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6(a) Solve by method of variation of parameters

$$x''(t) - \frac{2x(t)}{t^2} = t, \quad \text{where } 0 < t < \infty$$

(15)

$$\left[D^2 - \frac{2}{t^2}\right] x(t) = t$$

i.e. $[t^2 D^2 - 2]\, x(t) = t^3$

Put $t = e^u \quad \therefore u = \log t$

$$D' = \frac{d}{du} = tD \quad ; \quad D'(D'-1) = t^2 D^2$$

$$\therefore \left[D'(D'-1) - 2\right] x = e^{3u}$$

$$(D'^2 - D' - 2)\, x = e^{3u} \quad \text{--- (1)}$$

Auxiliary Eqn: $D'^2 - D' - 2 = 0$

$$(D'-2)(D'+1) = 0$$

$$D' = 2, -1$$

$$C.F. = c_1' e^{2u} + c_2' e^{-u}$$

Now, we use the variation of parameters to find complete integral of D.E. (1)

Replacing c_1', c_2' by unknown functions A and B, the complete solution is

$$y = A e^{2u} + B e^{-u}$$

$$= A y_1 + B y_2$$

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where, $y_1 = e^{2u}$, $y_2 = e^{-u}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2u} & e^{-u} \\ 2e^{2u} & -e^{-u} \end{vmatrix}$$

$$= -e^{u} - 2e^{u} = -3e^{u} \neq 0$$

$\therefore$ y_1 & y_2 are independent.

$$A = -\int \frac{y_2 R}{W} du \quad , \quad R = e^{3u}$$

$$= -\int \frac{e^{-u} \cdot e^{3u}}{-3e^{u}} du$$

$$= \frac{1}{3} \int e^{u} du = \frac{e^{u}}{3} + c_1$$

$$B = \int \frac{y_1 R}{W} du$$

$$= \int \frac{e^{2u} \cdot e^{3u}}{-3e^{u}} du$$

$$= -\frac{1}{3} \int e^{4u} du = \frac{-e^{4u}}{12} + c_2$$

$\therefore$ Complete Solution,

$$y = Ay_1 + By_2$$

$$= \left(\frac{e^{u}}{3} + c_1\right) e^{2u} + \left(\frac{-e^{4u}}{12} + c_2\right) e^{-u}$$

or $$y = \left(\frac{t}{3} + c_1\right) t^2 + \left(\frac{-t^4}{12} + c_2\right) \frac{1}{t}$$

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6(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$

(the pole being the centre of the force).

(15)

$r^2 = a^2 \cos 2\theta$ or $a^2 u^2 \cos 2\theta = 1$, $u = \frac{1}{r}$ — ①

Taking log,

$$2 \log a + 2 \log u + \log \cos 2\theta = 0$$

Differentiating w.r.t. θ

$$0 + \frac{2}{u} \cdot \frac{du}{d\theta} - \frac{2 \sin 2\theta}{\cos 2\theta} = 0$$

$$\frac{du}{d\theta} = u \tan 2\theta$$

$$\frac{d^2u}{d\theta^2} = 2u \sec^2 2\theta + \frac{du}{d\theta} \tan 2\theta$$

$$= 2u \sec^2 2\theta + u \tan^2 2\theta$$

$$\therefore \frac{d^2u}{d\theta^2} + u = 2u \sec^2 2\theta + u \tan^2 2\theta + u$$

$$= 3u \sec^2 2\theta = 3u (a^2 u^2)^2 \quad \text{— from ①}$$

$$= 3a^4 u^5$$

WKT DE of the central orbit in polar form is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2}$$

$$\therefore \frac{F}{h^2 u^2} = 3a^4 u^5 \Rightarrow F = 3h^2 a^4 u^7$$

$$= K \cdot \frac{1}{r^7} \quad \text{using ①.}$$

i.e. $\boxed{F \propto \frac{1}{r^7}}$

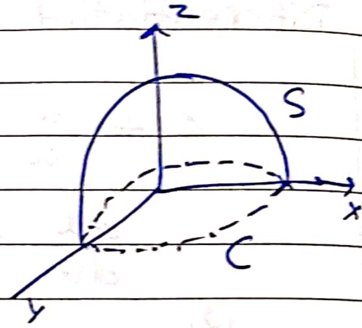
Hence the force varies inversely as the 7th power of the distance from the pole.

6(c) Verify Stokes' theorem for
$\vec{V} = (2x-y)i - yz^2 j - y^2 z k$,

where S is the upper half of the surface of sphere $x^2+y^2+z^2=1$ and C is its boundary. (10)

Stoke's Theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$$



Here, the boundary C of S is a circle in xy plane
$x^2+y^2=1$

Let $x=\cos t$, $y=\sin t$, $z=0$
$0\le t\le 2\pi$ be parametric Eqn of C.

$$\oint_C \vec{V}\cdot d\vec{r} = \oint_C [(2x-y)i - yz^2 j - y^2 z k]\cdot[dx\, i + dy\, j + dz\, k]$$

$$= \oint_C (2x-y)\,dx - yz^2\,dy - y^2 z\,dz$$

$$= \oint_C (2x-y)\,dx \quad [\because z=0 \ \& \ dz=0]$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt$$

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$$= -\int_0^{2\pi} \sin 2t - \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= -\left[\frac{-\cos 2t}{2} - \frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi}$$

$$= -\left[\left(\frac{-1}{2} - \frac{2\pi}{2} + 0\right) - \left(\frac{-1}{2} - 0 + 0\right)\right]$$

$$= \pi$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= i(-2yz + 2yz) + j(0-0) + k(0+1) \quad \text{--- } (1)$$

$$= k$$

Normal to surface S: $\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2+4y^2+4z^2}}$

$\hat{n} = xi + yj + zk$ $\quad [S: x^2+y^2+z^2-1=0]$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_D k \cdot (xi + yj + zk) \frac{dx\,dy}{z}$$

$$= \iint_D dx\,dy$$

[D: $x^2 + y^2 \le 1$
unit circle in xy plane
centred at origin]

= Area of circle D

$$= \pi(1)^2 = \pi \quad \text{--- } (2)$$

from (1) and (2) we see that

$$\oint_C \vec{V} \cdot dr = \iint_S (\nabla \times \vec{V}) \cdot \hat{n}\, dS$$