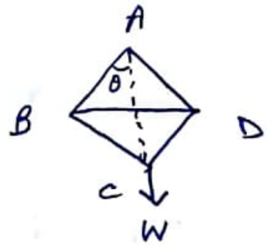
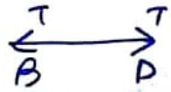
Virtual Work

(S) 5 weightless rods of equal length joined as rhombus ABCD. and diagonal BD. Weight W at C, suspend from A. Find thrust in BD.



Let length of rod be $2a$.
As system is in equilibrium and hanging from A, C must be directly below A.
Also, BD therefore is horizontal (Diagonal of rhombus bisect at 90°)

Let thrust in BD be T



$\angle BAC$ be θ.

$AC = 4a\cos\theta$.

Give a small displacement s.t $\theta \to \theta + \delta\theta$. (symmetrical)
Replace rod BD by two equal and opposite forces T.

$BD = (2a\sin\theta) \times 2$.

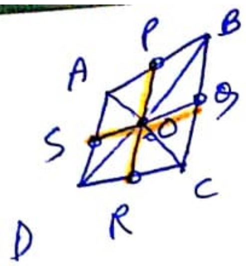
By principle of virtual work $\Rightarrow$

$$T\,\delta(4a\sin\theta) + W\,\delta(4a\cos\theta) = 0$$

$$T\cos\theta = W\sin\theta \Rightarrow T = W\tan\theta.$$

At equilibrium, $BD = 2a$, $\underline{\theta = \frac{\pi}{6}}$ (Equilateral Δr)

$$\text{So, } T = W\tan\frac{\pi}{6} = \frac{W}{\sqrt{3}}$$



$$\Delta OAD \equiv OA^2 + OB^2 = 2\left(OS^2 + AS^2\right) = 2\left(\left(\frac{1}{2} SQ\right)^2 + \left(\frac{1}{2} AD\right)^2\right)$$

$$\Delta OBC \equiv OB^2 + OC^2 = 2\left(\left(\frac{1}{2} SQ\right)^2 + \frac{1}{2} (BC)^2\right)$$

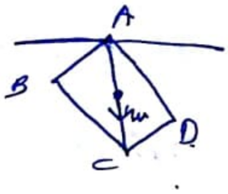
$$\Delta OAB \equiv OA^2 + OB^2 = \frac{1}{2}(PR)^2 + \frac{1}{2}(AB)^2$$

$$\Delta OCD \equiv OC^2 + OD^2 = \frac{1}{2}(PR)^2 + \left(\frac{1}{2} CD\right)^2$$

$$\left(\frac{1}{2} SQ\right)^2 + \left(\frac{1}{2} BC\right)^2 + \left(\frac{1}{2} SQ\right)^2 + \left(\frac{1}{2} AD\right)^2 = 2\left(\left(\frac{1}{2} PR\right)^2 + \left(\frac{1}{2} AB\right)^2 + \left(\frac{1}{2} CD\right)^2\right)$$

$$\frac{PR^2}{\cancel{2}} - \frac{SQ^2}{\cancel{2}} = \text{Constant} \Rightarrow \boxed{\frac{PR}{SQ} = \frac{S(BQ)}{S(PR)}}$$

Q Parallelogram ABCD by A, in shape by string AC. All rods equal weight w. Find Tension string



COM of ABCD can be at mid-point of AC

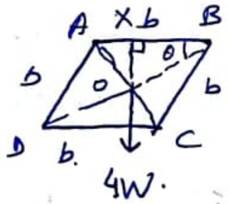
So, AC is vertical.

$$-T\,\delta(AC) + 4w\,\delta(AC/2) = 0$$

$$T = 2w\,\frac{\delta(AC)}{\delta(AC)} = 2w.\ \text{Ans}$$

*✓

Q Four rods, (b, W) form rhombus. String of length a is shorter diagonal. If one rods is supported in horizontal position, find T of string.

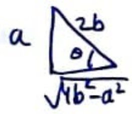


$$OB^2 = b^2 - \frac{a^2}{4}\ ;\ OB = AB\cos\theta.$$

$$OX = OB\sin\theta = AB\sin\theta\cos\theta = b\sin\theta\cos\theta.$$

$$AO = AB\sin\theta = b\sin\theta$$

$\sin\theta = \frac{a}{2b}$



$$-T\,\delta(2b\sin\theta) + 4W\,\delta(b\sin\theta\cos\theta) = 0$$

$$-\cancel{2}Tb\cos\theta\,\cancel{\delta\theta} + {}^2\frac{\cancel{4}Wb}{\cancel{2}}.\cancel{2}\cos2\theta\,\cancel{\delta\theta} = 0$$

$$T\cos\theta = 2W\cos2\theta \Rightarrow \boxed{T = 2W\left(\cos\theta - \sin\theta\tan\theta\right)}$$

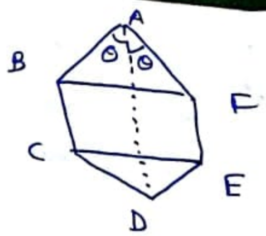
$$= 2W\left(\cos\theta - \frac{(1-\cos^2\theta)}{\cos\theta}\right)$$

$$= 2W\left(\frac{2\cos^2\theta - 1}{\cos\theta}\right)$$

$$= 2W\left(\frac{\frac{\cancel{2}.\,4b^2 - 2a^2 - 4b^2}{4b^2\,2b}}{\frac{\sqrt{4b^2-a^2}}{2b}}\right)$$

$$= \frac{\cancel{2}W(4b^2 - 2a^2)}{\cancel{2}b\sqrt{4a^2 - a^2}}$$

(8) Six equal heavy rods at A (W)
2 light rods BF, CE.
Find thrust in both these rods



Let each rod be of length $2a$.
T_1, T_2 be thrust in BF, CE

(1) Find T_1 ≡ Small displacement such that $\theta \to \theta + \delta\theta$ but BCDEF stays intact.

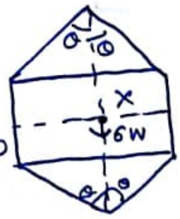
So only change in length of BF.
As unsymmetric, can't take all weights at 1pt.
(AB, AF)

$$2W\delta(a\cos\theta) + 2W\,\delta(a + 2a\cos\theta) + 2W\delta(k + 2a\cos\theta)$$
$$+ T_1\delta(4a\sin\theta) = 0$$

$$2W\cancel{a}\sin\theta + 4W\cancel{a}\sin\theta + 4W\cancel{a}\sin\theta = T_1 4\cancel{a}\cos\theta$$

$$T_1 = W \frac{10\sin\theta}{4\cos\theta} = \boxed{\frac{5}{2}W\tan\theta}$$

(2) Find T_2



As symmetric, consider $6W$ at X.
$$6W\delta(a + 2a\cos\theta) + T_1\delta(4a\sin\theta) + T_2\delta(4a\sin\theta) = 0$$

$$4T_2\cancel{a}\cos\theta = 12W\cancel{a}\sin\theta - 4\cancel{a}\cdot\frac{5}{2}W\tan\theta\cos\theta.$$

$$T_2 = 3W\tan\theta - \frac{5}{2}W\tan\theta = \frac{1}{2}W\tan\theta.$$

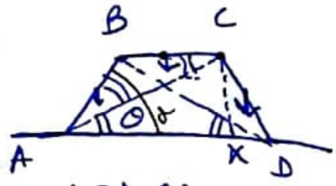
At equilibrium, $\theta = \pi/3$.

$$T_1 = \frac{5\sqrt{3}}{2}W \qquad T_2 = \frac{\sqrt{3}}{2}W. \quad \underline{\text{Ans}}$$

19 3 equal rods (w) joined at B and C (AB, BC and CA) and rest in a vertical plane. A and D on smooth table. AC and BD string support, AB and CD at α to horizontal. Show if W placed on m.p of BC, then find T in each string.

ME

Let rods be 2a. BC is horizontal



$CA = CX \operatorname{cosec}\theta = 2a\sin\alpha \operatorname{cosec}\theta = BD.$

$CA = 2a\sin\alpha \times \frac{1}{\sin\alpha/2} = \boxed{2a \cdot 2\cos\alpha/2}$

$ABD \cong ACD$

$CX = 2a\sin\alpha.$

$\angle CAD = \theta.$

$AB = BC$ – isosceles

$\angle BCA = \angle CAD$ $BC \parallel AC$

$\theta = \alpha/2.$

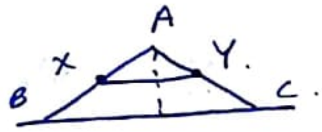
$-(W+w)\,\delta(2a\sin\alpha) - 2w\,\delta(a\sin\alpha) + 2T\,\delta(4a\cos\alpha/2) = 0$

(-ve as measured from table)

$+2(W+w)a\cos\alpha + 2wa\cos\alpha = +\frac{8}{2}aT\sin\frac{\alpha}{2}$

$$T = \frac{2W\cos\alpha + 4w\cos\alpha}{4\sin\alpha/2}.$$ Ans

19 2 uniform rods AB, AC.
M.points X, Y connected by string.
Show Tension in string $= \frac{W}{\tan B + \tan C}$ ← Combined weight.



As $AB = 2a$, $AC = 2b$

Weight of AB and AC acts somewhere in XY.

$a\sin B = b\sin C.$

$a\cos B\,\delta B = b\cos C\,\delta C$

$\boxed{\frac{a\cos B}{b\cos C} = \frac{\delta C}{\delta B}}$

$BC = 2a\cos B + 2b\cos C.$

$XY \parallel BC$ and $\frac{1}{2}BC = a\cos B + b\cos C.$

$-T\,\delta(a\cos B + b\cos C) - W\,\delta(a\sin B) = 0$

$+T(a\sin B\,\delta B + b\sin C\,\delta C) = Wa\cos B\,\delta B.$

$T\left(\cancel{a}\frac{\sin B}{\cos B} + \cancel{b}\sin C\cdot\frac{\cancel{a}\cancel{\cos B}}{\cancel{b}\cos C}\right) = W\cancel{a}\cancel{\cos B}$

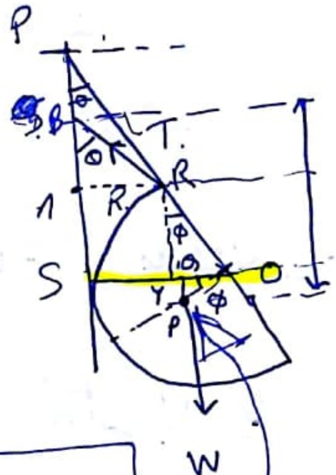
$T = \frac{W}{\tan B + \tan C}$

(8) Solid Hemisphere by a string to a vertical wall with curved surface in contact. θ, ϕ angle of string, base of hemisphere to vertical,

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

Taking length of string be l.

Radius of hemisphere $= r$.



$l\sin\theta = AR$

$l\sin\theta = PR\cos\phi$.

$RY = r\cos\phi$.

$PQ = \frac{2}{r} r\sin\phi$.

Distance W from B $= RY + PQ + BA$.

$= r\cos\phi + \frac{3}{8} r\sin\phi + l\cos\theta$.

(*) Displace such that $\theta \to \theta + \delta\theta$, $\phi \to \phi + \delta\phi$

$\angle PSO$ remains $90°$

Length of string QR remains fixed $(= l)$

$$W\,\delta\left(r\cos\phi + \frac{3}{8} r\sin\phi + l\cos\theta\right) = 0.$$

$$\left(-Wr\sin\phi + \frac{3}{8}Wr\cos\phi\right)\delta\phi - Wl\sin\theta\,\delta\theta = 0.$$

(*) $\Rightarrow OS = a = a\sin\phi + l\sin\theta$.

$$a\cos\phi\,\delta\phi + l\cos\theta\,\delta\theta = 0 \Rightarrow \frac{\delta\theta}{\delta\phi} = -\frac{a\cos\phi}{l\sin\theta}$$

$$+\cancel{r}\sin\phi \mp \frac{3}{8}\cancel{r}\cos\phi = \cancel{l\sin\theta}.\ \frac{+\cancel{a}\cos\phi}{\cancel{l\sin\theta}}$$

$$\boxed{\tan\phi = \frac{3}{8} + \tan\theta}$$

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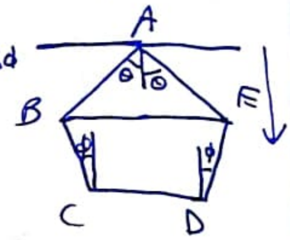
⑧ 5 equal rods form regular pentagon ABCDE. (W). BE is weightless rod, suspend by A. Thrust in BE ?



$4a\sin\theta = 2a + 4a\sin\phi \Rightarrow 4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$

AB, BE

$$2W\left(\delta(a\cos\theta)\right) + 2W\,\delta\left(2a\cos\theta + a\cos\phi\right)$$
$$+ W\,\delta\left(2a\cos\theta + 2a\cos\phi\right) + T\,\delta\left(4a\sin\theta\right) = 0.$$

$$-2Wa\sin\theta\,\delta\theta - 6Wa\sin\theta\,\delta\theta - 4Wa\sin\phi\,\delta\phi - \cancel{2Wa\sin\phi\,\delta\phi} =$$
$$-4Ta\cos\theta\,\delta\theta.$$

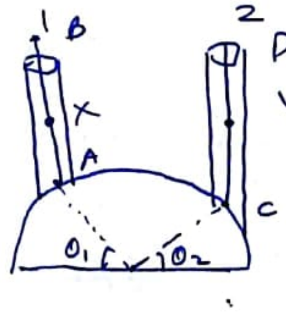
$$2\cancel{8}W\cancel{a}\overset{\tan\theta}{\cancel{\sin\theta}} + \cancel{4}W\cancel{a}\sin\phi\,\frac{\cos\theta}{\cos\phi} = \cancel{4}T\cancel{a}\cancel{\cos\theta}.$$

$$T = 2W\tan\theta + W\tan\phi.$$

Equilibrium = $\theta = \frac{3}{10}\pi$, $\phi = \frac{\pi}{10}$.

$$T = 2W\tan\frac{3\pi}{10} + W\tan\frac{\pi}{10}$$
$$= 2W\cot\left(\frac{\pi}{2} - \frac{3\pi}{10}\right) + W\cot\left(\frac{5\pi}{10} - \frac{\pi}{10}\right)$$
$$= 2W\cot\left(\frac{2\pi}{10}\right) + W\tan\left(\frac{\pi}{10}\right)$$
$$\cancel{2}W\frac{1-\tan^2\pi/10}{\cancel{2}\tan\pi/10} + \cancel{W\tan\frac{\pi}{10}} = \boxed{W\cot\frac{\pi}{10}}$$ Ans

[8] Plan semicircular board with plane vertical on a table is pressed at two pts of circumference by 2 beams which slide in smooth vertical tubes. If in equilibrium, find ratio of weights of beams



Let radius be r.

2 tubes $\equiv l_1, l_2$.

$W_1 \quad W_2$.

$$-W_1\,\delta\left(a\sin\theta_1 + \frac{l_1}{2}\right) - W_2\,\delta\left(a\sin\theta_2 + \frac{l_2}{2}\right) = 0.$$

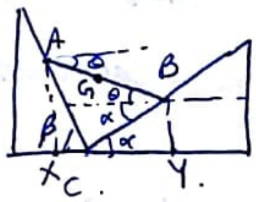
Also, $a\cos\theta_1 + a\cos\theta_2 = k \Rightarrow \cancel{a}\sin\theta_1\,\delta\theta_1 + \cancel{a}\sin\theta_2\,\delta\theta_2 = 0.$

$$W_1 \cancel{a}\cos\theta_1\,\delta\theta_1 + W_2\,\cancel{a}\cos\theta_2\,\delta\theta_2 = 0.$$

$$\frac{W_1}{W_2} = +\frac{\cos\theta_2}{\cos\theta_1}\;\frac{\sin\theta_1}{\sin\theta_2} = \frac{\tan\theta_1}{\cot\theta_2}$$

[9] Heavy rod (2a) lies on 2 inclined planes (α, β) to horizon. If θ be inclination of rod to horizon, prove

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$$



$\angle BAC = \beta - \theta$

$\angle ABC = \theta + \alpha$.

$\angle ACB = \pi - (\alpha+\beta)$

Height of G from base $= \frac{1}{2}(AX + BY)$

$$-W\delta\left(\frac{1}{2}\left(AC\sin\beta + BC\sin\alpha\right)\right) = 0$$

$$\frac{AC}{\sin(\alpha+\theta)} = \frac{BC}{\sin(\beta-\theta)} = \frac{2a}{\sin(\alpha+\beta)}.$$

$$-\frac{W}{2}\,\delta\left(\frac{2a\sin\beta}{\sin(\alpha+\beta)}\cdot\sin(\alpha+\theta) + \frac{2a\sin\alpha}{\sin(\alpha+\beta)}\sin(\beta-\theta)\right) = 0$$

$$-\frac{\cancel{W}}{\cancel{2}}\left(\sin\beta\cos(\alpha+\theta)\,\delta\theta + \sin\alpha\cos(\beta-\theta)\,\delta\theta\right) = 0$$

$\sin\beta\cos(\alpha+\theta) = \sin\alpha\cos(\beta-\theta)$

$\sin\beta\left(\cos\alpha\cos\theta - \sin\alpha\frac{\sin\theta}{\cos\theta}\right) = \sin\alpha\left(\cos\beta\cos\theta + \sin\beta\frac{\sin\theta}{\cos\theta}\right)$

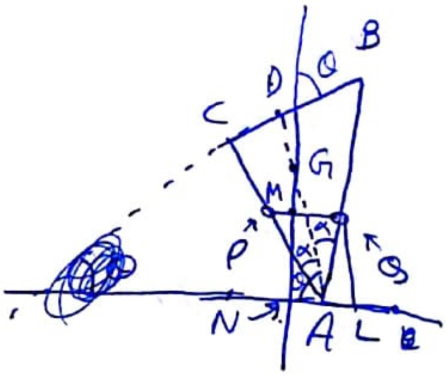
$\cos\alpha\sin\beta - \sin\alpha\cos\beta = 2\tan\theta\,(\sin\alpha\sin\beta)$

$\tan\theta = \frac{1}{2}\left(\cot\alpha - \cot\beta\right)$ Ans

9 Isosceles triangular lamina, vertex downwards between 2 pegs in same line. 2α vertical angle. Length of base = 3x dist pegs

MUG UP!

Find θ for equilibrium.



Let $PQ = a$, $BC = 3a$

D is middle point of BC.

G is centre of gravity of lamina

$AG = \frac{2}{3} AD$

$\angle QPA = \angle PAN = \theta - \alpha$. $\quad \angle QAL = \pi - (\theta + \alpha)$

Small displace $\Rightarrow \theta \to \theta + \delta\theta$.

Height of $G \equiv$ Above fixed PQ

$$GM = NG - NM = NG - LQ$$
$$= AG \sin\theta - AQ \sin(\pi - (\theta+\alpha))$$
$$= \frac{2}{3} AD \sin\theta - AQ \sin(\theta+\alpha)$$

$AD = CD \cot\alpha = \frac{3}{2} a \cot\alpha$.

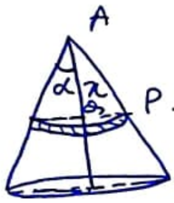
$\Delta AQP \equiv \dfrac{AQ}{\sin \angle APQ} = \dfrac{PQ}{\sin 2\alpha} \Rightarrow AQ = \dfrac{a \sin(\theta-\alpha)}{\sin 2\alpha}$.

$$MG = \frac{2}{3} \cdot \frac{3}{2} a \cot\alpha \sin\theta - \frac{a}{\sin 2\alpha} \sin(\theta-\alpha)\sin(\theta+\alpha)$$

Work $= -W\delta(MG) = 0$

$\cos\theta = 0 \Rightarrow \theta = \pi/2$. or $\sin\theta = \cos^2\alpha \Rightarrow \boxed{\theta = \sin^{-1}(\cos^2\alpha)}$ An

(Q) A elastic string with natural length $2\pi a$ placed round a cone. vertical half-angle (α). If $W \equiv$ weight of string, λ be modulus of elasticity of string.
Prove it is in equilibrium in a circle with radius $a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$



Let $AQ = x$.

Let length $= 2\pi x \tan\alpha$.

Tension. ~~Elastic Energy~~ in string $= \lambda . \frac{(2\pi x\tan\alpha - 2\pi a)}{2\pi a}$

$$-T\,\delta(2\pi x\tan\alpha) + W\,\delta(x) = 0.$$

$$-T\,2\pi\tan\alpha + W = 0 \Rightarrow W = 2\pi T\tan\alpha.$$

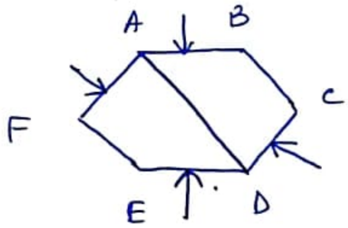
$$T = \frac{W\cot\alpha}{2\pi}$$

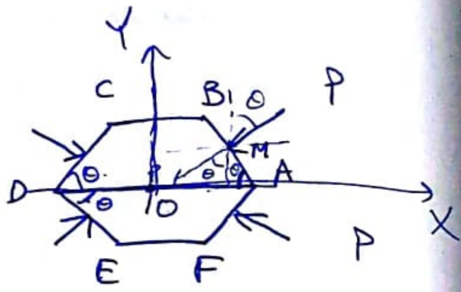
$$\frac{W\cot\alpha}{2\pi} = \lambda\frac{\cancel{2\pi}(x\tan\alpha - a)}{\cancel{2\pi} a}$$

$$\frac{Wa\cot\alpha}{2\pi\lambda} = x\tan\alpha - a \Rightarrow \underbrace{x\tan\alpha}_{\text{radius of loop}} = a\left(1+\frac{W\cot\alpha}{2\pi\lambda}\right)$$

Q ABCDEF is regular hexagon of light rods, with a diagonal rod AD. Four equal forces at m.p of AB, CD, DE, FA and ⊥r to sides. Find stress in AD.

ME





$\downarrow P\cos\theta$

$\leftarrow P\sin\theta$.

~~AB~~ $AD = 2a + 4a\cos\theta$.

Work by $P \equiv \cancel{X\delta t}$. $M = (a + a\cos\theta, a\sin\theta)$

$\rightarrow X = -P\sin\theta$, $Y = -P\cos\theta$.

$\rightarrow -P\sin\theta\,\delta(a + a\cos\theta) - P\cos\theta\,\delta(a\sin\theta)$

$= -aP\cos 2\theta\,\delta\theta$

By symmetry, total virtual work $\equiv -4aP\cos 2\theta\,\delta\theta$

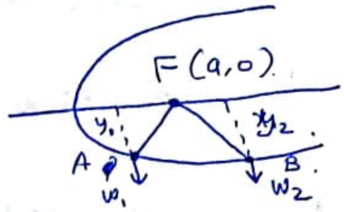
$-4aP\cos 2\theta\,\delta\theta + T\,\delta(2a + 4a\cos\theta) = 0$.

$T\sin\theta + P\cos 2\theta = 0 \Rightarrow T = \dfrac{-P\cos 2\theta}{\sin\theta}$

At equilibrium $\Rightarrow \theta = \dfrac{\pi}{3} \Rightarrow \boxed{T = \dfrac{P}{\sqrt{3}}}$ Ans

Q. 2 rings slide on parabolic wire, axis horizontal, connected by string via focus. Prove in equilibrium, ratio of weights is proportional to vertical depths below axis.

$y^2 = 4ax$. $(at^2, 2at)$



Length of string $= l$.

Give rings displacement s.t $y_1 \to y_1 + \delta y_1$
$y_2 \to y_2 + \delta y_2$.

As l is fixed $\Rightarrow$ work by Tension is 0.

Virtual work $\Rightarrow$

$$w_1 \delta y_1 + w_2 \delta y_2 = 0 \quad -(1)$$

$AF = x_1 + a$, $FB = x_2 + a$.

$FA + FB = 2a + x_1 + x_2$.

$$l = 2a + \frac{y_1^2}{4a} + \frac{y_2^2}{4a}.$$

$$y_1 \delta y_1 = -y_2 \delta y_2. \quad -(2)$$

Divide (1) by (2) $\Rightarrow$

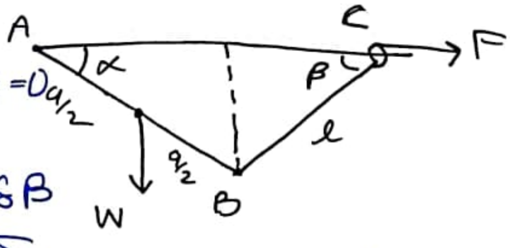
$$\boxed{\frac{w_1}{y_1} = \frac{w_2}{y_2}}$$

Ans

$l \equiv$ string, AB be rod. Find F to equilibrium.

Given α, β

Find F



$$W\delta\left(\frac{a}{2}\sin\alpha\right) + F\delta\left(a\cos\alpha + l\cos\beta\right) = 0$$

$$\Rightarrow a\left(\frac{W}{2}\cos\alpha - P\sin\alpha\right)\delta a = Pl\sin\beta\,\delta\beta$$

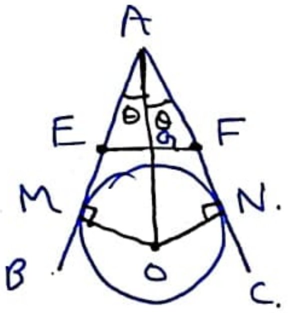
$a\sin\alpha = l\sin\beta$

$a\cos\alpha\,\delta\alpha = l\cos\beta\,\delta\beta$

$$\Rightarrow P = \frac{W\cos\alpha\cos\beta}{2\sin(\alpha+\beta)}.$$

• Two equal rods AB, AC each $2b$, freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be angle between them,

$$b\sin^3\theta = a\cos\theta$$



Take 2W at G.

$\angle OMA = \angle ONA = 90^\circ$.

The system is assumed symmetric.

Common Catenary

$\boxed{\cosh^2\left(\frac{x}{c}\right) - \sinh^2\left(\frac{x}{c}\right)}$

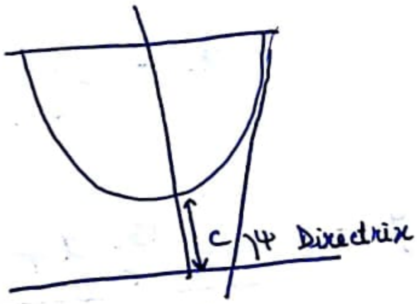
- $S = c \tan\psi$ $\qquad \frac{dy}{dx} = \tan\psi$
- $S = c \sinh(x/c)$
- $y = c \cosh(x/c)$ $\qquad$ • $\boxed{y = c\sec\psi}$
- $y^2 = c^2 + S^2$
- $x = c \log(\sec\psi + \tan\psi)$

$$\rho = \frac{ds}{d\psi} = c\sec^2\psi$$

$T = wy$

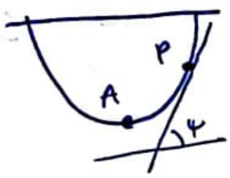
$T_x = wc$

$T_y = w(\overset{\frown}{AB})$
$\quad (s)$



Q. T be tension at any point P, T_0 at lowest point A.

$T^2 - T_0^2 = W^2$ ($W \equiv$ weight of arc AP)



$T\cos\psi = T_0 \qquad T\sin\psi = w\,\overset{\frown}{AP} = W$

$T^2 = T_0^2 + W^2$

$T^2 - T_0^2 = W^2$ Ans

[9] Find COM of catenary from vertex to $P(x,y)$.

$S = c\tan\psi$.

$$\bar{x} = \frac{\int x\,ds}{\int ds} = \frac{\int x\frac{ds}{d\psi}\,d\psi}{\int \frac{ds}{d\psi}\,d\psi}$$

$$= \frac{\int c\log(\sec\psi+\tan\psi)\cdot c\sec^2\psi\,d\psi}{\int c\sec^2\psi\,d\psi}$$

$$= \frac{c^2\log(\sec\psi+\tan\psi)\tan\psi\Big]_0^\psi - c^2\int \frac{\tan\psi\,(\cancel{\sec^2\psi+\sec\psi\tan\psi})}{\cancel{\sec\psi+\tan\psi}}\,d\psi}{c\tan\psi\Big]_0^\psi}$$

(with $\sec\psi$ written above the cancelled numerator)

$$= \frac{c^2\log(\sec\psi+\tan\psi)\tan\psi - c^2\sec\psi + c^2}{c\tan\psi}$$

$$= x - c\left(\frac{\sec\psi}{\tan\psi} - 1\right) = x - c\frac{(1-\cos\psi)}{\sin\psi} = x - c\tan\frac{\psi}{2}.$$

$$\bar{y} = \frac{\int y\frac{ds}{d\psi}\,d\psi}{\int\frac{ds}{d\psi}\,d\psi} = \frac{\int c^{\cancel{2}}\sec^3\psi\,d\psi}{\cancel{c}\tan\psi}$$

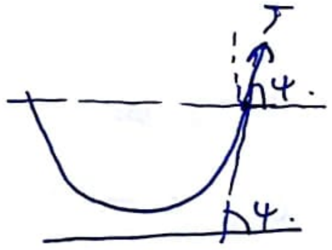
$$\cancel{= \frac{c\left(\sec\psi\tan\psi\Big]_0^\psi - \int_0^\psi \sec\psi\tan\psi\tan\psi\,d\psi\right)}{\tan\psi}}$$

$$\cancel{= c\sec\psi\tan\psi - \left[\sec\psi\tan\psi\right]_0^\psi}$$

$$= \frac{c^{\cancel{2}}\left[\frac{1}{2}\sec\psi\tan\psi + \frac{1}{2}\int_0^\psi \sec\psi\,d\psi\right]}{\tan\psi}$$

$$= \frac{1}{2\tan\psi}\left[c\sec\psi\tan\psi + c\log(\sec\psi+\tan\psi)\right]$$

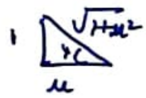
(5) End links of uniform chain slide along a rough horizontal rod. Ratio of max span to length of chain is $\mu \log\left(\frac{1+\sqrt{1+\mu^2}}{\mu}\right)$.



For equilibrium =

$$\mu T \sin\psi = \cancel{\mu}\, T\cos\psi.$$

$$\boxed{\tan\psi = \frac{1}{\mu}.}$$



Let length of chain be l.

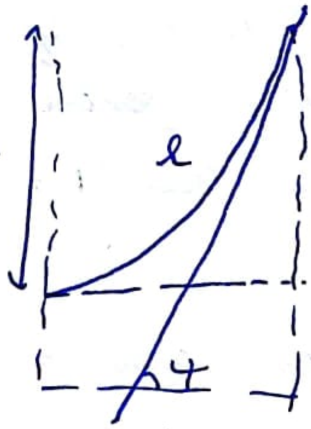
$$T\cos\psi = wc \qquad T\sin\psi = w\frac{l}{2}.$$

$$\frac{\text{Max span}}{\text{length}} = \frac{2.c\log(\sec\psi+\tan\psi)}{l}$$

$$= \frac{\cancel{2}.\cancel{T}\cos\psi\log\left(\frac{1}{\mu}+\frac{\sqrt{1+\mu^2}}{\mu}\right)}{\cancel{\mu}\,\cancel{2}.\cancel{T}\,\frac{\sin\psi}{\cancel{w}}}$$

$$= \mu\log\left(\frac{1}{\mu}+\frac{\sqrt{1+\mu^2}}{\mu}\right)$$ Ans.

9. Kite flying at height h from ground, string $= l$.
Find ψ, T_0, T_{top}.



$S = c \tan\psi$

$l = c\tan\psi.$ $\qquad h + c = c\sec\psi.$

$$(h+c)^2 = c^2 + l^2 \Rightarrow h^2 + 2hc = l^2.$$

$$\Rightarrow c = \frac{l^2 - h^2}{2h}.$$

$$T_0 = wc = \frac{w(l^2-h^2)}{2h}$$

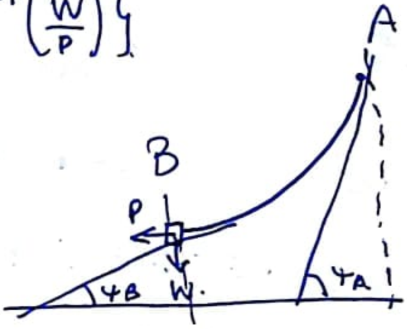
$$T_{top} = wy = w(c+h) = w.\frac{l^2+h^2}{2h}$$

(*) $\tan\psi = \dfrac{2hl}{l^2-h^2}$

$$\psi = \tan^{-1}\left(\frac{2h/l}{1 - h^2/l^2}\right) = 2\tan^{-1}(h/l)$$

(8) Weight W by a string (ω, l). Drawn aside by horizontal force P. Show in equilibrium distance of W from fixed point

$$\frac{P}{\omega}\left\{\sinh^{-1}\left(\frac{W+l\omega}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right\}$$



$$\boxed{S = c\sinh\frac{x}{c}}$$

At $B \Rightarrow \quad \omega c = P. \Rightarrow \boxed{c = P/\omega}.$

$T_Y^B = W \qquad W = \omega y_B \Rightarrow y_B = \frac{W}{\omega}.$

$T_B\cos\psi_B = P. \quad , \quad T_B\sin\psi_B = W \Rightarrow \tan\psi_B = \frac{W}{P}.$

$\tan\psi_A = \frac{W+l\omega}{P}$

$x = c\sinh^{-1}\left(\frac{S}{c}\right) \qquad \boxed{\frac{S}{c} = \tan\psi}$

$x_A - x_B = \frac{P}{\omega}\left\{\sinh^{-1}\left(\frac{W+l\omega}{P}\right) - \sinh^{-1}\left(\frac{W}{P}\right)\right\}$ Ans

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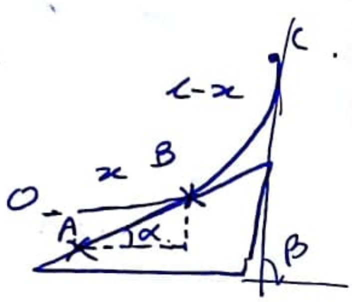
(3) A catenary rests partially on an incline (smooth)

(1) Prove directrix of portion not in touch with plane is line through end on the plane.



At B, Let tension be T_B.

$CB = l - x$, $AB = x$.

$T_B = w.AB\sin\alpha = w y_B$

$\Rightarrow y_B = AB\sin\alpha$. So, directrix is through A.

(2) Plane α, angle at C $= \beta$, l = length of string. Find length on plane.

$\cancel{y_C = c\sec\beta}$ $\qquad$ $\boxed{\cancel{y_B = c\sec\alpha}}$

$\cancel{y_C^2 = c^2 + l^2} \Rightarrow \boxed{\cancel{c^2 = l^2\cot^2\beta}}$

Let O be lowest point of catenary

$OB = c\tan\alpha$ $\qquad$ $OC = c\tan\beta$.

$\cancel{x = c\tan\alpha}$ $\qquad$ $l - x = c(\tan\beta - \tan\alpha)$

$T_B = w y_B = w x\sin\alpha = wc\sec\alpha$.

$x = c\sec\alpha\operatorname{cosec}\alpha$.

$$\frac{l-x}{x} = \frac{\tan\beta - \tan\alpha}{\sec\alpha\operatorname{cosec}\alpha} \Rightarrow \frac{\sin(\beta-\alpha)\sin\alpha}{\cos\beta} \Rightarrow a = \frac{l\cos\beta}{\cos\alpha\cos(\beta-\alpha)}$$ Ans

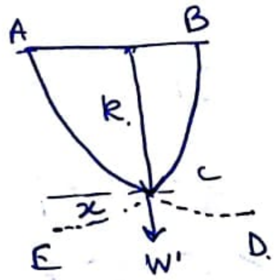
(B) Chain (l, W) hangs. W' is attached to lowest point. If k = sag in middle prove

$$\frac{Wk}{2l} + \frac{W'l}{4k} + \frac{Wl}{8k} = \text{Tension at either point of support}$$

As W' is placed at mid-point

Consider 2 catenary = BCE, ACD.

s.t W' is shared equally by both



$$y_B^2 = c^2 + (x + l/2)^2 \quad \Big\} \quad (y_c + k)^2 - y_c^2 = \frac{l^2}{4} + xl.$$

$$y_c^2 = c^2 + x^2 \quad \Big\} \quad 2y_c k = k^2 + \frac{l^2}{4} + \frac{W'l^2}{2W} \Rightarrow y_c = \frac{k}{2} + \frac{l^2}{8k} + \frac{W'l^2}{4Wk}$$

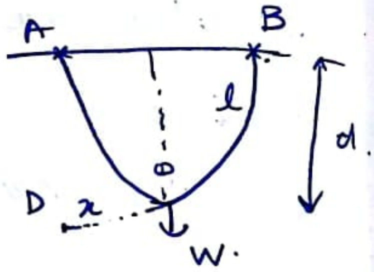
At $C \Rightarrow \quad T_c^y = \frac{W'}{2} = \frac{W}{l} x \Rightarrow \boxed{x = \frac{lW'}{2W}}$

$$T = \frac{W}{l} y_c = \frac{W}{l}\left(\frac{k}{2} + \frac{l^2}{8k} + \frac{W'l^2}{4Wk}\right)$$

$$= \boxed{\frac{Wk}{2l} + \frac{Wl}{8k} + \frac{W'l}{4k}} \quad \underline{\text{Ans}}$$

In all such problems, add arc x.

(8) A, E are 2 points on same horizontal line 2a apart. AO, OB are 2 equal heavy strings tied together at O and a weight at O. If l is length of each string, d depth of O below AB, find c $[l^2-d^2 = 2c^2[\cosh(a/c)-1]]$



$$y_B = c\cosh\frac{x_B}{c}$$

$$\cancel{y_B^2 = c^2 + (l+x)^2}$$
$$\cancel{y_0^2 = c^2 + x^2} \Big\} \cancel{(y_0+d)^2 - y_0^2 = l^2 + 2xl}$$
$$\cancel{\Rightarrow d^2 + 2dy_0 = l^2 + 2xl}$$

$$x = c\sinh\left(\frac{x_0}{c}\right) \qquad l + x = c\sinh\left(\frac{x_B}{c}\right)$$

$$l = c\left(\sinh\left(\frac{x_B}{c}\right) - \sinh\left(\frac{x_0}{c}\right)\right)$$
$$= c\left(\sinh\left(\frac{x_0+a}{c}\right) - \sinh\left(\frac{x_0}{c}\right)\right)$$

$$d = c\left(\cosh\left(\frac{x_0+a}{c}\right) - \cosh\left(\frac{x_0}{c}\right)\right)$$

$$l^2 - d^2 = c^2\left[\sinh^2\left(\frac{x_0+a}{c}\right) + \sinh^2\left(\frac{x_0}{c}\right) - \cosh^2\left(\frac{x_0+a}{c}\right) - \cosh^2\left(\frac{x_0}{c}\right) + 2\cosh\left(\frac{x_0+a}{c}\right)\cosh\left(\frac{x_0}{c}\right) - 2\sinh\left(\frac{x_0+a}{c}\right)\left(\frac{x_0}{c}\right)\right]$$

$$c^2\left(-2 + 2\cosh\left(\frac{x_0+a}{c} - \frac{x_0}{c}\right)\right) = 2c^2\left(\cosh\left(\frac{a}{c}\right) - 1\right) \text{ Ans}$$

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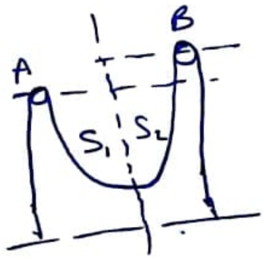
(5) A uniform string 90 inches long, on 2 pegs at different heights. Parts which hang are 30 and 33 inches. Prove divided by vertex in 4 : 5. Also find distance between pegs



As $T_A = \omega y_A$ and $T_B = \omega y_B$

$y_A = 30$ inch. $\quad y_B = 33$ inches.

$S_1 + S_2 = 27$

$S_1^2 = y_1^2 + c^2 \qquad S_2^2 = y_2^2 + c^2$

$S_1^2 - S_2^2 = y_1^2 - y_2^2 \Rightarrow \cancel{27}(S_1 - S_2) = 63 \times 3$

$S_2 - S_1 = 7.$

$S_2 = 17 \qquad S_1 = 10.$

$$\frac{l_1}{l_2} = \frac{30+10}{33+17} = \frac{4}{5}.$$

$C = -17^2 + 33^2 = 50 \times 16 = 800 = \boxed{20\sqrt{2}}$

$y = c\cosh\frac{x}{c} \Rightarrow x_1 + x_2 = c\cosh^{-1}\left(\frac{y_1}{c}\right) + c\cosh^{-1}\left(\frac{y_2}{c}\right)$

$\cancel{= 20}$

Tightly Stretched Wires

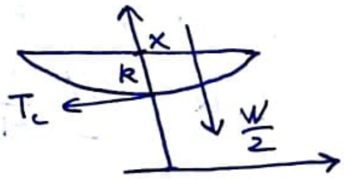
~~$y = c\sinh(x/c)$~~
~~$y = c\cosh(x/c)$~~

$$y = c\cosh\left(\frac{x}{c}\right) = c\left[1 + \frac{1}{2}\left(\frac{x}{c}\right)^2\right]$$

$$S = c\sinh(x/c)$$

$$2y = 2c + \frac{x^2}{c} \Rightarrow 2cy = 2c^2 + x^2 \Rightarrow \boxed{x^2 = 2c(y-c)}$$

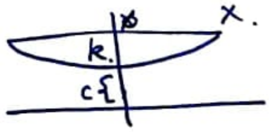
- Sag.



Taking horizontal span = l.
Torque at X

$$T_c k = \frac{W}{2}\cdot\frac{l}{4} \Rightarrow \boxed{T_c = \frac{Wl}{8k}}$$

Q Find T_{max} in wire, $w = 15lb/yard$, sag = 1 ft, span = 100 ft.



$$T_{max} = w\,y_x. \qquad y_x = c + k.$$

$$x^2 = 2c(y_x - c) \Rightarrow \cancel{(50)^2 = 2c}$$

$$\frac{x^2}{2c} + c = c + k \Rightarrow \frac{50^2}{2c} + c = c + 1 \Rightarrow c = 1250$$

$$T = w(c+1) \quad \underline{\text{Ans}}$$

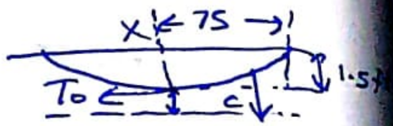
- $S = c\sinh(x/c)$

$$\Rightarrow S = c\left(\frac{x}{c} + \frac{1}{3!}\frac{x^3}{c^3}\right)$$

$$\Rightarrow \boxed{S - x = \frac{x^3}{6c^3}}$$

2000

- Tele wire $w = 0.04\ lb/ft$, span $= 150\ ft$, Sag $= 1.5\ ft$.
Find length of wire and T_{max}.



$y_x = C + 1.5$

$x^2 = 2c(y_x - C) \Rightarrow 75^2 = 2c \times 1.5 \Rightarrow C = 75 \times 25 = 1875$

$T = w(1875 + 1.5) = \boxed{75.06}$

$T_0 = \cancel{w(1.5)}\ w(1875)$

$k = \text{sag} = 1.5\ ft.$

$T_0 k = \frac{W}{2} \frac{l}{4}$

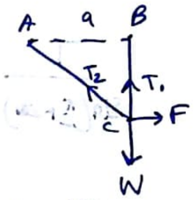
$\cancel{w}(1875) \times 1.5 = \frac{\cancel{w} l^2}{8} \Rightarrow l^2 = (12 \times 1875)\ ft^2$

$\boxed{l = 150\ ft}$

STATICS.

Q String of l is tied to A and B at same level, $AB = a$. Ring of W can slide on string and horizontal F is applied such that ring is vertically below B.

Prove $F = \frac{aW}{l}$, $T = \frac{W(l^2+a^2)}{2l^2}$.



C is in equilibrium under

F along AB
T_2 along CA
W-T along BC.

So,

$$\frac{F}{AB} = \frac{T_2}{CA} = \frac{W-T_1}{BC}$$

$$x^2 = (l-x)^2 + a^2$$

$$\frac{F}{a} = \frac{T_2}{x} = \frac{W-T_1}{l-x}.$$

$$2lx = l^2 + a^2 \Rightarrow x = \frac{l^2+a^2}{2l}.$$

As light string, $T_2 = T_1 = T$

$$T(l-x) = Wx - Tx$$

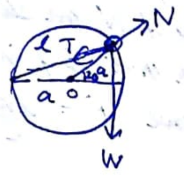
$$T = \frac{Wx}{l} = \frac{W(l^2+a^2)}{2l^2}$$

$$T = \frac{F(l^2+a^2)}{a \cdot 2l}$$

$$\cancel{F = \frac{a(W-T)2l}{l^2-a^2}}$$

$$F = \frac{aW(l^2+a^2)2l}{2l^2(l^2+a^2)} = \boxed{\frac{aW}{l}}$$ Ans

Q) A ring W free to slide on circle (a) by string $l\left(\begin{smallmatrix} > a\sqrt{2} \\ < 2a \end{smallmatrix}\right) \Rightarrow$
Find Tension in string

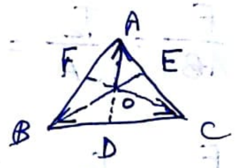


* As all forces not along edges of triangle.
So use Lami's Theorem.

$$\frac{T}{\sin\left(\frac{\pi}{2}+2\theta\right)} = \frac{W}{\sin(\pi-\theta)} = \frac{N}{\sin\left(\theta+\frac{\pi}{2}-2\theta\right)}$$

$$T = \frac{W\cos 2\theta}{\sin\theta} = \frac{W(l^2-2a^2)}{a\sqrt{4a^2-l^2}} \qquad \cos\theta = \frac{l}{2a}$$

Q) If O is the centroid



$P \parallel OA,\ Q \parallel OB,\ R \parallel OC.$

Show $P : Q : R = OA : OB : OC$

ar ΔOBC = ar ΔOCA = ar ΔOAB

$$\frac{1}{2} OB.OC \sin BOC = \frac{1}{2} OC.OA \sin COA = \frac{1}{2} OA.OB \sin AOB$$

Divide by $\frac{1}{2} OB.OC.OA$

$$\frac{\sin BOC}{OA} = \frac{\sin COA}{OB} = \frac{\sin AOB}{OC}$$

By Lami's $\Rightarrow \dfrac{P}{OA} = \dfrac{Q}{OB} = \dfrac{R}{OC}$ Ans

$$\Rightarrow \quad a_1X + b_1Y + c_1Z = 0$$
$$a_2X + b_2Y + c_2Z = 0$$

$$\frac{X}{b_1c_2 - b_2c_1} = \frac{Y}{c_1a_2 - c_2a_1} = \frac{Z}{a_1b_2 - a_2b_1}$$

Stability of Equilibrium

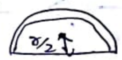
$$\text{STABLE} \Rightarrow \frac{1}{h} > \frac{1}{r} + \frac{1}{R}.$$

$$\frac{1}{h} \ngtr \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \sec\alpha.$$

Centre of Gravity

① Sphere

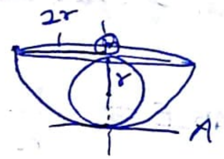
└ Hemisphere ─┬─ Hollow



└─ Solid. $= \frac{3}{8} r$

② Cone $= \frac{h}{4}$

Q. Show whatever m is equilibrium is stable
m is attached to top of sphere



Analysing at A ⇒

$$h = \frac{Wr + m2r}{W+m}$$

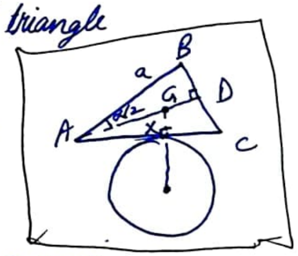
$$\frac{1}{h} > \frac{1}{r} - \frac{1}{2r} > \frac{1}{2r}$$ if $h < 2r$ then equilibrium is stable

$$h = r\frac{(W+2m)}{(W+m)} \Rightarrow \frac{(W+2m)\cancel{r}}{W+m} < 2\cancel{r} \Rightarrow W+2m < 2W+2m$$

$W < 2W$ is true

So, stable

Q. A lamina (isosceles triangle), vertical angle α on a sphere r with an equal side on sphere. Show equilibrium is stable if $\sin\alpha < 3r/a$, a is one of equal sides of triangle



For ABC, COM lies on AD = a median
For isosceles, AD is ⊥r to BC.

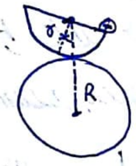
$AD = a\cos\alpha/2$ $\quad AG = \frac{2}{3}AD = \frac{2}{3}a\cos\alpha/2.$

$h = GX = AG\sin\frac{\alpha}{2} = \frac{\cancel{2}}{3}a\sin\alpha.$ $\quad (GX \perp AC)$

Stable if ⇒ $\frac{1}{h} > \frac{1}{r} \Rightarrow \frac{3}{a\sin\alpha} > \frac{1}{r}$

$$\boxed{\sin\alpha < \frac{3r}{a}}$$

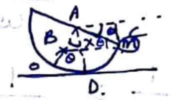
Q. A heavy hemisphere shell of radius r has particle on rim. Its curved surface in contact with rough sphere of radius R at highest point. Show $\frac{R}{r} > \sqrt{5} - 1$ equilibrium is stable irrespective of particle



Let h be height of CG

Stable if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$h < \frac{Rr}{R+r}$$

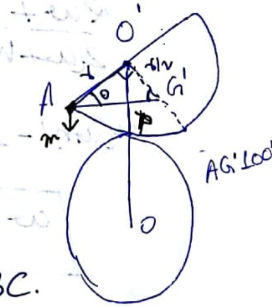


If even for maximum possible value of h this holds, then proved

From A ⇒

~~m is at a height = $r - r\sin\theta$.~~

~~B is at a height = $r - \frac{r}{2}\cos\theta$.~~



$$\cancel{h = \frac{m(r - r\sin\theta) + M(r - \frac{r}{2}\cos\theta)}{M + m}}$$

h is max if AX is minimum, if $AG \perp BC$.

If CXA is right angled. (or $\angle APO$

Let $\angle ACX = \theta$, then ΔCAB

$\tan\theta = \frac{r}{2r} = \frac{1}{2}$. $\left(\frac{AB}{AC}\right)$

$AX = r\sin\theta = \frac{r}{\sqrt{5}}$; max $h = r - \frac{r}{\sqrt{5}}$ $\left(OP = \frac{r}{\sqrt{5}}\right)$

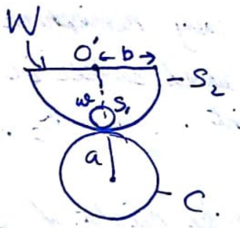
So, stable if $r - \frac{r}{\sqrt{5}} < \frac{Rr}{R+r}$

$$\frac{R}{r} < \sqrt{5} - 1$$

Q

Any disturbance in S_2 leads to S_1 moving such line of action of its weight w acts through O'.

Hence, assume w acts at O'.



$$h = \frac{wb + W\frac{b}{2}}{w+W}$$

$$\frac{1}{h} > \frac{1}{b} + \frac{1}{a} \Rightarrow h < \frac{ab}{a+b}$$

$$\frac{2w+W}{2(w+W)} < \frac{a\cancel{b}}{a+b} \Rightarrow \cancel{2wa - 2wab < 2Wab - Wb}$$

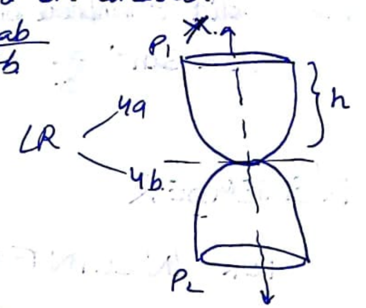
$$2w(a+b) - 2wa < 2Wa - Wa - Wb$$

$$\boxed{w < \frac{Wa - Wb}{2b}}$$

Q No sliding $\Rightarrow$ Fall of cube when A touches sphere

$\frac{\alpha}{2\pi}\gamma = \frac{\pi\gamma}{4} \Rightarrow \boxed{\alpha = \frac{\pi}{4}}$ Ans Total rock is $2 * \frac{\pi}{4} = \boxed{\frac{\pi}{2}}$

Q 2 solid paraboloids of revolution one on another. Check if equilibrium stable if $h < \frac{3ab}{a+b}$



Let $\equiv$ P_1: $\cancel{y^2 = 4ax}$ $x^2 = 4ay$, P_2: $\cancel{y^2 = -4bx}$ $x^2 = -4by$

To find COM of $P_1 \equiv$

$$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int y \pi x^2\,dy}{\int \pi x^2\,dy}$$

$$\cancel{= \frac{\int_0^h \frac{y^5}{4a^2}\,dy}{\int_0^h \frac{y^4}{(4a)^2}\,dy} = \left[\frac{5\,y^6}{6\,y^5}\right]_0^h = \left(\frac{5h}{6}\right)}$$

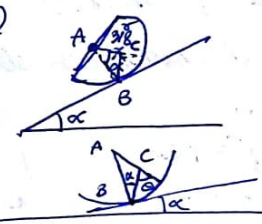
$$= \frac{\int_0^h 4a\pi y^2\,dy}{\int_0^h 4a\pi y\,dy} = \frac{2h^3}{3h^2} = \frac{2h}{3}.$$

$P \equiv \boxed{P_1 = 2a, \quad P_2 = -2b.}$ $\left[P = \lim \frac{x^2}{2y}.\right]$

$$\frac{3}{\cancel{2}h} > \frac{1}{\cancel{2}a} + \frac{1}{\cancel{2}b} \Rightarrow \boxed{h < \frac{3ab}{a+b}}$$ Ans

8 Solid Hemisphere on rough plane (α)

$\alpha < \sin^{-1}\frac{3}{8}$



REMEMBER

FOR INCLINED $\boxed{\frac{1}{h} > \left(\frac{1}{r_1} + \frac{1}{r_2}\right)\sec\alpha}$

$\Delta ABC =$

$$\frac{AC}{\sin B} = \frac{AB}{\sin(\pi-\theta)} \Rightarrow \sin\theta = \frac{8\cancel{r}}{3\cancel{r}}\sin\alpha.$$

$$\theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)$$

As, $\sin\alpha < \frac{3}{8}$ given $\Rightarrow \frac{8}{3}\sin\alpha < 1$

So for equilibrium = $\alpha < \sin^{-1}\left(\frac{3}{8}\right)$

Let $BC = h$,

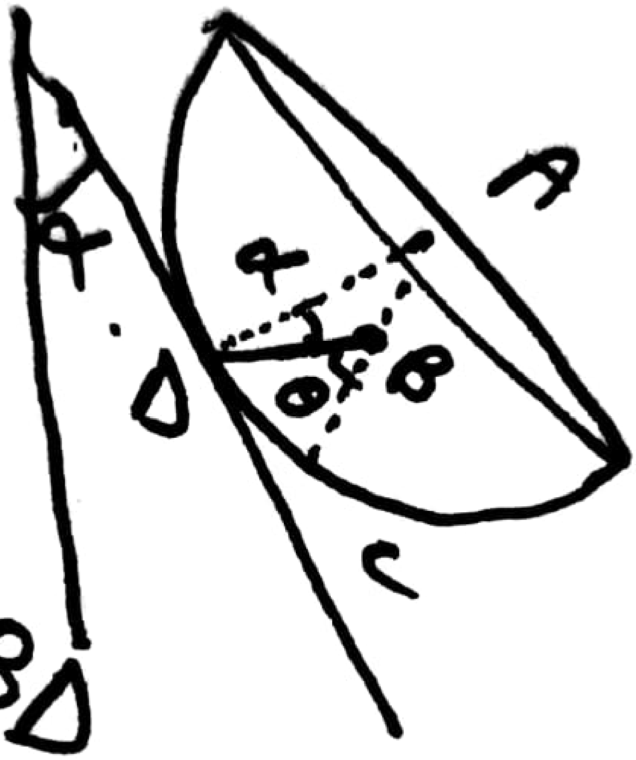
$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha} \Rightarrow h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha}$$

$P_1 = r$; $P_2 = \infty$.

$$h < r\cos\alpha \Rightarrow$$

(2) $-\sin\alpha\sqrt{9-64\sin^2\alpha} > 0$ As $\sin\alpha < \frac{3}{8}$

So, (2) holds

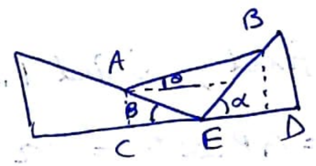


$\Delta ABD.$ → $\dfrac{r}{\sin(\pi-\theta)} = \dfrac{3r/8}{\sin\alpha}$

→ $\sin\theta = \frac{8}{3}\sin\alpha.$

Z-test

① Beam of length $2a$ on 2 inclines given by $\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$. Show equilibrium is unstable.



$$\frac{EA}{\sin(\alpha-\theta)} = \frac{EB}{\sin(\beta+\theta)} = \frac{2a}{\sin(\alpha+\beta)}$$

$$Z = \frac{1}{2}(AC + BD) = \frac{1}{2}\left(AE\sin\beta + BE\sin\alpha\right)$$

$$= \frac{1}{2}\left(\frac{2a\,(\sin\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)\sin\alpha}{\sin(\alpha+\beta)}\right)$$

$$\frac{dz}{d\theta} = 0 \Rightarrow -\cancel{2a}\cos(\alpha-\theta)\sin\beta + \cancel{2a}\,\overset{\cos\theta}{\cancel{\sin}}(\beta+\theta)\sin\alpha = 0$$

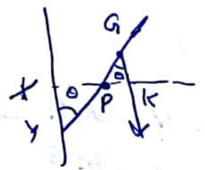
$$\cancel{\sin\beta}\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\right) = \sin\alpha\left(\sin\beta\cos\theta + \sin\theta\cos\beta\right)$$

$$\frac{\sin\theta}{\cos\theta}\left(\sin\beta\sin\alpha - \sin\alpha\cos\beta\right) = \left(\sin\alpha\sin\beta - \sin\beta\cos\alpha\right)$$

$$\tan\theta = \frac{1}{2}\left(\cot\beta - \cot\alpha\right)$$

$\boxed{\frac{d^2z}{d\theta^2} \text{ is shown } < 0}$ Ans

Q P is peg. Uniform heavy rod of 2a
Find position and nature of equilibrium.



PX is fixed, To find ~~P~~KG

$z = GK = GP\cos\theta = (a - PY)\cos\theta = (a - PX\,\mathrm{cosec}\,\theta)\cos\theta$

$\frac{dz}{d\theta} = -a\sin\theta + b\,\mathrm{cosec}^2\theta = 0$

$\sin^3\theta = \frac{b}{a}$

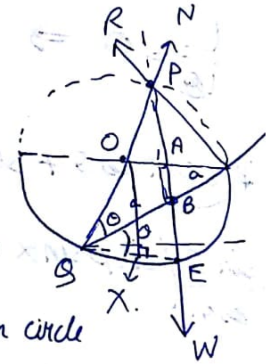
$\frac{d^2z}{d\theta^2} = -a\cos\theta + 2b\,\mathrm{cosec}\,\theta.(\mathrm{cosec}\,\theta)\cot\theta$

$= -a\cos\theta - 2\cancel{b}\cos\theta\,\frac{a}{\cancel{b}} = -3a\cos\theta$

$-$ve for all $\theta < 90°$

So unstably

Q R ⊥ to rod, N passes through.
∠ in semicircle is 90°



→ So, R, N meet at P
→ BE ⊥ QE ⇒ E must lie on circle

depth from O

AE = OX

$Z = AE - BE = OX - BE = (r\sin 2\theta - a\sin\theta)$

[CG at B]

3-D Equilibrium

(1) Invariants $\equiv X^2+Y^2+Z^2$ $\qquad LX+MY+NZ$

(2) pitch $\equiv \dfrac{LX+MY+NZ}{|X^2+Y^2+Z^2|}$

(3) Wrench $\equiv$ Set of forces reducible to equivalent force and a couple at any point.

$(G, R) \equiv$ If line of action of both are same

(4) Reduction to a single force if

- $LX+MY+NZ=0$
- $X^2+Y^2+Z^2 \neq 0$

Single Force $\equiv R=(X, Y, Z)$

Couple $\equiv G=(L, M, N)$

(5) Central Axis

$$\frac{L-yZ+zY}{X} = \frac{M-zX+xZ}{Y} = \frac{N-xY+yX}{Z} = \frac{LX+MY+NZ}{X^2+Y^2+Z^2}$$

Wrench $\equiv$ A combination of a single force R and couple G such that line of action of R and axis of G are same or $\boxed{R \parallel G}$

$|R| =$ intensity of wrench

$p = \frac{G}{R}$ pitch of wrench.

* $x_1 \quad y_1 \quad z_1 \qquad x_2 \quad y_2 \quad z_2$.

$l_1 \quad m_1 \quad n_1 \qquad l_2 \quad m_2 \quad n_2$.

$X_1 = F_1 l_1 \quad Y_1 = F_1 m_1 \quad Z_1 = F_1 n_1 \qquad X_2 = F_2 l_2 \quad Y_2 = F_2 m_2 \quad Z_2 = F_2 n_2$.

$X = \sum X_i$

$Y = \sum Y_i$

$Z = \sum Z_i$

$\rightarrow L, M, N$

$x_1 \quad y_1 \quad z_1 \qquad x_2 \quad y_2 \quad z_2$

$X_1 \quad Y_1 \quad Z_1 \qquad X_2 \quad Y_2 \quad Z_2$.

$L = \sum (y_i Z_i - z_i Y_i) \qquad M = \sum (z_i X_i - x_i Z_i) \qquad N = \sum (x_i Y_i - y_i X$

9. A single force is equivalent to component forces X, Y, Z along coordinate axes and to couples L, M, N about these axes. Prove magnitude of single force is $\sqrt{X^2+Y^2+Z^2}$ and find equations of its line of action.

→ Line of action of resultant force is

$$\frac{L-yZ+zY}{X} = \frac{M-zX+xZ}{Y} = \frac{N-xY+yX}{Z} = \frac{LX+MY+NZ}{X^2+Y^2+Z^2}$$

As system is reducible to single force, $LX+MY+NZ=0$.

So,

$$\frac{L-yZ+zY}{X} = \frac{M-zX+xZ}{Y} = \frac{N-xY+yX}{Z} = 0$$

$$yZ-zY=L,\quad zX-xZ=M,\quad xY-yX=N.$$

$$\frac{yZ-zY}{L} = \frac{zX-xZ}{M} = \frac{xY-yX}{N} = 1 \quad \text{(Line of action)}$$

Since X, Y, Z are components of single force along axes,

resultant $= \sqrt{X^2+Y^2+Z^2}$.

(8) Three equal forces (P) act at $(a,0,0) \parallel Y$, $(0,b,0) \parallel Z$, $(0,0,c) \parallel X$. Find resultant wrench.

$F_1 = 0\hat{i} + P\hat{j} + 0\hat{k}$ $\quad F_2 = 0\hat{i} + 0\hat{j} + P\hat{k}$ $\quad F_3 = P\hat{i} + 0\hat{j} + 0\hat{k}$

$(l_1, m_1, n_1) \equiv (0,1,0) \qquad (0,0,1) \qquad (1,0,0)$

$(x_i, y_i, z_i) \equiv (a,0,0) \qquad (0,b,0) \qquad (0,0,c)$

$(X_i, Y_i, Z_i) \equiv (0,P,0) \qquad (0,0,P) \qquad (P,0,0)$

$L = bP \qquad M = cP \qquad N = aP$

$X = P \qquad Y = P \qquad Z = P$

$$\frac{bP - yP + zP}{P} = \frac{cP - zP + xP}{P} = \frac{aP - xP + yP}{P}$$

$$b - y + z = c - z + x = a - x + y = \frac{(b+a+c)P^2}{3P^2}$$

$x + \frac{a+b+c}{3} = y + a \quad \Leftarrow 3y + 3a = 3x + a + b + c.$

$x + \frac{a+b+c}{3} = z + \frac{2a+2b+c}{3} \quad \Leftarrow 3x + a+b+c = 3z + 2a+2b+c.$

Axis is line through $\equiv \left[-\left(\frac{a+b+c}{3}\right), -a, -\left(\frac{2a+2b+c}{3}\right)\right]$

equally inclined to 3 axes

$\Rightarrow R$ (resultant of wrench) $= \sqrt{X^2+Y^2+Z^2} = \sqrt{3}P.$

$KR = LX + MY + NZ = (a+b+c)P^2$

So, $K = \frac{(a+b+c)P}{\sqrt{3}}$

resultant moment

Q 2 forces act along $y=0, z=0$ and $x=0, z=c$. As forces vary show surface generated by equivalent wrench

$$(x^2+y^2)z = cy^2$$

	P	Q
Pt	$0, 0, 0$	$0, 0, c.$
Direction	$1, 0, 0$	$0, 1, 0.$
	$P, 0, 0$	$0, Q, 0.$

$X=P$
$Y=Q$
$Z=0.$

$L = -cQ, \quad M = 0, \quad N = 0.$

$$\text{Line} \equiv \frac{-cQ - y.0 + z.Q}{P} = \frac{0 - zP + x.0}{Q} = \frac{-xQ + yP}{0} = \frac{-cPQ}{P^2+Q^2}.$$

$$\Downarrow$$

$$\boxed{\frac{x}{y} = \frac{P}{Q}}$$

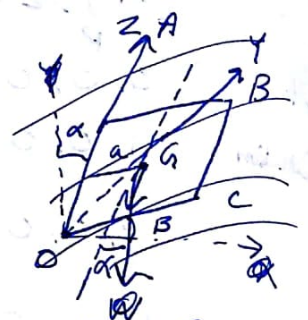
$$\frac{Q(z-c)}{P} = \frac{-zP}{Q} = -\frac{cPQ}{P^2+Q^2}.$$

$$y\frac{(z-c)}{x} = -\frac{zx}{y} \Rightarrow y^2z - cy^2 = -zx^2 \Rightarrow z(x^2+y^2) = cy^2$$

Do NOT USE $\frac{LX+MY+NZ}{X^2+Y^2+Z^2}$ term (Can be solved w/o it)

Q. A door of weight W turns about axis OA inclined at α to vertical. Find couple needed to keep it inclined at angle β to vertical plane.

$a \equiv$ distance of CG from OA



= Only external force on gate is W.

Components of W

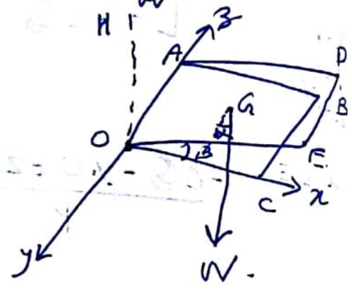
$\hookrightarrow Z \equiv -W\cos\alpha$.

$X \equiv \cancel{W\cos}\ W\sin\alpha\cos\beta$

$Y \equiv -W\sin\alpha\sin\beta$.

Coordinates of $G \equiv (a, 0, h)$.

height of door $= 2h$.



Door at OABC

Couple trying to rotate gate $\equiv$

$$xY - yX = a(-W\sin\alpha\sin\beta)$$

$$= -aW\sin\alpha\sin\beta$$

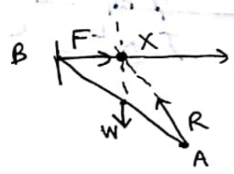
So, Torque needed is $aW\sin\alpha\sin\beta$

Equilibrium of a rigid body

≡ Equation of line of action of resultant → $xY - yX = M$.

M = sum of moments abt origin, $X, Y \equiv$ algebraic sum of forces

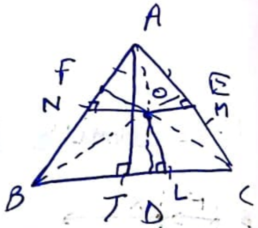
≡ Use Lami's Theorem + 3 concurrent forces for Rxn at hinge.



$\boxed{\text{To find } R \equiv R \text{ must pass } X.}$

Q Three forces P, Q, R along BC, CA, AB of Δ. If resultant passes through centroid of Δ ABC, prove

$$\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$$



$P.OL + Q.OM + R.ON = 0.$ — (1)
(for resultant through O)

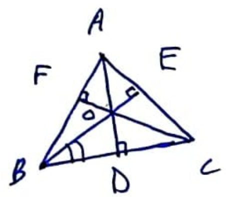
Draw $AT \perp BC$.
$\Delta DOL \sim \Delta DAT$
$\Rightarrow OL = AT.\frac{DO}{DA} = \frac{1}{3}AT.$

S is ar(ΔABC) then ≡ $S = \frac{1}{2}BC.AT \Rightarrow AT = \frac{2S}{a}.$

$OL = \frac{2S}{3a}$, IIIy $OM = \frac{2S}{3b}$, $ON = \frac{2S}{3c}$

Put in (1) ⇒
$\frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0 \Rightarrow \frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$

Q If resultant through orthocentre, prove
$P\sec A + Q\sec B + R\sec C = 0$



~~FS~~ If resultant via O,
$P.OD + Q.OE + R.OF = 0$ — (1).

$\angle CBE = \frac{\pi}{2} - C$, $OD = BD\cot C$.
$BD = AB\cos B = c\cos B.$
$OD = c\cos B\cot C = \frac{c}{\sin C}\cos C\cos B = k\cos B\cos C$

IIIy $OE = k\cos C\cos A$ $OF = k\cos A\cos B$

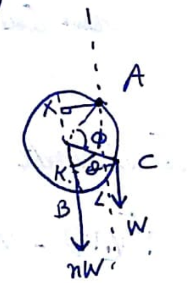
In (1) ⇒ $P.k\cos B\cos C + Q.k\cos C\cos A + Rk\cos A\cos B = 0$

$P\sec A + Q\sec B + R\sec C = 0$

Q Circular disc is ... Show

$AX =$
$CK =$
...

Q Circular disc of nW has a heavy particle of W at point C. If disc is suspended from A, B is lowest point and vice-versa. Show angle subtended by $AB = 2\sec^{-1}(2(n+1))$



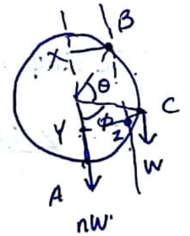
$AX = a\sin(\pi-\theta-\phi) = a\sin(\theta+\phi)$

$CK = a\sin\theta$. $\quad CL = CK - KL$

$= a\sin\theta - a\sin(\theta+\phi)$

$n\cancel{W}\cancel{a}\sin(\theta+\phi) = \cancel{W}(\cancel{a}\sin\theta - \cancel{a}\sin(\theta+\phi))$

$(n+1)\sin(\theta+\phi) = \sin\theta$.



$n\cancel{W}(BX) = \cancel{W}(CY-BX)$

$n\cancel{a}\sin(\theta+\phi) = \cancel{a}\sin\phi - \cancel{a}\sin(\theta+\phi)$

$(n+1)\sin(\theta+\phi) = \sin\phi$.

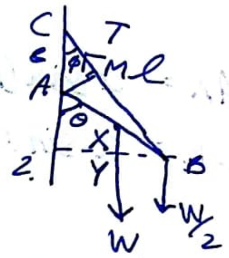
$\Rightarrow \sin\theta = \sin\phi \Rightarrow \theta = \phi$

$(n+1)\sin 2\theta = \sin\theta \Rightarrow \sin\theta\left(2(n+1)\cos\theta - 1\right) = 0$

$\cos\theta = \dfrac{1}{2(n+1)} \Rightarrow \theta = \sec^{-1}(2(n+1))$

Angle $\equiv 2\theta = 2\sec^{-1}(2(n+1))$ Ans

(8) A rod (W), with $W/2$ at one end. This end is tied by a string of length l to point at height c over hinge. Show Tension of string is Wl/c.



To eliminate reaction at A,

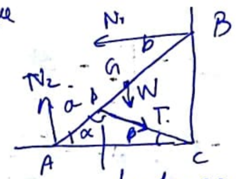
$T(AM) - W(YZ) - \frac{W}{2}(WZ) = 0.$

$T\,AM = W\left(YZ + \frac{1}{2} \cdot 2YZ\right)$ $\qquad YZ = a\sin\theta.$

$T c \sin\phi = 2Wa\sin\theta.$ $\qquad l\sin\phi = 2a\sin\theta.$

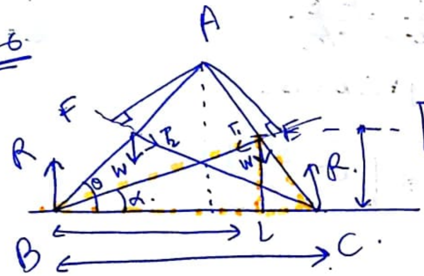
$T = \frac{2W\cancel{a}}{c} \cdot \frac{l}{2\cancel{a}} = \frac{Wl}{c}$ Ans

Solving Forces, Torque



⊥r for equilibrium.

Pg-46



Combined ≡ $2R = 2W$.

Relation between θ and α

$\Delta ELC \equiv EL = EC\sin\theta = a\sin\theta$

$BL = \frac{3}{4}BC = \frac{3}{4}\cdot 4a\cos\theta$

$\Delta BLE \to \boxed{\cot\alpha = \frac{BL}{LE} = 3\cot\theta}$

$\boxed{\text{Torque at } A}$