IFS MATHEMATICS MAIN EXAM., 2013

PAPER-I

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols and notations carry their usual meaning.*

Section-A

1. (*a*) Find the dimension and a basis of solution space W of the system

$x + 2y + 2z - s + 3t = 0,$

$x + 2y + 3z + s + t = 0,$

$3x + 6y + 8z + s + 5t = 0.$ **8**

(*b*) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. **8**

(*c*) Evaluate the integral $\int_0^\infty \int_0^x xe^{-x^2/y} dydx$ by changing the order of integration. **8**

(*d*) Find the surface generated by the straight line which intersects the lines $y = z = a$ and $x + 3z = a = y + z$ and is parallel to the plane $x + y = 0$. **8**

(*e*) Find C of the Mean value theorem, if $f(x) = x(x - 1)(x - 2)$, $a = 0$, $b = \frac{1}{2}$ and C has usual meaning. **8**

2. (*a*) Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$.

Let $F : V \to V$ be the linear map defined by $F(A) = MA$. Find a basis and the dimension of

(*i*) the kernel of W of F

(*ii*) the image U of F. **10**

(*b*) Locate the stationary points of the function $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature. **10**

(*c*) Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form

$q(x, y) = 2x^2 - 4xy + 5y^2$. **10**

(*d*) Discuss the consistency and the solutions of the equations

$x + ay + az = 1,\ ax + y + 2az = -4,$

$ax - ay + 4z = 2$

for different values of a. **10**

3. (*a*) Prove that if $a_0, a_1, a_2, \ldots\ldots, a_n$ are the real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0$$

then there exists at least one real number x between 0 and 1 such that

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.. + a_{n-1}x + a_n = 0$. **10**

(*b*) Reduce the following equation to its canonical form and determine the nature of the conic

$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$. **10**

(c) Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T? Find the nullity of T. 10

(d) Find the equation to the tangent planes to the surface $7x^2 - 3y^2 - z^2 + 21 = 0$, which pass through line $7x - 6y + 9 = 0$, $z = 3$. 10

4. (a) Evaluate:

$$\int_0^{\pi/2} \frac{x \sin x \cos x \, dx}{\sin^4 x + \cos^4 x}.$$ 10

(b) Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P' H \overline{P}$ is diagonal and also find its signautre. 10

(c) Find the magnitude and the equation of the line of shortest distance between the lines

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$ 10

(d) Find all the asymptotes of the curve $x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$. 10

Section-B

5. (a) Solve :

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$$ 10

(b) A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where OP = b with velocity v in the direction of OP. Find the time which elapses before it returns to P. 8

(c) $\bar{F}$ being a vector, prove that curl curl $\bar{F}$ = grad div $\bar{F} - \nabla^2 \bar{F}$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. 8

(d) A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical, the angle B being in the surface of the liquid, and the angle A not immersed. Find ρ/σ in terms of the lengths of the sides of the triangle. 8

(e) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium. 8

6. (a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$$

by changing the dependent variable. 13

(b) Evaluate $\int_S \bar{F} \cdot d\bar{s}$, where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} + z^2\bar{k}$ and s is the surface bounding the region $x^2 + y^2 = 4$, $z = 0$ and $z = 3$. 13

(c) Two bodies of weights w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other. 14

7. (a) Solve :

$$(D^3 + 1)y = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

where $D = \frac{d}{dx}$. 13

(b) A body floating in water has volumes ν_1, ν_2 and ν_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Find the value of:

$$\frac{\rho_2 - \rho_3}{\nu_1} + \frac{\rho_3 - \rho_1}{\nu_2} + \frac{\rho_1 - \rho_2}{\nu_3}.$$ **13**

(c) A particle is projected vertically upwards with a velocity u, in a resisting medium which produces a retardation $k\nu^2$ when the velocity is ν. Find the height when the particle comes to rest above the point of projection. **14**

8. (a) Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} - y = 2(1+e^x)^{-1}.$$ **13**

(b) Verify the Divergence theorem for the vector function

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$$

taken over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$ **14**

(c) A particle is projected with a velocity v along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time t. **13**

PAPER-II

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. (a) Evaluate :

$$\lim_{x \to 0}\left(\frac{e^{ax} - e^{bx} + \tan x}{x}\right).$$ **10**

(b) Prove that if every element of a group (G, 0) be its own inverse, then it is an abelian group. **10**

(c) Construct an analytic function $f(z) = u(x, y) + iv(x, y)$, where $v(x, y) = 6xy - 5x + 3$.

Express the result as a function of z. **10**

(d) Find the optimal assignment cost from the following cost matrix:

	A	B	C	D
I	4	5	4	3
II	3	2	2	6
III	4	5	3	5
IV	2	4	2	6

10

2. (a) Show that any finite integral domain is a field. **13**

(b) Every field is an integral domain–Prove it. **13**

(c) Solve the following Salesman problem:

	A	B	C	D
A	∞	12	10	15
B	16	∞	11	13
C	17	18	∞	20
D	13	11	18	∞

14

3. (a) Show that the function $f(x) = x^2$ is uniformly continuous in (0, 1) but not in $\mathbb{R}$. **13**

(b) Prove that :

(i) the intersection of two ideals is an ideal.

ANSWERS

PAPER-I

Section-A

1. (*a*) The augmented matrix for the given system is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -1 & 3 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 3 & 6 & 8 & 1 & 5 & 0 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -1 & 3 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & -1 & 3 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -1 & 3 & 0 \\ 1 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{array}\right]$$

Dimension = 2.

Since [1 2 2 –1 3] + 2[1 2 3 1 1] = [3 6 8 1 5]. the three equations have one degree of overlap. The solution space therefore has 3 degrees of freedom. The basis can be generated from the first two vectors (the third is redundunt).

1. (*b*) The characteristics equations of the matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2 - \lambda)(1 - \lambda)(2 - \lambda) - (1 - \lambda) = 0$$

or, $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ ...(*i*)

Cayley-Hamilton theorem states that every square matrix satisfies its characteristics equation.

So, By Cayley-Hamilton theorem, the matrix A must satisfy (*i*)

$\therefore$ We have, $A^3 - 5A^2 + 7A - 3I = 0$

$$A^3 = 5A^2 - 7A + 3I \quad ...(ii)$$

$$A^4 = 5A^3 - 7A^2 + 3A \quad ...(iii)$$

$$A^5 = 5A^4 - 7A^3 + 3A^2 \quad ...(iv)$$

$$A^6 = 5A^5 - 7A^4 + 3A^3 \quad ...(v)$$

$$A^7 = 5A^6 - 7A^5 + 3A^4 \quad ...(vi)$$

$$A^8 = 5A^7 - 7A^6 + 3A^5 \quad ...(vii)$$

or, $A^8 - 5A^7 + 7A^6 - 3A^5 = 0$.

Again from equation (*iii*)

$$A^4 - 5A^3 + 7A^2 - 3A = 0$$

From question,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= (A^8 - 5A^7 + 7A^6 - 3A^5) + (A^4 - 5A^3 + 7A^2 - 3A) + A^2 + A + I$$

$$= 0 + 0 + A^2 + A + I$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

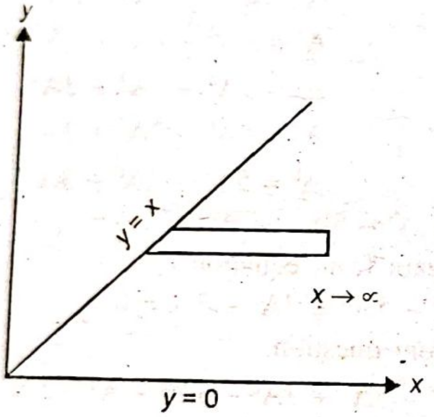
$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

1. (*c*) Let $I = \int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dy\,dx$

Here the limits of integration show that the integration is done first with respect to y from $y = 0$ to $y = x$ and then with respect to x from $x = 0$ and $x = \infty$, *i.e.*, the strip is taken parallel to y-axis in the region bounded by these curves.

On changing the order of integration, we find that the strip parallel to x-axis varies form $x = y$ to $x = \infty$ and then y varies from $y = 0$ to $y = \infty$ to cover the whole region (fig.) Hence on changing the order of integration, we have



figure

$$I = \int_0^{\infty}\int_{x=y}^{\infty} xe^{-x^2/y}\,dx\,dy$$

$$= \int_0^{\infty}\left[-\frac{y}{2}e^{-x^2/y}\right]_{x=y}^{\infty} dy$$

$$= \frac{1}{2}\int_0^{\infty} ye^{-y}\,dy$$

$$= \frac{1}{2}\left[y\frac{e^{-y}}{(-1)} - (1)(e^{-y})\right]_0^{\infty} = \frac{1}{2}.$$

1. (*d*) The equation of the given lines are

$y - a = 0,\ z - a = 0$ ....(*i*)

$x + 3z - a = 0,\ y + z - a = 0$ ...(*ii*)

The equation of any planes through the lines (*i*) and (*ii*) are

$(y - a) - \lambda_1 (z - a) = 0$

$\Rightarrow y - \lambda_1 z - a - a + a\lambda_1 = 0$ ...(*iii*)

and

$(x + 3z - a) - \lambda_2 (y + z - a) = 0$

or, $(x - \lambda_2 y) + (3 - \lambda_2)z - a + a\lambda_2 = 0$ ...(*iv*)

Any line intersecting the line (*i*) and (*ii*) is given by the intersection of the plane (*iii*) and (*iv*). Let λ, μ, ν are its d.r.'s., then

$0.\lambda + 1.\mu - \lambda_1 . \nu = 0$

and $1.\lambda - \lambda_2 . \mu + (3 - \lambda_2) . \nu = 0$

$$\therefore \quad \frac{\lambda}{3-\lambda_2-\lambda_1\lambda_2} = \frac{\mu}{-\lambda_1} = \frac{\nu}{-1}$$

Now, the line with d.r.'s λ, μ, ν is parallel to the plane $x + y = 0$, *i.e.*, this line is perpendicular to the normal to the plane $x + y = 0$, whose d.r.'s are 1, 1, 0.

So, we have

$1.(3 - \lambda_2 - \lambda_1\lambda_2) + 1(-\lambda_1) + 0.(-1) = 0$

or, $3 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2 = 0$ ...(*v*)

The required locus of the line is obtained by eliminating λ_1 and λ_2 between (*iii*), (*iv*) and (*v*) hence is given by

$$3 - \frac{y-a}{z-a} - \frac{x+3z-a}{y+z-a} - \frac{y-a}{z-a}.\frac{x+3z-a}{y+z-a} = 0$$

or, $3(y + z - a)(z - a) - (y - a)(y + z - a) - (z - a)(x + 3z - a) - (y - a)(x + 3z - a) = 0$

or, $-yz - y^2 + 2az - xz + 2ax - xy = 0$

or, $yz - y^2 + 2az - xz + 2ax - xy = 0$

or, $yz + y^2 + xz + xy = 2az + 2ax$

or, $(y + z)(x + y) = 2a(x + z)$.

1. (*e*) $f(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x$

$\therefore \quad f(a) = f(0) = 0$

and $\quad f(b) = f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)$

$$= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{8}$$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{4}$$

Also $\quad f'(x) = 3x^2 - 6x + 2$

so that $\quad f'(c) = 3c^2 - 6c + 2$

Substituting these values Lagrange's mean value theorem,

$$\frac{f(b)-f(a)}{b-a} = f'(c), \ (a < c < b)$$

$$\frac{3}{4} = 3c^2 - 6c + 2$$

or, $\quad 12c^2 - 24c + 5 = 0$

$$c = \frac{24 \pm \sqrt{(24)^2 - 4.12.5}}{2 \times 12}$$

$$= \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$= 1 \pm \frac{\sqrt{21}}{6}$$

$$\therefore \quad c = 1 - \frac{\sqrt{21}}{6}.$$

3. (*a*) From Binomial expension,

$$(1+x)^n = {}^nc_0x^n + {}^nc_1x^{n-1} + {}^nc_2x^{n-2} \ldots\ldots {}^nc_n$$

Integrating both sides, we get

$$\int (1+x)^n.dx = \frac{{}^nc_0x^{n+1}}{n+1} + \frac{{}^nc_1x^n}{n} + \frac{{}^nc_2x^{n-1}}{n-1} + \ldots\ldots \frac{{}^nc_nx}{1}$$

For $x = 1$,

$$\text{R.H.S.} = \frac{{}^nc_0}{n+1} + \frac{{}^nc_1}{n} + \frac{{}^nc_2}{n-1} + \ldots {}^nc_n$$

here $\quad {}^nc_0 = a_0, \ {}^nc_1 = a_1$

$\quad {}^nc_2 = a_2 \ldots {}^nc_n = a_n$ (say)

According to the question,

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} \ldots\ldots a_n = 0$$

Thus the given series

$$(1+x)^n = a_0x^n + a_1x^{n-1} + a2x^{n-2} \ldots\ a_n = 0$$

Converging for $x \in (0, 1)$.

3. (*b*) $4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$

General equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

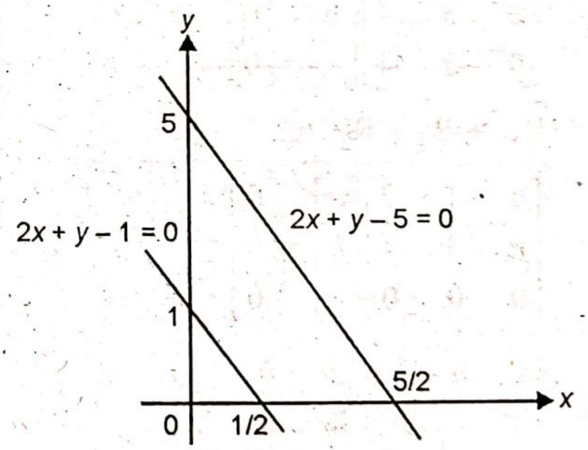
here $\quad a = 4, \ b = 1, \ c = 5$

$\quad g = -6, f = -3, h = 2$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 20 + 72 - 36 - 36 - 20$$

$$= 0$$



and $ab - h^2$

$$4 \times 1 - (2)^2 = 0$$

Hence, given equation will represent pair of parallel straight lines.

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

$$(2x+y)^2 - 6(2x+y) + 5 = 0$$

$$(2x+y-5)(2x+y-1) = 0$$

$$2x + y - 5 = 0$$

and $\quad 2x + y - 1 = 0$

3. (*c*) If $(a, b, c) \in \text{Ker T}$, then $T(a, b, c) = (0, 0, 0)$

$$\Rightarrow (a - b + 2c, 2a + b, -a - 2b + 2c) = (0, 0, 0)$$

$$\Rightarrow \quad a - b + 2c = 0$$
$$2a + b = 0$$
$$-a - 2b + 2c = 0$$

Since, $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{vmatrix} = 0$

The above equations have a non-zero solution.

Solving the equiations, we find

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a - b + 2c = 0$$
$$3b - 4c = 0$$

Since rank of coefficient matrix is 2, the number of L.I. solutions is $3 - 2 = 1$. If we take $c = k$, we get $a = -\frac{2k}{3}$, $b = \frac{4k}{3}$, $c = k$ as solution of the given equations. In other words a, b, c should satisfy the relation $\frac{a}{-2} = \frac{b}{4} = \frac{c}{4}$ for (a, b, c) to be in Ker T.

Now $(-2, 4, 3)$ is one member of Ker T and all other members would be multiples of this, *i.e.*, $\{(-2, 4, 3)\}$ generates ker T. Since $(-2, 4, 3)$ being non-zero is L.I. $\{(-2, 4, 3)\}$ forms a basis of Ker T or that dim Ker T = nullity T = 1.

Section-B

5. (*a*) $\frac{dy}{dx} + x.\sin 2y = x^3.\cos^2 y$

Dividing both sides by $\cos^2 y$, we have

$$\sec^2 y.\frac{dy}{dx} + \tan y.(2x) = x^3$$

Let $\qquad \tan y = t$

then, $\qquad \sec^2 y.\frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore \quad \frac{dt}{dx} + 2x.t = x^3$$

$$P = 2x,\ Q = x^3$$

$$\text{I.F.} = e^{\int p.dx} = e^{\int 2x.dx} = \frac{e^{2x}}{2}$$

$\therefore$ Solution of the differential equation is given as

$$t.(\text{I.F.}) = \int Q.(\text{I.F.})dx + c$$

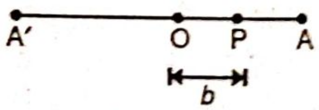
where c is integration constant

$$t.\frac{e^{2x}}{2} = \int x^3.\frac{e^{2x}}{2}.dx + c$$

$$= \frac{e^{2x}}{4}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c$$

$$2 \tan y\ e^{2x} = e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c.$$

5. (*b*) We have to find time taken from P to A and then A to P



$t = 2$ (time from A to P)

Date
Page

2(a) Let V be the vector space of 2×2 matrices over R and let $M=\begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix}$.
Let $T: V \to V$ be the linear map defined by $T(A)=MA$. Find a basis and the dimension of
i) the kernel, W of T
ii) the image, U of T.

Sol:
$$T\left(\begin{bmatrix}x & y\\z & w\end{bmatrix}\right)=\begin{bmatrix}1 & -1\\-2 & 2\end{bmatrix}\begin{bmatrix}x & y\\z & w\end{bmatrix}$$
$$=\begin{bmatrix}x-z & y-w\\-2x+2z & -2y+2w\end{bmatrix}$$
$$=(x-z)\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}+(y-w)\begin{bmatrix}0 & 1\\0 & -2\end{bmatrix}$$
$$=k_1\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}+k_2\begin{bmatrix}0 & 1\\0 & -2\end{bmatrix}$$
$k_1, k_2 \in R$

$\therefore$ Range(T)
$$W=\text{Span}\left\{\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & -2\end{bmatrix}\right\}$$

Dimension $(W)=2$.

($\because$ two vectors $\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix}$ and $\begin{bmatrix}0 & 1\\0 & -2\end{bmatrix}$ are not multiples of each other, hence independence.)

2(c)

For kernel,

$$T(A) = 0$$

ie. $T\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x-z & y-w \\ -2x+2z & -2y+2w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $x - z = 0$, $-2x+2z = 0$ & $y - w = 0$, $-2y+2w = 0$.

ie $x = z$ & $y = w$

$\therefore \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ x & y \end{bmatrix}$

$$= x\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

since vectors $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

are not multiples of each other, hence they are independence. Therefore they from the basis of kernel (T).

$$\text{Dim}(\text{Ker } T) = 2.$$

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$$\therefore \quad AC - B^2 = \left(-\frac{2}{3}a\right)\left(-\frac{2}{3}a\right) - \left(-\frac{a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{a^2}{3} > 0 \text{ and } A < 0$$

Thus f has a maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$\text{Max. value} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left[a - \frac{a}{3} - \frac{a}{3}\right] = \frac{a^2}{9}\left[\frac{a}{3}\right] = \frac{a^3}{27}$$

Example 4. ***Examine for extreme values $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.*** $\boxed{2(b)}$

[*M.D.U. 2018, 06; K.U. 2014*]

IFS 2013.

Solution. We have $\quad f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$\therefore \quad \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

and $$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\therefore \quad x^3 - x + y = 0 \qquad ...(1)$$

and $$y^3 + x - y = 0 \qquad ...(2)$$

Adding (1) and (2), we have

$$x^3 + y^3 = 0$$

or $$(x + y)(x^2 - xy + y^2) = 0$$

$\therefore$ For real x, $x + y = 0$ is the only possibility.

Putting $y = -x$ in (1), we get

$$x^3 - x - x = 0$$

or $$x^3 - 2x = 0$$

or $$x(x^2 - 2) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{2}$$

Hence the extreme points are $(0, 0)$; $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Now, $$A = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$B = \frac{\partial^2 f}{\partial y \partial x} = 4$$

and
$$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

At (0, 0) : $A = -4, \ B = 4, \ C = -4$

$\therefore$
$$AC - B^2 = 16 - 16 = 0$$

$\therefore$ At (0, 0), further investigation is required.

For small h, k and $h \neq k$, we have

$$f(h, k) - f(0, 0) = h^4 + k^4 - 2h^2 + 4hk - 2k^2$$

$$= -2(h-k)^2 < 0 \quad [\textit{Neglecting } h^4, k^4 \textit{ as } h, k \textit{ are small}]$$

For $h = k$, we have

$$f(h, k) - f(0, 0) = h^4 + h^4 - 2h^2 + 4h^2 - 2h^2$$

$$= 2h^4 > 0$$

As $f(h, k) - f(0, 0)$ does not keep the same sign for all small values of h and k, so the point (0, 0) is a saddle point.

At $(\sqrt{2}, -\sqrt{2})$: $A = 20, \ B = 4, \ C = 20$

$\therefore$
$$AC - B^2 > 0 \quad \text{and} \quad A > 0$$

$\Rightarrow$ f has a minimum at $(\sqrt{2}, -\sqrt{2})$.

Minimum value $= f(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4 = -8.$$

At $(-\sqrt{2}, \sqrt{2})$: $A = 20, \ B = 4, C = 20$

$\therefore$
$$AC - B^2 = 400 - 16 = 384 > 0 \quad \text{and} \quad A = 20 > 0$$

$\therefore$ $f(x, y)$ has a minimum at $(-\sqrt{2}, \sqrt{2})$

Minimum value $= f(-\sqrt{2}, \sqrt{2}) = -8.$

Example 5. ***Examine for maximum and minimum values the function***

$$\sin x + \sin y + \sin(x+y).$$ [K.U. 2016; M.D.U. 2015, 02, 01]

Solution. Let $f(x, y) = \sin x + \sin y + \sin(x + y)$

2(c) Find the orthogonal transformation of coordinates which diagonalizes the quadratic form

$$q(x,y) = 2x^2 - 4xy + 5y^2.$$

$$q(x,y) = 2x^2 - 2xy - 2yx + 5y^2$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

First we diagonalize this matrix by finding eigen-vectors.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda-5) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \boxed{\lambda = 1, 6}$$

$$\lambda = 1 \Rightarrow \begin{bmatrix} 2-1 & -2 \\ -2 & 5-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x - 2y = 0. \quad \text{ie } x = 2y$$

$$\therefore \text{Eigenvector } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= y \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

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2(d)

For $\lambda = 6$

$$\begin{bmatrix} 2-6 & -2 \\ -2 & 5-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x - y = 0 \Rightarrow y = -2x$$

Eigenvector,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Hence Diagonalizing Matrix is

$$M = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Such that

$$\boxed{M^{-1}AM = D}$$

Orthogonal ~~Linear~~ transformation is

$$x = \frac{2}{\sqrt{5}}u + \frac{1}{\sqrt{5}}v$$

$$y = \frac{1}{\sqrt{5}}u - \frac{2}{\sqrt{5}}v$$

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2(d) Discuss the consistency and the solutions of the equations

$$x + ay + az = 1$$
$$ax + y + 2az = -4$$
$$ax - ay + 4z = 2$$

for different values of a. (10)

Matrix Eqn : $AX = B$, where

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & 2a \\ a & -a & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\det(A) = 1(4+2a^2) - a(4a - 2a^2) + a(-a^2 - a)$$
$$= 4 + 2a^2 - 4a^2 + 2a^3 - a^3 - a^2$$
$$= a^3 - 3a^2 + 4$$
$$= (a+1)(a-2)^2$$

Case-1: When $a \neq -1$ and $a \neq 2$

$|A| \neq 0 \Rightarrow A^{-1}$ exist.

Hence system has unique solution.

Case-2: When $a = -1$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -1 & -1 & 1 \\ -1 & 1 & -2 & -4 \\ -1 & 1 & 4 & 2 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \quad x - y = 2 \quad , \quad z = 1$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y+2 \\ y \\ 1 \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Infinite many solutions.

Case-3: When $a = 2$.

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 2 & 1 & 4 & : & -4 \\ 2 & -2 & 4 & : & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & -3 & 0 & 6 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

Here, $\text{Rank}(A) = 2$

$\text{Rank}(A:B) = 3$

Both are not equal, hence system is inconsistent for $a = 2$.

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3(d) Find the equations to the tangent planes to the surface $7x^2-3y^2-z^2+21=0$, which pass through the line $7x-6y+9=0, z=3$. (10)

Eqn of a plane passing through given line

$$7x-6y+9+\lambda(z-3)=0$$
$$7x-6y+\lambda z+(9-3\lambda)=0 \quad \text{--- (1)}$$

Equation of ~~giv~~ tangent plane to given surface at a point (α,β,γ), lying on surface is

$$7\alpha x-3\beta y-\gamma z+21=0 \quad \text{--- (2)}$$

If (1) and (2) are same then

$$\frac{7\alpha}{7}=\frac{+3\beta}{+6}=\frac{-\gamma}{\lambda}=\frac{+21}{9-3\lambda}$$

ie. $\alpha=\frac{7}{3-\lambda}, \beta=\frac{14}{3-\lambda}, \gamma=\frac{-7\lambda}{3-\lambda}$

(α,β,γ) lies on given surface

$$\therefore 7\left(\frac{7}{3-\lambda}\right)^2-3\left(\frac{14}{3-\lambda}\right)^2-\left(\frac{-7\lambda}{3-\lambda}\right)^2+21=0.$$

$$2\lambda^2+9\lambda+4=0 \Rightarrow \lambda=-4, \frac{-1}{2}.$$

Hence, Eqn of tangent planes are—

$$\boxed{\begin{aligned}&7x-6y-4z+21=0.\\&14x-12y-z+21=0.\end{aligned}}$$

4(a) Evaluate $I = \int_0^{\pi/2} \frac{x \cdot \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ (10).

using formula $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} \cdot \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} - \underbrace{\int_0^{\pi/2} \frac{x \cdot \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx}_{= I}$$

$$\therefore\ 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\, dx$$

$$I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x}{1 + \tan^4 x}\, dx$$

(dividing by $\cos^4 x$ in num & deno.)

Put $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x\, dx = dt$

$$I = \frac{\pi}{4} \times \frac{1}{2} \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \frac{\pi}{8} \tan^{-1} t \Big|_0^{\infty}$$

$$= \frac{\pi}{8}\left(\frac{\pi}{2} - 0\right) = \boxed{\frac{\pi^2}{16}}.$$

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4(6) Let
$$H=\begin{bmatrix}1 & i & 2+i\\ -i & 2 & 1-i\\ 2-i & 1+i & 2\end{bmatrix}$$ be a Hermitian matrix. Find a non-singular matrix P such that $P^t H \bar{P}$ is diagonal and also find its signature.

Let $H = IHI$

$$\begin{bmatrix}1 & i & 2+i\\ -i & 2 & 1-i\\ 2-i & 1+i & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} H \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Row-operations applied on pre-factor and column operations on post-factor on RHS.

$R_2 \to R_2 + iR_1$, $R_3 \to R_3 + (-2+i)R_1$
$C_2 \to C_2 - iC_1$, $C_3 \to C_3 - (2+i)C_1$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & i\\ 0 & -i & -5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ i & 1 & 0\\ -2+i & 0 & 1\end{pmatrix} H \begin{pmatrix}1 & -i & -2-i\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$R_3 \to R_3 + iR_2$, $C_3 \to C_3 - iC_2$

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -6\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ i & 1 & 0\\ -3+i & i & 1\end{pmatrix} H \begin{pmatrix}1 & -i & -3-i\\ 0 & 1 & -i\\ 0 & 0 & 1\end{pmatrix}$$

$$P^T H \bar{P} = D$$

$$\Rightarrow P=\begin{bmatrix}1 & i & -3+i\\ 0 & 1 & i\\ 0 & 0 & 1\end{bmatrix}$$

Rank$(H) = 3$
Index$(H) = 2$ (Positive diagonal entries)
Signature(H) = No. of +ve − No. of −ve $= 2-1 = 1$.

4(c) **1.** Determine whether the following lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7} \quad \& \quad \frac{x-15}{3}=\frac{y-29}{8}=\frac{5-z}{5}$$

are intersecting or skew lines. In case they intersect, find the equation of the plane containing them. Or if they are skew lines find the magnitude and equation of the line of shortest distance between them. **12.5**

Two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and

$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Here, $\begin{vmatrix} 15-8 & 29-(-9) & 5-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = \begin{vmatrix} 7 & 38 & -5 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$

$= 1176 \neq 0$

Hence given two lines are not coplanar. As they are not parallel hence, not intersecting.

Let $A(3a+8, -16a-9, 7a+10)$ and $B(3b+15, 8b+29, -5b+5)$ are two general points on given line.

and $P(8,-9,10)$, $Q(15,29,5)$ are two given points on line.

D.r. of $AB = \langle 3a-3b-7, -16a-8b-38, 7a+5b+5 \rangle$

If AB is line of shortest distance, it will be perpendicular to both lines.

$$\therefore \quad 3(3a-3b-7)-16(-16a-8b-38)+7(7a+5b+5)=0$$

$$157a+77b+311=0.$$

$$\& \quad 3(3a-3b-7)+8(-16a-8b-38)-5(7a+5b+5)=0$$

ie $\quad 154a+98b+350=0.$

Solving, we get $\boxed{a=-1, \quad b=-2}$

$$\therefore \quad A(-3+8, 16-9, -7+10) \text{ ie. } (5,7,3)$$

$$B(-6+15, -16+29, 10+5) \text{ ie } (9,13,15)$$

$$\text{Dist }(AB) = \sqrt{(9-5)^2+(13-7)^2+(15-3)^2}$$

$$= \sqrt{16+36+144} = \sqrt{196} = 14$$

Eqn of AB,
$$\frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12}$$

ie $$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

4(d) Find all the asymptotes of the curve
$$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0.$$

Let eqn of Asymptote: $y = mx + C$.

$\phi_4 = x^4 - y^4$

Putting $x=1$, $y=m$

$\phi_4(m) = 1 - m^4$

$\phi_4(m) = 0 \Rightarrow m = 1, -1$

$\phi_3 = 3x^2y + 3xy^2$

$\phi_3(m) = 3m + 3m^2$

Also,
$$C = \frac{-\phi_3(m)}{\phi_4'(m)} = \frac{-3(m)(1+m)}{-4m^3}$$
$$= \frac{3(1+m)}{4m^2}$$

for, $m = 1 \Rightarrow C = 3/2$

$m = -1 \Rightarrow C = 0$

Hence, equations of asymptotes are
$$\boxed{\begin{array}{c} y = x + 3/2 \\ y = -x \end{array}}$$

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3. (c) If $(a, b, c) \in$ Ker T, then $T(a, b, c) = (0, 0, 0)$

$$\Rightarrow (a - b + 2c,\ 2a + b,\ -a - 2b + 2c) = (0, 0, 0)$$

$$\Rightarrow \quad a - b + 2c = 0$$
$$2a + b = 0$$
$$-a - 2b + 2c = 0$$

Since, $\begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{vmatrix} = 0$

The above equations have a non-zero solution.

Solving the equiations, we find

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a - b + 2c = 0$$
$$3b - 4c = 0$$

Since rank of coefficient matrix is 2, the number of L.I. solutions is $3 - 2 = 1$. If we take $c = k$, we get $a = -\frac{2k}{3}$, $b = \frac{4k}{3}$, $c = k$ as solution of the given equations. In other words a, b, c should satisfy the relation $\frac{a}{-2} = \frac{b}{4} = \frac{c}{4}$ for (a, b, c) to be in Ker T.

Now $(-2, 4, 3)$ is one member of Ker T and all other members would be multiples of this, *i.e.*, $\{(-2, 4, 3)\}$ generates ker T. Since $(-2, 4, 3)$ being non-zero is L.I. $\{(-2, 4, 3)\}$ forms a basis of Ker T or that dim Ker T = nullity T = 1.

Section-B

5. (a) $\frac{dy}{dx} + x.\sin 2y = x^3.\cos^2 y$

Dividing both sides by $\cos^2 y$, we have

$$\sec^2 y.\frac{dy}{dx} + \tan y.(2x) = x^3$$

Let $\tan y = t$

then, $\sec^2 y.\frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore \quad \frac{dt}{dx} + 2x.t = x^3$$

$$P = 2x,\ Q = x^3$$

$$\text{I.F.} = e^{\int p.dx} = e^{\int 2x.dx} = \frac{e^{2x}}{2}$$

$\therefore$ Solution of the differential equation is given as

$$t.(\text{I.F.}) = \int Q.(\text{I.F.})dx + c$$

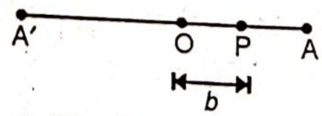
where c is integration constant

$$t.\frac{e^{2x}}{2} = \int x^3.\frac{e^{2x}}{2}.dx + c$$

$$= \frac{e^{2x}}{4}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c$$

$$2 \tan y\, e^{2x} = e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c.$$

5. (b) We have to find time taken from P to A and then A to P



$t = 2$ (time from A to P)

$$= 2\int_0^t dt = 2\int_a^p \frac{dx}{\sqrt{u}\sqrt{a^2 - x^2}}$$

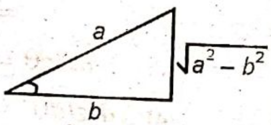
(Ignoring –ve sign) $\left(\frac{dx}{dt} = \sqrt{u}\sqrt{a^2 - x^2}\right)$

$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}\frac{a}{b}\right]$$

$$= \frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

$$\Rightarrow\ t = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{\sqrt{a^2 - b^2}}{b}\right)$$



$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

$$= \frac{2}{\frac{2\pi}{T}}\tan^{-1}\left[\frac{v}{b\left(\frac{2\pi}{T}\right)}\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left[\frac{vT}{2\pi b}\right].$$

$$\left[\begin{array}{ll} & v^2 = \mu(a^2 - b^2) \\ \Rightarrow & v = \sqrt{a}\sqrt{(a^2 - b^2)} \\ \Rightarrow & \frac{v}{\sqrt{\mu}} = \sqrt{a^2 - b^2} \end{array}\right]$$

$$\left[T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \frac{2\pi}{T}\right]$$

Proved.

5. (*c*) crul curl $\vec{F} = \nabla \times \nabla \times \vec{F}$

grad div $\vec{F} = \nabla\nabla.\vec{F}$

div grad $\vec{F} = \nabla.\nabla\vec{F} = \nabla^2\vec{F}$

$$\nabla \times (\nabla \times \vec{F}) = \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \nabla \times \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\right]\hat{i} +$$

$$\left[\frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]\hat{j}$$

$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\right]\hat{k}$$

$$= \left(\frac{\partial^2 F_1}{\partial y.\partial x} + \frac{\partial^2 F_3}{\partial z.\partial x}\right)\hat{i} + \left(\frac{\partial^2 F_3}{\partial z.\partial y} + \frac{\partial^2 F_1}{\partial x.\partial y}\right)\hat{j}$$

$$+\left(\frac{\partial^2 F_1}{\partial x.\partial z} + \frac{\partial^2 F_2}{\partial y.\partial z}\right)\hat{k} - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$-\left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\hat{j} - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2}\right)\hat{k}$$

$$= \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y.\partial x} + \frac{\partial^2 F_3}{\partial z.\partial x}\right)\hat{i}$$

$$+ \left(\frac{\partial^2 F_1}{\partial x.\partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z.\partial x}\right)\hat{j}$$

$$+ \left(\frac{\partial^2 F_1}{\partial x.\partial z} + \frac{\partial^2 F_2}{\partial y.\partial z} + \frac{\partial^2 F_3}{\partial z^3}\right)\hat{k}$$

$$- \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$- \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\hat{j}$$

$$- \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\hat{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y^2} + \frac{\partial F_3}{\partial z}\right)\hat{i}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y^2} + \frac{\partial F_3}{\partial z}\right)\hat{j}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\hat{k}$$

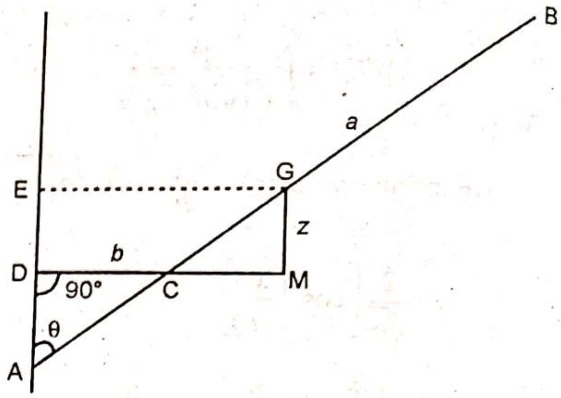
$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$$

$$= \nabla\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \nabla^2\bar{F}$$

$$= \nabla(\nabla.\bar{F}) - \nabla^2\bar{F}.$$

Proved.

5. (*e*) Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b, *i.e.*, CD = b.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C, *i.e.*, GM = z. We shall express z in terms of θ. We have,

$$z = GM = ED = AE - AD$$
$$= AG\cos\theta - CD\cot\theta$$
$$= a\cos\theta - b\cot\theta.$$

$\therefore \quad dz/d\theta = -a\sin\theta + b\,\text{cosec}^2\theta$

and $d^2z/d\theta^2 = -a\cos\theta - 2b\,\text{cosec}^2\theta\cot\theta$

For equilibrium of the rod, we have $dz/d\theta = 0$

i.e., $-a\sin\theta + b\,\text{cosec}^2\theta = 0$

or $\quad a\sin\theta = b\,\text{cosec}^2\theta,$

or $\quad \sin^3\theta = b/a$

or $\quad \sin\theta = (b/a)^{1/3},$

or $\quad \theta = \sin^{-1}.(b/a)^{1/3}.$

This gives the position of equilibrium of the rod.

Again $d^2z/d\theta^2 = -(a\cos\theta + 2b\,\text{cosec}^2\theta\cot\theta)$

= negative for all actute values of θ.

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

6. (*a*) We have,

$$\frac{d^2y}{dx^2} = 4x\frac{dy}{dx} + (4x^2 - 1)y$$

$$= -3e^{x^2}\sin 2x \qquad ...(i)$$

Here $p = -4x$, $Q = 4x^2 - 1$,

$$R = -3e^{x^2} \sin 2x$$

In order to remove the first derivative,

$$v = e^{-\frac{1}{2}\int p dx} = e^{-\frac{1}{2}\int -4x dx}$$

$$= e^{2\int x\,dx} = e^{x^2}$$

On putting $y = uv$, the normal equation is

$$\frac{d^2u}{dx^2} + Q_1 u = R_1 \qquad ...(ii)$$

where $Q_1 = Q - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4}$

$$= (4x^2 - 1) - \frac{1}{2}(-4) - \frac{16x^2}{4}$$

$$= 4x^2 - 1 + 2 - 4x^2 = 1$$

$$R_1 = \frac{R}{v} = \frac{-3e^{x^2} \sin 2x}{e^{x^2}} = -3\sin 2x$$

Equation (*ii*) becomes

$$\frac{d^2u}{dx^2} + u = -3 \sin 2x$$

$\Rightarrow$ $(D^2 + 1)u = -3 \sin 2x$

A.E. is $m^2 + 1 = 0$

$\Rightarrow$ $m = \pm i$

$\Rightarrow$ C.F. $= c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D_2 + 1}(-3\sin 2x)$$

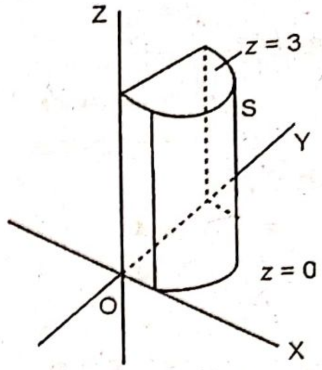
$$= \frac{-3\sin 2x}{-4+1} = \sin 2x$$

$u = c_1 \cos x + c_2 \sin x + \sin 2x$

$y = u.\ v = (c_1 \cos x + c_2 \sin x + \sin 2x)e^{x^2}$.

6. (*b*) We have,

$$\iint_s \vec{F}.d\vec{s} = \iint_s \vec{F}.\hat{n}\,ds \qquad ...(i)$$



where S = cylinder $x^2 + y^2 = 4$

Let $\phi = x^2 + y^2$

$\therefore$ $S = \phi = 4$, *i.e.*, level surface

and $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}(2x) + \hat{j}(2y) + 0 = 2(x\hat{i} + y\hat{j})$$

$\therefore$ $$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\hat{i} + y\hat{j})}{2\sqrt{x^2+y^2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{4}}$$

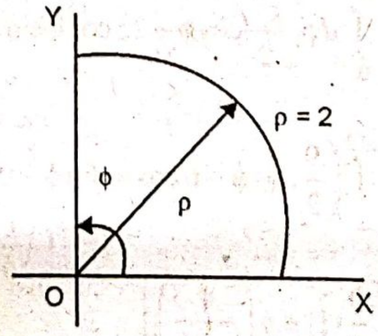
... $\because x^2 + y^2 = 4$ on S

$\therefore$ $\hat{n} = \frac{1}{2}(x\hat{i} + y\hat{j})$

$\therefore$ $\vec{F}.\hat{n} = (z\hat{i} + x\hat{j} - 3y^2z\hat{k}).\frac{1}{2}(x\hat{i} + y\hat{j})$

$$= \frac{1}{2}(zx + xy) \qquad ...(ii)$$

Use cylindrical polar co-ordinates, *i.e.*,



$x = \rho \cos \phi$

$y = \rho \sin \phi$

$$z = z$$

so that $x^2 + y^2 = \rho^2$

and $dv = dx\,dy\,dz$

$$= \rho\,d\rho\,d\phi\,dz$$

Now on S :

$$x^2 + y^2 = 4$$

i.e., $\rho^2 = 4$

i.e., $\rho = 2$

$\therefore$ $x = 2\cos\phi$

$$y = 2\sin\phi$$

$$z = z$$

$\because$ $dv = (ds)\,(d\rho)$

$\therefore$ $\rho\,d\rho\,d\phi\,dz = (ds)\,(d\rho)$

$\therefore$ $ds = \rho\,d\phi\,dz = 2d\phi dz$

$\therefore$ From (*i*),

$$\iint_s \vec{F}.d\vec{s} = \iint_S \frac{1}{2}(zx + xy)ds$$

...using (*ii*)

$$= \frac{1}{2}\int_{\phi=0}^{\pi/2}\int_{z=0}^{3}(z\,2\cos\phi + 2\cos\phi.2\sin\phi)2d\phi dz$$

.... $\because$ surface in 1st octant between $z = 0$ to $z = 3$

$$= 2\int_0^{\pi/2} d\phi \int_0^3 (z\cos\phi + 2\cos\phi\sin\phi)dz$$

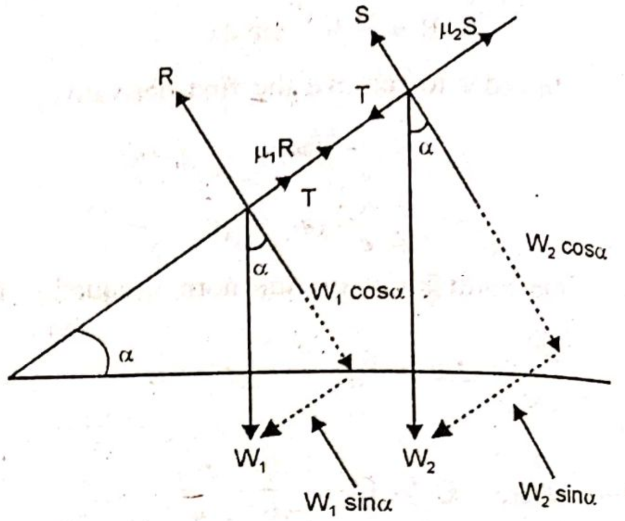
$$= 2\int_0^{\pi/2} d\phi\left[\frac{z^2}{2}\cos\phi + 2z\cos\phi\sin\phi\right]_0^3$$

$$= 2\int_0^{\pi/2}\left(\frac{9}{2}\cos\phi + 6\cos\phi.\sin\phi - 0\right).d\phi$$

$$= 2\left[\frac{9}{2}(1) + 6\left(\frac{1}{1+1}\right)\right]$$

$= 15.$

6. (*c*)



R and S are normal reactions and $\mu_1 R$ and $\mu_2 S$ are forces of friction. Let T be the tension in the string. Let α be the inclination of plane to the horizontal.

For W_1 : For limiting equilibrium,

<u>Horizontally</u>

$$\mu_1 R + T = W_1 \sin\alpha$$

$\Rightarrow$ $T = W_1\sin\alpha - \mu_1 R$ ....(*i*)

<u>Vertically</u>

$$R = W_1\cos\alpha \quad \text{....}(ii)$$

From (*i*) and (*ii*), we get

$$T = W_1\sin\alpha - \mu_1 W_1\cos\alpha \quad \text{...}(iii)$$

For W_2 : For limiting equilibrium,

<u>Horizontally</u>

$$T + W_2\sin\alpha = \mu_2 S$$

$\Rightarrow$ $T = \mu_2 S - W_2\sin\alpha$ ...(*iv*)

<u>Vertically,</u>

$$S = W_2\cos\alpha \quad \text{...}(v)$$

From (*iv*) and (*v*), we get

$$T = \mu_2 W_2\cos\alpha - W_2\sin\alpha \quad \text{...}(vi)$$

From (*iii*) and (*vi*), we get,

$$W_1\sin\alpha - \mu_1 W_1\cos\alpha$$
$$= \mu_2 W_2\cos\alpha - W_2\sin\alpha$$

$\Rightarrow$ $W_1\sin\alpha + W_2\sin\alpha$

$$= \mu_1 W_1\cos\alpha + \mu_2 W_2\cos\alpha$$

$\Rightarrow (W_1 + W_2) \sin\alpha$

$$= (\mu_1 W_1 + \mu_2 W_2)\cos\alpha$$

$$\Rightarrow \quad \tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}\right).$$

7. (*b*) Suppose the volume and the density of the body be V and ρ respectively.

Now, weight of the body = weight of air displaced + weight of water displaced

Hence,

$$V\rho g = V_1\rho_1 g + (V - V_1) \times 1 \times g \quad ...(i)$$

$$V\rho g = V_2\rho_2 g + (V - V_2) \times 1 \times g \quad ...(ii)$$

$$V\rho g = V_3\rho_3 g + (V - V_3) \times 1 \times g \quad ...(iii)$$

These relations give,

$$V_1 = \frac{\rho - 1}{\rho_1 - 1}V \text{ or } \frac{1}{V_1} = \frac{\rho_1 - 1}{(\rho - 1)V}$$

$$V_2 = \frac{\rho - 1}{\rho_2 - 1}V \text{ or } \frac{1}{V_2} = \frac{\rho_2 - 1}{(\rho - 1)V}$$

and $$V_3 = \frac{\rho - 1}{\rho_3 - 1}V \text{ or } \frac{1}{V_3} = \frac{\rho_3 - 1}{(\rho - 1)V}$$

$$\therefore \frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3}$$

$$= \frac{(\rho_1 - 1)}{(\rho - 1)V}(\rho_2 - \rho_3) + \frac{(\rho_2 - 1)}{(\rho - 1)V}(\rho_3 - \rho_1) + \frac{(\rho_3 - 1)}{(\rho - 1)V}(\rho_1 - \rho_2)$$

$$= \frac{1}{(\rho - 1)V}[(\rho_1 - 1)(\rho_2 - \rho_3) + (\rho_2 - 1)(\rho_3 - \rho_1) + (\rho_3 - 1)(\rho_1 - \rho_2)]$$

$= 0.$

7. (*c*) Equation of Motion is

$$\ddot{x} = -g - kv^2$$

For maximum/minimum velocity

$$\ddot{x} = 0 \Rightarrow v = \sqrt{g/k}$$

$$\therefore \quad \ddot{x} = -g\left(1 + \frac{v^2}{V^2}\right)$$

$$\text{or,} \quad v.\frac{dv}{dx} = -g\left(1 + \frac{v^2}{V^2}\right)$$

$$\frac{2g}{V^2}.x = \int \frac{2v.dv}{v^2 + V^2} + c$$

$$= -\text{Log}(v^2 + V^2) + c$$

For, $x = 0$, $v = u$

$$\therefore \quad c = \text{Log}(u^2 + V^2)$$

$$\text{i.e.,} \quad \frac{2g}{V^2}.x = \text{Log}(u^2 + V^2) - \text{Log}(v^2 + V^2)$$

$$= \text{Log}\left(\frac{u^2 + V^2}{v^2 + V^2}\right)$$

At highest point, $v = 0$, therefore the greatest height

$$h = \frac{V^2}{2g}.\text{Log}\left(\frac{u^2 + V^2}{V^2}\right)$$

$$\Rightarrow \quad = \frac{V^2}{2g}.\text{Log}\left(1 + \frac{u^2}{V^2}\right).$$

8. (*b*) To verify Gauss divergence theorem, we have to show that

$$\iiint_v div\,\bar{F}\,dv = \iint_s \bar{F}.\hat{n}.ds.$$

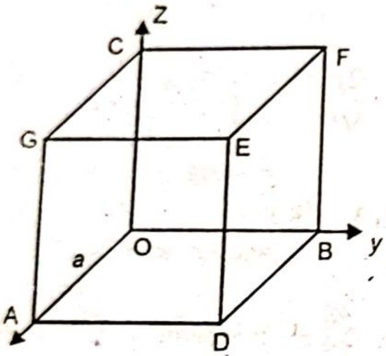
Firstly, $\iiint_v div\,\bar{F}\,dv$

$$= \int_0^c\int_0^b\int_0^a \left[\frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)\right]dxdydz$$

$$= \int_0^c \int_0^b \int_0^a 2(x+y+z)\,dx\,dy\,dz$$

$$= a^2bc + ab^2c + abc^2 = abc\,(a+b+c)$$

Now to calculate $\iint_s \bar{F}.n\,ds$, we divide the surface s of the parallelepiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ into six parts.



(*i*) For the face OADB, we have $\hat{n} = -\hat{k}$, $z = 0$,

Therefore, $\int\int_{OADB} \bar{F}.\hat{n}.ds$

$$= \int\int_{OADB} (x^2\hat{i} + y^2\hat{j} - xy\hat{k}).(-\hat{k})ds$$

$$= \int_0^b \int_0^a xy\,dx\,dy = \frac{a^2b^2}{4}.$$

(*ii*) For the face CGEF, we have $z = c$, $\hat{n} = \hat{k}$.

Therefore, $\int\int_{CGEF} \bar{F}.\hat{n}.ds$

$$= \int\int_{CGEF} \left[(x^2 - cy)\hat{i} + (y^2 - cx)\hat{j} + (c^2 - xy)\hat{k}\right].\hat{k}\,ds$$

$$= \int_0^b \int_0^a (c^2 - xy)\,dx\,dy = abc^2 - \frac{a^2b^2}{4}.$$

(*iii*) For the face ADEG, we have $\hat{n} = \hat{i}$, $x = a$ and $dx = 0$. Therefore,

$$\int\int_{ADEG} \bar{F}.n.ds = \int_0^c \int_0^b (a^2 - yz)\,dy\,dz$$

$$= a^2bc - \frac{b^2c^2}{4}$$

(*iv*) For the face OBFC, we have $\hat{n} = -\hat{i}$, $x = 0$, $dx = 0$, Therefore,

$$\int\int_{OBFC} \bar{F}.\hat{n}.ds = \int_0^a \int_0^b yz\,dy\,dz = \frac{b^2c^2}{4}.$$

(*v*) For the face OAGC, we have $\hat{n} = -\hat{j}$, $y = 0$, $dy = 0$, Therefore,

$$\int\int_{OAGC} \bar{F}.\hat{n}.ds = \int_0^a \int_0^b zx\,dz\,dx = \frac{a^2c^2}{4}.$$

(*vi*) For the face DBFE, we have $\hat{n} = \hat{j}$, $y = b$, $dy = 0$ Therefore,

$$\int\int_{DBFE} \bar{F}.\hat{n}.ds = \int_0^a \int_0^b (b^2 - zx)\,dz\,dx$$

$$= ab^2c - \frac{a^2c^2}{4}.$$

Hence adding the values of the above integrals, we get

$$\iint_s \bar{F}.\hat{n}.ds = abc\,(a+b+c).$$

Hence,

$$\iiint_V div\,\bar{F}\,dv = \iint_s \bar{F}.\hat{n}.ds,$$

which verifies the Gauss's divergence theorem.

8. (*c*) The distance described by the particle in time t, when medium resistance/mass is $(\mu) \times v^3$, where v is the velocity along the smooth horizontal plane

$$= \frac{1}{\mu v}\left[\sqrt{1+2\mu v^2 t} - 1\right]$$

here $\mu = 2$

$\therefore$ Distance described by the particle

$$= \frac{1}{2v}\left[\sqrt{1+4v^2 t} - 1\right].$$

PAPER-II

1. (*a*) $\lim_{x\to 0}\left(\frac{e^{ax} - e^{bx} + \tan x}{x}\right)$

this is in $\frac{0}{0}$ form, By L'Hospital rule,

$$\lim_{x\to 0}\left(\frac{a.e^{ax} - b.e^{bx} + \sec^2(x)}{1}\right)$$

$= a.e^{a(0)} - b.e^{b(0)} + \sec^2(0)$

$= 1 + a - b.$

1. (*b*) **Definition :** A group is a pair (G, •) for G a nonempty set and $\bullet : G \times G \to G$ satisfying:

(*i*) $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in G$;

(*ii*) there is $e_G \in G$ with $a \bullet e_G = a = e_G \bullet a$ for all $a \in G$; and

(*iii*) for each $a \in G$ there is $b \in G$ with $a \bullet b = b \bullet a = e_G$. If $g \bullet h = h \bullet g$ for all $g, h \in G$, then G is called **Abelian.**

The three group properties are called associativity, the existence of an identity, and the existence of an inverse for each element, respectively. If the multiplication "•" is clear, we write G instead of (G, •) and gh instead of $g \bullet h$. We say that G is **finite** when $|G|$ is finite and call $|G|$ the **order** of G. Otherwise G is called infinite and has infinite order. Consider the set of real numbers **R** that has both an addition and multiplication. Addition in **R** is associative and commutative, zero is an identity for addition, and the inverse of $r \in$ **R** is $-r$, so (**R**, +) is an Abelian group. Multiplication in **R** is associative and commutative, and 1 is an identity for it. The inverse for $r \in$ **R** using multiplication is $1/r$, requiring $r \neq 0$. Hence if $\mathbf{R}^* = \mathbf{R} - \{0\}$ then $(\mathbf{R}^*, \bullet)$ is an Abelian group. Other infinite Abelian groups are (**Q**, +), $(\mathbf{Q}^*, \bullet)$, $(\mathbf{Q}^+, \bullet)$, where $\mathbf{Q}^+ = \{q \in \mathbf{Q} \mid q > 0\}$, (**Z**, +). Although **Z** has a multiplication, the only integers with integer reciprocals are ± 1, so the only group we could define using the multiplication in **Z** is $\{\pm 1\}$, or $\{0\}$. In the latter case $0 \bullet 0 = 0$, so 0 is the identity element and its own inverse! This also holds for addition since $0 + 0 = 0$ so $(\{0\}, +)$ is a group.

If $n \in \mathbf{N}$ then $(\mathbf{Z}_n, +)$ is a finite Abelian group by Theorem. Just as for **R**, $(\mathbf{Z}_n, \bullet)$ is not a group since $[1]_n$ is the identity but $[0]_n$ has no inverse. For $[a]_n \in \mathbf{Z}_n$, $[a]_n[b]_n = [1]_n \Leftrightarrow [ab]_n = [1]_n \Leftrightarrow ab - 1 = nt$ for some $t \in \mathbf{Z} \Leftrightarrow (a, n) = 1$ by theorem. Thus when p is a prime, $[a]p \in \mathbf{Z}_p^* = \mathbf{Z}_p - \{[0]_p\}$ has an inverse under multiplication. Since $\mathbf{Z}_p^*$ is closed under multiplication by Theorem, since $[1]_p$ is an identity for multiplication, and since the multiplication in $\mathbf{Z}_p$ is associative and commutative by Theorem, $(\mathbf{Z}_p^*, \bullet)$ is an Abelian group.

1. (*c*) Here, $V = 6xy - 5x + 3$

$$\frac{\partial v}{\partial x} = 6y - 5 = \psi_2(x_1 y) \qquad \text{....}(i)$$

$$\frac{\partial v}{\partial y} = 6x = \psi_1(x_1 y) \qquad \text{...}(ii)$$