7 Mechanics and Fluid Dynamics

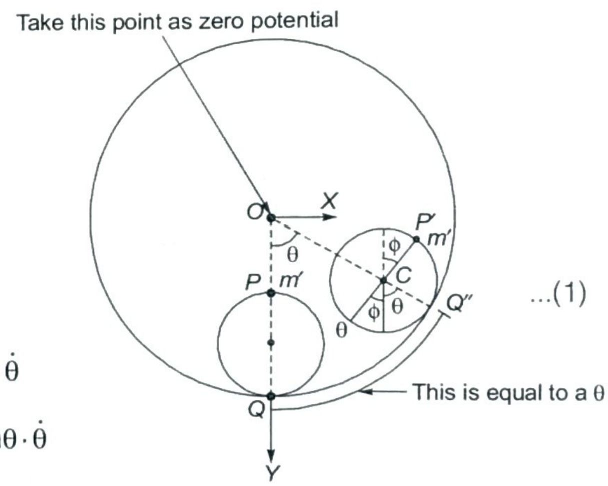
1. Generalised Coordinate

1.1 A perfectly rough sphere of mass m and radius b, rests on the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m' and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length

$$(a-b)\frac{4m' + \frac{7}{5}m}{m + m'\left(2 - \frac{a}{b}\right)}.$$

(2009 : 30 Marks)

Solution:



...(1)

Now, $\quad QQ'' = Q'Q''$

or $\quad a\theta = b(\theta + \phi)$

$$(a-b)\theta = b\phi$$

$$(a-b)\dot{\theta} = b\dot{\phi}$$

Now, total kinetic energy, T can be written as

$$T = \frac{1}{2}m'V_p^2 + \frac{1}{2}mV_{cm}^2 + \frac{1}{2}I_{cm}w^2$$

Now, $\quad x_c = (a-b)\sin\theta \Rightarrow \dot{x}_c = (a-b)\cos\theta\cdot\dot{\theta}$

$$y_c = (a-b)\cos\theta \Rightarrow \dot{y}_c = (a-b) - \sin\theta\cdot\dot{\theta}$$

$$V_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = (a-b)^2\dot{\theta}^2$$

$$I_{cm} = \frac{2}{5}ma^2 \text{ and } w = \dot{\phi}$$

$$x_p = (a-b)\sin\theta + b\sin\phi$$

$$\dot{x}_p = (a-b)\cos\theta\cdot\dot{\theta} + b\cos\phi\cdot\dot{\phi}$$

$$y_p = (a-b)\cos\theta - b\cos\phi$$

$$\dot{y}_p = (a-b)(-\sin\theta)\cdot\dot{\theta} + b\sin\phi\cdot\dot{\phi}$$

$$\dot{x}_p^2 + \dot{y}_p^2 = v_p^2 = (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}b(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}(1\cdot 1 - \theta\phi) \quad \text{(Ignoring the term } \theta\phi \text{ as it is very small)}$$

$$\approx (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)\dot{\theta}\dot{\phi}$$

Now potential energy, $\quad V = -mg(a-b)\cos\theta - m'g\{(a-b)\cos\theta - b\cos\phi\}$

Now lagrange, $\quad L = T - V$

$$L = \frac{1}{2}m'\{(a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2(a-b)b\dot{\theta}\dot{\phi}\} +$$

$$\frac{1}{2}m(a-b)^2\dot{\theta}^2 + \frac{1}{2}\times\frac{2}{5}ma^2\dot{\phi}^2 + mg(a-b)\cos\theta +$$
$$m'g\{(a-b)\cos\theta - b\cos\phi\}$$

As $(a-b)\theta = b\phi$

So, the generalised coordinate will only be one.

Convert θ to ϕ.

Also $(a-b)\dot{\theta} = b\dot{\phi}$

So,
$$L = \frac{1}{2}m'\{b\dot{\phi}^2 + b^2\dot{\phi}^2 + 2b^2\dot{\phi}^2\} + \frac{1}{2}mb^2\dot{\phi}^2 + \frac{ma^2}{5}\dot{\phi}^2 + mg(a-b)\cos\theta$$
$$+ m'g(a-b)\cos\theta - m'gb\cos\phi$$

$$L = \dot{\phi}^2\left(2m' + \frac{7m}{10}\right)b^2 + (m+m')g(a-b)\cos\frac{b\phi}{a-b} - m'gb\cos\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = 2\dot{\phi}\left(2m' + \frac{7m}{10}\right)b^2$$

$$\frac{\partial L}{\partial \phi} = \frac{b}{(a-b)}(m+m')g(a-b)(-1)\sin\frac{b\phi}{(a-b)} + m'gb\sin\phi$$

$$= -b(m+m')g\sin\frac{b\phi}{(a-b)} + m'gb\sin\phi$$

But $\sin\frac{b\phi}{a-b} \approx \frac{b\phi}{a-b}$ and $\sin\phi \approx \phi$

So,
$$\frac{\partial L}{\partial \phi} = -b(m+m')g\frac{b\phi}{(a-b)} + m'bg\phi$$

Now, using Lagrange's theorem

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

or
$$\ddot{\phi}\left(4m' + \frac{7m}{5}\right)b^2 + b(m+m')\frac{gb\phi}{a-b} - bm'g\phi$$

$$\ddot{\phi}\left(4m' + \frac{7m}{5}\right)b^2 + \phi bg\left[\frac{b(m+m') - m'(a-b)}{a-b}\right] = 0$$

$$\ddot{\phi} + \frac{gb}{\left(4m' + \frac{7m}{5}\right)b^2}\left(\frac{m'(2b-a) + bm}{(a-b)}\right)\phi = 0$$

This is S.H.M. equation.

or
$$\ddot{\phi} + w^2\phi = 0$$

So,
$$w^2 = \frac{g\left\{m'\left(2 - \frac{a}{b}\right) + m\right\}}{\left(4m' + \frac{7m}{5}\right)(a-b)} = \frac{g}{L}$$

or,
$$L = \frac{\left(4m' + \frac{7m}{5}\right)(a-b)}{m + m'\left(2 - \frac{a}{b}\right)}$$

So, oscillations are the same as those of a simple pendulum of above length

$$L = \frac{\left(4m' + \frac{7m}{5}\right)(a-b)}{m + m'\left(2 - \frac{a}{b}\right)}$$

1.2 **A sphere of radius a and mass m rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations.**

(2010 : 30 Marks)

Solution:

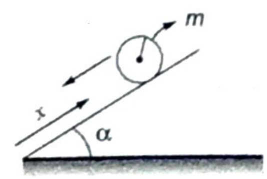


Figure-1

Figure-1 below depicts the situation given in problem.

$K \rightarrow$ Radius of gyration of given sphere

Kinetic energy,
$$T = \frac{1}{2}m(\dot{x}^2 + K^2\dot{\theta}^2)$$
$$= \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}a^2\dot{\theta}^2\right) \qquad \left(K = \frac{2}{a}a^2 \text{ for sphere}\right)$$
$$= \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}\dot{x}^2\right) = \frac{7}{10}m\dot{x}^2 \qquad (\text{In pure rolling, } \dot{x} = a\dot{\theta})$$

Potential energy,
$$V = -mfx\sin$$

$\therefore$
$$L = T - V = \frac{7}{10}m\dot{x}^2 + mfx\sin\alpha$$

Now,
$$P_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{10}m \times 2\dot{x} = \frac{7}{5}m\dot{x}$$

$\Rightarrow$
$$\dot{x} = \frac{5P_x}{7m}$$

$\therefore$ Hamiltonian,
$$H = -L + P_x \cdot \dot{x} = -\frac{7}{10}m\dot{x}^2 - mgx\sin\alpha + P_x \cdot \frac{5P_x}{7m}$$

$\Rightarrow$
$$H = \frac{-5P_x^2}{14m} - mgx\sin\alpha + \frac{5P_x^2}{7m} \qquad \left(\text{Putting } \dot{x} = \frac{5P_x}{7m}\right)$$

$\Rightarrow$
$$H = \frac{5}{14m}P_x^2 - mgx\sin\alpha$$

$\therefore$ One of the Hamiltonian's equation gives

$$\dot{P}_x = \frac{-\partial x}{\partial x} = +mf\sin\alpha$$

$\Rightarrow$
$$\frac{7}{5}m\ddot{x} = xf\sin\alpha$$

$\Rightarrow$
$$\ddot{x} = \frac{5}{7}g\sin\alpha$$

$\therefore$ Acceleration of sphere is $\frac{5}{7}g\sin\alpha$.

1.3 **Obtain the equations governing the motion of a spherical pendulum.**

(2012 : 12 Marks)

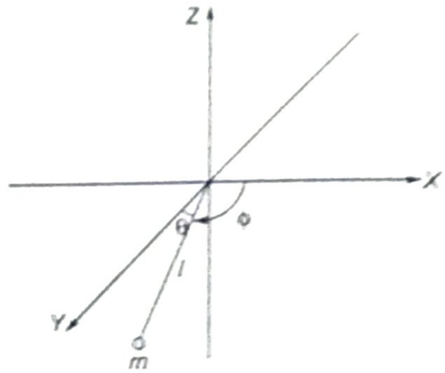
Solution:

Let 'm' be the mass of the bob of spherical pendulum, which can swing in any direction, traces out a sphere of constant length l.

Using polar co-ordinates θ and ϕ, the kinetic energy,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2)$$

and potential energy, $\quad V = -mgl\cos\theta$



Spherical Pendulum

For spherical pendulum, the length 'l' is constant.

$$\therefore \quad T = \frac{1}{2}m(0 + l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2)$$

$$= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

$$\therefore \quad \text{The Lagrangian } L = T - V$$

$$= \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2}ml^2(2\sin\theta\cos\theta\dot{\phi}^2) - mgl\sin\theta$$

$$= ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$\therefore$ Using Lagrangian equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\Rightarrow \quad ml^2\ddot{\theta} = ml^2\sin\theta\cos\theta\dot{\phi}^2 - mgl\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{g}{l}\sin\theta$$

Again, $\quad \dfrac{\partial L}{\partial \phi} = 0$

$$\frac{\partial L}{\partial \dot{\phi}} = 2\cdot\frac{1}{2}ml^2\sin^2\theta \times 2\dot{\phi} = ml^2\sin^2\theta\dot{\phi}$$

$$\therefore \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\Rightarrow \quad \frac{d}{dt}(ml^2\sin^2\theta\dot{\phi}) = 0$$

$$\Rightarrow \quad \dot{\phi} = \frac{K}{ml^2\sin^2\theta}, \ k \text{ is a constant}$$

$$\therefore \quad \ddot{\theta} = \sin\theta\cos\theta\left(\frac{k}{ml^2\sin^2\theta}\right)^2 - \frac{g}{l}\sin\theta$$

$$\Rightarrow \quad 2\dot{\theta}\ddot{\theta} = \frac{2k^2}{m^2l^4}\cot\theta\,\text{cosec}^2\theta\cdot\dot{\theta} - \frac{2g}{l}\sin\theta\dot{\theta}$$

Integrating both sides w.r.t. 't',

$$\dot{\theta}^2 = \frac{2k^2}{m^2l^4}\cdot\frac{(-\cot^2\theta)}{2} + \frac{2g}{l}\cos\theta + C$$

where C is a constant.

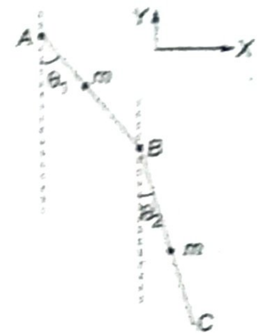
$$\Rightarrow \qquad \dot{\theta}^2 = \frac{k^2}{m^2 l^2}\cot^2\theta + \frac{2g}{l}\cot\theta + C$$

1.4 Two equal rods AB and BC each of length l, smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that the period of normal oscillations are $\frac{2\pi}{n}$ where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

(2013 : 15 Marks)

Solution:

The coordinates of midpoints of the rods :



$\left(\frac{l}{2}\sin\theta_1, \frac{-l}{2}\cos\theta_1\right)$ and $\left(\frac{l}{2}\sin\theta_1 + \frac{l}{2}\sin\theta_2, -l\cos\theta_1 - \frac{l}{2}\cos\theta_2\right)$ and their velocities.

$\left(\frac{l}{2}\cos\theta_1, \frac{l}{2}\sin\theta_1\right)$ and $\left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2, l\sin\theta_1 + \frac{l}{2}\sin\theta_2\right)$

$$T_1 = \frac{1}{2}I_1\dot{\theta}_1^2 \text{ where } I_1 \text{ is about } A$$

$$= \frac{1}{2}\frac{ml^2}{3}\dot{\theta}_1^2 = \frac{1}{6}ml^2\dot{\theta}_1^2$$

$$T_2 = \frac{1}{2}I_{CM}\dot{\theta}_2^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$= \frac{1}{2}\cdot\frac{ml^2}{12}\dot{\theta}_2^2 + \frac{1}{2}m\left[\left(l\cos\theta_1\dot{\theta}_1 + \frac{l}{2}\cos\theta_2\dot{\theta}_2\right)^2 + \left(l\sin\theta_1\dot{\theta}_1 + \frac{l}{2}\sin\theta_2\dot{\theta}_2\right)^2\right]$$

$$= \frac{ml^2}{24}\dot{\theta}_2^2 + \frac{1}{2}m\left[l^2\dot{\theta}_1^2 + \frac{l^2}{4}\dot{\theta}_2^2 + l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right]$$

Again, $\qquad V_1 = -mg\frac{l}{2}\cos\theta_1$

$$V_2 = -mg\left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)$$

$\therefore \qquad L = (T_1 + T_2) - (V_1 + V_2)$

$$= \frac{2}{3}ml^2\dot{\theta}_1^2 + \frac{1}{6}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + \frac{3}{2}mgl\cos\theta_1 + \frac{1}{2}mgl\cos\theta_2$$

The lagrangian equations are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = \frac{\partial L}{\partial\theta_1} \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) = \frac{\partial L}{\partial\theta_2} \text{ or } \frac{d}{dt}\left(\frac{4}{3}ml^2\dot{\theta}_1 + \frac{1}{2}ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_2\right)$$

$$= \frac{-3}{2}mgl\sin\theta_1 - \frac{1}{2}ml^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

i.e., $\frac{4}{3}ml^2\ddot{\theta}_1 + \frac{1}{2}ml^2\ddot{\theta}_2 = -\frac{3}{2}mgl\theta_1$

using small angle approximation

$$\cos(\theta_1 - \theta_2) = 1,\ \sin\theta_1 = \theta_1,\ \sin(\theta_1 - \theta_2) = 0$$

Similarly, other equations gives

$$\frac{d}{dt}\left[\frac{1}{3}ml^2\dot{\theta}_2+\frac{1}{2}ml^2\cos(\theta_1-\theta_2)\dot{\theta}_1\right] = -\frac{1}{2}mgl\sin\theta_2-\frac{1}{2}ml^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

$$\Rightarrow \qquad \frac{1}{2}ml^2\ddot{\theta}_1+\frac{1}{3}ml^2\ddot{\theta}_2 = -\frac{1}{2}mgl\theta_2 \text{ using small angle approximations.}$$

$$\therefore \qquad 8\ddot{\theta}_1+3\ddot{\theta}_2 = -9\frac{g}{l}\theta_1$$

$$3\ddot{\theta}_2+2\ddot{\theta}_2 = -3\frac{g}{l}\theta_2$$

Taking $\theta_1 = Ae^{iwt}$, $\theta_2 = Be^{iwt}$ as solution the equation are solvable if the determinant is zero, i.e.,

$$(8w^2-9g/l)(2w^2-3g/l)-9w^4 = 0$$

$$\Rightarrow \qquad 7w^4-42w^2\frac{g}{l}+27\frac{g^2}{l^2} = 0$$

$$w^2 = \frac{42\pm\sqrt{42^2-4\times7\times27}}{14}\frac{g}{l}$$

$$= \left(3\pm\frac{6}{\sqrt{7}}\right)\frac{g}{l}$$

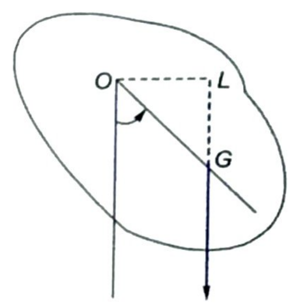
1.5 Find the equation of motion of a compound pendulum using Hamilton's equations.

(2014 : 10 Marks)

Solution:

At time t, let θ be the angle between the vertical plane through the fixed axis (plane fixed in space) and the plane through the centre of gravity 'G' and the fixed axis (plane fixed in the body).

Let $OG = h$



If T and v are the kinetic & potential energies of the pendulum then

$$T = \frac{1}{2}Mk^2.\theta^2$$

and $$V = -Mgh\cos\theta$$

(Negative sing is taken because G is below the fixed axis)

$$\therefore \qquad L = T-V$$

$$= \frac{1}{2}Mk^2.\theta^2+Mgh\cos\theta$$

Here θ is the only generalised co-ordinate.

$$\therefore \qquad p_\theta = \frac{\partial L}{\partial \theta} = Mk^2\theta \qquad \text{. . . (i)}$$

$$H = T+V = \frac{1}{2}Mk^2.\dot{\theta}^2-Mgh\cos\theta$$

$$= \frac{1}{2Mk^2}p_\theta^2-Mgh\cos\theta \qquad \text{(from (i))}$$

Hence the two Hamilton's equation are

$$\dot{p}_\theta = \frac{-2H}{\partial\theta} = -Mgh\sin\theta \qquad \text{. . . (ii)}$$

and $$\dot{\theta} = \frac{2H}{\partial P_\theta} = \frac{1}{Mk^2}p_\theta \quad \ldots \text{(iii)}$$

Differentiating (iii) and substituting from (i), we get

$$\ddot{\theta} = \frac{1}{Mk^r}\dot{p}_{\theta\cdot} = \frac{1}{Mk^2}(-Mgh\sin\theta)$$

$$\Rightarrow \quad \ddot{\theta} = \frac{-gh}{k^2}\sin\theta$$

which is the equation of correspond motion of pendulum.

1.6 A Hamitonian of a system with one degree of freedom has the form

$$H = \frac{P^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2}q^2e^{-\alpha t}(a + be^{-\alpha t}) + \frac{K}{2}q^2$$

where a, b, K are constants, q is the generalised coordinate and p is the corresponding generalized momentum.

(a) Find a lagrangian corresponding to this Hamiltonian.

(b) Find an equivalent lagrangian that is not explicitly dependent on time.

(2015 : 10+10 = 20 Marks)

Solution:

(a) Firstly, we try to get value of $\dot{q}$ end then eliminate P.

Given, $$H = \frac{P^2}{2\alpha} - bqpe^{-dt} + \frac{b\alpha}{2}q^2e^{-\alpha t}(\alpha + e^{-\alpha t}) + \frac{Kq^2}{2} \quad \ldots\text{(i)}$$

We know that

$$\dot{q} = \frac{\partial H}{\partial P} = \frac{2P}{2\alpha} - bqe^{-\alpha t} + 0 + 0 \quad \text{(from (i))}$$

$$\Rightarrow \quad \dot{q} = \frac{P}{\alpha} - bqe^{-\alpha t} \quad \ldots\text{(ii)}$$

$$\Rightarrow \quad \frac{P}{\alpha} = \dot{q} + bqe^{-\alpha t} \quad \ldots\text{(iii)}$$

Now, $H = p\dot{q} - L$, where L is the lagrangian.

$\therefore$ $L = p\dot{q} - H$

$$\Rightarrow \quad L = P\left(\frac{P}{\alpha} - bqe^{-\alpha t}\right) - \frac{P^2}{2\alpha} + bpqe^{-\alpha t} - \frac{b\alpha}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2}$$

(from (i) & (ii))

$$\Rightarrow \quad L = \frac{P^2}{\alpha} - bpqe^{-\alpha t} - \frac{P^2}{2\alpha} + bpqe^{-\alpha t} - \frac{b\alpha}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2}$$

$$\Rightarrow \quad L = \frac{P^2}{2\alpha} - \frac{b\alpha}{2}q^2e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2}$$

Putting value of P from (iii), we get

$$L = \frac{(\alpha\dot{q} + \alpha bq^{-\alpha t})^2}{2\alpha} - \frac{b\alpha q^2}{2}e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{kq^2}{2}$$

$$\Rightarrow \quad L = \frac{\alpha^2\dot{q}^2 + \alpha^2b^2q^2e^{-2\alpha t} + 2\alpha^2q\dot{q}be^{-\alpha t}}{2\alpha} - \frac{b\alpha^2q^2e^{-\alpha t}}{2} - \frac{b^2\alpha^2q^2e^{-2\alpha t}}{2} - \frac{kq^2}{2}$$

$$\Rightarrow \quad L = \frac{\alpha\dot{q}^2}{2} + \frac{\alpha b^2 q^2 e^{-2\alpha t}}{2} + \alpha q\dot{q}be^{-\alpha t} - \frac{b\alpha^2 q^2 e^{-\alpha t}}{2} - \frac{\alpha b^2 q^2 e^{-2\alpha t}}{2} - \frac{kq^2}{2}$$

$$\Rightarrow \quad L = \frac{\alpha\dot{q}^2}{2} - \frac{kq^2}{2} + \alpha q\dot{q}be^{-\alpha t} - \frac{\alpha^2 bq^2 e^{-\alpha t}}{2}$$

(b) In order to find lagrange independent of time, we need to find terms dependent on t in L and then assume equivalent lagrangian which does not depend on t.

$$L = \frac{\alpha\dot{q}^2}{2} - \frac{kq^2}{2} + \alpha q\dot{q}be^{-\alpha t} - \frac{\alpha^2 bq^2}{2}e^{-\alpha t}$$

$$\Rightarrow \quad L = \frac{\alpha\dot{q}^2}{2} - \frac{kq^2}{2} + \frac{d}{dt}\left(\frac{\alpha bq^2 e^{-\alpha t}}{2}\right)$$

Now, consider $$L' = L - \frac{d}{dt}\left(\frac{\alpha bq^2 e^{-\alpha t}}{2}\right)$$, it does not change motion.

$$\therefore \quad L' = \frac{\alpha\dot{q}^2}{2} - \frac{kq^2}{2} + \frac{d}{dt}\left(\frac{\alpha bq^2 e^{-\alpha t}}{2}\right) - \frac{d}{dt}\left(\frac{\alpha bq^2 e^{-\alpha t}}{2}\right)$$

$$\Rightarrow \quad L' = \frac{\alpha\dot{q}^2}{2} - \frac{kq^2}{2}$$ is equivalent lagrangian independent of time.

It is a one-dimensional harmonic oscillator.

1.7 Consider a single free particle of mass m, moving in space under no forces. If the particle starts from origin at $t = 0$ and reaches the position (x, y, z) at time z, find the Hamilton's characteristic function S as a function of x, y, z, τ.

(2016 : 10 Marks)

Solution:

Let the particle moves from origin to (x, y, z).

Kinetic energy, $$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Potential energy, $$V = 0$$

$$\therefore \quad H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = E = \text{Constant}$$

Now, Hamilton's characteristic function, S

$$\frac{\partial S}{\partial t} = -E = -\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\Rightarrow \quad S = -\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \times \tau \qquad (t = \tau)$$

$$\Rightarrow \quad S = -\frac{1}{2}m\left(\frac{x^2}{\tau^2} + \frac{y^2}{\tau^2} + \frac{z^2}{\tau^2}\right) \times \tau$$

$$\Rightarrow \quad S = -\frac{1}{2}m\frac{(x^2 + y^2 + z^2)}{\tau}$$

$\therefore$ Hamilton's characteristic function,

$$S = \frac{-1}{2}m\frac{(x^2 + y^2 + z^2)}{\tau}$$

1.8 A loop with radius r is rolling, without slipping, down an inclined plane of length l and with angle of inclination ϕ. Assign appropriate generalized coordinates to the system. Determine the coordinates, if any. Write down the Lagrangian equations for the system. Hence, or otherwise determine the velocity of the loop at the bottom of the inclined plane.

(2016 : 15 Marks)

Solution:

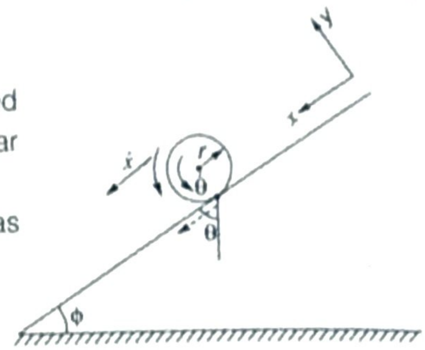


Figure shows motion of the loop down an inclined plane. Let the generalized coordinates be x and θ, where θ is angular displacement and x is linear displacement along the plane with x pointing downwards.

As the loop is not slipping, $\therefore$ velocity of point of contact should be O as plane is fixed.

$$\therefore \qquad \dot{x} = r\dot{\theta} \qquad \text{...(i)}$$

Let m be the mass of loop.

The kinetic energy of the loop is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \qquad \text{...(ii)}$$

where I is moment of inertia of loop.

The potential energy can be simply written as

$$V = -mg\,x\sin\phi \qquad \text{...(iii)}$$

$\therefore$ Lagregian,

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + mgx\sin\phi \qquad \text{(from (ii) \& (iii))}$$

From (i), we get

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 + mgx\sin\phi$$

$$\Rightarrow \qquad L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{r^2} + mgx\sin\phi$$

Now, we know that

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}\left(m\dot{x} + \frac{I\dot{x}}{r^2}\right) - mg\sin\phi = 0$$

$$\Rightarrow \qquad \left(m + \frac{I}{r^2}\right)\ddot{x} = mg\sin\phi, \text{ which is the equation of motion.}$$

Now, to find velocity at the end of motion, we use principle of conservation of total energy.

Let total energy, $\qquad E = T + V$

$$\Rightarrow \qquad E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 - mgx\sin\phi$$

$$\Rightarrow \qquad E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{\dot{x}}{r}\right)^2 - mgx\sin\phi \qquad \text{(from (i))}$$

$$\Rightarrow \qquad E = \left(m + \frac{I}{r^2}\right)\frac{\dot{x}^2}{2} - mgx\sin\phi$$

Now, initially, at $x = 0$, let velocity $= V_0$

$\therefore$
$$E_0 = \left(m + \frac{I}{r^2}\right)\frac{V_0^2}{2}$$

At the end of the plane $x = l$, velocity $= V_1$

$\therefore$
$$E_1 = \left(m + \frac{I}{r^2}\right)\frac{V_1^2}{2} - mgl\sin\phi$$

Now, by principle of conservation of energy
$$E_1 = E_0$$
$$\Rightarrow \quad \left(m + \frac{I}{r^2}\right)\frac{V_1^2}{2} - mgl\sin\phi = \left(m + \frac{I}{r^2}\right)\frac{V_0^2}{2}$$

The moment of inertia of loop, $I = mr^2$

$\therefore$ Equation becomes
$$\left(m + \frac{mr^2}{r^2}\right)\frac{V_1^2}{2} - mgl\sin\phi = \left(m + \frac{mr^2}{r^2}\right)\frac{V_0^2}{2}$$
$$\Rightarrow \quad mV_1^2 - mgl\sin\phi = mV_0^2$$
$$\Rightarrow \quad V_1^2 - gl\sin\phi = V_0^2$$
$$\Rightarrow \quad V_1^2 = V_0^2 + gl\sin\phi$$
$$\Rightarrow \quad V_1 = \sqrt{V_0^2 + gl\sin\phi}$$

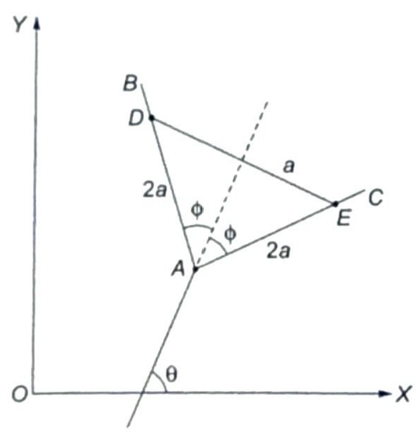
So, final velocity is $V_1 = \sqrt{V_0^2 + gl\sin\phi}$ where V_0 is initial velocity.

1.9 Two uniform rods AB, AC each of mass m and length $2a$, are smoothly hinged together at A and move on a horizontal plane. At time t, the mass centre of the rods is at the point (ξ, η) referred to fixed perpendicular axes, OX, OY in the plane, and the rods make angles $\theta \pm \phi$ with OX. Prove that the kinetic energy of the system is $m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right]$. Also derive Lagrange's equations of motion for the system if an external force with components $[X, Y]$ along the axes acts at A.

(2017 : 20 Marks)

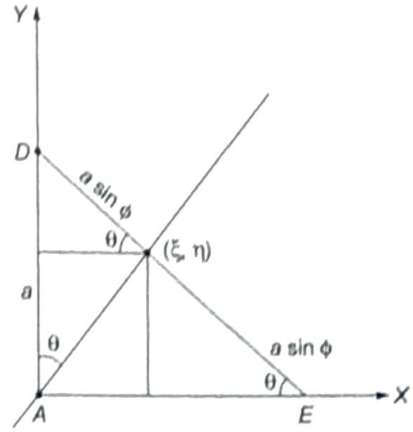
Solution:

For the 2 rods



$$T = \frac{1}{2}m\left[\dot{x}_D^2 + \dot{y}_D^2\right] + \frac{1}{2}\frac{mAa^2}{12}(\dot{\theta}^2 + \dot{\phi}^2) + \frac{1}{2}m\left[\dot{x}_E^2 + \dot{y}_E^2\right] + \frac{1}{2}m\left(\frac{mAa^2}{12}\right)(\dot{\theta}^2 - \dot{\phi}^2) \quad ...(i)$$

$G = (\xi, \eta)$ centre of mass



From the diagram,

$$\xi_D = \xi - a\sin\phi\sin\theta$$
$$y_D = \eta + a\sin\phi\cos\theta$$
$$y_E = \eta - a\sin\phi\cos\theta$$

$\therefore$

$$\dot{x}_D = \dot{\xi} - a\cos\phi\sin\theta\dot{\phi} - a\sin\phi\cos\theta\dot{\theta}$$
$$\dot{x}_E = \dot{\xi} + a\cos\phi\sin\theta\dot{\phi} + a\sin\phi\cos\theta\dot{\theta}$$
$$\dot{y}_D = \dot{\eta} + a\cos\phi\cos\theta\dot{\phi} - a\sin\phi\sin\theta\dot{\theta}$$
$$\dot{y}_E = \dot{\eta} - a\cos\phi\cos\theta\dot{\phi} + a\sin\phi\sin\theta\dot{\theta}$$
$$\dot{x}_D^2 + \dot{x}_E^2 = 2\dot{\xi}^2 + 2(a\cos\phi\sin\theta\dot{\phi} + a\sin\phi\cos\theta\dot{\theta})^2$$
$$= 2\dot{\xi}^2 + 2a\cos^2\phi\sin^2\theta\dot{\phi}^2 + 2a^2\sin^2\phi\cos^2\theta\dot{\theta}^2 + 4a^2\sin\phi\cos\phi\sin\theta\cos\theta\dot{\phi}\dot{\theta}$$

Similarly,

$$\dot{y}_0^2 + \dot{y}_E^2 = 2\dot{\eta}^2 + 2a^2\cos^2\phi\cos^2\theta\dot{\phi}^2 + 2a^2\sin^2\phi\sin^2\theta\dot{\theta}^2 - 4a^2\sin\phi\cos\phi\sin\theta\cos\theta\dot{\phi}\dot{\theta}$$

$\therefore$

$$\dot{x}_0^2 + \dot{x}_0^2 + \dot{y}_0^2 + \dot{y}_E^2 = 2[\dot{\xi}^2 + \dot{\eta}^2 + a^2\cos^2\phi\dot{\phi}^2 + a^2\sin^2\phi\dot{\theta}^2] \quad ...(ii)$$

Rearranging (i)

$$T = \frac{1}{2}m\left\{\dot{x}_0^2 + \dot{x}_E^2 + \dot{y}_D^2 + y_E^2\right\} + \frac{1}{3}\frac{ma^2}{2}\left[2\dot{\phi}^2 + 2\dot{\theta}^2\right]$$

Using (ii)

$$T = m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right].$$ **Proved.**

Now, constant force $[X, Y]$ acts at A.

$\therefore$

$$V = -\int_A F dr$$

Co-ordinates of A are $[\xi - a\cos\phi\cos\theta, \eta - a\cos\phi\sin\theta]$

$\therefore$

$$V = X(a\cos\phi\cos\theta - \xi) + Y(a\cos\phi\sin\theta - \eta) + \text{Constant}$$
$$L = T - V$$
$$= m\left[\dot{\xi}_1^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right] + X[\xi - a\cos\phi\cos\theta] + Y[\eta - a\cos\phi\sin\theta] + \text{Constant}$$

Equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_\alpha}\right)-\frac{\partial L}{\partial q_\alpha} = 0$$

for ξ, $$\frac{d}{dt}[2m\dot{\xi}]-X = 0$$

$\Rightarrow$ $$\ddot{\xi}-\frac{X}{2m} = 0 \quad \text{...(i)}$$

for η, $$\ddot{\eta}-\frac{Y}{2m} = 0 \quad \text{...(ii)}$$

for ϕ, $$\frac{d}{dt}\left[\left(\frac{1}{3}+\cos^2\phi\right)2a^2\dot{\phi}\right] - (2\sin\phi\cos\phi a^2\dot{\theta}^2-2\sin\phi\cos\phi a^2\dot{\phi}^2) + (Xa\sin\phi\cos\theta + 4a\sin\phi\sin\theta) = 0$$

$$2a^2\left(\frac{1}{3}+\cos^2\phi)\right)\ddot{\phi}-a^2(\sin2\phi\dot{\theta}^2-\sin^2\phi\dot{\phi}^2)-a\sin\phi(X\cos\theta+Y\sin\theta) = 0$$

$$2a^2\left(\frac{1}{3}+\cos^2\phi\right)\ddot{\phi}-a^2\sin2\phi(\dot{\theta}^2-\dot{\phi}^2)-a\sin\phi(X\cos\theta+Y\sin\theta) = 0 \quad \text{...(iii)}$$

For θ:

$$\frac{d}{dt}\left[\left(\frac{1}{3}+\sin^2\phi\right)2a^2\dot{\theta}\right]-(Xa\cos\phi\sin\theta-Ya\cos\phi\cos\theta) = 0$$

$$2a^2\left(\frac{1}{3}+\sin^2\phi\right)\ddot{\theta}-a\cos\theta(X\sin\theta-Y\cos\theta) = 0 \quad \text{...(iv)}$$

1.10 Suppose the Lagrangian of a mechanical system is given by $L = \frac{1}{2}m(a\dot{x}^2+2b\dot{x}\dot{y}+c\dot{y}^2)-\frac{1}{2}k(ax^2+2b....)$ where $a, b, c, m\,(>0), k(>0)$ are constants and $b^2 \neq ac$. Write down the Lagrengian equations of motion and identify the system.

(2018 : 20 Marks)

Solution:

Lagrangian x-equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x} = 0$$

$\Rightarrow$ $$\frac{\partial}{\partial t}(m(a\dot{x}+b\dot{y}))+k(\dot{a}x+by) = 0$$

$\Rightarrow$ $$m(a\ddot{x}+b\ddot{y}) = -k(ax+by)$$

$\Rightarrow$ $$a\ddot{x}+b\ddot{y} = \frac{-k}{m}(ax+by)$$

or $$\ddot{X} = \frac{-k}{m}X\text{, where } X = ax+by \quad \text{...(i)}$$

Lagrengian Y-Equations:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y} = 0$$

$\Rightarrow$ $$\frac{\partial}{\partial t}\left(\frac{1}{2}m(2b\dot{x}+2c\dot{y})\right)+k(bx+cy) = 0$$

$$\Rightarrow \quad b\ddot{x}+c\ddot{y} = \frac{-k}{m}(bx+cy)$$

or
$$\ddot{y} = \frac{-k}{m}y,$$

where
$$y = bx + cy \quad \text{...(ii)}$$

From (i) and (ii), it can be concluded that it is a 2-D harmonic oscillator.

1.11 The Hamiltonian of a mechanical system is given by $H = p_1q_1 - aq_1^2 + bq_2^2 - p_2q_2$, where a, b are the constants. Solve the Hamiltonian equations and show that $\frac{p_2 - bq_2}{q_1}$ = Constant.

(2018 : 20 Marks)

Solution:

Given :
$$H = p_1q_1 - aq_1^2 + bq_2^2 - p_2q_2$$

Now,
$$\dot{p}_1 = \frac{-\partial H}{\partial q_1} = -p_1 + 2aq_1,\ \dot{q}_1 = \frac{\partial H}{\partial p} = q_1 \quad \text{...(i)}$$

$$\dot{p}_2 = \frac{-\partial H}{\partial q_2} = -2bq_2 + p_2,\ \dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2 \quad \text{...(ii)}$$

From (i)
$\dot{q}_1 = q_1 \Rightarrow q_1 = c_1e^t$, where c_1 is a constant.

Putting this in equation,
$$\dot{p}_1 = -p_1 + 2aq_1 = -p_1 + 2ac_1e^t$$

$$\Rightarrow \quad \dot{p}_1 + p_1 = 2ac_1e^t$$

Integrating factor,
$$\text{I.F.} = e^{\int dt} = e^t$$

$$\therefore \quad p_1 \cdot e^t = \int 2ac_1e^{2t}dt = ac_1e^{2t} + c_2 \quad (c_2 \text{ is constant})$$

$$\therefore \quad p_1 = ac_1e^t + c_2e^{-t}$$

Similarly, from (ii)

$q_2 = c_3e^{-t}$, where c_3 is a constant.

$$\dot{p}_2 \mp p_2 = -2bc_3e^{-t}$$

Integrating factor,
$$\text{I.F.} = e^{-\int dt} = e^{-t}$$

$$\therefore \quad p_2 \cdot e^{-t} = -\int 2bc_3e^{-2t}dt = bc_3e^{-2t} + c_4 \quad (c_4 \text{ is a constant})$$

$$\therefore \quad p_2 = bc_3e^{-t} + c_4e^t$$

Now,
$$\frac{p_2 - bq_2}{q_1} = \frac{bc_3e^{-t} + c_4e^t - bc_3e^{-t}}{c_1e^t} = \frac{c_4}{c_1}$$

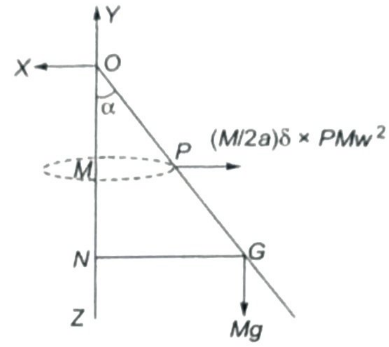
$$\Rightarrow \quad \frac{p - bq_2}{q_1} = \frac{c_4}{c_1} = \text{Constant}$$

1.12 A uniform rod OA, of length $2a$, free to turn about its end O, revolves with angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ; find the value of α.

(2019 : 10 Marks)

Solution:

Let the rod OA of length 2a and mass M revolve with uniform angular velocity w about the vertical OZ through O making a constant angle a to OZ. Let PQ = 5x be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}5x$.



This element PQ will make a circle in the horizontal plane with radius PM (= 2 sin a) and write at M. Since the rod revolve with uniform angular velocity, the only effective angular velocity, the only effective force on this element is $\frac{M}{2a}5x \cdot PM \cdot w^2$ along PM.

Thus, the reversed effective force on the element PQ is $\frac{M}{2a}5x \cdot x\sin\alpha . w'$ along MP.

Now, by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and reaction at 'O' are in equilibrium. To avoid reaction at O taking moment about O, we get

$$= \sum\left(\frac{M}{2a}5x \cdot w^2 \cdot \sin\alpha\right)\cdot OM - Mg - NG = 0$$

$$= \int_0^{2a}\frac{M}{2a}w^2x^2\sin\alpha\cos\alpha dx - Mg\cdot a\sin\alpha \qquad (\because OM = x\cos\alpha)$$

$$= \frac{M}{2a}w^2\left\{\frac{1}{3}(2a)^3\right\}\sin\alpha\cos\alpha - Mga\sin\alpha = 0$$

$$Mga\sin\alpha\left(\frac{4a}{3g}w^2\cos\alpha - 1\right) = 0$$

$\therefore$ either $\sin\alpha = 0$, i.e., α 0

$$\Rightarrow \qquad \frac{4a}{3g}w^2\cos\alpha - 1 = 0, \text{ i.e., } \cos\alpha = \frac{3g}{4aw^2}$$

Hence, the rod is inclined at an angle zero (or) $\cos^{-1}\left(\frac{3g}{4aw^2}\right)$.

1.13 A circular cylinder of radius *a* and radius of gyration *k* rolls without slipping inside a fixed hollow cylinder of radius *b*. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$.

(2019 : 20 Marks)

Solution:

Let *p* be the point of contact of the two cylinders at time *b* such that $\angle AOP = \theta$. Let f the angle which the line *CB* fixed in moving cylinder make with the vertical at time *t*. Here radius of fixed cylinder is *a* and that of moving cylinder is *a*. Since there is pure rolting therefore Arc *AP* = Arc *BP*.

$$\Rightarrow \qquad b\theta = (b-a)\theta$$

$$\therefore \qquad \ddot{\phi} = c\ddot{\theta} \qquad \text{...(1)}$$

where $\qquad c = (b-a)$

Let *R* be the normal reaction and *F* the friction at the point *p*.

∴ The centre C describes a circle of radius $OC = b - a = C$.

∴ Its acceleration along and perpendicular to CO are $C\dot{\theta}^2$ and $C\ddot{\theta}$ respectively.

∴ The equations of motion of the moving cylinder are

$$MC\dot{\theta}^2 = R - Mg\cos\theta \quad ...(2)$$

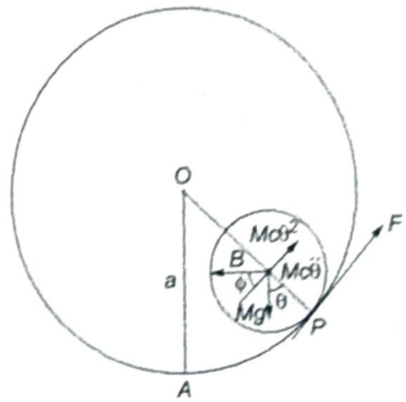
and
$$MC\ddot{\theta} = F - Mg\sin\theta \quad ...(3)$$

Also, for the motion relative to the center of inertia C,

$$MK^2\ddot{\phi} = \text{Moment of the forces about } C = -Fa \quad ...(4)$$

$$MK^2\frac{C}{a}\ddot{\theta} = -Fa$$

⇒
$$F = -MK^2\frac{C}{a^2}\ddot{\theta}$$



Substituting in (3), we get

$$MC\ddot{\theta} = -MK^2\frac{C}{a^2}\ddot{\theta} - Mg\sin\theta$$

$$= C\left(1+\frac{K^2}{a^2}\right)\ddot{\theta} = -g\sin\theta$$

$$= \ddot{\theta} = \frac{-g}{C\left(1+\frac{K^2}{a^2}\right)}\theta$$

$$= -\mu\theta$$

∵ θ is very small.

∴ Length of the simple equivalent pendulum is

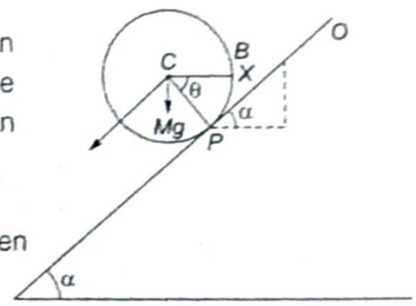
$$\frac{g}{\mu} = C\left(1+\frac{K^2}{a^2}\right) = (b-a)\left(1+\frac{K^2}{a^2}\right)$$

1.14 Using Hamilton's equation, find the acceleration for a sphere rolling down a rough inclined plane, if x be the distance of the point of contact of the spehre from a fixed point on the plane.

(2019 : 15 Marks)

Solution:

Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time b, let the sphere roll down a distance x and during this time let it turn through an angle θ. Since there is no slipping.



∴
$$x = OA = \text{arc } AB = a\theta$$

So that if T and V are the cinetic and potential energies of the sphere, then

$$T = \frac{1}{2}M\frac{2}{5}a^2\theta^2 + \frac{1}{2}M(a\dot{\theta})^2$$

$$= T = \frac{7}{10}M\dot{x}^2$$

and
$$V = -MgOL = -Mga\sin\alpha \text{ (since the sphere move down the plane)}$$

∴
$$L = T - V = \frac{7}{10}Mx^2 + Mgx\sin\alpha$$

Here x is the only generalised coordinate.

$\therefore \qquad p_x = \frac{\partial L}{\partial x} = \frac{7}{5}M\dot{x}$

Since L does not contain terplicity.

$$\therefore \qquad H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx \text{ find}$$

$$= \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx \text{ find}$$

$$= \frac{5}{14M}p_x^2 - Mgx\sin\alpha \qquad \text{...from (1)}$$

Hence, the two Hamilton's equations are

$$\dot{p}_x = \frac{-\partial H}{\partial x} = Mg\sin\alpha - (H_1)$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M}p_x - (H_2)$$

Differentiating (H_2) and using (H_1), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M}Mg\sin\alpha$$

$$\ddot{x} = \frac{5}{7}g\sin\alpha$$

which gives the required acceleration.

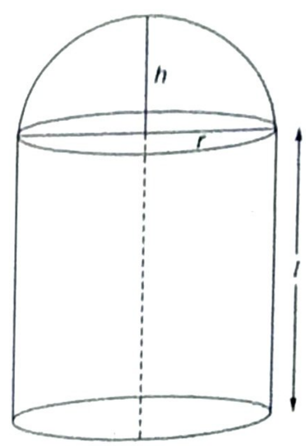
2. Moment of Inertia

2.1 The flat surface of a hemisphere of radius 'r' is cemented to one flat surface of a cylinder of the same radius and of the same material. If the length of the cylinder be l and the total mass be m, show that the moment of inertia of the combination about the axis of the cylinder is given by :

$$\frac{mr^2\left(\frac{l}{2} + \frac{4}{15}r\right)}{\left(l + \frac{2r}{3}\right)}$$

(2009 : 12 Marks)

Solution:



The combined moment of inertia about the axis of cylinder will be given by adding their moment of inertia separately about this axis, i.e.,

$$MI_{Total} = MI_H + MI_C \qquad \text{...(1)}$$

Now, let ρ be the density of materia and m_H, m_C be the masses of hemisphere and cylinder respectively.

So,

$$m = m_H + m_C$$

$$m = \rho \times \frac{2}{3}\pi r^3 + \rho \times \pi r^2 l$$

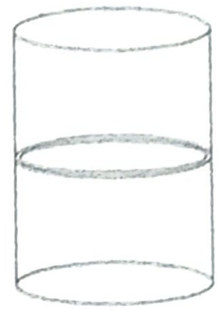
$$m = \rho\pi r^2\left(\frac{2}{3}r + l\right) \qquad \text{...(2)}$$

$$\rho = \frac{m}{\pi r^2\left(l + \frac{2r}{3}\right)} \qquad \text{...(3)}$$

Now, we know that moment of inertia of a spherical body (solid) or hemispherical body (solid) is given by $\frac{2}{5}MR^2$.

So,
$$MI_H = \frac{2}{5}mHr^2 = \frac{2}{5}\times\rho\times\frac{2}{3}\pi r^3 r^2$$
$$= \frac{4\pi\rho r^5}{15} \qquad ...(4)$$

Similarly, for calculating moment of inertia of cylinder (solid) can be calculated by assuming a disc of length/width dl.



Now
$$dMI_C = \frac{dmr^2}{2}$$

($\because$ MI of a disc about its axis is $\frac{MR^2}{2}$)

Now, mass of disc $= dm = \rho\times\pi r^2 dl$

So,
$$dMI_C = \frac{\rho\pi r^2 \cdot r^2 dl}{2}$$
$$MI_C = \frac{\rho\pi r^4}{2}\int_0^l dl = \frac{\rho\pi r^4 l}{2} \qquad ...(5)$$

From (1),
$$MI_{Total} = MI_H + M_IC$$
$$= \frac{4\pi\rho r^5}{15} + \frac{\rho\pi r^4 l}{2} \qquad \text{[from (4) and (5)]}$$
$$= \pi\rho r^4\left(\frac{4r}{15}+\frac{l}{2}\right) \qquad ...(6)$$

From (3), put value of ρ in (6)
$$MI_{Total} = \frac{\pi r^4 \times m}{\pi r^2\left(l+\frac{2r}{3}\right)}\left(\frac{l}{2}+\frac{4r}{15}\right)$$
$$= \frac{mr^2\left(\frac{l}{2}+\frac{4r}{15}\right)}{l+\frac{2r}{3}}$$

Hence, moment of inertia of the combination about the axis of the cylinder is given by :
$$\frac{mr^2\left(\frac{l}{2}+\frac{4r}{15}\right)}{\left(l+\frac{2r}{3}\right)}$$

2.2 **A uniform lamina is bounded by a parabolic arc of latus rectum 4*a* and a double ordinate at a distance *b* from the vertex. If $b = \frac{a}{3}(7+4\sqrt{7})$, show that two of the principal axes at the end of a lotus rectum are tangent and normal there.**

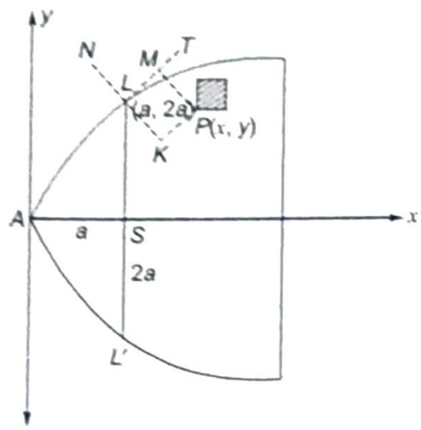
(2010 : 12 Marks)

Solution:

Let the equation of parabola be

$$y^2 = 4ax \quad \text{...(1)}$$

Coordinates of the end L of the latus rectum LL' are $(a, 2a)$ (see fig.)



Differentiating eqn. (1), we get

$$2y.y' = 4a \Rightarrow y' = \frac{2a}{y}$$

$\therefore$ At $L(a, 2a)$, $\frac{dy}{dy} = \frac{2a}{2a} = 1$

$\therefore$ Equation of the tangent LT at L is

$$y - 2a = 1.(x - a)$$

or $$y - x - a = 0 \quad \text{...(2)}$$

and equation of normal LN at L is

$$y - 2a = \frac{-1}{1}(x - a)$$

or $$y + x - 3a = 0 \quad \text{...(3)}$$

Consider an element $\delta x \delta y$ at the point $P(x, y)$ of the lamina.

PM = length of the perpendicular from P on the tangent LT given by (2)

$$= \frac{y - x - a}{\sqrt{1+1}} = \frac{y - x - a}{\sqrt{2}}$$

and PK = length of the perpendicular from P on the normal LN given by (3)

$$= \frac{y + x - 3a}{\sqrt{2}}$$

Product of inertia of the element about LT and LN

$$= PM.PK.\delta m = \frac{(y - x - a)}{\sqrt{2}} \frac{(y + x - 3a)}{\sqrt{2}} \cdot \rho \delta x \delta y$$

If the tangent and normal at L are the principal axes, then product of inertia of the lamina about these will be zero.

Hence, product of inertia of lamina about LT and LN is equal to zero, i.e.,

$$\int_{x=0}^{b} \int_{0}^{b} \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \{y^2 - 4ay + (3a^2 + 2ax - x^2)\} dx dy = 0$$

$$\Rightarrow \qquad \frac{\rho}{2}\int_0^b\int_{-2\sqrt{ax}}^{2\sqrt{ax}} \{y^2 - 4ay + (3a^2 + 2ax - x^2)\} dxdy = 0$$

$$\Rightarrow \qquad \int_0^b \left[\frac{1}{3}y^3 - 2ay^2 + (3a^2 + 2ax - x^2)\cdot y\right]_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$\Rightarrow \qquad 2\int_0^b \left\{\frac{8}{3}ax\sqrt{ax} + 2(3a^2 + 2ax - x^2)\sqrt{ax}\right\} dx = 0$$

$$\Rightarrow \qquad \int_0^b \left(\frac{8}{3}a^{3/2}x^{3/2} + 6a^{5/2}x^{1/2} + 4a^{3/2}x^{3/2} - 2a^{1/2}x^{5/2}\right) dx = 0$$

$$\Rightarrow \qquad \left[\frac{16}{15}a^{3/2}b^{5/2} + 4a^{5/2}b^{3/2} + \frac{8}{5}a^{3/2}b^{5/2} - \frac{4}{7}a^{1/2}b^{7/2}\right] = 0$$

$$\Rightarrow \qquad \frac{16}{15}ab + 4a^2 + \frac{8}{5}ab - \frac{4}{7}b^2 = 0$$

$$\Rightarrow \qquad b^2 - \frac{14}{3}ab - 7a^2 = 0$$

$$\Rightarrow \qquad b = \frac{\frac{14a}{3} \pm \sqrt{\frac{196}{9}a^2 + 28a^2}}{2} = \frac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right)a$$

$$\Rightarrow \qquad b = \frac{a}{3}(7 + 4\sqrt{7}) \text{ (–ve sign rejected as } b \text{ is always positive)}$$

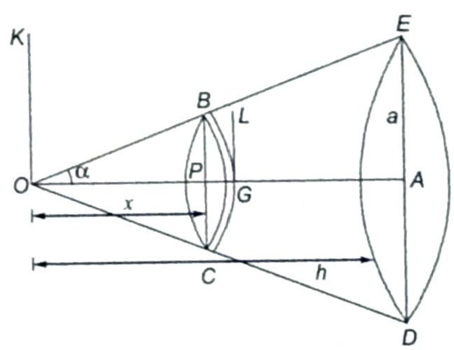
Hence, if $b = \frac{a}{3}(7 + 4\sqrt{7})$, then principal axes at L are the tangents and normal there.

2.3 Let a be the radius of the base of a right circular cone of height h and mass M. Find the moment of inertia of that right circular cone about a line through the vertex perpendicular to the axis.

(2011 : 12 Marks)

Solution:

Let ODE be the given cone whose semi-vertical angle is α and mass per unit volume is ρ.



Then
$$M = \frac{1}{3}\pi(h\tan\alpha)^2 \cdot h\rho$$

$$\Rightarrow \qquad \rho = \frac{3M}{\pi h^3 \tan^2\alpha} \qquad \text{...(i)}$$

Consider an elementary disc BC of thickness δx, parallel to the base DE and at a distance x from the vertex O.

Then
$$\delta m = \text{mass of elementary disc } BC$$
$$= \pi(x\tan\alpha)^2 \cdot \delta x \cdot \rho$$

∴ Moment of inertia of the elementary disc about the axis OA
$$= (\rho\pi x^2\tan^2\alpha\,\delta x)\cdot\frac{PB^2}{2}$$

($\because$ M.I. of a circular disc about a line through the centre and perpendicular to its plane $= \frac{1}{2}Ma^2$)
$$= (\rho\pi x^2\tan^2\alpha\,\delta x)(x\tan\alpha)^2\cdot\frac{1}{2} = \frac{1}{2}\rho\pi x^4\tan^4\alpha\,\delta x$$

Hence, M.I. of the cone ODE about its axis,
$$OA = \int_0^h \frac{1}{2}\rho\pi x^4\tan^4 x\,dx$$
$$= \frac{\rho\pi h^5\tan^4\alpha}{10}$$
$$= \frac{3M}{\pi h^3\tan^2\alpha}\times\frac{\pi h^5\tan^4\alpha}{10}$$
$$= \frac{3}{10}Mh^2\tan^2\alpha = \frac{3}{10}Ma^2 \text{ as } a = h\tan\alpha$$

Using the theorem of parallel axis, M.I. of the elementary disc about OK (which is a line through vertex O and perpendicular to OA)
$$= \text{M.I. of the elementary disc about its diameter} + x^2\cdot\delta m$$
$$= \pi x^2\tan^2\alpha\cdot\rho\left(\frac{1}{4}x^2\tan^2\alpha + x^2\right)\delta x$$

∴ M.I. of the cone ODE about OK
$$= \int_0^h \pi x^2\tan^2\alpha\cdot\rho\cdot\left(\frac{1}{4}x^2\tan^2\alpha + x^2\right)dx$$
$$= \frac{\pi\tan^2\alpha\cdot\rho\cdot h^5}{5}\left(\frac{\tan^2\alpha}{4}+1\right)$$
$$= \frac{\pi h^5\tan^2\alpha}{5}\times\frac{3M}{\pi h^3\tan^2\alpha}\left(\frac{\tan^2\alpha}{4}+1\right)$$
$$= \frac{3Mh^2}{5}\left(\frac{a^2}{4h^2}+1\right) \qquad \left(\because \tan\alpha = \frac{a}{h}\right)$$
$$= \frac{3M}{20}(a^2+4h^2)$$

2.4 A pendulum consists of a rod of length $2a$ and mass m; to one end of which a spherical bob of radius $a/3$ and mass 15 m is attached. Find the moment of inertia of the pendulum :

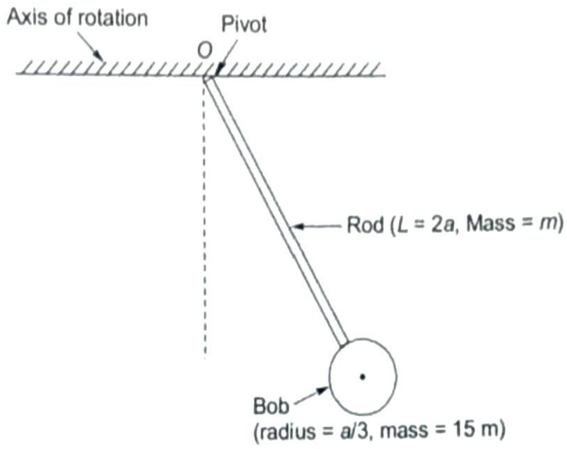
(a) about an axis through the other end of the rod and at right angles to the rod.

(2012 : 15 Marks)

(b) about a parallel axis through the centre of mass of the pendulum.
(Given : The centre of mass of the pendulum is $a/12$ above the centre of the sphere.]

(2012 : 15 Marks)

Solution:



Case I : Moment of inertia (M.I.) of the pendulum about an axis through the other end of the rod and at right angles to the rod

= M.I. of the rod about the given axis + M.I. of the spherical bob about the given axis

$$= \frac{1}{3}M_{rod}L_{rod}^2 + \frac{2}{5}M_{bob}(\text{Radius})^2 + M_{bob}(L_{rod}+R)^2$$

$$= \frac{1}{3}\cdot m\cdot(2a)^2 + \frac{2}{5}\cdot 15m\cdot\left(\frac{a}{3}\right)^2 + 15m\left(2a+\frac{a}{3}\right)^2$$

$$= \frac{4}{3}ma^2 + \frac{2}{3}ma^2 + \frac{245}{3}ma^2$$

$$= \frac{251}{3}ma^2$$

Case II : M.I. of the pendulum about a parallel axis through the centre of mass of the pendulum,

I_{CM} = $I_{\text{Case I}}$ – M.I. of mass (m + 15m) placed at O about the parallel axis.

$$= \frac{251}{3}ma^2 - 16m\times\left(\frac{9a}{4}\right)^2$$

$$= \frac{251}{3}ma^2 - 81ma^2$$

$$= \frac{251ma^2 - 243ma^2}{3} = \frac{8}{3}ma^2$$

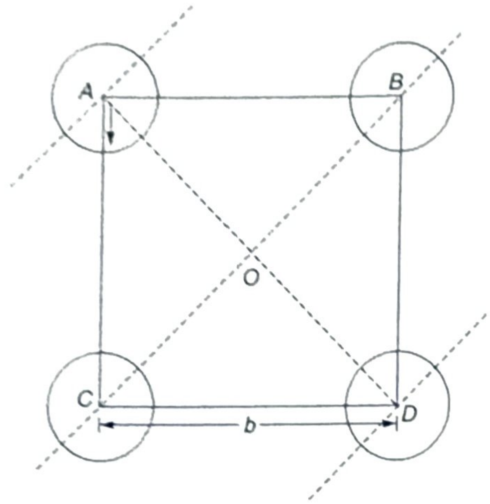
2.5 Four solid spheres A, B, C and D each of mass m and radius a, are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia about a diagonal of the square.

(2013 : 10 Marks)

Solution:

M.I. of sphere A about axis thrugh centre

$$= \frac{2}{5}ma^2$$



Let BC be the diagonal axis. Then M.I. of A about axis through centre parallel to $BC = \frac{2}{5}ma^2$

$\therefore$ M.I. of A about BC

$$= \frac{2}{5}ma^2 + m\cdot AO^2$$

$$= \frac{2}{5}ma^2 + m\cdot\left(\frac{b\sqrt{2}}{2}\right)^2$$

$$= m\left(\frac{2a^2}{5} + \frac{b^2}{2}\right)$$

Similarly, M.I. of D about BC

$$= m\left(\frac{2a^2}{5} + \frac{b^2}{2}\right)$$

$$\text{M.I. of } B \text{ (or } C\text{) about } BC = \frac{2}{5}ma^2$$

$\therefore$ M.I. of all 4 spheres about BC

$$= m\left(4\times\frac{2a^2}{5} + 2\times\frac{b^2}{2}\right) = m\left(\frac{8a^2}{5} + b^2\right)$$

2.6 Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, with mass m about the oz-axis.

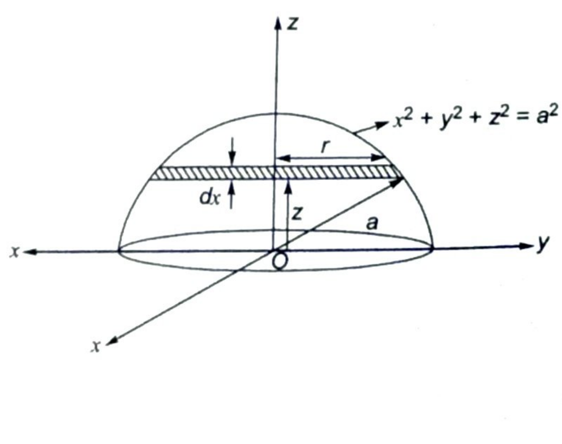
(2015 : 10 Marks)

Solution:

Figure shows solid hemisphere of radius a. Let mass of hemisphere be M, density ρ. We take disc of thickness dx of radius r at a distance x from the centre of hemisphere. Let dm be the mass of elementary disc.

$\therefore \qquad dm = \rho\times\pi r^2 dx$ ...(i)

Let moment of inertia of this elementary disc about oz is I_x.



$$\therefore \qquad I_x = \frac{r^2}{2}dm = \frac{r^2}{2}\rho\pi r^2 dx = \frac{\rho\pi r^4}{2}dx$$

Now, moment of inertia of hemisphere is

$$I = \int \frac{\rho\pi r^4}{2}dx \qquad \text{...(ii)}$$

Also, $r^2 = a^2 - x^2$

Putting this value in eqn. (ii), we get

$$I = \int \frac{\rho\pi r^4}{2}dx = \int_{x=0}^{a} \frac{\rho\pi}{2}(a^4dx + x^4dx - 2a^2x^2dx)$$

$$\Rightarrow \qquad I = \int_{x=0}^{a} \frac{\rho\pi}{2}(a^4dx + x^4dx - 2a^2x^2dx)$$

$$\Rightarrow \qquad I = \frac{\rho\pi}{2}\left[a^4[x]_0^a + \left[\frac{x^5}{5}\right]_0^a - 2a^2\left[\frac{x^3}{3}\right]_0^a\right]$$

$$\Rightarrow \qquad I = \frac{\rho\pi}{2}\left[a^5 + \frac{a^5}{5} - \frac{2a^5}{3}\right] = \frac{\rho\pi}{2}\times\frac{8a^5}{15} = \frac{4\rho\pi a^5}{15}$$

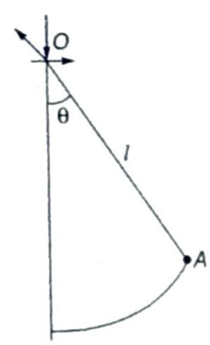
$$\rho = \text{Density} = \frac{M}{\frac{2}{3}\pi a^3}$$

$$\therefore \qquad I = \frac{4}{15}\times\frac{M}{\frac{2}{3}\pi a^3}\pi a^5 = \frac{2}{3}Ma^2$$

2.7 Solve the plane pendulum problem using Hamiltonian approach and show that it is a constant of motion.

(2015 : 15 Marks)

Solution:



Let OA be a pendulum of length l.

The kinetic energy of the pendulum,

$$T = \frac{1}{2}ml^2\theta^2$$

as velocity of A is $(l\dot{\theta})$

Potential energy of pendulum

$$V = mg(l - l\cos\theta)$$

$$V = mgl(1 - \cos\theta)$$

$$H = T + V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) \qquad \text{...(i)}$$

and

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) \qquad \text{...(ii)}$$

Now,

$$P = \frac{\partial L}{\partial\dot{\theta}} = \frac{\partial\left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)\right)}{2\dot{\theta}}$$

$$= \frac{1}{x} ml^2 + 2\dot{\theta} = ml^2\dot{\theta}$$

$$\Rightarrow \quad \dot{\theta} = \frac{P}{ml^2} \quad \text{...(iii)}$$

Using this value of $\dot{\theta}$, (i) becomes

$$H = \frac{1}{2} ml^2 \times \frac{P^2}{m^2 l^4} + mgl(1-\cos\theta)$$

$$\Rightarrow \quad H = \frac{P^2}{2ml^2} + mgl(1-\cos\theta) \quad \text{...(iv)}$$

Now,
$$\theta = \frac{\partial H}{\partial P} = \frac{\partial\left(\frac{P^2}{2ml^2} + mgl(1-\cos\theta)\right)}{\partial P}$$

$$\Rightarrow \quad \dot{\theta} = \frac{2P}{2ml^2} = \frac{P}{ml^2} \quad \text{...(v)}$$

and
$$\dot{P} = \frac{-\partial x}{\partial \theta} = \frac{-\partial}{\partial \theta}\left(\frac{P^2}{2ml^2} + mgl(1-\cos\theta)\right)$$

$$\dot{P} = -mgl \sin\theta \quad \text{...(vi)}$$

as from (v),
$$\dot{\theta} = \frac{P}{ml^2}$$

$$\Rightarrow \quad \ddot{\theta} = \frac{\dot{P}}{ml^2} = \frac{-mgl\sin\theta}{ml^2} \quad \text{(from (vi))}$$

$$\Rightarrow \quad \ddot{\theta} = \frac{-g}{l}\sin\theta$$, which is the equation of motion of plane perpendicular.

Now, from (iv)

$$H = \frac{P^2}{2ml^2} + mgl(1-\cos\theta)$$

$$\therefore \quad \frac{dH}{dt} = \frac{\partial H}{\partial P} \times \frac{\partial P}{\partial t} + \frac{\partial H}{\partial \theta} \times \frac{\partial \theta}{\partial t}$$

$$= \frac{\partial H}{\partial P} \times \dot{P} + \frac{\partial H}{\partial \theta} \times \dot{\theta}$$

$$\therefore \quad \frac{dH}{dt} = \frac{P}{ml^2} \times -mgl\sin\theta + mgl\sin\theta \times \frac{P}{ml^2} \quad \text{(from (v) \& (vi))}$$

$$\Rightarrow \quad \frac{dH}{dt} = 0$$

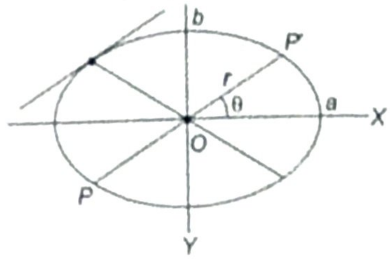
$\therefore$ H is a constant of motion.

2.8 Show that the Moment of Inertia (MoI) of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length 'r' is $\frac{1}{4}M\frac{a^2b^2}{r^2}$. Further prove that the MoI about a tangent is $\frac{5M}{4}p^2$, where p is the perpendicular distance from the centre of the ellipse to the tangent.

(2017 : 10 Marks)

Solution:

Here, let PP' be semi-diameter.



$$M_{PP'} = M_{ox}\cos^2\theta + M_{oy}\sin^2\theta$$

$$M_{ox} = \text{MoI about } OX = \frac{Mb^2}{4}$$

$$M_{oy} = \text{MoI about } OY = \frac{Ma^2}{4}$$

∴ $$M_{PP'} = \frac{Mb^2}{4}\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta$$

$$= \frac{M}{4}[b^2\cos^2\theta + a^2\sin^2\theta] \quad \text{...(i)}$$

Let $$P' = (r\cos\theta, r\sin\theta)$$

∴ $$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1 \quad \text{(as it satisfies equation of ellipse)}$$

⇒ $$b^2\cos^2\theta + a^2\sin^2\theta = \frac{a^2b^2}{r^2} \quad \text{...(ii)}$$

Using this in (i)

$$M_{PP'} = \frac{M}{4}\cdot\frac{a^2b^2}{r^2}$$

Part 2 :

Let the equation of the tangent be

$$y = mx + \sqrt{a^2m^2 + b^2}$$

$$m = \tan\theta \text{ so we get} \quad (\because TT' \parallel PP')$$

$$x\tan\theta - y + \sqrt{a^2\tan^2\theta + b^2} = 0 \quad \text{...(iii)}$$

Distance on (iii) from origin is

$$p = \frac{\sqrt{a^2\tan^2\theta + b^2}}{\sqrt{1+\tan^2\theta}} = \cos\theta\sqrt{a^2\tan^2\theta + b^2}$$

∴ $$p^2 = a^2\sin^2\theta + b^2\cos^2\theta \quad \text{...(iv)}$$

Using (iv) in (i), we get

$$M_{PP'} = \frac{M}{4}(b^2\cos^2\theta + a^2\sin^2\theta) = \frac{Mp^2}{4}$$

By parallel axis theorem :

$$M_{TT'} = M_{PP'} + M(\text{distance between } TT' \text{ and } PP')^2$$

$$= \frac{Mp^2}{4} + Mp^2 = \frac{5Mp^2}{4}$$

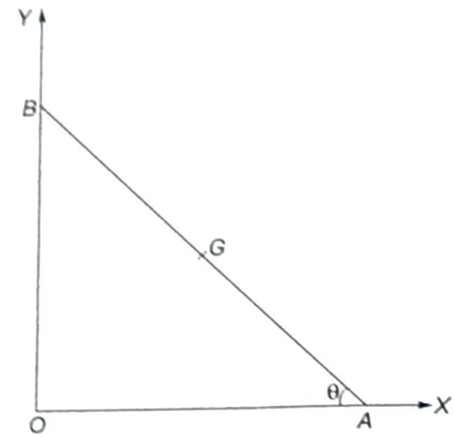
$$= \frac{5Mp^2}{4}. \text{ Hence Proved.}$$

3. Motion of Rigid Bodies in Two Dimensions

3.1 The ends of a heavy rod of length $2a$ are rigidly attached to two light rings which can respectively slide on the thin smooth fixed horizontal and vertical wires O_x and O_y. The rod starts at an angle α to the horizon with an angular velocity $\sqrt{[3g(1-\sin\alpha)/2a]}$ and moves downwards. Show that it will strike the horizontal wire at the end of time $-2\sqrt{a/(3g)}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cot\frac{\pi}{8}\right]$.

(2011 : 30 Marks)

Solution:



Let the rod AB whose centre of gravity is G be inclined at an angle θ to the horizontal at the time t.

Co-ordinates of G are $(a\cos\theta, a\sin\theta)$

$$\therefore \quad (\text{Velocity})^2 \text{ of } G = a^2\dot{\theta}^2$$

So at the time t, kinetic energy

$$= \frac{m}{2}\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right]$$

$$= \frac{1}{2}m\frac{4a^2\dot{\theta}^2}{3} \qquad \text{...(i)}$$

Initial angular velocity $= \sqrt{\dfrac{3g}{2a}(1-\sin\alpha)}$ (given)

$\therefore$ From (i),

$$\text{Initial kinetic energy} = \frac{1}{2}m\frac{4a^2}{3}\cdot\frac{3g}{2a}(1-\sin\alpha)$$

Hence, energy equation gives

$$\frac{1}{2}m\frac{4a^2}{3}\dot{\theta}^2 - \frac{1}{2}m\frac{4a^2}{3}\cdot\frac{3g}{2a}(1-\sin\alpha) = mg(a\sin\alpha - a\sin\theta)$$

$$\Rightarrow \quad \frac{2a^2}{3}\dot{\theta}^2 - ga(1-\sin\alpha) = ga(\sin\alpha - \sin\theta)$$

$$\Rightarrow \quad \dot{\theta}^2 = \frac{3g}{2a}(1-\sin\theta)$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -\sqrt{\frac{3g}{2a}(1-\sin\theta)}$$

(negative sign is taken because the motion is towards θ decreasing).

Integrating it between the limits of θ from α to 0, the required time is

$$t = -\sqrt{\frac{2a}{3g}}\int_{\alpha}^{0}\frac{1}{\sqrt{1-\sin\theta}}\,d\theta$$

$$= \sqrt{\frac{2a}{3g}}\int_{0}^{\alpha}\frac{d\theta}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)} = \sqrt{\frac{a}{3g}}\int_{0}^{\alpha}\frac{d\theta}{\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}$$

$$= \sqrt{\frac{a}{3g}}\int_{0}^{\alpha}\operatorname{cosec}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)d\theta$$

$$= -2\sqrt{\frac{a}{3g}}\left[\log\tan\left(\frac{\pi}{8}-\frac{\theta}{4}\right)\right]_0^{\alpha}$$

$$= +2\sqrt{\frac{a}{3g}}\cdot\log\left(\frac{\tan\frac{\pi}{8}}{\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)}\right)$$

$$= 2\sqrt{\frac{a}{3g}}\cdot\log\left[\tan\frac{\pi}{8}\cot\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\right]$$

$$= -2\sqrt{\frac{a}{3g}}\log\left[\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\cdot\cot\frac{\pi}{8}\right]$$

4. Equation of Continuity

4.1 An infinite mass of fluid is acted on by a force $\frac{\mu}{r^{3/2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = C$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} C^{5/4}$.

(2009 : 30 Marks)

Solution:

Here the motion of the fluid will take place in such a manner that each element of the fluid moves towards the centre. Hence, the free surface will be spherical. Thus, the velocity V' will be radial and hence V' will be a function of r' (radial distance from the centre of the sphere, which is taken as origin) and time t. Also, let v be the velocity at a distance R.

Using equation of continuity

$$r'^2V' = R^2V = F(t) \qquad ...(1)$$

$$F'(t) = \frac{r'^2\partial v'}{\partial t}$$

Using Fuler equation,

$$\frac{\partial v'}{\partial t}+\frac{v'\partial v'}{\partial r'} = F-\frac{1}{\rho}\frac{\partial p}{\partial r} \text{ where } F = \frac{-\mu}{r^{3/2}}$$

$$\frac{F'(t)}{r'^2}+\frac{1}{2}\frac{\partial v'^2}{\partial r} = \frac{-\mu}{r^{3/2}}-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{(from (1))}$$

Integrating with respect to r'

$$\frac{-F'(t)}{r'}+\frac{1}{2}v'^2 = \frac{-\mu r^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}-\frac{1\cdot P}{\rho}+C$$

$$\frac{-F'(t)}{r'}+\frac{v'^2}{2} = \frac{2\mu}{r'}-\frac{P}{\rho}+C \qquad ...(2)$$

Infinite mass,

at $r' = \infty$, $\quad P = 0,\ V' = 0$ (Put these values in (2))

$$0+0 = 0+0+C \Rightarrow C = 0$$

Now at $r' = R,\ v' = V,\ P = 0$ (due to cavity)

$$\frac{-F'(t)}{R} + \frac{V^2}{2} = \frac{2\mu}{\sqrt{R}} \quad \ldots(3)$$

Now

$$F(t) = R^2 V = R^2 \frac{dR}{dt} \quad \text{(from (1))}$$

$$F'(t) = 2R\left(\frac{dR}{dt}\right)^2 + R^2 \frac{d^2R}{dt^2}$$

$$= 2RV^2 + R^2 V \frac{dV}{dR} \quad \left[\because \frac{d^2R}{dt^2} = \frac{VdV}{dR}\right]$$

$$\frac{F'(t)}{R} = 2V^2 + RV\frac{dV}{dR}$$

Put this value in (3)

$$-\left(2V^2 + RV\frac{dV}{dR}\right) + \frac{V^2}{2} = \frac{2\mu}{\sqrt{R}}$$

$$\frac{-3V^2}{2} - RV\frac{dV}{dR} = \frac{2\mu}{\sqrt{R}}$$

$$3V^2 dR + 2RVdV = -4\mu\frac{dR}{\sqrt{R}}$$

Multiply by R^2

$$3V^2R^2dR + 2R^3VdV = \frac{-4\mu R^2 dR}{R^{1/2}}$$

$$d(R^3V^2) = -4\mu R^{3/2}dR$$

Integrating, we get

$$R^3V^2 = \frac{-4\mu R^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C'$$

$$R^3V^2 = \frac{-8\mu R^{\frac{5}{2}}}{5} + C' \quad \ldots(4)$$

Now, initially at $R = C,\ V = 0$

Put this value in (4)

$$C^3(O)^2 = \frac{-8\mu C^{\frac{5}{2}}}{5} + C' \Rightarrow C' = \frac{8\mu C^{\frac{5}{2}}}{5}$$

So (4) becomes

$$R^3V^2 = \frac{8\mu}{5}\left(C^{5/2} - R^{5/2}\right)$$

$$V^2 = \frac{8\mu}{5R^3}\left(C^{5/2} - R^{5/2}\right)$$

$$V = \sqrt{\frac{8\mu}{5}}\left(\frac{C^{5/2} - R^{5/2}}{R^{3/2}}\right)^{1/2} = \frac{-dR}{dt}$$

(Negative sign is taken, because R is decreasing with t)

$$\int_C^0 \frac{-R^{3/2}dR}{\left(C^{5/2}-R^{1/2}\right)^{1/2}} = \sqrt{\frac{8\mu}{5}}\int_0^T \qquad ...(5)$$

Put $\qquad C^{5/2} - R^{1/2} = x$

So, $\qquad dx = -\frac{5}{2}R^{3/2}dR$

$\Rightarrow \qquad \frac{2dx}{5} = -R^{3/2}dR$

So, $\qquad \frac{2}{5}\int \frac{dx}{x^{1/2}} = \frac{4}{5}\sqrt{x} + C$

So (5) becomes

$$\left|\frac{4}{5}(C^{5/2} - R^{5/2})^{1/2}\right|_C^0 = \frac{\sqrt{8\mu}}{\sqrt{3}}T$$

$$\frac{4}{5\sqrt{5}}C^{5/4} = \frac{\sqrt{8\mu}}{\sqrt{5}}T$$

$$T = \frac{4C^{5/4}}{\sqrt{2}\sqrt{\mu}\times\sqrt{5}} = \sqrt{\frac{2}{5\mu}}C^{5/4}$$

4.2 **A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the dusturbed stream.**

(2012 : 12 Marks)

Solution:

We may take the polar axis $0z$ to be in the direction of the given velocity.

Let us take the polar co-ordinates (r, θ, ϕ) with origin at the center of the fixed sphere.

The velocity of the fluid is given by

$$\vec{q} = -\text{grad}\,\psi, \text{ where}$$

(i) $\qquad \nabla^2\psi = 0$

(ii) $\qquad \frac{\partial\psi}{\partial r} = 0 \text{ or } r = a$

(iii) $\qquad \psi \sim -Vr\cos\theta = -Vr\,P_1\cos\theta \text{ as } r \to \infty$

The axially symmetrical function

$$\psi = \sum_{n=0}\left(A_x r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

satisfies (i).

Again, condition (ii) is satisfied if we take

$$nA_n a^{n-1} - (n+1)\frac{B_n}{a^{n+2}} = 0$$

i.e., if $\qquad B_n = na^{2n+1}\frac{A_n}{n+1}$

As $r \to \infty$, this velocity potential has the asymptotic form

$$\psi \sim \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$$

So to satisfy (iii), we take $A_1 = -V$ and all the other A's zero.

Hence, the required velocity potential is

$$\psi = -V\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

The components of velocity are

$$q_r = -\frac{d\psi}{dr} = V\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$q_\theta = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} = V\left(1 - \frac{a^3}{2r^3}\right)\sin\theta$$

4.3 Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$, determine the streamlines.

(2014 : 20 Marks)

Solution:

Velocity potential,

$$\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

To determine stream lines

$$\frac{-\partial\phi}{\partial x} = u = -\frac{\partial\varphi}{\partial y}; -\frac{\partial\phi}{\partial y} = v = \frac{\partial\Psi}{\partial x}$$

Hence,
$$\frac{\partial\phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \frac{\partial\phi}{\partial y} = -\frac{\partial\varphi}{\partial x}$$

Now,
$$\frac{\partial\varphi}{\partial y} = \frac{x+0}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

Integrating w.r.t. y

$$y = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) + f(x) \quad \text{...(i)}$$

where $f(x)$ is constant of integration. To ddetermine $f(x)$

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = \frac{-y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \quad \text{...(ii)}$$

$\Rightarrow$ by (i)
$$\frac{\partial\psi}{\partial x} = \frac{y}{(x+a)^2+y^2} + \frac{y}{(x+a)^2+y^2} + F'(x) \quad \text{...(iii)}$$

Equating (ii) and (iii)

$\Rightarrow$ $f'(x) = 0$

Integrating this, $f(x) =$ absolute constant hence neglected.

Since, it has no effect on the fluid motion.

Now (i), becomes,

$$\psi = \tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right)$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2} \quad \text{...(iv)}$$

Stream lines are given by, $\psi =$ constant

∴ i.e., $$\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right] = \text{constant}$$

or, $$\frac{y}{x^2-a^2+y^2} = \text{constant}$$

if we take, constant = 0; then

We get, $y = 0$, i.e. x-axis.

If we take, constant = ∞, then

We get circles, $x^2 - a^2 + y^2 = 0$

i.e., $x^2 + y^2 = a^2$

Thus, stream lines include x-axis and circle.

4.4 In an axisymmetric motion, show that the stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisifed by the stream function if the flow is irrotational.

(2015 : 20 Marks)

Solution:

Let the motion be two-dimensional with u and v as components of velocity in x and y direction respectively.

Now, by equation of continuity,

$$\frac{\partial y}{\partial x}+\frac{\partial v}{\partial y} = 0$$

⇒ $$\frac{\partial v}{\partial y} = \frac{-\partial y}{\partial x} \quad \text{...(i)}$$

Now, equation of stream lines is given by

$$\frac{dx}{u} = \frac{dy}{v}$$

⇒ $$vdx = udy$$

⇒ $$vdx - udy = 0 \quad \text{...(ii)}$$

Now, (i) shows that (ii) should be exact differential, say $d\psi$.

Thus, $$vdx - udy = d\psi = \left(\frac{\partial\psi}{\partial x}\right)dx+\left(\frac{\partial\psi}{\partial y}\right)dy \quad \text{...(iii)}$$

where ψ is called stream-function and is constant along a streamline.

∴ Stream-function exists due to (i), i.e., equation of continuity.

In eqn. (iii), comparing LHS and RHS, we get

$$v = \frac{\partial\psi}{\partial x}$$

and $$u = -\frac{\partial\psi}{\partial y} \quad \text{...(iv)}$$

which are components of velocity.

If the flow is irrotational, then

$$\nabla\times\vec{q} = 0$$

where $$\vec{q} = u\hat{i}+v\hat{j}$$

⇒ $$\nabla\times(u\hat{i}+v\hat{j}) = 0$$

$$\Rightarrow \quad \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\hat{k} = 0$$

$$\Rightarrow \quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial \frac{\partial \psi}{\partial x}}{\partial x}-\frac{\partial\left(-\frac{\partial \psi}{\partial y}\right)}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{(from (iv))}$$

4.5 A simple source of strength m is fixed at the origin O in a uniform stream of incompressible fluid moving with velocity $U\hat{i}$. Show that the velocity potential ϕ at any point P of the stream is $\frac{m}{r} - U_r \cos\theta$, where $OP = r$ and θ is the angle which $\overrightarrow{OP}$ makes with the direction $\hat{i}$. Find the differential equation of the streamlines and show that they lie on the surfaces $Ur^2 \sin^2\theta - 2m\cos\theta =$ Constant.

(2016 : 15 Marks)

Solution:

Given, strength of source $= m$

$\therefore$ at any point z, complex potential, $w = -m \log z$

$$\therefore \quad \frac{dw}{dz} = \frac{-m}{z} = \frac{-m}{re^{i\theta}} = \frac{-m}{r}e^{-i\theta}$$

$$\Rightarrow \quad \frac{dw}{dz} = \frac{-m}{r}(\cos\theta - i\sin\theta) = \frac{-m\cos\theta}{r}+\frac{im\sin\theta}{r}$$

$$\Rightarrow \quad \frac{dw}{dz} = \frac{-mx}{r^2}+i\frac{my}{r^2} = -u_m + iv_m$$

where u_m and v_m are velocity components due to source at m.

$$\therefore \quad u_m = \frac{mx}{r^2},\ v_m = \frac{my}{r^2}$$

Also, fluid is moving with velocity $U\hat{i}$.

$\therefore$ Total velocity,
$$\vec{q} = \left(U+\frac{mx}{r^2}\right)\hat{i}+\left(\frac{my}{r^2}\right)\hat{j} = u\hat{i}+\hat{j}v$$

Let ϕ be the velocity potential.

$$\therefore \quad u = \frac{-\partial\phi}{\partial x}$$

$$\Rightarrow \quad \frac{-\partial\phi}{\partial x} = U+\frac{mx}{r^2}$$

$$\Rightarrow \quad d\phi = \left(-U-\frac{mx}{r^2}\right)dx$$

Integrating both sides, we get

$$f = -Ux - m\log r + f(y) \qquad \text{...(i)}$$

where, $f(y)$ is arbitrary function of y.

Also,
$$\frac{-\partial\phi}{\partial y} = v = \frac{my}{r^2}$$

$$\Rightarrow \quad d\phi = \frac{-my}{r^2}dy$$

$\Rightarrow$ $$\phi = -m \log r + g(x) \quad ...(ii)$$

where $g(x)$ is an arbitrary function of x.

From (i) and (ii)

$$\phi = -Ux - m \log r$$

$\Rightarrow$ $$\phi = -Ur \cos\theta - m \log r$$

Now, equation of streamlines is given by

$$\frac{dx}{u} = \frac{dy}{v}$$

$\Rightarrow$ $$\frac{dx}{U + \frac{mx}{r^2}} = \frac{dy}{\frac{my}{r^2}}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{\frac{my}{r^2}}{\left(U + \frac{mx}{r^2}\right)},$$ which is the differential equation of streamlines.

4.6 **Let T be a closed curve in xy-plane and let S denote the region bounded by the curve T. Let**

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = f(x, y) \ \forall \ (x, y) \in S$$

If f is prescribed at each point (x, y) of S and W is prescribed on the boundary T of S, then prove that any solution $W = W(x, y)$, satisfying these conditions is unique.

(2017 : 10 Marks)

Solution:

Let $W_1(x, y)$ and $W_2(x, y)$ be two solutions satisfying,

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = f(x, y) \ \forall \ (x, y) \in S \quad ...(i)$$

S together with the prescribed boundary conditions on T.

Let $$W(x, y) = W_1(x, y) - W_2(x, y) \quad ...(ii)$$

Then, $$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = f - f = 0 \quad ...(iii)$$

Also, on T, $W = 0$, since $W_1 = W_2$ on T. Now consider integral.

$$I = \iint_S \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] dS$$

$$= \iint_S \left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + W\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)\right] dx\,dy$$

$$= \iint_S \left[\frac{\partial}{\partial x}\left(W\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial y}\left(W\frac{\partial W}{\partial y}\right)\right] dx\,dy$$

$$= \oint_\Gamma \left(W\frac{\partial W}{\partial x}dy - W\frac{\partial W}{\partial y}dx\right)$$

$= 0$. Since $W = 0$ on T.

Now, $$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 \geq 0, \text{ but } I = 0$$

Hence, $$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 0 \text{ on } \delta$$

which will be true only if

$$\frac{\partial W}{\partial x} = 0 = \frac{\partial W}{\partial y} \text{ at each point of } \delta.$$

This shows that W = constant in S. Since W = 0 on T, we infer from the continuity of W that W = 0 through out δ.

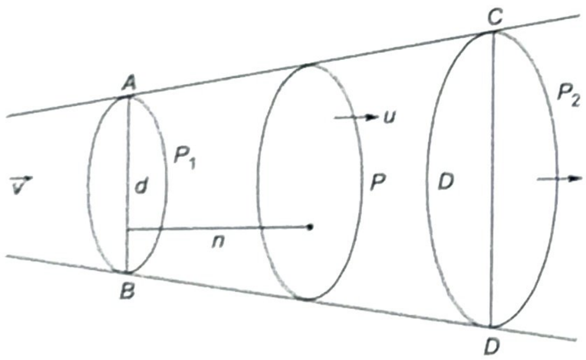
This gives $W_1 = W_2$ which proves our result.

4.7 A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are *D* and *d*. If *V* and *v* be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vertex of the cone, then prove that $\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2-V^2)/2K}$ where *K* is the pressure divided by the density and is constant.

(2017 : 15 Marks)

Solution:

Let P be the density, P be the pressure and u, the velocity at distance 'r' from AB.



Then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\cdot\frac{\partial P}{\partial r} \text{ (since motion is steady)}$$

or $$u\frac{\partial u}{\partial r} = -\frac{K}{\rho}\cdot\frac{\partial P}{\partial r} \qquad \left[\because K = \frac{P}{\rho}\right]$$

By integrating w.r.t. 'r', we get

$$\frac{u^2}{2} = -K\log P + C \qquad \text{...(i)}$$

Boundary conditions are :

$$\rho = \rho_1 \text{ when } u = v;$$

$$\therefore \quad \frac{v^2}{2} = -K\log \rho_1 + C$$

$$\rho = \rho_2 \text{ when } u = V;$$

$$\frac{V^2}{2} = -K\log \rho_2 + C$$

$$\Rightarrow \quad \frac{V^2 - v^2}{2} = K\log\frac{P_1}{P_2}$$

$$\Rightarrow \qquad \frac{\rho_1}{\rho_2} = e^{(V^2-v^2)/2K} \qquad ...(ii)$$

By the equation of continuity,

$$\text{flux at } A = \text{flux at } B$$

$$\pi\left(\frac{d}{2}\right)^2 \cdot V \cdot \rho_1 = \pi\left(\frac{D}{2}\right)^2 \cdot V \cdot \rho_2$$

or

$$\frac{\rho_1}{\rho_2} = \frac{V}{v} \cdot \frac{D^2}{d^2} \qquad ...(iii)$$

Using it in (ii)

$$\frac{V}{v} \cdot \frac{D^2}{d^2} = e^{(V^2-v^2)/2K}$$

or

$$\frac{v}{V} = \frac{D^2}{d^2} \cdot e^{-(V^2-v^2)/2K}$$

i.e.,

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2-V^2)/2K}$$

4.8 **If the velocity of an incompressible fluid at the point (x, y, z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$, $r^2 = x^2 + y^2 + z^2$, then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the streamlines.**

(2017 : 15 Marks)

Solution:

Given:

$$u = \frac{3xz}{r^5}, \; v = \frac{3yz}{r^5}$$

$$w = \frac{3z^2-r^2}{r^5} = \frac{3z^2}{r^5} - \frac{1}{r^3} \qquad ...(i)$$

Since,

$$r^2 = x^2 + y^2 + z^2 \qquad ...(ii)$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \; \frac{\partial r}{\partial y} = \frac{y}{r}, \; \frac{\partial r}{\partial z} = \frac{z}{r} \qquad ...(iii)$$

To prove that liquid motion is possible

$$\frac{\partial u}{\partial x} = 3z\left[\frac{1}{r^5} + \frac{(-5x)}{r^6} \cdot \frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2 z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6} \cdot \frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z}$$

$$= \frac{9z}{r^5} - \frac{15z^2}{r^7}$$

$$\therefore \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2)$$

$$= 0 \qquad \text{(using (ii))}$$

Since, equation of continuity is satisfied $\Rightarrow$ motion possible.

Let ϕ be the required velocity potential

$$\frac{\partial\phi}{\partial x} = -u = \frac{-3xz}{r^5}$$

Integrating w.r.t. x,

$$\phi = -\frac{3z}{2}\int (2x)(x^2+y^2+z^2)^{-5/2}dx$$

$$= \left(-\frac{3z}{2}\right)\left(-\frac{2}{3}\right)(x^2+y^2+z^2)^{-3/2}$$

$\therefore$

$$\phi = \frac{z}{(x^2+y^2+z^2)^{3/2}} = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$$

(on neglecting the constant of integration)

Streamlines : are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\underset{(I)}{\frac{dx}{3xz}} = \underset{II}{\frac{dy}{3yz}} = \underset{III}{\frac{dz}{3z^2-r^2}} - \underset{IV}{\frac{xdx+ydy+zdz}{2r^2z}}$$

$$\frac{dx}{3xz} = \frac{dy}{3yz}$$

$\Rightarrow$ $\log x = \log y + \log a \Rightarrow x = ay$ ...(v)

From (i) and (iv),

$$\frac{dx}{3x} = \frac{xdx+ydy+zdz}{2r^2}$$

or

$$\frac{4dx}{x} = \frac{3(2xdx+2ydy+2zdz)}{(x^2+y^2+z^2)}$$

Integrating, $4\log x = 3\log(x^2+y^2+z^2) + \log b$

$x^4 = b(x^2+y^2+z^2)^3$ ...(vi)

The required streamlines are the curves of intersections of (v) and (vi)

4.9 For an incompressible fluid flow, two components of velocity (u, v, w) are given by $u = x^2 + 2y^2 + 3z^2$, $v = x^2y - y^2z + zx$. Determine the third component w so that they satisfy equation of continuity. Also find z-component of acceleration.

(2018 : 10 Marks)

Solution:

Since the liquid satisfies equation of continuity

$\therefore$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

$\Rightarrow$

$$2x + x^2 - 2yz + \frac{\partial w}{\partial z} = 0$$

$\Rightarrow$

$$\frac{\partial w}{\partial z} = 2yz - x^2 - 2x$$

$\Rightarrow$

$$w = yz^2 - x^2z - 2xz + f(x, y)$$

where $f(x, y)$ is an arbitrary function.

Now, z-component of acceleration is

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$= (x^2+2y^2+3z^2)\left(-2x^2-2z+\frac{\partial f}{\partial x}\right)+(x^2y-y^2z+xz)\left(z^2+\frac{\partial f}{\partial y}\right)$$
$$+ (-x^2z - 2xz + yz^2 + f(x, y))(x^2 - 2x + 2yz)$$

4.10 For a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$.

Determine the velocity components along the directions x and y. Also determine the stream function y and check whether ϕ represents a possible case of flow or not.

(2018 : 15 Marks)

Solution:

Given : $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$, $q = u\hat{i} + v\hat{j} = -\nabla\phi$

Velocity component along x, $u = \frac{-\partial\phi}{\partial x}$

$\Rightarrow$ $$u = \frac{-\partial\left(x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)\right)}{\partial x} = -2xy + y^2 - x^2$$

Velocity component along y, $v = \frac{-\partial\phi}{\partial y}$

$\Rightarrow$ $$\frac{-\partial\left(x^2y - xy^2 + \frac{1}{3}(x^3 + y^3)\right)}{\partial y} = -x^2 + 2xy + y^2$$

Given, ψ is stream function.

So, $$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

$\Rightarrow$ $$\frac{\partial\psi}{\partial y} = x^2 - y^2 + 2xy$$

$\Rightarrow$ $$\psi = x^2y - \frac{y^3}{3} + xy^2 + f_1(x) \quad \text{...(i)}$$

Also, $$\frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x}$$

$\Rightarrow$ $$\frac{-\partial\psi}{\partial x} = x^2 - 2xy - y^2$$

$\Rightarrow$ $$\psi = \frac{-x^3}{3} + xy^2 + x^2y + f_2(y) \quad \text{...(ii)}$$

Comparing (i) and (ii), we get

$$y = \frac{-x^3}{3} - \frac{y^3}{3} + x^2y + xy^2$$

Again, if flow is possible, then it satisfies the equation of continuity, i.e.,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$\Rightarrow$ $$2x + 2y - 2y - 2x = 0$$

$\Rightarrow$ $$0 = 0$$

$\therefore$ Flow is possible.

4.11 A sphere of radius R, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, so that the pressure at the surface of the sphere at time t is $\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$.

(2019 : 15 Marks)

Solution:

Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence, the free surface would be spherical. Thus, the fluid velocity v^1 will be radial and hence v will be function of r^1 the radial distance from the centre of the sphere which is taken as origin and time t only. Let p be pressure at a distance r^1. Let p be the pressure on the surface of the sphere of radius R and V be the velocity there, then the equation of continuity is

$$r'V^1 = R^2V = F(t) \quad \text{...(1)}$$

from (1)

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{r'^2} \quad \text{...(2)}$$

Again equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{p}\frac{\partial p}{\partial r'}$$

$$= \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r^2}\left(\frac{1}{2}v'^2\right) = -p\frac{\partial p}{\partial r'} \quad \text{using (2)}$$

Integrating w.r.t. r^2, (3) reduces to

$$\frac{-F'(t)}{\partial'} + \frac{1}{2} = -\frac{p}{p} + C, \text{ C being an arbitrary constant.}$$

When $r' = \infty$, then $V' = 0$ and $p = \pi$ so that $C = \frac{\pi}{p}$, then we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\pi - p}{p}$$

$$p = \pi + \frac{1}{2}p\left[\frac{2F'(t)}{r'} - v'^2\right] \quad \text{...(4)}$$

But $p = P$ and $v' = V$ when $r' = R$. Hence (4) gives

$$P = \pi + \frac{1}{2}p\left[\frac{2}{R}\left(\frac{F'}{t}\right)_{r'=R} - V^2\right] \quad \text{...(5)}$$

Also, $V = \frac{dR}{dt}$. Hence, using (1), we have

$$\left(\frac{F'}{t}\right)_{r'=R} = \frac{d}{dt}(R^2V) = \frac{d}{dt}\left(R^2\frac{dR}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}, \frac{dR^2}{dt}\right)$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + \frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt}$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V and

$$\{F'(t)\}_{r'} = R \quad \text{...(5)}$$

reduces to

$$P = \pi + \frac{1}{2}P\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

$$P = \pi + \frac{1}{2}P\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

5. Euler's Equation of Motion for Inviscid Flow

5.1 Show that $\phi = xf(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.

(2012 : 30 Marks)

Solution:

Given, the velocity potential is

$$\phi = xf(r) \quad \text{...(i)}$$

$$\therefore \quad \vec{q} = -\nabla\phi = -\nabla[xf(r)]$$

$$= -[f(r)\nabla x + x\nabla f(r)] \quad \text{...(ii)}$$

Now $$r^2 = x^2 + y^2 + z^2 \Rightarrow 2r\frac{dr}{dx} = 2x$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{...(iii)}$$

But $$\nabla x = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]x = \vec{i}$$

$$\nabla f(r) = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]f(r)$$

$$= \vec{i}f'(r)\left(\frac{\partial r}{\partial x}\right) + \vec{j}f'(r)\left(\frac{\partial r}{\partial y}\right) + \vec{k}f'(r)\left(\frac{\partial r}{\partial z}\right)$$

$$= \vec{i}f'(r)\left(\frac{x}{r}\right) + \vec{j}f'(r)\left(\frac{y}{r}\right) + \vec{k}f'(r)\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}f'(r)(\vec{i}x + \vec{j}y + \vec{k}z) = \frac{1}{r}f'(r)\vec{r}$$

$$\therefore \quad \vec{q} = -f(r)\vec{i} - \frac{x}{r}f'(r)\vec{r} \text{ (from (ii))} \quad \text{...(iv)}$$

For a possible motion of an incompressible fluid, we have

$$\nabla\cdot\vec{q} = 0 \Rightarrow \nabla(-\nabla\phi) = 0 \Rightarrow \nabla^2\phi = 0$$

$$\Rightarrow \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(xf(r)) = 0 \quad \text{(from (i))} \left[\text{Note} : \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]$$

Now, $$\frac{\partial^2}{\partial x^2}[xf(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(xf(r)\right]$$

$$= \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$$

$$= 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$$

Similarly, $$\frac{\partial^2}{\partial y^2}[xf(r)] = x\frac{\partial^2 f}{\partial y^2}$$

and $$\frac{\partial^2}{\partial z^2}[xf(r)] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(xf(r)) = 0$$

$\Rightarrow$ $$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 \qquad \text{...(v)}$$

Using (iii), $$\frac{\partial f}{\partial x} = \frac{df}{dr}\cdot\frac{\partial r}{\partial x} = f'\cdot\frac{x}{r}$$

$$= \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$$

$$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x}$$

$$= \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}$$

$\Rightarrow$ $$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f'$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f'$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f'$$

$\Rightarrow$ $$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= 3\frac{f'}{r} + \frac{x^2 + y^2 + z^2}{r^2}\cdot f'' - \frac{x^2 + y^2 + z^2}{r^3}f'$$

$$= 3\frac{f'}{r} + f'' - \frac{f'}{r} \qquad [\because r^2 = x^2 + y^2 + z^2]$$

$$= 2\frac{f'}{r} + f''$$

$\therefore$ from (v), we have

$$\frac{2f'x}{r} + x\left(\frac{2f'}{r} + f''\right) = 0$$

$\Rightarrow$ $$f'' + 4\frac{f'}{r} = 0$$

$$\Rightarrow \qquad \frac{f''}{f'}+\frac{4}{r} = 0$$

$$\Rightarrow \qquad \log f' + 4\log r = \log C_1 \qquad \text{(Integration)}$$

$$\Rightarrow \qquad f' = C_1 r^{-4}$$

$$\Rightarrow \qquad f = -\frac{C_1}{3}r^{-3}+C_2$$

$\therefore$ from (iv),

$$\vec{q} = \left[\frac{C_1}{3r^3}-C_2\right]\vec{i}-\frac{C_1 x}{r^5}\vec{r}$$

But, it is given that $\vec{q}\to 0$ as $r\to\infty$,

$$\therefore \qquad C_2 = 0$$

$$\therefore \qquad \vec{q} = \frac{C_1}{3r^3}\left(\vec{i}-\frac{3x\vec{r}}{r^2}\right)$$

$$q^2 = \vec{q}\cdot\vec{q} = \frac{C_1^2}{9r^6}\left(\vec{i}-\frac{3x\vec{r}}{r^2}\right)\left(\vec{i}-\frac{3x\vec{r}}{r^2}\right)$$

$$= \frac{C_1^2}{9r^6}\left(\vec{i}\cdot\vec{i}-\frac{6x}{r^2}\vec{r}\cdot\vec{i}+\frac{9x^2}{r^4}\cdot\vec{r}\cdot\vec{r}\right)$$

$$= \frac{C_1^2}{9r^6}\left(1-\frac{6x^2}{r^2}+\frac{9x^2r^2}{r^4}\right) \qquad \left(\because \vec{r}\cdot\vec{r}=r^2 \text{ and } \vec{r}\cdot\vec{i}=x\right)$$

$$= \frac{C_1^2}{9r^6}\left(1+\frac{3x^2}{r^2}\right)$$

Hence, the required surfaces of constant speed are :

$$q^2 = \text{Constant}$$

or

$$\frac{C_1^2}{9r^6}\left(1+\frac{3x^2}{r^2}\right) = \text{Constant}$$

or

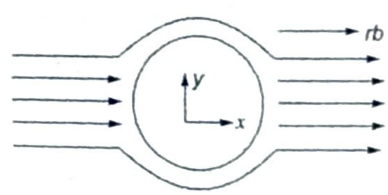
$$\frac{r^2+3x^2}{r^8} = \text{Constant}$$

5.2 Consider a uniform flow v_o in the positive x direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also, the stagnation points.

(2015 : 10 Marks)

Solution:

Figure shows flow past the fixed cylinder. Let the fluid be inviscid and incompressible. So, it is equivalent to superposition of uniform flow and a doublet.



As $\qquad$ Velocity $= v_o\hat{i}$

$\therefore$ Velocity potential, $\qquad \phi = v_o x+\dfrac{x\cos\theta}{r}$

As ϕ and ψ satisfy Cauchy-Riemann equation

$$\therefore \qquad \phi_x = \psi_y \qquad (\psi \text{ is stream function})$$

$\therefore \qquad \psi = V_o y - \dfrac{x \sin\theta}{r}$

$\therefore \qquad w = \phi + i\psi = U_o x - \dfrac{x\cos\theta}{r} + i\left(U_{oy} - \dfrac{x\sin\theta}{r}\right)$

$\Rightarrow \qquad w = V_0(x+iy) + \dfrac{x}{r}(\cos\theta - i\sin\theta)$

$\Rightarrow \qquad w = V_o z + \dfrac{x}{r^2}(\bar{z}) = V_o z + \dfrac{(z+\bar{z})}{2z\cdot\bar{z}}\cdot\bar{z}$

$\Rightarrow \qquad w = V_o z + \dfrac{1}{2}\left(1+\dfrac{\bar{z}}{z}\right)$

$\Rightarrow \qquad w = V_o z + \dfrac{1}{2}\left(1+\dfrac{1}{z^2}\right)$

$\therefore$ At stagnation points, $\dfrac{dw}{dz} = 0$

$\Rightarrow \qquad V_o + 0 - \dfrac{2}{2z^3} = 0$

$\Rightarrow \qquad \dfrac{1}{z^3} = V_0$

$\Rightarrow \qquad z = \left(\dfrac{1}{V_0}\right)^{1/3}$

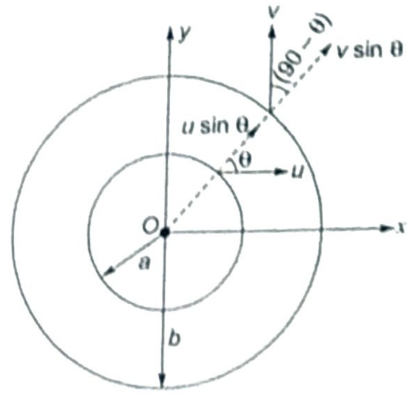
5.3 The space between two concentric spherical shells of radii a, b ($a < b$) is filled with a liquid of density ρ. If the shells are set in motion, the inner one with velocity U in the x-direction and the outer one with velocity V in the y-direction, then show that the initial motion of the liquid is given by velocity potential

$$\phi = \left\{\frac{a^3U\left(1+\frac{1}{2}b^3r^{-3}\right)x(-b^3V)\left(1+\frac{1}{2}a^3r^{-3}\right)y}{(b^3-a^3)}\right\}$$

where $r^2 = x^2 + y^2 + z^2$, the coordinates being rectangular. Evaluate the velocity at any point of the liquid.

(2016 : 20 Marks)

Solution:



As shown in figure, let O be the common centre and ϕ be velocity potential of initial motion. U and V be initial velocities of inner and outer shells in x and y direction respectively. The following two conditions should be satisfied.

(i) ϕ satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0$$

as there is symmetry.

(ii) ϕ satisfies following boundary conditions

$$\frac{-\partial\phi}{\partial r} = U\cos\theta \text{ (at } r = a) \qquad \text{...(i)}$$

$$\frac{-\partial\phi}{\partial r} = V\sin\theta \text{ (at } r = b) \qquad \text{...(ii)}$$

The above considerations suggest that ϕ must involve terms containing $\sin\theta$ and $\cos\theta$. So we assume that

$$\phi = \left(Ar+\frac{B}{r^2}\right)\cos\theta+\left(r+\frac{D}{r^2}\right)\sin\theta \qquad \text{...(iii)}$$

$$\therefore \quad \frac{-\partial\phi}{\partial r} = \left(-A+\frac{2B}{r^3}\right)\cos\theta+\left(-C+\frac{2D}{r^3}\right)\sin\theta \qquad \text{...(iv)}$$

Using boundary conditions for (i) and (ii) in (iv), we get

$$\left(-A+\frac{2B}{a^3}\right)\cos\theta+\left(-C+\frac{2D}{a^3}\right)\sin\theta = U\cos\theta \qquad \text{...(v)}$$

$$\left(-A+\frac{2B}{b^3}\right)\cos\theta+\left(-C+\frac{2D}{b^3}\right)\sin\theta = V\sin\theta \qquad \text{...(vi)}$$

Comparing coefficients of $\cos\theta$ and $\sin\theta$ in (v) and (vi), we get

$$\frac{-A+2B}{a^3} = U, \ \frac{-C+2D}{a^3} = 0 \qquad \text{...(vii)}$$

$$\frac{-A+2B}{b^3} = 0, \ \frac{-C+2D}{b^3} = V \qquad \text{...(viii)}$$

Solving (vii) and (viii), we get

$$A = \frac{Ua^3}{b^3-a^3}, \ B = \frac{Ua^3b^3}{2(b^3-a^3)}, \ C = \frac{-Ub^3}{b^3-a^3}, \ D = \frac{-Ua^3b^3}{2(b^3-a^3)}$$

Putting these values in (iii), we get

$$\phi = \left(\frac{Ua^3r}{b^3-a^3}+\frac{Ua^3b^3}{2(b^3-a^3)r^2}\right)\cos\theta+\left(\frac{-Ub^3}{b^3-a^3}\cdot r-\frac{Ua^3b^3}{2(b^3-a^3)r^2}\right)\sin\theta$$

$$\Rightarrow \quad \phi = \frac{\left\{Ua^3\left(1+\frac{b^3}{2r^3}\right)r\cos\theta-Vb^2\left(1+\frac{a^3}{2r^3}\right)r\sin\theta\right\}}{(b^3-a^3)}$$

$$\Rightarrow \quad \phi = \frac{\left\{a^3U\left(1+\frac{b^3r^{-3}}{2}\right)x-b^3V\left(1+\frac{a^3r^{-3}}{2}\right)y\right\}}{b^3-a^3}$$

Now, let velocity at any point $= \vec{q} = u\hat{i}+v\hat{j}$

Now,
$$u = \frac{-\partial\phi}{\partial x} = \frac{-a^3U\left(1+\frac{b^3r^{-3}}{2}\right)}{b^3-a^3}-\frac{a^3Ux\left(\frac{-3}{2}b^3r^{-4}\frac{x}{r}\right)}{(b^3-a^3)}$$
$$+\frac{b^3Vy\left(-\frac{3}{2}a^3r^{-4}\cdot\frac{x}{r}\right)}{b^3-a^3}$$

$$\Rightarrow \quad u = \frac{-a^3U\left(1+\frac{b^3r^{-3}}{2}\right)}{b^3-a^3}+\frac{\frac{3}{2}a^3b^3x^2Ur^{-5}}{(b^3-a^3)}-\frac{\frac{3}{2}a^3b^3xyVr^{-5}}{b^3-a^3}$$

$$\Rightarrow \qquad u = \frac{-a^3U(1+b^3r^{-3})}{(b^3-a^3)} + \frac{3a^3b^3x^2Ur^{-5}}{2(b^3-a^3)} - \frac{3a^3b^3xyVr^{-5}}{2(-b^3-a^3)} \qquad \text{...(ix)}$$

$$V = -\frac{\partial\phi}{\partial y} = \frac{-a^3Ux\left(-\frac{3}{2}b^3r^{-4}\frac{y}{r}\right)}{b^3-a^3} + \frac{b^3V\left(1+\frac{1}{2}a^3r^{-3}\right)}{b^3-a^3}$$

$$+ \frac{b^3Vy\left(-\frac{3}{2}a^3r^{-4}\frac{y}{r}\right)}{b^3-a^3}$$

$$\Rightarrow \qquad V = \frac{-3a^3b^3xyUr^{-5}}{2(b^3-a^3)} + \frac{b^3V\left(1+\frac{1}{2}a^3r^{-3}\right)}{b^3-a^3} - \frac{3a^3b^3y^2Vr^{-5}}{2(b^3-a^3)} \qquad \text{...(x)}$$

$$\therefore \qquad \text{Velocity} = \vec{q} = u\hat{i} + v\hat{j}$$

where u and v are given by (ix) and (x).

6. Source and Sink

6.1 Two sources each of strength m are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength 2 m is at the origin. Show that the streamlines are the curves :

$$(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$$

where λ is a variable parameter.

Show also that the fluid speed at any point is $\frac{2ma^2}{r_1r_2r_3}$, where r_1, r_2 and r_3 are the distances of the points from the sources and the sink.

(2009 : 12 Marks)

Solution:

The complex potential W at the any point $P(z)$ is given by :

$$W = -m\log(z+a) - m\log(z-a) + 2m\log z \qquad \text{...(1)}$$

$$W = -m\log z^2 - a^2 + m\log z^2$$

$$W = -m[\log\{x+iy)^2 - a^2\} - \log(x+iy)^2]$$

$$W = m[\log(x^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)] \qquad \text{...(2)}$$

Now, we know that

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

Also, $\qquad W = \phi + i\psi$

where streamlines will be given by ψ.

Using the above formula and (2)

$$\psi = m\left[\tan^{-1}\frac{2xy}{x^2-y^2} - \tan^{-1}\frac{2xy}{x^2-y^2-a^2}\right]$$

$$\psi = m\left[\frac{\tan^{-1}\dfrac{2xy}{x^2-y^2} - \dfrac{2xy}{x^2-y^2-a^2}}{1+\dfrac{4x^2y^2}{(x^2-y^2)(x^2-y^2-a^2)}}\right]$$

$$\psi = m\left[\frac{\tan^{-1}\dfrac{2xy(x^2-y^2-a^2-x^2+y^2)}{(x^2-y^2)(x^2-y^2-a^2)}}{\dfrac{(x^2-y^2)^2-(x^2-y^2)a^2+4x^2y^2}{(x^2-y^2)(x^2-y^2-a^2)}}\right]$$

$$\psi = \frac{m\tan^{-1}-2xya^2}{(x^2+y^2)^2-(x^2-y^2)a^2}$$

The required streamlines will be given by taking ψ = constant.

$$\text{Constant} = \frac{m\tan^{-1}-2xya^2}{(x^2+y^2)^2-(x^2-y^2)a^2}$$

$$(x^2+y^2)^2-(x^2-y^2)a^2 = C(-2xya^2)$$

($\because$ m is also constant and tangent of any constant is also constant)

$$(x^2+y^2)^2 = a^2(x^2-y^2-2cxy)$$

$$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$$

where $\lambda = -2c$ and it is a variable parameter.

Now, for calculation of fluid speed at any point is given by

$$q = \left|\frac{dw}{dz}\right|$$

From (1),

$$q = \left|\frac{-m}{z+a}-\frac{m}{z-a}+\frac{2m}{z}\right|$$

$$q = m\left|\frac{-(z-a)-(z+a)}{(z-a)(z+a)}+\frac{2}{z}\right|$$

$$q = m\left|\frac{-2z}{z^2-a^2}+\frac{2}{z}\right|$$

$$= 2m\left|\frac{-z^2+z^2-a^2}{(z)(z-a)(z+a)}\right|$$

$$= \frac{2ma^2}{|z|\,|z-a|\,|z+a|}$$

$$= \frac{2ma^2}{r_1r_2r_3} \qquad [r_1 = |z-a|;\ r_2 = |z+a|;\ r_3 = |z|]$$

6.2 **If fluid fills the region of space on the positive side of x-axis which is a rigid boundary and if there be a source m at the point (0, a) and an equal sink at (0, b) and if the pressure on the negative side be same as the pressure of infinity, show that the resultant pressure on the boundary is $\dfrac{\pi\rho m^2(a-b)^2}{\{2ab(a+b)\}}$ where ρ is the density of the fluid.**

(2013 : 15 Marks)

Solution:

The image system with respect to x-axis in the z-plane is

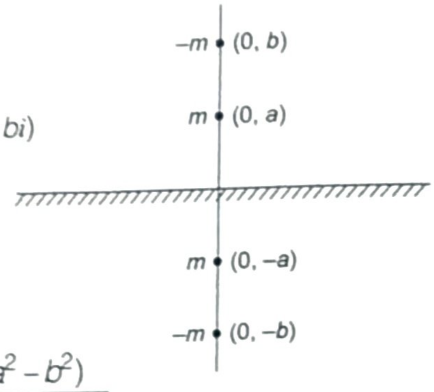
(i) Source of strength m at (0, a)

(ii) Source of strength m at (0, $-a$)

(iii) Sink of strength $-m$ at (0, b)

(iv) Sink of strength $-m$ at (0, $-b$)

This system does away witht he boundary. The complex potential of this entire system is given by



$$w = -m\log(z-ai) + m\log(z+bi) - m\log(z+ai) + m\log(z-bi)$$
$$w = -m\log(z^2+a^2) + m\log(z^2+b^2)$$

$$\text{Velocity} = \left|\frac{dw}{dz}\right| = -\frac{2zm}{z^2+a^2} + \frac{2zm}{z^2+b^2}$$

Velocity on the boundary, $y = 0$.

$$= \frac{-2xm}{x^2+a^2} + \frac{2xm}{x^2+b^2} = \frac{2xm(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

Let the pressure at infinity be p. By Bernoulli's theorem pressure at any point is given by

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \frac{1}{2}\times 0^2 + \frac{p_0}{\rho}$$

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

The resultant pressure on the boundary

$$= \int_0^\infty (p_0 - p)dx = \frac{1}{2}\rho\int_0^\infty q^2 dx$$

$$= 2\rho m^2\int_0^\infty \frac{x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}dx$$

$$= 2\rho m^2\int_0^\infty\left[-\frac{a^2+b^2}{b^2-a^2}\left\{\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right\} - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right]dx$$

On solving into partial fractions

$$= 2\rho m^2\left\{\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) - \frac{\pi}{4a} - \frac{\pi}{4b}\right\}$$

$$= \frac{\pi\rho m^2}{2ab}\cdot\left[\frac{2(a^2+b^2)-(a+b)^2}{(a+b)}\right] = \frac{\pi\rho m^2(a-b)^2}{2ab(a+b)}$$

6.3 **Two sources, each of strength m, are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at origin. Show that the stream lines are the curves $(x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter.**
Show also that the fluid speed at any point is $(2ma^2)/(r_1 r_2 r_3)$, where r_1, r_2 and r_3 are the distances of the points from the sources and the sink, respectively.

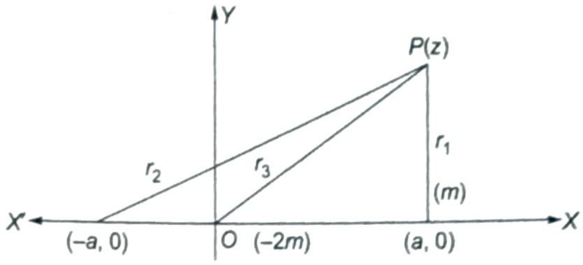
(2019 : 20 Marks)

Solution:

First Part : The complex potential w at any point $P(z)$ given by

$$w = -m\log(2-a) - m\log(2+a) + 2m\log 2 \quad \text{...(1)}$$
$$w = m[\log 2^2 - \log(2^2 - a^2)]$$

$\Rightarrow$
$$\phi + iw = m[\log(z^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)],\ \text{as} = x + iy$$



Equating the imaginary parts, we have

$$\varphi = \left[\tan^{-1}\left\{\frac{2xy}{(x^2-y^2)}\right\} - \tan^{-1}\{2xy\,|\,(x^2-y^2-a^2)\}\right]$$

$\therefore$

$$\varphi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)-a^2(x^2-y^2)}\right] \text{ on simplification.}$$

The detailed streamlines are given by u = Constant = $m\tan^{-1}\left(\frac{-2}{\lambda}\right)$

Then we obtain

$$\left(\frac{-2}{\lambda}\right) = \frac{(-2a^2xy)}{[(x^2+y^2)^2-a^2(x^2-y^2)]}$$

$$\Rightarrow \quad (x^2+y^2)^2 = a^2(x^2-y^2+\lambda 2y)$$

Second Part : From 0, we have

$$\frac{dw}{dz} = -\frac{m}{2-a}-\frac{m}{2+a}+\frac{2m}{2}$$

$$= \frac{2a^2m}{2(2-a)(2+a)}$$

$$q = \left|\frac{dw}{d2}\right| = \frac{2a^2m}{|2|\,|2-a|\,|2+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where $\quad r_1 = |z-a|,\ r_2 = |z+a|$ and $r_3 = |z|$

7. Vortex Motion

7.1 In an incompressible fluid, the vorticity at every point is constant in magnitude and direction; show that the components of velocity u, v, w are solutions of Laplace equation.

(2010 : 12 Marks)

Solution:

Let velocity, $\quad \vec{q} = u\hat{i}+v\hat{j}+w\hat{k}$

$\therefore$ Vorticity, $\quad \vec{w} = \nabla\times\vec{q}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)-\hat{j}\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)+k\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) = \text{Constant (given)}$$

$\therefore \qquad \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \text{ or } \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$

$\Rightarrow \qquad \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial z} \qquad ...(1)$

and $\qquad \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 v}{\partial z^2} \qquad ...(2)$

Similarly,

$$\frac{\partial w}{\partial x} = \frac{\partial y}{\partial z} \text{ Þ } \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial z} \qquad ...(3)$$

and $\qquad \frac{\partial^2 w}{\partial z \partial x} = \frac{\partial^2 u}{\partial z^2} \qquad ...(4)$

Also, $\qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

$\Rightarrow \qquad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 y}{\partial y^2} \qquad ...(5)$

and $\qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial x \partial y} \qquad ...(6)$

As fluid is incompressible, $\therefore$ by eqn. of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} = 0$

$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 y}{\partial z^2} = 0 \qquad$ (from (4) and (5))

So, u satisfies Laplace's equation.

In the same way,

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} = 0$$

$\Rightarrow \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0 \qquad$ (from (2) & (6))

and $\qquad \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2} = 0$

$\Rightarrow \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$

So, v and w also satisfy Laplace's equation.

$\therefore$ u, v, w are solutions of Laplace's equation.

7.2 When a pair of equal and opposite rectilinear vertices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation

$$(r^2 \sin 2\theta - b^2) = 4a^2 b^2 r^2 \sin^2\theta$$

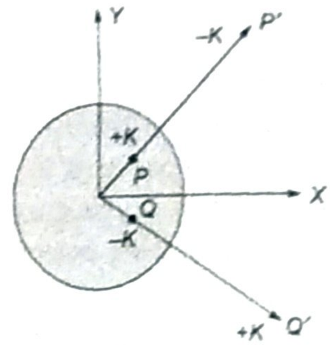
θ being measured from the line through the centre perpendicular to the joint of the vertices.

(2010 : 30 Marks)

Solution:

Let a pair of equal and opposite vertices of strength $+K$ and $-K$ are placed at the point $P(r, \theta)$ and $Q(r, \theta)$ respectively.

The image consists of



(i) A vertex of strength $-K$ at an inverse point $P'\left(\dfrac{a^2}{r}, \theta\right)$.

(ii) A vertex of strength $+K$ at an inverse point $Q'\left(\dfrac{a^2}{r}, -\theta\right)$.

The complex potential at any point z becomes

$$w = \frac{iK}{2\pi}\log(z - re^{\theta i}) - \frac{iK}{2\pi}\log\left\{z - \left(\frac{a^2}{r}\right)e^{\theta i}\right\} - \frac{iK}{2\pi}\log(z - re^{-\theta i}) + \frac{iK}{2\pi}\log\left\{z - \left(\frac{a^2}{r}\right)e^{-\theta i}\right\}$$

The motion of P is due to other vertices, thus for the motion P, the complex potential at any point $z(z = re^{\theta i})$ becomes

$$w_1 = \left[-\frac{iK}{2\pi}\log\left\{z - \left(\frac{a^2}{r}\right)e^{\theta i}\right\} - \frac{iK}{2\pi}\log(z - re^{-\theta i}) - \frac{iK}{2\pi}\log\left\{z - \left(\frac{a^2}{r}\right)e^{-\theta i}\right\}\right]_{z = re^{\theta i}}$$

or $$w_1 = -\frac{iK}{2\pi}\left[\log\left\{re^{\theta i} - \left(\frac{a^2}{r}\right)e^{\theta i}\right\}\log(re^{\theta i} - re^{-\theta i}) - \log\left\{re^{\theta i} - \left(\frac{a^2}{r}\right)e^{-\theta i}\right\}\right]$$

or $$w_1 = \frac{-iK}{2\pi}\left[\log\left\{\left(r\cos\theta - \left(\frac{a^2}{r}\right)\cos\theta\right) + i\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)\right\} + \log(2ir\sin\theta)\right.$$
$$\left. - \log\left\{r\cos\theta - \left(\frac{a^2}{r}\right)\cos\theta\right\} + i\left(r\sin\theta + \frac{a^2}{r}\right)\sin\theta\right]$$

or $$\psi = \frac{-K}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right\}\right]$$

or $$\psi = \frac{-K}{4\pi}\left[\log\left(r - \frac{a^2}{r}\cdot 2r\sin\theta\right)^2 - \log\left\{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right\}\right]$$

The stream lines are given by ψ = constant

i.e. $$\frac{-K}{4\pi}\left[\log\left\{4(r^2 - a^2)\right\}\sin^2\theta\right] - \log\left\{\frac{r^4 + a^4 2a^2 r^2\cos 2\theta}{r^2}\right\} = \text{Constant}$$

or $$\frac{r^2(r^2 - a^2)^2\sin^2\theta}{r^4 + a^4 - 2a^2r^2\cos 2\theta} = b^2 \quad \text{(let)}$$

or $$b^2\left\{(r^2 - a^2)^2 + 2a^2r^2(1 - \cos 2\theta)\right\} = r^2(r^2 - a^2)\sin^2\theta$$

or $$2a^2b^2r^2(1 - \cos 2\theta) = (r^2 - a^2)(r^2\sin^2\theta - b^2)$$

or $$4a^2b^2r^2\sin^2\theta = (r^2 - a^2)^2(r^2\sin 2\theta - b^2)$$

7.3 An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength K but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function.

(2011 : 30 Marks)

Solution:

Let the row of vertices be taken along the x-axis.

Let the points (0, 0), ($\pm 2a$, 0), ($\pm 4a$, 0), ... have vertices of strength k and the points ($\pm a$, 0), ($\pm 3a$, 0), ($\pm 5a$, 0), ... have vortices of strength $-k$.

The complete potential of the entire system is given by

$$W = \frac{ik}{2\pi}[\{\log z + \log(z-2a) + \log(z+2a) + \log(z-4a) + \log(z+4a) + ...\}$$
$$- \{\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + ...\}]$$
$$= \frac{ik}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2).....}{(z^2-a^2)(z^2-3^2a^2).....}$$
$$= \frac{ik}{2\pi}\log\frac{\frac{z}{2a}\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{4a}\right)^2\right].....}{\left[1-\left(\frac{z}{a}\right)^2\right]\left[1-\left(\frac{z}{3a}\right)^2\right].....} + a \text{ constant}$$

$\Rightarrow$
$$W = \frac{ik}{2\pi}\log\frac{\sin\left(\frac{\pi z}{2a}\right)}{\cos\left(\frac{\pi z}{2a}\right)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \quad ...(i)$$

which is the desired potential function that determines the velocity potential and stream function.

From (i),
$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x+iy) \quad ...(ii)$$

$\Rightarrow$
$$\phi - i\psi = -\frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x-iy) \quad ...(iii)$$

Subtracting (iii) from (ii), we have

$$2i\psi = \frac{ik}{2\pi}\left[\log\tan\frac{\pi}{2a}(x+iy) + \log\tan\frac{\pi}{2a}(x-iy)\right]$$

$\therefore$
$$\psi = \frac{k}{4\pi}\log\frac{\sin\frac{\pi}{2a}(x+iy)\sin\frac{\pi}{2a}(x-iy)}{\cos\frac{\pi}{2a}(x+iy)\cos\frac{\pi}{2a}(x-iy)}$$
$$= \frac{k}{4\pi}\log\frac{\cosh\left(\frac{\pi y}{a}\right) - \cos\left(\frac{\pi x}{a}\right)}{\cosh\left(\frac{\pi y}{a}\right) + \cos\left(\frac{\pi x}{a}\right)}$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity q_0 of the vortex at the origin is given by

$$q_0 = -\left[\frac{d}{dz}\left(\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a} - \frac{ik}{2a}\log z\right)\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\left[\frac{\sec^2\left(\frac{\pi z}{2a}\right)}{\tan\left(\frac{\pi z}{2a}\right)}\times\frac{\pi}{2a}-\frac{1}{2}\right]_{z=0}$$

$$= 0 \text{ (using L' Hospital's Rule)}$$

Hence, the vortex at the origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus, we find that the vortex row induces no velocity on itself.

7.4 **Prove that the necessary and sufficient conditions that the vortex lines may be at right angles to stream lines are :**

$$u, v, w = \mu\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)$$

where μ and φ are functions of x, y, z, t.

(2013 : 10 Marks)

Solution:

Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{...(i)}$$

and vortex lines by

$$\frac{dx}{\Omega x} = \frac{dy}{\Omega y} = \frac{dz}{\Omega z} \quad \text{...(ii)}$$

They are both perpendicular if

$$u\Omega x + v\Omega y + w\Omega z = 0$$

$$\Rightarrow u\left(\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\right)+v\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)+w\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right) = 0$$

which is the necessary and sufficient condition that $udx + vdy + wdz$ is a perfect differential.

$$\therefore \quad udx + vdy + wdz = \mu\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)$$

$$\therefore \quad u, v, w = \mu\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)$$

7.5 **If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid prove that the vortices will move around the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$. Find the velocity at any pointo f the liquid.**

(2013 : 20 Marks)

Solution:

Let the n vortices of strength k each be situated at points

$$z_m = ae^{2\pi im/n},\ m = 0, 1, 2, \ldots\ldots, n-1$$

Then complex potential due to these n vortices is given by

$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(7 - ae^{2\pi im/n})$$

$$= \frac{ik}{2\pi}\log\pi(z - ae^{2\pi im/n}) = \frac{ik}{2\pi}\log(z^n - a^n)$$

as $ae^{2\pi i m/n}$ is solution of $z^n = a^n$.

The fluid velocity q at any point out of all n vortices is given by

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\cdot\frac{nz^{n-1}}{z^n - a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n - a^n}\right|$$

Again, the velocity potential induced at A_1 $(z = a)$ by others is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n - a^n) - \frac{ik}{2\pi}\log(z-a)$$

$$= \frac{ik}{2\pi}\log\frac{(z^n - a^n)}{(z-a)}$$

$$= \frac{ik}{2\pi}\log(z^{n-1} + z^{n-2}a + \ldots\ldots + a^{n-1})$$

$$\therefore \quad \frac{dw'}{dz} = \frac{ik}{2\pi}\frac{(n-1)z^{n-1} + (n-2)z^{n-3}\cdot a + \ldots\ldots + a^{n-2}}{z^{n-1} + z^{n-2}a + \ldots\ldots + a^{n-1}}$$

$$\therefore \quad \left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi}\cdot\frac{(n-1) + (n-2) + \ldots\ldots + 1}{na}$$

$$= \frac{ik(n-1)}{4\pi a}$$

$$\therefore \quad u_1 - iv_1 = \left(\frac{-dw'}{dz}\right)_{z=a} = \frac{-ik(n-1)}{4\pi a}$$

$$u_1 = 0,\ v_1 = \frac{k(n-1)}{4\pi a}$$

Thus, the velocity at $z = a$ is along the tangential direction.

Since, all the vortices are symmetrical choice of co-ordinate system would have shown it to be true at all vortices.

$\therefore$ The vortices move around the circle.

$$\text{Time Period} = \frac{2\pi a}{v_\theta} = \frac{2\pi a}{\frac{k(n-1)}{4\pi a}}$$

$$= \frac{8\pi^2 a^2}{(n-1)k}$$

The velocity at any point has been found earlier.

7.6 Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r},\ 2y - 3z - \frac{2y}{r},\ x - 3y - \frac{2z}{r}\right]$ possess vorticity, where $\vec{q}(u, v, w)$ is the velocity in cartesian frame, $\vec{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$? What is the circulation in the circle $x^2 + y^2 = 9$, $z = 0$?

(2016 : 10 Marks)

Solution:

Given, velocity
$$\vec{q} = \left(z - \frac{2x}{r}\right)\hat{i} + \left(2y - 3z - \frac{2y}{r}\right)\hat{j} + \left(x - 3y - \frac{2z}{r}\right)\hat{k}$$

$$= u\hat{i} + v\hat{j} + w\hat{k}$$

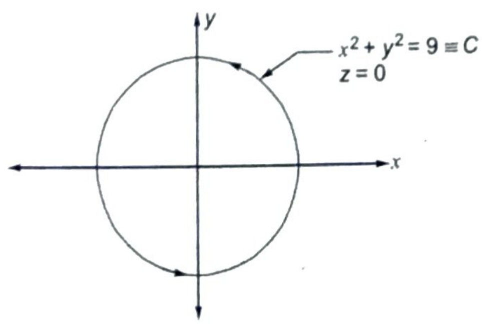
Now,
$$\nabla\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-\frac{2x}{r} & 2y-3z-\frac{2y}{r} & x-3y-\frac{2z}{r} \end{vmatrix}$$

$\Rightarrow$
$$\nabla\times\vec{q} = \hat{i}\left(-3+\frac{2z}{r^2}\times\frac{y}{r}+3-\frac{2y}{r^2}\times\frac{z}{r}\right)-\hat{j}\left(1+\frac{2z}{r^2}\times\frac{x}{\sigma}-1-\frac{2x}{r^2}\times\frac{z}{r}\right)$$
$$+\hat{k}\left(+\frac{2y}{r^2}\times\frac{x}{r}-\frac{2x}{r^2}\times\frac{y}{\sigma}\right)$$
$$= 0$$

as $\nabla\times\vec{q} = 0 \Rightarrow$ fluid is irrotational

$\therefore$ fluid does not possess vorticity.

Now,



Circulation around C,
$$\Gamma = \oint_C \vec{q}\cdot d\vec{r}$$

$\Rightarrow$
$$\Gamma = \oint_C\left\{\left(2-\frac{2x}{r}\right)i+\left(2y-3z-\frac{2y}{r}\right)\hat{j}+\left(x-3y-\frac{2z}{r}\right)\hat{k}\right\}\cdot(dx\hat{i}+dy\hat{j})$$

Also, $z = 0$, $dz = 0$

$\Rightarrow$
$$\Gamma = \oint_C -\frac{2x}{r}dx+\left(2y-\frac{2y}{r}\right)dy$$

Further,
$$r = 3 = \sqrt{9}$$

$\Rightarrow$
$$\Gamma = \oint_C \frac{-2x}{3}dx+\left(2y-\frac{2y}{3}\right)dy$$

$\Rightarrow$
$$\Gamma = \oint_C \frac{-2x}{3}dx+\left(\frac{4y}{3}\right)dy$$

$\Rightarrow$
$$\Gamma = 0 \text{ as fluid is irrotational.}$$

8. Navier-Stoke's Equation for Viscous Flow

8.1 Find Navier-Stokes equation for a steady laminar flow of a viscous incompressible fluid between two infinite parallel plates.

(2014 : 20 Marks)

Solution:

By Laminar flow, we mean that fluid moves in layers parallel to the plates.

We suppose that an incompressible fluid with constant viscosity is confined between two parallel plates,

$$y = \frac{a}{2}$$

and
$$y = -\frac{a}{2}$$

Let, the fluid be moving with velocity is parallel to x-axis with laminar flow. In order to maintain such a motion, the difference of pressure in x-direction must be balanced by shearing stresses.

Here :
$$q = q(u, 0, 0)$$

Equation of continuity

$$d = 0; \text{ so that } u = u(g, t)$$

Navier-strokes equation in absence of external force is

$$\frac{dq}{dt} = \frac{\partial q}{\partial t} + (q.\nabla)q = -\frac{1}{\rho}\nabla p + v\nabla^2 q$$

or,
$$i\frac{\partial u}{\partial t} + iu\frac{\partial u}{\partial x} = -\frac{1}{P}\nabla P + vi\nabla^2 u$$

or,
$$i\frac{\partial u}{\partial t} = -\frac{1}{\rho}\nabla P + vi\nabla^2 u \text{ as } \frac{\partial u}{\partial x} = 0$$

This,
$$\frac{\partial u}{\partial t} = \frac{1}{P}\frac{\partial P}{\partial x} + v\nabla^2 u \quad \text{...(i)}$$

The least two,
$$P = P(x, t) \quad \text{...(ii)}$$

Now (i)
$$d = s \quad \text{...(iii)}$$

Also,
$$P = p(x, t),\ u = u(y, t)$$

R.H.S of (iii) is constant or function of y, t.

Consequently (iii) declares that either $\frac{\partial p}{\partial x}$ is constant or function If 't'. Now consider the case of steady motion so that (iii) becomes.

$$\mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = \frac{dp}{dx}$$

or
$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Integrating,
$$\frac{du}{dy} = \frac{y}{\mu}\frac{dP}{dx} + A$$

or
$$u = \frac{y^2}{2\mu}\frac{dP}{dx} + dy + B \quad \text{...(iv)}$$

Case I : Plane Couette Flow;

In this case $\frac{dp}{dx} = 0$; the lower plate is stationary while the upper is moving with uniform velocity 'U' parallel to x-axis. The boundary conditions are

(i) $u = 0;\ y = -\frac{h}{2}$

(ii) $u = U = \text{constant}:\ y = \frac{h}{2}$

Substituting (iv) and (i) and (ii), we get.

$$0 = \frac{h^2}{8\mu}.0 + A\left(-\frac{h}{2}\right) + B$$

and
$$U = \frac{h^2}{8\mu}.0 + A.\frac{h}{2} + B$$

This
$$-Ah + 2B = 0 \quad ; \quad Ah + 2B = 2U$$

$\Rightarrow$
$$2B = U \quad ; \quad -Ah + U = 0$$

Now, (iv) becomes,

$$u = \frac{u}{h}y + \frac{u}{2} \qquad \text{. . . (v)}$$

Evidently; velocity distribution is linear.

Case II. Plane Poiseuille flow :

In this case $\frac{dp}{dx} = \text{constant} = a \neq 0$ and both the walls are at rest.

The boundary conditions are:

(i) $u = 0;\ y = -\frac{h}{2}$

(ii) $u = 0;\ y = \frac{h}{2}$

Subjecting (iv) to condition (i) and (ii),

$$\frac{ah^2}{8\mu} + A\left(-\frac{h}{2}\right) + B = 0$$

and
$$\frac{ah^2}{8\mu} + A\left(\frac{h}{2}\right) + B = 0$$

Subtracting we get,

$A = 0$ and $B = \frac{-ah^2}{8\mu}$.

Now (iv) becomes,
$$u = \frac{ay^2}{2\mu} - \frac{ah^2}{8\mu} = -\frac{h^2}{8\mu}\left(1 - \frac{4y^2}{h^2}\right)\frac{dP}{dx} \qquad \text{...(vi)}$$

$$u = u_n\left(1 - \frac{4y^2}{h^2}\right)$$

Where,
$$u_m = -\frac{h^2}{8\mu}\frac{dp}{dx} \qquad \text{...(vii)}$$

Is the maximum velocity in the flow accruing at $y = 0$; evidently, velocity distribution is parabolic. Drag (shear stress) at lower plate.

$$= \left(\mu \frac{du}{dy}\right)_{y=-\frac{h}{2}} = \mu\left(-\frac{8y}{h^2}\mu_m\right)_{y=-\frac{h}{2}} = \frac{4\mu\mu_m}{h}$$

∴ The average velocity distribution for the present flow is given by

$$u_a = \frac{1}{h}\int_{-h/2}^{h/2} u.dy(\text{using}(6)); \text{ we get}$$

$$u_a = \frac{1}{h}u_m\int_{-h/2}^{h/2}\left(1-\frac{4y^2}{h^2}\right)dy = \frac{2}{h}4m\int_{h}^{h/2}\left(1-\frac{4y^2}{h^2}\right)dy$$

$$u_a = \frac{2}{h}\left(-\frac{h^2}{8\mu}.a\right)\left[\frac{h}{2}\left(-\frac{4}{h^2}\right).\frac{1}{3}\left(\frac{h}{2}\right)^3\right] = \left(-\frac{ha}{4\mu}\right)\left(\frac{h}{3}\right)$$

$$u_a = \frac{2}{3}\left(-\frac{h^2a}{8\mu}\right) = \frac{2}{3}\mu_m$$

or $$u_a = \frac{2}{3}u_m \quad \text{...(viii)}$$

where; u_a = average velocity

$a = \frac{dp}{dx}$ = constant

u_m = maximum velocity.

Case III : Generalised Plane couette flow:

In this case $\frac{dp}{dx}$ = constant = $a \neq 0$; the lower plate is at rest while the upper plate is in motion with velocity U. The boundary condition are

(i) $u = 0;\ y = -\frac{h}{2}$

(ii) $u = U;\ y = \frac{h}{2}$

Substituting in (iv) from (i) and (ii)

$$\frac{ah^2}{8\mu} + A\left(-\frac{h}{2}\right) + B = 0$$

$$\frac{h^2}{8\mu} + A\left(\frac{h}{2}\right) + B = U$$

This, $$B = \frac{u}{2} - \frac{h^2}{8\mu}; A = \frac{u}{h}$$

Now (iv) becomes, $$u = \frac{dy^2}{2u} + \frac{u}{h}y + \frac{u}{2} - \frac{ah^2}{8\mu}$$

or, $$u = \frac{d}{8\mu}(4y^2 - h^2) + \frac{u}{2}\left(1+\frac{2y}{h}\right)$$

Evidently : velocity distribution is parabolic

$$\mu\frac{du}{dy} = \frac{a}{8\mu}(8y-0) + u.\frac{u}{2}\left(0+\frac{2}{h}\right)$$

$$= ay + \frac{\mu}{h}.u \quad \text{...(ix)}$$

Drag per unit area on boundaries.

$$= \mu \frac{du}{dy} \quad \text{at} \quad y = \pm \frac{h}{2}$$

$$= \mu \frac{U}{h} \pm \frac{h}{2} \frac{dP}{dx}$$

Total flux (flow) per unit breadth across a plane perpendicular to x-axis is Q.

$$= \int_{-h/2}^{h/2} u dy = \left[\frac{a}{8\mu} \left(\frac{4}{3} y^3 - h^2 y \right) + \frac{U}{2} \left(y + \frac{y^3}{h} \right) \right]_{y=\frac{-h}{2}}^{y=\frac{h}{2}}$$

or,

$$Q = U.\frac{h}{2} - \frac{h^3}{12} . \frac{a}{\mu}$$

Vorticity $w(\xi, \eta, \varsigma)$ at any point is given by

$$\xi = 0; \eta = 0$$

$$\varsigma = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = -\frac{1}{2} \frac{\partial u}{\partial y} = -\frac{1}{2} \frac{du}{dy}$$

$$\varsigma = -\frac{1}{2} \left(\frac{ay}{\mu} + \frac{U}{h} \right) \quad \text{by (ix)}$$

Rate D of dissipation of energy per unity area is given by

$$D = 4\mu \int_{-h/2}^{h/2} \xi^2 dy = \mu \int_{-h/2}^{h/2} \left[\frac{ay}{\mu} + \frac{U}{h} \right]^2 dy$$

$$D = \mu \int_{-h/2}^{h/2} \left(\frac{a^2 y^2}{\mu^2} + \frac{U^2}{h^2} + \frac{2ayU}{\mu h} \right) dy$$

$$D = \frac{a^2 h^3}{12\mu} + \frac{U^2 a}{h}$$

■■■■