UPSC Mathematics optional PDe 2019 Solution

Q5a) . Form a p d e of family of surfaces given by following expression

$$\psi(x^2+y^2+2z^2, y^2-2zx)=0$$

Given that : $\psi(x^2+y^2+2z^2, y^2-2zx)=0$

$\psi(u\,,v)=0$ ……(1) where $u=x^2+y^2+2z^2\,;v=\ y^2-2zx$

Differentiating Equation (1) partially w.r.t x

$\frac{\partial\psi}{\partial u}\left(\frac{\partial u}{\partial x}.\frac{\partial x}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right)+\frac{\partial\psi}{\partial v}\left(\frac{\partial v}{\partial x}.\frac{\partial x}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right)=0$ ………..(2)

Differentiating Equation (1) partially w.r.t y

$\frac{\partial\psi}{\partial u}\left(\frac{\partial u}{\partial y}.\frac{\partial y}{\partial y}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial y}\right)+\frac{\partial\psi}{\partial v}\left(\frac{\partial v}{\partial y}.\frac{\partial y}{\partial y}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}\right)=0$ ……………(3)

$\frac{\partial\psi}{\partial u}(2x+4zp)=-\frac{\partial\psi}{\partial v}(-2z-2px)$ …………….(4)

$\frac{\partial\psi}{\partial u}(2y+4zq)=-\frac{\partial\psi}{\partial v}(2y-2qx)$ ………………(5)

Divide equation 4 by 5 we get,

$\frac{x+2pz}{y+2qz}=\frac{z+px}{qx-y}$ hence $px(y+2z)+q(2z^2-x^2)=-y(x+z)$ is the pde.

Q6a) Solve the first order quasi linear partial differential equation by method of characteristics

$x\frac{\partial u}{\partial x}+(u-x-y)\frac{\partial u}{\partial y}=x+2y$ in

$x>0,\quad -\infty<y<\infty$ with $u=1+y$ on $x=1$ [15 marks]

Let $p=\frac{\partial u}{\partial x}\,,q=\frac{\partial u}{\partial y}$

$f(\,x\,,y,u,p,q)=xp+(u-x-y)q-(x+2y)$ …….(1)

Choosing $x_0(s)=s^0=1\;;y_0(s)=s\;;u_0(s)=s+1$

$u_0{}'(s)=p_0x_0{}'(s)+q_0y_0{}'(s)$

$1=p_0(0)+q_0(1)\quad;q_0=1$

$x_0p_0+(u_0-x_0-y_0)q_0-(x_0+2y_0)=0\;;p_0=2s+1$

$\frac{dx}{dt}=f_p=x$ ….(2) $\frac{dy}{dt}=f_q=u-x-y$ ……(3)

$\frac{dp}{dt}=-f_x-pf_u$ ……….(4) $\frac{dq}{dt}=-f_y-qf_u$ …………(5)

$\frac{du}{dt}=pf_p+qf_q=px+q(u-x-y)=x+2y$

$\frac{dx}{dt}=x\;;\int\frac{dx}{x}=\int dt\;;x=c_1e^t$; At $t=t_0=0\,;x_0=c_1e^0\quad;x=e^t$

$\frac{dy}{dt}+\frac{du}{dt}=y+u\;;\int\frac{d(y+u)}{y+u}=\int dt\,;y+u=c_2e^t\,;y_0+u_0=c_2e^0\;;t=t_0=0$

$y + u = (2s+1)e^t\ ; \frac{dy}{dt} = u - x - y\ ; \frac{dy}{dt} = (2s+1)e^t - y - e^t - y$

$\frac{dy}{dt} + 2y = 2se^t\ \ ; e^{2t}y = \frac{2se^{3t}}{3} + k\ ; y = \frac{2s}{3}e^t + \frac{s}{3}e^{-2t}$

$x = e^t\ \ ; y + u = (2s+1)e^t\ \ ; y + u = (2s+1)x$

$y = \frac{s}{3}(2e^t + e^{-2t})$ ....(6)

$2s + 1 = \frac{y+u}{x}\ ; s = \frac{y+u-x}{2x}$ Put value of s in equation (6)

$y = \frac{y+u-x}{6x}\left(2x + \frac{1}{x^2}\right)\ ;\ y = \frac{y+u-x}{6x^3}(2x^3 + 1)$

$4x^3y - 2x^3u + 2x^4 - y - u + x = 0$

Q7c) Reduce the following second order PDE to canonical form & find the general solution

$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} + 12x$ [20 marks]

Solution : $r - 2xs + x^2t = q + 12x$ ....(1)

Where $r = \frac{\partial^2 u}{\partial x^2}\ ; s = \frac{\partial^2 u}{\partial x \partial y}\ ; t = \frac{\partial^2 u}{\partial y^2}\ ; p = \frac{\partial u}{\partial x}\ ; q = \frac{\partial u}{\partial y}$

$Rr + Ss + Tt + f(x, y, u, p, q) = 0\ ; R = 1, S = -2x, T = x^2$

$S^2 - 4RT = 0$ So it is parabolic . $\lambda\ quadratic\ \ R\lambda^2 + S\lambda + T = 0$

$(\lambda - x)^2 = 0\ ; \lambda = x, x$. Characteristic equation

$\frac{dy}{dx} + \lambda = 0\ ; \frac{dy}{dx} + x = 0\ \ ; y + \frac{x^2}{2} = c$

Choose $m = y + \frac{x^2}{2}\ ; n = x$ such that m , n are independent function

$$\frac{\partial(m, n)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial m}{\partial x} & \frac{\partial m}{\partial y} \\ \frac{\partial n}{\partial x} & \frac{\partial n}{\partial y} \end{vmatrix} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

$p = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x} = x\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n}\ ;\ q = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y} = \frac{\partial u}{\partial m}$

$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial m}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial n}\right)$

$$= \frac{\partial u}{\partial m} + x\left[\frac{\partial}{\partial m}\left(\frac{\partial u}{\partial m}\right)\frac{\partial m}{\partial x} + \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial m}\right)\frac{\partial n}{\partial x}\right] + \left[\frac{\partial}{\partial m}\left(\frac{\partial u}{\partial n}\right)\frac{\partial m}{\partial x} + \frac{\partial}{\partial n}(\frac{\partial u}{\partial n})\frac{\partial n}{\partial x}\right]$$

$r = \frac{\partial u}{\partial m} + x^2\frac{\partial^2 u}{\partial m^2} + 2x\frac{\partial^2 u}{\partial m \partial n} + \frac{\partial^2 u}{\partial n^2}$

$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial q}{\partial y} = \frac{\partial}{\partial m}\left(\frac{\partial u}{\partial m}\right)\frac{\partial m}{\partial y} + \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial m}\right)\frac{\partial n}{\partial y} = \frac{\partial^2 u}{\partial m^2}$

$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial q}{\partial x} = \frac{\partial}{\partial m}\left(\frac{\partial u}{\partial m}\right)\frac{\partial m}{\partial x} + \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial m}\right)\frac{\partial n}{\partial x} = x\frac{\partial^2 u}{\partial m^2} + \frac{\partial^2 u}{\partial m \partial n}$

Putting the values the equation reduces to

$\frac{\partial^2 u}{\partial n^2} = 12x$, where $n = x$

$\frac{\partial^2 u}{\partial x^2} = 12x$; $\frac{\partial u}{\partial x} = 6x^2 + f(m)$; $u = 2x^3 + x\,f(m) + g(m)$

$u = 2x^3 + x\,f\left(y + \frac{x^2}{2}\right) + g\,(y + \frac{x^2}{2})$