IAS MATHEMATICS (OPT.)-2016

PAPER - 1 : SOLUTIONS

Que:- 1(a) IAS -2016. Using, elementary row operation's, find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

Sol^n:- We know that,

$$A = IA \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A.$$

using, Row transformation, converting L.H.S to identity matrix;

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1.$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A$$

$$R_2 \leftrightarrow R_3.$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot A$$

$$R_3 \to R_3 + \frac{1}{2} R_2.$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ \frac{-3}{2} & 1 & \frac{1}{2} \end{bmatrix} A.$$

$$R_2 \to \frac{R_2}{-2} \text{ then } R_1 \to R_1 - 2R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} A$$

$$= A^{-1}.A.$$

Hence,

$$A^{-1} = \begin{bmatrix} 1.5 & -1 & 0.5 \\ 0.5 & 0 & -0.5 \\ -1.5 & 1 & 0.5 \end{bmatrix}$$

ii) IF $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ then find $A^{14}+3A-2I$

Sol^n:- characterstic equation of 'A' is $|A-\lambda I| = 0$.

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)\left[(2-\lambda)(-3-\lambda)+0\right] -1\left[(-3-\lambda)5+12\right] +3\left[-5+2(2-\lambda)\right] \quad \text{—①}$$

$$-\lambda^3+0\lambda^2+0\lambda+0=0$$

$\Rightarrow \lambda^3 = 0$ By cayley-Hamilton $A^3 = 0$

$A^{14}+3A-2I.$

$= (A^3)^4 . A^2 + 3A - 2I = 0 + 3A - 2I.$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 15 & 4 & 18 \\ -6 & -3 & -11 \end{bmatrix} \Rightarrow \text{Required result.}$$

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

Q.1.(b) i) Using elementary row operation, find the condition that, linear eq^n have a sol^n.

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$$x-2y+z=a$$
$$2x+7y-3z=b$$
$$3x+5y-2z=c.$$

Sol^n:- For linear equation, to have a sol^n,

$$\rho(A)=\rho(A|B).$$

$$[A|B]=\begin{bmatrix}1 & -2 & 1\\ 2 & 7 & -3\\ 3 & 5 & -2\end{bmatrix} \longrightarrow \begin{matrix}R_2\to R_2-2R_1\\ R_3\to R_3-3R_1\end{matrix}$$

$$=\begin{bmatrix}1 & -2 & 1\\ 0 & 11 & -5\\ 0 & 11 & -5\end{bmatrix}$$

$$\Rightarrow \underline{\rho(A)=2}$$

$$[A|B]=\left[\begin{array}{ccc|c}1 & -2 & 1 & a\\ 2 & 7 & -3 & b\\ 3 & 5 & -2 & c\end{array}\right] \longrightarrow \begin{matrix}R_2\to R_2-2R_1\\ R_3\to R_3-3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & -2 & 1 & a\\ 0 & 11 & -5 & b-2a\\ 0 & 0 & 0 & c-a-b\end{bmatrix}$$

For linear eq^n to have solution

$$\rho(A)=\rho(A|B).$$

$$\rho(A)=2 \Rightarrow \rho(A|B)=2$$

$$\Rightarrow c-ab=0 \quad \boxed{c=a+b}$$

ii) If $W_1 = \{(x,y,z) \mid x+y-z=0\}$.

$W_2 = \{(x,y,z) \mid 3x+y-2z=0\}$

$W_3 = \{(x,y,z) \mid x-7y+3z=0\}$

then find $\dim(W_1 \cap W_2 \cap W_3)$ & $\dim(W_1+W_2)$

Solⁿ:- Basis of $W_1 = \{(1,0,1), (-1,1,0)\}$

Basis of $W_2 = \{(\frac{2}{3},0,1), (\frac{-1}{3},1,0)\}$

Basis of $W_3 = \{(-3,0,1), (7,1,0)\}$.

For $\dim(W_1+W_2)$

$$[W_1+W_2] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2/3 & 0 & 1 \\ -1/3 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_1 + R_2 . \\ R_3 \to R_3 - \frac{2}{3}R_1 . \\ R_4 \to R_4 + \frac{R_1}{3} . \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$\boxed{\therefore \dim(W_1+W_2) = 3.}$

let solve, 3 eqⁿ, we get $x=y$ & $z=2x$.

so,

basis of $(W_1 \cap W_2 \cap W_3) = \{(1,1,2)\}$.

so, $\dim(W_1 \cap W_2 \cap W_3) = 1$.

Q.1(c) Evaluate. $I = \int_0^1 \sqrt[3]{x \cdot \log \frac{1}{x}}\, dx$.

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Sol^n:- let us substitute,

$$\log\left(\frac{1}{x}\right) = t$$

$$\Rightarrow \frac{1}{1/x}\left(\frac{-1}{x^2}\right) dx = dt.$$

$$\Rightarrow \frac{-1}{x} \cdot dx = dt \Rightarrow dx = -e^{-t} dt$$

$$I = \int_0^1 x^{1/3} \cdot \left[\log\left(\tfrac{1}{x}\right)\right]^{\frac{1}{3}} dx.$$

$$= \int_\infty^0 -e^{-t/3} \cdot t^{1/3} \cdot e^{-t}\, dt.$$

$$I = \int_0^\infty e^{-4t/3} \cdot t^{1/3} \cdot dt$$

let, $\frac{4t}{3} = p \Rightarrow \frac{4}{3} dt = dp$

$$= \int_0^\infty e^{-p} \left(\frac{3p}{4}\right)^{1/3} \cdot \frac{3}{4}\, dp.$$

$$= \left(\frac{3}{4}\right)^{4/3} \int_0^\infty e^{-p} \cdot p^{1/3} dp = \left(\frac{3}{4}\right)^{4/3} \int_0^\infty e^{-p} \cdot p^{\frac{4}{3}-1} dp.$$

we know, $\int_0^\infty t^{x-1} \cdot e^{-t} dt = \Gamma x$

so, $I = \left(\frac{3}{4}\right)^{4/3} \Gamma 4/3$.

Q1(d) Find the eq^n of sphere which passes through the circle $x^2+y^2=4$; $z=0$ & is cut by the plane $x+2y+2z=0$ in circle of radius '3'

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Sol^n:-

The eq^n of sphere through the circle

$x^2+y^2=4$; $z=0$ is

$x^2+y^2+z^2-4+\lambda z=0$

Radius of sphere : $R=\sqrt{\frac{\lambda^2}{4}+4}$

centre of sphere : $(0,0,-\lambda/2)$

length of $\perp$er 'p' from centre of sphere to $x+2y+2z=0$ is

$$p=\frac{|2(-\lambda/2)|}{\sqrt{4+4+1}}=\frac{\lambda}{3}.$$

$\therefore$ radius of circle :-

$$r^2=R^2-p^2$$

$$=\frac{\lambda^2}{4}+4-\frac{\lambda^2}{9}=3^2$$

$$\Rightarrow \lambda=\pm 6.$$

$\therefore$ eq^n of sphere :-

$$x^2+y^2+z^2\pm 6z-4=0.$$

Q.1 (e) Find the S.D betn line $\frac{x-1}{2}=\frac{y-2}{4}=z-3$ and $y-mx=z=0$

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For what value 'm' will the two line intersect?

Soln:- given two lines

$$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{1}$$

$$\frac{x}{1}=\frac{y}{m}=\frac{z}{0}$$

let, 'l', m', n' be the d.c.s of the line of the Shortest distance.

It will be ⊥er to both of lines, then

$$2l'+4m'+n'=0$$

$$l'+mm'+0=0$$

$$\Rightarrow \frac{l'}{-m}=\frac{m'}{1}=\frac{n'}{2m-4}=\frac{\sqrt{l'^2+m'^2+n'^2}}{\sqrt{m^2+1+(2m-4)^2}}$$

$$=\frac{1}{\sqrt{5m^2+17-16m}}$$

So, d.c.s are : $\frac{-m}{\sqrt{m^2 5-16m+17}}, \frac{1}{\sqrt{5m^2+17-16m}},$

$$\frac{2m-4}{\sqrt{5m^2+17-16m}}$$

length of S.D = projection of join of (1,2,3) & (0,0,0) on line with above d.c.s.

$$= \frac{-m+2+6m-12}{\sqrt{5m^2-16m+17}}$$

$\therefore$ The shortest distance is

$$\frac{5m-10}{\sqrt{5m^2-16m+17}} \text{ units.}$$

for two lines to intersect,

shortest distance $= 0$

$$\frac{5m-10}{\sqrt{5m^2-16m+17}} = 0$$

$$\Rightarrow \boxed{m=2}$$

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2(c) IAS-2016 P-I

(i) If $M_2(\mathbb{R})$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T: M_2(\mathbb{R}) \to P_2(x)$ s.t

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$$

with respect to the standard bases of $M_2(\mathbb{R})$ and $P_2(x)$. further find the null space of T.

(ii) If $T: P_2(x) \to P_3(x)$ is such that

$$T(f(x)) = f(x)+5\int_0^x f(t)\,dt$$ then

choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T.

Sol Given that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$$

let $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

$T = \{1, x, x^2\}$ be the standard bases of $M_2(\mathbb{R})$ & $P_2(x)$.

$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = 1 + x + 0x^2 = 1(1) + 1x + 0(x^2)$

$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right) = 0 + 0x + x^2 = 0(1) + 0(x) + 1(x^2)$

$T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right) = 1 + 0x + 0x^2 = 1(1) + 0(x) + 0(x^2)$

$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right) = 0 - x + 0x^2 = 0(1) + (-1)x + 0(x^2)$

$\therefore$ The required matrix of linear transformation

$$[T : M_2, P_2] = \begin{bmatrix}1&0&1\\1&0&0\\0&1&0\end{bmatrix}.$$

If $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = 0 = 0 + 0x + 0x^2$

then $(a+c) + (a-d)x + (b+c)x^2 = 0 + 0x + 0x^2$

$\Rightarrow \left.\begin{array}{l} a+c=0 \\ a-d=0 \end{array}\right] \Rightarrow c+d=0$ — (3)

$a-d=0$ — (1)

$b+c=0$ — (2)

(3) − (2) $\equiv b-d=0 \Rightarrow \boxed{b=d}$

(1) $\equiv \boxed{a=d}$

(3) $\equiv \boxed{c=-d}$.

$\therefore$ The null space of T

$$N(T) = \left\{ \begin{bmatrix}a&b\\c&d\end{bmatrix} \middle/ \begin{array}{l} b=d \\ a=d \\ c=-d \end{array} \right\} = \left\{ \begin{bmatrix}d&d\\-d&d\end{bmatrix} \middle/ d \in \mathbb{R} \right\}$$

(ii) Given that

$$T(f(x)) = f(x) + 5\int_0^x f(t)\,dt$$

let $S = \{1, 1+x, 1-x^2\}$ and $\longrightarrow$ (1)

$T = \{1, x, x^2, x^3\}$ be the bases of

$P_2(x)$ & $P_3(x)$.

$\therefore T(1) = 1 + 5\int_0^x 1\,dt = 1 + 5x \longrightarrow$ (a)

$$T(1+x) = 1 + x + 5\int_0^x (1+t)\,dt$$
$$= 1 + x + 5\left[t + \frac{t^2}{2}\right]_0^x = 1 + x + 5x + \frac{5x^2}{2}$$
$$= 1 + 6x + \frac{5x^2}{2} \longrightarrow (b)$$

$$T(1-x^2) = 1 - x^2 + 5\int_0^x (1-t^2)\,dt$$
$$= 1 - x^2 + 5\left[t - \frac{t^3}{3}\right]_0^x$$
$$= 1 - x^2 + 5\left[x - \frac{x^3}{3}\right]$$
$$= 1 - x^2 + 5x - \frac{5x^3}{3}$$
$$= 1 + 5x - x^2 - \frac{5x^3}{3}. \longrightarrow (c)$$

$\therefore T(1) = 1 + 5(x) + 0x^2 + 0x^3$

$T(1+x) = 1(1) + 6(x) + \left(\frac{5}{2}\right)x^2 + 0x^3$

$T(1-x^2) = 1(1) + 5(x) + (-1)x^2 + \left(-\frac{5}{3}\right)x^3$,

The required matrix representation:

$$[T : P_2, P_3] = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 6 & 5 \\ 0 & 5/2 & -1 \\ 0 & 0 & -5/3 \end{bmatrix}.$$

Q(b) (i) If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find the eigen values and eigen vectors of A.

IAS -2014 P-II

(ii) prove that eigen values of Hermitian matrix are all real.

Sol We have $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 2\lambda\right] = 0$$

$$\Rightarrow \lambda(\lambda - 2)(1-\lambda) = 0$$

$$\Rightarrow \boxed{\lambda = 0, 1, 2.}$$

Let us find characteristic vectors corr. to each characteristic root:

corr. to $\lambda = 0$: $(A - 0I)X = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

we have $x+y=0 \Rightarrow x=-y$.

$z=0$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = yX_1 \text{ say.}$$

$\therefore$ eigen vector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

<u>corr. to $\lambda=1$:</u>

$$(A-I)X=0$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$\Rightarrow x=y=z=0$

$$\therefore X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

<u>corr. to $\lambda=2$:</u>

$$\therefore (A-2I)X=0$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow -x + y = 0 \Rightarrow x = y$$

$$-z = 0 \Rightarrow z = 0$$

$$\therefore X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = y X_3 \text{ say}$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2(C) IFoS -2016 P-I → If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a linear transformation $T: P_1(x) \to P_2(x)$ w.r.t bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$ then find T.

Sol $T(1-x) = 1(1) - 2(1+x) + 1+x^2$
$= 1-2-2x+1+x^2$
$= -2x+x^2$ —— (i)

$T(x(1-x)) = -1+1+x+2+2x^2$
$= 2+x+2x^2$ —— (ii)

$T(x(1+x)) = 2-1-x+3+3x^2$
$= 4-x+3x^2$ —— (iii)

we have
$\alpha = a+bx+cx^2 \in P_2(x)$
then $a+bx+cx^2 = a_1(1)+b_1(1+x)+c_1(1+x^2)$ —— (i)

$\Rightarrow a+bx+cx^2 = (a_1+b_1+c_1)+b_1x+c_1x^2$

$\Rightarrow a_1+b_1+c_1 = a,\ b_1 = b,\ c_1 = c$

$a_1 + b + c = a$

$$\boxed{a_1 = a - b - c}$$

$\therefore \alpha = a + bx + cx^2 = (a-b-c)(1) + b(1+x) + c(1+x^2)$ —— (v)

$\therefore T(1-x) = -2x + 2x^2$

$= -3(1) + (-2)(1+x) + (1)(1+x^2)$ —— (vi)

$T(x(1-x)) = 2 + x + 2x^2$

$= -1 + 1(1+x) + 2(1+x^2)$ —— (vii)

$T(x(1+x)) = 4 - x + 3x^2$

$= 2(1) + (-1)(1+x) + 3(1+x^2)$ —— (viii)

The required matrix representation:

$$[T:B] = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

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2016
3(a) Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the conditions

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \quad \text{and} \quad x+y-z=0$$

Sol:- Let $f_1(x,y,z) = x^2+y^2+z^2$

$$f_2(x,y,z) = \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$$

$$f_3(x,y,z) = x+y-z=0$$

Let $F(x,y,z) = f_1(x,y,z) + \lambda f_2(x,y,z) + \mu f_3(x,y,z)$

$$F(x,y,z) = x^2+y^2+z^2+\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}\right)+\mu(x+y-z)=0$$

$\Rightarrow \cancel{2(x^2+y^2+z^2)+2\lambda=0}$

For stationary point; $dF=0$

$$dx = 2x+\frac{2x}{4}\lambda+\mu=0 \quad \text{——— (1)}$$

$$dy = 2y+\frac{2y}{5}\lambda+\mu=0 \quad \text{——— (2)}$$

$$dz = 2z+\frac{2z}{25}\lambda-\mu=0 \quad \text{——— (3)}$$

Multiplying eq^n (1), (2), (3) with x, y, z respectively and adding, we get

$$2(x^2+y^2+z^2)+2\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}\right)+\mu(x+y-z)=0$$

$$\Rightarrow 2(x^2+y^2+z^2)+2\lambda=0 \quad \left[\begin{array}{l}\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \text{ — (4)}\\ x+y-z=0 \text{ — (5)}\end{array}\right.$$

$$\boxed{\therefore\ x^2+y^2+z=-\lambda} \text{ — (6)}$$

from ①, ② & ③

$$x = \frac{-\mu/2}{1+\lambda/4} \; ; \; y = \frac{-\mu/2}{1+\lambda/5} \; ; \; z = \frac{\mu/2}{1+\lambda/25}$$

$$\text{eq}^n \; ⑤ \Rightarrow \frac{1}{1+\lambda/4} + \frac{1}{1+\lambda/5} + \frac{1}{1+\lambda/25} = 0$$

$$\Rightarrow 17\lambda^2 + 245\lambda + 750 = 0$$

By solving

$$\lambda = -10, \; \frac{-75}{17}$$

$$\therefore \text{Max.}(x^2+y^2+z^2) = 10$$

$$\text{Min}(x^2+y^2+z^2) = \frac{75}{17}$$

from ⑥

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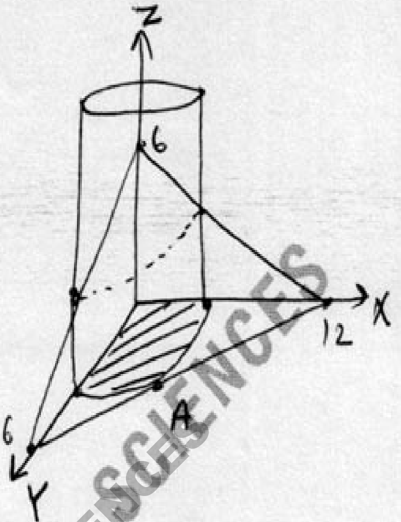
3(c) 2014 Find the surface area of the plane $x+2y+2z=12$ cut off by $x=0$, $y=0$ and $x^2+y^2=16$.

Ans. Plane $x+2y+2z=12$

or $\frac{x}{12}+\frac{y}{6}+\frac{z}{6}=1$

cuts the coordinate at a distance of 12, 6 and 6 from origin.



Cylinder : $x^2+y^2=16$

Planes : $x=0$, $y=0$

$$\text{Surface Area} = \iint_A \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$= \iint_A \sqrt{1+\left(-\frac{1}{2}\right)^2+(-1)^2}\,dx\,dy$$

$$= \frac{3}{2}\iint_A dx\,dy$$

$$= \frac{3}{2}\cdot\left[\frac{1}{4}\pi(4)^2\right]$$

$$= 6\pi$$

$\left(\because z = -\frac{x}{2} - y + 6\right.$

$\left. z_x = -\frac{1}{2},\ z_y = -1\right)$

(A : Projection of surface on xy-plane $x^2+y^2\le 16$ $x\ge 0, y\ge 0$)

3(b) Let $f(x,y) = \begin{cases} \frac{2x^4y-5x^2y^2+y^5}{(x^2+y^2)^2} & ,(x,y)\ne(0,0)\end{cases}$

Find a $\delta>0$ such that $|f(x,y)-f(0,0)|<0.01$, whenever $\sqrt{x^2+y^2}<\delta$.

Ans. This question is wrongly printed.

4(a). Find the surface generated by a line which intersects the lines $y=a=z$, $x+3z=a=y+z$ and parallel to the plane $x+y=0$.

2011

Ans. Equation of the required line :-

$$(y-a)+\lambda(z-a)=0 \text{ and}$$

$$(x+3z-a)+\mu(y+z-a)=0 \quad \text{— (1)}$$

$$\Rightarrow y+\lambda z-a-\lambda a=0$$

$$x+\mu y+(3+\mu)z-a-\mu a=0 \quad \text{— (2)}$$

If it's parallel to the plane $x+y=0$ — (3)

Then ratios of d.r.'s should be same.

$$\frac{x}{3+\mu-\lambda\mu}=\frac{y}{\lambda}=\frac{z}{-1}$$

So, dr's (x,y,z) are in the ratio of

$$((3+\mu-\lambda\mu), \lambda, -1)$$

As, it's parallel to $x+y=0$.

$$\therefore 3+\mu-\lambda\mu+\lambda=0 \quad \text{— (4)}$$

From (1), $\lambda=\dfrac{a-y}{z-a}$; $\mu=\dfrac{a-x-3z}{y+z-a}$

Substituting λ and μ in (4), we get

$$(y+z)(x+y)=2a(x+z)$$

(4)(b) Show that the cone $3yz - 2zx - 2xy = 0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ is a generator belonging to one such set, find the other two.

IAS 2011

Ans. Eqⁿ of the plane:

$$x + y + 2z = 0$$

Let lines of intersection be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

$$\Rightarrow l + m + 2n = 0$$

$$3mn - 2nl - 2lm = 0$$

Solving, $3mn - 2(n+m)(-m-2n) = 0$

$$\Rightarrow m = -4n, \ -\frac{n}{2}$$

for $m = -4n$, $l = 2n$

and for $m = -\frac{4n}{2}$, $l = -\frac{3}{2}n$

So, two lines dr's are:

$$\frac{l}{+2} = \frac{m}{-4} = \frac{n}{1} \quad \text{and} \quad \frac{l}{3} = \frac{m}{1} = \frac{n}{-2}$$

Therefore, the other generators are:

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{1} \quad \text{and} \quad \frac{x}{3} = \frac{y}{1} = \frac{z}{-2}.$$

(4) (c) Evaluate $\iint_R f(x,y)\,dx\,dy$ over the rectangle $R = [0,1; 0,1]$ where

$$f(x,y) = \begin{cases} x+y, & \text{if } x^2 < y < 2x^2 \\ 0, & \text{elsewhere} \end{cases}$$

Ans.

$$\int_0^1 \int_{x^2}^{2x^2} (x+y)\,dx\,dy$$

$$= \int_0^1 \left(xy + \frac{y^2}{2}\right)_{x^2}^{2x^2} dx$$

$$= \int_0^1 \left(x^3 + \frac{3}{2}x^4\right) dx$$

$$= \left[\frac{x^4}{4} + \frac{3}{2}\cdot\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{4} + \frac{3}{10} = \frac{11}{20}$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \frac{11}{20}$$

4(d) 2016

Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2+by^2+cz^2=1$.

Ans. The given conicoid is

$$ax^2+by^2+cz^2=1 \quad \text{—} (1)$$

Let $l_1x+m_1y+n_1z = \sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$ —(2)

$l_2x+m_2y+n_2z = \sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}}$ —(3)

and, $l_3x+m_3y+n_3z = \sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$ —(4)

be three mutually perpendicular tangent planes.

so that $\left.\begin{array}{l} l_1l_2+m_1m_2+n_1n_2=0 \text{ etc. and} \\ l_1l_3+m_1m_3+n_1n_3=0 \text{ etc. and} \\ l_1l_3+m_1m_3+n_1n_3=0 \text{ etc.}\end{array}\right\}$ —(5)

Also, $l_1^2+m_1^2+n_1^2=1$ etc. and $l_2^2+m_2^2+n_2^2=1$ etc.

The co-ordinates of the point of intersection satisfy the three eqns (2), (3), (4) and its locus is therefore obtained by eliminating l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 from equations

Squaring and adding (2), (3), (4)

We have

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$

$$=\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)+\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)+\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)$$

$$\Rightarrow x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3)^2+z^2(n_1^2+n_2^2+n_3^2)$$
$$+2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(m_1n_1+m_2n_2+m_3n_3)$$

$$+2zx(n_1l_1+n_2l_2+n_3l_3)=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

$$\Rightarrow x^2(1)+y^2(1)+z^2(1)+2xy(0)+2yz(0)+2zx(0)=\frac{1}{a}(1)+\frac{1}{b}(1)+\frac{1}{c}(1) \quad (\because \text{ from } (5))$$

$$\Rightarrow x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which is the required locus and is a sphere concentric with the conicoid and is known as the director sphere.

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

5 (a) Find a particular integral of $\frac{d^2y}{dx^2}+y=e^{x/2}\sin\frac{x\sqrt{3}}{2}$

Ans. 2016 P-15

Let, $\frac{d}{dx}=D$

$$(D^2+1)y=e^{x/2}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$\Rightarrow\ y=\frac{1}{(D^2+1)}e^{x/2}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$=e^{x/2}\frac{1}{\left(D+\frac{1}{2}\right)^2+1}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$=e^{x/2}\left[\frac{1}{D^2+D+\frac{5}{4}}\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$=e^{x/2}\left[\frac{1}{D+1/2}\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$=e^{x/2}\left[\frac{D-1/2}{D^2-1/4}\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$=e^{x/2}\left[\left(\frac{1}{2}-D\right)\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$=e^{x/2}\left[\frac{1}{2}\sin\left(\frac{x\sqrt{3}}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$\Rightarrow\ P.I.=\frac{e^{x/2}}{2}\left[\sin\left(\frac{\sqrt{3}x}{2}\right)-\sqrt{3}\cos\left(\frac{\sqrt{3}x}{2}\right)\right]$$

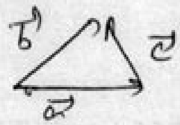
$$=e^{x/2}\left[\frac{1}{2}\sin\left(\frac{\sqrt{3}x}{2}\right)-\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}x}{2}\right)\right]$$

$$=e^{x/2}\left[\cos\frac{\pi}{3}\sin\left(\frac{\sqrt{3}x}{2}\right)-\underbrace{\frac{\sqrt{3}}{2}}_{\sin\frac{\pi}{3}}\cos\left(\frac{\sqrt{3}x}{2}\right)\right]$$

$$=-e^{x/2}\sin\left(\frac{\pi}{3}-\frac{\sqrt{3}x}{2}\right)$$

5 (b) 2016 P-II

Prove that the vectors $\vec{a} = 3\hat{i}+\hat{j}-2\hat{k}$, $\vec{b} = -\hat{i}+3\hat{j}+4\hat{k}$, $\vec{c} = 4\hat{i}-2\hat{j}-6\hat{k}$ can form the sides of a Δ, Find the length of the medians of the triangle.



Ans. Here,

$\vec{b}+\vec{c} = (-\hat{i}+3\hat{j}+4\hat{k}) + (4\hat{i}-2\hat{j}-6\hat{k})$

$= (3\hat{i}+\hat{j}-2\hat{k}) = \vec{a}$

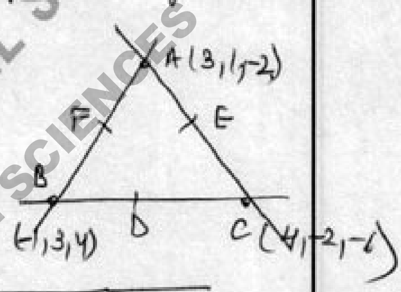
Hence $\vec{a}, \vec{b}, \vec{c}$ follows triangle law of addition

Therefore, $\vec{a}, \vec{b}$ and $\vec{c}$ are sides of a triangle.

Mid point of AB : $F(1,2,1)$

mid point of BC : $D(3/2, 1/2, -1)$

mid point of AC : $E\left(\frac{7}{2}, \frac{-1}{2}, -4\right)$



So,

Length of median $AD = \sqrt{(3-3/2)^2+(1-1/2)^2+(-2+1)^2}$

$= \sqrt{\frac{9}{4}+\frac{1}{4}+\frac{4}{4}} = \sqrt{7/2}$ units

Length of median $BE = \sqrt{\left(\frac{7}{2}+1\right)^2+\left(3+\frac{1}{2}\right)^2+(4+4)^2}$

$= \sqrt{\frac{81}{4}+\frac{49}{4}+64} = \sqrt{\frac{193}{2}}$ units

Length of median CF

$= \sqrt{(4-1)^2+(2+2)^2+(1+6)^2}$

$= \sqrt{9+16+49}$

$= \sqrt{74}$ units

5(c) Solve: $\frac{dy}{dx} = \frac{1}{1+x^2}\left(e^{\tan^{-1}x} - y\right)$

Ans. 2016 P-II

Given: $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$

Its of the standard form: $\frac{dy}{dx} + P_1 y = Q_1$

$$\Rightarrow P_1 = \frac{1}{1+x^2}$$

$$\therefore \int P_1\,dx = \int \frac{1}{1+x^2}\,dx = \tan^{-1}x$$

Integrating factor: $e^{\int P_1 dx} = e^{\tan^{-1}x}$

The required solⁿ of differential equation is:

$$y\,e^{\tan^{-1}x} = \int \frac{e^{2\tan^{-1}x}}{1+x^2}\,dx + c'$$

$$y\,e^{\tan^{-1}x} = \frac{e^{2\tan^{-1}x}}{2} + c', \quad c' \text{ is constant}$$

$$\Rightarrow \boxed{2y = e^{\tan^{-1}x} + ce^{-\tan^{-1}x}},$$

c is arbitrary constant.

P-1

5(c) IAS-2016

A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$.

Sol^n: Here the central acceleration varies inversely as the cube of the distance i.e, $P = \mu/r^3 = \mu u^3$, where μ is a constant.

If v is the velocity for a circle of radius a, then

$$\frac{v^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

$$\Rightarrow v = \sqrt{(\mu/a^2)}$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}\, v = \sqrt{2\mu/a^2}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u$$

Multiplying bothsides by $2(du/d\theta)$ and integrating we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \quad \text{--- (1)}$$

where A is a constant.

But initially when $r = a$ i.e. $u = 1/a$; $du/d\theta = 0$ (at an apse)

and $v = v_1 = \sqrt{2\mu/a^2}$.

$\therefore$ from (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

40

21

$\therefore\ h^2 = 2\mu$ and $A = \mu/a^2$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1-a^2u^2}{a^2}$$

$$\Rightarrow \sqrt{2}\,a\frac{du}{d\theta} = \sqrt{(1-a^2u^2)}$$

$$\Rightarrow \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1-a^2u^2)}}$$

Integrating, $\theta/\sqrt{2} + B = \sin^{-1}(au)$, where B is constant

But initially, when $u = 1/a$, $\theta = 0$.

$\therefore\ B = \sin^{-1} 1 = \frac{1}{2}\pi$

$\therefore\ \theta/\sqrt{2} + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$

$\Rightarrow au = a/r = \sin\left\{\frac{1}{2}\pi + (\theta/\sqrt{2})\right\}$

$\Rightarrow a = r\cos(\theta/\sqrt{2})$, which is the required equation of the path

7(a) A uniform rod AB of length $2a$ movable about a hinge at
IAS-2016 A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, Prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod.

Sol^n: Let a uniform rod AB of length $2a$ movable about the hinge at the end A rest with a smooth vertical wall CD.

Let W be the weight of the rod and G its middle point.

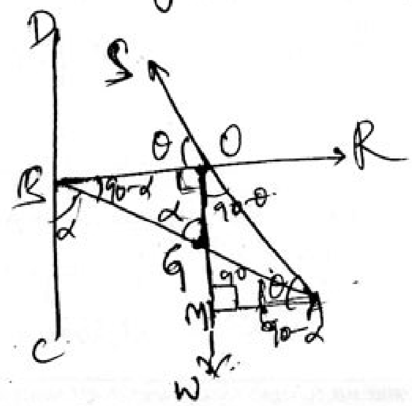
The rod is in equilibrium under the action of the following three forces only.

(i) R, the reaction of the wall at B acting at right angles to the wall.

(ii) S, the reaction of the hinge at A. and

(iii) W, the weight of the rod acting vertically downwards at its middle point G.

Since the force R and the line of action of W meet at O, therefore the reaction S of the hinge at A must also pass through O, as shown in the figure.



Let the rod AB and the reaction S make angles α and θ ~~respectively~~ with the vertical and horizontal respectively.

i.e,

$\angle ABC = \alpha$ and $\angle OAM = \theta$. and $\angle ABO = 90-\alpha$.

$\therefore$ $\angle OGB = \alpha$ and $\angle AOM = 90-\theta$.

In ΔOAB, by the trigonometrical theorem, we have

$$(AG+BG)\cot OGB = AG\cot AOG - BG\cot BOG.$$

$$(a+a)\cot\alpha = a\cot(90-\theta) - a\cot 90.$$

$$2a\cot\alpha = a\tan\theta - 0$$

$$\tan\theta = 2\cot\alpha. \quad \text{--- (1)}$$

$\therefore$ the reaction S at the hinge makes an angle $\theta = \tan^{-1}(2\cot\alpha)$ with the horizontal.

Now by Lami's theorem at the point O, we have

$$\frac{S}{\sin 90} = \frac{W}{\sin(180-\theta)} = \frac{R}{\sin(90+\theta)}$$

$$\therefore S = \frac{W}{\sin\theta} = W\operatorname{cosec}\theta = W\sqrt{1+\cot^2\theta}$$

$$= W\sqrt{\left(1+\frac{1}{4}\tan^2\alpha\right)}$$

$$= \frac{W}{2}\sqrt{4+\tan^2\alpha}.$$

2016 Q.8(c) A particle moves in a straight line, its accelaration directed towards a fixed point 'O' in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance 'x' from 'O'. If it starts from rest at a distance 'a' from 'O', show that it will arrive at 'O' with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.

Sol:- Take the centre of force 'O' as origin. Suppose a particle starts from rest at 'A', where OA=a. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time 't', where OP=x. The accelaration of the particle at 'P' is $\mu a^{5/3}.x^{-2/3}$ directed towards 'O'. Therefore, the equation of motion of the particle is –

$$\frac{d^2x}{dt^2} = -\mu a^{5/3}.x^{-2/3} \quad \text{———} (1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3}.x^{1/3}}{1/3} + K = -6\mu a^{5/3}.x^{1/3} + K$$

where. K is a constant.

At A ; $x=a$ and $\frac{dx}{dt}=0$; so that

$$-6\mu a^{5/3}.a^{1/3} + K = 0 \Rightarrow \boxed{K = 6\mu a^2}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -6\mu a^{5/3}.x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}(a^{1/3} - x^{1/3}) \quad (2)$$

which gives the velocity of particle at any distance x from the centre of force. Suppose, the particle arrives at O with the velocity v_1.

Then, at O ; $x=0$ and $\left(\frac{dx}{dt}\right)^2 = v_1^2$

So, from (2), we have

$$v_1^2 = 6\mu a^{5/3}(a^{1/3}-0) = 6\mu a^2$$

or $v_1 = a\sqrt{6\mu}$

Now, taking square root of (2), we get

$$\frac{dx}{dt} = -\sqrt{6\mu a^{5/3}}\sqrt{(a^{1/3}-x^{1/3})}$$

where the -ve sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$\boxed{dt = -\frac{1}{\sqrt{6\mu a^{5/3}}}\frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}} \quad \text{—— (3)}$$

let t_1 be the time from A to O. Then integrating (3) from A to O; we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{6\mu a^{5/3}}}\int_a^0 \frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}$$

$$\int_0^{t_1} = \frac{1}{\sqrt{6\mu a^{5/3}}}\int_0^a \frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}$$

Put $x = a\sin^6\theta$;

So that: $dx = 6a\sin^5\theta\cdot\cos\theta\, d\theta$; when $x=0$

$\theta = 0$ and when $x=a$, $\theta = \pi/2$

$$\therefore\ t_1 = \frac{1}{\sqrt{6\mu a^{5/3}}}\int_0^{\pi/2} \frac{6a\sin^5\theta\cdot\cos\theta}{a^{1/6}\cos\theta}\,d\theta$$

$$\therefore t_1 = \sqrt{\frac{6}{\mu}} \int_0^{\pi/2} \sin^5\theta \, d\theta$$

$$t_1 = \sqrt{\frac{6}{\mu}} \left[\frac{4 \cdot 2}{5 \cdot 3 \cdot 1} \right]$$

$$\boxed{t_1 = \frac{8}{15}\sqrt{\frac{6}{\mu}}}$$ Ans

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152