CSF 2019

(a) 1

$(D^2+1)y = x^2 \sin 2x$

Auxillary equation of given DE

$$m^2+1=0$$
$$m=\pm i$$

$$y_c = c_1\cos x + c_2 \sin x$$

$$y_p = \frac{1}{D^2+1} x^2 \sin 2x$$

$$= \text{Img} \frac{1}{D^2+1} x^2 e^{i2x}$$

$$= \text{Img} \left(e^{i2x} \frac{1}{(D+2i)^2+1} x^2 \right)$$

$$= \text{Img} \left(e^{i2x} \frac{1}{D^2+4Di-3} x^2 \right)$$

$$= \text{Img} \left[-\frac{e^{i2x}}{3} \left(1 - \left(\frac{D^2+4Di}{3}\right)\right)^{-1} x^2 \right]$$

(using binomial expression)

$$= \text{Img} \left[-\frac{e^{i2x}}{3} \left(1 + \frac{D^2+4Di}{3} + \frac{16}{9} D^2 i^2 \right) x^2 \right]$$

[neglecting higher order term]

$$= \text{Img} \left[-\frac{e^{i2x}}{3} \left(x^2 + \frac{2}{3} + \frac{4i}{3}(2x) - \frac{16}{9} \times 2 \right) \right]$$

$$= -\frac{1}{3} \left[\cos 2x \left(\frac{8x}{3}\right) + \sin 2x \left(x^2 + \frac{2}{3} - \frac{32}{9} \right) \right]$$

$$= -\frac{1}{27} \left[24x \cos 2x + (9x^2 - 26) \sin 2x \right]$$

$y = y_c + y_p$

$$\boxed{y = c_1\cos x + c_2\sin x - \frac{1}{27}\left[24x\cos 2x + (9x^2-26)\sin 2x\right]}$$

(b) 2

$$(px-y)(py+x) = h^2p \quad \text{where } p = y'$$

$$p^2xy + px^2 - py^2 - xy = h^2p \quad — (1)$$

let

$X = x^2 \quad Y = y^2$ — (ii) $\quad P = \frac{dY}{dX} = \frac{y}{x}p$ (ii) $\Rightarrow \quad p = \frac{Px}{y} = P\sqrt{\frac{X}{Y}}$ (iii)

combining (1) & (ii) & (iii), we have

$$P^2\frac{X}{Y}\sqrt{XY} + P\sqrt{\frac{X}{Y}}X - P\sqrt{\frac{X}{Y}}Y - \sqrt{XY} = h^2P\sqrt{\frac{X}{Y}}$$

$$P^2X + PX - PY - Y = h^2P$$

$$X(P^2+P) - Y(P+1) = h^2P$$

$$PX - Y = \frac{h^2P}{P+1}$$

$$Y = PX - \frac{h^2P}{P+1} \quad — (iv)$$

Equation (iv) is of Clairaut form

$$Y = PX + f(P)$$

so, replacing P by c

we have general solution

$$Y = cX - \frac{h^2c}{c+1}$$

Substituting $Y = y^2$, $X = x^2$ in above

required solution of differential equation:-

$$\boxed{y^2 = cx^2 - \frac{h^2c}{c+1}}$$

Scanned with CamScanner

(4) $\ddot{x} + 4y = \sin^2 2t$

$(D^2+4)\,x = \sin^2 2t$ where $D = \frac{dx}{dt}$

auxiliary equation: $m^2+4=0$

$m^2 = -4$

$m = \pm 2i$

$x_c = c_1 \cos 2t + c_2 \sin 2t$

$$x_p = \frac{1}{D^2+4} \sin^2 2t$$

$$= \frac{1}{D^2+4} \left[\frac{1-\cos 4t}{2}\right]$$

$$= \frac{1}{2} \left[\frac{1}{D^2+4} - \frac{1}{D^2+4} \cos 4t\right]$$

using $D^2 = -(4)^2$ in $\cos 4t$ term

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{\cos 4t}{-16+4}\right]$$

$$x_p = \frac{1}{2} \left[\frac{1}{4} + \frac{\cos 4t}{12}\right] = \frac{1}{8} + \frac{\cos 4t}{24}$$

$$x(t) = x_c + x_p$$

$$= c_1 \cos 2t + c_2 \sin 2t + \frac{1}{8} + \frac{1}{24} \cos 4t$$

at $t = \pi/8$ $x = 0$

$$0 = \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} + \frac{1}{8} + \frac{1}{24} \times 0$$

$$c_1 + c_2 = -\frac{1}{4\sqrt{2}} \quad \text{—(1)}$$

Scanned with CamScanner

at $t = \pi/8$ $\dot{x} = 0$

$$-2c_1\left(\frac{1}{\sqrt{2}}\right) + 2c_2\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{24}(-4) = 0$$

$$c_2 - c_1 = \frac{1}{6\sqrt{2}} \quad \text{— (ii)}$$

(i) + (ii)

$$2c_2 = \frac{1}{6\sqrt{2}} - \frac{1}{4\sqrt{2}}$$

$$2c_2 = \frac{-1}{12\sqrt{2}}$$

$$c_2 = \frac{-1}{24\sqrt{2}}$$

$$c_1 = \frac{-5}{24\sqrt{2}}$$

so $\boxed{x(t) = \frac{-1}{24\sqrt{2}}(5\cos 2t + \sin 2t) + \frac{1}{8} + \frac{1}{24}\cos 4t}$

(5)

$$(x-2)y'' - (4x-7)y' + (4x-6)y = 0$$

$$y'' - \frac{(4x-7)}{x-2}y' + \frac{(4x-6)}{x-2}y = 0$$

Comparing with $y'' + Py' + Qy = R$

$$P = -\frac{(4x-7)}{x-2} \quad Q = \frac{4x-6}{x-2} \quad R = 0$$

Casing

$$a^2 + aP + Q = 0$$

$$a^2 - a\frac{(4x-7)}{x-2} + \frac{4x-6}{x-2} = 0$$

$$x(a^2 - 4a + 4) - 2a^2 + 7a - 6 = 0$$

$$x(a-2)^2 - 2(a-2)(a - 3/2) = 0$$

$$\boxed{a = 2}$$

so $u = e^{2x}$ is one solution.

To find another solution v

we have

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

Let $\frac{dv}{dx} = z$

$$\frac{dz}{dx} + \left(-\frac{(x-1)}{x-2} + \frac{2}{e^{2x}} \cdot 2e^{2x}\right)z = 0$$

$$\frac{dz}{dx} - \frac{z}{x-2} = 0$$

$$\frac{dz}{z} = \frac{dx}{x-2}$$

$$\log\left(\frac{z}{x-2}\right) = c_1$$

$$\frac{dv}{dx} = z = c_1(x-2)$$

$$v = c_1\left(\frac{x^2}{2} - 2x\right) + c_2$$

general solution

$= u \cdot v$

$$\boxed{y = e^{2x}\left[c_1\left(\frac{x^2}{2} - 2x\right) + c_2\right]}$$

(3)

$$x''(t) - 2\frac{x(t)}{t^2} = t \quad , \quad 0 < t < \infty$$

comparing with $x''(t) + Px'(t) + Qx(t) = R$

$$P = 0 \quad Q = -\frac{2}{t^2} \quad R = t$$

Here using $m(m-1) + pmt + Qt^2 = 0$

for $\underline{m=2}$ $\quad 2 + 2Pt + Qt^2$

$$= 2 + 2 \times 0 \times t - \frac{2}{t^2} \times t^2$$

$$= 2 - 2 = 0$$

so one solution is $\boxed{u = t^2}$

Scanned with CamScanner

Now, to find another solution v

$$\frac{d^2v}{dt^2} + \left(P + \frac{2}{u}\frac{du}{dt}\right)\frac{dv}{dt} = \frac{R}{u}$$

Let $z = \frac{dv}{dt}$

$$\frac{dz}{dt} + \left(0 + \frac{2}{t^2}\cdot 2t\right) z = \frac{t}{t^2}$$

$$\frac{dz}{dt} + \frac{4}{t} z = \frac{1}{t}$$

$$I.F = e^{\int 4/t \, dt}$$

$$= e^{4\log t}$$

$$= t^4$$

$$z\cdot t^4 = \int \frac{1}{t}\cdot t^4 \, dt$$

$$z\cdot t^4 = \frac{t^4}{4} + c_1$$

$$\frac{dv}{dt} = z = \frac{1}{4} + c_1 t^{-4}$$

$$v = \int \frac{1}{4} + c_1 t^{-4}$$

$$= \frac{1}{4} + \frac{c_1 t^{-3}}{-3} + c_2$$

$$y_c = u\cdot v$$

$$= t^2\left[\frac{t}{4} + \left(\frac{c_1 t^{-3}}{-3}\right) + c_2\right]$$

$$= \frac{t^2}{4} - \frac{c_1 t^{-1}}{3} + c_2 t^2$$

$$y_c = \left(c_2 + \frac{1}{4}\right)t^2 - \frac{c_1}{3t}$$

comparing

$$x_c = c_1' u + c_2' v$$

where $c_1' = c_2 + 1/4$

$c_2' = -\frac{c_1}{3}$

$u(t) = t^2$

$v(t) = \frac{1}{t}$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} t^2 & 1/t \\ 2t & \frac{-1}{t^2} \end{vmatrix} = -1 - 2 = -3 \neq 0$$

so u & v are independent

from method of parameters

$$x_p(t) = A t^2 + \frac{B}{t}$$

$$A = -\int \frac{vR\,dt}{uv' - u'v} = -\int \frac{1/t \cdot t}{-3}\,dt = t/3$$

$$B = \int \frac{uR\,dt}{uv' - u'v} = \int \frac{t^2 \cdot t}{-3}\,dt = -\frac{t^4}{12}$$

$$x_p(t) = \frac{t^3}{3} - \frac{t^3}{12} = \frac{t^3}{4}$$

$$x = x_c(t) + x_p(t)$$

$$x = \left(c_2 + \frac{1}{4}\right) t^2 - \frac{c_1}{3t} + \frac{t^3}{4}$$

$$\boxed{x(t) = c_1' t^2 + \frac{c_2'}{t} + \frac{t^3}{4}}$$

$c_1' = c_2 + 1/4$

$c_2' = -c_1/3$