5 Dynamics & Statics

1. Rectilinear Motion

1.1 One end of a light elastic string of natural length l and modulus of elasticity 2 mg is attached to a fixed point O and the other end to a particle of mass m. The particle initially held at rest at O is bit fall. Find the greatest extension of the string during motion and show that the particle will reach O again after a time.

$$(\pi + 2 - \tan^{-1} 2)\sqrt{\frac{2l}{g}}$$

(2009 : 20 Marks)

Solution:

Let C be the equilibrium position of the body and $BC = d$.
In position of equilibrium

$$mg = 2mg \cdot \frac{d}{l} \Rightarrow d = \frac{l}{2}$$

When particle is dropped from A it free falls till B.

$$V_B^2 = 0 + 2g \times AB$$

$$\Rightarrow \quad V_B = \sqrt{2gl}$$

After B the tension in the string starts acting which is balanced at C. Beyond C the particles moves due to its velocity till it comes to stop at D.
At any point P with $CP = x$.

$$m\frac{d^2x}{dt^2} = mg - (2mg)\frac{d+x}{l}$$

$$= -2mg\frac{x}{l}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -\frac{2g}{l}x$$

So, the body performs SHM with centre C.

Multiplying with $2\frac{dx}{dt}$ and integrating

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{l}x^2 + C$$

At B, $$V_B = \sqrt{2gl}, \; x = -\frac{l}{2}$$

$$\therefore \quad 2gl = -\frac{2g}{l}\frac{l^2}{4} + C \Rightarrow C = \frac{5}{2}gl$$

$\therefore$ $$\left(\frac{dx}{dt}\right)^2 = \frac{5}{2}gl - \frac{2g}{l}x^2 \quad ...(i)$$

At D, $$\frac{dx}{dt} = 0$$

$\Rightarrow$ $$x^2 = \frac{5}{4}l^2 \Rightarrow x = \frac{\sqrt{5}}{2}l$$

So greatest distance through which particle falls

$$= AD = AB + BC + CD$$
$$= l + \frac{l}{2} + \frac{\sqrt{5}l}{2} = \frac{(3+\sqrt{5})l}{2}$$

$$\text{Greatest extension} = \frac{(1+\sqrt{5})l}{2}$$

From (i), $$\frac{dx}{dt} = \sqrt{\frac{2g}{l}}\left[\frac{5}{4}l^2 - x^2\right]^{1/2}$$

where positive sign is taken as particle is moving in direction of increasign x.

$$\sqrt{\frac{l}{2g}}\frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}} = dt$$

If t_1 is time from B to D

$$\sqrt{\frac{l}{2g}}\int_{-d/2}^{\sqrt{5}l/2}\frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}} = \int_0^t dt$$

$\Rightarrow$ $$\sqrt{\frac{l}{2g}}\left[\sin^{-1}\frac{x}{\sqrt{5}l/2}\right]_{-l/2}^{\sqrt{5}l/2} = t_1$$

$$t_1 = \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} - \sin^{-1} - \frac{1}{\sqrt{5}}\right] = \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \sin\frac{1}{\sqrt{5}}\right]$$
$$= \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \tan^{-1}\frac{1}{2}\right]$$
$$= \sqrt{\frac{l}{2g}}\left[\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}2\right]$$
$$= \sqrt{\frac{l}{2g}}\left[\pi - \tan^{-1}2\right]$$

Time in falling from A to B.

$$\frac{1}{2}gt_2^2 = l \Rightarrow t_2 = \sqrt{\frac{2l}{g}}$$

$\therefore$ Total time taken to come back to

$$O = 2\sqrt{\frac{l}{2g}}[\pi - \tan^{-1}2 + 2]$$
$$= \sqrt{\frac{2l}{g}}[\pi + 2 - \tan^{-1}2]$$

1.2 The velocity of a train increases from 0 to v at a constant acceleration f_1, then remains constant for an interval and again decreases to 0 at a constant retardation f_2. If the total distance described is x, find the total time taken.

(2011 : 10 Marks)

Solution:

(i) When velocity increases from 0 to v. Using $v = u + at$, $v = v$, $u = 0$, $a = f_1$, $t = t_1$,

we get,
$$v = 0 + f_1 t_1$$
$$\Rightarrow \quad t_1 = \frac{v}{f_1}$$

Using
$$S = ut + \frac{1}{2}at^2$$

We get,
$$d_1 = 0 + \frac{1}{2}\cdot f_1 \cdot t_1^2$$
$$= \frac{1}{2}\cdot f_1 \cdot \frac{v^2}{f_1^2} = \frac{v^2}{2f_1}$$

(ii) When velocity decreases from v to 0. Using $v = u + at$; $v = 0$, $u = v$, $a = -f_2$, $t = t_2$,

we get,
$$0 = v - f_2 t_2$$
$$\Rightarrow \quad t_2 = \frac{v}{f_2}$$

Using
$$S = ut + \frac{1}{2}at_1^2$$
$$d_2 = vt_2 - \frac{1}{2}\cdot f_2 t_2^2$$
$$= \frac{v^2}{f_2} - \frac{1}{2}\frac{v^2}{f_2} = \frac{v^2}{2f_2}$$

(iii) Since the total distance travelled is 'x', the distance travelled during the constant velocity phase
$$= x - \frac{v^2}{2f_1} - \frac{v^2}{2f_2}$$

Time taken to travel this distance, t_3
$$= \frac{x - \frac{v^2}{2f_1} - \frac{v^2}{2f_2}}{v}$$
$$= \frac{x}{v} - \frac{v}{2f_1} - \frac{v}{2f_2}$$

$$\therefore \quad \text{Total time taken} = t_1 + t_2 + t_3$$
$$= \frac{v}{f_1} + \frac{v}{f_2} + \frac{x}{v} - \frac{v}{2f_1} - \frac{v}{2f_2}$$
$$= \frac{x}{v} + \frac{v}{2f_1} + \frac{v}{2f_2}$$
$$= \frac{x}{v} + \frac{v}{2}\left(\frac{1}{f_1} + \frac{1}{f_2}\right)$$

1.3 A mass of 560 kg moving with a velocity of 240 m/sec strikes a fixed target and is brought to rest in $\frac{1}{100}$ sec. Find the impulse of the blow on the target and assuming the resistance to be uniform throughout the time taken by the body in coming to rest, find the distance through which it penetrates.

.(2011 : 20 Marks)

Solution:

Initial velocity, $u = 240$ m/s

Final velocity, $v = 0$ (as the mass finally comes to rest)

Time taken to come to rest, $t = 0.01$ s

Using, $v = u + at$, we get

$$0 = 240 + a \times 0.01$$

$$\Rightarrow \quad a = -24000 \text{ m/s}^2$$

The negative sign indicates that the velocity of the bullet is decreasing.

Using, $v^2 - u^2 = 2sa$, we have

$$0 - (240)^2 = 2 \times S \times (-24000)$$

$$\Rightarrow \quad S = \frac{-240 \times 240}{-2 \times 24000}$$

$$= 0.1 \text{ m}$$

Hence, the distance of penetration of the mass into the fixed target is 0.1 m.

Impulse of the blow on the target,

$$I = \text{Change in momentum of the mass}$$

$$= mv - m4$$

$$= m(v - 4) = 560 \times (-240) = -134400 \text{ kg m/s}$$

1.4 (i) After a ball has been falling under gravity for 5 seconds it passes through a pane of glass and loses half its velocity. If it now reaches the ground in 1 second, find the height of glass above the ground.

(2011 : 10 Marks)

(ii) A particle of mass m moves on straight line under an attractive force mn^2x towards a point O on the line, where x is the distance from O. If $x = a$ and $\frac{dx}{dt} = u$ when $t = 0$, find $x(t)$ for any time $t > 0$.

(2011 : 10 Marks)

Solution:

(i) Using $V = u + at$, the velocity of the ball after 5 seconds is given by

$$V = 0 + g \cdot 5$$

$$= 5g \text{ m/s}$$

Final velocity of the ball when it hits the ground

$$V = \frac{5g}{2} + g \cdot 1$$

(using $V = u + at$ and using the fact that the ball had lost half of its velocity)

$$\Rightarrow \quad V = \frac{7}{2}g \text{ m/s}$$

Let after losing half of velocity the ball falls through a height H, i.e., let H be the height of the glass pane.

Using $V^2 - u^2 = 2gH$

$$\left(\frac{7}{2}g\right)^2 - \left(\frac{5}{2}g\right)^2 = 2gH$$

$$\Rightarrow \qquad \frac{49}{4}g^2 - \frac{25}{4}g^2 = 2gH$$

$$\Rightarrow \qquad \frac{24}{4}g^2 = 2gH \Rightarrow 3g = H$$

$$\Rightarrow \qquad H = 3 \times 9.8$$
$$= 29.4 \text{ m}$$

(ii) Given : Force, $F = -mn^2x$, the negative sign is taken as the particle is moving towards the origin.

$$\therefore \qquad m\frac{d^2x}{dt^2} = -mn^2x$$

$$\Rightarrow \qquad \frac{d^2x}{dt^2} = -n^2x$$

$$\Rightarrow \qquad v\frac{dv}{dx} = -n^2x$$

$$\Rightarrow \qquad \frac{v^2}{2} = -n^2\frac{x^2}{2} + A \qquad \text{...(i)}$$

When $t = 0$, $x = a$, $\qquad v = \frac{dx}{dt} = u$

$$\therefore \qquad A = \frac{u^2}{2} + \frac{x^2a^2}{2}$$

$\therefore$ From (i), $$\frac{v^2}{2} = -\frac{x^2x^2}{2} + \frac{u^2}{2} + \frac{x^2a^2}{2}$$

$$\Rightarrow \qquad v^2 = -n^2x^2 + u^2 + n^2a^2$$

$$\Rightarrow \qquad \frac{dx}{dt} = v = -\sqrt{-n^2x^2 + u^2 + x^2a^2}$$

(the negative sign is taken because as t increases, x decreases)

$$\Rightarrow \qquad \frac{dx}{dt} = -n\sqrt{-x^2 + \frac{u^2 + n^2a^2}{n^2}}$$

$$\Rightarrow \qquad \frac{dx}{\sqrt{\frac{u^2 + n^2a^2}{n^2} - x^2}} = -ndt$$

$$\Rightarrow \qquad -\cos^{-1}\frac{x}{\sqrt{\frac{u^2 + n^2a^2}{x^2}}} = -nt + B \qquad \text{...(ii)}$$

Using, $t = 0$, $x = a$, we get

$$B = -\cos^{-1}\frac{a}{\sqrt{\frac{u^2 + n^2a^2}{x^2}}}$$

$\therefore$ From (ii) $$\cos^{-1}\frac{x}{\sqrt{\frac{u^2 + n^2a^2}{x^2}}} = xt + \cos^{-1}\frac{a}{\sqrt{\frac{u^2 + n^2a^2}{x^2}}}$$

$$\Rightarrow \qquad x = k\cos(xt + \cos^{-1} k)$$

where $$k = \sqrt{\frac{u^2 + n^2a^2}{n^2}}$$

1.5 A particle moves with an acceleration

$$\mu\left(x+\frac{a^4}{x^3}\right)$$

towards the origin. If it starts from rest at a distance a from the origin, find its velocity when its distance from the origin is $\frac{a}{2}$.

(2012 : 12 Marks)

Solution:

Given : $$v\frac{dv}{dx} = -\mu\left(x+\frac{a^4}{x^3}\right)$$

The negative sign is taken as the particle is moving towards the origin.

$$\Rightarrow \quad vdv = -\mu\left(x+\frac{a^4}{x^3}\right)dx$$

Integrating both sides, we have

$$\frac{v^2}{2} = -\mu\left(x^2+a^4\cdot\frac{x^{-2}}{-2}\right)+c \quad \text{...(i)}$$

It is given that when $x = a$, $v = 0$

$$\therefore \quad 0 = -\mu\left(a^2+\frac{a^4\cdot a^{-2}}{-2}\right)+c$$

$$\Rightarrow \quad c = \frac{\mu a^2}{2}$$

$\therefore$ from (i),

$$\frac{v^2}{2} = -\mu\left(x^2+\frac{a^4x^{-2}}{-2}\right)+\frac{\mu a^2}{2}$$

When $x = \frac{a}{2}$, we have

$$\frac{v^2}{2} = -\mu\left(\frac{a^2}{4}+\frac{a^4}{-2}\cdot\frac{4}{a^2}\right)+\frac{\mu a^2}{2}$$

After solving, we have

$$v^2 = \frac{9\mu a^2}{2}$$

or $v = -3a\sqrt{\frac{\mu}{a}}$, the negative sign is taken as the particle is moving towards the origin.

1.6 A body is performing SHM in a straight line OPQ. Its velocity is zero at points P and Q whose distance from O and x and y respectively and its velocity is V at the midpoint between P and Q. Find the time of one complete oscillation.

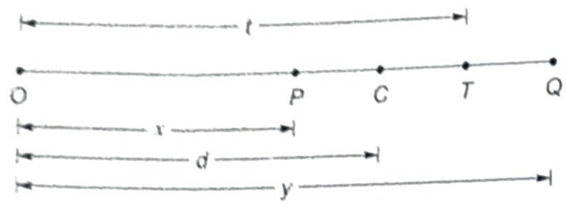
(2013 : 10 Marks)

Solution:

Let C be the centre of oscillation and T the position of point at any time

$$CT = (t-d) = r \text{ (let)}$$

Since body performs SHM about C



$$\frac{d^2r}{dt^2} = -\mu r$$

Multiplying by $2\frac{dr}{dt}$ and integrating

$$\left(\frac{dr}{dt}\right)^2 = -\mu r^2 + C = -\mu(t-d)^2 + C$$

Now velocity $= \frac{dr}{dt} = 0$ when $t = x$ and $t = y$

$$\therefore \left.\begin{array}{l} C-\mu(x-d)^2 = 0 \\ C-\mu(y-d)^2 = 0 \end{array}\right\} \mu(x+y-2d)(x-y) = 0$$

$$\Rightarrow \qquad d = \frac{x+y}{2}$$

So, the centre of oscillation is midpoint between P and Q.

Again $\qquad C = \mu\left(y-\frac{(x+y)}{2}\right)^2 = \mu\left(\frac{y-x}{2}\right)^2$

$$\therefore \qquad \left(\frac{dr}{dt}\right)^2 = \mu\left(\left(\frac{y-x}{2}\right)^2 - r^2\right)$$

At midpoint, $\qquad r = 0$ (as midpoint is the centre of oscillation)

and $\qquad \frac{dr}{dt} = v$

$$\therefore \qquad v^2 = \mu\left(\frac{y-x}{2}\right)^2 \Rightarrow \sqrt{\mu} = \frac{2v}{y-x}$$

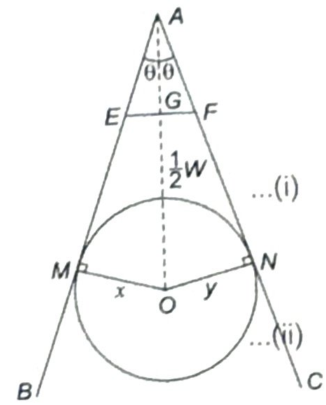
$$\text{Time period} = \frac{2\pi}{\sqrt{\mu}} = \frac{\pi(y-x)}{v}$$

1.7 A particle is performing a simple harmonic motion (S.H.M.) of period T about a centre O with amplitude a and it passes through a point P, when $OP = b$ in the direction OP. Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\cos^{-1}\left(\frac{b}{a}\right)$.

(2014 : 10 Marks)

Solution:

Let the equation of the S.H.M. with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$



The time period $T = \dfrac{2\pi}{\sqrt{\mu}}$

Let the amplitude be a.

Then $$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$$

when the particle passes through points velocity is given to be v in the direction $OP = b$. So putting $x = b$ and $dx/dt = v$ in (i), we get

$$v^2 = \mu(a^2 - b^2)$$

Let A be an extremely of the motion from P the particle comes to instaneous at A and then returns bank to P. In S.H.M. the time from P to A is equal to the time from A to P.

$\therefore$ The required time = 2. Time from A to P. Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)} \Rightarrow dt = \frac{-1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2 - x^2}}$$

Let t_1, be the time from A to P. then at $t = 0$, $x = a$ and at P, $t = t_1$ and $x = b$. Therefore integrating (iii) we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{a^2 - x^2}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{b}\right]_a^b$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}1\right] = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

Hence the required time $= 2t_1 = \dfrac{2}{\sqrt{\mu}}\cos^{-1}\dfrac{b}{a}$ $\left[\therefore\ T = \dfrac{2\pi}{\sqrt{\mu}} \text{ so that } \sqrt{\mu} = \dfrac{T}{2\pi}\right]$

$$= \frac{T}{\pi}\cos^{-1}\frac{b}{a}$$

1.8 A body moving under SHM has an amplitude 'a' and time period 'T'. If the velocity is trebled, when the distance from mean position is $\frac{2}{3}a$, the period being unaltered. Find the new amplitude.

(2015 : 10 Marks)

Solution:

Let $$x = a\sin\omega t,\ T = \frac{2\pi}{\omega}$$

$$\frac{dx}{dt} = a\omega\cos\omega t = \omega\sqrt{a^2 - x^2}$$

$\therefore$ $$V_{\frac{2}{3}a} = \omega\sqrt{a^2 - \frac{4}{9}a^2} = \frac{\sqrt{5}}{3}a\omega$$

New velocity,

$$V_1 = 3V_{\frac{2a}{3}} = 3\frac{\sqrt{5}}{3}a\omega = \sqrt{5}a\omega$$

$\Rightarrow$ $$\omega\sqrt{A^2 - \frac{4}{9}a^2} = \sqrt{5}a\omega$$

$\Rightarrow$ $$A^2 = 5a^2 + \frac{4}{9}a^2 = \frac{49}{9}a^2$$

$\therefore$ $$A = \frac{7}{3}a$$

1.9 A particle moving with simple harmonic motion in a straight line has velocities v_1 and v_2 at distances x_1 and x_2 respectively from centre of path. Find the period of its motion.

(2018 : 12 Marks)

Solution:

The equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} = -\mu x$$

Multiplying both sides by $\frac{2dx}{dt}$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = -\mu x^2 + c, \text{ where } c \text{ is a constant} \quad \ldots(i)$$

Given, $\frac{dx}{dt}$ at $x_1 = v_1$

and $\frac{dx}{dt}$ at $x_2 = v_2$

Putting these values in eqn. (i), we get

$$v_1^2 = -\mu x_1^2 + c \quad \ldots(ii)$$

$$v_2^2 = -\mu x_2^2 + c \quad \ldots(iii)$$

Subtracting (iii) from (ii), we get

$$v_1^2 - v_2^2 = -\mu(x_1^2 - x_2^2)$$

$$\Rightarrow \quad \mu = \frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}$$

$$\text{Time period} = \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

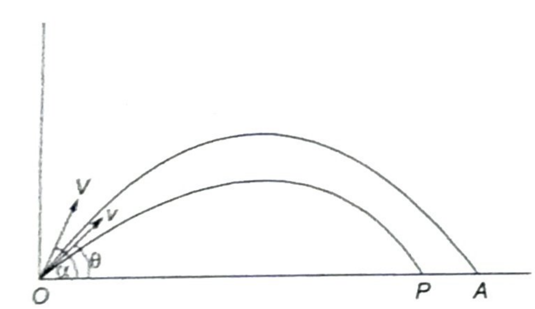
2. Motion in a Plane

2.1 A shot fired with a velocity V at an angle (elevation) α strikes a point P in a horizontal plane through the point of projection. If the point P is receding from the gun with velocity v show that the elevation must be changed to θ where

$$\sin 2\theta = \sin 2\alpha + \frac{2V}{V}\sin\theta$$

(2009 : 12 Marks)

Solution:



Let P be the original position of the point.

When P is at rest, $$OP = \frac{V^2 \sin 2\alpha}{g}$$

Let the elevation be changed to θ when the point recedes to hit it and let the point be hit at A.

Then $$\theta A = \frac{V^2 \sin 2\theta}{g}$$

The time period of travel $$= \frac{V^2 \sin 2\theta}{g \cdot V\cos\theta} = \frac{2V\sin\theta}{g}$$

∴ So the point must have travelled PA during this time.

$$\Rightarrow \qquad \frac{2V\sin\theta}{g}\cdot V = \frac{V^2 \sin 2\theta}{g} - \frac{V^2 \sin 2\alpha}{g}$$

$$\Rightarrow \qquad \sin 2\theta = \sin 2\alpha + \frac{2V}{V}\sin\theta$$

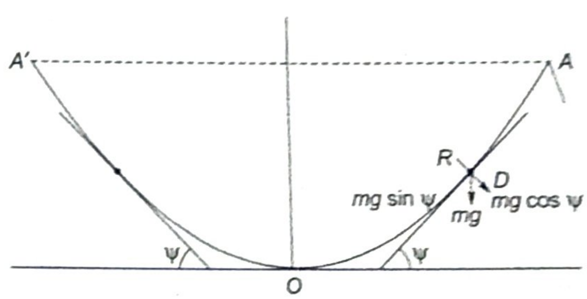
2.2 **A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc. Show that the time of reaching the vertex is**

$$2\sqrt{\frac{a}{g}}\cot^{-1}\left(\frac{V}{2\sqrt{ag}}\right)$$

where a is the radius of the generating circle.

(2009 : 10 Marks)

Solution:



Let particle be projected with velocity V from cusp A.
At any point P the equation of motion is

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \text{...(i)}$$

$$\frac{mv^2}{P} = R - mg\cos\psi \qquad \text{...(ii)}$$

where $s = OP$ is the arc length.
Using $s = 4a\sin\psi$ (equation of cycloid) (i) becomes

$$m\frac{d^2s}{dt^2} = -mg\frac{s}{4a} \Rightarrow \frac{d^2s}{dt^2} = -\frac{g}{4a}s$$

Multiplying by $2\frac{ds}{dt}$ and integrating

$$\left(\frac{ds}{dt}\right)^2 = \frac{-g}{4a}s^2 + C$$

At A, $$s = 4a \text{ and } \frac{ds}{dt} = V$$

$\Rightarrow$ $$V^2 = -4ag + C \Rightarrow C = V^2 + 4ag$$

$\therefore$ $$\left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2$$

$$\frac{ds}{dt} = -\sqrt{\frac{g}{4a}}\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]^{1/2}$$

Taking –ve sign as body moves in direction of decreasing s.

$$dt = -2\sqrt{\frac{a}{g}}\frac{ds}{\sqrt{\frac{4a}{g}(V^2 + 4ag) - s^2}}$$

$$\int_0^{t_1} dt = -2\sqrt{\frac{a}{g}}\int_{S=\frac{C}{a}}^{S=0}\frac{ds}{\sqrt{\frac{4a}{g}(V^2 + 4ag) - s^2}}$$

$$t_1 = 2\sqrt{\frac{a}{g}}\left[\sin^{-1}\frac{s}{\sqrt{\frac{4a}{g}(V^2 + 4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{\frac{a}{g}}\sin^{-1}\frac{\sqrt{4ag}}{\sqrt{V^2 + 4ag}}$$

$$= 2\sqrt{\frac{a}{g}}\cot^{-1}\frac{\sqrt{V^2}}{\sqrt{4ag}} = 2\sqrt{\frac{a}{g}}\cot^{-1}\frac{V}{2\sqrt{ag}}$$

2.3 On a rigid body, the forces $10(\hat{i} + 2\hat{j} + 2\hat{k})N$, $5(-2\hat{i} - \hat{j} + 2\hat{k})N$ and $6(2\hat{i} + 2\hat{j} - \hat{k})N$ are acting at points with position vector $\hat{i} - \hat{j}$, $2\hat{i} + 5\hat{k}$ and $4\hat{i} - \hat{k}$ respectively. Reduce this system to a single force $\vec{R}$ acting at the point $(4\hat{i} + 2\hat{j})$ together with a couple $\vec{G}$ whose axis passes through this point. Does the point $(4\hat{i} + 2\hat{j})$ lie on the central axis.

(2009 : 15 Marks)

Solution:

Let $A(\hat{i} - \hat{j})$, $B(2\hat{i} + 5\hat{k})$ and $C(4\hat{i} - \hat{k})$ be the given points and $O(4\hat{i} + 2\hat{j})$ the point about which the resultant has to found.

Then, $$\text{Resultant Force} = \Sigma\vec{F}$$

$$= 10(\hat{i} + 2\hat{j} + 2\hat{k}) + 5(-2\hat{i} - \hat{j} - 2\hat{k}) + 6(2\hat{i} + 2\hat{j} - \hat{k})$$

$$= 12\hat{i} - 27\hat{j} + 24\hat{k}$$

Positions vectors of A, B and C about O.

$$\overrightarrow{OA} = (\hat{i}-\hat{j})-(4\hat{i}+2\hat{j}) = -3\hat{i}-3\hat{j}$$

$$\overrightarrow{OB} = (2\hat{i}+5\hat{k})-(4\hat{i}+2\hat{j}) = -2\hat{i}-2\hat{j}+5\hat{k}$$

$$\overrightarrow{OC} = (4\hat{i}-\hat{k})-(4\hat{i}+2\hat{j}) = -2\hat{j}-\hat{k}$$

$\therefore$ $$\text{Resultant Couple} = \overrightarrow{OA}\times\vec{F}_1+\overrightarrow{OB}\times\vec{F}_2+\overrightarrow{OA}\times\vec{F}_3$$

$$\overrightarrow{OA}\times\vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -3 & 0 \\ 10 & 20 & 20 \end{vmatrix} = -60\hat{i}+60\hat{j}-30\hat{k}$$

$$\overrightarrow{OB}\times\vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -2 & 5 \\ -10 & -5 & 10 \end{vmatrix} = 5\hat{i}-30\hat{j}-10\hat{k}$$

$$\overrightarrow{OC}\times\vec{F}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & -1 \\ 12 & 12 & -6 \end{vmatrix} = -12\hat{j}+24\hat{k}$$

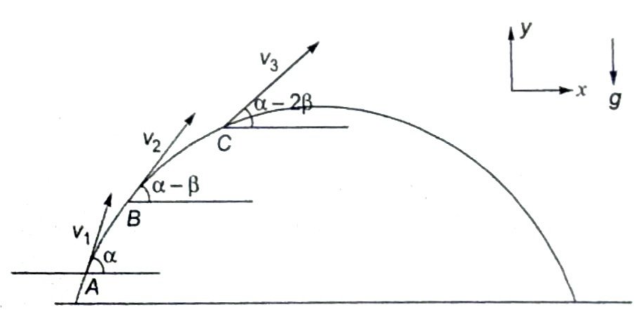
$\therefore$ $$\vec{G} = -55\hat{i}+18\hat{j}-16\hat{k} \text{ is the resultant moment.}$$

The resultant couple is not zero so the point does not lie on the control axis.

2.4 If v_1, v_2, v_3 are the velocities at three points A, B, C of the path of a projective, where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$ and if t_1, t_2 are the times describing the arcs AB, BC respectively, prove that $v_3 t_1 = v_1 t_2$ and $\dfrac{1}{v_1}+\dfrac{1}{v_3} = \dfrac{2\cos\beta}{v_2}$

(2010 : 12 Marks)

Solution:



Above figure shows the path of projectile given in question.

As in x-direction, there is no acceleration.

$\therefore$ Velocities in x-direction at A, B, C are equal.

$\therefore$ $$v_1\cos\alpha = v_2\cos(\alpha-\beta) = v_3\cos(\alpha-2\beta) \quad \text{...(1)}$$

Now, $$\frac{1}{v_1}+\frac{1}{v_3} = \frac{\cos\alpha}{v_2\cos(\alpha-\beta)}+\frac{\cos(\alpha-2\beta)}{v_2\cos(\alpha-\beta)} \quad \text{(from (1))}$$

$$= \frac{\cos\alpha+\cos(\alpha-2\beta)}{v_2\cos(\alpha-\beta)} = \frac{2\cos(\alpha-\beta)\cdot\cos\beta}{v_2\cos(\alpha-\beta)}$$

$$= \frac{2\cos\beta}{v_2}$$

$$\therefore \qquad \frac{1}{v_1}+\frac{1}{v_3} = \frac{2\cos\beta}{v_2}, \text{ which is the required result.}$$

Now, Newton's equation of motion is y-direction with g as acceleration.

$$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - gt_2 \text{ for } BC$$

and
$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - gt_1 \text{ for } AB$$

$$\therefore \qquad t_1 = \frac{v_1\sin\alpha - v_2\sin(\alpha-\beta)}{g} \qquad ...(2)$$

and
$$t_2 = \frac{v_2\sin(\alpha-\beta) - v_3\sin(\alpha-2\beta)}{g} \qquad ...(3)$$

So,
$$v_3 t_1 = \frac{v_3v_1\sin\alpha - v_3v_2\sin(\alpha-\beta)}{g} \qquad \text{(from (2))}$$

$$= \frac{\dfrac{v_1v_2\cos(\alpha-\beta)\sin\alpha}{\cos(\alpha-2\beta)} - \dfrac{v_1v_2\cos\alpha\sin(\alpha-\beta)}{\cos(\alpha-2\beta)}}{g} \qquad \text{(from (1))}$$

$$= \frac{v_1v_2}{g\cos(\alpha-2\beta)}\{\sin\alpha\cos(\alpha-\beta) - \cos\alpha\sin(\alpha-\beta)\}$$

$$= \frac{v_1v_2\sin\beta}{g\cos(\alpha-2\beta)} = \text{LHS}$$

Also,
$$v_1 t_2 = \frac{v_1\{v_2\sin(\alpha-\beta) - v_3\sin(\alpha-2\beta)\}}{g} \qquad \text{(from (3))}$$

$$= \frac{v_1v_2\sin(\alpha-\beta) - \dfrac{v_1v_2\cos(\alpha-\beta)\sin(\alpha-2\beta)}{\cos(\alpha-2\beta)}}{g} \qquad \text{(from (1))}$$

$$= \frac{v_1v_2}{g\cos(\alpha-2\beta)}[\sin(\alpha-\beta)\cos(\alpha-2\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)]$$

$$= \frac{v_1v_2}{g\cos(\alpha-2\beta)} \times \sin(\alpha-\beta-\alpha+2\beta) = \frac{v_1v_2\sin\beta}{g\cos(\alpha-2\beta)} = \text{RHS}$$

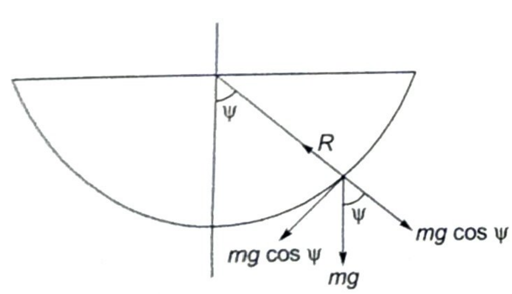
As LHS = RHS

$$\therefore \qquad v_3t_1 = v_1t_2, \text{ which is the required result.}$$

2.5 A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

(2010 : 20 Marks)

Solution:



Equation of motios are :

$$R - mg\cos\psi = \frac{mv^2}{r} \qquad ...(1)$$

and
$$\frac{md^2s}{dt^2} = -mg\sin\psi \qquad ...(2)$$

Also,
$$s = 4a\sin\psi \qquad ...(3)$$

By (2) and (3)

$$m\frac{d^2s}{dt^2} = -mg\times\frac{s}{4a}$$

$$\Rightarrow \quad \frac{d^2s}{dt^2} = -\frac{gs}{4a}$$

Multiply by $\frac{2ds}{dt}$ on both sides and integrate, we get

$$\left(\frac{ds}{dt}\right)^2 = -\frac{gs^2}{4a} + A$$

At $s = 4a$,
$$\frac{ds}{dt} = 0$$

$$\therefore \quad 0 = \frac{-g\times(4a)^2}{4a} + A$$

$$\Rightarrow \quad A = 4ag$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2 = \frac{-gs^2}{4a} + 4ag = \frac{g(16a^2 - s^2)}{4a}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{\frac{g}{4a}}\sqrt{16a^2 - s^2}$$

$$\Rightarrow \quad \frac{\sqrt{4a}}{\sqrt{16a^2 - s^2}}ds = -\sqrt{g}dt$$

Integrating both sides, we get

$$\int\sqrt{\frac{4a}{g}}\frac{ds}{\sqrt{1-\left(\frac{s}{4a}\right)^2}\times 4a} = -\int dt$$

$$\Rightarrow \quad \frac{1}{\sqrt{4ag}}\int\frac{ds}{\sqrt{1-\left(\frac{3}{4a}\right)^2}} = -\int dt$$

$$\Rightarrow \quad \sqrt{\frac{4a}{g}}\int\frac{ds}{\sqrt{1-\left(\frac{s}{4a}\right)^2}} = -\int dt$$

From 0 to t_1, s varies from $4a$ to $2\sqrt{2}a$.

$$\therefore \quad \sqrt{\frac{4a}{g}}\left[\sin^{-1}\left(\frac{s}{4a}\right)\right]_{4a}^{2\sqrt{2}a} = -t_1$$

$$\Rightarrow \quad t_1 = -\sqrt{\frac{4a}{g}}\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{\pi}{4}\sqrt{\frac{4a}{g}}$$

From t_1 to t_2, s varies from $2\sqrt{2}a$ to 0.

$$\therefore \quad \sqrt{\frac{4a}{g}}\left[\sin^{-1}\frac{s}{4a}\right]_{2\sqrt{2}a}^{0} = -[t]_{t_1}^{t_2}$$

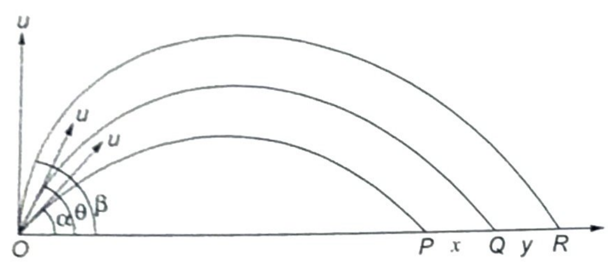
$$\Rightarrow \quad t_2 - t_1 = -\sqrt{\frac{4a}{g}}\times\left(0-\frac{\pi}{4}\right) = \frac{\pi}{4}\sqrt{\frac{4a}{g}}$$

$\therefore$ Time required in falling down in first half and second half is equal.

2.6 A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls x meter short of it when the angle of projection is a and goes y meter beyond when the angle of projection is β. If the velocity of projection is assumed same in all cases, find the correct angle of projection.

(2011 : 10 Marks)

Solution:



Let O be the point of projection of the projectile aimed at the mark Q such that

$$OQ = R \text{ (say)}$$

Let θ be the correct angle of projection and let u be the velocity of projection in each case.

When the angle of projection is α, the particle falls at P and when it is β, the particle falls at Q.

Then, $\quad OP = R - x$ and $OQ = R + y$

Using Range, $\quad R = \dfrac{u^2\sin 2\alpha}{g}$, we get

$$R - x = \frac{u^2\sin 2\alpha}{g} \text{ and } R + y = \frac{u^2\sin 2\alpha}{g}$$

$$\therefore \quad y(R-x) + x(R+y) = \frac{u^2}{g}(y\sin 2\alpha + x\sin 2\beta)$$

$$\Rightarrow \quad R(x+y) = \frac{u^2}{g}(y\sin 2\alpha + x\sin 2\beta) \qquad \text{...(i)}$$

Also, $\quad R = \dfrac{u^2\sin 2\theta}{g}$

$$\Rightarrow \quad \frac{u^2}{g} = \frac{R}{\sin 2\theta}$$

$\therefore$ from (i), we have

$$R(x+y) = \frac{R}{\sin 2\theta}(y\sin 2\alpha + x\sin 2\beta)$$

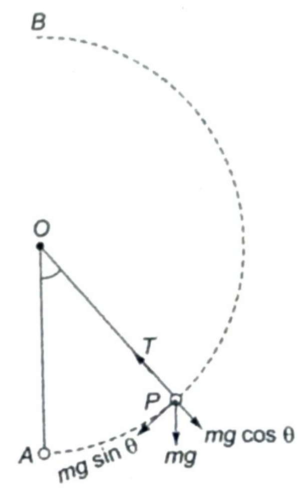
$$\Rightarrow \quad \sin 2\theta = \frac{y\sin 2\alpha + x\sin 2\beta}{x+y}$$

$$\Rightarrow \quad \theta = \frac{1}{2}\sin^{-1}\left(\frac{y\sin 2\alpha + x\sin 2\beta}{x+y}\right)$$

2.7 A particle of mass 2.5 kg hangs at the end of a string, 0.9 m long the other end of which is attached to a fixed point. The particle is projected horizontally with a velocity 8 m/sec. Find the velocity of the particle and tension in the string when the string is (i) horizontal (ii) vertically upward.

(2013 : 20 Marks)

Solution:



Let OA be the string of length l with mass m at the end.

From the equation of motion at any point P.

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad ...(i)$$

where $s = AP$ is the arc length.

and
$$T - mg\cos\theta = \frac{mv^2}{l} \qquad ...(ii)$$

$$s = l\theta \Rightarrow \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

$\therefore$
$$l\frac{d^2\theta}{dt^2} = -9\sin\theta$$

Multiplying by $2l\frac{d\theta}{dt}$ on both sides and integrating.

$$\left(l\frac{d\theta}{dt}\right)^2 = 2gl\cos\theta + B$$

$$V = \frac{ds}{dt} = l\frac{d\theta}{dt}$$

$\Rightarrow$
$$V^2 = 2gl\cos\theta + B$$

At point A,
$$\theta = 0,\ v = V$$

$$v^2 = 2gl + B \Rightarrow B = V^2 - 2gl$$

$\therefore$
$$\left(l\frac{d\theta}{dt}\right)^2 = v^2 = V^2 - 2gl(1-\cos\theta)$$

$$T - mg\cos\theta = \frac{mv^2}{l}$$

$$= \frac{mV^2}{l} - 2mg(1-\cos\theta)$$

$\Rightarrow$
$$T = \frac{mV^2}{l} - mg(2-3\cos\theta)$$

When string in horizontal $\theta = \frac{\pi}{2}$

$$v^2 = V^2 - 2gl = 8^2 - 2\times 9.8\times 0.9$$
$$= 46.36$$
$$v = 6.81 \text{ m/s}$$

$$T = \frac{2.5\times 8^2}{0.9} - 2\text{ mg} = 128.78 \text{ N}$$

When string is vertical $\theta = \pi$

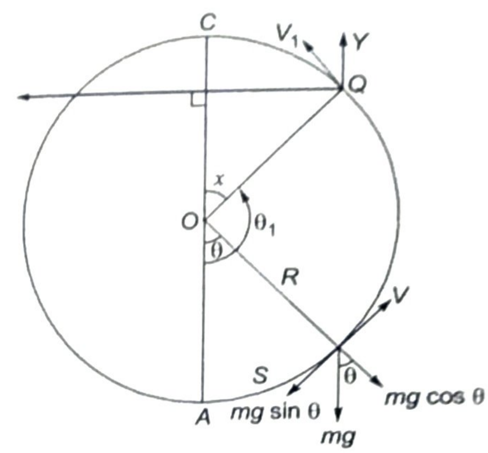
$$v^2 = V^2 - 4gl = 28.72 \Rightarrow v = 5.36 \text{ m/s}$$

$$T = \frac{mV^2}{l} - 5\text{ mg} = 55.28 \text{ N}$$

2.8 A particle of mass m, hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts to it a velocity $2\sqrt{gl}$. Find the velocity and height of the particle from the level of its initial position when the cord becomes slack.

(2014 : 15 Marks)

Solution:



Take $R = T$ (i.e., the tension in the string) let a particle tied to a cord OA of length l be struck by a horizontal blow which imports it a velocity $2\sqrt{gl}$. If P is the position of the particle at time t such that $\angle AOP = \theta$. Then the equations of motion are

$$m\frac{d^2s}{dt^2} = -m\sin\theta \qquad ...(i)$$

and

$$m\frac{v^2}{l} = T - mg\cos\theta \qquad ...(ii)$$

also

$$s = l\theta$$

from (i) and (iii), we have

$$l\frac{d^2\theta}{\phi t} = -g\sin\theta$$

Multiplying both sides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = s$$

But at the point A, $\theta = A$ and $v = 2\sqrt{gl}$

$\therefore$ $$4gl = 2lg + A \text{ so that } A = 2gl$$

$\therefore$ $$v^2 = 2lg(\cos\theta + 1) \qquad ...(iv)$$

from (ii) and (iv), we have

$$T = m/l\,(v^2 + gl\cos\theta) = mg(3\cos\theta + 2) \qquad ...(v)$$

If the cord becomes slack at the point Q, where $\theta = 0$,

$$T = 0 - mg(3\cos\theta_1 + 2)$$

giving as

$$\cos\theta_1 = -\frac{2}{3}$$

If $\angle COQ = \alpha$, than $\alpha = \pi - \theta$, and $\cos\alpha = \frac{2}{3}$

If v_1 is the velocity of the particle at Q, then $v = v$.

where $\theta = \theta$, therefore from (iv), we have

$$v_1^2 = 2lg(1+\cos\theta_1) = 2lg\left(1-\frac{2}{3}\right) = \frac{2lg}{3}$$

Now

$$OL = l\cos\alpha = \frac{2}{3}l$$

Thus the particle leaves the circular path at the point Q at a height $\frac{2}{3}l$ above the fixed point O with velocity

$v_1 = \sqrt{\left(\frac{2lg}{3}\right)}$ at an angle x to the horizontal and subsequently it describes a parabolic path.

2.9 A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2} and when $y = a$, it is projected parallel to the axis of x with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the path of the particle. Here μ is a constant.

(2014 : 15 Marks)

Solution:

Hence, we are given that

$$\frac{d^2y}{dt^2} = -\mu y^{-2} \quad \text{...(i)}$$

The negative sign has been taken because the force is in the direction of y increasing. Also, there is no force parallel to the axis of x.

Therefore,
$$\frac{d^2x}{dt^2} = 0 \quad \text{...(ii)}$$

Multiplying both sides of (i) by $2\frac{dy}{dt}$ and then integrating w.r.t., t, we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y}+A, \text{ where } A \text{ is a constant.}$$

Initially, when $y = a$,
$$\frac{dy}{dt} = 0 \text{ (note that initially there is no velocity parallel to } y\text{-axis)}$$

$\therefore$
$$A = -\frac{2\mu}{a}$$

$\therefore$
$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y}-\frac{2\mu}{a} = 2\mu\left(\frac{1}{y}-\frac{1}{a}\right) = \frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

$\Rightarrow$
$$\frac{dy}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)}\cdot\sqrt{\frac{a-y}{y}} \quad \text{...(iii)}$$

(–ve sign has been taken because the particle is moving in the direction of y decreasing).

Now integrating (ii), we have

$$\frac{dx}{dt} = B, \text{ where } B \text{ is a constant.}$$

Initially, when $y = a$,
$$\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$$

so that
$$B = \sqrt{\frac{2\mu}{a}}$$

$\therefore$
$$\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}} \quad \text{...(iv)}$$

Dividing (iii) by (iv), we have

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$\Rightarrow$
$$dx = -\sqrt{\frac{y}{a-y}}dy$$

Integrating,

$$\int dx = -\int \sqrt{\frac{y}{a-y}}dy$$

$$= 2a\int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\sin\theta d\theta + C$$

(Putting $y = a\cos^2\theta$, so that $dy = -2a\sin\theta\cos\theta\, d\theta$)

$$= a\int(1+\cos 2\theta)d\theta + C$$

$$= a\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a}{2}(2\theta + \sin 2\theta)C$$

Let us take $x = 0$, when $y = a$

i.e., when $a\cos^2\theta = a \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

Then $$0 = \frac{1}{2}a(2\theta + \sin 2\theta) \quad \text{...(v)}$$

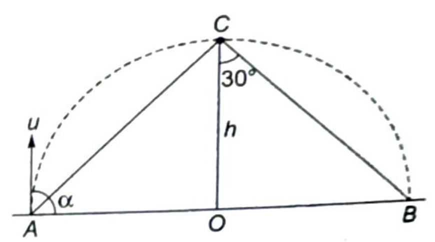
Also, $$y = a\cos^2\theta = \frac{a}{2}(1+\cos 2\theta) \quad \text{...(vi)}$$

The equation (v) and (vi) give us the path of the particle. But these are the parametric equation of a cycloid.

2.10 A particle is projected from the base of a hill whose slope is that of a right circular cone, whose axis is vertical. The projectile grazes the vertex and strikes the hill again at a point on the base. If the semi-vertical angle of the cone is 30°, 'h' is height, determine the initial velocity 'u' of the projection and its angle of projection.

(2015 : 13 Marks)

Solution:



Height, $OC = h$

$$OB = h\tan 30° = \frac{h}{\sqrt{3}}$$

$$AB = \frac{2h}{\sqrt{3}}$$

Range, $AB \Rightarrow$ $$\frac{u^2\sin^2\alpha}{g} = \frac{2h}{3} \quad \text{...(i)}$$

Height, $OC \Rightarrow$ $$\frac{u^2\sin^2\alpha}{2g} = h \quad \text{...(ii)}$$

Dividing (i) by (ii)

$$\frac{2\sin\alpha\cos\alpha}{(\sin^2\alpha)/2} = \frac{2h}{\sqrt{3}\times h}$$

$$\Rightarrow \quad \frac{4}{\tan\alpha} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad a = \tan^{-1}2\sqrt{3}$$

(ii) $\Rightarrow$ $$u^2 = \frac{2gh}{\sin^2\alpha} = \frac{2gh}{\left(\frac{2\sqrt{3}}{\sqrt{13}}\right)^2}$$

$$= \frac{13gh}{6}$$

$$\therefore \qquad u = \sqrt{\frac{13gh}{6}}$$

2.11 A particle moves with a central acceleration which varies inversely as the cube of the distance. If it is projected from an apse at a distance 'a' from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, then find the equation to the path.

(2016 : 10 Marks)

Solution:

Given : Acceleration $\propto \dfrac{1}{(\text{distance})^3}$

i.e., $$P = \frac{\mu}{r^3} = \mu u^3, \text{ where } \mu - \text{constant}$$

If V is the velocity for a circle of radius 'a', then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3} \text{ or } V = \sqrt{\frac{\mu}{a^2}}$$

$\therefore$ The velocity of projection

$$V_1 = \sqrt{2}V = \sqrt{\frac{2\mu}{a^2}}$$

The DE of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu \cdot u$$

Multiplying both sides by $2\dfrac{du}{d\theta}$ and integrating

$$V^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \qquad \text{...(i)}$$

where A is a constant.

But initially when $r = a$, i.e., $u = \dfrac{1}{a}$, $\dfrac{du}{d\theta} = 0$ and $V = V_1 = \sqrt{\dfrac{2\mu}{a^2}}$ (at an apse).

$\therefore$ from (i), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$$\therefore \qquad h^2 = 2\mu \text{ and } A = \frac{\mu}{a^2}$$

Substituting the values of h^2 and A in (i)

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu \cdot u^2 + \frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

or $$\sqrt{2}a\frac{du}{d\theta} = \sqrt{1-a^2u^2}$$

or $$\frac{d\theta}{\sqrt{2}} = \frac{a\cdot du}{\sqrt{1-a^2u^2}}$$

Integrating, $$\frac{\theta}{\sqrt{2}}+B = \sin^{-1}(au), \text{ where } B\text{-constant}$$

But initially, when $u = \frac{1}{a}$, $\theta = 0$

$\therefore$ $$B = \sin^{-1}1 = \frac{\pi}{2}$$

$\therefore$ $$\frac{\theta}{\sqrt{2}}+\frac{\pi}{2} = \frac{1}{2}\sin^{-1}(au)$$

or $$au = \frac{a}{r} = \sin\left[\frac{\pi}{2}+\frac{\theta}{\sqrt{2}}\right]$$

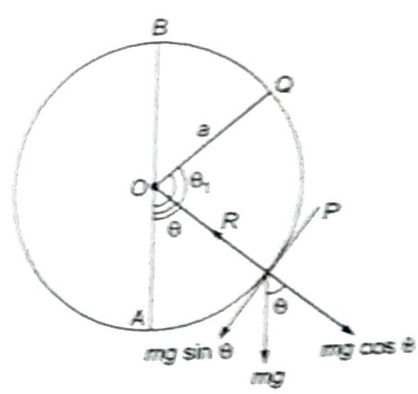
or $$a = r\cos\left(\frac{\theta}{\sqrt{2}}\right)$$, which is the required equation of the path.

2.12 A particle is free to move on a smooth vertical circular wire of radius a. At time $t = 0$ it is projected along the circle from its lowest point A with velocity just sufficient to carry it to the highest point B. Find the time T at which the reaction between the particle and the wire is zero.

(2017 : 17 Marks)

Solution:

If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $(AP) = S$, then the equation of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad ...(i)$$

$$m\frac{v^2}{a} = R-mg\cos\theta \quad ...(ii)$$

Also, $$S = a\theta \quad ...(iii)$$

From (i) and (iii), $$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying both sides by $2a\left(\frac{d\theta}{dt}\right)$ and integrating

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But, $v = 0$ at the highest point B, where $\theta = \pi$.

$\therefore$ $$0 = 2ag\cos\pi + A$$

$\Rightarrow$ $$A = 2ag$$

$\therefore$ $$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag \quad ...(iv)$$

From (ii) and (iv)

$$R = \frac{m}{a}(v^2+ag\cos\theta) = \frac{m}{a}(2ag+3ag\cos\theta) \quad ...(v)$$

If reaction $R = 0$ at the point Q where $\theta = \theta_1$ then from (v) we have

$$0 = \frac{m}{a}(2ag + 3ag\cos\theta_1)$$

$$\Rightarrow \qquad \cos\theta_1 = -\frac{2}{3} \qquad \text{...(vi)}$$

From (iv),

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta + 1) = 2ag \cdot 2\cos^2\frac{\theta}{2}$$

$$= 4ag\cos^2\frac{\theta}{2}$$

$$\therefore \qquad \frac{d\theta}{dt} = 2\sqrt{\frac{g}{a}}\cdot\cos\frac{\theta}{2}, \text{ taking +ve sign as } \theta \text{ increase with } t$$

$$dt = \frac{1}{2}\sqrt{\frac{a}{g}}\cdot\sec\frac{\theta}{2}d\theta$$

Integrating from $\theta_1 = 0$ to $\theta = \theta_1$,

$$t = \frac{1}{2}\sqrt{\frac{a}{g}}\int_0^{\theta_1}\sec\frac{\theta}{2}d\theta$$

$$= \sqrt{\frac{a}{g}}\left[\log\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_0^{\theta_1}$$

$$= \sqrt{\frac{a}{g}}\left[\log\left(\sec\frac{\theta_1}{2} + \tan\frac{\theta_1}{2}\right)\right]$$

From (vi),

$$2\cos^2\frac{\theta}{2} - 1 = -\frac{2}{3}$$

$$\Rightarrow \qquad \sec\frac{\theta_1}{2} = \sqrt{6}\,;\ \tan\frac{\theta_1}{2} = \sqrt{5}$$

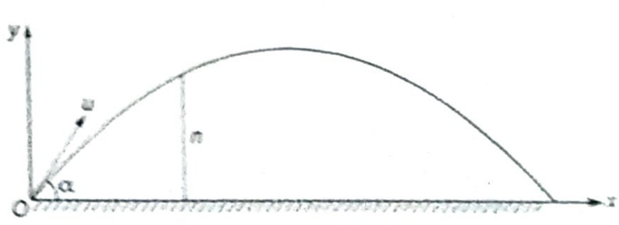
Hence,

$$t = \sqrt{\frac{a}{g}}\log\left(\sqrt{6} + \sqrt{5}\right)$$

2.13 A particle projected from a given point on the ground just clears a wall of height h at distance d from the point of projection. If particle moves in a vertical plane and if the horizontal range is R, find the elevation of projection.

(2018 : 10 Marks)

Solution:



Let u be the initial velocity and α be the angle of projection, g be the acceleration due to gravity.

Given, range is R $= \dfrac{u^2\sin 2\alpha}{g}$ ...(i)

Let t be the time taken to cover distance d, in horizontal direction.

$\therefore$
$$d = u\cos\alpha \times t + 0 \Rightarrow d = u\cos\alpha \times t$$

or
$$t = \frac{d}{u\cos\alpha} \qquad \text{...(ii)}$$

Given, h is the distance covered in vertical direction, therefore,

$$h = u\sin\alpha t - \frac{1}{2}gt^2$$

$$\Rightarrow \quad h = u\sin\alpha \times \frac{d}{u\cos\alpha} - \frac{1}{2}g \times \frac{d^2}{u^2\cos^2\alpha} \qquad \text{(from (ii))}$$

$$\Rightarrow \quad h = d\tan\alpha - \frac{1}{2}g\frac{d^2\sin^2\alpha}{Rg\cos^2\alpha} \qquad \text{(from (i))}$$

$$\Rightarrow \quad h = d\tan\alpha - \frac{1}{2}\frac{d^2}{R} \times \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha}$$

$$\Rightarrow \quad h = d\tan\alpha - \frac{d^2}{R}\tan\alpha$$

$$\Rightarrow \quad \tan\alpha = \frac{Rh}{Rd - d^2}$$

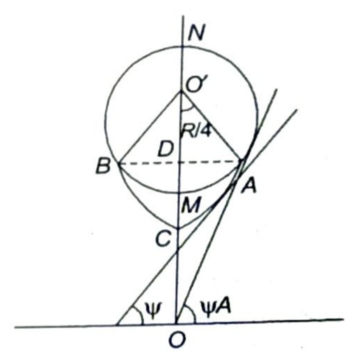
$$\Rightarrow \quad \alpha = \tan^{-1}\left[\frac{Rh}{d(R-d)}\right]$$

3. Constrained Motion

3.1 Find the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with three-fourth of the circumference of the pulley.

(2009 : 15 Marks)

Solution:



Since the chain is in contact with 3/4th of the pulley

$$ANB = \frac{3}{4} \times \text{Circumference of pulley}$$

$$= \frac{3}{4} \times 2\pi a = \frac{3\pi a}{2}$$

$$\angle AO'B = \frac{1}{4} \times 2\pi = \frac{\pi}{2}$$

$$\therefore \quad \angle AO'M = \angle BO'M = \frac{\pi}{4}$$

Let OX be the directrix and OY the axis of the caternary of the part of the chain hanging from the pulley. Let C be the parameter.

Then, $$OC = C$$

Tangent at A is perpendicular to radius.

So, the tangential angle at A, i.e., $\psi_A = \dfrac{\pi}{4}$

From $\Delta AO'D$

$$\Delta A = O'A\sin\frac{\pi}{4} = \frac{a}{\sqrt{2}}$$

$\therefore$ At A,
$$\psi = C\log(\tan\psi + \sec\psi)$$
$$\Rightarrow \quad \frac{a}{\sqrt{2}} = C\log\left(\tan\frac{\pi}{4}+\sec\frac{\pi}{4}\right)$$
$$= C\log\left(1+\sqrt{2}\right)$$
$$\therefore \quad C = \frac{a}{\sqrt{2}\log\left(1+\sqrt{2}\right)}$$
$$S = C\tan\psi \Rightarrow S_A = \frac{a}{\sqrt{2}\log\left(1+\sqrt{2}\right)}$$
$$\therefore \quad \text{Arc } AB = \frac{2a}{\sqrt{2}\log\left(1+\sqrt{2}\right)}$$
$$\text{Total length of chain} = \frac{\sqrt{2}a}{\log\left(1+\sqrt{2}\right)}+\frac{3\pi a}{2}$$

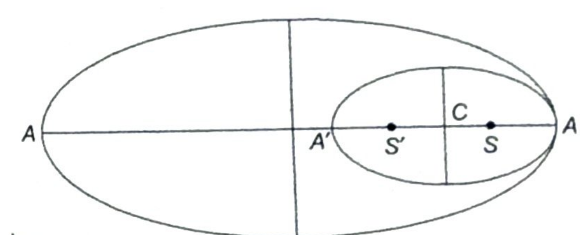
4. Work & Energy

4.1 A body is describing an ellipse of eccentricity e under the action of a central force directed towards a focus and when at the nearer apse the centre of the force is transferred to the other focus. Find the eccentrity of the new orbit in terms of the eccentricity of the original orbit.

(2009 : 12 Marks)

Solution:

Approach : The motion in an ellipse in under inverse square force and the velocity determines the eccentricity of the ellipse. So balance the velocity betweent he two orbits.



Let S and S' be the foci of the original orbits and the force initially be directed towards S. The nearer apse is at A.

Since the motion is in an ellipse the velocity at any point is given by
$$V^2 = \mu\left(\frac{2}{R}-\frac{1}{a}\right)$$
At A, $R = SA = CA - CS = a - ac = a(1-e)$
$$\therefore \quad V^2 = \mu\left(\frac{2}{a(1-e)}-\frac{1}{a}\right) = \frac{\mu(1+e)}{a(1-e)}$$
When the centre of force is changed to S', the velocity remains same. Let a' be the semi major axis of the new orbit.

Then,
$$V^2 = \frac{\mu(1+e)}{a(1-e)} = \mu\left(\frac{2}{R}-\frac{1}{a'}\right)$$
$$R = AS' = AC + CS' = a + ac = a(1+e)$$
$$\therefore \quad \frac{\mu(1+e)}{a(1-e)} = \mu\left(\frac{2}{a(1+e)}-\frac{1}{a'}\right)$$
$$\Rightarrow \quad \frac{1+e}{1-e} = \frac{2}{1+e}-\frac{a}{a'}$$
Again the direction of velocity is not changed, the velocity still being perpendicular to the direction of motion the point A is an apse for the new orbit as well.

$\therefore$ $$S'A = a'(1-e') = a(1+e)$$

$\Rightarrow$ $$\frac{a}{a'} = \frac{1-e'}{1+e}$$

$\Rightarrow$ $$\frac{1+e}{1-e} = \frac{2}{1+e} - \frac{1-e'}{1+e} = \frac{1+e'}{1+e}$$

$\Rightarrow$ $$e' = \frac{(1+e)^2}{1-e} - 1 = \frac{e^2+3e}{1-e} = \frac{e(e+3)}{1-e}$$

4.2 A particle moves with a central acceleration $\mu(r^5 - 9r)$, being projected from an apse at a distance $\sqrt{3}$ with velocity $3\sqrt{2\mu}$. Show that its path is the curve $x^4 + y^4 = 9$.

(2010 : 20 Marks)

Solution:

The equation of motion is

$$h^2\left(y + \frac{d^2y}{d\theta^2}\right) = \frac{P}{u^2} \text{ where } u = \frac{1}{r}$$

$$P = \text{central acceleration} = \mu(r^5 - 9r) \text{ given}$$

$$= \mu\left(\frac{1}{u^5} - \frac{9}{u}\right)$$

$\therefore$ Equation becomes $$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{\mu}{u^2}\left(\frac{1}{u^5} - \frac{9}{u}\right)$$

$\Rightarrow$ $$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu\left(\frac{1}{4^7} - \frac{9}{4^3}\right)$$

Multiplying both sides by $2\frac{du}{d\theta}$ and integrating, we get

$$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \mu\left(\frac{-2}{6u^6} + \frac{9\times 2}{2u^2}\right) + A \qquad (A \text{ is a constant})$$

$\Rightarrow$ $$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \mu\left(\frac{9}{u^2} - \frac{1}{3u^6}\right) + A = v^2 \qquad (v\text{-velocity})$$

At apse, $$v = 3\sqrt{2\mu} \qquad \text{(given)}$$

$$u = \frac{1}{r} = \frac{1}{\sqrt{3}}$$

$\therefore$ $$\mu\left(9\times 3 - \frac{1}{3}\times 27\right) + A = 18\mu$$

$\Rightarrow$ $$A + 18\mu = 18\mu \Rightarrow A = 0$$

$\therefore$ $$h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \mu\left(\frac{9}{u^2} - \frac{1}{3u^6}\right)$$

Also, at apse, $$\left(\frac{du}{d\theta}\right) = 0$$

$\therefore$ $$h^2\left(\frac{1}{3}\right) = \mu\left(9\times 3 - \frac{1}{3}\times 27\right) = 18\mu$$

$$\Rightarrow \qquad h^2 = 54\mu$$

$\therefore$ Equation becomes

$$54\mu\left(u^2+\left(\frac{du}{d\theta}\right)^2\right) = \mu\left(\frac{9}{4^2}-\frac{1}{3u^6}\right)$$

$$\Rightarrow \qquad u^2+\left(\frac{du}{d\theta}\right)^2 = \frac{1}{5u}\left(\frac{9}{u^2}-\frac{1}{3u^6}\right)$$

$$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{5u}\left(\frac{27u^4-1}{3u^6}\right)-u^2$$

$$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{27u^4-1-162u^8}{162u^6}$$

$$\Rightarrow \qquad \frac{du}{d\theta} = \sqrt{\frac{\frac{27}{162}u^4-\frac{1}{162}-u^8}{u^6}}$$

$$\Rightarrow \qquad \frac{u^3du}{\sqrt{\frac{u^4}{6}-\frac{1}{162}-u^8}} = d\theta$$

$$\Rightarrow \qquad \frac{u^3du}{\sqrt{-\frac{1}{162}-\left(4^4-\frac{1}{12}\right)^2+\frac{1}{12^2}}} = d\theta$$

$$\Rightarrow \qquad \frac{u^3du}{\frac{1}{36}\sqrt{1-36^2\left(4^4-\frac{1}{12}\right)^2}} = d\theta$$

Integrating both sides, we get

$$\int\frac{36u^3du}{\sqrt{1-(36u^4-3)^2}} = \int d\theta$$

$$\Rightarrow \qquad \frac{\sin^{-1}(36u^4-3)}{4} = \theta + B$$

Now, at $u = \frac{1}{r} = \frac{1}{\sqrt{3}}$, $\theta = 0$ (apse)

$$\therefore \qquad \frac{\sin^{-1}\left(\frac{36}{9}-3\right)}{4} = 0 + B$$

$$\Rightarrow \qquad \frac{\sin^{-1}(1)}{4} = B \Rightarrow B = \frac{\pi}{8}$$

$$\therefore \qquad \frac{\sin^{-1}(36u^4-3)}{4} = \theta+\frac{\pi}{8}$$

$$\Rightarrow \qquad \sin^{-1}(36u^4-3) = 4\theta+\frac{\pi}{2}$$

$$\Rightarrow \qquad 36u^4-3 = \sin\left(\frac{\pi}{2}+4\theta\right) = \cos 4\theta$$

$$\Rightarrow \quad (\cos^2 2\theta - \sin^2 2\theta) = 36u^4 - 3$$

$$\Rightarrow \quad (\cos^2 2\theta - \sin^2 2\theta) + (\cos 2 2\theta + \sin^2 2\theta) = 36u^4 - 2$$

$$\Rightarrow \quad 2\cos^2 2\theta = 36u^4 - 2$$

$$\Rightarrow \quad \cos^2 2\theta = 18u^4 - 1$$

$$\Rightarrow \quad (\cos^2\theta - \sin^2\theta)^2 + (\cos^2\theta + \sin^2\theta)^2 = 18u^4$$

$$\Rightarrow \quad \cos^4\theta + \sin^4\theta - 2\cos^2\theta\sin^2\theta + \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta = 18u^4$$

$$\Rightarrow \quad 2\cos^4\theta + 2\sin^4\theta = 18u^4$$

$$\Rightarrow \quad \cos^4\theta + \sin^4\theta = 9u^4 = \frac{9}{r^4}$$

$$\Rightarrow \quad r^4\cos^4\theta + r^4\sin^4\theta = 9$$

$$\Rightarrow \quad x^4 + y^4 = 9$$

$\therefore$ Path is the curve $x^4 + y^4 = 9$.

4.3 A particle moves in a plane under a force, towards a fixed centre, proportional to the distance. If the path of the particle has two apsidal distance a, b (a > b), then find the equation of the path.

(2015 : 13 Marks)

Solution:

$$P = u^2h^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$\Rightarrow \quad -\frac{\lambda}{u} = u^2h^2\left(u + \frac{d^2u}{d\theta^2}\right) \Rightarrow -\frac{\lambda}{u^3} = h^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow \quad -\frac{\lambda}{u^3}\cdot\frac{2du}{d\theta} = h^2\left[u\cdot 2\frac{du}{d\theta} + \frac{d^2u}{d\theta^2}\cdot\frac{2du}{d\theta}\right]$$

$$\Rightarrow \quad A + \frac{\lambda}{u^2} = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] \qquad \text{(Integrating)}$$

$$\text{or} \quad u^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\lambda}{u^2} + A$$

$$\text{At } u = \frac{1}{a}, \quad \frac{du}{d\theta} = 0 \Rightarrow A + \lambda a^2 = h\left[\frac{1}{a^2}\right]$$

$$\Rightarrow \quad A + \lambda a^2 = \frac{h^2}{a^2}$$

$$\text{Similarly,} \quad A + \lambda b^2 = \frac{h^2}{b^2}$$

$$\therefore \quad \lambda(a^2 - b^2) = h^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) \Rightarrow \frac{\lambda}{h^2} = \frac{-1}{a^2b^2}$$

$$\therefore \quad A + \lambda a^2 = \frac{-\lambda a^2b^2}{a^2} \Rightarrow A = -\lambda(a^2 + b^2)$$

$$\text{Thus,} \quad -\lambda a^2b^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\lambda(a^2 + b^2) + \frac{\lambda}{u^2}$$

$$\Rightarrow \qquad u^2\left(\frac{du}{d\theta}\right)^2 = \frac{\left(a^2+b^2-\frac{1}{u^2}\right)}{a^2b^2}$$

$$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{a^2+b^2}{a^2b^2} - \frac{1}{u^2a^2b^2} - u^2$$

$$= \frac{1}{u^2}\left[\frac{a^2+b^2}{a^2b^2}u^2 - \frac{1}{a^2b^2} - u^4\right]$$

$$= \frac{1}{u^2}\left[-\left(\frac{1}{2}\frac{a^2+b^2}{a^2b^2} - u^2\right)^2 + \frac{1}{4}\left(\frac{a^2+b^2}{a^2b^2}\right)^2 - \frac{1}{a^2b^2}\right]$$

Let
$$K_1^2 = \frac{1}{4}\left(\frac{a^2+b^2}{a^2b^2}\right)^2 - \frac{1}{a^2b^2} = \frac{1}{4(a^2b^2)^2}[(a^2+b^2)^2 - 4a^2b^2]$$

$$= \frac{(a^2-b^2)^2}{4(a^2b^2)^2} = \frac{1}{4}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2$$

and
$$K_2^2 = \frac{1}{2}\cdot\frac{a^2+b^2}{a^2b^2} = \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

$$\Rightarrow \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^2}(K_1^2 - (K_2^2 - u^2)^2)$$

$$\Rightarrow \qquad \frac{u\,du}{\sqrt{K_1^2 - (K_2^2 - u^2)^2}} = d\theta$$

Let
$$K_2^2 - u^2 = v$$
$$-2u\,du = dv$$

$$\therefore \qquad -\frac{1}{2}\cdot\frac{dv}{\sqrt{K_1^2 - v^2}} = d\theta \;\Rightarrow\; \theta + C = -\frac{1}{2}\sin^{-1}\frac{v}{K_1}$$

$$\Rightarrow \qquad \frac{v}{K_1} = \sin[-2(\theta + C)] \;\Rightarrow\; \frac{K_2^2 - u^2}{K_1} = -\sin 2(\theta + C)$$

$$\Rightarrow \qquad \frac{1}{2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) - u^2 = -\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)\sin 2(\theta + C)$$

$$u^2 = \frac{1}{2}\left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{b^2}\sin 2(\theta + C) - \frac{1}{a^2}\sin 2(\theta + C)\right]$$

Let $C_1 = C - \frac{\pi}{4}$ $\Rightarrow$
$$u^2 = \frac{\sin^2(\theta + C_1)}{a^2} + \frac{\cos^2(\theta + C_1)}{b^2}$$

4.4 A particle moves in a straight line. Its acceleration is directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance 'a' from O, then find the time, the particle will arrive at O.

(2016 : 15 Marks)

Solution:

Here, $$\frac{d^2x}{dt^2} = -\mu\left(\frac{a^5}{x^2}\right)^{1/3} \quad \text{(decreasing } x\text{)}$$

Multiplying both sides by $2\left(\frac{dx}{dt}\right)$ and integrating

$$v^2 = \left(\frac{dx}{dt}\right)^2 = -2\mu a^{5/3}(3x^{1/3}) + C$$

Initially, at $x = a$, $v = 0$

$$\therefore \quad 0 = -2\mu a^{5/3}(3a^{1/3}) + C$$

$$\Rightarrow \quad C = 6\mu a^2$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = -6\mu a^{5/3}x^{1/3} + 6\mu a^2$$

$$\therefore \quad \frac{dx}{dt} = -\sqrt{6\mu a^2 - 6\mu a^{5/3}\cdot x^{1/3}} \quad \text{(decreasing } x\text{)}$$

$$\frac{-dx}{\sqrt{6\mu a^2 - 6a^{5/3}\cdot \mu x^{1/3}}} = dt$$

Integrating, $$-\int_a^0 \frac{dx}{a\sqrt{6\mu}\sqrt{1-\left(\frac{x}{a}\right)^{1/3}}} = \int_0^t dt$$

Let $$\left(\frac{x}{a}\right)^{1/3} = \sin^2\theta \Rightarrow x^{2/3} = a^{2/3}\cdot\sin^4\theta$$

$$\therefore \quad \frac{1}{3a^{1/3}}\cdot x^{-\frac{2}{3}}dx = 2\cos\theta\sin\theta\, d\theta$$

i.e., $$dx = (6a\cos\theta\sin^5\theta)d\theta$$

Integration transforms to

$$\int_0^{\pi/2}\frac{6a\cdot\cos\theta\sin^5\theta d\theta}{a\sqrt{6\mu}\cdot\cos\theta} = \int_0^t dt$$

$$t = \sqrt{\frac{6}{\mu}}\cdot\int_0^{\pi/2}\sin^5\theta d\theta$$

$$= \sqrt{\frac{6}{\mu}}\cdot\frac{\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{5+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\frac{6}{\mu}}\cdot\frac{\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)}\cdot\frac{\sqrt{\pi}}{2} \qquad \left[\int_0^{\pi/2}\sin^n x dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\cdot\frac{\sqrt{\pi}}{2}\right]$$

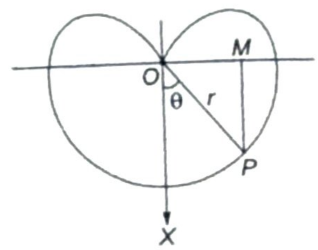
$$= \sqrt{\frac{6}{\mu}} \cdot \frac{2\times 1}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\cdot\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{8}{15} \cdot \sqrt{\frac{6}{\mu}} \qquad \begin{bmatrix} \Gamma(n+1) = n\Gamma(n) \\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \end{bmatrix}$$

4.5 A fixed wire is in the shape of the cardiod, $r = a(1 + \cos\theta)$, the initial line being the downward vertical. A small ring of mass 'm' can slide on the wire and is attached to the point $r = 0$ of the cardiod by an elastic string of natural length 'a' and modulus of elasticity 4 mg. The string is released from the rest when the string is horizongal. Show by using the laws of conservation of energy that $a\dot{\theta}^2(1+\cos\theta) - g\cos\theta(1-\cos\theta) = 0$, g being the acceleration due to gravity.

(2017 : 10 Marks)

Solution:



Here,

$$OP = r$$

$$PM = r\cos\theta$$

$$= a(1 + \cos\theta)\cos\theta$$

$$\text{K.E.} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$= \frac{1}{2}m(a^2\sin^2\theta + a^2(1+\cos\theta)^2)\dot{\theta}^2$$

$$= ma^2(1+\cos\theta)\dot{\theta}^2$$

where,

$$\dot{\theta} = \frac{d\theta}{dt},\ \dot{r} = \frac{dr}{dt}$$

$$v = \dot{r}\hat{I} + r\dot{\theta}\hat{J}$$

$$\hat{I} = \text{Radial Component}$$

$$\hat{J} = \text{Transverse Component}$$

PE due to gravity,

$$= -mg(PM)$$

$$= -mg\,a(1 + \cos\theta)\cos\theta$$

PE due to extension in string

$$= \frac{1}{2}\frac{4mg}{l}(r-a)^2$$

$$= \frac{1}{2}\cdot\frac{4mg}{l}(a\cdot\cos\theta)^2$$

$$= 2mg\,a\cos^2\theta$$

Thus, by conservation of Energy Principle, $KE + PE$ due to gravity $+ PE$ due to extension $= 0$.

$$ma^2(1+\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)\cos\theta + 2mga\cos^2\theta = 0$$

$$\Rightarrow \quad a(1+\cos\theta)\dot{\theta}^2 - g(\cos\theta + \cos^2\theta - 2\cos^2\theta) = 0$$

$$\Rightarrow \quad a(1+\cos\theta)\dot{\theta}^2 - g\cos\theta(1-\cos\theta) = 0.$$ Hence Proved.

[Equation of motion of spring is $4mg\cdot\frac{\text{Extension}}{\text{(Original Length)}}$

Integrating it we get the PE

$$= \frac{1}{2} \cdot \frac{4mg}{l} (\text{extension})^2]$$

4.6 A particle moving along the y-axis has an acceleration Fy towards the origin, where F is a positive and even function of y. The periodic time, when the particle vibrates between $y = -a$ and $y = a$ is T. Show that

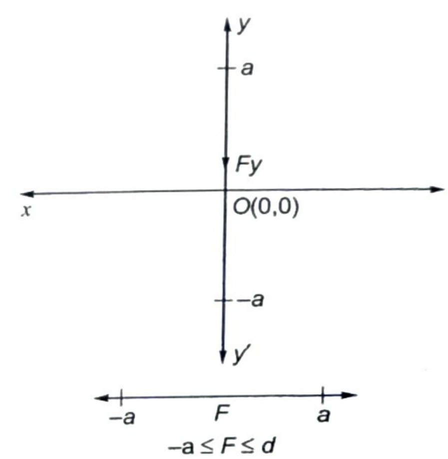
$$\frac{2\pi}{\sqrt{F_1}} < T < \frac{2\pi}{\sqrt{F_2}}$$

where F_1 and F_2 are the greatest and the least values of F within the range $[-a, a]$. Further, show that when a simple pendulum of length l oscillates through 30° on either side of the vertical line, T lies between $2\pi\sqrt{l/g}$ and $2\pi\sqrt{l/g}\sqrt{\pi/3}$.

(2019 : 20 Marks)

Solution:

A particle moving along y-axis has an acceleration Fy towards origin. The periodic time when the particle vibrates between $y = -a$ and $y = a$ is T.



$$w = \text{Frequency of oscillation}$$

$$w = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{w}$$

$$w_{max} = \sqrt{F_1}$$

$$w_{min} = \sqrt{F_2}$$

With the range $[-a, a]$

$$F_{max} = F_1$$
$$F_{min} = F_2$$

Let $w = \frac{2\pi}{T}$ in between w_{max} and w_{min}

$$w = \sqrt{\frac{g}{r}} \qquad \left[\because T = 2\pi\sqrt{\frac{L}{g}} \text{ and } w = \frac{2\pi}{T}\right]$$

Since, $T \propto \frac{1}{w}$ and $w_{max} = \sqrt{F_1}$; $w_{min} = \sqrt{F_2}$

$\therefore$ Time taken by F_1 is less than time taken by F_2 and hence

$$\frac{1}{\sqrt{F_1}} < \frac{1}{w} < \frac{1}{\sqrt{F_2}}$$

$$\frac{1}{\sqrt{F_1}} < \frac{1}{\frac{2\pi}{T}} < \frac{1}{\sqrt{F_2}}$$

$$\frac{2\pi}{\sqrt{F_1}} < \frac{1}{\frac{1}{T}} < \frac{2\pi}{\sqrt{F_2}}$$

$$\frac{2\pi}{\sqrt{F_1}} < T < \frac{2\pi}{\sqrt{F_2}}. \text{ Hence Proved.}$$

4.7 Prove that the path of a planet, which is moving so that its acceleration is always directed to a fixed point (star) and is equal to $\frac{\mu}{(\text{distance})^2}$, is a conic section. Find the conditions under which the path becomes (i) ellipse, (ii) parabola and (iii) hyperbola.

(2019 : 15 Marks)

Solution:

Here the force is always directed to a fixed point star so it is a case of central orbit. Also given that the central acceleration $p = \frac{\mu}{r^2}$, r = distance.

The differential equation of the path (in pedal from is)

$$\frac{h^2}{p^3}\frac{dp}{dr} = p = \frac{\mu}{r^2}$$

Multiplying both sides by –2, we have

$$\frac{-2h^2}{p^3}dp = \frac{-2\mu}{r^2}dr$$

Integrating, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + B \qquad ...(1)$$

where 'B' is a constant.

(Note that in a central orbit, $v = \frac{h}{p}$)

We know that referred to the focus as pole the pedal equations of ellipse, parabola and hyperbola (that branch which is never to focus taken as pole) are

$$\frac{b^2}{p^2} = \frac{2q}{r} - 1,\ p^2 = ar \text{ and } \frac{b^2}{p^2} = \frac{2q}{r} + 1$$

respectively, where in the case of ellipse $2a$ and $2b$ are the lengths of major and minor axes and in the case of parabola $4a$ is the length of latus rectum, and in the case of hyperbola $2a$ and $2b$ are the lengths of transverse and conjugate axes.

Now, since the equation (1) can be any of the above there forms, three cases arise here.

Case I : Elliptic Path

Comparing (1) with $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, the place equation of the ellipse, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{-1}$$

$$\therefore \qquad h^2 = \frac{\mu b^2}{a} \text{ and } B = -\frac{\mu}{a}$$

Substituting in (1), for elliptical path, we have

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

Obviously here, $\qquad v^2 < \frac{2\mu}{r}$

Case II : Parabolic Path

Comparing (1) with $p^2 = ar$ the pedal equation of a parabola, we have

$$\frac{h^2}{1} = \frac{2\mu}{1/a} = \frac{B}{0}$$

$$\therefore \quad h^2 = 2\mu a \text{ and } B = 0$$

Substituting in (1), for parabolic path we have

$$v^2 = \frac{2\mu}{r}$$

Case III : Hyperbolic Path

Comparing (1) with $\frac{b^2}{p^2} = \frac{2a}{r} + 1$ the pedal equation of a hyperbola, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{1}$$

$$\therefore \quad h^2 = \frac{\mu b^2}{a} \text{ and } B = \frac{\mu}{a}$$

Substituting in (1), for hyperbolic path, we have

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

Obviously, here $\quad v^2 > \frac{2u}{r}$

Thus, from the above three cases, we conclude that the equation (1) always represents a conic section whose focus is at the centre of force. Further, the path of the particle is an eclipse, parabola and hyperbola according as B is –ve, zero and positive. The sign of the value of the constant B depends upon the magnitude of the velocity of the particle at any point, we have found that

if (i) $\quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ or $v^2 < \frac{2\mu}{r}$, the path is elliptic,

if $\quad v^2 = \frac{2\mu}{r}$, then the path is parabolic.

if $u^2 > \frac{2\mu}{r}$ or $v^2 = \frac{2\mu}{r} + \frac{\mu}{a}$, then the path is hyperbolic. If is to be noted that in each of three cases magnitude of the velocity at any point is independent of the direction of the velocity at that time.

Also, we have found that :

$$h^2 = \frac{\mu b^2}{a} = \mu l \text{ in case of elliptic path.}$$

$$h^2 = 2\mu a = \mu l \text{ in case of parabolic path}$$

$$h^2 = \frac{\mu b^2}{a} = \mu l \text{ in case of hyperbolic path.}$$

Thus in all three cases, $h = \sqrt{\mu l}$, where 'l' is the length of the semi-eatus rectum.

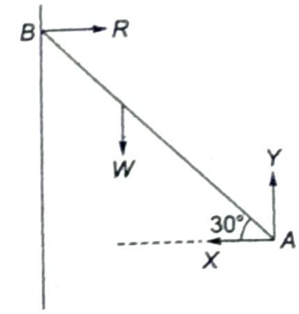
5. Equilibrium of a System of Particles

5.1 A uniform rod AB is movable about a hinge A and rests with one end in contact with a smooth vertical wall. If the rod is inclined at an angle 30° with the horizontal find the reaction at the hinge in magnitude and direction.

(2009 : 12 Marks)

Solution:

Approach : Balance forces and moments to get direction of the reaction at the hinge.



Let the reaction in the hinge in vertical and horizontal direction by Y and X respectively and the weight along the midpoint be W and the reaction at the wall (perpendicular to it) be R.

Balancing forces in horizontal and vertical direction

$$X = R \text{ and } Y = W$$

Let length of rod = $2a$

Now taking moments about the hinge to negate effect of reaction by hinge.

$$R \cdot 2a \sin 30° = W \cdot a \cdot \cos 30°$$

where $2a = AB$

$$\Rightarrow \quad R = \frac{W}{2}\cot 30° = \frac{\sqrt{3}W}{2}$$

$$\therefore \quad X = \frac{\sqrt{3}W}{2}, \; Y = W$$

$$\text{Net reaction} = \sqrt{X^2 + Y^2} = W\sqrt{\frac{3}{4} + 1}$$

$$= \frac{\sqrt{7}}{2}W$$

Let θ be angle of resultant from horizontal.

$$\tan\theta = \frac{\frac{\sqrt{3}W}{2}}{W} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \theta = \tan^{-1}\frac{\sqrt{3}}{2}$$

5.2 A ladder of weight W rests with one end against a smooth vertical wall and the other end rests on a smooth floor. If the inclination of the ladder to the horizon is 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from slipping down.

(2011 : 20 Marks)

Solution:

Let AB be the ladder.

Forces acting on the ladder are :

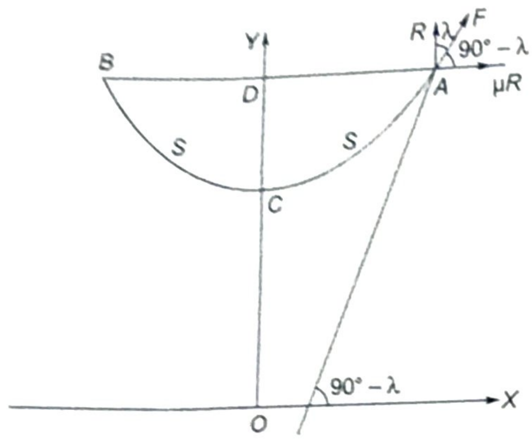
1. Normal reaction R at A perpendicular to AC.
2. Normal reaction S at B perpendicular to BC.
3. Weight W of rod acting vertically downwards through G, the mid-point of AB.

Let F be the required horizontal force. Resolving horizontally and vertically, we get

$$F = S \quad \text{...(i)}$$

and

$$R = W$$



Thus, for the point A of the catenary, we have

$$\psi = \psi_A = \frac{\pi}{2} - \lambda$$

$\therefore$ The length of the chain $= 2s$

$$= 2C \tan \psi_A$$

$$= 2C \tan\left(\frac{\pi}{2} - \lambda\right)$$

$$= 2C \cot \lambda$$

$$= \frac{2C}{\mu} \qquad (\because \tan \lambda = \mu)$$

If (x_A, y_A) are the co-ordinates of the point A, then the maximum span

$$AB = 2xA$$

$$= 2C \log(\tan \psi_A + \sec \psi_A)$$

$$= 2C \log\left(\tan\psi_A + \sqrt{1+\tan^2 \psi_A}\right)$$

$$= 2C \log\left(\cot\lambda + \sqrt{1+\cot^2 \lambda}\right) \qquad \left(\because \psi_A = \frac{\pi}{2} - \lambda\right)$$

$$= 2C \log\left(\frac{1}{\mu} + \sqrt{1+\frac{1}{\mu^2}}\right) = 2C \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)$$

Hence, the required ratio is

$$= \frac{2xA}{2S} = \frac{2C \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)}{\frac{2C}{\mu}}$$

$$= \mu \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)$$

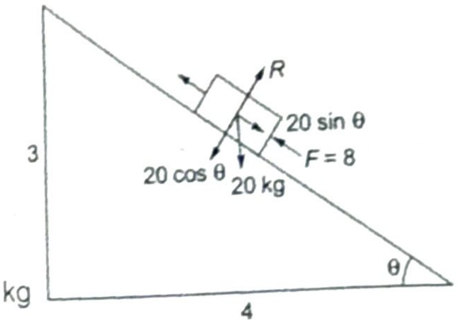
5.5 The base of an inclined plane in 4 metres in length and height is 3 metres. A force of 8 kg acting parallel to the plane will just prevent a weight of 20 kg from sliding down. Find the coefficient of friction between the plane and the weight.

(2013 : 10 Marks)

Solution:

The forces on the body are as shown in the diagram.

The limiting friction acts upward to prevent slipping



$$\tan\theta = \frac{3}{4}$$

Balance force along and perpendicular to the plane

$$R = 20\cos\theta = 20\times\frac{4}{5} = 16\text{ kg}$$

$$\mu R + F = 20\sin\theta = 20\times\frac{3}{4} = 12\text{ kg}$$

$$\Rightarrow \qquad \mu = \frac{12-8}{16} = \frac{4}{16} = 0.25$$

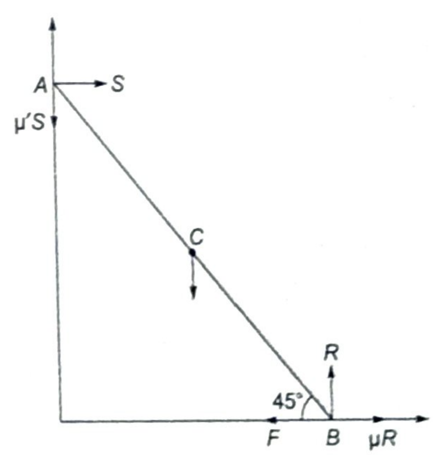
∴ Coefficient of friction is 0.25.

5.6 **A uniform ladder rests at an angle of 45° with the horizontal with its upper extremity against a rough vertical wall and its lower extremity on the ground. If μ and μ' are the coefficients of the limiting friction between the ladder and the ground and wall respectively then find the minimum horizontal force required to move the lower end of the ladder towards the wall.**

(2013 : 15 Marks)

Solution:

Let R and S be the normal reaction at the ground and the wall.



When the ladder is just about to move the friction at both ends will be limiting friction, i.e., μR and $u'S$. As the force F turns to move the ladder towards the wall friction at ground will be away from wall and on the wall downwards balancing forces in the horizontal and vertical directions.

$$F = \mu R + S \qquad \text{...(i)}$$

and
$$R = \mu' S + mg \qquad \text{...(ii)}$$

Let length of ladder $AB = l$

$$\therefore \qquad AC = \frac{l}{2}$$

Balancing moments about R.

$$\delta l\sin 45^\circ = mg\frac{l}{2}\cos 45^\circ + \mu'\delta l\cos 45^\circ$$

$$\Rightarrow \qquad \frac{\delta l}{\sqrt{2}} = \frac{mgl}{2\sqrt{2}} + \frac{\mu'\delta l}{\sqrt{2}}$$

$$\Rightarrow \quad 2(1-\mu')\delta l = mgl$$

$$\Rightarrow \quad \delta = \frac{mg}{2(1-\mu')}$$

$$\therefore \quad R = \mu'\delta + mg = mg\left[\frac{\mu'}{2(1-\mu')}+1\right]$$

$$= mg\frac{2-\mu'}{2(1-\mu')}$$

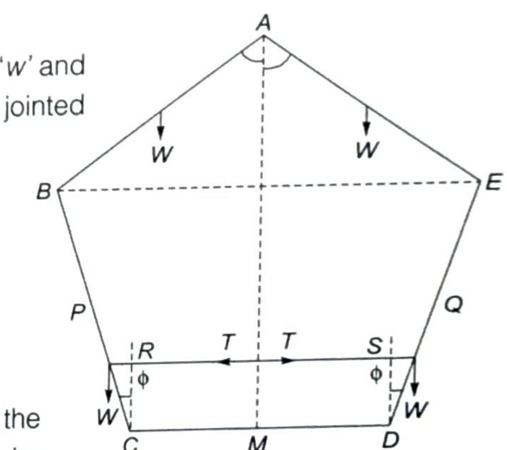
$$\therefore \quad F = \mu R + \delta = mg\left[\frac{2\mu-\mu\mu'}{2(1-\mu')}+\frac{1}{2(1-\mu')}\right]$$

$$= mg\left(\frac{2\mu-\mu\mu'+1}{2(1-\mu')}\right)$$

5.7 A regular pentagon ABCDE, formed of equal heavy uniform bars jointed together, is suspended from the joint A, and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod.

(2014 : 20 Marks)

Solution:



ABCDE is a pentagon formed of five equal rods each of weight 'w' and length '$2a$'. It is suspended from 'A' and midpoints of BC and ED is jointed by a weightless (light) rod PQ.

Let the thrust in the rod PQ. The line AM joining 'A' to the middle point 'M' of CD is vertical and PQ is horizontal. The weights of the rods AB, BC, CD, DE and EA act at their respective middle points.

In the portion of equilibrium the pentagon is a regular one so that each of the interior angle is 108° or $\frac{3\pi}{5}$ radians.

Let 'θ' be the angle which the upper slant rods AB and AE make with the vertical and ϕ be the angle which the lower slant rods CB and DE make with the vertical.

Replace the rod PQ by two equal and opposite forces T as shown in figure.

Give the system a small displacement about the vertical AM in which 'θ' changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\theta$.

The point A remains fixed. The lengths of the rods AB, BC etc. remains fixed. The length BE changes and the middle point of the rods AB, BC etc are slightly we have.

$$PS = PR + PS = SQ = PR = CD = SQ = a\sin\phi + 2a + a\sin\phi$$

$$PQ = 2a(1+\sin\phi)$$

The depth of the middle point of AB or AE below $A = a\cos\theta$

The depth of the middle point of BC or ED below $A = 2a\cos\phi + a\cos\theta)$ and depth of the middle point M of CD below $A = 2a(\cos\phi + \cos\theta)$

The equation of virtual work is

$$T[\delta[2a+2a\sin\phi]] + 2w\delta(a\cos\theta) + 2w\delta(2a\cos\theta + a\cos\phi) + w\delta(2a\cos\theta + 2a\cos\theta) = 0$$

$$\Rightarrow 2T\cos\phi\delta\phi - w\sin\theta\delta\theta - 4w\sin\theta\delta\theta - 2w\sin\phi\delta\phi - 2w\sin\phi\delta\theta - 2w\sin\theta\delta\phi = 0$$

$$\Rightarrow \quad [T\cos\phi - 2w\sin\phi]\delta\phi = 4w\sin\theta\delta\theta \qquad \text{...(i)}$$

From the figure, finding the length of BE in two ways; i.e., from the upper portion AE and from lower portion $BCDE$ we have.

$$4a\sin\theta = 2a + 4a\sin\phi$$

Differentiating, we get $\quad 4a\sin\theta\delta\theta = 2a\sin\phi\delta\phi$

or $\quad \cos\theta\delta\theta = \cos\phi\delta\phi \quad$ ...(ii)

giving (i) by (ii), we get

$$\Rightarrow \quad \frac{T\cos\phi - 2w\sin\phi}{\cos\phi} = \frac{4w\sin\theta}{\cos\theta} \Rightarrow T = 2w(\tan\phi + 2\tan\theta)$$

But in the position of equilibrium

$$\theta = \frac{1}{2}\cdot\frac{3\pi}{5} = \frac{3\pi}{10};\ \phi = \frac{3\pi}{5} - \frac{\pi}{2} = \frac{\pi}{10}$$

$$\therefore \quad T = 2w\left(\frac{\tan\pi}{10} + 2\frac{\tan 3\pi}{10}\right)$$

$$= 2w\left(\frac{\tan\pi}{10} + 2\cot + \frac{2\pi}{10}\right)$$

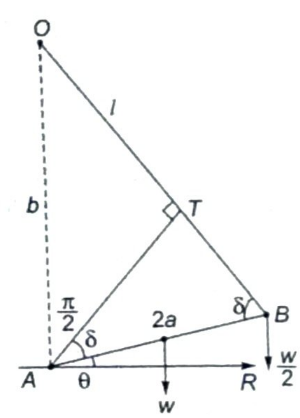
$$T = 2w\left(\frac{\tan\pi}{10}\right) + 2\left(\frac{1+\tan^2\left(\frac{\pi}{10}\right)}{2+\tan\frac{\pi}{10}}\right)$$

$$= 2w\cot\left(\frac{\pi}{10}\right)$$

5.8 A rod of 8 kg is movable in a vertical plane about a hinge at one end, another end is fastened a weight equal to half of the rod, this end is fastened by a string of length 'l' to a point at a height 'b' above the hinge vertically. Obtain the tension in the string.

(2015 : 10 Marks)

Solution:



Rod, $AB = 2a$, String $OB = l$

Moment about A:

$$a\cos\theta\cdot W + 2a\cos\theta\cdot\frac{W}{2} = 2a\cos\left(\frac{\pi}{2} - \delta\right)\cdot T$$

$$\Rightarrow \quad 2a\cos\theta\cdot W = 2a\sin\delta\cdot T$$

$$T = \frac{W\cos\theta}{\sin\delta}$$

Applying sine rule in ΔABO

$$\frac{l}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{b}{\sin\delta} \Rightarrow \frac{\cos\theta}{\sin\delta} = \frac{l}{b}$$

$$\therefore \quad T = \frac{Wl}{b} \Rightarrow T = \frac{8l}{b}$$

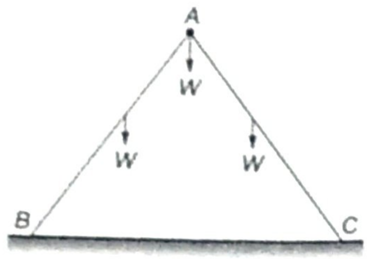
5.9 Two equal ladders of weight 4 kg each are placed so as to lean at A against each other with their ends resting on a rough floor, given the coefficient of friction is μ. The ladders at A make an angle 60° with each other. Find what weight on the top would cause them to slip.

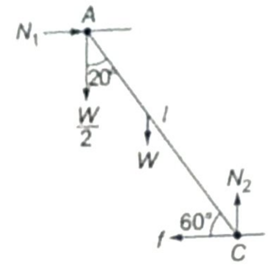
(2015 : 13 Marks)

Solution:

Ladders AB and AC

$$w = 4 \text{ kg}$$
$$W = ?$$





Taking equilibrium of one rod.

Force balanced in x-direction

$$N_1 = f \quad \text{...(i)}$$

$$N_2 = \frac{W}{2} + \omega = \frac{W}{2} + 4g \quad \text{...(ii)}$$

Moment balance about C,

$$N_1\, l \sin 60^\circ = \frac{W}{2} l \cos 60^\circ + 4g \cdot \frac{l}{2} \cos 60^\circ$$

$$\sqrt{3} N_1 = \frac{W}{2} + 2g \quad \text{...(iii)}$$

In the situation of slip, $f = \mu N_2$

Using (i), (ii) and (iii) in it

$$\frac{1}{\sqrt{3}}\left(\frac{W}{2} + 2g\right) = \mu\left(\frac{W}{2} + 4g\right)$$

$$\frac{W}{2}\left(\frac{1}{\sqrt{3}} - \mu\right) = g\left(4\mu - \frac{2}{\sqrt{3}}\right)$$

$\therefore$ Maximum weight at top,

$$W = \frac{4g(2\sqrt{3}\mu - 1)}{1 - \sqrt{3}\mu}$$

5.10 Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with the two-thirds of the circumference of the pulley.

(2015 : 12 Marks)

Solution:

$ABC \rightarrow$ Catenary

$$\phi_c = \angle COD = \frac{1}{2} \angle COA = \frac{\pi}{3}$$

$$AD = DC = OC \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} a$$

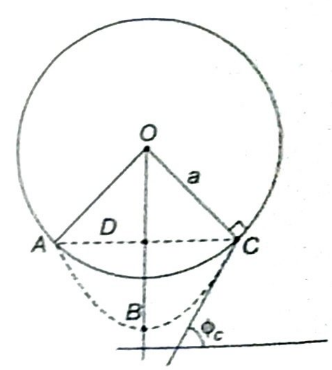
$$x = C \log(\tan\phi + \sec\varphi)$$

$$C: \frac{\sqrt{3}}{2} a = C\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$

$$= C\log(\sqrt{3} + 2)$$

$$\Rightarrow \quad C = \frac{a\sqrt{3}}{2\log(\sqrt{3}+2)}$$

$$s = C \tan\varphi$$



$\Rightarrow$
$$\text{arc } BC = C \times \sqrt{3} = \frac{3a}{2\log(\sqrt{3}+2)}$$

$$\text{Length of Chain} = \frac{2}{3}(2\pi a) + 2 \times (\text{Arc } BC)$$

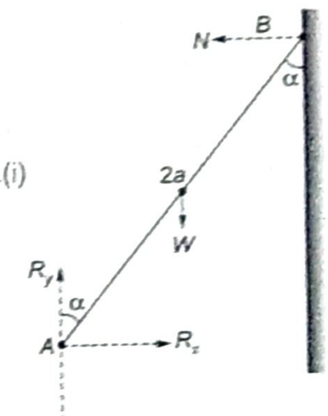
$$= \frac{4\pi a}{3} + 2 \cdot \frac{3a}{2\log(\sqrt{3}+2)} = \frac{4\pi a}{3} + \frac{3a}{\log(2+\sqrt{3})}$$

5.11 A uniform rod AB of length $2a$ movable about a hinge at A resits with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod.

(2016 : 15 Marks)

Solution:

In the adjacent figure,



Moment at A :

$$-W \times (a \sin\alpha) + N \times (2a \cos\alpha) = 0$$

$\therefore$
$$N = \frac{W}{2}\tan\alpha \qquad \text{...(i)}$$

Force Components :

$$R_x = N, \; R_y = W$$

$\therefore$
$$R_x = \frac{W}{2}\tan\alpha, \; R_y = W$$

$\therefore$
$$R = \sqrt{R_x^2 + R_y^2}$$

$$= \sqrt{\left(\frac{W}{2}\tan\alpha\right)^2 + W^2}$$

$\Rightarrow$
$$R = \frac{W}{2}\sqrt{4+\tan^2\alpha}$$

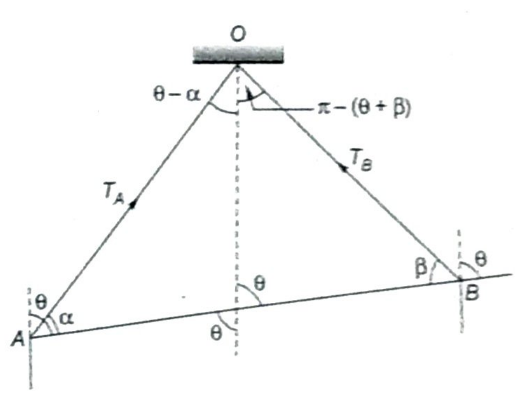
5.12 Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If the strings OA and OB make angles α and β with the rods AB, show that the angle Q which the rod makes with the vertical is given by

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}$$

(2016 : 15 Marks)

Solution:

In the adjacent figure, taking moment at O,



$$P \times OA\cos\left[\frac{\pi}{2} - (\theta - \alpha)\right] = Q \times OB\sin[\pi - (\theta + \beta)]$$

$$\Rightarrow \quad P{\cdot}OA\sin(\theta - \alpha) = Q{\cdot}OB\sin(\theta + \beta)$$

$$\Rightarrow \quad P{\cdot}OA(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = Q{\cdot}OB(\sin\theta\cos\beta + \cos\theta\sin\beta) \qquad ...(i)$$

Sine Rule : $$\frac{OB}{\sin\alpha} = \frac{OA}{\sin\beta} \qquad ...(ii)$$

Hence, (i) becomes

$P \times \sin\beta(\sin\theta\cos\theta - \cos\theta\sin\alpha) = \theta \times \sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta)$

Dividing both sides by $(\sin\alpha{\cdot}\sin\beta{\cdot}\cos\theta)$

$$P(\tan\theta\cot\alpha - 1) = Q(\tan\theta\cot\beta + 1)$$

$$\therefore \quad \tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}$$

5.13 A square $ABCD$, the length of whose sides is a, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l(> 4a)$ passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that the tension of the string is $\frac{W(l-3a)}{2\sqrt{l^2 - 6la + 8a^2}}$.

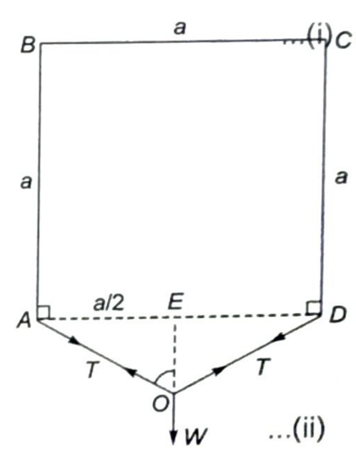
(2016 : 20 Marks)

Solution:

In the given set up, we take the vertical force, balanced at O.

$$2T\cos\theta = W \qquad ...(i)$$

$$\text{Length of } OA + OD = l - 3a$$



Consider ΔAEO,

$$OA^2 = AE^2 + EO^2$$

$$\therefore \quad EO^2 = \left(\frac{l-3a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$$

$$\Rightarrow \quad EO = \frac{1}{2}\sqrt{l^2 + 8a^2 + 6al}$$

$$\therefore \quad \cos\theta = \frac{EO}{AO} = \frac{\sqrt{l^2 + 8a^2 - 6al}}{(l-3a)} \qquad ...(ii)$$

Using (ii) in (i), we get

$$T = \frac{W}{2\cos\theta} = \frac{W(l-3a)}{2\sqrt{l^2 + 8a^2 - 6la}}$$

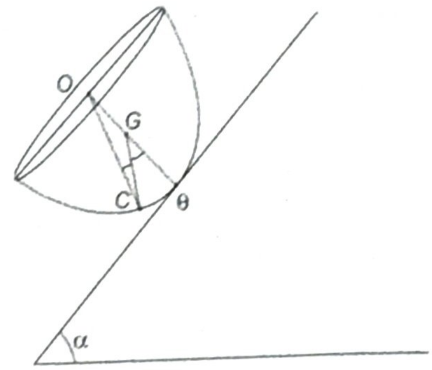
5.14 A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable?

(2017 : 17 Marks)

Solution:

Let O be the centre of the base of the hemisphere and 'r' be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC = r$.

Let G be the centre of gravity of hemisphere.



Then,
$$OG = \frac{3r}{8}$$

In the position of equilibrium the line CG must be vertical.

Since, OC is perpendicular to the inclined plane and CG is perpendicular to horizontal.

$\therefore$
$$\angle OGC = \alpha$$

Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From ΔOGC, we have

$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin\theta}, \text{ i.e., } \frac{3r/8}{\sin\alpha} = \frac{r}{\sin\theta}$$

$\therefore$
$$\sin\theta = \frac{8}{3}\sin\alpha \text{ or } \theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)$$

Since, $\sin\theta < 1$, hence,

$$\frac{8}{3}\sin\alpha < 1$$

i.e.,
$$\sin\alpha < \frac{3}{8}, \text{ i.e., } \alpha < \sin^{-1}\frac{3}{8} \quad \text{...(i)}$$

Thus, for the equilibrium to exist, we must have,

$$\alpha < \sin^{-1}\frac{3}{8}$$

To show that it is stable.

Let $CG = h$, then

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha}$$

$\Rightarrow$
$$h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha} \quad \text{...(*)}$$

Here, $P_1 = r$ and $P_2 = \infty$

The equilibrium will be stable if

$$h < \frac{P_1P_2\cos\alpha}{P_1+P_2}$$

i.e.,
$$\frac{1}{h} > \left(\frac{1}{P_1+P_2}\right)\sec\alpha, \text{ i.e., } \frac{1}{h} > \frac{1}{r}\sec\alpha$$

i.e.,
$$h < r\cos\alpha$$

Substituting in (*)

$$\frac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$$

or
$$3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$$

or
$$3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$$

or
$$8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{1-\frac{64}{9}\sin^2\alpha} < 8\sin\alpha\cos\alpha$$

or
$$-\sin\alpha\sqrt{9-64\sin^2\alpha} < 0 \quad \left(\because \sin\theta = \frac{8}{3}\sin\alpha\right)$$

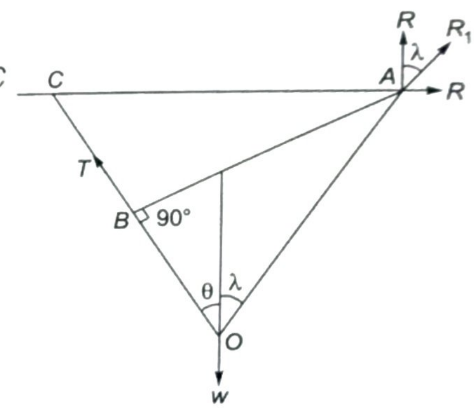
or $$\sin\alpha\sqrt{9-64\sin^2\alpha} > 0$$

From (i), $\sin\alpha < \frac{3}{8}$, hence relation is true and equilibrium is stable.

5.15 One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. When the rod is on the point of sliding, then $AC^2 - AB^2 = BC^2$. If θ is the angle between AB and the horizontal line, then prove that the coefficient of friction is $\frac{\cot\theta}{2+\cot^2\theta}$.

(2019 : 10 Marks)

Solution:



Let the lines of action of the weight 'w' and tension 'T' in the string BC meet in O. then the resultant reaction R_1 at A also passes through 'O'.
By "m-n theorem", in ΔAOB, we have

$$(a+a)\tan\theta = a\tan\lambda - a\tan(90°-\theta)$$
$$2a\tan\theta = a(\tan\lambda - \cot\theta)$$
$$2\tan\theta + \cot\theta = \tan\lambda$$
$$\frac{2}{\cot\theta} + \cot\theta = \tan\lambda$$
$$\tan\lambda = \frac{2+\cot^2\theta}{\cot\theta}$$
$$\mu = \frac{1}{\tan\lambda} = \cot\lambda = \frac{\cot\theta}{2+\cot^2\theta}$$

$\therefore$ $$\mu = \frac{\cot\theta}{2+\cot^2\theta}.$$ Hence the result.

5.16 A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius a. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.

(2019 : 15 Marks)

Solution:

AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C. Let H be the height OD of the cone and r be the radius OA or OC or OB of the hemisphere and the cone respectively.
Let G1 and G2 be the centres of gravity of the hemisphere and the cone respectively. Then

$$OG_1 = \frac{3r}{8} \text{ and } OG_2 = \frac{H}{4}$$

If 'h' be the height of the centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact 'C', then using the formula

$$x = \frac{w_1x_1 + w_2x_2}{w_1+w_2}, \text{ we have}$$

$$h = \frac{\frac{1}{3}\pi r^2H\cdot CG_2 + \frac{2}{3}\pi e^3 CG_2}{\frac{1}{3}\pi e^2H + \frac{2}{3}\pi e^3} = \frac{\frac{1}{3}\pi e^2H\left(e+\frac{1}{4}H\right) + \frac{2}{3}\pi e^3\cdot\frac{3}{8}e}{\frac{1}{3}\pi r^2H + \frac{2}{3}\pi e^3}$$

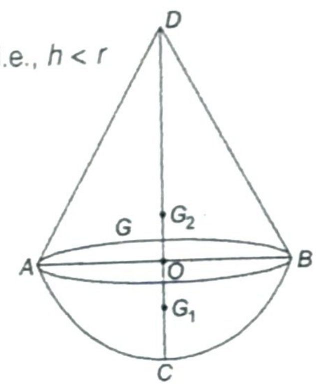
$$h = \frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r}$$

Here, P_1 = radius of curvature at the point of contact C of the upper body which is spherical = r.

P_2 = the radius of curvature of the lower body at the point of contact = ∞

∴ The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{P_1} + \frac{1}{P_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} + \frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{r}, \text{ i.e., } h < r$$



i.e.,
$$\frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r} < r$$

i.e.,
$$Hr + \frac{1}{4}H^2 + \frac{5}{4}H^2 < Hr + 2r^2$$

i.e.,
$$\frac{1}{4}H^2 < \frac{3}{4}r^2 \text{ i.e., } H^2 < 3r^2$$

i.e.,
$$H < r\sqrt{3}$$

$$r = a, \text{ i.e., } H < a\sqrt{3}$$

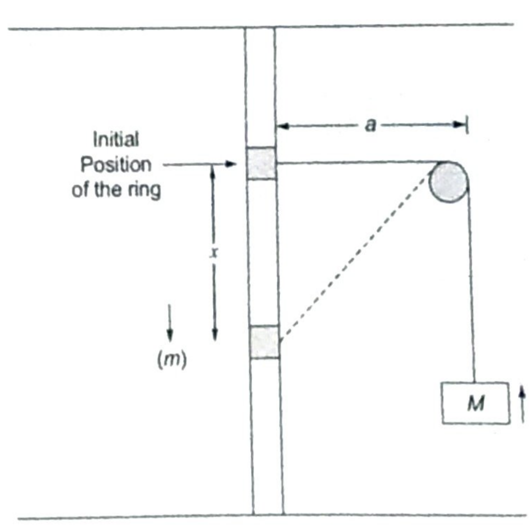
Hence, the greatest height of the cone consistent with the table, equilibrium of the body is $\sqrt{3}a$ ($\sqrt{3}$ times of the radius) of the hemisphere.

6. Work and Potential Energy

6.1 A heavy ring of mass m, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant a from the rod and has a mass $M (> m)$ fastened to its other end. Show that if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2 - m^2}$ before coming to rest.

(2012 : 20 Marks)

Solution:



By the law of conservation of energy, the ring will be in the rest position when loss in potential energy of ring is equal to the gain in potential energy of the block.

Let the distance travelled by the ring before it comes to rest be x.

Loss in potential energy of the ring = mgx ($\because$ P.E. = mgh)

When the ring falls down by the distance 'x', then the mass M will move up by length $\sqrt{a^2+x^2}-a$.

$\therefore$ Gain in P.E. of the mass $M = Mg\left(\sqrt{a^2+x^2}-a\right)$

$\because$ Loss in P.E. of the ring = Gain in P.E. of the block

$$\therefore \quad mgx = Mg\left(\sqrt{a^2+x^2}-a\right)$$

$$\Rightarrow \quad \frac{m}{M}x = \sqrt{a^2+x^2}-a$$

$$\Rightarrow \quad x = \frac{2Mma}{M^2-m^2}$$

6.2 A spherical shot of W gm weight and radius 'r' cm, lies at the bottom of cylinderical bucket of radius R cm. The bucket is filled with water upto a depth of 'h' cm ($h > 2r$). Show that the minimum amount of work done in lifting the shot just clear of the water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm g. W' gm is the weight of water displaced by the shot.

(2017 : 16 Marks)

Solution:

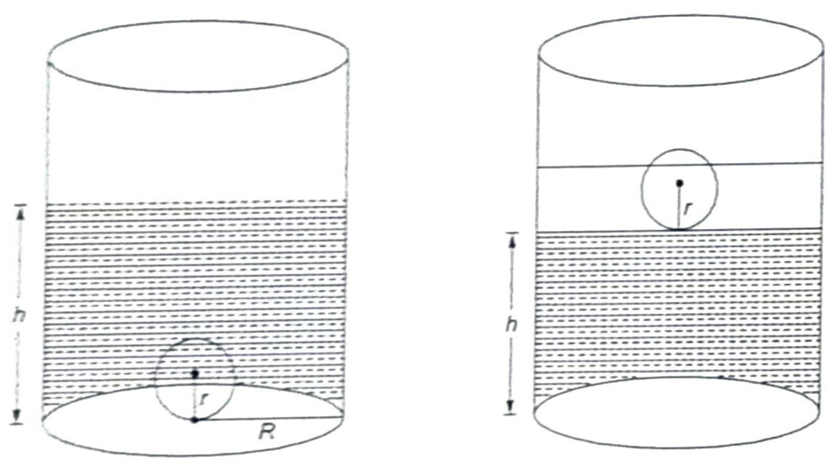
h' is the final height of water when spherical solid is out of water.

Volume of water + Volume of solid = Volume of cylinder with height h.

$$\pi R^2h' + \frac{4}{3}\pi r^3 = \pi R^2 h$$

$$\therefore \quad h' = h - \frac{4r^3}{3R^2} \qquad \text{...(i)}$$

Change in potential energy of water



$$\Delta PE_w = -W'\left[\left(\frac{h+h'}{2}\right)-r\right]$$

$$= -W'\left[h-\frac{2r^3}{3R^2}-r\right]$$

(final centre of mass and initial centre of mass)

Change in PE of solid,

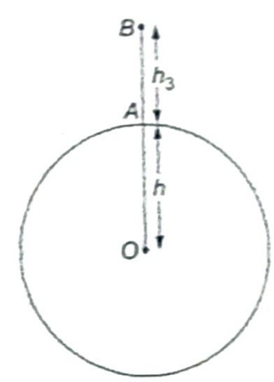
$$\Delta PE_s = W[(h'+r)-r] = Wh'$$
$$= W\left[h-\frac{4r^3}{3R^3}\right]$$

$\therefore$
$$\text{Work done} = \Delta PE_s + \Delta PE_w$$
$$= W\left[h-\frac{4r^3}{3R^2}\right]-W\left[h-\frac{2r^3}{3R^2}-r\right]$$
$$= W\left[h-\frac{4R^3}{3R^2}\right]+W\left[r-h+\frac{2r^3}{3R^2}\right]$$

6.3 The force of attraction of a particle by the earth is inversely proportional to the square of its distance from the earth's centre. A particle, whose weight on the surface of the earth is W, falls to the surface of the earth from a height $3h$ above it. Show that the magnitude of work done by the earth's attraction force is $\frac{3}{4}hW$, where h is the radius of the earth.

(2019 : 10 Marks)

Solution:



$$h = x = \text{radius of earth}$$
$$OA = h = x$$
$$AB = 3h$$

Hence, point B is at a distance of $4h$ from center of earth. The particle at B has the weight on the surface of the earth is w, falls to the surface of the earth from a height $3h$ above it.

The force of attraction of a particle by earth is inversely proportional to the square of its distance from earth's centers.

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^2} \qquad \text{(Given)}$$

At A;
$$\frac{d^2x}{dt^2} = -g, x = h$$

$\Rightarrow$
$$\mu = gh^2$$

Only force acting on particle is gravity (i.e., conservating force. No fraction or drag in air and initial and final velocity is zero.)

$$\text{Change in K.E.} = 0 \left(m = \frac{w}{g}\right)$$

$$w_{\text{conservative}} = -\int_{4h}^{h} \frac{w}{gh} \cdot \frac{\mu}{x^2} dx$$
$$= wh^2\left(\frac{1}{x}\right)_{4h}^{4}$$
$$= wh^2\left[\frac{1}{h}-\frac{1}{4h}\right]$$
$$= wh^2 \cdot \frac{3}{4h}$$

$$= w\frac{3}{4}h = \frac{3}{4}wh$$

$\therefore$ $$w_{conservative} = \frac{3}{4}wh. \text{ Hence the result.}$$

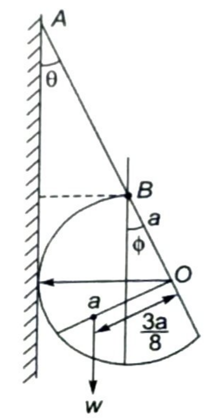
7. Principle of Virtual Work

7.1 A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ and ϕ are inclinations of the string and the plane base of the hemisphere to the vertical, prove by using the principle of virtual work that

$$\tan\phi = \frac{3}{8} + \tan\theta$$

(2010 : 20 Marks)

Solution:



a = radius of solid hemisphere

w = weight of solid hemisphere

l = length of string

$$l\sin\theta + a\sin\phi = a$$

$$\Rightarrow \quad l\cos\theta d\theta + a\cos\phi d\phi = 0$$

$$\Rightarrow \quad l\cos\theta d\theta = -a\cos\phi d\phi$$

The weight w acts at a distance $\frac{3a}{8}$ from centre.

$\therefore$ By principle of virtual work

$$\delta w\left(l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right) = 0$$

$$\Rightarrow \quad -l\sin\theta d\theta - a\sin\phi d\phi + \frac{3a}{8}\cos\phi d\phi = 0$$

$$\Rightarrow \quad a\cos\phi d\phi + \tan\theta - a\sin\phi d\phi + \frac{3a}{8}\cos\phi d\phi = 0$$

$$\Rightarrow \quad \tan\theta - \tan\phi + \frac{3}{8} = 0$$

$$\Rightarrow \quad \tan\phi = \frac{3}{8} + \tan\theta$$

7.2 Six rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Find the tension in the string.

(2013 : 15 Marks)

Solution:

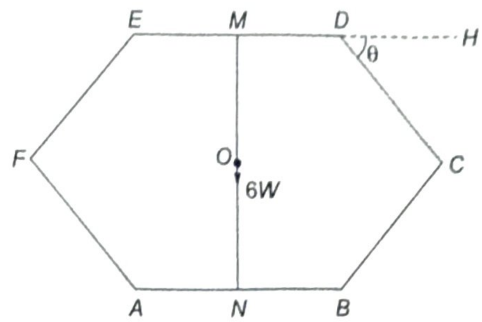
The middle points M and N are connected F by the string.

Let T be the tension in the string. Also let length of the rods be $2a$.

The weight of all the rods can be assumed to be acting at the middle point O.

Let $$\angle CDH = \theta$$

Also for a hexagon interior angle



$$= \frac{6\times\pi - 2\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$\therefore$ $$\theta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$$

Let it be given a small displacement in which the angle changes from θ to $\theta + \delta\theta$.

The line AB is fixed and so we measure all distances from it. The length MO and MN changes.

$$MN = 2MO = 2\cdot 2a \sin\theta = 4a\sin\theta$$

The depth of O above the fixed line = $N\theta = 2a\sin\theta$.

By principle of virtual work

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

$$-4aT\cos\theta\,\delta\theta + 16aW\cos\theta\, d\theta = 0$$

$$4a[-T + 3W]\cos\theta\, d\theta = 0$$

$\therefore$ $$T = 3W$$

7.3 Two equal uniform rods AB and AC, each of length l, are freely jointed at A and rest on a smooth fixed vertical circle of radius r. If 2θ is the angle between the rods, then find the relation between l, r and θ by using the principle of virtual work.

(2014 : 10 Marks)

Solution:

Let 'O' be the centre of the given fixed circle and 'w' be the weight each of the rods AB and AC. If 'E' and 'F' are the middle points of AB and AC, then the total weight '$2W$' of the two rods can be taken as acting at 'G'. Middle point of EF. The line AO is vertical we have

$$\angle BAO = \angle CAO = \theta$$

Also, $AB = l$, $AE = l/2$. If the rods AB touches the circle at M; then $\angle OMA = 90°$ and $OM = r$ the radius of circle.

Give the rods a small symmetrical displacement in which 'θ' changes to $\theta + \delta\theta$. The point O, remains fixed and the point 'G' is slightly displaced.

The $\angle AMO$ remains 90°, we have the height of G above the fixed point 'O'.

$\Rightarrow$ $$OG = OA - GA = OM\operatorname{cosec}\theta - AE\cos\theta$$

$$OG = r\operatorname{cosec}\theta \frac{-l}{2}\cos\theta$$

Equation of virtual work is

$\Rightarrow$ $$-2W\delta(OG) = \theta$$

Or $$\delta = (r\operatorname{cosec}\theta - l/2\cos\theta) = 0$$

$$= (-r\operatorname{cosec}\theta\cot\theta + l/2)\delta\theta = 0$$

$$m\operatorname{cosec}\theta\cot\theta = l/2\sin\theta$$

$$2r\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} = l\sin\theta$$

$$2r = \cos\theta = l\sin\theta$$

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