4b) Let $f_n(x) = \dfrac{x}{n+x^2}$, $x \in [0,1]$.
Show that the sequence $\langle f_n \rangle$ is uniformly convergent on $[0,1]$.

Pointwise limit function

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in (0,1]$$

Consider,

$$|f_n(x) - f(x)| = \left|\frac{x}{n+x^2} - 0\right| = \frac{x}{n+x^2}$$

$$\leq \frac{x}{n} \leq \frac{1}{n} \quad \text{on } x \in (0,1]$$

$$\left(\because x^2+n \geq n \Rightarrow \frac{1}{x^2+n} < \frac{1}{n}\right)$$

$$\lim_{n\to\infty} \operatorname{Sup} |f_n(x) - f(x)| \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

i.e. $\sup |f_n(x) - f(x)| \leq 0$

But $|f_n(x) - f(x)| \geq 0$

$\Rightarrow \operatorname{Sup} |f_n(x) - f(x)| = 0$

By M_n test $M_n = \operatorname{Sup} |f_n(x) - f(x)| = 0$

$\therefore f_n(x) = \dfrac{x}{n+x^2}$ is uniformly convergent on $[0,1]$