Real Analysis

① Sequences, Series, Convergences

(1) Show $(\sqrt{n+1} - \sqrt{n})$ is convergent.

– As no fractional elements and $\frac{U_n}{U_{n+1}}$ not very useful

$$f_n = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}}$$

Let $\varepsilon > 0$ s.t

$$\left|\frac{1}{2\sqrt{n}} - 0\right| < \varepsilon \Rightarrow \frac{1}{4n} < \varepsilon^2 \Rightarrow \boxed{n > \frac{1}{4\varepsilon^2}}$$

Let $m > 0$ and $m > 1/4\varepsilon^2$

Then for $\varepsilon > 0$ $\exists\, m$ s.t

$$|f_n - 0| \leq \varepsilon \quad \forall\, n \geq m$$

$\Rightarrow f_n$ is cvgt and $\lim_{n\to\infty} f_n = 0$

(2) Show $\langle r^n \rangle$ is cvgt if $|r| < 1$.

Let $f_n = r^n$. As $|r| < 1 \Rightarrow r = \frac{1}{1+h}$ $h \geq 0$

$$(1+h)^n = 1 + nh + {}^nC_2 h^2 \ldots$$

$$r^n = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + {}^nC_2 h^2 + \ldots} < \frac{1}{1+nh}$$

$$|f_n - 0| = |r^n - 0| < \frac{1}{1+nh} \Rightarrow |f_n - 0| < \varepsilon \text{ if } \frac{1}{1+nh} < \varepsilon \Rightarrow$$

$$\boxed{n > \frac{(1/\varepsilon - 1)}{h}}$$

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(iii) Show $\lim_{n\to\infty} n^{1/n} = 1$.

Let $f_n = \boxed{n^{1/n} = 1 + h_n}$ $\quad h_n \geq 0$.

$n = (1+h_n)^n = 1 + nh + {}^nC_2 h^2 + \dots \geq {}^nC_2 h^2$

$n \geq \frac{n(n-1)}{2} h^2 \Rightarrow h_n^2 < \frac{2}{n-1}$ $\quad (n>2)$.

$\lim_{n\to\infty} |f_n - 1| = \lim_{n\to\infty} |h_n| < \sqrt{\frac{2}{n-1}}$.

If $\sqrt{\frac{2}{n-1}} < \varepsilon \Rightarrow \frac{2}{\varepsilon^2} < n-1 \Rightarrow n > \frac{2}{\varepsilon^2} + 1$.

Let m be the int $> 2/\varepsilon^2 + 1$. Then,

$|f_n - 1| < \varepsilon \ \forall n \geq m. \Rightarrow \boxed{\lim n^{1/n} = 1}$

NOTE:

① For problems of $\sqrt{n!}$ $\equiv$ Use $\sqrt{n!} > \frac{n}{2}$.

(iv) Direct limits $\lim_{n\to\infty} f(n) = l. \Rightarrow \boxed{\text{Show } \lim_{n\to\infty} |f_n - l| = 0}$

• Show $\left\langle \frac{n^2+3n+5}{2n^2+5n+7} \right\rangle \to \frac{1}{2}$

$$\left| \frac{n^2+3n+5}{2n^2+5n+7} - \frac{1}{2} \right| = \lim_{n\to\infty} \left| \frac{n+3}{2(2n^2+5n+7)} \right| = \lim_{n\to\infty} \left| \frac{1/n + 3/n^2}{4 + \cdot} \right|$$

W.k.t $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ 0 Bans

(V). TYPE $\left[\frac{a+b+c+\cdots}{n}\right] \equiv f_n$

Cauchy's Theorem $\Rightarrow$ $\boxed{\lim_{n\to\infty} \frac{f_1+f_2+\cdots f_n}{n} = \lim_{n\to\infty} f_n}$

- $\lim_{n\to\infty} \left[\frac{1}{\sqrt{2n^2+1}} + \frac{1}{\sqrt{2n^2+2}} + \cdots \frac{1}{\sqrt{2n^2+n}}\right]$

Consider, $f_n = \frac{n}{\sqrt{2n^2+n}}$

$\lim_{n\to\infty} f_n = \frac{1/n}{\sqrt{2+1/n}} = \frac{1}{\sqrt{2}}$.

So, using Cauchy's Addition Theorem

given limit $\to \frac{1}{\sqrt{2}}$ Ans

(VI) $f_n > 0$, $\boxed{\lim f_n = \lim (f_1 \cdots f_n)^{1/n} = l.}$

- Corollary $\Rightarrow$ more useful. VVI helps remember

$\langle f_n \rangle$, $f_n > 0 \ \forall n \in N$

$\boxed{\lim \frac{f_{n+1}}{f_n} = l,\ l>0. \text{ Then } \lim_{n\to\infty} (f_n)^{1/n} = l.}$

- $\lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = e.$

$(f_n)^{1/n} = \left(\frac{n^n}{n!}\right)^{1/n} \Rightarrow$ Let $f_n = \frac{n^n}{n!}$

$\frac{f_{n+1}}{f_n} = \frac{(n+1)^{n+1}\ \cancel{n!}}{(n+1)!\ n^n} = \left(\frac{n+1}{n}\right)^n = e > 0.$

So, $\lim_{n\to\infty} (f_n)^{1/n} = e$ Proved

(VII) Comparison Tests for limits

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = l \qquad \begin{array}{l} |l|<1, \ \lim_{n\to\infty} f_n = 0 \\ l>1, \ \lim_{n\to\infty} f_n = \infty \end{array}$$

(IX) On a set of $\mathbb{R}$, Cauchy $\Leftrightarrow$ Convergence.
[Cauchy's Criterion of Convergence]

• Show $\sum 1/n$ is not convergent. ← series not sequence.

$n>m$, $|f_n - f_m| = \left|\frac{1}{m+1} + \cdots \frac{1}{n}\right| = \frac{1}{m+1} + \frac{1}{m+2} + \cdots \frac{1}{n}$

Put $2n = 2m \Rightarrow |f_{2m} - f_m| \geq \frac{m}{2m} = \frac{1}{2}$.

So, $|f_{2m} - f_m| > 1/2 \ \forall \ m$

So, $\langle f_n \rangle$ is not convergent

• Examine $f_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \frac{1}{n!}$

$n>m$,

$|f_n - f_m| = \frac{1}{(m+1)!} + \cdots \frac{1}{n!}$

Use $n! \geq 2^{n-1} \Rightarrow$

$$\frac{1}{(m+1)!} + \cdots \frac{1}{n!} \leq \frac{1}{2^m} + \cdots \frac{1}{2^{n-1}}$$

$$\leq \frac{1}{2^{m-1}} \frac{1-(1/2)^{n-m}}{\cancel{(1-1/2)}} = \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}}$$

$$\leq \frac{1}{2^{m-1}}$$

So, for $\varepsilon > 0$ $|f_n - f_m| < \varepsilon$ if $\frac{1}{2^{m-1}} < \varepsilon \Rightarrow m > \left[1 + \frac{\log(1/\varepsilon)}{\log 2}\right]$

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(*) f_n is sequence of +ve numbers

$f_n = \frac{1}{2}(f_{n-1} + f_{n-2}) \quad \forall n \geq 3$

Show $\langle f_n \rangle \to (f_1 + 2f_2)/3$

$\Rightarrow$ If $f_2 = f_1 \Rightarrow f_n = f_1 =$ constant $\Rightarrow$ obvious

If $f_2 \neq f_1 \Rightarrow |f_n - f_{n-1}| = \left|\frac{1}{2}(f_{n-1} + f_{n-2}) - f_{n-1}\right|$

$= \frac{1}{2}|f_{n-2} - f_{n-1}|$

So, $|f_n - f_{n-1}| = \frac{1}{2}|f_{n-2} - f_{n-1}|$

Continuing, $|f_n - f_{n-1}| = \frac{1}{2^{n-2}}|f_2 - f_1| \quad \forall n \geq 2.$

· Apply Cauchy's $\Rightarrow$

$|f_n - f_m| = |f_n - f_{n-1} + f_{n-1} - f_{n-2} + \dots - f_m|$

$\leq |f_n - f_{n-1}| + |f_{n-1} - f_{n-2}| + \dots |f_{m+1} - f_m|.$

$= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \dots \frac{1}{2^m} + \frac{1}{2^{m-1}}\right)|f_2 - f_1|.$

$= \frac{1}{2^{m-2}}\left(1 - \frac{1}{2^{n-m}}\right)|f_2 - f_1| < \frac{1}{2^{m-2}}|f_2 - f_1|.$

Let $\varepsilon > 0$, Then we can choose $m > 0$,

$\frac{1}{2^{m-2}}|f_2 - f_1| < \varepsilon \Rightarrow m > 2 + \frac{\log\left(\frac{|f_2 - f_1|}{\varepsilon}\right)}{\log 2}.$

So, we see f_n to be a Cauchy sequence $\Rightarrow$ Cvgt sequence.

Let $f_n \to l$

$f_3 = \frac{1}{2}(f_2 + f_1)$

$f_4 =$

$\vdots$

$f_n = \frac{1}{2}(f_{n-1} + f_{n-2})$

Adding $\Rightarrow$

$f_n + \frac{f_{n-1}}{2} = \frac{f_1}{2} + f_2.$

$\Rightarrow \frac{3l}{2} = \frac{f_1 + 2f_2}{2}$ Ans

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(x) Monotonic Sequences.

- Given a monotonic sequence,
  Show monotonic → Show bounded ⇒ Convergent.

- Show $\langle a_n \rangle$, $a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \frac{1}{n!}$ is convergent.
  Also $2 \leq \lim_{n\to\infty} (a_n) \leq 3$.

A. Monotonic

$$a_{n+1} - a_n = 1/(n+1)! > 0 \quad \forall n \in N$$

$\Rightarrow \langle a_n \rangle$ is monotonically increasing sequence

B. $a_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!}$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{1}{2^{n-1}} = 1 + \frac{1 - (1/2)^n}{1/2}$$

$$= 3 - (1/2)^{n-1}.$$

$a_n < 3 - (1/2)^{n-1} \Rightarrow a_n < 3.$

So, $\langle a_n \rangle$ is monotonically ↑ and bdd above and $\langle a_n \rangle$ converges to l.u.b which is ≤ 3.

$$\lim_{n\to\infty} a_n \leq 3$$

$a_n = 1 + 1 + \frac{1}{2!} + \cdots > 2 \Rightarrow \lim_{n\to\infty} a_n \geq 2$

So $\boxed{2 \leq \lim a_n \leq 3}$

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(*) $\langle f_n \rangle$, $f_n = (1 + 1/n)^n$ is convergent and $2 < \lim_{n\to\infty} (1+1/n)^n < 3$

$$f_n = 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots\cdots \frac{n!}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots \frac{1}{n!}(\,)\left(1 - \frac{n-1}{n}\right)$$

A.

$$f_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \cdots$$

Comparing termwise $\Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow 1 - \frac{1}{n+1} > 1 - \frac{1}{n} \cdots$

So $f_{n+1} > f_n$. $\Rightarrow$ Monotonic

B. $$f_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots$$

$$\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \frac{2}{n} = 3 - \frac{1}{2^{n-1}} < 3.$$

$f_n < 3$, $f_n > 2$ (obvious)

$2 \leq \lim_{n\to\infty} f_n \leq 3$ Bounded

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(X)

Subpart : Recursive sequences.

- $a_1 = 1, \quad a_{n+1} = \dfrac{4+3a_n}{3+2a_n}.$

A $a_2 = \frac{4}{3} \Rightarrow a_2 - a_1 > 0.$

Let $a_n - a_{n-1} > 0.$

$$a_{n+1} - a_n = \frac{4+3a_n}{3+2a_n} - \frac{4+3a_{n-1}}{3+2a_{n-1}}$$

$$= \frac{8a_{n-1} + 9a_n - 8a_n - 9a_{n-1}}{(3+2a_n)(3+2a_{n-1})}$$

$$= \frac{a_n - a_{n-1}}{(3+2a_n)(3+2a_{n-1})} > 0 \Rightarrow \text{Monoton}$$

B $a_{n+1} = \frac{3}{2} - \frac{1}{2(2a_n+3)} \Rightarrow a_{n+1} < \frac{3}{2}.$

So, $\langle a_n \rangle$ is bdd above

Hence, limit exists.

$$\lim_{n\to\infty} a_{n+1} = \frac{4+3a_n}{3+2a_n} \Rightarrow l(3+2l) = 4+3l$$

$$\Rightarrow l^2 = 2 \Rightarrow l = \sqrt{2} \text{ only } (+)$$

Others : (1) $f_1 = \sqrt{2} \quad f_{n+1} = \sqrt{2f_n} \rightarrow f_{n+1} > f_n \rightarrow 2f_{n+1} > 2f_n \Rightarrow \sqrt{2f_{n+1}} > \sqrt{2f}$

$\Rightarrow \boxed{f_{n+2} > f_{n+1}}$

$f_n < 2 \Rightarrow \sqrt{2f_n} < 2 \Rightarrow f_{n+1} < 2 \leftarrow$ bdd

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$* \; S_1 = a > 0 \;,\; S_{n+1} = \sqrt{\dfrac{ab^2 + S_n^2}{a+1}}$ $\quad (b > a)$

A $\quad S_2^2 - S_1^2 = \dfrac{ab^2 + \cancel{a^2} - a^3 - \cancel{a^2}}{a+1} = \dfrac{a(b^2 - a^2)}{a+1} > 0$

$S_2 > S_1$

Assume $\quad S_n > S_{n-1} \Rightarrow S_n^2 > S_{n-1}^2 \Rightarrow ab^2 + S_n^2 > ab^2 + S_{n-1}^2$

$\Rightarrow \sqrt{\dfrac{ab^2 + S_n^2}{a+1}} > \sqrt{\dfrac{ab^2 + S_{n-1}^2}{a+1}} \Rightarrow S_{n+1} > S_n$

So, S_n is monotonically increasing.

B $\quad$ Limit $\Rightarrow l^2(a+1) - \cancel{l^2} - ab^2 \Rightarrow l = \pm b \Rightarrow \underline{l = b}$

C Bounded $\quad S_1 < b \Rightarrow a < b$

$S_n < b \Rightarrow \sqrt{\dfrac{ab^2 + S_n^2}{a+1}} < \sqrt{\dfrac{ab^2 + b^2}{a+1}} \Rightarrow S_{n+1} < \sqrt{b^2}$

$\Rightarrow \underline{S_{n+1} < b}$

$* \; a_{n+1} = \sqrt{k + a_n} \quad k, a_1 > 0$. Prove $\langle a_n \rangle$ is $\uparrow$ or $\downarrow$

as $a_1 <, >$ +ve root of $x^2 = x + k$, and the limit is root.

A $\quad (a_{n+1})^2 - (a_n)^2 = a_n - a_{n-1}$. So, $a_n > a_{n-1} \Rightarrow a_{n+1} > a_n$

$a_n < a_{n-1} \Rightarrow a_{n+1} < a_n$.

So $\langle a_n \rangle$ is monotonic.

Increase or Decrease as $a_2 > a_1$, $a_2 < a_1$ $\quad \boxed{(x-\alpha)(x+\beta) = 0}$

$(\alpha, -\beta)$

$\leftarrow x^2 - x - k = 0 \Rightarrow \dfrac{1 \pm \sqrt{1+4k}}{2} \Rightarrow$ one root is +ve other is $\underline{-ve}$

$a_1 > 0 \Rightarrow$ +ve root $\Rightarrow \dfrac{1 + \sqrt{1+4k}}{2}$. $\quad \boxed{\alpha, \beta > 0}$

$x^2 - x - k = (x-\alpha)(x+\beta) = 0$

Let $a_1 > \alpha \Rightarrow (a_1 - \alpha)(a_1 + \beta) > 0$

$a_1^2 > a_1 + k \Rightarrow a_1^2 > a_2^2 \Rightarrow a_1 > a_2 \Leftrightarrow \langle a_n \rangle$ is $\downarrow$

Illy.

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- $a_1 = 3/2$, $a_{n+1} = 2 - (1/a_n)$ $\qquad a_2 = 4/3$, . . decreasing

$$a_n > 1 \Rightarrow -\frac{1}{a_n} > -1 \Rightarrow 2 - \frac{1}{a_n} > 2 - 1 \Rightarrow \boxed{a_{n+1} > 1}$$

$\sum \frac{1}{n^\lambda}$ is convergent for $\lambda > 1$

$\Rightarrow$. $S_n < S_{2n}$.

$$S_{2n} = \frac{1}{1^\lambda} + \frac{1}{2^\lambda} + \cdots \frac{1}{(2^n)^\lambda}.$$

$$S_{2^{n+1}-1} = \frac{1}{1^\lambda} + \left(\frac{1}{2^\lambda} + \frac{1}{3^\lambda}\right) + \left(\frac{1}{4^\lambda} + \frac{1}{5^\lambda} + \frac{1}{6^\lambda} + \frac{1}{7^\lambda}\right) + \left(\ldots \right)$$

$$+ \left(\frac{1}{(2^n)^\lambda} + \frac{1}{(2^n+1)^\lambda} + \cdots \frac{1}{(2^{n+1}-1)^\lambda}\right)$$

$$< \frac{1}{2^\lambda} + \frac{1}{2^\lambda} = \left(\frac{1}{2^{\lambda-1}}\right) \qquad < \frac{4}{4^\lambda} = \left(\frac{1}{2^{2(\lambda-1)}}\right). \qquad < \frac{2^{n+1} - 2^n}{(2^n)^\lambda} = \frac{1}{2^{n(\lambda-1)}}$$

$$S_{(2^{n+1}-1)} < \frac{1}{1^\lambda} + \frac{1}{2^{(\lambda-1)}} + \frac{1}{2^{2(\lambda-1)}} + \cdots \frac{1}{2^{n(\lambda-1)}}$$

$$< \frac{1\left[1 - \left(\frac{1}{2^{(\lambda-1)}}\right)^{n+1}\right]}{1 - \left(\frac{1}{2^{\lambda-1}}\right)} < \left(\frac{2^{\lambda-1}}{2^{\lambda-1} - 1}\right)$$

$$S_{2n} < S_{2^{n+1}-1} \Rightarrow S_n < S_{2n} < \left(\frac{2^{\lambda-1}}{2^{\lambda-1} - 1}\right).$$

S_n is bounded above

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2) Infinite Series (+ve)

(I) Using partial sums — Show bounded [For both cvgt, dvgt]

- $\sum_{1}^{\infty} \frac{1}{n(n+1)}$

$S_n = 1 - \frac{1}{n+1}$ $\quad \lim_{n\to\infty} S_n = 1$

$\langle S_n \rangle$ cvgs to 1 $\Rightarrow \sum U_n$ cvgt to 1.

- TS: Not converge (Show $\lim_{n\to\infty} U_n \neq 0$).

$\sum \sqrt{\frac{n}{2(n+1)}} \Rightarrow \lim_{n\to\infty} \sqrt{\frac{n}{2n+2}} = \frac{1}{\sqrt{2}} \neq 0.$

So, $\sum U_n$ is not cvgt.

$\boxed{\text{Thm: If } \sum U_n \text{ converges, then } U_n = 0 \text{ at } \infty}$

(II) Comparison Tests.

- $\sum r^n \qquad 0 \le r < 1$
- $\sum \frac{1}{n^\lambda} \qquad \lambda > 1$ [DON'T PROVE THIS IN EXAM]

– $\frac{U_n}{V_n} = l \neq 0 \Rightarrow U_n, V_n$ show similar behaviour.

[When comparing $\sum U_n$, $\sum V_n$]

$(1+x)^\alpha$ only when $|x| < 1$

- $(n^3+1)^{1/3} - n \Rightarrow n\left(\{(1+1/n^3)^{1/3} - 1\}\right) = n\left(1 + \frac{1}{3n^3} - \frac{1}{9n^6} + \cdots - 1\right)$

$= \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots$

Use $v = \frac{1}{n^2} \Rightarrow \frac{U_n}{V_n} = \frac{1}{3} \Rightarrow U_n$ is convergent

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- $1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdot \frac{1}{n^n} + \cdots = \Sigma U_n$

$n > 2$

$n^n > 2^n$

$\Rightarrow \frac{1}{n^n} < \frac{1}{2^n}$, $\frac{1}{2^n}$ is cvgt. So is $\underline{\Sigma U_n}$

- $\Sigma\left[(n^3+1)^{1/3} - n\right]$; $n\left\{(1+1/n^3)^{1/3} - 1\right\}$

$n\left\{\cancel{1} + \frac{1}{3}\frac{1}{n^3} - \frac{1}{3^2(n^3)^2} + + \cdots \cancel{-1}\right] = \frac{1}{3n^2} - \frac{1}{9n^5}$

Take $V_n = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{U_n}{V_n} = \frac{1}{3} \neq 0$.

So U_n is <u>cvgt</u>

A. TYPE II TESTS (Self ≡ No other series reqd)

- $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$ $\begin{cases} < 1 \Rightarrow \text{cvgt} \\ > 1 \Rightarrow \text{dvgt} \\ 1 \Leftrightarrow ?? \end{cases}$

- $\lim_{n\to\infty} (U_n)^{1/n} = l$ $\begin{cases} < 1 \Rightarrow \text{cvgt} \\ > 1 \Rightarrow \text{dvgt} \\ = 1 \Leftrightarrow ?? \end{cases}$

- $\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right]$ $\begin{cases} < 1 \Rightarrow \text{DIVGT} \\ > 1 \Rightarrow \text{cvgt} \\ = 1 \Leftrightarrow ?? \end{cases}$

$\Uparrow$

lly $\equiv \lim_{n\to\infty} \left[n \log \frac{U_n}{U_{n+1}}\right]$

$\lim_{n\to\infty} \left(n\left[\frac{U_n}{U_{n+1}} - 1\right] - 1\right) \log n$

$\lim_{n\to\infty} \left(n \log \frac{U_n}{U_{n+1}} - 1\right) \log n$

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Examples :

- $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \cdots \frac{x^n}{n(n+1)}$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \frac{x^n (n+2)(n+1)}{n(n+1)\, x^{n+1}} = \lim_{n\to\infty}\left[\frac{n+2}{n}\right]\frac{1}{x} = \frac{1}{x}.$$

Cvgt if $\frac{1}{x} > 1 \Rightarrow x < 1$; dvgt if $x > 1$.

$x = 1 \Rightarrow \langle U_n \rangle = \frac{1}{1\cdot 2} + \cdots \frac{1}{n(n+1)} + \cdots$

Comparison test with $v_n = \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{U_n}{v_n} = \frac{n^2}{n^2+n} = 1.$

As $\frac{1}{n^2}$ is cvgt, so U_n is cvgt.

$\boxed{\text{Cvgt } x \le 1, \text{ dvgt } x > 1}$

- $\sum \left(\frac{nx}{n+1}\right)^n$

$\lim_{n\to\infty} (U_n)^{1/n} = x.$ $\qquad x < 1 \to$ cvgt, > 1 dvgt.

$x = 1 \Rightarrow \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} \neq 0.$ $\boxed{\lim_{x\to\infty} U_n = 0}$ for cvgt

So, dvgt $\Rightarrow x < 1$ cvgt

(IMP) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\boxed{\log(1+x) = \cancel{1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}}}$

(IMP)

- $1 + \frac{x}{1!} + \cdots \frac{n^n x^n}{n!}$

$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} \Rightarrow \left(\frac{n}{n+1}\right)^n \times \frac{1}{x} = \frac{1}{(1+1/n)^n}\cdot\frac{1}{x} = \frac{1}{ex}.$ $\begin{cases} >1 & \text{cvgt} \\ <1 & \text{dvgt.} \end{cases}$

$(x = 1/e) \Rightarrow \frac{U_n}{U_{n+1}} = \cancel{\frac{n^n}{n!}} \left(\frac{n}{n+1}\right)^n \times e.$

-1 gives nothing
log only option.

$$\lim_{n\to\infty} n \log\left(\left(\frac{n}{n+1}\right)^n \times e\right) = n\left[n \log n - n\log(n+1) + 1\right]$$
$$= n\left[1 - n\left(\log(1+1/n)\right)\right] = n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right)\right]$$
$$= \frac{1}{2} < 1 \Rightarrow \text{dvgt at } \boxed{x = 1/e}$$

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$$\frac{(a+nx)^n}{n!}$$

$$\frac{U_n}{U_{n+1}} = \frac{(n+1)!}{n!}\,\frac{(a+nx)^n}{(a+nx+x)^{n+1}}$$

$$= \frac{(n+1)\cdot n^n\left[\frac{a}{n}+x\right]^n}{(n+1)^{n+1}\left[\frac{a}{n+1}+x\right]^{n+1}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \times \frac{x^n\left(1+\frac{a}{xn}\right)^n}{x^{n+1}\left(1+\frac{a}{x(n+1)}\right)^{n+1}}$$

$$= \frac{1}{ex} \times \frac{e^{a/x}}{e^{a/x}} = \frac{1}{ex}.$$

Cvgt $\Rightarrow x < 1/e$, dvgt if $x > 1/e$

$\boxed{x = 1/e}$

$$\frac{e\,(1+ea/n)^n}{(1+1/n)^n\left[1+ae/(n+1)\right]^{n+1}}$$

$$\log\frac{U_n}{U_{n+1}} = \log e + n\log\left(1+\frac{ea}{n}\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$= 1 + n\left[\frac{ea}{n} - \frac{e^2a^2}{n^2} + \cdots\right] - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) - (n+1)\left(\frac{ea}{n+1} - \frac{e^2a^2}{2(n+1)^2} + \cdots\right)$$

$$= \frac{1}{n}\left(\frac{1}{2} - \frac{e^2a^2}{2}\right) + \frac{e^2a^2}{2(n+1)} + \frac{1}{n^2} \cdots$$

$$n\log\frac{U_n}{U_{n+1}} = \frac{1}{2} \Rightarrow \underline{\text{dvgt}}$$

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(*) $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \dots$

$$U_n = \left[\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)}\right]^p$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \left(\frac{1+1/n}{1+1/2n}\right)^p = 1. \Rightarrow \text{Fails}$$

$$\log \frac{U_n}{U_{n+1}} = p\left[\log(1+1/n) - \log(1+1/2n)\right]$$

$$= p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{2n} + \frac{1}{8n^2} \dots\right]$$

$$= p\left[1/2n - 3/8n^2 + \dots\right]$$

$$\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}} = \boxed{\frac{p}{2}}$$

$p > 2$ cvgt
$p < 2$ dvgt.

$\boxed{p=2}$

$$n \log \frac{U_n}{U_{n+1}} = \overset{2}{p}\left[\frac{1}{2} - \frac{3}{8n} + \dots\right]$$

$$= \left(1 - \frac{3}{4n} + \frac{7}{12n^2} - \right)$$

$$\left(n \log \frac{U_n}{U_{n+1}} - 1\right) \log n \Rightarrow \left(-\frac{3}{4n} + \frac{7}{12n^2} - \right) \log n.$$

$$\lim_{n\to\infty} \left(-\frac{3}{4}\frac{\log n}{n} + \frac{7}{12}\frac{\log n}{n^2} - \dots\right)$$

$$\lim_{n\to\infty} = 0 < 1. \Rightarrow \text{dvgt}$$

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B Gauss' Test

- Expand $\frac{U_n}{U_{n+1}} = 1 + \frac{k}{n} + \frac{l}{n^2} + \cdots$

$k > 1 \rightarrow$ cvgt

$k \leq 1 \rightarrow$ divgt.

- $1 + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \cdots$

$\frac{U_n}{U_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p$ $\quad \lim \frac{U_n}{U_{n+1}} = 1 \rightarrow$ ratio fails

$= \left(1 + \frac{1}{n}\right)^p \left(1 + \frac{1}{2n}\right)^{-p} = \left\{1 + \frac{p}{n} + \frac{p(p-1)}{2} \times \frac{1}{n^2} + \cdots\right\}$

$\left\{1 - \frac{p}{2n} + \frac{p(p+1)}{2} \cdot \frac{1}{4n^2} - \cdots\right\}$

$= 1 + \frac{p}{2n} + O\left(\frac{1}{n^2}\right).$

By Gauss Test $\Rightarrow$ Cvgt if $p > 2$ <u>else divgt</u>

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C. Cauchy Integral Test (*) Also for n^p

$\Rightarrow \sum_{2}^{\infty} \frac{1}{n(\log n)^p}$.

$\frac{1}{n^p}$

$U_n = \frac{1}{n(\log n)^p} \Rightarrow U(x) = \frac{1}{x(\log x)^p}$

$U(x)$ is ① Non-negative ② Decreasing

③ Integrable Then $\sum_{1}^{\infty} U_n \sim \int U(x)\,dx$

$\Rightarrow$ Look at $\int_{2}^{t} U(x)\,dx$.

- $p = 1 \Rightarrow \int_{2}^{t} \frac{1}{x \log x}\,dx \Rightarrow \log\log t - \log\log 2$

$\lim_{t\to\infty} \log\log t - \log\log 2 = \infty \Rightarrow \int_{2}^{t} U(x)\,dx$ diverges.

- $p > 1 \Rightarrow \int_{2}^{t} \frac{1}{x(\log x)^p} = \left[-\frac{(\log x)^{1-p}}{p-1}\right]_{2}^{t}$. $\boxed{p-1>0}$

$\lim_{t\to\infty} \frac{1}{p-1}\left[\frac{1}{(\log 2)^{p-1}} - \cancel{\frac{1}{(\log t)^{p-1}}}^{\,0}\right]$

$\boxed{\frac{1}{p-1} \times \frac{1}{(\log 2)^{p-1}}}$ = finite $\Rightarrow$ cvgt.

- $0 < p < 1 \Rightarrow$ $\boxed{1-p>0}$

$\lim_{t\to\infty} \frac{1}{1-p}\left[(\log t)^{1-p} - (\log 2)^{1-p}\right] = \infty$.

So, cvgt $p>1$, dvgt elsewhere

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D. $\boxed{\text{Cauchy's Condensation Test}}$

$f(n) > 0$, Monotonic, $\downarrow$

Not $\Sigma f(n)$

$$\boxed{\Sigma f(n) \cong \Sigma a^n f(a^n)}$$ where $\boxed{a > 1}$

IMP

(Used in series with log terms + power on log.)

- $\frac{(\log 2)^2}{2^2} + \frac{(\log 3)^2}{3^2} + \cdot \frac{(\log n)^2}{n^2} + \cdots$

$$\left(\frac{\log n}{n}\right)^2 \Rightarrow > 0, \downarrow$$

Same as $\Rightarrow$ $\frac{\cancel{a^n}(\log a^n)^2}{\cancel{a^n} \cdot a^n} = \frac{n^2 (\log a)^2}{a^n} \equiv u_n.$

$$\frac{u_n}{u_{n+1}} = \frac{n^2 \cancel{(\log a)^2}\, a}{\cancel{a^n}\,(n+1)^2 \cancel{(\log a)^2}} = (a \cancel{\log a})^2 \left[\frac{1}{1+1/n}\right]^2$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = a > 1$$

So, $\Sigma f(n)$ <u>cvgt</u>

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- $\sum_{2}^{\infty} \frac{1}{n(\log n)^p}$ $\quad p > 1$ cvgt

$p \leq 1$ dvgt.

- Let $p > 0 \Rightarrow f(n)$ is +ve, monotonic, $\downarrow$.

Same as $\frac{a^n}{a^n(\log a^n)^p} = \frac{1}{n^p(\log a)^p}$.

$\frac{1}{n^p} \Rightarrow \left[\begin{array}{l} p > 1 \text{ cvgt} \\ p \leq 1 \text{ dvgt.} \end{array}\right\}$ Same for $f(n)$.

- Let $p \leq 0 \Rightarrow \frac{1}{n(\log n)^p} \geq \frac{1}{n} \quad \forall n > 1$

$\sum \frac{1}{n}$ is dvgt. So, for $p \leq 0 \Rightarrow$ <u>dvgt</u>

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③ Infinite Series (+ve, −ve both)

- Absolute Convergent $\Rightarrow$ Convergent.

$\langle U_n \rangle = U_1 - U_2 + U_3 - U_4 + \dots$

(1) Leibnitz Theorem (Proof)

(a) $U_n \geq 0$ (b) $U_{n+1} \leq U_n$ (c) $\lim_{n\to\infty} U_n = 0$

Then $\sum (-1)^{n-1} U_n$ is convergent

- $\sum \frac{(-1)^{n+1}}{\log(n+1)}$

$U_n = \frac{1}{\log(n+1)}$

$U_n \geq 0$

$\lim_{n\to\infty} U_n = 0$

$\log(n+2) > \log(n+1)$

$\Rightarrow \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$

$U_{n+1} < U_n.$

~~So $\sum U_n$ is~~

$\sum (-1)^{n+1} U_n$ is cvgt.

Abs Cvgt $\Rightarrow \sum_{2}^{\infty} \frac{1}{(\log n)} \rightarrow$ dvgt (How??)

* Leibnitz Proof

$\langle S_{2n} \rangle \rightarrow$ is monotonic, $\uparrow$ + S_{2n} is +ve + S_{2n} is bounded above

- $S_{2n} \equiv (U_1 - U_2) + (U_3 - U_4) + \dots - U_{2n}$
- Let $\lim_{n\to\infty} S_{2n} = l$

$\langle S_{2n+1} \rangle \rightarrow S_{2n} + U_{2n+1} \rightarrow \lim_{n\to\infty} = S_{2n} = l$

So, $\langle S_{2n} \rangle, \langle S_{2n+1} \rangle$ converge to same limit.

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- $\sum_{2}^{\infty} (-1)^k \frac{\log k}{k^2}$

$U_n \geq 0.$

$\lim_{n\to\infty} U_n = 0$

$U_n = \frac{\log(n+1)}{(n+1)^2}$

TS: $U_{n+1} < U_n \Rightarrow$ Show: $\boxed{\frac{\log x}{x^2} \text{ decreasing}}$

$\frac{1}{x^3} - \frac{2\log x}{x^3} \Rightarrow \frac{1-2\log x}{x^3} < 0 \ \forall x > e^{1/2}$

$\Rightarrow U(x)$ is decreasing

$\Rightarrow U_{n+1} < U_n.$

So, series is cvgt by Leibnitz Thm

- Show $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ converges iff $x \in [-1, 1]$

$|U_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$, $|U_{n+1}| = \left|\frac{x^{2n+1}}{2n+1}\right|$

$\frac{|U_n|}{|U_{n+1}|} = \frac{1}{x^2}$

$\sum |U_n|$ cvgt $\Rightarrow$ $1/x^2 > 1 \Rightarrow x^2 < 1 \Rightarrow |x| < 1$

dvgt $= |x| > 1$

At $|x| = 1 \rightarrow$ fails

$x = 1 \Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \Rightarrow$ cvgt

$x = -1 \Rightarrow -1 + \frac{1}{3} - \frac{1}{5}$ $\Rightarrow$ cvgt

Show by Leibnitz Test

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(II). If a series is absolutely convergent, then series of +ve and -ve terms only are also convergent

$\boxed{\text{If } \Sigma U_n \text{ – conditionally convergent, then series of +ve and -ve terms divergent}}$

(III) Dirichlet's Theorem

- $\langle U_n \rangle$ is +ve monotonically decreasing sequence with limit 0.
- $\sum^{\infty} a_n$ be an infinite series such that sequence of partial sums is bounded

Then $\Sigma a_n U_n$ is convergent.

(IV) Abel's Theorem

- $\langle U_n \rangle$ is +ve monotonically decreasing sequence
- Σa_n is a convergent series

Then $\Sigma a_n U_n$ is convergent.

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$\boxed{8}$ Test convergence $\sum \frac{(n+1)}{\log n}$.

$U_n = (n^3+1)^{1/3} - n$ $\qquad a_n = 1/\log n$

$$U_n = n\left[(1+1/n^3)^{1/3} - 1\right] = n\left[\left(\cancel{1} + \frac{1}{3n^3} - \frac{1}{9n^6} + \cdots\right) - \cancel{1}\right]$$

$$= \frac{1}{n^2}\left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right)$$

Take $V_n = 1/n^2$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right) = \frac{1}{3}$$

So $\sum U_n$ is cvgt. (Abel's Test)

$\langle a_n \rangle = 1/\log n \Rightarrow$ is +ve monotonically decreasing sequence.

$\boxed{9}$ Show $1 - \frac{1}{4.3} + \frac{1}{4^2.5} - \cdots$ is convergent.

$$\sum \frac{(-1)^{n-1}}{4^{n-1}} \times \frac{1}{(2n-1)}$$

$U_n = \frac{(-1)^{n-1}}{4^{n-1}}$, $a_n = \frac{1}{2n-1} \Rightarrow$ +ve, monotonically decreasing sequence

$|U_n| = \frac{1}{4^{n-1}}$ is convergent $\Rightarrow$

By Abel's Test

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$\boxed{8}$ Test convergence $\sum \frac{(n^3+1)^{1/3} - n}{\log n}$.

$U_n = (n^3+1)^{1/3} - n \qquad a_n = 1/\log n$

$$U_n = n\left[(1 + 1/n^3)^{1/3} - 1\right] = n\left[\left(\cancel{1} + \frac{1}{3n^3} - \frac{1}{9n^6} + \cdots\right) - \cancel{1}\right]$$

$$= \frac{1}{n^2}\left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right)$$

Take $V_n = 1/n^2$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{9n^3} + \cdots\right) = \frac{1}{3}$$

So $\sum U_n$ is cvgt. (Abel's Test)

$\langle a_n \rangle = 1/\log n \Rightarrow$ is +ve monotonically decreasing sequence.

$\boxed{9}$ Show $1 - \frac{1}{4.3} + \frac{1}{4^2.5} - \cdots$ is convergent.

$$\sum \frac{(-1)^{n-1}}{4^{n-1}} \times \frac{1}{(2n-1)}$$

$U_n = \frac{(-1)^{n-1}}{4^{n-1}}$, $a_n = \frac{1}{2n-1} \Rightarrow$ +ve, monotonically decreasing sequence

$|U_n| = \frac{1}{4^{n-1}}$ is convergent $\Rightarrow$

By Abel's Test

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(V) Re-arrangement of Series

$\boxed{\text{TIPS: LOOK FOR SIGN CHANGE TO BRACKET. CONSIDER } n \text{ brackets}}$

- $1+\frac{1}{2}+\cdots\frac{1}{n}-\log n=\gamma_n.$

$\boxed{5}$ Show $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\cdots=\frac{3}{2}(\log 2)$

$$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\cdots$$

$$\left(1-\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{4}+\frac{1}{7}\right)+\cdots$$

Consider n such brackets. $\boxed{1+\frac{1}{2}+\cdots\frac{1}{n}=\gamma_n+\log n}$

$$S=\left(1-\frac{1}{2}+\frac{1}{3}\right)+\cdots\left(\frac{1}{\ }-\frac{1}{2n}+\frac{1}{4n-1}\right)$$

$$\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{4n-1}+\frac{1}{4n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{2n}\right)$$

$$-\left(\frac{1}{2}+\frac{1}{4}+\cdots\frac{1}{4n}\right)$$

$$\gamma_{4n}+\log(4n)-\frac{1}{2}(\gamma_n+\log n)-\frac{1}{2}(\gamma_{2n}+\log 2n)$$

$$\gamma_{4n}-\frac{\gamma_n}{2}-\frac{\gamma_{2n}}{2}+\log\left[\frac{4n}{\sqrt{n}\sqrt{2n}}\right]=S.$$

As $n\to\infty$ $S=\log\left(\frac{4}{\sqrt{2}}\right)=\log(2\sqrt{2})=\boxed{\frac{3}{2}\log 2}$

* No change in Sum on rearrangement of absolutely convergent series.

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(4) Riemann Integrability

. Show $f(x) = \sin x$ is integrable on $[0, \frac{\pi}{2}]$

Partition the set as : $\overset{I_{\gamma}}{[0, \frac{\pi}{2n}]}\ [\frac{\pi}{2n}, \frac{2\pi}{2n}], \ldots [\frac{(n-1)\pi}{2n}, \frac{\pi n}{2n}]$

$\boxed{\delta_\gamma = \pi/2n}$

Let m_γ, M_γ be min, max of $\sin x$ in I_γ

As $\sin x$ is increasing of $[0, \frac{\pi}{2}]$

$$\text{On } I_\gamma \equiv \left[\frac{(\gamma-1)\pi}{2n}, \frac{\gamma\pi}{2n}\right] \begin{cases} m_\gamma = \sin\left(\frac{(\gamma-1)\pi}{2n}\right) \\ M_\gamma = \sin\left(\frac{\gamma\pi}{2n}\right). \end{cases}$$

$$\text{Upper Sums} = \sum_{1}^{n} M_\gamma \delta_\gamma.$$

$$= \frac{\pi}{2n}\left[\sum_{1}^{n} \sin\left(\frac{\gamma\pi}{2n}\right)\right]$$

$$= \frac{\pi}{2n} \times \frac{\sin\left(\frac{\pi}{2n} + \frac{(n-1)}{2}\frac{\pi}{2n}\right) \sin\left(\frac{n}{2}\frac{\pi}{2n}\right)}{\sin(\pi/4n)}$$

$\cancel{\sin\frac{\pi}{2n}}$

$$\boxed{\sin\alpha + \sin(\alpha+\beta) + \cdots \sin(\alpha + (n-1)\beta) = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin(\beta/2)}}$$

|Ly Calculate lower sum and equate at $n \to \infty$

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- $f(x) = \begin{cases} x + x^2 & x \text{ is rational} \\ x^2 + x^3 & x \text{ is irrational} \end{cases}$

Show $f(x)$ is not R-integral of $[0,2]$.

$$(x+x^2)-(x^2+x^3) = x - x^3 \geq 0$$

on $x \in [0,1]$.

So, $(x+x^2) > (x^2+x^3)$ on $x \in [0,1]$

$(x^2+x^3) > (x+x^2)$ $x \in [1,2]$.

Upper Integral $\equiv \int_0^1 x+x^2 + \int_1^2 x^2+x^3 = \frac{83}{12}$

Lower Integral $\equiv \int_0^1 x^2+x^3 + \int_1^2 x+x^2 = \frac{53}{12}$

So, not R-integrable

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① Summation of series.

$r/n \to x \qquad \frac{1}{n}\sum_{1}^{n} \to \int_0^1 dx.$

- $\lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) \cdots \left(1+\frac{4n}{n}\right)\right]^{1/n}.$

$$\log L = \frac{1}{n}\left[\sum_{1}^{4n} \log\left(1+\frac{r}{n}\right)\right]$$

$$= \int_0^4 \log(1+x)\,dx.$$

$$\boxed{\begin{array}{c} 1 - \frac{1}{1+x} \\ x - \log(1+x) \end{array}}$$

$$= x\log(1+x)\Big]_0^4 - \int_0^4 \frac{x}{1+x}\,dx$$

$$\cancel{= 4\log 5 - [\log 5] = 3\log 5.}$$

$$= 4\log 5 - [4 - \log 5] = 5\log 5 - 4.$$

$$L = \frac{e^{\log 5^5}}{e^4} = \boxed{\frac{5^5}{e^4}} \text{ Ans}$$

- $\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} = e. \qquad \left\{\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}\right\}$

$$L = \left(\frac{n}{1}\cdot\frac{n}{2}\cdots\frac{n}{n}\right)^{1/n} \Rightarrow \log L = \frac{1}{n}\left[\log\left(\frac{n}{1}\right) + \cdots \log\left(\frac{n}{n}\right)\right]$$

$$\log L = \int_0^1 \log\left(\frac{1}{x}\right) dx = \left[x\log\left(\frac{1}{x}\right) - \int_0^1 \cancel{x}\cdot\cancel{x}\,\frac{(-1)}{\cancel{x^2}}\,dx\right.$$

$$\log L = 1 \Rightarrow \boxed{L = e} \text{ Ans}$$

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② Integrability

- Continuous on R

OR

Only In $[a,b]$ • Bounded + finite pts of discontinuity

OR

In $[a,b]$ • Bounded + pts of discont have finite limit pts.

OR

- Monotonic in $[a,b]$ ← closed interval

③ Mean Value Theorems for integration

f is $C[a,b]$.

$$- \exists\, c \in (a,b) \text{ s.t } \int_a^b f(x)\,dx = f(c)(b-a)$$

✱ Generalized MVT.

- $\int_a^b f(x)\,dx$, $\int_a^b \varphi(x)\,dx$ both exist and φ has same sign in $[a,b]$, $\exists\, \mu$ s.t $m \le \mu \le M$, (m, M are bounds of f)

$$\int_a^b f\varphi\,dx = \mu \int_a^b \varphi\,dx$$

✱ 2nd MVT.

If $\int_a^b f(x)\,dx$ and $\int_a^b \varphi(x)\,dx$ exist and φ is monotonic, $\exists\, \xi \in [a,b]$

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(a)\int_a^{\xi} f(x)\,dx + \varphi(b)\int_{\xi}^{b} f(x)\,dx$$

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8 $f(x) = \begin{cases} 1/q & \text{when } x = p/q \text{ rational in lowest terms} \\ 0 & \text{when is irrational.} \end{cases}$

$\exists$ only finite integers s.t $\boxed{1/q > \varepsilon/2} \Rightarrow q < 2/\varepsilon$

Every interval encloses both rational and irrational points.

Oscillation of f in interval where no $q < 2/\varepsilon$ is $< \varepsilon/2$.

$\boxed{\frac{1}{q} < \varepsilon/2.}$

Enclose all points where $1/q > \varepsilon/2$ in interval sum of whose lengths. is $< \varepsilon/2$, so that oscillatory sum from these is $< \varepsilon/2$.

So, total oscillation $< \varepsilon \Rightarrow f$ is integrable

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$\boxed{9}$ $f(x) = \begin{cases} 1/q^2 & x = p/q \\ 1/q^3 & x = \sqrt{p/q} \\ 0 & \text{otherwise} \end{cases}$

Show f is Riemann-Integrable on $[a,b]$.

→ Consider any point p in $[a,b]$

(1) p is rational.
Then any interval $(p-\varepsilon, p+\varepsilon)$ has infinitely many irrational pts $\Rightarrow |f(x) - f(p/q)| = 1/q^2$
So, f is discont at p rational.

(2) d be irrational in $[a,b]$. $\Rightarrow f(d) \equiv$ either $1/q^3$ or 0
$\exists$ only finite values such that given $\varepsilon > 0$,
$1/q^2 > \varepsilon \Rightarrow q < 1/\sqrt{\varepsilon}$
So an interval around d can be found with no p/q s.t. $q < 1/\sqrt{\varepsilon}$.

In such an interval $\Rightarrow$

$$|f(x) - f(d)| \begin{cases} x \text{ irrational} = 0. \\ x = \sqrt{p/q} = |1/q^3| < \varepsilon. \\ x = p/q = |1/q^2| < \varepsilon \end{cases}$$

· d be $\sqrt{p/q}$

Illy $\begin{cases} x \text{ irrational } |1/q^3| < \varepsilon \\ x = \frac{p}{q} = |1/q^2 - 1/q^3| < \varepsilon \end{cases}$

So, f is continuous at all irrationals

Pts of Discont $\equiv$ Rationals $\equiv$ A set of measure 0. Thus f is R-integrable

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$\boxed{S}$ f is non negative continuous function on $[a,b]$ such that $\int_a^b f(x)\,dx = 0$. Show $f(x)=0$ $\forall$ $[a,b]$.

Let, if possible $f(c)>0$ for some $c \in [a,b]$

As f is continuous $\Rightarrow$

$$|[f(x)-f(c)]| < \varepsilon \quad \forall\, x \in [c-\delta, c+\delta]$$

Let $\varepsilon = f(c)/2$

Then on $x \in (c-\delta, c+\delta)$ $\quad f(x) > f(c) - \varepsilon$

$$f(x) > \tfrac{1}{2} \times f(c) > 0.$$

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b} f(x)\,dx.$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx \quad \text{as } f(x) \geq 0$$

$$\geq \frac{f(c)}{\cancel{2}} \cdot \cancel{2}\delta > 0. \rightarrow \leftarrow$$

So, $\boxed{f(c)=0} \Longrightarrow$

$\boxed{S}$ $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$

Let $f(x) = \sin \pi x$, $g(x) = \frac{1}{1+x^2}$

$f, g \geq 0$, are continuous and integrable.

g is monotonic decreasing $\in [\frac{1}{2} = \text{min}, 1 = \text{max}]$

By derivation of generalised MVT.

$$\frac{1}{2} \cdot \int_0^1 \sin \pi x\,dx \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq 1 \int_0^1 \sin \pi x\,dx.$$

$$\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \leq \frac{2}{\pi}$$ Proved

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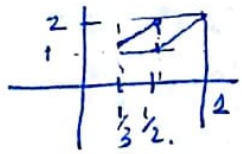
$\boxed{9}$

$F(x) = 2rx \qquad x \in \left(\frac{1}{r+1}, \frac{1}{r}\right) \quad r \in N$

Show F is integrable and $\int_0^1 F(x)\,dx = \frac{\pi^2}{6}$



As can be seen, pts of discontinuity as $S \equiv \left\{\frac{1}{r}\right\} \quad r \in \mathbb{N}$.

As set has only 1 limit pt at $x=0$. So f is integrable provided we define $F(x)$ at $x = \left\{\frac{1}{r}\right\}$ suitably keeping $F(x)$ bounded. (Take limits $1/n$ to be able to perform $\int$)

$$\int_{1/n}^{1} F(x)\,dx = \int_{1/2}^{1} 2x\,dx + \int_{1/3}^{1/2} 4x\,dx + \int_{1/4}^{1/3} 6x\,dx + \cdots$$

$$= 1 - \frac{1}{4} + \frac{1}{2} - \frac{2}{9} + \frac{1}{3} - \frac{3}{16} + \cdots$$

$$= \sum_{r=1}^{n-1} \frac{1}{r} - \frac{r}{(r+1)^2} = \sum_{1}^{n-1} \frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}$$

$$= 1 - \frac{1}{n}^{0} + \sum_{1}^{n-1} \frac{1}{(r+1)^2}$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \int_{1/n}^{1} f(x)\,dx = \left(1 + \frac{1}{2^2} + \cdots \frac{1}{n^2} + \cdots\right)$$

$$= \frac{\pi^2}{6}$$

$$\boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$

Examples – 13.48 onwards

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$$\int_0^x \frac{1}{1+t^4} = \int_0^x 1 - t^4 + t^8 \cdots (-1)^{n-1} t^{4n-4} + \boxed{\frac{(-1)^n t^{4n}}{1+t^4}} \quad \text{MUGT.LT}$$

$$= x - \frac{x^5}{5} + \frac{x^9}{9} + \cdots + (-1)^n \int_0^x \frac{t^{4n}}{1+t^4}\,dt$$

$$\lim_{n\to\infty} \left| \int_0^x \frac{t^{4n}}{1+t^4}\,dt \right| < \left| \int_0^x t^{4n}\,dt \right| = \left| \frac{x^{4n+1}}{4n+1} \right| < \frac{1}{4n+1} = 0$$

when $-1 < x \le 1$ $\quad \lim_{m\to\infty} \int_0^x \frac{t^m}{t+1}\,dt = 0$

$\forall x \in [0,1]$

$$0 \le \int_0^x \frac{t^m}{t+1}\,dt \le \int_0^x t^m\,dt = \frac{x^{m+1}}{m+1} < \frac{1}{m+1}$$

Illy $[-1 < x < 0]$, $t = -u$.

$$\left| \int_0^{-x} \frac{u^m}{1-u}\,du \right| < \left| \frac{1}{1+x} \int_0^{-x} u^m\,du \right| < \frac{1}{(m+1)(x+1)}$$

$\frac{\pi^2}{6}$

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$\langle f_n \rangle$

M_n test.

- Let $M_n = \sup\{|f(x) - f_n(x)|\}$

  $\lim_{n\to\infty} f_n(x) = f(x)$

  Then $\langle f_n \rangle$ converges uniformly on $[a,b]$ iff $M_n \to 0$ as $n\to\infty$

Let $f_n \xrightarrow{\text{uniformly}} f$ on $[a,b]$

Then for a given $\varepsilon > 0$, $\exists\, m > 0$ s.t.

$|f_n(x) - f(x)| < \varepsilon \quad \forall x \geq m$

$\Rightarrow \sup\{|f_n(x) - f(x)|\} : (x \in [a,b]) < \varepsilon \quad \forall n \geq m.$

$\Rightarrow M_n < \varepsilon \quad \forall n \geq m \qquad \Rightarrow M_n \to 0$ as $n \to \infty$

Sufficiency $\Rightarrow$ $M_n \to 0$ as $n \to \infty$

$|M_n - 0| < \varepsilon \quad \forall n \geq m$

$\Rightarrow \sup |f_n - f| < \varepsilon \quad \forall n \geq m \quad \Rightarrow |f_n - f| < \varepsilon.$

- Not uni. cont

  1. $f_n \to f$ and f is not continuous
  2. $\lim \int f_n \neq \int f\,dx.$
  3. If $\{f_n'\}$ converges uniformly to f' then $\{f_n\} \xrightarrow{\text{unif.}} f$

$\rightarrow$ Series

- $S_n \to S$ and S is not continuous
- $\int S_n \neq \int S \neq \sum \int f_n$ (Any one)

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(5) Uniform Convergence of Series of Functions

- $f_n = \frac{nx}{1+n^2x^2}$ is not uniformly convergent.

Suppose if it is uni. cvgt on $[a,b]$,

Let $\varepsilon > 0$ be given, then $\exists\, m$ s.t $\forall x \in [a,b], \forall n \geq m$

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \varepsilon.$$

pt wise limit [Evaluated by $\lim_{n\to\infty}$]

Let $\varepsilon = 1/4$, then $\exists\, k$ s.t $k \geq m$ and $1/k \in [a,b]$

Taking $n = k$, $x = 1/k$ we have.

$$\frac{\cancel{k}\cdot\frac{1}{\cancel{k}}}{1+\cancel{k^2}\cdot\frac{1}{\cancel{k^2}}} = \left|\frac{1}{2} - 0\right| \nless 1/4$$

We see it is not uniformly cvgt.

- $f_n = nxe^{-nx^2}$ is pt-wise cvgt but not uniformly cvgt on $[0,k]$ $k>0$.

$\lim f_n = 0$

Let f_n be uniformly cvgt. $\exists\, m$ s.t $\forall n \geq m$ and given $\varepsilon > 0$,

$$nxe^{-nx^2} < \varepsilon \qquad -(1)$$

Let $m_0 \geq m$ and $e^2\varepsilon^2$, ~~$x = 1/\sqrt{m_0}$~~

Then (1) holds for $\boxed{x = 1/\sqrt{m_0}}$ and $\boxed{n = m_0}$

$$\frac{\sqrt{m_0}}{e} < \varepsilon \Leftrightarrow m < e^2\varepsilon^2 \rightarrow\leftarrow$$

So, convergence is not uniform in $[0,k]$

THESE CAN BE DONE BY M_n Test

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(1)

Showing Uniformly convergent.

- M_n test $\equiv \langle f_n \rangle \to f \quad \forall x \in [a,b]$.

  $M_n = \sup\{|f_n(x) - f(x)|\} \Leftarrow$ Find maxima here

  $\langle f_n \rangle$ converges uniformly on $[a,b]$ $\boxed{\text{iff}}$ $M_n \to 0$ as $n \to \infty$

- Weierstrass M-test [For Sum of Series]

  Σf_n converges uniformly if $\exists$ convergent series ΣM_n of +ve numbers s.t $\boxed{|f_n(x)| \leq M_n}$ $\forall x \in [a,b]$.

- Abel's Test
  - $\Sigma f_n(x)$ is uniformly convergent in $[a,b]$
  - $\langle g_n(x) \rangle$ is monotonic decreasing $\forall x \in [a,b]$ $\oplus$ Bounded
  - $\exists\, k > 0$ such that $|g_n(x)| < k \;\forall x \in [a,b]$ and $n \in N$

  Then $\Sigma f_n(x) g_n(x)$ is uniformly convergent

- Dirichlet's
  - $\exists\, k > 0$ s.t $|S_n| = \left|\sum_{r=1}^{n} f_r(x)\right| < k \quad \forall n \in [a,b]$.
  - $\langle g_n(x) \rangle$ is +ve monotonic $\downarrow$ sequence converging ~~uniformly~~ to zero. (Not in Abel !)

  Then $\Sigma f_n g_n$ is uniformly convergent

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$\boxed{5}$ $f_n(x) = \dfrac{x}{1+nx^2}$ converges uniformly on any closed interval I.

$\cdot\ f(x) = \lim_{n\to\infty} f_n(x) = 0, \ \forall x \in I$

$\therefore \quad |f_n(x) - f(x)| = \left|\dfrac{x}{1+nx^2}\right| = |y|.$

$\dfrac{dy}{dx} = \dfrac{1-nx^2}{(1+nx^2)^2} \Rightarrow x = \pm\dfrac{1}{\sqrt{n}}.$

$\dfrac{d^2y}{dx^2} =$ at $x = \dfrac{1}{\sqrt{n}} = \dfrac{-\sqrt{n}}{2} < 0 \Rightarrow$ Maxima at $x = \dfrac{1}{\sqrt{n}}$.

$y = \dfrac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$. So $\langle f_n \rangle$ is uniformly cvgt

$\boxed{6}$ $\sum \dfrac{\sin(x^2+nx^2)}{n(n+1)}$

Weierstrass M Test
L generally in Trigo functions

$\left|\dfrac{\sin(x^2+nx^2)}{n(n+1)}\right| \le \dfrac{1}{n^2}$

$\langle 1/n^2 \rangle$ is convergent $\Rightarrow$ Given seq is uniformly cvgt

$\boxed{Q}$ $\sum_1^\infty n^2 x^n$ is uniformly cvgt in $[-\alpha, \alpha]$ when $0 < \alpha < 1$

Look at this series

$|f_n(x)| < \alpha^n n^2 = M_n$ as $|x| < \alpha$

$\lim_{n\to\infty} \dfrac{M_{n+1}}{M_n} = \alpha < (1 - \text{Given})$

Hence, by D'Alembert ratio $\Rightarrow M_n$ converges

$\Rightarrow$ Given series is uniformly cvgt

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$\boxed{Q}$ $\sum \frac{(-1)^{n-1}}{n} |x|^n$ is uniformly convergent in $[-1,1]$

$\Rightarrow f_n = \frac{(-1)^{n-1}}{n}$ $\quad g_n(x) = |x|^n$

- $\sum f_n(x)$ is convergent $\rightarrow$ By Leibnitz Test

As $f_n(x)$ is independent of x, then $\sum f_n(x)$ is uniformly convergent

Again, for $-1 \le x \le 1$, $|x|^n \le 1 \ \forall n \in N$

$|g_n(x)| = |x|^n \le 1.$

$\langle g_n(x) \rangle$ is monotonic decreasing and bounded on $[-1, 1] \ \forall n \in N.$

So, Abel's test $\Rightarrow \sum f_n g_n$ is uniformly convergent

$\boxed{Q}$ $\sum \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent $\forall x$.

- Dirichlet's Test $\left[\sum (-1)^{n-1}\right.$ neither ungt, nor monotonic$\left.\right]$

$\sum f_n(x) = \sum (-1)^{n-1}$

$\llcorner S_n(x) = \begin{cases} 1 & \text{odd} \\ 0 & \text{even} \end{cases} \Rightarrow S_n(x)$ is bounded

$\langle g_n(x) \rangle$ is +ve monotonic decreasing sequence converging to $0, \forall x.$

So $\sum f_n(x) g_n(x)$ is bounded

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(2) Showing Not uniformly convergent. ~~(Series)~~

$f_n \xrightarrow{\text{uniform}} f$ $\longrightarrow$ f is continuous on $[a,b]$

$\longrightarrow \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f(x)\,dx.$

. Check convergence of $f_n(x) = nx e^{-nx^2}$ (*) LEARN IT!

$$f(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \frac{nx}{1+\frac{nx^2}{1}+\frac{n^2x^4}{2!}+\cdots} = 0.$$

$$\int_0^1 f(x)\,dx = 0$$

$$\int_0^1 f_n(x)\,dx = \frac{(1-e^{-n})}{2}\ ;\ \lim_{n\to\infty} \frac{(1-e^{-n})}{2} = \frac{1}{2} \neq 0.$$

So $\langle f_n \rangle$ is not uniformly cvgt.

* For series $\Rightarrow$ $S(x) = \lim_{n\to\infty} S_n(x)$ is discont.
(Implies f_n not uniform continuous)

. $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots$

$$S_n = x^4 + \frac{x^4}{(1+x^4)} + \cdots \frac{x^4}{(1+x^4)^{n-1}} = 1+x^4 - \frac{1}{(1+x^4)^{n-1}}$$

$$\lim_{n\to\infty} S_n = \begin{cases} 0 & x=0 \\ 1+x^4 & x\in(0,1] \end{cases}$$

So, S_n is discont. at $x=0$ $\Rightarrow$ f_n is not uniformly convergent

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(3) Properties of Uniformly Convergent Series of functions

- If Σf_n be uniformly convergent series in $[a, b]$

$$\int_a^b \Sigma f_n(x)\,dx = \int_a^b S(x)\,dx = \Sigma \int_a^b f_n(x)\,dx.$$

For derivative equality only pt-wise cvgt (S_n is cvgt)

- Σf_n be pt-wise convergent series of derivable functions with continuous derivatives and $\Sigma f_n'$ is uniformly convergent then

$$\frac{d}{dx}(\Sigma f_n) = \Sigma f_n'$$

* TIP: DON'T BE SCARED OF USING ε-δ.
USE ε-δ when linear terms seen.

- TS: Term-by-term differentiable if.
  - Σf_n be pt-wise cvgt.
  - $\Sigma f_n'$ is uniformly cvgt [Find maximum $f'(x)$, $|f'(x)| < M \rightarrow$ Weierstrass Test]

IMP

5 $S_n = \frac{1}{2n^2} \times \log(1+n^4x^2)$

Show $\sum U_n'(x)$ does not converge uniformly but given series can be differentiated term-by-term.

TS: Show $S_n(x)$ is pt-wise cvgt.

$$S(x) \equiv \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{\log(1+n^4x^2)}{2n^2} = 0 \quad \forall x \in [0,1].$$

Hence, pt-wise differentiable

$\Rightarrow S'(x) = 0$

$$\lim_{n\to\infty} S_n'(x) = \frac{n^2x}{1+n^4x^2} = 0 = S'(x)$$ [So, term-by-term diff holds]

$\sum U_n'(x)$ is not uniformly cvgt $\Rightarrow$

- $S_n'(x)$ has 0 as pt of non-uniform convergen

How??

$|S_n(x) - 0| = y \qquad \frac{dy}{dx} = 0$ at $x = \frac{1}{n^2}$.

Value at $x = 1/n^2 \Rightarrow \frac{1}{1+1} = \frac{1}{2} \neq 0$.

So $\left\langle \frac{n^2x}{1+n^2x^2} \right\rangle$ is not uniformly cvgt at $x=0$

Post 15.22
Solve for practice

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To Show:

- Uniform convergence

$$|S_n(x) - S(x)| < \varepsilon$$

(Express n in terms of ε)

Q. $S_n = nx(1-x)^n$ is not uniformly cvgt on $[0,1]$

→ Can do M_n test ≡ Treat S_n as $\langle f_n \rangle$

$S_n \to 0$ as $n \to \infty$

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(6) Improper Integrals

$\boxed{Q}$ $\int_0^{\frac{\pi}{2}} \sin x \log \sin x.$

0 is a pt of ∞ discontinuity.

$$\int_{\varepsilon}^{\frac{\pi}{2}} \sin x \log \sin x \, dx = -\cos x \log \sin x \Big]_{\varepsilon}^{\pi/2} + \int_{\varepsilon}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x} dx$$

$$= +\cos\varepsilon \log \sin \varepsilon + \int_{\varepsilon}^{\frac{\pi}{2}} \operatorname{cosec} x - \sin x \, dx.$$

OR $\log(\operatorname{cosec} - \cot)$

$$= \cos\varepsilon \log \sin\varepsilon + \log\left(\tan\frac{x}{2}\right) + \cos x \Big]_{\varepsilon}^{\pi/2}$$

$$= \cos\varepsilon \log \sin\varepsilon + \log(1-\cos x) - \log \sin x + \cos x \Big]_{\varepsilon}^{\frac{\pi}{2}}.$$

$$= \cos\varepsilon \log \sin\varepsilon - \log(1-\cos\varepsilon) + \log \sin\varepsilon - \cos\varepsilon.$$

Convert all $\sin\varepsilon$ to $\varepsilon/2$

$$= \cos\varepsilon \log\left(2\cos\varepsilon/2\right) + \cos\varepsilon \log\left(\sin\frac{\varepsilon}{2}\right) - \log\left(2\sin^2\frac{\varepsilon}{2}\right) + \log\left(2\cos\varepsilon/2\right)$$

$$- \log 2 - 2\log\left(\sin\frac{\varepsilon}{2}\right)$$

$$+ \log\left(\sin\varepsilon/2\right) - \cos\varepsilon.$$

$$\cos\varepsilon \log\left(2\cos\varepsilon/2\right) + \cos\varepsilon \log\left(\sin\frac{\varepsilon}{2}\right) - \log\left(\sin\frac{\varepsilon}{2}\right) + \log\left(\cos\varepsilon/2\right) - \cos\varepsilon$$

$$-(1-\cos\varepsilon)\log \sin\varepsilon/2 \equiv \left(-2\sin^2\varepsilon/2 \log \sin\varepsilon/2\right)$$

$\lim = 0$

$$\left(\log 2 + 0 - 1\right) = \text{Ans}$$

$\boxed{\text{All non } \cos\varepsilon \text{ terms to } \varepsilon/2}$

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(1) Testing for Convergence. $\equiv$ Finite limits, ∞ Integrand

+ve

• Case I : Integrand is +ve $[f(x) > 0]$

$\llcorner$ Let $f(x) \le \phi(x)$.

Then $\int \phi$ cvgt $\Rightarrow \int f$ cvgt ; $\int f$ dvgt $\Rightarrow \int \phi$ dvgt.

Test :

• $\frac{f(x)}{\phi(x)} \to l$ $(\neq 0, \neq \infty)$. $\equiv$ Then $f(x) \equiv \phi(x)$.

$$\boxed{\text{STANDARD} \equiv \int_a \frac{1}{(x-a)^n} \text{ converges iff } n < 1}$$

• Case II : General $f(x)$ not necessarily ≥ 0

If $|f(x)|$ cvgt $\Rightarrow f(x)$ absolutely convergent

$$\boxed{\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}\,dx \quad \int_0^1 \left|\frac{1}{\sqrt{x}}\right| dx \text{ is cvgt}}$$

* Beta Function $\equiv \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$.

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$\boxed{8}$ $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}} = \int_\varepsilon^a + \int_a^n$ $\quad a \in (0,1)$.

0, 1 both are pts of infinite discontinuity

$\displaystyle\int_\varepsilon^a \frac{dx}{x^{1/2}(1-x)^{1/3}}$ Comparing with $\frac{1}{x^{1/2}} \Rightarrow \frac{f(x)}{\phi(x)} = \lim_{x\to 0} \frac{1}{(1-x)^{1/3}} = 1$

As $1/x^{1/2}$ is cvgt, So is $f(x)$ at $x=0$

$\displaystyle\int_a^n \frac{dx}{x^{1/2}(1-x)^{1/3}}$ Comparing with $\frac{1}{(1-x)^{1/3}} \Rightarrow \lim \frac{f(x)}{\phi(x)} = \frac{1}{x^{1/2}} = 1.$

As $(1-x)^{1/3}$ is cvgt, So is $f(x)$.

$\boxed{9}$ $\displaystyle\int_0^{\pi/2} x^m \operatorname{cosec}^n x \, dx$

∞ discontinuity at $x=0$.

$\displaystyle\int_\varepsilon^2 \frac{dx}{2x-x^2} = \left[\frac{1}{2}\log\frac{x}{2-x}\right]_\varepsilon^{2-\lambda}$

$\cancel{n \geq m \Rightarrow x^{n-m} = \operatorname{cosec}^{n-m} x \Rightarrow \lim_{x\to 0} = \infty}$

$\cancel{\operatorname{cosec}^m x}$

$\cancel{n = m \Rightarrow \frac{x^n}{\sin}}$

$\cancel{\frac{x^m}{\sin^m x} \operatorname{cosec}}$

$$f(x) \equiv \left(\frac{x^n}{\sin^n x}\right) \cdot x^{m-n} = \frac{x^n}{(\sin x)^n} \cdot \frac{1}{x^{n-m}}$$

Pt of ∞ discontinuity at $x=0$.

Comparing with $\phi = k x^{n-m} \Rightarrow \lim_{x\to 0} = 1$

$\frac{1}{x^{n-m}}$ is cvgt iff $n-m<1 \Rightarrow n < m+1$.

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$\int_0^1 x^{n-1} \log x$ Only 0 is discontinuity

As, $x^{n-1} \log x < 0$ in $(0,1)$,

So, consider $-x^{n-1} \log x \Rightarrow x^{n-1} \log\left(\frac{1}{x}\right)$

If $(n-1) > 0$, then $f(x) \to 0$ as $x \to 0$ (No discont)

Let $(n-1) \leq 0$

Let $m > 0$ s.t $\boxed{(m+n-1) > 0}$.

$x^m \cdot x^{n-1} \cdot \log\left(\frac{1}{x}\right) = x^{m+n-1} \log\left(\frac{1}{x}\right)$.

We have $\Rightarrow \lim_{x \to 0} \left(-x^{m+n-1} \log(x)\right) = 0$

s.t $\varepsilon > 0$ given $\Rightarrow$

$-x^{m+n-1} \log x < \varepsilon$

$-x^{n-1} \log x < \varepsilon / x^m$.

ε / x^m cvgt at $x = 0$ iff $\boxed{m < 1}$

$m+n-1 > 0$, $m < 1 \Rightarrow n > 0$

For $n = 0$, not cvgt

$n < 0$.

Convergent iff $\boxed{n > 0}$ Ans

ALITER

$\log \frac{1}{x} = t$

$x = e^{-t}$

$\int_0^\infty e^{(1-n)t} \, t \, dt$

$-\int_0^\infty \frac{t}{e^{nt}} \, dt$

At 0 cvgt

At ∞ cvgt

if $n > 0$

Cane make cases

x	< 0	> 1	
	$= 0$		
	$(0,1)$		

$\int_0^1 x^{n-1} \log x$ Only 0 is discontinuity

As, $x^{n-1} \log x < 0$ in $(0,1)$,

So, consider $-x^{n-1} \log x \Rightarrow x^{n-1} \log\left(\frac{1}{x}\right)$

If $(n-1) > 0$, then $f(x) \to 0$ as $x \to 0$ (No discont.)

Let $(n-1) \leq 0$

Let $m > 0$ s.t $\boxed{(m+n-1) > 0.}$

$x^{m} \cdot x^{n-1} \cdot \log\left(\frac{1}{x}\right) = x^{m+n-1} \log\left(\frac{1}{x}\right)$.

We have $\Rightarrow \lim_{x \to 0} \left(-x^{m+n-1} \log(x)\right) = 0$

s.t $\varepsilon > 0$ given $\Rightarrow$

$$-x^{m+n-1} \log x < \varepsilon$$

$$-x^{n-1} \log x < \varepsilon / x^{m}.$$

ε / x^{m} cvgt at $x = 0$ iff $\boxed{m < 1}$

$m+n-1 > 0$, $m < 1 \Rightarrow n > 0$

For $n = 0$, not cvgt

$n < 0$.

Convergent iff $\boxed{n > 0}$ Ans

ALITER

$\log \frac{1}{x} = t$

$x = e^{-t}$

$\int_0^\infty e^{(1-n)t} t \, dt$

$-\int_0^0 \frac{t}{e^{nt}} dt$

At 0 cvgt

At ∞ cvgt

if $n > 0$

Cano make cases

x	< 0	> 1
	$= 0$	
	$(0,1)$	
	1	

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$\boxed{8}$ $\int_0^{\pi/2} \log \sin x \, dx$. pt of discont at $x=0$.

As f is $-ve$ in $x \in (0, \frac{\pi}{2})$. Consider $-\log(\sin x)$

Take $\phi(x) = 1/x^n$ $\boxed{n>0}$

$$\lim_{x\to 0} \frac{-\log \sin x}{(x^{-n})} \cdot x^n \Rightarrow \lim_{x\to 0} \frac{\cot x}{n/x^{n+1}} = \lim_{x\to 0} \frac{x^n}{n} \frac{x}{\tan x}$$

$= 0$. if $n>0$ and $n<1$ $\Leftarrow$ Obv

So, I is convergent

$$\int_0^{\frac{\pi}{2}} \log \sin x + \log \cos x = 2I$$

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2x}{2}\right) dx = 2I$$

$$2I = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \frac{\pi}{2} \log 2.$$

$$2I = \int_0^{\pi} \frac{\log \sin t}{2} \, dt - \frac{\pi}{2} \log 2$$

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{\log \sin t}{2} \, dt - \frac{\pi}{2} \log 2 \Rightarrow \boxed{I = -\frac{\pi}{2} \log 2} \text{ Ans}$$

$\boxed{9}$ $\int_0^2 \frac{\log x}{\sqrt{2-x}} \, dx$

$\lim_{x\to 0} x^{\mu} \frac{\log x}{\sqrt{2-x}}$, take $1>\mu>0 \Rightarrow \lim = 0 \Rightarrow$ cvgt

$\lim_{x\to 2} \cancel{(2-x)^{1/2}} \frac{\log x}{\cancel{\sqrt{2-x}}} = \log 2 \Rightarrow$ cvgt

$$\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 \frac{dx}{\sqrt{1-x} \sqrt{1+x+x^2}} \Rightarrow \text{bdd} \Rightarrow \frac{1}{\sqrt{1-x}} \Rightarrow \text{cvgt}$$

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(1) A. When sign changes

$\int_0^1 \frac{\sin(1/x)}{\sqrt{x}}\,dx \rightarrow$ Show abs. convergence, which implies convergence.

$\left|\frac{\sin(1/x)}{\sqrt{x}}\right| \leq \frac{1}{\sqrt{x}} \rightarrow$ Convergent

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1) When limits are ∞

$$\int_2^\infty \frac{2x^2\,dx}{x^4-1} = \int_2^\infty \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)}\,dx = \int_2^\infty \frac{1}{x^2+1} + \frac{1}{x^2-1}\,dx$$

$$\tan^{-1}x\Big]_2^\infty + \frac{1}{2}\log\left(\frac{x-1}{x+1}\right)\Big]_2^\infty$$

$$\lim_{t\to\infty}\left(\cancel{\tan^{-1}t}^{\,\pi/2} + \cancel{\frac{1}{2}\log\frac{t-1}{t+1}}^{\,0}\right) - \frac{1}{2}\log\frac{1}{3} - \tan^{-1}2.$$

$$\int_{-1}^\infty \frac{dx}{\sqrt{x(1-x)}} \qquad \frac{1}{\sqrt{x-x^2}} = \frac{1}{\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}}$$

$$\int_{-1}^\infty \frac{dx}{\sqrt{(\frac{1}{2})^2-(x-\frac{1}{2})^2}} = \int_{-1}^{\varepsilon} + \int_{\varepsilon}^{t}$$

$$\lim_{\varepsilon\to 0}\ \sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\Big]_{-1}^{\varepsilon} + \frac{\sin^{-1}(x-\frac{1}{2})}{\frac{1}{2}}\Big]_{\varepsilon}^{t} \quad (*)$$

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} \qquad x = \tan^2\theta.$$

$$\int_0^{\pi/2} \frac{2\cancel{\tan\theta}\sec^2\theta}{\cancel{\sec^2\theta}\,\cancel{\tan\theta}}\,d\theta = \pi$$

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• Comparison Test.

$0 \le f(x) \le g(x)$

$\int_a^\infty g(x)$ is cvgt $\Rightarrow \int_a^\infty f(x)$ is convergent

• $f(x)/\phi(x) = l$.

$\neq 0, \neq \infty \rightarrow$ Same.

$= 0 \rightarrow \phi$ cvgt $\Rightarrow f$ cvgt.

$= \infty \rightarrow \phi$ dvgt $\Rightarrow f$ dvgt.

• $\int_a^\infty \frac{1}{x^n}\,dx \quad a>0$ converges iff $\boxed{n>1}$ (*)

only for ∞ limits

$\int_0^\infty \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x}\,dx \rightarrow \boxed{\lim_{x\to 0} = \frac{1}{6}}$

(*) $\left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} = \frac{e^x - e^{-x} - 2x}{x^2(e^x - e^{-x})} < \frac{e^x}{x^2(e^x - e^{-x})}$ $\quad x>0$

$< \frac{e^{2x}}{e^{2x}-1} \cdot \frac{1}{x^2}$

$\lim_{x\to\infty} \frac{e^{2x}}{e^{2x}-1} \rightarrow 1 \Rightarrow \exists\, k$ s.t $\forall x \ge k$. $\left(\frac{e^{2x}}{e^{2x}-1} < \frac{3}{2}\right)$

$\forall x \ge k \quad \left(\frac{1}{x} - \frac{1}{\sinh x}\right)\frac{1}{x} < \frac{3}{2}\cdot\frac{1}{x^2} \Rightarrow$ cvgt

$\Downarrow$

cvgt

So given integral is cvgt

$\boxed{Q}$ $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$

$x=0 \Rightarrow \lim_{x\to 0} \frac{\sin^2 x}{x^2} = 1 \Rightarrow 0$ is not pt of dis cont.

$x=\infty \Rightarrow \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \forall x>0.$ And $\frac{1}{x^2}$ is cvgt.

So by comparison f is cvgt

$\boxed{Q}$ Show gamma function is convergent.

$\int_0^\infty x^{n-1} e^{-x}\, dx$ is cvgt iff $n>0$

Do NOT USE (*) Here

At $x=0 \Rightarrow \varphi(x) = 1/x^{1-n} \Rightarrow \lim \frac{f}{\varphi} = 1 \Rightarrow 1-n<1 \Rightarrow$ $\boxed{n>0}$

$x=\infty \Rightarrow$ For sufficiently large $x \Rightarrow x^{n+1} < e^x \Rightarrow e^{-x}x^{n-1} < 1/x^2$

And $\int_a^\infty \frac{1}{x^2}$ is cvgt $\Rightarrow \int_a^\infty e^{-x}x^{n-1}$ is cvgt $\forall n$.

For $n>0$ $\int f$ is cvgt

$\boxed{Q}$ $\int_1^\infty \frac{\log x}{x^2}\, dx$

take $\frac{1}{x^{3/2}} \Rightarrow \frac{f}{\varphi} = \lim_{x\to\infty} \frac{\log x}{\sqrt{x}} = \frac{2}{\sqrt{x}} = 0$

$\Downarrow$

cvgt $\Rightarrow f$ is cvgt

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* IMP $\int_0^1 x^p \left(\log \frac{1}{x}\right)^q dx$.

Put $\log 1/x = t \Rightarrow x = e^{-t} \Rightarrow \int_0^\infty t^q e^{-(p+1)t} dt$.

At 0:- $\int_0^\alpha \frac{e^{-(p+1)t}}{t^{-q}} dt$. $\quad -q < 1 \Rightarrow \boxed{q > -1}$

At ∞: $\int_\alpha^\infty \frac{t^q}{e^{(p+1)t}} dt$ $\quad g(t) = 1/t^2$; $\frac{f(t)}{g(t)} = \frac{t^{q+2}}{e^{(p+1)t}} \to 0$ as $t \to \infty$

$\forall q$ and $\boxed{p+1 > 0}$

$p+1>0$

So, $\boxed{p > -1, q > -1}$ Ans

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(III) Product of functions

- φ be bounded and integrable in (a,b)
  $\int_a^\infty f(x)$ converges absolutely at ∞, then $\int_a^\infty f\varphi$ abs. converges

Eg $\int_1^\infty \frac{\sin x}{x^2}$ $\quad \left.\begin{array}{l}\varphi = \sin x \\ f = 1/x^2\end{array}\right\}$ So convergent

Illy $\boxed{\int_0^\infty e^{-ax}\cos x\,dx}$

- Abel Test

  $\int_a^\infty f(x)$ is convergent at ∞

  $\varphi(x)$ is bounded and monotonic for $x \geq a$.

  Then $\int_a^\infty f\varphi\,dx$ is convergent at ∞.

- Dirichlet

  φ : Bounded, monotonic, $\to 0$ as $x \to \infty$

  $\int_a^t f(x)$: Bounded $\forall\, t \geq a$.

  Then $\int_a^\infty f\varphi$ converges

To SELECT ABEL VS DIRICHLET:
If f/φ cvgt $\Rightarrow$ Abel else dirichlet.

Do Post 1628 for practice

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Use in $e^{-ax}\frac{\sin x}{x}$ also ← Abel's

$\boxed{3}$ $\int_0^\infty \frac{\sin x}{x}$ is conditionally convergent.

$\boxed{\int_1^\infty}$ ⟹ Convergent. [As $\sin x$, $1/x$ both not cvgt So, dirichlet]

$\phi(x) = \sin x$ is s.t $\int_a^t \sin x\, dx = (\cos a - \cos t)$ bounded

$|\cos a - \cos t| \le 2$

$f(x) = 1/x$ is monotonic, $\to 0$ as $x \to \infty$, bounded.

So $\int_1^\infty \frac{\sin x}{x}$ is cvgt by Dirichlet's Test.

$\boxed{\int_0^\infty}$ ⟹ $\lim_{x\to 0} \frac{\sin x}{x} = 1$ ⟹ 0 is not pt of ∞ discont

→ cvgt.

So, $\int_0^\infty \frac{\sin x}{x}$ is convergent.

$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx \text{ is convergent.}}$

TS: $\int_0^\infty \frac{|\sin x|}{x}\,dx$ is not convergent

Let $\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{r=1}^{\pi} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx.$

Put $x = (r-1)\pi + y$.

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} = \int_0^\pi \frac{\sin y}{(r-1)\pi + y}\,dy$$

As max value of D^r in $\int_0^\pi$ is $\cancel{r}\pi$ ⟹ $\int_0^\pi \frac{\sin y\,dy}{(r-1)\pi+y} \ge \frac{1}{r\pi}\int_0^\pi \sin y\,dy = \frac{2}{r}$

$$\int_0^{n\pi} \frac{|\sin x|}{x} \ge \sum \frac{2}{r\pi} = \frac{2}{\pi}\left(\sum \frac{1}{r}\right) \to \infty \text{ as } n \to \infty$$

So, $\int_0^{n\pi} \frac{|\sin x|}{x}\,dx \to \infty$ as $n \to \infty$ ⟹ $\int_0^\infty \frac{|\sin x|}{x}\,dx \to \infty$ ⟹ Not Converge

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$\boxed{8}$ Show $\int_0^\infty \frac{x\,dx}{1+x^6\sin^2 x}$ is convergent.

$$\int_0^{n\pi} \frac{x\,dx}{1+x^6\sin^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^6\sin^2 x}$$

$$\int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^6\sin^2 x} \le \int_{(r-1)\pi}^{r\pi} \frac{r\pi}{1+(r-1)^6\pi^6\sin^2 x} = a_r$$

In a_r, put $x = (r-1)\pi + y$

$$a_r = \int_0^{\pi} \frac{r\pi\,dy}{1+(r-1)^6\pi^6\sin^2 y} = 2\int_0^{\pi/2} \frac{r\pi\,dy}{1+(r-1)^6\pi^6\sin^2 y} \quad -(1)$$

$$\int_0^{\pi/2} \frac{dy}{1+A\sin^2 y} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2 y\,dy}{\operatorname{cosec}^2 y + A} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2 y}{A+1+\cot^2 y}\,dy \quad -(1)$$

$$\cot y = t \rightarrow \int_{\infty}^{0} \frac{(-dt)}{(A+1)+t^2} = \left[\frac{\tan^{-1}\left(\frac{t}{\sqrt{A+1}}\right)}{\sqrt{A+1}}\right]_0^{\infty}$$

$$\int_0^{\pi/2} \frac{dy}{1+A\cos^2 y} = \int_0^{\pi/2} \frac{dy}{1+A\sin^2 y} = \frac{\pi}{2\sqrt{A+1}} = \frac{1}{2}\int_0^{\pi} \frac{dx}{1+A\sin^2 x}$$

$$(1) \Rightarrow a_r = 2r\pi \times \frac{\pi}{2} \times \frac{1}{\sqrt{(r-1)^6\pi^6+1}} < \frac{r}{(r-1)^3}\cdot\frac{1}{\pi}$$

$$\frac{r}{(r-1)^3} = \underbrace{\frac{1}{(r-1)^2} + \frac{1}{(r-1)^3}}_{\text{both cvgt}} \Rightarrow \frac{r}{(r-1)^3}\frac{1}{\pi} \text{ is cvgt}$$

$\Rightarrow a_r$ is cvgt

$\Rightarrow \lim_{n\to\infty} \int_0^{n\pi} \frac{x\,dx}{1+x^6\sin^2 x}$ is cvgt

Divgt → Show
L Lower limit in N^r
Upper in D^r

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$$\int_1^\infty \frac{\sin x \log x}{x}$$

$\int f(x) = \sin x$. is bounded.

$\frac{\log x}{x}$ is montone, decrease, $\to 0$

So, by Dirichlet's test Cvgt

$$* \int_2^\infty \frac{\cos x}{\log x} \text{ conditionally cvgt} \to \text{Dirichlet}$$

$$I = \int_2^\infty \left|\frac{\cos x}{\log x}\right| dx = \int_2^{3\pi/2} + \int_{3\pi/2}^{5\pi/2} + \cdots \int_{(2n-1)\pi/2}^{(2n+1)\frac{\pi}{2}} + \int_{\pi/2}^{2} - \int_{\pi/2}^{2}$$

$$= \sum_{r=1}^{\infty} \int_{(2r-1)\pi/2}^{(2r+1)\pi/2} \frac{|\cos x|}{\log x} dx - \left(\int_{\pi/2}^{2} \frac{|\cos x|}{\log x} dx\right) \cdot \text{cvgt.}$$

$$\Downarrow$$

$$\geq \frac{1}{\log (2r+1)\frac{\pi}{2}} \left|\int \cos x\right| \geq \frac{2}{\log (2r+1)\frac{\pi}{2}} \quad -(1)$$

$\sum \frac{1}{\log x}$ is dvgt $\to$ then so is (1)

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Real Functions

(1) Revise $f(x) \to l$ as $x \to a$ iff $\lim f(x_n) = l$ as $x_n \to a$ for any sequence of $\langle x_n \rangle$.

Pf: $\Rightarrow$ Trivial

$\Leftarrow$ Suppose $\langle f(x_n) \rangle \to l$ for any seq. $\langle x_n \rangle \to a$

If possible, let $\boxed{\lim_{x \to a} f(x) \neq l}$

This means, $\exists\ \varepsilon > 0\ (\varepsilon_1)$ s.t $\forall$ no δ s.t.

$|f(x_n) - l| < \varepsilon \qquad 0 < |x-a| < \delta$

Hence, for any δ, $\exists\ x = x(\delta)$ such that,

$x(\delta) \in \{x : 0 < |x-a| < \delta\}$ and $|f(x) - l| \geq \varepsilon_1$

Standard Trick

Choose $\boxed{\delta = 1/2, 1/3 \cdots 1/n \Rightarrow}$

Then for each δ, there is a value x_n such that.

$0 < |x_n - a| < 1/n \qquad |f(x_n) - l| \geq \varepsilon_1$

$\Rightarrow x_n \to a$ but $f(x_n) \not\to l \longrightarrow\leftarrow$

$\boxed{\ominus}$ $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

Consider $\langle x_n \rangle = \frac{1}{2n\pi}$, $\langle x_n' \rangle = \frac{1}{2n\pi + \pi/2}$.

$\langle x_n \rangle, \langle x_n' \rangle \to 0$ as $n \to \infty$

$f(x_n) = 0 \qquad f(x_n') = 1$ Contradiction

$\max\{f, g\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$

$\min\{f, g\} = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

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② Discontinuity

- 1st Kind ≡ Jump LHL ≠ RHL
- 2nd Kind ≡ LHL, RHL both do not exist
- Mixed ≡ One limit does not exist, other does.
- ∞ discont ≡ One or more lt is ∞/−∞.

$\boxed{Q}$ $f(x) = x \quad -1 \le x \le 1$

$\quad\quad = f(x+2) \quad \forall x \in R.$

Show f is discontinuous at every odd integer.

⇒ Let $x = 2n+1$.

$LHL \Rightarrow f(2n+1-0) = \lim_{h\to 0} f(2n+1-h) = \lim_{h\to 0} f(1-h) = 1$

$RHL \Rightarrow \lim_{h\to 0} f(2n+1+h) = \lim_{h\to 0} (1+h) = -1$ Unequal

So, discont. at every odd integer

IMP THEOREMS (Do Proofs)

① f is Continuous $[a,b]$ —
- f is bounded
- f possesses greatest and least values
- $f(a)f(b) < 0 \Rightarrow \exists c$ s.t $f(c) = 0$ [Location of roots theorem]
- $f(a) \ne f(b) \Rightarrow f$ assumes all values in $[f(a), f(b)]$

- Fixed Point Theorem : $f \in C[a,b]$ and $f(x) \in [a,b]$ for every $x \in [a,b]$, then f has a fixed point, that is, there exists $c \in [a,b]$ such that $f(c) = c$.

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* A function $f: R \to R$ is continuous on R iff for each open set G in R, $f^{-1}(G)$ is open in R.

$\Rightarrow$ Let f be continuous on R, G is any open set

TS: $f^{-1}(G)$ is open.

If $f^{-1}(G)$ is empty $\Rightarrow$ open ✓

If not, $\alpha \in f^{-1}(G)$. Then $f(\alpha) \in G$.

As G is an open set with $f(\alpha)$ $\exists\, \varepsilon > 0$, $(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon) \subset G$

As f is continuous at a, so $\exists\, \delta > 0$ such that

$$|x-\alpha| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon.$$

$$\Rightarrow x \in (\alpha-\delta, \alpha+\delta) \Rightarrow f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$$

$$\Rightarrow f(x) \in G \Rightarrow x \in f^{-1}(G).$$

$$\Rightarrow x \in (\alpha-\delta, \alpha+\delta) \subset f^{-1}(G) \Rightarrow f^{-1}(G) \text{ is } \underline{\underline{\text{open}}}$$

$\Leftarrow$ Let $f^{-1}(G)$ be open, whenever G is open

TS: f is continuous on R.

$a \in R$, Let $\varepsilon > 0$ be given. $(f(a)-\varepsilon, f(a)+\varepsilon)$ be open set.

So, $f^{-1}((f(a)-\varepsilon), f(a)+\varepsilon)$ is also open containing a.

$\exists\, \delta > 0$, $(a-\delta, a+\delta) \subset f^{-1}(f(a)-\varepsilon, f(a)+\varepsilon)$

$$\Rightarrow f(a-\delta, a+\delta) \subset (f(a)-\varepsilon, f(a)+\varepsilon)$$

So, for a given $\varepsilon > 0$, we have $\delta > 0$ such that

$$x \in (a-\delta, a+\delta) \Rightarrow f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$$

$$\Rightarrow |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon.$$

$\Rightarrow$ f is continuous at a.

Thms from S. 37

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$\boxed{6}$ Show

$$f(x) = \begin{cases} x & x \text{ is rational} \\ 1-x & x \text{ is irrational.} \end{cases}$$

is continuous only at $x = 1/2$.

Let $a \neq 1/2$ be a rational, $\Rightarrow f(a) = a$.

We can find an irrational a_n for each $n \in N$,

$$a_n \in (a - 1/n, a + 1/n)$$

$$\Rightarrow |a_n - a| < 1/n \Leftrightarrow \langle a_n \rangle \to a$$

$$f(a_n) = 1 - a_n \quad \forall\, n \in N.$$

$$\lim_{n\to\infty} f(a_n) = 1 - \lim_{n\to\infty} a_n = 1 - a \neq a = f(a)$$

So $\langle f(a_n) \rangle$ does not converge to $f(a)$.

So, f is discont. at all rationals a

Let $b \neq 1/2$ be irrational, $f(b) = 1 - b$.

We can find rational $b_n \in (b - 1/n, b + 1/n)$ $\forall n \in N$

$$\Rightarrow |b_n - b| < 1/n \Rightarrow \langle b_n \rangle \to b$$

$$\lim_{n\to\infty} f(b_n) = b \neq 1 - b = f(b)$$

TS f is continuous at $x = 1/2$.

$$|f(x) - f(1/2)| = \begin{cases} |x - 1/2| & \text{rational} \\ |x - 1/2| & \text{irrational.} \end{cases}$$

So, given $\varepsilon > 0$, $\delta = \varepsilon > 0 \Rightarrow$

$$|f(x) - 1/2| < \varepsilon \text{ ; when } |x - 1/2| < \delta$$

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8 $f(x+y) = f(x) + f(y)$. f being continuous – Show $f(x) = \alpha x$

- $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$.
- $f(x) = f(1) + f(1) + \dots = x f(1)$. $(x \geq 0)$
- $f(-x) \Rightarrow f(x + (-x)) = 0 \Rightarrow f(x) + f(-x) = 0$. $(x < 0)$

$\Rightarrow \underbrace{f(-x)}_{+ve} = -f(x)$.

$\Rightarrow f(-x) = (-x) f(1) = -f(x)$.

$\Rightarrow f(x) = x f(1)$.

- $x = p/q \Rightarrow f(p) = p f(1) = f(p/q \cdot q) = q \cdot f(p/q)$

$\Rightarrow p f(1) = q f(p/q) \Rightarrow f(p/q) = p/q \, f(1)$

- x be any real.

Let $\langle x_n \rangle$ be a sequence of rationals $\equiv \langle x_n \rangle \to x$.

Then as f is continuous $f(x_n) \to f(x)$

$f(x_n) = \alpha x_n \to \boxed{\alpha x}$ $\alpha = \underline{\underline{f(1)}}$.

Hence <u>Proved</u>

9 Given $a > 0$ and $n \in N$, Show there exists one and only one +ve real ξ such that $\xi^n = a$.

→ Consider $f(x) = x^n$ and Let,

- $a < 1 \Rightarrow f(0) = 0$ $f(1) = 1$

f attains all values $\in [0, 1] \Rightarrow$ By IVT, $\exists\, \xi \in [0,1]$ s.t $\xi^n = a$.

If η is also a positive number $\eta^n = a$

$\xi < \eta \Rightarrow \xi^n < \eta^n \Rightarrow a < a$ $\xi > \eta \Rightarrow a > a \Rightarrow \xi = \eta$.

- $a > 1 \Rightarrow 1/a < 1 \Rightarrow \exists\, \xi$ s.t $\xi^n = 1/a \Rightarrow 1/\xi^n = a$

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③ Uniform continuity $\delta(\varepsilon)$ IMP

f: continuous on a closed interval $\Rightarrow$ uniform continuity

Proving $\equiv$ $|f(x)-f(y)| < \varepsilon$. $|x-y| < \delta$.

Q $\frac{1}{x}$ is not uniformly cont on $(0,1]$.

For any $\delta > 0$, $|x-y| < \delta$.

$|f(x)-f(y)| = \left|\frac{y-x}{xy}\right|$

(and let $1/n < \delta$. $n \in N$
Let $x = 1/n$, $y = \frac{1}{2n}$.
Then $x, y \in (0,1]$)

$= |-n| = n \geq 1/2$.

$|x-y| = \frac{1}{2n} < \frac{1}{n} < \delta$.

So, taking $\varepsilon = \frac{1}{2} > 0$, whenever $\delta > 0$ $\exists\, x, y \in (0,1]$ such that $|f(x)-f(y)| > \varepsilon$ when $|x-y| < \delta$

$\Rightarrow$ Choose Suitable (x, y)

Q x^2 not uniformly cont on $\mathbb{R}$.

TS: given $\delta > 0$, $|x-y| < \delta$, $\exists\, \varepsilon > 0$ $|f(x)-f(y)| > \varepsilon$

$|x-y| < \delta$ $|x^2-y^2| > \varepsilon \Rightarrow |(x-y)(x+y)| > \varepsilon$

For any $\delta > 0$, $n\delta^2 > \varepsilon$ such n can be found

$x_1 = n\delta$, $x_2 = n\delta + \frac{\delta}{2}$. $|x_1 - x_2| = \delta/2 < \delta$.

$|x_2^2 - x_1^2| = \underbrace{n\delta^2}_{> \varepsilon} + \frac{1}{4}\delta^2 > \varepsilon$.

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$\boxed{9}$ Show $f(x) = 1/x^2$ is uniformly continuous $[a, \infty)$, $a > 0$, but not $(0, \infty)$

$\Rightarrow$ $[a, \infty)$

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| \left|\frac{1}{x} + \frac{1}{y}\right| \qquad x, y \geq a.$$

$$= \left|\frac{y-x}{xy}\right| \left|\frac{1}{x} + \frac{1}{y}\right| \qquad \frac{1}{x}, \frac{1}{y} \leq \frac{1}{a}.$$

$$\leq \frac{2}{a} \left|\frac{y-x}{xy}\right| \leq \frac{2}{a^3} |y-x|.$$

$$|x-y| < \delta \Rightarrow \qquad \varepsilon = \frac{2}{a^3} \delta \Rightarrow \boxed{\delta = \frac{\varepsilon a^3}{2}}$$

$\Rightarrow (0, \infty)$. $\qquad x_1 = \frac{1}{\sqrt{n}}, \quad y = \frac{1}{\sqrt{n+1}}.$

$$|f(x) - f(y)| = 1 > \varepsilon. \left(\text{Take } \varepsilon = \tfrac{1}{2}\right)$$

$$|x-y| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| = \frac{(\sqrt{n+1} - \sqrt{n})}{\sqrt{n}\sqrt{n+1}}$$

$$= \frac{1}{\underbrace{(\sqrt{n+1} + \sqrt{n})}_{> 2\sqrt{n}} \underbrace{(\sqrt{n})(\sqrt{n+1})}_{> \sqrt{n}}} < \frac{1}{2\sqrt{n}\sqrt{n}} < \frac{1}{2n}$$

$\left(\text{Take } \delta \geq \frac{1}{2n}\right)$

So Not uniformly continuous

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1999 • $\tan x$ is not uniformly continuous in $(0, \frac{\pi}{2})$

Consider $x_n = \frac{\pi}{2} - \frac{1}{2^n}$, $y_n = \frac{\pi}{2} - \frac{1}{4^n}$.

$$|\tan x_n - \tan y_n| = \left|\frac{\sin(x_n - y_n)}{\cos x_n \cos y_n}\right| = \frac{\sin(2^{-n} - 4^{-n})}{\sin(2^{-n})\sin(4^{-n})}$$

$$\sim \frac{2^{-n} - 4^{-n}}{4^{-n} 2^{-n}} \to \infty$$

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① Boundedness Theorem

If f is $C[a,b]$ then it is bounded.

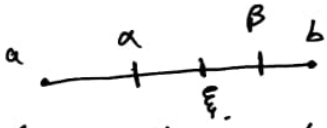
- Define Set $S = \{x : f \text{ is bounded in } [a,x]\}$

A. S is non-empty, since f is cont. at a, $\exists\, [a,c]$ in which f is bounded $\Rightarrow c \in S$.

B. S is bounded above as every member of $S \leq b$.

C. Order completeness of $R \Rightarrow$ Let ξ be l.u.b of S.

$\xi \neq b$ or $\xi = b$.



As f is cont. at $\xi \Rightarrow \exists\, [\alpha, \beta]$ s.t ξ lies within and f is bounded in $[\alpha, \beta]$. [Already f is bounded in $[a, \xi]$)

$\hookrightarrow \beta \in S \rightarrow \xi$ is not lub.

D. So, $\xi = b$. $\rightarrow$ f is bounded in $[a,b]$ ✓

② f is $C[a,b]$ then it possesses greatest and least values

A. f is $C[a,b] \Rightarrow f$ is bounded $\left[\begin{array}{l} \text{l.u.b} = M \\ \text{g.l.b} = m \end{array}\right.$

TS: $\exists\, \alpha, \beta$ s.t $f(\alpha) = M$, $f(b) = m$

B. If there does not exist x s.t $f(x) = M$

$\Rightarrow M - f(x) \neq 0 \Rightarrow \dfrac{1}{M - f(x)}$ is $C[a,b]$ and bounded.

C. Let k be given, As M is l.u.b $\Rightarrow \exists\, \xi \in [a,b]$

$f(\xi) > M - 1/k \Rightarrow \boxed{\dfrac{1}{M - f(\xi)} > k} \rightarrow$ not bounded.

So, f attains M for some $x_0 \in [a,b]$

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③ Bolzano's IVT

f is $C[a,b]$ and $f(a) \neq f(b)$, then f assumes every value between $f(a)$ and $f(b)$.

- Let $k \in (f(a), f(b))$
  Consider $\phi(x) = f(x) - k$. $\rightarrow$ $\phi(a), \phi(b)$ of opposite signs.

$\Rightarrow \exists\, \xi \in [a,b]$ s.t $\phi(\xi) = 0 \Rightarrow \boxed{f(\xi) = k}$

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Mean Value Theorems

Let M, m be l.ub, glb of f
c, d be $f(c) = M, f(d) = m$
One of c, d is different than a & b
let be c.
Show $f'(c) = 0$

① Rolle's Theorem

f with domain $[a, b]$

- $C[a,b]$
- $D(a,b)$
- $f(a) = f(b)$

Then $\exists\ c \in (a,b)$ such that $f'(c) = 0$

Q Between any two consecutive zeroes of P', there lies at most one zero of P.

Assume if 2 zeroes of P between zeroes of P' (α, β).
$\hookrightarrow c, d$ then $\exists\ \gamma \in (c, d)$ s.t $P'(\gamma) = 0$
Contradiction as α, β are consecutive

Q Between any two roots of $e^x \cos x = 1$ there lies $e^x \sin x - 1 = 0$.

Consider $F(x) = \cos x - e^{-x}$.

Roots $\alpha, \beta \Rightarrow \cos\alpha - e^{-\alpha} = 0 \Rightarrow e^{\alpha}\cos\alpha = 1.$
$\cos\beta - e^{-\beta} = 0 \Rightarrow e^{\beta}\cos\beta = 1.$ } Same roots.

Cont $[\alpha, \beta]$, $D(\alpha, \beta)$ So, $\exists\ \gamma \in [\alpha, \beta]$ s.t $F'(\gamma) = 0$.

$$-\sin x + e^{-x} = 0 \Rightarrow \boxed{e^{\gamma}\sin\gamma = 1} \Longrightarrow \text{Proved}$$

Q If f', g' exist and $g'(x) \neq 0$ then prove for some $c \in (a,b)$

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$

$$\boxed{\phi(x) = f(x)g(x) - f(a)g(x) - f(x)g(b).}$$

As f, g derivable so is ϕ

$\phi(a) = -f(a)g(b) = \phi(b).$

$\exists\ c \in (a,b)$ s.t $\phi'(c) = 0 \Rightarrow f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - f'(c)g(b) = 0.$

$$\Rightarrow \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)} \Longrightarrow$$

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* $p(x)$ is polynomial. Show between any two roots of $p(x)=0$ lies a root of $p'(x)+kp(x)=0$

$-\frac{p'(x)}{p(x)} = -k \Rightarrow e^{kx}p(x) = C.$ | Take $f(x) = e^{kx}p(x)$ and α, β be roots of $p(x)$

Then $f(\alpha) = f(\beta) = 0 \Rightarrow f'(c) = 0 \Rightarrow p'(c) + kp(c) = 0$ —

---

* Show $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c)$

Let $\boxed{\varphi(x) = f(x) + (a+h-x)f'(x) + \frac{1}{2}(a+h-x)^2 A}$

And $\varphi(a) = \varphi(a+h)$

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$\boxed{9}$ f satisfies $C[a, a+h]$, $D(a, a+h)$. Show c exists.

— $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(c)$

$\Rightarrow$ Construct $\Rightarrow \phi \equiv f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2} A$.

A is such that $\Rightarrow$ $\phi(a) = \phi(a+h)$.

Also by Rolle's $\Rightarrow$ $\phi'(c) = 0$

— $f(b) = f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 f''(c)$

Construct $\phi \equiv$ On putting $x_{(=a)}$ should obtain RHS, $f''(c)$ as A.

Same as previous.

$\phi(x) = f(x) + (b-x) f'(x) + \frac{1}{2}(b-x)^2 A$.

$\boxed{\phi(a) = \phi(b)} \Rightarrow f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 A = f(b)$

$\phi'(c) = 0 \Rightarrow \cancel{f'(c)} - \cancel{f'(c)} + (b-c) f''(c) + \frac{\cancel{2}}{\cancel{2}}(b-c) A = 0$

$f''(c) = -A = \frac{-2}{(b-a)^2}\left(f(b) - f(a) + (a-b) f'(a) \right)$

$-\frac{1}{2}(b-a)^2 f''(c) = f(a) - f(b) + (b-a) f'(a)$.

$$\boxed{f(b) = f(a) + (b-a) f'(a) + \frac{1}{2}(b-a)^2 f''(c)}$$

Ans

$\boxed{8}$ $f, g \in D(a,b)$, $f(a) = f(b) = 0$. Show $c \in (a,b)$
$f'(c) + f(c) g'(c) = 0$.

Take $F(x) = f(x) e^{g(x)}$

$\circledast$ $1 < 1 + \theta x \Rightarrow \frac{x}{1+\theta x} < x$.

$\llcorner$ 2

② LMVT

$f : [a,b] \to C[a,b]$, $D(a,b)$. ..(ms)

$\exists\, c \in (a,b)$ $\quad f'(c) = \dfrac{f(b)-f(a)}{b-a}$

To Prove $\Rightarrow$ $\phi(x) = f + Ax$
A s.t ϕ satisfies Rolle

⑤ f be such that $|f(x)-f(y)| < (x-y)^2 \;\forall\, x,y$.
Show f is constant.

Let $c \in R$ then for $x \neq c$.

$$\left|\frac{f(x)-f(c)}{x-c}\right| \leq |x-c|$$

Taking $|x-c| \leq \delta = \varepsilon$ given $\varepsilon > 0$. First given ε, then Select δ.

$$\left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon, \text{ when } |x-c| < \delta$$

$$\Rightarrow \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 0 \Rightarrow f'(c) = 0 \quad \forall c$$

⑥ Show $\dfrac{x}{1+x} < \log(1+x) < x$.

Take $f(x) = \log(1+x)$ in $[0,x]$ so that $f'(x) = 1/1+x$

$$\frac{f(x)-f(0)}{x-0} = f'(\theta x) \Rightarrow \frac{\log(1+x)}{x} = f'(c)$$

$$\boxed{f'(\theta x) = \frac{1}{1+\theta x}}$$

$\theta x < x$

$\dfrac{1}{1+\theta x} > \dfrac{1}{1+x}$

$\dfrac{x}{1+\theta x} > \dfrac{x}{1+x}$ ⊛

~~$f'(c) = \dfrac{1}{1+c}$ $\forall c \in (0,x)$~~

~~Consider $0 \leq c \leq x \Rightarrow$ $f'(c)$ being a decreasing function~~

~~$\dfrac{1}{1+x} \leq f'(c) \leq 1$ on $c \in [0,x]$~~ ①

~~$\dfrac{1}{1+x} \leq \dfrac{\log(1+x)}{x} \leq 1 \Rightarrow$~~ $\boxed{\dfrac{x}{1+x} < \log(1+x) < x}$

Ans

① + ② $\Rightarrow \dfrac{x}{1+x} < \dfrac{x}{1+\theta x} < x$

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$\boxed{Q}$ $f''(x) > 0 \quad \forall x \in R$ Show

$$f\left(\frac{1}{2}(x_1+x_2)\right) \leq \frac{1}{2}\left[f(x_1)+f(x_2)\right]$$

- $x_1 = x_2 \rightarrow$ Equality
- $x_1 \neq x_2$.

$$f''(x) > 0 \Rightarrow \cancel{\frac{f'(a) - f'(b)}{a-b} > 0 \quad \forall a,b \in R.} \quad \cancel{a > b}$$

Consider intervals,

$$\underset{c_1}{\left[x_1, \frac{x_1+x_2}{2}\right]} \quad \underset{c_2}{\left[\frac{x_1+x_2}{2}, x_2\right]}$$

$\exists\, c_1, c_2$

$$f(x_1+x_2/2) - f(x_1) = \left(\frac{1}{2}(x_1+x_2) - x_1\right) f'(c_1) \quad -(1)$$

$$f(x_2) - f\left(\frac{x_1+x_2}{2}\right) = \left(x_2 - \frac{1}{2}(x_1+x_2)\right) f'(c_2) \quad -(2)$$

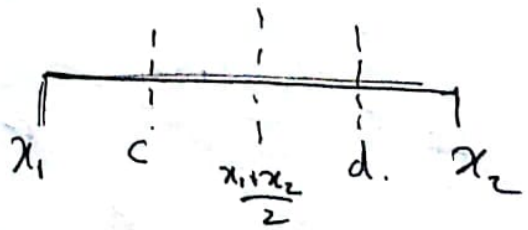
$(1) - (2) \Rightarrow$

$$2f\left((x_1+x_2)/2\right) - f(x_1) - f(x_2) = \left(\frac{x_2-x_1}{2}\right)\left[f'(c_1) - f'(c_2)\right]$$

(As $f'(c_2) - f'(c_1) = (c_2 - c_1) f''(d)$.
As $c_2 - c_1 > 0$ and $f''(d) > 0$.)

$\Rightarrow f'(c_1) - f'(c_2) < 0$. ; $\frac{x_2-x_1}{2} > 0 \Rightarrow$

$$\text{LHS} < 0 \Rightarrow \boxed{2f\left(\frac{x_1+x_2}{2}\right) < f(x_1) + f(x_2)}$$



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A. 2nd Way of Establishing Inequalities (Calculus)

$\boxed{Q}$ $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$ $x \geq 0$.

$f(x) = \log(1+x) - x + \frac{x^2}{2}$. at $x=0$ value $=0$
$x > 0$.

$f' \Rightarrow \frac{1}{1+x} - 1 + x$

$= \frac{x^2}{1+x} > 0 \quad \forall x \geq 0.$

So, increasing. Thus $\log(1+x) > x - \frac{x^2}{2}$

$g(x) = x - \frac{x^2}{2(1+x)} - \log(1+x)$

$g'(x) = 1 - \frac{2x}{2(1+x)} + \frac{x^2}{2(1+x)^2} - \frac{1}{1+x}.$

$\frac{\cancel{2} + \cancel{2x^2} + \cancel{2x} - \cancel{2x} - \cancel{2x^2} + x^2 - \cancel{2} - \cancel{2x}}{2(1+x)^2.} = \frac{x^2}{2(1+x)^2} > 0$

So, $\log(1+x) < x - \frac{x^2}{2(1+x)}$ Proved

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9 $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ $\quad 0 < x < \frac{\pi}{2}$.

$$f(x) = \frac{\tan x \sin x - x^2}{x \sin x}.$$

We need to find sign of $f(x)$ for $x > 0$,

$$x \sin x > 0 \ \forall x \in (0, \pi/2)$$

So, prove $\underbrace{\tan x \sin x - x^2}_{g(x)} > 0$

$$g'(x) = \sec^2 x \sin x + \cancel{\tan x \cos x}\ \sin x - 2x.$$

$$\cancel{g''(x) = 2\sec x \sec x \tan^2 x \sin x + \sec^2 x \cos x + \sec^2 x \cos x - \tan x \sin x} - 2.$$

$$= \cancel{2}$$

$$g''(x) = 2\sec x \tan^2 x + \sec x + \cos x - 2.$$

$$= \cancel{2\sec x(\sec^2 x - 1) + \sec x + \cos x} - 2.$$

$$= \cancel{2\sec^3 x - 2\sec x}$$

$$= \underbrace{2\sec^3 x \sin^2 x}_{>0} + \underbrace{(\sqrt{\sec} - \sqrt{\cos})^2}_{>0}$$

$\Rightarrow g'(x)$ is strictly increasing on $x \in (0, \pi/2)$.

$g'(0) = 0 \Rightarrow g'(x) > 0$

$\Rightarrow g(x) > 0$ Hence Proved

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9 CMVT.

F, f (i) $C[a,b]$ (ii) $D(a,b)$ (iii) $F'(x) \neq 0$

Then $\exists$ at least one point, $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}$$

$\Rightarrow$ Let $\phi(x) = f(x) + A F(x)$

$$\phi(a) = \phi(b) \Rightarrow \frac{f(a)-f(b)}{F(b)-F(a)} = A.$$

$\phi(a) = \phi(b) \rightarrow$ Using Rolle's

$$\phi'(c) = 0 \Rightarrow f'(c) + A F'(c) = 0 \Rightarrow \frac{f'(c)}{F'(c)} = -A \quad \text{Proven}$$

* Series

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots \frac{h^{n-1} f^{n-1}(a)}{(n-1)!} \quad \text{[Cauchy]} \quad \frac{h^n (1-\theta)^{n-1}}{(n-1)!} f^n$$

$$\frac{h^n}{n!} f^n(a+\theta h) \quad \text{[Lagrange]}$$

$$\text{Schlomilch} = \frac{h^n f^n(a+\theta h)(1-\theta)^{n-p}}{(n-1)!\, p}$$

$p = 1$ – Cauchy

$p = n$ – Lagrange.

TS: $R_n \to 0$ as $n \to \infty$

1. Generally Lagrange's form
2. $\log(1+x) \Rightarrow$ Cauchy's form

   $(1+x)^n \Rightarrow$ Cauchy's form

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B. 3rd Way of Establishing Inequalities (Series)

$\boxed{9}$ Show $x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$.

Let $x > 0$, Taylor's theorem about 0,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x) \quad (0 < \theta < 1)$$

$$\sin x = x - \frac{x^3}{6} \cos(\theta x).$$

$$x - \frac{x^3 \cos}{6} \geq x - \frac{x^3}{6} \Rightarrow \quad \sin x \geq x - \frac{x^3}{6}$$

Illy till 5 terms $\Rightarrow$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} \cos(\theta_1 x) \quad 0 < \theta_1 < 1$$

$$\sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}$$

Let $x < 0$, $(-x) > 0$ Put $y = -x$ in previous

$$x - \frac{x^3}{6} \geq \sin x \geq x - \frac{x^3}{6} + \frac{x^5}{120}$$

(Inequalities reverse for $x < 0$)

$\frac{x^n}{n!} f^n(\theta x)$

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1944 f, g, h continuous on $[a,b]$, $D'(a,b)$ show $\exists\ \xi, \eta \in [a,b]$ such that, $c \in (a,b)$

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{1}{2}(b-c)(c-a)(a-b) \begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix}$$

$\Rightarrow$ Consider $F(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}$

$$- \frac{(x-a)(x-b)}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

$F(a) = F(b) = 0 = F(c)$

$\exists\ x_1 \in (a,c)$, $x_2 \in (c,b)$

$F'(x_1) = F'(x_2) = 0$ $\exists$ Further Rolle $\Rightarrow \beta \in (x_1, x_2)$

$F''(\beta) = 0$

$$F''(\beta) = \begin{vmatrix} f(a) & f(b) & f''(\beta) \\ g(a) & g(b) & g''(\beta) \\ h(a) & h(b) & h''(\beta) \end{vmatrix} - \frac{2}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = 0$$

Consider

$$g(x) = \begin{vmatrix} f(a) & f(x) & f''(\beta) \\ g(a) & g(x) & g''(\beta) \\ h(a) & h(x) & h''(\beta) \end{vmatrix} - \frac{2(x-a)}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

$g(a) = g(b) = 0 \Rightarrow$ We get $g'(\alpha) = 0$ $\alpha \in (a,b)$

$$g'(\alpha) = 0 \Rightarrow \begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ g(a) & g'(\alpha) & g''(\beta) \\ h(a) & h'(\alpha) & h''(\beta) \end{vmatrix} = \frac{2}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$$

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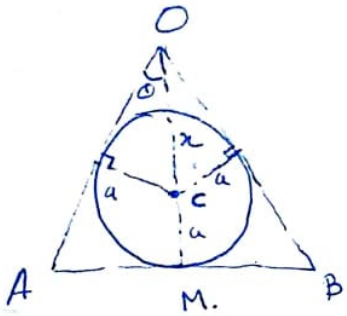
1999 Dimensions of Cone of max volume circumscribing a. sphere (a).



$\sin\theta = \frac{a}{x}$ $\quad AM = (a+x)\tan\theta.$

$$V = \frac{1}{3}\pi\,(a+x)^2\left(\frac{a}{(x^2-a^2)^{1/2}}\right)^2(a+x)$$

$$= \frac{\pi a^2}{3}\,\frac{(a+x)^2}{(x-a)}$$

$$\frac{dV}{dx} = 0 \Rightarrow \frac{\pi a^2}{3}\left(\frac{2(a+x)}{(x-a)} - \frac{(a+x)^2}{(x-a)^2}\right) = 0.$$

$$2x^2 - 2a^2 - a^2 - x^2 - 2ax = 0.$$

$$x^2 - 2ax - 3a^2 = 0 \Rightarrow \boxed{x = 3a}$$

$$\frac{+2a \pm \sqrt{16a^2}}{2}$$

$$\frac{2a \pm 4a}{2} = 3a, -a$$

$\boxed{Q}$ $\lim_{x\to\infty} \frac{x^n}{e^x} =$ diff n times $= \frac{n!}{e^x} \to 0.$

$\boxed{Q}$ $f(x) = e^{-1/x^2}\sin(1/x)$, $f(0)=0$. Show
$\llcorner$ f has differential coefficient, f' is cont. at $x=0$
$f' = 0$ at $x=0$ and at ∞ pts in every nbd of 0.

$$\Rightarrow f'(x) = e^{-1/x^2}\cdot\frac{2}{x^3}\sin(1/x) - \frac{e^{-1/x^2}\cos(1/x)}{x^2} = \frac{e^{-1/x^2}}{x^3}\left[2\sin(1/x) - x\cos(1/x)\right]$$

$$f'(0) = \lim_{x\to 0}\frac{e^{-1/x^2}\,2.\sin 1/x}{x^3}$$

$$e^{-1/x^2} > 1/x^2$$

$\Rightarrow$ Diff. Coefficient $\equiv$ Derivative

9.12 Q.12

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Real Numbers

① $\sqrt{8}$ is irrational.

As $2<\sqrt{8}<3$, Let $\sqrt{8}=p/q$ s.t

$$2q < p < 3q \Rightarrow 0 < p-2q < q.$$

As $p-2q$ is integer $<q \Rightarrow$

$$\frac{(p-2q)\,p}{q} \text{ is not an integer}$$

$$\frac{p^2}{q^2}q - 2p = 8q-2p \text{ is integer}$$

$\rightarrow\leftarrow$

ORDER COMPLETENESS $\Rightarrow$ Every set bdd above has lub (not for below) (need to prove

② $\mathbb{Q}$ is not order complete

$S: \{x : x\in\mathbb{Q},\ x\geq 0,\ x^2<2\}$

TS: Upper bounds of S has no least member

Let k be rational and l.u.b of $S \Rightarrow$

(i) $k^2=2$ (ii) $k^2>2$ (iii) $k^2<2$

$a_{n+1} = \frac{3+2a_n}{2+a_n}$ $a_0=0$

Show increasing, bounded limit is $\sqrt{3}$.

$\rightarrow k^2=2 \Rightarrow k$ is not a rational.

$\rightarrow$ Let $k^2>2$ and consider $y = \frac{4+3k}{3+2k}$

$$2-y^2 = \frac{2-k^2}{(3+2k)^2} < 0 \text{ as } k^2>2. \Rightarrow y^2>2 \rightarrow y \text{ is an ub of } S$$

$$k-y = \frac{2(k^2-2)}{3+2k} > 0 \Rightarrow y<k.$$

[We found y an upper bounded $<k$ (l u b) $\rightarrow\leftarrow$]

$\rightarrow$ Let $k^2<2$, same y.

$$2-y^2 = \frac{2-k^2}{(3+2k)^2} > 0 \qquad k-y = \frac{2(k^2-2)}{3+2k} < 0$$

Again $\rightarrow\leftarrow$

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③ Archimedean property of Reals

If x, y are reals with $x>0$, $\exists\, n \in N$ s.t $nx > y$.

$\Rightarrow$ Suppose if not possible $\Rightarrow nx \le y$.

Set $A = \{nx : n \in N\}$

A is bounded above with y an upper bound.

As R is order complete, A has a l.u.b $= \alpha$ — ①

$- \quad nx \le \alpha \quad \forall\, n \in N \Rightarrow (m+1)x \le \alpha \quad \forall\, m \in N$

$\Rightarrow mx \le \alpha - x \quad \forall\, m \in N \Rightarrow \alpha - x$ is an upper bound

$\boxed{\alpha - x < \alpha} \rightarrow\leftarrow$ ①

Thus, A is not bounded above $\Rightarrow nx > y$

(*)

④ Between any 2 real, there lies a rational

Let $x>0$, $x < r < y$

To find r: $\quad y > x \Rightarrow (y-x) > 0 \Rightarrow \boxed{\exists\, q \in n \text{ s.t } q(y-x) > 1}$ (*)

By Archimedean property of $\mathbb{R} \Rightarrow \exists$ natural n s.t $n > qx$

Let p be smallest such natural s.t $p > qx$

If $p = 1 \Rightarrow p-1 = 0 \Rightarrow p-1 < qx < 1 \Rightarrow p-1 < qx < p$

If $p \ge 2 \Rightarrow p-1$ is natural $\Rightarrow p-1 < qx$.

Now $qx < p\boxed{(p-1+1)} < qx + q(y-x)\boxed{qy}$

$\Rightarrow qx < p < qy \Rightarrow x < p/q < y$

$x \le 0$, $\exists\, k = k.1 > -x \Rightarrow k+x > 0$ (Arch.) $\quad k \in N$

$\exists$ a rational s, $k + x < s < k + y \Rightarrow x < s - k < y$

(*) $\frac{1}{q} < (y-x)$. Consider $\left(\frac{1}{q}, \frac{2}{q}, \dots \frac{n}{q}\right)$ s.t $\frac{n}{q}$ is greatest s.t $n/q \le x$

$\frac{n+1}{q} \ge x$; $\quad \frac{n+1}{q} < y \rightarrow$ if not. $y \le \frac{n+1}{q} \rightarrow \frac{1}{q} = \frac{n+1}{q} - \frac{n}{q} \ge (y-x)$

$\rightarrow\leftarrow$

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- Limit point of a set $S \rightarrow$ If every nbd of P contains a pt of $S-\{P\}$
  - Every nbd of a point p contains infinitely many points of S.

$\boxed{S}$ $S = \{1/n : n \in N\}$ has only one lt point 0.

0 is lt pt $\Rightarrow$ Consider $\varepsilon > 0$, $(-\varepsilon, \varepsilon)$

By Arch. property of reals $\rightarrow \exists\ m\varepsilon > 1 \Rightarrow \frac{1}{m} < \varepsilon$.

$-\varepsilon < 0 < \frac{1}{m} < \varepsilon$. So each nbd of 0 contains a pt of $S-\{0\} \Rightarrow$ Hence 0 is lt points

$p < 0 \Rightarrow \exists\ (-\infty, 0)$ of p s.t $(-\infty, 0) \cap S = \phi$

$p > 1 \Rightarrow \exists\ (1, \infty) \cap S = \phi$

$0 < p < 1 \Rightarrow \exists\ m$ s.t $\frac{1}{m+1} < p < 1/m$

$p = 1 \Rightarrow (1/2, 1)$ contains no other pt.

$p = 1/m \Rightarrow \left(\frac{1}{m+1}, \frac{1}{m-1}\right)$

- Irrationals with rational sup $\Rightarrow (\sqrt{n} - \sqrt{n-1})$.
- Irrational between rationals $(r_1, r_2) \Rightarrow$ $0 < 1/\sqrt{2} < 1 \Rightarrow 0 < \frac{1}{\sqrt{2}}(r_2 - r_1) < (r_2 - r_1)$

  $\Rightarrow r_1 < \underbrace{\frac{1}{\sqrt{2}}(r_2 - r_1) + r_1}_{\text{irrational}} < r_2$
- Infinite, Unbounded with limit point
  $\hookrightarrow \{1/2, 2, 1/3, 3, 1/4, 4 \ldots\}$.
- $(A \cup B)' = A' \cup B'$
- $(A \cap B)' \subset A' \cap B'$
  Example $(1,2)$, $(2,3)$

$*$ Union of arbitrary closed is not closed. $I_n = [1/n, 1]$

$\bigcup_{\infty} I_n = (0, 1]$

- Proofs of closed $\rightarrow$ Show complement is open.
- Open but not interval $(a,b) \cup (c,d)$.

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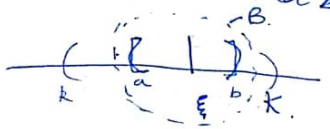
Bolzano - Weierstrass Theorem

Every bounded and infinite set has a limit point.

- Let A be infinite and bounded $\rightarrow A \subset [k, K]$



Define S: $\{x \in S$ iff it exceeds a finite # of elements of $A\}$

$k \in S$, $K \notin S$, So $S \neq \phi$, S is bdd above

Let ξ be l.u.b of S.

$(\xi - \varepsilon, \xi + \varepsilon)$ contains ∞ elems A.
So, ξ is limit point.

Consider an open interval (a, b) around S s.t

$(a, b) \subset B \leftarrow B$ is nbd of $\xi \subset S$.

As a is $<$ ξ which is lub, a exceeds at most finite S.

$b > \xi \Rightarrow b \notin S \Rightarrow b$ exceeds infinite S.

$\Rightarrow (a, b)$ contains ∞ pts of S, (a, b) contains ∞ pts of A.

$\Rightarrow \xi$ is a lt pt of S

TS: Neighbourhood $\equiv$ all points $(a-\varepsilon, a+\varepsilon) \in S$
Limit point $\equiv$ only one pt $(a-\varepsilon, a+\varepsilon) \in S$.

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⑥ Denumerable Set. $(A \sim N)$

If $\exists$ a 1-1 mapping $N \xrightarrow{onto} A$

- Countable → Finite or Denumerable

2° * Countable Union of Countable sets is countable.

$$\begin{array}{llll} A_1 = a_{11} & a_{12} & a_{13} & \cdots \\ A_2 = a_{21} & a_{22} & a_{23} & \\ A_3 = a_{31} & a_{32} & a_{33} & \end{array}$$

Count elements as per the sum of their suffices

$2 = a_{11}$ $\quad 3 - a_{12}, a_{21}$ $\quad 4 \to a_{13}, a_{31}, a_{22}$

So all elements can be counted out

* Set of rationals is countable

$$\begin{array}{lll} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} \end{array}$$

(Same)

2° * $[0, 1]$ is uncountable

Let it be countable

$x_1 = 0 \cdot a_{11} a_{12} \cdots$

$x_2 = 0 \cdot a_{21} a_{22} \cdots$

$x_3 = 0 \cdot a_{31} a_{32} a_{33} \cdots$

$x_4 = 0 \cdot a_{41} a_{42} a_{43} a_{44}$

$y = 0 \cdot b_1 b_2 b_3 \cdots$ where $b_n = 1$ where $a_{nn} \neq 1$, $b_n = 2$ o.w

So, y is not enumerated in any x_i.

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* $(0,1)$ and $[0,1]$ are equivalent

$(0,1) = A \cup [1/2, 1/3, 1/4 \ldots]$

$[0,1] = A \cup [0, 1, 1/2, 1/3 \ldots]$

Map : $f : [0,1] \to (0,1)$

$= x$ when $x \in A$

$= \frac{1}{n+2}$ $\quad x = 1/n \quad n \in N$

$= \frac{1}{2}$ $\quad x = 0.$

<u>f is 1-1 onto map</u>

* (a,b) equivalent to (c,d).

$$f(x) = c + \left(\frac{d-c}{b-a}\right)(x-a).$$

* R equivalent to $[0,1]$

Consider $\to (-\frac{\pi}{2}, \frac{\pi}{2})$ $\quad \tan x.$

$\tan x$ is 1-1, onto $(-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}.$

Map $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $[0,1]$

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Neighbourhoods of Set.

- A point p is a limit point iff every nbd of p contains infinitely many points of S.

2009 Every bounded and infinite set has a limit point

Let A be an infinite and bounded set.

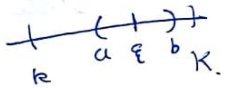
So $A \subset [k, K]$.

Let S be a set such that $S = \{x \mid x >$ finite elements of $A\}$

$S \neq \phi$ as $k \in S$. S is bounded above as K is an upper bound.

Let ξ be lub of S.

Consider a nbd of ξ B s.t $(a,b) \subset B$



Since $a \in S \Rightarrow a$ is $>$ finite pts of A.

$b > \xi \Rightarrow b$ is $> \infty$ pts of A. as $b \notin S$.

Thus, (a,b) contains an ∞ly many members of A

Thus $(a,b) \subset B$ nbd of $\xi \Rightarrow \xi$ is a lt point

2013 Every open set is a union of open intervals

$A \neq \phi$ be an open set, $a_\lambda \in A$

Consider $A = \bigcup_{\lambda \in \Lambda} \{a_\lambda\}$ as union of singletons

Since A is open, it is a neighbourhood of each of its points.

$\forall a_\lambda \in A$ there exists open interval I_{a_λ} s.t $a_\lambda \in I_{a_\lambda} \subset A$

$\Rightarrow \bigcup I_{a_\lambda} \subset A$, $\bigcup \{a_\lambda\} \subset \bigcup I_{a_\lambda}$

$\Rightarrow A \subset \bigcup I_{a_\lambda} \rightarrow \boxed{A = \bigcup_\lambda I_{a_\lambda}}$

Revisit Thms of Chapter 3

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* Every infinite set is equivalent to one of its proper sub-sets

→ Every infinite set has a countable sub-set.

$C = \{a_1, a_2, \ldots a_n, \ldots\}$ be countable sub-set of A

Let $D = A - \{a_1\}$ and $B = (A - C) \cup D$.

$a_1 \in A$ and $a_1 \notin B$ so B is proper subset of A.

$f : A \to B$

$f(x) = x \quad \{x \in A - C$ and $f(a_r) = a_{r+1} \ \forall r = 1, 2, 3\}$.

Since f is 1-1 from A to B, so $A \sim B$.

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* Between any two real lies a rational.

$\rightarrow$. $x, y \in R$, $x \neq y$.

Suppose $y > x$, $\exists\, z = y - x$.

By Archimedes' axiom $\rightarrow$ $\exists$ natural n such that.

$n(y-x) > 1 \Rightarrow \exists$ integer m such that.

$nx < m < ny \Rightarrow \boxed{x < \frac{m}{n} < y.}$ Ac

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① $x_n = \underbrace{\dfrac{x_{n-1}+y_{n-1}}{2}}$, $y_n = \sqrt{x_{n-1}\,y_{n-1}}$ HM < GM < AM

Let $x_1 > y_1$

$y_{n-1} < x_n < x_{n-1}$

$x_{n-1} > y_{n-1} \Rightarrow \sqrt{x_{n-1}y_{n-1}} > \sqrt{y_{n-1}^2} > y_{n-1}$

$\Rightarrow \boxed{y_n > y_{n-1}}$

$x_1 > x_2 \ldots > x_n > y_n > \ldots > y_1$

- $\dfrac{x_{n-1}+y_{n-1}}{2} < x_{n-1} \Rightarrow \boxed{x_n < x_{n-1}}$
- $AM > GM \Rightarrow \boxed{x_n > y_n}$

$x_n > y_n > y_{n-1} > y_{n-2} \ldots$

---

② $g(x) = xf(x)\ \forall x$. Show if f is cont at $x=0$, g is derivable.

$\rightarrow f(x)$ is cont $\Rightarrow \lim\limits_{x\to 0} f(x) = f(0)$

$$\frac{g(h)-g(0)}{h} = \lim_{h\to 0} \frac{\cancel{h}f(h)}{\cancel{h}} = f(0) \Rightarrow g \text{ is derivable.}$$

↳ If $f(x+y) = f(x)f(y)$ and $f(x) = 1 + xg(x)$ where $\lim\limits_{x\to 0} g(x) = 1$

Show $f'(x)$ exists and $f'(x) = f(x)$ for all x.

$$\rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)\,(f(h)-1)}{h} = \lim_{h\to 0} \frac{f(x)\,g(h)\,\cancel{h}}{\cancel{h}}$$

$$= f(x).1 = f(x)$$

---

③ If $f(x) = e^{-1/x^2}\sin(1/x)$, $f(0)=0$. Show

(i) f has at every point a differential coefficient and this is cont at 0

$\rightarrow$

$$f'(x) = \frac{2e^{-1/x^2}\sin(1/x)}{x^3} - \frac{e^{-1/x^2}\cos(1/x)}{x^2}$$

$$f'(0) = \lim_{x\to 0} \frac{e^{-1/x^2}}{x^3}\left(2\sin 1/x - x\cos 1/x\right) = 0.$$

TS: f' is cont at $x=0 \Rightarrow$

$$|f'(x) - f'(0)| = \left|\frac{e^{-1/x^2}}{x^3}\left(2\sin 1/x - x\cos 1/x\right)\right| \le \frac{e^{-1/x^2}}{|x|^3}(2+|x|)$$

$|f'(x) - f'(0)| \to 0$ as $x \to 0$ $\boxed{\text{So is continuous for } x=0}$

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(4) $\underline{\lim}\ \inf a_n = \sup \inf \cancel{\{a_n\}} \{a_n, a_{n+1}, \dots\}$ $a_n = (-1)^n/n^2$.

n even 1

$\overline{\lim}\ \sup a_n = \inf \sup \{a_n\}$ — Consider all $\geq n$.

<u>For Eg</u> $a_n = \frac{(-1)^n}{n^2}$

$$\underline{\lim}\ a_n = \sup \inf \left\{ \frac{(-1)^n}{n^2}, \frac{(-1)^{n+1}}{(n+1)^2}, \dots \right\}$$

$$= \sup_n \begin{cases} \frac{(-1)^n}{n^2} \text{ odd} \to 0 \\ \frac{(-1)^{n+1}}{(n+1)^2} \text{ even} \to 0 \end{cases} = 0.$$

(5) Show $b_n \to 0$

$$b_n = \left[\frac{1}{(n+1)^2} + \dots \frac{1}{(2n)^2} \right]$$

$$\underset{\to 0}{\frac{n}{(2n)^2}} \leq b_n \leq \underset{\to 0}{\frac{n}{n^2}}$$ Using Sandwich Theorem

(6) f is twice differentiable f s.t $f(a) = f(b) = 0$ and $f(c) > 0$ $a < c < b$. Prove $\exists\ \xi \in (a, b)$ s.t $f''(\xi) < 0$.

$\to$ Let $c \in (a, b) \to$

on $[a, c]$ $\quad \frac{f(c) - f(a)}{c - a} = f'(\xi_1)$, $\quad a < \xi_1 < c$.

$[c, b]$ $\quad \frac{f(b) - f(c)}{b - c} = f'(\xi_2)$, $\quad c < \xi_2 < b$

$$(>0)\ f'(\xi_1) = \frac{f(c)}{c - a}, \quad \underbrace{f'(\xi_2) = \frac{f(c)}{c - b}}_{(1)} \ (<0).$$

$a < \xi_1 < c < \xi_2 < b$ $\qquad \gamma \in (\xi_1, \xi_2)$

$$f'(x) \to \frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\gamma)$$

< 0 from (1)

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⑦ $\lim_{n\to\infty} \int_a^b f(x) \cos nx \, dx = 0$

Let $I_n = \int_a^b f(x) \cos nx \, dx$, $\varepsilon > 0$ be arbitrary

Choose $P = \{a = x_0, \ldots, x_p = b\}$ such that,

Osc. Sum $\Rightarrow$ $U(P, f) - L(P, f) = \sum_{i=1}^{p} (M_i - m_i) \Delta x_i < \varepsilon/2$

M_i, m_i bounds of f in Δx_i

$$I_n = \sum \int_{x_{i-1}}^{x_i} f(x) \cos nx \, dx = \sum f(x_{i-1}) \int_{x_{i-1}}^{x_i} \cos nx \, dx$$

$$+ \sum \int_{x_{i-1}}^{x_i} \{f(x) - f(x_{i-1})\} \cos nx \, dx$$

$$|I_n| \leq \sum_1^p |f(x_{i-1})| \left| \int_{x_{i-1}}^{x_i} \cos nx \, dx \right| + \sum_1^p \int_{x_{i-1}}^{x_i} (M_i - m_i) \, dx.$$

$$\leq \left|\frac{1}{n}(2)\right| \sum_1^p |f(x_{i-1})| \qquad < \varepsilon/2.$$

$$\boxed{|I_n| \leq \frac{2}{n} \sum_1^p |f(x_{i-1})| + \frac{\varepsilon}{2}.}$$

For P fixed, $f(x_{i-1})$ is also fixed

We can find s.t.

$$\frac{2}{n} \sum_1^p |f(x_{i-1})| < \varepsilon/2 \Rightarrow |I_n| < \varepsilon \quad \forall n \geq m$$

$$\lim_{n\to\infty} I_n = 0 \text{ Proved}$$

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