4 Ordinary Differential Equations

1. Formulation of Differential Equations

1.1 Find the differential equations of the family of circles in the xy-plane passing through $(-1, 1)$ and $(1, 1)$.

(2009 : 20 Marks)

Solution:

Approach : First use conditions to get the general equation of such a circle. Then get the differential equations.

General equation of circle in xy plane is

$$x^2 + y^2 + 2ax + 2by + d = 0 \qquad \text{...(i)}$$

It passes through $(-1, 1)$ and $(1, 1)$

$$\Rightarrow \quad 2 - 2a + 2b + d = 0 \Rightarrow 4a = 0$$

$$2 + 2a + 2b + d = 0 \Rightarrow d = -(2b + 2)$$

$\therefore$ General equation of circles passing through $(-1, 1)$ and $(1, 1)$ is

$$x^2 + y^2 + 2by - (2b + 2) = 0 \qquad \text{...(ii)}$$

where b is the single parameter.

Differentiating (ii) with respect to x

$$2x + 2y\frac{dy}{dx} + 2b\frac{dy}{dx} = 0$$

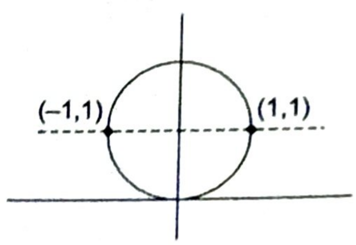
$$\Rightarrow \quad \frac{dy}{dx} = \frac{-x}{b+y} \Rightarrow b = \frac{-x}{dy/dx} - y$$

Putting this in (ii)

$$x^2 + y^2 + 2\left(\frac{-x}{dy/dx} - y\right)y - \left[2\left(\frac{-x}{dy/dx} - y\right) + 2\right] = 0$$

$$\Rightarrow \quad x^2 - y^2 - \frac{2xy}{dy/dx} + \frac{2x}{dy/dx} + 2y - 2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2 - 2y + 2}{2x(1-y)}$$



which is the required differential equation.

Alternatively : We can also use equation of circle $x^2 + (y-1)^2 + \lambda(y-1) = 0$ and proceed.

1.2 **Find the curve for which the part of the tangent cut-off by the axes is bisected at the point of tangency.**

(2014 : 10 Marks)

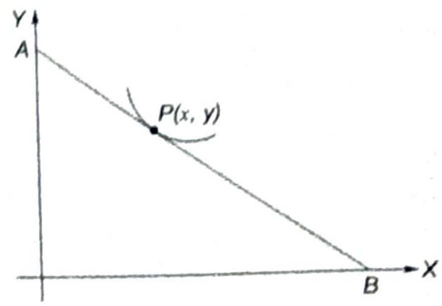
Solution:

Let equation of tangent line at point 'P' at

$$\frac{X-x}{Y-y} = \frac{dy}{dx} \qquad \text{...(i)}$$

Now, its point of intersection with co-ordinate axes are

$$A\left(0, y-\frac{x}{\frac{dy}{dx}}\right); B\left(x-y\frac{dy}{dx}, 0\right).$$



Given : 'P' is mid point of AB.

So,
$$\frac{x-y\frac{dy}{dx}}{2} = x \text{ and } \frac{\left(y-\frac{x}{\frac{dy}{dx}}\right)}{2} = y$$

$$\Rightarrow \quad x = -y\frac{dy}{dx} \text{ and } y = -\frac{x}{\frac{dy}{dx}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad ydy + xdx = 0$$

Integrating, we get

$x^2 + y^2 = C$ which the required curve.

1.3 Find the differential equation (DE) representing all the circles in the xy-plane.

(2017 : 10 Marks)

Solution:

Method I : General equation of circle

$$(x-a)^2 + (y-b)^2 = r^2$$

Differentiating w.r.t. x,

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

i.e.,
$$(x-a) + (y-b)y_1 = 0 \quad \text{...(i)}$$

Differentiating again w.r.t. x

$$1 + (y-b)y_2 + y_1^2 = 0 \quad \text{...(ii)}$$

Differentiating again w.r.t. x

$$(y-b)y_3 + y_1y_2 + 2y_1y_2 = 0$$

i.e.,
$$(y-b) = \frac{-3y_1y_2}{y_3}$$

Substituting it in (ii)

$$1 + \left(\frac{-3y_1y_2}{y_3}\right)y_2 + y_1^2 = 0$$

i.e.,
$$(1+y_1^2)y_3 - 3y_1y_2^2 = 0$$

i.e.,
$$(1+y_1^2)y_3 = 3y_1y_2^2$$

Method II : Using curvature-formula (K).

2. Equation of 1st Order and 1st Degree

2.1 Solve : $\frac{dy}{dx} = \frac{y^2(x-y)}{3xy^2 - x^2y - 4y^3}$, $y(0) = 1$

(2009 : 20 Marks)

Solution:

Approach : We check for exactness and find it to be so.

$$\frac{dy}{dx} = \frac{y^2(x-y)}{3xy^2 - x^2y - 4y^3}$$

$$\Rightarrow \quad y^2(y-x)dx + (3xy^2 - x^2y - 4y^3)dy = 0$$

Comparing to $Mdx + Ndy = 0$

$$M = y^2(y-x)$$
$$N = 3xy^2 - x^2y - 4y^3$$
$$\frac{\partial M}{\partial y} = 3y^2 - 2xy$$
$$\frac{\partial N}{\partial x} = 3y^2 - 2xy$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ so the equation is exact.}$$

Solution of an exact equation :

$$\int M(\text{treating } y \text{ as constant})dx + \int N(\text{terms not containing } x)dy = C$$

$$\int y^2(y-x)dx + \int -4y^3 dy = C$$

$$\Rightarrow \quad y^3x - \frac{y^2x^2}{2} - y^4 = C$$

$$y(0) = 1 \Rightarrow C = -1$$

$$\Rightarrow \quad y^3x - \frac{y^2x^2}{2} - y^4 + 1 = 0 \text{ is the final solution.}$$

2.2 Consider the differential equation

$$y' = \alpha x, \; x > 0$$

where α is a constant. Show that

(i) if $\phi(x)$ is any solution and $\psi(x) = \phi(x)e^{-\alpha x}$, then $\psi(x)$ is a constant.

(ii) if $\alpha < 0$, then every solution tends to zero as $x \to \infty$.

(2010 : 12 Marks)

Solution:

(i) Given, the differential equation is :

$$y' = \alpha x$$

$$\Rightarrow \quad \frac{dy}{dx} = \alpha x \Rightarrow \frac{dy}{dx} - \alpha x = 0$$

Integrating factor $\quad e^{\int -\alpha dx} = e^{-\alpha x}$

$\therefore$ Solution of equation is $\quad y.e^{-\alpha x} = \int 0 dx + \text{Constant}$

$$\Rightarrow \quad y.e^{-\alpha x} = \text{Constant}$$

Comparing above equation with

$$\psi(x) = \phi(x).e^{-\alpha x}$$

It can be concluded that $\psi(x)$ = Constant

(ii) Let c be the constant.

So, the solution is
$$y = c.e^{\alpha x}$$

$\therefore$ if $\alpha < 0$ and $x \to \infty$

$$e^{\alpha x} \to 0$$

$$\Rightarrow \qquad y \to 0$$

$\therefore$ Every solution tends to zero as $x \to \infty$ when $\alpha < 0$.

2.3 Show that the differential equation

$$(3y^2 - x) + 2y(y^2 - 3x)y' = 0$$

admits an integrating factor which is a function of $(x + y^2)$. Hence, solve the equation.

(2010 : 12 Marks)

Solution:

Given the equation is

$$(3y^2 - x) + 2y(y^2 - 3x)y' = 0$$

or
$$(3y^2 - x)dx + 2y(y^2 - 3x)dy = 0$$

Now, let
$$t = x + y^2$$

$\therefore$
$$y^2 = t - x \text{ and } 2ydy = dt - dx$$

$\therefore$ Equation becomes $(3(t-x)-x)dx + (t-x-3x) \times (dt-dx) = 0$

$$\Rightarrow \qquad (3t-4x)dx + (t-4x)(dt-dx) = 0$$

$\Rightarrow 3tdx - 4xdx + tdt - 4xdt - tdx + 4xdx = 0$

$$\Rightarrow \qquad 2tdx + (t-4x)dt = 0$$

Now, if integrating factor is a function of $x + y^2$, then equation becomes

$$2tf(t)dx + (t-4x)f(t)dt = 0 \qquad \text{(multiplying by } f(t))$$

$$\Rightarrow \qquad 2tf(t)dx + (t-4x)f(t)dt = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t}dt = df$$

Comparing LHS & RHS, we get

$$\frac{\partial f}{\partial x} = 2tf(t) \Rightarrow \frac{\partial^2 f}{\partial x \partial t} = 2f(t) + 2t\frac{\partial f}{\partial t}$$

and
$$\frac{\partial f}{\partial t} = (t-4x)f(t) \Rightarrow \frac{\partial^2 f}{\partial x \partial t} = -4f(t)$$

$$\therefore \qquad 2f(t) + 2t\frac{\partial t}{\partial t} = -4f(t)$$

$$\Rightarrow \qquad 6f(t) + 2t\frac{\partial f}{\partial t} = 0$$

$$\Rightarrow \qquad t\frac{\partial f}{\partial t} = -3f$$

$$\Rightarrow \qquad -3\frac{\partial t}{t} = \frac{\partial f}{f}$$

Integrating both sides, we get

$$f(t) = \frac{c}{t^3} = \frac{c}{(x+y^2)^3} \text{ where } c \text{ is a constant.}$$

$\therefore$ integrating factor is
$$f(x + y^2) = \frac{1}{(x+y^2)^3}$$

Multiplying eqn. by $\frac{1}{t^3}$, we get

$$\frac{2t}{t^3}dx + (t-4x)\frac{dt}{t^3} = 0$$

$$\Rightarrow \quad d\left(\frac{2x-t}{t^2}\right) = 0$$

$$\Rightarrow \quad \frac{2x-t}{t^2} = k, \text{ where } k \text{ is a constant.}$$

∴ Solution of equation is

$$2x - t = kt^2$$

$$\Rightarrow \quad 2x - t - kt^2 = 0$$

$$\Rightarrow \quad 2x - (x+y^2) - k(x+y^2)^2 = 0$$

$$\Rightarrow \quad x - y^2 - k(x+y^2)^2 = 0, \text{ where } k \text{ is a constant, is the required solution of the equation.}$$

2.4 Verify that :

$$\frac{1}{2}(Mx+Ny)d(\log_e xy) + \frac{1}{2}(Mx-My)d\left(\log_e \frac{x}{y}\right) = Mdx + Ndy$$

Hence show that :

(i) if the differential equation $Mdx + Ndy = 0$ is homogenous then $(Mx + Ny)^{-1}$ is an integrating factor unless $Mx + Ny = 0$.

(ii) if the differential equation $Mdx + Ndy = 0$ is not exact but is of the form

$$f_1(x, y)ydx + f_2(x, y)xdy = 0$$

then $(Mx - Ny)^{-1}$ is an integrating factor unless $Mx - Ny = 0$.

(2010 : 20 Marks)

Solution:

Given : $\frac{1}{2}(Mx+Ny)d(\log_e xy) + \frac{1}{2}(Mx-My)d\left(\log_e \frac{x}{y}\right) = Mdx + Ndy$

Taking LHS :

$$\frac{1}{2}(Mx+Ny)\times\frac{1}{xy}(xdy+ydx) + \frac{1}{2}(Mx-Ny)\times\frac{y}{x}\times\frac{(ydx-xdy)}{y^2}$$

$$= \frac{1}{2}(Mx+Ny)\times\left(\frac{dy}{y}+\frac{dx}{x}\right) + \frac{1}{2}(Mx-Ny)\times\left(\frac{dx}{x}-\frac{dy}{y}\right) \quad ...(1)$$

$$= \frac{1}{2}Mx\frac{dy}{y} + \frac{1}{2}Mx\frac{dx}{x} + \frac{Ny}{2}\frac{dy}{y} + \frac{Ny}{2}\frac{dx}{x} +$$

$$\frac{1}{2}Mx\frac{dx}{x} + \frac{1}{2}Ny\frac{dy}{y} - \frac{Mx}{2}\frac{dy}{y} - \frac{Ny}{2}\frac{dx}{x}$$

$$= Mdx + Ndy = \text{RHS}$$

Hence, given equation is verified.

(i) Let $Mdx + Ndy = 0$ be a homogeneous equation.

$$\therefore \quad Mdx + Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right] \quad \text{(from (1))}$$

If $\frac{1}{Mx+Ny}$ is an integrating factor, then equation becomes

$$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left[\frac{ydx+xdy}{xy}+\frac{Mx-Ny}{Mx+Ny}\times\frac{(ydx-xdy)}{xy}\right]$$

$$\Rightarrow \quad \frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left(d\log_o(xy)+f\left(\frac{y}{x}\right)d\log_e\left(\frac{x}{y}\right)\right)$$

where
$$f\left(\frac{y}{x}\right) = \frac{M-N\left(\frac{y}{x}\right)}{M+N\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad \frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left(d(\log_e xy)+f\left(e^{\log_e\left(\frac{y}{n}\right)}\right)d\left(\log_e\frac{x}{y}\right)\right)$$

$$\Rightarrow \quad \frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left(d(\log_e xy)+F\left(\log_e\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right)$$

which is an exact differential.

$\therefore$ $\dfrac{1}{Mx+Ny}$ is an integrating factor unless $Mx + Ny \neq 0$.

(ii) Rewriting $Mdx + Ndy$ as

$$Mdx + Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right] \quad \text{(from (1))}$$

if $(Mx - Ny)^{-1}$ is an integrating factor, then equation becomes

$$\frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}d(\log_e xy)+d\left(\log_e\frac{x}{y}\right)\right\}$$

as $\quad M = yf_1(xy)$ and $N = xf_2(xy)$ ...(given)

Let $\quad f(xy) = \dfrac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}$ and $g(x) = f(e^x)$, the above equation reduces to

$$\frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\left\{f(xy)d(\log_e xy)+d\left(\log_e\frac{x}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{g(\log_e xy)d(\log_e xy)+d\left(\log_e\frac{x}{y}\right)\right\}$$

which is an exact differential.

$\therefore$ $\left(\dfrac{1}{Mx-Ny}\right)$ is an integrating factor of given equation.

2.5 Show that the set of solutions of the homogenous linear differential equation

$$y' + p(x)y = 0$$

on an interval $I = [a, b]$ forms a vector subspace of continuous function on I. What is the dimension of W?

(2010 : 20 Marks)

Solution:

Given the equation is $\quad y' + p(x)y = 0$

$\therefore$ Integrating factor $= e^{\int p(x)dx}$

So, solution of equation is

$$y \cdot e^{\int p(x)dx} = \int 0 \cdot dx = c \text{ (where } c \text{ is a constant)}$$

$$\Rightarrow \quad y = c \cdot e^{-\int p(x)dx}$$

Let y_1 and y_2 are two solutions of given equations.

$$\alpha, \beta \in R$$

$$\therefore \quad y_1' + p(x)y_1 = 0 \quad \text{...(1)}$$

$$\text{and} \quad y_2' + p(x)y_2 = 0 \quad \text{...(2)}$$

$$\begin{aligned} \therefore \quad (\alpha y_1 + \beta y_2)' + p(x)(\alpha y_1 + \beta y_2) &= \alpha y_1' + \beta y_2' + p(x)(\alpha y_1 + \beta y_2) \\ &= \alpha(y_1' + p(x)y_1) + \beta(y_2' + p(x)y_2) \\ &= \alpha.0 + \beta.0 \quad \text{(from (1) and (2))} \\ &= 0 \end{aligned}$$

$\therefore$ $\alpha y_1 + \beta y_2$ is also a solution.

$\therefore$ Solutions form a vector subspace.

Now,

$$y_1 \in ce^{-\int p(x)dx}$$

$$y_2 \in ce^{-\int p(x)dx}$$

$$\therefore \quad \dim(w) = 1$$

2.6 Obtain the solution of the ordinary differential equation $\dfrac{dy}{dx} = (4x + y + 1)^2$, if $y(0) = 1$.

(2011 : 10 Marks)

Solution:

$$\frac{dv}{dx} = (4x + y + 1)^2 \quad \text{...(i)}$$

Put

$$4x + y + 1 = v$$

$$\Rightarrow \quad 4 + \frac{dv}{dx} = \frac{dv}{dx}$$

$\therefore$ from (i),

$$\frac{dv}{dx} - 4 = v^2$$

$$\Rightarrow \quad \frac{dv}{dx} = v^2 + 4 \Rightarrow \frac{dv}{v^2 + 4} = dx$$

$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\frac{v}{2} = x + C_1, \text{ where } C_1 \text{ is an arbitrary constant.}$$

$$\Rightarrow \quad v = 2\tan(2x + C), \; 2C_1 = C$$

$$\Rightarrow \quad 4x + y + 1 = 2\tan(2x + C) \quad \text{...(ii)}$$

Given $y(0) = 1 \Rightarrow y = 1$ when $x = 0$

$\therefore$ from (ii),

$$2 = 2\tan C \Rightarrow \tan C = 1$$

$$\Rightarrow \quad C = \frac{\pi}{4}$$

$\therefore$ The required solution is

$$4x + y + 1 = 2\tan\left(2x + \frac{\pi}{4}\right)$$

2.7 Determine the orthogonal trajectory of a family of curves represented by the polar equation

$$r = a(1 - \cos\theta),$$

(r, θ) being the plane polar coordinates of any point.

(2011 : 10 Marks)

Solution:

The given equation is

$$r = a(1 - \cos\theta) \qquad \text{...(i)}$$

$$\Rightarrow \quad \log r = \log[a(1-\cos\theta)]$$
$$= \log a + \log(1-\cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{(1-\cos\theta)}\sin\theta \qquad \text{...(ii)}$$

(ii) is the differential equation of the given family (i).

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (ii), we get the differential equation of the orthogonal trajectories.

$$\therefore \quad \frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \quad \frac{dr}{r} = \frac{-1}{\cot\frac{\theta}{2}}d\theta$$

$$\Rightarrow \quad \frac{dr}{r} = -\tan\frac{\theta}{2}d\theta$$

$$\Rightarrow \quad \log r = \frac{\log\left(\cos\frac{\theta}{2}\right)}{\frac{1}{2}} + \log C,$$

where C is an arbitrary constants.

$$\Rightarrow \quad \log\frac{r}{C} = \log\left(\cos^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad \frac{r}{C} = \cos^2\frac{\theta}{2} \Rightarrow \frac{r}{C} = \frac{1+\cos\theta}{2}$$

$$\Rightarrow \quad r = \frac{C}{2}(1+\cos\theta)$$

$$\Rightarrow \quad r = b(1 \cos\theta), \text{ where } \frac{C}{2} = b$$

This is the required orthogonal trajectory.

2.8 Solve :

$$\frac{dy}{dx} = \frac{2xye^{(x/y)^2}}{y^2(1+e^{(x/y)^2})+2x^2e^{(x/y)^2}}$$

(2012 : 12 Marks)

Solution:

Given :

$$\frac{dy}{dx} = \frac{2xye^{(x/y)^2}}{y^2(1+e^{(x/y)^2})+2x^2e^{(x/y)^2}} \qquad \text{...(i)}$$

Put $\frac{x}{y} = v$ or $x = vy$

Differentiate both sides w.r.t. 'y'

$$\frac{dx}{dy} = v \cdot 1 + y \cdot \frac{dv}{dy}$$

Substituting these values in (i)

$$\frac{1}{v + y\frac{dv}{dy}} = \frac{2vy^2e^{v^2}}{y^2(1+e^{v^2}) + 2y^2v^2e^{v^2}}$$

$$\Rightarrow \quad y^2(1+e^{v^2}) + 2y^2v^2e^{v^2} = 2v^2y^2e^{v^2} + 2vy^2e^{v^2}\frac{dv}{dy}$$

$$\Rightarrow \quad y^2(1+e^{v^2})dy = 2vy^3e^{v^2}dv$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{2ve^{v^2}}{1+e^{v^2}}dv \qquad \text{...(ii)}$$

Again put $\quad 1 + e^{v^2} = z$

and $\quad e^{v^2} \cdot 2vdv = dz$ in (ii)

We have $\quad \frac{dy}{y} = \frac{dz}{z}$, which on integration gives

$$\log y = \log z + \log c$$

or $\quad y = zc$ or $y = (1+e^{v^2})c$

or $\quad y = (1+e^{(x+y)^2})c$

2.9 Find the orthogonal trajectories of the family of curves $x^2 + y^2 = ax$.

(2012 : 12 Marks)

Solution:

The given family of curves is

$$x^2 + y^2 = ax \qquad \text{...(i)}$$

Differentiating (i) w.r.t. x, we get

$$2x + 2yy' = a \qquad \text{...(ii)} \left(\text{where } y' = \frac{dy}{dx}\right)$$

Eliminate 'a' between (i) and (ii), we have

$$x^2 + y^2 = 2x(x + yy')$$

or $\quad y^2 - x^2 = 2xy\frac{dy}{dx}$ which is the differential equation of given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, the differential equation of the required orthogonal trajectory is :

$$y^2 - x^2 = -2xy\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \qquad \text{...(iii)}$$

which is a homogeneous differential equation.

Put
$$y = vx$$
and
$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ in (iii)}$$
We have,
$$v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$$
or
$$\frac{dx}{x} = \frac{1-v^2}{v(1+v^2)}dv$$
$$= \left(\frac{1}{v} - \frac{2v}{1+v^2}\right)dv$$
On integrating both sides, we have
$$\log x = \log v - \log(1+v^2) + \log c$$
$\Rightarrow$
$$x = \frac{cv}{1+v^2}$$
Replacing v by $\frac{y}{x}$, we have
$$x^2 + y^2 = cy, \ c \text{ being a parameter.}$$

2.10 Show that the differential equation
$$(2xy\log y)dx + (x^2 + y^2\sqrt{y^2+1})dy = 0$$
is not exact. Find an integrating factor and hence, the solution of the equation.

(2012 : 20 Marks)

Solution:

Given :
$$(2xy\log y)dx + (x^2 + y^2\sqrt{y^2+1})dy = 0 \quad \text{...(i)}$$
This is of the form of $Mdx + Ndy = 0$

where
$$M = 2xy\log y$$
and
$$N = x^2 + y^2\sqrt{y^2+1}$$
$$\frac{\partial M}{\partial y} = 2x\log y + 2xy\cdot\frac{1}{y} = 2x\log y + 2x$$
$$\frac{\partial N}{\partial x} = 2x$$
Since
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \therefore \text{ (i) is not exact}$$
Now
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - (2x\log y + 2x)}{2xy\log y} = \frac{-1}{y}$$
which is a function of y alone.

$\therefore$ $e^{\int f(y)dy}$ is an integrating factor of $Mdx + Ndy = 0$, where $f(y) = \frac{1}{y}$

Multiplying eqn. (i) by $e^{\int f(y)dy}$
$$= e^{\int -\frac{1}{y}dy} = e^{[-\log y]} = \frac{1}{y}, \text{ we get}$$

$$\frac{1}{y}(2xy\log y)dx+\frac{1}{y}\left(x^2+y^2\sqrt{y^2+1}\right)dy = 0$$

$$\Rightarrow \quad 2x\log y\,dx+\left(\frac{x^2}{y}+y\sqrt{y^2+1}\right)dy = 0 \quad \text{...(ii)}$$

which is of the form of

$$M_1dx + N_1dy = 0$$

Now $$\frac{\partial M_1}{\partial y} = \frac{2x}{y} = \frac{\partial N_1}{\partial x}$$

$\therefore$ Eqn. (ii) is an exact differential equation and its solution is given by :

$$\underset{\substack{y \text{ is}\\ \text{constant}}}{\int} 2x\log y\,dx + \underset{\substack{\text{terms of}\\ N_1 \text{ not}\\ \text{containing}\\ x}}{\int} y\sqrt{y^2+1}dy = 0$$

$$\Rightarrow \quad x^2\log y+\frac{(y^2+1)^{3/2}}{2\cdot\frac{3}{2}} = c$$

$$\Rightarrow \quad x^2\log y+\frac{1}{3}(y^2+1)^{3/2} = c$$

2.11 y is a function of x, such that the differential coefficient $\frac{dy}{dx}$ is equal to $\cos(x + y) + \sin(x + y)$. Find out a relation between x and y, which is free from any derivative/differential. (2013 : 10 Marks)

Solution:

$$\frac{dy}{dx} = \sin(x+y)+\cos(x+y)$$

Let $v = x + y \Rightarrow$ $$\frac{dv}{dx} = 1+\frac{dy}{dx}$$

$$\therefore \quad \frac{dv}{dx}-1 = \cos v+\sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = (\cos v+\sin v+1)$$

$$\Rightarrow \quad \frac{dv}{(\cos v+\sin v+1)} = dx$$

$$\Rightarrow \quad \int\frac{dv}{1+\frac{1-\tan^2\frac{v}{2}}{1+\tan^2\frac{v}{2}}+\frac{2\tan\frac{v}{2}}{1+\tan^2\frac{v}{2}}} = \int dx$$

$$\Rightarrow \quad \int\frac{\sec^2\frac{v}{2}dv}{2\left(1+\tan\frac{v}{2}\right)} = x+c \Rightarrow \ln\left(1+\tan\frac{v}{2}\right) = x+c$$

$$\therefore \quad \ln\left(1+\tan\frac{x+y}{2}\right) = x+c$$

2.12 Obtain the equation of the orthogonal trajectory of the family of curves represented by $r^n = a \sin n\theta$, (r, θ) being the plane polar coordinates.

(2013 : 10 Marks)

Solution:

$$r^n = a \sin n\theta$$

Differentiating w.r.t. θ

$$nr^{n-1}\frac{dr}{d\theta} = an \cos n\theta$$

$$\Rightarrow \quad a = \frac{r^{n-1}}{\cos n\theta}\frac{dr}{d\theta}$$

$\therefore$ The differential equation is

$$r^n = \frac{r^{n-1}}{\cos n\theta}\frac{dr}{d\theta}\cdot \sin n\theta$$

$$\Rightarrow \quad r = \tan n\theta \frac{dr}{d\theta}$$

$\therefore$ Differential equation of orthogonal trajectory will be $f\left(r, \theta, -r^2\frac{d\theta}{dr}\right)$

$$\Rightarrow \quad r = \tan n\theta\left(-r^2\frac{d\theta}{dr}\right)$$

$$\Rightarrow \quad \frac{dr}{r} + \tan n\theta \, d\theta = 0$$

$$\Rightarrow \quad \int\frac{dr}{r} + \int\frac{\sin n\theta}{\cos n\theta}d\theta = \ln C$$

$$\Rightarrow \quad \ln r - \frac{1}{n}\ln(\cos n\theta) = \ln C$$

$$\Rightarrow \quad n \ln r - \ln(\cos n\theta) = n \ln C$$

$$\Rightarrow \quad r^n = C^n \cos n\theta$$

2.13 Solve the differential equation : $(5x^3 + 12x^2 + 6y^2)dx + 6xydy = 0$.

(2013 : 10 Marks)

Solution:

Comparing to $Mdx + Ndy = 0$

$$M = 5x^3 + 12x^2 + 6y^2; \; N = 6xy$$

$$\frac{\partial M}{\partial y} = 12y; \; \frac{\partial N}{\partial x} = 6y$$

$$\therefore \quad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{6y}{6xy} = \frac{1}{x} = f(x)$$

$$I.F = e^{\int f(x)dx} = e^{\ln x} = x$$

$\therefore$ $(5x^4 + 12x^3 + 6xy^2)dx + 6x^2ydy = 0$ is exact.

The solution for this integral is given by

$$\int Mdx \text{ (treating } y \text{ constant)} + \int N \text{ (terms not containing } x)dy = C$$

$$\Rightarrow \quad \int(5x^4 + 12x^3 + 6xy^2)dx + \int 0\,dy = 0$$

$$\Rightarrow \quad x^5 + 3x^4 + 3x^2y^2 = C$$

2.14 Justify that a differential equation of the form :

$$[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0,$$

where $f(x^2 + y^2)$ is an arbitrary function of $(x^2 + y^2)$, is not an exact differential equation and $\frac{1}{x^2 + y^2}$ is an integrating factor for it. Hence, solve this differential equation for $f(x^2 + y^2) = (x^2 + y^2)^2$.

(2014 : 10 Marks)

Solution:

Given that

$$y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0 \quad \text{...(i)}$$

which is in the form of $Mdx + Ndy = 0$

where
$$M = ytxf(x^2 + y^2),\ N = yf(x^2 + y^2) - x$$

$$\frac{\partial M}{\partial y} = 1+f'(x^2 + y^2)\cdot 2yx = 1 + 2xyf'(x^2 + y^2)$$

and
$$\frac{\partial N}{\partial x} = yf'(x^2 + y^2)(2x) - 1 = -1 + 2yxf'(x^2 + y^2)$$

Clearly,
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Given equation is not an exact differential equation.

To show $\frac{1}{x^2+y^2}$ is an integrating factor.

Multiplying equation (i) by $\frac{1}{x^2+y^2}$, then

$$M = \frac{y+xf(x^2+y^2)}{x^2+y^2} \text{ and } N = \frac{yf(x^2+y^2)-x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{[1+2xyf'(x^2+y^2)](x+y)-[y+xf(x^2+y^2)](2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy[(x^2+y^2)f'(x^2+y^2)-f(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{[-1+2xy-f'(x^2+y^2)](x^2+y^2)-[yf(x^2+y^2)-x](2a)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy[(x^2+y^2)f(x^2+y^2)-f(x^2+y^2)]}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow \frac{1}{x^2+y^2}$ is an integrating factor for equation (i).

Given :
$$f(x^2 + y^2) = (x^2 + y^2)^2$$

Then equation (i) becomes

$$[y + x^2(x^2 + y^2)^2]dx + [y(x^2 + y^2)^2 - x]dy = 0$$

This can be written as

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\Rightarrow \qquad \frac{ydx - xdy}{x^2+y^2} + (xdx + ydy)(x^2+y^2) = 0$$

$$d\left(\tan\left(\frac{x}{y}\right)\right) + \frac{x^2+y^2}{2} d(x^2+y^2) = 0$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\left(\frac{x^2+y^2}{2}\right) = C$$

which is the required solution.

2.15 Find the sufficient condition for the differential equation $M(x, y)dx + N(x, y)dy = 0$ to have an integrating factor as a function of $(x + y)$. What will be the integrating factor in that case? Hence find the integrating factor for the differential equation

$$(x^2 + xy)dx + (y^2 + xy)dy = 0$$

(2014 : 15 Marks)

Solution:

Let $$\text{I.F.} = F(x + y) \qquad ...(i)$$

Multiplying in the given equation

$$M(x, y)dx + N(x, y)dy = 0$$

we get

$$F(x + y)M(x, y)dx + F(x + y)N(x, y)dy = 0 \qquad ...(ii)$$

From (ii),

$$M' = F(x + y)M(x_1 y)$$

and $$N' = F(x + y)N(x_1 y)$$

For equation (ii) to be exact

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \qquad ...(iii)$$

$$\Rightarrow \qquad \frac{\partial M}{\partial y}F(x+y) + MF'(x+y) = \frac{\partial N}{\partial x}F(x+y) + NF'(x+y)$$

$$\Rightarrow \qquad F(x+y)\left[\frac{\partial M}{\partial y} - \frac{\partial M}{\partial x}\right] = (N - M)(F'(x + y))$$

$$\Rightarrow \qquad \frac{F'(x+y)}{F(x+y)} = \frac{1}{(N-M)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

Integrating, we get

$$\ln F(x + y) = \int \frac{1}{(N-M)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) d(x+y)$$

$$\therefore \qquad F(x + y) = e^{\int \frac{1}{(N-M)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) d(x+y)}$$

So, for F to be a function of $(x + y)$ $N \neq M$ and $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and $\frac{1}{(N-M)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of $(x + y)$.

Now, in case $$M = x^2 + xy$$

and $$N = y^2 + xy$$

$$\frac{\partial M}{\partial y} = x \text{ and } \frac{\partial N}{\partial x} = y$$

and
$$N - M = (y^2 - x^2) = (y - x)(y + x)$$

$\therefore$
$$\frac{1}{(N-M)}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(y-x)(y+x)}(x-y) = -\frac{1}{x+y}$$

So,
$$F(x+y) = e^{\int \frac{-1}{(x+y)} d(x+y)} = e^{-\ln(x+y)}$$
$$= \frac{1}{(x+y)}$$

So, in the given case the IF is $\frac{1}{(x+y)}$.

2.16 Find the constant 'a' so that $(x + y)a$ is the integrating factor (IF) of
$$(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0$$
and hence solve the DE.

(2015 : 12 Marks)

Solution:

As $(x + y)^a$ is an I.F., so
$$\frac{\partial}{\partial y}[(x+y)^a(4x^2+2xy+6y)] = \frac{\partial}{\partial x}[(x+y)^a(2x^2+9y+3x)]$$

$a(x + 4)^{a-1}(4x^2 + 2xy + 6y) + (x + y)^a(2x + 6) = a(x + 4)^{a-1}(2x^2 + 9y + 3x) + (x + 4)^a(4x + 3)$

Dividing both sides by $(x + y)^{a-1}$

$a(4x^2 + 2xy + 6y) + (x + 4)(2x + 6) = a(2x^2 + 9y + 3x) + (x + y)(4x + 3)$

$x^2(4a + 2 - 2a - 4) + xy(2a + 2 - 4) + y(6a + 6 - 9a + 3) + x(6 - 3a - 3) = 0$

$x^2(2a - 2) + xy(2a - 2) + y(3 - 3a) + x(3 - 3a) = 0$

This equation is satisfied universally for $a = 1$.

$\therefore$
$$IF = (x + y)$$

Multiplying DE by IF

$(4x^3 + 2x^2y + 6xy + 4yx^2 + 2xy^2 + 6y^2)dx + (2x^3 + 9xy + 3x^2 + 2x^2y + 9y^2 + 3xy)dy = 0$

$(4x^3 + 6x^2y + 2xy^2 + 6xy + 6y^2)dx + (2x^3 + 2x^2y + 12xy + 3x^2 + 9y^2)dy = 0$

$\int(4x^3 + 6x^2y + 2xy^2 + 6xy + 6y^2)dx + \int 9y^2 dy = 0$

$x^4 + 2x^3y + x^2y^2 + 3x^2y + 6xy^2 + 3y^3 = 0$

2.17 Solve : $\frac{dy}{dx} = \frac{1}{1+x^2}(e^{\tan^{-1}x} - y)$

(2016 : 10 Marks)

Solution:

Re-arranging the given ODE as
$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$$

It is of the form,
$$\frac{dy}{dx} + Py = Q$$

$$IF = e^{\int Pdx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$
$$y \cdot (IF) = \int (IF)Qdx$$

$$y \cdot e^{\tan^{-1}x} = \int e^{\tan^{-1}x} \cdot \frac{e^{\tan^{-1}x}}{1+x^2} dx$$

$$= \int \frac{e^{2t}}{1} dt$$

Put,

$$\tan^{-1}x = t$$

$$\frac{1}{1+x^2} dx = dt$$

$$= \frac{e^{2t}}{2} + C$$

$$= \frac{1}{2} e^{2\tan^{-1}x} + C$$

$\therefore$

$$y = \frac{1}{2} e^{2\tan^{-1}x} + C \cdot e^{-\tan^{-1}x}$$

2.18 Show that the family of parabolas $y^2 = 4cx + 4c^2$ is self-orthogonal.

(2016 : 10 Marks)

Solution:

Self-orthogonal : If the orthogonal trajectory of a system of curves is that system itself.

Here,
$$y^2 = 4cx + 4c^2 \quad \text{...(i)}$$

Differentiating,
$$2yy_1 = 4c$$

$\Rightarrow$
$$c = \frac{yy_1}{2}$$

Putting in (i) $\Rightarrow$
$$y^2 = 4 \cdot \frac{yy_1}{2} x + 4 \cdot \left(\frac{yy_1}{2}\right)^2$$

$\Rightarrow$
$$y^2 = 2xyy_1 + (yy_1)^2 \quad \text{...(ii)}$$

This is the DE of the given system of parabolas. To obtain the DE of orthogonal trajectories, we replace y_1 with $\frac{-1}{y_1}$.

$$y^2 = \frac{-2xy}{y_1} + \frac{y^2}{y_1^2} \Rightarrow y_1^2 y^2 = -2xyy_1 + y^2$$

$$y^2 = 2xyy_1 + (yy_1)^2 \quad \text{...(iii)}$$

This DE is same as obtained in (ii). Hence, the family of parabolas given by eqn. (i) is self-orthogonal.

2.19 Solve : $\{y(1 - x\tan x) + x^2 \cos x\}dx - xdy = 0$

(2016 : 10 Marks)

Solution:

Re-arranging the terms :

$$\frac{dy}{dx} + \left(\frac{-1}{x} + \tan x\right) y = x \cos x$$

This is a linear DE of the form

$$\frac{dy}{dx} + Py = Q$$

$$I = e^{\int P dx} = e^{\int \left(-\frac{1}{x} + \tan x\right) dx}$$

$$= e^{-\log x + \log \sec x} = \frac{\sec x}{x}$$

$$y(IF) = \int \theta \cdot (IF) dx$$

$$y \cdot \frac{\sec x}{x} = \int \left(\frac{\sec x}{x} \cdot x \cos x \right) dx$$

$$= \int dx = x + C$$

$\therefore$ $$y = \frac{x^2 + Cx}{\sec x}$$

i.e., $$y = (x^2 + Cx)\cos x$$

2.20 Suppose that the streamlines of the fluid flow are given by a family of curves $xy = C$. Find the equipotential lines, that is, the orthogonal trajectories of the family of curves representing the streamlines.

(2017 : 10 Marks)

Solution:

First we form the DE of family of curves,

$$xy = C$$

Differentiating w.r.t. x

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

To find the differential equation of orthogonal trajectory we replace $\frac{dy}{dx}$ by $\frac{-1}{dy/dx}$ i.e., $-\frac{dx}{dy}$.

$$x \cdot \left(-\frac{dx}{dy} \right) + y = 0$$

i.e., $$x\, dx = y\, dy$$

Solving this system, we get the orthogonal trajectory (i.e., equipotential lines)

$$\frac{x^2}{2} = \frac{y^2}{2} + C$$

i.e., $$x^2 - y^2 = C_1$$

2.21 If the growth rate of the population of bacteria at any time t is proportional to the amount present at that time and population doubles in one week, then how much bacteria can be expected after 4 weeks?

(2017 : 8 Marks)

Solution:

$$\frac{dx}{dt} = kx, \text{ where } x = \text{amount of bactera}$$

$\therefore$ $$\frac{dx}{x} = k\, dt$$

i.e., $$\log x = kt + C$$

i.e., $$x = Ae^{kt} \qquad (C = \log A)$$

At $t = 0$, $$x = x_0$$

At $t = 1$, $$x = 2x_0 \qquad (t \text{ in weeks})$$

$\therefore$ $$x_0 = A,\ 2x_0 = Ae^k$$

$\therefore$ $$k = \log 2$$

$\therefore$ $$x = x_0 e^{(\log 2)t} \qquad (\log 2 = 0.693)$$

After 4 weeks,

$$\log\frac{x}{x_0} = 4k = 4\ln 2 = \log 16$$

$\therefore$ $$\frac{x}{x_0} = 16$$

Population becomes 16 times.

2.22 Find α and β such that $x^\alpha y^\beta$ is an integrating factor of $(4y^2 + 3xy)dx - (3xy + 2x^2)dy = 0$ and solve the equation.

(2018 : 12 Marks)

Solution:

Given, the equation is $Mdx + Ndy = 0$

where
$$M = 4y^2 + 3xy$$
$$N = -(3xy + 2x^2)$$

If $x^\alpha y^\beta$ is an integrating factor, then the equation becomes

$$x^\alpha y^\beta M dx + x^\alpha y^\beta N dy = 0$$

Since, the equation has become perfect,

$$\frac{\partial x^\alpha y^\beta N}{\partial x} = \frac{\partial x^\alpha y^\beta M}{\partial y}$$

$$\Rightarrow \quad \frac{\partial(-x^\alpha y^\beta(3xy+2x^2))}{\partial x} = \frac{\partial x^\alpha y^\beta(4y^2+3xy)}{\partial y}$$

$$\Rightarrow \quad \frac{\partial(-3x^{\alpha+1}y^{\beta+1} - 2x^{\alpha+1}y^\beta)}{\partial x} = \frac{\partial(4x^\alpha y^{\beta+2} + 3x^{\alpha+1}y^{\beta+1})}{\partial y}$$

$$\Rightarrow \quad -3(\alpha+1)x^\alpha y^{\beta+1} - 2(\alpha+2)x^{\alpha+1}y^\beta = 4(\beta+2)x^\alpha y^{\beta+1} + 3(\beta+1)x^\alpha$$

$\Rightarrow$ Comparing LHS and RHS and coefficient of $x^\alpha y^{\beta+1}$ and $x^{\alpha+1}y^\beta$, we get

$$-3(\alpha+1) = 4(\beta+2) \Rightarrow -3\alpha - 3 = 4\beta + 8 \Rightarrow 3\alpha + 4\beta = -11 \qquad \text{...(i)}$$

Also,
$$-2(\alpha+2) = 3(\beta+1) \Rightarrow -2\alpha - 4 = 3\beta + 3 \Rightarrow 2\alpha + 3\beta = -7 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get

$$\alpha = -5, \beta = 1$$

The equation is

$$(4x^{-5}y^3 + 3x^{-4}y^2)dx - (3x^{-4}y^2 + 2x^{-3}y)dy = 0$$

Integrating it, we get

$$\int(4x^{-5}y^3 + 3x^{-7}y^2)dx + 0 = 0$$

$$\Rightarrow \quad \frac{-y^3}{x^4} - \frac{y^2}{x^3} + c = 0, \text{ where } c \text{ is a constant.}$$

$$\Rightarrow \quad c = \frac{y^3}{x^4} + \frac{y^2}{x^3}$$

2.23 Find $f(y)$ such that $(2xe^y + 3y^2)dy + (3x2 + f(y))dx$ is exact and hence solve.

(2018 : 12 Marks)

Solution:

Given, equation is $(2xe^y + 3y^2)dy + (3x^2 + f(y))dx$

If the equation is exact, then

$$\frac{\partial(2xe^y + 3y^2)}{\partial x} = \frac{\partial(3x^2 + f(y))}{\partial y}$$

$$\Rightarrow \qquad 2e^y + 0 = f'(y)$$

$$\Rightarrow \qquad f'(y) = 2e^y \Rightarrow f(y) = 2e^y + C, \text{ where } C \text{ is a constant.}$$

Now, since equation is exact, it can be integrated.

$$\int(3x^2 + f(y))dx + \int(2xe^y + 3y^2)dy = 0$$

$$\Rightarrow \qquad x^3 + 2xe^y + cx + y^3 + k = 0$$ (where k is constant)

$$\therefore \qquad f(y) = 2e^y + c$$

and solution of equation is $x^3 + 2xe^y + cx + y^3 + k = 0$

2.24 Solve the differential equation :

$$(2y \sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0$$

(2019 : 10 Marks)

Solution:

Given equation is

$(2y \sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0$ ...(1)

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = 2 \sin x + 12y^3 \sin x \cos x$$

$$\frac{\partial N}{\partial x} = 8y^3 \cos x \sin x + \sin x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \text{ Hence it is not exact.}$$

To make it exact, we need to find I.F.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1(\sin x(1+4y^3\cos x)}{-\cos x(4y^3\cos x + 1)}$$

$$= -\tan x$$

$$f - \tan x\ dx = \log \cos x$$

$$\text{I.F.} = e = e = \cos x$$

Then, multiply I.F. to equation (1), we get

$$\Rightarrow \qquad (2y \sin x \cos x + 3y^4 \sin x \cos^2 x)dx - (4y^3 \cos^3 x + \cos^2 x)dy = 0$$

$$\Rightarrow \qquad \left(y\sin 2x + \frac{3}{2}y^4 \sin 2x \cos x\right)dx - (4y^3\cos x + 1)\frac{(1+\cos 2x)}{2}dy = 0$$

$$\Rightarrow \left(y\sin 2x + \frac{3}{2}y^4 \sin^2 x \cos x\right)dx - \left[2y^3\cos x + 2y^3\cos x\cos 2x + \frac{1}{2} + \frac{\cos 2x}{2}\right]dy = 0$$

$$\text{Complete solution} = \int Mdx + \int Ndy$$

$$y = \text{Constant; } x \text{ terms not included}$$

$$\int\left(y\sin^2 x + \frac{3}{2}y^4 \cdot \sin 2x \cos x\right)dx + \int\frac{1}{2}dy = C$$

$$y = \text{Constant}$$

$$\Rightarrow \qquad -y\frac{\cos 2x}{2} + \frac{3}{4}y^4\int(\sin 3x + \sin x)dx + \frac{y}{2} = C$$

$$\Rightarrow \qquad -\left[\frac{y}{2}\cos 2x+\frac{3}{4}y^4\left(\frac{\cos^3 x}{3}-\cos 3x\right)\right]+\frac{y}{2} = C$$

Multiply whole equation by 4

$$\Rightarrow \qquad 2y\cos 2x + 3y^4 \cos x + y^4 \cos 3x + (-2y) = C^1$$

$$\Rightarrow \qquad 2y\cos 2x + y^4(\cos 3x + 3\cos x) - 2y = C^1$$

Hence, the required solution.

3. Equation of 1st Order but NOT of 1st Degree

3.1 Obtain Clairaut's form of the differential equation

$$\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right) = a^2\frac{dy}{dx}$$

Also find its general solution.

(2011 : 15 Marks)

Solution:

The given differential equation is

$$\left(x\frac{dy}{dx}-y\right)\left(y\frac{dy}{dx}+x\right) = a^2\frac{dy}{dx} \qquad \ldots(i)$$

Put $\quad \frac{dy}{dx} = P, x^2 = X, y^2 = Y$

$$\Rightarrow \qquad \partial x dx = dX, \partial y dy = dy$$

$$\Rightarrow \qquad \frac{dY}{dX} = \frac{2ydy}{2xdx} \Rightarrow P = \frac{y}{x}P, P = \frac{dY}{dX}$$

$\therefore$ from (i), $\quad \left(x\cdot\frac{x}{y}P-y\right)\left(y\cdot\frac{x}{y}P+x\right) = a^2\frac{x}{y}P$

$$\Rightarrow \qquad \left(\frac{Px^2-y^2}{y}\right)-x(P+1) = a^2\frac{x}{y}P$$

$$\Rightarrow \qquad PX-Y = \frac{a^2P}{P+1}$$

$$\Rightarrow \qquad Y = PX-\frac{a^2P}{P+1}$$

$\Rightarrow \qquad Y = PX + f(P)$ which is Clairaut's form.

$\therefore$ Its general solution is

$Y = CX + f(C)$ where C is an arbitrary constant.

$$\Rightarrow \qquad y^2 = Cx^2 + f(C)$$

3.2 Solve the DE : $x = py - p^2$ where $p = \frac{dy}{dx}$.

(2015 : 13 Marks)

Solution:

Differentiate w.r.t. y

$$\frac{1}{p} = p + y\frac{dp}{dy} - 2p\frac{dp}{dy}$$

$$\Rightarrow \quad 1 - p^2 = (py - 2p^2)\frac{dp}{dy}$$

$$\Rightarrow \quad (py - 2p^2)dp + (p^2 - 1)dy = 0 \qquad (Mdp + Ndy = 0)$$

$$\frac{\partial M}{\partial y} = p, \ \frac{\partial N}{\partial p} = 2p$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial p}\right) = \frac{-p}{p^2 - 1}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-p}{p^2-1}dp} = e^{-\frac{1}{2}\log(p^2-1)} = \frac{1}{\sqrt{p^2-1}}$$

$$\text{D.E.} = \left(\frac{py}{\sqrt{p^2-1}} - \frac{2p^2}{\sqrt{p^2-1}}\right)dp + \left(\sqrt{p^2-1}\right)dy = 0$$

$$\int \frac{py}{\sqrt{p^2-1}} - \int \frac{2p^2}{\sqrt{p^2-1}} = \text{Constant}$$

$$\Rightarrow \quad 2\sqrt{p^2-1} - 2\int p \cdot \frac{p}{\sqrt{p^2-1}} = C_1$$

$$\Rightarrow y\sqrt{p^2-1} - 2p\int \frac{p}{\sqrt{p^2-1}} + 2\int\int \frac{p}{\sqrt{p^2-1}} = C_1$$

$$\Rightarrow \quad y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + 2\int \sqrt{p^2-1} = C_1$$

$$\Rightarrow y\sqrt{p^2-1} - 2p\sqrt{p^2-1} + p\sqrt{p^2-1} - \log\left|(p + \sqrt{p^2-1}\right| = C_1$$

$$\Rightarrow \quad y = p + \frac{\log\left|p + \sqrt{p^2-1}\right| + C_1}{\sqrt{p^2-1}}$$

and

$$x = py - p^2 = p\frac{\left[\log\left|p + \sqrt{p^2-1}\right| + G\right]}{\sqrt{p^2-1}}$$

3.3 Solve the following simultaneous linear DE $(D + 1)y = z + e^x$ and $(D + 1)z = y + e^x$ where y and z are functions of independent variable x and $D = \frac{d}{dx}$.

(2017 : 8 Marks)

Solution:

$$(D+1)y = z + e^x \qquad \text{...(i)}$$

$$(D+1)z = y + e^x \qquad \text{...(ii)}$$

Applying $(D + 1)$ operator to (i)

$$(D+1)^2 y = (D+1)z + (D+1)e^x$$

$$(D^2 + 2D + 1)y = y + e^x + 2e^x \qquad \text{(using (ii))}$$

$$(D^2 + 2D)y = 3e^x$$

Solution to homogeneous equation

$$(D^2 + 2D)y = 0, \text{ i.e., } D(D+2)y = 0 \text{ is}$$

$$y_C = C_1 + C_2e^{-2x}$$

Particular integral,

$$PI = \frac{1}{D(D+2)}(3e^x) = \frac{3}{1(1+2)} \cdot e^x = e^x$$

$\therefore$

$$y = C_1 + C_2e^{-2x} + e^x$$

From (1),

$$z = (D+1)y - e^x$$

$$= -2C_2e^{-2x} + e^x + C_1 + C_2e^{-2x}$$

$\therefore$

$$z = C_1 - C_2e^{-2x} + e^x$$

3.4 Consider the DE, $xyp^2 - (x^2 + y^2 - 1)p + xy = 0$ where $p = \frac{dy}{dx}$. Substituting $u = x^2$ and $v = y^2$ reduce the equation to Clairaut's form in terms of u, v and $p' = \frac{dv}{du}$. Hence or otherwise solve the equation.

(2017 : 8 Marks)

Solution:

$$xyp^2 - (x^2 + y^2 - 1)p + xy = 0 \quad \text{...(i)}$$

$$u = x^2$$

$\Rightarrow$

$$du = 2xdx$$

$$v = y^2 \Rightarrow dv = 2ydy$$

$\therefore$

$$\frac{dv}{du} = \frac{y}{x}\frac{dy}{dx}$$

i.e.,

$$p' = \frac{y}{x}p \text{ or } p = \frac{x}{y}p'$$

Substituting in (i),

$$xy \cdot \frac{x^2}{y^2} \cdot p'^2 - (u+v-1)\frac{x}{y}p' + xy = 0$$

$\Rightarrow$

$$x^2 \cdot p'^2 - (u+v-1)p' + y^2 = 0$$

i.e.,

$$u^2p'^2 - (u + v - 1)p' + v = 0$$

$$(p' - 1)v = up'(p' - 1) + p'$$

$$v = up' + \frac{p'}{p'-1}$$

This is Clairaut's form

$\therefore$

$$v = uC + \frac{C}{C-1}$$

Solution to (1) $\Rightarrow$

$$y^2 = Cx^2 + \frac{C}{C-1}$$

3.5 Solve : $\left(\frac{dy}{dx}\right)^2 y + 2\frac{dy}{dx}x - y = 0$.

(2018 : 13 Marks)

Solution:

Given equation is

$$\left(\frac{dy}{dx}\right)^2 y+2\frac{dy}{dx}x-y = 0$$

Let $\frac{dy}{dx} = P$

∴ equation becomes $yp^2 + 2px - y = 0 \Rightarrow 2px = y - yp^2$

$$\Rightarrow \quad x = \frac{y}{2p}-\frac{py}{2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{1}{p}=\frac{1}{2p}-\frac{y}{2p^2}\frac{dp}{dy}-\frac{p}{2}-\frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} = \frac{-p}{2}-\frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2}+1\right)$$

$$\Rightarrow \quad \frac{1}{2}\frac{(p^2+1)}{p} = \frac{-y}{2}\frac{dp}{dy}\frac{(p^2+1)}{p^2}$$

$$\Rightarrow \quad \frac{p}{dp} = \frac{-dy}{y}$$

Integrating on both sides, we get

$\log p = \log y + \log c$, where c is a constant.

$$\Rightarrow \quad py = c \Rightarrow p = \frac{c}{y}$$

Putting this value in given differential equation, we get

$$2\cdot\frac{c}{y}x = y-y\frac{c^2}{y^x}$$

$$\Rightarrow \quad \frac{2cx}{y} = y-\frac{c^2}{y} \Rightarrow 2cx = y^2-c^2$$

$$\Rightarrow \quad y^2-2cx = c^2$$

3.6 Obtain the singular solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2\left(\frac{y}{x}\right)^2\cot^2\alpha-2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\operatorname{cosec}^2\alpha = 1$$

Also, find the complete primitive of the given differential equation. Give the geometrical interpretations of the complete primitive and singular solution.

(2019 : 15 Marks)

Solution:

Given :

$$\left(\frac{dy}{dx}\right)^2\left(\frac{y}{x}\right)^2\cot^2\alpha-2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\operatorname{cosec}^2\alpha = 1$$

which can be written as :

$$P^2y^2\cot^2\alpha-2pxy+y^2\operatorname{cosec}^2\alpha = x^2$$

$$P^2y^2\frac{\cos^2\alpha}{\sin^2\alpha}-2pxy+y^2\frac{1}{\sin^2\alpha} = x^2$$

$$\Rightarrow \quad p^2y^2\cos^2\alpha-2pxy\sin^2\alpha+y^2-x^2\sin^2\alpha = 0 \quad \text{...(A)}$$

$\Rightarrow$ $$(py)^2 - (2py)x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0$$

$\therefore$ $$py = \frac{2x\tan^2\alpha + \sqrt{4x^2\tan^4\alpha - 4(y^2\sec^2\alpha - x^2\tan^2\alpha)}}{2}$$

or $$py = x\tan^2\alpha \pm \sqrt{x^2\tan^2 d(\tan^2\alpha + 1) - y^2\sec^2\alpha}$$

$$py = x\tan^2\alpha \pm \sec\alpha\sqrt{x^2\tan^2\alpha - y^2}$$

$$ydy - x\tan^2\alpha\, dx = \pm\sec\alpha\sqrt{(x^2\tan^2 d - y^2)}\,dz$$

$\Rightarrow$ $$\pm\frac{x\tan^2\alpha dx - ydy}{\sqrt{(x^2\tan^2 d - y^2)}} = -\sec\alpha\, dx$$

Integrating, $$\pm\sqrt{x^2\tan^2\alpha - y^2} = C - x\sec\alpha$$

Squaring, $$x^2 + \tan^2\alpha - y^2 = C^2 - 2Cx\sec\alpha + x^2\sec^2 x$$
$$= x^2(\tan^2\alpha - \sec^2\alpha) - y^2 = C^2 - 2C\alpha\sec\alpha$$
$$= -x^2 - y^2 = C^2 = 2Cx\sec x \qquad [\because -1 = \tan^2 - \sec^2\alpha]$$

$\Rightarrow$ $$x^2 + y^2 - 2Cx\sec x + C^2 = 0 \qquad \text{...(B)}$$

From (A)
$$P^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2 a = 0$$

Since the given equation is quadratic in p, the p. disc, relation is
$$4x^2y^2\sin^4\alpha - 4y^2\cos^2\alpha(y^2 - x^2\sin^2\alpha) = 0$$
$$4y^2[x^2\sin^2\alpha(\sin^2\alpha + \cos^2\alpha) - y^2\cos^2\alpha) = 0$$
$$y^2[x^2\sin^2\alpha - y^2\cos^2\alpha] = 0$$
$$y^2\cos^2\alpha(x^2\tan^2\alpha - y^2) = 0 \qquad \text{...(C)}$$

Now, the general solution of (A) is
$$C^2 - 2Cx\sec\alpha + x^2 + y^2 = 0 \qquad \text{...(D)}$$

$\therefore$ The p. disc relation is
$$4x^2\sec^2\alpha - 4(x^2 + y^2) = 0$$
$$x^2(\sec^2\alpha - 1) - y^2 = 0$$
$$x^2\tan^2\alpha - y^2 = 0$$

$\therefore$ $$(x^2\tan\alpha - y)(x\tan\alpha + y) = 0 \qquad \text{...(E)}$$

So, $$x\tan\alpha - y = 0;\ x\tan\alpha + y = 0$$

i.e., the lines $y = \pm x\tan\alpha$ are singular solution envelope and $y = 0$ is tac-locus.

Now, the general solution (D) represent a family of circles all having their centres on x-axis. The circles of the system touch one and there on x-axis and $y = 0$, i.e., x-axis passes through the points of contact of non-consecutive circles which touch on x-axis the family of circles is being touched by $y = \pm x$ and which are equally inclined to the lie of centers of the circles, i.e., x-axis and pass through the origin.

4. Second and Higher Order Linear Equation with Constant Coefficients

4.1 Use the method of undetermined coefficients to find the particular solution of
$$y'' + y = \sin x + (1 + x^2)e^x$$
and hence find its general solution.

(2010 : 20 Marks)

Solution:

Given equation is : $y'' + y = \sin x + (1 + x^2)e^x$

Complementary Function (C.F.) :

The auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad y_c = c_1 e^{ix} + c_2 e^{-ix} \qquad (c_1, c_2 \text{ are constants})$$

$$= P\cos x + Q\sin x \qquad (P, Q \text{ are constants})$$

Particular Solution :

For this we will use method of undetermined coefficients.

As multiplicity of roots is 1.

Let $$y_p = x(A\cos x + B\sin x) + (Cx^2 + Dx + E)e^x \qquad ...(1)$$

$$\therefore \quad y_p' = (A\cos x + B\sin x) + x(-A\sin x + B\cos x) + (Cx^2 + Dx + E)e^x + (2Cx + D)e^x$$

$$\Rightarrow \quad y_p'' = (-A\sin x + B\cos x) + (-A\sin x + B\cos x) + x(-A\cos x - B\sin x) + (Cx^2 + Dx + E)e^x + (2Cx + D)e^x + (2Cx + D)e^x + 2Ce^x \,...(2)$$

$$\therefore \quad y_p'' + y_p = \sin x + (1 + x^2)e^x$$

Using values from (1) and (2), we get

$2(-A\sin x + B\cos x) + x(-A\cos x - B\sin x) + (Cx^2 + Dx + E)e^x + (4Cx + 2D)e^x + 2Ce^x + (Cx^2 + Dx + E)e^x + x(A\cos x + B\sin x) = \sin x + (1 + x^2)e^x$

$\Rightarrow 2(-A\sin x + B\cos x) + 2(Cx^2 + Dx + E)e^x + (4Cx + 2D)e^x + 2Ce^x = \sin x + (1 + x^2)e^x$

Comparing LHS & RHS, we get

$$B = 0,\ A = -\frac{1}{2}$$

$$2C = 1 \Rightarrow C = \frac{1}{2}$$

$$2D + 4C = 0 \Rightarrow 2D + 2 = 0 \Rightarrow D = -1$$

$$2E + 2D + 2C = 1 \Rightarrow 2E - 2 + 1 = 1 \Rightarrow E = 1$$

$$\therefore \quad y_p = \frac{-x}{2}\cos x + \left(\frac{x^2}{2} - x + 1\right)e^x$$

So, general solution of given equation is

$$y = P\cos x + Q\sin x - \frac{x}{2}\cos x + \left(\frac{x^2}{2} - x + 1\right)e^x$$

4.2 Obtain the general solution of the second order ordinary differential equation

$$y'' - 2y' + 2y = x + e^x\cos x,$$

where dashes denote derivatives w.r.t. x.

(2011 : 15 Marks)

Solution:

The given differential equation is

$$y'' + 2y' + 2y = x + e^x\cos x$$

The auxiliary equation is

$$m^2 + 2m + 2 = 0$$

$$\Rightarrow \quad m = -1 \pm i$$

$\therefore$ The complementary function is

$$y_C = e^{-x}(C_1\cos x + C_2\sin x), \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$

For particular integral,

$$y_P = \frac{x + e^x \cos x}{D^2 + 2D + 2}, D \equiv \frac{d}{dx}$$

$$= \frac{1}{D^2+2D+2}x + \frac{1}{D^2+2D+2}\cdot e^x \cos x$$

$$= \frac{1}{2\left[1+\frac{D^2+2D}{2}\right]}x + e^x \cdot \frac{1}{(D+1)^2+2(D+1)+2}\cdot \cos x$$

$$= \frac{1}{2}\left[1+\frac{D^2+2D}{2}\right]^{-1}\cdot x + e^x \cdot \frac{1}{D^2+4D+5}\cdot \cos x$$

$$= \frac{1}{2}\left[1-\frac{D^2+2D}{2}\right]\cdot x + e^x \cdot \frac{1}{-1+4D+5}\cdot \cos x$$

$$= \frac{1}{2}[x-1] + \frac{e^x}{4}\cdot\frac{1}{D+1}\cdot \cos x$$

$$= \frac{1}{2}(x-1) + \frac{e^x}{4}\frac{(D-1)}{D^2-1}\cos x$$

$$= \frac{x-1}{2} + \frac{e^x}{4}\cdot\frac{(-1)}{2}(-\sin x - \cos x)$$

$$= \frac{x-1}{2} + \frac{e^x}{8}(\sin x + \cos x)$$

$\therefore$

$$y = y_C + y_P$$

$$y = e^{-x}(C_1 \cos x + C_2 \sin x) + \frac{x-1}{2} + \frac{e^x}{8}(\sin x + \cos x)$$

is the required solution.

4.3 Find the general solution of the equation $y''' - y'' = 12x^2 + 6x$.

(2012 : 20 Marks)

Solution:

The given equation is

$$y''' - y'' = 12x^2 + 6x \quad \text{...(i)}$$

Denote $\frac{d}{dx}$ as D etc., the given equation reduces to

$$(D^3 - D^2)y = 12x^2 + 6x \quad \text{...(ii)}$$

The auxillary equation of (ii) is :

$$D^3 - D^2 = 0 \Rightarrow D = 0, 0, 1$$

4.4 Find a particular integral of

$$\frac{d^2y}{dx^2} + y = e^{x/2}\cdot \sin\frac{x\sqrt{3}}{2}$$

(2016 : 10 Marks)

Solution:

We write the given ODE as

$$(D^2+1)y = e^{\frac{x}{2}}\cdot \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^2+1}\left(e^{\frac{x}{2}}\cdot\sin\frac{\sqrt{3}x}{2}\right)$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{\left(D+\frac{1}{2}\right)^2+1}\sin\frac{\sqrt{3}}{2}x \qquad \left[\frac{1}{f(D)}e^{ax}\cdot V = e^{ax}\perp V\cdot f(D+a)\right]$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{D^2+D+\frac{5}{4}}\cdot\sin\frac{\sqrt{3}x}{2}$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{-\frac{3}{4}+D+\frac{5}{4}}\cdot\sin\frac{\sqrt{3}x}{2} \qquad \left[\because \frac{1}{f(D)}\sin(ax) = \frac{1}{f(-a^2)}\sin(ax)\right]$$

$$= e^{\frac{x}{2}}\cdot\frac{1}{D+\frac{1}{2}}\cdot\sin\frac{\sqrt{3}x}{2} = e^{\frac{x}{2}}\cdot\frac{D-\frac{1}{2}}{D^2-\frac{1}{4}}\cdot\sin\frac{\sqrt{3}x}{2}$$

$$= e^{\frac{x}{2}}\cdot\frac{D-\frac{1}{2}}{-\frac{3}{4}-\frac{1}{4}}\cdot\sin\frac{\sqrt{3}x}{2} = e^{\frac{x}{2}}\cdot\left(\frac{1}{2}-D\right)\cdot\sin\frac{\sqrt{3}x}{2}$$

$$= e^{\frac{x}{2}}\left[\frac{1}{2}\sin\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x\right]$$

$$= e^{\frac{x}{2}}\cdot\sin\left(\frac{\sqrt{3}x}{2}-\frac{\pi}{3}\right)$$

4.5 Solve : $y'' - y = x^2e^{2x}$

(2018 : 10 Marks)

Solution:

Given equation is $\qquad y'' - y = x^2e^{2x}$

or $\qquad (D^2-1)y = x^2e^{2x}$

C.F. : Auxiliary equation is $(D^2-1)y = 0$

or $\qquad m^2 = 1 \Rightarrow m = \pm 1$

$\therefore \qquad y_c = c_1e^x + c_2e^{-x}$, where c_1 and c_2 are constants.

P.I. : $\qquad y = \dfrac{1}{D^2-1}x^2e^{2x} = \dfrac{1}{D^2-1}e^{2x}\cdot x^2 = e^{2x}\dfrac{1}{(D+2)^2-1}\cdot x^2$

$$= e^{2x}\cdot\frac{1}{D^2+4D+3}\cdot x^2$$

$$= \frac{e^{2x}}{3}\cdot\frac{1}{1+\frac{D^2+4D}{3}}\cdot x^2 = \frac{e^{2x}}{3}\left[1+\frac{D^2+4D}{3}\right]^{-1}x^2$$

$$= \frac{e^{2x}}{3}\left[1-\frac{D^2}{3}-\frac{4D}{3}+\frac{(D^2+4D)^2}{9}+\ldots\right]x^2$$

$$= \frac{e^{2x}}{3}\left[x^2 - \frac{2}{3} - \frac{8x}{3} + \frac{32}{9}\right]$$

$$= \frac{e^{2x}}{3}\left[x^2 - \frac{8x}{3} + \frac{26}{9}\right]$$

$$y = y_c + y_p = c_1e^x + c_2e^{-x} + \frac{e^{2x}}{3}\left[x^2 - \frac{8x}{3} + \frac{26}{9}\right]$$

4.6 Solve $y''' - 6y'' + 12' - 8y = 12e^{2x} + 27e^{-x}$.

(2018 : 10 Marks)

Solution:

Given equation is $y''' - 6y'' + 12' - 8y = 12e^{2x} + 27e^{-x}$

or $(D^3 - 6D^2 + 12D - 8)y = 12e^{2x} + 27e^{-x}$

C.F. : Auxiliary equation is

$$m^3 - 6m^2 + 12m - 8 = 0$$

$\Rightarrow$ $(m-2)^3 = 0$

$\Rightarrow$ $m = 2, 2, 2$

$\therefore$ $y_c = 4e^{2x} + c_2xe^{2x} + c_3x^2e^x$, where c_1, c_2, c_3 are constants.

P.I. :

$$y_p = \frac{1}{(D-2)^3}(12e^{2x} + 27e^{-x})$$

$$= 12\times\frac{1}{(D-2)^3}e^{2x} + 27\times\frac{1}{(D-2)^3}e^{-x}$$

$$= 12\times\frac{x^3}{3!}e^{2x} + 27\times\frac{1}{(-3)^3}e^{-x}$$

$$= 2x^3e^{2x} - e^{-x}$$

$\therefore$ Solution is $y = y_c + y_p$

$\Rightarrow$ $y = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} + 2x^3e^{2x} - e^{-x}$

4.7 Solve : $y'' + 16y = 32 \sec 2x$

(2018 : 13 Marks)

Solution:

The given equation is

$$y'' + 16y = 32 \sec 2x$$

$\Rightarrow$ $(D^2 + 16)y = 32 \sec 2x$

C.F. : Auxiliary equation is

$$(D^2 + 16)y = 0$$

or $m^2 + 16 = 0$

$\Rightarrow$ $m = \pm 4i$

$\therefore$ $y_c = c_1e^{4in} + c_2c^{-4in} = A\cos 4x + B\sin 4x$

where A, B are constants.

P.I. : By method of variation of parameters

$$u = \cos 4x,\ v = \sin 4x$$

$$w = uv' - uv' = 4,\ R = 32 \sec 2x$$

$\therefore$ $c = \int\frac{-vR}{w}dx = \int\frac{-\sin 4x}{4}\times 32\sec 2x\,dx$

$$= -B\int \frac{2\sin 2x \cos 2x}{\cos 2x} dx$$

$$= -16 \times \frac{\cos 2x}{+2} = 8\cos 2x$$

$$D = \int \frac{uR}{w} dx = \int \frac{\cos 4x}{4} \times 32 \sec 2x\, dx$$

$$\Rightarrow \quad D = 8\int \frac{2\cos^2 2x - 1}{\cos 2x} dx = 8\int (2\cos 2x - \sec 2x) dx$$

$$= 16 \cdot \frac{\sin 2x}{2} - 8 \cdot \frac{\log(\sec 2x + \tan 2x)}{2}$$

$$= 8\sin 2x - 4\log(\sec 2x + \tan 2x)$$

$$\therefore \quad y = Y_c + C_u + D_v$$

$$= A\cos 4x + B\sin 4x + 8\cos 2x \cos 4x + 8\sin 2x \sin 4x$$
$$- 4\sin 4x \log(\sec 2x + \tan 2x)$$

4.8 Determine the complete solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x$$

(2019 : 10 Marks)

Solution:

Given : $\frac{d^2y}{dx^2} - u\frac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x$

For the equation, can be written as

$$y'' - 4y' + 4y = 3x^2 c^{2x} \sin 2x \quad \text{...(1)}$$

Auxillary equation of $(y'' - 4y' + 4)y = 0$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0, \text{ i.e., } m = 2, 2$$

$$\therefore \quad yC = (C_1 + xC_2)e^{2x} \quad \text{...(2)}$$

for, $$y_P = \frac{1}{D^2 - 4D + 4}(3x^2 \cdot e^{2x} \sin 2x)$$

$$= \frac{1}{(D-2)^2}(3x^2 e^{2x} \sin 2x)$$

$$= 3e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 3e^{2x} \frac{1}{D^2} x^2 \sin 2x \quad \text{...(2)}$$

Let $$I = \frac{1}{D^2} x^2 \sin 2x = \frac{1}{D^2} x^2 (\text{I.P.} e^{2xi})$$

$$= \text{I.P. of } \frac{1}{D^2} x^2 e^{2xi}$$

$$= \text{I.P. of } e^{2xi} \frac{1}{(D+2i)^2} x^2$$

$$= \text{I.P. of } \frac{e^{2xi}}{-4} \frac{\cdot 1}{-4\left(1+\frac{D}{2i}\right)^2} x^2$$

$$= \text{I.P. of } \frac{e^{2xi}}{-4}\left(1+\frac{D}{2i}\right)^{-2} x^2$$

$$= \text{I.P. of } \frac{-4}{4}(e^{2xi})\left(1-\frac{D}{i}+\frac{3D^2}{-4}+.....\right)x^2$$

$$= \text{I.P. of } -\frac{1}{4}(\cos 2x + i\sin 2x)\left[x^2 - \frac{2x}{i} - \frac{3}{4}x^2\right]$$

$$= \text{I.P. of } -\frac{1}{4}(\cos 2x + i\sin 2x)\left[x^2 + 2xi - \frac{3}{2}\right]$$

$$= \text{I.P. of } -\frac{x}{2}\cos 2x - \left(x^2 - \frac{3}{2}\right)\frac{1}{4}\sin 2x$$

$$\therefore \quad y_P = 3e^{2x}\left(\frac{-x}{2}\cos 2x - \frac{x^2}{4}\sin 2x + \frac{3}{8}\sin 2x\right)$$

$$y_P = -3e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right] \quad ...(3)$$

$$z = y_C + y_P$$

$$z = (C_1 + xC_2)e^{2x} + (-3)e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right]$$

$$z = (C_1 + xC_2)e^{2x} - 3e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right].$$

5. Wronskian

5.1 Find the Wronskian of the set of the function $\{3x^3, |3x^3|\}$ on the interval $[-1, 1]$ and determine whether the set is linearly dependent on $[-1, 1]$.

(2009 : 12 Marks)

Solution:

Approach : If functions are linearly dependent than the Wronskian as zero but the converse is not **quite true** as shown by these example.

$$|3x^3| = \begin{cases} 3x^3 & x \ge 0 \\ -3x^3 & x \le 0 \end{cases}$$

$$\frac{d}{dx}|3x^3| = \begin{cases} 9x^2 & x > 0 \\ -9x^2 & x < 0 \end{cases}$$

$\therefore$ From $x \in [-1, 0)$

$$\text{Wronskian} = \begin{vmatrix} 3x^3 & -3x^3 \\ 9x^2 & -9x^2 \end{vmatrix} = 0$$

and for $x \in (0, -1]$

$$\text{Wronskian} = \begin{vmatrix} 3x^3 & 3x^3 \\ 9x^2 & 9x^2 \end{vmatrix} = 0$$

i.e., Wronskian is identically zero on $[-1, 1]$.

But $\{3x^3, |3x^3|\}$ are not linearly dependant on $[-1, 1]$.

Because if they were, $\exists C_1, C_2$ both not zero such that

$$C_1 3x^3 + C_2|3x^3| = 0$$

$$\Rightarrow \quad \frac{C_1}{C_2} = \frac{-|3x^3|}{3x^3} = \begin{cases} 1 & x < 0 \\ -1 & x > 0 \end{cases}$$

So, $\quad C_1 = C_2,\ x < 0$

and $\quad C_1 = -C_2,\ x > 0$

So, such a single value of C_1 and C_2 does not exist except if $C_1 = C_2 = 0$.

$\therefore$ $3x^3$, $|3x^3|$ are linearly independent.

6. Second Order Linear Equations with Variable Coefficient

6.1 Solve the ordinary differential equation

$$x(x-1)y'' - (2x-1)y' + 2y = x^2(2x-3)$$

(2012 : 20 Marks)

Solution:

The given equation is :

$$x(x-1)y'' - (2x-1)y' + 2y = x^2(2x-3) \quad \ldots(i)$$

$$\Rightarrow \quad y'' - \frac{(2x-1)}{x(x-1)}y' + \frac{2y}{x(x-1)} = \frac{x^2(2x-3)}{x(x-1)}$$

which is of the form of

$$y'' + Py' + Qy = R$$

Here, $\quad P = \dfrac{-(2x-1)}{x(x-1)},\ Q = \dfrac{2}{x(x-1)}$

As $\quad 2 + 2Px + Qx^2 = 2 + 2x\cdot\dfrac{[-(2x-1)]}{x(x-1)} + x^2\cdot\dfrac{2}{x(x-1)} = 0$

$\Rightarrow 4 = x^2$ is a part of the complementary function.

Also $\quad V =$ Second part of C.F.

$$= u\int \frac{e^{-\int P dx}}{u^2}dx$$

$$= x^2\int \frac{1}{x^4} e^{\int \frac{2x-1}{x(x-1)}dx} dx = x^2\int \frac{1}{x^4}\cdot e^{\int\left(\frac{1}{x}+\frac{1}{x-1}\right)dx} dx$$

$$= x^2\int \frac{1}{x^4}\cdot(x^2 - x)dx = x^2\int\left(\frac{1}{x^2} - \frac{1}{x^3}\right)dx$$

$$= x^2\left(\frac{x^{-1}}{-1} - \frac{x^{-2}}{-2}\right)$$

$$= -x + 2$$

So, the complementary function is

$y_C = C_1 + C_2x + C_3e^x$ whre C_1, C_2 and C_3 are arbitrary constants.

Particular integral is

$$y_P = \frac{1}{(D^3 - D^2)}(12x^2 + 6x)$$

$$= \frac{-1}{D^2(1-D)}(12x^2+6x)$$

$$= \frac{-1}{D^2}(1-D)^{-1}(12x^2+6x)$$

$$= \frac{-1}{D^2}(1+D+D^2+.....)(12x^2+6x)$$

$$= \frac{-1}{D^2}(12x^2+30x+30)$$

$$= \frac{-1}{D}(4x^3+15x^2+30x)$$

$$= -(x^4+5x^3+15x^2)$$

∴ The required general solution is :

$$y = y_C + y_P$$
$$= C_1 + C_2x + C_3e^x - (x^4+5x^3+15x^2)$$

Particular Integral,

$$\text{P.I.} = u\int \frac{e^{-\int P dx}}{u^2}\left(\int R_4 e^{-\int P dx} dx\right)dx$$

$$= x^2\int \frac{(x^2-x)}{x^4}\left(\int \frac{x(2x-3)}{(x-1)}\cdot x(x-1)dx\right)dx \quad \left[\because e^{-\int P dx} = x(x-1)\right]$$

$$= x^2\int \frac{x^2-x}{x^4}\left[\int x^2(2x-3)dx\right]dx$$

$$= x^2\int \frac{x-1}{x^3}\left[\int (2x^3-3x^2)dx\right]dx$$

$$= \frac{x^2}{2}\int (x-1)(x-2)dx$$

$$= \frac{x^2}{2}\int (x^2-3x+2)dx = \frac{x^2}{2}\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)$$

∴ Complete solution of (i) is

$$y = C_1x^2 + C_2(2-x) + \frac{x^2}{2}\left(\frac{x^3}{3}-\frac{3x^2}{2}+2x\right)$$

6.2 Find the general solution of the equation :

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \ln x \sin(\ln x)$$

(2013 : 15 Marks)

Solution:

We have

$$(x^2D^2 + xD + 1)y = \ln x \sin(\ln x)$$

This is in the Cauchy-Euler form.

Let

$$z = \ln x$$

⇒

$$\frac{dz}{dx} = \frac{1}{x} \Rightarrow x\frac{d}{dx} = \frac{d}{dz}$$

$$\Rightarrow \quad xD = D_1 \text{ where } \frac{d}{dz} = D_1$$

Similarly, $x^2D^2 = D_1(D_1 - 1)$

$$\therefore \quad (D_1(D_1 - 1) + D_1 + 1)y = z \sin z$$

$$\Rightarrow \quad (D_1^2 + 1)y = z \sin z$$

Auxilliary equation is $m^2 + 1 = 0$

$$\Rightarrow \quad m = \pm i$$

$\therefore$ Complementary function (C.F.)

$$= C_1 \cos x + C_2 \sin x$$

$$P.I. = \frac{1}{(D^2+1)} z \sin z$$

$$= \text{Real part } \frac{1}{D^2+1} ze^{iz}$$

Now,

$$\frac{1}{D^2+1} ze^{iz} = e^{iz} \frac{1}{(D+i)^2+1} z = e^{iz} \frac{1}{D^2+2iD} z$$

$$= \frac{e^{iz}}{2i} \frac{1}{D}\left(1+\frac{D}{2i}\right)^{-1} z = \frac{e^{iz}}{2i} \cdot \frac{1}{D}\left(1-\frac{D}{2i}+\frac{D^2}{4i^2}\right) z$$

$$= \frac{e^{iz}}{2i} \frac{1}{D}\left(z-\frac{1}{2i}\right) = \frac{e^{iz}}{2i}\left(\frac{z^2}{2}-\frac{z}{2i}\right)$$

$$= \frac{e^{iz}}{4}(-iz^2+z)$$

$$\therefore \quad \text{R.P. of } \frac{1}{D^2+1} ze^{iz} = \text{R.P. of } \left\{\frac{e^{iz}}{4}(-iz^2+z)\right\}$$

$$= \frac{1}{4}(z\cos z + z^2 \sin z)$$

$$\therefore \quad P.I. = \frac{1}{4}(z\cos z + z^2 \sin z)$$

$$y = C_1 \cos z + C_2 \sin z + \frac{z}{4}\cos z + \frac{z^2}{4}\sin z$$

$$= \left(C_1 + \frac{\ln x}{4}\right)\cos(\ln x) + \left\{C_2 + \frac{(\ln x)^2}{4}\right\}\sin(\ln x)$$

6.3 Solve the differential equation :

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65 \cos(\log_e x)$$

(2014 : 20 Marks)

Solution:

Given that :

$$(x^3D^3 + 3x^2D^2 + xD + 8)y = 65 \cos \log x \qquad \text{...(i)}$$

Let $x = e^z$

$$\log x = z$$

$$\Rightarrow$$

and let $D_1 = \dfrac{d}{dr}$

Then

$$2D = D_1 \text{ and } x^2D^2 = D_1(D_1-1)$$
$$D^3 = D_1(D_1-1)(D_1-2)$$

∴ (i), we have

$(D_1(D_1-1)(D_1-2) + 3(D_1)(D_1-1) + D_1 + 8)y = 65 \cos z$

$$\Rightarrow \quad (D_1^3 + 8)y = 65 \cos z \quad \text{...(ii)}$$

Auxiliary equation of (ii) is

$$D_1^3 + 8 = 0$$
$$\Rightarrow \quad (D_1+2)(D_1 - 2D_1 + 4) = 0$$
$$\Rightarrow \quad D_1 = -2,\ 1 \pm \sqrt{3}i$$
$$\therefore \quad \text{C.F.} = y_C = C_1e^{-2z} + C_2e^z(C_2\cos\sqrt{3}z + C_3\sin\sqrt{3}z)$$

$$\text{P.I.} = \frac{1}{(D_1^3+8)}(65\cos z)$$
$$= \frac{65}{-D_1+8}\cos z$$
$$= \frac{(D_1+8)65}{-D_1^2+64}\cos z$$
$$= \frac{65}{65}(D_1+8)\cos z$$
$$= (D_1+8)\cos z$$
$$= -\sin z + 8\cos z$$

$$\therefore \quad y = y_C + y_P$$
$$y = C_1e^{-2z} + e^z\left(C_2\cos\sqrt{3}z + C_3\sin\sqrt{3}z\right) - \sin z + 8\cos z$$
$$= C_1x^{-2} + x(C_2\cos\sqrt{3}\log x + C_3\sin\sqrt{3}\log x) - \sin(\log x) + 8\cos(\log x)$$

which is the required solution.

6.4 Solve : $x^4\dfrac{d^4y}{dx^4} + 6x^3\dfrac{d^3y}{dx^3} + 4x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\cos(\log x)$.

(2015 : 13 Marks)

Solution:

Putting $x = e^z$, equation becomes

$[D(D-1)(D-2)(D-3) + 6D(D-1)(D-2) + 4D(D-1) - 2D - 4] = e^{2z} + 2\cos z$

$[D(D-1)(D-2)(D-3) + 6D(D-1)(D-2) + 2(D-2)(D+1)] = e^{2z} + 2\cos z$

$$\Rightarrow \quad (D^2+1)(D^2-4)y = e^z + 2\cos z$$

For homogeneous part,

$$y_H = C_1e^{2z} + C_2e^{-2z} + C_3\sin z + C_4\cos z$$

Particular Integral,

$$y_P = \frac{1}{(D^2+1)(D^2-4)}e^{2z} + \frac{1}{(D^2+1)(D^2-4)}2\cos z$$
$$= \frac{1}{5}\cdot\frac{1}{D^2-4}\cdot e^{2z} - \frac{2}{5}\frac{1}{(D^2+1)}\cos z$$
$$= \frac{z\cdot e^{2z}}{20} - \frac{z}{5}\sin z$$

∴ Complete solution : $$y = y_H + y_P$$

$$y = C_1e^{2z} + C_2e^{-2z} + C_3\sin z + C_4\cos z + \frac{z\cdot e^{2z}}{20} - \frac{z\sin z}{5}$$

$$y(x) = C_1x^2 + \frac{C_2}{x^2} + C_3\sin(\ln x) + C_4\cos(\ln x) + x^2\frac{\ln x}{20} - \frac{\ln x}{5}\sin\ln x$$

6.5 Find the general solution of the equation

$$x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$$

(2016 : 15 Marks)

Solution:

Multiplying both sides by x, we get

$$x^3\cdot\frac{d^3y}{dx^3} - 4x^2\cdot\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} = 4x$$

Using the operator, $$D = \frac{d}{dx}$$

$$[D(D-1)(D-2) - 4D(D-1) + 6D]y = 4x$$

$$[D^3 - 3D^2 + 2D - 4D2 + 4D + 6D]y = 4x$$

$$\Rightarrow \quad (D^3 - 7D^2 + 12D)y = 4x$$

Auxiliary eqn. : $$m^3 - 7m^2 + 12m = 0$$

i.e., $$m(m^2 - 7m + 12) = 0$$

i.e., $$m(m-3)(m-4) = 0 \Rightarrow m = 0, 3, 4$$

∴ $$\text{C.F.} = C_1e^{0x} + C_2e^{3x} + C_3e^{4x}$$

$$P.I. = \frac{1}{f(D)}\cdot Q = \frac{1}{D(D^2-7D+12)}\cdot 4x$$

$$= \frac{1}{12D\left[1+\frac{D^2}{12}-\frac{7}{12}D\right]}\cdot 4x = \frac{1}{12}\left[1+\frac{D^2}{12}-\frac{7D}{12}\right]^{-1}2x^2$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2}{12}-\frac{7D}{12}\right)+\ldots\right]x^2$$

$$= \frac{1}{6}\left[x^2 - \frac{1}{12}\cdot 2 + \frac{7}{12}\cdot 2x\right] \quad \text{(Higher derivative becomes zero)}$$

$$= \frac{1}{36}[6x^2 + 7x - 1]$$

∴ General Solution = CF + PI

$$y = C_1 + C_2e^{3x} + C_3e^{4x} + \frac{1}{36}(6x^2 + 7x - 1)$$

6.6 Solve the DE : $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$

(2017 : 8 Marks)

Solution:

Multiplying both sides by x,

$$x^2 \cdot \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 4x^4y = 8x^4 \sin(x^2) \quad \text{...(i)}$$

Let $x^2 = z \Rightarrow$
$$\frac{dz}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = 2x\frac{dy}{dz}$$
$$\Rightarrow \quad \frac{dy}{dx} = 2x\frac{dy}{dz} \quad \text{...(ii)}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right)$$
$$= 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$
$$\therefore \quad \frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2} \quad \text{...(iii)}$$

Using (ii) and (iii) in (i)

$$x^2\left[2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right] - x\left(2x\frac{dy}{dz}\right) - 4x^4y = 8x^4 \sin x^2$$
$$4x^4\left(\frac{d^2y}{dz^2} - y\right) = 8x^4 \sin(z^2)$$

i.e.,
$$\frac{d^2y}{dz^2} - y = 2\sin(z)$$

i.e.,
$$(D^2 - 1)y = 2\sin z,\ D^2 = \frac{d}{dz^2}$$
$$y_C = C_1e^{-x} + C_2e^{x} \quad \text{(Homogeneous part)}$$
$$y_P = \frac{1}{D^2-1}(2\sin z) = \frac{1}{-2}(2\sin z) = -\sin z$$
$$y = y_C + y_P$$

6.7 Solve : $(1 + x)^2y'' + (1 + x)y' + y = 4\cos(\log(1 + x))$.

(2018 : 13 Marks)

Solution:

Given the equation is
$$(1 + x)^2y'' + (1 + x)y' + y = 4\cos\log(1 + x)$$
Let
$$1 + x = e^z$$
The equation becomes
$$(D(D - 1) + D + 1)y = 4\cos z$$
$$\Rightarrow \quad (D^2 + 1)y = 4\cos z$$
C.F. : The auxiliary equation is
$$m^2 + 1 = 0 \Rightarrow m = \pm i$$
$\therefore$ $y_c = A\cos z + B\sin z$, where A and B are constant.

P.I. :
$$y_p = \frac{1}{D^2+1}\times 4\cos z = 4\times\frac{1}{D^2+1}\cos z$$

$$= \frac{4 \times z \sin z}{2} = 2z \sin z$$

$$\therefore \quad y = y_c + y_p = A\cos z + B\sin z + 2z\sin z$$

$$= A\cos\log(1+x) + B\sin\log(1+x) + 2\log(1+x)\sin(\log(1+x)$$

6.8 Solve the differential equation :

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

(2019 : 10 Marks)

Solution:

Given that :

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x \quad ...(1)$$

Clearly it is in the form of

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

where

$$P(x) = 3\sin x - \cot x$$

$$Q(x) = 2\sin^2 x$$

$$R(x) = e^{-\cos x}\sin^2 x$$

Let us solve (1) by changing the independent variable 'x' to the new independent variable z, where z is a function of 'x'.

Then the transformed equation is

$$\frac{\frac{d^2y}{dx^2} + \frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dz}\right)^2}\frac{dy}{dx} + \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} = \frac{R(x)}{\left(\frac{dz}{dx}\right)^2}$$

Let

$$\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2} = \text{Constant} \quad ...(2)$$

$$\Rightarrow \quad \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2} = 2 \text{ (say)} \Rightarrow \frac{dz}{dx} = -\sin x$$

$$\Rightarrow \quad z = \cos x$$

$\therefore$ from (2)

$$\frac{d^2y}{dz^2} + \frac{-\cos x + (3\sin x - \cot x)}{\sin^2 x}\left(\frac{dy}{dx}\right) + 2y = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + \frac{-\cos x - 3\sin^2 x + \cos x}{\sin^2 x}\left(\frac{dy}{dx}\right) + 2y = e^{-\cos x}$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} - 3\frac{dy}{dx} + 2y = e^{-z}$$

$$(D^2 - 3D + 2)y = e^{-z};\ D = \frac{d}{dz} \quad ...(3)$$

Its A.E. is
$$m^2 - 3m + 2 = 0$$
$$(m-2)(m-1) = 0$$
$$m = 1, 2$$
$\therefore$
$$y_C(z) = ae^z + be^{2z} \quad \text{...(4)}$$
and we have,
$$y_P(z) = \frac{1}{D^2 - 3D + 2} = \frac{1}{1+3+2}e^{-z}$$
$$y_P(z) = \frac{1}{6}e^{-z}$$
$\therefore$ The general solution of (1) is given by $y(z)$
$$= y_e(z) + y_P(z)$$
$$y(z) = ae^z + be^{2z} + \frac{1}{6}e^{-z}$$
$$y(k) = ae^{\cos x} + be^{2\cos x} + \frac{1}{6}e^{-\cos x}$$
which is the required general solution of the given equation.

7. Determination of Complete Solution when one solution is known and Method of Variation of Parameter

7.1 Using the method of variation of parameters, solve the second order differential equation
$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$
(2011 : 15 Marks)

Solution:

The given equation is
$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$
$\Rightarrow$
$$(D^2 + 4)y = \tan 2x, D \equiv \frac{d}{dx}$$
Auxiliary equation is $m^2 + y = 0$
$\Rightarrow$
$$m = \pm 2i$$
$\therefore$ Complementary function is
$$y_C = C_1 \cos 2x + C_2 \sin 2x, \text{ where } C_1 \text{ and } C_2 \text{ are arbitrary constants.}$$
Let
$$y_C = A\cos 2x + B\sin 2x$$
be particular integral of (i) where A and B are functions of x.

Then,
$$u(x) = \cos 2x,\ v(x) = \sin 2x \text{ and } R(x) = \tan 2x$$
Now, Wronskian of u and v
$$= w(u, v) = uv' - vu'$$
$$= \cos 2x \cdot 2\cos 2x - \sin 2x(-2\sin 2x)$$
$$= 2(\cos^2 2x + \sin^2 2x) = 2$$
$\therefore$
$$A = -\int \frac{vR}{w(u,v)}dx = -\int \frac{\sin 2x \cdot \tan 2x}{2}$$
$$= -\frac{1}{2}\int\left(\frac{1-\cos^2 2x}{\cos 2x}\right)dx$$

$$= -\frac{1}{2}\int(\sec 2x - \cos 2x)dx$$

$$= -\frac{1}{2}\left[\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right]$$

$$= -\frac{1}{4}\left[\log|\sec 2x + \tan 2x| - \sin 2x\right]$$

and

$$B = \int \frac{4R}{w(u,v)}dx$$

$$= \frac{1}{2}\int \cos 2x \cdot \tan 2x dx = \frac{1}{2}\int \sin 2x dx$$

$$= -\frac{1}{4}\cos 2x$$

$\therefore$

$$y_P = -\frac{1}{4}\left[\log|\sec 2x + \tan 2x| - \sin 2x\right]\cos 2x - \frac{1}{4}\cos 2x \cdot \sin 2x$$

$$= -\frac{1}{4}(\log|\sec 2x + \tan 2x|)\cos 2x$$

$\therefore$ The required general solution is

$$y = y_C + y_P$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4}(\log|\sec 2x + \tan 2x|)\cos 2x$$

7.2 Using the method of variation of parameter solve the differential equation :

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

(2013 : 10 Marks)

Solution:

Homogeneous part is

$$\frac{d^2y}{dx^2} + a^2y = 0 \Rightarrow (D^2 + a^2)y = 0$$

Auxilliary equation : $m^2 + a^2 = 0 \Rightarrow m = \pm ai$

$\therefore$

$$u = \cos ax$$

$$v = \sin ax$$

are two independent solutions of homogeneous equation.

Let $y = Au + Bv$ complete the solution.

Then,

$$A = \int \frac{-vR}{W}$$

$$B = \int \frac{uR}{W}$$

where

$$W = \text{Wronskian } (u, v)$$

$$= \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$\therefore$

$$A = -\int \frac{\sin ax \cdot \sec ax}{a}dx = -\frac{1}{a}\int \tan ax\, dx$$

$$= \frac{1}{a^2}\int \frac{-a\sin ax}{\cos ax}dx$$

$$= \frac{1}{a^2}\ln(\cos ax) + C_1$$

$$B = \int \frac{\cos ax \sec ax}{a} dx = \frac{x}{a} + C_2$$

$\therefore$

$$y = C_1 \cos ax + \frac{1}{a^2}\cos ax \times \ln(\cos(ax)) + C_2 \sin ax + \frac{x}{a}\sin ax$$

7.3 Solve by the method of variation of parameters :

$$\frac{dy}{dx} - 5y = \sin x$$

(2014 : 10 Marks)

Solution:

Given that

$$\frac{dy}{dx} - 5y = \sin x \quad ...(i)$$

Differentiating (i) w.r.t. x, we get

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = \cos x$$

i.e.,

$$(D^2 - 5D)y = \cos x \quad ...(ii)$$

Now consider auxiliary equation of (ii)

$$D(D-5) = 0$$

$\Rightarrow$

$$D = 0, 5$$

$\therefore$ C.F. of (ii) is

$$y_C = C_1 + C_2 e^{5x}$$

Let $y_p = Av + Bv$ be a particular integral of (ii) where A and B are functions of x and $u = 1$, $v = e^{5x}$

Now

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} = 5e^{5x} \neq 0$$

$\therefore$

$$A = -\int \frac{vR}{W} = -\int \frac{e^{5x} \cdot \cos x dx}{5e^{5x}} = -\frac{1}{5}\int \cos x dx = -\frac{1}{5}\sin x$$

$$B = \int \frac{uR}{W} = \int \frac{1.\cos x}{5e^{5x}} dx = \frac{1}{5}\int e^{-5x} \cos x dx$$

$$= \frac{1}{5}\frac{e^{-5x}}{25+1}[-5\cos x + \sin x]$$

$$= \frac{1}{5}\frac{e^{-5x}}{26}[-5\cos x + \sin x]$$

$\therefore$ The general solution of (ii) is

$$y = y_c + y_p$$

i.e.,

$$y = C_1 + C_2 e^{5x} - \frac{1}{5}\sin x \cdot 1 + \frac{1}{5}\frac{e^{-5x}}{26}[-5\cos x + \sin x]$$

$$y = C_1 + C_2 e^{5x} - \frac{1}{26}[\cos x + 5\sin x]$$

which is the required solution of given equation.

7.4 Solve the following differential equations :

$$x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^{2x},$$

when e^x is a solution to its corresponding homogeneous differential equation.

(2014 : 15 Marks)

Solution:

Given equation is

$$xy'' - 2(x+1)y' + (x+2)y = (x-2)e^{2x} \quad \text{...(i)}$$

It is given that e^x is a solution to its corresponding homogeneous differential equation, i.e., $y = u = e^x$ is the part of C.F. of (i).

Let the general solution of (i) is $y = uv$. Then v is given by

$$\frac{d^2v}{dx^2} + \left(p + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

where $$P = \frac{-2(1+x)}{x},\ Q = \frac{x+2}{x},\ R = \frac{(x-2)}{x}e^{2x}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left[-\frac{2}{x}(e+x) + \frac{2}{e^x}(e^x)\right]\frac{dv}{dx} = \left(\frac{x-2}{x}\right)\frac{e^{2x}}{e^x}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(-\frac{2}{x} - 2 + 2\right)\frac{dv}{dx} = \frac{(x-2)}{x}e^x \quad \text{...(ii)}$$

Let $\frac{dv}{dx} = q \Rightarrow$ $$\frac{d^2v}{dx^2} = \frac{dq}{dx}$$

$\therefore$ from (ii), we have

$$\frac{dq}{dx} + \left(-\frac{2}{x}\right)q = \frac{(x-2)}{x}e^x$$

which lies linear in q.

$$\text{I.F.} = e^{-\int\frac{2}{x}dx} = e^{-(2\log x)} = e^{\log x^{-2}} = x^{-2}$$

$$= \frac{1}{x^2}$$

$$\therefore \quad q(\text{IF}) = \int\left(\frac{x-2}{x}\right)e^x \cdot \text{I.F.} + C_1$$

$$q\left(\frac{1}{x^2}\right) = \int\left(\frac{x-2}{x}\right)e^x \cdot \frac{1}{x^2}dx + C_1$$

$$= \int x^{-2}e^x dx - 2\int x^{-3}e^x dx + C_1$$

$$= x^{-2}e^x - \int(-2)x^{-3}e^x - 2\int x^{-3}e^x dx + C_1$$

$$\frac{q}{x^2} = x^{-2}e^x + 2\int x^{-3}e^x dx - 2\int x^{-3}e^x dx + C_1$$

$$q = e^x + C_1x^2$$

$$\Rightarrow \quad dv = (e^x + C_1x^2)dx$$

$$\Rightarrow \quad v = e^x + \frac{1}{3}C_1x^3 + C_2$$

$$\therefore \quad y = uv = e^x\left(e^x + \frac{1}{3}C_1x^3 + C_2\right)$$ which is the required solution.

7.5 Using the method of variation of parameters, solve the DE

$$(D^2 + 2D + 1)y = e^{-x} \log(x)$$

(2016 : 15 Marks)

Solution:

For complimentary function (CF)

Auxiliary equation:

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$$

$$m = -1, -1$$

$\therefore$ $$\text{C.F.} = (C_1 + C_2 x)e^{-x} = C_1 e^{-x} + C_2 x e^{-x}$$

Taking e^{-x} and $x \cdot e^{-x}$ as y_1 and y_2.

For Particular Integral (PI)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^1 & y_2^1 \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix}$$

$$= -xe^{-2x} + e^{-2x} + xe^{-2x} = e^{-2x}$$

$$\text{PI} = Ay_1 + By_2,\ A, B \text{ are functions of } x.$$

$$= y_1\left(-\int \frac{y_2 \cdot Q}{W} dx\right) + y_2\left(\int \frac{y_1 \cdot Q}{W} dx\right)$$

$$= e^{-x}\int \frac{x \cdot e^{-x} \cdot e^{-x}\log x}{e^{-2x}} dx + xe^{-x}\int \frac{e^{-x} \cdot e^{-x}\log x}{e^{-2x}} dx$$

$$= e^{-x}\int x\log x dx + xe^{-x}\int \log x dx$$

$$= e^{-x}\left[\frac{x^2}{2}\log x - \frac{x^2}{4}\right] + xe^{-x}[x\log x - x]$$

$\therefore$ $$y = CF + PI$$

$$= e^{-x}\left[C_1 + C_2 x + \frac{3x^2}{2}\log x - \frac{5x^2}{4}\right]$$

7.6 Solve the following DE using method of variation of parameters :

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 44 - 76x - 48x^2$$

(2017 : 8 Marks)

Solution:

Homogeneous Part :

$$(D^2 - D - 2)y = 0$$

$$(D-2)(D+1)y = 0$$

$$y_C = C_1 e^{-x} + C_2 e^{2x}$$

Particular Integral,

$$y_P = Au + Bv$$ (A, B are function of x)

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^{x}$$

$$A = -\int \frac{vR}{W} dx = -1\int \frac{e^{2x}(44 - 76x - 48x^2)}{3e^{x}} dx$$

$$= -\frac{1}{3}\int e^x(44-76x-48x^2)dx$$

$$= -\frac{1}{3}[44e^x - 76(xe^x - e^x) - 48(x^2e^x - 2xe^x + 2e^x)]$$

$$= -\frac{1}{3}[-48x^2e^x + 20xe^x + 24e^x]$$

$$B = \int \frac{uR}{W}dx = \int \frac{e^{-x}}{3e^x}(44-76x-48x^2)dx$$

$$= \frac{1}{3}\left[-22e^{-2x} + 76\left[\frac{xe^{-2x}}{2} + \frac{e^{-2x}}{2}\right] + 48\left[\frac{x^2e^{-2x}}{2} + \frac{xe^{-2x}}{2} + \frac{e^{-2x}}{2}\right]\right]$$

$$= \frac{1}{3}[24x^2e^{-2x} + 62xe^{-2x} + 21e^{-2x}]$$

$$y_P = e^{-x}A + e^{2x}B = 24x^2 + 14x - 1$$

$$\therefore \quad y = y_C + y_P = C_1e^{-x} + C_2e^{2x} + 24x^2 + 14x - 1$$

7.7 Find the linearly independent solutions of the corresponding homogeneous differential equation of the equation $x^2y'' - 2xy' + 2y = x^3 \sin x$ and then find the general solution of the given equation by the method of variation of parameters.

(2019 : 15 Marks)

Solution:

Given :

$$x^2y'' - 2xy' + 2y = x^3 \sin x \quad \text{...(1)}$$

Divide equation (1) by x^2

$$y'' - \frac{2y'}{x} + \frac{2}{x^2}y = x \sin x \quad [\because R = x \sin x]$$

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0 \quad \text{(homogeneous equation)}$$

$$x^2y'' - 2xy' + 2y = 0$$

$$(x^2y'' - 2xy' + 2)y = 0$$

$$(x^2D^2 - 2xD + 2)y = 0 \quad \left[\because D = \frac{d}{dx}\right]$$

Put $x = e^z$ and $D_1 = \frac{d}{dz}$

Then, we get

$$(D_1(D_1 - 1) - 2(D_1) + 2)y = 0$$

$$(D_1^2 - 3D_1 + 2)y = 0$$

$$(D_1 - 2)(D_1 - 1) = 0$$

$$D = 2, 1$$

$$y_C = C_1e^z + C_2e^{2z}$$

In terms of x, $y_C = C_1x + C_2x^2$ $[\because x = e^z]$

where $u = x;\ v = x^2$

$$u = 1;\ V = 2x$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

For
$$P.I. = Au + Bv$$

where
$$A = -\int \frac{vR}{W}dx \text{ and } B = \int \frac{uR}{W}dz$$

$$A = -\int \frac{x^2 \cdot x \sin x}{x^2}dx$$

$$A = -(-x\cos x + \sin x) = x\cos x - \sin x + G$$

$$B = \int \frac{x \cdot x \sin x}{x^2}dx = \int \sin x\, dx$$

$$B = -\cos x + C_2$$

$$P.I. = x[x\cos x - \sin x] + x^2[-\cos x]$$

$$P.I. = x^2 \cos x - x\sin x + \left(\frac{-x^2}{\cos x}\right)$$

$$P.I. = -x \sin x$$

$$z = y_C + y_P$$

$$z = C_2 x + C_2 x^2 - x\sin x \text{ (required general solution)}$$

8. Laplace and Inverse Laplace Transformation and Properties

8.1 Find the inverse Laplace transform of $F(s) = \ln\left(\frac{S+1}{S+S}\right)$.

(2009 : 20 Marks)

Solution:

Approach : Use the differentiation property of inverse laplace transform.

Given :
$$F(s) = \ln\left(\frac{S+1}{S+S}\right)$$

By differentiation property we have

$$L^{-1}[F(S)] = \frac{1}{t}L^{-1}[F^1(S)]$$

$$\therefore \quad L^{-1}\left[\ln\left(\frac{S+1}{S+S}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{d}{ds}\ln\left(\frac{S+1}{S+S}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{d}{ds}\ln(S+1) - \ln(S+S)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{1}{S+1} - \frac{1}{S+S}\right]$$

$$= \frac{1}{t}\left\{L^{-1}\left(\frac{1}{S+1}\right) - L^{-1}\left(\frac{1}{S+S}\right)\right\} \text{ (Linearity)}$$

$$= \frac{1}{t}\{e^{-t} - e^{-5t}\}.$$

8.2 Use Laplace transform method to solve the following initial value problem :

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t, x(0) = 2 \text{ and } \left.\frac{dx}{dt}\right|_{t=0} = -1$$

(2011 : 15 Marks)

Solution:

The given initial value problem is

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t, x(0) = 2 \quad \text{...(i)}$$

$$\left(\frac{dx}{dt}\right)_{t=0} = -1$$

Taking Laplace transform of both sides of (i), we get

$$L\left(\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x\right) = L[e^t]$$

$$\Rightarrow \quad L\{x''(t)\} - 2L\{x'(t)\} + L\{x(t)\} = L\{e^t\}$$

$$\Rightarrow \quad S^2L\{x(t)\} - Sx(0) - x'(0) - 2[SL\{x(t)\} - x(0)] + L\{x(t)\} = \frac{1}{S}$$

$$\Rightarrow \quad L\{(t)\}[S^2 - 2S + 1] - S(2) - (-1) + 2(2) = \frac{1}{S}$$

$$\Rightarrow \quad L\{x(t)\}(S-1)^2 = \frac{1}{S} + 2S - 5$$

$$= \frac{1 + 2S^2 - 5S}{S}$$

$$\Rightarrow \quad L\{x(t)\} = \frac{1 + 2S^2 - 5S}{S(S-1)^2}$$

$$= \frac{1}{S-1} - \frac{2}{(S-1)^2} + \frac{1}{S} \quad \text{...(ii) (By using Partial Fractions)}$$

Taking inverse Laplace of (ii), we get

$$x(t) = L^{-1}\left\{\frac{1}{S-1} - \frac{2}{(S-1)^2} + \frac{1}{S}\right\}$$

$$\Rightarrow \quad x(t) = e^t - 2e^t t + 1$$

8.3 Using Laplace transforms, solve the initial value problem

$$y'' + 2y' + y = e^{-t}, y(0) = -1, y'(0) = 1$$

(2012 : 12 Marks)

Solution:

Given :

$$y'' + 2y' + y = e^{-t} \quad \text{...(i)}$$

$$y(0) = -1, y'(0) = 1$$

Taking Laplace transform of both sides of (i),

$$L(y''(t)) + 2L(y'(t)) + L(y) = L(e^{-t})$$

$$\Rightarrow p^2L(y) - py(0) - y'(0) + 2(pL(y) - y(0)) + L(y) = \frac{1}{p+1}$$

$$\Rightarrow L(y)(p^2 + 2p + 1) - p(-1) - 1 - 2(-1) = \frac{1}{p+1} \quad (\because y(0) = -1, y'(0) = 1)$$

$$\Rightarrow \quad L(y)(p+1)^2 + (p+1) = \frac{1}{p+1}$$

$\Rightarrow$

$$L(y) = \frac{1}{(p+1)^3} - \frac{1}{(p+1)} \quad \text{...(ii)}$$

Taking inverse Laplace transform on both sides of (ii), we have

$$y = L^{-1}\left[\frac{1}{(p+1)^3} - \frac{1}{(p+1)}\right]$$

$\Rightarrow$

$$y = e^{-t} \cdot \frac{t^2}{2} - e^{-t}$$

8.4 By using Laplace transform method solve the differential equation : $(D^2 + n^2)x = a \sin(nt + \alpha)$, $D^2 = \frac{d^2}{dt^2}$ subject to the initial conditions $x = 0$ and $\frac{dx}{dt} = 0$ at $t = 0$ in which a, n and α are constants.

(2013 : 15 Marks)

Solution:

$$(D^2 + n^2)x = a \sin(nt + \alpha)$$

Taking Laplace transform on both sides

$$L\left(\frac{d^2x}{dt^2}\right) + n^2 L(x) = aL[\sin nt \cos \alpha + \cos nt \sin \alpha]$$

$\Rightarrow$

$$\delta^2 L(x) - \delta x(0) - x'(0) + n^2 L(x) = a\left[\frac{n}{\delta^2 + n^2}\cos\alpha + \frac{s}{\delta^2 + n^2}\sin\alpha\right]$$

$\Rightarrow$

$$L(x) = a\left[\frac{n}{(\delta^2 + n^2)^2}\cos\alpha + \frac{s}{(\delta^2 + n^2)^2}\sin\alpha\right]$$

$\therefore$

$$x = a\left\{L^{-1}\left[\frac{n}{(\delta^2 + n^2)^2}\right]\cos\alpha + L^{-1}\left[\frac{s}{(\delta^2 + n^2)^2}\right]\sin\alpha\right\}$$

Now

$$\frac{d}{ds}\frac{1}{(\delta^2 + n^2)} = -\frac{2s}{(\delta^2 + n^2)^2}$$

$\therefore$

$$L^{-1}\left(-\frac{2s}{(\delta^2 + n^2)^2}\right) = (-1)tL^{-1}\left(\frac{1}{\delta^2 + n^2}\right)$$

$$= \frac{-t}{n} \cdot \sin nt$$

$\Rightarrow$

$$L^{-1}\left(\frac{s}{(\delta^2 + n^2)^2}\right) = \frac{t}{2n}\sin nt$$

Let

$$f(s) = \frac{1}{s^2 + n^2};\ g(s) = \frac{1}{s^2 + n^2}$$

$$F(t) = L^{-1}(f(s)) = \frac{1}{n}\sin nt$$

$$G(t) = L^{-1}(g(s)) = \frac{1}{n}\sin nt$$

$$L^{-1}(f(s)g(s)) = F*G = \int_0^t F(u)G(t-u)du$$

$$= \frac{1}{n^2}\int_0^t \sin nu \sin n(t-u)du$$

$$= \frac{1}{2n^2}\int_0^t (\cos n(t-2u)-\cos nt)du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{-\sin nt}{-2n} + \frac{\sin nt}{2n} - t\cos nt\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right]$$

$$\therefore \quad x = a\left\{\frac{1}{2n}\left[\frac{\sin nt}{n} - t\cos nt\right]\cos\alpha + \left(\frac{t}{2n}\sin nt \sin\alpha\right)\right\}$$

$$= a\left\{\frac{1}{2n^2}\sin nt\cos\alpha - \frac{t}{2n}\cos(nt+\alpha)\right\}$$

$$= \frac{a}{2n^2}\{\sin nt\cos\alpha - nt\cos(nt+\alpha)\}$$

8.5 Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t,\ y(0) = 0,\ y'(0) = 0$$

by using Laplace-transform.

(2014 : 20 Marks)

Solution:

Given equation is

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t$$

$$\Rightarrow \quad y'' + y = 8e^{-2t}\sin t \qquad \text{...(i)}$$

Taking Laplace transform of both sides of (i), we get

$$L(y'') + L(y) = 8L(e^{-2t}\sin t)$$

$$\Rightarrow \quad P^2L\{y(t)\} - Py(0) - y'(0) + L\{y(t) = \frac{8}{(P+2)^2+1}$$

$$\Rightarrow \quad P^2L\{y(t)\} + L\{y(t)\} = \frac{8}{P^2+4P+5}$$

$$\Rightarrow \quad L(y(t))(P^2+1) = \frac{8}{P^2+4P+5}$$

$$\Rightarrow \quad L\{y(t)\} = \frac{8}{(P^2+1)(P^2+4P+5)}$$

$$\Rightarrow \quad y(t) = L^{-1}\frac{8}{(P^2+1)(P^2+4P+5)}$$

$$y(t) = L^{-1}\left[\frac{-P+1}{P^2+1} + \frac{P+3}{P^2+4P+5}\right]$$

$$= L^{-1}\left(\frac{-P}{P^2+1}\right)+L^{-1}\left(\frac{1}{P^2+1}\right)+L^{-1}\left(\frac{(P+2)+1}{(P+2)^2+1}\right)$$

$$= -\cos t+\sin t+e^{-2t}L^{-1}\left(\frac{P+1}{P^2+1}\right)$$

$$= -\cos t+\sin t+e^{-2t}\left\{L^{-1}\left(\frac{P}{P^2+1}\right)+L^{-1}\left(\frac{1}{P^2+1}\right)\right\}$$

$$= -\cos t+\sin t+e^{-2t}\cos t+e^{-2t}\sin t$$

$$= (e^{-2t}-1)\cos t+(e^{-2t}+1)\sin t$$

which is the required solution.

8.6 (i) Obtain Laplace Inverse transform of

$$F(s) = \left\{\ln\left(1+\frac{1}{S^2}\right)+\frac{S}{S^2+25}e^{-\pi S}\right\}$$

(2015 : 6 Marks)

(ii) Using Laplace transform, solve :

$$y''+y = t,\ y(0) = 1,\ y'(0) = -2$$

(2015 : 6 Marks)

Solution:

(i)

$$F(s) = \log(s^2+1)-2\log s+\frac{s}{s^2+25}\cdot e^{-\pi s}$$

$$L^{-1}\left[\frac{s}{s^2+25}e^{-\pi s}\right] = u(t-\pi)\cos 5(t-\pi) = -u(t-\pi)\cos 5t$$

$$L^{-1}[\log(s^2+1)-2\log s] = f(t) \Rightarrow L(f(t)) = \log(1+s^2)-2\log s$$

$$L[tf(t)] = -\frac{d}{ds}(\log(1+5^2)-2\log s) = -\frac{2s}{1+s^2}+\frac{2}{s}$$

$$t\cdot f(t) = L^{-1}\left[-\frac{2s}{s^2+1}+\frac{2}{s}\right] = -2\cos t+2$$

$\Rightarrow$
$$f(t) = \frac{-2\cos t}{t}+\frac{2}{t}$$

$\therefore$
$$L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+2s}e^{-\pi s}\right] = \frac{2}{t}-\frac{2\cos t}{t}-u(t-\pi)\cos 5t$$

(ii) Taking Laplace :

$$s^2L-sy(0)-y'(0)+L = \frac{1}{s^2}$$

$$s^2L-s+2+L = \frac{1}{s^2}$$

i.e.,
$$L(1+s^2) = \frac{1}{s^2}+s-2$$

$$L = \frac{1}{s^2(1+s^2)}+\frac{s}{1+s^2}-\frac{2}{1+s^2}$$

$$= \frac{1}{s^2} - \frac{1}{1+s^2} + \frac{s}{1+s^2} - \frac{2}{1+s^2}$$

$$= \frac{1}{s^2} - \frac{3}{1+s^2} + \frac{s}{1+s^2}$$

$$y = L^{-1}\left[\frac{1}{s^2} - \frac{3}{1+s^2} + \frac{s}{1+s^2}\right]$$

$$= t - 3\sin t + \cos t$$

8.7 Using Laplace transformation, solve the following :

$$y'' - 2y' - 8y = 0,\ y(0) = 3,\ y'(0) = 6$$

(2016 : 10 Marks)

Solution:

Given DE :

$$y'' - 2y' - 8y = 0$$

Taking Laplace Transformation (LT) on both sides :

$$[s^2y(s) - sy(0) - y'(0)] - 2[sy(s) - y(0)] - 8y(s) = 0$$

$$\Rightarrow \quad s^2y(s) - 3s - 6 - 2sy(s) + 6 - 8y(s) = 0$$

$$\Rightarrow \quad s^2(y(s)) - 2s\cdot y(s) - 8y(s) = 3s$$

$$\therefore \quad y(s) = \frac{3s}{s^2-2s-8} = \frac{3s}{(s-4)(s+2)}$$

$$y(s) = \frac{2}{s-4} + \frac{1}{s+2}$$

Taking inverse L.T., we obtain

$$y(t) = 2e^{4t} + e^{-2t}$$

$$\left[\text{Definition : } L\{f(t)\} = \int_0^\infty e^{-st}\cdot f(t)dt\right]$$

8.8 Solve the following initial value problem using Laplace transform :

$$\frac{d^2y}{dx^2} + ay = r(x),\ y(0) = 0,\ y'(0) = 4$$

$$\text{where } r(x) = \begin{cases} 8\sin x, & \text{if } 0 < x < \pi \\ 0, & \text{if } x \geq \pi \end{cases}$$

(2017 : 17 Marks)

Solution:

$$\text{DE :} \quad \frac{d^2y}{dx^2} + ay = r(x) \quad \ldots(i)$$

$$L(y) = p$$

$$L(y'') = s^2p - sy(0) - y'(0)$$

$$= s^2p - 4 \quad \ldots(ii)$$

$$L(r(x)) = \int_0^\infty e^{-st}\cdot r(t)dt$$

$$= 8\int_0^\pi e^{-st}\cdot \sin t dt + 0 \quad \ldots(iii)$$

Let
$$I = \int_0^{\pi} e^{-st}\cdot \sin t dt$$

$$I = [-e^{-st}\cdot \cos t]_0^{\pi} - s\int_0^{\pi} e^{-st}\cdot \cos t dt$$

$$I = e^{-\pi s} + 1 - s\left[\left(e^{-st}\sin t\right)_0^{\pi} + s\int_0^{\pi} e^{-st}\cdot \sin t\, dt\right]$$

$$I = e^{-\pi s} + 1 - s^2\int_0^{\pi} e^{-st}\cdot \sin t dt$$

$$(1+s^2)I = e^{-\pi s} + 1$$

$\therefore$
$$I = \frac{e^{-\pi s}}{1+s^2} + \frac{1}{1+s^2} \qquad \text{...(iv)}$$

Applying Laplace to (i) and using (ii), (iii) and (iv)

$$s^2p - u + 9p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2}$$

$$(s^2+9)p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2} + 4$$

$$p = \frac{8e^{-\pi s}}{(1+s^2)(9+s^2)} + \frac{8}{(1+s^2)(9+s^2)} + \frac{4}{s^2+9}$$

$$= \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} - \frac{1}{s^2+9} + \frac{4}{s^2+9}$$

$$= \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} + \frac{3}{s^2+9}$$

$$L^{-1}(p) = u(t-\pi)\sin(t-\pi) - \frac{1}{3}u(t-\pi)\sin 3(t-\pi) + \sin t + \sin 3t$$

$\therefore$
$$y(x) = u(x-\pi)\sin(x-\pi) - \frac{1}{3}u(x-\pi)\sin(x-\pi) + \sin x + \sin 3x$$

$\therefore$
$$y(x) = \begin{cases} \sin x + \sin 3x, & 0 < x < \pi \\ \frac{4}{3}\sin 3x, & x \geq \pi \end{cases}$$

8.9 (i) Find the Laplace transform of $f(t) = \frac{1}{\sqrt{t}}$.

(ii) Find the inverse laplace transform of $\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}$.

(2018 : 10 Marks)

Solution:

(i)
$$L(f(t)) = \int_0^{\infty} e^{-st} f(t) dt$$

Here,
$$f(t) = \frac{1}{\sqrt{t}}$$

$\therefore$ $$L(f(t)) = \int_0^\infty e^{-st} t^{-1/2} dt$$

Let $st = p$, $\therefore$ $$s \cdot dt = dp$$

So, $$F(s) = \int_0^\infty e^{-p}\left(\frac{p}{s}\right)^{-1/2}\frac{dp}{s} = \frac{1}{\sqrt{s}}\int_0^\infty e^{-p}p^{-1/2}dp$$

Let $$\sqrt{p} = u$$

$\therefore$ $$\frac{1}{2\sqrt{p}}dp = du \Rightarrow p^{-1/2}dp = 2du$$

Putting this value, we get

$$F(s) = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u^2}\times 2du = \frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}du$$

We know that $$\int_0^\infty e^{-w^2}dw = \frac{\sqrt{\pi}}{2}$$

$\therefore$ $$F(s) = \frac{2}{\sqrt{s}}\times\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{5}}$$

$\therefore$ $$L(f(t)) = \sqrt{\frac{\pi}{5}}$$

(ii) Let $$\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}=\frac{C}{s+3} = F(s)$$

Comparing LHS and RHS, we get

$$A = 2, B = 2, C = 1$$

$\therefore$ $$\frac{5s^2+3s-16}{(s-1)(s-2)(s-3)} = \frac{2}{s-1}+\frac{2}{s-2}+\frac{1}{s+3}$$

$\therefore$ $$L^{-1}\left(\frac{5s^2+3s-16}{(s-1)(s-2)(s+3)}\right) = L^{-1}\left(\frac{2}{s-1}+\frac{2}{s-2}+\frac{1}{s+3}\right)$$

$$= L^{-1}\left(\frac{2}{s-1}\right)+L^{-1}\left(\frac{2}{s-2}\right)+L^{-1}\left(\frac{1}{s+3}\right) \quad \text{(by linearity)}$$

$$= 2L^{-1}\left(\frac{1}{s-1}\right)+2L^{-1}\left(\frac{1}{s-2}\right)+L^{-1}\left(\frac{1}{s+3}\right)$$

$$= 2e^t + 2e^{2t} + e^{-3t}$$

$\therefore$ $$L^{-1}(F(s)) = f(t) = 2e^t + 2e^t + e^{-3t}$$

8.10 Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$. Prove that the Laplace transform of $t^{n+\frac{1}{2}}$, where $n \in N$, is $\dfrac{\Gamma\left(n+1+\frac{1}{2}\right)}{s^{n+1+\frac{1}{2}}}$

(2019 : 10 Marks)

Solution:

(a) $$f(t) = t^{-1/2}$$

$$Lf(t) = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}dt$$

Now put $St = P \Rightarrow$
$$Sdt = dP$$

$\therefore$
$$dt = \frac{dP}{S}$$

$\Rightarrow$
$$\int_0^\infty e^{-P}\frac{\sqrt{S}}{\sqrt{P}}\cdot\frac{1}{S}dP = \frac{1}{\sqrt{5}}\int_0^\infty e^{-P}\cdot P^{-1/2}dP$$

$\Rightarrow$
$$\frac{1}{\sqrt{5}}\cdot\sqrt{\frac{1}{2}} = \frac{\sqrt{\pi}}{\sqrt{5}} = \sqrt{\frac{5}{\pi}}$$ (by gamma function)

(b)
$$f(t) = t^{1/2}$$

$$Lf(t) = \int_0^\infty e^{-St}\sqrt{t dt}$$

Put $St = P \Rightarrow$
$$Sdt = dP \Rightarrow dt = \frac{dP}{S}$$

$\Rightarrow$
$$\int_0^\infty e^{-P}\frac{\sqrt{P}}{\sqrt{5}}\frac{dP}{S} = S^{-3/2}\int e^{-P}P^{1/2}dP$$

$$= S^{-3/2}\int e^{-P}e^{\frac{3}{2}-1}dP$$ (by gamma function)

$$L(FCt) = \frac{1}{25}\sqrt{\frac{\pi}{5}}$$ (required solution)

(c)
$$F(t) = t^n$$

$\therefore$
$$F(s) = L(f(t)) = \lim_{A\to\infty}\int_0^A e^{-St}t^n dt$$

$$= \lim_{A\to\infty}\left\{t^n\frac{e^{-st}}{-S}\Bigg|_0^A - \int_0^A\frac{xt^{n-1}e^{-St}dt}{-S}\right\}$$

$$= 0+\frac{n}{S}\lim_{A\to\infty}\int_0^A e^{-St}t^{n-t}dt$$

$$L\{t^n\} = \frac{n}{S}L\{t^{n-1}\}$$

So recursive solution

$$L\{t^n\} = \frac{n}{2}\{L(t^{n-1})\}, n$$

So,
$$L\{t^{n-1}\} = \frac{n-1}{S}L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{S}L\{t^{3-2}\}$$

By Mathematical Induction, we get

$$L\{t^n\} = \frac{n}{S}\frac{(n-1)}{S}\frac{(n-2)}{S}\cdots\cdots\frac{1}{S}L\{1\}$$

$$= \frac{n!}{S^n} \cdot \frac{1}{S} = \frac{n!}{S^{n+1}} = \frac{\boxed{(n+1)}}{S^{n+1}}, (S > 0)$$

∴ from above formula

$$L\{t^{n+1/2}\} = \frac{\boxed{\left(n+1+\frac{1}{2}\right)}}{S^n+1+\frac{1}{2}}; n \in N. \text{ Hence the result.}$$

9. Application to Initial Value Problem for 2nd Order Linear Equation with Constant Coefficient

9.1 Solve the following initial value DEs :

$$20y'' + 4y' + y = 0,\ y(0) = 3.2 \text{ and } y'(0) = 0.$$

(2017 : 7 Marks)

Solution:

Auxiliary equation using Euler Cauchy equation

$$20m^2 + 4m + 1 = 0$$

$$m = \frac{-4 \pm \sqrt{16-80}}{40} = \frac{-4 \pm 8i}{40}$$

$$= \frac{-1}{10} \pm \frac{i}{5}$$

$$\therefore \quad y = e^{-K/10}\left[A\cos\frac{x}{5} + B\sin\frac{x}{5}\right]$$

$$y(0) = 3.2 \Rightarrow A = 3.2$$

$$y'(x) = e^{-\frac{x}{10}}\left[-\frac{A}{5}\sin\frac{x}{5} + \frac{B}{5}\cos\frac{x}{5}\right] - \frac{1}{10}e^{-\frac{x}{10}}\left[A\cos\frac{x}{5} + B\sin\frac{x}{5}\right]$$

$$y'(0) = \frac{B}{5} - \frac{A}{10} = 0 \qquad [\because y'(0) = 0]$$

$$\therefore \quad B = \frac{A}{2} = 1.6$$

Hence,

$$y(x) = e^{-\frac{x}{10}}\left[3.2\cos\frac{x}{5} + 1.6\sin\frac{x}{5}\right]$$

9.2 Solve the initial value problem :

$$y'' - 5y' + 4y = e^{2t}$$

$$y(0) = \frac{19}{12},\ y'(0) = \frac{8}{3}$$

(2018 : 13 Marks)

Solution:

Given equation can be written as

$$D^2y - 5Dy + 4y = e^{2t}$$

or

$$(D^2 - 5D + 4)y = e^{2t}$$

C.F. : The auxiliary equation is

$$m^2 - 5m + 4 = 0 \Rightarrow (m-4)(m-1) = 0$$

i.e., $m = 1$ or $m = 4$

So, $y_c = c_1e^t + c_2e^{4t}$, where c_1 and c_2 are constants.

P.I. :

$$y_p = \frac{1}{D^2 - D + 4}e^{2t} = \frac{1}{(D-4)(D-1)}e^{2t}$$

$$= \frac{1}{(D-4)} \times \frac{1}{1} \times e^{2t} = \frac{-e^{2t}}{2}$$

∴ $$y = y_c + y_p = c_1e^t + c_2e^{4t} - \frac{e^{2t}}{2}$$

Given : $$y(0) = \frac{19}{12} = c_1 + c_2 - \frac{1}{2}$$

⇒ $$c_1 + c_2 = \frac{25}{12} \quad \text{...(i)}$$

Also, $$y'(0) = \frac{8}{3} \Rightarrow c_1e^0 + c_2 \times 4e^0 - e^0 = \frac{8}{3}$$

⇒ $$c_1 + 4c_2 = \frac{11}{3} \quad \text{...(ii)}$$

Solving (i) and (ii), we get

$$c_2 = \frac{19}{36},\ c_1 = \frac{14}{9}$$

∴ $$y = \frac{14}{9}e^t + \frac{19}{36}e^{4t} - \frac{e^{2t}}{2}$$

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