2013

Form a pde by eliminating the arbitrary functions f and g from $z = yf(x) + xg(y)$

Solⁿ:- We have $z = yf(x) + xg(y)$ —— (1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) \quad \text{—— (2)}$$

Again differentiating (1) partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = f(x) + xg'(y) \quad \text{—— (3)}$$

Differentiating (2) partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \quad \text{—— (4)}$$

Differentiating (3) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial y \partial x} = f'(x) + g'(y) \quad \text{—— (5)}$$

From (4) and (5),

$$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial^2 z}{\partial y \partial x} = 0,$$ which is the required PDE.

$(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$ (Alternate solⁿ MD. RAIS)

A.E. $\equiv m^2 + m - 6 = 0 \Rightarrow m = 2, -3$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-3x)$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} x^2 \sin(x+y)$$

$$= \frac{1}{(D-2D')(D+3D')} x^2 e^{i(x+y)}$$

$$= e^{i(x+y)} \frac{1}{[(D+i) - 2(D'+i)][(D+i) + 3D' + 3i]} x^2$$

$$= e^{i(x+y)} \frac{1}{(D-2D'-i)(D+3D'+4i)} x^2$$

$$= e^{i(x+y)} \frac{1}{(-i)\,4i}\left[1 - \frac{D-2D'}{i}\right]^{-1}\left[1 + \frac{D+3D'}{4i}\right]^{-1} x^2$$

$$= \frac{e^{i(x+y)}}{4}\left[1 + \frac{D-2D'}{i} - (D-2D')^2 + \cdots\right]\left[1 - \frac{D+3D'}{4i} + \left(\frac{D+3D'}{4i}\right)^2 + \cdots\right] x^2$$

$$= \frac{e^{i(x+y)}}{4}\left[1 - i(D-2D') - (D^2 - 4DD' + 4D'^2) + \cdots\right]\left[1 + \frac{i}{4}(D+3D') - \frac{1}{16}(D^2 + 6DD' + 9D'^2) + \cdots\right] x^2$$

$$= \frac{e^{i(x+y)}}{4}\left[1 - i(D-2D') - (D^2 - 4DD' + 4D'^2) + \cdots\right]\left[x^2 + \frac{i}{4}\cdot 2x - \frac{1}{8}\right]$$

$$= \frac{e^{i(x+y)}}{4}\left[1 - i(D-2D') - (D^2 - 4DD' + 4D'^2) + \cdots\right]\left[x^2 + \frac{ix}{2} - \frac{1}{8}\right]$$

$$= \frac{e^{i(x+y)}}{4}\left[1 - iD + 2iD' - D^2 + 4DD' - 4D'^2 + \cdots\right]$$

$$= \frac{e^{i(x+y)}}{4}\left[\left(x^2+\frac{ix}{2}-\frac{1}{8}\right)-i\left(2x+\frac{i}{2}\right)-2\right]$$

$$= \frac{e^{i(x+y)}}{4}\left[\left(x^2-\frac{1}{8}+\frac{1}{2}-2\right)+i\left(\frac{x}{2}-2x\right)\right]$$

$$= \frac{1}{4}\left[\cos(x+y)+i\sin(x+y)\right]\left[\left(x^2-\frac{13}{8}\right)-\frac{3}{2}ix\right]$$

$$\therefore \text{P.I.} = \frac{1}{4}\left(x^2-\frac{13}{8}\right)\sin(x+y)-\frac{3x}{8}\cos(x+y)$$

$$\therefore z = \phi_1(y+2x)+\phi_2(y-3x)+\left(\frac{1}{4}x^2-\frac{13}{32}\right)\sin(x+y)-\frac{3x}{8}\cos(x+y).$$

Q. Reduce the equation $y\frac{\partial^2 z}{\partial x^2}+(x+y)\frac{\partial^2 z}{\partial x\partial y}+x\frac{\partial^2 z}{\partial y^2}=0$ to its canonical form when $x\neq y$.

Here $R=y,\ S=x+y,\ T=x$.

$\therefore\ S^2-4RT=(x+y)^2-4xy=(x-y)^2>0$, for $x\neq y$.

Thus the given equation is hyperbolic. The quadratic equation $R\lambda^2+S\lambda+T=0$ becomes $y\lambda^2+(x+y)\lambda+x=0$

$$\Rightarrow \lambda(y\lambda+x)+1(y\lambda+x)=0$$

$$\Rightarrow (\lambda+1)(y\lambda+x)=0$$

$$\Rightarrow \lambda=-1,\ -\frac{x}{y}.$$

For these values of λ, the equations $\frac{dy}{dx}+\lambda_i(x,y)=0$ become

$$\frac{dy}{dx}-1=0 \text{ and } \frac{dy}{dx}-\frac{x}{y}=0$$

$$\Rightarrow y-x=c_1 \text{ and } \frac{y^2}{2}-\frac{x^2}{2}=c_2$$

Let us take $u=y-x,\ v=\frac{y^2}{2}-\frac{x^2}{2}$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}(-1)+\frac{\partial z}{\partial v}(-x)=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}(1)+\frac{\partial z}{\partial v}(y)=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}$$

$$r=\frac{\partial^2 z}{\partial x^2}=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial z}{\partial v}-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=-\frac{\partial^2 z}{\partial u^2}(-1)-\frac{\partial^2 z}{\partial v\partial u}(-x)-\frac{\partial z}{\partial v}-x\left[\frac{\partial^2 z}{\partial u\partial v}(-1)+\frac{\partial^2 z}{\partial v^2}(-x)\right]$$

$$=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}(-1)+\frac{\partial^2 z}{\partial v\partial u}(-x)+y\left(\frac{\partial^2 z}{\partial u\partial v}(-1)+\frac{\partial^2 z}{\partial v^2}(-x)\right)$$

$$=-\frac{\partial^2 z}{\partial u^2}-(x+y)\frac{\partial^2 z}{\partial u\partial v}-xy\frac{\partial^2 z}{\partial v^2}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v\partial u}y+\frac{\partial z}{\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}y\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+y^2\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial v}$$

On substituting the above values, the given equation becomes

$$y\frac{\partial^2 z}{\partial u^2}+2xy\frac{\partial^2 z}{\partial u\partial v}+x^2y\frac{\partial^2 z}{\partial v^2}-y\frac{\partial z}{\partial v}-(x+y)\frac{\partial^2 z}{\partial u^2}-(x+y)^2\frac{\partial^2 z}{\partial u\partial v}$$

$$-xy(x+y)\frac{\partial^2 z}{\partial v^2}+x\frac{\partial^2 z}{\partial u^2}+2xy\frac{\partial^2 z}{\partial u\partial v}+xy^2\frac{\partial^2 z}{\partial v^2}+x\frac{\partial z}{\partial v}=0$$

$$\Rightarrow \{4xy-(x+y)^2\}\frac{\partial^2 z}{\partial u\partial v}-y\frac{\partial z}{\partial v}+x\frac{\partial z}{\partial v}=0$$

$$\Rightarrow (x-y)^2\frac{\partial^2 z}{\partial u\partial v}+(y-x)\frac{\partial z}{\partial v}=0$$

$$\Rightarrow u^2\frac{\partial^2 z}{\partial u\partial v}+u\frac{\partial z}{\partial v}=0$$

$\therefore \dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{1}{u}\dfrac{\partial z}{\partial v}=0$ is the required canonical form of the given equation.

2017 IFoS

CSE 2013 Find the surface which intersects the surfaces of the system $z(x+y)=c(3z+1)$, c being a constant orthogonally and which passes through the circle $x^2+y^2=1, z=1$.

Sol^n:- The given equation of surface is

$$f(x,y,z)\equiv\left\{\frac{z(x+y)}{(3z+1)}\right\}=c \quad \text{—— (1)}$$

$$\frac{\partial f}{\partial x}=\frac{z}{3z+1},\quad \frac{\partial f}{\partial y}=\frac{z}{3z+1}$$

$$\frac{\partial f}{\partial z}=\frac{(3z+1)(x+y)-z(x+y)\cdot 3}{(3z+1)^2}=\frac{(x+y)(3z+1-3z)}{(3z+1)^2}=\frac{x+y}{(3z+1)^2}$$

$\therefore$ The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{z}{3z+1}p+\frac{z}{3z+1}q=\frac{(x+y)}{(3z+1)^2}$$

$$\Rightarrow z(3z+1)p+z(3z+1)q=x+y \quad \text{—— (2)}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{z(3z+1)}=\frac{dy}{z(3z+1)}=\frac{dz}{x+y}$$

$\rightarrow$

contd.

Considering the first two fractions, we get

$dx = dy$

$\Rightarrow dx - dy = 0$

$\Rightarrow x - y = c_1$ [Integrating]

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction equals

$x\,dx + y\,dy - z(3z+1)\,dz = 0$

Integrating we get,

$\frac{x^2}{2} + \frac{y^2}{2} - z^3 - \frac{z^2}{2} = c_2$, c_2 being an arbitrary constant

$\Rightarrow x^2 + y^2 - 2z^3 - z^2 = c_3$, where $c_3 = 2c_2$

Hence the surface which is orthogonal to (1) is

$x^2 + y^2 - 2z^3 - z^2 = \phi(x-y)$, ϕ being an arbitrary function

In order to get the desired surface passing through the circle $x^2+y^2=1, z=1$, we choose $\phi(x-y) = -2$.

Thus, the required surface is

$x^2+y^2-2z^3-z^2 = -2$.

2013 A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating by giving each point a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.

Solution: The one dimensional wave equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{--- (1)}$$

Boundary conditions:

$$y(0,t) = y(l,t) = 0 \quad \text{--- (2)}$$

Initial conditions:

$$y(x,0) = f(x) = 0$$

$$y_t(x,0) = g(x) = \lambda x(l-x)$$

Let $y(x,t) = X(x)T(t)$ be the trial solution. (3)

$\Rightarrow$ (1) becomes $\frac{1}{X}X'' = \frac{1}{c^2 T}T''$

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu$, where μ is an arbitrary constant

$\Rightarrow X'' - \mu X = 0$ and $T'' - \mu c^2 T = 0$

Case-I: $\mu = 0$.

$X = Ax + B$, $T = Ct + D$

$\therefore y(x,t) = (Ax+B)(Ct+D) \quad \text{--- (4)}$

Case-II : $\mu = \alpha^2 > 0$.

$\therefore X(x) = Ae^{\pm \alpha x}, \quad T(t) = Be^{\pm \alpha c t}$

$y(x,t) = Ce^{\pm \alpha x \pm \alpha c t}$ — (5)

Case-III : $\mu = -\alpha^2 < 0$.

$X(x) = A\cos\alpha x + B\sin\alpha x, \quad T(t) = C\cos\alpha ct + D\sin\alpha ct$

$y(x,t) = (A\cos\alpha x + B\sin\alpha x)(C\cos\alpha ct + D\sin\alpha ct)$ — (6)

Using boundary conditions (2),

$y(0,t) = X(0)T(t) = 0$ for $t > 0$

$y(l,t) = X(l)T(t) = 0$ for $t > 0$

But $T(t) \neq 0$, since otherwise $y(x,t) = 0$, which contradicts (3)

$\therefore X(0) = 0 = X(l)$ — (7)

Using (7), cases I and II for $\mu \geq 0$ yield only trivial solution $y(x,t) = 0$. Hence we take case III $\mu = -\alpha^2 < 0$.

Now, $X(x) = A\cos\alpha x + B\sin\alpha x$.

Using (7), we get $A = 0$, $B\sin\alpha l = 0 \Rightarrow \sin\alpha l = \sin n\pi$ [$B \neq 0$]

$\Rightarrow \alpha l = n\pi$

$\Rightarrow \alpha = \alpha_n = \frac{n\pi}{l}, \quad n = 1, 2, 3, \ldots$

$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$

Also $T_n(t) = C_n \cos\alpha_n ct + D_n \sin\alpha_n ct$, where C_n and D_n are constants of integration.

Now from (3), we get

$y_n(x,t) = \left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right)$

where $a_n = C_n B_n$; $b_n = D_n B_n$ are new arbitrary constants and $n = 1, 2, 3, \ldots$

Since $\overline{=}n$ (1) is linear and homogeneous, the most general solution of (1) is obtained by the principle of superposition, in the form $y(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{l} + b_n \sin\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}$

Now from the initial conditions,

$y(x,0) = \sum a_n \sin\frac{n\pi x}{l}$

$\Rightarrow 0 = \sum a_n \sin\frac{n\pi x}{l} \Rightarrow a_n = 0$

$y_t(x,0) = g(x) = \left[\sum_{n=1}^{\infty} \left(-a_n \frac{n\pi c}{l} \sin\frac{n\pi ct}{l} + b_n \frac{n\pi c}{l} \cos\frac{n\pi ct}{l}\right) \sin\frac{n\pi x}{l}\right]_{t=0}$

$\Rightarrow g(x) = \left(\frac{\pi c}{l}\right) \sum_{n=1}^{\infty} (n b_n) \sin\left(\frac{n\pi x}{l}\right), \quad 0 \leq x \leq l$

By Fourier-sine series in $[0, l]$,

$\frac{\pi c}{l}(n b_n) = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$\Rightarrow b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$\therefore\ b_n = \frac{2}{n\pi c}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \begin{cases} 0 & \text{, if } n \text{ is even} \\ \dfrac{8kl^3}{cn^4\pi^4} & \text{, if } n \text{ is odd.} \end{cases}$$

$$\therefore\ y(x,t) = \frac{8kl^3}{c\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\left(\frac{(2n-1)\pi ct}{l}\right)\sin\left(\frac{(2n-1)\pi x}{l}\right)$$