Mains Test Series - 2019

Test - 06 (Paper-II)

Answer Key

Q.1) a) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find $O[b]$.

Solution

Given; $a^5 = e$; $aba^{-1} = b^2$ — (1)

$(aba^{-1})^2 = b^4$

$aba^{-1}aba^{-1} = b^4$

$abeba^{-1} = b^4$

$ab^2a^{-1} = b^4$

$aaba^{-1}a^{-1} = b^4$ — from (1)

$a^2ba^{-2} = b^4$

again squaring both side.

$(a^2ba^{-2})^2 = b^8$

$a^2ba^{-2}a^2ba^{-2} = b^8$

$a^2b^2a^{-2} = b^8$

$a^3ba^{-3} = b^8$ — from (i)

squaring both side.

$(a^3ba^{-3})^2 = b^{16}$

$a^3ba^{-3}a^3ba^{-3} = b^{16}$

$a^3b^2a^{-3} = b^{16}$

$a^4ba^{-4} = b^{16}$ — from (1)

squaring both side.

$(a^4 b a^{-4})^2 = b^{32}$

$a^4 b a^{-4} a^4 b a^{-4} = b^{32}$

$a^4 b e b a^{-4} = b^{32}$

$a^4 b^2 a^{-4} = b^{32}$

$a^4 a b a^{-1} a^{-4} = b^{32}$

$a^5 b a^{-5} = b^{32}$

as $a^5 = e$; i.e $a^{-5} = e$

Hence; $e\,b\,e = b^{32}$

$e = b^{31}$

$\Rightarrow \boxed{O[b] = 31}$ Ans

Q.1 (b) Show that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

Solution: To prove $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

let; $28 \in \mathbb{Z}[\sqrt{-3}]$ is a non unit, non zero element and we can express it as product of irreducable in two ways

$28 = (2)(14)$

$28 = (1 + 3\sqrt{-3})(1 - 3\sqrt{-3})$

But, 2 is not an associate of $1 + 3\sqrt{-3}$ or $1 - 3\sqrt{-3}$. Hence, $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

Now, to prove, that 2 is irreducable.

T.S. 2 is an irreducible element which is not prime.

Suppose, $2 = (a+\sqrt{-3}b)(c+\sqrt{-3}d)$

$a, b, c, d \in \mathbb{Z}$

Taking conjugates, we get

$$\bar{2} = (a-\sqrt{-3}b)(c-\sqrt{-3}b)$$

$$2\cdot\bar{2} = (a^2+3b^2)(c^2+3d^2)$$

$$4 = (a^2+3b^2)(c^2+3d^2)$$

$$a^2+3b^2 = 1, 2 \text{ or } 4$$

Now; $a^2+3b^2 = 2$ is not possible

as $a, b \in \mathbb{Z}$

If $a^2+3b^2=1$,

then $a=\pm1$, $b=0$

If $a^2+3b^2=4$, then

$c^2+3d^2=1$, i.e $c=\pm1$, $d=0$

Thus, if $a^2+3d^2=1$ then $a+\sqrt{-3}b = \pm1 =$ unit

and if $a^2+3d^2=4$ then $c+\sqrt{-3}d = \pm1 =$ unit

Hence, '2' is an irreducable element of $\mathbb{Z}[\sqrt{-3}]$.

Hence, $\mathbb{Z}[\sqrt{-3}]$ is a UFD.

Q.1) c) Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$, there is at least one real root of the equation $e^x \sin x + 1 = 0$.

Solution:

Given; $e^x \cos x + 1 = 0$

let, the interval be $[a, b]$

$$f(x) = \cos x + e^{-x}$$

and a, b are two real roots of the given equation

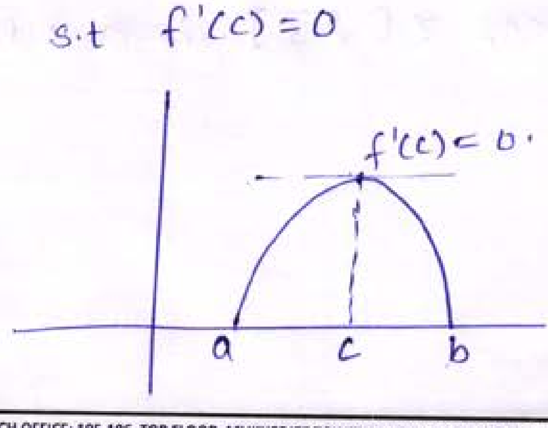
i) Such that $f(a) = f(b) = 0$

ii) $f(x)$ is continuous in the given interval $[a,b]$

iii) $f(x)$ is differentiable in (a,b)

so; $\exists$ a 'c' s.t $f'(c) = 0$

$f'(c) = 0$



$f(x) = \cos x + e^{-x}$.

$f'(x) = -\sin x - e^{-x}$

So,

$$f'(c) = -\sin c - e^{-c} = 0$$

$$\Rightarrow e^{c}\sin c + 1 = 0$$

So; $e^{x}\sin x + 1$ has one real root 'c' between any two real roots of the given equation.

Q.1) d) Prove that the function $f(z) = u + iv$,

where; $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ $(z \neq 0)$, $f(0) = 0$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Solution:

Given: $f(z) = u + iv = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$f(z) = u + iv = \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}$$

where $(z \neq 0)$

So; $u = \dfrac{x^3 - y^3}{x^2+y^2}$; $v = \dfrac{x^3+y^3}{x^2+y^2}$

Here, both u and v are rational and finite for all values of $z \neq 0$.

So, u and v are continuous at all those points for which $z \neq 0$.

Hence, $f(z)$ is continuous where $z \neq 0$.

At origin; $(x,y)=(0,0)$

$\therefore u=0$, $v=0$ [Since $f(0)=0$]

Hence, u and v are both continuous at origin; therefore $f(z)$ is continuous at the origin;

Now, at $(x,y)=(0,0)$. or at origin

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\left(\frac{x}{x}\right)=1.$$

$$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\left(\frac{-y}{y}\right)=-1.$$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\left(\frac{x}{x}\right)=1.$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y}=\lim_{y\to 0}\left(\frac{y}{y}\right)=1.$$

$\Rightarrow$ from above four equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Cauchy-Riemann equations are satisfied at $z=0$.

Again,

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z} \qquad z=x+iy.$$

$$=\lim_{z\to 0}\left[\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2).(x+iy)}\right]$$

let $z=0$ along $y=x$, then we have

$$f'(0)=\lim_{z\to 0}\frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}=\lim_{z\to 0}\frac{2i}{2(1+i)}$$

$$f'(0) = \frac{i}{(1+i)} = \frac{i(1-i)}{(1+i)(1-i)}$$

$$= \frac{i+1}{1^2 - i^2} = \frac{1+i}{1+1} = \frac{1+i}{2}$$

Further, let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{z \to 0} \frac{x^3(1+i)}{x^2(x)} = \frac{x^3(1+i)}{x^3}$$

$$f'(0) = 1+i.$$

Hence, $f'(0)$ is not unique.

Thus, $f'(z)$ does not exist at the origin.

Q.1) e) Obtain the dual of the LP program:

Min. $Z = x_1 + x_2 + x_3$; subject to the constraints:

$$x_1 - 3x_2 + 4x_3 = 5$$
$$x_1 - 2x_2 \leq 3$$
$$2x_2 - x_3 \geq 4$$

$x_1, x_2 \geq 0$ and x_3 is unrestricted.

Solve:- Transform the given LPP into the standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \geq 0$, $x_3'' \geq 0$.

Max. $Z_x' = -x_1 - x_2 - (x_3' - x_3'')$; $Z_x' = -Z$

Subject to constraints:
$$x_1 - 3x_2 + 4(x_3' - x_3'') \leq 5$$
$$-x_1 + 3x_2 - 4(x_3' - x_3'') \leq -5$$
$$x_1 - 2x_2 \leq 3$$
$$-2x_2 + (x_3' - x_3'') \leq -4$$
$$x_1, x_2, x_3', x_3'' \geq 0$$

Let ; $\omega_1', \omega_1'', \omega_2, \omega_3$ be the dual variables.
The dual problem of above standard primal is obtained as :

$$\text{Min } z_W' = 5(\omega_1' - \omega_1'') + 3\omega_2 - 4\omega_3$$

subject to constraints :

$$(\omega_1' - \omega_1'') + \omega_2 + 0\omega_3 \geq -1$$
$$-3(\omega_1' - \omega_1'') + (-2)\omega_2 - 2\omega_3 \geq -1$$
$$4(\omega_1' - \omega_1'') + 0\omega_2 + \omega_3 \geq -1.$$
$$-4(\omega_1' - \omega_1'') + 0\omega_2 - \omega_3 \geq 1.$$
$$\omega_1', \omega_1'', \omega_2, \omega_3 \geq 0$$

This dual can be written in more compact form as:

$$\text{Max } z_W = -5\omega_1 - 3\omega_2 + 4\omega_3$$

subject to constraints.

$$-\omega_1 - \omega_2 \leq 1$$
$$3\omega_1 + 2\omega_2 + 2\omega_3 \leq 1$$
$$-4\omega_1 - \omega_3 = 1$$

$\omega_2, \omega_3 \geq 0$ and ω_1 is unrestricted
which is required dual.

Q.2) a) Let R be a commutative ring with unity. An ideal M of R is maximal ideal of 'R' iff $\frac{R}{M}$ is a field.

Proof let, M be Maximal ideal of R.

Since, R is commutative ring with unity,

$\frac{R}{M}$ is also a commutative ring with unity.

Thus, all we need to prove is that non-zero elements of $\frac{R}{M}$ have multiplicative inverse.

Let; $x+M \in \frac{R}{M}$ be any non-zero element

then $\quad x+M \neq M \implies x \notin M$

Let, $\quad xR = \{xr \mid r \in R\}$.

It is easy to verify, that xR is an ideal of R.

Since, sum of two ideals is an ideal,

$\therefore$ $M+xR$ be an ideal of R.

Again, as $\quad x = 0 + x \cdot 1 \in M+xR \quad$ and $x \notin M$

we find, $\quad M \subset M+xR \subseteq R$

M, maximal $\Rightarrow M+xR = R$.

Thus; $\quad 1 \in R \Rightarrow 1 \in M+xR$

$$\Rightarrow 1 = m + xr \ ; \ \exists \ m \in M, r \in R$$

$$\begin{aligned} \Rightarrow 1+M &= (m+xr)+M \\ &= (m+M)+(xr+M) \\ &= xr+M \\ &= (x+M)(r+M). \end{aligned}$$

$\Rightarrow (r+M)$ is multiplicative inverse of $x+M$

Hence, $\frac{R}{M}$ is a field.

Conversely, let $\frac{R}{M}$ is a field.

Let, I be any ideal of R; st; $M \subset I \subseteq R$

then $\exists$ some $a \in I$; s.t $a \notin M$

Now, $a \notin M \Rightarrow a+M \neq M$

$\Rightarrow a+M$ is a non-zero element of $\frac{R}{M}$,

which being a field, means.

$a+M$ has multiplicative inverse.

Let, $b+M$ be its inverse, Then.

$$(a+M)(b+M) = 1+M$$

$$\Rightarrow ab+M = 1+M$$

$$\Rightarrow ab-1 \in M$$

$\Rightarrow ab-1 = m$ for some $m \in M$

$\Rightarrow 1 = ab-m \in I$ (using def. of ideal)

$\Rightarrow I = R$ (ideal containing unity, equals the ring)

Hence, M is maximal ideal of R.

Q.2) b) Let R_1 and R_2 be two Rings. Show that $R_1 \times R_2$ is an integral domain if and only if any one of them is an integral domain and other contains only a zero element.

Solution:

Given; R_1 and R_2 be two rings.

Let; R_1 is ID and $R_2 = \{0\}$.

To prove $R_1 \times R_2$ is ID.

Let $a \in R_1$ & $0 \in R_2$.

& $b \in R_1$

$(a,0) \in R_1 \times R_2)$ & $(b,0) \in R_1 \times R_2$

$(a,0)(b,0) = (ab,0) \in R_1 \times R_2$.

As, we know: R_1 is ID

R_1 has no zero divisors

$a \in R_1$, $b \in R_1$

and for $ab = 0$

either $a=0$ or $b=0$

So; $R_1 \times R_2$ is an ID.

Converse,

given $R_1 \times R_2$ is ID

To prove: R_1 is ID & $R_2 = \{0\}$.

Let, (a,b) & $(c,d) \in R_1 \times R$.

$(a,b)(c,d) = (0,0)$

either $(a,b) = (0,0)$ or $(c,d) = (0,0)$

as $R_1 \times R_2$ is ID, it has no zero divisors

let; $(c,d) = 0$

so, $R_2 = \{0\}$.

and $(a,b) \in R_1 \times R_2$

$ab = 0 \Rightarrow$

either $a = 0$ or $b = 0$

so; R_1 is ID.

Q.2) c) Prove that the intersection of arbitrary family of closed sets is closed, and the union of a finite family of closed is closed. Is it true for the union of an arbitrary family of closed sets? support your answer by example.

Solution:-

1) let $S_1, S_2, S_3 \ldots$ be closed set;

then $S_1^c, S_2^c, S_3^c \ldots$ be open sets

let $S = S_1 \cap S_2 \cap S_3 \ldots$

$S^c = (S_1 \cap S_2 \cap S_3 \ldots)^c$

$S^c = S_1^c \cup S_2^c \cup S_3^c \ldots$

w.k.t union of arbitrary family of open set is open.

So; S^c is open $\Rightarrow$ S is closed set.

(2) Let, $S_1, S_2, S_3 \cdots\cdots S_n$ are closed sets

then,
$S_1^c, S_2^c, S_3^c \cdots\cdots S_n^c$ are open sets.

$$S = S_1 \cup S_2 \cup S_3 \cdots\cdots \cup S_n$$
$$S^c = S_1^c \cap S_2^c \cap S_3^c \cdots\cdots \cap S_n^c$$

Since, Intersection of open sets is open

So, S^c is open set

$\Rightarrow S$ is closed set.

(3) For arbitrary family

Let; $S_n = \left[\frac{1}{n}, 1\right] \ \forall\ n \in N$

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cdots\cdots \cup S_n$$
$$= \{1\} \cup \left[\tfrac{1}{2}, 1\right] \cup \left[\tfrac{1}{3}, 1\right] \cup \cdots\cdots$$
$$= (0, 1],$$

which is not closed set,

Bcoz, '0' is limit point.

and $D(S) = \{0\} \nsubseteq S_n$

So; union of arbitrary family of closed set need not be closed.

Q.2)d) ~~Solve~~. Prove that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}$$

where, $a>b>0$

Solution:- given; $I = \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{2\sin^2\theta}{a+b\cos\theta}\,d\theta$$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{a+b\cos\theta}\,d\theta.$$

$$e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{iz}.$$

$$\cos\theta = \frac{z^2+1}{2z} \quad \& \quad \cos 2\theta = \frac{z^4+1}{2z^2}$$

$$I = \frac{1}{2}\oint_C \frac{1-\frac{z^4+1}{2z^2}}{a+b\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}.$$

$$I = \frac{1}{2i}\oint \frac{2z^2-(z^4+1)}{2za+b(z^2+1)} \cdot \frac{\cancel{2z}}{2z^2\cdot\cancel{z}}\,dz$$

$$I = -\frac{1}{2i}\oint \frac{(z^4+1-2z^2)\,dz}{z^2(2za+b(z^2+1))}$$

$$I = \frac{-1}{2ib}\oint \frac{(z^2-1)^2\,dz}{z^2(z^2+1+2z(a/b))} = \frac{-1}{2ib}\oint f(z)\,dz.$$

$$f(z) = \frac{(z^2-1)^2}{z^2\left[z^2+1+2z(a/b)\right]}$$

Now, to find the value of z.

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2} - 4}}{2} \Rightarrow z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \quad ; \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since; $a > b > 0$ $\therefore |\beta| > 1$

also $|\alpha\beta| = 1$; $\therefore |\alpha| < 1$

Thus ; $z=\alpha$; is a simple pole and $z=0$ are double pole inside 'C' ; β is outside 'C'.

Residue at $z=\alpha$

$$= \lim_{z\to\alpha} (z-\alpha)\left[\frac{-1}{2ib}\left[\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}\right]\right]$$

$$= \lim_{z\to\alpha}\left[\frac{-(z^2-1)^2}{2ib \cdot z^2(z-\beta)}\right] = \frac{-1}{2ib}\frac{[\alpha^2-1]^2}{\alpha^2[\alpha-\beta]}$$

$$= \frac{-1}{2ib}\cdot\frac{\cancel{\alpha^2}[\alpha-1/\alpha]^2}{\cancel{\alpha^2}(\alpha-\beta)} \; ; \quad \text{since; } |\alpha\beta| = 1, \; \beta = \frac{1}{\alpha}$$

$$= \frac{-1}{2ib}\frac{[\alpha-\beta]^{\cancel{2}}}{\cancel{(\alpha-\beta)}} = \frac{(\beta-\alpha)}{2ib}$$

$$= \frac{-1}{\cancel{2}ib}\cdot\frac{\cancel{2}}{b}\sqrt{a^2-b^2} = \frac{i}{b^2}\sqrt{a^2-b^2}$$

and residue at $z=0$ is

coefficient of $\frac{1}{z}$ in $\frac{-(z^2-1)^2}{2ibz^2[z^2+2(a/b)z+1]}$

$=$ coefficient of $\frac{1}{z}$ in $\frac{-1}{2ibz^2}(1-2z^2+z^4)\left(1+\frac{2a}{b}z+z^2\right)^{-1}$

$= \text{coeffi. of } \frac{1}{z} \text{ in } \frac{-1}{2ibz^2}(1-2z^2+z^4)\left(1-\frac{2a}{b}z-\cdots\right]$

$= \frac{a}{ib^2} = -\frac{ai}{b^2}$

$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta = 2\pi i \left[\begin{array}{l}\text{Sum of the residues at} \\ \text{poles inside } C\end{array}\right]$

$= 2\pi i\left[\frac{i}{b^2}\sqrt{a^2-b^2} - \frac{ai}{b^2}\right]$

$= -2\pi i^2\left[\frac{a}{b^2} - \frac{\sqrt{a^2-b^2}}{b^2}\right]$

$= +\frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$

$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}d\theta = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$

proved

Q.3) a) consider the ring $Z_6 = \{0,1,2,3,4,5\}$ mod 6. 2 is a prime element in Z_6 ~~but~~ but not irreducible.

Solution: $Z_6 = \{0,1,2,3,4,5\}_6$

Here; 2 is of course, non-zero, non-unit and we need to prove that 2 is a prime element in Z_6 but not irreducible

- For 2 be a prime element.

Suppose $2 | a \otimes b$

Since; $ab = 6q + a \otimes b$ for some 'q'

and as $2 | 6q$, $2 | a \otimes b$,

we find $2 | ab$

$\Rightarrow 2 | a$ or $2 | b$

$\Rightarrow 2 | a$ or $2 | b$ in Z_6

Hence, 2 is a prime element.

Again, now to prove 2 is irreducable or not.

as $2 \otimes 4 = 2$,

neither 2 nor 4 is a unit,

we find 2 is not irreducable.

[Note: Z_6 is not an integral domain].

Q.3(b)) If $p > 1$, show that $\Sigma\, x/(n^p + n^q x^2)$ can be differentiated term by term when $3p > q+2$. But, if $3p \le q+2$, then $f'(0)$ does not exist, where $f(x) = \Sigma \frac{x}{(n^p + n^q x^2)}$.

Solution:- Given; $p > 1$,

To prove $\Sigma \frac{x}{n^p + n^q x^2}$ can be differentiated term by term when $3p > q+2$, we need to first prove $\Sigma \frac{x}{n^p + n^q x^2}$ is uniformly convergent for $p > 1$.

Here; $f_n(x) = \dfrac{x}{n^p + n^q x^2}$

$$\frac{df_n(x)}{dx} = \frac{(n^p+n^q x^2).1 - x.2n^q x}{(n^p+n^q x^2)^2} = \frac{n^p - n^q x^2}{(n^p+n^q x^2)^2}.$$

For max. or min. $\dfrac{df_n(x)}{dx} = 0$

$$\therefore\ n^p - n^q x^2 = 0 \Rightarrow n^p = n^q x^2$$

$$\Rightarrow x^2 = n^{p-q}$$

$$\Rightarrow x = n^{\frac{p-q}{2}}$$

Also, $\dfrac{d^2 f_n(x)}{dx^2} = \dfrac{(n^p+n^q x^2)^2.(-2n^q x) - (n^p - n^q x^2).2(n^p+n^q x^2).2n^q x}{(n^p+n^q x^2)^4}$

$$\frac{d^2 f_n(x)}{dx^2} = -\frac{2n^q.x\,[(n^p+n^q x^2) + 2(n^p - n^q x^2)]}{[n^p+n^q x^2]^3}$$

$$\left.\frac{d^2 f_n(x)}{dx_2}\right]_{x=n^{\frac{p-q}{2}}} = -\frac{2n^q.\,n^{\frac{p-q}{2}}\,(n^p+n^p)}{(n^p+n^p)^3} = \frac{-1}{2}n^{\frac{q-3p}{2}} < 0$$

$$= \frac{-1}{2}\frac{1}{n^{\frac{3p-q}{2}}}$$

as $\dfrac{-q+3p}{2} > 0 \Rightarrow 3p > q+2$

$\Rightarrow f_n(x)$ is maximum at $x = n^{\frac{p-q}{2}}$, and the max. value of $f_n(x)$ is $\dfrac{n^{\frac{p-q}{2}}}{n^p+n^p} = \dfrac{1}{2n^{\frac{p+q}{2}}}$

$$\Rightarrow |f_n(x)| = \left|\frac{x}{n^p+n^q x^2}\right| \le \frac{1}{2n^{\frac{p+q}{2}}} < \frac{1}{n^{\frac{p+q}{2}}} = M_n.$$

Since. $\sum M_n = \sum \dfrac{1}{n^{\frac{p+q}{2}}}$ is convergent

if $\dfrac{p+q}{2} > 1$ or $3p > q+2$ or $\dfrac{3p-q}{2} > 0$

therefore, by Weirstress's M-test, the given series is convergent for all real x if $3p > q+2$.

Since, $f'(0)$ exist when $3p > q+2$.

when $3p \leq q+2$.

value of $f'(x) = $ constant.

$\left[\because \; n^{\frac{3p-q}{2}} = \text{constant} \right.$

as $\frac{3p-q}{2} = 0 \Rightarrow n^0 = 1$.

$\therefore \; f'(x) = -\frac{1}{2} \times 1 = -\frac{1}{2}$ (constant)

which is independent of value of 'x'

Hence; $f'(0)$ does not exist when $\underline{\underline{3p \leq q+2}}$

Q.3(c): Prove that the function $f(x) = \sin\frac{1}{x}$, $x \in (0,1)$ is not uniformly continuous on $(0,1)$.

Solution:-

given function; $f(x) = \sin\frac{1}{x}$;

Let us assume, that $f(x)$ is uniformly continuous on $(0,1)$.

Then, for every Cauchy sequence $\{x_n\}$ in $(0,1)$ the sequence $\{f\{x_n\}\}$ must be Cauchy sequence in R.

Let us, consider a sequence $\{x_n\}$.

where $x_n = \frac{2}{n\pi}$, $n \in N$.

This is a cauchy sequence in $(0,1)$

Then; The sequence $\{f(x_n)\}$

at $n=1$; $f(x_n) = \sin \pi/2 = 1$

at $n=2$; $f(x_n) = \sin \pi = 0$

at $n=3$; $f(x_n) = \sin 3\pi/2 = -1$

at $n=4$; $f(x_n) = \sin 2\pi = 0$

$\vdots$

$\therefore \{f(x_n)\} = \{1, 0, -1, 0, 1, 0 \ldots\}$

This is a divergent sequence, and therefore this is not a cauchy sequence in R. Thus, our assumption is wrong. Therefore, $f(x) = \sin 1/x$ is not uniformly continuous on $(0,1)$.

Q.3) d) Determine the optimum basic feasible solution to the following transportation problem.

To

	A	B	C	Available
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Required	4	2	2	

Solution:-

The initial basic feasible solution can be easily obtained by two different methods as follows:

(i) By Lowest cost Method

	1 (30)		1
2 (90)	1 (45)		3
2 (250)		2 (50)	4
4	2	2	

cost $z = 1\times30 + 2\times90 + 1\times45 + 2\times250 + 2\times50$

Cost $z = ₹ 855.$

(ii) By Vogel's Method:

1 (50)	(30)	(220)	1
3 (90)	(45)	(170)	3
(250)	2 (200)	2 (50)	4
4	2	2	

$\Sigma a_i = \Sigma b_j = 8$

$\therefore$ The problem is balanced.

No. of allocation $= m+n-1 = 5$

Cost $z = 1\times50 + 3\times90 + 2\times200 + 2\times50$

$= 50 + 270 + 400 + 100$

Cost $z. = ₹ 820$

Since, number of allocation = 5 and the cells allocated in Vogel's method is 4. Hence, the Initial basic solution of LC method is used; i.e

	A	B	C	
I		1(30)		1
II	2(90)	1(45)		3
III.	2(250)		2(50)	4
	4	2	2	

cost z = ₹ 855.

Now, to check the optimality of the solution using u-v method.

As, the max. no. of basic cells exists in 2^{nd} & 3^{rd} rows, we can start by putting either $u_2 = 0$ or $u_3 = 0$

Let, us put $u_2 = 0$

For basic cells we know

$\Delta_{ij} = u_i + v_j - c_{ij} = 0 \Rightarrow u_i + v_j = c_{ij}$

$u_2 + v_1 = 90 \Rightarrow v_1 = 90$

$u_2 + v_2 = 45 \Rightarrow v_2 = 45$

$u_3 + v_1 = 250 \Rightarrow u_3 = 160 \Rightarrow$

$u_3 + v_3 = 50 \Rightarrow v_3 = -110$

$u_1 + v_2 = 30 \Rightarrow u_1 = -15$

	A	B	C		u_{ij}
I		1(30)		1	-15
II	2(90)	1(45)		3	0
III	2(250)		2(50)	4	160
	4	2	2		
v_{ij}	90	45	-110		

For non-basic cells:

$\Delta_{ij} = c_{ij} - (u_i + v_j)$.

$\Delta_{11} = 50 - (-15 + 90) = -25$

$\Delta_{13} = 220 - (-15 + (-110)) = 345$

$\Delta_{23} = 170 - (0 + 110) = 280$

$\Delta_{32} = 200 - (205) = -5$

Table of c_{ij}

50		220
		170
	200	

Table of Δ_{ij} ⇒

-25		345
		280
	-5	

Here, most negative all are non-negative hence solution is not optimal.

Hence, most negative Δ_{ij} term is (-25) at cell (1,1). So, we need to alot maximum to it.

∴ As this necessiciate the shifting of 1 unit to the cell (1,1) from cell (1,2), as directed by the closed loop in below table:

Hence, cell (1,1) enters the solution and cell (1,2) leaves it, and it become empty or non-basic cell.

$+\theta$	$1-\theta$		1
$2-\theta$	$1+\theta$		3
			4
4	2	2	

Here : $\min [1-\theta, 2-\theta] = 1-\theta$

$1-\theta = 0$

$\boxed{\theta = 1 \text{ unit}}$

Now; the Improved basic table is

	A	B	C	
I	1(50)			1
II	1(90)	2(45)		3
III	2(250)		2(50)	4
	4	2	2	

Cost of $Z = 1\times50 + 1\times90 + 2\times45 + 2\times250 + 2\times50$
$= 50 + 90 + 90 + 500 + 100$

$\boxed{\text{Cost } Z = 830}$

Now, check the optimality of this solution.

Here; colum 1 has maximum basic cells.

Hence; either V_1 or V_2 or $V_3 = 0$

Let us take $V_1 = 0$

$V_1 + U_1 = 50 \Rightarrow U_1 = 50$ $\quad$ $V_2 + U_2 = 45 \Rightarrow V_2 = -45$

$V_1 + U_2 = 90 \Rightarrow U_2 = 90$ $\quad$ $V_3 + U_3 = 50 \Rightarrow V_3 = -200$

$V_1 + U_3 = 250 \Rightarrow U_3 = 250$

Matrix $[C_{ij}]$ for empty / non-basic cell

	30	220
		170
	200	

$\Rightarrow$ Matrix $(u_i + v_j)$ for empty cells

			u_i
	5	−150	50
		−110	90
	205		250
v_j: 0	−45	−200	

Hence, by solving

Matrix for $[\Delta_{ij} = C_{ij} - (u_i + v_j)] \Rightarrow$

Δ_{ij}

	25	370
		280
	−5	

Here, all are not non-negative, hence solution is not optimal.

Hence, most negative term of Δ_{ij} is −5 at cell (3,2). So, we need to alot maximum to it.

∴ Second iteration.

Since, the cell (3,2) enters the solution while cell (3,1) leaves the solution. (2,2) may also selected for leaving. Shown in table.

Here; $2 - \theta = 0$

$\boxed{\theta = 2}$

$1+\theta$	$2-\theta$	
$2-\theta$	$+\theta$	

Hence; improved basic cell table is –

	A	B	C	
I	1(50)			1
II	3(90)	0(45)		3
III		2(200)	2(50)	4
	4	2	2	

Cost $Z = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50$

Cost $Z = 50 + 270 + 400 + 100$

Cost $Z = 820$ = Cost Z of Vogel's IBFS.

Again proceed as earlier to test the next improved solution for optimality.

It has been observed that all cell evaluation are now non-negative. Hence, the solution under test is optimal.

$\boxed{Z = ₹820}$

This solution was initially obtained by Vogel's method.

Q.4) a) Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad \neq 0$, under matrix multiplication. Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$. Prove that

a) N is a normal subgroup of G.

b) G/N is an abelian.

Solution: Let us define a mapping

$f: G \longrightarrow \mathbb{R}^*$ given by

$f(B) = \det(B) \quad \forall \ B \in G$

consider ; $f(A) = f(B)$.

$\Rightarrow \det(A) = \det(B)$

$\Rightarrow \quad A = B$ (By property of matrices)

$\therefore f$ is well defined

Now, to show that f is a group homomorphism

i.e T.S.T $f(AB) = f(A) \cdot f(B)$.

consider $f(AB) = \det(AB) = \det(A) \cdot \det(B)$.

If A & B are both $n \times n$ matrices.

then $\det(AB) = \det(A) \cdot \det(B)$.

$\therefore$ f is group homomorphism.

Also ; $f(N) = \{1\}$.

$\because \{1\}$ is a nomal subgroup of $\mathbb{R}^*$

[$\mathbb{R}^*$ being an commutative abelian group, since, every subgroup of an abelian group is Normal.

$\Rightarrow$ N is a normal of G.

ii) To prove $\frac{G}{N}$ is an abelian.

Let $A, B \in G$

suppose ; $NANB = NBNA$

$$NAB = NBA$$

$$ABA^{-1}B^{-1} \in N$$

let; $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$; $B = \begin{bmatrix} m & n \\ 0 & p \end{bmatrix}$

$A^{-1} = \frac{1}{ad}\begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$; $B^{-1} = \frac{1}{mp}\begin{bmatrix} p & -n \\ 0 & m \end{bmatrix}$

So;

$$ABA^{-1}B^{-1} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} m & n \\ 0 & p \end{bmatrix}\begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} \\ 0 & \frac{1}{d} \end{bmatrix}\begin{bmatrix} \frac{1}{m} & \frac{-n}{mp} \\ 0 & \frac{1}{p} \end{bmatrix}$$

which is in the form of $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N$

So; $ABA^{-1}B^{-1} \in N$

Hence; $\frac{G}{N}$ is abelian.

Q4(b) Show that the function $f(x) = \sin\frac{1}{x}$; x is irrational
$= 0$; otherwise
is not Reimann integral on $[0,1]$.

Solution: Given;

$$f(x) = \sin\frac{1}{x} \text{ ; when } x \text{ is irrational}$$
$$= 0 \text{ ; otherwise}$$

we need to prove not reimann integral on $[0,1]$

Every subinterval of $[0,1]$ consists of both rational and irrational numbers 'x'.

Suppose; $x \in \left[\frac{2}{\pi}, 1\right]$

$f(x) = \sin\frac{1}{x}$ is monotonic on $\left[\frac{2}{\pi}, 1\right]$

$$U(P,f) > \left(1 - \frac{2}{\pi}\right)\sin 1$$

$$L(P,f) = 0$$

We know that $f \in R[a,b]$ iff

$\exists$ $\epsilon > 0$ such that

$$U(P,f) - L(P,f) < \epsilon$$

But ; $U(P,f) - L(P,f) > \left(1 - \frac{2}{\pi}\right)\sin 1$.

So; $f(x) = \sin\frac{1}{x}$ is not Reimann integral on $[0,1]$.

Q. 4(c) Using Cauchy's / Cauchy's Integral formula evaluate.

(i) $\int_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1| = 1$.

Solution: If $f(z) = \frac{z+4}{z^2+2z+5}$,

then the poles of $f(z)$ are given by.

$$z^2+2z+5 = 0$$

$$\therefore z = -1+2i \ , -1-2i$$

when $z = -1+2i$; then $|z+1| = |\cancel{-1}+2i+\cancel{1}| = 2 > 1$

$\therefore$ The pole $z = -1+2i$ lies outside the circle $|z+1| = 1$.

Since; $z = -1-2i$; then

$$|z+1| = |\cancel{-1}-2i+\cancel{1}| = 2 > 1.$$

This pole $z = -1-2i$ also lies outside the circle.

Hence, $f(z)$ is analytic everywhere within C.

Also $f'(z)$ is continuous within and on C.

By applying Cauchy's theorem, we get.

$$\int_C f(z)\,dz = 0$$

i.e $\int_C f(z)dz = \int_C \frac{z+4}{z^2+2z+5} dz = 0.$

(ii) $\oint_c \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$; If C is the circle $|z|=1$.

<u>Solution</u>:

given: $\oint_c \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$.

comparing the given interval with

$$\oint_c \frac{f(z)}{(z-z_0)^n}\,dz.$$

we get;

$$f(z) = \sin^6 z$$

$$\& \ z_0 = \pi/6$$

Since, $f(z)$ is an analytic in $|z|=1$.

and $z_0 = \pi/6$ is a point inside $|z|=1$.

we apply cauchy's Integral formula.

$$\oint_c \frac{f(z)\,dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0). \quad \text{---} (1)$$

Since, here ; $n=3$; $n-1=2$

and $f(z) = \sin^6 z = (\sin z)^6$

$$f'(z) = 6\sin^5 z \cdot \cos z$$

$$f''(z) = 6\left[\sin^5 z \cdot (-\sin z) + \cos z \cdot 5\sin^4 z \cdot \cos z\right]$$

$$f''(z) = 6\left[-\sin^6 z + \cos^2 z \cdot 5\sin^4 z\right]$$

$$f''(z) = 6\left[5\sin^4 z \cdot \cos^2 z - \sin^6 z\right]$$

put $z = z_0 = \pi/6$

$$f''(\pi/6) = 6\left[5 \times \frac{1}{16} \times \frac{3}{4} - \frac{1}{64}\right] = 6\left[\frac{15}{64} - \frac{1}{64}\right]$$

$$f''(\pi/6) = \frac{3\cancel{6} \times \cancel{14}\,7}{\cancel{64}\,\cancel{32}\,16} = \frac{21}{16}.$$

Put above value in (1)

$$\oint \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{\cancel{2}\pi i}{\cancel{2}} \times \frac{21}{16} = \frac{21\pi i}{16}.$$

Q.4(d) Maximize $z = 4x_1 + 5x_2 - 3x_3 + 50$

Subject to constraints:

$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 \geq 1.$$
$$2x_1 + 3x_2 + x_3 \leq 40$$
$$x_1, x_2, x_3 \geq 0$$

Solution:-

If any constant is included in the objective function; i.e $z = 4x_1 + 5x_2 - 3x_3 + 50$

It should be deleted in the beginning and finally adjusted in the optimum value of z.

And for equality in the contraints, then one variable can be eliminated from the inequalities with $\leq$ or $\geq$ sign. Now, the given equations become;

$$\text{Max. } z = 4x_1 + 5x_2 - 3x_3$$

S.t.C
$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 \geq 1$$
$$x_1 + 2x_2 \leq 30$$
$$x_1, x_2, x_3 \geq 0$$

Now, to introduce the slack, surplus and artificial variables, the problem becomes.

$$\text{Max } z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$$

Subject to constraints

$$x_1 + x_2 + x_3 = 10$$
$$x_1 - x_2 - x_4 + a_1 = 1$$
$$x_1 + 2x_2 + x_5 = 30$$
$$x_1, x_2, x_3, x_4, x_5, a_1 \geq 0$$

This is the standard form for simplex method.

		$C_j \rightarrow$	4	5	-3	0	-M	0	
Basic variable	C_B	x_B	x_1	x_2	x_3	x_4	a_1	x_5	Min. Ratio x_B/x_K
x_3	-3	10	1	1	1	0	0	0	10/1
$\leftarrow a_1$	-M	1	$\boxed{1}$	-1	0	1	1	0	$1/1 \leftarrow$
x_5	0	30	1	2	0	0	0	1	30/1
	$Z=-30-M$		$-7-M$	$-8+M$	0	M	0 $\downarrow$	0	
$\leftarrow x_3$	-3	9	0	$\boxed{2}$	1	1	X	0	$9/2 \leftarrow$
$\rightarrow x_1$	4	1	1	-1	0	-1	X	0	—
x_5	0	29	0	3	0	1	X	1	29/3
	$Z=9/2$		0	-15 $\uparrow$	0 $\downarrow$	-7	X	0	$\leftarrow \Delta_j$
$\rightarrow x_2$	5	9/2	0	1	1/2	1/2	X	0	
x_1	4	11/2	1	0	1/2	-1/2	X	0	
x_5	0	31/2	0	0	-3/2	-1/2	X	1	
	$Z=89/2$		0	0	15/2	1/2	X	0	$\leftarrow \Delta_j \geq 0$

Hence, we got the optimum solution.

Hence, the solution is; $x_1 = 11/2$
$x_2 = 9/2$
$x_5 = 31/2. \rightarrow$ ~~Not~~ used
$x_3 = 0$

$$\therefore \text{Max } Z = 4x_1 + 5x_2 - 3x_3 + 0x_4 - Ma_1 + 0x_5$$

$$= 4\times\frac{11}{2} + 5\times\frac{9}{2} - 3\times 0 + 0\times x_4 - Ma_1 + 0\times\frac{31}{2}$$

$$\text{Max } Z = \frac{89}{2}.$$

Optimal Solution $\rightarrow$ $\text{Max } Z = \frac{89}{2} + 50 = \frac{189}{2}$.

$$\boxed{\therefore \text{ Max } Z = 189/2}$$ A

Section - B

Q.5 (a) Find a complete integral of $z^2(p^2+q^2) = x^2+y^2$

i.e. $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2.$

Solution:

Given equation can be written as:

$$z^2\left[\frac{\partial z}{\partial x}\right]^2 + z^2\left[\frac{\partial z}{\partial y}\right]^2 = x^2+y^2$$

or

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2 \quad \text{--- (1)}$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$ — (2)

using (2), (1) becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2+y^2 \quad \text{or}$$

$$P^2 + Q^2 = x^2+y^2,$$

where $P = \frac{\partial Z}{\partial x}$, $Q = \frac{\partial Z}{\partial y}$

Separating P and x from Q and y, we get

$$P^2 - x^2 = -Q^2 + y^2$$

Equating each side to an arbitrary constant a^2

we get $P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$

So that $P = \sqrt{a^2+x^2}$ and $Q = \sqrt{y^2-a^2}$

Putting these values of P & Q in

$dZ = P\,dx + Q\,dy$, we have.

$$dz = (a^2+x^2)^{1/2}dx + (y^2-a^2)^{1/2}dy$$

Integrating z:

$$z = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\{x+\sqrt{a^2+x^2}\} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\log\{y+\sqrt{y^2-a^2}\} + \frac{1}{2}b$$

[Since, $\frac{z^2}{2}=z$]

Or $\Rightarrow z^2 = x\sqrt{a^2+x^2} + a^2\log\{x+\sqrt{a^2+x^2}\} + y\sqrt{y^2-a^2} - a^2\log\{y+\sqrt{y^2-a^2}\} + b.$

$$\therefore \boxed{z^2 = x\sqrt{x^2+a^2} + a^2\log\{x+\sqrt{x^2+a^2}\} + y\sqrt{y^2-a^2} + b - a^2\log\{y+\sqrt{y^2-a^2}\}.}$$

is required solution

Q.5 (b) Solve $[D^3-4D^2D'+5D'^2D-2D'^3]z = e^{y+2x}+(y+x)^{1/2}$

Solution :-

Given Partial D.E.

$$[D^3-4D^2D'+5DD'^2-2D'^3]z = e^{y+2x}+(y+x)^{1/2}$$

Here, auxillary equation is → putting $D=m$, $D'=1$.

$$m^3-4m^2+5m-2=0$$

$$(m-1)^2(m-2)=0$$

$$m=1,1,2.$$

$$\therefore C.F = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x).$$

Now; P.I. corresponding to

$$e^{y+2x}$$

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} \cdot e^{y+2x}$$

$$= \frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2} e^{y+2x}\right\}$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{(2-1)^2} \iint e^v \, dv\, dv \text{ ; where } v = y+2x.$$

$$= \frac{1}{D-2D'} \int e^v dv = \frac{1}{D-2D'} \cdot e^v$$

$$= \frac{1}{(1\cdot D - 2\cdot D')^1} e^{y+2x} = \frac{x}{1\cdot 1!} e^{y+2x} = xe^{y+x} \quad \text{(2)}$$

[Using formula; $a=2$, $b=1$, $m=1$.].

Finally P.I corresponding to $(y+x)^{1/2}$

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} (y+x)^{1/2}$$

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'} (y+x)^{1/2}\right\}$$

$\Rightarrow$ let $y+x = u$ then

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{1-2\cdot 1} \int u^{1/2} du\right\}$$

$$= -\frac{1}{(D-D')^2} \cdot \frac{2}{3} v^{3/2} = -\frac{2}{3} \cdot \frac{1}{(D-D')^2} (y+x)^{3/2}$$

$$= -\frac{\cancel{2}}{3} \cdot \frac{x^2}{1^2 \cdot \cancel{2}!} (y+x)^{3/2} = -\frac{x^2}{3} (y+x)^{3/2}$$

[where, $a=b=1$, $m=2$] — (3)

From ①, ② and ③, the required general solution is

$$\boxed{z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) + xe^{y+x} - \left(\frac{x^2}{3}\right)(y+x)^{3/2}}$$

which is required.

Q.5(c):- The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5.

Solution:

Here bacteria concentration $C = 0.5$,

so the given equation

$$C = 4e^{-2t} + e^{-0.1t} \quad \text{become}$$

$$0.5 = 4e^{-2t} + e^{-0.1t}$$

$$\therefore f(t) = 4e^{-2t} + e^{-0.1t} - 0.5$$

$$f'(t) = -8e^{-2t} - 0.1e^{-0.1t}.$$

Clearly, $f(t)$ and $f'(t)$ are continuous everywhere

let the initial approximation is $t_1 = 1$

and the Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(n)}{f'(n)} \; ; \; n = 0, 1, 2, 3, \ldots \quad \text{—①}$$

Now; put $t_1 = 1$.

$$t_2 = t_1 - \frac{f(t)}{f'(t)}$$

$$t_2 = 1 - \frac{f(1)}{f'(1)}$$

$$t_2 = 1 - \frac{0.94617}{-(1.17317)} = 1 + \frac{0.94617}{1.17317}$$

$$t_2 = 1.8065$$

Now;

$$t_3 = t_2 - \frac{f(t_2)}{f'(t_2)} = 1.8065 - \frac{0.4426}{-0.2992}$$

$$t_3 = 1.8065 + \frac{0.4426}{0.2992} = 3.28565$$

Now;

$$t_4 = t_3 - \frac{f(t_3)}{f'(t_3)} = 3.28565 - \frac{0.22555}{-(0.0832)}$$

$$t_4 = 3.28565 + \frac{0.22555}{0.0832} = 5.99681$$

Now:

$$t_5 = t_4 - \frac{f(t_4)}{f'(t_4)} = 5.99681 - \frac{0.04901}{-(0.05495)}$$

$$t_5 = 5.99681 + \frac{0.04901}{0.05495} = 6.888764$$

Now

$$t_6 = t_5 - \frac{f(t_5)}{f'(t_5)} = 6.888769 - \frac{0.00214}{-(0.05022)}$$

$$t_6 = 6.888769 + \frac{0.00214}{0.05022} = 6.93145$$

Now

$$t_7 = t_6 - \frac{f(t_6)}{f'(t_6)} = 6.93145 - \frac{0.000458}{-(0.05001)}$$

$$t_7 = 6.93145 + \frac{0.000458}{0.05001} = 6.93154$$

$$t_6 \approx t_7 = 6.931$$

Hence; the time required for bacteria concentration to be 0.5 is $t = 6.931$ seconds.

Ques. 5) d) Convert the following

(i) 7765_8 to decimal

Sol: $\Rightarrow 7\times8^3+7\times8^2+6\times8^1+5\times8^0$

$= 7\times512+7\times64+6\times8+5\times1$

$= 3584+448+48+5$

$= \underline{\underline{4085}}.$

ii) 199.3 to octal and then to binary.

Sol: $\cancel{1\times8^2+9\times8}$

$$\begin{array}{r|c|l} 8 & 199 & \\ \hline 8 & 24 & 7 \\ \hline & 3 & 0 \end{array}$$

$0.3\times8 = 2.4 \quad 2$
$0.4\times8 = 3.2 \quad 3$
$.2\times8 = 1.6 \quad 1$

$(0.3)_{10}\to(0.231)_8$

$(199)_{10}\to(307)_8$

$\therefore (199.3)_{10}\to(307.231)_8.$

Now to binary from octal.

3 0 7 . 2 3 1
011 000 111 . 010 011 001

$\Rightarrow (199.3)_{10}\to(307.231)_8\to(011000111.010011001)_2$

iii) $FB17_{16}$ to Binary.

F B 1 7
1111 1011 0001 0111 $\Rightarrow$

$(FB17)_{16}\to(1111\ 1011\ 0001\ 0111)_2.$

(iv) $641A_{16}$ to octal

$$\begin{array}{cccc} 6 & 4 & 1 & A \end{array}$$
$(0110\ 0100\ 0001\ 1010)_2$.

Now Binary to Octal.

$0\underline{110}\,\underline{010}\,\underline{000}\,\underline{011}\,\underline{010}$
$\quad 6 \quad 2 \quad 0 \quad 3 \quad 2$

$\Rightarrow (641A)_{16} \to (62032)_8$. Ans.

Q.5 (c) Show that $u = 2cxy$, $v = c(a^2+x^2-y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.

Solution:

Given; $u = 2cxy$ and $v = c(a^2+x^2-y^2)$ —(1)

Equation of continuity in xy-plane is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{—(2)}$$

from (1)

$$\frac{\partial u}{\partial x} = 2cy \quad ; \quad \frac{\partial v}{\partial y} = -2cy$$

putting these values in (2), we get

$$\boxed{2cy - 2cy = 0}$$

Hence, equation of continuity satisfied; hence u and v constitute a possible fluid motion.

Let, ψ be the required stream function. Then, we have

$$u = -(\partial\psi/\partial y) \quad \text{or} \quad \partial\psi/\partial y = -2cxy \quad \text{— (3)}$$

$$v = \partial\psi/\partial x \quad \text{or} \quad \partial\psi/\partial x = c(a^2+x^2-y^2) \text{ — (4)}$$

Integrating (3) w.r.t y

$$\psi = -2cx + \phi(x,t) \quad \text{— (5)}$$

where, $\phi(x,t)$ is an arbitrary function of 'x' & 't'.

Differentiate (5) partially, w.r.t 'x'

$$\frac{\partial\psi}{\partial x} = -cy^2 + \partial\phi/\partial x \quad \text{— (6)}$$

from (4) & (6)

$$-cy^2 + \partial\phi/\partial x = c(a^2+x^2-y^2)$$

$$\partial\phi/\partial x = c(a^2+x^2) - \cancel{cy^2} + \cancel{cy^2}$$

$$\boxed{\partial\phi/\partial x = c(a^2+x^2)} \quad \text{— (7)}$$

Integrating (7), partially w.r.t x

$$\phi(x,t) = c(a^2x + x^3/3) + \varphi(y,t)$$

where, $\varphi(y,t)$ is an arbitrary function of y & t.

Substituting the above value of $\phi(x,t)$, in (5), we get

$$\psi = -2cx + c(a^2x + x^3/3) + \varphi(y,t)$$

$$\boxed{\psi = c(a^2x - 2x + x^3/3) + \varphi(y,t)}$$

which is the required stream function.

Q.6) a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x^2+y^2+z^2, z^2-2xy)=0$

Solution: Given; $\phi(x^2+y^2+z^2, z^2-2xy)=0$ —(1)

Let; $u=x^2+y^2+z^2$ and $v=z^2-2xy$ —(2)

Then, (1) becomes $\phi(u,v)=0$ —(3)

Differentiate (3), partially, w.r.t 'x', we get.

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{—(4)}$$

where, $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$.

from (2), we have

$$\left.\begin{aligned}\frac{\partial u}{\partial x}=2x,\ \frac{\partial u}{\partial y}=2y,\ \frac{\partial u}{\partial z}=2z\\ \frac{\partial v}{\partial x}=-2y,\ \frac{\partial v}{\partial y}=-2x,\ \frac{\partial v}{\partial z}=2z\end{aligned}\right] \quad \text{—(5)}$$

Using (5), (4) reduces to

$$\frac{\partial\phi}{\partial u}(2x+2pz)+\frac{\partial\phi}{\partial v}(-2y+2pz)$$

or $$2(x+pz)\frac{\partial\phi}{\partial u}=2(y-pz)\frac{\partial\phi}{\partial v} \quad \text{—(6)}$$

Again; differentiate (3) partially, w.r.t 'y', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

or $$\frac{\partial\phi}{\partial u}(2y+2qz)+\frac{\partial\phi}{\partial v}(-2x+2qz)=0 \text{ — by (5)}$$

or $$2(y+qz)\frac{\partial\phi}{\partial u}=2(x-qz)\frac{\partial\phi}{\partial v} \quad \text{—(7)}$$

Dividing (6) by (7), we get

$$\frac{(x+pz)}{(y+qz)} = \frac{y-pz}{x-qz}$$

or $pz(y+x) - qz(y+x) = y^2 - x^2$

$$(p-q)(z\cancel{(y+x)}) = (y-x)\cancel{(y+x)}$$

$\boxed{(p-q)z = y-x}$ is required solution.

Ques 6 (b). Reduce $x\left(\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2 z}{\partial y^2} = x^2$, $(x>0)$ to canonical form.

Solution :-

given equation

$$x\left(\frac{\partial^2 z}{\partial x^2}\right) + \frac{\partial^2 z}{\partial y^2} = x^2 \quad , x>0$$

can be rewritten as

$$xr + t = x^2$$

$$xr + t - x^2 = 0 \ ; \ x>0 \quad \text{——— (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

Here, $R = x$, $S = 0$, $T = 1$, $f(x,y,z,p,q) = -x^2$

so that $S^2 - 4RT = -4x < 0$

showing that (1) is elliptic

The λ- quadratic equation $R\lambda^2 + S\lambda + T = 0$

reduces to $x\lambda^2 + 1 = 0$ or $\lambda^2 = \frac{-1}{x}$

$$\boxed{\lambda = \frac{i}{\sqrt{x}}, \frac{-i}{\sqrt{x}}}$$

The corresponding characteristic equations are given by

$$\frac{dy}{dx} + i x^{-1/2} = 0 \quad \text{and} \quad \frac{dy}{dx} - i x^{-1/2} = 0$$

Integrating these;

$$y + 2i x^{1/2} = c_1 \quad \text{and} \quad y - 2i x^{1/2} = c_2$$

choose; $u = y + 2i x^{1/2} = \alpha + i\beta$

$v = y - 2i x^{1/2} = \alpha - i\beta$

where $\alpha = y$ & $\beta = 2x^{1/2}$ —— (2)

are now two independent variables.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta}$ — (3) (By (2))

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}$ - by (2) — (4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial z}\left(x^{-1/2}\cdot\frac{\partial z}{\partial \beta}\right)$, using (3)

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\frac{1}{2}x^{-3/2}\cdot\frac{\partial z}{\partial \beta} + x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right)$$

$$= -\frac{1}{2x^{3/2}}\cdot\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \quad —— (5)$$

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$ (using 4) — (6)

using (5) & (6), in (1), the required canonical form is

$$x\left[-\frac{1}{2x^{3/2}}\cdot\frac{\partial z}{\partial \beta}+\frac{1}{x}\cdot\frac{\partial^2 z}{\partial \beta^2}\right)+\frac{\partial^2 z}{\partial \alpha^2}=x^2$$

$$= -\frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}+\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}=x^2$$

$$\therefore\ \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=x^2+\frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta} \qquad \text{--- (7)}$$

$\Rightarrow$ as $\beta = 2x^{1/2}$.

$$(7) \approx \boxed{\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\frac{\beta^4}{16}+\frac{1}{\beta}\left(\frac{\partial z}{\partial \beta}\right)}$$

which is required canonical form.

Q.6) d) Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length, whose one end is kept at $10°C$ and other end is insulated. Further, it is given that $y(x,0) = 1-x$, $0<x<1$.

Solution:-

Suppose the bar be placed along the x-axis with its one end (which is at $10°C$) at origin and the other end at $x=1$ (which is insulated so the flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity).

Then we are to solve

$$\frac{\partial y}{\partial t} = k\left(\partial^2 y/\partial x^2\right). \quad \text{——} (1)$$

with B.C. $y_x(1,t)=0$; $y(0,t)=10$ —— (2)

with I.C. $y(x,0)=1-x$; $0<x<1$ —— (3)

Let ; $y(x,t) = u(x,t)+10$ —— (4)

i.e $u(x,t) = y(x,t)-10$ —— (5)

using (4) or (5) in (1), (2) and (3) reduce to

$$\frac{\partial u}{\partial t} = k\left(\partial^2 u/\partial x^2\right) \quad \text{——} (6)$$

$u_x(1,t)=0$, $u(0,t)=0$ —— (7)

$u(x,0)=y(x,0)-10 = -(x+9)$ —— (8)

Suppose that (6) has solutions of the form

$$u(x,t) = X(x)T(t) \quad \text{——} (9)$$

substituting this value of 'u' in (6), we get

$$XT' = kX''T \text{ or } \boxed{\frac{X''}{X} = \frac{T'}{kT}} \quad \text{——} (10)$$

Since, x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ.

$$\therefore X'' - \mu X = 0 \quad \text{---} (11)$$

$$T' = \mu k T \quad \text{---} (12)$$

Using (7), (9) gives

$$X'(1)T(t) = 0 \text{ and } X(0)T(t) = 0 \quad \text{---} (13)$$

Since, $T(t) = 0$, leads to $u = 0$,
so we suppose that $T(t) \neq 0$

$$\therefore \text{from (13)} \quad X'(1) = 0 \text{ and } X(0) = 0 \quad \text{---} (14)$$

We now solve (11) under B.C. (14).
Three cases arises.

Case(i) let $\mu = 0$. Then solution of (11) is

$$X(x) = Ax + B \quad \text{---} (15)$$

from (15) $\quad X'(x) = A \quad \text{---} (15')$

Using B.C. (14), (15) and (15)' gives $0 = A$ & $0 = B$
So from (15), $X(x) \equiv 0$,
which lead to $u = 0$. So reject $\mu = 0$

Case ii let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{---} (16)$$

So that $\quad X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} \quad \text{---} (16)'$

Using B.C. (14), (16) & (16)', give

$$0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda} \text{ and } 0 = A + B$$

These give $A = B = 0$ so that $X(x) = 0$ and hence $u(x) \equiv 0$ and so we reject $\mu = \lambda^2$.

Case III) Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{---(17)}$$

so that $X'(x) = -A\lambda\sin x + B\lambda\cos x$ ---(17)'

using B.C. (14), (17) & (17)' give

$$0 = -A\lambda\sin x + B\lambda\cos x \text{ and } 0 = A$$

These gives ; $A = 0$ and $\cos\lambda = 0$ ---(18)

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u = 0$

Now, $\cos\lambda = 0 \Rightarrow \lambda = \frac{1}{2}(2n-1)\pi$; $n = 1, 2, 3 \ldots$

$$\therefore \mu = -\lambda^2 = -\frac{1}{4}(2n-1)^2\pi^2 \quad \text{---(19)}$$

Hence, non-zero solution $X_n(x)$ of (17) are given by

$$X_n(x) = B_n \sin\left\{\tfrac{1}{2}(2n-1)\pi x\right\}$$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4} T \text{ or } \frac{dT}{T} = -C_n^2 dt \quad \text{---(20)}$$

where ; $C_n^2 = +\frac{1}{4}(2n-1)^2\pi^2 K$ ---(21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ ---(22)

So; $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7). Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{---(23)}$$

Putting $t=0$ in (23) and using (8), we have

$$-(x+9)=\sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \quad \text{--- (24)}$$

Multiply both sides of (24) by $\sin\left\{\frac{1}{2}(2m-1)\pi x\right\}$ and then integrating w.r.t 'x' from 0 to 1, we get.

$$\Rightarrow -\int_0^1 (x+9)\sin\left\{\frac{1}{2}(2m-1)\pi x\right\}dx.$$

$$=\sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\cdot\sin\frac{(2m-1)\pi x}{2}\,dx. \quad \text{--- (25)}$$

But $\int_0^1 \sin\frac{(2n-1)\pi x}{2}\cdot\sin\frac{(2m-1)\pi x}{2}\,dx = \begin{cases}0, & \text{if } m\neq n\\ 1; & \text{if } m=n\end{cases}$ --- (26)

using (26), (25) gives.

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$$

$$\therefore E_m = -\int_0^1 (x+9)\sin\frac{(2n-1)}{2}\pi x\,dx$$

$$\therefore E_m = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\}-(1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2\pi^2}{4}}\right\}\right]_0^1$$

[on using chain rule of integration by parts].

$$\therefore E_m = \frac{8(-1)^n}{(2n-1)^2\pi^2}-\frac{36}{(2n-1)\pi}\left\{\begin{array}{l}\because \cos\frac{(2n-1)\pi}{2}=0\\ \text{and } \sin\frac{(2n-1)\pi}{2}=(-1)^{n-1}\end{array}\right\} \quad \text{--- (27)}$$

using (23) and (24), the required solution is given by

$$y(x,t)=10+\sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}\cdot e^{-c_n^2 t}$$

where C_n and E_n are given by (21) and (27) respectively.

Q.7 (a) Solve the equations:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss - Seidal iteration method.

Solution:

Rewriting the given equations as.

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{—(i)}$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{—(ii)}$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{—(iii)}$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3. \quad \text{—(iv)}$$

First iteration

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in (i) we get $x_1 = 0.3$

Putting $x_1 = 0.3$, $x_3 = 0$, $x_4 = 0$ in (ii) we get $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in (iii) we get $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in (iv) we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$, in (i) $\Rightarrow x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$, in (ii) $\Rightarrow x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$; in (iii) $\Rightarrow x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$; in (iv) $\Rightarrow x_4 = -0.0248$

third iteration

Putting ~~$x_1 = 0.8869$~~

$x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (i) $\Rightarrow x_1 = 0.9836$

$x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (ii) $\Rightarrow x_2 = 1.9899$

$x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$ in (iii) $\Rightarrow x_3 = 2.9924$

$x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$ in (iv) $\Rightarrow x_4 = -0.0042$

Fourth iteration:

Putting

$x_2 = 1.9899$, $x_3 = 2.9924$, $x_4 = -0.0042$ in (i) $\Rightarrow x_1 = 0.9968$

$x_1 = 0.9968$, $x_3 = 2.9924$, $x_4 = -0.0042$ in (ii) $\Rightarrow x_2 = 1.9982$

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_4 = -0.0042$ in (iii) $\Rightarrow x_3 = 2.9987$

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$ in (iv) $\Rightarrow x_4 = -0.0008$

fifth iteration

Putting

$x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$ in (i) $\Rightarrow x_1 = 0.9994$

$x_1 = 0.9994$, $x_3 = 2.9987$, $x_4 = -0.008$ in (ii) $\Rightarrow x_2 = 1.9997$

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_4 = -0.008$ in (iii) $\Rightarrow x_3 = 2.9997$

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$ in (iv) $\Rightarrow x_4 = -0.0001$

Sixth iteration

Putting

$x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001$ in (i) $\Rightarrow x_1 = 0.9999$

$x_1 = 0.9999$, $x_3 = 2.9997$, $x_4 = -0.0001$ in (ii) $\Rightarrow x_2 = 1.9999$

$x_1 = 0.9999$, $x_2 = 1.9999$, $x_4 = -0.0001$ in (iii) $\Rightarrow x_3 = 2.9999$

$x_1 = 0.9999$, $x_2 = 1.9999$, $x_3 = 2.9999$ in (IV) $\Rightarrow x_4 = -0.0001$

Hence the solution for the given equations is

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 0 \end{aligned}}$$

Q 7(b): The velocity 'v' of a particle at a distance 's' from a point on its path is given by the table:

s ft :	0	10	20	30	40	50	60
v ft/sec:	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

Solution: We know that

$$v = \frac{ds}{dt}$$

$$dt = \int \frac{ds}{v}$$

$$\Rightarrow t = \int dt = \int \frac{ds}{v} \quad \text{——} (1)$$

So,

S	0	10	20	30	40	50	60
$y = 1/v$	0.02127	0.01724	0.015625	0.015385	0.01639	0.01923	0.02631
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$h = \frac{b-a}{n} = \frac{60-0}{6} = 10$$

Using Simpson's 1/3 rule.

$$T = \frac{h}{3}\left[y_0 + y_6 + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$T = \frac{10}{3}\left[0.02127 + 0.02631 + 2(0.015625 + 0.01639) + 4(0.01724 + 0.015385 + 0.01923)\right]$$

$$T = \frac{10}{3}\left[0.04758 + 2 \times 0.032015 + 4 \times 0.051855\right]$$

$$\boxed{T_{1/3} = 1.06538 \text{ seconds}}$$

Now, Using Simpson's 3/8 rule.

$$T = \int y\,ds = \frac{3h}{8}\left[y_0 + y_6 + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$T = \frac{3\times 10}{8}\left[0.02127 + 0.02631 + 2\times(0.01538) + 3(0.01724 + 0.01639 + 0.015625 + 0.01923)\right]$$

$$T = \frac{30}{8}\left[0.04758 + 0.03076 + 3(0.052095) + 3\times 0.01639\right]$$

$$T = \frac{15}{4}\left[0.04758 + 0.03076 + 0.156285\right] + 3\times 0.01639$$

$$T = \frac{15}{4}\times\left[0.234625 + 0.04917\right] = \frac{15}{4}\times 0.283795$$

$T_{3/8} = 1.06423$ seconds.

$$\text{Difference} = T_{1/3} - T_{3/8} = 1.06538 - 1.06423$$

$$\boxed{\text{Difference} = 0.00115\text{ sec}}\ A.$$

Ques 7(c): Using Runge-kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.

Solution: we have; $f(x, y) = \frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$:

Here $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2\left[\frac{1-0}{1}\right] = 0.2$$

$$k_2 = h\,f\left[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right] = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = h\,f\left[x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2(.19672 + 0.1967) + 0.1891) = \underline{0.19599}$$

Hence, $y(0.2) = y_0 + k = \underline{\underline{1.196}}$

To find $y(0.4)$

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$

$k_1 = hf(x_1, y_1) = 0.2 f(0.2, 1.196) = 0.1891$

$k_2 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) = 0.2 f(0.3, 1.2906) = 0.1795$

$k_3 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = 0.2 f(0.3, 1.2858) = 0.1793$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 f(0.4, 1.3753) = 0.1688$

$k = \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$

$k = \frac{1}{6}[0.1891 + 2(0.1795 + 0.1793) + 0.1688]$

$\boxed{k = 0.1792}$

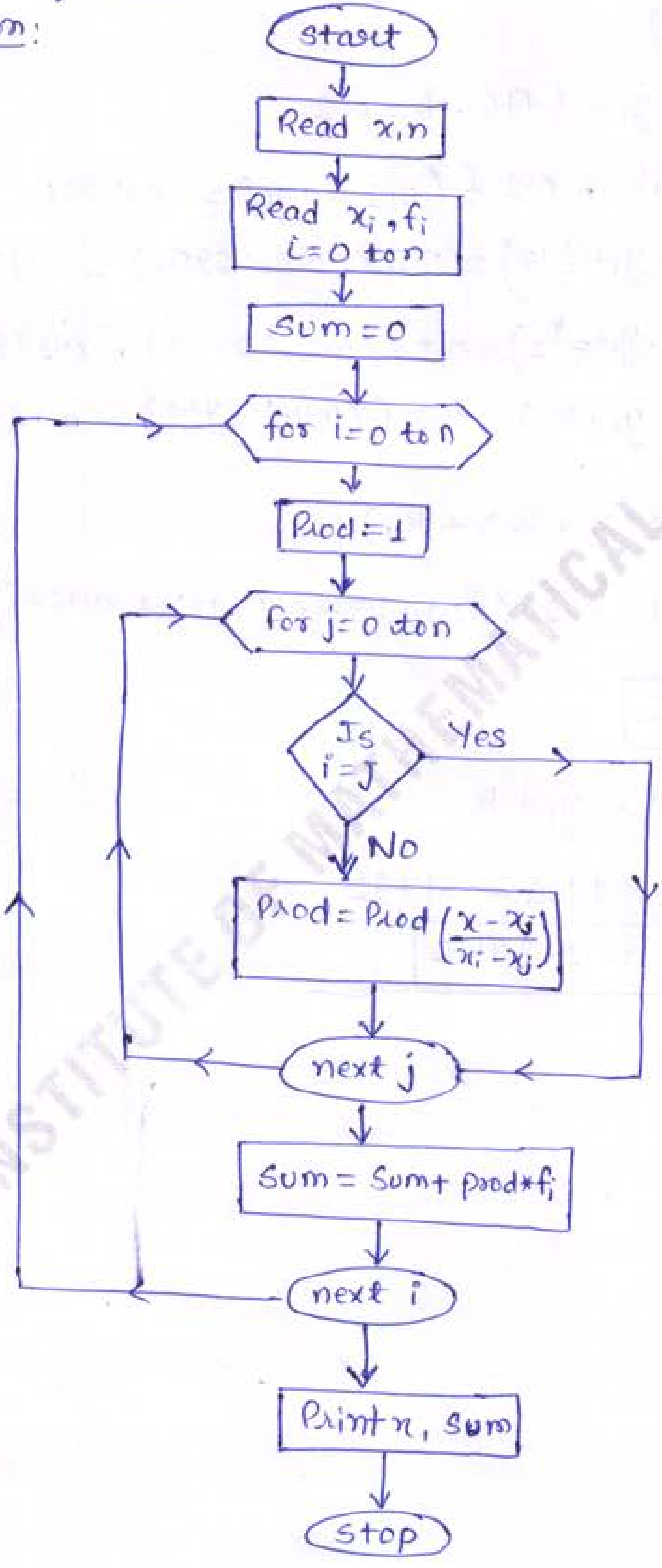
Hence; $y(0.4) = y_1 + k$

$= 1.196 + 0.1792$

$\boxed{y(0.4) = 1.3752}$

Q.7(d) Draw a flow chart for Lagrange's interpolation formula.

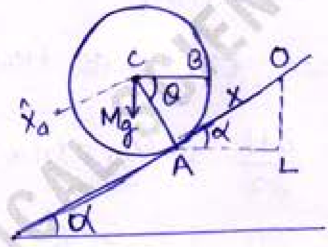
Solution:



Q.8) a) A sphere for radius 'a' and mass 'M' rolls down a rough plane inclined at an angle 'α' to the horizontal.
If 'x' be the distance of the point of contact of the sphere from a fixed point on the plane, find the acclaration by using Hamilton's equations.

Solution :-

Let a sphere of radius 'a' and mass 'M' rolls down a rough plane inclined at an angle 'α' starting initially from a fixed point 'O' of the plane.



In time 't', let the sphere roll down a distance 'x' and during this time let, it turn through an angle 'θ'. Since, there is no slipping.

$\therefore\ x = OA = \text{arc } AB = a\theta$

So, that; $x = a\theta$

If T and V are kinetic and potential energies of the sphere, then

$$T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\left(\frac{2}{5}a^2\dot{\theta}^2\right) + \frac{1}{2}M(a\dot{\theta})^2$$

or $\boxed{T = \frac{7}{10}M\dot{x}^2}$

and $V = -MgOL = -Mgx\sin\alpha$

(since, the sphere move down the plane).

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised coordinate.

$$\therefore P_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5} M\dot{x} \quad \text{——— (1)}$$

Since, L does not contain 't' explicitely,

$$\therefore H = T+V = \frac{7}{10} M\dot{x}^2 - Mgx\sin\alpha$$

or $$H = \frac{7}{10} M\left[\frac{5}{(7M)} P_x\right]^2 - Mgx\sin\alpha$$

$$H = \frac{5}{14M} P_x^2 - Mgx\sin\alpha \quad \text{— (from (1))}$$

Hence, the two Hamilton's equations are.

$$\dot{P}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \rightarrow (H_1)$$

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{5}{7M} P_x \quad \text{——} (H_2).$$

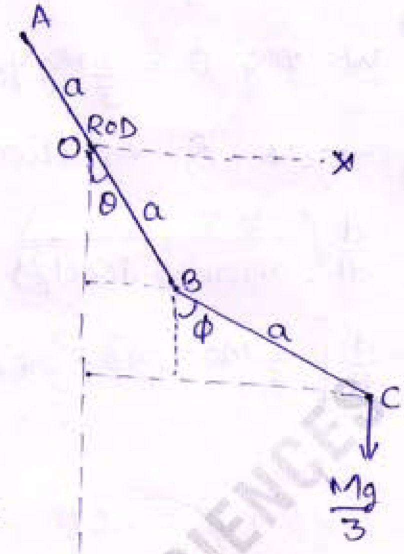
Differentiating (H_2) using (H_1), we get.

$$\ddot{x} = \frac{5}{7M}\dot{P}_x = \frac{5}{7M}\cdot Mg\sin\alpha$$

$$\boxed{\ddot{x} = \frac{5}{7} g\sin\alpha}$$ which gives the required accelaration.

Ques: 8(b) A uniform straight rod of length 2a is freely movable about its centre and a particle of mass one-third that of rod is attached by a light inextensible string of length 'a' to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{a/g}$.

Solution) 8(b)



Let, M be mass of rod AB of length 2a, BC the string and $\frac{M}{3}$ the Mass at 'C'.

At time 't', let the rod and the string make angles θ & φ to the vertical respectively.

Referred to the middle point 'O' of the rod 'AB' as origin, horizontal and verticle lines OX and OY through 'O' as axes, the co-ordinates of 'C' are given by

$$x_c = a\,(\sin\theta + \sin\phi)$$

$$y_c = a\,(\cos\theta + \cos\phi)$$

$$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2$$

$$v_c^2 = a^2(\cos\theta\,\dot\theta + \cos\phi\,\dot\phi)^2 + a^2(-\sin\theta\,\dot\theta - \sin\phi\,\dot\phi)^2$$

$$v_c^2 = a^2\left[\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi\cos(\theta-\phi)\right]$$

$$v_c^2 = a^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi) \quad [\because \theta, \phi \text{ are small}]$$

If 'T' be the total kinetic energy and 'W' the work function of the system, then

T = K.E of the rod + K.E of the particle at 'C'

$$T = \left[\frac{1}{2}M\cdot\frac{1}{3}a^2\dot\theta^2 + \frac{1}{2}Mv_0^2\right] + \frac{1}{2}\left(\frac{1}{3}M\right)v_c^2.$$

$$T = \frac{1}{6}Ma^2\dot\theta^2 + \frac{1}{2}Mv_0^2 + \frac{1}{6}Mv_c^2$$

$$T = \frac{1}{6}Ma^2\dot\theta^2 + \frac{1}{6}Ma^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)$$

$$\boxed{T = \frac{1}{6}Ma^2(2\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)} \qquad [\because v_0 = 0]$$

and

$$W = mg \cdot O = \frac{1}{3} Mg\, y_c + C = \frac{1}{3} Mg\, a(\cos\theta + \cos\phi) + C$$

Lagrange's θ - equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\text{down30 dedel}_{\dot\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e $\frac{d}{dt}\left[\frac{1}{6} Ma^2 (4\dot\theta + 2\dot\phi)\right] - 0 = \frac{1}{3} Mga(-\sin\theta) = -\frac{1}{3} Mga\theta$

[$\because \theta$ is small] ——— ①

or $\quad 2\ddot\theta + \ddot\phi = -c\theta \quad$ (where $c = g/a$)

and Lagrange's ϕ equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{6} Ma^2 (2\dot\phi + 2\dot\theta)\right] - 0 = \frac{1}{3} Mga(-\cos\phi)$

$= -\frac{1}{3} Mga\phi \quad$ [$\because \phi$ is small]. ——— ②

or $\quad \ddot\theta + \ddot\phi = -c\phi \quad$ (where $c = g/a$).

Equation ① and ② can be written as

$(2D^2 + c)\theta + D^2\phi = 0$ and

$D^2\theta + (D^2 + c)\phi = 0$

Eliminating ϕ between these two equations, we get

$[(D^2 + c)(2D^2 + c) - D^4]\theta = 0$

$(D^4 + 3cD^2 + c^2)\theta = 0$ ——— ③

let the solution of (3), be given by

$\theta = A\cos(pt + B)$.

$\therefore D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$

Substituting in (2), we get

$(p^4 - 3cp^2 + c^2)\theta = 0$ or $p^4 - 3cp^2 + c^2 = 0$

[$\because \theta \neq 0$]

$\therefore p^2 = \dfrac{3c \pm \sqrt{9c^2 - 4c^2}}{2} = \left(\dfrac{3 \pm \sqrt{5}}{2}\right) c$

$p^2 = \left(\dfrac{3 \pm \sqrt{5}}{2}\right) g/a$

$\therefore$ one value of p^2 is $p_1^2 = \left(\dfrac{3 - \sqrt{5}}{2}\right) \dfrac{g}{a}$

$\therefore$ one period of principal oscillation

$= \dfrac{2\pi}{p_1} = 2\pi \sqrt{\left[\dfrac{2}{3-\sqrt{5}} \cdot \dfrac{a}{g}\right]} = 2\pi \sqrt{\left[\dfrac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \cdot \dfrac{a}{g}\right]}$

$= 2\pi \sqrt{\left[\dfrac{(6+2\sqrt{5})}{4} \dfrac{a}{g}\right]} = 2\pi \sqrt{\left[\left(\dfrac{\sqrt{5}+1}{2}\right)^2 \cdot \dfrac{a}{g}\right]}$

$= \cancel{2}\pi \cdot \dfrac{\sqrt{5}+1}{\cancel{2}} \cdot \sqrt{\dfrac{a}{g}} = (\sqrt{5}+1)\pi\sqrt{a/g}$

$\boxed{\therefore \text{One period of principal oscillation} = (\sqrt{5}+1)\pi\sqrt{a/g}}$

Ques 8-C) A two-dimensional flow field is given by $\psi = xy$.

(a) show that the flow is irrotational.

(b) Find the velocity potential.

(c) Verify that ψ and ϕ satisfy the Laplace equation.

(d) Find the stream lines and potential lines.

Solution: The velocity components are given by

$$u = -\frac{\partial \psi}{\partial y} = -x \qquad \text{as } \psi = xy$$

$$v = +\frac{\partial \psi}{\partial x} = +y$$

So, that ; $q = u\hat{i} + v\hat{j}$

$q = -x\hat{i} + y\hat{j}$

and $\text{curl } q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -x & y & 0 \end{vmatrix}$

$\text{Curl } q = i(0-0) + j(0-0) + k(0-0)$

$\boxed{\text{Curl } q = 0}$

Hence, the flow is irrotational.

(b) we have; for velocity potential.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\therefore \phi = \int \left(\frac{\partial \psi}{\partial y}\right) dx + f_1(y) = \int x\, dx + f_1'(y).$$

$$\phi = \frac{x^2}{2} + f_1(y) \qquad \text{—— (1)}$$

and. $\phi = -\int \left(\frac{\partial \psi}{\partial x}\right) dy + f_2(x) = -\int y\, dy + f_2(x)$

$$\phi = -\frac{y^2}{2} + f_2(x). \qquad \text{—— (2)}$$

from (1) and (2),

$f_1(y) = -y^2/2 + \text{constant}$ &

$f_2(x) = x^2/2 + \text{constant}$.

so that

$$\boxed{\phi = \frac{x^2-y^2}{2} + \text{constant}}$$

is the required velocity potential

(c) To check ψ & ϕ satisfy Laplace equation –

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0+0$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 1-1 = 0$$

and as $\nabla^2\psi = \nabla^2\phi = 0$

Hence, ψ & ϕ satisfy the Laplace equation.

(d) The streamline (ψ = constant) and the potential lines (ϕ = constant) are given.

$\boxed{xy = c_1}$ and $\boxed{x^2-y^2 = c_2}$

respectively, where c_1 & c_2 are constants.