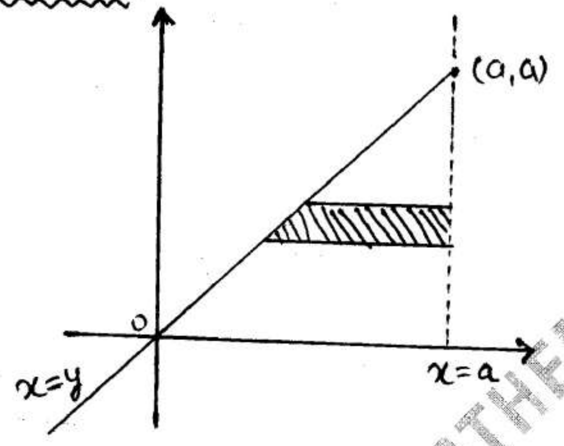
2008 - Paper - I :-

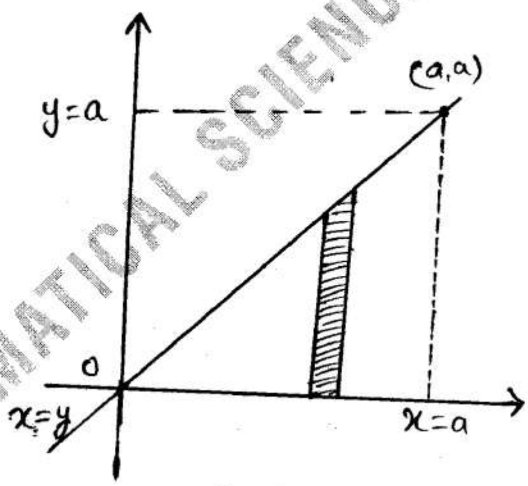
Ques: 3(b)) Evaluate the double integral

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$$

by changing the order of integration.

Solution:-





$$I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x\,dx\,dy}{x^2+y^2}$$

Integrating w.r.t y, keeping x constant

$$I = \int_0^a x \cdot \left[\frac{1}{x} \cdot \tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$I = \int_0^a x \left[\frac{1}{x} \tan^{-1} 1\right] dx = \int_0^a \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot \frac{\pi}{4}\,dx$$

$$I = \frac{\pi}{4}\left[x\right]_0^a = \frac{\pi}{4}a.$$

$$\boxed{\therefore\ I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \frac{\pi}{4}a}$$ Required result

Ques:-4(c)> Show that the enveloping cylinders of the ellipsoid $ax^2+by+cz^2=1$ with generators perpendicular to z axis meet the plane $z=0$ in parabolas.

Solution:-

let generator be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Since, it is $\perp^r$ to z-axis so

$$l\cdot 0+m\cdot 0+n\cdot 1=0 \quad \Rightarrow \boxed{n=0}$$

Any general point on generator can be $(lr+x_1,\ mr+y,\ z_1)$ put this in equation of ellipsoid.

$$a(lr+x_1)^2+b(mr+y_1)^2+cz_1^2=1$$

$$r^2(al^2+bm^2)+2r(alx_1+bmy_1)+ax_1^2+by_1^2+cz_1^2-1=0 \quad \text{--- (1)}$$

As generator touches ellipsoid so $r_1=r_2$ thus $B^2=4AC$ in equation (1)

$$4(alx_1+bmy_1)^2=4(al^2+bm^2)(ax_1^2+by_1^2+cz_1^2-1)$$

So equation of enveloping cylinder of ellipsoid is

$$\left[(alx+bmy)^2=(al^2+bm^2)(ax_1^2+by_1^2+cz_1^2-1)\right] \quad \text{--- (2)}$$

Enveloping cylinder meeting $z=0$ in parabola if $h^2=ab$ where $h=\frac{\text{coefficient of } xy}{2}$

a = coefficient of x and
b = coefficient of y

In (2)

a = coefficient of $x = a^2 l^2 - a(al^2 + bm^2) = -abm^2$

b = coefficient of $y = b^2m^2 - b(al^2 + bm^2)$

$= \cancel{b^2m^2} - bal^2 - \cancel{b^2m^2}$

$= -abl^2.$

$2h$ = coefficient of $xy = 2ablm$

These values satisfy $h^2 = ab$
thus enveloping cylinder meet $z=0$ in parabola

required result.