Q.1(c) Evaluate. $I = \int_0^1 \sqrt[3]{x \cdot \log \frac{1}{x}}\, dx$.

IAS 2016

Sol^n:- let us substitute,

$$\log\left(\frac{1}{x}\right) = t$$

$$\Rightarrow \frac{1}{1/x}\left(\frac{-1}{x^2}\right) dx = dt.$$

$$\Rightarrow \frac{-1}{x} \cdot dx = dt \Rightarrow dx = -e^{-t} dt$$

$$I = \int_0^1 x^{1/3} \cdot \left[\log\left(\tfrac{1}{x}\right)\right]^{\frac{1}{3}} dx.$$

$$= \int_\infty^0 -e^{-t/3} \cdot t^{1/3} \cdot e^{-t}\, dt.$$

$$I = \int_0^\infty e^{-4t/3} \cdot t^{1/3} \cdot dt$$

let, $\frac{4t}{3} = p \Rightarrow \frac{4}{3} dt = dp$

$$= \int_0^\infty e^{-p} \left(\frac{3p}{4}\right)^{1/3} \cdot \frac{3}{4}\, dp.$$

$$= \left(\frac{3}{4}\right)^{4/3} \int_0^\infty e^{-p} \cdot p^{1/3} dp = \left(\frac{3}{4}\right)^{4/3} \int_0^\infty e^{-p} \cdot p^{\frac{4}{3}-1} dp.$$

we know, $\int_0^\infty t^{x-1} \cdot e^{-t} dt = \Gamma x$

so, $I = \left(\frac{3}{4}\right)^{4/3} \Gamma 4/3$.

3-a) Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$, & $x+y-z=0$.

20 Marks

Let's consider a function

$$f = x^2+y^2+z^2 + \lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right) + \mu(x+y-z).$$

∴ The stationary points are found out by $df=0$

$\Rightarrow f_x dx + f_y dy + f_z dz = 0$

$\Rightarrow \underline{f_x=0}, \underline{f_y=0}, \underline{f_z=0}.$

$$2x + \frac{2x\lambda}{4} + \mu = 0$$
$$2y + \frac{2y\lambda}{5} + \mu = 0$$
$$2z + \frac{2z\lambda}{25} - \mu = 0$$

$$2x^2 + \frac{2x^2\lambda}{4} + \mu x = 0$$
$$2y^2 + \frac{2y^2}{5}\lambda + \mu y = 0$$
$$2z^2 + \frac{2z^2}{25}\lambda - \mu z = 0$$

adding we have $2(x^2+y^2+z^2) + 2\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}\right) + \mu(x+y-z) = 0$

$\Rightarrow 2r^2 + 2\lambda\times 1 + \mu\times 0 = 0$

$\Rightarrow \boxed{\lambda = -r^2}$ $\Rightarrow \boxed{x^2+y^2+z^2 = -\lambda}$

$$\boxed{2x\left(1+\frac{\lambda}{4}\right) = -\mu}$$
$$\boxed{2y\left(1+\frac{\lambda}{5}\right) = -\mu}$$
$$\boxed{2z\left(1+\frac{\lambda}{25}\right) = \mu}$$

$2(x-y) + 2\lambda\left(\frac{x}{4}-\frac{y}{5}\right) = 0$

$\Rightarrow \lambda = \frac{x-y}{\frac{y}{5}-\frac{x}{4}}$

$2x = \frac{-\mu}{1+\frac{\lambda}{4}}$, $2y = \frac{-\mu}{1+\frac{\lambda}{5}}$, $2z = \frac{\mu}{1+\frac{\lambda}{25}}$

$\Rightarrow x+y-z = 0 \Rightarrow \frac{1}{2}\left(\frac{-\mu}{1+\frac{\lambda}{4}} - \frac{\mu}{1+\frac{\lambda}{5}} - \frac{\mu}{1+\frac{\lambda}{25}}\right) = 0 \Rightarrow \mu

$$\frac{4}{4+\lambda}+\frac{5}{5+\lambda}+\frac{25}{25+\lambda}=0 \Rightarrow 4(25+\lambda)(5+\lambda)+5(4+\lambda)(25+\lambda)$$
$$+25(4+\lambda)(5+\lambda)=0$$

$$\Rightarrow 4(125+30\lambda+\lambda^2)+5(100+29\lambda+\lambda^2)+25(20+\lambda^2+9\lambda)=0$$

$$\Rightarrow 34\lambda^2+490\lambda+1500=0$$

$$\Rightarrow \lambda=\frac{-490\pm\sqrt{490^2-4\times34\times1500}}{2\times34}=\frac{-490\pm\sqrt{36100}}{68}$$

$$=\frac{-490\pm190}{68}=-10 \text{ or } \frac{-75}{17}$$

Now putting this value on the

$$\boxed{\frac{\mu}{1+\frac{\lambda}{4}}=-2x \qquad \frac{\mu}{1+\frac{\lambda}{5}}=-2y,\quad \frac{\mu}{1+\frac{\lambda}{25}}=+2z}$$

for $\lambda=-10$

~~$\mu = 2f$~~

$$x=\frac{-\mu}{2(1+\frac{\lambda}{4})}=\frac{-\mu}{2(1-\frac{10}{4})}=\frac{-\mu\times\cancel{2}}{-\cancel{6}}=\frac{\mu}{3}.$$

$$y=-\frac{1}{2}\frac{\mu}{1-\frac{10}{5}}=-\frac{1}{2}\times\frac{\mu\times5}{-5}=\frac{\mu}{2}$$

$$z=+\frac{1}{2}\frac{\mu}{1-\frac{10}{25}}=\frac{\mu}{2}\times\frac{\cancel{25}\,5}{\cancel{15}\,3}=\frac{5\mu}{6}$$

$$\therefore \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$$

$$\Rightarrow \frac{\mu^2}{9\times4}+\frac{\mu^2}{4\times5}+\frac{\cancel{25}\mu^2}{36\times\cancel{25}}=1$$

$$\Rightarrow \mu^2=\frac{180}{19}\Rightarrow \mu=\pm\sqrt{\frac{180}{19}}$$

1.05263 +2.3684 +6.5789 =10.

$\therefore$ The 4 stationary points are $\left(\frac{1}{3}\sqrt{\frac{180}{19}},\ \frac{1}{2}\sqrt{\frac{180}{19}},\ \frac{5}{6}\sqrt{\frac{180}{19}}\right)$

& $\left(-\frac{1}{3}\sqrt{\frac{180}{19}},\ -\frac{1}{2}\sqrt{\frac{180}{19}},\ -\frac{5}{6}\sqrt{\frac{180}{19}}\right)$

Similarly for $\lambda = \frac{-75}{17}$

$$x = \frac{-\mu}{2(1+\frac{\lambda}{4})} = \frac{-\mu}{2\left(1-\frac{75}{64}\right)} = \frac{-\mu}{2\left(\frac{-11}{64}\right)} = \frac{32\mu}{11}$$

$$y = \frac{-\mu}{2\left(1+\frac{\lambda}{5}\right)} = \frac{-\mu}{2\left(1-\frac{75}{5\times17}\right)} = \frac{-\mu\times 17}{2\times 10 \cdot 2}$$

$$\boxed{y = \frac{-\mu 17}{4}}$$

$$z = \frac{\mu}{2(1+\frac{\lambda}{25})} = \frac{\mu}{2\left(1-\frac{753}{17\times25}\right)} = \frac{\mu\times17}{2\times14}$$

$$\therefore \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25} = 1$$

$$\Rightarrow \frac{32\mu^2}{11^2\times4} + \frac{17^2\mu^2}{4^2\times5} + \frac{17^2\mu^2}{28^2\times25} = 1$$

$$\Rightarrow \mu = 0.5203$$

$$\therefore x = \frac{32}{11}\times0.52, \quad y = \frac{-17}{4}\times0.5203, \quad z = \frac{17}{28}\times0.52$$

$$\therefore x^2+y^2+z^2 = 1.5136^2 + 2.211275^2 + 0.3157^2$$

for $\lambda = \frac{-75}{17}$ $= \underline{\underline{7.28}}$

& $\underline{\underline{\lambda = -10}}$, $x^2+y^2+z^2 = \underline{\underline{10}}$

So the max^m value of the $x^2+y^2+z^2=10$ & min^n = $\underline{\underline{\frac{75}{17}}}$

3-b) Let $f(x,y) = \begin{cases} \dfrac{2x^4y - 5x^2y^2 + y^5}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Find a $\delta > 0$, such that $|f(x,y) - f(0,0)| < 0.01$ whenever $\sqrt{x^2+y^2} < \delta$.

[15 Marks]

Ans

$$f(x,y) = \frac{2x^4y - 5x^2y^2 + y^5}{(x^2+y^2)^2}.$$

$$|f(x,y) - f(0,0)| = \left|\frac{2x^4y - 5x^2y^2 + y^5}{(x^2+y^2)^2}\right|$$

$$= \left|\frac{y(2x^4+y^4) - 5x^2y^2}{(x^2+y^2)^2}\right| \leq \left|\frac{y(2x^4+y^4)}{(x^2+y^2)^2}\right|.$$

$$= \left|\frac{y\left((\sqrt{2}x^2+y^2)^2 - 2\sqrt{2}x^2y^2\right) - 5x^2y^2}{(x^2+y^2)^2}\right|$$

$$= \left|\frac{y(\sqrt{2}x^2+y^2)^2 - 2\sqrt{2}x^2y^3 - 5x^2y^2}{(x^2+y^2)^2}\right|$$

$$\leq \frac{y(\sqrt{2}x^2+y^2)^2}{(x^2+y^2)^2} \leq y\frac{(x^2+y^2)^2}{(x^2+y^2)^2} \leq y.$$

$x^4y + y^5$

$+x^4y$

$y(x^4+y^4) + x^4y$

$y(x^4+y^4+2x^2y^2$

$+x^4$

$-2x^2$

If $y < 0$ & $y > 0$

$$\leq \left|\frac{y(2x^4+y^4)}{(x^2+y^2)^2}\right| = \left|\frac{y\left[(\sqrt{2}x^2+y^2)^2 - 2\sqrt{2}x^2y^2\right]}{(x^2+y^2)^2}\right|.$$

$$\leq \left|\frac{y\left[(\sqrt{2}x^2+y^2)^2 - 2\sqrt{2}x^2y^2\right]}{(x^2+y^2)^2}\right|$$

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$$|f(x,y)| = \frac{|2x^4y + y^5 - 5x^2y^2|}{(x^2+y^2)^2}$$ Since the power is $\left(\frac{5}{2}\right)$

CONCEPT

$$= (x^2+y^2)^{5/2} = \frac{\left|y^5\left(1+\frac{2x^4}{y^4}-\frac{5x^2}{y^3}\right)\right|}{(x^2+y^2)^2}$$

$$\leq \frac{\left|y^5\left(1+\frac{2x^4}{y^4}\right)\right|}{(x^2+y^2)^2} \leq \frac{\left|y^5\left(1+\frac{5}{2}\frac{x^2}{y^2}+\frac{25}{4}\frac{x^4}{y^4}\right)\right|}{(x^2+y^2)^2}$$

$$\leq \frac{y^5\left(1+\frac{x^2}{y^2}\right)^{5/2}}{(x^2+y^2)^2} = \sqrt{(x^2+y^2)} < \epsilon = 0.01 = \underline{\underline{\delta}}.$$

3(c) 2014

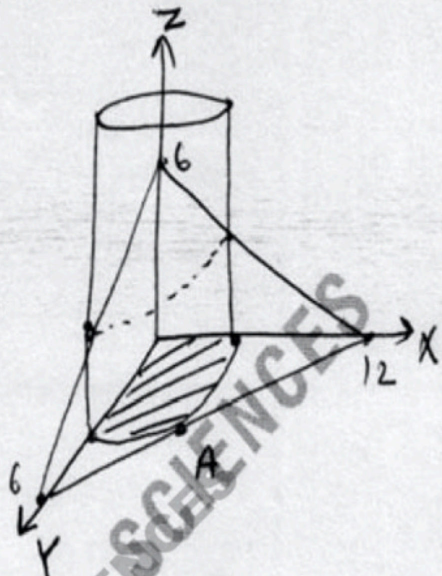
Find the surface area of the plane $x+2y+2z=12$ cut off by $x=0$, $y=0$ and $x^2+y^2=16$.

Ans.

Plane $x+2y+2z=12$

or $\frac{x}{12}+\frac{y}{6}+\frac{z}{6}=1$

cuts the coordinates at a distance of 12, 6 and 6 from origin.



Cylinder : $x^2+y^2=16$

Planes : $x=0$, $y=0$

$$\text{Surface Area} = \iint_A \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$= \iint_A \sqrt{1+\left(-\tfrac{1}{2}\right)^2+(-1)^2}\,dx\,dy$$

$\left(\because z = -\frac{x}{2} - y + 6\right.$

$\left. z_x = -\frac{1}{2},\ z_y = -1\right)$

$$= \frac{3}{2}\iint_A dx\,dy$$

$$= \frac{3}{2}\cdot\left[\frac{1}{4}\pi(4)^2\right]$$

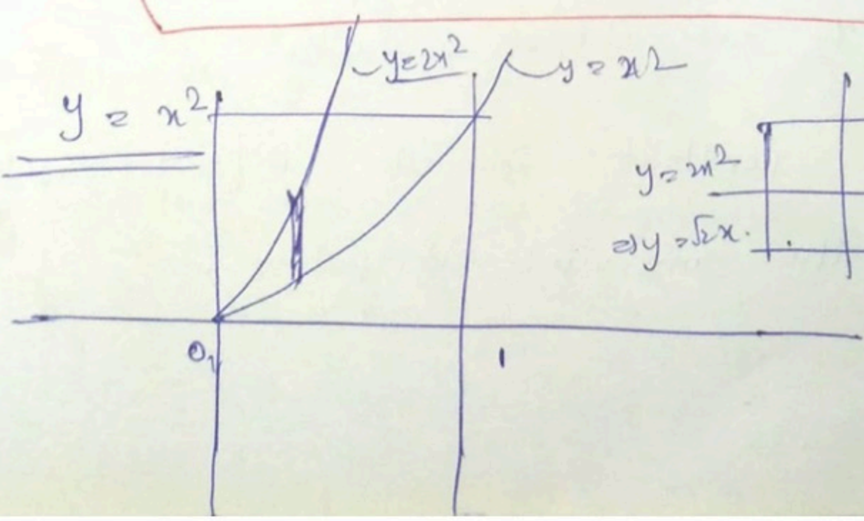
$$= 6\pi$$

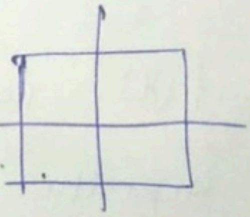
(A : Projection of surface on xy-plane
$x^2+y^2 \le 16$
$x \geq 0, y \geq 0$)

4(c) Evaluate $\iint_R f(x,y)\,dx\,dy$ over the rectangle $R = [0,1;0,1]$

where $f(x,y) = \begin{cases} x+y, & x^2 < y < 2x^2 \\ 0, & \text{elsewhere.} \end{cases}$

Ans// $y = x^2$





Area of the region:

$$\therefore \iint_R f(x,y)\,dx\,dy = \int_{x=0}^{1}\int_{x^2}^{2x^2} (x+y)\,dx\,dy = \int_0^1 yx + \frac{y^2}{2}\Big|_{x^2}^{2x^2} dx$$

$$= \int_0^1 \left[x(x^2) + \frac{1}{2}\left(4x^4 - x^4\right)\right] dx$$

$$= \frac{x^4}{4} + \frac{3}{2}\frac{x^5}{5}\Big|_0^1 = \frac{1}{4} + \frac{3}{10} = \frac{5+6}{20} = \frac{11}{20} \text{ (ans)}$$

2017:

IFoS - 2016 :

~~1-(a) prove that all bijective function from a non-empty set X onto itself is a a~~

1-(b) Show that $\frac{x}{1+x} < \log(1+x) < x \quad \forall x > 0$.

Ans Lets consider $f(x) = \frac{x}{1+x} - \log(1+x)$

$\Rightarrow f'(x) = \frac{(1+x) - x}{(1+x)^2} - \frac{1}{1+x} = \frac{1}{(1+x)^2} - \frac{1}{(1+x)} = \frac{\cancel{1} - \cancel{1} - x}{(1+x)^2}$.

$\Rightarrow f(x)$ is decreasing function

$\Rightarrow \underline{f(x) < f(0)}$ for $\underline{x > 0}$

$\Rightarrow \frac{x}{1+x} - \log(1+x) < 0 \Rightarrow \underline{\frac{x}{1+x} < \log(1+x)}$. — ①

Let $g(x) = x - \log(1+x)$

$\Rightarrow g'(x) = 1 - \frac{1}{1+x} > 0 \quad \forall \underline{x > 0}$

$\Rightarrow g(x)$ is increasing function.

$\Rightarrow g(x) > g(0)$, for $x > 0$

$\Rightarrow x - \log(1+x) > 0 \Rightarrow \underline{\log(1+x) < x}$ — ②

from 1 & 2 $\Rightarrow \boxed{\frac{x}{1+x} < \log(1+x) < x.}$ Ans.

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(c) Examine if the function $f(x,y) = \dfrac{xy}{x^2+y^2}$, $(x,y) \neq (0,0)$ and $f(0,0) = 0$ is continuous at $(0,0)$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin.

Sol

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Now Let's take $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore f(r,\theta) = \frac{r^2 \sin\theta\cos\theta}{r^2} = \frac{1}{2}\sin 2\theta$$

$$\therefore |f(x,y) - f(0,0)| = \left|\frac{\sin 2\theta}{2}\right| \leq \frac{1}{2}$$

Nilgiri
Special
4 - N.P.
3 → WLS
1

50 @ 100% 80 Chand Bali
30 @ 100%
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20 → 10 50:50

But if we take $\epsilon = \dfrac{1}{4}$ then $|f(r,\theta) - f(0,0)| > \epsilon$.

Hence the function is not continuous.

Now
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} (0) = 0.$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

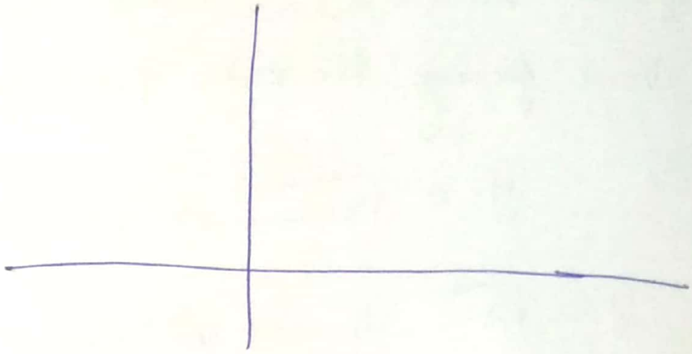
So the partial derivative exists & is 0 for points (x,y

2-(a) After changing the order of integration of

$$\int_0^\infty \int_0^\infty e^{-xy} \sin(nx)\, dx\, dy \text{ , show that } \int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}.$$

Ans ||

$$I = \int_{y=0}^{\infty} \int_{x=0}^{x=\infty} e^{-xy} \sin(nx)\, dx\, dy$$



$$= \int_0^\infty \sin(nx) \left(-\frac{1}{x}\right) e^{-xy} \Big|_{y=0}^{\infty} dx$$

$$= -\int_0^\infty \frac{\sin nx}{x}\, e^{-xy} \Big|_{y=0}^{y=\infty} dx$$

$$= -\int_0^\infty \frac{\sin nx}{x} (0-1)\, dx.$$

$$= \int_0^\infty \frac{\sin nx}{x}\, dx. \quad \text{—①}.$$

Now first integrating wrt x:

$$\int_0^\infty \left\{ -\frac{1}{y} e^{-xy} \sin(nx) \Big|_{x=0}^{\infty} + \int_0^\infty \frac{1}{y} e^{-xy}\, n\cos(nx)\, dx \right\} dy.$$

$$= \int_0^\infty \left\{ \frac{n}{y} \left(0 - \frac{1}{y} e^{-xy} \cos nx \Big|_{x=0}^{\infty} - \int_0^\infty \frac{e^{-xy}}{y}\, n \sin nx \right) \right\} dy$$

$$= \int_0^\infty \left\{ \frac{n}{y} \left(0 + \frac{1}{y} \right) - \frac{n}{y} I' \right\} dy.$$

$$\Rightarrow \frac{n}{y^2} - \frac{n^2}{y^2} I' = I' \Rightarrow \left(1 + \frac{n^2}{y^2}\right) I = \frac{n}{y^2} \Rightarrow I = \frac{n}{n^2+y^2}$$

$$= \int_0^\infty \frac{n}{n^2+y^2}\,dy = \frac{1}{n} n \tan^{-1}\left(\frac{y}{n}\right)\Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore I = \int_0^\infty \frac{\sin nx}{x}\,dx = \frac{\pi}{2} \text{ (Ans).}$$

Q-c) Using MVT, find a point on the curve $y = \sqrt{x-2}$, defined on $[2,3]$, where the tangent is parallel to the chord joining the end points of the curve.

Ans//

$$y = \sqrt{x-2} \Rightarrow y^2 = x-2 \Rightarrow x = y^2+2$$

$y = \sqrt{x-2}$ is $\Rightarrow$ continuous and differentiable in $[2,3]$.

Hence by Lagrange's MVT:

$$f'(c) = \frac{f(3) - f(2)}{3-2} \quad \text{where } c \in (2,3)$$

$$= \frac{1}{1} \Rightarrow f'(c) = 1$$

$f'(x) = \frac{1}{2\sqrt{x-2}} \Rightarrow f'(c) = 1 \Rightarrow \frac{1}{2\sqrt{c-2}} = 1 \Rightarrow \sqrt{c-2} = \frac{1}{2} \Rightarrow c-2 = \frac{1}{4}$

$\Rightarrow c = 2+\frac{1}{4} = 2.25 = \frac{9}{4} \in (2,3)$

Hence at $x = \frac{9}{4}$, the tangent is parallel to the chord joining the end points of the curve (Ans)

Q-3b) Using Lagrange's Multipliers, find the point on the plane $2x+3y+4z = 5$, which is closest to the point $(1,0,0)$.

Ans//
Let $P(x\,y\,z)$ be any point on the plane $2x+3y+4z=5$.

Such that $f(x\,y\,z) = (x-1)^2+y^2+z^2$ is the minimum.

Now let's consider the function

$F(x,y,z) = (x-1)^2 + y^2 + z^2 + \lambda(2x + 3y + 4z - 5)$.

$\because dF = 0 \Rightarrow F_x dx + F_y dy + F_z dz = 0$

$\Rightarrow F_x = 0 \Rightarrow \cancel{2}(x-1) + \cancel{2}\lambda = 0 \Rightarrow \boxed{x = 1-\lambda}$

$F_y = 0 \Rightarrow 2y + 3\lambda = 0 \Rightarrow \boxed{y = \frac{-3\lambda}{2}}$

$F_z = 0 \Rightarrow 2z + 4\lambda = 0 \Rightarrow \boxed{z = -2\lambda}$

$\therefore 2(1-\lambda) + 3\left(\frac{-3\lambda}{2}\right) + 4(-2\lambda) = 5$

$\Rightarrow 2 - 2\lambda - \frac{9\lambda}{2} - 8\lambda = 5$

$\Rightarrow -10\lambda - \frac{9\lambda}{2} = 3 \Rightarrow \boxed{\lambda = \frac{-6}{29}}$.

$\therefore x = 1 + \frac{6}{29} = \frac{35}{29}$.

$y = -\frac{3}{\cancel{2}} \times \frac{-\cancel{6}^{3}}{29} = \frac{9}{29}$

$z = -2 \times \frac{-6}{29} = \frac{12}{29}$

which is the reqd stationary points.

$d^2F = F_{xx}(dx)^2 + F_{yy}(dy)^2 + F_{zz}(dz)^2 + F_{xy}\,dx\,dy + F_{xz}\,dx\,dz$

$F_{yx}\,dx\,dy + F_{yz}\,dy\,dz + F_{zx}\,dx\,dz + F_{zy}\,dy\,dz$

> 0

Hence the function $F(x,y,z)$ is minimum at $\left(\frac{35}{29}, \frac{9}{29}, \frac{12}{29}\right)$

which is the required point. (Ans)

?

3-c) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$ CHECK

$x = r\cos\theta = 3(1 + \cos^2\theta)$

$y = r\sin\theta = 3(\tan\theta + \sin\theta\cos\theta)$.

$\Rightarrow \frac{x}{y} = \frac{\cos\theta}{\sin\theta} = \cot\theta$.

$\Rightarrow \boxed{y = x\tan\theta}$

$\frac{x}{3} = 1+\cos 2\theta$ $\qquad$ $\frac{y}{3} = \frac{\sin\theta}{\cos\theta}(1+\cos 2\theta)$.

$r = \frac{3(1+\cos 2\theta)}{\cos\theta}$.

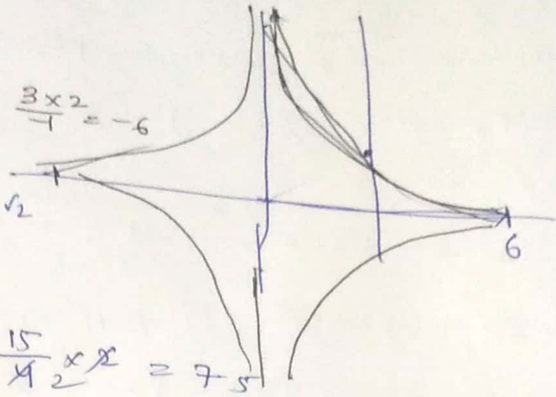
$\theta = 0 \Rightarrow r = 3\times 2 = 6$, $\theta = \pi \Rightarrow r = \frac{3\times 2}{-1} = -6$

$\theta = \frac{\pi}{2} \Rightarrow r = \infty$ $\qquad$ $\theta = \frac{3\pi}{2}$ =

$\theta = \frac{\pi}{4} \Rightarrow r = \frac{3(1+\frac{1}{2})}{\frac{1}{\sqrt{2}}} = \frac{9}{2}\times\sqrt{2} = \frac{9}{\sqrt{2}}$

$x = r\cos\theta = 3$

$\Rightarrow \theta = \frac{\pi}{3} = \frac{3(1+\frac{1}{4})}{\frac{1}{2}} = \frac{15}{4_2}\times 2 = 7.5$



$$\boxed{A = 4\times\int_{\theta=0}^{\pi/2}\int_{r=3\sec\theta}^{3(\sec\theta+\cos\theta)} r\,dr\,d\theta} = \int_0^{\pi/2} \frac{r^2}{2}\Big|_{3\sec\theta}^{3(\sec\theta+\cos\theta)} d\theta$$

asymptote $\Rightarrow \theta = \frac{\pi}{2}$

$$= \frac{1}{2}9\int_0^{\pi/2}\left\{(\sec\theta+\cos\theta)^2 - \sec^2\theta\right\}d\theta$$

$$= \frac{9}{2}\int_0^{\pi/2}(\cancel{\sec^2\theta} + \cos^2\theta + 2 - \cancel{\sec^2\theta})\,d\theta$$

$$= \frac{9}{2}\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = \frac{9}{2}\times\left(\frac{\pi}{2} + \frac{1}{2}\times\frac{\pi}{2}\right) = \frac{3\pi}{4}\times\frac{9}{2} = \frac{27\pi}{8}$$

So total area $= 4\times\frac{27\pi}{8} = \frac{27\pi}{2}$ (Ans).

4-b) Show that the integral $\int_0^\infty e^{-x}x^{(\alpha-1)}dx$, $\alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$, and $0<\alpha<1$.

Ans) ~~for $\alpha \geq 1$,~~ ~~$\int_0^\infty e^{-x}x^{\alpha-1}dx$ is an improper integral~~

~~Let $f(x) = e^{-x}x^{\alpha-1}$~~

Let $I = \int_0^\infty e^{-x} x^{\alpha-1} dx = \int_0^1 e^{-x} x^{\alpha-1} dx + \int_1^\infty e^{-x} x^{\alpha-1} dx.$

(I_1) (I_2)

for $\alpha \geq 1$: I_1 is a proper integral.

& I_2 is an improper integral.

$\therefore \int_1^\infty e^{-x} x^{\alpha-1} dx$ Let $f(x) = x^{\alpha-1} e^{-x}$.

Let's take $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\alpha-1}e^{-x}}{\frac{1}{x^2}} = \lim_{x\to\infty} x^{\alpha+1} e^{-x} = \lim_{x\to\infty} \frac{x^{\alpha+1}}{e^x} \left(\frac{\infty}{\infty}\right)$

$= \frac{(\alpha+1)x^\alpha}{e^x} \left(\frac{\infty}{\infty}\right) --- = \frac{(\alpha+1)!}{e^x} = \underline{\underline{0}}$.

Hence since $\int_1^\infty \frac{1}{x^2} dx$ exists $\Rightarrow \int_1^\infty \frac{dx}{e^x} x^{\alpha-1}$ is convergent & exists

for $\alpha \geq 1$.

for $0 < \alpha < 1$:

$I = \int_0^1 e^{-x} x^{\alpha-1} dx + \int_1^\infty e^{-x} x^{\alpha-1} dx$

(I_1) (I_2)

I_1 is an improper integral & I_2 is also an improper integral

& point of non-convergence $\Rightarrow \underline{x=0}$

$I_1 = \int_0^1 e^{-x} x^{\alpha-1} dx$ Let $f(x) = \frac{e^{-x}}{x^{1-\alpha}}$

Let's do the comparison with $g(x) = \frac{1}{x^{\frac{1}{2}}}$ where $\int_0^1 \frac{1}{x^{\mu}} dx$ is convergent $0<\mu<1$

$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha}} \times x^{\mu}. = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha-\mu}} = \underline{\underline{0}}$.

So the integral is convergent.

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For I_2: $\int_1^\infty e^{-x} x^{\alpha-1} dx$, $0<\alpha<1$

Let's take $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha}} x^2 = \frac{e^{-x}}{x^{1-\alpha-2}} = x^{(1+\alpha)} e^{-x} = \frac{x^{1+\alpha}}{e^x} \left(\frac{\infty}{\infty}\right)$

$= \frac{(1+\alpha) x^\alpha}{e^x} = 0$ as $x\to\infty$.

Hence since $\int_1^\infty \frac{dx}{x^2}$ is convergent, by comparison we have $\int_1^\infty e^{-x} x^{\alpha-1} dx$ is also convergent.

Hence the integral $\int_0^\infty e^{-x} x^{\alpha-1} dx$ exists for $\alpha>0$.

4-c) Prove that $\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\, \Gamma(n+\tfrac{1}{2})$

$\Gamma(n) = (n-1)!$

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(n+\tfrac{1}{2}) = \Gamma\{1+(n-\tfrac{1}{2})\} = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2}) = (n-\tfrac{1}{2})(n-\tfrac{3}{2})\,\Gamma(n-\tfrac{3}{2})$

$= \cdots = (n-\tfrac{1}{2})(n-\tfrac{3}{2})(n-\tfrac{5}{2}) \cdots \left(n-\frac{2n-1}{2}\right)\Gamma(\tfrac{1}{2})$.

$\therefore \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n)\, \Gamma(n+\tfrac{1}{2})$

$\Gamma(2n) = (2n-1)!$

$= \frac{2^{2n-1}}{\sqrt{\pi}} \times (n-1)! \times \frac{\sqrt{\pi}\ \tfrac{1}{2}\times\tfrac{3}{2}\times\tfrac{5}{2}\times \cdots (2n-5)(2n-3)(2n-1)}{2^n}$

$= 2^{2n-1-n} (n-1)! \,(1\times3\times5\times\cdots(2n-1))$

$= 2^{n-1} (n-1)!\ 1\cdot\times3\times5\times\cdots(2n-1)$

$= (2n-1)!\ \ 2^{n-1}$

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We know that $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m>0, n>0$

So let's take $m=n$. $\;m=n \Rightarrow 2n$

$$\therefore B(n,n) = \frac{(\Gamma(n))^2}{\Gamma(2n)} = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \int_0^1 x^{n-1}(1-x)^{n-1}dx$$

$$\boxed{\Rightarrow \frac{1}{2}\,\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta}$$

$$\frac{(\Gamma(n))^2}{\Gamma(2n)} = \int_0^1 x^{n-1}(1-x)^{n-1}dx$$

Let $x = \sin^2\theta$
$\Rightarrow dx = \sin 2\theta \, d\theta$

$$\Rightarrow \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \int_0^{\pi/2} 2\sin^{2n-2}\cos^{2n-2}\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2n-1}d\theta.$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}d\theta.$$

$$= \frac{2}{2^{2n-1}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}d\theta$$

Let $2\theta = \alpha$
$\Rightarrow 2d\theta = d\alpha$

$$\Rightarrow B(n,n) = \frac{1}{2^{2n-2}}\int_0^{\pi}(\sin\alpha)^{2n-1}\frac{d\alpha}{2}$$

$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

$$= \frac{1}{2^{2n-1}}\int_0^{\pi}\sin^{2n-1}\alpha\,d\alpha. = \frac{2}{2^{2n-1}}\int_0^{\pi/2}\sin^{2n-1}\alpha\,d\alpha$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}\alpha\cos^0\alpha\,d\alpha = \frac{1}{2^{2n-2}}\,\frac{\Gamma(n)\cdot\Gamma(\frac{1}{2})}{2\,\Gamma(n+\frac{1}{2})}$$

$2m-1 = 0 \Rightarrow m = \frac{1}{2}$

$$\Rightarrow \frac{\Gamma(n)\,\cancel{\Gamma(n)}}{\Gamma(2n)} = \frac{1}{2\cdot 2^{2n-2}}\,\frac{\sqrt{\pi}\,\cancel{\Gamma(n)}}{\Gamma(n+\frac{1}{2})} \Rightarrow \boxed{\Gamma(2n) = \Gamma(n)\,\Gamma(n+\tfrac{1}{2})\,\frac{2^{2n-1}}{\sqrt{\pi}}}$$

Proved