IAS MATHEMATICS (OPT.)-2011

PAPER - I : SOLUTIONS

1(a) IAS -2011 P-I

$\rightarrow$ Let A be a non-singular $n\times n$ matrix. S.T $A(\operatorname{adj} A) = |A| I_n$. Hence show that $|\operatorname{adj}(\operatorname{adj} A)| = |A|^{(n-1)^2}$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$

~~Then~~ $\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$

$= \begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ \vdots & & & & \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix}_{n\times n}$.

where A_{ij} is a minor of a_{ij}

we have

$A \operatorname{adj} A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$

$= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & |A| \end{bmatrix}_{n\times n}$

Here $a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} |A|; & i=j \\ 0; & i\neq j \end{cases}$

$\therefore$ ~~Adj A~~ $\cdot\ A\,\mathrm{adj}\,A = |A| \begin{vmatrix} 1 & 0 & - & - & 0 \\ 0 & 1 & - & - & 0 \\ \vdots & & & & \\ 0 & 0 & - & - & 1 \end{vmatrix}$

$= |A|\,|I|$,

$\therefore A\,\mathrm{adj}A = |A|\,|I_n|$

Now, to show that $|\mathrm{adj}(\mathrm{adj}\,A)| = |A|^{(n-1)^2}$

Consider $B = \mathrm{adj}\ A$ —— (2)

& $B = \frac{1}{|A|}\begin{vmatrix} |A| & 0 & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & & 0 \\ 0 & 0 & |A| & \cdots & & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & \cdots & & & |A| \end{vmatrix}_{n\times n}$

$= \frac{1}{|A|}\cdot|A|^n|I_n|$

$\left.\begin{array}{l} \therefore |B| = |A|^{n-1} \\ \Rightarrow |\mathrm{adj}\,A| = |A|^{n-1} \end{array}\right\}$ —(3)

using (1) $B.\,\mathrm{adj}\,B = |B|\cdot I_n$.

$|B.\mathrm{adj}\,B| = |B|^n.\,|I_n|$

$|B|\,|\mathrm{Adj}\,B| = |B|^n.\ 1$

$|\mathrm{Adj}\ B| = |B|^{n-1}$

$|\mathrm{Adj}\ (\mathrm{Adj}\ A)| = \left(|A|^{n-1}\right)^{n-1}$ [using (2) & (3)]

$\boxed{\therefore\ |\mathrm{Adj}\ (\mathrm{Adj}\ A)| = |A|^{(n-1)^2}}$

IAS 2011 Q.1(b)

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

Solve the system of equations; given by $AX=B$ using the above, also solve the system of equations. $A^T X = B$; where A^T denotes the transpose of matrix 'A'.

Sol: Given; $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

Solve of eq^n $\Rightarrow AX = B$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$R_3 \to 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 12 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 8 \\ 0 & 0 & -10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix}$$

$x - z = 2$ — (A)

$4y + 8z = 0$ — (B) $\Rightarrow$

$-10z = 10$ $\to$ (C)

By Solving

$z = -1$

$y = 2 \Rightarrow \cancel{X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}$

$x = 1$

$$\therefore \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Now, $A^T X = B$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 6 \\ -1 & 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \quad R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 6 \\ 0 & 8 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 7 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 6 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -5 \end{bmatrix}$$

$\Rightarrow \quad x + 3y = 2$

$4y + 6z = 6$

$-5z = -5 \Rightarrow z = 1.$

$4y + 6 = 6 \Rightarrow y = 0$

$x + 0 = 2 \Rightarrow x = 2.$

$\therefore \; x = 2, \; y = 0, \; z = 1.$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Q.2(i) (IAS-2011) Let $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ be the eigen values of $n\times n$ square matrix 'A' with corresponding eigen vectors. $X_1, X_2, \ldots X_n$; If B is a matrix similar to A. Show that the eigen values of 'B' are same as that of 'A'. Also find the relation between the eigen vectors of B and eigen vector of 'A'.

Sol:- Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be the eigen values of $n\times n$ of matrix 'A' and corresponding eigen vectors are; $X_1, X_2, X_3, \ldots X_n$;.

$$\therefore AX_i = \lambda_i X_i \quad (\forall\ i = 1,2,3,\ldots n) \quad \text{--- (1)}$$

$$\& \quad |A - \lambda_i I| = 0 \quad \text{--- (2)}$$

now; since 'B' matrix is similar to 'A'

$\Rightarrow \exists$ P an invertible matrix such that

$$B = P^{-1}AP \quad \text{--- (3)}$$

from (2) $\Rightarrow$ $|A - \lambda_i I| = 0$

$$|PP^{-1}APP^{-1} - \lambda_i PP^{-1}| = 0$$

$$\Rightarrow |P(P^{-1}AP - \lambda_i I)P^{-1}| = 0$$

$$|P|\,|P^{-1}AP - \lambda_i I|\,|P^{-1}| = 0$$

$$|P|\,|P^{-1}AP - \lambda_i I|\,\frac{1}{|P|} = 0$$

$$\Rightarrow |P^{-1}AP - \lambda_i I| = 0$$

$$= |B - \lambda_i I| = 0 \quad (\text{using (3)})$$

$\Rightarrow \lambda_i$ is an eigen value of B.

Hence, $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of 'B' also.

Now,

$$\begin{aligned} BP^{-1}X_1 &= (P^{-1}AP)P^{-1}X_1 \\ &= P^{-1}(A)(PP^{-1})X_1 \\ &= P^{-1}AIX_1 \\ &= P^{-1}AX_1 \end{aligned}$$

$$BP^{-1}X_1 = P^{-1}\lambda_1 X_1 \quad (\text{using } (3))$$

$\Rightarrow$ $P^{-1}X_1$ is an eigen vector corresponding to eigen values λ_1.

Hence, if X_i ; eigen vector of A corresponding to eigen values λ_i ($\forall\ i = 1, 2, \cdots n$).

then $P^{-1}X_i$ will be the eigen vector of B corresponding to eigenvalues λ_i ($\forall\ i = 1, 2, 3 \cdots n$).

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2(ii) Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$.

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Using this show that 'A' is non-singular and find A^{-1}?

Sol: Let λ be an eigen value of matrix 'A' then characteristic matrix.

$$|A-\lambda I| = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -5 & 1-\lambda \end{bmatrix}$$

Characteristic polynomial :-

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -5 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)^3 - \{-10 - 3(1-\lambda)\} = 0$$

$$(1-\lambda)^3 + 13 - 3\lambda = 0$$

$$(\lambda-1)^3 + 3\lambda - 13 = 0$$

$$\lambda^3 - 3\lambda^2 + 6\lambda - 14 = 0 \quad \text{——} (1)$$

$$A^2 = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} \quad \text{—} (2)$$

$$A^3 = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 15 & 0 \\ 0 & 11 & -6 \\ -30 & 0 & 2 \end{bmatrix} \quad \text{—} (3)$$

Putting values of A, A^2, A^3, in eqn — (1)

$$\lambda^3 - 3\lambda^2 + 6\lambda - 14 = 0$$

$$A^3 - 3A^2 - 6A - 14I = 0$$

$$A^3-3A^2+6A-14I = \begin{bmatrix} 2 & 15 & 0 \\ 0 & 11 & -6 \\ -30 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -15 & 6 \\ -12 & -3 & 6 \\ 12 & 30 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & -6 \\ 12 & 6 & 0 \\ 18 & -30 & 6 \end{bmatrix} + \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$

$$\therefore\ A^3 - 3A^2 + 6A - 14I = 0 \quad \text{——— (4)}$$

$\Rightarrow$ A satisfies equation (1) ; Characteristics polynomial

$\rightarrow$ Cayley-Hamilton theorem verified.

Multiply eq^n (4) by A^{-1}

$$\Rightarrow A^{-1}A^3 - 3A^{-1}A^2 + 6A^{-1}A - 14A^{-1}I = 0$$

$$\Rightarrow A^2 - 3A + 6I = 14A^{-1}$$

$$\Rightarrow 14A^{-1} = A^2 - 3A + 6I$$

$$\Rightarrow 14A^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 14A^{-1} = \begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix} \quad \underline{\text{Ans}}$$

2011

Q.2(b) (i) Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this space.

Sol:- Let, $B_1 = \{(1,1,-1),(1,0,1)\}$

$B_2 = \{(1,2,-3),(5,2,1)\}$

let, W be subspace s.t $W = L\{B_1\}$

V be subspace s.t $V = L\{B_2\}$

Consider, a matrix 'A', whose rows are vectors of B_1

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{——— (1)}$$

Now, consider; a matrix 'B', whose rows are vectors of B_2.

$$\Rightarrow B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 2 & +1 \end{bmatrix} \quad R_2 \to R_2 - 5R_1$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -8 & +16 \end{bmatrix} \quad R_2 \to R_2\left(+\frac{1}{8}\right)$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$B \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{——— (2)}$$

from (1) & (2); clearly rowspace of 'A' and row space of 'B' are same.

→ Therefore, All subspaces V and W are identical.

2.b(ii) Find the nullity and a basis of the null space of the linear transformation $A: \mathbb{R}^4 \to \mathbb{R}^4$ given by the matrix.

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol: $N(A) = \{(x,y,z,t) \in \mathbb{R}^4 \mid A(x,y,z,t) = \hat{0}\}$,

$A(x,y,z,t) = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$\sim R_3 \to R_3 - 3R_1$ & $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \\ \sim \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

corresponding system of equation is

$x + z + t = 0$ (2) & $y - 3z - t = 0$ --- (3)

Here two variables z, t and values of x & y depends upon z & t.

from (3) $y = 3z + t$ from (2) $x = -(z + t)$.

Put $z = 1$, $t = 0$ $\Rightarrow y = 3$, $x = -1$.

then; $(x,y,z,t) = (-1,3,1,0)$ --- (4)

Now put $z = 0$, $t = 1$ $\Rightarrow y = 1$, $x = -1$

then $(x,y,z,t) = (-1,1,0,1)$ --- (5)

clearly $(-1,3,1,0)$ & $(-1,1,0,1)$ are LI and generates $N(A)$.

Basis B of $N(a)$ is

$B = \{(-1,3,1,0), (1,1,0,1)\}$

$\boxed{\therefore \operatorname{Dim} N(A) = \eta(A) = 2}$

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Q.2(c) (i) Show that the vector $(1,1,1),(2,1,2)$ & $(1,2,3)$ are linearly independent in $\mathbb{R}^3$. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by-

$$T(x,y,z)=(x+2y+3z,\ x+2y+5z,\ 2x+4y+6z).$$

Show that the images of above vectors under T are linearly dependent. Give the reasons for the same.

Sol^n:- Consider, that $v_1=(1,1,1)$; $v_2=(2,1,2)$

$v_3=(1,2,3)$

Consider a matrix 'A' whose rows are v_1, v_2, v_3

$$\Rightarrow A=\begin{bmatrix}1&1&1\\2&1&2\\1&2&3\end{bmatrix} \quad \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-R_1\end{matrix}$$

$$A\sim\begin{bmatrix}1&1&1\\0&-1&0\\0&1&2\end{bmatrix} \quad \begin{matrix}R_2\to -R_2\\R_3\to R_3-R_2\end{matrix}$$

$$A\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&2\end{bmatrix} \sim R_2\to\frac{1}{2}R_3 \Rightarrow A\sim\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}$$

This echelon form of matrix 'A', in which all rows are non-zero

$\Rightarrow$ non rows can be written as linear combination of other ones.

$\Rightarrow$ all rows are L.I.

$\Rightarrow v_1, v_2, v_3$ are L.I.

now; given $T:\mathbb{R}^3\to\mathbb{R}^3$ by

$$T(x,y,z)=(x+2y+3z,\ x+2y+5z,\ 2x+4y+6z) \quad \text{—}(1)$$

$T(V_1) = T(1,1,1) = (6,8,12)$ using eqⁿ ①

$T(V_2) = T(2,1,2) = (10,14,20)$ f eqⁿ ①

$T(V_3) = T(1,2,3) = (14,20,28)$ — from eqⁿ ①

Consider, B matrix whose rows are $T(V_1)$ $T(V_2)$, $T(V_3)$ are.

$$B = \begin{bmatrix} 6 & 8 & 12 \\ 10 & 14 & 20 \\ 14 & 20 & 28 \end{bmatrix} = 2\begin{bmatrix} 3 & 4 & 6 \\ 5 & 7 & 10 \\ 7 & 10 & 14 \end{bmatrix}$$

$$|B| = 2\begin{vmatrix} 3 & 4 & 6 \\ 5 & 7 & 10 \\ 7 & 10 & 14 \end{vmatrix}$$

$|B| = 0 \Rightarrow$ B is linearly dependent

$\Rightarrow \{T(V_1), T(V_2), T(V_3)\}$ are L.D. —— (2)

Consider; co-efficient matrix of transformation

$$T \Rightarrow [T] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$|T| = \cancel{-8} + \cancel{8} + \cancel{3}(0)$$

$$|T| = 0$$

$\Rightarrow$ T is singular Matrix.

Hence; $\{T(V_1), T(V_2), T(V_3)\}$ are linearly dependent.

2.C.(ii) Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of matrix B^3, where $B = C^{-1}AC$.

Sol. $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(-(1+3)) + 2(-1-1) + 2(3-1)$$

$$= 2\times -4 + 2 \times \cancel{-2} \cancel{+ 2\times 2}$$

$$|A| = -8 \neq 0$$

→ A is linearly independent

→ A is diagonalizable ——— ①

let λ be an eigen value of A,

characteristic polynomial is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$0 = (2-\lambda)[(1-\lambda)^2 - 3] + 2[-1-\lambda-1] + 2[3-1+\lambda]$$

$$\Rightarrow (2-\lambda)(\lambda+2)(\lambda-2) = 0$$

$$\boxed{\lambda = 2, 2, -2} \text{ ——— ②}$$

∵ A is diagonalizable (from eqⁿ ①),

∃ P invertible matrix, such that

$A = P^{-1}DP$ where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\Rightarrow A = P^{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} P$

$A^2 = (P^{-1}DP)(P^{-1}DP)$

$A^2 = P^{-1}D(PP^{-1})DP$

$A^2 = P^{-1}DIDP$

$A^2 = P^{-1}D^2P$

Similarly, $A^3 = P^{-1}D^3P = P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} P$ —— (3)

Now, since,

$B = C^{-1}AC$

$B = C^{-1}P^{-1}DPC$

$B^2 = (C^{-1}P^{-1}DPC)(C^{-1}P^{-1}DPC)$

$B^2 = C^{-1}P^{-1}DP(CC^{-1})P^{-1}DPC$

$B^2 = C^{-1}P^{-1}DPIP^{-1}DPC = C^{-1}P^{-1}DPP^{-1}DPC$

$B^2 = C^{-1}P^{-1}DIDPC = C^{-1}P^{-1}D^2PC$

Similarly,

$B^3 = C^{-1}P^{-1}D^3PC$

$\therefore B^3 = C^{-1}P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} PC$

where, PC is invertible [$\because$ P & C both are non-singular $\Rightarrow$ invertible)

$\Rightarrow B^3$ is diagonalizable and diagonal elements are eigen values of B^3.

$\Rightarrow$ Eigen values of $B^3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 8, 8, -8$

5(a). Obtain the soluton of the ordinary differential equation $\frac{dy}{dx}=(4x+y+1)^2$, if y(0) = 1

SOLUTION

Let $4x + y + 1 = t$

differentiating w.r.t to X,

$$4+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx}-4$$

substituting in the given equation.

$$\frac{dt}{dx}-4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\int\frac{dt}{t^2+4} = \int dx + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Put the given condition y(0) = 1

$$\frac{1}{2}\tan^{-1}\left(\frac{0+1+1}{2}\right) = x + c$$

$$c = \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$c = \frac{\pi}{8}$$

$\therefore$ Required solution

$$= (4x+y+2) = 2\tan\left(2x+\frac{\pi}{4}\right)$$

$$\boxed{y=-4x-2+2\tan\left(2x+\frac{\pi}{4}\right)}$$

5(b). Determine the orthogonal trajectory of a family of curves represented by the polar equation $r = a(1 - \cos\theta)$, (r, θ) being the plane polar coordinates of any point.

SOLUTION

Given polar equation

$$r = a(1 - \cos\theta) \qquad .....(1)$$

differentiating w.r.t. to θ

$$\frac{dr}{d\theta} = a\sin\theta \qquad .....(2)$$

eliminating arbitary constant a

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

for orthogonal trajectories replace

$\frac{dr}{d\theta}$ by $-\frac{r^2}{\frac{dr}{d\theta}}$

$$\frac{1}{r}\left(-\frac{r^2}{\frac{dr}{d\theta}}\right) = \frac{\sin\theta}{1-\cos\theta}$$

$$\int\frac{-dr}{r} = \int\frac{(1-\cos\theta)d\theta}{\sin\theta} + c$$

$$-\ell n\, r = \int \tan\frac{\theta}{2}d\theta + c$$

$$-\ell n\, r = 2\ell n \sec\frac{\theta}{2} + c$$

$$r\sec^2\frac{\theta}{2} = c$$

Required solution

$$\boxed{r = c(1+\cos\theta)}$$

S(C) IAS 2011 P-I → The velocity of a train increases from 0 to v at a constant acceleration f_1, then remains constant for an interval and again decreases to 0 at a constant retardation f_2. If the total distance described is x, find the total time taken.

Sol^n → Let the train accelerate from the velocity 0 to v at this rate α in time t_1 and distance is d_1.

$$\alpha = V-0/t_1 = V/t_1$$

$$t_1 = V/\alpha$$

then $d_1 = V^2/2\alpha$

After that it moves with constant speed v in time t_2.

Then $d_2 = Vt_2$

Then it decelerates from v to zero in time t_3 with the rate β

$$t_3 = V/\beta$$

Now using

$$V^2 + u^2 = 2\beta d_3$$

$$0 + V^2 = 2\beta d_3$$

$$d_3 = V^2/2\beta$$

$$\text{Total distance } (d) = V^2/_{2\alpha} + Vt_2 + V^2/_{2\beta}$$

$$d = V^2/_{2\alpha} + Vt_2 + V^2/_{2\beta}$$

$$t_2 = d/_V - V/_2 + V/_2$$

Then total time $T = d/_V - V/_2 + V/_2 + V/_2 = d/_V + V/_2$

As the total distance is 1.

Then $T = 1/_V + V/_2\,[1/\alpha + 1/\beta]$.

5.(e) For two vectors $\vec{a}$ *and* $\vec{b}$ given respectively by $\vec{a}=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$ *and* $\vec{b}=\sin t\hat{i}-\cos t\hat{j}$

Determine: $(i)\dfrac{d}{dt}\left(\vec{a}\cdot\vec{b}\right)$ and $(ii)\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)$.

SOLUTION

Given vector are

$$\vec{a} = 5t^2\hat{i}+t\hat{j}-t^3\hat{k}$$

$$\vec{b} = \sin\hat{i}-\cos t\hat{j}$$

(i)
$$\frac{d}{dt}\left(\vec{a}\cdot\vec{b}\right) = \frac{d}{dt}\left(5t^2\sin t - t\cos t\right)$$

= 10 t sint + 5t² cost – cost + t sint

= 11 t sint + (5t² – 1) cost

(ii) $\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)$

$$\because \qquad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \hat{i}\left(-t^3\cos t\right)-\hat{j}\left(+t^3\sin t\right)+\hat{k}\left(-5t^2\cos t - t\sin t\right)$$

$$= -t^3\cos\hat{i}-t^3\sin\hat{j}-\left(5t\cos t+t\sin t\right)\hat{k}$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right) = \left(-3t^2\cos t+t^3\sin t\right)\hat{i}-\left(3t^2\sin t+t^3\cos t\right)\hat{j}-\left(10\cos t-5t^2\sin t+\sin t+t\cos t\right)\hat{k}$$

$$= \left(t^3\sin t-3t^2\cos t\right)\hat{i}-\left(t^3\cos t+3t^2\sin t\right)\hat{j}-\left\{11t\cos t+\left(1-5t^2\right)\sin t\right\}\hat{k}$$

is the required

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5.(f) If u and v are two scalar fields and $\vec{f}$ is a vector field, such that $u\vec{f} = grad\, v$, find the value of $\vec{f} \cdot curl\, \vec{f}$.

SOLUTION

Given that
u & v are scalar fields
& $\vec{f}$ is a vetor field

& $$u\vec{f} = \text{grad } v$$

i.e. $$u\vec{f} = \nabla v \quad .....(1)$$

We have to evaluate

$$\vec{f} \cdot \text{curl}\, \vec{f} = \vec{f}\left(\nabla \times \vec{f}\right) \quad .....(2)$$

$(1) \Rightarrow u\vec{f} = \nabla v$

Taking curl poth sides

$$\nabla \times \left(u\vec{f}\right) = \left(\nabla \times \nabla v\right)$$

$$\Rightarrow \sum i \frac{\partial}{\partial n} \times u\vec{f} = \left(\nabla \times \nabla\right) v$$

$$\Rightarrow \nabla \times \nabla v = \sum i \times \left(\frac{\partial u}{\partial n}\vec{f} + u\frac{\partial \vec{f}}{\partial n}\right)$$

$$= \sum \hat{i} \times \frac{\partial u}{\partial n}\vec{f} + \sum \hat{i} \times \frac{\partial u}{\partial n}$$

$$= \sum \frac{\partial u}{\partial n}\vec{f} \times \hat{i} - 4\sum \hat{i} \times \frac{\partial \vec{f}}{\partial n}$$

$$= \sum i \frac{\partial u}{\partial n} \times \vec{f} - 4\sum \frac{\partial}{\partial n}\hat{i} \times \vec{f}$$

$$\left(\nabla \times \nabla v\right) = \left(\nabla u\right) \times \vec{f} - u\left(\nabla \times \vec{f}\right)$$

$$\Rightarrow u\left(\nabla \times \vec{f}\right) = \nabla u \times \vec{f} - \nabla \times \nabla v$$

$$\Rightarrow u\vec{f} \cdot u\left(\nabla \times \vec{f}\right) = u\vec{f} \cdot \left(\nabla u \times \vec{f}\right) - u\vec{f} \cdot \left(\nabla \times \nabla v\right)$$

$$\Rightarrow u^2\vec{f} \cdot \left(\nabla \times \vec{f}\right) = 0 - u\vec{f} \cdot \left(\nabla \times \nabla v\right)$$

$$\Rightarrow u^2\vec{f}\left(\text{curl}\, \vec{f}\right) = \nabla v \cdot \left(\nabla \times \nabla v\right)$$

$$\Rightarrow \vec{f} \cdot \left(\text{curl}\, \vec{f}\right) = \frac{-\nabla v \cdot \left(\nabla \times \nabla v\right)}{u^2}$$

$$= 0$$

$\therefore$ $\boxed{\vec{f} \cdot \left(\text{curl}\, \vec{f} = 0\right)}$ is required result

6(a). Obtain Clairaut's form of the differential equation $\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+y\right)=a^2\frac{dy}{dx}$. Also find its general solution.

SOLUTION

Let $\frac{dy}{dx} = p$

Given equation $(xp - y)(yp + x) = a^2p$

Let $x^2 = u$

$y^2 = v$

$2x\,dx = du$

$2y\,dy = dv$

$\therefore \quad \frac{x}{y}\frac{dx}{dy} = \frac{du}{dv}$

$\Rightarrow \quad \frac{y}{x}p = P$

puting $p=\frac{x}{y}P$ in given equation

$$\left(\frac{x^2p}{y}-y\right)(xp+x) = a^2\frac{x}{y}p$$

$$(x^2p - y^2) = \frac{a^2p}{p+1}$$

Put $x^2 = u$

$y^2 = v$

$$pu - v = \frac{a^2p}{p+1}$$

$$\boxed{v = pu - \frac{a^2p}{p+1}}$$

This is clearly is clauraust form

$\therefore$ General solution of given differential equation is

$$\boxed{y^2 = cx^2 - \frac{a^2c}{c+1}}$$

6(b). Obtain the general solution of the second order ordinary differential equation $y''-2y'+2y=x+e^{x}\cos x$, where dashes denote derivatives w.r. to x.

SOLUTION

Given differential equation is

$$y''-2y'+2y = x + e^{x}\cos x$$

Auxillary equation of homogenous equation is

$$m^2 - 2m + 2 = 0$$

$\therefore$ $$m = 1 \pm i$$

$\therefore$ Complimentary function

$$y_c = c_1 e^{x}\cos x + c_2 e^{x}\sin x$$

particular integral

$$y_p = \frac{1\left(e^{x}\cos x + x\right)}{D^2-2D+2}$$

$$= \frac{1}{D^2-2D+2}e^{x}\cos + \frac{1}{2\left[1-\left(\frac{2D-D^2}{2}\right)\right]}x$$

$$= e^{x}\frac{\cos x}{(D+1)^2-2(D+1)+2} + \frac{1}{2}\left[1+\frac{\left(2D-D^2\right)}{2}\right]x$$

$$= \frac{e^{x}\cos x}{D^2+1} + \frac{1}{2}[x+1]$$

$$y_p = e^{x}\frac{x}{2}\sin x + \frac{(x+1)}{2}$$

$\therefore$ Solution $y = y_p + y_c$

$$\boxed{y = c_1e^{x}\cos x + c_2e^{x}\sin x + \frac{(x+1)}{2} + e^{x}\frac{x}{2}\sin x}$$

6(c). Using the method of variation of parameters, solve the second order differential equation $\frac{d^2y}{dx^2} + 4y = \tan 2x.$

SOLUTION

Given differential equations is

$$(D^2 + 4)\, y = \tan 2x$$

Auxillary equation is $m^2 + 4 = 0$

$$m = \pm 2$$

$\therefore$ Complimentary function

$$y_c = c_1 \sin 2x + c_2 \cos 2x$$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

Let $u(x) = \cos 2x$

$$v(x) = \sin 2x$$

$$w(u.v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0$$

$\therefore$ u.v all linearly independent

$\therefore$ Let $y_p(x) = A(x)\, u(x) + B(x)\, V(x)$

$$A(x) = -\int \frac{Rv}{W(u.v)} dx$$

$$= -\int \frac{\tan 2x \cdot \sin 2x}{2} dx$$

$$= -\int \frac{1-\cos^2 2x}{2\cos 2x} dx$$

$$= \frac{1}{2}\int [\cos 2x - \sec 2x] dx$$

$$= \frac{1}{2}\int \left[\frac{\sin 2x}{2} - \frac{\ell n(\tan 2x + \sec 2x)}{2}\right]$$

$$= \frac{-1}{4}\ell n(\sec 2x + \tan 2x) + \frac{\sin 2x}{4}$$

$$B(x) = \int \frac{uR}{w(u,v)} dx$$

$$= \int \frac{\cos 2x \cdot \tan 2x}{2} dx$$

$$= -\frac{\cos 2x}{4}$$

$\therefore$

$$y_p = A(x)\, u(x) + B(x)\, v(x)$$

$$= \frac{-\cos 2x}{4} \ell n(\sec 2x + \tan 2x)$$

General solution $= y_c + y_p$

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\cos 2x\, \ell n(\sec 2x + \tan 2x)$$

6(d) Use Laplace transform method to solve the following initial value problem:

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t,\ x(0) = 2 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = -1$$

SOLUTION

Given $x'' - 2x' + x = e^t$

Applying Laplace transform on both sides

$$s^2\mathcal{L}(x) - sx(0) - x'(0) - 2s\,\mathcal{L}(x) + 2x(0) + \mathcal{L}(x) = \frac{1}{s-1}$$

(Given $x(0) = 2$ and $x'(0) = -1$)

$$(s^2 - 2s + 1)\,\mathcal{L}(x) - 2s + 1 + 4 = \frac{1}{s-1}$$

$$\mathcal{L}(x) = \frac{1}{(s-1)(s-1)^2} + \frac{2s-5}{(s-1)^2}$$

$$\mathcal{L}(x) = \frac{1}{(s-1)^3} + \frac{2}{(s-1)} - \frac{3}{(s-1)^2}$$

$$X(t) = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^3} + \frac{2}{s-1} - \frac{3}{(s-1)^2}\right]$$

$$= e^t\left(\frac{t^2}{2} - 3t + 2\right)$$

8.(a) Examine whether the vectors $\nabla u, \nabla v$ *and* ∇w are coplanar, where u, v and w are the scalar functions defined by: $u=x+y+z$, $v=x^2+y^2+z^2$ and $w=yz+zx+xy$.

SOLUTION

Given that $\nabla u, \nabla v \& \nabla w$ are co-planer

$$\text{where} \qquad u = x + y + z$$

$$\therefore \qquad \nabla u = \sum \frac{\partial}{\partial x}\hat{i}(x+y+z)$$

$$= \hat{i}+\hat{j}+\hat{k}$$

$$v = x^2 + y^2 + z^2$$

$$\nabla v = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$w = yz + zn + xy$$

$$\nabla w = (y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$$

for $\nabla u, \nabla v \& \nabla w$ to be complanai; triple product must be zero. So evaluateing scatar triple product.

$$[\nabla u\ \nabla v\, \nabla w] = (\nabla u \times \nabla v)\cdot \nabla w$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+2 & z+n & x+y \end{vmatrix}$$

Operate R_3 ® (x + y +z) R_3

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ n & y & z \end{vmatrix}$$

$$= 0$$

By two rows are common-determinant must be zero.

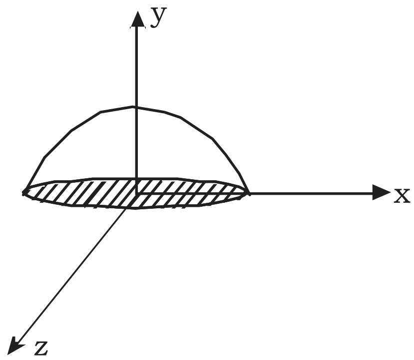
Hence ∇u, $\nabla v \& \nabla w$ are the coplanar vectors.

8.(b) If $\vec{u} = 4y\hat{i} + x\hat{j} - 2z\hat{k}$, calculate the double integral $\iint(\nabla \times \vec{u}) \cdot d\vec{s}$ over the hemisphere given by $x^2 + y^2 + z^2 = a^2, z \geq 0.$

SOLUTION

$$\vec{u} = 4y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$S: x^2+y^2+z^2 = a^2, z \geq 0$$



Evaluate

$$\iint(\nabla \times \vec{u}) \cdot d\vec{s} = \iint(\nabla \times \vec{u}) \cdot \hat{n}\, d\vec{s}$$

$$\hat{n} = \frac{\nabla s}{|\nabla s|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2 \times a} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$\because$
$$\nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial n} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & n & -2z \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(1-4)$$

$$= 3\hat{k}$$

$$(\nabla \times \vec{u}) \cdot \hat{n} = -3\hat{k} \cdot \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{a}$$

$$= -\frac{3z}{a}$$

dx = elementary syrface area

$$= \frac{dx\,dy}{|\hat{n}(-\hat{k})|} - \frac{dx\,dy}{(z/a)}$$

$\therefore$ Integral value $= \iint(\nabla \times \vec{u}) \cdot n\,ds$

$$= \iint \left(\frac{-32}{a}\right) \cdot \frac{dx\,dy}{(z/a)}$$

$$= \iint -3\,dx\,dy$$

$$= -3\iint dx\,dy$$

in xy plane

$$= -3\iint dx\,dy \quad x^2 + y^2 = a^2 \ \& \ z = 0$$

Let $x = \cos\theta$
$y = \sin\theta$

$$= -3\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,dr\,d\theta$$

$$= -3\left(\int_0^{\pi} d\theta\right)\left(\frac{r^2}{2}\right)_0^9$$

$$= -3\frac{a^2}{2} \times \pi = \frac{-3\pi a^2}{2} \text{ is require result}$$

8.(c) If $\vec{r}$ be the position vector of a point, find the value(s) of n for which the vector $r^n\vec{r}$ is (i) irrotational, (ii) solenoidal.

SOLUTION

$\vec{r}$ is a position vector of a point constidering $r^n\vec{r}$ vector

(i) For irrotational

$$\nabla \times r^n\vec{r} = 0$$

$$\Rightarrow \sum i\frac{\partial}{\partial n}\times r^n\vec{r} = 0$$

$$\Rightarrow \sum i\times\left(\frac{\partial r^n}{\partial n}\cdot\vec{r}+\frac{r^n\cdot\partial\vec{r}}{\partial x}\right) = 0$$

$$\Rightarrow \sum i\times\left(n\cdot r^{n-1}\cdot\frac{\partial r}{\partial x}\cdot\vec{r}+r^n\hat{k}\right) = 0$$

$$\Rightarrow \sum \hat{i}\times\left(n\cdot r^{n-1}\cdot\frac{2x}{2r}\cdot\vec{r}\right) = 0$$

$$\Rightarrow \sum \hat{i}\times\left(nr^{n-2}x\,\vec{r}\right) =$$

$$\Rightarrow \sum\left(\hat{i}\times\vec{r}\right)nr_n^{n-2} = 0$$

$$\Rightarrow nr^{n-2}\sum\left(y\hat{k}-z\hat{j}\right) = 0$$

$$\Rightarrow nr^{n-2}\sum\left(xy\hat{k}-z\hat{j}x\right)=0 = 0$$

Hence for real value of n the $r^n\vec{r}$ will be irrotational.

$\Rightarrow \boxed{0=0}$

(ii) For Solenoidal

$$\nabla\cdot\left(r^n\vec{r}\right) = 0$$

$$\Rightarrow \nabla\cdot r^n\vec{r}+r^n\nabla\cdot\vec{r} = 0$$

$$\Rightarrow nr^{n-1}\nabla r\cdot\vec{r}+r^n\nabla\cdot\vec{r} = 0$$

$$\Rightarrow nr^{n-1}\left(\frac{\vec{r}}{r}\right)\cdot\left(\vec{r}\right)+r^n\left(3\right) = 0$$

$$\Rightarrow nr^{n-1}\frac{r^2}{r}+3r^n = 0$$

$$\Rightarrow nr^n + 3r^n = 0$$

$$\Rightarrow (n + 3)\, r^n = 0$$

$$\Rightarrow n + 3 = 0$$

$\therefore \boxed{n=-3}$

8.(d) Verify Gauss Divergence Theorem for the vector $\vec{v} = x^2\hat{i} + y^2\hat{j} - z^2\hat{k}$ taken over the cube $0 \le x, y, z \le 1.$

SOLUTION

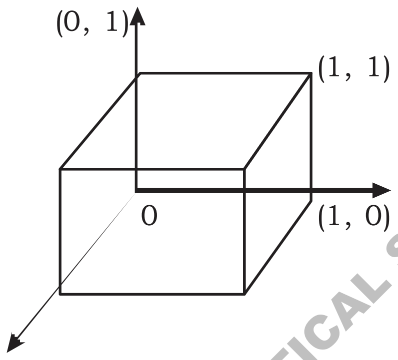
Gauss Divergence Theorem

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, ds$$

Where vector $\qquad v = x^2\hat{i} + y^2\hat{j} - z^2\hat{k}$

Taken over the cube $0 \le x, y, z \le 1$.

$$\text{Evaluate} = \iint_S \vec{v} \cdot \vec{n}\, ds \qquad .....(1)$$



$$\text{Evaluate} = \iiint_V (\nabla \cdot \vec{v})\, dv$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (2x + y - 2z)\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left(2xz + 2yz - z^2\right)_0^1 dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2x + 2y - 1)\, dx\, dy$$

$$= \int_{x=0}^{1} \left(2xy + y^2 - y\right)_0^1 dx$$

$$= \int_{x=0}^{1} (2x + 1 - 1)_0^1 dx$$

$$= x^2\Big|_0^1$$

$$= 1 \qquad .....(1)$$

$$\text{Now Evaluating} = \iint_S \left(\vec{F} \cdot \hat{n}\right) ds = \iint_{S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6} (\vec{v} \cdot \hat{n})\, ds$$

$$S = S_1+S_2+S_3+S_4+S_5+S_6$$

where

$S_1 : x = 0 \quad S_2 \Rightarrow x = 1$

$S_3 : y = 0 \quad S_4 : y = 1$

$S_5 : z = 0 \quad S_6 : z = 1$

Alongs : x = 0

$$\hat{n} = -\hat{i}$$

$$= \iint\left(x^2\hat{i}+y^2\hat{j}+z^2+z^2\hat{k}\right)\cdot\left(-\hat{i}\right)ds \text{ at } x = 0$$

$$= \iint -x^2 ds = 0 \qquad (\because x = 0)$$

Along $S_1 = 0$

$$\hat{n} = -\hat{i}$$

$$\iint_{S_1} = \iint x^2 \cdot \frac{dy\,dz}{\left|\hat{n}\cdot\hat{i}\right|} = \iint x^2\,dy\,dz \qquad (\because x = 1)$$

$$= \iint dy\,dz$$

$$= 1$$

Along $S_3 : y = 0$

$\hat{n} = -\hat{j}$

dx = dz dx

$$\iint_{S_3}\left(-y^2\right)\cdot ds = \iint 0 \times ds \qquad (\because s = 0)$$

$$= 0$$

Along $S_4 : y = 1$

$\hat{n} = \hat{j}$

$$\iint_{S_4} y^2\,ds = \iint ds \qquad (\because y = 1)$$

$$= \iint dz\,dx$$

= Surface area of square

= 1

Along $S_5 : z : 0$

$\hat{n} = -\hat{k}$

$$\iint_{S_5} +z^2\,ds = \iint (0)\,ds = 0 \qquad (\because z = 0)$$

Along $S_6 : z = 1$

$$\iint_{S_6}\left(x^2\hat{\imath}+y\hat{j}-z^2\hat{k}\right)\cdot\left(\hat{k}\right)ds = \iint_{S_6} -z^2 ds \qquad (\because z = 1)$$

$$= \iint -dx\,dy$$

$$= -\iint dx\,dy$$

$$= -1$$

$$\therefore \quad \iint_S (\vec{v}\cdot\vec{n})\,ds = \iint_{S_1} v(\hat{i})\,ds + \iint_{S_2} (v\cdot\hat{k})\,ds + \iint_{S_3} (v\cdot\hat{j})\,ds + \iint_{S_4} (v\cdot\hat{j})\,ds$$

$$+\iint_{S_5} (v\cdot\hat{k})\,ds + \iint_{S_6} (v\cdot\hat{k})\,ds$$

$$= 0 + 1 + 0 + 1 + 0 - 1$$

$\therefore$ L.H.S. = R.H.S. Hence Gauss Divergence Theorem