IFoS 2019 PDE

5a) Find the solution of the equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

$$\Rightarrow \quad (D^2 - D'^2) z = x - y$$

Auxiliary Eqn : $m^2 - 1 = 0$

$$\Rightarrow \quad m = +1, -1$$

$$\therefore \quad C.F. = \phi_1(y+x) + \phi_2(y-x)$$

Further,

$$P.I. = \frac{1}{D^2 - D'^2}(x-y)$$

$$= \frac{1}{D^2} \cdot \left(1 - \frac{D'^2}{D^2}\right)^{-1} (x-y)$$

$$= \frac{1}{D^2}\left[1 + \frac{D'^2}{D^2} + \ldots\right](x-y)$$

$$= \frac{1}{D^2}\left[(x-y) + \frac{1}{D^2} D'^2 (x-y)\right]$$

$$= \frac{1}{D^2}\left[(x-y) + \frac{1}{D^2}(0)\right]$$

$$= \frac{1}{D^2}(x-y) = \frac{1}{D}\left(\frac{x^2}{2} - xy\right)$$

$$= \frac{x^3}{6} - \frac{x^2 y}{2}$$

$\therefore$ Complete solution of PDE is

$$z = \phi_1(y+x) + \phi_2(y-x) + \frac{x^3}{6} - \frac{x^2 y}{2}$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $\quad f(x, y, z, p, q) = p^2x + q^2y - z = 0. \quad$ ...(1)

Charpit's auxiliary equations are
$$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or
$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2)
$$= \frac{2px\,dp + p^2dx}{2px(-p+p^2)+p^2(-2px)} = \frac{2qy\,dq + q^2dy}{2qy(-q+q^2)+q^2(-2qy)}$$

or
$$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\quad \log(p^2x) = \log(q^2y) + \log a \quad$ or $\quad p^2x = q^2ya. \quad$ ...(3)

Form (1) and (3), $\quad aq^2y + q^2y = z \quad$ or $\quad q = [z/(1+a)]^{1/2}. \quad$ ...(4)

Form (3) and (4),
$$p = q\left(\frac{ya}{x}\right)^{1/2} = \left\{\frac{za}{(1+a)x}\right\}^{1/2}.$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get
$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2}dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2}dy \qquad \text{or} \qquad (1+a)^{1/2}z^{-1/2}\,dz = \sqrt{a}\,x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $\quad (1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

6a) Test the integrability of the equation
$$z(z+y^2)dx + z(z+x^2)dy - xy(x+y)dz = 0$$
If integrable, then find its solution.

Comparing with, $P\,dx + Q\,dy + R\,dz = 0$

$$P = z(z+y^2) = z^2 + zy^2$$
$$Q = z(z+x^2) = z^2 + zx^2$$
$$R = -xy(x+y) = -(x^2y + xy^2)$$

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= P\left(2z + x^2 + x^2 + 2xy\right) + Q\left(-2xy + \cancel{y^2} - 2z - \cancel{y^2}\right)$$
$$+ R\left(2yz - 2zx\right)$$

$$= z(z+y^2)(2z + 2x^2 + 2xy)$$
$$+ z(z+x^2)(-2xy - 2z)$$
$$- (x^2y + xy^2)\,z\,(2y - 2x)$$

$$= 2z\Big[\cancel{z^2} + \cancel{zx^2} + \cancel{xyz} + y^2z + \cancel{x^2y^2} + \cancel{xy^3}$$
$$- \cancel{xyz} - \cancel{z^2} - \cancel{x^3y} - \cancel{x^2z}$$
$$- \cancel{x^2y^2} + \cancel{x^3y} - \cancel{xy^3} + x^2y^2\Big]$$

$$= 2z(y^2z + x^2y^2) \neq 0$$

Hence given system is not integrable.

classmate

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7c) Find the equations of the system of curves on the cylinder $2y=x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy=z+c$. (15m)

Given Surface is,

$$f(x,y,z) = 2y - x^2 \qquad \text{—(1)}$$

Hyperboloids of the one-parameter system is

$$xy = z + c \qquad \text{—(2)}$$

Then the system of D.Es. of the given curves of intersection of (1) and (2) is

$$-2x\,dx + 2\,dy = 0 \quad ;$$

$$y\,dx + x\,dy - dz = 0$$

Solving these equation for dx, dy, dz

$$\frac{dx}{-2-0} = \frac{dy}{0-2x} = \frac{dz}{-2x^2-2y}$$

$$\frac{dx}{1} = \frac{dy}{x} = \frac{dz}{x^2+y}$$

Hence the system of D.E. of the required orthogonal trajectories of the given curves is

$$-x\,dx + dy + 0\,dz = 0 \; ;$$

$$dx + x\,dy + (x^2+y)\,dz = 0$$

$$\Rightarrow \frac{dx}{(x^2+y)} = \frac{dy}{x(x^2+y)} = \frac{dz}{-x^2-1}$$

from the first two.

$$\frac{dx}{(x^2+y)} = \frac{dy}{x(x^2+y)}$$

$$\Rightarrow \quad x\,dx = dy$$

$$\Rightarrow \quad x^2/2 = y + c_1$$

Taking $c_1 = 0$, $\boxed{x^2 = 2y}$ — ($\star$)

from first and third, and using ($\star$)

$$\frac{dx}{x^2 + x^2/2} = \frac{dz}{-(x^2+1)}$$

$$\frac{2(x^2+1)}{x^2}\,dx = -3\,dz$$

$$2\left(1 + \frac{1}{x^2}\right) dx = -3\,dz$$

$$\Rightarrow \quad 2\left(x - \frac{1}{x}\right) = -3z + c'$$

c' being an arbitrary constant.

Hence required family of orthogonal trajectories is given by

$$x^2 = 2y \quad \text{and} \quad 3z + 2\left(x - \frac{1}{x}\right) = c'.$$