5.a) Find the Complementary function and particular integral for the Eqn

$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

and hence the general solution of the eqn.

$$(D^2-1)y = xe^x + \cos^2 x$$

$$m^2 - 1 = 0 \Rightarrow m = 1, -1$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{1}{D^2-1}(xe^x + \cos^2 x) = \frac{1}{D^2-1} xe^x + \frac{1}{D^2-1}\left(\frac{1+\cos 2x}{2}\right)$$

$$= e^x \cdot \frac{1}{(D+1)^2-1} x + \frac{1}{2}\frac{1}{D^2-1} \cdot 1 + \frac{1}{2}\frac{1}{D^2-1}\cos 2x$$

$$\left(\because \frac{1}{f(D)} e^{ax} \cdot V = e^x \cdot \frac{1}{f(D+a)} V\right)$$

$$= e^x \cdot \frac{1}{D(D+2)} x + \frac{1}{2}\frac{1}{(D^2-1)} e^{0x} + \frac{1}{2}\frac{1}{(-4-1)}\cos 2x$$

$$= e^x \cdot \frac{1}{D}\frac{1}{2}\left(\frac{D}{2}+1\right)^{-1} x + \frac{1}{2}\frac{e^{0x}}{(0-1)} - \frac{1}{10}\cos 2x$$

$$= e^x/2 \; \frac{1}{D}\left(1 - \frac{D}{2} + \dots\right) x - \frac{1}{2} - \frac{1}{10}\cos 2x$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left(x - \frac{1}{2}\right) - \frac{1}{2} - \frac{1}{10}\cos 2x$$

$$= \frac{e^x}{2}\int\left(x - \frac{1}{2}\right)dx - \frac{1}{2} - \frac{1}{10}\cos 2x$$

$$= \frac{e^x}{2}\left(\frac{x^2}{2} - \frac{x}{2}\right) - \frac{1}{2} - \frac{1}{10}\cos 2x$$

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5.b) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \log x, \quad x>0$

by the method of variation of parameters

$$(D^2 - 2D + 1)y = x \cdot e^x \log x$$

Auxiolary eqn:

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$$y_c = c_1 e^x + c_2 x \cdot e^x$$

$$y_p = u_1 y_1 + u_2 y_2$$

Here, $y_1 = e^x, \quad y_2 = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x \cdot e^x \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix}$$

$$= e^{2x}(1 + \cancel{x} - \cancel{x})$$

$$= e^{2x}$$

$$u_1 = -\int \frac{y_2 r(x)}{W} dx$$

$$= -\int \frac{x \cdot \cancel{e^x} \cdot x\cancel{e^x} \log x}{\cancel{e^{2x}}} dx$$

$$= -\int x^2 \log x \, dx$$

$$= -\int x^2 \log x \, dx$$

$$= -\left[(\log x) \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \right]$$

$$= -\frac{x^3}{3} \log x + \frac{x^3}{9} \quad \text{(by parts)}$$

$$u_2 = \int \frac{y_1 \, r(x)}{W} dx$$

$$= \int \frac{\cancel{e^x} \cdot x \cancel{e^x} \log x}{\cancel{e^{2x}}} dx$$

$$= (\log x) \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4}$$

$$\therefore y_p = e^x \left(-\frac{x^3}{3} \log x + \frac{x^3}{9} \right) + \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right)$$

$\times x \cdot e^x$

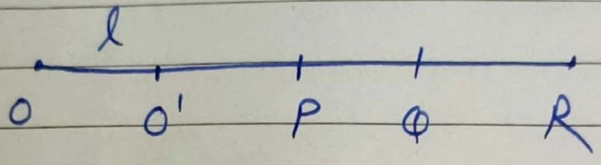
$$= \frac{x^3 \cdot e^x}{9} (1 - 3 \log x) + \frac{x^3 e^x}{4} (2 \log x - 1)$$

$$\therefore y = y_c + y_p$$

$$y = c_1 e^x + c_2 e^x \cdot x + \frac{x^3 e^x}{9} (1 - 3 \log x) + \frac{x^3 e^x}{4} (2 \log x - 1)$$

5.c) If the velocities in a SHM at distances a, b and c from a fixed point on the st line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by,

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$



Let $OO' = l$, $O'P = a$
$O'Q = b$, $O'R = c$

In SHM, $v^2 = \mu(A^2 - x^2)$

for Point P: $v = u$, $x = l + a$

$$\therefore \quad u^2 = \mu[A^2 - (l+a)^2]$$

$$\frac{u^2}{\mu} = A^2 - (l^2 + a^2 + 2al)$$

$$\frac{u^2}{\mu} + a^2 = (A^2 - l^2) - 2al$$

$$\left(\frac{u^2}{\mu} + a^2\right) + 2al + (l^2 - A^2) = 0 \quad \text{— (1)}$$

Similarly, for point at Q and R

$$\left(\frac{v^2}{\mu} + b^2\right) + 2lb + (l^2 - A^2) = 0 \quad \text{— (2)}$$

$$\left(\frac{w^2}{\mu}+c^2\right)+2lc+(l^2-A^2)=0 \quad \text{--- (3)}$$

We eliminate l and (l^2-A^2) from Eqns (1) (2) and (3)

$$\begin{vmatrix} \frac{u^2}{\mu}+a^2 & a & 1 \\ \frac{v^2}{\mu}+b^2 & b & 1 \\ \frac{w^2}{\mu}+c^2 & c & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} u^2/\mu & a & 1 \\ v^2/\mu & b & 1 \\ w^2/\mu & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

$$\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = +\mu(b-c)(c-a)(a-b)$$

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

$$\therefore \frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

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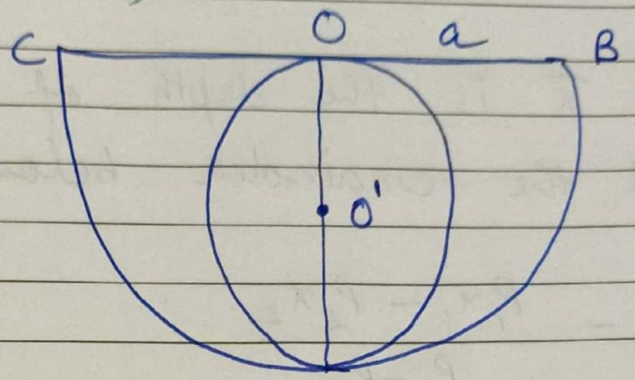
Q. From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Prove that the depth of the Centre of pressure (CP) of the remainder is $\frac{9\pi a}{8(16-3\pi)}$.



If x_1 is the depth of the C.P. of the semi-circle below O, then

$$x_1 = \frac{3\pi}{16} a$$

P_1 = Pressure on the semi-circle

$$= w \cdot \frac{1}{2}\pi a^2 \cdot \frac{4a}{3\pi} = \frac{2}{3} a^3 w.$$

Again depth of C.P. of the circle of radius $a/2$ below the centre O' is $A^2/4H$ where A is its radius $= a/2$

and H is the depth of the centre of the circle below the free surface

$$= OO' = a/2$$

$$\therefore \frac{A^2}{4H} = \frac{(a/2)^2}{4(a/2)} = a/8$$

$\therefore x_2$ = depth of C.P. of circle below O $= \frac{a}{2} + \frac{a}{8} = \frac{5a}{8}$

P_2 = Pressure on the circle

$= w \cdot \pi \left(\frac{a}{2}\right)^2 \cdot \frac{a}{2} = \frac{1}{8} w \pi a^3$

If $\bar{x}$ is the depth of the C.P. of the remainder below O, then

$$\bar{x} = \frac{P_1 x_1 - P_2 x_2}{P_1 - P_2}$$

$$= \frac{3a\pi}{64} \cdot \frac{24}{16-3\pi} = \frac{9\pi a}{8(16-3\pi)}$$

e) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $f(r)$ is differentiable, show that

$$\operatorname{div}\left(f(r)\,\vec{r}\right) = r f'(r) + 3 f(r)$$

Hence or otherwise show that:

$$\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = 0$$

We know that

$$\operatorname{div}(\phi A) = (\operatorname{grad}\phi)\cdot A + \phi \operatorname{div}(A)$$

$$\therefore\ \nabla\cdot\left(f(r)\,\vec{r}\right) = \left[\nabla f(r)\right]\cdot\vec{r} + f(r)\,\nabla\cdot\vec{r}$$

$$= \left[f'(r)\,\nabla r\right]\cdot\vec{r} + f(r)\left[1+1+1\right]$$

$$= \left[f'(r)\,\frac{\vec{r}}{r}\right]\cdot\vec{r} + 3f(r)$$

$$= r f'(r) + 3 f(r) \qquad \left[\because\ \vec{r}\cdot\vec{r} = r^2,\ r = |\vec{r}|\right]$$

Now, taking $f(r) = \frac{1}{r^3}$

$$\nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = r\left(\frac{-3}{r^4}\right) + 3\cdot\frac{1}{r^3}$$

$$= \frac{-3}{r^3} + \frac{3}{r^3} = 0.$$