Set - III

Infinite Series

→ If $\{x_n\}$ is a sequence of real numbers, then the expression $x_1 + x_2 + \cdots + x_n + \cdots$ (i.e, the sum of the terms of the sequence, which are infinite in number) is called an infinite series.

The infinite series $x_1 + x_2 + \cdots + x_n + \cdots$ is usually denoted by $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$.

The numbers $x_1, x_2, x_3 \cdots x_n, \cdots$ are called the first, second, third, $\cdots$ n^{th} term (or general term) $\cdots$ of the series.

Series of positive terms: If all the terms of the series $\sum x_n = x_1 + x_2 + x_3 + \cdots + x_n + \cdots$ are positive i.e, if $x_n > 0\ \forall n$, then the series $\sum x_n$ is called a series of positive terms.

Alternating Series:

A series in which the terms are alternatively +ve and −ve is called a alternating series.

i.e, $\sum (-1)^{n-1} \cdot x_n = x_1 - x_2 + x_3 - x_4 + \cdots + (-1)^{n-1} x_n + \cdots$

where $x_n > 0\ \forall n$ is an alternating series.

Partial Sums:

If $\sum x_n = x_1 + x_2 + \cdots + x_n + \cdots$ is an infinite series where the terms may be +ve or −ve then

$S_n = x_1 + x_2 + \cdots + x_n$ is called the n^{th} partial sum of $\sum x_n$.

∴ The n^{th} Partial Sum of an infinite Series is the sum of its first 'n' terms.

$S_1, S_2, S_3 \cdots$ are the first, second, third, $\cdots$ partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence called the sequence of partial sums of the infinite series. $\sum x_n$.

$\therefore$ To every infinite series $\sum x_n$, there corresponds a sequence of $\{S_n\}$ of its partial sums.

Nature of infinite series:

(i) The series $\sum x_n$ is said to be cgs if the sequence of its partial sums cgs.

i.e, $\sum x_n$ is cgt if $\lim\limits_{n\to\infty} S_n = l$ (finite).

(ii) If $\lim\limits_{n\to\infty} S_n = +\infty$ (or) $-\infty$ then the series $\sum u_n$ is called dgs.

(iii) The series $\sum x_n$ is neither cgt nor dgt, the series $\sum x_n$ is called oscillatory series.

→ The series $\sum x_n$ oscillates finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

i.e, $\sum x_n$ oscillates finitely if $\{S_n\}$ is bounded and neither cgt nor dgt.

→ The series $\sum x_n$ oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

i.e., $\sum x_n$ oscillates infinitely if $\{S_n\}$ is unbounded and neither cgs nor dgs.

Discuss the cgs and dgs.

(1) $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} n$

$= 1+2+3+\cdots+n+\cdots\cdots$

Let $S_n = 1+2+\cdots\cdots+n$

$= \frac{n(n+1)}{2}$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt to $+\infty$

$\therefore \sum x_n$ is dgt to $+\infty$.

(2) $\sum x_n = \sum n^2$

$= 1^2+2^2+\cdots+n^2+\cdots\cdots$

Let $S_n = 1^2+2^2+\cdots\cdots+n^2$.

$= \frac{n(n+1)(2n+1)}{6}$.

$\lim_{n\to\infty} S_n = +\infty$

$\therefore \{S_n\}$ is dgt.

$\therefore \sum x_n$ is dgt.

(3) $\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{3^{n-1}}+\frac{1}{3^n}+\cdots$

$S_n = \frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}$

$= 1+\frac{1}{3}+\frac{1}{3^2}+\cdots\cdots+\frac{1}{3^{n-1}}$

$= \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}$

$= \frac{3}{2}\left(1-\frac{1}{3^n}\right)$

$\lim_{n\to\infty} S_n = 3/2$

$\therefore \{S_n\}$ is cgt.

$\therefore \sum x_n$ is cgt.

→ $\sum_{n=1}^{\infty} \frac{1}{2^n}$

→ $\sum_{n=1}^{\infty} K$ (Constant) $= K + K + \cdots + K + \cdots$

$S_n = K + K + \cdots\cdots + K.$ (n terms)

$= nK.$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt.

$\therefore$ The series $\sum x_n$ is dgt.

Note:- Every constant sequence is cgt but the constant series is dgt.

→ $\sum \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$

Let $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right)$
$+ \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1 - 0$

$= 1$

$\therefore \{S_n\}$ is cgt to 1

$\therefore \sum x_n$ is cgt to 1.

Arithmetic Series:-

$\sum x_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) + \cdots$

Let $S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$

$S_n = \frac{n}{2}[a + (n-1)d]$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt.

$\therefore \sum x_n$ is dgt.

* Geometric series

$$\sum_{n=0}^{\infty} r^n = 1+r+r^2+\cdots+r^{n-1}+\cdots$$

(i) cgs if $-1<r<1$ i.e $|r|<1$

(ii) dgs if $r \geq 1$

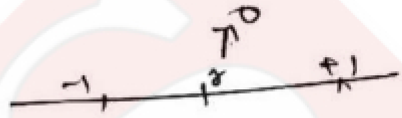
(iii) oscillates finitely if $r=-1$

(iv) oscillates infinitely if $r<-1$.

Sol Let $S_n = 1+r+r^2+\cdots+r^{n-1}$

then $S_n = \dfrac{1-r^n}{1-r}$

(i) if $-1<r<1$ i.e $|r|<1$



$\therefore r^n \to 0$ as $n\to\infty$

$$\underset{n\to\infty}{Lt}\, S_n = \underset{n\to\infty}{Lt}\frac{(1-r^n)}{1-r} = \frac{1}{1-r} \text{ (finite number)}$$

$\therefore \{S_n\}$ is cgt.

$\therefore \sum r^n$ is cgt.

(ii) If $r \geq 1$

subcase (i): If $r=1$

then $S_n = 1+1+\cdots+1$ (n times)

$= n(1)$

$= n$

now $\underset{n\to\infty}{Lt}\, S_n = \infty$

$\therefore \{S_n\}$ is dgt $\Rightarrow \sum r^n$ is dgt.

subcase (ii): If $r>1$

i.e $1<r\to\infty$

$\therefore r^n \to \infty$ as $n\to\infty$

$$\underset{n\to\infty}{Lt}\, S_n = \underset{n\to\infty}{Lt}\frac{(1-r^n)}{1-r} \text{ or } \underset{n\to\infty}{Lt}\frac{r^n-1}{r-1}$$

$= +\infty$

$\therefore \{S_n\}$ is dgt

$\sum r^n$ is dgt.

(iii) If $r=-1$

then $S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$\therefore Lt\ S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$\therefore \{S_n\}$ is oscillatory seq.

$\therefore \sum r^n$ is oscillatory series.

This oscillatory series is finite oscillatory series.

(IV) If $r < -1$ then $-r > 1$

$\Rightarrow x > 1$ where $x = -r$.

$\Rightarrow 1 < x^n \to \infty$ as $n \to \infty$

ie $(-r)^n \to \infty$ as $n \to \infty$.

$\therefore S_n = \dfrac{(1-(-r)^n)}{1-(-r)} = \dfrac{1-(-r)^n}{1+r}$

$= \begin{cases} \dfrac{1+r^n}{1+r} & \text{if } n \text{ is odd} \\ \dfrac{1-r^n}{1+r} & \text{if } n \text{ is even} \end{cases}$

Now $Lt\ S_n = \begin{cases} +\infty & \text{if } n \text{ is odd} \\ -\infty & \text{if } n \text{ is even.} \end{cases}$

$\therefore$ The series $\sum r^n$ is oscillatory series

This oscillates infinitely.

Note :- ① The g.s cgs only when common ratio is numerically less than 1.

② In an infinite series the terms are changed, a finite number of terms are added (or) omitted and each term of the series is multiplied and divided by the fixed non-zero number 'k' then the nature of the series does not change

Problems

→ * The geometric series

$\sum_{n=0}^{\infty} r^n = 1+r+r^2+ \cdots + r^{n-1}+ \cdots$

(i) converges if $-1<r<1$ i.e $|r|<1$

(ii) diverges if $r \geq 1$

(iii) oscillates finitely if $r=-1$

(iv) oscillates infinitely if $r<-1$.

→* **P-Test (or) P-Series:**

the series $\sum \frac{1}{n^p} =$

$\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots$

(i) converges if $p>1$

(ii) diverges if $p\leq 1$.

→* **The n^{th} term Test:**

If the series $\sum x_n$ converges then $\lim_{n\to\infty}(x_n)=0$.

Note:- (1) $\sum x_n$ converges $\Rightarrow \lim x_n = 0$.

(2) $\lim x_n = 0 \Rightarrow \sum x_n$ may (or) may not convergent.

(3) $\lim x_n \neq 0 \Rightarrow \sum x_n$ is not convergent

(4) the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is called the harmonic series.

→* A positive term series either converges (or) diverges to $+\infty$.

→* If $x_n>0 \ \forall n$ and $\lim_{n\to\infty} x_n \neq 0$ then $\sum x_n$ diverges to $+\infty$.

→ **Comparison Test:** Let $X=(x_n)$ and $Y=(y_n)$ be non-negative sequences of real numbers and suppose that for some $K\in\mathbb{N}$, we have $0\leq x_n\leq y_n$ for $n\geq K$.

then

(a) the convergence of $\sum y_n \Rightarrow$ the convergence of $\sum x_n$

(b) the divergence of $\sum x_n \Rightarrow$ the divergence of $\sum y_n$.

→ **Limit Comparison Test:-**

Suppose that $X=(x_n)$ and $Y=(y_n)$ are strictly +ve sequences and suppose that the following limit exists in $\mathbb{R}$:

$$r = \lim\left(\frac{x_n}{y_n}\right)$$

(a) If $r\neq 0$ (finite) then $\sum x_n$ is convergent (or divergent) iff $\sum y_n$ is convergent (or divergent).

(b) If $r=0$ and if $\sum y_n$ is convergent then $\sum x_n$ is convergent.

(c) If $r=\infty$ and if $\sum y_n$ diverges then $\sum x_n$ diverges.

Problems

→ the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ converges.

Sol'n:- Clearly the inequality.

$$0<\frac{1}{n^2+n}<\frac{1}{n^2} \quad \forall n$$

∴ which is in the form of $0<x_n<y_n \ \forall n$

where $x_n=\frac{1}{n^2+n}$ & $y_n=\frac{1}{n^2}$.

Now $\Sigma y_n = \Sigma \frac{1}{n^2}$ is of the form

$\Sigma \frac{1}{n^P}$ where $P = 2 > 1$

$\therefore$ By P-Test

Σy_n is convergent.

$\therefore$ By Comparison Test

Σx_n is convergent.

(Or)

Let $x_n = \frac{1}{n^2+n} = \frac{1}{n^2(1+\frac{1}{n})}$

Let $y_n = \frac{1}{n^2}$

then $\frac{x_n}{y_n} = \frac{1}{1+\frac{1}{n}}$

$\therefore \lim_{n\to\infty} \frac{x_n}{y_n} = 1 \neq 0.$

Since $\Sigma y_n = \Sigma \frac{1}{n^2}$ is convergent by P-Test.

$\therefore$ By Comparison test

$\therefore \Sigma x_n$ is convergent.

H.W. $\rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ is convergent

$\rightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent

* By using partial fractions, show that

(a) $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$

(b) $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0$ if $\alpha > 0$.

(c) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

sol'n: (a) Let $x_n = \frac{1}{(n+1)(n+2)}$

$= \frac{1}{n+1} - \frac{1}{n+2}$

$\therefore$ The n^{th} partial sum

$S_n = x_0 + x_1 + \cdots + x_{n-1}$

$= (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}) \cdots (\cancel{\tfrac{1}{n}} - \frac{1}{n+1})$

$= 1 - \frac{1}{n+1}$

$\therefore \lim S_n = 1$

$\therefore \Sigma x_n = 1$

(b) Let $x_n = \frac{1}{(\alpha+n)(\alpha+n+1)}$

$= \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}$

$\therefore S_n = x_0 + x_1 + \cdots + x_{n-1}$

$= \left(\frac{1}{\alpha} - \cancel{\frac{1}{\alpha+1}}\right) + \left(\cancel{\frac{1}{\alpha+1}} - \cancel{\frac{1}{\alpha+2}}\right) + \cdots + \left(\cancel{\frac{1}{\alpha+n-1}} - \frac{1}{\alpha+n}\right)$

$= \frac{1}{\alpha} - \frac{1}{\alpha+n}$

$\therefore \lim S_n = \frac{1}{\alpha} > 0$ if $\alpha > 0$.

(c) Let $x_n = \frac{1}{n(n+1)(n+2)}$

$= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

$\therefore S_n = x_1 + x_2 + \cdots + x_{n-1} + x_n.$

→ Discuss the Convergence or divergence of the following series.

(a) $\sqrt{1/4} + \sqrt{2/6} + \sqrt{3/8} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$

(b) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \cdots$

Sol'n: Let $\sum x_n = \sum \sqrt{\frac{n}{2(n+1)}}$

$$= \sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \cdots + \sqrt{\frac{n}{2(n+1)}} + \cdots$$

Here $x_n = \sqrt{\frac{n}{2(n+1)}}$

$$= \sqrt{\frac{1}{2(1+1/n)}}$$

$\therefore$ Lt x_n = Lt $\frac{1}{\sqrt{2(1+1/n)}}$

$$= \frac{1}{\sqrt{2}} \neq 0.$$

$\therefore$ Lt $x_n \neq 0$.

Since $x_n > 0 \ \forall n$ and Lt $x_n \neq 0$.

$\therefore \sum x_n$ diverges to $+\infty$.

(b) Given that

$$\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \cdots + \frac{1}{\sqrt{n(n+1)}} + \cdots = \sum \frac{1}{\sqrt{n(n+1)}}$$

Let $x_n = \frac{1}{\sqrt{n(n+1)}}$

$$= \frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}\right)$$

Let $y_n = 1/n$

then $x_n/y_n = \frac{1}{\sqrt{1+1/n}}$

$\therefore$ Lt $\frac{x_n}{y_n} = 1 \neq 0$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent. (by p-Test)

$\therefore$ By Comparison test the series $\sum x_n$ is divergent.

(c) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \cdots$

Sol'n: Let $x_n = \frac{n+1}{n^p}$

$$= \frac{1}{n^{p-1}}(1+1/n)$$

Let $y_n = \frac{1}{n^{p-1}}$

then $\frac{x_n}{y_n} = 1 + 1/n$.

$\therefore \underset{n\to\infty}{\text{Lt}}\left(\frac{x_n}{y_n}\right) = 1 \neq 0$.

Since $\sum y_n = \sum \frac{1}{n^{p-1}}$ is Convergent.

if $p-1 > 1$

i.e. $p > 2$

$\therefore$ By Comparison test $\sum x_n$ is convergent.

also $\sum y_n = \sum \frac{1}{n^{p-1}}$ is divergent.

if $p-1 \leq 1$

i.e. $p \leq 2$

$\therefore$ By comparison test $\sum x_n$ is divergent.

H.W.
→ $\sum (3n-1)^{-1}$
→ $\sum \left(n+\frac{1}{2}\right)^{-2}$
→ $\sum \left(\frac{n^2+n+1}{n^4+1}\right)$
→ $\sum \frac{1}{n^p(n+1)^p}$

→ $\sum\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$

Sol'n: Let $x_n=\sqrt{n^3+1}-\sqrt{n^3}$

$$=\left(\sqrt{n^3+1}-\sqrt{n^3}\right)\times\frac{\sqrt{n^3+1}+\sqrt{n^3}}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\frac{n^3+1-n^3}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\frac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$$

$$=\frac{1}{n^{3/2}\left(1+\sqrt{1+\frac{1}{n^3}}\right)}$$

Let $y_n=\frac{1}{n^{3/2}}$

then $\frac{x_n}{y_n}=\frac{1}{1+\sqrt{1+\frac{1}{n^3}}}$

$\therefore$ Lt $\frac{x_n}{y_n}=\frac{1}{2}\neq 0.$

Since $\sum y_n=\sum\frac{1}{n^{3/2}}$ is convergent by P-Test. Here $P=3/2>1$

$\therefore$ By Comparison test $\sum x_n$ is convergent.

→ $\sum\left(\sqrt{n^2+1}-n\right)$

→ $\sum\left(\sqrt{n^4+1}-n^2\right)$

→ $\sum\left(\sqrt[3]{n+1}-\sqrt[3]{n}\right)$

Sol'n: Let $x_n=\sqrt[3]{n+1}-\sqrt[3]{n}$

$$=(n+1)^{1/3}-n^{1/3}$$

$$=n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right]$$

$$=n^{1/3}\left[\left(1+\frac{1}{3}\cdot\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2}+\cdots\right)-1\right]$$

$$=n^{1/3}\left[\left(\cancel{1}+\frac{1}{3n}-\frac{1}{9n^2}+\cdots\right)-\cancel{1}\right]$$

$$=n^{1/3}\left[\frac{1}{3n}-\frac{1}{9n^2}+\cdots\right]$$

$$=\frac{1}{n^{2/3}}\left[\frac{1}{3}-\frac{1}{9n}+\cdots\right]$$

Let $y_n=\frac{1}{n^{2/3}}$

then $\frac{x_n}{y_n}=\frac{1}{3}-\frac{1}{9n}+\cdots$

$\therefore$ Lt $\frac{x_n}{y_n}=\frac{1}{3}\neq 0.$

Since $\sum y_n=\sum\frac{1}{n^{2/3}}$ is divergent by P-Test where $P=2/3<1$

$\therefore$ By comparison test $\sum x_n$ is divergent.

→ $\sum\left(\sqrt[3]{n^3+1}-n\right)$

Note: Rationalisation is effective only square roots are involved where as the method of Binomial expansion is general.

→ Test the convergence of the series.

(a) $\sum\sin\frac{1}{n}$ (b) $\sum\frac{1}{n}\sin\frac{1}{n}$

(c) $\sum\frac{1}{\sqrt{n}}\sin\frac{1}{n}$ (d) $\sum\cos\frac{1}{n}$

(e) $\sum\tan^{-1}\frac{1}{n}$ (f) $\sum\cot^{-1}n^2$

(g) $\sum\frac{1}{\sqrt{n}}\tan\frac{1}{n}$

Sol'n: (a) $\sum\sin\frac{1}{n}$

Let $x_n=\sin\frac{1}{n}$

and let $y_n=\frac{1}{n}$

then $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=\lim\limits_{n\to\infty}\frac{\sin 1/n}{1/n}$

$= \underset{1/n \to 0}{Lt} \dfrac{\sin 1/n}{1/n}$ $(\because n \to \infty \Rightarrow \frac{1}{n} \to 0)$

$= 1$

$\neq 0$.

Since $\Sigma y_n = \Sigma \frac{1}{n}$ is divergent.

by P-Test where $P = 1$.

$\therefore$ By comparison Test

Σx_n is divergent.

(b) Let $x_n = \frac{1}{n} \sin \frac{1}{n}$

Let $y_n = \frac{1}{n^2}$

then $\underset{n\to\infty}{Lt} \dfrac{x_n}{y_n} = \underset{n\to\infty}{Lt} \dfrac{\sin 1/n}{1/n}$

$= 1 \neq 0$.

Since $\Sigma y_n = \Sigma \frac{1}{n^2}$ is convergent

by P-Test where $P = 2 > 1$.

$\therefore$ By comparison test Σx_n is convergent.

(c) Let $x_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

Let. $y_n = \frac{1}{n^{3/2}}$

(d) Let $x_n = \cos \frac{1}{n}$

$= 1 - \dfrac{(1/n)^2}{2!} + \dfrac{(1/n)^4}{4!} - \dfrac{(1/n)^6}{6!} + \cdots$

$= 1 - \dfrac{1}{n^2 . 2!} + \dfrac{1}{n^4 . 4!} \cdots$

Let $y_n = 1/n$

then $\dfrac{x_n}{y_n} = n - \dfrac{1}{n.2!} + \dfrac{1}{n^3.4!} \cdots$

$\therefore Lt \dfrac{x_n}{y_n} = \infty$

Since $\Sigma y_n = \Sigma \frac{1}{n}$ is divergent

(by P-Test)

$\therefore$ By comparison test Σx_n is divergent.

(or)

$\underset{n\to\infty}{Lt} x_n = \underset{n\to\infty}{Lt} \cos 1/n$

$= 1 \neq 0$.

Since Σx_n is a series of +ve terms.

i.e. $x_n > 0 \ \forall n$.

and $Lt\, x_n \neq 0$.

$\therefore \Sigma x_n$ is divergent.

(e) Let $x_n = \tan^{-1} 1/n$

$= \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \frac{1}{7n^7} + \cdots$

(f) Let $x_n = \cot^{-1} n^2$

$= \tan^{-1}(1/n^2)$

$\because \cot^{-1} x = \theta$
$\Rightarrow x = \cot\theta$
$\Rightarrow x = \frac{1}{\tan\theta}$
$\Rightarrow \tan\theta = 1/n$
$\Rightarrow \theta = \tan^{-1}(1/n)$

(g) Let

$x_n = \frac{1}{\sqrt{n}} \tan^{-1}(1/n)$

Let $y_n = \frac{1}{n^{3/2}}$

$\rightarrow \Sigma \dfrac{1}{n^{1+1/n}}$

<u>Sol'n</u> – Let $x_n = \dfrac{1}{n.n^{1/n}}$

let $y_n = 1/n$ then $\dfrac{x_n}{y_n} = \dfrac{1}{n^{1/n}}$

$\therefore \underset{n\to\infty}{Lt} \dfrac{x_n}{y_n} = \underset{n\to\infty}{Lt} \dfrac{1}{n^{1/n}} = \dfrac{1}{1}$

$= 1$

Since $\Sigma y_n = \Sigma \frac{1}{n}$ is divergent by P-Test.

$\therefore$ By Comparison test Σx_n is divergent.

*** D'Alembert's Ratio Test :—**

If $\sum u_n$ is a series of +ve terms such that $\lim \frac{u_n}{u_{n+1}} = l$ then

(i) $\sum u_n$ converges if $l > 1$

(ii) $\sum u_n$ diverges if $l < 1$

(iii) $\sum u_n$ may converge (or) diverge if $l = 1$

(i.e. the test fails if $l = 1$).

Note: In general the ratio test is applied when fractionals & combination of powers involved.

* Discuss the convergence of the following series.

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$$

Sol^n:— Let $u_n = \frac{n!}{n^n}$ then

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Now $\frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$

$$= \frac{n(1+1/n)^{n+1}}{(n+1)}$$

$$= (1+1/n)^n$$

$\therefore \lim \frac{u_n}{u_{n+1}} = \lim (1+1/n)^n$

$$= e > 1$$

$\therefore$ By D'Alembert's Ratio test. $\sum u_n$ is convergent.

→ $\frac{1^2.2^2}{1!} + \frac{2^2.3^2}{2!} + \frac{3^2.4^2}{3!} + \cdots$

Sol^n:— Let $u_n = \frac{n^2(n+1)^2}{n!}$

then $u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!} \times \frac{(n+1)!}{(n+1)^2(n+2)^2}$

$$= \frac{(n+1)}{(1+\frac{2}{n})^2}$$

$$= n.\frac{(1+1/n)}{(1+2/n)^2}$$

$\therefore \lim\left(\frac{u_n}{u_{n+1}}\right) = \infty > 1$

$\therefore$ By D'Alembert's Ratio test $\sum u_n$ is convergent.

→ $\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$

Sol^n:— Let $u_n = \frac{2^n}{n^2+1}$

then $u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$

Now $\frac{u_n}{u_{n+1}} = \frac{2^n}{n^2+1} \times \frac{(n+1)^2+1}{2^{n+1}}$

$$= \frac{1}{2}\,\frac{(1+1/n)^2 + 1/n^2}{(1+1/n^2)}$$

$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$

$\therefore$ By D'Alembert's ratio test $\sum u_n$ is divergent.

→ $\sum \frac{x^n}{3^n.n^2}$, $x > 0$.

Sol^n:— Let $u_n = \frac{x^n}{3^n.n^2}$

then $u_{n+1} = \frac{x^{n+1}}{3^{n+1}.(n+1)^2}$

$\therefore \frac{u_n}{u_{n+1}} = \frac{x^n}{3^n.n^2} \times \frac{3^{n+1}.(n+1)^2}{x^{n+1}}$.

$$= \frac{3}{x}\left(1+\frac{1}{n}\right)^2$$

Now $\lim \frac{u_n}{u_{n+1}} = \frac{3}{x}$

$\therefore$ By D'Alembert's test

Σu_n converges if $3/x > 1$

i.e. $x < 3$

and diverges if $3/x < 1$ i.e. $x > 3$.

if $x = 3$ then $\lim \frac{u_n}{u_{n+1}} = 1$

$\therefore$ Ratio Test fails.

Now if $x = 3$, $u_n = \frac{3^n}{3^n . n^2} = \frac{1}{n^2}$

$\therefore \Sigma u_n = \Sigma \frac{1}{n^2}$ is Convergent by P-Test.

$\therefore$ Σu_n is Convergent if $x \le 3$ and divergent if $x > 3$.

$\longrightarrow$ $\Sigma \frac{x^n}{a+\sqrt{n}}$; $x > 0$

$\longrightarrow$ $\Sigma \sqrt{\frac{n+1}{n^3+1}} . x^n$; $x > 0$

$\longrightarrow$ $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots$; $x > 0$.

Sol'n :- Let $u_n = \frac{x^n}{2n}$ (leaving the first term)

then $u_{n+1} = \frac{x^{n+1}}{2(n+1)}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x^2}\left(1+\frac{1}{n}\right)$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ By D'Alembert's Ratio test

Σu_n is convergent if $\frac{1}{x^2} > 1$

i.e. $x^2 < 1$

i.e. if $x < 1$

and the series divergent if $\frac{1}{x^2} < 1$

i.e. $x^2 > 1$

i.e. $x > 1$

If $x^2 = 1$ i.e. $x = 1$, then the ratio test fails.

if $x = 1$ then $u_n = \frac{1}{2n}$

Let $v_n = \frac{1}{n}$ then

$$\frac{u_n}{v_n} = \frac{1}{2}$$

$$\therefore \lim \frac{u_n}{v_n} = \frac{1}{2} \neq 0.$$

Since $\Sigma v_n = \Sigma \frac{1}{n}$ is divergent by P-Test.

$\therefore$ By comparison test Σv_n is divergent.

$\therefore$ Σu_n is divergent if $x \ge 1$

and Σu_n is convergent if $x < 1$.

* Cauchy's Root Test :-

$\longrightarrow$ If Σu_n is a series of +ve terms such that

(a) $\lim (u_n)^{1/n} = l$ then

(i) Σu_n converges if $l < 1$

(ii) Σu_n diverges if $l > 1$.

(iii) Σu_n may converge or diverge if $l = 1$

(i.e. the test fails if $l = 1$).

(b) $\lim_{n\to\infty} (u_n)^{1/n} = \infty$ then Σu_n is divergent.

Note: – The root test is used when powers are involved.

Problems:

* Test the convergence of the following series:

→ (a) $\sum \left(\frac{n}{n+1}\right)^{n^2}$; (b) $\sum \frac{x^n}{n^n}, x>0$

(c) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(d) $\sum n^n x^n; x>0$ (e) $\sum \left(\frac{n+1}{3n}\right)^n$.

Sol'n: – (a) Let $x_n = \left(\frac{n}{n+1}\right)^{n^2}$

then $(x_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}$

$= \left(\frac{n}{n+1}\right)^n$

$= \left(\frac{n+1}{n}\right)^{-n}$

$= \left[\left(1+1/n\right)^{+n}\right]^{-1}$

Now $\lim_{n\to\infty} (x_n)^{1/n} = e^{-1}$

$= 1/e < 1$

∴ By Cauchy's root test

$\sum x_n$ converges.

(c). Let $x_n = \frac{1}{(\log n)^n}$

then $x_n^{1/n} = \frac{1}{\log n}$

$\therefore \lim_{n\to\infty} x_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$.

∴ By root test.

$\sum x_n$ is Convergent.

→ $\sum 5^{-n-(-1)^n}$

Sol'n: – Let $x_n = 5^{-n-(-1)^n}$

then $(x_n)^{1/n} = 5^{-1-\frac{(-1)^n}{n}}$

$= 5^{-1-\frac{1}{n}}$ if n is even

$= 5^{-1+\frac{1}{n}}$ if n is odd.

$\therefore \lim_{n\to\infty} x_n^{1/n} = 5^{-1}$

$= 1/5 < 1$

∴ By root test

$\sum x_n$ is Convergent.

→ $\sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$

→ $\sum_{n=1}^{\infty} 3^{-2n-5(-1)^n}$

Note: – Cauchy's root test is more general than D'Alembert's ratio test.

because:

(i) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists $\Rightarrow \lim_{n\to\infty} u_n^{1/n}$ exists.

and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} u_n^{1/n}$

(Cauchy's second theorem on limits)

∴ whenever ratio test is applicable, so is the root test.

(ii) If $\lim_{n\to\infty} u_n^{1/n}$ exists then

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ may not exist.

∴ when the ratio test fails the root test succeeds.

$\therefore$ The root test is more general than the ratio test.

→ Show that Cauchy's root test establishes the convergence of the series $\sum 3^{-n-(-1)^n}$ while D'Alembert's ratio test fails to do so.

Sol'n: Let $u_n = 3^{-n-(-1)^n}$

then $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} 3^{-1-\frac{(-1)^n}{n}}$

$= 3^{-1}$

$= \frac{1}{3} < 1$

$\therefore$ By root test $\sum u_n$ is Convergent.

Now if n is odd (so that $n+1$ is even)

$u_n = 3^{-n+1}$, $u_{n+1} = 3^{-(n+1)-1}$
$= 3^{-n-2}$

$\therefore \lim \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3^{-n+1}}{3^{-n-2}}$

$= \lim_{n\to\infty}\left(\frac{3^1}{3^{-2}}\right)$

$= 3^3$

$= 27 > 1$

when n is even

(so that $n+1$ is odd)

$\therefore u_n = 3^{-n-1}$, $u_{n+1} = 3$
$= 3^{-n}$

$\therefore \lim \frac{u_n}{u_{n+1}} = 3^{-1} = \frac{1}{3} < 1$.

$\therefore \lim \frac{u_n}{u_{n+1}}$ does not exist.

$\therefore$ D' Alembert's ratio test fails.

* Raabe's Test :-

If $\sum u_n$ is a series of +ve terms such that

$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then

(i) $\sum u_n$ Converges if $l > 1$

(ii) $\sum u_n$ diverges if $l < 1$.

(iii) The test fails if $l = 1$.

Note! - Raabe's test is stronger than D'Alembert's ratio test and may succeed where the ratio test fails.

For example:

$\sum \frac{1}{n^2}$

Let $u_n = \frac{1}{n^2}$

$u_{n+1} = \frac{1}{(n+1)^2}$

$\therefore \lim \frac{u_n}{u_{n+1}} = \lim \frac{(n+1)^2}{n^2}$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2$

$= 1$

Here $l = 1$,

$\therefore$ the ratio test fails.

But $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^2 - 1\right]$

$= \lim_{n\to\infty} n\left[\frac{(n+1)^2}{n^2} - 1\right]$

$$= \lim_{n\to\infty} n\left[\frac{n^2+1+2n-n^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}+2\right) = 2>1.$$

$\therefore$ By Raabe's test

Σu_n is Convergent.

Note:-

① If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \infty$ then Σu_n is Convergent.

② $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = -\infty$ then Σu_n is divergent.

③ In general Rabbe's test is used when D'Alemberts ratio test fails and the ratio $\frac{u_n}{u_{n+1}}$ doesnot involve the number 'e'

— when $\frac{u_n}{u_{n+1}}$ involves 'e' we apply logarithmic test after the ratio test and not Rabbe's test.

* Logarithmic test:

If Σu_n is a series of +ve terms such that $\lim_{n\to\infty} n \log\frac{u_n}{u_{n+1}} = l$,

then (i) Σu_n Converges if $l>1$

(ii) Σu_n diverges if $l<1$

(iii) The test fails if $l=1$.

2002 → $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$

Sol'n: Let $u_n = \frac{n^n \cdot x^n}{n!}$

then $u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n \cdot x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}x^{n+1}}$$

$$= \frac{(n+1)\cdot n^n}{(n+1)^{n+1}\cdot x}$$

$$= \frac{1}{(1+1/n)^n}\cdot\frac{1}{x}$$

Now $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$

$\therefore$ By D'Alembert's test, the series Σu_n Converges if $\frac{1}{ex}>1$

i.e. $ex<1$

i.e. $x<1/e$

and diverges if $1/ex<1$

i.e. $ex>1$

i.e. $x>1/e$.

If $x=1/e$ then the ratio test fails.

Now if $x=1/e$ then $\frac{u_n}{u_{n+1}} = \frac{1}{(1+1/n)^n}e$

Since $\frac{u_n}{u_{n+1}}$ involves the number 'e'.

$\therefore$ we apply the logarithmic test.

Now $\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{1}{(1+1/n)^n}e\right]$

$$= \log e - n\log(1+1/n)$$

③. If $\frac{u_n}{u_{n+1}}$ involves the number e, apply logarithmic test.

④. For application of Gauss Test, expand $\frac{u_n}{u_{n+1}}$ in powers of $1/n$ as

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right) \text{ where}$$

$O\left(\frac{1}{n^2}\right)$ stands for terms of order $1/n^2$ and higher powers of $1/n$.

* De Morgan's and Bertrand's Test

If $\sum u_n$ is a series of positive terms such that

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] = l$$

then (1) $\sum u_n$ converges if $l>1$.

(2) $\sum u_n$ diverges if $l<1$.

Note:- This test is to be applied when both D'Alembert's ratio test and Raabe's test fails.

* An alternative to Bertrand's Test:-

If $\sum u_n$ is a series of positive terms such that

$$\lim_{n\to\infty}\left[\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right] = l$$

then ① $\sum u_n$ converges if $l>1$.

② $\sum u_n$ diverges if $l<1$.

Note:- This test is to be applied when the logarithmic test fails.

Problems:-

→ $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \cdots\cdots$

Let $u_n = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n}$

then $u_{n+1} = \frac{1.3.5\cdots(2n-1)(2n+1)}{2.4.6\cdots(2n)(2n+2)}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n} \times \frac{2.4.6\cdots(2n)(2n+2)}{1.3.5\cdots(2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} = \frac{1+1/n}{1+\frac{1}{2n}}$$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$.

∴ D'Alembert's Ratio test fails.

Now we apply the Raabe's test

$$\therefore n\left[\frac{u_n}{u_{n+1}}-1\right] = n\left[\frac{2n+2}{2n+1}-1\right]$$

$$= \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = 1/2 < 1$$

∴ By Raabe's test, $\sum u_n$ diverges.

→ $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \cdots$

Sol'n:- Let $u_n = \frac{1^2.3^2.5^2\cdots(2n-1)^2}{2^2.4^2.6^2\cdots(2n)^2}$

then $u_{n+1} = \frac{1^2.3^2.5^2\cdots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\cdots(2n)^2(2n+2)^2}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{(1+1/n)^2}{(1+1/2n)^2}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = 1$$

$\therefore$ D'Alembert's Ratio test fails

Now apply Raabe's test

$$\therefore n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right]$$

$$= n\left[\frac{4n+3}{(2n+1)^2}\right]$$

$$= \frac{4n^2+3n}{(2n+1)^2}$$

$$= \frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1$$

$\therefore$ Raabe's test fails.

Now we apply Gauss test

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$= \left(1+\frac{1}{n}\right)^2 \left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2} - \cdots + \cdots\right)$$

$$= \left(1-\frac{\cancel{2}}{\cancel{2}n}+\frac{3}{4n^2} - \cdots\right) + \left(\frac{2}{n} - \frac{\cancel{4}2}{\cancel{2}n^2} + \frac{6}{4n^3} - \cdots\right) + \left(\frac{1}{n^2} - \frac{2}{2n^3} + \frac{3}{4n^4} - \cdots\right)$$

$$= 1+\frac{1}{n} - \frac{1}{4n^2} + \cdots$$

$$= 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Now composing with

$$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$$

we have $\lambda = 1$

$\therefore$ By Gauss Test

Σu_n is divergent.

Note:– when D' Almebert's test fails then we may directly apply Gauss test.

$\rightarrow$ $1+\frac{2^2}{3^2}+\frac{2^2 \cdot 4^2}{3^2 \cdot 5^2}+\frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2}+ \cdots$

Sol'n :– Omitting the first term, we have

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \cdots (2n+1)^2}$$

$\rightarrow$ $1+\frac{3}{7}x+\frac{3 \cdot 6}{7 \cdot 10}x^2+\frac{3 \cdot 6 \cdot 9}{7 \cdot 10 \cdot 13}x^3+\cdots$

Sol'n :– Leaving the first term, we have

$$u_n = \frac{3 \cdot 6 \cdot 9 \cdots (3n)}{7 \cdot 10 \cdot 13 \cdots (3n+4)} \cdot x^n$$

$$\Rightarrow u_{n+1} = \frac{3 \cdot 6 \cdot 9 \cdots (3n)(3n+3)}{7 \cdot 10 \cdot 13 \cdots (3n+4)(3n+7)} x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$= \frac{1+7/3n}{1+1/n} \cdot \frac{1}{x}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio test

$\sum u_n$ converges if $\frac{1}{x} > 1$ i.e. $x < 1$

and $\sum u_n$ diverges if $\frac{1}{x} < 1$ i.e. $x > 1$.

If $x = 1$ then the ratio test fails.

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

Now we apply Gauss Test.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

$$= \left(1+\frac{7}{3n}\right)\left(1+\frac{1}{n}\right)^{-1}$$

$$= \left(1+\frac{7}{3n}\right)\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+ - - -\right)$$

$$= \left(1-\frac{1}{n}+\frac{1}{n^2} - - - -\right)+\left(\frac{7}{3n}-\frac{7}{3n^2}+ - - -\right)$$

$$= 1+\frac{4}{3n}-\frac{4}{3n^2}+ - - -$$

$$= 1+\left(\frac{4}{3}\right)\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$$

where $\lambda = 4/3 > 1$

$\therefore$ By Gauss test,

$\therefore \sum u_n$ is Convergent.

$\therefore$ The given series converges if $x \le 1$ and diverges if $x > 1$.

$\rightarrow \frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + - - -$

Sol'n :- Neglecting the first term, we have

$$u_n = \frac{1.3.5.7 \ldots (2n-1)}{2.4.6 - - (2n)} \frac{x^{2n+1}}{(2n+1)}$$

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* Cauchy's Condensation Test:-

Let $\sum_{n=1}^{\infty} a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers.

$\sum_{n=1}^{\infty} a(n)$ Converges (or diverges) iff $\sum_{n=1}^{\infty} 2^n a(2^n)$ converges (or diverges).

Problems:-

(i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(iii) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$

(iv) $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$

Sol'n:-(i) Here given that

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

Put $\sum_{n=2}^{\infty} a(n) = \sum_{n=2}^{\infty} \frac{1}{\log n}$.

Here $a(n) = \frac{1}{\log n}$.

Since $(\log n)$ is an increasing Sequence.

$\therefore (a(n)) = \left(\frac{1}{\log n}\right)$ is a decreasing sequence.

$\therefore \sum_{n=2}^{\infty} 2^n a(2^n) = \sum_{n=2}^{\infty} 2^n \frac{1}{\log(2^n)}$

$= \sum_{n=2}^{\infty} 2^n \frac{1}{n \log 2}$

$= \frac{1}{\log 2} \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{n}$ —— (1)

Let $v_n = \frac{2^n}{n}$

then $v_n^{1/n} = \frac{2}{n^{1/n}}$

$\therefore \lim_{n\to\infty} v_n^{1/n} = \frac{2}{1}$

$= 2 > 1$

$\therefore$ By Cauchy's root test, Σv_n is divergent.

$\therefore \sum_{n=2}^{\infty} 2^n a(2^n)$ is divergent.

$\therefore$ By Cauchy's condensation test

$\Sigma a(n) = \Sigma \frac{1}{\ln n}$ is divergent.

(ii) Given that

$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \Sigma \frac{1}{n \log n}$

put $\Sigma a(n) = \Sigma \frac{1}{n \log n}$

Here $a(n) = \frac{1}{n \log n}$

Since $(n \log n)$ is an increasing sequence.

$\therefore (a(n)) = \frac{1}{(n \log n)}$ is a decreasing sequence.

$\therefore \sum_{n=2}^{\infty} 2^n a(2^n) = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \log(2^n)}$

$= \sum_{n=2}^{\infty} \frac{1}{n \log 2}$

$= \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$

is divergent by P-Test where $P=1$.

$\therefore$ By Cauchy's condensation test.

$\Sigma a(n) = \Sigma \frac{1}{n \ln n}$ is divergent.

(iii) Given $\sum_{n=3}^{\infty} \frac{1}{n (\ln n)(\ln \ln n)} = \sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)}$

Put $\Sigma a(n) = \Sigma \frac{1}{n(\log n)(\log \log n)}$

Here $a(n) = \frac{1}{n(\log n)(\log \log n)}$

Since $(n(\log n)(\log \log n))$ is an increasing sequence.

$\therefore (a(n))$ is a decreasing sequence

$\therefore \Sigma 2^n a(2^n) = \Sigma 2^n \frac{1}{2^n (\log 2^n)(\log \log 2^n)}$

$= \Sigma \frac{1}{(n \log 2) \log(n \log 2)}$

$= \Sigma x_n$ (say) —— (A)

Since $\log 2 < 1$

$\Rightarrow n\log 2 < n$

$\Rightarrow \log(n\log 2) < \log n$

$\Rightarrow \dfrac{1}{\log(n\log 2)} > \dfrac{1}{\log n}$

$\Rightarrow \dfrac{1}{n\log 2}\,\dfrac{1}{\log(n\log 2)} > \dfrac{1}{n\log 2}\,\dfrac{1}{\log n}$

$\Rightarrow x_n > \dfrac{1}{\log 2}\cdot\dfrac{1}{n\log n}$

$= y_n$ (say) ——— (B)

$\therefore x_n > y_n \ \forall n$

i.e. $y_n < x_n \ \forall n$ ——— (C)

Since $\Sigma y_n = \Sigma \dfrac{1}{\log 2}\cdot\dfrac{1}{n\log n}$

$= \dfrac{1}{\log 2}\Sigma\dfrac{1}{n\log n}$

diverges (by (ii))

$\therefore$ By comparison test,

Σx_n diverges.

$\therefore$ By Cauchy's Condensation test,

$\Sigma a(n)$ diverges.

(iv) Given that

$$\sum_{n=4}^{\infty}\frac{1}{n(\ln n)(\ln\ln n)(\ln\ln\ln n)}$$

$$= \sum_{n=4}^{\infty}\frac{1}{n(\log n)(\log\log n)(\log\log\log n)}$$

put $\sum_{n=4}^{\infty} a(n) =$

$$\sum_{n=4}^{\infty}\frac{1}{n(\log n)(\log\log n)(\log\log\log n)}$$

Here $a(n) = \dfrac{1}{n(\log n)(\log\log n)(\log\log\log n)}$

Since $(n(\log n)(\log\log n)(\log\log\log n))$ is an increasing sequence.

$\therefore (a(n))$ is a decreasing sequence

$\therefore \Sigma 2^n a(2^n) =$

$$\Sigma \cancel{2^n}\frac{1}{\cancel{2^n}(\log 2^n)(\log\log 2^n)(\log\log\log 2^n)}$$

$$= \Sigma\frac{1}{(n\log 2)(\log(n\log 2))(\log\log(n\log 2))}$$

$$= \frac{1}{\log 2}\Sigma\frac{1}{n(\log n\log 2)(\log\log(n\log 2))}$$

$= \Sigma x_n$ (say)

Since $\log 2 < 1$

$\Rightarrow \log n\log 2 < \log n$ ——— (A)

$\Rightarrow \log\log n\log 2 < \log\log n$

$\Rightarrow \dfrac{1}{\log\log n\log 2} > \dfrac{1}{\log\log n}$ ——— (B)

(A) $\equiv$

$\dfrac{1}{\log n\ \log 2} > \dfrac{1}{\log n}$ ——— (C)

But $\dfrac{1}{n} = \dfrac{1}{n}$ ——— (D) from ①, ©, ⓑ give,

$$\frac{1}{n(\log n\log 2)\cdot(\log\log n\log 2)} > \frac{1}{n(\log n)(\log\log n)}$$

$$= y_n \text{ say}$$

$\therefore x_n > y_n \ \forall n$

i.e. $y_n < x_n \ \forall n$

But by ⓑ,

$$\Sigma y_n = \Sigma \frac{1}{n(\log n)(\log\log n)}$$

diverges (by(iii))

$\therefore$ By comparison test

Σx_n also diverges.

$\therefore$ By Cauchy's Condensation test,

$\Sigma a(n)$ diverges.

→ If $c>1$ then show that the following series are convergent.

ⓐ $\Sigma \frac{1}{n(\log n)^c}$

ⓑ $\Sigma \frac{1}{n(\log n)(\log\log n)^c}$

Sol'n: (a) $\Sigma a(n) = \Sigma \frac{1}{n(\log n)^c}$

is decreasing for $c>1$.

$$\therefore \Sigma 2^n a(2^n) = \Sigma 2^n \frac{1}{2^n(\log 2^n)^c}$$

$$= \Sigma \frac{1}{(n\log 2)^c}$$

$$= \frac{1}{(\log 2)^c}\Sigma \frac{1}{n^c}$$

Since $\Sigma \frac{1}{n^c}$ is convergent for $c>1$.

$\therefore \Sigma 2^n a(2^n)$ is convergent.

$\therefore$ By Cauchys condensation test,

$\Sigma a(n)$ is convergent.

ⓑ $\Sigma a(n) = \Sigma \frac{1}{n(\log n)(\log\log n)^c}$

is decreasing for $c>1$.

$$\therefore \Sigma 2^n a(2^n) = \Sigma \frac{2^n}{2^n(\log 2^n)(\log\log 2^n)^c}$$

$$= \Sigma \frac{1}{(n\log 2)(\log n\log 2)^c}$$

$$= \Sigma x_n \text{ (say)}$$

since $\log 2 < 1$

$\Rightarrow n\log 2 < n$

$\Rightarrow \log(n\log 2) < \log n$

$\Rightarrow [\log(n\log 2)]^c < (\log n)^c$

$\Rightarrow n[\log(n\log 2)]^c < n(\log n)^c$

$\Rightarrow n\log 2\,[\log n\log 2]^c < \log 2\, n(\log n)^c$

$$\Rightarrow \frac{1}{n\log 2[\log n\log 2]^c} > \frac{1}{\log 2\cdot n(\log n)^c} = y_n \text{ (say)}$$

$\therefore x_n > y_n \ \forall n$

i.e. $y_n < x_n \ \forall n$

But $\Sigma y_n = \Sigma \frac{1}{\log 2\cdot n(\log n)^c}$

$$= \frac{1}{\log 2}\Sigma \frac{1}{n(\log n)^c}$$

is convergent for $c>1$. (by ⓐ).

$\therefore$ By Comparison test,

$\sum x_n$ is convergent.

$\therefore$ By Cauchy's Condensation test

$\sum a(n)$ is Convergent.

Comparison test :-

Note: Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let h & k be +ve real numbers such that

$$h v_n < u_n < k v_n \ \forall n$$

Then the series

$\sum u_n$ & $\sum v_n$ converge (or) diverge together.

$\rightarrow$ Examine the following series for Convergence.

(i) $\sum \frac{1}{(\log n)^{\log n}}$ (ii) $\sum \frac{1}{(\log\log n)^{\log n}}$

(iii) $\sum r^{\log n}$

Sol^n:- (i) Since $\lim_{n\to\infty} \log(\log n) = \infty$

$\therefore$ we can find n so large that $\log(\log n) > 2$

$\log n[\log(\log n)] > 2\log n$.

$\log(\log n)^{\log n} > \log n^2$

$\Rightarrow (\log n)^{\log n} > n^2$

$\Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$ ——— (1)

Since $\sum \frac{1}{n^2}$ is convergent (by P-Test)

$\therefore \sum \frac{1}{(\log n)^{\log n}}$ is convergent.

(ii) Since $\lim_{n\to\infty} \log(\log\log n) = \infty$

$\therefore$ we can find a n so large that

$\log(\log\log n) > 2$

$\Rightarrow \log n \cdot \log(\log\log n) > 2\log n$

$\Rightarrow \log(\log\log n)^{\log n} > \log n^2$

$\Rightarrow (\log\log n)^{\log n} > n^2$.

$\Rightarrow \frac{1}{(\log\log n)^{\log n}} < \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ is Convergent.

$\therefore$ By Comparison test.

$\sum \frac{1}{(\log\log n)^{\log n}}$ is convergent.

(iii) Since the multiplication of numbers is Commutative.

$\therefore \log n \log r = \log r \log n$

$\Rightarrow \log(r^{\log n}) = \log(n^{\log r})$

$\Rightarrow r^{\log n} = n^{\log r}$

$\therefore \sum r^{\log n} = \sum n^{\log r}$

$= \sum \frac{1}{n^{-\log r}}$

By P-Test it is convergent.

if $-\log r > 1$

i.e. if $\log r < -1$

i.e if $\log r < -\log e$

i.e. if $\log r < \log e^{-1}$

i.e. if $r < \frac{1}{e}$

$\therefore \sum r^{\log n}$ converges iff $r < 1/e$.

* Alternating Series :—

A Series with terms alternatively +ve and −ve is Called an alternative series.

i.e. $u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$

where $u_n > 0 \ \forall n$ is alternating series and is shortly written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n.$$

* Leibnitz's Test on Alternating Series:—

The alternating series

$$\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots,$$

$$u_n > 0 \ \forall n$$

Converges if (i) $u_n \geq u_{n+1} \ \forall n$ and

(ii) $\lim_{n\to\infty} u_n = 0$

Note:— The alternating series will not be Convergent if any one of the two Conditions is not satisfied.

* Absolute and Conditional Convergence :—

→ A series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is Convergent.

→ If $\sum_{n=1}^{\infty} u_n$ Converges but not absolutely.

i.e. $\sum_{n=1}^{\infty} |u_n|$ diverges then the Series $\sum_{n=1}^{\infty} u_n$ is Called Conditional Convergent (or) Semi-Convergent (or) non-absolutely Convergent.

Note:— Every absolutely convergent Series is convergent. but Convergent series need not be absolute Convergent.

Ex:—

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

Sol'n:— Let $u_n = 1/n$

then $u_n > 0 \ \forall n$

Since $\frac{1}{n} > \frac{1}{n+1} \ \forall n$.

$\Rightarrow u_n > u_{n+1} \ \forall n$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

$\therefore$ By Leibnitz's test,

$\sum \frac{(-1)^{n-1}}{n}$ Convergent.

But the series $\sum \left|\frac{(-1)^{n-1}}{n}\right|$

$= \sum \frac{1}{n}$ is divergent (by P-Test)

Problems :—

Test the Convergence and absolute convergence of the series.

(i) $1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots$

Sol'n:- The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $v_n = \frac{1}{2n-1} > 0 \ \forall n$

$$v_{n+1} = \frac{1}{2n+1} \ \forall n$$

Since $2n-1 < 2n+1 \ \forall n$

$$\frac{1}{2n-1} > \frac{1}{2n+1} \ \forall n$$

$\Rightarrow v_n > v_{n+1} \ \forall n$

and $\lim_{n\to\infty} v_n = 0$.

$\therefore$ By Leibnitz's test, the series is convergent.

Now $|u_n| = \frac{1}{2n-1}$

Since $\frac{1}{2n-1} > \frac{1}{2n} \ \forall n$

$\therefore \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ is divergent (by p-Test)

$\therefore$ By Comparison test,

$\sum \frac{1}{2n-1}$ is divergent.

$\therefore \sum |u_n|$ is divergent.

$\therefore$ The Series

$\sum \frac{(-1)^{n-1}}{n-1}$ is Conditional Convergent.

$\rightarrow \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$

$\rightarrow \frac{1}{1.3} - \frac{1}{2.4} + \frac{1}{3.5} - \frac{1}{4.6} + \cdots$

Sol'n:- The given series is

$$\sum v_n = \sum \frac{(-1)^{n-1}}{n.(n+2)} = \sum (-1)^{n-1} v_n.$$

It is an alternating series.

$\rightarrow \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

Sol'n:- The given series is

$$\sum u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\log(n+1)}$$

$$= \sum (-1)^{n-1} v_n$$

It is alternating series.

Here $v_n = \frac{1}{\log(n+1)}$

and $v_{n+1} = \frac{1}{\log(n+2)}$

Since $(n+1) < (n+2) \ \forall n$

$$\log(n+1) < \log(n+2) \ \forall n$$

$$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{\log(n+2)} \ \forall n$$

$\Rightarrow v_n > v_{n+1} \ \forall n.$

and $\lim v_n = \lim_{n\to\infty} \frac{1}{\log(n+1)}$

$$= 0$$

$\therefore$ By Leibnitz's test, the series is Convergent.

Now $|u_n| = \frac{1}{\log(n+1)}$

Since $\log(n+1) < (n+1) \ \forall n$

$$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{n+1} \ \forall n \quad \text{——} (A)$$

Since $\sum \frac{1}{n+1} = \sum x_n$ (say)

Let $x_n = \frac{1}{n+1} - \frac{1}{n(1+1/n)}$

and $y_n = 1/n$

then $\frac{x_n}{y_n} = \frac{1}{(1+1/n)}$

$\therefore \lim \frac{x_n}{y_n} = 1 \neq 0.$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent (by p-Test)

$\therefore$ By Comparison test

$\sum x_n = \sum \frac{1}{n+1}$ is divergent.

again by Comparison test

$\sum |u_n| = \sum \frac{1}{\log(n+1)}$ is divergent.

$\therefore \sum u_n = \sum \frac{(-1)^{n-1}}{n(n+1)}$

is conditional Convergent.

$\rightarrow$ Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$ is absolutely Convergent.

Sol'n:- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$

$|u_n| = \frac{2^n}{n!}$, $|u_{n+1}| = \frac{2^{n+1}}{(n+1)!}$

$\therefore \lim \left|\frac{u_n}{u_{n+1}}\right| = \lim \left(\frac{n+1}{2}\right) = \infty > 1$

$\therefore$ By D'Alembert's ratio test.

$\sum |u_n|$ is Convergent.

$\therefore$ The given alternating Series is absolutely Convergent.

Abel's Test :-

If $\sum_{n=1}^{\infty} a_n$ is Convergent and the sequence $\{b_n\}$ is Monotonic and bounded, then $\sum_{n=1}^{\infty} a_n \cdot b_n$ is Convergent.

Problem ①: Test the Convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (1+1/n)^n.$$

Sol'n:- Let $a_n = \frac{(-1)^{n-1}}{n}$ and

$b_n = \left(1+\frac{1}{n}\right)^n \ \forall n$

clearly $\sum a_n$ is Convergent (by Leibnitz's Test) and the sequence $\{b_n\}$ is monotone (increasing) and bounded.

Hence by Abel's Test, the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Problem ② Test the Convergence of

$$1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \cdots$$

Sol'n:- $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} + \frac{1}{7.4^2} + \cdots$

$$= \sum_n \frac{(-1)^{n-1}}{(2n-1)n^2}.$$

Let $a_n = \frac{(-1)^{n-1}}{n^2}$ and $b_n = \frac{1}{2n-1}$

Problem: show that the series

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3}-n}{\log n} \text{ is convergent.}$$

Sol'n :- Let $a_n = (n^3+1)^{1/3}-n$, $b_n = \frac{1}{\log n}$

then the series can be written as

$$\sum_{n=1}^{\infty} a_n b_n.$$

Now $a_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^6} + \cdots +\right]$$

$$= \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^2} + \cdots\right]$$

Take $c_n = \frac{1}{n^2}$ then

$$\frac{a_n}{c_n} = \frac{1}{3} - \frac{1}{9n^2} + \cdots$$

$\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{1}{3}$ which is finite and non-zero.

$\therefore$ By Comparison test, Σa_n and Σc_n converge (or) diverge together.

But $\Sigma c_n = \Sigma \frac{1}{n^2}$ is convergent.

$\therefore \Sigma a_n$ is convergent.

Also $\{b_n\}$ is a monotonically decreasing sequence of +ve terms and bounded below.

$\therefore$ By Abel's test the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

* Dirichlet's Test :-

If $\sum_{n=1}^{\infty} a_n$ is a series whose n^{th} partial sum $\{S_n\}$ is bounded and $\{b_n\}$ is a monotonic sequence converging to zero then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

$\rightarrow$ Note: Leibnitz's Test as a particular case of Dirichlet's Test:

The series $\sum_{n=1}^{\infty}(-1)^{n-1}$ has bounded partial sums,

Since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd.} \\ 0 & \text{if } n \text{ is even} \end{cases}$ $\left[\because S_n = 1+(-1)+1+(-1)+\cdots+(-1)^{n}\right]$

If $\{a_n\}$ is a monotonically decreasing sequence of +ve numbers convergent to '0' i.e. if (i) $a_n > 0$ ~~then~~

(ii) $a_n \geq a_{n+1} \ \forall n$

(iii) $a_n \to 0$ as $n \to \infty$

then by Dirichlet's test,

the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

i.e. the alternating series.

$a_1 - a_2 + a_3 - a_4 + \cdots$ is Convergent.

Problem :- Discuss the Convergence of

$$\Sigma \frac{(-1)^{n-1}}{n^p}, \ (p>0)$$

Sol'n :- Let $a_n = (-1)^{n-1}$ and $b_n = \frac{1}{n^p}$, $(p>0)$

then the series $\Sigma a_n = \Sigma(-1)^{n-1}$ has bounded partial sums,

Since $s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

and the sequence $\{b_n\} = \left\{\frac{1}{n^p}\right\}$ $(p>0)$ is monotonically decreasing sequence of +ve numbers convergent to '0'.

i.e. (i) $b_n > 0 \ \forall n$

(ii) $b_n \geq b_{n+1} \ \forall n$

(iii) $b_n \to 0$ as $n \to \infty$

Hence by Dirichlet's Test $\sum_{n=1}^{\infty} a_n b_n$ is Convergent.

* Rearrangement of Terms :-

A series $\sum_{n=1}^{\infty} b_n$ is said to arise from a series $\sum_{n=1}^{\infty} a_n$ by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series so that every a_n is some b_n and conversely.

for example, the series.

$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+ \cdots$

is a rearrangement of series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+ \cdots$

i.e. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+ \cdots$

on rearranging the terms so that each positive term is followed by two negative terms,

the series

$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+ \cdots$

— If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum arranged. But this is not so when infinite series are involved. An arrangement (or equally well derangement) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

Dirichlet's Theorem (1):-

(i) If $\sum_{n=1}^{\infty} a_n$ is a +ve term series converging to 's', then any derangement $\sum b_n$ of $\sum_{n=1} a_n$ also converges to 's'

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent positive term series then so also is $\sum_{n=1}^{\infty} b_n$.

Dirichlet's theorem (2):-

If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series then every derangement $\sum_{n=1}^{\infty} b_n$ also converges absolutely to the same sum as the original series.

* Riemann's Theorem :-

A conditionally convergent series can be made by derangement of terms. (i) to converge to any real number.

(ii) to diverge to any $+\infty$ or $-\infty$.

(iii) to oscillate finitely or infinitely.

Problem: ① Discuss the convergence of the series $1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}-\frac{1}{4^2}+\frac{1}{6^2}-\cdots$

Sol'n :- The given series is a rearrangement of the series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$$

which is absolutely convergent.

Hence by the Dirichlet's theorem, the given series is convergent.

Note :- Riemann's method is of theoretical importance only. For pratical applications, the method given by Pringsheim's is useful.

Pringsheim's Method :-

Let $f(n)$ be a +ve fn decreasing to zero as $n\to\infty$. Then by Leibnitz's test, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ is convergent.

Let the terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ be rearranged by taking alternatively α positive terms and β negative terms.

If $g=mf(m)$ and $k=\alpha/\beta$.

then the alteration ~~alteration~~ in the sum due to this rearrangement is $\frac{1}{2}g\log k$.

In particular, if $f(n)=1/n$.

then $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

$$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

then we know that the series is conditionally convergent and its sum is $\log 2$.

Also $g=mf(m)$

$$=m\cdot\frac{1}{m}=1$$

$\therefore$ If the terms are rearranged by taking alternately α +ve terms & β -ve terms.

then the sum of new series is

$\log 2+\frac{1}{2}g\log k=\log 2+\frac{1}{2}\log k$

$\log 2+\frac{1}{2}\log k$

Problems: ① find the sum of the series $1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\cdots$

Soln: The given series

$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\cdots$

is rearrangement of the series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

and is conditionally convergent and whose sum is $\log 2$.

Here the rearranged given series is formed by taking alternatively one +ve and two -ve terms.

Let α be the +ve terms & β be the -ve terms.

then $k=\alpha/\beta=1/2$.

and $g=mf(m)=m\cdot\frac{1}{m}=1$

$\therefore$ the sum of the rearranged given series is $\log 2 + \frac{1}{2} g \log k$.

$$= \log 2 + \tfrac{1}{2}\log \tfrac{1}{2}$$

$$= \log 2 - \tfrac{1}{2}\log 2$$

$$= \tfrac{1}{2}\log 2.$$

→ Find the sum of the series.

(i) $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}-----$

(ii) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{4}+----$

(iii) $1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}-\frac{1}{4}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}-\frac{1}{8}+----$

→ Investigate what derangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+-----$ will reduce its sum to zero.

Sol'n:- The given series is

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+--- = \sum(-1)^{n-1}\frac{1}{n}.$$

It is conditionally convergent with sum $\log 2$.

Let it be deranged by taking alternately α +ve & β -ve terms

so that $k = \alpha/\beta$

and $g = m f(m)$

$$= m\cdot\frac{1}{m} = 1$$

$\therefore$ The sum of the deranged series $= \log 2 + \frac{1}{2} g \log k$.

$$= \log 2 + \tfrac{1}{2}\log k.$$

But the sum is given to be zero.

$\therefore \log 2 + \frac{1}{2}\log k = 0$

$\Rightarrow \frac{1}{2}\log k = -\log 2$

$\Rightarrow \log k = -2\log 2$

$\Rightarrow \log k = \log \frac{1}{4}$

$\Rightarrow k = \frac{1}{4}$

$\Rightarrow \alpha/\beta = \frac{1}{4}$

$\Rightarrow \alpha = 1$ (one +ve term)

& $\beta = 4$ (four -ve terms).

$\therefore$ To get the sum zero, one +ve term should be followed by four -ve terms.

The deranged series.

$$1-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{3}-\tfrac{1}{10}-\tfrac{1}{12}-\tfrac{1}{14}-\tfrac{1}{16}+\tfrac{1}{5}------$$

H.W. ① → what derangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-----$ will reduce its sum to $\frac{1}{2}\log 2$.

P.-II $\boxed{2007}$ → H.W. Rearrange the series.

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ to converge to 1.

i.e. what derangement of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ will reduce its sum to 1.

Cauchy product of Two infinite series :-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series, then their product, called the cauchy' product, is defined as $\sum_{n=1}^{\infty} c_n$

where $c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \cdots\cdots + a_n b_1$

$= \sum_{r=1}^{n} a_r b_{n-r+1}$ for each $n \in N$

Thus $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$

$= (a_1 + a_2 + \cdots\cdots)\cdot(b_1 + b_2 + \cdots\cdots)$

$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots\cdots$

$= c_1 + c_2 + c_3 + \cdots\cdots$

The terms in the product are so arranged that all the terms which have the same sum of suffices are bracketed together.

Note:- (1) The cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined as $\sum_{n=0}^{\infty} c_n$.

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots\cdots + a_n b_0$

$= \sum_{r=0}^{n} a_r b_{n-r}$ for each $n \in N$.

(2) $c_n = \sum_{r=1}^{n} a_r b_{n-r+1} = \sum_{r=1}^{n} a_{n-r+1} b_r$

and $c_n = \sum_{r=0}^{n} a_r b_{n-r} = \sum_{r=0}^{n} a_{n-r} b_r$.

(3) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then it is not necessary that $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$ must converge.

→ $\sum_{n=1}^{\infty} c_n$ converges if

(i) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of non-negative terms

(or)

(ii) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent
(or)
(iii) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and one of them is absolutely convergent.

i.e

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of non-negative terms converging to A and B respectively then their cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

(ii) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two absolutely convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ Then their cauchy product $\sum_{n=1}^{\infty} c_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} c_n = AB$.

(iii) Merten's Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $\sum_{n=1}^{\infty} a_n$ converge absolutely. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then their product $\sum_{n=1}^{\infty} c_n$ converges to AB.

→ **Cesaro's Theorem:**

If two sequences (a_n) and (b_n) converge to 'a' and 'b' respectively, then the sequence (x_n) where $x_n = \dfrac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}$ converge to ab.

→ **Abel's test:**– Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. If their cauchy product $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = AB$.

problems

① → S.T the cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ with itself is not ~~absolutely~~ convergent. absolutely

Sol Given that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Let $a_n = b_n = \frac{(-1)^{n-1}}{n}, \forall n$

$\therefore$ By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The cauchy's product of the two series is $\sum_{n=1}^{\infty} c_n$

where $c_n = a_1 b_n + a_2 b_{n-1} + \cdots\cdots a_n b_1$

$$= \frac{(-1)^0}{1}\cdot\frac{(-1)^{n-1}}{n} + \frac{(-1)^1}{2}\cdot\frac{(-1)^{n-2}}{n-1} + \cdots\cdots$$

$$\cdots\cdots + \frac{(-1)^{n-1}}{n}\cdot\frac{(-1)^0}{1}.$$

$$|c_n| = \left|(-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2\cdot(n-1)} + \frac{1}{3\cdot(n-2)} + \cdots + \frac{1}{n\cdot 1}\right]\right|.$$

$$\geq \left|(-1)^{n-1}\left[\frac{1}{n\cdot n} + \frac{1}{n\cdot n} + \cdots + \frac{1}{n\cdot n}\right]\right|.$$

$\left(\because r \leq n \Rightarrow \frac{1}{r} \geq \frac{1}{n}\right)$ here $r = 1$ to n

$$\therefore \quad = \left|(-1)^{n-1}\left[\frac{1}{n\cdot n} \text{ (n times)}\right]\right|.$$

$$= \left|(-1)^{n-1}\left[\frac{n}{n\cdot n}\right]\right|$$

$$= \left|(-1)^{n-1}\left[\frac{1}{n}\right]\right|$$

$c_n \geq \frac{(-1)^{n-1}}{n}$

$\frac{(-1)^{n-1}}{n} \leq c_n$

$\sum \frac{(-1)^{n-1}}{n} \leq \sum c_n$

If dgt then dgt

$\therefore |c_n| \geq \left|(-1)^{n-1}\left(\frac{1}{n}\right)\right| \forall n$

$\Rightarrow |c_n| \geq \left|(-1)^{n-1}\frac{1}{n}\right| \forall n$

$\Rightarrow \frac{1}{n} \leq |c_n| \quad \forall n$

Since $\sum \frac{1}{n}$ is dgt $\Rightarrow \sum |c_n|$ is dgt (By comparison test)

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

Note:- The above example illustrates that the cauchy product of two conditionally convergent series need not be necessarily convergent.

(7) → Show that the cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself is not ~~convergent~~ absolutely convergent.

Sol Let $a_n = b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$, $n \in N$

By Leibnitz's test, the series $\sum_{n=1} a_n$ and $\sum_{n=1} b_n$ are both convergent (but not absolutely).

The cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + \cdots\cdots + a_n b_1$$

$$= \frac{(-1)^0}{\sqrt{1}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^{n-2}}{\sqrt{n-1}} + \cdots$$

$$\cdots\cdots + \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{(-1)^0}{\sqrt{1}}$$

$$|c_n| = \left|(-1)^{n-1}\left[\frac{1}{\sqrt{1\cdot n}} + \frac{1}{\sqrt{2(n-1)}} + \cdots + \frac{1}{\sqrt{n\cdot 1}}\right]\right|$$

$$|c_n| \geq \left|(-1)^{n-1}\left[\frac{1}{\sqrt{n\cdot n}} + \frac{1}{\sqrt{n\cdot n}} + \cdots\cdots + \frac{1}{\sqrt{n\cdot n}}\right]\right|$$

$$|c_n| = \left|(-1)^{n-1}\left[\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots\cdots + \frac{1}{n}\right]\right|$$

$$= \left|(-1)^{n-1}\left[\frac{n}{n}\right]\right|.$$

$$= |(-1)^{n-1}|.$$

$$\Rightarrow |c_n| \geq 1 \ \forall\, n \in N$$

$$\Rightarrow \lim_{n\to\infty} c_n \neq 0.$$

Hence $\sum_{n=1} c_n$ cannot converge.

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(3) → Show that the Cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ with itself is not absolutely convergent.

Sol Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, \forall n \in N$.

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$$

$$= \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \cdots + \frac{(-1)^n}{\sqrt{n+1}} \cdot \frac{(-1)^1}{\sqrt{2}}.$$

$$|c_n| = \left|(-1)^{n+1}\left[\frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{3n}} + \cdots + \frac{1}{\sqrt{(n+1)\cdot 2}}\right]\right|$$

$$|c_n| \geq \left|(-1)^{n+1}\left[\frac{1}{\sqrt{(n+1)(n+1)}} + \frac{1}{\sqrt{(n+1)(n+1)}} + \cdots + \frac{1}{\sqrt{(n+1)(n+1)}}\right]\right|$$

$$= \left|(-1)^{n+1} \cdot \left(\frac{n}{n+1}\right)\right|.$$

$$\therefore |c_n| \geq \left|(-1)^{n+1}\left(\frac{n}{n+1}\right)\right| \forall n \in N.$$

$$|c_n| \geq \frac{n}{n+1}, \forall n \in N.$$

Since $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent (By comparison test)

$\Rightarrow \sum |c_n|$ is divergent

$\Rightarrow \lim_{n\to\infty} c_n \neq 0.$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge

(4) → Show that the Cauchy product of two divergent series $\sum_{n=1}^{\infty} a_n = 2 + 2^1 + 2^2 + 2^3 + \cdots$ and $\sum_{n=1}^{\infty} b_n = -1 + 1 + 1 + 1 + 1 + \cdots$ is convergent.

Sol for $n \geq 2$,

$\sum a_n$ and $\sum b_n$ are geometric series with common ratios 2 and 1 respectively.

Since the geometric series $\sum r^n$ is divergent for $r \geq 1$

$\therefore$ the series $\sum a_n$ and $\sum b_n$ are both divergent

The cauchy product of the two given series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \cdots + a_{n-1} b_2 + a_n b_1.$$

$$= 2\cdot 1 + 2\cdot 1 + 2^2\cdot 1 + 2^3\cdot 1 + \cdots \cdots + 2^{n-2}\cdot 1 + 2^{n-1}\cdot(-1)$$

$$= 2 + (2 + 2^2 + 2^3 + \cdots + 2^{n-2}) - 2^{n-1}$$

$$= 2 + \frac{2(2^{n-2}-1)}{2-1} - 2^{n-1} \quad \left(\because r > 1 \Rightarrow \frac{a(r^n-1)}{r-1}\right)$$

$$= 2 + 2^{n-1} - 2 - 2^{n-1}$$

$$= 0 \quad \forall n \geq 1$$

and ~~$c_1 = a_1 b_1$~~

~~$= 2(-1)$~~

~~$= -2$~~

$\boxed{Lt_{n\to\infty} S_n = 2}$

Thus $\sum_{n=1}^{\infty} c_n = 2 + 0 + 0 + 0 + \cdots$

Clearly which cgs to 2.

(5) → Show that the cauchy product of two divergent series $\sum_{n=0}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \cdots\cdots$

and $\sum_{n=0}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \cdots\cdots$

is convergent.

Sol for $n \geq 1$, $\sum a_n$ is a geometric series with common ratio $3/2$ (>1). (20)

$\Rightarrow \sum a_n$ is divergent.

Also $\sum b_n$ is a series of positive terms and $b_n \geq 1$ $\forall n$ fal.

Since $\lim_{n\to\infty} b_n \neq 0$

$\therefore \sum b_n$ is divergent

The cauchy product of the two given series is $\sum_{n=0}^{\infty} c_n$, where

$$c_0 = a_0 b_0 = 1 \times 1 = 1$$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots\cdots + a_{n-1} b_1 + a_n b_0$$

$$= 1\cdot\left(\tfrac{3}{2}\right)^{n-1}\left(2^n + \tfrac{1}{2^{n+1}}\right) - \left(\tfrac{3}{2}\right)\cdot\left(\tfrac{3}{2}\right)^{n-2}\left(2^{n-1} + \tfrac{1}{2^n}\right)$$

$$- \left(\tfrac{3}{2}\right)^2\cdot\left(\tfrac{3}{2}\right)^{n-3}\left(2^{n-2} + \tfrac{1}{2^{n-1}}\right) - \cdots\cdots$$

$\frac{2^{n+1}}{2^{n-2}}$ $$\cdots\cdots - \left(\tfrac{3}{2}\right)^{n-1}\left(2 + \tfrac{1}{2^2}\right) - \left(\tfrac{3}{2}\right)^n\cdot 1$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\left(2^n + \tfrac{1}{2^{n+1}}\right) - \left(2^{n-1} + 2^{n-2} + \cdots\cdots + 2\right) - \left(\tfrac{1}{2^n} + \tfrac{1}{2^{n-1}} + \cdots\cdots + \tfrac{1}{2^2}\right)\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[2^n + \tfrac{1}{2^{n+1}} - \frac{2(2^{n-1}-1)}{2-1} - \frac{\frac{1}{2^2}\left(1 - \frac{1}{2^{n-1}}\right)}{1 - 1/2}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[2^n + \tfrac{1}{2^{n+1}} - 2^n + 2 - 1/2 + \tfrac{1}{2^n}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\tfrac{3}{2} + \tfrac{1}{2^{n+1}} + \tfrac{1}{2^n}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\tfrac{3}{2} + \tfrac{3}{2^{n+1}} - \tfrac{3}{2}\right] = \frac{3^n}{2^{2n}} = \left(\tfrac{3}{4}\right)^n$$

$\Rightarrow \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$

clearly which is a geometric series of positive terms with common ratio $\frac{3}{4}$ (<1) is absolutely convergent.

→ HW ⑥ S.T the cauchy product of two divergent series $\sum_{n=1}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \cdots$

and $\sum_{n=1}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \cdots$ is convergent.

(Note: In example ⑤ a_n is the $(n+1)^{th}$ term of $\sum_{n=0}^{\infty} a_n$ where as in example ⑥, a_n is the n^{th} term of $\sum_{n=1}^{\infty} a_n$).

→ prove that the cauchy product of the two series $3 + \sum_{n=1}^{\infty} 3^n$ and $-2 + \sum_{n=1}^{\infty} 2^n$ is absolutely convergent, although both the series are divergent.

Sol^n: Let $\sum_{n=0}^{\infty} a_n = 3 + 3 + 3^2 + 3^3 + \cdots = 3 + \sum_{n=1}^{\infty} 3^n$

and $\sum_{n=0}^{\infty} b_n = -2 + 2 + 2^2 + 2^3 + \cdots = -2 + \sum_{n=1}^{\infty} 2^n$

→ Show that

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{n\cdot 1}\right]$$

Sol^n: Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1} a_n$,

then $\sum_{n=1}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the cauchy product $\sum_{n=1}^{\infty} C_n$ of $\sum_{n=1}^{\infty} a_n$ with itself converges, (21)

then $\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} C_n$ —— (1)

Now,

$$C_n = 1\cdot\frac{(-1)^{n-1}}{n} - \frac{1}{2}\cdot\frac{(-1)^{n-2}}{n-1} + \cdots\cdots + \frac{(-1)^{n-2}}{n-2}\cdot\left(-\frac{1}{2}\right) + \frac{(-1)^{n-1}}{n}\cdot 1$$

$$= (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots\cdots + \frac{1}{(n-1)\cdot 2} + \frac{1}{n\cdot 1}\right] \text{ —— (2)}$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\frac{n+1}{1\cdot n} + \frac{n+1}{2(n-1)} + \cdots\cdots + \frac{n+1}{(n-1)\cdot 2} + \frac{n+1}{n\cdot 1}\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n-1}\right) + \cdots\cdots + \left(\frac{1}{n-1}+\frac{1}{2}\right) + \left(\frac{1}{n}+1\right)\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\left[2 + \frac{2}{2} + \cdots\cdots + \frac{2}{n-1} + \frac{2}{n}\right]$$

$$= (-1)^{n-1}\frac{2}{n+1}\left[1 + \frac{1}{2} + \cdots\cdots + \frac{1}{n-1} + \frac{1}{n}\right]$$

$$\therefore |C_n| = \frac{2}{n+1}\left[1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right]$$

$$= \frac{2n}{n+1}\left[\frac{1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}}{n}\right]$$

$$= \frac{2}{1+\frac{1}{n}}\left[\frac{1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}}{n}\right] \to 0 \text{ as } n\to\infty$$

(by cauchy's first theorem on limits)

Also
$$|C_{n+1}| - |C_n| = \frac{2}{n+2}\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}+\frac{1}{n+1}\right) - \frac{2}{n+1}\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right)$$

$$= \left(\frac{2}{n+2} - \frac{2}{n+1}\right)\left(1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}\right) + \frac{2}{(n+2)(n+1)}$$

$$= \frac{-2}{(n+2)(n+1)}\left(1+\frac{1}{2}+\cdots\cdots\frac{1}{n}-1\right)$$

$$= \frac{-2}{(n+2)(n+1)}\left(\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}\right) < 0$$

$\Rightarrow \quad |c_n| > |c_{n+1}|$

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots\cdots + \frac{1}{n\cdot 1}\right] \quad (\text{By } (2))$$

converges.

Hence, from (1), we have

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\cdots\right)^2 = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{1\cdot n}+\frac{1}{2(n-1)}+\cdots+\frac{1}{n\cdot 1}\right]$$

$\xrightarrow{\text{H.W}}$ Show that

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)^2 = 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\cdots\right]$$

$\xrightarrow{\text{H.W.}}$ Show that

$$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\cdots$$

* Infinite products :-

If (a_n) is a sequence, then the product $a_1 a_2 a_3 \cdots\cdots a_n \cdots\cdots$ is called an infinite product and is denoted by $\prod_{n=1}^{\infty} a_n$ or simply by $\prod a_n$.

i.e $\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \cdots\cdots a_n \cdots\cdots$

'a_n' is called the n^{th} factor of the product.

The product of first 'n' terms of the sequence (a_n) is called the n^{th} partial product and is denoted by P_n.

Thus $P_n = a_1 \cdot a_2 \cdot a_3 \cdots\cdots a_n$

$= \prod_{r=1}^{n} a_r$

— The sequence (P_n) is called the sequence of partial products of the sequence (a_n)

* Convergence of Infinite products :

Let $P_n = \prod_{r=1}^{n} a_r$ be the n^{th} partial product of the infinite product $\prod_{n=1}^{\infty} a_n$.

(i) If no factor a_n is 'zero', then the product $\prod_{n=1}^{\infty} a_n$ converges if the sequence (P_n) converges to a non-zero finite number P (say),

i.e if $\lim_{n\to\infty} P_n = P$ then P is called the value of the product and we write $\prod_{n=1}^{\infty} a_n = P$.

— If $\lim_{n\to\infty} P_n = \infty$ then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to ∞

— If $\lim_{n\to\infty} P_n = 0$, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to '0'.

(ii) If infinitely many factors 'a_n' are zero, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to '0'.

(iii) If finitely many factors 'a_n' are zero, then the product $\prod_{n=1}^{\infty} a_n$ said to converge if it converges when the zero factors are removed.

(iv) If a finite number of factors are negative, then there exists a positive integer 'm' such that $a_n > 0 \ \forall \ n > m$ and the product $\prod_{n=1}^{\infty} a_n$ is said to converge if the product $\prod_{n=m+1}^{\infty} a_n$ converges since $\prod_{n=1}^{\infty} a_n = a_1 a_2 \cdots a_m \prod_{n=m+1}^{\infty} a_n$.

(v) If the sequence (P_n) oscillates, then the product $\prod_{n=1}^{\infty} a_n$ is said to oscillate.

Note: ① It is usually convenient to write the factors of the infinite product as $1+a_n$ instead of a_n.

Thus an infinite product is usually written as $\prod_{n=1}^{\infty} (1+a_n)$ and $P_n = \prod_{r=1}^{n} (1+a_r)$.

② We shall assume throughout our discussion that $a_n > -1$ i.e. $1+a_n > 0 \ \forall n$ so that $\log(1+a_n)$ is defined for all n.

③ For $a_n > -1$,

let P_n denote the n^{th} partial product of $\prod_{n=1}^{\infty} (1+a_n)$, then

$$P_n = (1+a_1)(1+a_2) \cdots \cdots (1+a_n)$$

$$\Rightarrow \log P_n = \log(1+a_1) + \log(1+a_2) + \cdots + \log(1+a_n)$$
$$= S_n.$$

where $S_n = \sum_{r=1}^{n} \log(1+a_r)$ is the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \log(1+a_n)$

$$\Rightarrow P_n = e^{S_n}$$

If $\underset{n\to\infty}{Lt} S_n = S$ then $\underset{n\to\infty}{Lt} P_n = e^S$.

Thus, to say that the product $\prod_{n=1}^{\infty}(1+a_n)$ diverges to 'zero',

i.e $\underset{n\to\infty}{Lt} P_n = 0$ is equivalent to saying that the series $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to $-\infty$

i.e $\underset{n\to\infty}{Lt} S_n = -\infty$. $\left(\text{i.e } \underset{n\to\infty}{Lt} P_n = e^{-\infty} = 0.\right)$.

Problems:

→ Show that the infinite product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots$ converges to $\frac{1}{2}$.

Sol The given infinite product is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots = \prod_{n=1}^{\infty}\left(1-\frac{1}{(n+1)^2}\right)$$

$$= \prod_{n=1}^{\infty} \frac{(n+1)^2-1}{(n+1)^2}$$

$$= \prod_{n=1}^{\infty} \frac{n(n+2)}{(n+1)^2}$$

$$= \prod_{n=1}^{\infty}\left(\frac{n}{n+1}\cdot\frac{n+2}{n+1}\right)$$

$$\therefore P_n = \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\cdots\cdots\left(\frac{n}{n+1}\cdot\frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdots\cdots\frac{n}{n+1}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdots\cdots\frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{n+1}\right)\left(\frac{n+2}{2}\right) = \frac{1}{2}\left(1+\frac{1}{n+1}\right)$$

$$\Rightarrow \underset{n\to\infty}{Lt} P_n = \frac{1}{2}$$

Hence the given infinite product converges to $\frac{1}{2}$

i.e $\prod_{n=1}^{\infty}\left(1-\frac{1}{(n+1)^2}\right)=\frac{1}{2}$.

H.W → Show that the infinite product

$\left(1-\frac{2}{2\cdot 3}\right)\left(1-\frac{2}{3\cdot 4}\right)\left(1-\frac{2}{4\cdot 5}\right)\cdots\cdots$

converges to $\frac{1}{3}$

→ Show that the infinite products

(i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ and (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ are both divergent.

Sol (i) Given infinite product is

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)$

$=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdots\cdots\frac{(n+1)}{n}\cdots$

Let $P_n=\prod_{r=1}^{n}\left(\frac{k+1}{k}\right)$

$=\frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{5}}{\cancel{3}}\cdot\frac{7}{5}\cdots\cdots\frac{(n+1)}{\cancel{n}}$

$=n+1$

Now $\lim_{n\to\infty}P_n=\infty$.

∴ The given infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is dgt and dgs to ∞.

i.e $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)=\infty$.

(ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{\infty}\left(\frac{n-1}{n}\right)$

∴ $P_n=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdots\cdots\frac{\cancel{n}}{n+1}=\frac{1}{n+1}$

$\Rightarrow \lim_{n\to\infty}P_n=0$

∴ $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ dgs to '0'

→ Show that the infinite product

$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots$ is convergent.

Sol (already we have done (in the following method))

Let $P = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots$

~~then~~ $\log P = \log\left(1-\frac{1}{2^2}\right) + \log\left(1-\frac{1}{3^2}\right) + \cdots\cdots$

$$= \sum_{n=2}^{\infty} \log\left(1-\frac{1}{n^2}\right)$$

$$= \sum_{n=2}^{\infty} a_n \text{ (say)} \longrightarrow (1)$$

$\sum_{n=1}^{\infty} \log(1+\ldots)$ $= \log\left(1-\frac{1}{4}\right)$ $\log\frac{3}{4}$ $\log(0)$

Now $a_n = \log\left(1-\frac{1}{n^2}\right)$

$$= -\left[\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \cdots\right]$$

$$\left(\because \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\right)$$

$$= -\frac{1}{n^2}\left[1 + \frac{1}{2n^2} + \frac{1}{3n^4} + \cdots\right].$$

Let $b_n = \frac{1}{n^2}$ ~~$\forall n$~~.

~~then~~ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} -\left[1 + \frac{1}{2n^2} + \cdots\right]$

$$= -1$$
$$\neq 0.$$

~~Here~~ $\sum b_n = \sum \frac{1}{n^2}$ is cgt (by p-test)

$\therefore$ By comparison test,

$\sum a_n$ is cgt.

and is convergent to finite number 's' (say)

$\therefore$ from (1), $\log P$ = a finite number 's' when $n\to\infty$

i.e P = a finite number e^s when $n\to\infty$

$\therefore$ The given product is cgt.

H.W $\Rightarrow$ S.T the infinite product $\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right)\cdots\cdots$ is convergent

→ Show that the infinite product

$\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\cdots\frac{(2n-1)}{2n}\cdot\frac{(2n+1)}{2n}\cdots\cdots$

tends to a finite limit as $n\to\infty$.

→ S.T the infinite product

$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots\cdots$

converges to 1.

Sol Given infinite product is

$(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots\cdots$

Let $P=(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{4})(1-\frac{1}{5})\cdots\cdots(1+\frac{1}{2n})(1-\frac{1}{2n+1})\cdots\cdots$

then $\log P=\log\left[(1+\frac{1}{2})(1-\frac{1}{3})\right]+\log\left[(1+\frac{1}{4})(1-\frac{1}{5})\right]$

$+\cdots\cdots+\log\left[(1+\frac{1}{2n})(1-\frac{1}{2n+1})\right]+\cdots\cdots$

$$=\sum_{n=1}^{\infty}\log\left[(1+\frac{1}{2n})(1-\frac{1}{2n+1})\right]$$

$$=\sum_{n=1}^{\infty}\log\left[1+\frac{1}{2n}-(1+\frac{1}{2n})(\frac{1}{2n+1})\right]$$

$$=\sum_{n=1}^{\infty}\log\left[1+\left(\frac{1}{2n}-\frac{1}{2n+1}\right)-\frac{1}{2n(2n+1)}\right].$$

$$=\sum_{n=1}^{\infty}\log\left[1+\frac{1}{2n(2n+1)}-\frac{1}{2n(2n+1)}\right]$$

$$=\sum_{n=1}^{\infty}\log[1]=0.$$

$\therefore \log P=0 \Rightarrow P=e^0=1.$

∴ The given infinite product is convergent and is convergent to 1.

→ <u>A necessary condition for convergence:</u>

If the product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent, then

$\lim_{n\to\infty}a_n=0.$

proof Given that $\prod_{n=1}^{\infty}(1+a_n)$ is cgt

and it cgs to P (say)

$\therefore P \neq 0$; $\underset{n\to\infty}{Lt}\, P_n = P$ and $\underset{n\to\infty}{Lt}\, P_{n-1} = P$.

now $$\frac{P_n}{P_{n-1}} = \frac{\cancel{(1+a_1)(1+a_2)\cdots(1+a_{n-1})}(1+a_n)}{\cancel{(1+a_1)(1+a_2)\cdots(1+a_{n-1})}}$$

$$\frac{P_n}{P_{n-1}} = (1+a_n)$$

$$\therefore \underset{n\to\infty}{Lt}(1+a_n) = \underset{n\to\infty}{Lt}\frac{P_n}{P_{n-1}} = \frac{\cancel{P}}{\cancel{P}} = 1.$$

$$\underset{n\to\infty}{Lt}(1+a_n) = 1 \Rightarrow 1+\underset{n\to\infty}{Lt}\, a_n = 1$$

$$\Rightarrow \boxed{\underset{n\to\infty}{Lt}\, a_n = 0}$$

Note(1):- The converse of the above need not be true. i.e. if $\underset{n\to\infty}{Lt}\, a_n = 0$ i.e. $a_n \to 0$ as $n\to\infty$ then $\prod_{n=1}^{\infty}(1+a_n)$ need not be cgt.

For examples

The infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{\infty}(1+a_n)$

Here $a_n = \frac{1}{n} \to 0$ as $n\to\infty$

but the product is divergent. (already we have done)

Note(2): If $\underset{n\to\infty}{Lt}\, a_n \neq 0$ then $\prod_{n=1}^{\infty}(1+a_n)$ must not cgt.

General principle of convergence of an infinite product:

A necessary and sufficient condition for the convergence of the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is that for every $\epsilon > 0$, there exists a positive integer 'm' s.t $\left|\frac{P_{n+p}}{P_n} - 1\right| < \epsilon \;\forall n \geq m, p \geq 1$.

Note: In order to establish the convergence (or divergence) of infinite product, we now give the following statements:

→ If $a_n \geq 0$ then the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

→ If $-1 < a_n \leq 0$, then the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

→ If $0 \leq b_n < 1$ then $\prod_{n=1}^{\infty}(1-b_n)$ converges to non-zero finite limit, if $\sum_{n=1}^{\infty} b_n$ converges and <u>diverges to zero</u> if $\sum_{n=1}^{\infty} b_n$ diverges.

→ If the series $\sum_{n=1}^{\infty} a_n^2$ is convergent, then the product $\prod_{n=1}^{\infty}(1+a_n)$ and series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

→ If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then we have $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \log(1+a_n)$ converge or diverge together.

Also $\sum_{n=1}^{\infty} \log(1+a_n)$ and $\sum_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

∴ we have

→ if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \log(1+a_n)$ converges and therefore $\prod_{n=1}^{\infty}(1+a_n)$ converges.

→ if $\sum_{n=1}^{\infty} a_n$ diverges to ∞, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to ∞ and therefore $\prod_{n=1}^{\infty}(1+a_n)$ diverges to ∞.

→ if $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to $-\infty$ and therefore $\prod_{n=1}^{\infty}(1+a_n)$ diverges to zero.

Also, if $\sum_{n=1}^{\infty} a_n^2$ diverges and $\sum_{n=1}^{\infty} a_n$ converges or oscillates finitely, then $\prod_{n=1}^{\infty}(1+a_n)$ diverges to zero.

→ Absolute convergence of Infinite products:

Def The product $\prod_{n=1}^{\infty}(1+a_n)$ is said to be absolutely convergent if the product $\prod_{n=1}^{\infty}(1+|a_n|)$ is convergent

→ The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

→ The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff the series $\sum_{n=1}^{\infty} \log(1+a_n)$ is absolutely convergent.

→ Every absolutely convergent infinite product is convergent.
i.e If $\prod_{n=1}^{\infty}(1+a_n)$ is an absolutely convergent (i.e $\prod_{n=1}^{\infty}(1+|a_n|)$ is cgt)

~~then~~ $\prod_{n=1}^{\infty}(1+a_n)$ is cgt.

Note:- The factors of an absolutely convergent of infinite product may be rearranged in any order without affecting its convergence.

problems

→ Discuss the convergence of the infinite products:

(i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{3/2}}\right)$ (iii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)$; $\alpha>1$

(iv) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)$, $0<\alpha\le 1$ (v) $\prod_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$ (vi) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$

Sol. (i) The given product is

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \prod_{n=1}^{\infty}(1+a_n), \text{ where } a_n=\frac{1}{n^2}>0 \ \forall n$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt (by p-Test) here $p=2>1$

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$

is cgt.

→ Discuss the convergence of the infinite products:

(i) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ (iii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$

(iv) $\frac{3}{4}\cdot\frac{6}{7}\cdot\frac{9}{10}\cdots\cdots\frac{3n}{3n+1}\cdots\cdots = \prod_{n=1}^{\infty}\left(\frac{3n}{3n+1}\right)$

$$= \prod_{n=1}^{\infty}\left(1-\frac{1}{3n+1}\right)$$

Sol (i) The given product is

$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}(1-b_n)$, where $b_n = \frac{1}{n^2}$ and $n \geq 2$

so that $0 < b_n < 1$

$\therefore$ The product $\prod_{n=2}^{\infty}(1-b_n)$ and the series $\sum_{n=2}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{1}{n^2}$ is cgt (by p-Test)

$\therefore$ The given product is cgt.

(iv) The product is $\prod_{n=1}^{\infty}\left(1-\frac{1}{3n+1}\right) = \prod_{n=1}^{\infty}(1-b_n)$

where $b_n = \frac{1}{3n+1}$ and $n \geq 1$

so that $0 < b_n < 1$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{3n+1}$ dgs.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ dgs to zero.

→ Discuss the convergence of the infinite products: (27)

(i) $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+n\sin\frac{\theta}{n^2}\right)$

(iii) $\prod_{n=1}^{\infty}\left(1+\frac{x}{n^p}\right)$; where x is ~~the~~ real number.

Sol: The given product is $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)=\prod_{n=1}^{\infty}(1+a_n)$

where $a_n=\sin^2\frac{\theta}{n}\geq 0\ \forall n$.

∴ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum a_n$ converge or diverge together.

Now $a_n=\sin^2\frac{\theta}{n}=\left(\sin\frac{\theta}{n}\right)^2$

$$=\left(\frac{\theta}{n}-\frac{1}{3!}\cdot\frac{\theta^3}{n^3}+\frac{1}{5!}\cdot\frac{\theta^5}{n^5}-\cdots\right)^2$$

$(a-b)^2$ ...

$$=\frac{\theta^2}{n^2}-2\left(\frac{1}{3!}\cdot\frac{\theta^4}{n^4}\right)+\cdots\cdots$$

$$=\frac{1}{n^2}\left[\theta^2-2\left(\frac{1}{3!}\cdot\frac{\theta^4}{n^2}\right)+\cdots\right]$$

Take $b_n=\frac{1}{n^2}$

$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\theta^2$

Since $\sum_{n=1}^{\infty}b_n=\sum\frac{1}{n^2}$ is cgt

∴ By comparison test $\sum_{n=1}^{\infty}a_n$ is cgt.

~~Hence~~ the given product is cgt.

(iii) $\prod_{n=1}^{\infty}\left(1+\frac{a}{n^p}\right)=\prod_{n=1}^{\infty}(1+a_n)$

where $a_n=\frac{a}{n^p}>0\ \forall n$

∴ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum a_n$ converge or diverge together.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a}{n^p} = a \sum_{n=1}^{\infty} \frac{1}{n^p}$

which is cgt if $p>1$

and dgt if $p \leq 1$.

Hence the given product is also convergent if $p>1$ and dgt if $p \leq 1$.

→ S.T $(1+x)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\cdots\cdots$

dgs to $+\infty$ or to $-\infty$ according as $x>0$ or $x<0$.

Sol The given product is

$\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right) = \prod_{n=1}^{\infty}(1+a_n)$

where $a_n = \frac{x}{n}$.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{x}{n} = x \sum_{n=1}^{\infty} \frac{1}{n}$

which dgs to $+\infty$ if $x>0$

dgs to $-\infty$ if $x<0$.

Hence the given product dgs to ∞ if $x>0$

dgs to $-\infty$ if $x<0$.

→ Discuss the convergence of the products:

(i) $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right)$

(ii) $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots\cdots$

(iii) $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\cdots$

(iv) $\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\left(1-\frac{1}{\sqrt{5}}\right)\cdots\cdots$

(v) $\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{\sqrt{n}}\right)$.

Sol (i) The given product is
$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right) = \prod_{n=2}^{\infty}(1+a_n) \text{ where } a_n = \frac{(-1)^n}{n}$$

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n}$ is cgt by Leibnitz's test

and $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty}\frac{1}{n^2}$ is also cgt

$\therefore$ The given product is cgt.

→ Discuss the convergence of the infinite product $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\cdots$

Sol The given product is $\prod_{n=1}^{\infty}\left(1+(-1)^n\cdot\frac{1}{2}\right)$

$$= \prod_{n=1}^{\infty}(1+a_n)$$
where $a_n = (-1)^n\cdot\frac{1}{2}$.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{2}$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n$$

$$= \frac{1}{2}(-1+1-1+1-1\cdots)$$

which oscillates b/w $-\frac{1}{2}$ and 0.

$\therefore \sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty}\frac{1}{4} = \frac{1}{4}+\frac{1}{4}+\cdots\cdots$

which is dgt (to $+\infty$)

Hence the given product dgs to zero.

HW → Discuss the convergence of $\prod_{n=1}^{\infty}(1+(-1)^n)$.

→ Show that the infinite product $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^\alpha}\right)$ is convergent if $\alpha > \frac{1}{2}$.

Sol The given product is
$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^{\alpha}}\right)=\prod_{n=2}^{\infty}(1+a_n)$$
where $a_n=\frac{(-1)^n}{n^{\alpha}}$.

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^{\alpha}}$ cgs if $\alpha>0$ (by Leibnitz's test)

Also $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty}\frac{1}{n^{2\alpha}}$ cgs if $2\alpha>1$
i.e if $\alpha>1/2$

Hence the given product cgs if $\alpha>1/2$.

→ ① Discuss the convergence of the product
$$\prod_{n=1}^{\infty}\left[1+\left(\frac{nx}{n+1}\right)^n\right]$$

Sol Here $a_n=\left(\frac{nx}{n+1}\right)^n$

$\therefore a_n^{1/n}=\frac{nx}{n+1}=\frac{x}{1+1/n}$

$\therefore \lim_{n\to\infty} a_n^{1/n}=x$.

$\therefore$ By cauchy's root test, the series $\sum_{n=1}^{\infty} a_n$ is cgt if $x<1$ and
dgt if $x>1$.

Hence the given product is cgt if $x<1$
and dgt if $x>1$.

— If $x=1$ then $a_n=\left(\frac{n}{n+1}\right)^n=\frac{1}{(1+1/n)^n}$

$\therefore \lim_{n\to\infty} a_n=\frac{1}{e}\neq 0$ and $a_n>0$ $\forall n$.

$\therefore \sum a_n$ is dgt.

Hence $\prod_{n=1}^{\infty}(1+a_n)$ is dgt.

Thus the given product is cgt if $x<1$ and dgt if $x\geq 1$.

→ Discuss absolute convergence of the following infinite products:

(i) $\prod_{n=1}^{\infty} \cos\frac{\theta}{n}$ (ii) $\prod_{n=1}^{\infty}\left[\frac{\sin\frac{x}{n}}{\frac{x}{n}}\right]$.

Sol (i) Here $1+a_n = \cos\frac{\theta}{n}$.

$$1-\frac{1}{2!}\cdot\frac{\theta^2}{n^2}+\frac{1}{4!}\cdot\frac{\theta^4}{n^4}-\cdots$$

$$\Rightarrow a_n = -\frac{1}{2!}\frac{\theta^2}{n^2}+\frac{1}{4!}\frac{\theta^4}{n^4}-\cdots$$

$$= \frac{1}{n^2}\left(-\frac{\theta^2}{2!}+\frac{\theta^4}{4!\,n^2}-\cdots\right)$$

Now $|a_n| = \frac{1}{n^2}\left|-\frac{\theta^2}{2!}+\frac{\theta^4}{4!\,n^2}-\cdots\right|$.

Let $b_n = \frac{1}{n^2}$ then $\lim_{n\to\infty}\frac{|a_n|}{b_n} = \frac{\theta^2}{2}$, (a finite quantity)

But $\sum b_n = \sum\frac{1}{n^2}$ is cgt (by p-test).

$\therefore$ $\sum |a_n|$ is cgt (by comparison test).

$\therefore$ $\sum a_n$ is absolutely convergent

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\cos\frac{\theta}{n}$ is absolutely cgt.

(ii) Here $1+a_n = \dfrac{\sin\frac{x}{n}}{\frac{x}{n}}$

$$= \frac{1}{\left(\frac{x}{n}\right)}\left[\frac{x}{n}-\frac{1}{3!}\cdot\frac{x^3}{n^3}+\frac{1}{5!}\cdot\frac{x^5}{n^5}-\cdots\right]$$

$$= 1-\frac{1}{3!}\cdot\frac{x^2}{n^2}+\frac{1}{5!}\cdot\frac{x^4}{n^4}-\cdots$$

$$\Rightarrow a_n = -\frac{1}{3!}\frac{x^2}{n^2}+\frac{1}{5!}\frac{x^4}{n^4}-\cdots$$

$$= \frac{1}{n^2}\left(-\frac{x^2}{3!} + \frac{x^4}{5! n^2} - \cdots\right)$$ proceed in this way.

→ Prove that $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is absolutely convergent for any real x.

Sol Here $1+a_n = \left(1+\frac{x}{n}\right)e^{-x/n}$

$$= \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{x^2}{2! n^2}-\frac{x^3}{3! n^3}+\cdots\right)$$

$$= 1 - \frac{x^2}{n^2} + \frac{x^2}{2n^2} + \frac{x^3}{2n^3} - \frac{x^3}{6n^3} + \cdots$$

$$\Rightarrow a_n = \frac{-x^2}{2n^2} + \frac{x^3}{3n^3} - \cdots$$

$$= \frac{1}{n^2}\left(-\frac{x^2}{2} + \frac{x^3}{3n} - \cdots\right).$$

proceed in this way.

HW → P.T $\prod_{n=1}^{\infty}\left(1+\frac{x}{n\pi}\right)e^{-\frac{x}{n\pi}}$ is absolutely cgt for all values of x.

HW → P.T $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}$ is absolutely cgt.

→ Test the absolutely convergence of the infinite product $\prod_{n=1}^{\infty}\left(\frac{x+x^{2n}}{1+x^{2n}}\right)$.

Sol Here $1+a_n = \frac{x+x^{2n}}{1+x^{2n}} = \frac{(1+x^{2n})+(x-1)}{1+x^{2n}}$

$$\Rightarrow 1+a_n = 1+\frac{x-1}{1+x^{2n}}$$

$$\Rightarrow \boxed{a_n = \frac{x-1}{1+x^{2n}}}$$

Now, when $|x|>1$, we have

$$|a_n| = \left|\frac{x-1}{1+x^{2n}}\right| = \frac{|x-1|}{|1+x^{2n}|} = \frac{|x-1|}{1+x^{2n}} < \frac{|x-1|}{x^{2n}} \quad \text{---(1)}.$$

Now $\sum_{n=1}^{\infty} \frac{1}{x^{2n}} = \sum u_n$ (say).

Here $u_n = \frac{1}{x^{2n}}$ then.

$$u_n^{1/n} = \frac{1}{x^2}$$

$\therefore \lim_{n\to\infty} u_n^{1/n} = \frac{1}{x^2}\ (<1) \quad (\because |x|>1)$

$\therefore$ By Cauchy's n^{th} root test $\sum \frac{1}{x^{2n}}$ is cgt.

$\therefore$ By comparison test,

$\sum |a_n|$ is cgt

$\therefore \sum a_n$ is absolutely cgt.

Hence $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent.

Now, when $|x|<1$ (ie $-1<x<1$),

we have

$$1+a_n = \frac{x+x^{2n}}{1+x^{2n}} = \frac{x(1+x^{2n-1})}{1+x^{2n}} \longrightarrow x \quad \left(\because -1<x<1,\ \downarrow 0\right)$$

$\therefore 1+a_n \longrightarrow x$ as $n\longrightarrow\infty$.

$\Rightarrow a_n \longrightarrow x-1$ as $n\longrightarrow\infty$.

$\Rightarrow a_n$ does not tend to '0' $(\because -1<x<1)$

i.e $\lim_{n\to\infty} a_n \neq 0$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ is divergent.

Now, when $x=1$,

every factor is unity

$$\text{i.e } \prod_{n=1}^{\infty}\left(\frac{x+x^{2n}}{1+x^{2n}}\right) = \left(\frac{x+x^2}{1+x^2}\right)\left(\frac{x+x^4}{1+x^4}\right)\cdots\cdots$$

$$= \frac{2}{2}\cdot\frac{2}{2}\cdots\cdots$$

$$= 1\cdot 1\cdots\cdots$$

Hence the product is convergent.

Now, when $x=-1$,

every factor is zero.

Hence the product is divergent.

→ Discuss the convergence of the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$.

Sol Here $1+a_n = 1+\frac{x^n}{x^{2n}+1}$ so that $a_n = \frac{x^n}{x^{2n}+1}$

Now $a_{n+1} = \frac{x^{n+1}}{x^{2n+2}+1}$

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{x^n}{x^{2n}+1}\cdot\frac{x^{2n+2}+1}{x^{n+1}}\right|$$

$$= \left|\frac{x^{2n+2}+1}{x(x^{2n}+1)}\right| = \frac{|x^{2n+2}+1|}{|x||x^{2n}+1|}$$

$\therefore$ If $|x|<1$ (i.e. $-1<x<1$, $x\neq 0$),

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{|x|} > 1$$

$\therefore$ By ratio test, $\Sigma|a_n|$ cgs and hence $\Pi(1+a_n)$ cgs absolutely.

If $|x|>1$,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}+1}{x^{2n+1}+x}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x+\frac{1}{x^{2n+1}}}{1+\frac{1}{x^{2n}}}\right| = |x| > 1$$

$\therefore$ By ratio test, $\Sigma|a_n|$ cgs and hence $\Pi(1+a_n)$ cgs absolutely.

If $x=1$, $a_n = \frac{1}{2} \ \forall n$

$$\underset{n\to\infty}{Lt}\ a_n \neq 0.$$

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$ is dgt.

(Or)

If $x=1$, $a_n = \frac{1}{2} \ \forall n$.

$$\therefore \sum a_n = \sum_{n=1}^{\infty}\frac{1}{2} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

Let $S_n = \left(\frac{1}{2}+\frac{1}{2}+\cdots\cdots+\frac{1}{2}\right)$ (n times)

$$= \frac{n}{2}$$

$$\underset{n\to\infty}{Lt}\ S_n = \infty.$$

$\therefore \sum a_n$ is dgt.

hence the product $\prod_{n=1}^{\infty}(1+a_n)$ is dgt

If $x=-1$, the product $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$ becomes

$$\left(1-\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right)\left(1-\tfrac{1}{2}\right)\left(1+\tfrac{1}{2}\right)\cdots\cdots$$

which dgs to '0' (already we have done).

→ Show that $\prod_{n=2}^{\infty}\left[1-\left(1-\frac{1}{n}\right)^{-n}x^{-n}\right]$ cgs absolutely for $|x|>1$.

Sol Here $1+a_n = 1-\left(1-\frac{1}{n}\right)^{-n}x^{-n}$ so that

$$a_n = -\left(1-\frac{1}{n}\right)^{-n}x^{-n}.$$

Now

$$a_{n+1} = -\left(1-\frac{1}{n+1}\right)^{-n-1}\cdot x^{-n-1}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{-\left(1-\frac{1}{n}\right)^{-n}\cdot x^{-n}}{-\left(1-\frac{1}{n+1}\right)^{-n-1}\cdot x^{-n-1}} = \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n}\right)^{n}}\cdot x$$

$\therefore \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{e}{e}|x|$

$= |x| > 1.$

$\therefore$ By ratio test, $\sum |a_n|$ is cgt.

Hence the infinite product cgs absolutely.

→ Show that $\prod_{n=0}^{\infty}(1+x^{2^n})$ cgs $\frac{1}{1-x}$ if $|x|<1$.

Sol Given infinite product is

$$\prod_{n=0}^{\infty}(1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots\cdots(1+x^{2^{n-1}})(1+x^{2^n})\cdots\cdots$$

Let

$$P_n=\prod_{n=0}^{n-1}(1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots\cdots(1+x^{2^{n-1}})$$

$$=\left(\frac{1}{1-x}\right)\left[(1-x)(1+x)(1+x^2)(1+x^{2^2})(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^2)(1+x^2)(1+x^{2^2})\cdots(1+x^{2^{n-1}})\right]$$

$1-(x^4)^2$

$$=\frac{1}{1-x}\left[(1-(x^2)^2)(1+x^{2^2})(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{4(=2^2)})(1+x^4)(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-(x^4)^2)(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{8(=2^3)})(1+x^8)\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-(x^8)^2)(1+x^{2^4})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{16})(1+x^{2^4})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{2^4})(1+x^{2^4})\cdots(1+x^{2^{n-1}})\right]$$

$\cdots\cdots\cdots\cdots$

$$=\frac{1}{1-x}(1-x^{2^n}).$$

Now if $|x| < 1$ (i.e $-1 < x < 1$)

then $x^{2n} \to 0$ as $n \to \infty$.

$\therefore$ $P_n \to \frac{1}{1-x}$ as $n \to \infty$

$\therefore$ The infinite product $\prod_{n=0}^{\infty}(1+x^{2^n})$ cgs to

$\frac{1}{1-x}$.

HW $\rightarrow$ S.T $\prod_{n=0}^{\infty}\left[1+\left(\frac{1}{2}\right)^{2^n}\right]$ cgs to 2

(Hint: put $x = 1/2$).