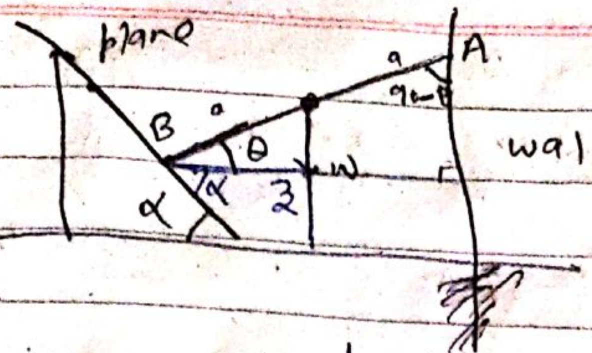
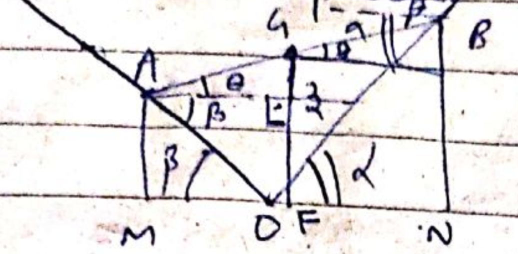
IFoS 2010

Q-57 (5C)



This is a special case of



$\angle ABO = \alpha - \theta$

$\angle AOB = \pi - (\beta+\theta) - (\alpha-\theta)$
$= \pi - (\alpha+\beta)$.

Let rod AB rests on 2 inclined planes with angle α & β with horizontal. The centre of gravity of rod AB is at height z from fixed horizontal line MN.

i.e. $z = \cancel{AG\sin\theta}\ \overset{GF}{}\ GL + \cancel{AM}\ \frac{1}{2}(AM + BN)$

$$z = \frac{1}{2}(AO\sin\beta + BO\sin\alpha)$$

In ΔAOB,

$$\frac{AO}{\sin(\alpha-\theta)} = \frac{BO}{\sin(\beta+\theta)} = \frac{AB}{\sin(\pi-(\alpha+\beta))}$$

i.e. $$\frac{AO}{\sin(\alpha-\theta)} = \frac{2a}{\sin(\alpha+\beta)} = \frac{BO}{\sin(\beta+\theta)}$$

i.e. $$z = \frac{1}{2}\left[\frac{2a}{\sin(\alpha+\beta)}\sin(\alpha-\theta)\sin\beta + \frac{2a}{\sin(\alpha+\beta)}\sin(\beta+\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta(\sin\alpha\cos\beta - \sin\beta\cos\alpha) + \cos\theta(\sin\alpha\sin\beta + \sin\alpha\sin\beta)\right]$$

$$z = \frac{a}{\sin(\alpha+\beta)}\left(\sin\theta(\sin(\alpha-\beta)) + 2\sin\alpha\sin\beta\cos\theta\right)$$

$$\frac{dz}{d\theta} = \frac{a}{\sin(\alpha+\beta)}\left[\cos\theta\sin(\alpha-\beta) - 2\sin\alpha\sin\beta\sin\theta\right]$$

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}\left[-\sin\theta\sin(\alpha-\beta) - 2\sin\alpha\sin\beta\cos\theta\right]$$

$$= -\frac{2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[1 + \frac{\sin\theta\sin\alpha\cos\beta - \sin\theta\cos\alpha\sin\beta}{2\sin\alpha\sin\beta\cos\theta}\right]$$

$$= -\frac{2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[1 + \frac{\tan\theta\cot\beta - \tan\theta\cot\alpha}{2}\right]$$

$$= -\frac{2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left(1 + \frac{1}{2}\tan\theta(\cot\beta - \cot\alpha)\right) \quad (*)$$

For equilibrium $\frac{dz}{d\theta} = 0 \Rightarrow$

$$\cos\theta\sin(\alpha-\beta) = 2\sin\alpha\sin\beta\sin\theta$$

$$\Rightarrow \tan\theta = \frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{2\sin\alpha\sin\beta}$$

$$= \frac{1}{2}(\cot\beta - \cot\alpha)$$

ROUGH

so (1) becomes,

$$\frac{d^2z}{d\theta^2} = \frac{-2\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[1+\frac{1}{2}\tan\theta \times 2\tan\theta\right]$$

$$= -\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\cos\theta \times \sec^2\theta$$

$$= -\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)} \quad = (1+\tan^2\theta) \times \frac{-2\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}$$

$\frac{d^2z}{d\theta^2} < 0$ because all angles α, β, θ are acute and $\alpha+\beta < \pi$

so z is maximum $\Rightarrow$ unstable.

But if $\beta = 90^\circ$ then,

$$\cancel{\frac{d^2z}{d\theta^2} = (1+\tan^2\theta) \times \frac{-2\sin\alpha\cos\theta}{\cos\alpha}}$$

$$\cancel{= -2\frac{\sin\alpha}{\cos\alpha} \times \cos\theta \times (1+\tan^2\theta)}$$

~~< 0 as α is acute, θ is acute~~

~~so equilibrium is unstable~~

and position of equilibrium is

$$\frac{dz}{d\theta} = 0 \Rightarrow \tan\theta = \frac{1}{2}(\cot 90^\circ - \cot\alpha)$$

ROUGH

$$\Rightarrow \tan\theta = \frac{1}{2}(-\cot\alpha)$$

$\Rightarrow \tan\theta$ is $-ve$ $\Rightarrow$ θ lies in 2^{nd} quadrant. $\Rightarrow \theta$ is obtuse

so $\frac{d^2 z}{d\theta^2} = (1+\tan^2\theta)x - 2\frac{\sin\alpha}{\cos\alpha}\cos\theta$

α is acute so $\sin\alpha, \cos\alpha > 0$

but θ is obtuse so,

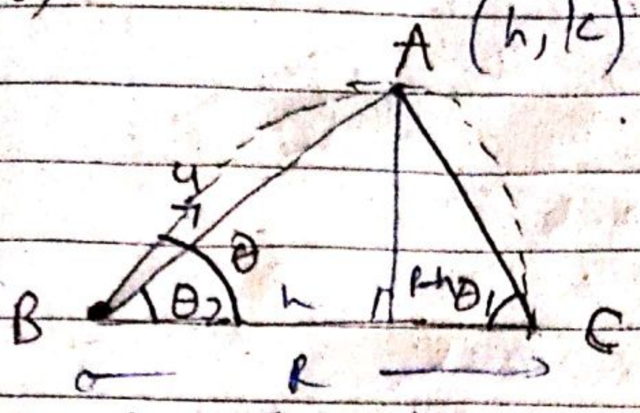
$\cos\theta < 0 \Rightarrow$

$\frac{d^2 z}{d\theta^2} > 0 \Rightarrow z$ is minimum

so equilibrium is stable

58 (5d)

(d)



Prove that —

$$\boxed{\tan\theta = \tan\theta_1 + \tan\theta_2}$$

Let particle projected from B with velocity u at angle θ grazing over vertex A with coordinates (h, k) landing at C other end of base

Let BC = R = Horizontal range

$$R = \frac{2u^2 \sin\theta \cos\theta}{g}$$

$$\boxed{\tan\theta_2 = \frac{k}{h}};$$

using $y = x\tan\alpha - \frac{1}{2} g \frac{x^2}{u^2\cos^2\alpha}$, we get

$$k = h\tan\theta - \frac{1}{2} g \frac{h^2}{u^2\cos^2\theta}$$

$$= h\tan\theta\left(1 - \frac{1}{2} \frac{g h^2}{u^2\cos^2\theta} \times \frac{\cos\theta}{\sin\theta}\right)$$

$$= h\tan\theta\left(1 - \frac{1 g h}{2u^2\sin\theta\cos\theta}\right)$$

$$\Rightarrow \frac{k}{h} = \tan\theta\left(1 - \frac{gh}{2u^2\sin\theta\cos\theta}\right) = \tan\theta\left(1 - \frac{h}{R}\right)$$

ROUGH

$$\Rightarrow \tan\theta_2 = \frac{K}{h} \Rightarrow \frac{R-h}{R}\tan\theta \quad \text{--- (1)}$$

Also $$\tan\theta_1 = \frac{K}{R-h} \quad \text{--- (2)}$$

using (1) $$\tan\theta = \frac{R}{R-h}\tan\theta_2$$

$$= \frac{R-h+h}{R-h}\tan\theta_2 = \left(1+\frac{h}{R-h}\right)\tan\theta_2$$

$$= \tan\theta_2 + \frac{h}{R-h}\tan\theta_2$$

$$= \tan\theta_2 + \frac{h}{R-h}\times\frac{K}{h}$$

$$= \tan\theta_2 + \frac{K}{R-h} = \tan\theta_2 + \tan\theta_1 \quad (\text{using } 2)$$

$$\Rightarrow \tan\theta = \tan\theta_1 + \tan\theta_2$$

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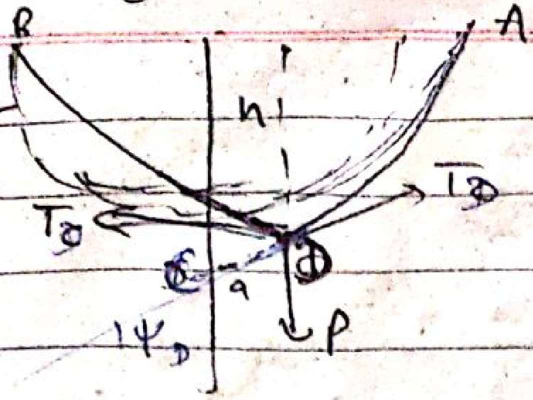
60 - 7(a)

60 (7a)



initially AB hangs under gravity.
But when load P is attached to middle point D such that

$AD = BD = l$;

Let T_D be tension at D along tangent at D to AD and BD.

Let C be the lowest point of catenary such that $CD = a$.

sag of catenary $= h$.

Then let ψ_D is the angle that T_D at D makes with horizontal

Then $2T_D \sin\psi_D = P$

Also, $T_D \sin\psi_D = ws$ $\left(\begin{array}{l} T_x = wc \\ T_y = ws \end{array}\right)$

$\boxed{w = \frac{W}{2l}}$ $\quad s = CD = a$ so,

$$T_D \sin\psi_D = \frac{Wa}{2l}$$

$$\Rightarrow \frac{P}{2} = \frac{Wa}{2l} \Rightarrow \boxed{a = \frac{P}{W} l}$$

Then at A, let y_A be the height, s_A = arc length and at D — y_D & s_D.

$\boxed{s_A = l + a}$, $(s_D = a)$

$y_D + h = y_A \Rightarrow \boxed{y_D = y_A - h}$

Also, $c^2+s^2=y^2$ given

$c^2+s_A^2=y_A^2$; $c^2+s_D^2=y_D^2$

$\Rightarrow y_A^2-y_D^2=s_A^2-s_D^2$

$\Rightarrow \quad =(l+a)^2-a^2$

$y_A^2-(y_A-h)^2=l^2+\cancel{a^2}+2al-\cancel{a^2}$

$\cancel{y_A^2}-\cancel{y_A^2}-h^2+2hy_A=l^2+2al$

$\Rightarrow \boxed{y_A=\frac{l^2+h^2+2al}{2h}}$

Also terminal tension at A or B

is $T=wy_A=\frac{W}{2l}\times\frac{l^2+h^2+2a\ldots}{2h}$

$=\frac{W}{4lh}\left[l^2+h^2+2\times\frac{P}{W}l^2\right]$

$=\frac{1}{2}\left[\frac{Pl}{h}+w\frac{l^2+h^2}{2lh}\right]$

61 (7b)

Q- $P =$ central acc $= \frac{\mu}{(\text{distance})^2} = \frac{\mu}{r^2}$

Projected with velocity V at distance R

TST path is rectangular hyperbola if angle of projection is $\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \frac{2\mu}{R}\right)^{1/2}}\right]$

We know that the path described is a hyperbola under central acc. $\frac{\mu}{(\text{distance})^2}$ if velocity v of particle at distance r from centre of force is given by →

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

Given $v = V$, $r = R$ initially →

$$V^2 = \frac{2\mu}{R} + \frac{\mu}{a}$$

$$\Rightarrow \boxed{\frac{\mu}{a} = V^2 - \frac{2\mu}{R}}$$

If α be the angle of projection then by $p = r\sin\phi$ becomes for $r = R$, $\phi = \alpha$ ⇒

$p = R\sin\alpha$

$h = \sqrt{\mu l} = \sqrt{\mu\frac{b^2}{a}} = \sqrt{\mu a}$ for rectangular hyperbola $b = a$

(b) If the particle describes a hyperbola under the central acceleration $\frac{\mu}{(\text{distance})^2}$, then the velocities V of the particle at distance r from centre of force is given by,

$$V^2 = \mu\left(\frac{2}{r}+\frac{1}{a}\right) \qquad \text{...(1)}$$

where $2a$ = transverse axis

As particle is projected with velocity V at distance R, then from (1), we have,

$$V^2 = \mu\left(\frac{2}{R}+\frac{1}{a}\right) \quad \text{or} \quad \frac{\mu}{a} = V^2 - \frac{2\mu}{R} \qquad \text{...(2)}$$

If α is required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$,

we have $h = Vp = VR\sin\alpha$. ... (3)

[$\because$ $p = r\sin\phi$ & initially $r = R$, $\phi = \alpha$]

Also, $h = \sqrt{\mu l} = \sqrt{\mu . b^2/a} = \sqrt{\mu a}$ ... (4)

[$b = a$ for rectangular hyperbola]

from (3) and (4) we have,

$$VR\sin\alpha = \sqrt{\mu a}$$

$$\Rightarrow \sin\alpha = \frac{\sqrt{\mu a}}{VR} = \frac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \frac{\mu}{VR\sqrt{\mu a}}$$

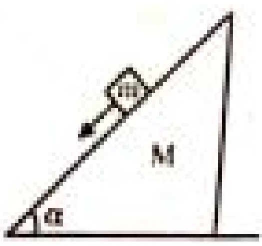
from (2)

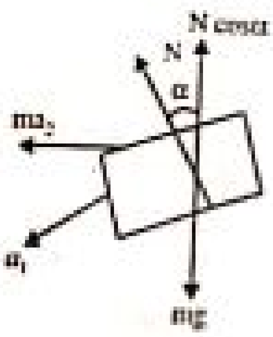
$$\Rightarrow \qquad \sin\alpha = \frac{\mu}{VR\sqrt{V^2 - \frac{2\mu}{R}}}$$

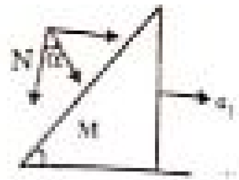
$$\Rightarrow \qquad \alpha = \sin^{-1}\left\{\frac{\mu}{VR\sqrt{V^2 - \frac{2\mu}{R}}}\right\}$$

which is required angle of projection.

(c) Drawing the free body diagram







Let a_1 and a_2 be the acceleration of m and M respectively.

Then from free body diagram.

$$mg - N\cos\alpha = ma_1\sin\alpha$$

$$ma_2 + N\sin\alpha = ma_1\cos\alpha \qquad \text{...(1)}$$

Also, $\quad N\sin\alpha = Ma_2$ ...(2)

(1) $\times\cos\alpha$ − (2) $\times\sin\alpha$ we get, ...(3)

$mg\cos\alpha - N - ma_2\sin\alpha = 0$

putting N from (3), we get

$$\Rightarrow mg\cos\alpha - \frac{Ma_2}{\sin\alpha} - ma_2\sin\alpha = 0$$

$$\Rightarrow a_2\,(M + m\sin^2\alpha) = mg\sin\alpha\cos\alpha$$

$$\therefore \; a_2 = \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

8.(a): Field F will be conservative then $\bar{\nabla}\times\bar{F} = 0$

$$i.e. \qquad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

$$\text{Now } \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}.0 - \hat{j}.(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

i.e. $\bar{\nabla}\times\bar{F} = 0 \;\Rightarrow\; \bar{F}$ is conservative field

Hence, $\bar{F}$ can be written as $\bar{F} = \bar{\nabla}U$

where U is scalar function.

Now, $\frac{\partial U}{\partial x} = 2xy + z^3 \Rightarrow U = x^2y + xz^3 + f_1(y, z)$

$\frac{\partial U}{\partial y} = x^2 \qquad \Rightarrow U = x^2y + f_2(x, z)$

$(3 \times 3) \times \frac{1}{2} \cdot 2 = 9 \cdot 3$

or $\quad F = 9$ grammes.

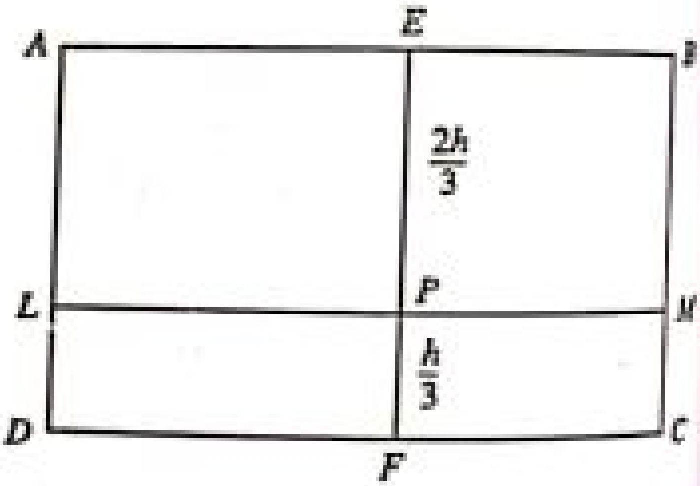
1/5/2020

Ex. 3. *Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which are in the ratio 4 : 5.*

Sol. LM is the horizontal line through P, the C.P. of rectangle $ABCD$ immersed in a liquid with the side AB in the surface.

Let $AB = a$ and $AD = h$.

Then $EP = \frac{2}{3} h$.



Now P = pressure on the area $ABCD$

= w .area of the rectangle $ABCD$. depth of its C.G. below the free surface

$= w \cdot ah \cdot \frac{1}{2} h = \frac{1}{2} wah^2,$

P_1 = pressure on the area $ALMB$

= w .area of the rectangle $ALMB$. depth of its C.G. below the free surface

$= w \cdot a \cdot \frac{2}{3} h \cdot \frac{1}{2}\left(\frac{2}{3} h\right) = \frac{2}{9} wah^2,$

P_2 = pressure on the area $LDCM = P - P_1 = wah^2\left(\frac{1}{2} - \frac{2}{9}\right)$

$= \frac{5}{18} wah^2.$

$\therefore \quad P_1 : P_2 = 4 : 5.$

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