Krishna's

TEXT BOOK on

Matrices

(*For B.A. and B.Sc. IInd year students of All Colleges affiliated to universities in Uttar Pradesh*)

As per U.P. UNIFIED Syllabus

(*w.e.f.* 2012-2013)

By

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to

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Preface

This book on **MATRICES** has been specially written according to the latest **Unified Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-II Students** of all Universities in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.* and **Mr. Sugam Rastogi**, *Executive Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

June, 2012 — ***Authors***

Syllabus

Matrices

U.P. UNIFIED (*w.e.f.* 2012-13)

B.A./B.Sc. IInd Year–Paper-Ist M.M. : 33 / 65

Section-A: Matrices

Unit-1: Symmetric and skew-symmetric matrices, Hermitian and skew-Hermitian matrices, Orthogonal and unitary matrices, Triangular and diagonal matrices, Rank of a matrix, Elementary transformations, Echelon and normal forms, Inverse of a matrix by elementary transformations.

Unit-2: Characteristic equation, Eigen values and eigen vectors of a matrix, Cayley-Hamilton's theorem and its use in finding inverse of a matrix, Application of matrices to solve a system of linear (both homogeneous and non-homogeneous) equations, Consistency and general solution, Diagonalization of square matrices with distinct eigen values, Quadratic forms.

Brief Contents

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B

MATRICES

Chapters

Chapter 1

Matrices

1.1 Some Basic Concepts

Consider the system of equations

$$2x + 9y + 7z = 4,\ \ 3x + 4y - 3z = 5,$$

$$6x + 8y - 3z = 1,\ \ 4x - 2y + z = 2.$$

Here x, y and z are unknowns and their coefficients are all numbers. Arranging the coefficients in the order in which they occur in the equations and enclosing them in square brackets, we obtain a rectangular array of the form

$$\begin{bmatrix} 2 & 9 & 7 \\ 3 & 4 & -3 \\ 6 & 8 & -3 \\ 4 & -2 & 1 \end{bmatrix}.$$

This rectangular array is an example of a **matrix.** The horizontal lines $(\rightarrow)$ are called **rows** or *row vectors*, and vertical lines $(\downarrow)$ are called **Columns** or *column vectors* of the matrix. There are 4 rows and 3 columns in this matrix. Therefore it is a matrix of the type 4×3. The numbers $3, 4, -3, 2$ etc. constituting this matrix are called its **elements.** The difference between a matrix and a number should be clearly understood. A matrix is not a number. It has got no numerical value. It is a new thing formed with the help of numbers. It is just an ordered collection of numbers arranged in the form of a rectangular array. Simply 7 is a number. But in our notation of matrices $[7]$ is a matrix of the type 1×1 and we cannot have $7 = [7]$. We cannot have a relation of equality between a matrix and a number.

We shall use capital letters (in bold type or in italic type) to denote matrices.

Thus $$\mathbf{A} = \begin{bmatrix} 5 & 0 & 1 \\ 6 & 1 & 7 \end{bmatrix}_{2 \times 3} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}$$

are both matrices. They are of the type 2×3 and 3×3 respectively.

Sometimes we also use the brackets ( ) or the double bars, $\| \ \|$, in place of square brackets [ ] to denote matrices.

Thus $$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 + 5i & 9 \\ -4 & 3 - 5i \end{pmatrix}, \mathbf{C} = \begin{Vmatrix} 7 & 7 \\ 7 & 7 \end{Vmatrix},$$

are all matrices each of the type 2×2.

1.2 Matrix

Definition: *A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called an $m \times n$ matrix [to be read as 'm by n' matrix.].*

An $m \times n$ matrix is usually written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by $\mathbf{A} = [a_{ij}], i = 1,2,\ldots,m, j = 1,2,\ldots n$ or simply by $[a_{ij}]_{m \times n}$. We write the general element of the matrix and enclose it in brackets of the type [ ] or of the type ( ).

The numbers a_{11}, a_{12} etc. of this rectangular array are called the **elements** of the matrix. The element a_{ij} belongs to the i^{th} row and the j^{th} column and is sometimes called the $(i, j)^{th}$ element of the matrix . Thus in the element a_{ij} the first suffix i will always denote the number of the row and the second suffix j, the number of the column in which the element occurs. In a matrix, the number of rows and the columns need not be equal.

1.3 Special Types of Matrices

(i) Square Matrix

Definition: *An $m \times n$ matrix for which $m = n$ (i.e., the number of rows is equal to the number of columns) is called a **square matrix** of order n.* It is also called an n-rowed square matrix. Thus in a square matrix, we have the same number of rows and columns. The elements a_{ij} of a square matrix $\mathbf{A} = [a_{ij}]_{n \times n}$, for which $i = j$ *i.e.*, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the **diagonal elements** and the line along which they lie is called the **principal diagonal** of the matrix. For example, the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 \\ 5 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}_{4 \times 4}$$

is a square matrix of order 4. The elements 0, 3, 1, 2 constitute the principal diagonal of this matrix.

(ii) Unit Matrix or Identity Matrix

Definition: *A square matrix each of whose diagonal elements is 1 and each of whose non-diagonal elements is equal to zero is called a **unit matrix** or an **identity matrix** and is denoted by $\mathbf{I}$. $\mathbf{I}_n$ will denote a unit matrix of order n. Thus a square matrix $\mathbf{A} = [a_{ij}]$ is a unit matrix if $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.*

For example, $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

are unit matrices of order 3 and 2 respectively.

(iii) Null Matrix or Zero Matrix

Definition: *The $m \times n$ matrix whose elements are all 0 is called the* ***null matrix*** *(or* ***zero matrix****) of the type $m \times n$. It is usually denoted by* **O** *or more clearly by $\mathbf{O}_{m, n}$. Often a null matrix is simply denoted by the symbol 0 read as 'zero'.*

For example, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3 \times 4}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3 \times 3}$

are zero matrices of the types 3×4 and 3×3 respectively.

(iv) Row Matrices and Column Matrices

Definition: *Any $1 \times n$ matrix which has only one row and n columns is called a* ***row matrix*** *or row vector. Similarly any $m \times 1$ matrix which has m rows and only one column is a* ***column matrix*** *or a* ***column vector***.

For example, $\mathbf{X} = [2 \quad 7 \quad -8 \quad 5 \quad 11]_{1 \times 5}$ is a row matrix of the type 1×5 while

$$\mathbf{Y} = \begin{bmatrix} 2 \\ -9 \\ 11 \end{bmatrix}_{3 \times 1}$$ is a column matrix of the type 3×1.

1.4 Submatrices of a Matrix

Definition: *Any matrix obtained by omitting some rows and columns from a given $(m \times n)$ matrix* **A** *is called a* ***submatrix*** *of* **A**.

The matrix **A** itself is a sub-matrix of **A** as it can be obtained from **A** by omitting no rows or columns.

A square submatrix of a square matrix **A** is called a **principal submatrix,** if its diagonal elements are also the diagonal elements of the matrix **A**. Principal submatrices are obtained only by omitting corresponding rows and columns.

Example: The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ is a submatrix of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 9 \\ 7 & 11 & 6 & 5 \\ 0 & 2 & 1 & 8 \end{bmatrix}$ as it can be obtained from **A** by omitting the second row and the fourth column.

1.5 Equality of Two Matrices

Definition: *Two matrices* $\mathbf{A} = [a_{ij}]$ *and* $\mathbf{B} = [b_{ij}]$ *are said to be equal, if*

(i) they are of the same size and

(ii) the elements in the corresponding places of the two matrices are the same i.e., $a_{ij} = b_{ij}$ *for each pair of subscripts i and j.*

If two matrices **A** and **B** are equal, we write **A** = **B.** If two matrices **A** and **B** are not equal, we write $\mathbf{A} \neq \mathbf{B}$. If two matrices are not of the same size, they cannot be equal.

1.6 Addition of Matrices

Definition: *Let* **A** *and* **B** *be two matrices of the same type* $m \times n$. *Then their sum (to be denoted by* **A** + **B***) is defined to be the matrix of the type* $m \times n$ *obtained by adding the corresponding elements of* **A** *and* **B.** *Thus if* $\mathbf{A} = [a_{ij}]_{m \times n}$ and $\mathbf{B} = [b_{ij}]_{m \times n}$, then $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]_{m \times n}$.

Note that **A** + **B** is also a matrix of the type $m \times n$.

More clearly we can say that, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$$

and
$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}_{m \times n}$$

then
$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots\dots & \dots\dots & \dots & \dots\dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}_{m \times n}$$

For example, if

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}_{2 \times 3} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & -2 & 7 \\ 3 & 2 & -1 \end{bmatrix}_{2 \times 3},$$

then $$\mathbf{A}+\mathbf{B}=\begin{bmatrix}3+1 & 2-2 & -1+7\\ 4+3 & -3+2 & 1-1\end{bmatrix}=\begin{bmatrix}4 & 0 & 6\\ 7 & -1 & 0\end{bmatrix}_{2\times 3}.$$

Important Note: It should be noted that addition is defined only for matrices which are of the same size. If two matrices **A** and **B** are of the same size, they are said to be **conformable for addition.** If the matrices **A** and **B** are not of the same size, we cannot find their sum.

1.7 Properties of Matrix Addition

(i) Matrix addition is commutative: *If* $\mathbf{A},\mathbf{B},\mathbf{C}$ *be two* $m\times n$ *matrices, then*

$$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}.$$

(ii) Matrix addition is associative: *If* $\mathbf{A},\mathbf{B},\mathbf{C}$ *be three matrices each of the* type $m\times n$, *then*

$$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C}).$$

(iii) Existence of additive identity: *If* $\mathbf{O}$ *be the* $m\times n$ *matrix each of whose elements is zero, then* $\mathbf{A}+\mathbf{O}=\mathbf{A}=\mathbf{O}+\mathbf{A}$ *for every* $m\times n$ *matrix* **A.**

(iv) Existence of the additive inverse.

Negative of matrix. Definition: *Let* $\mathbf{A}=[a_{ij}]_{m\times n}$. *Then the negative of the matrix* **A** *is defined as the matrix* $[-a_{ij}]_{m\times n}$ *and is denoted by* $-\mathbf{A}$.

The matrix $-\mathbf{A}$ is the additive inverse of the matrix **A.** Obviously,

$$-\mathbf{A}+\mathbf{A}=\mathbf{O}=\mathbf{A}+(-\mathbf{A}).$$

Here $\mathbf{O}$ is the null matrix of the type $m\times n$. It is identity element for matrix addition.

Subtraction of two matrices. Definition:

If **A** *and* **B** *are two* $m\times n$ *matrices, then we define* $\mathbf{A}-\mathbf{B}=\mathbf{A}+(-\mathbf{B})$.

Thus the difference $\mathbf{A}-\mathbf{B}$ is obtained by subtracting from each element of **A** the corresponding element of **B.**

(v) Cancellation laws hold good in the case of addition of matrices *i.e., if* $\mathbf{A},\mathbf{B},\mathbf{C}$ *are three* $m\times n$ *matrices, then*

$$\mathbf{A}+\mathbf{B}=\mathbf{A}+\mathbf{C}\Rightarrow\mathbf{B}=\mathbf{C} \qquad \text{(left cancellation law)}$$

and $$\mathbf{B}+\mathbf{A}=\mathbf{C}+\mathbf{A}\Rightarrow\mathbf{B}=\mathbf{C} \qquad \text{(right cancellation law)}$$

(vi) The equation $\mathbf{A}+\mathbf{X}=\mathbf{O}$ has a unique solution $\mathbf{X}=-\mathbf{A}$ in the set of all $m\times n$ matrices.

1.8 Multiplication of a Matrix by a Scalar

Definition: *Let* **A** *be any* $m \times n$ *matrix and* k *any complex number called scalar. The* $m \times n$ *matrix obtained by multiplying every element of the matrix* **A** *by* k *is called the scalar multiple of* **A** *by* k *and is denoted by* $k\mathbf{A}$ *or* $\mathbf{A}k$. *Symbolically, if* $\mathbf{A} = [a_{ij}]_{m \times n}$, *then* $k\mathbf{A} = \mathbf{A}k = [ka_{ij}]_{m \times n}$.

For example, if $k = 2$ and $\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}_{2 \times 3}$,

then $$2\mathbf{A} = \begin{bmatrix} 2 \times 3 & 2 \times 2 & 2 \times -1 \\ 2 \times 4 & 2 \times -3 & 2 \times 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 8 & -6 & 2 \end{bmatrix}_{2 \times 3}.$$

Properties of Multiplication of a Matrix by a Scalar.

Theorem 1: *If* **A** *and* **B** *are two matrices each of the type* $m \times n$, *then* $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$ *i.e., scalar multiplication of matrices distributes over the addition of matrices.*

Theorem 2: *If* p *and* q *are two scalars and* **A** *is any* $m \times n$ *matrix, then*

$$(p + q)\mathbf{A} = p\mathbf{A} + q\mathbf{A}.$$

Theorem 3: *If* p *and* q *are two scalars and* **A** *is any* $m \times n$ *matrix, then*

$$p(q\mathbf{A}) = (pq)\mathbf{A}.$$

Theorem 4: *If* **A** *be any* $m \times n$ *matrix and* k *be any scalar, then*

$$(-k)\mathbf{A} = -(k\mathbf{A}) = k(-\mathbf{A}).$$

Theorem 5: *If* **A** *be any* $m \times n$ *matrix, then*

(i) $1\,\mathbf{A} = \mathbf{A}$ (ii) $(-1)\mathbf{A} = -\mathbf{A}$.

Theorem 6: *If* **A** *and* **B** *are two* $m \times n$ *matrices, then* $-(\mathbf{A} + \mathbf{B}) = -\mathbf{A} - \mathbf{B}$.

1.9 Multiplication of Two Matrices

Definition: *Let* $\mathbf{A} = [a_{ij}]_{m \times n}$ *and* $\mathbf{B} = [b_{jk}]_{n \times p}$ *be two matrices such that the number of columns in* **A** *is equal to the number of rows in* **B**. *Then the* $m \times p$ *matrix* $\mathbf{C} = [c_{ik}]_{m \times p}$ *such that*

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}$$

[Note that the summation is with respect to the repeated suffix j *] is called the product of the matrices* **A** *and* **B** *in that order and we write* $\mathbf{C} = \mathbf{AB}$.

In the product **AB,** the matrix **A** is called the **pre-factor** and the matrix **B** is called the **post-factor.** Also we say that the matrix **A** has been post-multiplied by the matrix **B**. and the matrix **B** has been pre-multiplied by the matrix **A**.

Explanation to understand the above definition. The product **AB** of two matrices **A** and **B** exists if and only if the number of columns in **A** is equal to the number of rows in **B.** Two such matrices are said to be **conformable for multiplication.** If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then **AB** is an $m \times p$ matrix. Further, if $\mathbf{A} = [a_{ij}]_{m \times n}$ and $\mathbf{B} = [b_{jk}]_{n \times p}$, then $\mathbf{AB} = [c_{ik}]_{m \times p}$, where

$$c_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk} = a_{i1} b_{1k} + b_{i2} b_{2k} + \dots a_{in}\, b_{nk}$$

i.e., the $(i, k)^{th}$ element c_{ik} of the matrix **AB** is obtained by multiplying the corresponding elements of the i^{th} row of **A** and the k^{th} column of **B** and then adding the products. The rule of multiplication is **row-by-column multiplication** *i.e.*, in the process of multiplication we take the rows of **A** and the columns of **B**. The element c_{11} of the matrix **AB** is obtained by adding the products of the corresponding elements of the first row of **A** and the first column of **B**. The element c_{12} of the matrix **AB** is obtained by adding the products of the corresponding elements of the first row of **A** and the second column of **B.** Similarly the element c_{21} of the matrix **AB** is obtained by adding the products of the corresponding elements of the second row of **A** and the first column of **B**. In this way we multiply two matrices **A** and **B**.

For example, if $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3 \times 2}$, $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2 \times 2}$

then $\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}_{3 \times 2}.$

Important Note: If the product **AB** exists, then it is not necessary that the product **BA** will also exist. For example, if **A** is a 4×5 matrix and **B** is a 5×3 matrix, then the product **AB** exists while the product **BA** does not exist.

Example : *If* $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ *and* $\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$ *then find* **AB**. *Does* **BA** *exist ?*

Solution : The matrix **A** is of the type 3×3 and the matrix **B** is of the type 3×4. Since the number of columns of **A** is equal to the number of rows of **B** therefore **AB** is defined *i.e.*, the product **AB** exists and it will be a matrix of the type 3×4.

Let
$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}.$$

Then c_{11} = the sum of the products of the corresponding elements of the first row of **A** and the first column of **B**.

c_{12} = the sum of products of the corresponding elements of the first row of **A** and the second column of **B.**

c_{13} = the sum of the products of the corresponding elements of the first row of **A** and the third column of **B**.

c_{23} = the sum of the products of the corresponding elements of the second row of **A** and the third column of **B**.

c_{32} = the sum of the products of the corresponding elements of the third row of **A** and the second column of **B**, and so on.

Therefore by the row by column rule of multiplication (Rows of **A** multiplied by the columns of B), we have

$$\mathbf{AB} = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3 \times 3} \times \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}_{3 \times 4}$$

$$= \begin{bmatrix} 2.1 + 1.2 + 0.3 & 2.2 + 1.0 + 0.1 & 2.3 + 1.1 + 0.0 & 2.4 + 1.2 + 0.5 \\ 3.1 + 2.2 + 1.3 & 3.2 + 2.0 + 1.1 & 3.3 + 2.1 + 1.0 & 3.4 + 2.2 + 1.5 \\ 1.1 + 0.2 + 1.3 & 1.2 + 0.0 + 1.1 & 1.3 + 0.1 + 1.0 & 1.4 + 0.2 + 1.5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}_{3 \times 4.}$$

Since the number of the columns of **B** is not equal to the number of rows of **A**, therefore the product **BA** does not exist.

1.10 Properties of Matrix Multiplication

(i) *Matrix multiplication is associative, if conformability is assured i.e.,* **A (BC) = (AB) C** *if* **A**, **B**, **C** *are $m \times n$, $n \times p$, $p \times q$ matrices respectively.*

(ii) *Multiplication of matrices is distributive with respect to addition of matrices i.e.,*

$$\mathbf{A}\,(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC},$$

where **A, B, C** *are any three,* $m\times n, n\times p, n\times p$ *matrices respectively.*

(iii) *The multiplication of matrices is not always commutative.*

Whenever $\mathbf{AB}=\mathbf{BA}$, *the matrices* **A** *and* **B** *are said to* ***commute***.

If $\mathbf{AB}=-\,\mathbf{BA}$ *the matrices* **A** *and* **B** *are said to* ***anti-commute.***

(iv) *If* **A** *be any* $m\times n$ *matrix and* $\mathbf{O}_{n,p}$ *be an* $n\times p$ *null matrix, then* $\mathbf{AO}_{n,p}=\mathbf{O}_{m,p}$ *where* $\mathbf{O}_{m,p}$ *is an* $m\times p$ *null matrix.*

Similarly if $\mathbf{O}_{m,n}$ *be an* $m\times n$ *null matrix and* **A** *be any* $n\times p$ *matrix, then* $\mathbf{O}_{m,n}\mathbf{A}=\mathbf{O}_{m,p}$.

If **A** *be any n-rowed square matrix and* **O** *be an n-rowed null matrix, then* $\mathbf{AO}=\mathbf{OA}=\mathbf{O}$.

(v) *The equation* $\mathbf{AB}=\mathbf{O}$ *does not necessarily imply that at least one of the matrices* **A** *and* **B** *must be a zero matrix.*

Or

The product of two matrices can be a zero matrix while neither of them is a zero matrix

(vi) *In the case of matrix multiplication if* $\mathbf{AB}=\mathbf{O}$, *then it does not necessarily imply that* $\mathbf{BA}=\mathbf{O}$.

(vii) *If* **A** *be an* $m\times n$ *matrix,* $\mathbf{I}_n$ *denotes the n-rowed unit matrix, it can be easily seen that* $\mathbf{AI}_n=\mathbf{A}=\mathbf{I}_m\mathbf{A}$.

1.11 Triangular, Diagonal and Scalar Matrices

(i) Upper Triangular Matrix. Definition : *A square matrix* $\mathbf{A}=[a_{ij}]$ *is called an upper triangular matrix if* $a_{ij}=0$ *whenever* $i>j$.

Thus in an upper triangular matrix all the elements below the principal diagonal are zero.

For example $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots\dots & 0 \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$ *is an upper triangular matrix of the type* $n\times n$.

(ii) Lower Triangular Matrix. Definition: *A square matrix* $\mathbf{A}=[a_{ij}]$ *is called a lower triangular matrix if* $a_{ij}=0$ *whenever* $i<j$.

Thus in a lower triangular matrix all the elements above the principal diagonal are zero.

For example $\begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots & 0 \\ a_{31} & a_{32} & a_{33} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{bmatrix}_{n \times n}$ *is a lower triangular matrix of the size* $n \times n$.

A triangular matrix $\mathbf{A} = [a_{ij}]_{n \times n}$ is called **strictly triangular** if $a_{ii} = 0$ for $i = 1, 2, \ldots n$.

(iii) Diagonal matrix. Definition: *A square matrix* $\mathbf{A} = [a_{ij}]_{n \times n}$ *whose elements above and below the principal diagonal are all zero, i.e.,* $a_{ij} = 0$ *for all* $i \neq j$, *is called a diagonal matrix.*

Thus a diagonal matrix is both upper and lower triangular. An n-rowed diagonal matrix whose diagonal elements in order are $d_1, d_2, d_3, \ldots d_n$ will often be denoted by the symbol Diag. $[d_1, d_2, \ldots, d_n]$.

(iv) Scalar Matrix. Definition: *A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.*

If
$$\mathbf{S} = \begin{bmatrix} k & 0 & \ldots & 0 \\ 0 & k & \ldots & \vdots \\ 0 & 0 & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & k \end{bmatrix}$$

is an n-rowed scalar matrix each of whose diagonal elements is equal to k *and* $\mathbf{A}$ *is any n-rowed square matrix, then*

$$\mathbf{AS} = \mathbf{SA} = k\mathbf{A}.$$

i.e., the pre-multiplication or the post-multiplication of **A** by **S** has the same effect as the multiplication of **A** by the scalar k. This is perhaps the motivation behind the name 'scalar matrix'.

1.12 Idempotent, Involutary, Nilpotent and Periodic Matrices

Indempotent Matrix. A matrix such that $\mathbf{A}^2 = \mathbf{A}$ is called **idempotent matrix.** Example,

$$\mathbf{A} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Involutary Matrix. A matrix **A** is said to be an **involutary matrix** if $\mathbf{A}^2 = \mathbf{I}$ (unit matrix). Example,

$$\mathbf{A} = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

Since $\mathbf{I}^2 = \mathbf{I}$ always, we conclude that a **Unit matrix is always involutary.**

Nilpotent Matrix. A matrix is said to be a **nilpotent matrix if $\mathbf{A}^k = \mathbf{0}$** if (null matrix) where k is a +ve integer; if however k is the least +ve integer for which $\mathbf{A}^k = \mathbf{0}$, then k is called as the **index** of the *nilpotent matrix.* Examples,

$$\mathbf{A} = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \text{ is nilpotent of index 2.}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \text{ is nilpotent of index 2.}$$

Periodic Matrix. A matrix **A** is said to be a **periodic matrix** if $\mathbf{A}^{k+1} = \mathbf{A}$, where k is a +ve integer. If k is the least +ive integer for which $\mathbf{A}^{k+1} = \mathbf{A}$ then k is said to be **period** of **A**. If we choose $k = 1$, then $\mathbf{A}^2 = \mathbf{A}$ and we call it to be an **idempotent matrix.**

1.13 Determinant of a Square Matrix

Let **A** be any square matrix. The determinant formed by the elements of **A** is said to be the determinant of matrix **A**. This is denoted by $|\mathbf{A}|$ or det **A**. Since in a determinant the number of rows is equal to the number of columns, therefore **only square matrices can have determinants.**

Hence, if $\mathbf{A} = \begin{bmatrix} 12 & 0 & 13 \\ 15 & 12 & 11 \\ 13 & 11 & 14 \end{bmatrix}$, then $\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 12 & 0 & 13 \\ 15 & 12 & 11 \\ 13 & 11 & 14 \end{vmatrix}$

Difference between a matrix and a determinant.

(i) A matrix "**A**" cannot be reduced to a number whereas the determinant can be reduced to a number.

(ii) The number of rows may or may not be equal to number of columns in a matrix while in a determinant the number of rows is equal to the number of columns.

(iii) Interchanging the rows and columns, a different matrix is formed while in a determinant, an interchange of rows and columns does not change the value of the determinant.

1.14 Non-singular and Singular Matrices

Definition. *A square matrix* **A** *is said to be non-singular or singular according as*

$$|\mathbf{A}| \neq 0 \quad \text{or} \quad |\mathbf{A}| = 0$$

1.15 Transpose of a Matrix

Definition: *Let* $\mathbf{A} = [a_{ij}]_{m \times n}$. *Then the* $n \times m$ *matrix obtained from* **A** *by changing its rows into columns and its columns into rows is called the transpose of* **A** *and is denoted by the symbol* $\mathbf{A}'$ *or* $\mathbf{A}^T$.

The operation of interchanging rows with columns is called transposition. Symbolically if

$$\mathbf{A} = [a_{ij}]_{m \times n},$$

then $\qquad \mathbf{A}' = [b_{ji}]_{n \times m}$, where $b_{ji} = a_{ij}$,

i.e., the $(j, i)^{th}$ element of $\mathbf{A}'$ is the $(i, j)^{th}$ element of **A**.

For example, the transpose of the 3×4 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 2 & 1 \end{bmatrix}_{3 \times 4} \text{ is the } 4 \times 3 \text{ matrix } \mathbf{A}' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 1 & 1 \end{bmatrix}_{4 \times 3}.$$

The first row of **A** is the first column of $\mathbf{A}'$. The second row of **A** is the second column of $\mathbf{A}'$. The third row of **A** is the third column of $\mathbf{A}'$.

Theorems: *If* $\mathbf{A}'$ *and* $\mathbf{B}'$ *be the transposes of* **A** *and* **B** *respectively, then*

(i) $(\mathbf{A}')' = \mathbf{A}$;

(ii) $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$, **A** *and* **B** *being of the same size.*

(iii) $(k\mathbf{A})' = k\mathbf{A}'$, *k being any complex number.*

(iv) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$, **A** *and* **B** *being conformable to multiplication.*

The above law (iv) is called the reversal law for transposes *i.e.*, *the transpose of the product is the product of the transposes taken in the reverse order.*

1.16 Orthogonal Matrix

Orthogonal Matrix. Definition: *A square matrix* **A** *is said to be orthogonal if* $\mathbf{A}' \mathbf{A} = \mathbf{I}$. **(Kanpur 2009)**

If **A** is an orthogonal matrix, then $\mathbf{A}' \mathbf{A} = \mathbf{I}$

$\Rightarrow \quad |\mathbf{A}' \mathbf{A}| = |\mathbf{I}| \Rightarrow |\mathbf{A}'| \cdot |\mathbf{A}| = 1 \qquad [\because \det(\mathbf{AB}) = (\det \mathbf{A}).(\det \mathbf{B})]$

$\Rightarrow \quad |\mathbf{A}| \cdot |\mathbf{A}| = 1 \qquad [\because |\mathbf{A}'| = |\mathbf{A}|]$

$\Rightarrow \quad |\mathbf{A}|^2 = 1 \Rightarrow |\mathbf{A}| = \pm 1 \Rightarrow |\mathbf{A}| \neq 0$

$\Rightarrow$ **A** is invertible.

Also then $\mathbf{A}' \mathbf{A} = \mathbf{I} \Rightarrow \mathbf{A}' = \mathbf{A}^{-1}$ which in turn implies $\mathbf{AA}' = \mathbf{I}$.

Thus **A** is an orthogonal matrix if and only if

$$\mathbf{A}' \mathbf{A} = \mathbf{I} = \mathbf{AA}'.$$

Theorem. *If* **A**, **B** *be n-rowed orthogonal matrices,* **AB** *and* **BA** *are also orthogonal matrices.*

Proof: Since **A** and **B** are both n-rowed square matrices, therefore **AB** is also an n-rowed square matrix.

Since $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$ and $|\mathbf{A}| \neq 0$, also $|\mathbf{B}| 1 \neq 0$, therefore $|\mathbf{AB}| \neq 0$. Hence, **AB** is a non-singular matrix.

Now $\quad (\mathbf{AB})' = \mathbf{B}' \mathbf{A}'.$

$\therefore \quad (\mathbf{AB})' (\mathbf{AB}) = (\mathbf{B}' \mathbf{A}') (\mathbf{AB})$

$= \mathbf{B}' (\mathbf{A}' \mathbf{A}) \mathbf{B}$

$= \mathbf{B}' \mathbf{IB} \qquad [\because \mathbf{A}' \mathbf{A} = \mathbf{I}]$

$= \mathbf{B}' \mathbf{B}$

$= \mathbf{I}. \qquad [\because \mathbf{B}' \mathbf{B} = \mathbf{I}]$

$\therefore$ **AB** is orthogonal. Similarly we can prove that **BA** is also orthogonal.

Illustrative Examples

Example 1: *Determine the values of* α, β, γ *when* $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ *is orthogonal.*

Solution : Let $$\mathbf{A}=\begin{bmatrix}0 & 2\beta & \gamma\\ \alpha & \beta & -\gamma\\ \alpha & -\beta & \gamma\end{bmatrix}.$$

Then $$\mathbf{A}'=\begin{bmatrix}0 & \alpha & \alpha\\ 2\beta & \beta & -\beta\\ \gamma & -\gamma & \gamma\end{bmatrix}$$

If **A** is orthogonal, then **AA′ = I.**

$$\begin{bmatrix}0 & 2\beta & \gamma\\ \alpha & \beta & -\gamma\\ \alpha & -\beta & \gamma\end{bmatrix}\begin{bmatrix}0 & \alpha & \alpha\\ 2\beta & \beta & -\beta\\ \gamma & -\gamma & \gamma\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

or $$\begin{bmatrix}4\beta^2+\gamma^2 & 2\beta^2-\gamma^2 & -2\beta^2+\gamma^2\\ 2\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2\\ -2\beta^2+\gamma^2 & \alpha^2-\beta^2-\gamma^2 & \alpha^2+\beta^2+\gamma^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Now equating the corresponding elements, we get

$$4\beta^2+\gamma^2=1 \quad \ldots(1)$$

$$2\beta^2-\gamma^2=0 \quad \ldots(2)$$

$$\alpha^2+\beta^2+\gamma^2=1. \quad \ldots(3)$$

From (1) and (2), we get $\beta=\pm\frac{1}{\sqrt{6}},\gamma=\pm\frac{1}{\sqrt{3}}$.

From (3), we get $\alpha=\pm\frac{1}{\sqrt{2}}$.

Hence $$\alpha=\pm\frac{1}{\sqrt{2}},\beta=\pm\frac{1}{\sqrt{6}},\gamma=\pm\frac{1}{\sqrt{3}}.$$

Example 2: *Show that the matrix*

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \textit{ is orthogonal.}$$

Solution: Let $$\mathbf{A}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}.$$

Then $$\mathbf{A}'=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

We have $$\mathbf{AA'} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Hence the matrix **A** is orthogonal.

Example 3: *Verify that the matrix*

$$\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \text{ is orthogonal.}$$

Solution: Let $$\mathbf{A} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}.$$

Then $$\mathbf{A'} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

We have $$\mathbf{AA'} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3.$$

Hence, the matrix **A** is orthogonal.

1.17 Conjugate of a Matrix

If $i = \sqrt{(-1)}$, then $z = x + iy$ is called a complex number where x and y are any real numbers. If $z = x + iy$, then $\bar{z} = x - iy$ is called the **conjugate** of the complex number z.

We have $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$ *i.e.*, is real.

Also if $z = \bar{z}$, then $x + iy = x - iy$ *i.e.*, $2iy = 0$ *i.e.*, $y = 0$ *i.e.*, z is real.

Conversely, if z is real then $\bar{z} = z$.

If $z = x + iy$, then $\bar{z} = x - iy$. $\therefore$ $\overline{(\bar{z})} = x + iy = z$.

If z_1 and z_2 are two complex numbers, then it can be easily seen that

(i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ and (ii) $\overline{z_1 z_2} = \overline{(z_1)}\,\overline{(z_2)}$.

Conjugate of a Matrix

Definition: *The matrix obtained from any given matrix* **A** *on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of* **A** *and is denoted by* $\overline{\mathbf{A}}$.

Thus if $\mathbf{A} = [a_{ij}]_{m \times n}$, then $\overline{\mathbf{A}} = [\overline{a}_{ij}]_{m \times n}$ where $\overline{a}_{ij}$ denotes the conjugate complex of a_{ij}.

If **A** be a matrix over the field of **real numbers**, then obviously $\overline{\mathbf{A}}$ coincides with **A**.

Example: If $\mathbf{A} = \begin{bmatrix} 2+3i & 4-7i & 8 \\ -i & 6 & 9+i \end{bmatrix}$, then $\overline{\mathbf{A}} = \begin{bmatrix} 2-3i & 4+7i & 8 \\ i & 6 & 9-i \end{bmatrix}$.

Theorem. *If* $\overline{\mathbf{A}}$ *and* $\overline{\mathbf{B}}$ *be the conjugates of* **A** *and* **B** *respectively, then*

(i) $\overline{(\overline{\mathbf{A}})} = \mathbf{A}$

(ii) $\overline{(\mathbf{A}+\mathbf{B})} = \overline{\mathbf{A}} + \overline{\mathbf{B}}$, **A** *and* **B** *being of the same size*

(iii) $\overline{(k\,\mathbf{A})} = \overline{k}\,\overline{\mathbf{A}}$, *k being any complex number*

(iv) $\overline{(\mathbf{AB})} = \overline{\mathbf{A}}\,\overline{\mathbf{B}}$, **A** *and* **B** *being conformable to multiplication.*

1.18 Transposed Conjugate of a Matrix

Definition: *The transpose of the conjugate of a matrix* **A** *is called transposed conjugate of* **A** *and is denoted by* $\mathbf{A}^{\theta}$ *or by* $\mathbf{A}^*$.

Obviously the conjugate of the transpose of **A** is the same as the transpose of the conjugate of **A** *i.e.*,

$$(\overline{\mathbf{A}'}) = (\overline{\mathbf{A}})' = \mathbf{A}^{\theta}.$$

If $\quad \mathbf{A} = [a_{ij}]_{m \times n}$, then $\mathbf{A}^{\theta} = [b_{ji}]_{n \times m}$

where $b_{ji} = \overline{a}_{ij}$ *i.e.*, the $(j, i)^{th}$ element of $\mathbf{A}^{\theta}$ = the conjugate complex of the $(i, j)^{th}$ element of **A**.

Example: If

$$\mathbf{A} = \begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix},$$

then

$$\mathbf{A}' = \begin{bmatrix} 1+2i & 4-5i & 8 \\ 2-3i & 5+6i & 7+8i \\ 3+4i & 6-7i & 7 \end{bmatrix}$$

and
$$\overline{(\mathbf{A}')} = \mathbf{A}^{\theta} = \begin{bmatrix} 1-2i & 4+5i & 8 \\ 2+3i & 5-6i & 7-8i \\ 3-4i & 6+7i & 7 \end{bmatrix}.$$

Theorem: *If* $\mathbf{A}^{\theta}$ *and* $\mathbf{B}^{\theta}$ *be the transposed conjugates of* **A** *and* **B** *respectively, then*

(i) $(\mathbf{A}^{\theta})^{\theta} = \mathbf{A}$

(ii) $(\mathbf{A} + \mathbf{B})^{\theta} = \mathbf{A}^{\theta} + \mathbf{B}^{\theta}$, **A** *and* **B** *being of the same size*

(iii) $(k\mathbf{A})^{\theta} = \bar{k}\ \mathbf{A}^{\theta}$, *k being any complex number*

(iv) $(\mathbf{AB})^{\theta} = \mathbf{B}^{\theta}\ \mathbf{A}^{\theta}$, **A** *and* **B** *being conformable to multiplication.*

1.19 Symmetric and Skew-symmetric Matrices

Symmetric Matrix. Definition: *A square matrix* $\mathbf{A} = [a_{ij}]$ *is said to be symmetric if its* $(i, j)^{th}$ *element is the same as its* $(j, i)^{th}$ *element i.e., if* $a_{ij} = a_{ji}$ *for all i, j.*

For example,
$$\begin{bmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & s \end{bmatrix}, \begin{bmatrix} 1 & i & -2i \\ i & -2 & 4 \\ -2i & 4 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$$

are symmetric matrices.

Theorem 1: *A necessary and sufficient condition for a matrix* **A** *to be symmetric is that* **A** *and* **A′** *are equal.*

Skew-symmetric matrix. Definition: *A square matrix* $\mathbf{A} = [a_{ij}]$ *is said to be skew-symmetric if the* $(i, j)^{th}$ *element of* **A** *is the negative of the* $(j, i)^{th}$ *element of* **A** *i.e., if* $a_{ij} = -a_{ji}$ *for all i, j.*

If **A** is a skew-symmetric matrix, then

$$a_{ij} = -a_{ji} \qquad \text{[by definition]}$$

$\therefore$ $a_{ii} = -a_{ii}$, for all values of i.

$\therefore$ $2a_{ii} = 0$ or $a_{ii} = 0$.

Thus the diagonal elements of a skew-symmetric matrix are all zero.

For example, the matrices $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -3i & -4 \\ 3i & 0 & 8 \\ 4 & -8 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Theorem: *A necessary and sufficient condition for a matrix* **A** *to be skew-symmetric is that*

$$\mathbf{A}' = -\mathbf{A}.$$

1.20 Hermitian and skew-Hermitian Matrices

Hermitian Matrix. Definition: *A square matrix* $\mathbf{A} = [a_{ij}]$ *is said to be Hermitian if the* $(i, j)^{th}$ *element of* **A** *is equal to the conjugate complex of the* $(j, i)^{th}$ *element of* **A** *i.e., if* $a_{ij} = \bar{a}_{ji}$ *for all* i *and* j. **(Lucknow 2006)**

For example, $\begin{bmatrix} a & b + ic \\ b - ic & d \end{bmatrix}$, $\begin{bmatrix} 1 & 2 - 3i & 3 + 4i \\ 2 + 3i & 0 & 4 - 5i \\ 3 - 4i & 4 + 5i & 2 \end{bmatrix}$ are Hermitian matrices.

If **A** is a Hermitian matrix, then $a_{ii} = \bar{a}_{ii}$, by definition.

$\therefore$ a_{ii} is real for all i. Thus every diagonal element of a Hermitian matrix must be real.

A Hermitian matrix over the field of real numbers is nothing but a real symmetric matrix.

Theorem: *A necessary and sufficient condition for a matrix* **A** *to be Hermitian is that* $\mathbf{A} = \mathbf{A}^{\theta}$.

Skew-Hermitian Matrix. Definition: *A square matrix* $\mathbf{A} = [a_{ij}]$ *is said to be skew-Hermitian if the* $(i, j)^{th}$ *element of* **A** *is equal to the negative of the conjugate complex of the* $(j, i)^{th}$ *element of* **A** *i.e.,*

$$a_{ij} = -\bar{a}_{ji} \text{ for all } i \text{ and } j.$$

If **A** is a skew-Hermitian matrix, then

$$a_{ii} = -\bar{a}_{ii}, \text{ by definition.}$$

$$\therefore \quad a_{ii} + \bar{a}_{ii} = 0$$

i.e., a_{ii} must be either a pure imaginary number or must be zero.

Thus the diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zero.

(Lucknow 2008, 09)

For example, the matrices $\begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix}, \begin{bmatrix} -i & 3+4i \\ -3+4i & 0 \end{bmatrix}$

are skew-Hermitian matrices. A skew-Hermitian matrix over the field of real numbers is nothing but a real skew-symmetric matrix.

Theorem. *A necessary and sufficient condition for a matrix* **A** *to be skew-Hermitian is that*

$$\mathbf{A}^{\theta} = -\mathbf{A}.$$

Illustrative Examples

Example 4: *If* **A** *is a symmetric (skew-symmetric) matrix, then show that k***A** *is also symmetric (skew-symmetric).*

Solution: (i) Let **A** be a symmetric matrix. Then $\mathbf{A}' = \mathbf{A}$.

We have $(k\mathbf{A})' = k\mathbf{A}'$

$= k\mathbf{A}.$ $[\because \mathbf{A}' = \mathbf{A}]$

Since $(k\mathbf{A})' = k\mathbf{A}$, therefore $k\mathbf{A}$ is a symmetric matrix.

(ii) Let **A** be a skew-symmetric matrix. Then $\mathbf{A}' = -\mathbf{A}$.

We have $(k\mathbf{A})' = k\mathbf{A}' = k(-\mathbf{A})$ $[\because \mathbf{A}' = -\mathbf{A}]$

$= -(k\mathbf{A}).$

Since $(k\mathbf{A})' = -(k\mathbf{A})$, therefore $k\mathbf{A}$ is a skew-symmetric matrix.

Example 5: *If* **A** *is a Hermitian matrix, show that i***A** *is skew-Hermitian.*

Solution: Let **A** be a Hermitian Matrix. Then $\mathbf{A}^{\theta} = \mathbf{A}$.

We have $(i\mathbf{A})^{\theta} = \overline{i}\ \mathbf{A}^{\theta}$ $[\because (k\mathbf{A})^{\theta} = \overline{k}\ \mathbf{A}^{\theta}]$

$= (-i)\mathbf{A}^{\theta}$ $[\because \overline{i} = -i]$

$= -(i\mathbf{A}^{\theta})$

$= -(i\mathbf{A})$ $[\because \mathbf{A}^{\theta} = \mathbf{A}]$.

Since $(i\mathbf{A})^{\theta} = -(i\mathbf{A})$, therefore $i\mathbf{A}$ is a skew-Hermitian matrix.

Example 6: *If* **A** *and* **B** *are symmetric matrices, then show that* **AB** *is symmetric if and only if* **A** *and* **B** *commute i.e.,* **AB = BA.** **(Lucknow 2008)**

Solution: It is given that **A** and **B** are two symmetric matrices. Therefore $\mathbf{A}' = \mathbf{A}$ and $\mathbf{B}' = \mathbf{B}$.

Now suppose that **AB** = **BA.**

Then to prove that **AB** is symmetric.

We have $(\mathbf{AB})' = \mathbf{B}'\,\mathbf{A}'$

$= \mathbf{BA}$ $\quad [\because\ \mathbf{A}' = \mathbf{A},\ \mathbf{B}' = \mathbf{B}]$

$= \mathbf{AB}.$ $\quad [\because\ \mathbf{AB} = \mathbf{BA}]$

Since $(\mathbf{AB})' = \mathbf{AB}$, therefore **AB** is a symmetric matrix.

Conversely suppose that **AB** is a symmetric matrix. Then to prove that

$\mathbf{AB} = \mathbf{BA}.$

We have $\mathbf{AB} = (\mathbf{AB})'$ $\quad [\because$ **AB** is a symmetric matrix]

$= \mathbf{B}'\,\mathbf{A}' = \mathbf{BA}.$

Example 7: *If* **A** *be any matrix, then prove that* **AA′** *and* **A′ A** *are both symmetric matrices.*

Solution: Let **A** be any matrix.

We have $(\mathbf{AA}')' = (\mathbf{A}')'\,\mathbf{A}'$ $\quad$ [By the reversal law for transposes]

$= \mathbf{AA}'$ $\quad [\because\ (\mathbf{A}')' = \mathbf{A}].$

Since $(\mathbf{AA}')' = \mathbf{AA}'$, therefore **AA′** is a symmetric matrix.

Again $(\mathbf{A}'\,\mathbf{A})' = \mathbf{A}'\,(\mathbf{A}')' = \mathbf{A}'\,\mathbf{A}.$

Since $(\mathbf{A}'\,\mathbf{A})' = \mathbf{A}'\,\mathbf{A}$, therefore **A′ A** is a symmetric matrix.

Example 8: *Show that the matrix* **B′ AB** *is symmetric or skew-symmetric according as* **A** *is symmetric or skew-symmetric.* **(Lucknow 2005)**

Solution: **Case I.** Let **A** be a symmetric matrix. Then $\mathbf{A}' = \mathbf{A}.$

Now $(\mathbf{B}'\,\mathbf{AB})' = \mathbf{B}'\,\mathbf{A}'\,(\mathbf{B}')'$, by the reversal law for the transposes

$= \mathbf{B}'\,\mathbf{A}'\,\mathbf{B}$ $\quad$ [since $(\mathbf{B}')' = \mathbf{B}$]

$= \mathbf{B}'\,\mathbf{AB}.$

Hence **B′ AB** is symmetric.

Case II. Let **A** be a skew-symmetric matrix.

Then $\mathbf{A}' = -\,\mathbf{A}.$

Now $(\mathbf{B}'\,\mathbf{AB})' = \mathbf{B}'\,\mathbf{A}'\,(\mathbf{B}')' = \mathbf{B}'\,\mathbf{A}'\,\mathbf{B} = \mathbf{B}'\,(-\,\mathbf{A})\,\mathbf{B}$

$= -\,(\mathbf{B}'\,\mathbf{A})\,\mathbf{B} = -\,\mathbf{B}'\,\mathbf{AB}.$

Hence **B′ AB** is a skew-symmetric matrix.

Example 9: *Show that the matrix*

$$\begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$$

is skew-Hermitian.

Solution: Let us denote the given matrix by **A**.

Then $\overline{\mathbf{A}}$ = conjugate of the matrix **A**

$$= \begin{bmatrix} -i & 3-2i & -2+i \\ -3-2i & 0 & 3+4i \\ 2+i & -3+4i & +2i \end{bmatrix}$$

$$\therefore \qquad \mathbf{A} = (\overline{\mathbf{A}})' = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix}$$

$$= - \begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix} = -\mathbf{A}.$$

Since $\mathbf{A}^{\theta} = -\mathbf{A}$, therefore the matrix **A** is skew-Hermitian.

Example 10: *If* $\mathbf{A} = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$, *prove that* **A** *is a Hermitian matrix.*

(Rohilkhand 2008, 10)

Solution: Let us denote the given matrix by **A**.

Then $$\overline{\mathbf{A}} = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix}.$$

$$\therefore \qquad \mathbf{A}^{\theta} = (\overline{\mathbf{A}})' = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix} = \mathbf{A}$$

Since $\mathbf{A}^{\theta} = \mathbf{A}$, therefore the matrix **A** is Hermitian.

Example 11: *Express* $\begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$ *as the sum of a Hermitian and a skew-Hermitian matrix.*

Solution. If $\mathbf{A}$ is any square matrix, then we can write

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\theta}) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\theta}),$$

where $\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\theta})$ is a Hermitian matrix and $\frac{1}{2}(\mathbf{A} - \mathbf{A}^{\theta})$ is a skew-Hermitian matrix.

Let
$$\mathbf{A} = \begin{bmatrix} -2+3i & 1-i & 2+i \\ 3 & 4-5i & 5 \\ 1 & 1+i & -2+2i \end{bmatrix}$$

Then $\overline{\mathbf{A}} =$ conjugate of the matrix $\mathbf{A}$.

$$= \begin{bmatrix} -2-3i & 1+i & 2-i \\ 3 & 4+5i & 5 \\ 1 & 1-i & -2-2i \end{bmatrix}$$

$\therefore$
$$\mathbf{A}^{\theta} = (\overline{\mathbf{A}})' = \begin{bmatrix} -2-3i & 3 & 1 \\ 1+i & 4+5i & 1-i \\ 2-i & 5 & -2-2i \end{bmatrix}$$

Now
$$\frac{1}{2}(\mathbf{A} + \mathbf{A}^{\theta}) = \frac{1}{2}\begin{bmatrix} -4 & 4-i & 3+i \\ 4+i & 8 & 6-i \\ 3-i & 6+i & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2-\frac{1}{2}i & \frac{3}{2}+\frac{1}{2}i \\ 2+\frac{1}{2}i & 4 & 3-\frac{1}{2}i \\ \frac{3}{2}-\frac{1}{2}i & 3+\frac{1}{2}i & -2 \end{bmatrix},$$

which is a Hermitian matrix.

Again
$$\frac{1}{2}(\mathbf{A} - \mathbf{A}^{\theta}) = \frac{1}{2}\begin{bmatrix} 6i & -2-i & 1+i \\ 2-i & -10i & 4+i \\ -1+i & -4+i & 4i \end{bmatrix}$$

$$= \begin{bmatrix} 3i & -1-\frac{1}{2}i & \frac{1}{2}+\frac{1}{2}i \\ 1-\frac{1}{2}i & -5i & 2+\frac{1}{2}i \\ -\frac{1}{2}+\frac{1}{2}i & -2+\frac{1}{2}i & 2i \end{bmatrix},$$

which is a skew-Hermitian matrix.

Example 12: *Express the following matrix as the sum of a symmetric and a skew-symmetric matrix :*

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}.$$

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix}$; so that $\mathbf{A}' = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix}$.

$\therefore$ $$\mathbf{A} + \mathbf{A}' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 10 & 9 \\ 3 & 9 & 6 \end{bmatrix}$$

or $$\frac{1}{2}(\mathbf{A} + \mathbf{A}') = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 5 & 9/2 \\ 3/2 & 9/2 & 3 \end{bmatrix},$$ which is a symmetric matrix.

Again $$\mathbf{A} - \mathbf{A}' = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 5 & 3 \\ -1 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ 2 & 5 & 6 \\ 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 5 \\ -4 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}$$

or $$\frac{1}{2}(\mathbf{A} - \mathbf{A}') = \begin{bmatrix} 0 & 2 & 5/2 \\ -2 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix},$$ which is a skew-symmetric matrix.

Thus $$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 5 & 9/2 \\ 3/2 & 9/2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5/2 \\ -2 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix},$$

where the first matrix is symmetric and the second matrix is skew-symmetric.

Example 13: *Show that every square matrix is uniquely expressible as the sum of a symmetric matrix and a skew-symmetric matrix.*

(Purvanchal 2006, 09; Rohilkhand 07; Lucknow 11)

Solution. Let **A** be any square matrix. We can write

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') + \frac{1}{2}(\mathbf{A} - \mathbf{A}') = \mathbf{P} + \mathbf{Q}, \text{ say}$$

where $$\mathbf{P} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') \quad \text{and} \quad \mathbf{Q} = \frac{1}{2}(\mathbf{A} - \mathbf{A}')$$

We have $$\mathbf{P}' = \left\{\frac{1}{2}(\mathbf{A} + \mathbf{A}')\right\}' = \frac{1}{2}(\mathbf{A} + \mathbf{A}')' \qquad [\because (k\mathbf{A})' = k\mathbf{A}']$$

$$= \frac{1}{2}\{\mathbf{A}' + (\mathbf{A}')'\} \qquad [\because (\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}']$$

$$= \frac{1}{2}(\mathbf{A}' + \mathbf{A}) \qquad [\because (\mathbf{A}')' = \mathbf{A}]$$

$$= \frac{1}{2}(\mathbf{A} + \mathbf{A}') = \mathbf{P}.$$

Therefore **P** is a symmetric matrix.

Again $$\mathbf{Q}' = \left\{\frac{1}{2}(\mathbf{A} - \mathbf{A}')\right\}' = \frac{1}{2}(\mathbf{A} - \mathbf{A}')' = \frac{1}{2}\{\mathbf{A}' - (\mathbf{A}')'\}$$

$$= \frac{1}{2}(\mathbf{A}' - \mathbf{A}) = -\frac{1}{2}(\mathbf{A} - \mathbf{A}') = -\mathbf{Q}.$$

Therefore **Q** is a skew-symmetric matrix.

Thus we have expressed the square matrix **A** as the sum of a symmetric and a skew-symmetric matrix.

To prove that the representation is unique, let $\mathbf{A} = \mathbf{R} + \mathbf{S}$ be another such representation of **A,** where **R** is symmetric and **S** skew-symmetric. Then to prove that $\mathbf{R} = \mathbf{P}$ and $\mathbf{S} = \mathbf{Q}$.

We have $$\mathbf{A}' = (\mathbf{R} + \mathbf{S})' = \mathbf{R}' + \mathbf{S}' = \mathbf{R} - \mathbf{S} \qquad [\because \mathbf{R}' = \mathbf{R} \text{ and } \mathbf{S}' = -\mathbf{S}]$$

$\therefore$ $$\mathbf{A} + \mathbf{A}' = 2\mathbf{R} \text{ and } \mathbf{A} - \mathbf{A}' = 2\mathbf{S}.$$

This gives $$\mathbf{R} = \frac{1}{2}(\mathbf{A} + \mathbf{A}') \text{ and } \mathbf{S} = \frac{1}{2}(\mathbf{A} - \mathbf{A}').$$

Thus $$\mathbf{R} = \mathbf{P} \text{ and } \mathbf{S} = \mathbf{Q}.$$

Therefore the representation is unique.

1.21 Adjoint or Adjugate of a Square Matrix

Definition: *Let* $\mathbf{A} = [a_{ij}]_{n \times n}$ *be any* $n \times n$ *matrix. The transpose* $\mathbf{B}'$ *of the matrix* $\mathbf{B} = [A_{ij}]_{n \times n}$*, where* $\mathbf{A}_{ij}$ *denotes the cofactor of the element* a_{ij} *in the determinant* $|\mathbf{A}|$*, is called the adjoint of the matrix* $\mathbf{A}$ *and is denoted by the symbol adj* $\mathbf{A}$.

Thus the adjoint of a matrix **A** is the transpose of the matrix formed by the cofactors of **A** *i.e.*, if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

then Adj $\mathbf{A}$ = the transpose of the matrix $\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}$

$$= \text{the matrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

Note: Sometimes the adjoint of a matrix is also called the **adjugate** of that matrix.

Theorem: *If* **A** *is any square matrix of order n, then*

$$\mathbf{A}\,(adj\ \mathbf{A}) = |\mathbf{A}|\,\mathbf{I_n} = (adj\ \mathbf{A})\,\mathbf{A}.$$

1.22 Invertible Matrices

Inverse or reciprocal of a matrix.

Definition: *Let* **A** *be any n-rowed square matrix. Then a matrix* **B**, *if it exists, such that* $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$ *is called inverse of* **A**.

Note: For the products **AB, BA** to be both defined and be equal, it is necessary that **A** and **B** are both square matrices of the same order. Thus **non-square matrices cannot possess inverse.**

Existence of the Inverse. Theorem. *The necessary and sufficient condition for a square matrix* **A** *to possess the inverse is that* $|\mathbf{A}| \neq 0$.

Important. *If* $\mathbf{A}$ *be an invertible matrix, then the inverse of* $\mathbf{A}$ *is* $\frac{1}{|\mathbf{A}|}$ *Adj.* $\mathbf{A}$. *It is usual to denote the inverse of* $\mathbf{A}$ *by* $\mathbf{A}^{-1}$.

Thus, $$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \text{Adj} \,.\, \mathbf{A}, \text{ provided } |\mathbf{A}| \neq 0.$$

Thus the necessary and sufficient condition for a matrix to be invertible is that it is non-singular.

1.23 Reversal Law for the Inverse of a Product

Theorem 1: *If* $\mathbf{A}$, $\mathbf{B}$ *be two n-rowed non-singular matrices, then* $\mathbf{AB}$ *is also non-singular and* $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$, *i.e., the inverse of a product is the product of the inverses taken in the reverse order.*

Theorem 2: *If* $\mathbf{A}$ *be an* $n \times n$ *non-singular matrix, then* $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$ *i.e., the operations of transposing and inverting are commutative.*

Theorem 3: *If* $\mathbf{A}$ *be an* $n \times n$ *non-singular matrix, then* $(\mathbf{A}^{-1})^{\theta} = (\mathbf{A}^{\theta})^{-1}$.

Example : *Find the :*

(i) Transpose *(ii) Adjoint* *(iii) Inverse of the matrix :*

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}.$$

Solution: (i) We have the transpose of $\mathbf{A} = \mathbf{A}^T = \begin{bmatrix} 2 & -5 & -3 \\ -1 & 3 & 2 \\ 3 & 1 & 3 \end{bmatrix}$.

(ii) We have $$|\mathbf{A}| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

$$= 2(9-2) - (-1)(-15+3) + 3(-10+9)$$

$$= 14 - 12 - 3 = -1.$$

Let $\mathbf{A}_{ij}$ denote the cofactor of the element a_{ij} of $|\mathbf{A}|$.

Then $$A_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 7,\ A_{12} = -\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} = 12,$$

$$A_{13} = \begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} = -1,$$

$$A_{21} = -\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} = 9,\ A_{22} = \begin{vmatrix} 2 & 3 \\ -3 & 3 \end{vmatrix} = 15,$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = -1,$$

$$A_{31} = \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} = -10,\ A_{32} = -\begin{vmatrix} 2 & 3 \\ -5 & 1 \end{vmatrix} = -17,$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ -5 & 3 \end{vmatrix} = 1.$$

$\therefore$ The matrix **B** formed of the cofactors of the elements of $|\mathbf{A}|$ is,

$$\mathbf{B} = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}.$$

Now adj. **A** (= the transpose of the matrix **B**) $= \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$.

(iii) We have $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|}$ adj. $\mathbf{A} = \dfrac{1}{(-1)} \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & 1 & 1 \end{bmatrix}$

$$= (-1) \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}.$$

Note: To check that the answer is correct the students should verify that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

1.24 Unitary Matrix

Definition: *A square matrix* **A** *is said to be unitary if* $\mathbf{A}^{\theta}\,\mathbf{A} = \mathbf{I}$.

Since $|\mathbf{A}^{\theta}| = |\overline{\mathbf{A}}|$ and $|\mathbf{A}^{\theta}\,\mathbf{A}| = |\mathbf{A}^{\theta}|\,|\mathbf{A}|$, therefore if $\mathbf{A}^{\theta}\mathbf{A} = \mathbf{I}$, we have $|\mathbf{A}|\,|\overline{\mathbf{A}}| = 1$.

Thus the determinant of a unitary matrix is of unit modulus.

If **A** is a unitary matrix, then $|\mathbf{A}||\mathbf{A}^\theta| = 1$ and so $|\mathbf{A}| \neq 0$ *i.e.*, **A** is non-singular and so invertible.

Hence $\mathbf{A}^\theta \mathbf{A} = \mathbf{I}$ implies $\mathbf{A}\mathbf{A}^\theta = \mathbf{I}$.

Thus **A** is a unitary matrix iff $\mathbf{A}^\theta \mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^\theta$.

Theorem: *If* **A**, **B** *be n-rowed unitary matrices,* **AB** *and* **BA** *are also unitary matrices.*

Proof: Since **A** and **B** are both n-rowed square matrices, therefore **AB** is also an n-rowed square matrix.

Since $\quad |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ and $|\mathbf{A}| \neq 0$, also $|\mathbf{B}| \neq 0$, therefore $|\mathbf{AB}| \neq 0$.

Hence **AB** is a non-singular matrix.

Now $\qquad (\mathbf{AB})^\theta = \mathbf{B}^\theta \mathbf{A}^\theta$.

$$\begin{aligned} \therefore \qquad (\mathbf{AB})^\theta (\mathbf{AB}) &= (\mathbf{B}^\theta \mathbf{A}^\theta)(\mathbf{AB}) \\ &= \mathbf{B}^\theta (\mathbf{A}^\theta \mathbf{A}) \mathbf{B} \\ &= \mathbf{B}^\theta \mathbf{IB} && [\because \mathbf{A}^\theta \mathbf{A} = \mathbf{I}] \\ &= \mathbf{B}^\theta \mathbf{B} \\ &= \mathbf{I}. && [\because \mathbf{B}^\theta \mathbf{B} = \mathbf{I}] \end{aligned}$$

$\therefore$ **AB** is unitary. Similarly we can prove that **BA** is also unitary.

Illustrative Examples

Example 14: *Prove that*

$$\mathbf{B} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \text{ is unitary.}$$

(Bundelkhand 2006)

Solution: We have $\quad \overline{\mathbf{B}} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$

$$\therefore \qquad \mathbf{B}^\theta = (\overline{\mathbf{B}})' = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

Now $$\mathbf{B}^{\theta}\ \mathbf{B} = \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1.1+(1+i)(1-i) & 1.(1+i)+(1+i)(-1) \\ (1-i).1+(-1).(1-i) & (1-i)(1+i)+(-1)(-1) \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}.$$

$\therefore$ **B** is unitary.

Example 15: *Show that the matrix* $\mathbf{A} = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ *is a unitary matrix, if*

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

Solution: We have $$\mathbf{A} = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$\therefore \quad \mathbf{A}' = \begin{bmatrix} \alpha + i\gamma & \beta + i\delta \\ -\beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$\mathbf{A}^{\theta} = \overline{\mathbf{A}}' = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}.$$

A square matrix **A** is said to be unitary if $\mathbf{A}\mathbf{A}^{\theta} = \mathbf{I}$.

$$\therefore \quad \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}\begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & \alpha\beta - i\alpha\delta + i\beta\gamma + \gamma\delta - \alpha\beta - i\beta\alpha + i\alpha\delta - \delta\gamma \\ \alpha\beta - i\beta\gamma + i\alpha\delta + \gamma\delta - \alpha\beta - i\alpha\delta + i\beta\gamma - \delta\gamma & \beta^2 + \delta^2 + \alpha^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or $$\begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if $$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

Example 16: *If* **A** *is a unitary matrix, show that* $\mathbf{A}^{-1}$ *is also unitary.*

Solution: We have $\mathbf{A}\mathbf{A}^{\theta} = \mathbf{A}^{\theta}\mathbf{A} = \mathbf{I}$, since **A** is a unitary matrix.

$$\Rightarrow \qquad (\mathbf{AA}^{\theta})^{-1} = (\mathbf{A}^{\theta}.\mathbf{A})^{-1} = (\mathbf{I})^{-1} \quad \text{[Taking inverse]}$$

$$\Rightarrow \qquad (\mathbf{A}^{\theta})^{-1}.\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{\theta})^{-1} = \mathbf{I}$$

$$\Rightarrow \qquad (\mathbf{A}^{-1})^{\theta}.\mathbf{A}^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1})^{\theta} = \mathbf{I}.$$

Hence $\mathbf{A}^{-1}$ is a unitary matrix.

Comprehensive Exercise 1

1. (i) Show that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

(ii) Verify that the matrix.

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is orthogonal.}$$

2. (i) Show that the matrix $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

(ii) Prove that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary.

(Rohilkhand 2010)

3. (i) Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$, find $\mathbf{A} + \mathbf{A}'$ and $\mathbf{A} - \mathbf{A}'$ and hence express $\mathbf{A}$ as the sum of a symmetric and a skew-symmetric matrix.

(ii) Write the following matrix as the sum of a symmetric and a skew-symmetric matrix : $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$.

(Lucknow 2010)

4. (i) Show that the matrix $\begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ -7 & -8 & 0 \end{bmatrix}$ is skew-symmetric.

(ii) Show that the matrix **A**, where $\mathbf{A} = \begin{bmatrix} 1/6 & -2/6 & 1/6 \\ -2/6 & 4/6 & -2/6 \\ 1/6 & -2/6 & 1/6 \end{bmatrix}$ is symmetric.

(Rohilkhand 2010)

5. If $\mathbf{A} = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$, prove that **A** is a Hermitian matrix.

(Luckhnow 2011; Bundelkhand 06)

6. Prove that the matrix $\mathbf{A}^2$ is symmetric if either **A** is symmetric or **A** is skew-symmetric.

7. If **A** be any square matrix then show that $\mathbf{A} + \mathbf{A}'$ is symmetric and $\mathbf{A} - \mathbf{A}'$ is skew-symmetric.

8. If **A** is a skew-Hermitian matrix, then show that $i\mathbf{A}$ is Hermitian.

9. If **A**, **B** are symmetric (skew-symmetric) matrices of the same order, then so is also $\mathbf{A} + \mathbf{B}$.

10. Show that the matrix $\mathbf{B}^{\theta}\,\mathbf{AB}$ is Hermitian or skew-hermitian according as **A** is Hermitian or skew-Hermitian.

11. Show that all positive integral powers of a symmetric matrix are symmetric.

12. If **A** and **B** are symmetric matrices of order n, then show that $\mathbf{AB} + \mathbf{BA}$ is symmetric and $\mathbf{AB} - \mathbf{BA}$ is skew-symmetric.

(Luckhnow 2008)

13. If **A** be any square matrix, prove that $\mathbf{A} + \mathbf{A}^{\theta}, \mathbf{AA}^{\theta}, \mathbf{A}^{\theta}\,\mathbf{A}$ are all Hermitian and $\mathbf{A} - \mathbf{A}^{\theta}$ is skew-Hermitian.

14. Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a skew-Hermitian matrix.

(Purvanchal 2007, 10; Lucknow 09)

15. Show that every square matrix **A** can be uniquely expressed as $\mathbf{P} + i\mathbf{Q}$ where **P** and **Q** are Hermitian matrices.

(Lucknow 2007, 10)

Answers 1

3. (i) $$\mathbf{A}+\mathbf{A}'=\begin{bmatrix}4 & 0 & 2\\0 & 4 & 5\\2 & 5 & 2\end{bmatrix};\ \mathbf{A}-\mathbf{A}'=\begin{bmatrix}0 & 2 & -4\\-2 & 0 & -3\\4 & 3 & 0\end{bmatrix};$$

$$\mathbf{A}=\frac{1}{2}\begin{bmatrix}4 & 0 & 2\\0 & 4 & 5\\2 & 5 & 2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 2 & -4\\-2 & 0 & -3\\4 & 3 & 0\end{bmatrix}$$

(ii) $$\mathbf{A}=\begin{bmatrix}1 & 3 & 5\\3 & 5 & 3\\5 & 3 & 0\end{bmatrix}+\begin{bmatrix}0 & -1 & -2\\1 & 0 & 3\\2 & -3 & 0\end{bmatrix}$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writting the corresponding letter from (a), (b), (c) *and* (d).

1. Every diagonal element of skew-symmetric matrix is

(a) unity (b) zero

(c) non-zero (d) purely imaginary **(Rohilkhand 2005)**

2. The necessary and sufficient condition for a matrix to be invertible is that it is

(a) singular (b) non-singular

(c) zero matrix (d) none of these **(Rohilkhand 2007)**

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Every diagonal element of a Hermitian matrix must be ......

2. A square matrix **A** is said to be unitary if ....

3. A necessary and sufficient condition for a matrix **A** to be symmetric is that **A** and **A**′ are ....

True or False

Write 'T' for true and 'F' for false statement.

1. The diagonal elements of a skew-Hermitian matrix must be pure imaginary numbers or zero.

2. A necessary and sufficient condition for a matrix **A** to be skew-symmetric is that $\mathbf{A}' = -\mathbf{A}$

Answers

Multiple Choice Questions

1. (b) 2. (b) 3. real

Fill in the Blank(s)

1. $\mathbf{A}^{\theta}\mathbf{A} = \mathbf{I}$ 2. equal

True or False

1. *T* 2. *T*

□

Chapter 2

Rank of a Matrix

2.1 Submatrix of a Matrix

Suppose **A** is any matrix of the type $m \times n$. Then a matrix obtained by leaving some rows and columns from **A** is called a **submatrix** of **A**. In particular the matrix **A** itself is a sub-matrix of **A** because it is obtained from **A** by leaving no rows or columns.

Minors of a matrix. We know that every square matrix possesses a determinant. If **A** be an $m \times n$ matrix, then the determinant of every square sub-matrix of **A** is called a **minor** of the matrix **A.** If we leave $m - p$ rows and $n - p$ columns from **A**, we shall get a square submatrix of **A** of order p. The determinant of this square submatrix is called a p-rowed minor of **A**.

For example, let

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 9 & 1 \\ 0 & 5 & 2 & 5 & 2 \\ 1 & 9 & 7 & 3 & 4 \\ 3 & -2 & 8 & 1 & 8 \end{bmatrix}_{4 \times 5}.$$

In a determinant the number of rows is equal to the number of columns. Therefore there can be no 5-rowed minor of **A**.

If we leave any columns from **A**, we shall get a square sub-matrix of **A** of order 4.

Thus $$\begin{vmatrix} 2 & 4 & 1 & 9 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 7 & 3 \\ 3 & -2 & 8 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 4 & 9 & 1 \\ 0 & 5 & 5 & 2 \\ 1 & 9 & 3 & 4 \\ 3 & -2 & 1 & 8 \end{vmatrix}, \text{ etc.}$$

are 4-rowed minors of **A.**

If we leave two columns and one row from **A**, we shall get a square submatrix of **A** of order 3.

Thus $$\begin{vmatrix} 2 & 4 & 1 \\ 0 & 5 & 2 \\ 1 & 9 & 7 \end{vmatrix}, \begin{vmatrix} 4 & 1 & 9 \\ 5 & 2 & 5 \\ 9 & 7 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 2 & 5 \\ 9 & 7 & 3 \\ -2 & 8 & 1 \end{vmatrix}, \text{ etc.}$$

are 3-rowed minors of **A**.

If we leave three columns and two rows from **A**, we shall get a square submatrix of **A** of order 2.

Thus $$\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix}, \begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix}, \begin{vmatrix} 5 & 2 \\ 9 & 7 \end{vmatrix}, \text{ etc. are 2-rowed minors of } \mathbf{A}.$$

The numbers 2, 4, 1, 9, 1, 0, 5 etc. are all 1-rowed minors of **A**.

2.2 Rank of a Matrix

(Lucknow 2006)

Definition: *A number r is said to be the rank of a matrix* **A** *if it possesses the following two properties :*

(i) There is at least one square submatrix of **A** *of order r whose determinant is not equal to zero.*

(ii) If the matrix **A** *contains any square submatrix of order* $r + 1$, *then the determinant of every square submatrix of* **A** *of order* $r + 1$ *should be zero.*

In short the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

Thus the rank of a matrix **A** *is the order of any highest order square submatrix of* **A** *whose determinant is not equal to zero.*

We shall denote the rank of a matrix **A** by the symbol $\rho(\mathbf{A})$.

It is obvious that the rank r of an $(m \times n)$ matrix can at most be equal to the smaller of the numbers m and n , but it may be less.

If there is a matrix **A** which has at least one non-zero minor of order n and there is no minor of **A** of order $n + 1$, then the rank of **A** is n. **Thus the rank of every non-singular matrix of order** n is n. The rank of a square matrix **A** of order n can be less than n if and only if **A** is singular *i.e.*, $|\mathbf{A}| = 0$.

Note 1 : Since the rank of every non-zero matrix is ≥ 1, *we agree to assign the rank, zero, to every null matrix :*

Note 2 : Every $(r + 1)$-rowed minor of a matrix can be expressed as a linear combination of its r-rowed minors. Therefore if all the r-rowed minors of a matrix are equal to zero, then obviously all its $(r + 1)$-rowed minors will also be equal to zero.

Important : The following two simple results will help us very much in finding the rank of a matrix:

(i) *The rank of a matrix is* $\leq r$, *if all* $(r + 1)$ - *rowed minors of the matrix vanish.*

(ii) *The rank of a matrix is* $\geq r$, *if there is at least one r-rowed minor of the matrix which is not equal to zero.*

Example :

(a) Let $\mathbf{A} = \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a unit matrix of order 3.

We have $|\mathbf{A}| = 1$. Therefore **A** is a non-singular matrix. Hence rank $\mathbf{A} = 3$. In particular, the rank of a unit matrix of order n is n.

(b) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Since **A** is a null matrix, therefore rank $\mathbf{A} = 0$.

(c) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$.

We have $|\mathbf{A}| = 1(6 - 8) - 2(4 - 6) = 2 \neq 0$. Thus **A** is a non-singular matrix. Therefore rank $\mathbf{A} = 3$.

(d) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$.

We have $|\mathbf{A}| = 1(24-25) - 2(18-20) + 3(15-16) = 0$

Therefore the rank of **A** is less than 3. Now there is at least one minor of **A** of order 2, namely $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ which is not equal to zero. Hence rank $\mathbf{A} = 2$.

(e) Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ 3 & 1 & 2 \end{bmatrix}$.

We have $|\mathbf{A}| = 0$, since the first two columns are identical.

Also each 2-rowed minor of **A** is equal to zero. But **A** is not a null matrix. Hence rank $\mathbf{A} = 1$.

(f) Let $\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 5 & 1 & 4 \end{bmatrix}$.

Here we see that there is at least one minor of **A** of order 2 *i.e.*, $\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ which is not equal to zero. Also there is no minor of **A** of order greater than 2. Hence the rank of $\mathbf{A} = 2$.

Echelon form of a matrix. Definition. *A matrix* **A** *is said to be in Echelon form if :*

(i) Every row of **A** *which has all its entries* 0 *occurs below every row which has a non-zero entry.*

(ii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Important result. *The rank of a matrix in* **Echelon form** *is equal to the number of non-zero rows of the matrix.*

Example: *Find the rank of the matrix* $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The matrix **A** has one zero row. We see that it occurs below every non-zero row.

Further the number of zeros before the first non-zero element in the first row is one. The number of zeros before the first non-zero element in the second row is two. Thus the number of zeros before the first non-zero element in any row is less than the number of such zeros in the next row.

Thus the matrix **A** is in Echelon form.

$\therefore$ rank **A** = the number of non-zero rows of **A** = 2.

2.3 Theorem

The rank of the transpose of a matrix is the same as that of the original matrix.

Proof. Let **A** be any matrix and **A′** be the transpose of the matrix **A**. Let rank **A** = r and rank **A′** = s. Then to prove that $r = s$.

We have rank **A** = r

$\Rightarrow$ there exists at least one r-rowed square submatrix, say **R**, of **A** such that

$$|\mathbf{R}'| = |\mathbf{R}| \neq 0$$

$\Rightarrow$ there exists at least one r-rowed square submatrix **R′** of **A′** such that

$$|\mathbf{R}'| = |\mathbf{R}| \neq 0$$

$$\Rightarrow \quad \text{rank}\, \mathbf{A}' \geq r \Rightarrow s \geq r. \qquad \text{...(1)}$$

Since $(\mathbf{A}')' = \mathbf{A}$, therefore interchanging the roles of **A** and **A′** in the above result (1), we have

$$r \geq s. \qquad \text{...(2)}$$

From (1) and (2), we conclude that $r = s$. Hence the result.

Illustrative Examples

Example 1: *Find the ranks of the following matrices :*

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, *(ii)* $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$

(Bundelkhand 2009)

Solution : (i) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

We have $|\mathbf{A}| = 1(2-0) - 2(4-0) + 3(2-0)$, expanding along the first row

$= 2 - 8 + 6 = 0.$

But there is at least one minor of order 2 of the matrix $\mathbf{A}$, namely $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ which is not equal to zero. Hence rank $\mathbf{A} = 2$.

(ii) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

Here there is at least one minor of order 2 of the matrix $\mathbf{A}$, namely $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ which is not equal to 0. Also there is no minor of the matrix $\mathbf{A}$ or order greater than 2. Hence rank $\mathbf{A} = 2$.

Example 2: *Under what conditions the rank of the following matrix is 3*

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{bmatrix}?$$

Solution: The rank of the given matrix $\mathbf{A} = 3$.

$\therefore$ The minor of order 3 of matrix $\mathbf{A} \neq 0$

i.e., $\begin{vmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & x \end{vmatrix} \neq 0$

or $2(x-0) - 4(2x-2) + 2(0-1) \neq 0$

or $2x - 8x + 8 - 2 \neq 0$

or $-6x + 6 \neq 0$

or $x \neq 1.$

Example 3: *For which value of 'b' the rank of the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$ *is 2?*

Solution: The rank of the given matrix $\mathbf{A} = 2$.

$\therefore$ The minor of order 3 of matrix $\mathbf{A} = 0$

i.e.,
$$\begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

or
$$1(30-26) - 5(0-2b) + 4(0-3b) = 0$$

or
$$4 + 10b - 12b = 0$$

or
$$4 - 2b = 0$$

or
$$b = 2.$$

Example 4: *Find the values of a so that rank* $(\mathbf{A}) < 3$, *where* $\mathbf{A}$ *is the matrix*

$$\mathbf{A} = \begin{bmatrix} 3a-8 & 3 & 3 \\ 3 & 3a-8 & 3 \\ 3 & 3 & 3a-8 \end{bmatrix}.$$

Solution: The rank of the given matrix $\mathbf{A} < 3$

$\therefore$ The minor of order 3 of matrix $\mathbf{A} = 0$

i.e.,
$$\begin{bmatrix} 3a-8 & 3 & 3 \\ 3 & 3a-8 & 3 \\ 3 & 3 & 3a-8 \end{bmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3a-2 & 3a-2 & 3a-2 \\ 3 & 3a-8 & 3 \\ 3 & 3 & 3a-8 \end{vmatrix} = 0$$

or
$$(3a-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3a-8 & 3 \\ 3 & 3 & 3a-8 \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$(3a-2)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 3a-11 & 0 \\ 3 & 0 & 3a-11 \end{vmatrix} = 0$$

or $$(3a-2)(3a-11)(3a-11) = 0$$

or $$a = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}.$$

Example 5: *Prove that the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear if and only if the rank of the matrix.*

$$\mathbf{A} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \text{ is less than three.}$$

(Kanpur 2001; Bundelkhand 07)

Solution: **The condition is necessary:**

Given that the points $(x_i, y_i); i = 1, 2, 3$ are collinear.

We are to prove that the rank of matrix $\mathbf{A} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than three.

The given points are collinear

$\Rightarrow$ The area of the triangle formed by these points is zero.

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$\Rightarrow$ The rank of matrix $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is less than 3.

Hence the condition is necessary.

The condition is sufficient:

The rank of the given matrix $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ is less than 3.

We are to prove that the points $(x_i, y_i); i = 1, 2, 3$ are collinear.

$$\text{rank } \mathbf{A} < 3$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The given points are collinear.

Hence the condition is sufficient.

Example 6: ***$\mathbf{A}$ is a non-zero column and $\mathbf{B}$ a non-zero row matrix, show that rank $(\mathbf{AB}) = 1$.***

Solution : Let $\mathbf{A} = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \dots \\ a_{m1} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \end{bmatrix}$

be two non-zero column and row matrices respectively.

We have $$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & \dots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & \dots & a_{21}b_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1}b_{11} & a_{m1}b_{12} & a_{m1}b_{13} & \dots & a_{m1}b_{1n} \end{bmatrix}.$$

Since $\mathbf{A}$ and $\mathbf{B}$ are non-zero matrices, therefore the matrix $\mathbf{AB}$ will also be non-zero. The matrix $\mathbf{AB}$ will have at least one non-zero element obtained by multiplying corresponding non-zero elements of $\mathbf{A}$ and $\mathbf{B}$.

All the two-rowed minors of $\mathbf{A}$ obviously vanish. But $\mathbf{A}$ is a non-zero matrix. Hence rank $\mathbf{A} = 1$.

Comprehensive Exercise 1

1. Determine the rank of each of the following matrices :

(i) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 5 & 10 \\ 3 & 6 \end{bmatrix}$ **(Meerut 2006B)**

(v) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 2 & -7 & 5 \\ 0 & 5 & 0 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}$.

2. Show that the rank of a matrix is $\geq$ the rank of every sub-matrix thereof.

3. Show that the rank of a matrix does not alter on affixing any number of additional rows or columns of zeros.

4. If $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find the rank of $\mathbf{A}$ and $\mathbf{A}^2$.

5. Under what conditions the rank of the following matrix is 3 ? Is it possible for the rank to be 1? Why ? $\mathbf{A} = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 4 \\ 1 & 0 & x \end{bmatrix}$.

(Kanpur 2010)

6. $\mathbf{A}$ is an n-rowed square matrix of rank $(n-1)$, show that Adj. $\mathbf{A}$ is not a null matrix.

Answers 1

1. (i) 4 (ii) 1 (iii) 0 (iv) 1 (v) 1
(vi) 3 (vii) 2

4. Rank $\mathbf{A} = 3$, rank $\mathbf{A}^2 = 2$

5. $x \neq \frac{7}{5}$; No, because one minor of order 2 of **A** is non-zero

2.4 Elementary Operations or Elementary Transformations of a Matrix

An elementary transformation (or, an *E*-transformation) is an operation of any one of the following types :

1. *The interchange of any two rows(or columns).*
2. *The multiplication of the elements of any row (or column) by any non-zero number.*
3. *The addition to the elements of any other row (or column) the corresponding elements of any other row (or column) multiplied by any number.*

An elementary transformation is called a **row transformation** or a **column transformation** according as it applies to rows or columns.

2.5 Symbols to be Employed for the Elementary Transformations

The following notation will be used to denote the six elementary transformations :

1. The interchange of i^{th} and j^{th} rows will be denoted by $R_i \leftrightarrow R_j$.
2. The multiplication of the i^{th} row by a non-zero number k will be denoted by $R_i \to k\ R_i$.
3. The addition of k times the j^{th} row to the i^{th} row will be denoted by $R_i \to R_i + kR_j$.

The corresponding column transformations will be denoted by writing C, in place of R *i.e.*, by $C_i \leftrightarrow C_j, C_i \to k\ C_i, C_i \to C_i + kC_j$ respectively.

Important. It is quite obvious that if a matrix **B** is obtained from **A** by an elementary transformation, **A** can also be obtained from **B** by an elementary transformation of the same type.

For example, let

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 2 & 9 \\ 2 & 5 & 1 & 3 \\ 3 & 7 & 8 & 4 \end{bmatrix}_{3 \times 4}.$$

The elementary transformation $R_2 \to R_2 + 2R_3$ transforms **A** into a matrix **B**, where

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 2 & 9 \\ 8 & 19 & 17 & 11 \\ 3 & 7 & 8 & 4 \end{bmatrix}_{3 \times 4}.$$

Now if we apply the elementary transformation $R_2 \to R_2 - 2R_3$ to the matrix **B**, we see that the matrix **B** transforms to the matrix **A**.

Again suppose we apply the elementary transformation $R_2 \to 3R_2$ to the matrix **A**.

Then **A** transforms into a matrix **C**, where

$$\mathbf{C} = \begin{bmatrix} 1 & 4 & 2 & 9 \\ 6 & 15 & 3 & 9 \\ 3 & 7 & 8 & 4 \end{bmatrix}.$$

Now if we apply the elementary transformation $R_2 \to \frac{1}{3} R_2$ to the matrix **C**, we see that the matrix **C** transforms back to the matrix **A**.

2.6 Elementary Matrices

Definition: *A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix (or E-matrix). For example,*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are the elementary matrices obtained from $\mathbf{I}_3$ *by subjecting it to the elementary operations* $C_1 \leftrightarrow C_3, R_2 \to 4R_2, R_1 \to R_1 + 2R_2$ *respectively.*

It may be worthwhile to note that an E-matrix can be obtained from **I** by subjecting it to a row transformation or a column transformation. We shall use the following symbols to denote elementary matrices of different types :

(i) E_{ij} will denote the E-matrix obtained by interchanging the i^{th} and j^{th} rows of a unit matrix. The students can easily see that the matrices obtained by interchanging the i^{th} and j^{th} rows or the i^{th} and j^{th} columns of a unit matrix are the same. Therefore E_{ij} will also denote the elementary matrix obtained by interchanging the i^{th} and j^{th} columns of a unit matrix.

(ii) $\mathbf{E}_i\,(k)$ will denote the E-matrix obtained by multiplying the i^{th} row of a unit matrix by a non-zero number k. It can be easily seen that the matrices obtained by multiplying the i^{th} row or the i^{th} column of a unit matrix by k are the same. Therefore $\mathbf{E}_i\,(k)$ will also denote the elementary matrix obtained by multiplying the i^{th} column of a unit matrix by a non-zero number k.

(iii) $\mathbf{E}_{ij}\,(m)$ will denote the elementary matrix, obtained by adding to the elements of the i^{th} row of a unit matrix, the products by any number m of the corresponding elements of the j^{th} row. It may be easily seen that the E-matrix, $\mathbf{E}_{ij}\,(m)$ can also be obtained by adding to the elements of the j^{th} column of a unit matrix, the products by m of the corresponding elements of the i^{th} column.

It can be easily seen that

$$|\,\mathbf{E}_{ij}| = -1,\ |\,\mathbf{E}_i\,(k)\,| = k \neq 0,\ |\,\mathbf{E}_{ij}\,(m)\,| = 1.$$

Thus all the elementary matrices are non-singular.

Therefore each elementary matrix possesses inverse.

Now it is very interesting to note that the elementary transformations of a matrix can also be obtained by algebraic operations on the same by the corresponding elementary matrices. In this connection we have the following theorem.

2.7 Theorem

Every elementary row (column) transformation of a matrix can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix.

We shall first prove that every elementary row transformation of a product **AB** of two matrices **A** and **B** can be obtained by subjecting the pre-factor **A** to the same elementary row transformation. Similarly every elementary column transformation of a product **AB** of two matrices **A** and **B** can be obtained by subjecting the post-factor **B** to the same elementary column transformation.

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{jk}]$ be two $m \times n$ and $n \times p$ matrices respectively so that the product **AB** is defined.

Let $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \ldots, \mathbf{R}_m$ denote the row vectors of the matrix **A** and $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_p$ denote the column vectors of the matrix **B**.

We can then write, $\mathbf{A} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \\ \dots \\ \mathbf{R}_m \end{bmatrix}, \mathbf{B} = [\mathbf{C}_1 \ \mathbf{C}_2 \ \mathbf{C}_3 \dots \mathbf{C}_p]$.

$$\therefore \qquad \mathbf{AB} = \begin{bmatrix} \mathbf{R}_1\mathbf{C}_1 & \mathbf{R}_1\mathbf{C}_2 & \mathbf{R}_1\mathbf{C}_3 & \dots & \mathbf{R}_1\mathbf{C}_p \\ \mathbf{R}_2\mathbf{C}_1 & \mathbf{R}_2\mathbf{C}_2 & \mathbf{R}_2\mathbf{C}_3 & \dots & \mathbf{R}_2\mathbf{C}_p \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{R}_m\mathbf{C}_1 & \mathbf{R}_m\mathbf{C}_2 & \mathbf{R}_m\mathbf{C}_3 & \dots & \mathbf{R}_m\mathbf{C}_p \end{bmatrix}$$

Now if σ denotes any elementary row transformation, it is quite obvious from the above representation that $(\sigma \ \mathbf{A}) \ \mathbf{B} = \sigma \ (\mathbf{AB})$. For example, if σ denotes the elementary row transformation $R_1 \leftrightarrow R_2$, it is quite obvious that $(\sigma\mathbf{A}) \ \mathbf{B} = \sigma \ (\mathbf{AB})$.

Similarly it is quite obvious that if the columns $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_p$ of $\mathbf{B}$ be subjected to any elementary column transformation, the columns of $\mathbf{AB}$ are also subjected to the same elementary column transformation. Hence the result.

Now to prove our main theorem, if $\mathbf{A}$ be an $m \times n$ matrix, we can write

$$\mathbf{A} = \mathbf{I}_m \ \mathbf{A}.$$

If σ denotes any elementary row transformation, we have

$$\sigma \ \mathbf{A} = \sigma \ (\mathbf{I}_m \ \mathbf{A}) = (\sigma \ \mathbf{I}_m) \ \mathbf{A} = \mathbf{EA},$$

where $\mathbf{E}$ is the E-matrix corresponding to the same row transformation σ.

Similarly, we can write $\mathbf{A} = \mathbf{A} \ \mathbf{I}_n$.

If σ denotes any elementary column transformation, we have

$$\sigma \ (\mathbf{A}) = \sigma \ (\mathbf{AI}_n) = \mathbf{A}\sigma \ (\mathbf{I}_n) = \mathbf{AE}_1,$$

where $\mathbf{E}_1$ is the E-matrix corresponding to the same column transformation σ.

Example: Let $\qquad \mathbf{A} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}.$

The E-transformation $R_1 \to R_1 + 2R_3$ transforms $\mathbf{A}$ into $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}.$$

Also if we apply the row transformation $R_1 \to R_1 + 2R_3$ to the unit matrix

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then the E-matrix $\mathbf{E}$ thus obtained is $\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now $$\mathbf{EA} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 0\cdot 2 + 2\cdot 3 & 1\cdot 4 + 0\cdot 7 + 2\cdot 8 & 1\cdot 2 + 0\cdot 1 + 2\cdot 4 \\ 0\cdot 1 + 1\cdot 2 + 0\cdot 3 & 0\cdot 4 + 1\cdot 7 + 0\cdot 8 & 0\cdot 2 + 1\cdot 1 + 0\cdot 4 \\ 0\cdot 1 + 0\cdot 2 + 1\cdot 3 & 0\cdot 4 + 0\cdot 7 + 1\cdot 8 & 0\cdot 2 + 0\cdot 1 + 1\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix} = \mathbf{B}.$$

Similarly we can see that a column transformation of $\mathbf{A}$ can be effected by post-multiplying $\mathbf{A}$ with the corresponding elementary matrix.

Non-singularity and Inverses of the Elementary Matrices

(i) The elementary matrix corresponding to the E-operation $R_1 \leftrightarrow R_j$ is its own inverse.

Let $\mathbf{E}_{ij}$ denote the elementary matrix obtained by interchanging the i^{th} and j^{th} rows of a unit matrix.

The interchange of the i^{th} and j^{th} rows of $\mathbf{E}_{ij}$ will transform $\mathbf{E}_{ij}$ to the unit matrix. But every elementary row transformation of a matrix can be brought about by pre-multiplication with the corresponding elementary matrix. Therefore the row transformation which changes $\mathbf{E}_{ij}$ to $\mathbf{I}$ can be effected on pre-multiplication by $\mathbf{E}_{ij}$.

Thus $\mathbf{E}_{ij}\,\mathbf{E}_{ij} = \mathbf{I}$ or $(\mathbf{E}_{ij})^{-1} = \mathbf{E}_{ij}$.

Hence $\mathbf{E}_{ij}$ is its own inverse.

Similarly, *we can show that the elementary matrix corresponding to the E-operation $C_i \leftrightarrow C_j$ is its own inverse.*

(ii) The inverse of the E-matrix corresponding to the E-operation $R_i \to kR_i$, $(k \neq 0)$, is the E-matrix corresponding to the E-operation $R_i \to k^{-1} R_i$.

Let $\mathbf{E}_i(k)$ denote the elementary matrix obtained by multiplying the elements of the i^{th} row of a unit matrix $\mathbf{I}$ by a non-zero number k.

The operation of the multiplication of the i^{th} row of $\mathbf{E}_i(k)$, by k^{-1} will transform $\mathbf{E}_i(k)$ to the unit matrix $\mathbf{I}$. This row transformation of $\mathbf{E}_i(k)$ can be effected on pre-multiplication by the corresponding elementary matrix $\mathbf{E}_i(k^{-1})$.

Thus $\mathbf{E}_i(k^{-1})\,\mathbf{E}_i(k) = \mathbf{I}$ or $\{\mathbf{E}_i(k)\}^{-1} = \mathbf{E}_i(k^{-1})$.

Similarly, *we can show that the inverse of the E-matrix corresponding to the E-operation $C_i \to kC_i, k \neq 0,$ is the E-matrix corresponding to the E-operation $C_i \to k^{-1} C_i$.*

(iii) The inverse of the E-matrix corresponding to the E-operation $R_i \to R_i + kR_j$ is the E-matrix corresponding to the E-operation $R_i \to R_i - kR_j$.

Let $\mathbf{E}_{ij}(k)$ denote the elementary matrix obtained by adding to the elements of the i^{th} row of a unit matrix $\mathbf{I}$, the products by any number k of the corresponding elements of the j^{th} row of $\mathbf{I}$.

If we add to elements of the i^{th} row of $\mathbf{E}_{ij}(k)$, the products by $-k$ of the corresponding elements of its j^{th} row, then this row operation will transform $\mathbf{E}_{ij}(k)$ to the unit matrix $\mathbf{I}$. Now this row transformation of $\mathbf{E}_{ij}(k)$ can be effected on pre-multiplication by the corresponding elementary matrix $\mathbf{E}_{ij}(-k)$.

Therefore $\mathbf{E}_{ij}(-k)\,\mathbf{E}_{ij}(k) = \mathbf{I}$ or $\{\mathbf{E}_{ij}(k)\}^{-1} = \mathbf{E}_{ij}(-k)$.

Similarly, *we can show that the inverse of the E-matrix corresponding to the E-operation $C_i \to C_i + kC_j$ is the E-matrix corresponding to the E-operation $C_i \to C_i - kC_j$.*

From the above theorem, we thus conclude that the **inverse of an elementary matrix is also an elementary matrix of the same type.**

2.8 Invariance of Rank Under Elementary Transformations

Theorem: *Elementary transformations do not change the rank of a matrix.*
(Lucknow 2005, 06)

Proof: Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix of rank r. We shall prove the theorem in three stages.

Case I. *Interchange of a pair of rows does not change the rank.*

Let **B** be the matrix obtained from the matrix **A** by the *E*-transformation $R_p \leftrightarrow R_q$. Let r be the rank of **A** and s be the rank of **B.** Then to prove that $r = s$.

We have rank $\mathbf{A} = r \Rightarrow$ there exists at least one r-rowed square submatrix, say **R**, of **A** such that $|\mathbf{R}| \neq 0$.

Let **S** be the r-rowed square submatrix of **B** which has the same rows as are in **R** though they may be in different relative positions.

Then either $|\mathbf{S}| = |\mathbf{R}|$ or $|\mathbf{S}| = -|\mathbf{R}|$.

$\therefore \qquad |\mathbf{R}| \neq 0 \Rightarrow |\mathbf{S}| \neq 0.$

$\therefore \qquad \text{rank } \mathbf{B} \geq r \Rightarrow \mathbf{S} \geq r.$

Again, as **A** can also be obtained from **B** by an interchange of rows, we have $r \geq s$.

Hence $r = s$.

Case II. *Multiplication of the elements of a row by a non-zero number does not change the rank.*

Let **B** be the matrix obtained from the matrix **A** by the *E*-transformation $R_p \to k\, R_p, (k \neq 0)$, and let s be the rank of the matrix **B** .

Now if $|\mathbf{B}_0|$ be any $(r+1)$-rowed minor of **B** there exists a uniquely determined minor $|\mathbf{A}_0|$ of **A** such that

$|\mathbf{B}_0| = |\mathbf{A}_0|$ (this happens if the p^{th} row of **B** is one of those rows which are struck off to obtain $\mathbf{B}_0$ from **B**),

or $\qquad |\mathbf{B}_0| = k\,|\mathbf{A}_0|$

(this happens when p^{th} row is retained while obtaining $\mathbf{B}_0$ from **B**).

Since the matrix **A** is of rank r, therefore every $(r+1)$-rowed minor of **A** vanishes *i.e.*, $|\mathbf{A}_0| = 0$. Hence $|\mathbf{B}_0| = 0$. Thus we see that every $(r+1)$-rowed minor of **B** also vanishes. Therefore, s (the rank of **B**) cannot exceed r (the rank of **A**).

$$\therefore \qquad s \le r.$$

Also, since **A** can be obtained from **B** by E-transformation of the same type *i.e.*, $R_p \to (1/k)\, R_p$, therefore, by interchanging the roles of **A** and **B** we find that

$$r \le s.$$

Thus $\qquad r = s.$

Case III. *Addition to the elements of a row the products by any number k of the corresponding elements of any other row does not change the rank .*

Let **B** be the matrix obtained from the matrix **A** by the E-transformation $R_p \to R_p + kR_q$ and let s be the rank of the matrix **B**.

Let $\mathbf{B}_0$ be any $(r+1)$-rowed square sub-matrix of **B** and $\mathbf{A}_0$ be the correspondingly placed sub-matrix of **A**.

The transformation $R_p \to R_p + kR_q$ has changed only the p^{th} row of the matrix **A.** Also the value of a determinant does not change if we add to the elements of any row the corresponding elements of any other row multiplied by some number k.

Therefore, if no row of the sub-matrix $\mathbf{A}_0$ is part of the p^{th} row of **A**, or if two rows of $\mathbf{A}_0$ are parts of the p^{th} and q^{th} rows of **A,** then $|\mathbf{B}_0| = |\mathbf{A}_0|$.

Since the rank of **A** is r, therefore $|\mathbf{A}_0| = 0$, and consequently $|\mathbf{B}_0| = 0$.

Again, if a row of $\mathbf{A}_0$ is a part of the p^{th} row of **A**, but no row is a part of the q^{th} row, then $|\mathbf{B}_0| = |\mathbf{A}_0| + k\,|\mathbf{C}_0|$, where $\mathbf{C}_0$ is an $(r+1)$ -rowed square matrix which can be obtained from $\mathbf{A}_0$ by replacing the elements of $\mathbf{A}_0$ in the row which corresponds to the p^{th} row of **A** by the corresponding elements in the q^{th} row of **A**. Obviously all the $r+1$ rows of the matrix $\mathbf{C}_0$ are exactly the same as the rows of some $(r+1)$-rowed square sub-matrix of **A**, though arranged in some different order. Therefore $|\mathbf{C}_0|$ is ± 1 times some $(r+1)$-rowed minor of **A**. Since the rank of **A** is r, therefore, every $(r+1)$-rowed minor of **A** is zero, so that $|\mathbf{A}_0| = 0, |\mathbf{C}_0| = 0$, and consequently $|\mathbf{B}_0| = 0$.

Thus we see that every $(r+1)$-rowed minor of **B** also vanishes. Hence, s (the rank of **B**) cannot exceed r (the rank of **A**).

$$\therefore \qquad s \le r.$$

Also, since **A** can be obtained from **B** by an E-transformation of the same type *i.e.*, $R_p \to R_p - kR_q$, therefore, interchanging the roles of **A** and **B**, we have $r \le s$.

Thus $\qquad r = s.$

We have thus shown that rank of a matrix remains unaltered by any E-row transformation. Therefore we can also say that the rank of a matrix remains unaltered by a series of elementary row transformations.

Similarly we can show that the rank of a matrix remains unaltered by a series of elementary column transformations.

Finally, we conclude that the rank of a matrix remains unaltered by a finite chain of elementary operations.

Corollary. We have already proved that every elementary row (column) transformation of a matrix can be effected by pre-multiplication (post-multiplication) with the corresponding elementary matrix. Combining this theorem with the theorem just established, we conclude the following important results :

The pre-multiplication or post-multiplication by any elementary matrix, and as such by any series of elementary matrices, does not alter the rank of a matrix.

2.9 Reduction to Normal Form

Theorem: *Every $m \times n$ matrix of rank r can be reduced to the form* $\begin{pmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}$ *by a finite chain of E-operations where $\mathbf{I}_r$, is the r-rowed unit matrix.*

Proof: Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix of rank r. If **A** is a zero matrix, then r is equal to zero and we have nothing to prove. So let us take **A** as a non-zero matrix.

Since **A** is a non-zero matix, therefore **A** has at least one element different from zero, say $a_{pq} = k \ne 0$.

By interchanging the p^{th} row with the first row and the q^{th} column with the first column respectively, we obtain a matrix **B** whose leading element is equal to k which is not equal to zero.

Multiplying the elements of the first row of the matrix **B** by $1/k$, we obtain a matrix **C** whose leading element is equal to unity.

Let $$\mathbf{C} = \begin{bmatrix} 1 & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & c_{m3} & \dots & c_{mn} \end{bmatrix}.$$

Subtracting suitable multiples of the first column of **C** from the remaining columns, and suitable multiples of the first row from the remaining rows, we obtain a matrix **D** in which all elements of the first row and first column except the leading element are equal to zero.

Let $$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & & & & & \\ 0 & & & \mathbf{A_1} & & \\ 0 & & & & & \\ \dots & & & & & \\ 0 & & & & & \end{bmatrix},$$

where $\mathbf{A}_1$ is an $(m-1) \times (n-1)$ matrix.

If now, $\mathbf{A}_1$ be a non-zero matrix, we can deal with it as we did with **A**. If the elementary operations applied to $\mathbf{A}_1$ for this purpose be applied to **D**, they will not affect the first row and the first column of **D**. Continuing this process, we shall finally obtain a matrix **M**, such that

$$\mathbf{M} = \begin{bmatrix} \mathbf{I}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

The matrix **M** has the rank k. Since the matrix **M** has been obtained from the matrix **A** by elementary transformations and elementary transformations do not alter the rank, therefore we must have $k = r$.

Hence every $m \times n$ matrix of rank r can be reduced to the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ by a finite chain of elementary transformations.

Note. The above form is usually called the **first canonical form** or **normal form** of a matrix.

Corollary 1: *The rank of an $m \times n$ matrix* **A** *is r if and only if (iff) it can be reduced to the form* $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ *by a finite chain of E-operations.*

The condition is necessary. The proof has been given in the above theorem.

The condition is also sufficient. The matrix **A** has been transformed into the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ by elementary transformations which do not alter the rank of the matrix.

Since the rank of the matrix $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ is r, therefore the rank of the matrix **A** must also be r.

Corollary 2: *If* **A** *be an $m \times n$ matrix of rank r, there exist non-singular matrices* **P** *and* **Q** *such that*

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

Proof: If **A** be an $m \times n$ matrix of rank r, it can be transformed into the form $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ by elementary operations. Since *E*-row (column) operations are equivalent to pre-(post)-multiplication by the corresponding elementary matrices, we have the following result :

If **A** be an $m \times n$ matrix of rank r, there exist *E*-matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots \mathbf{P}_s, \mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_t$ such that

$$\mathbf{P}_s \, \mathbf{P}_{s-1} \ldots \mathbf{P}_1 \mathbf{A} \, \mathbf{Q}_1 \mathbf{Q}_2 \ldots \mathbf{Q}_t = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

Now each elementary matrix is non-singular and the product of non-singular matrices is also non-singular. Therefore if $\mathbf{P} = \mathbf{P}_s \mathbf{P}_{s-1} \ldots \mathbf{P}_2 \mathbf{P}_1$ and $\mathbf{Q} = \mathbf{Q}_1 \, \mathbf{Q}_2, \ldots \mathbf{Q}_t$, then **P** and **Q** are non-singular matrices. Hence

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

2.10 Equivalence of Matrices

Definition. *If* **B** *be an $m \times n$ matrix obtained from an $m \times n$ matrix* **A** *by finite number of elementary transformations of* **A** *then* **A** *is called equivalent to* **B.** Symbolically, we write $\mathbf{A} \sim \mathbf{B}$, which is read as '**A** *is equivalent to* **B**.'

The following three properties of the relation '$\sim$' in the set of all $m \times n$ matrices are quite obvious :

(i) Reflexivity: If **A** is any $m \times n$ matrix, then $\mathbf{A} \sim \mathbf{A}$. Obviously **A** can be obtained from **A** by the elementary transformation $R_i \to kR_i$, where $k = 1$.

(ii) Symmetry: If $\mathbf{A} \sim \mathbf{B}$, then $\mathbf{B} \sim \mathbf{A}$. If **B** can be obtained from **A** by a finite number of elementary transformations of **A**, then **A** can also be obtained from **B** by a finite number of elementary transformations of **B.**

(iii) Transitivity: If $\mathbf{A} \sim \mathbf{B}$, $\mathbf{B} \sim \mathbf{C}$, then $\mathbf{A} \sim \mathbf{C}$.

If **B** can be obtained from **A** by a series of elementary transformations of **A** and **C** can be obtained from B by a series of elementary transformations of **B**, then **C** can also be obtained from **A** by a series of elementary transformations of **A.**

Therefore the relation '$\sim$' in the set of all $m \times n$ matrices is an equivalence relation.

Illustrative Examples

Example 7: *If* **A** *and* **B** *be two equivalent matrices, then show that rank* **A** = *rank* **B**.

Solution : If $\mathbf{A} \sim \mathbf{B}$, then **B** can be obtained from **A** by a finite number of elementary transformations of **A**. Now the elementary transformations do not change the rank of a matrix.

$\therefore$ If $\mathbf{A} \sim \mathbf{B}$, then rank **A** = rank **B**.

Example 8: *Show that if two matrices* **A** *and* **B** *have the same size and the same rank, they are equivalent.*

Solution: Let **A** and **B** be two $m \times n$ matrices of the same rank r. Then by 2.9, we have

$$\mathbf{A} \sim \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \text{ and also } \mathbf{B} \sim \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}.$$

By the symmetry of the equivalence relation,

$$\mathbf{B} \sim \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \text{ implies } \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \sim \mathbf{B}.$$

Now by the transitivity of the equivalence relation

$$\mathbf{A} \sim \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \sim \mathbf{B} \text{ implies } \mathbf{A} \sim \mathbf{B}.$$

Example 9: *(i) Use elementary transformations to reduce the following matrix* **A** *to triangular form and hence find rank* **A** *:*

$$\mathbf{A}=\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}.$$

(Bundelkhand 2010)

(ii) Find the rank of the matrix $\begin{bmatrix}8 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ -8 & -1 & -3 & 4\end{bmatrix}.$

Solution : (i) We have the matrix

$$\mathbf{A}\sim\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 5 & 3 & 14 & 4\end{bmatrix}\text{ by } R_1\leftrightarrow R_3$$

$$\sim\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 8 & 4 & 4\end{bmatrix}\text{ by } R_3\to R_3-5R_1$$

$$\sim\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & -12 & -4\end{bmatrix}\text{ by } R_3\to R_3-8R_2.$$

The last equivalent matrix is in Echelon form (or in triangular form). The number of non-zero rows in this matrix is 3. Therefore its rank is 3. Hence rank $\mathbf{A}=3$.

(ii) Let us denote the given matrix by $\mathbf{A}$. To find the rank of $\mathbf{A}$, we shall reduce it to Echelon form. Performing the column operation $C_1\to\frac{1}{8}C_1$, we get

$$\mathbf{A}\sim\begin{bmatrix}1 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ -1 & -1 & -3 & 4\end{bmatrix}\sim\begin{bmatrix}1 & 1 & 3 & 6\\ 0 & 3 & 2 & 2\\ 0 & 0 & 0 & 10\end{bmatrix}\text{ by } R_3\to R_3+R_1.$$

The last equivalent matrix is in Echelon form. The number of non-zero rows in this matrix is 3. Therefore its rank is 3. Hence rank $A=3$.

Example 10: *Is the matrix* $\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 2\\ 2 & 1 & -3\end{bmatrix}$ *equivalent to* $\mathbf{I}_3$*?*

(Meerut 2008)

Solution: Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$.

We have $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{vmatrix} R_2 + R_1, R_3 - 2R_1$

$$= -10 + 9 = -1 \quad i.e., \quad \neq 0.$$

Thus the matrix $\mathbf{A}$ is non-singular. Hence it is of rank 3. The rank of $\mathbf{I}_3$ is also 3. Since $\mathbf{A}$ and $\mathbf{I}_3$ are matrices of the same size and the same rank, therefore $\mathbf{A} \sim \mathbf{I}_3$.

Example 11: *Reduce the matrix,* $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ *to the normal form* $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$ *and hence determine its rank.* **(Meerut 2001, 09B, 10)**

Solution: We have the matrix $\mathbf{A}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \text{by } C_2 \to C_2 + C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 + 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & -8 & 14 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \text{by } R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 5 & -8 & 14 \end{bmatrix} \text{by } R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & -8 & 4 \end{bmatrix} \text{by } C_4 \to C_4 - 2C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -8 & 4 \end{bmatrix} \text{by } R_3 \to R_3 - 3R_2, R_4 \to R_4 - 5R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & -8 \end{bmatrix} \text{by } C_3 \leftrightarrow C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \text{by } C_3 \to -\frac{1}{2} C_3, C_4 \to -\frac{1}{8} C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{by } R_4 \to R_4 + 2R_3$$

$\sim \mathbf{I}_4$. Hence the matrix $\mathbf{A}$ is of rank 4.

Example 12: *Determine the rank of the following matrices :*

(i) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ **(Kanpur 2010)**

(ii) $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ **(Meerut 2003, 09 B, 10B)**

Solution : (i) Let us denote the given matrix by **A.** Performing the elementary operations $R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$, we see that

$$\mathbf{A} \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_4 \to R_4 - 2R_2$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 12 & 16 & 4 \\ 0 & 12 & 12 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_2 \to 4R_2,\ R_3 \to 3R_3$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 12 & 16 & 4 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_3 \to R_3 - R_2.$$

The last equivalent matrix is in Echelon form. The number of non-zero rows in this matrix is 3. Therefore its rank is 3. Hence rank $\mathbf{A} = 3$.

(ii) Let us denote the given matrix by **A.** Performing the elementary operation $R_1 \leftrightarrow R_2$, we see that

$$\mathbf{A} \sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{by } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \text{by } R_2 \to \frac{1}{3}R_2,\ R_3 \to -\frac{1}{2}R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2.$$

The last equivalent matrix is in Echelon form. The number of non-zero rows in this matrix is 2. Therefore rank $\mathbf{A} = 2$.

Example 13: *Find the rank of the matrix* $\mathbf{A} = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ *by reducing it to normal form.*

(Meerut 2008; Avadh 08; Rohilkhand 09)

Solution : Performing the operation $R_1 \to \frac{1}{2} R_1, R_2 \to \frac{1}{2} R_2$, we see that

$$\mathbf{A} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

by $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \text{by } C_2 \to C_2 + C_1, C_4 \to C_4 - 3C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \end{bmatrix} \text{by } R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \end{bmatrix} \text{by } R_2 \to (-1) R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix} \text{by } R_4 \to R_4 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix} \text{by } C_3 \to C_3 + C_2, C_4 \to C_4 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_4 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } C_3 \to \frac{1}{3} C_3, C_4 \to -\frac{1}{8} C_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } C_4 \to C_4 - C_3,$$

which is the normal form $\begin{bmatrix} \mathbf{I}_3 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$. Hence rank $\mathbf{A} = 3$.

Example 14: *Find two non-singular matrices* **P** *and* **Q** *such that* **PAQ** *is in the normal form where*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Also find the rank of the matrix **A.**

Solution: We write $\mathbf{A} = \mathbf{I}_3 \mathbf{A} \mathbf{I}_3$ *i.e.*,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now we go on applying E-operations on the matrix **A** (the left hand member of the above equation) until it is reduced to the normal form. Every E-row operation will also be applied to the pre-factor $\mathbf{I}_3$ (or its transform) of the product on the right hand member of the above equation and every E-column operation to the post-factor $\mathbf{I}_3$ (or its transform).

Performing $R_2 \to R_2 - R_1, R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing $R_2 \to -\frac{1}{2} R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing $R_3 \to R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Performing $C_3 \to C_3 - C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\therefore$ $$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix},$$

where $$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathbf{A} \sim \begin{bmatrix} \mathbf{I}_2 & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$, therefore rank $\mathbf{A} = 2$.

Comprehensive Exercise 2

Determine the rank of the following matrices :

1. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

2. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$ **(Bundelkhand 2008; Rohilkhand 08; Avadh 05; Lucknow 11)**

3. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(Avadh 2006, 09; Purvanchal 06)**

4. $\begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$ **(Lucknow 2007)**

5. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ **(Rohilkhand 2007)**

6. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$ **(Lucknow 08)**

8. $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

11. $\begin{bmatrix} 8 & 0 & 0 & 1 \\ 1 & 0 & 8 & 1 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 1 & 8 \end{bmatrix}$ **(Kumaun 2008)**

12. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(Avadh 2010)**

13. $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{bmatrix}$ **(Kumaun 2008)**

14. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{bmatrix}$

16. $\begin{bmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{bmatrix}$

(Meerut 2011)

19. With the help of elementary transformations find the rank of the following matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

(Rohilkhand 2010)

20. Reduce the following matrix to its Echelon form and find its rank :

$$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{bmatrix}$$

(Meerut 2004B; Rohilkhand 06, 10)

21. Find the rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ after reducing it to normal form.

22. Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ to normal form and find its rank.

(Meerut 2009; Bundelkhand 11; Avadh 11)

23. (i) Reduce the matrix $\begin{bmatrix} 9 & 7 & 3 & 6 \\ 5 & -1 & 4 & 1 \\ 6 & 8 & 2 & 4 \end{bmatrix}$ to normal form and find its rank.

(Agra 2007)

(ii) Reduce the following matrix **A** into normal form and hence find its rank:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}.$$

(Kanpur 2011)

24. Are the following pairs of matrices equivalent?

(i) $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 1 & 1 & 5 \end{bmatrix}$

25. Find the ranks of **A, B, A + B, AB** and **BA** where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}.$$

26. Show that if **A** and **B** are equivalent matrices, then there exist non-singular matrices **P** and **Q** such that **B = PAQ**.

27. Show that the rank of a matrix is not altered if a column of it is multiplied by a non-zero scalar.

28. (i) What is the rank of a non-singular matrix of order n?

(ii) What is the rank of an elementary matrix?

Answers 2

1. 3 **2.** 2 **3.** 3 **4.** 2 **5.** 2 **6.** 2 **7.** 3 **8.** 4 **9.** 3

10. 3 **11.** 4 **12.** 2 **13.** 2 **14.** 1 **15.** 2 **16.** 2 **17.** 3

18. rank (**A**) = 3 if $a \neq b \neq c$ and $a + b + c \neq 0$; rank (**A**) = 2 if $a \neq b \neq c$ and $a + b + c = 0$; Also rank (**A**) = 2 if $a = b \neq c$; and rank (**A**) = 1 if $a = b = c$

19. 3 **20.** 2 **21.** 2 **22.** 3 **23.** **(i)** 3 **(ii)** 2

24. **(i)** No, since the rank of the first matrix is 4 and that of the second matrix is 2.

(ii) No, since the marices are not of the same type

25. Rank **A** = 2; Rank **B** = 1; Rank **A** + **B** = 2;

Rank **AB** = 0; Rank **BA** = 1

28. **(i)** n. **(ii)** Equal to the order of the matrix

2.11 Row and Column Equivalence of Matrices

Definition: *A matrix* $\mathbf{A}$ *is said to be row equivalent to* $\mathbf{B}$ *if* $\mathbf{B}$ *is obtainable from* $\mathbf{A}$ *by a finite number of E-row transformations of* $\mathbf{A}$*. Symbolically, we then write* $\mathbf{A} \overset{R}{\sim} \mathbf{B}$*. Similarly a matrix* $\mathbf{A}$ *is said to be column equivalent to* $\mathbf{B}$ *if* $\mathbf{B}$ *is obtainable from* $\mathbf{A}$ *by a finite number of E-column transformations of* $\mathbf{A}$*. Symbolically, we then write* $\mathbf{A} \overset{C}{\sim} \mathbf{B}$.

2.12 Employment of Only Row Transformations

Theorem: *If* $\mathbf{A}$ *be an* $m \times n$ *matrix of rank* r*, then there exists a non-singular matrix* $\mathbf{P}$ *such that* $\mathbf{PA} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$*, where* $\mathbf{G}$ *is an* $r \times n$ *matrix of rank* r *and* $\mathbf{O}$ *is* $(m-r) \times n$.

Proof. Since $\mathbf{A}$ is an $m \times n$ matrix of rank r, therefore there exist non-singular matrices $\mathbf{P}$ and $\mathbf{Q}$, such that

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}. \qquad \ldots(1)$$

Now every non-singular matrix can be expressed as the product of elementary matrices. So let

$\mathbf{Q} = \mathbf{Q}_1 \mathbf{Q}_2 \ldots\ldots \mathbf{Q}_t$ where $\mathbf{Q}_1, \mathbf{Q}_2, \ldots\ldots\ldots, \mathbf{Q}_t$ are elementary matrices.

Thus the relation (1) can be written as

$$\mathbf{PAQ}_1 \mathbf{Q}_2 \ldots\ldots \mathbf{Q}_t = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}. \qquad \ldots(2)$$

Now every E-column transformation of a matrix, is equivalent to post-multiplication with the corresponding elementary matrix. Since no column transformation can affect the last $(m-r)$ rows of the right hand side of (2), therefore post-multiplying the L. H.S. of (2) by the elementary matrices $\mathbf{Q}_t^{-1}, \mathbf{Q}_{t-1}^{-1}, \ldots, \mathbf{Q}_2^{-1}, \mathbf{Q}_1^{-1}$ successively and affecting the corresponding column transformations in the right hand side of (2), we get a relation of the form

$$\mathbf{PA} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}.$$

Since elementary transformations do not alter the rank, therefore the rank of the matrix **PA** is the same as that of the matrix **A** which is r. Thus the rank of the matrix $\begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$ is r and therefore the rank of the matrix **G** is also r as the matrix **G** has r rows and last $m - r$ rows of the matrix $\begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$ consist of zeros only.

2.13 Employment of Only Column Transformations

Theorem: *If* **A** *be an* $m \times n$ *matrix of rank* r, *then there exists a non-singular matrix* **Q** *such that* $\mathbf{AQ} = [\mathbf{H} \;\; \mathbf{O}]$, *where* **H** *is an* $m \times n$ *matrix of rank* r *and* **O** *is* $m \times (n - r)$.

Proof. Since **A** is an $m \times n$ matrix of rank r, therefore there exist non-singular matrices **P** and **Q** such that

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}. \qquad \ldots(1)$$

Now every non-singular matrix can be expressed as the product of elementary matrices. So let

$$\mathbf{P} = \mathbf{P}_1 \, \mathbf{P}_2 \ldots\ldots \mathbf{P}_s \text{ where } \mathbf{P}_1, \mathbf{P}_2, \ldots\ldots, \mathbf{P}_s \text{ are elementary matrices.}$$

Thus the relation (1) can be written as

$$\mathbf{P}_1 \, \mathbf{P}_2 \ldots\ldots \mathbf{P}_s \mathbf{AQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}. \qquad \ldots(2)$$

Now every *E*-row transformation of a matrix is equivalent to pre-multiplication with the corresponding elementary matrix. Again no row transformation can affect the last $n - r$ columns of $\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$.

Therefore pre-multiplying the L.H.S. of (2) by the elementary matrices.

$$\mathbf{P}_1^{-1}, \mathbf{P}_2^{-1}, \ldots\ldots \mathbf{P}_s^{-1}$$

successively and affecting the corresponding row transformations in the R.H.S. of (2), we get a relation of the form $\mathbf{AQ} = [\mathbf{H} \;\; \mathbf{O}]$.

Now elementary transformations do not alter the rank. Therefore the rank of the matrix **AQ** is the same as that of **A** which is r. Thus the rank of the matrix $[\mathbf{H} \;\; \mathbf{O}]$ is r and therefore the rank of the matrix **H** is also r as the matrix **H** has r columns and the last $n - r$ columns of the matrix $[\mathbf{H} \;\; \mathbf{O}]$ consists of zero only.

2.14 The Rank of a Product

Theorem: *The rank of a product of two matrices cannot exceed the rank of either matrix.* **(Lucknow 2009)**

Let **A** and **B** be two $m \times n$ and $n \times p$ matrices respectively. Let r_1, r_2 be the ranks of **A** and **B** respectively and let r be the rank of the product **AB**.

To prove $r \le r_1$ and $r \le r_2$.

Since **A** is an $m \times n$ matrix of rank r_1, therefore there exists a non-singular matrix **P** such that $\mathbf{PA} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$, where **G** is an $r_1 \times n$ matrix of rank r_1 and **O** is $(m - r_1) \times n$.

$$\therefore \qquad \mathbf{PAB} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix} \mathbf{B}.$$

Since the rank of a matrix does not alter by multiplying it with a non-singular matrix, therefore

Rank $(\mathbf{PAB})$ = Rank $(\mathbf{AB}) = r$.

$\therefore$ Rank of the matrix $\begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix} \mathbf{B} = r$.

Since the matrix **G** has only r_1 non-zero rows, therefore the matrix $\begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$ **B** cannot have more than r_1 non-zero rows which arise by multiplying the r_1 non-zero rows of **G** with the columns of **B**.

$\therefore$ Rank of the matrix $\begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix}$ **B** is $\le r_1$ *i.e.*, $t \le r_1$

i.e., Rank $(\mathbf{AB}) \le$ Rank of the prefactor **A.**

Again r = Rank $(\mathbf{AB})$ = Rank $(\mathbf{AB})'$ = Rank $(\mathbf{B}' \mathbf{A}') \le$ Rank of the prefactor $\mathbf{B}'$

= Rank **B** $\qquad [\because$ Rank $\mathbf{B}$ = Rank $\mathbf{B}']$

$= r_2$.

$\therefore$ $r \le r_2$ *i.e.*, Rank $(\mathbf{AB}) \le$ Rank of the post-factor **B**.

2.15 Theorem

Every non-singular matrix is row equivalent to a unit matrix.

Proof. We shall prove the theorem by induction on n, the order of the matrix. If the matrix be of order 1 *i.e.*, if $\mathbf{A} = |a_{11}|$, the theorem obviously holds.

Let us assume that the theorem holds for all non-singular matrices of order $n - 1$.

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ non-singular matrix. The first column of the matrix $\mathbf{A}$ has at least one element different from zero, for otherwise we shall have $|\mathbf{A}| = 0$ and the matrix $\mathbf{A}$ will not be non-singular.

Let $$a_{p1} = k \neq 0.$$

By interchanging the p^{th} row with the first row (if necessary), we obtain a matrix $\mathbf{B}$ whose leading element is equal to k which is not equal to zero.

Multiplying the elements of the first row of the matrix $\mathbf{B}$ by $1/k$, we obtain a matrix $\mathbf{C}$ whose leading element is equal to unity.

Let $$\mathbf{C} = \begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix}.$$

Subtracting suitable multiples of the first row of $\mathbf{C}$ from the remaining rows, we obtain a matrix $\mathbf{D}$ in which all elements of the first column except the leading element are equal to zero.

Let $$\mathbf{D} = \begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1n} \\ 0 & & & & \\ 0 & & \mathbf{A}_1 & & \\ \cdots & & & & \\ 0 & & & & \end{bmatrix}.$$

where $\mathbf{A}_1$ is an $(n-1) \times (n-1)$ matrix. The matrix $\mathbf{A}_1$ is non-singular, for otherwise $|\mathbf{A}_1| = 0$ and so $|\mathbf{D}|$ is also equal to zero. Thus the matrix $\mathbf{D}$ will be non-singular, and therefore $\mathbf{A}$, which is row equivalent to $\mathbf{D}$, will also not be non-singular.

By the inductive hypothesis, $\mathbf{A}_1$ can be transformed to $\mathbf{I}_{n-1}$ by *E*-row operations. If these elementary row operations be applied to **D**, they will not affect the first row and the first column of **D** and we shall obtain a matrix **M** such that

$$\mathbf{M} = \begin{bmatrix} 1 & d_{12} & d_{13} & \dots & d_{1n} \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

By adding suitable multiples of the second, third, ..., n^{th} rows to the first row of **M**, we obtain the matrix $\mathbf{I}_n$.

Thus the matrix **A** has been reduced to $\mathbf{I}_n$ by *E*-row operations only.

The proof is now complete by induction.

Corollary 1: *If* **A** *be an n-rowed non-singular matrix, there exist E-matrices* $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_t$ *such that*

$$\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}_n.$$

If **A** be an n-rowed non-singular matrix, it can be reduced to $\mathbf{I}_n$ by *E*-row operations only. Since every *E*-row operation is equivalent to pre-multiplication by the corresponding *E*-matrix, therefore we can say that if **A** be an n -rowed non-singular matrix, there exist *E*-matrices $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_t$ such that

$$\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}_n.$$

Corollary 2: *Every non-singular matrix* **A** *is expressible as the product of elementary matrices.*

If **A** be an n-rowed non-singular matrix, there exist *E*-matrices $\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_t$ such that

$$\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}_n. \qquad \dots(1)$$

Pre-multiplying both sides of the relation (1) by $(\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1)^{-1}$, we get

$$(\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1)^{-1} (\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1)\mathbf{A} = (\mathbf{E}_t \mathbf{E}_{t-1} \dots \mathbf{E}_2 \mathbf{E}_1)^{-1} \mathbf{I}_n$$

or $$\mathbf{I}_n \mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \mathbf{E}_3^{-1} \dots \mathbf{E}_{t-1}{}^{-1} \mathbf{E}_t^{-1} \mathbf{I}_n$$

or $$\mathbf{A} = \mathbf{E}_1^{-1} \mathbf{E}_2^{-1} \dots \mathbf{E}_{t-1}^{-1} \mathbf{E}_t^{-1}.$$

Since the inverse of an elementary matrix is also an elementary matrix of the same type hence we get the result.

Corollary 3: *The rank of a matrix does not alter by pre-multiplication or post-multiplication with a non-singular matrix.*

Every non-singular matrix can be expressed as the product of elementary matrices. Also *E*-row (column) transformations are equivalent to pre-(post)-multiplication with the corresponding elementary matrices and elementary transformations do not alter the rank of a matrix. Hence we get the result.

2.16 Use of Elementary Transformations to Find the Inverse of a Non-Singular Matrix

Let $\mathbf{A}$ be a non-singular matrix of order n. It can be easily shown that $\mathbf{A}$ can be reduced to the unit matrix $\mathbf{I}_n$ by a finite number of *E*-row transformations only. Now each *E*-row transformation of a matrix is equivalent to pre-multiplication by the corresponding *E*-matrix. Therefore there exist elementary matrices, say, $\mathbf{E}_1, \mathbf{E}_2, \ldots\ldots, \mathbf{E}_t$ such that $(\mathbf{E}_t \mathbf{E}_{t-t} \ldots\ldots \mathbf{E}_2 \mathbf{E}_1)\ \mathbf{A} = \mathbf{I}_n$.

Post-multiplying both sides by $\mathbf{A}^{-1}$, we get

$$(\mathbf{E}_t \mathbf{E}_{t-1} \ldots\ldots \mathbf{E}_2 \mathbf{E}_1) \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n \mathbf{A}^{-1}$$

or $$(\mathbf{E}_t \mathbf{E}_{t-1} \ldots\ldots \mathbf{E}_2 \mathbf{E}_1) \mathbf{I}_n = \mathbf{A}^{-1} \qquad [\because \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_n, \mathbf{I}_n \mathbf{A}^{-1} = \mathbf{A}^{-1}]$$

or $$\mathbf{A}^{-1} = (\mathbf{E}_t \mathbf{E}_{t-1} \ldots\ldots \mathbf{E}_2 \mathbf{E}_1) \mathbf{I}_n.$$

Hence we get the following result :

If a non-singular matrix $\mathbf{A}$ *of order n is reduced to the unit matrix* $\mathbf{I}_n$ *by a sequence of E-row transformations only, then the same sequence of E-row transformations applied to the unit matrix* $\mathbf{I}_n$ *gives the inverse of* $\mathbf{A}$ *(i.e.,* $\mathbf{A}^{-1}$*).*

2.17 Working Rule for Finding the Inverse of a Non-Singular Matrix by E-Row Transformations

Suppose $\mathbf{A}$ is a non-singular matrix of order n. Then we write $\mathbf{A} = \mathbf{I}_n \mathbf{A}$.

Now we go on applying *E*-row transformations only to the matrix $\mathbf{A}$ and the pre-factor $\mathbf{I}_n$ of the product $\mathbf{I}_n \mathbf{A}$ till we reach the result $\mathbf{I}_n = \mathbf{B}\mathbf{A}$.

Then obviously $\mathbf{B}$ is the inverse of $\mathbf{A}$.

Illustrative Examples

Example 15: *Find the inverse of the matrix* $\mathbf{A}=\begin{bmatrix}1 & 2 & 1\\ 3 & 2 & 3\\ 1 & 1 & 2\end{bmatrix}$ *by using E-transformations.*

(Avadh 2006, 08; Purvanchal 09; Rohilkhand 09; Lucknow 09)

Solution : We write $\mathbf{A}=\mathbf{I}_3\mathbf{A}$, *i.e.*, $\begin{bmatrix}1 & 2 & 1\\ 3 & 2 & 3\\ 1 & 1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\mathbf{A}.$

Now we go on applying E-row transformations to the matrix $\mathbf{A}$ (the left hand member of the above equation) until it is reduced to the form $\mathbf{I}_3$. Every E-row transformation will also be applied to the prefactor $\mathbf{I}_3$ (or its transform) of the product on the right hand side of the above equation.

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix}1 & 2 & 1\\ 0 & -4 & 0\\ 0 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\mathbf{A}.$$

Now we should try to make 1 in the place of the second element of the second row of the matrix on the left hand side. So applying $R_2 \to -\frac{1}{4}R_2$, we get

$$\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \frac{3}{4} & -\frac{1}{4} & 0\\ -1 & 0 & 1\end{bmatrix}\mathbf{A}.$$

Now we shall make zeros in the place of the second elements of the first and third rows with the help of the second row. So applying $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + R_2$, we get

$$\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2} & 0\\ \frac{3}{4} & -\frac{1}{4} & 0\\ -\frac{1}{4} & -\frac{1}{4} & 1\end{bmatrix}\mathbf{A}.$$

Now the third element of the third row is already 1. So to make the third element of the first row zero, we apply $R_1 \to R_1 - R_3$, and we get

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix} \mathbf{A}.$$

Thus $\qquad \mathbf{I}_3 = \mathbf{BA}$, where $\mathbf{B} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}.$

$$\therefore \qquad \mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}.$$

Example 16: *Find the inverse of the matrix* $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ *by using E-transformations.*

Solution : We write $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}.$

Performing $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_3 \to R_3 - 2R_1, R_4 \to R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_4 \to R_4 - R_2, R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 1 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_4 \to R_4 - R_2, R_1 \to R_1 - R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_3 \to -\frac{1}{2} R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & -3 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_2 \to R_2 - 2R_3, R_4 \to R_4 + 3R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ -1 & 1 & -\frac{3}{2} & 1 \end{bmatrix} \mathbf{A}.$$

Performing $R_4 \to -2R_4$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 2 & -2 & 3 & -2 \end{bmatrix} \mathbf{A}.$$

Performing $R_3 \to R_3 - \frac{3}{2} R_4, R_2 \to R_2 + R_4, R_1 \to R_1 - R_4$ we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix} \mathbf{A}.$$

$$\therefore \qquad \mathbf{A}^{-1} = \begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}.$$

Comprehensive Exercise 3

1. Reduce the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ to $\mathbf{I}_3$ by E-row transformations only.

2. Compute the inverse of the following matrices by using elementary row-transformations :

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ **(ii)** $\begin{bmatrix} i & -1 & 2i \\ 2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

(Avadh 2007)

(v) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

(Rohilkhand 2011; Bundelkhand 08)

Answers 3

2. **(i)** $\dfrac{1}{157}\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ -2 & -1 & 0 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 0 & \dfrac{1}{4} & -\dfrac{1}{2} \\ -1 & \dfrac{3i}{4} & \dfrac{i}{2} \\ 0 & \dfrac{1}{4} & \dfrac{1}{2} \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$ **(iv)** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \\ -4 & 3 & -1 \\ \dfrac{5}{2} & -\dfrac{3}{2} & \dfrac{1}{2} \end{bmatrix}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b) , (c) *and* (d).

1. If **A** is a matrix such that there exists a square submatrix of order r which is non-singular and every square submatrix of order $r+1$ or more is singular, then

(a) rank $\mathbf{A} = r + 1$

(b) rank $\mathbf{A} = r$

(c) rank $\mathbf{A} > r$

(d) rank $\mathbf{A} \geq r + 1$

2. Let $\mathbf{A} = [a_{ij}]_{m \times n}$ be a matrix such that $a_{ij} = 1$ for all i, j. Then

(a) rank $\mathbf{A} > 1$

(b) rank$\mathbf{A} = 1$

(c) rank $\mathbf{A} = m$

(d) rank $\mathbf{A} = n$

3. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ is

(a) 1 (b) 2

(c) 3 (d) 4

4. If $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that rank of $\mathbf{A} = m$ and rank of $\mathbf{B} = n$, then

(a) rank $(\mathbf{A}\ \mathbf{B}) = m\ n$

(b) rank $(\mathbf{A}\ \mathbf{B}) \geq$ rank $\mathbf{A}$

(c) rank $(\mathbf{A}\ \mathbf{B}) \geq$ rank $\mathbf{B}$

(d) rank $(\mathbf{A}\ \mathbf{B}) \leq$ min (rank $\mathbf{A}$, rank $\mathbf{B}$) **(Agra 2007)**

5. If $\mathbf{A}$ is an invertible matrix and $\mathbf{B}$ is a matrix, then

(a) rank $(\mathbf{A}\ \mathbf{B}) =$ rank $\mathbf{A}$

(b) rank $(\mathbf{A}\ \mathbf{B}) =$ rank $\mathbf{B}$

(c) rank $(\mathbf{A}\ \mathbf{B}) >$ rank $\mathbf{A}$

(d) rank $(\mathbf{A}\ \mathbf{B}) >$ rank $\mathbf{B}$

6. The rank of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is

(a) 0 (b) 1

(c) 2 (d) 3 **(Bundelkhand 2001; Meerut 03)**

Fill in the Blank(s)

Fill in the blanks "......", so that the following statements are complete and correct.

1. The rank of a matrix is the ...... of any highest order non-vanishing minor of the matr ix.
2. The rank of every non-singular matrix of order n is ....
(Bundelkhand 2005)
3. The rank of a matrix in Echelon form is equal to the number of non-zero .... of the matrix.
(Bundelkhand 2008)
4. **A** is a non-zero column matrix and **B** is a non-zero row matrix, then rank (**AB**) = ...
5. If a matrix **B** is obtained from **A** by an elementary transformation, **A** can also be obtained from **B** by an elementary transformation of the ...... type.
6. A matrix obtained from a ...... by a single elementary transformation is called an elementary matrix.
7. If a matrix **A** of order $m \times n$ can be expressed as :
$$\mathbf{A} \sim \begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}, \text{ then rank of } \mathbf{A} \text{ is ....}$$
(Meerut 2001)
8. If **A** be an $m \times n$ matrix of rank r, there exist non-singular matrices **P** and **Q** such that **PAQ** = ...
9. If **B** be an $m \times n$ matrix obtained from an $m \times n$ matrix **A** by finite number of elementary transformations of **A**, then **A** is called .... to be **B**.
10. If two matrices **A** and **B** have the same size and the same rank, they are ....
11. The rank of a matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is ....
(Agra 2008)

True or False

Write 'T' for true and 'F' for false statement.

1. The rank of the transpose of a matrix is the same as that of the original matrix.
2. Every elementary row transformation of a matrix can be obtained by post-multiplication with the corresponding elementary matrix.
3. Elementary transformations change the rank of a matrix.
4. If **A** and **B** are two equivalent matrices, then rank **A** = rank **B**.

5. The rank of a matrix does not alter by pre-multiplication or post-multiplication with a non-singular matrix.

Answers

Multiple Choice Questions

1.	(b)	2.	(b)	3.	(c)
4.	(d)	5.	(b)	6.	(c)

Fill in the Blank(s)

1.	order	2.	n	3.	rows
4.	1	5.	same	6.	unit matrix
7.	r	8.	$\begin{bmatrix} \mathbf{I}_r & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}$	9.	equivalent
10.	equivalent	11.	2		

True or False

1.	T	2.	F	3.	F
4.	T	5.	T		

□

Chapter 3

Linear Equations

We shall devote this chapter to the study of the nature of solutions of a system of linear equations with the help of the theory developed in the preceding chapters. Before discussing the solutions of linear equations, we shall discuss concepts of **linearly dependent** and **linearly independent** sets of vectors.

3.1 Vectors

Definition: *Any ordered n-tuple of numbers is called an n-vector.* By an ordered n-tuple we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \ldots, x_n$ be any n numbers, then the ordered n -tuple $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ is called an n-vector. The ordered triad (x_1, x_2, x_3) is called a 3-vector. Similarly $(1, 0, 1, -1)$ and $(1, 8, -5, 7)$ are 4-vectors. The n numbers $x_1, x_2, \ldots, x_n$ are called components of the n-vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)$. A vector may be written either as a *row vector* or as a *column vector*. If $\mathbf{A}$ be a matrix of the type $m \times n$, then each row of $\mathbf{A}$ will be an n-vector and each column of $\mathbf{A}$ will be an m-vector. A vector whose components are all zero is called a *zero vector* and will be denoted by $\mathbf{O}$.

If k be any number and $\mathbf{X}$ be any vector, then relative to the vector $\mathbf{X}$, k is called a scalar.

Algebra of vectors. Since an n-vector is nothing but a row matrix or a column matrix, therefore we can develop an algebra of vectors in the same manner as the algebra of matrices.

Equality of two vectors. Two n-vectors **X** and **Y** where $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{Y} = (y_1, y_2, \ldots, y_n)$ are said to be **equal** if and only if their corresponding components are equal *i.e.*, if $x_i = y_i$, for all $i = 1, 2, .., n$.

For example if

$$\mathbf{X} = (1, 4, 7) \quad \text{and } \mathbf{Y} = (1, 4, 7),$$

then $\mathbf{X} = \mathbf{Y}$.

But if $\mathbf{X} = (1, 4, 7)$ and $\mathbf{Y} = (4, 1, 7)$, then $\mathbf{X} \neq \mathbf{Y}$.

Addition of two vectors. If $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{Y} = (y_1, y_2, \ldots, y_n)$ then by definition $\mathbf{X} + \mathbf{Y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Thus $\mathbf{X} + \mathbf{Y}$ is an n-vector whose components are the sums of corresponding components of **X** and **Y**.

If $\mathbf{X} = (2, 4, -7)$ and $\mathbf{Y} = (1, -3, 5)$,

then $\mathbf{X} + \mathbf{Y} = (2 + 1, 4 - 3, -7 + 5) = (3, 1, -2)$.

Multiplication of a vector by a scalar (number).

If k be any number and

$$\mathbf{X} = (x_1, x_2, \ldots, x_n),$$

then by definition

$$k\mathbf{X} = (kx_1, kx_2, \ldots, kx_n).$$

The vector $k\,\mathbf{X}$ is called the scalar multiple of the vector **X** by the scalar k.

If $\mathbf{X} = (1, 3, 8)$, then $4\mathbf{X} = (4, 12, 32)$

and $0\,\mathbf{X} = (0, 0, 0)$.

Properties of addition and scalar multiplication of vectors. If **X, Y, Z** be any three n-vectors and p, q be any two numbers, then obviously

(i) $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$.

(ii) $\mathbf{X} + (\mathbf{Y} + \mathbf{Z}) = (\mathbf{X} + \mathbf{Y}) + \mathbf{Z}$.

(iii) $p(\mathbf{X} + \mathbf{Y}) = p\mathbf{X} + p\mathbf{Y}$

(iv) $(p + q)\mathbf{X} = p\mathbf{X} + q\mathbf{X}$.

(v) $p(q\mathbf{X}) = (pq)\,\mathbf{X}$.

3.2 Linear Dependence and Linear Independence of Vectors

(Agra 2005)

Linearly dependent set of vectors. Definition:

A set of r n-vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r$ *is said to be linearly dependent if there exist r scalars (numbers)* $k_1, k_2, \ldots, k_r$, *not all zero, such that*

$$k_1 \mathbf{X}_1 + k_2 \mathbf{X}_2 + .. + k_r \mathbf{X}_r = \mathbf{O},$$

where, $\mathbf{O}$ *denotes the n-vector whose components are all zero.*

Linearly independent set of vectors. Definition:

A set of r n-vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r$ *is said to be linearly independent if every relation of the type*

$$k_1 \mathbf{X}_1 + k_2 \mathbf{X}_2 + \ldots + k_r \mathbf{X}_r = \mathbf{O}$$

implies $k_1 = k_2 = k_3 \ldots = k_r = 0$.

Illustrative Examples

Example 1: *Show that the vectors* $\mathbf{X}_1 = (1, 2, 4), \mathbf{X}_2 = (3, 6, 12)$ *are linearly dependent.*

Solution: By a little inspection, we see that

$$3\mathbf{X}_1 + (-1)\, \mathbf{X}_2 = (3, 6, 12) + (-3, -6, -12) = (0, 0, 0) = \mathbf{O}.$$

Thus there exist numbers $k_1 = 3, k_2 = -1$ which are not all zero such that

$$k_1 \mathbf{X}_1 + k_2 \mathbf{X}_2 = \mathbf{O}.$$

Hence the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly dependent.

Example 2: *Show that the set consisting only of the zero vector,* $\mathbf{O}$, *is linearly dependent.*

Solution : Let $\mathbf{X} = (0, 0, 0, \ldots., 0)$ be an n-vector whose components are all zero. Then the relation $k\mathbf{X} = \mathbf{O}$ is true for some non-zero value of the number k. For example, $1\mathbf{X} = \mathbf{O}$ and $1 \neq 0$.

Hence the vector $\mathbf{O}$ is linearly dependent.

Example 3: *Show that the vectors* $\mathbf{X}_1 = (1, 2, 3)$ *and* $\mathbf{X}_2 = (4, -2, 7)$ *are linearly independent.*

Solution: Let k_1 and k_2 be two numbers such that $k_1 \mathbf{X}_1 + k_2 \mathbf{X}_2 = \mathbf{O}$,

i.e., $$k_1 (1, 2, 3) + k_2 (4, -2, 7) = (0, 0, 0)$$

i.e., $$(k_1 + 4k_2, 2k_1 - 2k_2, 3k_1 + 7k_2) = (0, 0, 0).$$

Equating the corresponding components, we get

$$k_1 + 4k_2 = 0, 2k_1 - 2k_2 = 0, 3k_1 + 7k_2 = 0.$$

The only common values of k_1 and k_2 which satisfy these equations are $k_1 = 0, k_2 = 0$.

Thus $$k_1\mathbf{X}_1 + k_2\mathbf{X}_2 = \mathbf{O}, \quad \text{iff } k_1 = k_2 = 0.$$

Hence the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent.

Example 4: *Show that the set of three 3-vectors*

$$\mathbf{X}_1 = (1, 0, 0), \mathbf{X}_2 = (0, 1, 0), \mathbf{X}_3 = (0, 0, 1)$$

is linearly independent.

Solution : Let k_1, k_2, k_3 be three numbers such that

$$k_1\mathbf{X}_1 + k_2\mathbf{X}_2 + k_3\mathbf{X}_3 = \mathbf{O},$$

i.e., $$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0),$$

i.e., $$(k_1, 0, 0) + (0, k_2, 0) + (0, 0, k_3) = (0, 0, 0),$$

i.e., $$(k_1, k_2, k_3) = (0, 0, 0).$$

Obviously this relation is true if and only if $k_1 = 0, k_2 = 0, k_3 = 0$.

Hence the vectors $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are linearly independent.

3.3 A Vector as a Linear Combination of Vectors

Definition: *A vector* $\mathbf{X}$ *which can be expressed in the form*

$$\mathbf{X} = k_1\mathbf{X}_1 + k_2\mathbf{X}_2 + \ldots + k_r\mathbf{X}_r,$$

is said to be a linear combination of the set of vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r$.

Here $k_1, k_2, \ldots, k_r$ are any numbers.

The following two results are quite obvious :

(i) *If a set of vectors is linearly dependent, then at least one member of the set can be expressed as a linear combination of the remaining members.*

(ii) *If a set of vectors is linearly independent then no member of the set can be expressed as a linear combination of the remaining members.*

3.4 The n-Vector Space

The set of all n vectors of a field F is called the n-vector space over F. It is usually denoted by $V_n(F)$ or simply by V_n if the field is understood. Similarly the set of all 3-vectors is a vector space which is usually denoted by V_3. The elements of the field F are known as *scalars* relatively to the vectors.

3.5 Sub-space of an n-Vector Space V_n

Definition. A non-empty set, S, of vectors of V_n is called a vector sub-space of V_n, if $\mathbf{a} + \mathbf{b}$ belongs to S whenever $\mathbf{a}$, $\mathbf{b}$ belong to S and $k\,\mathbf{a}$ belongs to S, where k is any scalar.

It is important to note that every sub-space of V_n contains the zero vector, being the scalar product of any vector with the scalar zero.

Example: If $\mathbf{a} = (a_1, a_2, a_3)$ is any non-zero vector of V_3, then the set S of vectors $k\mathbf{a}$ is a subspace of V_3, where k is a variable scalar which can take any value. The sum of any two members $k_1\mathbf{a}, k_2\mathbf{a}$ of S, is the vector $k_1\mathbf{a} + k_2\mathbf{a}$

i.e., $\quad (k_1a_1, k_1a_2, k_1a_3) + (k_2a_1, k_2a_2, k_2a_3)$

i.e., $\quad (k_1 + k_2)\,\mathbf{a}$

which is also a member of S. Also the scalar multiple by any scalar x of any vector $k_1\mathbf{a}$ of S is the vector $(xk_1)\mathbf{a}$ which is again a member of S.

Hence the set S of vectors $k\mathbf{a}$ is a subspace of V_3.

3.6 Vector Subspace Spanned by a given System of Vectors

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be any three vectors of V_3. The set S of all vectors of the form $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, where x, y, z are any scalars, is a subspace of V_3. For, if $x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}$, $x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}$ be any two members of S, then

$$(x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}) + (x_2\mathbf{a} + y_2\mathbf{b} + z_2\mathbf{c}) = (x_1 + x_2)\mathbf{a} + (y_1 + y_2)\,\mathbf{b} + (z_1 + z_2)\mathbf{c},$$

which is also a member of S. Also if k be any scalar, then

$$k\,(x_1\mathbf{a} + y_1\mathbf{b} + z_1\mathbf{c}) = (kx_1)\mathbf{a} + (ky_1)\mathbf{b} + (kz_1)\mathbf{c},$$

which is again a member of S. Thus S is a vector subspace and we say that S is a spanned by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. More generally, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ be a set of r fixed vectors of V_n, then the set S of all n-vectors of the form $p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + \ldots p_r\mathbf{a}_r$ where $p_1, p_2, \ldots., p_r$ are any scalars is a vector subspace of V_n.

This vector space is said to be spanned by the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$. **Thus a vector space which arises as a set of all linear combinations of any given set of vectors, is said to be spanned by the given set of vectors.**

3.7 Basis and Dimension of a Subspace

A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_k$ belonging to the subspace S is said to be a basis of S, if

(i) the subspace S is spanned by the set $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ and

(ii) the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ are linearly independent.

It can be easily shown that every subspace, S, of V_n possesses a basis.

It can be easily shown that the vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \mathbf{e}_3 = (0, 0, 1, 0, \ldots, 0), \ldots,$$

$$\mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

constitute a basis of V_n.

We have already shown that these vectors are linearly independent. Moreover any vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of V_n is expressible as $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + \ldots + a_n\mathbf{e}_n$. Hence the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$ constitute a basis of V_n.

Theorem: *A basis of a subspace, S, can always be selected out of a set of vectors which span S.*

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ be a set of vectors which spans a subspace S. If these vectors are linearly independent, they already constitute a basis of S as they span S. In case they are linearly dependent, some member of the set is a linear combination of the members. Deleting this member we obtain another set which also spans.

Continuing in this manner we shall ultimately, in a finite number to steps arrive at a basis of S.

A vector subspace may (and in fact does) possess several bases

For examlple, if

$$\mathbf{a}_1 = (1, 0, 0), \mathbf{a}_2 = (0, 1, 0), \mathbf{a}_3 = (0, 0, 1), \quad \mathbf{b}_1 = (1, 1, 1), \mathbf{b}_2 = (1, 1, 0), \mathbf{b}_3 = (1, 0, 0),$$

then $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ constitute a basis of V_3. Also $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ constitute a basis of V_3.

But it is important to note that *the number of members in any one basis of a subspace is the same as in any other basis. This number is called the dimension of the subspace.*

We have already shown that one basis of V_n possesses n members. Therefore every basis of V_n must possess n members. Thus V_n is of dimension n. In a particular the dimension of V_3 is 3.

Also it can be easily shown that if r be the dimension of a subspace S and if $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_k$ be a linearly independent set of vectors belonging to S, then we can always find vectors $\mathbf{a}_{k+1}, \mathbf{a}_{k+2}, \dots. \mathbf{a}_r$ such that the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_r$ constitute a basis of S. In other words we can say that every linearly independent set of vectors belonging to a subspace S can always be extended so as to constitute a basis of S.

Moreover if r be the dimension of a subspace S, then every set of more than r members of S will be linearly dependent.

Intersection of subspaces. If S and T be two subspaces of V_n, then the vectors common to both S and T also constitute a subspace. This subspace is called the intersection of the subspaces S and T.

3.8 Row Rank and Column Rank of a Matrix

Row rank of a matrix. Let $\mathbf{A} = [a_{ij}]$ be any $m \times n$ matrix. Each of the m rows of $\mathbf{A}$ consists of n elements. Therefore the row vectors of $\mathbf{A}$ are n-vectors. These row vectors of $\mathbf{A}$ will span a subspace R of V_n. This subspace R is called the *row space* of the matrix $\mathbf{A}$. The dimension r of R is called the *row rank* of $\mathbf{A}$. In other words the row rank of a matrix $\mathbf{A}$ is equal to the maximum number of linearly independent rows of $\mathbf{A}$.

Row rank of a matrix. Definition: *The maximum number of linearly independent rows of a matrix* $\mathbf{A}$ *is said to be the* ***row rank*** *of the matrix* $\mathbf{A}$.

Left nullity of a matrix. Suppose $\mathbf{X}$ is an m-vector written in the form of a row vector. Then the matrix product $\mathbf{XA}$ is defined. The subspace S of V_m generated by the row vectors $\mathbf{X}$ belonging to V_m such that $\mathbf{XA} = \mathbf{O}$ is called the *row null* space of the matrix $\mathbf{A}$. The dimension s of S is called the left nullity or *row nullity* of the matrix $\mathbf{A}$.

We shall now prove that the *sum of the row rank and the row nullity of a matrix is equal to the number of rows, i.e.,*

$$r + s = m.$$

Proof: Since the row space R of $\mathbf{A}$ is spanned by the row vectors of $\mathbf{A}$, therefore it will be a set of all vectors of the form

$$x_1 (a_{11}, a_{12}, a_{13}, \dots, a_{1n}) + x_2 \, (a_{21}, a_{22}, a_{23}, \dots., a_{2n}) + \dots$$
$$\dots + x_m \, (a_{m1}, a_{m2}, a_{m3}, \dots, a_{mn})$$

i.e., of the form $(x_1a_{11}+x_2a_{21}+\ldots+x_ma_{m1}, x_1a_{12}+x_2a_{22}+\ldots.+x_ma_{m2},$

$$x_1a_{13}+x_2a_{23}+\ldots+x_ma_{m3},\ldots,x_1a_{1n}+x_2a_{2n}+\ldots+x_ma_{mn})$$

i.e., of the form $\mathbf{XA}$, where $\mathbf{X}=(x_1,x_2,\ldots,x_m)$ is an m-vector.

Let $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s$ be a basis of the subspace S of V_m generated by all vectors $\mathbf{X}$ such that $\mathbf{XA}=\mathbf{O}$. Then, we have

$$\mathbf{u}_1\mathbf{A}=\mathbf{u}_2\mathbf{A}=\ldots=\mathbf{u}_s\mathbf{A}=\mathbf{O}.$$

Since the vectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s$ belong to V_m and form a linearly independent set, therefore we can find vectors $\mathbf{u}_{s+1},\mathbf{u}_{s+2},\ldots,\mathbf{u}_m$ in V_m such that the vectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s,\mathbf{u}_{s+1},\ldots,\mathbf{u}_m$ constitute a basis of V_m. Then every vector $\mathbf{X}$ belonging to V_m can be expressed in the form

$$\mathbf{X}=h_1\mathbf{u}_1+h_2\mathbf{u}_2+\ldots.+h_m\mathbf{u}_m.$$

Now every member of the subspace R is expressible as $\mathbf{XA}$

i.e., as $\quad (h_1\mathbf{u}_1+h_2\mathbf{u}_2+\ldots+h_m\mathbf{u}_m)\,\mathbf{A}$

i.e., as $\quad h_1\mathbf{u}_1\mathbf{A}+h_2\mathbf{u}_2\mathbf{A}+\ldots+h_s\mathbf{u}_s\mathbf{A}+h_{s+1}\mathbf{u}_{s+1}\mathbf{A}+h_{s+2}\,\mathbf{u}_{s+2}\,\mathbf{A}+\ldots+h_m\mathbf{u}_m\mathbf{A}$

i.e., as $\quad h_{s+1}\mathbf{u}_{s+1}\mathbf{A}+h_{s+2}\mathbf{u}_{s+2}\mathbf{A}+\ldots+h_m\mathbf{u}_m\mathbf{A}.$

Therefore the $m-s$ n-vectors $\mathbf{u}_{s+1}\mathbf{A},\mathbf{u}_{s+2}\mathbf{A},\ldots,\mathbf{u}_m\mathbf{A}$ span R. In fact, these vectors form a basis of R. For any relation of the form

$$k_{s+1}\mathbf{u}_{s+1}\mathbf{A}+k_{s+2}\mathbf{u}_{s+2}\,\mathbf{A}+\ldots+k_m\mathbf{u}_m\mathbf{A}=\mathbf{O}$$

implies $\quad (k_{s+1}\mathbf{u}_{s+1}+k_{s+2}\mathbf{u}_{s+2}+\ldots+k_m\mathbf{u}_m)\mathbf{A}=\mathbf{O},$

which shows that $k_{s+1}\mathbf{u}_{s+1}+k_{s+2}\mathbf{u}_{s+2}+\ldots+k_m\mathbf{u}_m$ is a member of the subspace S and as such it can be linearly expressed in terms of the basis $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_s$ of S.

But the vectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_m$ are linearly independent. Therefore a relation of the form $k_{s+1}\mathbf{u}_{s+1}+k_{s+2}\mathbf{u}_{s+2}+\ldots+k_m\mathbf{u}_m=p_1\mathbf{u}_1+p_2\mathbf{u}_2+\ldots..+p_s\mathbf{u}_s$ will exist if and only if $k_{s+1}=k_{s+2}=\ldots=k_m=0$. Hence the vectors $\mathbf{u}_{s+1}\mathbf{A},\mathbf{u}_{s+2}\mathbf{A},\ldots,\mathbf{u}_m\mathbf{A}$ are linearly independent and form a basis of R. Thus the dimension of R is $m-s$.

Hence $\quad r=m-s \quad$ or $\quad r+s=m.$

3.9 Column Rank of a Matrix

Let $A = [a_{ij}]$ be any $m \times n$ matrix. Each of the n columns of **A** consists of m elements. Therefore the column vectors of **A** are m-vectors. These column vectors of **A** will span a subspace C of V_m. This subspace C is called the *column space* of the matrix **A.** The dimension c of C is called the *column rank* of **A**. In other words the column rank of a matrix **A** is equal to the maximum number of linearly independent columns of **A.**

Column rank of a matrix. Definition. *The maximum number of linearly independent columns of a matrix* **A** *is said to be the* **column rank** *of the matrix* **A**.

Right nullity of a matrix. Suppose **Y** is an n-vector written in the form of a column vector. Then the matrix product **AY** is defined. The subspace T of V_n generated by the column vectors **Y** belonging to V_n such that $\mathbf{AY} = \mathbf{O}$ is called the *column null space* of the matrix **A**. The dimension t of T is called the *right nullity* or *column nullity* of the matrix **A.**

As in 3.8, we can show that $c + t = n$.

3.10 Invariance of Row Rank under *E*-row Operations

Theorem. *Row equivalent matrices have the same row rank.*

Proof. Let **A** be any given $m \times n$ matrix. Let **B** be a matrix row equivalent to **A.** Since **B** is obtainable from **A** by a finite chain of E-row operations and every E-row operation is equivalent to pre-multiplication by the corresponding E-matrix, there exist E-matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ each of the type $m \times m$ such that

$$\mathbf{B} = \mathbf{E}_k \mathbf{E}_{k-1} \ldots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}, \ i.e., \mathbf{B} = \mathbf{PA},$$

where $\mathbf{P} = \mathbf{E}_k \mathbf{E}_{k-1} \ldots \mathbf{E}_2 \mathbf{E}_1$ is a non-singular matrix of the type $m \times m$.

Let us write

$$\mathbf{B} = \mathbf{PA} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \cdots \\ \cdots \\ \mathbf{R}_m \end{bmatrix} \qquad \ldots(1)$$

where the matrix **A** has been expressed as a matrix of its row sub-matrices $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_m$.

From the product of the matrices on the R.H.S. of (1), we observe that the rows of the matrix **B** are

$$p_{11}\mathbf{R}_1 + p_{12}\mathbf{R}_2 + \ldots + p_{1m}\mathbf{R}_m,$$
$$p_{21}\mathbf{R}_1 + p_{22}\mathbf{R}_2 + \ldots + p_{2m}\mathbf{R}_m,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$p_{m1}\mathbf{R}_1 + p_{m2}\mathbf{R}_2 + \ldots + p_{mm}\mathbf{R}_m.$$

Thus we see that the rows of **B** are all linear combinations of the rows $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_m$ of **A.** Therefore every member of the row space of **B** is also a member of the row space of **A**.

Similarly by writing $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}$ and giving the same reasoning we can prove that every member of the row space of **A** is also a member of the row space of **B**. Therefore the row spaces of **A** and **B** are identical.

Thus we see that elementary row operations do not alter the row space of a matrix. Hence the row rank of a matrix remains invariant under E-row transformations.

Note. From the above theorem we also conclude that *pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.*

3.11 Invariance of Column Rank under *E*-column Operations

Theorem. *Column equivalent matrices have the same column rank.*

Or

Post-multiplication by a non-singular matrix does not alter the column rank of a matrix.

Proof: Proceeding in the same way as in 3.10, we can show that post-multiplication with a non-singular matrix does not alter the column space and therefore the column rank of a matrix.

Note. Since every n-rowed E-matrix is obtained from $\mathbf{I}_n$ by single E-operation (row or column operation as may be desired), therefore the row rank and column rank of an E-matrix are equal to n.

3.12 Invariance of Column Rank under *E*-row Operations

Theorem: *Row equivalent matrices have the same column rank.*

Proof: Let **A** be any given $m \times n$ matrix and let **B** be a matrix row equivalent to **A**. Then there exists a non-singular matrix **P** such that $\mathbf{B} = \mathbf{PA}$.

For every column vector **X** such that $\mathbf{AX} = \mathbf{O}$, we have

$$\mathbf{BX} = (\mathbf{PA})\,\mathbf{X} = \mathbf{P}\,(\mathbf{AX}) = \mathbf{PO} = \mathbf{O}.$$

Since $\mathbf{B} = \mathbf{PA}$, therefore $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}$.

Therefore for every vector **X** such that $\mathbf{BX} = \mathbf{O}$, we have

$$\mathbf{AX} = (\mathbf{P}^{-1}\mathbf{B})\,\mathbf{X} = \mathbf{P}^{-1}\,(\mathbf{BX}) = \mathbf{P}^{-1}\mathbf{O} = \mathbf{O}.$$

Thus we see that the matrices **A** and **B** have the same right nullities and consequently their column ranks are equal.

Similarly we can prove that column equivalent matrices have the same row rank.

3.13 Theorem

If r be the row rank of an $m \times n$ matrix **A** *then there exists a non-singular matrix,* **P** *such that*

$$\mathbf{PA} = \begin{bmatrix} \mathbf{K} \\ \mathbf{O} \end{bmatrix},$$

where **K** *is an $r \times n$ matrix consisting of a set of r linearly independent rows of* **A**.

Proof. If the row rank r of **A** is zero, we have nothing to prove. Therefore let us assume that $r > 0$. The matrix **A** has then r linearly independent rows. By elementary row operations on **A** we can bring these linearly independent rows in the first r places. Since the last $m - r$ rows are now linear combinations of the first r rows, they can be made zero by *E*-row operations without altering the first r rows.

Thus we see that the matrix **A** is row equivalent to a matrix **B** such that

$$\mathbf{B} = \begin{bmatrix} \mathbf{K} \\ \mathbf{O} \end{bmatrix},$$

where **K** is an $r \times n$ matrix consisting of a set of r linearly independent rows of **A.**

Since every elementary row operation is equivalent to pre-multiplication by the corresponding E-matrix and the product of E-matrices is a non-singular matrix, therefore there exists a non-singular matrix **P** such that

$$\mathbf{PA} = \begin{bmatrix} \mathbf{K} \\ \mathbf{O} \end{bmatrix}.$$

Similarly considering column transformations instead of row transformations, we can show that *if c be the column rank of a matrix* **A**, *then there exists a non-singular matrix* **R** *such that*

$$\mathbf{AR} = [\mathbf{L} \quad \mathbf{O}],$$

where **L** *is an $m \times c$ matrix consisting of, c, linearly independent columns of* **A.**

3.14 Equality of Row Rank, Column Rank and Rank

Theorem 1. *The row rank of a matrix is the same as its rank.*

Proof: Let, s be the row rank and r, the rank of an $m \times n$ matrix **A.** Since the matrix **A** is of row rank s, therefore by 3.13, there exists a non-singular matrix **P** such that $\mathbf{PA} = \begin{bmatrix} \mathbf{K} \\ \mathbf{O} \end{bmatrix}$, where **K** is an $s \times n$ matrix.

Now we know that pre-multiplication by a non-singular matrix does not alter the rank of a matrix.

$\therefore \qquad$ Rank $(\mathbf{PA})$ = Rank $\mathbf{A} = r$.

But each minor of order $(s+1)$ of the matrix **PA** involves atleast one row of zeros.

$\therefore \qquad$ Rank $(\mathbf{PA}) \le s$.

$\therefore \qquad r \le s$

Again, since the rank of the matrix **A** is r, therefore by 2.12 of chapter 2 there exists a non-singular matrix **R** such that

$$\mathbf{RA} = \begin{bmatrix} \mathbf{G} \\ \mathbf{O} \end{bmatrix},$$

where **G** is an $r \times n$ matrix.

Now we know that pre-multiplication by a non-singular matrix does not alter the row rank of a matrix.

$\therefore$ Row rank (**RA**) = Row rank **A** = s.

But the matrix **RA** has only r non-zero rows. Therefore the row rank of **RA** can, at the most, be equal to r.

$\therefore$ $s \leq r$

Hence $r = s$.

Theorem 2. *The column rank of a matrix is the same as its rank.*

Proof: Let the matrix **A′** be the transpose of the matrix **A**. Then the columns of **A** are the rows of **A′**.

$\therefore$ the column rank of **A** = the row rank of **A′** = the rank of **A′** = the rank of **A**.

Thus from theorems 1 and 2, we conclude that the rank, row rank and column rank of a matrix are all equal. In other words we can say that *the maximum number of linearly* independent rows of a matrix is equal to the maximum number of its linearly independent columns and is equal to the rank of the matrix.

Thus we have proved that the row rank, the column rank and the rank of a matrix are all equal. Therefore sometimes *the rank of a matrix is also defined as the maximum number of linearly independent row vectors or column vectors.*

We shall now first consider systems of linear homogeneous equations and then proceed to discuss systems of non-homogeneous linear equations.

3.15 Homogeneous Linear Equations

Suppose
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\} \quad \ldots(1)$$

is a system of m homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$. Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n \times 1}, \quad \mathbf{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}_{m \times 1},$$

where **A, X, O** are $m \times n, n \times 1, m \times 1$ matrices respectively. Then obviously we can write the system of equations (1) in the form of a single matrix equation

$$\mathbf{AX} = \mathbf{O}. \qquad \ldots(2)$$

The matrix **A** is called the **coefficient matrix** of the system of equations (1).

Obviously $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ *i.e.*, $\mathbf{X} = \mathbf{O}$ is a solution of (1). It is a trivial (self-obvious) solution of (1).

Again suppose $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are two solutions of* (2). *Then their linear combination* $k_1\mathbf{X}_1 + k_2\mathbf{X}_2$, *where* k_1 *and* k_2 *are any arbitrary numbers, is also a solution of* (2).

We have
$$\mathbf{A}(k_1\mathbf{X}_1 + k_2\mathbf{X}_2) = k_1(\mathbf{AX}_1) + k_2(\mathbf{AX}_2)$$
$$= k_1\mathbf{O} + k_2\mathbf{O} = \mathbf{O}. \qquad [\because \mathbf{AX}_1 = \mathbf{O} \text{ and } \mathbf{AX}_2 = \mathbf{O}]$$

Hence $k_1\mathbf{X}_1 + k_2\mathbf{X}_2$ is also a solution of (2).

Therefore the collection of all the solutions of the system of equations $\mathbf{AX} = \mathbf{O}$ *forms a sub-space of the n-vector space* V_n.

Theorem: *The number of linearly independent solutions of the system of m homogeneous linear equations in n variables,* $\mathbf{AX} = \mathbf{O}$, *is* $(n - r)$, *where r is the rank of the matrix* **A**.

Proof:

Let
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}, \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n \times 1}.$$

Since the rank of the coefficient matrix **A** is r, therefore it has r linearly independent columns. Without loss of generality we can suppose that the first r columns from the left of the matrix **A** are linearly independent, because it amounts only to renaming the components of **X**.

The matrix **A** can be written as $\mathbf{A} = [\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r, \ldots \mathbf{C}_n]_{1 \times n}$, where $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ are the column vectors of the matrix **A** each of them being an m-vector.

The equation $\mathbf{AX} = \mathbf{O}$ can now be written as the vector equation

$$x_1\mathbf{C}_1 + x_2\mathbf{C}_2 + \ldots + x_r\mathbf{C}_r + x_{r+1}\mathbf{C}_{r+1} + \ldots + x_n\mathbf{C}_n = \mathbf{O}. \qquad \ldots(1)$$

Since each of the vectors $\mathbf{C}_{r+1}, \mathbf{C}_{r+2}, \ldots, \mathbf{C}_n$ is a linear combination of the vectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$, therefore we have relations of the type

$$\left.\begin{array}{l} \mathbf{C}_{r+1} = p_{11}\mathbf{C}_1 + p_{12}\mathbf{C}_2 + \ldots + p_{1r}\mathbf{C}_r, \\ \mathbf{C}_{r+2} = p_{21}\mathbf{C}_1 + p_{22}\mathbf{C}_2 + \ldots + p_{2r}\mathbf{C}_r, \\ \ldots \ldots \ldots \ldots \ldots \ldots \\ \mathbf{C}_n = p_{k1}\mathbf{C}_1 + p_{k2}\mathbf{C}_2 + \ldots + p_{kr}\mathbf{C}_r, \text{where } k = n - r \end{array}\right\} \quad \ldots(2)$$

The relations (2) can be written in the form

$$\left.\begin{array}{l} p_{11}\mathbf{C}_1 + p_{12}\mathbf{C}_2 + \ldots + p_{1r}\mathbf{C}_r - 1.\, \mathbf{C}_{r+1} + 0.\mathbf{C}_{r+2} + \ldots + 0.\mathbf{C}_n = \mathbf{O}, \\ p_{21}\mathbf{C}_1 + p_{22}\mathbf{C}_2 + \ldots + p_{2r}\mathbf{C}_r + 0.\mathbf{C}_{r+1} - 1.\mathbf{C}_{r+2} + \ldots + 0.\mathbf{C}_n = \mathbf{O}, \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ p_{k1}\mathbf{C}_1 + p_{k2}\mathbf{C}_2 + \ldots + p_{kr}\mathbf{C}_r + 0.\mathbf{C}_{r+1} + 0.\mathbf{C}_{r+2} + \ldots\ldots - 1.\mathbf{C}_n = \mathbf{O} \end{array}\right\} \quad \ldots(3)$$

Comparing (1) and (3), we find that the vectors

$$\mathbf{X}_1 = \begin{bmatrix} p_{11} \\ p_{12} \\ \ldots \\ \ldots \\ p_{1r} \\ -1 \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}, \mathbf{X}_2 = \begin{bmatrix} p_{11} \\ p_{22} \\ \ldots \\ \ldots \\ p_{2r} \\ 0 \\ -1 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}, \ldots, \mathbf{X}_{n-r} = \begin{bmatrix} p_{k1} \\ p_{k2} \\ \ldots \\ \ldots \\ p_{kr} \\ 0 \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \end{bmatrix}$$

are $(n - r)$ solutions of the equation $\mathbf{AX} = \mathbf{O}.$

The vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-r}$ form a linearly independent set. For, if we have a relation of type

$$l_1\mathbf{X}_1 + l_2\mathbf{X}_2 + \ldots + l_{n-r}\mathbf{X}_{n-r} = \mathbf{O}, \quad \ldots(4)$$

then comparing the $(r+1)^{th}, (r+2)^{th}, \ldots, n^{th}$ components on both sides of (4), we get

$$-l_1 = 0,\ -l_2 = 0, \ldots, -l_{n-r} = 0,$$

i.e., the vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-r}$ are linearly independent.

It can now be easily seen that every solution of the equation $\mathbf{AX} = \mathbf{O}$ is some suitable linear combination of these $n - r$ solutions $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-r}$.

Suppose the vector $\mathbf{X}$, with components $x_1, x_2, \ldots, x_n$ is any solution of the equation $\mathbf{AX} = \mathbf{O}$. The the vector

$$\mathbf{X} + x_{r+1}\mathbf{X}_1 + x_{r+2}\mathbf{X}_2 + \ldots + x_n\mathbf{X}_{n-r}, \qquad \ldots(5)$$

which, being a linear combination of solutions, is also a solution. It is quite obvious that the last $n - r$ components of the vector (5) are all equal to zero. Let $z_1, z_2, \ldots, z_r$ be the first r components of the vector (5). Then the vector whose components are $(z_1, z_2, \ldots, z_r, 0, 0, \ldots, 0)$ is a solution of the equation

$$\mathbf{AX} = \mathbf{O}.$$

Therefore from (1), we have

$$z_1\mathbf{C}_1 + z_2\mathbf{C}_2 + \ldots + z_r\mathbf{C}_r = \mathbf{O}.$$

But the vectors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$

are linearly independent. Therefore, we have $z_1 = 0, z_2 = 0, \ldots, z_r = 0$. Hence (5) is a zero vector.

Therefore

$$\mathbf{X} = - x_{r+1}\mathbf{X}_1 - x_{r+2}\mathbf{X}_2 - \ldots - x_n\mathbf{X}_{n-r}.$$

Thus every solution $\mathbf{X}$ is a linear combination of the $n - r$ linearly independent solutions $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-r}$.

Therefore the set of solutions

$$\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{n-r}\}$$

forms a basis of the vector space of all the solutions of the system of equations $\mathbf{AX} = \mathbf{O}$.

3.16 Some Important Conclusions About the Nature of Solutions of the Equation $\mathbf{AX} = \mathbf{O}$

Suppose we have m equations in n unknowns. Then the coefficient matrix $\mathbf{A}$ will be of the type $m \times n$. Let r be the rank of the matrix $\mathbf{A}$. Obviously r cannot be greater than n (the number of columns of the matrix $\mathbf{A}$). Therefore we have either $r = n$ or $r < n$.

Case I. If $r = n$, the equation $\mathbf{AX} = \mathbf{O}$ will have $n - n$ *i.e.*, no linearly independent solutions. In this case the zero solution will be the only solution. We know that zero vector forms a linearly dependent set.

Case II. If $r < n$, we shall have $n - r$ linearly independent solutions. Any linear combination of these $n - r$ solutions will also be a solution of $\mathbf{AX} = \mathbf{O}$. Thus in this case the equation $\mathbf{AX} = \mathbf{O}$ will have an infinite number of solutions.

Case III. Suppose $m < n$ *i.e.*, the number of equations is less than the number of unknowns. Since $r \leq m$, therefore r is definitely less than n. Hence in this case the given system of equations must possess a non-zero solution. The number of solutions of the equation $\mathbf{AX} = \mathbf{O}$ will be infinite.

3.17 Fundamental Set of Solutions of the Equation $\mathbf{AX} = \mathbf{O}$

Suppose the rank r of the coefficient matrix $\mathbf{A}$ is less than the number of the unknowns n. In this case the given equations have a set of $n - r$ linearly independent solutions and every possible solution is a linear combination of these $n - r$ solutions. This set of $n - r$ solutions is called a fundamental set of solutions of the equation $\mathbf{AX} = \mathbf{O}$.

Definition: *A set of linearly independent solutions* $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ *of the system of homogeneous equations* $\mathbf{AX} = \mathbf{O}$ *is called the fundamental system of solutions of* $\mathbf{AX} = \mathbf{O}$, *if every solution* $\mathbf{X}$ *of* $\mathbf{AX} = \mathbf{O}$ *can be written as a linear combination of these vectors i.e., in the form*

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \ldots + c_k\mathbf{X}_k,$$

where $c_1, c_2, \ldots, c_k$ *are suitable numbers.*

3.18 Working Rule for Finding the Solutions of the Equation $\mathbf{AX} = \mathbf{O}$

Reduce the coefficient matrix $\mathbf{A}$ to **Echelon form** by applying **elementary row transformations only.** This Echelon form will help us to know the rank of the matrix **A.** Suppose the matrix $\mathbf{A}$ is of the type $m \times n$ and its rank comes out to be r. If $r < m$, then in the process of reducing the matrix $\mathbf{A}$ to Echelon form, $(m - r)$ equations will be eliminated. The given system of m equations will thus be replaced by an equivalent system of r equations. Solving these r equations (by Cramer's rule or otherwise), we can express the values of some r unknowns in terms of the remaining $n - r$ unknowns. These $n - r$ unknowns can be given any arbitrarily chosen values.

If $r = n$, the zero solution (trivial solution) will be the only solution. If $r < n$, there will be an infinity of solutions.

Illustrative Examples

Example 5: *Does the following system of equations possess a common non-zero solution?*

$$x+2y+3z=0, 3x+4y+4z=0, 7x+10y+12z=0. \qquad \textbf{(Lucknow 2005)}$$

Solution : The given system of equations can be written in the form of the single matrix equation

$$\mathbf{AX}=\begin{bmatrix}1 & 2 & 3\\ 3 & 4 & 4\\ 7 & 10 & 12\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}=\mathbf{O}.$$

We shall start reducing the coefficient matrix **A** to triangular form by applying only E-row transformations on it. Applying $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 7R_1$, the given system of equations is equivalent to

$$\begin{bmatrix}1 & 2 & 3\\ 0 & -2 & -5\\ 0 & -4 & -9\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\mathbf{O}.$$

Here we find that the determinant of the matrix on the left hand side of this equation is not equal to zero. Therefore the rank of this matrix is 3. So there is no need of further applying E-row transformations on the coefficient matrix. The rank of the coefficient matrix **A** is 3, *i.e.*, equal to the number of unknowns. Therefore the given system of equations does not possess any linearly independent solution. The zero solution, *i.e.*, $x=y=z=0$ is the only solution of the given system of equations.

Example 6: *Solve completely the system of equations*

$$x+3y-2z=0, 2x-y+4z=0, x-11y+14z=0. \qquad \textbf{(Meerut 2005B)}$$

Solution : The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX}=\begin{bmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\mathbf{O}.$$

We shall reduce the coefficient matrix **A** to Echelon form by applying only E-row operations on it. Performing $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$, we have

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{O}.$$

Performing $R_3 \to R_3 - 2R_2$, we have

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{O}.$$

The coefficient matrix is now triangular. The coefficient matrix being of rank 2, the given system of equations possess $3-2=1$ linearly independent solution. We shall assign arbitrary values to $n-r=3-2=1$ variable and the remaining $r=2$ variables shall be found in terms of these. The given system of equations is equivalent to

$$x+3y-2z=0, -7y+8z=0.$$

Thus $\quad y=\frac{8}{7}z, x=-\frac{10}{7}z.$

Choose $\quad z=c.$

Then $\quad y=\frac{8}{7}c, x=-\frac{10}{7}c.$

Hence $\quad x=-\frac{10}{7}c, y=\frac{8}{7}c, z=c$

constitute the general solution of the given system, where c is an arbitrary parameter.

Remark. In matrix form, the general solution can be expressed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{10}{7}c \\ \frac{8}{7}c \\ c \end{bmatrix} = c\begin{bmatrix} -\frac{10}{7} \\ \frac{8}{7} \\ 1 \end{bmatrix}, \text{ where } c \text{ is an arbitrary number.}$$

Example 7: *Does the following system of equations possess a common non-zero solution?*

$$x+y+z=0, 2x-y-3z=0, 3x-5y+4z=0, x+17y+4z=0.$$

Solution: The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{O}.$$

We shall first find the rank of the coefficient matrix **A** by reducing it to Echelon form by applying elementary row transformations only.

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$, we get

$$\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -8 & 1 \\ 0 & 16 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & -24 & 3 \\ 0 & 48 & 9 \end{bmatrix} \text{by } R_3 \to 3R_3, R_4 \to 3R_4$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & -71 \end{bmatrix} \text{by } R_3 \to R_3 - 8R_2, R_4 \to R_4 + 16R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 43 \\ 0 & 0 & 0 \end{bmatrix} \text{by } R_4 \to R_4 + \frac{71}{43} R_3.$$

Above is the Echelon form of the coefficient matrix **A.** We have rank **A** = the number of non-zero rows in this Echelon form = 3. The number of unknowns is also 3. Since rank **A** is equal to the number of unknowns, therefore the given system of equations does not possess any linearly independent solution. Thus the given system of equations possesses no non-zero solution. Hence the zero solution *i.e.*, $x = y = z = 0$ is the only solution of the given system of equations.

Example 8: *Find all the solutions of the following system of equations :*

$$3x + 4y - z - 6w = 0, \; 2x + 3y + 2z - 3w = 0,$$

$$2x + y - 14z - 9w = 0, \; x + 3y + 13z + 3w = 0.$$

Solution: The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \mathbf{O}.$$

We shall first find the rank of the coefficient matrix **A** by reducing it to Echelon form by applying E-row transformations only.

Applying $R_4 \leftrightarrow R_1$, we get

$$\mathbf{A} \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -20 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix}$$

by $R_2 \to -\frac{1}{3} R_2,\ R_3 \to -\frac{1}{5} R_3, R_4 \to -\frac{1}{5} R_4$

$$\sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{by } R_3 \to R_3 - R_2,\ R_4 \to R_4 - R_2.$$

The rank of **A** is obviously 2 which is less than the number of unknowns 4. Therefore the given system of equations possesses $4 - 2$, *i.e.*, 2 linearly independent solutions. The given system of equations is equivalent to the equation

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the given system of four equations is equivalent to the system of two equations, *i.e.*,

$$\left. \begin{aligned} x + 3y + 13z + 4w &= 0 \\ y + 8z + 3w &= 0 \end{aligned} \right\}.$$

From these equations, we get

$$y = -8z - 3w, x = -3(-8z - 3w) - 13z - 3w$$

i.e., $$y = -8z - 3w, x = 11z + 6w.$$

Hence $x = 11c_1 + 6c_2, y = -8c_1 - 3c_2, z = c_1, w = c_2$ constitute the general solution of the given system of equations, where c_1 and c_2 are arbitrary numbers. Since we can give any arbitary values to c_1 and c_2, therefore the given system of equations have an infinite number of solutions.

Remark: In matrix form, the general solution of the given system of equations can be expressed as

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11c_1 + 6c_2 \\ -8c_1 - 3c_2 \\ 1c_1 + 0c_2 \\ 0c_1 + 1c_2 \end{bmatrix} = c_1 \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Here $$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

are two linearly independent solutions of the given system of equations and all their linear combinations will also be the solutions of the given system of equations.

Example 9: *Show that the only real value of* λ *for which the following equations have non-zero solutions is* 6 :

$$x + 2y + 3z = \lambda x,\ 3x + y + 2z = \lambda y,\ 2x + 3y + z = \lambda z.$$

(Kanpur 2011; Bundelkhand 10; Lucknow 10)

Solution: The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{O}.$$

If the given system of equations is to possess a non-zero solution, the coefficient matrix **A** must be of rank less than 3. If the matrix **A** is to be of rank less than 3 its determinant must be equal to zero. Thus we have

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0,$$

adding the second and the third rows to the first

or $$(6-\lambda) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

or $$(6-\lambda) \begin{vmatrix} 1 & 1 & 0 \\ 3 & -\lambda-2 & -1 \\ 1 & 1 & -\lambda-1 \end{vmatrix} = 0, C_2 - C_1, C_3 - C_1$$

or $$(6-\lambda)\,[(\lambda+2)\,(\lambda+1)+1] = 0$$

or $$(6-\lambda)\,[\lambda^2 + 3\lambda + 3] = 0.$$

The roots of the equation $\lambda^2 + 3\lambda + 3 = 0$ are $\lambda = \dfrac{-3 \pm \sqrt{(9-12)}}{2}$ *i.e.*, are imaginary.

Hence the only real value of λ for which the system of equations is to have a non-zero solution is 6.

Comprehensive Exercise 1

Find all the solutions of the following system of linear homogeneous equations :

1. $2x - 3y + z = 0, x + 2y - 3z = 0, 4x - y - 2z = 0.$
2. $x + y - 3z + 2w = 0, 2x - y + 2z - 3w = 0, 3x - 2y + z - 4w = 0,$
   $-4x + y - 3z + w = 0.$
3. $x + y + z = 0, 2x + 5y + 7z = 0, 2x - 5y + 3z = 0.$
4. $x + 2y + 3z = 0, 2x + 3y + 4z = 0, 7x + 13y + 19z = 0.$
5. $4x + 2y + z + 3u = 0, 6x + 3y + 4z + 7u = 0, 2x + y + u = 0.$
6. $2x - 2y + 5z + 3w = 0, 4x - y + z + w = 0, 3x - 2y + 3z + 4w = 0,$
   $x - 3y + 7z + 6w = 0.$
7. $x - 2y + z - w = 0, x + y - 2z + 3w = 0, 4x + y - 5z + 8w = 0,$
   $5x - 7y + 2z - w = 0$

Answers 1

1. $x = 0, y = 0, z = 0$
2. $x = 0, y = 0, z = 0, w = 0$
3. $x = 0, y = 0, z = 0$
4. $x = 2c, y = -2c, z = c$
5. $x = c_1, u = c_2, y = -2c_1 - c_2, z = -c_2$
6. $x = \frac{5}{9}c, y = 4c, z = \frac{7}{9}c, w = c$
7. $x = c_1 - \frac{5}{3}c_2, y = c_1 - \frac{4}{3}c_2, z = c_1, w = c_2$

3.19 System of Linear Non-homogeneous Equations

Sometimes we think that we can solve every two simultaneous equations of the type

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}.$$

But it is not so. For example, consider the simultaneous equations

$$\left.\begin{aligned} 3x + 4y &= 5 \\ 6x + 8y &= 13 \end{aligned}\right\}.$$

There is no set of values of x and y which satisfies both these equations. Such equations are said to be **inconsistent.**

Let us take another example. Consider the simultaneous equations

$$\left.\begin{array}{l} 3x + 4y = 5 \\ 6x + 8y = 10 \end{array}\right\}.$$

These equations are consistent since there exist values of x and y which satisfy both of these equations. We see that $x = -\frac{4}{3}c + \frac{5}{3}, y = c$ constitute a solution of these equations, where c is arbitrary. Thus these equations possess an infinite number of solutions.

Now we shall discuss the nature of solutions of a system of non-homogeneous linear equations :

Let
$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + .. + a_{mn}x_n = b_m \end{array}\right\} \qquad \ldots(1)$$

be a system of m non-homogeneous equations in n unknown $x_1, x_2, \ldots, x_n$.

If we write

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m \times n}, \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}_{n \times 1}, \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}_{m \times 1}$$

where **A, X, B** are $m \times n, n \times 1$ and $m \times 1$ matrices respectively, the above equations can be written in the form of a single matrix equation $\mathbf{AX} = \mathbf{B}$.

Any set of values of $x_1, x_2, \ldots, x_n$ which simultaneously satisfy all these equations is called a solution of the system (1). When the system of equations has one or more solutions, the equations are said to be ***consistent*** *otherwise they are said to be* ***inconsistent.***

The matrix

$$[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of the given system of equations.

3.20 Condition for Consistency

Theorem. *The system of equations* $\mathbf{AX} = \mathbf{B}$ *is consistent i.e., possesses a solution, if any only if the coefficient matrix* $\mathbf{A}$ *and the augmented matrix* $[\mathbf{A} \quad \mathbf{B}]$ *are of the same rank.*

[Meerut 2007B; Kanpur 11; Lucknow 05]

Proof. Let $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ denote the column vectors of the matrix $\mathbf{A}$. The equation $\mathbf{AX} = \mathbf{B}$ is then equivalent to

$$[\mathbf{C}_1, \mathbf{C}_2, \ldots., \mathbf{C}_n] \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix} = \mathbf{B} \quad i.e., \quad x_1\mathbf{C}_1 + x_2\mathbf{C}_2 + \ldots + x_n\mathbf{C}_n = \mathbf{B}. \qquad \ldots(1)$$

Let now r be the rank of the matrix $\mathbf{A}$. The matrix $\mathbf{A}$ has then r linearly independent columns and without loss of generality, we can suppose that the first r columns $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$ form a linearly independent set so that each of the remaining $n - r$ columns is a linear combination of these r columns.

The condition is necessary. If the given system of equations is consistent, there must exist n scalars (numbers) $k_1, k_2, \ldots, k_n$ such that

$$k_1\mathbf{C}_1 + k_2\mathbf{C}_2 + \ldots\ldots + k_n\mathbf{C}_n = \mathbf{B}. \qquad \ldots(2)$$

Since each of the $n - r$ columns $\mathbf{C}_{r+1}, \mathbf{C}_{r+2}, \ldots., \mathbf{C}_n$ is a linear combination of first r columns $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$ it is obvious from (2) that $\mathbf{B}$ is also a linear combination of $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$. Thus the maximum number of linearly independent columns of the matrix $[\mathbf{A} \quad \mathbf{B}]$ is also r. Therefore the matrix $[\mathbf{A} \quad \mathbf{B}]$ is also of rank r. Hence the matrices $\mathbf{A}$ and $[\mathbf{A} \quad \mathbf{B}]$ are of the same rank.

The condition is sufficient. Now suppose that the matrices $\mathbf{A}$ and $[\mathbf{A} \quad \mathbf{B}]$ are of the same rank r. The maximum number of linearly independent columns of the matrix $[\mathbf{A} \quad \mathbf{B}]$ is then r. But the first r columns $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_r$ of the matrix $[\mathbf{A} \quad \mathbf{B}]$ already form a linearly independent set. Therefore the column $\mathbf{B}$ should be expressed as a linear combination of the columns $\mathbf{C}_1, \mathbf{C}_2, \ldots., \mathbf{C}_r$.

Thus there exist r scalars $k_1, k_2, \ldots, k_r$ such that

$$k_1\mathbf{C}_1 + k_2\mathbf{C}_2 + \ldots + k_r\mathbf{C}_r = \mathbf{B}. \qquad \ldots(3)$$

Now (3) may be written as

$$k_1\mathbf{C}_1 + k_2\mathbf{C}_2 + \ldots + k_r\mathbf{C}_r + 0.\mathbf{C}_{r+1} + 0.\mathbf{C}_{r+2} + \ldots + 0.\mathbf{C}_n = \mathbf{B}. \qquad \ldots(4)$$

Comparing (1) and (4), we see that

$$x_1 = k_1, x_2 = k_2, \ldots., x_r = k_r, x_{r+1} = 0, x_{r+2} = 0, \ldots, x_n = 0$$

consitute a solution of the equation **AX** = **B**.

Therefore the given system of equations is consistent.

3.21 Condition for a System of n Equations in n Unknowns to have a Unique Solution

Theorem: *If* **A** *be an n-rowed non-singular matrix,* **X** *be an* $n \times 1$ *matrix,* **B** *be an* $n \times 1$ *matrix, the system of equations* **AX** = **B** *has a unique solution.*

Proof: If **A** be an n-rowed non-singular matrix, the ranks of the matrices **A** and [**A** **B**] are both n. Therefore the system of equations **AX** = **B** is consistent *i.e.*, possesses a solution.

Pre-multiplying both sides of **AX** = **B** by $\mathbf{A}^{-1}$, we have

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B} \quad i.e., \quad \mathbf{IX} = \mathbf{A}^{-1}\mathbf{B} \quad i.e., \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

is a solution of the equation **AX = B.**

To show that the solution is unique, let us suppose that $\mathbf{X}_1$ and $\mathbf{X}_2$ be two solutions of **AX** = **B**.

Then $$\mathbf{AX}_1 = \mathbf{B}, \mathbf{AX}_2 = \mathbf{B} \Rightarrow \mathbf{AX}_1 = \mathbf{AX}_2 \Rightarrow \mathbf{A}^{-1}\mathbf{AX}_1 = \mathbf{A}^{-1}\mathbf{AX}_2$$

$$\Rightarrow \quad \mathbf{IX}_1 = \mathbf{IX}_2 \Rightarrow \mathbf{X}_1 = \mathbf{X}_2.$$

Hence the solution is unique.

3.22 Working Rule for Finding the Solution of the Equation AX = B

Suppose the coefficient matrix **A** is of the type $m \times n$, *i.e.*, we have m equations in n unknowns. Write the augmented matrix [**A** **B**] and reduce it to Echelon form by applying only E-row transformations on it. This Echelon form will enable us to know the ranks of the augmented matrix [**A** **B**] and the coefficient matrix **A.** Then the following different cases arise :

Case I. Rank **A** < Rank [**A** **B**].

In this case the equations **AX** = **B** are inconsistent *i.e.*, they have no solution.

Case II. Rank **A** = Rank **[A B]** = r (say).

In this case the equations **AX** = **B** are consistent *i.e.*, they possess a solution. If $r < m$, then in the process of reducing the matrix [**A** **B**] to Echelon form, $(m - r)$ equations will be eliminated. The given system of m equations will then be replaced by an equivalent system of r equations. From these r equations we shall be able to express the values of some r unknowns in terms of the remaining $n - r$ unknowns which can be given any arbitrarily chosen values.

If $r = n$, then $n - r = 0$, so that no variable is to be assigned arbitrary values and therefore in this case there will be a unique solution.

If $r < n$, then $n - r$ variables can be assigned arbitrary values. So in this case there will be an infinite number of solutions. Only $n - r + 1$ solutions will be linearly independent and the rest of the solutions will be linear combinations of them.

If $m < n$, then $r \le m < n$. Thus in this case $n - r > 0$. Therefore when the number of equations is less than the number of unknowns, the equations will always have an infinite number of solutions, provided they are consistent.

Illustrative Examples

Example 10: *Show that the equations*

$$x + y + z = -3,\ \ 3x + y - 2z = -2,\ \ 2x + 4y + 7z = 7$$

are not consistent. **(Avadh 2008; Purvanchal 07)**

Solution : The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} = \mathbf{B}.$$

The augmented matrix $[\mathbf{A} \ \ \mathbf{B}] = \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 3 & 1 & -2 & : & -2 \\ 2 & 4 & 7 & : & 7 \end{bmatrix}.$

We shall reduce the augmented matrix [**A** **B**] to Echelon form by applying *E*-row transformations only. Applying

$$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 2R_1, \text{ we get}$$

$$[\mathbf{A} \quad \mathbf{B}] \sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 2 & 5 & : & 13 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 0 & 0 & : & 20 \end{bmatrix}, \text{aplying } R_3 \to R_3 + R_2.$$

Above is the Echelon form of the matrix $[\mathbf{A} \quad \mathbf{B}]$. We have rank $[\mathbf{A} \quad \mathbf{B}]$ = the number of non-zero rows in this Echelon form = 3

Also by the same *E*-row transformations, we get

$$\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously rank $\mathbf{A} = 2$.

Since Rank $\mathbf{A} \neq$ Rank $[\mathbf{A} \quad \mathbf{B}]$, therefore the given equations are inconsistent *i.e.*, they have no solution.

Remark: The inconsistency of the given equations can also be shown as below :

The given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 20 \end{bmatrix}.$$

This matrix equation is equivalent to the system of equations

$$\left.\begin{array}{l} x + y + z = -3 \\ 0x - 2y - 5z = 7 \\ 0x + 0y + 0z = 20 \end{array}\right\}.$$

The last equation shows that $0 = 20$, which is not possible. Hence the given equations are inconsistent.

Example 11: *Show that the equations*

$$x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30$$

are consistent and solve them. **(Meerut 2010 B; Avadh 09)**

Solution : The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = \mathbf{B}.$$

The augmented matrix $[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 1 & 2 & 3 & \vdots & 14 \\ 1 & 4 & 7 & \vdots & 30 \end{bmatrix}$.

We shall reduce the augmented matrix $[\mathbf{A} \quad \mathbf{B}]$ to Echelon form by applying elementary row transformations only. Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$[\mathbf{A} \quad \mathbf{B}] \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 3 & 6 & \vdots & 24 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 6 \\ 0 & 1 & 2 & \vdots & 8 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - 3R_2.$$

Above is the Echelon form of the matrix $[\mathbf{A} \quad \mathbf{B}]$. We have rank $[\mathbf{A} \quad \mathbf{B}]$ = the number of non-zero rows in this Echelon form $= 2$.

By the same elementary transformations, we get $\mathbf{A} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Obviously rank $\mathbf{A} = 2$. Since rank $\mathbf{A}$ = rank $[\mathbf{A} \ \ \mathbf{B}]$, therefore the given equations are consistent. Here the number of unknowns is 3. Since rank $\mathbf{A}$ is less than the number of unknowns therefore the given system will have an infinite number of solutions. We see that the given system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}.$$

This matrix equation is equivalent to the system of equations

$$\left.\begin{array}{l} x + y + z = 6 \\ y + 2z = 8 \end{array}\right\}.$$

$\therefore$ $$y = 8 - 2z, x = 6 - y - z = 6 - (8 - 2z) - z = z - 2.$$

Taking $z = c$, we see that $x = c - 2,\ y = 8 - 2c, z = c$ constitute the general solution of the given system, where c is an arbitrary constant.

Example 12: *Apply the test of rank to examine if the following equations are consistent :*

$$2x - y + 3z = 8, - x + 2y + z = 4, 3x + y - 4z = 0$$

and if consistent, find the complete solution. **(Meerut 2007)**

Solution : The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = \mathbf{B}.$$

The augmented matrix

$$[\mathbf{A} \quad \mathbf{B}] = \begin{bmatrix} 2 & -1 & 3 & \vdots & 8 \\ -1 & 2 & 1 & \vdots & 4 \\ 3 & 1 & -4 & \vdots & 0 \end{bmatrix}.$$

We shall reduce the augmented matrix to Echelon form by applying elementary row transformations only. Applying $R_1 \leftrightarrow R_2$, we get

$$[\mathbf{A} \quad \mathbf{B}] \sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 2 & -1 & 3 & \vdots & 8 \\ 3 & 1 & -4 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 7 & -1 & \vdots & 12 \end{bmatrix}, \text{ by } R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1$$

$$\begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 21 & -3 & \vdots & 36 \end{bmatrix}, \text{ by } R_3 \to 3R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & -38 & \vdots & -76 \end{bmatrix}, \text{ by } R_3 \to R_3 - 7R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 1 & \vdots & 4 \\ 0 & 3 & 5 & \vdots & 16 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}, \text{ by } R_3 \to -\frac{1}{38} R_3.$$

Above is the Echelon form of the matrix [**A** **B**]. We have the rank [**A** **B**] = the number of non-zero rows in this Echelon form = 3.

By the same transformations, we get $\mathbf{A} \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$.

Obviously rank **A** = 3. Since rank **A** = rank [**A** **B**], therefore the given equations are consistent.Here the number of unknowns is 3. Since rank **A** is equal to the number of unknowns, therefore the given equations have a unique solution. We see that the given equations are equivalent to the matrix equation

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 2 \end{bmatrix}.$$

This matrix equation is equivalent to the equations

$$-x + 2y + z = 4,\ 3y + 5z = 16, z = 2.$$

These give $z = 2, y = 2, x = 2.$

Example 13: *Show that the equations*

$$x + 2y - z = 3, 3x - y + 2z = 1, 2x - 2y + 3z = 2, x - y + z = -1$$

are consistent and solve them. **(Meerut 2006B, 09; Rohilkhand 06)**

Solution: The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \mathbf{B}.$$

The augmented matrix

$$[\mathbf{A}\ \mathbf{B}] = \begin{bmatrix} 1 & 2 & -1 & : & 3 \\ 3 & -1 & 2 & : & 1 \\ 2 & -2 & 3 & : & 2 \\ 1 & -1 & 1 & : & -1 \end{bmatrix}.$$

Performing $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - R_1$, we get

$$[\mathbf{A}\ \mathbf{B}] \sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -7 & 5 & \vdots & -8 \\ 0 & -6 & 5 & \vdots & -4 \\ 0 & -3 & 2 & \vdots & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & -6 & 5 & \vdots & -4 \\ 0 & -3 & 2 & \vdots & -4 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 5 & \vdots & 20 \\ 0 & 0 & 2 & \vdots & 8 \end{bmatrix}, \text{ by } R_3 \to R_3 - 6R_2,\ R_4 \to R_4 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 4 \\ 0 & 0 & 1 & \vdots & 4 \end{bmatrix}, \text{ by } R_3 \to \frac{1}{5}R_3,\ R_4 \to \frac{1}{2}R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & -1 & 0 & \vdots & -4 \\ 0 & 0 & 1 & \vdots & 4 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}, \text{ by } R_4 \to R_4 - R_3.$$

Thus the matrix [**A B**] has been reduced to Echelon form. We have rank [**A B**] = the number of non-zero rows in this Echelon form = 3. Also

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have rank **A** = 3. Since rank [**A B**] = rank **A**, therefore the given equations are consistent. Since rank **A** = 3 = the number of unknowns, therefore the given equations have unique solution. The given equations are equivalent to the equations

$$x + 2y - z = 3, -y = -4, z = 4.$$

These give $\qquad z = 4,\ y = 4,\ x = -1.$

Example 14: *State the conditions under which a system of non-homogeneous equations will have (i) no solution (ii) a unique solution (iii) infinity of solutions.* **(Lucknow 2008)**

Solution : Let $\mathbf{AX} = \mathbf{B}$ be a system of linear non-homogeneous equations, where $\mathbf{A}, \mathbf{X}, \mathbf{B}$ are $m \times n, n \times 1, m \times 1$ matrices respectively.

(i) These equations will have no solution if the coefficient matrix $\mathbf{A}$ and the augmented matrix $[\mathbf{A}\ \mathbf{B}]$ are not of the same rank.

(ii) These equations will possess a unique solution if the matrices $\mathbf{A}$ and $[\mathbf{A}\ \mathbf{B}]$ are of the same rank and the rank is equal to the number of variables. In particular if $\mathbf{A}$ is a square matrix, these equations will possess a unique solution if and only if the matrix $\mathbf{A}$ is non-singular.

(iii) These equations will have infinity of solutions if the matrices $\mathbf{A}$ and $[\mathbf{A}\ \mathbf{B}]$ are of the same rank and the rank is less than the number of variables.

Example 15: *By the use of matrices, solve the equations :*

$$x + y + z = 9, \quad 2x + 5y + 7z = 52, \quad 2x + y - z = 0.$$

(Bundelkhand 2011; Avadh 10; Rohilkhand 10)

Solution : The given system of equations is equivalent to the single matrix equation

$$\mathbf{AX} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix} = \mathbf{B}.$$

The augmented matrix

$$[\mathbf{A}\ \mathbf{B}] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 2 & 5 & 7 & \vdots & 52 \\ 2 & 1 & -1 & \vdots & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 0 & 3 & 5 & \vdots & 34 \\ 0 & -1 & -3 & \vdots & -18 \end{bmatrix}, \text{by } R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 0 & -1 & -3 & \vdots & -18 \\ 0 & 3 & 5 & \vdots & 34 \end{bmatrix}, \text{ by } R_2 \longleftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & 9 \\ 0 & -1 & -3 & \vdots & -18 \\ 0 & 0 & -4 & \vdots & -20 \end{bmatrix}, \text{ by } R_3 \to R_3 + 3R_2.$$

Above is the Echelon form of the matrix [**A B**]. We have rank [**A B**] = the number of non-zero rows in this Echelon form = 3.

Also by the same *E*-row transformations, we get

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix}.$$

$\therefore$ rank **A** = 3.

Since rank **A** = rank [**A B**], therefore the given equations are consistent. Also rank **A** = 3 and the number of unknowns is also 3. Hence the given equations will have a unique solution. To find the solution we see that the given system of equations is equivalent to the matric equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix}.$$

This matrix equation is equivalent to the system of equations

$$x + y + z = 9, - y - 3z = - 18, - 4z = - 20.$$

Solving these, we get $z = 5, y = 3, x = 1$.

Example 16: *Investigate for what values of* λ, μ *the simultaneous equations*

$$x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$$

have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

(Meerut 2006, 09B; Bundelkhand 09; Rohilkhand 07; Kanpur 09)

Solution : The matrix form of the given system of equations is

$$\mathbf{AX} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = \mathbf{B}.$$

The augmented matrix $[\mathbf{A}\ \ \mathbf{B}] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix} \text{by } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix} \text{by } R_3 \to R_3 - R_2.$$

If $\lambda \neq 3$, we have rank $[\mathbf{A}\ \ \mathbf{B}] = 3 =$ rank $\mathbf{A}$.

So in this case the given system of equations is consistent. Since rank $\mathbf{A}$ = the number of unknowns, therefore the given system of equations possesses a unique solution. Thus if $\lambda \neq 3$, the given system of equations possesses a unique solution for any value of μ.

If $\lambda = 3$ and $\mu \neq 10$, we have rank $[\mathbf{A}\ \ \mathbf{B}] = 3$ and rank $\mathbf{A} = 2$. Thus in this case rank $[\mathbf{A}\ \ \mathbf{B}] \neq$ rank $\mathbf{A}$ and so the given system of equations is inconsistent *i.e.*, possesses no solution.

If $\lambda = 3$ and $\mu = 10$, we have rank $[\mathbf{A}\ \ \mathbf{B}] = 2 =$ rank $\mathbf{A}$.

So in this case the given system of equations is again consistent. Since rank $\mathbf{A}$ < the number of unknowns, therefore in this case the given system of equations possesses an infinite number of solutions.

Example 17: *For what values of* η *the equations*

$$x + y + z = 1, x + 2y + 4z = \eta, x + 4y + 10z = \eta^2,$$

have a solution and solve them completely in each case.

(Meerut 2011; Agra 07; Kanpur 06, 08, 10)

Solution : The matrix form of the given system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta \\ \eta^2 \end{bmatrix}.$$

Performing $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta - 1 \\ \eta^2 - 1 \end{bmatrix}.$$

Performing $R_3 \to R_3 - 3R_2$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \eta - 1 \\ \eta^2 - 3\eta + 2 \end{bmatrix} \qquad \ldots(1)$$

Now the given equations will be consistent if and only if

$$\eta^2 - 3\eta + 2 = 0, \textit{ i.e.}, \text{ iff } (\eta - 2)(\eta - 1) = 0, \textit{ i.e.}, \text{ iff } \eta = 2 \text{ or } \eta = 1.$$

Case I. If $\eta = 2$, the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The above system of equations is equivalent to

$$y + 3z = 1, x + y + z = 1.$$

$$\therefore \qquad y = 1 - 3z, x = 2z.$$

Thus, $x = 2k, y = 1 - 3k, z = k$ constitute the general solution where k is an arbitrary constant.

Case II. If $\eta = 1$, the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The above system of equations is equivalent to

$$y + 3z = 0, x + y + z = 1.$$

$$\therefore \qquad y = -3z, x = 1 + 2z.$$

Thus $x = 1 + 2c, y = -3c, z = c$ constitute the general solution, where c is an arbitrary constant.

Comprehensive Exercise 2

1. Use the test of rank to show that the following equations are not consistent :

$$2x - y + z = 4, 3x - y + z = 6, 4x - y + 2z = 7, - x + y - z = 9.$$

(Rohilkhand 2009)

2. Show that the equations $-2x + y + z = a, x - 2y + z = b, x + y - 2z = c$ have no solution unless $a + b + c \neq 0$ **(Kanpur 2009)**

3. Apply the test of rank to examine if the following system of equations is consistent and if consistent, find the complete solution :

$$x + y + z = 6, x + 2y + 3z = 10, x + 2y + 4z = 1.$$

4. Use matrix method to solve the equations :

$$2x - y + 3z = 9, x + y + z = 6, x - y + z = 2.$$ **(Meerut 2005; Avadh 06,11)**

5. Solve completely the equations

$$x + 2y + 3z = 6, 2x + 4y + z = 7, 3x + 2y + 9z = 4.$$

6. Show that the equations $x + 2y - 5z = - 9, 3x - y + 2z = 5, 2x + 3y - z = 3,$ $4x - 5y + z = - 3$ are consistent and solve the same.

7. Show that the equations

$$x - 3y - 8z + 10 = 0, 3x + y - 4z = 0, 2x + 5y + 6z - 13 = 0$$

are consistent and solve the same. **(Bundelkhand 2007)**

8. Solve completely the equations $2x + 3y + z = 9, x + 2y + 3z = 6, 3x + y + 2z = 8.$

9. Solve completely the equations

$$x + y + z = 1, x + 2y + 3z = 4, x + 3y + 5z = 7, x + 4y + 7z = 10.$$ **(Lucknow 2009)**

10. Express the following system of equations into the matrix equation form **AX** = **B**:

$$x + 2y + z = - 1, 6x + y + z = - 4, 2x - 3y - z = 0 , - x - 7y - 2z = 7, x - y = 1.$$

Determine if this system of equations is consistent and if so find its solution.

11. Examine if the system of equations :

$$x + y + 4z = 6, 3x + 2y - 2z = 9, 5x + y + 2z = 13$$

is consistent. Find also the solution if it is consistent.

12. Prove, without actually solving, that the following system of equations has a unique solution :

$$5x + 3y + 14z = 4, y + 2z = 1, x - y + 2z = 0.$$

13. For what values of the parameter λ will the following equations fail to have a unique solution

$$3x - y + \lambda z = 1, 2x + y + z = 2, x + 2y - \lambda z = - 1?$$

Will the equations have any solutions for these values of λ?

14. Solve the equations

$$\lambda x + 2y - 2z - 1 = 0, 4x + 2\lambda y - z - 2 = 0, 6x + 6y + \lambda z - 3 = 0,$$

considering specially the case when $\lambda = 2$.

15. Discuss for all values of λ, the system of equations

$$x + y + 4z = 6, x + 2y - 2z = 6, \lambda x + y + z = 6,$$

as regards existence and nature of solutions.

16. Solve the following equations by matrix method :

$$5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5.$$

17. Solve the following system of linear equations by matrix method :

$$5x - 6y + 4z = 15, 7x + 4y - 3z = 19, 2x + y + 6z = 46.$$

18. Solve the following equations by matrix method :

$$x - y + 2z = 4, 3x + y + 4z = 6, x + y + z = 1.$$

19. Solve the following equations by matrix method :

$x - 2y + 3z = 6, 3x + y - 4z = -7, 5x - 3y + 2z = 5.$ **(Meerut 2010)**

Answers 2

3. $x = -7, y = 22, z = -9$

4. $x = 1, y = 2, z = 3$

5. $x = 1, y = 1, z = 1$

6. $x = \frac{1}{2}, y = \frac{3}{2}, z = \frac{5}{2}$

7. $x = -1 + 2c, y = 3 - 2c, z = c$, where c is arbitrary

8. $x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}$

9. $x = c - 2, y = 3 - 2c, z = c$

10. Consistent; $x = -1, y = -2, z = 4$

11. Consistent ; $x = 2, y = 2, z = \frac{1}{2}$

13. $\lambda \neq \frac{7}{2}$, the solution is unique; $\lambda = -\frac{7}{2}$, no solution

14. In case $\lambda = 2$, the general solution of the given system of equations is given by $x = \frac{1}{2} - c, y = c, z = 0$

15. $\lambda \neq \frac{7}{10}$, the solution is unique; $\lambda = -\frac{7}{10}$, no solution

16. $x = \frac{7}{11} - \frac{16}{11} c, y = \frac{3}{11} + \frac{1}{11} c, z = c$

17. $x = 3, y = 4, z = 6$

18. $x = \frac{5}{2} - \frac{3}{2} c, y = -\frac{3}{2} + \frac{1}{2} c, z = c$

19. $x = \frac{-8}{7} + \frac{5}{7} c, y = -\frac{25}{7} + \frac{13}{7} c, z = c$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. A set of r n-vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots\ldots\ldots, \mathbf{X}_r$ is said to be linearly independent if every relation of the type $k_1 \mathbf{X}_1 + k_2 \mathbf{X}_2 + \ldots\ldots\ldots + k_r \mathbf{X}_r = \mathbf{O}$ implies

   (a) $k_1 + k_2 + \ldots\ldots\ldots + k_r = 0$

   (b) $k_1 = k_2 = \ldots\ldots\ldots = k_r = 0$

   (c) $k_1 + k_2 + \ldots\ldots\ldots + k_r = 1$

   (d) none of these

2. Consider the system of equations

$$a_1 x + b_1 y + c_1 z = 0, a_2 x + b_2 y + c_2 z = 0, a_3 x + b_3 y + c_3 z = 0.$$

   If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the system has

   (a) more than two solutions

   (b) only trivial solution

   (c) no solution

   (d) none of these

3. The system of linear equations $x + y + z = 2, 2x + y - z = 3, 3x + 2y + kz = 4$ has a unique solution if

   (a) $k \neq 0$

(b) $-1 < k < 1$

(c) $-2 < k < 2$

(d) $k = 0$

4. The system of equations $x + 2y + 3z = 1, 2x + y + 3z = 2, 5x + 5y + 9z = 4$ has

(a) only one solution

(b) infinitely many solutions

(c) no solution

(d) none of these

(Meerut 2003)

Fill in the Blank(s)

Fill in the blanks ".........," so that the following statements are complete and correct.

1. The set consisting only of the zero vector, **O**, is linearly ....

2. The set of three vectors $\mathbf{X}_1 = (1, 0, 0), \mathbf{X}_2 = (0, 1, 0), \mathbf{X}_3 = (0, 0, 1)$ is linearly ....

3. The number of linearly independent solutions of the system of m homogeneous linear equations in n variables, $\mathbf{AX} = \mathbf{O}$, is ...., where r is the rank of the matrix **A.**

4. When the system of equations has one or more solutions, the equations are said to be ....., otherwise they are said to be ....

5. The sytem of m non-homogeneous linear equations in n unknowns, $\mathbf{AX} = \mathbf{B}$, is .... if and only if the coefficient matrix **A** and the ungmented matrix [**A B**] are of the same rank. **(Meerut 2001)**

6. If **A** be an n-rowed non-singular matrix, **X** be an $n \times 1$ matrix, **B** be an $n \times 1$ matrix, the system of equations **A X** = **B**, has a ...... solution.

7. If the number of equations is less than the number of unknowns, the equations will aways have an .... number of solutions, provided they are consistent.

True or False

Write 'T' for true and 'F' for false statement.

1. A set of r n-vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots\ldots, \mathbf{X}_r$, is said to be linearly dependent if there exist r scalars $k_1, k_2, \ldots\ldots\ldots, k_r$, not all zero, such that $k_1\mathbf{X}_1 + k_2\mathbf{X}_2 + \ldots\ldots\ldots + k_r\mathbf{X}_r = \mathbf{O}$, where **O** denotes the n-vector whose components are all zero.

2. If a set of vectors is linearly independent, then at least one member of the set can be expressed as a linear combination of the remaining members.

3. The zero solution *i.e.*, $x = y = z = 0$ is the only solution of the system of equations

$$x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0.$$

4. Let $\mathbf{AX} = \mathbf{B}$ be a system of m non-homogeneous linear equations in n unknowns. These equations will have no solution if the coeffcient matrix $\mathbf{A}$ and the augmented matrix $[\mathbf{A}\ \mathbf{B}]$ are not of the same rank.

5. The system of m non-homogeneous linear equations in n unknowns, $\mathbf{AX} = \mathbf{B}$ will possess a unique solution if the matrices $\mathbf{A}$ and $[\mathbf{A}\ \mathbf{B}]$ are of the same rank and the rank is less than the number of variables.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (a) 4. (a)

Fill in the Blank(s)

1. dependent
2. independent
3. $(n - r)$
4. consistent; inconsistent
5. consistent
6. unique
7. infinite

True or False

1. T 2. F 3. T
4. T 5. F

□

Chapter 4

Eigenvalues and Eigenvectors

4.1 Matric Polynomials

Definition: *An expression of the form*

$$F(\lambda) = \mathbf{A}_0 + \mathbf{A}_1\lambda + \mathbf{A}_2\lambda^2 + \ldots + \mathbf{A}_{m-1}\lambda^{m-1} + \mathbf{A}_m\lambda^m,$$

where $\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ *are all square matrices of the same order, is called a* **Matric polynomial** of *degree m provided* $\mathbf{A}_m$ *is not a null matrix. The symbol* λ is called *indeterminate*. If the order of each of the matric coefficients $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ is n, then we say that the matric polynomial is n-rowed. According to this definition of a matric polynomial, each square matrix can be expressed as a matric polynomial with zero degree. For example, if $\mathbf{A}$ be any square matrix, we can write $\mathbf{A} = \lambda^0 \mathbf{A}$.

Equality of Polynomials : Two matric polynomials are equal iff (if and only if), the coefficients of the like powers of λ are the same.

Theorem: *Every square matrix whose elements are ordinary polynomials in* λ*, can essentially be expressed as a matric polynomial in* λ *of degree m, where m is the highest power of* λ *occurring in any element of the matrix.* We shall illustrate this theorem by the following example.

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1+2\lambda+3\lambda^2 & \lambda^2 & 4-6\lambda \\ 1+\lambda^3 & 3+4\lambda^2 & 1-2\lambda+4\lambda^3 \\ 2-3\lambda+2\lambda^3 & 5 & 6 \end{bmatrix}$$

in which the highest power of λ occurring in any element is 3. Rewriting each element as a cubic in λ, supplying missing coefficients with zeros, we get

$$\mathbf{A} = \begin{bmatrix} 1+2.\lambda+3.\lambda^2+0.\lambda^3 & 0+0.\lambda+1.\lambda^2+0.\lambda^3 & 4-6.\lambda+0.\lambda^2+0.\lambda^3 \\ 1+0.\lambda+0.\lambda^2+1.\lambda^3 & 3+0.\lambda+4.\lambda^2+0.\lambda^3 & 1-2.\lambda+0.\lambda^2+4.\lambda^3 \\ 2-3.\lambda+0.\lambda^2+2.\lambda^3 & 5+0.\lambda+0.\lambda^2+0.\lambda^3 & 6+0.\lambda+0.\lambda^2+0.\lambda^3 \end{bmatrix}$$

Obviously $\mathbf{A}$ can be written as the matric polynomial

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 3 & 1 \\ 2 & 5 & 6 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 & -6 \\ 0 & 0 & -2 \\ -3 & 0 & 0 \end{bmatrix} + \lambda^2 \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda^3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}.$$

4.2 Characteristic Values and Characteristic Vectors of a Matrix

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be a given n-rowed square matrix. Let

$$\mathbf{A} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix}$$

be a column vector. Consider the vector equation

$$\mathbf{AX} = \lambda \mathbf{X}. \qquad ..(1)$$

where λ is a scalar (*i.e.,* number).

It is obvious that the zero vector $\mathbf{X} = \mathbf{O}$ is a solution of (1) for any value of λ. Now let us see whether there exist scalars λ and non-zero vectors $\mathbf{X}$ which satisfy (1).

If $\mathbf{I}$ denotes the unit matrix of order n, then the equation (1) may be written as

$$\mathbf{AX} = \lambda \mathbf{IX}$$

or $$(\mathbf{A} - \lambda \mathbf{I})\, \mathbf{X} = \mathbf{O}. \qquad ...(2)$$

The matrix equation (2) represents the following system of n homogeneous equations in n unknowns :

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)\, x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots \; + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \qquad \ldots(3)$$

The coefficient matrix of the equations (3) is $\mathbf{A} - \lambda\mathbf{I}$. The necessary and sufficient condition for equations (3) to possess a non-zero solution ($\mathbf{X} \neq \mathbf{O}$) is that the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ should be of rank less than the number of unknowns n. But this will be so if and only if the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular *i.e.*, if and only if $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Thus the scalars λ for which

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

are of special importance.

Definitions: **(Lucknow 2009)**

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be any n-rowed square matrix and λ an indeterminate. The matrix $\mathbf{A} - \lambda\mathbf{I}$ is called the characteristic matrix of $\mathbf{A}$ where $\mathbf{I}$ is the unit matrix of order n.

Also the determinant

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix},$$

which is an ordinary polynomial in λ of degree n, is called the **characteristic polynomial of** A. The equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is called the **characteristic equation** of **A** and the roots of this equation are called the **characteristic roots** or **characteristic values** or **eigenvalues** or **latent roots** or **proper values** of the matrix **A**. The set of the eigenvalues of **A** is called the **spectrum** of A.

If λ is a characteristic root of the matrix **A**, then

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

and the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular. Therefore there exists a non-zero vector **X** such that

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O} \qquad \text{or} \qquad \mathbf{AX} = \lambda\mathbf{X}.$$

Characteristic vectors. Definition: *If λ is a characteristic root of an $n \times n$ matrix* **A**, *then a non-zero vector* **X** *such that* $\mathbf{AX} = \lambda\mathbf{X}$ *is called a characteristic vector or eigenvector of* **A** *corresponding to the characteristic root* λ. **(Lucknow 2009)**

4.3 Certain Relations between Characteristic Roots and Characteristic Vectors

Theorem 1. *λ is a characteristic root of a matrix* **A** *if and only if there exists a non-zero vector* **X** *such that* $\mathbf{AX} = \lambda\mathbf{X}$.

Proof. Suppose λ is a characteristic root of the matrix **A**. Then $|\mathbf{A} - \lambda\mathbf{I}| = 0$ and the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular. Therefore, the matrix equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$ possesses a non-zero solution *i.e.*, there exists a non-zero vector **X** such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{X} = \mathbf{O}$ or $\mathbf{AX} = \lambda\mathbf{X}$.

Conversely suppose there exists a non-zero vector **X** such that $\mathbf{AX} = \lambda\mathbf{X}$ *i.e.*, $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$. Since the matrix equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{X} = \mathbf{O}$ possesses a non-zero solution, therefore the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ must be singular *i.e.*, $|\mathbf{A} - \lambda\mathbf{I}| = 0$. Hence λ is a characteristic root of the matrix **A.**

Theorem 2. *If* **X** *is a characteristic vector of a matrix* **A** *corresponding to the characteristic value λ, then k***X** *is also a characteristic vector of* **A** *corresponding to the same characteristic value λ. Here k is any non-zero scalar.*

Proof. Suppose **X** is a characteristic vector of **A** corresponding to the characteristic value λ. Then $\mathbf{X} \neq \mathbf{O}$ and $\mathbf{AX} = \lambda\mathbf{X}$.

If k is any non-zero scalar, then $k\ \mathbf{x} \neq \mathbf{O}$. Also

$$\mathbf{A}\,(k\mathbf{X}) = k(\mathbf{AX}) = k(\lambda\mathbf{X}) = \lambda(k\mathbf{X}).$$

Now $k\mathbf{X}$ is a non-zero vector such that $\mathbf{A}(k\mathbf{X}) = \lambda(k\mathbf{X})$. Hence $k\mathbf{X}$ is a characteristic vector of **A** corresponding to the characteristic value λ. Thus corresponding to a characteristic value λ, there corresponds more than one characteristic vectors.

Theorem 3: *If* **X** *is a characteristic vector of a matrix* **A**, *then* **X** *cannot correspond to more than one characteristic values of* **A**.

Proof: Let **X** be a characteristic vector of a matrix **A** corresponding to two characteristic values λ_1 and λ_2. Then $\mathbf{AX} = \lambda_1\mathbf{X}$ and $\mathbf{AX} = \lambda_2\mathbf{X}$. Therefore

$$\lambda_1\mathbf{X} = \lambda_2\mathbf{X}$$

$$\Rightarrow \quad (\lambda_1 - \lambda_2)\,\mathbf{X} = \mathbf{O}$$

$$\Rightarrow \qquad \lambda_1 - \lambda_2 = 0 \qquad [\because \ \mathbf{X} \neq \mathbf{O}]$$

$$\Rightarrow \qquad \lambda_1 = \lambda_2.$$

Hence the result.

4.4 Nature of the Characteristic Roots of Special Types of Matrices

Theorem 1: *The characteristic roots of a Hermitian matrix are real.*

(Meerut 2011; Lucknow 05, 11)

Proof. Suppose **A** is a Hermitian matrix, λ a characteristic root of **A** and **X** a corresponding eigenvector. Then

$$\mathbf{AX} = \lambda \mathbf{X}. \qquad ...(1)$$

Premultiplying both sides of (1) by X^{θ}, we get

$$\mathbf{X}^{\theta}\mathbf{AX} = \lambda \mathbf{X}^{\theta}\mathbf{X}. \qquad ...(2)$$

Taking conjugate transpose of both sides of (2), we get

$$(\mathbf{X}^{\theta}\mathbf{AX})^{\theta} = (\lambda \mathbf{X}^{\theta}\mathbf{X})^{\theta}$$

or $$\mathbf{X}^{\theta}\mathbf{A}^{\theta}(\mathbf{X}^{\theta})^{\theta} = \overline{\lambda}\mathbf{X}^{\theta}(\mathbf{X}^{\theta})^{\theta}$$

or $$\mathbf{X}^{\theta}\mathbf{AX} = \overline{\lambda}\mathbf{X}^{\theta}\mathbf{X} \qquad ...(3)$$

$$[\because (\mathbf{X}^{\theta})^{\theta} = \mathbf{X} \text{ and } \mathbf{A}^{\theta} = \mathbf{A}, \mathbf{A} \text{ being Hermitian}]$$

From (2) and (3), we have

$$\lambda \mathbf{X}^{\theta}\mathbf{X} = \overline{\lambda}\mathbf{X}^{\theta}\mathbf{X}$$

or $$(\lambda - \overline{\lambda})\mathbf{X}^{\theta}\mathbf{X} = \mathbf{O}.$$

But **X** is not a zero vector, therefore $\mathbf{X}^{\theta}\mathbf{X} \neq \mathbf{O}$.

Hence $\lambda - \overline{\lambda} = 0$, so that $\lambda = \overline{\lambda}$ and consequently λ is real.

Corollary 1. *The characteristic roots of a real symmetric matrix are all real.*

If the elements of a Hermitian matrix **A** are all real, then **A** is a real symmetric matrix. Thus a real symmetric matrix is Hermitian and therefore the result follows.

Corollary 2. *The characteristic roots of a skew-Hermitian matrix are either pure imaginary or zero.* **(Lucknow 2010)**

Suppose **A** is a skew-Hermitian matrix. Then $i\mathbf{A}$ is Hermitian. Let λ be a characteristic root of **A**. Then

$$\mathbf{AX} = \lambda \mathbf{X}$$

or $$(i\mathbf{A})\mathbf{X} = (i\lambda)\mathbf{X}.$$

From this it follows that $i\lambda$ is a characteristic root of $i\mathbf{A}$ which is Hermitian. Hence $i\lambda$ is real. Therefore either λ must be zero or pure imaginary.

Corollary 3. *The characteristic roots of a real skew-symmetric matrix are either pure imaginary or zero, for every such matrix is skew-Hermitian.*

Theorem 2. *The characteristic roots of a unitary matrix are of unit modulus.*

Proof. Suppose **A** is a unitary matrix. Then $\mathbf{A}^{\theta}\mathbf{A} = \mathbf{I}$.

Let λ be a characteristic root of **A** and **X** a corresponding eigenvector. Then

$$\mathbf{AX} = \lambda \mathbf{X}. \qquad \ldots(1)$$

Taking conjugate transpose of both sides of (1), we get

$$(\mathbf{AX})^{\theta} = (\lambda \mathbf{X})^{\theta}$$

or $$\mathbf{X}^{\theta}\mathbf{A}^{\theta} = \bar{\lambda}\mathbf{X}^{\theta}. \qquad \ldots(2)$$

From (1) and (2), we have

$$(X^{\theta}\mathbf{A}^{\theta})(\mathbf{AX}) = \bar{\lambda}\lambda\mathbf{X}^{\theta}\mathbf{X}$$

or $$\mathbf{X}^{\theta}(\mathbf{A}^{\theta}\mathbf{A})\,\mathbf{X} = \bar{\lambda}\lambda\mathbf{X}^{\theta}\mathbf{X} \qquad [\because\ \mathbf{A}^{\theta}\mathbf{A} = \mathbf{I}]$$

or $$\mathbf{X}^{\theta}\mathbf{IX} = \bar{\lambda}\lambda\mathbf{X}^{\theta}\mathbf{X}$$

or $$\mathbf{X}^{\theta}\mathbf{X} = \bar{\lambda}\lambda\mathbf{X}^{\theta}\mathbf{X}$$

or $$\mathbf{X}^{\theta}\mathbf{X}(\lambda\bar{\lambda} - 1) = \mathbf{O}. \qquad \ldots(3)$$

Since $\mathbf{X}^{\theta}\mathbf{X} \neq \mathbf{O}$, therefore, (3) gives

$$\lambda\bar{\lambda} - 1 = 0$$

or $$\lambda \bar{\lambda} = 1$$

or $$|\lambda|^2 = 1.$$

Corollary. *The characteristic roots of an orthogonal matrix are of unit modulus.*

We know that if the elements of a unitary matrix **A** are all real, then **A** is said to be an orthogonal matrix. Hence the result follows.

4.5 The Process of Finding the Eigenvalues and Eigenvectors of a Matrix

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be a square matrix of order n. First we should write the characteristic equation of the matrix **A** *i.e.*, the equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$. This equation will be of degree n in λ. So it will have n roots. These n roots will give us the eigenvalues of the matrix **A**. If λ_1 is an eigenvalue of **A**, then the corresponding eigenvectors of **A** will be given by the non-zero vectors

$$\mathbf{X} = [x_1, x_2, \ldots, x_n]'$$

satisfying the equation $\mathbf{AX} = \lambda_1 \mathbf{X}$

or $$(\mathbf{A} - \lambda_1 \mathbf{I})\, \mathbf{X} = \mathbf{O}.$$

Illustrative Examples

Example 1. *Determine the characteristic roots of the matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

(Meerut 2010)

Solution : The characteristic matrix of **A**

$$= \mathbf{A} - \lambda \mathbf{I}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - \lambda & 1 & 2 \\ 1 & 0 - \lambda & -1 \\ 2 & -1 & 0 - \lambda \end{bmatrix}.$$

It should be noted that in order to obtain the characteristic matrix of a matrix **A**, we should simply subtract λ from each of its principal diagonal elements.

The characteristic polynomial of **A**

$$= |\mathbf{A} - \lambda \mathbf{I}|$$

$$= \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 1) - 1(-\lambda + 2) + 2(-1 + 2\lambda)$$

$$= -\lambda^3 + \lambda + \lambda - 2 - 2 + 4\lambda$$

$$= -\lambda^3 + 6\lambda - 4.$$

$\therefore$ the characteristic equation of **A** is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

i.e., $$\lambda^3 - 6\lambda + 4 = 0$$

i.e., $$(\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0.$$

The roots of this equation are $\lambda = 2, -1 \pm \sqrt{3}$.

Hence the characteristic roots of the matrix **A** are $2, -1 \pm \sqrt{3}$.

Example 2: *Determine the eigenvalues of the matrix*

$$\mathbf{A} = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & c & c \end{bmatrix}.$$

(Meerut 2010B; Kanpur 09)

Solution: Here $$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a - \lambda & h & g \\ 0 & b - \lambda & 0 \\ 0 & c & c - \lambda \end{vmatrix}$$

$$= (a - \lambda)(b - \lambda)(c - \lambda).$$

The characteristic equation of **A** is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

i.e., $$(a - \lambda)(b - \lambda)(c - \lambda) = 0.$$

The roots of this equation are $\lambda = a, b, c$. Hence the eigenvalues of **A** are a, b, c.

Example 3: *Determine the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}.$$

(Bundelkhand 2006)

Solution: The characteristic equation of A is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

i.e., $$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

i.e., $$(5-\lambda)(2-\lambda) - 4 = 0$$

i.e., $$\lambda^2 - 7\lambda + 6 = 0.$$

The roots of this equation are $\lambda_1 = 6, \lambda_2 = 1$. Therefore the eigenvalues of **A** are 6, 1.

The eigenvectors $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of **A** corresponding to the eigenvalue 6 are given by the non-zero solutions of the equation

$$(\mathbf{A} - 6\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ applying } R_2 \to R_2 + R_1.$$

The coefficient matrix of these equations is of rank 1. Therefore these equations have $2 - 1$ *i.e.,* 1 linearly independent solution. These equations reduce to the single equation $-x_1 + 4x_2 = 0$. Obviously $x_1 = 4, x_2 = 1$ is a solution of this equation. Therefore $\mathbf{X}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 6. The set of all eigenvectors of **A** corresponding to the eigenvalue 6 is given by $c_1 \mathbf{X}_1$ where c_1 is any non-zero scalar.

The eigenvectors **X** of **A** corresponding to the eigenvalue 1 are given by the non-zero solutions of the equation

$$(\mathbf{A} - 1\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $$4x_1 + 4x_2 = 0, \quad x_1 + x_2 = 0.$$

From these $x_1 = -\,x_2$. Let us take $x_1 = 1, x_2 = -\,1$. Then $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 1. Every non-zero multiple of the vector $\mathbf{X}_2$ is an eigenvector of **A** corresponding to the eigenvalue 1.

Example 4: *Determine the characteristic roots and the corresponding characteristic vectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}.$$

(Purvanchal 2010; Bundelkhand 08; Rohilkhand 05; Agra 07; Kanpur 09; Avadh 05)

Solution : The characteristic equation of the matrix **A** is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

i.e., $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

or $$(8-\lambda)\{(7-\lambda)(3-\lambda) - 16\} + 6\{-6(3-\lambda) + 8\} + 2\{24 - 2(7-\lambda)\} = 0$$

or $$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

or $$\lambda(\lambda - 3)(\lambda - 15) = 0.$$

Hence the characteristic roots of **A** are 0, 3, 15.

The eigenvectors $\mathbf{X} = [x_1, x_2, x_3]'$ of **A** corresponding to the eigenvalue 0 are given by the non-zero solutions of the equation

$$(\mathbf{A} - 0\mathbf{I})\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_1 \longleftrightarrow R_3$$

or $$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_2 \to R_2 + 3R_1, R_3 \to R_3 - 4R_1$$

or $$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_3 \to R_3 + 2R_2.$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3 - 2 = 1$ linearly independent solution. Thus there is only one linearly independent eigenvector corresponding to the eigenvalue 0. These equations can be written as

$$2x_1 - 4x_2 + 3x_3 = 0, -5x_2 + 5x_3 = 0.$$

From the last equation, we get $x_2 = x_3$. Let us take $x_2 = 1, x_3 = 1$. Then the first equation gives $x_1 = \frac{1}{2}$. Therefore $\mathbf{X}_1 = \begin{bmatrix} \frac{1}{2} & 1 & 1 \end{bmatrix}'$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0. If c_1 is any non-zero scalar, then $c_1 \mathbf{X}_1$ is also an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

The eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 3 are given by the non-zero solutions of the equation

$$(\mathbf{A} - 3\mathbf{I})\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{by } R_1 \to R_1 + R_2$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & -8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 6R_1,\ R_3 \to R_3 + 2R_1$$

or $$\begin{bmatrix} -1 & -2 & -2 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + \frac{1}{2} R_2.$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3 - 2 = 1$ linearly independent solution. These equations can be written as

$$-x_1 - 2x_2 - 2x_3 = 0, 16x_2 + 8x_3 = 0.$$

From the second equation we get $x_2 = -\frac{1}{2} x_3$. Let us take $x_3 = 4, x_2 = -2$. Then the first equation gives $x_1 = -4$. Therefore $\mathbf{X}_2 = [-4 \ \ -2 \ \ 4]'$

is an eigenvector of **A** corresponding to the eigenvalue 3. Every non-zero multiple of the $\mathbf{X}_2$ is an eigenvector of **A** corresponding to the eigenvalue 3.

The eigenvectors of **A** corresponding to the eigenvalue 15 are given by the non-zero solutions of the equation

$$(\mathbf{A} - 15\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 2 & 6 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \to R_1 - R_2$$

or $$\begin{bmatrix} -1 & 2 & 6 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 6R_1,\ R_3 \to R_3 + 2R_1.$$

The coefficient matrix of these equations is of rank 2. Therefore these equations have $3 - 2 = 1$ linearly independent solution. These equations can be written as

$$-x_1 + 2x_2 + 6x_3 = 0, \; -20x_2 - 40x_3 = 0.$$

The last equation gives $x_2 = -2x_3$. Let us take $x_3 = 1$, $x_2 = -2$. Then the first equation gives $x_1 = 2$. Therefore

$$\mathbf{X}_3 = [2 \;\; -2 \;\; 1]'$$

is an eigenvector of **A** corresponding to the eigenvalue 15. If k is any non-zero scalar, then $k\mathbf{X}_3$ is also an eigenvector of **A** corresponding to the eigenvalue 15.

Example 5: *Determine the characteristic roots and the corresponding characteristic vectors of the matrix*

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

(Meerut 2006B, 09; Purvanchal 07)

Solution : The characteristic equation of **A** is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

or $$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

or $$\begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 2-\lambda \\ 2 & -1 & 2-\lambda \end{bmatrix} = 0, \text{ by } C_3 \to C_3 + C_2$$

or $$(2-\lambda)\begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 3-\lambda & 1 \\ 2 & -1 & 1 \end{bmatrix} = 0$$

or $$(2-\lambda)\begin{bmatrix} 6-\lambda & -2 & 0 \\ -4 & 4-\lambda & 0 \\ 2 & -1 & 1 \end{bmatrix} = 0, \text{ by } R_2 \to R_2 - R_3$$

or $$(2-\lambda)[(6-\lambda)(4-\lambda) - 8] = 0$$

or $$(2-\lambda)(\lambda^2 - 10\lambda + 16) = 0$$

or $$(2-\lambda)(\lambda - 2)(\lambda - 8) = 0.$$

Therefore the characteristic roots of **A** are given by $\lambda = 2, 2, 8$.

The characteristic vectors of **A** corresponding to the characteristic root 8 are given by the non-zero solutions of the equation

$$(\mathbf{A}-8\mathbf{I})\,\mathbf{X}=\mathbf{O}$$

or $$\begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1,\ R_3 \to R_3 + R_1$$

or $$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2.$$

The coefficient matrix of these equations is of rank 2. Therefore these equations possess $3-2=1$ linearly independent solution. These equations can be written as

$$-2x_1-2x_2+2x_3=0,$$

$$-3x_2-3x_3=0.$$

The last equation gives $x_2=-x_3$. Let us take $x_3=1, x_2=-1$. Then the first equation gives $x_1=2$. Therefore $\mathbf{X}_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 8. Every non-zero multiple of $\mathbf{X}_1$ is an eigenvector of **A** corresponding to the eigenvalue 8.

The eigenvectors of **A** corresponding to the eigenvalue 2 are given by the non-zero solutions of the equation

$$(\mathbf{A}-2\mathbf{I})\,\mathbf{X}=\mathbf{O}$$

or $$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_1 \longleftrightarrow R_2$$

or $$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 + 2R_1,\ R_3 \to R_3 + R_1.$$

The coefficient matrix of these equations is of rank 1. Therefore these equations possess $3 - 1 = 2$ linearly independent solutions. We see that these equations reduce to the single equation

$$-2x_1 + x_2 - x_3 = 0.$$

Obviously $$\mathbf{X}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

are two linearly independent solutions of this equation. Therefore $\mathbf{X}_2$ and $\mathbf{X}_3$ are two linearly independent eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 2. If c_1, c_2 are scalars not both equal to zero, then $c_1\mathbf{X}_2 + c_2\mathbf{X}_3$ gives all the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue 2.

Example 6: *Show that* 0 *is a characteristic root of a matrix if and only if the matrix is singular.*

Solution: We have 0 is an eigenvalue of $\mathbf{A} \Rightarrow \lambda = 0$ satisfies the equation

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow |\mathbf{A}| = 0 \Rightarrow \mathbf{A} \text{ is singular.}$$

Conversely, $\mathbf{A}$ is singular $\Rightarrow |\mathbf{A}| = 0$

$\Rightarrow$ $\lambda = 0$ satisfies the equation $|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow 0$ is an eigenvalue of $\mathbf{A}$.

Example 7: *If* $\lambda_1, \ldots\ldots\ldots, \lambda_n$ *are the eigenvalues of* $\mathbf{A}$, *then show that* $k\lambda_1, \ldots, k\lambda_n$ *are the eigenvalues of* $k\mathbf{A}$.

Solution: If $k = 0$, then $k\mathbf{A} = \mathbf{O}$ and each eigenvalue of $\mathbf{O}$ is 0. Thus $0\lambda_1, \ldots., 0\lambda_n$ are the eigenvalues of $k\mathbf{A}$ if $\lambda_1, \ldots, \lambda_n$ are the eigen-values of $\mathbf{A}$.

So let us suppose that $k \neq 0$.

We have $$|k\mathbf{A} - \lambda k\mathbf{I}| = |k(\mathbf{A} - \lambda\mathbf{I})|$$
$$= k^n |\mathbf{A} - \lambda\mathbf{I}|. \qquad [\because\ |k\mathbf{B}| = k^n|\mathbf{B}|]$$

$\therefore$ if $k \neq 0$, then $|k\mathbf{A} - \lambda k\mathbf{I}| = 0$ if and only if $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e., $k\lambda$ is an eigenvalue of $k\mathbf{A}$ if and only if λ is an eigenvalue of $\mathbf{A}$.

Thus $k\lambda_1, \ldots, k\lambda_n$ are the eigenvalues of $k\mathbf{A}$ if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\mathbf{A}$.

Example 8: *If* $\mathbf{A}$ *is non-singular, prove that the eigenvalues of* $\mathbf{A}^{-1}$ *are the reciprocals of the eigenvalues of* $\mathbf{A}$.

Solution: Let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{X}$ be a corresponding eigenvector. Then

$$\mathbf{AX} = \lambda \mathbf{X}$$

$$\Rightarrow \quad \mathbf{X} = \mathbf{A}^{-1}(\lambda \mathbf{X}) = \lambda(\mathbf{A}^{-1}\mathbf{X})$$

$$\Rightarrow \quad \frac{1}{\lambda}\mathbf{X} = \mathbf{A}^{-1}\mathbf{X} \qquad [\because \mathbf{A} \text{ is non-singular} \Rightarrow \lambda \neq 0]$$

$$\mathbf{A}^{-1}\mathbf{X} = \frac{1}{\lambda}\mathbf{X}$$

$\Rightarrow$ $\dfrac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$ and $\mathbf{X}$ is a corresponding eigenvector.

Conversely suppose that k is an eigenvalue of $\mathbf{A}^{-1}$. Since $\mathbf{A}$ is non-singular $\Rightarrow \mathbf{A}^{-1}$ is non-singular and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$, therefore, it follows from the first part of this question that $\dfrac{1}{k}$ is an eigenvalue of $\mathbf{A}$. Thus each eigenvalue of $\mathbf{A}^{-1}$ is equal to the reciprocal of some eigenvalue of $\mathbf{A}$.

Hence the eigenvalues of $\mathbf{A}^{-1}$ are nothing but the reciprocals of the eigenvalues of $\mathbf{A}$.

Example 9: *If the characteristic roots of* $\mathbf{A}$ *are* $\lambda_1, \lambda_2, \ldots \lambda_n$, *then the characteristic roots of* $\mathbf{A}^2$ *are* $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$. **(Kumaun 2008)**

Solution : Let λ be a characteristic root of the matrix $\mathbf{A}$. Then there exists a non-zero vector $\mathbf{X}$ such that

$$\mathbf{AX} = \lambda \mathbf{X}$$

$$\Rightarrow \quad \mathbf{A}(\mathbf{AX}) = \mathbf{A}(\lambda \mathbf{X})$$

$$\Rightarrow \quad \mathbf{A}^2\mathbf{X} = \lambda(\mathbf{AX})$$

$$\Rightarrow \quad \mathbf{A}^2\mathbf{X} = \lambda(\lambda \mathbf{X}) \qquad [\because \mathbf{AX} = \lambda \mathbf{X}]$$

$$\Rightarrow \quad \mathbf{A}^2\mathbf{X} = \lambda^2\mathbf{X}. \qquad \ldots(1)$$

Since $\mathbf{X}$ is a non-zero vector, therefore from the relation (1) it is obvious that λ^2 is a characteristic root of the matrix $\mathbf{A}^2$. Therefore if $\lambda_1, \ldots, \lambda_n$ are the characteristic roots of $\mathbf{A}$, then $\lambda_1^2, \ldots, \lambda_n^2$ are the characteristic roots of $\mathbf{A}^2$.

Example 10: *The characteristic roots of an idempotent matrix are either zero or unity.*

Solution: Let $\mathbf{A}$ be an idempotent matrix so that $\mathbf{A}^2 = \mathbf{A}$. Let λ be a characteristic root of the matrix $\mathbf{A}$. Then there exists a non-zero vector $\mathbf{X}$ such that

$$\mathbf{AX} = \lambda \mathbf{X} \qquad \ldots(1)$$

$\Rightarrow$ $\mathbf{A}(\mathbf{AX}) = \mathbf{A}(\lambda \mathbf{X})$

$\Rightarrow$ $\mathbf{A}^2 \mathbf{X} = \lambda (\mathbf{AX})$

$\Rightarrow$ $\mathbf{AX} = \lambda (\lambda \mathbf{X})$ $\quad [\because \mathbf{A}^2 = \mathbf{A} \text{ and } \mathbf{AX} = \lambda \mathbf{X}]$

$\Rightarrow$ $\mathbf{AX} = \lambda^2 \mathbf{X}.$ $\quad \ldots(2)$

From (1) and (2), we get $\lambda^2 \mathbf{X} = \lambda \mathbf{X}$

or $(\lambda^2 - \lambda)\mathbf{X} = \mathbf{O}$

or $\lambda^2 - \lambda = 0$ $\quad [\because \mathbf{X} \neq \mathbf{O}]$

or $\lambda(\lambda - 1) = 0.$

$\therefore$ $\lambda = 0$ or $\lambda = 1.$

Hence the characteristic roots of an idempotent matrix are either zero or unity.

Example 11: *The product of the characteristic roots of a square matrix of order n is equal to the determinant of the matrix.*

Solution: Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be a square matrix of order n. Then the characteristic polynomial $f(\lambda)$ of $\mathbf{A}$ is given by

$$f(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{22} & a_{22} - \lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

$$= (-1)^n [\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \ldots + p_n], \text{ say.}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic roots of $\mathbf{A}$, then $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

$$\therefore \quad |\mathbf{A} - \lambda \mathbf{I}| = (-1)^n [\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + \ldots + p_n]$$

$$= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n). \quad \ldots(1)$$

Putting $\lambda = 0$ on both sides of (1), we get

$$|\mathbf{A}| = (-1)^n (-\lambda_1)(-\lambda_2) \ldots (-\lambda_n)$$

$$= (-1)^n (-1)^n \lambda_1 \lambda_2 \ldots \lambda_n = (-1)^{2n} \lambda_1 \lambda_2 \ldots \lambda_n = \lambda_1 \lambda_2 \ldots \lambda_n$$

Hence $\lambda_1 \lambda_2 \ldots \lambda_n = |\mathbf{A}|$.

Example 12: *Any two characteristic vectors corresponding to two distinct characteristic roots of a:*

(i) Hermitian, (ii) Real symmetric, (iii) Unitary matrix are orthogonal.

Solution : **(i) A is Hermitian.**

We have $\mathbf{A}^\theta = \mathbf{A}$.

Also, we have $\mathbf{AX_1} = \lambda_1 \mathbf{X_1}$ and $\mathbf{AX}_2 = \lambda_2 \mathbf{X}_2$

or $\lambda_1 \mathbf{X}_1^\theta (\mathbf{X}_2^\theta)^\theta = \lambda_2 \mathbf{X}_1^\theta \mathbf{X}_2$; $[\lambda_1 \text{ is real} \Rightarrow \lambda_1^\theta = \lambda_1]$

or $\lambda_1 \mathbf{X}_1^\theta \mathbf{X}_2 = \lambda_2 \mathbf{X}_1^\theta \mathbf{X}_2$

or $(\lambda_1 - \lambda_2) \mathbf{X}_1^\theta \mathbf{X}_2 = \mathbf{O}$.

Since $\lambda_1 - \lambda_2 \neq 0$, we have $\mathbf{X}_1^\theta \mathbf{X}_2 = \mathbf{O}$.

$\Rightarrow$ $\mathbf{X}_1$ and $\mathbf{X}_2$ are orthogonal with respect to each other.

(ii) A is real symmetric. The real symmetric matrix is always Hermitian, so the result follows at once from (i).

(iii) A is unitary. We have $\mathbf{A}^\theta \mathbf{A} = \mathbf{I}$.

Now, $\mathbf{AX}_1 = \lambda_1 \mathbf{X}_1$ and $\mathbf{AX}_2 = \lambda_2 \mathbf{X}_2$,

where λ_1, λ_2 are characteristic roots of a unitary matrix which must be uni-modular

i.e., $\lambda_1 \bar{\lambda}_1 = 1, \ \lambda_2 \bar{\lambda}_2 = 1.$

Now, $$\mathbf{AX}_2 = \lambda_2 \mathbf{X}_2 \Rightarrow (\mathbf{AX}_2)^\theta = (\lambda_2 \mathbf{X}_2)^\theta \Rightarrow \mathbf{X}_2^\theta \, \mathbf{A}^\theta = \bar{\lambda}_2 \, \mathbf{A}_2^\theta$$

$\Rightarrow$ $$\mathbf{X}_2^\theta \, \mathbf{A}^\theta \, \mathbf{AX}_1 = \bar{\lambda}_2 \, \mathbf{X}_2^\theta \, \lambda_1 \, \mathbf{X}_1 = \bar{\lambda}_2 \, \lambda_1 \, \mathbf{X}_2^\theta \, \mathbf{X}_1$$

Again $\mathbf{A}^\theta \mathbf{A} = \mathbf{I}$, hence we get

$$\mathbf{X}_2^\theta \, \mathbf{X}_1 = \bar{\lambda}_2 \lambda_1 \, \mathbf{X}_2^\theta \, \mathbf{X}_1$$

$\Rightarrow$ $$(1 - \bar{\lambda}_2 \, \lambda_1) \, \mathbf{X}_2^\theta \, \mathbf{X}_1 = \mathbf{O}.$$

But $\bar{\lambda}_2 \, \lambda_1 \neq 1$; so $1 - \bar{\lambda}_2 \, \lambda_1 \neq 0$.

This implies that $\mathbf{X}_2^\theta \, \mathbf{X}_1 = \mathbf{O} \Rightarrow \mathbf{X}_1$ and $\mathbf{X}_2$ are orthogonal.

Example 13: *Show that the two matrices* $\mathbf{A}, \mathbf{C}^{-1}\mathbf{AC}$ *have the same characteristic roots.*

(Lucknow 2005, 07)

Solution: Let $\mathbf{B} = \mathbf{C}^{-1}\mathbf{AC}$.

Then $$\mathbf{B} - \lambda \mathbf{I} = \mathbf{C}^{-1}\mathbf{AC} - \lambda \mathbf{I}$$

$$= \mathbf{C}^{-1}\mathbf{AC} - \mathbf{C}^{-1} \lambda \mathbf{IC} \qquad [\because \mathbf{C}^{-1}(\lambda \mathbf{I})\,\mathbf{C} = \lambda \mathbf{C}^{-1}\mathbf{C} = \lambda \mathbf{I}]$$

$$= \mathbf{C}^{-1}(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{C}.$$

$\therefore$ $$|\mathbf{B} - \lambda \mathbf{I}| = |\mathbf{C}^{-1}|\,|\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{C}|$$

$$= |\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{C}^{-1}|\,|\mathbf{C}| = |\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{C}^{-1}\mathbf{C}|$$

$$= |\mathbf{A} - \lambda \mathbf{I}|\,|\mathbf{I}| = |\mathbf{A} - \lambda \mathbf{I}|.$$

Thus the two matrices **A** and **B** have the same characteristic determinants and hence the same characteristic equations and the same characteristic roots.

4.6 The Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation i.e., if for a square matrix **A** *of order* n,

$$|\mathbf{A} - \lambda \mathbf{I}| = (-1)^n \, [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n],$$

then the matrix equation $$\mathbf{X}^n + a_1 \mathbf{X}^{n-1} + a_2 \mathbf{X}^{n-2} + a_3 \mathbf{X}^{n-3} + \ldots a_n \mathbf{I} = \mathbf{O}$$

is satisfied by $\mathbf{X} = \mathbf{A}$ *i.e.*, $\mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \ldots + a_n\mathbf{I} = \mathbf{O}$.

(Meerut 2000, 03, 05B, 06B, 09B, 10B; Rohilkhand 05, 08; Agra 07; Avadh 05; Purvanchal 08; Lucknow 08, 09)

Proof. Since the elements of $\mathbf{A} - \lambda\mathbf{I}$ are at most of the first degree in λ, the elements of Adj $(\mathbf{A} - \lambda\mathbf{I})$ are ordinary polynomials in λ of degree $n-1$ or less. Therefore Adj $(\mathbf{A} - \lambda\mathbf{I})$ can be written as a matrix polynomial in λ, given by

$$\text{Adj}\,(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{B}_0\lambda^{n-1} + \mathbf{B}_1\lambda^{n-2} + \ldots + \mathbf{B}_{n-2}\,\lambda + \mathbf{B}_{n-1},$$

where $\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_{n-1}$ are matrices of the type $n \times n$ whose elements are functions of a_{ij}'s.

Now $\quad (\mathbf{A} - \lambda\mathbf{I})\ \text{Adj.}\,(\mathbf{A} - \lambda\mathbf{I}) = |\,\mathbf{A} - \lambda\mathbf{I}\,|\ \mathbf{I}. \qquad [\because\ \mathbf{A}\,.\,\text{Adj}\,\mathbf{A} = |\,\mathbf{A}\,|\ \mathbf{I}_n|\,]$

$$\therefore \quad (\mathbf{A} - \lambda\mathbf{I})\,(\mathbf{B}_0\lambda^{n-1} + \mathbf{B}_1\lambda^{n-2} + \ldots + \mathbf{B}_{n-2}\lambda + \mathbf{B}_{n-1})$$

$$= (-1)^n\,[\lambda^n + a_1\lambda^{n-1} + \ldots + a_n]\ \mathbf{I}.$$

Comparing coefficients of like powers of λ on both sides, we get

$$-\,\mathbf{IB}_0 = (-1)^n\ \mathbf{I},$$

$$\mathbf{AB}_0 - \mathbf{IB}_1 = (-1)^n a_1\mathbf{I},$$

$$\mathbf{AB}_1 - \mathbf{IB}_2 = (-1)^n a_2\mathbf{I},$$

....

....

$$\mathbf{AB}_{n-1} = (-1)^n\ a_n\mathbf{I}.$$

Pre-multiplying these successively by $\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots, \mathbf{I}$ and adding, we get

$$\mathbf{O} = (-1)^n\,[\mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \ldots + a_n\mathbf{I}].$$

Thus $\quad \mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \ldots + a_{n-1}\,\mathbf{A} + a_n\mathbf{I} = \mathbf{O}. \qquad \ldots(1)$

Corollary. 1. If $\mathbf{A}$ be a non-singular matrix, then $|\,\mathbf{A}\,| \neq 0$. Also $|\,\mathbf{A}\,| = (-1)^n\ a_n$ and therefore $a_n \neq 0$.

Pre-multiplying (1) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + a_2 \mathbf{A}^{n-3} + \ldots + a_{n-1} \mathbf{I} + a_n \mathbf{A}^{-1} = \mathbf{O}$$

or
$$\mathbf{A}^{-1} = -(1/a_n)\,[\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \ldots + a_{n-1} \mathbf{I}].$$

Corollary. 2. If m be a positive integer such that $m \geq n$, then multiplying the result (1) by $\mathbf{A}^{m-n}$, we get

$$\mathbf{A}^m + a_1 \mathbf{A}^{m-1} + \ldots + a_n \mathbf{A}^{m-n} = \mathbf{O},$$

showing that any positive integral power $\mathbf{A}^m$ $(m \geq n)$ of $\mathbf{A}$ is linearly expressible in terms of those of lower order.

Illustrative Examples

Example 14: *Find the characteristic equation of the matrix* $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ *and verify that it is satisfied by* $\mathbf{A}$ *and hence obtain* $\mathbf{A}^{-1}$. **(Meerut 2002, 05; Bundelkhand 05, 08, 10, 11; Kanpur 10; Lucknow 05,11)**

Solution : We have $|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-1(2-\lambda) + 1\} + 1\{1 - (2-\lambda)\}$$

$$= (2-\lambda)(3 - 4\lambda + \lambda^2) + (\lambda - 1) + (\lambda - 1)$$

$$= -\lambda^3 + 6\lambda^2 - 9\lambda + 4.$$

$\therefore$ the characteristic equation of the matrix $\mathbf{A}$ is $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0.$

We are now to verify that $\mathbf{A}^3 - 6\mathbf{A}^2 + 9\mathbf{A} - 4\mathbf{I} = \mathbf{O}.$ ...(1)

We have

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \mathbf{A}^2 = \mathbf{A} \times \mathbf{A} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix},$$

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}.$$

Now we can verify that $\mathbf{A}^3 - 6\mathbf{A}^2 + \mathbf{9A} - \mathbf{4I}$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$+ 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiplying (1) by $\mathbf{A}^{-1}$, we get $\mathbf{A}^2 - 6\mathbf{A} + 9\mathbf{I} - 4\mathbf{A}^{-1} = \mathbf{O}.$

$\therefore \qquad \mathbf{A}^{-1} = \frac{1}{4}(\mathbf{A}^2 - 6\mathbf{A} + 9\mathbf{I})$

Now $\qquad \mathbf{A}^2 - 6\mathbf{A} + 9\mathbf{I} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$$

$\therefore \qquad \mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}.$

Example 15: *Obtain the characteristic equation of the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ *and verify that it is satisfied by* $\mathbf{A}$ *and hence find its inverse.* **(Meerut 2001, 10B; Bundelkhand 09, 11; Kanpur 07; Purvanchal 10; Lucknow 07, 10)**

Solution : We have

$$|\mathbf{A}-\lambda\mathbf{I}| = \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda) + 2[0-2(2-\lambda)]$$

$$= (2-\lambda)[(1-\lambda)(3-\lambda)-4]$$

$$= (2-\lambda)[\lambda^2 - 4\lambda - 1]$$

$$= -(\lambda^3 - 6\lambda^2 + 7\lambda + 2).$$

$\therefore$ the characteristic equation of **A** is

$$\lambda^3 - 6\lambda^2 + 7\lambda + 2 = 0. \qquad \text{...(1)}$$

By the Cayley-Hamilton theorem

$$\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I} = \mathbf{O}. \qquad \text{...(2)}$$

Verification of (2). We have

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}.$$

Also $\mathbf{A}^3 = \mathbf{A}\,.\,\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}.$

Now $\mathbf{A}^3 - 6\mathbf{A}^2 + 7\mathbf{A} + 2\mathbf{I}$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{O}.$$

Hence Cayley-Hamilton theorem is verified. Now we shall compute $\mathbf{A}^{-1}$.

Multiplying (2) by $\mathbf{A}^{-1}$, we set $\mathbf{A}^2 - 6\mathbf{A} + 7\mathbf{I} + 2\mathbf{A}^{-1} = \mathbf{O}$.

$$\therefore \qquad \mathbf{A}^{-1} = -\frac{1}{2}(\mathbf{A}^2 - 6\mathbf{A} + 2\mathbf{I}) = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}.$$

Example 16: *Show that the matrix* $\mathrm{A} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ *satisfies Cayley-Hamilton theorem.*

(Meerut 2006B)

Solution : **We have** $\quad |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} \mathbf{0} - \lambda & \boldsymbol{c} & -\boldsymbol{b} \\ -\boldsymbol{c} & \mathbf{0} - \lambda & \boldsymbol{a} \\ \boldsymbol{b} & -\boldsymbol{a} & \mathbf{0} - \lambda \end{vmatrix}$

$$= -\lambda(\lambda^2 + a^2) - c(c\lambda - ab) - b(ac + b\lambda) = -\lambda^3 - \lambda(a^2 + b^2 + c^2).$$

$\therefore$ the characteristic equation of the matrix $\mathbf{A}$ is $\lambda^3 + \lambda(a^2 + b^2 + c^2) = 0$.

We are to verify that $\mathbf{A}^3 + (a^2 + b^2 + c^2)\mathbf{A} = \mathbf{O}$.

We have

$$\mathbf{A}^2 = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \times \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix}.$$

$$\therefore \qquad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{bmatrix} -c^2 - b^2 & ab & ac \\ ab & -c^2 - a^2 & bc \\ ac & bc & -b^2 - a^2 \end{bmatrix} \times \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -c^3 - b^2 c - a^2 c & bc^2 + b^3 + a^2 b \\ c^3 + a^2 c + b^2 c & 0 & -ab^2 - ac^2 - a^3 \\ -bc^2 - b^3 - a^2 b & ac^2 + ab^2 + a^3 & 0 \end{bmatrix}$$

$$= -(a^2 + b^2 + c^2) \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} = -(a^2 + b^2 + c^2) \mathbf{A}.$$

$\therefore \qquad \mathbf{A}^3 + (a^2 + b^2 + c^2)\,\mathbf{A} = -(a^2 + b^2 + c^2)\mathbf{A} + (a^2 + b^2 + c^2)\mathbf{A} = 0\,\mathbf{A} = \mathbf{O}.$

Hence $\mathbf{A}$ satisfies Cayley-Hamilton theorem.

Example 17: *Find the characteristic equation of the matrix* $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ *and hence find* $\mathbf{A}^{-1}$.

(Rohilkhand 2009, 10)

Solution : The characteristic equation of the matrix $\mathbf{A}$ is

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

or
$$\begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} = 0$$

or
$$(1-\lambda)(-1-\lambda)(-1-\lambda) - 2[2(-1-\lambda)] = 0$$

or
$$(1-\lambda)(1+\lambda)^2 + 4(1+\lambda) = 0$$

or
$$(1-\lambda)(1+2\lambda+\lambda^2) + 4 + 4\lambda = 0$$

or
$$1 + 2\lambda + \lambda^2 - \lambda - 2\lambda^2 - \lambda^3 + 4 + 4\lambda = 0$$

or
$$-\lambda^3 - \lambda^2 + 5\lambda + 5 = 0$$

or
$$\lambda^3 + \lambda^2 - 5\lambda - 5 = 0. \qquad \ldots(1)$$

Now by Cayley-Hamilton theorem the matrix A must satisfy its characteristic equation (1). Therefore we have

$$\mathbf{A}^3 + \mathbf{A}^2 - 5\mathbf{A} - 5\mathbf{I} = \mathbf{O}$$

or $$5\mathbf{I} = \mathbf{A}^3 + \mathbf{A}^2 - 5\mathbf{A} \quad \text{...(2)}$$

Pre-multiplying both sides of (2) by $\mathbf{A}^{-1}$, we have

$$5\mathbf{A}^{-1}\mathbf{I} = \mathbf{A}^{-1}\mathbf{A}^3 + \mathbf{A}^{-1}\mathbf{A}^2 - 5\mathbf{A}^{-1}\mathbf{A}$$

or $$5\mathbf{A}^{-1} = \mathbf{A}^2 + \mathbf{A} - 5\mathbf{I}$$

or $$\mathbf{A}^{-1} = \frac{1}{5}(\mathbf{A}^2 + \mathbf{A} - 5\mathbf{I}) \quad \text{...(3)}$$

Now $$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\therefore$ $$\mathbf{A}^2 + \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

Hence from (3), $\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$

Example 18: *State Cayley-Hamilton theorem. Use it to express* $2\mathbf{A}^5 - 3\mathbf{A}^4 + \mathbf{A}^2 - 4\mathbf{I}$ *as a linear polynomial in* $\mathbf{A}$, *when* $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}.$

(Lucknow 2005)

Solution : **Statement of Cayley-Hamilton theorem**. Every square matrix satisfies its characteristic equation.

Now let us find the characteristic equation of the matrix $\mathbf{A}$. We have

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) + 1 = \lambda^2 - 5\lambda + 7.$$

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A}-\lambda\mathbf{I}|=0$ *i.e.,* is

$$\lambda^2-5\lambda+7=0 \qquad \ldots(1)$$

By Cayley-Hamilton theorem,the matrix A must satisfy (1). Therefore we have

$$\mathbf{A}^2-5\mathbf{A}+7\mathbf{I}=\mathbf{O}. \qquad \ldots(2)$$

From (2), we get

$$\mathbf{A}^2=5\mathbf{A}-7\mathbf{I} \qquad \ldots(3)$$

Multiplying both sides of (3) by $\mathbf{A}$, we get

$$\mathbf{A}^3=5\mathbf{A}^2-7\mathbf{A} \qquad \ldots(4)$$

$\therefore$ $$\mathbf{A}^4=5\mathbf{A}^3-7\mathbf{A}^2 \qquad \ldots(5)$$

and $$\mathbf{A}^5=5\mathbf{A}^4-7\mathbf{A}^3 \qquad \ldots(6)$$

Now $$2\mathbf{A}^5-3\mathbf{A}^4+\mathbf{A}^2-4\mathbf{I}=2(5\mathbf{A}^4-7\mathbf{A}^3)-3\mathbf{A}^4+\mathbf{A}^2-4\mathbf{I}$$

[Substituting for $\mathbf{A}^5$ from (6)]

$$=7\mathbf{A}^4-14\mathbf{A}^3+\mathbf{A}^2-4\mathbf{I}=7(5\mathbf{A}^3-7\mathbf{A}^2)-14\mathbf{A}^3+\mathbf{A}^2-4\mathbf{I}$$

[by (5)]

$$=21\mathbf{A}^3-48\mathbf{A}^2-4\mathbf{I}=21(5\mathbf{A}^2-7\mathbf{A})-48\mathbf{A}^2-4\mathbf{I} \qquad \text{[by (4)]}$$

$$=57\mathbf{A}^2-147\mathbf{A}-4\mathbf{I}=57(5\mathbf{A}-7\mathbf{I})-147\mathbf{A}-4\mathbf{I} \qquad \text{[by (3)]}$$

$=138\mathbf{A}-403\mathbf{I}$, which is a linear polynomial in $\mathbf{A}$.

4.7 Diagonalization of Square Matrices with Distinct Eigen Values

Similarity of Matrices. Definition: *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be square matrices of order n. Then* $\mathbf{B}$ *is said to be similar to* $\mathbf{A}$ *if there exists a non-singular matrix* $\mathbf{P}$ *such that*

$$\mathbf{B}=\mathbf{P}^{-1}\mathbf{AP}.$$

Theorem 1: *Similarity of matrices is an equivalence relation.*

Theorem 2: *Similar matrices have the same determinant.*

Theorem 3: *Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If* $\mathbf{X}$ *is an eigenvector of* $\mathbf{A}$ *corresponding to the eigenvalue* λ *, then* $\mathbf{P}^{-1}\mathbf{X}$ *is an eigenvector of* $\mathbf{B}$ *corresponding to the eigenvalue* λ, *where*

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}.$$

Corollary. *If* $\mathbf{A}$ *is similar to a diagonal matrix* $\mathbf{D}$, *the diagonal elements of* $\mathbf{D}$ *are the eigenvalues of* $\mathbf{A}$.

Diagonalizable matrix. Definition: *A matrix* $\mathbf{A}$ *is said to be diagonalizable if it is similar to a diagonal matrix.*

Thus a matrix $\mathbf{A}$ is diagonalizable if there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix. Also the matrix $\mathbf{P}$ is then said to diagonalize $\mathbf{A}$ or transform $\mathbf{A}$ to diagonal form.

Theorem 1: *An* $n \times n$ *matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.*

Proof: Suppose $\mathbf{A}$ is diagonalizable. Then $\mathbf{A}$ is similar to a diagonal matrix $\mathbf{D}$ = dia. $[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Therefore there exists an invertible matrix $\mathbf{P} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n]$ such that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$$

i.e., $$\mathbf{AP} = \mathbf{PD}$$

i.e., $$\mathbf{A}[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] \text{ dia.} [\lambda_1, \lambda_2, \ldots, \lambda_n]$$

i.e, $$[\mathbf{AX}_1, \mathbf{AX}_2, \ldots, \mathbf{AX}_n] = [\lambda_1\mathbf{X}_1, \lambda_2\mathbf{X}_2, \ldots, \lambda_n\mathbf{X}_n]$$

i.e., $$\mathbf{AX}_1 = \lambda_1\mathbf{X}_1, \ \mathbf{AX}_2 = \lambda_2\mathbf{X}_2, \ldots, \mathbf{AX}_n = \lambda_n\mathbf{X}_n.$$

Therefore $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Since the matrix $\mathbf{P}$ is non-singular, therefore its column vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are linearly independent. Hence $\mathbf{A}$ possesses n linearly independent eigenvectors.

Conversely, suppose that $\mathbf{A}$ possesses n linearly independent eigenvectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues. Then $\mathbf{AX}_1 = \lambda_1\mathbf{X}_1, \mathbf{AX}_2 = \lambda_2\mathbf{X}_2, \ldots, \mathbf{AX}_n = \lambda_n\mathbf{X}_n$.

Let $\mathbf{P} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n]$ and $\mathbf{D}$ = dia. $[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

Then $$\mathbf{AP} = [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] = [\mathbf{AX}_1, \mathbf{AX}_2, \ldots, \mathbf{AX}_n]$$

$$= [\lambda_1 \mathbf{X}_1, \lambda_2 \mathbf{X}_2, \ldots, \lambda_n \mathbf{X}_n]$$

$$= [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n] \text{ dia. } [\lambda_1, \lambda_2, \ldots, \lambda_n] = \mathbf{PD}.$$

Since the column vectors $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ of the matrix **P** are linearly independent, therefore **X** is invertible and $\mathbf{X}^{-1}$ exists.

Therefore $\mathbf{AP} = \mathbf{PD} \Rightarrow \mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{PD}$

$\Rightarrow \mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$

$\Rightarrow$ **A** is similar to a diagonal matrix **D**

$\Rightarrow$ **A** is diagonalizable.

Remark: In the proof of the above theorem we have shown that if **A** is diagonalizable and **P** diagonalizes **A**, then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = \mathbf{D}$$

if and only if the j^{th} column of **P** is an eigenvector of **A** corresponding to the eigenvalue λ_j of **A**, $(j = 1, 2, \ldots, n)$. The diagonal elements of **D** are the eigenvalues of **A** and they occur in the same order as is the order of their corresponding eigenvectors in the column vectors of **P**.

Theorem 2: *If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.*

Proof: Let **A** be a square matrix of order n and suppose it has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots., \lambda_n$. We know that eigenvectors of a matrix corresponding to distinct eigenvalues are linearly independent. Therefore **A** has n linearly independent eigenvectors and so it is similar to a diagonal matrix

$$\mathbf{D} = \text{dia. } [\lambda_1, \lambda_2, \ldots, \lambda_n].$$

Corollary: *Two $n \times n$ matrices with the same set of n distinct eigenvalues are similar.*

Proof: Suppose **A** and **B** are two $n \times n$ matrices with the same set of n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{D} =$ dia. $[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Then both **A** and **B** are similar to **D.** Now **A** is similar to **D** and **D** is similar to **B** implies that **A** is similar to **B**.

Note that the relation of similarity is transitive.

Theorem 3: *The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues coincides with the algebraic multiplicity.*

Proof: The condition is necessary. Suppose **A** is similar to a diagonal matrix **D** = dia. $[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Then $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of **A** and there exists a non-singular matrix **P** such that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}.$$

Let α be an eigenvalue of **A** of algebraic multiplicity k. Then exactly k among $\lambda_1, \lambda_2, \ldots, \lambda_n$, are equal to α.

Let m = rank $(\mathbf{A} - \alpha\mathbf{I})$. Then the system of equations

$$(\mathbf{A} - \alpha\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

have $n - m$ linearly independent solutions and so $n - m$ will be the geometric multiplicity of α. We are to prove that $k = n - m$. We know that the rank of a matrix does not change on multiplication by a non-singular matrix. Therefore

$$\begin{aligned}
\text{rank}\,(\mathbf{A} - \alpha\mathbf{I}) &= \text{rank}\,[\mathbf{P}^{-1}(\mathbf{A} - \alpha\mathbf{I})\mathbf{P}] \\
&= \text{rank}\,[\mathbf{P}^{-1}\mathbf{AP} - \alpha\mathbf{I}] \\
&= \text{rank}\,[\mathbf{D} - \alpha\mathbf{I}] = \text{rank dia. } [\lambda_1 - \alpha, \lambda_2 - \alpha, \ldots, \lambda_n - \alpha] \\
&= n - k, \text{ since exactly } k \text{ elements of dia. } [\lambda_1 - \alpha, \lambda_2 - \alpha, \ldots, \lambda_n - \alpha] \\
&\qquad\qquad\qquad \text{are equal to zero.}
\end{aligned}$$

Thus rank $(\mathbf{A} - \alpha\mathbf{I}) = m = n - k$. Therefore $k = n - m$.

Thus there are exactly k linearly independent eigenvectors corresponding to the eigenvalue α.

The condition is sufficient. Suppose that the geometric multiplicity of each eigenvalue of **A** is equal to its algebraic multiplicity. Let $\lambda_1, \ldots, \lambda_p$ be the set of p distinct eigenvalues of **A** with respective multiplicities $r_1, \ldots, r_p$. We have

$$r_1 + \ldots + r_p = n.$$

To prove that **A** is diagonalizable.

Let $\left.\begin{matrix} \mathbf{C}_{11}, & \mathbf{C}_{12}, & \ldots, & \mathbf{C}_{1r_1} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{C}_{p1}, & \mathbf{C}_{p2}, & \ldots, & \mathbf{C}_{pr_p} \end{matrix}\right\}$ ...(1)

be linearly independent sets of eigenvectors corresponding to the eigenvalues $\lambda_1,\ldots,\lambda_p$ respectively. We claim that the n vectors given in (1) are linearly independent. Let

$$(a_{11}\mathbf{C}_{11} + a_{12}\,\mathbf{C}_{12} + \ldots + a_{1r_1}\,\mathbf{C}_{1r_1}) + \ldots + (a_{p1}\mathbf{C}_{p1} + \ldots + a_{pr_p}\mathbf{C}_{pr_p}) = \mathbf{O} \qquad \ldots(2)$$

The relation (3) may be written as

$$\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_p = \mathbf{O}, \qquad \ldots(3)$$

where $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_p$ denote the vectors written within brackets in (2) *i.e.*, $\mathbf{X}_1 = a_{11}\mathbf{C}_{11} + \ldots + a_{1r_1}\mathbf{C}_{1r_1}$, and so on.

Now $\mathbf{X}_1$ is a linear combination of eigenvectors of **A** corresponding to the eigenvalue λ_1. Therefore if $\mathbf{X}_1 \neq \mathbf{O}$, then $\mathbf{X}_1$ is also an eigenvector of **A** corresponding to the eigenvalue λ_1.

Similarly we can speak for $\mathbf{X}_2,\ldots,\mathbf{X}_p$.

In case some one of $\mathbf{X}_1,\ldots,\mathbf{X}_p$ is not zero, then the relation (3) implies that a system of eigenvectors of **A** corresponding to distinct eigenvalues of **A** is linearly dependent. But this is not possible. Hence each of the vectors $\mathbf{X}_1, \mathbf{X}_2 \ldots, \mathbf{X}_p$ must be zero.

Since $\mathbf{C}_{11}, \mathbf{C}_{12}, \ldots, \mathbf{C}_{1r_1}$ is a set of linearly independent vectors, therefore $\mathbf{O} = \mathbf{X}_1 = a_{11}\,\mathbf{C}_{11} + \ldots + a_{1r_1}\,\mathbf{C}_{1r_1}$ implies that

$$a_{11} = 0, \ldots, a_{1r_1} = 0.$$

Similarly we can show that each of the scalars in relation (2) is zero. Therefore the n vectors give in (1) are linearly independent. Thus **A** has n linearly independent eigenvectors. So it is similar to a diagonal matrix.

Illustrative Examples

Example 19: *Show that the rank of every matrix similar to* **A** *is the same as that of* **A.**

Solution: Let **B** be a matrix similar to **A.** Then there exists a non-singular matrix **P** such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. We know that the rank of a matrix does not change on multiplication by a non-singular matrix.

Therefore

$$\text{rank } (\mathbf{P}^{-1}\mathbf{AP}) = \text{rank } \mathbf{A} \Rightarrow \text{rank } \mathbf{B} = \text{rank } \mathbf{A}.$$

Example 20: *If* $\mathbf{U}$ *be a unitary matrix such that* $\mathbf{U}^{\theta}\mathbf{AU} = diag\,[\lambda_1,\ldots,\lambda_n]$, *show that* $\lambda_1,\ldots,\lambda_n$ *are the eigenvalues of* $\mathbf{A}$.

Solution: Let diag $[\lambda_1,\ldots,\lambda_n] = \mathbf{D}$. Since $\mathbf{U}$ is unitary, therefore $\mathbf{U}^{\theta} = \mathbf{U}^{-1}$. So

$$\mathbf{U}^{\theta}\mathbf{AU} = \mathbf{D} \Rightarrow \mathbf{U}^{-1}\mathbf{AU} = \mathbf{D}.$$

Thus $\mathbf{A}$ is similar to the diagonal matrix $\mathbf{D}$. But similar matrices have the same eigenvalues and eigenvalues of $\mathbf{D}$ are its diagonal elements. Therefore $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $\mathbf{A}$.

Example 21: *Prove that if* $\mathbf{A}$ *is similar to a diagonal matrix, then* $\mathbf{A}^T$ *is similar to* $\mathbf{A}$.

Solution: Suppose $\mathbf{A}$ is similar to a diagonal matrix $\mathbf{D}$. Then there exists a non-singular matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\,\mathbf{AP} = \mathbf{D}$$

$\Rightarrow$ $\quad \mathbf{A} = \mathbf{P\,D\,P}^{-1}$

$\Rightarrow$ $\quad \mathbf{A}^T = (\mathbf{P\,D\,P}^{-1})^T = (\mathbf{P}^{-1})^T\,\mathbf{D}^T\,\mathbf{P}^T$

$\Rightarrow$ $\quad \mathbf{A}^T = (\mathbf{P}^T)^{-1}\,\mathbf{D\,P}^T \qquad [\because \mathbf{D} \text{ is diagonal} \Rightarrow \mathbf{D}^T = \mathbf{D}]$

$\Rightarrow$ $\quad \mathbf{A}^T$ is similar to $\mathbf{D}$

$\Rightarrow$ $\quad \mathbf{D}$ is similar to $\mathbf{A}^T$.

Finally $\mathbf{A}$ is similar to $\mathbf{D}$ and $\mathbf{D}$ is similar to $\mathbf{A}^T$ implies that $\mathbf{A}$ is similar to $\mathbf{A}^T$.

Nilpotent Matrix. Definition.

A non-zero matrix $\mathbf{A}$ *is said to be nilpotent, if for some positive integer r,* $\mathbf{A}^r = \mathbf{O}$.

Example 22: *Show that a non-zero matrix is nilpotent if and only if all its eigenvalues are equal to zero.*

Solution: Suppose $\mathbf{A} \neq \mathbf{O}$ and $\mathbf{A}$ is nilpotent. Then

$$\mathbf{A}^r = \mathbf{O}, \text{ for some positive integer } r$$

$\Rightarrow$ the polynomial λ^r annihilates $\mathbf{A}$

$\Rightarrow$ the minimal polynomial $m(\lambda)$ of **A** divides λ^r

$\Rightarrow$ $m(\lambda)$ is of the type λ^s, where s is some positive integer

$\Rightarrow$ 0 is the only root of $m(\lambda)$

$\Rightarrow$ 0 is the only eigenvalue of **A**

$\Rightarrow$ all eigenvalues of **A** are zero.

Conversely, each eigenvalue of $\mathbf{A} = 0$

$\Rightarrow$ characteristic equation of **A** is $\lambda^n = 0$

$\Rightarrow$ $\mathbf{A}^n = \mathbf{O}$, since **A** satisfies its characteristic equation

$\Rightarrow$ **A** is nilpotent.

Example 23: *A square matrix* **A** *is defined by*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Find the modal matrix **P** *and the resulting diagonal matrix* **D** *of* **A**.

Solution: The characteristic equation of **A** is

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & 0-\lambda \end{vmatrix} = 0$$

or $\quad (1-\lambda)(-2\lambda + \lambda^2 + 1) - 2(-\lambda + 1) - 2(-1 + 2 - \lambda) = 0$

or $\quad (-\lambda + 1)(\lambda - 1)^2 + (4\lambda - 4) = 0$

or $\quad (-\lambda + 1)(\lambda + 1)(\lambda - 3) = 0$

The roots of this equation are $1, -1, 3$.

The eigenvectors **X** of **A** corresponding to the eigenvalue 1 are given by the equation

$$(\mathbf{A} - 1\,\mathbf{I})\,\mathbf{X} = \mathbf{O} \quad \text{or} \quad (\mathbf{A} - \mathbf{I})\,\mathbf{X} = \mathbf{O}$$

i.e.,
$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_1$$

The coefficient matrix of these equations is of rank 2. So these equations have 1 linearly independent solution. These equations can be written as

$$0x + 2y - 2z = 0 \ , \ x + y + z = 0$$

From these, we get $y = z = 1$, say. Then $x = -2$.

Therefore $\mathbf{X}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

Now the eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues -1 are given by $(\mathbf{A} + \mathbf{I})\, \mathbf{X} = \mathbf{O}$

i.e.,
$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to 2R_3 + R_1$$

The coefficient matrix of these equations is of rank 2. So these equations have 1 linearly independent solution.

These equations can be written as

$$2x + 2y - 2z = 0 \ , \ x + 3y + z = 0$$

Let us take $z = -1$, then $y = 1$ and $x = -2$. Therefore $\mathbf{X}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1.

Now the eigenvectors of **A** corresponding to the eigenvalue 3 are given by $(\mathbf{A} - 3\mathbf{I})\,\mathbf{X} = \mathbf{O}$

i.e.,
$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} -2 & 2 & -2 \\ -1 & -1 & -3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; R_2 \leftrightarrow R_3$$

Similarly, we have $x = 2,\ y = 1, z = -1.$

Therefore $\mathbf{X}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector of **A**

corresponding to the eigenvalue 3.

Let modal matrix
$$\mathbf{P} = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3] = \begin{bmatrix} -2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$\therefore$
$$\mathbf{P}^{-1} = \frac{\text{Adj}\,\mathbf{P}}{|\mathbf{P}|} = -\frac{1}{8}\begin{bmatrix} 0 & -4 & -4 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}.$$

The matrix **P** will transform **A** to diagonal form **D** which is given by the relation

$$\mathbf{P}^{-1}\mathbf{A}\ \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{D}$$

Example 24: *Show that the matrix*

$$\mathbf{A} = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

is diagonalizable. Also find the transforming matrix and diagonal matrix.

Solution: The characteristic equation of **A** is

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1-\lambda & -1+\lambda & -1+\lambda \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0, \text{ applying } R_1 - (R_2 + R_3)$$

or $$(1-\lambda)\begin{vmatrix} 1 & -1 & -1 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

or $$(1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 4 & 1-\lambda & 2 \\ 3 & -1 & 4-\lambda \end{vmatrix} = 0, \text{ applying } C_2 + C_1, C_3 + C_1$$

or $$(1-\lambda)[(1-\lambda)(4-\lambda)+2] = 0$$

or $$(1-\lambda)(\lambda^2 - 5\lambda + 6) = 0$$

or $$(1-\lambda)(\lambda-2)(\lambda-3) = 0$$

The roots of this equation are 1, 2, 3.

Since the eigenvalues of the matrix **A** are all distinct, therefore **A** is similar to a diagonal matrix. Since the algebraic multiplicity of each eigenvalue of **A** is 1, therefore there will be one and only one linearly independent eigenvector of **A** corresponding to each eigenvalue of **A.**

The eigenvectors **X** of **A** corresponding to the eigenvalue 1 are given by the equation

$$(\mathbf{A} - 1\mathbf{I})\,\mathbf{X} = \mathbf{O} \text{ or } (\mathbf{A} - \mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 7 & -8 & -2 \\ -3 & 4 & 0 \\ 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1$$

or $$\begin{bmatrix} 7 & -8 & -2 \\ -3 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2.$$

The matrix of coefficients of these equations has rank 2. Therefore these equations have only one linearly independent solution as it should have been because the algebraic multiplicity of the eigenvalue 1 is 1. Note that the geometric multiplicity cannot exceed the algebraic multiplicity. The above equations can be written as $7x_1 - 8x_2 - 2x_3 = 0,\ -3x_1 + 4x_2 = 0$. From the last equation, we get $x_1 = 4, x_2 = 3$. Then the first gives $x_3 = 2$. Therefore $\mathbf{X}_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

The eigenvectors $\mathbf{X}$ of $\mathbf{A}$ corresponding to the eigenvalue 2 are given by the equation

$$(\mathbf{A} - 2\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 6 & -8 & -2 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

applying $R_2 \to R_3 - R_1,\ R_3 \to R_3 - \frac{1}{2}R_1$.

These equations can be written as $6x_1 - 8x_2 - 2x_3 = 0,\ -2x_1 + 3x_2 = 0$. From these, we get $x_1 = 3, x_2 = 2, x_3 = 1$.

Therefore $\mathbf{X}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

The eigenvectors $\mathbf{X}$ of $\mathbf{A}$ corresponding to the eigenvalue 3 are given by the equation

$$(\mathbf{A} - 3\mathbf{I})\,\mathbf{X} = \mathbf{O}$$

or $$\begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} 5 & -8 & -2 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

applying $R_2 \to R_2 - R_1,\ R_3 \to R_3 + R_1 - 2R_2$.

These equations can be written as

$$5x_1 - 8x_2 - 2x_3 = 0,\ -x_1 + 2x_2 = 0.$$

From these, we get $x_1 = 2, x_2 = 1, x_3 = 1.$

$\therefore$ $\mathbf{X}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 3.

Let $$\mathbf{P} = [\mathbf{X}_1\ \mathbf{X}_2\ \mathbf{X}_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The columns of **P** are linearly independent eigenvectors of **A** corresponding to the eigenvalues 1, 2, 3 respectively. The matrix **P** will transform **A** to diagonal form **D** which is given by the relation

$$\mathbf{P}^{-1}\ \mathbf{A}\ \mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{D}.$$

Comprehensive Exercise 1

1. (i) State Cayley-Hamilton theorem.

(Bundelkhand 2005; Agra 05; Lucknow 07)

(ii) Find the characteristic roots of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$.

(Meerut 2009 B)

2. (i) If $a+b+c=0$, find the characteristic roots of the matrix $\mathbf{A}=\begin{bmatrix} a & c & b \\ b & b & a \\ c & a & c \end{bmatrix}$.

(Kumaun 2008; Kanpur 07, 08)

(ii) Prove that the matrices :

$$\mathbf{A}=\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} 0 & b & a \\ b & 0 & c \\ a & c & 0 \end{bmatrix} \text{ have}$$

same characteristic equation.

(Rohilkhand 2007; Purvanchal 06; Agra 05)

3. (i) Verify Cayley-Hamilton theorem for matrix $\mathbf{A}$, where $\mathbf{A}=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(Rohilkhand 2011)

(ii) Verify that the matrix $\mathbf{A}$ satisfies its characteristic equation, where $\mathbf{A}=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Express $\mathbf{A}^6-4\mathbf{A}^5+8\mathbf{A}^4-12\mathbf{A}^3+14\mathbf{A}^2$ as a linear polynomial in $\mathbf{A}$.

4. Find the characteristic roots o the matrix $\mathbf{A}=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrhix. Find the inverse of the matrix $\mathbf{A}$ and also express $\mathbf{A}^5-4\mathbf{A}^4-7\mathbf{A}^3+11\mathbf{A}^2-\mathbf{A}-10\mathbf{I}$ as a linear polynomial in $\mathbf{A}$.

5. Verify that the matrix $\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ satisfies its characteristic equation and compute $\mathbf{A}^{-1}$.

(Purvanchal 2009)

6. Find the characteristic roots of the matrix $\mathbf{A}=\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ and verify Cayley-Hamilton theorem.

(Agra 2008; Bundelkhand 11; Meerut 09B)

7. Determine the eigenvalues and eigenvectors of the matrix

$$\mathbf{A}=\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

8. Show that the matrix $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ satisfies Calyley-Hamilton theorem.

Also determine the characteristic roots (*i.e.*, latent roots) and the corresponding characteristic vectors of the matrix **A**. **(Meerut 2007)**

9. Verify Cayley-Hamilton theorem for the matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$.

Hence or otherwise evaluate $\mathbf{A}^{-1}$. **(Meerut 2003, 09; Avadh 05)**

10. Find the characteristic equation of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ and show that it is satisfied by **A**. Hence obtain the inverse of the given matrix **A**.

11. Verify that the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix? State the theorem that applies here.

12. Verify Cayley-Hamilton theorem for the following matrix :

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(Avadh 2008)

13. Find the characteristic roots and the characteristic spaces of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}.$$

(Meerut 2005; Rohilkhand 07)

14. Show that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

15. Let **A** and **B** be n-rowed square matrices and let **A** be non-singular. Show that the matrices $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{B}\mathbf{A}^{-1}$ have the same eigenvalues.

16. If **A** and **B** are non-singular matrices of order n, show that the matrices **AB** and **BA** are similar.

17. **A** and **B** are two $n \times n$ matrices with the same set of n distinct eigenvalues. Show that there exist two matrices **P** and **Q** (one of them non-singular) such that

$$\mathbf{A} = \mathbf{PQ},\ \mathbf{B} = \mathbf{QP}.$$

18. Prove that a non-zero nilpotent matrix cannot be similar to a diogonal matrix.

19. Find a matrix **P** which diagonalizes the matrix $\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$. Verify that $\mathbf{P}^{-1}\ \mathbf{AP} = \mathbf{D}$, where **D** is the diagonal matrix.

20. Reduce the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ to diagonal form.

Answers 1

1. (ii) 1, –4, 7

2. (i) $\lambda = 0, \pm \left\{ \frac{3}{2} (a^2 + b^2 + c^2) \right\}^{1/2}$

3. (ii) $-4\mathbf{A} + 5\mathbf{I}$.

4. $5, -1; \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$

5. Characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

$$\mathbf{A}^{-1} = \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$$

6. All the characteristic roots are zero.

7. Eigenvalues are $5, 0, 0$. Corresponding to the eigenvalue 2 an eigenvector is $[1, 0, -2]'$. Two linearly independent eigenvectors corresponding to the eigenvalue 0 are $[2, 0, 1]'$ and $[0, 1, 0]'$

8. 5, 1, 1. Corresponding to the characteristic root 5 a characteristic vector is $[1, 1, 1]'$. Two linearly independent characteristic vectors corresponding to the characteristic root 1 are $[1, 0, -1]'$ and $[1, 1, -3]'$

9. $\mathbf{A}^{-1} = \frac{1}{5} \begin{bmatrix} 4 & 1 & -1 \\ -12 & 2 & 3 \\ 5 & 0 & 0 \end{bmatrix}$

10. $\lambda^3 - 4\lambda^2 - 13\lambda - 40 = 0;$

$$\mathbf{A}^{-1} = \frac{1}{40} \begin{bmatrix} -4 & 11 & -5 \\ -4 & 1 & 25 \\ 8 & -2 & -10 \end{bmatrix}$$

13. Characteristic roots are 1, 1, 2. Corresponding to the characteristic root 1 a characteristic vector is $[1, 0, 0]'$. Corresponding to the characteristic root 2 a characteristic vector is $[2, 1, 0]'$. The characteristic space corresponding to the characteristic root 1 consists of the vector $c\,[1, 0, 0]'$, where c is any scalar. Similarly for the other characteristic space

19. 2,5. Corresponding to the characteristic root 2 a characteristic vector is $[1, -2]'$. Corresponding to the characteristic root 5 a characteristic vector is $[1, 1]'$.

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

20. 1, 2, 3 corresponding to the characteristic root 1 a characteristic vector is $[1, 0, 0]'$. Corresponding to the characteristic root 2 a characteristic vector is $[1, -1, 0]'$. Corresponding to the characteristic root 3 a characteristic vector is $[3, -2, 2]'$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If λ is a characteristic root of a matrix **A**, then a characteristic root of $\mathbf{A}^{-1}$ is

(a) $1/\lambda$ (b) λ

(c) λ^2 (d) $1/\lambda^2$

(Bundelkhand 2001)

2. The eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & c & c \end{bmatrix}$ are

(a) a, h, g (b) a, h, c

(c) a, g, c (d) a, b, c

(Kanpur 2009, 11)

3. If λ is a characteristic root of the matrix **A**, then a characteristic root of the matrix $\mathbf{A} + k\mathbf{I}$ is

(a) λ (b) $k + \lambda$

(c) $k - \lambda$ (d) none of these **(Agra 2007)**

4. The characteristic equation of $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is

(a) $\lambda^2 - 5\lambda + 7 = 0$ (b) $\lambda^2 - 3\lambda + 7 = 0$

(c) $\lambda^2 - 2\lambda + 7 = 0$ (d) none of these **(Bundelkhand 2007)**

Fill in the Blank(s)

Fill in the blanks "………", so that the following statements are complete and correct.

1. If **A** is any n-rowed square matrix and λ an indeterminate, then the equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$ is called the ……… of **A** and the roots of this equation are called the ……… of the matrix **A.**
2. If λ is a characteristic root of an $n \times n$ matrix **A**, then a non-zero vector **X** such that $\mathbf{AX} = \lambda \mathbf{X}$ is called a ……… of **A** corresponding to the characteristic root λ.
3. The characteristic roots of a Hermitain matrix are ……… **(Avadh 2005)**
4. The characteristic roots of a skew-Hermitian matrix are either ……… or ………
5. The matrices **A** and **A′** have the ……… eigenvalues. **(Agra 2008)**
6. If $\lambda_1, \ldots\ldots\ldots, \lambda_n$ are the eigenvalues of **A**, then $k\lambda_1, \ldots\ldots\ldots, k\lambda_n$ are the eigenvalues of ……… **(Lucknow 2011)**
7. If the characteristic roots of **A** are $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$, then the characteristic roots of ……… are $\lambda_1^2, \lambda_2^2, \ldots\ldots\ldots, \lambda_n^2$.
8. Every square matrix ….. its characteristic equation. **(Meerut 2001)**
9. The characteristic equation of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is ……… .

True or False

Write 'T' for true and 'F' for false statement.

1. The characteristic roots of a real symmetric matrix are all real.
2. The characteristic roots of a real skew-symmetric matrix are all pure imaginary.
3. The characteristic roots of a unitary matrix are of unit modulus.
4. The characteristic roots of a diagonal matrix are just the diagonal elements of the matrix.
5. Two matrices **A** and $\mathbf{C}^{-1}\mathbf{AC}$ do not have the same characteristic roots.
6. Cayley-Hamilton theorem states that 'Every square matrix satisfies its characteristic equation'. **(Rohilkhand 2006)**
7. The characteristic equation of

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

is $\lambda^2 - 3\lambda + 7 = 0$. **(Meerut 2003; Kumaun 08)**

Answers

Multiple Choice Questions

1. (a) 2. (d) 3. (b) 4. (a)

Fill in the Blank(s)

1. characteristic equations; characteristic roots
2. characteristic vector
3. real
3. pure imaginary; zero
4. same
5. $k\mathbf{A}$
7. $\mathbf{A}^2$
8. satisfies
9. $\lambda^3 - 2\lambda + 1 = 0$

True or False

1. *T*
2. *F*
3. *T*
4. *T*
5. *F*
6. *T*
7. *F*

Chapter 5

Quadratic Forms

5.1 Quadratic Forms

Definition: *An expression of the form* $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ *, where* a_{ij}*'s are elements of a field F, is called a quadratic form in the n variables* $x_1, x_2, \ldots\ldots, x_n$ *over a field F.*

Real Quadratic Form. Definition : As expression of the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j,$$

where a_{ij}*'s are all real numbers, is called a real quadratic form in the n variables* $x_1, x_2, \ldots, x_n$.

For example

(i) $2x^2 + 7xy + 5y^2$ is a real quadratic form in the two variables x and y.

(ii) $2x^2 - y^2 + 2z^2 - 2yz - 4zx + 6xy$ is a real quadratic form in the three variables x, y and z.

(iii) $x_1^2 - 2x_2^2 + 4x_3^2 - 4x_4^2 - 2x_1x_2 + 3x_1x_4 + 4x_2x_3 - 5x_3x_4$ is a real quadratic form in the four variables x_1, x_2, x_3 and x_4.

Theorem : *Every Quadratic form over a field F in n variables* $x_1, x_2, \ldots, x_n$ *can be expressed in the form* $\mathbf{X}'\,\mathbf{BX}$ *where*

$$\mathbf{X} = [x_1, x_2, \ldots\ldots, x_n]^T$$

is a column vector and $\mathbf{B}$ *is a symmetric matrix of order n over the field F.*

Proof : Let $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, ...(1)

be a quadratic form over the field F in the n variables $x_1, x_2, \ldots\ldots, x_n$.

In (1) it is assumed that $x_i x_j = x_j x_i$. Then the total coefficient of $x_i x_j$ in (1) is $a_{ij} + a_{ji}$. Let us assign half of this coefficient to x_{ij} and half to x_{ji}. Thus we define another set of scalars b_{ij}, such that $b_{ii} = a_{ii}$ and $b_{ij} = b_{ji} = \frac{1}{2}(a_{ij} + a_{ji}), i \neq j$. Then we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j.$$

Let $\mathbf{B} = [b_{ij}]_{n \times n}$. Then $\mathbf{B}$ is a symmetric matrix of order n over the field F since $b_{ij} = b_{ji}$.

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}$. Then $\mathbf{X}^T$ or $\mathbf{X}' = [x_1 \; x_2 \ldots\ldots x_n]$.

Now $\mathbf{X}^{\mathbf{T}}\mathbf{BX}$ is a matrix of the type 1×1. It can be easily seen that the single element of this matrix is $\sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j$. If we identify a 1×1 matrix with its single element *i.e.*, if we regard a 1×1 matrix equal to its single element, then we have

$$\mathbf{X}^T\mathbf{BX} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Hence the result.

4.2 Matrix of a Quadratic Form

Definition : *If* $\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ *is a quadratic form in n variables* $x_1, x_2, \ldots\ldots,$ *then there exists a unique symmetric matrix* **B** *of order n such that* $\phi = \mathbf{X}^T \mathbf{BX}$, *where* $\mathbf{X} = [x_1 \ x_2 \ \ldots\ldots \ x_n]^T$.

The symmetric matrix **B** *is called the matrix of the quadratic form* $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$.

Since every quadratic form can always be so written that matrix of its coefficients is a symmetric matrix, therefore we shall be considering quadratic forms which are so adjusted that the coefficient matrix is symmetric.

4.2 Quadratic Form Corresponding to a Symmetric Matrix.

Let $\mathbf{A} = [a_{ij}]_{n \times n}$ be symmetric matrix over the field F and let $\mathbf{X} = [x_1 \ x_2 \ \ldots\ldots \ x_n]^T$ be a column vector. Then $\mathbf{X}^T \mathbf{AX}$ determines a unique quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ in n variables $x_1, x_2, \ldots\ldots, x_n$ over the field F.

Thus we have seen that there exists a one-to-one correspondence between the set of all quadratic forms in n variables over a field F and the set of all n-rowed symmetric matrices over F.

Illustrative Examples

Example 1: *Write down the matrix of each of the following quadratic forms and verify that they can be written as matrix products* $\mathbf{X}^T \mathbf{AX}$ *:*

(i) $x_1^2 - 18 x_1 x_2 + 5 x_2^2$.

(ii) $x_1^2 + 2x_2^2 - 5x_3^2 - x_1 x_2 + 4 x_2 x_3 - 3 x_3 x_1$.

Solution: **(i)** The given quadratic form can be written as $x_1 x_1 - 9 x_1 x_2 - 9 x_2 x_1 + 5 x_2 x_2$.

Let **A** be the matrix of this quadratic form. Then $\mathbf{A}=\begin{bmatrix}1 & -9\\ -9 & 5\end{bmatrix}$.

Let $\mathbf{X}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$. Then $\mathbf{X}'=[x_1 \;\; x_2]$.

We have $\mathbf{X}'\,\mathbf{A}=[x_1 \;\; x_2]\begin{bmatrix}1 & -9\\ -9 & 5\end{bmatrix}=[x_1-9x_2 \quad -9x_1+5x_2]$.

$$\therefore \qquad \mathbf{X}'\,\mathbf{AX}=[x_1-9x_2 \quad -9x_1+5x_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$=x_1(x_1-9x_2)+x_2\,(-9x_1+5x_2)$$

$$=x_1^{\,2}-9x_1x_2-9x_2x_1+5x_2^{\,2}$$

$$=x_1^{\,2}-18x_1x_2+5x_2^{\,2}.$$

(ii) The given quadratic form can be written as

$$x_1\;x_1-\frac{1}{2}x_1x_2-\frac{3}{2}x_1x_3-\frac{1}{2}x_2x_1+2x_2x_2+2x_2x_3-\frac{3}{2}x_3x_1+2x_3x_2-5x_3x_3.$$

Let **A** be the matrix of this quadratic form. Then

$$\mathbf{A}=\begin{bmatrix}1 & -1/2 & -3/2\\ -1/2 & 2 & 2\\ -3/2 & 2 & -5\end{bmatrix}$$

Obviously **A** is a symmetric matrix.

Let $\mathbf{X}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$. Then $\mathbf{X}'=[x_1 \;\; x_2 \;\; x_3]$.

We have $\mathbf{X}'\,\mathbf{A}=[x_1 \;\; x_2 \;\; x_3]\begin{bmatrix}1 & -1/2 & -3/2\\ -1/2 & 2 & 2\\ -3/2 & 2 & -5\end{bmatrix}$

$$=\left[x_1-\frac{1}{2}x_2-\frac{3}{2}x_3 \quad -\frac{1}{2}x_1+2x_2+2x_3 \quad -\frac{3}{2}x_1+2x_2-5x_3\right].$$

$$\therefore \quad \mathbf{X'\,AX} = \left[x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3 \quad -\frac{1}{2}x_1 + 2x_2 + 2x_3 \quad -\frac{3}{2}x_1 + 2x_2 - 5x_3\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1\left(x_1 - \frac{1}{2}x_2 - \frac{3}{2}x_3\right) + x_2\left(-\frac{1}{2}x_1 + 2x_2 + 2x_3\right) + x_3\left(-\frac{3}{2}x_1 + 2x_2 - 5x_3\right)$$

$$= x_1^2 - \frac{1}{2}x_1x_2 - \frac{3}{2}x_1x_3 - \frac{1}{2}x_2x_1 + 2x_2^2 + 2x_2x_3 - \frac{3}{2}x_3x_1 + 2x_3x_2 - 5x_3^2$$

$$= x_1^2 + 2x_2^2 - 5x_3^2 - x_1x_2 + 4x_2x_3 - 3x_3x_1.$$

Example 2: *Obtain the matrices corresponding to the following quadratic forms*

(i) $ax^2 + 2hxy + by^2$

(ii) $2x_1x_2 + 6x_1x_3 - 4x_2x_3.$

(iii) $x_1^2 + 5x_2^2 - 7x_3^2$

(iv) $2x_1^2 - 7x_3^2 + 4x_1x_2 - 6x_2x_3.$

Solution: **(i)** The given quadratic form can be written as $ax^2 + hxy + hyx + by^2$.

$\therefore$ if **A** is the matrix of this quadratic form, then

$$\mathbf{A} = \begin{bmatrix} a & h \\ h & b \end{bmatrix}, \text{ which is a symmetric matrix of order 2.}$$

(ii) The given quadratic form can be written as

$$0x_1^2 + x_1x_2 + 3x_1x_3 + x_2x_1 + 0x_2^2 - 2x_2x_3 + 3x_3x_1 - 2x_3x_2 + 0x_3^2.$$

$\therefore$ if **A** is the matrix of this quadratic form, then

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{bmatrix}.$$

(iii) The given quadratic form can be written as

$$x_1^2 + 0x_1x_2 + 0x_1x_3 + 0x_2x_1 + 5x_2^2 + 0x_2x_3 + 0x_3x_1 + 0x_3x_2 - 7x_3^2.$$

$\therefore$ if **A** is the matrix of this quadratic form, then $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -7 \end{bmatrix}$.

(iv) The given quadratic form can be written as

$$2x_1^2 + 2x_1x_2 + 0x_1x_3 + 2x_2x_1 + 0x_2^2 - 3x_2x_3 + 0x_3x_1 - 3x_3x_2 - 7x_3^2.$$

$\therefore$ if **A** is the matrix of this quadratic form, then $\mathbf{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & -3 \\ 0 & -3 & -7 \end{bmatrix}$.

Example 3: *Find the matrix of the quadratic form*

$$x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 + 6x_1x_3 - 8x_2x_3$$

and verify that it can be written as a matrix product $\mathbf{X}'\mathbf{AX}$.

Solution: The given quadratic form can be written as

$$x_1^2 + 2x_1x_2 + 3x_1x_3 + 2x_2x_1 - 2x_2^2 - 4x_2x_3 + 3x_3x_1 - 4x_3x_2 - 3x_3^2.$$

Let **A** be the matrix of this quadratic form. Then

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}.$$

Let $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Then $\mathbf{X}' = [x_1 \ x_2 \ x_3]$.

We have $\mathbf{X}'\mathbf{A} = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{bmatrix}$

$$= [x_1 + 2x_2 + 3x_3 \quad 2x_1 - 2x_2 - 4x_3 \quad 3x_1 - 4x_2 - 3x_3].$$

$\therefore$ $$\mathbf{X}'\mathbf{AX} = [x_1 + 2x_2 + 3x_3 \quad 2x_1 - 2x_2 - 4x_3 \quad 3x_1 - 4x_2 - 3x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= (x_1 + 2x_2 + 3x_3)\, x_1 + (2x_1 - 2x_2 - 4x_3)\, x_2 + (3x_1 - 4x_2 - 3x_3) x_3$$

$$= x_1^2 - 2x_2^2 - 3x_3^2 + 4x_1x_2 - 8x_2x_3 + 6x_3x_1.$$

Example 4: *Write down the quadratic forms corresponding to the following matrices :*

(i) $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$ *(ii)* $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$.

Solution: (i) Let $\mathbf{X} = [x_1 \ x_2 \ x_2]^T$ and $\mathbf{A}$ denote the given symmetric matrix. Then $\mathbf{X}^T \mathbf{AX}$ is the quadratic form corresponding to this matrix. We have

$$\mathbf{X}^T \mathbf{A} = [x_1 \ x_2 \ x_3] \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

$$= [5x_2 - x_3 \quad 5x_1 + x_2 + 6x_3 \quad -x_1 + 6x_2 + 2x_3].$$

$$\therefore \qquad \mathbf{X}^T \mathbf{AX} = x_1(5x_2 - x_3) + x_2(5x_1 + x_2 + 6x_3) + x_3(-x_1 + 6x_2 + 2x_3)$$

$$= x_2^2 + 2x_3^2 + 10x_1x_2 - 2x_1x_3 + 12x_2x_3.$$

(ii) Let $\mathbf{X} = [x_1 \ x_2 \ x_3 \ x_4]^T$ and $\mathbf{A}$ denote the given symmetric matrix. Then $\mathbf{X}^T \mathbf{AX}$ is the quadratic form corresponding to this matrix. We have

$$\mathbf{X}^T \mathbf{A} = [x_1 \ x_2 \ x_3 \ x_4] \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

$$= [x_2 + 2x_3 + 3x_4 \quad x_1 + 2x_2 + 3x_3 + 4x_4$$

$$2x_1 + 3x_2 + 4x_3 + 5x_4 \quad 3x_1 + 4x_2 + 5x_3 + 6x_4]$$

$$\therefore \qquad \mathbf{X}^T \mathbf{AX} = x_1(x_2 + 2x_3 + 3x_4) + x_2(x_1 + 2x_2 + 3x_3 + 4x_4)$$

$$+ x_3(2x_1 + 3x_2 + 4x_3 + 5x_4) + x_4(3x_1 + 4x_2 + 5x_3 + 6x_4)$$

$$= 2x_2^2 + 4x_3^2 + 6x_4^2 + 2x_1x_2 + 4x_1x_3 + 6x_1x_4 + 6x_2x_3 + 8x_2x_4 + 10x_3x_4.$$

Comprehensive Exercise 1

1. *Obtain the matrices corresponding to the following quadratic forms :*

(i) $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6zx.$

(ii) $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$

(iii) $a_{11}x_1^{\ 2} + a_{22}x_2^{\ 2} + a_{33}x_3^{\ 2} + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1.$

(iv) $x_1^{\ 2} - 2x_2^{\ 2} + 4x_3^{\ 2} - 4x_4^{\ 2} - 2x_1x_2 + 3x_1x_4 + 4x_2x_3 - 5x_3x_4.$

(v) $x_1x_2 + x_2x_3 + x_3x_1 + x_1x_4 + x_2x_4 + x_3x_4.$

(vi) $x_1^{\ 2} - 2x_2x_3 - x_3x_4.$

(vii) $d_1x_1^{\ 2} + d_2x_2^{\ 2} + d_3x_3^{\ 2} + d_4x_4^{\ 2} + d_5x_5^{\ 2}.$

2. *Write down the quadratic forms corresponding to the following symmetric matrices :*

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & a & b & c \\ a & 0 & l & m \\ b & l & 0 & p \\ c & m & p & 0 \end{bmatrix}.$

(iii) diag. $[\lambda_1, \lambda_2, \ldots\ldots, \lambda_n].$

Answers

1. (i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

(iii) $\begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 & 0 & \frac{3}{2} \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -\frac{5}{2} \\ \frac{3}{2} & 0 & -\frac{5}{2} & -4 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$

(vii) diag. $[d_1, d_2, d_3, d_4, d_5]$

2. (i) $x_1^2 + x_3^2 + 4x_1x_2 + 6x_1x_3 + 6x_2x_3$

(ii) $2ax_1x_2 + 2bx_1x_3 + 2cx_1x_4 + 2lx_2x_3 + 2mx_2x_4 + 2px_3x_4$

(iii) $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots\ldots + \lambda_n x_n^2$

Objective Type Questions

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. An expression of the form $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, where a_{ij}'s are elements of a field F, is called a …… in the n variables $x_1, x_2, \ldots\ldots, x_n$ over a field F.

2. There exists a one-to-one correspondence between the set of all quadratic forms in n variables over a field F and the set of all n-rowed …… matrices over F.

3. The matrix **A** corresponding to the quadratic form

$d_1 x_1^2 + d_2 x_2^2 + d_3 x_3^2 + d_4 x_4^2 + d_5 x_5^2$ is **A** = …………

4. The quadratic form corresponding to the symmetric matrix diag. $[\lambda_1, \lambda_2, \ldots\ldots, \lambda_n]$ is … .

5. The quadratic form corresponding to the matrix $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ is ….

True or False

Write 'T' for true and 'F' for false statement.

1. The matrix corresponding to the quadratic form $d_1x_1^2 + d_2x_2^2 + d_3x_3^2 + d_4x_4^2$ is a diagonal matrix.

2. Every quadratic form over a field F in n variables $x_1, x_2, \ldots\ldots, x_n$ can be expressed in the form $\mathbf{X}'\,\mathbf{BX}$ where $\mathbf{X} = [x_1, x_2, \ldots\ldots, x_n]^T$ is a column vector and $\mathbf{B}$ is a skew symmetric matrix of order n over the field F.

Answers

Fill in the Blank(s)

1. quadratic form
2. symmetric
3. diag. $[d_1, d_2, d_3, d_4, d_5]$
4. $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots\ldots + \lambda_n x_n^2$
5. $2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 + 10x_1x_3 - 4x_2x_3$

True or False

1. T
2. F

□