① a] how many generators are there OF the cyclic group G OF order 8? explain.

b] Taking a Group $\{e, a, b, c\}$ OF order 4 where e is identity, construct composition table showing that one is cyclic while other is not.

⑤ + ⑤ = ⑩

Ans 1] G is a cyclic group of order 8

let $G = \langle a \rangle$, $a \in G$

we know that If G is cyclic group of order n the there are $\phi(n)$ generators in G. where $\phi(n) = \{a^r, (r,n) = 1\}$

$\therefore G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$

$8 = 2 \times 2 \times 2 = 2^3$

$\therefore \phi(8) = 8\left(1 - \frac{1}{2}\right)$

$= 8 \times \frac{1}{2} = 4$

$\therefore$ there are 4 generators of G.

namely,

a, a^3, a^5, a^7.

because this are of the type a^r where $(r, n) = 1$

b] $G = \{e, a, b, c\}$

$\therefore$ G can be isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$

if $G = \mathbb{Z}_4$

let $G = \langle a \rangle$

$= \{a, a^2 = b, a^3 = c, a^4 = e\}$

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$a \cdot b = a \cdot a^2 = a^3 = c$

$a \cdot c = a \cdot a^3 = e$

$b \cdot c = a^2 a^3 = a$

If $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ which is not cyclic.

then $a^2 = e$, $b^2 = c$, $c^2 = c$

$a \cdot b = c$ $\quad a \cdot c = b$ $\quad b \cdot c = a$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

by composition table this Group is not cyclic,

(b) Give an example of a ring having Identity but a subring of this having a different Identity. (10)

ans:- let $R = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ set of 2×2 matrices

where $a, b, c, d \in \mathbb{R}$

we can see

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is multiplicative identity element of R.

$$S = \left\{ \begin{bmatrix} x & x \\ x & x \end{bmatrix} : x \in \mathbb{R} \right\}$$

it is easy to check that S is subring

$$X \cdot Y = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in S$$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$\therefore \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \in S$ is identity element of S which is different from unity of R.

2) a] If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R' prove that $\phi(1)$ is the unit element of R'. (15)

proof:- Let $\langle R, +, \cdot \rangle$ and $\langle R', *, \circ \rangle$ be two rings. A mapping $\phi: R \to R'$ is called Homomorphism if

$$\phi(a+b) = \phi(a) * \phi(b)$$
$$\phi(a \cdot b) = \phi(a) \circ \phi(b)$$
$$a, b \in R$$

let $a' \in R'$ be arbitrary element

$\because$ $\phi: R \to R'$ is homomorphism and onto

$\therefore \exists$ $a \in R \ni$
$$\phi(a) = a'$$

Now $a' \cdot \phi(1)$
$$= \phi(a) \cdot \phi(1)$$
$$= \phi(a \cdot 1) \quad \text{... by homomorphism}$$
$$= \phi(a)$$
$$= a'$$

Similarly,
$$\phi(1) \cdot a' = \phi(1) \cdot \phi(a)$$
$$= \phi(1 \cdot a)$$
$$= \phi(a)$$
$$= a'$$

$\therefore$ $a' \cdot \phi(1) = \phi(1) \cdot a' = a'$

$\therefore$ $\phi(1)$ is unit element of R'.

4] a] Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are field:

(i) the set of numbers of the form $b\sqrt{2}$ with b rational. (5)

ii) the set of even integers. (6)

iii) The set of positive integers. (4)

Sol^n:

Integral Domain:-

A commutative ring R is called an integral domain if $ab = 0$ in $R \Rightarrow$ either $a = 0$ or $b = 0$.

Sol^n:-

1] $\{b\sqrt{2} : b \in \mathbb{Q}\}$

take $b, c \in \mathbb{Q}$

$b\sqrt{2} \cdot c\sqrt{2} = 2bc \notin b\sqrt{2}$ or $b\sqrt{2}$

$\therefore$ $\cdot$ is not a binary operation

$\therefore$ $\{b\sqrt{2}\}$ is not a ring.

hence not Integral domain.

2] the set of even integer.

$S = \{2a : a \in \mathbb{Z}\}$

let $2a \cdot 2b = 0$

$\Rightarrow 4a \cdot b = 0$

$\Rightarrow a \cdot b = 0$

but $\mathbb{Z}$ is I.D

$\Rightarrow$ either $a = 0$ or $b = 0$

$\Rightarrow$ either $2a = 0$ or $2b = 0$

hence S is I.D.

but let $a = 5$

$\therefore 2a = 10$

~~$\nexists$ any $2b \in S \ni$~~

~~$2a \cdot 2b =$~~

$\nexists$ any multiplicative identity for S.

$\therefore$ S is not a field.

3) $S = \{\mathbb{Z}\}$

$\mathbb{Z}$ is I.D.

but for $a \neq 1$ $\nexists b \in \mathbb{Z} \ni$

$a \cdot b = 1$ hence

$\mathbb{Z}$ is not field.