(1)

Mains Test Series - 2019

Test -05 (Paper - I Full Syllabus)

Answer Key

SECTION - A

Q.1. (a) → Solve the following system of linear equations. Sketch the set of solutions.

$$x_1 + 2x_2 + 3x_3 = 11$$
$$-x_2 + x_3 = -2$$
$$x_1 + x_2 + 4x_3 = 9.$$

Solution:
The given system of linear equations are

$$x_1 + 2x_2 + 3x_3 = 11$$
$$-x_2 + x_3 = -2$$
$$x_1 + x_2 + 4x_3 = 9$$

Now, we write a single augmented matrix,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & -1 & 1 & -2 \\ 1 & 1 & 4 & 9 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & -1 & 1 & -2 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$R_2 \to (-1) \times R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$R_3 \to R_3 + R_2$, $R_1 \to R_1 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 7 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(A|B) = 2 <$ no. of unknowns x_1, x_2, x_3.

$\therefore$ The given system is consistent and has infinite solutions.

Now, the echelon matrix equation is given by

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 5x_3 = 7$$
$$x_2 - x_3 = 2$$

$$\Rightarrow \quad x_1 = -5x_3 + 7$$
$$x_2 = x_3 + 2$$

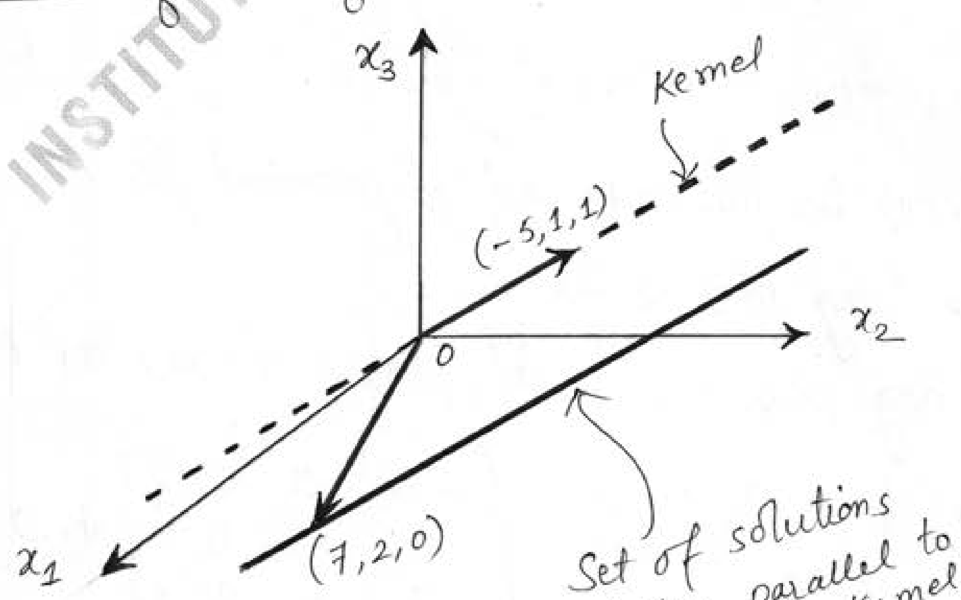
$\therefore$ The solutions are vectors of the form $(-5x_3 + 7,\ x_3 + 2,\ x_3)$ where $x_3 \in$ Field.

The part involving x_3 will be in the Kernel of the transformation defined by T, while the constant part will be a particular solution $\mathbf{x}_1$ to $T\mathbf{x} = \mathbf{y}$.

$$\underbrace{(-5x_3+7,\ x_3+2,\ x_3)}_{\text{arbitrary solution to } T\mathbf{x}=\mathbf{y}} = \underbrace{x_3(-5, 1, 1)}_{\text{Element of Kernel}} + \underbrace{(7, 2, 0)}_{\text{a particular solution to } T\mathbf{x}=\mathbf{y}}.$$

The set of solutions can be represented geometrically by sliding the kernel, namely the line defined by the vector $(-5, 1, 1)$, in the direction and distance defined by the vector $(7, 2, 0)$.

Sketch of set of solutions :-



Q.1. (b) → Let a, b and c be elements of a field F, and let $A = \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$.

Prove that the characteristic polynomial for A is $x^3 - ax^2 - bx - c$ and that this is also the minimal polynomial for A.

Solution:

The characteristic polynomial for A is

$$f(x) = |xI - A|$$

i.e. $$f(x) = \begin{vmatrix} x & 0 & -c \\ -1 & x & -b \\ 0 & -1 & x-b \end{vmatrix}$$

$$= x(x^2 - ax - b) - c$$

Hence, $f(x) = x^3 - ax^2 - bx - c$. — (i)

Let $p(x)$ be the minimal polynomial for A.

Then $\deg p(x) \leq 3$.

If $\deg p(x) = 1$, let $p(x) = x - \alpha$, $\alpha \in F$.

$$\therefore\ p(A) = A - \alpha I = \begin{bmatrix} -\alpha & 0 & c \\ 1 & -\alpha & b \\ 0 & 1 & a-\alpha \end{bmatrix} \neq 0$$

$(\because 1 \neq 0)$

If $\deg p(x) = 2$, let $p(x) = x^2 + \alpha x + \beta$; $\alpha, \beta \in F$.

$$\therefore\ p(A) = A^2 + \alpha A + \beta I$$

$$= \begin{bmatrix} 0 & c & ac \\ 0 & b & c+ab \\ 1 & a & b+a^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \alpha c \\ \alpha & 0 & \alpha b \\ 0 & \alpha & \alpha a \end{bmatrix} + \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

$$= \begin{bmatrix} \beta & c & ac+\alpha c \\ \alpha & b+\beta & c+ab+\alpha b \\ 1 & a+\alpha & b+a^2+\alpha a+\beta \end{bmatrix} \neq 0$$

$(\because 1 \neq 0)$

Hence, $\deg p(x) = 3$.
Since $p(x)$ divides $f(x)$, $p(x) = f(x)$.

Hence, the result.

Q.1. (c) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$, then evaluate:

$$\lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}.$$

Solution:

Here, $f(x+y) = f(x) + f(y)$

$$f(2) = f(1) + f(1) = 2$$

$$f(3) = f(1+2) = f(1) + f(2) = 3$$

$$f(4) = f(1+3) = f(1) + f(3) = 4$$

$$\vdots$$

$$f(x) = x \quad \forall\, x \in \mathbb{R}$$

$$\therefore f(\tan x) = \tan x, \quad f(\sin x) = \sin x$$

$$\Rightarrow \lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 \cdot f(\sin x)}$$

$$= \lim_{x\to 0} \frac{2^{\tan x} - 2^{\sin x}}{x^2 \cdot \sin x}$$

$$= \lim_{x\to 0} \frac{2^{\sin x} \cdot \{2^{\tan x - \sin x} - 1\}}{\sin x \cdot x^2} \times \frac{\tan x - \sin x}{\tan x - \sin x}$$

$$= \lim_{x\to 0} \left\{\frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x}\right\} \times \left\{\frac{\tan x - \sin x}{x^2 \sin x}\right\} \times 2^{\sin x}$$

$$= \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{1 - \cos x}{x^2 \cdot \cos x} \times 2^{\sin x}$$

$$= \lim_{x\to 0} \frac{2^{\tan x - \sin x} - 1}{\tan x - \sin x} \times \frac{2 \cdot \sin^2(x/2)}{4(x/2)^2} \times 2^{\sin x} \times \frac{1}{\cos x}$$

$$= \log 2 \times \frac{1}{2} \times 1 = \boxed{\frac{1}{2}\log 2}$$

Hence, the result.

Q. 1. (d) → Prove that

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3},$$

if $x > 0$.

Solution :

Let $f(x) = \log(1+x) - \left[x - \frac{x^2}{2} + \frac{x^3}{3(1+x)}\right]$

$$\therefore f'(x) = \frac{1}{1+x} - 1 + x - \frac{3x^2 + 2x^3}{3(1+x)^2}$$

$$= \frac{x^3}{3(1+x)^2}$$

$\Rightarrow f'(x) > 0$ if $x > 0$

$\Rightarrow$ f is increasing for $x > 0$

$\Rightarrow f(x) > f(0)$ for $x > 0$

$\Rightarrow f(x) > 0$ $\quad (\because f(0) = 0)$

$$\Rightarrow x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x), \text{ if } x > 0. \quad \text{---(i)}$$

Let $g(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \log(1+x)$.

$$\therefore g'(x) = 1 - x + x^2 - \frac{1}{1+x} = \frac{x^3}{1+x}$$

$\Rightarrow g'(x) > 0$ for $x > 0$

$\Rightarrow$ g is increasing for $x > 0$

$\Rightarrow$ $g(x) > g(0)$ for $x > 0$

$\Rightarrow$ $g(x) > 0$ $\quad (\because g(0) = 0)$

$\Rightarrow$ $\log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$, if $x > 0$. — (ii)

Thus, from (i) and (ii),

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}, \text{ if } x > 0$$

Hence, Proved.

Q.1. (c) → A point P moves on the plane $\frac{x}{a} + y/b + \frac{z}{c} = 1$ which is fixed, and the plane through P perpendicular to OP meets the axes in A, B, C. If the planes through A, B, C parallel to the co-ordinates planes meet in a point Q, show that the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}$$

Solution:

Let P be (α, β, γ), then as P lies on $\frac{x}{a} + y/b + z/c = 1$, so we have

$$\left(\alpha/a\right) + \left(\beta/b\right) + \left(\gamma/c\right) = 1 \quad \text{—— (1)}$$

Also d.c.'s of the line OP are α, β, γ.
therefore, the equation of the plane through P perpendicular to OP is

$$\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0$$

$$\Rightarrow \quad \alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2 \quad \text{—— (2)}$$

The plane (2) meets the coordinates axes in A, B, C whose co-ordinates are given by

$$\left\{\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right\}; \left\{0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right\}; \left\{0, 0, \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right\}$$

The equation of the plane through P parallel to yz-plane is $x = \dfrac{\alpha^2+\beta^2+\gamma^2}{\alpha}$.

Similarly, the equations of the planes through B and C parallel respectively to zx and xy-planes are

$$y = \frac{\alpha^2+\beta^2+\gamma^2}{\beta} \text{ and } z = \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

respectively.

The locus of Q, the point of intersection of these planes is obtained by eliminating α, β, γ between the equations of these planes and the relation (1).

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

$$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\cancel{\alpha^2+\beta^2+\gamma^2}}{(\alpha^2+\beta^2+\gamma^2)^{\cancel{2}}}=\frac{1}{(\alpha^2+\beta^2+\gamma^2)}$$

Also,

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{\alpha}{a(\alpha^2+\beta^2+\gamma^2)}+\frac{\beta}{b(\alpha^2+\beta^2+\gamma^2)}+\frac{\gamma}{c(\alpha^2+\beta^2+\gamma^2)}$$

$$=\frac{(\alpha/a)+(\beta/b)+(\gamma/c)}{(\alpha^2+\beta^2+\gamma^2)}$$

$$=\frac{1}{\alpha^2+\beta^2+\gamma^2} \quad \text{from (1)}$$

Hence, $\boxed{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}}$ is the required locus of Q.

Q.2. (a) → Find a basis for the column space of the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

<u>Solution:</u>

The transpose of A is

$$A' = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A'.

Now, we proceed further to find a basis for the row space of A' by obtaining the reduced echelon form of A'.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \underset{\substack{R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 2R_2}}{\sim} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly which is in reduced echelon form.
The non-zero row vectors of this echelon form

are $(1,0,5)$, $(0,1,-3)$ which form the basis for the row space of A'.

$\therefore$ These vectors in column form represent a basis for the column space of A.

Thus, $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ are a basis for the column space of A.

Hence, the result.

Q. 2. (b) → Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors $(1,2,3,4)$, $(-1,-1,-4,-2)$, $(3,4,11,8)$.

<u>Solution</u>:

We construct a matrix A having the given vectors as row vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Now, we proceed ahead to compute the reduced echelon form of A.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \quad R_2 \to R_2 + R_1, \; R_3 \to R_3 - 3R_1$$

$R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly, this is in echelon form.
The non-zero vectors of this reduced echelon form, namely $(1, 0, 5, 0)$ and $(0, 1, -1, 2)$ are a basis for the subspace V of $\mathbb{R}^4$.
The dimension of this subspace is two.

Hence, the result.

Q.2. (c) → Find the maximum and minimum values of $\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$, when $lx + my + nz = 0$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Interpret the result geometrically.

Solution:

Let $u = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}$; then

$$du = \frac{2x}{a^4}dx + \frac{2y}{b^4}dy + \frac{2z}{c^4}dz = 0 \quad \text{——} (1)$$

$$l dx + m dy + n dz = 0 \quad \text{——} (2)$$

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0 \quad \text{——} (3)$$

Multiplying (1), (2) and (3) by 1, λ_1, λ_2 respectively and adding, and then equating the co-efficients of dx, dy and dz to zero, we get

$$\frac{x}{a^4} + \lambda_1 l + \lambda_2 . \frac{x}{a^2} = 0, \quad \text{——} (4)$$

$$\frac{y}{b^4} + \lambda_1 m + \lambda_2 . \frac{y}{b^2} = 0 \quad \text{——} (5)$$

and $\frac{z}{c^4} + \lambda_1 n + \lambda_2 . \frac{z}{c^2} = 0 \quad$ —— (6)

Multiplying (4) by x, (5) by y, and (6) by z and adding, we get

$$u + \lambda_2 = 0 \text{ or } \lambda_2 = -u,$$

$$\therefore \frac{x}{a^4} + \lambda_1 l - \frac{x}{a^2}.u = 0 \text{ etc.}$$

Hence, $x = -\frac{\lambda_1 l a^4}{1 - a^2 u}$

Similarly, $y = \dfrac{-\lambda_1 m b^4}{1-b^2 u}$ and $z = \dfrac{-\lambda_1 n c^4}{1-c^2 u}$

Substituting these values of x, y, z in $lx+my+nz=0$, we get,

$$\boxed{\frac{l^2a^4}{1-a^2u} + \frac{m^2b^4}{1-b^2u} + \frac{n^2c^4}{1-c^2u} = 0} \quad \text{---}(7)$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical Interpretation:

The tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at (x', y', z') is $\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$

Perpendicular distance to the tangent plane from origin, i.e., p is given by

$$p^2 = \frac{1}{\frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}}$$

If the point (x', y', z') lies on the given plane $lx+my+nz=0$, the given problem consists of finding out the maximum or minimum value of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx+my+nz=0$ and the ellipsoid.

2(d)(i) A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is $p/x + q/y + r/z = 2$.

Sol'n: Let the coordinates of A, B and C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

The equation of the plane ABC is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ —— ①

If it passes through (p,q,r), then $p/a + q/b + r/c = 1$ —— ②

Also the equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0.$$

If its centre be (x_1, y_1, z_1) then $x_1 = \frac{1}{2}a$, $y_1 = \frac{1}{2}b$ & $z_1 = \frac{1}{2}c$

$$\Rightarrow a = 2x_1, \quad b = 2y_1 \ \& \ c = 2z_1$$

Substituting these values of a, b and c in ②, we get

$$p/2x_1 + q/2y_1 + r/2z_1 = 1$$

∴ Locus of the centre (x_1, y_1, z_1) of the sphere OABC is

$$p/x + q/y + r/z = 2 \qquad \text{Hence proved.}$$

2(d)(ii) Prove that the equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a cone if $u^2/a + v^2/b + w^2/c = d$.

Sol'n: Let $F(x,y,z,t) = ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$.

∴ $\frac{\partial F}{\partial x}=0$ for $t=1$ gives $2ax+2u=0 \Rightarrow x=-u/a$ —— ①

Similarly $\frac{\partial F}{\partial y}=0$ for $t=1$ gives $y=-v/b$ —— ②

$\frac{\partial F}{\partial z}=0$ for $t=1$ gives $z=-w/c$ —— ③

and $\partial F/\partial t = 0$ for $t=1$ gives $ux+vy+wz+d=0$ —— ④

Substituting the values x, y, z from ①, ②, ③ in ④ we get the required condition as

$$u(-u/a) + v(-v/b) + w(-w/c) + d = 0$$

$$\Rightarrow u^2/a + v^2/b + w^2/c = d \quad \text{Hence proved.}$$

Q.3. (a) → Determine the Kernel and the Range of the transformation defined by the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution:

A is a 3X3 matrix.

Thus, A defines a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$,

$$T(\mathbf{x}) = A\mathbf{x}$$

The elements of $\mathbb{R}^3$ are written in column matrix form for the purpose of matrix multiplication. For convenience, we express the elements of $\mathbb{R}^3$ in row form at all other times.

(i) Kernel:

The Kernel will consist of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{0}$.

Thus,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This matrix equation corresponds to the following system of linear equations.

$$x_1 + 2x_2 + 3x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$x_1 + x_2 + 4x_3 = 0$$

$\Rightarrow \quad x_2 = x_3$

$$x_1 = -x_2 - 4x_3 = -x_3 - 4x_3 = -5x_3$$

$\therefore$ The Kernel is thus the set of vectors of the form $(-5x_3, x_3, x_3)$.

$$\therefore \boxed{\text{Ker } T = \{(-5x_3, x_3, x_3) / x_3 \in \mathbb{R}\}}$$

Ker T is one-dimensional subspace of $\mathbb{R}^3$ with basis $(-5, 1, 1)$.

(ii) <u>Range</u>:

The range is spanned by the column vectors of A. We write these column vectors as rows of a matrix and compute an echelon form of the matrix. The non-zero ~~med~~ row vectors will give a basis for the range.

We get

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \underset{\substack{R_2 \to R_2 - 2R_1,\\ R_3 \to R_3 - 3R_1}}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_2 \to (-1)R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly it is in echelon form and the vectors $(1, 0, 1)$ and $(0, 1, 1)$ span the range of T.

An arbitrary vector in the range will be a linear combination of these vectors, $s(1, 0, 1) + t(0, 1, 1)$.

Thus, the range of T is

$$\boxed{\text{Range}(T) = \{(s, t, s+t) / s, t \in \mathbb{R}\}}$$

The vectors $(1, 0, 1)$ and $(0, 1, 1)$ are also linearly independent. Range (T) is a two-dimensional subspace of $\mathbb{R}^3$ with basis $\{(1, 0, 1), (0, 1, 1)\}$.

Hence the result

3(b) (i) Evaluate $\iiint x^{-1/2} y^{-1/2} z^{-1/2} (1-x-y-z)^{1/2}\, dx\,dy\,dz$ extended to all the value of the variables subject to the conditions $x+y+z<1$.

Sol^n: when the condition is $0<x+y+z<1$, we have the given integral.

$$= \iiint x^{1/2-1} y^{1/2-1} z^{1/2-1} \{1-(x+y+z)\}^{1/2} dx\,dy\,dz$$

$$= \frac{\Gamma(1/2)\Gamma(1/2)\Gamma(1/2)}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_0^1 (1-u)^{1/2} u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1} du$$

$$= \frac{\sqrt{(\pi)^3}}{\Gamma(3/2)} \int_0^1 (1-u)^{1/2} u^{1/2} du$$

$$= \frac{\sqrt{(\pi)^3}}{\frac{1}{2}\Gamma(\frac{1}{2})} \int_0^1 (1-u)^{3/2-1} u^{3/2-1} du$$

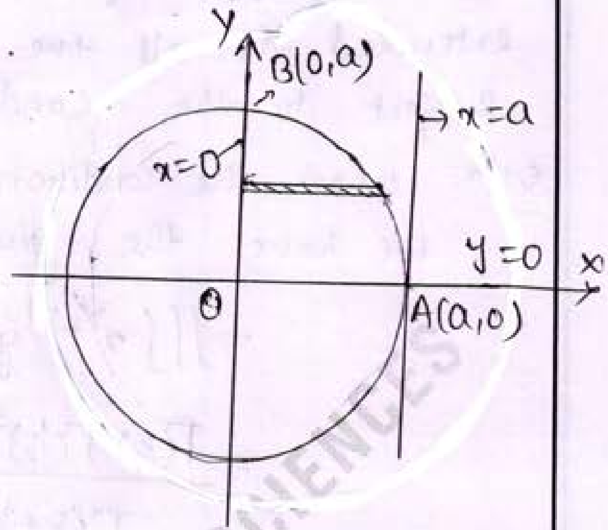
$$= \frac{(\sqrt{\pi})^3}{\frac{1}{2}\sqrt{\pi}} \cdot \frac{\Gamma(3/2)\Gamma(3/2)}{\Gamma(3)}$$

$$= 2\pi \frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2.1.1} = \boxed{\frac{\pi^2}{4}}$$

3(b) (ii) A right cone has its vertex at the centre and its axis coincident with the diameter passing through that point. Find the volume common to the cone and sphere.

Sol^n: Let a be the radius of the sphere and α the semivertical angle of the cone.

By the revolution of the area OPQ about OX, we get the required volume.



Consider an element of area $r\delta\theta\delta r$ at (r,θ), this area when revolved about OX, generates a ring of radius $r\sin\theta$ and volume of this elementry ring is

$$2\pi(r\sin\theta)\, r\delta\theta\delta r$$

$\therefore$ The required volume $= \int_0^{\alpha}\int_0^{a} 2\pi(r\sin\theta) \cdot r\, d\theta\, dr.$

[$\because$ the equation of circle with O as origin is $r=a$]

$$= 2\pi \int_0^{\alpha} \left[\frac{r^3}{3}\right]_0^a \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\left[-\cos\theta\right]_0^{\alpha}$$

$$= \boxed{\frac{2\pi a^3}{3}\left[1-\cos\alpha\right]}$$

3(C) Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid
$4x^2-y^2-z^2+2yz-8x-4y+8z-2=0$.

Sol'n: Here $a=4,\ b=-1,\ c=-1,\ f=1,\ g=0,\ h=0,\ u=-4$
$v=-2,\ w=4$ and $d=-2$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix}=0 \quad \text{—— (1)}$$

$$(4-\lambda)\left[(1+\lambda)^2-1\right]=0 \Rightarrow \lambda(\lambda+2)(\lambda-4)=0 \Rightarrow \lambda=0,-2,4$$

$\therefore$ Let $\lambda_1=-2,\ \lambda_2=4,\ \lambda_3=0$

Now putting $\lambda=0$ in the determinant given by (1) and associating each row with l_3, m_3, n_3 we have

$4l_3=0,\quad -m_3+n_3=0,\quad m_3-n_3=0 \Rightarrow l_3=0,\ m_3=n_3$

But $l_3^2+m_3^2+n_3^2=1$, so $0+m_3^2+m_3^2=1 \Rightarrow m_3=\frac{1}{\sqrt{2}}=n_3$

$\therefore$ we have $l_3=0,\ m_3=\frac{1}{\sqrt{2}},\ n_3=\frac{1}{\sqrt{2}}$

Now $k=ul_3+vm_3+wn_3=-4(0)-2\left(\frac{1}{\sqrt{2}}\right)+4\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}$

$\therefore$ Required reduced equation is $\lambda_1x^2+\lambda_2y^2+2kz=0$

$\Rightarrow -2x^2+4y^2+2\sqrt{2}z=0 \Rightarrow x^2-2y^2-2\sqrt{2}=0$

which represents a hyperbolic paraboloid as λ_1 is $-$ve and λ_2 is $+$ve.

Also if $F(x,y,z)=0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations.

$$\frac{(\partial F/\partial x)}{l_3} = \frac{(\partial F/\partial y)}{m_3} = \frac{(\partial F/\partial z)}{n_3} = 2k$$

and $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

i.e. any two of the equations

$$8x - 8 = 2\sqrt{2}(0) \text{ i.e. } x = 1$$

$$-2y + 2z - 4 = 2(\sqrt{2})\left(\tfrac{1}{\sqrt{2}}\right) \text{ i.e. } y - z + 3 = 0$$

$$-2z + 2y + 8 = 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right) \text{ i.e. } y - z + 3 = 0$$

with $\sqrt{2}\left[0.x + \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z\right] - 4x - 2y + 4z - 2 = 0$

$\Rightarrow x = 1,\ y - z + 3 = 0,\ 4x + y - 5z + 2 = 0$

solving these we get $x = 1,\ y = -9/4,\ z = 3/4$

$\therefore$ coordinates of the vertex are $(1, -9/4, 3/4)$

And the equations of its axis are

$$\frac{x-1}{l_3} = \frac{y-(-9/4)}{m_3} = \frac{z-(3/4)}{n_3}$$

$$\Rightarrow \frac{x-1}{0} = \frac{y+9/4}{1/\sqrt{2}} = \frac{z-3/4}{1/\sqrt{2}}$$

$$\Rightarrow \frac{x-1}{0} = \frac{4y+9}{1} = \frac{4z-3}{1}$$

Q.4. (a) (i) → Consider the linear transformation $T: P_2 \to P_1$ defined by $T(ax^2+bx+c) = (a+b)x - c$. Find a matrix of T with respect to the bases $\{u_1, u_2, u_3\}$ and $\{u_1', u_2'\}$ of P_2 and P_1 where $u_1 = x^2$, $u_2 = x$, $u_3 = 1$ and $u_1' = x$, $u_2' = 1$. Use this matrix to find the image of $u = 3x^2 + 2x - 1$.

Solution :

Consider the effect of T on each basis vector of P_2.

$$T(u_1) = T(x^2) = x = 1x + 0(1) = 1u_1' + 0u_2'$$

$$T(u_2) = T(x) = x = 1x + 0(1) = 1u_1' + 0u_2'$$

$$T(u_3) = T(1) = -1 = 0x + (-1)(1) = 0u_1' + (-1)u_2'$$

The coordinate vectors of $T(x^2)$, $T(x)$, and $T(1)$ are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

The matrix of T is thus,

$$\boxed{A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.} \qquad \text{——(i)}$$

Now, we find the image of $\mathbf{u} = 3x^2+2x-1$ using the matrix A.

The coordinate vector of $\mathbf{u}$ relative to the basis $\{x^2, x, 1\}$ is $\mathbf{a} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

We get,

$$\mathbf{b} = A\mathbf{a}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{T(\mathbf{u}) = 5\mathbf{u}_1' + 1\,\mathbf{u}_2' = 5x+1.} \quad \text{---(ii)}$$

Hence, the result.

Q.4. (a) (ii) → Consider the linear operator $T(x, y) = (3x+y, x+3y)$ on $\mathbb{R}^2$. Find a diagonal matrix representation of T. Determine the basis for this representation and give a geometrical interpretation of T.

<u>Solution:</u>

First of all, we find a matrix representation A relative to the standard basis $B = \{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

We get

$$T(1,0) = (3,1) = 3(1,0) + 1(0,1)$$

$$T(0,1) = (1,3) = 1(1,0) + 3(0,1)$$

The co-ordinate vectors of $T(1,0)$ and $T(0,1)$ relative to B are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

The matrix representation of T relative to the standard basis B is thus

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Now, we have to compute eigenvalues and eigenvectors of A.

The eigenvalues are obtained by $|A-\lambda I|=0$

$\Rightarrow \lambda^2-6\lambda+8=0$

$\Rightarrow \lambda = 4, 2$

The eigenvectors corresponding to eigenvalues

$\lambda_1 = 4$ is $v_1 = r\begin{bmatrix}1\\1\end{bmatrix}$;

$\lambda_2 = 2$ is $v_2 = s\begin{bmatrix}-1\\1\end{bmatrix}$ on solving.

The following matrix A' is thus a diagonal matrix representation of T.

$$\boxed{A' = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 2\end{bmatrix}} \quad \text{—— (i)}$$

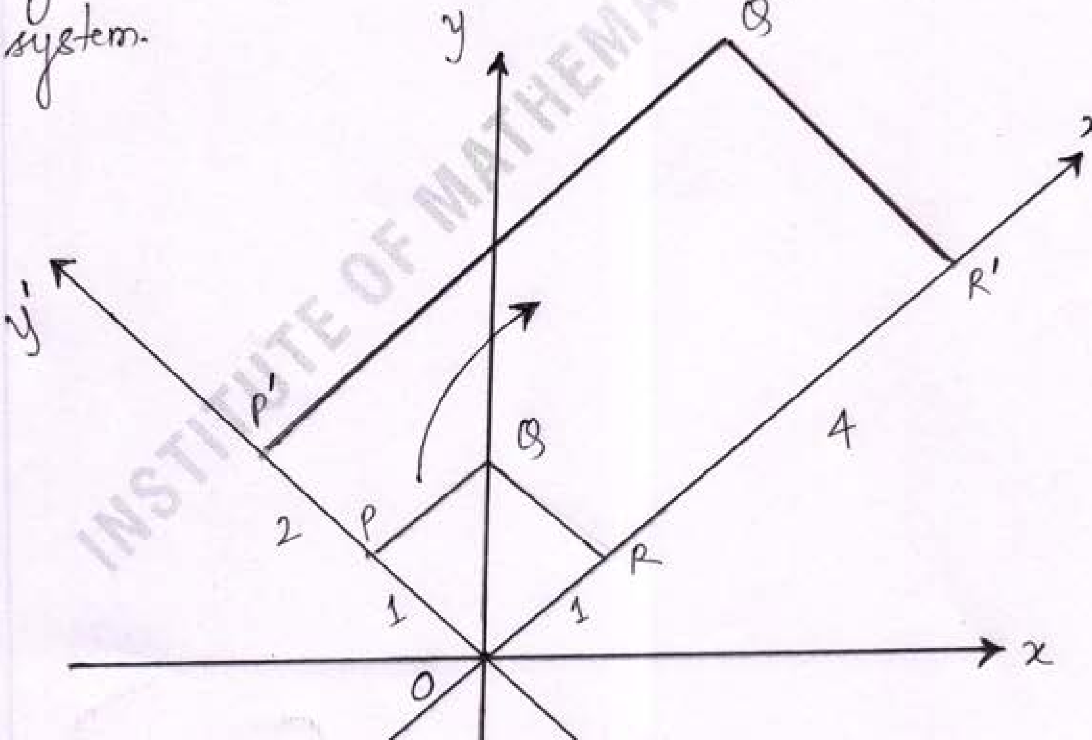
Now, we find the basis B' which gives this representation A'. Since A is a symmetric matrix, we select unit orthogonal eigenvectors for the coordinate vectors of B' relative to B. The transition matrix from B to B' will then be orthogonal, and the geometry will be preserved.

Let $\boxed{B' = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right); \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}}$

Also, the basis B' is obtained from the basis B by rotation through $\pi/4$. ——— (ii)

Geometrical Interpretation:

The standard basis B defines an xy co-ordinate system. Let the basis B' define an $x'y'$ coordinate system.



The matrix A' tells us that T is a scaling in the $x'y'$ coordinate system, with factor 4 in the x' direction and factor 2 in

the y' direction.
Thus, for example, T maps the square $PQRO$ into the rectangle $P'Q'R'O$.

4(b) → Find the maximum and minimum values of $x^2/a^4 + y^2/b^4 + z^2/c^4$, when $lx+my+nz=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Interpret the result geometrically.

Sol^n: Let $u = x^2/a^4 + y^2/b^4 + z^2/c^4$; then

$$du = \frac{2x}{a^4}dx + \frac{2y}{b^4}dy + \frac{2z}{c^4}dz = 0 \quad \text{— (1)}$$

$$l\,dx + m\,dy + n\,dz = 0 \quad \text{— (2)}$$

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 0 \quad \text{— (3)}$$

Multiplying (1), (2) & (3) by $1, \lambda_1, \lambda_2$ respectively and adding and then equating the coefficients of dx, dy and dz to zero, we get

$$\frac{x}{a^4} + \lambda_1 l + \lambda_2\frac{x}{a^2} = 0 \quad \text{— (4)}$$

$$\frac{y}{b^4} + \lambda_1 m + \lambda_2\frac{y}{b^2} = 0 \quad \text{— (5)}$$

and $$\frac{z}{c^4} + \lambda_1 n + \lambda_2\cdot\frac{z}{c^2} = 0 \quad \text{— (6)}$$

Multiplying (4) by x, (5) by y, and (6) by z and adding, we get

$$u + \lambda_2 = 0 \Rightarrow \lambda_2 = -u$$

$$\therefore \frac{x}{a^4} + \lambda_1 l - \frac{x}{a^2}u = 0 \text{ etc.}$$

Hence $x = -\dfrac{\lambda_1 l a^4}{1-a^2u}$

Similarly, $y = \dfrac{-\lambda_1 m b^4}{1-b^2u}$ and $z = \dfrac{-\lambda_1 n c^4}{1-c^2u}$

Substituting these values of x, y, z in $lx+my+nz=0$, we get

$$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0 \quad \text{——} (7)$$

the equation (7) gives the required maximum (or) minimum values of u.

Geometrical Interpretation:

The tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ at (x',y',z') is $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$.

$\perp$ar distance to the tangent plane from origin i.e. p is given by

$$p^2=\frac{1}{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}$$

If the point (x',y',z') lies on the given plane $lx+my+nz=0$, the given problem consists of finding out the maximum or minimum value of the $\perp$ar distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx+my+nz=0$ and the ellipsoid.

4(c) The generators through P of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meets the principal elliptic section of A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, show that P lies on the surface $z(\alpha x+\beta y)+\gamma(c^2+z^2)=0$

Sol^n:- Let the co-ordinates of P, A and B be (x_1, y_1, z_1), $(a\cos\theta, b\sin\theta, 0)$ and $(a\cos\phi, b\sin\phi, 0)$ respectively

Also the co-ordinates of F, the mid point of AB are $\left[\frac{1}{2}(a\cos\theta+a\cos\phi), \frac{1}{2}(b\sin\theta+b\sin\phi), 0\right]$

$= \left(a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, 0\right)$

$\therefore$ Direction ratios of the median PF through P are

$x_1-a\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right), \; y_1-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, \; z_1-0.$ ——(1)

The values of x_1, y_1, z_1 can be found as follows.

The equation of ~~the~~ tangent to the given hyperboloid at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$ and it meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0 \quad \text{——(i)}$$

which is the same as the line joining the points A and B.

i.e, $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2},\ z=0$ ——(ii)

Comparing (i) & (ii), we get

$$\frac{x_1/a^2}{\frac{1}{a}\cos\frac{\theta+\phi}{2}}=\frac{y_1/b^2}{\frac{1}{b}\sin\frac{\theta+\phi}{2}}=\frac{1}{\cos\frac{\theta-\phi}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos(\theta+\phi)/2}{\cos\frac{\theta-\phi}{2}},\quad \frac{y_1}{b}=\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \quad \text{——(iii)}$$

Again $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1$

$$\Rightarrow \frac{1}{\cos^2\frac{\theta-\phi}{2}}-\frac{z_1^2}{c^2}=1 \Rightarrow \frac{z_1^2}{c^2}=\sec^2\left(\frac{\theta-\phi}{2}\right)-1$$
$$=\tan^2\frac{\theta-\phi}{2}$$
$$\Rightarrow \frac{z_1}{c}=\pm\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\therefore P(x_1,y_1,z_1)=\boxed{\left(\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \frac{\pm c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right)}$$

$\therefore$ from (a), we have.

$$a\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\quad \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

$\Rightarrow \dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right), \quad \dfrac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),$

$$\dfrac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

$\Rightarrow a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2}, \quad b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},$

$c\tan\frac{\theta-\phi}{2}\operatorname{cosec}^2\frac{\theta-\phi}{2}$

$\Rightarrow$ $x_1, y_1, z_1 \operatorname{cosec}^2\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\cot^2\frac{\theta-\phi}{2}\right)$

where $\frac{z_1}{c} = \tan\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\frac{c^2}{z_1^2}\right)$

As PF is parallel to the plane

$\alpha x+\beta y+\gamma z=0$

$\therefore \alpha x_1+\beta y_1+\gamma z_1\left(1+\frac{c^2}{z_1^2}\right)=0$

$\Rightarrow (\alpha x_1+\beta y_1)z_1+\gamma(z_1^2+c^2)=0$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$

is $z(\alpha x+\beta y)+\gamma(z^2+c^2)=0$

— Hence proved

5(a) find the orthogonal trajectories of the following family of curve

$$r^n \sin n\theta = a^n.$$

Sol^n:- Given Equation of family of curve is

$$r^n \sin n\theta = a^n \text{, with 'a' as a parameter} \rightarrow (1)$$

from (1) $n\log r + \log \sin n\theta = \log a^n \rightarrow (2)$

Differentiating (2) w.r.t to θ we get

$$\frac{n}{r}\frac{dr}{d\theta} + n\cot n\theta = 0$$

$$\Rightarrow n\left[\frac{1}{r}\frac{dr}{d\theta} + \cot n\theta\right] = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} + \cot n\theta = 0. \rightarrow (3)$$

which is the differential Equation of the given family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2 d\theta/dr$ in (3) the differential equation of the required othogonal trajectory is.

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) + \cot n\theta = 0$$

$$-r\frac{d\theta}{dr} + \cot n\theta = 0.$$

$$\cancel{r} \cdot \frac{d\theta}{dr} = \cancel{} \cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = \frac{dr}{r}$$

Integrating on both sides then

$$\int \frac{d\theta}{\cot n\theta} = \int \frac{dr}{r} \Rightarrow \int \tan n\theta \, d\theta = \int \frac{dr}{r}.$$

$$\int \frac{dr}{r} = \int \frac{\sin n\theta}{\cos n\theta} d\theta$$

$$+\int \frac{dr}{r} = -\frac{1}{n} \int \frac{-n \sin n\theta}{\cos n\theta} d\theta.$$

$$\int \frac{dr}{r} = \frac{-1}{n} \int \frac{-n \sin n\theta}{\cos n\theta} d\theta$$

$$\log r = \frac{-1}{n} \log(\cos n\theta) + \frac{1}{n} \log c$$

$$n \log r = -\log(\cos n\theta) + \log c$$

$$n \log r + \log(\cos n\theta) = \log c.$$

$$\log r^n + \log(\cos n\theta) = \log c$$

$$\log r^n \cos n\theta = \log c$$

$$\therefore \boxed{r^n \cos n\theta = c} \text{ Ans.}$$

5(b) → solve and examine for singular solution of

$$x^2(y-xp)=yp^2.$$

Sol^n:

given ~~that~~ $x^2(y-xp)=yp^2$ —— (1)

$\Rightarrow y=\dfrac{x^3p}{x^2-p^2}$ —— (2) clearly it is solvable for y.

Diff. it wrt x, we get

$$\frac{dy}{dx}=\frac{(x^2-p^2)\left(3x^2p+x^3\frac{dp}{dx}\right)-(x^3p)\left(2x-2p\frac{dp}{dx}\right)}{(x^2-p^2)^2}$$

$$\Rightarrow p=\frac{(x^2-p^2)3x^2p-2x^4p+\left[(x^2-p^2)x^3+2x^3p^2\right]\frac{dp}{dx}}{(x^2-p^2)^2}$$

$$\Rightarrow p=\frac{x^4p-3x^2p^3+\left[x^5+x^3p^2\right]\frac{dp}{dx}}{(x^2-p^2)^2}$$

$$\Rightarrow x^4p-3x^2p^3-p(x^4+p^4-2x^2p^2)=(x^5+x^3p^2)\frac{dp}{dx}$$

$$\Rightarrow -x^2p^3-p^5=(x^5+x^3p^2)\frac{dp}{dx}$$

$$\Rightarrow -p^3(x^2+p^2)=x^3(x^2+p^2)\frac{dp}{dx}$$

$$\Rightarrow -p^3=x^3\frac{dp}{dx}$$

$$\Rightarrow \frac{1}{x^3}dx+\frac{1}{p^3}dp=0$$

$$\Rightarrow \frac{1}{x^2}+\frac{1}{p^2}=c_1$$ —— (3) The g.s of given by

The equations (2) & (3) together

let us examine the singular solution:

first of all, we find p-discriminant:

$(1) \equiv yp^2 + x^3 p - x^2 y = 0$. (4)

p-discri is $b^2 - 4ac = 0$

$$(x^3)^2 + 4y^2x^2 = 0.$$

$$x^2(x^4 + 4y^2) = 0.$$

$$\Rightarrow x = 0,\ x^4 + 4y^2 = 0.$$

(5)

clearly $x=0$ does not satisfy the given ODE.

We have

$$x^4 + 4y^2 = 0$$

$$\Rightarrow 4x^3 + 8y\frac{dy}{dx} = 0$$

$$\Rightarrow x^3 = -2y\frac{dy}{dx} \Rightarrow x^3 = -2yp$$

$$\Rightarrow \boxed{p = -\frac{x^3}{2y}}$$

$(2) \equiv \dfrac{x^3 p}{x^2 - p^2} = \dfrac{x^3\left(-\frac{x^3}{2y}\right)}{x^2 - \frac{x^6}{4y^2}} = -\dfrac{x^6}{2y} \times \dfrac{4y^2}{x^2 4y^2 - x^6}$

$$= \frac{-2x^6 y}{4x^2y^2 - x^6}$$

$$= \frac{-2x^4 y}{4y^2 - x^4}.$$

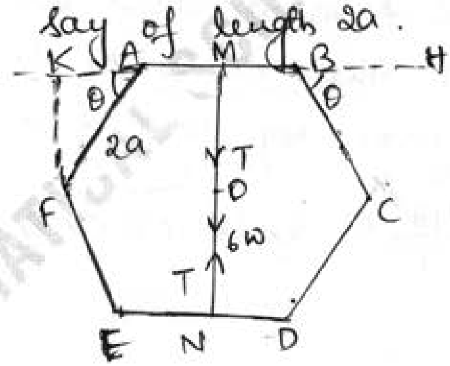
$$\neq y.$$

$\therefore x^4 + 4y^2 = 0$ also does not the given ODE.

clearly there exists no singular and there does not require to find c-discri.

5(C) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; Find the tension in the string.

Sol^n: ABCDEF is a hexagon formed of six equal rods each of weight W and say of length 2a. The rod AB is fixed in a horizontal position & the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN. The total weight 6W of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let



$$\angle FAK = \theta = \angle CBH$$

Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length MN changes and the point O also changes.

we have $MN = 2MO = 2KF = 2AF\sin\theta = 4a\sin\theta$

Also the depth of O below the fixed line AB

$$= MO = 2a\sin\theta$$

By the principle of virtual work, we have

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

$$\Rightarrow -4aT\cos\theta\,\delta\theta + 12aW\cos\theta\,\delta\theta = 0$$

$$\Rightarrow 4a[-T+3W]\cos\theta\,\delta\theta = 0$$

$$\Rightarrow -T+3W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$\Rightarrow T = 3W$$

5(d) If in a S.H.M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Sol^n: Let O and O' be the centre of force and the fixed point respectively on the line of motion and let $OO' = l$. Let u, v, w be the velocities of the particle at P, Q, R respectively where

$$O'P = a,\quad O'Q = b,\quad O'R = c$$

For a S.H.M of amplitude A, the velocity V at a distance x from the centre of force is given by

$$V^2 = \mu(A^2 - x^2) \quad \text{—— (1)}$$

At P, $x = OP = l+a$, $V = u$

at Q, $x = OQ = l+b$, $V = v$

and at R, $x = OR = l+c$, $V = w$

$\therefore$ from (1), we have

$$u^2 = \mu\{A^2 - (l+a)^2\}$$

$$\Rightarrow \frac{u^2}{\mu} = A^2 - l^2 - a^2 - 2al$$

$$\Rightarrow \left(\frac{u^2}{\mu} + a^2\right) + 2la + (l^2 - A^2) = 0 \quad \text{——— (2)}$$

Similarly $\left(\frac{v^2}{\mu} + b^2\right) + 2l.b + (l^2 - A^2) = 0$ ——— (3)

and $\left(\frac{w^2}{\mu} + c^2\right) + 2lc + (l^2 - A^2) = 0$ ——— (4)

eliminating $2l$ & $(l^2 - A^2)$ from (2) (3) & (4), we have

$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

$$\Rightarrow -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

$$\Rightarrow \mu \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \mu(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \text{--- (5)}$$

But the time period $T = \frac{2\pi}{\sqrt{\mu}}$ so that $\mu = \frac{4\pi^2}{T^2}$

Hence from (5), we have

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

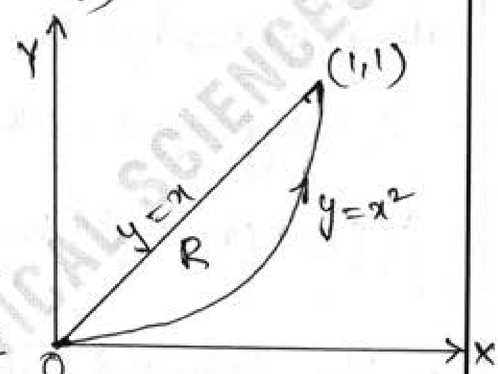
5(e) Verify Green's theorem in the plane for $\oint_C (xy+y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Sol'n: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = xy+y^2$, $N = x^2$

The curves $y=x$ and $y=x^2$ intersect at (0,0) and (1,1). The positive direction in traversing C



we have
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy\,dx = \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 \left[x^2 - x^2 - x^3 + x^4\right] dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Now let us evaluate the line integral along C.

Along $y=x^2 \Rightarrow dy = 2x\,dx$.

$\therefore$ along $y=x^2$, the line integral equals

$$\int_0^1 \left[\{(x)(x^2)+x^4\}dx + x^2(2x)dx\right]$$

$$= \int_0^1 (3x^3+x^4)\,dx = \frac{19}{20}$$

along $y=x \Rightarrow dy = dx$.

$\therefore$ along $y=x$, the line integral

$$\int_1^0 \left[\{(x)(x)+x^2\}dx + x^2 dx\right] = \int_1^0 3x^4\,dx = -1$$

$\therefore$ The required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$.

Hence the theorem is verified.

6(a) Solve $(x^3y^3+x^2y^2+xy+1)y\,dx+(x^3y^3-x^2y^2-xy+1)x\,dy=0$.

Sol^n: Comparing the given equation with

$$M\,dx+N\,dy=0, \text{ we get}$$

$$M=y(x^3y^3+x^2y^2+xy+1) \text{ and } N=x(x^3y^3-x^2y^2-xy+1)$$

$$\therefore Mx-Ny=xy(x^3y^3+x^2y^2+xy+1)-xy(x^3y^3-x^2y^2-xy+1)$$

$$=2xy(x^2y^2+xy)=2x^2y^2[xy+1]\neq 0.$$

Showing that I.F of the given equation

$$=\frac{1}{(Mx-Ny)}=\frac{1}{[2x^2y^2(xy+1)]}$$

On multiplying the given equation by its I.F. we have

$$\frac{x^2y^2(xy+1)+(xy+1)}{2x^2y^2(xy+1)}y\,dx+\frac{(x^2y+1)(x^2y^2-xy+1)-xy(xy+1)}{x^2y^2}x\,dy=0$$

$$\frac{x^2y^2+1}{x^2y^2}y\,dx+\frac{(x^2y^2-xy+1)-xy}{x^2y^2}x\,dy=0$$

$$\Rightarrow (y\,dx+x\,dy)+\frac{y\,dx+x\,dy}{x^2y^2}-\frac{2x^2y}{x^2y^2}dy=0$$

$$\Rightarrow d(xy)+\frac{d(xy)}{(xy)^2}-\frac{2}{y}dy=0.$$

$$\Rightarrow d(xy)+\frac{1}{z^2}dz-\frac{2}{y}dy=0 \quad \text{putting } xy=z$$

Integrating, $xy-\frac{1}{z}-2\log y=c \Rightarrow xy-\frac{1}{xy}-2\log y=c.$

6(b) Solve $\{(x+1)^4D^3+2(x+1)^3D^2-(x+1)^2D+(x+1)\}y=\frac{1}{(x+1)}$, $D\equiv\frac{d}{dx}$.

Sol^n: Dividing bothsides by $(x+1)$,

The given equation reduces to

$$\{(x+1)^3D^3+2(x+1)^2D^2-(x+1)D+1\}y=(1+x)^{-2} \quad \text{—— (1)}$$

Let $1+x=e^z \Rightarrow \log(1+x)=z$, Also let $D_1\equiv\frac{d}{dz}$ —— (2)

Then we have $xD=D_1$, $x^2D^2=D_1(D_1-1)$ &

$$x^3D^3=D_1(D_1-1)(D_1-2) \quad \text{—— (3)}.$$

Using (2) and (3), (1) reduces to

$$\{D_1(D_1-1)(D_1-2)+2D_1(D_1-1)-D_1+1\}y=e^{-2z}$$

$$\Rightarrow (D_1^3-D_1^2-D_1+1)y=e^{-2z} \Rightarrow (D_1-1)^2(D_1+1)y=e^{-2z} \quad \text{—— (4)}$$

Here auxiliary equation for (4) is $(D_1-1)^2(D_1+1)=0$

giving $D_1=1,1,-1$

$\therefore$ C.F $=(C_1+C_2z)e^z+C_3e^{-z}$, C_1, C_2, C_3 being arbitrary constants.

and $$P.I=\frac{1}{(D_1-1)^2(D_1+1)}e^{-2z}$$

$$=\frac{1}{(-2-1)^2(-2+1)}e^{-2z}=-\frac{1}{9}e^{-2z}$$

$\therefore$ The required solution is

$$y=(C_1+C_2z)e^z+C_3(e^z)^{-1}-\frac{1}{9}(e^z)^{-2}$$

$$y=\{C_1+C_2\log(1+x)\}(1+x)+C_3(1+x)^{-1}-\frac{1}{9}(1+x)^{-2}, \text{ using (2)}$$

6(c) → Using method of variation of parameters, solve $\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right) + y = xe^x \sin x$ with $y(0)=0$ and $\left(\frac{dy}{dx}\right)_{x=0} = 0$.

Sol'n: Given $(D^2-2D+1)y = xe^x \sin x$, where $D \equiv d/dx$ — ①

comparing ① with $y_2 + Py_1 + Qy = R$, here $R = xe^x \sin x$

consider $(D^2-2D+1)y=0 \Rightarrow (D-1)^2 y=0$, $D\equiv d/dx$ — ②

Auxiliary equation of ② is $(D-1)^2=0 \Rightarrow D=1,1$.

$\therefore$ C.F of ① $=(c_1+c_2x)e^x = c_1e^x + c_2xe^x$, c_1 and c_2 being arbitrary const. — ③.

let $u=e^x$, $v=xe^x$.

Also here $R = xe^x \sin x$ — ④

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix} = e^{2x} \neq 0$ — ⑤

Then, P.I of ① $= uf(x) + vg(x)$ — ⑥

where $f(x) = -\int \frac{vR}{W}dx = -\int \frac{xe^x(xe^x\sin x)}{e^{2x}}dx$

$= -\int x^2 \sin x\, dx$ by ④ & ⑤

$= -\{x^2(-\cos x) - (2x)(-\sin x) + 2\cos x\}$

and $g(x) = \int \frac{uR}{W}dx = \int \frac{e^x(xe^x\sin x)}{e^{2x}}dx = \int x\sin x\, dx$, by ④ & ⑤

$= (x)(-\cos x) - (1)(-\sin x) = \sin x - x\cos x$

$\therefore$ P.I of ① $= e^x(x^2\cos x - 2x\sin x - 2\cos x) + xe^x(\sin x - x\cos x)$ by ⑥

$= -xe^x \sin x - 2e^x\cos x$

Hence the general solution of ① is

$$y = C.F + P.I$$

i.e. $y = c_1e^x + c_2xe^x - xe^x\sin x - 2e^x\cos x$

$$= e^x(c_1 + c_2x - x\sin x - 2\cos x) \quad \text{—— ⑦}$$

Given that $y=0$ when $x=0$.

Hence ⑦ gives $0 = c_1 - 2 \Rightarrow c_1 = 2$

Putting $c_1 = 2$ in ⑦

$$y = e^x(2 + c_2x - x\sin x - 2\cos x) \quad \text{—— ⑧}$$

⑧ $\Rightarrow dy/dx = e^x(2 + c_2x - x\sin x - 2\cos x) + e^x\{c_2 - (\sin x + x\cos x) + 2\sin x\}$

Given that $dy/dx = 0$ when $x=0$.

So above equation gives $0 = c_2$

Putting $c_1 = 2$ & $c_2 = 0$ in ⑧, the required solution is $y = e^x(2 - x\sin x - 2\cos x)$.

6(d). By using Laplace Transform method

solve $(D^3-D^2+4D-4)y = 68e^t\sin 2t$, $y=1$, $Dy=-19$, $D^2y=-37$ at $t=0$.

Sol^n: Given equation $(D^3-D^2+4D-4)y = 68e^t\sin 2t$ ——(1).

Taking the Laplace transform of the given equation,

we have, $L\{y'''\}-L\{y''\}+4L\{y'\}-4L\{y\} = 68L\{e^t\sin 2t\}$

$$\Rightarrow p^3L\{y\}-p^2y(0)-py'(0)-y''(0)-[p^2L\{y\}-py(0)-y'(0)]$$
$$+4[pL\{y\}-y(0)]-4L\{y\} = 68.\frac{2}{(p-1)^2+4}$$

$$\left[\text{Since } L\{\sin 2t\}=\frac{2}{p^2+4}\right]$$

$$\Rightarrow (p^3-p^2+4p-4)L\{y\}-p^2+19p+37+p-19-4 = \frac{136}{p^2-2p+5}$$

$$\Rightarrow (p-1)(p^2+4)L\{y\} = p^2-20p-14+\frac{136}{p^2-2p+5}$$

$$\Rightarrow L\{y\} = \frac{p^2-20p-14}{(p-1)(p^2+4)}+\frac{136}{(p^2-2p+5)(p-1)(p^2+4)}$$

$$= -\frac{33}{5(p-1)}+\frac{38p-62}{5(p^2+4)}+\frac{34}{5(p-1)}-\frac{136(3p-7)}{85(p^2+4)}$$
$$-\frac{2p+14}{p^2-2p+5}$$

$$= \frac{1}{5(p-1)}+\frac{14p}{5(p^2+4)}-\frac{6}{5(p^2+4)}-\frac{2(p-1)}{(p-1)^2+4}-\frac{16}{(p-1)^2+4}$$

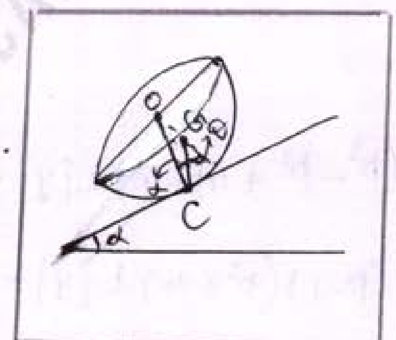
$$\therefore y = \frac{1}{5}L^{-1}\left\{\frac{1}{p-1}\right\}+\frac{14}{5}L^{-1}\left\{\frac{p}{p^2+4}\right\}-\frac{6}{5}L^{-1}\left\{\frac{1}{p^2+4}\right\}$$
$$-2L^{-1}\left\{\frac{p-1}{(p-1)^2+4}\right\}-16L^{-1}\left\{\frac{1}{(p-1)^2+4}\right\}$$

$\Rightarrow y = \frac{1}{5}e^t + \frac{14}{5}\cos 2t - \frac{6}{5}\cdot\frac{1}{2}\sin 2t - 2e^t\cos 2t - 16\cdot\frac{1}{2}e^t\sin 2t$

$= \frac{1}{5}\left(e^t + 14\cos 2t - 3\sin 2t\right) - 2e^t\left(\cos 2t + 4\sin 2t\right)$

7(a) A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1}(3/8)$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.

Sol^n: Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC = r$. Let G be the centre of Gravity of the hemisphere. Then $OG = 3r/8$. In the position of equilibrium the line CG must be vertical. From ΔOGC, we have.



$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin\theta} \quad \text{i.e.} \quad \frac{3r/8}{\sin\alpha} = \frac{r}{\sin\theta}.$$

$\therefore \sin\theta = \frac{8}{3}\sin\alpha \Rightarrow \theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)$

giving the position of equilibrium of the hemisphere.

Since $\sin\theta < 1$

$\therefore \frac{8}{3}\sin\alpha < 1$

$\Rightarrow \sin\alpha < \frac{8}{3} \Rightarrow \alpha < \sin^{-1}(3/8)$

Thus for the equilibrium to exist, we must have

$\alpha < \sin^{-1}(3/8)$

Now let $CG = h$. Then

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha}$$

So that $h = \dfrac{3r\sin(\theta-\alpha)}{8\sin\alpha}$

Here $\rho_1 = r$ and $\rho_2 = \infty$.

The equilibrium will be stable if $h < \dfrac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2}$

$\Rightarrow \dfrac{1}{h} > \dfrac{\rho_1+\rho_2}{\rho_1\rho_2}\sec\alpha$

$\Rightarrow \dfrac{1}{h} > \left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}\right)\sec\alpha$

$\Rightarrow 1/h > \dfrac{1}{r}\sec\alpha$

$\Rightarrow 1/h > \dfrac{1}{r}\sec\alpha$ $\quad [\because \rho_1 = r, \rho_2 = \infty]$

$\Rightarrow h < r\cos\alpha$

$\Rightarrow \dfrac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$

$\Rightarrow 3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$

$\Rightarrow 3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$

$\Rightarrow 8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{1-\frac{64}{9}\sin^2\alpha} < 8\sin\alpha\cos\alpha$

$[\because \sin\theta = 8/3\sin\alpha]$

$\Rightarrow -\sin\alpha\sqrt{(9-64\sin^2\alpha)} < 0$

$\Rightarrow \sin\alpha\sqrt{(9-64\sin^2\alpha)} > 0$ ——— (2)

But from (1),

$\sin\alpha < 3/8$ i.e. $64\sin^2\alpha < 9$ i.e. $\sqrt{(9-64\sin^2\alpha)}$ is a +ve real number.

$\therefore$ The Relation (2) is true. Hence the equilibrium is stable.

7(b) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then find its time, the particle will arrive at O.

Sol'n: Take the centre of force O as origin. Suppose a particle starts from rest at A, where $OA=a$. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after time t, where $OP=x$. The acceleration of the particle at P is $\mu a^{5/3}x^{-2/3}$ directed towards O.

$\therefore$ The equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu a^{5/3}x^{-2/3} \quad \text{——— (1)}$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3}x^{1/3}}{1/3} + k = -6\mu a^{5/3}x^{1/3} + k$$

$\cdot$ k = Constant.

At A, $x=a$ & $dx/dt=0$, so that

$$-6\mu a^{5/3}a^{1/3} + k = 0 \Rightarrow k = 6\mu a^2.$$

$$\therefore (dx/dt)^2 = -6\mu a^{5/3}x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}(a^{1/3} - x^{1/3}) \quad \text{— (2)}$$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x=0$ and $(dx/dt)^2 = v_1^2$. So from (2), we have

$$v_1^2 = 6\mu a^{5/3}(a^{1/3} - 0) = 6\mu a^2 \Rightarrow v_1 = a\sqrt{6\mu}$$

Now taking square root of (2), we get

$$dx/dt = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3}-x^{1/3})},$$

where the -ve sign has been taken because the particle moves in the direction of x decreating.

Separating the variables, we get

$$dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \cdot \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}} \quad \text{—(3)}$$

Let t_1 be the time from A to O. Then integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \cdot \int_a^0 \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}$$

$$= \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^a \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}$$

Put $x = a\sin^6\theta$, so that $dx = 6a\sin^5\theta\cos\theta\, d\theta$.

when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore t_1 = \frac{1}{\sqrt{6\mu a^{5/3}}} \int_0^{\pi/2} \frac{6a\sin^5\theta\cos\theta\, d\theta}{a^{1/6}\cos\theta}$$

$$= \sqrt{\frac{6}{\mu}} \int_0^{\pi/2} \sin^5\theta\, d\theta = \sqrt{\frac{6}{\mu}} \cdot \frac{4.2}{5.3.1}$$

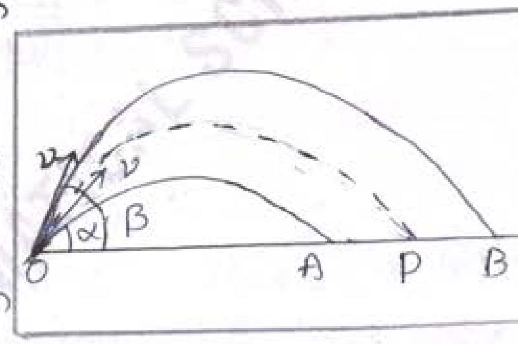
$$= \frac{8}{15}\sqrt{\frac{6}{\mu}}$$

7(C) A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the angle of projection is α and goes b metres beyond when the angle of projection is β. If the velocity of projection be the same in all cases, find the correct angle of projection.

Sol'n: Let O be the point of projection & v the velocity of projection in all the cases. Let P be the point in the horizontal plane through O required to hit from O. Let θ be the correct angle of projection to hit P from O. Then

OP = the range for the angle of projection $\theta = \frac{v^2 \sin 2\theta}{g}$

when the angle of projection is α the particle falls at A and when the angle of projection is β, it falls at B. we have



$OA = \frac{v^2 \sin 2\alpha}{g}$ & $OB = \frac{v^2 \sin 2\beta}{g}$

According to the Question

$AP = OP - OA = a$ and $PB = OB - OP = b$.

$$\therefore a = \frac{v^2 \sin 2\theta}{g} - \frac{v^2 \sin 2\alpha}{g} = \frac{v^2}{g}(\sin 2\theta - \sin 2\alpha) \quad \text{---(1)}.$$

$$\text{and } b = \frac{v^2 \sin 2\beta}{g} - \frac{v^2 \sin 2\theta}{g} = \frac{v^2}{g}(\sin 2\beta - \sin 2\theta) \quad \text{---(2)}$$

Dividing (1) by (2); we get $\frac{a}{b} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$

$$\Rightarrow a \sin 2\beta - a \sin 2\theta = b \sin 2\theta - b \sin 2\alpha$$

$$\Rightarrow (a+b)\sin 2\theta = a \sin 2\beta + b \sin 2\alpha$$

$$\Rightarrow \sin 2\theta = \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}$$

$$\therefore 2\theta = \sin^{-1} \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}$$

$$\Rightarrow \theta = \frac{1}{2} \sin^{-1} \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}$$

8(a). The acceleration a of a particle at any time $t \geq 0$ is given by $\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$. If the velocity v and displacement r are zero $t=0$, and v and r at any time.

Sol'n: $\vec{a} = \frac{dv}{dt} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\,\hat{k}$

Integrating we get

$\vec{v} = i\int e^{-t}dt - 6j\int (t+1)dt + 3\hat{k}\int \sin t\, dt$

$= -e^{-t}\hat{i} - 6j\left(\frac{t^2}{2}+t\right) + 3\hat{k}(-\cos t) + c$

when $t=0$, $v=0$

$\therefore 0 = -\hat{i} - 0\hat{j} + 3\hat{k}(-1) + c$

$\Rightarrow -3\hat{k} - \hat{i} + c = 0$

$\Rightarrow c = \hat{i} + 3\hat{k}$

$\therefore \vec{v} = (-e^{-t}+1)\hat{i} - 6\left(\frac{t^2}{2}+t\right)\hat{j} + (-3\cos t + 3)\hat{k}$

$\vec{v} = (1-e^{-t})\hat{i} - (3t^2+6t)\hat{j} - (3-3\cos t)\hat{k}$

Integrating we get

$\vec{r} = \hat{i}\int (1-e^{-t})dt - 3\hat{j}\int (3t^2+6t)dt - \hat{k}\int (3-3\cos t)dt$

$= (t+e^{-t})\hat{i} - 3\hat{j}(t^3+3t^2) - \hat{k}(3t - 3\sin t) + d$

when $t=0$, $\vec{r}=0$

$0 = \hat{i} - \hat{j}(0) - \hat{k}(0) + d \Rightarrow d = -\hat{i}$

$\therefore \vec{r} = (t-1+e^{-t})\hat{i} - (t^3+3t^2)\hat{j} + (3t-3\sin t)\hat{k}$

8(b) → Find κ and τ for the space curve $x=t,\ y=t^2,\ z=t^3$.

Sol: Given that $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$.

$$\frac{d\vec{r}}{dt} = 1\hat{i} + 2t\hat{j} + 3t^2\hat{k}.$$

$$\frac{d^2\vec{r}}{dt^2} = 0\hat{i} + 2\hat{j} + 6t\hat{k}. \text{ and } \frac{d^3\vec{r}}{dt^3} = 6\hat{k}.$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2+9t^4}.$$

Now
$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= \hat{i}(12t^2 - 6t^2) + \hat{j}(0-6t) + \hat{k}(2-0)$$

$$= 6t^2\hat{i} - 6t\hat{j} + 2\hat{k}.$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{36t^4+36t^2+4} = 2\sqrt{9t^4+9t^2+1}$$

$$\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] = \begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 1(12-0) + 2t(0-0) + 3t^2(0-0)$$

$$= 12$$

$$\text{curvature } (\kappa) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|dr/dt\right|^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(9t^4+4t^2+1)^{3/2}}$$

$$\text{Torsion } (\tau) = \frac{\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2} = \frac{12}{4(9t^4+9t^2+1)}$$

$$= \frac{3}{9t^4+9t^2+1}$$

Q.8. (c).(i) → If $\mathbf{A} = 2x^2\mathbf{i} - 3yz\mathbf{j} + xz^2\mathbf{k}$ and $\phi = 2z - x^3y$, find $\mathbf{A}\cdot\nabla\phi$ and $\mathbf{A}\times\nabla\phi$ at the point $(1,-1,1)$.

Solution :

Given $\vec{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and

$$\phi = 2z - x^3y.$$

$$\nabla\phi = \frac{\partial}{\partial x}(2z - x^3y)\hat{i} + \frac{\partial}{\partial y}(2z - x^3y)\hat{j} + \frac{\partial}{\partial z}(2z - x^3y)\hat{k}$$

$$\nabla\phi = -3x^2y\hat{i} - x^3\hat{j} + 2\hat{k}$$

$$\therefore \vec{A}\cdot\nabla\phi = -6x^4y + 3x^3yz + 2xz^2$$

i.e. $\boxed{\vec{A}\cdot\nabla\phi = 5}$ at $(1,-1,1)$. ——— (i)

Now,

$$\vec{A}\times\nabla\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix}$$

$$\therefore \vec{A}\times\nabla\phi = \hat{i}\left(-6yz+x^4z^2\right)-\hat{j}\left(4x^2+3x^3yz^2\right)$$
$$+\hat{k}\left(-2x^5-9x^2y^2z\right)$$

$$\therefore \vec{A}\times\nabla\phi = \left(x^4z^2-6yz\right)\hat{i}+\left(-4x^2-3x^3yz^2\right)\hat{j}$$
$$+\left(-2x^5-9x^2y^2z\right)\hat{k}.$$

i.e. $\boxed{\vec{A}\times\nabla\phi = 7\hat{i}-\hat{j}-11\hat{k}}$ at $(1,-1,1)$. —— (ii)

Hence, the result.

8(c)(iii) Find $\phi(r)$ such that $\nabla\phi = \frac{\vec{r}}{r^5}$ and $\phi(1)=0$.

Sol'n: Given $\nabla\phi = \frac{\vec{r}}{r^5}$ — ① and $\phi(1)=0$

Now $\nabla\phi(r) = \phi'(r)\nabla r$

$= \phi'(r)\left(\frac{\vec{r}}{r}\right)$ — ② $\left(\because \nabla r = \frac{\vec{r}}{r}\right)$

from ① and ②.

$$\phi'(r)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^5}$$

$$\Rightarrow \phi'(r) = \frac{1}{r^4}$$

Integrating, we get

$$\phi(r) = -\frac{1}{3r^3} + C \quad \text{— ③}$$

given $\phi(1)=0$

$\therefore$ ③ $\equiv \phi(1) = -\frac{1}{3} + C = 0 \Rightarrow C = \frac{1}{3}$

$\therefore \phi(r) = -\frac{1}{3r^3} + \frac{1}{3} = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$

8(C)(iii) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$

Sol: The given surfaces are

$$f_1 \equiv ax^2 - byz - (a+2)x = 0 \quad \text{--- (1)}$$

and $$f_2 \equiv 4x^2y + z^3 - 4 = 0 \quad \text{--- (2)}$$

The point $(1,-1,2)$ obviously lies on the surface (2). It will also lie on the surface (1) if

$a + 2b - (a+2) = 0 \Rightarrow 2b - 2 = 0 \Rightarrow \boxed{b = 1}$

Now $\text{grad } f_1 = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$

$\text{grad } f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2k.$

Then $n_1 = \text{grad } f_1$ at the point $(1,-1,2)$
$= (a-2)\hat{i} - 2bj + bk.$

and $n_2 = \text{grad } f_2$ at the point $(1,-1,2)$
$= -8\hat{i} + 4j + 12k.$

The vectors $\hat{n}_1$ and $\hat{n}_2$ are along the normal to the surfaces (1) and (2) at the point $(1,-1,2)$.
These surfaces will intersect orthogonally at the point $(1,-1,2)$. if the vectors $\hat{n}_1$ and $\hat{n}_2$ are perpendicular i.e, if $n_1 \cdot n_2 = 0$.

$\Rightarrow -8(a-2) - 8b + 12b = 0 \Rightarrow b - 2a + 4 = 0$ --- (3)

$\Rightarrow 1 - 2a + 4 = 0 \; (\because b = 1)$

$\Rightarrow a = 5/2$

$\therefore \boxed{a = 5/2, \; b = 1}$

8(d) → By using Divergence theorem of Gauss, evaluate the surface integral.

$\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds$, where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$, a, b & c being all +ve constants.

Sol'n: Let us first put the integral

$$\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds \text{ in the form}$$

$$\iint_S F.n\, ds,$$

where n is a unit normal vector to the closed surface S whose equation is $ax^2+by^2+cz^2=1$.

The normal vector to $\phi(x,y,z) \equiv ax^2+by^2+cz^2-1=0$ is

$$=\nabla\phi = 2ax\hat{i}+2by\hat{j}+2cz\hat{k}$$

$$\therefore n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\hat{i}+2by\hat{j}+2cz\hat{k}}{\sqrt{(4a^2x^2+4b^2y^2+4c^2z^2)}} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}}$$

Here we are to choose F such that

$$F.n = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \text{ on } S.$$

Obviously $F = x\hat{i}+y\hat{j}+z\hat{k}$, because then

$$F.n = \frac{ax^2+by^2+cz^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}} \text{ on } S.$$

Note that on S, $ax^2+by^2+cz^2=1$

Now $\iint_S \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}} ds$

$= \iint_S F \cdot n \, ds$, where $F = x\hat{i} + y\hat{j} + z\hat{k}$

$= \iiint_V (\nabla \cdot F) \, dv$ by divergence theorem; V is the volume enclosed by Surface S.

$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) \right] dV$

$= \iiint_V 3 \, dv$

$= 3V = 3 \cdot \frac{4}{3}\pi \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}}$

$= \boxed{\frac{4\pi}{\sqrt{abc}}}$