CSE – 2012 – Vector

Q1⇒ If $\vec{A} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$; $\vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$

Find the value of $\dfrac{\delta^2}{\delta x\,\delta y}(\vec{A}\times\vec{B})$ at $(1,0,-2)$.

Sol^n

$$\vec{A} = x^2yz\hat{i} - \cancel{2xz}\,2xz^3\hat{j} + xz^2\hat{k}, \quad \vec{B} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\hat{i} - \hat{j}(-x^4yz - 2xz^3) + \hat{k}(x^2y^2z + 2xz^4)$$

$$\vec{A}\times\vec{B} = (2x^3z^3 - xyz^2)\hat{i} + (x^4yz + 2xz^3)\hat{j} + (x^2y^2z + 2xz^4)\hat{k}$$

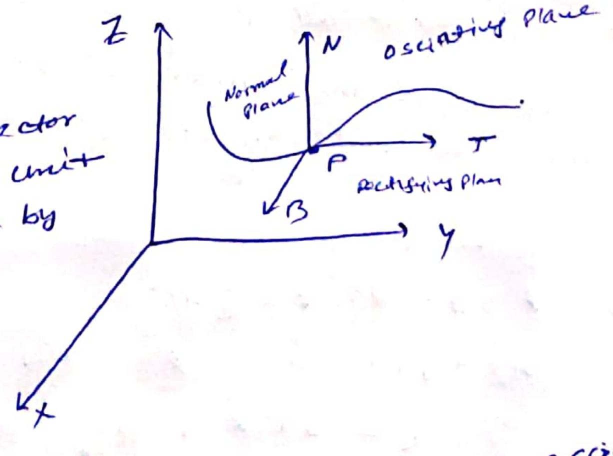
$$\cancel{\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y}} = \frac{\delta(\vec{A}\times\vec{B})}{\delta y} = -xz^2\hat{i} + x^4z\hat{j} + 2x^2yz\hat{k}$$

$$\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y} = -z^2\hat{i} + 4x^3z\hat{j} + 4xyz\hat{k}$$

$$\left.\frac{\delta^2(\vec{A}\times\vec{B})}{\delta x\,\delta y}\right|_{at\,(1,0,-2)} = -4\hat{i} + 4\times1\times-2\,\hat{j} = \boxed{-4\hat{i} - 8\hat{j}}$$

Q2⇒ Derive the Frenet-Serret Formulae. Define the curvature and torsion for a space curve. Compute them for the space curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$. Show that the curvature and torsion are equal for this curve.

Sol^n



Let $\vec{r}(t)$ be position vector of the point P; then the unit vector T at P is given by

$$\frac{d\vec{r}}{ds} = T$$

$\because |T| = 1$, we have

$$T\cdot\frac{dT}{ds} = 0$$

$\therefore$ T is perpendicular to T. $\frac{dT}{ds}$ lies in the osculating plane

$\therefore \frac{dT}{ds} \parallel N. \Rightarrow \boxed{\frac{dT}{ds} = kN}$, where k is curvature

$\cancel{\frac{dB}{ds}}$ $\because |B| = 1$

$\therefore B \cdot \frac{dB}{ds} = 0$

$\frac{dB}{ds} \perp$ to B.

We have, $B \cdot T = 0$

$\Rightarrow B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T = 0$

$B \cdot (KN) + \frac{dB}{ds} \cdot T = 0$

$K(B \cdot N) + \frac{dB}{ds} \cdot T = 0 \Rightarrow \frac{dB}{ds} \cdot T = 0$ $(B \cdot N = 0)$.

$\therefore T \cdot \frac{dB}{ds} = 0 \Rightarrow \frac{dB}{ds}$ is $\perp$ to T

$\frac{dB}{ds}$ lies in osculating plane. $\boxed{\therefore \frac{dB}{ds} = -\tau N}$

$\tau =$ torsion.

α

$\because B \times T = N$

$\Rightarrow B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$

$\Rightarrow B \times (KN) + (-\tau N) \times T = \frac{dN}{ds}$

$\Rightarrow \frac{dN}{ds} = K(B \times N) - \tau(N \times T)$

$= K(-T) - \tau(-B)$

$$\boxed{\frac{dN}{ds} = \tau B - KT.}$$

Curvature(K):- If T is the unit tangent vector to the curve $\vec{r}(s)$ at a point 'P' then the rate of change of T w.r.t 's' is called the curvature at 'P'.

Torsion(τ) :- If B is the Binormal vector to the curve $\vec{r}(s)$ at a point 'P', then the rate of change of B w.r.t. 's' is called torsion of the curve at 'P'.

Given, $x = t, y = t^2, z = \frac{2}{3}t^3$

$\vec{r}(t) = x\hat{i} + y\hat{j} + z\hat{k} = t\hat{i} + t^2\hat{j} + \frac{2}{3}t^3\hat{k}$

$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 2t^2\hat{k}$, $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+4t^2+4t^3}$

$\frac{d^2\vec{r}}{dt^2} = 2\hat{j} + 4t\hat{k}$

$\frac{d^3\vec{r}}{dt^3} = 4\hat{k}$

Mod:

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix} = \hat{i}(8t^2-4t^2) - \hat{j}(4t) + \hat{k}(2)$$

$$= 4t^2\hat{i} - 4t\hat{j} + 2\hat{k}$$

$$k = \frac{\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} = \frac{\sqrt{(4t^2)^2+(4t)^2+(2)^2}}{\left(\sqrt{1+4t^2+4t^4}\right)^3} \div \frac{2\sqrt{1+4t^2+4t^4}}{\left(\sqrt{1+4t^2+4t^4}\right)^3}$$

$$K = \frac{2}{1+4t^2+4t^4}$$

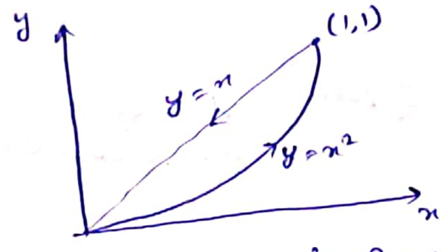
$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 1 & 2t & 2t^2 \\ 0 & 2 & 4t \\ 0 & 0 & 4 \end{vmatrix} = 1(8) - 0 + 0 \Rightarrow 8$$

$$T = \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|^2} = \frac{8}{4(1+4t^2+4t^4)} = \frac{2}{1+4t^2+4t^4}$$

$$\boxed{K = T}$$

Q 3. Verify Green's theorem in the plane for $\oint_C \{(xy+y^2)dx + x^2dy\}$, where C is the closed curve of the region bounded by $y=x$ & $y=x^2$.

Sol^n



Green's theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy$$

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy \quad \text{— (1)}$$

Comparing with (1) $\quad P = xy + y^2,\ Q = x^2$

Line integral along $y = x^2 \Rightarrow dy = 2x\,dx$,

$$\int_0^1 (x\cdot x^2 + x^4)dx + x^2\cdot 2x\,dx \Rightarrow \int_0^1 (3x^3+x^4)dx = \frac{3}{4}+\frac{1}{5} = \frac{19}{20}$$

Line integral along $y = x \Rightarrow dy = dx$

$$\int_1^0 (x\cdot x + x^2) + x^2dx \Rightarrow \int_1^0 3x^2\,dx = -1$$

$$\oint_C Pdx + Qdy = -1 + \frac{19}{20} = \boxed{-\frac{1}{20}}$$

$$\iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy = \int_{x=0}^{x=1}\int_{y=x^2}^{y=x}\left[\frac{\delta}{\delta x}(x^2) - \frac{\delta}{\delta y}(xy+y^2)\right]dy\,dx$$

$$= \int_{x=0}^{x=1}\int_{y=x^2}^{y=x}(2x-x-2y)\,dy\,dx = \int_{x=0}^{1}\left[xy-y^2\right]_{x^2}^{x}dx = \int_{x=0}^{1} x^4 - x^3\,dx$$

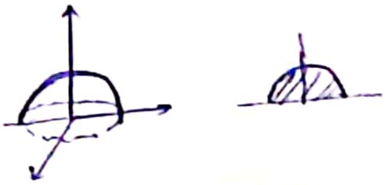
$$= \left[-\frac{x^4}{4}+\frac{x^5}{5}\right]_1^0 = -\frac{1}{4}+\frac{1}{5} = \boxed{-\frac{1}{20}}$$

$\therefore$ L.H.S = R.H.S.

Q4→ If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\vec{s}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above xy pl.

Sol^n

$\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$



$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & x - 2xz & -xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(-xy) - \frac{\delta}{\delta z}(x - 2xz)\right] - \hat{j}\left[\frac{\delta}{\delta x}(-xy) - \frac{\delta}{\delta z}(y)\right] + \hat{k}\left[\frac{\delta}{\delta x}(x - 2xz) - \frac{\delta}{\delta y}(y)\right]$$

$$\vec{\nabla} \times \vec{F} = (-x + 2x)\hat{i} - \hat{j}(-y) + \hat{k}(1 - 2z - 1) = x\hat{i} + y\hat{j} - 2z\hat{k}$$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}$$

$$(\nabla \times \vec{F}) \cdot \hat{n} = \left(\frac{x^2 + y^2 - 2z^2}{a}\right) \quad ; \quad ds = \frac{dx\,dy}{\hat{n} \cdot \hat{k}} = \frac{dx\,dy}{\frac{z}{a}} =$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, d\vec{s} = \iint_S \left(\frac{x^2 + y^2 - 2z^2}{a}\right) \frac{dx\,dy}{za} = \iint_S \frac{x^2 + y^2 - 2(a^2 - x^2 - y^2)}{\sqrt{a^2 - x^2 - y^2}}\, dx\,dy$$

$$\Rightarrow \iint \frac{3(x^2 + y^2) - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dx\,dy$$

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$r = 0$ to a, $\theta = 0$ to $\frac{\pi}{2}$

$$\Rightarrow \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\,dr\,d\theta$$

$$= \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}} \times \frac{\pi}{2} r\,dr \quad , \quad r = a\sin t, \; dr = a\cos t\, dt$$

$$\Rightarrow \frac{\pi}{2} \int_0^{\pi/2} \frac{3a^2\sin^2 t - 2a^2}{a\cos t} \times a\sin t \times a\cos t\, dt$$

$$\Rightarrow \frac{\pi}{2} \int_0^{\pi/2} a^2(3\sin^2 t - 2) \times a\sin t\, dt$$

$$\Rightarrow \frac{\pi}{2} \int_0^{\pi/2} (-a^3)(2 - 3\sin^2 t) \times a\sin t\, dt \Rightarrow \frac{\pi}{2} \int_0^{\pi/2} (-a^3)(3\cos^2 t - 1)\, a\sin t\, dt$$

$\cos t = p$

$-\sin t\, dt = dp$; $p = 1$ to 0

$$\Rightarrow \frac{\pi}{2} a^3 \int_1^0 (3p^2 - 1)\, dp \Rightarrow \frac{\pi}{2} a^3 \left[\frac{3p^3}{3} - p\right]_1^0 = 0$$

$$\therefore \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \cdot d\vec{s} = 0$$

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