Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

Partial Differential Equation

(1) $P = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $r = \frac{\partial^2 z}{\partial x^2}$ $S = \frac{\partial^2 z}{\partial x \partial y}$ $t = \frac{\partial^2 z}{\partial y^2}$

In case $z = f(x_1, x_2, \dots, x_n)$
then $P_1 = \frac{\partial z}{\partial x_1}$, $P_2 = \frac{\partial z}{\partial x_2}$ . . . . $P_n = \frac{\partial z}{\partial x_n}$

(2) Formation of PDE
(a) By eliminating arbitrary constants a, b, c etc.
→ If no. of arbitrary constants = ~~order~~ no. of independent variables
then only 1st order Partial derivatives are enough

But if no. of arbitrary constants are more, we have to go for higher order partial derivatives.

e.g. find $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ PDE By eliminating a, b, c.

→ Note we always assume soln is $z = f(x, y)$
∴ diff. w.r.t x & y ⇒ $\frac{2x}{a^2} + \frac{2z}{c^2} p = 0$

$$\frac{2y}{b^2} + \frac{2z}{c^2} q = 0$$

Diff. again ⇒ $\frac{1}{a^2} + \frac{1}{c^2} p^2 + \frac{z}{c^2} \frac{\partial p}{\partial x} = 0$

& $\frac{1}{b^2} + \frac{1}{c^2} q^2 + \frac{z}{c^2} \frac{\partial q}{\partial y} = 0$

∴ $zy \frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0$

Since 3 constants are needed one more level of P.D.

(b) By elimination of arbitrary function

U & V are functions of x, y, z & relation is

$f(u, v) = 0$ where f is arbitrary function.

~~Again we treat~~ z dependent & x, y independent.

If we have 2 arbitrary functions, we might need PDE of order 2 or higher.

e.g. Form PDE for $z = f(x+ay) + F(x-ay)$. (f & F arbit)

$\frac{\partial z}{\partial x} = f'(x+ay) + F'(x-ay)$ $\quad \frac{\partial z}{\partial y} = a f'(x+ay) - aF'(x-ay)$

$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay)$ $\quad \frac{\partial^2 z}{\partial y^2} = a^2 \left[f''(x+ay) + F''(x-ay)\right]$

$$\therefore \frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Generally in such 2 function questions, involving functions of $\alpha x + \beta y$ & $\alpha x - \beta y$, 2nd order gives answer.

e.g. $\phi(x+y+z, x^2+y^2-z^2) = 0$

This is different from above.

Let $u = x+y+z$ $\quad v = x^2+y^2-z^2$

Diff. w.r.t. x partially gives,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

w.r.t. $y \Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$

$$\therefore \frac{(x - zp)}{(1+p)} = \frac{(y - qz)}{1+q}$$

Seems natural but don't get confused at partial diff. w.r.t x; here also z is dependent & x, y independent.

(3) Some equations are solvable by direct integration. But here instead of arbitrary constants, we get arbitrary functions.

e.g. Solve $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

→ integrating twice w.r.t. x & keeping y constant:

$\frac{\partial^2 z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = x f(y) + g(y)$

integrating w.r.t. y & keeping x fixed, we get

$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\underbrace{\int f(y)dy}_{u(y)} + \underbrace{\int g(y)dy}_{v(y)} + w(x)$

$\therefore z + x^3y^2 - \frac{1}{4}\cos(2x-y) = x\,u(y) + v(y) + w(x)$

(4) Quasi-Linear PDE

Lagrange linear equation of 1^{st} order is $Pp + Qq = R$
where P, Q, R are functions of x, y, z.
This is quasi-linear equation.

The general solution is $f(u,v) = 0$
where f is an arbitrary function &
u & v are sol^n of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$

note that sol^n is $f(u,v) = 0$ & not $f(u,v) = c$.

(5) 3 methods of solving $\frac{dx}{P} = \frac{dy}{Q}$, $\frac{dx}{P} = \frac{dz}{R}$, $\frac{dy}{Q} = \frac{dz}{R}$

(a) If variable separable applicable separately, apply & get u, v

(b) If one sol^n can be found, find $u(x,y,z) = c_1$ & put it in other diff. eq. for simplifying.

$f(u,v)=0$ means you have to find some function involving u & v. So if you have to find a surface passing through given equations & fitting $f(u,v)=0$; try to manipulate given eq. & find a relationship for u & v.

e.g. $f(y^2+z,\ z^2+x)=0$ & surface passing through $x+z=2$ & $y^2+z^2=1$

→ adding these 2 we get $y^2+z+z^2+x=3$ → this easily is req. surface.

(C) Use componendo-dividendo

i.e. $\frac{dx}{P}=\frac{dy}{q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+mq+nR}=\frac{l_2dx+m_2dy+n_2dz}{l_2P+m_2q+n_2R}$

if $l_1dx+m_1dy+n_1dz=0$ great

o/w try $\frac{l\,dx+m\,dy+n\,dz}{lP+mq+nR}=d(\phi)$

e.g. (1) $P+3q=5z+\tan(y-3x)$

$\therefore \frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$

this gives $y-3x=C_1$

$\therefore \frac{dy}{3}=\frac{dz}{5z+\tan C_1}$ $\therefore \frac{1}{3}y-\frac{1}{5}\log\left[5z+\tan C_1\right]=C_2$

$\therefore f\left(y-3x,\ \frac{1}{3}y-\frac{1}{5}\log\left(5z+\tan(y-3x)\right)\right)=0$

(2) $x(y^2-z^2)P+y(z^2-x^2)q=z(x^2-y^2)$

$\rightarrow \frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$

$\therefore x\,dx+y\,dy+z\,dz=0$ & $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

(3) $\cos(x+y)P+\sin(x+y)q=z$

$\rightarrow \frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}$

$\therefore \frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\cos(x+y)-\sin(x+y)}$

$\therefore \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)=d(x-y)$

$\therefore \log\left[\cos(x+y)+\sin(x+y)\right]=x-y+C_1$ ... & so on.

(5) PDE with more than 2 independent variables

$P_1 p_1 + P_2 p_2 + P_3 p_3 + \dots + P_n p_n = R$

again sol^n is $f(u_1, u_2, \dots, u_n) = 0$ where $u_i(x_1, x_2, \dots, x_n, z) = c_i$ are solutions of

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

(6) Integral surfaces passing through a given curve

(a) Say curve is given in parametric from $x(t), y(t), z(t)$.

Then we put $u(x(t), y(t), z(t)) = c_1$

$v(x(t), y(t), z(t)) = c_2$

& by eliminating t, we get a relation bet^n c_1 & c_2.

Then replace c_1, c_2 by $u(x,y,z)$ & $v(x,y,z)$.

(b) They give curve & $\phi(x,y,z)=0$ & $\psi(x,y,z)=0$

we again get relation in c_1, c_2 & then put u, v.

e.g. (1) $(x-y)p + (y-x-z)q = z$ through circle $z=1$ & $x^2+y^2=1$

Sol^n we get are

$x+y+z = c_1$

& $\frac{y-x-z}{z^2} = c_2$ Putting $z=1$ & $x^2+y^2=1$, we get

$x+y = c_1 - 1$ & $y-x = c_2+1$

$\therefore c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$

$$\therefore (x+y+z)^2 + \frac{(y-x-z)^2}{z^4} - 2(x+y+z) + 2\frac{(y-x-z)}{z^2} = 0$$

PARTIAL DIFFERENTIAL EQUATION

① Types of solutions of first order PDE

ⓐ Complete Integral / Complete Solution

Let PDE be $F(x, y, z, p, q) = 0$ & sol^n $\phi(x, y, z, a, b) = 0$

The solution which contains as many arbitrary constants as are independent variables is called complete integral.

A particular solution is obtained by giving specific value to a & b.

ⓑ Singular Integral

Singular solution is obtained by eliminating a & b between 3 equations $f(x, y, z, a, b) = 0$

$$\frac{\partial f}{\partial a} = 0 \quad \& \quad \frac{\partial f}{\partial b} = 0$$

ⓒ General Integral

If $b = \phi(a)$, we get subfamily of solutions.

The envelope of this subfamily is the general integral & it is obtained by eliminating a between equations

$$f(x, y, z, a, \phi(a)) = 0 \quad \& \quad \frac{\partial f}{\partial a} = 0.$$

② Charpits Method for solving 1st order PDE

Charpits Auxillary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

Trick to remember :- When we have p or q above, denominator has derivatives in terms of x, y, z. When we got x, y, z above, denominator derivatives are of p or q.

Also last 3 terms are same as auxillary eq. for quasi-linear PDE which we have already done

Working Rule for Charpit's Method

(1) Use simplest possible equation from auxillary eq. to get p & q in terms of (x, y, z). Original PDE f is also used in this endeavour. Rather, lets say you got p as function of x, y, z then you should put that in f & try to find q rather than solving some other diff. eq for q.

This is bcoz by putting p function in f; you don't add unnecessary arbitrary constant which are added when you further integrate for q.

(2) Put these values of p&q in $dz = p\,dx + q\,dy$ & solve to get complete integral.

e.g. Find complete, singular & general integrals of
$$(p^2+q^2)\,y = qz$$
$$\therefore f \Rightarrow (p^2+q^2)y - qz = 0$$

eq. are
$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-2p^2y+qz-2q^2y}$$

first 2 eq. give $p^2+q^2 = a^2$

$$\therefore a^2y = qz \Rightarrow \boxed{q = \frac{a^2y}{z}} \Rightarrow \boxed{p = \frac{a}{z}\sqrt{z^2-a^2y^2}}$$

$$\therefore dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\therefore \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

$$\therefore z^2-a^2y^2 = (ax+b)^2$$

which is the complete integral.

Singular Integral $\Rightarrow$ diff. w.r.t. a & b $\Rightarrow$

$ax+b=0$ & $ax^2+bx+ay^2=0$

$\therefore$ $a=0$ & $b=0$ $\Rightarrow$ $z=0$ is the singular soln.

General integral :- Let $b=\phi(a)$ then

$$z^2-a^2y^2=\left[ax+\phi(a)\right]^2 \quad \text{--- (1)}$$

Diff. partially w.r.t. a we get

$$-2ay^2=2\left(ax+\phi(a)\right)\left(x+\phi'(a)\right) \quad \text{--- (2)}$$

(1) & (2) together give general integral by eliminating a.

(3) Clairaut Equation

Like ODE, very simple & elegant answer.

Let PDE be $z=px+qy+f(p,q)$

then $z=ax+by+f(a,b)$ is the required complete integral.

(4) A result of charpit's equations are if given PDE is $f(p,q)=0$; we get charpit eq. as

$$\frac{dp}{0}=\frac{dq}{0}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}$$

$\therefore$ $p=C_1$ & $q=C_2$ $\therefore$ $dz=C_1dx+C_2dy$ $\therefore$ $\boxed{z=C_1x+C_2y+C_3}$

We also have $f(C_1,C_2)=0$ thus express C_2 in terms of C_1 & get ~~C_1~~ $z=C_1x+\phi(C_1)y+C_3$.

$\therefore$ 2 constants as required.

(5) Reducing given eq. to $f(p,q)=0$ form (bcoz then life becomes easy)

e.g. $(x+y)(p+q)^2+(x-y)(p-q)^2=0$

in such case we put $X^2=x+y$ & $Y^2=x-y$

then $p=\frac{dz}{dx}=\frac{1}{2X}\frac{dz}{dX}+\frac{1}{2Y}\frac{dz}{dY}$ Similarly we find q

& then eq. becomes $P^2+Q^2=1$ so $f(P,Q)=0$ & so on.

LINEAR PDE WITH CONSTANT COEFFICIENTS

(1) $\frac{\partial z}{\partial x}$ is denoted by D & $\frac{\partial z}{\partial y}$ shown by D'.

linear PDE with constant coefficients

(a) Homogeneous linear PDE

→ Determining Complimentary Function

(1) Let eq. be $(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n) z = f(x,y)$

Replace D by m & D' by 1 to get auxillary eq.

$m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n = 0$

Depending on roots of above eq., following sol^n arise

(i) $m = m_1, m_2, \dots, m_n$ are all distinct

C.F. $= \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$

where ϕ_i are arbitrary functions

(ii) $m = m_1$ (repeated n times)

C.F. $= \phi_1(y+m_1x) + x\phi_2(y+m_1x) + \dots + x^{n-1}\phi_n(y+m_1x)$

(don't mistakenly write $\phi_1(y+m_1x)(1+x+x^2+\dots+x^{n-1})$ bcoz each term has a separate arbitrary function)

(iii) If D^m is factor of PDE then

C.F. $= \phi_1(y) + x\phi_2(y) + \dots + x^{m-1}\phi_m(y)$

(iv) If D'^m is factor of PDE then

C.F. $= \phi_1(x) + y\phi_2(x) + \dots + y^{m-1}\phi_m(x)$

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(6) Another ex. of reducing given eq to $f(p,q)$ form.

$(y-x)(qy-px) = (p-q)^2$

→ Here put $X = x+y$

& $Y = xy$

So if you see $(qy-px)$ try $X = x+y$ & $Y = xy$ combo.

if you see $(x+y)^2(p+1)^2$ try $X^2 = x+y$ & $Y^2 = x-y$.

if you see $(x^2+y^2)(p^2+q^2)$ go for polar i.e. (r,θ)

(2) Particular Integral $\frac{1}{F(D,D')} f(x,y)$

(a) If $F(D,D')$ is homogeneous function of degree n

(i) If $F(a,b) \neq 0$ then if $F(D,D')z = f(ax+by)$ type

Then $P.I. = \frac{1}{F(a,b)} \int\int\int \cdots \int f(v)\, dv \ldots dv$

n times integral of $f(ax+by)$ w.r.t. $ax+by$. n is degree of homogeneous PDE $F(D,D')$

(ii) When $F(a,b) = 0$, it implies $(bD - aD')$ is a factor of $F(D,D')$. Then remember

$$\frac{1}{(bD-aD')^n} \phi(ax+by) = \frac{x^n}{b^n n!} \phi(ax+by)$$

(iii) To find PI for $F(D,D')z = x^m y^n$
or algebraic function of x,y s.t. $(x^2+y^2$ etc.$)$

$\therefore P.I. = \frac{1}{F(D,D')} x^m y^n$. We expand denominator in terms of $\frac{D}{D'}$ if $m<n$ & $\frac{D'}{D}$ if $m>n$.

e.g. Solve ~~[illegible]~~ $r + t = \cos mx \cos ny$

→ C.F. is obvious → $\phi_1(y+ix) + \phi_2(y-ix)$

Then $P.I. = \frac{1}{D^2+D'^2} \frac{1}{2}\left[\cos(mx+ny) + \cos(mx-ny)\right]$

$$= \frac{1}{2}\left[\frac{1}{(D^2+D'^2)} \cos(mx+ny) + \frac{1}{(D^2+D'^2)} \cos(mx-ny)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(m^2+n^2)}\left(\int\int \cos(mx+ny)\right) + \frac{1}{m^2+n^2} \int\int \cos(mx-ny)\right]$$

$$= \frac{-1}{2}\left(\frac{1}{m^2+n^2}\right)\left[\cos(mx+ny) + \cos(mx-ny)\right]$$

e.g. Find P.I. $\frac{1}{(D+2D')(D+3D')}(y-2x)^{-1}$

$$= \frac{1}{(D+2D')}\left[\frac{1}{(D+3D')}(y-2x)^{-1}\right] = \frac{1}{(D+2D')}\left[\frac{1}{(-2+3)}\log(y-2x)\right]$$

$$= \frac{1}{(D+2D')}\log(y-2x) = \frac{x}{(1)1!}\log(y-2x) = x\log(y-2x)$$

(iv) A general method for P.I. → Useful for $\sin ax$ etc.

$$\frac{1}{(D-mD')}\phi(x,y) = \int \phi(x, a-mx)\,dx$$

After integration a is replaced by $y+mx$. This is important.

So if we can reduce $F(D,D')$ then this funda can be applied repeatedly for each factor

$$\frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')}\phi(x,y)$$

e.g. $(D^2+DD'-6D'^2)z = y\sin x$

$$P.I. = \frac{1}{D-2D'}\left[\frac{1}{D+3D'}(y\sin x)\right]$$

$$= \frac{1}{D-2D'}\left[\int (a+3x)\sin x\,dx\right] = \frac{1}{D-2D'}\left[-y\cos x + 3\sin x\right]$$

(reputting $a+3x=y$)

$$= \int -(b-2x)\cos x + 3\sin x\,dx = -y\sin x - \cos x$$

after putting $b = y+2x$

So remember this useful method for non std. functions.

(3) Non-Homogeneous Linear PDE with Constant Coefficients

case (i) Complimentary function when $F(D,D')$ can be resolved into linear factors

Say $F(D,D') \Rightarrow [(D-m_1D'-k_1)(D-m_2D'-k_2)\cdots(D-m_nD'-k_n)]Z=0$

Then

C.F is $Z = e^{k_1x}\phi_1(y+m_1x) + e^{k_2x}\phi_2(y+m_2x) + \cdots + e^{k_nx}\phi_n(y+m_nx)$

If $(D-mD'-k)$ appears r times then

G.S. $= e^{kx}[\phi_1(y+mx) + x\phi_2(y+mx) + \cdots + \phi_r(y+mx)x^{r-1}]$

case (ii) When linear factors of $F(D,D')$ are not possible, we use trial methods mostly e^{ax+by}

e.g. $(2D^4 - 3D^2D' + D'^2)Z = 0$

$\therefore (2D^2-D')(D^2-D')Z=0$

Let $Z = Ae^{hx+ky} \Rightarrow Ah^2e^{hx+ky} - Ake^{hx+ky} = 0 \quad (\because D^2-D'=0)$

$\therefore k = h^2$

$\therefore$ One sol^n is $Z = \sum Ae^{hx+h^2y}$

Similarly $(2D^2-D')=0$ gives

$Z = \sum Be^{(h'x+2h'^2y)}$

$\therefore$ G.S. $Z = \sum Ae^{hx+h^2y} + \sum Be^{h'x+2h'^2y}$

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(4) Particular Integral for Non-homo PDE

Very Similar to ODE

case (i) If $f(x,y) = e^{ax+by}$

then $P.I. = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(a,b)} e^{ax+by}$

case (ii) If $f(x,y) = \sin(ax+by)$

then $P.I. = \frac{1}{F(D,D')} \sin(ax+by)$ put $D^2 = -a^2$, $D'^2 = -b^2$ & $DD' = -ab$

case (iii) If $f(x,y) = x^m y^n$ then again expand $[F(D,D')]^{-1}$ in ascending powers of D or D'.

case (iv) $f(x,y) = e^{ax+by} g(x,y)$

$P.I. = \frac{1}{F(D,D')} e^{ax+by} g(x,y) = e^{ax+by} \frac{1}{F(D+a, D'+b)} g(x,y)$

(5) Equations reducible to linear form with constant coefficients

If $A_0 x^n \frac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + A_n y^n \frac{\partial^n z}{\partial y^n} = f(x,y)$

Put $x = e^{u}$ $y = e^{v}$ (like in ODE)

This reduces $x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n}$ to

$D(D-1) \dots (D-m+1) D'(D'-1) \dots (D'-n+1) z$

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6) Finding Characteristics of 2^{nd} order PDE

Let the 2^{nd} order PDE be $Rr + Ss + Tt + f(x,y,z,p,q) = 0$
where R, S, T are functions of x & y.

Then find solutions of quadratic

$$R\lambda^2 + S\lambda + T = 0$$

Let roots be λ_1 & λ_2. Then solve

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

The 2 solutions are characteristics of the 2^{nd} order PDE

e.g. Find characteristics of $y^2 r - x^2 t = 0$

$\longrightarrow$ $\therefore y^2\lambda^2 - x^2 = 0 \Rightarrow \lambda = x/y$ & $-x/y$

$$\therefore \frac{dy}{dx} + \left(\frac{x}{y}\right) = 0 \quad \& \quad \frac{dy}{dx} - \frac{x}{y} = 0$$

$$\therefore x^2 + y^2 = C_1 \quad \& \quad \cancel{x^2+y} \; x^2 - y^2 = C_2$$

$\therefore$ families of circles & hyperbolas are characteristics

Yogesh Kumbhejkar Notes

7) Finding surface that satisfies given conditions & PDE.

e.g. A surface is drawn satisfying $r + t = 0$ & touching $x^2 + z^2 = 1$ along its section by $y = 0$. Find its eq.

$\longrightarrow$ $(D^2 + D'^2)z = 0 \Rightarrow z = \phi_1(y + ix) + \phi_2(y - ix)$

$$\left.\begin{aligned} \therefore p &= i\phi_1'(y+ix) - i\phi_2'(y-ix) \\ q &= \phi_1'(y+ix) + \phi_2'(y-ix) \end{aligned}\right\} (1)$$

now, $x^2 + z^2 = 1 \Rightarrow \left. p = \dfrac{-x}{\sqrt{1-x^2}} \quad q = 0 \right\} (2)$

Imp funda $\rightarrow$ Since surface touches section by $y = 0$, values of p & q must be same on (1) & (2) for $y = 0$

$$\therefore i\phi_1'(ix) - i\phi_2'(-ix) = \frac{x}{\sqrt{1-x^2}} \quad \& \quad \phi_1'(ix) + \phi_2'(-ix) = 0$$

(d)

These give on integration
$$z = \frac{1}{2}\left[\sqrt{1+(y+ix)^2} + \sqrt{1+(y-ix)^2}\right] + C$$

Also, values of Z should be same for $y=0$
$\therefore \sqrt{1-x^2} = \sqrt{1-x^2} + C \quad \therefore C = 0$

So, in these problems equate Z for given conditions & also p and q.

(8) CANONICAL FORM

Let PDE be $\quad Rr + Ss + Tt + f(x, y, z, p, q) = 0$

This can be converted to cannonical form by changing variables x, y to u, v s.t. $u = u(x,y)$ $\quad v = v(x,y)$ (u & v are independent function)

Find u & v is simple.

(a) Solve quadratic equation $R\lambda^2 + S\lambda + T = 0$
Then if solⁿ are λ_1 & λ_2; Solve 2 equation
$$\frac{dy}{dx} + \lambda_1 = 0 \quad \& \quad \frac{dy}{dx} + \lambda_2 = 0$$

Remember this is similar to finding characteristics.

(b) Now let solutions be $f_1(x,y) = C_1$ & $f_2(x,y) = C_2$
Then we put $u = f_1(x,y)$ & $v = f_2(x,y)$ as new variables.

(c) Now we find p, q, r, s, t in terms of u, v, z & then write given PDE in terms of u, v, z.
This is ~~the~~ cannonical form.

(d) If you get distinct roots λ_1, λ_2, all well & good.
If we get repeated root λ, then u is found by $\frac{dy}{dx}+\lambda=0$
& we choose any v which is independent of u.

If we get complex roots say; $u = \alpha + i\beta$ $v = \alpha - i\beta$
then we choose α & β as our new coordinates find
p, q, r, s, t in terms of α, β & Z.

e.g. Reduce given eq. to cannonical form $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$

$\longrightarrow$ $\lambda^2 + x^2 = 0 \Rightarrow \lambda = ix, -ix$

$\therefore$ $u = y + \frac{1}{2} i x^2$ $\quad v = y - \frac{1}{2} i x^2$

$\therefore$ $\alpha = y$ & $\beta = \frac{1}{2}x^2$ our new coordinates.

$\therefore p = \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial \beta}$

$q = \frac{\partial z}{\partial \alpha}$ $\quad r = \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2}$

$t = \frac{\partial^2 z}{\partial \alpha^2}$

In reality, these calculations would be lengthy but do them dilligintely since answer depends on them.

Also, don't forget following while finding 2nd order terms.

$u = xe^y$ $\quad v = ye^x$ then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = e^y \frac{dz}{du} + ye^x \frac{dz}{dv}$

$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^y \frac{\partial z}{\partial u} + ye^x \frac{\partial z}{\partial v}\right)$

$= e^{2y}\frac{\partial^2 z}{\partial u^2} + 2ye^{x+y}\frac{\partial^2 z}{\partial u \partial v} + y^2 e^{2x}\frac{\partial^2 z}{\partial v^2} + ye^x \frac{\partial z}{\partial v}$

(don't forget $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$ would mean $\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x}$, don't just write first term)

9 PRODUCT METHOD OR SEPARATION OF VARIABLES METHOD

Very elegant method where we assume sol^n is $Z = X(x)Y(y)$

e.g. Solve by separation of variables

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

$\rightarrow$ Let sol^n be $Z = X(x)\,Y(y)$

$\therefore X''Y - 2X'Y + XY' = 0$

$\therefore \frac{X'' - 2X'}{X} = \frac{-Y'}{Y}$

Since x & y are independent variables; above each side should be equal to some constant a.

$\therefore \frac{X'' - 2X'}{X} = a \Rightarrow X'' - 2X' - Xa = 0$

$\therefore X = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}$

& $\frac{-Y'}{Y} = a \Rightarrow y = c_3 e^{-ay}$

$\therefore sol^n\ Z = \left(k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x}\right) e^{-ay}$

This method is used extensively in sol^n of wave eq, heat eq. & laplace eq.

This method can be used for higher than 2 variables also assuming $Z = X_1(x_1)X_2(x_2)\ldots X_n(x_n)$

String wave equation $\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{c^2}\frac{d^2y}{dt^2}$

1 dimensional heat eq $\Rightarrow \frac{d^2u}{dx^2} = \frac{1}{k}\frac{du}{dt}$

2 dimensional heat eq $\Rightarrow \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = \frac{1}{k}\frac{du}{dt}$

Applying PDE funde in Physics

(1) Tension :- Pulling force which is exerted by means of string or rod.

Thrust :- Pushing force which is exerted on a body by means of a rod & not string b'coz string is flexible.

(2) Fourier Sine Series :- Let $f(x)$ be defined in range $0 < x < l$ then fourier sine expansion is

$$f(x) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi x}{l} \quad \text{when } E_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

(3) Fourier Cosine Series

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos \frac{n\pi x}{l} \quad \text{where } b_0 = \frac{2}{l}\int_0^l f(x) dx$$

$$\& \quad b_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

Note above formulae are exactly similar to ~~taking~~ dot product with basis vectors of new system.

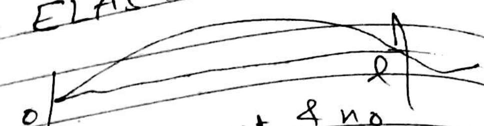
Also, $\frac{b_0}{2}$ in cosine series represent average value of function.

(4) Double fourier Sine Series :- for f defined on $0 \leq x \leq a$ & $0 \leq y \leq b$

$$f(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

(5) VIBRATIONS OF STRETCHED ELASTIC STRING



We assume string is tied on 2 ends & undergoes small vertical movement & no horizontal movements.

In that case eq. is $\frac{d^2y}{dt^2} = c^2 \frac{d^2y}{dx^2}$

let displacement $y(x,t) = X(x)\,T(t)$

$\therefore \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$

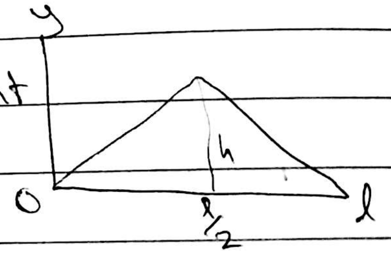
considering boundary conditions, $y(0,t) = y(l,t) = 0 \ \forall t$

Then required solution becomes

$$y(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi c t}{l} + F_n \sin\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l}$$

✯ In all these questions of wave, always use 2 conditions: initial displacement & velocity.

e.g. A taught string is given small displacement h as shown & released from rest. Find $y(x,t)$.



$$\rightarrow y(x,0) = \begin{cases} 2hx/l & 0 \le x \le l/2 \\ \frac{2h(l-x)}{l} & l/2 \le x < l \end{cases}$$

Also, initial velocity $y_t(x,0) = 0$

$$\therefore y(x,t) = \sum_{n=1} \sin\frac{n\pi x}{l}\cos\frac{n\pi c t}{l}$$

$$\& \ E_n = \frac{2}{l}\int_0^l y(x,0)\cdot \sin\frac{n\pi x}{l}\,dx$$

& find E_n coefficients.

HEAT EQUATION

① Let $u(x,t)$ be temperature of a rod. Then one dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

Again after assumming $u(x,t) = X(x)T(t)$ and discarding 2 cases of $K=0$ or $K>0$, we get

$$\boxed{u(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x) e^{-\lambda^2 K t}}$$

(note e^{-t} ensure as $t \to 0$, temp goes to 0 as expected)

Now, depending on initial & boundary conditions, we choose the solution $u(x,t)$.

② (A) When both ends of rod are kept at temperature zero & initial temperature is prescribed

$\therefore \quad u(0,t) = u(a,t) = 0 \ \forall t$

& $u(x,0) = f(x)$ is prescribed.

then we get

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{a}\right) e^{-c_n^2 t}$$

where $E_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

e.g. Solve boundary value problem if $u(x,0) = lx - x^2$

$\longrightarrow \quad E_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} \, dx$

& so on so forth.

(A2) When temp at ends is fixed but not at zero.

$u(0,t) = 2$ & $u(1,t) = 3$ & $u(x,0) = x(1-x)$

$\longrightarrow$ Let $g(x) = x+2$ Let $v(x,t) = u(x,t) - (x+2)$

now, our initial conditions for $v(x,t)$ become,

$V(0,t) = V(1,t) = 0$

& $V(x,0) = x(1-x) - x - 2 = -(x^2+2)$

& we know how to handle these problems.

(A3) Both ends of bar are insulated & initial temp. given

$\therefore u_x(0,t) = u_x(a,t) = 0 \quad \forall t$

Then we get $u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi x}{a} e^{-c_n^2 t}$

$E_0 = \frac{2}{a}\int_0^a f(x)dx$ & $E_n = \frac{2}{a}\int_0^a f(x) \cos \frac{n\pi x}{a} \cdot dx$

(A4) When one end is insulated & other is kept at constant temperature

e.g. One end of bar is kept at $10°c$ & other end is insulated. Also $y(x,0) = 1-x \quad 0<x<1$

$\rightarrow \quad y_x(1,t) = 0, \quad , \quad y(0,t) = 10, \quad y(x,0) = 1-x$

nice trick → Let $y(x,t) = u(x,t) + 10$

now, conditions for $u(x,t)$ are

$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $u_x(1,t) = 0 \qquad u(0,t) = 0$

$u(x,0) = y(x,0) - 10 = -(x+9)$

& now we assume $u(x,t) = X(x)T(t)$ & carry on as usual.

Here solution is $y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-c_n^2 t}$

(3) TWO-DIMENSIONAL HEAT EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k}\left(\frac{\partial u}{\partial t}\right)$$

Let sol^n be $u(x,y,t) = X(x)\,Y(y)\,T(t)$
then we assume

$$\frac{X''}{X} = -m^2 \qquad \frac{Y''}{Y} = -n^2 \qquad \frac{T'}{kT} = -(m^2+n^2)$$

$\therefore X_m(x) = A_m \cos mx + B_m \sin mx$

$Y_n(y) = A_n \cos ny + B_n \sin ny$

$$T_{mn}(t) = F_{mn}\, e^{-(m^2+n^2)kt}$$

e.g. Rectangular plate $x=0, x=a, y=0, y=b$ & these lines maintained at $0^\circ C$. Also $u(x,y,0) = f(x,y)$

$\longrightarrow$ Given that

$u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$

& $u(x,y,0) = f(x,y)$

Again let $u(x,y,t) = X(x)\,Y(y)\,T(t)$

We get $$u_{mn}(x,y,t) = F_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, e^{-\lambda_{mn}^2 t}$$

where $$F_{mn} = \frac{4}{ab}\int_0^a\!\!\int_0^b f(x,y)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,dx\,dy$$

(4) Similarly we can handle 3 dimensional heat eq.

Assume $u(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,T(t)$

Then $X_n(x) = A_n \cos nx + B_n \sin nx$

$Y_m(y) = C_m \cos my + D_m \sin my$

$Z_l(z) = E_l \cos lz + F_l \sin lz$

$$T_p(t) = H_{nml}\, e^{-(n^2+m^2+l^2)kt}$$

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TIPS / TRICKS of PDE

① It is important to remember in all these physics questions the difference between initial & boundary conditions.

Boundary conditions are valid for all time period. Hence we use them first to deduce which trigonometric function is useful in X_n, Y_m etc.

Whereas initial conditions are applicable at $t=0$. So they remove exponential part & allow us to compute coefficients E_n, F_n that we put in above step.

② Good question $P\cos(x+y) + q(\sin x+y) = Z$

$$\therefore \frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dZ}{Z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{d(x-y)}{\cos(\,)-\sin(\,)}$$

eq. ③ & ④ obv. give one sol^n.

Now trick is to use eq. ④ & ⑤.

$$d(x-y) = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)}\, d(x+y)$$

$$\therefore x-y = \log(\cos(\,) + \sin(\,)) + C$$ & so on.

You might get stuck after finding first integral if you don't remember this trick with ④ & ⑤.

Cauchy's Method of Characteristics

(1) We want to solve non-linear PDE here.
A curve is given in parametric form; we want to find integral surface passing through this curve.

Simple method, nothing to worry about

(a) Let given PDE be $f(x,y,z,p,q)=0$
Also let parametric curve be $x=f_1(\lambda)$ ~~z~~ $y=f_2(\lambda)$ $z=f_3(\lambda)$

(b) We first want to find initial values $x_0, y_0, z_0, p_0, q_0, t_0$
They are needed to determine arbitrary constants from the solutions of characteristic strip PDEs.
We take $x_0 = f_1(\lambda)$ $y_0 = f_2(\lambda)$ $z_0 = f_3(\lambda)$ then
p_0, q_0 are solutions of $f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$
& $f(x_0, y_0, z_0, p_0, q_0) = 0$

Once we have initial values, then we solve following 5 equations. (Just take negative of charpit's denominators)

$$\frac{dp}{dt} = -f_x - pf_z \qquad \frac{dq}{dt} = -f_y - qf_z$$

$$\frac{dx}{dt} = f_p \qquad \frac{dy}{dt} = f_q \qquad \frac{dz}{dt} = pf_p + qf_q$$

We find 5-6 solutions by combining different equations. With each solution, we get an arbitrary constant. It is determined by putting initial values x_0, y_0, z_0, p_0, q_0 that we found above. Sometimes we get t as part of above function; initial value t_0 is entirely our choice for convenience.

So, we get $x = \phi_1(\lambda, t)$ $y = \phi_2(\lambda, t)$ $Z = \phi_3(\lambda, t)$
Eliminate λ, t to get relation between x, y, z

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Orthogonal surfaces to given system of surfaces.

① Let given system be $f(x,y,z)=c$
Find orthogonal system of surfaces.

→ Normal is (f_x, f_y, f_z).
Let new system be $z = g(x,y)$
$\therefore g(x,y) - z = 0$
normal is $(g_x, g_y, -1)$ i.e. $(p, q, -1)$
$(\because p = g_x \;\&\; q = g_y)$

$\therefore$ we want $pf_x + qf_y = f_z$
This is std. quasi linear eq.

$\therefore$ Sol^n is solving $\dfrac{dx}{f_x} = \dfrac{dy}{f_y} = \dfrac{dz}{f_z}$

Remember that ~~consta~~ parameter c should completely be on one side & can't be part of f.

e.g. 2013 mains: Find orthogonal system & surface passing through $x^2+y^2=1, z=1$
given system $z(x+y) = c(3z+1)$

→ First take parameter to one side
$\dfrac{z(x+y)}{(3z+1)} = c$

$\therefore$ eq. becomes $\dfrac{dx}{\left(\frac{z}{3z+1}\right)} = \dfrac{dy}{\left(\frac{z}{3z+1}\right)} = \dfrac{dz}{\left(\frac{(x+y)}{(3z+1)^2}\right)}$

On solving we get sol^n as
$\phi\left((x-y), (x+y)^2 - 4z^3 - 2z^2\right) = 0$ ϕ arbitrary.
Given curve is $x^2+y^2=1$ & $z=1$; putting it gives
$(x-y)^2 + (x+y)^2 - 4z^3 - 2z^2 = -4$.

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