

IAS - 2009 Linear Algebra.
Paper-I

2009 12M (1) Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix $\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

Sol^n: Let $A = \begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

then $A+A^\Theta$ is Hermitian and

$A-A^\Theta$ is Skew-Hermitian.

$\therefore$ $\frac{1}{2}(A+A^\Theta)$ is Hermitian and

$\frac{1}{2}(A-A^\Theta)$ is skew-Hermitian.

Given that $A = \frac{1}{2}(A+A^\Theta) + \frac{1}{2}(A-A^\Theta)$

$= P+Q$ (say)

where P is Hermitian and

Q is skew-Hermitian.

To find P and Q:

Now $A^\Theta = (\bar{A})^T$

$$= \begin{bmatrix} -2i & 3 & -1 \\ 1 & 2-3i & 2 \\ +i+1 & 4 & -5i \end{bmatrix}^T$$

$$= \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}$$

we have

$$P = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} + \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & i \\ 4 & 4 & 6 \\ -i & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$$

now we have

$$Q = \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} - \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4i & 2 & -i-2 \\ -2 & 6i & -2 \\ -i+2 & 2 & 10i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 1 & -\frac{i}{2}-1 \\ -1 & 3i & -1 \\ -\frac{i}{2}+1 & 1 & 5i \end{bmatrix}$$

$\therefore$ The required Hermitian and Skew-Hermitian matrices are $\begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$ and

$\begin{bmatrix} 2i & 1 & -\frac{i}{2}-1 \\ -1 & 3i & -1 \\ -\frac{i}{2}+1 & 1 & 5i \end{bmatrix}$ respectively.

12M 2009 → 2. Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ which satisfy the equations $x_1+x_2+2x_3+x_4=0$ and $2x_1+3x_2-x_3+x_4=0$, is a subspace of $\mathbb{R}^4$. What is the dimension of this subspace? Find one of its bases.

Sol. Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) / x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the given vectorspace.

Let $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;/\; \begin{array}{l} x_1+x_2+2x_3+x_4=0 \\ 2x_1+3x_2-x_3+x_4=0; \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{array} \right\} \subseteq \mathbb{R}^4.$

Since $(0,0,0,0) \in \mathbb{R}^4$; $0+0+2(0)+0=0$ and $2(0)+3(0)-0+0=0$

$\therefore (0,0,0,0) \in V$.

$\therefore$ V is non-empty subset of $\mathbb{R}^4$.

Let $\alpha = (x_1, x_2, x_3, x_4)$

$\beta = (y_1, y_2, y_3, y_4) \in V$ then $x_1+x_2+2x_3+x_4=0$

$2x_1+3x_2-x_3+x_4=0;$

$y_1+y_2+2y_3+y_4=0$

$2y_1+3y_2-y_3+y_4=0.$

Let $a, b \in \mathbb{R}$ then we have

$$a\alpha+b\beta = (ax_1+by_1, ax_2+by_2, ax_3+by_3, ax_4+by_4). \quad \text{(1)}$$

Since $(ax_1+by_1)+(ax_2+by_2)+2(ax_3+by_3)+(ax_4+by_4)$

$= a(x_1+x_2+2x_3+x_4)+b(y_1+y_2+2y_3+y_4)$

$= a(0)+b(0)$

$= 0.$

and $2(ax_1+by_1)+3(ax_2+by_2) - (ax_3+by_3)+(ax_4+by_4)$

$= a(2x_1+3x_2-x_3+x_4)+b(2y_1+3y_2-y_3+y_4)$

$= a(0)+b(0) = 0.$

$\therefore$ from (I),

$a\alpha + b\beta \in V$

$\therefore$ V is a subspace of $\mathbb{R}^4$.

Now we have

$$x_1 + x_2 + 2x_3 + x_4 = 0 \quad \text{(i)}$$

$$2x_1 + 3x_2 - x_3 + x_4 = 0. \quad \text{(ii)}$$

$$\text{(i)} - \text{(ii)} \equiv -x_1 - 2x_2 + 3x_3 = 0.$$

$$\Rightarrow \boxed{x_1 = 2x_2 - 3x_3}$$

$$\text{(i)} \equiv 2x_2 - 3x_3 + x_2 + 2x_3 + x_4 = 0.$$

$$\Rightarrow 3x_2 - x_3 + x_4 = 0$$

$$\Rightarrow \boxed{x_4 = -3x_2 + x_3}.$$

$\therefore$ The subspace V of $\mathbb{R}^4$ is

$$V = \{(2x_2 - 3x_3, x_2, x_3, -3x_2 + x_3) \mid x_2, x_3 \in \mathbb{R}\}$$

Let $\delta = (2x_2 - 3x_3, x_2, x_3, -3x_2 + x_3) \in V$; $x_2, x_3 \in \mathbb{R}$

then $\delta = x_2(2, 1, 0, -3) + x_3(-3, 0, 1, 1)$

$\in L(S)$

where $S = \{(2, 1, 0, -3), (-3, 0, 1, 1)\} \subseteq V$

$\therefore$ $V \subseteq L(S)$ — (A)

since $S \subseteq V$

$\Rightarrow L(S) \subseteq V$ — (B)

from (A) & (B), $\boxed{V = L(S).}$

i.e. S spans the subspace of V.

since no vector of S is a scalar multiple other.

$\therefore$ S is basis of V.

Since the number of elements for a basis 's' is 2.

$\therefore \boxed{\dim V = 2}$.

20M 2009 → (3). Let $\mathcal{B} = \{(1,1,0), (1,0,1), (0,1,1)\}$ and $\mathcal{B}' = \{(2,1,1), (1,2,1), (-1,1,1)\}$ be the two ordered bases of $\mathbb{R}^3$. Then find a matrix representing the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which transforms $\mathcal{B}$ into $\mathcal{B}'$. Use this matrix representation to find $T(\bar{x})$, where $\bar{x} = (2,3,1)$.

Sol Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the given linear transformation.

Let $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ and

$B' = \{(2,1,1), (1,2,1), (-1,1,1)\}$ be two ordered bases of $\mathbb{R}^3$.

then we have

$$\left.\begin{aligned} T(1,1,0) &= (2,1,1) \\ T(1,0,1) &= (1,2,1) \\ T(0,1,1) &= (-1,1,1). \end{aligned}\right\} \text{—— (A)}.$$

Since B' is the basis of $\mathbb{R}^3$

let $x = (x, y, z) \in \mathbb{R}^3$ then

$$(x, y, z) = a(2,1,1) + b(1,2,1) + c(-1,1,1) \text{ —— (B)}$$

$$\Rightarrow \begin{aligned} 2a + b - c &= x \text{ —— (i)} \\ a + 2b + c &= y \text{ —— (ii)} \\ a + b + c &= z \text{ —— (iii)} \end{aligned}$$

from (ii) & (iii),

$$\boxed{b = y - z}$$

from (i) & (ii), we have

$$3a + 3b = x + y$$

$$\Rightarrow a = \frac{x+y}{3} - (y-z)$$

$$\boxed{a = \frac{x - 2y + 3z}{3}}$$

from (iii)

$$a + b + c = 0$$

$$\Rightarrow c = -(a+b)$$

$$= -\left[\frac{x - 2y + 3z}{3} + y - z\right]$$

$$\boxed{c = -\left[\frac{x+y}{3}\right]}$$

∴ from (B),

$$(x, y, z) = \left(\frac{x - 2y + 3z}{3}\right)(2,1,1) + (y-z)(1,2,1) + \left(\frac{x+y}{3}\right)(-1,1,1) \quad \text{(C)}$$

∴ from (A),

$$T(1,1,0) = \frac{2-2+3}{3}(2,1,1) + 0(1,2,1) + \left(\frac{2+1}{3}\right)(-1,1,1)$$

$$= 1(2,1,1) + 0(1,2,1) + 1(-1,1,1) \quad \text{(i)}$$

$$T(1,0,1) = \left(\frac{1-4+3}{3}\right)(2,1,1) + (2-1)(1,2,1) + \left(\frac{1+3}{3}\right)(-1,1,1)$$

$$= 0(2,1,1) + 1(1,2,1) + \frac{4}{3}(-1,1,1) \quad \text{(ii)}$$

$$T(0,1,1) = \frac{-1-2+3}{3}(2,1,1) + (1-1)(1,2,1) + \frac{-1+1}{3}(-1,1,1)$$

$$= 0(2,1,1) + 0(1,2,1) + 0(-1,1,1) \quad \text{(iii)}$$

Now the matrix of linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is

$$[T : B, B'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 4/3 & 0 \end{bmatrix}.$$

To find linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ explicitly by using this matrix:

~~Let~~ since S is the basis of $\mathbb{R}^3$ &

let $\alpha = (p, q, r) \in \mathbb{R}^3$: $p, q, r \in \mathbb{R}$

then we have

$$(p, q, r) = x(1,1,0) + y(1,0,1) + z(0,1,1) \quad \text{(D)}.$$

$$\Rightarrow x + y = p$$
$$x + z = q$$
$$y + z = r$$

After solving these equations, we get

$$x = \frac{q - r + p}{2}, \quad y = \frac{p - q + r}{2}, \quad z = \frac{q + r - p}{2}.$$

$\therefore$ from (D),

$$(p, q, r) = \frac{p + q - r}{2}(1,1,0) + \frac{p - q - r}{2}(1,0,1) + \frac{-p + q + r}{2}(0,1,1).$$

$$\Rightarrow T(p,q,r) = \frac{p+q-r}{2} T(1,1,0) + \frac{p-q-r}{2} T(1,0,1) + \frac{-p+q+r}{2} T(0,1,1) \quad (\because T \text{ is LT}).$$

$$= \frac{p+q-r}{2}(2,1,1) + \frac{p-q-r}{2}(1,2,1) + \frac{-p+q+r}{2}(-1,1,1)$$

$$\boxed{T(p,q,r) = \left(\frac{4p - 2r}{2}, \frac{2p + 2r}{2}, \frac{p + q + r}{2} \right)}$$

To find $T(\bar{x})$: where $\bar{x} = (2, 3, 1)$.

$$\therefore T(2,3,1) = \left(\frac{4(2) - 2(1)}{2}, \frac{2(2) + 2(1)}{2}, \frac{2 + 3 + 1}{2} \right)$$

$$= (1, 2, 3).$$

2011 (4) / 2009 Let $L: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$.

Then find the rank and nullity of L.

Also, determine null space and range space of L.

Sol^n: Given that $L: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation such that

$L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$. — (1)

Range space of $L = \{\beta \in \mathbb{R}^3 / T(\alpha) = \beta \text{ for } \alpha \in \mathbb{R}^4\}$

$\therefore$ The range space consists of all vectors of the type $(x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

$\therefore$ $\boxed{R(L) = \{(x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1) / x_1, x_2, x_3, x_4 \in \mathbb{R}\}}$. (A)

Let $\beta = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1) \in R(L)$

then $\beta = x_3(1,1,0) + x_4(1,0,1) + x_1(-1,0,-1) + x_2(-1,-1,0)$.

$\in L'(S)$ ($\because$ linear span of S).

where $S = \{(1,1,0), (1,0,1), (-1,0,-1), (-1,-1,0)\} \subseteq R(L)$.

$\therefore$ $\beta \in R(L) \Rightarrow \beta \in L'(S)$.

$\therefore$ $R(L) \subseteq L'(S)$. — (2).

Since $S \subseteq R(L)$

$\Rightarrow L'(S) \subseteq R(L)$ — (3).

$\therefore$ from (2) and (3), we have

$$L'(S) = R(L).$$

i.e S spans $R(L)$.

Now we construct a matrix, whose rows are vectors of the subset S' of $R(L)$ and convert into echelon form by using E-row transformations.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}_{4\times 3} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

clearly which is in echelon form and the number of non-zero rows of echelon form is 2.

$\therefore$ The set $S_1 = \{(1,1,0), (1,0,1)\}$ forms a basis of $R(L)$. and the number of elements of $S' = 2$.

$$\therefore \dim(R(L)) = 2.$$

$$\text{rank of } L = \rho(L)$$
$$= 2.$$

We know that rank of L + nullity of $L = \dim R^4$

$$\Rightarrow \quad 2 + \text{nullity of } L = 4$$

$$\Rightarrow \boxed{\text{nullity of } L = 2.}$$

Now we find nullspace of L:

Null space of $L = N(L)$

$= \{ \alpha \in \mathbb{R}^4 \mid T(\alpha) = (0,0,0) \text{ in } \mathbb{R}^3 \} \subseteq \mathbb{R}^4$.

Let $\alpha \in N(L)$

i.e $(x_1, x_2, x_3, x_4) \in N(L)$

$\implies L(x_1, x_2, x_3, x_4) = (0,0,0)$

$\implies (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1) = (0,0,0)$

$\implies x_3 + x_4 - x_1 - x_2 = 0$ — (i)

$x_3 - x_2 = 0$ — (ii)

$x_4 - x_1 = 0$ — (iii)

$\implies \boxed{x_3 = x_2}$

$\boxed{x_4 = x_1}$

$\therefore N(L) = \{ (x_1, x_2, x_2, x_1) \mid x_1, x_2 \in \mathbb{R} \} \subseteq \mathbb{R}^4$.

clearly which is the required nullspace of L.

20M (5) 2009 Find a 2×2 real matrix A which is both orthogonal and skew-symmetric. Can there exist a 3×3 matrix which is both orthogonal and skew-symmetric? Justify your answer.

12M. 2008 (1) Show that the matrix A is invertible if and only if the $\text{adj}(A)$ is invertible. Hence find $|\text{adj}(A)|$.

Sol. Let A be an invertible matrix of order $n \times n$.

$\Leftrightarrow$ A^{-1} exists

$\Leftrightarrow$ $A \cdot \frac{(\text{adj} A)}{|A|} = I_n$

$\Leftrightarrow$ $A \cdot \text{adj} A = |A| I_n$.

$\Leftrightarrow$ $|A \cdot \text{adj} A| = \big| |A| I_n \big|$.

$\Leftrightarrow$ $|A| |\text{adj} A| = \big||A|\big| |I_n|$

$\Leftrightarrow$ $|A| |\text{adj} A| = |A|^n \cdot 1$ $\left(\because |KA| = k^n |A| \ \& \ |I_n| = 1\right)$

$\Leftrightarrow$ $|\text{adj} A| = |A|^{n-1}$

$\Leftrightarrow$ $|\text{adj} A| \neq 0$ $(\because |A| \neq 0)$.

Clearly $|\text{adj} A| = |A|^{n-1}$.

12 M 2008 (2) Let S be a nonempty set and let V denote the set of all functions from S into $\mathbb{R}$. Show that V is a vector space with respect to the vector addition $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(cf)(x) = cf(x)$.

Sol Let $V = \{f \mid f: S \to \mathbb{R}\}$

Then we have to show that V is a vectorspace w.r.t to the vector addition (i) $(f+g)(x) = f(x) + g(x)$ $\forall x \in S$

scalar multiplication (ii) $(cf)(x) = cf(x)$ $\forall x \in S$, $c \in \mathbb{R}$.

$\forall f, g \in V \Rightarrow (f+g)(x) = f(x) + g(x)$ $\forall x \in S$ (By defn)

Since $f(x), g(x) \in \mathbb{R}$ and $\mathbb{R}$ is a field

$\Rightarrow f(x) + g(x) \in \mathbb{R}$

$\therefore (f+g)(x) = f(x) + g(x) \in \mathbb{R}$

$\therefore (f+g): S \to \mathbb{R}$

$\therefore f+g \in V$.

Internal composition is satisfied.

$\forall f \in V, c \in F \Rightarrow ((cf)(x) = cf(x)$ $\forall x \in S$ (by def)

since $f(x) \in \mathbb{R}$, $c \in \mathbb{R}$ and $\mathbb{R}$ is a field

$\therefore cf(x) \in \mathbb{R}$

$\therefore cf: S \to \mathbb{R}$

$\Rightarrow cf \in V \; \forall \; c \in \mathbb{R}, f \in V.$

$\therefore$ External composition is satisfied.

(I) (i) $\forall \; f, g \in V \Rightarrow f+g \in V$

$\therefore$ closure prop. is satisfied.

(ii) $\forall \; f, g, h \in V$

$\Rightarrow [(f+g)+h](x) = (f+g)(x) + h(x)$

$= f(x) + [g(x) + h(x)]$

$= [f + (g+h)](x)$ (by associative property of $\mathbb{R}$)

$\therefore (f+g)+h = f+(g+h)$

$\therefore$ Associative property is satisfied.

(iii) If $\forall \; x \in S, I(x) = 0 \in \mathbb{R}$ then $I \in V$ (i.e. $I : S \to \mathbb{R}$)

Now $(I+f)(x) = I(x) + f(x)$

$= 0 + f(x)$

$= f(x) \; \forall \; \cancel{f(x)} \in \mathbb{R}$

$\therefore I + f = f \; \forall \; f \in V.$

Similarly $f + I = f \; \forall \; f \in V$

$\therefore \forall \; f \in V, \exists \; I \in V$ such that

$I + f = f + I = f \; \forall \; f \in V$

$\therefore$ Identity element $= I \in V$.

(iv) if $f \in V$, then $-f = (-1)f \in V$.

now $[f+(-f)](x) = f(x) + (-f)(x)$
$= f(x) + [-1\, f(x)]$
$= f(x) - f(x)$
$= 0 = I(x)$

$\therefore f+(-f) = 0 = I$

Similarly $(-f)+f = 0 = I$

$\therefore f+(-f) = -f+f = 0 = I$

$\therefore$ Inverse of f is $-f$ in V.

(v) $\forall f, g \in V \Rightarrow (f+g)(x) = f(x)+g(x)$ (By definition)
$= g(x)+f(x)$ (By commutative property in $\mathbb{R}$ i.e, $f(x), g(x) \in \mathbb{R}$ i.e, $f(x)+g(x) = g(x)+f(x)$)
$= (g+f)(x)$

$\therefore f+g = g+f$

$\therefore$ Commutative property is satisfied.

II . $\forall a, b \in \mathbb{R}$, $f, g \in V$

(i) $[a(f+g)](x) = a(f+g)(x)$ (By definition)
$= a(f(x)+g(x))$
$= a f(x) + a g(x)$ (By left distributive law in $\mathbb{R}$)
$= (af)(x) + (ag)(x)$
$= (af+bg)(x)$

$\therefore a(f+g) = af+bg$

(ii) $[(a+b)f]x = (a+b)f(x)$
$= af(x) + bf(x) = (af)(x) + (bf)(x)$
$= (af+bf)(x)$.

$\therefore (a+b)f = af+bf$

(iii) $[(ab)f](x) = (ab)f(x)$ (by definition)
$= a(b\, f(x)) = a(bf)(x)$

$\therefore (ab)f = a(bf)$

(iv) $(1f)(x) = 1 \cdot f(x) = f(x)$ $\forall f(x) \in \mathbb{R}$.

$\therefore 1f = f$ $\forall f \in \mathbb{R}$

$V(\mathbb{R})$ is a vector space.

2007(3)/2008 Show that $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0)$, $T(1,1,0) = (1,1,1)$ and $T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$.

Sol Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be the given vectorspace over the field $\mathbb{R}$.

Let $B = \{(1,0,0),(1,1,0),(1,1,1)\} \subseteq \mathbb{R}^3$

To show that B is a basis of $\mathbb{R}^3$:-

We know that $\dim \mathbb{R}^3 = 3$.

Now we construct a matrix whose rows are given vectors of S and reduce it into echelon form.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

clearly which is in echelon form and the number of non-zero rows are equal to 3.

$\therefore$ The given three vectors of B are linearly independent vectors and $\dim \mathbb{R}^3 = 3$.

$\therefore$ B forms a basis of $\mathbb{R}^3$.

To find $T(x,y,z)$:

Since B is a basis of $\mathbb{R}^3$

$\therefore$

$\therefore$ there exists unique L.T $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1,0,0) = (1,0,0)$

$$T(1,1,0) = (1,1,1)$$
$$T(1,1,1) = (1,1,0).$$

Since $B = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of $\mathbb{R}^3$

$\therefore \alpha = (x,y,z) \in \mathbb{R}^3$; $x,y,z \in \mathbb{R}$,

$\exists$ $(x,y,z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$. ——— (1)

$a+b+c = x \Rightarrow \boxed{a = x-y}$; $a,b,c \in \mathbb{R}$.

$b+c = y \Rightarrow \boxed{b = y-z}$

$\boxed{c = z}$

$\therefore$ (1) $\equiv$

$$(x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1).$$

$$\Rightarrow T(x,y,z) = (x-y)\,T(1,0,0) + (y-z)\,T(1,1,0) + z\,T(1,1,1)$$
$(\because T \text{ is a LT})$.

$$= (x-y)(1,0,0) + (y-z)(1,1,1) + z(1,1,0)$$

$$= (x-\cancel{y}+\cancel{y}-\cancel{z}+\cancel{z},\ y-\cancel{z}+\cancel{z},\ y-z).$$

$\therefore$ $T(x,y,z) = (x,\ y,\ y-z)$.

which is the required explicitly condition of linear transformation.

2007(4)/2008 Let A be a non-singular matrix. Show that if $I + A + A^2 + A^3 + \cdots\cdots + A^{n-1} + A^n = 0$ then $A^{-1} = A^n$.

Sol. Given that A is non-singular matrix.

i.e $|A| \neq 0$

i.e A^{-1} exists.

and $I + A + A^2 + \cdots\cdots + A^{n-1} + A^n = 0$ ———— (1).

pre-multiply by A^{-1} on both sides we get

$A^{-1}(I + A + A^2 + \cdots\cdots + A^{n-1} + A^n) = A^{-1}(0)$.

$\Rightarrow A^{-1}I + A^{-1}A + A^{-1}\cdot A^2 + \cdots\cdots + A^{-1}\cdot A^{n-1} + A^{-1}\cdot A^n = 0$.

$\Rightarrow A^{-1} + I + A + A^2 + A^3 + \cdots\cdots A^{n-2} + A^{n-1} = 0$.

$\Rightarrow A^{n-1} = -A^{-1} - I - A - A^2 - A^3 - \cdots\cdots - A^{n-3} - A^{n-2}$. ———— (2).

substituting this in to (1), we get

$\cancel{I} + \cancel{A} + \cancel{A^2} + \cancel{A^3} + \cancel{A^4} + \cdots\cdots + \cancel{A^{n-4}} + \cancel{A^{n-3}} + \cancel{A^{n-2}}$
$- A^{-1} - \cancel{I} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \cancel{A^4} - \cdots - \cancel{A^{n-4}} - \cancel{A^{n-3}} - \cancel{A^{n-2}} + A^n = 0$.

$\Rightarrow -A^{-1} + A^n = 0$

$\Rightarrow \boxed{A^n = A^{-1}}$ which is the required ~~the~~ result.

2007 → 2008 (5) Find the dimesion of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. ~~Hence~~ find a basis for the subspace.

Sol.

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the given vectorspace over the field $\mathbb{R}$.

Let W be the subspace of $\mathbb{R}^4$ spanned by S where

$$S = \{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\} \subset W$$

Let us construct a matrix A whose rows are the given vectors of S and convert it into the echelon form.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

clearly which is in echelon form and the number of non-zero rows are equal to 3. corresponding these rows the vectors of S $(1,0,0,0), (0,1,0,0), (1,2,0,1)$

form a basis of W.

i.e $S' = \{(1,0,0,0), (0,1,0,0), (1,2,0,1)\}$

is a maximum number of linearly independent ~~subset~~ set of W.

and it forms a basis of W.

IAS - 2007

$\frac{12/7}{2007}$ → (1) Let S be the vector space of all polynomials $p(x)$, with real coefficients, of degree less than or equal to two considered over the real field $\mathbb{R}$, such that $p(0)=0$ and $p(1)=0$. Determine a basis for 'S' and hence its dimension.

Sol Let $S = \left\{ p(x) \,/\, p(x) = a_0 + a_1 x + a_2 x^2;\ a_0, a_1, a_2 \in \mathbb{R};\ p(0) = 0 \text{ and } p(1) = 0 \right\}$

be the given vectorspace of all polynomials with real coefficients, of degree less than or equal to two over the field $\mathbb{R}$ such that $p(0)=0$ and $p(1)=1$.

TO determine a basis for S:-.

given that $p(0)=0 \Rightarrow a_0 + a_1(0) + a_2(0)^2 = 0$

$\Rightarrow \boxed{a_0 = 0}$

and $p(1) = 0 \Rightarrow a_0 + a_1(1) + a_2(1)^2 = 0$

$\Rightarrow a_0 + a_1 + a_2 = 0$

$\Rightarrow 0 + a_1 + a_2 = 0$

$\Rightarrow \boxed{a_2 = -a_1}$

$\therefore$ The given vector space 'S' becomes

$S = \left\{ +a_1 x - a_1 x^2 \,/\, a_1 \in \mathbb{R} \right\}$

TO find a basis for 'S', $\exists$ a finite subset S' of S s.t (i) S' is LI

(ii) $L(S') = S$.

Let $p_1(x) = a_1 x - a_1 x^2 \in S$; $a_1 \in \mathbb{R}$

then $p_1(x) = a_1(x) + (-a_1)x^2$.

$\in L(S')$

where $S' = \{x, x^2\} \subseteq S$.

$\therefore S \subseteq L(S')$ —(i)

$\because S' \subseteq S$

$\Rightarrow L(S') \subseteq S$ —(ii)

from (i) and (ii), we have

$\boxed{L(S') = S.}$

i.e. S' spans S.

Let $a, b \in \mathbb{R}$ s.t. $a(x) + b(x^2) = 0$

$\Rightarrow a(x) + bx^2 = 0(x) + 0(x^2)$.

$\Rightarrow a = b = 0$.

$\therefore S'$ is LI subset of S.

$\therefore S'$ forms basis of S and the number of elements $= 2$.

$\therefore \boxed{\dim(S) = 2}$.

12M 2007 → (2) Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by

$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$

for each $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Determine a basis for the null space of T. What is the dimension of the Range space of T.

Sol. Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation defined by

$$T(x_1, x_2, x_3) = (2x_1 + x_2 + x_3,\ x_1 + x_2,\ x_1 + x_3,\ 3x_1 + x_2 + 2x_3)$$
$(x_1, x_2, x_3) \in \mathbb{R}^3$ ——— (1).

The null space of T $N(T) = \{ \alpha \in \mathbb{R}^3 \mid T(\alpha) = \hat{0} \text{ in } \mathbb{R}^4 \}$. ——— (2).

Let $\alpha \in N(T)$ then $T(\alpha) = \hat{0}$

$$\Rightarrow T(x_1, x_2, x_3) = (0, 0, 0, 0)$$

$$\Rightarrow (2x_1 + x_2 + x_3,\ x_1 + x_2,\ x_1 + x_3,\ 3x_1 + x_2 + 2x_3) = (0, 0, 0, 0)$$

$\Rightarrow 2x_1 + x_2 + x_3 = 0$ (i)

$x_1 + x_2 = 0$ (ii) $\Rightarrow x_2 = -x_1$

$x_1 + x_3 = 0$ (iii) $\Rightarrow x_3 = -x_1$.

$3x_1 + x_2 + 2x_3 = 0$ (iv)

$\therefore N(T) = \{ (x_1, -x_1, -x_1) \mid x_1 \in \mathbb{R} \}$.

To find a basis for $N(T)$,

$\exists$ a finite subset of 'S' such that

(i) S is L.I

(ii) $L(S) = N(T)$.

Now let $\alpha = (x_1, -x_1, -x_1) \in N(T)$; $x_1 \in \mathbb{R}$

then $\alpha = x_1 (1, -1, -1)$

$\in L(S)$ where $S = \{1, -1, 1\} \subseteq N(T)$

$\therefore \; N(T) \subseteq L(S)$ ——— (1)

Since $S \subseteq N(T)$

$\Longrightarrow L(S) \subseteq N(T)$ ——— (2)

$\therefore$ from (1) and (2), we have

$$\boxed{L(S) = N(T)}$$

Since the singleton non-zero vector of 'S' is linearly independent.

$\therefore$ S forms a basis of $N(T)$ and the number of elements $= 1$.

$\dim(N(T)) = 1$.

We know that $\dim(R(T)) + \dim(N(T)) = \dim(\mathbb{R}^3)$

$\Longrightarrow \dim(R(T)) + 1 = 3$.

$\Longrightarrow \dim(R(T)) = 3-1$

$\Longrightarrow \boxed{\dim R(T) = 2}$.

i.e dimension of range space of T $= 2$.

20M 2007 (3) Let W be the set of all 3×3 symmetric matrices over $\mathbb{R}$. Does it form a subspace of the vector space of the 3×3 matrices over $\mathbb{R}$? In case it does, construct a basis for this space and determine its dimension.

Sol Let $V = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}_{3\times3} \middle| \begin{matrix} x_1, x_2, x_3 \\ y_1, y_2, y_3 \\ z_1, z_2, z_3 \in \mathbb{R} \end{matrix} \right\}$ be the vectorspace of all 3×3 matrices over $\mathbb{R}$.

Let $W = \left\{ \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times3} \middle| a, b, c, h, g, f \in \mathbb{R} \right\}$ be the set of all 3×3 symmetric matrices over $\mathbb{R}$ and $W \subseteq V$.

Since $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in W \quad \therefore W \neq \phi$.

$\therefore W$ is non-empty subset of V.

Let $A = \begin{bmatrix} a_1 & h_1 & g_1 \\ h_1 & b_1 & f_1 \\ g_1 & f_1 & c_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & h_2 & g_2 \\ h_2 & b_2 & f_2 \\ g_2 & f_2 & c_2 \end{bmatrix}$ be two matrices in W and Let $x, y \in \mathbb{R}$

Then we have

$$xA + yB = \begin{bmatrix} xa_1 + ya_2 & xh_1 + yh_2 & xg_1 + yg_2 \\ xh_1 + yh_2 & xb_1 + yb_2 & xf_1 + yf_2 \\ xg_1 + yg_2 & xf_1 + yf_2 & xc_1 + yc_2 \end{bmatrix}$$

$= C_{3\times3} \in W$ ($\because C_{3\times3}$ is symmetric matrix)

$\therefore W$ forms subspace of V over the field $\mathbb{R}$.

To find a basis for W:-

For this, $\exists$ a finite sub set 'S' of W

s.t (i) S is LI
(ii) $L(S) = W$.

Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \in W$; $a, b, c, f, h, g \in \mathbb{R}$

then $A = a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + g\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
$+ b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\in L(S)$

where $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subseteq W$

$\therefore W \subseteq L(S)$ — (i)

since $S \subseteq W$
$\Longrightarrow L(S) \subseteq W$ — (ii)

$\therefore$ from (i) & (ii), we have
$L(S) = W$.
i.e. 'S' spans W.

Let $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}$ s.t

$$x_1\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + x_2\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} + x_3\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} + x_4\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + x_5\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix} + x_6\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1&x_2&x_3\\x_2&x_4&x_5\\x_3&x_5&x_6\end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\Rightarrow x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0.$

$\therefore$ S is linearly independent subset of W.

$\therefore$ S forms a basis of W and

~~the~~ number of elements in S = 6.

$\therefore \boxed{\dim W = 6}$.

20M 2007

4) Consider the vectorspace $X := \{P(x) \mid P(x)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$, over the real field $\mathbb{R}$. Define the map $D: X \to X$ by

$$D(p(x)) := p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the basis $\{1, x, x^2, x^3\}$ for X.

Sol'n: Let $p(x), q(x) \in X$, $a, b \in \mathbb{R}$

Given map $D: X \to X$ defined by

$$D(p(x)) = p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

i.e. $D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2$ —— (1)

Now $D[a p(x) + b q(x)] = D[a(p_0 + p_1 x + p_2 x^2 + p_3 x^3) + b(q_0 + q_1 x + q_2 x^2 + q_3 x^3)]$

$$= D[(ap_0 + bq_0) + (ap_1 + bq_1)x + (ap_2 + bq_2)x^2 + (ap_3 + bq_3)x^3]$$

$$= (ap_1 + bq_1) + 2(ap_2 + bq_2)x + 3(ap_3 + bq_3)x^2$$

$$= a(p_1 + 2p_2 x + 3p_3 x^2) + b(q_1 + 2q_2 x + 3q_3 x^2)$$

$$= aD(p(x)) + bD(q(x))$$

$\therefore D: X \to X$ is a linear transformation.

Now from ①,

$$D(1) = 0 = 0 + 0x + 0x^2 + 0x^3$$

$$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$$

$$D(x^2) = 2x = 0 + 2.x + 0x^2 + 0x^3$$

$$D(x^3) = 3x^2 = 0 + 0.x + 0.x^2 + 3x^2$$

Hence the matrix representation of D w.r.to the ordered basis $\{1, x, x^2, x^3\}$

is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

IAS – 2006

12M 2006 (1) Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V.

Sol Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2} \middle| \, a,b,c,d \in F \right\}$ be the given vector space over the field F.

To find a basis for V, ∃ a finite subset S of V such that (i) S is LI

(ii) $L(S) = V$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$; $a,b,c,d \in F$

then $A = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\in L(S)$

where $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq V$

$\therefore V \subseteq L(S)$ — (i)

Since $S \subseteq V$

$\Rightarrow L(S) \subseteq V$ — (ii)

from (i) and (ii), we have

$L(S) = V$.

i.e S spans V.

Let $x, y, z, t \in F$ s.t

$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ — (1).

$\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x = y = z = t = 0$.

$\therefore$ S is linearly independent subset of V.

$\therefore$ S forms a basis for V.

and the number of elements in S = 4.

$\boxed{\dim V = 4}$.

2006 12m. (2) State Cayley Hamilton theorem and using it, find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Sol'n: Statement: Every square matrix satisfies its characteristic equation.

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= 4 - 5\lambda + \lambda^2 - 6$$

$$= \lambda^2 - 5\lambda - 2$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

The given matrix A satisfies the characteristic equation

$$\therefore A^2 - 5A - 2I = 0 \quad \text{——} (1).$$

Now multiplying (1) by A^{-1}, we get

$$A - 5I - 2A^{-1} = 0$$

$$\Rightarrow 2A^{-1} = A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{2}[A - 5I] \quad \text{——} (2)$$

$$A-5I = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

$\therefore$ (2) $\equiv$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

15M 2006 (3) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x,y) = (2x-3y, x+y)$. compute the matrix of T relative to the basis $B = \{(1,2), (2,3)\}$.

Sol. Given that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by

$$T(x,y) = (2x-3y, x+y) \quad \text{(1)}.$$

To find matrix of T w.r.t the basis $B = \{(1,2), (2,3)\}$ of $\mathbb{R}^3$.

Now from (1), we have

$$T(1,2) = (2(1)-3(2), 1+2)$$

$$\therefore \boxed{T(1,2) = (-4, 3)} \quad \text{(A)}$$

$$T(2,3) = (4-9, 2+3)$$

$$\therefore \boxed{T(2,3) = (-5, 5)} \quad \text{(B)}$$

Since $B = \{(1,2), (2,3)\}$ is a basis of $\mathbb{R}^3$.

let $\alpha = (x,y) \in \mathbb{R}^3$;

$$(x,y) = a(1,2) + b(2,3). \quad \text{(2)}$$

$$\Rightarrow \quad a + 2b = x \text{ --- (i)}$$
$$2a + 3b = y \text{ --- (ii)}$$

$$2\times(i) - (ii) \equiv 4b - 3b = 2x - y$$

$$\Rightarrow \boxed{b = 2x - y}$$

$$(i) \equiv \quad a = x - 2b$$
$$= x - 2(2x - y)$$
$$\boxed{a = -3x + 2y}$$

$\therefore$ (2) $\equiv$ $(x, y) = (-3x+2y)(1,2) + (2x-y)(2,3)$.

Now from (A) and (B), we have

$T(1,2) = (-4,3) = 18(1,2) + (-11)(2,3)$.

$T(2,3) = (-5,5) = -5(1,2) + (-20)(2,3)$.

Now the matrix of the linear transformation T is $[T:B] = \begin{bmatrix} 18 & -5 \\ -11 & -20 \end{bmatrix}$.

15M 2006 (4) Using elementary row-operations, find the rank of the matrix $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.

Sol

Let $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Let us convert it into echelon form by using elementary row-operation.

$$A \sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 9 & 5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & 5 \\ 0 & 2 & 2 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 4R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_4 \to R_4 + 2R_3$$

Clearly which is echelon form and the number of non-zero rows of echelon form is the rank of A $\therefore \boxed{\text{rank of } A = 4}$.

15M 2006 (5) Investigate for what values of λ and μ the equations

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions.

Sol^n: Write the matrix equation of the given system.

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$R_3 \to$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad R_3 \to R_3 - R_2$$

If $\lambda = 3$ & $\mu \neq 10$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A/B) \neq \rho(A)$

$\therefore$ The given equations have no solutions

If $\lambda \neq 3$ and $\mu =$ any value then

$\rho(A/B) = \rho(A) = 3 =$ the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

If $\lambda = 3$ and $\mu = 10$ then

$\rho(A/B) = \rho(A) = 2 <$ The number of unknown variables

$\therefore$ The given equations are consistent and have infinite solutions.

2005

12th 1(b) Find the values of K for which the vectors $(1,1,1,1)$, $(1,3,-2,K)$, $(2, 2K-2, -K-2, 3K-1)$ and $(3, K+2, -3, 2K+1)$ are linearly independent in $\mathbb{R}^4$.

2005 2005

Sol^n: Form the matrix A whose rows are given vectors

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & K \\ 2 & 2K-2 & -K-2 & 3K-1 \\ 3 & K+2 & -3 & 2K+1 \end{bmatrix}$$

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & K \\ 2 & 2K-2 & -K-2 & 3K-1 \\ 3 & K+2 & -3 & 2K+1 \end{vmatrix} \neq 0$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 3R_1$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & K-1 \\ 0 & 2K-4 & -K-4 & 3K-3 \\ 0 & K-1 & -6 & 2K-2 \end{vmatrix} \neq 0$$

$$\Rightarrow 1 \begin{vmatrix} 2 & -3 & K-1 \\ 2K-4 & -K-4 & 3K-3 \\ K-1 & -6 & 2K-2 \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} 2 & -3 & K-1 \\ 2K-4 & -K-4 & 3(K-1) \\ K-1 & -6 & 2(K-1) \end{vmatrix} \neq 0$$

$$\Rightarrow (K-1) \begin{vmatrix} 2 & -3 & 1 \\ 2K-4 & -K-4 & 3 \\ K-1 & -6 & 2 \end{vmatrix} \neq 0$$

$$\Rightarrow (k-1)\begin{vmatrix} 2 & -3 & 1 \\ 2k-10 & -k+5 & 0 \\ k-5 & 0 & 0 \end{vmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\Rightarrow (k-1)(k-5)\begin{vmatrix} 2 & -3 & 1 \\ 2 & -1 & 0 \\ k-5 & 0 & 0 \end{vmatrix} \neq 0$$

$$\Rightarrow (k-1)(k-5)(k-5)(0+1) \neq 0$$

$$\Rightarrow (k-1)(k-5)^2 \neq 0$$

$$\Rightarrow k \neq 1, 5.$$

$\therefore$ The given vectors are linearly independent if $k \neq 1, 5$.

i.e i.e $k \in \mathbb{R} - \{1, 5\}$.

12M 2005 → (2) Let V be the vector space of polynomials in x of degree $\leq n$ over $\mathbb{R}$. prove that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x.

Sol^n: Let $V = \{p(x) \mid p(x)$ is a polynomial in x of degree $\leq n$ over $\mathbb{R}\}$.

Let $S = \{1, x, x^2, \ldots, x^n\} \subseteq V$

To show that 'S' is a basis of V :-

(i) To prove S is LI :-

Let $a_0, a_1, a_2, \ldots, a_n \in F$ then

$a_0(1) + a_1(x) + a_2 x^2 + \cdots + a_n(x^n) = 0$ (zero polynomial)

$\Rightarrow a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0(1) + 0x + \cdots + 0x^n.$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0.$

$\therefore$ S is LI.

(ii) <u>TO prove $L(S) = V$</u> :

we know that $L(S) \subseteq V$ —(i)

Let $p(x)$ be any polynomial of degree $\leq n$ over $\mathbb{R}$ i.e $p(x) \in V$.

then $p(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$ where $b_0, b_1, \cdots b_n \in \mathbb{R}$.

then $p(x) = b_0(1) + b_1(x) + b_2(x^2) + \cdots + b_n(x^n)$.
= linear combination of elements of S.
= $L(S)$

~~where~~

$\therefore p(x) \in L(S)$

$V \subseteq L(S)$ —(ii)

$\therefore$ from (i) & (ii), we have

S is a basis of V.

Let $F[x]$ be the vectorspace of all polynomials over the field $\mathbb{R}$.

Let $S' = \{1, x, x^2, \cdots x^n, \cdots\} \subseteq F[x]$.

since S is a finite subset of S' having 'n' vectors and S is LI.

$\therefore$ Every finite subset of S' is LI.

$\therefore S'$ is LI.

(ii) To prove $L(S') = F[x]$:—

we know that $L(S') \subseteq F[x]$ —— (i)

Let $f(x) \in F[x]$

i.e $f(x) = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$ be any polynomial of degree 'm' in $F[x]$.

$= b_0(1) + b_1(x) + b_2(x^2) + \dots b_m(x^m) + 0x^{m+1} + 0x^{m+2} + \dots$

$=$ linear combination of S'.

$\in L(S')$.

$\therefore f(x) \in L(S')$.

$\therefore F[x] \subseteq L(S')$ —— (ii)

from (i) & (ii), we have

$$L(S') = F[x].$$

$\therefore S'$ is a basis of $F[x]$.

i.e S' extends to form a basis of all polynomials vectorspace $F[x]$.

15M 2005 → (3) Let T be a linear transformation on $\mathbb{R}^3$, whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

find the matrix of T relative to the basis

$B = \{(1,1,1), (1,1,0), (0,1,1)\}$.

Sol Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$. then $T(1,0,0) = (2,1,3)$

$T(0,1,0) = (1,2,3)$

$T(0,0,1) = (1,2,4)$.

since S is a basis of $\mathbb{R}^3$

$\therefore \alpha = (x,y,z) \in \mathbb{R}^3$; $x,y,z \in \mathbb{R}$

$\therefore (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$.

$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$= x(2,1,3) + y(1,2,3) + z(1,2,4)$

$T(x,y,z) = (2x+y+z,\ x+2y+2z,\ 3x+3y+4z)$

which is the required explicitly condition of L.T. — (1)

<u>To find ~~the~~ matrix of T relative to the basis</u> $B = \{(1,1,1), (1,1,0), (0,1,1)\}$.

from (1),

$T(1,1,1) = (4, 5, 10)$

$T(1,1,0) = (3, 3, 6)$ — (A).

$T(0,1,1) = (2, 4, 7)$

since $B = \{(1,1,1), (1,1,0), (0,1,1)\}$ be a basis of $\mathbb{R}^3$.

Let $(a,b,c) \in \mathbb{R}^3$, $a,b,c \in \mathbb{R}$.

$$(a,b,c) = x(1,1,1) + y(1,1,0) + z(0,1,1) \quad \text{(B)}$$

$x,y,z \in \mathbb{R}$.

$$\Rightarrow x + y = a$$

$$x + y + z = b \Rightarrow \boxed{z = b - a}$$

$$x + z = c. \Rightarrow x = c - z$$

$$\Rightarrow x = c - (b - a)$$

$$\Rightarrow \boxed{x = a - b + c}$$

and $y = a - x$

$$\Rightarrow y = a - (a - b + c)$$

$$\Rightarrow \boxed{y = b - c}$$

$\therefore$ (B) $\equiv$

$$(a,b,c) = (a-b+c)(1,1,1) + (b-c)(1,1,0) + (b-a)(0,1,1).$$

Now from (A),

$$T(1,1,1) = (4,5,10) = 9(1,1,1) + (-5)(1,1,0) + 1(0,1,1)$$

$$T(1,1,0) = (3,3,6) = 6(1,1,1) + (-3)(1,1,0) + (-3)(0,1,1)$$

$$T(0,1,1) = (2,4,7) = 5(1,1,1) + (-3)(1,1,0) + 2(0,1,1).$$

$\therefore$ The matrix of linear transformation

$T: \mathbb{R}^2 \to \mathbb{R}^3$ is

$$[T:B] = \begin{bmatrix} 9 & 6 & 5 \\ -5 & -3 & -3 \\ 1 & -3 & 2 \end{bmatrix}.$$

2005

15M(4) Find the inverse of the matrix given below using Elementary row operations only. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol'n: Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ then

$$A = I_3 A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\sim \begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_1 \to R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A \quad R_2 \to R_2 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A \quad R_3 \to \frac{1}{3} R_3$$

$$I_3 = BA$$

where $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\therefore A^{-1} = B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

2005
15M (5) If S is skew-Hermitian matrix, then show that $A = (I+S)(I-S)^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided -1 is not an eigen value of A.

Sol'n: Given that 'S' is a skew-Hermitian matrix

$$\therefore S^{\theta} = -S \quad \text{——} \quad (1)$$

we know that the eigen values of a skew-Hermitian matrix 'S' are either purely imaginary (or) zero.

$\therefore$ Neither 1 nor -1 is a root of the equation

$$|S - \lambda I| = 0$$

$$\Rightarrow |S-I| \neq 0 \text{ and } |S+I| \neq 0$$

$$\Rightarrow |I-S| \neq 0 \text{ and } |I+S| \neq 0$$

$$(\because |-A| \neq 0 \Rightarrow |A| \neq 0)$$

$\therefore$ $I-S$ and $I+S$ are both non-singular matrices.

Now given that $A = (I+S)(I-S)^{-1}$

Consider $A^\theta = [(I+S)(I-S)^{-1}]^\theta$

$= [(I-S)^{-1}]^\theta (I+S)^\theta$

$= [(I-S)^\theta]^{-1} (I^\theta + S^\theta)$

$= (I^\theta - S^\theta)^{-1} (I-S)$ (by ①)

$= (I+S)^{-1}(I-S)$ (by ①)

$\therefore A^\theta A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$

$[\because (I-S)(I+S) = (I+S)(I-S)]$

$= I.I$

$= I$

$\therefore A^\theta A = I$

$\therefore$ A is a unitary matrix

IAS – 2004

12M 2004 (1) Let S be space generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,-2)\}$. What is the dimension of the space 'S'? find a basis for S.

Sol Given that S is a space generated by the set $S' = \{(0,2,6),(3,1,6),(4,-2,-2)\} \subseteq S$.

i.e $(S') = $

Let us construct a matrix A whose rows are the given vectors of S' and convert it into echelon form.

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 3 & 1 & 6 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 4 & -2 & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & \frac{-10}{3} & -10 \end{bmatrix} \quad R_3 \to R_3 - \frac{4}{3}R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + \frac{5}{3}R_2$$

Clearly which is in echelon form and the number of non-zero rows of echelon form $= 2$.

Corresponding to these non-zero rows the vectors $(0,2,6), (3,1,6)$ form a basis of S. and $\boxed{\dim(S) = 2}$.

12M; 2004 (2) Show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel? Find a basis for the kernel.

Soln: To show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation.

where $f(x,y,z) = 3x+y-z$ —— (1)

Let $\alpha, \beta \in \mathbb{R}^3$ s.t $\alpha = (x,y,z)$

$\beta = (x_1, y_1, z_1) \in \mathbb{R}^3$

let $a, b \in \mathbb{R}$ then $a\alpha + b\beta =$

$(ax + bx_1, ay + by_1, az + bz_1) \in \mathbb{R}^3$

$\therefore f(a\alpha + b\beta) = f(ax+bx_1, ay+by_1, az+bz_1)$

$= 3(ax+bx_1) + (ay+by_1) - (az+bz_1)$ (by defn).

$= a(3x+y-z) + b(3x_1+y_1-z_1)$

$f(a\alpha+b\beta) = a f(\alpha) + b f(\beta)$.

$\therefore f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation

Now the kernel of $f = \ker f$

$$= K = \{\alpha \in \mathbb{R}^3 \mid f(\alpha) = 0 \in \mathbb{R}\}$$

Let $\alpha = (x, y, z) \in K$.

then $f(\alpha) = 0$

$\Rightarrow f(x, y, z) = 0$

$\Rightarrow 3x + y - z = 0$

$\Rightarrow \boxed{z = 3x + y}$

$\therefore K = \{(x, y, 3x+y) \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

To find a basis of K, $\exists$ a finite subset S of K s.t. (i) S is LI

(ii) $L(S) = K$.

Let $\beta = (x, y, 3x+y) \in K$.

then $\beta = x(1,3) + y(1,1)$

$\in L(S)$ where $S = \{(1,3), (1,1)\} \subseteq K$

$\therefore K \subseteq L(S)$ —— (i)

Since $S \subseteq K$

$\Rightarrow L(S) \subseteq K$ —— (ii)

$\therefore$ from (i) & (ii), we have

$L(S) = K$.

Since no vector of S is a scalar multiple of other

$\therefore$ S is LI.

$\therefore$ S forms a basis of K and the number of elements of $S = 2$.

$\boxed{\dim(K) = 2}$

15M 2004 (3) Show that the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ which is represented by the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is one to one. Find a basis for its image.

Sol Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation which is represented by $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

ISM(4) 2004 → verify whether the following system of equations is consistent:

$$x + 3z = 5$$
$$-2x + 5y - z = 0$$
$$-x + 4y + z = 4.$$

<u>Sol</u> The given system is

$$\left.\begin{aligned} x + 3z &= 5 \\ -2x + 5y - z &= 0 \\ -x + 4y + z &= 4 \end{aligned}\right\} \longleftarrow (A)$$

we write the given matrix equation is $AX = B$

i.e $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$

where $A = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$.

Now the augmented matrix

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ -2 & 5 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{array}\right]$$

Let us convert into echelon form by using elementary row transformations:

$$[A \mid B] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & 4 & 4 & 9 \end{array}\right] \quad \begin{aligned} R_2 &\to R_2 + 2R_1 \\ R_3 &\to R_3 + R_1 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - \frac{4}{5}R_2$$

clearly which is in echelon form

$\therefore \rho(A) = 2$ & $\rho(A|B) = 3$.

$\therefore \rho(A) \neq \rho(A|B)$.

$\therefore$ The given system of equations is not consistent.

15M 2004 → (5) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$. Hence, find A^{-1} and A^{6}.

Sol Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}_{2\times2}$, $I_{2\times2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; λ is a scalar.

then $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$

$= (1-\lambda)(3-\lambda)+1.$

$\therefore$ Characteristic polynomial of A

is $|A - \lambda I| = 0$

i.e $(1-\lambda)(3-\lambda)+1 = 0.$

$\Rightarrow 3 - \lambda - 3\lambda + \lambda^2 + 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 4 = 0.$ ——— (1).

which is the required characteristic polynomial equation of A. $\overset{4}{\wedge}$ $^{-2}\ {}^{-2}$

We know that every square matrix satisfies its own characteristic equation.

$\therefore$ (1) $\equiv A^2 - 4A + 4I = 0.$ ——— (2).

since $|A| = 3+1$
$= 4$
$\neq 0.$

$\therefore A^{-1}$ exists.

(2) $\equiv \quad A - 4I + 4A^{-1} = 0.$

$$\Rightarrow 4A^{-1} = 4I - A$$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\therefore \quad 4A^{-1} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}.}$$

Again from (2), we have

$$A^6 - 4A^4 + 4A^3 = 0.$$

$$\Rightarrow A^6 = 4A^4 + 4A^3. \qquad \text{——— (3)}.$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 0 & 4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}$$

$$A^6 = A^3 A^3 = \begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}\begin{bmatrix} -4 & 12 \\ -12 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 160 & 192 \\ -192 & 256 \end{bmatrix}$$

P-I 2003) 12M → Let S be any non-empty subset of a vector space V over the field F.

Show that the set $\{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n : a_1, a_2 \cdots a_n \in F, \alpha_1, \alpha_2, \cdots, \alpha_n \in S, n \in N\}$ is the subspace generated by S.

Proof: Given that $V(F)$ is a vectorspace and $S \subseteq V$

Let $S = \{\alpha_1, \alpha_2, \cdots \alpha_n\} \subseteq V$

and $L(S) = \left\{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \middle/ \begin{array}{l}\alpha_1, \alpha_2 \cdots \alpha_n \in S \\ a_1, a_2 \cdots a_n \in F\end{array}\right\} \subseteq V$

Now $\forall\, a, b \in F$; $\alpha, \beta \in L(S)$

choose $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$

where a's, b's $\in F$ and α's $\in S$

$\Rightarrow a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$

$= (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bbn)\alpha_n$

$\in L(S)$

$(\because aa_1 + bb_1, aa_2 + bb_2, \cdots aa_n + bb_n \in F)$

$\therefore L(S)$ is a subspace of $V(F)$.

Let $\alpha_i \in S$, $i = 1, 2, \cdots n$

then $\alpha_i = 1\alpha_i$

$=$ linear Combination of α_i

$\in L(S)$

$\therefore \alpha_i \in L(S)$

$\therefore S \subseteq L(S)$.

Now let W be any subspace of $V(F)$ containing S.

$\therefore S \subseteq W$

if $\alpha \in L(S)$ then $\alpha =$ the linear Combination of a finite no. of elements of S

$\in W$ $(\because S \subseteq W)$

$\therefore$ If $\alpha \in L(S)$ then $\alpha \in W$

$\therefore L(S) \subseteq W$

$\therefore S \subseteq L(S) \subseteq W \subseteq V$

$\therefore L(S)$ is the smallest subspace of V containing S.

i.e, $L(S) = \{S\}$

P-I 2003 15M → (2) Prove that the eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent.

proof: Let $x_1, x_2, x_3 \ldots x_m$ be the characteristic vectors of a matrix A corresponding to distinct characteristic values $\lambda_1, \lambda_2, \ldots \lambda_m$.

Then $Ax_i = \lambda_i x_i$; $i = 1, 2, \ldots m$. —— ①

To prove that the vectors $x_1, x_2, \ldots x_m$ are linearly dependent.

then we can choose $r\ (1 \le r < m)$ such that $x_1, x_2, \ldots x_r$ are linearly independent and $x_1, x_2 \ldots x_r, x_{r+1}$ are linearly dependent.

$\therefore$ we can choose the scalars $k_1, k_2, \ldots k_{r+1}$ not all zeros such that

$$k_1x_1 + k_2x_2 + \cdots + k_rx_r + k_{r+1}x_{r+1} = 0 \quad \text{—— ②}$$

$$\Rightarrow A(k_1x_1 + k_2x_2 + \cdots + k_rx_r + k_{r+1}x_{r+1}) = A(0)$$

$$\Rightarrow k_1(Ax_1) + k_2(Ax_2) + \cdots + k_r(Ax_r) + k_{r+1}(Ax_{r+1}) = 0$$

$\Rightarrow k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) + \cdots + k_r(\lambda_r x_r) + k_{r+1}(\lambda_{r+1} x_{r+1}) = 0$ — (3)

(by using (1))

Now $(3) - \lambda_{r+1}(2) \equiv$

$k_1(\lambda_1 - \lambda_{r+1})x_1 + \cdots k_r(\lambda_r - \lambda_{r+1})x_r = 0$ — (4)

Since $x_1, x_2, \cdots x_r$ are linearly independent and $\lambda_1, \lambda_2, \cdots \lambda_r, \lambda_{r+1}$ are distinct

$\therefore k_1 = 0, \; k_2 = 0, \cdots k_r = 0$

Putting $k_1 = 0, k_2 \cdots k_r = 0$ in (2).

we get $k_{r+1} x_{r+1} = 0$

$\Rightarrow k_{r+1} = 0 \;\; (\because x_{r+1} \neq 0)$

$\therefore$ from (2), $k_1 = 0, k_2 = 0, \cdots k_r = 0, k_{r+1} = 0$

$\therefore$ which is Contradiction to our assumption that the scalars.

$k_1, k_2, \cdots k_r, k_{r+1}$ are not all zeros.

$\therefore$ our assumption that $x_1, x_2 \cdots x_m$ are linearly dependent is wrong.

$\therefore x_1, x_2, x_3 \cdots x_m$ are linearly independent.

$\therefore x_1, x_2, \cdots x_m$ which corresponds to distinct characteristic roots of A are linearly independent.

15M/2003 (3) If H is a Hermitian matrix, then show that $A = (H + iI)^{-1}(H - iI)$ is a unitary matrix. Also show that every unitary matrix can be expressed in this form, provided 1 is not an eigen value of A.

Solⁿ: Since H is a Hermitian matrix

$\therefore H^\theta = H$ -

12M 2003

(i) If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, then find the matrix represented by : $2A^{10} - 10A^9 + 14A^8 - 6A^7 - 3A^6 + 15A^5 - 21A^4 + 9A^3 + A - I$.

Sol^n: Given that

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Now $$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^4 = A^2 A^2$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

$$A^5 = A^4 A$$

$$= \begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix}$$

$$A^6 = A^5 A = \begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 364 & 365 \end{bmatrix}$$

$$A^7 = A^6 A = \begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 364 & 365 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix}$$

$$A^8 = A^7 A = \begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix}$$

$$A^9 = A^8 A = \begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix}$$

$$A^{10} = A^9 A = \begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 29525 & 29524 & 29524 \\ 0 & 1 & 0 \\ 29524 & 29524 & 29525 \end{bmatrix}$$

$$\therefore 2A^{10} - 10A^9 + 14A^8 - 6A^7 - 3A^6 + 15A^5 - 2A^4 + 9A^3 + A - I$$

$$= 2\begin{bmatrix} 29525 & 29524 & 29524 \\ 0 & 1 & 0 \\ 29524 & 29524 & 29525 \end{bmatrix} - 10\begin{bmatrix} 9842 & 9841 & 9841 \\ 0 & 1 & 0 \\ 9841 & 9841 & 9842 \end{bmatrix}$$

$$+ 14\begin{bmatrix} 3281 & 3280 & 3280 \\ 0 & 1 & 0 \\ 3280 & 3280 & 3281 \end{bmatrix} - 6\begin{bmatrix} 1094 & 1093 & 1093 \\ 0 & 1 & 0 \\ 1093 & 1093 & 1094 \end{bmatrix}$$

$$-3\begin{bmatrix} 365 & 364 & 364 \\ 0 & 1 & 0 \\ 364 & 365 & 365 \end{bmatrix} + 15\begin{bmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{bmatrix} - 2\begin{bmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{bmatrix}$$

$$+ 9\begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

IAS 2002 (1) Show that the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(a,b,c) = (a-b, b-c, a+c)$ is linear and non-singular.

Sol Given that $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t.

$$T(a,b,c) = (a-b, b-c, a+c) \quad \text{—— (1)}$$

Let $\alpha = (a,b,c)$ $(a,b,c) \in \mathbb{R}^3$.

$\beta = (a_1, b_1, c_1) \in \mathbb{R}^3$; $x, y \in \mathbb{R}$

then $x\alpha + y\beta = x(a,b,c) + y(a_1,b_1,c_1)$

$$= (xa + ya_1, bx + yb_1, cx + yc_1)$$

Now we have

$$T(x\alpha + y\beta) = T(xa + ya_1, bx + yb_1, cx + yc_1)$$

$$= (xa + ya_1 - bx - yb_1, bx + yb - cx - yc_1, xa + ya_1 + cx + yc_1)$$

$$= \big(x(a-b) + y(a_1 - b_1), x(b-c) + y(b_1 - c_1), x(a+c) + y(a_1 + c_1)\big)$$

$$= x(a-b, b-c, a+c) + y(a_1 - b_1, b_1 - c_1, a_1 + c_1)$$

$T(x\alpha + y\beta) = xT(\alpha) + yT(\beta)$.

$\therefore$ T is a linear transformation.

<u>To show that T is non-singular</u>:–

The null space $N(T) = \{\alpha \in \mathbb{R}^3 \,/\, T(\alpha) = \hat{0} \text{ in } \mathbb{R}^3\} \subseteq \mathbb{R}^3$.

Let $\alpha \in N(T)$ then $T(\alpha) = \hat{0}$

$$\Rightarrow (a-b, b-c, a+c) = (0,0,0)$$

$\Rightarrow a-b=0$ — (i)

$b-c=0$ — (ii)

$a+c=0$ — (iii)

$(i)+(ii) \equiv a-c=0$

$a+c=0.$

$\Rightarrow a=0=b=c.$

$\therefore$ ~~Ker~~ $N(T)=\{(0,0,0)\}\subseteq\mathbb{R}^3.$

$\therefore N(T)$ contains only zero vector.

$T:\mathbb{R}^3\to\mathbb{R}^3$ is non-singular.

12M 2000 → (2) ~~A~~ square matrix A is non-singular if and only if the constant term in ~~its~~ its characteristic polynomial is different from zero.

Sol

ISM 2002 (3) Let A be a real 3×3 symmetric matrix with eigen values 0,0,5. If the corresponding eigen vectors are (2,0,1), (2,1,1) and (1,0,−2) then find the matrix A.

Sol^n: Given that the matrix A is a real 3×3 symmetric matrix with eigen values 0, 0, 5 and the corresponding eigen vectors (2,0,1), (2,1,1) and (1,0,−2).

Let $x_1 = (2,0,1)^T$; $x_2 = (2,1,1)^T$; $x_3 = (1,0,-2)^T$

Now let us normalize the vectors x_1, x_2 & x_3 & for this

$\|x_1\| = \sqrt{5}$; $\|x_2\| = \sqrt{6}$; $\|x_3\| = \sqrt{5}$

$$\therefore \hat{x}_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{5}}(2,0,1)^T$$

$$= \left(\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)^T$$

$$\hat{x}_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{6}}(2,1,1)^T$$

$$= \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)^T$$

$$\hat{x}_3 = \frac{x_3}{\|x_3\|} = \frac{1}{\sqrt{5}}(1,0,-2)^T$$

$$= \left(\frac{1}{\sqrt{5}}, 0, \frac{-2}{\sqrt{5}}\right)^T$$

Let $P = [\hat{x}_1 \quad \hat{x}_2 \quad \hat{x}_3]$

$$= \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{6} & 1/\sqrt{5} \\ 0 & 1/\sqrt{6} & 0 \\ 1/\sqrt{5} & 1/\sqrt{6} & -2/\sqrt{5} \end{bmatrix}$$

which is the required orthogonal matrix.

Since P is orthogonal.

$\therefore P^T P = I$

P is non-singular

$\therefore P^T = P^{-1}$

$$\therefore P^{-1} = P^T = \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

Since every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

$\therefore P^{-1}AP = D = \text{dia}(0, 0, 5)$

$\Rightarrow AP = PD$

$\Rightarrow A = PDP^{-1}$

$\Rightarrow A = PDP^T$

$$\Rightarrow A = \begin{bmatrix} 2/\sqrt{5} & 2/\sqrt{6} & 1/\sqrt{5} \\ 0 & 1/\sqrt{6} & 0 \\ 1/\sqrt{5} & 1/\sqrt{6} & -2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 5/\sqrt{5} \\ 0 & 0 & 0 \\ 0 & 0 & -10/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$ which is the required symmetric matrix

12M 2001 (1) Show that the vectors $(1,0,-1)$, $(0,-3,2)$ and $(1,2,1)$ form a basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

Sol Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be the given vector space.

Let $S = \{(1,0,-1), (0,-3,2), (1,2,1)\} \subseteq \mathbb{R}^3$.

We know that $\dim(\mathbb{R}^3) = 3$.

Now we construct a matrix A whose rows are given vectors of S

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Now we have

$$|A| = 1(-7) - 0(-2) - 1(3)$$

$$= -7 - 3$$

$$= -10$$

$$\neq 0$$

$\therefore$ The given vectors of S are linearly independent and number of linearly independent vectors $= 3$.

$\therefore$ S forms a basis of $\mathbb{R}^3(\mathbb{R})$.

2001 12M (2) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

Sol'n: Since 'λ' is a characteristic root of a non-singular matrix, therefore $\lambda \neq 0$. Also λ is a characteristic root of A implies there exists a non-zero vector x such that $AX = \lambda x$

$$\Rightarrow (\text{Adj } A)(Ax) = (\text{Adj } A)(\lambda x)$$

$$\Rightarrow [(\text{Adj } A)A]x = \lambda (\text{Adj } A)x$$

$$\Rightarrow |A| I x = \lambda (\text{Adj } A)x \quad (\because (\text{Adj } A)A = |A| I)$$

$$\Rightarrow \frac{|A|}{\lambda} x = (\text{Adj } A)x$$

$$\Rightarrow (\text{Adj } A)x = \frac{|A|}{\lambda} x$$

$\Rightarrow \frac{|A|}{\lambda}$ is a characteristic root of the matrix Adj A.

2001
15m → (3) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, determine A^{50}.

Sol'n: If $n = 3$ then

$$A^3 = A + A^2 - I \quad \text{—— (1)}$$

Since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now $A + A^2 - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Suppose $A^n = A^{n-2} + A^2 - I$ is true for $n = k$

$\therefore A^k = A^{k-2} + A^2 - I$

Now for $n = k+1$;

$$A^{k+1} = A \cdot A^k$$
$$= A \cdot \left[A^{k-2} + A^2 - I\right]$$
$$= A^{k-1} + A^3 - A$$
$$= A^{k-1} + A + A^2 - I - A \quad (\text{from } 1)$$
$$= A^{k-1} + A^2 - I$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = k+1$

$\therefore$ By mathematical induction, it is true for every integer $n \geq 3$.

$$A^n = A^{n-2} + A^2 - I \quad \text{--- } (2)$$

Now $A^3 = A + A^2 - I$

$$\boxed{A^4 = 2A^4 - I}$$

$$A^6 = A^4 + A^2 - I$$

$$\boxed{A^6 = 3A^2 - 2I}$$

$$A^8 = A^6 + A^2 - I$$

$$\boxed{A^8 = 4A^2 - 3I}$$

$$A^{10} = A^8 + A^2 - I$$

$$\boxed{A^{10} = 5A^2 - 4I}$$

Similarly $A^{12} = 6A^2 - 5I$

$\left(\text{i.e. } A^{12} = \frac{12}{2} A^2 - \left(\frac{12}{2} - 1\right) I\right)$

$\vdots$

$A^{50} = 25A^2 - 24I$ $\quad \left(\text{i.e. } A^{50} = \frac{50}{2} A^2 - \left(\frac{50}{2} - 1\right) I\right)$

$$\therefore A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

15 M(4) 2001 → Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}.$$

Sol^n: The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (7-\lambda)\left[(8+\lambda)^2 - 1\right] + 4\left[4 + 4(8+\lambda)\right] - 4\left[-4 - 4(8+\lambda)\right]$$

$$\Rightarrow 729 + 81\lambda - 9\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow -\lambda^3 - 9\lambda^2 + 81\lambda + 729 = 0$$

$$\Rightarrow (\lambda - 9)(-\lambda^2 - 18\lambda - 81) = 0$$

$$\Rightarrow (\lambda - 9)(\lambda^2 + 18\lambda + 81) = 0$$

$$\Rightarrow (\lambda - 9)(\lambda + 9)^2 = 0$$

$$\Rightarrow \boxed{\lambda = 9, -9, -9.}$$

15M 2000 (1) Prove that a system $Ax=B$ of 'n' non-homogeneous equations 'n' unknowns has a unique solution provided the coefficient matrix is non-singular.

Sol^n: Since A is a non-singular matrix of order 'n'.

$\therefore |A| \neq 0.$

$\Rightarrow \rho(A) = n$ and $\rho(A|B) = n$

$\therefore \rho(A) = \rho(A|B)$

$\Rightarrow$ $AX = B$ is consistent and it has a solution.

Also A^{-1} exists $(\because |A| \neq 0)$

$\therefore A^{-1}(AX) = A^{-1}B$

$\Rightarrow (A^{-1}A)X = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow X = A^{-1}B$ is a solution of $Ax=B$.

If possible let x_1 & x_2 be two solutions of $Ax=B$.

$\therefore Ax_1 = B$ and $Ax_2 = B$

$\Rightarrow Ax_1 = Ax_2$

$\Rightarrow A^{-1}(Ax_1) = A^{-1}(Ax_2)$

$\Rightarrow (A^{-1}A)x_1 = (A^{-1}A)x_2$

$\Rightarrow Ix_1 = Ix_2$

$\Rightarrow x_1 = x_2$

$\therefore$ The solution $x = A^{-1}B$ of $Ax=B$ is unique.

ISM 2000 (2) Prove that two similar matrices have the same characteristic roots. Is its converse true? Justify your claim.

Soln: Let A and B be similar matrices.

Then $\exists$ an invertible matrix P such that

$$B = P^{-1}AP.$$

$$\therefore B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}AP - P^{-1}\lambda P$$
$$= P^{-1}AP - P^{-1}\lambda(IP)$$
$$= P^{-1}AP - P^{-1}(\lambda I)P$$
$$= P^{-1}(A - \lambda I)P$$

$$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|\,|P^{-1}P|$$
$$= |A - \lambda I|$$

$$\therefore |B - \lambda I| = |A - \lambda I|$$

$\therefore$ A and B have the same characteristic polynomial and hence same characteristic roots.

The converse of the above need not be true i.e., if the two matrices have the same characteristic roots then it is not necessary that they are similar.

For example :

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$$

have the same characteristic roots 'but are not similar.

2000

12 (3) 2000 Let V be a vectorspace over $\mathbb{R}$ and $T = \{(x,y) / x,y \in V\}$.
Define addition in T componentwise and scalar multiplication by a complex number $\alpha + i\beta$ by $(\alpha+i\beta)(x,y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall \alpha, \beta \in \mathbb{R}$

Show that T is a vectorspace over $\mathbb{C}$.

Sol Given that V is a vectorspace over the field $\mathbb{R}$.

$T = \{(x,y) / x,y \in V\}$

Define addition in T componentwise

(1) $(x,y) + (x_1,y_1) = (x+x_1, y+y_1)$ $\forall (x,y),(x_1,y_1) \in T$
$x,y; x_1,y_1 \in V$.

and scalar multiplication

(2) $(\alpha+i\beta)(x,y) = (\alpha x - \beta y, \beta x + \alpha y)$
$\forall \alpha, \beta \in \mathbb{R}$

To show that T is a vectorspace over the field $\mathbb{C}$.

Internal composition in T:-

by (1), $(x,y) + (x_1,y_1) = (x+x_1, y+y_1)$ $\forall (x,y),(x_1,y_1) \in T$
$x,y; x_1,y_1 \in V$.
$\in T$ $(\because x+x_1, y+y_1 \in V)$

$\therefore$ Internal composition in T is satisfied.

External composition in T over $\mathbb{C}$:-

by (2) $(\alpha+i\beta)(x,y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall \alpha, \beta \in \mathbb{R}$
$\in T$ $(\because \alpha x - \beta y, \beta x + \alpha y \in V$ (vector space over $\mathbb{R}$))

$\therefore$ External composition in T over $\mathbb{C}$ is satisfied.

(I) To show that $(T, +)$ is an abelian group:

(i) By internal composition in T,
T is closed.

(ii) $\forall (x,y), (x_1,y_1), (x_2,y_2) \in T$;

$[(x, y) + (x_1, y_1)] + (x_2, y_2) = (x+x_1, y+y_1) + (x_2, y_2)$ (by ①)

$= ((x+x_1)+x_2, (y+y_1)+y_2)$

$= (x+(x_1+x_2), y+(y_1+y_2))$ (by associative prop in V)

$= (x, y) + (x_1+x_2, y_1+y_2)$

$= (x, y) + [(x_1, y_1) + (x_2, y_2)]$ (by ①)

$\therefore$ associative property in V is satisfied.

(iii) Existence of right identity in T:

$\forall (x, y) \in T$, $\exists (0,0) \in T$; $0 \in V$ s.t

$(x, y) + (0,0) = (x+0, y+0)$ (by ①)

$= (x, y)$ (by identity in V).

$\therefore (0,0)$ is a right identity in T

(iv) Existence of right inverse in T:

for each $(x,y) \in T$, $\exists (-x, -y) \in T$; $-x, -y \in V$

s.t $(x, y) + (-x, -y) = (x+(-x), y+(-y))$

$= (0, 0)$ (by inverse prop. of V)
identity in T

$\therefore (-x, -y)$ is the right inverse of (x, y) in T.

(v) commutative property in T

$\forall (x, y) + (x_1, y_1) \in T$;

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1) \quad (\text{by } ①)$$
$$= (x_1 + x, y_1 + y) \quad (\text{by comm. in } V)$$
$$= (x_1, y_1) + (x, y) \quad (\text{by } ①)$$

$\therefore$ commutative prop in T is satisfied.

(II) Let $X = (x, y)$, $Y = (x_1, y_1) \in T$

$a = \alpha + i\beta$; $b = \alpha_1 + i\beta_1 \in C$; $\alpha, \beta, \alpha_1, \beta_1 \in \mathbb{R}$.

then we have

(i) $a(X + Y) = aX + aY$

(ii) $(a+b)X = aX + bX$

(iii) $(ab)X = a(bX)$

(iv) $1X = X$; $1 = 1 + i0 \in C$ (identity in C).

2000 P-I
12M (ii) If the matrix A is non-singular, then show that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

Sol^n: Since A is non-singular.

$\therefore A^{-1}$ exists.

Let λ be an eigen value of A and X be corresponding eigen vector of A.

then $AX = \lambda X$.

$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow X = \lambda(A^{-1}X)$

$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$ $\quad$ ($\because \lambda \neq 0$ A is non-singular)

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}.

and X is a corresponding eigen vector.

Conversely

Let K be an eigen value of A^{-1}.

A is non-singular

$\Rightarrow A^{-1}$ is non-singular.

and $(A^{-1})^{-1} = A$

$\therefore \frac{1}{K}$ is an eigen value of A.

$\therefore$ each eigen value of A^{-1} is the reciprocal of the eigen value of A.

$\therefore$ The eigen values of A^{-1} are nothing but the reciprocals of the eigenvalues of A.