ANALYTIC GEOMETRY

:IFoS-2010:

① (e) If a plane cuts the axes in A, B, C and (a, b, c) are coordinates of centroid of the triangle ABC, then show that the eqn of plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

→ Let the plane eqn be $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1$. —— ①

It cuts the axes at $A(p,0,0)$, $B(0,q,0)$, $C(0,0,r)$. Then centroid of ΔABC has coordinates $\left(\frac{p}{3},\frac{q}{3},\frac{r}{3}\right)$

given that (a,b,c) are coordinates of centroid,

$a=\frac{p}{3}$, $b=\frac{q}{3}$, $c=\frac{r}{3}$. $\Rightarrow$ $p=3a$, $q=3b$, $r=3c$.

① ≡ $\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1$ $\Rightarrow$ $\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3}$

② ① (f) Find the equations of the spheres passing through the circle $x^2+y^2+z^2-6x-2z+5=0$, $y=0$ and touching the plane $3y+4z+5=0$.

→ Sphere passing through the given circle is given by

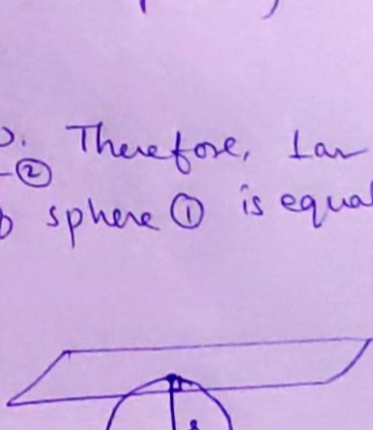
$x^2+y^2+z^2-6x+\lambda y-2z+5=0$ —— ①

It touches the plane $3y+4z+5=0$. Therefore, ⊥ar distance of plane ② from centre of sphere ① is equal to radius of the sphere

Centre of sphere ① is $\left(3,-\frac{\lambda}{2},1\right)$

Radius of sphere ① is $\sqrt{9+\frac{\lambda^2}{4}+1-5}$

$=\sqrt{5+\frac{\lambda^2}{4}}$



$\therefore \frac{3.0+3\left(-\lambda/2\right)+4.1+5}{\sqrt{0+9+16}}=\sqrt{5+\frac{\lambda^2}{4}}$

$$\Rightarrow \frac{\left(-\frac{3\lambda}{2}+9\right)^2}{(\sqrt{125})^2} = 5+\frac{\lambda^2}{4} \Rightarrow \frac{9\lambda^2}{4}+81-27\lambda = 125+\frac{25\lambda^2}{4}$$

$$\Rightarrow 4\lambda^2+27\lambda+44=0 \Rightarrow \lambda = -\frac{11}{4},\ \lambda=-4$$

$\therefore$ The eqns of spheres are

$$x^2+y^2+z^2-6x-\frac{11}{4}y-2z+5=0 \quad \text{and}$$

$$x^2+y^2+z^2-6x-4y-2z+5=0$$

4(a): Prove that the 2nd degree equation $x^2-2y^2+3z^2+5yz-6zx-4xy+8x-19y-2z-20=0$ represents a cone with vertex $(1,-2,3)$

$\rightarrow$ Let us make the given eqn homogeneous by introducing a new variable t. Then

$$F(x,y,z,t) = x^2-2y^2+3z^2+5yz-6zx-4xy+8xt-19yt-2zt-20t^2=0 \quad \text{--- (1)}$$

Diff. (1) partially wrt x, y, z & t and equating to zero,

$$\frac{\partial F}{\partial x} = 2x-4y-6z+8t=0$$

$$\frac{\partial F}{\partial y} = -4x-4y+5z-19t=0$$

$$\frac{\partial F}{\partial z} = -6x+5y+6z-2t=0$$

$$\frac{\partial F}{\partial t} = 8x-19y-2z-40t=0$$

Putting $t=1$ in these equations, we get

$$2x-4y-6z+8=0 \quad \text{--- (2)}, \qquad -4x-4y+5z-19=0 \quad \text{--- (3)}$$

$$-6x+5y+6z-2=0 \quad \text{--- (4)} \quad \text{and} \quad 8x-19y-2z-40=0 \quad \text{--- (5)}$$

(2), (3), (4) $\Rightarrow x=1,\ y=-2,\ z=3$

Putting in LHS of (5): $8\cdot 1-19\cdot(-2)-2\cdot(3)-40=0$.

$\therefore$ The given equation represents a cone with vertex at $(1,-2,3)$

④ (b) If feet of the normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, prove that the feet of other three normals lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$.

→ Let (α,β,γ) be the point P. Let (x_1,y_1,z_1) be any point on the ellipsoid. Then, tangent plane at (x_1,y_1,z_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1.$$

The normal at (x_1,y_1,z_1) is $\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}=\lambda$ (say)

It passes through (α,β,γ). Therefore,

$$\alpha-x_1=\frac{x_1\lambda}{a^2},\quad \beta-y_1=\frac{y_1\lambda}{b^2},\quad \gamma-z_1=\frac{z_1\lambda}{c^2}$$

$$\Rightarrow \alpha=\frac{a^2+\lambda}{a^2}x_1,\quad \beta=\frac{b^2+\lambda}{b^2}y_1,\quad \gamma=\frac{c^2+\lambda}{c^2}z_1$$

$$\Rightarrow x_1=\frac{a^2\alpha}{a^2+\lambda},\quad y_1=\frac{b^2\beta}{b^2+\lambda},\quad z_1=\frac{c^2\gamma}{c^2+\lambda}$$

The point (x_1,y_1,z_1) lies on the ellipsoid. Therefore,

$$\frac{\left(\frac{a^2\alpha}{a^2+\lambda}\right)^2}{a^2}+\frac{\left(\frac{b^2\beta}{b^2+\lambda}\right)^2}{b^2}+\frac{\left(\frac{c^2\gamma}{c^2+\lambda}\right)^2}{c^2}=1$$

$$\rightarrow \frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1 \quad \text{--- (1)}$$

It is a 6th degree eqⁿ in λ. Therefore, there are six feet of normals. whose coordinates are (x_1,y_1,z_1) which changes according to the value of λ.

Three feets of normals lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\therefore \frac{a^2\alpha}{a(a^2+\lambda)}+\frac{b^2\beta}{b(b^2+\lambda)}+\frac{c^2\gamma}{c(c^2+\lambda)}=1$$

$$\Rightarrow \frac{a\alpha}{a^2+\lambda}+\frac{b\beta}{b^2+\lambda}+\frac{c\gamma}{c^2+\lambda}=1 \quad \text{--- (2)}$$

Let the other three feet lie on the plane $\frac{x}{a_1}+\frac{y}{b_1}+\frac{z}{c_1}=p_1$

then : $\frac{a^2\alpha}{a_1(a^2+\lambda)}+\frac{b^2\beta}{b_1(b^2+\lambda)}+\frac{c^2\gamma}{c_1(c^2+\lambda)}=p_1$ — (3)

the (2) & (3) eqns combined form the eqn (1).

$\therefore$ Comparing coeff $\frac{a^2\alpha . a\alpha}{a_1(a^2+\lambda)^2}=\frac{a^2\alpha^2}{(a^2+\lambda)^2}$

$\Rightarrow \quad a_1=a. \Rightarrow \frac{1}{a_1}=\frac{1}{a}.$

Sly, $\frac{1}{b_1}=\frac{1}{b}, \frac{1}{c_1}=\frac{1}{c}$ & $p_1=-1$

$\therefore$ Reqd plane: $\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0}$

(4)(c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz-8xz-3xy=0$, find the eqns of the other two.

→ The other two generators lie on the plane $\perp$ar to the given line which has equation $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ — (1).

Eqn of plane $\perp$ar to this plane (1) is $x+2y+3z=0$ — (2)

Let any line on this plane be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (3).

Then this line lies on plane (2) $\Rightarrow l+2m+3n=0$

$\Rightarrow l=-(2m+3n)$ — (4)

Also, (3) is a generator of the cone. Therefore

$5mn-8nl-3lm=0 \Rightarrow 5mn+8n(2m+3n)+3m(2m+3n)=0$

$\Rightarrow$ $6m^2+30mn+24n^2=0 \Rightarrow \left(\frac{m}{n}\right)^2+5\frac{m}{n}+4=0$

$\Rightarrow$ $\frac{m}{n}+1=0, \quad \frac{m}{n}+4=0$

$\Rightarrow$ $\frac{m}{-1}=\frac{n}{1}$ & $\frac{m}{-4}=\frac{n}{1}$

$l = -(2m+3n)$ (from 4)

(i) $m = -n$:

$l = -(-2n+3n) = -n$

$\frac{l}{-1} = \frac{n}{1}$

(ii) $m = -4n$:

$l = -(-8n+3n) = 5n$

$\frac{l}{5} = \frac{n}{1}$

$\therefore \frac{l}{-1} = \frac{m}{-1} = \frac{n}{1}$ & $\frac{l}{5} = \frac{m}{-4} = \frac{n}{1}$

$\therefore$ Reqd lines: $\frac{x}{-1} = \frac{y}{-1} = \frac{z}{1}$ and $\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$

(4) (d) Prove that the locus of the point of intersection of 3 tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$.

→ Let $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ & $R(\alpha_3, \beta_3, \gamma_3)$ be three extremities of the 3 semi-conjugate diameters of ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2} = 1$ — ①

Then, diameter plane of P wrt ellipsoid ① is

$$\frac{x\alpha_1}{\alpha^2} + \frac{y\beta_1}{\beta^2} + \frac{z\gamma_1}{\gamma^2} = 0 .$$

Any plane parallel to this plane is

$$\frac{x\alpha_1}{\alpha^2} + \frac{y\beta_1}{\beta_2} + \frac{z\gamma_1}{\gamma^2} = k_1 \quad \text{— ②}$$

If this plane is tangent to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$,

thus $k_1^2 = \left(\frac{\alpha_1}{\alpha^2}\right)^2 a^2 + \left(\frac{\beta_1}{\beta^2}\right)^2 b^2 + \left(\frac{\gamma_1}{\gamma^2}\right)^2 c^2$

$\Rightarrow k_1^2 = \Sigma\left(\frac{\alpha_1 a}{\alpha^2}\right)^2$.

Sly, the planes parallel to diameter planes of Q & R are

$\frac{x\alpha_2}{\alpha^2} + \frac{y\beta_2}{\beta^2} + \frac{z\gamma_2}{\gamma^2} = k_2$ & $\frac{x\alpha_3}{\alpha^2} + \frac{y\beta_3}{\beta^2} + \frac{z\gamma_3}{\gamma^2} = k_3$ where

$k_2^2 = \Sigma\left(\frac{\alpha_2 a}{\alpha^2}\right)^2$. & $k_3^2 = \Sigma\left(\frac{\alpha_3 a}{\alpha^2}\right)^2$

Now: Squaring & adding the eqns of 3 planes,

$$\left(\frac{x\alpha_1}{\alpha^2} + \frac{y\beta_1}{\beta^2} + \frac{z\gamma_1}{\gamma^2}\right)^2 + \left(\frac{x\alpha_2}{\alpha^2} + \frac{y\beta_2}{\beta^2} + \frac{z\gamma_2}{\gamma^2}\right)^2 + \left(\frac{x\alpha_3}{\alpha^2} + \frac{y\beta_3}{\beta^2} + \frac{z\gamma_3}{\gamma^2}\right)^2 = k_1^2 + k_2^2 + k_3^2$$

$$\Rightarrow \Sigma \frac{x^2}{\alpha^4}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + \Sigma \frac{2xy}{\alpha^2\beta^2}(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3)$$

$$= \Sigma \frac{a^2}{\alpha^4}(\alpha_1^2 + \alpha_2^2 + \alpha_3^2).$$

Since the $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ & $(\alpha_3, \beta_3, \gamma_3)$ are semi-conjugate diameter extremities, $\Sigma\alpha_1^2 = \Sigma\beta_1^2 = \Sigma\gamma_1^2 \Rightarrow$

$\Sigma\alpha_1^2 = \alpha^2$, $\Sigma\beta_1^2 = \beta^2$ $\Sigma\gamma_1^2 = \gamma^2$ & $\Sigma\alpha_1\beta_1 = \Sigma\beta_1\gamma_1 = \Sigma\alpha_1\gamma_1 = 0$

$$\therefore \frac{x^2\alpha^2}{\alpha^4} + \frac{y^2\beta^2}{\beta^4} + \frac{z^2\gamma^2}{\gamma^4} = \frac{a^2\alpha^2}{\alpha^4} + \frac{b^2\beta^2}{\beta^4} + \frac{c^2\gamma^2}{\gamma^4}$$

$$\Rightarrow \boxed{\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}}$$ which is the reqd locus.