LINEAR ALGEBRA

: CSE-2011 :

1.(a) Let A be a non-singular $n\times n$ matrix. Show that $A\cdot(\text{adj}A) = |A|I_n$. Hence show that $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

$\longrightarrow$ Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ $\Rightarrow$ $\text{adj}A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}_{n\times n}$

where A_{ij} is the cofactor of the element in ith row and jth column in the matrix A.

Then, we have, $A\cdot(\text{adj}A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n} & a_{11}A_{21}+\cdots+a_{1n}A_{2n} & \cdots \\ a_{21}A_{11}+\cdots\cdots a_{2n}A_{1n} & a_{21}A_{21}+\cdots+a_{2n}A_{2n} & \cdots \\ a_{n1}A_{11}+\cdots+a_{nn}A_{1n} & a_{n1}A_{21}+\cdots+a_{nn}A_{2n} & \cdots \end{bmatrix}$$

WKT
$a_{11}A_{11}+a_{12}A_{12}+\cdots+a_{1n}A_{1n}=|A|$
$a_{21}A_{11}+a_{22}A_{12}+\cdots+a_{2n}A_{1n}=0$
& so on

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & \cdots & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & & |A| \end{bmatrix}$$

$$= |A|I_n.$$

$\therefore A\cdot(\text{adj}A) = |A|I_n.$

<u>Now:</u> Taking determinant both sides, we have

$|A(\text{adj}A)| = ||A|I_n|$

$|A|\,|\text{adj}A| = ||A||\,|I_n| = |A|^n \cdot 1.$

$\Rightarrow |\text{adj}A| = |A|^{n-1}.$

Replacing A with $\text{adj}A$, we have

$|\text{adj}(\text{adj}A)| = |\text{adj}A|^{n-1} = \left[|A|^{n-1}\right]^{n-1} = |A|^{(n-1)^2}$

$\Rightarrow |\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

1(b) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$. Solve the system of equations given by $AX = B$. Using the above, solve the system of equations $A^T X = B$ where A^T dénotes the transpose of A.

→ $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{vmatrix} = 1[28-30] - 1[18] = -20 \neq 0.$

$\therefore A^{-1}$ exists. finding A^{-1} using characteristic equation.

Char. eq^n of $A \Rightarrow |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 3 & 4-\lambda & 5 \\ 0 & 6 & 7-\lambda \end{vmatrix} = 0$

$\rightarrow (1-\lambda)[(4-\lambda)(7-\lambda) - 30] - 1[18] = 0$

$\rightarrow (1-\lambda)[-2 - 11\lambda + \lambda^2] - 18 = 0$

$\Rightarrow -2 + 2\lambda - 11\lambda + 11\lambda^2 + \lambda^2 - \lambda^3 - 18 = 0$

$\rightarrow \lambda^3 - 12\lambda^2 + 9\lambda + 20 = 0$ —— (1)

By Cayley Hamilton's Theorem, A satisfies (1)

$\therefore A^3 - 12A^2 + 9A + 20I = 0$

Premultiplying both sides with A^{-1}.

$A^{-1}.A^3 - 12A^{-1}A^2 + 9A^{-1}A + 20A^{-1}I = A^{-1}.0$

$\rightarrow A^2 - 12A + 9I + 20A^{-1} = 0$

$$20A^{-1} = -A^2 + 12A - 9I = -\begin{bmatrix}1&0&-1\\3&4&5\\0&6&7\end{bmatrix}\begin{bmatrix}1&0&-1\\3&4&5\\0&6&7\end{bmatrix} + 12\begin{bmatrix}1&0&-1\\3&4&5\\0&6&7\end{bmatrix} - 9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A^{-1} = \frac{1}{20}\begin{bmatrix}2&6&-4\\21&-7&8\\-18&6&-4\end{bmatrix}$$

<u>Now</u> The solution to system of equations $AX = B$ is given by $X = A^{-1}B$

$$\rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{1}{20}\begin{bmatrix}2&6&-4\\21&-7&8\\-18&6&-4\end{bmatrix}\begin{bmatrix}2\\6\\5\end{bmatrix} = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$$\therefore \boxed{x = 1,\ y = 2,\ z = -1}$$

<u>Now</u> <u>for $A^TX = B$.</u> we have $|A| = |A^T| \neq 0$.

$\therefore (A^T)^{-1}$ exists and $(A^T)^{-1} = (A^{-1})^T = \frac{1}{20}\begin{bmatrix}2&21&-18\\6&-7&6\\-4&8&-4\end{bmatrix}$

$$\therefore X = (A^T)^{-1}B = \frac{1}{20}\begin{bmatrix}2&21&-18\\6&-7&6\\-4&8&-4\end{bmatrix}\begin{bmatrix}2\\6\\5\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix}$$

$$\therefore \boxed{x = 2,\ y = 0,\ z = 1}$$

<u>2(a) (i)</u> Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of a $n \times n$ square matrix A with corresponding eigen vectors $X_1, X_2, \ldots X_n$. If B is a matrix similar to A, show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A.

$\longrightarrow$ B is similar to A $\Rightarrow \exists$ a non-singular matrix P such that $B = P^{-1}AP$.

For the eigen values of A and B to be same, their characteristic equation needs to be same.

Characteristic equation of B is given by $|B - \lambda I| = 0$.

<u>Now</u> $|B - \lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P|$

$= |P^{-1}AP - P^{-1}\lambda I P| = |P^{-1}(A - \lambda I)P|$

$= |P^{-1}|\,|A - \lambda I|\,|P|$

$\Rightarrow |B-\lambda I| = \frac{1}{|P|} |A-\lambda I| \, |P|$ $\quad [\because |P^{-1}| = |P|^{-1}]$

$|B-\lambda I| = |A-\lambda I|$

Hence, the characteristic equation of A and B are the same. Therefore, the eigen values of A and B are the same.

Now: $BP^{-1}X_i = (P^{-1}AP)P^{-1}X_i = P^{-1}APP^{-1}X_i \quad i\in[1,n].$

$= P^{-1}AIX_i = P^{-1}AX_i$

$\Rightarrow BP^{-1}X_i = P^{-1}AX_i = P^{-1}\lambda_i X_i \quad [AX_i = \lambda_i X_i]$

$\Rightarrow B(P^{-1}X_i) = \lambda_i(P^{-1}X_i)$

$\therefore$ For the eigen value λ_i $[i\in[1,n]]$, the eigen vector of A is X_i and the eigen vector of B is $P^{-1}X_i$.

2(a)(ii) Verify the Cayley-Hamilton's Theorem for the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$. Using this, show that A is a non-singular matrix and find A^{-1}.

$\longrightarrow$ Cayley-Hamilton's Theorem: Every square matrix satisfies its characteristic equation.

Char. equation of A is given by $|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -5 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(1-\lambda)^2 - 0] - 1[-10-3(1-\lambda)] = 0$

$\Rightarrow (1-\lambda)^3 + [13-3\lambda] = 0$

$\Rightarrow 1-\lambda^3 - 3\lambda + 3\lambda^2 + 13 - 3\lambda = 0$

$\rightarrow \lambda^3 - 3\lambda^2 + 6\lambda - 14 = 0$ —— ①

Putting A in place of λ in ①

$A^3 - 3A^2 + 6A - 14I = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence A satisfies its characteristic eqn. Hence, Cayley Hamilton Theorem is verified.

Now $A^3 - 3A^2 + 6A - 14I = 0$ ——— (2)

Premultiplying with A^{-1} on both sides, we have

$$A^{-1}.A^3 - 3A^{-1}A^2 + 6A^{-1}A - 14A^{-1}I = A^{-1}0$$

$\rightarrow$ $A^2 - 3A + 6I - 14A^{-1} = 0$

$\rightarrow$ $14A^{-1} = A^2 - 3A + 6I$ $\Rightarrow$ A^{-1} exists

$\Rightarrow$ $A^{-1} = \frac{1}{14}\left\{\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix} - 3\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix} + 6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$

$$A^{-1} = \frac{1}{14}\begin{bmatrix}1&5&1\\-2&4&-2\\-13&5&1\end{bmatrix}$$

2(b)(i) Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace.

$\longrightarrow$ The two given sets of vectors spans same subspace if their rowspace is identical.

Let $B = \begin{pmatrix}1&2&-3\\5&2&1\end{pmatrix} \sim \begin{pmatrix}-4&0&-4\\5&2&1\end{pmatrix} \xrightarrow{R_1\to R_1-R_2}$ $\overset{R_2\to R_2+R_1}{\sim} \begin{pmatrix}-4&0&-4\\1&2&-3\end{pmatrix}$

$\approx \begin{pmatrix}1&0&1\\1&2&-3\end{pmatrix} \xrightarrow{R_1\to \frac{R_1}{-4}} \sim \overset{R_1\leftrightarrow R_2}{} \begin{pmatrix}1&2&-3\\1&0&1\end{pmatrix} \overset{R_1\to R_1+R_2}{\sim} \begin{pmatrix}2&2&-2\\1&0&1\end{pmatrix} \overset{R_1\to R_1/2}{\sim} \begin{pmatrix}1&1&-1\\1&0&1\end{pmatrix}$

Thus, the rowspace of B is same as the rowspace of A where $A = \begin{pmatrix}1&1&-1\\1&0&1\end{pmatrix}$

$\therefore$ The span of the two given sets is the same.

Now $A = \begin{pmatrix}1&1&-1\\1&0&1\end{pmatrix}$. Reducing it to echelon form

$R_2\to R_2-R_1$ $\sim \begin{pmatrix}1&1&-1\\0&-1&2\end{pmatrix}$ $\rightarrow$ Echelon form.

Echelon form of A has 2 non-zero rows $\Rightarrow \rho(A) = 2$

$\therefore$ Dimension of subspace spanned by the vectors of A is 2

2(b)(ii) Find the nullity and basis of the nullspace of the linear transformation $A: \mathbb{R}^4 \to \mathbb{R}^4$.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

$\longrightarrow$ Let us consider the standard basis of $\mathbb{R}^4$ i.e.

$S = \{e_1, e_2, e_3, e_4\}$ where $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$ and $e_4 = (0,0,0,1)$

Then: $T(e_1) = 0e_1 + e_2 + 3e_3 + e_4 = (0,1,3,1)$

$T(e_2) = e_1 + 0e_2 + e_3 + e_4 = (1,0,1,1)$

$T(e_3) = -3e_1 + e_2 + 0e_3 - 2e_4 = (-3,1,0,-2)$

$T(e_4) = -e_1 + e_2 + 2e_3 + 0e_4 = (-1,1,2,0)$

$T(x,y,z,t) = T(xe_1 + ye_2 + ze_3 + te_4)$

$= \cancel{xT(e_1)}\ xT(e_1) + y\,T(e_2) + z\,T(e_3) + t\,T(e_4)$

$= x(0,1,3,1) + y(1,0,1,1) + z(-3,1,0,-2) + t(-1,1,2,0)$

$= (y-3z-t,\ x+z+t,\ 3x+y+2t,\ x+y-2z)$

Nullspace of T:

$N(T) = \{(x,y,z,t) \in \mathbb{R}^4 \mid T(x,y,z,t) = 0\}$

Let $(x,y,z,t) \in N_A(T)$. Then $T(x,y,z,t) = 0$.

$\Rightarrow (y-3z-t,\ x+z+t,\ 3x+y+2t,\ x+y-2z) = (0,0,0,0)$

$\Rightarrow y-3z-t = 0,\ x+z+t = 0,\ 3x+y+2t = 0,\ x+y-2z = 0$

~~Let~~ A $\Rightarrow$ $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\underset{R_1 \leftrightarrow R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim$$

$R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x + 0y + z + t = 0$ and $y - 3z - t = 0$

$x = -z - t$ & $y = 3z + t$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -z-t \\ 3z+t \\ z \\ t \end{bmatrix} = z \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Basis of $N_A(T) = \{(-1,3,1,0), (-1,1,0,1)\}$.

Nullity $(T) = \underline{\underline{2}}$

2(c)(i) Show that the vectors $(1,1,1), (2,1,2)$ and $(1,2,3)$ are linearly independent in $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$T(x,y,z) = (x+2y+3z,\ x+2y+5z,\ 2x+4y+6z)$.

Show that the images of above vectors under T are linearly dependent. Give reasons for the same.

$\longrightarrow$ Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. converting into echelon form

$R_2 \to R_2 - 2R_1$ — $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \text{Echelon form,}$$

Since the echelon form of A which contains the given three vectors has 3 non-zero rows, then, the given vectors $(1,1,1)$, $(2,1,2)$ and $(1,2,3)$ are L.I.

Now:

$T(1,1,1) = (6, 8, 12)$

$T(2,1,2) = (10, 14, 20)$

$T(1,2,3) = (14, 20, 28)$

Clearly:

~~T(1,2,3) = 2·T(1,1,1)~~

Let $B = \begin{bmatrix} 6 & 8 & 12 \\ 10 & 14 & 20 \\ 14 & 20 & 28 \end{bmatrix}$ <u>Converting into echelon form :</u>

$\sim$ $R_2 \to 6R_2 - 10R_1$
$R_3 \to 6R_3 - 14R_1$ $\quad R_3 \to R_3 - 2R_2$

$\begin{bmatrix} 6 & 8 & 12 \\ 0 & 4 & 0 \\ 0 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & 8 & 12 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to$ Echelon form

Since the echelon form of matrix B which contains the images of given vectors under T has only 2 non-zero rows, therefore, the three images are linearly dependent.

<u>Reason:</u> $T(x,y,z) = (x+2y+3z,\ x+2y+5z,\ 2x+4y+6z)$

Consider $C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ which is the coeff matrix of T.

then $|C| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 6 \end{vmatrix} = 1(12-20) + 2(10-6) + 3(4-4) = 0$

$\therefore$ T is ~~non~~-singular transformation.

<u>2(c) (ii)</u> Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ & C be a non-singular matrix of order 3x3. Find the eigen values of matrix B^3 where $B = C^{-1}AC$.

$\to$ $B^3 = (C^{-1}AC)(C^{-1}AC)(C^{-1}AC) = C^{-1}A(CC^{-1})A(CC^{-1})AC$
$= C^{-1}AIAIA \cdot C = C^{-1}A^3C$.

<u>Now</u> characteristic equation of B^3 is : $|B^3 - \lambda I| = 0$

$\to |B^3 - \lambda I| = |C^{-1}A^3C - \lambda I| = |C^{-1}A^3C - \lambda C^{-1}C|$
$= |C^{-1}A^3C - C^{-1}\lambda IC| = |C^{-1}(A^3 - \lambda I)C|$
$= |C^{-1}|\,|A^3 - \lambda I|\,|C| = \frac{1}{|C|}|A^3 - \lambda I|\,|C|$ $\quad [\because |C^{-1}| = |C|^{-1}]$
$= |A^3 - \lambda I|$.

$\therefore$ Eigen values of A^3 & B^3 are the same.

$$A^3 = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -8 & 8 \\ 12 & 0 & 8 \\ 12 & 8 & 0 \end{bmatrix}$$

Char. eq^n of $A^3 = |A^3 - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -8 & 8 \\ 12 & -\lambda & 8 \\ 12 & 8 & -\lambda \end{vmatrix} = 0 \Rightarrow (8-\lambda)[\lambda^2 - 64] + 8[-12\lambda - 96] + 8[96 + 12\lambda] = 0$$

$$\Rightarrow (8-\lambda)(\lambda - 8)(\lambda + 8) = 0$$

$\lambda = 8, 8, -8.$

$\therefore$ Eigen values of $A^3 = 8, 8, -8.$

Hence, eigen values of $B^3 = \underline{\underline{8, 8, -8}}$