IFS 2018 PAPER-I

(SeC-A)

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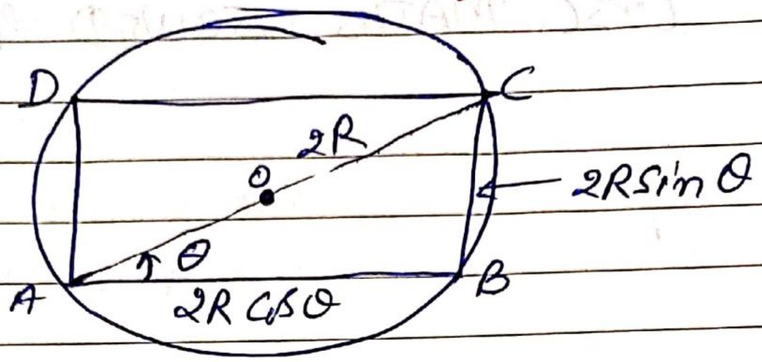
1 (a) Show that the maximum rectangle inscribed in a circle is a square. (8)



Let ABCD be the rectangle inscribed in a circle of radius R.

Let $\angle BAC = \theta$

$AB = 2R\cos\theta$, $BC = 2R\sin\theta$

Area, $A = (2R\cos\theta)(2R\sin\theta)$
$= 2R^2 \cdot \sin 2\theta$

for max area, $\frac{dA}{d\theta} = 0 \Rightarrow 4R^2\cos 2\theta = 0$

$\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ in $[0, 2\pi]$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ in $[0, 2\pi]$

But no rectangle is possible for $\theta = 3\pi/4$
so, we discard it

$\frac{d^2A}{d\theta^2} = -8R^2\sin 2\theta < 0$ at $\theta = \frac{\pi}{4}$

Hence, A is maximum when $\theta = \pi/4$

Then $AB = 2R\cos\frac{\pi}{4} = 2R\cdot\frac{1}{\sqrt{2}} = \sqrt{2}R = BC$

Hence, ABCD becomes square.

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1.(b) Given that $\operatorname{Adj} A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\det A = 2$.

find the Matrix A.

(8)

We know, $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$

$\Rightarrow \quad A = |A| (\operatorname{adj} A)^{-1}$

$|\operatorname{adj} A| = |A|^2 = 4$

We find the adjoint of the given matrix

$\operatorname{adj}(\operatorname{adj} A) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$

$\therefore \quad A = |A| (\operatorname{adj} A)^{-1}$

$= \cancel{2} \times \dfrac{1}{\cancel{4}_{\,2}} \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

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1.(c) If $f : [a,b] \to R$ be continuous in $[a,b]$ and derivable in (a,b), where $0<a<b$, show that for $c \in (a,b)$

$$f(b) - f(a) = c.f'(c) \log(b/a).$$ (8)

Cauchy's Mean Value Theorem

Two functions f and g are

i) cont on $[a,b]$ ii) derivable in (a,b)

iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$, then there exist atleast one point $c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Here, take $g(x) = \log x$ in $[a,b]$

$0<a<b$

Applying Cauchy's MVT

$\exists$ some $c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{\log b - \log a} = \frac{f'(c)}{(1/c)}.$$

$$\Rightarrow f(b) - f(a) = c.f'(c) \log b/a.$$

Hence proved.

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1.(d). Find the equations of the tangent planes to the ellipsoid

$$2x^2+6y^2+3z^2=27$$

which pass through the line

$$x-y-z=0=x-y+2z-9.$$

Any plane through the line $x-y-z=0=x-y+2z-9$ is

$$(x-y-z)+\lambda(x-y+2z-9)=0$$

$$x(1+\lambda)-(1+\lambda)y-(1-2\lambda)z=9\lambda \quad \text{--- (1)}$$

If it touches the conicoid $2x^2+6y^2+3z^2=27$

i.e. $$\frac{2}{27}x^2+\frac{2}{9}y^2+\frac{1}{9}z^2=1.$$

then $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

$$(ax^2+by^2+cz^2=1,\ lx+my+nz=p).$$

$$\frac{27}{2}(1+\lambda)^2+\frac{9}{2}(1+\lambda)^2+9(1-2\lambda)^2=(9\lambda)^2$$

$$3(\lambda^2+2\lambda+1)+(\lambda^2+2\lambda+1)+2(4\lambda^2-4\lambda+1)=2\times9\lambda^2$$

$$12\lambda^2+6=18\lambda^2$$

$$6\lambda^2=6 \Rightarrow \lambda=1,-1$$

Hence, from (1) required tangent planes are

$$2x-2y+z=9\ ;$$

$$z=3$$

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1.(e) Prove that the eigenvalues of a Hermitian matrix are all real. (8)

Let A be a Hermitian Matrix.

$\therefore \quad A^{\theta} = A$ —— ①

↳ conjugate transpose

Let λ be an eigen value of A and x be corresponding eigen vector of λ.

$\therefore \quad Ax = \lambda x$

$\Rightarrow \quad (Ax)^{\theta} = (\lambda x)^{\theta}$

$\Rightarrow \quad x^{\theta} \cdot A^{\theta} = \bar{\lambda} \cdot x^{\theta}$

$\Rightarrow \quad x^{\theta} \cdot A = \bar{\lambda} x^{\theta} \qquad [\because A^{\theta} = A$ by ①$]$

Post-multiplying x both sides.

$(x^{\theta} A) x = (\bar{\lambda} x^{\theta}) x$

$x^{\theta} (Ax) = \bar{\lambda} (x^{\theta} x)$

$x^{\theta} (\lambda x) = \bar{\lambda} (x^{\theta} x)$

$\lambda (x^{\theta} x) = \bar{\lambda} (x^{\theta} x) \qquad [\lambda$ is a scalar$]$

$(\lambda - \bar{\lambda}) (x^{\theta} x) = 0$

$\lambda - \bar{\lambda} = 0 \qquad [\because x \neq 0 \therefore x^{\theta} x \neq 0]$

$\therefore \quad \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is real.

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Q2(a) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2+y^2=4,\ z=2.$ (10)

Let $P(x_1, y_1, z_1)$ be any point on the cylinder then the eqn. of the generator through P are

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$$

This generator meets the plane $z=2$ in the point $\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{2-z_1}{3}$

i.e. $\left[\frac{3x_1-z_1+2}{3}, \frac{3y_1+2z_1-4}{3}, 2\right]$

$\therefore$ The generator intersect the given curve if

$$\frac{1}{9}(3x_1-z_1+2)^2 + \frac{1}{9}(3y_1+2z_1-4)^2 = 4$$

$\therefore$ The locus of P $P(x_1, y_1, z_1)$ or the required eqn of cylinder is

$$(3x-z+2)^2 + (3y+2z-4)^2 = 36$$

$$(9x^2+z^2+4 - 6xz - 4z + 12x) + (9y^2 + 4z^2 + 16 + 12yz - 16z - 24y) = 36$$

$$9x^2+9y^2+5z^2-6xz+12yz-20z-24y-12x = 16$$

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2(b) Show that the matrices A and B are congruent.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

(10)

* Sylvester's Law of Inertia

Two symmetric $n \times n$ matrices are congruent if and only if their diagonal representations have same rank, index and signature.

Rank = no. of non-zero eigen-values

index = no. of positive " "

Signature = no. of positive EV − no. of negative EV

* Also, two symmetric matrices (as well as skew-symmetric) are congruent if they have the same rank.

$$|A| = 1(6-1) - 1(3+1) \cancel{+} (1+2)$$

$$= 5 - 4 - 3 = -2 \neq 0$$

$$\therefore \rho(A) = 3$$

$$|B| = 1(0-4) + 3(0-6) = -4 - 18 = -22 \neq 0$$

$$\therefore \rho(B) = 3$$

Hence A and B are congruent.

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2(c) If ϕ and ψ be two functions derivable in $[a,b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a,b]$, there lies exactly one root of $\psi(x) = 0$. (10).

Let α and β be two consecutive roots of $\phi(x) = 0$ in $[a,b]$ and $\alpha < \beta$.

To prove that only one root of $\psi(x) = 0$ lies between α and β.

If possible, let $\psi(x) = 0$ has no root in (α, β).

Consider the function $F(x) = \dfrac{\phi(x)}{\psi(x)}$

$$F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0 \quad \& \quad F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0$$

$(\because \phi(\alpha) = 0 = \phi(\beta)$ and $\psi(\alpha) \neq 0,\ \psi(\beta) \neq 0)$

$\psi(x) \neq 0$ in $[\alpha, \beta]$

$\therefore$ $F(x)$ is continuous in $[\alpha, \beta]$

$F'(x) = \dfrac{\phi'(x)\psi(x) - \psi'(x)\phi(x)}{[\psi(x)]^2}$ exist in $[\alpha, \beta]$

$\therefore$ $F(x)$ satisfies all conditions of Rolle's Theorem in $[\alpha, \beta]$

$\therefore$ $F'(\gamma) = 0$ where $\alpha < \gamma < \beta$

But by given condition $\phi'(x)\psi(x) - \psi'(x)\phi(x) > 0$

$\therefore$ $F'(x) \neq 0$ in (α, β) and we get contradiction.

Hence $\psi(x)$ has atleast one root in (α, β)

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By similar arguement it can be shown that between two roots of $\psi(x)=0$ there is a root of $\phi(x)=0$.

Now, we prove that there is exactly one root of $\psi(x)=0$ between α, β.
If possible let γ and δ be two roots of $\psi(x)=0$ in (α,β) i.e. $\alpha<\gamma<\delta<\beta$.

Between γ and δ there would exist a root of $\phi(x)=0$ contradicting that α and β are consecutive roots of $\phi(x)=0$

Hence, there is only one root of $\psi(x)=0$ between α and β.

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2(d) Show that the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, $\alpha_3 = (0,-3,2)$ form a basis for R^3. Express each of the standard basis vectors as a L.C. of $\alpha_1, \alpha_2, \alpha_3$. (10)

Let $x_1(1,0,-1) + x_2(1,2,1) + x_3(0,-3,2) = (0,0,0)$

$(x_1 + x_2, 2x_2 - 3x_3, -x_1 + x_2 + 2x_3) = (0,0,0)$

where, $x_1, x_2, x_3 \in R$

Solving these, we get $x_1 = x_2 = x_3 = 0$

$\Rightarrow \alpha_1, \alpha_2, \alpha_3$ are L.I.

Again, let $\beta = (x,y,z) \in R^3$ and

$\beta = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$

$(x,y,z) = (a+b, 2b-3c, -a+b+2c)$

$$\left.\begin{aligned} a+b &= x \\ 2b-3c &= y \\ -a+b+2c &= z \end{aligned}\right] \Rightarrow \begin{aligned} a &= \tfrac{1}{10}(7x-2y-3z) \\ b &= \tfrac{1}{10}(3x+2y+3z) \\ c &= \tfrac{1}{5}(x-y+z) \end{aligned}$$

(~~Matrix method~~, elimentary row operations)

$\therefore (x,y,z) = \frac{1}{10}(7x-2y-3z)(1,0,-1) + \frac{1}{10}(3x+2y+3z)(1,2,1) + \frac{1}{5}(x-y+z)(0,-3,2)$ $\forall \beta \in R^3$

Using this, we write,

$(1,0,0) = 7/10\,(1,0,-1) + 3/10\,(1,2,1) + 1/5\,(0,-3,2)$

Similarly $(0,1,0)$ and $(0,0,1)$.

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3(a). Find the eqn of the tangent plane that can be drawn to the sphere

$$x^2+y^2+z^2-2x+6y+2z+8=0,$$

through the st line, $3x-4y-8=0=y-3z+2$ (10)

The eqn of any plane through given line

$$3x-4y-8+\lambda(y-3z+2)=0$$

$$3x-(4-\lambda)y-3\lambda z=8-2\lambda \quad \text{—(1)}$$

If this plane touches the sphere then length of $\perp$ from centre of sphere to plane = Radius of sphere

Centre $(1,-3,-1)$

Radius $=\sqrt{(1+9+1-8)}=\sqrt{3}$

$$\frac{3(1)-(4-\lambda)(-3)-3\lambda(-1)-8+2\lambda}{\sqrt{9+(4-\lambda)^2+9\lambda^2}}=\pm\sqrt{3}$$

$$-5+12-\cancel{3\lambda}+\cancel{3\lambda}+2\lambda=\pm\sqrt{3}\cdot\sqrt{9+16+\lambda^2-8\lambda+9\lambda^2}$$

$$(2\lambda+7)^2=3(10\lambda^2-8\lambda+25)$$

$$4\lambda^2+28\lambda+49=30\lambda^2-24\lambda+75$$

$$26\lambda^2-52\lambda+26=0$$

$$\lambda^2-2\lambda+1=0 \Rightarrow (\lambda-1)^2=0$$

$$\Rightarrow \lambda=1$$

Hence, Required Eqn of plane from (1)

$$3x-3y-3z=6$$

ie $x-y-z=2.$

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3(b). If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Chain Rule

$$\frac{\partial f(u,v)}{\partial (x)} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f(u,v)}{\partial x^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,

$$\frac{\partial^2 f(u,v)}{\partial y^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2\right] + \left[\frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$\cdot\; u = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x\cdot\cos y = u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} = u$$

$$\frac{\partial u}{\partial y} = -e^x\sin y = -v$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y} = -e^x\cos y = -u$$

$v = e^x \sin y$

$$\frac{\partial v}{\partial x} = e^x\sin y = v$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x} = v$$

$$\frac{\partial v}{\partial y} = e^x\cos y = u$$

$$\frac{\partial^2 v}{\partial y^2} = -e^x\sin y = -v$$

Using these values

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 f}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$+ \frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= \cancel{\frac{\partial f}{\partial u}(u-u)} + \frac{\partial^2 f}{\partial u^2}(u^2+v^2) + \cancel{\frac{\partial f}{\partial v}(v-v)} + \frac{\partial^2 f}{\partial v^2}(v^2+u^2)$$

$$= (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

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3(C) Let $T: V_2(R) \to V_2(R)$ be a LT defined by $T(a,b) = (a, a+b)$. Find the matrix of T, taking (e_1, e_2) as a basis for the domain and $\{(1,1), (1,-1)\}$ as a basis for range. (10)

Let $(x,y) = x_1(1,1) + x_2(1,-1)$

$$(x,y) = (x_1+x_2,\ x_1-x_2)$$

$$x_1 + x_2 = x \qquad \Rightarrow \qquad x_1 = \frac{x+y}{2}$$

$$x_1 - x_2 = y \qquad\qquad x_2 = x-y/2$$

$$\therefore\ (x,y) = \frac{(x+y)}{2}(1,1) + \left(\frac{x-y}{2}\right)(1,-1)$$

$$T(e_1) = T(1,0) = (1, 1+0) = (1,1)$$

$$\cancel{T(e_2)} = \frac{1+1}{2}(1,1) + \frac{1-1}{2}(1,-1).$$

$$= 1\cdot(1,1) + 0\cdot(1,-1)$$

$$T(e_2) = T(0,1) = (0, 0+1) = (0,1)$$

$$= \frac{0+1}{2}(1,1) + \frac{(0-1)}{2}(1,-1)$$

$$= \tfrac{1}{2}(1,1) - \tfrac{1}{2}(1,-1)$$

Matrix of LT is represented by writing coordinates of $T(e_1)$ and $T(e_2)$ as columns of matrix.

$$[T] = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

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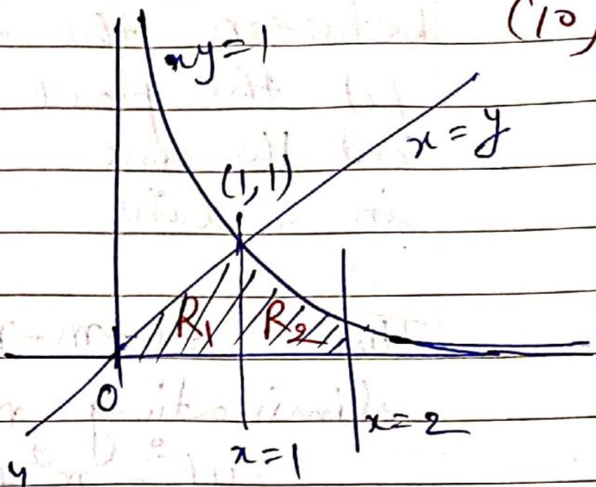
3(d) Evaluate $\iint_R (x^2+xy)\,dx\,dy$ over the region R bounded by $xy=1$, $y=0$, $y=x$ and $x=2$. (10)

We split the ~~area~~ region of integration in two parts.



$$I = \iint_R (x^2+xy)\,dx\,dy$$

$$= \iint_{R_1} (x^2+xy)\,dx\,dy + \iint_{R_2} (x^2+xy)\,dx\,dy$$

$$= \int_0^1 \int_0^x (x^2+xy)\,dy\,dx + \int_1^2 \int_0^{1/x} (x^2+xy)\,dy\,dx$$

$$= \int_0^1 \left. x^2 y + x\frac{y^2}{2}\right|_0^x dx + \int_1^2 \left. x^2 y + x\cdot\frac{y^2}{2}\right|_0^{1/x} dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{2}\right) dx + \int_1^2 \left(x^2\cdot\frac{1}{x} + \frac{x}{2}\cdot\frac{1}{x^2}\right) dx$$

$$= \int_0^1 \frac{3}{2}x^3\,dx + \int_1^2 \left(x + \frac{1}{2x}\right) dx$$

$$= \frac{3}{2}\times\frac{1}{4}\left[x^4\right]_0^1 + \left[\frac{x^2}{2} + \frac{1}{2}\log x\right]_1^2$$

$$= \frac{3}{8}\cdot 1 + \frac{4-1}{2} + \frac{1}{2}(\log 2 - \log 1)$$

$$= \frac{3+12}{8} + \frac{1}{2}\log 2 = \frac{15}{8} + \frac{1}{2}\log 2.$$

4(a). Find the equations of the st lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. Find the angle between the two st lines.

Let the plane $2x+y-z=0$ — ①
cut the cone $4x^2-y^2+3z^2=0$ — ②
in a line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — ③

Then $2l+m-n=0$ and $4l^2-m^2+3n^2=0$ — ④

eliminating n,
$$4l^2-m^2+3(2l+m)^2=0$$

$$16l^2+12lm+2m^2=0 \text{ or}$$
$$8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0.$$

Solving, we get, $l=-\frac{m}{4}$ or $l=-\frac{m}{2}$ — ⑤

From (4) $n=2l+m=2\left(-\frac{m}{4}\right)+m=\frac{m}{2}$

$\therefore l:m:n=-\frac{m}{4}:m:\frac{m}{2}$
$=-1:4:2$, when $l=-m/4$.

Again, when $l=-m/2$,
$n=2l+m=2(-m/2)+m=0.m$
$l:m:n=-\frac{m}{2}:m:0.m=-1:2:0$

Hence, lines are $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$

If θ is angle between these lines
$$\cos\theta=\frac{-1.(-1)+4.2+2.0}{\sqrt{1+16+4}\sqrt{1+4+0}}=\frac{9}{\sqrt{21}\cdot\sqrt{5}}=\frac{3\sqrt{105}}{35}$$

4(b) Show that the functions $u = x+y+z$, $v = xy+yz+zx$ and $w = x^3+y^3+z^3-3xyz$ are dependent and find the relation between them.

Given functions are dependent if their Jacobian vanishes.

$$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ y+z & x-y & x-z \\ 3x^2-3yz & 3y^2-3xz-3x^2+3yz & 3z^2-3xy-3x^2+3yz \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ y+z & x-y & x-z \\ x^2-yz & y^2-x^2+z(y-x) & z^2-x^2+y(z-x) \end{vmatrix}$$

$$= 3(y-x)(z-x)\left[-(z+x+y)-(-1)(y+x+z)\right]$$

$$= 0 \qquad \therefore u, v, w \text{ are dependent.}$$

Now

$$w = x^3+y^3+z^3-3xy+z$$
$$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$
$$= (x+y+z)\left[(x+y+z)^2-3(xy+yz+zx)\right]$$
$$= u\left[u^2-3v\right]$$

$\boxed{w = u^3-3uv}$ ⇒ required relation.

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4(c). Find the locus of the point of intersection of the $\perp$ generators of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$. (10)

The generators of λ and μ-systems are

$$\frac{x}{a} - \frac{y}{b} = \lambda z \quad , \quad \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda} \quad \text{—(1)}$$

and $$\frac{x}{a} - \frac{y}{b} = 2\mu \quad , \quad \frac{x}{a} + \frac{y}{b} = \mu z \quad \text{—(2)}$$

Eqn (1) can be re-written as

$$\frac{x}{a} - \frac{y}{b} - \lambda z = 0 \quad , \quad \frac{x}{a} + \frac{y}{b} + 0.z - \frac{2}{\lambda} = 0$$

$\therefore$ If l_1, m_1, n_1 be the d.r. of this generator then,

$$\frac{l_1}{a} - \frac{m_1}{b} - \lambda n_1 = 0, \quad \frac{l_1}{a} + \frac{m_1}{b} + 0.n_1 = 0.$$

Solving these simultaneously, we get

$$\frac{l_1}{\lambda/b} = \frac{m_1}{-\lambda/a} = \frac{n_1}{2/ab}$$

or $$\frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2} \quad \text{—(3)}$$

Again Eqn (2) can be written as

$$\frac{x}{a} - \frac{y}{b} + 0.z - 2\mu = 0, \quad \frac{x}{a} + \frac{y}{b} - \mu.z = 0$$

If l_2, m_2, n_2 be the d.r. of this generator

$$\frac{l_2}{a} - \frac{m_2}{b} + 0.n_2 = 0 \quad , \quad \frac{l_2}{a} + \frac{m_2}{b} - \mu n_2 = 0.$$

Solving simultaneously

$$\frac{l_2}{\mu/b} = \frac{m_2}{\mu/a} = \frac{n_2}{2/ab}$$

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or $\frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2}$ — (4)

As two above generator (1) & (2) are $\perp$, so

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or

$a\lambda \cdot a\mu + (-b\lambda) b\mu + 2 \cdot 2 = 0$

or $a^2 \lambda\mu - b^2 \lambda\mu + 4 = 0$ — (5)

Now, the point of intersection (x_1, y_1, z_1) of generators (1) and (2) are given as

$x_1 = \frac{a(\lambda+\mu)}{\lambda\mu}$, $y_1 = \frac{b(\mu-\lambda)}{\lambda\mu}$, $z_1 = \frac{2}{\lambda\mu}$

From (5), $(a^2 - b^2)\lambda\mu + 4 = 0$, we get

$(a^2 - b^2)\frac{2}{z_1} + 4 = 0$

$\therefore$ The locus of the point of Intersection (x_1, y_1, z_1) of the generators (1) & (2) is

$(a^2 - b^2)\frac{2}{z} + 4 = 0$

or $a^2 - b^2 + 2z = 0$

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4(d) If $(n+1)$ vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a L.D. set, then show that the vector α is a LC of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set. (10).

Sol: $\alpha_1, \alpha_2, \ldots, \alpha_n$ are Lin independent

$$\left.\begin{array}{l} \therefore \text{ if } a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0 \\ \Rightarrow a_1 = a_2 = \ldots = a_n = 0 \end{array}\right\} \quad \text{—(1)}$$

$\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a L.D. set

Hence there exist atleast one non-zero b_i $(1 \le i \le n+1)$ such that

$$b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n + b_{n+1}\alpha = 0 \quad \text{—(2)}$$

Now, b_{n+1} cannot be zero, otherwise we will get $b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n = 0$ with atleast one b_i non-zero.

which will be a contradiction to (1)

$$\therefore b_{n+1} \neq 0$$

Hence, from (2), $b_{n+1}\alpha = -b_1\alpha_1 - b_2\alpha_2 - \ldots - b_n\alpha_n$

$$\alpha = \left(\frac{-b_1}{b_{n+1}}\right)\alpha_1 + \left(\frac{-b_2}{b_{n+1}}\right)\alpha_2 + \ldots + \left(\frac{-b_n}{b_{n+1}}\right)\alpha_n$$

Hence, α is a L.C. of $\alpha_1, \alpha_2, \ldots, \alpha_n$.