or
$$T = 4W\left(\frac{2a\sin\theta - a}{a}\right)$$

or
$$T = 4W(2\sin\theta - 1) \quad \text{...(2)}$$

From (1) and (2), we have $2W = 4W(2\sin\theta - 1)$

or
$$W = 2W(2\sin\theta - 1)$$

or
$$4\sin\theta = 3 \Rightarrow \theta = \sin^{-1}\left(\frac{3}{4}\right)$$

Hence, the required angle between AB and BC $= 2\theta = 2\sin^{-1}\left(\frac{3}{4}\right)$.

Some Miscellaneous Solved Examples :

Example 12.

7.a IFS 2009 P-1

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, show that $\tan\phi = \frac{3}{8} + \tan\theta$. **[K.U. 2018, 17, 16, 15, 08, 05; M.D.U. 2011, 08]**

Solution. Let O be the point of suspension in the wall, AB the base of the hemisphere, C its centre, G its centre of gravity, M the point of contact of the hemisphere and the wall and OA the string. Let l be the length of the string OA and let a be the radius of the hemisphere.

$$\therefore \quad CA = a \quad \text{and} \quad CG = \frac{3a}{8}$$

Since O is a fixed point, so all the distances will be measured from this point O.

Let d be the depth of G below O

$$\therefore \quad d = OM + FG = OL + LM + CG\sin\phi$$

$$= l\cos\theta + AC\cos\phi + \frac{3a}{8}\sin\phi$$

$$\Rightarrow \quad d = l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi \quad \text{...(1)}$$

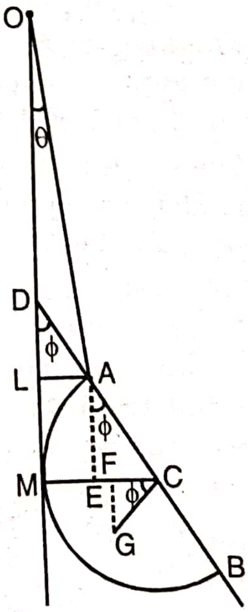

Fig. 8.21

The normal reaction at M is perpendicular to the wall.

$\therefore$ MC is horizontal

Let the system be given a small virtual displacement such that θ becomes $\theta + \delta\theta$ and ϕ becomes $\phi + \delta\phi$.

W, the weight of the hemisphere will be the only force doing work. The reaction at M does not appear in the equation of virtual work

$\therefore$ Equation of virtual work is $W.\delta(d) = 0$

or
$$\delta(d) = 0 \quad [\because W \neq 0]$$

Scanned with CamScanner

or $$\delta\left[l \cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right] = 0$$

or $$-l \sin\theta \,.\, \delta\theta - a \sin\phi\,\delta\phi + \frac{3a}{8}\cos\phi\,\delta\phi = 0$$

or $$l \sin\theta.\delta\theta = \left(\frac{3}{8}\cos\phi - \sin\phi\right).\, a\delta\phi \qquad ...(2)$$

Again, $$a = CM = CE + EM = CE + AL \qquad [\because EM = AL]$$
$$= CA \sin\phi + OA \sin\theta$$
$$= a \sin\phi + l \sin\theta$$

or $$l \sin\theta = a - a\sin\phi$$

Differentiating, $$l \cos\theta \,.\, \delta\theta = -\,a\cos\phi\,\delta\phi \qquad ...(3)$$

Dividing (2) by (3), we get $$\tan\theta = -\frac{3}{8} + \tan\phi$$

Hence $$\tan\phi = \frac{3}{8} + \tan\theta\,.$$

Scanned with CamScanner

Example 2. **A particle moves with a central acceleration $\mu\left(r+\frac{a^4}{r^3}\right)$, being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$.**

Prove that it describes the curve $r^2(2+\cos\sqrt{3}\theta)=3a^2$.

Solution. Here, $F=\mu\left(r+\frac{a^4}{r^3}\right)=\mu(u^{-1}+a^4u^3)$ ...(1)

where $u=\frac{1}{r}$

Differential equation of central orbit is

$$h^2\left(u+\frac{d^2u}{d\theta^2}\right)=\frac{F}{u^2}=\mu\frac{(u^{-1}+a^4u^3)}{u^2}=\mu(u^{-3}+a^4u) \quad ...(2)$$

Multiplying by $2\frac{du}{d\theta}$, we get

$$h^2\left[2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right]=2\mu[u^{-3}+a^4u]\frac{du}{d\theta}$$

Integrating, we get

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left[-\frac{1}{2}u^{-2}+\frac{a^4u^2}{2}\right]+c$$

or
$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]$$

$$=\mu(-u^{-2}+a^4u^2)+c \quad ...(3)$$

Initially, at an apse, $u=\frac{1}{a},\frac{du}{d\theta}=0$ and $v=2a\sqrt{\mu}$ *[Given]*

$\therefore$ From (3), $4a^2\mu=\frac{h^2}{a^2}=\mu(-a^2+a^2)+c$

$\therefore$ $h^2=4\mu a^4$ and $c=4\mu a^2$

Scanned with CamScanner

Putting the values of h^2 and c in (3), we get

$$4\mu a^4\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left(-u^{-2}+a^4u^2\right)+4\mu a^2$$

or $$4a^4\left(\frac{du}{d\theta}\right)^2=-\frac{1}{u^2}+a^4u^2-4a^4u^2+4a^2$$

$$=\frac{-1+a^4u^4-4a^4u^4+4a^2u^2}{u^2}$$

$$=\frac{-1-3a^4u^4+4a^2u^2}{u^2}$$

$$=\frac{-1-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2}{u^2}$$

$$=\frac{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}{u^2}$$

or $$2a^2\frac{du}{d\theta}=\pm\frac{\left[\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2\right]^{1/2}}{u}$$

or $$-\frac{2\sqrt{3}\,a^2u\,du}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}}=\sqrt{3}d\theta$$

[*Taking –ve sign*]

Put $\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}=t$ so that $2\sqrt{3}a^2u\,du=dt$

$\therefore$ $$-\frac{dt}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2-t^2}}=\sqrt{3}\,d\theta$$

Integrating, $\cos^{-1}(t\sqrt{3})=\sqrt{3}\,\theta+A$ ...(4)

Initially, when $u=\frac{1}{a}$, *i.e.*, $t=\frac{1}{\sqrt{3}}$, $\theta=0$

$\therefore$ $A=0$

$\therefore$ From (4), $\cos^{-1}(t\sqrt{3})=\sqrt{3}\,\theta$

or $$t\sqrt{3}=\cos\sqrt{3}\theta$$

or $$\sqrt{3}\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)=\cos\sqrt{3}\theta$$

or $$3a^2u^2-2=\cos\sqrt{3}\theta$$

or $$3a^2u^2=2+\cos\sqrt{3}\theta$$

Hence, $\mathbf{3a^2=r^2\,[2+\cos\sqrt{3}\,\theta]}$ $\left[\because u=\frac{1}{r}\right]$

which is the required path.

Scanned with CamScanner