[redacted]

IAS/IFoS

MATHEMATICS

Under the guidance *of* K. Venkanna

LINEAR ALGEBRA, CALCULUS AND THREE DIMENSIONAL GEOMETRY

TEST CODE: TEST-1: IAS(M)/05-JULY-2020

201/250

Time: 3 Hours | **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has **34 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: VISHAL KHATRI

Roll No.: vishalk2619@gmail.com

Test Centre: Online

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate: Vishal Khatri

I have verified the information filled by the candidate above

Signature of the Invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

QUESTION	No.
1	(a)
	(b)
	(c)
	(d)
	(e)
2	(a)
	(b)
	(c)
	(d)
3	(a)
	(b)
	(c)
	(d)
4	(a)
	(b)
	(c)
	(d)
5	(a)
	(b)
	(c)
	(d)
	(e)
6	(a)
	(b)
	(c)
	(d
7	(a
	(
	(
	(
8	(

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)				
	(b)			08	
	(c)			08	
	(d)			08	
	(e)			08	32
2	(a)			04	
	(b)			13	
	(c)			13	
	(d)			13	43
3	(a)			14	
	(b)			12	
	(c)			18	
	(d)				44
4	(a)				
	(b)				
	(c)				
	(d)				
5	(a)			08	
	(b)			08	
	(c)			08	
	(d)			08	
	(e)			08	40
6	(a)				
	(b)				
	(c)				
	(d)				
7	(a)			08	
	(b)			12	
	(c)			11	
	(d)			11	42
8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					201

DO NOT WRITE ON THIS SPACE

SECTION – A

1. (a) If S and T are subspace of IR^4 given by

$S=\{(x,y,z,w)\in IR^4 : 2x+y+3z+w=0\}$ and

$T=\{(x,y,z,w)\in IR^4 : x+2y+z+3w=0\}$, find $\dim(S\cap T)$. **[10]**

Consider the 2 vector spaces S & T as given

Since the intersection of 2 vector spaces is also a vector space, $S\cap T$ is a VS.

$$S\cap T = \text{Null space of } \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} = A$$

Conducting row operations on A

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -3 & 1 & -5 \end{bmatrix} \text{ in echelon form}$$

i.e. $\text{rank}(A) = 2$

By rank-nullity theorem; $\dim(\text{nullspace of } A) = 4-2 = 2$

Thus, $\boxed{\dim(S\cap T) = 2}$

~~Example of basis~~ : $\cancel{(1,0,3,-11)}$

$\cancel{(0,1,5,-16)}$

1. (b) If $T : P_2(x) \to P_3(x)$ is such that $T(f(x)) = f(x) + 5\int_0^x f(t)dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T. **[10]**

Consider the bases of $P_2(x) = \{\underset{u}{1}, \underset{v}{1+x}, \underset{w}{1-x^2}\}$

$$T(u) = 1 + 5\int_0^x 1\,dt = 1+5x = (1, 5, 0, 0) \text{ in } P_3(x)$$

$$T(v) = (1+x) + 5\int_0^x (1+t)\,dt = 1 + 6x + \frac{5x^2}{2}$$

$$= (1, 6, 5/2, 0) \text{ in } P_3(x)$$

$$T(w) = (1-x^2) + 5\int_0^x (1-t^2)\,dt = 1 + 5x - x^2 - \frac{5x^3}{3}$$

$$= (1, 5, -1, -5/3) \text{ in } P_3(x)$$

08

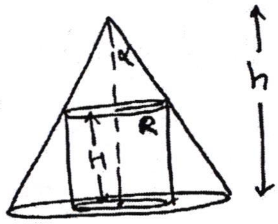
Thus, the matrix of the transformation $T : P_2 \to P_3$ is given by

$$\cancel{A =} \cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1/2 & -1 \\ 0 & 0 & -1/3 \end{bmatrix}} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 6 & 5 \\ 0 & 5/2 & -1 \\ 0 & 0 & -5/3 \end{bmatrix}$$

Scanned with CamScanner

1. (c) Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α. [10]

Consider a cylinder inscribed in the cone as shown with height H & radius R



$\therefore$ $\boxed{\tan\alpha = \dfrac{R}{h-H}}$

Volume of cylinder $= \pi R^2 H = \pi \left[\tan\alpha\,(h-H)\right]^2 H$

i.e. $V = \pi \tan^2\alpha \left[(h-H)^2 \times H\right]$

For maximization, $\dfrac{dV}{dH} = 0$

i.e. $\pi\tan^2\alpha \left[(h-H)^2 + 2H(h-H)\times -1\right] = 0$

i.e. $h-H = 2H \rightarrow \boxed{H = h/3}$

9.5

$\dfrac{d^2V}{dH^2} = -2(h-H) \cancel{+2h} - 4H \Big|_{H=h/3} = \dfrac{2h}{3} - \dfrac{4h}{3} < 0$

i.e. $\boxed{\dfrac{d^2V}{dH^2} < 0 \Rightarrow H = h/3 \text{ is a maxima}}$

Thus, $\boxed{\text{maximum volume of cylinder} = \dfrac{4\pi}{\cancel{9}\,27}\tan^2\alpha\, h^3}$

1. (d) Evaluate the following integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx.$$ [10]

Consider the definite integral

$$I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\,dx \quad \text{—— (1)}$$

We know that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$\therefore\ I = \int_{\pi/6}^{\pi/3} \frac{\left[\sin\left(\pi/2-x\right)\right]^{1/3}}{\left[\sin\left(\pi/2-x\right)\right]^{1/3}+\left[\cos\left(\frac{\pi}{2}-x\right)\right]^{1/3}}\,dx.$$

$$= \int_{\pi/6}^{\pi/3} \frac{(\cos x)^{1/3}}{(\cos x)^{1/3}+(\sin x)^{1/3}}\,dx \qquad \left[\begin{array}{l}\sin\left(\frac{\pi}{2}-x\right)=\cos x\\ \cos\left(\frac{\pi}{2}-x\right)=\sin x\end{array}\right]$$

i.e. $$I = \int_{\pi/6}^{\pi/3} \frac{(\cos x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\,dx \quad \text{—— (2)}$$

Adding (1) and (2), $2I = \int_{\pi/6}^{\pi/3} 1\,dx = \pi/6$

i.e. $\boxed{I = \frac{\pi}{12}}$

Thus $$I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\,dx = \frac{\pi}{12}$$

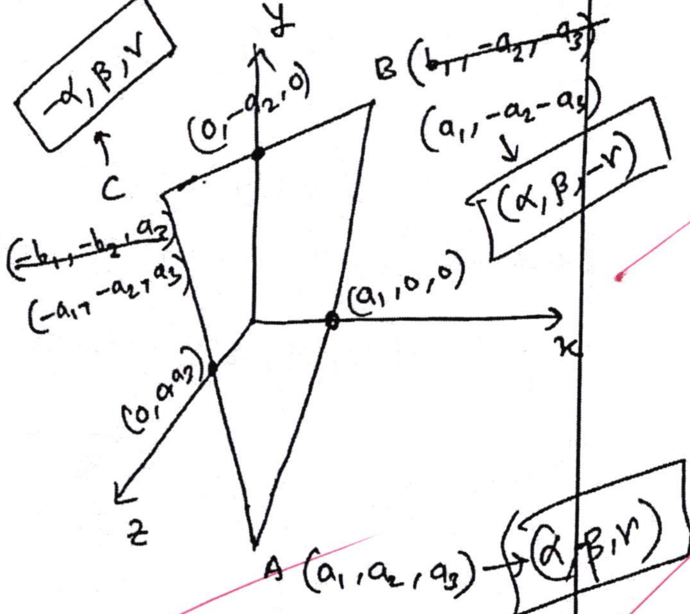
08

1. (e) A triangle, the length of whose sides are a, b and c is placed so that the middle points of the sides are on the axes. Show that the lengths α, β, γ intercepted on the axes are given by

$8\alpha^2 = b^2 + c^2 - a^2$, $8\beta^2 = c^2 + a^2 - b^2$, $8\gamma^2 = a^2 + b^2 - c^2$

and find the coordinates of its vertices **[10]**

Consider the figure shown so that midpt of AB on x-axis, BC on y-axis & AC on z-axis.



$\therefore A \equiv (a_1, a_2, a_3)$

$B \equiv (b_1, -a_2, -a_3)$

$C \equiv (-b_1, -b_2, a_3)$

midpoint of AC $\equiv \left(\frac{a_1 - b_1}{2}, \frac{a_2 - b_2}{2}, a_3\right)$ lies on z-axis

$\therefore a_1 = b_1$ and $a_2 = b_2$

$A \equiv (a_1, a_2, a_3)$ | $B \equiv (a_1, -a_2, -a_3)$ | $C \equiv (-a_1, -a_2, a_3)$

$L(BC) = \sqrt{4a_1^2 + 4a_3} = a \rightarrow a_1^2 + a_3^2 = \frac{a^2}{4}$

$L(AB) = \sqrt{4a_2^2 + 4a_3^2} = c \rightarrow a_2^2 + a_3^2 = \frac{c^2}{4}$

$a_1 = \alpha,\ a_2 = \beta,\ a_3 = \gamma$ $\qquad a_1^2 + a_2^2 = \frac{b^2}{4}$

$\therefore 2(\alpha^2 + \beta^2 + \gamma^2) = \frac{a^2 + b^2 + c^2}{4} \rightarrow \boxed{\Sigma \alpha^2 = \frac{a^2 + b^2 + c^2}{8}}$

$\therefore 8\alpha^2 = b^2 + c^2 - a^2$ | $8\beta^2 = c^2 + a^2 - b^2$ | $8\gamma^2 = a^2 + b^2 - c^2$

08

Scanned with CamScanner

2. (a) Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1, 2, 3) and (1, –1, 1). [06]

$V \equiv (a, b, 0)$

$W \equiv \alpha(1, 2, 3) + \beta(1, -1, 1)$

$\equiv (\alpha+\beta, 2\alpha - \beta, 3\alpha+\beta)$

let $(x, y, z) \in (V \cap W)$

$\therefore z = 0 \Rightarrow \boxed{\beta = -3\alpha}$

$\therefore x = -2\alpha \mid y = 5\alpha \mid z = 0$

$\therefore$ $\boxed{\text{Vector } (-2, 5, 0) \text{ generates } V \cap W}$

06

of V and W,
by the vectors
[06]

2. (b) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1\ 1\ 0\ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$

[14]

Eigenvalues

1) Since A is given to be singular,

$|A - 0I| = |A| = 0 \Rightarrow \boxed{\lambda_1 = 0}$

2) $\boxed{\lambda_2 = 4}$

3) $AX = \lambda_3 X \Rightarrow A[1\ 1\ 0\ 0]^T = \lambda_3 [1\ 1\ 0\ 0]^T$

i.e. $\boxed{\lambda_3 = 2}$

4) Sum of eigenvalues = trace (A) = 2

$\therefore\ 0 + 4 + 2 + \lambda_4 = 2$

$\therefore\ \boxed{\lambda_4 = -4}$

$\therefore$ Eigenvalues of A are 0, 4, 2, −4

Characteristic polynomial of A

$p(x) = (x-0)(x-4)(x-2)(x+4)$

$= x(x^2-16)(x-2)$

$= x(x^3-16x-2x^2+32) =$ ~~$x^4 + 2x^3$~~

13

Scanned with CamScanner

$P(x) = x^4 - 2x^3 - 16x^2 + 32x$

By cayley-hamiltan theorem, A satisfies $P(x)$.

$\therefore$

$P = -2$	$q = -16$	$r = 32$

2. (c) (i) If $z = \tan(y + ax) + (y - ax)^{3/2}$, find the value of
$$\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}.$$

(ii) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.

[15]

i) $z = \tan(y+ax) + (y-ax)^{3/2}$

$$\frac{\partial z}{\partial x} = a\sec^2(y+ax) - \frac{3a}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y+ax)\tan(y+ax) + \frac{3a^2}{4}(y-ax)^{-1/2} \quad (1)$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2} \quad (2)$$

13

$P(x)$.

Using eqn (1) & (2),

$$\boxed{\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0}$$

$\tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ is a homogenous function of degree $(1/2)$

Using Euler's formulae.

$$x \frac{\partial(\tan u)}{\partial x} + y \frac{\partial(\tan u)}{\partial y} = \frac{1}{2} \tan u$$

i.e. $x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$

[15]

i.e. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\tan u}{2 \sec^2 u}$

$$= \frac{\sin u}{2 \sec u}$$

$$= \frac{\sin u \cos u}{2}$$

$$= \frac{\sin 2u}{4}$$

$-a x)^{-1/2}$

(1)

$$\therefore \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\sin 2u}{4}}$$

$x)^{-1/2}$

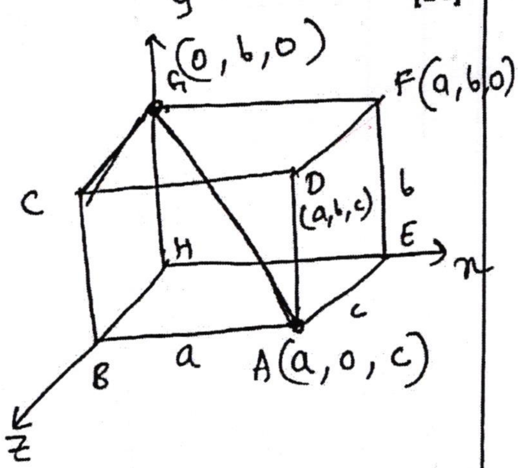
$-(2)$

2. (d) Prove that the S. D. between the diagonals of rectangular parallelopiped and the edges not meeting it are

$$\frac{bc}{\sqrt{(b^2+c^2)}}, \frac{ca}{\sqrt{(c^2+a^2)}}, \frac{ab}{\sqrt{(a^2+b^2)}}$$

where a, b, c are the lengths of the edges. **[15]**

Consider the diagonal AG and edge FD



$\vec{FD} = (-c)\hat{k}$

$\vec{AG} = (a)\hat{i} + (-b)\hat{j} + (c)\hat{k}$

The direction of shortest distance $= \vec{FD} \times \vec{AG}$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & -c \\ a & -b & c \end{vmatrix}$$

$= (-bc)\hat{i}\ (-ac)\hat{j}$

$\vec{v} = (-bc)\hat{i} + (-ac)\hat{j}$

13

Projection of AD on shortest distance

$\vec{AD} = (b)\cdot\hat{j}$

Shortest distance $= \left| \frac{\vec{AD} \cdot \vec{v}}{|v|} \right|$

$$= \left| \frac{(b\hat{j}) \cdot (-bc\hat{i} - ac\hat{j})}{|-bc\hat{i} - ac\hat{j}|} \right|$$

shortest distance (FD − AG) $= \left| \dfrac{-ab\cancel{c}}{\cancel{c}\sqrt{a^2+b^2}} \right|$

$= \dfrac{ab}{\sqrt{a^2+b^2}}$.

Thus, $\boxed{\text{shortest distance (FD − AG)} = \dfrac{ab}{\sqrt{a^2+b^2}}}$

By symmetry,

shortest distance (CD − AG) $= \dfrac{bc}{\sqrt{b^2+c^2}}$

shortest distance (FE − AG) $= \dfrac{ca}{\sqrt{a^2+c^2}}$

3. (a) (i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation:

$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$

$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$

$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$

[16]

i) $\boxed{\text{W is not a subspace of V}}$

Consider $w_1 \in V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$w_2 \in V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(w_1) = \det(w_2) = 0$

$\therefore \boxed{w_1, w_2 \in W}$

Consider $w_1 + w_2 \in V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(w_1 + w_2) = 1 \neq 0.$

Thus, $\boxed{w_1 + w_2 \notin W}$

i.e. $w_1, w_2 \in W$ but $w_1 + w_2 \notin W$

Hence, W is $\boxed{\text{not}}$ a subspace of V.

14

Consider $AX = 0$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$2x_3 + 5x_4 + x_5 = 0 \Rightarrow x_5 = -(2x_3 + 5x_4)$

$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \Rightarrow x_1 = -2x_2 - 3x_3 + 2x_4 + 8x_3 + 20x_4$

$= -2x_2 + 5x_3 + 22x_4$

Thus, $\boxed{\dim(W) = 3}$

$$\text{Basis}(W) = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 22 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix}$$

3. (b) Express $\int_0^1 x^m (1-x^n)^p\, dx$ interms of Gamma function and hence evaluate the integral.

$\int_0^1 x^6 \sqrt{(1-x^2)}\, dx.$ **[14]**

Consider the integral $I = \int_0^1 x^m (1-x^n)^p dx$

Putting $x^n = y \rightarrow n x^{n-1} = \frac{dy}{dx}$

$$I = \int_0^1 y^{m/n} (1-y)^p \frac{dy}{n\, y^{\left(\frac{n-1}{n}\right)}}$$

$$I = \int_0^1 \frac{y^{m/n} (1-y)^p}{n\, y^{(1-1/n)}} dy = \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n}-1\right)} (1-y)^p dy$$

$$I = \frac{1}{n} \beta\left[\left(\frac{m+1}{n}\right), p+1\right] =$$

$$\boxed{I = \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p+1\right)}}$$

12

Putting $m=6,\ n=2,\ p=1/2$

$$I = \frac{1}{2} \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(3/2\right)}{\Gamma(5)} = \frac{1}{2} \frac{\left(5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}\right)\left(1/2 \sqrt{\pi}\right)}{4!}$$

$$I = \frac{\cancel{15}^{5}\pi}{32\times\cancel{24}_{8}} = \frac{5\pi}{256}$$

Thus, $\boxed{\int_0^1 x^6 (1-x^2)^{1/2}\, dx = \frac{5\pi}{256}}$

3. (c) (i) Find the limiting points of the co-axial system of spheres determined by $x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 = 0$ and $x^2 + y^2 + z^2 - 18x + 27y - 36z + 29 = 0$.

(ii) Show that the plane $8x - 6y - z = 5$ touches the paraboloid $\left(\frac{x^2}{2}\right) - \left(\frac{y^2}{3}\right) = z$, and find the point of contact. **[20]**

i) $\boxed{\text{Co-axial system}}$

$$S_1 \equiv x^2+y^2+z^2-20x+30y-40z+29=0$$

$$S_2 \equiv x^2+y^2+z^2-18x+27y-36z+29=0$$

radical plane $P \equiv S_1 - S_2 = 0$

$$P \equiv 2x-3y+4z=0$$

18

Any sphere S in the system is given by

$$S \equiv S_1 + \lambda P = 0$$

$$S \equiv x^2+y^2+z^2+x(2\lambda-20)+y(30-3\lambda) + z(4\lambda-40)+29=0$$

$$S \equiv \left[x - (10-\lambda)\right]^2 + \left[y - \left(\frac{3\lambda-30}{2}\right)\right]^2 + \left[z - (20-2\lambda)\right]^2$$

$$= (10-\lambda)^2 + \left(\frac{30-3\lambda}{2}\right)^2 + (20-2\lambda)^2 - 29$$

For limiting point, radius of $S = 0$

i.e. $4(10-\lambda)^2 + (30-3\lambda)^2 + 4(20-2\lambda)^2 = 29 \times 4$ | -180 -320 -80

$29\lambda^2 - 580\lambda + 2900 = 29 \times 4$ | 400 900 1600

i.e. $\lambda^2 - 20\lambda + 100 = 4$ | 19 $\times 4$

$\lambda^2 - 20\lambda + 96 = 0$ | 96 $= 16 \times 6$ $= 8 \times 12$

$\lambda^2 - \cancel{28}\lambda - 12\lambda + 96 = 0$

$\boxed{\lambda = 8 \text{ or } 12}$

$\boxed{\text{Point of limiting} \equiv (2,-3,4) \text{ or } (-2,3,-4)}$

Paraboloid : $\frac{x^2}{2} - \frac{y^2}{3} - z = 0$

Gradient : $\left(x_0, -\frac{2y_0}{3}, -1\right)$

$\frac{8}{x_0} = \frac{-6}{-2y_0/3} = \frac{-1}{-1}$ $\Bigg\}$ $\boxed{\begin{array}{l} x_0 = 8 \\ y_0 = 9 \\ z_0 = 5 \end{array}}$ | 32 -27

Thus, $\boxed{8x - 6y - z = 5}$ is a tangent to the given paraboloid at $\boxed{(8, 9, 5)}$

Scanned with CamScanner

SECTION – B

5. (a) Find the condition on a, b, and c so that the following system in unknowns x, y and z has a solution.

$x + 2y - 3z = a$

$2x + 6y - 11z = b$

$x - 2y + 7z = c$

[10]

Consider the system of linear equations

$$\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{B}$$

Consider augmented matrix $[A|B]$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right]$$

08

Using row operations,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & c+2b-5a \end{array}\right]$$

The system has a solution if $\boxed{c + 2b - 5a = 0}$

56

10 marks

V vector space of 2x2 matrices over R

$M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. $F: V \to V$ is linear map

$F(A) = MA$. Find –

i) Basis & dimension of F

ii) Kernel W of F iii) Image U of F.

---

i) Basis of V = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Since they are linearly independent & span V.

$\boxed{\dim(V) = 4}$

ii) Kernel W of F: let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$

$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $a = c$ | $b = d$ |

Kernel W of F = $\begin{bmatrix} x & y \\ x & y \end{bmatrix}$ a 2-D subspace of 4-D V

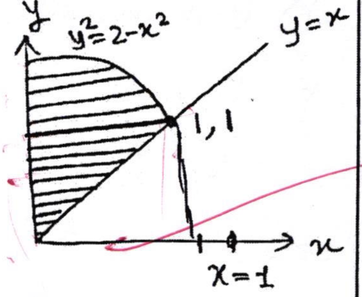
08

iii) Image U of F:

$$\boxed{U = \begin{bmatrix} a-c & b-d \\ -2(a-c) & -2(b-d) \end{bmatrix} = \begin{bmatrix} x & y \\ -2x & -2y \end{bmatrix}}$$ a 2D subspace of 4D V.

P.T.O.

5. (c) Evaluate the following integral $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}$ by changing the order of integration. **[10]**



Consider the given integral

$$I = \int_{x=0}^{1} \int_{y=x}^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

Changing the order of integration

$$I = \int_{y=1}^{\sqrt{2}} \int_{x=0}^{\sqrt{2-y^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} + \int_{y=0}^{1} \int_{x=0}^{y} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$

06

$$= \int_{y=1}^{\sqrt{2}} \left[x^2+y^2\right]^{1/2} \Big|_{x=0}^{2-y^2} dy + \int_{y=0}^{1} \left[x^2+y^2\right]^{1/2} \Big|_{x=0}^{y} dy$$

$$= \int_{y=1}^{\sqrt{2}} (\sqrt{2} - y)\,dy + \int_{y=0}^{1} (\sqrt{2}y - y)\,dy$$

$$= \sqrt{2} - \frac{3}{2} + (\sqrt{2}-1)\frac{1}{2}$$

$$\boxed{I = \frac{3}{2}\sqrt{2} - 2}$$

$\left(1 - \frac{1}{\sqrt{2}}\right)$ Ans

Scanned with CamScanner

5. (d) Show that the straight line whose direction cosines are given by the equations : $ul + vm + wn = 0$, $al^2 + bm^2 + cn^2 = 0$ are (α) perpendicular if $u^2(b + c) + v^2(c + a) + w^2(a + b) = 0$ and (β) parallel, if $(u^2/a) + (v^2/b) + (w^2/c) = 0$. **[10]**

Consider the lines given by DCs

$$ul + vm + wn = 0 \quad \Big| \quad aL^2 + bm^2 + cn^2 = 0$$

i.e. $$L = -\frac{(vm + wn)}{u} \Rightarrow \frac{a}{u^2}(vm + wn)^2 + bm^2 + cn^2 = 0$$

i.e. $$\left(\frac{m}{n}\right)^2\left(\frac{av^2}{u^2} + b\right) + 2\left(\frac{m}{n}\right)\left(\frac{avw}{u^2}\right) + \left(\frac{aw^2}{u^2} + c\right) = 0$$

$\boxed{\text{Parallel condition}}$ – only 1 unique root

$$B^2 = 4AC$$

i.e. $$\cancel{4}\left(\frac{avw}{u^2}\right)^2 = \cancel{4}\left(\frac{av^2}{u^2} + b\right)\left(\frac{aw^2}{u^2} + c\right)$$

i.e. $$a^2\cancel{v^2}w^2 = a^2\cancel{v^2 w^2} + ab\,u^2w^2 + cbu^4 + acu^2v^2$$

i.e. $$\boxed{\frac{w^2}{c} + \frac{u^2}{a} + \frac{v^2}{b} = 0}$$

$\boxed{\text{Perpendicular condition}}$ – $L_1L_2 + m_1m_2 + n_1n_2 = 0$

$$\frac{m_1m_2}{n_1n_2} = \cancel{2avw}\ \frac{aw^2 + cu^2}{av^2 + bu^2} \quad \Big| \quad \frac{L_1L_2}{m_1m_2} = \frac{cv^2 + bw^2}{aw^2 + cu^2}$$

$$L_1L_2 + m_1m_2 + n_1n_2 = k(cv^2 + bw^2 + aw^2 + cu^2 + av^2 + bu^2) = 0$$

i.e. $$\boxed{u^2(b + c) + v^2(c + a) + w^2(a + b) = 0}$$

5. (e) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle. [10]

Consider the circle given by –

$$S \equiv x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$$

$$P \equiv 2x + 3y + 4z - 8 = 0$$

Any sphere s' through this circle is given by $s' \equiv S + \lambda P = 0$ where λ is a parameter

i.e. $s' \equiv x^2+y^2+z^2 + 2\lambda x + (7+3\lambda)y + (4\lambda-2)z + 2 - 8\lambda = 0$

The given circle is a great circle of s' if & only if centre of s' lies on P.

i.e. $c \equiv \left(-\lambda, -\frac{(7+3\lambda)}{2}, 1-2\lambda\right)$ lies on P.

i.e. $-2\lambda - \frac{3}{2}(7+3\lambda) + 4(1-2\lambda) = 8$

i.e. $\boxed{\lambda = -1}$

$\boxed{s' \equiv x^2+y^2+z^2 - 2x + 4y - 6z + 10 = 0}$ is the required sphere.

08

7. (a) Find the values of a and b such that

$$\lim_{x\to 0}\frac{a\sin^2 x+b\log\cos x}{x^4}=\frac{1}{2}$$ [10]

Consider the limit $\lim_{x\to 0}\left[\frac{a\sin^2 x + b\log\cos x}{x^4}\right]$

It is of 0/0 form → use of L'Hospitale's rule

$$L = \lim_{x\to 0}\left(\frac{2a\sin x\cos x + \frac{b}{\cos x}(-\sin x)}{4x^3}\right) \quad (0/0 \text{ form})$$

$$= \lim_{x\to 0}\left[\frac{2a\cos 2x - b\sec^2 x}{12x^2}\right]$$

In a nbd of $x=0$, taylor series for

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} \dots$$

$$\sec^2 x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

$$L = \lim_{x\to 0}\left[\frac{2a(1-2x^2+\dots) - b(1+x^2+\dots)}{12x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2a-b) - x^2(4a+b)}{12x^2}\right] = 1/2$$

$\therefore b = 2a$ and $4a+b = -6 \Rightarrow \boxed{\begin{array}{l} a=-1 \\ b=-2 \end{array}}$

08

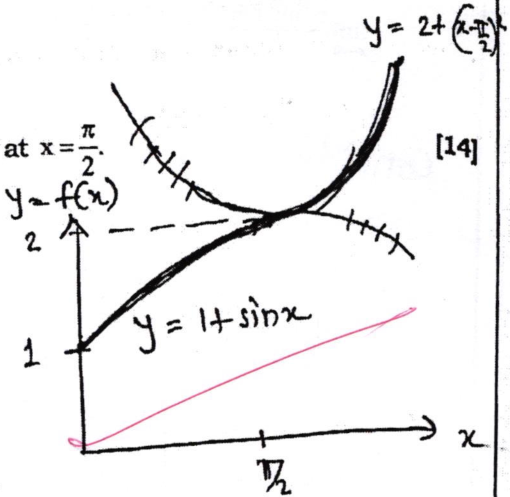
Scanned with CamScanner

7. (b) A function f(x) is defined as follows :

$$f(x) = 1 + \sin x \text{ for } 0 < x < \frac{\pi}{2}$$

$$f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2 \text{ for } x \geq \frac{\pi}{2}$$

Examine its continuity and derivability at $x = \frac{\pi}{2}$. **[14]**



Consider the given function

$$f(x) = \begin{cases} 1 + \sin x, & 0 < x < \pi/2 \\ 2 + (x - \pi/2)^2, & x \geq \pi/2 \end{cases}$$

$\boxed{\text{Continuity at } x = \pi/2}$

Left limit : $= \lim_{h \to 0^-} [f(x)] = \lim_{h \to 0^+} \left[f\left(\frac{\pi}{2} - h\right)\right]$

$= \lim_{h \to 0^+} \left[1 + \sin\left(\pi/2 - h\right)\right] = \lim_{h \to 0^+} (1 + \cos h) = 2$

12

Right limit $= \lim_{h \to 0^+} \left[f\left(\pi/2 + h\right)\right] =$

$= \lim_{h \to 0^+} \left[2 + \left(\frac{\pi}{2} + h - \pi/2\right)^2\right]$

$= \lim_{h \to 0^+} (2 + h^2) = 2$

Thus, at $x = \pi/2$, left limit $=$ right limit $= f(\pi/2)$
$= 2$

Thus, $\boxed{f(x) \text{ is continuos at } x = \pi/2}$

$\boxed{\text{Derivability}}$

Left limit : $\lim_{h\to 0^-}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0^+}\frac{f(\pi/2-h)-f(\pi/2)}{(-h)}$

$$= \lim_{h\to 0^+}\left[\frac{1+\sin(\pi/2-h) - 2+(\pi/2-h-\pi/2)^2}{(-h)}\right]$$

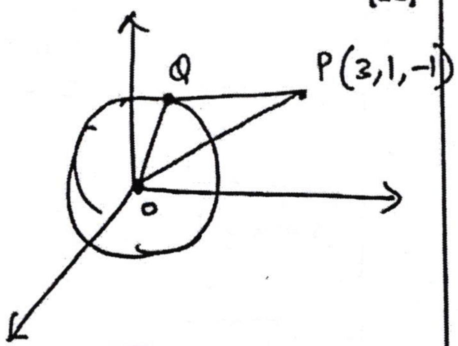
$$= \lim_{h\to 0^+}\left[\frac{1+\cos h - 2 + h^2}{(-h)}\right] = 0$$

Right limit : $\lim_{h\to 0^+}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0^+}\left[\frac{2+h^2-2}{h}\right] = 0$

Thus, left limit = right limit = 0

Hence, $\boxed{f(x) \text{ is derivable at } x=\pi/2 \text{ with } f'(\pi/2)=0}$

7. (c) Find the points on the sphere $x^2+y^2+z^2=4$ that are closest to and farthest from the point $(3, 1, -1)$. [13]



Consider any point Q on the sphere

By triangle inequality,

$OP \le PQ + OQ$

i.e. $\boxed{PQ \geq OP - OQ}$

Thus, minima occurs when $PQ = OP - OQ$ and O–Q–P are collinear.

i.e. $Q \equiv (3k, k, -k) \Rightarrow Q = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right)$

and $\boxed{PQ = \sqrt{11} - 2}$

From triangle inequality, we also have,

$$PQ \leq OP + OQ$$

Maxima occurs when $PQ = OP + OQ$ and

$Q - O - P$ are collinear

i.e. $Q \equiv \left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

$$\boxed{PQ = \sqrt{11} + 2}$$

Summary of results

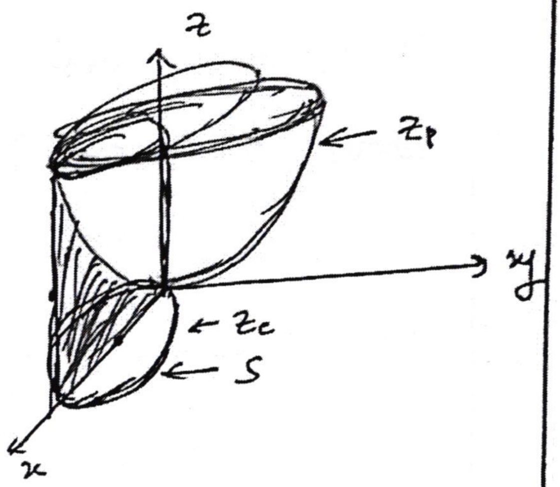
	Maxima of PQ	Minima of PQ
Value :	$\sqrt{11} + 2$	$\sqrt{11} - 2$
Point Q :	$\left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$	$\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right)$

7. (d) Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by xy-plane. [13]

Cylinder : $(x-1)^2+y^2=1$

Paraboloid : $x^2+y^2=2z$

As shown in the figure



$$\text{Volume } V = \iint_S (z_p - z_c)\,dx\,dy$$

$$= \iint_S \left(\frac{x^2+y^2}{2} - 0\right) dx\,dy$$

substituting $x = 1+r\cos\theta$, $y = r\sin\theta$

$$V = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} \frac{(1+r\cos\theta)^2+(r\sin\theta)^2}{2}\, r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{r=0}^{1}\int_{\theta=0}^{2\pi} (1+r^2+2r\cos\theta)\,r\,dr\,d\theta$$

$$= \frac{1}{\cancel{2}}\int_{r=0}^{1} (2\pi r + r^3 \times \cancel{2}\pi)\,dr$$

$$= \pi\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{3\pi}{4}$$

Scanned with CamScanner

Thus, the volume of the enclosed solid above xy-plane is $\boxed{V = \frac{3\pi}{4} \text{ units}}$