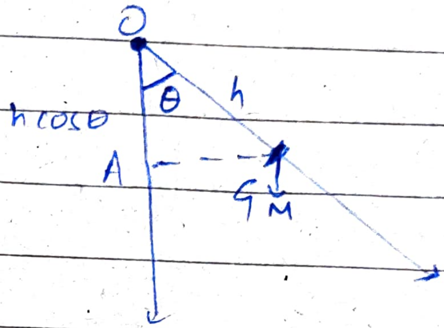
2014 - UPSC - CSE

24 Q-5 (e) eq. of motion of compound pendulum



$$T = \frac{1}{2} I\omega^2 = \frac{1}{2} Mk^2\dot{\theta}^2$$

(T = kinetic energy of pendulum)

let $OG = h$

V = potential energy $= -Mgh\cos\theta$

(−ive because below fixed point O)

so, by $L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta$ — (1)

~~since~~ Here θ is the only generalised co-ordinate;

since L is free of time so,

$$H = T + V = \frac{1}{2}Mk^2\dot{\theta}^2 - Mgh\cos\theta \quad (2)$$

From (1), $\frac{\partial L}{\partial \dot{\theta}} = p_\theta = Mk^2\dot{\theta}$

$$\Rightarrow \boxed{\dot{\theta} = \frac{p_\theta}{Mk^2}}$$

Now (2) becomes,

$$H = \frac{1}{2}Mk^2\frac{p_\theta^2}{M^2k^4} - Mgh\cos\theta$$

So Hamiltonian equations are-

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta \quad , \quad \frac{\partial H}{\partial p_\theta} = \dot{\theta} \quad \text{gives}$$

$$-\dot{p}_\theta = Mgh \sin\theta \quad ; \quad \dot{\theta} = \frac{p_\theta}{Mk^2}$$

$$\Rightarrow \ddot{\theta} = \frac{\dot{p}_\theta}{Mk^2} = -\frac{\cancel{M}gh\sin\theta}{\cancel{M}k^2} = -\frac{gh}{k^2}\sin\theta$$

$$\boxed{\ddot{\theta} = \frac{-gh}{k^2}\sin\theta}$$ is the equation of motion for compound pendulum.

Problem 19. *Show that velocity potential*

$$\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

ϕ $\nabla^2\phi = 0$

gives a possible motion. Determine the form of stream lines and the curves of equal speed.

Solution. Given, $\phi = \frac{1}{2}\log[(x+a)^2+y^2] - \frac{1}{2}\log[(x-a)^2+y^2]$. ...(1)

$\therefore$
$$\frac{\partial\phi}{\partial x} = \frac{x+a}{(x+a)^2+y^2} - \frac{(x-a)}{(x-a)^2+y^2}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{[(x+a)^2+y^2]-2(x+a)^2}{[(x+a)^2+y^2]^2} - \frac{[(x-a)^2+y^2]-2(x-a)^2}{[(x-a)^2+y^2]^2}$$

or
$$\frac{\partial^2\phi}{\partial x^2} = \frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2} - \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \qquad \dots(2)$$

By (1), $\dfrac{\partial\phi}{\partial y} = \dfrac{y}{(x+a)^2+y^2} - \dfrac{y}{(x-a)^2+y^2}$.

$$\frac{\partial^2\phi}{\partial y^2} = \frac{(x+a)^2+y^2-2y^2}{[(x+a)^2+y^2]^2} - \frac{(x-a)^2+y^2-2y^2}{[(x-a)^2+y^2]^2} \qquad \dots(3)$$

Adding (2) and (3), $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0$ or $\nabla^2\phi = 0$.

Thus the equation of continuity is satisfied and so (1) gives a possible liquid motion.

Second Part. To determine stream lines.

$$-\frac{\partial\phi}{\partial x} = u = -\frac{\partial\psi}{\partial y},\ -\frac{\partial\phi}{\partial y} = v = \frac{\partial\psi}{\partial x}.$$

Hence $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y},\ \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$

Now $\dfrac{\partial\psi}{\partial y} = \dfrac{x+a}{(x+a)^2+y^2} - \dfrac{x-a}{(x-a)^2+y^2}$

Integrating w.r.t. y,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + F(x) \dots \qquad \dots(4)$$

where $F(x)$ is constant of integration. To determine $F(x)$.

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \qquad \dots(5)$$

By (4), $\dfrac{\partial\psi}{\partial x} = -\dfrac{y}{(x+a)^2+y^2} + \dfrac{y}{(x-a)^2+y^2} + F'(x) \qquad \dots(6)$

Equating (5) to (6), $F'(x) = 0$. Integrating this

$F(x) =$ absolute const. and hence neglected.

Since it has no effect on the fluid motion.

Now (4) becomes

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} \qquad \dots(7)$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}.$$

Stream lines are given by $\psi =$ const., *i.e.*,

$$\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right] = \text{const. or } \frac{y}{x^2-a^2+y^2} = \text{const.}$$

If we take const. $= 0$, then we get $y = 0$, *i.e.*, x-axis.

If we take conts. $= \infty$, then we get circle $x^2 - a^2 + y^2 = 0$.

26 (8c)

Find Navier-Stokes equation for a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates.

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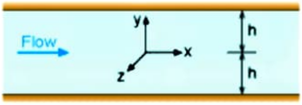

Flow between Fixed Parallel Plates

Flow between Fixed Parallel Plates

Consider steady, incompressible, laminar flow between two infinite parallel horizontal plates as shown in the figure. The flow is in the x- direction, hence there is no velocity component in either the y- or z- direction (i.e., v = 0 and w = 0). The steady-state continuity equation becomes

$$\frac{\partial u}{\partial x} = 0 \qquad [1]$$

From Eqn. 1, it can be concluded that the velocity u is a function of both y and z only. Since the plates are infinitely wide, it can be argued that the velocity u should not be a function of z, i.e., it must be a function of y only, u = u(y).

Applying the Navier-Stokes equations, along with the assumptions that v = 0, w = 0 and u = u(y), yields

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dy^2} \qquad [2]$$

$$\frac{\partial p}{\partial y} = -\rho g \qquad [3]$$

$$\frac{\partial p}{\partial z} = 0 \qquad [4]$$

Eqn. 4 indicates that the pressure is a function of x and y. Integrate Eqn. 3 to yield

$$p = -\rho g y + g_1(x)$$

Hence it can be concluded that $\frac{\partial p}{\partial x}$ is a function of x only. Now, integrate Eqn. 2 twice with respect to y, and treat $\frac{\partial p}{\partial x}$ as a constant (with respect to y) to give:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + c_1 y + c_2$$

Applying the no-slip conditions (i.e., the fluid is "stuck" to the

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u}{dy^2}$$

[2]

$$\frac{\partial p}{\partial y} = -\rho g$$

[3]

$$\frac{\partial p}{\partial z} = 0$$

[4]

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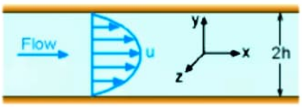
Applying the no-slip conditions (i.e., the fluid is "stuck" to the plates, or u = 0 at y = ±h) to determine the coefficients as follows:

$$c_1 = 0 \quad \text{and} \quad c_2 = -\frac{h^2}{2\mu}\frac{\partial p}{\partial x}$$



Velocity Profile

The velocity profile becomes

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - h^2\right)$$

which is a parabola. The total volumetric flow per linear depth can be obtained integrating the velocity to give

$$q = \int_{-h}^{h} u\,dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

Note, q is per linear depth, which is different than Q which is the total volumetric flow rate.

Also note that the flow is negative, i.e. to the left, for a positive pressure gradient, dp/dx. This is due to the gradient definition where decreasing pressure to the right is negative.