Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

Vector Analysis

(1) If $\vec{P}$ & $\vec{Q}$ are parallel $\bar{P} \times \bar{Q} = 0$

$$\vec{P} \times \vec{Q} = \begin{vmatrix} i & j & k \\ P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \end{vmatrix}$$

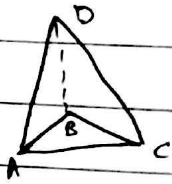
(2) If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ then either $\vec{a} = \vec{b}$
or $\vec{a} - \vec{b}$ is $\perp$ to $\vec{c}$

(3) Scalar triple product $a \cdot (\vec{b} \times \vec{c}) = [a \ b \ c] = [b \ c \ a]$
$= [c \ a \ b] = -[c \ b \ a]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [a \ b \ c] = 0$

If $\vec{a}, \vec{b}, \vec{c}$ are adjacent sides of a parallelopiped then $vol^n = [a \ b \ c]$



A tetrahedron has 4 triangular faces &
Vol^n of tetrahedron
$$= \frac{1}{6} [\vec{AB} \quad \vec{AC} \quad \vec{AD}]$$

(3) Vector Triple Product

$$(\vec{a} \times \vec{b}) \times \vec{c} = (a \cdot c)\vec{b} - (b \cdot c)\vec{a}$$
$$a \times (\vec{b} \times \vec{c}) = (a \cdot c)\vec{b} - (a \cdot b)\vec{c}$$

Tip to remember, however be the bracket, Vector $\vec{B}$ (middle vector) component is always +ve & other vector in bracket component is -ve.

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

one row to a & b each.
one column to c & d each

(4) Reciprocal system of vectors
Set of vectors $(\vec{a},\vec{b},\vec{c})$ & $(\vec{a}',\vec{b}',\vec{c}')$ are reciprocal
if $\vec{a}\cdot\vec{a}' = \vec{b}\cdot\vec{b}' = \vec{c}\cdot\vec{c}' = 1$
& $a\cdot b' = a'\cdot b = b\cdot c' = b'\cdot c = a\cdot c' = a'\cdot c = 0$

$$a' = \frac{\vec{b}\times\vec{c}}{[a\ b\ c]} \qquad b' = \frac{\vec{c}\times\vec{a}}{[a\ b\ c]} \qquad c' = \frac{\vec{a}\times\vec{b}}{[a\ b\ c]}$$

(5) Scalar field :- At each point, a scalar value is defined
$\phi(x,y,z)$
Vector field : At each point a vector value is defined
$\vec{F}(x,y,z)$.

(6) Let $\vec{A},\vec{B},\vec{C}$ be diff. vector function of parameter t &
ϕ a scalar function of t. then

(1) $\frac{d}{dt}(\phi\vec{A}) = \frac{d\phi}{dt}\vec{A} + \phi\frac{d\vec{A}}{dt}$

(2) $\frac{d}{dt}[A\ B\ C] = \left[\frac{dA}{dt}\ B\ C\right] + \left[A\ \frac{dB}{dt}\ C\right] + \left[A\ B\ \frac{dC}{dt}\right]$

(3) $\frac{d}{dt}[A\times(B\times C)] = \frac{d\vec{A}}{dt}\times(B\times C) + A\times\left(\frac{dB}{dt}\times C\right) + A\times\left(B\times\frac{dC}{dt}\right)$

(7) 2 ways of representing a curve
(a) Intersection of 2 surfaces $F_1(x,y,z)=0$
& $F_2(x,y,z)=0$
(b) Parametric representation $x=f_1(t)$ $y=f_2(t)$ $z=f_3(t)$

(8) Vector eq. of a line passing through point $\vec{a}$ & direction $\vec{t}$
$\vec{r} = \vec{a} + b\vec{t}$ $b\in\mathbb{R}$

(9) Plane curve :- lies in a single plane

Twisted curve :- Not lieing in a single ~~curve~~ plane.

(10) Unit tangent Vector $\vec{T} = \frac{d\vec{r}}{ds}$ (s is curve length)

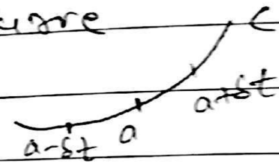
& $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$

(11) $\boxed{\text{Serret Frenet Formula}}$

let T be unit tangent vector & N unit principle normal.

How to find plane of N? → It is osculating Plane

Osculating Plane – To find osc. plane at a point a on curve C, consider 3 points on the curve a, a−δt & a+δt. Consider plane passing through these 3 points say $P_{\delta t}$



Osculating Plane $= \lim_{\delta t \to 0} P_{\delta t}$

~~Bvector~~ Binormal $B = T \times N$. remember as TaNB (Sitla's Tanvi)

$T \times N = B$

Now, the formulae are

$$\frac{dT}{ds} = kN \qquad \frac{dB}{ds} = -\tau N \qquad \frac{dN}{ds} = \tau B - kT$$

k = curvature – measure of curve (here also NBT order, cyclic to TNB, $\frac{dN}{ds}$ = Tanvi + kT Gadi)

τ = torsion – measure of how much curve is going out of current plane.

$\sigma = \frac{1}{\tau}$ = radius of torsion

$\rho = \frac{1}{k}$ = radius of curvature

(12) Obv. $\left|\frac{dT}{ds}\right| = k$ & $\left|\frac{dB}{ds}\right| = \tau$

Given $\vec{r} = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$

Then $T = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$

then $\frac{dT}{ds} = \frac{dT}{dt} \Big/ \frac{ds}{dt}$ & we know $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$

$\therefore$ we know $\frac{dT}{ds}$ so $\left|\frac{dT}{ds}\right| = k$

& $\widehat{\frac{dT}{ds}} = N$ (unit vector)

& $B = T \times N$.

now, $$k = \frac{\left[\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right]}{\left|\frac{dr}{dt}\right|^3}$$

$$\tau = \frac{\left[\frac{dr}{dt} \;\; \frac{d^2r}{dt^2} \;\; \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

As k & τ are dimensionless, so are above formulae. This helps in remembering denominators. Also term in numerator has power 1 & has higher terms. Denominator has more than 1 power.

Proof is very easy. Prepare it.

GRADIENT / DIVERGENCE / CURL

① Let $\vec{r} = f(x,y,z)$ then total differential is given by

$$d\vec{r} = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy + \frac{\partial r}{\partial z} dz$$

② Vector differential operator ∇

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

We can treat it like a simple vector operator for the 3 main operations gradient / divergence / curl

Gradient $\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$ (f is scalar)

Divergence $\nabla \cdot \vec{f} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\left(f_1 i + f_2 j + f_3 k\right)$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Curl $\nabla \times \vec{f} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$

Laplacian ϕ scalar $\rightarrow$ $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

$\vec{F}$ vector $\rightarrow$ $\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$

Laplace equation $\nabla^2 \phi = 0$

function that satisfies Laplace eq. is harmonic function.

③ Properties of Gradient

① A scalar function is constant $\Leftrightarrow \nabla\phi = 0$

② $\nabla(fg) = f\nabla g + g\nabla f$

③ $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

④ Divergence $\nabla \cdot \vec{F} = \sum i \cdot \frac{dF}{dx}$

Curl $\nabla \times \vec{F} = \sum i \times \frac{\partial \vec{F}}{\partial x}$

Try to use Σ form for solving some questions as it can be more efficient than actual arithmatic

⑤ Solenoidal vector $\nabla \cdot \vec{F} = 0$

Irrotational vector $\nabla \times \vec{F} = 0$

⑥ Regarding questions involving functions of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ & $r = |\vec{r}|$

Remember this trick $r^2 = x^2 + y^2 + z^2$

$$\frac{dr}{dx} = \frac{x}{r}, \quad \frac{dr}{dy} = \frac{y}{r}, \quad \frac{dr}{dz} = \frac{z}{r}$$

⑦ Divergence Properties :- Curl Properties:

ϕ scalar A & B vector function

ⓐ $\nabla \cdot (\phi\vec{A}) = \nabla\phi \cdot \vec{A} + \phi(\nabla \cdot \vec{A})$

ⓑ $\nabla \times (\phi\vec{A}) = (\nabla\phi) \times \vec{A} + \phi \text{ curl } A$

ⓒ $\text{div}(A \times B) = B \cdot \text{curl } A - A \cdot \text{curl } B$

ⓓ $\text{curl}(A \times B) = (\nabla \cdot B + B \cdot \nabla) A - (\nabla \cdot A + A \cdot \nabla) B$

Generally in $\nabla \cdot (\ )$ or $\nabla \times (\ )$ questions, Σ are more efficient.

(8) Curl of a gradient is zero. (∵ irrotational)
Divergence of a curl is zero.

(9) $\vec{r} \cdot \frac{d\vec{r}}{dt} = 0 \iff r(t)$ has constant magnitude

$\vec{r} \times \frac{d\vec{r}}{dt} = 0 \iff r(t)$ has constant direction.

(10) Level Surface
Given by $f(x,y,z) = C$ (constant)

∇f is a vector normal to the surface $f(x,y,z) = C$

(11) Directional Derivative in direction $\hat{a}$ is given by $\frac{df}{ds} = \nabla f \cdot \hat{a}$

$\text{Grad } f = \frac{df}{dn} \hat{n}$

So, grad f is in direction where maximum value of directional derivative $\frac{df}{ds}$ occurs.

(12) Equation of tangent plane on curve is given by

$(R - r_0) \cdot \nabla f = 0$ where r_0 is a point on ~~curve~~ surface at which tangent plane is being drawn & ∇f is gradient at r_0.

Equation of normal at r_0 to a given surface

$(R - r_0) \times \nabla f = 0$.

INVARIANCE

(13) Under rotation of axes with origin remaining same, ∇ (del) vector operator remains invariant.

(a) If scalar function ϕ is invariant under rotation
$\rightarrow$ grad ϕ is invariant

(b) If vector function V is invariant under rotation,
$\rightarrow$ div. V is invariant.
$\rightarrow$ Curl V is invariant

As scalar or vector function is defined at a point, obviously it shouldn't matter on your frame of reference for how the function value is changing in space.

(14) Jacobian in changing coordinates & finding ds
Say from (x, y) to (r, θ)

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$\therefore$ $dx\,dy = r\,dr\,d\theta$

(15) In questions asking relationship betn curl, grad, div etc., sometimes using just basic defn also works.

e.g. Prove $\nabla \times (\nabla \times A) = -\nabla^2 A + \nabla(\nabla \cdot A)$

Use basic defn of curl = $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$ then again apply it.

You get the answer.

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~~Frenet~~ Frenet - Serret Formula

(i) $\frac{dT}{ds} = kN$ (ii) $\frac{dB}{ds} = -\tau N$ (iii) $\frac{dN}{ds} = \tau B - kT$

→ (Concept used is if magnitude of $\vec{R}$ moving on a curve is constant, then $\vec{R} \cdot \frac{d\vec{R}}{ds} = 0$ (quite intuitive)

We also use that
B lies in osculating plane & $B \cdot T = 0$
& $\frac{dT}{ds}, \frac{dB}{ds}$ lies in osculating plane & $T \cdot \frac{dT}{ds} = 0$ ($\because |T| = 1$)

Using these we essentially show that $\frac{dT}{ds}$ & $\frac{dB}{ds}$ are perpendicular to T & since they lie in osculating plane, ‖ to

third result $\left(\frac{dN}{ds}\right)$ comes directly by using $N = B \times T$ & taking derivative.

Very simple.

→ $\frac{d\vec{r}}{ds} = T$ & $|T| = 1$ $\therefore$ $T \cdot \frac{dT}{ds} = 0$ (constant magnitude) —(1)

We know $\frac{dT}{ds}$ lies in osculating plane & ~~from~~ (1) it is ⊥ to T.
$\therefore$ it is parallel to N $\therefore \frac{dT}{ds} = kN$. By convention we take +ve k.

→ $B \cdot T = 0$ taking derivative $\left(\frac{dB}{ds} \cdot T\right) + \left(B \cdot \frac{dT}{ds}\right) = 0$

$\therefore \left(\frac{dB}{ds} \cdot T\right) + (B \cdot Nk) = 0$ $\therefore$

but $B \cdot N = 0$ $\therefore$ $\frac{dB}{ds} \cdot T = 0$

$\therefore \frac{dB}{ds}$ ⊥ to T (& already) lies in osculating plane $\therefore$ ‖ to N.
Convention $\frac{dB}{ds} = -\tau N$.

→ $N = B \times T$ $\therefore \frac{dN}{ds} = \left(B \times \frac{dT}{ds}\right) + \left(\frac{dB}{ds} \times T\right) = (B \times kN) - (\tau N \times T)$
$= -kT - (-\tau B)$
$= \tau B - kT$.

So simple.

GREEN / GAUSS / STROKE THM

(1) Simply Connected Region :- If any closed curve in the region can be contracted to a point without going out of region, then it is simply connected.

Otherwise it is multiply connected region.

Simply Connected : Multiply Connected

(2) Line integral $\int F \cdot dr = \int F_1 dx + F_2 dy + F_3 dz$

Flux $= \iint_S \vec{F} \cdot n \, dS$ flux of $\vec{F}$ over surface S

direction of differential surface $d\vec{S}$ is same as $\hat{n}$ & remember it is given by grad ϕ for level surface

so $\iint_S F \cdot n \, dS = \iint_R F \cdot n \dfrac{dx\,dy}{|n \cdot k|}$

where R is projection of surface S on plane XY.

Although we are now working in projection R, don't assume Z becomes 0 in integrand. It's just that you have to express Z in terms of x & y and take appropriate limits on x & y.

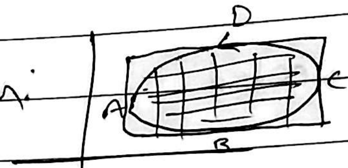
i.e. say you have to find integral on hemisphere $x^2+y^2+z^2=1$ above xy-plane, then you take projection on plane XY & if Z appears in $F \cdot n$, you put it as $\sqrt{1-x^2-y^2}$

Only when the surface intersects with Z=0 plane can we say putting Z=0 gives projection on XY plane.

(3) GREEN's THM In 2-D

$$\oint_C P\,dx + Q\,dy = \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy$$

All 3 proofs of Green / Stroke / Gauss are based on same funda of cutting given curve or surface in 2 halfs.



& ~~calculate~~ name upper curve (surface) as function over x say $f(x)$ & lower as $g(x)$

Then finding higher level integral (always start with double integral for Green / triple for Gauss) boils down to one level below on above curve (surface) as you are reducing y as $f(x)$ or $g(x)$.

& then you get your equation.

Green's thm is Special condition of stroke's thm

$$\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot k\, ds$$

(4) Area enclosed by simple curve using Green's thm

$$\oint_C \frac{x\,dy - y\,dx}{2} \quad \left(\because = \iint_S ds = \text{area}\right)$$

(5) GAUSS DIVERGENCE THM

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint \vec{F}\cdot\hat{n}\,ds$$

i.e. $$\iiint\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz = \iint\left(F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy\right)$$

STOKE'S THM

$$\oint_C F\cdot dr = \iint_S (\nabla\times F)\cdot n\,ds$$

In case of any confusion, just remember there is one ∇ in both formula. Now Gauss formula is known as divergence
$\therefore$ no reason for further confusion
~~& obv. term with ∇ needs higher~~
integral & ∇ appears only in surface integral.

Using Stoke & Gauss to simplify life.

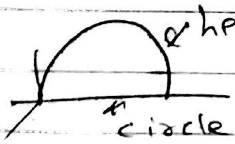
for Gauss we need a closed region & Stokes has an open surface with boundary.

If we consider a closed region & surface S

$$\iint_S (\nabla\times F)\cdot n\,ds = \iiint_V \text{div.}(\nabla\times F)\,dV = \iiint_V 0\,dv = 0$$

(By Gauss)

So, if we have to find $\iint(\nabla\times F)\cdot n\,ds$ over a difficult surface, rather find it on a simple surface that forms a closed region with this surface



So, instead of hemisphere over XY plane, choose circle in XY plane & use

$$\iint_{\text{hemisphere}} F\cdot n\,ds = -\iint_{\text{circle}} F\cdot n\,ds$$

(Be careful of sign of $\hat{n}$, it is outward always)

Potential / Conservative vector field

1. The line integral $\int_C f\,dx + g\,dy + h\,dz$ is independent of path iff. integrand is an exact differential iff curl $\vec{R} = 0$ where $\vec{R} = f\hat{i} + g\hat{j} + h\hat{k}$

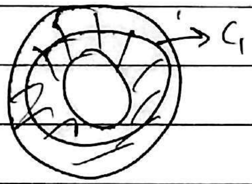
i.e. $\oint_C F \cdot dr = 0$ for any closed curve

i.e. F is a conservative vector field

i.e. F is irrotational

2. ~~Here region R for which C is boundary must be simply connected.~~ O/w above results don't hold

e.g. $F = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$ (undefined at origin)



so $\oint_{C_1} F \cdot dr = 2\pi$ & not 0 due to problem at origin.

3. Sample Question

Check if following form is exact & if yes find the potential

(a) $x\,dx - y\,dy + z\,dz$

$\rightarrow$ let $\vec{F} = x\hat{i} - y\hat{j} + z\hat{k}$

then $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & -y & z \end{vmatrix} = 0$ $\therefore$ Conservative

$\therefore$ exact

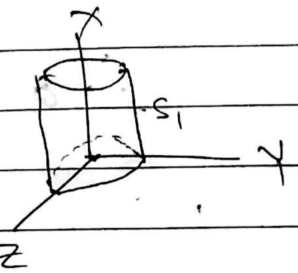
let $\frac{d\phi}{dx} = x$. $\therefore$ $\phi = \frac{x^2}{2} + f_1(y, z)$ — (1)

$$\frac{\partial \phi}{\partial y} = -y \qquad \phi = \frac{-y^2}{2} + f_2(x, z) \qquad -(2)$$

$$\frac{\partial \phi}{\partial z} = z \qquad \phi = \frac{z^2}{2} + f(y, x) \qquad -(3)$$

(1), (2) & (3) gives. $\phi = \dfrac{x^2 - y^2 + z^2}{2}$

Q. Verify divergence thm for $A = 2x^2 y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ in first octant region bounded by $y^2 + z^2 = 9$ & $x = 2$



→ Calculating $\iiint \nabla \cdot A \, dv$ simple.

Now for $\iint A \cdot n \, ds$; ideally there are five surfaces for given 1^{st} octant region.

$X=0$, $X=2$, $Y=0$, $Z=0$ & curved surface S_1.

Now, for some reason, Sir doesn't consider $Y=0$ & $Z=0$ planes. Could be bcoz they are not part of original surface of cylinder. luckily in this question, $A \cdot n$ comes 0 for both these.

So in exam, first find $\iiint \nabla \cdot A \, dv$.

There will be no ambiguity here.

Then first find surface integral for 3 surfaces that we anyways have to consider i.e. $X=0$, $X=2$ & curved surface.

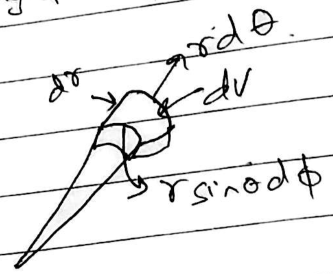
Add these 3 & if it matches div. good.

o/w check $Y=0$ & $Z=0$ surfaces also.

Different Coordinate Systems.

r = radial distance θ = polar angal ϕ = azimutual angle

① Polar coordinates



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$dv = r^2 \sin\theta \, dr\, d\theta\, d\phi$ easily imaginable
& dS would depend on which surface we are talking about.

If it is spherical surface, then r = constant

$\therefore$ ~~dv~~ $dS = r\sin\theta\, d\phi \cdot r\, d\theta = r^2 \sin\theta\, d\theta\, d\phi$

(1) If it is cone, then θ = constant

then $dS = r\sin\theta\, d\phi \cdot dr = r\sin\theta\, dr\, d\phi$

(2) If it is plane through origin (z axis) then ϕ = constant

then $dS = r\, dr\, d\theta$

(3) It is simply understanding what are differential lengths in each direction i.e. dr, $r\,d\theta$, $r\sin\theta\, d\phi$

The unit vectors in this coordinate system is given by

$\vec{e}_r$ ~~dr~~ = vector normal to surface r = constant
$\vec{e}_\theta$ ~~dθ~~ = vector normal to surface θ = constant
$\vec{e}_\phi$ = vector normal to surface ϕ = constant

(2) Cylindrical coordinates (should use ρ instead of r)

$x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad z = z$
$\quad(\rho) \qquad\qquad (\rho)$

so differential lengths in each direction are

dr, $r\,d\theta$ & dz

$\therefore$ differential volume becomes $dv = r\,dr\,d\theta\,dz$

Surface differentials

(1) If it is in XY plane or $\parallel$ $\rightarrow$ z = constant

$dS = r\,dr\,d\theta$ $\quad$ i.e. $dS = \rho\,d\rho\,d\theta$

(2) If it is plane passing through origin (z axis) i.e. θ = constant (ϕ)

Then

$dS = d\rho\,dz$

(3) If it is curved surface of a cylinder i.e. ρ = constant

Then $dS = \rho\,d\phi\,dz$

Same as polar coordinates, e_ρ, e_z, e_ϕ is given by normals to corresponding constant surfaces.

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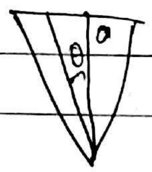
(3) Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{c^2}{z^2}=1$

use $x = a\, r \sin\theta \cos\phi$
$y = b\, r \sin\theta \sin\phi$
$z = c\, r \cos\theta$

$dv = abc\, r^2 \sin\theta \, dr\, d\theta\, d\phi$

(4) A cone capped on top by $z=$ constant say a

Let cone be $x^2+y^2=z^2$



In this case limits will be given by

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a/\cos\theta} \_\_\_\_ \, dr\, d\theta\, d\phi$$

GREEN'S IDENTITIES : Very simple

(1) Green's thm or Green's 2^{nd} identity. (comes from Gauss divergence)

For well behaved ϕ, ψ (cont. etc.)

$$\iiint_V \left(\psi \nabla^2\phi - \phi\nabla^2\psi\right) dv = \iint \left(\psi\nabla\phi - \phi\nabla\psi\right).n\, ds$$

(so elegant & simple!! Similar to Gauss divergence).

→ Consider $F = \phi\nabla\psi$.

Divergence theorem gives $\boxed{\text{Green's First Identity}}$

$$\iiint_V \left(\phi\nabla^2\psi + (\nabla\phi).(\nabla\psi)\right) dv = \iint_S (\phi\nabla\psi).n\, ds \quad -(1)$$

Interchang ϕ & ψ ; get

$$\iiint_V \left(\psi \nabla^2 \phi + (\nabla\phi)\cdot(\nabla\psi)\right) dv = \iint (\psi \nabla \phi)\cdot n \, ds \quad -②$$

① - ② gives (GREEN'S SECOND IDETITY)

$$\iiint_V \left(\psi \nabla^2\phi - \phi \nabla^2\psi\right) dv = \iint_S \left(\psi \nabla\phi - \phi \nabla\psi\right) \cdot n \, ds$$

② now $\nabla\psi = \frac{\partial\psi}{\partial n} \cdot n \quad \therefore \quad \nabla\psi \cdot n = \frac{\partial\psi}{\partial n}$

$\therefore$ Green's 2^{nd} identity is also shown as

$$\iiint \left(\psi \nabla^2\phi - \phi \nabla^2\psi\right) dv = \iint \left(\psi \frac{\partial\phi}{\partial n} - \phi \frac{\partial\psi}{\partial n}\right) ds$$

③ Some very simple problems.

P. T. $\iiint_V \nabla\phi \, dv = \iint_S \phi n \, ds$ (ϕ is scalar $\therefore$ can't use Gauss directly)

$\longrightarrow$ let $F = \phi \vec{C}$ $\quad$ C is a constant vector

$\therefore \nabla \cdot F = \nabla \cdot (\phi \vec{C}) = \nabla\phi \cdot \vec{C} + \phi \cdot \nabla \vec{C} = \nabla\phi \cdot \vec{C}$ $\quad (\nabla \cdot \vec{C} = 0)$

& then apply Gauss div. & take out C.

Similarly Proving $\iiint \nabla \times B \, dv = \iint_S n \times B \, ds$.

$\longrightarrow$ Here take $F = B \times C$ $\quad$ (C is arbitrary constant vector)

$\nabla \cdot F = \text{curl } B \cdot C - B \cdot \text{curl } c = (\text{curl } B) C$ & so on