IAS MATHEMATICS (OPT.)-2008

PAPER - I : SOLUTIONS

12M. 2008 (i) Show that the matrix A is invertible if and only if the adj(A) is invertible. Hence find $|adj(A)|$.

1.(a)

Sol. Let A be an invertible matrix of order $n \times n$.

$\Leftrightarrow A^{-1}$ exists

$\Leftrightarrow A \cdot \frac{(adj A)}{|A|} = I_n$

$\Leftrightarrow A \cdot adj A = |A| I_n$.

$\Leftrightarrow |A \cdot adj A| = \big| |A| I_n \big|$.

$\Leftrightarrow |A| \, |adj A| = \big||A|\big| \, |I_n|$

$\Leftrightarrow |A| \, |adj A| = |A|^n \cdot 1 \quad (\because |KA| = k^n |A| \ \& \ |I_n| = 1)$

$\Leftrightarrow |adj A| = |A|^{n-1}$

$\Leftrightarrow |adj A| \neq 0 \quad (\because |A| \neq 0)$.

Clearly $|adj A| = |A|^{n-1}$.

12M 2008 (2) Let S be a nonempty set and let V denote the set of all functions from S into $\mathbb{R}$. Show that V is a vector space with respect to the vector addition $(f+g)(x) = f(x)+g(x)$ and scalar multiplication $(cf)(x) = cf(x)$.

1.(b)

Sol Let $V = \{f \mid f: S \to \mathbb{R}\}$

Then we have to show that V is a vectorspace w.r.t to the vector addition (i) $(f+g)(x) = f(x)+g(x)\ \forall x \in S$

scalar multiplication (ii) $(cf)(x) = cf(x)\ \forall x \in S$, $c \in \mathbb{R}$.

$\forall f, g \in V \Rightarrow (f+g)(x) = f(x)+g(x)\ \forall x \in S$ (By defn)

since $f(x), g(x) \in \mathbb{R}$ and $\mathbb{R}$ is a field

$\Rightarrow f(x)+g(x) \in \mathbb{R}$

$\therefore (f+g)(x) = f(x)+g(x) \in \mathbb{R}$

$\therefore (f+g) : S \to \mathbb{R}$

$\therefore f+g \in V$.

Internal composition is satisfied.

$\forall f \in V, c \in F \Rightarrow (cf)(x) = cf(x)\ \forall x \in S$ (by def)

since $f(x) \in \mathbb{R}$, $c \in \mathbb{R}$ and $\mathbb{R}$ is a field

$\therefore cf(x) \in \mathbb{R}$

$\therefore cf : S \to \mathbb{R}$

$\Rightarrow cf \in V \ \forall \ c \in \mathbb{R}, f \in V.$

$\therefore$ External composition is satisfied.

(I) (i) $\forall \ f, g \in V \Rightarrow f+g \in V$

$\therefore$ closure prop. is satisfied.

(ii) $\forall \ f, g, h \in V$

$\Rightarrow [(f+g)+h](x) = (f+g)(x)+h(x)$

$= f(x)+[g(x)+h(x)]$

$= [f+(g+h)](x)$ (by associative property of $\mathbb{R}$)

$\therefore (f+g)+h = f+(g+h)$

$\therefore$ Associative property is satisfied.

(iii) If $\forall x \in S$, $I(x) = 0 \in \mathbb{R}$ then $I \in V$ (i.e. $I : S \to \mathbb{R}$)

Now $(I+f)(x) = I(x) + f(x)$

$= 0 + f(x)$

$= f(x) \ \forall \ x \in \mathbb{R}$

$\therefore I+f = f \ \forall \ f \in V.$

Similarly $f+I = f \ \forall f \in V$

$\therefore \forall \ f \in V, \exists \ I \in V$ such that

$I+f = f+I = f \ \forall \ f \in V$

$\therefore$ Identity element $= I \in V$.

(iv) If $f \in V$ then $-f \in (-1)f \in V$

now $[f+(-f)](x) = f(x) + (-f)(x)$
$= f(x) + [-1 f(x)]$
$= f(x) - f(x)$
$= 0 = I(x)$

$\therefore f+(-f) = 0 = I$

Similarly $(-f)+f = 0 = I$

$\therefore f+(-f) = -f+f = 0 = I$

$\therefore$ Inverse of f is $-f$ in V.

(v) $\forall f, g \in V \Rightarrow (f+g)(x) = f(x) + g(x)$ (By definition)
$= g(x) + f(x)$ (By commutative property in $\mathbb{R}$ i.e. $f(x), g(x) \in \mathbb{R}$ i.e. $f(x)+g(x) = g(x)+f(x)$)
$= (g+f)(x)$

$\therefore f+g = g+f$

$\therefore$ Commutative property is satisfied.

$\boxed{\text{II}}$. $\forall a, b \in \mathbb{R}$, $f, g \in V$

(i) $[a(f+g)](x) = a(f+g)(x)$ (By definition)
$= a(f(x) + g(x))$
$= af(x) + ag(x)$ (By left distributive law in $\mathbb{R}$)
$= (af)(x) + (ag)(x)$
$= (af+bg)(x)$

$\therefore a(f+g) = af+bg$

(ii) $[(a+b)f]x = (a+b)f(x)$
$= af(x) + bf(x) = (af)(x) + (bf)(x)$
$= (af+bf)(x)$.

$\therefore (a+b)f = af+bf$

(iii) $[(ab)f](x) = (ab)f(x)$ (by definition)
$= a(b f(x)) = a(bf)(x)$

$\therefore (ab)f = a(bf)$

(iv) $(1.f)(x) = 1.f(x) = f(x)$ $\forall f(x) \in \mathbb{R}$.

$\therefore 1f = f$ $\forall f \in \mathbb{R}$

$V(\mathbb{R})$ is a vector space.

12M
2008 → Find the value of $\lim_{x \to 1} \ln(1-x) \cot \frac{\pi x}{2}$

1.(c)

Sol^n: $\lim_{x \to 1} \ln(1-x) \cot\frac{\pi x}{2} = \lim_{x\to 1} \frac{\ln(1-x)}{\tan\frac{\pi x}{2}}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

$= \lim_{x\to 1} \frac{\frac{1}{1-x}(-1)}{\sec^2\frac{\pi x}{2} \cdot \frac{\pi}{2}}$ (by L'Hospital rule)

$= \lim_{x\to 1} \frac{\frac{1}{x-1}}{\frac{\pi}{2}\sec^2\frac{\pi x}{2}}$

$= \lim_{x\to 1} \frac{\frac{2}{\pi}\cos^2\frac{\pi}{2}x}{x-1}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 1} \frac{\frac{2}{\pi}\cdot 2\cos\frac{\pi}{2}x\left(-\sin\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}}{1}$ (by L'Hospital rule)

$= \lim_{x\to 1} -\sin\pi x$

$= 0$

2008

12M

1.(e) The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$; find the equation of the plane in its new position.

Sol'n: The given planes are $x-2y+3z=0$ —— ①

$2x+3y-4z+5=0$ —— ②

Now the equation of any plane through the intersection of ① and ② is

$(x-2y+3z)+k(2x+3y-4z+5)=0$ —— ③

If this be the equation of the plane ① in its new position, then the planes ① and ③ are at right angles.

Now the d.r's of the normals to the planes ① and ③ are

$1,-2,3$ and $1+2k,\ -2+3k,\ 3-4k$.

Since the two planes ① and ③ are perpendicular.

$\therefore$ Their normals are also perpendicular.

Hence $1(1+2k)+(-2)(-2+3k)+3(3-4k)=0$

$\Rightarrow 1+2k+4-6k+9-12k=0$

$\Rightarrow 14-16k=0$

$\therefore \boxed{k=7/8}$.

From ③, we have

$x-2y+3z+\frac{7}{8}(2x+3y-4z+5)=0$

$8x-16y+24z+14x+21y-28z+35=0$

$\Rightarrow 22x+5y-4z+35=0$

which is the required equation of the plane.

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1.(f) find the equations (in symmetric form) of the tangent line to the sphere $x^2+y^2+z^2+5x-7y+2z-8=0$, $3x-2y+4z+3=0$ at the point $(-3,5,4)$.

Sol'n: The given sphere is

$$x^2+y^2+z^2+5x-7y+2z-8=0 \quad \text{——} \quad (1).$$

and plane of the circle is $3x-2y+4z-3=0$ —— (2)

Now equation of the tangent plane at $P(-3,5,4)$ to the sphere (1) is

$$x(-3)+y(5)+z(4)+\tfrac{5}{2}(x-3)-\tfrac{7}{2}(y+5)+1(z+4)-8=0$$

$$\Rightarrow -x+3y+10z-58=0$$

$$\Rightarrow x-3y-10z+58=0 \quad \text{——} \quad (3)$$

The equations (2) & (3) taken together represent the equation of the tangent line to the circle given by (1) & (2).

To find the d.r's of the tangent line:

omitting the constant terms in (2) & (3), the equations are

$$3x-2y+4z=0$$
$$x-3y-10z=0$$

$$\therefore \frac{x}{20+12}=\frac{y}{4+30}=\frac{z}{-9+2}$$

$$\Rightarrow \frac{x}{32}=\frac{y}{34}=\frac{z}{-7}$$

$\therefore$ The d.r's of the tangent line are $32, 34, -7$.

Also the tangent line passes through the given point $P(-3,5,4)$.

$\therefore$ The equations of the tangent line to the circle at $P(-3,5,4)$ are $\dfrac{x+3}{32}=\dfrac{y-5}{34}=\dfrac{z-4}{-7}$

2013, 2008

2.(a) (3) Show that $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0)$, $T(1,1,0) = (1,1,1)$ and $T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$.

Sol Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be the given vectorspace over the field $\mathbb{R}$.

Let $B = \{(1,0,0),(1,1,0),(1,1,1)\} \subseteq \mathbb{R}^3$

To show that B is a basis of $\mathbb{R}^3$:

We know that $\dim \mathbb{R}^3 = 3$.

Now we construct a matrix whose rows are given vectors of S and reduce it into echelon form.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Clearly which is in echelon form and the number of non-zero rows are equal to 3.

$\therefore$ The given three vectors of B are linearly independent vectors and $\dim \mathbb{R}^3 = 3$.

$\therefore$ B forms a basis of $\mathbb{R}^3$.

To find $T(x,y,z)$:

Since B is a basis of $\mathbb{R}^3$

$\therefore$

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$\therefore$ there exists unique LT $T: \mathbb{R}^3 \to \mathbb{R}^3$

such that $T(1,0,0) = (1,0,0)$

$T(1,1,0) = (1,1,1)$

$T(1,1,1) = (1,1,0)$.

Since $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $\mathbb{R}^3$

$\therefore \alpha = (x,y,z) \in \mathbb{R}^3$; $x,y,z \in \mathbb{R}$,

$\Rightarrow (x,y,z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$. ——— (1)

$a,b,c \in \mathbb{R}$.

$a+b+c = x \Rightarrow \boxed{a = x-y}$

$b+c = y \Rightarrow \boxed{b = y-z}$

$\boxed{c = z}$

$\therefore$ (1) $\equiv$

$(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$.

$\Rightarrow T(x,y,z) = (x-y)\,T(1,0,0) + (y-z)\,T(1,1,0) + z\,T(1,1,1)$ $(\because T$ is a LT$)$

$= (x-y)(1,0,0) + (y-z)(1,1,1) + z(1,1,0)$

$= (x-\cancel{y}+\cancel{y}-\cancel{z}+\cancel{z},\ y-\cancel{z}+\cancel{z},\ y-z)$.

$\therefore T(x,y,z) = (x, y, y-z)$.

which is the required explicitly condition of linear transformation. ✓

I 2008 20M

2.(b) Determine the maximum and minimum distances of the origin from the curve given by the equation 1.

$3x^2+4xy+6y^2=140$

Sol^n: We have to find the maximum and minimum value of x^2+y^2 (the square of the distance from the origin to any point in the xy plane) subject to the constraint

$$3x^2+4xy+6y^2=140$$

Consider the function

$$F = x^2+y^2+\lambda(3x^2+4xy+6y^2-140)$$

$$\therefore F_x = 2x+\lambda(6x+4y),\quad F_y = 2y+\lambda(4x+12y).$$

For stationary values, $F_x=0$, $F_y=0$

$$\therefore (1+3\lambda)x+2\lambda y=0 \qquad \text{—— (1)}$$

$$2\lambda x+(1+6\lambda)y=0 \qquad \text{—— (2)}$$

From (1) and (2), since $(x,y)\neq 0$; We must have

$$\begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix}=0 \text{ or } (1+3\lambda)(1+6\lambda)-4\lambda^2=0$$

or $$14\lambda^2+9\lambda+1=0 \Rightarrow (7\lambda+1)(2\lambda+1)=0$$

$$\lambda = \frac{-1}{2}, -\frac{1}{7}$$

putting $\lambda=-\frac{1}{2}$ in (1) gives $x+2y=0$ or $x=-2y$

putting $x=-2y$ in $3x^2+4xy+6y^2=140$ gives $y^2=14$

and $x^2=4\times 14=56 \Rightarrow x^2+y^2=70$

putting $\lambda = -\frac{1}{7}$ in (1) gives $y = 2x$

putting $y = 2x$ in $3x^2 + 4xy + 6y^2 = 140$ gives

$x^2 = 4$ and $y^2 = 4 \times 4 = 16 \Rightarrow x^2 + y^2 = 20$

Hence the maximum value $= 70$, minimum value $= 20$.

2.(c)

Q.) A sphere S has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere having the ~~ends~~ intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

A.) Diameter's end points $\rightarrow (0,1,0), (3,-5,2)$

Diameter form of sphere 'S' $\rightarrow (x-0)(x-3)+(y-1)(y+5)$
$$+(z-0)(z-2)=0$$

Equation of sphere 'S' $\rightarrow x^2-3x+y^2+4y-5+z^2-2z=0$

$$S:\ x^2+y^2+z^2-3x+4y-2z-5=0$$

Equation of Plane, $P:\ 5x-2y+4z+7=0$

Let the new sphere passing through the intersection of sphere 'S' and plane 'P', be S_1.

$$\therefore S_1 = S+\lambda P = x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z$$
$$+(-5+7\lambda)=0$$

Center of the sphere 'S_1' : $\left(\frac{3-5\lambda}{2}, \frac{-4+2\lambda}{2}, \frac{2-4\lambda}{2}\right)$

Since the plane 'P' is the great circle, then the center of S_1 lies on plane 'P'.

$$\therefore 5\left(\frac{3-5\lambda}{2}\right)-2\left(\frac{-4+2\lambda}{2}\right)+4\left(\frac{2-4\lambda}{2}\right)+7=0$$

$$\therefore 15-25\lambda+8-4\lambda+8-16\lambda+14=0.$$

$$\therefore 45-45\lambda=0 \Rightarrow \boxed{\lambda=1}$$

$$\therefore \boxed{\text{Equation of Sphere } S_1:\ x^2+y^2+z^2+2x+2y+2z+2=0}$$

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2017(4) 2008 Let A be a non-singular matrix. Show that if $I + A + A^2 + A^3 + \cdots + A^{n-1} + A^n = O$ then $A^{-1} = A^n$.

3.(a)

Sol. Given that A is non-singular matrix.

i.e $|A| \neq 0$

i.e A^{-1} exists.

and $I + A + A^2 + \cdots + A^{n-1} + A^n = 0$ ——— (1).

pre-multiply by A^{-1} on both sides we get

$A^{-1}(I + A + A^2 + \cdots + A^{n-1} + A^n) = A^{-1}(0)$.

$\Rightarrow A^{-1}I + A^{-1}A + A^{-1}.A^2 + \cdots + A^{-1}A^{n-1} + A^{-1}.A^n = O$.

$\Rightarrow A^{-1} + I + A + A^2 + A^3 + \cdots A^{n-2} + A^{n-1} = O$.

$\Rightarrow A^{n-1} = -A^{-1} - I - A - A^2 - A^3 - \cdots - A^{n-3} - A^{n-2}$ ——— (2).

substituting this into (1), we get

$\cancel{I} + \cancel{A} + \cancel{A^2} + \cancel{A^3} + \cancel{A^4} + \cdots + \cancel{A^{n-2}} - A^{-1} - \cancel{I} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \cancel{A^4} - \cdots - \cancel{A^{n-3}} - \cancel{A^{n-2}} + A^n = O$.

$\Rightarrow -A^{-1} + A^n = O$

$\Rightarrow \boxed{A^n = A^{-1}}$ which is the required the result.

2008 20M

3.(c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.

Sol^n: The given cone is

$$5yz-8zx-3xy=0 \quad \text{——} \ (1).$$

and one of its three ⊥lar generators is

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \quad \text{——} \ (2)$$

The other two ⊥lar generators are the lines which plane through the vertex (0,0,0) and ⊥lar to line (2)

i.e. the plane $x+2y+3z=0$ —— (3)

Let a line of section of (1) & (3) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{——} \ (4)$$

Since (4) lies in the plane (3)

∴ It is ⊥lar to the normal to the plane.

$$\therefore \ l+2m+3n=0 \quad \text{——} \ (5)$$

Also (4) lies on cone (1)

∴ i.e. the direction cosines of (4) satisfies the equation of cone.

$$\therefore \ 5mn-8nl-3lm=0 \quad \text{——} \ (6)$$

$$(5) \equiv l=-(2m+3n)$$

$$\therefore (6) \equiv 5mn+8n(2m+3n)+3m(2m+3n)=0$$

$$\Rightarrow 6m^2+30mn+24n^2=0$$

$$\Rightarrow m^2+5mn+4n^2=0$$

$$\Rightarrow (m+n)(m+4n)=0$$

$\Rightarrow m+n=0$

$\Rightarrow 0l+m+n=0$

Also (5) $\equiv l+2m+3n=0$

$\Rightarrow m+4n=0$

$\Rightarrow 0l+m+4n=0$

Also $l+2m+3n=0$

solving

$\frac{l}{3-2}=\frac{m}{1-0}=\frac{n}{0-1}$

$\frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$

$\frac{l}{3-8}=\frac{m}{4-0}=\frac{n}{0-1}$

$\frac{l}{-5}=\frac{m}{4}=\frac{n}{-1}$

$\therefore$ (4) $\equiv \frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ & $\frac{x}{-5}=\frac{y}{4}=\frac{z}{-1}$

which are the other two generators.

2008 **4.(a)** Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\}$. Hence find a basis for the subspace.

Sol.

Let $\mathbb{R}^4 = \{(x_1,x_2,x_3,x_4) \mid x_1,x_2,x_3,x_4 \in \mathbb{R}\}$ be the given vector space over the field $\mathbb{R}$.

Let W be the subspace of $\mathbb{R}^4$ spanned by S where

$S=\{(1,0,0,0),(0,1,0,0),(1,2,0,1),(0,0,0,1)\}$

Let us construct a matrix A whose rows are the given vectors of S and convert it into the echelon form.

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&2&0&1\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&0&1\\0&0&0&1\end{bmatrix}\quad R_3\to R_3-R_1$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&1\end{bmatrix}\quad R_3\to R_3-2R_2$$

$$\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\quad R_4\to R_4-R_3$$

clearly which is in echelon form and the number of non-zero rows are equal to 3. corresponding these rows the vectors of S $(1,0,0,0),(0,1,0,0),(1,2,0,1)$

form a basis of W.

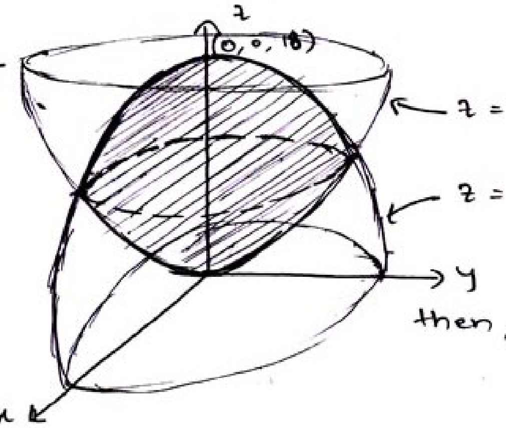
i.e $S' = \{(1,0,0,0), (0,1,0,0), (1,2,0,1)\}$ is a maximum number of linearly independent subset of W.

and it forms a basis of W.

4.(b)

Q.) Obtain the volume bounded by the elliptic paraboloids given by the equations : $z = x^2 + 9y^2$ & $z = 18 - x^2 - 9y^2$

Soln.



The two curves intersects then, $z = 18 - (x^2 + 9y^2)$

$z = 18 - z \Rightarrow \boxed{z = 9}$

Volume bounded by elliptic paraboloids, $V = \iiint_V dx\,dy\,dz$

Let us change the system of coordinates to

$x = r\cos\theta$, $y = \frac{r\sin\theta}{3}$, $z = z$

Jacobian, $J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \frac{\sin\theta}{3} & \frac{r\cos\theta}{3} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{r}{3}$

Now, the limits are :

z : 0 to 9 and z : 9 to 18

r : 0 to $\sqrt{z}$ r : 0 to $\sqrt{18 - z}$

θ : 0 to 2π θ : 0 to 2π

$\therefore V = \iiint_V dx\,dy\,dz = \iiint \frac{r}{3} \cdot dr\,d\theta \cdot dz$

$= \int_{z=0}^{9}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{z}} \frac{r}{3} \cdot dr\,d\theta \cdot dz + \int_{z=9}^{18}\int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{18-z}} \frac{r}{3} \cdot dr\,d\theta\,dz$

$= \int_0^{2\pi} d\theta \int_0^9 \left[\int_0^{\sqrt{z}} \frac{r\,dr}{3}\right] \cdot dz + \int_0^{2\pi} d\theta \cdot \int_9^{18} \left[\int_0^{\sqrt{18-z}} \frac{r\,dr}{3}\right] \cdot dz$

$= \frac{\cancel{2}\pi}{3} \cdot \int_0^9 \frac{z}{\cancel{2}} \cdot dz + \frac{\cancel{2}\pi}{3} \cdot \int_9^{18} \frac{(18 - z)}{\cancel{2}} \cdot dz = \frac{\pi}{3} \frac{(9^2 - 0^2)}{2} + \frac{\pi}{3}\left(18(9) - \frac{(18^2 - 9^2)}{2}\right)$

$= \frac{\pi \times 81}{3 \times 2} + \frac{\pi}{3 \times 2} \times 9^2 = \frac{27\pi}{2} + \frac{27\pi}{2} \Rightarrow \boxed{V = 27\pi}$

5(a) Solve the differential equation

$ydx + \left(x + x^3y^2\right)dy = 0$.

SOLUTION

$y\,dx + (x + x^3y^2)\,dy = 0$

standard form of exact differential equation is $M\,dx + N\,dy = 0$

$\therefore$ here $M = y;$

$N = x + x^3y^2$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1 + 3x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ given equation not exact

Given equation in the form of $yf_1(xy)\,dx + xf_2(xy)\,dy = 0$

then integrating factor is $\frac{1}{Mx - Ny}$

$$Mx - Ny = xy - xy - x^3y^3 = -x^3y^3$$

$\therefore$ integrating factor $-\frac{1}{x^3y^3}$

multiplying with

I.F. $+\frac{1}{x^3y^2}dx + \left(\frac{1}{x^2y^3} + \frac{1}{y}\right)dy = 0$

General solution $= \int Mdx + Ndy$

y constant y term not containing X

$$\int \frac{1}{x^3y^2}dx + \frac{1}{y}dy = c$$

$$\boxed{\frac{-1}{2x^2y^2} + \ell ny = c}$$

5(b). Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4 \sin x$

SOLUTION

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = -x^2 \sin x \qquad .....(1)$$

which is a Hamogeneous = n of the form

$$y'' + p(x)y' + Q(x)y = R(x) \qquad .....(2)$$

where $P(x) = \frac{-4}{x}$;

$Q(x) = \frac{6}{x^2}$;

$R(x) = -x^2 \sin x$

$\because \quad 2 + 2Px + Qx^2 = 0$

$u(x) = x^2$

$\because \quad 6 + 3Px + Qx^2 = 0$

$v(x) = x^3$

Therefore $yc(x) = c_1x^2 + c_2x^3$ .....(3)

where c_1, c_2 are constants.

Wronskian of us v is given by

$$w(u, v) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^2 - 2x^4 = x^4 \neq 0$$

$y_p(x) = A(x)\, x^2 + B(x)\, x^3$; A & B are parameters .....(4)

Where
$$A(x) = -\int \frac{vR}{uv' - vu'} \cdot dx$$
$$= -\int \frac{x^3 \times (-x^2 \sin x)}{x^4} \cdot dx$$
$$= +\int x \sin x \cdot dx$$
$$= -x \cos x + \int \cos x \cdot dx$$
$$= -x \cos x + \sin x. \qquad .....(5)$$

and
$$B(x) = \int \frac{uR}{uv' - vu'} \cdot dx$$
$$= \int \frac{x^2 \times (-x^2 \sin x)}{x^4} dx$$
$$= -\int \sin x \cdot dx$$
$$= \cos x. \qquad ......(6)$$

Put (5) & (6) in (4)

$y_p(x) = -x^3 \cos x + x^2 \sin x + x^3 \cos x$

$= x^2 \sin x$ .....(7)

Its general solution is given by

$$y = y_c(x) + y_p(x)$$
$$= c_1x^2 + c_2x^3 + x^2 \sin x$$

(from (3) and (7)

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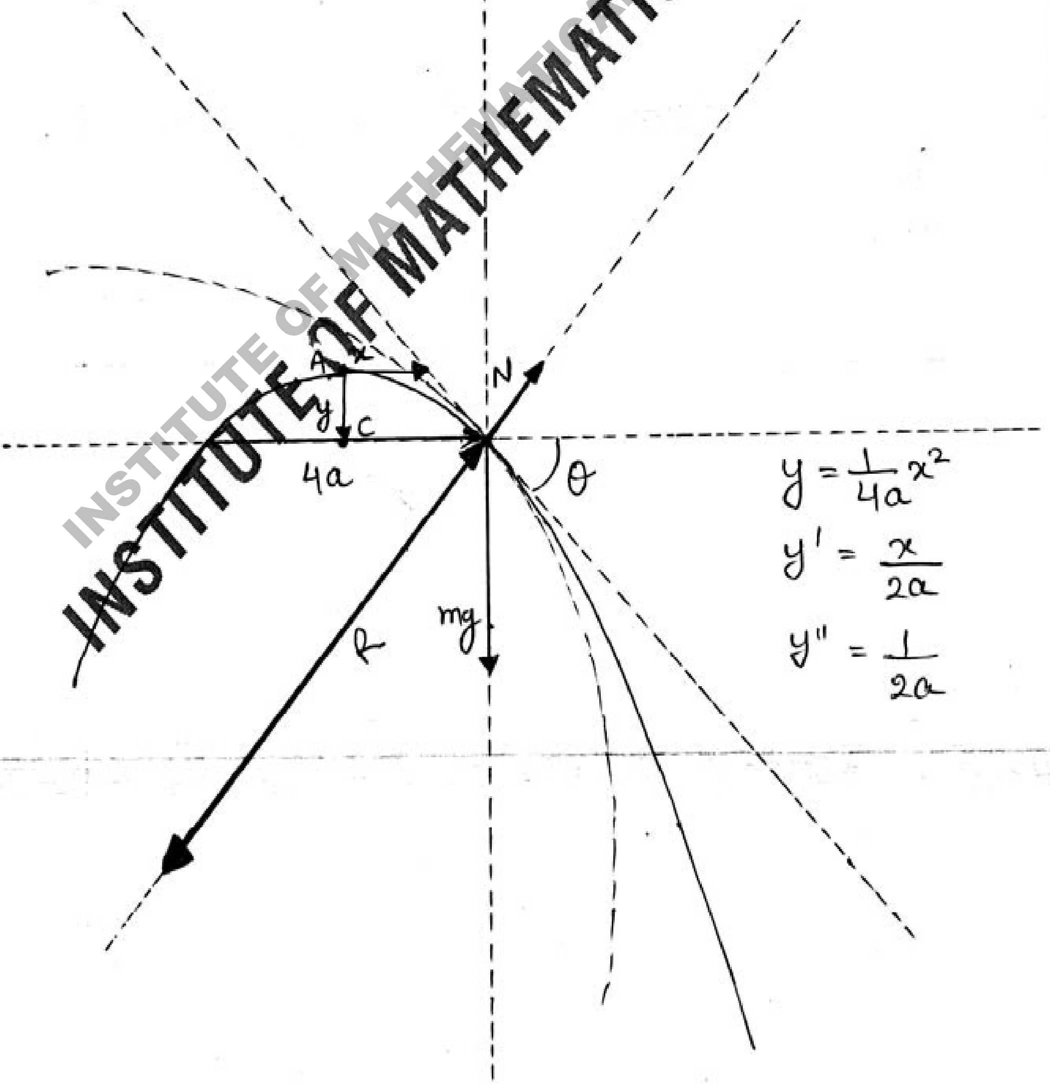
5(c) IAS 2008 P-I → A smooth parabolic tube is placed with vertex downwards in a vertical plane. A particle slides down the tube from rest under the influence of gravity. Prove that in any position, the reaction of the tube is equal to $2w(h+a)/\rho$, where w is the weight of the particle, ρ the radius of curvature of the tube, $4a$ the latus rectum and h the initial vertical height of the particle above the vertex of the tube.

Sol^n → The particle at a particular instant.



$$y = \frac{1}{4a}x^2$$

$$y' = \frac{x}{2a}$$

$$y'' = \frac{1}{2a}$$

The radius of curvature, R, can be calculated as-

$$R = \left| \frac{(1+y'^2)^{3/2}}{y''} \right| = \frac{(4a^2+x^2)^{3/2}}{4a^2} = \frac{2}{\sqrt{a}}(y+a)^{3/2}$$

And the speed of the particle at a given point:

$$v = \sqrt{2gy}$$

The particle is moving on a parabola, writing the forces perpendicular to the surface

$$mg\cos\theta - N = m\frac{v^2}{R}$$

where

$$\cos\theta = \sqrt{\frac{1}{1+\tan^2\theta}} = \sqrt{\frac{1}{1+y'^2}}$$

$$= \sqrt{\frac{a}{a+y}} = \frac{2(y+a)}{R}$$

Now calculating, we get.

$$N = 2mg\frac{y+a}{R} - 2mg\frac{y}{R} = \frac{2mga}{R}$$

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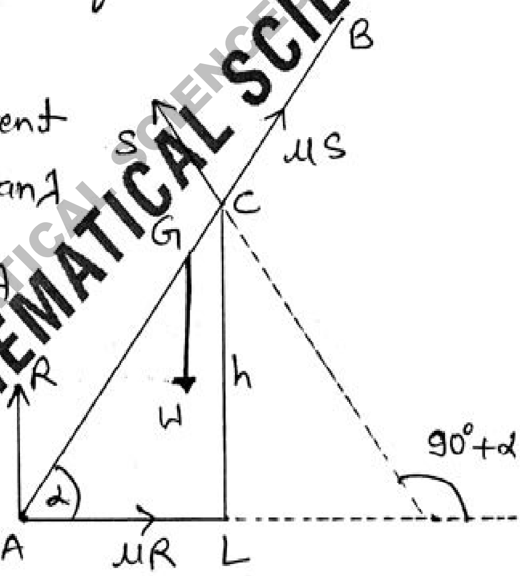
5(d) IFS 2008 P-I

A straight uniform beam of length '2h' rests in limiting equilibrium, in contact with a rough vertical wall of height 'h' with one end on a rough horizontal plane and with the other end projecting beyond the wall. If both the wall and the plane be equally rough, prove that 'λ' the angle of friction, is given by $\sin 2\lambda = \sin\alpha \sin 2\alpha$, 'α' being the inclination of the beam to the horizon.

Sol^n →

Let μ be the co-efficient of friction, then $\mu = \tan\lambda$



Taking moments about A

$$S \cdot AC = w \cdot AG \cos\alpha$$

or $S \cdot h \operatorname{cosec}\alpha = w \cdot h \cos\alpha$

$$\therefore\ S = w \sin\alpha \cos\alpha$$

$$= \frac{w}{2}\sin 2\alpha \quad \text{——} (1)$$

Resolving horizontally,

$$\mu R + \mu S \cos\alpha + S\cos(90^\circ+\alpha) = 0$$

or $\mu R + \mu S \cos\alpha = S \sin\alpha$ —— (2)

Resolving vertically $R + \mu S \sin\alpha + S\sin(90+\alpha) = W$

or, $R + \mu S \sin\alpha + S\cos\alpha = W$ —— (3)

Multiplying (3) by μ and substracting (2) from the result

$$\mu^2 S \sin\alpha = \mu w - S\sin\alpha$$

or $\quad u^2 \cdot \frac{\omega}{2} \sin 2\alpha \sin\alpha = u\,\omega - \frac{\omega}{2} \sin 2\alpha \sin\alpha$

$(\because$ from ①$)$

or, $\quad (1+u^2)\cdot\frac{1}{2}\sin 2\alpha \sin\alpha = u$

or, $\quad \sin 2\alpha \sin\alpha = \frac{2u}{1+u^2} = \frac{2\tan\lambda}{1+\tan^2\lambda}$

or $\quad \sin 2\alpha \sin\alpha = \sin 2\lambda.$

5(e) IAS-2008 P-I

Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Sol^n: The given surfaces are

$f_1 = ax^2 - byz - (a+2)x = 0$ —— (1)

and $f_2 = 4x^2y + z^3 - 4 = 0$ —— (2)

The point $(1,-1,2)$ obviously lies on the surface (2). It will also lie on the surface (1) if

$a + 2b - (a+2) = 0 \Rightarrow 2b - 2 = 0$

$\Rightarrow \boxed{b = 1}$

Now $\text{grad} f_1 = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$.

and $\text{grad} f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$.

Then $n_1 = \text{grad} f_1$ at the point $(1,-1,2)$

$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$

and $n_2 = \text{grad} f_2$ at the point $(1,-1,2)$

$= -8\hat{i} + 4\hat{j} + 12\hat{k}$

The vectors n_1 and n_2 are along the normals to the surfaces (1) and (2) at the point $(1,-1,2)$.

These surfaces will intersect orthogonally at the point $(1,-1,2)$ if the vectors n_1 and n_2

perpendicular i.e. if $n_1 \cdot n_2 = 0$

$\Rightarrow -8(a-2) - 8b + 12b = 0$

$\Rightarrow b - 2a + 4 = 0$

$\Rightarrow 1 - 2a + 4 = 0 \qquad (\because b = 1)$

$\Rightarrow \boxed{a = 5/2}$

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Q.5(f) (2008) Show $\vec{F} = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative force field. Find the scalar potential for $\vec{F}$ and the work done in moving an object from $(1,-2,1)$ to $(3,1,4)$?

Sol:
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$\nabla \times \vec{F} = i(0-0) - j(3z^2 - 3z^2) + k(2x - 2x)$$

$$\therefore \nabla \times \vec{F} = 0$$

$\Rightarrow \vec{F}$ is a coservative force field scalar potential for $\vec{F}$ is given by

$$\vec{F} = \nabla\phi$$

$$\Rightarrow (2xy + z^3)i + x^2 j + 3xz^2 k = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3$$

$$\Rightarrow \phi = yx^2 + xz^3 + f_1(y,z) \quad \text{———} (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 \quad \Rightarrow \phi = x^2 y + f_2(x,z) \quad \text{———} (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f_3(x,y) \quad \text{———} (3)$$

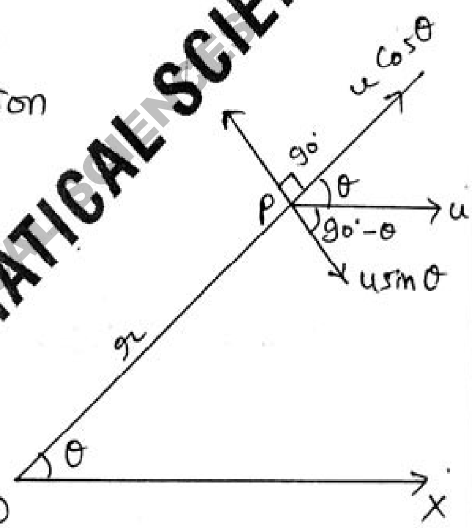
From (1), (2) and (3)
we get; $\phi = x^2 y + xz^3$
is the scalar potential for $\vec{F}$.

6(a) IAS 2008 P.I

A particle P moves in a plane such that it is acted on by two constant velocities u and v respectively along the direction OX and along the direction perpendicular to OP, where O is some fixed point, that is the origin. Show that the path traversed by P is a conic section with focus at O and eccentricity u/v.

Sol^n Take the fixed point O as pole and the fixed direction as the initial line OX.

Let $P(r, \theta)$ be the position of the particle at any time t. Then according to the question P possesses two constant velocities :



(i) u, in the fixed direction OX and

(ii) v perpendicular to OP as shown in the fig.

Resolving the velocities of P along and perpendicular to the radius vector OP, we have

the radial velocity $= \dfrac{dr}{dt} = u\cos\theta$ ——— (1)

and the transverse velocity $= r\dfrac{d\theta}{dt} = v - u\sin\theta$ ——— (2)

Dividing (1) by (2), we have

$$\frac{dr}{r\,d\theta} = \frac{u\cos\theta}{v - u\sin\theta}.$$

or $\frac{dr}{r} = \frac{u\cos\theta}{v - u\sin\theta}\,d\theta$

integrating, $\log r = -\log(v - u\sin\theta) + \log c$

or $\log\left(\frac{c}{r}\right) = \log(v - u\sin\theta)$

or $\frac{c}{r} = v - u\sin\theta$

or $\frac{c}{r} = v + u\cos\left(\frac{1}{2}\pi + \theta\right)$

or $\frac{c/v}{r} = 1 + \frac{u}{v}\cos\left(\frac{1}{2}\pi + \theta\right) \longleftarrow (3)$

Which is the path of the particle.

The equation (3) is of the form $l/r = 1 + e\cos\theta$ which is a conic whose focus is the pole O and eccentricity e is u/v.

Hence the path of P is a conic whose focus is O and eccentricity is u/v.

6(b). Using Laplace transform, solve the initial value problem $y''-3y'+2y=4t+e^{3t}$ with y(0) = 1, $y'(0)=-1$.

SOLUTION

$y''-3y'+2y=4t+e^{3t}$ with y(0) = 1 & y′(0) = 1

$[s^2 L\{y\} - sy(0) - y'(0)] - 3[sL\{y\} - y(0)] + 2L\{y\} = 4\,L\{t\} + L\{e^{3t}\}$

[Using formula $L\{y^n(t)\} = p^n L\{y(t)\} - P^{n-1}\, y(0) \ldots\ldots F^{n-1}(0)$]

$(s^2 - 3s + 2)\,L\{y\} - s + 1 + 3 = \dfrac{4}{s^2}+\dfrac{1}{s-3}$

[using formula $L\{t\} = 1/s^2$ and $L\left\{e^{at}\right\}=\dfrac{1}{s-a}$]

$$(s^2 - 3s + 2)\,L\{y\} = \frac{4}{s^2}+\frac{1}{s-3}+s-4$$

$$L\{y\} = \frac{4}{s^2(s^2-3s+2)}+\frac{1}{(s-3)(s^2-3x+2)}+\frac{s}{s^2-3s+2}-\frac{4}{s^2-3s+2}$$

$$L\{y\} = \underset{(1)}{\frac{4}{s^2(s-1)(s+2)}}+\underset{(2)}{\frac{1}{(s-3)(s-1)(s-2)}}\ \underset{(3)}{\frac{s}{(s-1(s-2)}}+\underset{(4)}{\frac{-4}{(s-1)(s-2)}}$$

$$(1)\Rightarrow \frac{4}{s^2(s-1)(s-2)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}$$

4 = As(s–1)(s–2)+B(s–1)(s–2)+C(s–2)s^2 +D(s–1)s^2

Put s = 0, B = 2

Put s = 1, C = –4

Put s = 2, D = 1

A = 3

$$(1)\Rightarrow L^{-1}\left[\frac{3}{s}\right]+L^{-1}\left[\frac{2}{s^2}\right]+L^{-1}\left[\frac{-4}{s-1}\right]+L^{-1}\left[\frac{1}{s-2}\right]$$

$(1) \Rightarrow 3 + 2t - 4e^t + e^{2t}$

$$(2)\Rightarrow \frac{1}{(s-3)(s-1)(s-2)}=\frac{A}{s-3}+\frac{B}{s-1}+\frac{C}{s-2}$$

1 = A(s–1)(s–2)+B(s–3)(s–2)+C(s–3)(s–1)

Put s = 1, B = $\dfrac{1}{2}$

Put s = 2, c = –1

Put s = 3, A = $\dfrac{1}{2}$

$$(2)\Rightarrow L^{-1}\left[\frac{1}{s-3}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right]-1L^{-1}\left[\frac{1}{s-2}\right]\ \frac{1}{2}e^{3t}+\frac{1}{2}e^{t}-e^{2t}$$

$$(3) \Rightarrow \frac{s}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$s = A(s-2)+B(s-1)$$

$$s = AS - 2A + Bs - B$$

Compairng s & constant terms

$$A+B = 1 \quad .....(1)$$

$$-2A - B = 0$$

$$B = -2A \quad .....(2)$$

Put (2) in (1)

$$A - 2A = 1$$

$$-A = 1$$

$$A = -1$$

Put the value of A in (1) B = 2

$$(3) \Rightarrow L^{-1}\left[\frac{-1}{s-1}\right] + 2L^{-1}\left[\frac{1}{s-2}\right]$$

$$= -e^{t} + 2e^{2t}$$

$$(4) \Rightarrow \frac{-4}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$-4 = A(s-2)+B(s-1)$$

Put s = 1, A = 4

Put s = 2, B = –4

$$(4) \Rightarrow L^{-1}\left[\frac{4}{s-1}\right] + -4L^{-1}\left[\frac{1}{s-2}\right]$$

Combining (1), (2), (3) and (4), we have the required

$$y(t) = 3 + 2t - 4e^{t} + e^{2t} + \frac{1}{2e}e^{3t} + \frac{1}{2}e^{t} - e^{2t} - e^{t} + 2e^{2t} + 4e^{t} - 4e^{2t}$$

$$y(t) = \frac{1}{2}e^{3t} - 2e^{2t} - \frac{1}{2}e^{t} + 2t + 3$$

P-1

2008 Q.6(c) Solve the differential equation

$$x^3y'' - 3x^2y' + xy = \sin(\ln x) + 1.$$

Ans. Given $x^3y'' - 3x^2y' + xy = \sin(\ln x) + 1$

$$\Rightarrow x^2y'' - 3xy' + y = \frac{\sin(\ln x)}{x} + \frac{1}{x}$$

$$\ln x = z \Rightarrow x = e^z$$

$$\frac{d}{dz} = D$$

$$\Rightarrow [D(D-1) - 3D + 1]y = \frac{\sin z + 1}{e^z}$$

$$[D^2 - 4D + 1]y = \frac{\sin z + 1}{e^z}$$

Auxillary Equation:

$$m^2 - 4m + 1 = 0$$

$$m = 2 \pm \sqrt{3}$$

The solution is $e^{2z}[c_1\cos\sqrt{3}z + c_2\sin\sqrt{3}z]$

The particular integral is given by –

$$\Rightarrow y = \frac{1}{(D^2-4D-1)}[(\sin z)e^{-z} + e^{-z}]$$

$$y = e^{-z}\cdot\frac{\sin z}{(D-1)^2 - 4(D-1) + 1} + \frac{e^{-z}}{(-1)^2 - 4(-1) + 1}$$

$$y = e^{-z}\cdot\frac{\sin z}{D^2 - 6D + 6} + \frac{e^{-z}}{6}$$

$$y = e^{-z}\frac{\sin z}{-1 - 6D + 6} + \frac{e^{-z}}{6}$$

$$= e^{-z}\frac{(5+6D)\sin z}{(5-6D)(5+6D)} + \frac{e^{-z}}{6}$$

$$\Rightarrow y_P = e^{-z}\frac{(5\sin z + 6\cos z)}{61} + \frac{e^{-z}}{6}$$

$\therefore$ The general solution is :-

$$y = y_c + y_p$$

$$y = c_1 e^{2z}\cdot\cos\sqrt{3}\,z + c_1 e^{2z}\sin\sqrt{3}\,z + \frac{e^{-z}}{61}[5\sin z + 6\cos z] + \frac{e^{-z}}{6}$$

$$\Rightarrow y = c_1 x^2\cos(\sqrt{3}\ln x) + c_2 x^2\sin(\sqrt{3}\ln x) + \frac{1}{61x}[5\sin(\ln x) + 6\cos(\ln x)) + \frac{1}{6x}$$

6(d). Solve the equation $y-2xp+yp^2=0$ where $p=\frac{dy}{dx}$.

SOLUTION

Given equation is y – 2xp + yp²

Lets take $y^2 = Y$

$$2y\frac{dy}{dx} = \frac{dY}{dx}$$

$$p = \frac{P}{2y}$$

$$\sqrt{Y}-2x\frac{P}{2\sqrt{Y}}+\frac{\sqrt{Y}\,P^2}{4Y} = 0$$

$$Y-xP+\frac{P^2}{4} = 0$$

$$Y = xP-\frac{P^2}{4}$$

Clearly equatins is Clairaut form

$\therefore$ $$Y = cx-\frac{P^2}{4}$$

Replacing the original variables

$$\boxed{y^2 = cx-\frac{c^2}{4}}$$

7(a) IAS-2008 P-I

A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$, $a>b$ and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu/(a+b)}$. Show that the equation of its path is $r = a+b\cos\theta$.

Sol'n: Here the central acceleration

$$P = \mu\{3au^4 - 2(a^2-b^2)u^5\}$$

∴ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\{3au^4-2(a^2-b^2)u^5\}$$

$$\Rightarrow h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \mu\{3au^2-2(a^2-b^2)u^3\}$$

Multiplying bothsides by $2\left(\frac{du}{d\theta}\right)$ and integrating, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\mu\{au^3 - 2(a^2-b^2)u^4\} + A$$

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\{2au^3-(a^2-b^2)u^4\}+A \quad \text{—①}$$

where A is a constant.

But initially at an apse $r = a+b$, $u = \frac{1}{(a+b)}$, $\frac{du}{d\theta}=0$ and $v = \sqrt{\mu}/(a+b)$

from ① we have

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right]+A$$

∴ $h^2=\mu$ and $A=0$.

Substituting the values of h^2 and A in ①, we have

$$-\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2}+\frac{2a}{r^3}-\frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2+2ar-(a^2-b^2)\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2 = b^2 - (r^2 - 2ar + a^2)$$

$$= b^2 - (r-a)^2$$

$$\therefore \frac{dr}{d\theta} = \sqrt{\{b^2 - (r-a)^2\}} \Rightarrow d\theta = \frac{dr}{\sqrt{\{b^2-(r-a)^2\}}}$$

Integrating $\theta + B = \sin^{-1}\left(\frac{r-a}{b}\right)$ ——— (3)

But initially when $r = a+b$, let us take $\theta = 0$.

Then from (3), $B = \sin^{-1}(1) = \pi/2$

Substituting in (3), we have

$$\theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r-a}{b}\right) \Rightarrow r - a = b\sin\left(\tfrac{1}{2}\pi + \theta\right)$$

$r = a + b\cos\theta$, which is required equation of the path.

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7(b)
IAS-2008
P-I

A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s is the distance apart, in the tube of the masses after a time t, show that the work done by the explosion is $\frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\cdot\frac{s^2}{t^2}$.

Sol^n: Since the shell is lying in the tube, its velocity before explosion is zero. Let u_1 and u_2 be the velocities of the masses m_1 and m_2 respectively after explosion. Then the relative velocity of the masses after explosion is u_1+u_2. Since the tube is smooth and horizontal, u_1+u_2 will remain constant.

$$\therefore (u_1+u_2)t = s \quad \text{---} (1)$$

Also by the principle of conservation of linear momentum, we have

$$m_1u_1 - m_2u_2 = 0$$

$$\Rightarrow m_1u_1 = m_2u_2 \quad \text{---} (2)$$

Substituting for u_2 from (2) in (1), we get

$$\left(u_1 + \frac{m_1u_1}{m_2}\right)t = s$$

$$\Rightarrow u_1\left(\frac{m_1+m_2}{m_2}\right)t = s$$

$$\Rightarrow u_1 = \frac{m_2 s}{(m_1+m_2)t}$$

$$\therefore u_2 = \frac{m_1}{m_2}u_1 = \frac{m_1}{m_2}\cdot\frac{m_2 s}{(m_1+m_2)t} = \frac{m_1 s}{(m_1+m_2)t}$$

Now the workdone by the explosion

= the kinetic energy released due to the explosion

$$= \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2$$

$$= \frac{1}{2}m_1\cdot\frac{m_2^2 s^2}{(m_1+m_2)^2t^2} + \frac{1}{2}m_2\frac{m_1^2 s^2}{(m_1+m_2)t^2}$$

$$= \frac{1}{2} \frac{s^2}{t^2} \frac{1}{(m_1+m_2)^2} \left(m_1 m_2^2 + m_2 m_1^2\right)$$

$$= \frac{1}{2} \frac{s^2}{t^2} \frac{m_1 m_2 (m_1+m_2)}{(m_1+m_2)^2}$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1+m_2} \cdot \frac{s^2}{t^2}$$

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IAS - 2008 - PI - 8(a)

Q.) Prove that $\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial f}{\partial r}$ where $r = (x^2+y^2+z^2)^{1/2}$.

Hence, find $f(r)$ such that $\nabla^2 f(r) = 0$

Soln. :- We know, $r^2 = x^2+y^2+z^2$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{dr}{\partial z} = \frac{z}{r}$$

$$\nabla f(r) = \left(\frac{\partial}{\partial x}\cdot \hat{i} + \frac{\partial}{\partial y}\cdot \hat{j} + \frac{\partial}{\partial z}\cdot \hat{k}\right)\cdot f(r)$$

$$= \frac{\partial f}{\partial r}\cdot \frac{\vec{r}}{r} = \frac{\partial f}{\partial r}\cdot \hat{r}$$

$$\nabla^2 f(r) = \nabla(\nabla f(r)) = \nabla\left(\frac{\partial f}{\partial r}\cdot \hat{r}\right) = \nabla\left(\frac{\partial f}{\partial r}\right)\cdot \hat{r} + \nabla(\hat{r})\cdot \frac{\partial f}{\partial r}$$

$$= \left(\frac{\partial^2 f}{\partial r^2}\cdot \hat{r}\right)\cdot \hat{r} + \frac{\partial f}{\partial r}\cdot \nabla\left(\frac{\vec{r}}{r}\right)$$

$$\nabla\left(\frac{\vec{r}}{r}\right) = \frac{\partial}{\partial x}\cdot\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\cdot\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}\cdot\left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\cdot(y) + \frac{\partial}{\partial z}\cdot(z)\right] + x\frac{\partial}{\partial x}\cdot\left(\frac{1}{r}\right) + y\cdot\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + z\cdot\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \frac{3}{r} + x\left(\frac{-1}{r^2}\right)\cdot\left(\frac{x}{r}\right) + y\cdot\left(\frac{-1}{r^2}\right)\cdot\left(\frac{y}{r}\right) + z\left(\frac{-1}{r^2}\right)\cdot\left(\frac{z}{r}\right)$$

$$= \frac{3}{r} - \frac{(x^2+y^2+z^2)}{r^3} = \frac{3}{r} - \frac{\cancel{r^2}}{r\cancel{^3}} = \frac{3-1}{r} = \frac{2}{r}$$

$$\therefore \nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2}(\hat{r}\cdot\hat{r}) + \frac{\partial f}{\partial r}\cdot\left(\frac{2}{r}\right)$$

$$\therefore \boxed{\nabla^2 f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial f}{\partial r}}$$

If $\nabla^2 f(r) = 0$, then $\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial f}{\partial r} = 0$

Integrating factor, I.F. $= e^{\int 2/r \cdot dr} = e^{2\log r} = r^2$

$$\therefore \left(r^2\cdot\frac{\partial f}{\partial r}\right) = C_1 \Rightarrow \frac{\partial f}{\partial r} = \frac{C_1}{r^2} \Rightarrow \boxed{f(r) = -\frac{C_1}{r} + C_2}$$

8(b) Show that for the space curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$ the curvature and torsion are same at every point.

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we have $\vec{r} = (x, y, z)$

$= (t, t^2, \tfrac{2}{3}t^3)$

$\vec{r}' = (1, 2t, 2t^2)$; $|\vec{r}'| = (1 + 4t^2 + 4t^4)^{1/2}$

$\vec{r}'' = (0, 2, 4t)$

$\vec{r}''' = (0, 0, 4)$

$$|\vec{r}' \times \vec{r}''| = \left[\,\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}\,\right] = (4 + 16t^2 + 16t^4)^{1/2}$$

$$= 2(1 + 4t^2 + 4t^4)^{1/2}$$

$$\text{Curvature } \kappa = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{2(1 + 4t^2 + 4t^4)^{1/2}}{(1 + 4t^2 + 4t^4)^{3/2}}$$

$$= \frac{2}{(1 + 4t^2 + 4t^4)} \rightarrow (1)$$

$$[\vec{r}'\ \vec{r}''\ \vec{r}'''] = \begin{vmatrix} 1 & 2t & 2t^2 \\ 0 & 2 & 4t \\ 0 & 0 & 4 \end{vmatrix} = 8$$

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Torsion $\tau = \dfrac{[\vec{r'}\ \vec{r''}\ \vec{r'''}]}{|\vec{r'} \times \vec{r''}|^2}$

$$\tau = \frac{8}{4(1+4t^2+4t^4)}$$

$$= \frac{2}{1+4t^2+4t^4} \rightarrow (2)$$

from (1) and (2) it is clear that the curvature and torsion for the given curve are the same at every point.

8(C) IFoS-2008 P-I

Evaluate $\int_C \vec{A}\cdot d\vec{r}$ along the curve $x^2+y^2=1$, $z=1$ from $(0,1,1)$ to $(1,0,1)$ if $\vec{A}=(yz+2x)\hat{i}+xz\hat{j}+(xy+2z)\hat{k}$.

Sol^n:- using Stoke's theorem

$$\int_C \vec{A}\cdot\vec{dr} = \iint_S (\nabla\times\vec{A})\,\hat{n}\cdot ds$$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+2x & xz & xy+2z \end{vmatrix}$$

$$= \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z)$$

$$= 0.$$

$$\therefore \int_C \vec{A}\cdot\vec{dr} = \iint_S (\nabla\times\vec{A})\,\hat{n}\cdot ds = 0$$

Thus $\vec{A}$ is a conservative field and hence

$$\int_C \vec{A}\cdot d\vec{r} = 0.$$

8(d). Evaluate $\iint_S F.n\,ds$, where $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the closed surface consisting of the cylinder $x^2+y^2=4$ and the circular disc $z=0$ and $z=3$.

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Sol^n: Here the surface S consists of three surfaces: (i) the surface S_1 of the base i.e. the plane face $z=0$ of the cylinder (ii) the surface S_2 of the top i.e. the plane face $z=3$ of the cylinder and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 i.e. $z=0$, $F = 4x\hat{i} - 2y^2\hat{j}$, putting $z=0$ in F.

A unit vector n along the outward drawn normal to S_1 is obviously $-k$.

$$\therefore \iint_{S_1} F.n\,ds = \iint_{S_1} (4x\hat{i} - 2y^2\hat{j}).(-\hat{k})\,ds$$

$$= \iint_{S_1} 0\,ds = 0$$

For the surface S_2 i.e. $z=3$, $F = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$, putting $z=3$ in F.

A unit vector n along the outward drawn normal to S_2 is obviously k.

$$\therefore \iint_{S_2} F.n\,ds = \iint_{S_2} (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}).\hat{k}\,ds$$

$$= \iint_{S_2} 9\,ds = 9\iint_{S_2} ds = 9.2.\pi.2 = 36\pi$$

[$\because$ area of the plane face S_2 of the cylinder $= 2\pi r = 2\pi.2$]

For the convex portion S_3. i.e., $x^2+y^2=4$, a vector normal to S_3 is given by $\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$.

$\therefore n =$ a unit vector along outward drawn normal at any point of S_3.

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(4x^2+4y^2)}} = \frac{x\hat{i}+y\hat{j}}{2}, \text{ since } x^2+y^2 = 4 \text{ on } S_3$$

$\therefore$ on S_3, $F.n = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}).\left\{\frac{1}{2}(x\hat{i}+y\hat{j})\right\}$

$$= 2x^2 - y^3$$

Also ds = elementary area on the surface S_3

$= 2\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} F.n\,ds = \iint_{S_3} (2x^2 - y^3)\,2\,d\theta\,dz \quad \text{where } x = 2\cos\theta,\ y = 2\sin\theta$$

$$= \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 2\int_{\theta=0}^{2\pi} 8(\cos^2\theta - \sin^3\theta)\left[z\right]_{z=0}^{3} d\theta$$

$$= 48\left[\int_0^{2\pi}\cos^2\theta\,d\theta - \int_0^{2\pi}\sin^3\theta\,d\theta\right]$$

$$= 48\left[4\int_0^{\pi/2}\cos^2\theta\,d\theta - 0\right] \quad \left[\because \sin^3(2\pi-\theta) = -\sin^3\theta\right]$$

$$= 192 \cdot \frac{1}{2}\cdot\frac{\pi}{2} = 48\pi$$

Hence the required surface integral

$$= \iint_S F.n\,ds = \iint_{S_1} F.n\,ds + \iint_{S_2} F.n\,ds + \iint_{S_3} F.n\,ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$