set-II

* Numerical Analysis *

Solution of Algebraic and Transcendental Equations:

Introduction: In this chapter, we shall discuss some numerical methods for solving algebraic and transcendental equations.

The equation $f(x)=0$ is said to be algebraic if $f(x)$ is purely, a polynomial in x. If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then equation $f(x)=0$ is called a Transcendental equation.

The equations $x^3-7x+8=0$ and $x^4+4x^3+7x^2+6x+3=0$ are algebraic.

The equations $3\tan 3x = 3x+1$, $x-2\sin x=0$ and $e^x=4x$ are transcendental.

→ Algebraically, the real number α is called the real root (or zero of the function $f(x)$) of the equation $f(x)=0$ if and only if $f(\alpha)=0$ and geometrically the real root of an equation $f(x)=0$

is the value of x where the graph of $f(x)$ meets the x-axis in rectangular co-ordinate system.

We shall assume that the equation

~~$f(x)$~~ $= 0$ ——— (1) has only isolated roots, that is for each root of the equation there is a neighbourhood which does not contain any other roots of the equation.

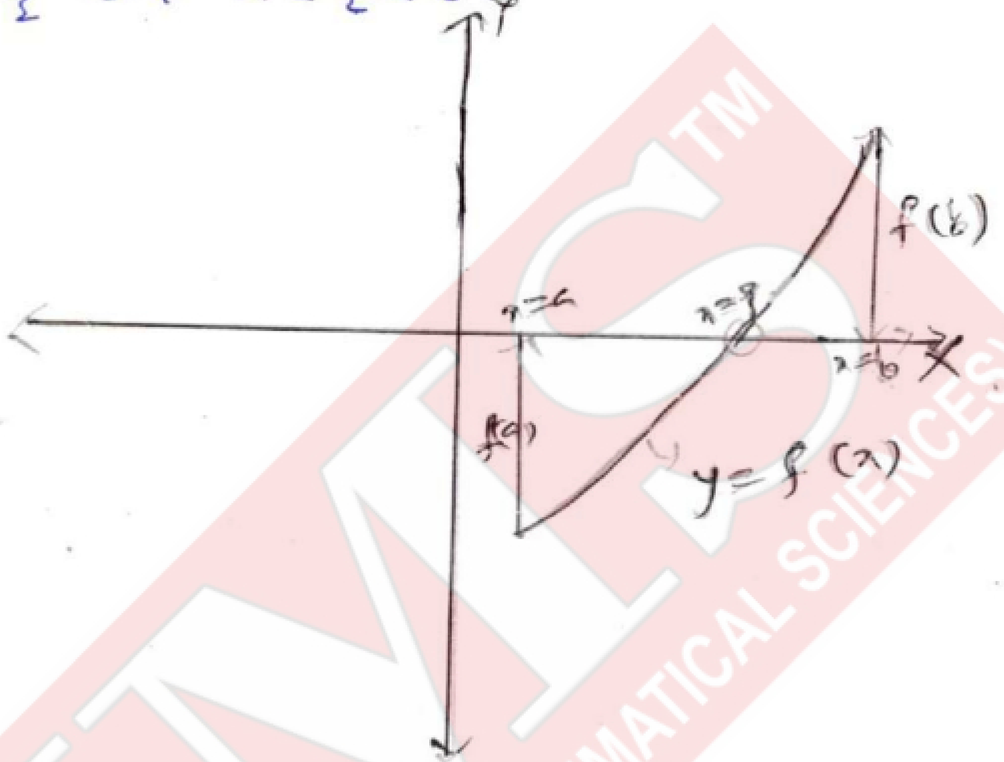
Approximately the isolated roots of the equation (1) has two stages.

(1) Isolating the roots that is finding ~~the~~ smallest possible interval (a, b) containing one and only one root of the equation (1)

(2) Improving the values of the approximate roots to the specified degree of accuracy. Now we state a very useful theorem of mathematical analysis without proof.

Theorem: Intermediate value property:
If $f(x)$ is a real valued continuous in the closed interval $a \leq x \leq b$. If $f(a)$ and $f(b)$ have opposite signs,

then the graph of the function $y = f(x)$ crosses the x-axis at least once; that is $f(x) = 0$ has at least one root ξ s.t $a < \xi < b$.



Broadly speaking, all the known numerical methods for solving either a transcendental equation (or) an algebraic equation can be classified into two groups: <u>direct methods</u> and <u>iterative methods</u>.

– Direct methods require no knowledge of the initial approximation of a root of the equation $f(x) = 0$, while iterative methods do require first approximation to initiate iteration.

(Iteration means repeated application of a numerical process or a pattern of action)

How to get the first approximation?
We can find the approximate value of the root of $f(x)=0$ either by a graphical method (or) by an analytical method as explained below:

Graphical method:

The real root of the equation

$f(x)=0$ ——— (1) can be determined approximately as the abscissas of the points of intersection of the graph of the function $y=f(x)$ with the x-axis. If $f(x)$ is simple, we shall draw the graph of $y=f(x)$ w.r.t a rectangular axis $x'ox$ and $y'oy$. The points at which the graph meets the x-axis are the location of the roots of (1).

If $f(x)$ is not simple we replace equation (1) by an equivalent equation say $\phi(x)=\psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ are simpler than $f(x)$. Then the x-co-ordinate of the point of intersection of the graphs gives the crude approximation of the real roots of the equation (1).

Problem (1)

Solve the equation $x\log_{10}^{x} = 1$, graphically. —— (1)

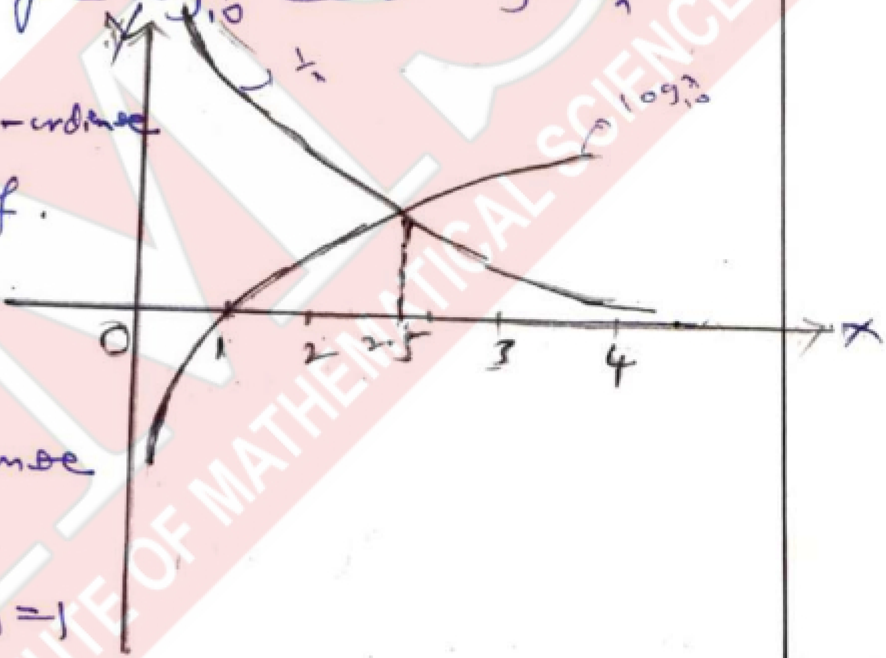
Sol The given equation (1)

$x\log_{10}^{x} = 1$ can be written

as $\log_{10}^{x} = \frac{1}{x}$ —— (2)

where $\log_{10}^{x}$ and $\frac{1}{x}$

simpler than $x\log_{10}^{x} = 1$, constructing the curves $y = \log_{10}^{x}$ and $y = \frac{1}{x}$



we get x-co-ordinate of the point of intersection as 2.5.

$\therefore$ The approximate value of the root of $x\log_{10}^{x} = 1$ is $\xi = 2.5$

HW → solve $x^2 + x - 1 = 0$ graphically

HW → solve $-e^{2x} + 2x + 0.1 = 0$ graphically.

→ solve $x - \sin x - 1 = 0$, graphically.

Sol Let the given equation be

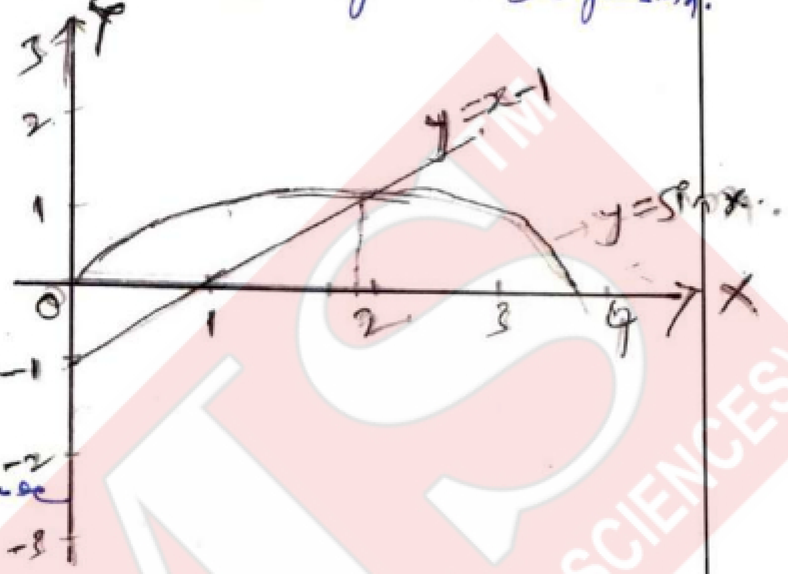
$f(x) = x - \sin x - 1 = 0$ —— (1)

It can be written as

$$x-1 = \sin x$$

where $x-1$ and $\sin x$ simpler than ①.

constructing the curves $y=x-1$ & $y=\sin x$.



we get x-co-ordinate of the point of the inter-section as 1.9

$\therefore$ The approximate value of the root of ① is $\xi = 1.9$.

* Analytical method:

This method is based on 'intermediate value property'. we shall illustrate it through an example.

Let $f(x) = 3x - \sqrt{1+\sin x} = 0$ —— ①

now $f(0) = -1$,

$$f(1) = 3 - \sqrt{1+\sin(1)}$$
$$= 3 - \sqrt{1+0.84147}$$
$$= 1.64299$$

$\therefore f(0) < 0$ & $f(1) > 0$

ie $f(0)$ and $f(1)$ are opposite signs.

$\therefore$ By intermediate value property, there is atleast one root b/w $x=0$ and $x=1$.

This method is often used to find the first approximation to a root of either transcendental equation or algebraic eqn.

Hence, in analytical method, we must always start with an initial interval (a,b), so that $f(a)$ and $f(b)$ have opposite signs.

* Bisection Method :—

This method is due to Bolzano.

suppose, we wish to locate the root of an equation $f(x)=0$ in an interval, say (x_0, x_1). Let $f(x_0)$ and $f(x_1)$ are of opposite signs, such that $f(x_0)\cdot f(x_1) < 0$.

Then the graph of the function crosses the x-axis between x_0 and x_1, which guarantees the existence of atleast one root in the interval (x_0, x_1).

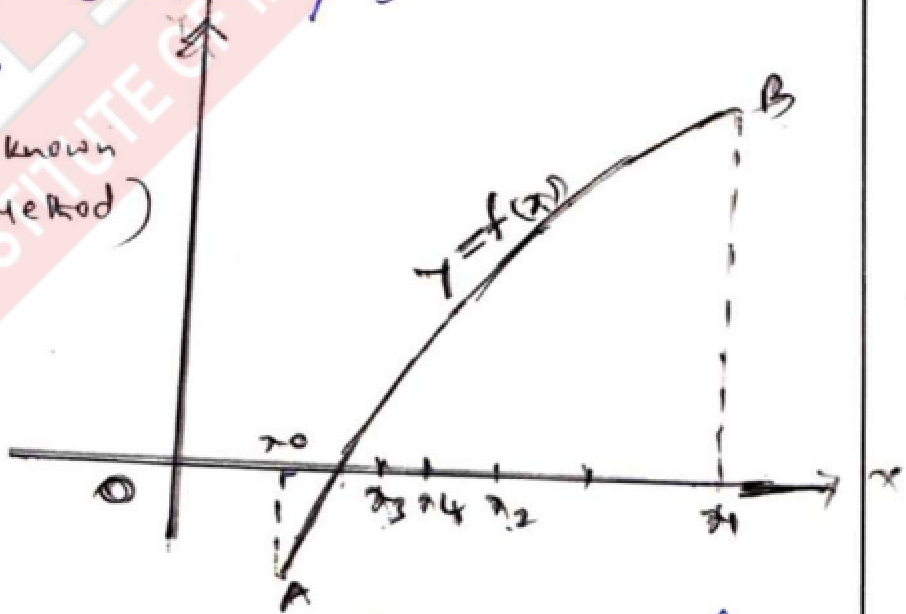
The desired root is approximately defined by the midpoint $x_2 = \frac{x_0+x_1}{2}$.

If $f(x_2)=0$, then x_2 is the desired root of $f(x)=0$. However, if $f(x_2)\neq 0$, then the root may be between x_0 and x_2 (or) x_2 and x_1.

Now, we define the next approximation by $x_3=\frac{x_0+x_2}{2}$ provided $f(x_0).f(x_2)<0$, then the root may be found b/w x_0 and x_2 (or) by $x_3=\frac{x_1+x_2}{2}$ provided $f(x_1)\,f(x_2)<0$, then the root lies b/w x_1 and x_2 etc.

Thus, at each step, we either find the desired root to the required accuracy or narrow the range to half the previous interval as depicted in the given figure. This process of halving the intervals is continued to determine a small and smaller interval within which the desired root lies. Continuation of this process eventually gives us the desired root.

(This method is known as an Iteration Method)



Geometrical illustration of bisection method.

→ solve $x^3-9x+1=0$ for the root b/w $x=2$ and $x=4$ by the bisection method.

Sol Let $f(x)=x^3-9x+1$.

Since $f(2)=-9<0$, $f(4)=29>0$

$\therefore f(2).f(4)<0$

Hence the root lies b/w 2 and 4.

Let $x_0=2$, $x_1=4$. Then the first approximation to the root is $x_2=\frac{x_0+x_1}{2}$

$=\frac{2+4}{2}=3 \Rightarrow \boxed{x_2=3}$

Since $f(x_2)=f(3)=1.(>0)$

$\therefore f(2).f(3)<0$ i.e $f(x_0).f(x_2)<0$

Hence the root lies b/w 2 and 3

the second approximation to the root is $x_3=\frac{x_0+x_2}{2}=\frac{2+3}{2}=\frac{5}{2}=2.5$

$\therefore \boxed{x_3=2.5}$

Since $f(x_3)=f(2.5)<0$

$\therefore f(x_2).f(x_3)<0$

i.e $f(3).f(2.5)<0$.

Hence the root lies b/w 3 & 2.5,

the third approximation to the root is $x_4=\frac{x_2+x_3}{2}=\frac{3+2.5}{2}=2.75$

i.e $\boxed{x_4=2.75}$

sly, we can find the $\boxed{x_5=2.875}$ and $\boxed{x_6=2.9375}$

and the process can be continued until the root is obtained to the desired accuracy.

Now, we can write for the table

n	x_n	$f(x_n)$
2	3	1.0
3	2.5	−5.875
4	2.75	−2.9531
5	2.875	−1.1113
6	2.9375	−0.0901.

→ Solve the equation x^3-9x+1 for the root lying b/w 2 and 3, correct to three significant figures.

Sol Let $f(x)=x^3-9x+1$

Since $f(2) = 8-18+1$
$= -9 < 0$ and

$f(3) = 27-27+1$
$= 1 > 0.$

$\therefore$ $f(2)\cdot f(3) < 0.$

Hence the root lies b/w 2 & 3.

Let $a_0 = 2$, $b_0 = 3$.

n	a_n (−ve)	b_n (+ve)	$x_{n}=\frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	−5.8 (<0)
1	2.5	3	2.75	−2.9 (<0)
2	2.75	3	2.88	−1.03 (<0)
3	2.88	3	2.94	−0.05 (<0)
4	2.94	3.	2.97	0.47 (>0)
5	2.94	2.97	2.955	0.21 (>0)
6	2.94	2.955	2.9475	0.08 (>0)
7	2.94	2.9475	2.9438	0.017 (>0)
8	2.94	2.9438	2.9419	−0.016 (<0)
check 9	2.9419	2.9438	2.9428	0.003 (>0)

(6)

In the 8th step a_n, b_n and x_{n+1} are equal upto three significant figures.
we can take 2.94 as a root upto three significant figures.

$\therefore$ The root of $x^3-9x+1=0$ is 2.94.

→ find a root of the equation $x^3-4x-9=0$, using the bisection method in four stages.

Sol) Let $f(x) = x^3-4x-9$.

since $f(2)$ is $-ve$ and

$f(3)$ is $+ve$

$\therefore f(2)\,f(3) < 0$

Hence the root lies b/w 2 and 3.

$\therefore$ first approximation to the root

is $x_1 = \frac{2+3}{2} = \boxed{2.5}$

Now $f(x_1) = f(2.5)$

$= (2.5)^3 - 4(2.5) - 9$

$= -3.375$

$= -ve$

$\therefore f(3)\cdot f(x_1) < 0$

Hence the root lies b/w x_1 and 3.

$\therefore$ second approximation to the root is $x_2 = \frac{x_1+3}{2}$

$= \frac{2.5+3}{2}$

$= \boxed{2.75}$

Now $f(x_2) = f(2.75)$

$= (2.75)^3 - 4(2.75) - 9$

$= 0.7969$

i.e. $+ve$.

$\therefore f(x_1)\,f(x_2) < 0$

Hence the root lies b/w x_1 and x_2

The third approximation to the root is $x_3 = \frac{x_1+x_2}{2}$

$= \boxed{2.625}$

Now $f(x_3) = (2.625)^3 - 4(2.625) - 9$

$= -1.4121$

i.e. $-ve$

$\therefore f(x_2)\, f(x_3) < 0$

Hence the root lies b/w x_2 and x_3.

$\therefore$ The fourth approximation to the root is $x_4 = \frac{1}{2}(x_2+x_3)$

$= \boxed{2.6875}$

Hence the root is 2.6875 approximately.

→ Find the real root to four decimals of the equation $x^6 - x^4 - x^3 - 1 = 0$ which lies b/w 1 and 2.

Sol Let $f(x) = x^6 - x^4 - x^3 - 1$

since $f(1) = -2 < 0$ &

$f(2) = 39 > 0$.

$\therefore f(1) \cdot f(2) < 0$.

Hence the root lies b/w 1 & 2

The first approximation to the root is $x_1 = \frac{1+2}{2} = \boxed{1.5}$

Now $f(x_1) = f(1.5)$

$= +ve$

$\therefore f(1) \cdot f(x_1) < 0$

Hence the root lies b/w 1 & x_1

The second approximation to the root is $x_2 = \frac{1+x_1}{2} = \boxed{1.25}$

Now $f(x_2) = f(1.25)$
$= -ve.$

$\therefore f(x_1) \cdot f(x_2) < 0.$

Hence the root lies b/w x_2 & x_1.

The third approximation to the root is $x_3 = \frac{x_2 + x_1}{2}$

$= \frac{1.25 + 1.5}{2} = \boxed{1.375}$

Now $f(1.375)$ is $-ve$.

$\therefore f(x_3) \cdot f(x_1) < 0.$

Hence the root lies b/w x_3 & x_1.

The fourth approximation to the root is $x_4 = \frac{x_3 + x_1}{2}$

$= \frac{1.375 + 1.5}{2}$

$= \boxed{1.4375}$

Now $f(x_4) = f(1.4375)$

$= +ve$

$\therefore f(x_3) \, f(x_4) < 0$

Hence the root lies b/w x_3 & x_4.

Now the fifth approximation to the root is $x_5 = \frac{x_3 + x_4}{2}$

$= \frac{1.375 + 1.4375}{2}$

$= \boxed{1.40625}$

Now $f(x_5) = +ve.$

$\therefore f(x_1) \cdot f(x_5) < 0$

Hence the root lies b/w x_1 and x_5

$\therefore$ The sixth approximation to the root is $x_6 = \frac{x_3 + x_5}{2}$

$= \frac{1.375 + 1.40625}{2}$

$= \boxed{1.390625}$

Now $f(x_6) = -ve$

$\therefore f(x_5)\, f(x_6) < 0$

Hence the root lies b/w x_6 & x_5

the 7th approximation to the root is $x_7 = \frac{x_6 + x_5}{2}$

$= \frac{1.390625 + 1.40625}{2}$

$= \boxed{1.3984375}$

Now $f(x_7) = f(1.3984375)$

$= -ve.$

$\therefore f(x_7) \cdot f(x_5) < 0$

Hence the root lies b/w x_7 & x_6

The 8th approximation to the root is $x_8 = \frac{x_7 + x_5}{2}$

$= \frac{1.3984375 + 1.40625}{2}$

$= \boxed{1.40234375}$

Now $f(x_8) < 0$.

$\therefore x_9 = \frac{x_7 + x_5}{2}$

$= \frac{1.40234375 + 1.40625}{2}$

$= 1.4043$ (nearly)

8

now $f(x_9) > 0$

$\therefore x_{10} = \frac{x_8 + x_9}{2}$

$= \frac{1.40234375 + 1.4043}{2}$

$= \boxed{1.4033}$

now $f(x_{10}) < 0$

$\therefore x_{11} = \frac{x_{10} + x_9}{2}$

$= \frac{1.4033 + 1.4043}{2}$

$= \boxed{1.4038}$

now $f(x_{11}) = +ve.$

$\therefore x_{12} = \frac{x_{10} + x_{11}}{2}$

$= \frac{1.4033 + 1.4038}{2}$

$= \boxed{1.40355}$

Hence the root to four decimals of $x^6 - x^4 - x^3 - 1 = 0$ lying b/w 1 and 2 is 1.4036 (approximately).

HW → find to three decimals a root of the equation $3x - \cos x - 1 = 0$.

$\boxed{\text{Ans: } 0.607}$

HW → compute one root of $e^x - 3x = 0$ correct to two decimal places.

$\boxed{\text{Ans: } 1.51}$

HW → find the root of $\tan x + x = 0$ upto two decimal places which lies b/w 2 and 2.1

$\boxed{\text{Ans: } 2.03}$

HW → find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places by using bisection method.

$\boxed{\text{Ans: } 2.7065}$

HW → compute one +ve root of $2x - 3\sin x - 5 = 0$, by bisection method, correct to three significant figures.

$\boxed{\text{Ans: } 2.88}$

HW → compute one root of $x + \log x - 2 = 0$ correct to two decimal places which lies b/w 1 and 2

$\boxed{\text{Ans: } 1.56}$

Note: ① while applying bisection method we must be careful to check that $f(x)$ is continuous.

For example, we may come across functions like $f(x) = \frac{1}{x-1}$. If we consider the interval $(0.5, 1.5)$, then $f(.5)\, f(1.5) < 0$. In this case we may be tempted to use bisection method. But we cannot use the method here because $f(x)$ is not defined at middle point $x=1$. We can overcome these difficulties by taking $f(x)$ to be continuous throughout the initial bisecting interval (Note that, if f is a continuous function on $[a,b]$ and $f(a) \neq f(b)$ then f assumes every value b/w $f(a)$ and $f(b)$)

Therefore we should always examine the continuity of the function in the initial interval before attempting ~~the~~ bisection method.

Note(2): It may happen that a function has more than one root in an interval. The bisection method helps us in determining one root only. We can determine the other roots by properly choosing the initial intervals.

— A numerical process starts with an initial approximation and iteration improves this approximation until we get the desired accurate value of the root.

Let us consider another iteration method now:

* Regula - Falsi Method:—

— This method is also known as the method of false position.

— The Latin word Regula falsi means rule of falsehood. It doesnot mean that the rule is a false statement. But it conveys that the roots that we get according to the rule are approximate roots and not necessarily exact roots. This method is similar to the bisection method.

— The bisection method for finding approximate roots has a drawback that it makes use of only the signs of $f(a)$ and $f(b)$. It doesnot use the values $f(a)$, $f(b)$ in the computations.

For example -
If $f(a) = 100$ and $f(b) = -0.1$, then by the bisection method the first approximate value of a root of $f(x)$ is the mid value x_0 of the interval (a, b). But at x_0, $f(x_0)$ is nowhere near 0.

$\therefore$ In this case it makes more sense to take a value near to -0.1 than the middle value as the approximation to the root.

This drawback is to some extent overcome by the Regula-falsi method.

Geometrically, suppose we want to find a root of the eqn $f(x)=0$ where $f(x)$ is a continuous function. As in the bisection method, we first find an interval (a,b) such that $f(a)f(b)<0$.

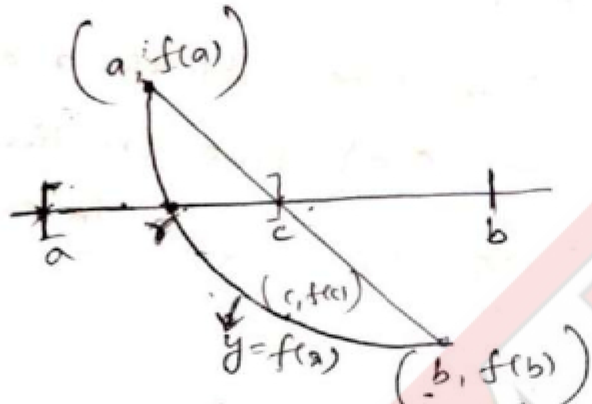


fig (1)

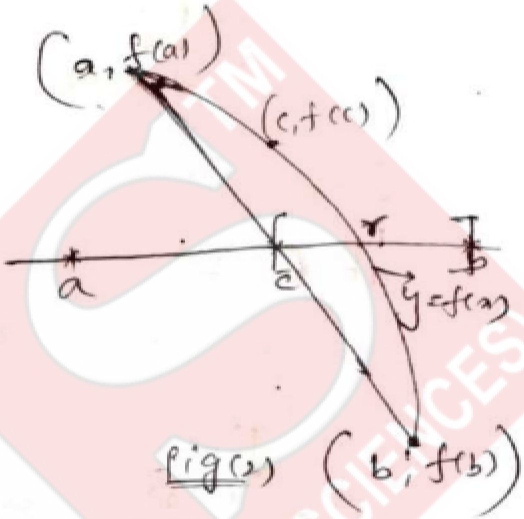


fig (2)

The condition $f(a)f(b)<0$ means that the points $(a, f(a))$ and $(b, f(b))$ lie on the opposite sides of the x-axis.

The line joining $(a, f(a))$ and $(b, f(b))$ crosses the x-axis at some point $(c, 0)$.

Then we take the x-coordinate of that point as the first approximation.

– If $f(c)=0$, then $x=c$ is the required root.

If $f(a)f(c)<0$, then the root lies in (a,c) (fig(1)). In this case the graph of $y=f(x)$ is concave near the root r. Otherwise if $f(a)f(c)>0$, the root lies in (c,b) (fig(2)). In this case the graph of $y=f(x)$ is convex near the root.

Having fixed the interval in which the root lies, we repeat the above procedure.

In mathemetrical form,

The formula for the line joining the two points $(a, f(a))$ and $(b, f(b))$ is given by

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a).$$

$$\Rightarrow \frac{y-f(a)}{f(b)-f(a)} = \frac{x-a}{b-a} \quad \text{—— (1)}$$

Since the stright line intersects the x-axis at $(c, 0)$, the point $(c, 0)$ lies on the straight-line. putting $x=c$, $y=0$ in eqn(1), we get

$$\frac{-f(a)}{f(b)-f(a)} = \frac{c-a}{b-a}$$

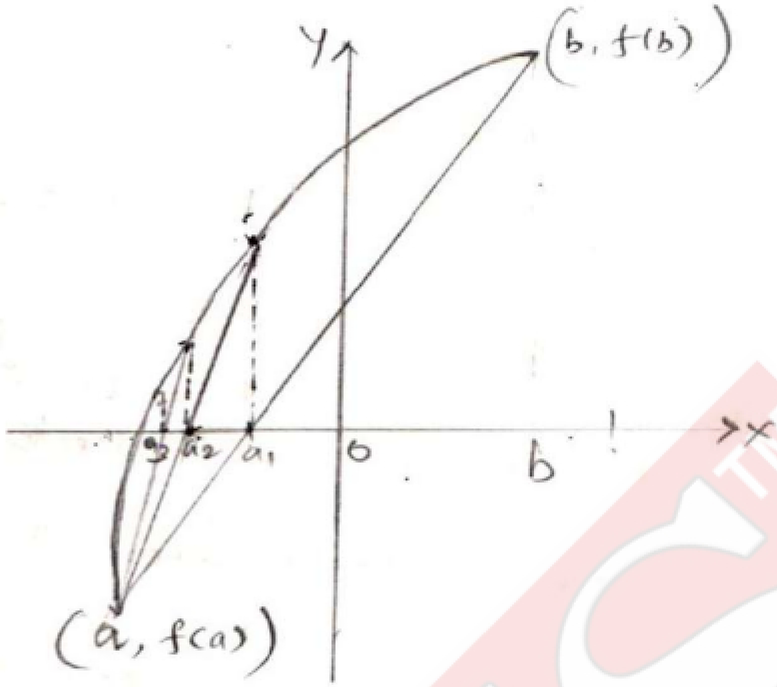
$$\Rightarrow \frac{c}{b-a} - \frac{a}{b-a} = \frac{-f(a)}{f(b)-f(a)}$$

$$\Rightarrow \quad c = a - \frac{f(a)}{f(b)-f(a)}(b-a)$$

$$\Rightarrow \quad c = \frac{af(b) - bf(a)}{f(b)-f(a)} \quad \text{—— (2)}$$

This expression for 'c' gives an approximate value of a root of $f(x)$.

Now, examine the sign of $f(c)$ and decide in which interval (a, c) or (c, b) the root lies. we thus obtain a new interval such that $f(x)$ is of opposite signs at the end points of this interval. By repeating this process, we get a sequence of intervals (a, b), (a, a_1), (a, a_2), ...



We stop the process when either of the following holds.

(i) The interval containing the zero of f(x) is of sufficiently small length.

(or)

(ii) The difference between two successive approximations is negligible.

In the iteration format, the method is usually written as

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{———} \boxed{\text{I}}$$

where (x_0, x_1) is the interval in which the root lies.

→ We now summarise this method in algorithm form.

<u>Step 1</u>: Find numbers x_0 and x_1 such that $f(x_0) f(x_1) < 0$.

<u>Step 2</u>: Set $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

This gives the first approximation.

Step3: If $f(x_2)=0$ then x_2 is the required root. If $f(x_2)\neq 0$ and $f(x_0)\,f(x_2)<0$, then the next approximation lies in (x_0, x_2). Otherwise it lies in (x_2, x_1).

Step4: Repeat the process till the magnitude of the difference between two successive iterated values x_i and x_{i+1} is less than the accuracy required.

Note: $|x_{i+1}-x_i|$ gives the error after i^{th} iteration.

2002 → Find a real root of the eqn $x^3-2x-5=0$ by the method of false position to three decimal places.

Sol^n: Let $f(x)=x^3-2x-5$.

So that $f(2)=-1$ and $f(3)=16$

$\therefore f(2)\,f(3)<0$

Hence the root lies b/w 2 and 3.

Take $x_0=2$, $x_1=3$.

$f(x_0)=-1$, $f(x_1)=16$

By the method of false position, we get

$$x_2 = x_0 - \frac{(x_1-x_0)}{f(x_1)-f(x_0)} f(x_0) \quad \text{——— (1)}$$

$$= 2 - \frac{3-2}{16+1}(-1)$$

$$= 2+\frac{1}{17} = \frac{35}{17} = 2.0588$$

Now $f(x_2) = -0.3908$.

$\therefore f(2.0588)\,.\,f(3) < 0$

Hence the root lies between 2.0588 and 3.

Take $x_0 = 2.0588$, $x_1 = 3$

$\therefore f(x_0) = -0.3908$, $f(x_1) = 16$.

from ①
$$x_3 = 2.0588 - \frac{3-2.0588}{16+0.3908}(-0.3908)$$
$$= 2.0813.$$

Now repeating this process, the successive approximations are given by

$x_4 = 2.0862$, $x_5 = 2.0915$, $x_6 = 2.0934$,

$x_7 = 2.0941$, $x_8 = 2.0943$ etc.

$\therefore$ The approximate root is 2.094 correct to 3 decimal places.

→ The equation $x^3+7x^2+9=0$ has a root b/w -8 and -7. Use the Regula - Falsi method to obtain the root rounded off to 3 decimal places. Stop the iteration when $|x_{i+1}-x_i| < 10^{-4}$.

Sol: Let $f(x) = x^3+7x^2+9$.

Take $x_0 = -8$ and $x_1 = -7$.

$f(x_0) = f(-8) = -55 < 0$

$f(x_1) = f(-7) = 9 > 0$.

By method of false position, we get
$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1)-f(x_0)} f(x_0)$$
$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1)-f(x_0)} \quad \text{——— ①}$$
$$= \frac{(-8)(9)-(-7)(-55)}{9+55}$$
$$x_2 = -7.1406$$

$\therefore$ The first approximation to the root is $x_2 = -7.1406$.

Now $f(x_1) = 1.862856 > 0$

and $f(x_0) f(x_1) = f(-8) f(-7.1406) < 0$

Hence the root lies between -8 and -7.1406.

Take $x_0 = -8$ and $x_1 = -7.1406$.

$\therefore f(x_0) = -55$ and $f(x_1) = 1.862856$

$\therefore$ from (1)

$$x_3 = \frac{(-8)(1.862856) + (7.1406)(-55)}{1.862856 + 55}$$

$$= -7.168174.$$

$\therefore$ The second approximation to the root is $x_3 = -7.168174$.

Now repeating this process, the successive approximations are given by

$x_4 = -7.1735649$, $x_5 = -7.1745906$

$x_6 = -7.1747855$, $x_7 = -7.1748226$.

The absolute value of the difference between the 5^{th} and 7^{th} iterated values is $|7.1748226 - 7.1747855| = 0.0000371 < 10^{-4}$.

$\therefore$ we stop the iteration here.

Further, the value of $f(x)$ at 6^{th} iterated value is $0.00046978 = 4.6978 \times 10^{-4}$ which is close to zero.

Hence -7.175 is an approximate root of $x^3 + 7x^2 + 9 = 0$ rounded-off to 3 decimal places.

→ Determine an approximate root of the equation $\cos x - xe^x = 0$ using Regula-False method. correct to 4 decimal places (13)

*-2008

Sol^n: $f(x) = \cos x - xe^x$.

So that $f(0) = 1$ and $f(1) = \cos 1 - e$
$= -2.17798$

$\therefore f(0)\, f(1) < 0$

Hence the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$.

$\therefore f(x_0) = 1$ and $f(x_1) = -2.17798$

By the method of false position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{—— (1)}$$

$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$

$$= 0.31467$$

$\therefore$ The first approximation to the root is

$$x_2 = 0.31467$$

Now $f(x_2) = 0.51987 > 0$.

$\therefore f(x_2)\, f(x_1) < 0$.

$\therefore$ The root lies b/w 0.31467 and 1

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (1),

$$x_3 = \frac{(0.3146)(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987}$$

$$x_3 = 0.44673$$

$\therefore$ The 2nd approximation to the root is

$$x_3 = 0.44673$$

Now repeating this process, the successive approximations are

$x_4 = 0.49402$, $x_5 = 0.50995$,

$x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$.

$x_9 = 0.51767$, $x_{10} = 0.51775$ etc.

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places

→ Find a real root of the eqn $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places. $\boxed{\text{Ans: } 2.7406}$

→ Use the method of false position to find the fourth root of 32 correct to three decimal places.

Soln: Let $x = (32)^{1/4}$ then $x^4 = 32 \Rightarrow x^4 - 32 = 0$

Let $f(x) = x^4 - 32$. $\boxed{\text{Ans: } 2.378}$

2007 12M. → Use the method of false-position to find a real root of $x^3 - 5x - 7 = 0$ lying between 2 and 3 correct to 3 places of decimals.

→ Use the Regula-falsi method to compute a real root of the eqn $x^3 - 9x + 1 = 0$

(i) if the root lies b/w 2 and 4

(ii) if the root lies b/w 2 and 3.

Comment on the results.

2010 → Use Regula-falsi method to find a real root of the eqn $\log x - \cos x = 0$ accurate to four decimal places after three successive approximations. $\boxed{\text{Ans: } 1.3030}$

1998 → Use Regula-falsi method to show that the real root of $x \log_{10} x = 1.2$ lies b/w 3 and 2.740646.

Note: In regula-falsi method, at each stage we find an interval (x_0, x_1) which contains a root and then apply iteration formula $\boxed{I}$. This procedure has a disadvantage. To over come this, regula-falsi method is modified.

The modified method is known as Secant-method.

In this method we choose x_0 and x_1 as any two approximations of the root. The interval (x_0, x_1) need not contain the root. Then we apply formula $\boxed{I}$ with $x_0, x_1, f(x_0)$ and $f(x_1)$.

The iterations are now defined as:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

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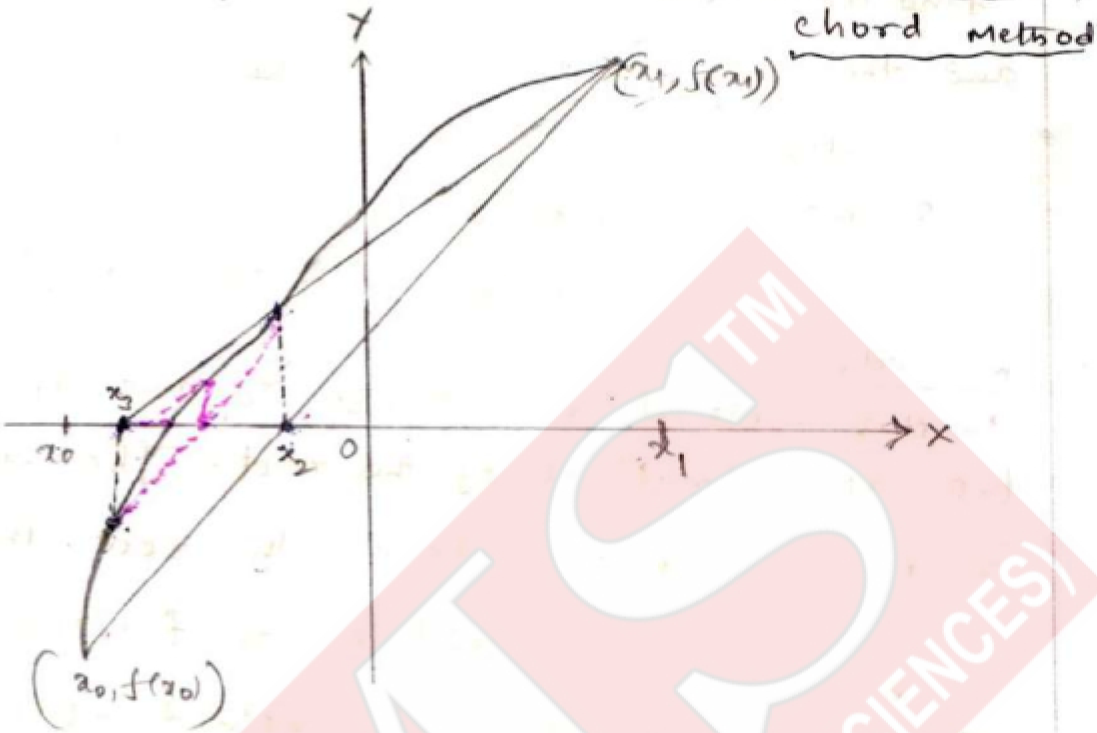
$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \quad \text{—— (1')}$$

→ Geometrically, in Secant method, we replace the graph of $f(x)$ in the interval (x_n, x_{n+1}) by a straight line joining two points $(x_0, f(x_0))$ $(x_1, f(x_1))$ on the curve and take the point of intersection with x-axis as the approximate value of the root.

→ Any line joining two points on the curve is called a secant line. That is why this method is known as secant method (or) chord method



→ Determine an approximate root of the eqn $x^2-2x-1=0$ using secant method starting with $x_0=2.6$ and $x_1=2.5$, rounded-off to 5 decimal places. Compare the result with the exact root $1+\sqrt{2}$.

Sol^n: Let $f(x)=x^2-2x-1$

Starting with $x_0=2.6$ and $x_1=2.5$ the successive approximations are

$$x_2=\frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}$$

$$=\frac{2.6 f(2.5)-2.5 f(2.6)}{f(2.5)-f(2.5)}$$

$$=\frac{2.6(0.25)-(2.5)(.56)}{0.25-0.56}=2.41935484$$

and $f(x_2) = 0.0145682$

To find the next approximation, we compute

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(2.5)(0.0145682) - (2.41935484)(0.25)}{(0.0145682) - (0.56)}$$

$$= 2.41436464.$$

proceeding similarly, we get

$x_4 = 2.41421384$ and $x_5 = 2.41421356.$

Since x_4 and x_5 rounded-off to 5 decimal places are the same, we stop the process here.

∴ The required root rounded-off to 5 decimal places is 2.41421.

The exact value of the root $1+\sqrt{2} = 2.41421$, which is rounded-off to 5 decimal places. Hence the computed root and exact root are the same when we round off to five decimal places.

H.W → Determine an approximate root of the eqn $\cos x - xe^x = 0$ using secant method starting with the two initial approximations as $x_0 = 0$ and $x_1 = 1$, correct to 4 decimal places

→ find an approximate root of the cubic eqn $x^3 + x^2 - 3x - 3 = 0$ using

(a) (i) regula-falsi method correct to 3 decimal places

(ii) secant method starting with $x_0 = 1$, $x_1 = 2$, rounded-off to 3 decimal places

b) Compare the results obtained by (i) & (ii) in part(a).

Sol^n: Let $f(x) = x^3 + x^2 - 3x - 3$

Take $x_0 = 1$ and $x_1 = 2$.

$f(x_0) = -4 < 0$ and $f(x_1) = 3 > 0$

$\therefore$ The root lies between 1 and 2.

By the method of false-position,
the ~~ap~~ first approximation is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{—①}$$

$$= \frac{1(3) - 2(-4)}{3 - (-4)} = \frac{11}{7} = 1.57142$$

Now $f(x_2) = -1.36449 < 0$ and $f(x_1) f(x_2) < 0$

$\therefore$ The root lies between 1.57142 ~~and~~ 2

Take $x_0 = 1.57142$ and $x_1 = 2$

$f(x_0) = -1.36449$ and $f(x_1) = 3$

$\therefore$ from ①

$$x_3 = \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449}$$

$x_3 = 1.70540.$

$f(x_3) = -ve < 0$
~~the~~ root lies b/w
1.7054083, ...
we get $x_4 = 1.72788$

Now Repeating this process, the successive approximations is given by

$x_4 = 1.72788$, $x_5 = 1.73140$, and $x_6 = 1.73194$.

Since x_5 and x_6 are correct to 3 decimal places are same.

$\therefore$ we stop the process here.

Hence the root correct to 3 decimal places is 1.731.

(ii). Secant method. (16)

Starting with $x_0 = 1$, $x_1 = 2$ the successive approximations are

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(3) - 2(-4)}{3-(-4)} = \frac{11}{7} = 1.57142$$

To calculate the next approximation, take $x_1 = 2$ and $x_2 = 1.57142$. we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{(1.57142)(3) - 2(-1.36449)}{1.57142 + 1.36449} = 1.70540$$

To find the 3rd approximation

let $x_2 = 1.57142$ and $x_3 = 1.70540$

$$x_4 = \frac{(1.57142) f(1.70540) - (1.70540) f(1.57142)}{f(1.70540) - f(1.57142)}$$

$$= \frac{(1.57142)(-0.24784) - (1.70540)(-1.36449)}{-0.2478 + 1.36449}$$

$$= 1.73513$$

Repeating this process,

we get $x_5 = 1.73199$, $x_6 = 1.73205$

Since x_5 and x_6 rounded-off to 3 decimal places are the same, we stop here.

Hence the root is $\underline{\underline{1.732}}$, rounded-off 3 decimal places.

(b). W.K.T $|x_{i+1} - x_i|$ gives the error after the i^{th} iteration.

In Regula-falsi method, the error after 5^{th} iteration is

$$|x_6 - x_5| = |1.73194 - 1.73140|$$
$$= 0.00011.$$

Whereas in secant method, the error after 5^{th} iteration is

$$|x_6 - x_5| = |1.73205 - 1.73199|$$
$$= 0.00006$$

This shows that the error in the case of secant method is smaller than that in Regula-falsi method for the same number of iterations.

Newton - Raphson Method:

→ This method is one of the most useful method for finding roots of an algebraic equation.

→ Suppose we want to find an approximate root of the eqn $f(x)=0$.

If $f(x)$ is continuous, then we can apply either bisection method or regula-falsi method to find approximate roots.

Now if $f(x)$ and $f'(x)$ are continuous, then we can use a new iteration method called Newton-Raphson method. This method gives the result more faster than bisection or regula-falsi methods.

— The underlying idea of the method is due to mathematician Isac Newton. But the method as now used is due to the mathematician Raphson.

→ Suppose we want to find a root of the equation $f(x)=0$ where $f(x)$ and $f'(x)$ are continuous.

Let x_0 be an initial approximation and assume that x_0 is close to the exact root α and $f'(x_0) \neq 0$.

Let $\alpha = x_0 + h$ where h is a small quantity.

Hence $f(\alpha) = f(x_0+h) = 0$.

Now, expanding $f(x_0+h)$ by Taylor's theorem, we get

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

Since h is small, neglecting the terms containing h^2 and higher powers, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\Rightarrow h = \frac{-f(x_0)}{f'(x_0)}.$$

This gives a new approximation to α as

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Now the iteration can be defined by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

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$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{———} \quad (1).$$

which is the Newton Raphson formula.

Geometrical:- Interpretation

Suppose the graph of the function $y=f(x)$ crosses the x-axis at α then $x=\alpha$ is the root of the eqn $f(x)=0$

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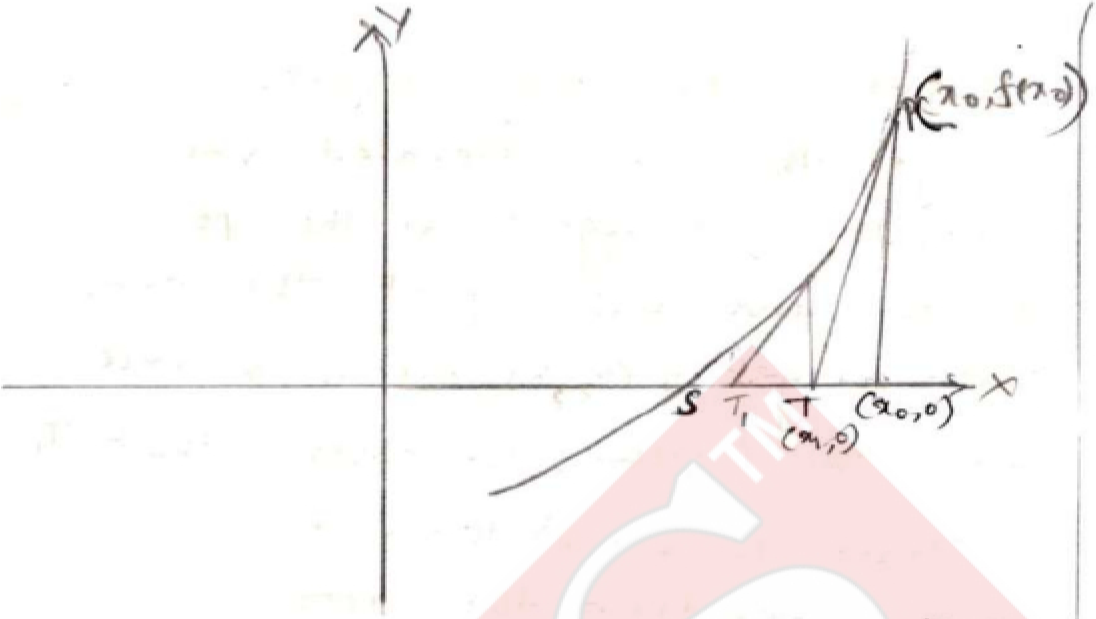

If x_0 is an initial approximation to the root α, then the corresponding point on the graph is $P(x_0, f(x_0))$. we draw a tangent to the curve at P., it intersect the x-axis at T. Let x_1 be the co-ordinate of T.
Let $S(\alpha, 0)$ denote the point on the x-axis where the curve cuts the x-axis, where α is the root of the eqn $f(x)=0$.

we take x_1 as the new approximation which may be closer to α than x_0.

Now let us find the tangent at $P(x_0, f(x_0))$.
The slope of the tangent at P is given by $f'(x_0)$.

$\therefore$ By the point-slope form of the expression for a tangent to a curve is

$$y - f(x_0) = f'(x_0)(x - x_0)$$

The tangent passes through the point $T(x_1, 0)$.

$$\therefore \; 0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$\Rightarrow \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This x_1 is the first iterated value.

To get the second iterated value we again consider a tangent at the point $p(x_1, f(x_1))$ on the curve and repeat the process.

Then we get $T(x_2, 0)$ on the x-axis.

From the figure, we observe that T_1 is more closer to $S(\alpha, 0)$ than T. Therefore after each iteration the approximation is coming closer and closer to the actual root.

Ex-1 Find a real root of the eqn $x^3-4x+1=0$ using Newton-Raphson method, starting with $x_0=0$ rounded-off to 4 decimal places.

Sol^n: Let $f(x) = x^3-4x+1$.

$$f'(x) = 3x^2-4.$$

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

The initial approximation is $x_0=0$

The Newton's iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2,\ldots \quad \text{——— (1)}$$

putting $n=0$, in (1) the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 - \frac{1}{(-4)} = \frac{1}{4} = 0.25$$

putting $n=1$ in ①,

the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.25 - \frac{f(0.25)}{f'(0.25)}$$
$$= 0.25 - \frac{0.015625}{(-3.8125)}$$
$$= 0.254098.$$

Similarly, we get

$$x_3 = 0.254101.$$

Since x_2 and x_3 rounded-off to four decimal places are the same, we stop the iteration here.

Hence the root is 0.2541

Ex 2 → Using Newton-Raphson method, find the real root of the eqn $x^3-6x+4=0$ lying between 0 and 1 correct to 4 decimal places.

Soln: We have $f(x) = x^3-6x+4.$

$$f'(x) = 3x^2-6.$$

Clearly $f(x)$ and $f'(x)$ are continuous $[0,1]$

we have $f(0) = 4$ and $f(1) = -1$

$\therefore f(0)f(1) < 0$

$\therefore$ The root lies b/w 0 & 1.

The value of the root is nearer to 1.

Let $x_0 = 0.7$ be the approximation to the root.

Now $f(x_0) = f(0.7) = 0.143$

and $f'(x_0) = f'(0.7) = -4.53$

Then by newton's iteration formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.7 - \frac{0.143}{(-4.53)}$$

$$= 0.7316$$

Now $f(x_1) = f(0.7316) = 0.0019805$

and $f'(x_1) = f'(0.7316) = -4.39428$

∴ The second approximation of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7316 + \frac{0.0019805}{4.39428}$$

$$= 0.73250699.$$

→ Find the smallest positive root of $2x - \tan x = 0$ by Newton Raphson method, correct to 5 decimal places. $\boxed{\text{Ans: } 1.16556.}$ let $x_0 = 1$

→ By using the newton Raphson method, find an approximate root of $2x - 2 - \sin x = 0$ in the interval $[0, \pi]$ with error less than 10^{-5} start with $x_0 = 1.5$. $\boxed{\text{Ans: } 1.498701}$

→ Find a real root of the eqn $x^3 - x - 1 = 0$ using Newton-Raphson method, correct to four decimal places. [Hint: $f(1) < 0$ & $f(2) > 5$]
Ans: 1.3247.

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→ IAS-2017 Find the real root of the eqn $3x = \cos x + 1$ by using Newton's-Raphson method. [Ans: 0.6071]
(root lies b/w 0 & 1)

→ Find the real root of the eqn $x \log_{10} x = 1.2$ correct to five decimal places.
[Ans: 2.74065] [root lies b/w 2 & 3]

→ IAS-2014 Apply Newton-Raphson's method to determine a root of the eqn $f(x) = \cos x - xe^x = 0$ such that $|f(x^*)| < 10^{-8}$, where x^* is the approximation to the root.

[Ans: $x^* = 0.51775736$] Let $x_0 = 1$

Here $f(x^*) = -0.2910 \times 10^{-10}$.
$\therefore |f(x^*)| < 10^{-8}$.

→ We shall now consider an application of Newton-Raphson formula.

W.K.T finding the square root of a number is not easy unless we use a calculator. Calculators use some algorithm to obtain this value.

We shall now illustrate how Newton-Raphson method enables us to obtain such an algorithm for calculating square roots.

Ex: Find an approximate value of $\sqrt{2}$ using the Newton-Raphson formula.

Sol^n: Let $x = \sqrt{2}$.

$\Rightarrow x^2 - 2 = 0$.

Let $f(x) = x^2 - 2$.

then $f'(x) = 2x$.

clearly $f(x)$ and $f'(x)$ are continuous everywhere.

$\therefore$ $f(x)$ satisfies all the conditions for Newton-Raphson method.

$\therefore$ Choose $x_0 = 1$ be the initial approximation to the root. [$\because \sqrt{1} < \sqrt{2} < \sqrt{4}$
$1 < \sqrt{2} < 2.$)
The root is nearest to 1]

The iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 2}{2x_n}$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{2}{x_n}\right] \quad \text{—— (1)}$$

putting $n = 0, 1, 2, 3, \ldots$, we get

$$x_1 = \frac{1}{2}\left[x_0 + \frac{2}{x_0}\right]$$

$$\Rightarrow x_1 = \frac{1}{2}\left[1 + \frac{2}{1}\right]$$

$$= \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2}\left[1.5 + \frac{2}{1.5}\right] = 1.4166667$$

$$x_3 = \frac{1}{2}\left[1.4166667 + \frac{2}{1.4166667}\right.$$

$$= 1.41242157.$$

Similarly, we get

$x_4 = 1.4142136$

$x_5 = 1.4142136.$

Thus the value of $\sqrt{2}$ correct to seven decimal places is 1.4142136.

Note (1) The method used in the above example is applicable for finding square root of any +ve real number.

For example. we want to find an approximate value of $\sqrt{N}$ where N is a positive real number. Then we consider eqn $x^2 - N = 0$

The iterated formula is

$$\boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]}$$

→ 2. From the above example and examples (1)&(2) we find that Newton-Raphson method gives the root very fast.

One reason for this is that the derivative $|f'(x)|$ is large compared to $|f(x)|$ for any $x = x_n$. The quantity $\left|\frac{f(x)}{f'(x)}\right|$ which is the difference between two iterated values

$\left[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow \frac{f(x_n)}{f'(x_n)} = x_n - x_{n+1}\right]$ is small in this case.

→ In general we can say that if $|f'(x_i)|$ is large compared to $|f(x_i)|$, then we can obtain the desired root very fast by this method.

→ The Newton-Raphson method has some limitations. Some of the difficulties are as given below.

1. Suppose $f'(x_i)$ is zero in a neighbourhood of the root, then it may happen that $f'(x_n) = 0$ for some x_n. In this case we cannot apply Newton-Raphson formula, since division by zero is not allowed.

[2]. Another difficulty is that it may happen that $f'(x)$ is zero only at the roots.
This happens in either of the situations.

(i) $f(x)$ has multiple root at α i.e, a polynomial function $f(x)$ has a multiple root α of order p, then $f(x)$ can be written as

$$f(x) = (x-\alpha)^p \, h(x)$$

where $h(x)$ is a function such that $h(\alpha) \neq 0$.

— For a general function $f(x)$, this means

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \cdots\cdots = f^{p-1}(\alpha) = 0$$

and $f^p(\alpha) \neq 0$.

(ii) $f(x)$ has a stationary point (point of maximum or minimum) at the root.
i.e, $f'(x) = 0$ at some point $x = x_n$.

H.W. Using Newton - Raphson method find the
(i) square root of 8. $\boxed{\text{Ans : } 2.828427}$
(ii) square root of $\sqrt{28}$ $\boxed{\text{Ans : } 5.2915}$

→ Using Newton Raphson method prove that

1997 (i) Iterative formula for $\frac{1}{N}$ is $x_{n+1} = x_n(2 - Nx_n)$

(ii) Iterative formula for $\frac{1}{\sqrt{N}}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$

(iii) Iterative formula for $\sqrt[k]{N}$ is $x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right]$

Sol^n: (i) Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x}$

$$\Rightarrow \frac{1}{x} - N = 0$$

Let $f(x) = \frac{1}{x} - N.$

then $f'(x) = -\frac{1}{x^2} = -x^{-2}$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\left(-x_n^{-2}\right)}$$

$$= x_n + \left(\frac{1}{x_n} - N\right) x_n^2$$

$$= x_n + x_n - N x_n^2$$

$$= 2x_n - N x_n^2$$

$$\boxed{x_{n+1} = x_n (2 - N x_n)}.$$

which is the required result

(ii) Let $x = \frac{1}{\sqrt{N}}$

$\Rightarrow x^2 = \frac{1}{N}$

$\Rightarrow x^2 - \frac{1}{N} = 0$

Let $f(x) = x^2 - \frac{1}{N}$

then $f'(x) = 2x$.

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left(x_n^2 - \frac{1}{N}\right)}{2x_n}$$

$$= \frac{\left(2x_n^2 - x_n^2 + \frac{1}{N}\right)}{2x_n}$$

$$= \frac{x_n^2 + \frac{1}{N}}{2x_n} = \frac{1}{2}\left[x_n + \frac{1}{N x_n}\right]$$

(iii) Let $x = \sqrt[k]{N}$. $\Rightarrow x^k = N$

$\Rightarrow x^k - N = 0$

Let $f(x) = x^k - N$.

Then $f'(x) = kx^{k-1}$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^k - N}{k x_n^{k-1}}$$

$$= \frac{k x_n^k - x_n^k + N}{k x_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n^k + \frac{N}{x_n^{k-1}}\right]$$

→ Evaluate the following (correct to four decimal places) by Newton-Raphson method.

(i) $1/31$ (ii) $\frac{1}{\sqrt{14}}$ (iii) $\sqrt[3]{24}$ (iv) $(30)^{-1/5}$ [Hint: put $k=-5$ in formula(iii)]

$\boxed{\text{Ans: } 0.0323}$ $\boxed{\text{Ans: } 0.2673}$ $\boxed{\text{Ans: } 2.8845}$ $\boxed{\text{Ans: } 0.5065}$

2000 / 8M → Using Newton-Raphson's method, show that the iteration formula for finding the reciprocal of the p^{th} root of N is

$$x_{n+1} = \frac{x_n(p+1-Nx_n^p)}{p}.$$

Sol^n: $x = \frac{1}{\sqrt[p]{N}} \Rightarrow x = \frac{1}{N^{1/p}} \Rightarrow x = N^{-1/p}$

$\Rightarrow x^{-p} = N.$

$\Rightarrow x^{-p} - N = 0.$

Let $f(x) = x^{-p} - N$

$f'(x) = -p\,x^{-p-1}.$

By Newton-Raphson iteration formula, if x_n denotes the n^{th} iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^{-p} - N}{-p\,x_n^{-p-1}}$$

$$= \frac{p x_n + (x_n^{-p} - N)x_n^{p+1}}{p}$$

$$= \frac{p x_n + x_n - N x_n^{p+1}}{p} = \frac{x_n(p+1-Nx_n^p)}{p}$$

which is the required result

Convergence criterion

We shall now introduce a new concept called convergence criterion related to an iteration process. This criterion gives us an idea of ~~how many successive~~ how many successive iterations have to be carried out to obtain the desired accuracy.

Definition: Let $x_0, x_1, \ldots, x_n, \ldots$ be the successive approximations of an iteration process. We denote the sequence of these approximations as $\{x_n\}_{n=0}^{\infty}$. We say that $\{x_n\}_{n=0}^{\infty}$ converges to a root α with order $p \geq 1$ if

$$|x_{n+1} - \alpha| \leq \lambda |x_n - \alpha|^p \quad \text{--- (1)}$$

for some number $\lambda > 0$. p is called the order of convergence and λ is called the ~~asyptotic~~ asymptotic error constant.

For each n, we denote by $\epsilon_n = x_n - \alpha$. Then the eqn (1) be written as

$$|\epsilon_{n+1}| \leq \lambda |\epsilon_n|^p \quad \text{--- (2)}$$

This inequality shows the relationship between the error in successive approximations.

For example:

Suppose $p = 2$ and $|\epsilon_n| \simeq 10^{-2}$ for some n, then we can expect that $|\epsilon_{n+1}| \simeq \lambda 10^{-4}$.

Thus if p is large, the iteration converges rapidly.

- When p takes the values 1, 2, 3 then we say that the convergence is linear, quadratic and cubic respectively.

- In the case of linear convergence (i.e $p=1$), then we require that $\lambda < 1$.

∴ Eqn ① becomes

$$|x_{n+1} - \alpha| \le \lambda |x_n - \alpha| \text{ for all } n \geq 0. \quad \text{---} ③$$

If this condition is satisfied for an iteration process then we say that the iteration process converges linearly.

Setting $n=0$, in the inequality ③, we get

$$|x_1 - \alpha| \le \lambda |x_0 - \alpha|.$$

for $n=1$, we get

$$|x_2 - \alpha| \le \lambda |x_1 - \alpha|$$
$$\le \lambda^2 |x_0 - \alpha|$$

for $n=2$

$$|x_3 - \alpha| \le \lambda |x_2 - \alpha|$$
$$\le \lambda^2 |x_1 - \alpha|$$
$$\le \lambda^3 |x_0 - \alpha|.$$

Using induction on 'n', we get

$$|x_n - \alpha| \le \lambda^n |x_0 - \alpha| \quad \text{---} ④$$

If either of the inequalities ③ or ④ is satisfied, then we conclude that $\{x_n\}_{n=0}^{\infty}$ cgs to the root.

Convergence of bisection method:

Suppose that we apply the bisection method on the interval $[a_0, b_0]$ for the eqn $f(x)=0$.
In this method we construct intervals $[a_0,b_0] \supset [a_1,b_1] \supset [a_2,b_2] \supset \ldots\ldots$ each of which contains the required root of the given eqn.
In each step the interval width is reduced by $\frac{1}{2}$.

i.e.,
$$b_1 - a_1 = \frac{b_0-a_0}{2}$$

$$b_2 - a_2 = \frac{b_1-a_1}{2} = \frac{b_0-a_0}{2^2}$$

$$\ldots\ldots\ldots$$

$$b_n - a_n = \frac{b_0-a_0}{2^n} \quad \text{——— (5)}$$

Clearly the eqn $f(x)=0$ has a root in $[a_0,b_0]$.
Let α be the root of the eqn. Then α lies in ~~all the in~~ all the intervals $[a_i, b_i]$, $i = 0,1,2,\ldots$
For any n, let $c_n = \frac{a_n+b_n}{2}$ denote the middle point of the interval $[a_n, b_n]$. Then $c_0, c_1, c_2, \ldots$ are taken as successive approximations to the root α.

Let us check the inequality (3) for $\{c_n\}_{n=0}^{\infty}$.

For each n, α lies in the interval $[a_n, b_n]$.

$\therefore$ we have
$$|c_{n+1} - \alpha| \le \frac{|c_n - \alpha|}{2}$$

Thus $\{c_n\}_{n=0}^{\infty}$ cgs to the root α. Hence we can

Say that the bisection method always cgs.

→ for practical purposes, we should be able to decide at what stage we can stop the iteration to have an acceptably good approximate value of α. The number of iterations required to achieve a given accuracy for the bisection method can be obtained.

– Suppose that we want an approximate solution with an error bound of 10^{-M}.

Taking logarithms on both sides of eqn ⑤.

we find the number of iterations required, say n, approximately given by

$\frac{b_0-a_0}{2^n} \le 10^{-M}$

$$n = \text{int}\left[\frac{\log(b_0-a_0) - \log 10^{-M}}{\log 2}\right] \quad \text{⑥}$$

$\log_e\left[\frac{b_0-a_0}{2^n}\right] = \log_e 10^{-M}$

where the symbol 'int' stands for the integral part of the number in the bracket and $[a_0, b_0]$ is the initial interval in which a root lies.

since 'n' is integer, we take n as the next nearest integer.

Ex: Suppose that the bisection method is used to find a zero of $f(x)$ in the interval $[0,1]$. How many times this interval be bisected to guarantee that we have an approximate root with absolute error less than or equal to 10^{-5}.

Sol: Let 'n' denote the required number.

$a_0 = 0, b_0 = 1$ and $M = 5$.

from eqn ⑥

$$n = \text{int}\left[\frac{\log(b_0-a_0) - \log 10^{-M}}{\log 2}\right]$$

$$n = \text{int}\left[\frac{\log 1 - \log 10^{-5}}{\log 2}\right]$$

$$= \text{int}\left[\frac{11.51292547}{0.69314718}\right]$$

$$= \text{int}[16.60964047]$$

$$n = 17. \text{ (approximately)}.$$

→ The following table gives the minimum no. of iterations required to find an approximate root in the interval [0, 1] for various acceptable errors.

E	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24

– This table shows that for getting an approximate value with an absolute error bounded by 10^{-5}, we have to perform 17 iterations.

– Thus even though the bisection method is simple to use, it requires a large no. of iterations to obtain a reasonably good approximate root. This is one of the disadvantages of the bisection method.

Convergence criteria for Secant Method :-

Let $f(x)=0$ be the given eqn. Let α denote a simple root of the eqn $f(x)=0$. Then we have $f'(\alpha)\neq 0$.

The iteration formula for the Secant method is

$$x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i)-f(x_{i-1})} f(x_i) \quad \text{---(i)}$$

For each i, set $\epsilon_i = x_i - \alpha$.

$\Rightarrow x_i = \epsilon_i + \alpha$.

Substituting $x_i = \epsilon_i + \alpha$ in eqn (i)

$$\epsilon_{i+1}+\alpha = \epsilon_i + \alpha - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i+\alpha) - f(\epsilon_{i-1}+\alpha)} f(\epsilon_i+\alpha)$$

$$\epsilon_{i+1} = \epsilon_i - \frac{\epsilon_i - \epsilon_{i-1}}{f(\epsilon_i+\alpha)-f(\epsilon_{i-1}+\alpha)} f(\epsilon_i+\alpha) \quad \text{---(ii)}$$

Now expanding $f(\epsilon_i+\alpha)$ and $f(\epsilon_{i-1}+\alpha)$ using Taylor's theorem about the point $x=\alpha$

we get

$$f(\epsilon_i+\alpha) = f(\alpha) + \frac{f'(\alpha)}{1}\epsilon_i + \frac{f''(\alpha)}{2}\epsilon_i^2 + \cdots\cdots$$

$$= f'(\alpha)\epsilon_i + \frac{f''(\alpha)}{2}\epsilon_i^2 + \cdots\cdots \quad (\because f(\alpha)=0)$$

$$= f'(\alpha)\left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\cdots\right] \quad \text{---(iii)}$$

Similarly

$$f(\epsilon_{i-1}+\alpha) = f'(\alpha)\left[\epsilon_{i-1} + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_{i-1}^2 + \cdots\cdots\right] \quad \text{---(iv)}$$

$$\therefore f(\epsilon_i+\alpha) - f(\epsilon_{i-1}+\alpha) = f'(\alpha)\left[(\epsilon_i - \epsilon_{i-1}) + (\epsilon_i^2 - \epsilon_{i-1}^2)\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\cdots\right]$$

$$= f'(\alpha)(\epsilon_i - \epsilon_{i-1})\left[1 + (\epsilon_i+\epsilon_{i-1})\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\cdots\right] \quad \text{---(v)}$$

Substituting eqns (iii) & (v) in eqn (ii), we get

$$\epsilon_{i+1} = \epsilon_i - \frac{(\epsilon_i - \epsilon_{i-1})\, f'(\alpha)\left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\right]}{f'(\alpha)(\epsilon_i - \epsilon_{i-1})\left[1 + (\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{2f'(\alpha)} + \cdots\right]}$$

$$= \epsilon_i - \left[\epsilon_i + \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_i^2 + \cdots\right]\left[1 + \frac{1}{2}(\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \epsilon_i - \left[\epsilon_i + \frac{1}{2}\epsilon_i^2\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 - \frac{1}{2}(\epsilon_i + \epsilon_{i-1})\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$= \epsilon_i - \left[\epsilon_i + \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}(\cancel{\epsilon_i^2} - \cancel{\epsilon_i^2} - \epsilon_i\epsilon_{i-1}) + \cdots\right]$$

By neglecting the terms involving $\epsilon_i\epsilon_{i-1}^2 + \epsilon_i^2\epsilon_{i-1}$ in the above expression, we get

$$\epsilon_{i+1} \approx \epsilon_i\epsilon_{i-1}\left[\frac{f''(\alpha)}{2f'(\alpha)}\right] \quad \text{—— (vi)}$$

This relationship between the errors is called error eqn. This relationship holds only if α is a simple root.

Now using eqn (vi) we will find the numbers p and λ such that

$$\epsilon_{i+1} = \lambda\epsilon_i^p; \quad i = 0, 1, 2, \cdots \quad \text{—— (vii)}$$

setting $i = j-1$, we obtain

$$\epsilon_j = \lambda\epsilon_{j-1}^p$$

(or) $\epsilon_i = \lambda\epsilon_{i-1}^p$ —— (viii)

Taking p^{th} root on both sides of (viii), we get

$$\epsilon_i^{1/p} = \lambda^{1/p}\epsilon_{i-1}$$

$$\Rightarrow \epsilon_{i-1} = \lambda^{-1/p}\epsilon_i^{1/p} \quad \text{—— (ix)}$$

from eqns (vi) & (vii), we have

$$\lambda \epsilon_i^p = \epsilon_i \epsilon_{i-1} \frac{f''(\alpha)}{2f'(\alpha)}.$$

$$\Rightarrow \lambda \epsilon_i^p = \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i \lambda^{-1/p} \epsilon_i^{1/p} \quad (\text{by eqn (ix)})$$

$$\Rightarrow \lambda \epsilon_i^p = \frac{f''(\alpha)}{2f'(\alpha)} \lambda^{-1/p} \epsilon_i^{1+\frac{1}{p}} \quad \text{——— (x)}$$

equating the powers of ϵ_i on both sides of eqn(x)

we get $p = 1 + \frac{1}{p}$

$\Rightarrow p^2 - p - 1 = 0$

which gives $p = \frac{1 \pm \sqrt{5}}{2}$ ($\because$ p cannot be $-$ve). neglecting the minus sign

$$\boxed{p = \frac{1+\sqrt{5}}{2} \simeq 1.618}$$

Now, to get the number λ, we equate the constant terms on both sides of eqn(x),

we get $\lambda = \frac{f''(\alpha)}{2f'(\alpha)} \lambda^{-1/p}$

$$\Rightarrow \lambda^{1+\frac{1}{p}} = \frac{f''(\alpha)}{2f'(\alpha)}$$

$$\Rightarrow \boxed{\lambda = \left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{p/p+1}}$$

Hence the order of convergence of the secant method is $p = 1.62$ and the asymptotic error constant is $\left[\frac{f''(\alpha)}{2f'(\alpha)}\right]^{p/p+1}$

Ex: The following are the five successive iterations obtained by secant method to find the root $\alpha = -2$ of the eqn $x^3 - 3x + 2 = 0$.

$x_1 = -2.6$, $x_2 = -2.4$, $x_3 = -2.106598985$

$x_4 = -2.022641412$ and $x_5 = -2.000022537$

Compute the asymptotic error constant and show that $\epsilon_5 \simeq \lambda \epsilon_4$.

Solⁿ: Let $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$ and $f''(x) = 6x$

$\therefore f'(-2) = 9$ and $f''(-2) = -12$

we have $\lambda = \left[\dfrac{f''(\alpha)}{2f'(\alpha)}\right]^{p/1+p}$

$\lambda = \left(\dfrac{-12}{18}\right)^{\frac{1.62}{1+1.62}} = \left(-\dfrac{2}{3}\right)^{\frac{1.62}{2.62}}$

$= \left(-\dfrac{2}{3}\right)^{0.618}$

$\lambda = -0.778351205$

Now $\epsilon_5 = |x_5 - \alpha|$

$= |-2.000022537 + 2|$

$= 0.000022537$

and $\epsilon_4 = |x_4 - \alpha|$

$= |-2.022641412 + 2|$

$= 0.022641412$

Then $\lambda\epsilon_4 = 0.778351205 \times 0.022641412$

$= 0.000021246$

$\simeq 0.00002253.$

$\simeq \epsilon_5$

$\therefore \lambda\epsilon_4 \simeq \epsilon_5.$

Convergence of Newton-Raphson Method:

Newton-Raphson iteration formula is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{——} ①$$

To obtain the order of convergence, assume that α is a simple root of $f(x)=0$.

Let $x_i - \alpha = \epsilon_i$, $i = 0, 1, 2, \ldots$

Also $x_{i+1} - \alpha = \epsilon_{i+1}$

$\therefore$ from ①

$$\epsilon_{i+1} + \alpha = \epsilon_i + \alpha - \frac{f(\epsilon_i + \alpha)}{f'(\epsilon_i + \alpha)}$$

$$\Rightarrow \epsilon_{i+1} = \epsilon_i - \frac{f(\epsilon_i+\alpha)}{f'(\epsilon_i+\alpha)} = \frac{\epsilon_i f'(\epsilon_i+\alpha) - f(\epsilon_i+\alpha)}{f'(\epsilon_i+\alpha)}$$

Now expanding $f(\epsilon_i+\alpha)$ and $f'(\epsilon_i+\alpha)$, using Taylor's theorem, about the point α,

we obtain

$$\epsilon_{i+1} = \frac{\epsilon_i\left[f'(\alpha) + \epsilon_i f''(\alpha) + \frac{\epsilon_i^2}{2} f'''(\alpha) + \cdots\right] - \left[f(\alpha) + \epsilon_i f'(\alpha) + \frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right]}{f'(\alpha) + \epsilon_i f''(\alpha) + \epsilon_i^2 f'''(\alpha) + \cdots}$$

But $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

$$\therefore \epsilon_{i+1} = \frac{\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots}{f'(\alpha)\left[1 + \epsilon_i \frac{f''(\alpha)}{f(\alpha)} + \cdots\right]}$$

$$= \left[\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right] \frac{1}{f'(\alpha)} \left[1 + \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$= \frac{1}{f'(\alpha)} \left[\frac{\epsilon_i^2}{2} f''(\alpha) + \cdots\right]\left[1 - \epsilon_i \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

On neglecting ϵ_i^3 and higher power of ϵ_i, we get $\epsilon_{i+1} = \frac{f''(\alpha)}{2f'(\alpha)} \epsilon_i^2$

This shows that the errors satisfy the inequality $|\epsilon_{i+1}| \le \lambda |\epsilon_i|^p$ with

$$p=2 \text{ and } \lambda = \frac{f''(\alpha)}{2f'(\alpha)}.$$

Hence Newton Raphson method is of order 2

i.e, the Newton Raphson method has second order convergence.

and the error is proportional to the square of the previous errors in each step.

<u>Note</u>: If α is a multiple root i.e, $f'(\alpha)=0$, then the convergence is not quadratic, but only linear.

<u>For example</u>:

Let $f(x)=(x-2)^4=0$. Starting with the initial approximation $x_0 = 2.1$, compute the iterations x_1, x_2, x_3 and x_4 using Newton-Raphson method. Is the sequence converging quadratically or linearly?

<u>Solⁿ</u>:- Let $f(x)=(x-2)^4$. The given function has multiple roots at $x=2$ is of order 4.

Newton Raphson iteration formula for the given equation is

$$x_{i+1} = x_i - \frac{(x_i-2)^4}{4(x_i-2)^3}$$

$$= x_i - \frac{1}{4}(x_i-2) = \frac{1}{4}(3x_i+2)$$

Starting with $x_0 = 2.1$, the iterations are given by

$$x_1 = \frac{1}{4}(6.3+2) = \frac{8.3}{4} = 2.075$$

Similarly, $x_2 = 2.05625$

$x_3 = 2.0421875$

$x_4 = 2.031640625$

Now $\epsilon_0 = x_0 - 2 = 0.1$

$\epsilon_1 = x_1 - 2 = 0.075$

$\epsilon_2 = x_2 - 2 = 0.05625$

$\epsilon_3 = x_3 - 2 = 0.0421875$

$\epsilon_4 = x_4 - 2 = 0.031640625$

we have $\epsilon_1 = 0.075$

$= \frac{3}{4} \times 0.1$

$= \frac{3}{4}\epsilon_0$

$\therefore \epsilon_1 = \frac{3}{4}\epsilon_0$

and $\epsilon_2 = \frac{3}{4}\epsilon_1$

$\epsilon_3 = \frac{3}{4}\epsilon_2$

$\epsilon_4 = \frac{3}{4}\epsilon_3$

$\epsilon_{n+1} = \lambda(\epsilon_n)^p$ here $p = 1$ type.

i.e, the convergence is linear in this case.

Also the error is reduced by a factor of $\frac{3}{4}$ with each iteration.

H.W. → The quadratic eqn $x^4 - 4x^2 + 4 = 0$ has a double root at $x = \sqrt{2}$. Starting with $x_0 = 1.5$, compute three successive iterations to the root by Newton-Raphson Method. Does the result converge quadratically or linearly?

Example 2.1 The equation

$$8x^3 - 12x^2 - 2x + 3 = 0$$

has three real roots. Find the intervals each of unit length containing each one of these roots.

We prepare a table of the values of the function $f(x)$ for various values of x (Table 2.1).

Table 2.1 Values of f(x)

x	-2	-1	0	1	2	3
$f(x)$	-105	-15	3	-3	15	105

From the table, we find that the equation $f(x) = 0$ has roots in the intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$. The exact roots are -0.5, 0.5 and 1.5.

Example 2.2 Obtain an interval which contains a root of the equation

$$f(x) = \cos x - x e^x = 0.$$

We prepare a table of the values of the function $f(x)$ for various values of x (Table 2.2).

Table 2.2 Values of $f(x)$

x	0	0.5	1	1.5	2
$f(x)$	1	0.0532	-2.1780	-6.6518	-15.1942

From the table we find that the equation $f(x) = 0$ has at least one root in the interval (0.5, 1). The exact root correct to ten decimal places is 0.5177573637.

which is same as (2.14).

Example 2.7 Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

The smallest positive root lies in the interval (0, 1). Take the initial approximation as $x_0 = 0.5$. We have

$$f(x) = x^3 - 5x + 1,\ f'(x) = 3x^2 - 5.$$

Using the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5},\ k = 0, 1, \cdots$$

Starting with $x_0 = 0.5$, we obtain

$$x_1 = 0.176471,\ x_2 = 0.201568.$$
$$x_3 = 0.201640,\ x_4 = 0.201640.$$

The exact value correct to six decimal places is 0.201640.

Perform four iterations of the Newton-Raphson method to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation $x_0 = 2$.

Let $x = (17)^{1/3}$. We obtain $x^3 = 17$ and $f(x) = x^3 - 17 = 0$. Using the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},\ k = 0, 1, \cdots$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 17}{3x_k^2} = \frac{2x_k^3 + 17}{3x_k^2},\ k = 0, 1, \cdots$$

Starting with $x_0 = 2$, we obtain

$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = 2.75, \qquad x_2 = \frac{2x_1^3 + 17}{3x_1^2} = 2.582645$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = 2.571332, \quad x_4 = \frac{2x_3^3 + 17}{3x_3^2} = 2.571282.$$

The exact value correct to six decimal places is 2.571282.

Apply Newton-Raphson's method to determine a root of the equation

$$f(x) = \cos x - xe^x = 0$$

such that $|f(x^*)| < 10^{-8}$, where x^* is the approximation to the root. Take the initial approximation as $x_0 = 1$.

We write (2.14) in the form

$$x_{k+1} = x_k - \Delta x_k,\ k = 0, 1, 2, \cdots$$

where $$\Delta x_k = \frac{f(x_k)}{f'(x_k)} = \frac{(\cos x_k - x_k e^{x_k})}{(-\sin x_k - x_k e^{x_k} - e^{x_k})}$$

Starting with $x_0 = 1$, we get

$$\Delta x_0 = \frac{\cos x_0 - x_0 e^{x_0}}{-\sin x_0 - x_0 e^{x_0} - e^{x_0}} = \frac{-2.17797952}{-6.27803464} = 0.34692060$$

$$x_1 = x_0 - \Delta x_0 = 1 - 0.34692060 = 0.65307940$$

$$\Delta x_1 = \frac{\cos x_1 - x_1 e^{x_1}}{-\sin x_1 - x_1 e^{x_1} - e^{x_1}} = \frac{-0.46064211}{-3.78394215} = 0.12173603$$

$$x_2 = x_1 - \Delta x_1 = 0.53134337.$$

The results obtained are given in Table 2.7.

Approximations to the Root by the Newton-Raphson Method

k	x_k	Δx_k	x_{k+1}	$f(x_{k+1})$
0	1.0	0.3469	0.65307940	− 0.4606
1	0.65307940	0.1217	0.53134337	− 0.4180(− 1)
2	0.53134337	0.1343(− 1)	0.51790991	− 0.4641(− 3)
3	0.51790991	0.1525(− 3)	0.51775738	− 0.5926(− 7)
4	0.51775738	0.1948(− 7)	0.51775736	− 0.2910(− 10)

2.5 RATE OF CONVERGENCE

We now study the rate at which the iteration method converges if the initial approximation to the root is sufficiently close to the desired root.

Definition 2.4 An iterative method is said to be of **order** p or has the rate of **convergence** p, if p is the largest positive real number for which there exists a finite constant $C \neq 0$ such that

$$|\varepsilon_{k+1}| \leq C|\varepsilon_k|^p \tag{2.34}$$

where $\varepsilon_k = x_k - \xi$ is the error in the kth iterate.

The constant C is called the **asymptotic error constant** and usually depends on derivatives of $f(x)$ at $x = \xi$.

Secant Method

We assume that ξ is a simple root of $f(x) = 0$. Substituting $x_k = \xi + \varepsilon_k$ in (2.12) we obtain

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})f(\xi+\varepsilon_k)}{f(\xi+\varepsilon_k) - f(\xi+\varepsilon_{k-1})} \tag{2.35}$$

Expanding $f(\xi + \varepsilon_k)$ and $f(\xi + \varepsilon_{k-1})$ in Taylor's series about the point ξ and noting that $f(\xi) = 0$, we get

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})\left[\varepsilon_k\, f'(\xi) + \frac{1}{2}\varepsilon_k^2\, f''(\xi) + \cdots\right]}{(\varepsilon_k - \varepsilon_{k-1})\, f'(\xi) + \frac{1}{2}(\varepsilon_k^2 - \varepsilon_{k-1}^2)\, f''(\xi) + \cdots}$$

$$\varepsilon_{k+1} = C^*\, \varepsilon_k \tag{2.39}$$

where $C^* = C\varepsilon_0$ is the asymptotic error constant. Hence, the Regula-Falsi method has linear rate of convergence.

Newton-Raphson Method

On substituting $x_k = \xi + \varepsilon_k$ in (2.14) and expanding $f(\xi + \varepsilon_k)$, $f'(\xi + \varepsilon_k)$ in Taylor's series about the point ξ, we obtain

$$\varepsilon_{k+1} = \varepsilon_k - \frac{\left[\varepsilon_k\, f'(\xi) + \frac{1}{2}\varepsilon_k^2\, f''(\xi) + \cdots\right]}{f'(\xi) + \varepsilon_k\, f''(\xi) + \cdots}$$

$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]\left[1 + \frac{f''(\xi)}{f'(\xi)}\varepsilon_k + \cdots\right]^{-1}$$

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + O(\varepsilon_k^3).$$

On neglecting ε_k^3 and higher powers of ε_k, we get

$$\varepsilon_{k+1} = C\, \varepsilon_k^2 \tag{2.40}$$

where
$$C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}.$$

Thus, the Newton-Raphson method has second order convergence.

Muller Method

On substituting $x_j = \xi + \varepsilon_j$, $j = k-2, k-1, k$ and expanding $f(\xi + \varepsilon_j)$ in Taylor's series about the point ξ in (2.25*a*) and using $f(\xi) = 0$, we get

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$$= \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\varepsilon_k^2 \frac{f''(\xi)}{f'(\xi)} + \cdots\right]\left[1 + \frac{1}{2}(\varepsilon_{k-1} + \varepsilon_k)\frac{f''(\xi)}{f'(\xi)} + \cdots\right]^{-1}$$

or
$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k \varepsilon_{k-1} + O(\varepsilon_k^2 \varepsilon_{k-1} + \varepsilon_k \varepsilon_{k-1}^2)$$

or
$$\varepsilon_{k+1} = C\varepsilon_k \varepsilon_{k-1} \tag{2.36}$$

where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and higher powers of ε_k are neglected.

The relation of the form (2.36) is called the **error equation**. Keeping in view the definition of the rate of convergence, we seek a relation of the form

$$\varepsilon_{k+1} = A\varepsilon_k^p \tag{2.37}$$

where A and p are to be determined.
From (2.37) we have

$$\varepsilon_k = A\varepsilon_{k-1}^p \text{ or } \varepsilon_{k-1} = A^{-1/p}\varepsilon_k^{1/p}$$

Substituting the values of ε_{k+1} and ε_{k-1} in (2.36) we obtain

$$\varepsilon_k^p = CA^{-(1+1/p)}\varepsilon_k^{1+1/p}. \tag{2.38}$$

Comparing the powers of ε_k on both sides, we get

$$p = 1 + \frac{1}{p}$$

which gives
$$p = \frac{1}{2}(1 \pm \sqrt{5}).$$

Neglecting the minus sign, we find that the rate of convergence for the secant method (2.12) is $p = 1.618$.

From (2.38), we also obtain $A = C^{p/(p+1)}$.

Regula – falsi Method.

If the function $f(x)$ in the equation $f(x) = 0$ is convex in the interval (x_0, x_1) that contains the root, then one of the points x_0 or x_1 is always fixed and the other point varies with k. If the point x_0 is fixed, then the function $f(x)$ is approximated by the straight line passing through the points (x_0, f_0) and (x_k, f_k), $k = 1, 2, \ldots$. The error equation (2.36) becomes

$$\varepsilon_{k+1} = C\,\varepsilon_0\,\varepsilon_k$$

where $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$ and $\varepsilon_0 = x_0 - \xi$ is independent of k. Therefore, we can write

Example 2.18 Show that the following two sequences have convergence of the second order with the same limit $\sqrt{a}$.

(i) $x_{n+1} = \frac{1}{2}x_n\left(1 + \frac{a}{x_n^2}\right)$, (ii) $x_{n+1} = \frac{1}{2}x_n\left(3 - \frac{x_n^2}{a}\right)$

If x_n is a suitably close approximation to $\sqrt{a}$, show that the magnitude of the error in the first formula for x_{n+1} is about one-third of that in the second formula, and deduce that the formula

(iii) $x_{n+1} = \frac{1}{8}x_n\left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$

gives a sequence with third-order convergence.

Taking the limits as $n \to \infty$ and noting that $\lim_{n\to\infty} x_n = \xi$, $\lim_{n\to\infty} x_{n+1} = \xi$, where ξ is the exact root, we obtain from all the three methods $\xi^2 = a$. Thus all the three methods determine $\sqrt{a}$, where a is any positive real number.

Substituting $x_n = \xi + \varepsilon_n$, $x_{n+1} = \xi + \varepsilon_{n+1}$ and $a = \xi^2$, we get

(i) $$\xi + \varepsilon_{n+1} = \frac{1}{2}(\xi + \varepsilon_n)\left[1 + \frac{\xi^2}{(\xi+\varepsilon_n)^2}\right]$$

$$= \frac{1}{2}(\xi + \varepsilon_n)\left[1 + \left(1 + \frac{\varepsilon_n}{\xi}\right)^{-2}\right] = \frac{1}{2}(\xi + \varepsilon_n)\left[2 - \frac{2\varepsilon_n}{\xi} + \frac{3\varepsilon_n^2}{\xi^2} - \cdots\right]$$

$$= \frac{1}{2}\left[2\xi_n + (2-2)\varepsilon_n + (3-2)\frac{\varepsilon_n^2}{\xi} + \cdots\right]$$

Therefore,

$$\varepsilon_{n+1} = \frac{1}{2\xi}\varepsilon_n^2 + O(\varepsilon_n^3). \tag{2.47}$$

Hence, the method has second order convergence, with the error constant $C = 1/(2\xi)$.

(ii) $$\xi + \varepsilon_{n+1} = \frac{1}{2}(\xi + \varepsilon_n)\left[3 - \frac{1}{\xi^2}(\xi+\varepsilon_n)^2\right] = (\xi + \varepsilon_n)\left(1 - \frac{\varepsilon_n}{\xi} - \frac{\varepsilon_n^2}{2\xi^2}\right)$$

Therefore,

$$\varepsilon_{n+1} = -\frac{3}{2\xi}\varepsilon_n^2 + O(\varepsilon_n^3). \tag{2.48}$$

Hence, the method has second order convergence with the error constant $C^* = -3/(2\xi)$.

Therefore, the magnitude of the error in the first formula is about one-third of that in the second formula.

(iii) If we multiply (2.47) by 3 and add to (2.48), we find that

$$\varepsilon_{n+1} = O(\varepsilon_n^3). \tag{2.49}$$

It can be verified that $O(\varepsilon_n^3)$ term in (2.49) does not vanish.

Adding 3 times the first formula to the second formula, we obtain the new formula

$$x_{n+1} = \frac{1}{8}x_n\left(6 + \frac{3a}{x_n} - \frac{x_n^2}{a}\right)$$

which has third order convergence.

Example 2.19 Let the function $f(x)$ be four times continuously differentiable and have a simple zero ξ. Successive approximations x_n, $n = 1, 2, \cdots$ to ξ are computed from

$$x_{n+1} = \frac{1}{2}(x'_{n+1} + x''_{n+1})$$

where
$$x'_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ x''_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},\ g(x) = \frac{f(x)}{f'(x)}$$

Prove that if the sequence $\{x_n\}$ converges to ξ, then the convergence is cubic.

[Lund. Univ., Sweden, BIT 8 (1968), 59)

We have

$$g(x) = \frac{f(x)}{f'(x)},\ g'(x) = \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$$

$$x'_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x''_{n+1} = x_n - \frac{f(x_n)/f'(x_n)}{1 - f(x_n)f''(x_n)/(f'(x_n))^2}$$

$$= x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{f(x_n)f''(x_n)}{(f'(x_n))^2} + \left\{\frac{f(x_n)f''(x_n)}{(f'(x_n))^2}\right\}^2 + \cdots\right]$$

From the formula

$$x_{n+1} = \frac{1}{2}(x'_{n+1} + x''_{n+1})$$

we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]^2 \frac{f''(x_n)}{f'(x_n)} - \frac{1}{2}\left[\frac{f(x_n)}{f'(x_n)}\right]^3 \left[\frac{f''(x_n)}{f'(x_n)}\right]^2 + \cdots \qquad (2.50)$$

Using $x_n = \xi + \varepsilon_n$, and

$$C_i = \frac{f^{(i)}(\xi)}{f'(\xi)},\ i = 1, 2, 3, \cdots$$

we get

$$\frac{f(x_n)}{f'(x_n)} = \varepsilon_n - \frac{1}{2}C_2\varepsilon_n^2 + \left(\frac{1}{2}C_2^2 - \frac{1}{3}C_3\right)\varepsilon_n^3 + \cdots$$

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$$\frac{f''(x_n)}{f'(x_n)} = C_2 + (C_3 - C_2^2)\,\varepsilon_n + \cdots$$

Using these expressions in (2.50), we obtain the error equation, on simplification, as

$$\begin{aligned}\varepsilon_{n+1} &= \varepsilon_n - \left[\varepsilon_n - \frac{1}{2}C_2\,\varepsilon_n^2 + \left(\frac{1}{2}C_2^2 - \frac{1}{3}C_3\right)\varepsilon_n^2 + \cdots\right] \\ &\quad - \frac{1}{2}\,[\varepsilon_n^2 - C_2\,\varepsilon_n^3 + \cdots]\,[C_2 + (C_3 - C_2^2)\varepsilon_n + \cdots] \\ &\quad - \frac{1}{2}\,[\varepsilon_n^3 + \cdots]\,[C_2^2 + 2C_2\,(C_3 - C_2^2)\varepsilon_n + \cdots] + \cdots \\ &= -\frac{1}{6}C_3\,\varepsilon_n^3 + O(\varepsilon_n^4).\end{aligned}$$

Hence the method has cubic convergence.

Example 2.20 The equation $x^4 + x = \varepsilon$, where ε is a small number, has a root which is close to ε. Computation of this root is done by the expression

$$\xi = \varepsilon - \varepsilon^4 + 4\varepsilon^7$$

(i) Find an iterative formula $x_{n+1} = F(x_n)$, $x_0 = 0$, for the computation. Show that we get the expression above after three iterations when neglecting terms of higher order.

(ii) Give a good estimate (of the form $N\,\varepsilon^k$, where N and k are integers) of the maximal error when the root is estimated by the expression above.

(Inst. Tech., Stockholm, Sweden, BIT 9(1969), 87)

We write the given equation $x^4 + x = \varepsilon$ in the form

$$x = \frac{\varepsilon}{x^3 + 1}$$

and consider the formula

$$x_{n+1} = \frac{\varepsilon}{x_n^3 + 1}.$$

Starting with $x_0 = 0$, we obtain

$$x_1 = \varepsilon$$

$$\begin{aligned}x_2 &= \frac{\varepsilon}{1+\varepsilon^3} \\ &= \varepsilon(1 + \varepsilon^3)^{-1} = \varepsilon(1 - \varepsilon^3 + \varepsilon^6 + \cdots) \\ &= \varepsilon - \varepsilon^4 + \varepsilon^7 \quad \text{(neglecting higher powers of } \varepsilon\text{)}\end{aligned}$$

$$x_3 = \frac{\varepsilon}{1+\left(\varepsilon-\varepsilon^4+\varepsilon^7\right)^3}$$

$$= \varepsilon - \varepsilon^4 + 4\varepsilon^7 \quad \text{(neglecting higher powers of } \varepsilon\text{)}.$$

Taking $\xi = \varepsilon - \varepsilon^4 + 4\varepsilon^7$, we find that

$$\begin{aligned} \text{Error} &= \xi^4 + \xi - \varepsilon \\ &= (\varepsilon - \varepsilon^4 + 4\varepsilon^7)^4 + (\varepsilon - \varepsilon^4 + 4\varepsilon^7) - \varepsilon \\ &= 22\varepsilon^{10} + \text{higher powers of } \varepsilon. \end{aligned}$$

Example 2.21 How should the constant α be chosen to ensure the fastest possible convergence with the iteration formula

$$x_{n+1} = \frac{\alpha x_n + x_n^{-2} + 1}{\alpha + 1}$$

(Uppsala Univ., Sweden, BIT 11 (1971), 225)

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \xi$, we get

$$\xi = \frac{1}{\alpha+1}\left(\alpha\,\xi + \frac{1}{\xi^2} + 1\right), \text{ or } (\alpha + 1)\xi^3 = (\alpha\xi^3 + \xi^2 + 1)$$

or

$$\xi^3 - \xi^2 - 1 = 0.$$

Thus, the formula is being used to find a root of the equation

$$f(x) = x^3 - x^2 - 1 = 0.$$

Substituting $x_n = \xi + \varepsilon_n$, $x_{n+1} = \xi + \varepsilon_{n+1}$, we obtain

$$(1 + \alpha)\,(\xi + \varepsilon_{n+1}) = \alpha(\xi + \varepsilon_n) + \frac{1}{\xi^2}\left(1 + \frac{\varepsilon_n}{\xi}\right)^{-2} + 1$$

which gives

$$(1 + \alpha)\varepsilon_{n+1} = \left(\alpha - \frac{2}{\xi^3}\right)\varepsilon_n + O(\varepsilon_n^2).$$

For fastest convergence, we must have $\alpha = 2/\xi^3$.

We can find the approximate value of ξ by using Newton-Raphson method to determine a root of $x^3 - x^2 - 1 = 0$. We obtain $\xi = 1.4656$ and hence $\alpha = 0.6353 \approx 0.64$.

Example 2.22 Determine α_1 and α_2 so that the order of the iterative method

$$x_{k+1} = x_k - \alpha_1\, W_1(x_k) - \alpha_2 W_2(x_k)$$

where $\quad W_1(x_k) = f(x_k)/f'(x_k)$

and $\quad W_2(x_k) = f(x_k)/f'(x_k + \beta W_1(x_k)), \quad \beta \neq 0.$

for finding a simple root of the equation $f(x) = 0$ becomes as high as possible.

We substitute $x_k = \xi + \varepsilon_k$ and $f(\xi) = 0$ in $W_1(x_k)$ and $W_2(x_k)$ to get

$$W_1(\xi + \varepsilon_k) = \frac{f(\xi+\varepsilon_k)}{f'(\xi+\varepsilon_k)}$$

$$= \varepsilon_k - \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\,\varepsilon_k^2 + O(\varepsilon_k^3)$$

$$\begin{aligned} f'[x_k + \beta W_1(x_k)] &= f'[\xi + \varepsilon_k + \beta\{\varepsilon_k + O(\varepsilon_k^2)\}] \\ &= f'[\xi + (1+\beta)\,\varepsilon_k + O(\varepsilon_k^2)] \\ &= f'(\xi) + (1+\beta)\,\varepsilon_k f''(\xi) + O(\varepsilon_k^2) \end{aligned}$$

$$W_2(x_k) = \left[\varepsilon_k f'(\xi) + \frac{1}{2}\varepsilon_k^2 f''(\xi) + \cdots\right] \times$$

$$\frac{1}{f'(\xi)}\left[1 + (1+\beta)\frac{f''(\xi)}{f'(\xi)}\varepsilon_k + O(\varepsilon_k^2)\right]^{-1}$$

$$= \left[\varepsilon_k + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]\left[1 - (1+\beta)\frac{f''(\xi)}{f'(\xi)}\varepsilon_k + O(\varepsilon_k^2)\right]$$

$$= \varepsilon_k - \frac{1}{2}(1+2\beta)\frac{f''(\xi)}{f'(\xi)}\,\varepsilon_k^2 + O(\varepsilon_k^3).$$

Using these expressions in the iteration method, we get

$$\varepsilon_{k+1} = \varepsilon_k - \alpha_1\left[\varepsilon_k - \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right] - \alpha_2\left[\varepsilon_k - \frac{1}{2}(1+2\beta)\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + \cdots\right]$$

$$= (1 - \alpha_1 - \alpha_2)\,\varepsilon_k + \frac{1}{2}\left[\alpha_1 + \alpha_2(1+2\beta)\right]\frac{f''(\xi)}{f'(\xi)}\,\varepsilon_k^2 + \cdots$$

Equating the coefficients of ε_k and ε_k^2 to zero, we obtain

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 + \alpha_2(1+2\beta) = 0.$$

If $\beta \neq 0$, then we have

$$\alpha_1 = \frac{1+2\beta}{2\beta}, \quad \alpha_2 = -\frac{1}{2\beta}.$$

Thus, the order of the iteration method is three for arbitrary $\beta \neq 0$. For $\beta = -1/2$, we obtain the method (2.31).

Theorem 2.2 *If $\phi(x)$ is a continuous function in some interval $[a, b]$ that contains the root and $|\phi'(x)| \le c < 1$ in this interval, then for any choice of $x_0 \in [a, b]$, the sequence $\{x_k\}$ determined from*

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, 2, \cdots$$

converges to the root ξ of $x = \phi(x)$.

Proof: The exact solution ξ satisfies the equation

$$\xi = \phi(\xi).$$

We have

$$\xi - x_{k+1} = \phi(\xi) - \phi(x_k), \quad k = 0, 1, 2, \cdots$$

Using the Mean Value Theorem, we get

$$\xi - x_{k+1} = (\xi - x_k)\, \phi'(\xi_k), \quad x_k < \xi_k < \xi$$

or

$$\begin{aligned} \varepsilon_{k+1} &= \varepsilon_k\, \phi'(\xi_k) \\ &= \varepsilon_{k-1}\, \phi'(\xi_k)\, \phi'(\xi_{k-1}) \\ &= \varepsilon_0\, \phi'(\xi_k)\, \phi'(\xi_{k-1}) \cdots \phi'(\xi_0) \end{aligned}$$

where $x_0 < \xi_0 < \xi$, $x_1 < \xi_1 < \xi$, $\cdots$, $x_{k-1} < \xi_{k-1} < \xi$.

If $|\phi'(\xi_r)| \le c$, $r = 0, 1, 2, \cdots, k$, then

$$|\varepsilon_{k+1}| \le |\varepsilon_0|\, c^{k+1}$$

If $c < 1$, the right hand side goes to zero as k becomes large. Thus, the iteration method converges if $|\phi'(x)| \le c < 1$. This condition is same as the Lipschitz condition and c is the Lipschitz constant.

Second Order Method

Here $a_1 = 0$, $a_2 \ne 0$ and the equation (2.54) becomes

$$\varepsilon_{k+1} = a_2\, \varepsilon_k^2, \quad k = 0, 1, 2 \cdots \tag{2.56}$$

We have,

$$\varepsilon_k = a_2\,\varepsilon_{k-1}^2 = a_2\,(a_2\,\varepsilon_{k-2}^2)^2 = a_2^{2^2-1}\,(\varepsilon_{k-2})^{2^2}$$
$$= a_2^{2^2-1}\,(a_2\,\varepsilon_{k-3}^2)^{2^2} = a_2^{2^3-1}\,\varepsilon_{k-3}^{2^3} = \cdots = a_2^{2^k-1}\,\varepsilon_0^{2^k}.$$

If $|a_2| \approx 1$, ε_0 is small, then the iteration method (2.51) converges and has second order convergence, that means, each successive iteration approximately doubles the number of significant digits of accuracy.

High Order Methods

Definition 2.5 The iteration method (2.51) is said to be of the p-th order if

$$\phi'(\xi) = \phi''(\xi) = \cdots = \phi^{(p-1)}(\xi) = 0,$$
$$\phi^{(p)}(\xi) \neq 0$$

where ξ is the solution of $x = \phi(x)$.

The equation (2.54) for a p-th order method becomes

$$\varepsilon_{k+1} = \frac{1}{p!}\,\phi^{(p)}(\xi)\,\varepsilon_k^p + O(\varepsilon_k^{p+1}). \qquad (2.57)$$

Thus, the number of significant digits of accuracy at each step is approximately p-times the number of significant digits of accuracy of the previous step.

Note that for the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \phi(x_k)$$

we have

$$\phi(x) = x - \frac{f(x)}{f'(x)}$$

$$\phi'(x) = 1 - \frac{[f'(x)]^2 - f(x)\,f''(x)}{[f'(x)]^2} = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

$$\phi''(x) = \frac{1}{[f'(x)]^3}\,[f'(x)\,\{f'(x)\,f''(x) + f(x)\,f'''(x)\} - 2f(x)\,(f''(x))^2]$$

Since ξ is an exact root, we have $f(\xi) = 0$ and $f'(\xi) \neq 0$. We find that

$$\phi(\xi) = \xi,\ \phi'(\xi) = 0 \text{ and } \phi''(\xi) = f''(\xi)/f'(\xi) \neq 0.$$

Therefore, the Newton-Raphson method is a second order method.

For the Chebyshev method (2.29)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{1}{2}\left[\frac{f(x_k)}{f'(x_k)}\right]^2 \frac{f''(x_k)}{f'(x_k)} = \phi(x_k)$$

we have

$$\phi(x) = x - \frac{f(x)}{f'(x)} - \frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2 \frac{f''(x)}{f'(x)}.$$

Using $f(\xi) = 0, f'(\xi) \neq 0$, it can be easily verified that

$$\phi(\xi) = \xi,\ \phi'(\xi) = \phi''(\xi) = 0 \text{ and } \phi'''(\xi) \neq 0.$$

Therefore, the Chebyshev method is a third order method.

Example 2.23 The equation

$$f(x) = 3x^3 + 4x^2 + 4x + 1 = 0$$

has a root in the interval $(-1, 0)$. Determine an iteration function $\phi(x)$, such that the sequence of iterations obtained from

$$x_{k+1} = \phi(x_k),\ x_0 = -0.5,\ k = 0, 1, \cdots$$

converges to the root.

We write the given equation as

$$x = x + \alpha\,(3x^3 + 4x^2 + 4x + 1) = \phi(x)$$

where α is an arbitrary constant to be determined such that

$$|\phi'(x)| = |1 + \alpha\,(9x^2 + 8x + 4)| < 1$$

for all $x \in (-1, 0)$. Since, $g(x) = 9x^2 + 8x + 4 > 0$ for $x \in (-1, 0)$, we find that $\alpha < 0$. The condition $|\phi'(x)| < 1$ must also be satisfied at the initial approximation $x_0 = -0.5$. Using this condition, we get

$$|\phi'(-0.5)| = \left|1 + \frac{9\alpha}{4}\right| < 1$$

or

$$-1 < 1 + \frac{9\alpha}{4} < 1 \text{ or } -\frac{8}{9} < \alpha < 0.$$

The range of α depends on x_0.

For example, when $\alpha = -1/2$, we obtain the iteration method

$$x_{k+1} = x_k - \frac{1}{2}\,(3x_k^3 + 4\,x_k^2 + 4x_k + 1) = -\frac{1}{2}\,(3x_k^3 + 4x_k^2 + 2x_k + 1) = \phi(x_k).$$

Starting with $x_0 = -0.5$, we obtain

$$x_1 = -0.3125,\quad x_2 = -0.337036,\ x_3 = -0.332723,$$
$$x_4 = -0.333435,\ x_5 = -0.333316.$$

We can verify that $|\phi'(x_j)| < 1$ for all j. The exact root is $-1/3$.

Acceleration of the Convergence

The linear convergence of an iterative method of the form (2.51) can be improved with the help of the **Aitken** Δ^2 method. The error in two successive approximations, using (2.55), may be written as

$$\varepsilon_{k+1} = a_1\, \varepsilon_k$$
$$\varepsilon_{k+2} = a_1\, \varepsilon_{k+1}. \tag{2.58}$$

Eliminating a_1, we get

$$\varepsilon_{k+2}\, \varepsilon_k = \varepsilon_{k+1}^2$$

or $$(\xi - x_{k+2})(\xi - x_k) = (\xi - x_{k+1})^2.$$

Solving for ξ, we get

$$\xi = \frac{x_k x_{k+2} - x_{k+1}^2}{x_{k+2} - 2x_{k+1} + x_k}$$
$$= x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k} = x_k - \frac{(\Delta x_k)^2}{\Delta^2 x_k} \tag{2.59}$$

where $\Delta x_k = x_{k+1} - x_k$ and $\Delta^2 x_k = x_{k+2} - 2x_{k+1} + x_k$ are the first and the second order forward differences respectively (see Chapter 4).

The number ξ gives an improved value of the approximation x_{k+2}. The computational procedure can be written as follows. Choose an initial approximation x_0 and calculate $x_1 = \phi(x_0)$, $x_2 = \phi(x_1)$, and determine the sequence $\{x_k\}$ from the following:

$$x_{3k+2}^* = x_{3k} - \frac{(\phi(x_{3k}) - x_{3k})^2}{\phi(\phi(x_{3k})) - 2\phi(x_{3k}) + x_{3k}},\ k = 0, 1, 2, \cdots$$

Example 2.24 Perform two iterations of the linear iteration method followed by one iteration of the Aitken Δ^2 method to find the root of the equations

(i) $f(x) = x^3 - 5x + 1 = 0$, $x_0 = 0.5$.
(ii) $f(x) = x - e^{-x} = 0$, $x_0 = 1$.

Repeat the process two times in each case.

(i) The root lies in the interval (0, 1). We write the given equation as

$$x = \frac{1}{5}(x^3 + 1) = \phi(x).$$

Now, $\phi'(x) = 3x^2/5$ and $|\phi'(x)| < 1$ in the interval (0, 1). Therefore, the iteration method

$$x_{k+1} = \phi(x_k) = \frac{1}{5}(x_k^3 + 1),\ k = 0, 1, \cdots$$

will converge to the root. Starting with $x_0 = 0.5$, we obtain

$$x_1 = \phi(x_0) = 0.225$$
$$x_2 = \phi(x_1) = 0.202278$$
$$x_0^* = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 0.5 - \frac{(-0.275)^2}{0.252278} = 0.200232$$
$$x_1^* = \phi(x_0^*) = 0.201606$$
$$x_2^* = \phi(x_1^*) = 0.201639$$

Chapter 2

$$x_0^{**} = x_0^* - \frac{(x_1^* - x_0^*)^2}{x_2^* - 2x_1^* + x_0^*} = 0.200232 - \frac{(0.001375)^2}{-0.001342}$$

$$= 0.201640.$$

ii) The root lies in the interval (0, 1). We write the given equation as

$$x = e^{-x} = \phi(x).$$

Now, $\phi'(x) = -e^{-x}$ and $|\phi'(x)| < 1$ in the interval (0, 1).

Therefore, the iteration method

$$x_{k+1} = \phi(x_k) = e^{-x_k},\ k = 0, 1, \cdots$$

will converge to the root. Starting with $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 0.367879$$

$$x_2 = \phi(x_1) = 0.692201$$

$$x_0^* = x_0 - \frac{(x_1 - x_0)^2}{x_2 - 2x_1 + x_0} = 1 - \frac{(-0.632121)^2}{0.956443} = 0.582226$$

$$x_1^* = \phi(x_0^*) = 0.558653$$

$$x_2^* = \phi(x_1^*) = 0.571979$$

$$x_0^{**} = x_0^* - \frac{(x_1^* - x_0^*)^2}{x_2^* - 2x_1^* + x_0^*} = 0.582226 - \frac{(-0.023573)^2}{0.036899}$$

$$= 0.567166.$$