Partial Differential Equation

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5(a) Solve $(D^2-2DD'+D'^2)z = e^{x+2y}+x^3+\sin 2x$.

where $D\equiv \frac{\partial}{\partial x}$, $D'\equiv\frac{\partial}{\partial y}$, $D^2\equiv\frac{\partial^2}{\partial x^2}$, $D'^2\equiv\frac{\partial^2}{\partial y^2}$ (10)

Sol: Given equation can be written as

$$(D-D')^2 z = e^{x+2y}+x^3+\sin 2x \quad \cdots (1)$$

Its auxiliary equation is

$(m-1)^2=0$ so that $m=1,1$

$\therefore$ C.F $= \phi_1(y+x)+x\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions

(complementary function)

NOW, ~~P.I to~~ Particular Integral corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}e^{x+2y} = e^{x+2y}$$

Particular Integral corresponding to x^3

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3$$

$$= \frac{1}{D^2}(1+\ldots)x^3 = \frac{1}{D^2}x^3 = \frac{1}{D}\frac{x^4}{4} = \frac{x^5}{20}$$

Particular Integral corresponding to $\sin 2x$

$$= \frac{1}{(D-D')^2}\sin 2x = \frac{1}{(D-D')^2}\sin(2x+0y)$$

$$= \frac{1}{(2-0)^2}\iint \sin v\, dv\, dv, \quad \text{where } v = 2x+0y$$

$$= -\frac{1}{4}\int \cos v\, dv = -\frac{1}{4}\sin v = -\frac{1}{4}\sin 2x$$

Hence, the required general solution is

$$\boxed{z = \phi_1(y+x)+x\phi_2(y+x)+e^{x+2y}+\frac{x^5}{20}-\frac{1}{4}\sin 2x}$$

6(a) Find a complete Integral of the Partial Differential equation
$2(pq + yp + qx) + x^2 + y^2 = 0$ (15)

Ex. 29. *Find a complete integral of* $2(pq + yp + qx) + x^2 + y^2 = 0$. **[Kanpur 1993]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0. \quad$ ...(1)

Charpit's auxiliary equations are
$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or
$$\frac{dp}{2q+2x}=\frac{dq}{2p+2y}=\frac{dz}{-p(2q+2y)-q(2p+2x)}=\frac{dx}{-(2q+2y)}=\frac{dy}{-(2p+2x)}, \text{ by (1)}$$

Each of these above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$
$= (dp + dq + dx + dy)/0$

This $\Rightarrow$ $dp + dq + dx + dy = 0$ so that $(p + x) + (q + y) = a$. ...(2)

Re-writing (1), $2(p + x)(q + y) + (x - y)^2 = 0$ or $(p + x)(q + y) = -(x - y)^2/2$. ...(3)

Now, $\quad (p + x) - (q + y) = \sqrt{\{(p+x)^2+(q+y)\}^2-4(p+x)(q+y)}$

$\therefore \quad (p + x) - (q + y) = \sqrt{a^2+2(x-y)^2}$, using (2) and (3) ...(4)

Adding (2) and (4), $\quad 2(p + x) = a + \sqrt{a^2+2(x-y)^2}$.

Substracting (4) from (2), $\quad 2(q + y) = a - \sqrt{a^2+2(x-y)^2}$.

These give $\quad p = -x + \dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}, \quad q = -y + \dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$

Substituting the above values of p and q, $dz = p\,dx + q\,dy$ becomes
$$dz = -(x\,dx + y\,dy) + (a/2) \times (dx + dy) + (1/2)\times\sqrt{a^2+2(x-y)^2}\ (dx - dy)$$

or
$$dz = -\frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\ d(x - y) \quad ...(5)$$

Put $\quad x - y = t \quad$ so that $\quad d(x - y) = dt$. Then (5) becomes
$$dz = -(1/2) \times d(x^2 + y^2) + (a/2) \times d(x + y) + (1/\sqrt{2}) \times \sqrt{(a/\sqrt{2})^2+t^2}\ dt.$$

$$\therefore \quad z = -\frac{x^2+y^2}{2}+a\frac{x+y}{2}+\frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{(a/\sqrt{2})^2+t^2}+\frac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

Putting the value of t, the required complete integral is
$$z = -\frac{x^2+y^2}{2}+\frac{a(x+y)}{2}+\frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}+\frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right]+b.$$

7(a) Reduce the equation

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to canonical form and hence solve it.

Sol: Given equation is

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y} \quad \ldots (1)$$

Eqn (1) can be rewritten as

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0 \quad \ldots (2)$$

comparing (2) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

here $R = y^2$, $S = -2xy$, $T = x^2$ so that $S^2 - 4RT = 0$

showing that (1) is parabolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

or, $(y\lambda - x)^2 = 0$ so that $\lambda = \frac{x}{y}, \frac{x}{y}$

The corresponding characteristic eqn is

$$\frac{dy}{dx} + \frac{x}{y} = 0$$

or, $x\,dx + y\,dy = 0$ so that $\frac{x^2}{2} + \frac{y^2}{2} = c_1$

c_1 being an arbitrary function

Let us choose

$$u = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{and} \quad v = \frac{x^2}{2} - \frac{y^2}{2} \quad \ldots (3)$$

in such a way that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -2xy \neq 0$$

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$, using (3) ... (4)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \quad \text{using (3)} \quad \ldots (5)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by } (4)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right), \text{ using (3)} \quad \ldots (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ by (5)}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots (7)$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

or, $$s = xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots (8)$$

using (4), (5), (6), (7) and (8) in (2) and simplifying, we get

$$4x^2y^2\left(\frac{\partial^2 z}{\partial v^2}\right) = 0 \quad \text{so that} \quad \boxed{\frac{\partial^2 z}{\partial v^2} = 0} \quad \text{-- (9)}$$

which is the required canonical form

Integrating (9) partially w.r.t. 'v',

$$\frac{\partial z}{\partial v} = \phi(u), \quad \phi \text{ being arbitrary function} \ldots (10)$$

Integrating (10) partially w.r.t. 'v',

$$z = v\,\phi(u) + \psi(u), \quad \psi \text{ being arbitrary function}$$

or $$z = \left[\frac{(x^2-y^2)}{2}\right]\phi\left\{\frac{(x^2+y^2)}{2}\right\} + \psi\left\{\frac{(x^2+y^2)}{2}\right\}, \text{ using (3)}$$

or $$\boxed{z = (x^2-y^2)\,F(x^2+y^2) + G(x^2+y^2)}. \quad F, G \text{ being arbitrary functions.}$$

8(a) Given one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}; \quad t > 0$$

where $c^2 = \frac{T}{m}$, T is the constant tension in the string and m is the mass per unit length of the string.

(i) Find the appropriate solution of the above wave equation. (20)

(ii) find also the solution under the conditions

$$y(0,t) = 0, \quad y(l,t) = 0 \quad \text{for all } t$$

and $$\left[\frac{\partial y}{\partial t}\right]_{t=0} = 0, \quad y(x,0) = a\sin\frac{\pi x}{l}, \quad 0 < x < l, \quad a > 0$$

i.e. $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\qquad \delta x \qquad \right]$$

Taking limits as $Q \to P$, *i.e.* $\delta x \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation.*

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $$\frac{\partial^2 y}{\partial t^2} = X \cdot T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' \cdot T$$

Substituting these in (1), we get $XT'' = c^2 X''T$ *i.e.* $$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad ...(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \qquad ...(3)$$

and $$\frac{d^2T}{dt^2} - kc^2T = 0 \qquad ...(4)$$

Solving (3) and (4), we get

and $$\frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say. $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero.* $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.* of the form $y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$ ...(8)

is the only suitable solution of the *wave equation.* (*Assam, 1999*)

Example 18·3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l)$. (*S.V.T.U., 2007 ; Kerala, 2005 ; U.P.T.U., 2004*)

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Sol. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad \text{...(ii)}$$

and

$$y(l, t) = 0 \quad \text{...(iii)}$$

Since the initial transverse velocity of any point of the string is zero,

therefore,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{...(iv)}$$

Also

$$y(x, 0) = a \sin(\pi x / l) \quad \text{...(v)}$$

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{...(vi)}$$

By (*ii*), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt)$ ...(*vii*)

and $\frac{\partial y}{\partial t} = C_2 \sin px \,\{C_3(-cp \cdot \sin cpt) + C_4 (cp \cdot \cos cpt)\}$

$\therefore$ By (*iv*), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \cdot (C_4\, cp) = 0$, whence $C_2 C_4 cp = 0$.

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For this to be true for all time, $C_1 = 0$.

Hence $y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt)$

and $\frac{\partial y}{\partial t} = C_2 \sin px \{C_3(-cp \cdot \sin cpt) + C_4 (cp \cdot \cos cpt)\}$ ...(vii)

$\therefore$ By (iv), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \cdot (C_4 cp) = 0$, whence $C_2 C_4 cp = 0$.

If $C_2 = 0$, (vii) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (vii) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt$ ...(viii)

$\therefore$ By (iii), $y(l, t) = C_2 C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore pl = n\pi$, *i.e.* $p = n\pi/l$, where n is an integer.

Hence (i) reduces to $y(x, t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$.

[These are the solutions of (i) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum**.]

Finally, imposing the last condition (v), we have $y(x, 0) = C_2 C_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$ ...(ix)

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