ANALYTIC GEOMETRY

:IFoS-2013:

① (d) Find the surface generated by the straight line which intersects the lines $y = z = a$ and $x + 3z = a = y + z$ and is parallel to the plane $x + y = 0$

$\Rightarrow$ $L_1: \dfrac{x}{1} = \dfrac{y-a}{0} = \dfrac{z-a}{0}$ $\qquad L_2 = \dfrac{x-a}{3} = \dfrac{z}{-1} = \dfrac{y-a}{1}$

Any point on L_1 is (r_1, a, a)

Any point on L_2 is $(3r_2 + a, a + r_2, -r_2)$

Any line joining these points is

$$\frac{x - r_1}{3r_2 - r_1 + a} = \frac{y-a}{r_2} = \frac{z-a}{-(r_2+a)} \quad \text{—— ①}$$

This line is parallel to $y + x = 0$ plane.

$\therefore \cancel{3r_2 - r_1 + a - r_2} = 0$

$\cancel{r_1 = 2r_2}$

$3r_2 - r_1 + a + r_2 = 0 \quad \Rightarrow r_1 = 4r_2 + a$

$$\therefore ① \equiv \frac{x - (4r_2 + a)}{3r_2 - (4r_2 + a) + a} = \frac{y-a}{r_2} = \frac{z-a}{-(r_2+a)}$$

$$\Rightarrow \frac{x - (4r_2 + a)}{-r_2} = \frac{y-a}{r_2} = \frac{z-a}{-(r_2+a)}$$

$$\frac{y-a}{r_2} = \frac{z-a}{-(r_2+a)} \Rightarrow -(r_2)(y-a) - ay + a^2 = zr_2 - ar_2$$

$$\Rightarrow zr_2 - ar_2 + r_2 y - ar_2 = a^2 - ay$$

$$\Rightarrow r_2 = \frac{a(a-y)}{y + z - 2a}$$

$$① \equiv \frac{x - (4r_2 + a)}{-\cancel{r_2}} = \frac{y-a}{\cancel{r_2}} \Rightarrow x - 4r_2 - a = -y + a$$

$$\Rightarrow x + y - \frac{4a(a-y)}{y+z-2a} = 2a$$

$$\Rightarrow (x+y)(y+z-2a) - 4a(a-y) = 2a(y+z-2a)$$

$xy + y^2 + xz + yz - 2ax - 2ay - ya^2 + yay = 2ay + 2az - ya^2$

$\Rightarrow \boxed{y^2 + yz + zx + xy - 2ax - 2az = 0}$ which is the reqd surface

③(b) Reduce the following equation to its canonical form & determine the nature of the conic $4x^2+4xy+y^2-12x-6y+5=0$

$\rightarrow$ Comparing with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

we have $a=4, b=1, c=0, f=0, g=0, h=2, u=-6, v=-3, w=0,$ $d=5$.

Now, $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

~~$Au + Bv + H$~~

$Au + Hv + Gw = (bc-f^2)u + (ac-g^2)v + (ab-h^2)w$
$= (0-0)u + (0-0)v + (4-4)w = 0$

$A = bc - f^2 = 0 - 0 = 0.$

$fu = 0 \quad , gv = 0$

$\therefore$ The given eqn. can be rewritten as:

$(2x+y)^2 = 12x + 6y - 5$

$\Rightarrow (2x+y+\lambda)^2 = 12x + 6y - 5 + \lambda^2 + 4x\lambda + 2y\lambda$

Choosing λ such that the planes $2x+y+\lambda = 0$ and $12x + 6y - 5 + \lambda^2 + 4x\lambda + 2y\lambda = 0$ are $\perp$ar

i.e. $2\cdot(12+4\lambda) + 1\cdot(6+2\lambda) = 0$

$\Rightarrow 24 + 8\lambda + 6 + 2\lambda = 0$

$\Rightarrow 10\lambda = -30 \Rightarrow \lambda = -3.$

$\therefore (2x+y-3)^2 = 12x + 6y - 5 + 9 - 12x - 6y$
$= 4$

~~$\Rightarrow 5\left[\frac{2x+y-3}{\sqrt{5}}\right]^2 = 4$~~ $\Rightarrow 2x+y-3 = \pm 2.$

$\Rightarrow 2x+y = 5$ or $2x+y = 1$ which represents a pair of planes, 3 (d) Find the equation to the tangent planes to the surface $7x^2-3y^2-z^2+21=0$ which pass through the line $7x-6y+9=0, z=3$.

$\hookrightarrow$ Any plane through the given line is $7x-6y+9+\lambda(z-3)=0$

$\Rightarrow 7x-6y+\lambda z+(9-3\lambda)=0$

Comparing with $lx+my+nz=p$, $l=7, m=-6, n=\lambda, p=3\lambda-9$

Given conicoid is $7x^2-3y^2-z^2+21=0$

Comparing with $ax^2+by^2+cz^2=1$, $a=-\frac{1}{3}$, $b=\frac{1}{7}$, $c=\frac{1}{21}$

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The condition for tangency is $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

$\rightarrow 49 \times -3 + 36 \times 7 + 21\lambda^2 = (3\lambda - 9)^2$

$\Rightarrow 105 + 21\lambda^2 = 9\lambda^2 + 81 - 54\lambda$

$\Rightarrow 12\lambda^2 + 54\lambda + 24 = 0 \Rightarrow 4\lambda^2 + 18\lambda + 8 = 0$

$\Rightarrow 2\lambda^2 + 9\lambda + 4 = 0 \Rightarrow 2\lambda^2 + 8\lambda + \lambda + 4 = 0$

$\Rightarrow (\lambda + 4)(2\lambda + 1) = 0 \Rightarrow \lambda = -4, \lambda = -\frac{1}{2}.$

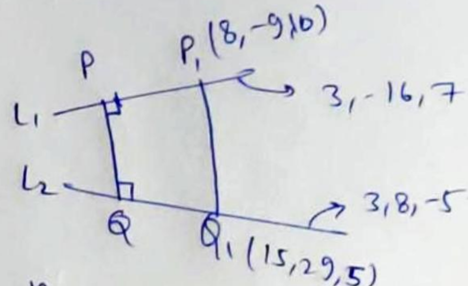
$\therefore$ Reqd tangent planes: $7x - 6y + 9 - 4(z-3) = 0$
& $7x - 6y + 9 - \frac{1}{2}(z-3) = 0$

$\Rightarrow 7x - 6y - 4z + 21 = 0$ & $14x - 12y - z + 21 = 0$

Q find the magnitude and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ — ①

and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$ — ②

$\rightarrow$ Let the SD line be PQ. Then
$PQ \perp L_1$ & $PQ \perp L_2$.



Let l, m, n be drs of PQ. Then,

$3l - 16m + 7n = 0$
$3l + 8m - 5n = 0$
$\Rightarrow \frac{l}{24} = \frac{m}{36} = \frac{n}{72} \Rightarrow \frac{l}{2} = \frac{m}{3} = \frac{n}{6} = \frac{1}{7}$

$l = \frac{2}{7}, m = \frac{3}{7}, n = \frac{6}{7}$

Then $S.D. = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$
$= \frac{1}{7}[2(15-8) + 3(29+9) + 6(5-10)]$
$= \frac{98}{7} = 14$ units.

Eqn of plane containing L_1 & PQ

$$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

$\Rightarrow 117x + 4y - 41z = 490$

Eqn of plane containing L_2 & PQ

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

$\Rightarrow 63x - 28y - 10z = 63.$

Eqn of SD line is

$117x + 4y - 41z - 490 = 0 = 63x - 28y - 10z - 63$

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