

# MATHEMATICS

## PAPER - I

### SECTION A

1. Answer any four of the following:

- (a) Let  $S_1, S_2, \dots, S_k$  be subspaces of a vector space  $V(F)$ . Show that the following statements are equivalent: (10)

- (i)  $S_1 + S_2 + \dots + S_k$  is a direct sum of  $V(F)$ .
- (ii)  $(S_1 + S_2 + \dots + S_i) \cap S_{i+1} = \{0\}$ , for  $i = 1, 2, \dots, k-1$ .
- (iii)  $x_1 + x_2 + \dots + x_k = 0, x_i \in S_i, i = 1, 2, \dots, k \Rightarrow x_i = 0$  for  $i = 1, 2, \dots, k$
- (iv)  $d(S_1 + S_2 + \dots + S_k) = d(S_1) + d(S_2) + \dots + d(S_k)$ .

(b) Given  $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \\ 2 & 2 \end{bmatrix}$

For which values of  $\alpha$  does the vector sequence  $\{y_n\}_0^\infty$  defined by

$$y_n = (I + \alpha A + \alpha^2 A^2) y_{n-1}, n = 1, 2, \dots \text{ converge to } 0 \text{ as } n \rightarrow \infty?$$

- (c) Find the extremum values of  $x^2 + y^2$  subject to the condition  $3x^2 + 4xy + 6y^2 = 140$ . (10)

- (d) Show that any two circular sections of an ellipsoid of opposite systems lie on a sphere. (10)

- (e) Prove that

$$2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right) \sqrt{\pi} \Gamma(2x). \quad (10)$$

2. (a) Show that if  $\lambda$  is a non-zero eigen value of the non-singular  $n$ -square matrix  $A$ , then  $\frac{|A|}{\lambda}$  is an eigen value of  $\text{adj } A$ .

Also given an example to prove that the eigen values of  $AB$  are not necessarily the product of eigen values of  $A$  and that of  $B$ . (10)

- (b) Given the liner transformation

$$Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -2 & 3 & 5 \end{bmatrix} X,$$

show that it is singular and the images of the linearly independent vectors

$$X_1 = [1, 1, 1]'$$

$$X_2 = [2, 1, 2]'$$

$$X_3 = [1, 2, 3]'$$

are linearly dependent.

(10)

- (c) (i) Calculate  $f(A) = e^A - e^{-A}$  for

$$A = \begin{bmatrix} 2 & 4 & 0 \\ 6 & 0 & 8 \\ 0 & 3 & -2 \end{bmatrix}$$

(5)

- (ii) The  $n \times n$  matrix A satisfies

$$A^4 = -1.6A^2 - 0.64I.$$

Show that  $\lim_{n \rightarrow \infty} A^n$  exists and determine this limit.

(5)

- (d) Reduce  $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$  to normal form N and compute the matrices P and Q such that  $AQ=N$ .

(10)

3. (a) Find the volume and centroid of the region in the first octant bounded by  $6x+3y+2z=6$ .

(10)

- (b) If  $f(x) = e^{-\frac{x^2}{2}}$  and  $g(x) = xf(x)$  for all x, prove that

$$f(y) = \sqrt{\frac{2}{\pi}} \int_0^y f(x) \cos xy dx$$

$$g(y) = \sqrt{\frac{2}{\pi}} \int_0^y g(x) \sin xy dx$$

(10)

- (c) If  $\sin^{-1} x + \sin^{-1} y$  and

$$v = x\sqrt{1-y^2} + \sqrt{1-x^2},$$

determine whether there is a functional relationship between u and v, and if so find it.

(10)

- (d) If  $f(x)$  is monotonic in the interval  $0 < x < a$ , and the integral  $\int_0^a x^r f(x) dx$  exists, then show that  $\lim_{x \rightarrow 0} x^{r+1} f(x) = 0$ .

(10)

4. (a) Prove that the locus of the line of intersection of perpendicular tangent planes to the cone

$ax^2 + by^2 + cz^2 = 0$  is the cone.

$$a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0$$

(10)

- (b) Prove that two normals to the ellipsoid  $\frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{2} = 1$  lie in the plane  $x + 2y + 3z = 0$ , and the line joining their feet has direction cosines proportional to 12, -9, 2.

(10)

- (c) Prove that the shortest distances between the diagonals of a rectangular parallelopiped and edges not meeting them are

$$\frac{bc}{\sqrt{b^2 + c^2}}, \frac{ca}{\sqrt{c^2 + a^2}} \text{ and } \frac{ab}{\sqrt{a^2 + b^2}} \text{ where } a, b, c \text{ are lengths of the edges of a parallelopiped.}$$

- (d) Find the equation of the enveloping cylinder of the sphere  $x^2 + y^2 + z^2 - 2y - 4z = 1$  having its generator parallel to the line  $x = 2y = 2z$ .

(10)

## SECTION B

5. Answer any four of the following:

- (a) If  $(D-a)^n e^{ax}$  is denoted by  $z$ , prove that  $z, \frac{\partial z}{\partial n}, \frac{\partial^2 z}{\partial n^2}, \frac{\partial^3 z}{\partial n^3}$  all vanish when  $n = a$ . Hence, show that  $e^{ax}, xe^{ax}, x^2e^{ax}, x^3e^{ax}$  are all solutions of  $(D-a)^n y = 0$ . Here D stand for  $\frac{d}{dx}$ .

(10)

- (b) Solve  $4x(D^2 + (3x+1)^2) = 0$  and examine for singular solutions and extraneous loci.  
Interpret the results geometrically.

(10)

- (c) Find the curvature and torsion of the curve

$$x = \frac{2t+1}{t-1}, y = \frac{t^2}{t-1}, z = t+2. \text{ interpret your answer.}$$

(10)

- (d) An ellipse is just immersed with its major axis vertical. Show that the centre of pressure coincides with the lower focus, the eccentricity of the ellipse being  $\frac{1}{4}$ .

(10)

- (e) A particle falls towards the earth from infinity. Show that its velocity on reaching the earth is the same as it would have acquired in falling with constant acceleration  $g$  through a distance equal to the earth's radius.

(10)

6. (a) (i) From the differential equation whose primitive is

$$y = A \left( \sin x + \frac{\cos x}{x} \right) + B \left( \cos x - \frac{\sin x}{x} \right)$$

(5)

- (ii) Prove that the orthogonal trajectory of system of parabolas belongs to the system itself.

(5)

- (b) Using variation of parameters solve the differential equation

$$\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1)y = 3e^x \sin 2x.$$

- (c) (i) Solve the equation by finding an integrating factor of

$$(x+2)\sin y \, dx + x \cos y \, dy = 0$$

(5)

- (ii) Verify that  $\phi(x) = x^2$  is a solution of  $y'' - \frac{2}{x^2}y = 0$  and find a second independent solution.

- (d) Show that the solution of  $(D^{2n+1} - 1)y = 0$ , consists of  $Ae^x$  and  $n$  pairs of terms of the form  $e^{ax}(b \cos \alpha x + c \sin \alpha x)$ ,

Where  $a = \cos \frac{2\pi r}{2n+1}$  and

$$\alpha = \sin \frac{2\pi r}{2n+1}, r = 1, 2, \dots, n \text{ and } b, c_r \text{ are arbitrary constants.}$$

(10)

7. (a) A particle moves with a central acceleration  $\frac{\mu}{(\text{distance})^3}$  and is projected from an apse at a distance  $a$  with velocity equal to  $a$  times that would be acquired in falling from infinity. Show that the other apsidal distance is  $\frac{a}{\sqrt{n^2 - 1}}$ .

(10)

- (b) A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find the positions of equilibrium and show that its length must be at least

$$\frac{3a}{4} + \frac{\alpha}{4}\sqrt{1+8e^2}, \text{ where } 2a \text{ is the major axis and } e \text{ the eccentricity.}$$

(10)

- (c) A spherical raindrop, falling freely, receives in each instant an increase of volume equal to  $n$  times its surface at that instant. Find the velocity at the end of time  $t$ , and the distance fallen through in that time if  $r_0$  be the initial radius of the raindrop.

(10)

- (d) If a planet were suddenly stopped in its orbit, supposed circular, show that it would fall into the sun in a time which is  $\frac{\sqrt{2}}{8}$  times the period of the planet's revolution. (10)
8. (a) State Stoke's theorem and then verify it for  $\vec{A} = (x^2 + 1)\vec{i} + xy\vec{j}$  integrated round the square in the plane  $z = 0$  whose sides are along the lines  $x = 0, y = 0, x = 1, y = 1$ . (10)
- (b) Prove that
- $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla}) \vec{A} - \vec{B} (\vec{\nabla} \cdot \vec{A}) - (\vec{A} \cdot \vec{\nabla}) \vec{B} + \vec{A} (\vec{\nabla} \cdot \vec{B})$  (6)
  - $\text{Curl } \frac{\vec{a} \times \vec{r}}{r^3} = -\frac{\vec{a}}{r^3} + \frac{3\vec{r}}{r^5} (\vec{a} \cdot \vec{r}),$   
 $\vec{a} = \text{constant vector}$  (4)
- (c) Show that if  $A \neq \vec{0}$  and both of the conditions  $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$  and  $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$  hold simultaneously then  $\vec{B} = \vec{C}$ , but if only one of these conditions holds then  $\vec{B} \neq \vec{C}$  necessarily. (10)
- (d) Prove the following:
- If  $u_1, u_2, u_3$  are general coordinates, then  
 $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3}$  and  $\vec{\nabla} u_1, \vec{\nabla} u_2, \vec{\nabla} u_3$  are  
 Reciprocal system of vectors. (5)
  - $\left( \frac{\partial \vec{r}}{\partial u_1} \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} \right) \left( \vec{\nabla} u_1, \vec{\nabla} u_2 \times \vec{\nabla} u_3 \right) = 1$  (5)

# MATHEMATICS

## PAPER - II

### SECTION A

1. Answer any four parts : (10 × 4 = 40)
- Show that every group consisting of four or less than four elements is abelian.
  - If  $f(z)$  has a simple pole with residue  $K$  at the origin and is analytic in  $0 < |z| \leq 1$ , show that
 
$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)(z-b)} dz = \frac{f(a)-f(b)}{a-b} + \frac{K}{ab}$$
 where  $0 < a, b < 1$  and  $C$  is the circle  $|z| = 1$ .
  - Evaluate  $\iiint_V \sqrt{1-z} dx dy dz$  though the volume bounded by the surfaces  $x = 0, y = 0, z = 0$ , and  $x + y + z = 1$ .
  - Explain a basic solution and a basic feasible solution of a linear programming problem. Determine the basic matrices and find the basic feasible solutions of the linear equations
 
$$\begin{aligned}x_1 + 2x_2 + x_3 &= 4 \\2x_1 + x_2 + 5x_3 &= 5.\end{aligned}$$
  - Define a compact set. Prove that the range of a continuous function defined on a compact set is compact.
2. (a) In the symmetric group  $S_n$  of permutations of  $n$  symbols, find the number of even permutation. Show that the set  $A_n$  of even permutations forms a finite group. Identify  $S_n$  and  $A$  when  $n = 4$ . (14)
- A function  $f$  is defined in  $[0, 1]$  as
 
$$f(x) = (-1)^{r-1}, \frac{1}{r+1} < x < \frac{1}{r}$$
 When  $r$  is a positive integer.  
 Show that  $f(x)$  is Riemann-integrable in  $[0, 1]$  and find its Riemann-integral.
  - A function  $f$  is defined as
 
$$f(x, y) = \frac{x_1 y_1}{x^2 + y^2}, (x, y) \neq (0, 0)$$

$$= 0 \quad , \quad \text{otherwise.}$$
 Prove that  $f_{x_1}(0, 0) = f_{y_1}(0, 0)$  but neither  $f_{x_1}$  nor  $f_{y_1}$  is continuous at  $(0, 0)$ . (13)
3. (a) If  $F$  is the finite field and  $\alpha, \beta$  are two non-zero elements of  $F$ , then show that there exist elements  $a$  and  $b$  if  $F$  such that  $1 + \alpha a^2 + \beta b^2 = 0$ . (14)

(b) If  $f(z) = \oint_C \frac{3z^2 + 7z + 1}{z - a} dz$

Where C is the circle  $|z| = 2$ ; find

- (i)  $f(1-i)$ ;
- (ii)  $f''(1-i)$ ;
- (iii)  $f(1+i)$ .

(3 × 4 = 12)

- (c) Under the bilinear transformation

$$w = \frac{3-z}{z-2}$$

find the images of

(i)  $\left|z - \frac{5}{2}\right| = \frac{1}{2}$  and

(ii)  $\left|z - \frac{5}{2}\right| < \frac{1}{2}$

in the w-plane.

(14)

4. (a) Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true.  
 (b) Use simplex method to solve the following linear programming problem:

$$\text{Max. } Z = 5x_1 + 7x_2$$

Subject to the constraints,

$$2x_1 + 3x_2 \leq 18$$

$$3x_1 + 2x_2 \leq 12$$

$$x_i \geq 0; i = 1, 2$$

(14)

- (c) A company has three factories and four warehouses. The production capacities of the factories are 7, 9, 18 respectively and the warehouses require 5, 8, 7, 14 respectively. The per unit transportation cost is given in the matrix below:

		Warehouse				Factory capacity
		H <sub>1</sub>	H <sub>2</sub>	H <sub>3</sub>	H <sub>4</sub>	
Factory	I	15	30	50	10	7
	II	70	30	40	60	9
	III	40	8	70	20	18
Requirements		5	8	7	14	

Find the allocation so that the transportation cost is minimum.

(13)

## SECTION B

5. Answer any four parts: (10 × 4 = 40)

- (a) Solve completely:

$$\frac{x}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}.$$

- (b) From the data given below:

x	0	1	2	4	5	6
f(x)	1	14	15	5	6	19

using Lagrange's interpolation formula calculate  $f(3)$

- (c) Convert  $(135.34)_8$  to decimal number and then convert the resulting decimal number to binary number.  
 (d) Find the moment of inertia of an elliptic lamina of axes  $2a$  and  $2b$  about a line OP passing through the centre O and inclined at  $\theta$  to major axis  $2a$ . Also find the moment of inertia about a tangent to the ellipse parallel to OP.  
 (e) Show that the velocity potential

$$\phi = \frac{1}{2} (x^2 + y^2 - 2z^2)$$

satisfies Laplace equation. Determine the stream lines of the flow.

6. (a) Using Charpit's method solve completely  $p^2 - q^2 = (x+y)^2$ .

(13)

- (b) Obtain the general solution of the following equation:

(13)

$$\frac{\partial^2 z}{\partial x^2} - 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = (2+4x)e^{x+2y}.$$

- (c) A uniform heavy rod of length  $2a$  and mass  $M$  is free to turn about an end which is fixed. Using Lagrange's equations of motion obtain the ordinary differential equations for the oscillation of the rod.

(14)

7. (a) Express Lagrange's equations of motion in generalised coordinates under conservative system of forces. If the geometrical equations do not contain time explicitly, deduce the principle of Energy  $T + V = \text{constant}$ .

(13)

- (b) Examine whether the motion specified by  $\ddot{q} = \frac{K^2(x\dot{y} - y\dot{x})}{x^2 + y^2}$ ,  $K$  a constant is a possible liquid motion. Also show that the motion is of potential type.

(13)

- (c) Solve the following system of equations by Gauss's elimination method:

$$10x - 7y + 3z + 5w = 6$$

$$-6x + 8y - z - 4w = 5$$

$$3x + y + 4z + 11w = 2$$

$$5x - 9y - 2z + 4w = 7$$

(14)

8. (a) Write a BASIC program to find

- (i) The sum of first N natural numbers
- (ii) The factorial of a given number

(2 × 7=14)

- (b) Given the differential equation

$$\frac{dy}{dx} = xy ; y = 2, \text{ when } x = 1,$$

use Runge-Kutta fourth order rule to find y at  $x = 1.2$  taking the step length  $h = 0.2$ .

(13)

- (c) Show that the velocity field

$$u(x, y) = \frac{K(x^2 - y^2)}{(x^2 + y^2)^2}, v = \frac{2 Kxy}{(x^2 + y^2)}, w = 0$$

where K is a constant, satisfies the equations of motion for steady, inviscid, incompressible flows in the absence of external forces.

(13)