IFoS MATHEMATICS (OPT.)-2009

PAPER - I : SOLUTIONS

1 (a). Let V be the vectorspace of polynomials over $\mathbb{R}$. Let U and W be the subspaces generated by $\{t^3+4t^2-t+3,\ t^3+5t^2+5,\ 3t^3+10t^2-5t+5\}$ and $\{t^3+4t^2+6,\ t^3+2t^2-t+5,\ 2t^3+2t^2-3t+9\}$ respectively. Find (i) $\dim(U+W)$
(ii) $\dim(U\cap W)$.

Sol

Let $S=\{t^3+4t^2-t+3,\ t^3+5t^2+5,\ 3t^3+10t^2-5t+5\}$
$=\{\alpha_1, \alpha_2, \alpha_3\}$ (say)

and $T=\{t^3+4t^2+6,\ t^3+2t^2-t+5,\ 2t^3+2t^2-3t+9\}$
$=\{\beta_1, \beta_2, \beta_3\}$ (say)

Since U and W are spanned by the sets S and T of all polynomials of degree 3.

$\therefore$ U and W are subspaces of the vectorspace $V(\mathbb{R})$ of all real polynomials of degree ≤ 3.

We know that the set $S_1=\{1, t, t^2, t^3\}$ is a standard basis for $V(\mathbb{R})$

Now the co-ordinate vectors of α_1, α_2 & α_3 w.r.t the above basis S_1 are

$(3, -1, 4, 1)$, $(5, 0, 5, 1)$ and $(5, -5, 10, 3)$

Again the co-ordinate vectors of β_1, β_2 & β_3 w.r.t the above basis S_1 are

$(6, 0, 4, 1)$, $(5, -1, 2, 1)$ and $(9, -3, 2, 2)$.

since U and W are two subspace of $V(\mathbb{R})$

$\therefore$ U+W is also subspace of $V(\mathbb{R})$

$\therefore$ U+W is the space generated by all the six co-ordinate vectors.

Now form the matrix A whose rows are the given six ~~vectors~~ co-ordinate vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \\ 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & -10 & 10 & 4 \\ 0 & 2 & -4 & -1 \\ 0 & 2 & -14 & -2 \\ 0 & 0 & -10 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to 3R_2 - 5R_1 \\ R_3 \to 3R_3 - 5R_1 \\ R_4 \to R_4 - 2R_1 \\ R_5 \to 3R_5 - 5R_1 \\ R_6 \to R_6 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -6 \\ 0 & 0 & -10 & -1 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_4 \to 5R_4 - 2R_2 \\ R_5 \to 5R_5 - 2R_2 \end{array}$$

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$$\sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \longleftrightarrow R_6.$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_4 \to R_4 - R_3 \\ R_5 \to R_5 - 6R_3 \end{matrix}$$

$\therefore$ clearly which is in echelon form.

$\therefore$ The echelon matrix of A has three non-zero rows.

$$\therefore \boxed{\dim(U+W) = 3.}$$

Now we form the matrix 'A' whose rows are the co-ordinates vectors of 'S' and reduce it to an echelon form

$$A = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & -10 & 10 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to 3R_2 - 5R_1 \\ R_3 \to 3R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$\therefore$ The echelon matrix of A has two non-zero rows.

$$\therefore \boxed{\dim(U) = 2}$$

Again form the matrix 'A' whose rows are the co-ordinate vectors of 'T' and reduce it to an echelon matrix

$$A = \begin{bmatrix} 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -18 & -24 & 3 \end{bmatrix} \begin{array}{l} R_2 \to 6R_2 - 5R_1 \\ R_3 \to 6R_3 - 9R_1 \end{array}$$

$$\sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -6 & -8 & 1 \end{bmatrix} R_3 \to \frac{1}{3} R_3$$

$$\sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \cancel{5R_3 + 6R_2} \; R_3 - R_2$$

$\therefore$ clearly which is echelon matrix of A and has ~~three~~ two non-zero rows.

$\therefore \dim(W) = 2$

Since $\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2 + 2 - 3$

$= 1$.

$\therefore \boxed{\dim(U \cap W) = 2}$

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1(b) Find a linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose image is generated by $(1,2,0,-4)$ and $(2,0,-1,-3)$

Sol Given that $R(T)$ spanned by $\{(1,2,0,-4),(2,0,-1,-3)\}$.

Let us include a vector $(0,0,0,0)$ in this set which will not effect the spanning property.

So that $S=\{(1,2,0,-4),(2,0,-1,-3),(0,0,0,0)\}$.

Let $B=\{\alpha_1,\alpha_2,\alpha_3\}$ be the standard basis of $\mathbb{R}^3$.

We know that, there exists a transformation 'T' such that $T(\alpha_1)=(1,2,0,-4)$

$T(\alpha_2)=(2,0,-1,-3)$ and

$T(\alpha_3)=(0,0,0,0)$.

Now if $\alpha \in \mathbb{R}^3 \Rightarrow \alpha=(a,b,c)$

$= a\alpha_1+b\alpha_2+c\alpha_3$.

$\therefore T(\alpha)=T(a\alpha_1+b\alpha_2+c\alpha_3)$

$=aT(\alpha_1)+bT(\alpha_2)+cT(\alpha_3)$ ($\because T$ is a LT)

$=a(1,2,0,-4)+b(2,0,-1,-3)+c(0,0,0,0)$

$=(a+2b,\ 2a,\ -b,\ -4a-3b)$

$\therefore T(a,b,c)=(a+2b,\ 2a,\ -b,\ -4a-3b)$ which is the required transformation.

1.(c) (i) find the difference between the maximum and the minimum of the function $\left(a-\frac{1}{a}-x\right)(4-3x^2)$ where a is a constant and greater than zero.

sol. Let $f(x)=\left(a-\frac{1}{a}-x\right)(4-3x^2)$ — (1).
where a is a constant and greater than zero.

Now $f'(x)=\left(a-\frac{1}{a}-x\right)(-6x)+(-1)(4-3x^2)$.

$\Rightarrow f'(x)=-6\left(a-\frac{1}{a}\right)x+6x^2-4+3x^2$

$=9x^2-6\left(a-\frac{1}{a}\right)x-4$ — (2).

For maximum or minimum

$f'(x)=0.$

$9x^2-6\left(a-\frac{1}{a}\right)x-4=0.$

$$\Rightarrow x=\frac{6\left(a-\frac{1}{a}\right)\pm\sqrt{36\left(a-\frac{1}{a}\right)^2+36(4)}}{18}$$

$$\Rightarrow x=\frac{\left(a-\frac{1}{a}\right)\pm\sqrt{\left(a-\frac{1}{a}\right)^2+4}}{3}$$

$$\Rightarrow x=\frac{\left(a-\frac{1}{a}\right)\pm\left(a+\frac{1}{a}\right)}{3}$$

$$\Rightarrow \boxed{x=\frac{2a}{3}} \text{ and } \boxed{x=-\frac{2}{3a}}$$

Now from (2),

$f''(x)=18x-6\left(a-\frac{1}{a}\right)$. — (3)

(4)

When $x = \frac{2a}{3}$:— $f''\left(\frac{2a}{3}\right) = \frac{18(2a)}{3} - 6\left(a - \frac{1}{a}\right)$

$$= 12a - 6a + 6/a$$

$$= 6a + 6/a$$

$$> 0 \quad (\because a > 0).$$

$\therefore$ f is minimum at $x = \frac{2a}{3}$.

$$\therefore f_{min} = f\left(\frac{2a}{3}\right)$$

$$= \left(a - \frac{1}{a} - \frac{2a}{3}\right)\left(4 - 3\left(\frac{4a^2}{9}\right)\right)$$

$$= \left(\frac{a}{3} - \frac{1}{a}\right)\left(4 - \frac{4a^2}{3}\right)$$

$$= \frac{4a}{3} - \frac{4}{a} - \frac{4a^3}{9} + \frac{4a}{3}$$

$$\boxed{f_{min} = \frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9}} \quad \text{— (4)}$$

When $x = \frac{-2}{3a}$:— $f''\left(\frac{-2}{3a}\right) = 18\left(\frac{-2}{3a}\right) - 6\left(a - \frac{1}{a}\right)$

$$= -\frac{12}{a} - 6a + 6/a$$

$$= -\frac{6}{a} - 6a$$

$$= -6\left(\frac{1}{a} + a\right)$$

$$< 0 \quad (\because a > 0)$$

$\therefore$ f is maximum at $x = \frac{-2}{3a}$.

$$\therefore f_{max} = f\left(\frac{-2}{3a}\right)$$

$$\boxed{f_{max} = 4a - \frac{8}{3a} + \frac{4}{9a^3}} \quad \text{— (5)}$$

Now the difference between the maximum and the minimum value of the function is

$$\left(4a - \frac{8}{3a} + \frac{4}{9a^3}\right) - \left(\frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9}\right)$$

$$= \frac{4a}{3} + \frac{4}{3a} + \frac{4}{9}\left(\frac{1}{a^3} + a^3\right)$$

$$= \frac{4}{3}\left(a + \frac{1}{a}\right) + \frac{4}{9}\left[\left(a + \frac{1}{a}\right)\left(a^2 + 1 + \frac{1}{a^2}\right)\right]$$

$$= \left(a + \frac{1}{a}\right)\left[\frac{4}{3} + \frac{4}{9}\left(a^2 + \frac{1}{a^2} + 1\right)\right]$$

$$= \left(a + \frac{1}{a}\right)\left[\left(a^2 + \frac{1}{a^2}\right)\frac{4}{9} + \frac{16}{9}\right]$$

$$= \frac{4}{9}\left(a + \frac{1}{a}\right)\left[\left(a^2 + \frac{1}{a^2}\right) + 4\right].$$

(ii) If $f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h)$, $0 < \theta < 1$.

find θ, when $h = 1$ and $f(x) = (1-x)^{5/2}$.

Sol: Given that $f(x) = (1-x)^{5/2} \Rightarrow \boxed{f(h) = (1-h)^{5/2}}$

and $f'(x) = -\frac{5}{2}(1-x)^{3/2} \Rightarrow \boxed{f'(0) = -\frac{5}{2}}$

$\Rightarrow f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow \boxed{f''(\theta h) = \frac{15}{4}(1-\theta h)^{1/2}}$

$$\therefore f(h) = f(0) + h f'(0) + \frac{h^2}{2!} f''(\theta h).$$
$$0<\theta<1.$$

$$\Rightarrow (1-h)^{5/2} = 1 + h\left(-\tfrac{5}{2}\right) + \frac{h^2}{2!}\left(\frac{15}{4}\right)(1-\theta h)^{1/2}.$$

when $h = 1$

$$\therefore 0 = 1 + 1\left(-\tfrac{5}{2}\right) + \frac{1}{2}\left(\frac{15}{4}\right)(1-\theta)^{1/2}$$

$$\Rightarrow 0 = 1 - \tfrac{5}{2} + \frac{15}{8}(1-\theta)^{1/2}$$

$$\Rightarrow 0 = \frac{3}{8} - (1-\theta)^{1/2}$$

$$\Rightarrow (1-\theta)^{1/2} = 3/8$$

$$\Rightarrow 1-\theta = \frac{9}{64}$$

$$\Rightarrow 1 - \frac{9}{64} = \theta$$

$$\Rightarrow \frac{64-9}{64} = \theta$$

$$\Rightarrow \boxed{\frac{55}{64} = \theta}$$

$$\Rightarrow \theta = \frac{55}{64} \in (0,1).$$

1.(d) Evaluate

(i) $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x + \cos x}$ (ii) $\displaystyle\int_1^{\infty} \frac{x^2}{(1+x^2)^2}\,dx.$

Sol (i) Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx$. (Clearly, which is a proper integral)

Then
$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\,dx.$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx + \int_0^{\pi/2} \frac{\cos^2 x}{\sin x+\cos x}$$

$$= \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\,dx.$$

$$= \int_0^{\pi/2} \frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}\,dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sin\frac{\pi}{4}\sin x + \cos\frac{\pi}{4}\cos x}\,dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos\left(x-\frac{\pi}{4}\right)}\,dx.$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sec\left(x-\frac{\pi}{4}\right)dx.$$

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$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\left[\sec\left(\frac{-\pi}{4}\right)+\tan\left(\frac{-\pi}{4}\right)\right]\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1)-\log(\sqrt{2}-1)\right] \qquad \left(\because \sec\left(-\frac{\pi}{4}\right) = \sec\frac{\pi}{4} = \sqrt{2}\right).$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$= \sqrt{2}\log(\sqrt{2}+1)$$

$$\therefore 2I = \sqrt{2}\log(\sqrt{2}+1)$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

(ii) $$\int_1^{\infty}\frac{x^2}{(1+x^2)^2}dx = \lim_{t\to\infty}\int_1^t \frac{x^2}{(1+x^2)^2}dx \quad \text{———} (1).$$

Now putting $x = \tan\theta$

$$\Rightarrow \boxed{dx = \sec^2\theta\, d\theta}$$

$$\therefore \int\frac{x^2}{(1+x^2)^2}dx = \int\frac{\tan^2\theta}{(1+\tan^2\theta)^2}\cdot\sec^2\theta\, d\theta.$$

$$= \int\frac{\tan^2\theta}{(\sec^2\theta)^2}\cdot\sec^2\theta\, d\theta$$

$$= \int\frac{\tan^2\theta}{\sec^2\theta}d\theta.$$

$$= \int\frac{\sin^2\theta}{\cancel{\cos^2\theta}}\times\frac{\cancel{\cos^2\theta}}{1}d\theta$$

$$= \int\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$= \frac{1}{2}\int(1-\cos 2\theta)\,d\theta$

$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right).$

$= \frac{1}{2}\left(\tan^{-1}x - \frac{1}{2}\cdot\frac{2\tan\theta}{1+\tan^2\theta}\right)$ $(\because \theta = \tan^{-1}x)$

$= \frac{1}{2}\left(\tan^{-1}x - \frac{x}{1+x^2}\right).$

Now from (1),

$$\int_1^\infty \frac{x^2}{(1+x^2)^2}dx = \underset{t\to\infty}{Lt}\int_1^t \frac{x^2}{(1+x^2)^2}dx$$

$$= \underset{t\to\infty}{Lt}\left[\frac{1}{2}\left(\tan^{-1}x - \frac{x}{1+x^2}\right)\right]_1^t.$$

$$= \frac{1}{2}\underset{t\to\infty}{Lt}\left[\left(\tan^{-1}t - \frac{t}{1+t^2}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0 - \frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right].$$

$= \frac{1}{4} + \frac{\pi}{8}$. which is finite.

$\therefore \int_1^\infty \frac{x^2}{(1+x^2)^2}dx$ is convergent and its value is $\frac{1}{4} + \frac{\pi}{8}$

→ **1(e).** Show that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z=2$ in a circle of radius unity and find the equation of the sphere which has this circle as one of its great circles.

Sol^n: The given sphere

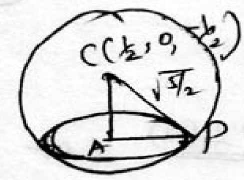
$$x^2+y^2+z^2-x-z-2=0 \quad \text{——} (1)$$

and the plane

$$x+2y-z-4=0 \quad \text{——} (2)$$

Centre of the sphere (1) is

$$C\left(\tfrac{1}{2}, 0, -\tfrac{1}{2}\right)$$



and its radius is

$$CP = \sqrt{\tfrac{1}{4}+0+\tfrac{1}{4}+2}$$

$$= \sqrt{5/2}$$

$CA = \perp^r$ distance from $C(1/2, 0, -1/2)$ to the plane (2)

$$= \frac{\left|\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4\right|}{\sqrt{1+4+1}}$$

$$= \frac{3}{\sqrt{6}} = \sqrt{3/2}.$$

$\therefore$ Radius of circle

$$AP = \sqrt{CP^2 - CA^2}$$

$$= \sqrt{5/2 - 3/2}$$

$$= \sqrt{1} = 1.$$

$\therefore$ The plane (2) meets the sphere (1) in a circle of radius unity.

Now any sphere through the intersection of (1) & (2) is

$x^2+y^2+z^2-x+z-2+k(x+2y-z-4)=0$ —— (3)

If the circle of intersection of (1) & (2) is a great circle of sphere (3);

then the centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ lies on the plane (2).

$\therefore \frac{1-k}{2}+2(-k)-\left(\frac{k-1}{2}\right)-4=0$

$\Rightarrow \boxed{k=-1}$

$\therefore$ (3) $\equiv x^2+y^2+z^2-2x-2y+2z+2=0.$

2(a) Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x,y,z)=(2x, 4x-y, 2x+3y-z)$.

(i) Show that T is invertible.

(ii) Find a formula for T^{-1}.

Sol^n: (i) Let $(x,y,z) \in \ker T$ be arbitrary.

Then $T(x,y,z)=(0,0,0)$

$\Rightarrow (2x, 4x-y, 2x+3y-z)=(0,0,0)$

$\Rightarrow 2x=0,\ 4x-y=0$ & $2x+3y-z=0.$

$\Rightarrow x=0,\ y=0,\ z=0.$

and so $(x,y,z)=(0,0,0)$

$\Rightarrow \ker T=\{(0,0,0)\}$

Hence T is invertible.

(ii) Now we shall find T^{-1}.

Since T invertible.

$\therefore$ T is onto.

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for any $(a,b,c) \in R^3$, there exists some $(x,y,z) \in R^3$ such that

$T(x,y,z) = (a,b,c)$

$\Rightarrow (2x, 4x-y, 2x+3y-z) = (a,b,c)$

$\Rightarrow 2x = a,\ 4x-y = b,\ 2x+3y-z = c.$

$\Rightarrow \boxed{x = \frac{a}{2}},\ \boxed{y = 2a-b}$

from $2x+3y-z=c$

$2\left(\frac{a}{2}\right)+3(2a-b)-z=c$

$a+6a-3b-z=c$

$\Rightarrow \boxed{z = 7a-3b-c}$

Hence $T(x,y,z) = (a,b,c)$

$\Rightarrow T^{-1}(a,b,c) = (x,y,z)$

$\Rightarrow T^{-1}(a,b,c) = \left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$, $\forall (a,b,c) \in R^3$.

2(b)

$\longrightarrow$ Find the rank of the matrix:

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

Sol^n: $A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}$ $\quad R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 + R_2 \end{array}$$

which is in echelon form.

The number of non-zero rows of this echelon form is 2.

$\therefore \ \rho(A) = 2.$

→ 2(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Is A is similar to a diagonal matrix? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Sol^n: The characteristic equation of A is

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[-(5+\lambda)(4-\lambda)+18\right] + 3\left[3(4-\lambda)-18\right] + 3\left[-18+6(5+\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-(20-\lambda-\lambda^2)+18\right] + 3\left[-6-3\lambda\right] + 3\left[12+6\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[-2+\lambda+\lambda^2\right] + 18 + 9\lambda = 0.$$

$$\Rightarrow (1-\lambda)\left[(\lambda+2)(\lambda-1)\right] + 9(\lambda+2) = 0$$

$$\Rightarrow (\lambda+2)\left[-(\lambda-1)^2 + 9\right] = 0$$

$$\Rightarrow (\lambda+2)(-\lambda^2+2\lambda+8) = 0.$$

(9)

$\Rightarrow (\lambda+2)[-\lambda^2+4\lambda-2\lambda+8]=0$

$\Rightarrow (\lambda+2)[-\lambda(\lambda-4)-2(\lambda-4)]=0$

$\Rightarrow (\lambda+2)[(-\lambda-2)(\lambda-4)]=0$

$\Rightarrow (\lambda+2)(\lambda+2)(\lambda-4)=0$

$\Rightarrow \lambda=-2,-2,4.$

$\therefore$ The characteristic roots of A are $-2,-2,4$.

The eigen vectors X of A corresponding to the eigen value -2 are given by

$(A-(-2)I)X=0.$

$$\Rightarrow \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-R_1$
$R_3 \to R_3-2R_1$

$$\sim \begin{bmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of coefficient matrix $=1$.

$\therefore$ The equations have $3-1=2$ L.I solutions corresponding to eigen value '-2'.

$\therefore$ we have

$3x_1-3x_2+3x_3=0$

$\Rightarrow x_1-x_2+x_3=0$

Let $x_2=k_1$ and $x_3=k_2$; where k_1, k_2 are arbitrary constants.

$\therefore x_1=k_1-k_2$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X = k_1 X_1 + k_2 X_2$$

Here $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ & $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are LI vectors of A corresponding to characteristic root -2.

$\therefore$ The geometric multiplicity of eigenvalue is equal to its algebraic multiplicity.

Now the eigen vectors X of A corresponding to the eigen value 4 are given by

$$(A - 4I)X = 0$$

$$\Rightarrow \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The rank of the coefficient matrix $= 2$

$\therefore$ The equations have $3 - 2 = 1$ LI solution.

(10)

$\therefore$ we have

$-3x_1 - 3x_2 + 3x_3 = 0$

$-12x_2 + 6x_3 = 0$

$\Rightarrow -x_1 - x_2 + x_3 = 0$

$-2x_1 + x_3 = 0$

$2x_1 = x_3.$

Take $x_1 = 1$ then $x_3 = 2$

and $x_2 = 1.$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 4.

$\therefore$ The geometric multiplicity of eigen value 4 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigen value of A is equal to its algebraic multiplicity.

$\therefore$ A is similar to diagonal matrix.

Let $P = [X_1, X_2, X_3]$

$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The columns of P are L.I eigen vectors of A corresponding to the eigen values $-2, -2, 4$ respectively.

The matrix P will transform A to the diagonal form D is given by the relation $P^{-1}AP = D$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

The transforming matrix. $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$

(11)

→ 4(a) obtain the equations of the planes which pass through the point (3,0,3), touch the sphere $x^2+y^2+z^2=9$ and are parallel to the line $x=2y=-z$.

Sol The given line is

$$x = 2y = -z$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \quad \text{--- (1)}$$

Any line parallel to (1) and passing through (3,0,3) is $\frac{x-3}{2} = \frac{y-0}{1} = \frac{z-3}{-2}$ --- (2).

Now the general form of the line (2) is given by

$$x-3 = 2y, \; -2y = z-3$$

$$\Longrightarrow x-2y-3=0 = 2y+z-3 \quad \text{--- (3)}$$

Now any plane through the line (3) is

$$(x-2y-3)+\lambda(2y+z-3)=0 \quad \text{--- (4)}$$

$$\Rightarrow x+(-2+2\lambda)y+\lambda z+(-3-3\lambda)=0$$

clearly, it will be the tangent plane to the given sphere $x^2+y^2+z^2-9=0$ if the perpendicular distance of the plane from centre (0,0,0) of the sphere is equal to the radius of the sphere.

i.e $\frac{|0+0+0-3-3\lambda|}{\sqrt{1+(-2+2\lambda)^2+\lambda^2}} = 3$ ($\because$ radius $=3$).

$$1+\lambda = \sqrt{5+5\lambda^2-8\lambda}$$

$$\Rightarrow (1+\lambda)^2 = 5+5\lambda^2-8\lambda$$

$$\Rightarrow \lambda^2+2\lambda+1 = 5\lambda^2-8\lambda+5$$

$$\Rightarrow 4\lambda^2-10\lambda+4=0$$

$$\Rightarrow 2\lambda^2-5\lambda+2=0$$

$$\Rightarrow 2\lambda(\lambda-2)-1(\lambda-2)=0.$$

$$\Rightarrow (2\lambda-1)(\lambda-2)=0$$

$$\Rightarrow \boxed{\lambda=2}, \boxed{\lambda=\tfrac{1}{2}.}$$

If $\lambda=2$: then from (4) we get,

$$x-2y-3+2(2y+z-3)=0$$

$$\Rightarrow \boxed{x+2y+2z-9=0} \quad \text{—— (5)}$$

If $\lambda=\tfrac{1}{2}$ then from (4) we get,

$$x-2y-3+\tfrac{1}{2}(2y+z-3)=0.$$

$$\Rightarrow \boxed{2x-2y+z-9=0.} \quad \text{—— (6).}$$

$\therefore$ The equations (5) & (6) are required planes, which are parallel to the given line.

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→ **4(b)** The section of the cone whose vertex is P and guiding curve is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$.

Sol^n: Let the vertex P be (α,β,γ) and given guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ —— (1)

Now the equation of any line through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—— (2)}$$

it meets the plane $z=0$

$$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\Rightarrow x-\alpha=-\frac{l\gamma}{n},\quad y-\beta=-\frac{\gamma m}{n},\quad z=0.$$

$$\Rightarrow x=\alpha-\frac{l}{n}\gamma,\quad y=\beta-\frac{m}{n}\gamma,\quad z=0$$

This point lies on the ellipse (1)

$$\therefore \frac{1}{a^2}\left(\alpha-\frac{l}{n}\gamma\right)^2+\frac{1}{b^2}\left(\beta-\frac{m}{n}\gamma\right)^2=1 \quad \text{—— (3)}$$

Now eliminating l,m,n from (2) & (3).

$$\frac{1}{a^2}\left(\alpha-\frac{(x-\alpha)\gamma}{z-\gamma}\right)^2+\left(\beta-\frac{(y-\beta)\gamma}{z-\gamma}\right)^2=1$$

$$\Rightarrow \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-y\gamma)^2=(z-\gamma)^2 \quad \text{—— (4)}$$

which is the required equation of the cone.

This meets the plane $x=0$.

$$\therefore (4) \equiv \frac{1}{a^2}(\alpha z-0)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$$

$$\Rightarrow \frac{\alpha^2 z^2}{a^2}+\frac{\beta^2 z^2+\gamma^2 y^2-2\beta\gamma zy}{b^2}=z^2+\gamma^2-2z\gamma.$$

This will be a rectangular hyperbola in yz-plane

if coefficient of y^2 + coefficient of $z^2=0$

$$\Rightarrow \frac{\gamma^2}{b^2}+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1=0$$

$$\Rightarrow \frac{\alpha^2}{a^2}+\frac{\beta^2+\gamma^2}{b^2}-1=0$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

... of the poles of the

→ **4(c)** Prove that the locus of the poles of the tangent planes of the conicoid $ax^2+by^2+cz^2=1$ with respect to the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$ is the conicoid $\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1$.

Sol^n: Let $lx+my+nz=p$ — (i) be the tangent plane to the conicoid

$$ax^2+by^2+cz^2=1 \quad \text{— (ii)}$$

Then $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$ — (iii)

Let (a',b',c') be the pole of the plane (i) w.r.t $\alpha x^2+\beta y^2+\gamma z^2=1$

Then we have $a'\alpha x+b'\beta y+c'\gamma z=1$. — (iv)

comparing (i) & (iv), we get

$$\frac{a'\alpha}{l}=\frac{b'\beta}{m}=\frac{c'\gamma}{n}=\frac{1}{p}. \quad \text{— (v)}$$

Eliminating l,m,n between (iii) and (v), we get

$$\frac{(a'\alpha p)^2}{a}+\frac{(b'\beta p)^2}{b}+\frac{(c'\gamma p)^2}{c}=p^2.$$

$$\frac{(a'\alpha)^2}{a}+\frac{(b'\beta)^2}{b}+\frac{(c'\gamma)^2}{c}=1.$$

∴ The required locus of (a',b',c') is

$$\frac{(\alpha x)^2}{a}+\frac{(\beta y)^2}{b}+\frac{(\gamma z)^2}{c}=1.$$

4(d) Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$ at its points of intersection with the planes $lx+my+nz=p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

Sol^n: Let (α,β,γ) be the point of intersection of the given conicoid and given plane,

then we have $a\alpha^2+b\beta^2+c\gamma^2=1$ ——(i)

and $l\alpha+m\beta+n\gamma=p$ ——(ii)

Also the equations of the normals to the given conicoid at (α,β,γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}.$$

∴ The equations of the line through the origin parallel to this line are

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma} \text{ ——(iii)}$$

from (i) & (ii), we have

$$a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{p}\right)^2$$

$$\Rightarrow p^2(a\alpha^2+b\beta^2+c\gamma^2)=(l\alpha+m\beta+n\gamma)^2$$

$$\Rightarrow p^2\left(\frac{a^2\alpha^2}{a}+\frac{b^2\beta^2}{b}+\frac{c^2\gamma^2}{c}\right)=\left(\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right)^2$$

$$\Rightarrow p^2\left[\frac{(a\alpha)^2}{a}+\frac{(b\beta)^2}{b}+\frac{(c\gamma)^2}{c}\right]=\left(\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right)^2$$

$$\Rightarrow p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

eliminating α,β,γ from (iii)

Hence the line (iii) generates the above cone.

Section – B

→ 5(a) solve $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$.

Sol^n: Given that $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$ ——— ①

put $\tan y = t$

$\Rightarrow \sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ ① $\equiv$ $\frac{dt}{dx} + 2xt = x^3$

which is linear in t and x.

$I.F = \int e^{2x} = e^{x^2}$.

$\therefore$ General solution of ① is

$t e^{x^2} = \int x^3 e^{x^2} dx + C$

$= \frac{1}{2}\int z e^z dz + C$ Let $x^2 = z$ $\Rightarrow 2x\,dx = dz$ $\Rightarrow x\,dx = \frac{1}{2}dz$

$= \frac{1}{2} e^z (z-1) + C$.

$\therefore \tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + C$ $(\because t = \tan y)$

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5.(b) Find the 2nd order ODE for which e^x and x^2e^x are solutions.

Sol Let $y_1 = x$ and $y_2 = x^2e^x$. Then their wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2e^x \\ 1 & x^2e^x + 2xe^x \end{vmatrix}$$

$$= x^3e^x + 2x^2e^x - x^2e^x$$

$= x^3e^x + x^2e^x$. which is not identically equal to zero on $\mathbb{R} = (-\infty, \infty)$.

To find the required differential equation:-

The general solution of the required differential equation may be written as

$y = c_1y_1 + c_2y_2$ where c_1 and c_2 are arbitrary contents.

$$\Rightarrow y = c_1e^x + c_2x^2e^x \quad \text{---}(1)$$

Differentiating (1) w.r.t 'x' weget

$$y' = c_1e^x + c_2x^2e^x + c_2(2xe^x).$$

$$\Rightarrow y' = y + 2c_2xe^x. \text{ (from (1))}$$

$$\Rightarrow y' - y = 2c_2xe^x \quad \text{---}(2)$$

again differentiating w.r.t. 'x', weget

$$y'' - y' = 2c_2xe^x + 2c_2e^x.$$

$$y'' - y' = y' - y + 2c_2e^x$$

$$\Rightarrow y'' - 2y' + y = 2c_2e^x. \quad \text{---}(3)$$

Now we eliminate c_1 and c_2 from ①, ② and ③. for this, substitute ① in ②, we get

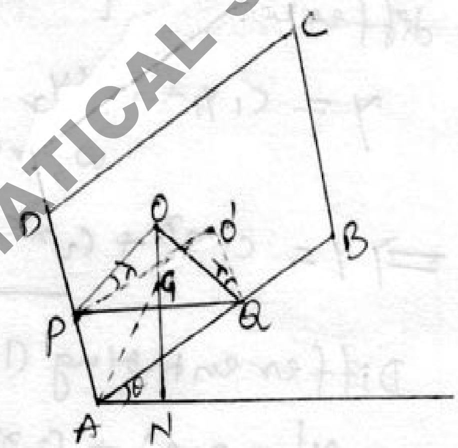
$$y' - y = x(y'' - 2y' + y)$$

$$\Rightarrow xy'' - (2x+1)y' + (x+1)y = 0$$

which is the required differential equation.

→ 5(c): A uniform rectangular board, whose sides are 2a and 2b, rests in limiting equilibrium in contact with two rough pegs in the same horizontal line at a distance d apart. Show that the inclination θ of the side 2a to the horizontal is given by the equation $d\cos\lambda[\cos(\lambda+2\theta)] = a\cos\theta - b\sin\theta$ where λ is the angle of friction.

Sol:



Let ABCD be the rectangle resting on the two pegs P and Q.

Suppose that the resultants of the reactions and the frictional forces at P and Q meet at O. Then the centre of gravity G of the rectangle must be vertically below O.

Let AN be the perpendicular from A on OG.

Suppose that the normals at P and Q meet at O'.

(15)

The angles OPO' and OQO' are equal, hence O, P, Q, O' are concyclic. Again O', P, A, Q are concyclic, hence O, O', A, P are concyclic.

It follows that $\angle OAO' = \lambda$, $\angle O'AQ = \angle AQP = \angle QAN = \theta$.

and $\angle O'OA = \angle O'PA = \frac{\pi}{2}$.

Also from the rectangle $O'PAQ$, $O'A = PQ$.

We will now find AN in two ways.

Firstly $AN = OA\cos(\lambda+2\theta)$
$= O'A\cos\lambda\cos(\lambda+2\theta)$
$= d\cos\lambda\cos(\lambda+2\theta)$.

Again $AN = AG\cos(GAQ + QAN)$
$= AG\cos GAQ\cos\theta - AG\sin GAQ\sin\theta$
$= a\cos\theta - b\sin\theta$.

Hence $d\cos\lambda\cos(\lambda+2\theta) = a\cos\theta - b\sin\theta$.

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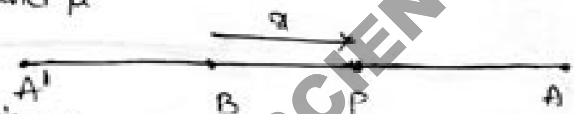
(d) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'. The particle is slightly displaced towards one of them, show that the time of small oscillation is $\frac{2\pi}{\sqrt{\mu+\mu'}}$.

Sol'n :- Suppose A and A' are the two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium



at B under the attraction of these two centres. If $AB = a$ and $A'B = a'$, the forces of attraction at B due to the centres A and A' are $m\mu a$ & $m\mu' a'$ respectively in opposite directions. As these two forces balance, we have

$$m\mu a = m\mu' a' \quad \text{——— (1)}$$

Now suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP = x$.

The attraction at P due to the centre A is $m\mu . AP$ or $m\mu(a-x)$ in the direction PA i.e, in the direction of x increasing. Also the attraction P due to the centre A' is $m\mu' . A'P$ or $m\mu'(a'+x)$ in the direction PA' i.e., in the direction of x decreasing. Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\left(d^2x/dt^2\right) = m\mu(a-x) - m\mu'(a'+x) \quad \text{——— (2)}$$

where the force in the direction of x increasing has been taken with +ve sign and the force in the direction of x decreasing has been taken with -ve sign.

Simplifying the equation (2), we get

$$m\left(d^2x/dt^2\right) = m\left(\mu a - \mu x - \mu' a' - \mu' x\right)$$

$$\text{(or)} \quad d^2x/dt^2 = -(\mu+\mu')x \quad [\because \text{by (1)}, m\mu a = m\mu' a']$$

this is the equation of a S.H.M with Centre at the origin. Hence the motion of the particle is Simple Harmonic with centre at B and its time period is $2\pi/\sqrt{\mu+\mu'}$.

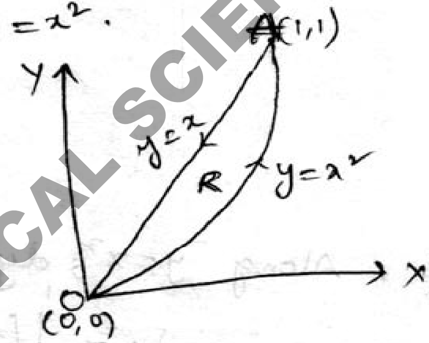
5(e) verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Sol: By Green's theorem in plane,

we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy.$$

Here $M = xy + y^2$ and $N = x^2$.



The curves $y=x$ and $y=x^2$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in the figure.

we have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy.$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{1} \left(xy - y^2\right)_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left(\cancel{x^2} - \cancel{x^2} - x^3 + x^4\right) dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\, dx$$

$$= \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Now let us evaluate the line integral along 'C'.

The line integral along C = Line integral along $y = x^2$ (from O to A) + line integral along $y = x$ (from A to O)

$= I_1 + I_2$.

Along $y = x^2$; $dy = 2x\,dx$.

$$\therefore I_1 = \int_0^1 [x(x^2) + x^4]\,dx + x^2(2x)\,dx$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}.$$

Along $y = x$, $dy = dx$

$$\therefore I_2 = \int_1^0 \{x(x) + x^2\}\,dx + x^2\,dx$$

$$= \int_1^0 3x^2\,dx$$

$$= 3\frac{x^3}{3} = 0 - 1 = -1.$$

$$\therefore I_1 + I_2 = \frac{19}{20} - 1 = -\frac{1}{20}.$$

Hence the theorem is verified

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6(a)

Solve $(y^3-2yx^2)dx+(2xy^2-x^3)dy=0$

Sol^n: Given that $(y^3-2yx^2)dx+(2xy^2-x^3)dy=0$ —— ①

clearly ① is a homogeneous diff equation.

comparing ① with $Mdx+Ndy=0$,

we have $M=y^3-2x^2y$; $N=2xy^2-x^3$

$\therefore Mx+Ny = y^3x-2x^3y+2xy^3-x^3y$

$= 3xy^3-3x^3y$

$= 3xy(y^2-x^2)\neq 0.$

$\Rightarrow \dfrac{1}{Mx+Ny} = \dfrac{1}{3xy(y^2-x^2)}.$

Multiplying ① by $\dfrac{1}{3xy(y^2-x^2)}$,

we get

$$\frac{y(y^2-2x^2)}{3xy(y^2-x^2)}dx+\frac{(2y^2-x^2)x}{3xy(y^2-x^2)}dy=0$$

$$\Rightarrow \frac{y^2-2x^2}{3x(y^2-x^2)}dx+\frac{(2y^2-x^2)}{3y(y^2-x^2)}dy=0. \quad \text{—— ②}$$

comparing ② with $Pdx+Qdy=0$

$\therefore P=\dfrac{y^2-2x^2}{3xy^2-3x^3}$, $Q=\dfrac{2y^2-x^2}{3y^3-3x^2y}$

$\dfrac{\partial P}{\partial y}=\dfrac{6xy}{(3y^2-3x^2)^2}$, $\dfrac{\partial Q}{\partial x}=\dfrac{6xy}{(3y^2-3x^2)^2}.$

$\therefore \dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}.$

$\therefore$ ② is an exact.

and its solution is given by

$$\int_{y\text{-constant}} P\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = C_1$$

$$\Rightarrow \int \frac{y^2-2x^2}{3x(y^2-x^2)}\,dx + \int \frac{1}{3y}\,dy = C_1$$

$$\Rightarrow \int \frac{y^2-x^2-x^2}{3x(y^2-x^2)}\,dx + \int \frac{1}{3y}\,dy = C_1$$

$$\Rightarrow \int \frac{1}{3x}\,dx + \int \frac{x}{3(y^2-x^2)}\,dx + \int \frac{1}{3y}\,dy = C_1$$

$$\Rightarrow \frac{1}{3}\log x + \frac{1}{6}\log(y^2-x^2) + \frac{1}{3}\log y = \log C_2$$

where $C_1 = \log C_2$

$$\Rightarrow (xy)^{1/3}(y^2-x^2)^{1/6} = C_2$$

$$\Rightarrow (xy)(y^2-x^2)^{1/2} = C_2^3$$

$$\Rightarrow (x^2y^2)(y^2-x^2) = C_2^6$$

$$\Rightarrow x^2y^2(y^2-x^2) = C \quad \text{where } C = C_2^6$$

which is the required solution

→ 6(b) Solve $\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}\cosh x + 1 = 0.$

Sol^n The given equation is

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}\cosh x + 1 = 0$$

$\therefore p^2 - 2p\cosh x + 1 = 0$ where $\frac{dy}{dx} = p$.

Solving for p,

$$p = \frac{2\cosh x \pm \sqrt{4\cosh^2 x - 4}}{2}$$

$$= \cosh x \pm \sqrt{\cosh^2 x - 1}$$

$$= \cosh x \pm \sinh x.$$

$\Rightarrow p = \cosh x + \sinh x$; $p = \cosh x - \sinh x$.

$\Rightarrow \frac{dy}{dx} = \cosh x + \sinh x$; $\frac{dy}{dx} = \cosh x - \sinh x$.

Integrating

$y = \sinh x + \cosh x + c$; $y = \sinh x - \cosh x + c$.

Hence ~~the~~ general solution is

$$(y - \sinh x + \cosh x - c)(y - \sinh x + \cosh x - c) = 0$$

$$\Rightarrow \left(y - \frac{(e^x - e^{-x})}{2} - \frac{e^x + e^{-x}}{2} - c\right)\left(y - \frac{(e^x - e^{-x})}{2} + \frac{e^x + e^{-x}}{2} - c\right) = 0$$

$$\Rightarrow (y - e^{+x} - c)(y + e^{-x} - c) = 0.$$

$\therefore$ The general solution is

$$\boxed{(y - e^x - c)(y + e^{-x} - c) = 0}$$

6(c)

Solve $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=x^2e^{-x}$.

Sol^n: The given can be written as

$$(D^3+3D^2+3D+1)y=x^2e^{-x}.$$

The auxiliary equation is

$$D^3+3D^2+3D+1=0$$

$$\Rightarrow (D+1)^3=0$$

$$\Rightarrow D=-1,-1,-1.$$

$\therefore$ C.F is $y_c=(c_1+c_2x+c_3x^2)e^{-x}$.

Particular Integral:

$$y_p=\frac{1}{D^3+3D^2+3D+1}x^2e^{-x}$$

$$=\frac{1}{(D+1)^3}x^2e^{-x}$$

$$=e^{-x}\frac{1}{(D-1+1)^3}x^2$$

$$=e^{-x}\frac{1}{D^3}x^2$$

$$y_p=e^{-x}\frac{x^5}{60}.$$

$\therefore$ $y=y_c+y_p$

$$y=(c_1+c_2x+c_3x^2)e^{-x}+\frac{e^{-x}x^5}{60}$$

is the general solution of the given differential equation.

→ 6(d)

Show that e^{x^2} is a solution of

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-2)y = 0$$

Find a second independent solution.

Sol^n: Given that $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-2)y = 0$ ——— (1)

To show: $y = e^{x^2}$ is a solution of (1).

$y' = e^{x^2}.2x$

and $y'' = 2[e^{x^2} + 2x^2 e^{x^2}]$

$= 2e^{x^2}[1+2x^2]$

with these values of y, y', y''

L.H.S. of (1) $= 2e^{x^2}(1+2x^2) - 4x(e^{x^2}.2x) + (4x^2-2)e^{x^2}$

$= \cancel{2e^{x^2}} + \cancel{4x^2e^{x^2}} - \cancel{8x^2e^{x^2}} + \cancel{4x^2e^{x^2}} - \cancel{2e^{x^2}}$

$= 0$

Showing that $y = e^{x^2}$ is a solution of (1)

Since $y = e^{x^2}$ is a solution of (1)

$\therefore y = u = e^{x^2}$ is in a part of C.F of (1).

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

$\therefore P = -4x;\ Q = 4x^2-2;\ R = 0$

Let $y = uv$ be the general solution of (1).

then v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \quad \text{——— (2)}$$

Since $u = e^{x^2} \Rightarrow \frac{du}{dx} = 2xe^{x^2}$

$\therefore P + \frac{2}{u}\frac{du}{dx} = -4x + \frac{2}{e^{x^2}}(2xe^{x^2})$

$= -4x + 4x$

$= 0$

→ 7(c): A shell, lying in a straight smooth horizontal tube, suddenly explodes and breaks into portions of masses m and m'. If 'd' is the distance apart of the masses after a time 't', show that the workdone by the explosion is $\frac{1}{2}\frac{mm'}{m+m'}\frac{d^2}{t^2}$.

Sol^n: Since the shell is lying in the tube, its velocity before explosion is zero.

Let u_1 and u_2 be the velocities of the masses m, and m' respectively after explosion. Then the relative velocity of the masses after explosion is u_1+u_2. Since the tube is smooth and horizontal, u_1+u_2 will remain constant.

$$\therefore (u_1+u_2)t = d \quad \text{---} (1)$$

Also by the principle of conservation of linear momentum, we have $mu_1 - m'u_2 = 0$

$$\Rightarrow mu_1 = m'u_2 \quad \text{---} (2)$$

Substituting for u_2 from (2) in (1),

we get $\left(u_1 + \frac{mu_1}{m'}\right)t = d.$

$$u_1\left(\frac{m'+m}{m'}\right)t = d$$

$$\Rightarrow u_1 = \frac{m'd}{(m+m')t}$$

$$\therefore \text{from (2) } u_2 = \frac{m}{m'}u_1$$

$$= \frac{m}{m'}\frac{m'd}{(m+m')t}$$

$$u_2 = \frac{md}{(m+m')t}.$$

explosion

Now the workdone by the explosion

= the kinetic energy released due to the explosion

$$= \frac{1}{2}mu_1^2 + \frac{1}{2}m'u_2^2$$

$$= \frac{1}{2}m\left[\frac{m'^2d^2}{(m+m')^2t^2}\right] + \frac{1}{2}m'\left[\frac{m^2d^2}{(m+m')^2t^2}\right]$$

$$= \frac{1}{2}\frac{d^2}{t^2}\frac{1}{(m+m')^2}\left[mm'^2 + m'm^2\right]$$

$$= \frac{1}{2}\frac{d^2}{t^2}\frac{mm'}{(m+m')^2}\left[m+m'\right]$$

$$= \frac{1}{2}\frac{d^2}{t^2}\left(\frac{mm'}{m+m'}\right)$$

Now from ②, we have

$$\frac{d^2v}{dx^2} + (0)\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \frac{dv}{dx} = c_1$$

$$\Rightarrow v = c_1x + c_2$$

∴ General solution of ① is $y = uv$.

$$\Rightarrow y = e^{x^2}(c_1x + c_2)$$

$$= c_1xe^{x^2} + c_2e^{x^2}.$$

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8(a) Show that $\bar{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ is irrotational. Find a scalar function ϕ such that $\bar{A} = \text{grad}\,\phi$.

Sol^n: Given that $\bar{A} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$.

$$\text{curl}\,\bar{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix}$$

$$= \hat{i}(-1+1) + \hat{j}(3z^2-3z^2) + \hat{k}(6x-6x)$$

$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= 0.$$

$\therefore$ The vector $\bar{A}$ is irrotational.

Let $\bar{A} = \text{grad}\,\phi$

i.e, $\bar{A} = \nabla\phi$.

$$\Rightarrow (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

Then $\dfrac{\partial\phi}{\partial x} = 6xy+z^3$ —— (1)

$\dfrac{\partial\phi}{\partial y} = 3x^2-z$ —— (2)

$\dfrac{\partial\phi}{\partial z} = 3xz^2-y$. —— (3)

(1) partially w.r.t x treating y, z as constants.

$\therefore$ (1) $\equiv$ $\phi = 3x^2y + z^3x + f_1(y,z)$ —— (4)

(2) partially w.r.t y treating x, z as constants

$\therefore$ (2) $\equiv$ $\phi = 3x^2y - zy + f_2(x,z)$ —— (5)

(3) partially w.r.t z treating x, y as constants

$\therefore$ (3) $\equiv$ $\phi = xz^3 - yz + f_3(x,y)$ —— (6)

(20)

(4), (5), (6) each represents ϕ. These agree if we choose $f_1(y,z) = -yz$; $f_2(x,z) = xz^3$, $f_3(x,y) = 3x^2y$.

$$\therefore \phi = 3x^2y + xz^3 - yz + c$$

where c is any arbitrary constant.

8.6 → Let $\psi(x, y, z)$ be a scalar function. find grad ψ and $\nabla^2\psi$ in spherical co-ordinates.

sol). we know that

$$\nabla\psi = \text{grad}\,\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}e_3 \quad \text{——(1)}$$

for spherical co-ordinates (r, θ, ϕ)

$u_1 = r,\ u_2 = \theta,\ u_3 = \phi$;

$e_1 = e_r,\ e_2 = e_\theta,\ e_3 = e_\phi$;

$h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_\phi = r\sin\theta$.

∴ from (1)

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi$$

$$= \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi.$$

we know that.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{(r\sin\theta)(1)}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

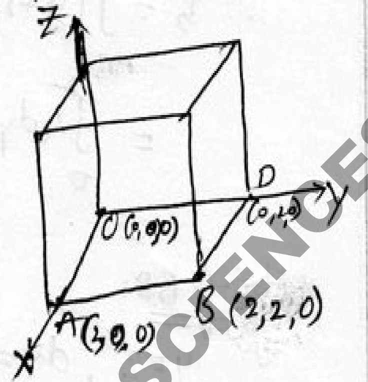
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$= \cancel{\frac{1}{r^2\sin\theta}\left[\sin\theta\right.}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.$$

8(c) → verify Stoke's theorem for
$\bar{A}=(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}$,
where S is the surface of the cube $x=0, y=0$, $z=0$, $x=2$, $y=2$, $z=2$ above the xy-plane.



Sol^n:
The xy-plane cuts the surface of the cube in a square.
Thus the curve C bounding the surface S is the square, say, OABD, in the xy-plane whose vertices in the xy-plane are the points.

$O(0,0)$, $A(2,0)$, $B(2,2)$, $D(0,2)$.

Then $\oint \vec{F}\cdot dr = \int_C [(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$

$= \int_C (y-z+2)\,dx+(yz+4)\,dy-xz\,dz$

$= \int_C (y+2)\,dx+4dy$ (∵ on C, $z=0$. and $dz=0$)

$= \int_{OA}+\int_{AB}+\int_{BD}+\int_{DO}$.

$= I_1+I_2+I_3+I_4$. ——— (1)

Along OA;
$y=0 \Rightarrow dy=0$.
and x varies from 0 to 2.

$$\therefore I_1 = \int_{OA} (y+2)\,dx + 4\,dy$$
$$= \int_0^2 2\,dx$$
$$= [2x]_0^2 = 4.$$

Along AB:

$x=2$, $dx=0$ and y varies from 0 to 2

$$\therefore I_2 = \int_{AB} (y+2)\,dx + 4\,dy$$
$$= \int_0^2 4\,dy = [4y]_0^2 = 8.$$

Along BD:

$y=2$, $dy=0$ & x varies from 2 to 0.

$$\therefore I_3 = \int_{BD} (y+2)\,dx + 4\,dy$$
$$= \int_2^0 4\,dx = [4x]_2^0$$
$$= -8$$

Along DO:

$x=0$, $dx=0$ & y varies from 2 to 0.

$$\therefore I_4 = \int_{DO} (y+2)\,dx + 4\,dy$$
$$= \int_2^0 4\,dy = [4y]_2^0 = -8$$

$\therefore$ (1) $\equiv$

$$\int \bar{F}\cdot d\bar{r} = I_1 + I_2 + I_3 + I_4$$
$$= 4 + \cancel{8} - \cancel{8} - 8$$
$$= -4 \qquad (2)$$

Now $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$= \hat{i}(0-y) + \hat{j}(-1+z) + \hat{k}(0-1)$

$= -y\hat{i} + (-1+z)\hat{j} - \hat{k}.$

$\hat{n}$ = unit normal vector to $S = \hat{k}$.

$\therefore dS = \dfrac{dx\,dy}{|n\cdot k|} = dx\,dy.$

$(\nabla \times F)\cdot\hat{n} = \left[(-y\hat{i} + (-1+z)\hat{j} - \hat{k}\right]\cdot\hat{k}$

$= -1.$

$$\iint_S (\nabla\times F)\cdot\hat{n}\,dS = \int_{x=0}^{2}\int_{y=0}^{2}(-1)\,dx\,dy$$

$= -\int_{x=0}^{2} [y]_0^2\,dx$

$= -\int_{x=0}^{2} 2\,dx$

$= -2\int_0^2 dx$

$= -2[x]_0^2$

$= -2[2]$

$= -4.$ —— (2)

$\therefore$ from (2) & (3)

$$\iint_S (\nabla\times F)\cdot\hat{n}\,dS = \int_C \vec{F}\cdot d\vec{r} = -4.$$

Hence the Stoke's theorem is verified

8.(d) Show that, if $\bar{r} = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$ is a space curve, $\frac{d\bar{r}}{ds} \cdot \frac{d^2\bar{r}}{ds^2} \times \frac{d^3\bar{r}}{ds^3} = \frac{\tau}{\rho^2}$, where τ is the torsion and ρ is the radius of curvature.

Sol: we know that $T = \frac{d\vec{r}}{ds}$

and $kN = \frac{d^2\vec{r}}{ds^2}$

Here k is the curvature.

Now $\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} = T \times kN$

$= k(T \times N)$

$= kB \quad (\because T \times N = B)$

$\therefore k = \left| \frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} \right|$

$$\frac{d^3\vec{r}}{ds^3} = \frac{d}{ds}\left(\frac{d^2\vec{r}}{ds^2}\right) = \frac{d}{ds}(kN)$$

$= k\frac{dN}{ds} + \frac{dk}{ds}N$

$= k(\tau B - kT) + \frac{dk}{ds}N$

$\left(\because \frac{dN}{ds} = \tau B - kT\right)$

$= k\tau B - k^2 T + \frac{dk}{ds}N.$

$$\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) = T \cdot \left[kN \times \left(k\tau B - k^2 T + \frac{dk}{ds}N\right)\right]$$

$= T \cdot \left[k^2(N \times \tau B) - k^3(N \times T) + k\frac{dk}{ds}(N \times N)\right]$

$= T \cdot \left[k^2\tau(N \times B) - k^3(-B) + k\frac{dk}{s}(0)\right]$

$(\because N \times T = -B \;\&\; N \times N = 0)$

$= T \cdot \left(k^2\tau(T) + k^3 B\right)$

$(\because N \times B = T)$

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$= K^2 \tau (T \cdot T) - K^3 (T \cdot B)$

$= K^2 \tau (1) - K^3 (0)$ $\left(\because T \cdot T = 1 \text{ \& } T \cdot B = 0 \right)$

$= K^2 \tau$

w.k.T radius of the curvature 'ρ' is the reciprocal of curvature K

i.e, $\rho = \frac{1}{K}$

$\Rightarrow K = \frac{1}{\rho}$

$$\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \right) = \frac{1}{\rho^2} \tau.$$