Q 1 ⇒

2012 − ODE − CSE

Solve $\frac{dy}{dx} = \frac{2xy\, e^{(x/y)^2}}{y^2\left[1 + e^{(x/y)^2}\right] + 2x^2 e^{(x/y)^2}}$

Sol^n

$\because$ Given eq^n is homogeneous.

let $x = vy \Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{y^2\left[1+e^{v^2}\right] + 2y^2v^2e^{v^2}}{2vy^2e^{v^2}}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{1+e^{v^2} + 2v^2e^{v^2}}{2ve^{v^2}}$$

$$\Rightarrow \quad y\frac{dv}{dy} = \frac{1+e^{v^2}}{2ve^{v^2}}$$

$$\Rightarrow \quad \frac{2ve^{v^2}}{1+e^{v^2}} = \frac{dy}{y} \qquad \left[\begin{array}{l}\text{take } 1+e^{v^2} = t \\ 2ve^{v^2}\,dv = dt\end{array}\right]$$

$$\ln(t) = \ln y + \ln(c)$$

$$\log(1+e^{v^2}).c = \log y$$

$$\boxed{y = \left\{1 + e^{\left(\frac{x}{y}\right)^2}\right\}.c}$$

Q 2 ⇒ Find the orthogonal Trajectory of the Family of Curves $x^2+y^2 = ax$.

Sol^n

Given $x^2+y^2 = ax$.

Diff. w.r.t. x. $\quad 2x + 2y\frac{dy}{dx} = a$

$$\Rightarrow \quad (x^2+y^2) = \left(2x + 2y\frac{dy}{dx}\right)x$$

for orthogonal replace $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}}$

$$\Rightarrow \quad (x^2+y^2) = 2x^2 + 2xy\frac{(-1)}{\frac{dy}{dx}}$$

$$\Rightarrow 2xy\frac{dx}{dy} = (x^2-y^2)$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^2-y^2}{2xy} \quad \text{—— (1)}$$

Clearly this eq^n is homogenous.

so let $x = vy$

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

replacing in eq^n (1)

$$v + y\frac{dv}{dy} = \frac{(v^2-1)y^2}{2vy^2}$$

$$v + y\frac{dv}{dy} = \frac{v^2-1}{2v} \Rightarrow y\frac{dv}{dy} = \frac{v^2-1}{2v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{v^2-1-2v^2}{2v} \Rightarrow y\frac{dv}{dy} = -\left(\frac{1+v^2}{2v}\right)$$

$$\Rightarrow \left(\frac{2v}{1+v^2}\right)dv = -\left(\frac{dy}{y}\right)$$

$$\log(1+v^2) = -\log(y) + \log c$$

$$(1+v^2)y = c$$

$$\left(1+\frac{x^2}{y^2}\right)y = c$$

$$\boxed{\therefore x^2+y^2 = cy}$$

Q3 ⇒ Show that the differential Equation $(2xy\log y)dx + (x^2+y^2\sqrt{y^2+1})dy = 0$ is not exact. Find the general ~~solution of the equation~~ an integrating factor and solution of the equation.

hence, the solution of the equation.

Sol^n Given eq^n $(2xy\log y)dx + (x^2+y^2\sqrt{y^2+1})dy = 0$ —(1)

Comparing with
$M\,dx + N\,dy = 0$

$$\frac{\delta M}{\delta y} = 2x\log y + 2x \Rightarrow 2x(1+\log y)$$

$$\frac{\delta N}{\delta x} = 2x$$

$$\therefore \frac{\delta M}{\delta y} \neq \frac{\delta N}{\delta x}$$

∴ Given eq^n is not Exact equation.

$$\because \frac{\left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right)}{M} = \frac{2x - 2x - 2x\log y}{2xy\log y} = -\frac{1}{y} = f(y).$$

$$\therefore \text{I.F.} = e^{\int f(y)dy} = e^{\int -\frac{1}{y}dy} = \frac{1}{y}$$

$\therefore$ multiply Equation ① by $\frac{1}{y}$

$$(2x\log y)\,dx + \left(\frac{x^2}{y} + y\sqrt{y^2+1}\right)dy = 0 \quad \text{— ②}$$

$\therefore$ Now $\frac{\delta M}{\delta y} = \frac{2x}{y}$ & $\frac{\delta N}{\delta x} = \frac{2x}{y}$

$\therefore \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$; $\therefore$ eqn ② is exact diff.

Integrating eqn ②; keeping y constant in first term and Elemenating the term containing in ~~2nd term~~ x in 2nd term.

$$\therefore \int (2x\log y)\,dx + \int y\sqrt{y^2+1}\,dy = c$$

$$\boxed{x^2\log y + \frac{(y^2+1)^{3/2}}{3} = c}$$

Q4⇒ Find the General solution of the Equation

$y''' - y'' = -12x^2 + 6x.$

Solⁿ Given $y''' - y'' = 12x^2 + 6x$ — ①

Homogeneous eqn of ① is $D^3 - D^2 = 0$

Auxiallary Eqn $m^3 - m^2 = 0$

$m = 0, 0, 1.$

$$\therefore \text{C.F.} = (c_1 + c_2x)e^{0x} + c_3e^x$$

$$\text{C.F.} = c_1 + c_2x + c_3e^x.$$

Particular integral $\text{P.I.} = \frac{(12x^2 + 6x)}{D^3 - D^2}$

$$= \frac{-1}{D^2}[1 - D]^{-1}[12x^2 + 6x]$$

$$\Rightarrow \frac{-1}{D^2}\left[1+D+D^2 \cdots\right]\left[12x^2+6x\right]$$

$$\Rightarrow -\frac{1}{D^2}\left[12x^2+6x+24x+6+24\right]$$

$$y_P = -x^4-5x^3-15x^2$$

$\therefore$ General sol^n $\boxed{= c_1+c_2x+c_3e^x-x^4-5x^3-15x^2}$

Q5⇒ Solve the ordinary differential equation $x(x-1)y''-(2x-1)y'+2y = x^2(2x-3)$.

Sol^n Given Eq^n is $x(x-1)y''-(2x-1)y'+2y = x^2(2x-3)$

$$y'' - \frac{(2x-1)}{x(x-1)}y' + \frac{2}{x(x-1)}y = \frac{x(2x-3)}{(x-1)} \quad \text{—— (1)}$$

Compare Eq^n(1) with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$$P = -\frac{(2x-1)}{x(x-1)} \;;\; Q = \frac{2}{x(x-1)} \;;\; R = \frac{x(2x-3)}{x-1}$$

$\because$ $2+2Px+Qx^2=0$

$\therefore$ $u=x^2$ is part of complementary function.

let General solution be $y=uv$

$u=x^2$.

$\therefore$ New Eq^n.

$$\frac{d^2v}{dx^2} + \left(P+\frac{2u'}{u}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left[\frac{-(2x-1)}{x(x-1)} + \frac{4x}{x^2}\right]\frac{dv}{dx} = \frac{(2x-3)}{x(x-1)}$$

$$\frac{d^2v}{dx^2} + \frac{(2x-3)}{x(x-1)}\frac{dv}{dx} = \frac{2x-3}{x(x-1}$$

$$\frac{dv}{dx} = p$$

Solve $\frac{dt}{dx} + \frac{(2x-3)}{x(x-1)} t = \frac{2x-3}{x(x-1)}$

$$\frac{dt}{dx} = \frac{2x-3}{x(x-1)}(1-t)$$

$$\int \frac{dt}{(1-t)} = \int \left[\frac{3}{x} + \left(\frac{-1}{x-1}\right) \right] dx + c$$

$$\log(1-t) = 3\log x - \log(x-1) + \log c.$$

$$(1-t) = \frac{c_1(x-1)}{x^3}$$

$$1 - \frac{dv}{dx} = \frac{c_1}{x^2} - \frac{c_1}{x^3}$$

$$\frac{dv}{dx} = 1 + \frac{c_1}{x^2} - \frac{c_1}{x^3}$$

$$v = x + \frac{c_1}{x} - \frac{c_1}{2x^2} + c_2$$

$\because y = uv$

$\therefore y = \left(x + \frac{c_1}{x} - \frac{c_1}{2x^2} + c_2\right) x^2$

$$\boxed{y = x^3 + c_1 x - \frac{c_1}{2} + c_2 x^2}$$

Q6 $\Rightarrow$ ~~$y'' + 2y' + y = e^{t}$~~ Using Laplace transform, solve the initial value Problem

$$y'' + 2y' + y = e^{-t}; \quad y(0) = -1, \; y'(0) = 1.$$

Sol^n

Given $y'' + 2y' + y = e^{-t}$

Applying Laplace transform

$$L(y'' + 2y' + y) = L(e^{-t})$$

$$p^2 L(y) - p\,y(0) - y'(0) + 2p\,L(y) - 2y(0) + L(y) = \frac{1}{1+p}$$

$\because$ Given $y(0) = -1, \; y'(0) = 1$

$$p^2 L(y) + p - 1 + 2pL(y) + 2 + L(y) = \frac{1}{1+p}$$

$$(p^2 + 2p + 1) L(y) + (p+1) = \frac{1}{1+p}$$

$$(1+p)^2 L(y) = \frac{1}{1+p} - (p+1)$$

$$L(y) = \frac{1}{(1+p)^3} - \frac{1}{1+p}$$

Apply inverse Laplace transform

$$y(t) = e^{-t}\frac{t^2}{2} - e^{-t}$$

$$\boxed{y(t) = \left(\frac{t^2}{2} - 1\right)e^{-t}}$$