2018 CSE

5(a). Solve, $y'' - y = x^2 e^{2x}$

$$(D^2 - 1) y = x^2 e^{2x}$$

Auxiliary Eqn: $m^2 - 1 = 0 \Rightarrow m = 1, -1$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$P.I. = \frac{1}{D^2-1} x^2 . e^{2x}$$

$$= e^{2x} \frac{1}{(D+2)^2 - 1} x^2$$

$$= e^{2x} \frac{1}{3\left(1 + \frac{D^2+4D}{3}\right)} x^2$$

$$= e^{2x} \frac{1}{3} \left(1 + \frac{D^2+4D}{3}\right)^{-1} x^2$$

$$= \frac{e^{2x}}{3} \left[1 - \left(\frac{D^2+4D}{3}\right) + \left(\frac{D^2+4D}{3}\right)^2 - \dots\right] x^2$$

$$= \frac{e^{2x}}{3} \left[1 - \frac{4D}{3} + \frac{13}{9} D^2 - \dots\right] x^2$$

$$= \frac{1}{3} e^{2x} \left(x^2 - \frac{8}{3} x + \frac{26}{9}\right)$$

General Solution,

$$Y = Y_c + Y_p$$

$$Y = c_1 e^x + c_2 e^{-x} + \frac{e^{2x}}{3} \left[x^2 - \frac{8x}{3} + \frac{26}{9}\right]$$

Scanned with CamScanner

5(c) Solve:

$y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$

$(D^3 - 6D^2 + 12D - 8)y = 12e^{2x} + 27e^{-x}$

$(D-2)^3 y = 12e^{2x} + 27e^{-x}$

Auxiliary Eqn: $(m-2)^3 = 0 \Rightarrow m = 2, 2, 2$

$y_c = (c_1 + c_2 x + c_3 x^2) e^{2x}$

$$y_p = \frac{1}{(D-2)^3}(12e^{2x} + 27e^{-x})$$

$$= \frac{1}{(D-2)^3} 12e^{2x} + \frac{1}{(D-2)^3} 27e^{-x}$$

$$= 12 \cdot \frac{x^3}{3!} e^{2x} + \frac{1}{(-1-2)^3} 27e^{-x}$$

$$= 2x^3 e^{2x} - e^{-x}$$

General Solution,

$y = y_c + y_p$

$y = (c_1 + c_2 x + c_3 x^2) e^{2x} + 2x^3 e^{2x} - e^{-x}$

Scanned with CamScanner

5(d) i) Find the Laplace transform of $f(t) = \frac{1}{\sqrt{t}}$.

$$Lf(t) = \int_0^{\infty} e^{-st}\, t^{-1/2}\, dt$$

Put $st = y$

$dt = \frac{dy}{s}$

$$= \int_0^{\infty} e^{-y} \left(\frac{y}{s}\right)^{-1/2} \frac{dy}{s}$$

$$= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-y} \cdot y^{-1/2}\, dy$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\frac{-1}{2}+1\right) = \frac{\sqrt{\pi}}{\sqrt{s}} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

ii) Find the inverse Laplace transform of $\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}$

$$\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3}$$

$\therefore A = 2, \quad B = 2, \quad C = 1$

$$\therefore L^{-1}\left[\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right] = L^{-1}\left[\frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s+3}\right]$$

$$\Rightarrow f(t) = 2e^{t} + 2e^{2t} + e^{-3t}$$

$$\left[\because L(e^{at}) = \frac{1}{s-a}, \; s > a\right]$$

6(a) Solve, $\left(\frac{dy}{dx}\right)^2 y + 2\left(\frac{dy}{dx}\right)x - y = 0.$

$$yp^2 + 2px - y = 0 \quad -(1) \qquad \left(p = \frac{dy}{dx}\right)$$

Solving for x,

$$x = \frac{1}{2}\cdot\frac{y}{p} - \frac{1}{2}\cdot yp \quad -(2)$$

Differentiating w.r.t. y

$$\frac{dx}{dy} = \frac{1}{2}\,\frac{1\cdot p - y\cdot \frac{dp}{dy}}{p^2} - \frac{1}{2}\left(p + y\frac{dp}{dy}\right)$$

ie. $$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - p - y\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} + p = -y\cdot\frac{dp}{dy}\left(\frac{1}{p^2} + 1\right)$$

ie. $$\cancel{\left(\frac{1}{p^2}+1\right)} p = -y\,\frac{dp}{dy}\cancel{\left(\frac{1}{p^2}+1\right)}$$

$$\therefore \quad \frac{1}{p}\,dp = \frac{-1}{y}\,dy$$

Integrating both sides

$$\log p = -\log y + \log C$$

ie. $py = C$ ie $p = C/y \quad -(3)$

Using (3) in Eqn (2)

$$x = \frac{1}{2}\cdot y \times \frac{y}{C} - \frac{1}{2}\,y \times \frac{C}{y}$$

$$\boxed{x = \frac{y^2}{2C} - \frac{C}{2}}$$

6(c) Solve $y'' + 16y = 32\sec 2x$

$$(D^2+16)y = 32\sec 2x$$

Auxiliary Eqn : $m^2+16=0 \Rightarrow m = \pm 4i$

$$y_c = c_1 \cos 4x + c_2 \sin 4x$$

$$P.I. = \frac{1}{D^2+16} 32\sec 2x = 32 \cdot \frac{1}{(D+4i)(D-4i)} \sec 2x$$

$$= \frac{32}{8i}\left[\frac{1}{D-4i}\sec 2x - \frac{1}{D+4i}\sec 2x\right]$$

$$\frac{1}{D-4i}\sec 2x = e^{4ix}\int e^{-4ix}\sec 2x\, dx$$

$$= e^{4ix}\int (\cos 4x - i\sin 4x)\sec 2x\, dx$$

$$= e^{4ix}\int (2\cos^2 2x - 1 - i2\sin 2x\cos 2x)\sec 2x\, dx$$

$$= e^{4ix}\int (2\cos 2x - 2i\sin 2x - \sec 2x)\, dx$$

$$= e^{4ix}\left[\sin 2x + i\cos 2x - \log(\sec 2x + \tan 2x)\times\frac{1}{2}\right]$$

$$\frac{1}{D+4i}\sec 2x = e^{-4ix}\int e^{4ix}\sec 2x\,dx$$

$$= e^{-4ix}\int (\cos 4x + i\sin 4x)\sec 2x\,dx$$

$$= e^{-4ix}\int (2\cos^2 2x - 1 + 2i\sin 2x\cos 2x)\sec 2x\,dx$$

$$= e^{-4ix}\int (2\cos 2x + 2i\sin 2x - \sec 2x)\,dx$$

$$= e^{-4ix}\left[\sin 2x - i\cos 2x - \log(\sec 2x + \tan 2x)\times\frac{1}{2}\right]$$

Hence, General Solution,

$$y = y_c + y_p$$

$$y = c_1\cos 4x + c_2\sin 4x - 4i\,e^{4ix}\left(\sin 2x + i\cos 2x - \frac{1}{2}\log(\sec 2x + \tan 2x)\right) - 4i\,e^{-4ix}\left[\sin 2x - i\cos 2x - \frac{1}{2}\log(\sec 2x + \tan 2x)\right]$$

7(a) Solve, $(1+x)^2 y'' + (1+x) y' + y = 4\cos(\log(1+x))$

Put $\log(1+x) = z$, $D_1 = \frac{d}{dz}$

Then given D.E. can be written as

$$D_1(D_1 - 1) y + D_1 y + y = 4\cos z$$

$$(D_1^2 + 1) y = 4\cos z$$

The Auxiliary Eqn : $m_z^2 + 1 = 0$

$$m_z = \pm i$$

$$y_c(z) = c_1 \cos z + c_2 \sin z$$

$$y_c = c_1 \cos\log(1+x) + c_2 \sin\log(1+x)$$

$$y_p = P.I. = \frac{1}{D_1^2 + 1} 4\cos z$$

$$= 4 \frac{z}{\frac{d}{dD_1}(D_1^2+1)} \cos z$$

($\because$ Replacing D_1^2 with -1, denominator becomes zero)

$$= 2z \cdot \frac{1}{D_1} \cos z$$

$$= 2z \sin z = 2\log(1+x) \cdot \sin\log(1+x)$$

Gen Sol : $y = y_c + y_p$

$$y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x) \cdot \sin\log(1+x)$$

Solve the initial value problem

$$y'' - 5y' + 4y = e^{2t}$$

7(c) $\quad y(0) = 19/12, \quad y'(0) = 8/3.$

$$y'' - 5y' + 4y = e^{2t}$$

Taking laplace transform on both sides

$$\left[s^2 y(s) - s\,y(0) - y'(0)\right] - 5\left[s\,y(s) - y(0)\right] + 4y(s) = \mathcal{L}(e^{2t})$$

$$\left(s^2 y(s) - \frac{19}{12}s - \frac{8}{3}\right) - 5s\,y(s) + \frac{95}{12} + 4y(s) = \frac{1}{s-2}$$

$$y(s)\left[s^2 - 5s + 4\right] = \frac{1}{s-2} + \frac{19s}{12} - \frac{63}{12}$$

$$y(s)(s-1)(s-4) = \frac{19s^2 - 101s + 138}{12(s-2)}$$

$$y(s) = \frac{19s^2 - 101s + 138}{12(s-1)(s-2)(s-4)}$$

$$= \frac{1}{12}\left[\frac{56}{3}\cdot\frac{1}{(s-1)} - \frac{12}{2}\cdot\frac{1}{(s-2)} + \frac{38}{6}\frac{1}{(s-4)}\right]$$

Taking inverse laplace transformations

$$y(t) = \frac{14}{9}e^{t} - \frac{1}{2}e^{2t} + \frac{19}{36}e^{4t}$$

7(d)

Find α and β such that $x^{\alpha}y^{\beta}$ is an integrating factor of $(4y^2+3xy)dx-(3xy+2x^2)dy=0$ and solve the equation.

Multiplying the given DE with integrating factor $x^{\alpha}y^{\beta}$.

$$\left(4x^{\alpha}y^{\beta+2}+3x^{\alpha+1}\cdot y^{\beta+1}\right)dx-\left(3x^{\alpha+1}y^{\beta+1}+2x^{\alpha+2}\cdot y^{\beta}\right)dy=0$$

$$M\,dx+N\,dy=0. \quad \text{—} \ (1)$$

If this is exact D.E., then

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$4(\beta+2)x^{\alpha}y^{\beta+1}+3(\beta+1)x^{\alpha+1}\cdot y^{\beta}$$
$$=-3(\alpha+1)x^{\alpha}\cdot y^{\beta+1}-2(\alpha+2)x^{\alpha+1}\cdot y^{\beta}$$

$\therefore \ 4(\beta+2)=-3(\alpha+1) \Rightarrow 3\alpha+4\beta=-11$

$3(\beta+1)=-2(\alpha+2) \Rightarrow 2\alpha+3\beta=-7$

Solving for α and β we get

$$\alpha=-5, \quad \beta=1$$

Hence, the integrating factor is : $x^{-5}\cdot y^{1}$

i.e. $\dfrac{y}{x^5}$.

① becomes

$$\left(4x^{-5}y^{3}+3x^{-4}y^{2}\right)dx+\left(-3x^{-4}y^{2}-2x^{-3}y\right)dy=0.$$

This is exact D.E., its solution is given by

$$\int\left(4\cdot\frac{y^{3}}{x^{5}}+3\cdot\frac{y^{2}}{x^{4}}\right)dx+\int \cancel{rd}\,dy=0.$$

y-constant   excluding terms of x.

$$\frac{y^{3}}{x^{4}}+\frac{y^{3}}{x^{3}}=c$$

ie. $\boxed{y^{3}(1+x)=cx^{4}}$

Scanned with CamScanner

8(d) Find $f(y)$ such that

$$(2xe^y + 3y^2)dy + (3x^2 + f(y))dx = 0$$

is exact and hence solve.

Comparing with $M dx + N dy = 0$

$$M = 3x^2 + f(y)$$

$$N = 2xe^y + 3y^2$$

If D.E. is exact $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$f'(y) = 2e^y$$

$$\Rightarrow f(y) = 2e^y$$

$\therefore$ D.E becomes: $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$

Solution is given by

$$\underset{y = \text{constant}}{\int M\,dx} + \underset{\text{excluding terms of } x}{\int N\,dy} = 0$$

$$\int (3x^2 + 2e^y)dx + \int 3y^2\,dy = 0$$

$$\boxed{x^3 + 2xe^y + y^3 = c.}$$