IFOS-2014

① Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

C.F

$$(D^3 - 3D^2 + 4D - 2)y = 0$$

$$(D-1)(D^2 - 2D + 2)y = 0$$

$$D^3 - 2D^2 + 2D - D^2 + 2D - 2$$

$$D^3 - 3D^2$$

$$\frac{2 \pm \sqrt{4-8}}{2}$$

$$\frac{2 \pm 2i}{2} \quad 1 \pm i$$

$$D = 1,\ 1+i,\ 1-i$$

$$\boxed{y = Ae^x + Be^x \cos(x + c)} \cancel{+ De^x \cos(x + c)}$$

P.I. $\frac{e^x}{D^3 - 3D^2 + 4D - 2} + \frac{\cos x}{D^3 - 3D^2 + 4D - 2}$

$$\frac{x e^x}{3D^2 - 6D + 4} + \frac{\cos x}{D^3 + 3 + 4D - 2}$$

$$\frac{x e^x}{3 - 6 + 4} + \frac{\cos x}{-D + 3 + 4D - 2}$$

$$\boxed{xe^x + \frac{3\sin x + \cos x}{10}}$$

$$\frac{x e^x}{} + \frac{\cos x\,(3D-1)}{(3D+1)(3D-1)}$$

$$xe^x + \frac{(3D-1)\cos x}{9D^2 - 1}$$

$$xe^x + \frac{(-3\sin x - \cos x)}{-10}$$

$$\boxed{y = Ae^x + Be^x\cos(x+c) + xe^x + \frac{3\sin x + \cos x}{10}}$$

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② $\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x \tan\frac{y}{x^2}$.

$y = mx^2$.

$\frac{dy}{dx} = 2mx + x^2\frac{dm}{dx}$.

$2mx + x^2\frac{dm}{dx} = 2mx + \frac{x}{m} + x\tan m$.

$x\frac{dm}{dx} = \frac{1}{m} + \tan m$.

$= \frac{dx}{x}$

$\frac{m\cos m\, dm}{\cos m + m\sin m}$.

~~$m\cos m = t$~~.

$\cos m + m\sin m = t$

$dm(-\sin m + \sin m + m\cos m) = dt$.

$(m\cos m)\, dm = dt$.

$\log|\cos m + m\sin m| = \ln x + c$.

$\frac{\cos m + m\sin m}{x} = C$.

$$\boxed{\frac{\cos\left(\frac{y}{x^2}\right) + \left(\frac{y}{x^2}\right)\sin\left(\frac{y}{x^2}\right)}{x} = C}$$

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③ $\frac{d^4y}{dx^4} - 16y = x^4 + \sin x$

$\boxed{\text{C-Function.}}$

$(D^4 - 16)y = 0$

$(D^2+4)(D^2-4)y = 0.$

$D = \pm 2, \pm 2i$

$$\boxed{y = Ae^{2x} + Be^{-2x} + C\cos(2x + D)}$$

$\boxed{\text{P-Integral}} \Rightarrow (D^4 - 16)y = x^4 + \sin x$

$y = \frac{x^4}{D^4 - 16} + \frac{\sin x}{D^4 - 16} \cdot \quad D^2 = -1$

$\Rightarrow y = \frac{-1}{16}\left(1 - \frac{D^4}{16}\right)^{-1} x^4 + \frac{\sin x}{(-15)}$

$y = \frac{-1}{16}\left(1 + \frac{D^4}{16} + \dots\right)x^4 - \frac{\sin x}{15}$

$y = \frac{-x^4}{16} - \frac{1}{16 \cdot 16_2}(4 \cdot 3 \cdot 2 \cdot 1) - \frac{\sin x}{15}$

$y = \frac{-4}{2 \cdot 16} - \frac{3}{32} - \frac{\sin x}{15}$

$y = \frac{-5}{32} - \frac{\sin x}{15}.$

$$\boxed{y = Ae^{2x} + Be^{-2x} + C\cos(2x + D) - \frac{5}{32} - \frac{\sin x}{15}.}$$

$$\boxed{y = Ae^{2x} + Be^{-2x} + C\cos(2x + D) - \frac{x^4}{16} - \frac{3}{32} - \frac{\sin x}{15}}$$

④ Solve by method of variation of parameters

$y'' + 3y' + 2y = x + \cos x$

$D^2 + 3D + 2 = 0$

$\frac{-3 \pm \sqrt{9-8}}{2}$

$\frac{-3+1}{2} ; \frac{-3-1}{2}$

$\frac{-2}{2} ; \frac{-4}{2} \qquad -1 ; -2$

C. Function $\Rightarrow$ $\boxed{y = c_1 e^{-x} + c_2 e^{-2x}}$

$u = e^{-x} \qquad v = e^{-2x}$

$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \Rightarrow -e^{-3x} \neq 0$

hence independent.

P. Integral: $u f(x) + v g(x)$

$f(x) = -\int \frac{vR}{w} dx = -\int \frac{e^{-2x}(x + \cos x)}{(-e^{-3x})} dx \Rightarrow$

$\int e^{x}(x + \cos x) dx$

$f(x) = \int x e^{x} dx + \int e^{x} \cos x \, dx = x e^{x} - e^{x} + \frac{1}{2} e^{x}(\cos x + \sin x)$

$\Rightarrow \boxed{f(x) = x e^{x} - e^{x} + \frac{1}{2} e^{x}(\cos x + \sin x)}$

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$$g(x) = \int \frac{uR}{W}\,dx = \int \frac{e^{-x}(x+\cos x)}{-e^{-3x}} = -\int e^{2x}(x+\cos x)\,dx.$$

$$g(x) = -\int x e^{2x}\,dx - \int e^{2x}\cos x\,dx$$

$$= -\left[\frac{x}{2}e^{2x} - \frac{e^{2x}}{4} + \frac{1}{5}e^{2x}(2\cos x + \sin x)\right]$$

P.S. $\Rightarrow y = uf(x) + vg(x)$.

$$y = e^{-x}\left(xe^{x} - e^{x} + \frac{e^{x}}{2}(\cos x + \sin x)\right) + e^{-2x}\left(\frac{-x}{2}e^{2x} + \frac{e^{2x}}{4} - \frac{e^{2x}}{5}(2\cos x + \sin x)\right)$$

$$y = x - 1 + \frac{\cos x + \sin x}{2} - \frac{x}{2} + \frac{1}{4} - \frac{(2\cos x + \sin x)}{5}.$$

$$y = \frac{x}{2} - \frac{3}{4} + \frac{3\sin x + \cos x}{10}.$$

$$\boxed{y = c_1 e^{-x} - c_2 e^{-2x} + \frac{x}{2} - \frac{3}{4} + \frac{1}{10}(3\sin x + \cos x)}$$

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⑤ $p^2y + 2px - y = 0$

$y(p^2-1) = -2px$

$y(1-p^2) = 2px$

$y = \frac{2px}{1-p^2}$

differentiating, we get

$\frac{dy}{dx} = \frac{(1-p^2)(2p+2x\frac{dp}{dx}) - (2px)(-2p\frac{dp}{dx})}{(1-p^2)^2}$

$p(1-p^2)^2 = (1-p^2)(2p+2x\frac{dp}{dx}) + 4p^2x\frac{dp}{dx}$

$p(1-p^2)^2 = 2p+2x\frac{dp}{dx} - 2p^3 - 2xp^2\frac{dp}{dx} + 4xp^2\frac{dp}{dx}$

$\cancel{p(1+p^4-2p^2)}$

$p(1-p^2)^2 = 2p - 2p^3 + 2x\frac{dp}{dx} + 2xp^2\frac{dp}{dx}$

$p(1-p^2)^2 = 2p(1-p^2) + 2x\frac{dp}{dx}(1+p^2)$

$p(1-p^2)(1-p^2-2) = 2x\frac{dp}{dx}(1+p^2)$

$-p(1-p^2)(1+p^2) = 2x\frac{dp}{dx}(1+p^2)$

$(1+p^2)\left(p(p^2-1) - 2x\frac{dp}{dx}\right) = 0$

$p(p^2-1) - 2x\frac{dp}{dx} = 0$

$\Rightarrow \frac{dx}{x} = \frac{2dp}{p(p^2-1)}$

$\Rightarrow \frac{dx}{x} = -2\left(\frac{1}{p} - \frac{p}{p^2-1}\right)$

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$$\frac{dx}{x} = 2\left(\frac{p}{p^2-1} - \frac{1}{p}\right) dp$$

$$\ln x = \ln(p^2-1) - 2\ln p$$

$$\ln x = \ln\left(\frac{p^2-1}{p^2}\right) + c.$$

$$\frac{p^2-1}{p^2} = cx$$

$$p^2(1-cx) = 1$$

$$p = \sqrt{\frac{1}{1-cx}}$$

Putting this in question we get,

$$\cancel{\frac{y}{1-cx}} \cancel{+2px} \quad \frac{y}{1-cx} + \frac{2x}{\sqrt{1-cx}} = y$$

$$y\left[1 - \frac{1}{1-cx}\right] = \frac{2x}{\sqrt{1-cx}}$$

$$y\left(\frac{1-cx-1}{1-cx}\right) = \frac{2x}{\sqrt{1-cx}}$$

$$\frac{-cy}{\sqrt{1-cx}} = 2x$$

$$\boxed{c^2y^2 = 4x^2(1-cx)}$$