$

then by applying arguments essentially the same as above, the surface integral of a bounded function f over S can be expressed in terms of a double integral, by the relation

$$\left.\begin{aligned}\iint_S f(x, y, z)\, dS &= \iint_D f(X, Y, Z)\sqrt{K}\, du\, dv \\ \text{or} \qquad &= \iint_D f(X, Y, Z)\sqrt{AC - B^2}\, du\, dv\end{aligned}\right\} \quad \text{...(7)}$$

where the domain D in uv-plane corresponds to S, and

$$K = \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2$$

$$A = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$B = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$C = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Note: We have considered surface integrals of scalar function f. This notion can be easily generalized to a vector function $\mathbf{F}$ defined on S. Let

$$\mathbf{F}(x, y, z) = \mathbf{i}\mathbf{P}(x, y, z) + \mathbf{j}\mathbf{Q}(x, y, z) + \mathbf{k}\mathbf{R}(x, y, z)$$

where P, Q, R are scalar functions. We define the surface integral of $\mathbf{F}$ over S by the relation

$$\iint_S \mathbf{F}\, dS = \mathbf{i}\iint_S P\, dS + \mathbf{j}\iint_S Q\, dS + \mathbf{k}\iint_S R\, dS \quad \text{...(8)}$$

where the surface integrals of scalar functions on the right side can be expressed as double integrals, and call it the surface integral of the first type of the vector function $\mathbf{F}$ over the surface S.

We now proceed to discuss the theory of *Surface integrals of the second type*. To understand the theory, let us first discuss the question of choosing a side of a surface, which is analogous to the problem of introducing an orientation of a curve.

4.3 Oriented Surfaces, Positive and Negative Sides

A surface is said to be two sided (*bilateral*) if it is possible to distinguish one of its sides from the other. We assume that a surface has two distinct sides in such a way that it is impossible to pass from one side to the other along a continuous path which lies on the surface and which does not cross one of the bounding curves. However, all surfaces are not two-sided. The simplest example of a unilateral (one-sided) surface is provided by the well known *Möbius strip*, shown in the figure (c) which may be obtained by taking a rectangular strip $ABCD$ of paper and pasting its on two sides BC and AD after giving a half twist, *i.e.*, in such a way that the point A coincides with the point C, and B with D.

The concept of a side of a surface is closely related to the orientation of its bounding curve. A side of a surface (a region) is said to be positive or positively oriented if the orientation of its boundary is positive.

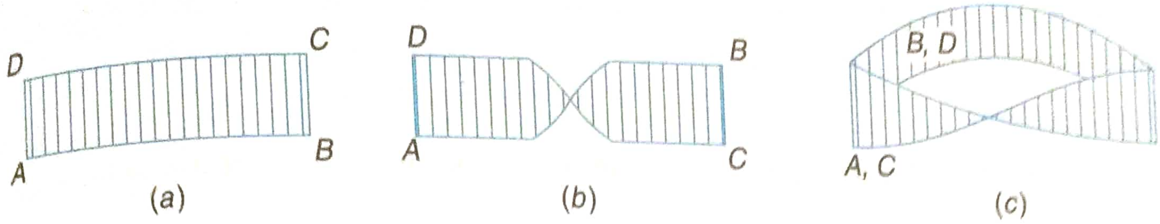

Fig. 4

Let S be a two-sided smooth surface and C its bounding curve. Take a small area σ bounded by a curve Γ on the surface. Take a point P in σ and the normal PNP' to the surface at P. The half lines PN and PN', which do not pierce the surface, are drawn in opposite directions from the surface; one of these directions, say that of PN, is chosen as the positive direction of the normal at P, and PN may be called the positive normal, PN' the negative normal. That side of the surface σ which faces the positive direction of the normal at P will be called the positive (or upper) side of the surface; the other side will be the negative (or lower) side.

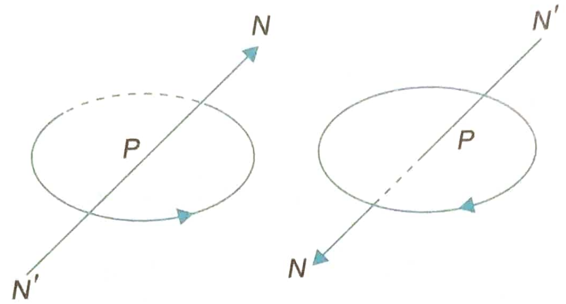

Fig. 5

The concept of a side of a surface is closely related to the orientation of its contour. A side of a surface is said to be positive or positively oriented if the orientation of its boundary is positive.

To correlate these two ideas, we introduce for each contour entering into the boundary its orientation according to the following rule:

'A direction in which the contour C is described is considered to be positive if the surface S is always kept on the left of a person who is on the surface and walking round the contour in this direction, so that the normal PN goes from his feet to his head. The opposite side is referred to as the negative one.'

Thus the side of the surface with this orientation of the boundary is called the positive side and the other the negative side. The two sides of the surface are then called *orientable* and the process of choosing a certain side of a two-sided surface is referred to as the *orientation* of the surface. The one-sided surfaces are thus non-orientable.

Plane surfaces are oriented in the same way. Accordingly the positive side of xy-plane is that which faces the positive direction of z-axis.

In what follows, it will always be assumed that the positive side of a surface projects on the positive side of the coordinate planes.

4.4 Surface Integral of the Second Type: Flux Across a Surface

Definition. Let S be a smooth (or piecewise smooth) two-sided (oriented) surface, bounded by a smooth (or piecewise smooth) contour C. Let a bounded vector-valued function $F = (f, g, h)$ be defined for all points of a certain side, say positive side of the surface. The projection (or resolved part) F_n of F in the direction of the normal to the surface at arbitrary point (x, y, z) can be written as

$$F_n = f\cos\alpha + g\cos\beta + h\cos\gamma$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the normal to the surface at (x, y, z).

The surface integral

$$\iint_S (f\cos\alpha + g\cos\beta + h\cos\gamma)\, dS \qquad ...(1)$$

is called the *Surface integral of the second type* of the vector valued function $F = (f, g, h)$ over the surface S (or strictly speaking, over the chosen side of the surface) and will be denoted as

$$\iint_S (f\, dy\, dz + g\, dz\, dx + h\, dx\, dy) \qquad ...(2)$$

Thus by definition, we have the relation

$$\iint_S (f\cos\alpha + g\cos\beta + h\cos\gamma)\, dS = \iint_S f\, dy\, dz + g\, dz\, dx + h\, dx\, dy. \qquad ...(3)$$

Note: The surface integral of the second type of F is the same as the surface integral of first type of F_n (the component of F along the normal to the surface).

Remark: We have defined the surface integral of the second type on the basis of the notion of the surface integral of the first type. But it can be defined directly also as follows:

For brevity, let us first consider only one of the components, say h, of the vector valued functions. Let D be the projection of S on the xy-plane. Since S is positively oriented, D should also be oriented positively. Clearly D is a plane surface on the xy-plane. Let S be partitioned into the sub-areas of D. Choose an arbitrary point (ξ_i, η_i, ζ_i) in each sub-area ΔS_i and consider the sum

$$\sum_{i=1}^{n} h(\xi_i, \eta_i, \zeta_i)\, \Delta\sigma_i \qquad ...(4)$$

If the limit of the sum exists, when the norm of the partition tends to zero (or $n \to \infty$) (which always exists for continuous function h and smooth surface S), the limit is equal to the integral

$$\iint_S h(x, y, z)\, dx\, dy \qquad ...(5)$$

We can similarly define by means of corresponding sums of the integrals

$$\iint_S f(x, y, z)\, dy\, dz \ \text{ and } \iint_S g(x, y, z)\, dz\, dx$$

and consequently, write the sum of these integrals,

$$\iint_S f\, dy\, dz + g\, dz\, dx + h\, dx\, dy \qquad ...(6)$$

called the *surface integral of the second type* of $F = (f, g, h)$ over S.

Notes:

1. The sum (4) can also be written as

$$\sum_{i=1}^{n} h(\xi_i, \eta_i, \zeta_i)\cos\gamma\, \Delta S_i \qquad ...(7)$$

which represents the flux of h across the surface S, and in the limit, represents the surface integral.

$$\iint_S h \cos\gamma \, dS$$

Thus, in the limit, the sum (4) represents the surface integral, $\iint_S h \, dx \, dy$ as well as $\iint_S h \cos\gamma \, dS$. Proceeding, similarly, for the other two functions f and g, we establish the justification of relation (3).

2. The surface integral of the second type, given by (1), is the flux of $F = (f, g, h)$ across the surface S and that is the reason, the surface integral of the second type is also called the flux of a function across a surface.

4.5 Reducing a Surface Integral of the Second Type to a Double Integral

The definition of the surface integral of the second type implies the following results:

(*i*) Let S be a smooth surface determined by an equation

$$Z = \Psi(x, y)$$

and let $h(x, y, z)$ be a bounded function defined on S. Then for the surface integral of the second type taken over the positive side of S we have the relation (definition, relation (4))

$$\iint_S h(x, y, z) \, dx \, dy, = \iint_D h(x, y, \Psi(x, y)) \, dx \, dy \qquad \text{...(8)}$$

where D is the projection of S on the $z = 0$ plane, and since S is positively oriented, the sign before the double integral is to be positive if D is positively oriented, otherwise negative.

If the integral is taken over the other side (negative) of S, the sign before the double integral is to be negative if D is positively oriented, otherwise positive.

We, similarly derive the formula

$$\iint_S f(x, y, z) \, dy \, dz = \iint_{D_1} f(\theta(y, z), y, z) \, dy \, dz \qquad \text{...(9)}$$

$$\iint_S g(x, y, z) \, dz \, dx = \iint_{D_2} g(x, \phi(z, x), z) \, dz \, dx \qquad \text{...(10)}$$

where D_1, D_2 are the projections of S on yz and zx planes respectively. So finally we have

$$\iint_S f \, dy \, dz + g \, dz \, dx + h \, dx \, dy = \iint_{D_1} f(\theta(y, z), y, z) \, dy \, dz$$

$$+ \iint_{D_2} g(x, \phi(z, x), z) \, dz \, dx + \iint_D h(x, y, \Psi(x, y)) \, dx \, dy \qquad \text{...(11)}$$

(*ii*) Let S be a smooth surface represented as

$$x = X(u, v), \quad y = Y(u, v), \quad z = Z(u, v); (u, v) \in D$$

where D is the surface in uv-plane corresponding to the surface S in the xy-plane.

Let $F = (f, g, h)$ be a bounded vector valued function defined on S. Then for the surface integral of the second type taken over the positive side of S, we have the relations (definition, relation (4))

$$\iint_S h(x, y, z) \, dx \, dy = \iint_D h(x, y, z) \, dx \, dy = \iint_D h(X, Y, Z) \frac{\partial(X, Y)}{\partial(u, v)} \, du \, dv.$$

$$\iint_S f(x, y, z)\, dy\, dz = \iint_D f(X, Y, Z) \frac{\partial (Y, Z)}{\partial (u, v)}\, du\, dv$$

$$\iint_S g(x, y, z)\, dz\, dx = \iint_D g(X, Y, Z) \frac{\partial (Z, X)}{\partial (u, v)}\, du\, dv$$

where D is also oriented in the same sense as S.

Thus, finally we have

$$\iint_S f\, dy\, dz + g\, dz\, dx + h\, dx\, dy$$

$$= \iint_D \left[f \frac{\partial (Y, Z)}{\partial (u, v)} + g \frac{\partial (Z, X)}{\partial (u, v)} + h \frac{\partial (X, Y)}{\partial (u, v)} \right] du\, dv \qquad ...(12)$$

Note: Vectorial formation

Let

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

be the position vector of any point on the surface S, and

$$\mathbf{F}(x, y, z) = \mathbf{i}P(x, y, z) + \mathbf{j}Q(x, y, z) + \mathbf{k}R(x, y, z)$$

be a vector function, with P, Q, R as its components.

Let $\mathbf{n}$ denote the unit vector along the normal at any point on the side of the surface under consideration, so that

$$\mathbf{n} = \mathbf{i} \cos \alpha + \mathbf{j} \cos \beta + \mathbf{k} \cos \gamma$$

$$\therefore \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma)\, dS \qquad ...(13)$$

Thus the surface integral of the second type of $\mathbf{F}$ (or the surface integral of the first type of $\mathbf{F} \cdot \mathbf{n}$) over S is $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Ex. Show that relation (12) satisfies (13).

4.6 Properties of Surface Integrals

1. The surface integrals (of the second type) taken over the opposite sides of a surface have different signs, *i.e.*,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -\iint_{S'} \mathbf{F} \cdot \mathbf{n}\, dS$$

where S and S' are the two sides of the surface.

2. If a surface S is broken up into m parts $S_1, S_2, \ldots, S_m$, the integral over the whole surface S (say, over its positive side) is equal to the sum of the integrals taken over the corresponding (*i.e.*, positive) sides of the surfaces $S_1, S_2, \ldots, S_m$.

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS + \ldots + \iint_{S_m} \mathbf{F} \cdot \mathbf{n}\, ds \qquad ...(14)$$

4.7 Relation Between the Two Types of Surface Integrals

Let S be a smooth (or piecewise smooth) oriented surface and C its bounding curve. Then the surface integral of the second type of a function P, taken over a certain side (say, positive) of the surface is

$\iint_S f\, dy\, dz$, which is equivalent to $\iint_S f \cos\alpha\, dS$. The latter being the surface integral of the first type of the function $f \cos\alpha$.

Similarly, the surface integral of the second type of the function $F = (f, g, h)$ is

$$\iint_S (f\, dy\, dz + g\, dz\, dx + h\, dx\, dy) \qquad \ldots(1)$$

which is equivalent to

$$\iint_S (f \cos\alpha + g \cos\beta + h \cos\gamma)\, dS \qquad \ldots(2)$$

The latter being a surface integral of the first type of the function $(f \cos\alpha + g \cos\beta + h \cos\gamma)$. Also $(f \cos\alpha + g \cos\beta + h \cos\gamma)$ is the projection or the component of the function F along the normal to the surface.

Thus the surface integral of the second type of a function is same as the surface integral of the first type of the normal component of the function.

Note: In symbolic representation of the surface integral,

$$\iint_S f\, dS, \quad \iint_S f \cos\alpha\, dS \quad \text{or} \quad \iint_S f\, dy\, dz$$

the meanings are well understood and so there is no need to mention the type.

But when symbols are not being employed, mention the type (of the integral) has to be made. Thus for $\iint_S f\, dS$, we say 'the surface integral of the first type of f', and for $\iint f \cos\alpha\, dS$, 'the surface integral of the first type of $f \cos\alpha$ or the surface integral of the second type of f'.

Example 8. Find the value of the surface integral $\iint_S \frac{dS}{r}$, where S is the portion of the surface of the hyperbolic paraboloid $z = xy$ cut off by the cylinder $x^2 + y^2 = a^2$, and r is the distance from a point on the surface to the z-axis.

▪ For any point (x, y) on the surface, $r = \sqrt{x^2 + y^2}$.

Now

$$\iint_S \frac{dS}{\sqrt{x^2 + y^2}} = \iint_D \frac{1}{\sqrt{x^2 + y^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

where D is the projection of E on xy-plane

$$= \iint_D \frac{\sqrt{1 + x^2 + y^2}}{\sqrt{x^2 + y^2}}\, dx\, dy = \int_0^{2\pi} d\theta \int_0^a \frac{\sqrt{1 + r^2}}{r}\, r\, dr$$

$$= \pi\left[a\sqrt{a^2 + 1} + \log\left(a + \sqrt{a^2 + 1}\right)\right]$$

Example 9. Evaluate the surface integral

$$\iint_S p(x^4 + y^4 + z^4)\, dS$$

where S is the surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and p is the perpendicular from the origin to the tangent plane at the point (x, y, z).

■ Symmetry of the integrand and the surface S show that its value is 8 times the value of the integral over the portion of the surface in the first octant.

$$\text{Let} \quad \left.\begin{aligned} x &= a \cos\theta \cos\phi \\ y &= b \cos\theta \sin\phi \\ z &= c \sin\theta \end{aligned}\right\}, \quad \begin{aligned} 0 \le \phi \le \pi/2 \\ 0 \le \theta \le \pi/2 \end{aligned}$$

Then

$$\frac{\partial(x, y)}{\partial(\theta, \phi)} = -\, ab \cos\theta \sin\theta$$

$$\frac{\partial(y, z)}{\partial(\theta, \phi)} = -\, bc \cos^2\theta \cos\phi$$

$$\frac{\partial(z, x)}{\partial(\theta, \phi)} = -\, ac \cos^2\theta \sin\phi$$

so that

$$\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2$$

$$= a^2b^2c^2 \cos^2\theta \left\{\frac{\cos^2\theta \cos^2\phi}{a^2} + \frac{\cos^2\theta \sin^2\phi}{b^2} + \frac{\sin^2\theta}{c^2}\right\}$$

and

$$p = 1\Big/\sqrt{\frac{\cos^2\theta \cos^2\phi}{a^2} + \frac{\cos^2\theta \sin^2\phi}{b^2} + \frac{\sin^2\theta}{c^2}}$$

The surface integral becomes

$$\iint_S p(x^4 + y^4 + z^4)\, dS$$

$$= 8 \int_0^{\pi/2} d\theta \int_0^{\pi/2} abc \cos\theta [a^4 \cos^4\theta \cos^4\phi + b^4 \cos^4\theta \sin^4\phi + c^4 \sin^4\theta]\, d\phi$$

$$= 8abc \int_0^{\pi/2} \cos\theta \left[a^4 \cos^4\theta \cdot \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} + b^4 \cos^4\theta \cdot \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} + c^4 \sin^4\theta \cdot \frac{\pi}{2}\right] d\theta$$

$$= 8abc \cdot \frac{\pi}{2}\left[\frac{3}{8}(a^4 + b^4)\frac{4\cdot 2}{5\cdot 3} + \frac{c^4}{5}\right] = \frac{4\pi abc(a^4 + b^4 + c^4)}{5}.$$

Example 10. Evaluate

$$I = \iint_S (x\,dy\,dz + dz\,dx + xz^2\,dx\,dy),$$

where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

- Let us denote the projections of the surface S on the coordinate planes yz, zx and xy by D_1, D_2, D_3 respectively. These are quarter circles of radius 1. Then we have

$$I_1 = \iint_S x\,dy\,dz = \iint_{D_1} \sqrt{1 - y^2 - z^2}\,dy\,dz$$

$$I_2 = \iint_S dz\,dx = \iint_{D_2} dz\,dx$$

$$I_3 = \iint_S xz^2\,dx\,dy = \int_{D_3} x(1 - x^2 - y^2)\,dx\,dy.$$

The second integral I_2 is simply the area of the domain D_2, (quarter circle $x^2 + z^2 = 1$), *i.e.*, equal to $\pi/4$.

$$I_1 = \int_0^{\pi/2} d\theta \int_0^1 \sqrt{1 - r^2}\; r\,dr = \frac{\pi}{6}$$

$$I_3 = \int_0^{\pi/2} \cos\theta\,d\theta \int_0^1 r(1 - r^2)\,r\,dr = \frac{2}{15}$$

Thus,

$$I = I_1 + I_2 + I_3 = \frac{\pi}{6} + \frac{\pi}{4} + \frac{2}{15} = \frac{5\pi}{12} + \frac{2}{15}.$$

Example 11. Evaluate the surface integral $\iint z\cos\gamma\,dS$, over the outer side of the sphere $x^2 + y^2 + z^2 = 1$, where γ is the inclination of the normal at any point of the sphere with the z-axis.

- Here the z-coordinate can be expressed as a single-valued function of x and y for the whole surface. Let us break it into two parts—the upper hemisphere S_1, above the xy-plane, and the lower hemisphere S_2, below it.

Accordingly their equations are

$$z = \sqrt{1 - x^2 - y^2},\quad z = -\sqrt{1 - x^2 - y^2}$$

We can write

$$\iint_S z\cos\gamma\,dS = \iint_{S_1} z\cos\gamma\,dS + \iint_{S_2} z\cos\gamma\,dS$$

Here, of the two integrals on the right hand side, the first is over the upper side of the upper hemisphere in the upward direction and the second is over the lower side of the lower hemisphere in the downward direction. The second integral will be negative and will therefore have to be taken with a negative sign.

$$\therefore \qquad \iint_{S_1} z\cos\gamma\, dS = \iint_{S_1} z\, dx\, dy = \iint_D \sqrt{1-x^2-y^2}\, dx\, dy$$

$$\iint_{S_2} z\cos\gamma\, dS = \iint_{S_1} z\, dx\, dy = -\iint_D -\sqrt{1-x^2-y^2}\, dx\, dy$$

where D is the circle $x^2 + y^2 = 1$.

Thus

$$\iint_S z\cos\gamma\, dS = 2\iint_D \sqrt{1-x^2-y^2}\, dx\, dy = 2\int_0^{2\pi} d\theta \int_0^1 \sqrt{1-r^2}\, r\, dr = \frac{4\pi}{3}.$$

Note: The surface integral is the flux of a function through the surface S. In the above example, the flux through the whole surface is equal to the sum of the fluxes through the two hemisphere, ignoring the direction.

Example 12. Show that

$$I = \iint_S (yz\, dy\, dz + zx\, dz\, dx + xy\, dx\, dy) = \frac{3}{8}$$

where S is the other surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

■ Let $\left.\begin{aligned} x &= \cos\theta\cos\phi \\ y &= \cos\theta\sin\phi \\ z &= \sin\theta \end{aligned}\right\},\ \begin{aligned} 0 \le \theta \le \pi/2 \\ 0 \le \phi \le \pi/2 \end{aligned}$

so that

$$\frac{\partial(y,z)}{\partial(\theta,\phi)} = -\cos^2\theta\cos\phi,\quad \frac{\partial(z,x)}{\partial(\theta,\phi)} = -\cos^2\theta\sin\phi,\quad \frac{\partial(x,y)}{\partial(\theta,\phi)} = -\sin\theta\cos\theta$$

the negative signs show that the correspondence is inverse and so the double integrals are to be taken with negative signs.

Thus we get

$$I = 3\int_0^{\pi/2} \cos^3\theta\sin\theta\, d\theta \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = \frac{3}{8}$$

Ex. Do the above example by the method explained in solved example 11.

Example 13. Evaluate $\iint_S x\, dS$, where S is the entire surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$, $z = x + 2$.

■ As shown in the figure, the surface consists of three parts: S_1, the circular base in the xy-plane, S_2, the elliptic plane section, *i.e.*, part of the plane $z = x + 2$ inside the cylinder $x^2 + y^2 = 1$, and S_3, the lateral surface of the cylinder.

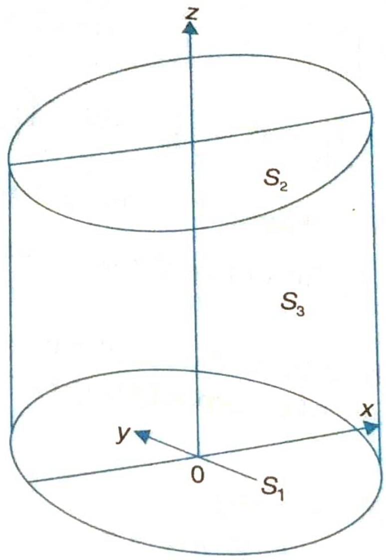

Fig. 6

On S_1, we have

$$z = 0,\ x^2 + y^2 = 1$$

$$\therefore \qquad \iint_{S_1} x\, dS = \iint_{S_1} x\, dx\, dy = \int_{-1}^{1} x\, dx \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} dy = 0$$

On S_2,

$$z = x + 2,\ x^2 + y^2 = 1$$

$$\therefore \qquad \iint_{S_2} x\, dS = \iint_{S_1} x\sqrt{1+1}\, dx\, dy = \sqrt{2} \iint_{S_1} x\, dx\, dy = 0$$

On S_2,

$$x^2 + y^2 = 1,\ 0 \le z \le x + 2$$

put

$$x = \cos\theta,\ y = \sin\theta,\ z = z$$

$$-\pi \le \theta \le \pi,\ 0 \le z \le 2 + \cos\theta$$

so that

$$\frac{\partial(y, z)}{\partial(\theta, z)} = \cos\theta,\ \frac{\partial(z, x)}{\partial(\theta, z)} = \sin\theta,\ \frac{\partial(x, y)}{\partial(\theta, z)} = 0$$

$$\therefore \qquad \iint_{S_2} x\, dS = \iint_{D} \cos\theta \sqrt{\cos^2\theta + \sin^2\theta}\, d\theta\, dz.$$

where D is the corresponding domain in the θz-plane.

$$= \int_{-\pi}^{\pi} \cos\theta \, d\theta \int_0^{2+\cos\theta} dz = \int_{-\pi}^{\pi} \cos\theta (2 + \cos\theta)\, d\theta = \pi$$

$$\iint_S x \, dS = \iint_{S_1} x \, dS + \iint_{S_2} x \, dS + \iint_{S_3} x \, dS = \pi \, .$$

Example 14. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$, where S is the entire surface of the solid formed by $x^2 + y^2 = 1$, $z = 0$, $z = x + 2$ and $\mathbf{n}$ is the outward drawn unit normal and the vector function $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$.

■ The surface S consists of three sub-surfaces, the circular base S_1, the elliptic plane section S_2 and the lateral circular section S_3. (figure of example 13).

On S_1,

$$x^2 + y^2 = 1, \; z = 0$$

$$\mathbf{n} = -\mathbf{k}, \; \mathbf{F} \cdot \mathbf{n} = -z$$

$$\therefore \qquad \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} -z \, dS = \iint_{S_1} -z \, dx \, dy = 0$$

On S_2,

$$z = x + 2, \;\; x^2 + y^2 = 1$$

$$-\frac{1}{\sqrt{2}} \mathbf{i} + 0\mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{2}} (-2x + z) = \frac{1}{\sqrt{2}} (-x + 2)$$

$$\therefore \qquad \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{\sqrt{2}} \iint_{S_2} (-2x + z) \, dS$$

$$= \frac{1}{\sqrt{2}} \iint_{S_1} (-2x + x + 2) \sqrt{1 + 1} \, dx \, dy$$

$$= \iint_{S_1} (2 - x) \, dx \, dy$$

where S_1 is the circular base, $x^2 + y^2 = 1$, $z = 0$.

$$\iint_{S_2} F.n \, dS = \int_{-\pi}^{\pi} d\theta \int_0^1 (2 - r\cos\theta) \, r \, dr = \int_{-\pi}^{\pi} (1 - \frac{1}{3}\cos\theta) d\theta = 2\pi$$

On S_3,

$$x^2 + y^2 = 1, \; 0 \le z \le x + 2$$

Let

$$x = \cos\theta, \; y = \sin\theta, \;\; z = z$$

$$-\pi \le \theta \le \pi,\ 0 \le z \le 2 + \cos\theta$$

$$\mathbf{F}\cdot\mathbf{n} = (\mathbf{i}2\cos\theta - \mathbf{j}3\sin\theta + \mathbf{k}z)\cdot(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$$

$$= 2\cos^2\theta - 3\sin^2\theta$$

$$\therefore \quad \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= \int_{-\pi}^{\pi}\int_0^{2+\cos\theta} (2\cos^2\theta - 3\sin^2\theta)\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta\, dz$$

$$= \int_{-\pi}^{\pi} (2 - 5\sin^2\theta)(2 + \cos\theta) = -2\pi$$

$$\therefore \quad \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= 0 + 2\pi - 2\pi = 0\ .$$

EXERCISE

1. Evaluate the integrals

(i) $\iint_S (z + 2x + \frac{4}{3}y)\, dS$, over the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$, lying in the first octant.

(ii) $\iint_S xyz\, dS$, over the portion of $x + y + z = 1$, lying in the first octant.

(iii) $\iint_S x\, dS$ where S is the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant.

2. Compute the integrals:

(i) $\iint_S \sqrt{a^2 - x^2 - y^2}\, dS$, where S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

(ii) $\iint_S x^2y^2\, dS$, where S is the hemisphere $z = \sqrt{R^2 - x^2 - y^2}$.

(iii) $\iint_S \frac{dS}{r^2}$, where S is the cylinder $x^2 + y^2 = a^2$, bounded by the planes $z = 0$, $z = h$, and r is the distance between a point on the surface and the origin.

3. Evaluate the surface integrals:

(i) $\iint_S x^2y^2 z\, dS$, where S is the positive side of the lower half of the sphere $x^2 + y^2 + z^2 = a^2$.

(ii) $\iint_S z^2 dx\, dy$, where S is the outer side of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(iii) $\iint_S (x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy)$, where S is the outer surface of the sphere $x^2 + y^2 + z^2 = 1$.

(iv) $\iint_S (xz\,dx\,dy + xy\,dy\,dz + yz\,dz\,dx)$, where S is the outer side of the pyramid formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

4. Evaluate the surface integral

$$\iint_S (x \cos \alpha + y \cos \beta + z \cos \gamma)\,dS,$$

$\cos \alpha$, $\cos \beta$, $\cos \gamma$ being the direction cosines of the outward drawn normal of the surface S, where

(i) S is the positive side of the cube formed by the planes $x = 0$, $y = 0$, $z = 0$, and $x = 1$, $y = 1$, $z = 1$.

(ii) S is the outer surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lying above the xy-plane.

5. Evaluate $\iint_E (x + y + z)(lx + my + nz)\,dS$, where E is the surface of the region $x^2 + y^2 \le a^2, 0 \le z \le h$.

6. Find the value of the surface integral

$$\iint_S (yz\,dx\,dy + xz\,dy\,dz + xy\,dx\,dz),$$

where S is the outer side of the surface situated in the first octant and formed by the cylinder $x^2 + y^2 = a^2$ and the planes $x = 0$, $y = 0$, $z = 0$, $z = h$.

7. Evaluate $\iint_S z^2\,dS$, where S is the part of the outer surface of the cylinder $x^2 + y^2 = 4$ between the planes $z = 0$, $z = x + 3$.

8. Compute the integral

$$\iint_S (y^2 z\,dx\,dy + xz\,dy\,dz + x^2 y\,dx\,dz),$$

where S is the outer side of the surfaces situated in the first octant and formed by the paraboloid of revolution $z = x^2 + y^2$, cylinder $x^2 + y^2 = 1$ and the coordinate planes.

[See solved Example No. 24 for another method.]

9. Compute $\iint_S \mathbf{F} \cdot \mathbf{n}\,dS$, when $\mathbf{n}$ is the outward drawn unit normal of the surface, and the vector function $\mathbf{F}$ and the oriented surface S are given in each case.

(i) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, S is part cone $z^2 = x^2 + y^2, 1 \le z \le 2, \mathbf{n} \cdot \mathbf{k} > 0, \mathbf{n} \cdot \mathbf{i} > 0$.

(ii) $\mathbf{F} = y^2\mathbf{i} + z\mathbf{j} - x\mathbf{k}$, S is part of cylinder $y^2 = 1 - x$ between the planes $z = 0, z = x;\ x \ge 0$ with $\mathbf{n} \cdot \mathbf{i} > 0$.

ANSWERS

1. (i) $4\sqrt{61}$ (ii) $\sqrt{3}/120$ (iii) $\pi a^3/4$
2. (i) πa^3 (ii) $2\pi R^6/15$ (iii) $2\pi \tan^{-1}(h/a)$
3. (i) $2\pi a^7/105$ (ii) 0 (iii) $56\pi a^2/9$ (iv) $1/8$
4. (i) 3 (ii) $2\pi abc$
5. $\frac{1}{3}\pi ah[3(l + m)a^2 + 3nah + 2nh^2]$
6. $a^2h(2a/3 + \pi h/8)$
7. π
8. $\pi/8$
9. (i) $15\pi/4$ (ii) $4/15$

5. STOKES' THEOREM (*First generalization of Green's theorem*)

We recall that Green's Theorem expresses a relation between a double integral over a plane region and a line integral taken round its plane boundary. There are two ways to generalise this in R^3. One of these extensions, known as *Stokes' Theorem*, relates a surface integral taken over a surface to a line integral taken around the boundary curve of the surface. This generalisation is due to an English mathematician, *George Gabriel Stokes* (1819–1903).

A second generalisation arises when the double integral is replaced by a triple integral, and the line integral by a surface integral. This generalisation is named as *Gauss's Theorem* and will be taken up later.

Stokes' Theorem. *If S is a smooth oriented surface bounded by a curve C oriented in the same sense, and f, g, h are three functions which along with their first order partial derivatives are continuous in a three dimensional domain containing S, then*

$$\int_C (f\,dx + g\,dy + h\,dz) = \iint_S \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy\,dz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz\,dx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy\right]$$

Let the oriented surface be represented as

$$x = x(u, v),\ \ y = y(u, v),\ \ z = z(u, v),\ \ (u, v) \in D$$

where D is an oriented surface in uv-plane. Also, let its boundary be an oriented curve Γ represented by

$$u = u(t),\ \ v = v(t),\ \ a \le t \le b$$

The proof of the theorem involves the following steps:

1. The line integral along C is expressed as an ordinary integral,
2. The ordinary integral is expressed as a line integral along Γ,
3. The line integral along Γ is then expressed, by Green's Theorem, as double integral over D, and finally,
4. The double integral along D is expressed as a surface integral over S.

Now

$$\int_C (f\,dx + g\,dy + h\,dz)$$

$$= \int_a^b \left[f\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) dt + g\left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right) dt + h\left(\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\right) dt\right]$$

$$= \int_\Gamma \left(f\frac{\partial x}{\partial u} + g\frac{\partial y}{\partial u} + h\frac{\partial z}{\partial u}\right) du + \left(f\frac{\partial x}{\partial v} + g\frac{\partial y}{\partial v} + h\frac{\partial z}{\partial v}\right) dv$$

$$= \iint_D \left[\frac{\partial}{\partial u}\left(f\frac{\partial x}{\partial v} + g\frac{\partial y}{\partial v} + h\frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(f\frac{\partial x}{\partial u} + g\frac{\partial y}{\partial u} + h\frac{\partial z}{\partial u}\right)\right] du\,dv \qquad \ldots(1)$$

But

$$\frac{\partial}{\partial u}\left(f\,\frac{\partial x}{\partial v}\right)=\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v}+f\,\frac{\partial^2 x}{\partial u\partial v}$$

Writing down similar expressions for the other terms of the integrand and rearranging, the double integral on the right hand side of (1) becomes

$$\iint_D\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\frac{\partial(z,x)}{\partial(u,v)}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}\right]du\,dv$$

$$=\iint_S\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dy\,dz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dz\,dx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dx\,dy \qquad \text{...(2)}$$

Hence the proof.

Also by the definition of surface integral, relation (2) is equivalent to

$$\int_C (f\,dx+g\,dy+h\,dz)=\iint_S\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\cos\alpha+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\cos\beta\right.$$

$$\left.+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\cos\gamma\right]dS \qquad \text{...(3)}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the normal at any point to the surface.

Notes:

1. (*Vectorial formulation*). Let

$$\mathbf{r}=\mathbf{i}\,x+\mathbf{j}\,y+\mathbf{k}\,z$$

be the position vector of any point on the surface S, and

$$\mathbf{F}(x,y,z)=\mathbf{i}P(x,y,z)+\mathbf{j}Q(x,y,z)+\mathbf{k}R(x,y,z)$$

be a vector function defined on S.

Let $\mathbf{n}$ denote the unit normal at any point of the surface under consideration, so that

$$\mathbf{n}=\mathbf{i}\cos\alpha+\mathbf{j}\cos\beta+\mathbf{k}\cos\gamma$$

$$\therefore\quad \text{curl }\mathbf{F}\cdot\mathbf{n}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma$$

and

$$\mathbf{F}\cdot d\mathbf{r}=P\,dx+Q\,dy+R\,dz$$

so that by (3) Stokes' theorem can be written as

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \text{curl }\mathbf{F}.\mathbf{n}\,dS$$

2. If the surface S is a piece of a plane parallel to the xy-plane then $dz=0$ and we get Green's Theorem as special case of Stokes' Theorem.

5.1 Deductions from Stokes' Theorem

Stokes' theorem has various applications in mathematical analysis. Here we are going to establish only one such deduction: the conditions for a line integral to be independent of the path of integration. These conditions, in fact, generalise the results obtained from Green's theorem (§ 4.1 Ch. 17) concerning the question of path independence of an integral over a plane curve. With that view, we introduce the following concept.

Definition. A three-dimensional domain V is said to be *simply connected* if, for any closed contour belonging to V there exists a surface, with the contour as its boundary, entirely lying in V.

A sphere (ball), the whole space, the domain lying between two concentric spheres are examples of a *simply connected* space. An example of a domain which is not simply connected (*referred to as multiply connected*) is a ball with a cylindrical tunnel passing through it.

Now we proceed to establish the following result analogous to § 4.1, Chapter 17.

Theorem 2. *If three functions $f(x, y, z)$, $g(x, y, z)$, and $h(x, y, z)$, defined in a bounded closed simply connected domain V, are continuous along with their first order partial derivatives in the domain, then the following four assumptions are equivalent to each other.*

1. *The line integral $\int f\,dx + g\,dy + h\,dz$ taken along any closed contour lying inside V is equal to zero.*
2. *The line integral $\int_{AB} f\,dx + g\,dy + h\,dz$ is independent of the path of integration connecting two arbitrary points A and B.*
3. *The expression $f\,dx + g\,dy + h\,dz$ is the total differential of a single valued function defined in V.*
4. *The conditions*

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}, \frac{\partial h}{\partial y} = \frac{\partial g}{\partial z}, \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x}$$

are fulfilled at each point of the domain V.

The theorem is proved according to the scheme $1 \to 2 \to 3 \to 4 \to 1$ which we followed while proving § 4.1 Ch. 16. We leave the proof to the reader with the only hint that to deduce condition 1 from condition 4, one must take an arbitrary closed contour Γ lying within V and consider a surface entirely lying in V whose boundary is Γ, such a surface exists because of the condition that V is a simply connected domain. Then the application of Stoke's theorem to the line integral along Γ shows that condition 4 implies the relation $\int_\Gamma f\,dx + g\,dy + h\,dz = 0$.

Example 15. Use Stokes' theorem to find the line integral

$$\int_C x^2 y^3\,dx + dy + z\,dz$$

where C is the circle $x^2 + y^2 = a^2, z = 0$.

■ Now, by Stokes' theorem

$$\int_C x^2y^3\,dx + dy + z\,dz = \iint_S \left(\frac{\partial z}{\partial y} - \frac{\partial .1}{\partial z}\right) dy\,dz + \left(\frac{\partial x^2y^3}{\partial z} - \frac{\partial z}{\partial x}\right) dz\,dx$$

$$+\left(\frac{\partial .1}{\partial x} - \frac{\partial x^2y^3}{\partial y}\right) dx\,dy = -\iint_S 3x^2y^2\,dx\,dy$$

where S is the circle $x^2 + y^2 = a^2$ in the xy-plane. Changing to polars.

$$= -3\int_{-\pi}^{\pi}\int_0^a r^4 \cos^2\theta \sin^2\theta\, r\,dr\,d\theta$$

$$= -\frac{3a^6}{6}\int_{-\pi}^{\pi} \cos^2\theta \sin^2\theta\,d\theta = -\frac{\pi a^6}{8}$$

Example 16. Show that

$$\iint_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy = a^3\pi$$

where S is the portion of the surface $x^2 + y^2 - 2ax + az = 0,\ z \geq 0$.

■ By Stokes' theorem

$$\iint_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy$$

$$= \frac{1}{2}\int_C (y^2+z^2)\,dx + (z^2+x^2)\,dy + (x^2+y^2)\,dz$$

where C is the curve

$$(x-a)^2 + y^2 = a^2,\quad z = 0$$

On putting $x = a + a\cos\theta = a(1+\cos\theta)$, $y = a\sin\theta$, the line integral becomes

$$= \frac{1}{2}\int_{-\pi}^{\pi} [a^2\sin^2\theta(-a\sin\theta) + a^2(1+\cos\theta)^2\, a\cos\theta]\,d\theta$$

$$= \frac{a^3}{2}\int_{-\pi}^{\pi} (-\sin^3\theta + \cos\theta + 2\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= a^3\int_0^{\pi} (2\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= a^3\int_0^{\pi/2} (2\cos^2\theta + \cos^3\theta)\,d\theta + a^3\int_0^{\pi/2} (2\sin^2\theta - \sin^3\theta)\,d\theta = \pi a^3.$$

Note: The method employed to convert a surface integral into a line integral is not general.

EXERCISE

1. Using Stokes' theorem, show that

$$\int_C y\,dx + z\,dy + x\,dz = -\iint_S (\cos\alpha + \cos\beta + \cos\gamma)\,dS.$$

2. Show, using Stokes' theorem, that

$$\int_\Gamma (y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$$

where Γ is the circle $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$.

3. Using Stokes' theorem, prove that

$$\int_\Gamma y\,dx + z\,dy + x\,dz = -2\pi a^2\sqrt{2}$$

where Γ is the curve $x^2 + y^2 + z^2 - 2ax - 2ay = 0$, $x + y = 2a$.

4. Apply Stokes' theorem to transform the integral

$$\int_C (y^2+z^2)\,dx + (x^2+z^2)\,dy + (x^2+y^2)\,dz$$

taken along a smooth curve C to a certain integral over a smooth oriented surface with C as its boundary.

5. Verify Stokes' theorem for the integral

$$\int_C x^2\,dx + yx\,dy$$

where C is a square in the $z = 0$ plane with sides along the lines, $x = 0$, $y = 0$, $x = a$, $y = a$.

6. Verify Stokes' theorem in each case

(i) $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

S is the part of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$, $\mathbf{n}\cdot\mathbf{k} > 0$.

(ii) $\mathbf{F} = y^2\mathbf{i} + xy\mathbf{j} - 2xz\mathbf{k}$

S is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$ with $n\cdot\mathbf{k} > 0$.

6. THE VOLUME OF A CYLINDRICAL SOLID BY DOUBLE INTEGRALS

We have shown earlier that the volume of a cylindrical solid can be found with the help of double integrals.

Let a cylindrical solid be bounded above by a surface $z = \Psi(x, y)$, below by a plane region D (on the xy-plane) and on the sides by lines parallel to z-axis. Its volume V is given by

$$V = \iint_D \Psi(x, y)\,dx\,dy, \text{ in cartesian coordinates}$$

or

$$= \iint_D \Psi(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \text{ in polar coordinates}$$

or

$$= \iint_S z\cos\gamma\,dS, \text{ as a surface integral}$$

where S is the surface of the solid.

If the equation of the surface is given in the form

$$x = \theta(y, z), \text{ or } y = \phi(z, x)$$

then the corresponding formulas for calculating the volumes are of the form

$$V = \iint_{D_1} \theta(y, z)\, dy\, dz \quad \text{or} \quad \iint_{S_1} x \cos\alpha\, dS$$

$$V = \iint_{D_2} \phi(z, x)\, dz\, dx \quad \text{or} \quad \iint_{S_2} y \cos\beta\, dS$$

where D_1, D_2 are the domains in the *yz*-plane and *zx*-plane, in which the given surface is projected.

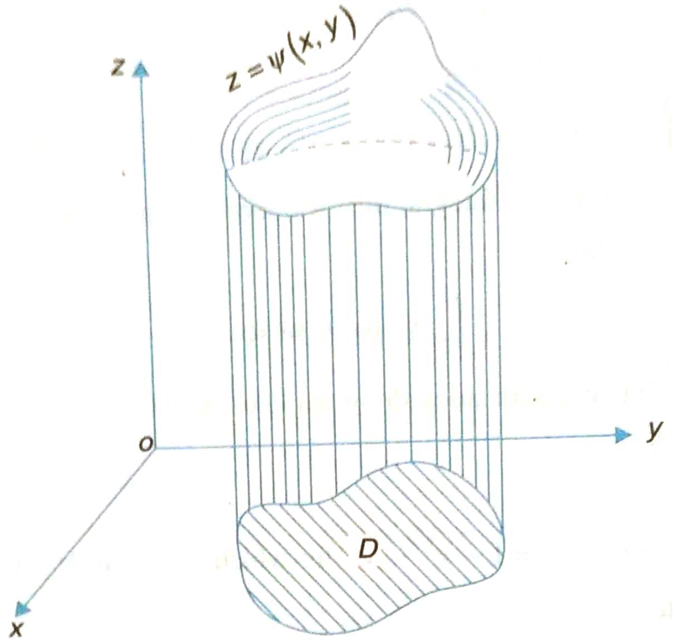

Fig. 7

Notes:

1. Clearly D is projection on the *xy*-plane, of the portion of the surface $z = \Psi(x, y)$, cut off by the lateral cylindrical surface.
2. The function ψ is assumed to be continuous and single-valued so that the surface is met by a line parallel to z-axis not more than one point.
3. If the function ψ changes sign in D, then we divide the domain D into two parts. (*i*) the sub-domain D_1 where $\Psi \geq 0$, and (*ii*) the sub-domain D_2 where $\Psi \leq 0$. The double integral over D_1 will be positive and equal to the volume of the solid lying above the *xy*-plane. The integral over D_2 will be negative and equal, in absolute value, to the volume of the solid lying below the *xy*-plane. Thus the integral over D will express the difference between the corresponding volumes. The sum of the absolute values of the two integrals, over D_1 and D_2, will give the volume of the solid.
4. *Volume by iterated integral interpreted geometrically.*

$$V = \iint_D \Psi\, dxdy = \int_a^b dx \int_{\phi_1(x)}^{\phi_2(x)} \Psi\, dy$$

$$= \int_a^b S(x)\, dx\,.$$

where

$$S(x) = \int_{\phi_1(x)}^{\phi_2(x)} \Psi \, dy = \text{area of a cross section parallel to } zx\text{-plane}$$

$\therefore$ V = volume of the solid by parallel cross-sections

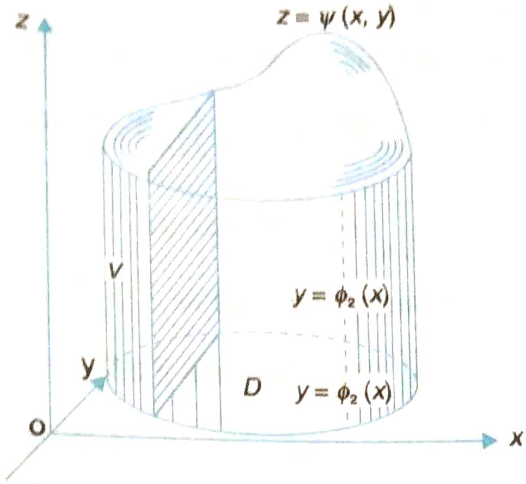

Fig. 8

6.1 Volume Enclosed by Two Surfaces

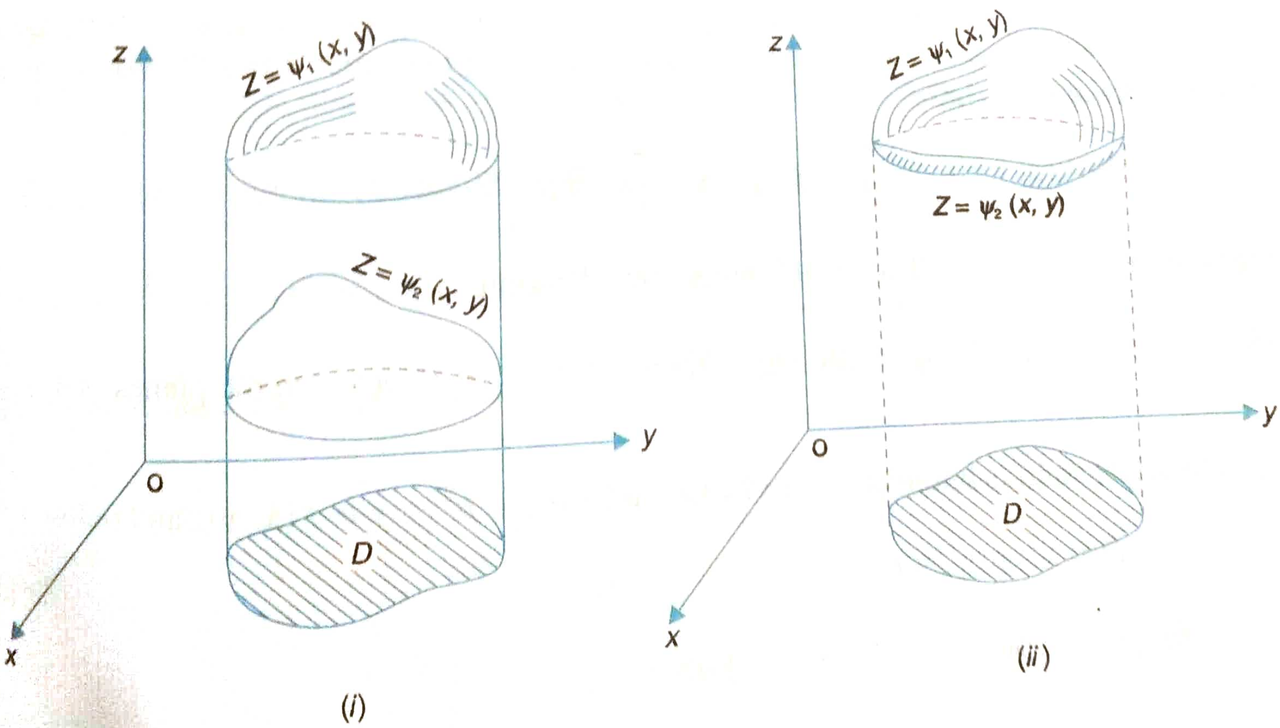

Fig. 9

If a solid, the volume of which is being found, is bounded above by the surface $z = \Psi_1(x, y) \ge 0$ and below by the surface $z = \Psi_2(x, y) \ge 0$, and domain D is the projection of both the surfaces (Fig. (i) and (ii)) on the xy-plane, then the volume, V, of this solid is equal to the difference between the volumes of the two 'cylindrical bodies'; the first of these cylindrical bodies has the domain D for its lower base and the surface $z = \Psi_1(x, y)$ for the upper; the second body also has D for its lower base and the surface $z = \Psi_2(x, y)$ for its upper base.

Therefore the required volume V is equal to the difference between the two double integrals.

$$\therefore \qquad V = \iint_D \Psi_1 \, dx\, dy - \iint_D \Psi_2 \, dx\, dy$$

$$= \iint_D (\Psi_1 - \Psi_2)\, dx\, dy$$

which may be expressed in terms of a surface integral as

$$= \iint_S (\Psi_1 - \Psi_2) \cos\gamma \, dS$$

It may be easily verified that the formula holds true not only for the case where Ψ_1 and Ψ_2 are non-negative, but also where Ψ_1 and Ψ_2 are any continuous, single-valued function that satisfy the relationship

$$\Psi_1(x, y) \ge \Psi_2(x, y) \text{ over } D$$

6.2 Volume Enclosed by a Closed Surface

Let a closed surface S be such that any straight line parallel to the z-axis cut it in not more than two points. Let the outer normal be the positive direction of the normal.

The surface may be divided into two parts: the upper and the lower. Let their equations be $z = \Psi_1(x, y)$, $z = \Psi_2(x, y)$. If D is the projection of S on xy-plane, then since the normal to the lower surface is downward, we have

$$\iint_S z \cos\gamma \, dS = \iint_D (\Psi_1 - \Psi_2)\, dx\, dy$$

which represents the volume of the solid under consideration.

Example 17. Find the volume within the cylinder $x^2 + y^2 = a^2$ between the planes $y + z = b^2$, and $z = 0$.

■ The cylindrical solid is bounded above by the surface $z = b^2 - y \equiv f(x, y)$, and below by the disc

$$D \equiv x^2 + y^2 = a^2.$$

$$\therefore \qquad \text{Required volume} = \iint_D (b^2 - y)\, dx\, dy$$

$$= \int_{-a}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} (b^2 - y)\, dy = \pi a^2 b^2 .$$

Example 18. Find the volume of

(*i*) the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and the plane $z = 0$.

(*ii*) the sphere $x^2 + y^2 + z^2 = a^2$, using polar coordinates.

▪ (*i*) The solid in question is a segment of the elliptical paraboloid lying above the *xy*-plane. The paraboloid cuts the *xy*-plane along the ellipse $4x^2 + y^2 = 1$, which forms the base D of the solid. Thus the solid, without lateral cylindrical surface, is bounded above by $z = 1 - 4x^2 - y^2$ and below by the ellipse $4x^2 + y^2 = 1$. Moreover the solid being symmetrical, its volume V is four times the volume lying in the first octant.

$$\therefore \quad V = 4\int_0^{1/2} dx \int_0^{\sqrt{1-4x^2}} (1 - 4x^2 - y^2)\, dy = \frac{8}{3}\int_0^{1/2} (1 - 4x^2)^{3/2}\, dx$$

$$= \frac{4}{3}\int_0^{\pi/2} \cos^4 t\, dt = \frac{\pi}{4}.$$

(*ii*) As in Part (*i*) the solid under consideration is bounded above by $z^2 = a^2 - (x^2 + y^2)$, or by $z = \sqrt{a^2 - r^2}$, on changing to cylindrical polar coordinates. The sphere cuts the *xy*-plane in the circle $x^2 + y^2 = a^2$ or $r^2 = a^2$ and has no lateral cylindrical surface. Again because of symmetry, its volume

$$V = 2\int_0^{2\pi} d\theta \int_0^a \sqrt{a^2 - r^2}\, r\, dr = \frac{4}{3}\pi a^3.$$

Example 19. Compute the volume V, common to the ellipsoid of revolution $x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ and the cylinder $x^2 + y^2 - ay = 0$.

▪ The required volume is double the volume that lies above the *xy*-plane.

The solid under consideration is bounded above by $z = \dfrac{b}{a}\sqrt{a^2 - x^2 - y^2}$ and below by the circular base $D \equiv x^2 + y^2 - ay = 0$ on the *xy*-plane.

In polar coordinates, the upper boundary is $z = \dfrac{b}{a}\sqrt{a^2 - r^2}$, and the lower base is $r = a \sin\theta$.

Thus,

$$V = 2\int_0^{\pi/2} d\theta \int_0^{a\sin\theta} \frac{b}{a}\sqrt{a^2 - r^2}\, r\, dr$$

$$= \frac{4a^2b}{3}\int_0^{\pi/2} (1 - \cos^3\theta)\, d\theta = \frac{2}{9}a^2b(3\pi - 4).$$

Example 20. Compute the volume of the solid bounded by the cylindrical surfaces $z = 4 - y^2$ and $y = x^2/2$ and the plane $z = 0$.

The upper boundary of the solid is the surface with equation $z = 4 - y^2$. The domain of integration D is bounded by the parabola $y = x^2/2$ and the line of intersection of the cylinder $z = 4 - y^2$ with $z = 0$ plane, *i.e.*, the straight line $y = 2$.

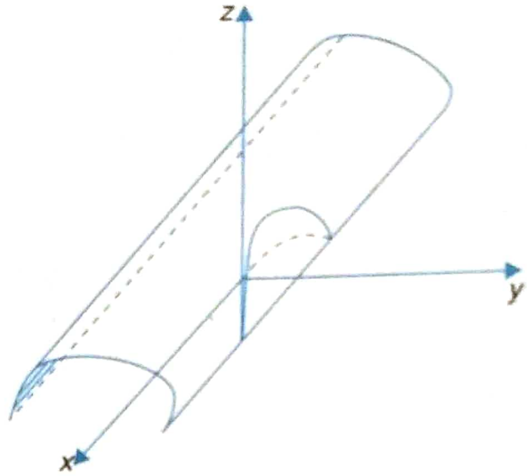

Fig. 10

The solid being symmetrical about the yz-plane, its volume

$$V = 2\int_0^2 dx \int_{x^2/2}^2 (4 - y^2)\, dy$$

$$= 2\int_0^2 \left(8 - \frac{8}{3} - 2x^2 + \frac{x^6}{24}\right) dx = \frac{256}{21}.$$

Example 21. Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$.

■ The two surfaces intersect in a space curve, whose projection on the xy-plane is the ellipse

$$x^2 + 4y^2 = 4$$

or

$$x^2/4 + y^2 = 1$$

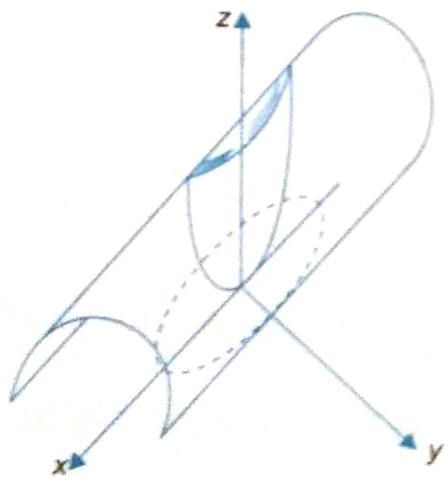

Fig. 11

which on putting

$$x = 2r\cos\theta,\quad y = r\sin\theta$$

becomes $r^2 = 1$.

The difference,

$$f_1(x, y) - f_2(x, y) = 4 - 4y^2 - x^2 = 4(1 - r^2)$$

The Jacobian $J = 2r$.

Making use of the symmetry of the solid, the required volume

$$= 4\int_0^{\pi/2}\int_0^1 4(1 - r^2)2r\,dr\,d\theta = 4\pi\,.$$

EXERCISE

1. Show that the volume of the solid bounded above by $z = 2 - r$ and below by the plane region, $0 \le r \le 2\cos\theta,\ -\pi/2 \le \theta \le \pi/2$ is $2(9\pi - 16)/9$.
2. Prove that the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$ is $2(3\pi - 4)a^3/9$.
3. Show that the volume common to the surface $y^2 + z^2 = 4ax$ and $x^2 + y^2 = 2ax$ is $\frac{2}{3}(3\pi + 8)a^3$.
4. Show that the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $16a^3/3$.
5. A sphere of radius a is pierced by a circular cylinder of radius b $(b < a)$, the axis of the cylinder passing through the centre of the sphere. Prove that the volume of the sphere that lies inside the cylinder is $\frac{4}{3}\pi\,[a^3 - (a^2 - b^2)^{3/2}]$ and the surface of the sphere inside the cylinder is $4\pi a\{a - (a^2 - b^2)^{1/2}\}$.
6. The part of the volume of a sphere of radius a that lies inside a right circular cone of semi-vertical angle α whose vertex is on the sphere and whose axis is a diameter of the sphere is $\frac{4}{3}\pi a^3\,(1 - \cos^4\alpha)$.
7. One loop of the curve $r^2\cos^2\theta = a^2\cos 2\theta$ makes a complete revolution about the initial line; show that the volume of the solid generated is $\dfrac{\pi}{6}(10 - 3\pi)a^3$.
8. If V is the volume of a solid bounded by a surface σ then show that
$$\iint_\sigma (x\cos\alpha + y\cos\beta + z\cos\gamma)\,d\sigma = 3V\,.$$
9. Find $\iint_S z\,dx\,dy$, where S is the external surface of the sphere $x^2 + y^2 + z^2 = a^2$.
10. The sphere $x^2 + y^2 + z^2 = a^2$ is pierced by the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, prove that the volume of the sphere inside the cylinder is $\dfrac{8}{3}\left(\dfrac{\pi}{4} + \dfrac{5}{3} - \dfrac{4\sqrt{2}}{3}\right)a^3$, and that the area of the spherical surface inside the cylinder is $8\left(\dfrac{\pi}{4} + 1 - \sqrt{2}\right)a^2$.

7. VOLUME INTEGRALS (*Triple Integrals*)

In the foregoing chapter we introduced the notion of the double integral. Here we are going to define the integral of a function of three independent variables, the so-called *volume integral*, also known as *triple integral*, in R^3.

Triple integrals are a straight and simple extension of the idea of double integrals and are in many respects almost completely analogous to them. We shall, therefore, only briefly indicate the various stages of the development of the theory and omit those proofs which do not essentially differ from the corresponding proofs of the theory of the double integrals.

7.1 Partition of a Rectangular Parallelepiped

A region in R^3, enclosed by the inequalities (including its boundary),

$$a \le x \le b; \quad c \le y \le d; \quad g \le z \le h$$

and denoted by $R = [a, b; c, d; g, h]$ is called a *rectangular parallelepiped* (a cuboid), or a *rectangle* in R^3. Its volume $V = (b-a)(d-c)(h-g)$.

If $P_1 = \{a = x_0,\ x_1, \ldots, x_l = b\}$, $P_2 = \{c = y_0,\ y_1, \ldots, y_m = d\}$, $P_3 = \{g = z_0,\ z_1, \ldots, z_n = h\}$ are respectively the partitions of $[a, b]$, $[c, d]$ and $[g, h]$, then planes drawn parallel to the coordinate planes through the points of P_1, P_2, P_3 give rise to a *partition* P of the parallelepiped R into lmn sub-parallelepipeds $[x_{i-1}, x_i;\ y_{j-1}, y_j;\ z_{k-1}, z_k]$ denoted as ΔR_{ijk}. The symbol ΔR_{ijk} will denote the sub-parallelepiped as also its volume $(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$.

Clearly $P = P_1 \times P_2 \times P_3$, the Cartesian product of P_1, P_2, P_3.

If $\mu(P_1) = \Delta x_r$, $\mu(P_2) = \Delta y_s$, $\mu(P_3) = \Delta z_t$ be the norms of P_1, P_2, P_3 respectively, then $\Delta R_{rst} = [x_{r-1}, x_r;\ y_{s-1}, y_s;\ z_{t-1}, z_t]$ is called the *norm* of P, denoted as $\mu(P)$.

Clearly, the volume of each sub-parallelepiped tends to zero as $\mu(P)$ tends to zero.

7.2 Triple Integration over a Parallelepiped

Let f be a bounded function of x, y, z on a parallelepiped $R = [a, b; c, d; g, h]$, and P, a partition of R. As in double integrals, we form the *Darboux sums*

$$U(P, f) = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} M_{ijk}\, \Delta R_{ijk}, \text{ the } \textit{upper sum}$$

$$L(P, f) = \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} m_{ijk}\, \Delta R_{ijk}, \text{ the } \textit{lower sum}$$

where m_{ijk}, M_{ijk} are the lower and the upper bounds of f in ΔR_{ijk}.

It may be easily shown that

$$m(b-a)(d-c)(h-g) \le L(P, f) \le U(P, f) \le M(b-a)(d-c)(h-g)$$

The infimum of the set of upper sums and the supremum of the set of lower sums, for all partitions of R, are respectively known as the *upper* and the *lower integrals* of f over R, and are denoted as

$$\overline{\iiint}_R f\,dx\,dy\,dz \quad \text{and} \quad \underline{\iiint}_R f\,dx\,dy\,dz$$

In case the two integrals are equal, f is said to be integrable—the common value denoted by

$$\iiint_R f\,dx\,dy\,dz \quad \text{or} \quad \int_a^b \int_c^d \int_g^h f\,dx\,dy\,dz$$

is called the *triple integral* of f over R.

7.3 The following *necessary and sufficient* condition for the existence of the triple integral is proved by applying arguments similar to those for the double and single integrals.

Theorem 3. *A bounded function f defined on a parallelepiped R is integrable on R if and if for every $\varepsilon > 0$ there is a partition P of R such that*

$$U(P, f) - L(P, f) < \varepsilon$$

The criterion implies all theorems (including Darboux theorem) similar to those of § 2.6 and 2.7 for double integrals. Three of the theorems may be stated as follows:

1. **Darboux Theorem.** *If f is a bounded function on R then to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that*

$$U(P, f) < I^u + \varepsilon, \quad L(P, f) > I_l - \varepsilon$$

for every partition P of R with norm $\mu(P) < \delta$.

2. *Every continuous function is integrable.*
3. *A bounded function is integrable if its discontinuities are finite in number or if, when infinite in number, they all lie on a finite number of surfaces, which can therefore be enclosed in a finite number of parallelepipeds whose total volume can be made arbitrarily small.*

7.4 Reduction to Iterated Integrals
(Calculation of a triple integral over a parallelepiped)

At iterated integrals is an integral of the form

$$\iint_{R_1} dx\,dy \int_g^h f(x, y, z)\,dz \quad \text{or} \quad \iint_{R_1} \left[\int_g^h f(x, y, z)\,dz\right] dx\,dy$$

where $R_1 = [a, b; c, d]$ is the projection of $R = [a, b; c, d; g, h]$ on the xy-plane.

The proof of the theorem relating to the reduction of a triple integral to iterated integrals and is similar to that for double integrals will therefore be only briefly indicated.

Theorem 4. *If*

(i) *the triple integral* $\iiint_R f(x, y, z)\,dx\,dy\,dz$ *exists over* $R = [a, b; c, d; g, h]$,

(ii) *the integral* $\Psi(x, y) = \int_g^h f(x, y, z)\,dz$ *exists for every fixed point* (x, y) *of* $R_1 = [a, b; c, d]$,

then the integral $\iint_{R_1} \left[\int_g^h f(x, y, z)\,dz\right] dx\,dy$ *also exists, and*

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \iint_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \qquad ...(1)$$

Let I^u and I_l denote the upper and the lower triple integrals of f over R.

Let $\varepsilon > 0$ be an arbitrary number.

There exists a partition P of R such that (with usual notation)

$$\sum_i \sum_j \sum_k M_{ijk}\, \Delta x_i\, \Delta y_j\, \Delta z_k < I^u + \varepsilon$$

For each fixed (x, y) in $R_1 = [a, b; c, d]$ we have

$$\overline{\iint}_{R_1}\left[\overline{\int_g^h} f(x, y, z)\, dz\right] dx\, dy \leq \overline{\iint}_{R_1}\left[\sum_k M_{ijk}\, \Delta z_k\right]$$

$$\leq \sum_i \sum_j \left(\sum_k M_{ijk}\, \Delta z_k\right) \Delta x_i\, \Delta y_j < I^u + \varepsilon$$

But by hypothesis, $\overline{\int_c^d} f(x, y, z)\, dz = \int_c^d f(x, y, z)\, dz$, and since ε is an arbitrary positive number,

$$\therefore \qquad \overline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \leq I^u + \varepsilon \qquad ...(2)$$

$$\leq I^u = \overline{\iiint}_R f(x, y, z)\, dx\, dy\, dz$$

It can be similarly shown that

$$\underline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \geq I_l = \underline{\iiint}_R f(x, y, z)\, dx\, dy\, dz \qquad ...(3)$$

Again, since $I^u = I_l = I$, as the triple integral exists, we get from equations (2) and (3)

$$I \leq \underline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \leq \overline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \leq I$$

$$\Rightarrow \qquad \underline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy = \overline{\iint}_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy = I$$

$$\Rightarrow \qquad \iint_{R_1}\left[\int_g^h f(x, y, z)\, dz\right] dx\, dy \text{ exists and equals } \iiint_R f(x, y, z)\, dx\, dy\, dz$$

Hence the theorem.

Corollary 1. If, further we assume that the double integral can be reduced to iterated integrals (*i.e.*, $\phi(x) = \int_c^d \Psi(x, y)\, dy$ exists for each fixed x in $[a, b]$), then

$$\iint_{R_1} \Psi(x, y)\, dx\, dy = \int_a^b dx \int_c^d \Psi(x, y)\, dy$$

$$= \int_a^b dx \int_c^d \left[\int_g^h f(x, y, z)\, dz\right] dy$$

and so we deduce that

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_c^d dy \int_g^h f(x, y, z)\, dz \qquad \text{...(4)}$$

The formula reduces the evaluation of a triple integral over a parallelepiped R to successive separate integrations with respect to each variable. The integration is performed first with respect to z, then with respect to y and finally with respect to x.

Similarly, if the integrals

$$\Psi_1(y, z) = \int_a^b f(x, y, z)\, dx \quad \text{and} \quad \int_c^d \Psi_1(y, z)\, dy$$

exist, then we derive the analogous formula

$$\iiint_R f(x, y, z)\, dx\, dy\, dz = \int_g^h dz \int_c^d dy \int_a^b f(x, y, z)\, dx \qquad \text{...(5)}$$

Similarly, on condition that the corresponding single and double integrals exist, we can establish analogous formulas reducing the triple integral to iterated integrals with respect to x, y and z in various orders.

Corollary 2. In particular, if the function f is continuous, the triple and all the possible double and single integrals are sure to exist and therefore the triple integral can be evaluated by expressing it as an iterated integral of x, y, z in any order.

However, in the general case of an arbitrary integrable function f, the orders are not always interchangeable.

7.5 Triple Integral over Regions (*Bounded domains*)

Definition 1. Let a bounded function f (of three variables) be defined on a region E of volume V.

Let a partition, consisting of a finite number of surfaces, divide the region E into n sub-regions of elementary volumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_n$.

Clearly the sum of these volumes is $\Sigma\, \Delta V_i = V$.

With, usual notation form the sums

$$U(P, f) = \sum_i M_i\, \Delta V_i, \quad \text{and} \quad L(P, f) = \sum_i m_i\, \Delta V_i$$

respectively called the upper and the lower (Darboux) sums. It can be easily shown that

$$mV \le L(P, f) \le U(P, f) \le MV$$

so that the two sets of sums, the upper and the lower sum (corresponding to all the partitions of E) are bounded.

The infimum of the set of upper sums, and the supremum of the set of lower sums are respectively called the *upper* and the *lower integrals* denoted as

$$I^u = \overline{\iiint_E} f\, dV \quad \text{or} \quad \overline{\iiint_E} f\, dx\, dy\, dz$$

and

$$I_l = \underline{\iiint_E} f\, dV \quad \text{or} \quad \underline{\iiint_E} f\, dx\, dy\, dz$$

When these two integrals are equal, f is said to be integrable and the common value, called the *integral* of f over E, is denoted as

$$I = \iiint_E f\, dV \quad \text{or} \quad \iiint_E f\, dx\, dy\, dz \qquad \text{...(1)}$$

Remark: (*Volume of a solid*). Taking $f = 1$ in (1) we deduce that the volume enclosed by a closed region E (of any shape) is given by

$$V = \iiint_E dx\, dy\, dz \qquad \text{...(2)}$$

Definition 2. (*Integral as a limit of the sums*). If (ξ_i, η_i, ζ_i) is any point of ΔV_i, then the limit

$$\lim_{\mu(P)\to 0} S(P, f) = \lim_{\mu(P)\to 0} \sum_i f(\xi_i, \eta_i, \zeta_i)\, \Delta V_i$$

if it exists, for all partitions P of E and for all positions of the point (ξ_i, η_i, ζ_i) in ΔV_i, is called the triple integral of f over E. Thus

$$\lim_{\mu(P)\to 0} S(P, f) = \iiint_E f\, dV = \iiint_E f\, dx\, dy\, dz \qquad \text{...(3)}$$

7.6 Some Theorems

The basic properties of triple integrals are completely analogous to those of the double integrals and as such all theorems of double integrals mentioned in § 2.3, 2.6, 2.7 of Chapter 17 and those of triple integrals over parallelepipeds (§ 7.3) hold good for triple integrals over any region E.

We may enumerate some of them here:

1. Darboux Theorem.
2. The necessary and sufficient conditions for integrability.

   The statements and proofs are exactly same as for triple integrals over parallelepiped except that R is replaced by E.

   If functions f_1, f_2 and f_3 of x, y, z are integrable over a region E in R^3, then
3. (Linearity)

$$\iiint_E (k_i f_1 + k_2 f_2)\, dx\, dy\, dz = k_1 \iiint_E f_1\, dx\, dy\, dz + k_2 \iiint_E f_2\, dx\, dy\, dz$$

where k_1, k_2 are constants.

4. (Additivity). If E is union of two regions E_1, E_2 with no interior points in common and f is integrable over E_1 and E_2 then f is integrable over E and
$$\iiint_E f\,dx\,dy\,dz = \iiint_{E_1} f\,dx\,dy\,dz + \iiint_{E_2} f\,dx\,dy\,dz.$$
5. (Monotonicity). If $f_1(x, y, z) \geq f_2(x, y, z)$, then
$$\iiint_E f_1\,dx\,dy\,dz \geq \iiint_E f_2\,dx\,dy\,dz.$$
6. When f is integrable over E, so is $|f|$ and
$$\left|\iiint_E f\,dx\,dy\,dz\right| \leq \iiint_E |f|\,dx\,dy\,dz.$$
7. (Mean Value Theorems). If $m \leq f(x, y, z) \leq M$, and V is the volume of E, then
$$mV \leq \iiint_E f\,dx\,dy\,dz \leq MV.$$
If f is continuous, then there exists a point (ξ, η, ζ) of E such that
$$\iiint_E f\,dx\,dy\,dz = V \cdot f(\xi, \eta, \zeta).$$
8. Every continuous function is integrable.
9. A bounded function is integrable if its discontinuities are finite in number or if, when infinite in number, they all lie on a finite number of surfaces, which can therefore be enclosed in a finite number of volumes whose total volume can be made arbitrarily small.

7.7 Volume of Solids by Triple Integrals

Volume V of a solid of any shape in R^3, (§ 7.5) is given by
$$V = \iiint_E dx\,dy\,dz \qquad ...(1)$$
It may be noted that whereas the formulas of § 6 are helpful to find the volume of *cylindrical* solids only, the present formula enables us to find the volume of a solid of *any shape* in R^3.

7.8 Regular (or Quadratic) Domain

Definition. *A three-dimensional domain is called regular (or quadratic) with respect to z-axis if every straight line parallel to the z-axis and passing through a point (not lying on the boundary) of the domain, cuts its surface in not more than two points.*

A domain E bounded above and below by the surfaces
$$z = \phi(x, y), \quad z = \Psi(x, y), \quad (\phi(x, y) \geq \Psi(x, y))$$
and on the sides by a lateral cylindrical surface, is *regular with respect to z-axis.*

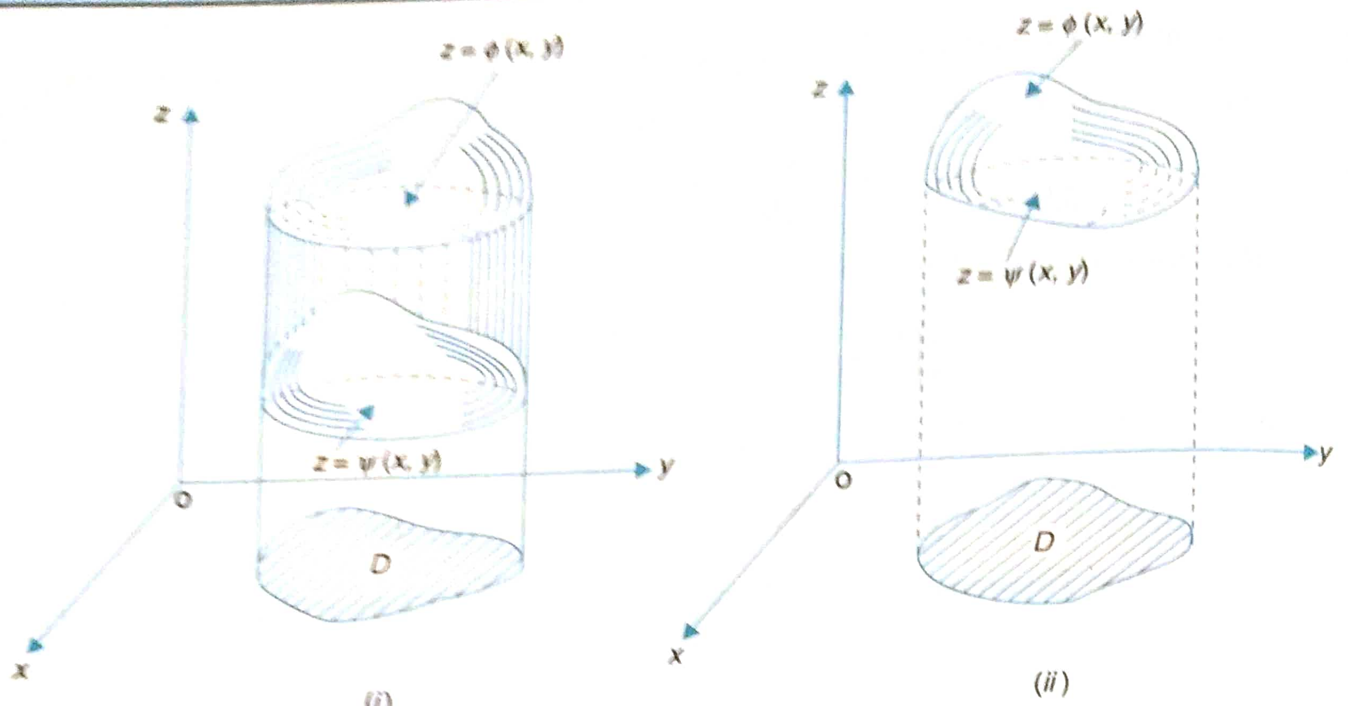

Fig. 12

Such domains are projectible on the xy-plane into a regular (two dimensional) domain D. Further any part of such a domain cut off by a plane parallel to any of the coordinate planes is also regular with respect to z-axis.

Domains regular with respect to the axis of x or y are similarly defined.

A domain which is regular with respect to all the axes is called a *regular domain*.

A domain which can be divided into a finite number of domains each of which is regular with respect to an axis is called *piecewise regular* with respect to that axis.

It may be noted that the definition also includes the cases where there can be no lateral surface. For instance, a three dimensional sphere (ball) is considered to be a domain regular with respect to z-axis (all the axes) which is bounded above and below by the upper and the lower hemispheres and whose lateral surface degenerates into the equator.

7.9 Reduction to Iterated Integrals (Calculation of a triple integral over any region in R^3)

Theorem 5. *If a triple integral $\iiint_E f\, dx\, dy\, dz$ exists for a function f defined on a closed (regular) domain E bounded above and below by the surfaces*

$$z = \phi(x, y), \quad z = \Psi(x, y), \quad (\Psi(x, y) \geq \phi(x, y))$$

and on the sides by a cylindrical surface, and if the integral $\int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz$ exists for each fixed point (x, y) belonging to D (the projection of E on xy-plane), then the iterated integral $\iint_D \left[\int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz\right] dx\, dy$ also exists and

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iint_D \left[\int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz\right] dx\, dy$$

Let a parallelepiped $R = [a, b; c, d; g, h]$ encloses the bounded domain E. Then obviously the projection R_1 of R on xy-plane encloses D.

Let us define a function F over R such that

$$F(x, y, z) = \begin{cases} f(x, y, z), & \text{at points of } E \\ 0 & \text{outside } E \end{cases}$$

Then

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iiint_R F(x, y, z)\, dx\, dy\, dz$$

$$\iint_{R_1} \left[\int_g^h F(x, y, z)\, dz\right] dx\, dy \qquad \text{...(1)}$$

The function $F(x, y, z)$ vanishes outside E, therefore we have

$$\int_g^h F(x, y, z)\, dz = \int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz \qquad \text{...(2)}$$

The expression (2) is a function of x and y which is equal to zero outside the domain D. Therefore, the double integral of the expression taken over R_1 coincides with its double integral over D. Hence (1) and (2) give

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iint_D \left[\int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz\right] dx\, dy \qquad \text{...(3)}$$

Hence the theorem.

Remarks:

1. If further, the conditions for the reduction of the double integral to the iterated integrals are satisfied, *i.e.*, if

$$\iint_D I(x, y)\, dx\, dy = \int_a^b dx \int_{\phi_1(x)}^{\Psi_1(x)} I(x, y)\, dy$$

where
$$I(x, y) = \int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz, \text{ then}$$

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{\phi_1(x)}^{\Psi_1(x)} dy \int_{\phi(x,y)}^{\Psi(x,y)} f(x, y, z)\, dz \qquad \text{..(4)}$$

It is this final formula that reduces the triple integral to an iterated integral.
The variables x, y, z can be interchanged if the corresponding conditions hold.

2. When deriving formula (3) we have taken the domain to be regular with respect to z-axis. If the domain is of a more complicated form, we break up the domain into parts such that each of them is regular with respect to some axis.

A Rule for Limits of Integration. We follow the following steps to reduce a triple integral to an iterated one.

1. Break the domain, if necessary, into sub-domains such that a line parallel to z-axis has at the most two common points with its boundary. In what follows we mention only one such sub-domain.
2. Fix arbitrary x and y and let a line parallel to z-axis cut the boundary of the given domain E at two points with z coordinates $\phi(x, y)$ and $\Psi(x, y)$. The expressions $\phi(x, y)$ and $\Psi(x, y)$ should be taken as the limits of integration with respect to z.
3. The domain of definition of the function of x, y (obtained after integration with respect to z) is now D, the projection of the given domain E on the xy-plane. Let a line parallel to the y-axis cut the bounding curve of D at two points with y coordinates $\phi_1(x)$ and $\Psi_1(x)$. These expressions, $\phi_1(x)$ and $\Psi_1(x)$ form the limits of integration with respect to y.
4. The limits of integral with respect to x can be easily determined.

For example, the triple integral of a function f taken over the sphere $x^2 + y^2 + z^2 = a^2$ is of the form

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \int_{-a}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{\sqrt{(a^2-x^2-y^2)}} f(x, y, z)\, dz$$

7.10 Change of Variables in Triple Integrals

We have already dealt with the method of changing the variables for the double integrals (§ 5 Ch. 17). The theory underlying the change of variables in triple integrals is exactly the same except that it is more laborious. Here we shall simply indicate the method to be adopted in practical problems.

Let the functions

$$x = X(u, v, w), \quad y = Y(u, v, w), \quad z = Z(u, v, w)$$

map in one-to-one manner, a domain E in cartesian coordinates x, y, z onto a domain E' in the new coordinates u, v, w.

Let $f(x, y, z) = f(X(u, v, w), Y(u, v, w), Z(u, v, w)) = F(u, v, w)$

Then

$$\iiint_E f(x, y, z)\, dx\, dy\, dz = \iiint_{E'} F(u, v, w)\, |J|\, du\, dv\, dw$$

where the Jacobian $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

The following two transformations, because of their frequent occurrence, deserve special mention.

(*i*) **Cylindrical polar coordinates.**

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Here, the Jacobian $J = r$.

(ii) Spherical polar coordinates.

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$

Jacobian $J = r^2 \sin\theta$.

8. GAUSS'S THEOREM (or *Divergence theorem*) (*Second generalisation of Green's theorem*)

If a three-dimensional regular (or piecewise regular) domain E is bounded by a smooth (or piecewise smooth) oriented surface S, and f, g, h are three functions which along with their partial derivatives f_x, g_y, h_z are continuous at each point of E and S, then

$$\iiint_E (f_x + g_y + h_z)\, dx\, dy\, dz = \iint_S (f\, dy\, dz + g\, dz\, dx + h\, dx\, dy)$$

where the surface integral is taken over the exterior of S.

Let us first consider the domain E (Figs. (*i*) and (*ii*) of § 7.8) to be regular with respect to z-axis, and bounded, above and below, by smooth oriented surfaces S_1, S_2 determined by the equations

$$Z = \phi(x, y) \quad \text{and} \quad Z = \Psi(x, y), \quad (\Psi(x, y) \geq \phi(x, y)) \qquad \ldots(1)$$

and by a lateral cylindrical surface S_3 (which may reduce to the common curve of S_1 and S_2 as in Fig. (*ii*)), with generators parallel to the z-axis. The union of S_1, S_2, S_3 forms the surface S. Let D be the projection of S (or the common projection of S_1 and S_2) on the xy-plane.

As we are considering the outer side of the surface S, the outward drawn normals of S_1 and S_2 are in the opposite directions and so S_1 and S_2 are of opposite orientation. Accordingly, if D is the region on the xy-plane on which S_1 projects, then $(-D)$ is the region on which S_2 projects.

Thus, we have

$$\iiint_E h_z\, dx\, dy\, dz = \iint_D \left[\int_{\phi(x,y)}^{\Psi(x,y)} h_z dz \right] dx\, dy$$

$$= \iint_D [h(x, y, \Psi) - h(x, y, \phi)]\, dx\, dy$$

$$= \iint_D h(x, y, \Psi)\, dx\, dy - \iint_D h(x, y, \phi)\, dx\, dy$$

$$= \iint_D h(x, y, \Psi)\, dx\, dy + \iint_{-D} h(x, y, \phi)\, dx\, dy$$

$$= \iint_{S_1} h(x, y, z)\, dx\, dy + \iint_{S_2} h(x, y, z)\, dx\, dy$$

$$= \iint_{S_1} h\, dx\, dy + \iint_{S_2} h\, dx\, dy + \iint_{S_3} h\, dx\, dy$$

where the last surface integral (which is obviously equal to zero) is taken over the outer side of the lateral surface S_3. Thus

$$\iiint_E h_z\, dx\, dy\, dz = \iint_S h\, dx\, dy \qquad ...(2)$$

We claim that the relation (2) is also valid for any domain which is piecewise regular with respect to z-axis. For, if the domain E is divided, by surfaces into sub-domains each of which is regular with respect to z-axis, relation (2) holds for each sub-domain. When all these relations are added, we get on the left hand side the volume integral over E, and on the right hand side, the surface integral over S (the surface integrals over the dividing surface being taken twice, once over each side, cancel each other).

Since E is regular with respect to all the coordinate axes, considering the domain regular with respect to x-axis and y-axis in turn, we deduce that

$$\iiint_E f_x\, dx\, dy\, dz = \iint_S f\, dy\, dz \qquad ...(3)$$

and

$$\iiint_E g_y\, dx\, dy\, dz = \iint_S g\, dz\, dx \qquad ...(4)$$

Adding equations (2), (3) and (4), we get

$$\iiint_E (f_x + g_y + h_z)\, dx\, dy\, dz = \iint_S (f\, dy\, dz + g\, dz\, dx + h\, dx\, dy) \qquad ...(5)$$

which holds for any regular or piecewise regular domain with a smooth boundary S.

Relation (5) may be expressed in the form

$$\iiint_E (f_x + g_y + h_z)\, dx\, dy\, dz = \iint_S (f \cos\alpha + g \cos\beta + h \cos\gamma)\, dS \qquad ...(6)$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outward drawn unit normal of S.

Note: (*Vectorial formulation*). Let $\mathbf{F} = \mathbf{i}P + \mathbf{j}Q + \mathbf{k}R$ be a vector function defined on E and S, then

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Let $\mathbf{n} = \mathbf{i} \cos\alpha + \mathbf{j} \cos\beta + \mathbf{k} \cos\gamma$

be an outward drawn unit normal of the surface S, then

$$\mathbf{F} \cdot \mathbf{n} = P \cos\alpha + Q \cos\beta + R \cos\gamma$$

Hence, Gauss's theorem may be expressed as

$$\iiint_E \text{div } \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad ...(7)$$

where dV denotes an element of volume.

Relation (7) shows that Gauss's theorem may be stated as:

"The integral of the divergence of a vector function **F** *over some volume is equal to the vector flux throuh the surface bounding the given volume."*

8.1 Applications of Gauss's Theorem

I. Evaluation of surface integrals. We know that a surface integral can generally be evaluated by reducing it to the corresponding double integral. But there are cases where this is inconvenient. In such cases, it is generally advisable to reduce a surface integral over a closed surface to a triple integral by means of Gauss's Theorem.

ILLUSTRATIONS

1. Evaluate the surface integral

$$I = \iint_S (x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy)$$

over the sphere $x^2 + y^2 + z^2 = a^2$.

By Gauss's Theorem, we have

$$I = 3 \iiint_E (x^2 + y^2 + z^2)\, dx\, dy\, dz$$

where the domain E is the sphere $x^2 + y^2 + z^2 \leq a^2$.

Changing to spherical polars, we get

$$I = 3 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{a} r^4 \sin\theta\, dr = \frac{12}{5} \pi a^5$$

2. Evaluate the surface integral

$$I = \iint_S z\, dy\, dz + x\, dz\, dx + y\, dx\, dy$$

over a closed surface S.

By Gauss's Theorem, the integral reduces to the triple integral (over the domain bounded by the surface S) whose integrand is identically equal to zero. Hence $I = 0$ for any closed surface S.

II. Volume of a solid by a surface integral. Just as Green's Theorem enables us to express the area of a plane figure as a line integral along its boundary, the Gauss' Theorem helps us to find an expression of the volume of a solid in the form of a surface integral over the closed surface bounding the solid.

Let us choose three functions f, g, h so that

$$f_x + g_y + h_z = 1$$

Then we obtain,

$$\iint_S f\ y\ z + g\ dz\ dy\ dx + h\ dx\ dy = \iiint_E dx\ dy\ dz = V$$

where V is the volume of the domain E bounded by S, and the surface integral is taken over the outer side of the surface S.

In particular, putting

$$f(x, y, z) = x,\ g = 0,\ h = 0$$

we get

$$V = \iint_S x\, dy\, dz \qquad \ldots(8)$$

Similarly

$$V = \iint_S y\,dz\,dx, \quad V = \iint_S z\,dx\,dy \qquad ...(9)$$

Adding these results, or putting

$$f(x, y, z) = \frac{1}{3}x, \quad g(x, y, z) = \frac{1}{3}y, \quad h(x, y, z) = \frac{1}{3}z$$

we get

$$V = \frac{1}{3}\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) \qquad ...(10)$$

the integration being taken over the outer side of S in each case.

ILLUSTRATION

The volume V of the sphere $x^2 + y^2 + z^2 = a^2$ is given by

$$V = \frac{1}{3}\iint_S (x\,dy\,dz - y\,dz\,dx + z\,dx\,dy)$$

$$= 2\iint_D \sqrt{a^2 - x^2 - y^2}\,dx\,dy$$

where D is the circle $x^2 + y^2 \le a^2$

$$= 2\iint_0^{2\pi} d\theta \int_0^a (\sqrt{a^2 - r^2})\,r\,dr = \frac{4}{3}\pi a^2$$

Example 22. Compute the integral

$$\iiint_E xyz\,dx\,dy\,dz$$

over a domain bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

■ The domain is regular and bounded, above and below by $z = 1 - x - y$ and $z = 0$. Its projection D on the xy-plane is a triangle bounded by $x = 0$, $y = 0$, $y = 1 - x$.

$$\therefore \quad \iiint_E xyz\,dx\,dy\,dz = \iint_D \left[\int_0^{1-x-y} xyz\,dz\right] dx\,dy$$

$$= \int_0^1 dx \int_0^{1-x} \frac{1}{2}xy\,(1 - x - y)^2\,dy$$

$$= \int_0^1 \frac{x}{24}(1 - x)^4\,dx = \frac{1}{720}$$

Example 23. Evaluate

$$I = \iint_S (x\cos\alpha + y\cos\beta + z^2\cos\gamma)\,dS,$$

where S denotes the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$; and $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of the outward drawn normal of S.

- By Gauss's Theorem

$$I = \iiint_E (2 + 2z)\, dx\, dy\, dz$$

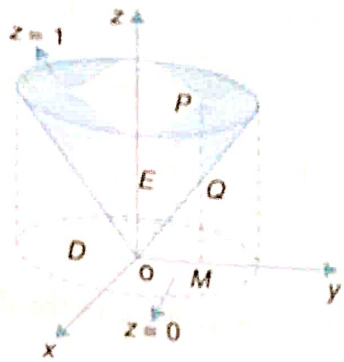

Fig. 13

where E is the domain bounded by $x^2 + y^2 = z^2$ and $z = 1$

$$= \iint_D dx\, dy \int_{\sqrt{(x^2+y^2)}}^{1} (2 + 2z)\, dz$$

$$= \iint_D [3 - 2\sqrt{x^2 + y^2} - (x^2 + y^2)]\, dx\, dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 (3 - 2r - r^2)\, r\, dr = \frac{7\pi}{6}$$

Note: The given surface integral is same as

$$\iint_S (x\, dy\, dz + y\, dz\, dx + z^2\, dx\, dy)$$

Example 24. Evaluate the surface integral

$$I = \iint_S (y^2 z\, dx\, dy + xz\, dy\, dz + x^2 y\, dz\, dx),$$

where S is the outer side of the surface situated in the first octant and formed by the paraboloid of revolution $z = x^2 + y^2$, cylinder $x^2 + y^2 = 1$ and the coordinate planes.

- Using Gauss's Theorem

$$I = \iiint_V (x^2 + y^2 + z)\, dx\, dy\, dz$$

where V is the volume enclosed by S.

$$= \iint_D \left[\int_0^{x^2+y^2} (x^2 + y^2 + z)\, dz\right] dx\, dy$$

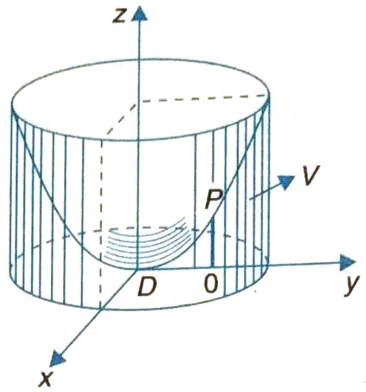

Fig. 14

where $D(x^2 + y^2 \le 1)$ is the projection on the xy-plane of the domain enclosed by S.

$$= \frac{3}{2} \iint_D (x^2 + y^2)^2 \, dx \, dy$$

$$= \frac{3}{2} \int_0^{\pi/2} d\theta \int_0^1 r^4 r \, dr = \frac{\pi}{8}$$

Example 25. Compute the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

■ The ellipsoid is bounded above and below by the surfaces

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \text{ and } z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

and its projection on the xy-plane is the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\therefore \qquad \text{Volume} = \iiint_E dx \, dy \, dz$$

$$= \iint_D dx \, dy \int_{-c\sqrt{(1-x^2/a^2-y^2/b^2)}}^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dz$$

$$= 2c \iint_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dx \, dy$$

Putting $\qquad x/a = r\cos\theta, \; y/b = r\sin\theta,$

$$J = abr$$

$$\therefore \qquad \text{Volume} = 2\,abc \int_0^{2\pi} d\theta \int_0^1 (\sqrt{1 - x^2}) \, r \, dr$$

$$= 2abc \cdot 2\pi \left[-\frac{1}{3}(1 - r^2)^{3/2} \right]_0^1 = \frac{4}{3}\pi abc$$

Example 26. Compute the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the surface of the paraboloid $x^2 + y^2 = 3z$.

▪ The two surfaces intersect at $z = 1$.

The domain E, under consideration, is bounded, above and by the two surfaces $z = \sqrt{4 - x^2 - y^2}$, and $z = \frac{1}{3}(x^2 + y^2)$, and its projection D on the xy-plane is the circle.

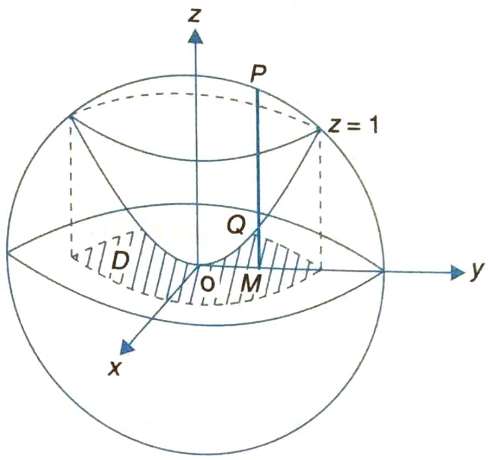

Fig. 15

$$x^2 + y^2 \leq 3$$

$$\therefore \quad \text{Volume} = \iiint_E dx\, dy\, dz$$

$$= \iint_D dx\, dy \int_{\frac{1}{3}\sqrt{(x^2+y^2)}}^{\sqrt{(4-x^2-y^2)}} dz$$

$$= \iint_D \left[\sqrt{4 - x^2 - y^2} - \frac{1}{3}(x^2 + y^2)\right] dx\, dy$$

Changing to polars, (D being the circle, $x^2 + y^2 \leq 3$),

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{4 - r^2} - \frac{r^2}{3}\right) r\, dr = \frac{19}{6}\pi$$

EXERCISE

1. Compute the integrals

(i) $\int_0^a dx \int_0^x dy \int_0^y xyz\, dz$ (ii) $\int_0^a dx \int_0^x dy \int_0^{xy} x^3 y^3 z\, dz$

2. Compute $\iiint_E xy\, dx\, dy\, dz$, when E is the domain bounded by $z = xy$, $x + y = 1$, $z = 0$ $(z \geq 0)$.

3. Evaluate $\iiint_E y \cos(z + x)\, dx\, dy\, dz$, E being the domain bounded by the cylinder $y = \sqrt{x}$ and the planes, $y = 0$, $z = 0$, $x + z = \pi/2$.

4. Compute the integral $\iiint \frac{dx\, dy\, dz}{(x + y + z + 1)^3}$, over the volume bounded by the planes, $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

5. Find the volume of a circular cone with height h and radius a.

6. Evaluate the volume of enclosed region:
   (i) Cone $x^2 + y^2 = z^2$ and the plane $z = 1$,
   (ii) Parabolic cylinder $z = 4 - x^2$ and the planes, $x = 0$, $y = 0$, $z = 0$, $y = 6$.

7. Calculate the volume of the solid bounded by a surface with the equation $(x^2 + y^2 + z^2)^2 = a^3 x$.

8. Find the volume of the solid bounded by
   (i) The paraboloid $z = x^2 + y^2$, cylinder $y = x^2$ and the planes $y = 1$, $z = 0$.
   (ii) The cylinders $z = 4 - y^2$, $y = \frac{1}{2}x^2$, and the plane $z = 0$.
   (iii) The cylinders $x^2 + y^2 = a^2$, $z = x^3/b^2$, and the plane $z = 0$, $(x \geq 0)$.
   (iv) The cylinder $x^2 + y^2 = 2ax$ and the paraboloid $y^2 + z^2 = 4ax$.

Use Gauss's Theorem to evaluate the integrals.

9. $\iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$, S being the outer side of the cube $[0, a; 0, a; 0, a]$.

10. $\iint_S [(x^3 - yz) \cos\alpha - 2x^2 y \cos\beta + 2 \cos\gamma]\, dS$; taken over the outer surface of the cube bounded by the planes $x = 0$, $x = a$; $y = 0$, $y = a$; $z = 0$, $z = a$. $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outward drawn normal.

11. $\iint_S (ax \cos\alpha + by \cos\beta + cz \cos\gamma)\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$, and α, β, γ have this usual meaning.

12. $\iint_S (y^2 z^2\, dy\, dz + z^2 x^2\, dz\, dx + x^2 y^2\, dx\, dy)$, S being the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane.

13. $\iint_S (x^2\, dy\, dz + y^2\, dz\, dx + z^2\, dx\, dy) = 0$, taken over the surface of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

14. $\iint_S (x^3 \cos\alpha + y^3 \cos\beta + z^3 \cos\gamma)\, dS$, where S is the surface of a sphere of radius a with centre at the origin and α, β, γ are the angles of inclination to the coordinate axes of the normal to the surface.

15. $\iint (xz\, dx\, dy + xy\, dy\, dz + yz\, dz\, dx)$, taken over the pyramid formed by the planes, $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

16. $\iint_S (xz\, dy\, dz + xy\, dz\, dx + yz\, dx\, dy)$, where S is the outer surface situated in the first octant and formed by the cylinder $x^2 + y^2 = a^2$ and the planes, $x = 0$, $y = 0$, $z = 0$, $z = h$.

ANSWERS

1. (i) $a^6/48$ (ii) $a^{11}/110$
2. $1/180$
3. $(\pi^2 - 8)/16$
4. $\frac{1}{2}\log 2 - 5/16$
5. $\pi a^2 h/3$
6. $\pi/3$
7. $\pi a^3/3$
8. (i) $88/105$ (ii) $256/21$ (iii) $4a^5/15b^2$ (iv) $2a^3(3\pi + 8)/3$
9. $3a^3$
10. $\frac{1}{3}a^5$
11. $4\pi(a + b + c)/3$
12. $\pi/24$
14. $12\pi a^5/5$
15. $1/8$
16. $a^2h(2a/3 + \pi h/8)$.

Example 27. Compute

$$I = \iiint \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dx\, dy\, dz$$

taken over the region $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

▪ Let us change to spherical polar coordinates, where

$$x/a = r \sin\theta \cos\phi,\ y/b = r \sin\theta \sin\phi,\ z/c = r\cos\theta$$
$$0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$$

so that

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = abcr^2 \sin\theta$$

$$\therefore \qquad I = abc \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^1 (\sqrt{1 - r^2})r^2\, dr = \frac{\pi^2 abc}{4}.$$

Example 28. Show that

$$I = \iiint x^{l-1} y^{m-1} z^{n-1} (1 - x - y - z)^{p-1}\, dx\, dy\, dz,\quad (l, m, n, p \ge 1)$$

taken over the tetrahedron bounded by the planes. $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ is $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l + m + n + p)}$.

▪ The given integral is same as

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1} (1 - x - y - z)^{p-1}\, dz.$$

First method. Let us put $x + y + z = u$, $x + y = uv$, $x = uvw$, *i.e.*, $x = uvw$, $y = uv(1 - w)$, $z = u(1 - v)$

It may be seen that when x, y, z are positive and $x + y + z \le 1$, then each of u, v, w lie between 0 and 1, and conversely. So the given region is fully described when $0 \le u \le 1, 0 \le v \le 1, 0 \le w \le 1$.

Also, then

$$|J| = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| = |-u^2 v| = u^2 v$$

$$\therefore \quad I = \int_0^1 u^{l+m+n-1}(1-u)^{p-1}\, du \int_0^1 v^{l+m-1}(1-v)^{n-1}\, dv$$

$$\int_0^1 x^{l-1}(1-w)^{m-1}\, dw$$

$$= \beta(l+m+n, p)\cdot\beta(l+m, n)\cdot\beta(l, m) = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)}.$$

Second method. Put $x = u$, $y = (1 - u)v$, $z = (1 - u)(1 - v)w$ so that

$1 - x - y - z = (1 - u)(1 - v)(1 - w)$ and

$$|J| = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| = (1-u)^2(1-v)$$

$$\therefore \quad I = \int_0^1 u^{l-1}(1-u)^{m+n+p-1}\, du \int_0^1 v^{m-1}(1-v)^{n+p-1}\, dv$$

$$\int_0^1 w^{n-1}(1-w)^{p-1}\, dw$$

$$= \beta(l, m+n+p)\cdot\beta(m, n+p)\cdot\beta(n, p) = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)}$$

Note: The usefulness of the two sets of substitution lies in the fact that the new integrals are with constant limits (ref. solved example 26, Ch.17)

Example 29. Evaluate

$$\iiint_E z^2\, dx\, dy\, dz$$

taken over the region common to the surfaces

$$x^2 + y^2 + z^2 = a^2, \text{ and } x^2 + y^2 = ax$$

■ The region is bounded, above and below by the surfaces

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = -\sqrt{a^2 - x^2 - y^2}$$

and its projection on the *xy*-plane is the circular domain $D \equiv x^2 + y^2 \le ax$.

$$\therefore \quad \iiint_E z^2\, dx\, dy\, dz = \iint_D dx\, dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{\sqrt{(a^2-x^2-y^2)}} z^2\, dz$$

$$= \frac{2}{3} \iint_D (a^2 - x^2 - y^2)^{3/2} \, dx \, dy$$

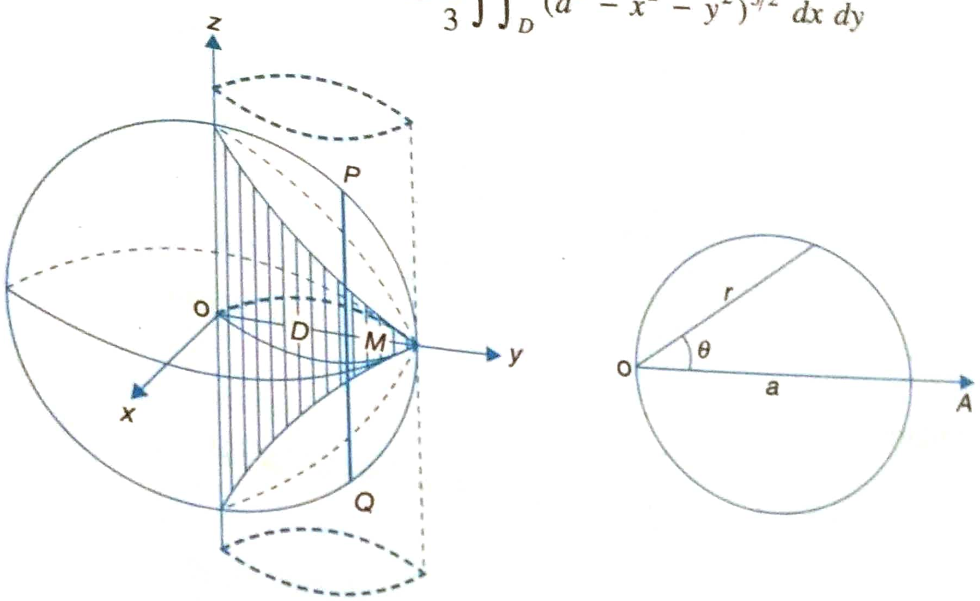

Fig. 16

Changing to polars, the region D becomes the circle,

$$r = a \cos \theta$$

where, $$0 \le \theta \le \pi, 0 \le r \le a \cos \theta$$

$$= \frac{2}{3} \int_0^{\pi} d\theta \int_0^{a \cos \theta} (a^2 - r^2)^{3/2} r \, dr$$

$$= \frac{2}{15} a^5 \int_0^{\pi/2} (1 - \sin^5 \theta) \, d\theta = \frac{2a^5(15\pi - 16)}{225}.$$

Example 30. Evaluate

$$I = \iiint_E (y^2z^2 + z^2x^2 + x^2y^2) \, dx \, dy \, dz$$

taken over the domain bounded by the cylinder $x^2 + y^2 = 2ax$, and the cone $z^2 = k^2(x^2 + y^2)$.

- The domain E is bounded above and below by the surface

$$z = k\sqrt{x^2 + y^2}$$

and $$z = -k\sqrt{x^2 + y^2}$$

and its projection on the xy-plane is the circular domain D, $x^2 + y^2 \le 2ax$.

$$\therefore \quad I = \iint_D dx \, dy \times \int_{-k\sqrt{(x^2+y^2)}}^{k\sqrt{(x^2+y^2)}} \{(x^2 + y^2)z^2 + x^2y^2\} \, dz$$

$$= 2 \iint_D [\tfrac{1}{3}(x^2 + y^2)^2 k^2 + x^2y^2] k\sqrt{x^2 + y^2} \, dx \, dy.$$

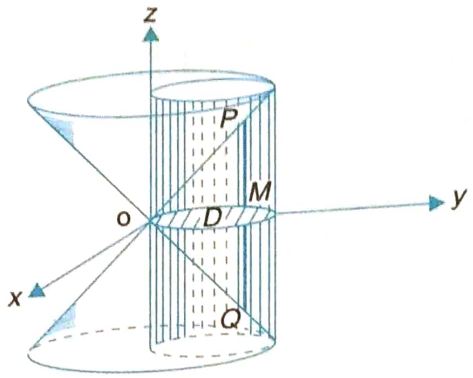

Fig. 17

Changing to polars,

$$= \frac{4k}{3} \int_0^{\pi/2} d\theta \int_0^{2a\cos\theta} (k^2 + 3\cos^2\theta \sin^2\theta)\, r^6\, dr$$

$$= \frac{512}{21} ka^7 \int_0^{\pi/2} (k^2 \cos^7\theta + 3\cos^9\theta \sin^2\theta)\, d\theta$$

$$= \frac{8192}{735} ka^7 \left(k^2 + \frac{8}{33}\right)$$

Example 31. Integrate $1/xyz$ throughout the volume enclosed by the six spheres, $x^2 + y^2 + z^2 = ax$, $a'x$; by, $b'y$; cz, $c'z$; a, a'; b, b'; c, c' being all positive.

■ Let

$$\frac{(x^2 + y^2 + z^2)}{x} = u, \frac{(x^2 + y^2 + z^2)}{y} = v, \frac{(x^2 + y^2 + z^2)}{z} = w$$

so that

$$x = \frac{1}{u\sum u^{-2}}, y = \frac{1}{v\sum u^{-2}}, z = \frac{1}{w\sum u^{-2}}.$$

If we take u, v, w as the new coordinate system, the new domain of integration is the parallelepiped, $[a, a'; b, b'; c, c']$.

The Jacobian,

$$|J| = \begin{vmatrix} \frac{2u^{-2} - \sum u^{-2}}{u^2(\sum u^{-2})^2} & \frac{2v^{-2}}{uv(\sum u^{-2})^2} & \frac{2w^{-2}}{uw(\sum u^{-2})^2} \\ \frac{2u^{-2}}{uv(\sum u^{-2})^2} & \frac{2v^{-2} - \sum u^{-2}}{v^2(\sum u^{-2})^2} & \frac{2w^{-2}}{vw(\sum u^{-2})^2} \\ \frac{2u^{-2}}{uw(\sum u^{-2})^2} & \frac{2v^{-2}}{vw(\sum u^{-2})^2} & \frac{2w^{-2} - \sum u^{-2}}{w^2(\sum u^{-2})^2} \end{vmatrix}$$

$$= \frac{1}{u^2 v^2 w^2 (\Sigma u^{-2})^3}$$

$\therefore$
$$\iiint_E \frac{1}{xyz}\, dx\, dy\, dz = \int_a^{a'} \frac{1}{u}\, du \int_b^{b'} \frac{1}{v}\, dv \int_c^{c'} \frac{1}{w}\, dw$$

$$= \log \frac{a'}{a} \log \frac{b'}{b} \log \frac{c'}{c}.$$

Example 32. Evaluate $\iint \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the entire surface $x^2 + y^2 = 1$, $z = 0$, $z = x + 2$, and $\mathbf{n}$ is the outward drawn unit normal, and

$$\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}.$$

- By Gauss's Divergence Theorem

(This is solved as surface integral in Ex. 14).

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \text{div } \mathbf{F}\, dv$$

$$= \iiint \left(\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-3y) + \frac{\partial}{\partial z}(z) \right) dx\, dy\, dz = 0.$$

This shows that not only Gauss' Theorem is very useful in changing volume integral to surface integral but also simplifies the problem by changing surface integral to volume integral.

Example 33. (Same as example 11). Evaluate the surface integral

$$\iint z \cos \gamma\, dS,$$

over the outer side of the sphere $x^2 + y^2 + z^2 = 1$, where γ is the inclination of the normal at any point of the sphere with z-axis.

- Since $\cos \gamma = \mathbf{k} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward drawn unit normal to the surface.

$\therefore$ By Gauss' Divergence theorem, we have

$$\iint_S z \cos \gamma\, dS = \iint_S z\, \mathbf{k} \cdot \mathbf{n}\, dS = \iiint_V \text{div } (z\, \mathbf{k})\, dx\, dy\, dz$$

$$= \iiint_V 1 \cdot dx\, dy\, dz$$

$$= \frac{4}{3} \pi (1)^3 \text{ (volume of the sphere with unit radius)}$$

$$= 4\pi/3.$$

Example 34. Evaluate $\int (x^2 + y^2)\, ds$ and $\int (x^2 + y^2)\, dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

- Taking, $\mathbf{n} = \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right)$, and $\mathbf{F} = (ax, ay, 0)$, we have

$$\int (x^2 + y^2)\, dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

$$= \iiint_V \operatorname{div} \mathbf{F}\, dV, \text{ by Gauss' theorem}$$

$$= \iiint_V (a + a)\, dx\, dy\, dz = 2a\, \frac{4}{3} \pi a^3 = \frac{8}{3} \pi a^4,$$

Now

$$\int_S (x^2 + y^2)\, dS = \int_S (x^2 + y^2)\, \mathbf{n}\, ds$$

$$= \iint_S (x^2 + y^2)\, (\mathbf{i}\, dy\, dz + \mathbf{j}\, dz\, dx + \mathbf{k}\, dx\, dy)$$

Taking the parametric co-ordinates,

$$x = a \sin\theta \cos\phi,\ y = a \sin\theta \sin\phi,\ z = a\cos\theta,$$

where $0 \le \theta \le \pi, 0 \le \phi \le 2\pi$, on the sphere $x^2 + y^2 + z^2 = a^2$.

$$\therefore \quad \int_S (x^2 + y^2)\, dS = \int_0^{\pi} \int_0^{2\pi} a^2 \sin^2\theta \left[\mathbf{i}\, \frac{\partial(y, z)}{\partial(\theta, \phi)} + \mathbf{j}\, \frac{\partial(z, x)}{\partial(\theta, \phi)} + \mathbf{k}\, \frac{\partial(x, y)}{\partial(\theta, \phi)} \right] d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} a^2 \sin^2\theta\, (\mathbf{i}\, a^2 \sin^2\theta \cos\phi + \mathbf{j}\, a^2 \sin^2\theta \sin\phi \quad + \mathbf{k}\, a^2 \sin\theta \cos\theta)\, d\theta\, d\phi$$

$$= 0.$$

EXERCISE

1. Evaluate the integrals by passing over to cylindrical or spherical polar coordinates.

(i) $\displaystyle\int_0^1 dx \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} dy \int_0^a dz$

(ii) $\displaystyle\int_0^2 dx \int_0^{\sqrt{(2x-x^2)}} dy \int_0^a z\sqrt{x^2 + y^2}\, dz$

(iii) $\displaystyle\int_{-a}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dy \int_0^{\sqrt{(a^2-x^2-y^2)}} (x^2 + y^2)\, dz$

(iv) $\displaystyle\int_0^1 dx \int_0^{\sqrt{(1-x^2)}} dy \int_0^{\sqrt{(1-x^2-y^2)}} \sqrt{x^2 + y^2 + z^2}\, dz$

2. Compute $\iiint_E (x^2 + y^2)\, dx\, dy\, dz$, where E is specified by $z \ge 0, a^2 \le x^2 + y^2 + z^2 \le b^2$.

3. Compute $\displaystyle\iiint_E \frac{dx\, dy\, dz}{\sqrt{x^2 + y^2 + (z-2)^2}}$, where E is the sphere $x^2 + y^2 + z^2 \le 1$.

4. Show that

$$\iiint_E (ax + by + cz)^2\, dx\, dy\, dz = \frac{4}{15}\pi\,(a^2 + b^2 + c^2)$$

where domain E is the sphere $x^2 + y^2 + z^2 \le 1$.

5. Show that

$$\iiint (lx^2 + my^2 + nz^2)\, dx\, dy\, dz$$

taken throughout the sphere $x^2 + y^2 + z^2 = a^2$, is $4\pi(l + m + n)a^5/15$.

6. Prove that

$$\iiint_E z\, dx\, dy\, dz = \frac{\pi}{4} h^4 \cot\alpha \cot\beta,$$

where E is the domain bounded by the cone $z^2 = x^2 \tan^2\alpha + y^2 \tan^2\beta$ and the planes $z = 0$, $z = h$.

Further, if E_1 is the part of the domain E for which x, y and z are positive, show that

$$\iiint_{E_1} xyz\, dx\, dy\, dz = \frac{1}{48} h^6 \cot^2\alpha \cot^2\beta.$$

7. Show that

$$\iiint \frac{dx\, dy\, dz}{\sqrt{1 - x^2 - y^2 - z^2}} = \frac{\pi^2}{8}$$

integral being extended to all the positive values of the variables for which the expression is real.

8. Show that

$$\iiint e^{\sqrt{(x^2/a^2 + y^2/b^2 + z^2/c^2)}}\, dx\, dy\, dz = 4\pi abc(e - 2)$$

integral being taken over the region $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

9. Prove that

$$\iiint_E \sqrt{\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}}\, dx\, dy\, dz = \frac{\pi}{8}\left[\beta\left(\frac{3}{4}, \frac{1}{2}\right) - \beta\left(\frac{5}{4}, \frac{1}{2}\right)\right],$$

where E is the domain for which x, y, z are all positive and $x^2 + y^2 + z^2 \le 1$.

10. Evaluate $\iiint z\, dx\, dy\, dz$, over the region $x^2 + y^2 \le z^2$, $x^2 + y^2 + z^2 \le 1$, $z \ge 0$.

11. Compute the volumes of the solids bounded by

(i) The cylinders $z = 4 - y^2$, and $z = y^2 + 2$, and the planes $x = -1$, $x = 2$

(ii) The paraboloids $z = x^2 + y^2$, and $z = x^2 + 2y^2$, and the planes $y = x$, $y = 2x$, $x = 1$.

12. (i) The paraboloids $(x - 1)^2 + y^2 = z$, and planes $2x + z = 2$.
[*Projection of the solid on the xy-plane is a circle.*]

(ii) The sphere $x^2 + y^2 + z^2 = a^2$, and the cone $z^2 \sin^2 \alpha = (x^2 + y^2)\cos^2 \alpha$, where α is a constant such that $0 \le \alpha \le \pi$.

[In spherical polar coordinates, the surfaces are $r^2 = a^2$ and $\theta = \alpha$]

13. Show that the volume of the region bounded by the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, its asymptotic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, and the planes, $z = z_1, z = z_2\ (z_2 > z_1)$ is $\pi ab(z_2 - z_1)$.

14. Find the volume of the solid bounded by the six cylinders, $z^2 = y, z^2 = 2y; x^2 = z, x^2 = 2z; y^2 = x, y^2 = 2x$.

15. Show that the volume enclosed by the paraboloid $x^2 - y^2 = 2az$, the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$, and the planes $z = 0$ is $a^3/6$.

16. Show that $\iiint_E (x^3 + y^3 + z^3)\, dx\, dy\, dz = \frac{32}{5}\pi a^6$, where E is the interior of the sphere

$$x^2 + y^2 + z^2 - 2a(x + y + z) + 2a^2 = 0.$$

17. Show that $\iiint_E xyz\, dx\, dy\, dz = \frac{2}{15}(\lambda^{-2} + \mu^{-2} + \nu^{-2})^{-3}$, where E is the volume common to the spheres

$$x^2 + y^2 + z^2 = 2\lambda x,\ \ x^2 + y^2 + z^2 = 2\mu y,\ \ x^2 + y^2 + z^2 = 2\nu z.$$

18. Evaluate $\iiint x^l y^m z^n (1 - ax - by - cz)^p\, dx\, dy\, dz$ over the interior of the tetrahedron formed by the planes $x = 0,\ y = 0,\ z = 0,\ ax + by + cz = 1$.

19. Evaluate the triple integral $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$, where $l, m, n \ge 1$ and the variables are all positive such that

(i) $x/a + y/b + z/c \le h$

(ii) $(x/a)^p + (y/b)^q + (z/c)^r \le 1$

[*Hint*: (i) Put $x/a + y/b + z/c = hu,\ x/a + y/b = huv,\ x/a = huvw\ \ 0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1$

(ii) Put $(x/a)^p + (y/b)^q + (z/c)^r = u,\ (x/a)^p + (y/b)^q = uv,\ (x/a)^p = uvw,$

so that $(x/a)^p = uvw,\ (y/b)^q = uv(1 - w),\ (z/c)^r = u(1 - v)$

$0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1$].

20. Evaluate $\iiint x^{l-1} y^{m-1} z^{n-1}\ f[(x/a)^p + (y/b)^q + (z/c)^r]\, dx\, dy\, dz$ over the region in which x, y, z take positive values subject to the condition that $(x/a)^p + (y/b)^q + (z/c)^r \le h$.

[*Hint*: Put $(x/a)^p + (y/b)^q + (z/c)^r = hu,\ (x/a)^p + (y/b)^q = huv,\ (x/a)^p = huvw$

where $0 \le u \le 1,\ 0 \le v \le 1,\ 0 \le w \le 1$]

21. Prove that

$$\iiint_E z^{-3}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\, dx\, dy\, dz = \frac{\pi}{pq}$$

where E is the smaller region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and the half-cone $z^2 = p^2x^2 + q^2y^2, z > 0$.

22. Show that the entire volume enclosed by the solid $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ is $4\pi abc/35$.

23. Prove that the volume in the positive octant bounded by the surface, $(z/c)^m = (x/a)^m + (y/b)^m$, and the planes $x = 0,\ y = 0,\ z = h$ is equal to

$$\frac{abh^3[\Gamma(1/m)]^2}{6mc^2\ \Gamma(2/m)}.$$

24. Prove that the volume of a cone which extends from the origin to the surface $x = f(u, v), y = g(u, v), z = h(u, v)$ is given by $\frac{1}{3}\iint \Delta\, du\, dv$, where

$$\Delta = \begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

Hence prove that the volume of one octant of the cone $x^4 + y^4 = z^4 \tan^2 \alpha$ between its vertex and the surface $x^4 + y^4 + z^4 = 1$ is

$$\frac{\Gamma\left(\frac{1}{4}\right)^2}{24\Gamma\left(\frac{1}{2}\right)}\int^{\alpha}\frac{dt}{\sqrt{\cos t}}.$$

[*Hint*: Put $x^2 = \sin\theta \cos\varphi$, $y^2 = \sin\theta \cdot \sin\varphi$, $z^2 = \cos\theta$, where $0 \le \theta \le \alpha, 0 \le \varphi \le \pi/2$.]

ANSWERS

1. (*i*) $\frac{1}{2}\pi a$, (*ii*) $8a^2/9$, (*iii*) $4\pi a^5/15$, (*iv*) $\pi/8$.

2. $4\pi(b^5 - a^5)/15$ 3. $2\pi/3$ 10. $\pi/8$ 11. (*i*) 8 (*ii*) 7/12

12. (*i*) $\frac{1}{2}\pi$ (*ii*) $2\pi a^3(1 - \cos\alpha)/3$ 14. 1/7

18. $\dfrac{\Gamma(l+1)\,\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(p+1)}{a^{l+1}b^{m+1}c^{n+1}\ \Gamma(l+m+n+p+4)}$

19. (*i*) $a^l b^m c^n (h)^{\Sigma l}\ \dfrac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(\Sigma l + 1)}$, where $\Sigma l = l + m + n$

(*ii*) $\dfrac{a^l b^m c^n}{pqr}\ \dfrac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}$

20. $\dfrac{a^l b^m c^n}{pqr}(h)^{\Sigma(l/p)}\ \dfrac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r)}\displaystyle\int_0^1 u^{\Sigma(l/p)} f(hu)\, du$,

where $\Sigma(l/p) = l/p + m/q + n/r$.