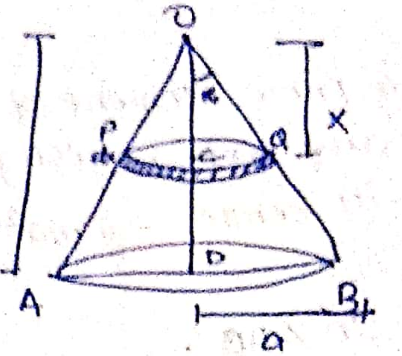
(A) let OAB be right circular cone of vertex O at a distance h from circular base with AB as diameter having radius 'a'



Now $M = \rho V$

$= \rho \times \frac{1}{3}\pi a^2 h$ (volume of cone $= \frac{1}{3}\pi a^2 h$)

$= \rho \times \frac{1}{3}\pi \times (h\tan\alpha)^2 \times h$ $(\because \tan\alpha = a/h)$

$= \frac{1}{3}\pi \rho h^3 \tan^2\alpha$ — (I)

If we consider an elementary ~~mass~~ disc PQ of thickness dx parallel to base AB and at a distance x from vertex O

$dm = \rho \times (\pi x^2 \times \tan^2\alpha) \times dx$

M.I of elemantary disc about axis OD of cone

$= \frac{1}{2}(dm)\, CP^2$

$= \frac{1}{2} \times (\rho \times \pi x^2 \times \tan^2\alpha \times dx)\,(x\tan\alpha)^2$

$dI = \frac{1}{2}\rho\pi\, x^4 \tan^4\alpha\, dx$

$I = \int dI = \int_0^h \frac{1}{2}\rho\pi\, x^4 \tan^4\alpha\, dx$

$= \frac{1}{2}\rho\pi \tan^4\alpha \cdot \frac{h^5}{5}$

$I = \frac{\rho\pi h^5 \tan^4\alpha}{10}$ — (II)

Substituting ~~tan α~~ ρ from (I)

$I = \frac{M}{\frac{\pi h^3 \tan^2\alpha}{3}} \times \frac{\pi}{10} \times h^5 \times \tan^4\alpha$

$= \frac{3M}{10} h^2 \tan^2\alpha = \boxed{\frac{3M}{10} a^2}$

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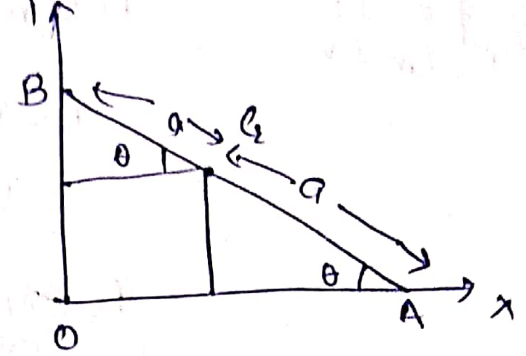
let AB be rod of length '2a' & mass M.

at time t, let rod be inclined at angle θ with horizontal



G $(a\cos\theta, a\sin\theta)$

velocity of G is given by

$$v_G^2 = (-a\sin\theta\,\dot{\theta})^2 + (a\cos\theta\,\dot{\theta})^2 = a^2\dot{\theta}^2$$

K.E of rod at time t

$$= \frac{1}{2}M\left(\frac{1}{3}a^2\dot{\theta}^2\right) + \frac{1}{2}M(a^2\dot{\theta}^2) = \frac{2}{3}Ma^2\dot{\theta}^2$$

But initial $\dot{\theta}$ given $= \sqrt{\dfrac{3g(1-\sin\alpha)}{2a}}$

$$\text{Initial K.E} = \frac{2}{3}Ma^2 \times \frac{3g}{2a}(1-\sin\alpha)$$

$$= Mag\,(1-\sin\alpha)$$

Change in K.E = work done by gravity

$$\frac{2}{3}Ma^2\dot{\theta}^2 - Mag(1-\sin\alpha) = Mg(a\sin\alpha - a\sin\theta)$$

$$\frac{2}{3}Ma^2\dot{\theta}^2 = Mag\,(1-\sin\theta)$$

$$\dot{\theta}^2 = \frac{3g}{2a}(1-\sin\theta)$$

$$\dot{\theta} = \frac{d\theta}{dt} = -\sqrt{\frac{3g}{2a}(1-\sin\theta)}$$

-ve sign indicating motion towards θ decreasing.

$$dt = +\sqrt{\frac{2a}{3g}}\ \frac{d\theta}{\sqrt{1-\sin\theta}}$$

$$t = +\sqrt{\frac{2a}{3g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{1-\sin\theta}}$$

$$= +\sqrt{\frac{2a}{3g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{\cos^2\theta/_2+\sin^2\theta/_2-2\sin\theta/_2\cos\theta/_2}}$$

$$= +\sqrt{\frac{2a}{3g}}\int_0^{\alpha}\frac{d\theta}{\sqrt{(\cos\theta/_2-\sin\theta/_2)}}$$

$$= +\sqrt{\frac{2a}{3g}}\cdot\frac{1}{\sqrt{2}}\int_0^{\alpha}\frac{d\theta}{\sin(\pi/_4-\theta/_2)}$$

$$= \sqrt{\frac{a}{3g}}\int_0^{\alpha}\operatorname{cosec}(\pi/_4-\theta/_2)\,d\theta$$

$$= \sqrt{a/_{3g}}\left[-2\log\tan(\pi/8-\theta/4)\right]_0^{\alpha}$$

$$= 2\sqrt{\frac{a}{3g}}\left[\log\tan\pi/_8-\log\tan(\pi/_8-\theta/_4)\right]$$

$$\boxed{t = -2\sqrt{\frac{a}{3g}}\left[\log(\tan\pi/_8-\theta/_4)\cot\pi/_8\right]}$$

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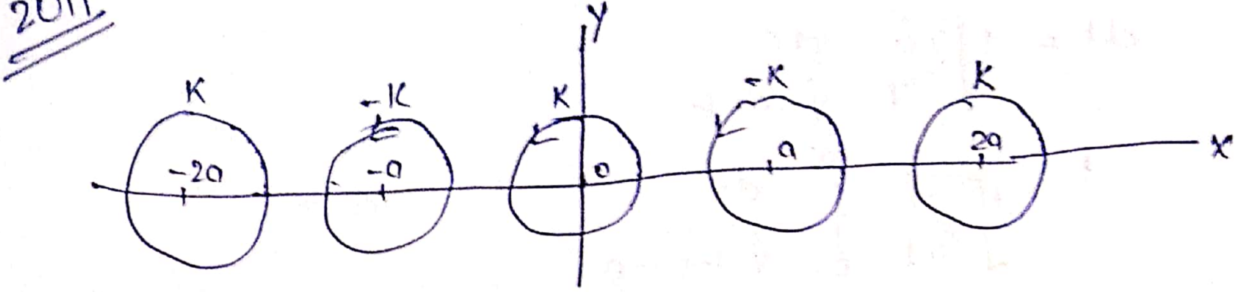

Let row of vortices be taken along x-axis.

So, we have vortices of strength K each at points (0,0), (±2a,0), (±4a,0) .... and those of strength -K each at points (±a,0), (±3a,0) ....

complex potential of entire system is given by

$$W = \frac{iK}{2\pi}\Big[\log z + \log(z-2a) + \log(z+2a) + \cdots - \{\log(z-a) + \log(z+a) + \log(z-3a) + \cdots\Big]$$

$$= \frac{iK}{2\pi} \log \frac{z\,[z^2-(2a)^2]\,[z^2-(4a)^2]\cdots}{(z^2-a^2)\,(z^2-(3a)^2)\cdots}$$

$$= \frac{iK}{2\pi} \log \frac{\frac{z}{2a}\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{4a}\right)^2\right]\cdots}{\left[1-\left(\frac{z}{a}\right)^2\right]\left[1-\left(\frac{z}{3a}\right)^2\right]\cdots}$$

$$= \frac{iK}{2\pi} \log \frac{\sin \pi z/2a}{\cos(\pi z/2a)}$$

$$\boxed{W = \frac{iK}{2\pi} \log \tan\left(\frac{\pi z}{2a}\right)}$$

This will determine velocity potential & stream function.