IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

IAS MATHEMATICS (OPT.)-2014

PAPER - I : SOLUTIONS

1(a) LA-2014 → Find one vector in $\mathbb{R}^3$ which generates the intersection of V and W, where V is the xy-plane and W is the space generated by the vectors (1,2,3) and (1,−1,1)

Sol^n: Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be the given vectorspace.

Given that V is xy-plane

$$\therefore V = \{(x,y,z) \in \mathbb{R}^3 / z=0 ; x,y,z \in \mathbb{R}\} \rightarrow (1).$$

and given that W is the space generated by the vectors (1,2,3) and (1,−1,1).

For this we find a homogeneous system whose solution set W is generated by

$$S = \{(1,2,3), (1,-1,1)\}$$

$$W = \{\alpha(1,2,3) + \beta(1,-1,1) ; \alpha, \beta \in \mathbb{R}^3\}$$

$$= \{(\alpha+\beta, 2\alpha-\beta, 3\alpha+\beta) ; \alpha, \beta \in \mathbb{R}^3\}$$

$$V = \{(x,y,0) ; x,y \in \mathbb{R}^3\}$$

Now for $V \cap W$

$x = \alpha+\beta$, $y = 2\alpha-\beta$ and $3\alpha+\beta = 0$

$\Rightarrow \beta = -3\alpha$

$\Rightarrow x = -2\alpha$, $y = 5\alpha$ and $z = 0$

$\therefore V \cap W = \{-2\alpha, 5\alpha, 0 ; \alpha \in \mathbb{R}^3\}$

clearly $V \cap W$ is spanned by $(-2,5,0) \in \mathbb{R}^3$

1(b) LA-2014 → Using elementary row or column operations, find the rank of the matrix.

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol'n: Given matrix is

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_4 \text{ and } R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -3 & 1 \end{bmatrix} \quad R_2 \rightarrow R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad R_4 \rightarrow 2R_4 + R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in a echelon form and has 3 independent rows.

So, rank of matrix $= 3$.

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1(c) Prove that between two real roots of $e^x \cos x + 1 = 0$ a real root of $e^x \sin x + 1 = 0$ lies.

Solⁿ: Let $x=a$ and $x=b$ be two distinct roots of the given equation $e^x \cos x + 1 = 0$.

$\therefore e^a \cos a = -1$ and $e^b \cos b = -1$

$\Rightarrow \cos a = -e^{-a}$ and $\cos b = -e^{-b}$. —— ①

Let $f(x) = -\cos x - e^{-x}$ $\forall x \in [a,b]$.

(i) Since $\cos x$ and e^{-x} are continuous in $[a,b]$

$\therefore f(x)$ is continuous in $[a,b]$

(ii) $f'(x) = \sin x + e^{-x}$

which exists for all $x \in (a,b)$.

$\therefore f$ is derivable in (a,b)

(iii) $f(a) = -\cos a - e^{-a} = 0$ (by ①)

$f(b) = -\cos b - e^{-b} = 0$ (by ①)

$\therefore f(a) = f(b) = 0$

$\therefore$ The conditions of Rolle's theorem are satisfied.

$\therefore \exists$ atleast one point $c \in (a,b)$ such that $f'(c) = 0$.

$\Rightarrow f'(c) = \sin c + e^{-c} = 0$

$\Rightarrow \sin c = -e^{-c}$

$\Rightarrow e^c \sin c = -1$

$\Rightarrow e^c \sin c + 1 = 0$.

$\Rightarrow x = c \in (a,b)$ is a root of the equation $e^x \sin x + 1 = 0$.

$\therefore e^x \sin x + 1$ has a real root between any two roots of the equation $e^x \cos x + 1 = 0$

1(d) Evaluate $\int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx$.

Sol^n: Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

put $x=\tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$.

when $x=0$, $\theta=0$ and when $x=1$, $\theta=\pi/4$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \quad \text{——— (1)}$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]$$

$$\left[\text{by the property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\right]$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1+\tan\theta)\}\, d\theta$$

$$= \log 2\int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$I = \log 2.\,[\theta]_0^{\pi/4} - I \quad \text{(from (1))}$$

$$\Rightarrow 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow I = \frac{\pi}{8}\log 2.$$

$$\therefore \int_0^{\pi/4} \frac{\log(1+x)}{1+x^2}\, dx = \frac{\pi}{8}\log 2$$

1(e) → Examine whether the plane $x+y+z=0$ cuts the
3D-2014 cone $xy+zx+yz=0$ in perpendicular lines.

Sol'n: From the equation of plane and the cone it is clear that the lines of intersection passes through origin.

Let equation of lines be

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c} \quad \text{--- (1)}.$$

(1) must satisfy equation of plane and cone

So, $a+b+c=0$ --- (2)

$ab+bc+ca=0$ --- (3)

from (2) & (3)

$ab+(a+b)\times -(a+b)=0$

$\Rightarrow (a+b)^2-ab=0$

$\Rightarrow a^2+b^2+ab=0$

$\Rightarrow \left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)+1=0$ --- (4).

$\therefore \frac{a_1a_2}{b_1b_2}=1$; Similarly $\frac{a_1a_2}{c_1c_2}=1$

$\therefore \sum a_1a_2=3a_1a_2=3b_1b_2=3c_1c_2$

So, only those lines of intersection which are in yz, xz (or) xy planes will be perpendicular.

2(a) Let V and W be the following subspaces of $\mathbb{R}^4$:
$V = \{(a,b,c,d) : b-2c+d=0\}$ and
$W = \{(a,b,c,d) : a=d,\ b=2c\}$.
Find a basis and the dimension of (i) V (ii) W (iii) $V \cap W$.

Sol^n: We observe that
$(a,b,c,d) \in V \Leftrightarrow b-2c+d=0$
$\Leftrightarrow (a,b,c,d) = (a,b,c,2c-b)$
$= (a,0,0,0) + (0,b,0,-b) + (0,0,c,2c)$
$= a(1,0,0,0) + b(0,1,0,-1) + c(0,0,1,2)$

This shows that every vector in V is a linear combination of the three linearly independent vectors $(1,0,0,0), (0,1,0,-1)$, $(0,0,1,2)$.
Thus, a basis of V is
$A = \{(1,0,0,0), (0,1,0,-1), (0,0,1,2)\}$
Hence $\boxed{\dim V = 3}$

Now $(a,b,c,d) \in W \Leftrightarrow a=d,\ b=2c$
$\Leftrightarrow (a,b,c,d) = (a,2c,c,a)$
$= (a,0,0,a) + (0,2c,c,0)$
$= a(1,0,0,1) + c(0,2,1,0)$
which shows that W is generated by the linearly independent set $\{(1,0,0,1), (0,2,1,0)\}$
$\therefore$ A basis for W is
$B = \{(1,0,0,1), (0,2,1,0)\}$
and $\boxed{\dim W = 2}$

$(a,b,c,d) \in V \cap W \Leftrightarrow (a,b,c,d) \in V$ and $(a,b,c,d) \in W$
$\Leftrightarrow b-2c+d=0,\ a=d,\ b=2c$
$\Leftrightarrow (a,b,c,d) = (0,2c,c,0)$
$= c(0,2,1,0)$
Hence a basis of $V \cap W$ is $\{(0,2,1,0)\}$
and $\boxed{\dim(V \cap W) = 1}$

2(b)(i) Investigate the values of λ and μ so that the
LA-2016 equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$
have (1) no solution (2) a unique solution,
(3) an infinite number of solutions.

Sol^n: write the matrix equation of the given system.

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] R_3 \to R_3 - R_2$$

$\rightarrow$ If $\lambda=3$ & $\mu\neq 10$ then
$\rho(A|B)=3$ & $\rho(A)=2$
$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ The given equations have no solutions.

$\rightarrow$ If $\lambda \neq 3$ and μ = any value then
$\rho(A|B) = \rho(A) = 3 =$ the number of unknown variables.

$\therefore$ The equations are are consistent and have unique solution.

$\rightarrow$ If $\lambda=3$ and $\mu=10$ then
$\rho(A|B)=\rho(A)=2 <$ the number of unknown variables.

$\therefore$ the given equations are consistent and have infinite solutions.

2(b)(iii) Find the characteristic roots of the matrix
LA-2014
$A=\begin{bmatrix}1&4\\2&3\end{bmatrix}$ and verify Caley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

Sol: The characteristic equation of the matrix A is $|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix}1-\lambda & 4\\ 2 & 3-\lambda\end{vmatrix}=0$

$\Rightarrow (1-\lambda)(3-\lambda)-8=0$ — ①

$\Rightarrow \lambda^2-4\lambda-5=0$

$\Rightarrow (\lambda-5)(\lambda+1)=0$

$\Rightarrow \lambda=5,-1.$

The roots of this equation are $\lambda=5,-1$ and these are characteristic roots of A

By Caley-Hamilton theorem, the matrix A must satisfy its characteristic equation ①

so we must have

$A^2-4A-5I=0$ — ②

verification:-

we have $A^2=\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1&4\\2&3\end{bmatrix}=\begin{bmatrix}9&16\\8&17\end{bmatrix}$

Now $A^2-4A-5I=\begin{pmatrix}9&16\\8&17\end{pmatrix}-\begin{pmatrix}4&16\\8&12\end{pmatrix}-\begin{pmatrix}5&0\\0&5\end{pmatrix}$

$=\begin{pmatrix}0&0\\0&0\end{pmatrix}=0$

Now multiplying (2) by A^{-1}, we get

$A^2A^{-1} - 4AA^{-1} - 5IA^{-1} = 0A^{-1}$

$\Rightarrow A - 4I - 5A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$

Now $A - 4I = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$

The characteristic equation of A is $\lambda^2 - 4\lambda - 5 = 0$. Dividing the polynomial $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = 0$ by the polynomial $\lambda^2 - 4\lambda - 5$, we get

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$

$\therefore A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I$.

But $A^2 - 4A - 5I = 0$

Therefore we have

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$ which is a linear polynomial in A.

2(c) By using the transformation $x+y=u$, $y=uv$, evaluate the integral $\iint \{xy(1-x-y)\}^{1/2}\,dx\,dy$ taken over the area enclosed by the straight lines $x=0$, $y=0$ and $x+y=1$.

Sol^n: Given $x+y=u$, $y=uv$.

$$\Rightarrow x = u - uv$$
$$x = u(1-v)$$

Now $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$$= u - vu + uv$$
$$= u$$

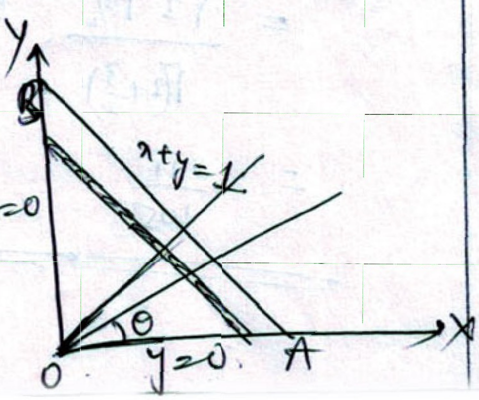
But $dx\,dy = \dfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv$

$\therefore dx\,dy = u\,du\,dv.$

and $\sqrt{xy(1-x-y)} = \sqrt{u(1-v)uv(1-u)}$

$$= uv^{1/2}\sqrt{(1-u)(1-v)}$$

Clearly the region of integration is OAB.



The integration formulae are $x+y=u$, $y=uv$
$$= (x+y)v$$
$$\Rightarrow y = \frac{v}{1-v}x.$$

i.e., clearly the area for new variables is to be divided by the lines parallel to $x+y=1$ and by lines $y=\dfrac{v}{1-v}x$.

i.e., $y = x\tan\theta$

where $\tan\theta = \dfrac{v}{1-v}$

where θ varies from 0 to $\pi/2$ and so

v varies from 0 to 1. and $u = x+y$,

varies from 0 to 1.

i.e, limits of u are 0 to 1.

Hence the given integral

$$= \int_0^1\int_0^1 uv^{1/2}\sqrt{(1-u)(1-v)}\, u\,du\,dv$$

$$= \int_0^1 u^2(1-u)^{1/2}\,du \int_0^1 v^{1/2}(1-v)^{1/2}\,dv$$

$$= \int_0^1 u^{3-1}(1-u)^{\frac{3}{2}-1}\,du \int_0^1 v^{\frac{3}{2}-1}(1-v)^{\frac{3}{2}-1}\,dv$$

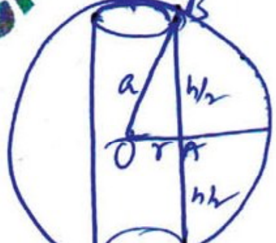
$$= B\left(3,\tfrac{3}{2}\right) B\left(\tfrac{3}{2},\tfrac{3}{2}\right)$$

$$= \frac{\Gamma 3\, \Gamma 3/2}{\Gamma\left(3+\frac{3}{2}\right)} \cdot \frac{\Gamma 3/2\, \Gamma 3/2}{\Gamma 3}$$

$$= \frac{2\pi}{105}.$$

3(a) Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

Sol: Let O be the centre of the sphere of radius 'a'.
Let 'h' be the height and 'r' be the radius of the cylinder.



Then $OA = r = \sqrt{a^2 - h^2/4}$

The volume V of the cylinder $= \pi r^2 h$

$= \pi\left(a^2 - \frac{h^2}{4}\right) h$

$\therefore \frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right)$

For V to be maximum or minimum, we must have

$\frac{dV}{dh} = 0$.

i.e, $\pi\left(a^2 - \frac{3}{4}h^2\right) = 0 \Rightarrow h = \frac{2a}{\sqrt{3}}$

Also, $\frac{d^2V}{dh^2} = -\frac{6h}{4} = -\frac{3}{2}h < 0$ at $h = \frac{2a}{\sqrt{3}}$.

Hence V is maximum when $\boxed{h = \frac{2a}{\sqrt{3}}}$.

and maximum value $= \pi\left(a^2 - \frac{h^2}{4}\right) h$

$= \pi\left[a^2 - \frac{1}{4}\left(\frac{2a}{\sqrt{3}}\right)^2\right] \frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{3\sqrt{3}}$.

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3(b) Find the maximum (or) minimum values of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.
CAL-2014

Soln: Given that

$$f(x,y,z) = x^2+y^2+z^2 \quad \text{---} (1)$$

Subject to the conditions

$$ax^2+by^2+cz^2=1 \text{ --- } (2) \text{ and } lx+my+nz=0 \text{ --- } (3)$$

Let us consider a function F of independent variables x, y, z.

where

$$F = x^2+y^2+z^2+\lambda_1(ax^2+by^2+cz^2-1)+\lambda_2(lx+my+nz) \quad \text{---} (4)$$

$$\therefore dF = (2x+2ax\lambda_1+l\lambda_2)dx+(2y+2by\lambda_1+m\lambda_2)dy$$
$$+(2z+2\lambda_1 cz+n\lambda_2)dz \quad (\because dF = F_x dx+F_y dy+F_z dz) \quad \text{---} (5)$$

At stationary points, $dF=0$.

$$\left.\begin{aligned} \therefore F_x=0 &\Rightarrow 2x+2ax\lambda_1+l\lambda_2=0 \\ F_y=0 &\Rightarrow 2y+2by\lambda_1+m\lambda_2=0 \\ F_z=0 &\Rightarrow 2z+2cz\lambda_1+n\lambda_2=0 \end{aligned}\right\} \text{---} (6)$$

Multiplying (6) by x, y, z respectively and adding we get

$$2(x^2+y^2+z^2)+2(ax^2+by^2+cz^2)\lambda_1+(lx+my+nz)\lambda_2=0$$

$$\Rightarrow 2u+2(1)\lambda_1+0(\lambda_2)=0 \quad \text{where } u=x^2+y^2+z^2$$

$$\Rightarrow \boxed{\lambda_1=-u}$$

from (6), we have

$$2x+2ax(-u)+l\lambda_2=0 \Rightarrow x=\frac{-l\lambda_2}{2(1-au)}$$

$$2y+2by(-u)+m\lambda_2=0 \Rightarrow y=\frac{-m\lambda_2}{2(1-bu)}$$

$$2z+2cz(-u)+n\lambda_2=0 \Rightarrow z=\frac{-n\lambda_2}{2(1-cu)}$$

$$(3) \equiv l\left(\frac{-l\lambda_2}{2(1-au)}\right)+m\left(\frac{-m\lambda_2}{2(1-bu)}\right)+n\left(\frac{-n\lambda_2}{2(1-cu)}\right)=0$$

$$\Rightarrow \lambda_2\left[\frac{l^2}{1-au}+\frac{m^2}{1-bu}+\frac{n^2}{1-cu}\right]=0 \quad \text{--- (7)}$$

If $\lambda_2=0$ then we get $x=y=z=0$.

but $(x,y,z)=(0,0,0)$ doesnot satisfy one of the condition of the constraint (1).

So that $\lambda_2\neq 0$.

from (7), we have

$$\frac{l^2}{1-au}+\frac{m^2}{1-bu}+\frac{n^2}{1-cu}=0$$

which gives the maxima and minima of u

i.e. $u=x^2+y^2+z^2$

3(c)(i) LA-2014 → Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors.

Sol^n: First we find the eigen values

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda+2)\left[\lambda(\lambda-1)-12\right] + 4(\lambda+3) + 3(\lambda+3) = 0$$

$$\Rightarrow -(\lambda+2)(\lambda^2-\lambda-12) + 7(\lambda+3) = 0$$

$$\Rightarrow -(\lambda+2)(\lambda-4)(\lambda+3) + 7(\lambda+3) = 0$$

Solving we get

$$\lambda = -3, -3, 5$$

So, eigen values are $-3, -3, 5$

finding eigen vector

For $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x + 2y - 3z = 0 \Rightarrow z = \frac{x+2y}{3}$$

So, two eigen vectors corresponding to $\lambda = -3$ are

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving we get $x = -z$; $y = -2z$

So, $v_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

So, eigen vectors are.

$$\begin{bmatrix} 1 \\ 0 \\ \frac{1}{3} \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix} \& \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

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3(C)(ii) Prove that the eigen values of a unitary matrix have absolute value 1.
LA IAS-2014

Sol'n: Let A be an unitary matrix

$$\therefore A^{\theta}A = I \quad \text{--- (1)}$$

Let λ be a characteristic root of A.

$\therefore \exists$ a non-zero column matrix.

i.e. characteristic vector x such that

$$Ax = \lambda x \quad \text{--- (2)}.$$

Taking conjugate transpose on the sides, we get

$$(Ax)^{\theta} = (\lambda x)^{\theta}$$

$$\Rightarrow x^{\theta}A^{\theta} = \bar{\lambda}x^{\theta} \quad \text{--- (3)}$$

from (2) & (3), we have

$$\Rightarrow x^{\theta}A^{\theta} = (\bar{\lambda}x^{\theta})(\lambda x)$$

$$\Rightarrow x^{\theta}(A^{\theta}A)x = (\bar{\lambda}\lambda)(x^{\theta}x)$$

$$\Rightarrow x^{\theta}(I)x = |\lambda|^2(x^{\theta}x) \quad (\text{by } (1))$$

$$(\because \bar{\lambda}\lambda = |\lambda|^2)$$

$$\Rightarrow x^{\theta}x = |\lambda|^2(x^{\theta}x)$$

$$\Rightarrow (1-|\lambda|^2)(x^{\theta}x) = 0.$$

$$\Rightarrow 1-|\lambda|^2 = 0 \quad (\because x \neq 0 \Rightarrow x^{\theta}x \neq 0)$$

$$\Rightarrow |\lambda|^2 = 1$$

$$\Rightarrow |\lambda| = 1$$

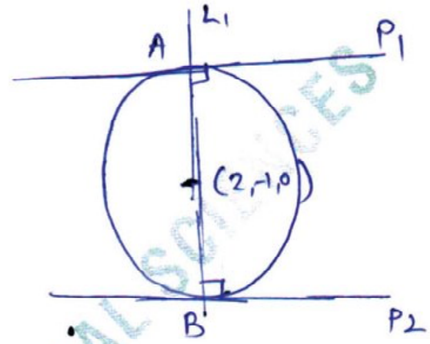
4(a)(i) 3D-2014 Find the coordinates of the points on the sphere $x^2+y^2+z^2-4x+2y=4$, the tangent planes at which are parallel to the plane $2x-y+2z=1$.

Sol'n:

let, the eq^n of planes P_1 & P_2 parallel to

$$2x-y+2z=1$$

$$2x-y+2z+\lambda=0$$



now, of $2x-y+2z+\lambda=0$ be tangent to sphere.

length $\perp$er to P_1 & P_2 = radius of sphere.

$$\left|\frac{2(2)-1(-1)+2(0)+\lambda}{\sqrt{4+1+4}}\right| = 3$$

So, $\lambda = 14, -4$

Now, to find points of contact of tangent plane P_1, P_2 & sphere. [pt. A & B in diag].

eq^n of line 'L_1' normal to tangent plane & passing through centre $(2,-1,0)$ is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r$$

For A & B $\Rightarrow r = \pm 3$

so, $\frac{x-2}{2/3} = \frac{y+1}{-1/3} = \frac{z}{2/3} = \pm 3$

OR $(x,y,z) = (4,-2,2), (0,0,-2)$

4(a)(ii) Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$, represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$.

Sol^n: Let $f(x,y,z,t) = ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$

$\therefore \frac{\partial f}{\partial x}=0$ for $t=1$ gives $2ax+2u=0$

$\Rightarrow x = -u/a$ — (1)

Similarly $\frac{\partial f}{\partial y}=0$ for $t=1$ gives $y=-v/b$ — (2)

$\frac{\partial f}{\partial z}=0$ for $t=1$ gives $z=-w/c$ — (3)

and $\frac{\partial f}{\partial t}=0$ for $t=1$ gives

$ux+vy+wz+d=0$ — (4)

Substituting the values x, y, z from (1), (2), (3) in (4) we get the required condition as

$u(-u/a)+v(-v/b)+w(-w/c)+d=0$

$\Rightarrow \boxed{\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d}$

Which is the required result.

4(b) Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$, at its points of intersection with the plane $lx+my+nz=p$ generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

Sol^n: Let (α, β, γ) be the point of intersection of the given conicoid and the given plane, then we have $a\alpha^2+b\beta^2+c\gamma^2=1$ ——— ①

and $l\alpha+m\beta+n\gamma=p$ ——— ②

Also the equations of the normals to the given conicoid at (α,β,γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$$

∴ The equations of the line through the origin parallel to this line are

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma} \quad ——— ③$$

from ① and ②, we have

$$a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{p}\right)^2$$

$$\Rightarrow p^2(a\alpha^2+b\beta^2+c\gamma^2)=(l\alpha+m\beta+n\gamma)^2$$

$$\Rightarrow p^2\left(\frac{(a\alpha)^2}{a}+\frac{(b\beta)^2}{b}+\frac{(c\gamma)^2}{c}\right)=\left[\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right]^2$$

$$\Rightarrow p^2\left[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right]=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

from ③ eliminating α,β,γ

Hence the line ③ generates the above cone.

Hence proved.

4(c) Find the equations of the two generating lines through any point $(a\cos\theta, b\sin\theta, 0)$ of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ of the hyperboloid by the plane $z=0$.

Sol'n:

Any point on the elliptic section of the hyperboloid is $(a\cos\theta, b\sin\theta, 0)$.

$\therefore$ Equation's of any line through this point is

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}=r \text{ (say)} \quad \text{—— (i)}$$

Any point on this line is $(lr+a\cos\theta, mr+b\sin\theta, nr)$.

and it lies on given hyperboloid, if

$$\frac{(lr+a\cos\theta)^2}{a^2}+\frac{(mr+b\sin\theta)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

OR $$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2+2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r=0. \quad \text{—— (ii)}$$

If the line (i) is generator of given hyperboloid, then (i) lies wholly on the hyperboloid and the condition for which from (ii) are.

$$\left(\frac{l^2}{a^2}\right)+\left(\frac{m^2}{b^2}\right)-\left(\frac{n^2}{c^2}\right)=0 \quad \text{—— (iii)}$$

$$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0 \quad \text{—— (iv)}$$

From (iv) we get,

$$\frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} \quad \text{OR} \quad \frac{(l/a)}{\sin\theta} = \frac{(m/-b)}{\cos\theta}.$$

$$\Rightarrow \frac{(l/a)}{\sin\theta} = \frac{(m/-b)}{\cos\theta} = \frac{\sqrt{(l^2/a^2)+(m^2/b^2)}}{\sqrt{\sin^2\theta+\cos^2\theta}} = \frac{\sqrt{n^2/c^2}}{1} \text{ From(iii)}$$

$$\Rightarrow \frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} = \frac{n}{\pm c}$$

$\therefore$ The equation to the required generator From (i) are

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c}$$

5(a) Justify that a differential equation of the form:

$[y + x f(x^2+y^2)]\,dx + [y f(x^2+y^2) - x]\,dy = 0,$

where $f(x^2+y^2)$ is an arbitrary function of (x^2+y^2), is not an exact differential equation and $\frac{1}{x^2+y^2}$ is an integrating factor for it.

Hence solve this differential equation for

$f(x^2+y^2) = (x^2+y^2)^2.$

sol^n: Given that

$[y + x f(x^2+y^2)]\,dx + [y f(x^2+y^2) - x]\,dy = 0$ —— ①

which is in the form of $Mdx + Ndy = 0$

where $M = y + x f(x^2+y^2)$, $N = y f(x^2+y^2) - x$

$\frac{\partial M}{\partial y} = 1 + f'(x^2+y^2)\cdot 2yx$ & $\frac{\partial N}{\partial x} = y f'(x^2+y^2)(2x) - 1.$

$= 1 + 2xy f'(x^2+y^2)$ $\qquad = -1 + 2yx f'(x^2+y^2).$

Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ Given equation is not an exact differential equation.

To show $\frac{1}{x^2+y^2}$ is an integrating factor.

Multiplying equation ① by $\frac{1}{x^2+y^2}$.

Then

$M = \frac{y + x f(x^2+y^2)}{x^2+y^2}$ & $N = \frac{y f(x^2+y^2) - x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = \frac{[1 + 2xy f'(x^2+y^2)](x^2+y^2) - [y + x f(x^2+y^2)](2y)}{(x^2+y^2)^2}$

$= \frac{x^2 - y^2 + 2xy\left[(x^2+y^2) f'(x^2+y^2) - f(x^2+y^2)\right]}{(x^2+y^2)^2}$

$$\frac{\partial N}{\partial x} = \frac{\left[-1+2xy\,f'(x^2+y^2)\right](x^2+y^2) - \left[y\,f(x^2+y^2)-x\right](2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy\left[(x^2+y^2)f'(x^2+y^2) - f(x^2+y^2)\right]}{(x^2+y^2)^2}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\Rightarrow \frac{1}{x^2+y^2}$ is an integrating factor for equation (1)

Given $f(x^2+y^2) = (x^2+y^2)^2$.

Then equation (1) becomes

$$\left[y + x^2(x^2+y^2)^2\right]dx + \left[y(x^2+y^2)^2 - x\right]dy = 0$$

This can be written as

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\Rightarrow \frac{y\,dx - x\,dy}{x^2+y^2} + (x\,dx + y\,dy)(x^2+y^2) = 0$$

$$\Rightarrow d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) + \frac{x^2+y^2}{2}\,d(x^2+y^2) = 0$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\frac{(x^2+y^2)^2}{2} = C.$$

which is the required solution

5(b). Find the curve for which the part of the tangent cut-off by the axes is bisected at the point of tangency.

Solution: Let $\frac{x}{a}+\frac{y}{b}=1$ be the tangent. It cuts the axes at $(a,0)$ and $(0,b)$. So the mid point of the part of tangent cut-off by the axes is $(\frac{a}{2},\frac{b}{2})$. The slope of this tangent is $\frac{-b}{a}$. $(Since\ y=\frac{-b}{a}x+b)$.

Let the slope of the required curve at point (x,y) given by $\frac{dy}{dx}=f(x,y)$.

So we can say that

$$f(\frac{a}{2},\frac{b}{2})=\frac{-b}{a}\Rightarrow f(\frac{a}{2},\frac{b}{2})=\frac{-b/2}{a/2}.$$

$$\Rightarrow f(x,y)=\frac{-y}{x}$$

Now

$$f(x,y)=\frac{dy}{dx}=\frac{-y}{x}\Rightarrow\frac{dy}{y}=-\frac{dx}{x}\Rightarrow\log.$$

$$(y)=-\log(x)+\log(c)\Rightarrow xy=c$$

5(c) IAS-2014. A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP; Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\cos^{-1}\left(\frac{b}{a}\right)$.

Sol^n: Let the equation of the S.H.M with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$

The time period $T = 2\pi/\sqrt{\mu}$

Let the amplitude be a. Then $\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$ —— (1)

when the particle passes through P its velocity is given to be v in the direction OP. Also $OP = b$. So putting $x = b$ and $dx/dt = v$ in (1), we get

$$v^2 = \mu(a^2 - b^2) \text{ —— (2)}$$

Let A be an extremity of the motion. From P the particle comes to instaneous rest at A and then returns back to P.

In S.H.M the time from P to A is equal to the time from A to P

$\therefore$ the required time $= 2$. time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2-x^2)} \Rightarrow dt = \frac{-1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2-x^2}}$$

Let t_1 be the time from A to P. Then at $t=0$, $x = a$ and at P, $t = t_1$ and $x = b$. Therefore integrating (3) we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{a^2-x^2}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1}b/a - \cos^{-1}1\right] = \frac{1}{\sqrt{\mu}}\cos^{-1}b/a$$

Hence the required time $= 2t_1 = \frac{2}{\sqrt{\mu}}\cos^{-1}b/a$

$$= \frac{T}{\pi}\cos^{-1}b/a$$

$\left[\because T = \frac{2\pi}{\sqrt{\mu}} \text{ so that } \sqrt{\mu} = \frac{T}{2\pi}\right]$

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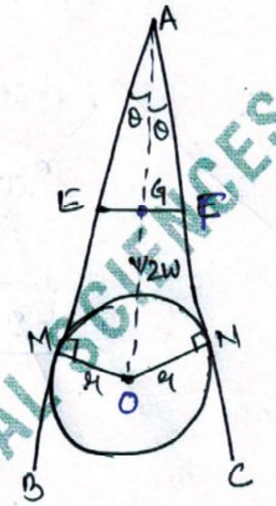
5(d) Ques. Two equal uniform rods AB and AC, each of length 'l' are freely jointed at 'A' and rest on a smooth fixed verticle circle of radius 'r'. If 2θ is the angle b/w the rods, then find the relation between l, r and θ, by using principle of virtual work.



Sol: Let 'O' be the centre of the given fixed circle and 'W' be the weight each of the rods AB and AC. If 'E' and 'F' are the middle pts of AB and AC, then the total weight '2W' of the two rods can be taken as acting at 'G', middle point of EF. The line AO is vertical. we have;

$$\angle BAO = \angle CAO = \theta$$

Also, $AB = l$, $AE = l/2$. If the rod AB touches the circle at M; then $\angle OMA = 90^\circ$ and '$OM = r$' the radius of circle.

Give the rods a small symmetrical displacement in which 'θ' changes to $\theta + \delta\theta$. The point O, remains fixed and the point 'G' is slightly displaced.

The $\angle AMO$ remains 90°, we have.

the height of G above the fixed point 'O'

$$\Rightarrow OG = OA - GA = OM \operatorname{cosec}\theta - AE\cos\theta$$

$$OG = r\operatorname{cosec}\theta - \frac{l}{2}\cos\theta$$

Equation of Virtual work is –

$$\Rightarrow -2W\delta(OG) = 0$$

or $\Rightarrow \quad \delta(r\operatorname{cosec}\theta - l/2 \cos\theta) = 0$

$$= (-r\operatorname{cosec}\theta\cot\theta + l/2\sin\theta)\delta\theta = 0$$

$$r\operatorname{cosec}\theta\cot\theta = l/2\sin\theta$$

$$2r \cdot \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} = l\sin\theta.$$

$$\boxed{2r\cdot\cos\theta = l\sin^3\theta}$$

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5(e) Find the curvature vector at any point of the curve $\vec{r}(t) = t\cos t\,\hat{i} + t\sin t\,\hat{j}$, $0 \le t \le 2\pi$. Give its magnitude also.

Solⁿ: The position vector $\vec{r}$, of any point on the given curve is

$$\vec{r} = t\cos t\,\hat{i} + t\sin t\,\hat{j}$$

i.e., $\vec{r} = (t\cos t,\ t\sin t)$

$$\therefore \frac{d\vec{r}}{dt} = (\cos t - t\sin t,\ \sin t + t\cos t)$$

Now $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\cos^2 t + t^2\sin^2 t - 2t\cos t\sin t + \sin^2 t + t^2\cos^2 t + 2t\cos t\sin t}$

$$= \sqrt{1+t^2}$$

Hence $T = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{1}{\sqrt{1+t^2}}(\cos t - t\sin t,\ \sin t + t\cos t)$

Differentiating this w.r.t. s we get

$$\frac{dT}{ds} = kN = \frac{dT/dt}{ds/dt}$$

$$= \frac{1}{\sqrt{1+t^2}} \cdot \frac{dT}{dt}$$

$$= \frac{1}{\sqrt{1+t^2}}\left[\left((-\sin t - \sin t - t\cos t)\sqrt{1+t^2} - \frac{1}{2\sqrt{1+t^2}}(2t)(\cos t - t\sin t),\ (\cos t + \cos t - t\sin t)\sqrt{1+t^2} - \frac{1}{2\sqrt{1+t^2}}(2t)(\sin t + t\cos t)\right)\right]\cdot\frac{1}{(1+t^2)}$$

$$= \frac{1}{\sqrt{1+t^2}}\cdot\frac{1}{(1+t^2)}\left(\frac{(-2\sin t - t\cos t)(1+t^2) - t(\cos t - t\sin t)}{\sqrt{1+t^2}},\ \frac{(2\cos t - t\sin t)(1+t^2) - t(\sin t + t\cos t)}{\sqrt{1+t^2}}\right)$$

$$kN = \frac{1}{(1+t^2)^2}\left(-(2+t^2)(\sin t + \cos t),\ (2+t^2)(\cos t - \sin t)\right)$$

$$kN = -\frac{(2+t^2)}{(1+t^2)^2}(\sin t + \cos t)\,\hat{i} + \left(\frac{2+t^2}{(1+t^2)^2}(\cos t - \sin t)\right)\hat{j}$$

which is the required curvature vector.

$\therefore\ k_1 = |kN|$

$$= \sqrt{\left[\frac{2+t^2}{(1+t^2)^2}\right]^2 \left((\sin t+\cos t)^2 + (\cos t-\sin t)^2\right)}$$

$$= \frac{2+t^2}{(1+t^2)^2}\sqrt{2\sin^2 t+2\cos^2 t}$$

$$= \frac{\sqrt{2}(2+t^2)}{(1+t^2)^2}$$

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6(a) Solve by the method of variation of parameters:

$$\frac{dy}{dx} - 5y = \sin x.$$

Sol^n: Given that $\frac{dy}{dx} - 5y = \sin x$ — (1)

Differentiating (1) w.r.t x, we get

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = \cos x$$ — (2)

i.e., $(D^2 - 5D)y = \cos x$

$\Rightarrow D(D-5)y = \cos x.$ — (2)

Now consider Auxiliary equation of (2) is

$$D(D-5) = 0$$

$$\Rightarrow D = 0, 5$$

$\therefore$ C.F. of (2) is $y_c = C_1 + C_2 e^{5x}$

Let $y_p = Au + Bv$ be a particular integral of (2)

where A and B are functions of x

and $u = 1$, $v = e^{5x}$.

Now $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} = 5e^{5x} \neq 0.$

$\therefore A = -\int \frac{vR}{W} = -\int \frac{e^{5x}\cdot \cos x\, dx}{5e^{5x}} = -\frac{1}{5}\int \cos x\, dx = -\frac{1}{5}\sin x$

$B = \int \frac{uR}{W} = \int \frac{1\cdot \cos x}{5e^{5x}}\, dx = \frac{1}{5}\int e^{-5x}\cos x\, dx$

$= \frac{1}{5}\frac{e^{-5x}}{25+1}\left[-5\cos x + \sin x\right]$

$= \frac{1}{5}\frac{e^{-5x}}{26}\left[-5\cos x + \sin x\right]$

$\therefore$ The general solution of (2) is $y = y_c + y_p$

i.e., $y = C_1 + C_2 e^{5x} - \frac{1}{5}\sin x \cdot 1 + \frac{1}{5}\frac{e^{-5x}}{26}\left[-5\cos x + \sin x\right]$

$y = C_1 + C_2 e^{5x} - \frac{1}{26}\left[\cos x + 5\sin x\right]$

which is the Required solution of given equ

6(b) Solve the differential equation

$$x^2\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log_e x)$$

Sol^n: Given that

$$\left(x^2D^3+3x^2D^2+xD+8\right)y=65\cos\log x \quad \text{—(1)}$$

Let $x=e^z$

$\Rightarrow \log x = z$

and let $D_1=\frac{d}{dz}$

Then $xD=D_1$ and $x^2D^2=D_1(D_1-1)$

$x^3D^3=D_1(D_1-1)(D_1-2)$

$\therefore$ (1) $\equiv$ we have

$$\left[D_1(D_1-1)(D_1-2)+3(D_1)(D_1-1)+D_1+8\right]y=65\cos z$$

$$\Rightarrow \left(D_1^3+8\right)y=65\cos z \quad \text{—(2)}$$

Auxiliary equation of (2) is

$$D_1^3+8=0$$

$$\Rightarrow (D_1+2)(D_1^2-2D_1+4)=0$$

$$\Rightarrow D_1=-2,\ 1\pm\sqrt{3}\,i$$

$$\therefore C.F = y_c = c_1e^{-2z}+e^{z}\left(c_2\cos\sqrt{3}z+c_3\sin\sqrt{3}z\right)$$

$$P.I=\frac{1}{(D_1^3+8)}(65\cos z)$$

$$=\frac{65}{-D_1+8}\cos z$$

$$=\frac{(D_1+8)65}{-D_1^2+64}\cos z$$

$$=\frac{65}{65}(D_1+8)\cos z$$

$$=(D_1+8)\cos z.$$

$= -\sin z + 8\cos z.$

$\therefore\ y = y_c + y_p$

$y = c_1 e^{-2z} + e^{z}\left(c_2 \cos\sqrt{3}\, z + c_3 \sin\sqrt{3} z\right) - \sin z + 8\cos z$

$= c_1 x^{-2} + x\left(c_2 \cos\sqrt{3}\log x + c_3 \sin\sqrt{3}\log x\right)$

$- \sin(\log x) + 8\cos(\log x)$

which is the required solution

6(c). Evaluate by stoke's theorem $\int_{\Gamma} (y\,dx + z\,dy + x\,dz)$ where Γ is the curve given by $x^2+y^2+z^2-2ax-2ay=0, x+y=2a$, starting from $(2a,0,0)$ and then going below the z-plane.

Sol^n: The centre of the sphere $x^2+y^2+z^2-2ax-2ay=0$ is the point $(a,a,0)$. Since the plane $x+y=2a$ passes through the point $(a,a,0)$. therefore the circle C is great circle of this sphere.

$\therefore$ Radius of the circle C = radius of the sphere

$= \sqrt{(a^2+a^2)} = a\sqrt{2}$.

Now $\int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\hat{i} + z\hat{j} + x\hat{k})\cdot d\bar{r}$

$= \iint_S [\text{Curl}\,(y\hat{i}+z\hat{j}+x\hat{k})]\cdot n\,ds$

where S is any surface of which circle C is boundary [stoke's theorem]

Now $\text{Curl}\,(y\hat{i}+z\hat{j}+x\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= -\hat{i}-\hat{j}-\hat{k} = -(\hat{i}+\hat{j}+\hat{k})$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle C. then a vector normal to S is grad $(x+y) = \hat{i}+\hat{j}$

$\therefore n$ = unit normal to $S = \frac{1}{\sqrt{2}}(\hat{i}+\hat{j})$

$\therefore \int_C (y\,dx + z\,dy + x\,dz) = \iint_C -(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{j}\right)ds$

$= \frac{-2}{\sqrt{2}}\iint_S ds = -\frac{2}{\sqrt{2}}$ (area of the circle of radius $a\sqrt{2}$)

$= -\sqrt{2}\,(2\pi a^2)$

7(a) Solve the following differential equation:

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^{2x},$$

when e^x is a solution to its corresponding homogeneous differential equation.

Sol: Given equation is

$$xy'' - 2(x+1)y' + (x+2)y = (x-2)e^{2x} \quad \text{---(1)}$$

$$y'' - \frac{2}{x}(x+1)y' + \frac{1}{x}(x+2)y = \left(\frac{x-2}{x}\right)e^{2x}$$

It is given that e^x is a solution to its corresponding homogeneous differential equation.

i.e., $u = e^x$ is the part of C.F of (1).

Let the general solution of (1) is $y = uv$.

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

where $P = \frac{-2(1+x)}{x}$, $Q = \frac{x+2}{x}$, $R = \frac{(x-2)}{x}e^{2x}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\frac{2}{x}(1+x) + \frac{2}{e^x}(e^x)\right]\frac{dv}{dx} = \left(\frac{x-2}{x}\right)\frac{e^{2x}}{e^x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-\frac{2}{x} - \cancel{2} + \cancel{2}\right)\frac{dv}{dx} = \left(\frac{x-2}{x}\right)e^x \quad \text{---(2)}$$

Let $\frac{dv}{dx} = q \Rightarrow \frac{d^2v}{dx^2} = \frac{dq}{dx}$

$\therefore$ from (2), we have

$$\frac{dq}{dx} + \left(-\frac{2}{x}\right)q = \left(\frac{x-2}{x}\right)e^x$$

which is linear in q.

$$I.F = e^{-\int \frac{2}{x}dx} = e^{-(2\log x)} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$$\therefore q(I.F) = \int\left(\frac{x-2}{x}\right)e^x \cdot I.F + c_1$$

$$\frac{e^{x}}{x^2}q\left(\frac{1}{x^2}\right) = \int\left(\frac{x-2}{x}\right)e^{x}\cdot\frac{1}{x^2}dx+C_1$$

$$= \int x^{-2}e^{x}dx-2\int x^{-3}e^{x}dx+C_1$$

$$= x^{-2}e^{x}-\int(-2)x^{-3}e^{x}-2\int x^{-3}e^{x}dx+C_1$$

$$q/x^2 = x^{-2}e^{x}+2\int x^{-3}e^{x}dx-2\int x^{-3}e^{x}dx+C_1 = x^{-2}e^{x}+C_1$$

$$q = e^{x}+C_1x^2$$

$$\Rightarrow dv = (e^{x}+C_1x^2)dx$$

$$\Rightarrow v = e^{x}+\frac{1}{3}C_1x^3+C_2.$$

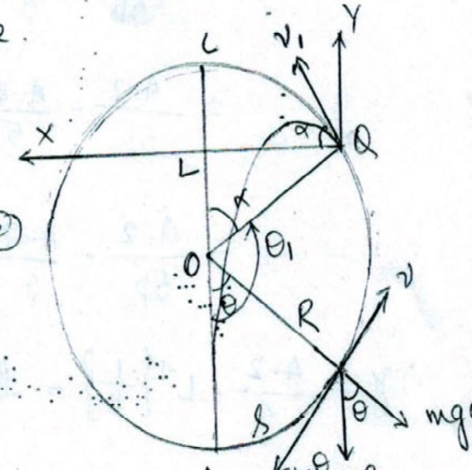
$$\therefore y = uv = e^{x}\left(e^{x}+\frac{1}{3}C_1x^3+C_2\right)$$ which is the required solution.

7(b) IAS-2014 D4

A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$, Prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.

Sol^n: Take $R=T$ (i.e. the tension in the string)

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are.



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{—— (1)}$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{—— (2)}$$

Also $s = l\theta$

From (1) & (3), we have

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying bothsides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at the point A, $\theta=0$ and $v=2\sqrt{gl}$

$$\therefore 4gl = 2lg + A \quad \text{so that } A = 2gl$$

$$\therefore v^2 = 2lg(\cos\theta+1) \quad \text{—— (4)}$$

from (2) and (4), we have

$$T = \frac{m}{l}\left(v^2 + gl\cos\theta\right) = mg(3\cos\theta+2) \quad \text{—— (5)}$$

If the cord becomes slack at the point Q, where $\theta=\theta_1$, then from (5), we have

$$T = 0 - mg(3\cos\theta_1 + 2)$$

giving as $\cos\theta_1 = -2/3$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta$, and $\cos\alpha = 2/3$

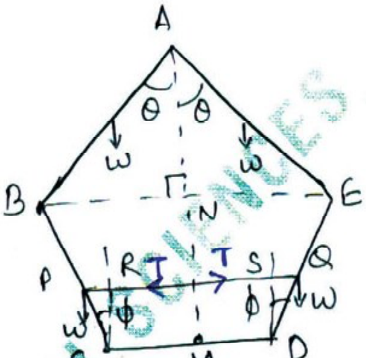
If v_1 is the velocity of the particle at Q, then $v = v_1$ where $\theta = \theta_1$. Therefore from (4), we have

$$v_1^2 = 2lg(1+\cos\theta_1) = 2lg(1-2/3) = 2lg/3$$

Now $OL = l\cos\alpha = \frac{2}{3}l$

Thus the particle leaves the circular path at the point Q at a height $\frac{2}{3}l$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

7(c)
Ques) A regular pentagon ABCDE. formed of equal heavy uniform bars jointed together, is suspended from point 'A'. and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod.



Sol: ABCDE is a pentagon formed of five equal rods each of weight 'w' and length '2a'.
It is suspended from 'A' and midpoints of BC and ED is jointed by a weightless (light) rod PQ.
Let T, The thrust in the rod PQ. The line AM joining 'A' to the middle point 'M' of CD is verticle and PQ is horizontal. The weights of the rods AB, BC, CD, DE & EA act at their respective middle points. In the portion of equilibrium the pentagon is a regular one so that each of the interior angles is $108°$ or $\frac{3\pi}{5}$ radians.

Let 'θ' be the angles which the upper slat rods AB and AE make with the verticle and φ be the angle which the lower slant rods CB and DE make with the verticle.
Replace the rod PQ by two equal and opposite forces T as shown in figure.
Give the system a small displacement about the vertical AM in which 'θ' changes to $\theta+\delta\theta$ and φ changes to $\phi+\delta\phi$.

The point A remains fixed. The lengths of the rods AB, BC etc remains fixed, the length BE changes and the middle point of the rods AB, BC etc are slightly displaced. The $\angle ANB$ remains 90°.

we have.

$$PQ = PR + PS + SQ = PR + CD + SQ = a\sin\phi + 2a + a\sin\phi$$

$$PQ = 2a(1 + \sin\phi).$$

The depth of the middle pnt. of AB or AE below A $= a\cos\theta$

The depth of the middle pt. of BC or ED below A $= 2a\cos\theta + a\cos\phi$

and depth of the middle pt M of CD below A $= 2a(\cos\phi + \cos\theta)$

The equation of virtual work is -

$$T[\delta[2a + 2a\sin\phi]] + 2w\delta(a\cos\theta) + 2w\delta(2a\cos\theta + a\cos\phi)$$
$$+ w\delta(2a\cos\theta + 2a\cos\phi) = 0$$

$$\Rightarrow 2T\cos\phi\,\delta\phi - 2w\sin\theta\,\delta\theta - 4w\sin\theta\,\delta\theta - 2w\sin\phi\,\delta\phi$$
$$- 2w\sin\theta\,\delta\theta - 2w\sin\phi\,\delta\phi = 0$$

$$\Rightarrow [T\cos\phi - 2w\sin\phi]\delta\phi = 4w\sin\theta\,\delta\theta \quad \text{——— (1)}$$

from the figure, finding the length of BE in two ways; i.e from the upper portion AE and from lower portion BCDE, we have.
$$4a\sin\theta = 2a + 4a\sin\phi$$

Differentiating, we get $4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$

or $\cos\theta\,\delta\theta = \cos\phi\,\delta\phi$ ——(2)

Dividing (1) by (2), we get.

$$\Rightarrow \frac{T\cos\phi - 2w\sin\phi}{\cos\phi} = \frac{4w\sin\theta}{\cos\theta} \Rightarrow \underline{T = 2W(\tan\phi + 2\tan\theta)}$$

But in the position of equilibrium

$$\theta = \frac{1}{2}\cdot\frac{3\pi}{5} = \frac{3\pi}{10} \;;\; \phi = \frac{3\pi}{5} - \frac{\pi}{2} = \frac{\pi}{10}$$

$$\therefore T = 2W(\tan\pi/10 + 2\tan 3\pi/10) = 2W(\tan\pi/10 + 2\cot 2\pi/10)$$

$$T = 2W\left[\tan\pi/10 + 2\cdot\left(\frac{1-\tan^2(\pi/10)}{2\tan\pi/10}\right)\right] = 2W\cot(\pi/10)$$

8(a). Find the sufficient condition for the differential equation $M(x, y)\ dx + N(x, y)\ dy = 0$ to have an integrating factor as a function of $(x+y)$. What will be the integrating factor in that case? Hence find the integrating factor for the differential equation

$(x^2 + xy)\ dx + (y^2+xy)\ dy = 0$ and solve it.

SOLUTION

Let IF $= f(x + y)$

$M(x, y)\ dx + N(x, y)\ dy = 0$ .....(1)

Multiplying in the given equations with $f(x + y)$, we get

$$\left.\begin{aligned} &f(x + y)\ M(x, y)\ dx + f(x + y)\ N(x, y)\ dy = 0 \\ &\text{or} \quad M'(x, y)dx + N'(x, y)dy = 0 \end{aligned}\right\} \quad .....(2)$$

from (2)

$$\frac{\partial M'}{\partial y} = f'(x+y)\cdot M + \frac{\partial M}{dy} f(x+y)$$

$$\frac{\partial N'}{\partial x} = f'(x+y)\cdot N + \frac{\partial N}{dy} f(x+y)$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

as the equation is to be made exact

$$f'(x+y)M + \frac{\partial M}{\partial y} f(x+y) = f'(x+y)N + \frac{\partial N}{\partial y} f(x+y)$$

$$f(x+y)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = (N-M)f'(x+y)$$

$$\frac{f'(x+y)}{f(x+y)} = \frac{1}{(N-M)}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

integrating we get,

$$\ell n(f(x+y)) = \int \frac{1}{(N-M)}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

$$\therefore \quad f(x + y) = e^{\int \frac{1}{N-M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] d(x+y)}$$

for f to be fuction of (x+y)

$$N \neq M,\ \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ and } \frac{1}{N-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is a fuction of (x+y).

Now $M = x^2 + xy$ and $N = y^2 + xy$

$$\frac{\partial M}{\partial y} = x \text{ and } \frac{\partial N}{\partial x} = y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$M - N = x^2 - y^2 \neq 0$$

$$\therefore \quad \frac{1}{N-M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1(x-y)}{(y-x)(y+x)} = \frac{-1}{(x+y)}$$

If $\quad F(x+y) = e^{\int \frac{-1}{x+y} d(x+y)}$

$\boxed{IF = \left(\frac{1}{x+y}\right)}$ Required I.F.

$\therefore$ above question has $\frac{1}{x+y}$ as the I.F. multiplying with IF

$$\frac{(x^2+xy)dx}{x+y} + \frac{(y^2+xy)dy}{x+y} = 0$$

$$xdx + ydy = 0$$

$\therefore \quad \boxed{x^2 + y^2 = C}$

8(b) A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2} and when $y=a$, it is projected parallel to the axis of x with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the path of the particle. Here μ is a constant.

Sol^n: Here we are given that

$$\frac{d^2y}{dt^2} = -\mu y^{-2} \quad \text{—— (1)}$$

the negative sign has been taken because the force is in the direction of y increasing.

the force is in the direction of y increasing. Also there is no force parallel to the axis of x.

Therefore $\frac{d^2x}{dt^2} = 0$ —— (2)

Multiplying both sides of (1) by $2\frac{dy}{dt}$ and then integrating w.r.t. t, we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + A, \quad \text{where } A \text{ is a constant.}$$

Initially, when $y=a$, $\frac{dy}{dt}=0$ (Note that initially there is no velocity ||lel to y-axis)

$$\therefore A = -\frac{2\mu}{a}$$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} - \frac{2\mu}{a} = 2\mu\left(\frac{1}{y}-\frac{1}{a}\right) = \frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

$$\Rightarrow \frac{dy}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)} \cdot \sqrt{\frac{a-y}{y}} \quad \text{—— (3)}$$

(−ve sign has been taken because the particle is moving in the direction of y decreasing).

Now integrating (2), we have

$\frac{dx}{dt} = B$, where B is a constant.

Initially, when $y = a$, $\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$.

so that $B = \sqrt{\frac{2\mu}{a}}$.

$\therefore \frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$ —— (4)

Dividing (3) by (4), we have

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$$\Rightarrow dx = -\sqrt{\frac{y}{a-y}}\, dy$$

Integrating,

$$\int dx = -\int\sqrt{\frac{y}{a-y}}\, dy$$

$$= 2a\int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\sin\theta\, d\theta + C.$$

(putting $y = a\cos^2\theta$, so that $dy = -2a\sin\theta\cos\theta\, d\theta$)

$$= a\int(1+\cos 2\theta)\, d\theta + C$$

$$= a\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a}{2}(2\theta + \sin 2\theta) + C.$$

Let us take $x = 0$, when $y = a$

i.e., when $a\cos^2\theta = a \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

Then $0 = \frac{1}{2}a(0+0) + C \Rightarrow C = 0$

$\therefore x = \frac{1}{2}a(2\theta + \sin 2\theta)$ —— (5)

Also $y = a\cos^2\theta = \frac{a}{2}(1+\cos 2\theta)$ —— (6)

The equations (5) & (6) give us the path of the particle. But these are the parametric equations of a cycloid.

8(c) Solve the initial value problem

$$\frac{d^2y}{dt^2}+y=8e^{-2t}\sin t,\ y(0)=0,\ y'(0)=0$$

by using Laplace-transform.

Sol^n: Given equation is $\frac{d^2y}{dt^2}+y=8e^{-2t}\sin t$

$$\Rightarrow y''+y=8e^{-2t}\sin t \quad \text{--- (1)}$$

Taking Laplace transform of bothsides of (1),

we get

$$L(y'')+L(y)=8L(e^{-2t}\sin t)$$

$$\Rightarrow p^2L\{y(t)\}-py(0)-y'(0)+L\{y(t)\}=\frac{8}{(p+2)^2+1}$$

$$\Rightarrow p^2L\{y(t)\}+L\{y(t)\}=\frac{8}{p^2+4p+5}$$

$$\Rightarrow L(y(t))(p^2+1)=\frac{8}{p^2+4p+5}$$

$$\Rightarrow L\{y(t)\}=\frac{8}{(p^2+1)(p^2+4p+5)}$$

$$\Rightarrow y(t)=L^{-1}\frac{8}{(p^2+1)(p^2+4p+5)}$$

$$y(t)=L^{-1}\left[\frac{-p+1}{p^2+1}+\frac{p+3}{p^2+4p+5}\right]$$

$$=L^{-1}\left(\frac{-p}{p^2+1}\right)+L^{-1}\left(\frac{1}{p^2+1}\right)+L^{-1}\left(\frac{(p+2)+1}{(p+2)^2+1}\right)$$

$$=-\cos t+\sin t+e^{-2t}L^{-1}\left(\frac{p+1}{p^2+1}\right)$$

$$=-\cos t+\sin t+e^{-2t}\left\{L^{-1}\left(\frac{p}{p^2+1}\right)+L^{-1}\left(\frac{1}{p^2+1}\right)\right\}$$

$$=-\cos t+\sin t+e^{-2t}\cos t+e^{-2t}\sin t$$

$$=(e^{-2t}-1)\cos t+(e^{-2t}+1)\sin t.$$

which is the required solution