IAS MATHEMATICS (OPT.)-2015

PAPER - II : SOLUTIONS

Q.1(a) (i) How many generators are there of the cyclic group G of order 8? Explain.

We have $G = \langle a \rangle$, $a^8 = e$

All the generators of G are a^1, a^3, a^5, a^7

i.e $G = \langle a \rangle = \langle a^3 \rangle = \langle a^5 \rangle = \langle a^7 \rangle$

[$\because$ 1, 3, 5, 7 are positive integers less than 8 and prime to 8]

(ii) Taking a group {e, a, b, c} of order 4, where 'e' is the identity, construct composition tables showing ~~that~~ one is cyclic while the other is not.

<u>sol</u> <u>let us construct composition tables:-</u>

<u>table (1):</u>

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

<u>table (2):</u>

	e	a	b	c
e	a	b	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Clearly table (2) represents cyclic while table (1) is not

1(b) Give an example of a ring having identity but a subring of this having a different identity.

Sol. let $S = \{0, 2, 4, 6, 8\}$ is a subring of Z_{10}

Hence;

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

from the tables

$\forall\ a, b \in S \Rightarrow a - b \in S$
and $a \times b \in S$

Since, $0, 4 \in S \Rightarrow 0 - 4 = -4 = 6 \in S$ etc.
and $0 \times 4 = 0 \in S$ etc.

$\therefore S$ is subring of Z_{10}; where unity is '6'

whereas; unity of Z_{10} is 1.

Hence; A ring Z_{10} having identity '1' but a subring of Z_{10} i.e 'S' having a different identity '6'.

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Q.1(c) Test the convergence and absolute convergence of the series: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^2+1}$.

Sol:- We check the convergence of the given series by applying the Leibnit's Test:-

$\Sigma (-1)^{n+1} \cdot u_n$; $u_n > 0 \ \forall n$; is an alternating series, which converges if

(i) $u_n \geqslant u_{n+1} \ \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

Given series is an alternating series

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n}{n^2+1}$. ; Here; $u_n = \frac{n}{n^2+1}$

we consider; $f(x) = \frac{x}{x^2+1}$

then $f'(x) = \frac{1-x^2}{1+x^2} < 0 \ \forall \ x > 1$.

Thus, 'f' is a decreasing function and.

hence; $u_n \geqslant u_{n+1} \longrightarrow$ [1st condition of Leibnitz test is satisfied]

Now; $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$

Hence; 2nd condition of Leibnitz test satisfied.

Since, both conditions of Leibnitz's test are satisfied; $\therefore$ given series is convergent.

We have $\Sigma \left| (-1)^{n+1} \frac{n}{n^2+1} \right| = \Sigma \frac{n}{n^2+1}$

$= \Sigma$ Ven sky

Here $v_n = \frac{n}{n^2+1}$ term ~~b~~.

$= \frac{1}{n(1+\frac{1}{n^2})}$ term —— (A)

Let $v_n' = \frac{1}{n}$ th.

we have $\lim_{n\to\infty} \frac{v_n}{v_n'} = 1 \neq 0$

$\therefore$ By comparison test

$\sum v_n$ & $\sum v_n'$ are converge or diverge together.

$\because \sum v_n' = \sum \frac{1}{n}$ is divergent by p-test where $p=1$

$\therefore \sum v_n$ is also divergent

$\therefore \sum (-1)^{n+1} \frac{n}{n^2+1}$ is convergent but not absolutely convergent

Q.1(d) Show that the function $v(x,y) = \ln(x^2+y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x,y)$. Also find the corresponding analytic function $f(z) = u + iv$ in terms of z.

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Sol:- given; $v(x,y) = \ln(x^2+y^2) + x + y$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{-(2x^2-2y^2)}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = 0$$

Hence, it is a harmonic function.

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1 = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \frac{2y}{x^2+y^2} + 1$$

integrating w.r.t x

$$u = \tan^{-1}(x/y) + x + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+(x/y)^2} + xy + f'(y)$$

$$\frac{\partial u}{\partial y} = \frac{y^2}{y^2+x^2} + xy + f'(y)$$

$$f'(y) = \frac{du}{dy} - \frac{y^2}{y^2+x^2} - xy$$

$$f'(y) = \frac{2x}{x^2+y^2} + 1 - \frac{y^2}{y^2+x^2} - xy$$

$$f'(y) = \frac{2x - y^2}{x^2+y^2} + 1 - xy$$

$$f(y) = \tan^{-1}(y/x) + y - xy^2 - \int\left[\frac{\cancel{x^2+y^2}}{\cancel{x^2+y^2}} - \frac{x^2}{x^2+y^2}\right]$$

$$f(y) = \tan^{-1}(y/x) + \cancel{y} - xy^2 - \cancel{y} + \frac{x^{\cancel{2}}}{2\cancel{x}}\tan^{-1}(y/x)$$

$$f(y) = \tan^{-1}(y/x) - xy^2 + \frac{x}{2}\tan^{-1}(y/x)$$

Hence; u becomes.

$$u = \tan^{-1}(y/x) + x + \tan^{-1}(y/x) + \frac{x}{2}\tan^{-1}(y/x) - xy^2$$

$$u = 2\tan^{-1}(y/x) + \frac{x}{2}\tan^{-1}(y/x) + x(1-y^2)$$

$$\boxed{u = \tan^{-1}(y/x)\left[2 + x/2\right] + x(1-y^2)}$$

$$f = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \frac{2y}{x^2+y^2} + 1 + i\left(\frac{2x}{x^2+y^2} + 1\right)$$

Put $x = z$, $y = 0$

$$f'(z) = 1 + i\left(\frac{2z}{z^2} + 1\right) = 1 + i\,(2/z + 1)$$

$$f(z) = z + i(2\ln z + z) \pm c.$$

$$\boxed{f(z) = z + i(z + \ln z^2) + c}$$

Q.1(e) Solve the following assignment problem to maximize the sales:

Territories

$$\begin{array}{cc} & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \text{Salesman} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 15 & 13 & 7 & 6 \\ 6 & 13 & 12 & 5 & 11 \\ 7 & 12 & 15 & 8 & 5 \\ 8 & 13 & 10 & 6 & 9 \end{bmatrix} \end{array}$$

Sol: Upon conversion to minimisation, the problem becomes

$$\begin{bmatrix} -3 & -4 & -5 & -6 & -7 \\ -4 & -15 & -13 & -7 & -6 \\ -6 & -13 & -12 & -5 & -11 \\ -7 & -12 & -15 & -8 & -5 \\ -8 & -13 & -10 & -6 & -9 \end{bmatrix}$$

Subtracting the minimum element from each row

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 11 & 0 & 2 & 8 & 9 \\ 7 & 0 & 1 & 8 & 2 \\ 8 & 3 & 0 & 7 & 10 \\ 5 & 0 & 3 & 7 & 4 \end{bmatrix}$$

By. subtracting the minimum element from each column we get

0	3	2	0	0
7	0	2	7	9
3	0	1	7	2
4	3	0	6	10
1	0	3	6	4

Now covering all zeros with minimum no. of lines

0	3	2	0	0
7	0	2	7	9
3	0	1	7	2
4	3	0	6	10
1	0	3	6	4

The no. of lines (3) is less than the order of table (5).

$\therefore$ Subtracting minimum uncovered element (1) from every uncovered element and adding to its intersection elements:

0	4	2	0	0
6	0	1	6	8
2	0	0	6	1
4	4	0	6	10
0	0	2	5	3

$r = 4 < 5 = n.$

Repeat the above process

0	5	3	0	0
5	0	1	5	7
1	0	0	5	0
3	4	0	5	9
0	1	3	5	3

$r = n = 5 \Rightarrow$ optimality is obtained.

To find optimal Solution

~~0~~	5	3	0	~~0~~
5	0	1	5	7
1	~~0~~	~~0~~	5	0
3	4	0	5	9
0	1	3	5	3

Hence: max. sales are

$= 8 + 15 + 15 + 6 + 11$

$= \underline{\underline{55}}$

2(a). If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R', Prove that $\phi(1)$ is the unit element of R'?

Sol:- Let $(R,+,\cdot)$ be a ring with unity, and $(R',+,\cdot)$ be a ring.

Let $f: R \rightarrow R'$ be a homomorphism and onto

$\therefore$ R' is the homomorphic image of R.

i.e. $R' = f(R)$

Let $a', b' \in R'$

$\therefore \exists$ elements $a, b \in R$; such that

$f(a) = a'$, $f(b) = b'$

Since, R is ring unity

$\therefore \forall\, a \in R$; $\exists\, 1 \in R$, s.t

$1 \cdot a = a = a \cdot 1$

Now, since $1 \in R$ (unity in R)

we shall show that $f(1)$ is the unity in R'.

We have

$a' \cdot f(1) = f(a) \cdot f(1)$

$= f(a \cdot 1)$ [$\because f$ is homo]

$= f(a)$ [$\because a \cdot 1 = a$ in R]

$a' \cdot f(1) = a'$

Similarly ; $f(1) \cdot a' = a'$

$\therefore \forall\, a' \in R'$ $\exists$ $f(1) \in R'$ s.t.

$f(1) \cdot a' = a' \cdot f(1) = a'$

$\therefore R'$ is a ring with unity.

<u>Note</u>: The converse of this need not be true i.e. if the homomorphic image of a ring 'R' is ring with unity then the ring 'R' need not be ring with unity.

<u>For example</u>:-

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle/ a, b \in Z \right\}$$ is a ring without unity and Z is a ring with unity.

The mapping $f: R \to Z$ defined as $f\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right\} = a$ is an onto homomorphism.

2(b)
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P-II

→ Is the function $f(x)=\begin{cases}\frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n}\\ 0; & x=0\end{cases}$

Riemann integrable ?

If yes, obtain the value of $\int_0^1 f(x)\,dx$.

Sol Given that

$$f(x)=\begin{cases}\frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n}\\ 0; & x=0.\end{cases}$$

$= 1; \frac{1}{2}<x\le 1$

$= \frac{1}{2}; \frac{1}{3}<x\le\frac{1}{2}$

$= \frac{1}{3}; \frac{1}{4}<x\le\frac{1}{3}$

$\vdots$

$= \frac{1}{n-1}; \frac{1}{n}<x\le\frac{1}{n-1}$.

$\vdots$

$= 0; x=0$

$f(x)$ is bounded and continuous on $[0,1]$ except at the points.

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n+1}, \cdots\cdots$

The set of points of discontinuity.

if f on $[0,1]$ is $\{1/2, 1/3, \frac{1}{4}, \dots\}$ which has one limit point '0'. The set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore$ f is integrable on $[0,1]$.

Now $$\int_{1/n+1}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \dots + \int_{\frac{1}{n+1}}^{1/n} f(x)\,dx$$

$$= \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} \tfrac{1}{2}\,dx + \int_{1/4}^{1/3} \tfrac{1}{3}\,dx + \dots \int_{1/n+1}^{1/n} \tfrac{1}{n}\,dx.$$

$$= (1-1/2) + 1/2(1/2 - 1/3) + 1/3(1/3 - 1/4) + \dots + 1/n\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots \frac{1}{n(n+1)}\right)$$

$$= \left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \left[(1-1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \dots + (1/n - 1/n+1)\right]\right.$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)$$

$$\therefore \int_{1/n+1}^{1} f(x)\,dx = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)$$

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Now taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_{\frac{1}{n+1}}^{1} f(x)\,dx = \lim_{n\to\infty} \left[\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2} \right) - \left(1 - \frac{1}{n+1} \right) \right]$$

$$\Rightarrow \int_0^1 f(x)\,dx = \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) - \lim_{n\to\infty} \left(1 - \frac{1}{n+1} \right)$$

$$= \frac{\pi^2}{6} - 1$$

$\left(\because \text{The series } \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots \text{ converges to } \frac{\pi^2}{6} \right)$.

2.(C) Find all possible Taylor's and Laurent's series expansion of $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about $z=0$?

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Sol. $f(z) = \dfrac{2z-3}{z^2-3z+2} = \dfrac{2z-3}{(z-1)(z-2)}$

By using partial fraction

$$\frac{2z-3}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$= A(z-2) + B(z-1)$$

$\Rightarrow A+B=2$ & $2A+B=+3$

$\therefore$ By Solving ; we get

$A=1,\ B=1$

$$\therefore f(z) = \frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$f_1(z) = \frac{1}{(z-1)} = \frac{1}{z(1-1/z)} = \frac{1}{z}\left(1-1/z\right)^{-1}$$

$$f_1(z) = \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

Since $\left|\frac{1}{z}\right| < 1$ hence $|z|>1$. so valid for $|z|>1$.

Again;

$$f_2(z) = \frac{1}{(z-2)} = \frac{1}{z(1-2/z)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f_2 = \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right] = \left[\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right]$$

Since ; $\left|\frac{2}{z}\right| < 1$; Hence valid for $|z|>2$.

Now, the Laurent's Series

$$(f_1+f_2)z = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\left[\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right]$$

$$(f_1+f_2)z = \frac{2}{z}+\frac{3}{z^2}+\frac{5}{z^3}+\frac{9}{z^4}+\cdots \quad \text{is valid for } |z|>2.$$

is the required Laurent's series.

For Taylor's Series

$$f_1(z) = \frac{1}{(z-1)} = \frac{-1}{(1-z)} = -1(1-z)^{-1}$$

$$f_1(z) = -(1+z+z^2+z^3+\cdots) \quad ; \quad |z|>1.$$

$$f_2(z) = \frac{1}{(z-2)} = \frac{-1}{2(1-z/2)} = \frac{-1}{2}(1-z/2)^{-1}$$

$$f_2(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] \quad ; \quad |z|>2.$$

$$\therefore (f_1+f_2)z = -1(1+z+z^2+\cdots)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

$$(f_1+f_2)z = -\left[(1+z+z^2+z^3+\cdots)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)\right]$$

is the required Taylor's Series.

3(a). State Cauchy's residue theorem. Using it, evaluate the integral. $\int_C \frac{e^z+1}{z(z+1)(z-i)^2}dz$; $C: |z|=2$.

Sol: Given; $\int_C \frac{e^z+1}{z(z+1)(z-i)^2} dz$

Over the region $C: |z|=2$

$$f(z) = \frac{e^z+1}{z(z+1)(z-i)^2}$$

Poles are $z=0$, $z=-1$ of order 1.
$z=i$ of order 2

Now; residue at $z=0$

$$\Rightarrow \lim_{z\to 0} z\cdot f(z) = \lt_{z\to 0} \cancel{z}\cdot \frac{e^z+1}{\cancel{z}(z+1)(z-i)^2}$$

$$= \frac{e^0+1}{(0+1)(0-i)^2} = \frac{2}{i^2} = -2.$$

Residue at $z=-1$.

$$\Rightarrow \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1} \cancel{(z+1)}\cdot\frac{e^z+1}{z\cancel{(z+1)}(z-i)^2}$$

$$= \frac{e^{-1}+1}{(-1)(-1-i)^2} = \frac{-(e^{-1}+1)}{(1+i)^2}$$

Residue at $z=i$ of order 2.

$$\Rightarrow \lt_{z\to i} \frac{d}{dz}\cdot(z-i)^2\cdot f(z) = \lt_{z\to i}\frac{d}{dz}\,\cancel{(z-i)^2}\cdot\frac{e^z+1}{z(z+1)\cancel{(z-i)^2}}$$

$$= \lt_{z\to i}\frac{d}{dz}\cdot\frac{e^z+1}{z(z+1)}$$

$$= \lim_{z\to i} \frac{e^z(z^2+z)-(e^z+1)(2z+1)}{z^2(z+1)^2}$$

$$= \lim_{z\to i} \frac{e^z\cdot z^2+e^z\cdot z-[2e^z\cdot z+e^z+2z+1]}{z^2(z+1)^2}$$

$$= \lim_{z\to i} \frac{e^z[z^2+z-2z-1]-2z-1}{z^2(z+1)^2}$$

$$= \lim_{z\to i} \frac{e^z[z^2-z-1]-2z-1}{z^2(z+1)^2} = \frac{e^i(i^2-i-1)-2i-1}{i^2(1+i)^2}$$

$$= \cancel{-}\frac{(2e^i+2i+1+ie^i)}{\cancel{(-1)}(1+i)^2} = \frac{(2e^i+2i+1+ie^i)}{(1+i)^2}$$

$$= \frac{e^i(2+i)+(2i+1)}{(1+i)^2}$$

$\therefore \oint_c f(z)\,dz = 2\pi i\{\text{residue at } z=0 + \text{residue at } z=-1 + \text{residue at } z=i\}$

$$= 2\pi i\left[-2-\frac{(e^{-1}+1)}{(1+i)^2}+\frac{e^i(2+i)+(2i+1)}{(1+i)^2}\right]$$

$$\oint_c f(z)\,dz = -2\pi i\left[2+\frac{(e^{-1}+\cancel{1})-e^i(2+i)-2i-\cancel{1}}{(1+i)^2}\right]$$

$$\oint_c f(z)\,dz = -2\pi i\left[2+\left[\frac{e^{-1}-e^{-i}(2+i)-2i}{(1+i)^2}\right]\right]$$

A

2015 Q. 3(c). Consider the following LPP.

Max. $Z = x_1 + 2x_2 - 3x_3 + 4x_4$ subject to

$$x_1 + x_2 + 2x_3 + 3x_4 = 12$$
$$x_2 + 2x_3 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Sol: Since, there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero. and then solving the resulting equations. Also the total numbers of basic solution $= {}^4C_2 = 6$. The characteristics of the various basic solutions are given below.

No. of basic Solution	Basic Variables	Non-basic Variables	Values of basic variables	Is the Solution feasible? (Are all $x_j \geq 0$)	Value of Z	Is the Solution optimal?
1	x_1, x_2	$x_3=0$ $x_4=0$	$x_1+x_2=12$ $\therefore x_2=8$ $x_1=4.$	Yes	20	Yes
2.	x_1, x_3	$x_2=0$ $x_4=0$	$x_1+2x_3=12$ $2x_3=8$ $x_3=4, x_1=4$	Yes	-8	—
3.	x_1, x_4	$x_2=0$ $x_3=0$	$x_1+3x_4=12$ $x_4=8$ $x_1=-12$ $x_4=8.$	NO	-20	No.

4	x_2, x_3	$x_1 = 0$ $x_4 = 0$	$x_2 + 2x_3 = 12$ $x_2 + 2x_3 = 8$			No
5.	x_2, x_4	$x_1 = 0$ $x_3 = 0$	$x_2 + 3x_4 = 12$ $x_2 + x_4 = 8$ $\therefore x_4 = 2$ $x_2 = 6$	Yes	6	No
6.	x_3, x_4	$x_1 = 0$ $x_2 = 0$	$2x_3 + 3x_4 = 12$ $2x_3 + x_4 = 8$ $x_4 = 2$ $x_3 = 3$	Yes	-1.	No

Hence, the optimal basic feasible solution is ; $x_1 = 4$, $x_2 = 8$

$x_3 = 0$, $x_4 = 0$

and the Max. value of $Z = \underline{\underline{20}}$.

4(a) IAS-2015

Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields?

(i) $b\sqrt{2}$ → The set of numbers of the form $b\sqrt{2}$ with b rational.

let $F = \{b\sqrt{2} / b \text{ is rational}\}$.

Sol: let $a\sqrt{2}, b\sqrt{2} \in F$; $a, b \in Q$

$\Rightarrow a\sqrt{2} \cdot b\sqrt{2} = ab(2) \notin F$

$\therefore x^n$ is not a binary operation on F

$\therefore$ F is not a ring.

(ii) The set of even integers.

(iii) The set of positive integers.

Sol (ii) let $I_E = \{\ldots -4, -2, 0, 2, 4 \ldots\}$

clearly I_E is an integral domain but not a field. because every non-zero element of I_E doesnot possesses multiplicative inverse.

(iii) let $I^+ = N = \{1, 2, 3, \ldots\}$.

clearly N is not a ring because N is not having identity element w.r.t addition.

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2015 4(c) (6)

2) Solve the LPP by simplex problem. Write its dual Also write optimal soln of dual from optimal table of given prob.

$\therefore$ Max. $Z = 2x_1 - 4x_2 + 5x_3$

S.T $x_1 + 4x_2 - 2x_3 \le 2$

$-x_1 + 2x_2 + 3x_3 \le 1$ $\therefore x_i \geq 0$.

Soln:- Solving by simplex method.

$\therefore$ Standard form of given problm.

Max $Z = 2x_1 - 4x_2 + 5x_3 + 0S_1 + 0S_2$

S.T. $\left.\begin{array}{l} x_1 + 4x_2 - 2x_3 + S_1 + 0S_2 = 2 \\ -x_1 + 2x_2 + 3x_3 + 0S_1 + S_2 = 1 \end{array}\right] -(1)$

$x_1, x_2, S_1, S_2, x_3 \geq 0$ $\because S_1, S_2$ = slack variable

$\therefore$ From (1) IBFS $= (x_1, x_2, x_3; S_1, S_2) = (0, 0, 0, 2,$

Now, I.V = Introducing variable $\therefore$ O.V = outgoing variable

C_B	C_j / Basic	2 / x_1	−4 / x_2	5 / x_3	0 / S_1	0 / S_2	B	θ	
0	S_1	1	4	−2	1	0	2	—	3 = key elem
0	S_2	−1	2	3	0	1	1	1/3 ←	I.V = x_3, O.V = S_2
Z_j	=	0	0	0	0	0			
C_j	=	2	−4	5 ↑	0	0			
0	S_1	2/3	14/3	0	1/2	1/3	4/3	2	2/3 = key element
5	x_3	−1/3	2/3	1	0	1/3	1/3	—	$\therefore$ I.V = x_1, $\therefore$ O.V = S_1
Z_j	=	−5/3	10/3	5	0	5/3			
C_j	=	↑ 11/3	−22/3	0	0	−5/3			
2	x_1	1	7	0	3/4	1/2	2		
5	x_3	0	9	3	3/4	3/2	3		
Z_j	=	2	59	15	21/4	17/2	19 → Max Z.		
C_j	=	0	−63	−10	−21/4	−17/2			

$\therefore C_j\text{'s} \leq 0$.

$\therefore$ Optimal feasible soln. From simplex table we get $(x_1, x_2, x_3, s_1, s_2) = (2, 0, 3, 0, 0)$

$\therefore$ Max $= z = 2x_1 - 4x_2 + 5x_3$
$= 2(2) - 4(0) + 5(3) = \underline{19}$.

$\therefore$ Now <u>Dual of given problem:</u>

$\text{Min } W = 2w_1 + w_2$

s.c $w_1 - w_2 \geq 2$

$4w_1 + 2w_2 \geq -4$

$-2w_1 + 3w_2 \geq 5$

$w_1, w_2, w_3 \geq 0$.

Min $W = 19$.

OBFS: (w_1, w_2)
$= \left(\frac{21}{4}, \frac{17}{2}\right)$

2015
P-2
Q. 5(a)

Solve the partial differential equation

$$(y^2+z^2-x^2)p - 2xyq + 2xz = 0$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$?

Sol: Given $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

Above eqn can be rewritten as

$$(y^2+z^2-x^2)p - 2xyq = -2xz \quad \text{——} (1)$$

Here, the lagrange's Auxillary Equation of the given equation are –

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad \text{——} (2)$$

Taking the last two fractions of (2), we have.

$$\frac{1}{y}dy = \frac{1}{z}dz$$

So that $\frac{1}{y}dy - \frac{1}{z}dz = 0$

$$\Rightarrow \log y - \log z = \log c_1$$

$$\Rightarrow \log(y/z) = \log c_1$$

$$\boxed{y = zc_1} \quad \text{——} (3)$$

Choosing x, y, z, as multipliers, each fraction of (2)

$$= \frac{xdx+ydy+zdz}{xy^2+xz^2-x^3-2xy^2-2xz^2} = \frac{xdx+ydy+zdz}{-x(x^2+y^2+z^2)} \quad (4)$$

combining (1) and (4), we have,

$$\frac{xdx+ydy+zdz}{\cancel{-x}(x^2+y^2+z^2)} = \frac{dz}{-2\cancel{x}z}$$

$$\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} - \frac{dz}{z} = 0$$

Integrating

$$\log(x^2+y^2+z^2) - \log z = \log c_2$$

$$\log\left(\frac{x^2+y^2+z^2}{z}\right) = \log c_2$$

$$\frac{x^2+y^2+z^2}{z} = c_2 \quad \text{——} (5)$$

from (3) and (5), the general solution is given by

$$\phi\left(y/z, \frac{x^2+y^2+z^2}{z}\right) = 0;$$ ϕ being an arbitrary function.

1) Solve $p\cos(x+y) + q\sin(x+y) = z$

6(a) IAS-2014

Sol: Lagrange's auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$

$$\therefore \frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)}$$

$$= \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

Taking 3rd & 4th fractions

$$\frac{dz}{z} = \frac{dx+dy}{\sqrt{2}\sin\left(x+y+\frac{\pi}{4}\right)}$$

$\because \sin\frac{\pi}{4}\cos(x+y) + \cos\frac{\pi}{4}\sin(x+y) = \sin\left((x+y)+\frac{\pi}{4}\right)$

$$\Rightarrow \sqrt{2}\frac{dz}{z} = \operatorname{cosec}\left(t+\frac{\pi}{4}\right)dt$$

by putting $x+y=t$, $dx+dy=dt$

Integrating, we get

$$\sqrt{2}\log z = \log\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)dt + \log c_1$$

$$\frac{\log z^{\sqrt{2}}}{\tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right)} = \log c_1$$

$$u = z^{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) = c_1$$

Again taking 4th & 5th fractions

$$dx-dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy)$$

Integrating,

$$x-y = \log\{\cos(x+y)+\sin(x+y)\}(dx+dy)$$

$$e^{x-y} = \log\{\cos(x+y)+\sin(x+y)\} - \log c_2$$

$$v = e^{y-x}\log\{\cos(x+y)+\sin(x+y)\} = c_2$$

Hence, the required general solution is

$f(u, v) = 0$

i.e, $f\left[z\sqrt{2}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right),\ e^{y-x}\left\{\cos(x+y)+\sin(x+y)\right\}\right] = 0$

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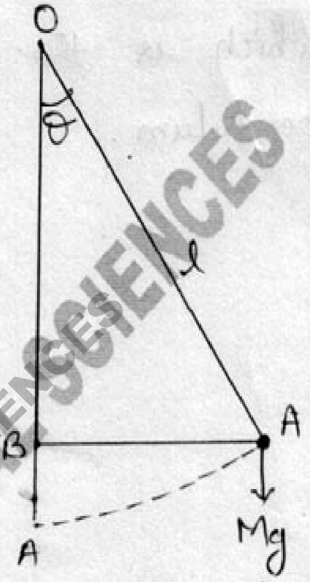
6(b) IAS-2015 P-II → Solve the plane pendulum problem using the Hamiltonian approach and show that H is a constant of motion.

Sol^n → Let l be the length of the pendulum and M be the mass of the bob. At time t, let θ be the inclination of the string to the downward vertical. Then, if T and V are the kinetic and potential energies of the pendulum, then,



$$T = \frac{1}{2} M (l\dot{\theta})^2 = \frac{1}{2} M l^2 \dot{\theta}^2$$

and V = work done against $Mg = Mg\,A'B$

$$= Mgl(1-\cos\theta)$$

$$\therefore L = T - V = \frac{1}{2} Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta) \quad \text{——} (1)$$

Here θ is the only generalised co-ordinate

$$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta} \quad \text{——} (2)$$

Since L does not contain t explicity,

$$\therefore H = T + V = \frac{1}{2} Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta)$$

$$\therefore H = p_\theta^2/(2Ml^2) + Mgl(1-\cos\theta) \quad [\text{from } (2)]$$

Here the two Hamilton's equations are.

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \quad \text{i.e.} \quad \dot{p}_\theta = -Mgl\sin\theta \quad \text{——} (3)$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \text{i.e.} \quad \dot{\theta} = p_\theta/(Ml^2) \quad \text{——} (4)$$

Differentiating (4), we get.

$\ddot{\theta} = \dot{p}_\theta/(ml^2) = -(mgl \sin\theta)/(ml^2)$ [from (3)]

or $\ddot{\theta} = -(g/l)\sin\theta$.

which is the equation of motion of a simple pendulum.

6(c) IAS-2015 Find the lagrange's interpolating polynomial that fits the following data

x	-1	2	3	4
$f(x)$	-1	11	31	69

Find $f(1.5)$?

Sol: Lagrange's Interpolating polynomial is given by -

$$p(x) = \sum_{k=0}^{n} l_k f(x_k)$$

where $l_k = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$

In the given problem, we have,

$$p(x) = \sum_{k=0}^{3} l_k f(x_k).$$

$$p(x) = f(x_0)\frac{(x-2)(x-3)(x-4)}{(-1-2)(-1-3)(-1-4)} + f(x_1)\frac{(x+1)(x-3)(x-4)}{(2+1)(2-3)(2-4)}$$

$$f(x_2)\frac{(x+1)(x-2)(x-4)}{(3+1)(3-2)(3-4)} + f(x_3)\frac{(x+1)(x-2)(x-3)}{(4+1)(4-2)(4-3)}$$

$$p(x) = (+1)\frac{(x-2)(x-3)(x-4)}{+60} + 11\frac{(x+1)(x-3)(x-4)}{+6}$$

$$+31\frac{(x+1)(x-2)(x-4)}{-4} + 69\frac{(x+1)(x-2)(x-3)}{10}$$

$$p(x) = (x-3)(x-4)\left[\frac{x-2}{60} + \frac{11(x+1)}{6}\right]$$

$$+(x+1)(x-2)\left[\frac{31(x-4)}{-4} + \frac{69}{10}(x-3)\right]$$

$$p(x) = (x^2 - 7x + 12)\left[\frac{111x + 108}{60}\right] + (x^2 - x - 2)\left[\frac{-17x + 206}{20}\right]$$

$$= x^3\left[\frac{111}{60} - \frac{17}{20}\right] + x^2\left[\frac{108}{60} - \frac{777}{60} + \frac{206}{20} + \frac{17}{20}\right]$$

$$+ x\left[\frac{111}{5} - \frac{756}{60} - \frac{206}{20} + \frac{34}{20}\right] + \left[\frac{108}{5} - \frac{206}{10}\right]$$

$$p(x) = x^3\left[\frac{111 - 51}{60}\right] + x^2\left[\frac{-\cancel{669} + \cancel{669}}{60}\right]$$

$$+ x\left[\frac{576 - 516}{60}\right] + \left[\frac{216 - 206}{10}\right]$$

$$p(x) = x^3\left[60/60\right] + x^2(0) + x(60/60) + 10/10$$

$$p(x) = x^3 + x + 1$$

Now put $x = 1.5$

$$f(1.5) \sim p(1.5) = (1.5)^3 + 1.5 + 1$$

$$= 5.875.$$

$$\boxed{\therefore\ f(1.5) \sim 5.875}$$

Q. 7(b) Solve the initial value problem
$\frac{dy}{dx} = x(y-x)$, $y(2) = 3$; in the interval $[2, 2.4]$ using the Runge-kutta fourthorder method with step size $h = 0.2$?

Sol:
$$\frac{dy}{dx} = x(y-x)$$
$$x_0 = 2,\quad y_0 = 3,\quad h = 0.2$$

RK. 4th order

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_i + h, y_i + k_3).$$

$\therefore$ 1st Iteration
$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = hf(x_0, y_0) = 0.2f(2,3) = 0.4$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(2.1, 3.2) = 0.462.$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(2.1, 3.231) = 0.47502$$
$$k_4 = hf\left(x_0 + \frac{h}{8}, y_0 + k_3\right) = 0.2f(2.2, 3.47502) = 0.56101.$$

$$\therefore y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_1 = 3 + \frac{1}{6}(0.4 + 2\times 0.462 + 2\times 0.47502 + 0.56101)$$
$$y_1 = 3.47251$$

$\therefore$ ~~At $x_2 = 2.2$~~

$\therefore$ At $x_1 = 2.2$; $y_1 = 3.47251$

2nd Iteration :-

$K_1 = hf(x_1, y_1) = 0.2\, f(2.2, 3.47251) = 0.55990$

$K_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right) = 0.2\, f(2.3, 3.75246)$

$K_2 = 0.66813$

$K_3 = hf\left(x_1+h/2, y_1+\frac{K_2}{2}\right) = 0.2\, f[2.3, 3.80658]$

$K_3 = 0.69303$

$K_4 = hf(x_1+h, y_1+K_3) = 0.2\, f(2.4, 4.16554]$

$K_4 = 0.84746.$

$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y_2 = 3.47251 + \frac{1}{6}[0.5599 + 2(0.66813 + 0.69303) + 0.84746]$

$\boxed{y_2 = 4.16079}$

$\boxed{\therefore \text{At } x = 2.4,\ y = 4.16079}$

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8(a) Reduce the second-order partial differential equation $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol Let $x = e^X$, $y = e^Y$

so that $X = \log x$, $Y = \log y$ — ①

Also let $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$ and $D_1 = \frac{\partial}{\partial X}$, $D_1' = \frac{\partial}{\partial Y}$

Then the given equation becomes

$$[D_1(D_1-1) - 2D_1D_1' + D_1'(D_1'-1) + D_1 + D_1']u = 0$$

$$(D_1^2 - 2D_1D_1' + D_1'^2)u = 0$$

$$(D_1 - D_1')^2 u = 0$$

Hence the required general solution is

$$C.F = \phi_1(Y+X) + X\phi_2(Y+X)$$

$$= \phi_1(\log y + \log x) + \log x\ \phi_2(\log y + \log x)$$

$$= \phi_1(\log xy) + \log x \cdot \phi_2(\log xy)$$

$$= F_1(xy) + \log x\ F_2(xy)$$

Q. 8(b) Solve the equations:-

P-II, IAS 2015

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9.$$

by Gauss-Seidel iteration method.

Sol:- Rewriting the given equations as -

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{--- (1)}$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{--- (2)}$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{--- (3)}$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \quad \text{--- (4)}$$

First iteration

Putting; $x_2 = 0$; $x_3 = 0$, $x_4 = 0$ in (1); we get
$x_1 = 0.3$.

Putting; $x_1 = 0.3$; $x_3 = 0$, $x_4 = 0$ in (2), we get $x_2 = 1.56$

Putting; $x_1 = 0.3$; $x_2 = 1.56$; $x_4 = 0$ in (3); we get $x_3 = 2.886$

Putting: $x_1 = 0.3$; $x_2 = 1.56$; $x_3 = 2.886$ in (4); we get $x_4 = -0.1368$

Second Iteration:-

Putting $x_2 = 1.56$; $x_3 = 2.886$; $x_4 = -0.1368$
in (1) we obtain $x_1 = 0.8869$.

Putting; $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$
in (2) we get $x_2 = 1.9523$

Putting: $x_1 = 0.8869$; $x_2 = 1.9523$, $x_4 = -0.1368$
in (3) we get $x_3 = 2.9566$.

Putting $x_1 = 0.8869$; $x_2 = 1.9523$; $x_3 = 2.9566$
in (4) ; we get $x_4 = -0.0248$.

Third iteration:-

Putting $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$
in (1) we get $x_1 = 0.9836$

Putting $x_1 = 0.9836$, $x_3 = 2.9566$; $x_4 = -0.0248$
in (2) ; we get $x_2 = 1.9899$

Putting $x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$
in (3) ; we get $x_3 = 2.9924$.

Putting $x_1 = 0.9836$; $x_2 = 1.9899$, $x_3 = 2.9924$.
in (4) ; we get $x_4 = -0.0042$.

Fourth iteration

Putting $x_2 = 1.9899$, $x_3 = 2.9924$, $x_4 = -0.0042$
in (1) we get $x_1 = 0.9968$

Putting $x_1 = 0.9968$, $x_3 = 2.9924$, $x_4 = -0.0042$
in (2) we get $x_2 = 1.9982$

Putting $x_1 = 0.9968$, $x_2 = 1.9982$, $x_4 = -0.0042$
in (3) we get $x_3 = 2.9987$

Putting $x_1 = 0.9968$; $x_3 = 1.9982$; $x_3 = 2.9987$
in (4) we get $\underline{x_4 = -0.0008}$.

$\therefore$ After 4^{th} Iteration.

$x_1 = 0.9968$; $x_2 = 1.9982$; $x_3 = 2.9987$; $x_4 = -0.0008$