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2016 (IFOS)

Q1. Laurent Series

$$f(z) = \frac{1}{1-z^2} \quad \text{centre } z=1$$

$$f(z) = \frac{1}{1-z^2} = \frac{1}{2}\left[\frac{1}{1-z} + \frac{1}{1+z}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z-1}\right]$$

Let $u = z-1$

$$f(z) = \frac{1}{2}\left[\frac{1}{u+2} - \frac{1}{u}\right] \qquad \text{If } |u| < 2$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(1+\frac{u}{2}\right)^{-1} - \frac{1}{u}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8} - \cdots\right) - \frac{1}{u}\right]$$

$$= \frac{-1}{2u} + \frac{1}{4}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+\cdots\right)$$

$$\boxed{f(z) = \frac{-1}{2(z-1)} + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n}$$

↳ Laurent Series when $|z-1| < 2$

If $|u| > 2$

$$f(z) = \frac{1}{2}\left[\frac{1}{u}\left(1+\frac{2}{u}\right)^{-1} - \frac{1}{u}\right]$$

$$= \frac{1}{2u}\left[\left(1+\frac{2}{u}\right)^{-1} - 1\right]$$

$$= \frac{1}{2u}\left[1 - \frac{2}{u} + \frac{4}{u^2} - \frac{8}{u^3} + \cdots - 1\right]$$

$\Rightarrow -\frac{1}{u^2} + \frac{2}{u^3} - \frac{4}{u^4} + \frac{8}{u^5} - \cdots$

$$\boxed{f(z) = \frac{1}{2(z-1)} \left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z-1}\right)^n - 1 \right]}$$

↳ Laurent series when $|z-1| > 2$

Q. $u = e^{-x}[(x^2-y^2)\cos y + 2xy \sin y]$

$\Rightarrow \frac{\partial u}{\partial x} = e^{-x} 2x \cos y - e^{-x}(x^2-y^2)\cos y - e^{-x} 2xy \sin y + 2e^{-x} y \sin y$

$\Rightarrow \frac{\partial u}{\partial y} = e^{-x}(-x^2 \sin y + y^2 \sin y + 2xy \cos y + 2x \sin y)$

$\Rightarrow \left(\frac{\partial u}{\partial x}\right)_{(z,0)} = 2ze^{-z} - e^{-z}z^2$

$\Rightarrow \left(\frac{\partial u}{\partial y}\right)_{(z,0)} = 0$

$$f'(z) = 2ze^{-z} - e^{-z}z^2$$

$$f(z) = \int (2z - z^2)e^{-z}\,dz$$

$\Rightarrow -2ze^{-z} - 2e^{-z} + z^2e^{-z} + 2ze^{-z} + 2e^{-z}$

$\Rightarrow \underline{z^2e^{-z} + c}$

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Q3. $\int_0^{\pi} \frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}$ contour integration.

$\Rightarrow$ $I = \int_0^{\pi} \frac{d\theta}{(1+\frac{1}{2}\cos\theta)^2}$

$= \frac{1}{2}\int_0^{2\pi} \frac{4\,d\theta}{(2+\cos\theta)^2}$

$= \int_0^{2\pi} \frac{2\,d\theta}{(2+\cos\theta)^2}$

Let the contour 'c' be the unit circle $|z| = 1$ with centre at the origin.

$z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$d\theta = \frac{dz}{iz}$

$I = \int_0^{2\pi} \frac{2\,dz}{iz\left(2+\frac{z^2+1}{2z}\right)^2}$

$= \frac{1}{i}\int_0^{2\pi} \frac{8z}{(z^2+4z+1)^2}\,dz$

$= \frac{8}{i}\int_c \frac{z}{(z^2+4z+1)^2}\,dz$

$= \frac{8}{i}\int_c f(z)\,dz$. $f(z) = \frac{z}{(z^2+4z+1)^2}$

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Now the poles are given by.

$$(z^2+4z+1)^2 = 0$$

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm \sqrt{12}}{2}$$

$$= -2 \pm \sqrt{3}$$

Let $\alpha = -2 + \sqrt{3}$

$\beta = -2 - \sqrt{3}$

$|\beta| > 1$, $|\alpha\beta| = 1$

$\Rightarrow |\alpha| < 1$

pole inside c is at $z = \alpha$ of order 2.

$$\therefore \int_c f(z)\,dz = \int \frac{z\,dz}{(z^2+4z+1)^2}$$

$$= 2\pi i \text{ (Residue at } z = \alpha)$$

$$\lim_{z \to \alpha} \frac{d}{dz}(z-\alpha)^2 \frac{z}{(z^2+4z+1)^2} = \lim_{z \to \alpha} \frac{d}{dz} \frac{z}{(z-\beta)^2}$$

$$\lim_{z \to \alpha} \frac{-(\beta+z)}{(z-\beta)^3}$$

$$\Rightarrow \frac{-(\alpha+\beta)}{(\alpha-\beta)^3} \Rightarrow \frac{-(-4)}{(2\sqrt{3})^3} = \frac{4}{8 \cdot 4(3)} = \frac{1}{3}$$

$$\int_c f(z)\,dz = 2\pi i\left(\frac{1}{3}\right) = \frac{2\pi i}{3}$$

$$\int_0^{2\pi} \frac{2\,d\theta}{(2+\cos\theta)^2} = \frac{8}{i}\left(\frac{2\pi i}{3}\right) = \frac{16\pi}{3}.$$