Differential Equations

Differential eqn: An equation involving derivatives of a dependent variable w.r.t one or more independent variables, is called a differential eqn.

Ex: (1) $\frac{dy}{dx} = x\log x$

(2) $\frac{d^2y}{dx^2} + 3x\left(\frac{dy}{dx}\right)^2 - 5y = \log x$

(3) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$

(4) $\left(\frac{d^3y}{dx^3}\right)^{2003} + P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)\, y = S(x).$

(5) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(6) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

Note: $\frac{dy}{dx} = y'$ (or) $y^{(1)}$ (or) y_1 ; $\frac{d^2y}{dx^2} = y''$ (or) $y^{(2)}$ (or) y_2

$\frac{d^3y}{dx^3} = y'''$ (or) $y^{(3)}$ (or) y_3 .... $\frac{d^ny}{dx^n} = y^{(n)}$ (or) y_n

Types of Differential equations:

(i) **Ordinary Diff. eqn:** An eqn involving the derivatives of a dependent variable w.r.t a single independent variable, is called an ordinary diff. eqn.

The above examples (1), (2), (3), & (4) are ordinary diff. eqns.

(ii) **Partial Diff. eqn:** An equation involving the derivatives of a dependent variable w.r.t more than one independent variable, is called a partial diff. eqn.

The above examples (5) & (6) are partial diff. eqns.

Order of a Diff. eqn: The order of the highest order derivative involving in a differential eqn is called the order of the diff. eqn.

Ex: (1) $\frac{d^2y}{dx^2} + 4y = e^x$. is of 2nd order.

(2) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3$ is of second order.

(3) $\frac{d^2y}{dx^2} = k\left[1 + \left(\frac{dy}{dx}\right)^3\right]^{5/3}$ is of 2nd order.

(4) $\log\left(\frac{dy}{dx}\right) = ax + by$ is of 1st order.

(5) $\sin\left(\frac{dy}{dx}\right) = x^{100}$
order = 1

(6) $\cos\left(\frac{dy}{dx}\right) = x^{100}$
order = 1

Note [1]. A differential eqn of order one is of the form $f\left(x, y, \frac{dy}{dx}\right) = 0$

[2]. A diff. eqn of order two is of the form
$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

[3]. In general, diff. eqn of order 'n' is of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0$

Degree of a diff. eqn: The degree (i.e, power) of the highest order derivative involving in a diff. eqn, when the derivatives are made free from radicals (roots) and fractions, is called the degree of the diff. eqn.

Ex: (1) $x\left(\frac{d^2y}{dx^2}\right)^3 + y^2\left(\frac{dy}{dx}\right)^4 + xy = 0$ is of order 2 and degree 3.

(2) $\frac{d^2y}{dx^2} = k\left[1+\left(\frac{dy}{dx}\right)^3\right]^{5/3}$ (radical form)

cubing on both sides, we get,

$$\left(\frac{d^2y}{dx^2}\right)^3 = k^3\left(1+\left(\frac{dy}{dx}\right)^3\right)^5$$ order = 2
Degree = 3.

(3) $y\left(\frac{dy}{dx}\right) = \sqrt{x} + \frac{k}{dy/dx}$ (fractions form)

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2 = \sqrt{x}\left(\frac{dy}{dx}\right) + k$$

$\therefore$ order = 1
Degree = 2

(4) $y = x\frac{dy}{dx}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow y^2 = x^2\left(\frac{dy}{dx}\right)^2 + x^2\left(\frac{dy}{dx}\right)^4.$$

Order = 1
Degree = 4

(5) $\frac{d^3y}{dx^3} = \sqrt{1+\left(\frac{dy}{dx}\right)^5}$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2 = 1+\left(\frac{dy}{dx}\right)^5$$

order = 3
degree = 2

(6) $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$ fractions form

$$\Rightarrow \rho\left(\frac{d^2y}{dx^2}\right) = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\Rightarrow \rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

$\therefore$ Order $=2$

Degree $=2$

(7) $$(y''')^{4/3} + \sin x\left(\frac{dy}{dx}\right) + xy = x$$

$$\Rightarrow (y''')^{4/3} = x - \sin x\left(\frac{dy}{dx}\right) - xy$$

$$\Rightarrow (y''')^4 = \left(x - \sin x\left(\frac{dy}{dx}\right)^3 - xy\right)^3$$

order $=3$

Degree $=4$

(8) $$(y''')^{1/2} - 2(y')^{1/4} + xy = 0$$

$$\Rightarrow (y''')^{1/2} + xy = 2(y')^{1/4}$$

$$\Rightarrow \left((y''')^{1/2} + xy\right)^4 = 2^4 y'$$

$$\Rightarrow \left[\left((y''')^{1/2} + xy\right)^2\right]^2 = 16y'$$

$$\Rightarrow \left[y''' + x^2y^2 + 2xy(y''')^{1/2}\right]^2 = 16y'$$

$$\Rightarrow (y''')^2 + x^4y^4 + 4x^2y^2y''' + 2x^2y^2y''' + 4x^3y^3(y''')^{1/2}$$
$$+ 4xy(y''')^{3/2} = 16y'$$

$$\Rightarrow 4xy(y''')^{1/2}\left[y''' + x^2y^2\right] = \left[16y' - (y''')^2 - x^4y^4 - 6x^2y^2y'''\right]$$

Squaring on both sides

$$16x^2y^2y'''\left(y''' + x^2y^2\right)^2 = \left(16y' - (y''')^2 - x^4y^4 - 6x^2y^2y'''\right)^2$$

$$\Rightarrow 16x^2y^2y'''\left[(y''')^2 + x^4y^4 + 2x^2y^2y'''\right] = (16y')^2 + (y''')^4 + \cdots$$

$\therefore$ order $=3$ & Degree $=4$

(9) $(y''')^{4/3} + (y')^{1/5} + y = 0$

Order $= 3$

Degree $= 60$

(10) $(y''')^{3/2} + (y''')^{2/3} = 0$

Order $= 3$

Degree $= 9$

Note:

$\boxed{1}$ $y = \sin\left(\frac{dy}{dx}\right)$

Order $= 1$

Degree = not defined.

Because $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

$$= \frac{dy}{dx} - \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \frac{1}{5!}\left(\frac{dy}{dx}\right)^5 - \frac{1}{7!}\left(\frac{dy}{dx}\right)^7 + \ldots\ldots$$

Here $x = \frac{dy}{dx}$

Similarly $\cos\left(\frac{dy}{dx}\right)$; $\tan\left(\frac{dy}{dx}\right)$; $\cot\left(\frac{dy}{dx}\right)$, $\sec\left(\frac{dy}{dx}\right)$ and $\operatorname{cosec}\left(\frac{dy}{dx}\right)$ degrees donot exist (or not defined.

$\boxed{2}$. $y = x\left(\frac{dy}{dx}\right) + \sin\left(\frac{dy}{dx}\right)$

Order $= 1$

Degree = not defined.

(3) $\frac{d^2y}{dx^2} + 2e^{x\frac{dy}{dx}} - 3y = x$

$\Rightarrow 2e^{x\frac{dy}{dx}} = x + 3y - \frac{d^2y}{dx^2}$

$$\Rightarrow x\frac{dy}{dx} = \log\left[\frac{1}{2}\left(x+3y - \frac{d^2y}{dx^2}\right)\right]$$

$\therefore$ order $= 2$

Degree = not defined.

(4) $3x^2 \dfrac{d^3y}{dx^3} - \sin \dfrac{d^2y}{dx^2} - \cos(xy) = 0$

(5) $(y''')^{1/3} + xy'' = 2005$

$$\Rightarrow (y''')^{1/3} = -xy'' + 2005$$

$$\Rightarrow y''' = (2005 - xy'')^3$$

Order $= 3$

Degree $= 1$

(6) $[y'' - 4(y')^2]^{5/2} = ay''$

$$[y'' - 4(y')^2]^5 = a(y'')^2$$

Order $= 2$

degree $= 5$

Linear Differential Eqn:

A differential equation is said to be linear if (i) the dependent variable say 'y' and all its derivatives occur in the first degree only
(ii) no product of dependent variables (or) derivatives occur.

Ex: (1) $\frac{dy}{dx} = x + \sin x$

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

Non-Linear diff. eqn:

A diff. eqn which is not linear is called a non-linear diff. eqn.

Ex (1) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$.

(2) $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{dy/dx}$

(3) $k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(4) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

Note: In general, a linear diff eqn of n^{th} order is of the form

$$\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = Q(x).$$

Solution of a diff. eqn: Any relation between the dependent and independent variables which when we substituted in the diff. eqn reduces it to an identity is called a solution (or) integral (or) primitive of the diff. eqn.

Ex: $y = ce^{2x}$ is a solⁿ of the diff. eqn $y' = 2y$ — ①

because $y = ce^{2x} \Rightarrow y' = 2ce^{2x}$

$\therefore$ ① $\equiv$ $2ce^{2x} = 2[ce^{2x}]$ is an identity

General solⁿ: The solⁿ of a diff. eqn in which the number of arbitrary constants is equal to the order of the diff. eqn.

Ex: $y = ce^{2x}$ is G.S. of the diff eqn $y' = 2y$

$\therefore$ Arbitrary constants = order of the diff. eqn.

Particular solution: A solution obtained by giving particular values to arbitrary constants in the general solution, is called a particular solⁿ (or) particular integral.

Ex: In the above example taking $c = 1$

$y = e^{2x}$ is a particular solⁿ of $y' = 2y$

Singular solution:

An eqn $\psi(x, y) = 0$ is called singular solution of the diff. eqn $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0$ if (i) $\psi(x, y) = 0$ is a solution of the given diff. eqn.

(ii) $\psi(x, y) = 0$ doesnot contain arbitrary constants.

and (iii) $\psi(x, y) = 0$ is not obtained by giving particular values to arbitrary constants in the general solⁿ.

Arbitrary constants: The complete solⁿ of a diff. eqn of the nᵗʰ order contains exactly 'n' arbitrary constants.

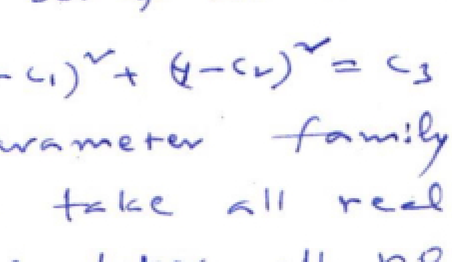
Family of plane curves:

For each given set of real numbers $c_1, c_2, c_3, \ldots\ldots c_n$, the equation $\phi(x, y, c_1, c_2, \ldots c_n) = 0$ represents a curve in xy-plane.

For different sets of real values of $c_1, c_2, \ldots\ldots c_n$, the equation $\phi(x, y, c_1, c_2 \ldots c_n) = 0$ represents infinitely many curves.

The set of all these curves is called n-parameter family of curves. $c_1, c_2, \ldots\ldots c_n$ are called parameters of the family.

Ex:- (1) the set of concentric circles defined by $x^2+y^2=c$ is one parameter family if 'c' takes all non-negative values.



(2) the set of all circles, defd by $(x-c_1)^2+(y-c_2)^2=c_3$ is a three-parameter family if c_1, c_2 take all real values and c_3 takes all non-negative real values.

Formation of Diff. eqns:—

working rule:

To form the diff. eqn from a given eqn is x and y, containing arbitrary constants:

Step 1: write down the given eqn.

Step 2: Differentiate w.r.t 'x' successively as many as the number of arbitrary constants.

Step 3: Eliminate the arbitrary constants from the given eqns of above two steps.

∴ The resulting is the required diff. eqn.

Problems:

(1) Find the diff. eqn of $y = Ae^{2x} + Be^{-3x}$; (A, B are arbitrary constants)

Soln: $y' = 2Ae^{2x} - 3Be^{-3x}$ —— ①

$y'' = 4Ae^{2x} + 9Be^{-3x}$ —— ②

$① + ② \equiv y' + y'' = 6Ae^{2x} + 6Be^{-3x}$

$= 6(Ae^{2x} + Be^{-3x})$

$= 6y.$

$\therefore y' + y'' = 6y$

$\Rightarrow y'' + y' - 6y = 0$

which is the required diff. eqn.

(or)

$$\begin{vmatrix} y & -e^{2x} & -e^{-3x} \\ y' & -2e^{2x} & 3e^{-3x} \\ y'' & -4e^{2x} & -9e^{-3x} \end{vmatrix} = 0 \Rightarrow e^{-2x}e^{-3x}\begin{vmatrix} y & -1 & -1 \\ y' & -2 & 3 \\ y'' & -4 & -9 \end{vmatrix} = 0$$

$$\Rightarrow y(18+12)+1(-9y'-3y'')-1(-4y'+2y'')=0$$

$$\Rightarrow 30y-5y'-5y''=0 \qquad (\because e^{2x}e^{-3x}\neq 0.)$$

$$\Rightarrow y''+y'-6y=0.$$

(2) Find the diff eqn of the family of curves $y=a(x-a)^2$ ——① where a is an arbitrary constant.

Soln: Differentiate ① w.r.t x we get,

$$y'=2a(x-a)$$

Now $\frac{①}{②} \Rightarrow \frac{1}{2}(x-a)=\frac{y}{y'}$

$$\Rightarrow 2y=y'(x-a)$$

$$\Rightarrow ay'=xy'-2y$$

$$\Rightarrow a=\frac{xy'-2y}{y'}$$

$\therefore ① \equiv$ $$y=\left(\frac{xy'-2y}{y'}\right)\left[x-\left(\frac{xy'-2y}{y'}\right)\right]^2$$

$$=\left(\frac{xy'-2y}{y'}\right)\left(\frac{2y}{y'}\right)^2$$

$$y=(xy'-2y)\frac{4y^2}{(y')^3}$$

$$\Rightarrow (y')^3=(xy'-2y)4y.$$

which is the required diff. eqn.

(3) Find the diff eqn of $y=Ae^{ax}+Be^{bx}$; A, B are arbitrary constants.

Ans $\boxed{y''-(a+b)y'+aby=0}$

(4) Find the diff-eqn of $y=Ae^{-x}+Be^{2x}$; A, B are arbitrary constants.

$$y''-(-1+2)y'+(-1)(2)y=0$$

$$\Rightarrow y''-y'-2y=0.$$

(5) Find the diff. eqn of $y = Ae^{ax} + Be^{bx} + Ce^{cx}$; (A, B, C)

$$\boxed{y''' - (a+b+c)y'' + (ab+bc+ca)y' - abcy = 0}$$

(6) Find the diff. eqn of $y = ae^{x} + be^{3x} + ce^{5x}$; (a, b, c).

$$y''' - (1+3+5)y'' + (1\cdot3 + 3\cdot5 + 5\cdot1)y' - (1\cdot3\cdot5)y = 0$$

$$\Rightarrow y''' - 9y'' + 23y' - 15y = 0$$

(7) form the diff. eqn of $y = ae^{x} + be^{2x} + ce^{3x}$ where a, b, c arbitrary constants.

$$y''' - 6y'' + (2+6+3)y' - 6y = 0$$

$$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$$

(8) $y = ae^{2x} + be^{-3x} + ce^{x}$; (a, b, c)

$$y''' - (2-3+1)y'' + (2\cdot(-3) + (-3)(1) + 1(2))y' - (2\cdot -3\cdot 1)y = 0$$

$$\Rightarrow y''' - 0y'' + (-6-3+2)y' + 6y = 0$$

$$\Rightarrow y''' - 7y' + 6y = 0$$

(9) form the diff. eqn $y = ae^{3x} + be^{5x}$

Ans: $y'' - 8y' + 15y = 0$

(10) form the diff. eqn of $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$ — (1)

(c_1, c_2 are arbitrary constants)

$$y' = e^{ax}(c_1 b\cos bx - c_2 \sin bx) + ae^{ax}(c_1 \sin bx + c_2 \cos bx)$$

$$\Rightarrow y' = e^{ax}(c_1 b\cos bx - c_2 \sin bx) + ay \quad \text{(from (1))}$$

$$\Rightarrow y' - ay' = e^{ax}(c_1 b\cos bx - c_2 b\sin bx) \quad \text{——(2)}$$

$$\Rightarrow y'' - ay' = e^{ax}(-c_1 b^2 \sin bx - c_2 b^2 \cos bx) + a e^{ax}(c_1 b\cos bx - c_2 b\sin bx)$$

$$= -b^2 y + a(y' - ay) \quad (\text{by (1) \& (2)})$$

$$\Rightarrow y'' - ay' = -b^2 y + ay' - a^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2+b^2)y = 0}$$

(11) Form the diff. eqn of $y = e^{ax}(c_1 \sin bx - c_2 \cos bx)$ ① where c_1, c_2 are arbitrary constants

$$y' = e^{ax}(c_1 b\cos bx + c_2 b\sin bx) + a e^{ax}(c_1 \sin bx - c_2 \cos bx)$$

$$\Rightarrow y' = e^{ax}(c_1 b\cos bx + c_2 b\sin bx) + ay \quad (\text{by (1)})$$

$$\Rightarrow y' - ay = e^{ax}(c_1 b\cos bx + c_2 b\sin bx) \quad \text{——(2)}$$

$$\Rightarrow y'' - ay' = a e^{ax}(c_1 b\cos bx + c_2 b\sin bx) + e^{ax}(-c_1 b^2 \sin bx + c_2 b^2 \cos bx)$$

$$\Rightarrow y'' - ay' = a(y' - ay) - b^2 y$$

$$\Rightarrow \boxed{y'' - 2ay' + (a^2+b^2)y = 0}$$

(12) $y = e^x(c_1 \cos x + c_2 \sin x)$; (c_1, c_2)

$$y'' - 2(1)y' + (1^2+1^2)y = 0$$

$$\Rightarrow y'' - 2y' + 2y = 0$$

(13) $y = c_1 \cos 2x + c_2 \sin 2x$

$$y'' - 2(0)y' + (0^2+2^2)y = 0$$

$$\Rightarrow y'' - 4y = 0$$

(14) Find the diff. eqn of $Ax^2+By^2=1$ ———(1) ; A, B are arbitrary constants.

Sol^n:
$$\begin{vmatrix} x^2 & y^2 & -1 \\ 2x & 2yy' & 0 \\ 2 & 2(yy''+(y')^2) & 0 \end{vmatrix} = 0$$

$$\Rightarrow -1\begin{vmatrix} 2x & 2yy' \\ 2 & 2\left(yy''+(y')^2\right) \end{vmatrix} = 0$$

$$\Rightarrow -1\left(4x(yy''+(y')^2)-4yy'\right)=0$$

$$\Rightarrow 4xyy''+4x(y')^2-4yy'=0$$

$$\Rightarrow x(yy''+(y')^2)-yy'=0$$

(OR)

$$2Ax+2Byy'=0$$

$$\Rightarrow \frac{yy'}{x} = \frac{-A}{B}$$

$$\Rightarrow \frac{x((y')^2+yy'')-yy'}{x^2}=0$$

$$\Rightarrow x((y')^2+yy'')-yy'=0$$

(15) Find the diff. eqn of the family of ellipses whose axes coincide with the axes of co-ordinates and centres at the origin.

i.e, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ———(1) ; a, b are arbitrary constants

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{yy'}{x} = \frac{-b^2}{a^2} \quad \text{———(2)}$$

$$\frac{yy' - x\left((y')^2 + yy''\right)}{x^2} = 0$$

$$\Rightarrow x\left((y')^2 + yy''\right) - yy' = 0$$

(16) 1992 $xy = ae^x + be^{-x} + x^2$; (a, b)

$$\Rightarrow xy - x^2 = ae^x + be^{-x} \quad \text{—①}$$

$$\Rightarrow xy' + y - 2x = ae^x - be^{-x}$$

$$\Rightarrow xy'' + y' + y' - 2 = ae^x + be^{-x}$$

$$\Rightarrow xy'' + 2y' - 2 = xy - x^2 \quad (\text{from ①})$$

(or)

$$(xy - x^2)'' - (0)(xy - x^2)' + (-1)(xy - x^2) = 0$$

$$\Rightarrow (xy' + y - 2x)' - xy + x^2 = 0$$

$$\Rightarrow xy'' + y' - 2 - xy + x^2 = 0$$

(17) Form the diff. eqn of the family of circles, given by $x^2 + y^2 + 2ax + 2by + c = 0$; a, b, c arbitrary constants —①

Sol:

$$2x + 2yy' + 2a + 2by' = 0$$

$$x + yy' + a + by' = 0 \quad \text{—②}$$

$$1 + yy'' + (y')^2 + by'' = 0$$

$$\Rightarrow -b = \frac{1 + yy'' + (y')^2}{y''} \quad \text{—③}$$

$$\frac{y''\left(yy''' + y'y'' + 2y'y''\right) - y'''\left(1 + yy'' + (y')^2\right)}{(y'')^2} = 0$$

$$\Rightarrow y''\left(yy''' + 3y'y''\right) - y'''\left(1 + yy'' + (y')^2\right) = 0$$

$$\Rightarrow y'''\left(yy'' - 1 - yy'' - (y')^2\right) + 3y'(y'')^2 = 0$$

$$\Rightarrow y'''\left(1 + (y')^2\right) = 3y'(y'')^2$$

(18) Find the diff. eqn of family of the curves $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$ where λ is parameter. (9)

Sol^n $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$ —— ①

$$\frac{2x}{a^2}+\frac{2yy'}{a^2+\lambda}=0$$

$$\Rightarrow -\frac{1}{a^2+\lambda}=\frac{x^2}{a^2(yy')}$$

$$\Rightarrow \boxed{\frac{1}{a^2+\lambda^2}=\frac{-x^2}{a^2(yy')}}$$

$$\therefore \text{①} \equiv \frac{x^2}{a^2}-\frac{x^2y^2}{a^2yy'}=1$$

$$\Rightarrow \left(1-\frac{y}{y'}\right)=\frac{a^2}{x^2}$$

$$\Rightarrow y'-y=\frac{a^2}{x^2}y'$$

$$\Rightarrow \left(1-\frac{a^2}{x^2}\right)y'-y=0.$$

which is the required diff. equation.

19 $y=ax^2+bx$; (a,b) —— ①

Sol $y'=2ax+b$ —— ②

$y''=2a$ —— ③

Eliminating the arbitrary constants from ①, ② and ③, we have

$$\begin{vmatrix} y & -x^2 & -x \\ y' & -2x & -1 \\ y'' & -2 & 0 \end{vmatrix}=0 \Rightarrow x^2y''+2xy'-2x^2y''-2y=0$$

$$\Rightarrow x^2y''-2xy'+2y=0.$$

20 $y = ax + be^x$; (a, b) —— ①

Sol $y' = a + be^x$ —— ②

$y'' = be^x$ —— ③

Now eliminating, a & b from ①, ② & ③, we get

$$\begin{vmatrix} y & -x & -e^x \\ y' & -1 & -e^x \\ y'' & 0 & -e^x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y & -x & -1 \\ y' & -1 & -1 \\ y'' & 0 & -1 \end{vmatrix} = 0 \quad (\because e^{+x} \neq 0)$$

$$\Rightarrow y(1) + x(-y' + y'') - 1(y'') = 0$$

$$\Rightarrow (x-1)y'' - xy' + y = 0.$$

HW

21 $y = ax^2 + be^{-x}$; (a, b)

22) $x = A\sin t + B\cos t + t\sin t$; (A, B) —— ①

Sol $x - t\sin t = A\sin t + B\cos t$ —— ②

$\Rightarrow x' - t\cos t - \sin t = A\cos t - B\sin t$ —— ③

$\Rightarrow x'' - \cos t + t\sin t - \cos t = -A\sin t - B\cos t$

$\Rightarrow x'' - 2\cos t + t\sin t = -(A\sin t + B\cos t)$

$\Rightarrow x'' - 2\cos t + t\sin t = x - t\sin t$ (by ②)

$\Rightarrow \boxed{x'' - 2\cos t + 2t\sin t = x}$

23 $y = A\sin(mx + \alpha)$ —— ① (α, A)

Sol $y' = [A\cos(mx+\alpha)]m$ —— ②

$y'' = -[A\sin(mx+\alpha)]m^2$

$\boxed{y'' = -ym^2}$ which is the reqd diff. eqn.

24 $x = A\cos(pt-\alpha)$; (A, α)

Ans: $\boxed{x'' = -p^2 x}$

25 $y^2 = 4ax$; 'a' is parameter. —— ①

$2yy' = 4a$

$\therefore$ ① $\equiv$ $\boxed{y^2 = (2yy')x}$.

26 $y = a\cos(x+3)$; a is parameter —— ①

$y' = -a\sin(x+3)$ —— ②

$\frac{①}{②} \equiv \frac{y}{y'} = -\cot(x+3)$

$\Rightarrow \frac{y'}{y} = -\tan(x+3)$

$\Rightarrow \boxed{y' + y\tan(x+3) = 0}$

27 Find the diff. equation of the family of circles $(x-h)^2+(y-k)^2 = r^2$ where h and k are parameters & r is fixed constant.

Sol: $(x-h)^2+(y-k)^2 = r^2$ —— ①

$(x-h) + (y-k)y' = 0$ —— ②

$\Rightarrow 1 + (y-k)y'' + (y')^2 = 0$

$\Rightarrow \boxed{(y-k) = -\frac{[1+(y')^2]}{y''}}$.

(2) $\equiv$ $x-h = \frac{[1+(y')^2]}{y''} y'$

$\therefore$ (1) $\equiv$ $\frac{[1+(y')^2]^2}{(y'')^2}(y')^2 + \frac{[1+(y')^2]^2}{(y'')^2} = r^2$

$\Rightarrow \frac{[1+(y')^2]^2}{(y'')^2}[(y')^2+1] = r^2$

$\Rightarrow \boxed{[1+(y')^2]^3 = r^2 (y'')^2}$

28. $y = ae^x + be^{-x} + c\cos x + d\sin x$ (a, b, c, d) —— (1)

Sol: $y' = ae^x - be^{-x} - c\sin x + d\cos x$ —— (2)

$y'' = ae^x + be^{-x} - c\cos x - d\sin x$ —— (3)

$y''' = ae^x - be^{-x} + c\sin x - d\cos x$ —— (4)

$y^{iv} = ae^x + be^{-x} + c\cos x + d\sin x$ —— (5)

$\Rightarrow \boxed{y^{iv} = y}$ (from (1)).

which is the required diff. eqn.

HW

29. Find the diff. eqn of the family of straight lines $y = mx + \frac{a}{m}$ where 'm' is the parameter.

HW

30. Find the diff. eqn of family of all circles of fixed radius 'r' and centres on y-axis i.e. $x^2 + (y-k)^2 = r^2$ (k is parameter, r is constant).

31 find the diff. eqn of all circles of fixed radius 'r' and centres on x-axis is $(x-h)^2+y^2=r^2$; h is parametre.

HW

32 The family of all circles touching x-axis at the origin is

$$x^2+y^2-2ay=0.$$

HW

33 The family all circles touching y-axis at the origin is $x^2+y^2-2ax=0$. ('a' is parametre).

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* Solution of Differential equations of the first order and first degree:-

Defn: A diff. eqn of first order and first degree is an eqn of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ (or) $M\,dx+N\,dy=0$ where M and N are functions of x & y.

The first order, first degree diff. eqns solving into four methods

(i) Variables seperable (ii) Homogenous equations (iii) Exact eqns (iv) Linear equations.

(i) Variables seperable:

If in an eqn, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to anotherside, then the variables are said to be seperable.

Working rule:-

Step(1): Consider the equation $\frac{dy}{dx} = XY$; where X is a function of x only and Y is a function of y only.

Step(2): $\frac{dy}{Y} = X\,dx$; i.e. the variables have been seperated.

Step(3): Integrating on both sides,

$\int \frac{dy}{Y} = \int X\,dx + c$, where 'c' is an arbitrary constant.

Note:- (1) Never forget to add an arbitrary constant on one side (only).

A solution without this arbitrary constant is wrong, for it is not a general solution.

(2). The nature of the arbitrary constant depends upon the nature of the Problem.

Problems:

→ Solve the following diff. eqns.

[1] $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ ——— ①.

$\frac{dy}{dx} = e^x \cdot e^y + x^2 e^y$

$\Rightarrow \frac{dy}{dx} = e^y (e^x + x^2)$

$\Rightarrow \frac{1}{e^y} dy = (e^x + x^2)\,dx$

i.e. the variables have been seperated.

Now integrating on both sides, we get

$$\int \frac{dy}{e^y} = \int (e^x + x^2)\, dx + c$$

$$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c.$$

$$\Rightarrow \boxed{e^{-y} + e^x + \frac{x^3}{3} + c = 0}$$ which is the reqd g.s.

[12] $\frac{dy}{dx} = e^{x-y} + e^{2\log x - y}$

$$\Rightarrow \frac{dy}{dx} = e^x \cdot e^{-y} + e^{2\log x} \cdot e^{-y}$$

$$\frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \frac{dy}{dx}$$

$$\Rightarrow e^y dy = (e^x + x^2)\, dx$$

integrating on both sides, we get

$$\boxed{e^y = e^x + \frac{x^3}{3} + c}$$ which is the reqd g.s.

[13] $y\frac{dy}{dx} = xe^{x^2+y^2}$

$$y\frac{dy}{dx} = \frac{xe^{x^2}}{e^{-y^2}} \Rightarrow ye^{-y^2} dy = xe^{x^2}$$

integrating on both sides, we get

$$\int ye^{-y^2} dy = \int xe^{x^2} dx + c$$

$$\Rightarrow -\frac{e^{-y^2}}{2} = \frac{e^{x^2}}{2} + \frac{c}{2}$$ put $y^2 = t$, $y\,dy = \frac{dt}{2}$

$$\Rightarrow \boxed{e^{-y^2} + e^{x^2} + c = 0}$$

14 $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

$\Rightarrow \frac{dy}{dx} = -\left[\frac{1+y^2}{1+x^2}\right]$

$\Rightarrow \frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0$

$\Rightarrow \boxed{\tan^{-1}y + \tan^{-1}x = \tan^{-1}c}$

15 $y\,dx - x\,dy = xy\,dx$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = dx$

integrating on both sides, we get

$\log x - \log y = x + \log c$

$\Rightarrow \log\left(\frac{x}{yc}\right) = x$

$\Rightarrow \frac{x}{yc} = e^x \Rightarrow \boxed{\frac{x}{y} = e^x c}$

16 $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

$\Rightarrow \frac{\sec^2 y}{\tan y}dy = -\frac{\sec^2 x}{\tan x}dx$

$\Rightarrow \log(\tan y) = -\log(\tan x) + \log c.$

$\Rightarrow \log(\tan x . \tan y) = \log c$

$\Rightarrow \boxed{\tan x . \tan y = c}$

17 $\log\left(\frac{dy}{dx}\right) = ax + by$

$\Rightarrow \frac{dy}{dx} = e^{ax}.e^{by}$

$$\Rightarrow \frac{dy}{e^{by}} = e^{ax}\, dx$$

Integrating on both sides, we get,

$$\boxed{\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + e^{c}}$$

[8] $3e^{x} \tan y\, dx + (1-e^{x}) \sec^2 y\, dy = 0.$

Sol

$$\frac{3e^{x}}{1-e^{x}}\, dx + \frac{\sec^2 y}{\tan y}\, dy = 0.$$

Integrating on both sides, we get,

$$-3\log(1-e^{x}) + \log(\tan y) = \log c.$$

$$\Rightarrow \boxed{\frac{\tan y}{(1-e^{x})^3} = c.}$$

[9] $\left(y - x\frac{dy}{dx}\right)x = y$

$$\Rightarrow \cancel{xy} - x^2\frac{dy}{dx} = \cancel{y}\, y$$

$$\Rightarrow (x-1)y = x^2\frac{dy}{dx}$$

$$\Rightarrow \frac{x-1}{x^2}\, dx = \frac{1}{y}\, dy$$

$$\Rightarrow \frac{1}{y}\, dy = \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$\Rightarrow \log y = \log x + \frac{1}{x} + \log c.$$

$$\Rightarrow \boxed{y = xc\, e^{1/x}}$$

HW [10] $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

HW [11] $\frac{dy}{dx} = xy + x + y + 1.$

Sol

$$\Rightarrow \frac{dy}{dx} = x(y+1)+(y+1)$$
$$= (y+1)(x+1)$$

$$\Rightarrow \frac{dy}{(y+1)} = (x+1)\,dx$$

$$\boxed{\log(y+1) = \frac{x^2}{2}+x+c}$$

[12] $x(1+y^2)\,dx + y(1+x^2)\,dy = 0$

$$\Rightarrow \frac{x}{1+x^2}dx + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{\log(1+x^2)}{2} + \frac{\log(1+y^2)}{2} = \log c$$

$$\Rightarrow \log\left[(x^2+1)(y^2+1)\right] = \log c$$

$$\Rightarrow \boxed{(x^2+1)(y^2+1) = c}$$

[13] $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integrating on both sides, we get

$$\boxed{\sin^{-1}y + \sin^{-1}x = \sin^{-1}c}$$

which is the reqd. g.s.

[14] Find the equation of the curve passing through the point (1,1) whose diff. equation is $(y-yx)\,dx + (x+xy)\,dy = 0$ ——— (1).

Sol $y(1-x)\,dx + x(1+y)\,dy = 0$

$$\Rightarrow \frac{1+y}{y}dy + \frac{1-x}{x}dx = 0$$

Integrating, we get

$$\log y + \log x - x + y = c$$

$$\Rightarrow \boxed{\log(xy) + (y-x) = c} \quad (2)$$

which is reqd eqn of the curve.

Since the pt (1, 1), passing through (1), we get

$$\log(1 \cdot 1) + (1-1) = c$$

$$\Rightarrow \boxed{c = 0}$$

∴ from (2), we have

$$\boxed{\log(xy) + (y-x) = 0}$$

HW

15 Find the equation of the curve that passes through the point (1, 2) and satisfies the equation $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$.

* SECOND FORM:

Equations reducible to the form in which variables can be seperated:

Equations of the form $\frac{dy}{dx} = f(ax+by+c)$ (1)

can be reduced to the form in which the variables are seperable.

put $ax + by + c = z$

diff. w.r.t x, we get,

$$a + b\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$$\therefore (1) \equiv \frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

$\Rightarrow \dfrac{dz}{(a+bf(z))} = dx$

The variables have been seperated.

Integrating on both sides, we get solution.

problems

→ solve the following diff. equations:

1. $\dfrac{dy}{dx} = (3x+y+4)^2$ ——— ①

put $3x+y+4 = z$

diff. w.r.t. x, we get

$3+y' = \dfrac{dz}{dx}$

$\Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{dz}{dx} - 3}$

∴ from ①, we have,

$\dfrac{dz}{dx} - 3 = z^2$

$\Rightarrow \dfrac{dz}{dx} = z^2+3$

$\Rightarrow \dfrac{dz}{z^2+(\sqrt{3})^2} = dx$

integrating on both sides, we get,

$\Rightarrow \dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{z}{\sqrt{3}}\right) = x + c$

$\int \dfrac{1}{x^2+a^2}\,dx = \dfrac{1}{a}\tan^{-1}(x/a)$

$\Rightarrow \dfrac{1}{\sqrt{3}} \tan^{-1}\left(\dfrac{3x+y+4}{\sqrt{3}}\right) = x+c$

$\Rightarrow \tan^{-1}\left(\dfrac{3x+y+4}{\sqrt{3}}\right) = \sqrt{3}(x+c)$

$\Rightarrow 3x+y+4 = \sqrt{3}\tan[(\sqrt{3})(x+c)]$.

2 $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ —— (1)

Sol put $x+y = z$

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$$

∴ from (1), we have,

$$\frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{dx} = (1 + \cos z) + \sin z$$

$$= 2\cos^2(z/2) + 2\sin z/2 \cos z/2$$

$$= 2\cos^2(z/2)\left[1 + \tan(z/2)\right]$$

$$\Rightarrow \frac{dz}{2\cos^2(z/2)\left(1+\tan(z/2)\right)} = dx$$

$$\Rightarrow \int \frac{\sec^2(z/2)}{2\left[1+\tan(z/2)\right]} dz = \int dx$$

$$\Rightarrow \log\left[1 + \tan(z/2)\right] = x + c$$

$$\Rightarrow \boxed{\log\left[1 + \tan\left(\frac{x+y}{2}\right)\right] = x + c.}$$

3 $\frac{dy}{dx} + 1 = e^{x+y}$

$$\Rightarrow \frac{dy}{dx} = e^{x+y} - 1$$ —— (1)

put $x+y = z$

$$\boxed{\frac{dy}{dx} = dz/dx - 1}$$

$\therefore$ ① $\equiv \frac{dz}{dx} - 1 = e^z - 1$

$\Rightarrow \frac{dz}{dx} = e^z$

$\Rightarrow -e^{-z} = x + c$

$\Rightarrow -e^{-(x+y)} = x + c.$

$\Rightarrow \boxed{e^{-(x+y)} + x + c = 0}$

4 $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

$\Rightarrow \frac{\cos x}{\sin x}dx + \frac{e^y}{e^y+1}dy = 0$

$\Rightarrow \log(\sin x) + \log(e^y+1) = \log c$

$\Rightarrow \boxed{(\sin x)(e^y+1) = c}$

5 $\frac{dy}{dx} = \sec(x+y)$ ——— ①

put $x+y = z$

$\boxed{\frac{dy}{dx} = \frac{dz}{dx} - 1}$

$\therefore$ ① $\equiv \frac{dz}{dx} - 1 = \sec z.$

$\frac{dz}{1+\sec z} = dx$

$\Rightarrow \left(\frac{\cos z}{1+\cos z}\right)dz = dx$

$\Rightarrow \left(1 - \frac{1}{1+\cos z}\right)dz = dx$

$\Rightarrow \left(1 - \frac{1}{2\cos^2 z/2}\right)dz = dx$

$\Rightarrow \left(1 - \frac{1}{2}\sec^2 z/2\right)dz = dx$

$$\Rightarrow z - \tan\left(\frac{z}{2}\right) = x + c$$

$$\Rightarrow x + y - \tan\left(\frac{x+y}{2}\right) = x + c$$

$$\Rightarrow \boxed{y - \tan\left(\frac{x+y}{2}\right) = c}$$

* Third Form :-

Differential eqns of the form

$$\frac{dy}{dx} = \frac{(ax+by)+c}{m(ax+by)+c_1} \quad (\text{or}) \quad \frac{dy}{dx} = \frac{m(ax+by)+c}{ax+by+c_1} \quad \text{——— (1)}.$$

put $ax+by = z$

$$\Rightarrow a + by' = z'$$

$$\Rightarrow y' = \frac{1}{b}(z' - a)$$

$$\therefore (1) \equiv \frac{1}{b}\left(\frac{dz}{dx} - a\right) = \frac{z+c}{mz+c_1}$$

$$\Rightarrow \frac{dz}{dx} = \frac{b(z+c)}{mz+c_1} + a$$

Now seperate the variables, which can be easily solved.

Problems.

→ solve the following diff. eqns:

[1] $\dfrac{dy}{dx} = \dfrac{x-y+3}{2x-2y+5}$

Sol $\dfrac{dy}{dx} = \dfrac{x-y+3}{2(x-y)+5}$ ——— (1)

put $x - y = z$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$\therefore (1) \equiv 1 - \frac{dz}{dx} = \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = \frac{z+2}{2z+5}$

$\Rightarrow \frac{2z+5}{z+2}\, dz = dx$

$\Rightarrow \left(2 + \frac{1}{z+2}\right) dz = dx$

$$z+2 \overline{\big)\, 2z+5 \,(}\, 2 \quad (Q)$$
$$\underline{2z+4}$$
$$1\,(R)$$

$Q + \frac{R}{D}$

Integrating on both sides, we get

$\Rightarrow 2z + \log(z+2) = x + c$

$\Rightarrow 2(x-y) + \log(x-y+2) = x + c.$

$\Rightarrow \boxed{x - 2y + \log(x-y+2) = c}$

$\boxed{2}$ $\frac{dy}{dx} = \frac{2y-x-3}{2x-4y+5}$

$= \frac{2y-x-3}{2(x-2y)+5}$ —— (1)

put $2y - x = z$

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{dz}{dx} + 1\right]$

$\therefore (1) \equiv \frac{1}{2}\left[\frac{dz}{dx} + 1\right] = \frac{z-3}{-2z+5}$

$$4z-11 \overline{\big)\, -2z+5 \,(}\, -\tfrac{1}{2}$$
$$\underline{+2z - \tfrac{11}{2}}$$
$$-\tfrac{1}{2}$$

$\Rightarrow \frac{dz}{dx} = \frac{4z-11}{-2z+5}$

$\Rightarrow \left[\frac{-2z+5}{4z-11}\right] dz = dx$

$\Rightarrow \int \left[-\frac{1}{2} - \frac{1}{2}\frac{1}{(4z-11)}\right] dz = \int dx + c$

$$\Rightarrow -\tfrac{1}{2}z - \tfrac{1}{2}\frac{\log(4z-11)}{4} = x + c$$

$$\Rightarrow \boxed{-\tfrac{1}{2}[2y-x] - \tfrac{1}{8}\log[4(2y-x)-11] = x + c}$$

which is the reqd eqn

* Homogeneous Differential eqns:

* Homogeneous function :—

A function $f(x,y)$ is said to be a homogeneous function of degree 'n' in x & y. if $f(kx, ky) = k^n f(x,y)$ $\forall$ n & k is const.

Ex: (1) $f(x,y) = \dfrac{x^2+y^2}{x^3+y^3} \Rightarrow f(kx,ky) = \dfrac{k^2x^2+k^2y^2}{k^3x^3+k^3y^3}$

$$= k^{-1}\frac{(x^2+y^2)}{(x^3+y^3)}$$

$$= k^{-1} f(x,y).$$

$\therefore f(x,y)$ is a homogeneous function of degree -1 in x & y.

(2) $f(x,y) = \dfrac{\sqrt[3]{x} + \sqrt[3]{y}}{x+y} \Rightarrow f(kx,ky) = \dfrac{\sqrt[3]{kx} + \sqrt[3]{ky}}{kx+ky}$

$$= k^{1/3-1}\left(\frac{\sqrt[3]{x}+\sqrt[3]{y}}{x+y}\right)$$

$$= k^{-2/3} f(x,y).$$

$\therefore f(x,y)$ is homo. function of degree $-2/3$.

(3) $f(x,y) = \cos x + \tan y$

$\Rightarrow f(kx,ky) = \cos kx + \tan ky$

$\neq k^n f(x,y)$.

$\therefore f(x,y)$ is not a homogeneous function.

(4) $f(x,y) = \dfrac{(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}}$

$\Longrightarrow f(kx, ky) = k^0 f(x,y)$

$\therefore$ $f(x,y)$ is a homogenous fn. of degree zero in x & y

Note:- If $f(x,y)$ is a homogenous fn of degree zero then $f(x,y)$ is a function of y/x (or) $\frac{x}{y}$.

* Homogeneous Diff. equation :-

A diff. eqn is said to be homogeneous, if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$.

where f, g are homogenous functions of same degree in x & y.

Working rule:-

step(1): put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

step(2): put the above values in the given diff. eqn

step(3): seperate the variables and integrate.

step(4): Replace v by y/x to get the reqd solution.

Problems:

problems

* Solve the following diff. eqns

→ $x^2y\,dx-(x^3+y^3)\,dy=0$.

$\Rightarrow \dfrac{dy}{dx}=\dfrac{x^2y}{x^3+y^3}$ —(1)

clearly (1) is homo. diff. eqn.

put $y=vx$

$y'=v+x\dfrac{dv}{dx}$

$(1)\equiv v+x\dfrac{dv}{dx}=\dfrac{x^2vx}{x^3+v^3x^3}$

$=\dfrac{v}{1+v^3}$

$\therefore x\dfrac{dv}{dx}=\dfrac{\cancel{v}-\cancel{v}-v^4}{1+v^3}$

$\Rightarrow \dfrac{1+v^3}{v^4}\,dv=-\dfrac{1}{x}\,dx$

$\Rightarrow \int\left(\dfrac{1}{v^4}+\dfrac{1}{v}\right)dv=-\int\dfrac{1}{x}\,dx+c$

$\Rightarrow -\dfrac{1}{3v^3}+\log v=-\log x+c.$

$\Rightarrow -\dfrac{1}{3}\left(\dfrac{x^3}{y^3}\right)+\log\left(\dfrac{y}{x}\right)+\log x=c$ $\left(\because \dfrac{y}{x}=v\right)$

$\Rightarrow \boxed{-\dfrac{1}{3}\left(\dfrac{x^3}{y^3}\right)+\log y=c.}$

→ $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-x/y\right)dy=0$

Sol $\dfrac{dx}{dy}=\dfrac{-e^{x/y}\left(1-x/y\right)}{1+e^{x/y}}$ —(1)

put $x=vy$

$\Rightarrow \boxed{x'=v+y\dfrac{dv}{dy}}$

$(1)\equiv v+y\dfrac{dv}{dy}=\dfrac{-e^{+v}(1-v)}{1+e^{v}}$

$\Rightarrow y\dfrac{dv}{dy}=\dfrac{-e^{+v}\cancel{+ve^{v}}-v-\cancel{ve^{v}}}{1+e^{v}}$

$\Rightarrow y\dfrac{dv}{dy}=-\dfrac{\left[v+e^{v}\right]}{1+e^{v}}$

$\Rightarrow \int\left[\dfrac{1+e^{v}}{v+e^{v}}\right]dv=-\int\dfrac{1}{y}\,dy+\log c$

$\Rightarrow \log\left(v+e^{v}\right)=-\log y+\log c$

$\Rightarrow \log\left(\dfrac{x}{y}+e^{x/y}\right)=-\log y+\log c$ $\left(\because \dfrac{x}{y}=v\right)$

$\Rightarrow \log\left[\left(\dfrac{x}{y}+e^{x/y}\right)y\right]=\log c$

$\Rightarrow \boxed{x+y\,e^{x/y}=c}$

→ $\dfrac{dy}{dx}=\dfrac{y}{x}+e^{y/x}$ —(1)

put $y=vx$

$\Rightarrow y'=v+x\dfrac{dv}{dx}$

$(1)\equiv v+x\dfrac{dv}{dx}=v+e^{v}$

$\Rightarrow x\dfrac{dv}{dx}=e^{v}$

$\Rightarrow \int e^{-v}\,dv=\int\dfrac{1}{x}\,dx+c$

$\Rightarrow -e^{-v}=\log x+c$

$\Rightarrow \boxed{-e^{-\frac{y}{x}}=\log x+c}$

→ $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$ —(1)

put $y=vx\Rightarrow y'=v+x\dfrac{dv}{dx}$

$(1)\equiv v+x\dfrac{dv}{dx}=\dfrac{x+vx}{x-vx}$

$\Rightarrow x\dfrac{dv}{dx}=\dfrac{1+v}{1-v}-v$

$=\dfrac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$

$=\dfrac{1+v^2}{1-v}$

$\Rightarrow \frac{1-v}{1+v^2}$

$\Rightarrow \int\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right) dv = \int \frac{1}{x} dx + c$

$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log x + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log\left[\left(\sqrt{1+v^2}\right)\cdot x\right] + c$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log\left[\sqrt{\left(1+\frac{y^2}{x^2}\right)}\, x\right] + c.$

* Non-Homogeneous Diff. eqns :-

(or)

Eqns Reducible to Homo form :-

Diff. eqns of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{--- (1)}$$

Case (i) $\frac{a}{a_1} \neq \frac{b}{b_1}$ i.e $ab_1 - a_1b \neq 0$

working rule:

put $x = X+h$; $y = Y+k$ where h & k are constants.

$dx = dX$; $dy = dY$

$\Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$

$\therefore (1) \equiv \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a_1(X+h)+b_1(Y+k)+c_1}$

$= \frac{aX+bY+(ah+bk+c)}{a_1X+b_1Y+(a_1h+b_1k+c_1)}$ ⟶ (2)

choosing h & k s.t

$ah+bk+c=0$

$a_1h+b_1k+c_1=0$

solving these equations ~~we get~~

we the values of h & k

i.e $\frac{h}{bc_1-b_1c} = \frac{k}{ca_1-ac_1} = \frac{1}{ab_1-a_1b}$

$\Rightarrow h = \frac{bc_1-b_1c}{ab_1-a_1b}$, & $k = \frac{ca_1-ac_1}{ab_1-a_1b}$.

$(2) \equiv \frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$

which is clearly homogeneous.

This eqn can be solved by putting $Y = VX$

finally replacing Y by $y-k$ and X by $x-h$.

Case (ii): $\frac{a}{a_1} = \frac{b}{b_1}$ i.e $ab_1 - a_1b = 0$

$\therefore$ h & k both become infinite.

Hence the method fails.

Now $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{m}$ (say)

$\Rightarrow a_1 = am$; $b_1 = bm$

$\therefore (1) \equiv \frac{dy}{dx} = \frac{ax+by+c}{max+mby+c_1}$

$= \frac{(ax+by)+c}{m(ax+by)+c_1}$

put $ax+by = z$

This can be easily solved by variable seperable problems

$\longrightarrow \frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ — (1)

$\frac{1}{1} \neq \frac{1}{-1}$ i.e $\frac{a}{a_1} \neq \frac{b}{b_1}$

clearly (1) is not homo.

put $x = X+h$; $y = Y+k$

$dx = dX$; $dy = dY$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$

$\therefore (1) \equiv \frac{dY}{dX} = \frac{Y+k+X+h-2}{Y+k-X-h-4}$

$= \frac{(X+Y)+(h+k-2)}{(-X+Y)+(-h+k-4)}$ — (2)

choosing h, k s.t

$h+k-2=0$ — (3)

$-h+k-4=0$ — (4)

$2k = 6$

$\Rightarrow \boxed{k=3}$

$\boxed{h=-1}$

$\therefore (2) \equiv \frac{dY}{dX} = \frac{X+Y}{-X+Y} \rightarrow (5)$

put $Y = vX$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$\therefore (5) \equiv v + X\frac{dv}{dX} = \frac{X(1+v)}{X(-1+v)}$

$\Rightarrow X\frac{dv}{dX} = \frac{1+v}{-1+v} - v$

$\Rightarrow X\frac{dv}{dX} = \frac{1+v+v-v^2}{-1+v}$

$= \frac{1+2v-v^2}{v-1}$

$\Rightarrow \frac{v-1}{1+2v-v^2}\,dv = \frac{1}{X}dX$

$\Rightarrow \int \left[\frac{-\frac{1}{2}(-2v+2)}{-v^2+2v+1}\right] dv = \int \frac{1}{X}dX + c_1$

$\Rightarrow -\frac{1}{2}\log[-v^2+2v+1] = \log X + \log C$

$\Rightarrow \log\left(\frac{1}{[-v^2+2v+1]^{1/2}}\right) = \log(XC)$

$\Rightarrow \frac{1}{\left[-\frac{Y^2}{X^2}+2\frac{Y}{X}+1\right]^{1/2}} = XC$

$\Rightarrow \frac{\cancel{X}}{[-Y^2+2YX+X^2]^{1/2}} = \cancel{X}C$

$\Rightarrow \frac{1}{[-(y-k)^2+2(y-k)(x-h)+(x-h)^2]^{1/2}} = C$

$\Rightarrow \frac{1}{[-(y-3)^2+2(y-3)(x+1)+(x+1)^2]^{1/2}} = C$

HW $\rightarrow \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$.

Exact Differential equations:

Defn: The diff. equation $M(x,y)\,dx + N(x,y)\,dy = 0$ where M & N are functions of x & y, is called an exact diff. equation if $Mdx + Ndy = 0$ is an exact derivative of x & y.

i.e, $Mdx + Ndy = du$, where u is a function of x & y.

$$Mdx + Ndy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

$xy = c$

$\frac{d}{dx}(xy) = 0$

$x\frac{dy}{dx} + y = 0$

$xdy + ydx = 0$

$\therefore d(xy) = xdy + ydx$

Ex (1) $x\,dy + y\,dx = 0$ is an exact.

Because $xdy + ydx = d(xy)$

$$\Rightarrow xdy + ydx = \frac{\partial}{\partial x}(xy)\,dx + \frac{\partial}{\partial y}(xy)\,dy$$

(2) $\frac{1}{x}dy - \frac{y}{x^2}dx = 0$ is an exact.

Because $\frac{1}{x}dy - \frac{y}{x^2}dx = \frac{xdy - ydx}{x^2} = d(y/x)$

Note: The diff. eqn $Mdx + Ndy = 0$ is an exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

working rule:

(1) The diff. eqn $Mdx + Ndy = 0$ is an exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(2) The G.S. is

$$\int\limits_{y\text{-constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

Problems Solve the following diff. equations.

(1) $(x + 2y - 2)\,dx + (2x - y + 3)\,dy = 0$ —— ①

Comparing ① with $Mdx + Ndy = 0$

we have $\frac{\partial M}{\partial y} = 2$; $\frac{\partial N}{\partial x} = 2$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

G.S is $\int (x + 2y - 2)\,dx + \int(-y + 3)\,dy = \int 0$

$$\Rightarrow \frac{x^2}{2} + 2xy - 2x - \frac{y^2}{2} + 3y = c.$$

(2) $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

$\Rightarrow (ax+hy+g)\,dx + (hx+by+f)\,dy = 0$ — ①

$\frac{\partial M}{\partial y} = h \;;\; \frac{\partial N}{\partial x} = h$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

G.S of ① is

$\int (ax+hy+g)\,dx + \int (by+f)\,dy = \int 0$

$\Rightarrow a\frac{x^2}{2} + hxy + gx + b\frac{y^2}{2} + fy = c$

$\Rightarrow ax^2 + by^2 + 2gx + 2fy + 2hxy = 2c.$

Hw $\boxed{3}$ $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xy\,e^{xy^2} - 3y^2\right)dy = 0$

$\boxed{4}$ $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Sol^n: $(e^y+1)\cos x\,dx + e^y \sin x\,dy = d\left[(e^y+1)\sin x\right]$

$\therefore$ G.S. is $(e^y+1)\sin x = c$

96 $\boxed{5}$. $y\sin 2x\,dx - (y^2+\cos^2 x+1)\,dy = 0$ — ①

Sol^n: $M = y\sin 2x \;;\; N = -y^2+\cos^2 x+1$

$\Rightarrow \frac{\partial M}{\partial y} = \sin 2x \;;\; \frac{\partial N}{\partial x} = 2\cos x\sin x = \sin 2x.$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

G.S. of ① is

$\int y\sin 2x\,dx - \int (y^2+1)\,dy = \int 0$

$\Rightarrow -y\frac{\cos 2x}{2} - \frac{y^3}{3} - y = c_1$

$\Rightarrow y\frac{\cos 2x}{2} + \frac{y^3}{3} + y = c$ where $c = -c_1$

93 [6] $\left\{y\left(1+\frac{1}{x}\right)+\cos y\right\}dx+\left(x+\log x-x\sin y\right)dy=0$ — (1)

Sol^n $M = y\left(1+\frac{1}{x}\right)+\cos y$; $N = x+\log x - x\sin y$

$\frac{\partial M}{\partial y} = 1+\frac{1}{x}-\sin y$; $\frac{\partial N}{\partial x} = 1+\frac{1}{x}-x\sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ G.S. of (1) is

$$\int\left[y\left(1+\tfrac{1}{x}\right)+\cos y\right]dx+\int 0\,dy=\int 0$$

$$\Rightarrow \boxed{y(x+\log x)+x\cos y = c}$$

[7]. $(2ax+by)y\,dx+(ax+2by)x\,dy=0$

Sol^n $M = 2axy+by^2$; $N = ax^2+2byx$

$\frac{\partial M}{\partial y} = 2ax+2by$; $\frac{\partial N}{\partial x} = 2ax+2by$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

G.S. is $\int(2axy+by^2)dx+\int 0\,dy=\int 0$

$$\Rightarrow \boxed{ax^2y+bxy^2=c}$$

[8]. $(x^2-ay)dx-(ax-y^2)dy=0$

Sol^n $M = x^2-ay$; $N = -(ax-y^2)$

$\frac{\partial M}{\partial y} = -a$; $\frac{\partial N}{\partial x} = -a$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

G.S. is $\int(x^2-ay)dx+\int y^2dy=\int 0$

$$\Rightarrow \frac{x^3}{3}-axy+\frac{y^3}{3}=c_1$$

$$\Rightarrow \boxed{x^3-3axy+y^3=c}.$$ where $c=3c_1$

10. $\frac{dy}{dx} = \frac{2x-y}{x+2y-5}$

Sol^n $\Rightarrow (2x-y)dx \quad (x+2y-5)dy = 0$ —— ①

$M = 2x-y \quad ; \quad N = -(x+2y-5)$

$\frac{\partial M}{\partial y} = -1 \quad ; \quad \frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

G.S is $\int(2x-y)\,dx - \int(2y-5)\,dy = \int 0$

$\Rightarrow \boxed{x^2 - xy - y^2 + 5y = c}$

H.W 11. $(x^2+y^2+a^2)y\,dy + (x^2+y^2-a^2)x\,dx = 0$

H.W 12. $x(x^2+3y^2)dx + y(y^2+3x^2)dy = 0$

H.W 13. $(a^2-2xy-y^2)dx - (x+y)^2dy = 0$

14. $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol^n: $M = 1+e^{x/y} \quad ; \quad N = e^{x/y}\left(1-\frac{x}{y}\right)$

$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) \quad ; \quad \frac{\partial N}{\partial x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$

$= -e^{x/y}\cdot\frac{x}{y^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ G.S is

$\int(1+e^{x/y})\,dx + \int 0\,dy = \int 0$

$\Rightarrow x + \frac{e^{x/y}}{1/y} = c$

$\Rightarrow \boxed{x + y\,e^{x/y} = c}$

Integrating factor:

Some times $Mdx+Ndy=0$ is not exact but it can be made exact by multiplying throughout by a suitable non-zero function $\mu(x,y)$. This multiplier is called the integrating factor.

Note: If the given diff. eqn can be transformed into the following formulas then the equations are exact.

(1) $d(xy) = xdy + ydx.$

(2) $d(y/x) = \frac{xdy-ydx}{x^2}$

(3) $d(x/y) = \frac{ydx-xdy}{y^2}$

(4) $d(y^2/x) = \frac{2xydy-y^2dx}{x^2}$

(5) $d(x^2/y) = \frac{2xydx-x^2dy}{y^2}$

(6) $d(y^2/x^2) = \frac{2x^2ydy-2xy^2dx}{x^4}$

(7) $d\left(\frac{x^2}{y^2}\right) = \frac{2y^2xdx-2x^2ydy}{y^4}$

(8) $d(e^y/x) = \frac{xe^ydy-e^ydx}{x^2}$

(9) $d(e^x/y) = \frac{ye^xdx-e^xdy}{y^2}$

(10) $d[\log|xy|] = \frac{xdy+ydx}{xy}$

(11) $d\left[\log\left|\frac{y}{x}\right|\right] = \frac{xdy-ydx}{xy}$

(12) $d\left[\log\left|\frac{x}{y}\right|\right] = \frac{ydx-xdy}{xy}$

(13) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy-ydx}{x^2+y^2}$

(14) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{ydx-xdy}{x^2+y^2}$

(15) $d\left[\log\sqrt{x^2+y^2}\right]$ (or) $d\left[\frac{1}{2}\log(x^2+y^2)\right] = \frac{xdx+ydy}{x^2+y^2}$

(16) $d\left(\frac{-1}{xy}\right) = \frac{xdy+ydx}{x^2y^2}$

Ex:① $2y\,dx + x\,dy = 0$ —— ①

$M = 2y$; $N = x$

$\frac{\partial M}{\partial y} = 2$; $\frac{\partial N}{\partial x} = 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Multiplying ① with x, we get

$2xy\,dx + x^2 dy = 0$ —— ②

$\frac{\partial M}{\partial y} = 2x$; $\frac{\partial N}{\partial x} = 2x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ eqn ② is an exact

$\therefore x$ is an integrating factor of

$2y\,dx + x\,dy = 0$

and $\underline{2xy\,dx + x^2 dy} = d(x^2 y)$.

Ex:-② $y\,dx - x\,dy = 0$ —— ①

$\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ ① is not exact

Multiplying ① by $\frac{1}{x^2}$ we get,

$\frac{y}{x^2}dx - \frac{1}{x}dy = 0$ —— ②

$\frac{\partial M}{\partial y} = \frac{1}{x^2}$; $\frac{\partial N}{\partial x} = \frac{1}{x^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ ② is an exact.

$\therefore \frac{1}{x^2}$ is I.F. of ①.

and $-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$

$\Rightarrow \frac{-y\,dx + x\,dy}{x^2} = d(y/x)$

Also $\dfrac{y\,dx - x\,dy}{y^2} = d(x/y)$;

$\dfrac{y\,dx - x\,dy}{xy} = d\left[\log(x/y)\right]$;

and $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}(x/y)\right)$

$\therefore$ $\dfrac{1}{y^2}, \dfrac{1}{xy}, \dfrac{1}{x^2+y^2}$ are Integrating factors of $y\,dx - x\,dy = 0$

From the above example we observe that a diff. eqn has more than one I.F.

Problems

[1]. $x\,dy - y\,dx + 2x^3\,dx = 0$

Sol^n $\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$

$\Rightarrow d(y/x) + 2x\,dx = 0$

Integrating, we get

$$\boxed{y/x + x^2 = c}$$

[2] $\dfrac{x\,dy - y\,dx}{x^2+y^2} = x\,dx$

Sol^n: $\Rightarrow d\left(\tan^{-1}(y/x)\right) = x\,dx$

Integrating we get

$$\boxed{\tan^{-1}(y/x) = \frac{x^2}{2} + c}$$

[3] $x\,dy - y\,dx = xy^2\,dx$

Sol^n $\Rightarrow \dfrac{y\,dx - x\,dy}{y^2} + x\,dx = 0 \Rightarrow d(x/y) + x\,dx = 0$

Integrating, we get

$$\boxed{\frac{x}{y} + \frac{x^2}{2} = c}$$

[4] $y\,dx - x\,dy + (1+x^2)\,dx + x^2 \sin y\,dy = 0$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} - \left(\dfrac{1}{x^2}+1\right)dx - \sin y\,dy = 0$

$\Rightarrow d(y/x) - \left(\dfrac{1}{x^2}+1\right)dx - \sin y\,dy = 0$

Integrating, we get

$\boxed{y/x - \left(x - \dfrac{1}{x}\right) + \cos y = c}$

[5]. $y \sin 2x\,dx = (1+y^2+\cos^2 x)\,dy$

$\Rightarrow \cos^2 x\,dy - 2y \sin x \cos x\,dx + (1+y^2)\,dy = 0$

$\Rightarrow d(y\cos^2 x) + (1+y^2)\,dy = 0$

Integrating, we get

$\boxed{y\cos^2 x + y + \dfrac{y^3}{3} = c}$

[6]. $y(2x^2 y + e^x)\,dx - (e^x + y^3)\,dy = 0$

$\Rightarrow e^x y\,dx - e^x dy + 2x^2 y^2 dx - y^3 dy = 0$

$\Rightarrow \dfrac{e^x y\,dx - e^x dy}{y^2} + 2x^2 dx - y\,dy = 0$

$\Rightarrow d(e^x/y) + 2x^2 dx - y\,dy = 0$

Integrating, we get

$\boxed{e^x/y + \dfrac{2}{3}x^3 - \dfrac{y^2}{2} = c}$

[7]. $x\,dy = [y + x\cos^2(y/x)]\,dx$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2} = \dfrac{1}{x}\cos^2(y/x)\,dx$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2 \cos^2 y/x} = \dfrac{1}{x}\,dx$

$\Rightarrow \sec^2(y/x)\left[\dfrac{x\,dy - y\,dx}{x^2}\right] = \dfrac{1}{x}\,dx$

$\Rightarrow d[\operatorname{Tan}(y/x)] = \dfrac{1}{x}\,dx$.

Integrating, we get $\boxed{\operatorname{Tan}(y/x) = \log x + c}$

8. $\left(y+\cos y+\frac{1}{2\sqrt{x}}\right)dx+(x-x\sin y-1)\,dy=0$

$\Rightarrow (y\,dx+x\,dy)+(\cos y\,dx - x\sin y\,dy)+\frac{1}{2\sqrt{x}}dx-dy=0$

$\Rightarrow \quad d(xy)+d(x\cos y)+\frac{1}{2\sqrt{x}}-dy=0$

Integrating

$$\boxed{xy+x\cos y+\sqrt{x}-y=c}$$

9. $(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)dy=0$

$\Rightarrow xy^2(1+2xy)\,dx+x^2y\,(1-xy)\,dy=0$

$\Rightarrow \quad y(1+2xy)\,dx+x(1-xy)\,dy=0$

$\Rightarrow \quad (y\,dx+x\,dy)+2xy^2dx-x^2y\,dy=0$

$\Rightarrow \quad \frac{y\,dx+x\,dy}{x^2y^2}+\frac{2}{x}dx-\frac{1}{y}dy=0$

$\Rightarrow \quad \frac{d(xy)}{x^2y^2}+\frac{2}{x}dx-\frac{1}{y}dy=0$ —— (2)

Clearly (2) is an exact.

Integrating (2), we get

$-\frac{1}{xy}+2\log x-\log y=\log c.$

$\Rightarrow -\frac{1}{xy}+\log\left(\frac{x^2}{yc}\right)=0$

$\Rightarrow \quad \log\left(\frac{x^2}{yc}\right)=\frac{1}{xy}$

$\Rightarrow \quad \boxed{x^2/yc=e^{\frac{1}{xy}}.}$

10. $(x^2+y^2-a^2)y\,dy+x(x^2+y^2-b^2)dx=0$ —— (1)

$\Rightarrow (x^2+y^2)[2y\,dy+2x\,dx]-2a^2y\,dy-2x\,b^2dx=0$

$\Rightarrow \quad (x^2+y^2)\,d(x^2+y^2)-2a^2y\,dy-2xb^2dx=0$

Integrating.

$\int z\,dz-2a^2\int y\,dy-2b^2\int x\,dx=\int 0$; where $x^2+y^2=z$

$\Rightarrow \frac{z^2}{2} - a^2y^2 - b^2x^2 = c_1$

$\Rightarrow \boxed{(x^2+y^2)^2 - 2a^2y^2 - 2b^2x^2 = c}$ where $c = 2c_1$

11. $x\,dy - (y-x)\,dx = 0$ —— ①

$\Rightarrow x\,dy - y\,dx + x\,dx = 0$

$\Rightarrow \frac{x\,dy - y\,dx}{x^2} + \frac{1}{x}dx = 0$

$\Rightarrow d(y/x) + \frac{1}{x}dx = 0$ —— ②

$\Rightarrow \int d(y/x) + \int \frac{1}{x}dx = \int 0$

$\Rightarrow \boxed{y/x + \log x = c}$

H.W.

12. $x\,dy - y\,dx = xy^2\,dx.$

13. $x\,dx + y\,dy + (x^2+y^2)\,dy = 0$

$\Rightarrow \frac{x\,dx + y\,dy}{x^2+y^2} + dy = 0$

$\Rightarrow d(\log\sqrt{x^2+y^2}) + dy = 0$

Integrating, we get,

$\boxed{\log\sqrt{x^2+y^2} + y = c.}$

14. $x\,dy - y\,dx = (x^2+y^2)\,dx$

$\Rightarrow \frac{x\,dy - y\,dx}{x^2+y^2} = dx$

$\Rightarrow d(\mathrm{Tan}^{-1}(y/x)) = dx$

Integrating, we get

$\boxed{\mathrm{Tan}^{-1}(y/x) = x + C}$

15. $y\,dx - x\,dy + \log x\,dx = 0$

$\Rightarrow -\frac{y\,dx - x\,dy}{x^2} = \frac{\log x}{x^2}$ $\quad \log x = t \Rightarrow \frac{1}{x}dx = dt$ $\Rightarrow x = e^t$

$\int d(y/x) = \int t e^{-t}\,dt + c$

$\Rightarrow y/x = -e^{-t}(t+1) + c$

$\Rightarrow y/x = -\frac{1}{x}(\log x + 1) + c.$

16. $(x^2+y^2-2y)\,dy = 2x\,dx.$

$\Rightarrow (x^2+y^2)\,dy = 2x\,dx + 2y\,dy$

$\Rightarrow \frac{2x\,dx + 2y\,dy}{x^2+y^2} = dy$

$\Rightarrow d(\log(x^2+y^2)) = dy$

Integrating, we get

$\boxed{\log(x^2+y^2) = y + C}$

H.W.

17. $y\,dx - x\,dy = 3x^2 e^{x^3} y^2\,dx.$

18. $(y - xy^2)\,dx - (x + x^2 y)\,dy = 0$

$\Rightarrow y\,dx - x\,dy - xy(y\,dx + x\,dy) = 0$

$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy - d(xy) = 0$

Integrating, we get

$\boxed{\log x - \log y - xy = C}$

19. $y(2xy + e^x)\,dx = e^x\,dy$

$\Rightarrow ye^x\,dx - e^x\,dy + 2xy^2\,dx = 0$

$\Rightarrow d(e^x/y) + 2x\,dx = 0$

Integrating, we get

$\boxed{e^x/y + x^2 = C}$

20. Solve $x^2\left(\frac{dy}{dx}\right) + xy = \sqrt{1 - x^2y^2}$

$\Rightarrow x\left[\frac{x\,dy + y\,dx}{dx}\right] = \sqrt{1-(xy)^2}$

$\Rightarrow \frac{x\,dy + y\,dx}{\sqrt{1-(xy)^2}} = \frac{dx}{x}$

$\Rightarrow \frac{d(xy)}{\sqrt{1-(xy)^2}} = \frac{dx}{x}.$

Integrating, we get

$\boxed{\sin^{-1}(xy) - \log x = C}$

Methods for finding integrating factors:

Method 1: If $Mdx+Ndy=0$ is homogeneous and $Mx+Ny\neq 0$, then $\frac{1}{Mx+Ny}$ is an I.F.

Problems

Find I.F and solve the following equations.

1. $x^2y\,dx-(x^3+y^3)dy=0$ — (1)

Sol^n Comparing (1) with $Mdx+Ndy=0$

$M=x^2y$; $N=-(x^3+y^3)$.

$\frac{\partial M}{\partial y}=x^2$; $\frac{\partial N}{\partial x}=-3x^2$.

$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

(1) is not exact.

$Mx+Ny=x^3y-x^3y-y^4=-y^4\neq 0$

$\Rightarrow \frac{1}{Mx+Ny}=-\frac{1}{y^4}$

Multiplying (1) by $-\frac{1}{y^4}$, we get,

$-\frac{x^2}{y^3}dx+\left(\frac{x^3}{y^4}+\frac{1}{y}\right)dy=0$ — (2)

Comparing (2) with $Pdx+Qdy=0$

$P=-x^2/y^3$; $Q=\frac{x^3}{y^4}+\frac{1}{y}$

$\frac{\partial P}{\partial y}=\frac{3x^2}{y^4}$; $\frac{\partial Q}{\partial x}=\frac{3x^2}{y^4}$.

$\because \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

$\therefore$ (2) is an exact.

it's solution is given by

$\int\left(-\frac{x^2}{y^3}\right)dx+\int\frac{1}{y}dy=\int 0$

$\Rightarrow -\frac{x^3}{3y^3}+\log y=\log c$.

$\Rightarrow \log(y/c)=\frac{x^3}{3y^3}$

$\Rightarrow \boxed{y=ce^{x^3/3y^3}}$

2. $y^2 dx + (x^2 - xy - y^2) dy = 0$

3. $(x^2 + y^2) dx - 2xy\, dy = 0$

4. $xy\, dx - (x^2 + 2y^2) dy = 0$

5. $(x^2 y - 2xy^2) dx - (x^3 - 3x^2 y) dy = 0$

6. $(3xy^2 - y^3) dx - (2x^2 y - xy^2) dy = 0$

7. $(y^3 - 2x^2 y) dx + (2xy^2 - x^3) dy = 0$

Method 2: If $M dx + N dy = 0$ is such that $M = y f_1(x, y)$ and $N = x f_2(x, y)$ i.e, $y f_1(x, y) dx + x f_2(x, y) = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx - Ny}$ is an Integrating factor.

problems:

find I.F & solve.

1. $y(1 + 2xy) dx + x(1 - 2xy) dy = 0$ —— ①

Sol: Comparing ① with $M dx + N dy = 0$.

$M = y(1 + 2xy)$; $N = x(1 - 2xy)$

clearly ① is of the form $y f_1(x, y) dx + x f_2(x, y) dy = 0$

$\therefore Mx - Ny = xy + 2x^2y^2 - xy + 2x^2y^2$

$= 4x^2y^2 \neq 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{4x^2y^2}$$

Multiplying ① by $\frac{1}{4x^2y^2}$

$$\left(\frac{1}{4x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{4xy^2} - \frac{1}{2y}\right) dy = 0 \quad \text{—— ②}$$

Clearly ② is an exact.

Integrating, we get

$$-\frac{1}{4xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c_1$$

$$\Rightarrow \boxed{-\frac{1}{xy} + 2\log(x/y) = c} \quad \text{where } c = 4c_1$$

2. $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$ —— ①

Soln: Comparing ① with $M\,dx + N\,dy = 0$

$M = (xy\sin xy + \cos xy)y$; $N = (xy\sin xy - \cos xy)x$

Clearly ① is of the form $y f_1(x,y)\,dx + x f_2(x,y)\,dy = 0$

$\therefore Mx - Ny = x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy$

$= 2xy\cos xy \neq 0$

$\therefore$ I.F $= \frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$

Multiplying ① by $\frac{1}{2xy\cos xy}$

$\left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0$ —— ②

Clearly ② is an exact.

Integrating, we get

$y\frac{\log|\sec xy|}{y} + \log x - \log y = \log c$

$\Rightarrow \log|\sec xy| + \log\frac{x}{y} = \log c$

$\Rightarrow \log\left|\frac{x}{y}\sec xy\right| = \log c$

$\Rightarrow \frac{x}{y}\sec xy = c$

$\Rightarrow \boxed{x\sec xy = cy.}$

HW

3. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$

4. $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

5. $y(1-xy)\,dx - x(1+xy)\,dy = 0$

6. $y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$

7. $y(1+xy)\,dx + x(1-xy)\,dy = 0$

8. $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$

Method 3: If $Mdx + Ndy = 0$ is such that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ or } k \text{ (constant)}$$

then $I.F = e^{\int f(x)dx}$ or $e^{\int k\,dx}$.

problems:

find I.F & solve the following diff. eqns.

1. $(x^2+y^2+2x)dx + 2y\,dy = 0$ —— ①

Sol: $M = x^2+y^2+2x$; $N = 2y$

$\frac{\partial M}{\partial y} = 2y$; $\frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Now $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2y}(2y) = 1$ (constant)

$I.F. = e^{\int dx} = e^{x}$.

Multiplying ① by e^x.

$(x^2e^x + e^xy^2 + 2xe^x)dx + 2ye^x dy = 0$ —— ②

Clearly ② is an exact.

$\int e^x(x^2+y^2+2x)\,dx = \int 0$

$\Rightarrow (x^2+y^2+2x)e^x - \int (2x+2)e^x\,dx = c$

$\Rightarrow (x^2+y^2+2x)e^x - 2\left[e^x(x-1)+e^x\right] = c$

$\Rightarrow (x^2+y^2+2x)e^x - 2xe^x = c$

$\Rightarrow \boxed{e^x(x^2+y^2) = c}$

(or) $\frac{2x\,dx + 2y\,dy}{x^2+y^2} + dx = 0$

$d\left[\log(x^2+y^2)\right] + dx$

integrating.

$\log(x^2+y^2) + x = \log c \Rightarrow \frac{x^2+y^2}{c} = e^{-x}$

$\Rightarrow e^x(x^2+y^2) = c$

2. $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(x+xy^2)dy=0$ ——— ①

$M = y+\frac{1}{3}y^3+\frac{1}{2}x^2$; $N = \frac{1}{4}(x+xy^2)$

$\frac{\partial M}{\partial y} = 1+y^2$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{4}(1+y^2)$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)} = \frac{3/4}{1/4\, x} = \frac{3}{x}$$

$$I.F = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

Multiplying ① by x^3

$$\left(x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right)dx+\frac{1}{4}(x^4+x^4y^2)dy=0 \text{ ——— ②}$$

Clearly ② is an exact.

Integrating, we get

$$\boxed{\frac{x^4y}{4}+\frac{x^4y^3}{12}+\frac{x^6}{12}=c}$$

3. $(x^2+y^2+1)dx-2xy\,dy=0$

4. $(3xy-2ay^2)dx+(x^2-2axy)dy=0$

5. $(x^2+y^2+x)dx+xy\,dy=0$

6. $(2y^3+2)dx+3xy^2dy=0.$

Method 4: If $Mdx+Ndy=0$ such that $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=f(y)$ (or) k(const)

then $I.F. = e^{\int f(y)dy}$ or $e^{\int k dy}$.

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

2. $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

3. $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$

4. $(y+y^2)dx+xy\,dy=0.$

Method 5: If $Mdx+Ndy=0$ can be put in the form of $x^\alpha y^\beta(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}(m'y\,dx+n'x\,dy)=0$ where α,β; α',β'; m',n'; m,n are constants then the given eqn has an I.F. $x^h y^k$ where h & k must be obtained by applying the condition that the given eqn must become exact after multiplying by $x^h y^k$..

Linear Differential Equations

The first order diff. eqns of the form $\frac{dy}{dx} + P(x)\cdot y = Q(x)$ where $P(x)$ & $Q(x)$ are functions of x only (or) constants

(OR) $\frac{dx}{dy} + P(y)\cdot x = Q(y)$ where $P(y)$ & $Q(y)$ are functions of y only (or) constants. is called a linear diff. eqn.

Note: [1] There are two types of linear diff. eqns.

[2]. To solve linear diff eqns, we take a factor is called "Integrating factor."

working rule:

Type 1: (i) $\frac{dy}{dx} + P(x)\,y = Q(x)$

(ii) Find $I.F = e^{\int P dx}$

(iii) G.S is $y(I.F) = \int Q\,(I.F)\cdot dx + C$

Type 2: (i) $\frac{dx}{dy} + P(y)\,x = Q(y)$

(ii) Find $I.F = e^{\int P dy}$

(iii) G.S is $x\,(I.F) = \int Q\,(I.F)\,dy + C$.

Problems:

[1] Find I.F. of $\sin x\frac{dy}{dx} + 3y = \cos x$ —— (1)

$\Rightarrow \frac{dy}{dx} + (3\operatorname{cosec} x)\,y = \cot x.$

$I.F = e^{\int 3\operatorname{cosec} x\,dx} = e^{3\log\tan(x/2)}$

$= \tan^3(x/2).$

[2]. Solve $\frac{dy}{dx} + y\cot x = 2\cos x.$

$I.F = e^{\int \cot x} = e^{\log\sin x}.$

$\therefore$ G.S. is $y\sin x = \int 2\cos x\cdot\sin x\,dx + C$

$= \int \sin 2x\,dx + C$

$$\boxed{y\sin x = -\frac{\cos 2x}{2} + C}$$

3. $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

$\Rightarrow \dfrac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x.$

$I.F = e^{\int \sec^2 x\, dx}$

$= e^{\tan x}.$

$\therefore$ G.S is

$y\, e^{\tan x} = \int \tan x \sec^2 x\, e^{\tan x} dx + c$

$= \int t e^t dt + c$ $\quad \left\| \begin{array}{l} \tan x = t \\ \Rightarrow \sec^2 x\, dx = dt \end{array} \right.$

$= e^t (t-1) + c$

$= e^{\tan x} (\tan x - 1) + c.$

$$\boxed{y e^{\tan x} = e^{\tan x} (\tan x - 1) + c}$$

4. $\dfrac{dy}{dx} + \dfrac{y}{x} = x^n$

$I.F = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

G.S is

$y x = \int x^{n+1} dx + c$

$= \dfrac{x^{n+2}}{n+2} + c$

5. $x\dfrac{dy}{dx} - 2y = x^2$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{2}{x} y = x$

$I.F = e^{-\int \frac{2}{x} dx} = e^{-2\log x} = \dfrac{1}{x^2}$

G.S is $\quad y\dfrac{1}{x^2} = \int \dfrac{1}{x^2} \cdot x\, dx + c$

$= \int \dfrac{1}{x} dx + c$

$$\boxed{y/x^2 = \log x + c}$$

6. $\dfrac{dy}{dx} + \dfrac{2y}{x} = \sin x.$

$I.F = e^{\int \frac{2}{x} dx} = e^{2\log x} = x^2$

G.S is $yx^2 = \int x^2 \sin x\,dx + c$

$= -x^2\cos x + \int 2x\cos x\,dx + c$

$= -x^2\cos x + 2[x\sin x + \cos x] + c$

$\boxed{7}$. $x\log x\frac{dy}{dx} + y = 2\log x.$

$\Rightarrow \frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$

$I.F = e^{\int\frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x.$

G.S is $y\log x = \int \frac{2}{x}\log x\,dx + c$

$= 2\frac{(\log x)^2}{2} + c$

$= (\log x)^2 + c.$

H.W. $\boxed{8}$. $x\frac{dy}{dx} + 2y = x^2\log x.$

$\boxed{9}$. $(\sin 2x)\frac{dy}{dx} = y + \tan x.$

$\Rightarrow \frac{dy}{dx} - (\operatorname{cosec} 2x)y = \frac{1}{2}\sec^2 x$

$I.F = e^{-\int\operatorname{cosec}2x\,dx} = e^{-\frac{\log(\tan x)}{2}}$

$= e^{\log\frac{1}{\sqrt{\tan x}}}$

$= \frac{1}{\sqrt{\tan x}}$

G.S is

$y\frac{1}{\sqrt{\tan x}} = \int \frac{1}{2}\sec^2 x\frac{1}{\sqrt{\tan x}}dx + c$

$= \frac{1}{2}\int\frac{1}{\sqrt{t}}dt + c$ $\qquad \tan x = t$, $\sec^2 x\,dx = dt$

$= \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + c$

$= \sqrt{\tan x} + c.$

H.W. $\boxed{10}$ $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}.$

11. $\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x-\sqrt{1-x^2}}{(1-x^2)^2}$

$I.F = e^{\int \frac{1}{(1-x^2)^{3/2}} dx}$ let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$= e^{\int \sec^2\theta\, d\theta}$

$= e^{\tan\theta} = e^{\frac{x}{\sqrt{1-x^2}}}$

G.S is $y e^{\frac{x}{\sqrt{1-x^2}}} = \int \frac{x-\sqrt{1-x^2}}{(1-x^2)^2} \cdot e^{\frac{x}{\sqrt{1-x^2}}} dx + c$

$= \int \frac{\sin\theta - \cos\theta}{\cos^4\theta} e^{\tan\theta} d\theta + c.$

$= \int e^{\tan\theta} (\sec^2\theta \tan\theta + \sec^2\theta) d\theta$

$= \int [t e^t dt + e^t] dt$ where $\tan\theta = t$

$= e^t(t-1) + e^t + c$

$= e^{\tan\theta}(\tan\theta - 1) + e^{\tan\theta} + c$

12. $x\cos x \frac{dy}{dx} + (x\sin x + \cos x) y = 1$

$\Rightarrow \frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{\sec x}{x}$

$I.F = e^{\int(\tan x + \frac{1}{x}) dx} = e^{\log|\sec x| + \log x}$

$= x\sec x.$

G.S is

$y x\sec x = \int \sec^2 x\, dx + c$

$= \tan x + c$

13. $(x+2y^3) \frac{dy}{dx} = y$

$y \frac{dx}{dy} = x + 2y^3$

$\Rightarrow \frac{dx}{dy} - \left(\frac{1}{y}\right) x = 2y^2$

$I.F = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$

$\therefore$ G.S is $\frac{x}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c$

$= y^2 + c$

14. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$\Rightarrow \frac{dx}{dy} + \frac{1}{(1+y^2)}x = \frac{\tan^{-1}y}{1+y^2}$

$I.F = e^{\tan^{-1}y}$

G.S is $x e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2} dy + c.$

$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$

15. $y^2 + \left(x - \frac{1}{y}\right)\frac{dy}{dx} = 0$

$\Rightarrow y^2 \frac{dx}{dy} + x = \frac{1}{y}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$

$I.F = e^{\int \frac{1}{y^2} dy} = e^{-1/y}$

G.S is $e^{-1/y} \cdot x = \int \frac{1}{y^3} e^{-1/y} dy + c$

$= e^{-1/y}\left[\frac{1}{y} - 1\right] + c$

Let $\frac{1}{y} = t$

$-\frac{1}{y^2}dy = dt$

$\Rightarrow \frac{1}{y^2}dy = -dt$

16. $(x+y+1)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dx}{dy} = x + y + 1$

$\Rightarrow \frac{dx}{dy} - x = y + 1$

$I.F = e^{-\int dy} = e^{-y}$

G.S is $x e^{-y} = \int (y+1) e^{-y} dy + c$

$= \int y e^{-y} dy + \int e^{-y} dy + c$

$= + \int t e^{t} dt + \int e^{t}(-dt) + c$

$-y = t$

$dy = -dt$

$= +e^{t}(t-1) - e^{t} + c$

$= +e^{-y}(-y-1) - e^{-y} + c$

$= -e^{-y}(y+2) + c$

[17]. $(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$

[18]. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = 0$ ——— ①

putting $\frac{dy}{dx} = p$

$\Rightarrow x\frac{dp}{dx} + p + 1 = 0$ ——— ②

$\Rightarrow \frac{dp}{dx} + \left(\frac{1}{x}\right)p = -\frac{1}{x}$

I.F $= e^{\int \frac{1}{x}dx} = x.$

G.S of ① is

$px = \int -\frac{1}{x}\cdot x\,dx + c$

$xp = -x + c$

$\Rightarrow x\frac{dy}{dx} = -x + c$

$\Rightarrow dy = \left(-1 + \frac{c}{x}\right)dx$

$\Rightarrow y = -x + c\log x + c'$

$\Rightarrow \boxed{x + y = c\log x + c'}$

Equations reducible to linear form:

I. $f'(y)\frac{dy}{dx} + P f(y) = Q$
where P & Q are functions of x only.
putting $f(y) = v$

$$\Rightarrow f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{dv}{dx} + Pv = Q$$

Clearly which is linear.

II. $f'(x)\frac{dx}{dy} + P f(x) = Q$:
where P & Q are functions of y only.

putting $f(x) = v$

$$\Rightarrow f'(x)\frac{dx}{dy} = \frac{dv}{dy}$$

$$\therefore \frac{dv}{dy} + Pv = Q.$$

which is linear.

(III) Bernoulli's equation.

An equation of the form $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$ where P & Q are functions of x alone (or) constants and 'n' is constant such that $n \neq 0$ & $n \neq 1$; is called Bernoulli's diff. equation.

(or)

$\frac{dx}{dy} + P(y)\,x = Q(y)\,x^n$
where P & Q are functions of y alone (or) constants and 'n' is constant such that $n \neq 0$ & $n \neq 1$; is called Bernoulli's diff. eqn.

working rule:

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \quad \text{--- (1)}$$

put $y^{1-n} = z$

$$y^{-n}(1-n)\frac{dy}{dx} = \frac{dz}{dx}.$$

$\Rightarrow y^{-n}\frac{dy}{dx} = \left(\frac{1}{1-n}\right)\frac{dz}{dx}$.

$① \equiv \frac{dz}{dx} + (1-n)\,P(x)z = Q(x)$

Clearly which is linear.

Solve the following diff. eqns.

(1) $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3$ — ①

put $\tan y = t$

$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$① \equiv \frac{dt}{dx} + 2xt = x^3$ — ②

$I.F. = e^{\int 2x\,dx} = e^{x^2}$.

G.S. of ① is

$t\cdot e^{x^2} = \int x^3 e^{x^2}\,dx + c$

$= \frac{1}{2}\int z e^{z}\,dz + c$ Let $x^2 = z$

$2x\,dx = dz$.

$= \frac{1}{2}e^{z}(z-1) + c$ $x\,dx = \frac{1}{2}dz$.

$\tan y \cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

[2]. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing by e^y we get

$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2}$ — ①

But $e^{-y} = t \Rightarrow e^{-y}\frac{dy}{dx} = -\frac{dt}{dx}$

$\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ — ②

$I.F. = \frac{1}{x}$

G.S is

$t\,\frac{1}{x} = \int\left(-\frac{1}{x^2}\right)\cdot\frac{1}{x}\,dx + c$

$= \frac{1}{2x^2} + c$

$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + c$

3. $x\frac{dy}{dx}+y=y^2\log x$.

4. $\frac{dy}{dx}+\frac{xy}{1-x^2}=x\sqrt{y}$

$y^{-1/2}\frac{dy}{dx}+\frac{x}{1-x^2}y^{1/2}=x$ — ①

put $y^{1/2}=t \Rightarrow \frac{1}{2}y^{-1/2}\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow \boxed{y^{-1/2}\frac{dy}{dx}=2\frac{dt}{dx}}$

① ≡ $\frac{dt}{dx}+\frac{x}{2(1-x^2)}t=\frac{x}{2}$

$I.F=e^{\int\frac{x}{2(1-x^2)}dx}=e^{-\frac{1}{4}\int\frac{-2x}{1-x^2}dx}$

$=e^{-\frac{1}{4}\log(1-x^2)}$

$=e^{-\frac{1}{4}\log(1-x^2)}$

$=\frac{1}{(1-x^2)^{1/4}}$

∴ G.S of ① is

$t\frac{1}{(1-x^2)^{1/4}}=\int\frac{1}{(1-x^2)^{1/4}}\left(\frac{x}{2}\right)dx+c$

$=-\frac{1}{4}\int\frac{-2x}{(1-x^2)^{1/4}}dx+c$

$=-\frac{1}{4}\frac{(1-x^2)^{3/4}}{3/4}+c$

$y^{1/2}(1-x^2)^{-1/4}=-\frac{1}{3}\frac{1}{(1-x^2)^{-3/4}}+c$

6. $\frac{dy}{dx}=e^{x-y}(e^x-e^y)$

$=e^{2x}e^{-y}-e^x$

$\Rightarrow \frac{dy}{dx}+e^x=e^{2x}e^{-y}$

$\Rightarrow e^y\frac{dy}{dx}+e^xe^y=e^{2x}$ — ①

put $e^y=t \Rightarrow e^y\frac{dy}{dx}=\frac{dt}{dx}$

$\frac{dt}{dx}+e^xt=e^{2x}$

$I.F=e^{\int e^xdx}=e^{e^x}$.

G.S is

$$te^{e^x} = \int e^{2x} e^{e^x} dx + c$$

$$= \int z e^{z} dz + c \qquad e^x = z$$

$$e^x dx = dz.$$

$$= e^{z}(z-1) + c$$

$$e^y e^{e^x} = e^{e^x}(e^x - 1) + c$$

7. $\frac{dy}{dx}(x^2y^3 + xy) = 1$

$$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$$

$$\Rightarrow \frac{dx}{dy} - yx = x^2y^3$$

$$\Rightarrow x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad \text{--- (1)}$$

put $x^{-1} = t$

$$-x^{-2}\frac{dx}{dy} = \frac{dt}{dy}$$

(1) $\equiv$ $-\frac{dt}{dy} - yt = y^3$

$$\Rightarrow \frac{dt}{dy} + yt = -y^3$$

$$IF = e^{\int y\,dy} = e^{y^2/2}$$

G.S. is

$$te^{y^2/2} = -\int y^3 e^{y^2/2} dy + c$$

$$= -\int 2z e^{z} dz + c \qquad y^2/2 = z$$

$$y^2 = 2z$$

$$= -2e^{z}(z-1) + c \qquad 2y\,dy = 2dz$$

$$y\,dy = dz$$

$$x^{-1}e^{y^2/2} = -2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$$

8. $\frac{dy}{dx} + \frac{y}{x} = y^2$

9. $\frac{dy}{dx} = x^3y^3 - xy$

10. $\frac{dy}{dx} + 1 = e^{x-y}$

11. $xy^2\frac{dy}{dx} - 2y^3 = 2x^3$

12. $x\frac{dy}{dx} + y\log y = xy\,e^{x}.$

13. $2y\cos y^2 \frac{dy}{dx} - \frac{2\sin y^2}{x+1} = (x+1)^3$ —①

put $\sin y^2 = t$

$2y\cos y^2 \frac{dy}{dx} = \frac{dt}{dx}$

①≡ $\frac{dt}{dx} - \frac{2t}{x+1} = (x+1)^3$

I.F. $= e^{-\int \frac{2}{x+1}dx} = e^{-2\log(x+1)} = \frac{1}{(x+1)^2}$

G.S. $t\frac{1}{(x+1)^2} = \int (x+1)\,dx + c$

$$\boxed{\sin y^2 \frac{1}{(x+1)^2} = \frac{x^2}{2} + x + c}$$

14. $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$\Rightarrow (\cos^2 y)^{-1}\frac{dy}{dx} + x\frac{2\sin y\cos y}{\cos^2 y} = x^3$

$\Rightarrow (\cos^2 y)^{-1}\frac{dy}{dx} + 2x\tan y = x^3$ —①

put $\tan y = t$

$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

①≡ $\frac{dt}{dx} + 2xt = x^3$

I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$te^{x^2} = \int e^{x^2} \cdot x^3\,dx + c$

$= \frac{1}{2}\int e^z z\,dz + c$ $\qquad x^2 = z$, $2x\,dx = dz$, $x\,dx = \frac{dz}{2}$

$= \frac{1}{2}e^z(z-1) + c$

$\tan y\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$

H.W. 15. Find the eqn of the curve which passes through the point $(1,-1)$ and satisfies the diff. eqn

$x\frac{dy}{dx} + y = x^2y^2$.

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2001 → $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ —— ①

Dividing by $z(\log z)^2$, we get

$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x}(\log z)^{-1} = \frac{1}{x^2}$ —— ②

put $(\log z)^{-1} = t$

$\frac{(-1)(\log z)^{-2}}{z}\frac{dz}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{1}{z(\log z)^2}\frac{dz}{dx} = -\frac{dt}{dx}$

②≡ $-\frac{dt}{dx} + \frac{1}{x}t = \frac{1}{x^2}$

$\Rightarrow \frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ —— ③

I.F. $= e^{-\int\frac{1}{x}dx} = e^{-\log x} = \frac{1}{x}$

∴ G.S of ③ is

$\frac{1}{x}t = \int -\frac{1}{x^2}\cdot\frac{1}{x}dx + c$

$= -\int x^{-3}dx + c$

$= -\frac{x^{-2}}{-2} + c$

$= \frac{1}{2x^2} + c$

$\therefore \boxed{\frac{1}{x}(\log z)^{-1} = \frac{1}{2x^2} + c}$

Miscellaneous

34(i)

To find the N.C and S.C that the equation $Mdx+Ndy=0$ may be exact.

proof **part 1**: let $Mdx+Ndy=0$ be an exact. ——(1)

then by defn, $Mdx+Ndy=du$

where u is a function of x & y.

and $Mdx+Ndy=du$
$$= \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\Rightarrow M=\frac{\partial u}{\partial x} \;\Big|\; N=\frac{\partial u}{\partial y}.$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x} \;;\; \frac{\partial N}{\partial x}=\frac{\partial^2 u}{\partial x\partial y}.$$

Hence $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ $\left(\because \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}\right)$.

part 2: let $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

To P.T $Mdx+Ndy=0$ is exact.

Let $\int Mdx=u$ ——(2)

where integration has been performed by treating y as constant.

$$\therefore \frac{\partial}{\partial x}\left[\int Mdx\right]=\frac{\partial u}{\partial x}.$$

$$\Rightarrow M=\frac{\partial u}{\partial x}. \quad \text{——(3)}$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial^2 u}{\partial y\partial x}. \quad \text{——(4)}$$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

$\textcircled{4} \equiv \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

integrating both sides w.r.t x by treating y as const.

$\therefore N = \frac{\partial u}{\partial y} +$ a function of y.

$= \frac{\partial u}{\partial y} + f(y)$ say. —— (5)

from (3) & (5), we have

$Mdx + Ndy \equiv \frac{\partial u}{\partial x} dx + \left[\frac{\partial u}{\partial y} + f(y)\right] dy$

$= \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right] + f(y)\, dy$

$= du + f(y)\, dy$

$= d\left[u + \int f(y)\, dy\right]$. —— (6)

which is an exact diff.

Hence $Mdx + Ndy = 0$ is an exact diff. equation.

Cor: solution of an exact diff. eq.

If the equation $Mdx + Ndy = 0$ is exact.

then $Mdx + Ndy = d[u + \int f(y) dy]$

$\Rightarrow Mdx + Ndy = 0 \Rightarrow d[u + \int f(y) dy] = 0$ (by using (6))

integrating both sides, we get

$u + \int f(y) dy = c$.

$\textcircled{2} \equiv u = \int_{y-\text{const}} M dx$

$\textcircled{5} \equiv f(y) =$ terms in N not containing x

$\therefore \int_{y\text{-const}} M dx + \int$ (terms in N containing x) $dy = c$.

34(ii)

Proofs

Method 1:- If $Mdx+Ndy=0$ is a homogeneous and $Mx+Ny\neq 0$ then $\frac{1}{Mx+Ny}$ is an I.F.

Proof. Given that $Mdx+Ndy=0$ —— (1)

where M & N are homogeneous functions of the same degree in x & y.

By Euler's theorem on partial differentiation

$$\left.\begin{aligned} x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y} &= nM \ \& \\ x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y} &= nN \end{aligned}\right\} \quad (2)$$

Now

$$(1)\times\frac{1}{Mx+Ny} \equiv \frac{M}{Mx+Ny}dx+\frac{N}{Mx+Ny}dy=0. \quad (3)$$

It will be exact if

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right)=\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right)$$

i.e
$$\frac{(Mx+Ny)\frac{\partial M}{\partial y}-M\left(x\frac{\partial M}{\partial y}+y\frac{\partial N}{\partial y}+N\right)}{(Mx+Ny)^2}$$

$$=\frac{(Mx+Ny)\frac{\partial N}{\partial x}-N\left(M+x\frac{\partial M}{\partial x}+y\frac{\partial N}{\partial x}\right)}{(Mx+Ny)^2}$$

i.e $\cancel{Mx\frac{\partial M}{\partial y}}+Ny\frac{\partial M}{\partial y}-\cancel{Mx\frac{\partial M}{\partial y}}-My\frac{\partial N}{\partial y}-\cancel{MN}$

$=Mx\frac{\partial N}{\partial x}-Ny\frac{\partial N}{\partial x}-\cancel{MN}-Nx$

$$\Rightarrow N\left(x\frac{\partial M}{\partial x}+\frac{\partial M}{\partial y}y\right)=M\left(x\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)$$

$\Rightarrow N \cdot nM = M \cdot nN$ (by using ②).

which is true. Hence the result.

Alternate Method :—

The given equation is $Mdx + Ndy = 0$ — ①

Where M & N are homogeneous functions of the same degree in x & y.

Now we have

$$Mdx + Ndy \equiv \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$\Rightarrow Mdx + Ndy \equiv \frac{1}{2}\left[(Mx+Ny)\,d(\log xy) + (Mx-Ny)\,d\left(\log\frac{x}{y}\right)\right]$$

Now dividing by $Mx+Ny$ (which is $\neq 0$)

$$\frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2}\left[d(\log(xy)) + \frac{Mx-Ny}{Mx+Ny}\,d\left(\log\frac{x}{y}\right)\right]$$

Since M & N are homogeneous functions of the same degree in x & y,

the expression $\frac{Mx-Ny}{Mx+Ny}$ is homogeneous and equal to a function of $\frac{x}{y}$ say $f\left(\frac{x}{y}\right)$

$$\therefore \frac{Mdx+Ndy}{Mx+Ny} \equiv \frac{1}{2}\,d(\log xy) + \frac{1}{2} f\left(\frac{x}{y}\right) d\left(\log\frac{x}{y}\right)$$

$\Rightarrow \frac{x}{y} = e^{\log(\frac{x}{y})}$,

$\Rightarrow f(\frac{x}{y}) = f\left(e^{\log\frac{x}{y}}\right) = F\left(\log\frac{x}{y}\right)$

$\Rightarrow \frac{M\,dx + N\,dy}{Mx+Ny} \equiv \frac{1}{2}\, d(\log xy) + \frac{1}{2} F(\log\frac{x}{y})\, d(\log\frac{x}{y})$

which is an exact differential.

$\Rightarrow \frac{M\,dx + N\,dy}{Mx+Ny} = 0$

$\Rightarrow \frac{M}{Mx+Ny}\,dx + \frac{N}{Mx+Ny}\,dy = 0$

is an exact diff. eqn.

Method 2: If the equation $M\,dx + N\,dy = 0$ is of the form $y f_1(xy)\,dx + x f_2(xy)\,dy = 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor (I.F.), provided $Mx-Ny \neq 0$.

proof The given equation is $M\,dx + N\,dy = 0$ — (1)

where $M = y f_1(xy)$; $N = x f_2(xy)$.

Now we have

$$M\,dx + N\,dy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right)\right]$$

$\Rightarrow M\,dx + N\,dy \equiv \frac{1}{2}\left[(Mx+Ny)\, d(\log xy) + (Mx-Ny)\, d(\log\frac{x}{y})\right]$

Dividing by $Mx-Ny$; $(Mx-Ny \neq 0)$ we get,

$\frac{M\,dx + N\,dy}{Mx-Ny} \equiv \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\, d(\log(xy)) + d\left(\log(\frac{x}{y})\right)\right]$.

$$= \frac{1}{2}\left[\frac{xy f_1(xy) + xy f_2(xy)}{xy f_1(xy) - xy f_2(xy)} d(\log(xy)) + d\left(\log\left(\tfrac{x}{y}\right)\right)\right]$$

$$= \frac{1}{2}\left[f(xy)\, d(\log(xy)) + d(\log x/y)\right]$$

Since $xy = e^{\log xy}$

$$\Rightarrow f(xy) = f(e^{\log xy}) = F(\log xy)$$

$$\therefore \frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2} F(\log xy)\, d(\log xy) + \frac{1}{2} d(\log x/y)$$

which is an exact diff.

$$\Rightarrow \frac{Mdx+Ndy}{Mx-Ny} = 0$$

$$\Rightarrow \frac{M}{Mx-Ny}dx + \frac{N}{Mx-Ny}dy = 0$$ is an exact diff. eqn.

$\therefore \frac{1}{Mx-Ny}$ is an ~~exact diff.~~ integrating factor

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Method 3: If in the equation $Mdx+Ndy=0$

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only $= f(x)$ say (or) K(constant) then $e^{\int f(x)dx}$ is an integrating factor.

Proof: The given equation is $M\,dx + N\,dy = 0$ ——— (1) gu(iv)

where $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ ——— (2)

Multiplying (1) by $e^{\int f(x)dx}$, we get

$$M e^{\int f(x)dx} dx + N e^{\int f(x)dx} dy = 0$$

$\therefore$ It will be exact if

$$\frac{\partial}{\partial y}\left(M e^{\int f(x)dx}\right) = \frac{\partial}{\partial x}\left[N e^{\int f(x)dx}\right]$$

$$\Rightarrow e^{\int f(x)dx} \cdot \frac{\partial M}{\partial y} = e^{\int f(x)dx} \cdot \frac{\partial N}{\partial x} + N e^{\int f(x)dx} \cdot f(x)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + N \cdot f(x).$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x).$$

which is true from (2).

$\therefore e^{\int f(x)dx}$ is an integrating factor of (1)

provided (2) is true.

Method 4: If in the equation $Mdx+Ndy=0$

$\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ is a function of y only $= f(y)$ say (or) K(const.) then $e^{\int f(y)dy}$ is an integrating ~~factor~~ (I.F).

Proof: The given eqn is $Mdx+Ndy=0$ ——— (1)

where $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = f(y)$ ——— (2)

multiplying (1) by $e^{\int f(y)dy}$, we get

$Me^{\int f(y)dy}dx + Ne^{\int f(y)dy}dy = 0$ ——— (3)

$\therefore$ It will be exact if

$\frac{\partial}{\partial y}\left[Me^{\int f(y)dy}\right] = \frac{\partial}{\partial x}\left[Ne^{\int f(y)dy}\right]$

$\Rightarrow Me^{\int f(y)dy}\cdot f(y) + \frac{\partial M}{\partial y}e^{\int f(y)dy} = \frac{\partial N}{\partial x}e^{\int f(y)dy}$.

$\Rightarrow M f(y) + \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = M\cdot f(y)$

$\Rightarrow \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = f(y)$.

$\therefore$ which is true from (2)

and $e^{\int f(y)dy}$ is an integrating factor of (1) and provided (2) is true.