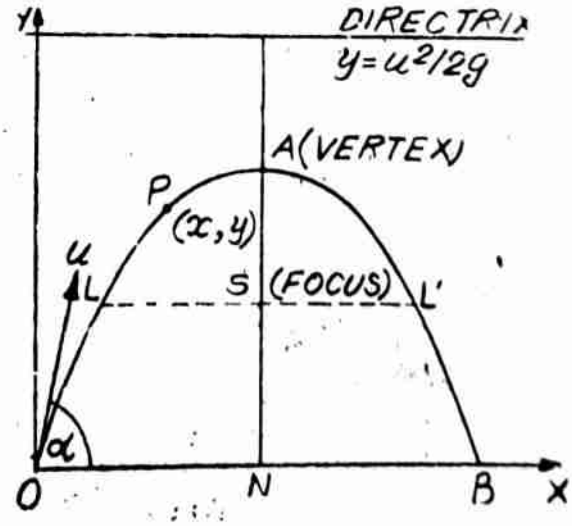
Projectiles

§ 1. **Introduction**. If we throw a ball into the air (not vertically upwards), it describes a curved path. The body so projected is called a projectile and the curved path described by the body is called its trajectory. In this chapter, we shall study the motion of a projectile in a vertical plane through the point of projection, assuming that air offers no resistance and that the acceleration due to the attraction of the earth is constant and is equal to g *i.e.*, its value on the surface of the earth.

§ 2. **The Motion of a Projectile and its Trajectory.** *A particle of mass m is projected, in a vertical plane through the point of projection, with velocity u in a direction making an angle α with the horizontal; to show that the path of the projectile in vaccum is a parabola.* **[Meerut 1977, 78, 81, 83, 83S, 84S, 86; Kanpur 84]**

Take the point of projection O as the origin, the horizontal line OX in the plane of projection as the x-axis and the vertical line OY as the y-axis. Let $P(x, y)$ be the position of the particle at any time t.

There is no force acting upon the particle in the direction of x-axis. The only external force acting upon the particle is its weight mg acting vertically downwards *i.e.*, parallel to the y-axis in the direction of y-decreasing. Therefore the equations of motion of the particle at P are



$$d^2x/dt^2=0 \qquad \ldots(1)$$

and $$d^2y/dt^2=-g \qquad \ldots(2)$$

Integrating (1), we get $dx/dt=$constant.

But initially at the point of projection O, we have $dx/dt=$the horizontal component of the velocity at $O=u\cos\alpha$.

$\therefore$ throughout the motion of the projectile, we have

$$dx/dt=u\cos\alpha. \qquad \ldots(3)$$

Thus the horizontal velocity of a projectile remains constant *i.e.*, $u \cos \alpha$ throughout the motion.

Integrating (3), we get

$$x=(u \cos \alpha).t+A, \text{ where } A \text{ is a constant.}$$

But at the point O, we have $x=0$ and $t=0$. $\therefore$ $A=0$.

$$\therefore \quad x=(u \cos \alpha).t. \qquad \text{...(4)}$$

The equation (4) gives the horizontal displacement of the particle in time t.

Again integrating (2), we get

$$dy/dt=-gt+C, \text{ where } C \text{ is a constant.}$$

But initially at O, $t=0$ and dy/dt = the vertical component of the velocity at $O=u \sin \alpha$.

$$\therefore \; u \sin \alpha=0+C \text{ or } C=u \sin \alpha.$$

$$\therefore \quad dy/dt=u \sin \alpha-gt. \qquad \text{...(5)}$$

The equation (5) gives the vertical component of the velocity of the projectile at any time t.

Now integrating (5), we get

$$y=(u \sin \alpha)\, t-\tfrac{1}{2}gt^2+B, \text{ where } B \text{ is a constant.}$$

But initially at the point O, $y=0$ and $t=0$ so that $B=0$.

$$\therefore \quad y=(u \sin \alpha)\, t-\tfrac{1}{2}gt^2. \qquad \text{...(6)}$$

The equation (6) gives the vertical displacement of the projectile from the point of projection in time t.

For a given value of y, say h, the equation (6) is a quadratic in t and will give two values of t. If the values of t are real and distinct, the smaller value of t will give the time for the projectile to be at a height h while rising upwards and the larger value will give the time for the projectile to be at a height h while falling downwards.

The equations (3), (4), (5) and (6) determine completely the motion of the projectile.

The equations (4) and (6) may be looked upon as the equations of the trajectory in parametric form, the parameter being t.

Eliminating t between (4) and (6), we get

$$y=(u \sin \alpha).\frac{x}{u \cos \alpha}-\tfrac{1}{2}g\left(\frac{x}{u \cos \alpha}\right)^2$$

or

$$y=x \tan \alpha-\tfrac{1}{2}g\frac{x^2}{u^2 \cos^2 \alpha}, \qquad \text{...(7)}$$

as the cartesian form of the equation of the trajectory. The equation (7) is a second degree equation in x and y in which the second degree terms are in a perfect square and hence it represents a parabola.

If v is the resultant velocity of the projectile at P at time t, we have

$$v=\sqrt{[(dx/dt)^2+(dy/dt)^2]}$$
$$=\sqrt{\{(u\cos\alpha)^2+(u\sin\alpha-gt)^2\}}$$
$$=\sqrt{(u^2-2ugt\sin\alpha+g^2t^2)}.$$

The direction of the velocity v is along the tangent to the trajectory at the point P. If this direction makes an angle θ with the horizontal, we have

$$\tan\theta=\frac{dy/dt}{dx/dt}=\frac{u\sin\alpha-gt}{u\cos\alpha}.$$

§ 3. Latus Rectum, Vertex, Focus and Directrix of the Trajectory. [Meerut 1971, 77, 86]

As found in the preceding article, referred to OX and OY as the coordinate axes, the equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}. \qquad \text{...(1)}$$

The equation (1) can be put in the form

$$\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}-x\tan\alpha=-y$$

or $$x^2-\frac{2u^2\cos^2\alpha\tan\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y$$

or $$x^2-\frac{2u^2\cos\alpha\sin\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y$$

or $$\left(x-\frac{u^2\cos\alpha\sin\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}y+\frac{u^4\cos^2\alpha\sin^2\alpha}{g^2}$$

or $$\left(x-\frac{u^2\cos\alpha\sin\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right). \qquad \text{...(2)}$$

If we shift the origin to the point $\left(\dfrac{u^2\cos\alpha\sin\alpha}{g},\dfrac{u^2\sin^2\alpha}{2g}\right)$, the coordinate axes remaining parallel to their original directions, the equation (2) becomes

$$x^2=-\frac{2u^2\cos^2\alpha}{g}y. \qquad \text{...(3)}$$

This is the standard equation of the parabola in the form $x^2=-4ay$ with its vertex at the new origin and its axis AN along the negative direction of the new y-axis. From the equation (3) it is clear that

the latus rectum of the trajectory

$$=\frac{2}{g}u^2\cos^2\alpha=\frac{2}{g}(\text{horizontal velocity})^2.$$

Vertex. If A is the vertex of the trajectory, then A is the new origin. So referred to the original coordinate axes OX and OY, the coordinates of the vertex of the parabola (1) are

$$\left(\frac{\mathbf{u^2 \sin \alpha \cos \alpha}}{\mathbf{g}}, \frac{\mathbf{u^2 \sin^2 \alpha}}{\mathbf{2g}}\right).$$

Focus. Let S be the focus of the trajectory. Then S is a point on the axis of the parabola. We shall find the coordinates of S with respect to the original coordinate axes OX and OY.

Obviously the x-coordinate of S=the x-coordinate of A

$$=\frac{u^2 \sin \alpha \cos \alpha}{g}=\frac{u^2 \sin 2\alpha}{2g}.$$

Again the y-coordinate of S

=the y-coordinate of $A-\frac{1}{4}$ latus rectum

$$=\frac{u^2 \sin^2 \alpha}{2g}-\frac{1}{4}\cdot\frac{2u^2 \cos^2 \alpha}{g}=\frac{u^2 \sin^2 \alpha}{2g}-\frac{u^2 \cos^2 \alpha}{2g}$$

$$=-\frac{u^2}{2g}(\cos^2 \alpha-\sin^2 \alpha)=-\frac{u^2}{2g}\cos 2\alpha.$$

$\therefore$ the coordinates of the focus of the parabola (1) are

$$\left(\frac{\mathbf{u^2}}{\mathbf{2g}}\ \mathbf{\sin 2\alpha}, \frac{\mathbf{-u^2}}{\mathbf{2g}}\ \mathbf{\cos 2\alpha}\right)$$

We observe that the y-coordinate $(-u^2/2g) \cos 2\alpha$ of the focus is positive, zero or negative according as

$$2\alpha > \text{ or } = \text{ or } < \tfrac{1}{2}\pi$$

i.e., $$\alpha > \text{ or } = \text{ or } < \tfrac{1}{4}\pi.$$

If $\alpha=\frac{1}{4}\pi$, the y-coordinate of the focus becomes zero and then the focus is in the horizontal line OX.

Directrix. The directrix of the trajectory is a line perpendicular to the axis of the parabola and so it is a horizontal line.

The height of the directrix above the point of projection O

=the height of the vertex A above $O+\frac{1}{4}$ latus rectum

$$=\frac{u^2 \sin^2 \alpha}{2g}+\frac{1}{4}\cdot\frac{2u^2 \cos^2 \alpha}{g}=\frac{u^2}{2g}(\cos^2 \alpha+\sin^2 \alpha)=\frac{u^2}{2g}.$$

Therefore the equation of the **directrix** of the parabola (1) is

$$\mathbf{y=u^2/2g}.$$

We observe that the equation of the directrix is independent of the angle of projection α.

Therefore *the trajectories of all the particles projected in the same vertical plane from the same point with the same velocity in different directions have the same directrix.*

§ 4. Time of flight, Horizontal range and Maximum height.

Time of flight. *The time taken by the particle from the point of projection to reach the horizontal plane through the point of projection again is called the time of flight.* It is usually denoted by T. In the figure of § 2, the time of flight T is the time from O to B.

Initial vertical velocity at O is $u \sin \alpha$ in the upward direction and the acceleration in the vertical direction is g acting vertically downwards. When the particle strikes the horizontal plane through O at the point B, its vertical displacement from O is zero. So considering the vertical motion from O to B and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$0=(u \sin \alpha)\, T-\tfrac{1}{2}gT^2 \quad \text{or} \quad T\,[u \sin \alpha-\tfrac{1}{2}gT]=0$$

or $$\mathbf{T=\frac{2u \sin \alpha}{g}}. \qquad [\because \ T \neq 0]$$

[Meerut 75, 83; Agra 85]

This gives the time of flight.

Horizonzal range. *If B is the point where the projectile after projection from O, strikes the ground again, then OB is called the horizontal range.* The horizontal range is usually denoted by R.

To find the horizontal range R we consider the horizontal motion from O to B. The horizontal velocity remains constant and equal to $u \cos \alpha$ during the motion from O to B. Also the time from O to B is T. Therefore

$$R=(u \cos \alpha).T=u \cos \alpha.\frac{2u \sin \alpha}{g}.$$

Thus $$\mathbf{R=\frac{2u^2 \sin \alpha \cos \alpha}{g}=\frac{u^2 \sin 2\alpha}{g}}. \qquad ...(1)$$

[Meerut 1983; Agra 85]

Maximum horizontal range. It is the greatest horizontal range for a given velocity of projection, say u. If u is given, then from (1), we see that R depends upon the angle of projection α. Obviously R is maximum when $\sin 2\alpha$ is maximum

i.e., when $\sin 2\alpha=1$ or $2\alpha=\frac{1}{2}\pi$ or $\alpha=\frac{1}{4}\pi$.

Thus for a given velocity of projection the horizontal range is maximum when the angle of projection is 45°. Also **the maximum horizontal range$=u^2/g$.** **[Meerut 1970, 75, 77]**

For the maximum horizontal range, the angle of projection $\alpha=\pi/4$. So in the case of maximum horizontal range, the y-coordinate of the focus of the trajectory

$=\frac{-u^2 \cos 2\alpha}{2g}=\frac{-u^2 \cos \frac{1}{2}\pi}{2g}=0$ *i.e.*, the focus lies on the horizontal line OX.

Thus in the case of maximum horizontal range the focus lies in the range itself.

Again from (1), we observe that the expression for the range remains unchanged if we replace α by $\frac{1}{2}\pi-\alpha$. Therefore to obtain a given horizontal range for a given velocity of projection, there are two possible directions of projection. The inclinations, say α_1 and α_2, of these two directions of projection to the horizontal are complementary angles. Thus

$$\alpha_1+\alpha_2=\tfrac{1}{2}\pi=\tfrac{1}{4}\pi+\tfrac{1}{4}\pi \quad \text{or} \quad \alpha_2-\tfrac{1}{4}\pi=\tfrac{1}{4}\pi-\alpha_1,$$

showing that *the two possible directions of projection for a given range are equally inclined to the direction of projection for the maximum range*.

Greatest height. *The greatest vertical height reached by the projectile during its motion is called the greatest height.* It is usually denoted by H. If A is the highest point of the trajectory, then at A the vertical component of the velocity is zero. Let H be the height of A above the point of projection O. Considering the vertical motion from O to A and using the formula $v^2=u^2+2fs$, we have

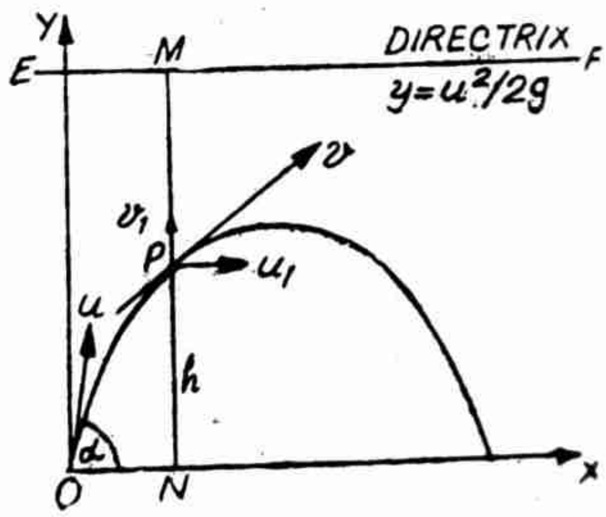
$$0=u^2 \sin^2 \alpha-2gH$$

or

$$H=\frac{u^2 \sin^2 \alpha}{2g}.$$

[Agra 1985]

§ **4. Velocity at any point of the trajectory.** *The velocity of a projectile at any point of its path is that due to a fall from the directrix to that point.* **[Meerut 1990S; Agra 76, 77, 79]**



Suppose a particle is projected from O with velocity u at an angle α to the horizontal. Take O as origin, the horizontal line OX in the plane of motion as x-axis and the vertical line OY as y-axis.

Let v be the velocity of the projectile at any point P of its path. Let the height PN of P above O be h. Suppose u_1 and v_1 are the horizontal and vertical components of v. We have

$$u_1 = u \cos \alpha.$$

Also considering the vertical motion from O to P and using the formula $v^2 = u^2 + 2fs$, we have

$$v_1^2 = u^2 \sin^2 \alpha - 2gh.$$

Now $v^2 = u_1^2 + v_1^2 = u^2 \cos^2 \alpha + u^2 \sin^2 \alpha - 2gh = u^2 - 2gh$. Thus

$$\mathbf{v^2 = u^2 - 2gh}. \qquad \ldots(1)$$

(Remember)

The relation (1) gives the velocity of the projectile at a height h above the point of projection.

The equation of the directrix EF of the trajectory is $y = u^2/2g$. The depth of P below the directrix $= MP = (u^2/2g) - h$. If a particle falls freely under gravity from M to P, let V be the velocity gained by it at P. Then

$$V^2 = 0 + 2g \cdot MP = 2g\,[(u^2/2g) - h] = u^2 - 2gh. \qquad \ldots(2)$$

From (1) and (2), we observe that $v = V$. Hence the velocity of a projectile at any point of its path is that due to a fall from the directrix to that point.

§ 6. **Locus of the focus and vertex of the trajectory.** *Particles are projected in the same verticle plane from the same point with the same velocity in different directions. To find the locus of the foci and also that of the vertices of their paths.*

Refer the figure of § 2.

Take the point of projection O as origin, the horizontal line OX lying in the plane of projection as the x-axis and the vertical line OY as the y-axis. Let u be the velocity of projection for each trajectory. Let S be the focus and A be the vertex of any trajectory for which α is the angle of projection. Here α is a parameter and we are to find the loci of the points S and A for varying values of α.

Locus of the focus. Let (x_1, y_1) be the co-ordinates of the focus S. Then

$$x_1 = \frac{u^2 \sin 2\alpha}{2g}, \quad y_1 = -\frac{u^2 \cos 2\alpha}{2g}.$$

Eliminating α between these two relations, we get

$$x_1^2 + y_1^2 = u^4/4g^2. \qquad \ldots(1)$$

Generalising (1) to get the locus of the point (x_1, y_1), we have $\qquad x^2 + y^2 = u^4/4g^2.$

This is the locus of the foci and is obviously a circle whose centre is the point of projection O and radius is $u^2/2g$.

Locus of the vertex. Let (h, k) be the co-ordinates of the vertex A. Then $h=\dfrac{u^2 \sin \alpha \cos \alpha}{g}$, ...(2)

and $k=\dfrac{u^2 \sin^2 \alpha}{2g}$. ...(3)

To find the locus of the point (h, k) for varying values of α, we have to eliminate α between (2) and (3).

From (3), $\sin^2 \alpha=2gk/u^2$.

Squaring both sides of (2), we get

$$h^2=\frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2}=\frac{u^4 \sin^2 \alpha}{g^2}(1-\sin^2 \alpha) \quad ...(4)$$

Putting $\sin^2 \alpha=2gk/u^2$ in (4), we get

$$h^2=\frac{u^4}{g^2} \cdot \frac{2gk}{u^2}\left(1-\frac{2gk}{u^2}\right)=\frac{2u^2k}{g}\left(1-\frac{2gk}{u^2}\right)$$

or $gh^2=2u^2k-4gk^2$ or $g(h^2+4k^2)=2u^2k$

or $h^2+4k^2=2u^2k/g$.

Generalising (h, k), we get the locus of the vertex as the ellipse $x^2+4y^2=2u^2y/g$.

§ 7. Some geometrical properties of a parabola. The following geometrical properties of a parabola will be often used while solving the problems on projectiles.

1. The distance of any point on a parabola from its focus is equal to its distance from the directrix.

2. The tangents at the extremities of any focal chord of a parabola intersect at right angles on the directrix.

3. The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular drawn from the point to the directrix.

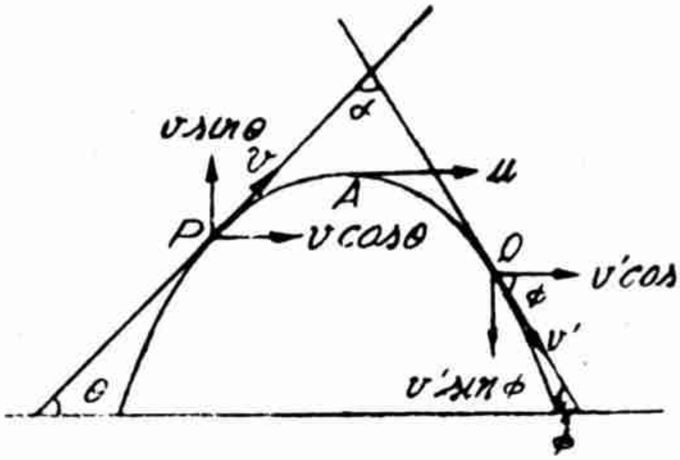
4. The line joining the point of intersection of the tangents at the extremities of any chord of a parabola to the middle point of the chord is parallel to the axis of the parabola.

Illustrative Examples

Ex. 1. *If α be the angle between the tangents at the extremities of any arc of a parabolic path, v and v' the velocities at these extremities and u the velocity at the vertex of the path, show that the time for describing the arc is* $(vv' \sin \alpha)/(gu)$. **[Lucknow 1977]**

Sol. PQ is an arc of a parabolic path and A is its vertex. Suppose the tangents at the points P and Q to the parabola make angles θ and ϕ respectively with the horizontal as shown in the figure. Since α is given to be the angle between the tangents at P and Q, therefore

$$\theta+\phi+\alpha=\pi \quad \text{or} \quad \theta+\phi=\pi-\alpha.$$



The velocity of the particle at P is v and is along the tangent at P. The velocity at Q is v' and is along the tangent at Q. The velocity at the vertex A is u and is along the tangent at A which is a horizontal line.

Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v \cos \theta=u=v' \cos \phi. \qquad \ldots(1)$$

The vertical velocity at $P=v \sin \theta$, vertically upwards and the vertical velocity at $Q=v' \sin \phi$, vertically downwards.

Let t be the time from P to Q. Considering the vertical motion from P to Q and using the formula $v=u+ft$, we have

$$-v' \sin \phi=v \sin \theta-gt, \text{ or } gt=v \sin \theta+v' \sin \phi.$$

$$\therefore \quad t=\frac{v \sin \theta+v' \sin \phi}{g}=\frac{uv \sin \theta+uv' \sin \phi}{gu}$$

[multiplying the Nr. and the Dr. by u]

$$=\frac{vv' \sin \theta \cos \phi+vv' \cos \theta \sin \phi}{gu}$$

[substituting suitably for u from (1)]

$$=\frac{vv' \sin (\theta+\phi)}{gu}=\frac{vv' \sin (\pi-\alpha)}{gu}=\frac{vv' \sin \alpha}{gu}.$$

Ex. 2. *If at any instant the velocity of a projectile be u, and its direction of motion θ to the horizontal, then show that it will be moving at right angles to this direction after time (u/g) cosec θ.*

[**Kanpur 1983**]

Sol. Draw figure as in Ex. 1 by taking $\alpha=\pi/2$ and $\phi=\frac{1}{2}\pi-\theta$.

The velocity of the projectile at the point P is u and its direction makes an angle θ with the horizontal. Let v be the velocity of the projectile at the point Q when it is moving at right angles to its direction at P. Obviously the tangent at Q to the path makes an angle $\frac{1}{2}\pi-\theta$ with the horizontal.

Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$u \cos \theta=v \cos (\tfrac{1}{2}\pi-\theta)=v \sin \theta. \qquad \text{...(1)}$$

The vertical velocity at P is $u \sin \theta$, vertically upwards and the vertical velocity at Q is $v \sin (\frac{1}{2}\pi-\theta)$ *i.e.*, $v \cos \theta$, vertically downwards. Let t be the time from P to Q. Considering the vertical motion from P to Q and using the formula $v=u+ft$, we have

$$-v \cos \theta=u \sin \theta-gt \quad \text{or} \quad gt=u \sin \theta+v \cos \theta.$$

$$\therefore \quad t=\frac{1}{g}(u \sin \theta+v \cos \theta)=\frac{1}{g}\left(u \sin \theta+\frac{u \cos \theta}{\sin \theta} \cos \theta\right)$$

[substituting for v from (1)]

$$=\frac{u}{g \sin \theta}(\sin^2 \theta+\cos^2 \theta)=\frac{u}{g \sin \theta}=\frac{u}{g} \operatorname{cosec} \theta.$$

Ex. 3. *If v_1, v_2 be the velocities at the ends of a focal chord of a projectile's path and u the velocity at the vertex of the path, then show that* $\dfrac{1}{v_1{}^2}+\dfrac{1}{v_2{}^2}=\dfrac{1}{u^2}$. **[Gorakhpur 1981; Kanpur 84]**

Sol. Let P and Q be the extremities of a focal chord PSQ of a projectile's path. [Draw the figure as in Ex. 1]. Suppose the tangent at P to the path makes an angle θ with the horizontal. Since the tangents at the extremities of a focal chord cut at right angles, therefore the tangent at Q to the path makes an angle $\frac{1}{2}\pi-\theta$ with the horizontal.

The velocity at P is v_1 and is along the tangent at P. The velocity at Q is v_2 and is along the tangent at Q. The velocity at the vertex of the path is u and is in a horizontal direction. Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v_1 \cos \theta=u=v_2 \cos (\tfrac{1}{2}\pi-\theta)=v_2 \sin \theta.$$

$$\therefore \quad \cos \theta=\frac{u}{v_1} \text{ and } \sin \theta=\frac{u}{v_2}$$

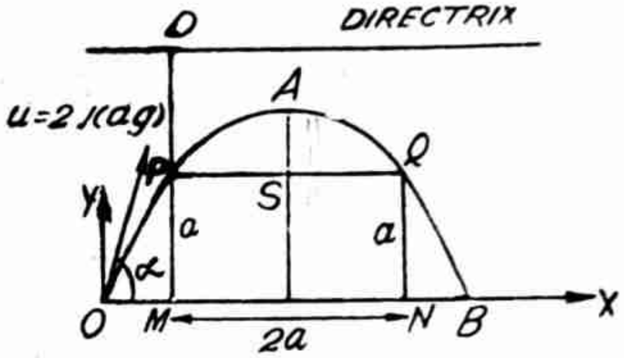
Squaring and adding, we get

$$\frac{u^2}{v_1{}^2}+\frac{u^2}{v_2{}^2}=1 \quad \text{or} \quad u^2\left(\frac{1}{v_1{}^2}+\frac{1}{v_2{}^2}\right)=1 \quad \text{or} \quad \frac{1}{v_1{}^2}+\frac{1}{v_2{}^2}=\frac{1}{u^2}.$$

Ex. 4. *A particle is projected with a velocity $2\sqrt{(ga)}$ so that it just clears two walls of equal height a which are at a distance 2a from each other. Show that the latus rectum of the path is equal to 2a, and that the time of passing between the walls is $2\sqrt{(a/g)}$.*

[Kanpur 1985]

Sol. PM and QN are two vertical walls each of height a and $MN=PQ=2a$. A particle is projected from O with velocity $u=2\sqrt{(ga)}$ at an angle, say α. The particle just clears the walls PM and QN.



Let S be the middle point of PQ. The chord PSQ is perpendicular to the axis of the parabola and the point S is on the axis. Also $PS=\frac{1}{2}PQ=a$.

The height of the directrix of the trajectory above the point of projection $O=DM=\dfrac{u^2}{2g}=\dfrac{4ga}{2g}=2a$.

$\therefore$ the perpendicular distance of P from the directrix $=PD=DM-PM=2a-a=a=PS.$

Thus S is a point on the axis of the parabola such that $PD=PS$. Therefore S is the focus of the trajectory and consequently PSQ is the latus rectum of the path.

$\therefore$ the length of the latus rectum of the path $=PQ=2a$.

But the length of the latus rectum of the path $=\dfrac{2}{g}(u\cos\alpha)^2$.

$\therefore\quad \dfrac{2}{g}(u\cos\alpha)^2=2a$ or $(u\cos\alpha)^2=ag$ or $u\cos\alpha=\sqrt{(ag)}$.

Thus the horizontal velocity of the particle is $\sqrt{(ag)}$.

Let t_1 be the time of passing between the walls *i.e.*, the time from P to Q. Since the horizontal velocity of a projectile remains constant throuhout the motion, therefore considering the horizontal motion from P to Q, we have $2a=(u\cos\alpha)\,t_1$.

$$\therefore\quad t_1=\frac{2a}{u\cos\alpha}=\frac{2a}{\sqrt{(ag)}}=2\sqrt{(a/g)}.$$

Ex. 5. *A body is projected at an angle α to the horizontal so as to clear two walls of equal height a at a distance 2a from each other. Show that the range is equal to $2a\cot\frac{1}{2}\alpha$.*

Sol. Draw figure as in Ex. 4.

Take the point of projection O as the origin, the horizontal line through O in the plane of motion as the x-axis and the vertical line through O as the y-axis. Let u be the velocity of projection and α be the angle of projection.

The equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}. \qquad \ldots(1)$$

The particle just clears two walls PM and QN each of height a and at a distance $2a$ from each other.

The y-co-ordinate of each of the points P and Q is a. Putting $y=a$ in (1), we get

$$a=x\tan\alpha-\tfrac{1}{2}\,\frac{gx^2}{u^2\cos^2\alpha}$$

or $$gx^2-2u^2x\sin\alpha\cos\alpha+2au^2\cos^2\alpha=0. \qquad \ldots(2)$$

Let x_1 and x_2 be the x-co-ordinates of the points P and Q respectively. Then $x_1=OM$ and $x_2=ON$.

Let R be the range of the particle *i.e.*, let $OB=R$. From the symmetry of the path about the axis of the parabola, we have

$$NB=OM=x_1.$$

Now $R=OB=ON+NB=x_2+x_1$.

Obviovsly x_1 and x_2 are the roots of the quadratic (2) in x.

We have $$x_1+x_2=\frac{2u^2\sin\alpha\cos\alpha}{g}=R$$

i.e., $$u^2=\frac{gR}{2\sin\alpha\cos\alpha} \qquad \ldots(3)$$

and $$x_1x_2=\frac{2au^2\cos^2\alpha}{g}.$$

But the distance between the walls $=2a=x_2-x_1$.

$$\therefore\quad 4a^2=(x_2-x_1)^2=(x_2+x_1)^2-4x_1x_2$$

$$=R^2-\frac{8au^2\cos^2\alpha}{g}=R^2-\frac{8a\cos^2\alpha}{g}\cdot\frac{gR}{2\sin\alpha\cos\alpha}$$

[substituting for u^2 from (3)]

or $$R^2-(4a\cot\alpha)\,R-4a^2=0.$$

$$\therefore\quad R=\frac{4a\cot\alpha\pm\sqrt{(16a^2\cot^2\alpha+16a^2)}}{2}.$$

Neglecting the $-$ive sign because R cannot be **$-$ive, we have**

$$R=2a\cot\alpha+2a\operatorname{cosec}\alpha=2a(\cot\alpha+\operatorname{cosec}\alpha)$$

$$=2a\,\frac{\cos\alpha+1}{\sin\alpha}=2a\,\frac{2\cos^2\tfrac{1}{2}\alpha}{2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha}=2a\cot\tfrac{1}{2}\alpha.$$

Ex. 6. *Two bodies are projected from the same point in directions making angles α_1 and α_2 with the horizontal and strike at the same point in the horizontal plane through the point of projection. If t_1, t_2 be their times of flight, show that*

$$\frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_1+\alpha_2)}.$$

Sol. Let u_1 be the velocity of projection of the body projected at an angle α_1 and u_2 be that of the body projected at an angle α_2. Since the horizontal ranges in the two cases are given to be equal, therefore

$$\frac{2u_1^2\sin\alpha_1\cos\alpha_1}{g}=\frac{2u_2^2\sin\alpha_2\cos\alpha_2}{g}$$

or $$\frac{u_1^2}{u_2^2}=\frac{\sin\alpha_2\cos\alpha_2}{\sin\alpha_1\cos\alpha_1}. \qquad ...(1)$$

Also $t_1=2u_1\sin\alpha_1/g$ and $t_2=2u_2\sin\alpha_2/g$.

$$\therefore \quad \frac{t_1^2}{t_2^2}=\frac{u_1^2\sin^2\alpha_1}{u_2^2\sin^2\alpha_2}=\frac{\sin\alpha_2\cos\alpha_2\sin^2\alpha_1}{\sin\alpha_1\cos\alpha_1\sin^2\alpha_2} \qquad \text{[from (1)]}$$

i.e., $$\frac{t_1^2}{t_2^2}=\frac{\sin\alpha_1\cos\alpha_2}{\cos\alpha_1\sin\alpha_2}.$$

Applying componendo and dividendo, we have

$$\frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{\sin\alpha_1\cos\alpha_2-\cos\alpha_1\sin\alpha_2}{\sin\alpha_1\cos\alpha_2+\cos\alpha_1\sin\alpha_2}=\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_1+\alpha_2)}.$$

Ex. 7. *If R be the range of a projectile on a horizontal plane and h its maximum height for a given angle of projection, show that the maximum horizontal range with the same velocity of projection is $2h+(R^2/8h)$.* **[Gorakhpur 1976]**

Sol. Let u be the velocity of projection and α be the angle of of projection. Then

$$h=\frac{u^2\sin^2\alpha}{2g} \quad \text{and} \quad R=\frac{2u^2\sin\alpha\cos\alpha}{g}.$$

We have $$2h+\frac{R^2}{8h}=\frac{2u^2\sin^2\alpha}{2g}+\frac{4u^4\sin^2\alpha\cos^2\alpha}{g^2}\cdot\frac{1}{8}\frac{2g}{u^2\sin^2\alpha}$$

$$=\frac{u^2\sin^2\alpha}{g}+\frac{u^2\cos^2\alpha}{g}=\frac{u^2}{g}(\sin^2\alpha+\cos^2\alpha)=\frac{u^2}{g}$$

= the maximum horizontal range for the velocity of projection u.

Ex. 8. *If R be the horizontal range and h the greatest height of a projectile, prove that the initial velocity is*

$$\left[2g\left(h+\frac{R^2}{16h}\right)\right]^{1/2}.$$

Sol. Let u be the velocity of projection and α the angle of projection. Then

$$R=\frac{2u^2 \sin\alpha\cos\alpha}{g} \quad \ldots(1)$$

and $$h=\frac{u^2\sin^2\alpha}{2g} \quad \ldots(2)$$

To obtain the required values of u we have to eliminate α between (1) and (2). Squaring both sides of (1), we get

$$R^2=\frac{4u^4\sin^2\alpha\cos^2\alpha}{g^2}=\frac{4u^4\sin^2\alpha\,(1-\sin^2\alpha)}{g^2}.$$

Substituting for $\sin^2\alpha$ from (2), we have

$$R^2=\frac{4u^4}{g^2}\cdot\frac{2gh}{u^2}\left(1-\frac{2gh}{u^2}\right)=\frac{8u^2h}{g}\left(1-\frac{2gh}{u^2}\right)=\frac{8u^2h}{g}-16h^2.$$

$\therefore\ (8u^2h)/g=16h^2+R^2$

or $$u^2=\frac{g}{8h}(16h^2+R^2)=\frac{g.16h}{8h}\left(h+\frac{R^2}{16h}\right)=2g\left(h+\frac{R^2}{16h}\right).$$

$$\therefore\ u=\left[2g\left(h+\frac{R^2}{16h}\right)\right]^{1/2}.$$

Ex. 9. *A number of particles start simultaneously from the same point in all directions in a vertical plane, with the same speed u. Show that after time t, they will all lie on a circle of radius ut.*

Show also that the centre of the circle descends with acceleration g.

Sol. The common velocity of projection for all the particles is given to be u. If α be the angle of projection for a particle which after time t has co-ordinates (x, y), then

$$x=(u\cos\alpha).t \text{ and } y=(u\sin\alpha).t-\tfrac{1}{2}gt^2.$$

Eliminating α, all the particles at time t lie on the curve

$$x^2+(y+\tfrac{1}{2}gt^2)^2=u^2t^2, \quad \ldots(1)$$

which is a circle of radius ut.

The co-ordinates of the centre of the circle (1) are $(0, -\frac{1}{2}gt^2)$. If (X, Y) be the centre of the circle (1), we have

$$X=0,\ Y=-\tfrac{1}{2}gt^2. \quad \ldots(2)$$

To find the acceleration of the centre, we differentiate the equations (2) with respect to t. Thus, we have

$$dX/dt=0 \text{ and } d^2X/dt^2=0.$$

Also $dY/dt=-\frac{1}{2}g.2t=-gt$ and $d^2Y/dt^2=-g$.

The x-component of the acceleration of the centre of the circle is 0 and the y-component is $-g$. So the centre of the circle descends with acceleration g.

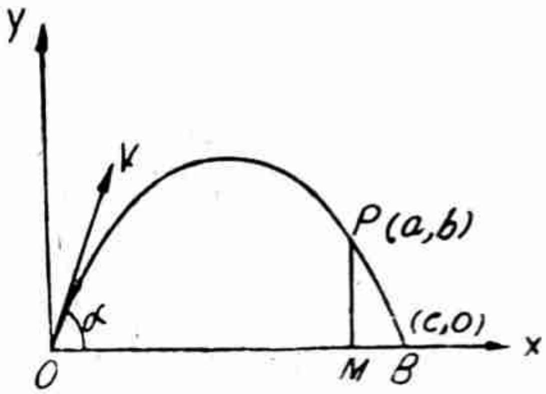
Ex. 10. *A particle just clears a wall of height b at a distance a and strikes the ground at a distance c from the point of projection. Prove that the angle of projection is*

$$\tan^{-1}\left\{\frac{bc}{a\,(c-a)}\right\},$$

and the velocity of projection V is given by

$$\frac{2V^2}{g}=\frac{a^2\,(c-a)^2+b^2c^2}{ab\,(c-a)}.$$

Sol. Let the particle be projected from O with a velocity V at an angle α to the horizontal. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes.



The equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2}{V^2\cos^2\alpha}. \qquad \ldots(1)$$

The particle just clears the wall PM of height b at a distance a from O and strikes the ground at the point B at a distance c from O. Thus both the points (a, b) and $(c, 0)$ lie on the curve (1).

Therefore $$b=a\tan\alpha-\tfrac{1}{2}g\,\frac{a^2}{V^2\cos^2\alpha} \qquad \ldots(2)$$

and $$0=c\tan\alpha-\tfrac{1}{2}g\,\frac{c^2}{V^2\cos^2\alpha}. \qquad \ldots(3)$$

To eliminate V^2, we multiply (2) by c^2 and (3) by a^2 and subtract. Thus we get

$$bc^2=ac^2\tan\alpha-a^2c\tan\alpha \quad \text{or} \quad bc^2=ac\tan\alpha\,(c-a).$$

$$\tan\alpha=\frac{bc}{a\,(c-a)}. \qquad \ldots(4)$$

Now from (3), $$\frac{2V^2}{g}=\frac{c\sec^2\alpha}{\tan\alpha}=\frac{c\,(1+\tan^2\alpha)}{\tan\alpha}.$$

Substituting the value of $\tan\alpha$ from (4), we have

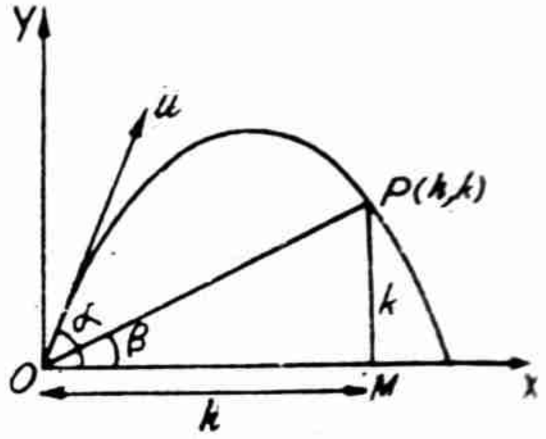
$$\frac{2V^2}{g}=\frac{c\,[1+\{b^2c^2/a^2\,(c-a)^2\}]}{bc/\{a\,(c-a)\}}=\frac{a^2\,(c-a)^2+b^2c^2}{ab\,(c-a)}.$$

Ex. 11. *A particle is projected from O at an elevation α and after t seconds it appears to have an elevation β as seen from the* *point of projection. Prove that the initial velocity was*

$$\frac{gt \cos \beta}{2 \sin (\alpha-\beta)}.$$

[Gorakhpur 1980]

Sol. Let u be the velocity of projection at O. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes. Let $P(h, k)$ be the position of the particle after time t. Considering the motion from O to P in the horizontal and vertical directions, we have



$$h=(u \cos \alpha).t$$

and $$k=(u \sin \alpha).t-\tfrac{1}{2}gt^2.$$

Now according to the question, $\angle POX=\beta$.

$$\therefore \quad \tan \beta=\frac{k}{h} \quad \text{or} \quad \frac{\sin \beta}{\cos \beta}=\frac{(u \sin \alpha)\, t-\frac{1}{2}gt^2}{(u \cos \alpha).t}$$

or $$\frac{\sin \beta}{\cos \beta}=\frac{u \sin \alpha-\frac{1}{2}gt}{u \cos \alpha} \quad [\because \quad t \neq 0]$$

or $$u \cos \alpha \sin \beta=u \sin \alpha \cos \beta-\tfrac{1}{2}gt \cos \beta$$

or $$u (\sin \alpha \cos \beta-\cos \alpha \sin \beta)=\tfrac{1}{2}gt \cos \beta$$

or $$u \sin (\alpha-\beta)=\tfrac{1}{2}gt \cos \beta$$

or $$u=\frac{gt \cos \beta}{2 \sin (\alpha-\beta)}.$$

Ex. 12. *A particle is projected under gravity with a velocity u in a direction making an angle α with the horizontal. Show that the amount of deviation D in the direction of motion of the particle is given by*

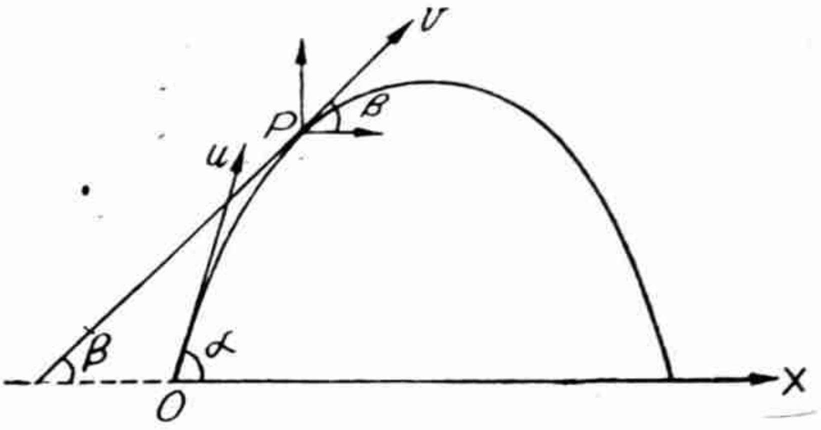
$$\tan D=\frac{gt \cos \alpha}{u-gt \sin \alpha}.$$

[Meerut 1988]

Sol. Let O be the point of projection, u the velocity of projection and u the angle of projection. Let P be the position of the particle at any time t. Suppose v is the velocity of the particle at P. Let β be the inclination to the horizontal of the direction of the velocity at P.

Since the horizontal component of the velocity remains constant throughout the motion, therefore

$$v \cos \beta=u \cos \alpha. \quad \ldots(1)$$



Considering the vertical motion from O to P, we have

$$v \sin \beta = u \sin \alpha - gt. \quad \ldots(2)$$

Dividing (2) by (1), we get

$$\tan \beta = \frac{u \sin \alpha - gt}{u \cos \alpha} = \tan \alpha - \frac{gt}{u \cos \alpha} \quad \ldots(3)$$

i.e.,

$$\tan \alpha - \tan \beta = \frac{gt}{u \cos \alpha}. \quad \ldots(4)$$

Now the deviation D = the angle between the tangents at the points O and $P = \alpha - \beta$.

We have $\tan D = \tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$$= \frac{gt/(u \cos \alpha)}{1 + \tan \alpha \left(\tan \alpha - \dfrac{gt}{u \cos \alpha}\right)} \qquad \text{[from (3) and (4)]}$$

$$= \frac{gt/(u \cos \alpha)}{1 + \tan^2 \alpha - \dfrac{gt \sin \alpha}{u \cos^2 \alpha}} = \frac{gt/(u \cos \alpha)}{\sec^2 \alpha - \dfrac{gt \sec^2 \alpha \sin \alpha}{u}}$$

$$= \frac{gt/(u \cos \alpha)}{\dfrac{\sec^2 \alpha}{u}(u - gt \sin \alpha)} = \frac{gt \cos \alpha}{u - gt \sin \alpha}.$$

Ex. 13. *At any instant a projectile is moving with a velocity u in a direction making an angle α to the horizontal. After an interval of time t, the direction of its path makes an angle β with the horizontal. Prove that*

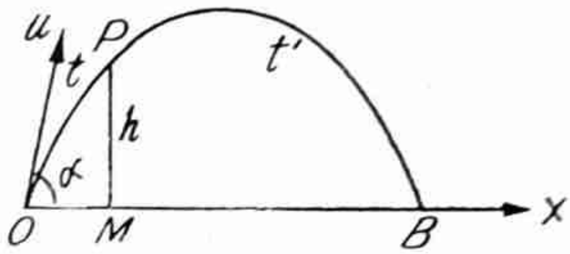
$$u \cos \alpha = \frac{gt}{\tan \alpha - \tan \beta}.$$

Sol. Proceed as in Ex. 12. Here the velocity of the projectile at the point O is u and its direction makes an angle α with the

horizontal. After an interval of time t the projectile is at the point P.

Ex. 14. *If t be the time in which a particle reaches a point P in its path and t' the time from P till it reaches the horizontal plane through the point of projection, show that the height of P above the horizontal plane is $\frac{1}{2}gtt'$.*

Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let OB be the horizontal range. Let P be a point on the path such that the time from O to P is t and the time from P to B is t'. Obviously $t+t'$ is the time of flight.



$$\therefore \quad t+t'=\frac{2u \sin \alpha}{g}. \qquad ...(1)$$

Let h be the height of P above the horizontal plane through O. Considering the vertical motion from O to P and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$h=(u \sin \alpha)\, t-\tfrac{1}{2}gt^2$$
$$=\tfrac{1}{2}g\,(t+t')\, t-\tfrac{1}{2}gt^2 \qquad \text{[substituting for } u \sin \alpha \text{ from (1)]}$$
$$=\tfrac{1}{2}gt^2+\tfrac{1}{2}gtt'-\tfrac{1}{2}gt^2=\tfrac{1}{2}gtt'.$$

Ex. 15. *Two particles are projected from the same point in the same vertical plane with equal velocities. If t_1 and t_2 be the times taken to reach the common point of their paths and T_1 and T_2 the times for their highest points, then prove that $(t_1T_1+t_2T_2)$ is independent of the directions of projection.*

[Gorakhpur 1981 ; Lucknow 81]

Sol. Let O be the common point of projection and u the common velocity of projection. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes. Let $P\,(h, k)$ be the other common point of the two paths.

Let α_1, α_2 be the directions of projection of the two particles. Let T_1, T_2 be the respective times to reach their greatest heights and t_1, t_2 be the respective times to reach the common point P.

We have $T_1=\dfrac{u \sin \alpha_1}{g}$ and $T_2=\dfrac{u \sin \alpha_2}{g}$. $\qquad ...(1)$

Considering the horizontal motion of the two particles from O to P, we have

$$h=(u \cos \alpha_1)\, t_1=(u \cos \alpha_2)\, t_2.$$

$$\therefore \quad t_1=\frac{h}{u \cos \alpha_1} \text{ and } t_2=\frac{h}{u \cos \alpha_2} \quad \ldots(2)$$

From (1) and (2), we have

$$t_1T_1+t_2T_2=\frac{h}{u \cos \alpha_1} \cdot \frac{u \sin \alpha_1}{g}+\frac{h}{u \cos \alpha_2} \cdot \frac{u \sin \alpha_2}{g}$$

$$=\frac{h}{g}(\tan \alpha_1+\tan \alpha_2). \quad \ldots(3)$$

Since the point (h, k) lies on both the trajectories, therefore

$$k=h \tan \alpha_1-\frac{1}{2}\frac{gh^2}{u^2 \cos^2 \alpha_1} \quad \ldots(4)$$

and $$k=h \tan \alpha_2-\frac{1}{2}\frac{gh^2}{u^2 \cos^2 \alpha_2}. \quad \ldots(5)$$

Subtracting (5) from (4), we get

$$h(\tan \alpha_1-\tan \alpha_2)-\frac{1}{2}\frac{gh^2}{u^2}(\sec^2 \alpha_1-\sec^2 \alpha_2)=0$$

or $(\tan \alpha_1-\tan \alpha_2)-\frac{1}{2}g\frac{h}{u^2}(\tan^2 \alpha_1-\tan^2 \alpha_2)=0$ $\quad [\because h \neq 0]$

or $(\tan \alpha_1-\tan \alpha_2)\,[1-\frac{1}{2}g\,(h/u^2)(\tan \alpha_1+\tan \alpha_2)]=0$

or $1-\frac{1}{2}g\,(h/u^2)(\tan \alpha_1+\tan \alpha_2)=0$ $\quad [\because \tan \alpha_1 \neq \tan \alpha_2]$

or $(\tan \alpha_1+\tan \alpha_2)=2u^2/gh.$

Substituting this value of $(\tan \alpha_1+\tan \alpha_2)$ in (3), we have

$t_1T_1+t_2T_2=\frac{h}{g} \cdot \frac{2u^2}{gh}=\frac{2u^2}{g^2}$, which is independent of α_1 and α_2.

Ex. 16. *Obtain the equation of the path of a projectile and show that it may be written in the form*

$$\frac{yR}{xR-x^2}=\tan \alpha,$$

where R is the horizontal range and α the angle of projection.

[Lucknow 1980 ; Allahabad 77]

Sol. For the first part refer § 2. Thus referred to the point of projection O as origin, the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes, the equation of the path of a projectile is

$$y=x \tan \alpha-\frac{1}{2}g\frac{x^2}{u^2 \cos^2 \alpha}, \quad \ldots(1)$$

where u is the velocity of projection.

If R is the horizontal range, then

$$R=\frac{2u^2 \sin\alpha \cos\alpha}{g}. \qquad ...(2)$$

Substituting for u^2 from (2) in (1), we have

$$y=x \tan\alpha-\tfrac{1}{2}g \frac{x^2}{\cos^2\alpha} \cdot \frac{2 \sin\alpha \cos\alpha}{Rg}$$

$$=x \tan\alpha-\frac{x^2}{R} \tan\alpha=\tan\alpha\left(x-\frac{x^2}{R}\right)$$

$\therefore \quad \dfrac{y}{x-(x^2/R)}=\tan\alpha$ or $\dfrac{yR}{xR-x^2}=\tan\alpha$, is the equation of the path in the required form.

Ex. 17. *A particle is projected in a direction making an angle θ with the horizontal. If it passes through the points (x_1, y_1) and (x_2, y_2) referred to horizontal and vertical axes through the point of projection, then prove that*

$$\tan\theta=\frac{x_2^2 y_1-x_1^2 y_2}{x_1 x_2 (x_2-x_1)}.$$

Sol. The equation of the trajectory is

$$y=x \tan\theta-\tfrac{1}{2}\frac{gx^2}{u^2 \cos^2\theta}. \qquad ...(1)$$

Since the curve (1) passes through the points (x_1, y_1) and (x_2, y_2), therefore

$$y_1=x_1 \tan\theta-\tfrac{1}{2}g\, x_1^2/(u^2 \cos^2\theta) \qquad ...(2)$$

and

$$y_2=x_2 \tan\theta-\tfrac{1}{2}g\, x_2^2/(u^2 \cos^2\theta). \qquad ...(3)$$

Multiplying (2) by x_2^2 and (3) by x_1^2 and subtracting, we get

$$y_1 x_2^2-y_2 x_1^2=x_1 x_2^2 \tan\theta-x_2 x_1^2 \tan\theta=x_1 x_2 (x_2-x_1) \tan\theta.$$

$$\therefore \quad \tan\theta=\frac{y_1 x_2^2-y_2 x_1^2}{x_1 x_2 (x_2-x_1)}.$$

Ex. 18. *A gun is firing from the sea level out to sea. It is then mounted in a battery h feet higher up and fired at the same elevation α. Show that the range is increased by*

$$\tfrac{1}{2}\left\{\left(1+\frac{2gh}{v^2 \sin^2\alpha}\right)^{1/2}-1\right\}$$

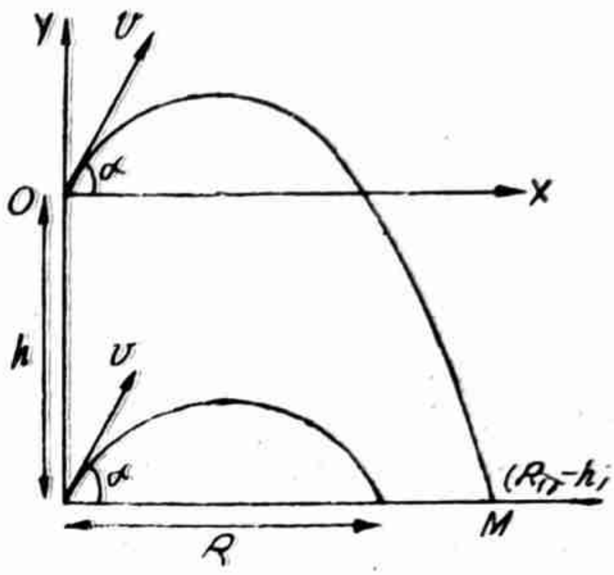
of itself, v being the velocity of projection.

[Gorakhpur 1979; Meerut 81, 85]

Sol. Let R be the original range. Then

$$R=\frac{2v^2 \sin\alpha \cos\alpha}{g}. \qquad ...(1)$$

Let O be a point at a height h above the water level. Let R_1 be the range on the sea when the shot is fired from O.



Referred to the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes, the co-ordinates of the point M where the shot strikes the water are $(R_1, -h)$.

The point $(R_1, -h)$ lies on the curve

$$y = x \tan \alpha - \frac{1}{2} \frac{gx^2}{v^2 \cos^2 \alpha}.$$

$$\therefore \quad -h = R_1 \tan \alpha - \frac{1}{2} \frac{gR_1^2}{v^2 \cos^2 \alpha}$$

or $$R_1^2 - \frac{2}{g} v^2 \sin \alpha \cos \alpha \, R_1 - \frac{2}{g} v^2 h \cos^2 \alpha = 0$$

or $$R_1^2 - RR_1 - \frac{2}{g} v^2 h \cos^2 \alpha = 0 \text{ or } R_1^2 - RR_1 = \frac{2}{g} v^2 h \cos^2 \alpha$$

or $$(R_1 - \tfrac{1}{2}R)^2 = \frac{1}{4} R^2 + \frac{2}{g} v^2 h \cos^2 \alpha = \frac{R^2}{4}\left[1 + \frac{1}{R^2} \cdot \frac{8}{g} v^2 h \cos^2 \alpha\right]$$

$$= \frac{R^2}{4}\left[1 + \frac{g^2}{4v^4 \sin^2 \alpha \cos^2 \alpha} \cdot \frac{8}{g} v^2 h \cos^2 \alpha\right], \quad \text{[by (1)]}$$

$$= \frac{R^2}{4}\left[1 + \frac{2gh}{v^2 \sin^2 \alpha}\right].$$

$$\therefore \quad R_1 - \tfrac{1}{2}R = \tfrac{1}{2}R\left(1 + \frac{2gh}{v^2 \sin^2 \alpha}\right)^{1/2}$$

so that $$R_1 - R = \tfrac{1}{2}R\left(1 + \frac{2gh}{v^2 \sin^2 \alpha}\right)^{1/2} - \tfrac{1}{2}R$$

$$= \tfrac{1}{2}\left\{\left(1 + \frac{2gh}{v^2 \sin^2 \alpha}\right)^{1/2} - 1\right\} R.$$

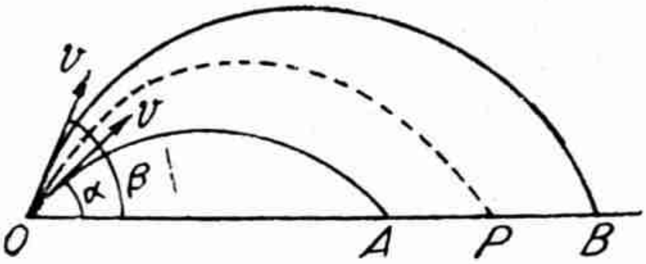
Hence the range is increased by $\frac{1}{2}\left\{\left(1 + \frac{2gh}{v^2 \sin^2 \alpha}\right)^{1/2} - 1\right\}$ of its former value.

Ex. 19. *A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is* $\frac{1}{2} \sin^{-1} \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}$.

[Gorakhpur 1976; Meerut 82, 85P, 86P]

Sol. Let O be the point of projection and v the velocity of projection in all the cases. Let P be the point in the horizontal plane through O required to be hit from O. Let θ be the correct angle of projection to hit P from O. Then

$$OP = \text{the range for the angle of projection } \theta = \frac{v^2 \sin 2\theta}{g}.$$



When the angle of projection is α, the particle falls at A and when the angle of projection is β, it falls at B. We have

$$OA = \frac{v^2 \sin 2\alpha}{g} \text{ and } OB = \frac{v^2 \sin 2\beta}{g}.$$

According to the question,

$$AP = OP - OA = a \quad \text{and} \quad PB = OB - OP = b.$$

$$\therefore \quad a = \frac{v^2 \sin 2\theta}{g} - \frac{v^2 \sin 2\alpha}{g} = \frac{v^2}{g} (\sin 2\theta - \sin 2\alpha), \qquad ...(1)$$

and $$b = \frac{v^2 \sin 2\beta}{g} - \frac{v^2 \sin 2\theta}{g} = \frac{v^2}{g} (\sin 2\beta - \sin 2\theta). \qquad ...(2)$$

Dividing (1) by (2), we get

$$\frac{a}{b} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$$

or $$a \sin 2\beta - a \sin 2\theta = b \sin 2\theta - b \sin 2\alpha$$

or $$(a+b) \sin 2\theta = a \sin 2\beta + b \sin 2\alpha$$

or $$\sin 2\theta = \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}.$$

$$\therefore \quad 2\theta = \sin^{-1} \frac{a \sin 2\beta + b \sin 2\alpha}{a+b} \text{ or } \theta = \tfrac{1}{2} \sin^{-1} \frac{a \sin 2\beta + b \sin 2\alpha}{a+b}.$$

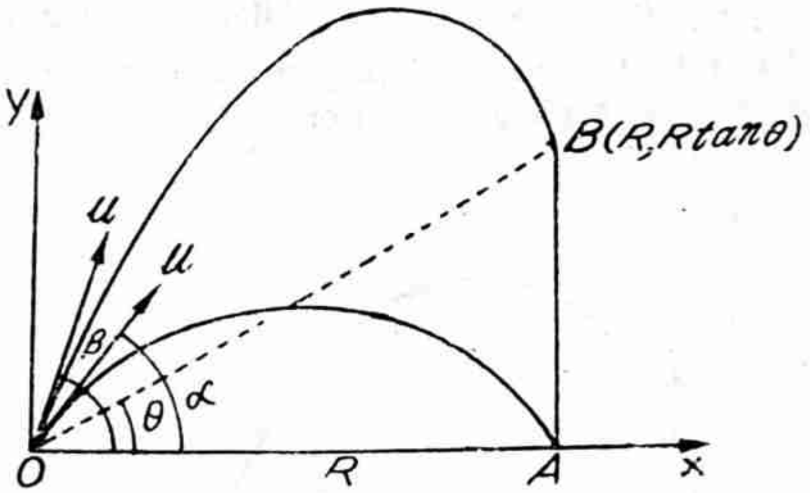
Ex. 20. *A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is* $\frac{1}{2}[\theta+\sin^{-1}(\sin\theta+\sin 2\alpha\cos\theta)]$.

Sol. Let AB be the tower and O the point of projection. It is given that $\angle AOB=\theta$.

Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB. Let $OA=R$.

Then $$R=\frac{u^2\sin 2\alpha}{g}.$$

Referred to the horizontal and vertical lines OX and OY lying in the plane of motion as the co-ordinate axes, the co-ordinates of the top B of the tower are $(R, R\tan\theta)$.



If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y=x\tan\beta-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\beta}.$$

$$\therefore\quad R\tan\theta=R\tan\beta-\tfrac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

or $$\tan\theta=\tan\beta-\tfrac{1}{2}g\frac{R}{u^2\cos^2\beta}. \qquad [\because\ R\neq 0]$$

Substituting the value of R from (1), we get

$$\tan\theta=\tan\beta-\tfrac{1}{2}g\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

or $$\tan\theta=\tan\beta-\frac{\sin 2\alpha}{2\cos^2\beta}$$

or $\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sin\beta}{\cos\beta}-\dfrac{\sin 2\alpha}{2\cos^2\beta}$.

Multiplying both sides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta=2\sin\beta\cos\beta\cos\theta-\cos\theta\sin 2\alpha$$

or $(1+\cos 2\beta)\sin\theta=\sin 2\beta\cos\theta-\cos\theta\sin 2\alpha$

or $\sin 2\beta\cos\theta-\cos 2\beta\sin\theta=\sin\theta+\cos\theta\sin 2\alpha$

or $\sin(2\beta-\theta)=\sin\theta+\cos\theta\sin 2\alpha$

or $2\beta-\theta=\sin^{-1}(\sin\theta+\cos\theta\sin 2\alpha)$

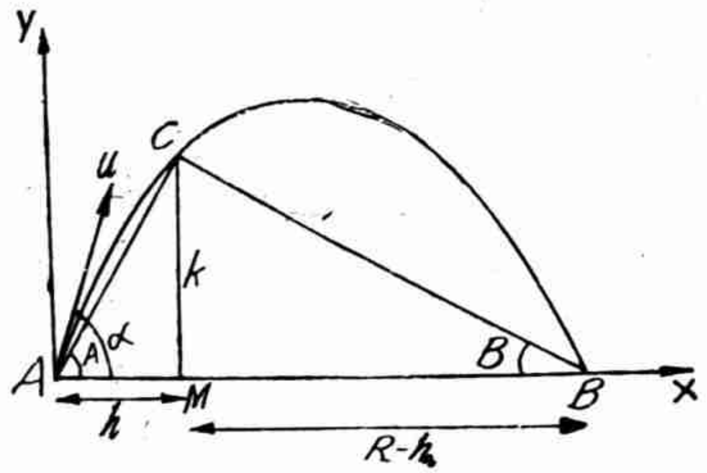
or $2\beta=\theta+\sin^{-1}(\sin\theta+\cos\theta\sin 2\alpha)$

or $\beta=\frac{1}{2}[\vartheta+\sin^{-1}(\sin\theta+\sin 2\alpha\cos\theta)]$.

Ex. 21. *A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, prove that*

$\tan\alpha=\tan A+\tan B$. **[Allahabad 1975, Agra 85]**

Sol. Let A be the point of projection, u the velocity of projection and α the angle of projection.



The particle while grazing over the vertex C falls at the point B. If $AB=R$, then $R=\dfrac{2u^2\sin\alpha\cos\alpha}{g}$. ...(1)

Take the horizontal line AB as the x-axis and the vertical line AY as the y-axis. Let the co-ordinates of the vertex C be (h, k). Then the point (h, k) lies on the trajectory whose equation is

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}.$$

$$\therefore\quad k=h\tan\alpha-\tfrac{1}{2}g\,\frac{h^2}{u^2\cos^2\alpha}=h\tan\alpha\left[1-\frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$$=h \tan \alpha \left[1-\frac{h}{R}\right] \qquad \text{[by (1)]}$$

$$\therefore \frac{k}{h}=\tan \alpha \left(\frac{R-h}{R}\right)$$

or $$\tan A=\tan \alpha \left(\frac{R-h}{R}\right). \left[\because \text{ from } \triangle CAM, \tan A=\frac{k}{h}\right]$$

$$\therefore \quad \tan \alpha=\tan A \left(\frac{R}{R-h}\right)=\tan A\left[\frac{(R-h)+h}{R-h}\right]$$

$$=\tan A\left[1+\frac{h}{R-h}\right]=\tan A+\tan A \frac{h}{R-h}$$

$$=\tan A+\frac{k}{h}\cdot\frac{h}{R-h} \qquad \left[\because \tan A=\frac{k}{h}\right]$$

$$=(\tan A)+k/(R-h).$$

But from the $\triangle CMB$, $\tan B=k/(R-h)$.

$\therefore \quad \tan \alpha=\tan A+\tan B$.

Ex. 22. *Two particles are projected simultaneously in the same vertical plane from the same point with velocities u and v at angles α and β to the horizontal. Prove that,*

(i) *The line joining them moves parallel to itself.*

(ii) *The time that elapses when their velocities are parallel, is*

$$\frac{uv \sin(\alpha-\beta)}{g\,(v \cos\beta-u\cos\alpha)}.$$

(iii) *The interval between their transits through the other common point to their paths is*

$$\frac{2uv \sin(\alpha-\beta)}{g\,(u \cos\alpha+v\cos\beta)}.$$

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of motion as the co-ordinate axes.

(i) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the respective positions of the two particles after time t. Then

$$x_1=(u\cos\alpha)\,t,\; y_1=(u\sin\alpha)\,t-\tfrac{1}{2}gt^2$$

and $$x_2=(v\cos\beta)\,t,\; y_2=(v\sin\beta)\,t-\tfrac{1}{2}gt^2.$$

The gradient (slope) of the line PQ

$$=\frac{y_2-y_1}{x_2-x_1}=\frac{(v\sin\beta-u\sin\alpha)\,t}{(v\cos\beta-u\cos\alpha)\,t}=\frac{v\sin\beta-u\sin\alpha}{v\cos\beta-u\cos\beta},$$

which is independent of the time t. Hence the line PQ moves parallel to itself.

(ii) Let θ_1 and θ_2 be the respective directions of motion of the two particles at time t. Then

$$\tan\theta_1=\frac{u\sin\alpha-gt}{u\cos\alpha} \quad \text{and} \quad \tan\theta_2=\frac{v\sin\beta-gt}{v\cos\beta}.$$

The two directions of motion will be parallel if

$$\theta_1=\theta_2 \quad i.e., \quad \text{if } \frac{u\sin\alpha-gt}{u\cos\alpha}=\frac{v\sin\beta-gt}{v\cos\beta}$$

i.e., if $uv\sin\alpha\cos\beta-tgv\cos\beta=uv\cos\alpha\sin\beta-tgu\cos\alpha$

i.e., if $tg(v\cos\beta-u\cos\alpha)=uv(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

i.e., if $t=\dfrac{uv\sin(\alpha-\beta)}{g(v\cos\beta-u\cos\alpha)}$.

(iii) Let (h, k) be the co-ordinates of the other common point, say C, of their paths.

Let t_1 and t_2 be the respective times taken by the two particles to reach the common point (h, k). Considering their horizontal motion from O to C (horizontal distance for both is h), we have

$$h=(u\cos\alpha)\,t_1=(v\cos\beta)\,t_2.$$

$$\therefore \quad t_1=\frac{h}{u\cos\alpha} \quad \text{and} \quad t_2=\frac{h}{v\cos\beta}.$$

$$\therefore \quad t_1-t_2=h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right). \qquad ...(1)$$

Since the point (h, k) lies on both the paths, therefore

$$k=h\tan\alpha-\tfrac{1}{2}\frac{gh^2}{u^2\cos^2\alpha}$$

and
$$k=h\tan\beta-\tfrac{1}{2}\frac{gh^2}{v^2\cos^2\beta}.$$

Subtracting these two equations, we have

$$\tfrac{1}{2}gh\left(\frac{1}{u^2\cos^2\alpha}-\frac{1}{v^2\cos^2\beta}\right)=\tan\alpha-\tan\beta$$

or
$$\tfrac{1}{2}gh\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)\left(\frac{1}{u\cos\alpha}+\frac{1}{v\cos\beta}\right)=\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}$$

or
$$h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)=\frac{2}{g}\frac{uv\sin(\alpha-\beta)}{(u\cos\alpha+v\cos\beta)}. \qquad ...(2)$$

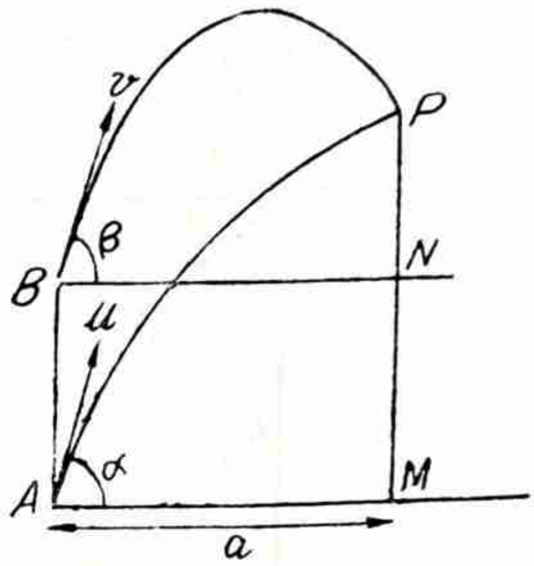
From (1) and (2), we have

$$t_1-t_2=\frac{2uv\sin(\alpha-\beta)}{g(u\cos\alpha+v\cos\beta)}.$$

Ex. 23. *Shots fired simultaneously from the bottom and top of a vertical cliff with elevations α and β respectively, strike an object* *simultaneously. Show that if a be the horizontal distance of the object from the cliff, the height of the cliff is*

$$a\ (\tan\alpha-\tan\beta).$$

Sol. AB is a vertical cliff. A shot is fired from A, say with velocity u, at an elevation α. At the same time a shot is fired from B, say with velocity v, at an elevation β. The two shots strike an object P simultaneously. Let t be the time taken by each shot to reach P. The horizontal distance of P from the cliff AB is a.



Considering the horizontal motion of each shot from its point of projection upto the point P, we have

$$a=(u\cos\alpha)\ t=(v\cos\beta)\ t. \qquad \ldots(1)$$

Considering the vertical motion of each shot from its point of projection upto the point P, we have

$$MP=(u\sin\alpha)\ t-\tfrac{1}{2}gt^2$$

and $$NP=(v\sin\beta)\ t-\tfrac{1}{2}gt^2.$$

The height of the cliff. $AB=MN=MP-NP$

$$=(u\sin\alpha)\ t-(v\sin\beta)\ t=(ut)\sin\alpha-(vt)\sin\beta$$

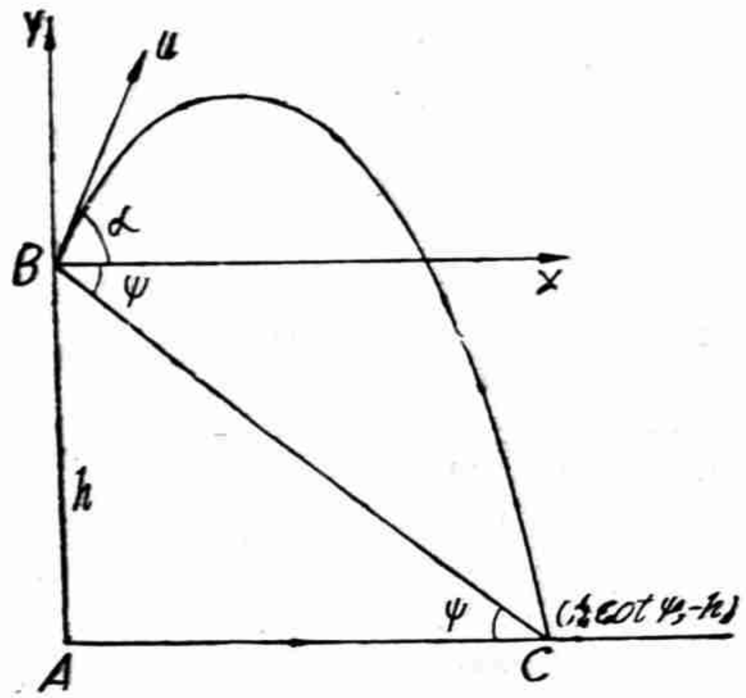
$$=\frac{a}{\cos\alpha}.\sin\alpha-\frac{a}{\cos\beta}\cdot\sin\beta$$

$$=a\ (\tan\alpha-\tan\beta).$$

Ex. 24. *From a tower an object was observed on the ground at a depression ϕ below the horizon. A gun was fired at an elevation α, but the shot missing the object struck the ground at a point whose depression was ψ. Prove that the correct elevation θ of the gun is given by*

$$\frac{\cos\theta\ \sin(\theta+\phi)}{\cos\alpha\ \sin(\alpha+\psi)}=\frac{\cos^2\phi\ \sin\psi}{\cos^2\psi\ \sin\phi}.$$

Sol. Let AB be a tower of height h. Let u be the velocity of projection of the shot. When projected at any elevation α from B, suppose the shot strikes the ground at C whose depression is ψ. Take the horizontal and vertical lines BX and BY as the co-



ordinate axes. The co-ordinates of the point C are $(h \cot \psi, -h)$. The point C lies on the trajectory whose equation is

$$y = x \tan \alpha - \tfrac{1}{2} \frac{gx^2}{u^2 \cos^2 \alpha}.$$

$$\therefore \quad -h = h \cot \psi \tan \alpha - \tfrac{1}{2} \frac{gh^2 \cot^2 \psi}{u^2 \cos^2 \alpha}$$

or $$1 + \cot \psi \tan \alpha = \tfrac{1}{2} g \frac{h \cot^2 \psi}{u^2 \cos^2 \alpha} \qquad [\because \; h \neq 0]$$

or $$1 + \frac{\cos \psi \sin \alpha}{\sin \psi \cos \alpha} = \tfrac{1}{2} g \frac{h \cos^2 \psi}{u^2 \sin^2 \psi \cos^2 \alpha}$$

or $$\frac{\sin (\alpha + \psi)}{\cos \alpha \sin \psi} = \tfrac{1}{2} g \frac{h \cos^2 \psi}{u^2 \sin^2 \psi \quad s^2}$$

or $$\cos \alpha \sin (\alpha + \psi) = \tfrac{1}{2} g \frac{h \cos^2 \psi}{u^2 \sin \psi}. \qquad \ldots(1)$$

Again when θ is the angle of projection, the shot strikes the point on the ground whose depression is ϕ. Therefore replacing α be θ and ψ by ϕ, we have from (1)

$$\cos \theta \sin (\theta + \phi) = \tfrac{1}{2} g \frac{h \cos^2 \phi}{u^2 \sin \phi}. \qquad \ldots(2)$$

Dividing (2) by (1), we have

$$\frac{\cos \theta \sin (\theta + \phi)}{\cos \alpha \sin (\alpha + \psi)} = \frac{\cos^2 \phi \sin \psi}{\cos^2 \psi \sin \phi}.$$

Ex. 25. *If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that* $\quad v_3t_1=v_1t_2$

and
$$\frac{1}{v_1}+\frac{1}{v_3}=\frac{2\cos\beta}{v_2}.$$

Sol. Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v_1\cos\alpha=v_2\cos(\alpha-\beta)=v_3\cos(\alpha-2\beta). \qquad \text{...(1)}$$

Considering the vertical motion from P to Q and then from Q to R and using the formula $v=u+ft$, we get

$$v_2\sin(\alpha-\beta)=v_1\sin\alpha-gt_1 \qquad \text{...(2)}$$

and
$$v_3\sin(\alpha-2\beta)=v_2\sin(\alpha-\beta)-gt_2. \qquad \text{...(3)}$$

From (2) and (3), we have

$$\frac{t_1}{t_2}=\frac{v_1\sin\alpha-v_2\sin(\alpha-\beta)}{v_2\sin(\alpha-\beta)-v_3\sin(\alpha-2\beta)}$$

$$=\frac{v_1\sin\alpha-\dfrac{v_1\cos\alpha}{\cos(\alpha-\beta)}\sin(\alpha-\beta)}{\dfrac{v_3\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}\sin(\alpha-\beta)-v_3\sin(\alpha-2\beta)}$$

[substituting suitably for v_2 from (1)]

$$=\frac{v_1[\sin\alpha\cos(\alpha-\beta)-\cos\alpha\sin(\alpha-\beta)]}{v_3[\sin(\alpha-\beta)\cos(\alpha-2\beta)-\cos(\alpha-\beta)\sin(\alpha-2\beta)]}$$

$$=\frac{v_1\sin\{\alpha-(\alpha-\beta)\}}{v_3\sin\{(\alpha-\beta)-(\alpha-2\beta)\}}=\frac{v_1\sin\beta}{v_3\sin\beta}=\frac{v_1}{v_3}.$$

$\therefore\quad v_3t_1=v_1t_2$. This proves the first result.

Again from (1), we have

$$\frac{1}{v_1}=\frac{1}{v_2}\frac{\cos\alpha}{\cos(\alpha-\beta)} \quad\text{and}\quad \frac{1}{v_3}=\frac{1}{v_2}\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}.$$

$$\therefore\quad \frac{1}{v_1}+\frac{1}{v_3}=\frac{1}{v_2}\frac{\cos\alpha+\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

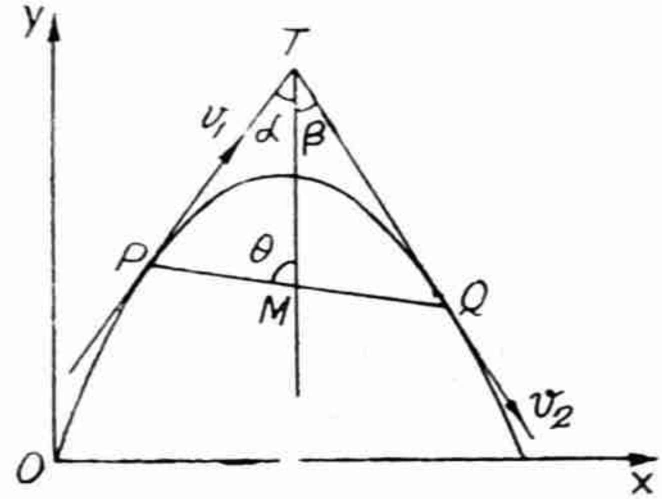
$$=\frac{1}{v_2}\frac{2\cos(\alpha-\beta)\cos\beta}{\cos(\alpha-\beta)}=\frac{2\cos\beta}{v_2}.$$

This proves the second result.

Ex. 26. *If v_1 and v_2 are the velocities at two points P and Q of a parabolic trajectory, and PT and QT the corresponding tangents, prove that*

$$\frac{v_1}{v_2}=\frac{PT}{QT}$$

Sol. Let M be the middle point of the chord PQ. By a geometrical property of a parabola, the line TM is parallel to the axis of the parabola. But the axis of the parabola is a vertical line and so the line TM must also be vertical.



Let the tangents PT and QT make angles α and β respectively with the vertical line TM and let $\angle TMP=\theta$.

Since the horizontal velocity of projectile remains constant throughout the motion, therefore

$$v_1 \sin \alpha = v_2 \sin \beta$$

or $$\frac{v_1}{v_2}=\frac{\sin \beta}{\sin \alpha}. \qquad \ldots(1)$$

In $\triangle TPM$, we have

$$\frac{PT}{\sin \theta}=\frac{PM}{\sin \alpha}.$$

$\therefore \quad PT \sin \alpha = PM \sin \theta. \qquad \ldots(2)$

Again in $\triangle TQM$, we have

$$\frac{QT}{\sin (\pi-\theta)}=\frac{QM}{\sin \beta}.$$

$\therefore \quad QT \sin \beta = QM \sin \theta. \qquad \ldots(3)$

But $PM=QM$, M being the middle point of PQ. Therefore from (2) and (3), we have

$$PT \sin \alpha = QT \sin \beta$$

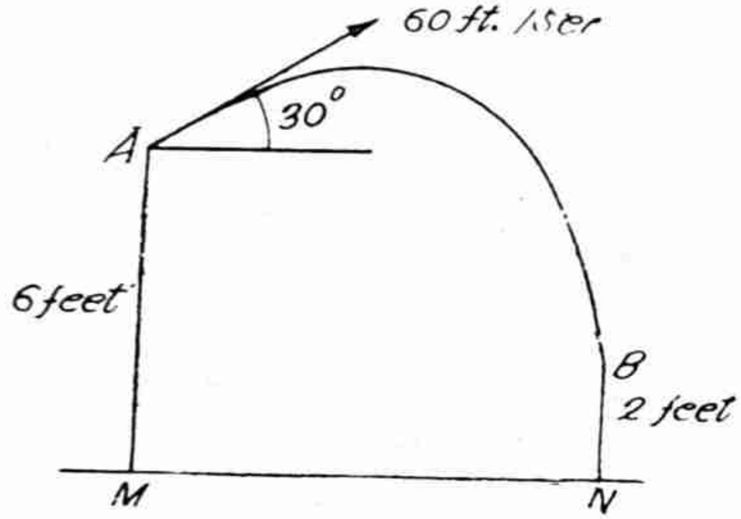
or $$\frac{\sin \beta}{\sin \alpha}=\frac{PT}{QT} \qquad \ldots(4)$$

Now from (1) and (4), we have

$$\frac{v_1}{v_2}=\frac{PT}{QT}.$$

Ex. 27. *A circket ball is thrown from a height of 6 feet at an angle of 30° to horizontal, with a speed of 60 feet/sec. It is caught by another fieldsman at a height of 2 feet from the ground. How far apart were the two men ?*

Sol. Suppose a cricket ball is thrown from the point A with velocity 60 feet/sec. at an angle of 30°. The height AM of A above



the ground is 6 feet. The ball is caught at B whose height BN above the ground is 2 feet. The horizontal distance between A and B is MN.

Let T be the time taken by the ball to travel from A to B. Considering the vertical motion from A to B and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$-4=(60 \sin 30°)\, T-\tfrac{1}{2}\times 32\times T^2$$

or $$-4=30T-16T^2 \quad \text{or} \quad 8T^2-15T-2=0$$

or $$(8T+1)\,(T-2)=0.$$

$\therefore \quad T=-\frac{1}{8},\ 2.$

Neglecting the negative value of T, we have $T=2$ seconds.

Now considering the horizontal motion from A to B, we get $MN=(60 \cos 30°)\, T$ feet$=60.(\sqrt{3}/2).2$ feet$=60\sqrt{3}$ feet.

Hence the two fieldsmen were at a distance $60\sqrt{3}$ feet apart.

Ex. 28. *If the maximum horizontal range of a particle is R, show that the greatest height attained is $\frac{1}{4}R$.*

A boy can throw a ball 60 metres. How long is the ball in the air and what height does it attain?

Sol. If the ball is projected with velocity u at an angle α with the horizontal, then

the horizontal range$=\dfrac{2u^2 \sin \alpha \cos \alpha}{g}$

and the greatest height$=\dfrac{u^2 \sin^2 \alpha}{2g}$.

The horizontal range is maximum when $\alpha=\frac{1}{4}\pi$. Thus

$$R=u^2/g.$$

If H be the greatest height attained in this case, then

$$H=\frac{u^2 \sin^2 \frac{1}{4}\pi}{2g}=\frac{u^2}{4g}=\frac{1}{4}\left(\frac{u^2}{g}\right)=\frac{1}{4}R.$$

This proves the first part of the problem. In the second part of the problem it is given that a boy can throw a ball 60 metres. So in this case $R=60$ metres.

$\therefore$ in this case greatest height attained

$$H=\frac{1}{4}R=\frac{1}{4}\times 60 \text{ metres}=15 \text{ metres}.$$

Hence the ball attains a height of 15 metres.

Also in the case of a projectile, the time of flight T is given by

$$T=\frac{2u \sin \alpha}{g}.$$

Since in this particular case $\alpha=\frac{1}{4}\pi$, therefore

$$T=(u/g)\sqrt{2}$$

or $$T^2=\frac{2u^2}{g^2}=\frac{2}{g}\left(\frac{u^2}{g}\right)=\frac{2}{g}\times 60 \quad \left[\because \frac{u^2}{g}=R=60 \text{ metres}\right]$$

$$=\frac{2\times 60}{9 \cdot 8}.$$

$$\therefore\ T=\sqrt{\left(\frac{2\times 60}{9\cdot 8}\right)} \text{ seconds}=3\cdot 5 \text{ seconds approximately.}$$

Hence the ball remains in the air for about 3·5 seconds.

Ex. 29. *Three particles are projected from the same point in the same vertical plane with velocities u, v, w at elevations α, β, γ respectively. Prove that the foci of their paths will lie on a straight line if*

$$\frac{\sin 2(\beta-\gamma)}{u^2}+\frac{\sin 2(\gamma-\alpha)}{v^2}+\frac{\sin 2(\alpha-\beta)}{w^2}=0.$$

Sol. Take the point of projection as the origin and the horizontal and the vertical lines in the plane of motion as the co-ordinate axes. Co-ordinates of the foci of the three trajectories are

$$\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2 \cos 2\alpha}{2g}\right), \left(\frac{v^2 \sin 2\beta}{2g}, \frac{-v^2 \cos 2\beta}{2g}\right),$$

and $$\left(\frac{w^2 \sin 2\gamma}{2g}, \frac{-w^2 \cos 2\gamma}{2g}\right).$$

These points will lie on a straight line if

$$\begin{vmatrix} \dfrac{u^2 \sin 2\alpha}{2g} & -\dfrac{u^2 \cos 2\alpha}{2g} & 1 \\ \dfrac{v^2 \sin 2\beta}{2g} & \dfrac{-v^2 \cos 2\beta}{2g} & 1 \\ \dfrac{w^2 \sin 2\gamma}{2g} & \dfrac{-w^2 \cos 2\gamma}{2g} & 1 \end{vmatrix}=0$$

i.e., if $$\begin{vmatrix} u^2 \sin 2\alpha & u^2 \cos 2\alpha & 1 \\ v^2 \sin 2\beta & v^2 \cos 2\beta & 1 \\ w^2 \sin 2\gamma & w^2 \cos 2\gamma & 1 \end{vmatrix} = 0.$$

Expanding the determinant in terms of the third column, we get $$v^2w^2 \sin 2(\beta-\gamma)+w^2u^2 \sin 2(\gamma-\alpha)+u^2v^2 \sin 2(\alpha-\beta)=0.$$

Dividing throughout by $u^2v^2w^2$, we get

$$\frac{\sin 2(\beta-\gamma)}{u^2}+\frac{\sin 2(\gamma-\alpha)}{v^2}+\frac{\sin 2(\alpha-\beta)}{w^2}=0,$$

as the required condition.

Ex. 30. *Particles are projected from the same point in a vertical plane with velocities which vary as*

$$1/\sqrt{(\sin \theta)},$$

θ being the angle of projection; find the locus of the vertices of the parabolas described. [**Meerut 1983 P**]

Sol. Take the common point of projection O as the origin and the horizontal and vertical lines OX and OY in the plane of motion as the co-ordinate axes. Let (x_1, y_1) be the co-ordinates of the vertex of a trajectory for which the velocity of projection is u and the angle of projection is θ. Then

$$x_1=\frac{u^2 \sin \theta \cos \theta}{g} \qquad \ldots(1)$$

and $$y_1=\frac{u^2 \sin^2 \theta}{2g}. \qquad \ldots(2)$$

We are to find the locus of the point (x_1, y_1) for varying values of u and θ subject to the condition

$$u=\frac{\lambda}{\sqrt{(\sin \theta)}}, \qquad \text{where } \lambda \text{ is some constant.}$$

Putting $u=\lambda/\sqrt{(\sin \theta)}$ in (1) and (2), we get

$$x_1=\frac{\lambda^2}{\sin \theta} \cdot \frac{\sin \theta \cos \theta}{g}=\frac{\lambda^2}{g} \cos \theta \qquad \ldots(3)$$

and $$y_1=\frac{\lambda^2}{\sin \theta} \cdot \frac{\sin^2 \theta}{2g}=\frac{\lambda^2}{2g} \sin \theta. \qquad \ldots(4)$$

Now we shall eliminate θ between (3) and (4). We have

$$\cos \theta=\frac{x_1}{\lambda^2/g} \quad \text{and} \quad \sin \theta=\frac{y_1}{\lambda^2/2g}.$$

Squaring and adding, we get

$$\frac{x_1^2}{\lambda^4/g^2}+\frac{y_1^2}{\lambda^4/4g^2}=\cos^2 \theta+\sin^2 \theta=1.$$

Generalising (x_1, y_1), we get the required locus as

$$\frac{x^2}{\lambda^4/g^2}+\frac{y^2}{\lambda^4/4g^2}=1, \qquad \text{which is an ellipse.}$$

Ex. 31. *Particles are projected from the same point in a vertical plane with velocity* $\sqrt{(2gk)}$; *prove that locus of the vertices of their paths i the ellipse* $\quad x^2+4y(y-k)=0.$

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of projection as the co-ordinate axes. Here the velocity of projection u for each trajectory is $\sqrt{(2gk)}$. Let (x_1, y_1) be the co-ordinates of the vertex of a trajectory for which the angle of projection is α. Then $x_1=\dfrac{u^2 \sin \alpha \cos \alpha}{g}=\dfrac{2gk \sin \alpha \cos \alpha}{g}=2k \sin \alpha \cos \alpha,$

...(1)

and $\qquad y_1=\dfrac{u^2 \sin^2 \alpha}{2g}=\dfrac{2gk . \sin^2 \alpha}{2g}=k \sin^2 \alpha. \qquad$ ...(2)

We are to find the locus of the point (x_1, y_1) for varying values of α. For this we have to eliminate α between (1) and (2). Squaring both sides of (1), we get

$$x_1^2=4k^2 \sin^2 \alpha \cos^2 \alpha=4k^2 \sin^2 \alpha\,(1-\sin^2 \alpha). \qquad ...(3)$$

From (2), $\sin^2 \alpha=y_1/k$. Putting this value of $\sin^2 \alpha$ in (3), we get $\qquad x_1^2=4k^2 \dfrac{y_1}{k}\left(1-\dfrac{y_1}{k}\right)=4ky_1-4y_1^2$

or $\qquad x_1^2+4y_1^2-4ky_1=0 \qquad$ or $\qquad x_1^2+4y_1(y_1-k)=0.$

$\therefore$ the locus of the point (x_1, y_1) is

$$x^2+4y(y-k)=0, \qquad \text{which is an ellipse.}$$

Ex. 32. *Particles are projected simultaneously in the same vertical plane from the same point. Show that the locus of the foci of all the trajectories is a parabola when for each trajectory there is the same*

(*i*) *horizontal velocity* $\qquad$ (*ii*) *initial vertical velocity*

(*iii*) *time of flight.*

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of projection as the co-ordinate axes.

Let (x_1, y_1) be the co-ordinates of the focus of the trajectory for which the velocity of projection is u and the angle of projection is α. Then $\qquad x_1=\dfrac{u^2 \sin 2\alpha}{2g}, \qquad$ ...(1)

and $$y_1 = -\frac{u^2 \cos 2\alpha}{2g} \quad \ldots(2)$$

We are to find the locus of the point (x_1, y_1) for varying values of u and α subject to the three different given conditions.

(i) When the horizontal velocity for each trajectory is constant *i.e.*, when $u \cos \alpha = c$ (constant). ...(3)

We have to eliminate u and α between (1), (2) and (3).

From (1), $x_1 = \dfrac{u^2 \sin \alpha \cos \alpha}{g}$.

Putting $u \cos \alpha = c$ in this relation, we get

$$x_1 = \frac{cu \sin \alpha}{g} \quad \text{or} \quad u \sin \alpha = \frac{x_1 g}{c}.$$

Now from (2), we have

$$y_1 = -\frac{u^2}{2g}(\cos^2 \alpha - \sin^2 \alpha) = -\frac{1}{2g}(u^2 \cos^2 \alpha - u^2 \sin^2 \alpha).$$

Putting $u \cos \alpha = c$ and $u \sin \alpha = x_1 g/c$ in this relation, we get

$$y_1 = -\frac{1}{2g}\left(c^2 - \frac{x_1^2 g^2}{c^2}\right) = -\frac{c^2}{2g} + \frac{x_1^2 g}{2c^2}$$

or $$2gc^2 y_1 + c^4 = x_1^2 g^2.$$

$\therefore$ the locus of the point (x_1, y_1) is $x^2 g^2 = 2gc^2 y + c^4$

or $$x^2 g^2 = 2gc^2\left(y + \frac{c^2}{2g}\right) \quad \text{or} \quad x^2 = \frac{2c^2}{g}\left(y + \frac{c^2}{2g}\right),$$

which is obviously a parabola.

(ii) When the initial vertical velocity for each trajectory is constant *i.e.*, when $u \sin \alpha = c$ (constant). ...(4)

We have to eliminate u and α between (1), (2) and (4).

From (1), $x_1 = \dfrac{u^2 \sin \alpha \cos \alpha}{g}$. Putting $u \sin \alpha = c$ in this relation, we get

$$x_1 = \frac{cu \cos \alpha}{g} \quad \text{or} \quad u \cos \alpha = \frac{x_1 g}{c}.$$

From (2), we have $$y_1 = -\frac{1}{2g}(u^2 \cos^2 \alpha - u^2 \sin^2 \alpha).$$

Putting $u \sin \alpha = c$ and $u \cos \alpha = x_1 g/c$ in this relation, we get

$$y_1 = -\frac{1}{2g}\left(\frac{x_1^2 g^2}{c^2} - c^2\right) = -\frac{x_1^2 g}{2c^2} + \frac{c^2}{2g}.$$

$\therefore$ the locus of the point (x_1, y_1) is

$$y = -\frac{x^2 g}{2c^2} + \frac{c^2}{2g} \quad \text{or} \quad \frac{x^2 g}{2c^2} = -y + \frac{c^2}{2g}$$

or $$x^2=-\frac{2c^2}{g}\left(y-\frac{c^2}{2g}\right),$$ which is obviously a parabola.

(iii) When the time of flight T for each trajectory is constant *i.e.*, when

$$\frac{2u \sin \alpha}{g}=\text{constant} \qquad \left[\because \quad T=\frac{2u \sin \alpha}{g}\right]$$

or $$u \sin \alpha=\text{constant}=c, \text{ say.} \qquad \ldots(5)$$

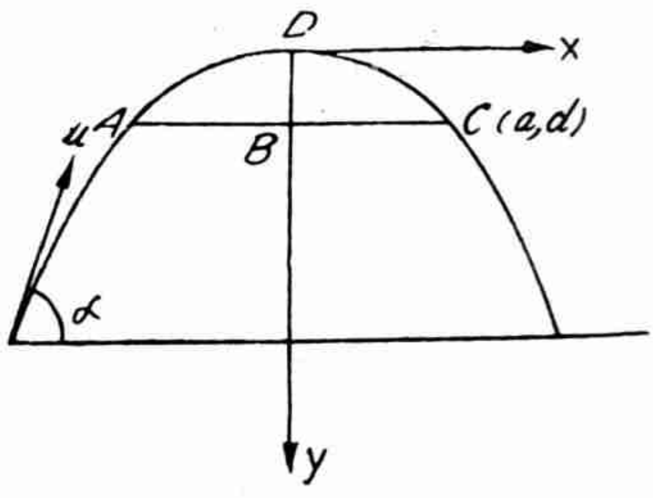
Now this part becomes exactly the same as part (ii).

Ex. 33. *A particle is to be projected so as just to pass through three equal rings, of diameter d, placed in parallel vertical planes at distances a apart, with their highest points in a horizontal straight line at a height h above the point of projection. Projection that the elevation must be* $\tan^{-1}\{2\sqrt{(hd)}/a\}$.

Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let A, B, C be the lowest points of the three rings and D the highest point of the middle ring.

According to the question the height of D above the point of projection O is h. Also $DB=d$ and $AB=BC=a$.

Now the particle just passes through the three rings. From the location of the rings it is obvious that the particle grazes over the lowest points A and C of the two side rings and just passes under the highest point D of the middle ring. Thus the particle is moving horizontally at D and the point D is the vertex of the parabolic path of the particle.



$\therefore$ $h=$ the height of the vertex D above the point of projection O

$=$ the greatest height attained by the particle

$$=\frac{u^2 \sin^2 \alpha}{2g}.$$

$$\therefore \quad u^2 \sin^2 \alpha=2gh. \qquad \ldots(1)$$

The latus rectum of the parabolic trajectory $=\dfrac{2}{g}u^2 \cos^2 \alpha.$

$\therefore$ referred to the vertex D as origin and the horizontal and vertical lines DX and DY as the co-ordinate axes, the equation of the parabolic trajectory is

$$x^2 = (\text{latus rectum})\ y$$

i.e., $$x^2 = \left(\frac{2}{g}\ u^2 \cos^2 \alpha\right) y. \qquad \ldots(2)$$

The point C whose co-ordinates are (a, d) lies on the curve (2).

$$\therefore \quad a^2 = \left(\frac{2}{g}\ u^2 \cos^2 \alpha\right) d$$

or $$u^2 \cos^2 \alpha = \frac{a^2 g}{2d}. \qquad \ldots(3)$$

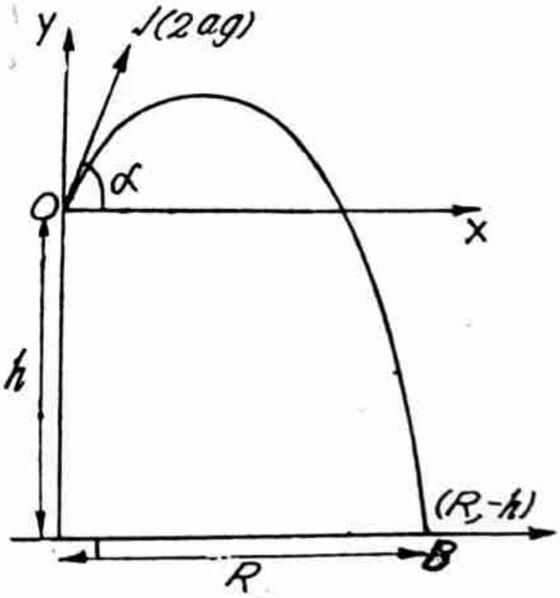
Dividing (1) by (3), we get $\tan^2 \alpha = 2gh . \dfrac{2d}{a^2 g} = \dfrac{4hd}{a^2}$.

$$\therefore \quad \tan \alpha = \frac{2\sqrt{(hd)}}{a} \quad \text{or} \quad \alpha = \tan^{-1}\left\{\frac{2\sqrt{(hd)}}{a}\right\}.$$

Ex. 34. *A particle is projected under gravity with velocity $\sqrt{(2ag)}$ from a point at a height h above a level plane. Show that the angle of projection α for the maximum range on the plane is given by $\tan^2 \alpha = a/(a+h)$, and that the maximum range is*

$$2\sqrt{\{a\,(a+h)\}}. \qquad \textbf{[Allahabad 1976]}$$

Sol. Take the point of projection O as the origin and the horizontal and the vertical lines OX and OY as the co-ordinate axes. The velocity of projection u is given to be $\sqrt{(2ag)}$.



When the particle is projected at an angle α suppose it hits the ground at the point B whose horizontal distance from the point of projection O is R. Then R is the range on the horizontal plane for the angle of projection α. The point $B\,(R, -h)$ lies on the trajectory whose equation is

$$y = x \tan \alpha - \tfrac{1}{2}\, \frac{gx^2}{2ag \cos^2 \alpha}. \qquad [\because\ u^2 = 2ag]$$

$$\therefore \quad -h = R \tan \alpha - \frac{1}{4a} R^2 \sec^2 \alpha. \qquad \ldots(1)$$

Now R is a function of α given by the equation (1). For R to be maximum we must have $dR/d\alpha=0$.

Differentiating both sides of (1) w.r.t. 'α', we get

$$0=\frac{dR}{d\alpha}\tan\alpha+R\sec^2\alpha-\frac{1}{4a}2R\frac{dR}{d\alpha}\sec^2\alpha-\frac{R^2}{4a}.2\sec^2\alpha\tan\alpha. \quad ...(2)$$

Putting $dR/d\alpha=0$ in (2), we see that for a maximum value of R we must have

$$R\sec^2\alpha-\frac{R^2}{2a}\sec^2\alpha\tan\alpha=0$$

or $$\tan\alpha=\frac{2a}{R} \text{ as } \sec\alpha\neq 0.$$

Putting this value of $\tan\alpha$ in (1), we have

$$-h=R.\frac{2a}{R}-\frac{R^2}{4a}\left(1+\frac{4a^2}{R^2}\right)$$

or $-h=2a-\dfrac{R^2}{4a}-a$ or $\dfrac{R^2}{4a}=a+h$ or $R^2=4a(a+h)$

or $R=2\sqrt{\{a(a+h)\}}$, which gives the maximum value of the range R.

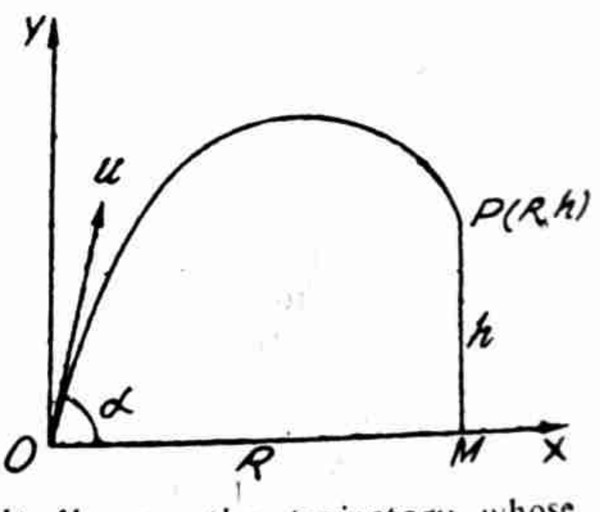
Also for this value of R, $\tan^2\alpha=\dfrac{4a^2}{R^2}=\dfrac{4a^2}{4a(a+h)}=\dfrac{a}{a+h}$.

Ex. 35. *A gun fires a shell with muzzle velocity u. Show that the farthest horizontal distance at which an aeroplane at a height h can be hit is* $(u/g)\sqrt{(u^2-2gh)}$, *and the gun's elevation then is*

$$\tan^{-1}\frac{u}{\sqrt{(u^2-2gh)}}.$$

[Lucknow 1975, 79; Kanpur 86]

Sol. Take the point of projection O as the origin and the horizontal and the vertical lines OX and OY as the co-ordinate axes. When the shell is projected at an angle α suppose it hits an aeroplane, which is at a height h above O, at the point P whose horizontal distance from O is R. The point P (R, h) lies on the trajectory whose equation is



$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}.$$

$$\therefore \quad h = R\tan\alpha - \tfrac{1}{2}g\frac{R^2}{u^2}\sec^2\alpha$$

or $$2u^2h = 2u^2R\tan\alpha - gR^2(1+\tan^2\alpha)$$

or $$gR^2\tan^2\alpha - 2u^2R\tan\alpha + gR^2 + 2u^2h = 0. \qquad \ldots(1)$$

The equation (1) is a quadratic in $\tan\alpha$. Its roots will be real

if $\quad 4u^2R^2 - 4gR^2(gR^2+2u^2h) \geqslant 0$

i.e., if $\quad u^4 - g(gR^2+2u^2h) \geqslant 0$ *i.e.*, if $g^2R^2 \leqslant u^4 - 2u^2gh$

i.e., if $\quad R^2 \leqslant \dfrac{u^2}{g^2}(u^2-2gh)$ *i.e.*, if $R \leqslant \dfrac{u}{g}\sqrt{(u^2-2gh)}$.

Hence the maximum value of R is $(u/g)\sqrt{(u^2-2gh)}$. For this value of R the equation (1) gives

$$g.\frac{u^2}{g^2}(u^2-2gh)\tan^2\alpha - 2u^2.\frac{u}{g}\sqrt{(u^2-2gh)}\tan\alpha + g.\frac{u^2}{g^2}(u^2-2gh) + 2u^2h = 0$$

or $$\frac{u^2}{g}(u^2-2gh)\tan^2\alpha - 2u^2.\frac{u}{g}\sqrt{(u^2-2gh)}\tan\alpha + \frac{u^4}{g} = 0$$

or $$\left[\frac{u\sqrt{(u^2-2gh)}}{\sqrt{g}}\tan\alpha - \frac{u^2}{\sqrt{g}}\right]^2 = 0.$$

$$\therefore \tan\alpha = \frac{u^2}{\sqrt{g}} \cdot \frac{\sqrt{g}}{u\sqrt{(u^2-2gh)}} = \frac{u}{\sqrt{(u^2-2gh)}}$$

or $$\alpha = \tan^{-1}\frac{u}{\sqrt{(u^2-2gh)}}.$$

Note. The above question can also be solved by the method given in Ex. 34.

Ex. 36. *A shell is fired vertically upwards. It bursts at a height a above the point of projection. Show that the fragments on reaching the ground, lie within a circle of radius* $(v/g)\sqrt{(v^2+2ag)}$, *assuming that the fragments start with the same velocity* v.

Sol. Suppose the shell bursts at the point O whose height above the ground is a. All the fragments start from O with velocity v but at different elevations. We have to find the maximum range of a fragment on the ground.

Now proceed as in Ex. 34.

Ex. 37. *A gun is fired from a moving platform and the ranges of the shot are observed to be R and S when the platform is moving forward and backward respectively with velocity v. Prove that the elevation of the gun is*

$$\tan^{-1}\left\{\frac{g}{4v^2}\,\frac{(R-S)^2}{R+S}\right\}.$$

Sol. Let α be the elevation of the gun. Then the angle of projection of the shot relative to the gun is also α. Let u be the velocity of projection of the shot relative to the gun.

Since at time of projection of the shot the gun moves horizontally, therefore the initial actual horizontal velocity of the shot is affected by the motion of the gun while the initial actual vertical velocity of the shot remains unaffected. Thus the initial actual vertical velocity of the shot is $u \sin \alpha$.

Now first consider the case when the gun moves forward. In this case the actual horizontal velocity of the shot becomes $u \cos \alpha + v$.

$$\therefore \text{ the range } R = \frac{2}{g} (u \cos \alpha + v) \, u \sin \alpha.$$

Next consider the case when the gun moves backward. In this case the actual horizontal velocity of the shot becomes $u \cos \alpha - v$.

$$\therefore \text{ the range } S = \frac{2}{g} (u \cos \alpha - v) \, u \sin \alpha. \quad \ldots(2)$$

From (1) and (2), we have

$R + S = (4/g) \, u^2 \cos \alpha \sin \alpha$ and $R - S = (4/g) \, uv \sin \alpha$.

$$\therefore \quad \frac{(R-S)^2}{R+S} = \frac{4v^2}{g} \tan \alpha$$

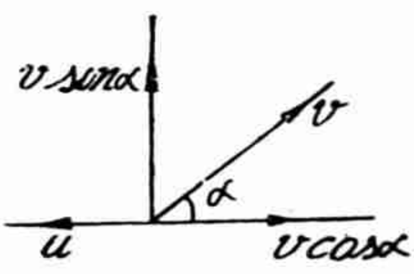
or

$$\tan \alpha = \frac{g}{4v^2} \frac{(R-S)^2}{(R+S)} \quad \text{or} \quad \alpha = \tan^{-1} \left\{ \frac{g \, (R-S)^2}{4v^2 \, (R+S)} \right\}.$$

Ex. 38. *A battleship is steaming ahead with velocity* u. *A gun is mounted on the ship so as to point straight backwards and is set at an angle of elevation* α. *If* v *be the velocity of projection relative to the gun, show that the range is* $(2v/g) \sin \alpha \, (v \cos \alpha - u)$, *and the angle of elevation for maximum range is*

$$\cos^{-1} \left[\frac{u + \sqrt{(u^2 + 8v^2)}}{4v} \right].$$

Sol. Since the ship is moving horizontally with a velocity u in a direction opposite that of the projection, therefore the initial actual horizontal velocity of the shot $= v \cos \alpha - u$.



Also the initial actual vertical velocity of the shot $= v \sin \alpha$.

$$\therefore \text{ the range } R = \frac{2}{g} \text{ (horizontal vel.) (initial vertical vel.)}$$

$$= \frac{2}{g} (v \cos \alpha - u) \, v \sin \alpha = \frac{2v}{g} \sin \alpha \, (v \cos \alpha - u).$$

Now R is a function of α. So R will be maximum when $dR/d\alpha=0$

i.e., when $\frac{2v}{g}(v\cos^2\alpha-v\sin^2\alpha-u\cos\alpha)=0$

i.e., when $2v\cos^2\alpha-u\cos\alpha-v=0$

i.e., when $\cos\alpha=\frac{u\pm\sqrt{(u^2+8v^2)}}{4v}$.

The negative sign before the radical is not admissible because it makes the value of $\cos\alpha$ negative or α obtuse.

$\therefore$ the angle of elevation for maximum range is

$$\cos^{-1}\left[\frac{u+\sqrt{(u^2+8v^2)}}{4v}\right].$$

Ex. 39. *A shot fired with velocity V at an elevation θ strikes a point P on the horizontal plane through the point of projection. If the point P is receding from the gun with velocity v, show that the elevation must be changed to ϕ, where*

$$\sin 2\phi=\sin 2\theta+\frac{2v}{V}\sin\phi.$$ **[Meerut 1973, 76]**

Sol. Let O be the point of projection. When the point P is stationary, then the original range $OP=\frac{V^2\sin 2\theta}{g}$.

When the point P recedes from O *i.e.*, moves away from O in the direction of motion of the shot, then to hit at P the angle of projection is changed to ϕ.

$\therefore$ the new range$=\frac{V^2\sin 2\phi}{g}$.

Also in this case the time of flight $T=\frac{2V\sin\phi}{g}$.

During this time P moves away from its original position a distance$=v.\frac{2V\sin\phi}{g}$.

In order to hit P, we should have

the new range$=$the original range$+$the distance moved by P in time T

i.e., $\frac{V^2\sin 2\phi}{g}=\frac{V^2\sin 2\theta}{g}+v\frac{2V\sin\phi}{g}$

i.e., $\sin 2\phi=\sin 2\theta+(2v/V)\sin\phi$.

Alternative solution. Let O be the point of projection. When the point P is stationary, then the original range $OP=\frac{V^2\sin 2\theta}{g}$.

When the point P recedes from O with velocity v, then to hit at P the angle of projection is changed to ϕ. In this case the initial horizontal velocity of the shot relative to P is $V\cos\phi-v$ and the initial vertical velocity of the shot relative to P is $V\sin\phi$. Therefore in this case the range of the shot relative to P

$$=(2/g)\,(V\cos\phi-v)\,V\sin\phi.$$

To hit P, we must have

$$\frac{2}{g}(V\cos\phi-v)\,V\sin\phi=\frac{V^2\sin 2\theta}{g}$$

i.e., $$\frac{V^2\sin 2\phi}{g}-\frac{2}{g}vV\sin\phi=\frac{V^2}{g}\sin 2\theta$$

i.e., $$\frac{V^2\sin 2\phi}{g}=\frac{V^2}{g}\sin 2\theta+\frac{2v}{g}V\sin\phi$$

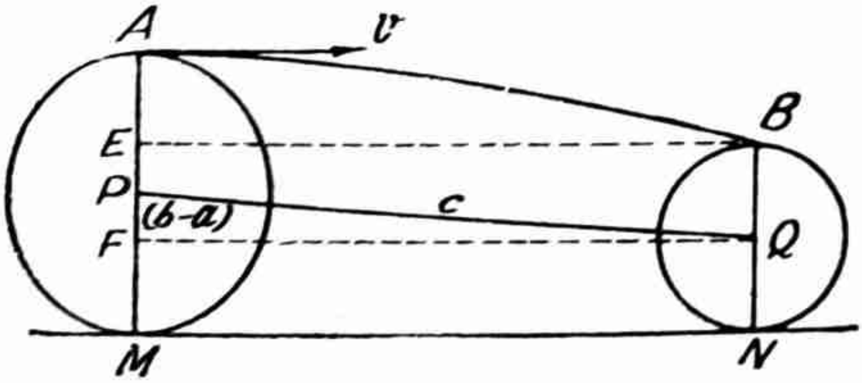
i.e., $$\sin 2\phi=\sin 2\theta+(2v/V)\sin\phi.$$

Ex. 40. *The distance between the axle-trees of the front and hind wheels of a carriage of radii a and b respectively is c. A particle of mud driven from the highest point of the hind wheel alights on the highest point of the front wheel. Show that the velocity of the carriage is*

$$\left[\frac{g\,(c+b-a)\,(c+a-b)}{4\,(b-a)}\right]^{1/2}.$$

[Meerut 1981S, 83S, 84, 85S, 86, 87S]

Sol. Let v be the velocity of the carriage. Then the velocity of the highest point A of the hind wheel is $2v$ horizontally. Therefore the actual velocity of the mud particle while driven from the highest point of the hind wheel is $2v$ horizontally. But the carriage is also moving horizontally with velocity v. Therefore the horizontal velocity of the mud particle relative to the carriage is $2v-v$ *i.e.*, v. The initial vertical velocity of the mud particle relative to the carriage is zero.



In coming to the highest point of the front wheel the vertical distance travelled by the particle $= AM - BN = 2b - 2a = 2(b-a)$.

In the figure P and Q are the centres of the hind and front wheels and $PQ = c$ is the distance between the axle trees.

Let T be the time taken by the mud particle to travel from the highest point of the hind wheel to the highest point of the front wheel. Then considering the vertical motion of the particle, we have $2(b-a) = 0.T + \frac{1}{2}gT^2$ [using the formula $s = ut + \frac{1}{2}ft^2$]

or $\quad T = 2\sqrt{\left(\dfrac{b-a}{g}\right)}$.

Also the horizontal distance moved by the particle relative to the carriage in time $T = EB = FQ = \sqrt{(PQ^2 - PF^2)} = \sqrt{\{c^2-(b-a)^2\}}$.

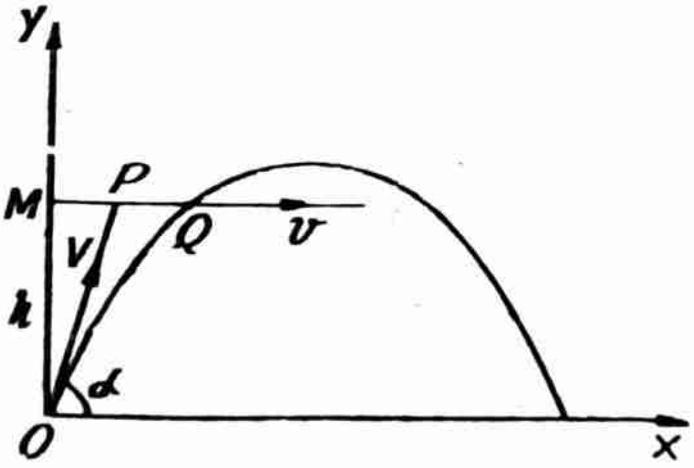
Considering the horizontal motion of the particle relative to the carriage, we have $\quad \sqrt{\{c^2-(b-a)^2\}} = vT$.

$$\therefore \quad v = \frac{\sqrt{\{c^2-(b-a)^2\}}}{T} = \frac{\sqrt{\{c^2-(b-a)^2\}}}{2} \cdot \sqrt{\left(\frac{g}{b-a}\right)}$$

$$= \sqrt{\left\{\frac{g\,(c+b-a)\,(c+a-b)}{4\,(b-a)}\right\}}.$$

Ex. 41. *An aeroplane is flying with constant velocity v at a constant height h. Show that, if a gun is fired point blank at the aeroplane after it has passed directly over the gun and when the angle of elevation as seen from the gun is* α, *the shell will hit the aeroplane, provided* $\quad 2\,(V\cos\alpha - v)\,v\tan^2\alpha = gh.$

Sol. Let O be the point of projection and OX and OY the horizontal and vertical lines through O in the plane of motion. Let P be the position of the aeroplane when the shot was fired from O. The gun is fired 'point blank' means that the initial velocity of the shot was along OP.



The path of the aeroplane is along the horizontal line PQ at a height h from O, and the path of the shot is the parabolic arc

OQ. The point Q is common to the two paths. The shot can hit the aeroplane if both reach the point Q at the same time. Suppose this happens after a time t from the moment of projection of the shot. The distance PQ moved by the aeroplane in time t is vt.

Considering the horizontal motion of the shot from O to Q, we have $$MQ=(V\cos\alpha)\,t.$$

But $MQ=MP+PQ=h\cot\alpha+vt$.

$$\therefore\quad h\cot\alpha+vt=(V\cos\alpha)\,t. \qquad ...(1)$$

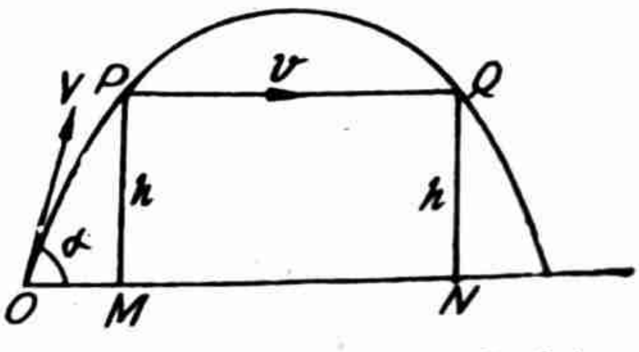
Considering the vertical motion of the shot from O to Q, we have $$h=(V\sin\alpha)\,t-\tfrac{1}{2}gt^2$$
or $$h=t\,[V\sin\alpha-\tfrac{1}{2}gt]. \qquad ...(2)$$

From (1), $t=\dfrac{h\cot\alpha}{V\cos\alpha-v}$.

Putting this value of t in (2), we get
$$h=\frac{h\cot\alpha}{V\cos\alpha-v}\left[V\sin\alpha-\tfrac{1}{2}g.\frac{h\cot\alpha}{V\cos\alpha-v}\right]$$
or $$2\,(V\cos\alpha-v)^2=\cot\alpha\,[2V\sin\alpha\,(V\cos\alpha-v)-gh\cot\alpha]$$
or $$2\,(V\cos\alpha-v)^2=2V\cos\alpha\,(V\cos\alpha-v)-gh\cot^2\alpha$$
or $$2\,(V\cos\alpha-v)\,[V\cos\alpha-(V\cos\alpha-v)]=gh\cot^2\alpha$$
or $$2\,(V\cos\alpha-v)\,v=gh\cot^2\alpha$$
or $$2\,(V\cos\alpha-v)\,v\tan^2\alpha=gh.$$

Ex. 42. *A shot is fired with velocity V at an elevation α so as to hit a bird sitting at the top of a pole of height h. However the bird immediately starts flying horizontally away from the gun with velocity v. Show that it will not escape being hit if*
$$(2V\cos\alpha-v)\,(V^2\sin^2\alpha-2gh)^{1/2}=vV\sin\alpha.$$

Sol. Let O be the point of projection of the shot and P the position of the bird at the top of a pole PM of height h. As the bird could be hit if it remained sitting at the top P, therefore the trajectory of the shot passes through P.



If the bird starts flying horizontally away from P and is hit at another position Q of the trajectory, it is necessary that the bird and the shot should reach Q at the same time.

Suppose the shot is at a height h after a time t of its projection from O. Then $h=(V\sin\alpha)\,t-\tfrac{1}{2}gt^2$

or $$gt^2-2V\sin\alpha.t+2h=0.$$

$$\therefore \quad t=\frac{2V\sin\alpha\pm\sqrt{(4V^2\sin^2\alpha-8gh)}}{2g}$$

$$=\frac{V\sin\alpha\pm\sqrt{(V^2\sin^2\alpha-2gh)}}{g}.$$

Let t_1 and t_2 be the two values of t. Then

$$t_1=\frac{V\sin\alpha-\sqrt{(V^2\sin^2\alpha-2gh)}}{g}$$

and $$t_2=\frac{V\sin\alpha+\sqrt{(V^2\sin^2\alpha-2gh)}}{g}.$$

Obviously t_1 is the time for the shot from O to P and t_2 is the time from O to Q. The horizontal distance $PQ=V\cos\alpha.(t_2-t_1)$.

Also the distance PQ is travelled by the bird in time t_2 with uniform velocity v.

$\therefore \quad PQ=vt_2.$

Hence $$V\cos\alpha.(t_2-t_1)=vt_2.$$

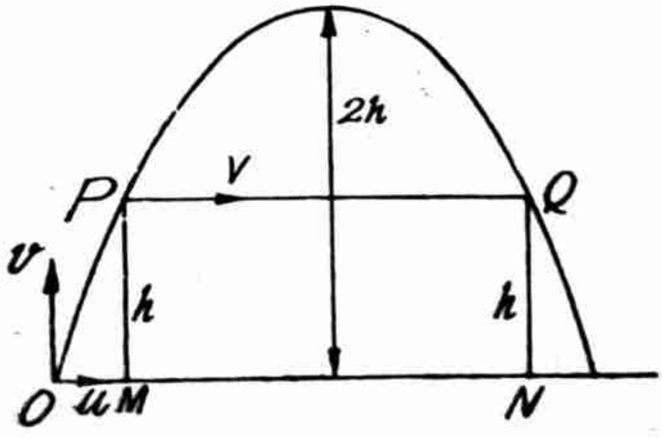
Substituting the values of t_2 and t_1 in this relation, we get

$$V\cos\alpha.\frac{2\sqrt{(V^2\sin^2\alpha-2gh)}}{g}=v\left[\frac{V\sin\alpha+\sqrt{(V^2\sin^2\alpha-2gh)}}{g}\right]$$

or $$(2V\cos\alpha-v)\sqrt{(V^2\sin^2\alpha-2gh)}=vV\sin\alpha.$$

Ex. 43. *A stone is thrown in such a manner that it would just hit a bird at the top of a tree and afterwards reach a height double that of the tree. If at the moment of throwing the stone the bird flies away horizontally, show that, not withstanding this, the stone will hit the bird if its horizontal velocity be to that of the bird as* $(\sqrt{2}+1):2$. **[Agra 1976, 79]**

Sol. Let O be the point of projection of the stone and P the top of the tree whose height above O is, say, h. Then according to the question the greatest height ever reached by the stone should be $2h$.



Let u and v be the initial horizontal and vertical components of the velocity of the stone and V the velocity of the bird which moves in the horizontal line PQ.

Since $2h$ is the greatest height of the trajectory, therefore

$$2h = v^2/2g$$

or
$$v^2 = 4gh. \qquad \ldots(1)$$

As the bird could be hit if it remained sitting at the top P, therefore the trajectory of the stone passes through P. If the bird starts flying horizontally away from P and is hit at another position Q of the trajectory, it is necessary that the bird and the stone should reach Q at the same time.

Suppose the stone is at a height h after a time t of its projection from O. Then

$$h = vt - \tfrac{1}{2}gt^2 \text{ or } gt^2 - 2vt + 2h = 0.$$

$$\therefore \quad t = \frac{2v \pm \sqrt{(4v^2 - 8gh)}}{2g} = \frac{v \pm \sqrt{(v^2 - 2gh)}}{g}$$

$$= \frac{2\sqrt{(gh)} \pm \sqrt{(2gh)}}{g}, \text{ substituting for } v \text{ from (1)}$$

$$= (2 \pm \sqrt{2})\sqrt{(h/g)}.$$

Let t_1 and t_2 be the two values of t. Then

$$t_1 = (2 - \sqrt{2})\sqrt{(h/g)} \quad \text{and} \quad t_2 = (2 + \sqrt{2})\sqrt{(h/g)}.$$

Obviously t_1 is the time for the stone from O to P and t_2 is the time from O to Q.

The horizontal distance PQ is travelled by the stone in time $t_2 - t_1$ with constant horizontal velocity u. Therefore $PQ = u(t_2 - t_1)$.

Also the distance PQ is travelled by the bird in time t_2 with uniform velocity V.

$$\therefore \quad PQ = Vt_2.$$

Hence $(t_2 - t_1)\, u = Vt_2$

or
$$\frac{u}{V} = \frac{t_2}{t_2 - t_1} = \frac{(2 + \sqrt{2})\sqrt{(h/g)}}{(2\sqrt{2})\sqrt{(h/g)}}$$

$$= \frac{2 + \sqrt{2}}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{2} + 1)}{2\sqrt{2}} = \frac{\sqrt{2} + 1}{2}.$$

Ex. 44. *Prove that when a shot is projected from a gun at any angle of elevation the shot as seen from the point of projection will appear to descend past a vertical target with uniform velocity.*

Sol. Let the shot be projected from O with velocity u at an angle α and let AB be the fixed vertical target at a given distance c from O *i.e.*, $OB = c$.

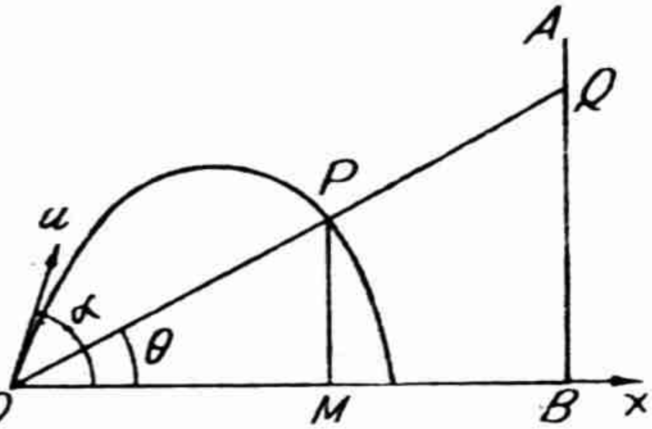
Let P be the position of the shot at any time t. Join OP and produce it to meet the target in a point Q. From the point of projection O the corresponding point on the vertical target as seen from O in the straight line OP is Q. In the question, we have to find the vertical velocity of Q.

Let $QB=y$ and $\angle QOB=\theta$. Let M be the foot of the perpendicular from P on the horizontal line OX. Considering the horizontal and vertical motion of the shot from O to P, we have

$$OM=(u\cos\alpha)\,t,$$

and

$$PM=(u\sin\alpha)\,t-\tfrac{1}{2}gt^2.$$



$$\therefore\quad \tan\theta=\frac{PM}{OM}=\frac{(u\sin\alpha)\,t-\frac{1}{2}gt^2}{(u\cos\alpha)\,t}=\tan\alpha-\frac{1}{2}\,\frac{gt}{u\cos\alpha}.$$

$$\text{Now } y=QB=OB\tan\theta=c\tan\theta=c\left[\tan\alpha-\frac{1}{2}\,\frac{gt}{u\cos\alpha}\right].$$

$$\therefore\quad \frac{dy}{dt}=\text{vertical velocity of } Q=-\frac{gc}{2u\cos\alpha}=\text{a negative constant.}$$

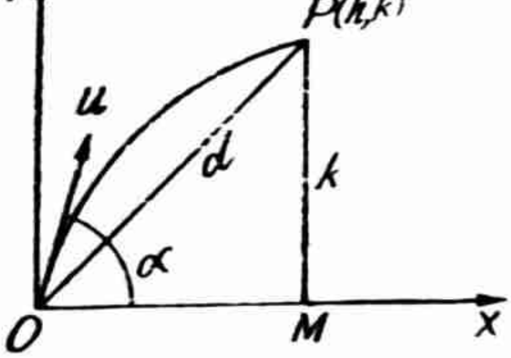
Since the value of dy/dt is a negative constant, therefore the shot as seen from the point of projection will appear to descend past a vertical target with uniform velocity.

§ 8. Projections to hit a given point.

(a) Two directions of projections to hit the given point.

[Gorakhpur 1980 ; Agra 78 ; Meerut 79]

Let a particle be projected from a given point O with a given velocity u so as to hit a given point P.



Referred to the horizontal and vertical lines OX and OY in the plane of motion as the co-ordinate axes, let the co-ordinates of the point P be (h, k). If the angle of projection is α, the equation of the trajectory is

$$y=x\tan\alpha-\frac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}. \qquad \ldots(1)$$

Since the point (h, k) lies on (1), therefore

$$k=h \tan \alpha-\tfrac{1}{2}g \frac{h^2}{u^2} \sec^2 \alpha$$

or $$k=h \tan \alpha-\frac{gh^2}{2u^2}(1+\tan^2 \alpha)$$

or $$\frac{2u^2}{gh^2}k=\frac{2u^2}{gh^2}.h \tan \alpha-(1+\tan^2 \alpha)$$

or $$\tan^2 \alpha-\frac{2u^2}{gh} \tan \alpha+\left(1+\frac{2u^2k}{gh^2}\right)=0. \qquad \ldots(2)$$

The equation (2) is a quadratic in $\tan \alpha$. Therefore it gives in general two values of $\tan \alpha$ or two values for the angle α. *Thus there are in general two directions in which a particle may be projected from a given point O with a given velocity u so as to pass through a given point P.*

(b) Least velocity of projection to hit the given point.

[Gorakhpur 1977; Lucknow 80; Meerut 79, 80, 82, 84P, 85, 85P, 87S]

In order to be able to hit the given point P from the given point O with the given velocity u, the two directions of projection given by equation (2) must be real.

The roots of the quadratic (2) in $\tan \alpha$ are real if its dircriminant is $\geqslant 0$

i.e., if $$\frac{4u^4}{g^2h^2}-4\left(1+\frac{2u^2k}{gh^2}\right)\geqslant 0$$

or $$u^4-g^2h^2\left(1+\frac{2u^2k}{gh^2}\right)\geqslant 0$$

or $$u^4-g^2h^2-2u^2gk\geqslant 0 \quad \text{or} \quad u^4-2u^2ghk\geqslant g^2h^2$$

or $$(u^2-gk)^2 \geqslant g^2h^2+g^2k^2 \quad \text{or} \quad (u^2-gk)^2 \geqslant g^2 (h^2+k^2)$$

or $$u^2-gk \geqslant g\sqrt{(h^2+k^2)} \quad \text{or} \quad u^2 \geqslant gk+g\sqrt{(h^2+k^2)}$$

or $$u^2 \geqslant g \{k+\sqrt{(h^2+k^2)}\}$$

or $$u \geqslant [g \{k+\sqrt{(h^2+k^2)}\}]^{1/2}.$$

Hence the least value of $u=[g \{k+\sqrt{(h^2+k^2)}]^{1/2}$

$$=\sqrt{\{g (k+d)\}}, \text{ where } d=OP=\sqrt{(h^2+k^2)}.$$

Thus **remember** *that the least velocity of projection to hit P from O is $\sqrt{\{g(k+OP)\}}$, where k is the vertical height of P above O.*

If the point P to be hit lies below the point of projection O, then replacing k by $-k$ in the above result, we see that the least velocity of projection to hit the point P is $\sqrt{\{g (OP-k)\}}$ where k is the vertical depth of P below O. **[Gorakhpur 1978]**

(c) Two times of flight to hit a given point.

Let a particle be projected from a given point O with a given velocity u, say at an angle α, so as to hit a given point P whose co-ordinates are (h, k). Since there can be two values of α to hit P, therefore α is a variable. If t is the time of flight from O to P, then considering the horizontal and vertical motions of the particle from O to P, we have $h=(u \cos \alpha)\, t$, ...(1)

and $k=(u \sin \alpha)\, t-\frac{1}{2}gt^2$. ...(2)

Eliminating α between (1) and (2), we have

$$h^2+(k+\tfrac{1}{2}gt^2)^2=u^2t^2$$

or $$4h^2+(2k+gt^2)^2=4u^2t^2$$

or $$4h^2+4k^2+4gkt^2+g^2t^4=4u^2t^2$$

or $$g^2t^4+4\,(gk-u^2)\,t^2+4\,(h^2+k^2)=0$$

or $$t^4+4\left(\frac{k}{g}-\frac{u^2}{g^2}\right)t^2+\frac{4}{g^2}\,(h^2+k^2)=0. \quad ...(3)$$

The equation (3) is a quadratic in t^2 and thus gives two values of t^2 and consequently two possible values of t to hit the given point. If corresponding to the two directions of projection to hit P the two possible times of flight are t_1 and t_2 then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. From the theory of equations, we have

$$t_1^2+t_2^2=-4\left(\frac{k}{g}-\frac{u^2}{g^2}\right)$$

and $$t_1^2t_2^2=\frac{4}{g^2}\,(h^2+k^2)=\frac{4}{g^2}\,OP^2, \text{ so that } t_1t_2=\frac{2}{g}.OP.$$

Illustrative Examples

Ex. 45. *If α, β are two possible directions to hit a given point (a, b), then show that $\tan(\alpha+\beta)=-a/b$.*

Sol. Let a particle be projected from a given point O with a given velocity u so as to hit a given point (a, b). If the angle of projection is θ, the equation of the trajectory is

$$y=x \tan\theta-\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\theta} \quad ...(1)$$

Since the point (a, b) lies on (1), therefore

$$b=a\tan\theta-\tfrac{1}{2}g\,\frac{a^2}{u^2}\sec^2\theta$$

or $$b=a\tan\theta-\tfrac{1}{2}g\,\frac{a^2}{u^2}\,(1+\tan^2\theta)$$

or $$\tan^2\theta-\frac{2u^2}{ga}\tan\theta+\left(1+\frac{2u^2b}{ga^2}\right)=0. \quad ...(2)$$

The equation (2) is a quadratic in $\tan\theta$ showing that there are in general two directions of projection to hit the given point (a, b).

If α, β are the two possible directions of projection, then $\tan\alpha$, $\tan\beta$ are the roots of the quadratic (2) in $\tan\theta$.

$$\therefore \quad \tan\alpha+\tan\beta=\frac{2u^2}{ga} \text{ and } \tan\alpha\tan\beta=1+\frac{2u^2b}{ga^2}.$$

$$\text{We have } \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2u^2/ga}{1-1-(2u^2b/ga^2)}$$

$$=-\frac{2u^2}{ga}\cdot\frac{ga^2}{2u^2b}=-\frac{a}{b}.$$

Ex. 46. *A particle is projected under gravity from A so as to pass through B; show that for a given velocity of projection there are two paths. Show that if B has horizontal and vertical co-ordinates x, y referred to A and the velocity of projection is $\sqrt{(2gh)}$, the angle between the two paths at B is a right angle if B lies on the ellipse $x^2+2y^2=2hy$.*

Sol. Let u be the velocity and α the angle of projection. Since the trajectory passes through the point B (x, y), therefore

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2\cos^2\alpha}$$

or
$$y=x\tan\alpha-\tfrac{1}{2}g\,(x^2/u^2)\sec^2\alpha \qquad \ldots(1)$$

or
$$y=x\tan\alpha-\tfrac{1}{2}g\,(x^2/u^2)\,(1+\tan^2\alpha). \qquad \ldots(2)$$

The equation (2) is a quadratic in $\tan\alpha$ showing that there are in general two directions of projection to hit B from A. Thus for a given velocity there are two paths from A to B.

Putting $u^2=2gh$ in (1), we have

$$y=x\tan\alpha-(x^2/4h)\sec^2\alpha. \qquad \ldots(3)$$

$$\text{From (3), } \frac{dy}{dx}=\tan\alpha-\frac{x}{2h}\sec^2\alpha=m \text{ (say)}. \qquad \ldots(4)$$

Then m is the gradient of the trajectory for the angle of projection α and the velocity of projection $\sqrt{(2gh)}$ at the point (x, y).

The equations (3) and (4) can be rearranged as

$$x^2\sec^2\alpha-4hx\tan\alpha+4hy=0, \qquad \ldots(5)$$

and
$$x\sec^2\alpha-2h\tan\alpha+2mh=0. \qquad \ldots(6)$$

Solving (5) and (6) for $\sec^2\alpha$ and $\tan\alpha$, we have

$$\frac{\sec^2\alpha}{-8mh^2x+8h^2y}=\frac{\tan\alpha}{4hxy-2mhx^2}=\frac{1}{-2hx^2+4hx^2}.$$

$$\therefore \quad \sec^2\alpha=\frac{8h^2(y-mx)}{2hx^2}=\frac{4h}{x^2}(y-mx)$$

and
$$\tan\alpha=\frac{2hx\,(2y-mx)}{2hx^2}=\frac{2y-mx}{x}.$$

Now the two paths depend upon the angle of projection α. So eliminating α from these, we get

$$\frac{4h}{x^2}(y-mx)=1+\frac{(2y-mx)^2}{x^2}$$

or $$4h(y-mx)=x^2+(2y-mx)^2$$

or $$m^2x^2-4mx(y-h)+x^2+4y^2-4hy=0. \qquad \ldots(7)$$

The equation (7) is a quadratic in m and so it gives us two values of m, say m_1 and m_2. Then m_1 and m_2 are the gradients of the two paths at B. Since m_1 and m_2 are the roots of the quadratic (7) in m, therefore

$$m_1m_2=\frac{x^2+4y^2-4hy}{x^2}.$$

The two paths at B are at right angles if $m_1m_2=-1$

i.e., if $$\frac{x^2+4y^2-4hy}{x^2}=-1$$

i.e., if $x^2+4y^2-4hy=-x^2$ *i.e.,* if $2x^2+4y^2=4hy$

i.e., if $x^2+2y^2=2hy$.

Hence the two paths at B are at right angles if B lies on the ellipse $x^2+2y^2=2hy$.

Ex. 47. *A stone is projected with velocity u from a height h to hit a point in the level at a horizontal distance R from the point of projection. Show that the angle of projection is given by*

$$R^2 \tan^2 \alpha-\frac{2u^2}{g} R \tan \alpha+R^2-\frac{2hu^2}{g}=0.$$

Hence deduce that the maximum range on the level for this velocity is $$\sqrt{\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)}$$

and that if R′ is this maximum range and α the angle of projection to give the maximum range, then

$$\tan \alpha=u^2/gR' \qquad \text{and} \qquad \tan 2\alpha=R'/h.$$

Sol. Referred to the point of projection O as the origin, the equation of the trajectory for the angle of projection α is

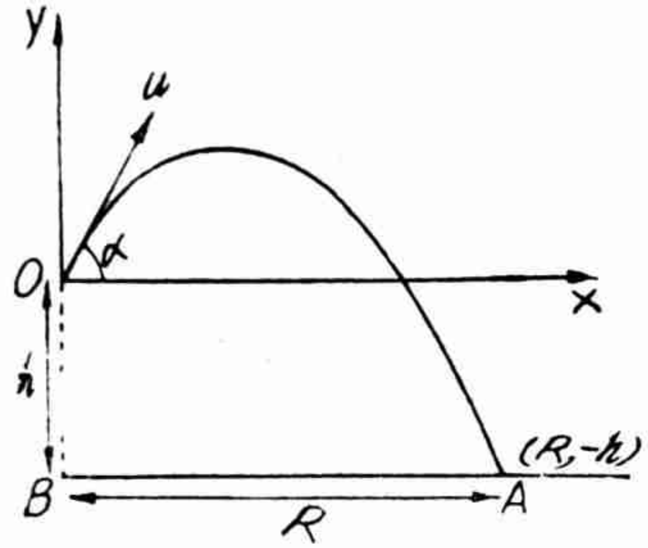
$$y=x \tan \alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2 \cos^2 \alpha}. \qquad \ldots(1)$$

Suppose the stone hits the ground at the point A whose coordinates are $(R, -h)$. Then the point $(R, -h)$ lies on the curve (1). Therefore

$$-h=R \tan \alpha-\tfrac{1}{2}g\,\frac{R^2}{u^2}(1+\tan^2 \alpha)$$

or $$R^2 \tan^2 \alpha - \frac{2u^2}{g} R \tan \alpha + R^2 - \frac{2hu^2}{g} = 0. \quad \ldots(2)$$

The equation (2) gives the values of $\tan \alpha$ and so the values of the angle of projection.



Now if u is given, then R is a function of α given by the equation (2). For R to be maximum we must have $dR/d\alpha = 0$.

. Differentiating both sides of (2) w.r.t. 'α', we get

$$2R \frac{dR}{d\alpha} \tan^2 \alpha + 2R^2 \tan \alpha \sec^2 \alpha - \frac{2u^2}{g} \frac{dR}{d\alpha} \tan \alpha - \frac{2u^2}{g} R \sec^2 \alpha$$

$$+ 2R \frac{dR}{d\alpha} = 0. \quad \ldots(3)$$

Putting $dR/d\alpha = 0$ in (3), we have

$$2R^2 \tan \alpha \sec^2 \alpha - \frac{2u^2}{g} R \sec^2 \alpha = 0$$

or $$2R \left(R \tan \alpha - \frac{u^2}{g} \right) \sec^2 \alpha = 0$$

or $$R \tan \alpha - (u^2/g) = 0 \quad [\because \sec \alpha \neq 0]$$

or $$\tan \alpha = u^2/gR. \quad \ldots(4)$$

The equation (4) gives the relation between the angle of projection and the maximum range. If R' is the maximum range, then replacing R by R' in (4), we have

$$\tan \alpha = u^2/gR'. \quad \ldots(5)$$

Putting $\tan \alpha = u^2/gR'$ and $R = R'$ in (2), the maximum range R' is given by

$$R'^2 \frac{u^4}{g^2 R'^2} - \frac{2u^2}{g} R' . \frac{u^2}{gR'} + R'^2 - \frac{2hu^2}{g} = 0$$

or $$\frac{u^4}{g^2} - \frac{2u^4}{g^2} + R'^2 - \frac{2hu^2}{g} = 0$$

or $$R'^2=\frac{u^4}{g^2}+\frac{2hu^2}{g} \qquad ...(6)$$

or $$R'=\sqrt{\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)}.$$

Now $$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{2u^2/gR'}{1-u^4/g^2R'^2} \qquad \text{[from (5)]}$$

$$=\frac{2u^2gR'}{g^2R'^2-u^4}$$

$$=\frac{2u^2gR'}{g^2\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)-u^4} \qquad \text{[from (6)]}$$

$$=\frac{2u^2gR'}{2hgu^2}=\frac{R'}{h}.$$

Ex. 48. *Determine the least velocity with which a ball can be thrown to reach the top of a cliff* 40 *metres high and* $40\sqrt{3}$ *metres away from the point of projection.* **[Meerut 1972, 76, 87, 88]**

Sol. We know that the least velocity of projection u to hit a point P from a point O is given by

$$u=[g\,\{k+\sqrt{(h^2+k^2)}\}]^{1/2}, \qquad ...(1)$$

where h and k are respectively the horizontal and vertical distances of P from O. [Refer § 8, part (b), page 48]

Here $h=40\sqrt{3}$ meters and $k=40$ metres. Substituting these values of h and k in (1) and putting $g=9\cdot 8$ metres/sec², the required least velocity of projection

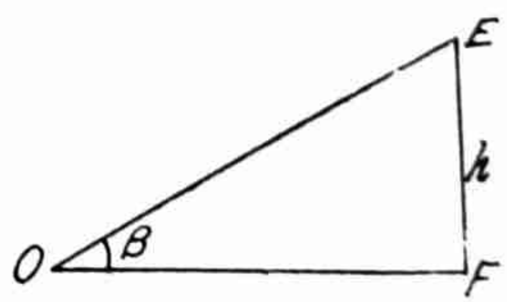
$$=[9\cdot 8\,\{40+\sqrt{(4800+1600)}\}]^{1/2} \text{ metres/sec.}$$

$$=[9\cdot 8\times 120]^{1/2} \text{ meters/sec.}=\sqrt{(1176)} \text{ metres/sec.}$$

$$=14\sqrt{6} \text{ metres/sec.}$$

Ex. 49. *The angular elevation of an enemy's position on a hill h metres high is* β. *Show that in order to shell it, the initial velocity of the projectile must not be less than* $\sqrt{[gh\,(1+\text{cosec}\,\beta)]}$.

[Agra 1978; Meerut 80(S)]



Sol. In the figure FE is a hill h metres high and E is the position of the enemy. If O is the point from which the enemy's position is to be shelled, then according to the question $\angle EOF=\beta$. Let u be the least velocity of projection to hit E from O. Then

$$u=\sqrt{\{g\,(OE+EF)\}}. \qquad \text{[Refer § 8, part (b), page 48]}$$

$=\sqrt{\{g\,(h \operatorname{cosec}\beta+h)\}}$ [$\because$ $OE=h \operatorname{cosec}\beta$]
$=\sqrt{\{gh\,(1+\operatorname{cosec}\beta)\}}$.

Ex. 50. *A boy can throw a ball vertically upwards to a height h_1. Show that he cannot clear a wall of height h_2 distant d from him if* $2h_1<h_2+\sqrt{(h_2^2+d^2)}$.

Sol. Since the boy can throw a ball vertically upwards to a height h_1, therefore if u is the maximum velocity with which the boy can throw the ball, we have

$0=u^2-2gh_1$ [using the formula $v^2=u^2+2fs$]

or $u^2=2gh_1$ or $u=\sqrt{(2gh_1)}$.

Now the vertical height of the top of the wall from the point of projection is h_2 and its horizontal distance from the point of projection is d. To hit the top of the wall from the point of projection with velocity u, we must have

$u\geqslant[g\,\{h_2+\sqrt{(d^2+h_2^2)}\}]^{1/2}$ [by the formula for the least velocity of projection]

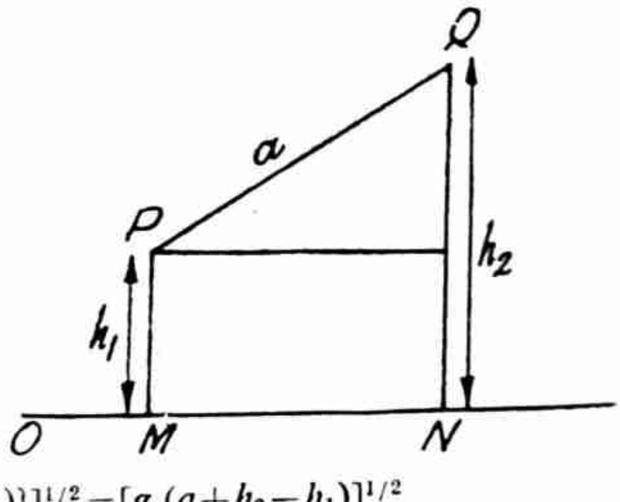
or $u^2\geqslant g\{h_2+\sqrt{(d^2+h_2^2)}\}$

or $2gh_1\geqslant g\,\{h_2+\sqrt{(d^2+h_2^2)}\}$ or $2h_1\geqslant h_2+\sqrt{(d^2+h_2^2)}$.

Therefore if $2h_1<h_2+\sqrt{(d^2+h_2^2)}$, the boy cannot clear the wall.

Ex. 51. *Two points P* and *Q are at a distance a apart, their heights above the ground being h_1 and h_2. Prove that the least velocity with which a particle can be thrown from the ground level, so as to pass through both the points, is* $\sqrt{[g\,(a+h_1+h_2)]}$.

Sol. Let O be the point of projection on the ground and u be the velocity of projection at O.



We have $PQ=a$ (given). Also the vertical height of Q above $P=h_2-h_1$.

If v be the least velocity of the projectile at P so as to hit Q, we must have $v=[g\,\{PQ+(h_2-h_1)\}]^{1/2}=[g\,(a+h_2-h_1)]^{1/2}$

or $v^2=g\,(a+h_2-h_1)$. ...(1)

Now if a particle is projected from O with velocity u and its velocity at P is v, we have

$v^2=u^2-2gh_1$ [Refer § 5, page 7]

or $u^2=v^2+2gh_1$. ...(2)

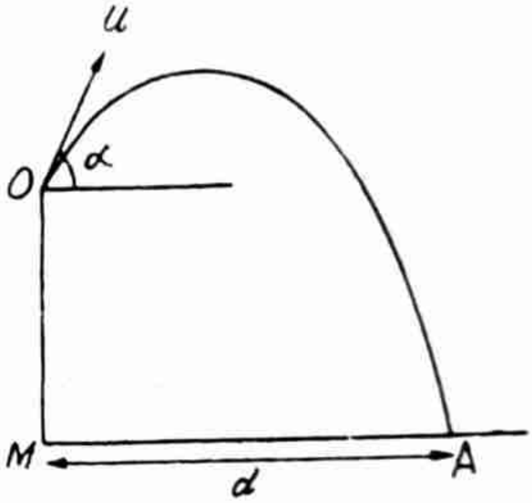
From (2) it is clear that u is least when v is least. So putting for v^2 from (1) in (2), the least value of u is given by

$$u^2=g\ (a+h_2-h_1)+2gh_1=g\ (a+h_1+h_2)$$

or $$u=\sqrt{[g\ (a+h_1+h_2)]}.$$

Ex. 52. *A shot is fired with velocity u from the top of a cliff of height h and strikes the sea at a distance d from the foot of the cliff. Show that the possible times of flight are the roots of the equation* $\frac{1}{4}g^2t^4-(gh+u^2)\ t^2+d^2+h^2=0.$

Sol. Let OM be a cliff of height h. A shot is fired from O with velocity u, say at an angle α. It strikes the sea at the point A whose distance from the foot of the cliff is h. Let t be the time of flight from O to A. Then considering the horizontal and vertical motions of the shot from O to A, we have



$$d=(u\cos\alpha)\ t, \qquad \ldots(1)$$

and $$-h=(u\sin\alpha)\ t-\tfrac{1}{2}gt^2$$

i.e., $$\tfrac{1}{2}gt^2-h=(u\sin\alpha)\ t. \qquad \ldots(2)$$

To eliminate α, squaring and adding (1) and (2), we get

$$d^2+(\tfrac{1}{2}gt^2-h)^2=u^2t^2$$

or $$\tfrac{1}{4}g^2t^4-(gh+u^2)\ t^2+d^2+h^2=0. \qquad \ldots(3)$$

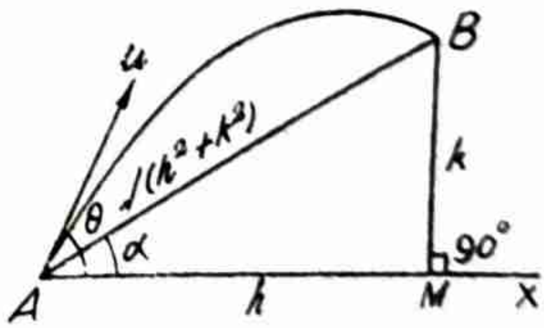
Hence the possible times of flight are the roots of the equation (3).

Ex. 53. *If t_1 and t_2 be the times of flight from A to B and α the inclination of AB to the horizontal, prove that*

$$t_1^2+2t_1t_2\sin\alpha+t_2^2$$

is independent of α. **[Meerut 1979]**

Sol. Let u be the velocity at A, its direction making an angle θ with the horizontal AX. Let t be the time of flight from A to B. It is given that $\angle BAM=\alpha$. Let $AM=h$ and $BM=k$; then



$$\sin\alpha=k/\sqrt{(h^2+k^2)}.$$

Considering the horizontal and vertical motions of the particle from A to B, we have

$$h=(u \cos \theta)\, t \qquad \ldots(1)$$

and $$k=(u \sin \theta)\, t-\tfrac{1}{2}gt^2$$

i.e., $$k+\tfrac{1}{2}gt^2=(u \sin \theta)\, t. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get

$$h^2+(k+\tfrac{1}{2}gt^2)^2=u^2t^2$$

or $$h^2+k^2+\tfrac{1}{4}g^2t^4+kgt^2-u^2t^2=0$$

or $$g^2t^4-4\,(u^2-kg)\, t^2+4\,(h^2+k^2)=0. \qquad \ldots(3)$$

If t_1 and t_2 are the two possible times of flight from A to B, then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. We have

$$t_1^2+t_2^2=\frac{4\,(u^2-kg)}{g^2} \text{ and } t_1^2t_2^2=\frac{4\,(h^2+k^2)}{g^2}.$$

Now $t_1^2+2t_1t_2 \sin \alpha+t_2^2=(t_1^2+t_2^2)+2t_1t_2 \sin \alpha$

$$=\frac{4\,(u^2-kg)}{g^2}+2.\frac{2}{g}\sqrt{(h^2+k^2)}.\frac{k}{\sqrt{(h^2+k^2)}}$$

$$=\frac{4u^2}{g^2}-\frac{4k}{g}+\frac{4k}{g}=\frac{4u^2}{g^2},$$

which is independent of h, k and is therefore independent of α.

Ex. 54. *Show that the product of the two times of flight from P to Q with a given velocity of projection is $(2PQ)/g$.* [**Kanpur 1986**]

Sol. Let u be the velocity of projection at P and θ be the angle of projection. Let t be the time of flight from P to Q. Suppose h and k are respectively the horizontal and vertical distances of Q from P. Then proceeding as in Ex. 53, we have

$$g^2t^4-4\,(u^2-kg)\, t^2+4\,(h^2+k^2)=0.$$

If t_1 and t_2 are the two possible times of flight from P to Q, then t_1^2 and t_2^2 are the roots of the above quadratic in t^2. We have

$$t_1^2t_2^2=\frac{4\,(h^2+k^2)}{g^2}=\frac{4}{g^2}\,(PQ)^2 \qquad [\because\ PQ^2=h^2+k^2]$$

so that $$t_1t_2=(2/g)\, PQ.$$

Ex. 55. *A shell bursts at a horizontal distance a from the foot of a hill of height h. Fragments of the shell fly in all directions with a velocity upto V. Find how long a man on the top of the hill will be in danger.*

Sol. Let u be the velocity and α the angle of projection for a fragment reaching the man. According to the question the greatest value of u can be V. If t be the time taken by this fragment to reach the man, then considering the horizontal and vertical motions of the fragment, we have $\quad a=(u \cos \alpha)\, t \qquad \ldots(1)$

and
$$h=(u \sin \alpha)\, t-\tfrac{1}{2}gt^2$$
i.e.,
$$h+\tfrac{1}{2}gt^2=(u \sin \alpha)\, t. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get
$$a^2+(h+\tfrac{1}{2}gt^2)^2=u^2t^2$$
or
$$a^2+h^2+\tfrac{1}{4}g^2t^4+ght^2=u^2t^2$$
or
$$g^2t^4-4\,(u^2-gh)\, t^2+4\,(a^2+h^2)=0. \qquad \ldots(3)$$

If t_1 and t_2 are two possible times of flight of the fragment to reach the man, then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. We have
$$t_1^2+t_2^2=\frac{4\,(u^2-gh)}{g^2} \quad \text{and} \quad t_1^2t_2^2=\frac{4\,(a^2+h^2)}{g^2}.$$

The period in which the man will be in danger on account of this fragment
$$=t_1-t_2=\sqrt{(t_1-t_2)^2}=\sqrt{\{(t_1^2+t_2^2)-2t_1t_2\}}$$
$$=\sqrt{\left\{\frac{4\,(u^2-gh)}{g^2}-2.\frac{2\sqrt{(a^2+h^2)}}{g}\right\}}$$
$$=\frac{2}{g}\,\{u^2-gh-g\sqrt{(a^2+h^2)}\}^{1/2}. \qquad \ldots(4)$$

From the result (4), we observe that the period t_1-t_2 increases as u increases. But the greatest value taken by u is V. Hence the man on the top of the hill will be in danger for a period
$$(2/g)\,\{V^2-gh-g\sqrt{(a^2+h^2)}\}^{1/2}.$$

Ex. 56. *A shell bursts on contact with the ground and pieces from it fly in all directions all with velocities upto* 80 *feet/sec. Show that a man* 100 *feet away is in danger for* $\frac{5}{2}\sqrt{2}$ *seconds.*

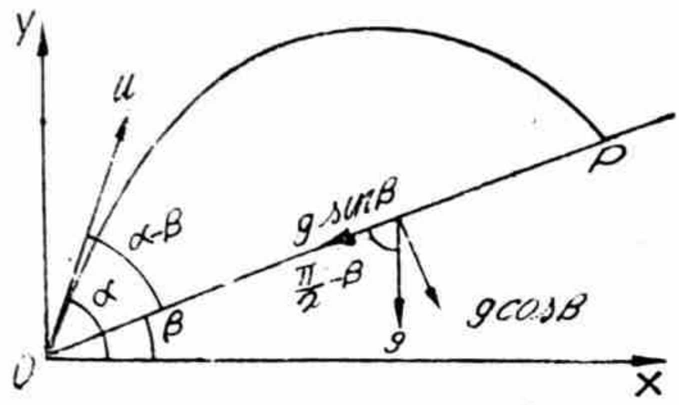
[Meerut 1987 P]

Sol. Proceed as in Ex. 55 by taking $a=100$ feet, $h=0$ and $V=80$ feet/sec. Note that here the man is on the ground and so $h=0$. The required period in which the man is in danger is
$$=\frac{2}{g}\,\{80^2-100g\}^{1/2} \text{ seconds}$$
$$=\tfrac{2}{32}\,\{6400-100\times 32\}^{1/2} \text{ seconds}=\tfrac{2}{32}\sqrt{(3200)} \text{ seconds}$$
$$=\tfrac{2}{32}\times 10\times 4\times \sqrt{2} \text{ seconds}=\tfrac{5}{2}\sqrt{2} \text{ seconds}.$$

§ 9. Range and time of flight on an inclined plane.

A particle is projected with velocity u at an angle α to the horizontal from a point O on an inclined plane of inclination β to the horizontal. The particle is projected up the inclined plane to move in the vertical plane through the line of greatest slope. If the particle strikes the inclined plane, to determine the range and the time of flight.

Let O be the point of projection and u the velocity of projection making an angle α with the horizontal OX.



Suppose the particle strikes the inclined plane at P, where $OP=R$. Then R is the range up the inclined plane. Let T be the time of flight from O to P.

Initial velocity at O along the inclined plane

$$=u \cos(\alpha-\beta), \text{ up the plane}$$

and initial velocity at O perpendicular to the inclined plane

$$=u \sin(\alpha-\beta),$$

along the upward normal to the plane.

The resolved part of the acceleration g along the inclined plane$=g \sin\beta$, down the plane and the resolved part of g perpendicular to the inclined plane$=g \cos\beta$, along the downward normal to the plane.

While moving from O to P the displacement of the particle perpendicular to the inclined plane is zero. So considering the motion of the particle from O to P perpendicular to the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$0=u \sin(\alpha-\beta).T-\tfrac{1}{2}g \cos\beta.T^2.$$

But $T=0$ gives the time from O to O. Therefore the **time of flight** T from O to P is given by

$$\mathbf{T=\frac{2u \sin(\alpha-\beta)}{g \cos\beta}}. \qquad ...(1)$$

Now considering the motion of the particle from O to P along the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$R=u \cos(\alpha-\beta).T-\tfrac{1}{2}g \sin\beta T^2$$

$$=T[u \cos(\alpha-\beta)-\tfrac{1}{2}g \sin\beta T]$$

$$=\frac{2u \sin(\alpha-\beta)}{g \cos\beta}\left[u \cos(\alpha-\beta)-\tfrac{1}{2}g \sin\beta \frac{2u \sin(\alpha-\beta)}{g \cos\beta}\right]$$

[substituting for T from (1)]

$$=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\cdot\frac{u\{\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta\}}{\cos\beta}$$

$$=\frac{2u^2\sin(\alpha-\beta)\cos\{(\alpha-\beta)+\beta\}}{g\cos^2\beta}$$

i.e., $$\mathbf{R}=\frac{2\mathbf{u}^2\sin(\alpha-\beta)\cos\alpha}{\mathbf{g}\cos^2\beta}. \quad ...(2)$$

This gives range up the inclined plane.

Maximum range up the inclined plane. From the formula (2), we observe that if u and β are given, then the range R depends upon the angle of projection α. We can write

$$R=\frac{u^2}{g\cos^2\beta}[\sin(\alpha-\beta+\alpha)+\sin(\alpha-\beta-\alpha)]$$

$$=\frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta].$$

Obviously for given u and β, R is maximum when $\sin(2\alpha-\beta)$ is maximum *i.e.*, when $\sin(2\alpha-\beta)=1$

i.e., when $2\alpha-\beta=\frac{1}{2}\pi$, ...(3)

i.e., when $\alpha=\frac{1}{4}\pi+\frac{1}{2}\beta$.

Also the maximum range

$$=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}=\frac{u^2}{g}\cdot\frac{1-\sin\beta}{1-\sin^2\beta}$$

$$=\frac{u^2}{g}\frac{(1-\sin\beta)}{(1+\sin\beta)(1-\sin\beta)}=\frac{u^2}{g(1+\sin\beta)}.$$

Thus the maximum range up the inclined plane

$$=\frac{\mathbf{u}^2}{\mathbf{g}(1+\sin\beta)}.$$ **[Agra 1986]**

From (3) we observe that the range on the inclined plane is maximum when $\quad 2\alpha-\beta=\frac{1}{2}\pi$

i.e., when $\quad \alpha-\beta=\frac{1}{2}\pi-\alpha$

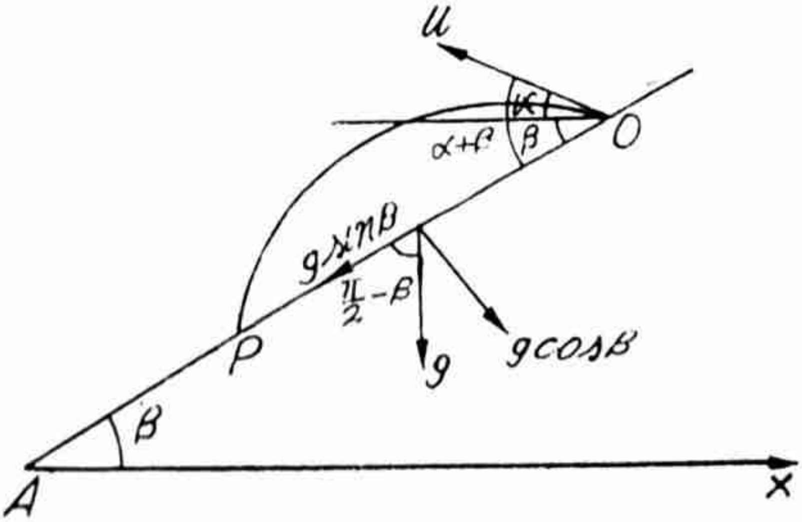
i.e., when the angle between the direction of projection and the inclined plane is the same as the angle between the direction of projection and the vertical.

Hence in the case of maximum range on the inclined plane the direction of projection bisects the angle between the vertical and the inclined plane. **[Lucknow 1977]**

Now the direction of projection at O is along the tangent to the parabolic path at O. Also the vertical through O is perpendicular to the directrix of the path. In the case of a parabola the tangent at any point bisects the angle between the focal distance of that point and the perpendicular from that point to the directrix. Therefore in the case of maximum range on the inclined plane the range OP coincides with the line joining O to the focus of the parabola. *Hence in the case of maximum range on an inclined plane the focus of the path lies in the range itself.*

§ 10. **Range and time of flight down an inclined plane.**

Let O be a point on an inclined plane whose inclination to the horizontal is β. Suppose a particle is projected from O down the inclined plane. Let u be the velocity of projection making an angle α with the horizontal through O. Suppose the particle strikes the inclined plane at P, where $OP=R$. Then R is the range down the inclined plane. Let T be the time of flight from O to P.



Initial velocity at O along the inclined plane $=u\cos(\alpha+\beta)$ down the plane

and initial velocity at O perpendicular to the inclined plane

$$=u\sin(\alpha+\beta),$$

along the upward normal to the plane.

Resolved part of the acceleration g along the inclined plane is $g\sin\beta$ and perpendicular to the inclined plane is $g\cos\beta$ as shown in the figure.

While moving from O to P in time T the displacement of the particle perpendicular to the inclined plane is zero. So considering the motion of the particle from O to P perpendicular to the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$0=u\sin(\alpha+\beta)\,T-\tfrac{1}{2}g\cos\beta\,T^2$$

i.e., $$T=\frac{2u\sin(\alpha+\beta)}{g\cos\beta} \quad \ldots(1)$$

Now considering the motion of the particle from O to P along the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$R=u\cos(\alpha+\beta).T+\tfrac{1}{2}g\sin\beta.T^2.$$

$$=T\,[u\cos(\alpha+\beta)+\tfrac{1}{2}g\sin\beta\,.\,T]$$

$$=\frac{2u\sin(\alpha+\beta)}{g\cos\beta}\left[u\cos(\alpha+\beta)+\tfrac{1}{2}g\ \sin\beta.\frac{2u\sin(\alpha+\beta)}{g\cos\beta}\right]$$

$$=\frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}. \qquad ...(2)$$

To find the maximum range down the inclined plane, we can write (2) as $\qquad R=\dfrac{u^2\,[\sin(2\alpha+\beta)+\sin\beta]}{g\cos^2\beta}.$

$\therefore$ for given u and β, R is maximum when $\sin(2\alpha+\beta)=1$.

Also the maximum value of R

$$=\frac{u^2(1+\sin\beta)}{g\cos^2\beta}=\frac{u^2(1+\sin\beta)}{g(1-\sin^2\beta)}=\frac{u^2}{g(1-\sin\beta)}.$$

Thus for motion down the inclined plane,

time of flight$=\dfrac{2u\sin(\alpha+\beta)}{g\cos\beta}$, range$=\dfrac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$,

and maximum range$=\dfrac{u^2}{g(1-\sin\beta)}$. **[Agra 1980]**

We observe that if we replace β by $-\beta$ in the results for motion up the inclined plane, we get the corresponding results for motion down the inclined plane.

Illustrative Examples

Ex. 57. *A particle is projected with velocity u from a point on a plane inclined at an angle β to the horizontal. If r and r' are its maximum ranges up and down the plane, prove that $1/r+1/r'$ is independent of the inclination of the plane.*

Sol. Here the inclination of the inclined plane to the horizontal is β.

$\therefore\quad r=$the maximum range up the inclined plane

$$=\frac{u^2}{g(1+\sin\beta)}$$

and $r'=$the maximum range down the inclined plane

$$=\frac{u^2}{g(1-\sin\beta)}.$$

Now $\dfrac{1}{r}+\dfrac{1}{r'}=\dfrac{g}{u^2}[(1+\sin\beta)+(1-\sin\beta)]=\dfrac{2g}{u^2}$, which is independent of the inclination β of the plane.

Ex. 58. *For a given velocity of projection the maximum range down an inclined plane is three times the maximum range up the inclined plane; show that the inclination of the plane to the horizontal is* $30°$ **[Agra 1975]**

Sol. Let u be the velocity of projection and β the inclination of the inclined plane to the horizontal. Then the maximum ranges up and down the inclined plane are respectively

$$\frac{u^2}{g\,(1+\sin\beta)} \text{ and } \frac{u^2}{g\,(1-\sin\beta)}.$$

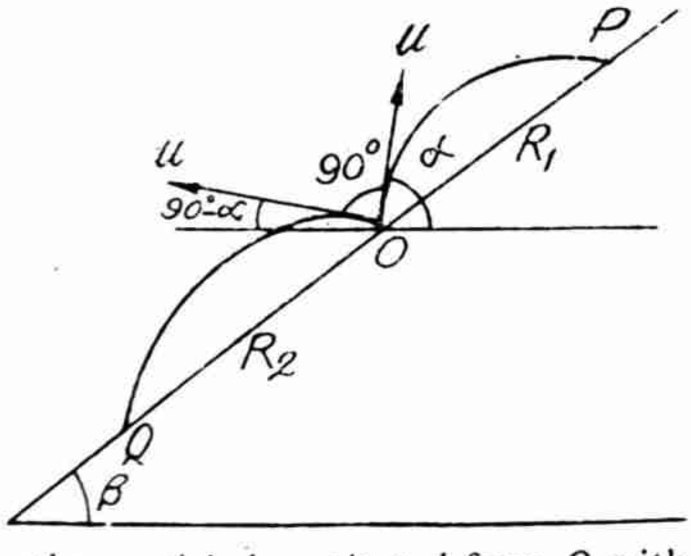
According to the question, $\dfrac{u^2}{g\,(1-\sin\beta)}=3\,\dfrac{u^2}{g\,(1+\sin\beta)}$.

$\therefore\ 1+\sin\beta=3-3\sin\beta$ or $4\sin\beta=2$ or $\sin\beta=1/2$ or $\beta=30°$.

Ex. 59. *If from a point on the side of a hill two bodies are projected in the vertical plane through the line of greatest slope with the same velocity but in directions at right angles to each other, show that the difference of their ranges is independent of their angles of projection.*

Sol. Let β be the inclination of the hill to the horizontal and O the point of projection. Suppose a particle is projected from O up the hill with velocity u making an angle α with the horizontal through O. If R_1 is the range of this particle on the hill, then using the formula for the range up an inclined plane, we have

$$R_1=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}. \quad \ldots(1)$$



Now the other particle is projected from O with velocity u in a direction at right angles to the direction of projection of the first particle. Therefore this particle moves down the hill and its direction of projection makes an angle $\frac{1}{2}\pi-\alpha$ with the horizontal through O. If R_2 be the range of this particle on the hill, then using the formula for the range down an inclined plane, we have

$$R_2=\frac{2u^2\cos(\frac{1}{2}\pi-\alpha)\sin\{(\frac{1}{2}\pi-\alpha)+\beta\}}{g\cos^2\beta}$$

$$=\frac{2u^2 \sin \alpha \sin \{\frac{1}{2}\pi-(\alpha-\beta)\}}{g \cos^2 \beta}$$

$$=\frac{2u^2 \sin \alpha \cos (\alpha-\beta)}{g \cos^2 \beta}. \quad \ldots(2)$$

From (1) and (2), we have

$$R_2-R_1=\frac{2u^2}{g \cos^2 \beta}\left[\sin \alpha \cos (\alpha-\beta)-\cos \alpha \sin (\alpha-\beta)\right]$$

$$=\frac{2u^2}{g \cos^2 \beta} \sin \{\alpha-(\alpha-\beta)\}=\frac{2u^2 \sin \beta}{g \cos^2 \beta},$$

which is independent of the angle of projection α.

Ex. 60. *Show that if a gun be situated on an inclined plane, the maximum range in a direction at right angles to the line of greatest slope is a harmonic mean between the maximum ranges up and down the plane respectively.*

Sol. Let β be the inclination of the inclined plane to the horizontal, O the point of projection and u the velocity of projection.

If R_1 and R_2 are the maximum ranges up and down the inclined plane respectively, then

$$R_1=\frac{u^2}{g(1+\sin \beta)} \text{ and } R_2=\frac{u^2}{g(1-\sin \beta)}.$$

Now the line of greatest slope through O is the line lying in the inclined plane and at right angles to the line in which the inclined plane meets the horizontal. Therefore the direction through O at right angles to the line of greatest slope is a horizontal direction. If R_3 be the maximum range in this direction, then

$R_3=$ the maximum range in a horizontal direction with velocity of projection $u=u^2/g$.

Now $$\frac{1}{2}\left[\frac{1}{R_1}+\frac{1}{R_2}\right]=\frac{1}{2}\frac{g}{u^2}\left[(1+\sin \beta)+(1-\sin \beta)\right]$$

$$=g/u^2=1/R_3.$$

$\therefore$ $1/R_3$ is the arithmetic mean of $1/R_1$ and $1/R_2$

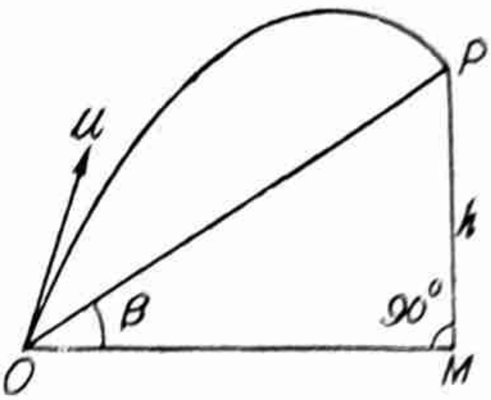
i.e., R_3 is the harmonic mean of R_1 and R_2.

Ex. 61. *The angular elevation of an enemy's position on a hill h metres high is β. Show that in order to shell it, the initial velocity of the projectile must not be less than*

$$\sqrt{[gh(1+\operatorname{cosec} \beta)]}.$$ **[Agra 1978; Meerut 80, 83]**

Sol. Let O be the point of projection and P the enemy's position. Then as given in the question, $PM=h$ metres and $\angle POM=\beta$.

Let u be the least velocity of projection to hit P from O. Then for the velocity of projection u at O, OP is the maximum range up the inclined plane OP.



$$\therefore\ OP=\frac{u^2}{g(1+\sin\beta)}. \quad ...(1)$$

But from $\triangle PMO$, we have $OP=PM \operatorname{cosec}\beta=h\operatorname{cosec}\beta$. ...(2)

From (1) and (2), we have

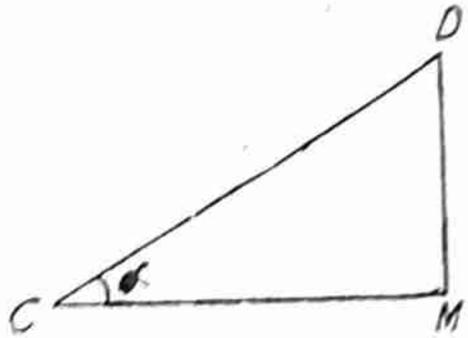
$$\frac{u^2}{g(1+\sin\beta)}=h\operatorname{cosec}\beta$$

i.e., $u^2=gh\operatorname{cosec}\beta\,(1+\sin\beta)=gh(\operatorname{cosec}\beta+1)$

i.e., $u=\sqrt{[gh(1+\operatorname{cosec}\beta)]}$.

Ex. 62. *The line joining C to D is inclined at an angle α to the horizontal. Show that the least velocity required to shoot from C to D is $\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$ times the least velocity required to shoot from D to C.*

Sol. Let u be the least velocity of projection to hit D from C. Then for the velocity of projection u at C, CD is the maximum range up the inclined plane CD.



$$\therefore\ CD=\frac{u^2}{g(1+\sin\alpha)}. \quad ...(1)$$

Again let v be the least velocity to shoot C from D. Then for the velocity of projection v at D, DC is the maximum range down the inclined plane DC.

$$\therefore\ DC=\frac{v^2}{g(1-\sin\alpha)}. \quad ...(2)$$

From (1) and (2), we have

$$\frac{u^2}{g(1+\sin\alpha)}=\frac{v^2}{g(1-\sin\alpha)}.$$

$$\therefore\ \frac{u^2}{v^2}=\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{1-\cos(\frac{1}{2}\pi+\alpha)}{1+\cos(\frac{1}{2}\pi+\alpha)}=\frac{2\sin^2(\frac{1}{4}\pi+\frac{1}{2}\alpha)}{2\cos^2(\frac{1}{4}\pi+\frac{1}{2}\alpha)}$$

$$=\tan^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha).$$

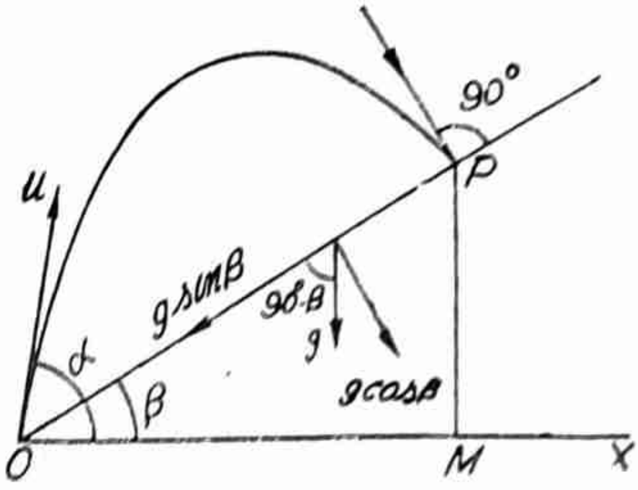
$\therefore\ u/v=\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$ or $u=v\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$, as was to be proved.

Ex. 63. *A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is β. Show that it will strike the plane at right angles if*

$$\cot \beta = 2 \tan (\alpha - \beta).$$

[Lucknow 1978, 79; Meerut 77, 78, 79, 83S, 84S]

Sol. Let O be the point of projection, u the velocity of projection and P the point where the particle strikes the plane.



Let T be the time of flight from O to P. Then by the usual formula for time of flight on an inclined plane, we have

$$T = \frac{2u \sin (\alpha - \beta)}{g \cos \beta}. \qquad \ldots(1)$$

Since in this question the particle strikes the inclined plane at right angles at P, therefore the direction of the velocity of the particle at P is perpendicular to the inclined plane. Consequently the resolved part of the velocity of the particle at P along the inclined plane is zero. Also the resolved part of the velocity of the particle at O along the inclined plane is $u \cos (\alpha - \beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g \sin \beta$ downwards. So considering the motion of the particle from O to P along the inclined plane and using the formula $v = u + ft$, we have $\quad 0 = u \cos (\alpha - \beta) - g \sin \beta \, T$

i.e.,
$$T = \frac{u \cos (\alpha - \beta)}{g \sin \beta}. \qquad \ldots(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u \sin (\alpha - \beta)}{g \cos \beta} = \frac{u \cos (\alpha - \beta)}{g \sin \beta}$$

or
$$\frac{2 \sin (\alpha - \beta)}{\cos (\alpha - \beta)} = \frac{\cos \beta}{\sin \beta}$$

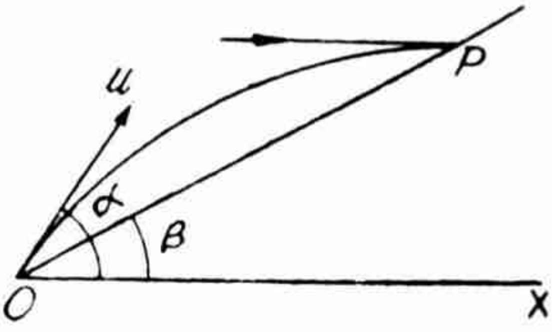
or
$$2 \tan (\alpha - \beta) = \cot \beta.$$

Ex. 64. *A shot is fired at an angle α to the horizontal up an hill of inclination β to the horizontal. Show that it strikes the hill :*

(a) *horizontally if $\tan \alpha = 2 \tan \beta$,*

(b) *normally if $\tan \alpha = 2 \tan \beta + \cot \beta$.* **[Agra 1986]**

Sol. (a). Let O be the point of projection, u the velocity of projection and P the point where the shot strikes the plane. Let T be the time of flight from O to P. Then by the usual formula for the time of flight on an inclined plane, we have



$$T=\frac{2u \sin (\alpha-\beta)}{g \cos \beta}. \quad \ldots(1)$$

Now according to the question the particle strikes the inclined plane horizontally at P *i.e.*, the direction of the velocity of the particle at P is horizontal. So the vertical velocity of the particle at P is zero. Also the vertical velocity of the particle at O is $u \sin \alpha$ upwards and the acceleration in the vertical direction is g downwards. So considering the vertical motion of the particle from O to P and using the formula $v=u+ft$, we have

$$0=u \sin \alpha-gT$$

i.e. $$T=\frac{u \sin \alpha}{g} \quad \ldots(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u \sin (\alpha-\beta)}{g \cos \beta}=\frac{u \sin \alpha}{g}$$

or $$2 \sin (\alpha-\beta)=\sin \alpha \cos \beta$$

or $$2 \sin \alpha \cos \beta-2 \cos \alpha \sin \beta=\sin \alpha \cos \beta$$

or $$\sin \alpha \cos \beta=2 \cos \alpha \sin \beta$$

or $$\frac{\sin \alpha}{\cos \alpha}=\frac{2 \sin \beta}{\cos \beta}$$

or $$\tan \alpha=2 \tan \beta.$$

(b) Proceeding as in Ex. 63, we get the condition for striking the inclined plane normally at P as

$$\cot \beta=2 \tan (\alpha-\beta).$$ [Derive it here.]

$$\therefore \quad \cot \beta=\frac{2 (\tan \alpha-\tan \beta)}{1+\tan \alpha \tan \beta}$$

or $$\cot \beta (1+\tan \alpha \tan \beta)=2 \tan \alpha-2 \tan \beta$$

or $$\cot \beta+\tan \alpha=2 \tan \alpha-2 \tan \beta$$

or $$\tan \alpha=2 \tan \beta+\cot \beta.$$

Ex. 65. *A particle is projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β, and strikes it at right angles. Show that*

(i) *the time of flight is* $\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$,

[Lucknow 1978; Meerut 81, 84, 85, 87S]

(ii) *the range on the inclined plane is* $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$.

[Gorakhpur 1978; Agra 80, 86; Meerut 81, 84, 85, 87S, 88P]

and (iii) *the vertical height of the point struck, above the point of projection is* $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$.

Sol. Refer figure of Ex. 63, page 65.

Let O be the point of projection, u the velocity of projection, α the angle of projection and P the point where the particle strikes the plane at right angles.

Let T be the time of flight from O to P. Then by the formula for the time of flight on an inclined plane, we have

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \qquad \ldots(1)$$

Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particle at P along the inclined plane is zero. Also the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g\sin\beta$ downwards. So considering the motion of the particle from O to P along the inclined plane and using the formula $v=u+ft$, we have $\quad 0=u\cos(\alpha-\beta)-g\sin\beta T$

or $$T=\frac{u\cos(\alpha-\beta)}{g\sin\beta}. \qquad \ldots(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta}=\frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

or $$\tan(\alpha-\beta)=\tfrac{1}{2}\cot\beta, \qquad \ldots(3)$$

as the condition for striking the plane at right angles.

(i) From (2),

$$T=\frac{u}{g\sin\beta\sec(\alpha-\beta)}=\frac{u}{g\sin\beta\sqrt{\{1+\tan^2(\alpha-\beta)\}}}$$

$$=\frac{u}{g\sin\beta\sqrt{\{1+\frac{1}{4}\cot^2\beta\}}}, \text{ substituting for } \tan(\alpha-\beta) \text{ from (3)}$$

$$=\frac{2u\sin\beta}{g\sin\beta\sqrt{(4\sin^2\beta+\cos^2\beta)}}=\frac{2u}{g\sqrt{(\sin^2\beta+\cos^2\beta+3\sin^2\beta)}}$$

$$=\frac{2u}{g\sqrt{(1+3\sin^2\beta)}}.$$

(ii) Let R be the range on the inclined plane; then $R=OP$. Considering the motion from O to P along the inclined plane and using the formula $v^2=u^2+2fs$, we have

$$0=u^2\cos^2(\alpha-\beta)-2g\sin\beta R$$

or
$$R=\frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta}=\frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$=\frac{u^2}{2g\sin\beta\{1+\tan^2(\alpha-\beta)\}}$$

$$=\frac{u^2}{2g\sin\beta\{1+\frac{1}{4}\cot^2\beta\}} \qquad \text{[from (3)]}$$

$$=\frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)}=\frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}.$$

(iii) The vertical height of P above $O=PM$

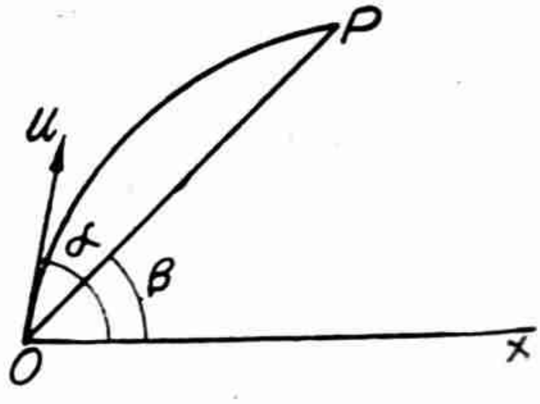
$$=OP\sin\beta=R\sin\beta=\frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

Ex. 66. *Prove that if a particle is projected from O at an elevation α and after time t the particle is at P, then*

$$2\tan\beta=\tan\alpha+\tan\theta,$$

where β and θ are the inclinations to the horizontal of OP and of the direction of motion of the particle when at P.

Sol. Let O be the point of projection, u the velocity of projection and t the time of flight from O to P. It is given that $\angle POX=\beta$, where OX is the horizontal through O in the plane of motion. We can regard t as the time of flight on the inclined pane OP whose inclination to the horizontal is β.



$$\therefore \quad t=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \qquad \ldots(1)$$

Since θ is the inclination to the horizontal of the direction of motion at P, therefore

$$\tan\theta=\frac{\text{vertical velocity at } P}{\text{horizontal velocity at } P}$$

$$=\frac{u\sin\alpha-gt}{u\cos\alpha}=\tan\alpha-\frac{g}{u\cos\alpha}t$$

$=\tan\alpha-\dfrac{g}{u\cos\alpha}\cdot\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$, substituting for t from (1)

$=\tan\alpha-\dfrac{2\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\dfrac{2(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{\cos\alpha\cos\beta}$

$=\tan\alpha-2(\tan\alpha-\tan\beta)=\tan\alpha-2\tan\alpha+2\tan\beta$

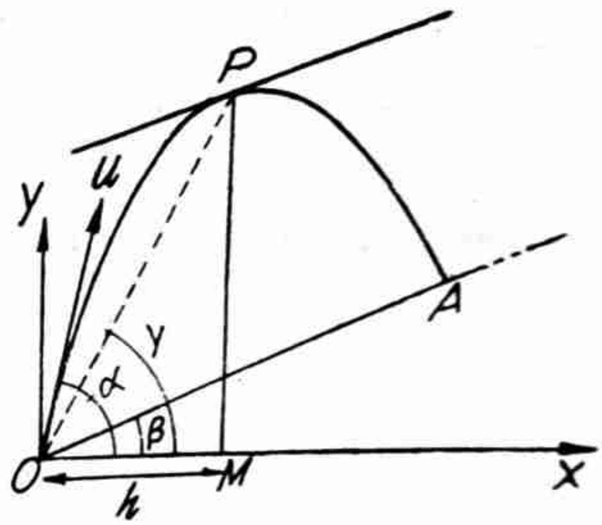
$=2\tan\beta-\tan\alpha.$

$\therefore\quad 2\tan\beta=\tan\alpha+\tan\theta.$

Ex. 67. *A stone is thrown at an angle α with the horizontal from a point in a plane whose inclination to the horizontal is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if γ be the elevation of that point of the path which is most distant from the inclined plane, then*

$$2\tan\gamma=\tan\alpha+\tan\beta.$$

[Allahabad 1979; Meerut 81S, 88S]



Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let P be the point of the trajectory which is most distant from the inclined plane. Then the tangent at P to the trajectory is parallel to the line OA. Referred to the horizontal and verifical lines OX and OY in the plane of motion as the coordinate axes, let the coordinates of P be (h, k). It is given that $\angle POM=\gamma$. Therefore

$$\tan\gamma=k/h. \qquad \ldots(1)$$

The equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}.$$

$\therefore\quad \dfrac{dy}{dx}=\tan\alpha-\dfrac{gx}{u^2\cos^2\alpha}$, which gives the slope of the tangent to the horizontal at any point (x, y) of the trajectory.

Since the tangent to the trajectory at the point $P(h, k)$ makes an angle β with the horizontal line OX, therefore

$$\left(\frac{dy}{dx}\right)_{(h,\,k)}=\tan\beta$$

i.e., $$\tan\alpha-\frac{gh}{u^2\cos^2\alpha}=\tan\beta. \quad ...(2)$$

Alos the point (h, k) lies on the trajectory. Therefore, we have

$$k=h\tan\alpha-\tfrac{1}{2}\frac{gh^2}{u^2\cos^2\alpha} \quad \text{or} \quad k=h\left[\tan\alpha-\tfrac{1}{2}\frac{gh}{u^2\cos^2\alpha}\right]$$

or $$\frac{k}{h}=\tan\alpha-\tfrac{1}{2}\frac{gh}{u^2\cos^2\alpha}. \quad ...(3)$$

But from (1), $\dfrac{k}{h}=\tan\gamma$ and from (2), $\dfrac{gh}{u^2\cos^2\alpha}=\tan\alpha-\tan\beta.$

Substituting these in (3), we get

$$\tan\gamma=\tan\alpha-\tfrac{1}{2}(\tan\alpha-\tan\beta)$$

or $$2\tan\gamma=2\tan\alpha-\tan\alpha+\tan\beta$$

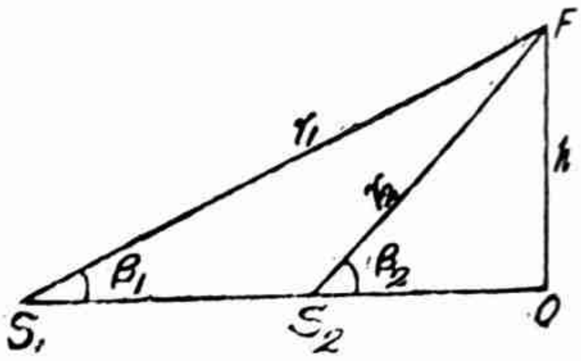
or $$2\tan\gamma=\tan\alpha+\tan\beta,$$

which proves the required result.

Ex. 68. *A fort is on the edge of a cliff of height h. Show that there is an annular region of area $8\pi hk$ in which the fort is out of range of the ship, but the ship is not out of range of the fort, where $\sqrt{(2gk)}$ is the velocity of the shells used by both.*

[Meerut 1980, 90S]

Sol. Let F be the fort on the top of a cliff OF whose height is h. Let S_1 be the farthest position of the ship where it can be hit from the fort with velocity of projection $\sqrt{(2gk)}$. Then $\sqrt{(2gk)}$ is the least velocity of projection to hit S_1 from F and consequently for the velocity of projection $\sqrt{(2gk)}$ at F, FS_1 is the maximum range down the inclined plane FS_1. Let $\angle FS_1O=\beta_1$ and $FS_1=r_1$. By the formula for the maximum range down an inclined plane, we have



$$r_1=FS_1=\frac{u^2}{g(1-\sin\beta_1)}, \text{ where } u \text{ is the velocity of projection}$$

$$=\frac{2gk}{g(1-\sin\beta_1)} \qquad [\because \ u^2=2gk]$$

$$=\frac{2k}{1-\sin\beta_1}.$$

$\therefore \quad r_1(1-\sin\beta_1)=2k \quad$ or $\quad r_1-r_1\sin\beta_1=2k$

or $$r_1-h=2k \qquad [\because \text{ from } \triangle FS_1O,\ h=r_1\sin\beta_1]$$

or $$r_1=2k+h. \quad ...(1)$$

Again let S_2 be the farthest position of the ship from where the fort can be hit with velocity of projection $\sqrt{(2gk)}$. Then for the velocity of projection $\sqrt{(2gk)}$ at S_2, S_2F is the maximum range up the inclined plane S_2F. Let $\angle FS_2O=\beta_2$ and $FS_2=r_2$. By the formula for the maximum range up an inclined plane, we have

$$r_2=S_2F=\frac{u^2}{g(1+\sin\beta_2)}, \text{ } u \text{ being the velocity of projection}$$

$$=\frac{2gk}{g(1+\sin\beta_2)} \qquad [\because \; u^2=2gk]$$

$$=\frac{2k}{1+\sin\beta_2}.$$

$$\therefore \; r_2(1+\sin\beta_2)=2k \quad \text{or} \quad r_2+r_2\sin\beta_2=2k$$

or $$r_2+h=2k \qquad [\because \text{ from } \triangle FS_2O,\ h=r_2\sin\beta_2]$$

or $$r_2=2k-h. \qquad ...(2)$$

Now if the ship is anywhere between S_1 and S_2, then the fort cannot be shelled from the ship while the ship can be shelled from the fort. If the line OS_1 revolves about O, then there is an annular region bounded by the concentric circles with centre at O and radii as OS_1 and OS_2 in which the fort is out of range of the ship while the ship is not out of range of the fort. The area of this annular region $=\pi(OS_1^2-OS_2^2)=\pi[(r_1^2-h^2)-(r_2^2-h^2)]$

$[\because$ from $\triangle FS_1O$, $OS_1^2=r_1^2-h^2$, etc.$]$

$$=\pi(r_1^2-r_2^2)$$
$$=\pi[(2k+h)^2-(2k-h)^2] \qquad [\text{from (1) and (2)}]$$
$$=8\pi hk.$$

Ex. 69. *A fort and a ship are both armed with guns which give their projectiles a muzzle velocity $\sqrt{(2gh)}$ and guns in the fort are at a height k above the ship. If d_1 and d_2 are greatest horizontal ranges at which the fort and ship, respectively, can engage, prove that* $$\frac{d_1}{d_2}=\sqrt{\left(\frac{h+k}{h-k}\right)}.$$ **[Lucknow 1981; Gorakhpur 80]**

Sol. Proceed exactly in the same way as in Ex. 68. Here $OF=k$. Thus replacing h by k and k by h in the results of Ex. 68, we get

$$r_1=2h+k \quad \text{and} \quad r_2=2h-k.$$

According to this question $OS_1=d_1$ and $OS_2=d_2$

From $\triangle OS_1F$, $OS_1=\sqrt{(FS_1^2-OF^2)}=\sqrt{(r_1^2-k^2)}$.

$\therefore \; d_1=\sqrt{\{(2h+k)^2-k^2\}}=\sqrt{(4h^2+4hk)}=2\sqrt{h}\sqrt{(h+k)}$.

Again from $\triangle OS_2F$, $OS_2=\sqrt{(FS_2^2-OF^2)}=\sqrt{(r_2^2-k^2)}$.

$\therefore \quad d_2 = \sqrt{\{(2h-k)^2 - k^2\}} = \sqrt{(4h^2 - 4hk)} = 2\sqrt{h}\sqrt{(h-k)}.$

$\therefore \quad \dfrac{d_1}{d_2} = \dfrac{2\sqrt{h}\sqrt{(h+k)}}{2\sqrt{h}\sqrt{(h-k)}} = \sqrt{\left(\dfrac{h+k}{h-k}\right)}.$

Ex. 70. *If u be the velocity of projection and v_1 the velocity of striking the plane when projected so that range up the plane is maximum and v_2 the velocity of striking the plane when projected so that range down the plane is maximum, prove that* $\quad u^2 = v_1 v_2.$

Sol. Let β be the inclination of the plane to the horizontal.

For the velocity of projection u, the maximum range up the inclined plane $= \dfrac{u^2}{g\,(1+\sin\beta)}.$

$\therefore$ the height of the point of striking the plane above the point of projection $= \dfrac{u^2}{g\,(1+\sin\beta)} . \sin\beta = h_1$ (say).

Since the velocity of the projectile at this vertical height h_1 above the point of projection is given to be v_1, therefore

$$v_1^2 = u^2 - 2gh_1 \qquad \text{[Refer § 5, page 7]}$$

$$= u^2 - 2g . \frac{u^2 \sin\beta}{g\,(1+\sin\beta)} = u^2 \frac{1-\sin\beta}{1+\sin\beta}. \qquad \text{...(1)}$$

Again for the velocity of projection u, the maximum range down the inclined plane $= \dfrac{u^2}{g\,(1-\sin\beta)}.$

$\therefore$ the depth of the point of striking the plane below the point of projection $= \dfrac{u^2}{g\,(1-\sin\beta)} . \sin\beta = h_2$ (say).

Since the velocity of the projectile at this vertical depth h_2 below the point of projection is given to be v_2, therefore

$$v_2^2 = u^2 + 2gh_2 = u^2 + 2g . \frac{u^2 \sin\beta}{g\,(1-\sin\beta)}$$

$$= u^2 \frac{1+\sin\beta}{1-\sin\beta}. \qquad \text{...(2)}$$

From (1) and (2), we have

$$v_1^2 v_2^2 = u^4 \qquad i.e. \qquad v_1 v_2 = u^2.$$

Ex. 71. *Show that the greatest range up an inclined plane through the point of projection is equal to the distance through which a particle could fall freely during the corresponding time of flight.*

Sol. Let β be the inclination of the plane to the horizontal. If α is the angle of projection, then for maximum range up the plane,

$$\alpha = \tfrac{1}{4}\pi + \tfrac{1}{2}\beta.$$

Time of flight up the inclined plane is

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

When $\alpha=\frac{1}{4}\pi+\frac{1}{2}\beta$, $T=\dfrac{2u\sin\{(\frac{1}{4}\pi+\frac{1}{2}\beta)-\beta\}}{g\cos\beta}$

$$=\frac{2u\sin(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g\cos\beta}.$$

The vertical distance fallen freely under gravity by a particle during this time T

$$=0.T+\tfrac{1}{2}gT^2=\tfrac{1}{2}g.\ \frac{4u^2\sin^2(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g^2\cos^2\beta}$$

$$=\frac{2u^2\sin^2(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g\cos^2\beta}=\frac{u^2\{1-\cos(\frac{1}{2}\pi-\beta)\}}{g\cos^2\beta}=\frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$=\dfrac{u^2}{g(1+\sin\beta)}$ = the maximum range up the inclined plane.

Ex. 72. *Two inclined planes intersect in a horizontal plane, their inclinations to the horizon being α and β; if a particle is projected at right angles to the former from a point in it so as to strike the other at right angles, the velocity of projection is*

$$\sin\beta\left\{\frac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right\}^{1/2}$$

a being the distance of the point of projection from the intersection of the planes.

Sol. Let OA and OB be the two inclined planes and P the point of projection so that $OP=a$. The particle is projected from P at right angles to OA, say, with velocity u. Let PN be perpendicular from P to BO produced and PM be drawn parallel to OB.

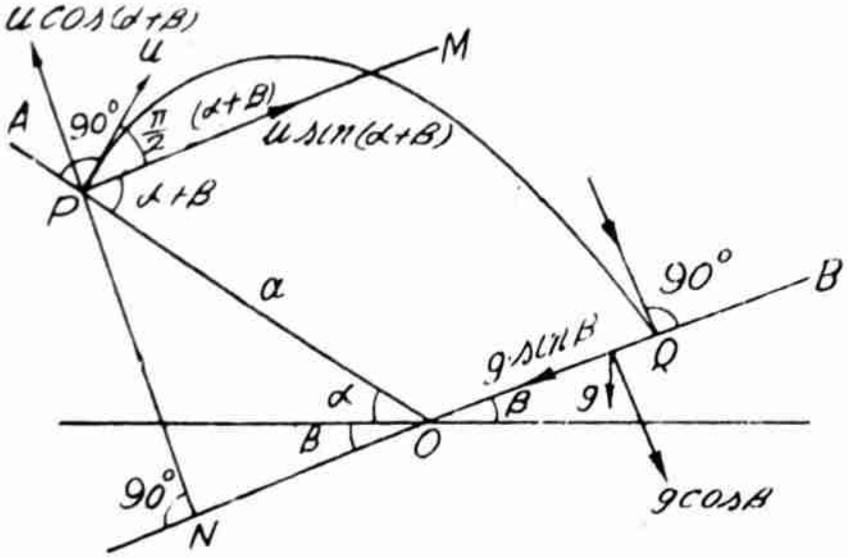
We have $PN=a\sin(\alpha+\beta)$. Also $\angle MPO=\angle PON=\alpha+\beta$, being the alternate angles. Thus the velocity of projection u makes an angle $\frac{1}{2}\pi-(\alpha+\beta)$ with PM.

The resolved part of the velocity at P along PM *i.e.*, parallel to OB $\quad=u\cos\{\frac{1}{2}\pi-(\alpha+\beta)\}=u\sin(\alpha+\beta)$

and the resolved part of the velocity at P along NP *i.e.*, perpendicular to OB $\quad=u\cos(\alpha+\beta)$.

The resolved parts of the acceleration g due to gravity along and perpendicular to OB are $g\sin\beta$ and $g\cos\beta$ as shown in the figure.

Let t be the time of flight from P to Q. Since the particle strikes the inclined plane OB at right angles at Q, therefore the velocity of the particle at Q along OB is zero. So considering the motion of the particle from P to Q parallel to OB and using



the formula $v=u+ft$, we have

$$0=u\sin(\alpha+\beta)-g\sin\beta\, t$$

or
$$t=\frac{u\sin(\alpha+\beta)}{g\sin\beta}. \qquad \ldots(1)$$

Again the displacement from P to Q perpendicular to OB is $PN=a\sin(\alpha+\beta)$, in the downward direction. So considering the motion from P to Q perpendicular to OB and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$-a\sin(\alpha+\beta)=u\cos(\alpha+\beta).t-\tfrac{1}{2}g\cos\beta.t^2$$

$$=t\{u\cos(\alpha+\beta)-\tfrac{1}{2}g\cos\beta.t\}$$

$$=\frac{u\sin(\alpha+\beta)}{g\sin\beta}\left\{u\cos(\alpha+\beta)-\tfrac{1}{2}g\cos\beta.\ \frac{u\sin(\alpha+\beta)}{g\sin\beta}\right\}$$

[substituting for t from (1)]

$$=\frac{u^2\sin(\alpha+\beta)}{2g\sin^2\beta}\left\{2\cos(\alpha+\beta)\sin\beta-\sin(\alpha+\beta)\cos\beta\right\}$$

i.e.,

$$a=\frac{u^2}{2g\sin^2\beta}.\left\{\sin(\alpha+\beta)\cos\beta-\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\beta)\sin\beta\right\}$$

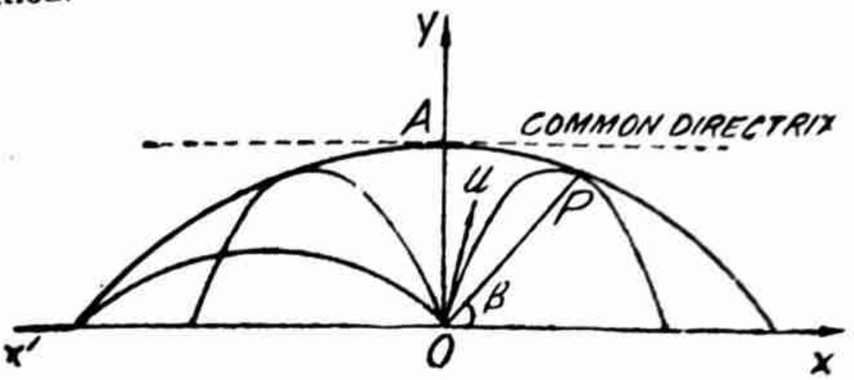
$$=\frac{u^2}{2g\sin^2\beta}\left[\sin\{(\alpha+\beta)-\beta\}-\sin\beta\cos(\alpha+\beta)\right]$$

$$=\frac{u^2}{2g\sin^2\beta}\left[\sin\alpha-\sin\beta\cos(\alpha+\beta)\right].$$

$$\therefore\ u^2=\frac{2ag\sin^2\beta}{[\sin\alpha-\sin\beta\cos(\alpha+\beta)]}$$

or $$u=\sin\beta\left\{\frac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right\}^{1/2}.$$

§ 11. Envelope of the trajectories with the same velocity of projection. **[Gorakhpur 1978]**



Let u be the given velocity of projection and O the given point of projection. Take the horizontal and vertical lines OX and OY lying in the plane of motion as the co-ordinate axes. The equation of the family of trajectories is

$$y=x\tan\alpha-\tfrac{1}{2}g\,\frac{x^2\sec^2\alpha}{u^2}, \qquad \ldots(1)$$

where the angle of projection α is the variable parameter.

To find the envelope of the family of curves (1), differentiating (1) partially w.r.t. the parameter 'α', we get

$$0=x\sec^2\alpha-\tfrac{1}{2}g\,\frac{x^2}{u^2}\cdot 2\sec^2\alpha\tan\alpha. \qquad \ldots(2)$$

Eliminating α between (1) and (2), we get the envelope of (1).

Since $\sec^2\alpha\neq 0$, therefore from (2), $\tan\alpha=u^2/gx$.

Putting $\tan\alpha=u^2/gx$ in (1), we have

$$y=x\,\frac{u^2}{gx}-\tfrac{1}{2}\,\frac{gx^2}{u^2}\left(1+\frac{u^4}{g^2x^2}\right)\quad[\because\ \sec^2\alpha=1+\tan^2\alpha]$$

$$=\frac{u^2}{g}-\tfrac{1}{2}\,\frac{gx^2}{u^2}-\frac{u^2}{2g}=\frac{u^2}{2g}-\frac{gx^2}{2u^2}$$

or $$\frac{gx^2}{2u^2}=-y+\frac{u^2}{2g}=-\left(y-\frac{u^2}{2g}\right)$$

or $$x^2=-\frac{2u^2}{g}\left(y-\frac{u^2}{2g}\right), \qquad \ldots(3)$$

which is the equation of the envelope of the family of trajectories (1).

If h be the height of the common directrix of all the trajectories above the point of projection O, we have $h=u^2/2g$ *i.e.*, $u^2=2gh$. Hecce the equation (3) of the envelope becomes

$$\mathbf{x^2=4h\,(y-h)}. \qquad \ldots(4)$$

The equation (4) represents a parabola whose axis is the line $x=0$ *i.e.*, the vertical line through the point of projection and whose vertex is the point $A\ (0, h)$ *i.e.*, the point where the common directrix of all the trajectories meets the vertical through O. The length of the latus rectum of the parabola (4) is $4h$ and so the focus of (4) is a point on OY whose depth below A is $\frac{1}{4}\,(4h)$ *i.e.*, h. Thus the point of projection O is the focus of the enveloping parabola (4).

Geometrical interpretation of the enveloping parabola. The enveloping parabola touches all the trajectories externally and thus it encloses all the possible trajectories of the particles projected from O with velocity u. No projectile can go beyond the enveloping parabola and so this parabola is an outer boundary to the region that can be reached by projectiles starting from O with the given velocity u. *Hence in order to find the* **maximum range** *in a given direction, we have to simply find the point where this direction meets the enveloping parabola.* Thus if a line making an angle β with the horizontal meets the enveloping parabola at the point P, then OP is the maximum range in the direction OP for a velocity of projection u at O. Again if we want to find the maximum horizontal range at a height k above O, then we should find the value of x from the equation (4) after putting $y=k$ in it.

Illustrative Examples

Ex. 73. *A particle is projected under gravity with velocity $\sqrt{(2ga)}$ from a point at a height h above the level plane. Show that the angle of projection θ for the maximum range on the plane is given by $\tan^2\theta=a/(a+h)$, and that the maximum range is*

$$2\sqrt{[a\,(a+h)]}.$$ **[Allahabad 1976]**

Sol. Take the point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of motion as the co-ordinate axes. Since the velocity of projection is $\sqrt{(2ga)}$, therefore the height of the common directrix of all the trajectories above the point of projection O is $[\sqrt{(2ga)}]^2/2g$ *i.e.*, a. Hence the equation of the enveloping parabola of all the trajectories is

$$x^2=-4a\,(y-a). \qquad \ldots(1)$$

To find the maximum range on the horizontal, we should find the value of x for the point where the envelope (1) meets the horizontal line $y=-h$. Putting $y=-h$ in (1), we get

$$x^2=-4a\,(-h-a)=4a\,(a+h)$$

or $\qquad x=2\sqrt{[a(a+h)]}$, which gives the maximum horizontal distance of the point of striking the plane from the point of projection O.

The co-ordinates of this point of striking are $(2\sqrt{[a(a+h)]}, -h)$. Let θ be the angle of projection which gives this maximum range on the plane. Then the point $(2\sqrt{[a(a+h)]}, -h)$ lies on the curve

$$y=x\tan\theta-\frac{1}{2}\frac{gx^2}{u^2\cos^2\theta}, \text{ where } u^2=2ga$$

i.e., on the curve $y=x\tan\theta-\dfrac{x^2(1+\tan^2\theta)}{4a}$.

$$\therefore\quad -h=2\sqrt{[a(a+h)]}\tan\theta-\frac{4a(a+h)}{4a}(1+\tan^2\theta)$$

or $$(a+h)\tan^2\theta-2\sqrt{[a(a+h)]}\tan\theta+a=0$$

or $$\tan^2\theta-2\sqrt{\left(\frac{a}{a+h}\right)}\tan\theta+\frac{a}{a+h}=0$$

or $$\left[\tan\theta-\sqrt{\left(\frac{a}{a+h}\right)}\right]^2=0$$

or $$\tan\theta=\sqrt{\left(\frac{a}{a+h}\right)} \quad\text{or}\quad \tan^2\theta=\frac{a}{a+h}.$$

Ex. 74. *A rocket fired vertically upwards bursts at a height a above the point of projection. Show that the fragments on reaching the ground lie within a circle of radius $(u/g)\sqrt{(u^2+2ag)}$, assuming that the fragments start with the same velocity u.*

Sol. Let O be the point at a height a above the ground where the rocket bursts. Referred to O as the origin, the equation of the envelope of the trajectories of the fragments is

$$x^2=-\frac{2u^2}{g}\left(y-\frac{u^2}{2g}\right). \qquad \ldots(1)$$

The ground is at a depth a below O. So on the ground, we have $\qquad y=-a.$

Putting $y=-a$ in (1), we have

$$x^2=-\frac{2u^2}{g}\left(-a-\frac{u^2}{2g}\right)=\frac{u^2}{g^2}(u^2+2ag).$$

$\therefore\quad$ maximum range on the ground is $x=(u/g)\sqrt{(u^2+2ag)}$.

Hence all the fragments will fall within a circle of radius

$$(u/g)\sqrt{(u^2+2ag)}.$$

§ 12. Particles suffered to describe parabolic paths.

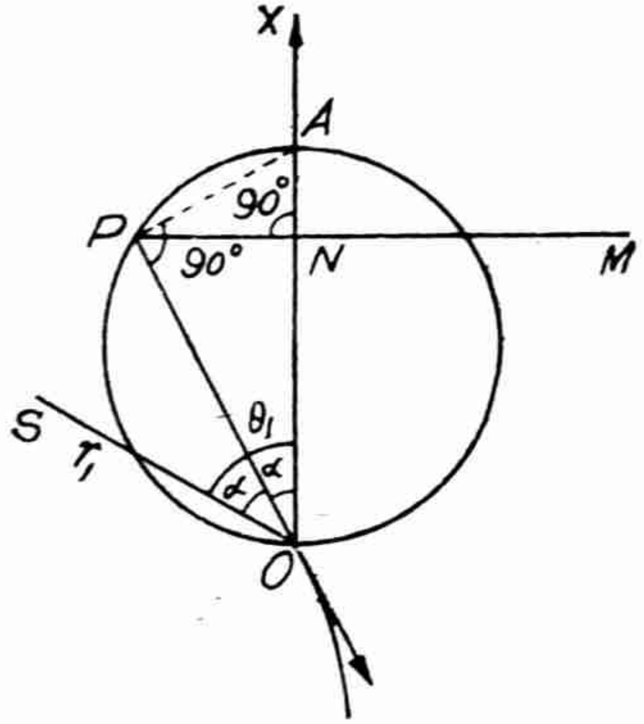
Now we shall discuss a few problems in which the particles are suffered to describe parabolic paths. In these problems we shall make use of the following two properties of parabolic trajectories :

(i) The velocity of a projectile at any point of its path is that due to a fall from the directrix to that point.

(ii) The tangent at any point of a parabola bisects the angle between the focal distance of the point and the perpendicular from that point to the directrix.

Illustrative Examples

Ex. 75. *Particles slide down the chords of a vertical circle terminating in the lowest point; show that the locus of the foci of the paths subsequently described is a cardioid.* **[Gorakhpur 1977, 78]**



Sol. Let O be the lowest point of the circle and OA the vertical diameter. Let PO be any chord of the circle terminating at O. Let $\angle POA=\alpha$; then α varies as the chord PO varies.

A particle slides from P along the chord PO. The velocity of the particle on reaching the point O
$=\sqrt{(2g \times \text{vertical depth of } O \text{ below } P)}$, and its direction is along the line PO.

After leaving the chord at O the particle moves freely under gravity and so its motion is that of a projectile. Thus after O the particle describes a parabolic path and the line PO is along the tangent to this parabola at the point O. Since the point O is on the parabolic trajectory described after O, therefore the velocity of the particle at O is that due to a fall from the directrix of the parabola to the point O. But as already mentioned the velocity of the particle at $O=\sqrt{(2g\times ON)}$, where N is the point where the horizontal line through O meets the diameter OA. Hence the horizontal line PM is the directrix of the parabola described after O.

Let S be the focus of the parabola described after O. Since the tangent PO to the parabola at the point O bisects the angle between the focal distance OS and the perpendicular ON from O to the directrix, therefore $\angle NOP=\angle POS$. Also $ON=OS$, by the definition of a parabola.

Let (r_1, θ_1) be the co-ordinates of the focus S referred to O as pole and the line OA as the initial line. Then $\theta_1=\angle SOA=2\alpha$.

Also $r_1=OS=ON=OP\cos\alpha$.

But from the right angled triangle APO, we have

$OP=OA\cos\alpha=2a\cos\alpha$, where a is the radius of the circle.

$$\therefore \quad r_1=2a\cos\alpha\cos\alpha=2a\cos^2\alpha=a\,(1+\cos 2\alpha)$$
$$=a\,(1+\cos\theta_1). \qquad [\because \quad 2\alpha=\theta_1]$$

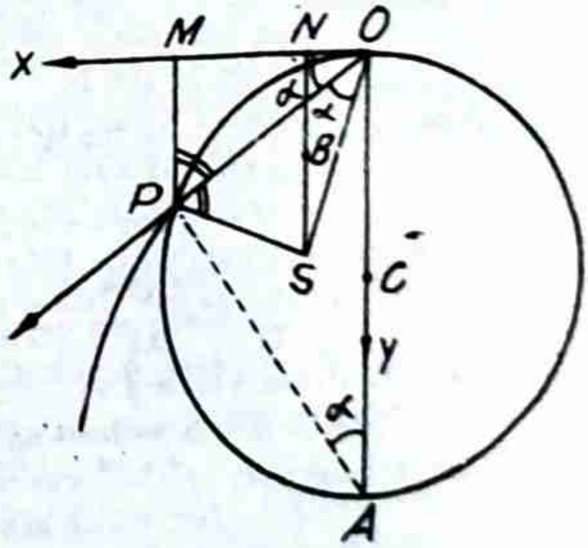
Generalising (r_1, θ_1), the locus of the point S is the curve

$$r=a\,(1+\cos\theta), \text{ which is a cardioid.}$$

Ex. 76. *Particles slide down along chords of a vertical circle from rest at the highest point. If they are allowed to move freely after leaving the chords, obtain the locus of the foci, and locus of the vertices of the subsequent parabolic paths.*

Sol. Let O be the highest point of the circle, OA the vertical diameter and OX the horizontal line through O. Let OP be any chord of the circle drawn from O. Let $\angle POX=\alpha$; then α varies as the chord OP varies.

A particle slides from O along the chord OP. After leaving the chord at P, the particle describes a parabolic path. The direction of velocity at P *i.e.*, the line OP is tangent to this parabola at the point P.



While sliding along the chord OP, the velocity of the particle on reaching the point $P=\sqrt{(2g\times \text{vertical depth of } P \text{ below } O)}=\sqrt{(2g\times PM)}$. Also the velocity in the parabolic path at the point $P=\sqrt{(2g\times \text{vertical depth of } P \text{ below the directrix of the parabola})}$. Hence the horizontal line OX is the directrix of the parabola described after P.

Let S be the focus of the parabola described after P. Then by geometrical properties of a parabola, $PS=PM$ and $\angle OPS=\angle OPM$. Thus the triangles OPS and OPM are congruent, so that $OS=OM$ and $\angle SOP=\angle MOP=\alpha$.

Locus of the focus. Referred to the point O as the pole and the line OX as the initial line, let, (r_1, θ_1) be the polar co-ordinates of the focus S. Then $\theta_1=\angle SOX=2\alpha$.

Also $r_1=OS=OM=OP\cos\alpha$.

But from the right angled triangle OPA, we have

$$OP=OA\sin\angle OAP=2a\sin\alpha,$$

where a is the radius of the circle.

$$\therefore\quad r_1=2a\sin\alpha\cos\alpha=a\sin 2\alpha=a\sin\theta_1.$$

Generalising (r_1, θ_1), the locus of the focus S is the curve $r=a\sin\theta$ which is a circle passing through the pole O and diameter of length a. Also the diameter of this circle through O makes an angle $\pi/2$ with OX. Thus OC is the diameter of this circle where C is the centre of the given circle.

Locus of the vertex. Draw SN perpendicular from the focus S to the directrix OX. Then the middle point B of NS is the vertex of the parabola. Referred to OX as x-axis and OA as y-axis let the cartesian co-ordinates of the point B be (h, k).

We have $h=$the perpendicular distance of B from OA

$$\begin{aligned}
&=ON=OS\cos 2\alpha=OM\cos 2\alpha && [\because\ OS=OM]\\
&=OP\cos\alpha\cos 2\alpha && [\because\ OM=OP\cos\alpha]\\
&=OA\sin\alpha\cos\alpha\cos 2\alpha && [\because\ OP=OA\sin\alpha]\\
&=2a\sin\alpha\cos\alpha\cos 2\alpha,\text{ where } OA=2a\\
&=a\sin 2\alpha\cos 2\alpha. && \ldots(1)
\end{aligned}$$

Also

$$\begin{aligned}
k&=BN=\tfrac{1}{2}SN=\tfrac{1}{2}(OS\sin 2\alpha)=\tfrac{1}{2}OM\sin 2\alpha\\
&=\tfrac{1}{2}OP\cos\alpha\sin 2\alpha=\tfrac{1}{2}.2a\sin\alpha\cos\alpha\sin 2\alpha\\
&=\tfrac{1}{2}a\sin 2\alpha\sin 2\alpha=\tfrac{1}{2}a\sin^2 2\alpha. && \ldots(2)
\end{aligned}$$

Now we shall eliminate α between (1) and (2). From (1), we have $\quad h^2=a^2\sin^2 2\alpha\cos^2 2\alpha=a^2\sin^2 2\alpha\,(1-\sin^2 2\alpha)$

$$=a^2.\frac{2k}{a}\left(1-\frac{2k}{a}\right),\text{ substituting for } \sin^2 2\alpha \text{ from (2).}$$

$$\therefore\quad h^2=2k\,(a-2k)=2ak-4k^2\quad\text{or}\quad h^2+4k^2=2ak.$$

Generalising (h, k), the locus of the vertex B is

$$x^2+4y^2=2ay,\text{ which is an ellipse.}$$