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Multiple Integrals
(Double and Triple Integrals and Change of Order of Integration)

§1. Double Integrals.

The concept of double integral is an extension of the concept of definite integral to the case of two arguments (*i.e.* a two dimensional space). Let a function $f(x, y)$ of the independent variables x and y be continuous inside some domain (region) A and on its boundary. Divide the domain A into n subdomains $A_1, A_2, \ldots, A_n$ of areas $\delta A_1, \delta A_2, \ldots,$ δA_n. Let (x_r, y_r) be any point inside the rth elementary area δA_r. Form the sum

$$S_n = f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_r, y_r)\,\delta A_r + \ldots + f(x_n, y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r. \qquad \ldots(1)$$

Now take the limit of the sum (1) as $n \to \infty$ in such a way that the largest of the areas δA_r approaches to zero. This limit, if it exists, is called the **double integral** of the function $f(x, y)$ over the domain A. It is denoted by $\iint_A f(x, y)\, dA$ and is read as "the double integral of $f(x, y)$ over A".

Suppose the domain (region) A is divided into rectangular partitions by a network of lines parallel to the coordinate axes. Let dx be the length of a sub-rectangle and dy be its width so that $dx\, dy$ is an element of area in Cartesian coordinates. The integral $\iint f(x, y)\, dA$ is written as $\iint_A f(x, y)\, dx\, dy$ and is called the *double integral* of $f(x, y)$ over the region A.

§2. Properties of a double integral.

(I) If the region A is partitioned into two parts, say A_1 and A_2, then

$$\iint_A f(x, y)\, dx\, dy = \iint_{A_1} f(x, y)\, dx\, dy + \iint_{A_2} f(x, y)\, dx\, dy.$$

Similarly for a sub-division of A into three or more parts.

(II) The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term. Thus

$$\iint_A [f_1(x,y)+f_2(x,y)+f_3(x,y)+\dots]\, dx\, dy$$
$$=\iint_A f_1(x,y)\, dx\, dy+\iint_A f_2(x,y)\, dx\, dy+\iint_A f_3(x,y)\, dx\, dy+\dots$$

(iii) A constant factor may be taken outside the integral sign.

Thus $$\iint_A m f(x,y)\, dx\, dy = m \iint_A f(x,y)\, dx\, dy,$$

where m is a constant.

§ 3. Evaluation of Double Integrals.

(a) If the region A be given by the inequalities
$$a \le x \le b,\ c \le y \le d,$$
then the double integral
$$\iint_A f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dx\, dy$$
$$= \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx, \qquad \text{...(1)}$$
or $$\iint_A f(x,y)\, dx\, dy = \int_c^d \int_a^b f(x,y)\, dy\, dx$$
$$= \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy \qquad \text{...(2)}$$

i.e., in this case the order of integration is immaterial, provided the limits of integration are changed accordingly.

Important Note : In formula (1) the definite integral $\int_c^d f(x,y)\, dy$ is calculated first. During this integration x is regarded as a constant. While in the formula (2) the definite integral $\int_a^b f(x,y)\, dx$ is calculated first and during this integration y is regarded as a constant.

(b) If the region A is bounded by the curves
$$y=f_1(x),\ y=f_2(x),\ x=a \text{ and } x=b, \text{ then}$$
$$\iint_A f(x,y)\, dx\, dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dx\, dy$$
$$= \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\right] dx,$$
where the integration with respect to y is performed first treating x as a constant.

Similarly, if the region A is bounded by the curves
$x=f_1(y),\ x=f_2(y),\ y=c,\ y=d$, we have
$$\iint_A f(x,y)\, dx\, dy = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x,y)\, dy\, dx$$

$$= \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\, dx\right] dy,$$

where the integration with respect to x is performed first treating y as constant.

Remember. While evaluating double integrals, first integrate w.r.t. the variable having variable limits (treating the other variable as constant) and then integrate w.r.t. the variable with constant limits.

Remark. In the double integral $\int_a^b \int_c^d f(x,y)\, dx\, dy$, it is generally understood that the limits of integration c to d are those of y and the limits of integration a to b are those of x. However this is not a standard convention. Some authors regard these limits in the reverse order *i.e.* they regard the limits c to d as those of x and the limits a to b as those of y. So it is better to write this double integral as $\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dx\, dy$ so that there is no confusion about the limits. However in the double integral $\int_a^b \int_{f_1(x)}^{f_2(x)} F(x,y)\, dx\, dy$, there is no confusion about the limits. Obviously the variable limits are those of y because they are in terms of x and so the constant limits must be those of x. Here the first integration must be performed with respect to y regarding x as constant.

Solved Examples

Ex. 1. *Evaluate the following double integrals :*

(i) $\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$; (Kanpur 1980; Meerut 96P)

(ii) $\int_1^a \int_1^b \dfrac{dx\, dy}{xy}$;

(iii) $\int_1^2 \int_0^x \dfrac{dx\, dy}{x^2+y^2}$; (Gorakhpur 1983)

(iv) $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\, dy\, dx$; (Meerut 1993, 98)

(v) $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy$; (vi) $\int_1^2 \int_0^{3y} y\, dy\, dx$. (Agra 1974)

(vii) $\int_0^2 \int_0^{2x-4} \dfrac{y-1}{x+1}\, dx\, dy$ (Kanpur 1980)

Sol. (i) We have
$$\int_0^a \int_0^b (x^2+y^2)\, dx\, dy = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^{b} dx.$$
(integration w.r.t. y treating x as constant)

$= \int_0^a \left[bx^2 + \frac{b^3}{3}\right] dx = \left[b\frac{x^3}{3} + \frac{b^3}{3}x\right]_0^a = \frac{ba^3}{3} + \frac{b^3 a}{3}$

$= \frac{1}{3}(ab)(a^2+b^2).$

(ii) $\int_1^a \int_1^b \frac{dx\,dy}{xy} = \int_1^a \frac{1}{x}\left[\log y\right]_{y=1}^b dx,$

(inegrating w.r.t. y treating x as constant)

$= \int_1^a \frac{(\log b - \log 1)}{x} dx$

$= \log b \int_1^a \frac{1}{x} dx = (\log b)\left[\log x\right]_1^a = (\log b)(\log a - \log 1)$

$= (\log b).(\log a).$

(iii) $\int_1^2 \int_0^x \frac{dx\,dy}{x^2+y^2} = \int_1^2 \left[\int_0^x \frac{dy}{x^2+y^2}\right] dx$

$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{y=0}^x dx$

(integrating w.r.t. y treating x as constant)

$= \int_1^2 \left[\frac{1}{x}(\tan^{-1}1 - \tan^{-1}0)\right] dx = \frac{\pi}{4}\int_1^2 \frac{dx}{x} = \frac{\pi}{4}\left[\log x\right]_1^2$

$= \frac{1}{4}\pi\left[\log 2 - \log 1\right] = \frac{1}{4}\pi \log 2.$

(iv) $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx = \int_0^{\pi/2}\left[\int_{\pi/2}^{\pi} \cos(x+y)\,dx\right] dy$

$= \int_0^{\pi/2} \left[\sin(x+y)\right]_{x=\pi/2}^{\pi} dy,$

(integrating w.r.t. x treating y as constant)

$= \int_0^{\pi/2} \left[\sin(\pi+y) - \sin(\frac{1}{2}\pi + y)\right] dy$

$= \int_0^{\pi/2} (-\sin y - \cos y)\, dy$

$= \left[\cos y - \sin y\right]_0^{\pi/2} = (0-1) - (1-0) = -2.$

(v) $\int_0^1\int_0^{x^2} e^{y/x}\,dx\,dy = \int_0^1 \left[xe^{y/x}\right]_{y=0}^{x^2} dx,$

(integrating w.r.t. y treating x as constant)

$= \int_0^1 \left[xe^{x^2/x} - xe^{0/x}\right] dx = \int_0^1 (xe^x - x)\, dx$

$= \left[xe^x\right]_0^1 - \int_0^1 e^x dx - \left[\frac{x^2}{2}\right]_0^1$

$= e - \left[e^x\right]_0^1 - \frac{1}{2} = e - (e-1) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$

(vi) We have

$\int_1^2\int_0^{3y} y\,dy\,dx = \int_1^2 y\left[x\right]_0^{3y} dy,$

integrating w.r.t. x regarding y as a constant

$= \int_1^2 y\,[3y - 0]\,dy = 3\int_1^2 y^2 dy = 3\left[\frac{y^3}{3}\right]_1^2 = \left[y^3\right]_1^2 = 8 - 1 = 7.$

(vii) The given integral

$I = \int_{x=0}^2 \int_{y=0}^{2x-4} \frac{2y-1}{x+1}\,dx\,dy$

$= \int_0^2 \frac{1}{x+1}\left[y^2 - y\right]_{y=0}^{2x-4} dx,$

integrating w.r.t. y treating x as constant

$= \int_0^2 \frac{1}{x+1}\left[(2x-4)^2 - (2x-4)\right] dx = \int_0^2 \frac{4x^2 - 18x + 20}{x+1} dx$

$= \int_0^2 \left[4x - 22 + \frac{42}{x+1}\right] dx,$ dividing the Nr. by the Dr.

$= \left[2x^2 - 22x + 42\log(x+1)\right]_0^2$

$= 8 - 44 + 42\log 3 = -36 + 42\log 3.$

Ex. 2. *Evaluate*

(i) $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy.$

(ii) $\int_0^1\int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}.$

(Meerut 1991; Rohilkhand 79; Kanpur 84; Agra 50, 88)

(iii) $\int_0^2\int_0^{\sqrt{(4+x^2)}} \frac{dx\,dy}{4+x^2+y^2}.$

(iv) $\int_0^1\int_0^{\sqrt{(1-y^2)}} 4y\,dy\,dx.$

(v) $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy.$

(vi) $\int_2^3\int_0^{y-1} \frac{dy\,dx}{y}.$

Sol. (i) $\int_0^3\int_1^2 xy(1+x+y)\,dx\,dy$

$= \int_0^3 \left[x\cdot\frac{y^2}{2} + x^2\cdot\frac{y^2}{2} + x\cdot\frac{y^3}{3}\right]_{y=1}^2 dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_0^3 \left[\frac{x}{2}(4-1) + \frac{x^2}{2}(4-1) + \frac{x}{3}(8-1)\right] dx$$

$$= \int_0^3 \left[\left(\frac{3}{2} + \frac{7}{3}\right)x + \frac{3}{2}x^2\right] dx = \left[\frac{23}{6}\cdot\frac{x^2}{2} + \frac{3}{2}\cdot\frac{x^3}{3}\right]_0^3$$

$$= \frac{23}{6}\cdot\frac{9}{2} + \frac{27}{2} = \frac{123}{4} = 30\frac{3}{4}.$$

(ii) $$\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}$$

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(1+x^2)}}\right]_{y=0}^{\sqrt{(1+x^2)}} dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}}\left[\tan^{-1}1 - \tan^{-1}0\right] dx = \frac{\pi}{4}\int_0^1 \frac{dx}{\sqrt{(1+x^2)}}$$

$$= \frac{\pi}{4}\left[\log\{x + \sqrt{(1+x^2)}\}\right]_0^1 = \frac{\pi}{4}\log(1+\sqrt{2}).$$

(iii) The given integral

$$I = \int_{x=0}^2 \int_{y=0}^{\sqrt{(4+x^2)}} \frac{dx\,dy}{(4+x^2)+y^2}$$

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}}\left[\tan^{-1}\frac{y}{\sqrt{(4+x^2)}}\right]_{y=0}^{\sqrt{(4+x^2)}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}}[\tan^{-1}1 - \tan^{-1}0]\, dx = \frac{\pi}{4}\int_0^2 \frac{dx}{\sqrt{(4+x^2)}}$$

$$= \frac{\pi}{4}\left[\log\{x + \sqrt{(4+x^2)}\}\right]_0^2 = \frac{\pi}{4}[\log(2+2\sqrt{2}) - \log 2]$$

$$= \frac{\pi}{4}\log\frac{2+2\sqrt{2}}{2} = \frac{\pi}{4}\log(1+\sqrt{2}).$$

(iv) The given integral

$$I = \int_{y=0}^1 \int_{x=0}^{\sqrt{(1-y^2)}} 4y\,dy\,dx$$

$$= \int_0^1 4y\left[x\right]_{x=0}^{\sqrt{(1-y^2)}} dy,$$

integrating w.r.t. x treating y as constant

$$= 4\int_0^1 y\sqrt{(1-y^2)}\,dy = 4\int_0^1 \left(-\tfrac{1}{2}\right).(1-y^2)^{1/2}(-2y)\,dy$$

$$= -2\cdot\frac{2}{3}\left[(1-y^2)^{3/2}\right]_0^1,$$

by power formula

$$[0 - 1] = \tfrac{4}{3}.$$

(v) The given integral

$$I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[x^2y + \tfrac{1}{3}y^3\right]_{y=x}^{\sqrt{x}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^1 \left[x^2\sqrt{x} + \tfrac{1}{3}x\sqrt{x} - x^3 - \tfrac{1}{3}x^3\right] dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{1}{3}x^{3/2} - \frac{4}{3}x^3\right] dx$$

$$= \left[\frac{2}{7}x^{7/2} + \frac{1}{3}\cdot\frac{2}{5}x^{5/2} - \frac{1}{3}x^4\right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30+14-35}{105} = \frac{3}{35}.$$

(vi) The given integral

$$I = \int_{y=2}^3 \int_{x=0}^{y-1} \frac{dy\,dx}{y}$$

$$= \int_2^3 \frac{1}{y}\left[x\right]_{x=0}^{y-1} dy,$$

integrating w.r.t. x treating y as constant

$$= \int_2^3 \frac{y-1}{y}\,dy = \int_2^3 \left(1 - \frac{1}{y}\right) dy = \left[y - \log y\right]_2^3$$

$$= 3 - \log 3 - 2 + \log 2 = 1 - \log\tfrac{3}{2}.$$

Ex. 3. *Evaluate*

(i) $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\,dy\,dx.$

(Meerut 1991 P, 93P, 97; Gorakhpur 87; Kanpur 86; Agra 86)

(ii) $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} (a^2-x^2-y^2)\,dy\,dx.$ **(Rohilkhand 1990; Agra 84)**

(iii) $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} (x+y)\,dx\,dy.$

Sol. (i) Here the variable limits are those of x and so the first integration must be performed w.r.t. x taking y as constant.

$$\therefore \int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\,dy\,dx$$

$$= \int_0^a \left[\int_0^{\sqrt{(a^2-y^2)}} \sqrt{\{(a^2-y^2)-x^2\}}\,dx\right] dy$$

$$= \int_0^a \left[\frac{x\sqrt{(a^2-y^2-x^2)}}{2} + \frac{(a^2-y^2)}{2}\sin^{-1}\frac{x}{\sqrt{(a^2-y^2)}}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^a \left[0 + \frac{a^2 - y^2}{2} \cdot \frac{\pi}{2}\right] dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{1}{6}\pi a^3$$

(ii) The given integral

$$I = \int_{y=0}^a \int_{x=0}^{\sqrt{(a^2-y^2)}} [(a^2 - y^2) - x^2]\, dy\, dx$$

$$= \int_0^a \left[(a^2 - y^2)x - \tfrac{1}{3}x^3\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy,$$

integrating w.r.t. x treating y as constant

$$= \int_0^a [(a^2 - y^2)^{3/2} - \tfrac{1}{3}(a^2 - y^2)^{3/2}]\, dy$$

$$= \tfrac{2}{3}\int_0^a (a^2 - y^2)^{3/2}\, dy$$

$$= \tfrac{2}{3}\int_0^{\pi/2} a^3 \cos^3\theta . a\cos\theta\, d\theta, \text{ putting } y = a\sin\theta \text{ so that } dy = a\cos\theta\, d\theta$$

$$= \tfrac{2}{3}a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= \frac{2}{3}a^4 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= \pi a^4/8.$$

(iii) The given integral

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{(a^2-x^2)}} (x + y)\, dx\, dy$$

$$= \int_0^a \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{\sqrt{(a^2-x^2)}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^a [x\sqrt{(a^2 - x^2)} + \tfrac{1}{2}(a^2 - x^2)]\, dx$$

$$= \int_0^a [-\tfrac{1}{2}(a^2 - x^2)^{1/2}(-2x) + \tfrac{1}{2}(a^2 - x^2)]\, dx$$

$$= \left[-\tfrac{1}{2} \cdot \tfrac{2}{3}(a^2 - x^2)^{3/2}\right]_0^a + \tfrac{1}{2}\left[a^2 x - \tfrac{1}{3}x^3\right]_0^a, \quad \text{by power formula}$$

$$= 0 + \tfrac{1}{3}a^3 + \tfrac{1}{2}[a^3 - \tfrac{1}{3}a^3] = \tfrac{2}{3}a^3.$$

Ex. 4. *Evaluate* $\int_0^2 \int_0^{\sqrt{(2x - x^2)}} x\, dx\, dy.$

Sol. Here the variable limits are those of y and so the first integration must be performed w.r.t. y regarding x as constant.

$$\therefore \int_0^2 \int_0^{\sqrt{(2x-x^2)}} x\, dx\, dy = \int_0^2 x\left[y\right]_0^{\sqrt{(2x-x^2)}} dx$$

$$= \int_0^2 x\sqrt{(2x - x^2)}\, dx = \int_0^2 x\sqrt{\{1 - (1 - x)^2\}}\, dx. \quad \text{(Note)}$$

Now put $(1 - x) = t$ so that $-dx = dt$

Also when $x = 0$, $t = 1$ and when $x = 2$, $t = -1$.

$\therefore$ the required integral $= \int_{-1}^1 (1 - t)\sqrt{(1 - t^2)}\, dt$

$$= \int_{-1}^1 \sqrt{(1 - t^2)}\, dt - \int_{-1}^1 t\sqrt{(1 - t^2)}\, dt$$

$$= 2\int_0^1 \sqrt{(1 - t^2)}\, dt - 0,$$

the second integral vanishes because the integrand is an odd function of t

$$= 2\left[\frac{t}{2}\sqrt{(1 - t^2)} + \tfrac{1}{2}\sin^{-1} t\right]_0^1$$

$$= 2[0 + \tfrac{1}{2} \cdot \tfrac{1}{2}\pi] = \tfrac{1}{2}\pi.$$

Ex. 5. *Evaluate* $\displaystyle\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{\{(1 - x^2)(1 - y^2)\}}}.$

Sol. We have $\displaystyle\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{\{(1 - x^2)(1 - y^2)\}}}$

$$= \int_{y=0}^1 \frac{1}{\sqrt{(1 - y^2)}}\left[\int_{x=0}^1 \frac{1}{\sqrt{(1 - x^2)}}\, dx\right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{(1 - y^2)}}\left[\sin^{-1} x\right]_0^1 dy,$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^1 \frac{\pi}{2\sqrt{(1 - y^2)}}\, dy = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}.$$

Ex. 6. *Show that*

$$\int_1^2 \int_3^4 (xy + e^y)\, dx\, dy = \int_3^4 \int_1^2 (xy + e^y)\, dy\, dx.$$

Sol. Integral on the L.H.S.

$$= \int_1^2 \left[\int_3^4 (xy + e^y)\, dy\right] dx$$

$$= \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2 [8x + e^4 - \tfrac{9}{2}x - e^3]\, dx$$

$$= \int_1^2 \left[\frac{7}{2}x + e^4 - e^3\right] dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \tfrac{7}{4} - (e^4 - e^3) = \tfrac{21}{4} + e^4 - e^3.$$

And the integral on the R.H.S.

$$= \int_3^4 \left[\int_1^2 (xy + e^y)\, dx\right] dy$$

$$= \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy = \int_3^4 \left[2y + 2e^y - \frac{y}{2} - e^y\right] dy$$

$$= \int_3^4 \left[\frac{3y}{2} + e^y\right] dy = \left[\frac{3y^2}{4} + e^y\right]_3^4$$

$= 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$. Hence the result.

Ex. 7. (a). *Show that*

$$\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2\int_0^{x/2} x\,dx\,dy.$$

(Agra 1987; Kanpur 79; Meerut 96 BP)

Sol. We have

$$\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2 \left[y\int_0^{y/2} dx\right] dy = \int_1^2 y\,\left[x\right]_0^{y/2} dy,$$

(integrating w.r.t. x treating y as a constant)

$$= \int_1^2 y\left[\frac{y}{2} - 0\right] dy = \frac12\int_1^2 y^2\,dy = \frac12\left[\frac{y^3}{3}\right]_1^2 = \frac16[8-1] = \frac76 \quad ...(1)$$

Again $\int_1^2\int_0^{x/2} x\,dx\,dy = \int_1^2 x\left[\int_0^{x/2} dy\right]dx = \int_1^2 x\,\left[y\right]_0^{x/2} dx,$

(integrating w.r.t. y treating x as a constant)

$$= \int_1^2 x\left[\frac{x}{2}-0\right]dx = \frac12\int_1^2 x^2\,dx = \frac12\left[\frac{x^3}{3}\right]_1^2 = \frac16(8-1) = \frac76 \quad ...(2)$$

From (1) and (2), we see that

$$\int_1^2\int_0^{y/2} y\,dy\,dx = \int_1^2\int_0^{x/2} x\,dx\,dy.$$

Ex. 7. (b) *Show that*

$$\int_0^1 dx\int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy\int_0^1\frac{x-y}{(x+y)^3}\,dx.$$

Find the values of the two integrals. **(Kanpur 1978, Gorakhpur 82)**

Sol. The integral on the L.H.S.

$$= \int_0^1 dx\int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,dy = \int_0^1 dx\int_0^1\left\{\frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2}\right\}dy$$

$$= \int_0^1\left[\frac{-x}{(x+y)^2} + \frac{1}{x+y}\right]_0^1 dx,$$

(integrating w.r.t. y regarding x as constant)

$$= \int_0^1\left[-\frac{x}{(1+x)^2} + \frac1x + \frac{1}{1+x} - \frac1x\right]dx = \int_0^1\frac{dx}{(1+x)^2} = \left[\frac{-1}{1+x}\right]_0^1$$

$= -\frac12 + 1 = \frac12.$

And the integral on the R.H.S.

$$= \int_0^1 dy\int_0^1 \frac{(x+y)-2y}{(x+y)^3}\,dx$$

$$= \int_0^1 dy\int_0^1\left\{\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right\}dx = \int_0^1\left[\frac{-1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \int_0^1\left[\frac{-1}{1+y} + \frac1y + \frac{y}{(1+y)^2} - \frac1y\right]dy = -\int_0^1\frac{dy}{(1+y)^2} = \left[\frac{1}{1+y}\right]_0^1$$

$= \frac12 - 1 = -\frac12.$ Thus the two integrals are not equal.

Examples on the region of integration (*Double Integration*)

Ex. 8. *Evaluate* $\iint x^2y^2\,dx\,dy$ *over the region* $x^2+y^2 \le 1$.

Sol. Let R denote the region $x^2 + y^2 \le 1$. Then R is the region in the xy-plane bounded by the circle $x^2+y^2=1$. The limits of integration for this region can be expressed either as

$$-1 \le x \le 1,\ -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$$

or as

$$-\sqrt{(1-y^2)} \le x \le \sqrt{(1-y^2)},\ -1 \le y \le 1.$$

Because from the equation of the circle $x^2+y^2=1$, we have $x^2 = 1-y^2$ so that $x = \pm\sqrt{(1-y^2)}$. Thus for a fixed value of y, x varies from $-\sqrt{(1-y^2)}$ to $\sqrt{(1-y^2)}$ in the area bounded by the circle $x^2+y^2=1$. Also y varies from -1 to 1 to cover the whole area of the circle $x^2+y^2=1$. Therefore if the first integration is to be performed w.r.t. x regarding y as constant, then

$$\iint_R x^2y^2\,dx\,dy = \int_{y=-1}^{1}\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2y^2\,dx\,dy$$

$$= \int_{y=-1}^{1} y^2\left[\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2\,dx\right]dy$$

$$= \int_{-1}^{1} y^2\left[2\int_{x=0}^{\sqrt{(1-y^2)}} x^2\,dx\right]dy = \int_{-1}^{1} 2y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}} dy$$

$$= \int_{-1}^{1}\tfrac23 y^2(1-y^2)^{3/2}\,dy = 2\cdot\tfrac23\int_0^1 y^2(1-y^2)^{3/2}\,dy.$$

Put $y = \sin\theta$ so that $dy = \cos\theta\,d\theta$;

when $y=0$, $\theta=0$ and when $y=1$, $\theta=\pi/2$.

$$\therefore\ \iint_R x^2y^2\,dx\,dy = \tfrac43\int_0^{\pi/2}\sin^2\theta\,(1-\sin^2\theta)^{3/2}.\cos\theta\,d\theta$$

$$= \frac43\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta = \frac43\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

Ex. 9. *Evaluate* $\iint x^2y^3\,dx\,dy$ *over the circle* $x^2+y^2=a^2$.

Sol. If the first integration is to be performed w.r.t. y regarding x as constant, then the region of integration R can be expressed as

$$-a \le x \le a,\ -\sqrt{(a^2-x^2)} \le y \le \sqrt{(a^2-x^2)}.$$

$$\therefore\ \iint_R x^2y^3\,dx\,dy = \int_{x=-a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} x^2y^3\,dx\,dy = 0.$$

[$\because y^3$ is an odd function of y] **(Note)**

Ex. 10. *Find by double integration the area of the region bounded by the circle* $x^2 + y^2 = a^2$.

Sol. The area of a small element situated at any point (x, y) is $dx\,dy$. To find the area bounded by the circle $x^2 + y^2 = a^2$, the region of integration R can be expressed as

$$-a \le y \le a,\ -\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)},$$

where the first integration to be performed w.r.t. x regarding y as constant.

$\therefore$ the required area

$$= \iint_R dx\,dy = \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} 1.\,dx\,dy$$

$$= \int_{-a}^{a} \left[2 \int_0^{\sqrt{(a^2-y^2)}} 1.\,dx\right] dy = 2 \int_{-a}^{a} \left[x\right]_0^{\sqrt{(a^2-y^2)}} dy$$

$$= 2 \int_{-a}^{a} \sqrt{(a^2 - y^2)}\,dy = 2.2 \int_0^a \sqrt{(a^2 - y^2)}\,dy$$

$$= 4 \left[\frac{y\sqrt{(a^2 - y^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}\right]_0^a = 4 \left[0 + \frac{a^2}{2} \sin^{-1} 1\right] \qquad \textbf{(Note)}$$

$$= 4.\tfrac{1}{2} a^2.\tfrac{1}{2}\pi = \pi a^2.$$

Ex.11. *Evaluate* $\iint (x + y + a)\,dx\,dy$ *over the circular area* $x^2 + y^2 \le a^2$.

Sol. Here the region of integration R can be expressed as

$$-a \le y \le a,\ -\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)},$$

where the first integration is to be performed w.r.t. x regarding y as constant.

$$\therefore \quad \iint_R (x + y + a)\,dx\,dy$$

$$= \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (x + y + a)\,dx\,dy$$

$$= \int_{-a}^{a} \left[\frac{x^2}{2} + (y + a)x\right]_{-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} dy,$$

[integrating w.r.t. x treating y as a constant]

$$= \int_{-a}^{a} \left[\frac{a^2 - y^2}{2} + (y + a)\sqrt{(a^2 - y^2)} - \frac{a^2 - y^2}{2} + (y + a)\sqrt{(a^2 - y^2)}\right] dy$$

$$= \int_{-a}^{a} 2(y + a)\sqrt{(a^2 - y^2)}\,dy$$

$$= \int_{-a}^{a} 2y.\sqrt{(a^2 - y^2)}\,dy + 2a \int_{-a}^{a} \sqrt{(a^2 - y^2)}\,dy$$

$$= 0 + 2a.\,2 \int_0^a \sqrt{(a^2 - y^2)}\,dy,$$

first integral vanishes because the integrand is an odd function of y

$$= 4a \left[\frac{y\sqrt{(a^2 - y^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a}\right]_0^a$$

$$= 4a \left[0 + \tfrac{1}{2} a^2 \sin^{-1} 1 - 0\right]$$

$$= 4a.\tfrac{1}{2} a^2.\tfrac{1}{2}\pi = \pi a^3.$$

Ex. 12 (a). *Evaluate* $\iint x^2y^2\,dx\,dy$ *over the region bounded by* $x = 0,\ y = 0$ *and* $x^2 + y^2 = 1$.

Sol. The given region for integration is the area of the positive quadrant of the circle $x^2 + y^2 = 1$ in the xy-plane. This region R can be expressed either as

$$0 \le x \le \sqrt{(1 - y^2)},\ 0 \le y \le 1$$

$$0 \le y \le \sqrt{(1 - x^2)},\ 0 \le x \le 1$$

$$\therefore \quad \iint_R x^2y^2\,dx\,dy = \int_{y=0}^{1} \int_{x=0}^{\sqrt{(1-y^2)}} x^2y^2\,dx\,dy,$$

the first integration to be performed w.r.t. x regarding y as constant

$$= \int_0^1 y^2 \left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}} dy = \int_0^1 \tfrac{1}{3} y^2 (1 - y^2)^{3/2}\,dy.$$

Put $y = \sin\theta$ so that $dy = \cos\theta\,d\theta$.

When $y = 0,\ \theta = 0$ and when $y = 1,\ \theta = \pi/2$.

$$\therefore \quad \iint_R x^2y^2\,dx\,dy = \int_0^{\pi/2} \tfrac{1}{3} \sin^2\theta\,(1 - \sin^2\theta)^{3/2}.\cos\theta\,d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^2\theta \cos^4\theta\,d\theta = \frac{1}{3} \cdot \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{\pi}{96}.$$

Ex. 12 (b). *Evaluate* $\iint x^2y^2\,dx\,dy$ *over the region* $x^2 + y^2 \le 1$. **(Gorakhpur 1984)**

Sol. Here the given region of integration R is the whole area of the circle $x^2 + y^2 = 1$. This region R can be expressed as

$$-\sqrt{(1 - y^2)} \le x \le \sqrt{(1 - y^2)},\ -1 \le y \le 1.$$

$$\therefore \quad \iint_R x^2y^2\,dx\,dy = \int_{y=-1}^{1} \int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2y^2\,dx\,dy$$

$$= 2 \int_{y=-1}^{1} \int_{x=0}^{\sqrt{(1-y^2)}} x^2y^2\,dx\,dy,$$

by a property of definite integrals because x^2 is an even function of x

$$= 2 \int_{-1}^{1} \tfrac{1}{3} y^2 (1 - y^2)^{3/2}\,dy, \text{ proceeding as in Ex. 12 (a)}$$

$$= \tfrac{4}{3} \int_0^1 y^2 (1 - y^2)^{3/2}\,dy,$$

because $y^2 (1 - y^2)^{3/2}$ is an even function of y

$= \frac{\pi}{24}$, proceeding as in Ex. 12 (a).

Ex. 13. *Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$, by double integration.*

Sol. From the equation of the ellipse, we have

$$\frac{y}{b} = \pm \sqrt{\left\{1 - \frac{x^2}{a^2}\right\}}.$$

So the region of integration R to cover the area of the ellipse can be considered as bounded by

$$y = -b\sqrt{(1 - x^2/a^2)},\ y = b\sqrt{(1 - x^2/a^2)},\ x = -a \text{ and } x = a.$$

Therefore the required area of the ellipse

$$= \iint_R dx\, dy = \int_{x=-a}^{a} \int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} 1.\, dx\, dy$$

$$= \int_{-a}^{a} \left[2\int_0^{b\sqrt{(1-x^2/a^2)}} 1.\, dy\right] dx = 2\int_{-a}^{a} \left[y\right]_0^{b\sqrt{(1-x^2/a^2)}} dx$$

$$= 2\int_{-a}^{a} b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}\, dx = 2.2 \int_0^a b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}\, dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{(a^2 - x^2)}\, dx = \frac{4b}{a}\left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\{\sin^{-1} 1 - \sin^{-1} 0\}\right] = \frac{4b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{b} = \pi ab.$$

Ex. 14. *Compute the value of $\iint_R y\, dx\, dy$, where R is the region in the first quadrant bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

Sol. If the first integration is to be performed w.r.t. y regarding x as a constant, then the given region of integration can be expressed as

$$0 \le x \le a,\ 0 \le y \le b\sqrt{(1 - x^2/a^2)}.$$

$$\therefore \iint_R y\, dx\, dy = \int_{x=0}^{a} \int_{y=0}^{b\sqrt{(1-x^2/a^2)}} y\, dx\, dy$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{b\sqrt{(1-x^2/a^2)}} dx = \frac{1}{2}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2 - x^2)\, dx = \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{b^2}{2a^2} \cdot \frac{2a^3}{3} = \frac{ab^2}{3}.$$

Ex. 15. *Evaluate $\iint (x + y)^2\, dx\, dy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. Hence find the mass of an elliptic plate whose density per unit area is given by $\rho = k(x + y)^2$.*

Sol. The region of integration can be considered as bounded by

$$y = -b\sqrt{(1 - x^2/a^2)},\ y = b\sqrt{(1 - x^2/a^2)},\ x = -a \text{ and } x = a.$$

$$\iint (x+y)^2\, dx\, dy = \int_{-a}^{a}\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} (x^2 + y^2 + 2xy)\, dx\, dy,$$

first integration to be performed w.r.t. y regarding x as a constant

$$= \int_{-a}^{a} 2\int_0^{b\sqrt{(1-x^2/a^2)}} (x^2 + y^2)\, dx\, dy,$$

[$\because$ $2xy$ being an odd function of y, its integration under the given limits of y is 0]

$$= 2\int_{-a}^{a}\left[x^2y + \frac{y^3}{3}\right]_0^{b\sqrt{(1-x^2/a^2)}} dx$$

$$= 2\int_{-a}^{a}\left\{x^2b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} + \frac{b^3}{3}\left(1 - \frac{x^2}{a^2}\right)^{3/2}\right\} dx$$

$$= 4\int_0^a \left\{x^2b\sqrt{\left(1 - \frac{x^2}{a^2}\right)} + \frac{b^3}{3}\left(1 - \frac{x^2}{a^2}\right)^{3/2}\right\} dx$$

$$= 4b\int_0^{\pi/2}\left\{a^2\sin^2\theta\cos\theta + \frac{b^2}{3}\cos^3\theta\right\} a\cos\theta\, d\theta,$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$

$$= 4ab\int_0^{\pi/2}\left\{a^2\sin^2\theta\cos^2\theta + \frac{b^2}{3}\cos^4\theta\right\} d\theta$$

$$= 4ab\left[a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{b^2}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right]$$

$$= 4ab\left[a^2 \cdot \frac{1.1}{4.2} \cdot \frac{\pi}{2} + \frac{b^2}{3} \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}\right], \text{ [by Walli's formula]}$$

$$= 4ab\left[\frac{1}{16}\pi a^2 + \frac{1}{16}\pi b^2\right] = \frac{1}{4}\pi ab(a^2 + b^2).$$

The mass of an elliptic plate whose density is given by $\rho = k(x + y)^2$

$= \iint_A k(x + y)^2\, dx\, dy$, where the integration is to be performed over the area A of the ellipse

$= k.\frac{1}{4} \cdot \pi ab(a^2 + b^2)$.

Ex. 16. *Evaluate $\iint xy\, dx\, dy$ over the region in the positive quadrant for which $x + y \le 1$.* **(Kanpur 1983)**

Sol. The region of integration is the area bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

To cover this region of integration R, x varies from 0 to 1 and y varies from 0 to $1 - x$.

$$\therefore \iint_R xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\,dx\,dy = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \tfrac{1}{2}\int_0^1 x(1-x)^2\,dx = \tfrac{1}{2}\int_0^1 x(1-2x+x^2)\,dx = \tfrac{1}{2}\left[\frac{x^2}{2} - 2\cdot\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right] = \frac{1}{24}.$$

Ex. 17. *Evaluate* $\iint e^{2x+3y}\,dx\,dy$ *over the triangle bounded by* $x=0,\ y=0$ *and* $x+y=1$. **(Kanpur 1985)**

Sol. The given region of integration R can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1-x,$$

where the first integration is to be performed w.r.t. y regarding x as a constant.

$$\therefore \iint_R e^{x+3y}\,dx\,dy = \int_0^1\int_0^{1-x} e^{2x+3y}\,dx\,dy$$

$$= \int_0^1 \left[\frac{e^{2x+3y}}{3}\right]_0^{1-x} dx = \tfrac{1}{3}\int_0^1 [e^{3-x} - e^{2x}]\,dx$$

$$= \tfrac{1}{3}\left[-e^{3-x} - \frac{e^{2x}}{2}\right]_0^1 = -\tfrac{1}{3}[(e^2-e^3) + \tfrac{1}{2}(e^2-e^0)]$$

$$= -\tfrac{1}{3}[-e^2(e-1) + \tfrac{1}{2}(e+1)(e-1)] = \tfrac{1}{3}(e-1)[e^2 - \tfrac{1}{2}(e+1)]$$

$$= \tfrac{1}{6}(e-1)(2e^2-e-1) = \tfrac{1}{6}(e-1)\{(e-1)(2e+1)\}$$

$$= \tfrac{1}{6}(e-1)^2(2e+1).$$

Ex. 18 (a). *Evaluate* $\iint (x^2+y^2)\,dx\,dy$ *over the region in the positive quadrant for which* $x+y \le 1$. **(Meerut 1990 S; Gorakhpur 82)**

Sol. The region of integration R is the area bounded by the coordinate axes and the straight line $x+y=1$. Therefore the region R is bounded by $y=0,\ y=1-x$ and $x=0,\ x=1$.

Therefore

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dx\,dy,$$

the first integration to be performed w.r.t. y regarding x as constant

$$= \int_0^1 \left[x^2y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{3\times 4}\right]_0^1 = \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{12}\right] = \frac{1}{6}.$$

Ex. 18 (b). *Evaluate* $\iint_A (x^2+y^2)\,dx\,dy$, *where* A *is the region bounded by* $x=0,\ y=0,\ x+y=1$. **(Agra 1985; Kanpur 81, 82)**

Sol. The region A is bounded by $y=0,\ y=1-x$ and $x=0$. Now proceed as in Ex. 18 (a).

Ex. 19. *Evaluate* $\iint xy(x+y)\,dx\,dy$ *over the area between* $y=x^2$ *and* $y=x$. **(Gorakhpur 1985)**

Sol. Draw the given curves $y=x^2$ and $y=x$ in the same figure. The two curves intersect at the points whose abscissae are given by $x^2=x$ or $x(x-1)=0$ i.e., $x=0$ or 1. When $0<x<1$, we have $x>x^2$. So the area of integration can be considered as lying between the curves $y=x^2,\ y=x,\ x=0$ and $x=1$.

Therefore the required integral

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dx\,dy = \int_0^1\left[\int_{x^2}^{x}(x^2y+xy^2)\,dy\right]dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx = \int_0^1 \left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right) - \left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx = \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{9}{168} = \frac{3}{56}.$$

Ex. 20 (a). *Find by double integration the area lying between the parabola* $y=4x-x^2$ *and the line* $y=x$. **(Meerut 1989; Agra 80, 87)**

Sol. Solving $y=4x-x^2$ and $y=x$ for x, we have

$$4x-x^2=x \text{ or } x^2-3x=0 \text{ or } x(x-3)=0 \text{ i.e., } x=0 \text{ or } 3.$$

Thus the curves $y=4x-x^2$ and $y=x$ interesect at the points where $x=0$ and $x=3$. When $0<x<3$, we have $4x-x^2>x$.

So the required area can be considered as lying between the curves $y=x,\ y=4x-x^2,\ x=0$ and $x=3$.

Therefore the required area $= \int_{x=0}^{3}\int_{y=x}^{4x-x^2} dx\,dy$

$$= \int_0^3 \left[y\right]_x^{4x-x^2} dx = \int_0^3 (4x-x^2-x)\,dx = \int_0^3 (3x-x^2)\,dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = 27\cdot\frac{1}{6} = \frac{9}{2}.$$

Ex. 20 (b). *Prove by the method of double integration that the area lying between the parabolas* $y^2=4ax$ *and* $x^2=4ay$ *is* $\frac{16}{3}a^2$.

(Kanpur 1984, 88; Agra 79; U.P. P.C.S. 94)

Sol. Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissae are given by $(x^2/4a)^2 = 4ax$ i.e., $x(x^3-64a^3)=0$ i.e., $x=0$ and $x^3=64a^3$. Thus the two parabolas intersect at the points where $x=0$ and $x=4a$.

Now the area of a small element situated at any point $(x, y) = dx\,dy$.

$\therefore$ the required area $= \int_{x=0}^{4a} \int_{y=x^2/4a}^{\sqrt{(4ax)}} dx\,dy = \int_0^{4a} \left[y\right]_{x^2/4a}^{\sqrt{(4ax)}} dx$

$= \int_0^{4a} \left[2\sqrt{a}.x^{1/2} - \frac{1}{4a}.x^2\right] dx = \left[2\sqrt{a}.x^{3/2}.\frac{2}{3} - \frac{1}{4a}.\frac{x^3}{3}\right]_0^{4a}$

$= \frac{4}{3}\sqrt{a}.(4a)^{3/2} - \frac{1}{12a}.64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$

Ex. 21 (a). *Evaluate* $\iint y\,dx\,dy$ *over the area between the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$.

Sol. The two parabolas intersect at the points whose abscissae are given by $(\frac{1}{4}x^2)^2 = 4x$ or $x(x^3 - 64) = 0$ *i.e.*, $x = 0$ or 4. When $0 < x < 4$, we have $2\sqrt{x} > \frac{1}{4}x^2$. Therefore the given region of integration can be expressed as

$0 \le x \le 4, \frac{1}{4}x^2 \le y \le 2\sqrt{x}$.

$\therefore$ the required integral $= \int_{x=0}^{4} \int_{y=x^2/4}^{2\sqrt{x}} y\,dx\,dy$

$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx = \int_0^4 \left[2x - \frac{x^4}{32}\right] dx = \left[\frac{2x^2}{2} - \frac{x^5}{32 \times 5}\right]_0^4$

$= 16 - \frac{32}{5} = \frac{48}{5}$.

Ex. 21 (b). *When the region of integration A is the triangle given by* $y = 0$, $y = x$ *and* $x = 1$, *show that*

$$\iint_A \sqrt{(4x^2 - y^2)}\,dx\,dy = \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

(Agra 1986)

Sol. In the diagram draw the straight lines $y = 0$, $y = x$ and $x = 1$. Then we observe that the region of integration A can be expressed as

$0 \le y \le x, 0 \le x \le 1$.

$\therefore \iint_A \sqrt{(4x^2 - y^2)}\,dx\,dy = \int_{x=0}^{1} \int_{y=0}^{x} \sqrt{(4x^2 - y^2)}\,dx\,dy$

$= \int_0^1 \left[\frac{y}{2}\sqrt{(4x^2 - y^2)} + 2x^2 \sin^{-1}\frac{y}{2x}\right]_{y=0}^{x} dx,$

integrating w.r.t. y treating x as constant

$= \int_0^1 \left[\frac{x}{2}\sqrt{(4x^2 - x^2)} + 2x^2 \sin^{-1}\frac{1}{2} - 0\right] dx$

$= \int_0^1 \left[\frac{\sqrt{3}}{2}x^2 + \frac{\pi}{3}x^2\right] dx = \left[\frac{\sqrt{3}}{2}.\frac{x^3}{3} + \frac{\pi}{3}.\frac{x^3}{3}\right]_0^1$

$= \frac{1}{3}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$.

Ex. 21 (c). *Evaluate* $\iint \frac{xy}{\sqrt{(1 - y^2)}}\,dx\,dy$ *over the positive quadrant of the circle* $x^2 + y^2 = 1$.

Sol. Here the region of integration R is the area of the circle $x^2 + y^2 = 1$ lying in the positive quadrant. This region of integration can be expressed as

$0 \le x \le \sqrt{(1 - y^2)}, 0 \le y \le 1$.

$\therefore \iint_R \frac{xy}{\sqrt{(1 - y^2)}}\,dx\,dy = \int_{y=0}^{1} \int_{x=0}^{\sqrt{(1-y^2)}} \frac{xy}{\sqrt{(1 - y^2)}}\,dx\,dy$

$= \int_0^1 \frac{y}{\sqrt{(1 - y^2)}} \left[\frac{x^2}{2}\right]_{x=0}^{\sqrt{(1-y^2)}} dy,$

integrating w.r.t. x treating y as constant

$= \frac{1}{2}\int_0^1 y\sqrt{(1 - y^2)}\,dy = \frac{1}{2}\int_0^1 -\frac{1}{2}.(1 - y^2)^{1/2}(-2y)\,dy$

$= -\frac{1}{4}.\frac{2}{3}\left[(1 - y^2)^{3/2}\right]_0^1$, by power formula

$= \frac{1}{6}$.

Ex. 21 (d). *Evaluate the double integral*

$$\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} x^2 y\,dx\,dy.$$

Mention the region of integration involved in this double integral.

Sol. The given integral

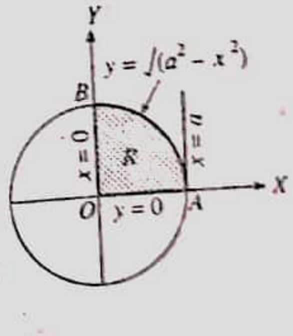
$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2 - x^2)}} x^2 y\,dx\,dy$

$= \int_0^a x^2 \left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{(a^2 - x^2)}} dx,$ integrating w.r.t. y treating x as constant

$= \frac{1}{2}\int_0^a x^2(a^2 - x^2)\,dx = \frac{1}{2}\int_0^a (a^2x^2 - x^4)\,dx$

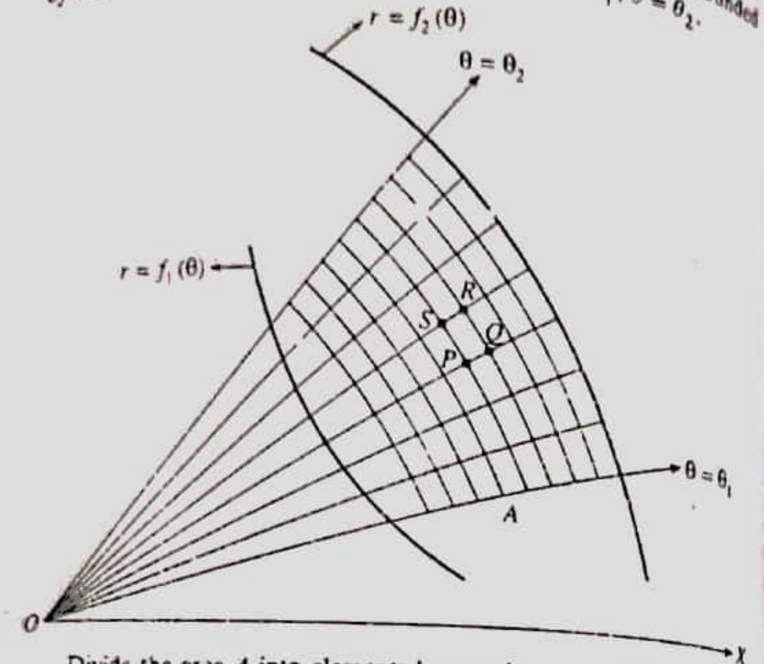
$= \frac{1}{2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{1}{15}a^5.$

From the limits of integration it is obvious that the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 = a^2$ between the lines $x = 0$, $x = a$ and lying above the line $y = 0$ *i.e.*, the axis of x. Thus the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.



§ 4. To express a double integral in terms of polar coordinates.

Let a function $f(r, \theta)$ of the polar coordinates (r, θ) be continuous inside some region A and on its boundary. Let the region A be bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.



Divide the area A into elements by a series of concentric circular arcs with centre at origin and successive radii differing by equal amounts and a series of straight lines drawn through the origin at equal intervals of angles. Let δr be the distance between two consecutive circles and $\delta\theta$ be the angle between two consecutive lines. There is thus a network of elementary areas (say n in number) of which a typical one is $PQRS$. If P is the point (r, θ), the area of the element $PQRS$ situated at the point P is $\frac{1}{2}(r + \delta r)^2 \delta\theta - \frac{1}{2} r^2 \delta\theta = r\delta\theta\,\delta r$, by neglecting the term $\frac{1}{2}(\delta r)^2 \delta\theta$ being an infinitesimal of higher order.

Now by the definition of the double integral of $f(r, \theta)$ over the region A, we have

$$\iint_A f(r, \theta)\, dA = \lim_{\substack{n \to \infty \\ \delta r \to 0,\, \delta\theta \to 0}} \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\, \delta\theta\, \delta r,$$

where $r_k\, \delta\theta \delta r$ is the area of the element situated at the point (r_k, θ_k).

Using the area of integration, this double integral is generally written as

$$\int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta)\, d\theta\, dr, \text{ or } \int_{\theta_1}^{\theta_2} d\theta \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta)\, dr.$$

The first integration is performed with respect to r, keeping θ as constant. After substituting the limits for r, the second integration with respect to θ is performed.

Remark. The area of the typical element $PQRS$ situated at the point $P(r, \theta)$ can also be found as below :

We have $OP = r$, $OQ = r + \delta r$ so that $PQ = \delta r$. Also PS is the arc of circle of radius r subtending an angle $\delta\theta$ at the centre of the circle and so arc $PS = r\,\delta\theta$. Therefore the area of the element $PQRS$ is $r\delta\theta$ i.e., $r\,\delta\theta\,\delta r$.

Ex. 22. *Evaluate* $\int_0^{\pi} \int_0^{a \sin\theta} r\, d\theta\, dr$.

Sol. Here the limits of r are variable and those of θ are constant. Therefore first integration shall be performed w.r.t. r regarding θ as a constant. We have

$$\int_0^{\pi} \int_0^{a \sin\theta} r\, d\theta\, dr = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a \sin\theta} d\theta = \tfrac{1}{2} \int_0^{\pi} a^2 \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2} . 2 \int_0^{\pi/2} \sin^2\theta\, d\theta = \frac{a^2}{2} . 2 . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Ex. 23. *Evaluate* $\int_0^{\pi/2} \int_0^{a \cos\theta} r \sin\theta\, d\theta\, dr$. **(Agra 1974)**

Sol. We have

$$\int_0^{\pi/2} \int_0^{a \cos\theta} r \sin\theta\, d\theta\, dr = \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_0^{a \cos\theta} d\theta.$$

integrating first w.r.t. r regarding θ as a constant

$$= \tfrac{1}{2} \int_0^{\pi/2} \sin\theta\, .\, a^2 \cos^2\theta\, d\theta = \frac{a^2}{2} \int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta$$

$$= \tfrac{1}{2} a^2 . \frac{1.1}{3.1} = \frac{1}{6} a^2.$$

Ex. 24. *Evaluate* $\int_0^{\pi} \int_0^{a(1 + \cos\theta)} r^2 \cos\theta\, d\theta\, dr$.

Sol. We have

$$\int_0^{\pi} \int_0^{a(1 + \cos\theta)} r^2 \cos\theta\, d\theta\, dr = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a(1 + \cos\theta)} d\theta$$

$$= \tfrac{1}{3} \int_0^{\pi} \cos\theta\, .\, a^3 (1 + \cos\theta)^3\, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} \cos\theta\, (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta$$

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$$= \frac{a^3}{3} \int_0^{\pi} [\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta]\, d\theta$$

$$= 2 \cdot \frac{a^3}{3} \int_0^{\pi/2} [3\cos^2\theta + \cos^4\theta]\, d\theta,$$

$$\left[\because \int_0^{\pi} \cos^n\theta\, d\theta = 0 \text{ or } 2\int_0^{\pi/2} \cos^n\theta\, d\theta \text{ according as } n \text{ is odd or even}\right]$$

$$= \frac{2a^3}{3}\left[3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3.1}{4.2} \cdot \frac{\pi}{2}\right] = \frac{2a^3}{3} \cdot \frac{3\pi}{4}\left[1 + \frac{1}{4}\right]$$

$$= \frac{2a^3}{3} \cdot \frac{3\pi}{4} \cdot \frac{5}{4} = \frac{5\pi a^3}{8}.$$

Ex. 25 (a). *Evaluate* $\iint r^2\, d\theta\, dr$ *over the area of the circle* $r = a\cos\theta$.

Sol. The circle $r = a\cos\theta$ passes through the pole and the diameter through the pole is initial line. The region of integration can be covered by radial strips originating from $r = 0$ and terminating at $r = a\cos\theta$. From the equation of the circle, we have $r = 0$ when $\cos\theta = 0$ *i.e.*, $\theta = \pm\pi/2$. Therefore for the given area θ varies from $-\pi/2$ to $\pi/2$. Therefore the required integral

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a\cos\theta} r^2\, d\theta\, dr = \int_{-\pi/2}^{\pi/2} \left[\int_0^{a\cos\theta} r^2\, dr\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{a^3\cos^3\theta}{3}\, d\theta = \frac{2a^3}{3}\int_0^{\pi/2} \cos^3\theta\, d\theta = \frac{2a^3}{3} \cdot \frac{2}{3.1} = \frac{4a^3}{9}.$$

Ex. 25 (b). *Evaluate* $\iint_R r^2 \sin\theta\, d\theta\, dr$ *where R is the circle* $r = 2a\cos\theta$. **(Meerut 1995)**

Sol. Proceed as in Ex. 25 (a).

The given integral $I = \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^2 \sin\theta\, d\theta\, dr$

$$= \int_{-\pi/2}^{\pi/2} \left[\int_{r=0}^{2a\cos\theta} r^2\, dr\right] \sin\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2a\cos\theta} \sin\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{8a^3}{3} \cos^3\theta \sin\theta\, d\theta$$

$= 0$ because $\cos^3\theta\sin\theta$ is an odd function of θ.

[Note that $\int_{-a}^{a} f(x)\, dx = 0$ if $f(-x) = -f(x)$]

Ex. 26 (a). *Evaluate* $\iint \frac{r\, d\theta\, dr}{\sqrt{(a^2 + r^2)}}$ *over one loop of the lemniscate* $r^2 = a^2\cos 2\theta$.

Sol. In the equation of the lemniscate $r^2 = a^2\cos 2\theta$, putting $r = 0$, we get $\cos 2\theta = 0$ *i.e.*, $2\theta = \pm\pi/2$ *i.e.*, $\theta = \pm\pi/4$. Therefore for one loop of the given lemniscate θ varies from $-\pi/4$ to $\pi/4$ and r varies from 0 to $a\sqrt{(\cos 2\theta)}$.

Therefore the required integral

$$= \int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \frac{r\, d\theta\, dr}{\sqrt{(a^2 + r^2)}}$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \tfrac{1}{2}(a^2 + r^2)^{-1/2}(2r)\, d\theta\, dr$$

$$= \int_{-\pi/4}^{\pi/4} \left[(a^2 + r^2)^{1/2}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [a(1 + \cos 2\theta)^{1/2} - a]\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [(2\cos^2\theta)^{1/2} - 1]\, d\theta = 2a\int_0^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= 2a\int_0^{\pi/4} [(2\cos^2\theta)^{1/2} - 1]\, d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4} = 2a\left[\sqrt{2} \cdot \frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= 2a\left[1 - \frac{\pi}{4}\right] = \frac{a}{2}(4 - \pi).$$

Ex. 26 (b). *Find the mass of a loop of the lemniscate* $r^2 = a^2\sin 2\theta$ *if density* $\rho = kr^2$.

Sol. In the equation of the lemniscate $r^2 = a^2\sin 2\theta$, putting $r = 0$, we get $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \frac{1}{2}\pi$. Therefore for one loop of the given lemniscate θ varies from 0 to $\pi/2$ and r varies from 0 to $a\sqrt{(\sin 2\theta)}$.

$\therefore$ mass of a loop of the lemniscate

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin 2\theta)}} \rho r\, d\theta\, dr = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin 2\theta)}} kr^2 . r\, d\theta\, dr$$

$$= k\int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin 2\theta)}} r^3\, d\theta\, dr = k. \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{a\sqrt{(\sin 2\theta)}} d\theta$$

$$= \frac{ka^4}{4}\int_0^{\pi/2} \sin^2 2\theta\, d\theta = \frac{ka^4}{8}\int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$$

$$= \frac{ka^4}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{ka^4}{8} \cdot \frac{\pi}{2} = \frac{\pi k a^4}{16}.$$

Ex. 27. *Integrate* $r\sin\theta$ *over the area of the cardioid* $r = a(1 + \cos\theta)$ *lying above the initial line.*

Sol. For the area of the cardioid $r = a(1 + \cos\theta)$ above the initial line θ varies from 0 to π. Also for the required area r varies from $r = 0$ to $r = a(1 + \cos\theta)$. If A denotes the region consisting of the area of the cardioid lying above the initial line, then the required integral

$$= \iint_A r\sin\theta \, dA = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta \, r \, d\theta \, dr$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta = \frac{a^3}{3}\int_0^{\pi} \sin\theta\,(1+\cos\theta)^3 \, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi/2} \sin\phi\cos^7\phi \, . \, 2d\phi, \text{ putting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2d\phi$$

$$= 32 . \frac{a^3}{3}\left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2} = \frac{32a^3}{3}\left[0 + \frac{1}{8}\right] = \frac{4a^3}{3}.$$

Ex. 28 (a). *Find by double integration the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

(Meerut 1984; U.P. P.C.S. 93)

Sol. Eliminating r between the given equations of the cardioid $r = a(1 + \cos\theta)$ and the circle $r = a$, we have

$$a = a(1 + \cos\theta) \quad \text{or} \quad \cos\theta = 0 \text{ i.e., } \theta = \pm\pi/2.$$

Thus the region of integration A is enclosed by

$$r = a, r = a(1 + \cos\theta), \theta = -\pi/2, \theta = \pi/2.$$

$\therefore$ the required area $= \iint_A r\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,d\theta\,dr$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [a^2(1+\cos\theta)^2 - a^2]\,d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos^2\theta + 2\cos\theta - 1)\,d\theta$$

$$= \frac{a^2}{2} . 2\int_0^{\pi/2} [\cos^2\theta + 2\cos\theta]\,d\theta$$

$$= a^2\left[\frac{1}{2}\cdot\frac{1}{2}\pi + 2\left\{\sin\theta\right\}_0^{\pi/2}\right] = a^2\left[\frac{1}{4}\pi + 2\right] = \frac{a^2}{4}(\pi + 8).$$

Ex. 28 (b). *Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 + \cos\theta)$.*

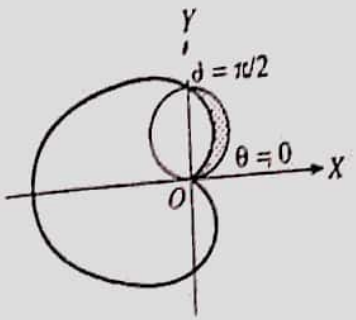
(Meerut 1985, 94, 96)

Sol. The given circle is $r = a\sin\theta$ and the cardioid is $r = a(1 - \cos\theta)$. Note that the given circle passes through the pole and the diameter through the pole makes an angle $\pi/2$ with the initial line. Eliminating r between the two equations, we have

$a\sin\theta = a(1 - \cos\theta)$ or $1 = \dfrac{\sin\theta}{1-\cos\theta} = \dfrac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\dfrac{\theta}{2}$

... $\frac{1}{2}\theta = \frac{1}{4}\pi$ i.e., $\theta = \pi/2$.

Thus the two curves meet at the ... where $\theta = \pi/2$. Also for both the ... $r = 0$ when $\theta = 0$ and so the two ... also meet at the pole O where ... 0. To cover the required area the ... of integration for r are ... $(1-\cos\theta)$ to $a\sin\theta$ and for θ are 0 ... $\pi/2$. Therefore the required area



$$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} [a^2\sin^2\theta - a^2(1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} [\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\pi}{2} + 2.1 - \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right] = \frac{a^2}{4}(4 - \pi).$$

Ex. 28 (c). *Find by double integration the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.*

(Kanpur 1987)

Sol. Eliminating r between the given equations of the cardioid and the parabola, we have

$$(1 + \cos\theta) = 1/(1 + \cos\theta) \quad \text{or} \quad (1 + \cos\theta)^2 = 1$$

or $\cos^2\theta + 2\cos\theta = 0$ or $\cos\theta(2 + \cos\theta) = 0$

or $\cos\theta = 0$, because $\cos\theta$ cannot be equal to -2

or $\theta = \pm\pi/2$.

Thus the two curves intersect at the points where $\theta = -\pi/2$ and $\theta = \pi/2$.

Therefore the required area is enclosed by $r = 1/(1 + \cos\theta)$, $r = (1 + \cos\theta)$, $\theta = -\pi/2$, $\theta = \pi/2$.

Hence the required area

$$= \int_{-\pi/2}^{\pi/2}\int_{1/(1+\cos\theta)}^{(1+\cos\theta)} r\,d\theta\,dr = \int_{-\pi/2}^{\pi/2}\left[\frac{1}{2}r^2\right]_{1/(1+\cos\theta)}^{(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

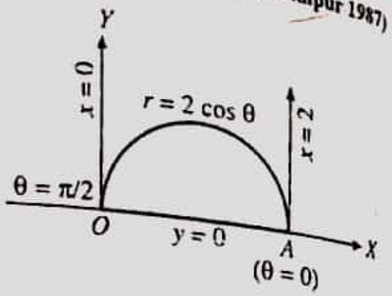
$$= 2\cdot\tfrac{1}{2}\int_0^{\pi/2}\left[(1+2\cos\theta+\cos^2\theta)-\frac{1}{(2\cos^2\frac{1}{2}\theta)^2}\right]d\theta$$
$$=\int_0^{\pi/2}(1+2\cos\theta)\,d\theta+\int_0^{\pi/2}\cos^2\theta\,d\theta-\tfrac{1}{4}\int_0^{\pi/2}\sec^4\tfrac{1}{2}\theta\,d\theta$$
$$=\Big[\theta+2\sin\theta\Big]_0^{\pi/2}+\tfrac{1}{2}\cdot\frac{\pi}{2}-\tfrac{1}{4}\int_0^{\pi/2}(1+\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\,d\theta$$
$$=\frac{\pi}{2}+2+\frac{\pi}{4}-\tfrac{1}{4}\int_0^{\pi/2}[\sec^2\tfrac{1}{2}\theta+2(\tan^2\tfrac{1}{2}\theta)(\tfrac{1}{2}\sec^2\tfrac{1}{2}\theta)]\,d\theta$$
$$=\frac{3\pi}{4}+2-\tfrac{1}{4}\Big[2\tan\tfrac{1}{2}\theta+\tfrac{2}{3}\tan^3\tfrac{1}{2}\theta\Big]_0^{\pi/2}$$
$$=\frac{3\pi}{4}+2-\tfrac{1}{4}\Big[2+\frac{2}{3}\Big]=\frac{3\pi}{4}+2-\frac{2}{3}=\frac{3\pi}{4}+\frac{4}{3}=\frac{(9\pi+16)}{12}.$$

Ex. 29 (a). *Transform the integral*

$$\int_0^2\int_0^{\sqrt{(2x-x^2)}}\frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}$$

by changing to polar coordinates and hence evaluate it. **(Gorakhpur 1987)**

Sol. From the limits of integration it is obvious that the region of integration is bounded by $y=0$, $y=\sqrt{(2x-x^2)}$ and $x=0$, $x=2$ *i.e.*, the region of integration is the area of the circle $x^2+y^2-2x=0$ between the lines $x=0$, $x=2$ and lying above the axis of x *i.e.*, the line $y=0$.



Putting $x=r\cos\theta$, $y=r\sin\theta$ the corresponding polar equation of the circle is

$$r^2(\cos^2\theta+\sin^2\theta)-2r\cos\theta=0,\text{ or } r=2\cos\theta.$$

From the figure it is obvious that r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\pi/2$. Note that at the point A of the circle $\theta=0$ and at the point O, $r=0$ and so from $r=2\cos\theta$, we get $\theta=\pi/2$ at O.

The polar equivalent of elementary area $dx\,dy$ is $r\,d\theta\,dr$.

$$\therefore\quad \iint_A f(x,y)\,dx\,dy=\iint_A f(r\cos\theta,r\sin\theta)\,r\,d\theta\,dr,$$

where A is the region of integration.

Hence transforming to polar coordinates, the given double integral

$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta}\frac{r\cos\theta}{r}\,r\,d\theta\,dr$$

$$=\int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta}d\theta$$
$$=\int_0^{\pi/2}\tfrac{1}{2}\cos\theta\,.\,4\cos^2\theta\,d\theta=2\int_0^{\pi/2}\cos^3\theta\,d\theta=2.\tfrac{2}{3}=\tfrac{4}{3}.$$

Ex. 29 (b). *Transform the following double integrals to polar coordinates and hence evaluate them.*

(i) $\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}(a^2-x^2-y^2)\,dx\,dy.$ **(Rohilkhand 1984, 90; Meerut 97)**

(ii) $\int_0^1\int_x^{\sqrt{(2x-x^2)}}(x^2+y^2)\,dx\,dy$ **(Gorakhpur 1988)**

(iii) $\int_0^a\int_0^{\sqrt{(a^2-x^2)}}y^2\sqrt{(x^2+y^2)}\,dx\,dy$ **(Gorakhpur 1986, 89; Kanpur 87)**

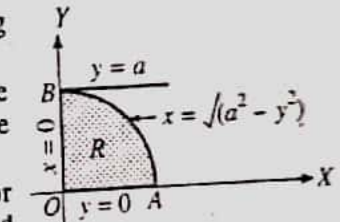
Sol. (i) The given double integral

$$I=\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}}[a^2-(x^2+y^2)]\,dx\,dy.$$

From the limits of integration it is obvious that the region of integration R is bounded by

$$x=0,\ x=\sqrt{(a^2-y^2)}\text{ and }y=0,\ y=a.$$

Thus the region of integration is the area OAB of the circle $x^2+y^2=a^2$ lying in the positive quadrant.



Putting $x=r\cos\theta$, $y=r\sin\theta$ the corresponding polar equation of the circle is $r=a$.

From the figure it is obvious that for the area OAB, r varies from 0 to a and θ varies from 0 to $\pi/2$. Also the polar equivalent of $dx\,dy$ is $r\,d\theta\,dr$. $[\because x^2+y^2=r^2]$

$$\therefore\quad I=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}(a^2-r^2)\,r\,d\theta\,dr,$$
$$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a}[a^2r-r^3]\,d\theta\,dr=\int_0^{\pi/2}\left[\frac{a^2r^2}{2}-\frac{r^4}{4}\right]_{r=0}^{a}d\theta$$
$$=\int_0^{\pi/2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]d\theta=\frac{a^4}{4}\int_0^{\pi/2}d\theta=\frac{a^4}{4}\Big[\theta\Big]_0^{\pi/2}=\frac{\pi a^4}{8}.$$

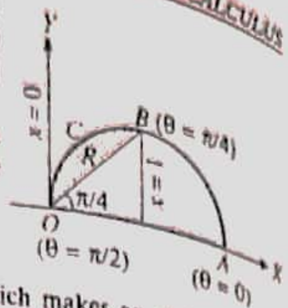
(ii) The given double integral

$$I=\int_{x=0}^{1}\int_{y=x}^{\sqrt{(2x-x^2)}}(x^2+y^2)\,dx\,dy.$$

Here the region of integration R is bounded by

$$y=x,\ y=\sqrt{(2x-x^2)}\text{ and }x=0,\ x=1$$

i.e., the region of integration is the area $OBCO$ of the circle $x^2 + y^2 - 2x = 0$ bounded by the lines $y = x$, $x = 0$ and $x = 1$.



Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is

$$r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0,$$

or $r = 2\cos\theta$.

The point B is on the line $y = x$ which makes an algle $\pi/4$ with OX and so, at B, $\theta = \pi/4$. At the point O of the circle $r = 2\cos\theta$, we have $r = 0$ and so $\theta = \pi/2$. Thus for the region R, r varies from 0 to $2\cos\theta$ and θ varies from $\pi/4$ to $\pi/2$. Also the polar equivalent of $dx\,dy$ is $r\,d\theta\,dr$.

Hence transforming to polar coordinates, we have

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 . r\,d\theta\,dr = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^3\,d\theta\,dr$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{2\cos\theta} d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2} \cos^4\theta\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \left[\frac{3}{2}\theta + 2.\frac{\sin 2\theta}{2} + \frac{1}{2}\cdot\frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2}$$

$$= \left[\frac{3\pi}{4} - \frac{3\pi}{8} - 1\right] = \frac{3\pi}{8} - 1.$$

(iii) The given double integral

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} y^2\sqrt{(x^2+y^2)}\,dx\,dy.$$

Here the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$. Thus the region of integration R is the area of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant. The polar equation of this circle is $r = a$ and for the region R, r varies from 0 to a and θ varies from 0 to $\pi/2$. Putting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dx\,dy$ by $r\,d\theta\,dr$, we have

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2\sin^2\theta . r . r\,d\theta\,dr$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^4\sin^2\theta\,d\theta\,dr$$

$$= \int_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a \sin^2\theta\,d\theta = \frac{a^5}{5}\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= \frac{a^5}{5}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^5}{20}.$$

Ex. 29 (c). *Change the following integrals into polar coordinates.*

(i) $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} (x^2 + y^2)\,dy\,dx.$ **(Meerut 1995 BP)**

(ii) $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy.$ **(Meerut 1994)**

Sol. (i) Let $I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{(a^2-y^2)}} (x^2 + y^2)\,dx\,dy.$

Proceed as in Ex. 29 (b) part (i).

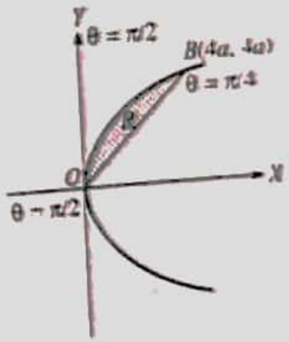
Changing to polar coordinates, we have

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 . r\,d\theta\,dr$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3\,d\theta\,dr.$$

(ii) Let $I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy.$

Here the region of integration R is bounded by $x = y^2/4a$, $x = y$, $y = 0$ and $y = 4a$ *i.e.*, the region of integration is the area of the parabola $y^2 = 4ax$ cut off by the straight line $y = x$.



Changing to polar coordinates, the equation $y^2 = 4ax$ becomes

$$(r\sin\theta)^2 = 4a(r\cos\theta)$$

or
$$r = \frac{4a\cos\theta}{\sin^2\theta}.$$

At the point B, $\theta = \pi/4$.

At the point O of the parabola $r = \frac{4a\cos\theta}{\sin^2\theta}$, we have $r = 0$ and so $\theta = \pi/2$.

Thus for the region R, r varies from 0 to $\frac{4a\cos\theta}{\sin^2\theta}$ and θ varies from $\pi/4$ to $\pi/2$. Also the polar equivalent of $dx\,dy$ is $r\,d\theta\,dr$.

Hence transforming to polar coordinates, we have

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{(4a\cos\theta)/\sin^2\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\, r\, d\theta\, dr$$

$$= \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{(4a\cos\theta)/\sin^2\theta} r\cos 2\theta\, d\theta\, dr.$$

Ex. 30 (a). *Transform to polar coordinates and integrate*

$$\iint \sqrt{\left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)}\, dx\, dy$$

the integral being extended over all positive values of x and y subject to $x^2+y^2 \le 1$. **(Kanpur 1989)**

Sol. Here the region of integration R of the given double integral is the area of the circle $x^2+y^2=1$ lying in the positive quadrant. The polar equation of this circle is $r=1$ and for the region R, r varies from 0 to 1 and θ varies from 0 to $\pi/2$. Also $dx\,dy = r\,d\theta\,dr$.

Hence transforming to polar coordinates, the given double integral

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \sqrt{\left(\frac{1-r^2}{1+r^2}\right)}\, r\, d\theta\, dr$$

$$= \int_{r=0}^{1} r\sqrt{\left(\frac{1-r^2}{1+r^2}\right)} \left[\theta\right]_{\theta=0}^{\pi/2} dr,$$

first integrating w.r.t. θ taking r as constant

$$= \frac{\pi}{2}\int_0^1 \frac{r(1-r^2)}{\sqrt{(1-r^4)}}\, dr$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{(1-\sin t)}{\cos t}\cdot \tfrac{1}{2}\cos t\, dt, \quad \text{putting } r^2 = \sin t \text{ so that } 2r\,dr = \cos t\, dt$$

$$= \frac{\pi}{4}\left[t + \cos t\right]_0^{\pi/2} = \frac{\pi}{4}\left[\frac{\pi}{2} + 0 - 0 - 1\right] = \frac{\pi}{8}(\pi - 2).$$

Ex. 30 (b). *By changing to polar coordinates, evaluate*

$$\iint xy(x^2+y^2)^{n/2}\, dx\, dy,\ n+3>0,$$

over the positive quadrant of the circle $x^2+y^2=a^2$.
Deduce the value of

$$\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$$

over the positive quadrant of the ellipse $x^2/a^2 + y^2/b^2 = 1$. **(Kanpur 1988)**

Sol. Let $I = \iint_R xy(x^2+y^2)^{n/2}\, dx\, dy$,

where the region of integration R is the area of the circle $x^2+y^2=a^2$ lying in the positive quadrant.

We have $x = r\cos\theta$, $y = r\sin\theta$ and $x^2+y^2=a^2$. The polar equation of the circle $x^2+y^2=a^2$ is $r=a$ and for the region R, r varies from 0 to a and θ varies from 0 to $\pi/2$. Also $dx\,dy = r\,d\theta\,dr$.

$\therefore$ transforming to polar coordinats, we have

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta \,.\, r\sin\theta \,.(r^2)^{n/2}\, r\, d\theta\, dr$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^{n+3}\cos\theta\sin\theta\, d\theta\, dr$$

$$= \int_0^{\pi/2}\left[\frac{r^{n+4}}{n+4}\right]_{r=0}^{a} \cos\theta\sin\theta\, d\theta$$

$$= \frac{a^{n+4}}{n+4}\int_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= \frac{a^{n+4}}{n+4}\cdot\frac{1}{2} = \frac{a^{n+4}}{2(n+4)}.$$

Now let $I_1 = \iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) dx\, dy$, the integral being extended over the positive quadrant of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Put $\frac{x}{a}=u$, $\frac{y}{b}=v$. Then $dx = a\,du$, $dy = b\,dv$.

$\therefore \quad I_1 = \iint au\, bv\,.(u^2+v^2)\, ab\, du\, dv$, the integral being extended to all positive values of u and v subject to the condition $u^2+v^2 \le 1$

$$= a^2b^2 \iint uv(u^2+v^2)\, du\, dv.$$

Now putting $a=1$ and $n=2$ in the value of I, we have

$$\iint uv(u^2+v^2)\, du\, dv = \tfrac{1}{12}.$$

$$\therefore \quad I_1 = a^2b^2\cdot\frac{1}{12} = \frac{a^2b^2}{12}.$$

§ 5. Triple integrals.

Let the function $f(x,y,z)$ of the point $P(x,y,z)$ be continuous for all points within a finite region V and on its boundary. Divide the region V into n parts; let $\delta V_1, \delta V_2, \ldots, \delta V_n$ be their volumes. Take a point in each part and form the sum

$$S_n = f(x_1,y_1,z_1)\,\delta V_1 + f(x_2,y_2,z_2)\,\delta V_2 + \ldots + f(x_n,y_n,z_n)\,\delta V_n \quad \ldots(1)$$

$$= \sum_{r=1}^{n} f(x_r,y_r,z_r)\,\delta V_r.$$

Then the limit to which the sum (1) tends when n tends to infinity and the dimensions of each sub-division tend to zero, is called the triple integral of the function $f(x, y, z)$ over the region V. This is denoted by $\iiint_V f(x, y, z)\, dV$ or $\iiint_V f(x, y, z)\, dx\, dy\, dz$.

§ 6. Evaluation of triple integrals.

(a) If the region V be specified by the inequalities
$$a \le x \le b,\ c \le y \le d,\ e \le z \le f,$$
then the triple integral
$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_c^d \int_e^f f(x, y, z)\, dx\, dy\, dz$$
$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\, dz.$$

Here the order of integration is immaterial and the integration with repsect to any of x, y and z can be performed first.

(b) If the limits of z are given as functions of x and y, the limits of y as functions of x while x takes the constant values say from $x = a$ to $x = b$, then
$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz.$$

The integration with respect to z is performed first regarding x and y as constants, then the integration w.r.t. y is performed regarding x as a constant and in the last we perform the integration w.r.t. x.

Ex. 31 (a). *Evaluate* $\int_{y=0}^{3} \int_{x=0}^{2} \int_{z=0}^{1} (x + y + z)\, dz\, dx\, dy$.

Sol. The given integral
$$= \int_{y=0}^{3} \int_{x=0}^{2} \left\{ \int_0^1 (x + y + z)\, dz \right\} dx\, dy$$
$$= \int_{y=0}^{3} \int_{x=0}^{2} \left[xz + yz + \frac{z^2}{2} \right]_0^1 dx\, dy = \int_0^3 \left\{ \int_0^2 \left(x + y + \tfrac{1}{2}\right) dx \right\} dy$$
$$= \int_0^3 \left[\frac{x^2}{2} + xy + \frac{x}{2} \right]_0^2 dy = \int_0^3 (3 + 2y)\, dy = \left[3y + \frac{2y^2}{2} \right]_0^3 = 18.$$

Ex. 31 (b). *Show that*
$$\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=1}^{2} x^2 yz\, dz\, dy\, dx = 1.$$

Sol. We have $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=1}^{2} x^2 yz\, dz\, dy\, dx$
$$= \int_{x=0}^{1} \int_{y=0}^{2} \left\{ \int_1^2 x^2 yz\, dz \right\} dy\, dx$$
$$= \int_{x=0}^{1} \int_{y=0}^{2} \left[x^2 y . \frac{z^2}{2} \right]_1^2 dy\, dx = \frac{1}{2} \int_0^1 \left[\int_0^2 (3x^2 y)\, dy \right] dx$$

$$= \frac{3}{2} \int_0^1 \left[x^2 . \frac{y^2}{2} \right]_0^2 dx = \frac{3}{4} \int_0^1 4x^2\, dx = 3 \left[\frac{x^3}{3} \right]_0^1 = 3 . \frac{1}{3} = 1.$$

Ex. 32. *Evaluate* $\int_0^1 \int_0^1 \int_0^1 e^{x + y + z}\, dx\, dy\, dz$.

Sol. $\int_0^1 \int_0^1 \int_0^1 e^{x + y + z}\, dx\, dy\, dz = \int_0^1 \int_0^1 \left[\int_0^1 e^{x + y + z}\, dx \right] dy\, dz$
$$= \int_0^1 \int_0^1 \left[e^{x + y + z} \right]_0^1 dy\, dz = \int_0^1 \left[\int_0^1 \left\{ e^{1 + y + z} - e^{y + z} \right\} dy \right] dz$$
$$= \int_0^1 \left[e^{1 + y + z} - e^{y + z} \right]_0^1 dz$$
$$= \int_0^1 \{(e^{2 + z} - e^{1 + z}) - (e^{1 + z} - e^z)\}\, dz$$
$$= \int_0^1 (e^{2 + z} - 2e^{1 + z} + e^z)\, dz = \int_0^1 (e^2 - 2e + 1)\, e^z\, dz$$
$$= (e^2 - 2e + 1) \int_0^1 e^z\, dz = (e - 1)^2 \left[e^z \right]_0^1 = (e - 1)^2 (e - e^0)$$
$$= (e - 1)^2 (e - 1) = (e - 1)^3.$$

Ex. 33. *Evaluate* $\int_{-1}^{1} \int_0^z \int_{x - z}^{x + z} (x + y + z)\, dy\, dx\, dz$.

Sol. Here $x - z$ to $x + z$ are the limits of integration of y, 0 to z are those of x and -1 to 1 are those of z. The given triple integral
$$= \int_{-1}^{1} \int_0^z \left[\int_{x - z}^{x + z} (x + y + z)\, dy \right] dx\, dz$$
$$= \int_{-1}^{1} \int_0^z \left[xy + \frac{y^2}{2} + zy \right]_{x - z}^{x + z} dx\, dz$$
$$= \int_{-1}^{1} \int_0^z \left[x(x + z) + \frac{(x + z)^2}{2} + z(x + z) - x(x - z) - \frac{(x - z)^2}{2} - z(x - z) \right] dx\, dz$$
$$= \int_{-1}^{1} \left[\int_0^z (4xz + 2z^2)\, dx \right] dz = \int_{-1}^{1} \left[2zx^2 + 2z^2 x \right]_0^z dz$$
$$= \int_{-1}^{1} (2z . z^2 + 2z^2 . z)\, dz = 4 \int_{-1}^{1} z^3\, dz$$
$$= 4 \left[\frac{z^4}{4} \right]_{-1}^{1} = 1 . [1 - 1] = 0.$$

Ex. 34. *Evaluate the following integrals.*

(i) $\int_0^1 \int_0^{1 - x} \int_0^{1 - x - y} xyz\, dx\, dy\, dz$; **(Kanpur 1979; Meerut 96 BP)**

(ii) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$; **(Rohilkhand 1977)**

(iii) $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$; **(Kanpur 1984; Rohilkhand 80; Agra 75)**

(iv) $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dy\,dx\,dz$. **(Kanpur 1980; U.P. P.C.S. 91)**

(v) $\displaystyle\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{dx\,dy\,dz}{(1+x+y+z)^3}$. **(Kanpur 1985; Agra 77)**

(vi) $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}} xyz\,dz\,dy\,dx$ **(Meerut 1994P, 95)**

(vii) $\int_1^e\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dx\,dy$. **(Meerut 1995 BP)**

Sol. (i) We have

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dx\,dy\,dz$$

$$=\int_0^1\int_0^{1-x} xy\left[\frac{z^2}{2}\right]_0^{1-x-y}dx\,dy,$$

integrating w.r.t. z regarding x and y as constants

$$=\tfrac{1}{2}\int_0^1\int_0^{1-x} xy\,\{(1-x)-y\}^2\,dx\,dy$$

$$=\tfrac{1}{2}\int_0^1\int_0^{1-x} x\,[y\,(1-x)^2-2\,(1-x)\,y^2+y^3]\,dx\,dy$$

$$=\tfrac{1}{2}\int_0^1 x\left[\frac{(1-x)^2y^2}{2}-\frac{2\,(1-x)\,y^3}{3}+\frac{y^4}{4}\right]_0^{1-x}dx,$$

integrating w.r.t. y regarding x as constant

$$=\tfrac{1}{24}\int_0^1 x\,[6\,(1-x)^4-8\,(1-x)^4+3\,(1-x)^4]\,dx$$

$$=\tfrac{1}{24}\int_0^1 x\,(1-x)^4\,dx$$

$$=\tfrac{1}{24}\int_0^{\pi/2}\sin^2\theta\cos^8\theta.\,2\sin\theta\cos\theta\,d\theta,$$

putting $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

$$=\frac{1}{12}\int_0^{\pi/2}\sin^3\theta\cos^9\theta\,d\theta=\frac{1}{12}\cdot\frac{2.8.6.4.2}{12.10.8.6.4.2}=\frac{1}{720}.$$

(ii) Here the integrand $x^2+y^2+z^2$ is a symmetrical expression in x, y and z and therefore the limits of integration can be assigned at pleasure. We have the given integral

$$=\int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=-a}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=2\int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=0}^{a}(x^2+y^2+z^2)\,dx\,dy\,dz,$$

because $x^2+y^2+z^2$ is an even function of x

$$=2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{x^3}{3}+(y^2+z^2)\,x\right]_0^a dy\,dz,$$

integrating w.r.t. x regarding y and z as constants

$$=2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{a^3}{3}+ay^2+az^2\right]dy\,dz$$

$$=4\int_{z=-c}^{c}\int_0^{b}\left[\frac{a^3}{3}+az^2+ay^2\right]dy\,dz,$$

because $\frac{a^3}{3}+az^2+ay^2$ is an even function of y

$$=4\int_{z=-c}^{c}\left[\frac{a^3}{3}y+az^2y+\frac{ay^3}{3}\right]dz,$$

integrating w.r.t. y regarding z as constant

$$=4\int_{z=-c}^{c}\left[\frac{a^3b}{3}+abz^2+\frac{ab^3}{3}\right]dz=8\int_0^{c}\left[\frac{a^3b}{3}+abz^2+\frac{ab^3}{3}\right]dz$$

$$=8\left[\frac{a^3b}{3}z+ab\,\frac{z^3}{3}+\frac{ab^3}{3}z\right]_0^c$$

$$=\frac{8}{3}(a^3bc+abc^3+ab^3c)=\frac{8}{3}abc\,(a^2+b^2+c^2).$$

(iii) We have $\int_0^{\log 2}\int_0^x\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$

$$=\int_0^{\log 2}\int_0^x\left[e^{x+y+z}\right]_0^{x+\log y}dx\,dy,$$

integrating w.r.t. z regarding x and y as constants

$$=\int_0^{\log 2}\int_0^x\left[e^{x+y+x+\log y}-e^{x+y}\right]dx\,dy$$

$$=\int_0^{\log 2}\int_0^x\left[e^{2x}e^y e^{\log y}-e^x e^y\right]dx\,dy \qquad [\because e^{\log y}=y]$$

$$=\int_0^{\log 2}\int_0^x\left[e^{2x}ye^y-e^xe^y\right]dx\,dy,$$

$$=\int_0^{\log 2}\left[\int_0^x e^{2x}y\,e^y\,dy-\int_0^x e^xe^y\,dy\right]dx$$

$$=\int_0^{\log 2}\left[e^{2x}\left\{y\,e^y\right\}_0^x-e^{2x}\int_0^x e^y\,dy-e^x\left\{e^y\right\}_0^x\right]dx,$$

integrating w.r.t. y regarding x as a constant; to integrate ye^y we have applied integration by parts

$$=\int_0^{\log 2}\left[e^{2x}.\,x\,e^x-e^{2x}\left\{e^y\right\}_0^x-e^x\,(e^x-1)\right]dx$$

$$=\int_0^{\log 2}\left[xe^{3x}-e^{2x}\,(e^x-1)-e^{2x}+e^x\right]dx$$

$= \int_0^{\log 2} [x e^{3x} - e^{3x} + e^x]\, dx$

$= \int_0^{\log 2} x e^{3x}\, dx - \int_0^{\log 2} e^{3x}\, dx + \int_0^{\log 2} e^x\, dx$

$= \frac{1}{3} \left[x e^{3x}\right]_0^{\log 2} - \frac{1}{3} \int_0^{\log 2} e^{3x}\, dx - \int_0^{\log 2} e^{3x}\, dx + \int_0^{\log 2} e^x\, dx$

$= \frac{1}{3} (\log 2)\, e^{3 \log 2} - \frac{4}{3} \left[\frac{e^{3x}}{3}\right]_0^{\log 2} + \left[e^x\right]_0^{\log 2}$

$= \frac{1}{3} (\log 2)\, e^{\log 8} - \frac{4}{9} (e^{3 \log 2} - 1) + (e^{\log 2} - 1)$

$= \frac{8}{3} \log 2 - \frac{4}{9} (8 - 1) + (2 - 1) = \frac{8}{3} \log 2 - \frac{28}{9} + 1$

$= \frac{8}{3} \log 2 - \frac{19}{9}$.

(iv) We have $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dy\, dx\, dz$

$= \int_0^1 \int_{y^2}^1 x \left[z\right]_0^{1-x} dy\, dx$,

integrating w.r.t. z regarding x and y as constants

$= \int_0^1 \int_{y^2}^1 x (1 - x)\, dy\, dx$

$= \int_0^1 \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{y^2}^1 dy$,

integrating w.r.t. x regarding y as constant

$$= \int_0^1 \left[\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy = \int_0^1 \left[\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy$$

$$= \left[\frac{1}{6} y - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1 = \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}.$$

(v) We have $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2 (1 + x + y + z)^2}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[-\frac{1}{4} + \frac{1}{(1 + x + y)^2}\right] dx\, dy$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{1}{4} y - \frac{1}{(1 + x + y)}\right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{1}{4} (1 - x) - \frac{1}{2} + \frac{1}{(1 + x)}\right] dx$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{3}{4} + \frac{1}{4} x + \frac{1}{(1 + x)}\right] dx$$

$= \frac{1}{2} \left[-\frac{3}{4} x + \frac{1}{4} \cdot \frac{x^2}{2} + \log (1 + x)\right]_0^1$

$= \frac{1}{2} \left[-\frac{3}{4} + \frac{1}{8} + \log 2\right] = \frac{1}{2} (\log 2 - \frac{5}{8})$.

(vi) The given integral I

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{(1-x^2)}} \int_{z=0}^{\sqrt{(1-x^2-y^2)}} xyz\, dz\, dy\, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{(1-x^2)}} xy \left[\frac{z^2}{2}\right]_{z=0}^{\sqrt{(1-x^2-y^2)}} dy\, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{(1-x^2)}} \frac{1}{2} xy (1 - x^2 - y^2)\, dy\, dx$$

$$= \int_{x=0}^1 \frac{1}{2} x \left[(1 - x^2) \frac{y^2}{2} - \frac{y^4}{4}\right]_{y=0}^{\sqrt{(1-x^2)}} dx$$

$$= \int_0^1 \frac{1}{2} x \left[\frac{1}{2} (1 - x^2)^2 - \frac{1}{4} (1 - x^2)^2\right] dx$$

$= \int_0^1 \frac{1}{2} x \cdot \frac{1}{4} (1 - x^2)^2\, dx$

$= \frac{1}{8} \int_0^{\pi/2} \sin \theta \,.\, \cos^4 \theta \cos \theta\, d\theta$,

putting $x = \sin \theta$ so that $dx = \cos \theta\, d\theta$

$$= \frac{1}{8} \int_0^{\pi/2} \sin \theta \cos^5 \theta\, d\theta = \frac{1}{8} \cdot \frac{1.4.2}{6.4.2} = \frac{1}{48}.$$

(vii) The given integral I

$= \int_{y=1}^e \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z\, dz\, dx\, dy$

$= \int_{y=1}^e \int_{x=1}^{\log y} [z \log z - z]_{z=1}^{e^x}\, dx\, dy$

$= \int_{y=1}^e \int_{x=1}^{\log y} [x e^x - e^x + 1]\, dx\, dy$

$= \int_{y=1}^e [x e^x - 2e^x + x]_{x=1}^{\log y}\, dy$

$= \int_1^e [y \log y - 2y + \log y - e + 2e - 1]\, dy$ $\qquad [\because e^{\log y} = y]$

$= \int_1^e [y \log y + \log y - 2y + e - 1]\, dy$

$= \left[\frac{y^2}{2} \log y - \frac{y^2}{4} + y \log y - y - y^2 + ey - y\right]_1^e$

$= \frac{1}{2} e^2 - \frac{1}{4} e^2 + e - e - e^2 + e^2 - e + \frac{1}{4} + 1 + 1 - e + 1$

$= \frac{1}{4} e^2 - 2e + \frac{13}{4}$.

Ex. 35. *Evaluate* $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{(xy)}} xyz\, dx\, dy\, dz$.

Sol. The given triple integral is

$$\int_1^3\int_{1/x}^1\left[\int_0^{\sqrt{(xy)}} xyz\,dz\right]dx\,dy = \int_1^3\int_{1/x}^1\left[xy.\frac{z^2}{2}\right]_0^{\sqrt{(xy)}}dx\,dy$$

$$=\frac12\int_1^3\left[\int_{1/x}^1 x^2y^2\,dy\right]dx=\frac12\int_1^3\left[x^2.\frac{y^3}{3}\right]_{1/x}^1 dx$$

$$=\frac16\int_1^3\left[x^2-\frac1x\right]dx=\frac16\left[\frac{x^3}{3}-\log x\right]_1^3$$

$$=\tfrac16[(9-\log 3)-(\tfrac13-\log 1)]=\tfrac16[(9-\tfrac13)-\log 3]$$

$$=\tfrac16[\tfrac{26}{3}-\log 3].$$

Ex. 36. *Evaluate* $\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}dr\int_0^{(a^2-r^2)/a} r\,dz$.

(Kanpur 1986; Meerut 94)

Sol. The given triple integral is

$$=\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}dr\left[rz\right]_0^{(a^2-r^2)/a}$$

$$=\int_0^{\pi/2}d\theta\int_0^{a\sin\theta}\frac{r(a^2-r^2)}{a}dr$$

$$=\frac1a\int_0^{\pi/2}\left[\frac{a^2r^2}{2}-\frac{r^4}{4}\right]_0^{a\sin\theta}d\theta=\frac{a^3}{4}\int_0^{\pi/2}(2\sin^2\theta-\sin^4\theta)\,d\theta$$

$$=\frac{a^3}{4}\left[2\cdot\frac12\cdot\frac\pi2-\frac34\cdot\frac12\cdot\frac\pi2\right]=\frac{5a^3\pi}{64}.$$

Ex. 37. *Evaluate* $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\,dx\,dy\,dz$.

Sol. The given triple integral is

$$=\int_0^a\int_0^x\left[\int_0^{x+y}e^{x+y+z}\,dz\right]dx\,dy=\int_0^a\int_0^x\left[e^{x+y+z}\right]_{z=0}^{x+y}dx\,dy$$

$$=\int_0^a\int_0^x\left[e^{2(x+y)}-e^{(x+y)}\right]dx\,dy=\int_0^a\left[\tfrac12e^{2(x+y)}-e^{(x+y)}\right]_0^x dx$$

$$=\int_0^a\left[\tfrac12(e^{4x}-e^{2x})-(e^{2x}-e^x)\right]dx=\int_0^a\left(\tfrac12e^{4x}-\tfrac32e^{2x}+e^x\right)dx$$

$$=\left[\tfrac12\cdot\tfrac14e^{4x}-\tfrac34\cdot\tfrac12e^{2x}+e^x\right]_0^a$$

$$=\left[(\tfrac18e^{4a}-\tfrac34e^{2a}+e^a)-(\tfrac18e^0-\tfrac34e^0+e^0)\right]$$

$$=\tfrac18e^{4a}-\tfrac34e^{2a}+e^a-(\tfrac18-\tfrac34+1)=\tfrac18(e^{4a}-6e^{2a}+8e^a-3).$$

Ex. 38. *Evaluate* $\int_0^4\int_0^{2\sqrt z}\int_0^{\sqrt{(4z-x^2)}}dz\,dx\,dy$.

(Kanpur 1987; Rohilkhand 79; Meerut 96 P)

Sol. The given triple integral is

$$=\int_0^4\int_0^{2\sqrt z}\left[\int_0^{\sqrt{(4z-x^2)}}dy\right]dz\,dx=\int_0^4\int_0^{2\sqrt z}\left[y\right]_0^{\sqrt{(4z-x^2)}}dz\,dx$$

$$=\int_0^4\left[\int_0^{2\sqrt z}\sqrt{(4z-x^2)}\,dx\right]dz$$

$$=\int_0^4\left[\frac x2\sqrt{(4z-x^2)}+\frac{4z}{2}\sin^{-1}\frac{x}{2\sqrt z}\right]_0^{2\sqrt z}dz$$

$$=\int_0^4\left[0+\frac{4z}{2}\sin^{-1}\frac{2\sqrt z}{2\sqrt z}\right]dz=\int_0^4 2z.\frac\pi2\,dz=\int_0^4\pi z\,dz$$

$$=\pi\left[\frac{z^2}{2}\right]_0^4=\frac\pi2[16]=8\pi.$$

Ex. 39 (a). *Evaluate* $\int_0^a\int_0^{a-x}\int_0^{a-x-y}x^2\,dx\,dy\,dz$.

(Rohilkhand 1976, 80; Kanpur 78; Meerut 96)

Sol. The given triple integral

$$=\int_0^a\int_0^{a-x}\int_0^{a-x-y}x^2\,dx\,dy\,dz$$

$$=\int_0^a\int_0^{a-x}x^2\left[z\right]_0^{a-x-y}dx\,dy,$$

integrating w.r.t. z regarding x and y as constants

$$=\int_0^a\int_0^{a-x}x^2[a-x-y]\,dx\,dy=\int_0^a\int_0^{a-x}x^2[(a-x)-y]\,dx\,dy$$

$$=\int_0^a x^2\left[(a-x)y-\tfrac12y^2\right]_0^{a-x}dx,$$

integrating w.r.t. y regarding x as constant

$$=\int_0^a x^2\left[(a-x)^2-\tfrac12(a-x)^2\right]dx$$

$$=\int_0^a x^2.\tfrac12(a-x)^2\,dx=\tfrac12\int_0^a x^2(a^2-2ax+x^2)\,dx$$

$$=\tfrac12\int_0^a(x^2a^2-2ax^3+x^4)\,dx=\tfrac12\left[a^2\frac{x^3}{3}-2a\frac{x^4}{4}+\frac{x^5}{5}\right]_0^a$$

$$=\tfrac12\left[\tfrac13a^5-\tfrac12a^5+\tfrac15a^5\right]=\tfrac12(\tfrac13-\tfrac12+\tfrac15)a^5=\tfrac1{60}a^5.$$

Ex. 39 (b). *Evaluate the triple integral of the function $f(x,y,z)=x^2$ over the region V enclosed by the planes $x=0, y=0, z=0$ and $x+y+z=a$.*

Sol. The given region V is bounded by the co-ordinate planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=a$. To cover the region V, let the values of x,y lie within the triangle bounded by x-axis, the y-axis and the line $(x+y=a, z=0)$. Then for any point $(x,y,0)$ within this triangle, z varies from $z=0$ to $z=a-x-y$ in the region V.

But the values of x and y vary within the triangle formed in the xy-plane. Therefore x varies from 0 to a and for any intermediary value of x, y varies from 0 to $a - x$.

Therefore the region of integration V can be expressed as

$$0 \le x \le a,\ 0 \le y \le a - x,\ 0 \le z \le a - x - y.$$

Hence the required triple integral

$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\, dx\, dy\, dz$. Now proceed as in Ex. 39 (a).

Ex. 40 (a). *Find the volume of the tetrahedron bounded by the coordinate planes and the plane* $x + y + z = 1$.

Sol. As in Ex. 39, here the region of integration V to cover the volume of the tetrahedron can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y.$$

Therefore the required volume of the tetrahedron

$$= \iiint_V dx\, dy\, dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \left[z\right]_0^{1-x-y} dx\, dy = \int_0^1 \int_0^{1-x} (1 - x - y)\, dx\, dy \qquad \text{(Note)}$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\, dx = \tfrac{1}{2}\left[\frac{(1-x)^3}{3.(-1)}\right]_0^1 = -\frac{1}{6}\,[0 - 1] = \frac{1}{6}.$$

Ex. 40 (b). *Find the volume of the tetrahedron bounded by the plane* $x/a + y/b + z/c = 1$ *and the coordinate planes.* **(Kanpur 1971)**

Sol. Here the region of integration V to cover the volume of the given tetrahedron can be expressed as

$$0 \le x \le a,\ 0 \le y \le b\,(1 - x/a),\ 0 \le z \le c\,(1 - x/a - y/b).$$

Therefore the required volume of the tetrahedron

$$= \iiint_V dx\, dy\, dz = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dx\, dy\, dz.$$

Now proceed as in Ex. 40 (a). The required volume $= \dfrac{abc}{6}$.

Ex. 40 (c). *Evaluate the integral*

$$\iiint xyz\, dx\, dy\, dz$$

over the volume enclosed by three coordinate planes and the plane $x + y + z = 1$. **(Kanpur 1981)**

Sol. The region of integration V enclosed by the three coordinate planes and the plane $x + y + z = 1$ can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y.$$

$\therefore$ the required triple integral $\iiint_V xyz\, dx\, dy\, dz$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} xyz\, dx\, dy\, dz.$$

Now proceed as in Ex. 34 (i).

Ex. 40 (d). *Find the volume of a sphere of radius a by triple integral.* **(Kanpur 1982; Meerut 94, 95)**

Sol. Referred to centre as origin the equation of a sphere of radius a is

$$x^2 + y^2 + z^2 = a^2. \qquad \text{...(1)}$$

The sphere (1) is symmetrical in all the eight octants.

$\therefore$ volume of the sphere (1) $= 8$. (the volume of the part of the sphere lying in the positive octant).

Now for the region consisting of the volume of the sphere (1) lying in the positive octant, we have

$$0 \le x \le a,\ 0 \le y \le \sqrt{(a^2 - x^2)},\ 0 \le z \le \sqrt{(a^2 - x^2 - y^2)}.$$

$\therefore$ the required volume of a sphere of radius a

$$= 8\int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} \int_{z=0}^{\sqrt{(a^2-x^2-y^2)}} dx\, dy\, dz$$

$$= 8\int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} \left[z\right]_{z=0}^{\sqrt{(a^2-x^2-y^2)}} dx\, dy$$

$$= 8\int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{\{(a^2 - x^2) - y^2\}} + \frac{a^2 - x^2}{2}\sin^{-1}\frac{y}{\sqrt{(a^2 - x^2)}}\right]_{y=0}^{\sqrt{(a^2-x^2)}} dx$$

$$= 8\int_0^a \left[0 + \frac{a^2 - x^2}{2}\cdot\frac{\pi}{2} - 0 - 0\right] dx$$

$$= 8\cdot\frac{\pi}{4}\int_0^a (a^2 - x^2)\, dx = 2\pi\left[a^2 x - \frac{x^3}{3}\right]_0^a = 2\pi\left[a^3 - \frac{a^3}{3}\right] = \frac{4}{3}\pi a^3.$$

Ex. 41. *Evaluate* $\iiint (x + y + z)\, dx\, dy\, dz$ *over the tetrahedron* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.

Sol. The region of integration V for the given tetrahedron can be expressed as $0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y$.

Hence the required triple integral $= \iiint_V (x + y + z)\, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \left[(x + y)z + \frac{z^2}{2}\right]_0^{1-x-y} dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \left[(x + y)(1 - x - y) + \frac{(1 - x - y)^2}{2}\right] dx\, dy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left(x+y+\frac{1-x-y}{2}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \tfrac{1}{2}(1-x-y)(1+x+y)\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} [1-(x+y)^2]\,dx\,dy = \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left(1 - x - \frac{1}{3} + \frac{x^3}{3}\right) dx = \frac{1}{2}\int_0^1 \left(\frac{2}{3} - x + \frac{x^3}{3}\right) dx \quad \text{(Note)}$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{3 \times 4}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

Ex. 42. *Evaluate* $\iiint \dfrac{dx\,dy\,dz}{(x+y+z+1)^3}$ *over the region* $x \ge 0, y \ge 0, z \ge 0, x+y+z \le 1$.

Sol. The given region of integration R can be expressed as $0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y$.

Hence the required triple integral $= \iiint_R \dfrac{dx\,dy\,dz}{(x+y+z+1)^3}$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3}\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} (x+y+z+1)^{-3}\,dz\right] dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y} dx\,dy$$

$$= -\tfrac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right] dx\,dy$$

$$= -\tfrac{1}{2}\int_0^1 \left[\frac{y}{4} + \frac{1}{(x+y+1)}\right]_0^{1-x} dx$$

$$= -\tfrac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{(x+1)}\right] dx$$

$$= -\tfrac{1}{2}\left[\frac{(1-x)^2}{2 \times 4 \times (-1)} + \tfrac{1}{2}x - \log(x+1)\right]_0^1$$

$$= -\tfrac{1}{2}[\{0 + \tfrac{1}{2} - \log 2\} - \{-\tfrac{1}{8} + 0 - 0\}] = -\tfrac{1}{2}[\tfrac{1}{2} - \log 2 + \tfrac{1}{8}]$$

$$= -\tfrac{1}{2}[\tfrac{5}{8} - \log 2] = \tfrac{1}{2}[\log 2 - \tfrac{5}{8}].$$

Ex. 43. *Evaluate* $\iiint xyz\,dx\,dy\,dz$ *over the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(Agra 1988)**

Sol. Here the region of integration can be expressed as

$-a \le x \le a, -b\sqrt{\{1-(x^2/a^2)\}} \le y \le b\sqrt{\{1-(x^2/a^2)\}}$

$-c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}} \le z \le c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}$.

$\therefore$ the required triple integral

$$= \int_{-a}^{a} \int_{-b\sqrt{\{1-(x^2/a^2)\}}}^{b\sqrt{\{1-(x^2/a^2)\}}} \left[\int_{-c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}}^{c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}} (xy).z\,dz\right] dx\,dy$$

$= 0$, [$\because z$ is an odd function of z and xy is treated as constant while integrating w.r.t. z]

Ex. 44. *Evaluate* $\iiint z^2\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Sol. Here the region of integration can be expressed as

$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$,

$-\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}$.

$\therefore$ the required triple integral

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} z^2\,dx\,dy\,dz$$

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left[\frac{z^3}{3}\right]_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} dx\,dy$$

$$= \tfrac{1}{3}\int_{-1}^{1} \left[\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} 2(1-x^2-y^2)^{3/2}\,dy\right] dx$$

$$= \frac{2}{3}\int_{-1}^{1} \left[\int_{-\pi/2}^{\pi/2} [(1-x^2)\cos^2\theta]^{3/2}.\sqrt{(1-x^2)}.\cos\theta\,d\theta\right] dx$$

[putting $y = \sqrt{(1-x^2)}\sin\theta$ so that $dy = \sqrt{(1-x^2)}\cos\theta\,d\theta$; also when $y = 0, \theta = 0$ and when $y = \sqrt{(1-x^2)}, \theta = \pi/2$]

$$= \frac{2}{3}\int_{-1}^{1} \left[2.\int_0^{\pi/2} (1-x^2)^2\cos^4\theta\,d\theta\right] dx$$

$$= \frac{4}{3}\int_{-1}^{1} (1-x^2)^2.\ \frac{3.1}{4.2}\cdot\frac{\pi}{2}\,dx = \frac{\pi}{4}\int_{-1}^{1} (1-x^2)^2\,dx$$

$$= \frac{\pi}{4}.2\int_0^1 (1-2x^2+x^4)\,dx = \frac{\pi}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{\pi}{2}\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{\pi}{2}\cdot\frac{8}{15} = \frac{4\pi}{15}.$$

Ex. 45. *Evaluate* $\iiint (z^5 + z)\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Sol. The given region of integration can be expressed as

$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$,

$-\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}$.

Hence the required triple integral

$$=\int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\left[\int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}}(z^5+z)\,dz\right]dx\,dy$$

$=0$. [∵ (z^5+z) is an odd function of z]

Ex. 46. *Evaluate* $\iiint_R u^2 v^2 w\,du\,dv\,dw$, *where* R *is the region* $u^2+v^2\le 1,\ 0\le w\le 1$. **(Meerut 1977)**

Sol. Here the limits of integration to cover the region R can be taken as

$$-1\le u\le 1,\ -\sqrt{(1-u^2)}\le v\le \sqrt{(1-u^2)},\ 0\le w\le 1,$$

where the first integration is to be performed with respect to v.

$$\therefore\ \iiint_R u^2v^2w\,du\,dv\,dw=\int_0^1\int_{-1}^1\int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}}u^2v^2w\,dw\,du\,dv$$

$$=\int_0^1\int_{-1}^1 u^2w\left[\int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}}v^2\,dv\right]dw\,du,$$

because the first integration is to be performed w.r.t. v regarding u and w as constants

$$=\int_0^1\int_{-1}^1\left[2u^2w\int_0^{\sqrt{(1-u^2)}}v^2\,dv\right]dw\,du,$$

because v^2 is an even function of v

$$=\int_0^1\int_{-1}^1 2u^2w\left[\frac{v^3}{3}\right]_0^{\sqrt{(1-u^2)}}dw\,du$$

$$=\frac{2}{3}\int_0^1\int_{-1}^1 wu^2(1-u^2)^{3/2}\,dw\,du$$

$$=\frac{2}{3}\int_0^1\left[w.2\int_0^1 u^2(1-u^2)^{3/2}\,du\right]dw$$

$$=\frac{4}{3}\int_0^1 w\left[\int_0^{\pi/2}\sin^2\theta.\cos^3\theta.\cos\theta\,d\theta\right]dw,\ \text{putting } u=\sin\theta$$

$$=\frac{4}{3}\int_0^1 w\left[\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta\right]dw=\frac{4}{3}\int_0^1 w.\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2}\,dw$$

$$=\frac{\pi}{24}\int_0^1 w\,dw=\frac{\pi}{24}\left[\frac{w^2}{2}\right]_0^1=\frac{\pi}{48}[1-0]=\frac{\pi}{48}.$$

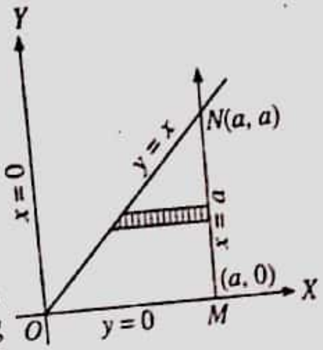
§ 7. Change of Order of Integration.

If in a double integral the limits of integration of both x and y are constant, we can generally integrate $\iint f(x,y)\,dx\,dy$ in either order. But if the limits of y are functions of x, we must first integrate w.r.t. y regarding x as constant and then integrate w.r.t. x. In this case the order

of integration can be changed only if we find the new ... functions of y and the new constant limits of y. This is usually best obtained from geometrical considerations as will be clear from the examples that follow.

Ex. 47 (a). *Change the order of integration in the double integral*

$$\int_0^a\int_0^x f(x,y)\,dx\,dy.$$



Sol. In the given integral the limits of integration are given by the straight lines $y=0,\ y=x,\ x=0$ and $x=a$. Draw these lines bounding the region of integration in the same figure. We observe that the region of integration is the area ONM.

In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as constant and then w.r.t. x.

To reverse the order of integration, we have to integrate first w.r.t. x regarding y as constant and then w.r.t. y. This is done by dividing the area ONM into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line ON (*i.e.*, $y=x$) and terminating on the line MN (*i.e.*, $x=a$). Thus for this region ONM, x varies from y to a and y varies from 0 to a.

Hence by changing the order of integration, we have

$$\int_0^a\int_0^x f(x,y)\,dx\,dy=\int_0^a\int_y^a f(x,y)\,dy\,dx.$$

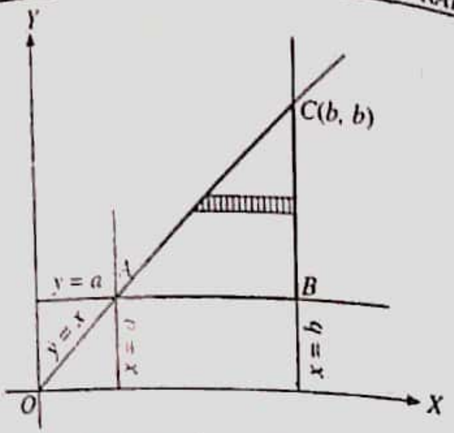
Ex. 47 (b). *Prove that* **(Agra 1977)**

$$\int_a^b dx\int_a^x f(x,y)\,dy=\int_a^b dy\int_y^b f(x,y)\,dx.$$

Sol. Let $I=\int_a^b dx\int_a^x f(x,y)\,dy$.

We are required to change the order of integration in the integral I. In the integral I the limits of integration of y are given by the straight lines $y=a$ and $y=x$. Also the limits of integration of x are given by $x=a$ and $x=b$. Draw the straight lines $y=a,\ y=x,\ x=a$ and $x=b$, bounding the region of integration, in the same figure. We observe that the region of integration is the area of the triangle ABC.

In the integral I we are required to integrate first w.r.t. y and then w.r.t. x. To reverse the order of integration we have to integrate first w.r.t. x and then w.r.t. y. This is done by dividing the area ABC into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line AC (*i.e.*, $y=x$) and terminating on the line



BC (*i.e.*, $x = b$). Thus for the region ABC, x varies from y to b and y varies from a to b. Hence by changing the order of integration, we have

$$\int_a^b dx \int_a^x f(x, y)\, dy = \int_a^b dy \int_y^b f(x, y)\, dx.$$

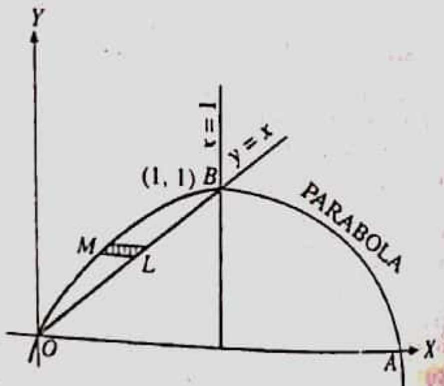
Ex. 48 (a). *Change the order of integration in*

$$\int_0^1 \int_x^{x(2-x)} f(x, y)\, dx\, dy.$$

Sol. In the given integral the limits of integration of y are given by $y = x$, which is a straight line passing through the origin, and $y = x(2 - x)$ or $y = 2x - x^2$ or $(x - 1)^2 = -(y - 1)$ which is a parabola with vertex $(1, 1)$ and passing through the origin.

Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = 1$ which is a straight line parallel to the y-axis at a distance 1 from the origin.

We draw the curves $y = x$, $(x - 1)^2 = -(y - 1)$, $x = 0$ and $x = 1$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OLBMO$.



In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as a constant and then w.r.t. x.

...we want to reverse the order of integration, we have to first ...ate w.r.t. x regarding y as a constant and then we integrate w.r.t. ...is done by covering the area of integration $OLBMO$ by drawing ...aight lines $y =$ constant *i.e.*, by dividing this area into strips ...el to the x-axis.

So divide the region $OLBMO$ into strips parallel to the x-axis ...ng from the arc OMB of the parabola and terminating on the line ...

For the point B, $x = 1$. Putting $x = 1$ in the equation of the line ..., we get $y = 1$. So the y-coordinate of the point B is also 1.

For the region $OMBLO$, the lower limit of x is the value of x ...d in terms of y from the equation $(x - 1)^2 = 1 - y$ and the upper ...t of x is the value of x found in terms of y from the equation ... From the equation $(x - 1)^2 = 1 - y$, we get ...$-1 = \pm\sqrt{(1 - y)}$ or $x = 1 \pm \sqrt{(1 - y)}$. Since in the region ...BLO, x takes values less than 1, therefore we take ...$= 1 - \sqrt{(1 - y)}$.

Thus in the region $OMBLO$, x varies from $1 - \sqrt{(1 - y)}$ to y and ...aries from 0 to a.

Hence by changing the order of integration, we have the given ...tegral

$$= \int_0^1 \int_{1-\sqrt{(1-y)}}^{y} f(x, y)\, dy\, dx.$$

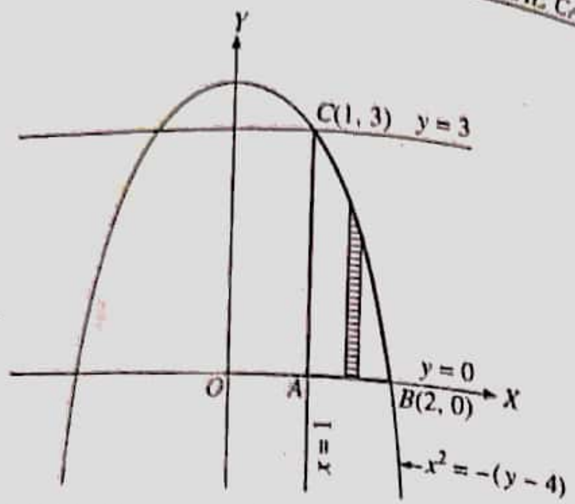
Ex. 48 (b). *Change the order of integration in the integral*

$$\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y)\, dy\, dx.$$ **(Meerut 1993; Agra 79)**

Sol. In the given integral the limits of integration of x are given by the straight line $x = 1$ and the curve $x = \sqrt{(4 - y)}$ *i.e.*, $x^2 = 4 - y$ *i.e.*, $x^2 = -(y - 4)$ which is a parabola, symmetrical about the y-axis, with vertex at the point $(0, 4)$ and existing in the region $y \le 4$. Again the limits of integration of y are given by the straight lines $y = 0$ (*i.e.*, the x-axis) and $y = 3$.

We draw the curves $x = 1$, $x^2 = -(y - 4)$, $y = 0$ and $y = 3$, giving the limits of integration in the same figure. Putting $x = 1$ in the equation $x^2 = -(y - 4)$, we get $y = 3$. Thus the straight line $y = 3$ passes through the point of intersection C of $x = 1$ and $x^2 = -(y - 4)$. Also at the point of intersection B of the parabola $x^2 = -(y - 4)$ and the x-axis (*i.e.*, the line $y = 0$), we have $x = 2$. We observe that the region of integration is the area $ABCA$.

In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y.



If we want to change the order of integration, we have to first integrate w.r.t. y regarding x as a constant and then we integrate w.r.t. x. This is done by covering the area $ABCA$ by strips drawn parallel to the y-axis. These strips start from the line AB (*i.e.*, $y = 0$) and terminate on the arc BC of the parabola $x^2 = 4 - y$. Therefore for the region $ABCA$, y varies from 0 to $4 - x^2$ and x varies from 1 to 2. Hence by changing the order of integration, we have the given integral

$$= \int_1^2 \int_0^{4-x^2} (x + y)\, dx\, dy.$$

Ex. 49 (a). *Change the order of integration in*

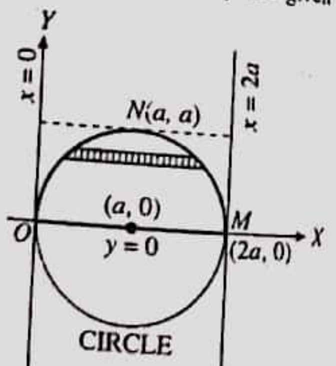
$$\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} f(x, y)\, dx\, dy.$$

(Meerut 1991)

Sol. In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = \sqrt{(2ax - x^2)}$ *i.e.*, $y^2 = 2ax - x^2$ *i.e.*, $(x - a)^2 + y^2 = a^2$ which is a circle with centre $(a, 0)$ and radius a. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = 2a$.

Draw the curves

$(x - a)^2 + y^2 = a^2$, $y = 0$, $x = 0$ and $x = 2a$, bounding the region of integration, in the same figure. From figure we observe that the area of integration is $OMNO$.



... given integral we ... w.r.t. x.
... y as a constant and then w.r.t. x. ... reverse the order of integration, divide the area $OMNO$ into ... parallel to the x-axis. These strips will have their extremities on ... ON and NM of the circle.

... the equation of circle $(x - a)^2 + y^2 = a^2$ for x, we get ... $= a^2 - y^2$ *i.e.*, $x - a = \pm \sqrt{(a^2 - y^2)}$ *i.e.*, $x = a \pm \sqrt{(a^2 - y^2)}$.

... for the region $OMNO$, x varies from $a - \sqrt{(a^2 - y^2)}$ to ... $- y^2)$ and y varies from 0 to a.

... therefore, changing the order of integration, the given double ... transforms to

$$\int_0^a \int_{a - \sqrt{(a^2 - y^2)}}^{a + \sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx.$$

Ex. 49 (b). *Change the order of integration*

$$\int_0^a \int_{a - \sqrt{(a^2 - y^2)}}^{a + \sqrt{(a^2 - y^2)}} dy\, dx.$$

(Meerut 1995)

Sol. In the given integral the limits of integration of x are given ... $= a - \sqrt{(a^2 - y^2)}$ and $x = a + \sqrt{(a^2 - y^2)}$ and those of y are given ... $= 0$ and $y = a$.

When $x = a - \sqrt{(a^2 - y^2)}$ or $x = a + \sqrt{(a^2 - y^2)}$, we have $(x - a)^2 = a^2 - y^2$ or $(x - a)^2 + y^2 = a^2$ or $y^2 = 2ax - x^2$ ... is a circle with centre $(a, 0)$ and radius a.

So the region of integration is the area $OMNO$ as shown in the ... of Ex. 49 (a).

To reverse the order of integration, we divide the area $OMNO$ ... strips parallel to the y-axis. These strips will have their extremities ... the x-axis and on the circular arc given by $y = \sqrt{(2ax - x^2)}$. Also x ... go from 0 to $2a$.

Hence changing the order of integration, the given double integral ... transforms to

$$\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} dx\, dy.$$

Ex. 50. *Change the order of integration in the double integral*

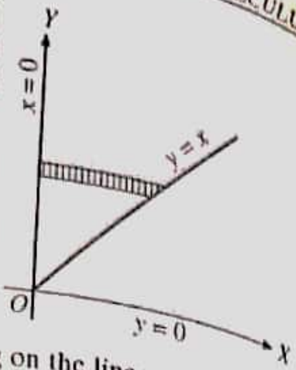
$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$$

and hence find its value. **(Agra 1984; Gorakhpur 85; Meerut 92)**

Sol. In the given integral the limits of integration are given by the lines $y = x$, $y = \infty$, $x = 0$ and $x = \infty$. Therefore the region of integration is bounded by $x = 0$, $y = x$ and, an infinite boundary. In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y

regarding x as constant and then we integrate w.r.t. x. This is done by first integrating w.r.t. y along a strip drawn parallel to the y-axis and then integrating w.r.t. x along all such strips so drawn as to cover the whole region of integration.

If we want to reverse the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by dividing this area into strips parallel to the x-axis. So we take strips parallel to the x-axis starting from the line $x = 0$ and terminating on the line $y = x$. Now the limits for x are 0 to y and the limits for y are 0 to ∞.



Hence by changing the order of integration, we have

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dy\,dx$$

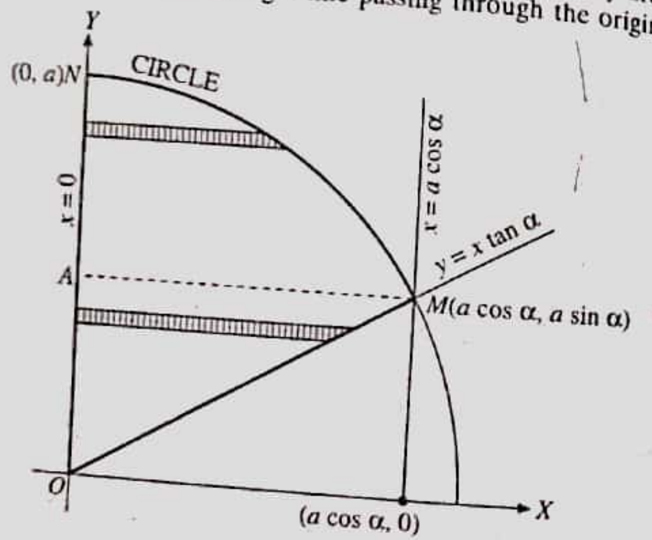
$$= \int_0^\infty \frac{e^{-y}}{y}\left[x\right]_0^y dy = \int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy = \int_0^\infty e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = 1.$$

****Ex. 51 (a).** *Change the order of integration in the integral*

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

(Agra 1989; Kanpur 86; Gorakhpur 88)

Sol. In the given integral the limits of integration of y are given by $y = x\tan\alpha$ which is a straight line passing through the origin and



$y = \sqrt{(a^2 - x^2)}$ *i.e.*, $y^2 = a^2 - x^2$

or $x^2 + y^2 = a^2$

which is a circle of radius a with centre at the origin $(0, 0)$.

Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = a\cos\alpha$ which is a straight line parallel to the y-axis at a distance $a\cos\alpha$ from the origin.

We draw the curves $y = x\tan\alpha$, $x^2 + y^2 = a^2$, $x = 0$ and $x = a\cos\alpha$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OMNO$.

In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of integration $OMNO$ by drawing the straight lines $x =$ constant *i.e.*, by dividing this area into strips parallel to the y-axis.

If we want to reverse the order of integration, we have to first integrate with respect to x regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration $OMNO$ by drawing the straight lines $y =$ constant *i.e.*, by dividing this area into strips parallel to the x-axis.

Now if we take strips prallel to the x-axis starting from the line $x = 0$, some of these strips end on the line OM while the others end on the arc MN of the circle $x^2 + y^2 = a^2$. So we draw the line of demarcation MA dividing the area $OMNO$ into two portions OMA and AMN.

For the point M, $x = a\cos\alpha$. Putting $x = a\cos\alpha$ in the equation of the line $y = x\tan\alpha$, we get $y = a\sin\alpha$. So the y-coordinate of the point M is $a\sin\alpha$ and the equation of the line of demarcation MA is $y = a\sin\alpha$.

For the region OMA, x varies from 0 to $y\cot\alpha$ and y varies from 0 to $a\sin\alpha$.

For the region AMN, x varies from 0 to $\sqrt{(a^2 - y^2)}$ and y varies from $a\sin\alpha$ to a.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dy\,dx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx.$$

Ex. 51 (b). *Change the order of integration in the integral*

$$\int_0^a\int_0^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

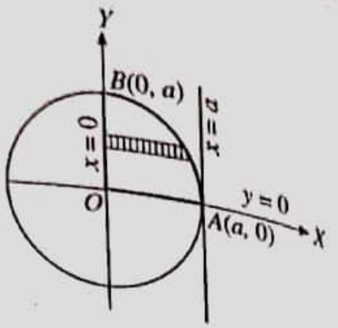
(Agra 1978; Meerut 98)

Sol. In the given integral the limits of integration of y are given by the straight line $y = 0$ (*i.e.*, the x-axis) and the curve

$$y = \sqrt{(a^2 - x^2)} \text{ i.e., } y^2 = a^2 - x^2 \text{ i.e., } x^2 + y^2 = a^2$$

which is a circle with centre at the origin and radius a. Again the limits of integration of x are given by the lines $x = 0$ and $x = a$.

We draw the curves $y = 0, x^2 + y^2 = a^2, x = 0$ and $x = a$, giving the limits of integration, in the same very figure and we observe that the region of integration is the area OAB of the quadrant of the circle $x^2 + y^2 = a^2$.



To change the order of integration in the given integral, we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covering the area OAB by strips drawn parallel to the x-axis. These strips start from the line OB (i.e., $x = 0$) and terminate on the arc AB of the circle $x^2 + y^2 = a^2$. So on these strips x varies from 0 to $\sqrt{(a^2 - y^2)}$. Also to cover the area OAB, y varies from 0 to a. Hence by changing the order of integration, we have the given integral

$$= \int_0^a \int_0^{\sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx.$$

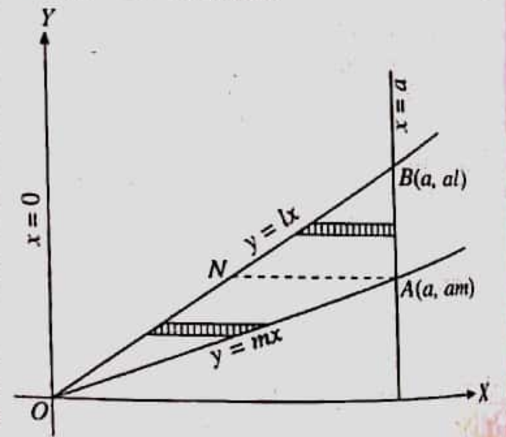
Ex. 52. *Change the order of integration in*

$$\int_0^a \int_{mx}^{lx} f(x, y)\, dx\, dy.$$

(Agra 1985)

Sol. Here the area of integration is bounded by the straight lines $y = mx, y = lx, x = 0$ and $x = a$. Drawing all these lines in one figure, we observe that the area of integration is $OABO$.

To reverse the order of integration, cover this area $OABO$ by strips parallel to the axis of x. Draw the straight line AN parallel to the x-axis and thus divide the area $OABO$ into two portions OAN and NBA according to the character of the strips.



For the point $A, x = a$. Putting $x = a$ in the equation of the line $y = mx$, we get $y = ma$. Also for the point $B, x = a$; therefore putting $x = a$ in the equation of the line $y = lx$, we get $y = la$.

Now for the area ONA, x varies from the line $y = lx$ to $y = mx$ i.e., x varies from y/l to y/m and y varies from 0 to am. Again for the area NBA, x varies from the line NB $(y = lx)$ to the line $x = a$ i.e., x varies from y/l to a and y varies from am to al.

Therefore, by changing the order of integration the given integral transforms to

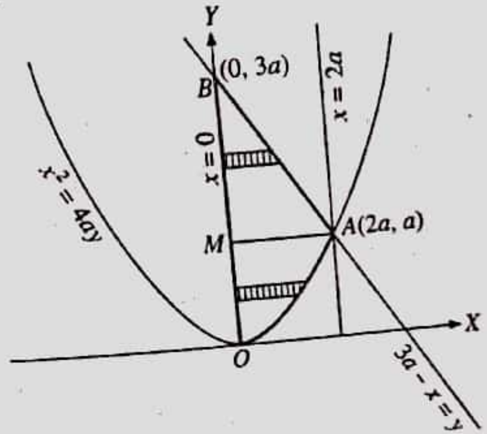
$$\int_0^{am} \int_{y/l}^{y/m} f(x, y)\, dy\, dx + \int_{am}^{al} \int_{y/l}^{a} f(x, y)\, dy\, dx.$$

Ex. 53 (a). *Change the order of integration in*

$$\int_0^{2a} \int_{x^2/4a}^{3a - x} f(x, y)\, dx\, dy.$$

(Meerut 1991 P)

Sol. In the given integral the limits of integration are given by $x^2/4a = y$ i.e., $x^2 = 4ay$, (which is a parabola passing through the origin), and the lines $y = 3a - x$, $x = 0$, and $x = 2a$. Drawing these curves in one figure we observe that the region of integration is the area $OABMO$.



To change the order of integration, first we divide the region of integration into two portions OAM and MAB, by drawing the line AM parallel to the x-axis. Now to reverse the order of integration, cover the whole region $OABMO$ by strips parallel to the x-axis starting from the line $x = 0$. Some of these strips end on the arc OA while others end on the line AB.

For the point A, we have $x = 2a$. Putting $x = 2a$ in the equation of the line $y = 3a - x$, we get $y = a$.

For the region OAM, x varies from 0 to $\sqrt{(4ay)}$ and y varies from 0 to a. Again for the region MAB, x varies from 0 to $3a - y$ and y varies from a to $3a$.

Hence the transformed integral is given by

$$\int_0^a \int_0^{\sqrt{(4ay)}} f(x, y)\, dy\, dx + \int_a^{3a} \int_0^{3a-y} f(x, y)\, dy\, dx.$$

Ex. 53 (b). *Change the order of integration*

$$\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\, dy.$$

(Meerut 1994)

Sol. Proceed as in Ex. 53 (a).

In the given integral the limits of integration are given by $x^2/a = y$ *i.e.*, $x^2 = ay$ (which is a parabola passing through the origin), and the straight lines $y = 2a - x$, $x = 0$ and $x = a$.

Draw figure as in Ex. 53 (a).

Here the coordinates of A are (a, a) and those of B are $(0, 2a)$.

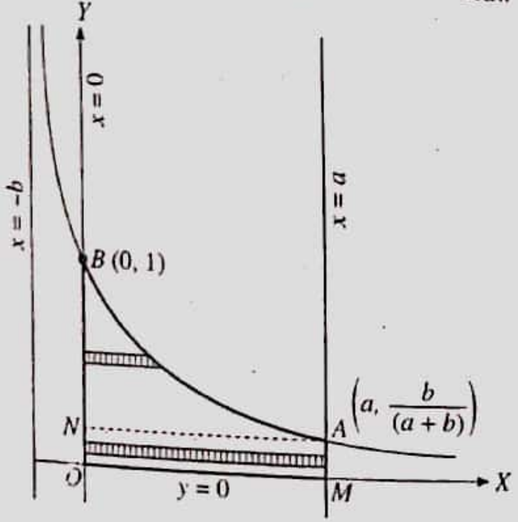
The transformed integral is given by

$$\int_0^a \int_0^{\sqrt{(ay)}} xy\, dy\, dx + \int_a^{2a} \int_0^{2a-y} xy\, dy\, dx.$$

Ex. 54. *Change the order of integration in the double integral*

$$\int_0^a \int_0^{b/(b+x)} f(x, y)\, dx\, dy.$$

Sol. In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = b/(b+x)$ *i.e.*, $y(b+x) = b$ which is a rectangular hyperbola having for its asymptotes the straight lines $y = 0$ and $x = -b$. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$. We draw the curves



…$(b+x) = b$, $y = 0$, $x = 0$ and $x = a$, giving the limits of integration, …same figure. We observe that the region of integration is the …$OMABO$.

…the given integral we are required to integrate first w.r.t. y and …w.r.t. x. To change the order of integration, we have to first …w.r.t. x regarding y as constant and then we integrate w.r.t. … This is done by covering the area of integration $OMABO$ by drawing …straight lines $y =$ constant *i.e.*, by dividing this area into strips …parallel to the x-axis.

Now if we take strips parallel to the x-axis originating from the …$x = 0$, some of these strips terminate on the line AM while the …terminate on the arc AB. So according to the character of the …we divide the region of integration into two portions namely …A and NAB, by drawing the line AN parallel to the axis of x.

For the point B, $x = 0$. Putting $x = 0$ in the equation …$(b+x) = b$, we get $y = 1$. So the coordinates of the point …are $(0, 1)$.

Similarly putting $x = a$ in the equation $y(b+x) = b$, we get …$b/(a+b)$ and thus the coordinates of the point A are …$(a, b/(a+b))$.

For the area $NOMA$, x varies from 0 to a and y varies from 0 to …$/(a+b)$.

For the area NBA, x varies from 0 to $b(1-y)/y$ and y varies from …$/(a+b)$ to 1.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{b/(a+b)} \int_0^a f(x, y)\, dy\, dx + \int_{b/(a+b)}^1 \int_0^{b(1-y)/y} f(x, y)\, dy\, dx.$$

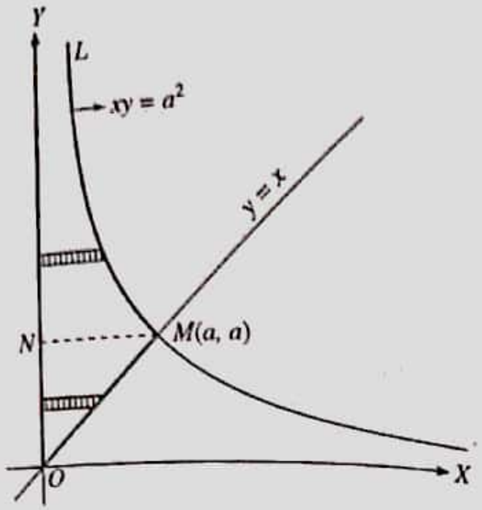
Ex. 55. *Change the order of integration in*

$$\int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy.$$

Sol. In the given integral the limits of integration of y are given by $y = x$ which is a straight line passing through the origin equally inclined to both the axes and $y = a^2/x$ or $xy = a^2$ which is a rectangular hyperbola. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$.

We draw the curves $y = x$, $xy = a^2$, $x = 0$, and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $LMOY$ ... extended upto infinity on the above side.

In the given integral we are required to integrate first w.r.t. y and then w.r.t. x. If we want to change the order of integration, we have to



first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration by strips parallel to the x-axis.

Now if we take strips parallel to the x-axis starting from the line $x = 0$, some of these strips end on the line OM while the others end on the arc ML of the rectangular hyperbola. So we divide the region of integration into two portions, the triangle OMN and the area $YNML$ which extends upto infinity, by drawing the line MN parallel to the axis of x.

For the point $M, x = a$. Putting $x = a$ in the equation of the line $y = x$ or the rectangular hyperbola $xy = a^2$, we get $y = a$.

So the y-coordinate of the point M is a and the equation of the line of demarcation MN is $y = a$.

For the area OMN, x varies from 0 to y and y varies from 0 to a.

For the area $YMNL$ ..., x varies from 0 to a^2/y and y varies from a to ∞.

Hence by changing the order of integration, we have the given integral

$$= \int_0^a \int_0^y f(x,y)\, dy\, dx + \int_a^\infty \int_0^{a^2/y} f(x,y)\, dy\, dx.$$

Ex. 56. *Change the order of integration in*

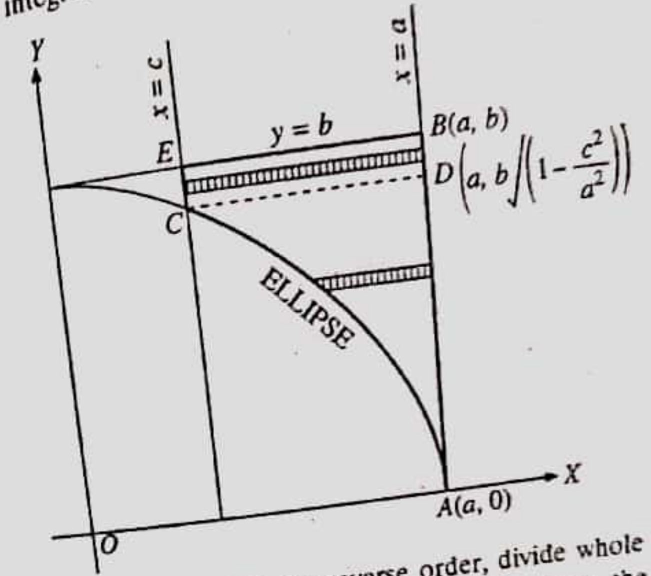
$$\int_c^a \int_{(b/a)\sqrt{(a^2-x^2)}}^b f(x,y)\, dx\, dy, \text{ where } c < a.$$

Sol. In the given integral the limits of integration ...

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)} \text{ i.e., } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

...h is an ellipse with centre (0, 0) and the straight line $y = b$. Again the limits of integration of x are given by the straight lines ...c and $x = a$.

Draw the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the straight lines $y = b$, ...c and $x = a$, bounding the region of integration, in the same figure. ...e observe that the region of integration is the area $ABECA$. In the ...en integral, the limits of integration of y being variable, we are ...quired to integrate first w.r.t. y and then w.r.t. x.



In order to integrate in the reverse order, divide whole the area into strips parallel to the x-axis originating either from the line EC (i.e., $x = c$) or from the arc AC of the ellipse and terminating on the line BA (i.e., $x = a$). While integrating we must first obviously divide the region of integration $ABECA$ into two portions CAD and $ECDB$ according to the character of the strips. For the point $C, x = c$. Putting $x = c$ in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $y = b\sqrt{\{1 - (c^2/a^2)\}}$ which is the y-coordinate of the point C. The equation of the line of demarcation CD is thus $y = b\sqrt{\{1 - (c^2/a^2)\}}$.

For the area CAD, x varies from $a\sqrt{\{1 - (y^2/b^2)\}}$ to a and y varies from 0 to $b\sqrt{\{1 - (c^2/a^2)\}}$.

For the area $ECDB$, x varies from c to a and y varies from $b\sqrt{\{1-(c^2/a^2)\}}$ to b.

Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{b\sqrt{(1-(c^2/a^2))}}\int_{a\sqrt{(1-(y^2/b^2))}}^{a} f(x,y)\,dy\,dx + \int_{b\sqrt{(1-(c^2/a^2))}}^{a}\int_c^a f(x,y)\,dy\,dx.$$

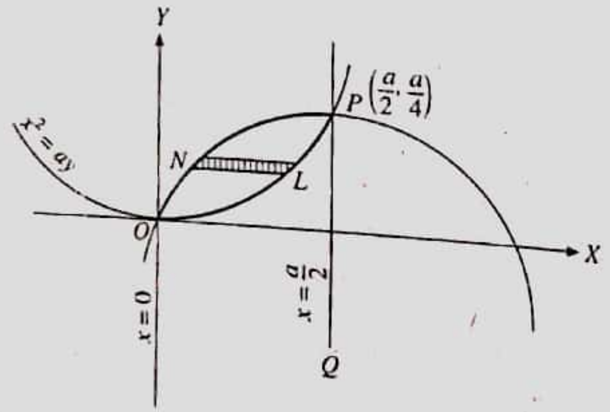
Ex. 57. *Change the order of integration in*

$$\int_0^{a/2}\int_{x^2/a}^{x-(x^2/a)} f(x,y)\,dx\,dy.$$

(Meerut 1982, Gorakhpur 86, Kanpur 89)

Sol. In the given integral the limits of integration of y are given by $y = x^2/a$ i.e., $x^2 = ay$ which is a parabola with vertex $(0,0)$ and $x - x^2/a = y$ i.e., $ax - x^2 = ay$ i.e., $(x-\frac{1}{2}a)^2 = -a(y-\frac{1}{4}a)$ which is also a parabola with vertex $(\frac{1}{2}a, \frac{1}{4}a)$.

The points of intersection of the two parabolas are $(0,0)$ and $(\frac{1}{2}a, \frac{1}{4}a)$.



Again the limits of integration of x are given by $x = 0$ i.e., the y-axis and $x = a/2$ which is a straight line parallel to the y-axis at a distance $a/2$ from the origin.

Draw the two parabolas

$$x^2 = ay \text{ and } (x-\tfrac{1}{2}a)^2 = -a(y-\tfrac{1}{4}a)$$

intersecting at $O(0,0)$ and $P(\frac{1}{2}a, \frac{1}{4}a)$ along with the lines $x = 0$ and $x = a/2$ in the same figure. We observe that the region of integration is $ONPLO$. In the given integral we are required to

...grate first w.r.t. y ($\because$ the limits of integration of y are variable) and ...w.r.t. x. To reverse the order of integration, draw strips parallel ...the x-axis originating from the arc ONP of the parabola ...$x^2 = ay$ and terminating on the arc OLP of the parabola ...$= ay$. Then for the region $ONPLO$, the limits of integration for x ...given by $ax - x^2 = ay$ and $x^2 = ay$. Solving $ay = ax - x^2$ i.e., ...$-ax + ay = 0$ for x, we get

$$x = \tfrac{1}{2}[a \pm \sqrt{(a^2 - 4ay)}]$$

$$x = \tfrac{1}{2}[a - \sqrt{(a^2 - 4ay)}],$$

...ing the +ive sign since x cannot be greater than $\frac{1}{2}a$ in the region ...PLO.

Thus the limits of x are

$$x = \tfrac{1}{2}[a - \sqrt{(a^2 - 4ay)}] \text{ and } x = \sqrt{(ay)}.$$

Clearly for this region y varies from 0 to $\frac{1}{4}a$.

Hence by changing the order of integration, we have

$$\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} f(x,y)\,dx\,dy = \int_0^{a/4}\int_{\frac{1}{2}[a-\sqrt{(a^2-4ay)}]}^{\sqrt{(ay)}} f(x,y)\,dy\,dx.$$

Ex. 58 (a). *Change the order of integration in the double integral*

$$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x,y)\,dx\,dy.$$

(Kanpur 1976; Gorakhpur 87)

Sol. In the given integral the limits of integration of y are given by

$$y = \sqrt{(2ax - x^2)}$$

i.e., $$y^2 = 2ax - x^2$$

i.e., $$(x-a)^2 + y^2 = a^2$$

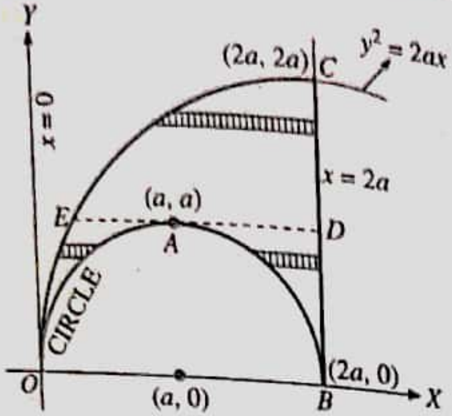
which is a circle with centre $(a, 0)$ and radius a and $y = \sqrt{(2ax)}$ i.e., $y^2 = 2ax$ which is a parabola with vertex $(0,0)$ and the x-axis as its axis. Again the limits of integration of x are given by $x = 0$ i.e., the y-axis and $x = 2a$, a line parallel to the y-axis at a distance $2a$ from the origin.

We draw the curves

$$(x-a)^2 + y^2 = a^2, y^2 = 2ax, x = 0$$

and $x = 2a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OABCO$.

To reverse the order of integration, cover this area of integration $OABCO$ by strips parallel to the x-axis. Through A, draw the line EAD parallel to the x-axis (i.e., tangent to the circle at A) so that the region of integration is divided into three portions OEA, ABD and ECD.



For the point A, $x = a$. Putting $x = a$ in $(x-a)^2 + y^2 = a^2$, we get $y = a$ as the y-coordinate of A.

For the point C, $x = 2a$; therefore from $y^2 = 2ax$, we get $y = 2a$ at C.

Now from the equation of the circle $(x-a)^2 + y^2 = a^2$, we have $x = a \pm \sqrt{(a^2 - y^2)}$ *i.e.*, x for the arc OA is given by $a - \sqrt{(a^2 - y^2)}$ and for the arc AB, x is given by $a + \sqrt{(a^2 - y^2)}$.

Now for the region OEA, x varies from $y^2/2a$ (which is the value of x on the arc OE of the parabola $y^2 = 2ax$) to $a - \sqrt{(a^2 - y^2)}$ which is the value of x on the arc OA of the circle and y varies from 0 to a. For the region ABD, x varies from the arc AB of the circle to the straight line BD (*i.e.*, x varies from $a + \sqrt{(a^2 - y^2)}$ to $2a$ and y varies from 0 to a.

And for the region ECD, x varies from the arc EC of the parabola to the straight line $x = 2a$ *i.e.*, x varies from $y^2/2a$ to $2a$ and y varies from a to $2a$.

Hence the transformed integral is

$$= \int_0^a \int_{y^2/2a}^{a - \sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx$$

$$+ \int_0^a \int_{a + \sqrt{(a^2 - y^2)}}^{2a} f(x, y)\, dy\, dx + \int_a^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dy\, dx.$$

Ex. 58 (b). *Change the order of integration in*

$$\int_0^a \int_{\sqrt{(ax - x^2)}}^{\sqrt{(ax)}} f(x, y)\, dx\, dy.$$

Sol. Proceeding exactly as in Ex. 58 (a) above, the given double integral transforms to

$$\int_0^{a/2} \int_{y^2/a}^{\frac{1}{2}[a - \sqrt{(a^2 - 4y^2)}]} f(x, y)\, dy\, dx$$

$$+ \int_0^{a/2} \int_{\frac{1}{2}[a + \sqrt{(a^2 - 4y^2)}]}^{a} f(x, y)\, dy\, dx + \int_{a/2}^{a} \int_{y^2/a}^{a} f(x, y)\, dy\, dx.$$

Ex. 58 (c). *Change the order of integration in* $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\, dx\, dy$ *and hence find its value.* **(Meerut 1993 P)**

Sol. In the given integral the limits of integration of y are given by $y = \sqrt{x}$ and $y = 1$.

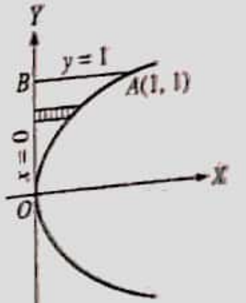
When $y = \sqrt{x}$, we have $y^2 = x$ which is a parabola with vertex at $(0, 0)$ and the axis of x as its axis. Also the limits of integration of x are given by $x = 0$ and $x = 1$.

The region of integration is the area $OABO$.

To reverse the order of integration, we divide the area $OABO$ into strips parallel to the x-axis.

Changing the order of integration, the given double integral I transforms to

$$I = \int_{y=0}^{1} \int_{x=0}^{y^2} e^{x/y}\, dy\, dx.$$



We have $$I = \int_{y=0}^{1} \left[y e^{x/y}\right]_{x=0}^{y^2} dy$$

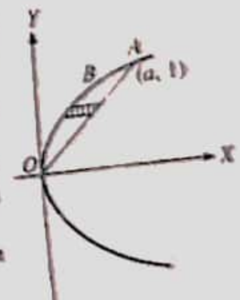
$$= \int_0^1 (y e^y - y)\, dy$$

$$= \left[y e^y - e^y - \frac{y^2}{2}\right]_0^1$$

$$= e - e - \tfrac{1}{2} + 1 = \tfrac{1}{2}.$$

Ex. 58 (d). *Change the order of integration* **(Meerut 1994 P, 95 BP)**

$$\int_0^a \int_{x/a}^{\sqrt{(x/a)}} (x^2 + y^2)\, dx\, dy.$$



Sol. In the given integral the limits of integration of y are given by $y = x/a$ and $y = \sqrt{(x/a)}$.

$y = x/a$ is a straight line passing through the origin.

When $y = \sqrt{(x/a)}$, we have $y^2 = x/a$ and which is a parabola with vertex at $(0, 0)$ and x-axis as its axis.

The straight line $y = x/a$ meets $y^2 = x/a$ at the point $A(a, 1)$.

The limits of integration of x are given by $x = 0$ and $x = a$. Thus the region of integration is the area $OABO$.

To reverse the order of integration, we divide the area $OABO$ into strips parallel to x-axis.

Changing the order of integration, the given double integral I transforms to

$$I = \int_{y=0}^{1} \int_{x=ay^2}^{ay} (x^2 + y^2)\, dy\, dx.$$

Ex. 59. *Change the order of integration in*

$$\int_0^a \int_{\sqrt{(a^2 - x^2)}}^{x + 2a} f(x, y)\, dx\, dy.$$

Sol. Here the area of integration is bounded by the curves

$$y = \sqrt{(a^2 - x^2)}$$

i.e.,

$$x^2 + y^2 = a^2$$

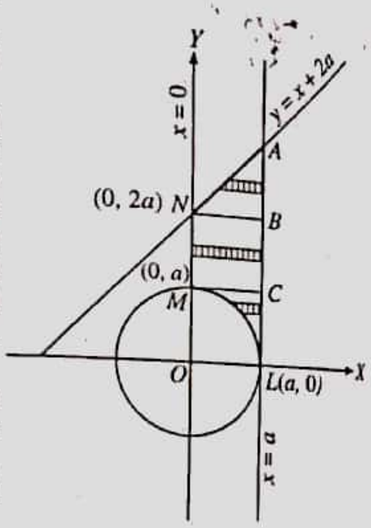
which is a circle with centre $(0, 0)$ and radius a, $y = x + 2a$ which is a straight line passing through $(0, 2a)$, $x = 0$ *i.e.*, the y-axis and the line $x = a$ which is a line parallel to the y-axis at a distance a from the origin.

We draw the curves

$$x^2 + y^2 = a^2,\ y = x + 2a,\ x = 0$$

and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $MLANM$.

To reverse the order of integration, cover this area of integration $MLANM$ by strips parallel to the x-axis. Draw the lines MC and NB parallel to the x-axis so that the region of integration $MLANM$ is divided into three portions MLC, $NMCB$ and NAB.



For the region MLC, x varies from the arc ML of the circle $x^2 + y^2 = a^2$ to the line $x = a$ *i.e.*, x varies from $\sqrt{(a^2 - y^2)}$ to a and y varies from 0 to a.

For the region $NMCB$, x varies from 0 to a and y varies from a to $2a$.

For the region NBA, x varies from $y - 2a$ to a and y varies from $2a$ to $3a$.

Therefore, changing the order of integration, the given integral transforms to

$$\int_0^a \int_{\sqrt{(a^2 - y^2)}}^a f(x, y)\, dy\, dx + \int_a^{2a} \int_0^a f(x, y)\, dy\, dx$$
$$+ \int_{2a}^{3a} \int_{y - 2a}^a f(x, y)\, dy\, dx.$$

Ex. 60. *Change the order of integration in the double integral*

$$\int_0^{ab/\sqrt{(a^2 + b^2)}} \int_0^{(a/b)\sqrt{(b^2 - y^2)}} f(x, y)\, dy\, dx.$$

Sol. In the given integral the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and

$$x = (a/b)\sqrt{(b^2 - y^2)}\ i.e., x^2/a^2 + y^2/b^2 = 1$$

which is an ellipse with centre as origin. **(Note)**

Again the limits of integration of y are given by $y = 0$ *i.e.*, the x-axis and $y = ab/\sqrt{(a^2 + b^2)}$ which is a straight line parallel to the x-axis at a distance $ab/\sqrt{(a^2 + b^2)}$ from the origin.

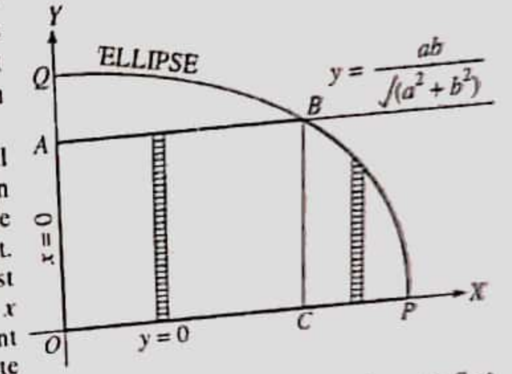
We draw the curves

$$x = 0,\ x^2/a^2 + y^2/b^2 = 1,\ y = 0$$

and

$$y = ab/\sqrt{(a^2 + b^2)},$$

giving the limits of integration, in the same figure. We observe that the region of integration is the area $OPBAO$.



In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y.

If we want to reverse the order of integration, we have to first integrate w.r.t. y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of integration $OPBAO$ by strips parallel to the y-axis. Now if we take strips parallel to the y-axis starting from the line $y = 0$, some of these strips end on the line AB while the others end on the arc BP of the ellipse. So we draw the line of demarcation BC dividing the area $OPBAO$ into two portions $OCBA$ and BCP. For the point B, $y = ab/\sqrt{(a^2 + b^2)}$. Putting this value of y in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get

$x = ab/\sqrt{(a^2+b^2)}$. For the region $OCBA$, y varies from 0 to $ab/\sqrt{(a^2+b^2)}$ and x varies from 0 to $ab/\sqrt{(a^2+b^2)}$.

For the region BCP, y varies from 0 to $(b/a)\sqrt{(a^2-x^2)}$ and x varies from $ab/\sqrt{(a^2+b^2)}$ to a.

Hence the given integral transforms to

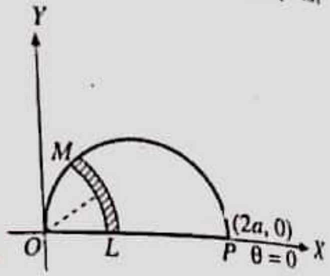
$$\int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x,y)\,dx\,dy$$

$$+\int_{ab/\sqrt{(a^2+b^2)}}^{0} \int_a^{(b/a)\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$$

Ex. 61. *Change the order of integration in*

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,d\theta\,dr.$$

Sol. Here the region of integration is bounded by the polar curves $r=0$ (the pole), $r=2a\cos\theta$ (a circle of diameter $2a$ passing through the pole), $\theta=0$ (the initial line) and $\theta=\pi/2$ (a line through the pole perpendicular to initial line).



We draw the curves $r=0$, $r=2a\cos\theta$, $\theta=0$ and $\theta=\pi/2$, giving the limits of integration, in the same figure.

We observe that the region of integration is the area of the semi-circle $OMPO$.

In the given integral the limits of integration of r are variable while those of θ are constant. Thus we have to first integrate with respect to r regarding θ as a constant and then we integrate w.r.t. θ.

If we want to reverse the order of integration, we have to first integrate with repsect to θ regarding r as constant and then we integrate w.r.t. r. This is done by covering the area of integration $OMPO$ by circular arcs with centre as pole. On these arcs θ varies and r remains constant. Thus for the area $OMPO$, for a fixed value of r, θ varies from the initial line (*i.e.*, $\theta=0$ to a point on the arc OMP of the circle $r=2a\cos\theta$ *i.e.*, to a point for which $\theta=\cos^{-1}(r/2a)$ and r varies from 0 to $2a$.

Hence by chainging the order of integration, we have

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,d\theta\,dr$$

$$=\int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)\,dr\,d\theta.$$

...Change of variables in a double integral.

Sometimes, the evaluation of a double integral becomes more convenient by a suitable change of variables from one system to another system.

Let the variables in the double integral $\iint_A f(x,y)\,dx\,dy$ be changed from x,y to u,v where $x=\phi(u,v)$ and $y=\psi(u,v)$.

Then on substituting for x and y, the double integral is transformed to $\iint_{A'} F(u,v)\,J\,du\,dv$, where $J(u,v)$ is the Jacobian of x,y w.r.t. u,v *i.e.*,

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix},$$

and A' is the region in the uv-plane corresponding to the region A in the xy-plane. Thus remember that $dx\,dy = J\,du\,dv$.

Special case. Change to polar coordinates from the cartesian co-ordinates.

To change the variables from cartesian to polar coordinates we put $x=r\cos\theta$, $y=r\sin\theta$. In this case

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r.$$

and therefore $dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr$.

This change is specially useful when the region of integration is a circle or a part of a circle.

Ex. 62. *Transform* $\iint f(x,y)\,dx\,dy$ *by the substitution* $x+y=u$, $y=uv$.

Sol. We have $\quad x+y=u \quad$ and $\quad y=uv$. ...(1)

From these, we have ...(2)

$$x=u-y=u-uv \quad\text{and}\quad y=uv.$$

$$\therefore\quad \frac{\partial x}{\partial u}=1-v,\ \frac{\partial x}{\partial v}=-u,\ \frac{\partial y}{\partial u}=v \quad\text{and}\quad \frac{\partial y}{\partial v}=u.$$

$$\therefore\quad J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & -u\\ v & u\end{vmatrix}=u.$$

$$\therefore\quad dx\,dy = J\,du\,dv = u\,du\,dv.$$

Hence the given integral transforms to

$$\iint F(u,v)\,u\,du\,dv.$$

Ex. 63. *Transform $\iint f(x,y)\,dx\,dy$ to polar coordinates.*

Sol. We have $x = r\cos\theta,\ y = r\sin\theta$.

Now $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore\quad dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr$.

Hence the given integral transforms to

$$\iint F(r,\theta)\,r\,d\theta\,dr.$$

Ex. 64. *Transform $\int_0^a \int_0^{a-x} f(x,y)\,dx\,dy$, by the substitution*

$$x + y = u, \qquad y = uv.$$

Sol. As shown in Ex. 62,

$$\iint f(x,y)\,dx\,dy = \iint F(u,v)\,u\,du\,dv.$$

(Prove it here.)

Now in the given integral, the region of integration is bounded by the lines $y = 0$, $y = a - x$, $x = 0$ and $x = a$.

Put $x = u - y = u - uv = u(1-v)$ and $y = uv$.

Then in the uv-plane the four straight lines become $uv = 0$, $uv = a - u(1-v)$, $u(1-v) = 0$ and $u(1-v) = a$, giving $v = 0$, $v = 1$, $u = 0$ and $u = a$.

Hence for the given region, v varies from 0 to 1 and u varies from 0 to a.

Therefore, by changing the variables, the given double integral transforms to $\int_0^a \int_0^1 F(u,v)\,u\,du\,dv$.

Ex. 65. *By using the transformation $x + y = u$, $y = uv$, show that*

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dx\,dy = \tfrac{1}{2}(e-1).$$

Sol. As proved in Ex. 62, we have

$dx\,dy = u\,du\,dv$. *(Prove it here)*

Here the region of integration is bounded by the lines

$$y = 0,\ y = 1 - x,\ x = 0 \text{ and } x = 1.$$

Changing these equations to new variables u and v by using the relations $x = u - y = u - uv = u(1-v)$ and $y = uv$, we have

$uv = 0$, $uv = 1 - u(1-v)$, $u(1-v) = 0$ and $u(1-v) = 1$,

giving $v = 0$, $v = 1$, $u = 0$ and $u = 1$.

Hence for the given region v varies from 0 to 1 and u varies from 0 to 1.

Further $e^{y/(x+y)} = e^{uv/u} = e^v$. $\quad [\because x + y = u, y = uv]$

Therefore, changing the variables to u, v, the given integral becomes

$$\int_0^1 \int_0^1 e^v\,.\,u\,du\,dv = \int_0^1 \left[e^v\right]_0^1 u\,du = \int_0^1 (e^1 - e^0)\,u\,du$$

$$= (e-1)\int_0^1 u\,du = (e-1)\,.\left[\frac{u^2}{2}\right]_0^1 = \tfrac{1}{2}(e-1).$$

Ex. 66. *By using the transformation $x + y = u$, $y = uv$, prove that $\iint \{xy(1-x-y)\}^{1/2}\,dx\,dy$ taken over the area of the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ is $2\pi/105$.*

Sol. As proved in Ex. 65, we have $dx\,dy = u\,du\,dv$; u varies from 0 to 1 and also v varies from 0 to 1.

Now

$$\{xy(1-x-y)\}^{1/2} = [xy\{1-(x+y)\}]^{1/2}$$
$$= [u(1-v)\,.\,uv\,.\,(1-u)]^{1/2}$$
$$[\because x = u(1-v), y = uv]$$
$$= u(1-u)^{1/2}\,.\,v^{1/2}(1-v)^{1/2}.$$

Hence the given double integral transforms to

$$\int_0^1 \int_0^1 u(1-u)^{1/2}\,.\,v^{1/2}(1-v)^{1/2}\,.\,u\,du\,dv$$

$$= \left[\int_0^1 u^2(1-u)^{1/2}\,du\right]\cdot\left[\int_0^1 v^{1/2}(1-v)^{1/2}\,dv\right]$$

$$= \left[\int_0^1 u^{3-1}(1-u)^{3/2-1}\,du\right]\cdot\left[\int_0^1 v^{3/2-1}(1-v)^{3/2-1}\,dv\right]$$

$= B(3, \tfrac{3}{2})\,.\,B(\tfrac{3}{2}, \tfrac{3}{2})$, [by the def. of Beta function]

$$= \frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(3+\frac{3}{2})}\cdot\frac{\Gamma\frac{3}{2}\,.\,\Gamma\frac{3}{2}}{\Gamma(\frac{3}{2}+\frac{3}{2})} = \frac{2\,.\,[\frac{1}{2}\sqrt{\pi}]^3}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\,.\,2} = \frac{2\pi}{105}.$$

Ex. 67. *Evaluate $\iint (x^2+y^2)^{7/2}\,dx\,dy$ over the circle $x^2 + y^2 = 1$.*

Sol. Here the region of integration is a circle. Therefore we shall change the given double integral to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore\quad dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr$.

Clearly the region of integration is the circle $x^2 + y^2 = 1$ i.e., the circle with centre (0, 0) and radius 1.

Changing to polar coordinates, the region of integration is covered when r varies from 0 to 1 and θ varies from 0 to 2π.

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$$\therefore \iint_{x^2+y^2\le 1} (x^2+y^2)^{7/2}\,dx\,dy = \int_0^{2\pi}\int_0^1 (r^2)^{7/2} J\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^1 r^7 . r .\,d\theta\,dr = \int_0^{2\pi}\int_0^1 r^8\,d\theta\,dr = \int_0^{2\pi}\left[\frac{r^9}{9}\right]_0^1 d\theta$$

$$= \tfrac{1}{9}\int_0^{2\pi} d\theta = \tfrac{1}{9}\left[\theta\right]_0^{2\pi} = \tfrac{2}{9}\pi .$$

Ex. 68. *Evaluate* $\iint xy\,(x^2+y^2)^{3/2}\,dx\,dy$ *over the positive quadrant of the circle* $x^2+y^2=1$.

Sol. Changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$, we have $J = r$ so that $dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr$.

The given region of integration is the area lying in the positive quadrant of the circle $x^2+y^2=1$.

Changing to polar coordinates, this region of integration is covered when r varies 0 to 1 and θ varies from 0 to $\pi/2$.

$\therefore$ the required integral

$$\iint xy\,(x^2+y^2)^{3/2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 r\cos\theta . r\sin\theta . (r^2)^{3/2} . r\,d\theta\,dr$$

$$= \int_0^{\pi/2}\int_0^1 r^6 \sin\theta\cos\theta\,d\theta\,dr = \int_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^1 \sin\theta\cos\theta\,d\theta$$

$$= \tfrac{1}{7}\int_0^{\pi/2} \tfrac{1}{2}\sin 2\theta\,d\theta = \tfrac{1}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = -\tfrac{1}{28}[-1-1] = \tfrac{1}{14}.$$

Ex. 69. *Evaluate* $\iint \sqrt{(a^2-x^2-y^2)}\,dx\,dy$ *over the semi-circle* $x^2+y^2 = ax$ *in the positive quadrant.*

Sol. Here the region of integration is a semi-circle. Therefore, for the sake of convenience, changing to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2+y^2=ax$, we have

$$r^2\cos^2\theta + r^2\sin^2\theta = a\,r\cos\theta \text{ or } r^2(\sin^2\theta+\cos^2\theta) = ar\cos\theta$$

or $\qquad r = a\cos\theta .$

The equation $r = a\cos\theta$ represents a circle passing through the pole and diameter through the pole along the initial line.

For the given region r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\pi/2$.

$$\therefore \iint \sqrt{(a^2-x^2-y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{(a^2-r^2)} . r\,d\theta\,dr,$$

$$[\because x^2+y^2 = r^2 \text{ and } dx\,dy = r\,d\theta\,dr]$$

$$= \int_0^{\pi/2}\left[\int_0^{a\cos\theta} -\tfrac{1}{2}(a^2-r^2)^{1/2} . (-2r)\,dr\right] d\theta \qquad \text{(Note)}$$

$$= \int_0^{\pi/2}\left[-\tfrac{1}{2}.\tfrac{2}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\tfrac{1}{3}\int_0^{\pi/2} (a^2\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\left[\frac{2}{3.1} - \frac{\pi}{2}\right]$$

$$= \tfrac{1}{3}a^3\left(\tfrac{1}{2}\pi - \tfrac{2}{3}\right).$$

Ex. 70. *Evaluate* $\iint e^{-(x^2+y^2)}\,dx\,dy$ *over the circle* $x^2+y^2=a^2$.

Sol. Changing to polar coordinates, the equation $x^2+y^2=a^2$ transforms to $r^2\cos^2\theta + r^2\sin^2\theta = a^2$ i.e., $r = a$.

Hence for the given region r varies from 0 to a and θ varies from 0 to 2π.

Also $\qquad dx\,dy = r\,d\theta\,dr .$

$\therefore$ the required integral

$$\iint e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2} r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{a^2} e^{-t} . \tfrac{1}{2}\,d\theta\,dt, \text{ putting } r^2 = t \text{ so that } 2r\,dr = dt$$

$$= \tfrac{1}{2}\int_0^{2\pi}\left[\frac{e^{-t}}{-1}\right]_0^{a^2} d\theta = -\tfrac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta$$

$$= -\tfrac{1}{2}(e^{-a^2}-1)\left[\theta\right]_0^{2\pi} = \tfrac{1}{2}(1-e^{-a^2}) . 2\pi = \pi(1-e^{-a^2}). \qquad \square$$