$\therefore$ We have 4 possibilities for x.

Total number of homomorphism $= 4 + 1 = 5$.

30. Show that the alternating group on four letters A_4 has no subgroup of order 6. (15)

Sol.: Consider the alternating group A_4.

$$o(A_4) = \frac{o(S_4)}{2} = \frac{\lfloor 4}{2} = 12$$

We show although $6 \mid 12$, A_4 has no subgroup of order 6. Suppose H is a subgroup of A_4 and $o(H) = 6$.

By previous problem the number of distinct 3-cycles in S_4 is

$$\frac{1}{3}\frac{4!}{(4-3)!} = \frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 1} = 8.$$

Again, as each 3-cycle will be even permutation all these 3-cycles are in A_4. Obviously then, at least ~~one 3-cycle~~, say σ, does not belong to $H(o(H)=6)$.

Now $\sigma \in H \Rightarrow \sigma^2 \in H$, because if $\sigma^2 \in H$

Then $\sigma^4 \in H$

$\Rightarrow \quad \sigma \in H$

As $\quad \sigma^3 = I$ as $o(\sigma) = 3$.

Let

$K = <\sigma> = \{I, \sigma, \sigma^2\}$ then, $oK = 3 (= o(\sigma))$

and $\quad H \cap K = \{I\} \quad (\sigma, \sigma^2 \notin H)$

$$\Rightarrow \quad o(HK) = \frac{o(H)\cdot o(K)}{o(H\cap K)} = \frac{6.3}{1} = 18, \text{ not}$$

possible as $HK \subseteq A_4$ and $o(A_4) = 12$.

Year-2010

31. Let $G = \mathbb{R} - \{-1\}$ be the set of all real numbers omitting -1. Define the binary relation* on G by $a * b = a + b + ab$. Show $(G, *)$ is a group and it is abelian. (12)

Sol. Here $G =$ the set of all real numbers other than -1.

$\therefore$ a, b, c are real number other than -1.

(i) Closure property

Since a and b are real number so $a + b + ab$ is also a real number

If $a + b + ab = -1$ then $a + b + ab + 1 = 0$

$\Rightarrow \quad a + ab + b + 1 = 0$

$\Rightarrow \quad a(1+b) + (1+b) = 0$

$\Rightarrow \quad (a+1)(b+1) = 0$

$\Rightarrow \quad a = -1$ or $b = -1$, which is not true

$\therefore \quad a + b + ab \neq -1 \Rightarrow a + b + ab \in G$.

$\therefore$ closure property holds in G.

(ii) Associative property

$(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c + (a + b + ab)c$

$= a + b + c + ab + ac + bc + abc$

$a * (b * c) = a * (b + c + bc)$

$= a + b + c + bc + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$\therefore \quad (a * b) * c = a * (b * c)$

$\therefore$ associative law holds in G.

(iii) Existence of identity

Let $e \in G$ be the identity element of G.

Then

$a * e = a$

$a + e + ae = a$

$(1 + a)e = 0$

$e = 0 \qquad (\because a \neq -1 \Rightarrow 1 + a \neq 0)$

(iv) Existence of inverse

For any $a \in G$ Let

$a * b = 0$

$\Rightarrow a + b + ab = 0$

$\Rightarrow a + (1 + a)b = 0$

$\Rightarrow (1 + a)b = -a$

$\Rightarrow b = -\dfrac{a}{1+a} \in G.$

(v) Commutative property

$a * b = a + b + ab$

$= b + a + ba$

$= b * a$

$\therefore$ $(G, *)$ is an abelian group.

32. Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group

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of order 2 and a cyclic group of order 3. Can you generalize this? Justify. (12)

Sol. Let A and B cyclic group of order 2 and order 3 respectively.

i.e., $A = \langle a \rangle, B = \langle b \rangle$

$O(A) = O(a) = 2, O(B) = O(b) = 3.$

As $O(a), O(b)$ are relatively prime.

We will show $A \times B$ is cyclic.

i.e., for $a \in A, b \in B$.

$O(a,b) = 2.3 = O(A \times B).$

Consider,

$(a,b)^6 = (a^6, b^6)$

$= ((a^2)^3, (b^3)^2)$

$= (e_1, e_2) =$ identity of $A \times B$.

Let

$(a,b)^r = (e_1, e_2)$

$\Rightarrow (a^r, b^r) = (e_1, e_2)$

$\Rightarrow a^r = e_1, b^r = e_2$

$\Rightarrow O(a)/r, O(b)/r$

$\Rightarrow 2/r, 3/r.$

$\Rightarrow 6/r$ $\quad (\because \text{g.c.d } (2,3) = 1)$

$\Rightarrow O(a,b) = 6 = O(A \times B).$

Hence, $A \times B$ is cyclic group of order 6. And every cyclic group of order n is isomorphic to $\mathbb{Z}_n$.

$\therefore \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

We will show A, B be finite cyclic groups of order m and n respectively, then $A \times B$ is cyclic if and only if m and n are relatively prime.

Let $A = \langle a \rangle, B = \langle b \rangle$

$o(A) = o(a) = m, o(B) = o(b) = n$

Suppose $A \times B$ is cyclic.

Let $A \times B = \langle (x, y) \rangle, x \in A, y \in B$

$o(A \times B) = mn = o(x, y)$

Let g.c.d. of m and n be d

$\therefore \frac{m}{d}$ and $\frac{n}{d}$ are relatively prime integers.

Consider $(x,y)^{\frac{mn}{d}} = \left(x^{\frac{mn}{d}}, y^{\frac{mn}{d}}\right)$

$= \left(e_1^{\frac{n}{d}}, e_2^{\frac{m}{d}}\right), e_1 =$ identity of A

$e_2 =$ identity of B

$= (e_1, e_2)$

$=$ identity of $A \times B$

$\therefore o(x,y) \Big| \frac{mn}{d}$

$\Rightarrow mn \Big| \frac{mn}{d}$

$\Rightarrow d.\frac{mn}{d} \Big| \frac{mn}{d}$

$\Rightarrow d | 1 \Rightarrow d = 1.$

$\therefore m$ and n are relatively prime.

Conversely, let m and n be relatively prime. We show $A \times B$ is cyclic, generated by (a,b). For that we prove $o(a,b) = mn = o(A \times B)$.

Consider $(a,b)^{mn} = (a^{mn}, b^{mn})$

$= \left((a^m)^n, (b^n)^m\right)$

$= (e_1, e_2) =$ identity of $A \times B$

Let $(a,b)^r = (e_1, e_2)$

$\Rightarrow (a^r, b^r) = (e_1, e_2)$

$\Rightarrow a^r = e_1, b^r = e_2$

$\Rightarrow o(a) = m | r, o(b) = n | r$

$\Rightarrow mn | r$ as m, n are relatively prime.

$\Rightarrow mn \le r$

$\therefore o(a,b) = mn = o(A \times B)$

Hence $A \times B = \langle (a,b) \rangle =$ cyclic group generated by (a,b).

33. Let $(\mathbb{R}^*, \cdot)$ be the multiplicative group of non-zero real and $(GL(n, \mathbb{R}), \times)$ be the multiplicative group of $n \times n$ non-singular real matrices. Show that the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ and $(\mathbb{R}^*, \cdot)$ are isomorphic where $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / \det A = 1\}$.

What is the centre of $GL(n, \mathbb{R})$? **(15)**

Sol.: Let $(\mathbb{R}^*, \cdot)$ be group of non-zero real number.

We have to show $\dfrac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})} \cong (\mathbb{R}^*, \cdot)$

Define $f : GL(n, \mathbb{R}) \to (\mathbb{R}^*, \cdot)$

$f(A) = |A|$

We will show f is a homomorphism.

Consider, $A, B \in GL(n, \mathbb{R})$

$f(A \cdot B) = |A \cdot B|$

$= |A||B| = f(A) f(B).$

$\therefore f$ is a homomorphism.

$Ker\, f = \{A \in GL(n, \mathbb{R}) / f(A) = 1\}$.

$= \{A \in GL(n, \mathbb{R}) / |A| = 1\}$

$= SL(n, \mathbb{R})$.

We will show f is onto.

Let $r \in \mathbb{R}^*$

$$\text{Define } A = \begin{bmatrix} r & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{n \times n}$$

As $|A| = r \neq 0$

And A is $n \times n$ matrix with $|A| \neq 0$

$\therefore \quad A \in GL(n, \mathbb{R})$.

Hence, for any $r \in \mathbb{R}^*$

$\exists\, A \in GL(n, \mathbb{R})$ such that

$f(A) = r$

$\therefore f$ is onto.

By first group homomorphism theorem;

$$\frac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})} \cong f(GL(n, \mathbb{R})) = (\mathbb{R}^*, \cdot)$$

We have to find centre of $GL(n, \mathbb{R})$.

As we know scalar matrix is the only matrix which commutes with every matrix.

$\therefore \quad Z(GL(n,\mathbb{R})) = \{A \in GL(n, \mathbb{R}) / A \cdot B = B \cdot A \;\forall\, B \in GL(n, \mathbb{R})\}$

$$\left\{ A = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & \dots & a \end{bmatrix} \middle/ AB = BA \;\forall\, B \in GL(n, \mathbb{R}) \right\}$$

$$\text{As } A = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & \dots & a \end{bmatrix}_{n \times n} \in GL(n, \mathbb{R})$$

$\therefore \quad |A| = a^n \neq 0.$

i.e., $\quad a \neq 0.$

$\therefore$

$$Z(GL(n,\mathbb{R})) = \left\{ \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \dots & \dots & a \end{bmatrix}_{n \times n} : a \in \mathbb{R}^* \right\}.$$

Year-2011

34. Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of Complex numbers defined by

$f_1(z) = z,\; f_2(z) = 1 - z$,

$f_3(z) = \dfrac{z}{(z-1)},\; f_4(z) = \dfrac{1}{z},\; f_5(z) = \dfrac{1}{(1-z)}$ and

$f_6(z) = \dfrac{(z-1)}{z}$

is a non-abelian group of order 6 w.r.t. composition of mappings. **(12)**

Sol.

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	z	$1-z$	$\frac{z}{z-1}$	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$
f_2	$1-z$	z	$\frac{1}{1-z}$	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$
f_3	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$
f_4	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$	z	$1-z$	$\frac{z}{z-1}$
f_5	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z
f_6	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$	$1-z$	z	$\frac{1}{1-z}$

From table, its clear that the given set G is a non-abelian group of order 6 w.r.t. composition of mappings.

35. (i) Prove that a group of Prime order is abelian. (06)

(ii) How many generators are there of the cyclic group $(G, \cdot)$ of order 8? (06)

Sol. (i) Let G be a group of prime order.

i.e. $O(G) = \text{prime} = p \text{ (say)}; p > 1.$

and we know order of an element divides the order of group.

Let $a \in G$.

$\therefore \quad \frac{O(a)}{O(g)} \to O(a) / O(G)$

$\frac{O(a)}{p} \to O(a) / p$

$\Rightarrow \quad O(a) = 1 \text{ or } O(a) = p.$

and identity element is the only element of order 1.

If $a \neq e$.

$\therefore \quad O(a) = p$

Let $H = (a)$

$\therefore \quad H < G.$

and $O(H) = p; H \subseteq G$

$\therefore \quad H = G = (a)$

$\therefore \quad G$ is cyclic

Hence, G is a abelian $\quad (\because \text{cydic} \Rightarrow \text{abelian})$

(ii) Let G be a cyclic group of order 8.

$\therefore \quad \exists$ an element $'a' \in G$ and that $O(0) = 8$. $\quad (\because G \text{ is cyclic})$

We know,

$$O(a^k) = \frac{o(a)}{(0(a), k)}$$

$\therefore \quad O(a^k) = O(a) \Leftrightarrow (O(a), k) = 1.$

Number of elements of G whose order co-prime to $O(a) = \phi(O(a))$.

Here, $O(a) = 8$

$\therefore \quad$ Number of such elements $= \phi(8) = 4$

$\left(\because \phi(p^n) = p^n - p^{n-1}\right)$

$\therefore$ Number of generators of cyclic group of order $8 = 4$.

36. Given an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic. (15)

Sol. Consider $G = K_4 = \{e, a, b, c\}$

Here, $a^2 = e = b^2 = c^2$ and $ab = c$.

If $H(K_4)$ then $\frac{O(H)}{O(G)} \to O(H) / O(G)$

$\therefore O(H) / G$

$(O(H) = 1, 2 \text{ or } 4.$

If $O(H) = 1$.

i.e., $H = \{e\}$

$\therefore \quad H$ is cyclic.

If $O(H) = 2$.

i.e. We have 3 possibilities,

$H = \{e, a\}$ or $H = \{e, b\}$ or $\{e, c\}$

and we know,

every group of prime order is cyclic as 2 is prime.

H is cyclic.

If $O(H) = 4$ and $H < K$

$\therefore \quad H = K$

and, as a finite group is cyclic iff of an element whose order is equal to order of group.

Here $\nexists$ any element $'p'$ such that

$o(p) = 4 = o(K_4)$.

$\therefore \quad K_4$ is not cyclic.

37. Let a and b be elements of a group, with $a^2 = e, b^6 = e$ and $ab = b^4 a$. Find the order of ab, and express its inverse in each of the form $a^m b^n$ and $b^m a^n$. (20)

Sol. Let $a, b \in G$

With $a^2 = e, b^6 = e$

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and $ab = b^4 a$.

We have to find order of ab.

Consider,

$ab = b^4 a$

$\Rightarrow \quad aba^7 = b^4$

$b^8 = b^4 \cdot b^4$

$= aba^4 aba^4$

$= ab^2 a^8$

Further

$b^{16} = b^8 \cdot b^8$

$\Rightarrow \quad b^{16} = ab^2 a^{-1} ab^2 a^4$

$\Rightarrow \quad b^{16} = ab^4 a^{-1}$

$b^{16} = aaba^{-1}a^{-1}$

$b^{16} = a^2 b (a^2)^{-1}$

$b^{16} = ebe$

$b^{16} = b$

$\Rightarrow \quad b^{15} = e$.

$\therefore \quad O(b) = 1, 3, 5$ or 15 ...(1)

and $\quad b^6 = e$

$O(b) = 1, 2, 3$ or 6. ...(2)

From (1) and (2);

$O(b) = 3$.

And $ab = b^4 a$.

$\Rightarrow ab = b^3 ba$.

$\Rightarrow ab = ba$

And for any $a, b \in G$, Such that $ab = ba$ and $(O(a), O(b)) = 1$ then $(O(a), O(b)) = 1$

$\therefore O(ab) = 6$

And $(ab)^6 = e$

$\Rightarrow (ab)^5 = (ab)^{-1}$

$\Rightarrow a^5 b^5 = (ab)^{-1}$

Similarly $(ab)^{-1} = b^5 a^5$ (as $ab = ba$).

Year-2011

38. Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of Complex numbers defined by

$f_1(z) = z,\ f_2(z) = 1 - z$

$f_3(z) = \frac{z}{(z-1)},\ f_4(z) = \frac{1}{z},\ f_5(z) = \frac{1}{(1-z)}$ and

$f_6(z) = \frac{(z-1)}{z}$

is a non-abelian group of order 6 w.r.t. composition of mappings. (12)

Sol.

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	z	$1-z$	$\frac{z}{z-1}$	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$
f_2	$1-z$	z	$\frac{1}{1-z}$	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$
f_3	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$
f_4	$\frac{1}{z}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$	z	$1-z$	$\frac{z}{z-1}$
f_5	$\frac{1}{1-z}$	$\frac{1}{z}$	$1-z$	$\frac{z}{z-1}$	$\frac{z-1}{z}$	z
f_6	$\frac{z-1}{z}$	$\frac{z}{z-1}$	$\frac{1}{z}$	$1-z$	z	$\frac{1}{1-z}$

From table, its clear that the given set G is a non-abelian group of order 6 w.r.t. composition of mappings.

39. (i) Prove that a group of Prime order is abelian. (06)

(ii) How many generators are there of the cyclic group $(G, \cdot)$ of order 8? (06)

Sol. (i) Let G be a group of prime order.

i.e. $O(G) = \text{prime} = p$ (say); $p > 1$.

and we know order of an element divides the order of group.

Let $a \in G$.

$\therefore \quad \frac{O(a)}{O(g)} \rightarrow O(a) / O(G)$

$\frac{O(a)}{p} \Rightarrow O(a) / p$

$\Rightarrow \quad O(a) = 1$ or $O(a) = p$.

and identity element is the only element of order 1.

$\therefore\quad O(b)=1,3,5$ or 15 ...(1)

and $b^6=e$

$O(b)=1,2,3$ or 6. ...(2)

From (1) and (2);

$O(b)=3$.

And $ab=b^4a$.

$\Rightarrow ab=b^3ba$.

$\Rightarrow ab=ba$

And for any $a,b\in G$. Such that $ab=ba$ and $(O(a),O(b))=1$ then $(O(a),O(b))=1$

$\therefore O(ab)=6$

And $(ab)^6=e$

$\Rightarrow (ab)^5=(ab)^{-1}$

$\Rightarrow a^5b^5=(ab)^{-1}$

Similarly $(ab)^{-1}=b^5a^5$ (as $ab=ba$).

Year-2012

42. How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1}=a^{-1}$. (12)

Sol. As $O(G)=16$.

and $O(a)=8, O(b)=2$.

and $bab^{-1}=a^{-1}$

on multiplying b from both sides

$\Rightarrow\quad b^2ab^{-1}=ba^{-1}$

$\Rightarrow\quad ab^{-1}=ba^{-1}$

$\Rightarrow\quad ab=ba^{-1}\qquad (\because b=b^{-1})$

and G is generated by a,b.

$O(a)=8, O(b)=2, bab^{-1}=a^{-1}$

$\therefore\quad G=D_8$

$\therefore\quad \left(D_8=b^ia^j, b^2=a^8=e;\quad \begin{matrix} i=0 \text{ to } 1\\ j=1 \text{ to } n-1\end{matrix}\right)$

Consider, $H=\langle a\rangle$

$O(a)=8$.

$\therefore\ H$ is cyclic group of order 8.

In finite cyclic group, for each divisor $k/n=O(G)$ there exist exactly $\phi(k)$ element of order k.

As $2/8$

$\therefore\quad \exists$ exist $\phi(2)$ elements of order 2.

$\therefore$ 1 element of order 2.

D_8 consist of 8 rotation and 8 reflection.

Since order of each reflection is 2.

$\therefore$ Number of elements of order $2=8+1=9$ elements.

43. How many conjugacy classes does the permutation group S_5 of permutation 5 numbers have? Write down one element in each class (preferably in terms of cycles) (15)

Sol. We know, two permutation are said to be conjugate if they have same cyclic decomposition.

$\therefore$ Number of cyclic decomposition = 7.

They are:

(i) $1+1+1+1+1$

(ii) $1+1+1+2$

(iii) $1+1+3$

(iv) $1+4$

(v) 5

(vi) $2+3$

(vii) $1+2+2$

Number of conjugate classes = 7.

Element in 1st class = (1)

Element in 2nd class = (1 2)

Element in 3rd class = (1 2 3)

Element in 4th class = (1 2 3 4)

Element in 5th class = (1 2 3 4 5)

Element in 6th class = (1 2 3) (4 5)

Element in 7th class = (1 2) (3 4).

Year-2014

44. Let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y\\ 0 & z\end{bmatrix}$, where $xz\neq 0$. Show that G is a group under matrix multiplication. Let N denote the subset $\left\{\begin{bmatrix} 1 & a\\ 0 & 1\end{bmatrix}: a\in\mathbb{R}\right\}$. Is N a normal subgroup of G? Justify your answer.

Sol. Let $G=\left\{\begin{bmatrix} x & y\\ 0 & z\end{bmatrix};\ xz\neq 0;\ x,y,z\in\mathbb{R}\right\}$.

To show: G is a group under matrix multiplication.

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