3-D Geometry

CSE-2019:

① (e) Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ —① and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ —② intersect. Find the coordinate of point of intersection and the equation of plane containing them.

→ The condition for intersection is $\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$ where —③

$(\alpha_1,\beta_1,\gamma_1)$ & $(\alpha_2,\beta_2,\gamma_2)$ are points on the first line & second line respectively and l_1,m_1,n_1 & l_2,m_2,n_2 are drs of line ① & line ②.

Line ① passes through $(-1,3,-2)$ and has drs $-3,2,1$.

Line ② passes through $(0,7,-7)$ and has drs $1,-3,2$.

Putting in LHS of ③ $= \begin{vmatrix} -1-0 & 3-7 & -2+7 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} -1 & -4 & 5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$

$= -1[4+3]-4[1+6]+5[9-2]=0$

∴ The two given lines are intersecting lines.

Any point on line ① is $(-3r_1-1,\ 2r_1+3,\ r_1-2)$ & on the second line is $(r_2,\ -3r_2+7,\ 2r_2-7)$. If these points be the points of intersection, then,

$-3r_1-1=r_2$; $2r_1+3=-3r_2+7$; $r_1-2=2r_2-7$ —④

$\Rightarrow 3r_1+r_2+1=0$
$2r_1+3r_2-4=0$ $\Rightarrow \frac{r_1}{-7}=\frac{r_2}{14}=\frac{1}{7}$

$\Rightarrow r_1=-1,\ r_2=2$

Putting in ④ LHS:

r_1-2-2r_2+7

$\Rightarrow -1-2-4+7=0=$ RHS

∴ The point of intersection is $(-3r_1-1,\ 2r_1+3,\ r_1-2)$
$=(2,1,-3)$

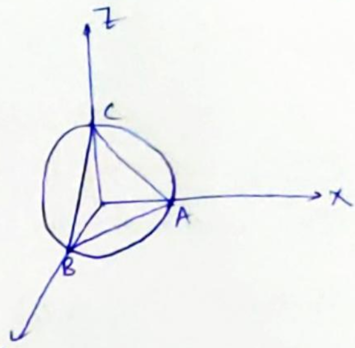
The equation of plane containing the two lines is given by $\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$

$\Rightarrow (x+1)7+(y-3)(7)+(z+2)(7)=0$

$\Rightarrow x+y+z=0$

② The planes $x+2y+3z=12$ cuts the axes of coordinates in A, B & C.
ⓒ(i) Find the equation of the circle circumscribing the triangle ABC.

→ The plane equation can be re-written as



$\frac{x}{12}+\frac{y}{6}+\frac{z}{4}=1 \longrightarrow$ Intercept form.

∴ It's intercepts on x, y, z axes are $12, 6, 4$.

∴ $A(12,0,0)$, $B(0,6,0)$ & $C(0,0,4)$.

Equation of sphere: ~~through A, B and C is~~

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$.

It passes through A, B, C. Then

$A(12,0,0) \equiv \quad 144+24u+d=0 \Rightarrow u=\frac{d-144}{24}=\frac{d}{24}-6$

$B(0,6,0) \equiv \quad 36+12v+d=0 \Rightarrow v=\frac{d-36}{12}=\frac{d}{12}-3$

$C(0,0,4) \equiv \quad 16+8w+d=0 \Rightarrow w=\frac{d-16}{8}=\frac{d}{8}-2$

Putting back in eqⁿ of sphere, we get

$x^2+y^2+z^2+2\left[\frac{d}{24}-6\right]x+2\left[\frac{d}{12}-3\right]y+2\left[\frac{d}{8}-2\right]z+d=0$

$\Rightarrow x^2+y^2+z^2+\frac{d}{12}-12x+\frac{d}{6}-6y+\frac{d}{4}-4z+d=0$

$\Rightarrow x^2+y^2+z^2-12x-6y-4z+\frac{3}{2}d=0 \quad$ ① where d can take any value

The given plane is $x+2y+3z+12=0$. This plane and the sphere ① gives the equation of required circle.

② Prove that the plane $z=0$ cuts the enveloping cone of the
ⓒ(ii) sphere $x^2+y^2+z^2=11$ which has vertex at $(2,4,1)$ in a rectangular hyperbola.

→ Let $S = x^2+y^2+z^2-11$ ~~=0~~. The ~~poin~~ vertex is given as $(2,4,1)$.
Let $x_1=2$, $y_1=4$, $z_1=1$

Let $S_1 \equiv x_1^2+y_1^2+z_1^2-11 = 4+16+1-11=10$

Tangent plane to the given sphere at (x_1,y_1,z_1) is given by

$T \equiv xx_1+yy_1+zz_1-11=0 \Rightarrow T \equiv 2x+4y+z-11$.

Then, the enveloping cone of the sphere $x^2+y^2+z^2=11$ with vertex $(2,4,1)$ is given by

$T^2 = SS_1$.

$(2x+4y+z-11)^2 = 10(x^2+y^2+z^2-11)$

$\Rightarrow$ $4x^2+16y^2+z^2+16xy+8yz+4xz+121-44x-88y-22z$
$= 10(x^2+y^2+z^2-11)$

$\Rightarrow$ $6x^2-6y^2+9z^2-16xy-8yz-4xz+44x+88y+22z-\cancel{242}\;231=0$

It represents a rectangular hyperbola iff coeff of x^2 + coeff of y^2 is equal to zero when it meets the plane $z=0$.

It meets plane $z=0$, then

$$6(x^2-y^2)-16xy+44x+88y-231=0 \quad \text{—— ①}$$

coeff of $x^2=6$, coeff of $y^2=-6$. $\Rightarrow$ coeff of x^2 + coeff of y^2 $= 6-6=0$.

$\therefore$ The equation ① represents a rectangular hyperbola.

③ (b) Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2+y^2=2az$, but if the point lies on the surface $27a(x^2+y^2)+8(a-z)^3=0$, then, two of the three normals coincide.

$\rightarrow$ Given paraboloid: $x^2+y^2=2az$.

Let the given point be (α,β,γ). Now, the equation of tangent plane at any point (x_1,y_1,z_1) to the given paraboloid is

$xx_1+yy_1=a(z+z_1) \Rightarrow xx_1+yy_1-az=az_1$.

$\therefore$ DRs of normal to the tangent plane at (x_1,y_1,z_1) is $x_1, y_1, -a$.

Then, equation of normal to the paraboloid at (x_1,y_1,z_1) is

$$\frac{x-x_1}{x_1}=\frac{y-y_1}{y_1}=\frac{z-z_1}{-a} \quad \text{—— ②}.$$

If this normal passes through (α,β,γ), then

$$\frac{\alpha-x_1}{x_1}=\frac{\beta-y_1}{y_1}=\frac{\gamma-z_1}{-a}=r \text{ (say)} \Rightarrow \alpha=x_1(1+r),\ \beta=y_1(1+r),\ \gamma=z_1-ar.$$

$\therefore$ $x_1=\frac{\alpha}{1+r}$, $y_1=\frac{\beta}{1+r}$, $z_1=\gamma+ar$.

(x_1,y_1,z_1) lie on the given paraboloid. Hence

$$(x_1)^2+(y_1)^2=2az_1 \Rightarrow \frac{\alpha^2}{(1+r)^2}+\frac{\beta^2}{(1+r)^2}=2a(\gamma+ar). \quad \text{—— ③}$$

The equation ③ is a cubic in r and hence gives three values of 'r' i.e. 3 feet of perpendicular from (α,β,γ) to the given paraboloid.

$\therefore$ Three normals can be drawn to the given paraboloid from (α,β,γ).

Now: If the two normals coincide, we have two values of 'r' are equal.

Let ③ be written as

$$f(r) = 2a(1+r)^2(\gamma + ar) - (\alpha^2+\beta^2) = 0 \quad — ④$$

The condition that $f(r)$ has two equal roots is obtained by eliminating r between $f(r)=0$ and $f'(r)=0$

$$f'(r) = 4a(1+r)(\gamma+ar) + 2a^2(1+r)^2 = 0.$$

$$= (1+r)[4a(\gamma+ar) + 2a^2(1+r)] = 0.$$

$$= \cancel{2\gamma + 2a} \quad \cancel{2a} \quad 2(\gamma+ar) + a(1+r) = 0$$

$$= (2a+a)r = -(a+2\gamma) \Rightarrow r = -\frac{(a+2\gamma)}{3a}.$$

Putting in $f(r)$: $2a\left(1-\frac{a+2\gamma}{3a}\right)^2\left(\gamma - a\left(\frac{a+2\gamma}{3a}\right)\right) - (\alpha^2+\beta^2) = 0$

$$\Rightarrow 2[2a-2\gamma]^2[\gamma - a] - 27(\alpha^2+\beta^2) = 0$$

$$\Rightarrow 27(\alpha^2+\beta^2) + 8(a-\gamma)^3 = 0$$

$\therefore$ Required locus of (α,β,γ) is:

$$27(x^2+y^2) + 8(a-z)^3 = 0 \quad — ⑤$$

$\therefore$ The two normals coincide if the point lies on surface ⑤.

4(b) Find the length of the normal chord through a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and prove that if it is equal to $4PG_3$ where G_3 is the point where the normal chord through P meets the xy-plane, then P lies on the cone

$$\frac{x^2}{a^6}(2c^2-a^2) + \frac{y^2}{b^6}(2c^2-b^2) + \frac{z^2}{c^4} = 0.$$

$$\Rightarrow \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} = 2c^2\left[\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right] \quad [\text{from } ③]$$

$$\Rightarrow \frac{\alpha^2 a^2}{a^6} - 2c^2\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6}b^2 - 2c^2\frac{\beta^2}{b^6} + \frac{\gamma^2}{c^4} - \frac{2\gamma^2}{c^4} = 0$$

$$\Rightarrow \frac{\alpha^2}{a^6}(a^2 - 2c^2) + \frac{\beta^2}{b^6}(b^2 - 2c^2) - \frac{\gamma^2}{c^4} = 0$$

$$\Rightarrow \frac{\alpha^2}{a^6}(2c^2 - a^2) + \frac{\beta^2}{b^6}(2c^2 - b^2) + \frac{\gamma^2}{c^4} = 0.$$

$\therefore$ Required locus of $P(\alpha, \beta, \gamma)$ is:

$$\frac{x^2}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0. \quad ⑤$$

$\therefore$ P lies on the cone given by ⑤