Mains Test Series - 2019

Test - 4 (Paper - II)

Answer Key

PDE, Numerical Analysis, Computer Programming, Mechanics & Fluid Dynamics

SECTION - A

Q.1. (a) Solve $x^2p^2 + y^2q^2 = z^2$

Solution:

The given equation can be rewritten as

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1$$

(or) $$\left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{--- (1)}$$

Put $\frac{1}{x}\,dx = dX$, $\frac{1}{y}\,dy = dY$ and $\frac{1}{z}\,dz = dZ$ --- (2)

$\Rightarrow \log x = X$, $\log y = Y$ and $\log z = Z$ --- (3)

Using (2), (1) becomes $\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dZ}{dY}\right)^2 = 1$

(or) $P^2 + Q^2 = 1$, --- (4)

where $P = \frac{dZ}{dX}$ and $Q = \frac{dZ}{dY}$.

(4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $Z = aX + bY + c$, — (5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$,

putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c$$

$$\Rightarrow Z = X\cos\alpha + Y\sin\alpha + \log c'$$

[Taking $a = \cos\alpha$ and $c = \log c'$]

$$\Rightarrow \log z = \cos\alpha \log x + \sin\alpha \log y + \log c'$$

$$\Rightarrow \boxed{z = c' x^{\cos\alpha} y^{\sin\alpha}} \quad \text{— (6)}$$

<u>To determine singular integral,</u>

Differentiating (6) partially w.r.t. α and c' successively, we obtain

$$0 = c'\cos\alpha . x^{\cos\alpha} y^{\sin\alpha} \log y - c'\sin\alpha . x^{\cos\alpha} y^{\sin\alpha} \log x \quad \text{— (7)}$$

and $0 = x^{\cos\alpha} y^{\sin\alpha}$ — (8)

Eliminating α and c' from (6), (7) and (8), the singular solution is $z = 0$.

To determine general integral :

Putting $c' = \phi(\alpha)$, where ϕ is an arbitrary function, (6) becomes

$$z = \phi(\alpha) . x^{\cos\alpha} y^{\sin\alpha} \quad \text{——— (9)}$$

Differentiating (9), partially, w.r.t. 'α', we get

$$0 = \phi'(\alpha) x^{\cos\alpha} y^{\sin\alpha} + \phi(\alpha) \left\{ x^{\cos\alpha} y^{\sin\alpha} \cos\alpha - y^{\sin\alpha} x^{\cos\alpha} \sin\alpha \right\} \quad \text{——— (10)}$$

The required general integral is obtained by eliminating α from (9) and (10).

Q.1. (b) → Solve $(D^2 - 4D'^2) z = (4x/y^2) - (y/x^2)$

Solution:

Here A.E. is $m^2 - 4 = 0$

$\Rightarrow m = 2, -2.$

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I. = \frac{1}{(D+2D')(D-2D')} \left(\frac{4x}{y^2} - \frac{y}{x^2} \right)$$

$$= \frac{1}{D+2D'} \int \left\{ \frac{4x}{(c-2x)^2} - \frac{c-2x}{x^2} \right\} dx,$$

where $c = y + 2x$.

$$= \frac{1}{D+2D'} \int \left\{ -\frac{2}{c-2x} + \frac{2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x} \right\} dx$$

$$= \frac{1}{D+2D'} \left\{ \log(c-2x) + \frac{c}{c-2x} + \frac{c}{x} + 2\log x \right\}$$

$$= \frac{1}{D+2D'} \left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + 2\log x \right]$$

$$= \int \left\{ \log(c'+2x) + 1 + 2 \cdot \frac{x}{c'+2x} + \frac{c'+2x}{x} + 2 + 2\log x \right\} dx$$

[Taking $c' = y - 2x$]

$$= x\log(c'+2x) + 5x + c\log x + 2x\log x - 2x$$

$$= x\log y + y\log x + 3x, \quad \text{as } c' = y - 2x$$

$$\therefore \boxed{z = \phi_1(y+2x) + \phi_2(y-2x) + x\log y + y\log x + 3x.}$$

Hence, the result.

Q.1. (c) → Obtain the Newton-Raphson extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}$$

for the root of the equation $f(x)=0$.

Solution:

Expanding $f(x)$ in the neighbourhood of x_0 by Taylor's series, we get

$$0 = f(x) = f(x_0 + x - x_0) = f(x_0) + (x-x_0)f'(x_0)$$

$$\Rightarrow x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Hence, the first approximation to the root is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$.

Again by Taylor's series, we obtain

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2}(x-x_0)^2 f''(x_0)$$

$$\therefore \quad f(x_1) = f(x_0) + (x_1-x_0)f'(x_0) + \frac{1}{2}(x_1-x_0)^2 f''(x_0).$$

Since x_1 is an approximation to the root so $f(x_1) = 0$.

Hence,

$$f(x_0) + (x_1-x_0)f'(x_0) + \frac{1}{2}(x_1-x_0)^2 f''(x_0) = 0$$

$$\Rightarrow f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2} \frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^2} = 0$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}$$

This formula is known as Chebyshev formula of third order.

Hence, proved.

Q.1. (d) → (i) Simplify the expression $AB + \overline{AC} + A\bar{B}C\,(AB+C)$.

(ii) Simplify the given Boolean expression

$$Y = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C}.$$

Solution:

(i)

$$AB + \overline{AC} + A\bar{B}C(AB+C) = AB + \overline{AC} + A\bar{B}C\cdot AB + A\bar{B}C\cdot C$$

$$= AB + \overline{AC} + A\bar{B}C$$

$$[\because C\cdot C = C \text{ and } \bar{B}\cdot B = 0]$$

$$= AB + \bar{A} + \bar{C} + A\bar{B}C$$

$$= AB + \bar{A} + \bar{C} + \bar{B}C$$

$$[\because A + \bar{A}B = A + B]$$

$$= \bar{A} + AB + \bar{C} + C\bar{B}$$
$$= \bar{A} + B + \bar{C} + \bar{B}$$
$$= \bar{A} + \bar{C} + 1 \quad [\because B + \bar{B} = 1]$$
$$= \bar{A} + 1$$
$$= 1$$

i.e. $\boxed{AB + \bar{A}\bar{C} + A\bar{B}C(AB + C) = 1}$ —— (i)

(ii)

$$Y = \bar{A}\bar{B}\bar{C} + \bar{A}BC\bar{} + A\bar{B}\bar{C} + AB\bar{C}$$
$$= \bar{A}\bar{C}(B + \bar{B}) + A\bar{C}(B + \bar{B})$$
$$= \bar{A}\bar{C} + A\bar{C} \quad [\because B + \bar{B} = 1]$$
$$= \bar{C}(\bar{A} + A)$$
$$= \bar{C} \cdot 1 \quad [\because \bar{A} + A = 1]$$
$$= \bar{C}$$

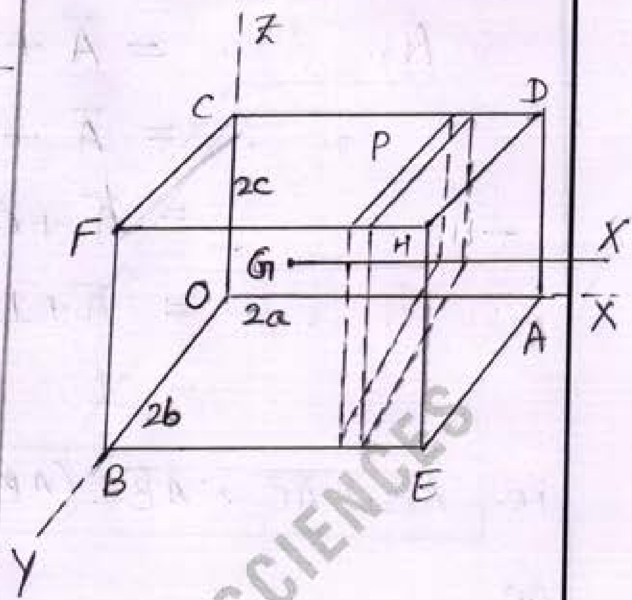
i.e. The given Boolean expression $Y = \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C}$ is equivalent to $\underline{\bar{C}}$.

Q.1. (c) → Find the M.I. of a rectangular parallelopiped about an edge.

Solution:

Let $2a$, $2b$, $2c$ be the lengths of the edges of a rectangular parallelopiped of mass M.

$\therefore$ M.I. of the rectangular parallelopiped about the edge OA = M.I. of the rectangular parallelopiped about a parallel axis GX′ through its C.G. 'G' + M.I. of total mass M at C.G. 'G' about OA.



$$= \frac{M}{3}(b^2+c^2) + M(\text{perpendicular distance of } G \text{ from } OA)^2$$

$$= \frac{M}{3}(b^2+c^2) + M(b^2+c^2)$$

$$= \boxed{\frac{4}{3}M(b^2+c^2)}.$$

Alternatively,

Consider an element $\delta x\, \delta y\, \delta z$ at the point P whose co-ordinates referred to the rectangular axes along edges OA, OB, OC are (x, y, z).

$\therefore$ M.I. of this element about OA $= (\rho\, \delta x\, \delta y\, \delta z)\cdot(y^2+z^2)$

$\therefore$ M.I. of the rectangular parallelopiped about OA $= \int_{x=0}^{2a}\int_{y=0}^{2b}\int_{z=0}^{2c} \rho(y^2+z^2)\, dx\, dy\, dz$

$$= \boxed{\frac{4}{3}(b^2+c^2)M} \qquad \left[\because \rho = \frac{M}{8abc}\right]$$

Hence, the result.

Q.2. (a) → Reduce the equation $u_{xx} + xu_{yy} = 0$, $x \neq 0$ for all x, y to canonical form.

Solution:

The discriminant $B^2 - 4AC = -4x$. Hence the given PDE is of mixed type: hyperbolic for $x<0$ and elliptic for $x>0$.

Case I: In the half-plane $x<0$, the characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2-4AC}}{2A} = \frac{-2\sqrt{-x}}{2}$$

$$= -\sqrt{-x}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2-4AC}}{2A} = \sqrt{-x}$$

Integration yields

$$y = \frac{2}{3}(-x)^{3/2} + c_1$$

$$y = -\frac{2}{3}(-x)^{3/2} + c_2$$

Therefore, the new coordinates are

$$\xi(x,y) = \frac{3}{2}y - (\sqrt{-x})^3 = c_1$$

$$\eta(x,y) = \frac{3}{2}y + (\sqrt{-x})^3 = c_2$$

which are cubic parabolas.

In order to find the canonical equations, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$= -\frac{9}{4}x + 0 + \frac{9}{4}x = 0$$

$$\bar{B} = 9x, \quad \bar{C} = 0, \quad \bar{D} = -\frac{3}{4}(-x)^{-1/2} = -\bar{E},$$

$$\bar{F} = \bar{G} = 0$$

Thus, the required canonical equation is

$$9x\,u_{\xi\eta} - \frac{3}{4}(-x)^{-1/2}u_\xi + \frac{3}{4}(-x)^{-1/2}u_\eta = 0$$

or

$$\boxed{u_{\xi\eta} = \frac{1}{6(\xi-\eta)}(u_\xi - u_\eta)}$$

Case II:

In the half-plane $x > 0$, the characteristic equations are given by

$$\frac{dy}{dx} = i\sqrt{x}, \quad \frac{dy}{dx} = -i\sqrt{x}$$

On integration, we have

$$\xi(x,y) = \frac{3}{2}y - i(\sqrt{x})^3,$$

$$\eta(x,y) = \frac{3}{2}y + i(\sqrt{x})^3$$

Introducing the second transformation

$$\alpha = \frac{\xi+\eta}{2}, \quad \beta = \frac{\xi-\eta}{2i}$$

we obtain

$$\alpha = \frac{3}{2}y, \quad \beta = -(\sqrt{x})^3$$

The corresponding normal or canonical form is

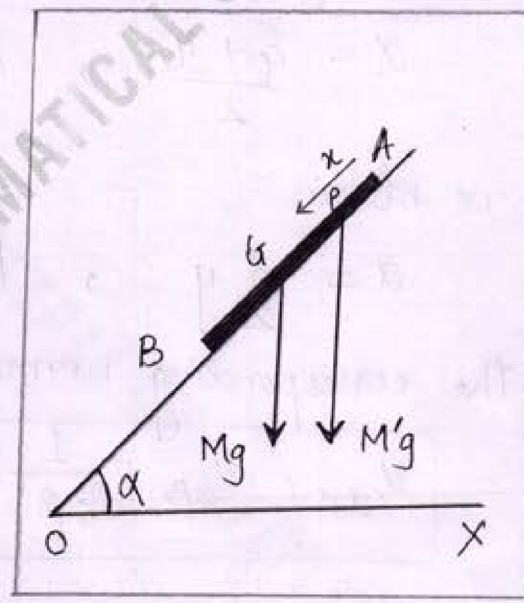
$$\boxed{u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\beta}u_\beta = 0}$$

Hence the result.

Q. 2. (b) → A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M' starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time

$$\sqrt{\left\{\frac{2M'a}{(M+M')g\sin\alpha}\right\}},$$ where 'a' is the length of the plane.

Solution:



Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an angle α to the horizon.

A man of mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance $AP = x$ in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C.G. of the plank and the man from A in this position, then

$$\bar{x} = \frac{M \cdot AG + M' \cdot AP}{M+M'} = \frac{M \cdot (a/2) + M' \cdot x}{M+M'}$$

Differentiating twice w.r.t. 't', we have,

$$\ddot{\bar{x}} = \frac{M'}{M+M'} \ddot{x}, \quad \text{———} (1)$$

Now the total weight $(M+M')g$ will act vertically downwards at the C.G. of the system.

$\therefore$ The equation of motion of the C.G. of the system is given by

$$(M+M')\ddot{\bar{x}} = (M+M')g \sin\alpha \quad \text{———} (2)$$

$\therefore$ from (1) and (2), we get,

$$M'\ddot{x} = (M+M')g\sin\alpha$$

Integrating, we get $M'\dot{x} = (M+M')g\sin\alpha . t + c_1$

But initially when $t=0$, $\dot{x}=0$

$\therefore c_1 = 0$.

$\therefore M'\dot{x} = (M+M')g\sin\alpha . t$

Integrating again, we get $M'x = M+M' g \sin\alpha \cdot \frac{t^2}{2} + c_2$.

Initially, when $t=0$, $x=0$ $\therefore c_2 = 0$

$\therefore M'x = (M+M')g\sin\alpha . \frac{1}{2}t^2$

$$\Rightarrow \quad t = \sqrt{\left\{\frac{2M'x}{(M+M')g\sin\alpha}\right\}}$$

Putting $x = AB = a$, the time to reach the other end B of the plank is given by

$$t = \sqrt{\left\{\frac{2M'a}{(M+M')g\sin\alpha}\right\}}$$

Hence, proved.

Q.2. (c) → Four equidistant values u_{-1}, u_0, u_1 and u_2 being given, a value is interpolated by Lagrange's formula. Show that it may be written in the form

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0 \quad \text{where } x+y=1.$$

Solution:

We have

$$\Delta^2 u_{-1} = (E-1)^2 u_{-1} = (E^2 - 2E + 1)u_{-1}$$

$$= u_1 - 2u_0 + u_{-1}.$$

Similarly,

$$\Delta^2 u_0 = (E^2 - 2E + 1) u_0 = u_2 - 2u_1 + u_0$$

Now the R.H.S.

$$= y u_0 + x u_1 + y\frac{(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

$$= (1-x)u_0 + x u_1 + \frac{(1-x)x(x-2)}{3!}(u_1 - 2u_0 + u_{-1})$$

$$+ \frac{x(x+1)(x-1)}{3!}(u_2 - 2u_1 + u_0)$$

$$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \Big[-(x-1) + \tfrac{1}{3}x(x-1).$$

$$(x-2) + \tfrac{1}{6}(x-1)x(x+1)\Big]u_0 +$$

$$\Big[x - \frac{x(x-1)(x-2)}{6} - \frac{(x-1)x(x+1)}{3}\Big]u_1 +$$

$$\frac{(x-1)x(x+1)}{6}u_2$$

$$= -\frac{x(x-1)(x-2)}{6}u_{-1} + \frac{(x-2)(x-1)(x+1)}{2}u_0$$

$$- \frac{(x-2)x(x+1)}{2}u_1 + \frac{(x-1)x(x+1)}{6}u_2 \quad \text{——— (1)}$$

For u_{-1}, u_0, u_1, u_2 as known values, by Lagrange's formula, we have

$$u_x = \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} u_{-1} + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} u_0$$

$$+ \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} u_1 + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} u_2$$

$$= -\frac{x(x-1)(x-2)}{6} u_{-1} + \frac{(x+1)(x-1)(x-2)}{2} u_0$$

$$-\frac{(x+1)x(x-2)}{2} u_1 + \frac{(x+1)x(x-1)}{6} u_2. \quad \text{—(2)}$$

from (1) & (2), we have

$$u_x = y u_0 + x u_1 + \frac{y(y^2-1)}{3!} \Delta^2 u_{-1} + \frac{x(x^2-1)}{3!} \Delta^2 u_0.$$

Hence, proved.

Q.3. (a) → Find the characteristics of the PDE $p^2+q^2=2$ and determine the integral surface which passes through $x=0$, $z=y$.

Solution:

The initial data curve is

$$x_0(s)=0, \quad y_0(s)=s, \quad z_0(s)=s.$$

Using this data, the given PDE becomes

$$p_0^2 + q_0^2 - 2 = 0 = F \quad \text{——— (1)}$$

and the strip condition gives

$$1 = p_0(0) + q_0(1)$$

$$\Rightarrow q_0 - 1 = 0 \quad \text{——— (2)}$$

Hence,

$$q_0 = 1, \quad p_0 = \pm 1 \quad \text{——— (3)}$$

Now, the characteristic equations for the given PDE are given by

$$\left.\begin{aligned} &\frac{dx}{dt} = 2p, \ \frac{dy}{dt} = 2q, \ \frac{dz}{dt} = 2p^2 + 2q^2 = 4 \\ &\frac{dp}{dt} = 0, \ \frac{dq}{dt} = 0 \end{aligned}\right\} \quad \text{——— (4)}$$

On integration, we get,

$$\left.\begin{aligned} &p = c_1, \ q = c_2, \ x = 2c_1 t + c_3 \\ &y = 2c_2 t + c_4, \ z = 4t + c_5 \end{aligned}\right\} \quad \text{——— (5)}$$

Taking into account the initial conditions,

$$x_0 = 0, \ y_0 = s, \ z_0 = s, \ p_0 = \pm 1, \ q_0 = 1,$$

we find

$$\left.\begin{aligned} &p = \pm 1, \ q = 1, \ x = \pm 2t, \\ &y = 2t + s, \ z = 4t + s \end{aligned}\right\} \quad \text{——— (6)}$$

The last - three equations of (6) are parametric

equations of the desired integral surface.
Eliminating the parameters s and t, we get

$$\boxed{z = y \pm x}$$

Hence, the result.

Q.3. (b) → If the velocity of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$, $r^2 = x^2+y^2+z^2$, then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$.
Further, determine the streamlines.

Solution:
Given $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$, $w = \frac{3z^2-r^2}{r^5}$

Since $r^2 = x^2+y^2+z^2$ hence $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Step I: To prove that liquid motion is possible.
i.e. To prove that the equation of continuity

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ is satisfied.

$$\frac{\partial u}{\partial x} = \frac{3z}{r^{10}}\left(r^5 - 5r^3x^2\right),$$

$$\frac{\partial v}{\partial y} = \frac{3z}{r^{10}}\left(r^5 - 5r^3y^2\right),$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}\left[(6z-2z)r^5 - 5r^3(3z^2-r^2)z\right]$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3z}{r^{10}}\left[2r^5 - 5r^3(r^2-z^2)\right] + \frac{1}{r^{10}}\left(9zr^5 - 15r^3z^3\right) = 0$$

Hence, the result.

Step II: To show that $\phi = \frac{z}{r^3}$.

$$\partial\phi = \frac{\partial \phi}{\partial x}.dx + \frac{\partial \phi}{\partial y}.dy + \frac{\partial \phi}{\partial z}.dz$$

$$= -u\,dx - v\,dy - w\,dz$$

$$= -\frac{1}{r^5}\left[3xz\,dx + 3yz\,dy + (3z^2-r^2)\,dz\right]$$

$$= -\frac{1}{r^5}\left[3z\,(x\,dx + y\,dy + z\,dz) - r^2\,dz\right]$$

$$= \frac{-1}{r^5}\left[3z\,d\left(\frac{r^2}{2}\right) - r^2\,dz\right]$$

$$= -\frac{3z}{r^4}\,dr + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right)$$

Integrating,

$$\boxed{\phi = \frac{z}{r^3}}$$ (neglecting the constant of integration).

Hence, the result.

* Alternatively,

$$\frac{\partial\phi}{\partial x} = -u = -\frac{3xz}{r^5}$$

Integrating w.r.t. 'x', we have,

$$\phi = -\frac{3z}{2}\int (2x)(x^2+y^2+z^2)^{-5/2}\,dx$$

$$= \left(-\frac{3z}{2}\right)\left(\frac{-2}{3}\right)(x^2+y^2+z^2)^{-3/2}$$

$$\phi = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

i.e. $\boxed{\phi = \frac{z}{r^3}}$

Hence, the result.

Step III:

Streamlines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Putting the values of respective terms,

$$\underset{(1)}{\frac{dx}{3xz}} = \underset{(2)}{\frac{dy}{3yz}} = \underset{(3)}{\frac{dz}{3z^2-r^2}} = \underset{(4)}{\frac{x\,dx+y\,dy+z\,dz}{3z(x^2+y^2+z^2)-r^2z}}$$

Taking the ratios (1) and (2), $\frac{dx}{x} = \frac{dy}{y}$

Integration yields the result,

$$\log x = \log y + \log a$$

$$\Rightarrow \boxed{x = ay} \quad \text{—— (5)}$$

By the ratios (1) and (4),

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2r^2}$$

$$\Rightarrow \frac{4\,dx}{x} = 3\left(\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}\right)$$

Integrating, $4\log x = 3\log(x^2+y^2+z^2) + \log b$

$$\Rightarrow \boxed{x^4 = b(x^2+y^2+z^2)^3} \quad \text{—— (6)}$$

The equations (5) and (6) represent stream lines.

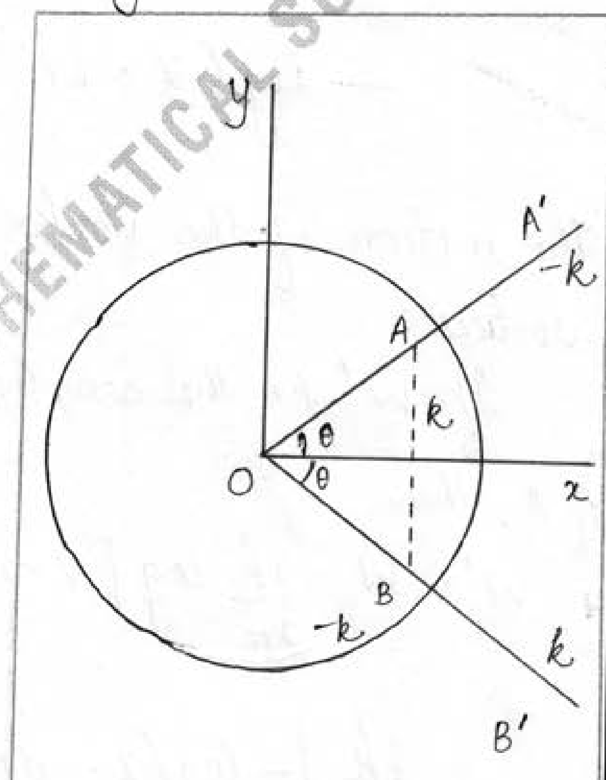
Hence, the result.

Q.4. (b) → When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that path of each vortex is given by the equation

$$(r^2\sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta,$$

θ being measured from the line through the centre perpendicular to the join of the vortices.

<u>Solution</u>:



Let x-axis be the axis of the cylinder.

Consider the vortices $+k$ at $A(r,\theta)$ and $-k$ at $B(r,-\theta)$ inside the cylinder such that distances of A and B from the axis are equal. Evidently, AB is perpendicular to x-axis.

The image of vortex $+k$ at A w.r.t. the cylinder is a vortex $-k$ at A', the inverse point of A. Similarly the image of vortex $-k$ at B is a vortex $+k$ at B'.

$$OB \cdot OB' = a^2 = OA \cdot OA',$$

where a is the radius of the cylinder.

Then $OB' = \frac{a^2}{r} = OA'$ as $OB = OA = r$.

The complex potential due to this system at $P(z)$ is

$$W = \frac{ik}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right)\right.$$

$$\left. - \log(z - re^{-i\theta}) + \frac{ik}{2\pi}\log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

The motion of the vortex at A is due to other vortices.

If W' be the complex potential for the motion of A, then

$$W' = W - \frac{ik}{2\pi}\log\left[z - re^{i\theta}\right] \quad \text{at } z = re^{i\theta}$$

$$= \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2e^{i\theta}}{r}\right) - \log(z - re^{-i\theta})\right.$$

$$\left. - \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right] \text{ at } z = re^{i\theta}$$

$$W' = -\frac{ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log\left(re^{i\theta} - re^{-i\theta}\right)\right.$$

$$- \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\Big]$$

$$= \frac{-ik}{2\pi}\left[\log(r^2-a^2)e^{i\theta} - \log r + \log(2ir\sin\theta)\right]$$

$$- \left[\log\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\} + \log r\right]$$

$$\therefore\ \Psi = \frac{-k}{2\pi}\Big[\log|(r^2-a^2)e^{i\theta}| + \log|2ir\sin\theta|$$

$$- \log|\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\}|$$

$$= -\frac{k}{2\pi}\Big[\log(r^2-a^2) + \log 2r\sin\theta\Big] - \frac{1}{2}\log$$

$$\left\{(r^2-a^2)^2\cos^2\theta + \sin^2\theta(r^2+a^2)^2\right\}\Big]$$

Stream lines are given by $\Psi = \text{constant}$,

i.e., $$\log\left\{\frac{(r^2-a^2)^2(2r\sin\theta)^2}{(r^2-a^2)^2\cos^2\theta + (r^2+a^2)^2\sin^2\theta}\right\} = \text{constant}$$

$$= \log 4b^2$$

$$\Rightarrow (r^2-a^2)^2 r^2\sin^2\theta = b^2\left[r^4 + a^4 - 2r^2a^2\cos 2\theta\right]$$

$$\Rightarrow (r^2-a^2)^2 r^2\sin^2\theta = b^2\left[(r^2-a^2)^2 + 2r^2a^2.2\sin^2\theta\right]$$

$$\Longrightarrow \boxed{(r^2-a^2)^2\left[r^2\sin^2\theta - b^2\right] = 4r^2a^2b^2\sin^2\theta}$$

Hence proved.

Q.5. (a) → Find the integral surface of the equation $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$, through the curve $xz = a^3$, $y = 0$.

Solution:

The given differential equation is

$$(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z \quad \text{—— (1)}$$

The Lagrange's subsidiary equations are

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$$

Taking 1^{st} and 2^{nd} fractions,

$$x^2dx + y^2dy = 0$$

Integrating,

$$u \equiv x^3 + y^3 = c_1 \quad \text{—— (2)}$$

Again, $$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}$$

$$\Rightarrow \quad \frac{dx-dy}{(x-y)} = \frac{dz}{z}$$

Integrating, $\log(x-y) = \log z + \log c_2$

$$\Rightarrow \quad v \equiv \frac{(x-y)}{z} = c_2 \quad \text{—— (3)}$$

The integral surface of (1) given by (2), (3) is required to pass through the curve.

$$xz = a^3, \quad y = 0 \qquad \text{——— (4)}$$

for which we shall eliminate x, y, z between (2), (3) and (4).

Putting $y = 0$ from (4) in (2) and (3), we get

$$x^3 = c_1 \qquad \text{——— (5)}$$

and $x/z = c_2$

$$\Rightarrow x\left(\frac{x}{a^3}\right) = c_2 \quad (\text{using } 4)$$

$$\Rightarrow x^2 = a^3 c_2 \qquad \text{——— (6)}$$

from (5) and (6),

$$(x^3)^2 = (x^2)^3$$

$$\Rightarrow (c_1)^2 = (a^3 c_2)^3$$

$$\Rightarrow c_1^2 = a^9 c_2^3 \qquad \text{——— (7)}$$

Substituting the values of c_1 and c_2 from (2) and (3) in (7) the required integral surface is

$$(x^3 + y^3)^2 = a^9 \left[(x-y)/z\right]^3$$

$$\Rightarrow \boxed{z^3(x^3 + y^3)^2 = a^9 (x-y)^3}.$$

Hence, the result.

Q.5. (b) → Solve

$$\left.\begin{array}{l} 10x - 7y + 3z + 5u = 6 \\ -6x + 8y - z - 4u = 5 \\ 3x + y + 4z + 11u = 2 \\ 5x - 9y - 2z + 4u = 7 \end{array}\right\}$$ by Gauss's Elimination method.

Solution:

We can rewrite the given system as

$$\left.\begin{array}{l} x - 0.7y + 0.3z + 0.5u = 0.6 \\ -6x + 8y + z - 4u = 5 \\ 3x + y + 4z + 11u = 2 \\ 5x - 9y - 2z + 4u = 7 \end{array}\right\} \quad \text{---(1)}$$

First, we eliminate x from the second, third and fourth equations, using the first equation.

Subtracting (-6) times the first equation from the second equation, 3 times the first equation from the third equation, and 5 times the first equation from the fourth equation, we get the new system as

$$\left.\begin{array}{l} x - 0.7y + 0.3z + 0.5u = 0.6 \\ 3.8y + 0.8z - u = 8.6 \\ 3.1y + 3.1z + 9.5u = 0.2 \\ -5.5y - 3.5z + 1.5u = 4 \end{array}\right\} \quad \text{---(2)}$$

Since the numerically largest y-coefficient is in the fourth equation of system (2), so we permute the second and fourth equations. After that, y is eliminated from the third and fourth equations by using the second equation. The new system becomes

$$\left.\begin{aligned} x - 0.7y + 0.3z + 0.5u &= 0.6 \\ y + 0.63636z - 0.27273u &= -0.72727 \\ -1.61818z + 0.03636u &= 11.36364 \\ 1.12727z + 10.34545u &= 2.45455 \end{aligned}\right\} \quad \text{--- (3)}$$

Now, elimination of z gives,

$$\left.\begin{aligned} x - 0.7y + 0.3z + 0.5u &= 0.6 \\ y + 0.63636z + 0.27273u &= -0.72727 \\ z - 0.02247u &= -7.02247 \\ 10.37079u &= 10.37079 \end{aligned}\right\} \quad \text{--- (4)}$$

Thus, the final solution is given by;

$$\boxed{u = 1,\ z = -7,\ y = 4 \text{ and } x = 5.}$$

Hence, the result.

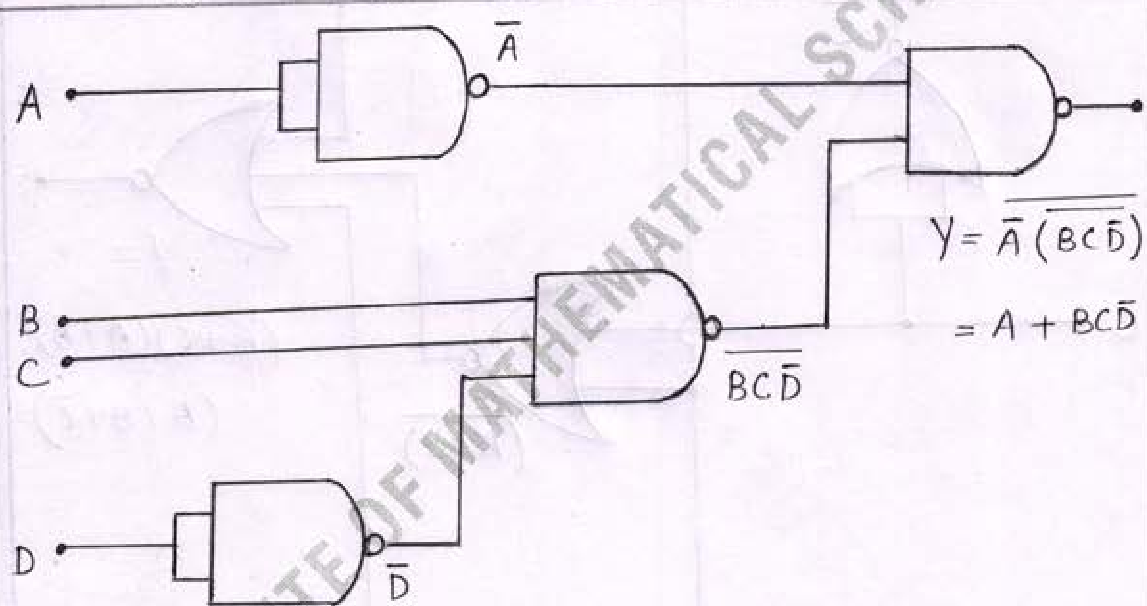
Q.5. (c) → Realise

(a) $Y = A + BC\bar{D}$ using NAND gates and

(b) $Y = (A+C)(A+\bar{D})(A+B+\bar{C})$ using NOR gates.

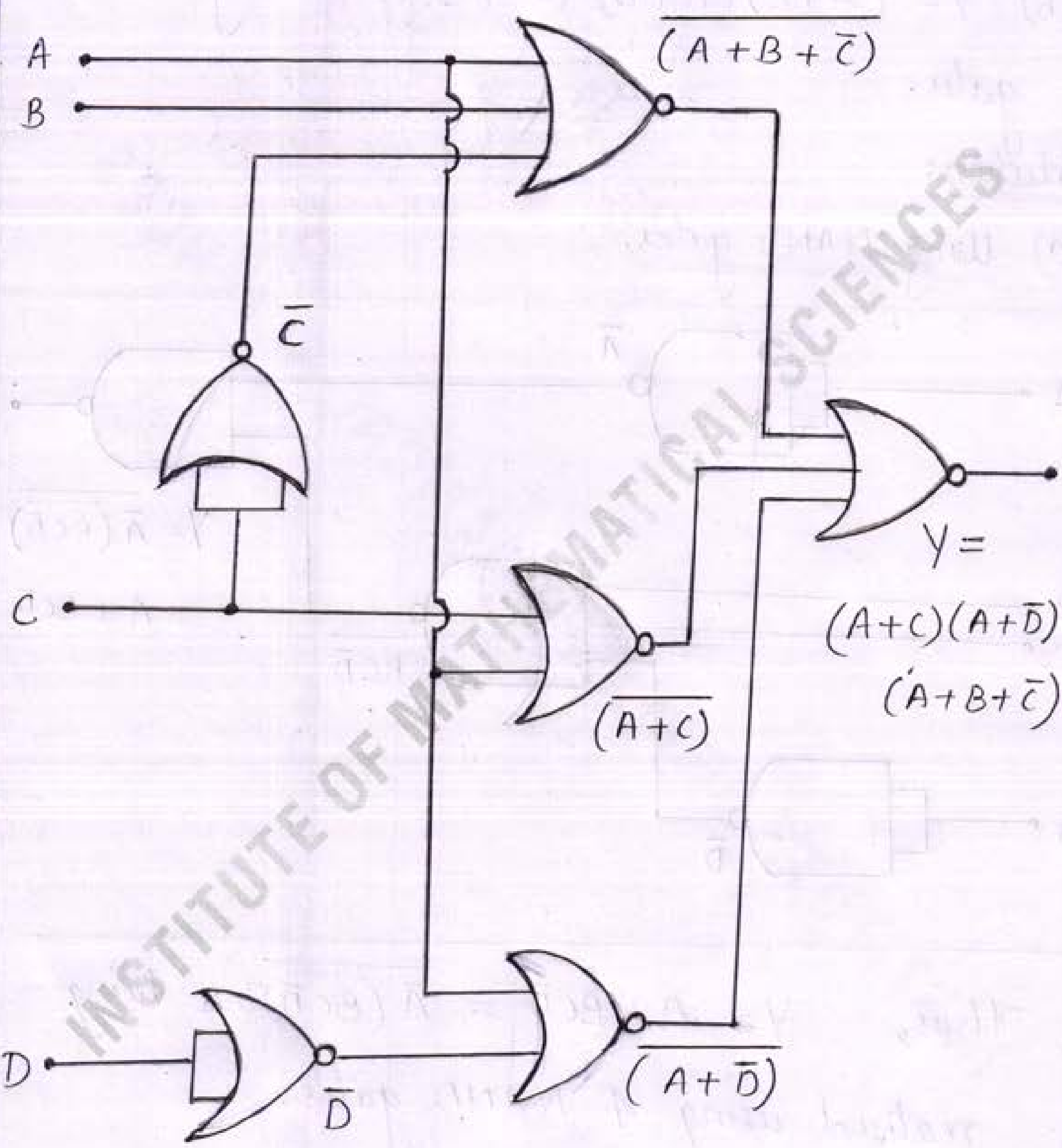
Solution:

(a) Using NAND gates.



Thus, $Y = A + BC\bar{D} = \overline{\bar{A}\,(\overline{BC\bar{D}})}$ is being realised using 4 NAND gates. — (i)

(b) Using NOR gates



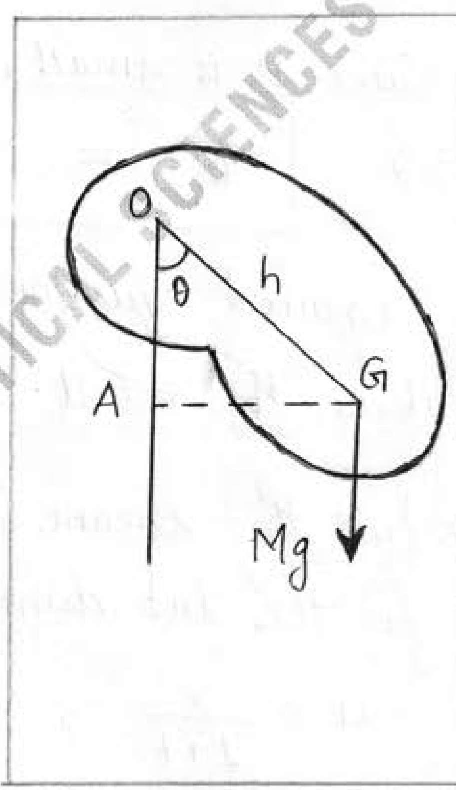
Thus, $Y = (A+C)(A+\bar{D})(A+B+\bar{C})$ is being realised using 6 NOR gates. —(ii)

Hence, the result.

Q.5. (d). Use Lagrange's equations to find the equation of motion of the compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.

Solution:

Let the vertical plane through the C.G. of the pendulum meet the horizontal axis of rotation at O. Let $OG = h$. Let OG make an angle θ to the vertical at time t. Thus, θ is the only generalised coordinate. If K is the radius of gyration of the pendulum about the axis of rotation through O, then



K.E., $T = \frac{1}{2} MK^2 \dot{\theta}^2$

And the potential energy relative to the horizontal plane through O is $V = -Mgh \cos\theta$.

$\therefore$ Lagrange's θ equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

i.e., $\frac{d}{dt}(Mk^2\dot{\theta}) = -Mgh\sin\theta$

$\Rightarrow \cancel{M}k^2\ddot{\theta} = -\cancel{M}gh\sin\theta$

$\Rightarrow \ddot{\theta} = -\left(\frac{gh}{k^2}\right)\sin\theta.$

Since θ is small,

$\Rightarrow \boxed{\ddot{\theta} = -\left(\frac{gh}{k^2}\right)\theta}$ which is the required equation of motion.

Hence, the result.

Q.5. (c) → Find the stream lines and paths of the particles for the two dimensional velocity field:

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Solution:

We have

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Step I: To determine stream lines.

Stream lines are the solution of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Putting the value $\frac{(1+t)}{x}\,dx = \frac{dy}{y} = \frac{dz}{0}$

$$\Rightarrow \left(\frac{1+t}{x}\right)dx = \frac{dy}{y}, \quad \frac{dy}{y} = \frac{dz}{0}$$

$$\Rightarrow (1+t)\log x = \log y + \log a, \quad dz = 0$$

$$\Rightarrow \boxed{x^{1+t} = ay, \quad z = b} \quad \text{—— (i)}$$

These two equations represent stream lines.

Step II: To determine path lines:

Path lines are solutions of

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = dt, \quad dz = 0$$

Integrating,

$$\log x = \log(1+t) + \log a,$$

$$\log y = t - \log b, \quad z = c$$

$$\Rightarrow x = a(1+t), \quad (y/b) = e^t, \quad z = c$$

$$\Rightarrow \boxed{y = be^{[(x/a)-1]}, \quad z = c} \quad \text{—— (ii)}$$

These two equations represent path lines.
Hence, the result.

Q.6. (a) Form a partial differential equation by eliminating arbitrary function f and g from
$$z = f(x^2-y) + g(x^2+y).$$

Solution:

Given $z = f(x^2-y) + g(x^2+y)$ —— (1)

Differentiating (1) partially w.r.t. x and y, we get,

$$\frac{\partial z}{\partial x} = 2x f'(x^2-y) + 2x g'(x^2+y) \quad \text{—— (2)}$$

and $$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y) \quad \text{—— (3)}$$

Differentiating (2) partially w.r.t. x and (3) partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2-y) + g'(x^2+y)\} + 4x^2\{f''(x^2-y) + g''(x^2+y)\} \quad \text{—— (4)}$$

and $$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad \text{—— (5)}$$

From (2), we have $f'(x^2-y) + g'(x^2+y) = \frac{1}{2x}\frac{\partial z}{\partial x}$ —— (6)

Substituting from (5) and (6) in (4), we have,

$$\frac{\partial^2 z}{\partial x^2} = \cancel{2}\cdot\frac{1}{\cancel{2}x}\frac{\partial z}{\partial x} + 4x^2\cdot\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \boxed{x r = p + 4x^3 t}$$

which is the required partial differential equation of order 2.

Hence, the result.

Q.6.(b) $\rightarrow$ Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

<u>Solution:</u>

The given equation can be re-written as

$$(D-D')(D+D'-3)z = xy + e^{x+2y}$$

$$\therefore C.F. = \phi_1(y+x) + e^{3x}\phi_2(y-x),$$

ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to xy

$$= \frac{1}{(D-D')(D+D'-3)}xy = \frac{-1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)^{-1}xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\right]xy$$

$$= \frac{-1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left(1+\frac{D+D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= \frac{-1}{3D}\left(1+\frac{D}{3}+\frac{D'}{3}+\frac{D'}{D}+\frac{D'}{3}+\frac{2DD'}{9}+\cdots\right)xy$$

$$= \frac{-1}{3D}\left(xy+\frac{y}{3}+\frac{2x}{3}+\frac{1}{D}x+\frac{2}{9}\right)$$

$$= -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{2}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D+D'-3)(D-D')}e^{x+2y} = \frac{1}{(D+D'-3)}\cdot\frac{1}{(1-2)}e^{x+2y}$$

$$= \frac{-1}{(D+D'-3)}e^{1x+2y}\cdot 1$$

$$= -e^{1.x+2y}\cdot\frac{1}{(D+1)+(D'+2)-3}\cdot 1$$

$$= -e^{x+2y}\cdot\frac{1}{D+D'}\cdot 1$$

$$= -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}\cdot 1$$

$$= -e^{x+2y}\frac{1}{D}(1+\cdots)\cdot 1$$

$$= -xe^{x+2y}$$

Hence,
the required general solution is

$$z = C.F + P.I.$$

i.e.,
$$\boxed{z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - (1/6)x^2y - (1/6)xy - (1/9)x^2 - (1/18)x^3 - (2/27)x - xe^{x+2y}.}$$

Hence the result.

Q.6. (c) → The temperature of a bar 50 cm long with insulated sides is kept at 0° at one end and 100° at the other end until steady conditions prevail. The two ends are then suddenly insulated so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

Solution:

Let the rod lie along the axis of x with one end which is kept at $0°C$ be at the origin.

The temperature distribution $u(x,t)$ in the bar at any time t at any distance x is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——— (A)}$$

Initially $u = 0°C$ at $x = 0$ and $u = 100°C$ at $x = 50$.

In the steady state, u is independent of t.

$$\therefore \frac{\partial u}{\partial t} = 0$$

$\therefore$ from (A), we have $\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B$

$\because$ at $x = 0$, $u = 0°C$ and at $x = 50$, $u = 100°C$

$\therefore \quad 0 = 0 + B$ and $100 = 50A + B$

$\Rightarrow \quad \boxed{B = 0}$ and $\boxed{A = 2}$

$$\therefore \quad u = 2x$$

i.e., initial temperature distribution in the bar is given by $u = 2x$ i.e., $u(x,0) = 2x$

Now let $u(x,t)$ represent the temperature distribution in the rod at time t measured from the instant the two ends of the bar are suddenly insulated after the rod reaches to steady state.

Thus, we are required to solve (A) under the following conditions:

Boundary conditions: $u_x(0,t)=0$ and —(B) (C)

$u_x(50,t)=0$ —(B)

and Initial conditions: $u(x,0)=2x$ —(C) (D)

Let $u(x,t) = F(x)\,T(t) = FT_{(say)}$ —(1)

where F is function of x alone and T is function of t alone.

be solution of (A).

Differentiating (1) and substituting in (A), we get,

$$F\frac{dT}{dt} = c^2 T\frac{d^2F}{dx^2}$$

(or) $\dfrac{1}{F}\dfrac{d^2F}{dx^2} = \dfrac{1}{c^2T}\dfrac{dT}{dt} = 0$

(or) k^2 or $-k^2$

Thus, we get three sets of two ordinary differential equations.

(i) $\dfrac{d^2F}{dx^2} = 0$ and $\dfrac{dT}{dt} = 0$

(ii) $\dfrac{d^2F}{dx^2} - k^2F = 0$ and $\dfrac{dT}{dt} = k^2c^2T$

(iii) $\dfrac{d^2F}{dx^2} + k^2F = 0$ and $\dfrac{dT}{dt} = -k^2c^2T$

The general solutions in these three cases are

(a) $F = A_1x + A_2$ and $T = A_3$

$\therefore\ u = FT = A_1'x + A_2'$ where $A_1' = A_1A_3$ and $A_2' = A_2A_3$

(b) $F = B_1e^{kx} + B_2e^{-kx}$ and $T = B_3e^{k^2c^2t}$

$\therefore\ u = FT = (B_1'e^{kx} + B_2'e^{-kx})e^{k^2c^2t}$,

where $B_1' = B_1B_3$ and $B_2' = B_2B_3$

(c) $F = C_1\cos kx + C_2\sin kx$ and $T = C_3e^{-k^2c^2t}$

$\therefore\ u = FT = (C_1'e^{kx} + C_2'e^{-kx})e^{-k^2c^2t}$

Here in the present problem, the solution given by (b) is inadmissible as in this

temperature $u(x,t)$ increases infinitely with time.
The solution given in (c) itself is inadequate to give the complete solution since in this case the temperature $u(x,t) \to 0$ as $t \to \infty$.
Thus, our solution will be combinations of solutions given in (a) and (c).

From the solution in (a),

$$u(x,t) = A_1' x + A_2$$

$$\therefore \frac{\partial u}{\partial x} = A_1'$$

Now, $u_x(0,t) = 0$ and $u_x(l,t) = 0$

$\Rightarrow A_1' = 0$

$\therefore u(x,t) = A_2$ (constant) is the solution of (A), corresponding to (a). —— (2)

Again if the solution of the heat equation (A) is given by (c), then

$$u(x,t) = (C_1' \cos kx + C_2' \sin kx)\, e^{-k^2c^2t} \quad \text{——(3)}$$

$$\therefore u(x,t) = k(-C_1' \sin kx + C_2' \cos kx)\, e^{-k^2c^2t}$$

$\therefore$ from (3), $u_x(0,t) = 0 \Rightarrow kC_2' e^{-k^2c^2t} = 0$

$$\Rightarrow C_2' = 0 \quad \because e^{-k^2c^2t} \neq 0$$

$\therefore\ u_x(l,t) = 0 \Rightarrow -kC_1' \sin kl.\, e^{-k^2c^2t} = 0,$

$\because C_2' = 0$

$\Longrightarrow \sin kl = 0 \qquad \because C_1' \neq 0$ and $e^{-k^2c^2t} \neq 0$

otherwise $u(x,0) = 0$

$\Rightarrow \quad kl = n\pi$, where n is any integer

i.e., $\quad k = \frac{n\pi}{l}$.

$\therefore$ from (3), for all integral values of n, the solution of (A) is given by

$$u_n(x,t) = D_n \cos\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2\pi^2c^2t/l^2} \qquad \text{--- (4)}$$

where $D_n = C_1'$ is new arbitrary constant, depending on n.

From (2) and taking $n = 1, 2, 3, \ldots,$ in (4), the complete solution of equation (A) satisfying conditions (B) and (C) can be taken as

$$u(x,t) = A_2' + \sum_{n=1}^{\infty} u_n(x,t)$$

i.e.,
$$u(x,t) = A_2' + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2\pi^2c^2t/l^2} \qquad \text{--- (5)}$$

Now from (5), the condition (D),

i.e. $u(x,0) = 2x$

$$\Rightarrow \quad 2x = A_2' + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{--- (6)}$$

Now to find the values of A_2' and D_n, we proceed as follows.

Multiplying both sides of (6) by $\cos\left(\frac{n\pi x}{l}\right)$ and integrating between the limits 0 to l, we get,

$$D_n \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx + A_2' \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^l 2x \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow D_n \cdot \frac{1}{2} \int_0^l \left(1 + \cos\frac{2n\pi x}{l}\right) dx + A_2' \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \int_0^l \overset{2x}{f(x)} \cos\left(\frac{n\pi x}{l}\right) dx$$

$$\Rightarrow \frac{1}{2} \cdot D_n \cdot l + A_2' \cdot 0 = \int_0^{\infty} 2x \cdot \cos\frac{n\pi x}{l}\, dx$$

$$\Rightarrow \quad D_n = \frac{2}{l} \int_0^{\infty} 2x \cdot \frac{\cos n\pi x}{l}\, dx \quad \text{--- (7)}$$

Again integrating both sides of (6) between the limits 0 to l, we get,

$$A_2' \int_0^l dx + \sum_{n=1}^{\infty} D_n \int_0^l \cos\frac{n\pi x}{l}\, dx = \int_0^l 2x\, dx$$

$$\Rightarrow \quad A_2' \cdot l + 0 = \int_0^l 2x\, dx$$

$$\Rightarrow \quad A_2' = \frac{1}{l}\int_0^l 2x\, dx$$

$$\Rightarrow \quad A_2' = \frac{1}{2} D_0 \qquad \text{—— (8)}$$

Hence from (5), the complete solution of (A) satisfying the conditions (B), (C) and (D) is given by

$$u(x,t) = \frac{1}{2} D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) \cdot e^{-\left(\frac{n\pi c}{l}\right)^2 t} \qquad \text{—— (9)}$$

where D_n is given by

$$D_n = \frac{2}{l}\int_0^l 2x \cdot \cos\left(\frac{n\pi x}{l}\right) dx$$

i.e. $$D_n = \frac{2}{\cancel{50}_{25}}\int_0^{50} 2x \cdot \cos\left(\frac{n\pi x}{50}\right) dx$$

[Here $l = 50$cm]

$$= \frac{1}{25}\left[2x \cdot \frac{50}{n\pi}\sin\left(\frac{n\pi x}{50}\right) - 2\left(\frac{50}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{50}\right)\right]_0^{50}$$

$$= \frac{2}{25}\left(\frac{50}{n\pi}\right)^2 (\cos n\pi - 1)$$

$$= \frac{200}{n^2\pi^2}\left((-1)^n - 1\right)$$

Obviously, $D_n = 0$ when $n = 2m$ (even)

and $D_n = -\frac{400}{n^2\pi^2}$ when $n = (2m-1)$ odd.

i.e., $D_{2m} = 0$ and $D_{2m-1} = \frac{-400}{(2m-1)^2\pi^2}$ —— (10)

and $D_0 = \frac{2}{50}\int_0^{50} 2x\, dx = \frac{1}{25}\left[x^2\right]_0^{50} = 100$ —— (11)

Now using (10) and (11), the required solution from (9) is given by

$$u(x,t) = \frac{1}{2}(100) + \sum_{n=1}^{\infty} \frac{-400}{(2m-1)^2\pi^2} \cdot \cos\frac{(2m-1)\pi x}{50} \cdot e^{-[(2m-1)\pi c/50]^2 t}$$

i.e.,

$$\boxed{u(x,t) = 50 - \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \cos\frac{(2n-1)\pi x}{50} \cdot e^{-(2n-1)^2\pi^2c^2t/2500}} \quad \text{--- (12)}$$

Hence the result.

Q.7. (a) → Use Euler's modified method to compute y for $x = 0.05$ and $x = 0.1$. Given that $\frac{dy}{dx} = x+y$ with the initial conditions $x_0 = 0$, $y_0 = 1$. Give the correct result upto four decimal places.

Solution:

The given differential equation is

$$\frac{dy}{dx} = f(x,y) = x+y. \quad \text{——— (1)}$$

$$\therefore \left(\frac{dy}{dx}\right)_0 = f(x_0, y_0) = x_0 + y_0 = 0+1 = 1.$$

Take $h = 0.05$, then we have,

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.05)(1)$$
$$\therefore y_1^{(1)} = 1.05$$

Now, $\left(\frac{dy}{dx}\right)_1^{(1)} = x_1 + y_1^{(1)} = 0.05 + 1.05 = 1.10.$

The second approximation to y_1 is given by

$$y_1^{(2)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(1)}}{2} \cdot h$$

$$= 1 + \frac{1 + 1.10}{2} \times 0.05$$

$$= 1.0525$$

The second approximation to $\left(\frac{dy}{dx}\right)_1$ is

$$\left(\frac{dy}{dx}\right)_1^{(2)} = x_1 + y_1^{(2)} = 0.05 + 1.0525 = 1.1025.$$

Now the third approximation to y_1 is

$$y_1^{(3)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(2)}}{2} \cdot h$$

$$= 1 + \frac{1 + 1.1025}{2} \times 0.05$$

$$= 1.05256.$$

The third approximation to $\left(\frac{dy}{dx}\right)_1$ is

$$\left(\frac{dy}{dx}\right)_1^{(3)} = x_1 + y_1^{(3)} = 0.05 + 1.05256 = 1.10256.$$

The fourth approximation to y_1 is

$$y_1^{(4)} = y_0 + \frac{\left(\frac{dy}{dx}\right)_0 + \left(\frac{dy}{dx}\right)_1^{(3)}}{2} \cdot h$$

$$= 1 + \frac{1 + 1.10256}{2} \times 0.05$$

$$= 1.105256.$$

Since $y_1^{(3)} = y_1^{(4)}$, so, we stop here.

Hence, we take $\underline{y_1 = 1.0526}$, $\underline{\left(\frac{dy}{dx}\right)_1 = 1.1026}$

As a first approximation to y_2, by using the relation

$$y_{n+1} = y_{n-1} + 2h \cdot y_n',$$

we have

$$y_2^{(1)} = y_0 + 2h\left(\frac{dy}{dx}\right)_1$$

$$= 1 + 2\times(0.05)\times(1.1026)$$

$$= 1.1103$$

Now, $\left(\frac{dy}{dx}\right)_2^{(1)} = 0.1 + 1.1103 = 1.2103.$

The second approximation to y_2 is

$$y_2^{(2)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(1)}}{2} \cdot h$$

$$= 1.0526 + \left(\frac{1.1026 + 1.2103}{2}\right) \times 0.05$$

$$= 1.1104$$

and $\left(\frac{dy}{dx}\right)_2^{(2)} = 0.1 + 1.1104 = 1.2104$

Then $y_2^{(3)} = y_1 + \frac{\left(\frac{dy}{dx}\right)_1 + \left(\frac{dy}{dx}\right)_2^{(2)}}{2} \cdot h$

$$= 1.0526 + \frac{(1.1026 + 1.2104)}{2} \times 0.05$$

$$= 1.1104$$

Since $y_2^{(2)} = y_2^{(3)}$ so we take

$$\underline{y_2 = 1.1104}, \quad \underline{\left(\frac{dy}{dx}\right)_2 = 1.2104}$$

The following table shows the results in tabular form:

x	y	dy/dx
0.00	1.0000	1.0000
0.05	1.0526	1.1026
0.10	1.1104	1.2104

Hence, the result.

7(b) Ques:) Evaluate the integrals

(i) $I = \int_0^2 \frac{dx}{3+4x}$ (ii) $\int_0^2 \frac{dx}{x^2+2x+10}$

by Gauss - Legendre two point and three point formulas.

Solution:-

(i) $I = \int_0^2 \frac{dx}{3+4x}$

Convert the limit (0,2) to (-1,1) for gauss-legendre formula.

$$x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$$

$$x = \frac{1}{2}(2-0)u + \frac{1}{2}(2+0) = u+1$$

and $dx = du$.

Now, two-point formula:

$$I = \int_{-1}^{1} \frac{dx}{f(x)} = 1 \cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1 \cdot f\left(\frac{1}{\sqrt{3}}\right)$$

Here; $I = \int_{-1}^{1} \frac{du}{3+4(u+1)} = \int_{-1}^{1} \frac{du}{4u+7}$

$$\therefore I = 1 \cdot \frac{1}{4\left(\frac{-1}{\sqrt{3}}\right)+7} + 1 \cdot \frac{1}{\frac{4}{\sqrt{3}}+7}$$

$$= \sqrt{3}\left[\frac{1}{7\sqrt{3}-4} + \frac{1}{7\sqrt{3}+4}\right] = \sqrt{3}\left[\frac{14\sqrt{3}}{(7\sqrt{3})^2 - 4^2}\right]$$

$$= \frac{42}{147-16} = \frac{42}{131} = 0.32061.$$

Now, three point formula.

$$I = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$\therefore\ I = \int_{-1}^{1}\frac{du}{4u+7} = \frac{8}{9}\times\frac{1}{7} + \frac{5}{9}\left[\frac{1}{-\frac{4\sqrt{3}}{\sqrt{5}}+7} + \frac{1}{7+\frac{4\sqrt{3}}{\sqrt{5}}}\right]$$

$$= \frac{8}{63} + \frac{5\sqrt{5}}{9}\left[\frac{1}{7\sqrt{5}-4\sqrt{3}} + \frac{1}{7\sqrt{5}+4\sqrt{3}}\right]$$

$$= \frac{8}{63} + \frac{5\sqrt{5}}{9}\times\frac{14\sqrt{5}}{(7\sqrt{5})^2-(4\sqrt{3})^2}$$

$$= \frac{8}{63} + \frac{350}{9\times 197} = 0.32439.$$

Hence; for $I = \int_0^2 \frac{dx}{3+4x}$

$$\boxed{\begin{array}{ll}\text{Two point value} & I_2 = 0.32061\\ \text{Three point value} & I_3 = 0.32439\end{array}}$$

Which is required result.

(ii) $$I = \int_0^2 \frac{dx}{x^2+2x+10} = \int_{-1}^{1}\frac{du}{u^2+4u+13}$$

$\because$ Converting $(0,2)$ to $(-1,1)$ $\Rightarrow x = u+1$
$dx = du.$

put $x = u+1$ in $x^2+2x+10$
we get $= (u+1)^2+2(u+1)+10 = u^2+4u+13$

Now; two point formula.

$$I = 1\cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1\cdot f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 1\cdot\frac{1}{\left(\frac{-1}{\sqrt{3}}\right)^2+4\left(\frac{-1}{\sqrt{3}}\right)+13} + 1\cdot\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2+\frac{4}{\sqrt{3}}+13}$$

$$I = \frac{3\sqrt{3}}{40\sqrt{3}-12} + \frac{3\sqrt{3}}{40\sqrt{3}+12}$$

$$I = \frac{3\sqrt{3} \times 80\sqrt{3}}{(40\sqrt{3})^3-(12)^2} = \frac{720}{4800-144}$$

$$\boxed{I_2 = \frac{720}{4656} = 0.15464}$$

Now, three point formula,

$$I_3 = \frac{8}{9}f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$I_3 = \frac{8}{9} \times \frac{1}{13} + \frac{5}{9}\left[\frac{1}{\left(-\sqrt{\frac{3}{5}}\right)^2 + 4\times -\sqrt{\frac{3}{5}} + 13} + \frac{1}{\left(\sqrt{\frac{3}{5}}\right)^2 + \frac{4\sqrt{3}}{\sqrt{5}} + 13}\right]$$

$$I_3 = \frac{8}{117} + \frac{5}{9}\left[\frac{5\sqrt{5}}{68\sqrt{5}-20\sqrt{3}} + \frac{5\sqrt{5}}{68\sqrt{5}+20\sqrt{3}}\right]$$

$$I_3 = \frac{8}{117} + \frac{25\sqrt{5}}{9}\left[\frac{136\sqrt{5}}{(68\sqrt{5})^2-(20\sqrt{3})^2}\right]$$

$$I_3 = \frac{8}{117} + \frac{25\sqrt{5} \times 136\sqrt{5}}{9 \times (23120-1200)}$$

$$I_3 = \frac{8}{117} + \frac{1700\cancel{0}}{9 \times 2192\cancel{0}}$$

$$I_3 = 0.06837 + 0.08617$$

$$\boxed{I_3 = 0.154546}$$

Hence, for, $I = \int_0^2 \frac{dx}{x^2+2x+10}$

$$\boxed{\begin{array}{l} \text{Two point value} \rightarrow I_2 = 0.15464 \\ \text{Three point value} \rightarrow I_3 = 0.154546 \end{array}}$$

Q.7. (c) → The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45−50 and 50−55.

Age group :	25-35	35-45	45-55	55-65
Deaths :	13229	18139	24225	31496

Solution:

First we shall form the cumulative frequency table.

Age above 25 and below 65 x	No. of deaths $f(x)$
35	13229
45	31368
55	55593
65	87089

The difference table is as under:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

First, we shall find $f(50)$ i.e. the no. of deaths above the age 25 and below 50.

Here $u = \frac{x-a}{h} = \frac{50-35}{10} = 1.5$

By Newton's forward interpolation formula, we get,

$$f(50) = f(35) + (1.5)\Delta f(35) + \frac{(1.5)(1.5-1)}{2!}\cdot \Delta^2 f(35) + \frac{(1.5)(1.5-1)(1.5-2)}{3!}\cdot \Delta^3 f(35)$$

$$= 13229 + (1.5)(18139) + \frac{(1.5)(0.5)}{2} \times 6086 + \frac{(1.5)(0.5)(-0.5)}{6} \times (1185)$$

$$= 13229 + 27208.5 + 2282.25 - 74.0625$$

$$= 42646.$$

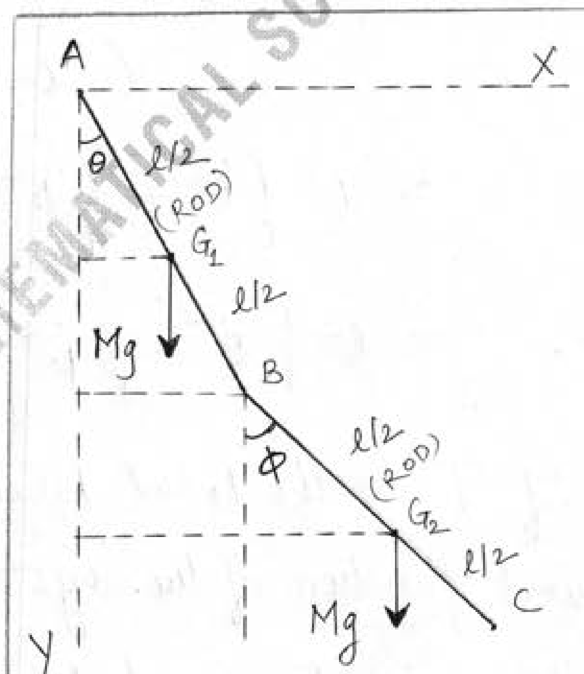
Hence, the required number of deaths between 45 and 50 $= 42646 - 31368 = \underline{\underline{11278}}$.

Therefore, the number of deaths between 50 and 55 $= 24225 - 11278 = \underline{\underline{12947}}$.

Hence, the result.

Q. 8. (a) → Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$

Solution:



Let AB and BC be the rods of equal length 'l' and mass M.

At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred A as origin, horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by

$$x_{G_1} = \frac{1}{2}l\sin\theta, \quad y_{G_1} = \frac{1}{2}l\cos\theta;$$

$$x_{G_2} = l\sin\theta + \frac{1}{2}l\sin\phi, \quad y_{G_2} = l\cos\theta + \frac{1}{2}l\cos\phi.$$

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2, then

$$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\tfrac{1}{2}l\cos\theta.\dot{\theta}\right)^2 + \left(-\tfrac{1}{2}l\sin\theta.\dot{\theta}\right)^2$$

$$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta.\dot{\theta} + \tfrac{1}{2}l\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \tfrac{1}{2}l\sin\phi.\dot{\phi}\right)^2$$

$$= l^2\left[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

$$= l^2\left[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right] \quad [\because \theta, \phi \text{ are small}]$$

If T be the total kinetic energy and W the work function of the system, then

T = K.E. of rod AB + K.E. of rod BC

$$= \left[\frac{1}{2}M.\frac{1}{3}\left(\frac{1}{2}l\right)^2\dot{\theta}^2 + \frac{1}{2}M.v_{G_1}^2\right] + \left[\frac{1}{2}m.\frac{1}{3}\left(\frac{1}{2}l\right)\dot{\phi}^2 + \frac{1}{2}M.v_{G_2}^2\right]$$

$$= \frac{1}{2}M\left[\frac{1}{12}l^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\theta}^2\right] + \frac{1}{2}M\left[\frac{1}{12}l^2\dot{\phi}^2 + l^2\left(\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)\right]$$

$$= \frac{1}{2} Ml^2\left(\frac{4}{3}\dot{\theta}^2 + \frac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$$

and $W = Mg y_{G_1} + Mg y_{G_2} + C$

$$= Mg\left[\frac{1}{2} l\cos\theta + l\cos\theta + \frac{1}{2} l\cos\phi\right] + C$$

$$= \frac{1}{2} Mgl\,(3\cos\theta + \cos\phi).$$

$\therefore$ Lagrange's θ- equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2}Ml^2\left(\frac{8}{3}\dot{\theta} + \dot{\phi}\right)\right] - 0 = \frac{1}{2}Mgl(-3\sin\theta)$

$= -\frac{3}{2} Mgl\theta \qquad [\because \theta \text{ is small}]$

or $8\ddot{\theta} + 3\ddot{\phi} = -9c\theta$, (where $c = g/l$). ——— (1)

Equations (1) and (2) can be written as

$(8D^2 + 9c)\theta + 3D^2\phi = 0$ and

$3D^2 + \theta + (2D^2 + 3c)\phi = 0.$

Eliminating ϕ between these two equations, we get,

$[(2D^2 + 3c)(8D^2 + 9c) - 9D^4] = 0$

or $(7D^4 + 42cD^2 + 27c^2)\theta = 0$ —— (3)

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$$\theta = A\cos(nt + B)$$

$\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$

Substituting in (3), we get,

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

(or) $7n^4 - 42cn^2 + 27c^2 = 0$ $\because \theta \neq 0$

$$\therefore n^2 = \frac{42c \pm \sqrt{[42c]^2 - 4.7.27.c^2}}{2\cdot 7}$$

$$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)c$$

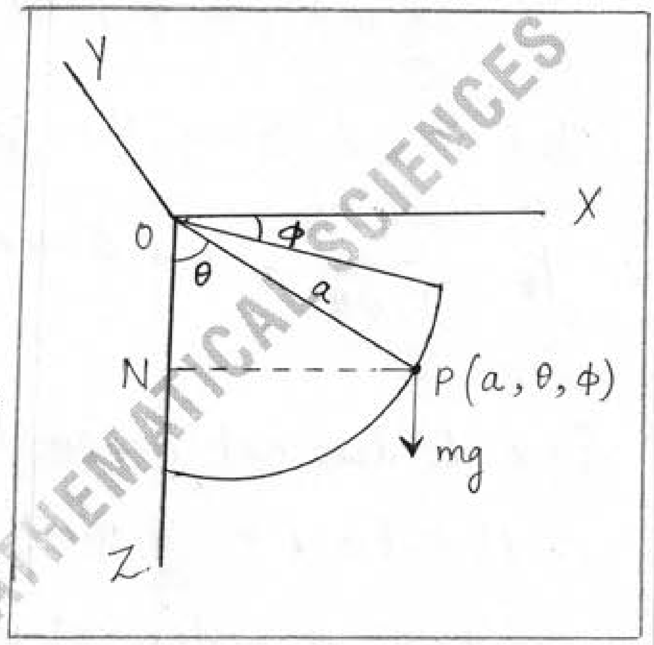
$$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l} \quad \left[\because c = g/l\right]$$

Hence, Proved.

Q. 8. (b) → Determine the motion of a spherical pendulum by using Hamilton's equations.

Solution :

Let m be the mass of the bob suspended by a light rod of length a.



In a spherical pendulum of length a, the path of the motion of the bob is the surface of a sphere of radius a and centre at the fixed point O.

At time t, let $P(a, \theta, \phi)$ be the position of the bob. If (x, y, z) are the cartesian coordinates of P then

$$x = a\sin\theta\cos\phi,$$
$$y = a\sin\theta\sin\phi,$$
$$z = a\cos\theta.$$

$$\therefore \text{ K.E., } T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

and potential $V = -mgz$
$= -mga\cos\theta$.

(Since m is below the fixed point O).

$\therefore L = T - V$
$$= \frac{1}{2} ma^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + mga\cos\theta$$

Here θ and ϕ are the generalised coordinates.

$$\therefore p_\theta = -\frac{\partial L}{\partial \theta} = ma^2\dot{\theta} \text{ and } p_\phi = -\frac{\partial L}{\partial \phi} = ma^2\dot{\phi}\sin^2\theta \quad \text{--- (1)}$$

Since L does not contain t explicitly,

$$\therefore H = T + V = \frac{1}{2} ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - mga\cos\theta$$

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations (1), we get

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \operatorname{cosec}^2\theta\, p_\phi^2\right) - mga\cos\theta$$

Hence the <u>four Hamilton's equations</u> are:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{ma^2}\operatorname{cosec}^2\theta\cot\theta\, p_\phi^2 - mga\sin\theta \quad \text{--- } (H_1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2} p_\theta \quad \text{--- } (H_2)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{--- } (H_3)$$

and $\dot{\phi} = \partial H/\partial p_\phi = (1/ma^2)\operatorname{cosec}^2\theta\, p_\phi$ —— (H_4)

from (H_3), Integrating,

$$p_\phi = c \text{ (Constant)}$$

$\therefore$ from (H_4), we have

$$\dot{\phi} = \frac{1}{ma^2} \cdot C \operatorname{cosec}^2\theta$$

i.e. $$\boxed{\dot{\phi} = \frac{A}{\sin^2\theta}} \qquad \left(\text{where } A = \frac{C}{ma^2}\right) \text{ --- (2)}$$

Also from (H_1) and (H_2), we have,

$$\ddot{\phi} = \frac{1}{ma^2}\dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2} \cdot \frac{\cos\theta}{\sin^3\theta} \cdot p_\phi^2 - mga\sin\theta\right]$$

$$= -\frac{1}{(ma^2)^2} C^2 \frac{\cos\theta}{\sin^3\theta} - \left(g/a\right)\sin\theta, \quad \because p_\phi = C$$

$$= A^2 \frac{\cos\theta}{\sin^3\theta} - \left(g/a\right)\sin\theta \qquad \left(\because A = C/ma^2\right)$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get,

$$\boxed{\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B} \quad \left[B \text{ is a constant}\right] \text{ --- (3)}$$

Equations (2) and (3) represent the required motion

Hence, the result.

Q.8.(c) → Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$

determine the streamlines.

Solution:

The given velocity potential

$$\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

To determine the stream lines, we proceed as follows:

$$-\frac{\partial\phi}{\partial x} = u = -\frac{\partial\psi}{\partial y}, \quad -\frac{\partial\phi}{\partial y} = v = \frac{\partial\psi}{\partial x}.$$

Hence,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial y}\ x$$

Now, $$\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

Integrating w.r.t. y,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + F(x) \quad \text{---}(4)$$

where $F(x)$ is constant of integration.

To determine $F(x)$.

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \quad \text{——— (5)}$$

By (4),

$$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} + F'(x) \quad \text{——— (6)}$$

Equating (5) to (6), $F'(x) = 0$

Integrating this,
$F(x)$ = absolute constant and hence neglected
Since it has no effect on the fluid motion

Now (4) becomes,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} \quad \text{——— (7)}$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}$$

Stream lines are given by ψ = constant

i.e., $\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right]$ = constant

or $\frac{y}{x^2-a^2+y^2}$ = constant.

If we take constant = 0, then we get $y=0$,

i.e. x-axis.

If we take constant = ∞, then we get circle

$$x^2 - a^2 + y^2 = 0,$$

i.e. $x^2 + y^2 = a^2$.

Thus, stream lines include x-axis and circle.

Hence, the result.