15 Functions of Several Variables

So far attention has mainly been directed to functions of a single independent variable and the application of the differential calculus to such functions has been considered. In this chapter, we shall be mainly concerned with the application of differential calculus to functions of more than one variable. The characteristic properties of a function of n independent variables may usually be understood by the study of a function of two or three variables and this restriction of two or three variables will be generally maintained. This restriction has the considerable advantage of simplifying the formulae and of reducing the mechanical labour.

1. EXPLICIT AND IMPLICIT FUNCTIONS

If we consider a set of n independent variables $x, y, z, \ldots, t$ and one dependent variable u, the equation

$$u = f(x, y, z, \ldots, t) \qquad \ldots(1)$$

denotes the functional relation. In this case if $x_1, y_1, z_1, \ldots, t_1$, are the n arbitrarily assigned values of the independent variables, the corresponding values of the dependent variable u are determined by the function relation.

The function represented by equation (1) is an *explicit* function but where several variables are concerned it is rarely possible to obtain an equation expressing one of the variables explicitly in terms of the others. Thus most of the functions of more than one variable are *implicit* functions, that is to say we are given a functional relation

$$\phi(x, y, z, \ldots, t) = 0 \qquad \ldots(2)$$

connecting the n variables $x, y, z, \ldots, t$, and is not in general possible to solve this equation to find an *explicit* function which expresses one of these variables say x, in terms of the other $n-1$ variables.

In this chapter we shall be mainly concerned with the *explicit* functions.

1.1 An Explicit Function of Two Variables

If x, y are two independent variables and a variable z depends for its values on the values of x, y by a functional relation

$$z = f(x, y) \qquad \ldots(3)$$

then we say z is a *function of* x, y. The ordered pair of numbers (x, y) is called a *point* and the aggregate of the pairs of numbers (x, y) is said to be the *domain* (or region) *of definition* of the function.

When the domain of definition is bounded by a closed curve C, it is said to be *closed* if f is defined for all points within and on the curve C; but *open or unclosed* when the function is defined for points within but not on the curve C.

1.2 The Neighbourhood of a Point

The set of values x_1, y_1 other than a, b that satisfy the conditions

$$|x_1 - a| < \delta, |y_1 - b| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point (a, b). Thus a neighbourhood is the square

$$(a - \delta, a + \delta; b - \delta, b + \delta)$$

where x takes any value from $a - \delta$ to $a + \delta$ except a, and y from $b - \delta$ to $b + \delta$ except b.

This is not the only way of specifying a neighbourhood of a point. There can be many other, though equivalent ways; for example the points inside the circle $x^2 + y^2 = \delta^2$ may be taken as a neighbourhood of the point (0, 0).

1.3 Limit Point

A point (ξ, η) is called a *limit point* or *a point of condensation* of a set of points S, if for every neighbourhood of (ξ, η) contains an infinite number of points of S. The limit point itself may or may not be a point of the set. For example, the point (0, 0) is a limit point of the set $\{(1/m, 1/n) : m, n \in \mathbf{N}\}$.

1.4 The Limit of a Function

A function f is said to tend to a limit l as a point (x, y) tends to the point (a, b) if for every arbitrarily small positive number ε, there corresponds a positive number δ, such that

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y), [different from (a, b)] which satisfies

$$|x - a| < \delta, |y - b| < \delta$$

In other words, *a function f tends to a limit l, when (x, y) tends to (a, b) if for every positive number ε, there corresponds a neighbourhood N of (a, b) such that*

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y) other than (a, b) of the neighbourhood N.

Symbolically we then write

$$\lim_{(x, y)\to(a, b)} f(x, y) = l.$$

l is the *limit* (the *double limit* or the *simultaneous limit*) of f when x, y tend to a & b simultaneously.

Remark: The above definition implies that there must be no assumption of any relation between the independent variables as they tend to their respective limits.

For instance take $f(x, y)$ where

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

and find the limit when $(x, y) \to (0, 0)$.

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If we put $y = m_1 x$ and let $x \to 0$, we get the limit to be equal to $\frac{m_1}{1+m_1^2}$, while putting $y = m_2 x$ leads to a limit $\frac{m_2}{1+m_2^2}$. Similarly letting $x \to 0$, while y remains constant or vice-versa leads to zero limit. Thus, we are led to erroneous results. Geometrically speaking when we approach the point (0, 0) along different paths, first along lines with slopes m_1 and m_2 and then along lines parallel to the coordinate axes, the function reaches different limits. The simultaneous limit postulates that by whatever path the point is approached, the function f attains the same limit. In general the determination whether a simultaneous limit exists or not is a difficult matter but very often a simple consideration enables us to show that the *limit does not exist.*

It may however be noted that

$$\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = l \;\Rightarrow\; \lim_{x\to a} f(x,\,b) = l = \lim_{y\to b} f(a,\,y)$$

Non-existence of limit. The above remark makes it amply clear that if $\lim_{(x,\,y)\to(a,\,b)} f(x,\,y) = l$ and if $y = \phi(x)$ is any function such that $\phi(x) \to b$, when $x \to a$, then $\lim_{x\to a} f(x,\,\phi(x))$ must exist and should be equal to l.

Thus, if we can find two functions $\phi_1(x)$ and $\phi_2(x)$ *such that the limits of* $f(x,\,\phi_1(x))$ *and* $f(x,\,\phi_2(x))$ *are different, then the simultaneous limit in question does not exist.*

Example 1(*a*). Let

$$f(x,\,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x + y = 0 \end{cases}$$

If we approach the origin along any axis, $f(x,\,y) \equiv 0$.

If we approach (0, 0) along any line $y = mx$, then

$$f(x,\,y) = f(x,\,mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \to 0, \quad \text{as } x \to 0$$

So any straight line approach gives,

$$\lim_{(x,\,y)\to(0,\,0)} f(x,\,y) = 0$$

But putting $y = mx^2$,

$$\lim_{(x,\,y)\to(0,\,0)} f(x,\,y) = \lim_{x\to 0} f(x,\,mx^2) = \frac{m}{1+m^2}$$

which is different for the different m selected.

Hence, $\lim_{(x,\,y)\to(0,\,0)} f(x,\,y)$ does not exist.

Thus, the function possesses no limit at the origin, but a straight line approach gives the limit zero.

Example 1 (b). Show that

$$\lim_{(x, y)\to(0, 0)} \frac{2xy^2}{x^2 + y^4} \text{ does not exist.}$$

- If we put $x = my^2$ and let $y \to 0$, we get

$$\lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4} = \frac{2m}{1+m^2}$$

which is different for different values of m.

Hence, the limit does not exist.

Remark: It is pointed out earlier also that the determination of a simultaneous limit is a difficult matter but a simple consideration, as shown above, very often, enables us to show that the limit does not exist. We now show that sometimes it is possible to determine the simultaneous limit by changing to polars.

Example 2 (a). Show that

$$\lim_{(x, y)\to(0, 0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

- Put $x = r\cos\theta$, $y = r\sin\theta$

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} \right| = \left| r^2 \sin\theta \cos\theta \cos 2\theta \right|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right| \le \frac{r^2}{4} = \frac{x^2 + y^2}{4} < \varepsilon,$$

if

$$\frac{x^2}{4} < \frac{\varepsilon}{2}, \frac{y^2}{4} < \frac{\varepsilon}{2}$$

or if

$$|x| < \sqrt{2\varepsilon} = \delta, \ |y| < \sqrt{2\varepsilon} = \delta$$

Thus for $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ when } |x| < \delta, |y| < \delta$$

$$\Rightarrow \lim_{(x, y)\to(0, 0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

Example 2 (b). Show that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=0$$

■ Since x, y are small

$$\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=\frac{(1+x^2y^2)^{1/2}-1}{x^2+y^2}\simeq\frac{\frac{1}{2}x^2y^2}{x^2+y^2}$$

Now changing to polars, we can show, as in the above example, that

$$\lim_{(x,\,y)\to(0,\,0)} \frac{\frac{1}{2}x^2y^2}{x^2+y^2}=0$$

Hence the required result.

Ex. 1. Show that

(*i*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \left(\frac{1}{|x|}+\frac{1}{|y|}\right)=\infty$, (*ii*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{xy^2}{x^2+y^2}=0$,

(*iii*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} (x+y)=0$, (*iv*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} (1/xy)\sin(x^2y+xy^2)=0$

Ex. 2. Show that the limit, when $(x, y)\to(0, 0)$ does not exist in each case

(*i*) $\displaystyle\lim \frac{2xy}{x^2+y^2}$, (*ii*) $\displaystyle\lim \frac{xy^3}{x^2+y^6}$,

(*iii*) $\displaystyle\lim \frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$, (*iv*) $\displaystyle\lim \frac{x^3+y^3}{x-y}$

[*Hint*: (*iv*) Put $y=x-mx^3$].

Ex. 3. Show that the limit, when $(x, y)\to(0, 0)$ exist in each case.

(*i*) $\displaystyle\lim \frac{xy}{\sqrt{x^2+y^2}}$, (*ii*) $\displaystyle\lim \frac{x^3y^3}{x^2+y^2}$,

(*iii*) $\displaystyle\lim \frac{x^3-y^3}{x^2+y^2}$, (*iv*) $\displaystyle\lim \frac{x^4+y^4}{x^2+y^2}$.

1.5 Algebra of Limits

If f and g are two functions with a domain N, we define four functions, $f\pm g$, fg, f/g on N by settin

$$(f+g)(x, y)=f(x, y)+g(x, y)$$

$$(f - g)(x, y) = f(x, y) - g(x, y)$$

$$f \cdot g\,(x, y) = f(x, y) \cdot g(x, y)$$

$$(f/g)(x, y) = f(x, y)/g(x, y), \text{ if } g(x, y) \neq 0, \text{ for } (x, y) \in N.$$

Theorem 1. *If f, g be two functions defined on some neighbourhood of a point (a, b) such that* $\lim f(x, y) = l$, $\lim g(x, y) = m$, *when* $(x, y) \to (a, b)$, *then*

(*i*) $\lim(f + g) = \lim f + \lim g = l + m$

(*ii*) $\lim(f - g) = \lim f - \lim g = l - m$

(*iii*) $\lim(f \cdot g) = \lim f \cdot \lim g = l \cdot m$

(*iv*) $\lim \dfrac{f}{g} = \dfrac{\lim f}{\lim g} = \dfrac{l}{m}$, *provided* $m \neq 0$, *when* $(x, y) \to (a, b)$

The proofs are exactly similar to those of the corresponding theorems for a single variable.

Example 3 (*a*). Prove that

$$\lim_{(x, y) \to (1, 2)} (x^2 + 2y) = 5$$

- *Method* 1. (Using definition of limit). We have to show that for any $\varepsilon > 0$, we can find $\delta > 0$, such that

$$\left| x^2 + 2y - 5 \right| < \varepsilon, \text{ when} \left| x - 1 \right| < \delta, \left| y - 2 \right| < \delta$$

If $|x - 1| < \delta$, and $|y - 2| < \delta$, then

$$1 - \delta < x < 1 + \delta \text{ and } 2 - \delta < y < 2 + \delta, \text{ excluding } x = 1, y = 2$$

Thus

$$1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2$$

and

$$4 - 2\delta < 2y < 4 + 2\delta$$

Adding

$$5 - 4\delta + \delta^2 < x^2 + 2y < 5 + 4\delta + \delta^2$$

or

$$-4\delta + \delta^2 < x^2 + 2y - 5 < 4\delta + \delta^2$$

Now if $\delta \le 1$, it follows that

$$-5\delta < x^2 + 2y - 5 < 5\delta$$

i.e.,

$$\left| x^2 + 2y - 5 \right| < 5\delta = \varepsilon$$

so that $\delta = \varepsilon/5$ (or $\delta = 1$ whichever is smaller).

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$$\therefore \qquad |x^2+2y-5|<\varepsilon \text{ when } |x-1|<\delta, |y-2|<\delta$$

$$\therefore \qquad \lim_{(x,\,y)\to(1,\,2)} (x^2+2y)=5$$

Method 2. Using above theorem on algebra of limits,

$$\lim_{(x,\,y)\to(1,\,2)} (x^2+2y)=\lim_{(x,\,y)\to(1,\,2)} x^2+\lim_{(x,\,y)\to(1,\,2)} 2y=1+4=5.$$

Example 3 (*b*). Show that

(*i*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}=0$, (*ii*) $\displaystyle\lim_{(x,\,y)\to(2,\,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}=\frac{1}{3}$.

■ (*i*) $\displaystyle\lim_{(x,\,y)\to(0,\,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}=\underbrace{\lim_{(x,\,y)\to(0,\,0)} x}\cdot\underbrace{\lim_{(x,\,y)\to(0,\,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}}=0.1=0$

(*ii*) $\displaystyle\lim_{(x,\,y)\to(2,\,1)} \frac{\sin^{-1}(xy-2)}{\tan^{-1}(3xy-6)}=\lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}3t}$, where $t=xy-2=\displaystyle\lim_{t\to 0}\frac{1/\sqrt{1-t^2}}{3/(1+9t^2)}=\frac{1}{3}$

Ex. 1. Show that $\displaystyle\lim_{(x,\,y)\to(0,\,1)} \tan^{-1}(y/x)$, does not exist.

$\left[\textit{Hint}: \text{Limit from the left is } -\dfrac{\pi}{2} \text{ and that from the right } \dfrac{\pi}{2}\right]$.

Ex. 2. Show, by using the definition that

$$\lim_{(x,\,y)\to(1,\,2)} 3xy=6$$

Ex. 3. Prove that

(*i*) $\displaystyle\lim_{(x,\,y)\to(4,\,\pi)} x^2\sin\frac{y}{8}=8\sqrt{2}$, (*ii*) $\displaystyle\lim_{(x,\,y)\to(0,\,1)} e^{-1/x^2(y-1)^2}=0$,

(*iii*) $\displaystyle\lim_{(x,\,y)\to(0,\,1)} \frac{x+y-1}{\sqrt{x}-\sqrt{1-y}}=0,\ x>0,\ y<1.$

1.6 Repeated Limits

If a function f is defined in some neighbourhood of (a, b), then the limit

$$\lim_{y\to b} f(x,y),$$

if it exists, is a function of x, say $\phi(x)$. If then the limit $\lim_{x\to a}\phi(x)$ exists and is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lambda$$

and say that λ is a *repeated limit* of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \lambda' \text{ (say)}$$

when first $x \to a$, and then $y \to b$.

These two limits may or may not be equal.

Note: In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However *if the repeated limits are not equal, the simultaneous limit cannot exist.*

Example 4. (*i*) Let

$$f(x, y) = \frac{xy}{x^2 + y^2}, \text{ then}$$

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} (0) = 0,$$

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = 0.$$

Thus, the repeated limits exist and are equal. But the simultaneous limit does not exist which may be seen by putting $y = mx$.

(*ii*) Let

$$f(x, y) = \frac{y - x}{y + x} \cdot \frac{1 + x}{1 + y}, \text{ then}$$

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \left(-\frac{1 + x}{1} \right) = -1,$$

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} \left(\frac{1}{1 + y} \right) = 1.$$

Thus, the two repeated limits exist but are unequal, consequently the simultaneous limit cannot exist, which may be verified by putting $y = mx$.

Example 5. Show that the limit exists at the origin but the repeated limits do not, where

$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) + y \sin\left(\frac{1}{x}\right), & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Here $\lim_{y \to 0} f(x, y)$, $\lim_{x \to 0} f(x, y)$ do not exist and therefore $\lim_{x \to 0} \lim_{y \to 0} f(x, y)$; $\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ do not exist.

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Again

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < |x| + |y| \le 2(x^2 + y^2)^{1/2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{4}, \ y^2 < \frac{\varepsilon^2}{4}$$

or

$$|x| < \frac{\varepsilon}{2} = \delta, \ |y| < \frac{\varepsilon}{2} = \delta$$

Thus for $\varepsilon > 0, \exists\, \delta > 0$ such that

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| < \varepsilon, \text{ when } |x| < \delta, \ |y| < \delta$$

$$\Rightarrow \qquad \lim_{(x,\, y) \to (0,\, 0)} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x} \right) = 0.$$

Example 6. Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where

$$f(x, y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}$$

■ Here

$$\lim_{y \to 0} f(x, y) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\therefore \qquad \lim_{x \to 0} \lim_{y \to 0} f(x, y) = 1$$

Similarly,

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = 1$$

Hence, the repeated limits exist and are equal.

Again, since there are points arbitrarily near (0, 0) at which f is equal to 0 and points arbitrarily near (0, 0) at which f is equal to 1, therefore, there is an $\varepsilon > 0$, such that

$$|f(x, y) - f(0, 0)| = |f(x, y)| \nless \varepsilon,$$

for all points in any neighbourhood of (0, 0).

Hence, $\lim_{(x,\, y) \to (0,\, 0)} f(x, y)$ does not exist.

Ex. 1. Show that $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x, y)$ exist, but $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ does not, where

$$f(x, y) = \begin{cases} y + x\sin\left(\dfrac{1}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

Ex. 2. Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x, y)$ exists, but the other repeated limit and the double limit do not exist at the origin, when

$$f(x, y) = \begin{cases} y\sin(1/x) + xy/(x^2 + y^2), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Ex. 3. Show that the repeated limits exist but the double limit does not when $(x, y) \to (0, 0)$:

(*i*) $f(x, y) = \dfrac{x - y}{x + y}$, (*ii*) $f(x, y) = \dfrac{x^2 y^2}{x^4 + y^4 - x^2 y^2}$

(*iii*) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$ (*iv*) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & x \neq y \\ 0, & x = y \end{cases}$

Ex. 4. Show that the limit and the repeated limits exist when $(x, y) \to (0, 0)$:

$$f(x, y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

2. CONTINUITY

A function f is said to be *continuous* at a point (a, b) of its domain of definition, if

$$\lim_{(x, y)\to(a, b)} f(x, y) = f(a, b)$$

In other words, a function f is said to be *continuous* at a point (a, b) of its domain of definition if for $\varepsilon > 0$, there exists a neighbourhood N of (a, b) such that

$$|f(x, y) - f(a, b)| < \varepsilon, \quad \text{for all } (x, y) \in N$$

Note: The definition of continuity of a function f at a point (a, b) requires that besides (a, b), f is defined in a certain neighbourhood of (a, b) and moreover the limit of f when $(x, y) \to (a, b)$ exists and equals to the value $f(a, b)$.

A function which is not continuous at a point is said to be *discontinuous* there at.

Remark: A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of a single variable. To be more specific if a function f of two variables x, y is continuous at (a, b) then $f(x, b)$ is a continuous function of x at $x = a$ and $f(a, y)$ that of y at $y = b$.

The converse however is not true, *i.e.*, a function may be a continuous function of one variable when the others remain constant and yet not be a continuous function of all the variables.

For instance, consider a function f, where

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y}, & (x, y) \neq (0, 0) \\ 0, & \text{at } (0, 0) \end{cases}$$

The function is not continuous at (0, 0) for $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist. But

$$\lim_{x \to 0} f(x, 0) = 0 = f(0, 0), \text{ and } \lim_{y \to 0} f(0, y) = 0 = f(0, 0)$$

so that f is continuous at (0, 0), when considered as a function of a single variable x or that of y.

A function is said to be continuous in a region if it is continuous at every point of the same.

As in limits, it can be easily proved that the sum, difference, product and quotient (provided the denominator does not vanish) of two continuous functions are also continuous.

The theorems on continuity for functions of a single variable can be easily extended to functions of several variables; the proofs for some of them, except for verbal changes, are the same while for others the method is not quite the same. However, within the scope of the present work, it is not possible to discuss all of them here.

Example 7. Investigate the continuity at (0, 0) of

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

■ Since $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist, therefore the function is not continuous at (0, 0).

Example 8. Investigate for continuity at (1, 2)

$$f(x, y) = \begin{cases} x^2 + 2y, & (x, y) \neq (1, 2) \\ 0, & (x, y) = (1, 2) \end{cases}$$

■ Here

$$\lim_{(x, y) \to (1, 2)} f(x, y) = 5 \neq f(1, 2).$$

Hence, the function is not continuous at (1, 2).

The point (1, 2) is a *point of discontinuity* of the function.

However, if the function has the value 5 at (1, 2), it was then continuous at the point.

Remark: If, as in the above example, it is possible to so redefine the value of the function at a point of discontinuity that the new function is continuous, we say that the point is a *removable discontinuity* of the original function.

Example 9. Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at the origin.

■ Let $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore$
$$\left|\frac{xy}{\sqrt{x^2 + y^2}}\right| = r|\cos\theta \sin\theta| \leq r = \sqrt{x^2 + y^2} < \varepsilon,$$

if

or, if
$$x^2 < \frac{\varepsilon^2}{2}, \quad y^2 < \frac{\varepsilon^2}{2}$$

Thus
$$|x| < \frac{\varepsilon}{\sqrt{2}}, \quad |y| < \frac{\varepsilon}{\sqrt{2}}$$

$$\left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| < \varepsilon, \text{ when } |x| < \frac{\varepsilon}{\sqrt{2}}, |y| < \frac{\varepsilon}{\sqrt{2}}$$

$\Rightarrow$
$$\lim_{(x, y) \to (0, 0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0$$

$\therefore$
$$\lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0)$$

Hence, f is continuous at (0, 0).

EXERCISE

1. Show that the following functions are discontinuous at the origin:

(i) $f(x, y) = \begin{cases} \dfrac{1}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(ii) $f(x, y) = \dfrac{x^4 - y^4}{x^4 + y^4}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

(iii) $f(x, y) = \dfrac{(x^2 y^2)}{(x^4 + y^4)}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$

2. Show that the following functions are continuous at the origin:

(i) $f(x, y) = \dfrac{x^2 y^2}{(x^2 + y^2)}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

(ii) $f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

3. Show that the following functions are discontinuous at (0, 0).

(i) $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$

(iii) $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$, $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

4. Discuss the following functions for continuity at (0, 0).

(i) $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^3 + y^3}, & x^2 + y^2 \neq 0 \\ 0, & x + y = 0 \end{cases}$

(ii) $f(x, y) = \begin{cases} 2xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(iii) $f(x, y) = \begin{cases} 0, & (x, y) = (2y, y) \\ \exp\{|x - 2y|/(x^2 - 4xy + 4y^2)\}, & (x, y) \neq (2y, y). \end{cases}$

5. Show that f has a removable discontinuity at (2, 3):

$$f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2, 3) \\ 6, & (x, y) = (2, 3) \end{cases}$$

Suitably redefine the function to make it continuous.

6. Show that the function f is continuous at the origin, where

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

7. Can the given functions be appropriately defined at (0, 0) in order to be continuous there?

(*i*) $f(x, y) = |x|^y$, (*ii*) $f(x, y) = \sin\dfrac{x}{y}$,

(*iii*) $f(x, y) = \dfrac{x^3 + y^3}{x^2 + y^2}$, (*iv*) $f(x, y) = x^2 \log(x^2 + y^2)$.

3. PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the *partial derivative* of the function with respect to the variable. Partial derivative of $f(x, y)$ with respect to x is generally denoted by $\partial f/\partial x$ or f_x or $f_x(x, y)$, while those with respect to y are denoted by $\partial f/\partial y$ or f_y or $f_y(x, y)$.

$$\therefore \quad \frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

when these limits exist.

The partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial f}{\partial x}\right]_{(a, b)}, \quad \frac{\partial f(a, b)}{\partial x} \quad \text{or} \quad f_x(a, b)$$

and

$$\left[\frac{\partial f}{\partial y}\right]_{(a, b)}, \quad \frac{\partial f(a, b)}{\partial y} \quad \text{or} \quad f_y(a, b)$$

$$\therefore \quad f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}$$

in case the limit exists.

Example 10. If $f(x, y) = 2x^2 - xy + 2y^2$, then find $\partial f/\partial x$ and $\partial f/\partial y$ at the point (1, 2).

■ Now

$$\frac{\partial f}{\partial x} = 4x - y = 2, \text{ at } (1, 2)$$

$$\frac{\partial f}{\partial y} = -x + 4y = 7, \text{ at } (1, 2)$$

Note: $f_x(1, 2)$ and $f_y(1, 2)$ have been respectively obtained from $f_x(x, y)$ and $f_y(x, y)$ by replacing (x, y) by $(1, 2)$. The procedure, though simple, is not always possible. The reader has to be on his guard.

Example 11. If

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

show that both the partial derivatives exist at (0, 0) but the function is not continuous there at.

- Putting $y = mx$, we see that

$$\lim_{x \to 0} f(x, y) = \frac{m}{1 + m^2}$$

so that the limit depends on the value of m, *i.e.*, on the path of approach and is different for the different paths followed and therefore does not exist. Hence the function $f(x, y)$ is not continuous at (0, 0).
Again

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0 \cdot 0 + k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0.$$

Notes:

1. Unlike the situation for functions of one variable, the existence of the first partial derivatives at a point does not imply continuity at the point. The explanation lies in the fact that the information given by the existence of the two first partial derivatives at a point is limited. The values of f_x and f_y at a point (a, b) depend only on the values of f along two lines through (a, b) respectively parallel to the coordinate axes. This information is incomplete and tells us nothing at all about the behaviour of the function f as the point (a, b) is approached along lines not parallel to the axes. On the other hand, the continuity of f at (a, b) requires the function to tend to its value $f(a, b)$ by whatever path the point (a, b) is approached. Therefore, there is nothing surprising in the fact that the *partial derivatives may exist at a point at which the function is not even continuous.*
2. In the above example, if $x \neq y$,

$$f_x = \frac{y^3 - x^2 y}{(x^2 + y^2)^2},$$

$$f_y = \frac{x^3 - xy^2}{(x^2 + y^2)^2},$$

and $f_x(0, 0), f_y(0, 0)$ cannot be computed from them by letting $x = 0, y = 0$.

EXERCISE

1. If $f(x, y) = x^3 y + e^{xy^2}$, find f_x and f_y.
2. If $f(x, y) = xy\dfrac{(x^2 - y^2)}{(x^2 + y^2)}$, when $x^2 + y^2 \neq 0$, and $f(0, 0) = 0$, show that

$$f_x(x, 0) = 0 = f_y(0, y)$$

$$f_x(0, y) = -y, f_y(x, 0) = x.$$

3. If $f(x, y) = \begin{cases} \dfrac{x^2 - xy}{x + y}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$, find $f_x(0, 0)$ and $f_y(0, 0)$.

4. If $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$, show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin.

5. If $f(x, y) = \begin{cases} xy \tan(y/x), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$, show that $xf_x + yf_y = 2f$.

6. Calculate f_x, f_y, $f_x(0, 0)$, $f_y(0, 0)$ for the following:

(i) $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & x \neq 0, y \neq 0 \\ 0, & x = 0 = y \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$

7. Show that the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

8. If $f(x, y) = \sqrt{|xy|}$, find $f_x(0, 0)$, $f_y(0, 0)$.

3.1 A Mean Value Theorem

If f_x exists throughout a neighbourhood of a point (a, b) and $f_y(a, b)$ exists then for any point $(a + h, b + k)$ of this neighbourhood,

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + k[f_y(a, b) + \eta]$$

where $0 < \theta < 1$, and η is a function of k, tending to zero with k.

Now

$$f(a + h, b + k) - f(a, b) = f(a + h, b + k) - f(a, b + k) + f(a, b + k) - f(a, b) \quad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), therefore by Lagrange's mean value theorem,

$$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k), 0<\theta<1 \quad ...(2)$$

Also $f_y(a, b)$ exists, so that

$$\lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

$$\Rightarrow \quad f(a, b+k) - f(a, b) = k[f_y(a, b) + \eta] \quad ...(3)$$

where η is a function of k and tends to zero as $k \to 0$.

From equations (1), (2) and (3) we get the required result.

3.2 A Sufficient Condition for Continuity

A sufficient condition that a function f be continuous at (a, b) is that one of the partial derivatives exists and is bounded in a neighbourhood of (a, b) and that the other exists at (a, b).

Let f_x exists and be bounded in a neighbourhood of (a, b) and let $f_y(a, b)$ exists, then for any point $(a+h, b+k)$ of this neighbourhood we have (§ 3.1)

$$f(a+h, b+k) - f(a, b) = hf_x(a+\theta h, b+k) + k[f_y(a, b) + \eta]$$

where $0<\theta<1$, and $\eta \to 0$ as $k \to 0$.

Proceeding to limits as $(h, k) \to (0, 0)$, since $f_x(a+\theta h, b+k)$ is bounded, we have

$$\lim_{(h, k) \to (0, 0)} f(a+h, b+k) = f(a, b)$$

$\Rightarrow$ f is continuous at (a, b).

Note: A sufficient condition that a function be continuous in a *closed region* is that both the partial derivatives exist and are bounded throughout the region.

4. DIFFERENTIABILITY

Let (x, y), $(x+\delta x, y+\delta y)$ be two neighbouring points in the domain of definition of a function f. The change δf in the function as the point changes from (x, y) to $(x+\delta x, y+\delta y)$ is given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y)$$

The function f is said to be *differentiable* at (x, y) if the change δf can be expressed in the form

$$\delta f = A\,\delta x + B\,\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\,\psi(\delta x, \delta y) \quad ...(1)$$

where A and B are constants independent of δx, δy and ϕ, ψ are functions of δx, δy tending to zero as δx, δy tend to zero simultaneously.

Also, $A\delta x + B\delta y$ is then called the *differential* of f at (x, y) and is denoted by df. Thus

$$df = A\delta x + B\delta y$$

From (1) when $(\delta x, \delta y) \to (0, 0)$, we get

or
$$f(x+\delta x, y+\delta y) - f(x, y) \to 0$$

$$f(x+\delta x, y+\delta y) \to f(x, y)$$

$\Rightarrow$ The function f is continuous at (x, y)

Thus every *differentiable function is continuous.*

Again from (1), when $\delta y = 0$ (*i.e.*, y remains constant)

$$\delta f = A\,\delta x + \delta x\,\phi(\delta x, 0)$$

Dividing by δx and proceeding to limits as $\delta x \to 0$, we get

$$\frac{\partial f}{\partial x} = A$$

Similarly,

$$\frac{\partial f}{\partial y} = B$$

Thus, the constants A and B are respectively the partial derivatives of f with respect to x and y.

Hence, *a function which is differentiable at a point possesses the first order partial derivatives there at.*

Converse, of course is not true, so that functions exist which are continuous and may even possess partial derivatives at a point but are not differentiable there at (see example 12),

Again the differential of f is given by

$$df = A\,\delta x + B\,\delta y = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

Taking $f = x$, we get $dx = \delta x$.

Similarly taking $f = y$, we obtain $dy = \delta y$.

Thus, the differentials dx, dy of x, y are respectively δx and δy, and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x\,dx + f_y\,dy \qquad ...(2)$$

is the differential of f at (x, y).

Notes:

1. If we replace δx, δy, by h, k in equation (1) we say that the function is differentiable at a point (a, b) of the domain of definition if df can be expressed as

$$\begin{aligned} df &= f(a+h, b+k) - f(a, b) \\ &= Ah + Bk + h\phi(h, k) + k\psi(h, k) \qquad ...(3) \end{aligned}$$

where $A = f_x$, $B = f_y$ and ϕ, ψ are function of h, k tending to zero as h, k tend to zero simultaneously.

2. We have seen that a function differentiable at a point is necessarily continuous and possesses partial derivatives there at. Not only that, we talk of differentiability at a point of a function only when it is continuous and has partial derivatives there at, for it is only then that it can be expressed in the form of equation (1).

Let a function f and its partial derivatives f_x, f_y be continuous at a point (x, y) of its domain of definition, and let

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}$$

Using Lagrange's mean value theorem of one variable, we get

$$\delta f = \delta x\, f_x(x + \theta_1 \delta x, y + \delta y) + \delta y\, f_y(x, y + \theta_2 \delta y)$$

where $0 < \theta_1 < 1$, $0 < \theta_2 < 1$.

Since f_x, f_y are continuous at (x, y) therefore when $(\delta x, \delta y) \to (0, 0)$, we get

$$\delta f = (f_x + \phi)\, \delta x + (f_y + \psi)\, \delta y$$

when ϕ and ψ tend to zero as $(\delta x, \delta y) \to (0, 0)$

$$\therefore \quad \delta f = f_x \delta x + f_y \delta y + \delta x \phi + \delta y \psi$$

We now give an example to show that a function may be continuous and possess partial derivatives at a point and still may not be differentiable there at.

Example 12. Let

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Put $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore \quad \left|\frac{x^3 - y^3}{x^2 + y^2}\right| = \left|r(\cos^3\theta - \sin^3\theta)\right| \le 2|r| = 2\sqrt{x^2 + y^2} < \varepsilon,$$

if

$$x^2 < \frac{\varepsilon^2}{8}, \quad y^2 < \frac{\varepsilon^2}{8}$$

or, if

$$|x| < \frac{\varepsilon}{2\sqrt{2}}, \quad |y| < \frac{\varepsilon}{2\sqrt{2}}$$

$$\therefore \quad \left|\frac{x^3 - y^3}{x^2 + y^2} - 0\right| < \varepsilon, \text{ when } |x| < \frac{\varepsilon}{2\sqrt{2}}, |y| < \frac{\varepsilon}{2\sqrt{2}}$$

$$\Rightarrow \quad \lim_{(x, y) \to (0, 0)} \frac{x^3 - y^3}{x^2 + y^2} = 0$$

$$\Rightarrow \quad \lim_{(x, y) \to (0, 0)} f(x, y) = f(0, 0)$$

Hence the function is continuous at (0, 0).

Again,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

Thus, the function possesses partial derivatives at (0, 0).

If the function is differentiable at (0, 0), then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \qquad ...(1)$$

when A and B are constants $(A = f_x(0, 0) = 1,\ B = f_y(0, 0) = -1)$ and ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$, $k = \rho \sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi \cos\theta + \psi \sin\theta \qquad ...(2)$$

For arbitrary $\theta = \tan^{-1}(h/k)$, $\rho \to 0$ implies that $(h, k) \to (0, 0)$. Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

or

$$\cos\theta \sin\theta (\cos\theta - \sin\theta) = 0$$

which is plainly impossible for arbitrary θ.

Thus, the function is not differentiable at the origin.

Note: The method used to show that the function is not differentiable, can also be used to show that the function is not continuous at (0, 0); for example,

The function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is not differentiable at the origin because it is discontinuous there at.

Example 13. Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is continuous, possesses partial derivative but is not differentiable at the origin.

▪ As was shown in Example 9, f is continuous at the origin. Also it may be easily shown that

$$f_x(0, 0) = 0 = f_y(0, 0)$$

If the function is differentiable at the origin, then by definition

$$df = f(h, k) - f(0, 0) = Ah + Bk + h\phi + k\psi \qquad ...(1)$$

CHAPTER 15

where $A = f_x(0, 0) = 0$, $B = f_y(0, 0) = 0$, and ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

$$\therefore \quad \frac{hk}{\sqrt{h^2 + k^2}} = h\phi + k\psi \qquad \ldots(2)$$

Putting $k = mh$ and letting $h \to 0$, we get

$$\frac{m}{\sqrt{1 + m^2}} = \lim_{h \to 0} (\phi + m\psi) = 0$$

which is impossible for arbitrary m.

Hence, the function is not differentiable at (0, 0).

Note: If we put $h = r\cos\theta$, $k = r\sin\theta$ in (2) we get

$$\cos\theta \sin\theta = \phi\cos\theta + \psi\sin\theta$$

For arbitrary θ, $r \to 0$ implies $(h, k) \to (0, 0)$.

Thus when $r \to 0$, we get

$$\cos\theta \cdot \sin\theta = 0$$

which is impossible for arbitrary θ. So f is not differentiable at the origin.

Ex. 1. Show that the function f, where

$$f(x, y) = \begin{cases} x\sin 1/x + y\sin 1/y, & xy \neq 0 \\ x\sin 1/x, & y = 0, x \neq 0 \\ y\sin 1/y, & x = 0, y \neq 0 \\ 0, & x = 0 = y \end{cases}$$

is continuous but not differentiable at the origin.

Ex. 2. Show that the function $|x| + |y|$ is continuous, but not differentiable at the origin.

Ex. 3. Discuss the following functions for continuity and differentiability at the origin.

(*i*) $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

(*ii*) $f(x, y) = y\sin 1/x$, if $x \neq 0$, $f(0, y) = y$.

4.1 A Sufficient Condition for Differentiability

Theorem 2. *If (a, b) be a point of the domain of definition of a function f such that*

(i) f_x is continuous at (a, b),

(ii) f_y exists at (a, b),

then f is differentiable at (a, b).

The condition (*i*) implies that f_x exists in a certain neighbourhood $(a - \delta, a + \delta; b - \delta, b + \delta)$ of (a, b). Let $(a + h, b + k)$ be a point of this neighbourhood. Thus

$$df = f(a+h, b+k) - f(a, b)$$

$$= f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b)$$

Since f_x exists in $(a-\delta, a+\delta; b-\delta, b+\delta)$, applying Lagrange's mean value theorem, we get

$$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k) \quad \text{...(1)}$$

where $0 < \theta < 1$, and depends on h and k.

Again, since f_x is continuous at (a, b), therefore

$$\lim_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a, b) \quad \text{...(2)}$$

so that we can write

$$f_x(a+\theta h, b+k) = f_x(a, b) + \phi(h, k)$$

where $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$. ...(3)

Again, since by condition (*ii*), $f_y(a, b)$ exists, therefore

$$\lim_{k\to 0} \frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b)$$

so that we can write

$$\frac{f(a, b+k) - f(a, b)}{k} = f_y(a, b) + \psi(k)$$

where $\psi(k) \to 0$ as $k \to 0$. ...(4)

$\therefore$ From (1), (2), (3) and (4), we get

$$df = hf_x(a, b) + kf_y(a, b) + h\phi(h, k) + k\psi(k)$$

$\Rightarrow$ f is differentiable at (a, b).

Note: In a similar way it can be shown that f is differentiable at (a, b), if f_x exists and f_y is continuous at (a, b). In fact, one of the partial derivatives is to be continuous and the other merely to exist at the point.

Remark: We have shown that the condition of existence of one partial derivative and the continuity of the other is sufficient to ensure that the function is differentiable but with the help of an example (Example I below) we now show that the condition of continuity is not necessary so that function may be differentiable even though none of the partial derivatives is continuous. However, if the function is not differentiable at a point, the partial derivatives cannot be continuous there at (Example II).

Example I. Consider the function

$$f(x, y) = \begin{cases} x^2 \sin 1/x + y^2 \sin 1/y, & \text{if } xy \neq 0 \\ x^2 \sin 1/x, & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin 1/y, & \text{if } x = 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

■ The partial derivatives,

$$f_x(x, y) = \begin{cases} 2x \sin 1/x - \cos 1/x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} 2y \sin 1/y - \cos 1/y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

are discontinuous at the origin, so that both the partial derivatives exist at the origin, but none is continuous there at.

Let us show that the function is differentiable at the origin. Here,

$$f(h, k) - f(0, 0) = h^2 \sin 1/h + k^2 \sin 1/k$$

$$= 0h + 0k + h(h \sin 1/h) + k(k \sin 1/k)$$

Now $(h \sin 1/h)$ and $(k \sin 1/k)$ both tend to zero when $(h, k) \to (0, 0)$ so that f is differentiable at the origin.

Example II. Prove that the function

$$f(x, y) = \sqrt{|xy|}$$

is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

■ Now at (0, 0),

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

If the function is differentiable at (0, 0) then by definition

$$f(h, k) - f(0, 0) = 0h + 0k + h\phi + k\psi$$

where ϕ and ψ are functions of h and k, and tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$, $k = \rho \sin\theta$ and dividing by ρ, we get

$$|\cos\theta \sin\theta|^{1/2} = \phi \cos\theta + \psi \sin\theta$$

Now for arbitrary θ, $\rho \to 0$ implies that $(h, k) \to (0, 0)$.

Taking the limit as $\rho \to 0$, we get

$$|\cos\theta \sin\theta|^{1/2} = 0,$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at (0, 0) and consequently the partial derivatives f_x, cannot be continuous at (0, 0), for otherwise the function would be differentiable there at, Let us now see that it is actually so

For $(x, y) \neq (0, 0)$.

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{|x+h||y|} - \sqrt{|x||y|}}{h}$$

$$= \lim_{h\to 0} \sqrt{|y|} \frac{|x+h| - |x|}{h\left[\sqrt{|x+h|} + \sqrt{|x|}\right]}$$

Now as $h \to 0$, we can take $x + h > 0$, *i.e.*, $|x+h| = x + h$, when $x > 0$ and $x + h < 0$ or $|x+h| = -(x+h)$, when $x < 0$.

$$\therefore \qquad f_x(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|y|}{|x|}}, & \text{when } x < 0 \end{cases}$$

Similarly,

$$f_y(x, y) = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y > 0 \\ -\dfrac{1}{2}\sqrt{\dfrac{|x|}{|y|}}, & \text{when } y < 0 \end{cases}$$

which are, obviously, not continuous at the origin.

Example 14. Show that the function f, where

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is differentiable at the origin.

- It may be easily shown that

$$f_x(0, 0) = 0 = f_y(0, 0)$$

Also when $x^2 + y^2 \neq 0$,

$$|f_x| = \frac{|x^4 y + 4x^2 y^3 - y^5|}{(x^2 + y^2)^2} \leq \frac{6(x^2 + y^2)^{5/2}}{(x^2 + y^2)^2} = 6(x^2 + y^2)^{1/2}$$

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Evidently

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = 0 = f_x(0, 0)$$

Thus f_x is continuous at (0, 0) and $f_y(0, 0)$ exists.

$\Rightarrow$ f is differentiable at (0, 0).

4.2 Algebra of Differentiable Functions

If f and g are two functions differentiable at (a, b), then $f \pm g$, fg are differentiable at (a, b); f/g is differentiable at (a, b), if $g(a, b) \neq 0$, and

$$d(f \pm g) = df \pm dg$$

$$d(fg) = gdf + fdg$$

$$d(f/g) = (gdf + fdg)/g^2.$$

5. PARTIAL DERIVATIVES OF HIGHER ORDER

If a function f has partial derivatives of the first order at each point (x, y) of a certain region, then f_x, f_y are themselves functions of x, y and may also possess partial derivatives. These are called *second order partial derivatives of* f and are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

In a similar manner higher order partial derivatives are defined. For example $\dfrac{\partial^3 f}{\partial x \partial x \partial y} = f_{xxy}$ and so on.

The second order partial derivatives at a particular point (a, b) are often denoted by

$$\left[\frac{\partial^2 f}{\partial x^2}\right]_{(a,b)}, \quad \frac{\partial^2 f(a, b)}{\partial x^2}, \quad f_{xx}(a, b) \text{ or } f_{x^2}(a, b)$$

$$\left[\frac{\partial^2 f}{\delta x \partial y}\right]_{(a,b)}, \quad \frac{\partial^2 f(a, b)}{\partial x \partial y} \text{ or } f_{xy}(a, b)$$

and so on.

Thus

$$f_{xx}(a,b)=\lim_{h\to 0}\frac{f_x(a+h,b)-f_x(a,b)}{h}$$

$$f_{xy}(a,b)=\lim_{h\to 0}\frac{f_y(a+h,b)-f_y(a,b)}{h}$$

$$f_{yx}(a,b)=\lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k}$$

$$f_{yy}(a,b)=\lim_{k\to 0}\frac{f_y(a,b+k)-f_y(a,b)}{k}$$

in case the limits exist.

5.1 Change in the Order of Partial Derivation

In most of the cases that occur in practice, a partial derivative has the same value in whatever order the different operations are performed. Thus, for example, it is usually found that

$$f_{xy}=f_{yx},\quad f_{xyx}=f_{xxy},\quad f_{xyxy}=f_{xxyy}$$

and one is often tempted to believe that it is always so. But it is not the case and there is no *a priori* reason why they should be equal. Let us now see why f_{xy} may be different from f_{yx} at some point (a, b) of the region.

Now

$$f_{xy}(a,b)=\lim_{h\to 0}\frac{f_y(a+h,b)-f_y(a,b)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\lim_{k\to 0}\frac{f(a+h,b+k)-f(a+h,b)}{k}-\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}\right]$$

$$=\lim_{h\to 0}\lim_{k\to 0}\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk}=\lim_{h\to 0}\lim_{k\to 0}\frac{\phi(h,k)}{hk}$$

where $\phi(h,k)=f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)$.

Similarly,

$$f_{yx}(a,b)=\lim_{k\to 0}\lim_{h\to 0}\frac{\phi(h,k)}{hk}$$

Thus we see that $f_{xy}(a, b)$ and $f_{yx}(a, b)$ are the repeated limits of the same expression taken in different orders. There is therefore no *a priori* reason why they should always be equal.

Let us consider an example to show that f_{xy} may be different from f_{yx}.

Example 15. Let

$$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2},\ (x,y)\neq(0,0),\ f(0,0)=0,\ \text{then}$$

show that at the origin $f_{xy}\neq f_{yx}$.

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■ Now

$$f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

$$f_y(h, 0) = \lim_{k\to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k\to 0} \frac{hk(h^2 - k^2)}{k \cdot (h^2 + k^2)} = h$$

$\therefore$
$$f_{xy} = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

Again

$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$$

But

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = 0$$

$$f_x(0, k) = \lim_{h\to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h\to 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = -k$$

$\therefore$
$$f_{yx}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

$\therefore$
$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

Example 16. Examine the equality of f_{xy} and f_{yx}, where

$$f(x, y) = x^3 y + e^{xy^2}$$

■ Now

$$f_y = x^3 + 2xye^{xy^2}$$

$\therefore$
$$f_{xy} = 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2}$$

Again

$$f_x = 3x^2y + y^2e^{xy^2}$$

$$f_{yx} = 3x^2 + 2ye^{xy^2} + 2xy^3e^{xy^2}$$

$\Rightarrow$
$$f_{xy} = f_{yx}.$$

EXERCISE

1. Verify that $f_{xy} = f_{yx}$ for the functions:

(a) $\dfrac{2x-y}{x+y}$, (b) $x \tan xy$, (c) $\cosh (y+\cos x)$, (d) x^y

indicating possible exceptional points and investigate these points.

2. Show that $z = \log \{(x-a)^2 + (y-b)^2\}$ satisfies $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$, except at (a, b).

3. Show that $z = x \cos (y/x) + \tan (y/x)$ satisfies $x^2 z_{xx} + 2xyz_{xy} + y^2 z_{yy} = 0$, except at points for which $x = 0$.

4. Prove that $f_{xy} \neq f_{yx}$ at the origin for the function:

$$f(x, y) = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y),\ x \neq 0,\ y \neq 0$$
$$f(x, y) = 0, \text{ elsewhere.}$$

5. If $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

6. Examine for the change in the order of derivation at the origin for the functions:

(i) $f(x, y) = e^x (\cos y + x \sin y)$

(ii) $f(x, y) = \sqrt{x^2 + y^2} \sin 2\phi$,

where $f(0, 0) = 0$ and $\phi = \tan^{-1}(y/x)$,

(iii) $f(x, y) = \left| x^2 - y^2 \right|$.

7. Examine the equality of $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function:

$$f(x, y) = (x^2 + y^2) \tan^{-1}(y/x),\quad x \neq 0,\ f(0, y) = \pi y^2/2.$$

8. Given $u = e^x \cos y + e^y \sin z$, find all first partial derivatives and verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x};\ \frac{\partial^2 u}{\partial x \partial z} = \frac{\partial^2 u}{\partial z \partial x};\ \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y}.$$

5.2 Sufficient Conditions for the Equality of f_{xy} and f_{yx}

As was said earlier there is no *a priori* reason why f_{xy} and f_{yx} should always be equal. We now give two theorems the object of which is to set out precisely under what conditions it is safe to assume that $f_{xy} = f_{yx}$ at a point, *i.e.*, **sufficient conditions** for the equality of f_{xy} and f_{yx}.

Theorem 3. ***Young's theorem.*** *If f_x and f_y are both differentiable at a point (a, b) of the domain of definition of a function f, then*

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The differentiability of f_x and f_y at (a, b) implies that they exist in a certain neighbourhood of (a, b) and that all the second order partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ exist at (a, b).

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We prove the theorem by taking equal increment h both for x and y and calculating $\phi(h, h)$ in two different ways.

Let $(a+h, b+h)$ be a point of this neighbourhood. Consider

$$\phi(h, h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a, b)$$

$$G(x) = f(x, b+h) - f(x, b)$$

so that

$$\phi(h, h) = G(a+h) - G(a) \quad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a+h[$ and therefore by Lagrange's mean value theorem, we get from (1),

$$\phi(h, h) = hG'(a+\theta h), \quad 0 < \theta < 1$$

$$= h\{f_x(a+\theta h, b+h) - f_x(a+\theta h, b)\} \quad ...(2)$$

Again, since f_x is differentiable at (a, b), we have

$$f_x(a+\theta h, b+h) - f_x(a, b) = \theta h f_{xx}(a, b) + h f_{yx}(a, b) + \theta h \phi_1(h, h) + h\psi_1(h, h) \quad ...(3)$$

and

$$f_x(a+\theta h, b) - f_x(a, b) = \theta h f_{xx}(a, b) + \theta h \phi_2(h, h) \quad ...(4)$$

where ϕ_1, ψ_1, ϕ_2 all tend to zero as $h \to 0$.

From equations (2), (3), and (4), we get

$$\phi(h, h)/h^2 = f_{yx}(a, b) + \theta\phi_1(h, h) + \psi_1(h, h) - \theta\phi_2(h, h) \quad ...(5)$$

By a similar argument, on considering

$$H(y) = f(a+h, y) - f(a, y)$$

we can show that

$$\phi(h, h)/h^2 = f_{xy}(a, b) + \phi_3(h, h) + \theta'\psi_2(h, h) - \theta'\psi_3(h, h) \quad ...(6)$$

where ϕ_3, ψ_2, ψ_3 all tend to zero as $h \to 0$.

On taking the limit as $h \to 0$, we obtain from equations (5) and (6)

$$\lim_{h \to 0} \frac{\phi(h, h)}{h^2} = f_{xy}(a, b) = f_{yx}(a, b).$$

Note: An alternative set of conditions which involves only the existence of one of the second order partial derivatives of f at (a, b) provided we assume also its continuity, is made in the following next theorem.

Theorem 4. Schwarz's theorem. *If f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f, and f_{yx} is continuous at (a, b), then $f_{xy}(a, b)$ exists and is equal to $f_{yx}(a, b)$.*

Under the given conditions, f_x, f_y, and f_{yx} all exist in a certain neighbourhood of (a, b). Let $(a+h, b+k)$ be a point of this neighbourhood.

Consider

$$\phi(h, k) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$
$$G(x) = f(x, b + k) - f(x, b)$$

so that

$$\phi(h, k) = G(a + h) - G(a) \qquad ...(1)$$

Since f_x exists in a neighbourhood of (a, b), the function $G(x)$ is derivable in $]a, a+h[$, and therefore by Lagrange's mean value theorem, we get from (1)

$$\phi(h, k) = hG'(a + \theta h), \ 0 < \theta < 1$$
$$= h\{f_x(a + \theta h, b + k) - f_x(a + \theta h, b)\} \qquad ...(2)$$

Again, since f_{yx} exists in a neighbourhood of (a, b), the function f_x is derivable with respect to y in $]b, b+k[$, and therefore by Lagrange's mean value theorem, we get from (2)

$$\phi(h, k) = hkf_{yx}(a + \theta h, b + \theta' k), \ 0 < \theta' < 1$$

or

$$\frac{1}{h}\left\{\frac{f(a + h, b + k) - f(a + h, b)}{k} - \frac{f(a, b + k) - f(a, b)}{k}\right\} = f_{yx}(a + \theta h, b + \theta' k)$$

Proceeding to limits when $k \to 0$, since f_y and f_{yx} exist in a neighbourhood of (a, b), we get

$$\frac{f_y(a + h, b) - f_y(a, b)}{h} = \lim_{k\to 0} f_{yx}(a + \theta h, b + \theta' k)$$

Again, taking limits as $h \to 0$, since f_{yx} is continuous at (a, b), we get

$$f_{xy}(a, b) = \lim_{h\to 0}\lim_{k\to 0} f_{yx}(a + \theta h, b + \theta' k) = f_{yx}(a, b)$$

Notes:

1. *If f_{xy} and f_{yx} are both continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$*, for the assumption of continuity of both these derivatives is a wider assumption than those required for proving either Theorem 3 or Theorem 4.
2. If the conditions of Young's or Schwarz's theorem are satisfied then $f_{xy} = f_{yx}$ at a point (a, b). But if the conditions are not satisfied, we cannot draw any conclusion regarding the equality of f_{xy} and f_{yx} they may or may not be equal (see examples 17 and 18). Thus the conditions are *sufficient* but *not necessary*.

Example 17. Show that for the function

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f_{xy}(0, 0) = f_{yx}(0, 0)$, even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.

▪ Now

$$f_x(0, 0) = \lim_{x\to 0}\frac{f(x, 0) - f(0, 0)}{x} = 0$$

Similarly, $f_y(0, 0) = 0$.

Also, for $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{(x^2 + y^2) \cdot 2xy^2 - x^2y^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2x^4y}{(x^2 + y^2)^2}$$

Again

$$f_{yx}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = 0$$

and

$f_{xy}(0, 0) = 0$, so that $f_{xy}(0, 0) = f_{yx}(0, 0)$

For $(x, y) \neq (0, 0)$, we have

$$f_{yx}(x, y) = \frac{8xy^3(x^2 + y^2)^2 - 2xy^4 \cdot 4y(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{8x^3y^3}{(x^2 + y^2)^3}$$

and it may be easily shown (by putting $y = mx$) that

$$\lim_{(x, y) \to (0, 0)} f_{yx}(x, y) \neq 0 = f_{yx}(0, 0)$$

so that f_{yx} is not continuous at (0, 0), *i.e.*, the conditions of Schwarz's theorem are not satisfied.

Let us now show that the conditions of Young's theorem are also not satisfied.

Now

$$f_{xx}(0, 0) = \lim_{x \to 0} \frac{f_x(x, 0) - f_x(0, 0)}{x} = 0$$

Also f_x is differentiable at (0, 0) if

$$f_x(h, k) - f_x(0, 0) = f_{xx}(0, 0) \cdot h + f_{yx}(0, 0) \cdot k + h\phi + k\psi$$

or

$$\frac{2hk^4}{(h^2 + k^2)^2} = h\phi + k\psi$$

where ϕ, ψ tend to zero as $(h, k) \to (0, 0)$.

Putting $h = \rho \cos\theta$ and $k = \rho \sin\theta$, and dividing by ρ, we get

$$2 \cos\theta \sin^4\theta = \cos\theta \cdot \phi + \sin\theta\psi$$

and $(h, k) \to (0, 0)$ is same thing as $\rho \to 0$ and θ is arbitrary. Thus proceeding to limits, we get

$$2 \cos\theta \sin^4\theta = 0$$

which is impossible for arbitrary θ.

$\Rightarrow$ f_x is not differentiable at (0, 0)

Similarly, it may be shown that f_y is not differentiable at (0, 0).

Thus the conditions of Young's theorem are also not satisfied but, as shown above,

$$f_{xy}(0, 0) = f_{yx}(0, 0).$$

Example 18. Show that the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

does not satisfy the conditions of Schwarz's theorem and

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

- It may be shown, as in example 15, that

$$f_{xy}(0, 0) = 1, \ f_{yx}(0, 0) = -1$$

so that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

Now, for $(x, y) \neq (0, 0)$ we have

$$f_x(x, y) = \frac{(x^2 + y^2)y(3x^2 - y^2) - 2x^2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^2}$$

$$\therefore \quad f_{yx}(x, y) = \frac{(x^2 + y^2)^2\{x^4 + 12x^2y^2 - 5y^4\} - 4y^2(x^2 + y^2)\{x^4 + 4x^2y^2 - y^4\}}{(x^2 + y^2)^4}$$

$$= \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2 + y^2)^3}.$$

By putting $y = mx$ or $x = r\cos\theta$, $y = r\sin\theta$, it may be shown that

$$\lim_{(x, y)\to(0, 0)} f_{yx}(x, y) \neq -1 = f_{yx}(0, 0).$$

Thus f_{yx} is not continuous at (0, 0).

It may similarly be shown that f_{xy} is also not continuous at (0, 0).

Thus, the conditions of Schwarz's theorem are not satisfied.

6. DIFFERENTIALS OF HIGHER ORDER

Let $z = f(x, y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x, y) of the domain. The first differential of z at (x, y), denoted by dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad ...(1)$$

If dx and dy are regarded as constants and if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable at (x, y) then dz is a function of x and y and is itself differentiable at (x, y). The differential of dz, called the *second differential* of z, is denoted by d^2z and is calculated in the same way as the first.

$$\therefore \quad d^2z = d(dz) = d\left(\frac{\partial z}{\partial x}\right)dx + d\left(\frac{\partial z}{\partial y}\right)dy \quad \text{...(2)}$$

Replacing z by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in (1), we get

$$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy$$

$$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy$$

Also by Young's theorem, since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are differentiable, we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

$$\therefore \quad d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}dy^2 \quad \text{...(3)}$$

where, of course, $dx^2 = dx \cdot dx = (dx)^2$, $dy^2 = (dy)^2$

In abbreviated notation, it may be written as

$$d^2z = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 z \quad \text{...(4)}$$

Again d^2z is differentiable at (x, y) if all the second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are differentiable at (x, y). This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y, and so

$$d^3z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2 \partial y}dx^2\,dy + 3\frac{\partial^3 z}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}\partial y^3 = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^3 z \quad \text{...(5)}$$

Proceeding in the manner, we can define the successive differentials $d^4z, d^5z, \ldots$ Thus the differential of nth order, d^nz exists if $d^{n-1}z$ is differentiable, which implies that all the partial derivatives of the $(n-1)$th order are differentiable. This condition also ensures the legitimacy of inverting the order of the partial derivatives with respect to x and with respect to y in the partial derivatives of order n. Thus it may be shown by Mathematical induction that

$$d^n z = \frac{\partial^n z}{\partial x^n} dx^n + n \frac{\partial^n z}{\partial x^{n-1} \partial y} dx^{n-1} dy + \frac{n(n-1)}{2!} \frac{\partial^n z}{\partial x^{n-2} \partial y^2} dx^{n-2} dy^2 + \ldots + \frac{\partial^n z}{\partial y^n} dy^n$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n z.$$

Note: In the above discussion, x and y are *Independent Variables* and so dx and dy may be treated as constants. The reason for this being so is that the differentials of independent variables are the arbitrary increments of these variables, $dx = \delta x$, $dy = \delta y$.

7. FUNCTIONS OF FUNCTIONS

So far we have considered functions of the form

$$z = f(x, y, \ldots)$$

where the variables $x, y, \ldots$ are the independent variables. We now consider functions

$$z = f(x, y, \ldots)$$

where $x, y, \ldots$ are not independent variables, but are themselves functions of other independent variables $u, v, \ldots$, so that

$$x = g(u, v, \ldots) \text{ and } y = h(u, v, \ldots)$$

To fix the ideas, we consider only two variables x and y as functions of two independent variables u and v. The method of proof is, however, general.

Theorem 5. *If $z = f(x, y)$ is a differentiable function of x, y and $x = g(u, v)$, $y = h(u, v)$ are themselves differentiable functions of the independent variables u, v, then z is a differentiable function of u, v and*

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

just as though x, y were the independent variables.

Let (u, v), $(u + \delta u, v + \delta v)$ be two neighbouring points of the domain of definition of x and y, and (x, y), $(x + \delta x, y + \delta y)$ the corresponding points of the domain of definition of z, so that

$$\delta x = g(u + \delta u, v + \delta v) - g(u, v)$$

$$\delta y = h(u + \delta u, v + \delta v) - h(u, v)$$

The differentiability, and hence the continuity of g and h imply that

$$\delta x \to 0, \delta y \to 0, \text{ as } (\delta u, \delta v) \to (0, 0)$$

Again, since g and h are differentiable function of u and v,

$$\delta x = g_u \delta u + g_v \delta v + \phi_1 \delta u + \psi_1 \delta v \quad \ldots(1)$$

$$\delta y = h_u \delta u + h_v \delta v + \phi_2 \delta u + \psi_2 \delta v, \quad \ldots(2)$$

where $\phi_1, \phi_2, \psi_1, \psi_2$ are functions of δu, δv, and tend to zero as,

$$(\delta u, \delta v) \to (0, 0).$$

Also, $dx = g_u\, du + g_v\, dv$, $dy = h_u\, du + h_v\, dv$.

Also, since f is a differentiable function of x, y, we have

$$\delta z = f_x\, \delta x + f_y\, \delta y + \phi_3\, \delta x + \psi_3\, \delta y, \quad ...(3)$$

where ϕ_3, ψ_3 are functions of δx, δy, and tend to zero as $(\delta x, \delta y) \to (0, 0)$.

From equations (1), (2), and (3) we get

$$\delta z = (f_x g_u + f_y h_u)\, \delta u + (f_x g_v + f_y h_v)\, \delta v + F_1\, \delta u + F_2\, \delta v$$

where

$$F_1 = f_x \phi_1 + f_y \phi_2 + \phi_3 g_u + \phi_3 \phi_1 + \psi_3 h_u + \psi_3 \phi_2$$

$$F_2 = f_x \psi_1 + f_y \psi_2 + \phi_3 g_v + \phi_3 \psi_1 + \psi_3 h_v + \psi_3 \psi_2$$

Since the coefficients F_1 and F_2 of $\delta u, \delta v$ tend to zero as $(\delta u, \delta v) \to (0, 0)$, therefore z is a *differentiable function of* u, v and

$$\begin{aligned} dz &= (f_x g_u + f_y h_u)\, du + (f_x g_v + f_y h_v)\, dv \\ &= f_x(g_u\, du + g_v\, dv) + f_y(h_u\, du + h_v\, dv) \\ &= f_x\, dx + f_y\, dy \end{aligned}$$

$$\therefore \quad dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

Remark: The theorem establishes a fact of fundamental importance that *the first differential of a function is expressed always by the same formula, whether the variables concerned are independent or whether they are themselves functions of other independent variables.*

Note: The differential dz is sometimes referred to as the *total differential.*

7.1 Differentials of Higher Order of a Function of Functions

If $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are differentiable functions of x, y so that they are also differentiable functions of u, v, and dx, dy are differentiable functions of u, v, then from the preceding theorem we have

$$d^2 z = d(dz) = d\left(\frac{\partial z}{\partial x}\right) dx + \frac{\partial z}{\partial x} d^2 x + d\left(\frac{\partial z}{\partial y}\right) dy + \frac{\partial z}{\partial y} d^2 y$$

and on comparison with (2) and (3) of § 6, we see that

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial y} d^2 y \quad ...(1)$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^2 z + \frac{\partial z}{\partial x} d^2 x + \frac{\partial z}{\partial x} d^2 y \quad ...(2)$$

The differentials of higher orders can be written in the same manner, but their formation becomes more and more complicated and lengthy, and no simple general formula for $d^n z$ can be given.

The introduction of more than two *intermediary variables** causes no fresh difficulty. Thus, when $z = f(x_1, x_2, x_3)$ and x_1, x_2, x_3 are not the independent variables,

$$d^2 z = \left(\frac{\partial}{\partial x_1} dx_1 + \frac{\partial}{\partial x_2} dx_2 + \frac{\partial}{\partial x_3} dx_3\right)^2 z + \frac{\partial z}{\partial x_1} d^2 x_1 + \frac{\partial z}{\partial x_2} d^2 x_2 + \frac{\partial z}{\partial x_3} d^2 x_3$$

Note: If x, y are linear functions of independent variables u and v, *i.e.*, x and y are of the form $x = a + bu + cv$, $y = a' + b'u + c'v$ then dx and dy are constants, and so d^2x, d^2y and all higher differentials of x and y are zero, and therefore

$$d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n z,$$

the form being same as for independent x and y.

7.2 The Derivation of Composite Functions (*The chain rule*)

From the preceding theorem we deduce two important results:

I. *If*

(*i*) *x, y be differentiable functions of a single variable, and*

(*ii*) *z is differentiable function of x and y,*

then z possesses continuous derivative with respect to t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Because of (*i*),

$$dx = \frac{dx}{dt} \cdot dt, \text{ and } dy = \frac{dy}{dt} \cdot dt$$

Since z is a differentiable function of x and y, and x, y are differentiable functions of t, we deduce from § 7, that z is a differentiable function of t.

$$\therefore \quad dz = \frac{dz}{dt} \cdot dt \qquad ...(1)$$

Also

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial z}{\partial x}\frac{dx}{dt} dt + \frac{\partial z}{\partial y}\frac{dy}{dt} dt \qquad ...(2)$$

From equations (1) and (2),

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad ...(3)$$

* Variables like x, y which are functions of independent variables u, v are called *intermediary variables*.

Again because of conditions (*i*) and (*ii*), $\frac{dz}{dt}$ is a continuous function of t.

Corollary. *If $z = f(x, y)$ possesses nth order partial derivatives, and x, y are linear functions of a single variable t, i.e., $x = a + ht$, $y = b + kt$, where a, b, h, k are constants, then*

$$\frac{d^n z}{dt^n} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n z$$

Now

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)z \qquad \text{...(1)}$$

Replacing z by $\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$ in (1), we get

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = h\frac{\partial}{\partial x}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right) + k\frac{\partial}{\partial y}\left(h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right)$$

$$= h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial x \partial y} + k^2\frac{\partial^2 z}{\partial y^2} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 z$$

By induction, we may obtain the required expression for $\frac{d^n z}{dt^n}$.

II. *If*

(*i*) *x, y are differentiable functions of two independent variables u and v, and*

(*ii*) *z is a differentiable function of x and y,*

then z possesses continuous partial derivatives with respect to u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Because of (*i*)

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \end{aligned}\right\} \qquad \text{...(1)}$$

Since z is a differentiable function of x and y and x, y are differentiable functions of u and v, we deduce from § 7, that z is a differentiable function of u, and v, and

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \qquad \text{...(2)}$$

Also $$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$
$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv \qquad ...(3)$$

Hence, from equations (2) and (3), we get
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Again, because of conditions (*i*) and (*ii*) we see that $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ are continuous functions of u, v.

Note: In (1) when x is a function of a single variable t, we have $dx = \frac{dx}{dt}dt$, so that the derivative $\frac{dx}{dt}$ appears as the coefficient of a differential and that is precisely the reason why the derivative is also called the *differential coefficient.*

Example 19. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

■ $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2e^{xy^2})(\cos t - t\sin t) + (2xye^{xy^2})(\sin t + t\cos t)$$

At $t = \frac{\pi}{2} \Rightarrow x = 0,\ y = \frac{\pi}{2}$.

$$\therefore \qquad \left[\frac{dz}{dt}\right]_{t=\pi/2} = \frac{\pi^2}{4}\left(-\frac{\pi}{2}\right) = -\frac{\pi^3}{8}.$$

Example 20. If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

■ $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (3y^2 - x)\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (3x^2 - y)(-r\sin\theta) + (3y^2 - x)\,r\cos\theta.$$

Example 21. Show that $z = f(x^2y)$, where f is differentiable, satisfies
$$x\left(\frac{\partial z}{\partial x}\right) = 2y\left(\frac{\partial z}{\partial y}\right).$$

■ Let $x^2y = u$, so that $z = f(u)$. Thus

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$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = f'(u) \cdot 2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = f'(u) \cdot x^2$$

$\therefore$ $$x\frac{\partial z}{\partial x} = f'(u)\, 2x^2 y = 2y\frac{\partial z}{\partial y}$$

Aliter. $dz = f'(u)\,du = f'(x^2y)\,(2xy\,dx + x^2\,dy)$

Also $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial x}dy$

Then $\frac{\partial z}{\partial x} = 2xyf'(x^2y),\ \frac{\partial z}{\partial y} = x^2f'(x^2y)$

The result now follows as above.

Example 22. If for all values of the parameter λ, and for some constant n, $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ (F is then called a *homogeneous function* of degree n), identically where F is assumed differentiable, prove that $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = nF$. Hence show that, for $F(x, y) = x^4y^2 \sin^{-1}\frac{y}{x}$,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 6F.$$

■ Let $\lambda x = u$, $\lambda y = v$. Then

$$F(u, v) = \lambda^n F(x, y) \qquad ...(1)$$

The derivative w.r.t. λ of the left side of (1) is

$$\frac{\partial F}{\partial \lambda} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial \lambda} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial \lambda} = x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}$$

The derivative w.r.t. λ of the right side of (1) is $n\lambda^{n-1}F(x, y)$. Then

$$x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v} = n\lambda^{n-1}F$$

The result follows for $\lambda = 1$, then $u = x$, $v = y$.

Again, since $F(\lambda x, \lambda y) = (\lambda x)^4(\lambda y)^2 \sin^{-1} y/x = \lambda^6 F(x, y)$, the result follows for $n = 6$. That it is so, can also be shown by direct differentiation.

Example 23. If z is given as a function of two independent variables x and y, change the variables so that x becomes the function, and z and y the independent variables, and express the first and second order partial derivatives of x with respect to z and y in terms of the derivatives of z with respect to x and y.

- When x and y are independent variables and z the dependent, a usual notation (which will be often employed) is

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$$

We know

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy \quad \text{...(1)}$$

Again, when z and y are independent and x the function,

$$dx = \frac{\partial x}{\partial z}dz + \frac{\partial x}{\partial y}dy \quad \text{...(2)}$$

From equation (1),

$$dx = \frac{1}{p}dz - \frac{q}{p}dy \quad \text{...(3)}$$

Comparing the coefficients in equation (2) and equation (3), we get

$$\frac{\partial x}{\partial z} = \frac{1}{p}, \quad \frac{\partial x}{\partial y} = -\frac{q}{p} \quad \text{...(4)}$$

Taking the differential of the first, we have

$$d\left(\frac{\partial x}{\partial z}\right) = d\left(\frac{1}{p}\right)$$

or

$$\frac{\partial^2 x}{\partial z^2}dz + \frac{\partial^2 x}{\partial y \partial z}dy = -\frac{1}{p^2}dp \quad \text{...(5)}$$

But p is a function of x and y.

$$\therefore \quad dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = r\,dx + s\,dy$$

Hence from equation (5),

$$\frac{\partial^2 x}{\partial z^2}dz + \frac{\partial^2 x}{\partial y \partial z}dy = -\frac{1}{p^2}(r\,dx + s\,dy) = -\frac{r}{p^3}dz + \frac{rq - sp}{p^3}dy \quad \text{[using (3)]}$$

In this equation, we have only the differentials of independent variables and can therefore equate the coefficients of dz and dy, hence

$$\frac{\partial^2 x}{\partial z^2} = -\frac{r}{p^3}, \quad \frac{\partial^2 x}{\partial y \partial z} = \frac{rq - sp}{p^3} \quad \text{...(6)}$$

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In the same way, the second equation of (4) gives

$$\frac{\partial^2 x}{\partial z \partial y} dz + \frac{\partial^2 x}{\partial y^2} dy = -\frac{p\,dq - q\,dp}{p^2} \qquad ...(7)$$

But $dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s\,dx + t\,dy$, substituting, as before for dp, dq and using (3), we have from (7)

$$\frac{\partial^2 x}{\partial z \partial y} dz + \frac{\partial^2 x}{\partial y^2} dy = \frac{rq - sp}{p^3} dz + \frac{2pqs - tp^2 - rq^2}{p^3} dy$$

and, therefore

$$\frac{\partial^2 x}{\partial y^2} = \frac{2pqs - tp^2 - rq^2}{p^3}$$

The value of $\partial^2 x/\partial z \partial y$ being the same as $\partial^2 x/\partial y \partial z$.

Note: $$\frac{\partial^2 x}{\partial z^2} \cdot \frac{\partial^2 x}{\partial y^2} - \left(\frac{\partial^2 x}{\partial z \partial y}\right)^2 = \frac{rt - s^2}{p^4}$$

8. CHANGE OF VARIABLES

In problems involving change of variables it is frequently required to transform a particular expression involving a combination of derivatives with respect to a set of variables, in terms of derivatives with respect to another set of variables. A general method, illustrating the principles involved is given below, but it can often be modified so as to reduce the algebraic work.

We shall consider derivatives up to second order only. The higher derivatives may be obtained by exactly the same method; fortunately they are not often required. The algebra of the transformation is tedious but the method seems simple.

Problem. *If z is a function $f(x, y)$ of the independent variables x, y, and if x, y are changed to new independent variables u, v by the substitutions $x = \phi(u, v)$, $y = \psi(u, v)$, it is required to express the derivatives of z with respect to x, y in terms of u, v and the derivatives of z with respect to u, v.*

It is understood that f, ϕ, ψ are differentiable (or possesses continuous partial derivatives) with respect to the corresponding variables.

But rule II of § 7.2, we have

$$\left.\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}\right\} \qquad ...(1)$$

Solving these for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get

$$\left.\begin{aligned}\frac{\partial z}{\partial x} &= A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\\ \frac{\partial z}{\partial y} &= C\frac{\partial z}{\partial u} + D\frac{\partial z}{\partial v}\end{aligned}\right\} \quad \text{...(2)}$$

where
$$A = \frac{\partial y}{\partial v}\Big/J,\ B = -\frac{\partial y}{\partial u}\Big/J,\ C = -\frac{\partial x}{\partial v}\Big/J,\ D = \frac{\partial x}{\partial u}\Big/J \text{ and}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}, \text{ called the } Jacobian,$$

are functions of u and v. (Refer to § 2 of Chapter 16, for properties of Jacobians.)

Thus,

$$\frac{\partial z}{\partial x} = -\frac{\partial(y, z)}{\partial(u, v)}\Big/\frac{\partial(x, y)}{\partial(u, v)} \text{ and } \frac{\partial z}{\partial y} = -\frac{\partial(z, x)}{\partial(u, v)}\Big/\frac{\partial(x, y)}{\partial(u, v)}$$

Equation (2) expresses $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $A, B, C, D, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ which are all functions of u, v, and not contain x, y explicitly.

From (2),

$$\frac{\partial}{\partial x} = A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v},\quad \frac{\partial}{\partial y} = C\frac{\partial}{\partial u} + D\frac{\partial}{\partial v}$$

Replacing z by $\frac{\partial z}{\partial x}$ in equation (2), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) = \left(A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v}\right)\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$

$$= A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) + B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$

$$= A^2\frac{\partial^2 z}{\partial u^2} + 2AB\frac{\partial^2 z}{\partial u\partial v} + B^2\frac{\partial^2 z}{\partial v^2} + \left(A\frac{\partial A}{\partial u} + B\frac{\partial A}{\partial v}\right)\frac{\partial z}{\partial u} + \left(A\frac{\partial B}{\partial u} + B\frac{\partial B}{\partial v}\right)\frac{\partial z}{\partial v}$$

The values of $\frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial^2 z}{\partial x\partial y}$ may be found in the same way.

Remark: The expression for $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ can also be found as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \quad ...(3)$$

To obtain $\frac{\partial u}{\partial x}$, differentiate $x = \phi(u, v)$ and $y = \psi(u, v)$ with respect to x,

$$\therefore \quad 1 = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x}, \text{ and } 0 = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}$$

and these give

$$\frac{\partial u}{\partial x} = A, \frac{\partial v}{\partial x} = B \quad ...(4)$$

Differentiating $x = \phi(u, v)$, $y = \psi(u, v)$ with respect to y, we get

$$\frac{\partial u}{\partial y} = C, \frac{\partial v}{\partial y} = D \quad ...(5)$$

Equation (3) now gives the required result.

Note: In the 'change of variables' the variables x, y which are functions of u, v are called the *Intermediate variables*, while u, v are independent variables.

Example 24. If $u = F(x, y, z)$, and $z = f(x, y)$, find a formula for $\partial^2 u/\partial x^2$ in terms of the derivatives of F and the derivatives of z.

■ In the expression for F we consider x, y, z as intermediate variables, while in the expression for f we consider x and y as independent variables.

Now
$$\frac{\partial u}{\partial x} = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x},$$

but since x and y are independent, $\frac{\partial y}{\partial x} = 0$.

Also,
$$\frac{\partial x}{\partial x} = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = F_x + F_z\frac{\partial z}{\partial x}$$

Differentiating a second time, we get

$$\frac{\partial^2 u}{\partial x^2} = F_{xx}\frac{\partial x}{\partial x} + F_{yx}\frac{\partial y}{\partial x} + F_{zx}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x}\left(F_{xz}\frac{\partial x}{\partial x} + F_{yz}\frac{\partial y}{\partial x} + F_{zz}\frac{\partial z}{\partial x}\right) + F_z\frac{\partial^2 z}{\partial x^2}$$

$$= F_{x^2} + 2F_{zx}\frac{\partial z}{\partial x} + F_{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + F_z\frac{\partial^2 z}{\partial x^2}$$

of course, F and f are supposed to be differentiable.

Example 25. Show that $f(xy, z-2x)=0$ satisfies, under suitable conditions, the equation $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2x$. What are these conditions?

▪ Let $u = xy$, $v = z - 2x$; then $f(u, v) = 0$, and

$$df = f_u\,du + f_v\,dv = f_u(x\,dy + y\,dx) + f_v(dz - 2\,dx) = 0$$

Taking z as dependent variable and x and y as independent variables, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

∴ $$df = f_u(x\,dy + y\,dx) + f_v\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy - 2\,dx\right) = 0$$

or

$$\left\{yf_u + f_v\left(\frac{\partial z}{\partial x} - 2\right)\right\}dx + \left\{xf_u + f_v\frac{\partial z}{\partial y}\right\}dy = 0$$

But, since x and y are independent, we have

$$yf_u + f_v\left(\frac{\partial z}{\partial x} - 2\right) = 0, \text{ and } xf_u + f_v\frac{\partial z}{\partial y} = 0.$$

Finding f_u from one equation and putting in the other, we get

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x, \text{ provided } f_v \neq 0$$

Thus, the result holds when f is differentiable and $f_v \neq 0$ (and then $f_u \neq 0$).

Example 26. Prove that, by the transformations $u = x - ct$, $v = x + ct$, the partial differential equation

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} \text{ reduces to } \frac{\partial^2 z}{\partial u\partial v} = 0$$

▪ In this problem, we consider z as a function of u and v which are linear functions of the independent variables x and t.

Now by § 7.1,

$$d^2z = \frac{\partial^2 z}{\partial u^2}du^2 + 2\frac{\partial^2 z}{\partial u\partial v}du\,dv + \frac{\partial^2 z}{\partial v^2}dv^2 + \frac{\partial z}{\partial u}d^2u + \frac{\partial z}{\partial v}d^2v$$

But $u = x - ct$, and $v = x + ct$

∴ $$du = dx - c\,dt,\ dv = dx + c\,dt$$

$$d^2u = 0,\ d^2v = 0$$

∴ $$d^2z = \frac{\partial^2 z}{\partial u^2}(dx - c\,dt)^2 + 2\frac{\partial^2 z}{\partial u\partial v}(dx - c\,dt)(dx + c\,dt) + \frac{\partial^2 z}{\partial v^2}(dx + c\,dt)^2$$

CHAPTER 15

$$= \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)dx^2 + B\,dx\,dt + c^2\left\{\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right\}dt^2 \quad \text{...(1)}$$

the coefficient of $dx\,dt$ is written as B since its actual value is not required for this problem.

Again regarding z as a function of independent variables x and t, we have (by § 6)

$$d^2 z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial t}dx\,dt + \frac{\partial^2 z}{\partial t^2}dt^2 \quad \text{...(2)}$$

From equations (1) and (2) by comparing the coefficients

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{...(3)}$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\left\{\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right\} \quad \text{...(4)}$$

$$\therefore \quad \frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} \Rightarrow \frac{\partial^2 z}{\partial u \partial v} = 0$$

Note: $\frac{\partial^2 z}{\partial t^2} + c^2\frac{\partial^2 z}{\partial x^2}$ reduces to $2c^2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

Example 27. Prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$ is invariant for change of rectangular axes.

■ The change of origin does not affect the expression, for then $x = a + x'$, $y = b + y'$, and $dx = dx'$, $dy = dy'$, and all the partial derivatives V remain of the same form.

Let the axes turn through an angle α, so that

$$x = x' \cos\alpha - y' \sin\alpha, \text{ and } y = x' \sin\alpha + y' \cos\alpha,$$

where α is a constant.

We have

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$$

$$= \frac{\partial V}{\partial x}(dx' \cos\alpha - dy' \sin\alpha) + \frac{\partial V}{\partial y}(dx' \sin\alpha + dy' \cos\alpha)$$

$$= \left(\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\sin\alpha\right)dx' + \left(-\frac{\partial V}{\partial x}\sin\alpha + \frac{\partial V}{\partial y}\sin\alpha\right)dy'$$

Also $$dV = \frac{\partial V}{\partial x'}dx' + \frac{\partial V}{\partial y'}dy'$$

$$\therefore \quad \frac{\partial V}{\partial x'} = \frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\sin\alpha \text{ and } \frac{\partial V}{\partial y'} = -\frac{\partial V}{\partial x}\sin\alpha + \frac{\partial V}{\partial y}\cos\alpha \quad \ldots(1)$$

These give

$$\left.\begin{aligned}\frac{\partial V}{\partial x} &= \frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha \\ \frac{\partial V}{\partial y} &= \frac{\partial V}{\partial x'}\sin\alpha + \frac{\partial V}{\partial y'}\cos\alpha\end{aligned}\right\} \quad \ldots(2)$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial}{\partial x'}\cos\alpha - \frac{\partial}{\partial y'}\sin\alpha\right)\left(\frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\cos^2\alpha - 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\sin^2\alpha$$

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial}{\partial x'}\sin\alpha + \frac{\partial}{\partial y'}\cos\alpha\right)\left(\frac{\partial V}{\partial x'}\sin\alpha + \frac{\partial V}{\partial y'}\cos\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\sin^2\alpha + 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\cos^2\alpha$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2}$$

Thus, the expression remains invariant.

Note: If we write $x' = x\cos\alpha + y\sin\alpha$, $y' = -x\sin\alpha + y\cos\alpha$ the procedure is slightly simplified.

Example 28. If V is a function of two variables x and y and $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}$$

■ We have

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y}$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}$$

Solving these, we get

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta} \Rightarrow \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial V}{\partial y}=\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial y}\Rightarrow\frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Hence,

$$\frac{\partial^2 V}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)$$

$$=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial V}{\partial r}-\frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 V}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2}=\sin^2\theta\frac{\partial^2 V}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 V}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}$$

$$\therefore\quad\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}+\frac{1}{r}\frac{\partial V}{\partial r}$$

Deduction 1.

$$\frac{\partial^2 V}{\partial x\partial y}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial V}{\partial r}+\frac{\cos\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$=\cos\theta\sin\theta\frac{\partial^2 V}{\partial r^2}+\frac{\cos 2\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}-\frac{\cos 2\theta}{r^2}\frac{\partial V}{\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial V}{\partial r}$$

Also

$$x^2-y^2=r^2\cos 2\theta,\text{ and } 4xy=2r^2\sin 2\theta$$

$$\therefore\quad(x^2-y^2)\left(\frac{\partial^2 V}{\partial x^2}-\frac{\partial^2 V}{\partial y^2}\right)+4\,xy\frac{\partial^2 V}{\partial x\partial y}=r^2\frac{\partial^2 V}{\partial r^2}-r\frac{\partial V}{\partial r}-\frac{\partial^2 V}{\partial\theta^2}$$

Deduction 2. To show that $\frac{\partial^2\theta}{\partial x\partial y}=-\frac{\cos 2\theta}{r^2}$.

Here $x=r\cos\theta,\ y=r\sin\theta$

Differentiating w.r.t. $r,\ \theta$ are (functions of x and y)

$$\left.\begin{aligned}1&=\frac{\partial r}{\partial x}\cos\theta-r\sin\theta\frac{\partial\theta}{\partial x}\\0&=\frac{\partial r}{\partial x}\sin\theta-r\cos\theta\frac{\partial\theta}{\partial x}\end{aligned}\right\}\quad\therefore\quad\begin{aligned}\frac{\partial r}{\partial x}&=\cos\theta\\\frac{\partial\theta}{\partial x}&=-\frac{\sin\theta}{r}\end{aligned}$$

Differentiating w.r.t. y,

$$\left.\begin{aligned}0 &= \frac{\partial r}{\partial y}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial y}\\ 1 &= \frac{\partial r}{\partial y}\sin\theta - r\cos\theta\frac{\partial\theta}{\partial y}\end{aligned}\right\} \quad \therefore \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

Now $\dfrac{\partial^2\theta}{\partial x\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\theta}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\cos\theta}{r}\right) = -\dfrac{\cos\theta}{r^2}\dfrac{\partial r}{\partial x} - \dfrac{\sin\theta}{r}\dfrac{\partial\theta}{\partial x} = -\dfrac{\cos 2\theta}{r^2}$

Similarly, it may be shown that

$$\frac{\partial^2\theta}{\partial y\partial x} = -\frac{\cos 2\theta}{r^2}$$

$$\frac{\partial^2 r}{\partial x\partial y} = -\frac{\sin\theta\cos\theta}{r} = \frac{\partial^2 r}{\partial y\partial x}$$

Note: Using the method of Ded. 2, we get

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}$$

and

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta}$$

which are same as in the above example.

Example 29. Given that F is a function of x and y and that $x = e^u + e^{-v}$, $y = e^v + e^{-u}$, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\partial v} + \frac{\partial^2 F}{\partial v^2} = x^2\frac{\partial^2 F}{\partial x^2} - 2xy\frac{\partial^2 F}{\partial x\partial y} + y^2\frac{\partial^2 F}{\partial y^2} + x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}$$

■ Here F is a function of the intermediary variables x and y; and u, v are the independent variables.
Now $x = e^u + e^{-v}$, and $y = e^v + e^{-u}$

$$\therefore \quad dx = e^u\,du - e^{-v}\,dv, \qquad dy = e^v\,dv - e^{-u}\,du$$

$$d^2x = e^u\,du^2 + e^{-v}\,dv^2, \qquad d^2y = e^v\,dv^2 + e^{-u}\,du^2$$

$$d^2F = \frac{\partial^2 F}{\partial x^2}dx^2 + 2\frac{\partial^2 F}{\partial x\partial y}dx\,dy + \frac{\partial^2 F}{\partial y^2}dy^2 + \frac{\partial F}{\partial x}d^2x + \frac{\partial F}{\partial y}d^2y$$

$$= \frac{\partial^2 F}{\partial x^2}(e^u\,du - e^{-v}\,dv)^2 + 2\frac{\partial^2 F}{\partial x\partial y}(e^u\,du - e^{-v}\,dv)\,(e^v\,dv - e^{-u}\,du)$$

$$+ (e^v\,dv - e^{-u}\,du)^2\frac{\partial^2 F}{\partial y^2} + \frac{\partial F}{\partial x}(e^u\,du^2 + e^{-v}\,dv^2) + \frac{\partial F}{\partial y}(e^v\,dv^2 + e^{-u}\,du^2)$$

$$= \left(e^{2u} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{-2u} \frac{\partial^2 F}{\partial y^2} + e^u \frac{\partial F}{\partial x} + e^{-u} \frac{\partial F}{\partial y} \right) du^2$$

$$+ 2 \left[-e^u e^{-v} \frac{\partial^2 F}{\partial x^2} + (e^u e^v + e^{-u} e^{-v}) \frac{\partial^2 F}{\partial x \partial y} - e^v e^{-u} \frac{\partial F}{\partial y^2} \right] du\, dv$$

$$+ \left(e^{-2v} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{2v} \frac{\partial^2 F}{\partial y^2} + e^{-v} \frac{\partial F}{\partial x} + e^v \frac{\partial F}{\partial y} \right) dv^2$$

Also $d^2 F = \dfrac{\partial^2 F}{\partial u^2} du^2 + 2 \dfrac{\partial^2 F}{\partial u \partial v} du\, dv + \dfrac{\partial^2 F}{\partial v^2} dv^2$

Comparing the coefficients, we get

$$\frac{\partial^2 F}{\partial u^2} = e^{2u} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{-2u} \frac{\partial^2 F}{\partial y^2} + e^u \frac{\partial F}{\partial x} + e^{-u} \frac{\partial F}{\partial y}$$

$$\frac{\partial^2 F}{\partial u \partial v} = -e^{-u} e^{-v} \frac{\partial^2 F}{\partial x^2} + (e^u e^v + e^{-u} e^{-v}) \frac{\partial^2 F}{\partial x \partial y} - e^v e^{-u} \frac{\partial^2 F}{\partial y^2}$$

$$\frac{\partial^2 F}{\partial v^2} = e^{-2v} \frac{\partial^2 F}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} + e^{2v} \frac{\partial^2 F}{\partial y^2} + e^{-v} \frac{\partial F}{\partial x} + e^v \frac{\partial F}{\partial y}$$

$$\therefore \quad \frac{\partial^2 F}{\partial u^2} - 2 \frac{\partial^2 F}{\partial u \partial v} + \frac{\partial^2 F}{\partial x^2} = x^2 \frac{\partial^2 F}{\partial x^2} - 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y}$$

Remark: The four examples 26 to 29 give different methods for solving problems of this type. However any one method could be used to solve all such problems. The reader is advised to try.

EXERCISE

1. If $x = u \cos\alpha - v \sin\alpha$ and $y = u \sin\alpha + v \cos\alpha$, where α is a constant, show that

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial u}\right)^2 + \left(\frac{\partial V}{\partial v}\right)^2$$

2. If $2axz + 2byz + cz^2 = k$, $ax + by + cz = R$, prove that

$$R^3 \frac{\partial^2 z}{\partial x^2} = a^2 k,\ R^3 \frac{\partial^2 z}{\partial x \partial y} = abk,\ R^3 \frac{\partial^2 z}{\partial y^2} = b^2 k$$

3. If $z^3 + 3(ax + by)z = c^3$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{2z(ax + by)^3}{(ax + by + z^2)^3}$$

4. If $ax^3 + by^3 + cz^3 + 3hxyz = k$, show that

$$(hxy + cz^2)^3 \frac{\partial^2 z}{\partial x \partial y} = hk(hxy - cz^2) - 2(abc + h^3)x^2y^2z.$$

5. Given that $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, verify that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$$

6. If $z = a \tan^{-1}(y/x)$, show that

(i) $(1 + q^2)r - 2pqs + (1 + p^2)t = 0$

(ii) $(rt - s^2)/(1 + p^2 + q^2)^2 = -a^2/(x^2 + y^2 + a^2)^2$

where p, q, r, s, t have their usual meaning as in Example 23.

7. If $u = y^2 + z^{-2}$, $v = z^2 + x^{-2}$, $w = x^2 + y^{-2}$ and if V is a function of x, y, z; prove that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} + 2\left(u\frac{\partial V}{\partial u} + v\frac{\partial V}{\partial v} + w\frac{\partial V}{\partial w}\right) = 4\left(y^2\frac{\partial V}{\partial u} + z^2\frac{\partial V}{\partial v} + x^2\frac{\partial V}{\partial w}\right)$$

8. If $z = f[(ny - mz)/(nx - lz)]$, prove that

$$(nx - lz)\frac{\partial z}{\partial x} + (ny - mz)\frac{\partial z}{\partial y} = 0$$

9. If $u = f(x + 2y) + g(x - 2y)$, show that

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

10. If $u = \phi(x + at) + \psi(x - at)$, show that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$

Prove that if $y = x + at$, $z = x - at$, the equation becomes $\dfrac{\partial^2 u}{\partial y \partial z} = 0$.

11. Given that $u = F(x, y, z)$ and $z = f(x, y)$, find $\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial^2 u}{\partial y \partial x}$ in terms of the derivatives of F and f (as in Example 24).

12. If $V = F(x, y)$ and $x = e^u \cos t$, $y = e^u \sin t$, show that

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial t^2} = e^{2u}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right).$$

13. If $V = F(x, y)$ and $x = \frac{1}{2}u(e^v + e^{-v})$, $y = \frac{1}{2}u(e^v - e^{-v})$, show that

$$V_{xx} - V_{yy} = V_{uu} + \frac{1}{u}V_u + \frac{1}{u^2}V_{vv}.$$

14. If z is a function of u and v, and $u = x^2 - y^2 - 2xy$, $v = y$, prove that the equation

$$(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = 0 \text{ is equivalent to } \frac{\partial z}{\partial v} = 0.$$

15. If $x = u + v$, $y = uv$ and z is a function of x and y, prove that

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial z}{\partial y}$$

16. If $x = r\cos\theta$, $y = r\sin\theta$, prove that the equation

$$xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - (x^2 - y^2)\frac{\partial^2 u}{\partial x \partial y} = 0$$

becomes

$$r\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial u}{\partial \theta} = 0.$$

17. If $x = c\cosh u \cos v$, $y = c\sinh u \sin v$, and F is a function of x and y, show that

$$F_{uu} + F_{vv} = \frac{1}{2}c^2(\cosh 2u - \cos 2v)\,(F_{xx} + F_{yy}).$$

18. If V is a function of u, v, and $u = x^2 - y^2$, $v = 2xy$, prove that

$$4(u^2 + v^2)\frac{\partial^2 V}{\partial u \partial v} + 2u\frac{\partial V}{\partial v} + 2v\frac{\partial V}{\partial u} = xy\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y^2}\right) + \frac{1}{2}(x^2 - y^2)\frac{\partial^2 V}{\partial x \partial y}$$

19. Given that f is a function of x and y and that $x = u^2 v$, $y = uv^2$, prove that

$$2x^2 f_{x^2} + 2y^2 f_{y^2} + 5xyf_{xy} = uvf_{uv} - \frac{2}{3}(uf_u + vf_v)$$

20. Prove that, if in the equation

$$\frac{\partial^2 z}{\partial x^2} + 2xy^2\frac{\partial z}{\partial x} + 2(y - y^3)\frac{\partial z}{\partial y} + x^2 y^2 z^2 = 0,$$

the variables x, y are changed to u, v, where $x = uv$, $y = \dfrac{1}{v}$, the new equation is obtained by writing u for x and v for y, then z is the same function of u, v as of x, y.

21. If the variables x, y in the equations

$$(x^2 + y)^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + 4xy\frac{\partial^2 z}{\partial x \partial y} + 2x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} = 0$$

are changed to u, v, where $2x = e^u + e^v$, $2y = e^u - e^v$, show that the new equation is

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0$$

22. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, prove that

$$\frac{\partial r}{\partial x} = \sin\theta\cos\phi, \quad \frac{\partial r}{\partial y} = \sin\theta\sin\phi, \quad \frac{\partial r}{\partial z} = \cos\theta,$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{r}\cos\theta\cos\phi,\ \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta\sin\phi,\ \frac{\partial \theta}{\partial z} = -\frac{1}{r}\sin\theta,$$

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta},\ \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta},\ \frac{\partial \phi}{\partial z} = 0.$$

Find also the derivatives of x, y, z with respect to r, θ, ϕ.

23. If u is a function of x, y, z, prove using values in Ex. 22, that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)^2.$$

9. TAYLOR'S THEOREM

If $f(x, y)$ is a function which possesses continuous partial derivatives of order n in any domain of a point (a, b), and the domain is large enough to contain a point $(a+h, b+k)$ with it, then there exists a positive number $0 < \theta < 1$, such that

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \dots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n,$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0 < \theta < 1.$

Let $x = a + th,\ y = b + tk$, where $0 \le t \le 1$ is a parameter, and

$$f(x, y) = f(a + th, b + tk) = \phi(t)$$

Since the partial derivatives of $f(x, y)$ of order n are continuous in the domain under consideration, $\phi^{(n)}(t)$ is continuous in [0, 1], and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$\vdots$$

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$$

therefore by Maclaurin's theorem

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \ldots + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) + \frac{t^n}{n!}\phi^{(n)}(\theta t),$$

where $0 < \theta < 1$.

Now on putting $t = 1$, we get

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \ldots + \frac{1}{(n-1)!}\phi^{(n-1)}(0) + \frac{1}{n!}\phi^{(n)}(0)$$

But $\phi(1) = f(a+h, b+k)$, and $\phi(0) = f(a, b)$

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a, b)$$

$$\phi''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$\vdots$$

$$\phi^{(n)}(\theta) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

$$\therefore \quad f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a, b)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b) + \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n,$$

where $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), 0 < \theta < 1.$

R_n is called the *remainder after n terms*, and the theorem, *Taylor's theorem with remainder* or *Taylor's expansion* about the point (a, b).

If we put $a = b = 0$; $h = x$, $k = y$, we get

$$f(x, y) = f(0, 0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0, 0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \ldots + \frac{1}{(n-1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n-1} f(0, 0) + R_n$$

where $R_n = \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f(\theta x, \theta y),\ 0 < \theta < 1,$ is called the *Maclaurin's theorem* or *Maclaurin's expansion.*

It is easy to see that Taylor's theorem can also be put in the form:

$$f(a+h, b+k) = f(a, b) + df(a, b) + \frac{1}{2!}d^2 f(a, b) + \ldots$$
$$+ \frac{1}{(n-1)!}d^{n-1} f(a, b) + \frac{1}{n!}d^n f(a + \theta h, b + \theta k)$$

The reasoning in the general case of several variables is precisely the same and so the theorem can be easily extended to any number of variables.

9.1 The Theorem can be Stated in Still another Form

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a, b)$$
$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a, b) + \ldots$$
$$+ \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a, b) + R_n,$$

where $R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a + (x-a)\theta, b + (y-b)\theta),\ 0 < \theta < 1,$ called the Taylor's expansion of $f(x, y)$ about the point (a, b) in powers of $x - a$ and $y - b$.

Example 30. Expand $x^2y + 3y - 2$ in powers of $x - 1$ and $y + 2$.

■ Let us use Taylor's expansion with $a = 1$, $b = -2$. Then

$f(x, y) = x^2y + 3y - 2,$ $\quad f(1, -2) = -10$

$f_x(x, y) = 2xy,$ $\quad f_x(1, -2) = -4$

$f_y(x, y) = x^2 + 3,$ $\quad f_y(1, -2) = 4$

$f_{xx}(x, y) = 2y,$ $\quad f_{xx}(1, -2) = -4$

$f_{xy}(x, y) = 2x,$ $\quad f_{xy}(1, -2) = 2$

$f_{yy}(x, y) = 0,$ $\quad f_{yy}(1, -2) = 0$

$f_{xxx}(x, y) = 0 = f_{yyy}(x, y),$ $\quad f_{yxx}(1, -2) = 2 = f_{xxy}(1, -2)$

CHAPTER 15

All higher derivatives are zero.

$$\therefore \quad x^2y + 3y - 2 = -10 - 4(x-1) + 4(y+2) + \frac{1}{2}[-4(x-1)^2 + 4(x-1)(y+2)]$$

$$+ \frac{1}{3!}3(x-1)^2(y+2)(2) + 0$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

Example 31. If $f(x, y) = \sqrt{|xy|}$, prove that Taylor's expansion about the point (x, x) is not valid in any domain which includes the origin.

■ As was shown earlier in Example II § 4.1,

$$f_x(0, 0) = 0 = f_y(0, 0)$$

$$f_x(x, y) = \begin{cases} \frac{1}{2}\sqrt{|y/x|}, & x > 0 \\ -\frac{1}{2}\sqrt{|y/x|}, & x < 0 \end{cases}$$

$$f_y(x, y) = \begin{cases} \frac{1}{2}\sqrt{|x/y|}, & y > 0 \\ -\frac{1}{2}\sqrt{|x/y|}, & y < 0 \end{cases}$$

$$\therefore \quad f_x(x, x) = f_y(x, x) = \begin{cases} \frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases}$$

Now Taylor's expansion about (x, x) for $n = 1$, is

$$f(x+h, x+h) = f(x, x) + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)]$$

or

$$|x+h| = \begin{cases} |x| + h, & \text{if } x + \theta h > 0 \\ |x| - h, & \text{if } x + \theta h < 0 \\ |x|, & \text{if } x + \theta h = 0 \end{cases} \qquad ...(1)$$

If the domain $(x, x; x+h, x+h)$ includes the origin, then x and $x+h$ must be of opposite signs, that is either

$$|x+h| = x+h, \quad |x| = -x$$

or

$$|x+h| = -(x+h), \quad |x| = x$$

But under these conditions none of the inequalities (1) holds. Hence the expansion is not valid.

Ex. 1. Expand $x^4 + x^2y^2 - y^4$ about the point $(1, 1)$ up to terms of the second degree. Find the form of R_2.

Ex. 2. Find the expansion of $\sin x \sin y$ about $(0, 0)$ up to and including the terms of the fourth degree in (x, y). Compare the result with that you get by multiplying the series for $\sin x$ and $\sin y$.

Ex. 3. Expand $e^x \tan^{-1} y$ about $(1, 1)$ up to the second degree in $(x-1)$ and $(y-1)$.

Ex. 4. Show that the expansion of $\sin(xy)$ in powers of $(x-1)$ and $(y-\pi/2)$ up to and including second degree terms is

$$1-\frac{1}{8}\pi^2(x-1)^2-\frac{1}{2}\pi(x-1)\left(y-\frac{\pi}{2}\right)-\frac{1}{2}\left(y-\frac{\pi}{2}\right)^2$$

Ex. 5. Show that, for $0<\theta<1$,

$$\sin x \sin y = xy - \frac{1}{6}[(x^3+3xy^2)\cos\theta x \sin\theta y + (y^3+3x^2y)\sin\theta x \cos\theta y]$$

Ex. 6. Prove that the first four terms of the Maclaurin expansion of $e^{ax}\cos by$ are

$$1+ax+\frac{a^2x^2-b^2y^2}{2!}+\frac{a^3x^3-3ab^2xy^2}{3!}.$$

Ex. 7. Prove that for $0<\theta<1$,

$$e^{ax}\sin by = by + abxy + \frac{1}{6}[(a^3x^3-3ab^2xy^2)\sin(b\theta y)$$
$$+(3a^2bx^2y-b^3y^3)\cos(b\theta y)]\, e^{a\theta x}$$

Ex. 8. Show that if f, f_x, f_y are all continuous in a domain D of (a, b), and D is large enough to contain the point $(a+h, b+k)$, within it, then for $0<\theta<1$,

$$f(a+h, b+k) = f(a, b) + hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k).$$

If $f(x, y) = x\sqrt{x^2+y^2}$, $a=b=-1, h=k=3$, verify that the above conditions are satisfied and find the value of θ.

10. EXTREME VALUES: MAXIMA AND MINIMA

The theory of extreme values (maximum or minimum) for functions of one variable was considered in an earlier chapter. We now investigate the theory for explicit functions of more than one variable. That for implicit functions will be discussed in the next chapter.

Let (a, b) be a point in the domain of definition of a function f. Then $f(a, b)$ is an *extreme value* of f, if for every point (x, y), [other than (a, b)] of some neighbourhood of (a, b), the difference

$$f(x, y) - f(a, b) \qquad ...(1)$$

keeps the same sign.

The extreme value $f(a, b)$ is called *a maximum or a minimum value* according as the sign of (1) is negative or positive.

10.1 A Necessary Condition

A necessary condition for $f(x, y)$ *to have an extreme value at* (a, b) *is that* $f_x(a, b) = 0, f_y(a, b) = 0,$ *provided these partial derivatives exist.*

If $f(a, b)$ is an extreme value of the function $f(x, y)$ of two variables, then it must also be an extreme value of both the functions, $f(x, b)$ and $f(a, y)$ of one variable. But a necessary condition that these have extreme value at $x = a$ and $y = b$ respectively, is

$$f_x(a, b) = 0, \; f_y(a, b) = 0.$$

Notes:

1. The function $f(x, y) = |x| + |y|$ has an extreme value at (0, 0) even though the partial derivatives f_x and f_y do not exist at (0, 0).
2. If $f(x, y) = 0$, if $x = 0$ or $y = 0$, and $f(x, y) = 1$ elsewhere, then both the partial derivatives exist (each equal to zero) at the origin, but $f(0, 0)$ is not an extreme value. Thus the conditions obtained above are *only necessary and not sufficient.*
3. Point at which $f_x = 0, f_y = 0$ (or $df = 0$) are called *Stationary points.*

10.2 Sufficient Conditions for $f(x, y)$ to have an Extreme Value at (a, b)

Let $f_x(a, b) = 0 = f_y(a, b)$. Further, let us suppose that $f(x, y)$ possesses continuous second order partial derivatives in a certain neighbourhood of (a, b) and that these derivatives at (a, b) viz. $f_{xx}(a, b)$, $f_{xy}(a, b), f_{yy}(a, b)$ are not all zero.

Let $(a+h, b+k)$ be a point of this neighbourhood.

Let us write

$$A = f_{xx}(a, b), B = f_{xy}(a, b), C = f_{yy}(a, b)$$

By Taylor's theorem, we have for $0 < \theta < 1$,

$$f(a+h, b+k) = f(a, b) + [hf_x(a, b) + kf_y(a, b)]$$

$$+ \frac{1}{2!}[h^2 f_{xx}(a+\theta h, b+\theta k) + 2hkf_{xy}(a+\theta h, b+\theta k) + k^2 f_{yy}(a+\theta h, +b+\theta k)]$$

But $f_x(a, b) = 0 = f_y(a, b)$, and

Since the second order partial derivatives are continuous at (a, b), we write

$$f_{xx}(a+\theta h, b+\theta k) - f_{xx}(a, b) = \rho_1$$

$$f_{xy}(a+\theta h, b+\theta k) - f_{xy}(a, b) = \rho_2$$

$$f_{yy}(a+\theta h, b+\theta k) - f_{yy}(a, b) = \rho_3$$

where ρ_1, ρ_2, ρ_3 are functions of h and k, and $\to 0$ as $(h, k) \to (0, 0)$.

$$\therefore \quad f(a+h, b+k) - f(a, b) = \frac{1}{2}[Ah^2 + 2Bhk + Ck^2 + \rho]$$

where $\rho = \rho_1 + \rho_2 + \rho_3 \to 0$ as $(h, k) \to (0, 0)$ and is of unknown sign.

Let $G = Ah^2 + 2Bhk + Ck^2$, so that

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}[G + \rho] \qquad \ldots(1)$$

There are several cases to consider.

(*i*) G never vanishes and keeps a constant sign. Since $\rho \to 0$ when $(h, k) \to (0, 0)$, therefore ρ is a small number and the sign of $G + \rho$ is same as of G, *i.e.*, $G + \rho$ remains negative or positive according as G is negative or positive. Thus the difference,

$$f(a+h, b+k) - f(a, b) \lesseqgtr 0 \text{ according as } G \lesseqgtr 0$$

But we know by definition that $f(x, y)$ has a maxima or a minima at (a, b) according as the difference $f(a+h, b+k) - f(a, b)$ is negative or positive for all (h, k) except $(0, 0)$.

Thus $f(a, b)$ will be a maximum or a minimum value according as G is negative or positive.

(*ii*) If G can change sign, since $f(a+h, b+k) - f(a, b)$ and G have the same sign when ρ is small, $f(a, b)$ will not be an extreme value.

(*iii*) If G, without ever changing sign, vanishes for certain values of (h, k), the sign of $f(a+h, b+k) - f(a, b)$ will depend upon ρ, which is of unknown sign, and so no conclusion can be drawn. This is the *doubtful case* and requires further investigation.

Let us first take $A \neq 0$.

G may be written in the form:

$$G = \frac{(Ah + Bk)^2 + k^2(AC - B^2)}{A}$$

(1) If $AC - B^2 > 0$, the numerator of G is the sum of two positive quantities and it never vanishes except when $k = 0$, $h = 0$, simultaneously, which is not permissible [see(*i*)]. Hence, G never vanishes and has the same sign as A.

Thus, $f(a, b)$ has a maximum value if $A < 0$, and a minimum value if $A > 0$.

(2) If $AC - B^2 < 0$, the sign of the numerator of G may be positive or negative according as $(Ah + Bk)^2 >$ or $< k^2(B^2 - AC)$, *i.e.*, according to the values of (h, k). Hence, G does not keep the same sign for all values of (h, k), and therefore, $f(a, b)$ is not an extreme value.

(3) If $AC - B^2 = 0$, the numerator of G is a perfect square but may vanish for values of (h, k) for which $Ah + Bk = 0$. Thus G, without changing sign, may vanish for certain values of (h, k).

This is the doubtful case in which the sign of $f(a+h, b+k) - f(a, b)$ depends upon ρ and requires further investigation.

If $A = 0$, then

$$G = 2Bhk + Ck^2 = k(2Bh + Ck)$$

(4) If $A = 0$, $B \neq 0$, G changes sign with k and $(2Bh + Ck)$, and there is no extreme value.

(5) If $A = 0$, $B = 0$, G does not change sign but may vanish when $k = 0$ (without $h = 0$). This is therefore the doubtful case and requires further investigation.

Rule. *$f(a, b)$ is an extreme value of $f(x, y)$, if $f_x(a, b) = 0 = f_y(a, b)$, and*

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0,$$

and this extreme value is a maximum or a minimum according as $f_{xx}(a, b)$ [or $f_{yy}(a, b)$] is negative or positive.

Further investigation is necessary, if

$$f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

Remark: Since at (a, b),

$$df = hf_x(a, b) + kf_y(a, b)$$

and

$$d^2f = h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b) = Ah^2 + 2Bhk + Ck^2$$

so $f(a, b)$ is an extreme value of $f(x, y)$ if at (a, b), $df = 0$, and d^2f keeps the same sign for all values of $(h, k) \neq (0, 0)$.

Note: Discussion of the doubtful case involves the consideration of terms of higher order than the second in the Taylor expansion of $f(a + h, b + k)$ but this is generally not easy and will not be considered here.

However, it is sometimes possible to decide whether f has a maxima or a minima at (a, b) by algebraic or geometric considerations.

Example 32. Find the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

■ We have

$$f_x(x, y) = 3x^2 - 3 = 0, \text{ when } x = \pm 1$$

$$f_y(x, y) = 3y^2 - 12 = 0, \text{ when } y = \pm 2$$

Thus, the function has four stationary points:

$$(1, 2), (-1, 2), (1, -2), (-1, -2)$$

Now

$$f_{xx}(x, y) = 6x, f_{xy}(x, y) = 0, f_{yy}(x, y) = 6y$$

At $(1, 2)$

$$f_{xx} = 6 > 0, \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence, $(1, 2)$ is a point of minima of the function.

At $(-1, 2)$

$$f_{xx} = -6, \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = -72 < 0$$

Hence, the function has neither a maxima nor a minima at $(-1, 2)$.

At $(1, -2)$,

$$f_{xx} = 6, \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = -72 < 0.$$

Hence, the function has neither maximum nor minimum at $(1, -2)$.

At (−1, −2),

$$f_{xx} = -6, \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence, the function has a maximum value at (−1, −2).

Note: Stationary points like (−1, 2), (1, −2) which are not extreme points are called the *saddle points.*

Example 33. Show that the function

$$f(x, y) = 2x^4 - 3x^2y + y^2$$

has neither a maximum nor a minimum at (0, 0), where

$$f_{xx}f_{yy} - (f_{xy})^2 = 0.$$

■ Now

$$f_x(x, y) = 8x^3 - 6xy, \; f_y(x, y) = -3x^2 + 2y$$

∴ $$f_x(0, 0) = 0 = f_y(0, 0)$$

Also

$$f_{xx}(x, y) = 24x^2 - 6y = 0, \text{ at } (0, 0)$$
$$f_{xy}(x, y) = -6x = 0, \text{ at } (0, 0)$$
$$f_{yy}(x, y) = 2, \text{ at } (0, 0)$$

Thus at (0, 0), $f_{xx}(0, 0) \cdot f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0$.

So that it is a *doubtful* case, and thus requires further examination.

Again

$$f(x, y) = (x^2 - y)(2x^2 - y); \; f(0, 0) = 0$$

or

$$f(x, y) - f(0, 0) = (x^2 - y)(2x^2 - y)$$
$$> 0, \text{ for } y < 0 \text{ or } x^2 > y > 0$$
$$< 0, \text{ for } y > x^2 > \frac{y}{2} > 0$$

Thus $f(x, y) - f(0, 0)$ does not keep the same sign near the origin. Hence f has neither a maximum nor a minimum value at the origin.

Example 34. Show that

$f(x, y) = y^2 + x^2y + x^4$, has a minimum at (0, 0).

■ It can be easily verified that at the origin,

$$f_x = 0, f_y = 0, f_{xx} = 0, f_{xy} = 0, f_{yy} = 2.$$

Thus at the origin $f_{xx}f_{yy} - (f_{xy})^2 = 0$, so that it is a *doubtful* case and requires further investigation.

But we can write

$$f(x, y) = \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4$$

and

$$f(x, y) - f(0, 0) = \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4$$

which is greater than zero for all values of (x, y). Hence f has a minimum value at the origin.

EXERCISE

1. Examine the following for extreme values:
   (*i*) $4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$
   (*ii*) $x^3y^2(12 - 3x - 4y)$
   (*iii*) $y^2 + 4xy + 3x^2 + x^3$
   (*iv*) $(x^2 + y^2 - 4)^2 - x^2$
   (*v*) $(x^2 + y^2)\, e^{6x + 2x^2}$
   (*vi*) $(x - y)^2 (x^2 + y^2 - 2)$
   (*vii*) $x^3 + y^3 - 63(x + y) + 12xy.1$
2. Investigate the maxima and minima of the functions,
   (*i*) $21x - 12x^2 - 2y^2 + x^3 + xy^2$
   (*ii*) $2(x - y)^2 - x^4 - y^4$
   (*iii*) $x^2 + 3xy + y^2 + x^3 + y^3$
   (*iv*) $x^2 + 4xy + 4y^2 + x^3 + 2x^2y + y^4$
   (*v*) $x^2y^2 - 5x^2 - 8xy - 5y^2$
   (*vi*) $x^2 - 2xy + y^2 + x^3 - y^3 + x^5$
3. Show that the function $(y - x)^4 + (x - 2)^4$ has a minimum at (2, 2).
4. Prove that the function $f(x, y) = x^2 - 2xy + y^2 + x^4 + y^4$ has a minima at the origin.
5. Find and classify the extreme values (if any) of the functions:
   (*i*) $x^2 + y^2 + x + y + xy$
   (*ii*) $x^2 + xy + y^2 + ax + by$
   (*iii*) $y^2 - x^3$
   (*iv*) $x^4 + y^4 - 6(x^2 + y^2) + 8xy$
6. A rectangular box, open at the top, is to have a volume of 32 cu ft. What must be the dimensions so that the total surface is a minimum.
7. Show that the function $f(x, y) = x^2 - 3xy^2 + 2y^4$ has neither a maximum nor a minimum value at the origin.

   [*Hint*: $f(x, y) = (x - y^2)(x - 2y^2)$].

ANSWERS

1. (*i*) min. at (0, 0), max. at $(\pm 3/2, \mp 3/2)$;
   (*ii*) max. at (2, 1);
   (*iii*) min. at (2/3, –4/3);
   (*iv*) min. at $\left(\pm 3\sqrt{2}/2, 0\right)$;
   (*v*) extremes at (0, 0), (–1, 0), (–1/2, 0);
   (*vi*) min. at $\left(\mp 1/\sqrt{2}, \pm 1/\sqrt{2}\right)$;
   (*vii*) min. at (3, 3), max. at (–7, –7), and neither max. nor min. at (5, –1) and (–1, 5)
2. (*i*) min. at (7, 0), max. at (1, 0);
   (*ii*) max at (0, 0);
   (*iii*) max. at (–5/3, –5/3);
   (*iv*) min. at (0, 0);
   (*v*) max. at (0, 0);
   (*vi*) neither max. nor min. at (0, 0);

5. (i) min. at (–2/3, –2/3); (ii) min. at $\left[\frac{1}{3}(b-2a), \frac{1}{3}(a-2b)\right]$;

(iii) neither at (0, 0);

(iv) max. at (0, 0), min at $(\pm\sqrt{5}, \mp\sqrt{5})$, neither at $(\pm 1, \mp 1)$.

11. FUNCTIONS OF SEVERAL VARIABLES

We conclude the chapter by referring briefly—in fact very briefly, to the functions of several variables.

An ordered set $(a_1, a_2, ..., a_n)$ of n real numbers is called a **point** in a space of n dimensions. The aggregate of points $(x_1, x_2, ..., x_n)$ when $x_1, x_2, ..., x_n$ range over the entire set of real numbers is referred to as the *space of n dimensions*, denoted by $\boldsymbol{R}^n$.

Neighbourhoods. The set of values $x_1, x_2, ..., x_n$ other than $a_1, a_2, ..., a_n$ that satisfy the conditions

$$|x_1 - a_1| < \delta, |x_2 - a_2| < \delta, ..., |x_n - a_n| < \delta$$

where δ is an arbitrarily small positive number, is said to form a *neighbourhood* of the point $(a_1, a_2, ..., a_n)$. The neighbourhood may however be specified in other, though equivalent ways.

The rectangle

$$(a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ..., a_n - h_n, a_n + h_n)$$

where $h_1, h_2, ..., h_n$ are arbitrarily small positive numbers, is said to be rectangular neighbourhood of $(a_1, a_2, ..., a_n)$.

The points inside the sphere $x_1^2 + x_2^2 + ... + x_n^2 = \delta^2$ may be taken as a neighbourhood of the point (0, 0, ..., 0), called a spherical *nbd.*

Continuity. A function $f(x_1, x_2, ..., x_n)$ is said to be *continuous at a point* $P(a_1, a_2, ..., a_n)$, if to every positive number ε, there corresponds a neighbourhood of P such that for every point $(x_1, x_2, ..., x_n)$ of this neighbourhood

$$|f(x_1, x_2, ..., x_n) - f(a_1, a_2, ..., a_n)| < \varepsilon$$

A function which is continuous at every point of a region is said to be *continuous in that region.*

Partial Derivatives

The derivative of a function with respect to one variable, when all others are kept constant, is called the *partial derivative* of the function with respect to that variable. Thus the partial derivative of f with respect to x_1 at $(a_1, a_2, ..., a_n)$ is

$$\lim_{h_1 \to 0} \frac{f(a_1 + h_1, a_2, ..., a_n) - f(a_1, a_2, ..., a_n)}{h_1}$$

and is denoted by $f_{x_1}(a_1, a_2, ..., a_n)$.

The other partial derivatives of the first, second or higher orders may be defined similarly.

Differentiability. A function f is said to be *differentiable* at $(a_1, a_2, ..., a_n)$, if the change δf in the value of the function, when the point changes from $(a_1, a_2, ..., a_n)$ to $(a_1 + h_1, a_2 + h_2, ..., a_n + h_n)$, can be expressed in the form

$$\delta f = A_1h_1 + A_2h_2 + ... + A_nh_n + h_1\phi_1 + h_2\phi_2 + ... + h_n\phi_n,$$

where $\phi_1, \phi_2, ..., \phi_n$ are functions of $h_1, h_2, ..., h_n$ and tend to zero as $(h_1, h_2, ..., h_n) \to (0, 0, ..., 0)$.

A function which is differentiable at every point of a region, is said to be differentiable over the region.

The differentials $df, d^2f, ...,$ may now be easily defined as in the case of functions of two variables.

11.1 Extreme Values of a Function of *n* Variables

A point $(a_1, a_2, ..., a_n)$ is said to be an *extreme point*, and $f(a_1, a_2, ..., a_n)$ an *extreme value* of a function f, if for every point $(x_1, x_2, ..., x_n)$, other than $(a_1, a_2, ..., a_n)$ of some neighbourhood of $(a_1, a_2, ..., a_n)$, the difference,

$$f(x_1, x_2, ..., x_n) - f(a_1, a_2, ..., a_n) \quad ...(1)$$

keeps the same sign. The extreme value is a **maximum** or a **minimum value** according as the sign is negative or positive.

If $f(a_1, a_2, ..., a_n)$ is an extreme value of the function f of n variables, then it must also be an extreme value of the function $f(x_1, a_2, ..., a_n)$ of one variable x_1 and therefore the partial derivative $f_{x_1}(a_1, a_2, ..., a_n)$, in case it exists, must be zero. The same is true for all the other variables $x_2, x_3, ..., x_n$.

Thus, the **necessary conditions** for $f(a_1, a_2, ..., a_n)$ to be an extreme value of the function f are that all the partial derivatives $f_{x_1}, f_{x_2}, ..., f_{x_n}$, in case they exist, vanish at $(a_1, a_2, ..., a_n)$.

Since these are only necessary and not sufficient conditions, therefore points which satisfy these conditions may not be extreme points. A point $(a_1, a_2, ..., a_n)$ is called a **stationary point** if all the first order partial derivatives of the function vanish at that point. Thus the stationary points are determined by solving the following n equations simultaneously.

$$f_{x_1}(x_1, x_2, ..., x_n) = 0$$
$$f_{x_2}(x_1, x_2, ..., x_n) = 0$$
$$\vdots$$
$$f_{x_n}(x_1, x_2, ..., x_n) = 0$$

For a function f of n independent variables $x_1, x_2, ..., x_n$ the condition can be given in a more compact form.

Thus, if $(a_1, a_2, ..., a_n)$ is a stationary point, then

$$df(a_1, a_2, ..., a_n) = 0$$

i.e., the differential of the function vanishes at a stationary point. For, at the stationary point all the partial derivatives vanish and therefore,

$$df(a_1, a_2, ..., a_n) = f_{x_1}(a_1, a_2, ..., a_n)dx_1 + f_{x_2}(a_1, a_2, ..., a_n)dx_2$$
$$+ ... + f_{x_n}(a_1, a_2, ..., a_n)dx_n = 0$$

Conversely, when $df = 0$, the coefficients of the differentials $dx_1, dx_2, ..., dx_n$ of independent variables, are separately equal to zero.

Further investigations are necessary to decide whether a stationary point is an extreme point or not, or whether it is a maxima or a minima. We now state a rule (without giving a proof which is beyond the scope of this book) for a function of three variables but is applicable to a function of any number of variables.

The Rule

For a function $f(x, y, z)$ of three independent variables, sufficient conditions for (a, b, c) to be an extreme point are that

(i) $df(a, b, c) = f_x dx + f_y dy + f_z dz = 0$, *so that* $f_x = f_y = f_z = 0$, *and*

(ii) $d^2 f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}dz\,dx$,

keep the same sign for arbitrary values of dx, dy, dz; the extreme point being a maxima or a minima according as the sign of d^2f is negative or positive. The point will be neither a maxima nor a minima if d^2f does not keep the same sign; and requires further investigation, if d^2f keeps the same sign but vanishes at some points of a nbd of (a, b, c).

The conditions that d^2f keeps the same sign may be stated (without proof) in terms of matrices, as follows:

Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}.$$

d^2f will always be positive if and only if the three principal minors

$$f_{xx}, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all positive, and d^2f will be always negative if and only if their signs are alternatively negative and positive.

Example 35. Show that

$$f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$$

has a minima at $(1, 1, 1)$ and a maxima at $(-1, -1, -1)$.

We have

$$f_x = 3(x + y + z)^2 - 24yz - 3$$
$$f_y = 3(x + y + z)^2 - 24zx - 3$$
$$f_z = 3(x + y + z)^2 - 24xy - 3$$

Hence, the stationary points are given by

$$\left.\begin{aligned}(x+y+z)^2-8yz-1&=0\\(x+y+z)^2-8zx-1&=0\\(x+y+z)^2-8xy-1&=0\end{aligned}\right\}$$

Subtracting second equation from the first,

$$z(x-y)=0.$$

Similarly, $x(y-z)=0$, $y(z-x)=0$.

Therefore, either $x=0, y=0, z=0$ or $x=y=z$

Hence, stationary points are $(1, 1, 1)$, and $(-1, -1, -1)$.

Again, we have

$$\begin{aligned}f_{xx}&=6(x+y+z)=f_{yy}=f_{zz}\\f_{xy}&=6(x+y+z)-24z=f_{yx}\\f_{yz}&=6(x+y+z)-24x=f_{zy}\\f_{zx}&=6(x+y+z)-24x=f_{xz}\end{aligned}$$

At $(1, 1, 1)$,

$$f_{xx}=f_{yy}=f_{zz}=18, f_{xy}=f_{yz}=f_{zx}=-6$$

$$\therefore \quad d^2f=18(dx^2+dy^2+dz^2)-12(dx\,dy+dy\,dz+dz\,dx)$$

$$=6[(dx^2+dy^2+dz^2)+(dx-dy)^2+(dy-dz)^2+(dz-dx)^2]$$

which is positive for all values of dx, dy, dz and does not vanish for

$$(dx, dy, dz)\neq(0,0,0)$$

Thus $(1, 1, 1)$ is a point of minima of the function.

At $(-1, -1, -1)$,

$$f_{xx}=f_{yy}=f_{zz}=-18, f_{xy}=f_{yz}=f_{zx}=6$$

$$\therefore \quad d^2f=-18(dx^2+dy^2+dz^2)+12(dx\,dy+dy\,dz+dz\,dx)$$

$$=-6[(dx^2+dy^2+dz^2)+(dx-dy)^2+(dy-dz)^2+(dz-dx)^2]$$

which is negative for all dx, dy, dz and never vanishes. Hence the function has a maximum value at $(-1, -1, -1)$.

Example 36. Show that the minimum and the maximum values of

$$f(x, y, z)=(ax+by+cz)\,e^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2} \text{ are}$$

$$-\sqrt{\frac{1}{2}(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2})/e} \text{ and } \sqrt{\frac{1}{2}(a^2\alpha^{-2}+b^2\beta^{-2}+c^2\gamma^{-2})/e}$$

■ We have

$$\begin{aligned}f_x&=[a-2\alpha^2x\,\Sigma ax]\exp(-\Sigma\alpha^2x^2)\\f_y&=[b-2\beta^2y\,\Sigma ax]\exp(-\Sigma\alpha^2x^2)\\f_z&=[c-2\gamma^2z\,\Sigma ax]\exp(-\Sigma\alpha^2x^2)\end{aligned}$$

At the stationary point, since exp $(-\Sigma\alpha^2x^2) \neq 0$, we have

$$\left.\begin{aligned} a - 2\alpha^2 x\, \Sigma ax = 0 \\ b - 2\beta^2 y\, \Sigma ax = 0 \\ c - 2\gamma^2 z\, \Sigma ax = 0 \end{aligned}\right\} \quad \ldots(1)$$

$$\therefore \quad \left.\begin{aligned} x\Sigma ax = a/2\alpha^2 \\ x\Sigma ax = b/2\beta^2 \\ x\Sigma ax = c/2\gamma^2 \end{aligned}\right\}$$

Multiplying by a, b, c and adding,

$$(\Sigma ax)^2 = \frac{1}{2}\Sigma a^2\alpha^{-2}$$

$$\therefore \quad \Sigma ax = \sqrt{\frac{1}{2}\Sigma a^2\alpha^2} = \pm k, \text{ say}$$

Hence from (1), the stationary points are

$$\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right), \left(-\frac{a}{2\alpha^2 k}, -\frac{b}{2\beta^2 k}, -\frac{c}{2\gamma^2 k}\right)$$

Again, we have

$$f_{xx} = -2a^2x[a - 2\alpha^2 x\Sigma ax] \exp(-\Sigma\alpha^2x^2) - 2\alpha^2[\Sigma ax + ax] \exp(-\Sigma\alpha^2x^2)$$

$$f_{xy} = -2a^2x[b - 2\beta^2 y\Sigma ax] \exp(-\Sigma\alpha^2x^2) - 2\beta^2 ay \exp(-\Sigma\alpha^2x^2)$$

and similar expressions for $f_{yy}, f_{zz}, f_{yz}, f_{zx}$,

At the stationary point $\left(\frac{a}{2\alpha^2 k}, \frac{b}{2\beta^2 k}, \frac{c}{2\gamma^2 k}\right)$, we have $\Sigma\alpha^2x^2 = \frac{1}{2}$,

$$f_{xx} = 0 - 2\alpha^2\left[k + \frac{a^2}{2\alpha^2 k}\right]e^{-1/2} = -\frac{1}{\sqrt{e}}\left[2\alpha^2 k + \frac{a^2}{k}\right] = -\frac{2\alpha^2k^2 + a^2}{k\sqrt{e}}$$

$$f_{yy} = -\frac{2\beta^2k^2 + b^2}{k\sqrt{e}}, \quad f_{zz} = -\frac{2\gamma^2k^2 + c^2}{k\sqrt{e}}$$

$$f_{xy} = 0 - \frac{ab}{k\sqrt{e}}, \quad f_{yz} = -\frac{bc}{k\sqrt{e}}, \quad f_{zx} = -\frac{ca}{k\sqrt{e}}$$

$$\therefore \quad d^2 f = \frac{-1}{k\sqrt{e}}[(2\alpha^2k^2 + a^2)dx^2 + (2\beta^2k^2 + b^2)dy^2 + (2\gamma^2k^2 + c^2)dz^2]$$

$$-\frac{2}{k\sqrt{e}}[ab\, dxdy + bc\, dydz + ca\, dzdx]$$

CHAPTER 15

Now $f_{xx} < 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\dfrac{2\alpha^2k^2+\alpha^2}{k\sqrt{e}} & -\dfrac{ab}{k\sqrt{e}} \\ -\dfrac{ab}{k\sqrt{e}} & -\dfrac{2\beta^2k^2+b^2}{k\sqrt{e}} \end{vmatrix} = \frac{1}{k^2e}\begin{vmatrix} 2\alpha^2k^2+a^2 & ab \\ ab & 2\beta^2k^2+b^2 \end{vmatrix}$$

$$= \frac{2}{e}(2\alpha^2\beta^2k^2+\alpha^2b^2+a^2\beta^2) > 0$$

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = -\frac{1}{k^3e^{3/2}}\begin{vmatrix} 2\alpha^2k^2+a^2 & ab & ac \\ ab & 2\beta^2k^2+b^2 & bc \\ ac & bc & 2\gamma^2k^2+c^2 \end{vmatrix}$$

$$= -\frac{1}{k^3e^{3/2}}[(2\gamma^2k^2+c^2)2k^2(2\alpha^2\beta^2k^2+\alpha^2b^2+a^2\beta^2)+a^2c^2(-2\beta^2k^2)-2\alpha^2b^2c^2k^2]$$

$$= -\frac{4k}{e^{3/2}}[2\alpha^2\beta^2\gamma^2k^2+\alpha^2b^2c^2+\alpha^2\beta^2c^2+a^2b^2\gamma^2] < 0.$$

Thus, the three principal minors have alternatively negative and positive signs and so d^2f is always negative, and hence $(a/2\alpha^2k, b/2\beta^2k, c/2\gamma^2k)$ is a point of maxima, and the maximum value $= ke^{-1/2} = \sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}$.

At the point $(-a/2\alpha^2k, -b/2\beta^2k, -c/2\gamma^2k)$, it may be shown as above that $\Sigma\alpha^2x^2 = \frac{1}{2}$, and

$$f_{xx} = \frac{2\alpha^2k^2+a^2}{k\sqrt{e}},\ f_{yy} = \frac{2\beta^2k^2+b^2}{k\sqrt{e}},\ f_{zz} = \frac{2\gamma^2k^2+c^2}{k\sqrt{e}}$$

$$f_{xy} = \frac{ab}{k\sqrt{e}},\ f_{yz} = \frac{bc}{k\sqrt{e}},\ f_{zx} = \frac{ca}{k\sqrt{e}}$$

and the three principal minors are of positive sign, so that d^2f is positive, and hence the point in question is a minima and the minimum value of the function $= -ke^{-1/2} = -\sqrt{\frac{1}{2}\Sigma a^2\alpha^{-2}/e}$.

EXERCISE

1. Show that the following functions have a minima at the points indicated:

   (i) $x^2+y^2+z^2+2xyz$ at (0, 0, 0)

   (ii) $8z+2x^2+3y^2+4z^2-3xy$ at (0, 0, −1)

   (iii) $x^4+y^4+z^4-4xyz$ at (1, 1, 1).

2. Show that the function

$$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$$

has 5 stationary points but has a minimum value only at (1, 2, 0).

3. Show that the function

$$3 \log (x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3, \ (x, y, z) \neq (0, 0, 0)$$

has only one extreme value at $\log (3/e^2)$.

4. Show that the function

$$(y + z)^2 + (z + x)^2 + xyz$$

has no maximum or minimum value.

5. If all the letters are denoted by positive numbers, show that the maximum value of

$$xy(z - h)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) \text{is } (2h/5)^5 ab/c^4 .$$