Applications of PDE Continuation

2. State Laplace equation, wave equation and heat conduction in cartesian coordinates and explain the method of separation of variables for solving these equations.

1.20. Use of plane polar co-ordinates for solution of the two-dimensional Laplace's Equation.

Ex. 1. *Solve two dimensional Laplace's equation in plane polar coordinates* (r, θ). **[Delhi B.Sc. (H) 1994]**

Sol. *In plane polar coordinates* (r, θ) *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad ...(1)$$

reduces to
$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0. \qquad ...(2)$$

Suppose (1) has solutions of the form

$$u(r, \theta)=R(r)\,\Theta(\theta) \qquad ...(3)$$

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (2), we get

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0 \qquad ...(4)$$

or
$$(r^2R''+rR')/R=-\Theta''/\Theta, \qquad ...(5)$$

where the dashes denote derivatives with respect to the relevant variables. Since r and θ are independent, (5) is true only if each side is equal to the constant. Careful inspection of physical conditions of the actual problems guides us towards the correct choice of the proposed constant. On physical grounds we require $u(r,\theta)=u(r,\theta+2\pi)$. Hence solution must involve trigonometric functions. For this purpose we choose each side of (5) equal to n^2, where n is an integer. Hence (5) gives

$$r^2R''+rR'-n^2R=0 \qquad ...(6)$$

and
$$\Theta''+n^2\Theta=0 \qquad ...(7)$$

From (6),
$$r^2\frac{d^2R}{dr^2}+r\frac{d^2R}{dr^2}-n^2R=0 \qquad ...(8)$$

which is linear homogeneous differential equation.

Let $r=e^z$ and $D_1 \equiv r\dfrac{d}{dr}\equiv\dfrac{d}{dz}$. Then as usual (8) reduces to

$$[D_1(D_1-1)+D_1-n^2]R=0 \quad or \quad (D_1{}^2-n^2)R=0 \qquad ...(9)$$

Its auxiliary equation is $D_1{}^2-n^2=0$ so that $D_1=n, -n$.

Hence solution of (9) is

$$R=A_n e^{nz}+B_n e^{-nz}$$

or
$$R(r)=A_n r^n+Br^{-n} \quad (\because e^z=r) \qquad ...(10)$$

From (7),
$$\Theta(\theta)=C_n\cos n\theta+D_n\sin n\theta \qquad ...(11)$$

From (10) and (11), a solution of (2) is

$$u_n(r,\theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad ...(12)$$

or, by superposition, a more general solution of (2) is

$$u(r,\theta) = \sum_n (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad ...(13)$$

where the summation over n may be finite of infinite.

In the special case when $n = 0$, (6) and (7) become

$$rR'' + R' = 0 \qquad ...(14)$$

and

$$\Theta'' = 0 \qquad ...(15)$$

Re-writing (14), $\quad \dfrac{d}{dr}(rR') = 0 \qquad ...(16)$

Integrating (16), $\quad rR' = E$ or $dR = (E/r)\,dr \qquad ...(17)$

Integrating (17), $\quad R(r) = E \log r + F \qquad ...(18)$

From (15), $\quad \Theta(\theta) = G\theta + H \qquad ...(19)$

Thus for $n = 0$, a solution of (2) is

$$u(r,\theta) = (E \log r + F)(G\theta + H) \qquad ...(20)$$

Hence *a* more general solution can be obtained by combining the R.H.S. of (13) with a term in the form of the R.H.S. of (20). When the region under consideration includes of the origin, at which $r = 0$, terms involving $\log r$ have to be excluded by setting $B_n = 0$ and $E = 0$. This will lead to non-singular solution of (2).

Note 1. Solution (12) is also expressed as $(Ar^n + Br^{-n})\,e^{\pm in\theta}$.

Note 2. The solution (12) is known as circular harmonics of degree n. (19) is called circular harmonics of degree zero.

Ex. 2. *Solve the differential equation*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions: (*i*) u is finite when $r \to 0$

(*ii*) $u = \Sigma C_n \cos n\theta$ when $r = a$. **(Delhi B.Sc. (H) 1996, 2005)**

Sol. Given $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$

Suppose (1) has a solution of the form

$$u(r,\theta) = R(r)\,\Theta(\theta) \qquad ...(2)$$

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (1), we get

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

or

$$(r^2R'' + rR')/R = -\Theta''/\Theta, \qquad ...(3)$$

where the dashes denote derivatives with respect to the relevant variables. Since r and θ are independent, (3) is true only if each side in equal to the same constant. Since the required solution involves trigonometric function $\cos n\theta$, we take each side of (5) equal to n^2, where n is an integer. Hence (3) gives

$$r^2 R'' + rR' - n^2 R = 0 \qquad ...(4)$$

and

$$\Theta'' + n^2\Theta = 0 \qquad ...(5)$$

From (4), $$r^2 \frac{d^2R}{d\theta^2} + r\frac{dR}{d\theta} - n^2 R = 0, \qquad ...(6)$$

which is linear homogeneous differential equation.

Let $r = e^z$ so that $D_1 \equiv r\dfrac{d}{dr} \equiv \dfrac{d}{dz}$

Then (6) becomes

$$[D_1(D_1 - 1) + D_1 - n^2]\,R = 0 \quad \text{or} \quad (D_1{}^2 - n^2)\,R = 0 \qquad ...(7)$$

Its anxiliary equation is $D_1{}^2 - n^2 = 0$ so that $D_1 = n, -n$.

Hence solution os (7) is

$$R = A_n e^{nz} + B_n e^{-nz}$$

or

$$R(r) = A_n r^n + B_n r^{-n} \quad (\because \ e^z = r) \qquad ...(8)$$

From (5), $$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta \qquad ...(9)$$

From (8) and (9), a solution of (1) is

$$u_n(r, \theta) = (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta) \qquad ...(10)$$

So by the principal of superposition, a more general solution of (1) is

$$u(r, \theta) = \sum_n (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta), \qquad ...(11)$$

where the summation over n may be finite or infinite

From given B.C. (i), u is finite when $r \to 0$. Hence we must take $B_n = 0$ in (11). Then (11) reduces to

$$u(r,\theta) = \Sigma A_n r^n (C_n \cos n\theta + D_n \sin n\theta)$$

i.e.

$$u(r,\theta) = \Sigma r^n (E_n \cos n\theta + F_n \sin n\theta) \qquad ...(12)$$

where $E_n (= A_n C_n)$ and $F_n (= A_n D_n)$ are new arbitary constants.

Re-writing given B.C. (ii), $u(a,\theta) = \Sigma C_n \cos n\theta \qquad ...(13)$

Putting $r = a$ in (12) and using (13), we get

$$\Sigma C_n \cos n\theta = \Sigma a^n (E_n \cos n\theta + F_n \sin n\theta)$$

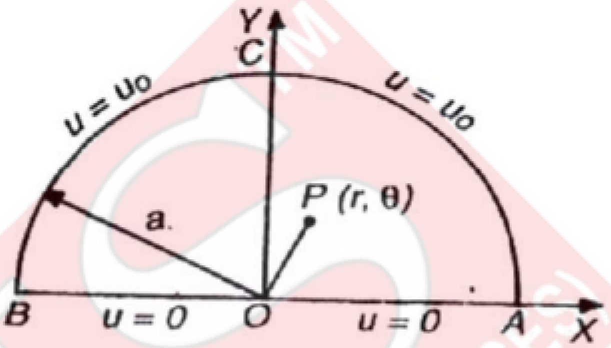
so that $F_n = 0$ and $a^n E_n = C_n$ *i.e.* $E_n = C_n / a^n$. Hence the required solution becomes

$$u(r, \theta) = \Sigma C_n (r/a)^n \cos n\theta.$$

Ex. 3. *Obtain steady temperature distribution in a semi circular plate of radius a, insulated on both faces, with its curved boundary kept at a constant temperature u_0 and its boundary diameter kept at zero temperature.*

Sol. The partial differential equation governing the steady temperature distribution is the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$



For the present broblem for a semi-circular plate the cartesian form of Laplace's equaion is not suitable. So we use polar form of it, namely,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$$

Here we wish to solve (2) under the following boundary conditions

(*i*) $u(r, 0) = 0, \quad u(r, \pi) = 0 \qquad$ (*ii*) $u(a, \theta) = u_0$.

Here conditions (i) express the fact that temperature $u(r, \theta)$ is zero along OA $(\theta = 0)$ and OB $(\theta = \pi)$ so that temperature is zero along the bounding diameter AB. Again condition (ii) expresses the fact that temperature on the boundary ACB (where $r = a$) is u_0.

Suppose (1) has a solution of the form

$$u(r, \theta) = R(r)\,\Theta(\theta), \qquad ...(2)$$

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (1), we get

$$R''\,\Theta + \frac{1}{r} R'\,\Theta + \frac{1}{r^2} R\,\Theta'' = 0$$

or
$$(r^2 R'' + rR')/R = -\Theta''/\Theta\,, \qquad ...(3)$$

where the deshes denote derivatives with respect to the relevant variables. Since r and θ are independent, (3) is true only if each side is equal to the same constant. On physical grounds we require $u(r, \theta) = u(r, \theta + 2\pi)$. Hence the solution must involve trigonometric functions. For this we choose each side of (3) equal to n^2 and obtain

$$\Theta'' + n^2\,\Theta = 0 \qquad ...(4)$$

and
$$r^2 R'' + r R' - n^2 R = 0 \qquad ...(5)$$

Now, putting $\theta = 0$ and $\theta = \pi$ in (1) and using B.C. (i), we have

$$0 = R(r)\,\Theta(0) \qquad \text{and } 0 = R(r)\,\Theta(\pi)$$

so that $\Theta(0) = 0$ and $\Theta(\pi) = 0.$
where we have taken $R(r) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy given B.C. (ii).

The general solution of (4) is

$$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta \quad \text{...(7)}$$

Using B.C. (6), (7) reduces to

$$0 = C_n \text{ and } 0 = C_n \cos n\pi + D_n \sin n\pi$$

so that $C_n = 0$ and $\sin n\pi = 0,$

where we have taken $D_n \neq 0$, since otherwise $u \neq 0$ which does not satisfy given B.C. (ii).

Now, $\sin n\pi = 0 \Rightarrow n$ is an integer.

With the above values of C_n and n, we get

$$\Theta(\theta) = D_n \sin n\theta \quad \text{...(8)}$$

Next, from (5), $r^2 \dfrac{d^2R}{dr^2} + r\dfrac{dR}{dr} - n^2 R = 0,$ ...(9)

which is linear homogeneous differential equation.

Let $r = e^z$ so that $r\dfrac{d}{dr} \equiv \dfrac{d}{dz} \equiv D_1$

Then (9) becomes

$$[D_1(D_1 - 1) + D_1 - n^2]\,R = 0 \quad \text{or} \quad (D_1{}^2 - n^2)\,R = 0 \quad \text{...(10)}$$

Its anxiliary equation is $D_1{}^2 - n^2 = 0$ giving $D_1 = n, -n.$

Hence solution of (10) is

$$R = A_n e^{nz} + B_n e^{-nz}$$

or

$$R(r) = A_n r^n + B_n r^{-n} \qquad (\because e^z = r) \quad \text{...(11)}$$

As $r \to 0$, $r^{-n} \to \infty$ so that $R(r) \to \infty$ and hence $u(r, \theta) \to \infty$. But this contradicts the physical problem in hand. So we take $B_n = 0$ in (11) and obtain

$$R(r) = A_n r^n \quad \text{...(12)}$$

From (8) and (12), a solution of (1) is

$$u_n(r, \theta) = A_n D_n r^n \sin n\theta = E_n r^n \sin n\theta \quad \text{...(13)}$$

where $E_n (= A_n D_n)$ is another constant. These give solution satisfying B.C. (i).

In order to satisfy B.C. (ii) also, we consider a more general solution

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta \quad \text{...(14)}$$

Putting $r = a$ in (14) and using B.C. (ii), we have

$$u_0 = \sum_{n=1}^{\infty} E_n a^n \sin n\theta$$

which is Fourier sine series and hence

$$E_n a^n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin n\theta \, d\theta$$

$$= \frac{2u_0}{\pi} \left[-\frac{\cos n\theta}{n} \right]_0^{\pi} = \frac{2u_0}{n\pi} \left[1 - (-1)^n \right]$$

$$\therefore \qquad E_n = \begin{cases} \dfrac{4u_0}{(2m-1)\pi a^{2m-1}}, & \text{if } n = 2m-1 \\ 0, & \text{if } n = 2m \end{cases}$$

With these values of E_n, (14), reduces to

$$u(r, \theta) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \left(\frac{r}{a} \right)^{2m-1} \sin (2m-1)\theta$$

which gives the required temperature distribution.

Ex. 4. *u is a function of r and θ satisfying the equation*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a$, $r = b$, $\theta = 0$, $\theta = \pi/2$. Its value along the boundary $r = a$ is $\theta(\pi/2 - \theta)$, and it value along the other boundary is zero. Prove that

$$u = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}} \frac{\sin (4m-2)\theta}{(2m-1)^3}$$ **[Delhi B.Sc. (H) 98, 99]**

Sol. Given $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r} \dfrac{\partial u}{\partial \theta} + \dfrac{1}{r} \dfrac{\partial^2 u}{\partial \theta^2} = 0$ ...(1)

We are to solve (1) under the following boundary conditions :

[**Note**—Here shaded area of the figure given on the next page is being considered.]

(i) $u(r, 0) = 0$, $u(r, \pi/2) = 0$

(ii) $u(b, \theta) = 0$ (iii) $u(a, \theta) = \theta(\frac{1}{2}\pi - \theta)$.

Suppose (1) has a solution of the form

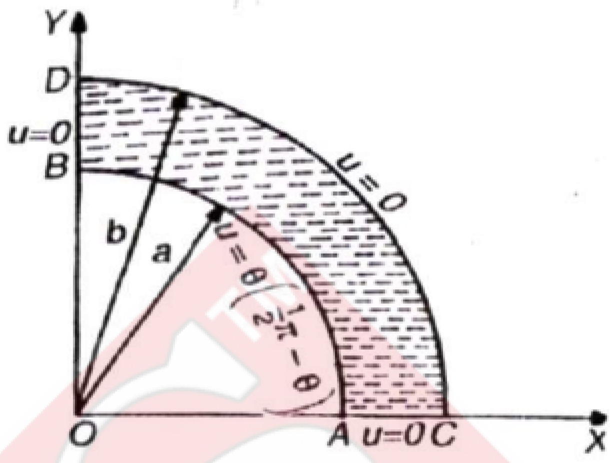
$$u(r, \theta) = R(r)\,\Theta(\theta), \qquad ...(2)$$

where R and Θ are functions of r and θ respectively.

Substituting this value of u in (1), we get

$$R''\Theta + (1/r) R'\Theta = -\Theta''/\Theta, \qquad ...(3)$$

where the dashes denote derivatives with respect to the releant variables. Since r and θ are independent, (3) is true only if each side in equal to the same constant. On physical grounds we require



$$u(r, \theta) = u(r, \theta + 2\pi).$$

Hence the solution must involve trigonometric functions. For this we choose each side of (3) equal to n^2 and obtain

$$\Theta'' + n^2 \Theta = 0 \qquad \ldots(4$$

and $$r^2R'' + rR' - n^2 R = 0, \qquad \ldots(5$$

Now, putting $\theta = 0$ and $\theta = \pi/2$ in (2) and using B.C. (i), we have

$$0 = R(r)\,\Theta(0) \quad \text{and } 0 = R(r)\,\Theta(\pi/2)$$

so that $$\Theta(0) = 0 \text{ and } \Theta(\pi/2) = 0, \qquad \ldots(6$$

where we have taken $R(r) \neq 0$, since otherwise $u = 0$ which does not satis given B.C. (iii).

The general solution of (4) is

$$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta \qquad \ldots(7$$

Using B.C. (6), (7) reduces to

$$0 = C_n \text{ and } 0 = C_n \cos(n\pi/2) + D_n \sin(n\pi/2)$$

so that $\quad C_n = 0$ and $\sin(n\pi/2) = 0$

where we have taken $D_n \neq 0$, since otherwise $u = 0$ which does not sati B.C. (iii).

Now, $\sin(n\pi/2) = 0 \Rightarrow \frac{1}{2} n\pi = (2m-1)\pi$.

$$\therefore \qquad n = 4m - 2 \qquad ..$$

With the above values of C_n and n, (7) becomes

$$\Theta(\theta) = D_m \sin(4m-2)\theta \qquad ..$$

Next, from (5) $$r\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0, \qquad \ldots$$

which is linear homogeneous differential equation.

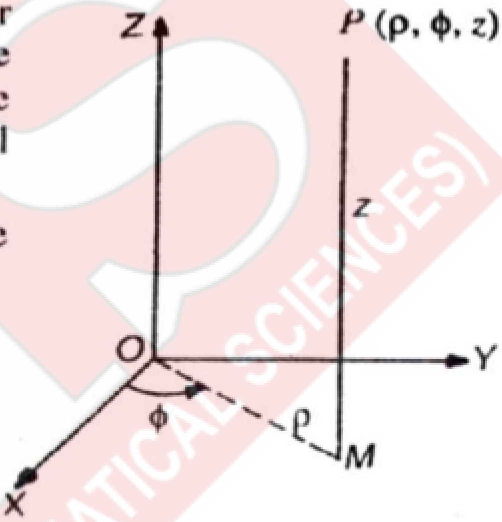
Let $\qquad r = e^z$ so that $r\dfrac{d}{dr} = \dfrac{d}{dz} \equiv D_1$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta,\ A_n = \frac{1}{a^n\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta,$$

$$B_n = \frac{1}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta \text{ where } n = 1, 2, 3,...$$

1.21. Solution of boundary value problems in cylindrical coordinates.

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a cylindrical surface (for example, the problem of evaluating the temperature in a cylindrical rod), it will be convenient to make use of cylindrical coordinates.



Cylindrical coordinates (ρ, φ, z) are defined by means of equations

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi\ ;\ z = z \qquad ...(1)$$

where $\rho \geq 0;\ 0 \leq \varphi \leq 2\pi;\ -\infty < z < \infty.$

The Laplacian operator ∇^2 in cylindrical coordinates, reduces to

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} \qquad ...(2)$$

We produce below some useful equations in cylindrical coordinates (ρ, φ, z) :

I. Laplace's equations $\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

II. Heat (Diffusion) equation $\nabla^2 u = (1/k)(\partial u/\partial t)$

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

III. Wave equation $\nabla^2 u = (1/c^2)(\partial^2 u/\partial t^2)$

$$\frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Note. Some authors use r and θ for ρ and φ respectively while writing the above equations in cylindrical coordinates. The corresponding changes must be made if r and θ are used in any problem.

List of Useful Results

	Solution
1. $\rho^2 R'' + \rho R' + (m^2\rho^2 - n^2) R = 0$. Bessel's equation, where n is an integer	$R(\rho) = A_{mn} J_n(m\rho) + B_{mn} Y_n(m\rho)$, where J_n and Y_n are Bessel's functions of first and second kind respectively. Also note that $Y_n \to \infty$ and $\rho \to 0$
2. $\rho^2 R'' + \rho R' + m^2\rho^2 R = 0$. Bessel's equation of zeroth order	$R(\rho) = A_m J_0(m\rho) + B_m Y_0(m\rho)$ where $Y_0 \to \infty$ as $\rho \to 0$
3. $\rho^2 R'' + \rho R' + (m^2\rho^2 - n^2) R = 0$, where n is not an integer	$R(\rho) = A_{mn} J_n(m\rho) + B_{mn} J_{-n}(m\rho)$ where J_n and J_{-n} are Bessel's functions of first kind and $J_{-n} \to 0$ as $\rho \to 0$
4. $\rho^2 R'' + \rho R' - n^2 R = 0$ (Homogeneous linear equation)	$R(\rho) = A_n r^n + B_n r^{-n}$.

Fourier-Bessel Series

If
$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r),\ 0 \le r \le a,$$

then
$$A_n = \frac{2}{a^2 [J_1(\lambda_n a)]^2} \int_0^a r J_0(\lambda_n r) f(r)\, dr$$

1.22. Solution of Laplace's equation.

Ex. 1. (*i*) Obtain solution of Laplace's equation in cylindrical polar coordinates. **[Delhi B.Sc. (H) 1997, 2000; Ravishankar 95]**

(*ii*) Show that in a cylindrical coordinates Laplace's equation has solution of the form $R(\rho)\, e^{\pm mz \pm in\varphi}$ where $R(\rho)$ is a soution of Bessel's equation. ***[Delhi B.Sc. (Hons) 2000]***

(*iii*) Obtain solution of the three dimensional Laplace equation which is finite on the axis. **[Delhi B.Sc. (H) 1995]**

Sol. (*i*) Laplace's equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(1)}$$

Let a solution of (1) be of the form

$$u(\rho, \varphi, z) = R(\rho)\, \Phi(\varphi)\, Z(z), \qquad \text{...(2)}$$

where R, Φ and Z are functions of ρ, φ, and z respectively.

Then (10) becomes

$$[D_1(D_1-1)+D_1-n^2]R=0 \quad \text{or} \quad (D^2-n^2)R=0 \qquad \ldots(11)$$

Its auxiliary equation is $D_1{}^2-n^2=0$ giving $D_1=n,-n$

Hence solution of (11) is

$$R=A_n e^{nz}+B_n e^{-nz}$$

or
$$R(r)=A_n r^n+B_n r^{-n} \qquad [\because \quad e^z=r] \qquad \ldots(12)$$

Now, putting $r=b$ in (2) and using B.C. (ii), we get

$$0=R(b)\,\Theta(\theta)$$

so that
$$R(b)=0. \qquad \ldots(13)$$

where we have taken $\Theta(\theta)\neq 0$, otherwise $u=0$ which does not satisfy B.C. (iii).

Putting $r=b$ in (12) and using (13), we get

$$0=A_n b^n+B_n b^{-n} \text{ so that } B_n=-A_n b^{2n}$$

Using this value of B_n in (12), we get

$$R(r)=A_n(r^n-b^{2n}r^{-n})=A_n(r^n-b^{2n}/r^n)$$

or
$$R(r)=A_m(r^{4m-2}-b^{8m-4}/r^{4m-2}) \qquad \ldots(14)$$

[using (8)]

From (9) and (14), a solution of (1) satisfying. B.C. (i) and (ii) is

$$u_m(r,\theta)=A_mD_m\left(r^{4m-2}-\frac{b^{8m-4}}{r^{4m-2}}\right)\sin(4m-2)\theta$$

or
$$u_m(r,\theta)=E_m\left(r^{4m-2}-\frac{b^{8m-4}}{r^{4m-2}}\right)\sin(4m-2)\theta \qquad \ldots(15)$$

where $E_m\,(=A_mD_m)$ is new constant.

In order to satisfy B.C. (iii) also, we consider a more general solution

$$u(r,\theta)=\sum_{m=1}^{\infty}E_m\left(r^{4m-2}-\frac{b^{8m-4}}{r^{4m-2}}\right)\sin(4m-2)\theta \qquad \ldots(16)$$

Putting $r=a$ in (16) and using B.C. (iii), we have

$$\theta\left(\frac{\pi}{2}-\theta\right)=\sum_{m=1}^{\infty}E_m\left(a^{4m-2}-\frac{b^{8m-4}}{a^{4m-2}}\right)\sin(4m-2)\theta$$

$\therefore \quad E_m\,(a^{4m-2}-b^{8m-4}/a^{4m-2})$

$$=\frac{4}{\pi}\int_0^{\pi/2}\theta\left(\frac{\pi}{2}-\theta\right)\sin(4m-2)\theta\;d\theta$$

$$= \frac{4}{\pi}\left[\ \theta\left(\frac{\pi}{2}-\theta\right)\cdot\left\{-\frac{\cos(4m-2)\theta}{4m-2}\right\}-\left(\frac{\pi}{2}-2\theta\right)\right.$$
$$\left.\times\left\{-\frac{\sin(4m-2)\theta}{(4m-2)^2}\right\}+(-2)\left\{\frac{\cos(4m-2)\theta}{(4m-2)^3}\right\}\right]_0^{\pi/2}$$

$$= \frac{4}{\pi}\left[-\frac{2\cos(2m-1)\pi}{8(3m-1)^3}+\frac{2}{8(2m-1)^3}\right]$$

$$= \frac{1-(-1)^{2m-1}}{\pi(2m-1)^3} = \frac{2}{\pi(2m-1)^3}$$

$$\therefore\ E_m = \frac{2}{\pi(2m-1)^3(a^{4m-2}-b^{8m-4}/a^{4m-2})}$$

With this value, (16) reduces to

$$u(r,\theta) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{r^{4m-2}-\dfrac{b^{8m-4}}{r^{4m-2}}}{a^{4m-2}-\dfrac{b^{8m-4}}{a^{4m-2}}}\ \frac{\sin(4m-2)\theta}{(2m-1)^3}$$

$$\therefore\qquad u(r,\theta) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(r/b)^{4m-2}-(b/r)^{4m-2}}{(a/b)^{4m-2}-(b/a)^{4m-2}}\ \frac{\sin(4m-2)\theta}{(2m-1)^3}$$

Exercise (H)

1. Write down the Laplace's equation in two-dimensional polar coordinates. Obtain a solution by separating variables. What conditions the arbitrary constants must satisfy if

(i) the solution is regular at $r=0$

(ii) the solution exists for all values of θ ?

2. Show that $(\partial^2 V/\partial r^2)+(1/r)\ (\partial V/\partial r)+(1/r^2)\ (\partial^2 V/\partial\theta^2)=0$ has solutions of the form $(Ar^n+Br^{-n})\,e^{\pm in\theta}$, where A, B and n are constants. Determine V satisfying the above equation in the region $0\le r\le a$, $0\le\theta\le 2\pi$ and satisfying the boundary conditions

(*i*) V remains finite as $r\to 0$.

(*ii*) $V=\sum_n C_n\cos n\theta$ on $r=a$. **[Delhi B.Sc. (H) 96; Meerit 96]**

3. The edge $r=a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.

Ans. $u(r,\theta)=\sum_{n=0}^{\infty} r^n(A_n\cos n\theta+B_n\sin\theta)$, where

Substituting this value of u in (1), we get

$$R''\,\Phi Z + \frac{1}{\rho} R'\,\Phi Z + \frac{1}{\rho^2} R\,\Phi'' Z + R\Phi Z'' = 0$$

where the dashes denote derivative with respect to the relevant variables. Dividing throughout by $R\Phi Z$, we obtain

$$\frac{R''}{R} + \frac{R'}{\rho R} + \frac{\Phi''}{\rho^2 \Phi} = -\frac{Z''}{Z} \qquad ...(3)$$

Since the right hand side depends only on z while the left hand side depends only on ρ and φ, (3) can only be true if each side is equal to the same constant, * say $-m^2$. Then (3) gives

$$Z'' - m^2 Z = 0 \qquad ...(4)$$

and

$$\frac{R''}{R} + \frac{R'}{\rho R} + \frac{\Phi''}{\rho^2 \Phi} = -m^2 \qquad ...(5)$$

Re-writing (5) by separating variables, we have

$$\rho^2 \left(\frac{R''}{R} + \frac{R'}{\rho R} + m^2 \right) = -\frac{\Phi''}{\Phi} \qquad ...(6)$$

Since ρ and φ are independent, (6) is true only when each side is equal to the same constant. On physical grounds of actual problems, we must involve trigonometric functions in solutions. So we choose the proposed constant to be n^2. Furthermore, in order to satisfy the physical condition $u(\rho, \varphi) = u(\rho, \varphi + 2\pi)$, we suppose that n is an integer. Then (6) reduces to

$$\Phi'' + n^2 \Phi = 0 \qquad ...(7)$$

and

$$\rho^2 R'' + \rho R' + (m^2 \rho^2 - n^2) R = 0 \qquad ...(8)$$

Solutions of (4), (7) and (8) are

$$Z(z) = A_m e^{mz} + B_m e^{-mz} \qquad ...(9)$$

$$\Phi(\varphi) = C_n \cos n\varphi + D_n \sin n\varphi \qquad ...(10)$$

$$R(\rho) = E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho) \qquad ...(11)$$

Hence a general solution of (1) is

$$u(\rho, \varphi, z) = \sum_n \sum_m (A_m e^{mz} + B_m e^{-mz})(C_n \cos n\varphi + D_n \sin n\varphi)$$

$$\times [E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho)] \qquad ...(12)$$

Note. Multiplying (9), (10) and (11) we get a solution

$(A_m e^{mz} + B_m e^{-mz})(C_n \cos n\varphi + D_n \sin n\varphi)[E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho)]$ of (1). This is called a cylindrical harmonic.

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* We select $-m^2$ because the resulting equation (4) gives solutions which are generally useful for practical problems to be discussed later.

Part (ii) Proceed like part (i) upto (11). Now solutions (9) and (10) can also be re-written as $e^{\pm mz}$ and $e^{\pm in\varphi}$ respectively. Further, $R(\rho)$ is solution of (8) which is Bessel's equation. Hence we get a solution of the form

$$R(\rho)\, e^{\pm mz \pm in\varphi}.$$

Part (iii) Proceed like part (i) upto (11). We know that $Y_n \to \infty$ as $\rho \to 0$. Now on the axis (*i.e.*, z-axis), $\rho \to \mathbf{0}$ and hence we must take $F_{mn} = 0$ in (11) so that solution may remain finite on the axis. Thus the resulting general solution is given by

$$u(\rho, \varphi, z) = \sum_n \sum_m E_{mn} J_n(m\rho)(A_m e^{mz} + B_m e^{-mz}) \times (C_n \cos n\varphi + D_n \sin n\varphi) \qquad \text{...(14)}$$

which is also expressed as

$$u(\rho, \varphi, z) = \sum_n \sum_m E_{mn} J_n(m\rho)\, e^{\pm mz \pm in\varphi} \qquad \text{...(15)}$$

Ex. 2. *(i) Obtain the axially symmetrical solutions of the three dimensional Laplace equation.*

(ii) Obtain the general solution of Laplace's equation in three dimension which remains finite on the axis of z and is symmetrical about it.

Sol. (i). Laplace's equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{...(1)}$$

Since the required solution is symmetrical about the axis of z, we have $\partial^2 u/\partial\varphi^2 = 0$ so that (1) reduces to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{...(2)}$$

Let a solution of (1) be of the form

$$u(\rho, z) = R(\rho)\, Z(z). \qquad \text{...(3)}$$

Substituting this value of u in (2), we get

$$R''Z + (1/\rho)R'Z + RZ'' = 0 \ \text{ or } \ \frac{1}{R}\left(R'' + \frac{R'}{\rho}\right) = -\frac{Z''}{Z}. \qquad \text{...(4)}$$

Since ρ and z are independent, (4) can only be true if each side is equal to the same constant, say $-m^2$. Then (4) gives

$$Z'' - m^2 Z = 0 \qquad \text{...(5)}$$

and $$R'' + (R'/\rho) = -m^2 R \ \text{ or } \ \rho^2 R'' + \rho R' + m^2\rho^2 R = 0 \qquad \text{...(6)}$$

Solving (5), $$Z(z) = A_m e^{mz} + B_m e^{-mz}. \qquad \text{...(7)}$$

(6) is a Bessel's equation of zeroth order and its solution is

$$R(\rho) = C_m J_0(m\rho) + D_m Y_0(m\rho), \qquad \text{...(8)}$$

where J_0 and Y_0 are function of first and second kind respectively, From (7) and (8), a general axially symmetrical solution is given by

$$u(\rho, z) = \sum_m (A_m e^{mz} + B_m e^{-mz}) [C_m J_0(m\rho) + D_m Y_0(m\rho)] \qquad ...(9)$$

Part (ii) First do like part (*i*) above. Since $Y_0 \to \infty$ as $\rho \to 0$, we take $D_m = 0$ in (9) in order that solution may remain finite on the axis (*i.e.*, axis of z where $\rho \to 0$ at origin). Hence taking $D_m = 0$ in (9) and writing $C_m A_m = E_m$ and $C_m B_m = F_m$, the required general solution is

$$u(\rho, z) = \sum_m J_0(m\rho) \ (E_m e^{mz} + F_m e^{-mz}).$$

Ex. 3. *Find the potential function $\psi(\rho, z)$ in the region $0 \le \rho \le 1$, $z \ge 0$ satisfying the conditions, (i) $\psi \to 0$ as $z \to \infty$, (ii) $\psi = 0$ on $\rho = 1$ (iii) $\psi = f(\rho)$ on $z = 0$ for $0 \le \rho \le 1$.*

Sol. We know that potential function $\Psi(\rho, z)$ is a solution of Laplace's equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0. \qquad ...(1)$$

We are to solve (1) subject to the following boundary conditions :

$$\psi(\rho, z) \to 0 \text{ as } z \to \infty, \qquad ...(2)$$

$$\psi(1, z) = 0 \qquad ...(3)$$

$$\psi(\rho, 0) = f(\rho), \quad 0 \le \rho \le 1. \qquad ...(4)$$

Let a solution of (1) be of the form

$$\psi(\rho, z) = R(\rho) Z(z). \qquad ...(5)$$

Substituting this value of Ψ in (1), we get

$$R''Z + (1/\rho) R'Z + RZ'' = 0$$

or

$$\frac{1}{R}\left(R'' + \frac{1}{\rho} R'\right) = -\frac{Z''}{Z}$$

Since ρ and z are independent, the above equation is true if each side in equal to the same constant, say $-m^2$. Hence the above equation gives

$$Z'' - m^2 Z = 0$$

and $\quad R'' + (R''/\rho) = -m^2\rho$ or $\rho^2 R'' + \rho R' + m^2\rho^2 R = 0$

Solving these equations we obtain respectively,

$$Z(z) = A_m e^{mz} + B_m e^{-mz} \qquad ...(6)$$

and

$$R(\rho) = C_m J_0(m\rho) + D_m Y_0(mp) \qquad ...(7)$$

We must take $A_m = 0$ in (6), since otherwise $Z(z) \to \infty$ as $z \to \infty$ and hence $\psi(\rho, z) \to \infty$ as $z \to \infty$ which contradicts B.C. (2).

Hence (6) reduces to

$$Z(z) = B_m e^{-mz}. \quad \text{...(8)}$$

Again we must take $D_m = 0$, since otherwise $R(\rho) \to \infty$ as $\rho \to 0$

$(\because Y_0(m\rho) \to \infty$ as $\rho \to 0)$ and hence $\psi(\rho, z) \to \infty$ as $\rho \to 0$ which contradicts the fact that potential is finite on the axis of z.

Hence (7) reduces to

$$R(\rho) = C_m J_0(m\rho). \quad \text{...(9)}$$

Putting $\rho = 1$ in (5) and using (3), we get

$$R(1) Z(z) = 0 \quad \text{so that } R(1) = o. \quad \text{...(10)}$$

where we have taken $Z(z) \neq 0$, since otherwise $\psi(\rho, z) \equiv 0$ which does not satisfy (4).

Putting $\rho = 1$ in (9) and using (10), we get

$$J_0(m) = 0. \quad \text{...(11)}$$

Let m_i $(i = 1, 2, 3, ...)$ denote the ith positive root of (11). From (8), (9) and (5), the function $E_m J_0(m\rho) e^{-mz}$ is a solution of (1) satisfying (2) and (3). Here $E_m (= C_m B_m)$ are new arbitrary constants. In view of (11), the functions

$$\psi(\rho, z) = F_i J_0(m_i\rho) e^{-m_i z}, i = 1, 2, 3, ...$$

are solutions of (1), satisfying (2) and (3) ; here we write $E_{m_i} = F_i$. In order to obtain a solution also satisfying (4), we consider a more general solution by superposition

$$\psi(\rho, z) = \sum_{i=1}^{\infty} \psi_i(\rho, z) = \sum_{i=1}^{\infty} F_i J(m_i\rho) e^{-m_i z} \quad \text{...(12)}$$

Putting $z = 0$ in (12) and using (4), we get

$$f(\rho) = \sum_{i=1}^{\infty} F_i J_0(m_i \rho)$$

which is Fourier-Bessel series for $f(\rho)$ and so we get

$$F_i = \frac{2}{[J_1(m_i)]^2} \int_0^1 \rho f(\rho) J_0(m_i \rho) \, d\rho, \quad i = 1, 2, 3, ... \quad \text{...(13)}$$

Hence the required potential functional $\psi(\rho, z)$ is given by (12) wherein F_i is given by (13).

Exercise (I)

1. Derive the solution of $\partial^2 V/\partial r^2 + (1(r)(\partial V/\partial r) + \partial^2 V/\partial z^2 = 0$ for the region $r \geq 0$, $z \geq 0$, with boundary condition:

$V \to 0$ as $z \to \infty$ and $r \to \infty$.

$V = f(r)$ on $z = 0$, $z \geq 0$.

2. Obtain the general solution of Laplace's equation in cylindrical coordinates for the case of axial symmetry, namely

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \textbf{(I.A.S. 1984)}$$

1.23. Solution of Heat (diffusion) equation.

Ex. 1. *Solve the heat (diffusion) equation in cylindrical coordinates.* **[Delhi B.Sc. (H) 1998]**

OR

Obtain the solution of the diffusion equation $\nabla^2 u = (1/k)(\partial u/\partial t)$ *in cylindrical coordinates* (ρ, φ, z) *in the form*

$$\sum_{\lambda, \mu, \gamma} A_{\lambda\mu\gamma} J_{\pm\gamma} \{\sqrt{(\lambda^2 + \mu^2)}\, \rho\} \exp(\pm \mu z - \lambda^2 kt \pm i\gamma \varphi).$$

Sol. Heat (diffusion) equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \text{...(1)}$$

Let a solution of (1) be of the form

$$u(\rho, \varphi, z, t) = R(\rho)\,\Phi(\varphi)\,Z(z)\,T(t) \qquad \text{...(2)}$$

Substituting this value of u in (1) and then dividing by $R\Phi ZT$,

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = \frac{1}{k}\frac{T'}{T} \qquad \text{...(3)}$$

Since the right hand side depends only on t while the left hand side depends only on ρ, Φ and z, (3) can only be true if each side is equal to the same constant, say $-\lambda^2$. We have chosen this special value of constant so that the resulting solution may tend to zero as $t \to \infty$ (Note that with $-\lambda^2$, (3) gives $T' = -k\lambda^2 T$ so that $T(t) = Ae^{-k\lambda^2 t}$ showing that $T(t) \to 0$ as $t \to \infty$ and hence $u \to 0$ at $t \to \infty$). Hence (3) reduces to

$$T' = -\lambda^2 kT \qquad \text{...(4)}$$

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \lambda^2 = -\frac{Z''}{Z} \qquad \text{...(5)}$$

Since the right hand side of (5) depends on z only while the left hand side depends only on ρ and ϕ, (5) can only be true if each side is equal to the same constant, say $-\mu^2$ (as explained in Ex. 1 of Art.1.22). So (5) gives

$$Z'' - \mu^2 Z = 0 \qquad ...(6)$$

and
$$p^2\left(\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \lambda^2 + \mu^2\right) = -\frac{\Phi''}{\Phi} \qquad ...(7)$$

Since ρ and φ are independent, (7) can only be true if each side is equal to the same constant, say γ^2 (as in Ex. 1 of Art. 1.22). Hence (7) gives

$$\Phi'' + \gamma^2 \Phi = 0 \qquad ...(8)$$

and
$$\frac{1}{R}(\rho^2 R'' + \rho R') + (\lambda^2 + \mu^2)\rho^2 = \gamma^2$$

i.e.
$$\rho^2 R'' + \rho R' + \{(\lambda^2 + \mu^2)\rho^2 - \gamma^2\} R = 0 \qquad ...(9)$$

Solutions of (4), (6), (8) and (9) are

$$T_\lambda(t) = A_\lambda e^{-k\lambda^2 t} \qquad ...(10)$$

$$Z_\mu(z) = B_\mu e^{\mu z} + C_\mu e^{-\mu z} \qquad ...(11)$$

$$\Phi_\gamma(\varphi) = D_\gamma \cos\gamma\varphi + E_\gamma \sin\gamma\varphi \qquad ...(12)$$

and
$$R_{\lambda\mu\gamma}(\rho) = F_{\lambda\mu\gamma} J_\gamma\{\sqrt{(\lambda^2+\mu^2)}\,\rho\} + G_{\lambda\mu\gamma} J_{\gamma}(\{\sqrt{(\lambda^2+\mu^2)}\,\rho\}. \qquad ...(13)$$

General solution of (1) is given by

$$u(\rho, \varphi, z, t) = \sum_{\lambda,\mu,\gamma} T_\lambda(t)\, Z_\mu(t)\, \Phi_\gamma(\varphi)\, R_{\lambda\mu\gamma}(\rho)$$

which can also be expressed as

$$u(\rho, \varphi, z, t) = \sum_{\lambda,\mu,\gamma} A_{\lambda\mu\gamma} J_{\pm\gamma}\{\sqrt{(\lambda^2+\mu^2)}\,\rho\}\, e^{\pm\mu z - \lambda^2 kt \pm i\gamma\varphi}$$

Ex. 2. *Determine the temperature distribution in the infinite cylinder* $0 \le \rho \le a$ *when the initial temperature is* $\theta(\rho, 0) = f(\rho)$ and the surface $\rho = a$ is maintained at zero temperature. **[*Delhi B.Sc. (H) 2001, 2003*]**

OR

A circular plate of radius a has its plane faces insulated. If the initial temperature is $f(\rho)$ *and if the rim is kept at temperature zero, find the* temperature of the plate at any time. **[*Delhi B.Sc. (H) Phy. 1999*]**

Sol. Heat equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\theta}{\partial\varphi^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{k}\frac{\partial\theta}{\partial t} \qquad ...(1)$$

Due to the physical conditions of the problem, we have

$$\partial^2\theta/\partial\varphi^2 = 0 \text{ and } \partial^2\theta/\partial z^2 = 0$$

So temperature $\theta(\rho, t)$ in the present problem is governed by

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} = \frac{1}{k}\frac{\partial\theta}{\partial t} \qquad ...(2)$$

We are to solve (2) under the following conditions :

$$\theta(a, t) = 0 \qquad ...(3)$$

$$\theta(\rho, 0) = f(\rho) \qquad ...(4)$$

Let solution of (2) be of the from

$$\theta(\rho, t) = R(\rho)\, T(t) \qquad ...(5)$$

Substituting this value of θ in (2) and dividing by RT, we get

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} = \frac{1}{k}\frac{T'}{T} \qquad ...(6)$$

Since ρ and t are independent, (6) can only be true if each side is equal to the same constant, say $-\lambda^2$. Then (6) gives

$$T' = -\lambda^2 kT \qquad ...(7)$$

and
$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} = -\lambda^2$$

i.e.
$$\rho^2 R'' + \rho R' + \lambda^2\rho^2 R = 0 \qquad ...(8)$$

Solving (7),
$$T(t) = A_\lambda e^{-\lambda^2 kt} \qquad ...(9)$$

(8) is Bessel's equation of zeroth order and hence its solution is

$$R(\rho) = B_\lambda J_0(\lambda\rho) + C_\lambda Y_0(\lambda\rho) \qquad ...(10)$$

We take $C_\lambda = 0$ in (10), since otherwise

$$R(\rho) \to \infty \text{ as } \rho \to 0 \qquad (\because \quad Y_0(\lambda\rho) \to \infty \text{ as } \rho \to 0)$$

and hence $\quad \theta(\rho, t) \to \infty \text{ as } \rho \to 0$

which contradicts the fact that temperature is finite on the axis of z. Hence (10) reduces to

$$R(\rho) = B_\lambda J_0(\lambda\rho) \qquad ...(11)$$

Putting $\rho = a$ in (5) and using B.C. (3), we get

$$R(a)T(t) = 0 \text{ so that } R(a) = 0, \qquad ...(12)$$

where we have taken $T(t) \neq 0$, since otherwise $\theta(\rho, t) \equiv 0$ which does not satisfy (4).

Putting $\rho = a$ in (11) and using (12), we get

$$J_0(\lambda a) = 0 \qquad (\because B_\lambda \neq 0) \qquad ...(13)$$

Let λ_i $(i = 1, 2, 3, ...)$ denote the ith positive roots of (13). From (5), (9) and (11), the function

$$D_\lambda J_0(\lambda\rho)\, e^{-\lambda^2 kt}, \text{ where } D_\lambda = A_\lambda B_\lambda,$$

is a solution of (2) satisfying (3). In view of (13), the functions

$$\theta_i(\rho, t) = E_i J_0(\lambda_i \rho)\, e^{-\lambda_i^2 kt},\ i = 1, 2, 3, ...$$

are solutions of (2), satisfying (3). Here we write $D_{\lambda_i} = E_i$. In order to obtain a solution also satisfying (4), we consider a more general solution (by superposition)

$$\theta(\rho, t) = \sum_{i=1}^{\infty} \theta_i(\rho, t) = \sum_{i=1}^{\infty} E_i J_0(\lambda_i \rho) e^{-\lambda_i^2 kt} \quad \ldots(14)$$

Putting $t = 0$ in (14) and using (4), we get

$$f(\rho) = \sum_{i=1}^{\infty} E_i J_0(\lambda_i \rho)$$

which is Fourier-Bessel series and hence

$$E_i = \frac{2}{a^2 [J_1(\lambda_i a)]^2} \int_0^a \rho f(\rho) J_0(\lambda_i \rho)\, d\rho,\ i = 1, 2, 3, \ldots \quad \ldots(15)$$

Hence the required temperature distribution $\theta(\rho, t)$ is given by (14) wherein the coefficients E_i are determined by (15).

Exercise (J)

1. Show that in cylindrical coordinates Laplace's equation has solution of the form $R(\rho)\, e^{\pm mz = \pm in\varphi}$, where $R(\rho)$ is solution of Bessel's equation. Hence find the temperature distribution $\theta(\rho, \varphi, z)$ in a cylindrical rod $0 \le \rho \le a, 0 \le z \le b$ with boundary conditions $\theta = 0$ at $0 \le \rho \le \le a, z = b$ and $\theta(a, \varphi, z) = f(\varphi, z)$.

2. A circular cylinder of radius a has its surface kept at a constant temperature θ_0. If the initial temperature is zero throughout the cylinder, prove that for $t>0$

$$\theta(r, t) = \theta_0 \left\{ 1 - \frac{2}{a} \sum_{i=1}^{\infty} \frac{J_0(\lambda_i a)}{\lambda_{i,} J_1(\lambda_i a)} e^{-\lambda_i^2 kt} \right\}$$

where $\lambda_i, i = 1, 2, 3, \ldots$ are the positive roots of $J_0(\lambda a) = 0$ and k is the diffusivity.

3. An infinite homogeneous solid circular cylinder of radius a is lagged round the outside to prevent heat escape. At any time t, the temperature $u(r, t)$ at a distance r from the axis of symmetry $(0 \le r \le a)$ is given by the heat diffusion equation (axial symmetry).

$$k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial t}, \quad (0 \le r \le a;\ t > 0)$$

where k is a constant. At time $t = 0$, the intial temperature distribution at r from the axis is known to be $f(r)$. Find the temperature distribution at any subsequent time.

Ans. $u(r, t) = \sum_{i=1}^{\infty} A_i J_0(\lambda_i r) e^{-k\lambda_i^2 t}$, where $A_i = \dfrac{2}{a^2 [J_0(\lambda_i a)]^2} \int_0^a r f(r) J_0(\lambda_i r)\, dr$

and λ_i, $i = 1, 2, 3, \ldots$ are positive roots of $J_0(\lambda a) = 0$.

1.24. Solution of wave equation.

Ex. 1. *Obtain a solution of wave equation in cylindrical coordinates by the method of separation variables* **[Delhi B.Sc. (H) 1998, 99, 2001]**

Sol. Wave equation in cylindrical coordinates (ρ, φ, z) is

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{...(1)}$$

Let a solution of (1) be of the form

$$u(\rho, \varphi, z, t) = R(\rho)\, \Phi(\varphi)\, Z(z)\, T(t) \qquad \text{...(2)}$$

Substituting this value of u in (1) and then dividing $RZ\Phi T$, we obtain

$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + \frac{Z''}{Z} = \frac{1}{c^2}\frac{T''}{T} \qquad \text{...(3)}$$

Since the right hand side depends only on t while the left hand side depends on ρ, φ and z, (3) can only be true if each side is equal to the same constant, say $-k^2$. Then (3) gives

$$T'' + c^2 k^2 T = 0 \qquad \text{...(4)}$$

and
$$\frac{R''}{R} + \frac{1}{\rho}\frac{R'}{R} + \frac{1}{\rho^2}\frac{\Phi''}{\Phi} + k^2 = -\frac{Z''}{Z} \qquad \text{...(5)}$$

Since the right hand side of (5) depends on z only while the left hand side depends only on ρ and φ, (5) can only be true if each side is equal to the same constant, say γ^2. Then (5) gives

$$Z'' + \gamma^2 Z = 0 \qquad \text{...(6)}$$

and
$$\frac{1}{R}\left(R'' + \frac{1}{\rho}R'\right) + k^2 - \gamma^2 = -\frac{1}{\rho^2}\frac{\Phi''}{\Phi} \qquad \text{...(7)}$$

i.e.,
$$\frac{1}{R}(\rho^2 R'' + \rho R') + (k^2 - \gamma^2)\rho^2 = -\frac{\Phi''}{\Phi}$$

i.e.,
$$\frac{1}{R}(\rho^2 R'' + \rho R') + \omega^2 \rho^2 = -\frac{\Phi''}{\Phi} \qquad \text{...(8)}$$

where
$$\omega^2 = k^2 - \gamma^2 \qquad \text{...(8)}'$$

Since ρ and φ are independent, (8) is true only if each side is equal to the same constant, say m^2. Then (8) gives

$$\Phi'' + m^2 \Phi = 0 \qquad \text{...(9)}$$

and
$$\rho^2 R'' + \rho R' + (\omega^2 \rho^2 - m^2) R = 0 \qquad \text{...(10)}$$

Solving (4), (6), (9) and (10), we get

$$T(t) = A_k \cos ckt + B_k \sin ckt \quad ...(11)$$

$$Z(z) = C_\gamma \cos \gamma z + D_\gamma \sin \gamma z \quad ...(12)$$

$$\Phi(\varphi) = E_m \cos m\varphi + F_m \sin m\varphi \quad ...(13)$$

and
$$R(\rho) = G_{k\gamma m} J_m(\omega\rho) + H_{k\gamma m} Y_m(\omega\rho) \quad ...(14)$$

We choose $H_{k\gamma m} = 0$ in (14) so that $R(\rho)$ and hence $u(\rho, \varphi, z, t)$ may be finite at $\rho = 0$ (along the z-axis). Then (14) reduces to

$$R(\rho) = G_{k\gamma m} J_m(\omega\rho) \quad ...(15)$$

Hence the most general solution is

$$u(\rho, \varphi, z, t) = \sum_{k, \gamma, m} G_{k\gamma m} J_m(\omega\rho)(A_k \cos ckt + B_k \sin ckt)$$

$$\times (C_\gamma \cos \gamma z + D_\gamma \sin \gamma z)(E_m \cos m\varphi + F_m \sin m\varphi)$$

which can also be expressed as

$$u(\rho, \varphi, z, t) = \sum_{k, \gamma, m} G_{k\gamma m} J_m(\omega\rho)\, e^{ikct - i\gamma z \pm im\varphi}$$

Ex. 2. *Discuss the solution of transverse vibrations of a thin membrane bounded by circle of radius a in a xy-plane described by the function z (x, y, t) satisfyin the wave equation,*

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad \textbf{[Delhi B.Sc. (H) 1997]}$$

with conditions $z = 0$ on $r = a$ and $z = f(r)$, $\partial z/\partial t = 0$ at $t = 0$. **(Meerut 98)**

Sol. Due to axial symmetry, $\partial^2 z/\partial\theta^2 = 0$. Hence we are to determine deflection $z(r, t)$ satisfying

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad ...(1)$$

which is subject to the following boundary conditions

$$z(a, t) = 0 \quad ...(2)$$

$$\left(\frac{\partial z}{\partial t}\right)_{t=0} = 0 \quad ...(3)$$

$$z(r, 0) = f(r) \quad ...(4)$$

Let a solution of (1) be of the form

$$z(r, t) = R(r)\, T(t) \quad ...(5)$$

Substituting this value of z in (1) and dividing by RT, we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{c^2}\frac{T''}{T} \quad ...(6)$$

Since r and t are independent, (6) can only be true if each side is equal to the same constant, say $-k^2$. Then (6) gives

$$T'' + c^2k^2T = 0 \quad \text{...(7)}$$

and $$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$$

i.e., $$r^2 R'' + r R' + k^2r^2 R = 0 \quad \text{...(8)}$$

Solving (7) and (8), we have

$$T(t) = A_k \cos(ckt) + B_k \sin(ckt) \quad \text{...(9)}$$

$$R(r) = C_k J_0(kr) + D_k Y_0(kr) \quad \text{...(10)}$$

We choose $D_k = 0$ so that $R(r)$ and hence $z(r, t)$ may be finite at $r = 0$ (along the z-axis). Then (10) reduces to

$$R(r) = C_k J_0(kr) \quad \text{...(11)}$$

Differentiating with respect to t, (5) and (9) give

$$\partial z/\partial t = R(r)\, T'(t) \quad \text{...(12)}$$

and $$T'(t) = -A_k ck \sin(ckt) + B_k ck \cos(ckt) \quad \text{...(13)}$$

Putting $t = 0$ in (12) and using (3), we get

$$0 = R(r)\, T'(0) \qquad \text{so that } T'(0) = 0 \quad \text{...(14)}$$

where we have taken $R(r) \not\equiv 0$, since otherwise $z(r, t) \equiv 0$ which does not satisfy (4). Putting $t = 0$ in (13) an using (14), we get

$$0 = B_k ck \text{ so that } B_k = 0.$$

Hence (9) reduces to $T(t) = A_k \cos(ckt)$ ...(15)

From (11) and (15), a solution of (1) is $A_k C_k J_0(kr) \cos(ckt)$

i.e. $$E_k J_0(kr) \cos(ckt), \quad \text{...(16)}$$

where $E_k\ (= A_k C_k)$ is a new arbitrary constant

Putting $r = a$ in (5) and using (2) we get $0 = R(a)\, T(t)$

so that $R(a) = 0$ as $T(t) \neq 0$.

Putting $r = a$ in (11) and using $R(a) = 0$, we get

$$0 = C_k J_0(ka) \qquad \text{so that} \qquad J_0(ka) = 0 \quad \text{...(17)}$$

where we have taken $C_k \neq 0$, since otherwise $R(r) \equiv 0$ so that $z(r, t) \equiv 0$ which does not satisfy (4).

Let k_i, $i = 1, 2, 3, \ldots$ be the positive roots of (17). Hence the functions [using (16)]

$$z_i(r, t) = F_i J_0(k_i r) \cos(ck_i t),\ i = 1, 2, 3, \ldots$$

are solutions of (1), satisfying (2) and (3). Here we write $E_{k_i} = F_i$. In order to obtain a solution also satisfying (4), we consider a more general solution (by superposition)

$$z(r, t) = \sum_{i=1}^{\infty} z_i(r, t) = \sum_{i=1}^{\infty} F_i J_0 (k_i r) \cos (ck_i t) \qquad ...(18)$$

Putting $t = 0$ in (18) an using (4), we get

$$f(r) = \sum_{i=1}^{\infty} F_i J_0 (k_i r)$$

which is Fourier-Bessel series for $f(r)$ and so we get

$$F_i = \frac{2}{a^2 [J_1(k_i a)]^2} \int_0^a r f(r) J_0 (k_i r)\, dr, \ i = 1, 2, 3, ... \qquad ...(19)$$

Hence the required solution is given by (18) wherein F_i is given by (19).

Ex. 3. *Consider vibrations of a circular membrane of radius a which is fixed along the boundary r = a. Let initial deflection be f(r) and initial velocity be g(r). Obtain radially symmetric solution u(r, t) of*

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

with
$$u (a, t) = 0 \qquad ...(2)$$

$$(\partial z/\partial t)_{t=0} = g(r) \qquad ...(3)$$

and
$$u(r, 0) = f(r) \qquad ... (4)$$

[*Delhi B.Sc. Hons Physics 2001*]

Sol. Let a solution of (1) be of the form

$$z(r,t) = R(r)T(t) \qquad ...(5)$$

Substituting this value of z in (1) and dividing by RT, we get

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \frac{1}{c^2}\frac{T''}{T} \qquad ...(6)$$

Since r and t are independent, (6) can only be true if each side is equal to the same constant, say $-k^2$. Then (6) gives

$$T'' + c^2k^2 T = 0 \qquad ...(7)$$

and
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -k^2$$

i.e.
$$r^2R'' + r R' + k^2r R = 0 \qquad ...(8)$$

Solving (7) and (8), we have

$$T(t) = A_k \cos (ckt) + B_k \sin (ckt) \qquad ...(9)$$

and
$$R(r) = C_k J_0 (kr) + D_k Y_0(kr) \qquad ...(10)$$

We choose $D_k = 0$ so that $R(r)$ and hence $z(r, t)$ may be finite at $r = 0$ (along the z-axis). The (10) reduces to

$$R(r) = C_k J_0(kr) \qquad ...(11)$$

Particular Case :—If initial velocity is zero so that

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(r) = 0,$$

then $G_i = 0$ from (17). Hence deflection is given by

$$u(r, t) = \sum_{i=1}^{\infty} H_i \cos(ck_i t)\, J_0(k_i r)$$

where H_i is given by (16). This has already been obtained in Ex. 2 in an alternative way.

Exercise (K)

1. Show that one possible type of vibration of a uniform circular membrane of radius a whose boundary is fixed is given by $u(r, \theta, t) = J_n(kr) \cos n\theta \cos ckt$ where k is a root of $J_n(ak) = 0$, and c is a constant depending on the membrane.

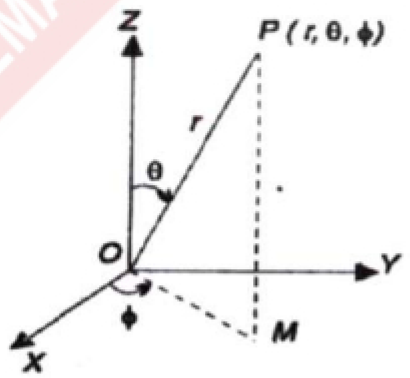
2. Find a solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \text{ for } r > 1, t > 0,$$

given $u(r, \theta, 0) = f(r, \theta)$, $(\partial u/\partial t)_{t=0} = 0$, $u(1, \theta, t) = 0$.

1.25. Solution of boundary value problems in spherical coordinates.

While solving boundary value problems, appropriate choice of coordinate system is very useful. In physical problems that involve a spherical surface (for example, the problem of evaluating temperature in a solid sphere) it will be natural to make use of spherical coordinates.



Spherical coordinates (r, θ, φ) are defined by means of equations

$$x = r \sin\theta \cos\varphi,\ y = r \sin\theta \sin\varphi,\ z = r\cos\theta \quad \text{...(1)}$$

where $r \ge 0$, $0 \le \theta \le \pi$, $0 \le \varphi < 2\pi$. The Laplacian operator ∇^2 in spherical coordinates reduces to

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2} \quad \text{...(2)}$$

which may also be re-written as

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \varphi^2} \quad \text{...(3)}$$

We produce below some useful equations in spherical coordinates (r, θ, φ):

I. Laplace's Equation $\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0$$

II. Heat (Diffusion) equation $\nabla^2 u = (1/k)\,(\partial u/\partial t)$

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=\frac{1}{k}\frac{\partial u}{\partial t}$$

III. Wave equation $\nabla^2 u = (1/c^2)\,(\partial^2 u/\partial t^2)$

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

List of Useful Results

Equation	Solution
1. $\Theta''+\cot\theta\,\Theta'+\{n(n+1)-(m^2/\sin^2\theta)\}\,\Theta=0$	$\Theta(\theta)=A_{mn}P_n^m(\cos\theta)+B_{mn}Q_n^m(\cos\theta)$
OR	OR
$(1-\mu^2)\dfrac{d^2\Theta}{d\mu^2}-2\mu\dfrac{d\Theta}{d\mu}+\left\{n(n+1)-\dfrac{m^2}{1-\mu^2}\right\}\Theta=0$ where $\mu=\cos\theta$. (Legendre's associated equation)	$\Theta(\mu)=A_{mn}P_n^m(\mu)+B_{mn}Q_n^m(\mu)$ where P_n^m and Q_n^m are associated Legendre's functions of first and second kind respectively. Not that $Q_n^m\to\infty$ as $\theta\to 0$
2. $\Theta''+\cot\theta\,\Theta'+n(n+1)\,\Theta=0$	$\Theta(\theta)=A_nP_n(\cos\theta)+B_nQ_n(\cos\theta)$
OR	OR
$(1-\mu^2)\dfrac{d^2\Theta}{d\mu^2}-2\mu\dfrac{d\Theta}{d\mu}+n(n+1)\,\Theta=0,$ where $\mu=\cos\theta$. (Legendre's equation)	$\Theta(\mu)=A_nP_n(\mu)+B_nQ_n(\mu)$ where P_n ans Q_n are Legendre's functions of first and second kind respectively. Note that $Q_n\to\infty$ as $\theta\to 0$

3.	$r^2R'' + 2rR' + \{\lambda^2 r^2 - n(n+1)\}R = 0$	$R(r) = (\lambda)^{-1/2}[A_n J_{n+1/2}(\lambda r) + B_n J_{-(n+1/2)}(\lambda r)]$ where $J_{\pm(n+1/2)}$ are spherical Bessel functions.
4.	$r^2R'' + 2rR' - n(n+1)R = 0$ Homogeneous equation.	$R(r) = A_n r^n + \dfrac{B_n}{r^{n+1}}$

Legendre series Expansion of f (μ).

If $$f(\mu) = \sum_{n=0}^{\infty} A_n P_n(\mu),$$

then $$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(\mu)\, P_n(\mu)\, d\mu .$$

1.26. Solution of Laplace's equation.

Ex. 1. (*i*) *Find the solution of Laplace's equation in spherical coordinates by the method of separation of variables.*

(*ii*) *Show that in spherical polar coordinates Laplace's equation possesses solution of the form*

$$\left(Ar^n + \frac{B}{r^{n+1}}\right) P_n^m(\cos\theta)\exp(\pm im\varphi),$$

where $P_n(\cos\theta)$ i.e. $P_n(\mu)$ satisfies the differential equation

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

[*Delhi B.Sc. (H) 1998, 2001, 2003*]

Sol. (*i*) Laplace's equation is spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0 \qquad ...(1)$$

Let a solution of (1) be of the form

$$u(r, \theta, \varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi) \qquad ...(2)$$

where R, Θ and Φ are functions of r, θ and φ respectively. Substituting this value of u in (1) and dividing by $R\,\Theta\,\Phi$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = 0$$

or $$\left[\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Theta}(\Theta'' + \cot\theta\,\Theta')\right] r^2\sin^2\theta = -\frac{\Phi''}{\Phi} \qquad ...(3)$$

Since the right hand side depends only on φ while the left hand side depends only on r and θ, (3) can only be true if each side is equal to the same constant, say m^2. In many physical problems, we require that solution should involve trigonometric function involving φ. Hence we have chosen m^2 as the constant of separation in (3). Thus (3) reduces to

$$\Phi'' + m^2 \Phi = 0 \qquad ...(4)$$

and
$$\frac{1}{R}(r^2R'' + 2rR') + \frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') = \frac{m^2}{\sin^2\theta}$$

i.e.,
$$\frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}(r^2R'' + 2rR') \qquad ...(5)$$

Since r and θ are independent, (5) can only be true if each side of (5) is equal to the same constant, say $-n(n+1)$ (not carefully). Then (5) gives

$$r^2R'' + 2rR' - n(n+1)R = 0 \qquad ...(6)$$

and
$$\Theta'' + \cot\theta\,\Theta' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \qquad ...(7)$$

which is associated Legendre's equation.

Putting $\cos\theta = \mu$, (7) reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0 \qquad ...(8)$$

Solving (4), (6) and (8), we have

$$\Phi(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi \qquad ...(9)$$

$$R(r) = C_n r^n + \frac{D_n}{r^{n+1}} \qquad ...(10)$$

and
$$\Theta(\theta) = E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta) \qquad ...(11)$$

$$\therefore \quad u_{mn}(r,\theta,\varphi) = (A_m \cos m\varphi + B_m \sin m\varphi)(C_n r^n + D_n / r^{n+1})$$
$$\times [E_{mn} P_n^m(\cos\theta) + F_{mn} Q_n^m(\cos\theta)] \qquad ...(12)$$

are solutions of (1). A more general solution of (1) is given by

$$u(r,\theta,\varphi) = \sum_m \sum_n u_{mn}(r,\theta,\varphi) \qquad ...(13)$$

where $u_{mn}(r,\theta,\varphi)$ is given by (12)

Solution (13) can also be expressed as

$$\Sigma (C_n r^n + D_n / r^{n+1})\,\Theta(\cos\theta)\,e^{\pm im\varphi}$$

where $\Theta(\mu)$ is solution of associated Legendre's equation

Part (ii) Do upto equation (12) as in part (*i*). In many real problems, we know that on physical grounds the functions Θ remains finite (and hence u remains finite) along the polar axis $\theta = 0$. But $\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow Q_n^m(\cos\theta) \to \infty$. Hence we take $F_{mn} = 0$. then solution (12) may be expressed in the desired form.

$$(C_n r^n + D_n / r^{n+1})\,P_n^m(\cos\theta)\exp(\pm im\varphi).$$

Remarks. A solution of (1) of the form $R(r)\,\Theta(\theta)\,\Phi(\varphi)$ given by (12) is called a spherical harmonic and a solution of the form $\Theta(\theta)\,\Phi(\varphi)$ is called a surface harmonic.

Ex. 2. *Potential* $u(r, \theta)$ *in the exterior of a unit sphere satisfies the equation*

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0.$$

Obtain the expression for $u(r, \theta)$ *if*

(*i*) $u(1, \theta) = \cos 2\theta$.

(*ii*) $u(1, \theta) = \cos 3\theta - 1$.

Sol. Part. (*i*) Re-writing given equation, we have

$$r^2\frac{\partial^2 u}{\partial r^2}+2r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial\theta^2}+\cot\theta\frac{\partial u}{\partial\theta}=0. \quad \text{...(1)}$$

We require potential $u(r, \theta)$ at a point (r, θ) outside the given unit sphere (*i.e.*, sphere of radius unity). On physical grounds, potential at infinity is zero.

$$\therefore \quad \lim_{r\to\infty} u(r, \theta) = 0. \quad \text{...(2)}$$

Further, given $\quad u(1, \theta) = \cos 2\theta$. ...(3)

We now solve (1) subject to the B.C. (2) and (3). Let a solution of (1) be of the form

$$u(r, \theta) = R(r)\,\Theta(\theta) \quad \text{...(4)}$$

Substituting this value of u in (1) and dividing by $R\Theta$, we get

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}=0.$$

i.e.
$$\frac{1}{\Theta}(\Theta''+\cot\theta\,\Theta')=-\frac{1}{R}(r^2R''+2rR') \quad \text{...(5)}$$

Since r and θ are independent, (5) can only be true if each side is equal to a constant, say $-n(n+1)$. Then (5) gives

$$r^2R''+2rR'-n(n+1)R=0 \quad \text{...(6)}$$

and
$$\Theta''+\cot\theta\,\Theta'+n(n+1)\Theta=0 \quad \text{...(7)}$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+n(n+1)\Theta=0 \quad \text{...(8)}$$

which is Legendre's equation.

Solution of (6) and (8) are

$$R(r)=A_nr^n+B_n/r^{n+1} \quad \text{...(9)}$$

and
$$\Theta(\theta)=C_nP_n(\cos\theta)+D_nQ_n(\cos\theta) \quad \text{...(10)}$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along polar axis $\theta = 0$. Again, we take $A_n = 0$ in (9), since otherwise $R(r)\to\infty$ which contradicts (2). With these values of A_n and D_n, the functions

$$u_n(r, \theta) = E_n r^{-(n+1)} P_n(\cos\theta)$$

with $E_n = B_n C_n$, are solutions of (1), satisfying B.C. (2). In order to obtain a solution also satisfying (3), we consider a more general solution by superposition, namely,

$$u(r, \theta) = \sum_{n=0}^{\infty} u_n(r, \theta) = \sum_{n=0}^{\infty} E_n r^{-(n+1)} P_n(\cos\theta) \quad ...(11)$$

Putting $r = 1$ in (11) and using (3), we get

$$\cos 2\theta = \sum_{n=0}^{\infty} E_n P_n(\cos\theta)$$

or $$2\cos^2\theta - 1 = \sum_{n=0}^{\infty} E_n P_n(\cos\theta)$$

or $$2\mu^2 - 1 = \sum_{n=0}^{\infty} E_n P_n(\mu), \quad ...(12)$$

[Taking $\mu = \cos\theta$]

which is Legendre series expansion of $2\mu^2 - 1$. Hence, we have

$$E_n = \frac{2n+1}{2} \int_{-1}^{1} (2\mu^2 - 1) P_n(\mu)\, d\mu. \quad ...(13)$$

Putting $n = 0$ in (13) and noting that $P_0(\mu) = 1$, we get

$$E_0 = \frac{1}{2} \int_{-1}^{1} (2\mu^2 - 1)\, \mu\, d\mu = -\tfrac{1}{3} \quad ...(14)$$

Putting $n = 1$ in (13) and noting that $P_1(\mu) = \mu$, we get

$$E_1 = \frac{3}{2} \int_{-1}^{1} (2\mu^2 - 1)\, \mu\, d\mu = 0 \quad ...(15)$$

Putting $n = 2$ in (13) and noting that $P_2(\mu) = (3\mu^2 - 1)/2$, we get

$$E_2 = \frac{5}{2} \int_{-1}^{1} \frac{(2\mu^2 - 1)(3\mu^2 - 1)}{2}\, d\mu = \frac{5}{4} \int_{-1}^{1} (6\mu^4 - 5\mu^2 + 1)\, d\mu$$

$$\therefore \quad E_2 = \frac{5}{4}\left[\frac{6\mu^5}{5} - \frac{5\mu^3}{3} + \mu\right]_{-1}^{1} = \frac{4}{3} \quad ...(16)$$

Again, we know that

$$\int_{-1}^{1} \mu^m P_n(\mu)\, d\mu = 0 \text{ if } m < n \quad ...(17)$$

$$\int_{-1}^{1} P_n(\mu)\, d\mu = 0 \text{ for } n \geq 1. \quad ...(18)$$

and $P_n(\mu)$ is a polynomial of degree n. ...(19)

In view of (17), (18) and (19), (13) shows that

$$E_n = 0 \text{ for each } n \geq 3. \qquad \text{...(20)}$$

Using (14), (15), (16) and (20), (11) reduces to

$$u(r, \theta) = \frac{E_0 P_0 (\cos \theta)}{r} + 0 + \frac{E_2 P_2 (\cos \theta)}{r^2}$$

i.e.,
$$u(r, \theta) = -\frac{1}{3r} + \frac{4}{3r^3} \times \frac{3\cos^2\theta - 1}{2}$$

i.e.,
$$u(r, \theta) = \frac{2(3\cos^2\theta - 1) - r^2}{3r^3},$$

which is the required potential.

Part (ii) Left as an exercise for the reader.

Ex. 3. Determine the potential outside and insde a spherical surface which is kept at a fixed distribution of electrical potential of the form $u = F(\theta)$. It is assumed that the space inside and outside the surface is free of charges. **(Meerut 98)**

Sol. We know that potential u satisfies a Laplace equation. Since we have a spherical surface, u is supposed to satisfy

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

Due to spherical symmetry of the problem, $\partial^2 u / \partial \varphi^2 = 0$.

Hence we are to solve the resulting Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0. \qquad \text{...(1)}$$

Let a be the radius of the sphere. Then, by problem

$$u(a, \theta) = F(\theta). \qquad \text{...(2)}$$

Again, the potential $u,(r, \theta)$ is finite along the polar axis $\theta = 0$ so that

$$u(r, 0) = \text{ finite quantity}. \qquad \text{...(3)}$$

Let a solution of (1) be of the form

$$u(r, \theta) = R(r)\,\Theta(\theta) \qquad \text{...(4)}$$

Substituting this value of u in (1) and dividing by $R\Theta$, we have

$$r^2 \frac{R''}{R} + 2r\frac{R'}{R} + \frac{\Theta''}{\Theta} + \cot\theta \frac{\Theta'}{\Theta} = 0,$$

i.e.,
$$\frac{1}{\Theta}(\Theta'' + \cot\theta\, \Theta') = -\frac{1}{R}(r^2 R' + 2r R') \qquad \text{...(5)}$$

Since r and θ are independent, (2) can only be true it each side in equal to a constant, say $-n(n+1)$. Then (5) gives

$$r^2 R'' + 2r R' - n(n+1) R = 0 \qquad \text{...(6)}$$

and
$$\Theta'' + \cot\theta\, \Theta' + n(n+1)\Theta = 0 \qquad \text{...(7)}$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + n(n+1)\Theta = 0, \qquad \text{...(8)}$$

which is Legendre's equation.

Solutions of (6) and (8) are

$$R(r) = A_n r^n + \frac{B_n}{r^{n+1}} \qquad \text{...(9)}$$

and
$$\Theta(\theta) = C_n P_n(\cos\theta) + D_n Q_n(\cos\theta) \qquad \text{...(10)}$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along the polar axis (*i.e.*, (3) is satisfied). Hence (10) reduces to

$$\Theta(\theta) = C_n P_n(\cos\theta) \qquad \text{...(11)}$$

Hence a general solution of (1) is

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$$

i.e.,
$$u(r,\theta) = \sum_{n=0}^{\infty}\left(E_n r^n + \frac{F_n}{r^{n+1}}\right) P_n(\cos\theta) \qquad \text{...(12)}$$

where $E_n = C_n A_n$ and $F_n = C_n B_n$. Two cases arise :

(i) Potential outside the sphere.

Since the potential at $r = \infty$ is zero on physical grounds, we take $E_n = 0$ in (12). Hence the general solution of (1) for $r > a$ (*i.e.*, outside the sphere) is given by

$$u(r,\theta) = \sum_{n=1}^{\infty}\frac{F_n}{r^{n+1}} P_n(\cos\theta) \qquad \text{...(13)}$$

Putting $r = a$ in (13) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty}\frac{F_n}{a^{n+1}} P_n(\cos\theta)$$

or
$$F(\cos^{-1}\mu) = \sum_{n=0}^{\infty}\frac{F_n}{a^{n+1}} P_n(\mu), \qquad \left[\begin{array}{ll}\because & \mu = \cos\theta \\ \therefore & \theta = \cos^{-1}\mu\end{array}\right]$$

which is Legendre's series expansion for $F(\cos^{-1}\mu)$ and hence

$$\frac{F_n}{a^{n+1}} = \frac{2n+1}{2} \int_{-1}^{1} F(\cos^{-1}\mu)\, P_n(\mu)\, d\mu)$$

$\therefore$ $$F_n = \frac{2n+1}{2} a^{n+1} \int_{\pi}^{0} F(\theta)\, P_n(\cos\theta)\, (-\sin\theta\, d\theta$$

$$[\because \mu = \cos\theta \text{ so that } d\mu = -\sin\theta\, d\theta\,]$$

or $$F_n = \frac{2n+1}{2} a^{n+1} \int_{0}^{\pi} F(\theta)\, P_n(\cos\theta) \sin\theta\, d\theta \quad ...(14)$$

Hence the required potential is given by (13) wherein F_n is given by (14)

(ii) Potential inside the sphere.

Since the potential at $r=0$ is finite on physical grounds, we take $F_n = 0$ in (12).

Hence the general solution of (1) for $r < a$ (*i.e.*, inside the sphere) is given by

$$u(r, \theta) = \sum_{n=0}^{\infty} E_n r^n P_n(\cos\theta) \quad ...(15)$$

Putting $r = a$ in (15) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty} E_n a^n P_n(\cos\theta)$$

or $$F(\cos^{-1}\mu) = \sum_{n=0}^{\infty} E_n a^n P_n(\mu), \text{ as before}$$

which is Legendre series expansion for $F(\cos^{-1}\mu)$ and hence

$$E_n a^n = \frac{2n+1}{2} \int_{-1}^{1} F(\cos^{-1}\mu)\, P_n(\mu)\, d\mu$$

$\therefore$ $$E_n = \frac{2n+1}{2a^n} \int_{\pi}^{0} F(\theta)\, P_n(\cos\theta)\, (-\sin\theta\, d\theta)$$

or $$E_n = \frac{2n+1}{2a^n} \int_{0}^{\pi} F(\theta)\, P_n(\cos\theta) \sin\theta\, d\theta \quad ...(16)$$

Hence the required potential is given by (15) wherein E_n is given by (16).

Ex. 4. *Find the steady state temperature in a uniform solid sphere of radius a when its surface is maintained at temperature $F(\theta)$.*

Sol. Steady state temperature is governed by Laplace's equation (which we take in spherical coordinates due to the geometry of the present problem).

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}=0$$

Due to spherical symmetry of the problem in hand, we must have $\partial^2 u/\partial\varphi^2 = 0$ (*i.e.*, temperature u depends only on r and θ) and so temperature u satisfies the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0 \qquad ...(1)$$

with boundary conditions. $u(a, \theta) = F(\theta)$ ...(2)

Again the temperature is finite along the polar axis $\theta = 0$ on physical grounds. Hence

$$u(r, 0) = \text{ finite quantity} \qquad ...(3)$$

Let a solution of (1) be of the form

$$u(r, \theta) = R(r)\,\Theta(\theta) \qquad ...(4)$$

Substituting this value of u in (1) and dividing by $R\,\Theta$, we have

$$r^2\frac{R''}{R}+2r\frac{R'}{R}+\frac{\Theta''}{\Theta}+\cot\theta\frac{\Theta'}{\Theta}=0,$$

i.e.

$$\frac{1}{\Theta}(\Theta''+\cot\theta\,\Theta')=-\frac{1}{R}(r^2R''+2rR') \qquad ...(5)$$

Since r and θ are independent, (5) can only be true if each side is equal to a constant, say $-n(n+1)$. Then (5) gives

$$r^2R''+2rR'-(n+1)R=0 \qquad ...(6)$$

$$\Theta''+\cot\theta\,\Theta'+n(n+1)\Theta=0. \qquad ...(7)$$

Putting $\mu = \cos\theta$, (7) reduces to

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+n(n+1)\Theta=0, \qquad ...(8)$$

which is Legendre's equation.

Solutions of (6) and (8) are

$$R(r)=A_n r^n+\frac{B_n}{r^{n+1}} \qquad ...(9)$$

and

$$\Theta(\theta)=C_n P_n(\cos\theta)+D_n Q_n(\cos\theta) \qquad ...(10)$$

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite along the polar axis (*i.e.* (3) is satisfied). Hence (10) reduces to

$$\Theta(\theta)=C_n P_n(\cos\theta) \qquad ...(11)$$

Hence a general solution of (1) is

$$u(r,\theta)=\sum_{n=0}^{\infty} C_n\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

i.e.
$$u(r, \theta) = \sum_{n=0}^{\infty} \left(E_n r^n + \frac{F_n}{r^{n+1}} \right) P_n(\cos\theta) \qquad ...(12)$$

where $E_n (= C_n A_n)$ and $F_n (= C_n B_n)$ are new arbitrary constants.

Now, since the temperature at $r = 0$ is finite on physical grounds, we take $F_n = 0$ in (12) and obtain

$$u(r, \theta) = \sum_{n=0}^{\infty} E_n r^n P_n(\cos\theta) \qquad ...(13)$$

Putting $r = a$ in (13) and using (2), we get

$$F(\theta) = \sum_{n=0}^{\infty} E_n a^n P_n(\cos\theta)$$

or
$$F(\cos^{-1}\mu) = \sum_{n=0}^{\infty} E_n a^n P_n(\mu), \qquad \begin{bmatrix} \because & \mu = \cos\theta \\ \therefore & \theta = \cot^{-1}\mu \end{bmatrix}$$

which is Legendre series expansion for $F(\cos^{-1}\mu)$ and hence

$$E_n a^n = \frac{2n+1}{2} \int_{-1}^{1} F(\cos^{-1}\mu) P_n(\mu)\, d\mu$$

$$\therefore \qquad E_n = \frac{2n+1}{2a^n} \int_{\pi}^{0} F(\theta) P_n(\cos\theta)(-\sin\theta\, d\theta)$$

$$[\because \ \mu = \cos\theta \ \therefore \ d\mu = -\sin\theta\, d\theta]$$

or
$$E_n = \frac{2n+1}{2a^n} \int_{0}^{\pi} F(\theta) P_n(\cos\theta) \sin\theta\, d\theta. \qquad ...(14)$$

Hence the required temperature is given by (13) wherein E_n is given by (14).

Exercise (L)

1. Find the potential $u(r, \theta)$ (a) interior to and (b) exterior to a hollow sphere of unit radius if half of its surface is charged to potential u_0 and the other half to potential zero.

Ans. (a) $u(r, \theta) = \frac{1}{2}u_0 [1 + (3r/2) P_1(\cos\theta) - (7/8) r^3 P_3(\cos\theta) + (1/16) r^5 P_5(\cos\theta) + ...]$

(b) $u(r, \theta) = (u_0/2r) [1 + (3/2r) P_1(\cos\theta) - (7/8r^3) P_3(\cos\theta) + (11/16r^5) P_5(\cos\theta) + ...]$

2. Find the steady state temperature in a uniform solid sphere of radius unity when one half of the surface is kept at the constant temperature 0°C and the other half at the constant temperature 1°C.

Ans. $u(r, \theta) = \frac{1}{2} + \frac{3}{4} r P_1(\cos\theta) - (7/16) r^3 P_3(\cos\theta) + (11/32) r^5 P_5(\cos\theta) - ......$

3. Show that a solution of Laplace's equation $\nabla^2 u = 0$ which is independent of φ is given by

$$u(r, \theta) = \left(A_n r^n + \frac{B_n}{r^{n+1}}\right)[C_n P_n(\mu) + D_n Q_n(\mu)], \text{ where } \mu = \cos\theta.$$

4. Write Laplace's equation in cartesian, cylindrical and spherical coordinates. Solve Laplace's equation for spherically symmetric case.

5. Solve the equation : $\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial u}{\partial r}\right) + \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial u}{\partial \theta}\right) = 0.$

with the boundary conditions $\partial u/\partial r = 0$, $r = a$; $\partial u/\partial r \to u_0 \cos\theta$, $r \to \infty$ where u_0 is a constant.

6. Show that a solution of a Laplace's equation in three dimensions $\nabla^2 u = 0$, can be put in the form

$$u = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta),$$

where A_n and B_n are constants, and deduce that if the potential has a constant value u_0 on a spherical surface of radius a, and the space inside the outside the surface is free of charge, then $u = u_0$ at all interior points and $u = u_0 a/r$ at each exterior point.

7. A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the disturbed steam.

8. Find a solution of the equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \varphi}{\partial \theta}\right) = 0$$

of the form $\varphi = f(r)\cos\theta$ given that $-\partial\varphi/\partial r = v\cos\theta$ when $r = a$; $-\partial\theta/\partial r = 0$ when $r = \infty$.

1.27. Solution of heat (diffusion) equation.

Ex. 1. *Solve the heat (diffusion) equation in spherical polar coordinates.*

Sol. Heat (diffusion) equation in spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{k}\frac{\partial u}{\partial t} \qquad \ldots(1)$$

Let a solution of (1) be of the form

$$u(r, \theta, \varphi, t) = R(r)\ \Theta\ (\theta)\,\Phi(\varphi)\,T(t) \qquad \ldots(2)$$

Substituting this value of u in (1) and then dividing by $R\Theta\Phi T$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = \frac{1}{k}\frac{T'}{T} \qquad \ldots(3)$$

Since the right hand side depends only on t while the left hand side depends only on r, θ and φ, (3) can only be true if each side is equal to the same constant. On physical grounds temperature $u \to 0$ as $t \to \infty$ *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take the constant of separation as $-\lambda^2$. Thus (3) gives

$$T'' = -k\lambda^2 T \qquad ...(4)$$

and $$\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Theta}(\Theta'' + \cot\theta\,\Theta') + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} = -\lambda^2$$

i.e., $$\left[\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Theta}(\Theta'' + \cot\Theta\,\Theta') + \lambda^2\right] r^2\sin^2\theta = -\frac{\Phi''}{\Phi} \qquad ...(5)$$

Since the right hand side depends only on φ while the left hand side depends only on r and θ, (5) can only be true if each side is equal to the same constant, say m^2. W choose m^2 as the constant of separation so that the resulting solution may involve trigonometric functions containing φ. Thus, (5) gives

$$\Phi'' + m^2\Phi = 0 \qquad ...(6)$$

and $$\frac{1}{R}(rR'' + 2r\,R') + \frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta}$$

i.e., $$\frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}(r^2R'' + 2rR') - \lambda^2 r^2 \qquad ...(7)$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant, say $-n(n+1)$. Then (7) gives

$$r^2R'' + 2r\,R' + \{\lambda^2r^2 - n(n+1)\}R = 0 \qquad ...(8)$$

and $$\Theta'' + \cot\theta\,\Theta' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \qquad ...(9)$$

Putting $\mu = \cos\theta$, (9) can be written as

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0, \qquad ...(10)$$

which is associated Legendre's equations.

Solutions of (4), (6), (8) and (10) are

$$T_\lambda(t) = A_\lambda e^{-k\lambda^2 t} \qquad ...(11)$$

$$\Phi_m(\varphi) = B_m\cos m\varphi + C_m\sin m\varphi \qquad ..(12)$$

$$R_{\lambda n}(r) = (\lambda r)^{-1/2}\left[D_{\lambda n}\,J_{n+1/2}(\lambda r) + E_{\lambda n}\,J_{-(n+1/2)}(\lambda r)\right] \qquad ...(13)$$

$$\Theta_{mn}(\theta) = F_{mn}P_n^m(\cos\theta) + G_{mn}Q_n^m(\cos\theta) \qquad ...(14)$$

A general solution is given by

$$u(r, \theta, \varphi, t) = \sum_n \sum_m \sum_\lambda T_\lambda(t)\, \Phi_m(\varphi)\, R_{\lambda n}(r)\, \Theta_{mn}(\theta),$$

where $T_\lambda, \Phi_m, R_{\lambda n}$ and Φ_{mn} are given by (11), (12), (13) and (14) respectively.

Ex. 2. *Find temperature in a sphere of radius a when its surface is maintained at zero temperature and its initial temperature is* (r, θ).

Sol. Heat equation in spherical coordinates (r, θ, φ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

Due to spherical symmetry in a sphere, $\partial^2 u/\partial \varphi^2 = 0$ so that the resulting heat equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = \frac{1}{k}\frac{\partial u}{\partial t} \quad \text{...(1)}$$

We are to solve (1) subject to the conditions :

Boundary Condition : $\quad u(a, \theta, t) = 0 \quad$ ...(2)

Initial Condition : $\quad u(r, \theta, 0) = f(r, \theta). \quad$ ...(3)

Let a solution of (1) be of the form

$$u(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t) \quad \text{...(4)}$$

Substituting this value of u in (1) and then dividing by $R\Theta T$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} = \frac{1}{k}\frac{T'}{T} \quad \text{...(5)}$$

Since the right hand side depends only on t while the left hand depends only on r and θ, (5) can only be true if each side is equal to the same constant, say $-\lambda^2$. Then (5) gives

$$T' = -k\lambda^2 T \quad \text{...(6)}$$

and $$\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Theta}(\Theta'' + \cot\theta\, \Theta') = -\lambda^2$$

i.e., $$\frac{1}{\Theta}(\Theta'' + \cot\theta\, \Theta') = -\frac{r^2}{R}\left(R'' + \frac{2}{R}R'\right) - \lambda^2 r^2 \quad \text{...(7)}$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant. say $-n(n+1)$. Then (7) gives

$$r^2 R'' + 2rR' + \{n^2 r^2 - n(n+1)\}\, R = 0 \quad \text{...(8)}$$

and $$\Theta'' + \cot\theta\, \Theta' + n(n+1)\, \Theta = 0 \quad \text{...(9)}$$

Solutions o (6), (8) and (9) are

$$T(t) = A_\lambda e^{-k\lambda^2 t} \qquad \ldots(10)$$

$$R(r) = (\lambda r)^{-1/2} [B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)] \qquad \ldots(11)$$

$$\Theta(\theta) = D_n P_n(\cos\theta) + E_n Q_n(\cos\theta) \qquad \ldots(12)$$

In order to satisfy physical condition $u(r, \theta + 2\pi, t) = u(r, \theta, t)$, n must be an integer. We choose $C_{\lambda n} = 0$ in (11) so that $R(r)$ and hence $u(r, \theta, t)$, may remain finite at $r = 0$ (note that $J_{-(n+1/2)}(\lambda r) \to \infty$ as $r \to 0$).

Again, we choose $E_n = 0$ in (12) so that Θ and hence $u(r, \theta, t)$ may remain finite at $\theta = 0$ (polar axis) on physical grounds.

Using these values of $C_{\lambda n}$ and E_n, (11) and (12) reduce to

$$R(r) = B_{\lambda n} (\lambda r)^{-1/2} J_{n+1/2}(\lambda r) \qquad \ldots(13)$$

$$\Theta(\theta) = D_n P_n(\cos\theta) \qquad \ldots(14)$$

Putting $r = a$ in (4) and using B.C. (2), we have

$$R(a)\,\Theta(\theta)\,T(t) = 0 \text{ so that } R(a) = 0 \qquad \ldots(15)$$

where we have taken $\Theta(\theta) \not\equiv 0$ and $T(t) \not\equiv 0$, since otherwise $u \equiv 0$ which does not satisfy (3).

Now putting $r = a$ in (13) and using (15), we get

$$B_{\lambda n} J_{n+1/2}(\lambda a) = 0 \text{ so that } J_{n+1/2}(\lambda a) = 0, \qquad \ldots(16)$$

where we have taken $B_{\lambda n} \neq 0$, since otherwise $R(r) \equiv 0$ so that $u \equiv 0$ which does not satisfy (3).

Let λ_{ni} $(i = 1, 2, 3, \ldots)$ be the positive roots of (16). Then for each value of λ_{ni}, (10) and (13) may be written as

$$T(t) = F_i e^{-k\lambda_{ni}^2 t} \qquad \ldots(17)$$

and

$$R(r) = G_{ni} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r) \qquad \ldots(18)$$

where F_i and G_{ni} are new arbitrary constants.

From (14), (17) and (18), the functions

$$u_{ni}(r, \theta, t) = H_{ni} P_n(\cos\theta) e^{-k\lambda_{ni}^2 t} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r)$$

where $H_{ni} = D_n F_i G_{ni}$, are solutions of (1) satisfying (2). In order to obtain a solution also satisfying (3), we consider a more general solution

$$u(r, \theta, t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} P_n(\cos\theta) e^{-k\lambda_{ni}^2 t} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r). \qquad \ldots(19)$$

Putting $t = 0$ in (19) and using B.C. (3), we have

$$f(r, \theta) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} P_n (\cos \theta) (\lambda_{ni} r)^{-1/2} J_{n+1/2} (\lambda_{ni} r) \qquad ...(20)$$

With the help of theory of Bessel functions and Legendre polynomials, (20) gives

$$H_{ni} = \frac{(2n+1) \lambda_{ni}^{1/2}}{a^2 [J'_{n+1/2} (\lambda_{ni} a)]^2} \int_0^a r^{3/2} J_{n+1/2} (r\lambda_{ni}) \, dr \int_{-1}^{1} P_n (\mu) f(r, \theta) \, d\mu$$

where $\mu = \cos \theta$.

Exercise (M)

1. Obtain the solution of the diffusion equation $k\nabla^2 \psi = \partial\psi/\partial t$ in sperical polar coordinates. Use your solution to find the temperature in a sphere of radius a when its surface is maintained at zero temperature and its initial temperature is $f(r, \theta)$.

2. Show that the solutions of the diffusion equation

$$k \nabla^2 \psi = \partial \psi / \partial t$$

are of the from $\displaystyle\sum_{m, n, \lambda} C_{mn\lambda} (\lambda r)^{-1/2} J_{n+1/2} (\lambda r) P_n^m(\mu) e^{\pm im\varphi} e^{-\lambda^2 kt}$,

where $\mu = \cos \theta$.

1.23. Solution of wave equation.

Ex. 1. *Solve three dimensional wave equation in spherical polar coordinates.* **[Delhi B.Sc. (H) 1995, 2000]**

Sol. Wave equation in spherical polar coordinates (r, θ, φ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \qquad ...(1)$$

Let a solution of (1) be of the form

$$u(r, \theta, \varphi, t) = R(r) \Theta (\theta) \Theta (\varphi) T(t) \qquad ...(2)$$

Substituting this value of u in (1) and then dividing by $R\Theta \Phi T$, we have

$$\frac{R''}{R} + \frac{2}{r} \frac{R'}{R} + \frac{1}{r^2} \frac{\Theta''}{\Theta} + \frac{\cot \theta}{r^2} \frac{\Theta'}{\Theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\Phi''}{\Phi} = \frac{1}{c^2} \frac{T''}{T} \qquad ...(3)$$

Since the right hand side depends only on t while the left hand side depends only on r, θ and φ, (3) can only be true if each side is equal to the same constant, say $-k^2$. The (3) gives

$$T'' + k^2c^2 T = 0 \qquad ...(4)$$

and $$\left[\frac{1}{R} (R'' + \frac{1}{r} R') + \frac{1}{r^2\Theta} (\Theta'' + \cot \theta \, \Theta') + k^2\right] r^2 \sin^2 \theta = - \frac{\Phi''}{\Phi} \qquad ...(5)$$

Since the right hand side depends only on φ while the left hand side depends only on r and θ, (5) can only be true if each side is equal to the same constant, say m^2. Then (5) gives

$$\Phi'' + m^2 \Phi = 0 \qquad \text{...(6)}$$

and
$$\frac{1}{R}(r^2 R'' + 2r R') + k^2 r^2 + \frac{1}{\Theta}(\Theta'' + \cot\theta\, \Theta') = \frac{m^2}{\sin^2\theta}$$

i.e.,
$$\frac{1}{\Theta}(\Theta'' + \cot\theta\, \Theta') - \frac{m^2}{\sin^2\theta} = -\frac{1}{R}(r^2 R'' + 2rR') - k^2 r^2 \qquad \text{...(7)}$$

Since r and θ are independent, (7) can only be true if each side is equal to the same constant, say $-n(n+1)$. Then (7) gives

$$r^2 R'' + 2r R' + \{k^2 r^2 - n(n+1)\} R = 0 \qquad \text{...(8)}$$

and
$$\Theta'' + \cot\theta\, \Theta' + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0 \qquad \text{...(9)}$$

Putting $\mu = \cos\theta$, (9) may be re-written as

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu\frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0, \qquad \text{...(10)}$$

which is associated Legendre's equation.

Solutions of (4), (6), (8) and (10) are

$$T_k(t) = A_k \cos(ckt) + B_k \sin(ckt) \qquad \text{...(11)}$$

$$\Phi_m(\varphi) = C_m \cos m\varphi + D_m \sin m\varphi \qquad \text{...(12)}$$

$$R_{kn}(r) = (kr)^{-1/2}[E_{kn} J_{n+1/2}(kr) + F_{kn} J_{-(n+1/2)}(kr)] \qquad \text{...(13)}$$

$$\Theta_{mn}(\theta) = G_{mn} P_n^m(\cos\theta) + H_{mn} Q_n^m(\cos\theta) \qquad \text{...(14)}$$

A more general solution is given by

$$u(r, \theta, \varphi, t) = \sum_k \sum_m \sum_n T_k(t)\, \Phi_m(\varphi)\, R_{kn}(r)\, \Theta_{mn}(\theta),$$

where T_k, Φ_m, R_{kn} and Θ_{mn} are given by (11), (12), (13) and (14) respectively.

Exericse (N)

1. Prove that

$$\psi(r, \theta, \varphi, t) = r^{-1/2} J_{\pm(n+1/2)}(kr)\, P_n^m(\cos\theta)\, e^{\pm im\phi \pm ickt}$$

is solution of the wave equation

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

2. Show that the wave equation $\nabla^2\psi = \partial^2\psi/\partial t^2$ has solutions of the form $\psi = S(\theta, \varphi)\, R(r, t)$ where r, θ, φ are spherical polar coordinates,

$$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+l(l+1)\right\}S=0 \text{ and}$$

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{l(l+1)}{r^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\}R=0.$$

[Here l is a constant integer]

1.29. Solution of telegraph or transmission line equations given in Art. 1.6.

Ex. 1. *Explain the method of separation of variables to solve a second order parital differential equation. Using this method find the current I and voltage V in a transmission line of length l, t seconds after the ends are suddenly grounded given that $I(x, 0) = I_0$ and $V(x, 0) = V_0 \sin(\pi x/l)$ and R and G are negligible.*

Sol. For first part refer Art. 1.8.

Since R and G are neglected, the transmission line equations become

$$\partial V/\partial x = -L\,(\partial I/\partial t) \qquad ...(1)$$

and
$$\partial I/\partial x = -C\,(\partial V/\partial t) \qquad ...(2)$$

Diff. (1) w.r.t. 'x' and (2) w.r.t 't', we get

$$\partial^2 V/\partial x^2 = -L\,(\partial^2 I/\partial x\,\partial t) \text{ and } \partial^2 I/\partial t\,\partial x = -C\,(\partial^2 V/\partial t^2)$$

Eliminating $\partial^2 I/\partial x\partial t$ from these, we obtain

$$\partial^2 V/\partial x^2 = LC\,(\partial^2 V/\partial t^2) \qquad ...(3)$$

Since the ends are suddenly grounded, we get

Boundary conditions : $V(0, t) = 0,\ V(l, t) = 0$ ...(4)

Now $I(x, 0) = I_0 \Rightarrow (\partial I/\partial x)_{t=0} = 0 \Rightarrow (\partial V/\partial t)_{t=0} = 0$, by (2)

$\therefore$ Initial conditions are given by

$$V(x, 0) = V_0 \sin(\pi x/l) \qquad ...(5)$$

and
$$(\partial V/\partial t)_{t=0} = 0 \qquad ...(6)$$

Let $V(x, t) = X(x)\,T(t)$ be a solution of (3). Substituting in (1), we obtain

$$X''/X = CL\,T''/T = -k^2, \text{ say} \qquad ...(7)$$

where we have taken $-k^2$ because the initial conditions suggest that $V(x, t)$ must be periodic and hence it must contain trigonometric functions.

From (7), $\quad X'' + k^2 X = 0$ and $CL\,T'' + k^2 T = 0$

Solving these and using $V = XT$, we get

$$V(x, t) = (c_1\cos kx + c_2\sin kx)\left(c_3\cos\frac{kt}{\sqrt{(LC)}} + c_2\sin\frac{kt}{\sqrt{(LC)}}\right)$$

Using the boundary conditions (4), we get

$c_1 = 0$ and $k = n\pi/l$, n being any integer

$$\therefore \qquad V(x,t) = c_2 \sin\frac{n\pi x}{l}\left(c_3 \cos\frac{n\pi t}{l\sqrt{(LC)}} + c_4 \sin\frac{n\pi t}{l\sqrt{(LC)}}\right) \qquad ...(9)$$

From (9),

$$\frac{\partial V}{\partial t} = c_2 \sin\frac{n\pi x}{l} \times \frac{n\pi}{l\sqrt{(LC)}}\left(-c_3 \sin\frac{n\pi t}{l\sqrt{(LC)}} + c_4 \cos\frac{n\pi t}{l\sqrt{(LC)}}\right)$$

Putting $t = 0$ and using (6), we get $c_2 c_4 = 0$

$\therefore$ *From*(9), $V(x,t) = A_n \sin(n\pi x/l)\cos[n\pi t/l\sqrt{(LC)}]$,

where $A_n = c_2 c_3$. By superposition, the general solution is

$$V(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/l)\cos[n\pi t/l\sqrt{(LC)}] \qquad ...(10)$$

Putting $t = 0$ and using (5), we get

$$V_0 \sin(\pi x/l) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/l)$$

which suggest that $A_1 = V_0$ and $A_n = 0$ for remaining values of n. Hence, from (10), the required solution is

$$V(x,t) = V_0 \sin(\pi x/l)\cos[\pi t/l\sqrt{(LC)}] \qquad ...(11)$$

From (1), (2) and (11), we have

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial x} = -\frac{V_0\pi}{lL}\cos\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{(LC)}} \qquad ...(12)$$

and
$$\frac{\partial I}{\partial x} = -c\frac{\partial V}{\partial t} = \frac{cV_0\pi}{l\sqrt{(LC)}}\sin\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} \qquad ...(13)$$

Integrating (12) and (13), we obtain

$$I(x,t) = -V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} + f(x)$$

and
$$I(x,t) = -V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}} + g(t)$$

Since $I(x,0) = I_0$, so we get $f(x) = g(t) = I_0$.

$$\therefore \qquad I(x,t) = I_0 - V_0\left(\frac{C}{L}\right)^{1/2}\cos\frac{\pi x}{l}\sin\frac{\pi t}{l\sqrt{(LC)}}. \qquad ...(14)$$

Hence the required solutions are given by (11) and (14).

Exercise 1 (O)

1. Neglecting R and G, find the *e.m.f.* $V(x,t)$ in a line a km long, t second after the ends were suddenly grounded, if initially $I(x,0) = I_0$ and

(a) $V(x,0) = E_1 \sin(\pi x/a) + E_7 \sin(7\pi x/a)$ (b) $V(x,0) = Ex/a$

Ans. (*a*) $V = E_1 \sin\frac{\pi x}{a} \cos\frac{\pi t}{a\sqrt{(LC)}} + E_7 \sin\frac{7\pi x}{a} \cos\frac{7\pi t}{a\sqrt{(LC)}}$

(*b*) $V = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} \cos\frac{n\pi t}{a\sqrt{(LC)}}$, where $A_n = \frac{2E}{a^2}\int_0^a x \sin\frac{n\pi x}{a}\, dx$

2. A transmission line 1000 km long, is initially under steady state conditions with potential 1300 volts at the sending end ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential $E(x, t)$.

Ans. $E(x, t) = 1300 - 1{\cdot}3\, x$

$$+ \frac{2400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2 t)/(l^2 RC)}$$

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ADDITIONAL PROBLEMS ON BOUNDARY VALUE PROBLEMS

Ex. 1. *Transform Laplace differential equation $(d^2 v/dx^2) + (d^2 v/dy^2) = 0$ into polar form r, θ in R^2 where $v \equiv v(x, y)$.*

Sol. If (x, y) be the cartesian coordinates of the point P whose polar coordinates are (r, θ), then

$$x = r\cos\theta \text{ and } y = r\sin\theta. \quad \ldots(1)$$

Then, $$r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}(y/x). \quad \ldots(2)$$

From (2), $2r(\partial r/\partial x) = 2x$ and $2r(\partial r/\partial y) = 2y$

so that $\partial r/\partial x = x/r = \cos\theta$ and $\partial r/\partial y = y/r = \sin\theta$. ...(3)

Also, $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\left(\frac{-y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}. \quad \ldots(4)$$

and $$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y^2/x^2)}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}. \quad \ldots(5)$$

Given that $$(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) = 0. \quad \ldots(6)$$

Now, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}. \quad \ldots(7)$$

Then, (7) $\Rightarrow$ $$\frac{\partial}{\partial x} \equiv \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \quad \ldots(8)$$

$\therefore$ $$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

[using (7) and (8)]

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \cos\theta\left[\cos\theta\frac{\partial^2 v}{\partial r^2} - \sin\theta\left(-\frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\partial\theta}\right)\right]$$

$$- \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial v}{\partial r} + \cos\theta\frac{\partial^2 v}{\partial\theta\,\partial r} - \frac{1}{r}\left\{\cos\theta\frac{\partial v}{\partial\theta} + \sin\theta\frac{\partial^2 v}{\partial\theta^2}\right\}\right]$$

Thus, $$\frac{\partial^2 v}{\partial x^2} = \cos^2\theta\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \quad \ldots(9)$$

Next, $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y} = \sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta} \quad \ldots(10)$$

Then, (10) $\Rightarrow \dfrac{\partial}{\partial y} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial\theta}$. ...(11)

$$\therefore\ \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial^2 v}{\partial r^2} + \cos\theta\left\{-\frac{1}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}\right\}\right] + \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial v}{\partial r}\right.$$

$$\left. + \sin\theta\frac{\partial^2 v}{\partial\theta\,\partial r} + \frac{1}{r}\left\{-\sin\theta\frac{\partial v}{\partial\theta} + \cos\theta\frac{\partial^2 v}{\partial\theta^2}\right\}\right].$$

Thus, $$\frac{d^2 v}{\partial y^2} = \sin^2\theta\frac{\partial^2 v}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial v}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \qquad ...(12)$$

Adding (1) and (2), we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} \qquad ...(13)$$

Hence, by (6), the Laplace's equation in polar form is

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} = 0. \qquad ...(14)$$

Ex.2. *Transform Laplace equation* $(\partial^2 v/\partial x^2) + (\partial^2 v/\partial y^2) + (\partial^2 v/\partial z^2) = 0$ *into cylindrical coordinates* (r, θ, z). **(Nagpur 96)**

Sol. If (x, y, z) are the cartesian coordinates of the point P whose cylindrical coordinates are (r, θ, z), then (refer page 70 for figure where we take $\rho = r$, $\phi = \theta$ for the present problem).

$$x = r\cos\theta,\ y = r\sin\theta \text{ and } z = z.$$

Take $x = r\cos\theta$ and $y = r\sin\theta$ and proceed exactly as in Ex. 1 upto euation (13). Now adding $\partial^2 v/\partial z^2$ on both sides of (13), we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial z^2}.$$

Hence the given Laplace equation reduces to

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Ex. 3. *Transform Laplace equation* $(\partial v/\partial x^2) + (\partial^2 v/\partial y^2) + (\partial^2 v/\partial z^2) = 0$ *into spherical polar coordinates.*

Sol. If (x, y, z) are the carterian coordinates of the point P whose spherical polar coordinates are (r, θ, ϕ), then (for figure see page 85)

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta. \quad ...(1)$$

$$\therefore \quad r^2 = x^2 + y^2 + z^2,\ \tan\theta = (x^2+y^2)^{1/2}/z,\ \tan\phi = y/x$$

or
$$r^2 = x^2 + y^2 + z^2,\ \theta = \tan^{-1}\{(x^2+y^2)^{1/2}/z\},\ \phi = \tan^{-1}(y/x). \quad ...(2)$$

From (1) and (2), we have the following relations

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi,\ \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi,\ \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta, \quad ...(3)$$

$$\frac{\partial\theta}{\partial x} = \frac{\cos\theta\cos\phi}{r},\ \frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\phi}{r},\ \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}. \quad ...(4)$$

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta},\ \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \text{ and } \frac{\partial\phi}{\partial z} = 0. \quad ...(5)$$

Now, $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial v}{\partial\phi}\frac{\partial\phi}{\partial x}$$

$$= \sin\theta\cos\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial v}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}, \text{ by (3), (4), (5)}$$

$$\therefore \quad \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\therefore \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$\times\left(\sin\theta\cos\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial v}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$= \sin^2\theta\cos^2\phi\frac{\partial^2 v}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 v}{\partial r\,\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial v}{\partial\theta}$$
$$- \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\partial\phi} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial v}{\partial r}$$
$$+ \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 v}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi} + \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}$$
$$+ \frac{\sin^2\phi}{r}\frac{\partial v}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial v}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}. \quad ...(6)$$

Next, $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial v}{\partial\phi}\frac{\partial\phi}{\partial y}$$

$$= \sin\theta\sin\phi\frac{\partial v}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial v}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}, \text{ by (3), (4) and (5)}$$

$$\therefore \quad \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\sin\phi\frac{\partial v}{\partial r}+\frac{\cos\theta\sin\phi}{r}\frac{\partial v}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial v}{\partial\phi}\right)$$

$$=\sin^2\theta\sin^2\phi\frac{\partial^2 v}{\partial r^2}+\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial v}{\partial\theta}$$

$$+\frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 v}{\partial r\,\partial\phi}-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial v}{\partial\phi}+\frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial v}{\partial r}$$

$$+\frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 v}{\partial\theta^2}+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 v}{\partial\theta\,\partial\phi}-\frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}$$

$$+\frac{\cos^2\phi}{r}\frac{\partial v}{\partial r}+\frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}+\frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial v}{\partial\phi}. \qquad ...(7)$$

Finally, $\dfrac{\partial v}{\partial z}=\dfrac{\partial v}{\partial r}\dfrac{\partial r}{\partial z}+\dfrac{\partial v}{\partial\theta}\dfrac{\partial\theta}{\partial z}+\dfrac{\partial v}{\partial\phi}\dfrac{\partial\phi}{\partial z}=\cos\theta\dfrac{\partial v}{\partial r}-\dfrac{\sin\theta}{r}\dfrac{\partial v}{\partial\theta}$, by (3), (4), (5)

$$\therefore \qquad \frac{\partial}{\partial z}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

$$\therefore \quad \frac{\partial^2 v}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 v}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 v}{\partial r\,\partial\theta}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial v}{\partial\theta}$$

$$+\frac{\sin^2\theta}{r}\frac{\partial v}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 v}{\partial\theta^2}. \qquad ...(8)$$

Adding (6), (7) and (8), we see that $(\partial^2v/\partial x^2)+(\partial^2v/\partial y^2)+(\partial^2v/\partial z^2)$ $=0$ is transformed to $\dfrac{\partial^2 v}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial v}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial\theta^2}+\dfrac{\cot\theta}{r^2}\dfrac{\partial v}{\partial\theta}+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 v}{\partial\phi^2}=0$ which is the Laplace's differential equation in spherical polar coordinates as desired in the given problem.

Ex.4. *Obtain partial differential equation of the transverse vibrations of an elastic string fixed at the end points.* **(Nagpur 1995, 96)**

6(c)

Sol. Refer Art 1.2 on page 1.

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Ex. 5. (*a*) *A tightly stretched elastic string of length l, with fixed end points $x = 0$ and $x = l$ is initially in the position given by $y = C\sin^3(\pi x/l)$*, C being constant. It is released from the position of rest. Find the displacement y (*x*, *t*).

(*b*) *A tightly strecthed elastic string of length* π, *with fixed end points $x = 0$ and $x = 1$ is initially in the position given by $y = C\sin^3 x$, C being constant. It is released from the position of rest. Find the displacement $y(x, t)$.* **(Nagpur, 1996)**

Sol. (*a*) The partial differential equation of the transverse vibrations of the given elastic string in given by

Additional Problems

$$\partial^2 y/\partial x^2 = (1/a^2)(\partial^2 y/\partial t^2) \qquad ...(1)$$

where $y(x, t)$ is the deflection of the string and a in a constant. Given boundary and initial conditions are:

Boundary condition (B.C.): $y(0, t) = y(l, t) = 0$ for all t ... (2)

Initial conditions (I.C.): $y(x, 0) = C\sin^3(\pi x/l)$ (Initial deflection) ...3(a)

$(\partial y/\partial t)_{t=o} = y_t(x, 0) = 0$ (Initial velocity). ...3(b)

Let solution of (1) be of the form $y(x, t) = X(x)\,T(t)$. ...(4)

Substituting this value of y in (1), we have

$$X''T = \frac{1}{a^2}XT'' \quad \text{or} \quad \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T}. \qquad ...(5)$$

Since x and t are independent, (5) can only be true if each side is equal to the same constant, say μ. Then (5) gives

$$X'' - \mu X = 0 \qquad ...(6)$$

and $$T'' - \mu a^2 T = 0. \qquad ...(7)$$

Using (2), (4) gives $X(0)\,T(t) = 0$ and $X(l)\,T(t) = 0$. ...(8)

Since T (t) = 0 leads to $y = 0$, hence we assume that $T(t) \neq 0$.

Then (8) gives $X(0) = 0$ and $X(l) = 0$. ...(9)

We now solve (6) under B.C. (9). Three cases arise :

Case I. Let $\mu = 0$. Then solution of (6) is given by

$$X(x) = Ax + B. \qquad ...(10)$$

Using B.C. (9), (10) gives O=B and $O = Al + B$. These give $A = B = O$ so that $X(x) = 0$. With the help of (4), this leads to $y \equiv 0$, which does not satisfy 3(a) and 3(b). So we reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}. \qquad ...(11)$$

Using B.C. (9), (11) given $A + B = 0$ and $Ae^{l\lambda} + Be^{-l\lambda} = 0$ ...(12)

Solving (12), $A = B = 0$ and so $X(x) = 0$. Hence as in case I, we reject the possibility $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \qquad ...(13)$$

Using B.C. (9), (13) gives A = 0 and $A\cos\lambda l + B\sin\lambda l = 0$ so that $A = 0$ and $\sin\lambda l = 0$, where we have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $y = 0$ which does not satisfy 3(a) and 3(b).

Now , $\sin\lambda l = 0 \Rightarrow \lambda l = n\pi \Rightarrow \lambda = n\pi/l,\ n = 1, 2, 3, ...$ ...(14)

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$X_n(x) = B_n \sin(n\pi x/l). \qquad ...(15)$$

Using (14), (7) reduces to

$$(d^2T/dt^2) + (n^2\pi^2a^2/l^2)\,T = 0, \qquad [\because \mu = -\lambda^2 = -(n^2\pi^2)/l^2]$$

whose general solution is $T_n(t) = C_n \cos\dfrac{n\pi at}{l} + D_n \sin\dfrac{n\pi at}{l}$. ... (16)

$$\therefore\ y_n(x, t) = X_n(t)\,T_n(t) = \left(E_n \cos\frac{n\pi at}{l} + F_n \sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

are solutions of (1) satisfying (2) for $n = 1, 2, 3, \ldots$. Here $E_n\ (= C_n B_n)$ and $F_n\ (= B_n D_n)$ are new arbitrary constants. In order to obtain a solution also satisfying 3(*a*) and 3(*b*), we consider most general solution of the form

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi at}{l} + F_n \sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}. \quad \ldots(17)$$

Differentiating (17) partially w.r.t. '*t*', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{n\pi aE_n}{l}\sin\frac{n\pi at}{l} + \frac{n\pi aF_n}{l}\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}. \quad \ldots(18)$$

Putting $t = 0$ in (17) and (18) and using 3(a) and 3(b), we get

$$C\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{and} \quad 0 = \sum_{n=1}^{\infty}\left(\frac{n\pi aF_n}{l}\right)\sin\frac{n\pi x}{l}$$

which are Fourier sine series. Accordingly, we have

$$E_n = \frac{2}{l}\int_0^l C\sin^3\left(\frac{\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx \quad \ldots(19)$$

and $$\frac{n\pi aF_n}{l} = \frac{2}{l}\int_0^l 0\,.\sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow F_n = 0. \quad \ldots(20)$$

Now, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = (3\sin\theta - \sin 3\theta)/4$.

$$\therefore \quad \sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] \quad \ldots(21)$$

$$\therefore\ (19) \Rightarrow E_n = \frac{2C}{l}\int_0^l \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\sin\frac{n\pi x}{l}\,dx$$

or $$E_n = \frac{3C}{2l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx - \frac{C}{2l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\,dx. \quad \ldots(22)$$

We now show that

$$I = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} l/2, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \quad \ldots(23)$$

If $m = n$, then, we have

$$I = \int_0^l \sin^2\frac{n\pi x}{l}\,dx = \int_0^l \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{l}\right)dx = \frac{1}{2}\left[x - \frac{l}{2n\pi}\sin\frac{2n\pi\pi}{l}\right]_0^l$$

$\therefore\ \ I = l/2$ when $m = n$

If $m \neq n$, then we have

$$I = \frac{1}{2}\int_0^l 2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \int_0^l\left(\cos\frac{(n-m)\pi x}{l} - \cos\frac{(n+m)\pi x}{l}\right)dx$$

$$= \frac{1}{2}\left[\frac{l}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l} - \frac{l}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]_0^l = 0$$

Using (23), from (22), we have

$$\left.E_1 = \left(\frac{3C}{2l}\right)\left(\frac{l}{2}\right) = \frac{3C}{4},\ E_3 = -\left(\frac{C}{2l}\right)\left(\frac{l}{2}\right) = -\frac{C}{4}\ \right\} \qquad \ldots(24)$$

Also $E_n = 0$ for $n \neq 1$ and $n \neq 3$.

Using (20) and (24), (17) reduces to

$$y(x,t) = E_1\cos\frac{\pi at}{l}\sin\frac{\pi x}{l} + E_3\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}$$

or $$y(x,t) = \frac{C}{4}\left[3\cos\frac{\pi at}{l}\sin\frac{\pi x}{l} - \cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}\right] \qquad \ldots(25)$$

(*b*) This is a particular case of part (*a*) with $l = \pi$. So reproduce the entire solution of part (*a*) by replacing l by π.

Ans. $y(x,t) = (C/4)(3\cos at\sin x - \cos 3at\sin 3x)$

Ex. 6. *A stretched string with fixed ends at $x = 0$ and $x = 1$ initially is in equilibrium position. It is set vibrating by giving to each point a velocity $(\partial y/\partial t)_{t=0} = V_0\sin^3(\pi x/l)$, find $y(x,t)$ if $(\partial^2 y/\partial t^2) = a^2(\partial^2 y/\partial x^2)$.*

(Nagpur, 1995)

Sol. Proceed as in Ex. 5(*a*). Here initial conditions have been changed. So you have to take now

$y(x,0) = 0$ [Initial deflection] ...3(*a*)

$(\partial y/\partial t)_{t=0} = V_0\sin^3\dfrac{\pi x}{l}$ [Initial velocity] ...3(*b*)

With these new values of initial conditions, now $E_n = 0$ for all n and $F_n = 0$ the $n \neq 1$ and $n \neq 3$. Compute F_1 and F_3 as before to get solution.

Ex. 7. *A uniform rod 20cm in length is insulated over its sides. Its ends are kept at $0°C$. Its initial temperature is sin $(\pi x/20)$ at a distance x from an end, find temperature $u(x,t)$ at time t. Given that $\partial u/\partial t = a^2(\partial^2 u/\partial x^2)$.* **(Nagpur 1996)**

Sol. Given $\partial u/\partial t = a^2(\partial^2 u/\partial x^2)$. ...(1)

Boundary Conditions (B.C.) : $u(0,t) = u(20,t) = 0$ for all t ...(2)

Initial Conditions (I.C.) : $u(x,0) = \sin(\pi x/20)$. ...(3)

Let a solution of (1) be $u(x,t) = X(x)\,T(t)$. ...(4)

Substituting this value of u in (1), we have

$$XT' = a^2X''T \quad \text{or} \quad X''/X = T'/a^2T. \qquad \ldots(5)$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant say μ.

$$\therefore \qquad X'' - \mu X = 0 \qquad \text{...(6)}$$

and $$T' = \mu a^2 T \qquad \text{...(7)}$$

Using (2), (4) gives $X(0)\, T(t) = 0$ and $X(20)\, T(t) = 0$. ...(8)

Since $T(t) \neq 0$, so (8) $\Rightarrow$ $X(0)$ and $X(20) = 0$. ...(9)

We now solve (6) under B.C. (9). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is

$$X(x) = Ax + B. \qquad \text{...(10)}$$

Using B.C. (9), (10) gives $B = 0$ and so $20A + B = 0$ so that $A = B = 0$. Hence $X(x) \equiv 0$ so that $u = 0$ which does not satisfy (3). Hence reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A e^{\lambda x} + B e^{-\lambda x}. \qquad \text{...(11)}$$

Using B.C. (9), (11) given $A + B = 0$ and $A e^{20\lambda} + B e^{-20\lambda} = 0$ ...(12)

Solving (12), $A = B = 0$ so that $X(x) \equiv 0$ and hence $u \equiv 0$, which does not satisfy (3). So we also reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. then solution of (6) is

$$X(x) = A \cos \lambda x + B \sin \lambda x. \qquad \text{...(13)}$$

Using B.C. (9), (13) gives $A = 0$ and $A \cos 20\lambda + B \sin 20\lambda = 0$ giving $\sin 20\lambda = 0$. We have taken $B \neq 0$, since otherwise $X \equiv 0$ so that $u \equiv 0$ which does not satisfy (3). Now, we have

$$\sin 20\lambda = 0 \Rightarrow 20\lambda = n\pi \Rightarrow \lambda = n\pi/20, \text{ where } n = 1, 2, 3 \text{ ...(14)}$$

Hence non-zero solution $X_n(x)$ of (6) are given by

$$X_n(x) = B_n \sin(n\pi x/20). \qquad \text{...(15)}$$

Using (14), (7) reduces to

$$\frac{dT}{T} = -\frac{n^2\pi^2 a^2}{400}\, dt \qquad \left[\because \mu = -\lambda^2 = -\frac{n^2\pi^2}{400}\right]$$

whose general solution is $T_n(t) = D_n e^{-(n^2\pi^2 a^2/400)t}$. ...(16)

$$\therefore u_n(x, t) = X_n(x)\, T_n(t) = E_n \sin(n\pi x/20)\, e^{-(n^2\pi^2 a^2/400)t},$$

are solutions of (1), satisfying (2). Here E_n $(= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (3), we consider the most general solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{20}\, e^{-(n^2\pi^2 a^2/400)t}. \qquad \text{... (17)}$$

Putting time $t = 0$ in (17) and using (3), we get

$$\sin\frac{\pi x}{20} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{20}, \qquad \ldots(18)$$

which is Fourier sine series. So the constants E_n are given by

$$E_n = \frac{2}{20}\int_0^{20} \sin\frac{\pi x}{20}\sin\frac{n\pi x}{20}\,dx,\ n = 1, 2, 3, \ldots \qquad \ldots(19)$$

Case I. If $n \neq 1$, then (19) gives

$$E_n = \frac{1}{20}\int_0^{20}\left[\cos\frac{(n-1)\pi x}{20} - \cos\frac{(n+1)\pi x}{20}\right]dx$$

$$= \frac{1}{20}\left[\frac{20}{(n-1)\pi}\sin\frac{(n-1)\pi x}{20} - \frac{20}{(n+1)\pi}\sin\frac{(n+1)\pi x}{20}\right]_0^{20} = 0$$

Thus, $E_n = 0$ for $n = 2, 3, 4$ ...(20)

Case II. If $n = 1$, then (11) gives

$$E_1 = \frac{1}{20}\int_0^{20}\left(2\sin^2\frac{\pi x}{20}\right) = \frac{1}{20}\int_0^{20}\left(1 - \cos\frac{2\pi x}{20}\right)dx$$

$$= \frac{1}{20}\left[x - \frac{10}{x}\sin\frac{\pi x}{10}\right]_0^{20} = 1. \qquad \ldots(21)$$

Using (20) and (21), (17) reduces to

$$u(x,t) = E_1 \sin\frac{\pi x}{20} e^{-(\pi^2 a^2/400)\,t} = \sin\frac{\pi x}{20} e^{-(\pi^2 a^2/400)\,t}.$$

Ex. 8. *Explain the method of separation of variables for finding solutions of second order linear partial differential equations. Hence, find the solution of one-dimensional diffusion equation $\partial^2 z/\partial x^2 = (1/k)(\partial z/\partial t)$ which tends to zero as $t \to \infty$.* **[Delhi, B.Sc. (H), 1997]**

Sol. ***Method of separation of variables.*** A powerful method of solving the following second-order linear partial differential equation can be used in certain circumstances.

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad \ldots(1)$$

where $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$, $t = \partial^2 z/\partial y^2$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ and R, S, T, P, Q, Z and F are functions of x and y.

Let (1) possess a solution of the form $z = X(x)\,Y(y)$, ...(2)

where X is a function of x alone and Y is a function of y alone.

Substituting the above value of z in (1), (1) reduces to

$$(1/X)\,f(D)X = (1/Y)\,g(D')\,Y, \qquad \ldots(3)$$

where $f(D)$, $g(D')$ are quadratic functions of $D = \partial/\partial x$ and $D' = \partial/\partial y$, respectively. When (1) reduces to (3), we say that the equation (1) is separable in the variables x, y. Now, the L.H.S. of (3) is a function of x alone whereas the R.H.S. is a function of y alone and the two can be equal only if each is equal to a constant, λ say. Then (3) gives the following pair of second-order linear ordinary differential equations

$$f(D)X = \lambda X \text{ and } g(D')Y = \lambda Y. \quad ...(4)$$

The problem of finding solutions of the form (2) of (1) reduces to solving two equations given by (4).

Required solution of given one-dimensional diffusion equation.

Given $\partial^2 z/\partial x^2 = (1/k)(\partial z/\partial t)$ ...(5)

Let a solution of (5) be of the form $z = X(x)\, T(t)$, ...(6)

where X is a function of x alone and T is a function of t alone.

From (6), $\partial z/\partial x = X'T$, $\partial^2 z/\partial x^2 = X''T$ and $\partial z/\partial t = XT'$, ...(7)

where dashes denote derivatives with respect to the relevant variable.

Substituting (7) in (5), we get $(1/X)\, X'' = (1/kT)\, T'$. ...(8)

Here the L.H.S. of (8) is a function of x alone and the R.H.S. is a function of t alone and so the two can be equal only if each side is equal to a constant, λ say. Then (8) leads to

$$(d^2X/dx^2) - \lambda X = 0 \quad \text{and} \quad dT/dt = \lambda kT. \quad ...(9)$$

Since we require a solution of (5) which tends to zero as $t \to \infty$, we choose $\lambda = -n^2$, where n is a non-zero constant. Then, solving (9) we get,

$$X = a_n \cos(nx + e_n),\ T = c_n e^{-kn^2 t},\ a_n, e_n, c_n \text{ being arbitrary constants.}$$

$\therefore$ By (6), $z(x, t) = d_n \cos(nx + e_n)\, e^{-kn^2 t}$, (taking $d_n = a_n c_n$) ...(10)

is a solution of (5) for all values of n; d_n and e_n being new arbitrary constants.

Note. Solution (10) can also be re-written as

$$z(x, t) = (D_n \cos nx + E_n \sin nx)\, e^{-kn^2 t}. \quad ...(11)$$

Ex. 9. *Solve the one-dimensional diffusion equation* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *in the region* $0 \le x \le \pi$, $t \ge 0$ *when (i)* u *remains finite as* $t \to \infty$; *(ii)* $u = 0$ *if* $x = 0$ *or* π, *for all values of* t; *(iii) At* $t = 0$, $u = x$ *for* $0 \le x \le \pi/2$, *and* $u = \pi - x$ *for* $\pi/2 < x \le \pi$.

[This is unsolved Ex. 3 on page 23.]

Sol. Given that $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$. ...(1)

Also, $u(x, t) =$ finite quantity as $t \to \infty$. ...(2)

Boundary conditions are $u(0, t) = u(\pi, t) = 0$ for each t. ...(3)

Initial condition is $u(x, 0) = \begin{cases} x, & \text{when } 0 \le x \le \pi/2 \\ \pi - x, & \text{when } \pi/2 \le x \le \pi. \end{cases}$ ...(4)

Let a solution of (1) be $u(x, t) = X(x)\, T(t)$. ...(5)

Substituting (5) in (1), we get $(1/X)\, X'' = (1/kT)\, T'$. ...(6)

Since the L.H.S. of (6) is a function of x alone whereas the R.H.S. is a function of t alone, hence the two can be equal only if each side is equal to a constant, λ say. In view of given condition (2), we choose $\lambda = -n^2$, where n is a non-zero quantity. Then (6) gives

$$(1/X)\, X'' = -n^2 \text{ so that } (D^2 + n^2)\, X = 0, \text{ where } D \equiv d/dx$$

and $(1/kT)\, T' = -n^2$ so that $(1/T)\, dT = -n^2 k\, dt$.

Additional Problems

Solving these, $X_n(x) = A_n \cos nx + B_n \sin nx$ and $T_n(t) = C_n e^{-n^2kt}$.

$\therefore$ By (5), $u(x,t) = (D_n \cos nx + E_n \sin nx)\, e^{-n^2kt}$, ...(7)

where D_n $(= A_nC_n)$ and E_n $(= B_nC_n)$ are new arbitrary constants. (7) is a solution of (1) for all values of n. Putting $x = 0$ and $x = \pi$ in (7), and using the boundary conditions (3), we get $0 = D_n$ and $0 = D_n \cos n\pi + E_n \sin n\pi$. These give $D_n = 0$ and $\sin n\pi = 0$ [taking $E_n \neq 0$ for non-trivial solution $u(x, t)$ of (1)]. Now $\sin n\pi = 0 \Rightarrow n$ must be an integer. Hence, we take $n = 1, 2, 3, \ldots$ in (7). Thus, from (7), we have

$$u_n(x,t) = E_n \sin nx\, e^{-n^2kt}, n = 1, 2, 3, \ldots \qquad \text{...(8)}$$

which are solutions of (1) satisfying the boundary conditions (3). To find a solution satisfying the initial conditions (4), we consider the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin nx\, e^{-n^2kt}. \qquad \text{...(9)}$$

Putting $t = 0$ in (9), gives

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin nx, n = 1, 2, 3, \ldots \qquad \text{...(10)}$$

which is half range Fourier sine series in $(0, \pi)$ and so E_n is given by

$$E_n = \frac{2}{\pi}\int_0^{\pi} u(x,0) \sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} u(x,0) \sin nx\, dx + \int_{\pi/2}^{\pi} u(x,0) \sin nx\, dx\right]$$

$$= \int_0^{\pi/2} \frac{2x}{\pi} \sin nx\, dx + \int_{\pi/2}^{\pi} \left(\frac{2}{\pi}\right)(\pi - x) \sin nx\, dx, \text{ using (4)}$$

$$= \left[\left(\frac{2x}{\pi}\right)\left(-\frac{\cos nx}{n}\right) - \left(\frac{2}{\pi}\right)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}.$$

$$+\left[\left\{\frac{2}{\pi}(\pi - x)\right\}\left(-\frac{\cos nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi};$$

$$= -(1/n)\cos(n\pi/2) + (2/\pi n^2)\sin(n\pi/2)$$

$$+ (1/n)\cos(n\pi/2) + (2/\pi n^2)\sin(n\pi/2)$$

$$\therefore E_n = \frac{4}{\pi n^2} \sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4(-1)^{m+1}/\pi(2m-1)^2, \text{ if } n = 2m - 1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of E_n in (9), the required solution is

$$u(x,t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin(2m-1)x\; e^{-(2m-1)^2 kt}, m = 1, 2, 3, \ldots$$

Ex. 10. *Solve the one-dimensional diffusion equation* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *in the range* $0 \le x \le 2\pi$, $t \ge 0$ *subject to the boundary conditions :*

$u(x, 0) = \sin^3 x$ *for* $0 \le x \le 2\pi$ *and* $u(0, t) = u(2\pi, t) = 0$ *for* $t \ge 0$.

[*Delhi, B.Sc. (H), 1998, 99, 2004*]

[This is unsolved Ex. 16 on page 25.]

Sol. Proceed upto equation (18) as in Ex. 2 on page 14 by taking $a = 2\pi$ and $f(x) = \sin^3 x$. Then, equation (18) for the present problem reduces to

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{nx}{2}\right) e^{-(n^2 kt)/4}, n = 1, 2, 3, \ldots \qquad \ldots(18)$$

Putting $t = 0$ in (18) and using given condition $u(x, 0) = \sin^3 x$, we get

$$\sum_{n=1}^{\infty} E_n \sin(nx/2) = \sin^3 x = (3/4)\sin x - (1/4)\sin 3x$$

$$\left[\because \sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = (1/4)(3\sin x - \sin 3x)\right]$$

or $\quad E_1 \sin(x/2) + E_2 \sin x + E_3 \sin(3x/2) + E_4 \sin 2x + E_5 \sin(5x/2)$

$$+ E_6 \sin 3x + E_7 \sin(7x/2) + \ldots = (3/4)\sin x - (1/4)\sin 3x. \qquad \ldots(19)$$

Equating the coefficients of the like terms on both sides of (19), we get $E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ when $n \ne 2$ or $n \ne 6$. Substituting these values in (18), the required solution is

$$u(x,t) = E_2 \sin x\, e^{-kt} + E_6 \sin 3x\, e^{-9kt} = (1/4)\left[3\sin x\, e^{-kt} - \sin 3x\, e^{-9kt}\right].$$

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Ex. 11. *Solve the boundary value problem* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *satisfying the conditions* $u(0, t) = u(l, t) = 0$ *and* $u(x, 0) = x$ *when* $0 \le x \le l/2$; $u(x, 0) = l - x$ *when* $l/2 \le x \le l$.

[This is unsolved Ex. 7 on page 23]

Sol. Proceed upto equation (18) as in Ex. 2 on page 14 by taking $a = l$. Then, the equation (18) for the present problem reduces to

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} e^{-(n^2\pi^2 kt)/l^2}. \qquad \ldots(18)$$

Given that $\quad u(x,0) = \begin{cases} x, \text{ when } 0 \le x \le l/2 \\ l - x, \text{ when } l/2 \le x \le l. \end{cases} \qquad \ldots(19)$

Putting $t = 0$ in (18), we get $\qquad u(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$,

which is half range Fourier sine series in $(0, l)$. So, E_n is given by

$$E_n = \frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2} u(x,0)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} u(x,0)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \int_0^{l/2} \frac{2x}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2}{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \left[\left(\frac{2x}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_0^{l/2}$$

$$+\left[\left(\frac{2(l-x)}{l}\right)\left(-\frac{\cos(n\pi x)/l}{(n\pi)/l}\right) - \left(-\frac{2}{l}\right)\left(-\frac{\sin(n\pi x)/l}{(n\pi)^2/l^2}\right)\right]_{l/2}^{l}$$

$$= -(l/n\pi)\cos(n\pi/2) + \left(2l/n^2\pi^2\right)\sin(n\pi/2) + (l/n\pi)\cos(n\pi/2) + \left(2l/n^2\pi^2\right)\sin(n\pi/2)$$

$$\therefore\ E_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4l/(2m-1)^2\pi^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

$$\therefore\ \text{by (18)},\ u(x,t) = \frac{4l}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\, e^{-\{(2m-1)^2\pi^2 kt\}/l^2}.$$

Ex. 12. *Solve the boundary value problem $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ satisfying the conditions $u(0, t) = u(l, t) = 0$ and $u(x, 0) = lx - x^2$.*

Sol. Proceed upto equation (19) as in Ex. 2 on page 14 by taking $a = l$ and $f(x) = u(x, 0) = lx - x^2$. Then (18) and (19) give

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2 kt)/l^2}, \qquad \ldots(18)$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l \left(lx - x^2\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left(lx-x^2\right)\left\{\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right\} - (l-2x)\left\{\frac{-\sin(n\pi x)/l}{(n\pi)^2/l^2}\right\} + (-2)\left\{\frac{\cos(n\pi x)/l}{(n\pi)^3/l^3}\right\}\right]_0^l$$

$$= (2/l)\left\{-\left(2l^3/n^3\pi^3\right)\cos n\pi + \left(2l^3/n^3\pi^3\right)\right\} = \left(4l^2/n^3\pi^3\right)\left\{1-(-1)^n\right\}$$

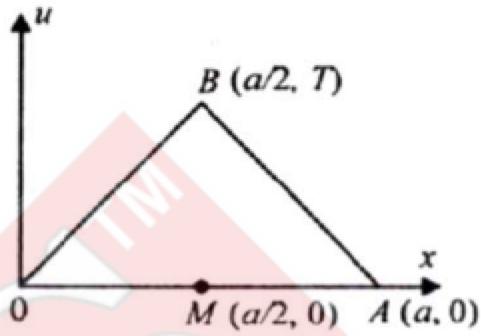
$$\therefore\ E_n = \begin{cases} \left(8l^2\right)/(2m-1)^3\pi^3, \text{ if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3, \ldots \\ 0, \text{ if } n = 2m \text{ (even) where } m = 1, 2, 3, \ldots \end{cases}$$

So, by (18), we have

$$u(x,t) = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\, e^{-\{(2m-1)^2\pi^2 kt\}/l^2}.$$

Ex. 13. *A homogeneous rod of conducting material of length a has its* *ends kept at zero temperature. The temperature at the centre is T and falls uniformly to zero at the two ends. Find the temperature function u (x, t).*

Sol. We know that $u(x, t)$ is the solution of heat equation $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$. The boundary conditions are $u(0, t) = u(a, t) = 0$ for all $t \geq 0$. Let OA be the given rod and M be its middle point. Given that the temperature at the centre M is T and falls uniformly to zero at the two ends O and A of the rod. Hence, the temperature distribution at $t = 0$ is as given in the adjoining figure. The equations of straight lines OB and BA respectively are given by



$$u-0=\frac{T-0}{(a/2)-0}(x-0) \quad \text{and} \quad u-0=\frac{T-0}{(a/2)-a}(x-a)$$

So, here $f(x)=u(x,0)=\begin{cases}(2Tx)/a, \text{ for } 0\leq x\leq a/2 \\ \{2T(a-x)\}/a, \text{ for } a/2\leq x\leq a.\end{cases}$ ...(*i*)

Now, proceed upto equation (19) as in Ex. 2 on page 14. Then, (18) and (19) become

$$u(x,t)=\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}e^{-(n^2\pi^2 kt)/a^2}, \qquad ...(18)$$

where $$E_n=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx \qquad ...(19)$$

or $$E_n=\frac{2}{a}\left[\int_0^{a/2} f(x)\sin\frac{n\pi x}{a}dx+\int_{a/2}^{a} f(x)\sin\frac{n\pi x}{a}dx\right]$$

$$=\frac{2}{a}\int_0^{a/2}\frac{2Tx}{a}\sin\frac{n\pi x}{a}dx+\frac{2}{a}\int_{a/2}^{a}\frac{2T(a-x)}{a}\sin\frac{n\pi x}{a}dx, \text{ using } (i)$$

$$=\int_0^{a/2}\frac{4Tx}{a^2}\sin\frac{n\pi x}{a}dx+\int_{a/2}^{a}\frac{4T(a-x)}{a^2}\sin\frac{n\pi x}{a}dx$$

$$=\left[\left(\frac{4Tx}{a^2}\right)\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_0^{a/2}$$

$$+\left[\left\{\frac{4T(a-x)}{a^2}\right\}\left(-\frac{\cos(n\pi x)/a}{(n\pi)/a}\right)-\left(-\frac{4T}{a^2}\right)\left(-\frac{\sin(n\pi x)/a}{(n\pi)^2/a^2}\right)\right]_{a/2}^{a}$$

$$=-(2T/n\pi)\cos(n\pi/2)+\left(4T/n^2\pi^2\right)\sin(n\pi/2)$$
$$+(2T/n\pi)\cos(n\pi/2)+\left(4T/n^2\pi^2\right)\sin(n\pi/2)$$

$\therefore\ E_n = (8T/n^2\pi^2)\sin(n\pi/2)$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 8(-1)^{m+1}\, T/(2m-1)^2\pi^2, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

So by (18), we have

$$u(x,t) = \frac{8T}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{a}\, e^{-\{(2m-1)^2\pi^2 kt\}/a^2}$$

Ex. 14. (*a*) *An insulated rod of length l has its ends A and B kept at a° celsius and b° celsius respectively until steady state conditions prevail. The temperature at each end is suddenly reduced to zero degree celsius and kept so. Find the resulting temperature at any point of the rod taking the end A as origin.*

(*b*) *A rod 30 cm long has its ends A and B kept at 20° and 80° respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0° and kept so. Find the resulting function u (x, t) taking x = 0 at A.*

(*c*) *The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.*

Sol. (*a*) The temperature function $u(x, t)$ is a solution of heat equation

$$\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t). \qquad \ldots(i)$$

When the steady state condition prevails, $\partial u/\partial t = 0$ and therefore (*i*) reduces to ordinary differential equation $d^2u/dx^2 = 0$. ...(*ii*)

Integrating (*ii*), $\quad du/dx = c_1$, ...(*iii*)

Integrating (*iii*), $\quad u(x) = c_1 x + c_2$, ...(*iv*)

where c_1 and c_2 are arbitrary constants. Putting $x = 0$ and $x = l$ in (*iv*) and using the fact that $u = a$ when $x = 0$ and $u = b$ when $x = l$, we have

$a = (c_1 \cdot 0) + c_2$ and $b = c_1 l + c_2$ so that $c_1 = (b-a)/l$ and $c_2 = a$

So, by (*iv*), here $\quad u(x, 0) = f(x) = \{(b-a)x\}/l + a$. ...(*v*)

Also, given boundary conditions : $u(0, t) = u(l, t) = 0$ for all $t \geq 0$.

Proceed upto equation (18) and (19) as in Ex. 2 on page 14 (replace a by l for the present problem). Then

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\, e^{-(n^2\pi^2 kt)/l^2} \qquad \ldots(18)$$

where $\quad E_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$ ...(19)

$$\therefore\quad E_n = \frac{2}{l}\int_0^l \left\{\frac{(b-a)x}{l} + a\right\}\sin\frac{n\pi x}{l}\,dx, \text{ by } (v)$$

$$=\left[\frac{2}{l}\left\{\frac{(b-a)x}{l}+a\right\}\left(\frac{-\cos(n\pi x)/l}{(n\pi)/l}\right)-\frac{2(b-a)}{l^2}\left(\frac{-\sin(n\pi x)/l}{(n\pi/l)^2}\right)\right]_0^l$$

$$=-(2b/n\pi)\cos n\pi+(2a/n\pi)=(2/n\pi)\left\{a-b(-1)^n\right\}$$

So, by (18), $u(x,t)=\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{a-b(-1)^n}{n}\sin\frac{n\pi x}{l}\,e^{-(n^2\pi^2kt)/l^2}$ **Ans.**

(*b*) Proceed as in part (*a*) taking $l = 30$, $a = 20$ and $b = 80$.

Ans. $u(x,t)=\dfrac{40}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1-4(-1)^n}{n}\sin\frac{n\pi x}{30}\,e^{-(n^2\pi^2kt)/900}$

(*c*) Proceed as in part (*a*) taking $l = 50$, $a = 0$ and $b = 100$.

Ans. $u(x,t)=\dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{50}\,e^{-(n^2\pi^2kt)/2500}$

Ex. 15. *Solve* $k\,(\partial^2u/\partial x^2)=\partial u/\partial t$ *for* $0<x<\pi$, $t>0$, *if* $u_x(0,t)=u_x(\pi,t)=0$ *and* $u(x,0)=\sin x$.

[It is unsolved Ex. 8 on page 23.]

Sol. Given that $\partial^2u/\partial x^2=(1/k)\,(\partial u/\partial t)$. ...(1)

Boundary conditions (B.C.) : $u_x(0,t)=u_x(\pi,t)=0$ for all $t\ge 0$ ...(2)

Initial condition (I.C.) : $u(x,0)=\sin x$, $0<x<\pi$. ...(3)

Let a solution of (1) be of the form $u(x,t)=X(x)\,T(t)$. ...(4)

Substituting (4) in (1), $X''T=(1/k)XT'$ or $X''/X=T'/kT$. ...(5)

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Then, (5) gives

$$X''-\mu X=0 \text{ or } (D^2-\mu)X=0, \text{ where } D=d/dx \quad ...(6)$$

and $T'=\mu kT$ or $(1/T)\,dT=\mu k\,dt$ so that $T=ce^{\mu kt}$. ...(7)

Diff. (4) partially w.r.t. 'x' $\partial u/\partial x=u_x(x,t)=X'(x)\,T(t)$. ...(8)

Putting $x=0$ and $x=\pi$ by turn in (8) and using (2), we get

$0=X'(0)\,T(t)$ and $0=X'(\pi)\,T(t)$ so that $X'(0)=0$ and $X'(\pi)=0$ ...(9)

where have chosen $T(t)\neq 0$ for getting a non-trivial solution $u(x,t)$.

We now solve (6) using conditions (9). Three cases arise :

Case I. Let $\mu=0$. Then solution of (6) in $X(x)=Ax+B$. ...(10)

Differentiating (10), $X'(x)=A$. ...(11)

Putting $x=0$ and $x=\pi$ by turn in (11) and using (9), we get $A=0$.

Hence, (10) reduces to $X(x)=B$. ...(12)

Case II. Let $\mu=\lambda^2$, where $\lambda\neq 0$. Then solution of (6) is

$$X(x)=Ae^{\lambda x}+Be^{-\lambda x} \quad ...(13)$$

Differentiating (13), $X'(x)=A\lambda\,e^{\lambda x}-B\lambda\,e^{-\lambda x}$. ...(14)

Putting $x = 0$ and $x = \pi$ by turn in (14) and using (9), we get

$$0 = A\lambda - B\lambda \text{ and } 0 = A\lambda\, e^{\lambda x} - B\lambda\, e^{-\lambda x} \text{ so that } A = B = 0.$$

Then (13) gives $X(x) = 0$ and hence $u(x, t) = 0$, which does not satisfy (3), so we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \qquad ...(15)$$

Differentiating (15), $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x.$ ...(16)

Putting $x = 0$ and $x = \pi$ by turn in (16) and using (9), we get

$$0 = B\lambda \text{ and } 0 = -A\lambda\sin\lambda\pi + B\lambda\cos\lambda\pi.$$

$\Rightarrow$ $B = 0$ and $A\sin\lambda\pi = 0$ $\Rightarrow$ $B = 0$ and λ is an integer.

We have taken $A \neq 0$, since otherwise $X = 0$ so that $u(x, t) = 0$ which does not satisfy (3). Then (15) reduces to

$X(x) = A\cos\lambda x$, where λ is non-zero integer. ...(17)

Thus, we get non-trivial solution in cases I and III. Note that solution (12) of case I is included in solution (17) if we take $\lambda = 0, 1, 2, 3, ...$

Hence, non-zero solutions $X_n(x)$ of (6) are given by

$$X_\lambda(x) = A_\lambda\cos\lambda x, \quad \lambda = 0, 1, 2, 3, ...$$

Also, with $\mu = -\lambda^2$, the corresponding solution of (7) becomes

$$T_\lambda(t) = C_\lambda\, e^{-\lambda^2 kt}, \quad \lambda = 0, 1, 2, 3, ...$$

$\therefore$ By (4), $u_\lambda(x, t) = X_\lambda(x)\; T_\lambda(t) = D_\lambda\cos\lambda x\, e^{-\lambda^2 kt}$

are solutions of (1) satisfying (2). Here D_λ $(= A_\lambda C_\lambda)$ are new arbitrary constants. In order to obtain a solution also satisfying (3), we consider more general solution

$$u(x, t) = \sum_{\lambda=0}^{\infty} u_\lambda(x, t) = \sum_{\lambda=0}^{\infty} D_\lambda\cos\lambda x\, e^{-\lambda^2 kt}, \quad \lambda = 0, 1, 2, 3, ... \qquad ...(18)$$

Substituting $t = 0$ in (18) and using (3), we get

$$\sin x = \sum_{\lambda=0}^{\infty} D_\lambda\cos\lambda x \qquad ...(19)$$

which Fourier cosine series. So, the constants D_λ are given by

$$D_0 = \frac{1}{\pi}\int_0^\pi \sin x\, dx \text{ and } D_\lambda = \frac{2}{\pi}\int_0^\pi \sin x\cos\lambda x\, dx, \quad \lambda = 0, 1, 2, 3, ... \qquad ...(20)$$

Now, $$D_0 = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi} \qquad ...(21)$$

With $\lambda = 1$, (20) reduces to

$$D_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\, dx = \frac{1}{\pi}\int_0^\pi \sin 2x\, dx = \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^\pi = 0.$$

For $\lambda > 1$, (20) reduces to

$$D_\lambda = \frac{1}{\pi}\int_0^\pi [\sin(\lambda+1)x - \sin(\lambda-1)x]dx = \frac{1}{\pi}\left[-\frac{\cos(\lambda+1)x}{\lambda+1} + \frac{\cos(\lambda-1)x}{\lambda-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{\lambda+1}}{\lambda+1} + \frac{(-1)^{\lambda-1}}{\lambda-1} + \frac{1}{\lambda+1} - \frac{1}{\lambda-1}\right] = \frac{1}{\pi}\left[\frac{1-(-1)^{\lambda+1}}{\lambda+1} + \frac{(-1)^{\lambda-1}-1}{\lambda-1}\right].$$

If $\lambda = 2m-1$ (odd) with $m = 1, 2, 3, \ldots$. Then, $D_\lambda = 0$.

If $\lambda = 2m$ (even) with $m = 1, 2, 3, \ldots$. Then, D_{2m} is given by

$$D_{2m} = \frac{2}{\pi}\left(\frac{1}{2m+1} - \frac{1}{2m-1}\right) = -\frac{4}{\pi\left(4m^2-1\right)}.$$

Substituting the above values in (18), we get

$$u(x,t) = D_0 + \sum_{m=1}^{\infty} D_{2m} \cos 2mx \, e^{-4m^2kt} = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}e^{-4m^2kt}.$$

Ex. 16. *Determine u such that* $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$ *and satisfy the conditions (i)* $u \to 0$ *as* $t \to \infty$ *(ii)* $u = \sum_n c_n \cos nx$ for $t = 0$.

Sol. Given $\partial^2 u/\partial x^2 = (1/k)(\partial u/\partial t)$. **[Delhi B.Sc. (H) 2004]** ...(1)

Also given that $u \to 0$ as $t \to \infty$. ...(2)

and $$u(x,0) = \sum_n c_n \cos nx. \quad ...(3)$$

Let a solution of (1) be $u(x, t) = X(x)\,T(t)$, ...(4)

where $X(x)$ is a function of x alone and $T(t)$ is a funciton of t alone.

Substituting (4) in (1), we get $(1/X)X'' = (1/kT)T'$. ...(5)

Since the L.H.S. of (5) is a function of x alone and the R.H.S. of (6) is a function of t alone, hence the two sides of (6) can be equal only if each side is equal to a constant, λ say. In view of condition (2), we choose $\lambda = -n^2$, where n is a non-zero constant. Then (6) gives

$$(1/X)X'' = -n^2 \text{ so that } \left(D^2+n^2\right)X = 0, \text{ where } D \equiv d/dx.$$

and $$(1/kT)T' = -n^2 \text{ so that } (1/T)\,dT = -n^2 k\,dt.$$

Solving these $X_n(x) = a_n \cos nx + b_n \sin nx$ and $T_n(t) = e_n\, e^{-n^2kt}$.

Keeping (3) and (4) in view, the most general solution of (1) may be written as

$$u(x,t) = \sum_n u_n(x,t) = \sum_n X_n(x)\,T_n(t)$$

or $$u(x,t) = \sum_n (c_n \cos nx + d_n \sin nx)\,e^{-n^2kt}, \quad ...(1)$$

where $c_n (= a_n e_n)$ and $d_n (= b_n e_n)$ are new arbitrary constants.

Putting $t = 0$ in (1) and using (3), we have

$$\sum_n c_n \cos nx = \sum_n (c_n \cos nx + d_n \sin nx),$$

showing that for the present problem $d_n = 0$. Then, from (1) the required solution is

Additional Problems

$$u(x,t) = \sum_n c_n \cos nx\, e^{-n^2 kt}.$$

Ex. 17. *Explain D'Alembert's solution of wave equation.*

or

Obtain general solution of wave equation $\partial^2u/\partial x^2 = (1/c^2)(\partial^2u/\partial t^2)$, given that initial deflection $= u(x, 0) = f(x)$ and initial velocity $= (\partial u/\partial t)_{t=0} = g(x)$,

Sol. Consider the wave equation $\partial^2u/\partial x^2 = (1/c^2)(\partial^2u/\partial t^2)$ ...(1)

Let $D \equiv \partial/\partial t$ and $D' \equiv \partial/\partial x$. Then (1) becomes $(D^2 - c^2D'^2)\,y = 0$. whose auxiliary equation is $m^2 - c^2 = 0$ so that $m = c, -c$.

Hence, solution of (1) is [Refer Art. 3.4, page 117 of Part II of this book.]

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \quad ...(2)$$

where ϕ and ψ are arbitrary functions. Solution (2) is known as the D'Alembert's solution of the wave equation (1).

The function ϕ and ψ can be obtained by using the given initial conditions:

$$u(x,0) = f(x), \text{ for all } x. \quad ...(3)$$

and
$$(\partial u/\partial t)_{t=0} = g(x) \text{ for all } x. \quad ...(4)$$

From (2), $\partial u/\partial t = c\phi'(x+ct) - c\psi'(x-ct)$. ...(5)

Putting $t = 0$ in (2) and using (3), we get

$$\phi(x) + \psi(x) = f(x). \quad ...(6)$$

Putting $t = 0$ in (4) and using (4), we get

$$c\phi'(x) - c\psi'(x) = g(x) \text{ or } \phi'(x) - \psi'(x) = (1/c)\,g(x). \quad ...(7)$$

Integrating (7), $\phi(x) - \psi(x) = \dfrac{1}{c}\displaystyle\int_a^x g(\theta)\,d\theta,$ ...(8)

where a is arbitrary. Solving (6) and (8), we have

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_a^x g(\theta)\,d\theta \text{ and } \psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_a^x g(\theta)\,d\theta.$$

$$\therefore \quad \phi(x+ct) = \frac{f(x+ct)}{2} + \frac{1}{2c}\int_a^{x+ct} g(\theta)\,d\theta,$$

$$\textit{and} \quad \psi(x-ct) = \frac{f(x-ct)}{2} - \frac{1}{2c}\int_a^{x-ct} g(\theta)\,d\theta.$$

Substituting these values in (2), we have

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_a^{x+ct} g(\theta)\,d\theta - \int_a^{x-ct} g(\theta)\,d\theta\right]$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_a^{x+ct} g(\theta)\,d\theta + \int_{x-ct}^{a} g(\theta)\,d\theta\right]$$

$$\therefore \quad u(x,t) = \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\theta)\,d\theta. \quad ...(9)$$

Particular case. If the string is at rest when $t = 0$ so that $g = 0$, then the required solution (9) reduces to

$$u(x,t) = (1/2)\{f(x+ct) + f(x-ct)\}. \quad ...(10)$$

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Ex. 18. (*a*) *A string of length l has its ends x = 0 and x = l fixed. It is released from rest in the position* $y = \{4\lambda x(l-x)\}/l^2$. *Find an expression for the displacement of the string at any subsequent time.*

[This is unsolved Ex. 8 on page 40]

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(*b*) *A tant string of length l has its ends x = 0 and x = l fixed. The mid-point is taken to a small height h and released from rest at time t = 0. Find the displacement function y (x, t).*

[This is unsolved Ex. 1 on page 39]

(*c*) *A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form* $y = y_0 \sin(\pi x/l)$ *from which it is released at time t = 0. Find the displacement at any point at a distance x from one end at time t.*

[This is unsolved Ex. 2 on page 40]

(*d*) *A tightly stretched elastic string of length l, with fixed end points x = 0 and x = l is initially in the position given by* $y = y_0 \sin^3(\pi x/l)$, y_0 *being constant. It is released from the position of rest. Find the displacement y (x, t).*

[This is unsolved Ex. 5 on page 40 and solved Ex 5 (*a*) on page 108]

(*e*) *A tightly stretched elastic string of length π, with fixed end points x = 0 and x = π is initially in the position given by* $y = y_0 \sin^3 x$, y_0 *being constant. Find the displacement y (x, t).* **[*Nagpur*, 1996]**

[This is solved Ex. 5 (*b*) on page 108]

Note. In all the above problems, we shall apply the following result of Ex. 2 on page 31. For the present problems, replace u by y and a by l.

Result. Solution of wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$. ...(1)

subject to boundary conditons $y(0, t) = y(l, 0) = 0$ for all t. ...(2)

$$\left.\begin{array}{l}\text{with initial velocity, } (\partial u/\partial t)_{t=0} = 0 \\ \text{and with initial displacement } u(x, 0) = f(x)\end{array}\right\} \quad ...(3)$$

is $$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \quad ...(4)$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \quad ...(5)$$

Sol. (*a*) The displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad ...(1)$$

subject to boundary conditions :

$$y(0, t) = y(l, t) = 0 \text{ for all } t \geq 0. \qquad ...(2)$$

and initial condition, namely,

Initial velocity $= (\partial y/\partial t)_{t=0} = 0$ for $0 \leq x \leq l$. ...(3a)

and initial displacement $= y(x, 0) = f(x) = \{4\lambda x(l-x)\}/l^2$. ...(3b)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}, \qquad ...(4)$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \qquad ..(5)$$

Substituting the value of $f(x)$ given by (3) in (5), we get

$$E_n = \left(\frac{2}{l}\right)\times\left(\frac{4\lambda}{l^2}\right)\int_0^l \left(lx - x^2\right)\sin\frac{n\pi x}{l}\,dx.$$

$$\Rightarrow \frac{8\lambda}{l^3}\left[\left(lx-x^2\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)+(-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[-\frac{2l^3(-1)^n}{n^3\pi^3}+2\frac{l^3}{n^3\pi^3}\right] = \frac{16\lambda}{n^3\pi^3}\left[1-(-1)^n\right], \text{ as } \cos n\pi = (-1)^n$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) with } m = 1, 2, 3, \ldots \\ (32\lambda)/(2m-1)^3\pi^3, \text{ if } n = 2m-1 \text{ (odd) with } m = 1, 2, 3, \ldots \end{cases}$$

Substituting the value of E_n in (4), the required expression is

$$y(x,t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}.$$

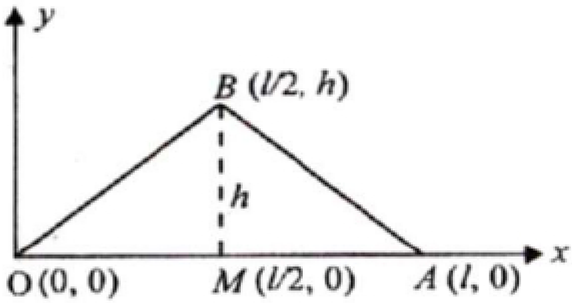
(*b*) The displacement function $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \qquad ...(1)$$

subject to the boundary conditons :

$$y(0, t) = y(l, t) = 0 \text{ for all } t \geq 0. \qquad ...(2)$$

Initial position of the string at $t = 0$ is made up of two straight line segments OB and BA as shown in the figure and the string is released from rest.



The equation of OB is given by

$$y - 0 = \frac{h-0}{(l/2)-0}(x-0) \text{ or } y = (2hx)/l \text{ for } 0 \leq x \leq l/2.$$

The equation of BA is given by

$$y - 0 = \frac{h-0}{(l/2)-l}(x-l) \text{ or } y = \{2h(l-x)\}/l \text{ for } l/2 \leq x \leq l.$$

Hence, the initial displacement is given by

$$= u(x, 0) = f(x) = \begin{cases} (2hx)/l, & 0 \le x \le l/2 \\ \{2h(l-x)\}/l, & l/2 \le x \le l. \end{cases} \qquad \text{...(3A)}$$

and the initial velocity $= (\partial u/\partial t)_{t=0} = 0$ ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}, \qquad \text{...(4)}$$

where $$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \qquad \text{...(5)}$$

$$\therefore \quad E_n = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}\,dx\right], \text{ using (3A)}$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{4h}{l^2}\int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{4h}{l^2}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+ \frac{4h}{l^2}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + \frac{4h}{l^2}\left(\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= (8h/n^2\pi^2)\sin(n\pi/2)$$

$$= \begin{cases} \{8(-1)^{m+1}h\}/(2m-1)^2\pi^2, & \text{if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3, \dots \\ 0, & \text{if } n = 2m \text{ (even) and } m = 1, 2, 3, \dots \end{cases}$$

[Note that for $n = 2m - 1$, $\sin(n\pi/2) = \sin(2m-1)(\pi/2) = \sin(m\pi - \pi/2)$
$= \sin m\pi \cos(\pi/2) - \cos m\pi \sin(\pi/2) = (-1)^{m+1}$]

Substituting the above value of E_n in (4), the required displacement function is given by

$$y(x,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}$$

(*c*) The required displacement $y(x, t)$ of the string is the solution of the wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)

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subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

Initial velocity $= (\partial u/\partial t)_{t=0} = 0$ for $0 \leq x \leq l$. ...(3A)

and initial displacement $= y(x,0) = y_0 \sin(\pi x/l)$, $0 \leq x \leq l$. ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \qquad ...(4)$$

Putting $t = 0$ in (4) and using (3B), we have

$$y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

$$\therefore\ E_1 \sin(\pi x/l) + E_2 \sin(2\pi x/l) + E_3 \sin(3\pi x/l) + \ldots = y_0 \sin(\pi x/l)$$

Comparing the coefficients of the like terms on both sides, we have $E_1 = y_0$ and $E_n = 0$ for $n \neq 1$. With these values, (4) reduces to

$$y(x,t) = y_0 \sin(\pi x/l)\cos(\pi ct/l).$$

(*d*) The required displacement $y(x, t)$ of the string is the solution of the wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)
subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

Initial velocity $= (\partial u/\partial t)_{t=0} = 0$ for $0 \leq x \leq l$. ...(3A)

and initial displacement $= y(x,0) = y_0 \sin^3(\pi x/l)$, $0 \leq x \leq l$. ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \qquad ...(4)$$

Putting $t = 0$ in (4) and using (3B), we have

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

or $$E_1 \sin(\pi x/l) + E_2 \sin(2\pi x/l) + E_3 \sin(3\pi x/l) + E_4 \sin(4\pi x/l) + \ldots$$
$$= (y_0/4)\{3\sin(\pi x/l) - \sin(3\pi x/l)\}$$

$$[\because\ \sin 3\theta = 3\sin\theta - 4\sin^3\theta \text{ so that } \sin^3\theta = (1/4)(3\sin\theta - \sin 3\theta)]$$

Comparing the coefficients of the like terms on both sides, we have

$$E_1 = 3(y_0/4);\ E_2 = 0,\ E_3 = -(y_0/4),\ E_n = 0 \text{ for all } n \geq 4.$$

Substituting these values in (4), the required displacement is given by

$$y(x,t) = (3y_0/4)\sin(\pi x/l)\cos(\pi ct/l) - (y_0/4)\sin(3\pi x/l)\cos(3\pi ct/l).$$

(*e*) The required displacement $y(x, t)$ of the string is the solution of the wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)
subject to the boundary conditons : $y(0, t) = y(\pi, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

Initial velocity $= (\partial u/\partial t)_{t=0} = 0$ for $0 \leq x \leq \pi$. ...(3A) and initial displacement $= y(x, 0) = y_0 \sin^3 x,\ 0 \le x \le \pi.$ ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x, t) = \sum_{n=1}^{\infty} E_n \sin(nx) \cos(nct). \qquad \text{...(4)}$$

[Comparing this part (*e*) with part (*d*), we find that here $l = \pi$. Hence, we have result (4) in the above form.]

Putting $t = 0$ in (4) and using (3B), we have

$$y_0 \sin^3 x = \sum_{n=1}^{\infty} E_n \sin nx$$

or $E_1 \sin x + E_2 \sin 2x + E_3 \sin 3x + E_4 \sin 4x + \ldots = (y_0/4)(3 \sin x - \sin 3x)$

so that $E_1 = 3y_0/4,\ E_2 = 0,\ E_3 = -y_0/4,\ E_n = 0$ for all $n \ge 4$. So, (4) gives

$$y(x, t) = (3y_0/4) \sin x \cos(ct) - (y_0/4) \sin(3x) \cos(3ct).$$

Ex. 19. (*a*) *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l - x)$, find its displacement.* **(I.A.S. 2000)**

[This is unsolved Ex. 3 and Ex. 7 on page 40] **[Delhi (H) 2000]**

(*b*) *A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity v given by $v = kx$ if $0 \le x \le l/2$ and $v = k(l - x)$ if $l/2 \le x \le l$. Find the displacement function $y(x, t)$.*

(*c*) *A string of length l is initially at rest in its equilibrium position and each of its points is given the velocity $(\partial y/\partial t)_{t=0} = v_0 \sin^3 (\pi x/l)$ where $0 < x < l$. Find the displacement function.*

[This is unsolved Ex. 9 on page 41 and Ex. 6 on page 111]

(*d*) *If the string of length l is initially at rest in equilibrium position and each of its points is given the velocity $v_0 \sin(3\pi x/l) \cos(2\pi x/l)$ where $0 < x < l$ at $t = 0$. Find the displacement function.*

Note. In all the above problems, we shall apply the following result of Ex. 2 of page 31. For the present problems, replace u by y and a by l.

Result: Solution of wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 u/\partial t^2)$ ...(1)
subject to the boundary conditons : $y(0, t) = y(l, 0) = 0$ for all t. ...(2)
with given initial conditions, namely :

initial displacement $= y(x, 0) = f(x) = 0$ ...(3A)
and initial velocity $= (\partial y/\partial t)_{t=0} = g(x)$ ...(3B)

is
$$y(x, t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}, \qquad \text{...(4)}$$

where
$$F_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\frac{n\pi x}{l}\, dx. \qquad \text{...(5)}$$

Sol. (*a*) The required displacement $y(x, t)$ of the string is the solution of the wave equation $\quad \partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)

subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

initial displacement $= y(x, 0) = f(x) = 0$ ...(3A)

and initial velocity $= (\partial y/\partial t)_{t=0} = g(x) = k(lx - x^2)$. ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}, \quad \text{...(4)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx = \frac{2k}{n\pi c}\int_0^l \left(lx - x^2\right)\sin\frac{n\pi x}{l}\,dx. \quad \text{...(5)}$$

$$\therefore \quad F_n = \frac{2k}{n\pi c}\left[\left(lx-x^2\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2k}{n\pi c}\left[-\frac{2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right] = \frac{4kl^3}{cn^4\pi^4}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) and } m = 1, 2, 3, \ldots \\ \left(8kl^3\right)/c\pi^4(2m-1)^4, \text{ if } n = 2m-1 \text{ (odd) and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting these values in (4), the required displacement is given by

$$y(x,t) = \frac{8kl^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}.$$

(*b*) The required displacement $y(x, t)$ is the solution of the wave equation

$$\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{...(1)}$$

subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

initial displacement $= y(x, 0) = f(x) = 0$ ...(3A)

and initial velocity $= (\partial y/\partial t)_{t=0} = g(x) = \begin{cases} kx, & 0 \leq x \leq l/2 \\ k(l-x), & l/2 \leq x \leq l. \end{cases}$ ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}, \quad \text{...(4)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \quad \text{...(5)}$$

$$= \frac{2}{n\pi c}\left[\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} g(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{n\pi c}\int_0^{l/2} x \sin\frac{n\pi x}{l}\,dx + \frac{2k}{n\pi c}\int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx, \quad \text{by } (3B)$$

$$= \frac{2k}{n\pi c}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+\frac{2k}{n\pi c}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2k}{n\pi c}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$= (4kl^2/cn^2\pi^3)\sin(n\pi/2)$

$$=\begin{cases}0, & \text{if } n=2m \text{ and } m=1,2,3,\ldots\\ (-1)^{m+1}\left(4kl^2\right)/c\pi^3(2m-1)^3, & \text{if } n=2m-1 \text{ and } m=1,2,3,\ldots\end{cases}$$

$$\left[\because \text{ when } n=2m-1, \sin\frac{n\pi}{2}=\sin\frac{\pi}{2}(2m-1)=\sin\left(m\pi-\frac{\pi}{2}\right)=-(-1)^m=(-1)^{m+1}\right]$$

With these values of E_n in (4), the required displacement function is

$$y(x,t)=\frac{4kl^2}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}.$$

(c) The required displacement function $y(x, t)$ is the solution of the wave equation $\quad \partial^2y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)
subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)
and the given initial conditions, namely :

initial displacement $= y(x, 0) = f(x) = 0$ ...(3A)
and initial velocity $= (\partial y/\partial t)_{t=0} = g(x) = v_0 \sin^3(\pi x/l)$, $0 \leq x \leq l$. ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x,t)=\sum_{n=1}^{\infty}F_n\sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}, \qquad \text{...(4)}$$

Differentiating both sides of (4) partially w.r.t. 't', we have

$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}F_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad \text{...(5)}$$

Putting $t = 0$ in (5) and using initial condition (5), we have

$$\sum_{n=1}^{\infty}F_n\left(\frac{n\pi c}{l}\right)\sin\frac{n\pi x}{l}=\left(\frac{\partial y}{\partial t}\right)_{t=0}=v_0\sin^3\frac{\pi x}{l}=\frac{v_0}{4}\left[3\sin\frac{\pi x}{l}-\sin\frac{3\pi x}{l}\right]$$

$$\left[\because \sin^3\theta=(1/4)(3\sin\theta-\sin 3\theta)\Rightarrow \sin^3(\pi x/l)(v_0/4)\{3\sin(\pi x/l)-\sin(3\pi x/l)\}\right]$$

or $(\pi cF_1/l)\sin(\pi x/l)+(2\pi cF_2/l)\sin(2\pi x/l)+(3\pi cF_3/l)\sin(3\pi x/l)+\ldots$

$$=(3v_0/4)\sin(\pi x/l)-(v_0/4)\sin(3\pi x/l) \qquad \text{...(6)}$$

Additional Problems

Equating the coefficients of like terms on both sides of (6), we get

$$(\pi c F_1)/l = 3v_0/4,\ (2\pi c F_2)/l = 0,\ (3\pi c F_3)/l = -(v_0/4), \ldots$$

so that $B_1 = (3v_0 l)/4\pi c$, $B_2 = 0$, $B_3 = -(v_0 l)/12\pi c$, $E_n = 0$ for all $n \geq 4$.

Substituting these values in (4), the required displacement is

$$y(x, t) = \frac{3v_0 l}{4\pi c} \sin\frac{\pi x}{l} \sin\frac{\pi ct}{l} - \frac{v_0 l}{12\pi c} \sin\frac{3\pi x}{l} \sin\frac{3\pi ct}{l}.$$

(*d*) The required displacement function $y(x, t)$ is the solution of the wave equation $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ...(1)

subject to the boundary conditons : $y(0, t) = y(l, t) = 0$ for all $t \geq 0$. ...(2)

and the given initial conditions, namely :

initial displacement $= y(x, 0) = f(x) = 0$ ...(3A)

and initial velocity $= (\partial y/\partial t)_{t=0} = g(x) = v_0 \sin(3\pi x/l) \cos(2\pi x/l)$ ...(3B)

The solution of (1) satisfying the above boundary and initial conditions is

$$y(x, t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{l} \sin\frac{n\pi ct}{l}. \qquad \text{...(4)}$$

Differentiating both sides of (4) partially w.r.t. '*t*', we have

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l}. \qquad \text{...(5)}$$

Putting $t = 0$ in (5) and using initial condition (5), we have

$$\sum_{n=1}^{\infty} F_n \left(\frac{n\pi c}{l}\right) \sin\frac{n\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin\frac{3\pi x}{l} \cos\frac{2\pi x}{l} = \frac{v_0}{2}\left[\sin\frac{5\pi x}{l} + \sin\frac{\pi x}{l}\right]$$

$$[\because\ 2\sin A \cos B = \sin(A+B) + \sin(A-B)]$$

or $\quad (\pi c F_1/l)\sin(\pi x/l) + (2\pi c F_2/l)\sin(2\pi x/l) + \ldots$

$$= (v_0/2)\sin(\pi x/l) + (v_0/2)\sin(5\pi x/l).$$

Equating the coefficients of like terms, we have

$$(\pi c F_1)/l = v_0/2,\ (5\pi c F_5)/l = v_0/2,\quad F_2 = F_3 = F_4 = F_6 = F_7 = \ldots = 0.$$

$$\Rightarrow\ F_1 = (l v_0)/2\pi c,\ F_5 = (l v_0)/5\pi c,\ F_2 = F_3 = F_4 = F_6 = F_7 = \ldots = 0.$$

With these values, (4) gives the required displacement

$$y(x, t) = (l v_0/2\pi c)\sin(\pi x/l)\sin(\pi ct/l)$$

$$+ (l v_0/5\pi c)\sin(5\pi x/l)\sin(5\pi ct/l).$$

Ex. 20. *Define Laplace's or harmonic equation. Find solution of two-dimensional Laplace's equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$.

Sol. ***Laplace's equation. Definition.*** The Laplace's equation is $\nabla^2 u = 0$, where ∇^2 is usual Laplacian operator. Since function u is frequently a potential function, this equation is also known as the potential equation.

A solution of Laplace's equation is known as *harmonic function.*

Solution of two-dimensional Laplace's equation namely :

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \qquad ...(1)$$

Suppose that (1) has solutions of the form

$$u(x, y) = X(x)\, Y(y) \qquad ...(2)$$

where X and Y are functions of x and y respectively.

From (2), $\partial^2 u/\partial x^2 = X''Y$ and $\partial^2 u/\partial y^2 = XY''$. Hence, (2) reduces to

$$X''Y + XY'' = 0 \text{ or } (1/X)X'' = -(1/Y)Y''. \qquad ...(3)$$

Since the L.H.S. of (3) depends only on x and the R.H.S. only on y, both sides of (3) must be equal to same constant, say μ. This leads to two ordinary differential equations

$$X'' - \mu X = 0 \text{ and } Y'' + \mu Y = 0, \qquad ...(4)$$

whose solutions depend only on the value of μ. Three cases arise :

Case I. When $\mu = 0$. Then (4) reduces to $X'' = 0$ and $Y'' = 0$.

Solving these, $\quad X = A_1 x + B_1$ and $Y = C_1 y + D_1$.

Then a solution of (1) is $u(x, y) = (A_1 x + B_1)(C_1 y + D_1)$. $\quad$ ...(5)

Case II. When $\mu = \lambda^2$, *i.e.*, positive. Here $\lambda \neq 0$. Then (4) reduces to

$$X'' - \lambda^2 X = 0 \quad \text{and} \quad Y'' + \lambda^2 Y = 0.$$

Solving these, $X = A_2 e^{\lambda x} + B_2 e^{-\lambda x}$ and $Y = C_2 \cos \lambda y + D_2 \sin \lambda y$.

Then a solution of (1) is

$$u(x, y) = \left(A_2 e^{\lambda x} + B_2 e^{-\lambda x}\right)(C_2 \cos \lambda y + D_2 \sin \lambda y). \qquad ...(6)$$

Case III. When $\mu = -\lambda^2$, *i.e.*, negative. Here $\lambda \neq 0$. Then (4) reduces to

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad Y'' - \lambda^2 Y = 0.$$

Solving these, $X = A_3 \cos \lambda x + B_3 \sin \lambda x$ and $Y = C_3 e^{\lambda y} + D_3 e^{-\lambda y}$.

Then a solution of (1) is

$$u(x, y) = (A_3 \cos \lambda x + B_3 \sin \lambda x)\left(C_3 e^{\lambda y} + D_3 e^{-\lambda y}\right). \qquad ...(7)$$

Out of the above mentioned three types of solutions (5), (6) and (7), we must select an appropriate solution which suits the physical nature of the problem and given boundary conditions.

Ex. 21. *Solve the Laplace's equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ *subject to the following boundary conditions :* $u(x, 0) = u(x, b) = 0$ *for* $0 \le x \le a$, $u(0, y) = 0$ *and* $u(a, y) = f(y)$ *for* $0 \le y \le b$.

[This example is exactly similar to solved Ex. 1. on page 45]

Sol. Given Laplace's equation is $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. $\quad$ ...(1)

Also given $\quad u(x, 0) = u(x, b) = 0$ for $0 \le x \le a$, $\quad$ ...(2)

$$u(0, y) = 0 \text{ for } 0 \le y \le b, \qquad ...(3a)$$

and $\quad u(a, y) = f(y)$ for $0 \le y \le b$. $\quad$ ...(3b)

Let a solution of (1) be of the form $\quad u(x, y) = X(x)\, Y(y)$. $\quad$ ...(4)

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Using (4), (1) reduces to

$$X''Y + XY'' = 0 \text{ or } (1/X)\,X'' = -(1/Y)\,Y''. \qquad ...(5)$$

Since the L.H.S. of (5) depends only on x and the R.H.S. depends only on y, each side of (5) must be equal to the same constant, say μ. Then, (5) leads to

$$X'' - \mu X = 0. \qquad ...(6)$$

and

$$Y'' + \mu Y = 0. \qquad ...(7)$$

Using (2), (4) gives $X(x)\,Y(0) = 0$ and $X(x)\,Y(b) = 0$

so that $\quad Y(0) = 0$ and $Y(b) = 0, \qquad ...(8)$

where we have taken $X(x) \neq 0$, since otherwise $u = 0$ does not satisfy (3*b*).

We now solve (7) under boundary conditions (8). Three cases arise :

Case I. Let $\mu = 0$. Then, solution of (7) is $Y(y) = Ay + B. \qquad ...(9)$

Using B.C. (8), (9) gives $0 = B$ and $0 = Ab + B$. These give $A = B = 0$ so that $Y(y) \equiv 0$. This leads to $u = 0$ which does not satisfy 3(*b*). So we reject $\mu = 0$.

Case II. Let $\mu = -\lambda^2$ (negative). Here $\lambda \neq 0$. Then solution of (7) is

$$Y(y) = Ae^{y\lambda} + Be^{-y\lambda}. \qquad ...(10)$$

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{b\lambda} + Be^{-b\lambda}. \qquad ...(11)$

(11) $\Rightarrow A = B = 0$ so that $X(x) = 0$. This leads to $u = 0$ which does not satisfy 3(*b*). So we reject $\mu = -\lambda^2$.

Case III. Let $\mu = \lambda^2$ (positive). Here $\lambda \neq 0$. Then solution of (7) is

$$Y(y) = A\cos\lambda y + B\sin\lambda y. \qquad ...(12)$$

Using B.C. (8), (12) gives $0 = A$ and $0 = A\cos\lambda b + B\sin\lambda b$ so that $A = 0$ and $\sin\lambda b = 0$, where we have taken $B \neq 0$, since otherwise we shall get $Y(y) \equiv 0$. This leads to $u = 0$ which does not satisfy 3(*b*).

Now $\sin\lambda b = 0 \Rightarrow \lambda b = n\pi \Rightarrow \lambda = n\pi/b,\ n = 1, 2, 3, \ldots$

Hence, non-zero solutions $Y_n(y)$ of (7) are given by

$$Y_n(y) = B_n\sin(n\pi y/b), n = 1, 2, 3, \ldots \qquad ...(13)$$

Also, then $\mu = -\lambda^2 = -n^2\pi^2/b^2$. Hence, (6) reduces to

$$X'' - (n^2\pi^2/b^2)\,X = 0,\ n = 1, 2, 3, \ldots \qquad ...(14)$$

whose solutions are $\quad X_n(x) = C_n e^{n\pi x/b} + D_n e^{-n\pi x/b}. \qquad ...(15)$

Using 3 (*a*), (4) gives $0 = X(0)\,Y(y)$ so that $X(0) = 0$, where we have taken $Y(y) \neq 0$, since otherwise $u = 0$ which does not satisfy (3*b*).

Now $X(0) = 0 \Rightarrow X_n(0) = 0$. Putting $n = 0$ in (15) and using $X_n(0) = 0$, we get $0 = C_n + D_n$ so that $D_n = -C_n$. Then (15) reduces to

$$X_n(x) = C_n\left(e^{n\pi x/b} - e^{-n\pi x/b}\right) = 2C_n\sinh(n\pi x/b). \qquad ...(16)$$

$$[\because\ e^{\theta} - e^{-\theta} = 2\sinh\theta]$$

From (4), (13) and (16), we see that a solution $u_n(x, y)$ of (1) is

$$u_n(x, y) = X_n(x)\,Y_n(y) = E_n\sinh(n\pi x/b)\sin(n\pi y/b),\ n = 1, 2, 3, \ldots \qquad ...(17)$$

where $E_n(=2B_nC_n)$ are new arbitrary constants. In order to obtain a solution also satisfying B.C. (3*b*), we consider more general solution

$$u.(x,y)=\sum_{n=1}^{\infty} E_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \qquad ...(18)$$

Putting $x = a$ in (18) and using B.C. (3b), we have

$$f(y)=\sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b},$$

which is the half range Fourier sine series of $f(y)$ in $(0, b)$. Hence

$$E_n \sinh\frac{n\pi a}{b}=\frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$$

or

$$E_n=\frac{2}{b\sinh(n\pi a/b)}\int_0^b f(y)\sin\frac{n\pi y}{b}. \qquad ...(19)$$

Hence, (18) is the required solution wherein E_n is given by (19).

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Ex. 22. *A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 10$ and $y = 10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x, 10) = x(10 - x)$ while the other three faces are kept at 0°C. Find the steady state temperature in the plate.*

Sol. The steady state temperature $u(x, y)$ is the solution of

$$\partial^2 u/\partial x^2+\partial^2 u/\partial y^2=0 \qquad ...(1)$$

subject to boundary conditions

$$u(0,y)=u(10,y)=0,\ 0\le y\le 10 \qquad ...(2)$$

$$u(x,0)=0,\ 0\le x\le 10 \qquad ...(3a)$$

and

$$u(x,10)=10x-x^2,\ 0\le x\le 10. \qquad ...(3b)$$

Now proceed as in solved Ex. 3 on page 50 upto equation (20). Note that in the present problem condition 3(*b*) has been changed to $u(x, 10) = 10x - x^2$ in place of $u(x, b) = 100$. Also, we have $a = b = 10$ in the present problem. So on reproducing all steps as in Ex. 3 on page 50, equation (20) will take the following form

$$u(x,y)=\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{10}\sinh\frac{n\pi y}{10}. \qquad ...(20)$$

Putting $y = 10$ in (20) and using 3(*b*), we have

$$10x-x^2=\sum_{n=1}^{\infty}(E_n\sinh n\pi)\sin\frac{n\pi x}{10},$$

which is the half range Fourier sine series of $(10x - x^2)$ in $(0, 10)$. So we get

$$E_n\sinh n\pi=\frac{2}{10}\int_0^{10}(10x-x^2)\sin\frac{n\pi x}{10}\,dx$$

Additional Problems

$$= \frac{1}{5}\left[(10x - x^2)\left(-\frac{10}{n\pi}\right)\cos\frac{n\pi x}{10} - (10-2x)\left(-\frac{100}{n^2\pi^2}\right)\sin\frac{n\pi x}{10}\right.$$

$$\left. + (-2)\left(\frac{1000}{n^3\pi^3}\right)\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5}\left[-\frac{2000\,(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3}\right] = \frac{400}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore \qquad E_n = \frac{400\,\text{cosech}\,n\pi}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$\therefore \quad E_n = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ \{800\,\text{cosech}\,(2m-1)\,\pi\}/(2m-1)^3\pi^3, \text{ if } n = 2m-1,\ m = 1, 2, 3 \end{cases}$$

Putting this value of E_n in (20), the required temperature is given by

$$u(x, y) = \frac{800}{\pi^3}\sum \frac{1}{(2m-1)^3}\sin\frac{(2m-1)\,\pi x}{10}\sinh\frac{(2m-1)\,\pi y}{10}\,\text{cosech}\,(2m-1)\,\pi.$$

Ex. 23. *Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditons*

$$u(0, y) = u(a, y) = 0 \quad \textit{for} \quad 0 \le y \le b$$

and $$u(x, 0) = 0 \quad \textit{and} \quad u(x, b) = f(x) \quad \textit{for} \quad 0 \le y \le a.$$

Sol. The heat flow in a body for a two-dimensional case is governed by the equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)\,(\partial u/\partial t)$. When the steady state condition prevails, temperature u is independent of t so that $\partial u/\partial t = 0$. Then, the above equation reduces to Laplace's equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \qquad \ldots(1)$$

Given that $\quad u(0, y) = u(a, y) = 0$ for $0 \le y \le b \qquad \ldots(2)$

Also $\quad u(x, 0) = 0$ for $0 \le x \le a \qquad \ldots(3a)$

and $\quad u(x, b) = f(x)$ for $0 \le x \le a \qquad \ldots(3b)$

Now proceed exactly as in solved Ex. 3 on page 50 upto equation (20). Note that in the present problem condition (3b) has been taken as $u(x, b) = f(x)$ in place of $u(x, b) = 100$ occurring in Ex. 3 on page 50. Reproduce all steps and obtain

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}. \qquad \ldots(20)$$

Putting $y = b$ in (20) and using (3b), we have

$$f(x) = \sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a},$$

which is the half range Fourier sine series of $f(x)$ in $(0, a)$. Hence,

$$E_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi b}{a}\,dx$$

or
$$E_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx. \qquad \text{...(21)}$$

Hence, (20) is the required solution wherein E_n is given by (21).

Ex. 24. *Find the steady state temperature distribution in a rectangular plate of sides a and b insulated at the lateral surface and satisfying the boundary conditons*

$$u(0, y) = u(a, y) = 0 \quad \textit{for} \quad 0 \le y \le b$$

and
$$u(x, b) = 0 \quad \textit{and} \quad u(x, 0) = x(a - x) \quad \textit{for} \quad 0 \le x \le a.$$

Sol. This example is a particular case of solved Ex. 1 on page 45. Here $f(x) = x(a-x) = ax - x^2$. So, proceeding as in Ex. 1, page 45, the required temperature is given by

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi(b-y)}{a}. \qquad \text{...(20)}$$

where $E_n = \dfrac{2}{a\sinh(n\pi b/a)}\displaystyle\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$ [Refer eq. (21) on page 47]

$$\therefore \quad E_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a (ax - x^2)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{2}{a\sinh(n\pi b/a)}\left[(ax - x^2)\left(-\frac{a}{n\pi}\right)\cos\frac{n\pi x}{a} - (a - 2x)\left(-\frac{a^2}{n^2\pi^2}\right)\sin\frac{n\pi x}{a} + (-2)\left(\frac{a^3}{n^3\pi^3}\right)\cos\frac{n\pi x}{a}\right]_0^a$$

$$= \frac{2}{a\sinh(n\pi b/a)}\left[-\frac{2a^3(-1)^n}{n^3\pi^3} + \frac{2a^3}{n^3\pi^3}\right] = \frac{4a^2}{n^3\pi^3}\left[1 - (-1)^n\right]\operatorname{cosech}\frac{n\pi b}{a}$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ (even) and } m = 1, 2, 3, \dots \\ \left\{8a^2/\pi^3(2m-1)^3\right\}\operatorname{cosech}\{(2m-1)\pi b/a\}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \dots \end{cases}$$

With this value of E_n, (20) reduces to

$$u(x, y) = \frac{8a^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)(b-y)\pi}{a}\operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

✓ **Ex. 25.** *Find the steady state temperature distribution in a rectangular plate bounded by the lines x = 0, x = a, y = 0 and y = b if the edge*

$y = 0$ is insulated, the edges $x = 0$ and $x = a$ are kept at $0°C$ and the edge $y = b$ is kept at temperature $f(x)$.

[**Note.** This is similar to solved Ex. 4 on page 52]

Sol. The required temperature $u(x, y)$ in steady state in two-dimensional plate is governed by the Laplace equation

$$\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2 = 0. \qquad ...(1)$$

Given that $u(0, y) = u(a, y) = 0$ for $0 \le y \le b$. ...(2)

$(\partial u / \partial y)_{y=0} = 0$ as the edge $y = 0$ is insulated for $0 \le x \le a$. ...(3a)

and $u(x, b) = f(x)$ for $0 \le x \le a$. ...(3b)

Let (1) has a solution of the form $u(x, y) = X(x)\, Y(y)$. ...(4)

Using (4), (1) gives $X''Y + XY'' = 0$ or $(1/X)\, X'' = -(1/Y)\, Y''$. ...(5)

Since the L.H.S. of (5) depends only on x and the R.H.S. depends only on y, both sides of (5) must be equal to the same constant, say μ.

Then (5) gives $\quad X'' - \mu X = 0$ ...(6)

and $\quad Y'' + \mu Y = 0.$ ...(7)

Using (2), (4) gives $X(0)\, Y(y) = 0$ and $X(a)\, Y(y) = 0$ so that

$$X(0) = 0 \text{ and } X(a) = 0, \qquad ...(8)$$

where we have taken $Y(y) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy (3b). We now solve (6) under B.C. (8). Three cases arise.

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$. ...(9)

Using B.C. (8), (9) gives $0 = B$ and $0 = Aa + B$. These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0$ which does not satisfy 3(b). So reject $\mu = 0$.

Case II. Let $\mu = \lambda^2$ (positive), where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}. \qquad ...(10)$$

Using B.C. (8), (10) gives $0 = A + B$ and $0 = Ae^{\lambda a} + Be^{-\lambda a}$. ...(11)

These give $A = B = 0$ so that $X(x) \equiv 0$. This leads to $u \equiv 0$ which does not satisfy (3b). So we reject $\mu = \lambda^2$.

Case III. Let $\mu = -\lambda^2$ (negative), where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A \cos \lambda x + B \sin \lambda x. \qquad ...(12)$$

Using B.C. (8), (12) gives $0 = A$ and $0 = A \cos \lambda a + B \sin \lambda a$ so that $A = 0$ and $\sin \lambda a = 0$, where we have taken $B \neq 0$, since otherwise we shall get $X(x) \equiv 0$ and hence $u \equiv 0$ which does not satisfy (3b).

Now, $\sin \lambda a = 0 \Rightarrow \lambda a = n\pi \Rightarrow \lambda = n\pi / a$, $n = 1, 2, 3, \ldots$ ...(13)

Hence, non-zero solutions $X(x)$ of (6) are given by

$$X_n(x) = B_n \sin(n\pi x / a). \qquad ...(14)$$

Then $\mu = -\lambda^2 = -(n^2\pi^2 / a^2)$ and hence (7) reduces to

$$Y'' - (n^2\pi^2 / a^2)\, Y = 0. \qquad ...(15)$$

whose general solution is $\qquad Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}$. ...(16)

From (4), $\partial u/\partial y = X(x)\,Y'(y)$ so that $(\partial u/\partial y)_{y=0} = X(x)\,Y'(0)$ ...(17)

Using (3*a*), (17) gives $X(x)\,Y'(0)=0$ so that $Y'(0)=0$,

where we have taken $X(x) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy 3(*b*).

Now, $\qquad Y'(0)=0 \;\Rightarrow\; Y_n'(0)=0.$ ...(18)

Differentiating both sides of (16) partially, w.r.t. '*y*', we get

$$Y_n'(y) = (n\pi/a)\,C_n e^{n\pi y/a} - (n\pi/a)\,D_n e^{-n\pi y/a}. \qquad ...(19)$$

Putting $y = 0$ in (19) and using (18), we have

$0 = (n\pi/a)(C_n - D_n)$ so that $D_n = C_n$. Then (16) reduces to

$$Y_n(y) = C_n\left(e^{n\pi y/a} + e^{-n\pi y/a}\right) = 2C_n \cosh(n\pi y/a), \text{ as } e^{\theta}+e^{-\theta} = 2\cosh\theta. \;...(20)$$

$$\therefore \quad u_n(x,y) = X_n(x)\,Y_n(n) = E_n \sin(n\pi x/a)\cosh(n\pi y/a)$$

are solutions of (1) satisfying (2) and (3*a*). Here $E_n\ (= 2B_nC_n)$ are new arbitrary constants. In order to obtain a solution also satisfying (3*b*), we consider more general solution, namely,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}. \qquad ...(21)$$

Putting $y = b$ in (21) and using (3*b*), we have

$$f(x) = \sum_{n=1}^{\infty}\left(E_n \cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a},$$

which is the half range Fourier sine series of $f(x)$ in $(0, a)$. Hence, we get

$$E_n \cosh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx$$

or
$$E_n = \frac{2}{a\cosh(n\pi b/a)}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx. \qquad ...(22)$$

The required temperature is given by (21) wherein E_n is given by (22).

IFS-13

Ex. 26. *A rectangular metal plate is bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. The three sides $x = 0$, $x = a$ and $y = b$ are insulated and the side $y = 0$ is kept at temperature $u_0 \cos(\pi x/a)$. Find the steady state temperature at any point of the plate.*

Sol. The steady state temperature $u(x, y)$ is the solution of the equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \qquad ...(1)$$

Since the sides $x = 0$, $x = a$ and $y = b$ are insulated, we have

$$(\partial u/\partial x)_{x=0} = 0,\ (\partial u/\partial x)_{x=a} = 0. \qquad ...(2)$$

and
$$(\partial u/\partial y)_{y=b} = 0 \qquad ...3(a)$$

Also, given that $\qquad u(x,a) = u_0\cos(\pi x/a).$ ...3(*b*)

Suppose (1) has a solution of the form $u(x, y) = X(x)\, Y(y)$. ...(4)

Substituting (4) in (1), we get

$$X''Y + XY'' = 0 \quad \text{or} \quad (1/X)\, X'' = -(1/Y)\, Y''. \qquad ...(5)$$

Since the L.H.S. of (5) depends only on x and the R.H.S. depends only on y, both sides of (5) must be equal to the same constant, say μ.

Then, (5) yields $\quad X'' - \mu X = 0$ ...(6)

and $\quad Y'' + \mu Y = 0.$ ...(7)

From (4), $\quad \partial u / \partial x = X'(x)\, Y(y).$ ...(8)

Putting $x = 0$ and $x = a$ by turn in (8) and using (2), we have

$$0 = X'(0)\, Y(y) \text{ and } X'(a)\, Y(y) \text{ so that}$$

$$X'(0) = 0 \text{ and } X'(a) = 0, \qquad ...(9)$$

where we have taken $Y(y) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy 3(*b*). We now solve (6) and B.C. (9). Three cases arise :

Case I. Let $\mu = 0$. Then solution of (6) is $X(x) = Ax + B$. ...(10)

Differentiating (10) w.r.t. 'x', we get $\quad X'(x) = A.$ ...(11)

Putting $x = 0$ and $x = a$ in (11) by turn and using (9), we get $A = 0$,

With $A = 0$, (10) reduces to $\quad X(x) = B.$ ...(12)

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \text{ so that } X'(x) = \lambda\left(Ae^{\lambda x} - Be^{-\lambda x}\right). \qquad ...(13)$$

Using B.C. (9), (13) gives $0 = \lambda(A - B)$ and $0 = \lambda\left(Ae^{\lambda a} - Be^{-\lambda a}\right)$.

These give $A = B = 0$ so that $X(x) = 0$. This leads to $u \equiv 0$, which does not satisfy 3(*b*).

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A \cos \lambda x + B \sin \lambda x. \qquad ...(14)$$

From (14), we have $X'(x) = -A\lambda \sin \lambda x + B\lambda \cos \lambda x.$ ...(15)

Using B.C. (9), (14) gives

$$0 = B\lambda \text{ and } 0 = -A\lambda \sin \lambda a + B\lambda \cos \lambda a. \qquad ...(16)$$

Since $\lambda \neq 0$ and $A \neq 0$ for otherwise $u(x, y)$ would be a trivial solution,

(16) $\Rightarrow B = 0$ and $\sin \lambda a = 0 \Rightarrow B = 0$ and $\lambda a = n\pi$.

Thus, $B = 0$ and $\lambda = n\pi/a$ for $n = 1, 2, 3, \ldots$ ...(17)

Hence, non-zero solutions of (14) are given by

$$X(x) = A \cos(n\pi x / a), \; n = 1, 2, 3, \ldots \qquad ...(18)$$

With $n = 0$, (18) reduces to $X(x) = A$ = constant. Also $n = 0 \Rightarrow \lambda = 0$.

So we can include non-zero solution (12) in (18) by taking $n = 0, 1, 2, 3, \ldots$

Hence, all non-zero solution of (6) are given by

$$X_n(x) = A_n \cos(n\pi x / a), \; n = 0, 1, 2, 3, \ldots \qquad ...(19)$$

Now, $\mu = -\lambda^2 = -n^2\pi^2 / a^2$, $n = 0, 1, 2, 3, \ldots$ Then (7) becomes

$$Y'' - \left(n^2\pi^2 / a^2\right) Y = 0, \text{ whose general solution is}$$

$$Y_n(y) = C_n e^{n\pi y/a} + D_n e^{-n\pi y/a}. \quad ...(20)$$

From (4), $\quad \partial u/\partial y = X(x)\, Y'(y) \quad ...(21)$

Putting $y = b$ in (21) and using B.C. 3(*a*) gives $0 = X(x)\, Y'(b)$ so that $Y'(b) = 0$. ($\because$ $X(x) \neq 0$ for otherwise $u(x, y)$ would be a trivial solution.

Now, $\quad Y'(b) = 0 \Rightarrow Y_n'(b) = 0. \quad ...(22)$

From (20), $Y_n'(y) = C_n(n\pi/a)\, e^{n\pi y/a} - D_n(n\pi/a)\, e^{-n\pi y/a}. \quad ...(23)$

Putting $y = b$ in (23) and using (22), we have

$$0 = (n\pi/a)\left(C_n e^{n\pi b/a} - D_n e^{-n\pi b/a}\right) \Rightarrow D_n = \left(C_n e^{n\pi b/a}\right)/e^{-n\pi b/a}.$$

With this value of D_n, (20) reduces to

$$Y_n(y) = C_n\left(e^{n\pi y/a}\, e^{-n\pi b/a} + e^{-n\pi y/a}\, e^{n\pi b/a}\right)/e^{-n\pi b/a}$$

$$= C_n\, e^{n\pi b/a}\left\{e^{n\pi(b-y)/a} + e^{-n\pi(b-y)/a}\right\} = 2C_n e^{n\pi b/a} \cosh\{n\pi(b-y)/a\}$$

$$\therefore \quad u_n(x, y) = X_n(x)\, Y_n(y) = E_n \cos(n\pi x/a) \cosh\{n\pi(b-y)/a\}$$

are solutions of (1) satisfying (2) and 3(*a*). Hence $E_n\left(= 2A_n C_n e^{n\pi b/a}\right)$ are new arbitrary constants. In order to obtain a solution also satisfying 3(*b*), we consider the most general solution, namely,

$$u(x, y) = \sum_{n=0}^{\infty} u_n(x, y) = \sum_{n=0}^{\infty} E_n \cos(n\pi x/a) \cosh\{n\pi(b-y)/a\} \quad ...(24)$$

Putting $y = a$ in (24) and using B.C. 3(*b*), we get

$$u_0 \cos(\pi x/a) = \sum_{n=0}^{\infty} E_n \cos(n\pi x/a) \cosh\{n\pi(b-a)/a\}$$

or

$$u_0 \cos(\pi x/a) = E_0 + E_1 \cos(\pi x/a) \cosh\{\pi(b-a)/a\} + E_2 \cos(2\pi x/a) \cosh\{2\pi(b-a)/a\}$$
$$+ E_3 \cos(3\pi x/a) \cosh\{3\pi(b-a)/a\} + \ldots \quad ...(25)$$

Equating the coefficients of like terms on both sides of (25), we get

$$E_0 = 0, \; E_1 \cosh\{\pi(b-a)/a\} = u_0 \text{ and } E_n = 0 \text{ for } n \geq 2$$

Putting the above value of E_1 in (24), the required temperature is

$$u(x, y) = u_0 \operatorname{sech}\{(b-a)\pi/a\} \cos(\pi x/a) \cosh\{(b-y)\pi/a\}.$$

Ex. 27. *A rectangular plate with insulated surfaces 8 cm wide and so long compared to its width that it can be considered infinite in the length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by $u(x, 0) = 100 \sin(\pi x/8)$ in $0 < x < 8$ while the two long edges $x = 0$ and $x = 8$ as well as the other short edges are kept at 0°C. Find the steady state temperature function $u(x, y)$.*

Sol. The steady state temperature $u(x, y)$ is the solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \quad ...(1)$$

subject to the boundary conditions :

$$u(0, y) = u(8, y) = 0 \text{ for } 0 < y < \infty \quad ...(2)$$

$$u(x, y) \to 0 \text{ as } y \to \infty \text{ for } 0 < x < 8 \quad ...3(a)$$

and $$u(x, 0) = 100 \sin(\pi x/8) \text{ for } 0 < x < 8. \quad ...3(b)$$

Take $a = 8$ and proceed as in solved Ex. 2 on page 48 upto eqn. (18).

$$u(x, y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/8)\, e^{-n\pi y/8}. \quad ...(18)$$

Putting $y = 0$ in (18) and using B.C. 3(*b*), we have

$$100 \sin(\pi x/8) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/8) = E_1 \sin(\pi x/8) + E_2 \sin(2\pi x/8) + \ldots$$

Comparing the coefficients of like terms, $E_1 = 100$ and $E_n = 0$ for $n \geq 2$.

So, from (18), the required temperature is $u(x, y) = 100 \sin(\pi x/8)\, e^{-\pi y/8}$.

Ex. 28. *An infinitely long uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. The end is maintained at 100°C at all points and the other edges are at 0°C. Find the steady state temperature function u (x, y).*

Sol. The steady state temperautre $u(x, y)$ is the solution of

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0. \quad ...(1)$$

subject to the B.C. :

$$u(0, y) = u(\pi, y) = 0 \text{ for all } y \geq 0 \quad ...(2)$$

$$u(x, y) \to 0 \text{ as } y \to \infty \text{ for } 0 \leq x \leq \pi \quad ...3(a)$$

and $$u(x, 0) = f(x) = 100 \text{ for } 0 \leq x \leq \pi \quad ...3(b)$$

Take $a = \pi$ and proceed as in solved Ex. 2 on page 48 upto eqn. (19).

$$\therefore \quad u(x, y) = \sum_{n=1}^{\infty} E_n \sin nx\, e^{-ny} \quad ...(18)$$

where $$E_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx. \quad ...(19)$$

Here, $E_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} 100 \sin nx\, dx = \dfrac{200}{\pi}\left[-\cos nx\right]_0^{\pi}$, using 3(*b*).

$$= \frac{200}{\pi}\left[1-(-1)^n\right] = \begin{cases} 0, & \text{if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 400/(2m-1)\pi, & \text{if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting this value of E_n in (18), the required temperature function is

$$u(x, y) = \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2m-1)x}{2m-1}\, e^{-(2m-1)y}.$$

Ex. 29. *By separating the variables show that* $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ *has solutions of the form* $A \exp.(\pm nx \pm iny)$, *where A and n are constants.*

Deduce that the functions of the form $V(x, y) = \sum_r A_r \sin(r\pi y/a)e^{-(r\pi x/a)}$ $x \geq 0,\ 0 \leq y \leq \infty$ *where* A_r *are constants, are plane harmonic functions satisfying the conditions* $V(x,0) = V(x,a) = 0$ *and* $V(x,y) \to 0$ *as* $x \to \infty$.

[*Meerut, 1994*]

OR

Find $V(x, y)$ *such that* $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ *satisfies the conditions* (*i*) $V \to 0$ *as* $x \to \infty$ (*ii*) $V(x, 0) = V(x, a) = 0$.

Sol. Given that $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$. ...(1)

First part. Suppose that (1) has a solution of the form

$$V(x,y) = X(x)\,Y(y). \quad ...(2)$$

Substituting this value of V in (1),

$$X''Y + XY'' = 0 \quad \text{or} \quad (1/X)\,X'' = -(1/Y)\,Y''. \quad ...(3)$$

Since the L.H.S. of (3) depends only on x whereas the R.H.S. depends only on y, the two sides of (3) must be equal to the same constant, say n^2. Then (3) leads to the following two equations

$$X'' - n^2 X = 0 \quad \text{and} \quad Y'' + n^2 Y = 0$$

whose solutions are $X = Ae^{nx} + Be^{-nx}$ and $Y = C\cos ny + D\sin ny$.

Using (2), we see that (1) has solutions of the form

$$V(x, y) = \left(Ae^{nx} + Be^{-nx}\right)(C\cos ny + D\sin ny), \quad ...(4)$$

which may be put in compact form

$$V(x, y) = A\exp.(\pm nx \pm iny). \quad ...(5)$$

Second part. We shall now solve (1) under the given boundary conditions:

$$V(x,y) \to 0 \text{ as } x \to \infty \quad ...(6)$$

and $$V(x,0) = V(x,a) = 0. \quad ...(7)$$

B.C. (6) is satisfied by (4) by taking $A = 0$. Then (4) may be re-written as (writing $BC = E$ and $BD = F$).

$$V(x, y) = (E\cos ny + F\sin ny)\,e^{-nx}. \quad ...(8)$$

Putting $y = 0$ in (8) and using B.C. (7), we have

$0 = Ee^{-nx}$ for all x so that $E = 0$.

Next, putting $y = a$ in (8) and using B.C. (7), we have

$$0 = F\sin na\, e^{-nx} \text{ for all } x \Rightarrow \sin na = 0,$$

where we taken $F \neq 0$, for otherwise $E = F = 0$ would lead to trivial solution for $V(x, y)$.

Now, $\sin na = 0 \Rightarrow na = r\pi \Rightarrow n = r\pi/a,\ r = 1, 2, 3, \ldots$

Hence, non-zero solutions of (1) are given by (8), in the form

$$V(x, y) = F\sin(r\pi y/a)\,e^{-(r\pi x/a)},\ r = 1, 2, 3, \ldots$$

Hence, the most general solution of (1) is of the form

$$V(x, y) = \Sigma A_r \sin(r\pi y/a)\,e^{-(r\pi x/a)}, \text{ where } A_r \text{ are constants.}$$

Ex. 30. *Find solution of two-dimensional Laplace's equation* $r^2\left(\partial^2 u/\partial r^2\right)+r\left(\partial u/\partial r\right)+\left(\partial^2 u/\partial \theta^2\right)=0$ *in polar co-ordinates.*

Or

Solve heat equation in steady state in two-dimensional polar co-ordinates. **[Delhi B.Sc. (H) Physics 2001]**

Sol. Given $r^2\left(\partial^2 u/\partial r^2\right)+r\left(\partial u/\partial r\right)+\left(\partial^2 u/\partial \theta^2\right)=0.$ ...(1)

Suppose (1) has a solutions of the form $u(r,\theta)=R(r)\Theta(\theta)$, ...(2)
where R is a function of r only and Θ is a function of θ only.

From (2), $\partial u/\partial r=R'\Theta$, $\partial^2 u/\partial r^2=R''\Theta$, $\partial^2 u/\partial\theta^2=R\Theta''$. ...(3)

Using (3), (1) reduces to $r^2R''\Theta+rR'\Theta+R\Theta''=0$.

or $\left(r^2R''+rR'\right)\Theta=-R\Theta''$ or $\left(r^2R''+rR'\right)/R=-\Theta''/\Theta$. ...(4)

Since the L.H.S. of (4) is a function of r only and the R.H.S. is a function of θ only, both sides of (4) must be equal to the same constant, say μ. Then, (5) leads to the following ordinary differential equations :

$$r^2\left(d^2R/dr^2\right)+r\left(dR/dr\right)-\mu R=0 \quad ...(5)$$

and
$$d^2\Theta/d\theta^2+\mu\Theta=0. \quad ...(6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Re-writing (5), $\left(r^2D^2+rD-\mu\right)R=0$, where $D\equiv d/dr$. ...(7)

Let $r=e^z$ (or $z=\log r$) and $D_1=d/dz$. Then, we know that

$$rD=D_1 \text{ and } r^2D^2=D_1(D_1-1).$$

Substituting these in (7), we get

$$\{D_1(D_1-1)+D_1-\mu\}R=0 \text{ or } \left(D_1^2-\mu\right)R=0. \quad ...(8)$$

Again, let $D_2\equiv d/d\theta$. Then (6) may be re-written as

$$\left(D_2^2+\mu\right)\Theta=0. \quad ...(9)$$

The solutions of (8) and (9) depend on μ. Three cases arise :

Case I. Let $\mu=0$. Then (8) and (9) reduce to

$$d^2R/dz^2=0 \text{ and } d^2\Theta/d\theta^2=0.$$

Solving these, $R=A_1z+B_1=A_1\log r+B_1$ and $\Theta=C_1\theta+D_1$.

Hence, from (2), the required solution is of the form

$$u(r,\theta)=\left(A_1\log r+B_1\right)\left(C_1\theta+D_1\right). \quad ...(10)$$

Case II. Let $\mu=\lambda^2$, where $\lambda\neq 0$, *i.e.*, μ is positive. Then (8) and (9) become $\left(D_1^2-\lambda^2\right)R=0$ and $\left(D_2^2+\lambda^2\right)\Theta=0$.

Solving these, $R=A_2e^{\lambda z}+B_2e^{-\lambda z}=A_2\left(e^z\right)^{\lambda}+B_2\left(e^z\right)^{-\lambda}=A_2r^{\lambda}+B_2r^{-\lambda}$

and $\Theta=C_2\cos\lambda\theta+D_2\sin\lambda\theta$.

Hence, from (2), the required solution is of the form

$$u(r,\theta)=\left(A_2r^{\lambda}+B_2r^{-\lambda}\right)\left(C_2\cos\lambda\theta+D_2\sin\lambda\theta\right). \quad ...(11)$$

Case III. Let $\mu = -\lambda^2$, where $\lambda \neq 0$, *i.e.*, μ is negative. Then (8) and (9) become $\left(D_1^2 + \lambda^2\right) R = 0$ and $\left(D_2^2 - \lambda^2\right) \Theta = 0.$

Solving these, $R = A_3 \cos z + B_3 \sin z = A_3 \cos(\log r) + B_3 \sin(\log r)$

and $\Theta = C_3 e^{\lambda\theta} + D_3 e^{-\lambda\theta}$.

Hence, from (2), the required solution is of the form

$$u(r,\theta) = \left\{A_3 \cos(\log r) + B_3 \sin(\log r)\right\}\left(C_3 e^{\lambda\theta} + D_3 e^{-\lambda\theta}\right). \quad ...(12)$$

The correct choice of most suitable solution among (10), (11) and (12) depends upon the physical nature of the problem under consideration and the given boundary conditions.

Ex. 31. (*a*) *A thin semi-circular plate of radius a has its boundary diameter kept at 0°C and its circumference at f (θ). Find the temperature distribution in the steady state.*

(*b*) *A thin semi-circular plate of radius a has its boundary diameter kept at 0°C and its circumference at 100°C. If u (r, θ) is the steady state temperature, find u (a/4, π/2).*

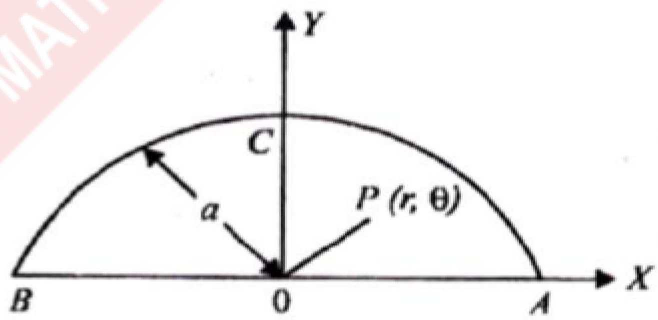
(*c*) The *boundary diameter of a semi-circular plate of radius 10 cm is kept at 0°C and its temperature along the semi-circular boundary is given by*

$$u(10,\theta) = \begin{cases} 50\,\theta, & \text{for } 0 \le \theta \le \pi/2 \\ 50(\pi - \theta), & \text{for } \pi/2 \le \theta \le \pi. \end{cases}$$

Find the steady state temperature u (r, θ) at point in the plate.

(*d*) *A semi-circular plate of radius a is kept at temperature u_0 along the bounding diameter and u_1 along the circumference. Find the steady state temperature at any point of the plate.*

Sol. Consider a thin semi-circular plate of radius a whose surfaces are insulated. Its bounding diameter AB is kept at temperature 0°C and its circumference ACB at temperature $f(\theta)$.



The steady state temperature $u(r, \theta)$ is the solution of the Laplace's equation in polar co-ordinates (r, θ)

$$r^2\left(\partial^2 u/\partial r^2\right) + r\left(\partial u/\partial r\right) + \partial^2 u/\partial\theta^2 = 0. \quad ...(1)$$

We are to solve (1) under the following boundary conditions :

$$u(r,0) = u(r,\pi) = 0 \quad ...(2)$$

and

$$u(a,\theta) = f(\theta). \quad ...(3)$$

Note that B.C. (2) expresses the fact that the temperature $u(r,\theta)$ is zero along OA ($\theta = 0$) and OB ($\theta = \pi$) so that temperature is zero along the bounding diameter AB. Again, the B.C. (3) expresses the fact that the temperature on the circumference ACB (where $r = a$) is $f(\theta)$.

Suppose (1) has a solution of the form $u(r,\theta) = R(r)\,\Theta(\theta)$, ...(4)

where R and Θ are functions of r and θ respectively.

Using (4), (1) becomes $\quad R''\,\Theta+(1/r)\,R'\,\Theta+\left(1/r^2\right)R\,\Theta''=0$

or $$\left(r^2R''+rR'\right)/R=-\Theta''/\Theta, \qquad \text{...(5)}$$

where the dashes denote derivatives with respect to the relevant variables. Since the L.H.S. of (5) is function of r alone whereas the R.H.S. is a function of θ alone, hence (5) is true only if each of its side is equal to the same constant.

For the solution of (1) satisfying the given boundary conditions, we choose this constant as λ^2. Then (5) gives

$$r^2R''+rR'-\lambda^2R=0 \text{ or } r^2\left(d^2R/dr^2\right)+r(dR/dr)-\lambda^2R=0. \qquad \text{...(6)}$$

and $$\Theta''+\lambda^2\Theta=0 \qquad \text{...(7)}$$

Putting $\theta=0$ and $\theta=\pi$ by turn in (4) and using B.C. (2), we get

$$0=R(r)\,\Theta(0) \text{ and } 0=R\,(r)(\Theta)(\pi) \Rightarrow \Theta(0)=0 \text{ and } (\Theta)(\pi)=0, \qquad \text{...(8)}$$

where we have taken $R(r)\neq 0$, since otherwise $u\equiv 0$ which does not satisfy B.C. (3). We now solve (7) under B.C. (8).

Solving (7), $$\Theta(\theta)=A\cos\lambda\theta+B\sin\lambda\theta. \qquad \text{...(9)}$$

Putting $\theta=0$ and $\theta=\pi$ by turn in (9) and using (8), we get

$$0=A \text{ and } 0=B\sin\lambda\pi \text{ so that } A=0 \text{ and } \sin\lambda\pi=0, \qquad \text{...(10)}$$

where we have taken $B\neq 0$ for getting non-trivial solution $u\,(r,\theta)$.

Now, $\quad \sin\lambda\pi=0 \Rightarrow \lambda\pi=n\pi \Rightarrow \lambda=n$, an integer.

Hence, (9) reduces to $$\Theta(\theta)=B\sin n\theta. \qquad \text{...(11)}$$

We now solve linear homogeneous equation (6). Let $D\equiv d/dr$. With $\lambda=n$, (6) may be re-written as

$$\left(r^2D^2+rD-n^2\right)R=0. \qquad \text{...(12)}$$

Let $r=e^z$ (or $z=\cos r$) and $D_1=d/dz$. Then (12) gives

$[D_1(D_1+1)+D_1-n^2]\,R=0$ or $\left(D_1^2-n^2\right)R=0$, whose solution is

$$R(r)=Ce^{nz}+De^{-nz}=C\left(e^z\right)^n+D\left(e^z\right)^{-n}=Cr^n+Dr^{-n}. \qquad \text{...(13)}$$

Due to physical nature of the given problem, $u\,(r,\theta)$ must be finite when $r\to 0$. Hence, $R\,(r)$ must be finite when $r\to 0$. For this purpose, we take $D=0$ in (13), for otherwise $r^{-n}\to\infty$ as $r\to 0$.

Then (13) reduces to $$R(r)=Cr^n. \qquad \text{...(14)}$$

From (4), (11) and (14), non-zero solutions of (1) are given by

$$u_n(r,\theta)=B\,Cr^n\sin n\theta = E_nr^n\sin\theta, \text{ taking } E_n=\text{BC}.$$

Hence, the most general solution of (1) satisfying B.C. (2) is

$$u(r,\theta)=\sum_{n=1}^{\infty}u_n(r,\theta)=\sum_{n=1}^{\infty}E_n\,r^n\sin n\theta. \qquad \text{...(15)}$$

Putting $r=a$ in (15) and using B.C. (3), we have

$$f(\theta) = \sum_{n=1}^{\infty} \left(E_n a^n\right) \sin n\theta,$$

which is the Fourier half range sine series of $f(\theta)$ in $(0, \pi)$. So, we get

$$E_n a^n = \frac{2}{\pi}\int_0^\pi f(\theta) \sin n\theta \, d\theta \quad \text{or} \quad E_n = \frac{2}{\pi a^n}\int_0^\pi f(\theta) \sin n\theta \, d\theta. \quad ...(16)$$

The required solution is given by (15) wherein E_n is given by (16).

(*b*) This is a particular case of part (*a*). Here $f(a, \theta) = f(\theta) = 100$.

Hence as in part (*a*), $\quad u(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta,$ where $\quad ...(15)$

$$E_n = \frac{2}{\pi a^n}\int_0^\pi f(\theta) \sin n\theta \, d\theta = \frac{2}{\pi a^n}\int_0^\pi 100 \sin n\theta \, d\theta = \frac{200}{\pi a^n}\left[-\cos n\theta\right]_0^\pi$$

$$= \frac{200}{n\pi a^n}\left[1-(-1)^n\right] = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 400/(2m-1)\,\pi a^{2m-1}, \text{ if } n = 2m-1, m = 1, 2, 3, \ldots \end{cases}$$

Substituting the above value of E_n in (15), we get

$$u(r, \theta) = \frac{400}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}.$$

Putting $r = a/4$ and $\theta = \pi/2$ in above relation, we get

$$u(a/4, \pi/2) = \frac{400}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}\left(\frac{1}{4}\right)^{2m-1}$$

$$\left[\because \ \sin(2m-1)(\pi/2) = \sin(m\pi - \pi/2) = (-1)^{m+1}\right]$$

$$= \frac{400}{\pi}\left[\frac{1}{4} - \frac{1}{3}\left(\frac{1}{4}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{1}{4}\right)^5, \ldots\right]$$

(*c*) This is a particular case of part (*a*). Here $a = 10$ and

$$f(\theta) = u(10, \theta) = \begin{cases} 50\,\theta, \text{ for } 0 \le \theta \le \pi/2 \\ 50(\pi - \theta), \text{ for } \pi/2 \le \theta \le \pi. \end{cases} \quad ...(i)$$

As in part (*a*), $\quad u(r, \theta) = \sum_{r=1}^{\infty} E_n r^n \sin n\theta \quad ...(15)$

where $\quad E_n = \dfrac{2}{\pi(10)^n}\displaystyle\int_0^\pi f(\theta) \sin n\theta$

$$= \frac{2}{\pi(10)^n}\left[\int_0^{\pi/2} f(\theta) \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} f(\theta) \sin n\theta \, d\theta\right]$$

$$= \frac{2}{\pi(10)^n}\left[\int_0^{\pi/2} 50\,\theta \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta) \sin n\theta \, d\theta\right], \quad \text{using } (i)$$

$$= \frac{100}{\pi (10)^n} \left[\theta \left(-\frac{\cos n\theta}{n} \right) - (1) \left(-\frac{\sin n\theta}{n^2} \right) \right]_0^{\pi/2}$$

$$+ \frac{100}{\pi (10)^n} \left[(\pi - \theta) \left(-\frac{\cos n\theta}{n} \right) - (-1) \left(-\frac{\sin n\theta}{n^2} \right) \right]_{\pi/2}^{\pi}$$

$$= \frac{100}{\pi (10)^n} \left[-\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{200}{\pi n^2} \sin \frac{n\pi}{2} = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ \left\{ 200 / \pi (2m-1)^2 \right\} (-1)^{m+1}, \text{ if } n = 2m - 1 \end{cases}$$

[Note that $\sin \{(2m-1)\pi/2\} = \sin(m\pi - \pi/2) = (-1)^{m+1}$]

Substituting the above value of E_n in (15), we get

$$u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{10} \right)^{2m-1} (-1)^{m-1} \frac{\sin (2m-1)\theta}{(2m-1)^2}$$

(*d*) Steady state temperature $u(r, \theta)$ is the solution of

$$r^2 (\partial^2 u / \partial r^2) + r(\partial u / \partial r) + \partial^2 u / \partial \theta^2 = 0 \qquad \ldots(i)$$

Given boundary conditions are

$$u(r, 0) = u(r, \pi) = u_0, \; 0 < r < a \qquad \ldots(ii)$$

and $\quad u(a, \theta) = u_1, \; 0 \le \theta < \pi. \qquad \ldots(iii)$

Let $\quad v(r, \theta) = u(r, \theta) - u_0 \qquad \ldots(iv)$

so that $\quad u(r, \theta) = v(r, \theta) + u_0 \qquad \ldots(v)$

Substituting the value of u from (*v*) in (*i*), we get

$$r^2 (\partial^2 v / \partial r^2) + r(\partial v / \partial r) + \partial^2 v / \partial \theta^2 = 0 \qquad \ldots(1)$$

Again, using (*ii*) and (*iii*), (*iv*) gives

$$v(r, 0) = v(r, \pi) = 0, \; 0 < r < a \qquad \ldots(2)$$

and $\quad v(a, \theta) = u_1 - u_0 = f(\theta)$, say $\qquad \ldots(3)$

Note that (1), (2) and (3) are exactly the same as the respective equations (1), (2) and (3) of part (*a*) of the present exercise.

Here, we have $u = v$ and $f(\theta) = u_1 - u_0$. So, as in part (*a*), we have

$$v(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta, \qquad \ldots(15)$$

where $E_n = \dfrac{2}{\pi a^n} \displaystyle\int_0^{\pi} f(\theta) \sin n\theta \, d\theta = \dfrac{2}{\pi a^n} \displaystyle\int_0^{\pi} (u_1 - u_0) \sin n\theta \, d\theta$

$$= \frac{2(u_1 - u_0)}{\pi a^n} \left[-\frac{\cos n\theta}{n} \right]_0^{\pi} = \frac{2(u_1 - u_0)}{\pi n a^n} \left[1 - (-1)^n \right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 4(u_1 - u_0) / \pi (2m-1) a^{2m-1}, \text{ if } n = 2m-1, \; m = 1, 2, 3, \ldots \end{cases}$$

Then, (15) gives $v(r, \theta) = \dfrac{4(u_1 - u_0)}{\pi} \sum_{m=1}^{\infty} \left(\dfrac{r}{a}\right)^{2m-1} \dfrac{\sin(2m-1)\theta}{2m-1}$.

Substituting this value in (v), the required temperature $u(r, \theta)$ is

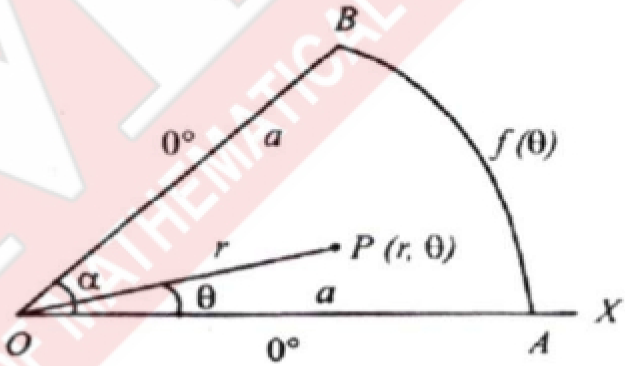
$$u(r, \theta) = u_0 + \frac{4(u_1 - u_0)}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}.$$

Ex. 32. (*a*) *A circular sector is determined by* $0 \le r \le a$, $0 \le \theta \le \alpha$. *The temperature is kept* $0°C$ *along the straight edges and at* $f(\theta)$ *along the curved edge. Find the steady state temperature at any point of the sector with its surface insulated.*

(*b*) *Find the steady state temperature at the points in the sector given by* $0 \le \theta \le \pi/4$, $0 \le r \le a$ *of a circular plate if the temperature is maintained at* $0°C$ *along the side edges and at a constant temperature* $u_0 C$ *along the curved edge.*

(*c*) *A circular sector is determined by* $0 \le r \le a$, $0 \le \theta \le \pi/2$. *The temperature is kept at* $0°C$ *along the straight edges and at temperature* $50(\pi\theta - 2\theta^2)$ *along the curved edge. Find the steady state temperature at* $(a/2, \pi/4)$.

Sol. (*a*) Consider a thin circular sector OAB of radius a such that $\angle AOB = \alpha$. The steady state temperature $u(r, \theta)$ is the solution of the Laplace's equation in polar co-ordinates (r, θ)



$$r^2(\partial^2 u/\partial r^2) + r(\partial u/\partial r) + \partial^2 u/\partial \theta^2 = 0 \qquad ...(1)$$

We are to solve (1) under the following boundary conditions :

$$u(r, 0) = u(r, \alpha) = 0, \ 0 < r < a \qquad ...(2)$$

and
$$u(a, \theta) = f(\theta), \ 0 < \theta < \alpha. \qquad ...(3)$$

Note that B.C. (2) expresses the fact that the temperature $u(r, \theta)$ is zero along the straight edges OA and OB of the given sector OAB and the B.C. (3) expresses the fact that the temperature on curved edge AB of the sector is $f(\theta)$.

Suppose (1) has a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$, ...(4)

where R and Θ are functions of r and θ respectively.

Using (4), (1) becomes; $R''\Theta + (1/r)R'\Theta + (1/r^2)R\Theta'' = 0$

or
$$(r^2 R'' + rR')/R = -\Theta''/\Theta, \qquad ...(5)$$

where the dashes denote derivatives with respect to the relevant variables. Since the L.H.S. of (5) is a function of r alone whereas the R.H.S. is a function of θ alone, hence (5) is true only if each of its side is equal to the same constant. For the solution of (1) satisfying the given boundary conditions, we choose this constant as λ^2. Then, (5) gives

$$r^2R'' + rR' - \lambda^2 R = 0 \text{ or } r^2\left(d^2R/dr^2\right) + r\left(dR/dr\right) - \lambda^2 R = 0 \qquad ...(6)$$

and $$\Theta'' + \lambda^2\,\Theta = 0. \qquad ...(7)$$

Putting $\theta = 0$ and $\theta = \alpha$ by turn in (4) and using B.C. (2), we get

$$0 = R(r)\Theta(0) \text{ and } 0 = R(r)\Theta(\alpha) \Rightarrow \Theta(0) = 0 \text{ and } \Theta(\alpha) = 0, \qquad ...(8)$$

where we have taken $R(r) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy B.C. (3). We now solve (7) under B.C. (8).

Solving (7), $$\Theta(\theta) = A\cos\lambda\theta + B\sin\lambda\theta. \qquad ...(9)$$

Putting $\theta = 0$ and $\theta = \alpha$ by turn in (9) and using (8), we get

$$0 = A \text{ and } 0 = B\sin\lambda\alpha \text{ so that } A = 0 \text{ and } \sin\lambda\alpha = 0, \qquad ...(10)$$

where we have taken $B \neq 0$ for getting non-trivial solution $u(r, \theta)$.

Now, $\sin\lambda\alpha = 0 \Rightarrow \lambda\alpha = n\pi \Rightarrow \lambda = n\pi/\alpha$, n being an integer.

Hence, (9) reduces to $$\Theta(\theta) = B\sin(n\pi\theta/\alpha) \qquad ...(11)$$

We now solve linear homogeneous equation (6). Let $D \equiv d/dr$. Then (6) may be re-written $$\left(r^2D^2 + rD - \lambda^2\right)R = 0 \qquad ...(12)$$

Let $r = e^z$ (or $z = \log r$) and $D_1 \equiv d/dz$. Then (12) reduces to

$$\left[D_1(D_1 - 1) + D_1 - \lambda^2\right]R = 0 \text{ or } \left(D_1^2 - \lambda^2\right)R = 0 \text{ whose solution is}$$

$$R(r) = Ce^{\lambda z} + De^{-\lambda z} = C\left(e^z\right)^{\lambda} + D\left(e^z\right)^{-\lambda} = Cr^{\lambda} + Dr^{-\lambda} \qquad ...(13)$$

or $$R(r) = Cr^{n\pi/\alpha} + Dr^{-n\pi/\alpha}, \text{ using (10).}$$

Due to physical nature of the given problem, $u(r, \theta)$ must be finite when $r \to 0$. Hence, $R(r)$ must be finite when $r \to 0$. For this problem, we take $D = 0$ in (13), for otherwise $r^{-n\pi/\alpha} \to 0$ as $r \to 0$.

Then (13) reduces to $$R(r) = Cr^{n\pi/\alpha}. \qquad ...(14)$$

From (4), (1) and (14), non-zero solutions of (1) are given by

$$u_n(r, \theta) = BCr^{n\pi/\alpha}\sin(n\pi/\alpha) = E_n r^{n\pi/\alpha}\sin(n\pi/\alpha), \text{ taking } E_n = BC$$

Hence, the most general solution of (1) satisfying (2) is

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} E_n r^{n\pi/\alpha}\sin(n\pi\theta/\alpha) \qquad ...(15)$$

Putting $r = a$ in (15) and using B.C. (3), we get

$$f(\theta) = \sum_{n=1}^{\infty}\left(E_n a^{n\pi/\alpha}\right)\sin(n\pi\theta/\alpha),$$

which is the Fourier half range sine series of $f(\theta)$ in $(0, \alpha)$. Hence, we get

$$E_n a^{n\pi/\alpha} = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\frac{n\pi\theta}{\alpha}\,d\theta \text{ or } E_n = \frac{2}{\alpha a^{n\pi/\alpha}}\int_0^{\alpha} f(\theta)\sin\frac{n\pi\theta}{\alpha}\,d\theta. \quad ...(16)$$

The required temperature $u(r, \theta)$ is given by (15) wherein the constants E_n are given by (16).

Particular case. If $\alpha = \pi$, the thin plate reduces to a semi-circular plate and we get results (15) and (16) of Ex. 31 (*a*).

(*b*) This is a particular case of part (*a*). Here $f(a,\theta) = f(\theta) = u_0$ and $\alpha = \pi/4$. Proceed as in part (*a*). Then, from (15), we have

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^{4n} \sin 4n\theta \qquad \text{...}(i)$$

From (16), we have $E_n = \dfrac{2}{(\pi/4)\, a^{4n}} \displaystyle\int_0^{\pi/4} u_0 \sin 4n\theta \, d\theta$

$$= \frac{8u_0}{\pi a^{4n}} \left[-\frac{\cos 4n\theta}{4n} \right]_0^{\pi/4} = \frac{2u_0}{n\pi a^{4n}} \left[1-(-1)^n \right]$$

$$\therefore \quad E_n = \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ (4u_0)/\pi(2m-1)\, a^{4(2m-1)}, \text{ if } n = 2m-1, \; m = 1, 2, 3, \ldots \end{cases}$$

Substituting this in (*i*), the required temperature distribution is

$$u(r, \theta) = \frac{4u_0}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{4(2m-1)} \frac{\sin 4(2m-1)\theta}{2m-1}.$$

(*c*) This is a particular case of part (*a*). Here $f(a, \theta) = 50\left(\pi\theta - 2\theta^2\right)$ and $\alpha = \pi/2$. Proceed as in part (*a*). Then, from (15), we have

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^{2n} \sin 2n\theta. \qquad \text{..}(i)$$

From (16), we have $E_n = \dfrac{2}{(\pi/2) a^{2n}} \displaystyle\int_0^{\pi/2} 50\left(\pi\theta - 2\theta^2\right) \sin 2n\theta \, d\theta$

$$= \frac{200}{\pi a^{2n}} \int_0^{\pi/2} \left(\pi\theta - 2\theta^2\right) \sin 2n\theta \, d\theta$$

$$E_n = \frac{200}{\pi a^{2n}} \left[\left(\pi\theta - 2\theta^2\right)\left(-\frac{\cos 2n\theta}{2n} \right) - (\pi - 4\theta)\left(-\frac{\sin 2n\theta}{4n^2} \right) + (-4)\left(\frac{\cos 2n\theta}{8n^3} \right) \right]_0^{\pi/2}$$

$$= \frac{200}{\pi a^{2n}} \left[-\frac{4\cos n\pi}{8n^3} + \frac{4}{8n^3} \right] = \frac{100}{\pi n^3 a^{2n}} \left[1-(-1)^n \right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ 200/\pi(2m-1)^3\, a^{2(2m-1)}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

$$\therefore \quad (i) \text{ gives } u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{2(2m-1)} \frac{\sin 2(2m-1)\theta}{(2m-1)^3}. \qquad \text{...}(ii)$$

Putting $r = a/2$ and $\theta = \pi/4$ in (*ii*), we get

Additional. Problems

$$u\left(\frac{a}{2}, \frac{\pi}{4}\right) = \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{4m-2}\frac{\sin(m\pi - \pi/2)}{(2m-1)^3}$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2^{4m-2}}\frac{1}{(2m-1)^3}$$

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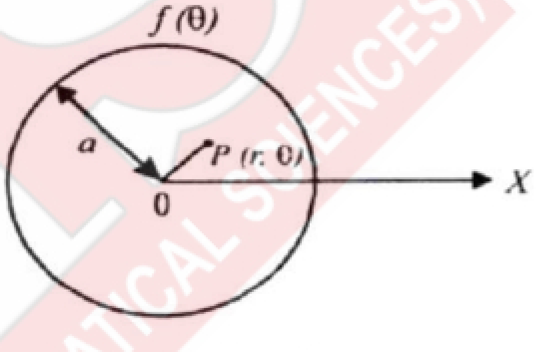
$$= (200/\pi)\left[\left(1/2^2\right) - \left(1/2^6\right)\left(1/3^3\right) + \left(1/2^{10}\right)\left(1/5^3\right) - \ldots\ldots\right].$$

Ex. 33. (*a*) ***Find the steady state temperature in a circular plate of radius a whose circumference is kept at temperature f (θ).***

(*b*) *Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at constant temperature u_0C.*

(*c*) *Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at 100°C.*



Sol. (*a*) Consider a thin circular plate of radius a with its surface insulated. The steady state temperature $u(r, \theta)$ is the solution of the Laplace's equation in polar co-ordinates (r, θ)

$$r^2\left(\partial^2 u/\partial r^2\right) + r\left(\partial u/\partial r\right) + \partial^2 u/\partial\theta^2 = 0. \qquad \ldots(1)$$

We are to solve (1) under the given boundary condition, namely

$$u(a, \theta) = f(\theta),\ 0 \le \theta \le 2\pi. \qquad \ldots(2)$$

The required solution $u(r, \theta)$ must be periodic in θ and is finite when $r \to 0$. Keeping these properties of $u(r, \theta)$, we now solve (1).

Suppose (1) has a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$, ...(3)

where R and Θ are function of r and θ respectively.

Using (3), (1) reduces to $r^2 R''\Theta + rR'\Theta + R\Theta'' = 0$

or $\quad \left(r^2R'' + rR'\right)\Theta = -R\Theta''$ or $\left(r^2R'' + rR'\right)/R = -\Theta''/\Theta.$ ...(4)

Since the L.H.S. of (4) is a function of r only and the R.H.S. is a function of θ only, the two sides of (4) must be equal to the same constant, say μ. Then, (5) gives

$$r^2\left(d^2R/dr^2\right) + r\left(dR/dr\right) - \mu R = 0 \qquad \ldots(5)$$

or

$$d^2\Theta/d\theta^2 + \mu\,\Theta = 0. \qquad \ldots(6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Rewritting (5), $\left(r^2D^2 + rD - \mu\right)R = 0$, where $D \equiv d/dr$. ...(7)

Let $r = e^z$ (or $z = \log r$) and $D_1 = d/dz$. Then, we know that $rD = D_1$ and $r^2D^2 = D_1(D_1 - 1)$. Substituting these in (7), we get

$$\{D_1(D_1-1)+D_1-\mu\}R=0 \text{ or } \left(D_1^2-\mu\right)R=0. \quad ...(8)$$

Again, let $D_2 = d/d\theta$. Then, (6) may be re-written as

$$\left(D_2^2+\mu\right)\Theta=0. \quad ...(9)$$

The solutions of (8) and (9) depend on μ. Consider following cases :

Case I. Let $\mu = 0$. Then (8) and (9) reduces to

$$d^2R/dz^2=0 \text{ and } d^2\Theta/d\theta^2=0.$$

Solving these, $R(r) = A_1z + B_1 = A_1 \log r + B_1$ and $\Theta = C_1\theta + D_1$.

Hence, from (3), a solution of (1) is of the form

$$u(r,\theta)=(A_1\log r+B_1)(C_1\theta+D_1). \quad ...(10)$$

Since $u(r, \theta)$ is periodic in θ and is finite when $r \to 0$, so we must taken $A_1 = 0$ and $C_1 = 0$. Then, (10) reduces to

$$u(r,\theta)=B_1D_1=(1/2)E_0 \quad ...(11)$$

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then (8) and (9) become

$$\left(D_1^2-\lambda^2\right)R=0 \text{ and } \left(D_2^2+\lambda^2\right)\Theta=0 \quad ...(12)$$

[Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $\left(D_2^2-\lambda^2\right)\Theta=0$ whose solution will not contain trigonometric functions and hence periodic nature of $u(r, \theta)$ will not be attained.]

$$\left.\begin{aligned} &\text{Solving (12), } R(r)=A_2e^{\lambda z}+B_2e^{-\lambda z}=A_2r^{\lambda}+B_2r^{-\lambda} \\ &\text{and } \quad \Theta(\theta)=C_2\cos\lambda\theta+D_2\sin\lambda\theta. \end{aligned}\right\} \quad ...(13)$$

Since $u(r, \theta)$ is periodic in θ with period 2π, we must take $\lambda = n$, where $n = 1, 2, 3, ...$ Using (13), (3) reduces to

$$u(r,\theta)=\left(A_2r^n+B_2r^{-n}\right)(C_2\cos n\theta+D_2\sin n\theta). \quad ...(14)$$

Furthermore, since $u(r, \theta)$ is finite when $r \to 0$, we must take $B_2 = 0$. Hence, (14) reduces to

$$u(r,\theta)=A_2r^n(C_2\cos n\theta+D_2\sin n\theta)$$

$$\text{or} \quad u(r,\theta)=r^n(E_n\cos n\theta+F_n\sin n\theta), \quad ...(15)$$

where $E_n (= A_2C_2)$ and $F_n (= A_2D_2)$ are new arbitrary constants.

Keeping (11) and (15) in view, the most general solution of (1) is

$$u(r,\theta)=\frac{E_0}{2}+\sum_{n=1}^{\infty}r^n(E_n\cos n\theta+F_n\sin n\theta). \quad ...(16)$$

Putting $r = a$ in (16) and using B.C. (2), we get

$$f(\theta)=\frac{E_0}{2}+\sum_{n=1}^{\infty}\left(E_na^n\cos n\theta+F_na^n\sin n\theta\right), \quad ...(17)$$

which is usual expansion of $f(\theta)$ as Fourier series in $(0, 2\pi)$. So, we get

$$\left.\begin{aligned} E_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta, \\ E_n &= \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \text{ and } F_n = \frac{1}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta. \end{aligned}\right\} \quad ...(18)$$

Hence, the required temperature is given by (16) wherein the constants E_0, E_n and F_n are given by (18).

(*b*) This is a particular case of part (*a*). Here, B.C. (2) is modified as

$$u(a,\theta) = f(\theta) = \begin{cases} u_0, & \text{for } 0 < \theta < \pi \\ 0, & \text{for } \pi < \theta < 2\pi \end{cases} \quad ...(2)$$

Now proceed exactly as in part (*a*). Solution is given by (16). To find constants E_0, E_n, F_n, we use (18) and (2). Thus, we get

$$E_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{\pi}\left[\int_0^{\pi} f(\theta)\,d\theta + \int_\pi^{2\pi} f(\theta)\,d\theta\right] = \frac{1}{\pi}\int_0^{\pi} u_0\,d\theta, \text{ by (2)}$$

$$= (1/\pi)\,[u_0\,\theta]_0^{\pi} = (1/\pi)\,[u_0\,\pi] = u_0$$

$$E_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{a^n\pi}\left[\int_0^{\pi} f(\theta)\cos n\theta\,d\theta + \int_\pi^{2\pi} f(\theta)\cos n\theta\,d\theta\right]$$

$$= \frac{1}{a^n\pi}\int_0^{\pi} u_0\cos n\theta\,d\theta = \frac{u_0}{a^n\pi}\left[\frac{\sin n\theta}{n}\right]_0^{\pi} = 0$$

$$F_n = \frac{1}{a^n\pi}\int_0^{\pi} u_0\sin n\theta\,d\theta = \frac{u_0}{a^n\pi}\left[-\frac{\cos n\theta}{n}\right]_0^{\pi} = \frac{u_0}{a^n n\pi}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n = 2m \text{ and } m = 1, 2, 3, \ldots \\ (2u_0)/\pi\,(2m-1)\,a^{2m-1}, \text{ if } n = 2m-1 \text{ and } m = 1, 2, 3, \ldots \end{cases}$$

Substituting these values in (16), the required temperature is

$$u(r,\theta) = \frac{u_0}{2} + \frac{2u_0}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}. \quad ...(i)$$

(*c*) This is a particular case of part (*b*). Hence $u_0 = 100$. Hence, from (*i*)

$$u(r,\theta) = 50 + \frac{200}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{a}\right)^{2m-1}\frac{\sin(2m-1)\theta}{2m-1}.$$

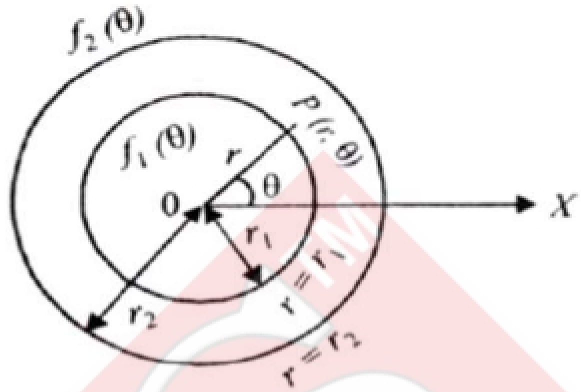
Ex. 34. (*a*) *Consider a circular annulus of inner radius r_1 and outer radius r_2. Let the surface of the annulus be insulated. Find the steady state temperature at any point (r, θ) in the annulus, given that the temperature distribution along the inner circle $r = r_1$ and the outer circle $r = r_2$ are maintained as $u(r_1, \theta) = f_1(\theta)$ and $u(r_2, \theta) = f_2(\theta)$.*

(*b*) *A plate in the form of a ring is bounded by the circles $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature $u(r, \theta)$ along the boundary are $u(2, \theta) = 6\cos\theta + 10\sin\theta$ and $u(4, \theta) = 15\cos\theta + 17\sin\theta$. Find the steady state temperature $u(r, \theta)$ in the ring.*

(*c*) *Along the inner boundary of a circular annulus of radii 10 cm and*

20 cm the temperature is maintained as u (10, θ) = 15 cos θ and along the outer boundary the temperature u (20, θ) = 30 sin is maintained. Find the steady state temperature at an arbitrary point (r, θ) in the annulus.

Sol. We are given a circular annulus whose inner and outer radii are r_1 and r_2 respectively. The steady state temperature $u(r, \theta)$ at any point $P(r, \theta)$ of the annulus is the solution of the Laplace's equation in polar co-ordinates (r, θ), namely



$$r^2\left(\partial^2 u/\partial r^2\right) + r\left(\partial u/\partial r\right) + \partial^2 u/\partial\theta^2 = 0. \qquad ...(1)$$

Since temperatures along the inner ($r = r_1$) and outer boundary ($r = r_2$) are maintained at $f_1(\theta)$ and $f_2(\theta)$ respectively, we have

$$u(r_1, \theta) = f_1(\theta) \text{ and } u(r_2, \theta) = f_2(\theta),\ 0 \le \theta \le 2\pi. \qquad ...(2)$$

Clearly the temperature function $u(r, \theta)$ must be periodic in θ of period 2π. Accordingly, we now proceed to solve (1).

Suppose (1) has a solution of the form $u(r, \theta) = R(r)\Theta(\theta)$, ...(3)
where R and Θ are functions of r and θ respectively.

Using (3), (1) reduces to $r^2 R''\Theta + rR'\Theta + R\Theta'' = 0$

or $$\left(r^2 R'' + rR'\right)\Theta = -R\Theta'' \text{ or } \left(r^2 R'' + rR'\right)/R = -\Theta''/\Theta. \qquad ...(4)$$

Since the L.H.S. of (4) is a function of r only and the R.H.S. is a function of θ only, the two sides of (4) must be equal to the same constant, say μ. Then, (5) gives

$$r^2\left(d^2R/dr^2\right) + r\left(dR/dr\right) - \mu R = 0 \qquad ...(5)$$

and $$d^2\Theta/d\theta^2 + \mu\Theta = 0. \qquad ...(6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Rewriting (5), $\left(r^2D^2 + rD - \mu\right)R = 0$, where $D \equiv d/dr$. ...(7)

Let $r = e^z$ (or $z = \log r$) and $D_1 \equiv d/dz$. Then, we know that $rD_1 = D_1$ and $r^2D^2 = D_1(D_1 - 1)$. Substituting these in (7), we get

$$\{D_1(D_1 - 1) + D_1 - \mu\}R = 0 \text{ or } \left(D_1^2 - \mu\right)R = 0. \qquad ...(8)$$

Again, let $D_2 \equiv d/d\theta$. Then, (6) may be re-written as

$$\left(D_2^2 + \mu\right)\Theta = 0. \qquad ...(9)$$

The solutions of (8) and (9) depend on μ. Consider following cases

Case I. Let $\mu = 0$. Then (8) and (9) reduce to

$$d^2R/dz^2 = 0 \text{ and } d^2\Theta/d\theta^2 = 0$$

Solving these, $R(r) = A_1 z + B_1 = A_1 \log r + B_1$ and $\Theta = C_1\theta + D_1$.

Hence, from (3), a solution of (1) is of the form

$$u(r,\theta) = (A_1 \log r + B_1)(C_1\theta + D_1) \qquad \text{...(10)}$$

Since $u(r, \theta)$ is periodic in θ, we must take $C_1 = 0$. Then the above solution (10) becomes

$$u(r,\theta) = (A_1 \log r + B_1) D_1 = (1/2)(a_0 \log r + b_0), \qquad \text{...(11)}$$

where $a_0 (= 2A_1D_1)$ and $b_0 (= 2B_1D_1)$ are new arbitrary constants.

Case II. Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then (8) and (9) become

$$\left(D_1^2 - \lambda^2\right) R = 0 \text{ and } \left(D_2^2 + \lambda^2\right)\Theta = 0. \qquad \text{...(12)}$$

[Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $\left(D_2^2 - \lambda^2\right)\Theta = 0$ whose solution will not contain trigonometric functions and hence periodic nature of $u(r, \theta)$ will not be attained.]

Solving (12), $R(r) = A_2 e^{\lambda z} + B_2 e^{-\lambda z} = A_2\left(e^z\right)^{\lambda} + B_2\left(e^z\right)^{-\lambda} = A_2 r^{\lambda} + B_2 r^{-\lambda}$

and $\Theta(\theta) = C_2 \cos\lambda\theta + D_2 \sin\lambda\theta$.

Hence, from (3), a solution of (1) is of the form

$$u(r,\theta) = \left(A_2 r^{\lambda} + B_2 r^{-\lambda}\right)(C_2 \cos\lambda\theta + D_2 \sin\lambda\theta). \qquad \text{...(13)}$$

Since $u(r, \theta)$ is periodic in θ with period 2π, we must take $\lambda = n$, where $n = 1, 2, 3, \ldots$. Hence, (13) takes the form

$$u(r,\theta) = \left(A_2 r^n + B_2 r^{-n}\right)(C_2 \cos n\theta + D_2 \sin n\theta), \; n = 1, 2, 3, \ldots \qquad \text{...(14)}$$

With help of (11) and (14), the most general solution of (1) is

$$u(r,\theta) = \frac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r^n + b_n r^{-n}\right)\cos n\theta + \left(c_n r^n + d_n r^{-n}\right)\sin n\theta\right], \qquad \text{...(15)}$$

which holds for $r_1 \leq r \leq r_2$. Here $a_n (= A_2C_2)$, $b_n (= B_2C_2)$, $c_n (= A_2D_2)$ and $d_n (= B_2D_2)$ are new arbitrary constants.

Putting $r = r_1$ and $r = r_2$ by turn in (15) and using B.C. (2), we have

$$f_1(\theta) = \frac{a_0 \log r_1 + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r_1^n + b_n r_1^{-n}\right)\cos n\theta + \left(c_n r_1^n + d_n r_1^{-n}\right)\sin n\theta\right]. \qquad \text{...(16)}$$

$$f_2(\theta) = \frac{a_0 \log r_2 + b_0}{2} + \sum_{n=1}^{\infty}\left[\left(a_n r_2^n + b_n r_2^{-n}\right)\cos n\theta + \left(c_n r_2^n + d_n r_2^{-n}\right)\sin n\theta\right]. \qquad \text{...(17)}$$

(16) and (17) are usual expansions of $f_1(\theta)$ and $f_2(\theta)$ as Fourier series in $(0, 2\pi)$. Hence, we have

$$a_0 \log r_1 + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\,d\theta,\ a_0 \log r_2 + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\,d\theta \qquad ...(18)$$

$$a_n r_1^n + b_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\cos n\theta\,d\theta,\ a_n r_2^n + b_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\,d\theta \qquad ...(19)$$

$$c_n r_1^n + d_n r_1^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\sin n\theta\,d\theta,\ c_n r_2^n + d_n r_2^{-n} = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\,d\theta \qquad ...(20)$$

Solving pair of equations in (18), (19) and (20) we can get a_0, b_0; a_n, b_n and c_n, d_n respectively. Substituting these values in (15), we obtain the required temperature distribution.

(*b*) This is a particular case of part (*a*). Here $r_1 = 2$, $r_2 = 4$, $f_1(\theta) = 6\cos\theta + 10\sin\theta$ and $f_2(\theta) = 15\cos\theta + 17\sin\theta$. Now, proceed as in part (*a*) upto equation (17).

Putting the given values of r_1, r_2, $f_1(\theta)$ and $f_2(\theta)$, (16) and (17) reduce to

$$6\cos\theta + 10\sin\theta = (1/2)(a_0 \log 2 + b_0)$$

$$+\sum_{n=1}^{\infty}\left(a_n 2^n + b_n 2^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 2^n + d_n 2^{-n}\right)\sin n\theta \qquad ...(i)$$

$$15\cos\theta + 17\sin\theta = (1/2)(a_0 \log 4 + b_0)$$

$$+\sum_{n=1}^{\infty}\left(a_n 4^n + b_n 4^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(c_n 4^n + d_n 4^{-n}\right)\sin n\theta. \qquad ...(ii)$$

Equating the constants in (*i*) and (*ii*), we get

$(1/2)(a_0 \log 2 + b_0) = 0$ and $(1/2)(a_0 \log 4 + b_0) = 0 \Rightarrow a_0 = b_0 = 0$. ...(*iii*)

Equating the coefficients of $\cos\theta$ in (*i*) and (*ii*), we get

$2a_1 + 2^{-1}b_1 = 6$ and $4a_1 + 4^{-1}b_1 = 15$ so that $a_1 = 4$ and $b_1 = -4$. ...(*iv*)

Equating the coefficients of $\sin\theta$ in (*i*) and (*ii*), we get

$2c_1 + 2^{-1}d_1 = 10$ and $4c_1 + 4^{-1}d_1 = 17$ so that $c_1 = 4$ and $d_1 = 4$. ...(*v*)

Also, we note that $a_n = b_n = c_n = d_n = 0$ for all $n \geq 2$. ...(*vi*)

Using (*ii*), (*iii*), (*iv*), (*v*) and (*vi*) in (15) of part (*a*), we get

$$u(r,\theta) = \left(4r - 4r^{-1}\right)\cos\theta + \left(4r + 4r^{-1}\right)\sin\theta$$

(*c*) This is a particular case of part *a*. Here $r_1 = 10$, $r_2 = 20$, $f_1(\theta) = 15\cos\theta$ and $f_2(\theta) = 30\sin\theta$. Now, proceed as in part (*a*) upto equation (17). Putting the given values of r_1, r_2, $f_1(\theta)$ and $f_2(\theta)$, (16) and (17) reduce to

$$15\cos\theta = (1/2)(a_0 \log 10 + b_0)$$

$$+\sum_{n=1}^{\infty} \left(a_n 10^n + b_n 10^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty} \left(c_n 10^n + d_n 10^{-n}\right)\sin n\theta \quad ...(i)$$

$$30\sin\theta = (1/2)(a_0 \log 20 + b_0)$$

$$+\sum_{n=1}^{\infty} \left(a_n 20^n + b_n 20^{-n}\right)\cos n\theta + \sum_{n=1}^{\infty} \left(c_n 20^n + d_n 20^{-n}\right)\sin n\theta \quad ...(ii)$$

Equating the constants in (*i*) and (*ii*), we get

$(a_0 \log 10 + b_0)/2 = 0$ and $(a_0 \log 20 + b_0)/2 = 0 \Rightarrow a_0 = b_0 = 0.$ ...(*iii*)

Equating the coefficients of cos θ on both sides of (*i*) and (*ii*), we get

$10a_1 + 10^{-1}b_1 = 15$ and $20a_1 + 20^{-1}b_1 = 0 \Rightarrow a_1 = -1/2$ and $b_1 = 200$ ....(*iv*)

Equating the coefficients of sin θ on both sides of (*i*) and (*ii*), we get

$10c_1 + 10^{-1}d_1 = 10$ and $20c_1 + 20^{-1}d_1 = 30 \Rightarrow c_1 = 2$ and $d_1 = -200.$ ...(*v*)

Also, we have $a_n = b_n = c_n = d_n = 0$ for all $n \geq 2.$ ...(*vi*)

Using (*iii*), (*iv*), (*v*) and (*vi*), equation (15) of part (*a*) reduces to

$$u(r,\theta) = \left\{-(1/2)r + 200r^{-1}\right\}\cos\theta + \left(2r - 200r^{-1}\right)\sin\theta.$$

Ex. 35. *Obtain solution of Laplace's equation in cylindrical co-ordinates satisfying the following conditions :*

(*i*) *Solution is symmetrical about the z-axis and remains finite along the line ρ = 0.*

(*ii*) *Solution is symmetrical about the z-axis and tends to zero as ρ → 0 and as z → ∞.*

Sol. (*i*) Proceed as in solved Ex. 2 on page 73 upto equation (9).

$$\therefore \quad u(\rho, z) = \sum_m \left(A_m e^{mz} + B_m e^{-mz}\right)\left[C_m J_0(m\rho) + D_m Y_0(m\rho)\right], \quad ...(9)$$

which is solution of Laplace's equation when the solution is symmetrical about the z-axis. If the required solution remains finite along the line $\rho = 0$, we must take $D_m = 0$ because $Y_0(m\rho) \to \infty$ as $\rho \to 0$. So, (9) reduces to

$$u(\rho, z) = \sum_m \left(A_m e^{mz} + B_m e^{-mz}\right) C_m J_0(m\rho)$$

$$= \sum_m \left(E_m e^{mz} + F_m e^{-mz}\right) J_0(m\rho), \quad ...(10)$$

where E_m $(= A_m C_m)$ and F_m $(= B_m C_m)$ are new arbitrary constants.

Solution (10) can also be expressed as

$$u(\rho, z) = \sum_m G_m J_0(m\rho)\, e^{\pm mz} \quad ...(11)$$

(*ii*) This is a particular case of part (*i*). Here, $u(\rho, z) \to 0$ as $z \to \infty$ and hence we must take $E_m = 0$ in (*iv*). Then, (10) reduces to

$$u(\rho, z) = \sum_m F_m e^{-mz} J_0(m\rho). \qquad \text{...(12)}$$

Ex. 36. *Show that solutions of Laplace's equation $\nabla^2 u = 0$ in cylindrical co-ordinates satisfying the conditions (i) $u \to 0$ as $z \to \infty$. (ii) u is finite as $\rho \to 0$ are of the form.*

$$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm in\phi}.$$

Sol. Proceed as in solved Ex. 1 on page 72 upto equation (11)

Given that $u \to 0$ as $z \to \infty$, so we take $A_m = 0$ in (9).

Given that u is finite as $\rho \to 0$ so we take $F_{mn} = 0$ in (11), because we know that $Y_n(m\rho) \to \infty$ as $\rho \to 0$.

Hence, required general solution is of the form

$$u = \sum_m \sum_n B_m e^{-mz} (C_n \cos n\phi + D_n \sin n\phi) E_{mn} J_n(m\rho)$$

or $$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm in\phi}, \text{ where } G_{mn} = B_m E_{mn}$$

Ex. 37. (*a*) *Solve three dimensional wave equation in spherical polar co-ordinates* **(Delhit B.Sc. (H) 2005)**

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \qquad \text{...(1)}$$

(*b*) *If u remains finite for $\theta = 0$, then find the solution of (1).*

(*c*) *If there is axial symmetry (taking z-axis as the axis of symmetry), then find the solution of (1) which remains finite for $\theta = 0$.*

(*d*) *If there is a spherical symmetry, then find the solution of (1).*

Sol. (*a*) For part (*a*) refer solved Ex. 1 on page 100. Then the function

$$u = r^{-1/2} J_{\pm(n+1/2)}(kr)\left[G_{mn} P_n^m(\cos\theta) + H_{mn} Q_n^m(\cos\theta)\right] e^{\pm im\phi \pm ikct}. \qquad \text{...(2)}$$

is a solution of the wave equation (1). If on physical grounds we require the above solution to have the symmetry properties

$$u(r, \theta + \pi, \phi) = u(r, \theta, \phi) \text{ and } u(r, \theta, \phi + 2\pi) = u(r, \theta, \phi)$$

then we must take m and n to be integers.

(*b*) If u remains finite for $\theta = 0$, we must take $H_{mn} = 0$ because $Q_n^m(\cos\theta) \to \infty$ as $\theta \to 0$. Hence, from (2) a solution of (1) is of the form

$$u = r^{-1/2} J_{\pm(n+1/2)}(kr) P_n^m(\cos\theta) e^{\pm im\phi \pm ikct}. \qquad \text{...(3)}$$

(*c*) If u is symmetrical about z-axis, then u must be independent of ϕ and hence we must put $m = 0$ in solution (3) of (1). Then, noting that $P_n^m(\cos\theta) = P_n(\cos\theta)$, the required solution of (1) is of the form

$$u = r^{-1/2} J_{\pm(n+1/2)}(kr) P_n(\cos\theta) e^{\pm ickt}. \qquad \text{...(4)}$$

(*d*) When there is spherical symmetry, u will be independent of θ and ϕ both. Hence, we must put $m = 0$ and $n = 0$ in (3). Hence, a solution of (1) is of the form

$$u = r^{-1/2} J_{\pm(n+1/2)}(kr)\, e^{\pm ickt}.$$

Ex. 38. *Show that if there is spherical symmetry the solution of the wave equation can be obtained in the form*

$$\psi = (1/r)\{f(r-ct)+g(r+ct)\},$$

where f and g are arbitrary functions. **[Delhi, B.Sc. (Hons.), 1996, 2004]**

Sol. Wave equation in spherical polar co-ordinates (r, θ, ϕ) is

$$\frac{\partial^2\psi}{\partial r^2}+\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \quad ...(1)$$

Since there is spherical symmetry, ψ must be independent of θ and ϕ both. Hence, ψ must satisfy the equation (1) in modified form

$$\partial^2\psi/\partial r^2+(2/r)(\partial\psi/\partial r)=(1/c^2)(\partial^2\psi/\partial t^2). \quad ...(2)$$

We now re-write (2) by putting $\psi = \eta/r$. ...(3)

From (3), $$\frac{\partial\psi}{\partial r}=-\frac{1}{r^2}\eta+\frac{1}{r}\frac{\partial\eta}{\partial r},\quad \frac{\partial^2\psi}{\partial r^2}=\frac{2}{r^3}\eta-\frac{2}{r^2}\frac{\partial\eta}{\partial r}+\frac{1}{r}\frac{\partial^2 n}{\partial r^2}, \quad ...(4)$$

$$\frac{\partial\psi}{\partial t}=\frac{\partial}{\partial t}\left(\frac{\eta}{r}\right)=\frac{1}{r}\frac{\partial\eta}{\partial t} \text{ and } \frac{\partial^2\psi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial\eta}{\partial t}\right)=\frac{1}{r}\frac{\partial^2\eta}{\partial t^2}. \quad ...(5)$$

Substituting the above values in (2), we get

$$\frac{2\eta}{r^3}-\frac{2}{r^2}\frac{\partial\eta}{\partial r}+\frac{1}{r}\frac{\partial^2\eta}{\partial r}+\frac{2}{r}\left[-\frac{\eta}{r^2}+\frac{1}{r}\frac{\partial\eta}{\partial r}\right]=\frac{1}{c^2 r}\frac{\partial^2\eta}{\partial t^2}$$

or $\quad \partial^2\eta/\partial r^2=(1/c^2)(\partial^2\eta/\partial t^2)$

or $\quad (D^2-c^2D'^2)\eta=0$, where $D\equiv\partial/\partial t$, $D'\equiv\partial/\partial r$

or $\quad (D-cD')(D+cD')\eta=0$,

whose general solution is $\quad \eta=f(r-ct)+g(r+ct)$

or $\quad \psi r=f(r-ct)+g(r+ct)$ or $\psi(1/r)\{f(r-ct)+g(r+ct)\}$

where f and g are arbitrary functions.

Ex. 39. *Solve the diffusion equation in case of spherical symmetry.*

Sol. Diffusion (heat) equation in spherical co-ordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=\frac{1}{k}\frac{\partial u}{\partial t}. \quad ...(1)$$

Due to spherical symmetry, u will be independent of θ and ϕ and hence (1) reduces to

$$\partial^2 u/\partial r^2+(2/r)(\partial u/\partial r)=(1/k)(\partial u/\partial t). \quad ...(2)$$

Let a solution of (2) be of the form $u(r,t)=R(r)\,T(t)$. ...(3)

Substituting (3) in (2), we get $R''T+(2/r)R'T=(1/k)RT'$

or $\quad \{R''+(2/r)R'\}/R=(1/kT)\,T'$. ...(4)

Since the L.H.S. of (4) depends only on r and the R.H.S. depends only on t, hence, the two sides of (4) must be equal to the same constant, μ say. On physical grounds we require $u \to 0$ as $t \to \infty$, *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take the constant $\mu = -\lambda^2$. Then, (4) leads to the following two ordinary differential equations.

$$(1/kt)(dT/dt) = -\lambda^2 \quad \text{or} \quad (1/T)\,dT = -\lambda^2 k\,dt. \qquad ...(5)$$

and
$$\{(d^2R/dr^2) + (2/r)(dR/dr)\}/R = -\lambda^2$$

or
$$r^2(d^2R/dr^2) + 2r(dR/dr) + \lambda^2 r^2 R = 0. \qquad ...(6)$$

Solutions of (5) and (6) are $T(t) = A_\lambda e^{-\lambda^2 kt}$

and $\quad R(r) = (\lambda r)^{-1/2}\left[B_\lambda J_{1/2}(\lambda r) + C_\lambda J_{-1/2}(\lambda r)\right]$

Hence, the most general solution of (2) is

$$u = \sum_\lambda (\lambda r)^{-1/2}\left[D_\lambda J_{1/2}(\lambda r) + E_\lambda J_{-1/2}(\lambda r)\right]e^{-\lambda^2 kt}, \qquad ...(7)$$

where $D_\lambda (= A_\lambda B_\lambda)$ and $E_\lambda (= A_\lambda C_\lambda)$ are new arbitrary constants.

Ex. 40. *Solve the diffusion equation in case of a axial symmetry.*

Sol. Diffusion (heat) equation in spherical co-ordinates (r, θ, ϕ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k}\frac{\partial u}{\partial t}. \qquad ...(1)$$

Let u be symmetrical about z-axis. Then, u must be independent of ϕ and hence (1) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = \frac{1}{k}\frac{\partial u}{\partial t}. \qquad ...(2)$$

Let a solution of (2) be of the form $u = R(r)\Theta(\theta)T(t)$. $\qquad ...(3)$

Substituting (3) in (2) and then dividing by $R\,\Theta\,T$, we have

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} = \frac{1}{k}\frac{T'}{T}. \qquad ...(4)$$

Since the L.H.S. of (4) depends only on r and θ whereas the R.H.S. depends only on t, hence the two sides of (4) must be equal to the some constant, μ say. On physical grounds we require $u \to 0$ as $t \to \infty$, *i.e.*, $T(t) \to 0$ as $t \to \infty$. Accordingly, we take $\mu = -\lambda^2$. Then, (4) gives

$$T'' = -k\lambda^2 T \qquad ...(5)$$

and
$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{\cot\theta}{r^2}\frac{\Theta'}{\Theta} = -\lambda^2$$

i.e.,
$$\left(\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \lambda^2\right)r^2 = -\frac{1}{\Theta}(\Theta'' + \cot\theta\,\Theta') \qquad ...(6)$$

Since L.H.S. of (6) depends on r only and R.H.S. depends on θ only, the two sides of (6) must be equal to the same constant, say $n(n+1)$. Then, (6) gives

$$r^2 R'' + 2rR' + \left\{\lambda^2 r^2 - n(n+1)\right\} R = 0. \quad ...(7)$$

and
$$\Theta'' + \cot\theta\,\Theta' + n(n+1)\Theta = 0. \quad ...(8)$$

Solutions of (5), (7) and (8) are given by

$$T = A_\lambda e^{-k\lambda^2 t}$$

$$R = (\lambda r)^{-1/2}\left[B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)\right]$$

and
$$\Theta = D_n P_n(\cos\theta) + E_n Q_n(\cos\theta)$$

Hence, a solution of (1) in presence of axial symmetry is of the form

$$u = A_\lambda e^{-k\lambda^2 t}(\lambda r)^{-1/2}\left[B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)\right] \times \left[D_n P_n(\cos\theta) + E_n Q_n(\cos\theta)\right]$$

Ex. 41. *A gas is contained in a rigid sphere of radius a. Show that if c is the velocity of sound in the gas, the frequencies of purely radical oscillations are $c\xi_i/a$, where $\xi_1, \xi_2, \ldots$ are the positive roots of the equation $\tan\xi = \xi$.* **[*Delhi, B.Sc. (Hons.), 1996, 2003*]**

Sol. The wave equation in spherical polar co-ordinates is

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}. \quad ...(1)$$

The required wave equation ψ for the present problem must have spherical symmetry, *i.e.*, ψ is a function of only r and t, then (1) becomes

$$\partial^2\psi/\partial r^2 + (2/r)(\partial\psi/\partial r) = (1/c^2)(\partial^2\psi/\partial t^2). \quad ...(2)$$

Now the conditions to be satisfied by the wave function ψ are : (*i*) ψ satisfies (2) (*ii*) ψ remains finite at the origin (*iii*) $u = 0$, $\partial\psi/\partial r = 0$ at $r = a$.

In order to satisfy conditions (*i*) and (*ii*), we choose

$$\psi = A(1/r)\sin(kr)e^{ikct}, \text{ where A is a constant.} \quad ...(3)$$

From (3), $u = -\dfrac{\partial\psi}{\partial r} = A\left[\dfrac{k\cos(kr)}{r} - \dfrac{\sin(kr)}{r^2}\right]e^{ikct}$. ...(4)

To satisfy condition (*iii*), putting $u = 0$ and $r = a$ in (4), we get

$$A\left[\frac{k\cos(ka)}{a} - \frac{\sin(ka)}{a^2}\right]e^{ikct} = 0 \text{ so that } \tan(ka) = ka. \quad ...(5)$$

Hence, we have to choose k which satisfies (5).

The possible frequencies are therefore given by the expression $c\xi_i/a$ (i = 1, 2, 3, ...) where $\xi_1, \xi_2, \ldots$ are the positive roots of the transcendental equation $\tan\xi = \xi$.

Ex. 42. *Harmonic sound waves of period $2\pi/kc$ and small amplitude are propagated along a circular wave guide bounded by the cylinder $\rho = a$. Prove that solutions independent of the angle variable ϕ are of the form $u = e^{i(kct-\gamma z)} J_0(\xi_n \rho/a)$, where ξ_n is a zero of $J_1(\xi)$ and $\gamma^2 = k^2 - (\xi_n^2/a^2)$. Show that this mode is propagated only if $k > \xi_n/a$.*

Sol. The wave equation in cylindrical coordinates (ρ, ϕ, z) is

$$\partial^2 u/\partial\rho^2 + (1/\rho)(\partial u/\partial\rho) + (1/\rho^2)(\partial^2 u/\partial\phi^2) + \partial^2 u/\partial z^2 = (1/c^2)(\partial^2 u/\partial t^2)$$

Since u is independent of ϕ, this reduces to

$$\partial^2 u/\partial\rho^2 + (1/\rho)(\partial u/\partial\rho) + \partial^2 u/\partial z^2 = (1/c^2)(\partial^2 u/\partial t^2) \quad \text{... (1)}$$

The boundary condition is that the velocity of the gas vanishes on the cylinder $\rho = a$ so that $\quad (\partial u/\partial\rho)_{\rho=a} = 0 \quad$ ... (2)

Let a solution of (1) be of the form

$$u(\rho, z, t) = R(\rho)\, Z(z)\, T(t) \quad \text{... (3)}$$

Substituting this value of u in (1) and then dividing by $R(\rho) Z(z) T(t)$,

we obtain $\quad (1/R) R'' + (1/\rho)(R'/R) + Z''/Z = (1/c^2)(T''/T) \quad$ ... (4)

Since the R.H.S. of (4) depends on t while its L.H.S. depends on ρ and z. (3) can be true only if each side is equal to the same constant $-k^2$ (say).

Then (4) gives $\quad T'' + c^2k^2T = 0 \quad$ ... (5)

and $\quad R''/R + (1/\rho)(R'/R) + k^2 = -(Z''/Z) \quad$ ... (6)

Since the R.H.S. of (6) depends on z while its L.H.S. depends on ρ, (6) can be true only if each side is equal to the same constant, say γ^2, then

(6) gives $\quad Z'' + \gamma^2 Z = 0 \quad$ ... (7)

and $\quad \rho^2 R'' + \rho R' + (k^2 - \gamma^2)R = 0$ or $\rho^2 R'' + \rho R' + w^2 R = 0, \quad$ ... (8)

where $\quad w^2 = k^2 - \gamma^2 \quad$ or $\quad \gamma^2 = k^2 - w^2 \quad$ ... (9)

For the present problem, solution of (5) and (7) are taken in the form

$$T(t) = e^{ikct} \quad \text{and} \quad Z(z) = e^{-i\gamma z} \quad \text{... (10)}$$

Again, solution of Bessel's equation (8) is

$$R(\rho) = A\, J_0(w\rho) + BY_0(w\rho) \quad \text{... (11)}$$

We know that $Y_0(w\rho) \to \infty$ as $\rho \to 0$. Hence for finite solution of (1), we choose $B = 0$ in (11). Then solution (8) is of the form

$$R(\rho) = J_0(w\rho) \quad \text{... (12)}$$

From (10) and (12), a solution of (1) is of the form

$$u(\rho, z, t) = e^{ikct} e^{-i\gamma z} J_0(w\rho) = e^{i(kct-\gamma z)} J_0(w\rho) \quad \text{... (13)}$$

Differentiating, (13) partially w.r.t. ρ and using B.C. (2) we get

$$[J_0'(w\rho)]_{\rho=a} = 0 \quad \text{or} \quad [-J_1(w\rho)]_{\rho=a} = 0 \qquad [\because\ J_0'(x) = -J_1(x)]$$

or $$J_1(wa) = 0 \qquad \text{... (14)}$$

Let $\xi_n = wa$. Then (14) reduces to $J_1(\xi_n) = 0$,

i.e., $\xi_1, \xi_2, \ldots$ are the zeros of $J_1(\xi)$. Now, $\xi_n = wa$

gives $w = \xi_n / a$. Substituting this value of w in (13) and (9), we have

$$u(\rho, z, t) = e^{i(kct - \gamma z)} J_0(\xi_n \rho / a), \qquad \text{... (15)}$$

where $\gamma^2 = k^2 - (\xi_n^2 / a^2)$ or $\gamma = \{k^2 - (\xi_n^2 / a^2)\}^{1/2}$ ... (16)

For the mode given by equation (15) to be propagated we must have γ real *i.e.*, $(k^2 - \xi_n^2 / a^2) > 0$ *i.e.*, $k > \xi_n / a$ by (16)

Ex. 43. *Show that the solution $u(r, \theta)$ of Laplace's equation $\nabla^2 u = 0$ in the semi-circular region $r < a$, $0 < \theta < \pi$, which vanisher on $\theta = 0$ and takes the constant value u_0 on $\theta = \pi$ and on curved boundary $r = a$ is*

$$u(r, \theta) = (u_0 / \pi) \{\theta + 2 \sum_{n=1}^{\infty} (r/a)^n (1/n) \sin n\theta\}.$$

Hint. Proceed like solved Ex. 3 on page 64.

Ex. 44. *Show that if the two dimensional harmonic equation $\partial^2 v / \partial x^2 + \partial^2 v / \partial y^2 = 0$ is transformed to plane polar coordinates r and θ defined by $x = r\cos\theta$, $y = r\sin\theta$, it takes the form $\partial^2 v / \partial r^2 + (1/r)(\partial v / \partial r) + (1/r^2)(\partial^2 v / \partial \theta^2) = 0$ and deduce that it has solutions of the form $(Ar^n + Br^{-n}) e^{\pm in\theta}$, where A and B are constants.*

[Delhi B.Sc. (Hons) 1994]

Sol. For first part refer solved Ex. 1 on page 105.

Deduction. Proceed as in solved Ex. 1 on page 61 upto equation (10). Now complete the solution as given below.

Writing $A_n = A$ and $B_n = B$, equation (10) may be re-written as

$$R(r) = A\, r^n + B r^{-n}$$

Solution of equation (7) can be put in the form $\Theta(\theta) = e^{\pm in\theta}$.

Hence a solution of (2) can be expressed in the form

$(Ar^n + Br^{-n})\, e^{\pm in\theta}$, A and B being constants.

Ex. 45. *Show that the two dimensional diffusion equation $\partial^2\theta / \partial x^2 + \partial^2 \theta / \partial y^2 = (1/k)(\partial\theta / \partial t)$ can be put in the form, $\theta(x, y, t)$*

$$= \sum_{\lambda} \sum_{\mu} A_{\lambda\mu} \cos(\lambda x + \in_\lambda) \cos(\mu y + \in_\mu)\, e^{-(\lambda^2 + \mu^2)kt}$$ **[Delhi B.Sc. 1994]**

Sol. Suppose the given equation has solutions of the form

$$\theta(x, y, t) = X(x)\, Y(y)\, T(t). \qquad \text{... (1)}$$

Substitutins this value of θ in the given equation, we get

$X''YT + XY''T = (1/k)\times XYT'$ or $X''/X + Y''/Y = (1/k)(T'/T)$ ... (2)

Since x, y and t are independent variables, (2) is true if each term on each side is a canstant, such that

$$X''/X = -\lambda^2,\quad Y''/Y = -\mu^2 \quad \text{and} \quad T'/kT = -\nu^2 \qquad \text{... (3)}$$

where $$\nu^2 = \lambda^2 + \mu^2 \qquad \text{... (4)}$$

Solutions of equations in (3) can be put in the torm

$X(x) = \cos(\lambda x + \epsilon_\lambda)$, $Y(y) = \cos(\mu y + \epsilon_\mu)$, $T(t) = A_{\lambda\mu} e^{-(\lambda^2+\mu^2)kt}$

$\therefore$ By the principle of superposition, the required general solution is

$$\theta(x,\ y, t) = \sum_{\lambda} \sum_{\mu} A_{\lambda\mu} \cos(\lambda x + \epsilon_\lambda)\cos(\mu y + \epsilon_\mu)\, e^{-(\lambda^2+\mu^2)kt}$$

Ex. 46. Solve the two dimensional diffusion equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/k)\,(\partial u/\partial t)$, in the region $0 < x < a$, $0 < y < b$, $t > 0$ with the following boundary and initial conditions: $u(x, y, t) = 0$ on C where C is the boundary of the rectangle defined by $0 \le x \le a$ and $0 \le y \le b$ and $u(x, y, 0) = f(x, y)$, $0 < x < a$, $0 < y < b$.

Sol. This is exactly the same as solved Ex. 2 on page 26.

Ex. 47. *Solve one dimensional wave equation* $\partial^2 u/\partial x^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$. *Deduce the expression for u satisfying the boundary conditions* $u(0, t) = 0 = u(a, t)$. **[Delhi B.Sc. (Hons) 2000]**

Sol. For first part, refer solved Ex. 1 on page 30.

Deduction. Do exactly as in solved Ex. 2 on page 31 upto equation (17) with following changes. From this part of solution you have to only delete equations 3 (a) and 3 (b).

Ex. 48. *Solve the one-dimensional wave equation* $\partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$, $0 \le x \le 2\pi, t \ge 0$ *subject to the following initial and boundary conditions. (i)* $y(x, 0) = \sin^3 x$, $0 \le x \le 2\pi$ *(iii)* $(\partial y/\partial t)_{t=0} = 0$, $0 \le x \le 2\pi$ *(iii)* $y(0, t) = y\,(2\pi,\ t) = 0$, *for* $t \ge 0$.

Sol. [Refer result and solved Ex. 18 (d) and 18 (*e*) given on page 124]

The general solution of the given wave equation subject to the given initial condition $\partial y/\partial t = 0$ at $t = 0$ and boundary conditions $y(0, t) = y(2\pi, t) = 0$ is given by [Note that $l = 2\pi$ in result on page 124]

$$y(x, t) = \sum_{n=1}^{\infty} (E_n \sin(n\pi x/2\pi)\ \cos\ (n\pi ct/2\pi) \qquad \text{... (1)}$$

Putting $t = 0$ in (1) and using initial condition $y(x, 0) = \sin^3 x$ gives

$$\sin^3 x = \sum_{n=1}^{\infty} E_n \sin(nx/2)$$

or $$E_1 \sin(x/2) + E_2 \sin x + E_3 \sin(3x/2) + E_4 \sin 4x + E_5 \sin(5x/2) + E_6 \sin 3x + \dots = (1/4)\ (3\sin x - \sin 3x)$$

Comparing the coefficients of the like terms on both sides, we get $E_2 = 3/4$, $E_6 = -1/4$ and $E_n = 0$ for $n \neq 2$ or $n \neq 6$. With these values of E_2, E_6 etc, (1) reduces to $y(x,t) = (3/4)\sin x \cos ct - (1/4)\sin 3x \cos 3ct$

Ex. 49. *Solve the two-dimensional wave equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$ *by the method of separation of variables under the following initial and boundary conditions:*

$$u(x, y, 0) = f(x, y),\ (\partial u/\partial t)_{t=0} = 0,\ 0 \le x \le a,\ 0 \le y \le b$$

$$u(0, y, t) = 0 = u(a, y, t) \text{ for } t \ge 0$$

$$u(x, 0, t) = 0 = u(x, b, t), \text{ for } t \ge 0.$$ [Delhi B.Sc. (Hons) 1996]

Sol. Refer part (*ii*) of solves Ex 1 on page 41.

Ex. 50. *Show that the solution of three dimensional wave equation* $\nabla^2 u = (1/c^2)\,(\partial^2 u/\partial t^2)$ *can be put in the form*

$$exp\ \{\pm i\ (lx + my + nz + kct)\},\ \text{provided}\ k^2 = l^2 + m^2 + n^2.$$

Sol. Given $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/c^2)\,(\partial^2 u/\partial t^2)$ ... (1)

Suppose (1) has solutions of the form

$$u(x, y, z, t) = X(x)\, Y(y)\, Z(z)\, T(t) \quad \dots (2)$$

Substituting this in (1) and simplifying, we have

$$X''/X + Y''/Y + Z''/Z = (1/c^2)\,(T''/T) \quad \dots (3)$$

Since x, y, z and t are independent variables, (3) is true if each term on both sides is equal to a constant, that is,

$$X''/X = -l^2,\ Y''/Y = -m^2,\ Z''/Z = -n^2,\ T''/(c^2 T) = -k^2, \quad \dots (4)$$

provided $$k^2 = l^2 + m^2 + n^2 \quad \dots (5)$$

Solutions of differential equations of (4) are of the forms

$$X(x) = e^{\pm ilx},\ Y(y) = e^{\pm imy},\ Z(z) = e^{\pm inz},\ T(t) = e^{\pm ickt} \quad \dots (6)$$

Using (2) and (6), a solution of (1) can be put in the form

$$u(x, y, z, t) = e^{\pm i(lx + my + nz + ckt)}, \quad \text{where} \quad k^2 = l^2 + m^2 + n^2.$$

Ex. 51. *Use the method of separation of variables to determine the solution* $u(r, \theta)$ *of the problem which consists of Laplace equation* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ *and the boundary conditions:*

$$u(x, 0) = f(x),\ 0 \le x \le \pi;\ u(x, \pi) = 0,\ 0 \le x \le \pi,\ u(0, y) = u(\pi, y) = 0,$$

$0 \le y \le \pi$. *Here* $f(x)$ *is a given function of x.*

Sol. Proceed exactly as in solved Ex. 11 on page 45 by replacing a is π and b by π in the entire solution. Finally, we get

$$u(x,y)=\sum_{n=1}^{\infty} F_n \sin nx \sinh n(\pi-y),$$

where $$F_n=(2/\pi \sinh n\pi)\int_0^{\pi} f(x)\sin nx\,dx.$$

Ex. 52. *Find the solution of the three dimensional diffusion equation in the region* $0<x<a$, $0<y<b$, $0<z<c$, $t>0$, $\partial^2u/\partial u^2+\partial^2u/\partial y^2+\partial^2u/\partial z^2=(1/k)(\partial u/\partial t)$ *with the boundary and initial conditions:* $u(0,y,z,t)=0=u(a,y,z,t)$; $u(x,0,z,t)=0=u(x,b,z,t)$, $u(x,y,0,t)=0=u(x,y,c,t)$ *and* $u(x,y,z,0)=f(x,y,z)$.

Sol. Given $\partial^2u/\partial x^2+\partial^2u/\partial y^2+\partial^2u/\partial z^2=(1/k)(\partial u/\partial t)$ ... (1)

$$u(0,y,z,t)=0=u(a,y,z,t) \text{ for all } t>0 \quad \text{... (2)}$$

$$u(x,0,z,t)=0=u(x,b,z,t) \text{ for all } t>0 \quad \text{... (3)}$$

$$u(x,y,0,t)=0=u(x,y,c,t), \text{ for all } t>0 \quad \text{... (4)}$$

and $$u(x,y,z,0)=f(x,y,z),\ 0<x<a,\ 0<y<x,\ 0<z<c \quad \text{... (5)}$$

Let a solution of (1) be of the form

$$u(x,y,z,t)=X(x)\,Y(y)\,Z(z)\,T(t) \quad \text{... (6)}$$

Substituting this in (1), we have

$$X''/X+Y''/Y+Z''/Z=(1/k)(T'/T) \quad \text{... (7)}$$

Since x, y, z and t are independent variables, (7) is true if each term on each side is a negative constant. [We have chosen negative constants because non-negative constants will give rise to trivial solution of (1)] Thus, we have

$$X''/X=-\lambda_1^2,\ Y''/Y=-\lambda_2^2,\ Z''/Z=-\lambda_3^2 \text{ and } T'/kT=-\lambda^2, \quad \text{... (8)}$$

where $$\lambda^2=\lambda_1^2+\lambda_2^2+\lambda_3^2 \quad \text{... (9)}$$

Solutions of (8) are given by

$$X(x)=A\cos\lambda_1 x+B\sin\lambda_1 x \quad \text{... (10)}$$

$$Y(y)=C\cos\lambda_2 y+D\sin\lambda_2 y \quad \text{... (11)}$$

$$Z(z)=E\cos\lambda_3 z+\sin\lambda_3 z \quad \text{... (12)}$$

and $$T(t)=G\,e^{-\lambda^2 kt} \quad \text{... (13)}$$

Put $x=0$ and $x=a$ by turn in (6). Using (2) and the fact that $Y(y)\neq 0$, $X(z)\neq 0$, $T(t)\neq 0$ for non-trivial solution of (1), we get

$$X(0)=0 \text{ and } X(a)=0. \quad \text{... (14)}$$

Putting $x=0$ and $x=a$ by turn in (10) and using (14), we get

$$0=A \text{ and } 0=A\cos\lambda_1 a+B\sin\lambda_1 a$$

$$\Rightarrow A=0 \text{ and } \sin\lambda_1 a=0,$$

Additional Problems

where we have taken $B \neq 0$, since otherwise $X(x) = 0$ and hence we have trivial solution $u = 0$ for (1)

Now, $\sin\lambda_1 a = 0 \Rightarrow \lambda_1 a = l\pi \Rightarrow \lambda_1 = l\pi/a,\ l = 1,2,3$ ... (15)

and then from (10), $X_l(x) = B_l \sin(l\pi\ x/a),\ n = 1,2,3$ ... (16)

Similarly, boundary condition (3) and (11) give

$$\lambda_2 = m\pi/b \quad Y_m(y) = D_m \sin(m\pi y/b),\ m = 1, 2, 3 \quad \text{... (17)}$$

and finally, boundary condition (4) and (12) give

$$\lambda_3 = n\pi/c, \quad Z_n(z) = F_n \sin(n\pi z/c),\ n = 1, 2, 3 \quad \text{... (18)}$$

Using (15), (17) and (18), (9) reduces to

$$\lambda^2 = \lambda^2(l^2/a^2 + m^2/b^2 + n^2/c^2) \quad \text{... (19)}$$

Using (6), (13), (16), (17) and (18), a solution of (1) is given by

$$H_{lmn} \sin(l\pi x/a) \sin(m\pi y/b) \sin(n\pi z/c)\, e^{-\lambda^2 kt},$$

where $H_{lmn} (= B_l\ D_m\ F_n\ G)$ is another arbitrary constant Now, using the principle of superposition, the general solution of (1) is

$$u(x,y,z,t) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}\, e^{-\lambda^2 kt} \quad \text{... (20)}$$

where λ is given by (19),

Putting $t = 0$ in (20) and using (5), we have

$$f(x,y,z) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{lmn} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \quad \text{... (21)}$$

which is Fourier sine series for three variable x, y, z and so

$$H_{lmn} = \frac{8}{abc} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} f(x,y,z) \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c} \quad \text{... (22)}$$

The required solution of (1) is given by (21) and (22).

Ex. 53. *Obtain the solution of the wave equation* $\partial^2 u/\partial r^2 + (1/r)(\partial u/\partial r) + (1/r^2)(\partial^2 u/\partial\theta^2) = (1/c^2)(\partial^2 u/\partial t^2)$ *by the method of separation of the variables in the form*

$$\sum_{m,k} A_{m,k} J_m(kr)\, e^{\pm im\theta \pm ikct}$$

Sol. Let a solution of the given equation be of the form

$$u(r, \theta, t) = R(r)\, \Theta(\theta)\, T(t) \quad \text{... (1)}$$

Substituting this is the given equation, we get

$$R''/R + (1/r)(R'/R) + (1/r^2)(\Theta''/\Theta) = (1/c^2)(T''/T) \quad \text{... (2)}$$

Since r, θ and t are independent variables, (2) is true if its each side is equal to a constant, $-k^2$ (say). Thus, we get

$$T'' + k^2c^2T = 0 \quad \text{... (3)}$$

and $$R''/R+(1/r)(R'/R)+(1/r^2)(\Theta''/\Theta)=-k^2 \quad \text{... (4)}$$

Re-writing (4), $(r^2/R)\{R''+(1/r)R'+r^2R\}=-(\Theta''/\Theta)$ ... (5)

Since r and θ are independent variables, (5) is true if its each side is equal to a constant, say m^2. Then, we get

$$\Theta''+m^2\Theta=0 \quad \text{... (6)}$$

and $$r^2R''+\rho R'+(k^2r^2-m^2)R=0 \quad \text{... (7)}$$

Solving (3), (6) and (7), we have solutions of the forms

$$T(t)=e^{\pm ikct},\ \Theta(\theta)=e^{\pm im\theta}$$

and $$R(r)=A_{mk}J_m(kr)+B_{mk}Y_m(kr) \quad \text{... (9)}$$

We choose $B_{mk}=0$ in (9) so that $R(r)$ and hence $u(r,\theta,t)$ may be finite at $\rho=0$. Then (9) reduces to

$$R(r)=A_{mk}\ J_m(kr) \quad \text{... (10)}$$

Using (1), (8) and (10), a solution of the given equation is

$$A_{mk}\ J_m(kr)\ e^{\pm im\theta\pm ikct}$$

By the principle of superposition, general solution is given by

$$u(r,\theta,t)=\sum_{m,k} A_{mk}\ J_m(kr)\ e^{\pm im\theta\pm ikct}$$

Ex. 54. *Obtain the solution of the Laplace's equation* $\partial^2u/\partial r^2 + (2/r)(\partial u/\partial r)+(1/r^2)(\partial^2u/\partial\theta^2)+(\cot\theta/r^2)(\partial u/\partial\theta)=0$ *by the method of separation of variables in the form*

$$u(r,\theta)=\sum_{n=0}^{\infty}(A_n r^n+B_n/r^n)\ P_n(\cos\theta),$$

where $P_n(\mu)$ *are the Legendre's polynomials.*

Sol. Let a solution of the given equation be of the form

$$u(r,\theta)=R(r)\ \Theta(\theta). \quad \text{... (1)}$$

Substituting this in the given equation, we have

$$R''/R+(2/r)(R'/R)+(1/r^2)\ (\Theta''/\Theta)+(\cot\theta/r^2)(\Theta'/\Theta)=0$$

or $$-(1/R)(r^2R''+2rR')=(1/\Theta)\ (\Theta''+\cot\theta\ \Theta') \quad \text{... (2)}$$

Since r and θ are independent variables, (2) is true if each side is equal to the same constant, say $-n(n+1)$. Then, (2) gives

$$r^2R''+2r\ R'-n(n+1)\ R=0 \quad \text{... (3)}$$

and $$\Theta''+\cot\theta\ \Theta'+n(n+1)\ \Theta=0 \quad \text{... (4)}$$

Putting $\mu=\cos\theta$, (5) reduces to Legendre's equation

$$(1-\mu^2)(d^2\Theta/d\mu^2)-2\mu\ (d\Theta/d\mu)+n(n+1)\ \Theta=0 \quad \text{... (5)}$$

Solving (3), $R(r) = A'_n \ r^n + B'_n / r^{n+1}$ ... (6)

Solving (5), $\Theta(\theta) = C_n P_n(\cos\theta) + D_n Q_n(\cos\theta)$ ... (7)

We choose $D_n = 0$ if $\Theta(\theta)$ is to be finite on polar axis $\theta = 0$. Then (7) reduces to $\Theta(\theta) = C_n \ P_n(\cos\theta)$, ... (8)

where $P_n(\cos\theta)$ *i.e.*, $P_n(\mu)$ are the Legendre's polynomials

Using (1), (6) and (8) a solution of the given equation is

$$\{C_n A'_n \ r^n + C_n \ B'_n \ (1/r^{n+1})\} P_n(\cos\theta) \text{ or } (A_n \ r^n + B_n / r^{n+1}) \ P_n(\cos\theta),$$

where $A_n (= C_n \ A'_n)$ and $B_n (= C_n B'_n)$ are new arbitrary constants.

Using the principle of superposition, the required general solution is

$$u(r\ \theta) = \sum_{n=0}^{\infty} (A_n \ r^u + B_n / r^{n+1}) \ P_n(\cos\theta).$$

Ex. 55. *A rigid sphere of radius 'a' is placed in a stream of fluid whose velocity in the undisturbed state is V, determine the velocity of fluid at any point of the disturbed stream.*

Sol. Let z-axis be taken in the direction of the given velocity V. Let spherical polar coordinates of any point P of given fluid the (r, θ, ϕ) with origin at the centre of the fixed sphere of radius 'a'. We assume that there is perfect fluid having irrtational motion. Then know that the velocity q of fluid at P is given by*

$$\boldsymbol{q} = -\text{grad } \Psi = -\nabla\Psi, \quad \text{... (1)}$$

where Ψ is the velocity potential satisfying the following three conditions:

(*i*) $\nabla^2\Psi = 0$ *i.e.*, $\dfrac{\partial^2\Psi}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial\Psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\Psi}{\partial\theta^2} + \dfrac{\cot\theta}{r^2}\dfrac{\partial\Psi}{\partial\theta} + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\Psi}{\partial\phi^2} = 0$ ... (2)

(*ii*) $(\partial\Psi/\partial r)_{r=a} = 0$, ... (3)

i.e. the normal component of the velocity varishes on the boundary of the fixed sphere

(*iii*) $-\partial\Psi/\partial z = V$, since z-axis is taken in the direction of velocity V

$\Rightarrow$ $\Psi = -Vz = -Vr\cos\theta = -VrP_1(\cos\theta)$, ... (4)

noting that $z = r\cos\theta$ in spherical polar coordinates and $P_1(\cos\theta) = \cos\theta$, P_1 being Legendre's polynomial.

Since the motion about the fixed sphere is axially symmetry it follows that velocity potential Ψ must be independent of the variable angle ϕ, *i.e.*, $\partial^2\Psi/\partial\phi^2 = 0$. Hence (2) reduces to

$$\partial^2\Psi/\partial r^2 + (2/r)(\partial\Psi/\partial r) + (1/r^2)(\partial^2\Psi/\partial\theta^2) + (\cot\theta/r^2)(\partial\Psi/\partial\theta) = 0 \quad \text{... (5)}$$

* Refer chapter 5 of Flud Dynamics by Dr. M.D. Raisinghania published by S. Chand & Co. Ltd. Delhi

Let a solution of (5) be of the form

$$\Psi(r, \theta) = R(r)\ \Theta(\theta) \quad \dots (6)$$

Substituting (6) in (5), we have

$$R''/R + (2/r)(R'/R) + (1/r^2)(\Theta''/\Theta) + (\cot\theta/r^2)(\Theta'/\Theta) = 0$$

or

$$-(1/R)(r^2 R'' + 2rR') = (1/\Theta)(\Theta'' + \cot\theta\ \Theta') \quad \dots (7)$$

Since r and θ are independent variables, (7) is true if its each side is equal to the same constant $(= -n(n+1)$, say). Then (7) gives

$$r^2 R'' + 2r\ R' - n(n+1)\ R = 0 \quad \dots (8)$$

and

$$\Theta'' + \cot\theta\ \Theta' + n(n+1)\ \Theta = 0 \quad \dots (9)$$

Putting $\mu = \cos\theta$, (9) reduces to Legendre's equation

$$(1-\mu^2)(d^2\Theta/d\mu^2) - 2\mu(d\Theta/d\mu) + n(n+1)\ \Theta = 0 \quad \dots (10)$$

Solutions of (8) and (10) are given by

$$R(r) = A_n\ r^n + B_n/r^{n+1} \quad \dots (11)$$

and

$$\Theta(\theta) = C_n P_n(\cos\theta + D_n Q_n(\cos\theta) \quad \dots (12)$$

We choose $D_n = 0$ in (12) if $\Theta(\theta)$ and hence Ψ is be finite on z-axis (where $\theta = 0$). Hence (12) reduces to

$$\Theta(\theta) = C_n\ \ P_n\ (\cos\theta) \quad \dots (13)$$

Differentiating (6) partially w.r.t. 'r', we get

$$\partial\Psi/\partial r = R'(r)\ \Theta(\theta) \quad \dots (14)$$

Putting $r = a$ in (14) using (3), $R'(a) = 0$ ... (15)

Differentiating (11), $R'(r) = nA_n\ r^{n-1} - (n+1)\ B_n\ r^{-n-2}$ ... (16)

Putting $r = a$ in (16) and using (15), we get

$$0 = nA_n\ a^{n-1} - (n+1)\ B_n a^{-n-2} \quad \text{or} \quad B_n = \{n/(n+1)\}\ A_n\ a^{2n+1}$$

Substituting this value of B_n in (11), we get

$$R(r) = A_n\ \{r^n + na^{2n+1}/(n+1)\ r^{n+1}\} \quad \dots (17)$$

Using (6), (13) and (17), a solution of (5) is

$$A_n C_n\{r^n + na^{2n+1}/(n+1)\ r^{n+1}\} \quad \text{or} \quad E_n\{r^n + n\ a^{2n+1}/(n+1)\ r^{n+1}\},$$

where $E_n (= A_n C_n)$ are new arbitrary constants. Now by principle of superposition, the general solution of (5) is

$$\Psi(r, \theta) = \sum E_n\ \{r^n + na^{2n+1}/(n+1)r^{n+1}\}\ P_n(\cos\theta) \quad \dots (18)$$

As $r \to \infty$, this velocity potential has the asymptotic form*

$$\Psi \sim \sum E_n\ r^n\ P_n(\cos\theta) \quad \dots (19)$$

Hence in order to satisfy the condition (4), we must take $E_1 = -V$ and all other E's zero. Thus, we take $n = 1$ in (18) and take the remaining E's equal to zero and obtain

---

* ~ is symbol which indicates that a quantity can be equated approximately (asymptotically)

$$\Psi(r,\theta) = E_1\,(r + a^3/2r^2)\,P_1(\cos\theta) = -V(r + a^3/2r^2)\cos\theta,$$

giving the velocity potential of the fluid motion.

The components of the velocity are, therefore, given by

$$q_r = -\partial\Psi/\partial r = V(1 - a^3/r^3)\cos\theta$$

$$q_\theta = -(1/r)\,(\partial\Psi/\partial\theta) = -V(1 + a^3/2r^3)\sin\theta.$$

Ex. 56. *Obtain the solution of the three dimensional diffusion equation in spherical polar coordinates*

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = \frac{1}{k}\frac{\partial\Psi}{\partial t}$$

by the method of separation of variables in the form

$$r^{-1/2} J_{n+1/2}(\lambda r)\, P_n^m(\cos\theta)\, e^{\pm im\phi - \lambda^2 kt}$$

Sol. On simplifying the given diffusion equation, we have

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\Psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = \frac{1}{k}\frac{\partial\Psi}{\partial t} \quad \text{... (1)}$$

Proceed as in solved Ex. 1 on page 96 upto equation (10). Now, in order to get the solution of the required form complete the solution after equation (10) as given below.

Solutions of (4), (6), (7) and (10) are of the forms

$$T(t) = e^{-k\lambda^2 t},\ \Phi(\Phi) = e^{\pm im\phi} \quad \text{... (12)}$$

$$R(r) = r^{-1/2}\{A\, J_{n+1/2}(\lambda r) + B\, J_{-(n+1/2)}(\lambda r)\} \quad \text{... (13)}$$

and $$\Theta(\theta) = CP_n^m(\cos\theta) + D\,\Theta_n^m\,(\cos\theta) \quad \text{... (14)}$$

We know that $J_{-(n+1/2)}(\lambda r) \to \infty$ as $r \to 0$ and $Q_n^m\,(\cos\theta) \to 0$ as $\theta \to 0$. Hence for getting a finite solution valid for $r \to 0$ and $\theta \to 0$, we must take $B = 0$ in (13) and $D = 0$ in (14). Thus, we have

$$R(r) = r^{-1/2}\ J_{n+1/2}(\lambda r) \text{ and } \Theta\,(\theta) = P_n^m(\cos\theta) \quad \text{... (15)}$$

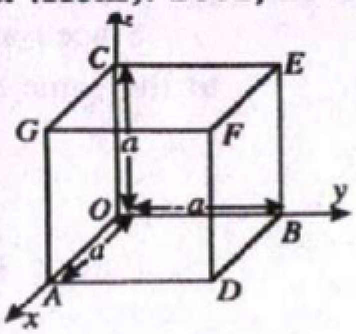
Using (2), (12) and (15), a solution of (1) is of the form

$$r^{-1/2} J_{n+1/2}(\lambda r)\, P_n^m(\cos\theta)\, e^{\pm im\phi - \lambda^2 kt}$$

Ex. 57. A *gas is contained in a cubical box of side a. Show that if c is the velocity of sound in the gas, the period of free oscillations are* $2a/c\,(n_1^2 + n_2^2 + n_3^2)^{1/2}$, *where* n_1, n_2, n_3 *are integers.*

[Delhi B.Sc. (Hons). 2001; 2003]

Sol. Let one corner O of the given cubical box (OBDA, CEFG) be taken as origin as shown in figure. Since side of the box is a, we can take its six faces as $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$. Thus, the gas is contained in the region defined by $0 \le x < a,\ 0 \le y \le a,\ 0 \le z \le a$.



We now solve three-dimensional wave equation

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = (1/c^2)\ (\partial^2 u/\partial t^2) \quad \dots (1)$$

which is to be solved under the boundary condition that at all the six faces of the cube $\partial u/\partial n = 0$, where n is the unit outward drawn normal to surface. Thus, we are to solve (1) subject to the following boundary conditions:

$$(\partial u/\partial x)_{x=0} = (\partial u/\partial x)_{x=a} = 0, \quad \text{for all } t \ge 0 \quad \dots (2)$$

$$(\partial u/\partial y)_{y=0} = (\partial u/\partial y)_{y=a} = 0, \text{ for all } t \ge 0, \quad \dots (3)$$

and $$(\partial u/\partial z)_{z=0} = (\partial u/\partial z)_{z=a} = 0, \text{ for all } t \ge 0 \quad \dots (4)$$

Since the initial velocity of the gas is zero, we have

$$(\partial u/\partial t)_{t=0} = 0, \text{ for } 0 \le x \le 0,\ 0 \le y \le a,\ 0 \le z \le a \quad \dots (5)$$

Let a solution of (1) be of the form

$$u(x, y, z, t) = X(x)\ Y(y)\ Z(z)\ T(t). \quad \dots (6)$$

Substituing (6) in (1), we have

$$X''/X + Y''/Y + Z''/Z = (1/c^2)\ (T''/T). \quad \dots (7)$$

Since x, y, z and t are independent variables, (7) is true if each term of both sides is equal to a constant. Since non-negative constants lead to a trivial solution of (1), hence we take

$$X''/X = -\mu_1^2,\ Y''/Y = -\mu_2^2,\ Z''/Z = -\mu_3^2,\ T''/c^2T = -\mu^2 \quad \dots (8)$$

where $$\mu^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 \quad \dots (9)$$

Solving differential equations of (8), we have

$$X(x) = A\cos\mu_1 x + B\sin\mu_1 x, \quad \dots (10)$$

$$Y(y) = C\cos\mu_2 y + D\sin\mu_2 y, \quad \dots (11)$$

$$Z(z) = E\cos\mu_3 z + F\sin\mu_3 z \quad \dots (12)$$

and $$T(t) = G\ \cos(\mu ct) + H\sin(\mu ct) \quad \dots (13)$$

Differentiating (6) partially w.r.t. 'x', we have

$$\partial u/\partial x = X'(x)\ Y(y)\ Z(z)\ T(t) \quad \dots (14)$$

Putting $x = 0$ and $x = a$ by form in (14) and using (2), we have

$$X'(0) = 0 \text{ and } X'(a) = 0 \quad \dots (15)$$

Differentiating (10), $X'(x) = -A\mu_1 \sin\mu_1 x + B\mu_1 \cos\mu_1 x \quad \dots (16)$

Putting $x = 0$ and $x = a$ by turn in (16) and using (15), we get

$$0 = B\mu_1 \text{ and } 0 = -A\mu_1 \sin\mu_1 a + B\mu_1 \cos\mu_1 a$$

These $\Rightarrow$ $B = 0$ and $\sin\mu_1 a = 0$, as $\mu_1 \ne 0$ and $A \ne 0$

Now $\sin\mu_1 a = 0 \Rightarrow \mu_1 a = n_1\pi \Rightarrow \mu_1 = n_1\pi/a$, $\quad \dots (17)$

where n_1 is an integer. With $B = 0$ and $\mu_1 = \mu_1\pi/a$, (10) gives

$$X_{n_1}(x) = A_{n_1} \cos(n_1\pi x/a). \quad \dots (18)$$

Similarly, (3) and (11) $\Rightarrow$ $\mu_2 = n_2\pi/a$ $Y_{n_2}(y) = D_{n_2}\cos(n_2\pi y/a)$ ... (19)

and (4) and (12) $\Rightarrow$ $\mu_3 = n_3\pi/a$, $Z_{n_3}(z) = F_{n_3}\cos(n_3\pi z/a)$ ... (20)

(9), (17), (18) and (20) $\Rightarrow$ $\mu^2 = (\pi^2/a^2)(n_1^2+n_2^2+n_3^2)$ ... (21)

From (13), (18), (19) and (20), general solution of (1) is given by

$$u(x,y,z,t) = \sum_{n_1,n_2,n_3} I_{n_1,n_2,n_3}\cos\left(\frac{n_1\pi x}{a}\right)\cos\left(\frac{n_2\pi y}{a}\right)\cos\left(\frac{n_3\pi z}{a}\right)[G_\mu\cos(\mu ct)$$
$$+H_\mu\sin(\mu ct)] \quad \text{... (22)}$$

where $I_{n_1,n_2,n_3}(=A_{n_1}D_{n_2}F_{n_3})$ is another arbitrary constant.

Differentiating (22) partially with respect to 't', we get

$$\frac{\partial u}{\partial t} = \sum_{n_1,n_2,n_3} I_{n_1 n_2 n_3}\cos\left(\frac{n_1\pi x}{a}\right)\cos\left(\frac{n_2\pi y}{a}\right)\cos\left(\frac{n_3\pi z}{a}\right)[-G_\mu\mu c(\sin(\mu ct)]$$
$$+H_\mu\mu c\cos(\mu ct)] \quad \text{... (23)}$$

Putting $t=0$ is (23) and using (5), we have $H_\mu = 0$. Hence (2) gives

$$u(x,y,z,t) = \sum_{n_1 n_2 n_3} J_{n_1 n_2 n_3}\cos\left(\frac{n_1\pi x}{a}\right)\cos\left(\frac{n_2\pi y}{a}\right)\cos\left(\frac{n_3\pi z}{c}\right)\cos(\mu ct) \quad \text{... (24)}$$

where $J_{n_1,n_2,n_3}(=I_{n_1,n_2,n_3}\ G_\mu)$ are new arbitrary constants. From (24), we see that the periods of the free oscillations of the gas are.

$$= \frac{2\pi}{\mu c} = \frac{2a}{c(n_1^2+n_2^2+n_3^2)}, \text{ using (23)}$$

Ex. 58. *Show that in cylindrical coordinates r, z, ϕ Laplace's equation has solutions of the form $R(\rho)\,e^{\pm mz \pm in\phi}$, where $R(\rho)$ is a solution of Bessel's equation* $d^2R/d\rho^2 + (1/\rho)(dR/d\rho) + (m^2 - n^2/\rho^2)R = 0$

If solution of Laplace's equation tends to zero as $z \to \infty$ and is finite when $\rho = 0$, show that, in general, in the usual notations for Bessel's functions, the appropriate solutions are made up of the terms of the form $J_n(m\rho)e^{-mz\pm in\phi}$ **[Delhi B. Sc (Hons) 2002]**

Sol. Proceed as in solved Ex. 1 on page 71 upto equation (8) then complete the solution as follows:

Re-writing (8), $d^2R/d\rho^2 + (1/\rho)(dR/d\rho) + (m^2 - n^2/\rho^2)R = 0$ ... (9)

Solutions of (4) and (7) can be written in the form $e^{\pm mz}$ and $e^{\pm im\phi}$ respectively. Hence (1) has a solution of the form

$$R(\rho)\,\Phi(\phi)\,Z(z) \;\; i.e. \;\; R(\rho)\,e^{\pm mz \pm in\phi}, \quad \text{... (10)}$$

where $R(\rho)$ is a solution of Bessel's equation (9).

Second part. Solution of (9) is given by

$$R(\rho) = A\,J_n(m\rho) + BY_n(m\rho) \quad \text{... (11)}$$

Since $Y_n(m\rho) \to \infty$ as $\rho \to 0$ hende for the solution of (1) to tend to zero as $\rho \to 0$, we must take $B = 0$ in (11). Then (11) becomes

$$R(\rho) = A\, J_n(m\rho) \qquad \text{... (12)}$$

Solution of (4) is $$Z(z) = C\, e^{mz} + D\, e^{-mz} \qquad \text{... (13)}$$

Since $e^{mz} \to \infty$ as $z \to \infty$, hence for the solution of (1) to tend to zero as $z \to \infty$, we must take $C = 0$ in (13). Hence (13) reduces to

$$Z(z) = D\, e^{-mz} \qquad \text{... (14)}$$

Hence the solution of (1), which tends to zero when $z \to \infty$ and $\rho \to 0$, is given by help of (10), (12) and (4) in the form $J_n(m\rho)\ e^{-mz \pm in\phi}$

Ex. 59. *Using the method of separation of variables, solve wave equation in spherical polar coordinates, namely,*

$$\frac{\partial^2 u}{\partial r} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

What happens to the solution if on physical grounds, we require the solution of the above equation to have symmetric properties for θ and ϕ?

[Delhi B.Sc. (Hons) 2002]

Sol. Proceed as in solved 1 on page 100 upto equation (10). Then complete the solution as follows:

Solutions of (4), (6), (8) and (10) are of the forms

$$T(t) = e^{\pm ickt},\ \Phi(\phi) = e^{\pm im\phi},\ R(r) = r^{-1/2} J_{\pm(n+1/2)}(kr)$$

and $$\Theta(\theta) = A\, P_n^m(\cos\theta) + B\, \Theta_n^m(\cos\theta)$$

Since $\Theta_n^m(\cos\theta) \to \infty$ as $\theta \to 0$, hence for a solution to remain finite for $\theta = 0$, we take $B = 0$. Then the solution of wave equation can be taken as

$$u(r, \theta, \phi, t) = r^{-1/2} J_{\pm(n+1/2)}(kr)\ P_n^m(\cos\theta)\ e^{\pm im\phi \pm ikct} \qquad \text{... (11)}$$

Second part. We now find the required solution having symmetric properties for θ and ϕ. Then solution will be independent of θ and ϕ and hence the corresponding solution can be obtained from (11) by taking $m = 0$ and $n = 0$ in (11) and is given by $r^{-1/2} J_{\pm(n+1/2)}(kr)\, e^{\pm ikct}$.

Ex. 60. *One end of a string ($x = 0$) is fixed and the point $x = a$ is made to oscillate, so that at time t the displacement is $g(t)$. Show that the displacement $u(x, t)$ of the point x at time t is given by*

$$u(x,t) = f(ct - x) - f(ct + x),$$

where f is a function satisfying the relation

$$f(t + 2a) = f(t) - g[(t + a)/c] \text{ for all } t.$$ **[I.A.S. 1999]**

Sol. The displacement $u(x, t)$ of the point x at time t is governed by one-dimensional wave equation

$$\partial^2 u/\partial x^2 = (1/c^2)\,(\partial^2 u/\partial t^2) \qquad \text{.... (1)}$$

We are to solve (1) subject to boundary conditions:

$$u(0, t) = 0 \quad \text{and} \quad u(a, t) = g(t) \qquad \dots (2)$$

Proceed now as in solved Ex. 6 on page 38 (replace ϕ by u in (1) and g by h in the solution obtained), and get

$$u(x, t) = f(x + ct) + h(x - ct). \qquad \dots (3)$$

Replacing x by 0 in (3) and using B.C. (2), we get

$$0 = f(ct) + h(-ct) \quad \text{or} \quad h(-ct) = -f(ct) \qquad \dots (4)$$

Replacing ct by $-ct$ on both sides of identity (4), we get

$$h(ct) = -f(-ct) \qquad \dots (5)$$

Replacing ct by $x - ct$ in (5), $h(x - ct) = -f(ct - x)$.

Hence (3) becomes $\quad u(x,t) = f(ct + x) - f(ct - x) \qquad \dots (6)$

Replacing x by a in (6) and using B.C. (2), we get

$$g(t) = f(ct + a) - f(ct - a) \qquad \dots (7)$$

Replacing t by $(t + a)/c$ on both sides of identity (7), we get

$$g\left(\frac{t+a}{c}\right) = f(t + a + a) - f(t + a - a) \quad \text{or} \quad f(t + 2a) = f(t) + g\left(\frac{t+a}{c}\right)$$

Ex. 61. *Solve* $\partial u/\partial t = \partial^2 u/\partial x^2,\ 0 < x < l,\ t > 0$ *given that* $u(0,t) = u(l,t) = 0$ *and* $u(x,0) = x(l - x),\ 0 \le x \le l$. **[I.A.S. 2002]**

Sol. Refer solved Ex. 12, page 117. Here $k = 1$ and hence the solution

reduces to $\quad u(x,t) = \dfrac{8l^2}{\pi^3} \displaystyle\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{l} e^{-\{(2m-1)^2 \pi^2 t / l^2}$.

Ex. 62. *A thin rod of length* π *is first immersed in boiling water so that its temperature is 100°C throughout. Then the rod is removed from water at* $t = 0$ *which is immediately put in ice so that the ends are kept at 0°C. Find w (x, t) if heat equation is* $a^2(\partial^2 w/\partial x^2) = \partial w/\partial t$. **[Naspur 2002]**

Sol. Refer solved example 3 on page 16. Here $u(x,t) = w(x,t)$, $k = a^2$, $l = \pi$, $u_0 = 100$.

$$\therefore \quad u(x,t) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m-1} \sin(2m-1)x \; e^{-(2m-1)^2 a^2 t}$$

Ex 63. *For the vibrating string if initial shape is given by y (x, 0) = C sin x, C being constant and string is released from rest, then find the displacement y (x, t).* **[Nagpur 2002]**

Hint. Proceed as in part (C) of Ex. 18, page 124.

Ex 64. *A uniform string of length l held tightly between x = 0 and x = l with no initial displacement, is struck at x = a, 0 < a < l with velocity* v_0. *Find the displacement of the string at any time t > 0.* **[I.A.S. 2004]**