particle at any time t, where $OP=x$. Since the displacement is small, therefore the tension in the string in any displaced position can be taken as T which is the tension in the string in the original position. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=-(T\cos\angle OPA+T\cos\angle OPB)$$

$$=-T\left(\frac{OP}{AP}+\frac{OP}{BP}\right)=-T\left(\frac{x}{\sqrt{(a^2+x^2)}}+\frac{x}{\sqrt{(b^2+x^2)}}\right)$$

$$=-T\left\{\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2}+\frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}\right\}$$

$$=-T\left[\frac{x}{a}\left(1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\dots\right)+\frac{x}{b}\left(1-\frac{1}{2}\cdot\frac{x^2}{b^2}+\dots\right)\right]$$

$=-T\left(\frac{x}{a}+\frac{x}{b}\right)$, neglecting higher powers of x/a and x/b which are very small

$$=-T\left(\frac{a+b}{ab}\right)x.$$

$\therefore\quad \frac{d^2x}{dt^2}=-\frac{T(a+b)}{mab}x=-\mu x$, where $\mu=\frac{T(a+b)}{mab}$.

This is the standard equation of a S. H. M. with centre at the origin. The time period

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left\{\frac{T(a+b)}{mab}\right\}}=2\pi\sqrt{\left\{\frac{mab}{T(a+b)}\right\}}.$$

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Ex. 46. *If in a S. H. M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation*

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a)=\begin{vmatrix} u & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

[Kanpur 1980, 85, 88; Meerut 84]

Sol. Let O and O' be the centre of force and the fixed point respectively on the line of motion and let

$OO'=l$. Let u, v, w be the velocities of the particle at P, Q, R respectively where

$$O'P=a,\ O'Q=b,\ O'R=c.$$

For a S.H.M. of amplitude A, the velocity V at a distance x from the centre of force is given by

$$V^2=\mu\,(A^2=x^2). \qquad \ldots(1)$$

At P, $x=OP=l+a$, $V=u$

at Q, $x=OQ=l+b$; $V=v$

and at R, $x=OR=l+c$, $V=w$.

$\therefore$ from (1), we have

$$u^2=\mu\,\{A^2-(l+a)^2\}$$

or
$$\frac{u^2}{\mu}=A^2-l^2-a^2-2al$$

or
$$\left(\frac{u^2}{\mu}+a^2\right)+2l.a+(l^2-A^2)=0. \qquad \ldots(2)$$

Similarly,

$$\left(\frac{v^2}{u}+b^2\right)+2l.b+(l^2-A^2)=0, \qquad \ldots(3)$$

and
$$\left(\frac{w^2}{\mu}+c^2\right)+2l.c+(l^2-A^2)=0. \qquad \ldots(4)$$

Eliminating $2l$ and (l^2-A^2) from (2), (3) and (4), we have

$$\begin{vmatrix} \dfrac{u^2}{\mu}+a^2 & a & 1 \\ \dfrac{v^2}{\mu}+b^2 & b & 1 \\ \dfrac{w^2}{\mu}+c^2 & c & 1 \end{vmatrix}=0$$

or
$$\begin{vmatrix} \dfrac{u^2}{\mu} & a & 1 \\ \dfrac{v^2}{\mu} & b & 1 \\ \dfrac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}=0$$

$$\text{or} \quad -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu} \begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

$$\text{or} \quad \mu \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{or} \quad \mu\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad \ldots(5$$

But the time period $T=\frac{2\pi}{\sqrt{\mu}}$, so that $\mu=\frac{4\pi^2}{T^2}$.

Hence from (5), we have

$$\frac{4\pi^2}{T^2}(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

§ 8. Hooke's Law :

Statement. *The tension of an elastic string is proportional the extension of the string beyond its natural length.*

If x is the stretched length of a string of natural length l, th by Hooke's law the tension T in the string is given by $T=\lambda.\frac{x}{}$ where λ is called the **modulus of elasticity** of the string. **Remem** that the direction of the tension is always opposite to extension.

semi-circles, then

$$\bar{x} = \frac{P_1 x_1 - P_2 x_2}{P_1 - P_2} = \frac{3\pi}{16} \cdot \frac{a^4 - b^4}{a^3 - b^3}.$$

Ex. 12. *From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Prove that the depth of the CP. of the remainder is* $\dfrac{9\pi a}{8(16 - 3\pi)}$.

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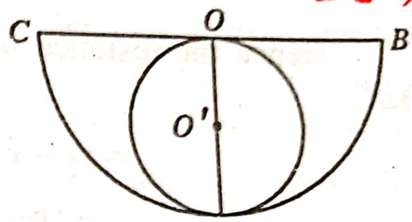
Sol. From Ex. 7, we have

x_1 = depth of the C.P. of the semi-circle below O

$$= \frac{3\pi}{16} a,$$



P_1 = pressure on the semi-circle

$$= w \cdot \frac{1}{2}\pi a^2 \cdot \frac{4a}{3\pi} = \frac{2}{3} a^3 w.$$

Again depth of the C.P. of the circle of radius $\frac{1}{2}a$ below the centre O' is $\dfrac{A^2}{4H}$, where A is its radius $= \dfrac{a}{2}$ and H is the depth of the centre of the circle below the free surface $= OO' = \dfrac{a}{2}$.

$$\therefore \quad \frac{A^2}{4H} = \frac{(a/2)^2}{4(a/2)} = \frac{a}{8}.$$

$$\therefore \quad x_2 = \text{depth of the C.P. of circle below } O = \frac{a}{2} + \frac{a}{8} = \frac{5a}{8},$$

$$P_2 = \text{pressure on the circle} = w \cdot \pi \left(\frac{a}{2}\right)^2 \cdot \frac{a}{2} = \frac{1}{8} w\pi a^3.$$

If $\bar{x}$ be the depth of the C.P. of the remainder below O, then

$$\bar{x} = \frac{P_1 x_1 - P_2 x_2}{P_1 - P_2} = \frac{3a\pi}{64} \cdot \frac{24}{(16 - 3\pi)} = \frac{9\pi a}{8(16 - 3\pi)}.$$

Ex. 13. *A circular disc of radius a is completely immersed with its plane vertical in a homogeneous fluid. If h is the depth of the centre below the free surface of the fluid, prove that the distance between the centres of pressure of the two halves into which the disc is divided by its horizontal diameter is*

$$\frac{6\pi a(4h^2 - a^2)}{9\pi^2 h^2 - 16a^2}.$$

... circular disc of radius a

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-ng\frac{AB+x}{l}=mg-ng\frac{AB}{l}-ng\frac{x}{l}$$

$$=-ng\frac{x}{l}, \qquad \left[\because \text{ from (1), } mg=ng\frac{AB}{l}\right].$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{ng}{lm}x, \qquad \text{...(2)}$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC.

Since $BC<AB$, therefore during the entire motion of the particle the string will not become slack.

Thus the entire motion of the particle is governed by the equation (2) and the particle will make oscillations in simple harmonic motion about the centre B.

The time of one oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(ng/lm)}}=2\pi\sqrt{\left(\frac{lm}{ng}\right)}.$$

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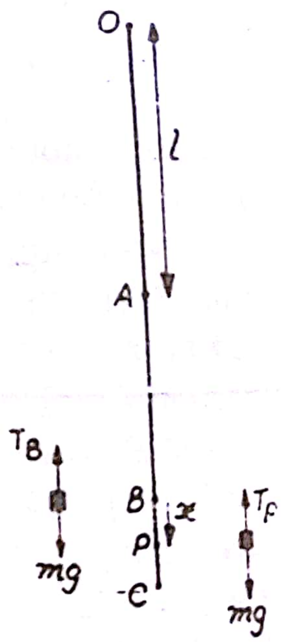
Ex. 52 (b). *A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that*

$$g\,(t_1^2-t_2^2)=4\pi^2\,(c_1-c_2). \qquad \text{[Rohilkhand 1985]}$$

Sol. One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m. Let $AB=d$.



In the equilibrium position of the particle of mass m at B, the tension $T_B=\lambda\,(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l}=mg \quad \text{or} \quad \frac{\lambda}{lm}=\frac{g}{d}. \qquad \text{...(1)}$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP=x$. When the particle is at P, the tension T_P in the string P is $\lambda\dfrac{d+x}{l}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2}=-\frac{\lambda\,(d+x)}{l}+mg,$$

[**Note** that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards *i.e.*, in the direction of x increasing.]

or $$m\,\frac{d^2x}{dt^2}=-\frac{\lambda d}{l}-\frac{\lambda x}{l}+mg$$

$$=-\frac{\lambda x}{l},\qquad \left[\because\ \frac{\lambda d}{l}=mg\right].$$

$$\therefore\ \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}\,x=-\frac{g}{d}\,x,\ \text{[from (1)]}.$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\dfrac{2\pi}{\sqrt{(g/d)}}$ *i.e.*, $2\pi\sqrt{\left(\dfrac{d}{g}\right)}$.

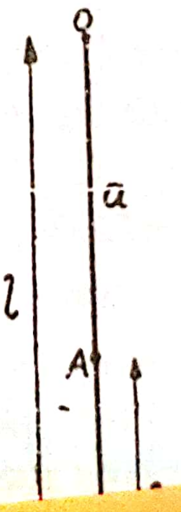
But according to the question, the period is t_1 when $d=c_1$ and the period is t_2 when $d=c_2$.

$$\therefore\ t_1=2\pi\sqrt{(c_1/g)}\ \text{and}\ t_2=2\pi\sqrt{(c_2/g)},$$

so that $$t_1^2-t_2^2=(4\pi^2/g)\,(c_1-c_2)$$

or $$g\,(t_1^2-t_2^2)=4\pi^2\,(c_1-c_2).$$

Ex. 53. *A mass m hangs from a light spring and is given a small vertical displacement. If l is the length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is $l-(g/4\pi^2n^2)$.*



Sol. Let $OA=a$ be the natural length of the spring which extends to a length $OB=l$ when a particle of mass m hangs in equilibrium. In the position of equilibrium of the particle at B, the tension T_B in the spring is $\lambda\{(l-a)/a\}$ and it balances the weight mg of the *length of the portion, show that the height of one extremity above the other is* $\dfrac{l \sin \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}$, *the two extremities being on one side of the vertex of the catenary.*

[Rohilkhand 88, 89; Gorakhpur 81; Agra 87; Luck. 80]

Sol. Let α, β be the inclinations to the horizontal of the tangents at the extremities P and Q (P lying above Q) of a portion PQ of a common catenary, the points P and Q being on the same side of the vertex of the catenary. Draw the figure by taking an arc PQ of a catenary lying only on one side of the vertex.

If y_P and y_Q are the ordinates of the points P and Q respectively, then from $y=c \sec \psi$, we have

$y_P=c \sec \alpha$ and $y_Q=c$

[$\because$

$\therefore$ Height of the extremity P ab

$=y_P-y_Q=c\,(\sec \alpha-\sec$

If C is the lowest point (*i.e.*, the vertex) of the catenary of which PQ is an arc, then from the formula $s=c \tan \psi$, we have

arc $CP=c \tan \alpha$ and arc $CQ=c \tan \beta$.

$\therefore$ $l=$ length of the arc $PQ=$ arc $CP-$ arc CQ

$=c\,(\tan \alpha-\tan \beta)$

or $$c=\frac{l}{(\tan \alpha-\tan \beta)}.$$

$\therefore$ from (1), the required height

$$=\frac{l\,(\sec \alpha-\sec \beta)}{(\tan \alpha-\tan \beta)}=\frac{l\,(\cos \beta-\cos \alpha)}{(\sin \alpha \cos \beta-\cos \beta \sin \alpha)}$$

$$=\frac{2l \sin \frac{1}{2}(\beta+\alpha) \sin \frac{1}{2}(\alpha-\beta)}{\sin (\alpha-\beta)}$$

$$=\frac{2l \sin \frac{1}{2}(\alpha+\beta) \sin \frac{1}{2}(\alpha-\beta)}{2 \sin \frac{1}{2}(\alpha-\beta) \cos \frac{1}{2}(\alpha-\beta)}=\frac{l \sin \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}.$$

Ex. 12. *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is*

$$\mu \log \left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$$

where μ *is the coefficient of friction.*

[Kanpur 85; Raj. T.D.C. 78, Meerut 90; Rohilkhand 88]

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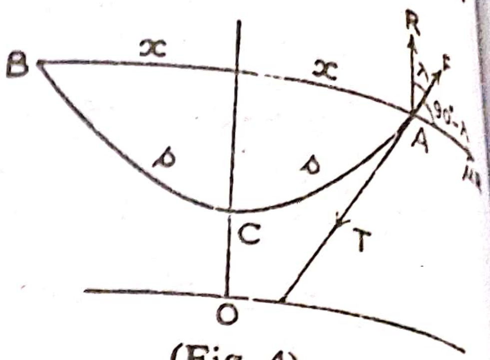
Sol. Let the end links A and B of a uniform chain slide along a fixed rough horizontal rod. If AB is the maximum span, then A and B are in the state of limiting equilibrium. Let R be the reaction of the rod at A acting perpendicular to the rod. Then the frictional force μR will act at A along the rod in the outward direction BA as shown in the figure. The resultant F of the forces R and μR at A will make an angle λ (where $\tan\lambda=\mu$) with the direction of R. For the equilibrium of A the resultant F of R and μR at A will be equal and opposite to the tension T at A.



(Fig. 4)

Since the tension at A acts along the tangent to the chain at A, therefore the tangent to the catenary at A makes an angle $\psi_A=\frac{1}{2}\pi-\lambda$ to the horizontal.

Thus for the point A of the catenary, we have $\psi=\psi_A=\frac{1}{2}\pi-\lambda$.

$\therefore$ the length of the chain

$$=2s=2c\tan\psi_A=2c\tan\left(\tfrac{1}{2}\pi-\lambda\right)$$

$$=2c\cot\lambda=\frac{2c}{\mu}. \quad [\because \tan\lambda=\mu]$$

If (x_A, y_A) are the coordinates of the point A, then the maximum span $AB=2x_A$

$$=2c\log(\tan\psi_A+\sec\psi_A)$$

$$=2c\log\{\tan\psi_A+\sqrt{(1+\tan^2\psi_A)}\}$$

$$=2c\log\{\cot\lambda+\sqrt{(1+\cot^2\lambda)}\} \quad [\because \psi_A=\tfrac{1}{2}\pi-\lambda]$$

$$=2c\log\left\{\frac{1}{\mu}+\sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\}=2c\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Hence the required ratio

$$=\frac{2x}{2s}=\frac{2c\log\left\{\dfrac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}}{(2c/\mu)}$$

$$=\mu\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Ex. 13. *If the ends of a uniform inextensible ... l hanging freely under gravity ...*

$$T = \frac{1}{2l \cos \alpha} [d(W+4w) \operatorname{cosec}^2 \alpha - 2l \sin \alpha (W+2w)]$$

or

$$T = \tan \alpha \left[\frac{d}{2l} (W+4w) \operatorname{cosec}^3 \alpha - (W+2w) \right].$$

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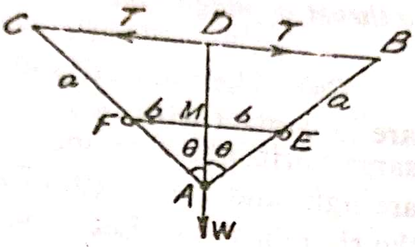
Ex. 41. *A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant 2b apart. A weight W is suspended from A, find the thrust in the rod BC.*

Sol. ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2b$. Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let



$$\angle BAD = \theta = \angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are : (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

We have,

$$BC = 2BD = 2AB \sin \theta = 2a \sin \theta.$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME \cot \theta = b \cot \theta.$$

The equation of virtual work is

$$T\delta (2a \sin \theta) + W\delta (b \cot \theta) = 0$$

or $$2aT \cos \theta \, \delta\theta - bW \operatorname{cosec}^2 \theta \, \delta\theta = 0$$

or $$(2aT \cos \theta - bW \operatorname{cosec}^2 \theta) \, \delta\theta = 0$$

or $$2aT \cos \theta - bW \operatorname{cosec}^2 \theta = 0 \qquad [\because \delta\theta \neq 0]$$

or $$2aT \cos \theta = bW \operatorname{cosec}^2 \theta$$

or $$T = \frac{Wb}{2a} \operatorname{cosec}^2 \theta \sec \theta.$$

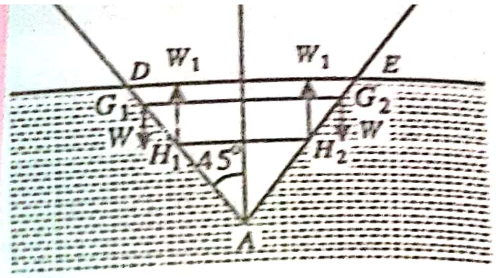
But in the position of equilibrium,

$$BC = \tfrac{3}{2}a \text{ and so } BD = \tfrac{3}{4}a.$$

Therefore $\sin \theta = \frac{BD}{AB} = \frac{\frac{3}{4}a}{a} = \frac{3}{4}$ $\qquad \therefore \cos\theta = \frac{\sqrt{7}}{4}$

$$\therefore T = \frac{32}{9\sqrt{7}} \cdot \frac{b}{a} W$$

W_1 be the weight of the displaced liquid in c
each rod of weight W.



For the sake of equilibrium of the syst

$$2W = 2W_1 \text{ or } W = W_1.$$

Now consider the forces acting on o
AB only.

(i) Weight of the rod W acting ve
downwards at G_1.

(ii) The force of buoyancy which is equal to the weight of the liquid displaced W_1 acting vertically upwards at H_1.

(iii) Tension T at B.

(iv) Reaction at the hinge A.

Taking moments about A, we get

$$T.AB\cos 45^\circ + W_1 . AH_1 \cos 45^\circ = W . AG \cos 45^\circ$$

or $$T.2a + W.b = W.a \quad \text{or} \quad T = \frac{W(a-b)}{2a}.$$

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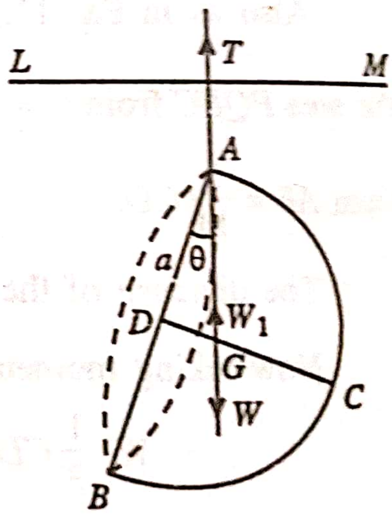
Ex. 14. *A solid hemisphere floats in a liquid completely immersed with a point of the rim joined to a fixed point by means of a string. Prove that the inclination of the base to the vertical is* $\tan^{-1}\frac{3}{8}$. *Also prove that the tension of the string is* $\frac{2}{3}\pi(\rho-\sigma)a^3 g$ *where* ρ *and* σ *are the densities of the solid and the liquid respectively and a is the radius of the hemisphere.*

Sol. Let the point A of the rim be joined to a fixed point by means of a string, and the base AB be inclined at an angle θ to the vertical. The forces acting on the hemisphere are :



(i) Wt. of the hemisphere $W = \frac{2}{3}\pi a^3 \rho g$, acting vertically downwards at G such that $DG = \frac{3}{8}a$.

(ii) The force of buoyancy *i.e.*, the weight of the liquid displaced $W_1 = \frac{2}{3}\pi a^3 \sigma g$, acting vertically upwards at G.

(iii) The tension T at A.

Since W and W_1 are acting in the same vertical line, therefore, for the equilibrium of the body the third force T will also act along the line of action of W and W_1 *i.e.*, the line of action of T will pass through G.

Here in ΔADG, $\tan\theta = \frac{DG}{DA} = \frac{3a}{8a} = \frac{3}{8}$.

$$\therefore \quad \theta = \tan^{-1}\frac{3}{8}.$$

Also $$T + W_1 = W \quad \text{or} \quad T = W - W_1 = \frac{2}{3}\pi a^3 \rho g - \frac{2}{3}\pi a^3 \sigma g = \frac{2}{3}\pi a^3(\rho - \sigma) g.$$

Ex. 15. *A semi-circular cylinder floats in water with its axis fixed in the surface of water. If this cylinder be movable about the fixed axis and its density be half that of water, show that it will be in equilibrium in any position.*