INSTITUTE FOR IAS/IFoS/CSIR/GATE EXAMINATIONS
MATHEMATICS by K. Venkanna

Mains Test Series - 2021

Test-V, Paper-I, full Syllabus

Answer Key

1.(a) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$x+2y-3z=a,\quad 2x+6y-11z=b,\quad x-2y+7z=c.$

Sol^n: The matrix form of the given system of equations is

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 6 & -11\\ 1 & -2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & -4 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c-a\end{bmatrix}\quad \begin{array}{l}R_2\to R_2-2R_1\\ R_3\to R_3-R_1\end{array}$$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c+2b-5a\end{bmatrix}\quad R_3\to R_3+2R_2$$

The system will have no solution if $c+2b-5a\neq 0$.

Thus the system will have atleast one solution if $c+2b-5a=0$ i.e. $5a=2b+c$

which is the required condition.

Note: In this case the system will have infinitely many solutions. In otherwords the system cannot have a unique solution.

1.(b) → Let $A=\begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$. Is A similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix?

Sol^n :- The characteristic polynomial of A is

$$|xI - A| = \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix} = (x-2)(x^2+1)$$

∴ The characteristic values of A are $2, \pm i$

Let A be similar to a diagonal matrix over R. Then there exists an invertible matrix P such that $P^{-1}AP = \text{diag}(a, b, c)$, where a, b, c are eigenvalues of A and $a, b, c \in R$.

But the characteristic values of A are $2, \pm i \in C$. So we arrive at a contradiction.

Hence A is not similar over the field R to a diagonal matrix. Since the characteristic values of A are $2, \pm i$, which are all distinct.

So A is similar over C to a diagonal matrix.

1(C) Let $f(x,y)$ be defined by

$$f(x,y)=\begin{cases}(x^2+y^2)\log(x^2+y^2) & ,\ (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$$

Prove that f_{xy} and f_{yx} are not continuous at $(0,0)$ but $f_{xy}(0,0)=f_{yx}(0,0)$.

Sol^n: Suppose $(x,y)\to(0,0)$ along the curve

$$y=x-mx^3.$$

Then

$$f(x,x-mx^3)=\frac{x^3+(x-mx^3)^3}{x-(x-mx^3)}$$

$$=\frac{x^3\left[1+(1-mx^2)^3\right]}{mx^3}$$

$$=\frac{2}{m}.$$

which is different for different values of m.

Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist and so the given function is discontinuous at $(0,0)$.

Now $\phi_x(0,0)=\lim\limits_{h\to0}\left[\phi(0+h),0)-\phi(0,0)\right]/h=\lim\limits_{h\to0}\dfrac{h^2-0}{h}=0,$

$\phi_y(0,0)=\lim\limits_{k\to0}\left[\phi(0,0+k)-\phi(0,0)\right]/k=\lim\limits_{k\to0}\dfrac{-k^2-0}{k}=0.$

$\therefore$ First order partial derivatives exist at the origin.

Hence the result.

1(d) A figure consists of a semi-circle with a rectangle on its diameter. Given that the perimeter of the figure is 20 feet, find its dimensions in order that its area may be maximum.

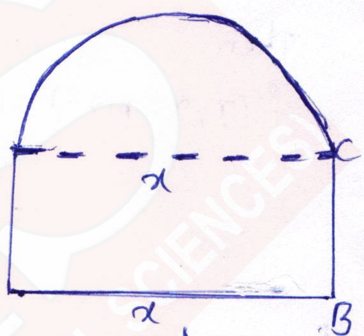
Sol^n: Let x be the breadth any y be the height of the rectangle. Then the diameter of the semi-circle is x. Therefore the perimeter of the fig.

$$= x + 2y + \pi x/2 = 20 \quad \text{—— (1)}$$

Let A be the area of the fig.

Then $A = xy + \frac{1}{2}\pi (x/2)^2$

$$= x\left(10 - x/2 - \pi x/4\right) + \frac{\pi x^2}{8}$$



$\left[\because \text{from (1)},\ y = 10 - \frac{x}{2} - \frac{\pi x}{4}\right]$

For a maximum or a minimum of A, we have

$$dA/dt = 10 - x - \pi x/4 = 0$$

$$\Rightarrow x(1 + \pi/4) = 10$$

$$\Rightarrow x = 40/(\pi + 4)$$

Now $d^2A/dx^2 = -1 - \frac{1}{4}\pi$, which is $-$ve when $x = 40/(\pi+4)$

Hence A is maximum when $x = 40/(\pi+4)$

when $x = 40/(\pi+4)$, we get from (1),

$$y = 10 - \frac{20}{(\pi+4)} - \frac{10\pi}{(\pi+4)} = \frac{20}{(\pi+4)}.$$

Hence the area of the figure is maximum when the radius of the semi-circle = the height of the rectangle $= \frac{20}{(\pi+4)}$ feet.

1.(e) Prove that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie.

Sol^n: Given lines are coplanar, if

$$\begin{vmatrix} (a-d)-(b-c) & a-b & (a+d)-(b+c) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

Adding 3rd column to first we get

$$\begin{vmatrix} 2(a-b) & a-b & (a+d)-(b+c) \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}=0$$

The first column being twice the second column, the determinant on the left vanishes, hence the given lines are coplanar.

Also the equation of the plane in which the two given lines lie is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}=0$$ adding 3rd column to the first.

$$\Rightarrow \begin{vmatrix} (x+z-2a)-2(y-a) & y-a & z-a-d \\ 2\alpha-2\alpha & \alpha & \alpha+\delta \\ 2\beta-2\beta & \beta & \beta+\gamma \end{vmatrix}=0$$ substracting twice second column from first.

$$\Rightarrow \begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix}=0 \Rightarrow (x+z-2y)[\alpha(\beta+\gamma)-\beta(\alpha+\delta)]=0$$

$$\Rightarrow x+z-2y=0.$$

2(a)(i) Let W be the vectorspace of 3×3 antisymmetric matrices over K. Show that $\dim W = 3$ by exhibiting a basis of W.

Solⁿ: (i) Let $$W(K) = \left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \Big/ a, b, c \in K \right\}$$

be the vectorspace of all 3×3 anti-symmetric matrices.

Let $A = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \in W(K)$ then

$$A = a\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in L(S)$$

where $$S = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq W(K)$$

$\therefore A \in W(K) \Rightarrow A \in L(S)$

$\therefore L(S) = W(K)$

Clearly S is linearly independent subset of $W(K)$

$\therefore S$ is a basis of W and $\underline{\dim(W) = 3}$.

2(a)(ii) → Find a basis and dimension of the subspace W of V spanned by the polynomials
$v_1 = t^3-2t^2+4t+1$, $v_2 = 2t^3-3t^2+9t-1$, $v_3 = t^3+6b-5$
$v_4 = 2t^3-5t^2+7t+5$.

Sol^n: Since W is spanned by polynomials of degree 3.

$\therefore$ W is a subspace of the space $V_3(\mathbb{R})$.
(the space of all real polynomials of degree ≤ 3)

w.r.t $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$

$\therefore$ The co-ordinate vectors of v_1, v_2, v_3, v_4 w.r.t the above basis are

$(1,4,-2,1)$, $(1,9,-3,2)$, $(-5,6,0,1)$ and $(5,7,-5,2)$

Now form the matrix A whose rows are these co-ordinate vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ -5 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 5R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + R_2 \end{matrix}$$

which is in the echelon form

The non-zero rows of the echelon form of A form a basis of the subspace W.

i.e. the vectors $(1,4,-2,1)$, $(0,13,-5,3)$ form a basis for W.

$\therefore$ A basis for W consists of polynomials t^3-2t^2+4t+1 and $3t^3-5t^2+13t$

$\dim W = 2$.

2(b)(i) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\sin 2u}{8}(2\cos^2 u - 3).$$

sol'n: $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \tan u$ ——— (1)

$$\therefore z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}\left(1+\sqrt{y/x}\right)} = x^{1/2} f(y/x)$$

So z is a homogeneous function in x & y of degree $\frac{1}{2}$.

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$

$$\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\frac{\partial u}{\partial y}\right) = \frac{1}{2}\tan u, \text{ by (1)}.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\tan u}{\sec^2 u} = \frac{1}{2}\sin u\cos u = \frac{1}{4}\sin 2u.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u \quad \text{——— (2)}$$

partially differentiating (2) w.r.t x, on both sides, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{2}\cos 2u\frac{\partial u}{\partial x}.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\,\partial y} = \frac{x}{2}\cos 2u\frac{\partial u}{\partial x}. \quad \text{——— (3)}$$

partially diff (3) w.r.t y, on both sides, we get

$$x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\cos 2u\frac{\partial u}{\partial y}.$$

$$\Rightarrow xy\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{y}{2}\cos 2u\frac{\partial u}{\partial y} \quad \text{——— (4)}$$

Adding (3) & (4), we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}.$$

$$=\frac{\cos 2u}{2}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

Using (1), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{1}{4}\sin 2u$$

$$=\frac{\cos 2u}{2}\cdot\frac{1}{4}\sin 2u.$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{4}\sin 2u\left[\frac{\cos 2u}{2}-1\right]$$

$$=\frac{1}{4}\sin 2u\left[\frac{2\cos^2 u-1}{2}-1\right]$$

$$=\frac{\sin 2u}{4}\left[\frac{2\cos^2 u-3}{2}\right]$$

$$=\frac{\sin 2u}{8}\left[2\cos^2 u-3\right]$$

2(b)(ii) Evaluate $\int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$.

Sol^n: put $x=\tan\theta$ so that $dx=\sec^2\theta\,d\theta$

$$\therefore \int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_0^{\pi/2}\log(\tan\theta+\cot\theta)\,d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{2}{2\sin\theta\cos\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{2}{\sin 2\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log 2\,d\theta - \int_0^{\pi/2}\log\sin 2\theta\,d\theta$$

$$= \frac{\pi}{2}\log 2 - \frac{1}{2}\int_{x=0}^{\pi}\log\sin x\,dx \quad \text{where } 2\theta=x$$

$$= \frac{\pi}{2}\log 2 - 2\times\frac{1}{2}\int_0^{\pi/2}\log\sin x\,dx.$$

$$= \frac{\pi}{2}\log 2 - \int_0^{\pi/2}\log\sin x\,dx = \frac{\pi}{2}\log 2 - I \quad (1)$$

$$\Rightarrow I = \int_0^{\pi/2}\log\sin x\,dx = \int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx$$

$$= \int_0^{\pi/2}\log\cos x\,dx$$

$$\therefore 2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$$

$$= \int_0^{\pi/2} \log \sin x \cos x \, dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

Let $2x = y$ so that $dx = \frac{1}{2} dy$.

$$\therefore 2I = \frac{1}{2} \int_{y=0}^{\pi} \log \sin y \, dy - \frac{\pi}{2} \log 2$$

$$= 2 \cdot \frac{1}{2} \int_0^{\pi/2} \log \sin y \, dy - \frac{\pi}{2} \log 2$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2$$

$$2I = I - \frac{\pi}{2} \log 2$$

$$\Rightarrow I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log \frac{1}{2} \quad \text{———} \quad (2).$$

Substituting (2) in eqn (1).

$$\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1+x^2} = \frac{\pi}{2} \log 2 + \frac{\pi}{2} \log 2$$

$$\therefore \int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1+x^2} = \pi \log 2$$

2(c)i) Find the equation of the sphere which passes through the points (1,0,0), (0,1,0) and (0,0,1) and has its radius as small as possible.

Sol: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad \text{——— (1)}$$

If it passes through (1,0,0), (0,1,0) and (0,0,1) then

$$1+2u+c=0,$$
$$1+2v+c=0,$$
$$1+2w+c=0.$$

or $\quad u=v=w=-\frac{1}{2}(1+c)$ ——— (2)

$\therefore$ If r be the radius of the sphere (1), then

$$r^2=u^2+v^2+w^2+c=R \text{ (say)}$$

or $\quad R=\frac{3}{4}(1+c)^2-c$, from (2)

If r is least then R is least.

Now $\frac{dR}{dc}=\frac{3}{2}(1+c)-1$ and $\frac{d^2R}{dc^2}=\frac{3}{2}$ = positive.

Equating $\frac{dR}{dc}$ to zero, we get

$$\frac{3}{2}c+\frac{1}{2}=0$$

or $c=-\frac{1}{3}$ and $\frac{d^2R}{dc^2}$ being positive R is least when $c=-\frac{1}{3}$.

$\therefore$ From (2) when R i.e. r^2 is least we have $u=v=w=-\frac{1}{2}\left(1-\frac{1}{3}\right)=-\frac{1}{3}$.

$\therefore$ From (1), the required equation is

$$x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$$

or $3(x^2+y^2+z^2)-2(x+y+z)-1=0$.

2(c)(ii). The section of a cone with vertex at P and guiding curve $x^2/a^2 + y^2/b^2 = 1,\ z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $x^2/a^2 + \{(y^2+z^2)/b^2\} = 1$.

Sol'n: Let the vertex P of the cone by (α, β, γ).

Any line through $P(\alpha,\beta,\gamma)$ is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ —— ①

This line meets the plane $z=0$ is $\left(\alpha - \dfrac{l\gamma}{n}, \beta - \dfrac{m\gamma}{n}, 0\right)$

and if this point lies on the given curve $x^2/a^2 + y^2/b^2 = 1$,

we have $\dfrac{1}{a^2}\left(\alpha - \dfrac{l\gamma}{n}\right)^2 + \dfrac{1}{b^2}\left(\beta - \dfrac{m\gamma}{n}\right)^2 = 1$ —— ②

Eliminating l, m, n between ① & ②, the equation of the cone is $\dfrac{1}{a^2}\left[\alpha - \left(\dfrac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + \dfrac{1}{b^2}\left[\beta - \left(\dfrac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 1$

$$\Rightarrow b^2(\alpha z - x\gamma)^2 + a^2(\beta z - y\gamma)^2 = a^2 b^2 (z-\gamma)^2 \quad \text{—— ③}$$

The section of this cone by the plane $x=0$ gives the conic on yz-plane as

$$b^2\alpha^2 z^2 + a^2(\beta z - y\gamma)^2 = a^2b^2(z-\gamma)^2$$

$$a^2y^2\gamma^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2)z^2 - 2a^2\beta\gamma yz + 2a^2b^2\gamma z - a^2b^2\gamma^2 = 0$$

If it represents a rectangular hyperbola on the yz-plane, then the sum of the coefficients of y^2 and z^2 must be zero.

i.e. $a^2\gamma^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2) = 0$

$$\Rightarrow \alpha^2/a^2 + [(\beta^2+\gamma^2)/b^2] = 1.$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ is

$$\frac{x^2}{a^2} + \frac{(y^2+z^2)}{b^2} = 1. \text{ Hence proved.}$$

3(a)(i) Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigen values of the matrix $B = M^2 - 2M + I$.

Sol^n.

$$M^2 = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)^2 & [(1+i)+(1-i)]2i & (i+3)(1+2i)+6i^2 \\ 0 & (1-i)^2 & 3i(1-i+i) \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 4i & 7i-5 \\ 0 & -2i & 3i \\ 0 & 0 & -1 \end{bmatrix}$$

Now

$$B = M^2 - 2M + I$$

$$= \begin{bmatrix} 2i & 4i & 7i-5 \\ 0 & -2i & 3i \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2+2i & 4i & 2i+6 \\ 0 & 2-2i & 6i \\ 0 & 0 & 2i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 5i-11 \\ 0 & -1 & -3i \\ 0 & 0 & -2i \end{bmatrix}$$

Let λ be the eigen value of B. Then, its augment matrix $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 & 5i-11 \\ 0 & -1-\lambda & -3i \\ 0 & 0 & -2i-\lambda \end{vmatrix} = 0$$

$$(1+\lambda)\left[(1+\lambda)(2i+\lambda)\right] = 0$$

$$\lambda = -1, \ \lambda = -1, \ \lambda = -2i$$

3.(a)(ii) Find the characteristic equation of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol: The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)[(1-\lambda)(2-\lambda)] - 1(0-0) + 1[0-(1-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)^2 - 1] = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2 - 4\lambda + 3] = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Caley-Hamilton theorem, the matrix A must satisfy its characteristic equation.

So we must have

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{--- (1)}$$

To evaluate

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$= A^5(0) + A(0) + A^2 + A + I \quad (\because \text{from (1)})$

$= A^2 + A + I$

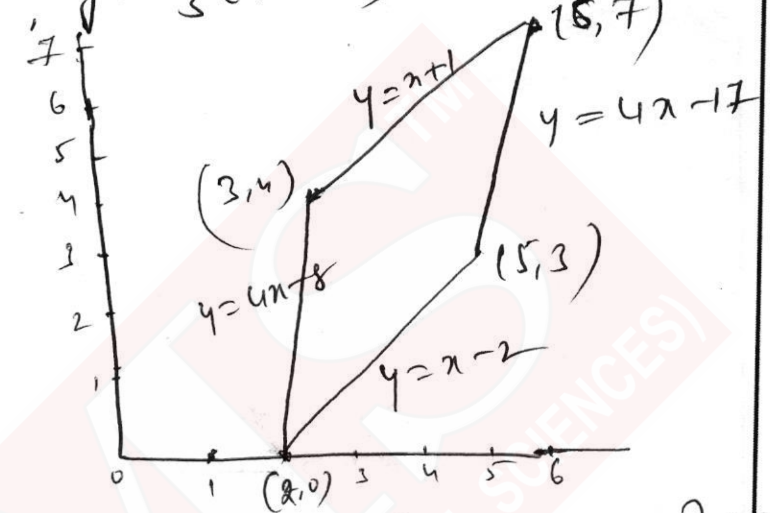
$$\therefore A^2 + A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

which is the required matrix

3(b) Evaluate $\iint_R 6x-3y \, dA$ where R is the parallelogram with vertices $(2,0)$, $(5,3)$, $(6,7)$ and $(3,4)$ using the transformation $x=\frac{1}{3}(v-u)$, $y=\frac{1}{3}(4v-u)$ to R.

Sol^n:



The equations of each of the boundaries of the region are given in the sketch

Now, let's transform each of the boundary curves.

$$y=x+1 \Rightarrow \frac{1}{3}(4v-u)=\frac{1}{3}(v-u)+1$$

$$\Rightarrow 3v=3$$

$$\Rightarrow v=1$$

$$y=4x-17 \Rightarrow \frac{1}{3}(4v-u)=\frac{4}{3}(v-u)-17$$

$$\Rightarrow 3u=-51 \Rightarrow u=-17$$

$$y=x-2 \Rightarrow \frac{1}{3}(4v-u)=\frac{1}{3}(v-u)-2$$

$$\Rightarrow 3v=-6 \Rightarrow v=-2$$

$y = 4x - 8 \Rightarrow \frac{1}{3}(4v-u) = \frac{4}{3}(v-u) - 8$

$\Rightarrow 3u = -24 \Rightarrow u = -8.$

Here is the sketch of the transformed region and note that the transformed region will be much easier to integrate over than the original region

So, the limits of transformed region are

$-17 \le u \le -8$
$-2 \le v \le 1$

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{vmatrix}$$

$$= -\frac{4}{9} + \frac{1}{9} = -\frac{1}{3}$$

$$\iint_R 6x-3y \, dA = \int_{-17}^{-8}\int_{-2}^{1} \left[6\left(\frac{1}{3}\right)(v-u) - 3\left(\frac{1}{3}\right)(4v-u)\right] \left|\frac{-1}{3}\right| dv\,du$$

$$= \int_{u=-17}^{-8}\int_{v=-2}^{1} -\frac{1}{3}(2v+u)\, dv\,du$$

$$= \int_{-17}^{-8} \left[-\frac{1}{3}(v^2+uv)\right]_{-2}^{1} du$$

$$= \int_{-17}^{-8} (1-u)\, du = \left(u - \frac{1}{2}u^2\right)_{-17}^{-8}$$

$$= \frac{243}{2}$$

3.(c)(i) Find the equations to the tangent planes to the hyperboloid $2x^2-6y^2+3z^2=5$ which pass through the line $x+9y-3z=0=3x-3y+6z-5$

Sol^n: Any plane through the given line is
$$(x+9y-3z)+\lambda(3x-3y+6z-5)=0$$
$$\Rightarrow (1+3\lambda)x+(9-3\lambda)y+(6\lambda-3)z=5\lambda \quad \text{--- (1)}$$
If this plane touches the given hyperboloid
$$2x^2-6y^2+3z^2=5$$
i.e., $$\frac{x^2}{(5/2)}+\frac{y^2}{(-5/6)}+\frac{z^2}{(5/3)}=1$$
then $a^2l^2+b^2m^2+c^2n^2=p^2$.
$$\Rightarrow \left(\frac{5}{2}\right)^2(1+3\lambda)^2+\left(\frac{-5}{6}\right)^2(9-3\lambda)+\left(\frac{5}{3}\right)^2(6\lambda-3)^2=(5\lambda)^2$$
$$\Rightarrow 15(9\lambda^2+6\lambda+1)-5(9\lambda^2-54\lambda+81)$$
$$+10(36\lambda^2-36\lambda+9)=150\lambda^2.$$
$$\Rightarrow 300\lambda^2-300=0$$
$$\Rightarrow \lambda^2=1$$
$$\Rightarrow \lambda=\pm 1$$
$\therefore$ From (1), the required tangent planes are
$$(1\pm 3)x+(9\mp 3)y+(\pm 6-3)z=\pm 5$$
i.e., $$4x+6y+3z=5$$
and $$2x-12y+9z=5.$$

3(c)(ii) Find the locus of the mid points of the chords of the conicoid $ax^2+by^2+cz^2=1$ which passes through (α, β, γ).

Sol^n: Let (x_1, y_1, z_1) be the mid point of the chord of the given conicoid. Then the locus of the chords of the given conicoid with (x_1, y_1, z_1) as mid-point is $\quad T = S_1$

where $T = axx_1 + byy_1 + czz_1 - 1$ and

$S_1 = ax_1^2 + by_1^2 + cz_1^2 - 1$

i.e. $axx_1 + byy_1 + czz_1 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1$

$\Rightarrow axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2$

If it passes through (α, β, γ). we have

$a\alpha x_1 + b\beta y_1 + c\gamma z_1 = ax_1^2 + by_1^2 + cz_1^2$

$\therefore$ The required locus of the mid point (x_1, y_1, z_1) of the chords of the given conicoid is

$ax^2 + by^2 + cz^2 = a\alpha x + b\beta y + c\gamma z$

$ax(x-\alpha) + by(y-\beta) + cz(z-\gamma) = 0.$

4(a). Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$

(a) verify that T is a linear transformation.

(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b and c that the vector be in the range of T? what is the rank of T?

(c) what are the conditions on $a, b, \& c$ that (a, b, c) be in the null space of T? what is the nullity of T?

Sol^n: Let $T : F^3 \to F^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3) \quad \text{—— (1)}$$

and F is a subfield of complex numbers.

(a) Let $\alpha, \beta \in F^3$ such that $\alpha = (x_1, x_2, x_3)$, $\beta = (y_1, y_2, y_3)$

and $a, b \in F$ Then we have

$$a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$
$$= (ax_1 + by_1,\ ax_2 + by_2,\ ax_3 + by_3)$$

Now we have

$$T(a\alpha + b\beta) = T(ax_1 + by_1,\ ax_2 + by_2,\ ax_3 + by_3)$$
$$= (ax_1 + by_1 - ax_2 - by_2 + 2ax_3 + 2by_3,$$
$$2ax_1 + 2by_1 + ax_2 + by_2,$$
$$-ax_1 - by_1 - 2ax_2 - 2by_2 + 2ax_3 + 2by_3)$$
$$= a(x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ 2x_1 + x_2,\ 2y_1 + y_2)$$
$$+ b(y_1 - y_2 + 2y_3,\ 2y_1 + y_2,\ -y_1 - 2y_2 + 2y_3)$$
$$= aT(\alpha) + bT(\beta)$$

$$\therefore\ T(a\alpha + b\beta) = aT(\alpha) + bT(\beta).$$

$\therefore T: F^3 \to F^3$ is a linear transformation.

Let $R(T) = \{ T(\alpha) / \alpha \in F^3 \}$

$= \{ (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3) / x_1, x_2, x_3 \in F \}$

be the range space.

Let $(a, b, c) \in R(T)$ if $(a, b, c) \in T(x_1, x_2, x_3)$ for some $(x_1, x_2, x_3) \in F^3$.

then
$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= a \\ 2x_1 + x_2 &= b \\ -x_1 - 2x_2 + 2x_3 &= c \end{aligned}\right\} \text{ --- } (1)$$

$\Rightarrow AX = B$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c+a \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 + R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ b+c-a \end{bmatrix} \quad R_3 \to R_3 + R_2$$

Hence $b + c - a = 0 \Rightarrow a = b + c$ is the required condition such that
$(a, b, c) \in$ Range T.

Let $(x_1, x_2, x_3) \in \ker T$ then
$T(x_1, x_2, x_3) = (0, 0, 0)$

using ①, we obtain
$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}\right\} \quad \text{— ②}$$

$$\Rightarrow Ax = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 + R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 + R_2 \quad \text{— ③}$$

Since the rank of the coefficient matrix is 2, the no. of L.I solutions of the system ② is

$n - r = 3 - 2 = 1$.

from ③, we have
$$\left.\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 3x_2 - 4x_3 &= 0 \end{aligned}\right\} \quad \text{— ④}$$

we can give an arbitrary value to x_3.

Taking $x_3 = 3$, we get $x_2 = 4$ and $x_1 = -2$

$\therefore (x_1, x_2, x_3) = (-2, 4, 3)$ is the only L.I element of $\ker T$ and so

$$\ker T = \{a(-2,4,3) / a \in \mathbb{R}\}$$

we have $\dim \ker T = 1$.

Since $\{(-2,4,3)\}$ is a basis of $\ker T$.

$\therefore$ Nullity $T = 1$

Since Rank T + Nullity T = dim F^3

$\Rightarrow$ Rank + 1 = 3

$\Rightarrow$ Rank T = 2

Let $(a,b,c) \in \ker T$ if $T(a,b,c) = (0,0,0)$

Then $a-b+2c=0$.

$2a+b=0$

$-a-2b+2c=0$

$\Rightarrow a-2b+2c=0, \quad 3b-4c=0$

Hence $a=-2k, \quad b=4k, \quad c=3k$

where k is arbitrary.

are the required conditions a, b, c

if $(a,b,c) \in \ker T$.

4(b) Find the maximum and minimum values of the function $f(x,y,z)=3x-y-2z$; subject to the constraints $x+y-z=0$, $x^2+2z^2=1$.

Sol^n: Let's define $g(x,y,z)=x+y-z$ and

$$h(x,y,z)=x^2+2z^2$$

So the problem is to find the maximum of $f(x,y,z)$ subject to the constraints $g(x,y,z)=0$ and $h(x,y,z)=1$.

we have $\nabla f = \lambda\nabla g + \mu\nabla h \Leftrightarrow (3,-1,-3)=\lambda(1,1,-1)+\mu(2x,0,4z)$.

$$\Rightarrow 3=\lambda+2\mu x \quad \text{——— (1)}$$

$$\Rightarrow -1=\lambda \quad \text{——— (2)}$$

$$\Rightarrow -3=-\lambda+4\mu z \quad \text{——— (3)}$$

$$x+y-z=0 \quad \text{——— (4)}$$

$$x^2+2z^2=1 \quad \text{——— (5)}$$

put $\lambda=-1$ in equation (1) and (3)

we obtain $x=\frac{2}{\mu}$ and $z=-\frac{1}{\mu}$ respectively.

Substitute x & z values in equation (5)

$$\left(\frac{2}{\mu}\right)^2+2\left(-\frac{1}{\mu}\right)^2=1 \Rightarrow \mu=\pm\sqrt{6}$$

Now from (4) we have $y=z-x$,

So we get

$$\mu=\sqrt{6} \Rightarrow x=\frac{2}{\sqrt{6}},\ z=-\frac{1}{\sqrt{6}},\ y=\frac{-3}{\sqrt{6}}$$

$$\mu=-\sqrt{6} \Rightarrow x=\frac{-2}{\sqrt{6}},\ z=\frac{1}{\sqrt{6}},\ y=\frac{3}{\sqrt{6}}$$

Since the intersection of $x+y-z=0$ and $x^2+2z^2=1$ is closed and bounded, all we need to do now is evaluate f at the critical points we have found.

$f\left(\frac{2}{\sqrt{6}}, \frac{-3}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = 2\sqrt{6}$ is the maximum value,

$f\left(-\frac{2}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -2\sqrt{6}$ is the minimum value.

4.(c) If A and A' are the extremities of the major axis of the principal elliptic section and any generator meets two generators of the same system through A and A' in P and P' respectively, then prove that

$$AP.\ A'P' = b^2 + c^2$$

Solⁿ: We know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \text{are given by}$$

$$x=\frac{a(1+\lambda\mu)}{\lambda+\mu},\ y=\frac{b(\lambda-\mu)}{\lambda+\mu},\ z=\frac{c(1-\lambda\mu)}{\lambda+\mu} \quad \text{--- (1)}$$

The extremities of the major axis of the principal elliptic section are $A(a,0,0)$ and $A'(-a,0,0)$.

At A and A' from (1) we have

$$\lambda-\mu=0,\quad 1-\lambda\mu=0$$

$$\Rightarrow \lambda=\mu \text{ and } 1-\lambda^2=0$$

$$\Rightarrow \lambda=\pm 1$$

Now consider the generator through $A(a,0,0)$ corresponding to $\lambda=1$ and then its points of intersection P with a generator of μ-system is obtained from (1) by putting $\lambda=1$ and is $\left[a, \frac{b(1-\mu)}{1+\mu}, \frac{c(1-\mu)}{1+\mu}\right]$

(or) (a, bt, ct) where $t=\frac{1-\mu}{1+\mu}$

$$\therefore AP^2=(a-a)^2+(bt-0)^2+(ct-0)^2=(b^2+c^2)t^2 \quad \text{--- (2)}$$

Again the generator through $A'(-a,0,0)$ corresponding to $\lambda=-1$ meets the generator of μ-system at P', whose coordinates are obtained from (1) by putting $\lambda=-1$ and is

$$\left[-a, \frac{b(1+\mu)}{1-\mu}, \frac{c(1+\mu)}{-(1-\mu)}\right]$$

(or) $\left(-a, \frac{b}{t}, -\frac{c}{t}\right)$ where $t=\frac{1-\mu}{1+\mu}$

$$\therefore (A'P')^2=(-a-a)^2+\left(\frac{b}{t}-0\right)^2+\left(-\frac{c}{t}-0\right)^2$$

$$=\frac{b^2+c^2}{t^2} \quad \text{--- (3)}$$

$\therefore$ from (2) and (3) we get

$$AP^2\cdot A'P'^2=(b^2+c^2)^2$$

$$\Rightarrow AP\cdot A'P' = b^2+c^2$$

5(a) Solve $dy/dx = (x+y-1)^2/4(x-2)^2$.

Sol^n: Given $dy/dx = (x+y-1)^2/4(x-2)^2$ —— (1)

put $x = X+h$ and $y = Y+k$

so that $dx = dX$ and $dy = dY$ —— (2)

Then from (1), $\frac{dY}{dX} = \frac{(X+h+Y+k-1)^2}{4(X+h-2)^2} = \frac{(X+Y+h-k-1)^2}{4(X+h-2)^2}$ —— (3)

Choose h and k such that $h-k-1=0$ and $h-2=0$

So that $h=2$ and $k=-1$.

Then (2) $\Rightarrow X = x-h = x-2$ and $Y = y-k = y+1$ —— (4)

Also, from (3) $\frac{dY}{dX} = (X+Y)^2/4X^2$ which is homogeneous.

Putting $Y = vX$ so that $\frac{dY}{dX} = v + X\left(\frac{dv}{dX}\right)$,

(5) becomes $v + X\frac{dv}{dX} = \frac{(X+vX)^2}{4X^2}$

$$\Rightarrow X\frac{dv}{dX} = \frac{(v+1)^2}{4} - v = \frac{(1-v)^2}{4}$$

Separating the variables,

$$4(1-v)^{-2}dv = \left(\frac{1}{X}\right)dX.$$

Integrating $4(1-v)^{-1} = \log X + c$

$$4\left(1-\frac{Y}{X}\right)^{-1} = \log X + c$$

$$\Rightarrow \frac{4X}{X-Y} = \log X + c$$

$$\Rightarrow \frac{4(x-2)}{x-y-3} = \log(x-2)+c, \text{ by (4)}$$

5(b) Solve $p^2 + 2py\cot x = y^2$.

Sol^n: Given $p^2 + (2y\cot x)p - y^2 = 0$

Solving it for p, we get

$$p = \{-2y\cot x \pm \sqrt{(4y^2\cot^2 x + 4y^2)}\}/2$$

$$\Rightarrow p = -y\cot x \pm y(\cot^2 x + 1)^{1/2} = -y(\cot x \pm \operatorname{cosec} x)$$

Its component equations are

$$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x) \quad \text{--- (1)}$$

and $$\frac{dy}{dx} = -y(\cot x - \operatorname{cosec} x) \quad \text{--- (2)}.$$

By (1), $$\frac{dy}{dx} = -y\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right] = -\frac{1+\cos x}{\sin x}y$$

$$= -\frac{2y\cos^2 x/2}{2\sin x/2 \cos x/2}$$

$$\Rightarrow \left(\frac{1}{y}\right)dy = -\cot(x/2)\,dx.$$

Integrating $\log y = \log c - 2\log\sin(x/2)$

$$\Rightarrow y = c\operatorname{cosec}^2(x/2) \quad \text{--- (3)}$$

By (2) $$\frac{dy}{dx} = -y\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \frac{1-\cos x}{\sin x}y = \frac{2y\sin^2(x/2)}{2\sin x/2\cos x/2}$$

$$\Rightarrow \frac{1}{y}dy = \tan x/2\,dx$$

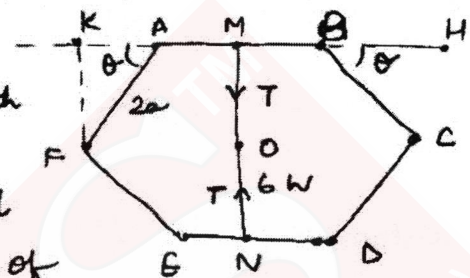
Integrating $\log y = \log c - 2\log\cos x/2 \Rightarrow y = c\sec^2 x/2$ --- (4)

$\therefore$ from (3) and (4), the combined solution is

$$(y - c\sec^2 x/2)(y - c\operatorname{cosec}^2 x/2) = 0$$

5(c) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is 3W.

Sol. ABCDEF is a hexagon formed of six equal rods each of weight W and say of length 2a. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string.



Let T be the tension in the string MN. The total weight 6W of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN.

Let $\angle FAK = \theta = \angle CBH$

Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rod AB, BC etc. remains fixed, the length MN changes and the point O also changes.

We have, $MN = 2MO = 2KF = 2AF\sin\theta = 4a\sin\theta$

Also, the depth of O below the fixed line AB
$= MO = 2a\sin\theta$

By the principle of virtual work, we have,

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

$$\Rightarrow -4aT\cos\theta\,\delta\theta + 12aW\cos\theta\,\delta\theta = 0$$

$$\Rightarrow 4a[-T+3W]\cos\theta\,\delta\theta = 0$$

$$\Rightarrow -T+3W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$\Rightarrow T = 3W$$

5(d) A particle whole mass is m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^3}\right]$ towards origin: if it starts from rest at a distance a, show that it will arrive at origin in time $\pi/4\sqrt{\mu}$.

Sol'n: Given $$\frac{d^2x}{dt^2} = -\mu\left[x+\frac{a^4}{x^3}\right] \quad \text{——} (1)$$

the −ve sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get $$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2+\frac{a^4}{x^2}\right]+C$$

when $x=a$, $dx/dt=0$, so that $C=0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4-x^4}{x^2}\right]$$

$$\Rightarrow \frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4-x^4)}}{x} \quad \text{——} (2)$$

the −ve sign is taken because the particle is moving in the direction of x decreasing.

If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{a^4-x^4}}\,dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4-x^4)}}$$

put $x^2=a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$. when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2}$$

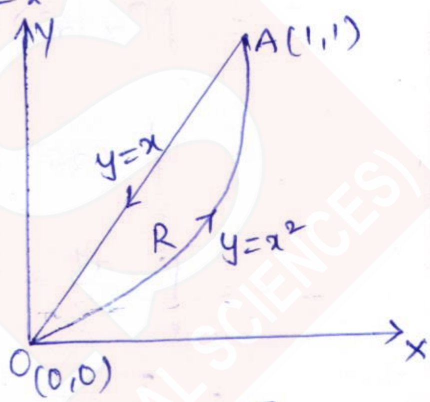
$$= \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}.$$

5(c) Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx + x^2\,dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

Sol'n: By Green's theorem in plane,

we have $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

Here $M = xy+y^2$ and $N = x^2$



The curves $y=x$ and $y=x^2$ intersect at $(0,0)$ and $(1,1)$.
The +ve direction in traversing C is as shown in the figure.

we have $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x-2y)\,dy\,dx$$

$$= \int_{x=0}^{1} (xy - y^2)_{y=x^2}^{x}\,dx$$

$$= \int_{x=0}^{1} (\cancel{x^2} - \cancel{x^2} - x^3 + x^4)\,dx$$

$$= \int_{x=0}^{1} (x^4 - x^3)\,dx = \left(\frac{x^5}{5} - \frac{x^4}{4}\right)_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}.$$

Now let us evaluate the line integral along 'C'.

The line integral along C = Line integral along $y=x^2$ (from O to A)

+ line integral along $y=x$ (from A to O).

$= I_1 + I_2$

Along $y=x^2$; $dy = 2x\,dx$

$$\therefore I_1 = \int_0^1 [x(x^2)+x^4]dx + x^2(2x)dx$$

$$= \int_0^1 (3x^3+x^4)dx$$

$$= \left[3\frac{x^4}{4}+\frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4}+\frac{1}{5} = \frac{19}{20}$$

Along $y=x$, $dy=dx$

$$\therefore I_2 = \int_1^0 \{x(x)+x^2\}dx + x^2dx$$

$$= \int_1^0 3x^2\,dx$$

$$= \frac{3x^3}{3} = 0-1 = -1$$

$$\therefore I_1 + I_2 = \frac{19}{20} - 1 = -\frac{1}{20}$$

Hence the theorem is verified.

6(a) Find the orthogonal trajectories of the family of circles passing through the points $(0,2)$ and $(0,-2)$.

Sol'n: The points $(0,2)$ and $(0,-2)$ lie on y-axis,

Therefore x-axis is the axis of symmetry for the circles passing through the points $(0,2)$, $(0,-2)$ and the centres of such circles lies on x-axis.

Let $(h,0)$ be the centres of the circles, (where h is arbitrary)

The one parameter family of circles are given by

$$(x-h)^2+(y-0)^2 = h^2+4$$

$$\Rightarrow x^2-2xh+y^2=4 \quad \text{——} \; (1)$$

Differentiating, we get $2x-2h+2yy'=0 \; (2)$ —— (2)

Eliminating h from (2), we get

$$2x-\frac{1}{x}(x^2+y^2-4)+2yy'=0$$

$$x^2-y^2+4+2xyy'=0 \quad \text{——} \; (3)$$

Equation (2) is the differential equation of the given family of curves

The differential equation of the orthogonal trajectories is

$$x^2-y^2+4+2xy\left(-\frac{1}{y'}\right)=0$$

$$\Rightarrow 2xy\,dx-(x^2-y^2+4)dy=0 \quad \text{——} \; (4)$$

The above equation is in the form $Mdx+Ndy=0$

$$M=2xy \; , \; N=y^2-x^2-4$$

$$\frac{\partial M}{\partial y}=2x \; , \; \frac{\partial N}{\partial x}=-2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation ② is not exact.

we have $\frac{1}{N}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{2xy}(-2x-2y)=-2/y$

The integration factor is $e^{-\int 2/y\,dy}=e^{-2\log y}=y^{-2}=1/y^2$

Multiplying ② by $1/y^2$, we get

$$\frac{2x}{y}dx-\frac{1}{y^2}(x^2-y^2+4)dy=0$$

$$dy+\frac{2xy\,dx-x^2dy}{y^2}-\frac{4}{y^2}dy=0$$

i.e. $dy+d\left(\frac{x^2}{y}\right)+d\left(\frac{4}{y}\right)=0$

Integrating, we get $y+x^2/y+4/y=c''=4c$ (say)

$$x^2+y^2+4=4cy$$

$$\Rightarrow x^2+y^2-4cy=-4$$

$$\Rightarrow x^2+y^2-4cy+4c^2=4c^2-4$$

$$\Rightarrow x^2+(y-2c)^2=4(c^2-1),\quad |c|>1$$

The orthogonal trajectories of the given family of circles is family of circles.

$$x^2+(y-2c)^2=4(c^2-1)$$

where centre is at $(0,2c)$ and is radius is $2\sqrt{c^2-1}$, $|c|>1$

where, $a>0$ is an arbitrary constant.

6(b) Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

Sol'n: Here the auxiliary equation is

$D^2-4D+4=0$ using $D=2,2$

$\therefore$ C.F $=(C_1+C_2x)e^{2x}$, C_1, C_2 being arbitrary Constants.

$$P.I = \frac{1}{(D-2)^2} 8x^2e^{2x}\sin 2x = 8e^{2x}\frac{1}{(D+2-2)^2}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{D^2}x^2\sin 2x$$

$$= 8e^{2x}\frac{1}{D}\int x^2\sin 2x\,dx$$

$$= 8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - \int(2x)\left(-\frac{\cos 2x}{2}\right)dx\right]$$

Integrating by parts

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1.\frac{\sin 2x}{2}dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right]$$

$$= 8e^{2x}\int\left(-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)dx$$

$$= 8e^{2x}\left[-\frac{1}{2}\int x^2\cos 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{4}\int\cos 2x\,dx\right]$$

$$= 8e^{2x}\int\left(-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)dx$$

$$= 8e^{2x}\left[-\frac{1}{2}\int x^2\cos 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{4}\int\cos 2x\,dx\right]$$

$$= 8e^{2x}\left[-\frac{1}{2}\left\{x^2\left(\frac{1}{2}\sin 2x\right) - \int 2x\left(\frac{1}{2}\sin 2x\right)dx\right\} + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x \, \cancel{dx} + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{2}\int x\sin 2x\,dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x + \int x\sin 2x\,dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x + x\left(-\frac{1}{2}\cos 2x\right) - \int 1.\left(-\frac{1}{2}\cos 2x\right)dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x - \frac{1}{2}x\cos 2x + \frac{3}{8}\sin 2x\right].$$

$\therefore$ The required solution is

$$y = (c_1 + c_2 x)e^{2x} + e^{2x}(3\sin 2x - 4x\cos 2x - 2x^2\sin 2x)$$

6(c) Solve by the method of variation of parameters

$$x^2y'' - 2x(1+x)y' + 2(x+1)y = x^3.$$

Sol^n: Re-writing the given equation in standard form, we get

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x \quad \text{—— (1)}$$

C.F of (1), i.e. solution of

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = 0 \quad \text{—— (2)}$$

Comparing (2) with $y'' + Py' + Qy = R$,

$$P = -\frac{2(1+x)}{x}, \quad Q = \frac{2(x+1)}{x^2} \text{ and } R = 0 \quad \text{—— (3)}$$

Here $P + xQ = 0$, showing that $u = x$ —— (4) is a part of C.F of (2)

Let the complete solution of (1) be $y = uv$ —— (5)

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\frac{2(1+x)}{x} + \frac{2}{x}\frac{dx}{dx}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0; \text{ using (3) and (4)}$$

$$\Rightarrow (D^2 - 2D)v = 0 \quad \text{where } D \equiv d/dx \quad \text{—— (6)}$$

Auxiliary equation of (6) is $D^2 - 2D = 0 \Rightarrow D = 0, 2$.

$\therefore$ Solution of (6) is $y = c_1e^{0.x} + c_2e^{2x} = c_1 + c_2e^{2x}$, c_1 & c_2 being arbitrary constants —— (7)

from (4), (5) and (7), the complete solution of (1), i.e. C.F of (1) is given by

$$y = x(c_1 + c_2e^{2x}) \Rightarrow y = c_1x + c_2xe^{2x} \quad \text{—— (8)}$$

Let $y = Ax + Bxe^{2x}$ ——— (9)

be the complete solution of (1). Then A and B are function of x which are so choosen that (1) will be satisfied. Differentiating (9), w.r.t 'x' we get

$$y' = A + A_1x + B(e^{2x} + 2xe^{2x}) + B_1xe^{2x} \quad \text{——— (10)}$$

where $A_1 = dA/dx$ and $B_1 = dB/dx$. choose A & B such that

$$A_1x + B_1xe^{2x} = 0 \quad \text{——— (11)}$$

Then (10) reduces to $y' = A + Be^{2x}(1+2x)$ ——— (12)

Differentiating (12), $y'' = A_1 + B_1e^{2x}(1+2x) + B\{2e^{2x}(1+2x) + 2e^{2x}\}$ ——— (13)

Substituting the values of y, y' and y'' given by (9), (12) and (13) is (1), we have

$$x^2\{A_1 + B_1e^{2x}(1+2x) + 4Be^{2x}(1+x)\} - 2x(1+x)\{A + Be^{2x}(1+2x)\}$$
$$+ 2(x+1)(Ax + Bxe^{2x}) = x^3$$

$$\Rightarrow A_1x^2 + x^2B_1e^{2x}(1+2x) = x^3 \Rightarrow A_1 + B_1(1+2x)e^{2x} = x \quad \text{——— (14)}$$

Solving (11) & (14) for A_1 and B_1, we have

$$A_1 = dA/dx = -\tfrac{1}{2} \text{ and } B_1 = \frac{dB}{dx} = \tfrac{1}{2}e^{-2x}$$

Integrating these $A = (-x/2) + c_1$ & $B = -\tfrac{1}{4}e^{-2x} + c_2$

Substituting the above values of A and B is (9), the required solution is

$$y = \{(-x/2) + c_1\}x + \{(-\tfrac{1}{4})e^{-2x} + c_2\}xe^{2x}$$

$$y = c_1x + c_2xe^{2x} - (x^2/2) - x/4$$

6(d) (i) If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find $L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}$.

Sol'n: Given $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$ —— ①

writing as for s in ① and using change of scale property, we have

$$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}\cdot\frac{1}{a}\cdot\frac{t}{a}\sin t/a \quad —— ②$$

put $a=4$ in ② and obtain

$$L^{-1}\left\{\frac{4s}{(4^2s^2+1)^2}\right\} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{t}{4}\sin\frac{t}{4}$$

$$\Rightarrow 8L^{-1}\left\{\frac{4s}{(16s^2+1)^2}\right\} = 8\cdot\frac{1}{32}t\sin\frac{t}{4}$$

$$\Rightarrow L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\} = \frac{1}{4}t\sin\frac{t}{4}.$$

(ii) Solve $(D^2+6D+9)y = \sin t$, where $y(0)=1$, $y'(0)=0$.

Sol'n: Given that $y''+6y'+9y = \sin t$ —— ①

with initial conditions: $y(0)=1$ and $y'(0)=0$.

Taking Laplace transform of both sides of ①, we get

$$L\{y''\}+6L\{y'\}+9L\{y\} = L\{\sin t\}$$

$$\Rightarrow s^2L\{y\}-sy(0)-y'(0)+6[sL\{y\}-y(0)]+9L\{y\} = \frac{1}{(s^2+1)}$$

$$\Rightarrow (s^2+6s+9)L\{y\}-s-6 = \frac{1}{(s^2+1)}, \text{ using ②}$$

$$\Rightarrow (s+3)^2L\{y\} = s+6+\frac{1}{(s^2+1)} = (s+3)+3+\frac{1}{(s^2+1)}$$

$$\Rightarrow L\{y\} = \frac{1}{(s+3)} + \frac{3}{(s+3)^2} + \frac{1}{(s+3)^2(s^2+1)} \quad \text{--- (3)}$$

Let $\frac{1}{(s+3)^2(s^2+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{Cs+D}{s^2+1}$ --- (4)

Multiply both sides of (4) by $(s+3)^2$ and let $s \to -3$,

then $B = \lim_{s\to -3} \frac{1}{s^2+1} = \frac{1}{9+1} = \frac{1}{10}$

Then, (4) $\Rightarrow \frac{1}{(s+3)^2(s^2+1)} = \frac{A}{s+3} + \frac{1}{10(s+3)^2} + \frac{Cs+D}{s^2+1}$ --- (5)

Multiplying bothsides of (5) by $(s+3)^2(s^2+1)$, we get

$$1 = A(s+3)(s^2+1) + \frac{1}{10}(s^2+1) + (Cs+D)(s+3)^2$$

$$\Rightarrow 1 = A(s+3)(s^2+1) + \frac{1}{10}(s^2+1) + (Cs+D)(s^2+6s+9)$$

$$\Rightarrow 1 = 3A + \frac{1}{10} + 9D + s(A+9C+6D) + s^2\left[3A + \frac{1}{10} + D + 6C\right] + s^3(A+C)$$

Equating the coefficients of s^3, s and constant terms on bothsides of the identity (6), $A+C=0$, $A+9C+6D=0$ and $3A+\frac{1}{10}+9D=1$.

solving these, $A = 3/50$, $C = -3/50$, $D = 4/50$

$\therefore$ (5) $\Rightarrow \frac{1}{(s+3)^2(s^2+1)} = \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}$.

$\therefore$ (3) $\Rightarrow L\{y\} = \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}$

$$L\{y\} = \frac{53}{50(s+3)} + \frac{31}{10(s+3)^2} - \frac{3s}{50(s^2+1)} + \frac{2}{25(s^2+1)}$$

$$\Rightarrow y = \frac{53}{50}e^{-3t} + \frac{31}{10}e^{-3t}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{50}\cos t + \frac{2}{25}\sin t$$

(Using first shifting theorem)

$$\Rightarrow y = \frac{53}{50}e^{-3t} + \frac{31}{10}te^{-3t} - \frac{3}{50}\cos t + \frac{2}{25}\sin t.$$

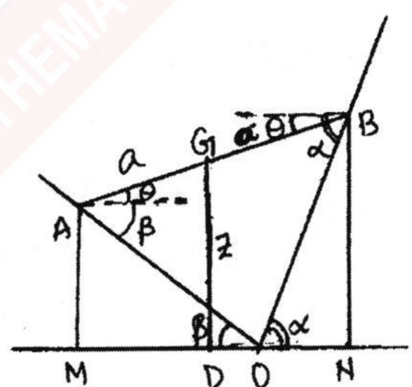
7.(a) A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in horizontal line. If the inclinations of the planes to the horizontal are α and β $(\alpha > \beta)$, show that the inclination of the beam to the horizontal in one of the equilibrium positions given by

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha).$$

Sol^n: Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. we have

$\angle AOM = \beta$ and $\angle BON = \alpha$

The centre of gravity of the beam AB is its middle point G. Let z be the height of G above the fixed horizontal line MN. we shall express z as a function of θ.



we have, $z = GD = \frac{1}{2}(AM + BN)$

$$= \frac{1}{2}(OA\sin\beta + OB\sin\alpha)$$

Now in the triangle OAB, $\angle OAB = \beta + \theta$, $\angle OBA = \alpha - \theta$ and $\angle AOB = \pi - (\alpha + \beta)$. Applying the sine theorem for the $\triangle OAB$, we have

$$\frac{OA}{\sin(\alpha-\theta)} = \frac{OB}{\sin(\beta+\theta)} = \frac{AB}{\sin[\pi-(\alpha+\beta)]} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\therefore OA = \frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}, \quad OB = \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}$$

Substituting for OA and OB in (1), we have

$$= \frac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta + \sin(\beta+\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right]$$

$\therefore$ For equilibrium of the beam, we have $\frac{dz}{d\theta} = 0$

i.e, $\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0.$

i.e, $2\sin\theta\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

(or) $$\frac{\sin\theta}{\cos\theta} = \frac{1}{2}\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$$

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha).$$

This gives the required position of equilibrium of the beam

7.(b) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$, Prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.

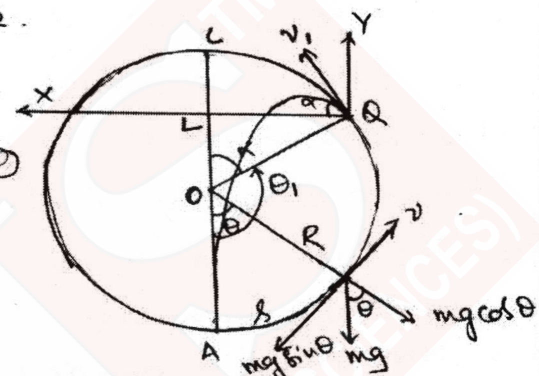
Sol^n: Take $R=T$ (i.e. the tension in the string)

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are.

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{---} (1)$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{---} (2)$$

Also $s = l\theta$



From (1) & (3), we have

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying bothsides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at the point A, $\theta=0$ and $v = 2\sqrt{gl}$

$\therefore 4gl = 2lg + A$ so that $A = 2gl$

$\therefore v^2 = 2lg(\cos\theta+1)$ ——— (4)

from (2) and (4), we have

$$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta+2) \quad \text{---} (5)$$

If the cord becomes slack at the point Q, where $\theta=\theta_1$, then from (5), we have

$$T = 0 - mg(3\cos\theta_1 + 2)$$

giving as $\cos\theta_1 = -\frac{2}{3}$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$ and $\cos\alpha = \frac{2}{3}$

If v_1 is the velocity of the particle at Q, then $v=v_1$ where $\theta=\theta_1$. Therefore from (4), we have

$$v_1^2 = 2lg(1+\cos\theta_1) = 2lg\left(1-\frac{2}{3}\right) = \frac{2lg}{3}$$

Now $OL = l\cos\alpha = \frac{2}{3}l$

Thus the particle leaves the circular path at the point Q at a height $\frac{2}{3}l$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

7.(C) → Discuss the motion of a particle falling under gravity in a medium whose resistance varies as the velocity.

Sol^n → Suppose a particle of mass m starts at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where OP = x and let v be the velocity of the particle at P.

The forces acting on the particle at P are-

(i) The force mkv due to the resistance acting vertically upwards i.e. against the direction of motion of the particle and

(ii) The weight mg of the particle acting vertically downwards. By Newton's second law of motion the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv$$

or, $$\frac{d^2x}{dt^2} = g - kv \quad \text{——} (1)$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0 = g - kV \quad \text{or} \quad k = g/V$$

putting $k = g/V$ in (1) we get

$$\frac{d^2x}{dt^2} = g - \frac{g}{V}v = \frac{g}{V}(V - v) \quad \text{——} (2)$$

Relation b/w v and x :-

The equation (2) can be written as,

$$v\frac{dv}{dx} = \frac{g}{V}(V-v)$$

or, $$dx = \frac{V}{g}\,\frac{v}{V-v}\,dv = -\frac{V}{g}\,\frac{-v}{V-v}\,dv$$

$$= \frac{-V}{g}\,\frac{(V-v)-V}{V-v}\,dv$$

$$= -\frac{V}{g}\left[1-\frac{V}{V-v}\right]dv$$

Integrating $$x = -\frac{V}{g}\left[v + V\log(V-v)\right] + A$$

where A is a constant.

But initially at O, $x=0$ and $v=0$

$$\therefore\ A = \frac{V^2}{g}\log V$$

$$\therefore\ x = -\frac{V}{g}v - \frac{V^2}{g}\log(V-v) + \frac{V^2}{g}\log V$$

or, $$x = -\frac{V}{g}v + \frac{V^2}{g}\log\frac{V}{V-v} \quad \text{—— (3)}$$

Which is the velocity of the particle at any position.

Relation b/w v ant t:-

The equation (2) can also be written as

$$\frac{dv}{dt} = \frac{g}{V}(V-v)$$

$$\therefore\ dt = \frac{V}{g}\,\frac{dv}{V-v}.$$

Integrating we have,

$$t = -\frac{V}{g}\log(V-v) + B \quad , \text{ where } B \text{ is constant.}$$

Initially at O, $t=0$ and $v=0$.

$$\therefore B = \frac{V}{g}\log V.$$

$$\therefore t = -\frac{V}{g}\log(V-v) + \frac{V}{g}\log V$$

or, $$t = \frac{V}{g}\log\frac{V}{V-v} \quad \text{——} \quad (4)$$

Which is the velocity of the particle at any time t.

<u>Relation b/w x and t:-</u>

From eqn (4), we have,

$$\log\frac{V}{V-v} = \frac{gt}{V} \quad \text{or} \quad \frac{V}{V-v} = e^{gt/V}$$

or, $$V - v = Ve^{-gt/V}$$

or, $$v = V[1 - e^{-gt/V}]$$

or, $$\frac{dx}{dt} = V[1 - e^{-gt/V}]$$

or, $$dx = V[1 - e^{-gt/V}]dt.$$

Integrating we get,

$$x = Vt + \frac{V^2}{g}e^{-gt/V} + c, \text{ where } c \text{ is constant.}$$

Initially at O, $x=0$ and $t=0$.

$$\therefore c = -\frac{V^2}{g}$$

$$\therefore x = Vt + \frac{V^2}{g}e^{-gt/V} - \frac{V^2}{g} \quad \text{or} \quad x = Vt + \frac{V^2}{g}(e^{-gt/V} - 1) \quad \text{——} \quad (5)$$

Which gives the distance fallen through in time t.

8(a)(i) For a solenoidal vector $\vec{F}$, show that $\text{Curl curl curl curl}\,\vec{F} = \nabla^4 \vec{F}$.

Sol^n: Since vector $\vec{F}$ is solenoidal, so $\text{div}\,\vec{F} = 0$ —— ①

we know that $\text{Curl curl}\,\vec{F} = \text{grad div}\,(\vec{F} - \nabla^2\vec{F})$ —— ②

using ① & ②, $\text{grad div}\,\vec{F} = \text{grad}(0) = 0$ —— ③

on putting the value of grad div F in ②, we get

$$\text{Curl curl}\,\vec{F} = -\nabla^2 \vec{F} \qquad \text{——— ④}$$

Now, $\text{Curl curl curl curl}\,\vec{F} = \text{curl curl}\,(-\nabla^2\vec{F})$ (using ④)

$= -\text{Curl curl}\,(\nabla^2\vec{F})$

$= -[\text{grad div}\,(\nabla^2\vec{F} - \nabla^2(\nabla^2\vec{F})]$ (using ②)

$= -\text{grad}\,(\nabla.\nabla^2\vec{F}) + \nabla^2(\nabla^2\vec{F})$

$= -\text{grad}\,(\nabla^2\nabla.\vec{F}) + \nabla^4\vec{F}$ $[\nabla.\vec{F} = 0]$

$= 0 + \nabla^4\vec{F} = \nabla^4\vec{F}$ (using ①)

(ii) Find the directional derivative of $\nabla(\nabla f)$ at the point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

Sol'n: Here, we have $f = 2x^3y^2z^4$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Directional derivative of $\nabla(\nabla f)$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(12xy^2z^4+4x^3z^4+24x^3y^2z^2\right)$$

$$=\left(12y^2z^4+12x^2z^4+72x^2y^2z^2\right)\hat{i}+\left(24xyz^4+48x^3yz^2\right)\hat{j}$$
$$+\left(48xy^2z^3+16x^3z^3+48x^3y^2z\right)\hat{k}$$

Directional derivative at $(1,-2,1)$

$$=(48+12+288)\hat{i}+(-48-96)\hat{j}+(192+16+192)\hat{k}$$
$$=348\hat{i}-144\hat{j}+400\hat{k}$$

Normal to $(xy^2z-3x-z^2)=\nabla\left(xy^2z-3x-z^2\right)$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(xy^2z-3x-z^2\right)$$
$$=(y^2z-3)\hat{i}+(2xyz)\hat{j}+(xy^2-2z)\hat{k}$$

Normal at $(1,-2,1)=\hat{i}-4\hat{j}+2\hat{k}$

Unit Normal vector $=\dfrac{\hat{i}-4\hat{j}+2\hat{k}}{\sqrt{1+16+4}}=\dfrac{1}{\sqrt{21}}(\hat{i}-4\hat{j}+2\hat{k})$

Directional derivative in the direction of normal

$$=(348\hat{i}-144\hat{j}+400\hat{k})\frac{1}{\sqrt{21}}(\hat{i}-4\hat{j}+2\hat{k})$$

$$=\frac{1}{\sqrt{21}}(348+576+800)=\frac{1724}{\sqrt{21}}$$

8(b) Find the curvature and torsion for the space curve $x = t - \frac{t^3}{3}$, $y = t^2$, $z = t + \frac{t^3}{3}$.

Sol'n: Given that $\vec{r} = \left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}$

$$\frac{d\vec{r}}{dt} = (1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}$$

$$\frac{d^2 r}{dt^2} = -2t\hat{i} + 2\hat{j} + 2t\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2}$$

$$= \sqrt{1 + t^4 - \cancel{2t^2} + 4t^2 + 1 + t^4 + \cancel{2t^2}}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{2 + 2t^4 + 4t^2}$$

Now, $$\left[\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} i & j & k \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \end{vmatrix}$$

$$= i\,(4t^2 - 2 - 2t^2) - j\,(2t - 2t^3 + 2t + 2t^3) + \hat{k}\,(2 - 2t^2 + 4t^2)$$

$$= 2(t^2-1)\hat{i} - 4t\hat{j} + \hat{k}\,2(t^2+1)$$

$$= 2(t^2-1)\hat{i} - 4t\hat{j} + 2(t^2+1)\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{4(t^2-1)^2 + 16t^2 + 4(t^2+1)^2}$$

$$= 2\sqrt{(t^2-1)^2 + 4t^2 + (t^2+1)^2}$$

$$= 2\sqrt{2 + 2t^4 + 4t^2}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = 2\left|\frac{d\vec{r}}{dt}\right|$$

$$\frac{d^3\vec{r}}{dt^3} = -2\hat{i} + 0\hat{j} + 2\hat{k}$$

$$\left[\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \\ -2 & 0 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 4\left[(1-t^2)1 - 2t(-t+t) + (1+t^2)(1)\right]$$

$$= 4\left[1-\cancel{t^2}+1+\cancel{t^2}\right] = 8$$

$$\text{Curvature} = K = \frac{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{2\cancel{\left|\frac{d\vec{r}}{dt}\right|}}{\left|\frac{d\vec{r}}{dt}\right|^{3\,2}}$$

$$K = \frac{2}{\sqrt{(2+2t^4+4t^2)^2}} = \frac{2}{2t^4+4t^2+2}$$

$$\boxed{K = \frac{1}{t^4+2t^2+1} = \frac{1}{(t^2+1)^2}}$$

$$\text{Torsion } (\tau) = \frac{\left[\frac{dr}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2r}{dt^2}\right|^2} = \frac{8}{4(2t^4+4t^2+2)}$$

$$\tau = \frac{\cancel{8}}{\cancel{8}(t^4+2t^2+1)} \Rightarrow \boxed{\tau = \frac{1}{(t^2+1)^2}}$$

Hence $\boxed{K = \tau = \frac{1}{(t^2+1)^2}}$

is the required solution.

8(C) (i) If r is the position vector of the point (x, y, z). show that $\text{curl}(r^n r) = 0$, where r is the module of r.

(ii) A vector function f is the product of a scalar function. show that $f \cdot \text{curl} f = 0$.

sol^n: (i) we know that :

$$\text{curl}(\phi A) = \nabla\phi \times A + \phi\, \text{curl}\, A.$$

putting $\phi = r^n$ and $A = r$ in this identity,

we get $\text{curl}(r^n r) = (\nabla r^n) \times r + r^n\, \text{curl}\, r$

$$= (n r^{n-1} \nabla r) + r^n 0$$

$[\because \nabla f(r) = f'(r) \nabla r$ and $\text{curl}\, r = \text{curl}(xi + yj + zk) = 0]$

$$= \left(n r^{n-1} \frac{1}{r} r\right) \times r \qquad \left[\because \nabla r = \frac{1}{r} r\right]$$

$$= n r^{n-2} (r \times r) \qquad [\because r \times r = 0]$$

$$= n r^{n-2}\, 0$$

$$= 0$$

(ii) Let $f = \psi\, \text{grad}\, \phi$, where ψ and ϕ are scalar functions.

we have $\text{curl} f = \text{curl}(\psi\, \text{grad}\, \phi)$.

we know that $\text{curl}(\phi A) = (\text{grad}\, \phi) \times A + \phi\, \text{curl}\, A$

$\therefore \text{curl}(\psi\, \text{grad}\, \phi) = (\text{grad}\, \psi) \times (\text{grad}\, \phi) + \psi(\text{curl grad}\, \phi)$

$$= (\text{grad}\, \psi) \times (\text{grad}\, \phi) \qquad [\because \text{curl grad}\, \phi = 0]$$

Now $f \cdot \text{curl} f = (\psi\, \text{grad}\, \phi) \cdot \{(\text{grad}\, \psi) \times (\text{grad}\, \phi)\}$

$$= [\psi\, \text{grad}\, \phi \;\; \text{grad}\, \psi \;\; \text{grad}\, \phi]$$

$$= \psi [\text{grad}\, \phi, \text{grad}\, \psi, \text{grad}\, \phi]$$

$= 0$, since the value of a scalar triple product is zero if two vectors are equal.

8(d) Use the Divergence theorem to evaluate $\iint_S F \cdot ds$ where $F = 2xz\,i + (1-4xy^2)\,j + (2z-z^2)\,k$ and S is the surface of the solid bounded by $z = 6-2x^2-2y^2$ and the plane $z=0$

Sol^n:

$$\operatorname{div} F = \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(1-4xy^2) + \frac{\partial}{\partial z}(2z-z^2)$$

$$= \cancel{2z} + (-8xy) + (2 - \cancel{2z})$$

$$= 2 - 8xy.$$

$$\iint_S F \cdot ds = \iiint_E \operatorname{div} F \, dv$$

$$= \iiint_E (2-8xy)\, dv.$$

E is the region, which is the intersection of the two surfaces

i.e, $6-2x^2-2y^2 = z$ and $z=0$

$$\Rightarrow 6-2x^2-2y^2 = 0$$

$$\Rightarrow x^2+y^2 = 3$$

Using the cylindrical coordinates, The cylindrical limits for the region E are, $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{3}$,

$$\therefore \iint_S F\cdot ds = \iiint_E \operatorname{div} F\, dv$$

$$= \iiint_E (2-8xy)\, dv$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} (2-8r^2\cos\theta\sin\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{6-2r^2} (2r-8r^3\cos\theta\sin\theta)\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (2r-8r^3\cos\theta\sin\theta)\,[z]_0^{6-2r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} 12r-4r^3-(48r^3-16r^5)\cos\theta\sin\theta\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} 6r^2-r^4-\left(12r^4-\frac{8}{3}r^5\right)\cos\theta\sin\theta\, d\theta$$

$$= \int_0^{2\pi} (9-36\cos\theta\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (9-18\sin 2\theta)\, d\theta$$

$$= [9\theta]_0^{2\pi} = 18\pi$$