4

Partial Differentiation

§ 1. Functions of two or more independent variables.

Functions in which several independent variables occur are very common. For example, the volume of a rectangular box depends upon three variables, *viz.* its length, breadth and depth. Similarly the area of a triangle depends upon two variables, *viz.*, the base and the altitude. In both these examples we observe that any of the variables may vary independently of the others.

A function of x, y and z may be written as $f(x, y, z)$ or $\phi(x, y, z)$, etc. Similarly, a function of x and y is generally denoted by the symbols $f(x, y)$ or $\phi(x, y)$ etc.

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a **partial derivative.** The operation of finding the partial derivatives of a function of more than one independent variables is called **partial differentiation.** The symbols $\partial/\partial x$, $\partial/\partial y$, etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$, etc., are respectively called the **partial differential coefficients** of u w.r.t. x, y, etc. Thus if $u = f(x, y, z)$, then the partial differential coefficient of u w.r.t. x *i.e.*, $\partial u/\partial x$ is obtained by differentiating u w.r.t. x keeping y and z as constants. Sometimes $\partial f/\partial x$ is also denoted by f_x.

Second order partial differential coefficients. If $u = f(x, y)$, then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves functions of x and y and can be again differentiated partially.

We call $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right), \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right), \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$ and $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$

as the second order partial derivatives of u and these are respectively denoted by $\dfrac{\partial^2 u}{\partial x\,\partial y}, \dfrac{\partial^2 u}{\partial y\,\partial x}, \dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$.

Note. If $u = f(x, y)$ and its partial derivatives are continuous (as is true in all ordinary cases), the order of differentiation is immaterial *i.e.* $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}\,\partial \mathbf{y}} = \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}\,\partial \mathbf{x}}$.

Solved Examples

Ex. 1. *Verify that* $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$ *for the following functions :*

(i) $z = x \log y$ **(Kanpur 1988)** **(ii)** $z = \sin^{-1}(x/y)$

(iii) $z = a \tan^{-1}(x/y)$ **(iv)** $z = x^y + y^x$

(v) $z = e^{ax} \sin by$ **(vi)** $z = \log(y \sin x + x \sin y)$.

Sol. (i) We have $z = x \log y$.

Differentiating z partially w.r.t. x taking y as constant, we have $\partial z/\partial x = \log y$.

Now $$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(\log y) = \frac{1}{y}.$$

Again differentiating z partially w.r.t. y taking x as constant, we have $\partial z/\partial y = x/y$.

Now $$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y},$$ treating y as constant.

Hence $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x$.

(ii) We have $z = \sin^{-1}(x/y)$.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{\sqrt{\{1 - (x/y)^2\}}} \cdot \frac{1}{y} = \frac{1}{\sqrt{(y^2 - x^2)}} = (y^2 - x^2)^{-1/2}.$$

Now $$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\{(y^2 - x^2)^{-1/2}\}$$

$$= -\tfrac{1}{2}(y^2 - x^2)^{-3/2} \cdot 2y = \frac{-y}{(y^2 - x^2)^{3/2}}.$$

Again $$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{\{1 - (x/y)^2\}}} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y}(y^2 - x^2)^{-1/2},$$

and $$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= -\frac{1}{y}(y^2 - x^2)^{-1/2} - \left(\frac{x}{y}\right) \cdot \{-\tfrac{1}{2}(y^2 - x^2)^{-3/2}\} \cdot (-2x)$$

$$= -\frac{1}{y(y^2 - x^2)^{1/2}} - \frac{x^2}{y(y^2 - x^2)^{3/2}} = -\frac{(y^2 - x^2) + x^2}{y(y^2 - x^2)^{3/2}}$$

$$= \frac{-y}{(y^2 - x^2)^{3/2}}.$$

Hence $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x$.

(iii) We have $z = a \tan^{-1}(x/y)$.

$$\therefore \quad \frac{\partial z}{\partial x} = a\frac{1}{1 + (x^2/y^2)} \cdot \frac{1}{y} = \frac{ay}{x^2 + y^2},$$

and $$\frac{\partial^2 z}{\partial y\,\partial x} = a \cdot \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = \frac{a(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Again $$\frac{\partial z}{\partial y} = a \cdot \frac{1}{1 + (x^2/y^2)} \cdot \left(-\frac{x}{y^2}\right) = \frac{-ax}{x^2 + y^2},$$

and $$\frac{\partial^2 z}{\partial x\,\partial y} = -a \cdot \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = -a \cdot \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{a(x^2 - y^2)}{(x^2 + y^2)^2}$$

Thus $\partial^2 z/\partial x\,\partial y = \partial^2 z/\partial y\,\partial x$.

(iv) We have $z = x^y + y^x$.

$\therefore$ $\partial z/\partial x = yx^{y-1} + y^x \cdot \log y$

and $\dfrac{\partial^2 z}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}[yx^{y-1} + y^x \log y]$

$= 1 \cdot x^{y-1} + yx^{y-1} \cdot \log x + xy^{x-1} \log y + y^x \cdot (1/y)$

$= x^{y-1}(1 + y \log x) + y^{x-1}(1 + x \log y)$.

Again $\dfrac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$ and $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}[x^y \log x + xy^{x-1}]$

$= yx^{y-1} \cdot \log x + x^y \cdot (1/x) + 1 \cdot y^{x-1} + xy^{x-1} \log y$

$= x^{y-1}(1 + y \log x) + y^{x-1}(1 + x \log y)$.

Thus $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

(v) We have $z = e^{ax} \sin by$.

$\therefore$ $\partial z/\partial x = ae^{ax} \sin by$,

$\dfrac{\partial^2 z}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}(ae^{ax} \sin by) = ae^{ax} \cdot b \cos by = ab\, e^{ax} \cos by$.

Again $\partial z/\partial y = e^{ax} \cdot b \cos by$;

$\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}(be^{ax} \cos by) = ab\, e^{ax} \cos by$.

Thus $\partial^2 z/\partial y\, \partial x = \partial^2 z/\partial x\, \partial y$.

(vi) We have $z = \log(y \sin x + x \sin y)$.

$\therefore$ $\dfrac{\partial z}{\partial x} = \dfrac{y \cos x + \sin y}{y \sin x + x \sin y}$,

and $\partial^2 z/\partial y\, \partial x$

$$= \frac{(y \sin x + x \sin y)(\cos x + \cos y) - (y \cos x + \sin y)(\sin x + x \cos y)}{(y \sin x + x \sin y)^2}$$

$$= \frac{x \sin y \cos x + y \sin x \cos y - \sin x \sin y - xy \cos x \cos y}{(y \sin x + x \sin y)^2}.$$

Again $\dfrac{\partial z}{\partial y} = \dfrac{\sin x + x \cos y}{y \sin x + x \sin y}$

and $\partial^2 z/\partial x\, \partial y$

$$= \frac{(y \sin x + x \sin y)(\cos x + \cos y) - (\sin x + x \cos y)(y \cos x + \sin y)}{(y \sin x + x \sin y)^2}$$

$$= \frac{x \sin y \cos x + y \sin x \cos y - \sin x \sin y - xy \cos x \cos y}{(y \sin x + x \sin y)^2}.$$

$\therefore$ $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

Ex. 2. *If* $u = x^2 \tan^{-1} \dfrac{y}{x} - y^2 \tan^{-1} \dfrac{x}{y}$, *find* $\dfrac{\partial^2 u}{\partial x\, \partial y}$.

(Andhra 1986; Meerut 89P; Kashmir 88; Utkal 89)

Sol. We have

$$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - 2y \tan^{-1} \frac{x}{y} - y^2 \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} + \frac{xy^2}{x^2 + y^2}$$

$$= \frac{x(x^2 + y^2)}{x^2 + y^2} - 2y \tan^{-1} \frac{x}{y} = x - 2y \tan^{-1} \frac{x}{y}.$$

Now $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 1 - 2y \dfrac{1}{1 + (x/y)^2} \cdot \dfrac{1}{y} = 1 - \dfrac{2y^2}{x^2 + y^2}$

$= \dfrac{x^2 + y^2 - 2y^2}{x^2 + y^2} = \dfrac{x^2 - y^2}{x^2 + y^2}$.

Ex. 3. *If* $z = f(y/x)$, *show that* $x(\partial z/\partial x) + y(\partial z/\partial y) = 0$.

(Garhwal 1983; Bundelkhand 82; U.P. P.C.S. 94)

Sol. We have

$\partial z/\partial x = [f'(y/x)](-y/x^2)$, (diff. partially w.r.t. x).

$\therefore$ $x(\partial z/\partial x) = -(y/x) f'(y/x)$. ...(1)

Again $\partial z/\partial y = [f'(y/x)] \cdot (1/x)$, (diff. partially w.r.t. y)

$\therefore$ $y(\partial z/\partial y) = (y/x) f'(y/x)$. ...(2)

Adding (1) and (2), we get $x(\partial z/\partial x) + y(\partial z/\partial y) = 0$.

****Ex. 4 (a)** *If* $u = \sin^{-1} \dfrac{x}{y} + \tan^{-1} \dfrac{y}{x}$, *show that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

(Meerut 1982S, 94P, 95, 96P; Agra 82; Allahabad 84; Gorakhpur 85; Lucknow 82; Kashmir 84; Rohilkhand 91)

Sol. Here $\dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{\{1 - (x/y)^2\}}} \cdot \dfrac{1}{y} + \dfrac{1}{1 + (y/x)^2} \cdot \left(-\dfrac{y}{x^2}\right)$,

(treating y as constant)

$= \dfrac{1}{\sqrt{(y^2 - x^2)}} - \dfrac{y}{(x^2 + y^2)}$.

$\therefore$ $x \dfrac{\partial u}{\partial x} = \dfrac{x}{\sqrt{(y^2 - x^2)}} - \dfrac{xy}{x^2 + y^2}$. ...(1)

Again $\dfrac{\partial u}{\partial y} = \dfrac{1}{\sqrt{\{1 - (x/y)^2\}}}\left(-\dfrac{x}{y^2}\right) + \dfrac{1}{1 + (y/x)^2} \cdot \dfrac{1}{x}$,

(treating x as constant)

$= -\dfrac{x}{y\sqrt{(y^2 - x^2)}} + \dfrac{x}{x^2 + y^2}$.

$\therefore$ $y \dfrac{\partial u}{\partial y} = -\dfrac{x}{\sqrt{(y^2 - x^2)}} + \dfrac{xy}{x^2 + y^2}$. ...(2)

Adding (1) and (2), we have $x(\partial u/\partial x) + y(\partial u/\partial y) = 0$.

Ex. 4 (b) *Find the value of*

$\dfrac{1}{a^2} \dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2} \dfrac{\partial^2 z}{\partial y^2}$ *when* $a^2x^2 + b^2y^2 - c^2z^2 = 0$. **(Meerut 1991)**

Sol. We have $z^2 = \dfrac{a^2}{c^2} x^2 + \dfrac{b^2}{c^2} y^2$. ...(1)

Differentiating (1) partially with respect to x taking y as constant, we have

$$2z\frac{\partial z}{\partial x}=2\frac{a^2}{c^2}x \text{ or } \frac{\partial z}{\partial x}=\frac{a^2}{c^2}\cdot\frac{x}{z}.$$

$$\therefore\ \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{a^2}{c^2}x\cdot\frac{1}{z}\right)=\frac{a^2}{c^2}\cdot\frac{1}{z}+\frac{a^2}{c^2}x\cdot\left(-\frac{1}{z^2}\right)\frac{\partial z}{\partial x}$$

$$=\frac{a^2}{c^2z}-\frac{a^2x}{c^2z^2}\cdot\left(\frac{a^2x}{c^2z}\right)=\frac{a^2}{c^2z}-\frac{a^4x^2}{c^4z^3}.$$

$$\therefore\ \frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2z}-\frac{a^2x^2}{c^4z^3} \qquad ...(2)$$

Again differentiating (1) partially with respect to y taking x as constant, we have

$$2z\frac{\partial z}{\partial y}=2\frac{b^2}{c^2}y \quad\text{or}\quad \frac{\partial z}{\partial y}=\frac{b^2}{c^2}\cdot\frac{y}{z}.$$

$$\therefore\ \frac{\partial^2 z}{\partial y^2}=\frac{b^2}{c^2}\cdot\frac{1}{z}+\frac{b^2}{c^2}y\cdot\left(-\frac{1}{z^2}\right)\frac{\partial z}{\partial y}$$

$$=\frac{b^2}{c^2z}-\frac{b^2y}{c^2z^2}\cdot\left(\frac{b^2y}{c^2z}\right)=\frac{b^2}{c^2z}-\frac{b^4y^2}{c^4z^3}.$$

$$\therefore\ \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2z}-\frac{b^2y^2}{c^4z^3}. \qquad ...(3)$$

Adding (2) and (3), we get

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}=\frac{2}{c^2z}-\frac{a^2x^2+b^2y^2}{c^4z^3}$$

$$=\frac{2}{c^2z}-\frac{c^2z^2}{c^4z^3}=\frac{2}{c^2z}-\frac{1}{c^2z}=\frac{1}{c^2z}.$$

Ex. 4 (c) *If* $u=\tan^{-1}\dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, *show that*

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{3/2}}. \qquad \textbf{(Agra 1987)}$$

Sol. We have

$$\frac{\partial u}{\partial x}=\frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\cdot$$

$$y\cdot\frac{1\cdot\sqrt{(1+x^2+y^2)}-x\cdot\frac{1}{2}(1+x^2+y^2)^{-1/2}(-2x)}{1+x^2+y^2}$$

$$=\frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\cdot y\cdot\frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$=\frac{y(1+y^2)}{(1+x^2)(1+y^2)(1+x^2+y^2)^{1/2}}=\frac{1}{1+x^2}\cdot\frac{y}{(1+x^2+y^2)^{1/2}}.$$

$$\therefore\ \frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$=\frac{1}{1+x^2}\cdot\frac{1.(1+x^2+y^2)^{1/2}-y\cdot\frac{1}{2}(1+x^2+y^2)^{-1/2}.2y}{1+x^2+y^2}$$

$$=\frac{1}{1+x^2}\cdot\frac{1+x^2+y^2-y^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$=\frac{1}{1+x^2}\cdot\frac{1+x^2}{(1+x^2+y^2)^{3/2}}=\frac{1}{(1+x^2+y^2)^{3/2}}.$$

***Ex. 5.** *If* $z=f(x+ay)+\phi(x-ay)$, *prove that* $\partial^2 z/\partial y^2=a^2(\partial^2 z/\partial x^2)$.

(Rohilkhand 1981; Bihar 87; Gorakhpur 80; Meerut 90, 85)

Sol. We have $z=f(x+ay)+\phi(x-ay)$.

$\therefore\ \partial z/\partial x=f'(x+ay)+\phi'(x-ay),$ (diff. partially w.r.t. 'x')

and $\partial^2 z/\partial x^2=f''(x-ay)+\phi''(x-ay)$. ...(1)

Again, $\partial z/\partial y=af'(x+ay)-a\phi'(x-ay)$.

$\therefore\ \partial^2 z/\partial y^2=a^2f''(x+ay)+a^2\phi''(x-ay)$. ...(2)

From (1) and (2), we get $\partial^2 z/\partial y^2=a^2(\partial^2 z/\partial x^2)$.

Ex. 6. *If* $u=\sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, *show that*

$x(\partial u/\partial x)+y(\partial u/\partial y)=\tan u$. **(Kanpur 1980; Allahabad 84)**

Sol. We have $\sin u=(x^2+y^2)/(x+y)$.

$\therefore\ \log\sin u=\log(x^2+y^2)-\log(x+y)$. ...(1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{1}{\sin u}\cos u.\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}-\frac{1}{x+y}.$$

$$\therefore\ (\cot u)\,x\frac{\partial u}{\partial x}=\frac{2x^2}{x^2+y^2}-\frac{x}{x+y}. \qquad ...(2)$$

Again differentiating (1) partially w.r.t. y, we get

$$\frac{\cos u}{\sin u}\cdot\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}-\frac{1}{x+y}.$$

$$\therefore\ (\cot u).y\frac{\partial u}{\partial y}=\frac{2y^2}{x^2+y^2}-\frac{y}{x+y}. \qquad ...(3)$$

Adding (2) and (3), we get

$$(\cot u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{2x^2+2y^2}{x^2+y^2}-\frac{x+y}{x+y}=2-1=1.$$

$\therefore\ x(\partial u/\partial x)+y(\partial u/\partial y)=1/\cot u=\tan u$.

Ex. 7. *If* $z=xf(x+y)+y\phi(x+y)$, *prove that* $(\partial^2 z/\partial x^2)+(\partial^2 z/\partial y^2)=2(\partial^2 z/\partial x\,\partial y)$.

Sol. We have $z=xf(x+y)+y\phi(x+y)$.

$\therefore\ \partial z/\partial x=f(x+y)+xf'(x+y)+y\phi'(x+y),$

and $\partial^2 z/\partial x^2=f'(x+y)+f'(x+y)+xf''(x+y)+y\phi''(x+y)$

$=2f'(x+y)+xf''(x+y)+y\phi''(x+y).$

Also $\partial z/\partial y = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$

and $\partial^2 z/\partial y^2 = xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)$
$= 2\phi'(x+y) + xf''(x+y) + y\phi''(x+y)$.

Again $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$
$= \partial/\partial x\,\{xf'(x+y) + \phi(x+y) + y\phi'(x+y)\}$
$= f'(x+y) + xf''(x+y) + \phi'(x+y) + y\phi''(x+y)$.

Now $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2)$
$= 2[f'(x+y) + \phi'(x+y) + xf''(x+y) + y\phi''(x+y)]$
$= 2(\partial^2 z/\partial x \partial y)$.

***Ex. 8.** *If* $z = (x^2+y^2)/(x+y)$, *show that*

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

(Kanpur 1978; Meerut 85; Agra 80; Allahabad 78)

Sol. We have $z = (x^2+y^2)/(x+y)$.

$\therefore\ \dfrac{\partial z}{\partial x} = \dfrac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \dfrac{x^2 - y^2 + 2xy}{(x+y)^2}$,

and $\dfrac{\partial z}{\partial y} = \dfrac{(x+y)2y - (x^2+y^2).1}{(x+y)^2} = \dfrac{y^2 - x^2 + 2xy}{(x+y)^2}$.

$\therefore\ \left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = \left[\dfrac{(x^2-y^2+2xy) - (y^2-x^2+2xy)}{(x+y)^2}\right]^2$

$= \left[\dfrac{2(x^2-y^2)}{(x+y)^2}\right]^2 = 4\left[\dfrac{x-y}{x+y}\right]^2$.

Also $1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 1 - \dfrac{x^2-y^2+2xy}{(x+y)^2} - \dfrac{y^2-x^2+2xy}{(x+y)^2}$

$= \dfrac{(x^2+y^2+2xy) - x^2 + y^2 - 2xy - y^2 + x^2 - 2xy}{(x+y)^2} = \dfrac{x^2 - 2xy + y^2}{(x+y)^2}$

$= \left(\dfrac{x-y}{x+y}\right)^2$.

Hence $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

****Ex. 9.** *If* $1/u = \sqrt{(x^2+y^2+z^2)}$, *show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u,$$

and
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(Allahabad 1981; Meerut 85, 87, 98; Agra 79; Garhwal 77; Kumayun 83; Gorakhpur 83; Vikram 85; Jhansi 89)

Sol. We have $u = (x^2+y^2+z^2)^{-1/2}$.

$\therefore\ \partial u/\partial x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.(2x) = -x(x^2+y^2+z^2)^{-3/2}$,

and $\partial^2 u/\partial x^2 = x.\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x) - 1(x^2+y^2+z^2)^{-3/2}$

$= (x^2+y^2+z^2)^{-5/2}[3x^2 - (x^2+y^2+z^2)]$
$= u^5(2x^2 - y^2 - z^2)$.

Similarly, by symmetry $\partial u/\partial y = -y(x^2+y^2+z^2)^{-3/2}$,
$\partial^2 u/\partial y^2 = u^5(2y^2 - x^2 - z^2)$, $(\partial u/\partial z) = -z(x^2+y^2+z^2)^{-3/2}$,

and $\partial^2 u/\partial z^2 = u^5(2z^2 - x^2 - y^2)$.

Now $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$

$= -x^2(x^2+y^2+z^2)^{-3/2} - y^2(x^2+y^2+z^2)^{-3/2} - z^2(x^2+y^2+z^2)^{-3/2}$

$= -(x^2+y^2+z^2)^{-3/2}(x^2+y^2+z^2) = -(x^2+y^2+z^2)^{-1/2} = -u$.

Also $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

$= u^5(2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2) = u^5.0 = 0$.

Ex. 10. *If* $u = \log(x^2+y^2+z^2)$, *show that*

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}.$$

Sol. We have $u = \log(x^2+y^2+z^2)$.

$\therefore\ \dfrac{\partial u}{\partial z} = \dfrac{2z}{x^2+y^2+z^2}$, [treating x and y as constants]

and $\dfrac{\partial^2 u}{\partial y \partial z} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial z}\right) = 2z.[-(x^2+y^2+z^2)^{-2}.2y]$
$= -4yz/(x^2+y^2+z^2)^2$.

Now $x\dfrac{\partial^2 u}{\partial y\,\partial z} = -4xyz/(x^2+y^2+z^2)^2$.

By symmetry, $y\dfrac{\partial^2 u}{\partial z\,\partial x} = z\dfrac{\partial^2 u}{\partial x\,\partial y} = -\dfrac{4xyz}{(x^2+y^2+z^2)^2}$.

Hence $x\dfrac{\partial^2 u}{\partial y\,\partial z} = y\dfrac{\partial^2 u}{\partial z\,\partial x} = z\dfrac{\partial^2 u}{\partial x\,\partial y}$.

****Ex. 11.** *If* $u = \log(x^3+y^3+z^3 - 3xyz)$, *show that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

and
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

(Meerut 1996, 94P, 83S, 81; Bundelkhand 78; Gorakhpur 75; Rohilkhand 83, 88, 89; Lucknow 80; Alld. 80; Kanpur 86)

Sol. We have $u = \log(x^3+y^3+z^3-3xyz)$.

$\therefore\ \dfrac{\partial u}{\partial x} = \dfrac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$, $\dfrac{\partial u}{\partial y} = \dfrac{3y^2 - 3zx}{x^3+y^3+z^3-3xyz}$

and $\dfrac{\partial u}{\partial z} = \dfrac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$.

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)}=\frac{3}{x+y+z}. \quad ...(1)$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \text{ from (1)}$$

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]=\frac{-9}{(x+y+z)^2}.$$

Ex. 12. *If $z=\tan(y+ax)+(y-ax)^{3/2}$, find the value of* $(\partial^2 z/\partial x^2)-a^2(\partial^2 z/\partial y^2)$.

Sol. Here $z=\tan(y+ax)+(y-ax)^{3/2}$.

$\therefore \quad (\partial z/\partial x)=\{\sec^2(y+ax)\}.a+\frac{3}{2}(y-ax)^{1/2}.(-a)$,

and $(\partial^2 z/\partial x^2)=2a^2\tan(y+ax)\sec^2(y+ax)+\frac{3}{4}a^2(y-ax)^{-1/2}$.

Again $(\partial z/\partial y)=\sec^2(y+ax)+\frac{3}{2}(y-ax)^{1/2}$,

and $(\partial^2 z/\partial y^2)=2\sec^2(y+ax)\tan(y+ax)+\frac{3}{4}(y-ax)^{-1/2}$.

Thus $(\partial^2 z/\partial x^2)-a^2(\partial^2 z/\partial y^2)=0$.

Ex. 13. *If $x=r\cos\theta$, $y=r\sin\theta$, prove that*

*(a) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$,

(Kanpur 81; Vikram 83; Gorakhpur 82)

(b) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$,

(c) $(\partial r/\partial x)^2+(\partial r/\partial y)^2=1$. **(Meerut 1983)**

Sol. (a). We have $x=r\cos\theta, y=r\sin\theta$.

Therefore $r^2=x^2+y^2$ ...(1)

Now $2r(\partial r/\partial x)=2x$; [diff. (1) partially w.r.t. 'x']

$\therefore \quad \partial r/\partial x=x/r$. ...(2)

Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^2 r}{\partial x^2}=\frac{r.1-x.\partial r/\partial x}{r^2}=\frac{r-x.x/r}{r^2}, \quad \text{[using (2)]}$$

$$=\frac{r^2-x^2}{r^3}=\frac{(x^2+y^2)-x^2}{r^3}=\frac{y^2}{r^3}. \quad ...(3)$$

Again differentiating (1) partially w.r.t. 'y', we get

$$2r\frac{\partial r}{\partial y}=2y; \quad \therefore \quad \frac{\partial r}{\partial y}=\frac{y}{r}. \quad ...(4)$$

Differentiating (4) partially w.r.t. y, we get

$$\frac{\partial^2 r}{\partial y^2}=\frac{r.1-y.\partial r/\partial y}{r^2}=\frac{r-y.y/r}{r^2}, \quad \left[\because \text{ from (4), } \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{r^2-y^2}{r^3}=\frac{(x^2+y^2)-y^2}{r^3}=\frac{x^2}{r^3}. \quad ...(5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}+\frac{x^2}{r^3}=\frac{x^2+y^2}{r^3}=\frac{1}{r}.$$

Also $\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}$.

$$\therefore \quad \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$$

(b) Differntiating (4) partially w.r.t. 'x', we get

$$\frac{\partial^2 r}{\partial x\,\partial y}=-\frac{y}{r^2}\cdot\frac{\partial r}{\partial x}=-\frac{xy}{r^3}, \quad \left[\because \text{ from (2), } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

Now $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}\cdot\frac{x^2}{r^3}$, [from (3) and (5)]

$$=\frac{x^2y^2}{r^6}=\left(\frac{-xy}{r^3}\right)^2=\left(\frac{\partial^2 r}{\partial x\,\partial y}\right)^2.$$

(c) From (2) and (4), on squaring and adding, we get

$$\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2=\frac{x^2}{r^2}+\frac{y^2}{r^2}=\frac{x^2+y^2}{r^2}=\frac{r^2}{r^2}=1.$$

Ex. 14. *If $x=r\cos\theta, y=r\sin\theta$, prove that*

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \textit{ except when } x=0, y=0.$$

(Meerut 1983, 87; Gorakhpur 81; Vikram 87; G.N.U. 88)

Sol. Given $x=r\cos\theta, y=r\sin\theta$.

$\therefore \quad r^2=x^2+y^2; \theta=\tan^{-1}(y/x)$.

Now $\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\{\tan^{-1}(y/x)\}=\frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right]=-\frac{y}{x^2+y^2}$.

$$\therefore \quad \frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right)=-y\cdot\frac{-2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}. \quad ...(1)$$

Also $\frac{\partial\theta}{\partial y}=\frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{(x^2+y^2)}$.

$$\therefore \quad \frac{\partial^2\theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right)=x\cdot\frac{-2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}. \quad ...(2)$$

Adding (1) and (2), we get $(\partial^2\theta/\partial x^2)+(\partial^2\theta/\partial y^2)=0$.

But at $x=0$, $y=0$ both $\partial^2\theta/\partial x^2$ and $\partial^2\theta/\partial y^2$ are of the indeterminate form 0/0.

$\therefore \quad (\partial^2\theta/\partial x^2)+(\partial^2\theta/\partial y^2)=0$, excpet when $x=0, y=0$.

Ex. 14 (a). *If $u=\tan^{-1}(y/x)$, then verify that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

(Meerut 1991 P, Gorakhpur 87)

Sol. For complete solution of this question refer Ex. 14.

****Ex. 15.** *If $u=f(r)$ where $r^2=x^2+y^2$, show that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

(Delhi 1982; Lucknow 82; Rohilkhand 82; Kanpur 85, 89; Meerut 97, 94, 82 S, 77; Allahabad 82, 77)

Sol. Differentiating $r^2=x^2+y^2$ partially w.r.t. x and y, we get

$$2r\frac{\partial r}{\partial x}=2x \text{ or } \frac{\partial r}{\partial x}=\frac{x}{r};\ 2r\frac{\partial r}{\partial y}=2y \text{ or } \frac{\partial r}{\partial y}=\frac{y}{r}. \quad ...(1)$$

Now $u=f(r)$. Therefore $\frac{\partial u}{\partial x}=\{f'(r)\}\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)$, [from (1)]

and
$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left[x\cdot\frac{1}{r}\cdot f'(r)\right]$$

$$=1\cdot\frac{1}{r}\cdot f'(r)+\{xf'(r)\}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\frac{x}{r}\{f''(r)\}\frac{\partial r}{\partial x}$$

$$=\frac{1}{r}f'(r)-\frac{x}{r^2}\cdot\frac{x}{r}f'(r)+\frac{x^2}{r^2}f''(r),$$

[$\because$ from (1), $\partial x/\partial r=x/r$]

$$=(1/r)f'(r)-(x^2/r^3)f'(r)+(x^2/r^2)f''(r). \quad ...(2)$$

Similarly, by symmetry, we have

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{r}f'(r)-\frac{y^2}{r^3}f'(r)+\frac{y^2}{r^2}f''(r). \quad ...(3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)-\frac{x^2+y^2}{r^3}f'(r)+\frac{x^2+y^2}{r^2}f''(r)$$

$=(2/r)f'(r)-(r^2/r^3)f'(r)+(r^2/r^2)f''(r)$, [$\because r^2=x^2+y^2$]

$=(2/r)f'(r)-(1/r)f'(r)+f''(r)=f''(r)+(1/r)f'(r).$

Ex. 16. *If $u=x\phi(y/x)+\psi(y/x)$, prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

(Agra 1978; Alld. 84)

Sol. We have $u=x\phi(y/x)+\psi(y/x)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$(\partial u/\partial x)=x\{\phi'(y/x)\}.(-y/x^2)+\phi(y/x)+\{\psi'(y/x)\}.(-y/x^2)$,

and $(\partial u/\partial y)=x\{\phi'(y/x)\}.(1/x)+\{\psi'(y/x)\}.(1/x)$.

$\therefore\ x(\partial u/\partial x)+y(\partial u/\partial y)=x\phi(y/x)$. ...(2)

Now differentiating (2) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=x\left\{\phi'\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right)+\phi\left(\frac{y}{x}\right).$$

and
$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=x\left\{\phi'\left(\frac{y}{x}\right)\right\}\cdot\frac{1}{x}.$$

Multiplying these equations by x and y respectively and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\phi(y/x)$$

or
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0, \quad \text{[from (2)].}$$

Ex. 17. *If* $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

(Meerut 1983; Kanpur 82; Gorakhpur 87)

Sol. From the given equation we observe that u is a function of three independent variables x, y and z. Differentiating the given equation partially w.r.t. 'x', we get

$$\frac{2x}{a^2+u}-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}=0.$$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{2x/(a^2+u)}{\Sigma\{x^2/(a^2+u)^2\}}.$$

Similarly, by symmetry, we can write the values of $\partial u/\partial y$ and $\partial u/\partial z$.

Now
$$\left(\frac{\partial u}{\partial x}\right)^2=\frac{4x^2/(a^2+u)^2}{[\Sigma\{x^2/(a^2+u)^2\}]^2}.$$

$$\therefore\quad \Sigma(\partial u/\partial x)^2=\frac{4\Sigma\{x^2/(a^2+u)^2\}}{[\Sigma\{x^2/(a^2+u)^2\}]^2}$$

$$=\frac{4}{\Sigma\{x^2/(a^2+u)^2\}} \quad ...(1)$$

Again
$$2x\frac{\partial u}{\partial x}=\frac{4x^2/(a^2+u)}{\Sigma\{x^2/(a^2+u)^2\}}.$$

$$\therefore\quad 2\Sigma x(\partial u/\partial x)=\frac{4\Sigma\{x^2/(a^2+u)\}}{\Sigma\{x^2/(a^2+u)^2\}}=\frac{4}{\Sigma\{x^2/(a^2+u)^2\}}. \quad ...(2)$$

[$\because \Sigma\{x^2/(a^2+u)\}=1$, from the given relation]

Now from (1) and (2), we have

$$\Sigma(\partial u/\partial x)^2=2\Sigma\{x(\partial u/\partial x)\}.$$

Ex. 18. *If $u=\log r$, where $r^2=(x-a)^2+(y-b^2)+(z-c)^2$, show that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}.$$

Sol. We have $r^2=(x-a)^2+(y-b)^2+(z-c)^2$. ...(1)

Differentiating (1) partially w.r.t. 'x', we have

$$2r\frac{\partial r}{\partial x}=2(x-a) \text{ or } \frac{\partial r}{\partial x}=\frac{1}{r}\left(\frac{x-a}{r}\right) \qquad ...(2)$$

Now $u=\log r$. $\therefore$ $\dfrac{\partial u}{\partial x}=\dfrac{1}{r}\dfrac{\partial r}{\partial x}=\dfrac{1}{r}\left(\dfrac{x-a}{r}\right)$. [from (2)]

Thus $(\partial u/\partial x)=(x-a)/r^2$.

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right)=\frac{r^2(1)-(x-a)\,.\,2r\,(\partial r/\partial x)}{r^4}$$

$$=\frac{r^2-2(x-a)^2}{r^4}, \qquad \left[\because \text{ from (2), } \frac{\partial r}{\partial x}=(x-a)/r\right]$$

Similarly, by symmetry

$$\frac{\partial^2 u}{\partial y^2}=\frac{r^2-2(y-b)^2}{r^4} \quad\text{and}\quad \frac{\partial^2 u}{\partial z^2}=\frac{r^2-2(z-c)^2}{r^4}.$$

Hence $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{3r^2-2\{(x-a)^2+(y-b)^2+(z-c)^2\}}{r^4}$$

$$=\frac{3r^2-2r^2}{r^4}, \{\text{using (1)}\}=\frac{r^2}{r^4}=\frac{1}{r^2}.$$

Ex. 19. *If $u=e^{xyz}$, show that*

$$\partial^3 u/\partial x\,\partial y\,\partial z=(1+3xyz+x^2y^2z^2)\,e^{xyz}.$$

(Lucknow 1983; Meerut 84; Gorakhpur 81)

Sol. Here $u=e^{xyz}$. $\therefore$ $\partial u/\partial z=xy\,e^{xyz}$.

Now $$\frac{\partial^2 u}{\partial y\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial}{\partial y}(xy\,e^{xyz})=x\frac{\partial}{\partial y}(y\,e^{xyz})$$

$$=x\,[y\,.\,xz\,e^{xyz}+e^{xyz}]=e^{xyz}(x^2yz+x).$$

Again $$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right)=\frac{\partial}{\partial x}[e^{xyz}(x^2yz+x)]$$

$$=e^{xyz}(2xzy+1)+yz\,e^{xyz}(x^2yz+x)$$

$$=e^{xyz}[2xyz+1+x^2y^2z^2+xyz]$$

$$=e^{xyz}[1+3xyz+x^2y^2z^2].$$

Ex. 20. *If $x^x y^y z^z=c$, show that at $x=y=z$,*

$$\partial^2 z/\partial x\partial y=-\{x\log(ex)\}^{-1}.$$

(Agra 1983; Meerut 1982, 84, 88 P, Alld. 80, 76)

Sol. We have $x^x\,.\,y^y\,.\,z^z=c$. From this equation we observe that we can regard z as a function of two independent variables x and y. Taking logarithms of both sides of the given equation, we get

$$x\log x+y\log y+z\log z=\log c. \qquad ...(1)$$

Now differentiating (1) partially w.r.t. x taking y as constant, we have

$$x\cdot\frac{1}{x}+1\,.\log x+\left[z\cdot\frac{1}{z}+1\,.\log z\right]\frac{\partial z}{\partial x}=0.$$

[Note that z is not a constant but is a function of x and y]

$$\therefore \quad \frac{\partial z}{\partial x}=-\frac{(1+\log x)}{(1+\log z)}. \qquad ...(2)$$

Similarly differentiating (1) partially w.r.t. y, we have

$$\frac{\partial z}{\partial y}=-\frac{(1+\log y)}{(1+\log z)}. \qquad ...(3)$$

Now $$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left[-\left(\frac{1+\log y}{1+\log z}\right)\right], \qquad \text{[from (3)]}$$

$$=-(1+\log y)\cdot\frac{\partial}{\partial x}[(1+\log z)^{-1}]$$

$$=-(1+\log y)\cdot\left[-(1+\log z)^{-2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

$$=\frac{(1+\log y)}{z(1+\log z)^2}\cdot\left[-\left(\frac{1+\log x}{1+\log z}\right)\right], \qquad \text{[from (2)].}$$

Hence, when $x=y=z$, we have

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{(1+\log x)^2}{x(1+\log x)^3},$$

[putting $y=z=x$ in the value of $(\partial^2 z/\partial x\partial y)$]

$$=-\frac{1}{x(1+\log x)}=-\frac{1}{x(\log e+\log x)}, \qquad [\because \log e=1]$$

$$=-\frac{1}{x\log(ex)}=-\{x\log(ex)\}^{-1}.$$

Ex. 21. (a) *If $u=(1-2xy+y^2)^{-1/2}$, prove that*

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=0.$$

(Meerut 1982)

Sol. Here $u=(1-2xy+y^2)^{-1/2}$.

$$\therefore \quad \frac{\partial u}{\partial x}=-\tfrac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)=yu^3,$$

and $$\frac{\partial u}{\partial y}=-\tfrac{1}{2}(1-2xy+y^2)^{-3/2}\,.\,(-2x+2y)=(x-y)\,u^3.$$

Now $$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}=\frac{\partial}{\partial x}\{(1-x^2)\,.\,yu^3\}$$

$$=y(-2x)\,u^3+y(1-x^2)\,.\,3u^2\frac{\partial u}{\partial x}=-2xy\,u^3+3y(1-x^2)\,u^2\,.\,yu^3$$

$$=-2xy\,u^3+3y^2u^5(1-x^2). \qquad ...(1)$$

Also $$\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=\frac{\partial}{\partial y}\{y^2(x-y)\,u^3\}=\frac{\partial}{\partial y}\{(y^2x-y^3)\,u^3\}$$

$$=(2xy-3y^2)\,u^3+(y^2x-y^3)\,.\,3u^2\frac{\partial u}{\partial y}$$

$$=(2xy-3y^2)\,u^3+y^2(x-y)\,.\,3u^2\,.\,(x-y)\,u^3$$

$$=(2xy-3y^2)\,u^3+y^2(x-y)^2\,.\,3u^5$$

$$=2xy\,u^3+3y^2u^5[(x-y)^2-u^{-2}]$$

$$= 2xy\,u^3 + 3y^2 u^5 [(x-y)^2 - (1 - 2xy + y^2)],$$

$$[\because u^{-2} = 1 - 2xy + y^2]$$

$$= 2xy\,u^3 + 3y^2 u^5 [x^2 - 1] = 2xy\,u^3 - 3y^2 u^5 (1 - x^2). \quad \ldots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$

Ex. 21. (b) *If* $\theta = t^n e^{-r^2/4t}$, *what value of n will make*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}?$$

(Meerut 1989)

Sol. We have $\dfrac{\partial \theta}{\partial r} = t^n . e^{-r^2/4t} . \left(-\dfrac{2r}{4t}\right) = -\dfrac{r}{2} t^{n-1} e^{-r^2/4t}.$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\tfrac{1}{2} r^3 t^{n-1} e^{-r^2/4t}.$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} - \tfrac{1}{2} r^3 t^{n-1} e^{-r^2/4t} . \left(-\frac{2r}{4t}\right)$$

$$= -\tfrac{3}{2} r^2 t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^4 t^{n-2} e^{-r^2/4t}.$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\tfrac{3}{2} t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$$

Also $\dfrac{\partial \theta}{\partial t} = n t^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} . \dfrac{r^2}{4t^2}$

$$= n t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$$

Now $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$

$$\Rightarrow \quad -\tfrac{3}{2} t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t} = n t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}$$

$$\Rightarrow \quad -\tfrac{3}{2} t^{n-1} e^{-r^2/4t} = n t^{n-1} e^{-r^2/4t},$$

for all possible values of r and t

$$\Rightarrow \quad n = -\tfrac{3}{2}.$$

§ 2. Homogeneous Functions.

An expression in which every term is of the same degree is called a homogeneous function. Thus

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} x y^{n-1} + a_n y^n$$

is a homogeneous function of x and y of degree n. This can also be written as

$$x^n \left\{a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \ldots + a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n\right\}$$

or $x^n f(y/x)$, where $f(y/x)$ is some function of (y/x).

Note 1. To test whether a given function f (x, y) is homogeneous or not we put tx for x and ty for y in it.

If we get $f(tx, ty) = t^n f(x, y)$,
the function $f(x, y)$ is homogeneous of degree n; otherwise $f(x, y)$ is not a homogeneous function.

Note 2. If u is a homogeneous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogeneous functions of x and y each being of degree n − 1.

Let $u = x^n f(y/x)$.

[$\because$ u is a homogeneous function of x and y of degree n]

Then $\dfrac{\partial u}{\partial x} = nx^{n-1} f(y/x) + x^n \{f'(y/x)\} . (-y/x^2)$

$= x^{n-1} [n f(y/x) - (y/x) f'(y/x)]$

$= x^{n-1}$. [some function of y/x]

= a homogeneous function of x and y of degree $(n-1)$.

Similarly, $\dfrac{\partial u}{\partial y} = x^n \{f'(y/x)\} \cdot \dfrac{1}{x} = x^{n-1} f'(y/x)$

$= x^{n-1}$. (some function of y/x)

= a homogeneous function of x and y of degree $(n-1)$.

**§ 3. Euler's Theorem on homogeneous functions.

If u is a homogeneous function of x and y of degree n, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

(Meerut 1982, 84, 86, 87S, 88, 88S, 89, 92, 96 BP; Agra 81; Gorakhpur 86, 88; Kanpur 78; Lucknow 79; Alld. 82, 78)

Proof. Since u is a homogeneous function of x and y of degree n, therefore u may be put in the form

$$u = x^n f(y/x). \quad \text{...(1)}$$

Differentiating (1) partially w.r.t. 'x', we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} [x^n f(y/x)]$$

$$= [f(y/x)] nx^{n-1} + x^n [f'(y/x)] (-y/x^2).$$

$$\therefore \quad x \frac{\partial u}{\partial x} = nx^n f(y/x) - x^{n-1} y . f'(y/x). \quad \text{...(2)}$$

Again differentiating (1) partially w.r.t. 'y', we have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} [x^n f(y/x)] = x^n [f'(y/x)] \cdot \frac{1}{x} = x^{n-1} f'(y/x).$$

$$\therefore \quad y \frac{\partial u}{\partial y} = y . x^{n-1} f'(y/x). \quad \text{...(3)}$$

Adding (2) and (3), we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f(y/x)=n\,u. \qquad \text{[from (1)]}$$

Hence $\quad \mathbf{x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=n\,u.}$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. *Thus if $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function of $x_1, x_2, \ldots, x_n$ of degree n, then*

$$\mathbf{x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\ldots+x_n\frac{\partial f}{\partial x_n}=nf.}$$

Solved Examples

Ex. 22. *Verify Euler's theorem in the following cases :*

(i) $u=x^4-3x^3y+5x^2y^2+4xy^3-2y^4$.

(ii) $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$, **(Meerut 1989P)**

(iii) $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$,

(iv) $u=axy+byz+czx$, **(Meerut 1987)**

(v) $u=x^n \log(y/x)$, **(Meerut 1989P)**

(vi) $u=1/\sqrt{(x^2+y^2)}$. **(Meerut 1989)**

Sol. (i) We have $u=x^4-3x^3y+5x^2y^2+4xy^3-2y^4$. Obviously u is a homogeneous function of x and y of degree 4. So by Euler's theorem, we must have $x(\partial u/\partial x)+y(\partial u/\partial y)=4u$. Let us verify it.

We have

$$(\partial u/\partial x)=4x^3-9x^2y+10xy^2+4y^3,$$

and

$$(\partial u/\partial y)=-3x^3+10x^2y+12xy^2-8y^3.$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x(4x^3-9x^2y+10xy^2+4y^3)+y(-3x^3+10x^2y+12xy^2-8y^3)$$

$$=4(x^4-3x^3y+5x^2y^2+4xy^3-2y^4)$$

$=4u$. This verifies Euler's theorem.

(ii) We have $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$ which is obviously a homogeneous function of x and y of degree $4-3$ *i.e.*, 1. Note that each term in the numerator is of degree 4 while each term in the denominator is of degree 3

In order to verify Euler's theorem we are to show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\,.\,u=u.$$

Now $\quad \log u=\log x+\log(x^3-y^3)-\log(x^3+y^3). \qquad \ldots(1)$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{3x^2}{x^3-y^3}-\frac{3x^2}{x^3+y^3} \qquad \ldots(2)$$

and

$$\frac{1}{u}\frac{\partial u}{\partial y}=0-\frac{3y^2}{x^3-y^3}-\frac{3y^2}{x^3+y^3}. \qquad \ldots(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$$

$$=1+3-3=1.$$

$\therefore\quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u.$ This verifies Euler's theorem.

(iii) Here u is a homogeneous function of x and y of degree $\frac{1}{4}-\frac{1}{5}$ *i.e.*, $\frac{1}{20}$. So by Euler's theorem we must have

$$x(\partial u/\partial x)+y(\partial u/\partial y)=\tfrac{1}{20}u.$$

Let us verify it. We have

$$\log u=\log(x^{1/4}+y^{1/4})-\log(x^{1/5}+y^{1/5}).$$

$$\therefore\quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{\frac{1}{4}x^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{\frac{1}{5}x^{-4/5}}{x^{1/5}+y^{1/5}}$$

and

$$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{\frac{1}{4}y^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{\frac{1}{5}y^{-4/5}}{x^{1/5}+y^{1/5}}.$$

$$\therefore\quad \frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{1}{4}\frac{x^{1/4}+y^{1/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{x^{1/5}+y^{1/5}}{x^{1/5}+y^{1/5}}=\frac{1}{4}-\frac{1}{5}=\frac{1}{20}.$$

$\therefore\quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{20}u.$ This verifies Euler's theorem.

(iv) We have $u=axy+byz+czx$, which is a homogeneous function of x, y and z of degree 2. So in order to verify Euler's theorem, we must show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=2u.$

Now $\dfrac{\partial u}{\partial x}=ay+cz$, $\dfrac{\partial u}{\partial y}=ax+bz$, and $\dfrac{\partial u}{\partial z}=by+cx$.

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x(ay+cz)+y(ax+bz)+z(by+cx)$$

$=2(axy+byz+czx)=2u$. This verifies Euler's theorem.

(v) Here u is a homogeneous function of x and y of degree n. So by Euler's theorem we must have

$$x(\partial u/\partial x)+y(\partial u/\partial y)=nu.$$

Now do the verification yourself.

(vi) Here $u=\dfrac{1}{\sqrt{(x^2+y^2)}}=\dfrac{1}{x\sqrt{[1+(y/x)^2]}}=x^{-1}\cdot\dfrac{1}{\sqrt{[1+(y/x)^2]}}$ is a homogeneous function of x and y of degree -1. Now proceed yourself.

Ex. 23. *If* $u = x^2y^2/(x+y)$, *show that*
$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = 3u.$$

Sol. We have $u = \dfrac{x^2y^2}{x+y} = \dfrac{x^3\,(y/x)^2}{[1+(y/x)]} = x^3 f(y/x)$, say.

Thus u is a homogeneous function of x and y of degree 3.

Therefore by Euler's theorem, we have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u.$

****Ex. 24.** *If* $u = \sin^{-1}\{(x^2+y^2)/(x+y)\}$, *show that*
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

(Meerut 1981, 88, 98; Lucknow 83; Delhi 80, 74; Gorakhpur 89; Agra 82, 81; Kanpur 80; Magadh 74; Jodhpur 76; Alld. 78; U.P. P.C.S. 90)

Sol. We have given one method for solving this question in Ex. 6 page 93. Here we shall give another method using Euler's theorem.

We have $\sin u = (x^2+y^2)/(x+y) = v$, say.

Obviously $v = (x^2+y^2)/(x+y)$ is a homogeneous function of x and y of degree $2-1$ *i.e.*, 1. Therefore by Euler's theorem, we have
$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1\,.\,v = v. \qquad ...(1)$$

Now $\quad v = \sin u.$

Therefore $\dfrac{\partial v}{\partial x} = \cos u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y} = \cos u\dfrac{\partial u}{\partial y}$.

Putting these values in (1), we get
$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = v \quad \text{or} \quad \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{v}{\cos u} = \frac{\sin u}{\cos u} = \tan u, \qquad [\because v = \sin u.]$$

Ex. 24. (a). *If* $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *show that*
$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \tfrac{1}{2}\tan u.$$

(Rohilkhand 1990; Agra 82; Kanpur 88)

Sol. We have
$$\sin u = (x+y)/(\sqrt{x}+\sqrt{y}) = v, \text{ say}.$$

Then v is a homogeneous function of x and y of degree $(1-\frac{1}{2})$ *i.e.* $\frac{1}{2}$. Applying Euler's theorem for v, we have
$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = \tfrac{1}{2}v$$

or $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \tfrac{1}{2}\sin u, \qquad [\because v = \sin u]$$

or $$x\cos u\,(\partial u/\partial x) + y\cos u\,(\partial u/\partial y) = \tfrac{1}{2}\sin u$$

or $$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \tfrac{1}{2}\tan u.$$

Ex. 24 (b). *If* $u = \cos^{-1}\{(x+y)/(\sqrt{x}+\sqrt{y})\}$, *show that* $x\,(\partial u/\partial x) + y\,(\partial u/\partial y) + \frac{1}{2}\cot u = 0.$

Sol. Proceed exactly as in Ex. 24 (a).

Ex. 24 (c). *If* $u = \log\dfrac{x^3+y^3}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

(Meerut 1990 P)

Sol. We have $\quad e^u = \dfrac{x^3+y^3}{x+y} = v$, say.

Obviously $v = (x^3+y^3)/(x+y)$ is a homogeneous function of x and y of degree $3-1$ *i.e.*, 2. Therefore by Euler's theorem, we have
$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2\,.\,v = 2v \qquad ...(1)$$

Now $v = e^u$. $\quad\therefore\quad \dfrac{\partial v}{\partial x} = e^u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y} = e^u\dfrac{\partial u}{\partial y}$.

Putting these values in (1), we get
$$xe^u\frac{\partial u}{\partial x} + y\,e^u\frac{\partial u}{\partial y} = 2e^u$$

or $$e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2e^u \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2.$$

****Ex. 25.** *If* $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, *show that*
$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \sin 2u.$$

(Meerut 1982, 83, 88 P, 90 S; Delhi 83, 81; Luck. 80; Gorakhpur 89; Rohilkhand 8!)

Sol. We have $\tan u = (x^3+y^3)/(x+y) = v$, say. Then v is a homogeneous function of x and y of degree $3-1$ *i.e.*, 2. Therefore by Euler's theorem, we have
$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = 2v. \qquad ...(1)$$

Now $\quad v = \tan u.$

$\therefore\quad \dfrac{\partial v}{\partial x} = \sec^2 u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y} = \sec^2 u\dfrac{\partial u}{\partial y}$.

Putting these values in (1), we get
$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2v}{\sec^2 u} = \frac{2\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$$

Ex. 25 (a). *If* $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, *show that*
$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \sin 2u.$ **(Kanpur 1985; Meerut 91S, 93)**

Sol. Proceed exacly as in Ex. 25.

Ex. 25 (b). *If* $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, *then prove that*

$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \frac{1}{2}\sin 2u.$ **(Gorakhpur 1982; Allahabad 82)**

Sol. Proceed exactly as in Ex. 25.

Ex. 26 (a). *If* $u = \sin^{-1}\{(\sqrt{x}-\sqrt{y})/(\sqrt{x}+\sqrt{y})\}$, *show that*

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

(Rohilkhand 1987; Alld. 77; Kurukshetra 83)

Sol. We have $\sin u = (\sqrt{x}-\sqrt{y})/(\sqrt{x}+\sqrt{y}) = v$, say. Then v is a homogeneous function of x and y of degree $\frac{1}{2}-\frac{1}{2}$ *i.e.*, 0.

Therefore by Euler's theorem, we have $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 0 \cdot v = 0.$ ...(1)

Now $v = \sin u.$

$$\therefore \quad \frac{\partial v}{\partial x} = \cos u\,\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = \cos u\,\frac{\partial u}{\partial y}.$$

Putting these values in (1), we get

$$x\cos u\,\frac{\partial u}{\partial x} + y\cos u\,\frac{\partial u}{\partial y} = 0 \quad\text{or}\quad \cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) = 0$$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$, since $\cos u \neq 0$. Hence $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}.$

Ex. 26 (b). *Use Euler's theorem to show that if*

$u = \tan^{-1}(y/x)$, *then*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

(Meerut 1989 S)

***Ex. 27.** *If u be a homogeneous function of x and y of degree n, show that*

$$x\,(\partial^2 u/\partial x^2) + y\,(\partial^2 u/\partial x\,\partial y) = (n-1)\,(\partial u/\partial x),$$

and $$x\,(\partial^2 u/\partial x\,\partial y) + y\,(\partial^2 u/\partial y^2) = (n-1)\,(\partial u/\partial y).$$

Hence deduce that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)\,u.$$

(Allahabad 1981; Vikram 83; Agra 84; Gorakhpur 83; I.C.S. 95; Jiwaji 85; Kurukshetra 88)

Sol. By Euler's theorem, we have

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = nu \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$

or $$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = (n-1)\frac{\partial u}{\partial x} \qquad ...(2)$$

Proved.

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}. \qquad ...(3)$$

Proved.

Now multiplying (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right] = (n-1)\,nu = n(n-1)\,u.$$

Ex. 28. *If* $u = x\phi(y/x) + \psi(y/x)$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

(Rohilkhand 1986; Agra 85)

Sol. Let $\quad u = z_1 + z_2$, where $z_1 = x\,\phi(y/x)$ and $z_2 = \psi(y/x)$.

Obviously z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree zero. Now

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(z_1+z_2) + y\frac{\partial}{\partial y}(z_1+z_2)$$

$$= \left(x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right) + \left(x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right) = 1\cdot z_1 + 0\cdot z_2.$$

(by Euler's theorem).

Thus $\quad x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = z_1 \qquad ...(1)$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial z_1}{\partial x}, \qquad ...(2)$$

and $$x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial z_1}{\partial y}. \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + z_1 = 1\cdot z_1,$$

[$\because x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = z_1$ by (1), and $x\,(\partial z_1/\partial x) + y\,(\partial z_1/\partial y) = 1\cdot z_1$ by Euler's theorem]

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

Ex. 29. *If* $z = xy\,f(y/x)$, *show that* $x\,(\partial z/\partial x) + y\,(\partial z/\partial y) = 2z$. *Show also that if z is a constant,*

$$\frac{f'(y/x)}{f(y/x)} = \frac{x\,\{y + x\,(dy/dx)\}}{y\,\{y - x\,(dy/dx)\}}.$$

Sol. We have, $z = x^2 . (y/x) f(y/x)$, so that z is a homogeneous function of x and y of degree 2.

Hence by Euler's theorem, we have

$$x(\partial z/\partial x) + y(\partial z/\partial y) = 2z.$$

If z be a constant, then differentiating $z = xy f(y/x)$ logarithmically, w.r.t. x, we get

$$0 = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + \frac{f'(y/x)}{f(y/x)} \cdot \frac{x(dy/dx) - y}{x^2}$$

$$= \frac{y + x(dy/dx)}{xy} + \frac{f'(y/x)}{f(y/x)} \cdot \frac{x(dy/dx) - y}{x^2}.$$

Hence $$\frac{f'(y/x)}{f(y/x)} = \frac{x\{y + x(dy/dx)\}}{y\{y - x(dy/dx)\}}.$$

Ex. 30. *If* $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, *find the value of*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$ **(U.P. P.C.S. 1993)**

Sol. First proceed as in Ex. 25 on page 107. Thus we get

$$x(\partial u/\partial x) + y(\partial u/\partial y) = \sin 2u \qquad \text{...(1)}$$

Now differentiating (1) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial x}, \qquad \text{...(2)}$$

and $$x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\frac{\partial u}{\partial y}. \qquad \text{...(3)}$$

Multiplying (2) by x, (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \sin 2u = 2\cos 2u \sin 2u, \text{ [by (1)]}$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$$

Ex. 31. *If* $f(x, y, z)$ *is a homogeneous function of the* n^{th} *degree in* x, y, z, *prove that*

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2yz\frac{\partial^2 f}{\partial y\,\partial z} + 2zx\frac{\partial^2 f}{\partial z\,\partial x} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} = n(n-1)f(x, y, z).$$

Sol. Here $f(x, y, z)$ is a homogeneous function of the n^{th} degree in x, y, z. Therefore $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ are homogeneous functions of the $(n-1)$th degree in x, y, z. So using Euler's theorem for $\partial f/\partial x$, we have

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) + z\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = (n-1)\frac{\partial f}{\partial x}$$

or $$x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y\partial x} + z\frac{\partial^2 f}{\partial z\partial x} = (n-1)\frac{\partial f}{\partial x}. \qquad \text{...(1)}$$

Similarly, $$x\frac{\partial^2 f}{\partial x\partial y} + y\frac{\partial^2 f}{\partial y^2} + z\frac{\partial^2 f}{\partial y\partial z} = (n-1)\frac{\partial f}{\partial y} \qquad \text{...(2)}$$

and $$x\frac{\partial^2 f}{\partial z\partial x} + y\frac{\partial^2 f}{\partial y\partial z} + z\frac{\partial^2 f}{\partial z^2} = (n-1)\frac{\partial f}{\partial z} \qquad \text{...(3)}$$

Multiplying (1) by x, (2) by y and (3) by z and adding, we get

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2yz\frac{\partial^2 f}{\partial y\,\partial z} + 2zx\frac{\partial^2 f}{\partial z\,\partial x} + 2xy\frac{\partial^2 f}{\partial x\,\partial y}$$

$$= (n-1)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right)$$

$$= (n-1)\,n f(x, y, z) = n(n-1) f(x, y, z).$$

§ 4. Total Derivatives.

If $u = f(x, y)$, *where* $x = \phi_1(t)$ *and* $y = \phi_2(t)$, *then*

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dt}}.$$

Here du/dt is called the **total differential coefficient** of u with respect to t while $\partial u/\partial x$ and $\partial u/\partial y$ are partial derivatives of u.

Proof. Let t be given a small increment δt, and let the corresponding changes in u, x and y be δu, δx and δy respectively. We have then

$$u = f(x, y), \qquad \text{...(1)}$$

and $$u + \delta u = f(x + \delta x, y + \delta y). \qquad \text{...(2)}$$

$\therefore$ $$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]. \qquad \textbf{[Note]}$$

$\therefore$ $$\frac{\delta u}{\delta t} = \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t}\right\} + \left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta t}\right\}$$

$$= \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\right\} \cdot \frac{\delta x}{\delta t} + \left\{\frac{f(x, y + \delta y) - f(x, y)}{\delta y}\right\} \cdot \frac{\delta y}{\delta t}. \qquad \text{...(3)}$$

Let $\delta t \to 0$ so that $\delta x \to 0$ and $\delta y \to 0$.

Now $$\lim_{\delta t \to 0}\frac{\delta u}{\delta t} = \frac{du}{dt}, \quad \lim_{\delta t \to 0}\frac{\delta x}{\delta t} = \frac{dx}{dt} \quad \text{and} \quad \lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Also $$\lim_{\delta y \to 0}\frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y},$$

and $$\lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}.$$

Since δx and δy tend to zero with δt and the functions involved are all supposed to be continuous, therefore the limit (3) becomes

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dt}}.$$

In the same way if $u = f(x, y, z)$, where x, y, z are all functions of some variable t, then

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{z}} \cdot \frac{\mathbf{dz}}{\mathbf{dt}}.$$

This result can be extended to any number of variables.

Cor. 1. *If u be a function of x and y, where y is a function of x, then*

$$\frac{\mathbf{du}}{\mathbf{dx}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dx}}.$$

This result follows immediately by taking $t = x$ in the formula of § 4.

Cor. 2. *If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then*

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{t}_1} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{t}_1} \quad \textit{and} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{t}_2} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{t}_2}.$$

Cor. 3. *If x and y are connected by an equation of the form $f(x, y) = 0$, then*

$$\frac{\mathbf{dy}}{\mathbf{dx}} = -\frac{\partial \mathbf{f}/\partial \mathbf{x}}{\partial \mathbf{f}/\partial \mathbf{y}}.$$

Since $f(x, y) = 0$, therefore by cor. 1, we get

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx},$$

from which the required result follows.

Ex. 32. (a). *If $(\tan x)^y + y^{\cot x} = a$, find dy/dx.*

Sol. Let $f(x, y) = (\tan x)^y + y^{\cot x} - a$. Then

$(\partial f/\partial x) = y(\tan x)^{y-1} \cdot \sec^2 x + y^{\cot x} \cdot \log y \cdot (-\operatorname{cosec}^2 x)$,

and $(\partial f/\partial y) = (\tan x)^y \log \tan x + (\cot x) \cdot y^{\cot x - 1}$

Now we are given that $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y(\tan x)^{y-1}\sec^2 x - y^{\cot x} \cdot \log y \cdot \operatorname{cosec}^2 x}{(\tan x)^y \log \tan x + \cot x \cdot y^{\cot x - 1}}.$$

Ex. 32. (b). *If $x^3y^3 + 3x \sin y = e^y$, find dy/dx.*

Sol. Let $f(x, y) = x^3y^3 + 3x \sin y - e^y$. Then we have $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(3x^2y^3 + 3\sin y)}{3x^3y^2 + 3x\cos y - e^y}.$$

Ex. 32. (c). *If $x^y + y^x = a^b$, find dy/dx.* **(Meerut 1981, 82)**

Sol. Let $f(x, y) = x^y + y^x - a^b$. Then we have $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

Ex. 32 (d). *If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, prove that*

$$\frac{dy}{dx} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

(Meerut 1987)

Sol. It is given that $\dfrac{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}{x-y} = a.$

Let $f(x, y) = \dfrac{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}{x-y} - a$. Then $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{\dfrac{\{\frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)\}(x-y) - 1 \cdot \{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\}}{(x-y)^2}}{\dfrac{\{\frac{1}{2}(1-y^2)^{-1/2} \cdot (-2y)\}(x-y) - (-1) \cdot \{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\}}{(x-y)^2}}$$

$$= -\frac{\dfrac{-x(x-y)}{\sqrt{(1-x^2)}} - \sqrt{(1-x^2)} - \sqrt{(1-y^2)}}{\dfrac{-y(x-y)}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)} + \sqrt{(1-y^2)}}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{-x^2 + xy - (1-x^2) - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-yx + y^2 + \sqrt{(1-x^2)}\sqrt{(1-y^2)} + 1 - y^2}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{xy - 1 - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-xy + 1 - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}$$

$$= \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

Ex. 33. *If $u = \log\{(x^2+y^2)/xy\}$, find du.*

Sol. We have $\quad u = \log(x^2+y^2) - \log x - \log y.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x} = \frac{2x^2 - x^2 - y^2}{x(x^2+y^2)} = \frac{x^2-y^2}{x(x^2+y^2)},$$

and $$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y} = \frac{2y^2 - x^2 - y^2}{y(x^2+y^2)} = \frac{(y^2-x^2)}{y(x^2+y^2)}.$$

Now $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{(x^2-y^2)}{x(x^2+y^2)}dx + \frac{(y^2-x^2)}{y(x^2+y^2)}dy$$

$$= \frac{x^2-y^2}{xy(x^2+y^2)}(y\,dx - x\,dy).$$

Ex. 34. *If $u = \sin(x^2+y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find du/dx.*

Sol. We have $$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}. \qquad \text{...(1)}$$

Now $\quad u = \sin(x^2+y^2).$

$\therefore \quad \dfrac{\partial u}{\partial x} = 2x \cos (x^2 + y^2)$ and $\dfrac{\partial u}{\partial y} = 2y \cos (x^2 + y^2)$.

Since $a^2x^2 + b^2y^2 = c^2$,

therefore $\quad 2a^2x + 2b^2y\,(dy/dx) = 0$ or $dy/dx = -(a^2x)/(b^2y)$.

$\therefore$ from (1), $\dfrac{du}{dx} = 2x \cos (x^2 + y^2) - [2y \cos (x^2 + y^2)]\,[(a^2x)/(b^2y)]$

$$= [2x \cos (x^2 + y^2)]\,[1 - (a^2/b^2)].$$

Ex. 35. (a) *If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that*

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$ **(Meerut 1983, 87; Alld. 79, 78)**

(b) *If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at point of contact*

$$\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial x}.$$

Sol. (a) From $f(x, y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$. ...(1)

From $\phi(y, z) = 0$, we have $\dfrac{dz}{dy} = -\dfrac{\partial \phi/\partial y}{\partial \phi/\partial z}$. ...(2)

Multiplying the respective sides of (1) and (2), we have

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}\right) \Big/ \left(\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}\right)$$

or

$$\frac{dz}{dx} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

(b) The curves will touch if at their common point they have the same value of dy/dx. Now for the curve $f(x, y) = 0$, we have

$\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$ and for the curve $\phi(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{\partial \phi/\partial x}{\partial \phi/\partial y}.$$

$\therefore$ the two curves touch if at their common point, we have

$$\frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-\partial \phi/\partial x}{\partial \phi/\partial y} \quad i.e., \quad \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial x}.$$

Ex. 36. Formula for the second differential coefficient of an implicit function. *If $f(x, y) = 0$ be an implicit function of x and y, find a formula for d^2y/dx^2.*

Sol. We have $\quad \dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{p}{q}$, ...(1)

where $\partial f/\partial x$ and $\partial f/\partial y$ have been denoted by p and q respectively. Also using the notations

$r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$ and $t = \dfrac{\partial^2 f}{\partial y^2}$, we have from (1),

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{p}{q}\right) = -\frac{q\,(dp/dx) - p\,(dq/dx)}{q^2} \quad ...(2)$$

But $\dfrac{dp}{dx} = \dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y} \cdot \dfrac{dy}{dx} = r + s\left(-\dfrac{p}{q}\right) = \dfrac{qr - sp}{q}$,

and $\dfrac{dq}{dx} = \dfrac{\partial q}{\partial x} + \dfrac{\partial q}{\partial y} \cdot \dfrac{dy}{dx} = s + t\left(-\dfrac{p}{q}\right) = \dfrac{qs - pt}{q}$.

Substituting in (2) the values of dp/dx and dq/dx, we get

$$\frac{d^2y}{dx^2} = -\left[q\left(\frac{qr - ps}{q}\right) - p\left(\frac{qs - pt}{q}\right)\right] \cdot \left[\frac{1}{q^2}\right]$$

$$= -(q^2r - 2pqs + p^2t)/q^3.$$ **(Remember)**

Ex. 37. *Prove that $\dfrac{d^2y}{dx^2} + \dfrac{2a^2x^2}{y^5} = 0$, where $y^3 - 3ax^2 + x^3 = 0$.* **(Delhi 1983)**

Sol. Let $\quad f(x, y) \equiv y^3 - 3ax^2 + x^3 = 0$.

Then $\quad p = \partial f/\partial x = -6ax + 3x^2, q = \partial f/\partial y = 3y^2$,

$r = \partial^2 f/\partial x^2 = -6a + 6x, s = \partial^2 f/\partial x \partial y = 0, t = \partial^2 f/\partial y^2 = 6y$.

Now $\quad \dfrac{d^2y}{dx^2} = -\dfrac{q^2r - 2pqs + p^2t}{q^3}$, refer Ex. 36

$$= -\frac{(-6a + 6x)(3y^2)^2 + 6y(-6ax + 3x^2)^2}{(3y^2)^3}$$

$$= -\frac{2(-a + x)y^3 + 2(-2ax + x^2)^2}{y^5}$$

$$= -\frac{2x(y^3 + x^3 - 3ax^2) - 2ay^3 + 8a^2x^2 - 2ax^3}{y^5}$$ **(Note)**

$$= -\frac{-2a(y^3 + x^3) + 8a^2x^2}{y^5}, \quad [\because y^3 + x^3 - 3ax^2 = 0]$$

$$= -\frac{-2a(3ax^2) + 8a^2x^2}{y^5} = -\frac{2a^2x^2}{y^5}.$$

Hence $\dfrac{d^2y}{dx^2} + \dfrac{2a^2x^2}{y^5} = 0$.

Ex. 38 (a). *If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, find d^2y/dx^2.*

(b) *If $ax^2 + 2hxy + by^2 = 1$, find d^2y/dx^2.* **(Delhi 1983)**

Sol. (a) Let $\quad F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Then with usual notation *i.e.*, $\partial F/\partial x = p$, $\partial F/\partial y = q$, etc., we have

$p = 2(ax + hy + g), q = 2(hx + by + f)$,

$r = 2a, s = 2h$ and $t = 2b$. $\quad$ Then from Ex. 36, we get

$$\frac{d^2y}{dx^2} = -8\,[(hx + by + f)^2 a - 2(ax + hy + g)(hx + by + f)h$$

$$+ b(ax + hy + g)^2]/\{8(hx + by + f)^3\}$$

$$= \frac{-[(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh]}{(hx+by+f)^3}$$

$$= \frac{-[(ab-h^2)(-c)+af^2+bg^2-2fgh]}{(hx+by+f)^3}$$

$$= [abc+2fgh-af^2-bg^2-ch^2]/(hx+by+f)^3.$$

(b) Proceed exactly as in part (a).

Ex. 39. *If $u = f(y-z, z-x, x-y)$, prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

(Meerut 1983 S, 77; Lucknow 79)

Ex. 40. *If $z = f(u, v)$, where $u = x^2 - 2xy - y^2$ and $v = y$, show that*

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=(x-y)\frac{\partial z}{\partial v}.$$

(Agra 1983)

Ex. 41. *If z be a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that*

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Ex. 42. *Transform the equation $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.* **(Allahabad 1980; Gorakhpur 82; Kanpur 86)**

Ex. 43. *Transform $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ into polars and show that $u = (Ar^n + Br^{-n})\sin n\theta$ satisfies the above equation.*

For complete solutions of exercises 39 to 43 refer chapter 9 on change of independent variables.

□