IFoS - 2017

Q1 ⇒ If $f(z) = u(x,y) + i v(x,y)$ is an analytic function of $z = x+iy$ and $u + 2v = x^3 - 2y^3 + 3xy(2x-y)$ then find $f(z)$ in terms of z.

Sol^n

~~Method~~

Given that $u+2v = x^3 - 2y^3 + 6x^2y - 3xy^2$

diff. w.r.t. x,

$$\frac{\delta u}{\delta x} + 2\frac{\delta v}{\delta x} = 3x^2 \cancel{+6y^2} + 12xy - 3y^2 \quad — ①$$

diff. w.r.t. y — ②

$$\frac{\delta u}{\delta y} + 2\frac{\delta v}{\delta y} = -6y^2 + 6x^2 - 6xy$$

Using C.R. eq^n $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$; & $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ in Eq^n ① & ②

$$\Rightarrow 5\frac{\delta v}{\delta y} = 15x^2 - 15y^2$$

$$\Rightarrow \frac{\delta v}{\delta y} = 3(x^2-y^2) \quad \Rightarrow v = 3x^2y - y^3 + \phi(x) \quad — ③$$

$$\Rightarrow \cancel{v = x^3 - 3y^2x + \phi(x)}$$

using Eq^n ① & ②

$$\frac{\delta u}{\delta y} = -6xy.$$

from Eq^n ③

$$\frac{\delta v}{\delta x} = 6xy + \phi'(x).$$

$\because$ C.R. Eq^n $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \Rightarrow \phi'(x) = 0$

$$\boxed{v = 3x^2y - y^3} \quad — ④$$

Acc. to Milne-thomson's method

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)\,dz] + c$$

from Eq^n ④

where $\psi_1(x,y) = \frac{\delta v}{\delta y} = 3x^2 - 3y^2$

$\psi_2(x,y) = \frac{\delta v}{\delta x} = 6xy$

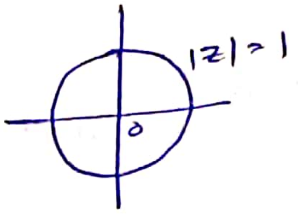
$$f(z) = \int [(3z^2 - 0) + i(0)]\,dz + c$$

$$\boxed{f(z) \Rightarrow z^3 + c}$$

Q 2 → Prove by the method of contour integration that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$

Sol^n ∵ $z = e^{i\theta}$ & $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$; $\frac{dz}{iz} = d\theta$

& let contour be unit circle



$$I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$$

⇒ replacing $d\theta$ & $\cos\theta$.

$$I = \int_C \frac{1+\left(z+\frac{1}{z}\right)}{5+2\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{2iz} = \frac{1}{2i}\int \frac{(z^2+z+1)\, dz}{z\,(2z^2+5z+2)}$$

$$I = \frac{1}{4i}\int_C \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\, dz$$

∵ $z = 0, -\frac{1}{2}, -2$ are poles but $z=-2$ ~~does not it~~ out of the region.

∴ By Cauchy-Residue Theorem $\int_C \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\, dz = 2\pi i \left[\text{Residue at } \left(-\tfrac{1}{2}\right) + \text{Residue at } 0\right]$ —— ①

Residue at $z=0$ is $= \lim_{z\to 0} (z-0)\frac{(z^2+z+1)}{z\left(z+\frac{1}{2}\right)(z+2)} = 1.$

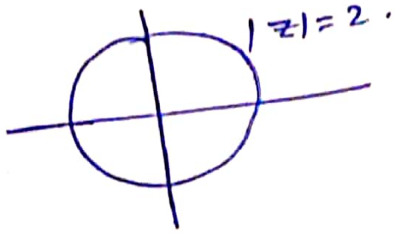
Residue at $z=-\frac{1}{2}$ $= \lim_{z\to -1/2} (z+1/2)\frac{(z^2+z+1)}{z(z+1/2)(z+2)} = -1.$

∴ from Eq^n ① $I = \frac{1}{4i}\int_C \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\, dz = 2\pi i\,[1-1] = 0$

Q3⇒ Find the sum of Residues of $f(z) = \frac{\sin z}{\cos z}$ at the Poles inside the circle $|z| = 2$.

Soln

given $f(z) = \frac{\sin z}{\cos z}$.



$\because \cos z = 0, \quad z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

$\therefore$ Poles of $f(z) = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Residue at $\left(\frac{\pi}{2}\right) \Rightarrow f\left(\frac{\pi}{2}\right) = \underset{z \to \frac{\pi}{2}}{Lt} \left(z - \frac{\pi}{2}\right) \cdot \frac{\sin z}{\cos z}$

$$\Rightarrow \underset{z \to \frac{\pi}{2}}{Lt} \cdot \sin z \cdot \lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right)}{\cos z}$$

$$\Rightarrow 1 \cdot \lim_{z \to \frac{\pi}{2}} \frac{\frac{d}{dz}\left(z - \frac{\pi}{2}\right)}{\frac{d}{dz} \cos z}$$

$$\Rightarrow 1 \times -1 = -1$$

Residue at $\left(-\frac{\pi}{2}\right) \Rightarrow f\left(-\frac{\pi}{2}\right) = \underset{z \to -\frac{\pi}{2}}{Lt} \left(z + \frac{\pi}{2}\right) \cdot \frac{\sin z}{\cos z}$

$$= -1$$

Hence, sum of Residues $= -1 + (-1) = -2$

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