Test Series – 2018
Answer Key - Test - 2 (Paper-II)
Modern Algebra, Real Analysis & CA and LPP

1(a) A semigroup $(S,*)$ is a group iff
(i) there exists $e \in S$ such that $e*a = a \ \forall a \in S$ and
(ii) $\forall a \in S$, there exists $b \in S$ such that $b*a = e$.

Sol^n: Suppose $(S,*)$ is a semi group that satisfies (i) and (ii).

Let a be any element of S. Then $\exists$ $b \in S$ such that $b*a = e$ by (ii).

For $b \in S$, $\exists$ $c \in S$ such that $c*b = e$ by (ii).

Now,
$a = e*a = (c*b)*a = c*(b*a) = c*e$.

and $a*b = (c*e)*b = c*(e*b) = c*b = e$.

Hence $a*b = e = b*a$.

Also
$a*e = a*(b*a) = (a*b)*a = e*a = a$.

Thus, $a*e = a = e*a$

This shows that e is the identity element of S.

Now since $a*b = e = b*a$
we have $b = a^{-1}$.

Therefore, $(S,*)$ is a group.

Let S be a group.
By the defn of a group, the conditions (i) & (ii) immediately follow.

1(b) Find the group of units in the ring Z_8.

Sol: we have
$$Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

Note that
$$1 \times_8 1 = 1$$
$$3 \times_8 3 = 1$$
$$5 \times_8 5 = 1$$
$$7 \times_8 7 = 1$$

$\Rightarrow$ 1, 3, 5, and 7 are the units of Z_8.

On the other hand 0, 2, 4 and 6 are not units of Z_8 as there are no such elements $i \in Z_8$ so that $k \times_8 i = 1$ for $k = 0, 2, 4, 6$.

$\therefore$ The group of units of Z_8 is given by
$$\{1, 3, 5, 7\}.$$

Note that this group is isomorphic to the Klein's 4-group.

1(d) Prove that $u(x,y) = 4xy - x^3 + 3xy^2$ is a harmonic function. Determine its harmonic conjugate, hence find corresponding analytic function $f(z)$ in terms of z.

Sol^n: we have

$$u(x,y) = 4xy - x^3 + 3xy^2 \quad \text{--- (1)}$$

$$\therefore \frac{\partial u}{\partial x} = 4y - 3x^2 + 3y^2$$

$$\& \quad \frac{\partial^2 u}{\partial x^2} = -6x$$

$$\frac{\partial u}{\partial y} = 4x + 6xy$$

$$\frac{\partial^2 u}{\partial y^2} = 6x$$

so that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$

i.e., u satisfies Laplace's equation.

Hence u is a harmonic function.

Let v be the harmonic conjugate to u.

Now consider

$$\frac{\partial u}{\partial x} = 4y - 3x^2 + 3y^2 = \frac{\partial v}{\partial y} \quad \text{(By Cauchy-Riemann eqns)}$$

$$\Rightarrow \frac{\partial v}{\partial y} = 4y - 3x^2 + 3y^2$$

Integrating w.r.t y, we get

$$v = 2y^2 - 3x^2 y + y^3 + \phi(x) \quad \text{--- (2)}$$

when $\phi(x)$ is a constant function of the integration.

Differentiating ② partially w.r.t x, we get

$$\frac{\partial v}{\partial x} = -6xy + \phi'(x)$$

By Cauchy-Riemann equation

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow 4x + 6xy = 6xy - \phi'(x)$$

$$\Rightarrow \phi'(x) = -4x.$$

$$\Rightarrow \phi(x) = -2x^2$$

$\therefore$ from ② $v = 2y^2 - 3x^2y + y^3 - 2x^2$.

which is the required harmonic conjugate.

Hence the corresponding analytic function is

$$f(z) = 4xy - x^3 + 3xy^2 + i\left[2y^2 - 3x^2y + y^3 - 2x^2\right]$$

$$= -x^3 + 3xy^2 - i3x^2y + iy^3 - 2i(x^2 - y^2) + 4xy$$

$$= -(x^3 - 3xy^2 + i3x^2y - iy^3) - 2i(x^2 + (iy)^2 + 2ixy)$$

$$= -(x+iy)^3 - 2i(x+iy)^2 + C_1$$

$$= -z^3 - 2iz^2 + C_1$$

11(e) The standard weight of a special purpose brick is 5kg and it contains two basic ingredients B_1 and B_2. B_1 costs Rs 5 per kg and B_2 costs Rs. 8 per kg. Strength considerations state that the brick contains not more than 4kg of B_1 and minimum of 2kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions

Sol^n: The formulation of the given problem is.

Min (total cost) $Z = 5x_1 + 8x_2$

Subject to the constraints

$x_1 \leq 4$

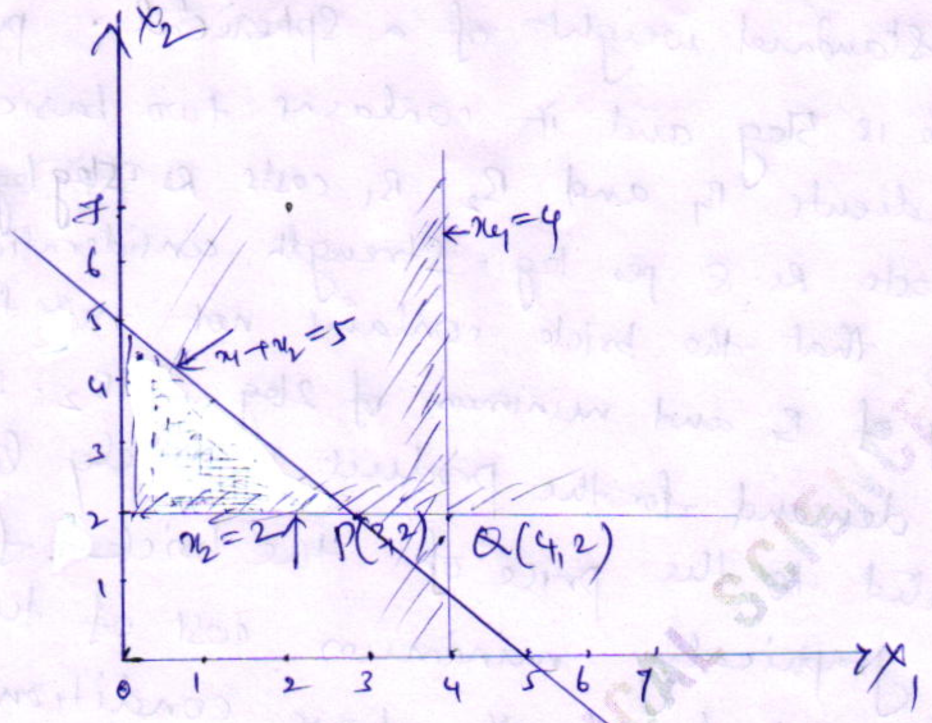
$x_2 \geq 2$

$x_1 + x_2 = 5$

$x_1, x_2 \geq 0$.

where x_1, x_2 are the amount of ingredients B_1 and B_2 (in kg) respectively

The given constraints are plotted on the graph as shown in the figure.



It may be observed that feasible region has two cornerpoints $P(3,2)$ and $Q(4,2)$.
The minimum value of Z is found at $P(3,2)$.

i.e $x_1=3$, $x_2=2$.

Hence the optimum product mix is to have 3kg of ingredient B_1 and 2kg of ingredient B_2 of a special cake brick in order to achieve the minimum cost of Rs. 31.

2(a) (i) Show that Z_9 is not a homomorphic image of Z_{16}.
(ii) Find the numbers of elements of order 5 in $Z_{15} \times Z_5$.

Sol'n: (i) Suppose there exists an epimorphism $f: Z_{16} \to Z_9$.
Then by the First Isomorphism theorem,
$Z_{16}/\ker f \simeq Z_9$. Hence $|Z_{16}/\ker f| = |Z_9| = 9$. This shows that

$$\frac{|Z_{16}|}{|\ker f|} = 9,$$

ie, $9 \cdot |\ker f| = 16$. This is absurd. Hence Z_9 is not a homomorphic image of Z_{16}.

(ii) We count the number of elements (a,b) in $Z_{15} \times Z_5$ with the property that $5 = o(a,b) = \text{lcm}\{o(a), o(b)\}$.
If $\text{lcm}\{o(a), o(b)\} = 5$, then we have following cases:

Case I : $o(a) = 5$, $o(b) = 5$
Case II : $o(a) = 5$, $o(b) = 1$
Case III : $o(a) = 1$, $o(b) = 5$

Case I: Since Z_{15} is cyclic, it contains only one subgroup of order 5 and Z_5 itself a subgroup of order 5 in Z_5.
In any subgroup of order 5, except identity element, every element is of order 5. Hence there are 4 choices of a and 4 choices of b. This gives 16 elements of order 5 in $Z_{15} \times Z_5$.

Case II: There are four choices of a and only one choice of b. This gives 4 elements of order 5 in $Z_{15} \times Z_5$

Case III : There is only one choice of a and four choices of b. This gives 4 elements of order 5 in $Z_{15} \times Z_5$.

Thus we find that $Z_{15} \times Z_5$ contains 24 elements of order 5.

2(b) Let $R = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in M_2(\mathbb{C}) \;/\; \bar{\alpha}, \bar{\beta} \text{ denote the Conjugates of } \alpha, \beta \right\}$

Define addition + and multiplication . in R by usual matrix addition and matrix multiplication. Show that R is a division ring but not a field.

Sol^n: Let $A = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ and $B = \begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$

Adding, A & B,

$$A+B = \begin{bmatrix} (a+r)+i(b+t) & (c+u)+i(d+v) \\ -(c+u)-i(d+v) & (a+r)-i(b+t) \end{bmatrix} \in R.$$

and $AB = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}\begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$

(property of matrices and complex numbers)

Here $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$ is zero element for any $A \in R$, we have $-A \in R$ such that

$$A + (-A) = O.$$

further, distributive properties hold.

Hence R is a ring with identity

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R.$$

Now $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \in R$

and $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \in R$

$$\begin{bmatrix} i & i \\ i & 1 \end{bmatrix}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

$\therefore$ R is a non-commutative ring.

Let $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ be a non-zero element of R.

Then either $a+ib \neq 0$ or $c+id \neq 0$

i.e. $a^2+b^2 \neq 0$ or $c^2+d^2 \neq 0$.

Hence $a^2+b^2+c^2+d^2 \neq 0$.

Let $k = a^2+b^2+c^2+d^2$.

Note that $\frac{1}{k}\begin{bmatrix} a-ib & c+id \\ -(c+id) & a+ib \end{bmatrix} \in R$.

is the inverse of $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \in R$.

Hence each non-zero element of R has an inverse in R. Hence R is a division ring.

But R is non-commutative; clearly R is not a field.

2(d) show by the method for Contour integration that $\int_0^\infty \frac{\cos mx}{(a^2+x^2)^2}dx = \frac{\pi}{4a^2}(1+ma)e^{-ma}$, $(a>0, m>0)$.

Sol'n: Consider the integral

$$\int_C f(z)\,dz, \quad \text{where } f(z) = \frac{e^{imz}}{(a^2+z^2)^2}$$

taken round the closed contour C consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.

Pole of $f(z)$ are given by $(a^2+z^2)^2=0$

i.e. $z=ia$ and $z=-ia$ are the two poles each of order two. The only pole which lies within the contour is at $z=ia$ of order 2.

Since the pole $z=ia$ is of order 2, hence to know the residue at $z=ia$, put $z=ia+t$ in $f(z)$, then it becomes

$$\frac{e^{im(ia+t)}}{\{a^2+(ia+t)^2\}^2} = \frac{e^{-ma}\cdot e^{imt}}{(2iat+t^2)^2} = \frac{e^{-ma}\cdot e^{imt}}{-4a^2t^2}\left[1+\frac{t}{2ia}\right]^{-2}$$

$$= \frac{e^{-ma}}{-4a^2t^2}(1+imt+\cdots)\left[1-\frac{2t}{2ia}+\cdots\right]$$

in which coefficient of $\frac{1}{t}$ is easily seen to be $\frac{-ie^{-ma}(1+ma)}{4a^3}$

which is therefore residue theorem,

Hence by Cauchy's residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{Sum of the residues within } C$$

i.e. $$\int_{-R}^{R} f(x)\,dx \times \int_{C_R} f(z)\,dz = 2\pi i \times \frac{-ie^{-ma}(1+ma)}{4a^3}$$

$$\Rightarrow \int_{-R}^{R} \frac{e^{imz}}{(a^2+x^2)^2}\,dx + \int_{C_R} \frac{e^{imz}}{(a^2+z^2)^2}\,dz = \frac{2\pi}{4a^3} e^{-ma}(1+ma) \quad \text{—— ①}$$

Now, $\left| \int_{C_R} \frac{e^{imz}}{(a^2+z^2)^2}\,dz \right| \le \int_{C_R} \frac{|e^{imz}|\,|dz|}{|a^2+z^2|^2}$

$$\le \int_{C_R} \frac{|e^{imz}|\,|dz|}{(|z|^2-a^2)^2} = \int_0^{\pi} \frac{e^{-mR\sin\theta}\cdot R\,d\theta}{(R^2-a^2)^2}$$

$$\le \frac{R}{(R^2-a^2)^2}\cdot 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$$

$$= \frac{\pi}{m(R^2-a^2)^2}\left(1-e^{-mR}\right)$$

which $\to 0$ as $R \to \infty$

Hence by making $R\to\infty$, relation ① becomes

$$\int_{-\infty}^{\infty} \frac{e^{ixm}}{(a^2+x^2)^2}\,dx = \frac{2\pi}{4a^3}(1+ma)e^{-ma}$$

equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx = \frac{2\pi}{4a^3}(1+ma)e^{-ma}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{(a^2+x^2)^2}\,dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}$$

3(a) Let F be the field of integers modulo 5. Show that the polynomial x^2+2x+3 is irreducible over F. Use this to construct a field containing 25 elements.

Sol'n: we have $F = \mathbb{Z}_5 = \{0,1,2,3,4\}$

Let $x^2+2x+3 = (x+a)(x+b)$; $a,b \in F$

Comparing the coefficients of x and constants on bothsides, we get

$$2 = a+b \quad \text{——} \quad (1)$$
$$3 = ab \quad \text{——} \quad (2)$$

(1) is satisfied for $(a,b) = (0,2), (1,1), (3,4), (2,0), (4,3)$.
For these values of a and b, $ab = 0,1,2,0,2$ i.e.
(2) is never satisfied. Consequently,
x^2+2x+3 is irreducible over F. Hence by theorem
[Let F be a field. the ideal $A = \langle p(x)\rangle = \{p(x)f(x) : f(x) \in F[x]\}$ in $F[x]$ is a maximal ideal if and only if $p(x)$ is an irreducible polynomial over F. Further $\frac{F[x]}{\langle p(x)\rangle}$ is a field]

$\frac{F[x]}{\langle x^2+2x+3\rangle}$ is a field. Any element of this field is
$f(x)+A$, where $f(x) \in F[x]$, $A = \langle x^2+2x+3\rangle$. By Division algorithm in $F[x]$, for $f(x) \in F[x]$, $x^2+2x+3 \in F[x]$, there exist $t(x), r(x) \in F[x]$ such that

$$f(x) = (x^2+2x+3)\, t(x) + r(x) \quad \text{——} \quad (1)$$

where $r(x) = 0$ (or) $\deg r(x) < \deg(x^2+2x+3) = 2$
we may take $r(x) = \alpha x + \beta$, where $\alpha, \beta \in F$

$\therefore f(x) + A = r(x) + (x^2+2x+3)\,t(x) + A$, by (1)

(or) $f(x)+A = r(x) + A = \alpha x + \beta + A$ —— (2)

Since $(x^2+2x+3)\,t(x) \in A = \langle x^2+2x+3\rangle$.
In (2), we see that $\alpha, \beta \in F = \mathbb{Z}_5$ and $o(\mathbb{Z}_5) = 5$. Consequently each of α & β can be selected in 5 ways. Hence by (2), the number of elements of the field $\frac{F[x]}{\langle x^2+2x+3\rangle}$ is $5^2 = 25$

3(b) Show that series $\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p}$ $(p>0)$, converges for all real x.

Soln.
$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p} = \sum_{n=1}^{\infty} \frac{\sin(n\pi + nx)}{n^p}$$
$$= \sum_{n=1}^{\infty} \frac{\sin n(\pi+x)}{n^p}$$

let $a_n = \sin n(\pi+x)$, $b_n = \frac{1}{n^p}$, $p>0$

then the given series can be written as
$$\sum_{n=1}^{\infty} a_n b_n$$

let S_n be the n^{th} partial sum of the series $\sum a_n$ then

$S_n = \sin(\pi+x) + \sin 2(\pi+x) + \sin 3(\pi+x) \cdots$
$\cdots + \sin n(\pi+x).$

$$= \frac{\sin\left[(\pi+x) + \frac{n-1}{2}(\pi+x)\right] \sin \frac{n(\pi+x)}{2}}{\sin\left(\frac{\pi+x}{2}\right)}$$

$$= \frac{\sin(n+1)\frac{\pi+x}{2} \sin \frac{n(\pi+x)}{2}}{\cos x/2}$$

$$\Rightarrow |S_n| = \left| \frac{\sin(n+1)\frac{\pi+x}{2} \sin \frac{n(\pi+x)}{2}}{\cos x/2} \right|$$

$$\leq \frac{1}{|\cos x/2|} = |\sec x/2|$$

$\Rightarrow$ $\langle S_n \rangle$ is bounded for all x for which $\cos\frac{x}{2} \neq 0$,

i.e for which $\frac{x}{2} \neq (2r+1)\pi/2$ (or) $x \neq (2r+1)\pi$, $r \in Z$

Also $\langle b_n \rangle$ is a decreasing sequence with

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^p} = 0$$

$\therefore$ By Dirichlet's test, $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^p}$, $(p>0)$

is convergent for all $x \neq (2r+1)\pi$, $r \in Z$.

when $x = (2r+1)\pi$, $\sin nx = 0$

$\therefore$ each term of the series is 0 and so the series is convergent.

Hence the given series is convergent for all real x.

3(c) For each of the following functions, locate and name the singularities in the finite z-plane and determine whether they are isolated singularities or not.

(i) $f(z) = \frac{z}{(z^2+4)^2}$ (ii) $f(z) = \sec\frac{1}{z}$ (iii) $f(z) = \frac{\ln(z-2)}{(z^2+2z+2)^4}$

Sol^n:

(i) $f(z) = \frac{z}{(z^2+4)^2} = \frac{z}{\{(z+2i)(z-2i)\}^2}$

$= \frac{z}{(z+2i)^2(z-2i)^2}$

Since $\lim_{z\to 2i}(z-2i)^2 f(z) = \lim_{z\to 2i}\frac{z}{(z+2i)^2} = \frac{1}{8i} \neq 0$.

$\therefore$ $z=2i$ is a pole of order 2.

Similarly $z=-2i$ is a pole of order 2.

Since we can find δ such that no singularity other than $z=2i$ lies inside the circle $|z-2i|=\delta$ (choose $\delta=1$), it follows that $z=2i$ is an isolated singularity.

Similarly $z=-2i$ is an isolated singularity

(ii) Since $\sec\frac{1}{z} = \frac{1}{\cos(\frac{1}{z})}$.

The singularities occur when $\cos\frac{1}{z} = 0$

i.e. $\frac{1}{z} = (2n+1)\frac{\pi}{2}$

$\Rightarrow z = \frac{2}{(2n+1)\pi}$

where $n = 0, \pm1, \pm2, \ldots$

Also, since $f(z)$ is not defined at $z=0$, it follows that $z=0$ is also a singularity.

Now, by L'Hospital's rule,

$$\lim_{z\to\frac{2}{(2n+1)\pi}} \left\{z-\frac{2}{(2n+1)\pi}\right\} f(z) = \lim_{z\to\frac{2}{(2n+1)\pi}} \frac{z-\frac{2}{(2n+1)\pi}}{\cos\left(\frac{1}{z}\right)}$$

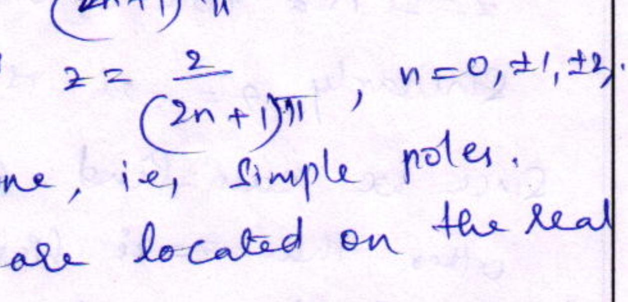
$$= \lim_{z\to\frac{2}{(2n+1)\pi}} \frac{1}{\sin\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)}$$

$$= \frac{\left\{2/(2n+1)\pi\right\}^2}{\sin(2n+1)\pi/2}$$

$$= \frac{4(-1)^n}{(2n+1)^2\pi^2} \neq 0.$$

Thus the singularities $z=\frac{2}{(2n+1)\pi}$, $n=0,\pm1,\pm2,\ldots$ are poles of order one, i.e., simple poles.

Note that these poles are located on the real axis at $z=\pm 2/\pi$, $\pm\frac{2}{3\pi}$, $\pm\frac{2}{5\pi}$, ... and that there are infinitely many in a finite interval which includes 0.



Since we can surround each of these by a circle of radius δ, which contains no other singularity, it follows that they are isolated singularities. It should be noted that the δ-required is smaller the closer the singularity is to the origin.

Since we cannot find any +ve integer n such that $\lim_{z\to 0}(z-0)^n f(z) = A \neq 0$, it follows that $z=0$ is an essential singularity. Also, since every circle of radius δ with centre at $z=0$ contains singular points other than $z=0$, no matter how small we take δ, we see that $z=0$ is a non-isolated singularity.

(iii) $f(z) = \dfrac{\ln(z-2)}{(z^2+2z+2)^4}$.

The point $z=2$ is a branch point and is non-isolated singularity.

Also, since $z^2+2z+2=0$ when

$\Rightarrow z = -1\pm i$.

It follows that

$z^2+2z+2 = (z+1+i)(z+1-i)$.

and that $z=-1\pm i$ are poles of order 4, which are isolated singularities.

3(d) Solve the following LPP

$\text{Max } z = 2x_1+x_2$

subject to $4x_1+3x_2 \le 12$, $4x_1+x_2 \le 8$, $4x_1-x_2 \le 8$ and $x_1, x_2 \ge 0$.

Solⁿ: Introducing the slack variables $s_1, s_2, s_3 \geq 0$, the problem becomes

$$\text{Max } z = 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to the constraints

$$4x_1 + 3x_2 + s_1 = 12$$
$$4x_1 + x_2 + s_2 = 8$$
$$4x_1 - x_2 + s_3 = 8$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Now the initial basic feasible solution is given by

setting $x_1 = x_2 = 0$ (non-basic)

$s_1 = 12,\ s_2 = 8,\ s_3 = 8$ (basic)

$\therefore$ The IBFS is $(0, 0, 12, 8, 8)$

for which $z = 0$.

put the above information in tableau form

	c_j	2	1	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	4	3	1	0	0	12	$12/4 = 3$
0	s_2	4	1	0	1	0	8	$8/4 = 2$
0	s_3	$\boxed{4}$	-1	0	0	1	8	$8/4 = 2$ →
$z_j = \Sigma C_B a_{ij}$		0	0	0	0	0	0	
$C_j = c_j - z_j$		2 ↑	1	0	0	0		

from the table,

x_1 is the incoming variable. But the two rows have the same ratio under θ-column.

This is an indication of degeneracy.

Using the procedure of degeneracy, compute

$\min\left[\frac{\text{elements of first column of unit matrix}}{\text{corresponding elements of key column}}\right]$

only for second and 3rd rows.

$\therefore \min\left[-, \frac{0}{4}, \frac{0}{4}\right]$ which is not unique.

So again compute

$\min\left[\frac{\text{elements of 2nd column of unit matrix}}{\text{corresponding element of key column}}\right]$

only for 2nd and 3rd rows.

$\therefore \min\left[-, \frac{1}{4}, \frac{0}{4}\right] = 0$ which occurs corresponding to the 3rd row. and therefore s_2 is outgoing variable

$\therefore$ the key element is 4.

Thus the new simplex table:

	c_j	2	1	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	0	4	1	0	-1	4	1
0	s_2	0	$\boxed{2}$	0	1	-1	0	$0/2 = 0 \rightarrow$
2	x_1	1	$-1/4$	0	0	$1/4$	2	—
$z_j = \Sigma C_B a_{ij}$		2	$-1/2$	0	0	$1/2$	4	
$C_j = c_j - z_j$		0	$3/2 \uparrow$	0	0	$-1/2$		

from the above table,

x_2 is the incoming variable, s_2 is the outgoing variable and (2) is the key element and making it into unity and all other elements in its column to zero.

	C_j	2	1	0	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	0	0	1	-2	(1)	4	$4 \rightarrow$
1	x_2	0	1	0	$1/2$	$-1/2$	0	–
2	x_1	1	0	0	$1/8$	$1/8$	2	16
$z_j = \Sigma C_B a_{ij}$		2	1	0	$3/4$	$-1/4$	4	
$C_j' = C_j - z_j$		0	0	0	$-3/4$	$1/4$ $\uparrow$		

from the above table,
S_3 is incoming variable, S_1 is the outgoing variable and $\underline{1}$ is the key element and making all other elements in its column to zero.

	C_j	2	1	0	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	S_3	b
0	S_3	0	0	1	-2	1	4
1	x_2	0	1	$1/2$	$-1/2$	0	2
2	x_1	1	0	$-1/8$	$3/8$	0	$3/2$
$z_j = \Sigma C_B a_{ij}$		2	1	$1/4$	$1/4$	0	5
$C_j' = C_j - z_j$		0	0	$-1/4$	$-1/4$	0	

from the above table,
all $C_j' \leq 0$.
so, the table gives the optimal solution. Hence an optimal basic feasible solution is
$x_1 = 3/2,\ x_2 = 2$ and $\underline{\text{Max } Z = 5}$

4(a) Every integral domain can be imbedded in a field.

Sol'n: Let D be an integral domain with atleast two elements.

Let us consider $S = \{(a,b) \mid a,b \in D ; b \neq 0\}$ then

$S \neq \phi$ and $S \subset D \times D$.

$\forall (a,b), (c,d) \in S$

define a relation '$\sim$' on 'S' as

$(a,b) \sim (c,d) \Leftrightarrow ad = bc$

we now prove that $\sim$ is an equivalance relation on S.

(1) for each $(a,b) \in S$

we have $ab = ba$.

which implies that $(a,b) \sim (a,b)$.

(2) for $(a,b), (c,d) \in S$

we have $(a,b) \sim (c,d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c,d) = (a,b)$.

(3) for $(a,b), (c,d), (e,f) \in S$

$(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$

$ad = bc$, $cf = de$

$\Rightarrow ad = bc$, $cf = de$

$\Rightarrow (ad)f = (bc)f$, $cf = de$

$\Rightarrow (af)d = b(de)$

$\Rightarrow (af)d = b(ed)$

$\Rightarrow (af)d = (be)d$

$\Rightarrow af = be \quad (\because d \neq 0)$

$\Rightarrow (a,b) \sim (e,f)$.

$\therefore \sim$ is an equivalence relation on 's'.

The equivalence relation '$\sim$' partitions the set 'S' into equivalence classes which are identical or disjoint.

For $(a,b) \in S$,

let $\frac{a}{b}$ be denote equivalence class of (a,b)

then $\frac{a}{b} = \{(x,y) \in S \mid (x,y) \sim (a,b)\}$

i.e. $\frac{a}{b} = \{(x,y) \in S / (x,y) \sim (a,b) \Leftrightarrow xb = ya\}$

If $\frac{a}{b}, \frac{c}{d}$ are the equivalence classes of $(a,b), (c,d) \in S$.

then either $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} \cap \frac{c}{d} = \phi$

It is evident that $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$.

Let F denote the set of all the equivalence classes or the set of quotients then $F = \{\frac{a}{b} / (a,b) \in S\}$.

Since D has at least two elements say $0, a \in D$,

we have quotients $\frac{0}{a}, \frac{a}{a} \in F$ and $\frac{0}{a} \neq \frac{a}{a}$.

$\therefore$ the set F has atleast two elements.

For $\frac{a}{b}, \frac{c}{d} \in F$, define addition (+) and multiplication (·) as

(i) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and

(ii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Since D is without zero divisors

$$\underset{\neq 0}{b}, \underset{\neq 0}{d} \in D \Rightarrow bd \neq 0$$

so $\frac{ad+bc}{bd}, \frac{ac}{bd} \in F$

Now we prove that the $+^n$ and $\times^n$ defined above are well-defined.

Let $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$

Then $ab' = a'b$ and $cd' = c'd$ ——— (I)

Now (I) $\Rightarrow ab'dd' = a'bdd'$ and $bb'cd' = bb'c'd$

$\Rightarrow ab'dd' + bb'cd' = a'bdd' + bb'c'd$

$\Rightarrow (ad+bc)\,b'd' = (a'd' + b'c')\,bd$

$\Rightarrow \frac{ad+bc}{bd} = \frac{a'd' + b'c'}{b'd'}$

Also (I) $\Rightarrow ab'cd' = a'bc'd$

$\Rightarrow (ac)(b'd') = (a'c')(bd)$

$\Rightarrow \frac{ac}{bd} = \frac{a'c'}{b'd'}$

$\therefore$ $+^n$ and $\times^n$ of quotients are well-defined binary operations on F.

<u>we now Prove that $(F,+,\cdot)$ is a field:</u>

(1) For $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$; $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd}+\frac{e}{f}$

$= \frac{(ad+bc)f + (bd)e}{(bd)f}$

$= \frac{a(df) + (cf+de)b}{b(df)}$

$= \frac{a}{b} + \frac{cf+de}{df}$

$= \frac{a}{b} + \left(\frac{c}{d}+\frac{e}{f}\right)$

$\therefore$ addition is associative.

(2) For $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$

$= \frac{bc+ad}{db}$

$= \frac{c}{d}+\frac{a}{b}$

$\therefore$ addition is commutative.

(3) For $u \neq 0 \in D$ we have $\frac{0}{u} \in F$ such that

$$\frac{0}{u} + \frac{a}{b} = \frac{0b+ua}{ub} = \frac{ua}{ub} = \frac{a}{b} \;\forall\; \frac{a}{b} \in F.$$

$\therefore \frac{0}{u} \in F$ is the zero element.

(4) Let $\frac{a}{b} \in F$. Then $\frac{-a}{b} \in F$ such that

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab+(-a)b}{b^2} = \frac{0}{b^2} = \frac{0}{u} \quad (\because 0u = 0b^2)$$

$\therefore$ every element in F has additive inverse.

(5) For $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$; $\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f}$

$$= \frac{(ac)e}{(bd)f}$$

$$= \frac{a}{b}\cdot\frac{ce}{df} = \frac{a}{b}\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

$\therefore$ multiplication is associative.

(6) For $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d}\cdot\frac{a}{b}$

$\therefore$ multiplication is commutative

(7) For $u \neq 0 \in D$ we have $\frac{u}{u} \in F$ such that

$$\frac{a}{b}\cdot\frac{u}{u} = \frac{au}{bu} = \frac{a}{b} \;\forall\; \frac{a}{b} \in F.$$

$\therefore \frac{u}{u} \in F$ is the unity element.

(8) Let $\frac{a}{b} \in F$ and $\frac{a}{b} \neq \frac{0}{u}$

then $au \neq 0$ which implies that $a \neq 0$ as $u \neq 0$.

$\therefore b \neq 0$ and $a \neq 0 \Rightarrow \frac{b}{a} \in F$.

$\therefore$ for $\frac{a}{b}\left(\neq \frac{0}{u}\right) \in F$ there exists $\frac{b}{a} \in F$ such that

$$\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ba}=\frac{u}{u} \quad (\because (ab)u=(ba)u)$$

$\therefore$ Every non-zero element in F has multiplicative inverse.

(9). For $\frac{a}{b},\frac{c}{d},\frac{e}{f}\in F$; $\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right)=\frac{a}{b}\cdot\frac{cf+de}{df}$

$$=\frac{a(cf+de)}{b(df)}$$

$$=\frac{(acf+ade)(bdf)}{(bdf)(bdf)} \quad \left[\because \frac{bdf}{bdf}=\frac{u}{u}\right]$$

$$=\frac{acf\,bdf+ade\,bdf}{bdf\,bdf}$$

$$=\frac{acf}{bdf}+\frac{ade}{bdf}$$

$$=\frac{ac}{bd}+\frac{ae}{bf}=\frac{a}{b}\cdot\frac{c}{d}+\frac{a}{b}\cdot\frac{e}{f}$$

Similarly we can prove that

$$\left(\frac{c}{d}+\frac{e}{f}\right)\cdot\frac{a}{b}=\frac{c}{d}\cdot\frac{a}{b}+\frac{e}{f}\cdot\frac{a}{b}.$$

$\therefore$ multiplication is distributive over addition.

In view of (1); (2), (3), (4), (5), (6), (7), (8) and (9) $(F,+,\cdot)$ is a field.

Now we have to prove that D is embedded in the field F, that is, we have to show that there exists an isomorphism of D into F.

Define the mapping $\phi: D\to F$ by

$$\phi(a)=\frac{ax}{x} \ \forall\ a\in D \text{ and } x(\neq 0)\in D.$$

$a,b\in D$ and $\phi(a)=\phi(b) \Rightarrow \frac{ax}{x}=\frac{bx}{x}$

$$\Rightarrow (ax)x=(bx)x$$

$$\Rightarrow (a-b)x^2=0$$

$$\Rightarrow a-b=0 \text{ since } x^2\neq 0.$$

$$\Rightarrow a=b$$

$\therefore \phi$ is one-one

For $a,b\in D$; $\phi(a+b)=\frac{(a+b)x}{x}=\frac{(a+b)xx}{xx}$

$$= \frac{axx + bxx}{xx}$$

$$= \frac{ax}{x} + \frac{bx}{x} = \phi(a) + \phi(b)$$

$$\phi(ab) = \frac{(ab)x}{x} = \frac{(ab)xx}{xx} = \frac{ax}{x} \cdot \frac{bx}{x} = \phi(a) \cdot \phi(b)$$

$\therefore \phi$ is a homomorphism.

Hence ϕ is an isomorphism of D into F.

$\therefore$ the integral domain D is embedded in the field F.

Note 1: Every element in the field F is in the form of a quotient of two elements in D. So, the field F is called "field of quotients of D".

2. The equivalence class of $(a,b) \in S$ is also denoted as

$[(a,b)]$ or $[a,b]$ or (a,b).

then $[(a,b)] = [(c,d)] \Leftrightarrow ad = bc$,

$[(a,b)] + [(c,d)] = [(ad+bc, bd)]$

$[(a,b)] \cdot [(c,d)] = [(ac, bd)]$

the zero element of $F = [(0,1)]$ and the unit element of $F = [(1,1)]$.

3. If D is the ring of integers then the field F, constructed in the above theorem, would be the field of rational numbers.

4(b) If $0<x<1$, show that $2x<\log\frac{1+x}{1-x}<2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$

Sol^n let $f(x)=\log\frac{(1+x)}{1-x}-2x,\ 0\le x<1 \rightarrow (1)$

let $c\in\,]0,1[$ Then f is continuous and derivable in $[0,c]$. Again, from (1) we have

$$f'(x)=\frac{1-x}{1+x}\times\frac{(1-x)+(1+x)}{(1-x)^2}-2$$

$$=\frac{2}{1-x^2}-2=\frac{2x^2}{1-x^2}\rightarrow (2)$$

from (2), we find that $f'(x)>0$ for $0\le x<c$

Hence f is strictly increasing in $[0,c]$

In particular, $f(c)>f(0)$ if $c>0$ Also $f(0)=0$, by (2)

Now $f(c)>0 \Rightarrow \log\frac{1+c}{1-c}-2c>0,\ 0<c<1$

$$\Rightarrow 2c<\log\frac{1+c}{1-c}\ \forall\, c\in\,]0,1[$$

Since 'c' is any point of $]0,1[$ it follows that

$$2x<\log\frac{(1+x)}{1-x}\ \text{for}\ 0<x<1\rightarrow (3)$$

let $g(x)=2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)-\log\frac{1+x}{1-x}\rightarrow (4)$

from (4)

$$g'(x) = 2+\frac{2}{3}\cdot\frac{(1-x^2)\cdot 3x^2+x^3\cdot 2x}{(1-x^2)^2}-\left(\frac{1}{1+x}+\frac{1}{1-x}\right)$$

$$g'(x) = 2+\frac{2(3x^2-x^4)}{3(1-x^2)^2}-\frac{2}{1-x^2}$$

$$= \frac{4x^4}{3(1-x^2)^2} \longrightarrow (5)$$

from (5) we find that $g'(x)>0$ for $0\le x<c$

Here f is strictly increasing in $[0,c]$. In particular, $g(c)>g(0)$ if $c>0$. Also $g(0)=0$ by (4)

Now, $g(c)>0 \Rightarrow 2c\left(1+\frac{1}{3}\cdot\frac{c^2}{1-c^2}\right)-\log\frac{1+c}{1-c}>0$.

for $0<c<1$

Since c is any point of $]0,1[$, it follows that:

$$2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)-\log\frac{1+x}{1-x}>0, \text{ for } 0<x<1.$$

$$\log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right) \text{ for } 0<x<1 \longrightarrow (6)$$

from (3) and (6) $2x<\log\frac{1+x}{1-x}<2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right)$, $0<x<1$.

4(c) Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^3}{\bar{z}^2} : \text{for } z \neq 0 \\ 0 \quad \text{for } z = 0 \end{cases}$$

satisfies the C-R equation at the origin, yet it is not differentiable there.

Sol^n: Given that $f(z) = \frac{z^3}{\bar{z}^2}$ for $z \neq 0$.

$$= \frac{z^3 \cdot z^2}{\bar{z}^2 \cdot z^2}$$

$$= \frac{z^5}{(\bar{z} \cdot z)^2}$$

$$= \frac{z^5}{[|z|^2]^2}$$

$$= \frac{z^5}{|z|^4}$$

∴ The given function f can be written as

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$$= \begin{cases} \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x^5/x^4}{x} = 1.$$

$$\frac{\partial u}{\partial y} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x} = \underset{x\to 0}{Lt} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{Lt} \frac{y^5/y^4}{y} = 1.$$

$\therefore$ we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence Cauchy-Riemann conditions are satisfied at $z=0$

Taking $h = re^{i\theta} \neq 0$ with $r \neq 0$.

it follows that for $h \neq 0$.

$$f'(0) = \frac{f(h)-f(0)}{h}$$

$$= \frac{h^4}{|h|^4} = e^{i4\theta}$$

and therefore, as $h\to 0$ along different paths, the difference quotient doesnot yeild a unique value.

Thus, f is not differentiable at the origin.

4(d) Give the dual of the linear programming problem.

$\text{Max } Z = 3x_1 - 2x_2$

subject to $x_1 + x_2 \le 5$

$x_1 \le 4$

$1 \le x_2 \le 6$

and $x_1, x_2 \ge 0$.

Sol^n: The given problem can be written in the standard primal form as:

$\text{Max } z = 3x_1 - 2x_2$

subject to

$x_1 + x_2 \le 5$

$x_1 \le 4$

$x_2 \ge 1 \Rightarrow -x_2 \le -1$

$x_2 \le 6$

$x_1, x_2 \ge 0$.

Let y_1, y_2, y_3 and y_4 be the dual variables associated with the above 4 constraints.

Then the dual is given by

$\text{Min } W = 5y_1 + 4y_2 - y_3 + 6y_4$

subject to

$y_1 + y_2 \cancel{+ 0y_3 + 0y_4} \ge 3$

$y_1 + 0y_2 - y_3 + y_4 \ge -2$

All $y_i \ge 0$.

5(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$; $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$

be elements of S_7.

(i) Write α as a product of disjoint cycles

(ii) Write β as a product of 2-cycles

(iii) Is β an even permutation

(iv) Is α^{-1} an even permutation.

Sol:

Here α is such that

$\alpha(1) = 6$, $\alpha(6) = 3$, $\alpha(3) = 7$, $\alpha(7) = 1$.

Now, we define $\alpha_1 : I_7 \to I_7$ by

$$\alpha_1 : \begin{array}{l} 1 \to 6 \\ 6 \to 3 \\ 3 \to 7 \\ 7 \to 1 \\ x \to x \end{array}$$

when $x \neq 1, 6, 3, 7$

Hence we have a cycle $\alpha_1 = (1\ 6\ 3\ 7)$.

Now, $2 \notin \{1, 6, 3, 7\}$ and we see that

$\alpha(2) = 4$, $\alpha(4) = 5$, $\alpha(5) = 2$

So define $\alpha_2 : I_7 \to I_7$ by

$$\alpha_2 : \begin{array}{l} 2 \to 4 \\ 4 \to 5 \\ 5 \to 2 \\ x \to x \end{array}$$

when $x \neq 2, 4, 5$.

Hence we have a cycle $\alpha_2 = (2\ 4\ 5)$.

As we see that, $\{1, 6, 3, 7\} \cup \{2, 4, 5\} = I_7$.

5(b) Given $f(x) = \begin{cases} (\cos x - \sin x)^{\operatorname{cosec} x}, & -\frac{\pi}{2} < x < 0 \\ a, & x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}}, & 0 < x < \frac{\pi}{2}. \end{cases}$

If $f(x)$ is continuous at $x=0$, find a and b.

Sol'n: $\lim\limits_{x\to(0+0)} f(x) = \lim\limits_{h\to 0} \dfrac{e^{1/h} + e^{2/h} + e^{3/h}}{ae^{2/h} + be^{3/h}}$

$= \lim\limits_{h\to 0} \dfrac{\frac{1}{e^{2/h}} + \frac{1}{e^{1/h}} + 1}{\frac{a}{e^{1/h}} + b}$

$= \frac{1}{b}$

$\lim\limits_{x\to(0-0)} f(x) = \lim\limits_{h\to 0} (\cos h + \sin h)^{-\operatorname{cosec} h}$

$= \lim\limits_{h\to 0} \left\{1 + (\cos h + \sin h - 1)\right\}^{-1/\sin h}$

$= e^{\lim\limits_{h\to 0}\left\{-2\sin^2 h/2 + 2\sin h/2 \cos h/2\right\}\left\{-\frac{1}{2\sin h/2 \cos h/2}\right\}}$

$\Rightarrow e^{\lim\limits_{h\to 0} \frac{\sin h/2 - \cos h/2}{\cos h/2}} = e^{-1}$

and $f(0) = a$

$\therefore a = e^{-1} = \frac{1}{b} \Rightarrow a = \frac{1}{e},\ b = e$

5(c) Let $\langle a_n \rangle$ be a sequence defined as: $a_1 = 3/2$, $a_{n+1} = 2 - \left(\frac{1}{a_n}\right)$ $\forall n \geq 1$. Show that $\langle a_n \rangle$ is monotonic and bounded and converges to 1.

Sol'n: Given that $a_1 = 3/2$ and $a_{n+1} = 2 - \left(\frac{1}{a_n}\right)$ $\forall n \geq 1$. — (1)

Using mathematical induction, we shall prove that

$a_n > 1\ \forall n \in N$ — (2)

we conclude that

$$\alpha = \alpha_1 \alpha_2 = (1\,6\,3\,7)(2\,4\,5)$$

(ii) we see that the permutation β can be written as

$$\beta : \begin{array}{l} 1 \longrightarrow 1 \\ 2 \longrightarrow 4 \\ 4 \longrightarrow 7 \\ 7 \longrightarrow 2 \\ 3 \longrightarrow 6 \\ 6 \longrightarrow 5 \\ 5 \longrightarrow 3 \end{array}$$

Hence $\beta = (1)(2\,4\,7)(3\,6\,5)$.

$= (2\,4\,7)(3\,6\,5)$.

$= (2\,7)(2\,4)(3\,5)(3\,6)$

(iii) Since β is a product of an even number of 2-cycles, β is an even permutation

(iv) from the given permutation α, we may calculate α^{-1} as follows.

$$\alpha : \begin{array}{l} 1 \rightarrow 6 \\ 2 \rightarrow 4 \\ 3 \rightarrow 7 \\ 4 \rightarrow 5 \\ 5 \rightarrow 2 \\ 6 \rightarrow 3 \\ 7 \rightarrow 1 \end{array} \quad \text{so that} \quad \alpha^{-1} = \begin{array}{l} 1 \rightarrow 7 \\ 2 \rightarrow 5 \\ 3 \rightarrow 6 \\ 4 \rightarrow 2 \\ 5 \rightarrow 4 \\ 6 \rightarrow 1 \\ 7 \rightarrow 3 \end{array}$$

Hence $\alpha^{-1} = (1\,7\,3\,6)(2\,5\,4) = (1\,6)(1\,3)(1\,7)(2\,4)(2\,5)$

As α^{-1} is a product of an odd no. of 2-cycles, we conclude that α^{-1} is not an even permutation from ①, $a_1 > 1$. so ② is true for $n=1$.

Let ② be true for some +ve integer m, ie. let

$$a_m > 1, \text{ where } m \in N \quad \text{—— ③}$$

Now, $a_m > 1 \Rightarrow \frac{1}{a_m} < 1 \Rightarrow -\frac{1}{a_m} > -1 \Rightarrow 2-\frac{1}{a_m} > 2-1$

$$\Rightarrow a_{m+1} > 1, \text{ by ①}$$

Thus, we find that if ② is true for $n=m$, then it is also true for $n=m+1$. Also ② is true for $n=1$. Hence by mathematical induction, ② is true $\forall n \in N$.

Now, from ①, $a_1 = 3/2$ and $a_2 = 2-\left(\frac{1}{a_1}\right) = 2-\frac{2}{3} = 4/3$

Thus $a_2 < a_1$ —— ④

using mathematical induction, we shall prove that

$$a_{n+1} < a_n \ \forall n \in N \quad \text{—— ⑤}$$

Let ⑤ be true for some +ve integer m, ie. let

$$a_{m+1} < a_m$$

Then, ⑥ $\Rightarrow \frac{1}{a_{m+1}} > \frac{1}{a_m} \Rightarrow -\frac{1}{a_{m+1}} < -\frac{1}{a_m} < 2-\frac{1}{a_{m+1}} < 2-\frac{1}{a_m}$

Thus, $a_{m+2} < a_{m+1}$,

showing that if ⑤ is true for $= n = m+1$. Since ⑤ is true for $n=1$, so by mathematical induction, ⑤ is true $\forall n \in N$

from ② & ⑤, we find that $\langle a_n \rangle$ is monotonically decreasing sequence and bounded below and so it is convergent.

Let $\lim a_n = l$ so that $\lim a_{n+1} = l$

then, $a_{n+1} = 2-\frac{1}{a_n} \Rightarrow \lim a_{m+1} = 2-\frac{1}{\lim a_n} \Rightarrow l = 2-\frac{1}{l}$

$\therefore l^2 - 2l + 1 = 0 \Rightarrow (l-1)^2 = 0 \Rightarrow l = 1$.

Thus, $\lim a_n = 1$.

5(d) prove that the equation $z^5+15z+1=0$ has one root in the disc $|z|<\frac{3}{2}$ and four roots in the annulus $\frac{3}{2}<|z|<2$

Soln: Consider the circle $C_1: |z|=2$

Let $f(z)=z^5$, $g(z)=15z+1$

Both $f(z)$ and $g(z)$ being polynomial are analytic with in and on C_1.

on C_1, we have $\frac{|g(z)|}{|f(z)|}=\frac{|15z+1|}{|z^5|}\leq\frac{15|z|+1}{|z^5|}$
$=\frac{31}{32}<1.$

Thus, $|g(z)|<|f(z)|$ on C_1.

Hence, by Rouche's theorem

$f(z)+g(z)=z^5+15z+1$ has same no. of zeros inside $|z|=2$ as $f(z)=z^5$.

But $f(z)=z^5$ has five zeros all located inside $|z|=2$ so that $z^5+15z+1$ has all the five zeros inside $|z|=2$.

Consider the circle $C_2: |z|=\frac{3}{2}$.

Let $f(z)=15z$, $g(z)=z^5+1$.

Both $f(z)$ and $g(z)$ being polynomial are analytic with in and on C_2.

On C_2,

we have

$$\frac{|g(z)|}{|f(z)|} = \frac{|z^5+1|}{|15z|} \leq \frac{|z|^5+1}{15|z|} = \frac{55}{144} < 1.$$

Thus $|g(z)| < |f(z)|$ on C_2.

Hence, by Rouché's theorem, $f(z)+g(z) = z^5+15z+1$ the same no. of zeros inside $|z| = \frac{3}{2}$ as $f(z) = 15z$.

But $f(z) = 15z$ has only one zero inside $|z| = \frac{3}{2}$. and hence the remaining four ~~zeros~~ must lie in the annulus $\frac{3}{2} < |z| < 2$.

5(e) Compute all the basic feasible solutions of the LPP:

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 - 7x_4$$

Such that

$$2x_1 + 3x_2 - x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

and choose that one which maximizes Z.

Sol^n: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions $= 4C_2 = 6$.

The Characteristics of the various basic solutions are as given below.

No. of basic solutions	Basic variables	Non-basic variables	Values of basic variables	Is the sol^n feasible?	value of Z	Is the solution optimal?
1	x_1, x_2	$x_3 = x_4 = 0$	$2x_1 + 3x_2 = 8$ $x_1 - 2x_2 = -3$ $\therefore x_1 = 1, x_2 = 2$	yes	8	NO.
2		$x_2 = x_4 = 0$	$2x_1 - x_3 = 8$ $x_1 + 6x_3 = -3$ $x_1 = -14/13$ $x_3 = -67/13$	NO	—	—
3	x_1, x_4	$x_2 = x_3 = 0$	$2x_1 + 4x_4 = 8$ $x_1 - 7x_4 = -3$ $x_1 = \frac{22}{9}, x_4 = \frac{7}{9}$	yes	-ve	NO

4.	x_2, x_3	$x_1 = x_4 = 0$	$3x_2 - x_3 = 8$ $-2x_2 + 6x_3 = -3$ $\therefore x_2 = 45/16$ $x_3 = 7/16$	Yes	10.2	Yes
5.	x_2, x_4	$x_3 = x_1 = 0$	$3x_2 + 4x_4 = 8$ $-2x_2 - 7x_4 = -3$ $\therefore x_2 = 132/29$ $x_4 = -7/13$	NO	—	—
6.	x_3, x_4	$x_1 = x_2 = 0$	$-x_3 + 4x_4 = 8$ $6x_3 - 7x_4 = -3$ $\therefore x_3 = 44/17$ $x_4 = 45/17$	Yes	-ve	NO

7(a). Show that the function $f(x) = 1/x^2$ is uniformly continuous on $[a,\infty)$ where $a>0$, but not uniformly continuous on $(0,\infty)$.

Sol: To show that $f(x) = 1/x^2$ is uniformly continuous on $[a,\infty)$ where $a>0$.

for $x, y \geq a > 0$, we obtain

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right|$$
$$= \left|\frac{1}{x} - \frac{1}{y}\right| \left|\frac{1}{x} + \frac{1}{y}\right|$$

$$\Rightarrow |f(x) - f(y)| = \left(\frac{y-x}{xy}\right)\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\leq \frac{2|x-y|}{a|xy|} \quad \left(\because x, y \geq a > 0 \Rightarrow \frac{1}{x} < \frac{1}{a} \;\&\; \frac{1}{y} < \frac{1}{a} \Rightarrow \frac{1}{x} + \frac{1}{y} < \frac{2}{a}\right)$$

$$\leq \frac{2}{a^3}|x-y| \quad \left(\because x \geq a, y \geq a \Rightarrow xy \geq a^2 \Rightarrow \frac{1}{xy} \leq \frac{1}{a^2}\right)$$

Let $\epsilon > 0$ be given.

Let $\delta = \epsilon a^3/2$.

Then $|f(x) - f(y)| < \epsilon$, when $|x-y| < \delta$ $\forall x, y \geq a$.

Hence, f is uniformly continuous on $[a,\infty)$

Now we show that f is not uniformly continuous in $(0,\infty)$.

Let $\epsilon = 1/2$ and δ be any positive number. We can always choose a positive integer n such that $n > 1/2\delta$ (or) $\frac{1}{2n} < \delta$. ——— (1)

Let $x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}} \in (0, \infty)$.

Then $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = |n-(n+1)| = 1 > \epsilon$

and $|x_1 - x_2| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| = \left|\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n(n+1)}}\right|$

$$= \frac{1}{\sqrt{n}\sqrt{n+1}\left[\sqrt{n+1}+\sqrt{n}\right]}$$

$$< \frac{1}{\sqrt{n}\cdot 2\sqrt{n}} \quad \left(\because \sqrt{n}\sqrt{n+1} > \sqrt{n}\sqrt{n} \ \& \ \sqrt{n+1}+\sqrt{n} > 2\sqrt{n}\right)$$

$$= \frac{1}{2n} < \delta \text{ by } (1)$$

Thus $|f(x_1) - f(x_2)| > \epsilon$, when $|x_1 - x_2| < \delta$.

Hence, f is not uniformly continuous on $(0, \infty)$.

7(b) Show that the series for which $S_n(x) = nx(1-x)^n$ can be integrated term by term on $[0,1]$, though it is not uniformly convergent on $[0,1]$.

Sol^n:

Here $S_n(x) = nx(1-x)^n$

when $0<x<1$, we have

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = 0$$

Also $S_n(x) = 0$ for $x = 0$ (or) 1

$\therefore$ $f(x) = 0$ for every x in $[0,1]$.

Now $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

and $\int_0^1 S_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$.

changing x to $1-x$.

$$= \int_0^1 n(1-x)x^n\,dx = \int_0^1 n(x^n - x^{n+1})\,dx$$

$$= n\left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{n}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \lim_{n\to\infty} \frac{n}{(n+1)(n+2)} = 0$$

Since $\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty} [S_n(x)]\,dx$

the series is integrable term by term on $[0,1]$ although $x=0$ is a point of non-uniform convergence of the series.

7(c) Discuss the convergence of the series

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

Sol Neglecting the first term, we have

$$u_n = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)}{2\cdot 4\cdot 6\cdots\cdots(2n)}\cdot\frac{x^{2n+1}}{2n+1}.$$

$$\Rightarrow u_{n+1} = \frac{1\cdot 3\cdot 5\cdots\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots\cdots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{2n+3}{2n+1}\cdot\frac{1}{x^2}$$

$$= \frac{2n(1+1/n)\cdot 2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)\cdot 2n\left(1+\frac{1}{2n}\right)}\cdot\frac{1}{x^2}$$

$$= \frac{(1+1/n)(1+3/2n)}{(1+1/2n)^2}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{(1+1/n)\left(1+\frac{3}{2n}\right)}{(1+1/2n)^2}\cdot\frac{1}{x^2}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ By Ratio Test, $\sum u_n$ is convergent if $\frac{1}{x^2}>1$

i.e if $x^2<1$ and divergent if $\frac{1}{x^2}<1$ i.e $x^2>1$.

if $x^2=1$, then Ratio test fails.

when $x^2=1$, we have $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$

$$= \frac{4n^2+10n+6}{4n^2+4n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$= \lim_{n\to\infty} \frac{6n^2+5n}{4n^2+4n+1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(6+\frac{5n}{n^2}\right)}{\cancel{n^2}\left(4+\frac{4n}{n^2}+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{6+5/n}{4+\frac{4}{n}+1/n^2}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

$\therefore$ By Raabe's test, the series converges.

Hence $\sum u_n$ is convergent if $x^2 \le 1$ and divergent if $x^2>1$.

7(d) prove that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = B(m,n)$

where m, n are both positive.

Sol^n. $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$.

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx. \quad \rightarrow (1)$$

In the second integral on R.H.S of (1) put

$x = 1/t$, so that $dx = -\frac{1}{t^2}\,dt$.

when $x=1, t=1$; when $x \to \infty, t \to 0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{(1/t)^{m-1}}{(1+1/t)^{m+n}}\left(-1/t^2\right)dt$$

$$= \int_0^1 \frac{1}{t^{m-1}} \cdot \frac{t^{m+n}}{(1+t)^{m+n}} \cdot \frac{1}{t^2}\,dt.$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

$\therefore$ from (1)

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\therefore\ B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

8(a) Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent's series valid for :

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $0<|z-2|<1$

(iv) $|z-1|>1$

Sol? Resolving into partial fractions

$$f(z) = \frac{z}{(z-1)(2-z)} = \frac{1}{z-1} + \frac{2}{2-z}$$

(i) If $|z|<1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{-1}{(1-z)} + \frac{2}{2\left(1-\frac{z}{2}\right)}$$

$$= -(1-z)^{-1} + \left(1-\frac{z}{2}\right)^{-1}$$

$$= -\left[1+z+z^2+\cdots\right] + \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= -\frac{z}{2} - \frac{3}{4}z^2 - \frac{7}{8}z^3 - \frac{15}{16}z^4 - \cdots$$

which is the required Laurent's expansion valid for both $|z|<1$ and $|z|<2$. i.e $|z|<1$.

(ii) $1<|z|<2$.

$$f(z)=\frac{1}{z-1}+\frac{2}{2-z}=\frac{1}{z(1-\frac{1}{z})}+\frac{1}{(1-\frac{z}{2})}$$

$$=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\left(1-\frac{z}{2}\right)^{-1}$$

[arranged suitably to make the binomial expansions valid for $1<|z|<2$.]

$$=\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]+\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$=\cdots+\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots$$

This represents a series in positive and negative powers of z. In other words it is expansion of $f(z)$ in Laurent's series in the annulus $1<|z|<2$.

(iii) $|z|>2$.

$$f(z)=\frac{1}{z-1}+\frac{2}{2-z}=\frac{1}{z(1-\frac{1}{z})}+\frac{2}{-z(1-\frac{2}{z})}$$

$$=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)-\frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$=\frac{1}{z}\left[\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)-2\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right)\right]$$

$$=\frac{1}{z}\left(-1-\frac{3}{z}-\frac{7}{z^2}-\frac{15}{z^3}-\cdots\right)$$

$$=-\frac{1}{z}-\frac{3}{z^2}-\frac{7}{z^3}-\frac{15}{z^4}-\cdots$$

which is the required Laurent's expansion valid for both $|z|>1$ and $|z|>2$

i.e. $|z|>2$.

(iv) $0<|z-2|<1$

Let $z-2=u$

$\Rightarrow z=u+2$

i.e. $0<|u|<1$

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1}+\frac{2}{2-z}$$

$$= \frac{1}{(u+1)}+\frac{2}{(-u)}$$

$$= (1+u)^{-1}-\frac{2}{u}$$

$$= (1-u+u^2+\cdots)-2u^{-1}$$

$$= 1-(z-2)+(z-2)^2-(z-2)^3-\cdots -2(z-2)^{-1}$$

$$= 1-2(z-2)^{-1}-(z-2)+(z-2)^2-(z-2)^3+(z-2)^4+\cdots$$

which is valid for $|u|<1$, $u\neq 0$. i.e. $0<|z-2|<1$

(v) $|z-1|>1$

Let $z-1=u$

$z=u+1$

$$\frac{z}{(z-1)(2-z)} = \frac{1}{z-1}+\frac{2}{2-z}$$

$$= \frac{1}{u}+\frac{2}{1-u}$$

$|u|>1$

$\left|\frac{1}{u}\right|<1$

$$= \frac{1}{u}+\frac{2}{-u[1-\frac{1}{u}]} = \frac{1}{u}-\frac{2}{u}\left[1-\frac{1}{u}\right]^{-1}$$

$$= \frac{1}{u}-\frac{2}{u}\left[1+\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u^3}+\cdots\right]$$

$$= -\frac{1}{u}-\frac{2}{u^2}-\frac{2}{u^3}-\frac{2}{u^4}-\cdots$$

$$= -(z-1)^{-1}-2(z-1)^{-2}-2(z-1)^{-3}-2(z-1)^{-4}-\cdots$$

which is valid for $|u|>1$, i.e. $|z-1|>1$

8(b) Use Cauchy's theorem and/or Cauchy integral formula

(i) $\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$.

(ii) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z|=3$

(iii) $\oint_C \frac{e^{-z}}{z+1} dz$, where C is the circle $|z|=\frac{1}{2}$

Sol:

(i) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{z-1}$

we have

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} dz - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} dz \quad \text{—(1)}$$

By Cauchy's integral formula

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} dz = f(a)$$

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} dz = 2\pi i\left\{\sin\pi(2)^2 + \cos\pi(2)^2\right\} = 2\pi i$$

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} = 2\pi i\left\{\sin\pi(1)^2 + \cos\pi(1)^2\right\} = 2\pi i\{-1\} = -2\pi i$$

Since $z=1$ and $z=2$ are inside C and $\sin\pi z^2 + \cos\pi z^2$ is analytic inside C. Then the required integral has the value

i.e. from (1) $2\pi i - (-2\pi i) = 4\pi i$.

(ii) Let $f(z) = e^{2z}$ and $a = -1$
in the Cauchy integral formula

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad \text{--- (2)}$$

If $n = 3$,
then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$

Hence (2) becomes

$$8e^{-2} = \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4} dz$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i e^{-2}}{6}$$

$$= \frac{\pi i e^{-2}}{3}$$

(iii) Here $f(z) = e^{-z}$ is analytic function.
The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's theorem, we have

$$\oint \frac{e^{-z}}{z+1} dz = 0$$