[G-20 MATHS]

COMPLEX ANALYSIS CSE ERROR FREE PYQs

All these questions are discussed /solved in Topicwise G-20 Modules

2020

1 (1d)

Evaluate the integral $\int_C (z^2 + 3z)dz$ counterclockwise from (2, 0) to (0, 2) along the curve C, where C is the circle $|z| = 2$. 10

2 (2c)

Using contour integration, evaluate the integral $\int_0^{2\pi} \frac{1}{3+2\sin\theta} d\theta$. 20

3 (4a)

If $v(r, \theta) = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0,$

then find an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ 15

2019

4 (1d)

Suppose $f(z)$ is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation $Im\, f(z) = (Re\, f(z))^2$, $Z \in D$. Show that $f(z)$ is constant in D. 10

5 (2d)

Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$

where $\phi(z)$ is analytic and non zero at z_0.

Moreover $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geqslant 1$. 15

6 (3c)

Evaluate the integral $\int_c Re(z^2)dz$ from 0 to $2+4i$ along the curve C where C is a parabola $y = x^2$. 10

7 (4b)

Obtain the first three terms of the Laurent series expansion of the function $f(z) = \dfrac{1}{(e^z - 1)}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$. 10

2018

8 (1c)

Prove that the function : $u(x, y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. 10

9 (3b)

Show by applying the residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$, $a > 0$. 15

10 (4b)

Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$ 15

2017

11 (1d)

Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. *10*

12 (2b)

Using contour integral method, prove that

$$\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2}\, dx = \frac{\pi}{2} e^{-ma}.$$ *15*

13 (3b)

Let $f = u + iv$ be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

at all points of D. *15*

14 (4a)

For a function $f : \mathbb{C} \to \mathbb{C}$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all k = 1, 2, 3, … . Show that f is a polynomial. *15*

2016

15 (1d)

Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function $u(x, y)$ and hence obtain the analytic function whose real and imaginary parts are u and v respectively. *10*

16 (3c)

Let

$\gamma : [0, 1] \to \mathbb{C}$ be the curve

$\gamma(t) = e^{2\pi it}, \quad 0 \le t \le 1.$

Find, giving justifications, the value of the contour integral *15*

$$\int_\gamma \frac{dz}{4z^2 - 1}$$

17 (4c)

Prove that every power series represents an analytic function inside its circle of convergence. *20*

2015

18 (1d)

Show that the function $v(x, y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z. 10

19 (2c)

Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about the point $z = 0$. 20

20 (3a)

State Cauchy's residue theorem. Using it, evaluate the integral

$$\int_C \frac{e^z+1}{z(z+1)(z-i)^2}\, dz; \quad C: |z| = 2$$

15

2014

21 (1c)

Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \quad z \neq 0; \; f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist. 10

22 (1d)

Expand in Laurent series the function $f(z) = \dfrac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$. 10

23 (3c)

Evaluate the integral $\displaystyle\int_0^{\pi} \frac{d\theta}{\left(1 + \frac{1}{2}\cos\theta\right)^2}$ using residues. 20

2013

24 (1d)

Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has n zeroes in the unit circle. *10*

25 (4b)

Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^{\pi} \sin^4 \theta \, d\theta$$ *15*

2012

26 (1c)

(c) Show that the function defined by

$$f(z) = \begin{cases} \dfrac{x^3 y^5 (x + iy)}{x^6 + y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. 12

27 (2c)

(c) Use Cauchy integral formula to evaluate

$$\int_c \frac{e^{3z}}{(z+1)^4}\,dz,$$ where c is the circle $|z| = 2$.

15

28 (3c)

(c) Expand the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1 < |z| < 3$ (ii) $|z| > 3$ (iii) $0 < |z+1| < 2$

(iv) $|z| < 1$ 15

29 (3d)

(d) Evaluate by contour integration

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2},\quad a^2 < 1.$$ 15

2011

30 (1c)

If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, find $f(z)$ subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$. 12

31 (2c)

If the function $f(z)$ is analytic and one valued in $|z - a| < R$, prove that for $0 < r < R$,

$f'(a) = \dfrac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}\,d\theta$, where $P(\theta)$ is the real part of $f(a + re^{i\theta})$. 15

32 (3c)

Evaluate by Contour integration,

$$\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}}$$ 15

33 (3d)

Find the Laurent series for the function

$f(z) = \dfrac{1}{1-z^2}$, with centre $z = 1$. 15

2010

34 (1e)

Show that

$u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function.

Find a harmonic conjugate of $u(x, y)$. Hence find the analytic function f for which $u(x, y)$ is the real part. 12

35 (4a)

(i) Evaluate the line integral $\int_c f(z)\, dz$ where $f(z) = z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.

(ii) Find the image of the finite vertical strip $R: \; x = 5$ to $x = 9$, $-\pi \leq y \leq \pi$ of z-plane under exponential function. 15

36 (4b)

Find the Laurent series of the function

$$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right] \text{ as } \sum_{n=-\infty}^{\infty} C_n z^n$$

for $0 < |z| < \infty$

where $C_n = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - \lambda \sin \phi)\, d\phi$,

$n = 0, \pm 1, \pm 2, \ldots..$

with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}(-\pi \leq \phi \leq \pi)$ as contour in this region. 15

Notes: