IFoS - 2013

Q) $\lim_{x \to 0} \dfrac{e^{ax} - e^{bx} + \tan x}{ax}$

Soln : Applying L - Hospital :

$$= \lim_{x \to 0} \frac{a e^{ax} - b e^{bx} + \sec^2 x}{1}$$

$$= a - b + 1$$

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8) Prove that $f(x) = x^2$ is uniformly continuous on $(0,1)$ but not in $\mathbb{R}$

Soln: $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$

$= |x_1 - x_2| \, |x_1 + x_2|$

Then,

$|f(x_1) - f(x_2)| < \epsilon$

for $|x_1 - x_2| < \delta = \frac{\epsilon}{2}$

$\because |x_1 + x_2| < 2$ on $(0,1)$

So, $f(x)$ is uniformly continuous on $(0,1)$.

on the domain $\mathbb{R}$:

$a_n = \sqrt{n}$

$b_n = \sqrt{n+1}$

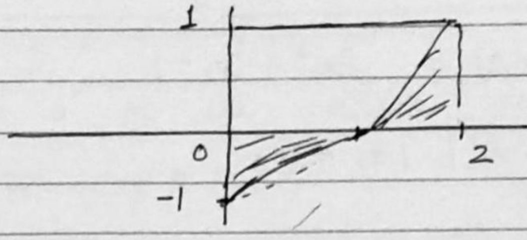
Then $|\langle a_n - b_n \rangle| \to 0$ as $n \to \infty$

But $|f(a_n) - f(b_n)| = 1 \not\to 0$ as $n \to \infty$

So, $f(x)$ is not uniformly continuous.

Q) Area b/w x-axis and $y = (x-1)^3$ from $x = 0$ to $x = 2$

Soln:



$$\text{Area} = \int_0^2 |y| \, dx$$

$$= \int_0^1 (1-x)^3 \, dx + \int_1^2 (x-1)^3 \, dx$$

$$= \frac{(1-x)^4}{-4}\Bigg|_0^1 + \frac{(x-1)^4}{4}\Bigg|_1^2$$

$$= \left(\frac{-1}{4}\right)(0-1) + \frac{1}{4}(1) = \frac{1}{2}$$

IFoS - 2012

(a) $f(x) = \begin{cases} 1 ; & x \in Q^c \\ -1 ; & x \in Q \end{cases}$

Show that $f(x)$ is discontinuous at every point in $\mathbb{R}$.

Soln: Take $a \in \mathbb{R}$

and $\langle a_n \rangle \to a ;\quad a_n \in Q$

$\langle b_n \rangle \to a ;\quad b_n \in Q^c$

$\Rightarrow f(a_n) = 1$

and $f(b_n) = 1$

$\Rightarrow \lim\limits_{x \to a} f(x)$ doesn't exist for $a \in \mathbb{R}$

$\Rightarrow f(x)$ is discontinuous $\forall \; \mathbb{R}$.

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Q) Show that

$$u = x^2 + y^2 + z^2$$
$$v = x + y + z$$
$$w = yz + zx + xy$$

are not independent of one another.

Soln: $$\frac{J(u, v, w)}{(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

So, u, v, w are dependent

$$\cancel{\frac{\partial u}{\partial x^2}} = 2\left[\cancel{\tan^{-1} y/x} + \quad x\right.$$

Q) $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1} x/y$

Show that :

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

Soln : $\frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} + x^2 \cdot \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)$

$$- y^2 \frac{y^2}{(x^2+y^2)} \cdot \frac{1}{y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} - \frac{x^2y + y^3}{(x^2+y^2)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} - \frac{y(x^2+y^2)}{(x^2+y^2)}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x\tan^{-1}\frac{y}{x} - y \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = x^2 \frac{x^2}{x^2+y^2} \times \frac{1}{x} - 2y\tan^{-1} x/y$$

$$- y^2 \frac{y^2}{(x^2+y^2)} \frac{(-x)}{y^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2y\tan^{-1}\frac{x}{y} + \frac{xy^2 + x^3}{x^2+y^2}$$

$$\Rightarrow \frac{\partial u}{\partial y} = -2y\tan^{-1}\frac{x}{y} + x \quad \text{— (ii)}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\tan^{-1}\frac{y}{x} + x \cdot \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} = 2x^2\tan^{-1}\frac{y}{x} - \frac{2x^3 y}{x^2+y^2} \quad - (A)$$

$$\frac{\partial^2 u}{\partial y^2} = -2\left[\tan^{-1}\frac{x}{y} + y\,\frac{y^2}{x^2+y^2}\left(-\frac{x}{y^2}\right)\right]$$

$$\Rightarrow y^2\frac{\partial^2 u}{\partial y^2} = -2y^2\tan^{-1}\frac{x}{y} + \frac{2xy^3}{x^2+y^2} \quad - (B)$$

$$\frac{\partial^2 u}{\partial x\partial y} = -1 + (2x)\frac{x^2}{(x^2+y^2)}\left(\frac{1}{x}\right)$$

$$\Rightarrow 2xy\frac{\partial^2 u}{\partial x\partial y} = -2xy + \frac{4x^3 y}{x^2+y^2}$$

$$\Rightarrow 2xy\frac{\partial^2 u}{\partial x\partial y} = \frac{-2x^3y - 2xy^3 + 4x^3 y}{x^2+y^2}$$

$$\Rightarrow 2xy\frac{\partial^2 u}{\partial x\partial y} = \frac{2xy(x^2-y^2)}{(x^2+y^2)} \quad - (C)$$

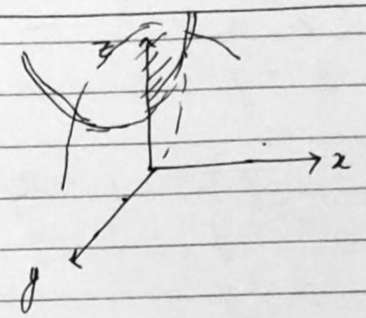
(A) + (B) + (C)

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y}$$

$$= 2x^2\tan^{-1}\frac{y}{x} - 2y^2\tan^{-1}\frac{x}{y} = 2u$$

Q) Volume bounded above by $z = 4 - y^2$
and bounded below by $z = x^2 + 3y^2$

Soln :



$$\text{Volume} = -\iint_D \left[x^2 + 3y^2 - (4 - y^2)\right] dx\, dy$$

where $D \equiv \quad x^2 + 3y^2 = 4 - y^2$

$$\Rightarrow x^2 + 4y^2 = 4$$

$$\Rightarrow \frac{x^2}{4} + y^2 = 1$$

So, $\text{Volume} = -\iint_D (x^2 + 4y^2 - 4)\, dx\, dy$

Putting $x = 2r\cos\theta$

$y = \cancel{2}\, r \sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = (2\cos\theta)\, r\cos\theta + (2r\sin\theta)\sin\theta = 2r$$

So, $\text{Volume} = -\int_0^{2\pi}\int_0^1 (4r^2\cos^2\theta + 4r^2\sin^2\theta - 4)\, 2r\, dr\, d\theta$

$$= -\int_0^{2\pi}\int_0^1 (4r^2 - 4)\, 2r\, dr\, d\theta$$

$$= -8 \times 2\pi \left[\frac{r^4}{4} - \frac{r^2}{2}\right]_0^1$$

$$= -16\pi \times -\frac{1}{4} = +4\pi$$

Q) $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$

Examine for uniform convergence. Also show that uniform convergence of $\sum u_n(x)$ is sufficient but not necessary for $S(x)$ to be continuous

Soln: $S_n(x) = \frac{x}{1+x^2} + \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$

$+ \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2} + \ldots$

$+ \frac{(n-1)x}{1+(n-1)^2x^2} - \frac{(n-2)x}{1+(n-2)^2x^2}$

$+ \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$

$= \frac{nx}{1+n^2x^2}$

Then $S(x) = \lim_{n\to\infty} S_n(x) = 0$

Define $M_n = \sup \left|\frac{nx}{1+n^2x^2}\right|$

Let $\phi(x) = \frac{nx}{1+n^2x^2}$

$\phi'(x) = \frac{(1+n^2x^2)n - 2n^3x^2}{(1+n^2x^2)^2} = 0$

$\Rightarrow n - n^3x^2 = 0 \Rightarrow x = \pm\frac{1}{n}$

for $x < \frac{1}{n}$; $\phi'(x) > 0$ & for $x > \frac{1}{n}$; $\phi'(x) < 0$.

So, $M_n = \left|\frac{n \cdot 1/n}{1+n^2\cdot 1/n^2}\right| = 1/2$

$\lim_{n\to\infty} M_n \neq 0 \Rightarrow \sum u_n x$ is not uniformly convergent.

But $S(x) = 0$ is continuous $\Rightarrow$ Uniform convergence is sufficient but not necessary condition.