SuccessClap

Online Coaching for UPSC MATHEMATICS

QUESTION BANK SERIES

PAPER 2 : 08 Complex Analysis

Content:

01 ANALYTIC FUNCTION

02 COMPLEX INTEGRATION

03 TAYLOR LAURENT SERIES

04 POLES RESIDUE

05 COUNTER INTEGRATION

06 ROUCHES THEOREM

07 SINGULARITY

08 POWER SERIES

SuccessClap : Question Bank for Practice

01 ANALYTIC FUNCTION

(1) If u + v = $\frac{2\sin 2x}{e^{2y}+e^{-2y}-2\cos 2x}$, find the corresponding analytic function f(z) = u+iv

(2) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) |\text{ Real } f(z)\,|^2 = 2\,|\,f'(z)\,|^2$ where $w = f(z)$is analytic.

(3) If w = f(z) is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ log |f'(z)|=0. If |f '(z)| is the product of a function of x and a function of y, show that f '(z) = exp ($\alpha z^2+\beta z+\gamma$) where α is a real and β, γ are complex constants.

(4) Show that a harmonic function satisfies the formal differential equation $\frac{\partial^2 u}{\partial z\ \partial \bar{z}}$ =0

(5) If φ and ψ are functions of x and y satisfying Laplace's equation, show that s+it is analytic, where x= $\frac{\partial\varphi}{\partial y}-\frac{\partial\psi}{\partial x}$ and $t=\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}$.

(6) Show that the function u = sin x cosh y +2 cos x sinh y+ $x^2 - y^2$+4xy satisfies Laplace's equation and find the corresponding analytic function u +iv.

(7) Show that the function f(z) = $\sqrt{|\,xy\,|}$ is not analytic at the origin although the Cauchy – Riemann equations are satisfied at that point.

(8) If f(z) = $\frac{x^3y(y-ix)}{x^6+y^2}$, $z \neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

(9) Show that the function f(z) = u+iv where f(z) = $\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and $f(0) = 0$ is continuous and that Cauchy – Riemann equations are satisfied at the origin, yet f' (0) does not exist.

(10) Examine the nature of the function f(z) = $\frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0, f(0) = 0$ in a region including the origin.

(11) Show that the function f(z) = $e^{-z^{-4}}, z \neq 0 \text{ and } f(0) = 0$ is not analytic at z =0, although Cauchy- Riemann equations are satisfied at this point.

(12) For what values of z the function w defined by
z = e^{-v}(cosu+i sin u) where w =u+iv ceases to be analytic?

(13) Find the orthogonal trajectory of the family of curves $x^2 - y^2 + x = c$.

(14) If f(z) = u+iv is an analytic function, regular in D, where f(z) $\neq$ 0, prove that the curves u =const, v = const, form two orthogonal families. Verify this in case of f(z) = sin z.

(15) Two functions u(x,y) and v(x,y) are harmonic conjugates of each other if and only if they are constants.

(16) Find the analytic function whose real part is given and hence find the imaginary part:
(i) e^x sin y (ii)sin x cosh y (iii) x^2-y^2

(17) Find the analytic function whose imaginary part is given and hence the real part:
(i) cos x cosh y (ii) $\frac{x-y}{x^2+y^2}$ (iii) $\tan^{-1}\frac{y}{x}$.

(18) Determine the analytic function f(z) = u+iv given that $3u+2v = y^2 - x^2+16x$.

(19) Show that e^{x^2} is entire. Find its derivative.

(20) Determine the analytic function w = u+iv where u = $\frac{2\cos x \cosh y}{\cos 2x+\cosh 2y}$ given that f(0) =1.

(21) If f(z) =u+iv is an analytic function of z, find f(z) if $2u+v = e^{2x}[(2x+y)\cos 2y+(x-2y)\sin 2y]$

(22) Find an analytic function f(z) such that $\text{Re}[f'(z)] = 3x^2-4y-3y^2$ and $f(1+i) = 0$.

(23) Find a and b if $f(z) = (x^2-2xy+ay^2) + i(bx^2-y^2+2xy)$ is analytic. Hence find f(z) in terms of z.

(24) Find the conjugate harmonic of $u = e^{x^2-y^2}\cos 2xy$. Hence find f(z) in tems of z.

(25) Determine p such that the function $f(z) = \frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

(26) Find the analytic function f(z) = u+iv if $u = a(1+\cos\theta)$

(27) Find a such that the function $f(z) = r^2\cos 2\theta + ir^2\sin a\theta$ is analytic.

(28) Find the orthogonal trajectories of the family of curves $x^3y-xy^3 = C$ = constant.

(29) If u is a harmonic function, show that $w = u^2$ is not a harmonic function unless u is a constant.

(30) Show that $\{\frac{\partial}{\partial x}\ |f(z)|\}^2 + \{\frac{\partial}{\partial y}\ |f(z)|\}^2 = \{f'(z)\}^2$.

(31) Prove that z^n (n is a positive integer) is analytic and hence find its derivative.

(32) Show that the curves $r^n = a\sec n\theta$ and $r^n = \beta\,\text{cosec}\, n\theta$ cut orthogonally.

(33) If $w = u+iv = z^3$ prove that $u = c_1$ and $v = c_2$ where c_1 and c_2 are constants, cut each other orthogonally.

(34) If $w = \exp(z^2)$, find u and v and prove that the curve $u(x,y) = c_1$ and $v(x,y) = c_2$ where c_1 and c_2 are constants cut orthogonally.

(35) Show that for the function $f(z) = \{ \frac{z^5}{|z|^4}, z \neq 0$

$0, z = 0$

Cauchy – Riemann equations are satisfied at z =0, but f(z) is not differentiable at 0.

(36) Find where the function (i) w = 1/z (ii) z/z-1
(iii) $w = z^3 - 4z + 1$ (iv) $w = z+2/z(z^2+1)$
(v) $z = e^{-v}(\cos u + I \sin u)$ ceases (fails) to be analytic.

(37) If w = log z, find $\frac{dw}{dz}$ and determine where w is non – analytic.

(or) Show that the real and imaginary parts of the function w = log z satisfy the C – R equations when z is not zero.

(38) Show that $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}, z \neq 0$

0, if z =0 is not analytic at z =0 although C –R equations are satisfied at the origin.

(39) Show that an analytic function of constant absolute value is constant.

(40) Show that f(z) = sin z is analytic everywhere in the complex plane and find f '(z).

(41) Find the values of a and b such that the function $f(z) = x^3 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is analytic. Also find f '(z).

(42) If f(z) = u+iv is analytic function and $u^{-v} = e^x.(\cos y - \sin y)$, find f(z) in terms of z.

(43) If f(z) = u+iv is an analytic function of z and u-v = $\frac{\cos x+\sin x-e^{-y}}{2\cos x-e^{y}-e^{-y}}$ $find\ f(z) subject\ to\ the\ condition\ f\left(\frac{\pi}{2}\right) = 0.$

(44) If f(z) = u+iv is an analytic function of z = x+iy and u-v = $\frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$.

(45) Find the analytic function of which the real part is $e^{-x}\{(x^2-y^2)\cos y + 2x \sin y\}$.

(46) If u = $e^x(x \cos y - y \sin y)$, find the analytic function u+iv.

(47) If u-v =$(x-y)(x^2+4xy+y^2)$ and f(z)= u+iv is an analytic function of z = x+iy, find f(z) in terms of z.

(48) If w =u+iv represents the complex potential for an electric field and v = $x^2-y^2+\frac{x}{x^2+y^2}$, determine the function u.

(49) (i) Prove that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\ \partial \bar{z}}$

(ii) If f(z) is a regular function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 |f'(z)|^2$

(50) If f(z) = u+iv is an analytic function of z, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^p = p^2 |f(z)|^{p-2} |f'(z)|^2$

(51) If f(z) = u+iv is an analytic function of z = x+iy, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2} |f'(z)|^2$.

(52) For what values of z do the functions w defined by the following equations cease to be analysis:

(i) z = sinh u cos v + i cosh u sin v, w = u =iv

(53) for what value of z, the function w defined by z = $\log \rho + i\varphi$ where $w = \rho(\cos\varphi + i \sin\varphi)$ ceases to be analytic?

SuccessClap : Question Bank for Practice

02 COMPLEX INTEGRATION

(1). Evaluate $I = \int_{(0,1)}^{(2,5)} (3x+y)dx + (2y-x)dy \ along$

a) The curve $y = x^2 + 1$.

b) The line joining $(0,1) \ and \ (2,5)$.

c) The line from $(0,1) \ to \ (0,5)$ and then from $(0,5) \ to \ (2,5)$.

(2). Evaluate $\int (\bar{z})^2 \ dz$ around the circle.

a) $|z| = 1$,

b) $|z-1| = 1$.

(3). Evaluate $\int_c f(z) \ dz$ where $f(z) = y - x - 3x^2 i \ and \ C$

a) Is the line segment from $z = 0 \ to \ z = 1 + i$.

b) Consists of two-line segments, one from $z = 0 \ to \ z = I$ and other from $z = i \ to \ z = 1 + i$.

(4). Evaluate $\int_c (x+2y)dx + (4-2x)dy \ \ around$ the ellipse

$$x = 4cos\theta, 3sin\theta, 0 \leq \theta \leq 2\pi$$

where the arc C is taken in the anticlockwise direction.

(5). Evaluate $\int_L \frac{dz}{z}$ where L represents the square described in the positive sense with sides parallel to the axes and of length 2a and having its centre at the origin.

(6). Poisson's Integral Formula for A Circle.

If $f(z)$ is analytic in the region $|z| \leq P \ and \ R$ is any number such that 0<R<P then prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\theta})}{R^2 - 2Rrcos(\theta - \emptyset) + r^2} d\emptyset$$

$where \ z = re^{i\theta}$ is any point of the domain $|z| < p$.

(7). Evaluate $\int_C \frac{1}{z(z-1)} dz$ Where C is the circle $|z| = 3$.

(8). Evaluate (without using integral formula)

a) $\int_C \frac{z}{z-z_0} dz.$

b) $\int_C \frac{z}{(z-z_0)^2} dz.$

$Where$ z_0 is any point within C.

(9). Verify Cauchy's Theorem for the function 5sin2z .. C is the square vertices at $1 \pm i, -1 \pm i.$

(10). Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$ Where C: $|z-i| = 2.$

(11). Evaluate $\int_C \frac{z-3}{z^2+2z+5} dz$ Where C is circle

a) $|z| = 1$ And,

b) $|z+1-i| = 2.$

(12). Evaluate the following integrals by using Cauchy's integrals formula

a) $\int_c \frac{sin\pi z^2 + cos\pi z^2}{(z-1)(z-2)} dz,$

b) $\frac{1}{2\pi i} \int_c \frac{e^{zt}}{z^2+1} dz \; t > 0,$

Where C represents the circle $|z| = 3.$

(13). Evaluate $\int_c \frac{z^2-4}{z(z^2+9)} dz,$ Where C represents the circle $|z| = 1.$

(14). Evaluate $\int_c \frac{e^{ax}}{(z^2+1)} dz,$ Where C represents the circle $|z| = 2.$

(15). Evaluate $\int_c \frac{e^{ax}}{(z+1)^4} dz,$ Where C represents the circle $|z| = 3.$

(16). Using Cauchy's integrals formula, calculate the following integrals:

a) $\int_c \frac{zdz}{(9-z^2)(z+i)}$, where C is the circle $|z| = 2$ described in positive sense.

b) $\int_C \frac{\cosh(\pi z)\, dz}{z(z^2+1)}$, where C is the circle $|z| = 2$.

c) $\int_C \frac{e^{az} dz}{(z-\pi i)}$, where C is the ellipse $|z-2| + |z+2| = 6$.

d) $\int_C \frac{dz}{z-2}$, where C is the circle $|z| = 3$.

(17).

I. Evaluate $\int_C \frac{\tan(z/2) dz}{(z-x_0)^2}$,

Where C is the boundary of the square whose sides lie along the lines $x = \pm 2, v = \pm 2$ and it is described in positive sense, where $|x_0| < 2$.

II. Evaluate $\int_C \frac{dz}{z^2+2z+2}$,

Where C is the square having vertices at (0,0), (-2,0), (-2, -2), (0, -2) oriented in anticlockwise direction.

III. Evaluate $\int_C \frac{\sin z\, dz}{\left(z-\frac{\pi}{4}\right)^3}$, Where C is $\left|z-\frac{\pi}{4}\right| = \frac{1}{2}$.

IV. If C is unit circle about the origin, described in positive sense, show that

a) $\int_C \frac{e^{-z}}{z^2} dz = -2\pi i$ and

b) $\int_C \frac{\sin z}{z} dz = 0$.

(18). Evaluate $\int_c f(z)\, dz$ where $f(z) = (z+2)/z$ where c is.

a) The semi-circle $z = 2e^{i\theta}$ where θ varies from 0 to π

b) The semi-circle $z = 2e^{i\theta}$ where θ varies from 0 to -π.

c) The semi-circle $z = 2e^{i\theta}$ where θ varies from -π to π.

(19). Evaluate $\int_c f(z)\, dz$ where $f(z) = z - I$ and C is the arc from z=0 and z=2;

a) The semi-circle z-1=$e^{i\theta}$ $(\leq \theta \leq \pi)$
b) The segment of the x-axis.

(20). Evaluate $\int_c (x-2y)dx\ +(y^2-x^2)dy$ where C is the boundary of the first quadrant of the circle $x^2+y^2=4$.

(21). Evaluate $\int_c (z^2+3z+2)dz$ where C is the arc of the cycloid $x=a(\theta+sin\theta).y=a(1-cos\theta)$ between the points (0,0) and (a$\pi,2a$).

(22). Evaluate $\int_{1+i}^{2+4i} z^2 dz$
a) Along the parabola $x=t, y=t^2\ where\ 1\leq t\leq 2.$
b) Along the straight line joining $I+i\ and\ 2+4i.$
c) Along straight lines from $I+i\ to\ 2+i\ and\ then\ to\ 2+4i.$

(23). Evaluate $\int_c (y^2+2xy)dx+(x^2-2xy)dy$ where C is the boundary of the region $y=x^2\ and\ x=y^2$.

(24). Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices -1, I, 1+I, -1+i.

(25). Verify Cauchy's theorem for the function $f(z)=3z^2+iz-4$ if C is the square with vertices at 1+I and -1+i.

(26). B is the positively oriented boundary of the region between the circle $|z|=4$ and the square with sides along the lines $x=\pm 1\ and\ y=\pm 1.$ Evaluate $\int_B f(z)dz$ for the following function.
a) $f(z)=\frac{1}{3z^2+1}$,
b) $f(z)=\frac{z+2}{sin(z/2)}$.
c) $f(z)=\frac{z}{1-e^z}$.

(27). Show that $\int_c \frac{x\,dx-y\,dy}{\sqrt{x^2+y^2}}$ is independent of any path of integration which does not pass through the origin.

(28). Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$ where C is described in the positive sense
Evaluate each of the following integrals.

a) $\int_C \frac{e^{-z}\, dz}{z-(\pi i)/2}$

b) $\int_C \frac{cosz\, dz}{z(z^2+8)}$

c) $\int_C \frac{z\, dz}{2z+1}$

d) $\int_C \frac{tan(z/2)}{(z-x_0)^2} dz \ (|x_0| < 2)$

e) $\int_C \frac{cosh\, z}{z^4} dz$

(29). Let C be the unit circle $z = exp(i\theta)$ described from $\theta = -\pi\ to\ \pi\ and\ k$ is any real constant. Show that $\int_C \frac{e^{kz}}{z} dz = 2\pi i$. Then write the integral in terms of θ to derive the formula $\int_c e^{kcos\theta} cos(ksin\theta) d\theta = \pi$.

(30). Evaluate using Cauchy's integral formula:

a) $\int_C \frac{sin^6 z}{(z-\pi/2)^3} dz,$

b) $\int_C \frac{sin^6 z}{(z-\pi/6)^3} dz,$

Around the circle C: $|z| = 1$.

(31). Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ Around the circle C: $|z - 1| = 3$.

(32). If $\emptyset(e_z) = \int_C \frac{3z^2+7z+1}{z-e_z}$ where C is the circle $x^2 + y^2 = 4, find\ \emptyset(3), \emptyset(1), \emptyset'(1 - i)\ and\ \emptyset''(1 - i)$.

Or

If $F(a) = \int_C \frac{3z^2+7z+1}{z-a} dz$ where C is $|z| = 2$ using Cauchy's integral formula find F(1), F(3), F''(1-i).

(33). Evaluate $\int_C \frac{z+4}{z^2+2z+5}$ where C is the circle

a) $|z| = 1$,
b) $|z + 1 - i| = 2$,
c) $|z + 1 + i| = 2$.

(34). Evaluate $\oint_C \frac{dz}{(z-a)^n}$ *if* $z = a$ *is* a point inside a simple closed curve C and n is an integer.

(35). Evaluate $\int_C \frac{z}{z^2+1} dz$ where C is $\left|z + \frac{1}{z}\right| = 2$.

(36). Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is $|z| = 4$.

(37). Evaluate $\int_C \frac{dz}{(z-2i)^2(z+2i)^2}$,C being the circumference of the ellipse $x^2 + 4(y-2)^2 = 4$.

(38). Evaluate using Cauchy's theorem $\int_C \frac{z^3e^{-z}}{(z-1)^3} dz$ where C is $|z - 1| = \frac{1}{2}$ using Cauchy's integral formula.

(39). Evaluate $\int_C \frac{z^3 - \sin 3z}{\left(z-\frac{\pi}{2}\right)^3} dz$ with C: $|z| = 2$ using Cauchy's integral formula.

(40). Evaluate $\oint \frac{\sin^2 z}{(z-\pi/6)^3} dz$, if C is the circle $|z| = 1$.

(41). Evaluate $\int_C \frac{ze^z}{(z+a)^3}$ where C is any simple closed curve enclosing the point $z = -a$ using Cauchy's integral formula.

(42). Evaluate $\int_C \left[\frac{e^z}{z^3} + \frac{z^4}{(z+i)^2}\right] dz$ where C: $|z| = 2$ using Cauchy's integral formula.

(43).

(44). Find. $f(4), f(i)\ , f(-1) and f''(-i)$where $f(\xi) = \int_C \frac{4z^2+z+5}{z-\xi} dz$ and C is the ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1.$

(45). If $f(\alpha) = \int_C \frac{5z^2-4z+3}{z-\alpha} dz$ where C is the ellipse $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1.$ Find

a) $f(4.5)$,
b) $f(2)$,
c) $f'(i)$,
d) $f''(-2i)$.

(46). Evaluate $\int_C \frac{z-3}{z^2+2z+5} dz$ where C is $|z+1-i| = 2$ using Cauchy's integral formula.

(47). Use Cauchy's integral formula to calculate $\int_C \frac{z^2-1}{z^2+1} dz$, where C is $|z-i| = 1.$

(48). Using Cauchy's integral formula, evaluate:

a) $\int_C \frac{2z+1}{z^2+z} dz$, where C is the circle $|z| = \frac{1}{2}$.

b) $\int_C \frac{z^3-2z+1}{(z-i)^2} dz$, where C is the circle $|z| = 2.$

c) $\int_C \frac{z\,dz}{(z+i)(9-z^2)}$, where C is $|z| = 2.$

(49). Using Cauchy's integral formula, evaluate

a) $\int_C \frac{z^2+1}{z(2z+1)} dz$, where C is $|z| = 1.$

b) $\int_C \frac{\cosh \pi z}{z(z^2+1)} dz$, where C is $|z| = 2.$

(50). Evaluate $\int_C \frac{\log z}{(z-1)^3} dz$, where C: $|z-1| = \frac{1}{2}$ using Cauchy's integral

(51). Evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)^3} dz$, where C is $|z| = 3$ using Cauchy's integral

(52). Evaluate $\int_c \frac{e^z}{z(1-z)^3} dz$, if

a) 0 lies inside C and I lie outside C.
b) I lie inside C and 0 lies outside c.
c) Both lie inside C.

(53). Using Cauchy's integral formula, evaluate $\int_c \frac{z^4}{(z+1)(z-i)^2} dz$, where C is the ellipse $9x^2 + 4y^2 = 36$.

(54). Evaluate $\int_c \frac{\cos z - \sin z}{(z+i)^3} dz$, with C: $|z| = 2$ using Cauchy's integral

(55). Evaluate $\int_c \frac{e^z \sin 2z - 1}{z^2(z+2)^2} dz$, where C is $|z| = \frac{1}{2}$ using Cauchy's integral formula.

(56). Evaluate $\int_c \frac{e^{-2z} z^2}{(z-1)^3(z+2)} dz$, where C is $|z + 2| = I$ using Cauchy's integral formula.

(57). Evaluate using Cauchy's integral formula $\int_c \frac{(z+1)dz}{z^2+2z+4}$, where C: $|z + 1 + i| = 2$.

(58). Evaluate $\int_c \frac{ze^z dz}{(z+2)^3}$, where C is $|z| = 3$ using Cauchy's integral formula.

(59). Evaluate $\int_c \frac{dz}{e^z(z-1)^3} dz$, where C is $|z| = 2$ using Cauchy's integral formula.

(60). Evaluate $\int_c \frac{e^{3z}dz}{(z+1)^4}$, where C is $|z| = 3$ using Cauchy's integral formula.

(61). Evaluate $\int_c z^{-2}dz$, where C is

a) $|z| = 1, and$

b) $|z-1| = 1.$

(62). Use Cauchy's integral formula to evaluate $\oint \frac{z^4-3z^2+6}{(z+i)^3} dz$, where C is the circle $|z| = 2.$

(63). Use Cauchy's integral formula to evaluate $\oint \frac{e^z}{(z+2)(z+1)^2} dz$, where C is the circle $|z| = 3.$

(64). Show that $\int_c e^{-2z}dz$, is independent of the path C joining the points $1-\pi i \ and \ 2+3\pi i$ and determine its value.

(65). Find the value of $\int_c \frac{sin^6 z}{(z-\pi/6)^3} dz$, if C is the circle $|z| = 1.$

(66). Evaluate $\int \frac{e^{ax}}{z^2+1} dz$, where C is the circle $|z| = 2.$

SuccessClap : Question Bank for Practice

03 TAYLOR LAURENT SERIES

(1) Expand f(z) = $\frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions

(i) |z| < 1 (ii) 1 < |z| < 3.

(iii)|z| > 3 (iv) 0 < |z+1| < 2.

(2) Show that when 0 < |z| < 4,

$\frac{1}{4z-z^2} = \sum_{u=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$.

(3) Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

(i) 0 < |z| <1 (ii) 1 < |z| < 2

(iii)|z| > 2.

(4) Expand $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ for

(i) |z| < 1 (ii) 1 < |z| < 4

(iii)|z| > 4.

(5) Find two Laurent's series expansions in powers of z of the function f(z) = $\frac{1}{z(1+z^2)}$.

(6) Express f(z) = $\frac{1}{z(z+1)^2(z+2)^3}$ in a Laurent's series in the region $\frac{5}{4} \leq |z| \leq \frac{7}{4}$.

(7) Obtain the Taylor's or Laurent's series which represents the function f(z) = $\frac{1}{(1+z^2)(z+2)}$ when (i) |z| < 1 (ii) 1<|z|<2

(ii) |z|> 2.

(8) Find the expansion of $\frac{1}{(z^2+1)(z^2+2)}$ in powers of z when (i) |z| <1 (ii) 1 < |z| < $\sqrt{2}$ (iii)|z| > $\sqrt{2}$.

(9) Find different developments of $\frac{1}{(z-1)(z-3)}$ in powers of z according to the position of the point in the z – plane. Expand the function in Taylor's series about z = 2 and indicate the circle of convergence.

(10) Represent the function f(z) = $\frac{4z+3}{z(z-3)(z+2)}$ in Laurent's series

(i) Within |z|=1

(ii) In the angular region between |z| = 2 and |z| =3

(iii) Exterior to |z| = 3.

(11) Expand f(z) = $\frac{z+3}{z(z^2-z-2)}$ in powers of z where

(i) |z| < 1 (ii) 1 < |z| < 2 (iii) |z| > 2.

(12) For the function f(z) = $\frac{2z^3+1}{z^2+z}$, find

(i) Taylor's series valid in the neighbourhood of the point z =i

(ii) A Laurent's series valid within the annulus of which centre is the origin.

(13) Show that log z = (z-1) - $\frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots$ when |z-1| < 1.

(14) Prove that $\tan^{-1}$ z = z - $\frac{z^3}{3} + \frac{z^5}{5} - \cdots$when |z| < 1.

(15) Find Taylor's series expansion of the function f(z) = $\frac{z}{z^4+9}$ around z =0. Find also radius of convergence.

(16) Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity z = -2. Specify the region of convergence.

(17) Expand f(z) = $\frac{z-1}{z+1}$ $as\ a\ Taylor's series\ about\ (i) z = 0\ and\ (ii) z = 1\ (iii) its\ Laurent's\ series\ for\ the\ domain\ 1 <| z |< \infty$.

(18) Find Laurent's series of the function f(z) = $\frac{1}{(z^2-4)(z+1)}$ valid in the region 1 < |z| < 2.

(19) Obtain the Taylor series expansion of f(z) = 1/z about the point z = I.

(20) (i) Expand e^z as Taylor's series about z =1.
(iii) Find the Taylor's series expansion of e^z about z =3.

(21) Expand f(z) =$\frac{1}{z^2}$
(a) In powers of z+1 [or, Prove that when |z+1|<1,z^{-2} = 1+$\sum_{n=1}^{\infty}(n+1)(z+1)^n$]
(b) In powers of z -2. State the region of validity of the expansion in each case.

(22) Expand sinh z by Taylor's series about z = πi.

(23) Within what circle does the Maclaurin's series for the function tanh z converge to the function.

(24) Expand f(z) = $\frac{z-1}{z+1}$ in Taylor's series about the point (i) z=0 and (ii) z =1.

(25) Expand f(z) = $\frac{z-1}{z^2}$ in a Taylor's series in powers of z-1 and determine the region of convergence.

(26) Find Taylor's expansion for the function f(z) = $\frac{1}{(1+z)^2}$ with centre at –i.

(27) Obtain the Taylor expansion of $e^{(1+z)}$ in the powers of (z-I).

(28) Find the Taylor's series expansion of $\frac{1}{z^2+z-6}$ about z = -1.

(29) Find the Taylor's series expansion of cosh z about z = πi.

(30) Expand log (1-z) when |z| < 1 using Taylor series.

(31) Find Taylor's expansion of f(z) = $\frac{2z^3+1}{z^2+z}$ about the point
(i) Z =i

(ii) Z = 1

(32) Expand f(z) = sin z in Taylor's series about
(i) Z = $\frac{\pi}{4}$.
(ii) Z = $\frac{\pi}{2}$.

(33) Obtain the Taylor's series expansion of f(z) = $\frac{e^z}{z(z+1)}$ about z = 2.

(34) Obtain the Taylor series expansion of f(z) = $\frac{1}{(z-1)(z-2)}$ about z = 0.

(35) Expand f(z) = $\frac{1}{z^2-z-6}$ about (i) z = -1 (ii) z = 1

(36) Obtain the expansion of $\frac{1}{(z-1)(z-3)}$ in a Taylor's series in powers of (z-4) and determine the region of convergence.

(37) Find the Taylor's series expansion of f(z) = $\frac{z-1}{z^2}$ about the point z = l. Find its region of convergence.

(38) Expand in Taylor's series $\frac{z}{z^4+9}$ about the point z = 0.

(39) Show that when |z+1| < 1, z^{-2} = 1+$\sum_{n=1}^{\infty}(n+1)(z+1)^n$

(40) Find Taylor's expansion of f(z) = $\frac{z+1}{(z-3)(z-4)}$ about the point z = 2. Determine the region of convergence.

(41) Expand log z by Taylor's series about z = 1.

(42) Find the Taylor series for z/z+2 about z = 1. Also find the region of convergence.

(43) Expand f(z) = $-\frac{2}{(2z+1)^3}$ about (i) z = 0 (ii) z = 2

(44) Expand ze^z by Taylor's series about z = 1.

(45) Expand $\frac{z}{(z+1)(z-2)}$ about z =1.

(46) Find the Taylor's series of log $(1+e^z)$ about z =0.

(47) Let f(z) = $\frac{1}{(1-z)(z-2)}$
Find (a) Maclaurin's series expansion of f(z).

(b)Laurent's series expansion in the annulas region 1 < |z| < 2.

(c) Laurent's series expansion in |z| > 2.

(48) Represent f(z) = $\frac{z+1}{z-1}$ by
(a) Maclaurin's series giving the region of validity and
(b) Laurent's series for the domain |z|> 1.

(49) Give two Laurent's series expansion in powers of z for f(z) = $\frac{1}{z^2(1-z)}$ and specify the regions in which these expansions are valid.
(or) Find Laurent's series for f(z) = $\frac{1}{z^2(1-z)}$ and find the region of convergence.

(50) Expand f(z) = $\frac{1}{z^2-3z+2}$ in the region (i) 0 < |z-1| < 1

(ii)1 < |z| < 2.

(51) Expand the function f(z) = $\frac{z-1}{z}$ as Laurent series for |z-1|>1.

(52) Expand f(z) = $e^{2z}/(z-1)^3$ about z =1 as a Laurent's series. Also find the region of convergence.

(53) Find the Laurent series expansion of the function f(z) = $\frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region 3 < |z+2| < 5.

(54) Expand f(z) = z/z^2+1 for |z-3| > 2.

(55) Expand $\frac{7z-2}{(z+1)z(z-2)}$ about the point z = -1 in the region 1 < |z+1| <3 as Laurent's series.

(56) Expand the Laurent series of $\frac{z^2-1}{(z+2)(z+3)}$, for |z| > 3.

(57) Find the Laurent series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ if 2 < |z| < 3.

(58) Expand f(z) = $\frac{z+3}{z(z^2-z-2)}$ $in\ powers\ of\ z\ where (i)\ |\ z\ | < 1\ \ (ii) 1 < |\ z\ | < 2\ \ (iii)\ |\ z\ | > 2$

(or) Expand f(z) = $\frac{z+3}{z(z^2-z-2)}$ in powers of z

(a) Within the unit circle about the origin

(b) Within the annular region between the concentric circles about the origin having radii 1 and 2 respectively.

(c) The exterior to the circle of radius 2.

(59) Obtain Laurent's expansion for f(z) = $\frac{1}{(z+2)(1+z)^2}$ in

(a) (i) |z| < 2 (ii) |1+z| > 1.

(b) (i) |z| < 1 (ii) 1 < |z| < 2

(60) Obtain all the Laurent series of the function $\frac{7z-2}{(z+1)(z)(z-2)}$ about z_0 = -1.

(61) Obtain the Laurent's series expansion of f(z) = $\frac{e^z}{z(1-z)}$ about z =1.

(62) Find the Laurent series of the function f(z) = $\frac{z}{(z+1)(z+2)}$ about z = -2.

(63) (i) Find the Laurent expansion of $\frac{1}{z^2-4z+3}$, for

(a) 1 < |z| < 3

(b) |z| < 1

(c) |z| > 3

(ii)Expand $\frac{1}{(z+3)(z+1)}$ in the annular region between |z| =1 and |z| = 3.

(64) Express f(z) = $\frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of (z-1).

(65) Expand $\frac{1}{(z^2+1)(z^2+2)}$ in a positive and negative powers of z if 1 < |z| < $\sqrt{2}$.

(66) Expand f(z) = $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ in the region (a) 1 < |z| < 4

(b)|z| < 1.

(67) Find the Laurent series expansion of the function $\frac{z^2-1}{z^2+5z+6}$ about z = 0 in the region 2 < |z| < 3.

(68) Expand f(z) = $\frac{1+2z}{z^2+z^3}$ in a series of positive and negative powers of z.

(69) Represent the function f(z) = $\frac{z}{(z-1)(z-3)}$ by a series of positive and negative powers of (z-1) which converges to f(z) when 0 < |z-1| < 2.

(70) Expand $\frac{1}{z(z-2)}$ when |z| < 2.

(71) Obtain the Taylor's series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$, in the region |z| < 2.

(72) Expand f(z) = $\frac{1}{(z+1)^2}$ in Taylor's series about z = -i.

(73) Expand sin z in a Taylor's series about z = $\frac{\pi}{4}$.

SuccessClap : Question Bank for Practice

04 POLES RESIDUE

(1) Find the residue of $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at z =1.

(2) Determine the poles of the function f(z) = $\frac{z^2}{(z-1)^2(z+2)}$ and the residue at each point. Hence evaluate $\int_c f(z)dz,$ where C is the circle |z| = 2.5

(3) Find the residues of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles in the finite plane.

(4) Find the residue of $\frac{1}{(z^2+a^2)^2}$ at z = ia.

(5) Find the residues of the function $\frac{\cot \pi z}{(z-a)^2}$.

(6) Find the residues of e^z $\operatorname{cosec}^2 z$ at all its poles in the finite plane.

(7) Find the residue of $\frac{z^3}{z^2-1}$ $at\ z = \infty$.

(8) Evaluate $\int_c \frac{z-3}{x^2+2x+5}$dz where C is the circle

(i) |z| =1 (ii) |z+1-i| =2 (iii) |z+1+i| =2.

(9) Find the residue of $\frac{z^2}{(z-a)(z-b)(z-c)}$ at infinity.

(10) Evaluate the residues of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at z = 1,2,3 and infinity and show that their sum is zero.

(11) Evaluate $\int_c \frac{sin\pi z^2+\cos \pi z^2}{(z-1)^2(z-2)}$ dz, where C is the circle |z| =3.

(12) Find the poles and residues of $\frac{1}{z^2-1}$.

(13) Expand f(z) = $\frac{e^z}{(z-1)^2}$ as a Laurent series about z =1 and hence find the residue at that point.

(14) Find zeros and poles of $(\frac{z+1}{z^2+1})^2$

(15) Determine the poles of the function (i) $\frac{z}{cosz}$ (ii) cot z.

(16) Determine the poles of the function f(z) where f(z) = $\frac{e}{z^2+\pi^2}$ and the residues at these poles.

(17) Find the residue at z =0 of the function f(z) = $\frac{1+e^z}{\sin z+z\cos z}$.

(18) Find the residues of the function f(z) = $\frac{1-e^{2x}}{z^4}$ at the poles.

(19) Find the residue of f(z) = $\frac{z^3}{(z-1)^4(z-2)(z-3)}$ at z = 1.

(20) Find the residue of f(z) = $\frac{z^3}{z^2-!}$ $at\ z = \infty$.

(21) Find the residue of $\frac{z^2}{z^4+1}$ at these singular points which lie inside the circle |z| =2.

(22) Find the residue of $\frac{z^2}{1-z^4}$ at these singular points which lie inside the circle |z| = 1.5

(23) Find the residue of (i) $\frac{z^2-2z}{(z+1)^2(z^2+1)}$ (ii) tan z at each pole.

(24) Calculate the residue of $e^z. z^{-5}$ at z =0.

(25) Find the poles and residues of $\frac{3z+1}{(z+1)(2z-1)}$.

(26) Find the poles of f(z) and the residues of the poles which lie on imaginary axis if f(z) = $\frac{z^2+2z}{(z+1)^2(z^2+4)}$.

(27) Determine the poles and residues of f(z) = $\frac{z^2}{(z+2)z(z-1)^2}$.

(28) Find the residue of $\frac{z^2}{(z-a)(z-b)(z-c)}$ $at\ z = \infty$.

(29) Find the residue of $\frac{z\,e^{z!}}{(z-3)^2}$ at its pole.

(30) Find the poles and residues at each pole of tanh z.

(31) Find the poles and residues at each pole of f(z) = $\frac{sin^2 z}{(z-\frac{\pi}{6})^2}$.

(32) Find the poles and residues at each pole of $\frac{\cot z \coth z}{z^3}$.

(33) Find the poles of $\frac{e^{i\,z}}{z^2+1}$ and corresponding residues.

(34) Find the poles and residues at each pole of $\frac{2z+1}{1-z^4}$.

(35) Find the poles and the residues at each pole of f(z) = $\frac{1-e^z}{z^4}$.

(36) Find the poles and corresponding residues of $\frac{1}{(z^2-1)^3}$.

(37) Find the poles and residues at each pole f(z) = $\frac{ze^z}{(z+2)^4(z-1)}$ where z = -2 is a pole of order 4.

(38) Find the poles and residue at each pole of $\frac{z}{z^2-4}$.

(39) Find the poles and residue at each pole of the function $\frac{z^2}{(z^4-1)}$.

(40) Find the poles and residues at each pole of the function $\frac{z\sin z}{(z-\pi)^3}$.

(41) Find the poles and corresponding residues of f(z) = $\frac{e^z}{(1+z)^2}$.

(42) Evaluate $\oint_c \frac{dz}{(z^2+1)\,(z^2-4)}$ where c is the circle |z| = 1.5

(43) Evaluate $\oint_c \tan z\, dz$ where c is the circle |z| =2.

(44) Evaluate $\oint_c \frac{dz}{(z^2+4)^2}$ where c: |z-i| =2.

(45) Obtain Laurent's series for the function f(z) = $\frac{1}{z^2\sinh z}$ and evaluate $\int_c \frac{dz}{z^2\sinh z}$, where C is the circle |z-1| = 2.

(46) Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ where c is |z-i| =2.

(47) Evaluate $\int_c \frac{z\cos z}{(z-\frac{\pi}{2})^3}$ dz where C is (i) |z-1|=1 (ii) |z| =2.

(48) Evaluate $\oint_c \frac{z-3}{z^2+2z+5}$ dz, where c is the circle given by
(i) |z| =1 (ii) |z+1-i| =2 (iii) |z+1+i| = 2.

(49) Evaluate $\int_c \frac{\coth z}{z-i}$ dz where C is |z| = 2.

(50) Evaluate $\int_c \frac{dz}{\sinh z}$, where C is the circle |z| =4, using residue theorem.

(51) Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}$ dz where C is the circle $|\ z\ | = 3$.

(52) Evaluate $\int_c \frac{12z-7}{(2x+3)(z-1)^2}$ dz where C is $x^2+y^2 = 4$.

(53) Evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2}$ dz where C is |z| =4.

(54) Evaluate $\int_c \frac{2e^z dz}{c^{z(z-3)}}$ where C is |z|= 2 by Residue theorem.

(55) Evaluate $\int_c \frac{(2z+1)^2}{4z^3+z}$ dz where C is the circle |z| =1 using Residue theorem.

(56) Evaluate $\int_c \frac{e^{2z}}{(z+1)^3}$ dz using Residue theorem, where c is |z| =2.

(57) Evaluate $\int_c \frac{cos\pi z^2}{(z-1)(z-2)}$ dz where c is |z| = 3/2.

(58) Evaluate $\int_c \frac{3\sin z}{z^2-\frac{\pi^2}{4}}$ dz where c is |z| = π *by residue* theorem.

(59) Evaluate $\int_c \frac{\sin z}{z\cos z}$ dz where c is |z| = π by Residue theorem.

(60) Evaluate $\int_c \frac{z}{(z-1)(z-2)^2}$ dz where c is the circle |z-2| = ½ using Residue theorem.

(61) Evaluate $\int_c \frac{ze^z dz}{(z^2+9)}$ using c is |z| = 5 by residue theorem.

(62) Evaluate $\int_c \frac{dz}{z^2e^x}$ where c is |z| =1.

(63) Evaluate $\int_c \frac{z^2+2z-2}{z(z-4)(z-1)}$ dz, where c is |z| = 1.5.

SuccessClap : Question Bank for Practice

05 COUNTER INTEGRATION

(1). Show by the method of residues, $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}} (a > b > 0)$.

(2). Show that

$$\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{\pi}{(a^2-b^2)^{3/2}} (a > b > 0).$$

(3). Prove that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right] where (a > b > 0).$$

(4). Show that

$$I = \int_0^{2\pi} \frac{1+4\cos\theta}{17+8\cos\theta} d\theta = 0.$$

(5). Show that

$$\int_0^{2\pi} \frac{d\theta}{4\cos^2\theta+\sin^2\theta} = \pi.$$

(6). Show that

$$\int_0^{\pi} \frac{d\theta}{a^2+\sin^2\theta} = \frac{\pi}{a\sqrt{1+a^2}} \; for\ a > 0.$$

or

Show that

$$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}} \quad a > 0$$

Using Residue theorem.

(7). Use the method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2acos\theta} = \frac{2a\pi}{1-a^2}, 0 < a < 1.$$

(8). Prove that

$$\int_0^{2\pi} \frac{cos2\theta}{1-2acos\theta+a^2} d\theta = \frac{2\pi a^2}{1-a^2}, (a^2 < 1).$$

(9). Evaluate

$$\int_0^{2\pi} \frac{d\theta}{(5-3sin\theta)^2}$$

Using Residue theorem.

(10). Show that

$$\int_0^{2\pi} \frac{d\theta}{a+bsin\theta} = \int_0^{2\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, a > b > 0$$

Using Residue theorem.

(11). Evaluate

$$\int_0^{2\pi} \frac{sin3\theta}{5-3cos\theta} d\theta.$$

Using Residue theorem.

(12). Show that

$$\int_0^{\pi} \frac{cos2\theta}{1-2acos\theta+a^2} d\theta = \frac{2\pi a^2}{1-a^2}, (a^2 < 1).$$

Using Residue theorem.

(13). Evaluate

$$\int_0^{\infty} \frac{dx}{(x^2-a^2)^2}.$$

(14). Evaluate

$$\int_0^{\infty} \frac{dx}{x^4+a^4}.$$

(15). Using the method of contour integration, prove that

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

or

Evaluate

$$\int_0^\infty \frac{dx}{x^6+1}$$

Using Residue theorem.

(16). Prove that

$$\int_{-\infty}^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a+b)} (a>0, b>0, a\neq b).$$

(17). Prove that

$$\int_{-\infty}^\infty \frac{x^2-x+2}{x^4+10x^2+9} dx = \frac{5\pi}{12}.$$

(18). Evaluate

$$\int_{-\infty}^\infty \frac{dx}{(x^2+9)(x^2+4)^2}$$

Using Residue theorem

(19). Using the method of contour integration, prove that

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+a^2)^3} dx.$$

(20). Prove that

$$\int_{-\infty}^\infty \frac{cosax}{(x^2+1)} dx = \pi e^{-a}, a \geq 0.$$

(21). Evaluate

$$\int_0^\infty \frac{xsinmx}{x^4+16} dx$$

Using Residue theorem

(22). Show by the method of contour integration that

$$\int_0^\infty \frac{cosmx}{(a^2+x^2)^3}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}(a>0, b>0)$$

(23). Evaluate

$$\int_{-\infty}^\infty \frac{cosx}{a^2+x^2}dx(a>0).$$

Using Residue theorem

(24). Evaluate

$$\int_0^\infty \frac{sinmx}{x}dx\ when\ (m>0).$$

Using Residue theorem

(25). Evaluate

$$\int_{-\infty}^\infty \frac{e^{ax}}{e^x+1}dx.$$

(26). Prove that

$$\int_0^{2\pi} \frac{cos^2 3\theta}{1-2pcos2\theta+p^2}d\theta = \pi\frac{1-p+p^2}{1-p}, (0<p<1).$$

(27). Evaluate

$$\int_0^{2\pi} \frac{sinn\theta}{1+2acos\theta+a^2}d\theta = \int_0^{2\pi} \frac{cosn\theta}{1+2acos\theta+a^2}d\theta.$$

$a^2 < 1\ and\ n$ is a positive integer.

(28). Prove that

$$\int_0^{2\pi} \frac{(1+2cos\theta)^n cosn\theta}{3+2cos\theta}d\theta = \frac{2\pi}{\sqrt{5}}\left(3-\sqrt{5}\right)^n n \in I_+.$$

(29). Show that

$$\int_0^\pi tan(\theta+ia)d\theta = i\pi, where\ R(a)>0.$$

(30). By the method of contour integration, prove that

$$\int_0^{2\pi} e^{cos\theta}.cos(n\theta - sin\theta)d\theta = \int_0^{2\pi} e^{cos\theta}.cos(sin\theta - n\theta)d\theta = \frac{2\pi}{n!},$$

Where n is a positive integer.

(31). Prove that

$$\int_0^{2\pi} e^{-cos\theta}.cos(n\theta + sin\theta)d\theta = (-1)^n \frac{2\pi}{n!},$$

Where n is a positive integer.

(32). Show that if m is real and $(-1 < a < 1)$,

a) $\int_0^{2\pi} \frac{e^{mcos\theta}[cos(msin\theta)-asin\theta(msin\theta+\theta]}{1+a^2-2asin\theta} d\theta = 2\pi cosma,$

b) $\int_0^{2\pi} \frac{e^{mcos\theta}[sin(msin\theta)-acos(msin\theta+\theta]}{1+a^2-2asin\theta} d\theta = 2\pi sinma.$

(33). Use calculus of residues to evaluate the integral

$$\int_0^{2\pi} cos^{2n}\theta\, d\theta,$$

Where n is a +ve integer.

(34). Prove that

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3} dx = \frac{\pi}{8a^3},$$

$provided\ that\ R(a)\ is\ +ve.$

What is the value of this integral when $R(a)$ is negative?

(35). Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)(x^2+c^2)^2} dx = \frac{\pi(b+2c)}{2bc^3(b+c)^3}, b > 0, c > 0.$$

(36). Show that if m and n are positive integers and m<n

$$\int_0^{\infty} \frac{x^{2m}}{x^{2n}+1} dx = \frac{\pi}{2nsin(2m+1)/2n\pi}.$$

(37). Prove by contour integration that

$$\int_0^\infty \frac{dx}{(a+bx^2)^n} = \frac{1}{2^n b^{1/2}} \cdot \frac{1.3.5 \ldots \ldots (2n-3)}{1.2.3 \ldots \ldots (2n-1)} \cdot \frac{1}{a^{(n-1)/2}}.$$

(38). Use the method of contour integration to prove that

$$\int_0^\infty \frac{dx}{(x^2+b^2)^{n+1}} = \frac{\pi}{(n!)^2}.$$

Where n is a positive integer.

(39). Prove that

$$\int_{-\infty}^\infty \frac{sinx}{x^2+4x+5} dx = -\frac{\pi}{e} sin2.$$

(40). By integrating $e^{iz}/(z-ai), (a>0)$ round a suitable contour, prove that

$$\int_{-\infty}^\infty \frac{acosx \ + xsinx}{x^2+a^2} dx = 2\pi e^{-a}.$$

(41). By integrating $\frac{e^{iz}}{z+ai}$ round a suitable contour, prove that

$$\int_{-\infty}^\infty \frac{-acosx \ + xsinx}{x^2+a^2} dx = 0.$$

(42). Evaluate

$$\int_0^\infty \frac{cosax}{(x^2+b^2)^2} dx, a>0, b>0.$$

(43). Prove that

$$\int_{-\infty}^\infty \frac{cosx\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right), a>0, b>0.$$

(44). Prove that when $m>0$,

$$\int_{-\infty}^\infty \frac{cosmx}{x^4+x^2+1} dx = \frac{2\pi}{\sqrt{3}} sin\left(\frac{m}{2}+\frac{\pi}{6}\right) e^{-(1/2)m\sqrt{3}}.$$

(45). Prove that when $m>0$,

$$\int_0^\infty \frac{cosmx}{x^4+a^2}dx = \frac{\pi}{2a^3}e^{-ma/\sqrt{2}}sin\left(\frac{ma}{\sqrt{2}}+\frac{\pi}{2}\right).$$

Deduce that

$$\int_0^\infty \frac{xsinmx}{x^4+a^2}dx = \frac{\pi}{2a^2}e^{-ma/\sqrt{2}}sin\frac{ma}{\sqrt{2}}.$$

(46). Prove by contour integration that

$$\int_0^\infty \frac{x^3sinmx}{x^4+a^2}dx = \frac{\pi}{2}e^{-ma/\sqrt{2}}cos\frac{ma}{\sqrt{2}}.$$

(47).

$$\int_0^\infty \frac{x^3sinx}{(x^2+a^2)(x^2+b^2)}dx, a>0, b>0.$$

(48). If $a \geq 4$, prove that

a) $\int_0^\infty \frac{(1+x^2)^3cosax}{1+x^2+x^4}dx = \frac{\pi}{\sqrt{3}}e^{-a(\sqrt{3}/2)}cos\frac{a}{2}$.

b) $\int_0^\infty \frac{xsinax}{1+x^2+x^4}dx = \frac{\pi}{\sqrt{3}}e^{-a(\sqrt{3}/2)}sin\frac{a}{2}$.

(49). Prove that

$$\int_{-\infty}^\infty \frac{sinx}{(1-x+x^2)^2}dx = \frac{2\pi(\sqrt{3}+2)}{3\sqrt{3}}e^{-\sqrt{3}/2}sin\frac{1}{2}.$$

(50). Prove that

$$\int_0^\infty \frac{cosx^2+sinx^2-1}{x^2}dx = 0.$$

(51). Prove by contour integration

$$\int_0^\infty \frac{log(1+x^2)}{1+x^2}dx = \pi log2.$$

(52). If $0<\alpha<\pi/2)$, prove by contour integration

$$\int_0^\infty \frac{\tan^{-1}x}{x^2-2xsin\alpha+1}dx = \frac{\pi\alpha}{2cos\alpha}$$

Where

$$f(z) = \frac{log(1+iz)}{z^2 - 2zsin\alpha + 1}.$$

(53). Apply the calculus of residues to evaluate

$$\int_0^\infty \frac{cosax - cosbx}{x^2} dx, a > b > 0.$$

(54). Prove that if $a > 0$

a) $p\int_{-\infty}^{\infty} \frac{cosx}{a^2-x^2} dx = \frac{\pi sina}{a}$.

b) $p\int_{-\infty}^{\infty} \frac{cosx}{a^2-x^2} dx = 0$.

(55). Prove that

$$\int_0^\infty \frac{sinx}{x(x^2+a^2)} dx = \frac{\pi}{2a^2}(1-e^{-a}), a > 0.$$

(56). Prove that

$$\int_0^\infty \frac{sinmx}{x(x^2+a^2)} dx = \frac{\pi}{2a^4} - \frac{\pi e^{-ma}}{4a^3}\left(m + \frac{2}{a}\right), m > 0, a > 0.$$

(57). Prove that

$$\int_0^\infty \frac{sin^2mx}{x^2(x^2+a^2)} dx = \frac{\pi}{4a^3}(e^{-2ma} - 1 + 2ma), m > 0, a > 0.$$

(58). Show that if a and m are positive

$$\int_0^\infty \frac{sin^2mx}{x^2(x^2+a^2)^2} dx = \frac{\pi}{8a^5}(e^{-2am}(2am+3) + 4am - 3).$$

(59). Evaluate

$$\int_0^\infty \frac{x - sinx}{x^3(x^2+a^2)} dx, a > 0.$$

(60). Prove that

$$\int_0^\infty \frac{1 - cosx}{x^2} dx = \frac{\pi}{2}.$$

(61). Prove that

$$\int_0^\infty \frac{sin\, \pi x}{x(1-x^2)} dx = \pi.$$

(62). Prove that

$$P\int_0^{\infty}\frac{x^4}{x^6-1}dx=\frac{\pi\sqrt{3}}{6}.$$

SuccessClap : Question Bank for Practice

06 ROUCHES THEOREM

(1) **(Fundamental theorem of Algebra)**
Let $P(z) = a_0 + a_1z + \ldots + a_nz^n$, where $n \geq 1$ and $a_n \neq 0$ so that P(z) is a polynomial of degree one or greater. Then the equation P(z) =0 has at least one root.

(2) Every polynomial equation $P(z) = a_0+a_1z+a_2z^2+\ldots a_nz^n = 0$, where $n \geq 1$, $a_n \neq 0$ *has exactly n roots.*

(3) Prove that all the roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$.

(4) Prove that one root of the equation $z^4+z^3+1=0$ lies in the first quadrant.

(5) Use Rouche's theorem to show that the equation $z^5+15z+1=0$ has one root in the disc $|z| < 3/2$ and four roots in the annulus $3/2 < |z| < 2$.

(6) If a > e, use Rouche's theorem to prove that the equation $e^z = az^n$ has n roots inside the circle $|z|=1$.

(7) Using Rouche's theorem determine the number of zeros of the polynomial $P(z) = z^{10}-6z^7+3x^3+1$ in $|z| < 1$.

(8) Apply Rouche's theorem to determine the number of roots of the equation $z^8-4z^5+z^2-1=0$ that lie inside the circle $|z|=1$.

(9) Show that the polynomial z^5+z^3+2z+3 has just one zero in the first quadrant of the complex plane.

(10) Show that the equation $z^4+4(1+i)z+1=0$ has one root in each quadrant.

(11) In which quadrant do the roots of the equation $z^4+4z^3+8z^2+8z+4 =0$ lie?

(12) Show that the equation $z^3+iz+1 =0$ has a root in each of the first, second and fourth quadrants.

SuccessClap : Question Bank for Practice

07 SINGULARITY

(1) Show that a function which has no singularity in the finite part of the plane and has a pole of order n at infinity is a polynomial of degree n.

(2) A polynomial of degree n has no singularities in the finite part of the plane but has a pole of order n at infinity.

(3) If a function f(z) is analytic for all finite values of z and as $|z| \to \infty$, $|f(z)| = a|z|^k$ then f(z) is a polynomial of degree $\leq$ k.

(4) If f(z) is a function such that for some positive integer m, a value $\phi(z_0)$ exists with $\phi(z_0) \neq 0$ such that the function $\phi(z) = (z-z_0)^m f(z)$ is analytic at z_0. Then f(z) has a pole of order m at z_0.

(5) If f(z) has a pole of order m at z_0, then the function ϕ defined by $\phi(z) = (z-z_0)^m f(z)$ has a removable singularity at z_0 and $\phi(z_0) \neq 0$. Also show that the residue at z_0 is given by $\frac{\varphi^{m-1}(z_0)}{(m-1)!}$.

(6) Show that the function $e^{1/z}$ actually takes every value except zero an infinite number of times in the neighbourhood of z =0.

(7) Show that the function e^z has an isolated essential singularity at z = ∞.

(8) Show that the function $(z^2+4)/e^z$ has an isolated essential singularity at z = ∞.

(9) What kind of singularity have the following functions:

(i) $\frac{\cot \pi z}{(z-a)^2}$ at z =0, z = ∞ (ii) sin $\frac{1}{1-z}$ at z =1.

(ii) sin z – cos z at z = ∞ (iv) cosec 1/z at z =0.

(10) What kind of singularity have the following functions:

(i) cot z at z = ∞ $(ii)\ \sec\frac{1}{z}\ at\ z = 0$

(11) Specify the nature of singularity at z = -2 of
F(z) = (z-3) sin $\frac{1}{z+2}$.

(12) (i) Find zeros and poles of $(\frac{z+1}{z^2+1})^2$

(ii) What kind of singularity has the function $\frac{e^z}{z^2+4}$?

(13) Find the kind of the singularities of the following function:

(i) $\frac{1-e^z}{1+e^z}\ at\ z = \infty$ $(ii)\ \frac{1}{sinz - \cos z}\ at\ z = \frac{\pi}{4}$.

$(iii)\ \sin\frac{1}{z} at\ z = 0$ $(iv)\ z\ cosec\ z\ at\ z = \infty$.

(14) Show that z = a is an isolated essential singularity of the function
$\frac{e^{c/(z-a)}}{e^{z/a}-1}$.

(15) Find the nature and location of the singularities of the function f(z) =
$\frac{1}{z(e^z-1)}. Prove\ that\ f(z) can\ be\ expanded\ in\ the\ form \frac{1}{z^2} -$
$\frac{1}{2z}$ +a_0+a_2z^2+a_4z^4+.... Where 0 < |z| < 2π and find the values of a_0 and a_2.

(16) Discuss the nature of singularities of the following function:

(i) Tan z (ii) $\frac{1}{z(1-z^2)}$ (iii) $\frac{z}{1+z^4}$ (iv) $\frac{\sin z}{z-\pi)^2}$

(17) Find the zeros and discuss the nature of singularities of f(z) =
$\frac{z-2}{z^2}\sin\frac{1}{z-1}$.

(18) Show that the function e^{-1/x^2}

(19) If f(z) = $\sum_{n=1}^{\infty} \frac{z^2}{4+n^2z^2}$, show that f(z) is finite and continuous for all real values of z but f(z) cannot be expanded in a Maclaurin's series. Show that f(z) possesses Laurent's expansion valid in a succession of the ring – shaped spaces.

(20) The only singularities of a single valued function f(z) are poles of order 2 and 1 at z =1 and z =2 with residues of these poles 1 and 3 respectively. If f(0) = 3/2 and f(-1) =1, determine the function.

(21) The function f(z) has a double pole at z =0 with residue 2, a simple pole at z =1 with residue 2, is analytic at all other finite points of the plane an is bounded as |z|→ ∞. $If\ f(2) = 5\ and\ f(-1) = 2, find\ f(z)$.

(22) Let ϕ(z) $and\ \psi(z) be\ analytic\ functions. If\ z = a\ is\ a\ once\ repeated\ root\ of\ \psi(z) = 0\ such\ that\ \varphi(a) \neq 0, the\ residue\ of\ \frac{\varphi(z)}{\psi(z)}\ at\ z = a\ is\ \frac{6\varphi'(a)\psi''(a) - 2\varphi(a)\psi'''(a)}{3[\psi''(a)]^2}$

(23) The only singularities of a single valued function f(z) are poles of order 1 and 2 at z = -1 and z =2 with residues 1 and 2 respectively at these poles. If f(0)= 7/4 , f(1) = 5/2, determine the function and expand it in a Laurent's series valid in 1 < |z| < 2.

(24) Show that near z =1, the function log(1+z^2) may be expanded in a series of the form log 2 + $\sum_{n=1}^{\infty} a_n (z-1)^n$ and find the value of a_n, the principal value of the logarithm being taken.

(25) Consider the singularities of the function represented by the series $\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{1+(2^n z)^2}$ and obtain the expansion by Laurent's theorem.

SuccessClap : Question Bank for Practice

08 POWER SERIES

(1) **(Abel's theorem).** If the power series $\sum a_n z^n$ converges for a particular value z_0 of z, then it converges absolutely for all values of z for which $|z| < |z_0|$.

(2) **(Cauchy – Hadamard Theorem).** For every power series $\sum_{n=0}^{\infty} a_n z^n$ *there exist a number* $R, 0 \leq R < \infty$, *called the radius of convergence with the following* properties:

(i) The series converges absolutely for all $|z| < R$.

(ii) If $0 \leq \rho < R$, *the series converges uniformly for* $|z| \leq \rho$.

(iii) If $|z| > R$, the terms of the series are unbounded so that the series is divergent.

(3) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and $\sum_{n=1}^{\infty} a_n z^{n-1}$ *be the power series obtained by* differentiating the series $\sum a_n z^n$ term by term. Then the derived series has the same radius of convergence as the original series.

(4) Find the radii of convergence of the following power series;

(i) $\sum \frac{(n+1)}{(n+2)(n+3)} z^n$ (ii) $\sum \frac{z^n}{n!}$ (iii) $\sum \frac{n!}{n^n} z^n$ (iv) $\sum \frac{1}{n^p} z^n$

(iv) $\sum (1 + \frac{1}{n})^{n^2} z^n$. (vi) $\sum \frac{(n!)^2}{2n!} z^n$.

(5) Find the radii of convergence of the following power series:

(i) $\sum (\log n)^n z^n$ (ii) $\sum (1 - \frac{1}{n})^{n^2} z^n$ (iii) $\sum \frac{z^n}{2^n + 1}$ (iv) $\sum \frac{i\, n+2}{2^n} + z^n$

(v) $\sum_2^{\infty} \frac{z^n}{\log n}$ (vi) $\sum \frac{(-1)^n}{n} (z - 2i)^n$

(6) Find the radii of convergence of the following power series:

(i) $\sum (3+4i)^n z^n$ (ii) $\sum \frac{n\sqrt{2}+i}{1+i\, 2n} z^n$

(7) Find the radius of convergence of the following series:

(i) $\sum \frac{2^{-n}}{1+in^2} z^n$ (ii) $\sum 2^{\sqrt{n}} z^n$ (iii) $\sum \frac{1}{n^n} z^n$.

(8) Show that the radius of convergence of the series

$$\frac{1}{2}z + \frac{1.3}{2.5}z^2 + \frac{1.3.5}{2.5.8}z^3 + \cdots is\ \frac{3}{2}.$$

(9) Prove that $1+\frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2\ c\ (c+1)}z^2$ +...has unit radius of convergence.

(10) Find the radius of convergence of the power series f(z) =
$\sum_0^\infty \frac{z^n}{2^n+1}$ $and\ prove\ that\ (2-z)f(z) - 2 \to 0\ at\ z \to 2.$

(11) Show that the domain of convergence of the series $\sum (\frac{iz-1}{2+i})^n$ is given by $|z+i| < \sqrt{5}$.

(12) Find the domain of convergence of the series $\sum n^2 (\frac{z^2+1}{1+i})^n$.

(13) Find the region of convergence of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!}$.

(14) Determine the set of values of z for which the series $\sum(-1)^n(z^n+z^{n+1})$ converges. Also find the sum of the series.

(15) For what values of v does the series $\sum_{n=1}^{\infty} \frac{1}{(z^2+1)^n}$ converge. Also find its sum.

(16) Find the domain of convergence of the following series :

(i) $\sum_1^\infty \frac{1.3.5...(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n$ (ii) $\sum_2^\infty \frac{z^n}{n(\log n)^2}$.

(17) Find the domain of convergence of the power series $\sum\left(\frac{2i}{z+i+1}\right)^n$.

(18) Examine the behavior of the following power series on the circle of convergence:

(i) $\sum_1^\infty \frac{z^n}{n}$ (ii) $\sum_2^\infty \frac{z^{4n}}{4n+1}$ (iii) $\sum_1^\infty (-1)^n \frac{z^n}{n}$.

(19) Find the region of convergence of the series

(i) $\sum_{n=1}^\infty n!\ z^n$ (ii) $\sum_{n=1}^\infty \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

(20) If R_1 and R_2 are the radii of convergence of the power series $\sum a_n z^n$ and $\sum b_n z^n$ respectively, then show that the radius of convergence of the power series $\sum a_n b_n z^n$ is R_1 and R_2.