IFOS-2012

Q1 If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = xy+yz+zx$
prove that grad u, grad v and grad w are coplanar.

Ans. $$\text{grad}\,u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

$$= \frac{\partial}{\partial x}(x+y+z)\hat{i} + \frac{\partial}{\partial y}(x+y+z)\hat{j} + \frac{\partial}{\partial z}(x+y+z)\hat{k}$$

$$= \hat{i} + \hat{j} + \hat{k}$$

$$\text{grad}\,v = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad}\,w = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy+yz+zx)$$

$$= (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

Scalar triple product is zero for co-planar vector.

$$\therefore [\text{grad}\,u \;\; \text{grad}\,v \;\; \text{grad}\,w] = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix} \overset{R_2 \to R_2+R_3}{=} 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & x+z & x+y \end{vmatrix}$$

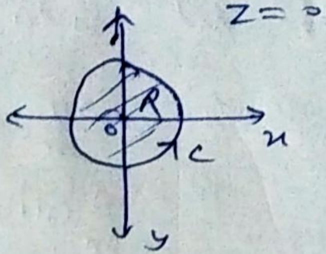
$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & x+z & x+y \end{vmatrix} = 2(x+y+z)(0) = 0$$

($\because$ Determinant is zero if two rows are same)

As $[\text{grad}\,u \;\; \text{grad}\,v \;\; \text{grad}\,w] = 0$

$\therefore$ They are coplanar.

Q3 Find value of line integral over circular path $x^2+y^2=a^2, z=0$ where $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$



Ans. $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

As $z=0$ $\quad dz=0$

$\therefore d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (\sin y\,\hat{i} + x(1+\cos y)\,\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_C \sin y\,dx + x(1+\cos y)\,dy$$

As the path C is closed & contains region R, therefore Applying Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$P = \sin y \quad \therefore \dfrac{\partial P}{\partial y} = \cos y$

$Q = x(1+\cos y) \quad \therefore \dfrac{\partial Q}{\partial x} = 1+\cos y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1+\cos y - \cos y = 1$$

$$\therefore \oint_C \sin y\,dx + x(1+\cos y)\,dy = \iint_R 1.\,dx\,dy = \iint_R dx\,dy$$

$= \text{Area of circle} = \pi a^2$

$\boxed{\therefore \text{Value of line integral} = \pi a^2}$