Q, If $\phi = xy^2 + yz^2 + x^2z$ Then Curl (grad ϕ) = ?

$$\text{Curl}\,(\text{grad}\,\phi) = 0$$

Q, $\nabla^2\left(\frac{1}{r}\right)$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{-2}{r^3} + \frac{2}{r}\left(\frac{-1}{r^2}\right)$$

$$= -\frac{2}{r^3} + \frac{2}{r^3}$$

$$= \cancel{\frac{-4}{r^3}}\ 0$$

$$f(r) = \frac{1}{r}$$

$$f'(r) = \frac{-1}{r^2}$$

$$f''(r) = \frac{+2}{r^3}$$

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Q> If $\vec{F} = yi + zj + xk$ and

$\nabla(\phi \vec{F}) = xy^2 + x^2 z + yz^2$ Then $\vec{F} \cdot (\nabla\phi) =$ ______

Wkt $\nabla \cdot (\phi \vec{F}) = \phi(\nabla \cdot \vec{F}) + \vec{F} \cdot (\nabla\phi)$

$\nabla \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(yi + zj + xk)$

$= 0 + 0 + 0 = 0$

$\vec{F} \cdot (\nabla\phi) = xy^2 + yz^2 + x^2 z$

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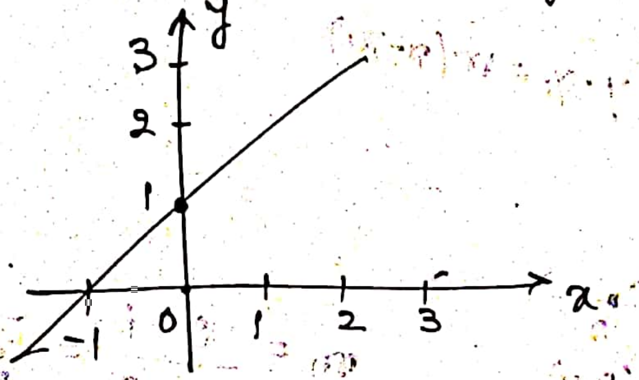
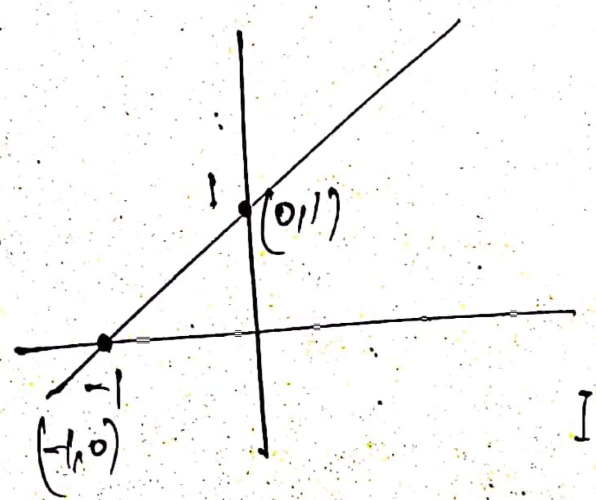
Q. The following plot shows a function y which varies linearly with x. The value of the integral $I = \int_1^2 y\,dx$ is



a) 1.0 b) 2.5 c) 4.0 d) 5.0

Soln



$$m = \frac{1-0}{0+1} = \frac{1}{1} = 1$$

$$y = mx + c$$
$$= x + 1$$

$$I = \int_1^2 (x+1)\,dx$$
$$= \left[\frac{x^2}{2} + x\right]_1^2$$
$$= \left(\frac{2^2}{2} + 2\right) - \left(\frac{1^2}{2} + 1\right)$$
$$= 4 - 3/2 = \frac{8-3}{2} = \frac{5}{2}$$

Ans 2.5

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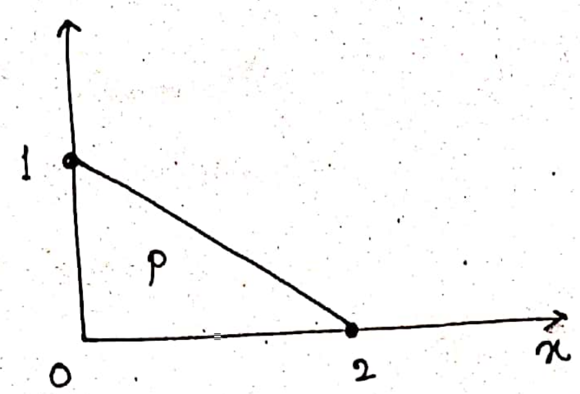
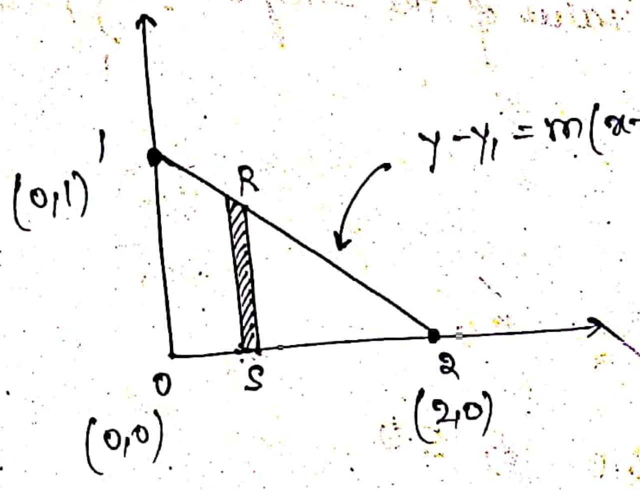
Q)



Find $\iint_P xy \, dx \, dy$

a) $\frac{1}{6}$ b) $\frac{2}{9}$ c) $\frac{1}{16}$ d) 1

Soln



$y - y_1 = m(x - x_1)$

$m = \frac{0-1}{2-0} = \frac{-1}{2}$

$$y - 1 = \frac{-1}{2}(x - 0)$$

$$y - 1 = \frac{-x}{2}$$

$$\boxed{y = 1 - \frac{x}{2}}$$

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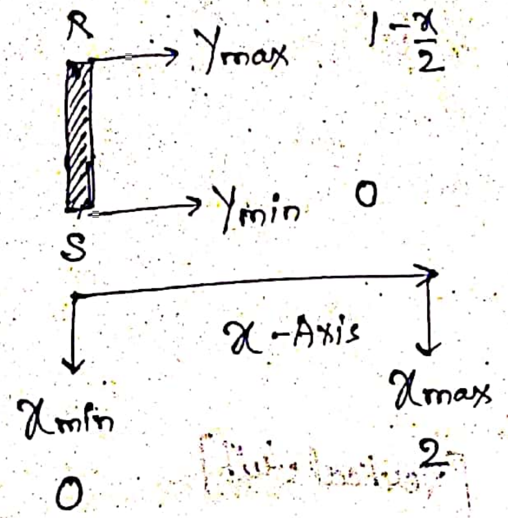
$$I = \iint_P xy \, dx\, dy$$



$$\int_0^2 \int_0^{1-\frac{x}{2}} xy \, dy\, dx$$

$$\int_0^2 \left\{ x \int_0^{1-\frac{x}{2}} y\, dy \right\} dx$$

$$\int_0^2 \left\{ x \left[\frac{y^2}{2}\right]_0^{1-\frac{x}{2}} \right\} dx$$

$$\int_0^2 \left\{ x \cdot \frac{\left(1-\frac{x}{2}\right)^2}{2} \right\} dx$$

$$\int_0^2 \frac{x \cdot \left[1 + \frac{x^2}{4} - 2 \cdot \frac{x}{2}\right]}{2}$$

$$\frac{1}{2} \int_0^2 \left[x + \frac{x^3}{4} - x^2\right]$$

$$\frac{1}{2} \left[\frac{x^2}{2} + \frac{x^4}{16} - \frac{x^3}{3}\right]_0^2$$

$$\frac{1}{2} \left[\frac{2^2}{2} + \frac{2^4}{16} - \frac{2^3}{3}\right]$$

$$\frac{1}{2} \left[2 + 1 - \frac{8}{3}\right]$$

$$\cancel{\frac{1}{2}\left[\frac{4}{1} - \frac{8}{3}\right] = \frac{1}{2}\left[\frac{12-8}{3}\right]}$$

$$\frac{4}{6}$$

$$\frac{1}{2}\left[3 - \frac{8}{3}\right] = \frac{1}{2}\left[\frac{1}{3}\right] = \frac{1}{6}$$

Ans

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Q) Changing the order of the integration in the double integral leads to

$$I = \int_0^8 \int_{x/4}^{2} f(x,y)\, dy\, dx$$ leads to $$J = \int_r^2 \int_p^q f(x,y)\, dy\, dx$$ What is q

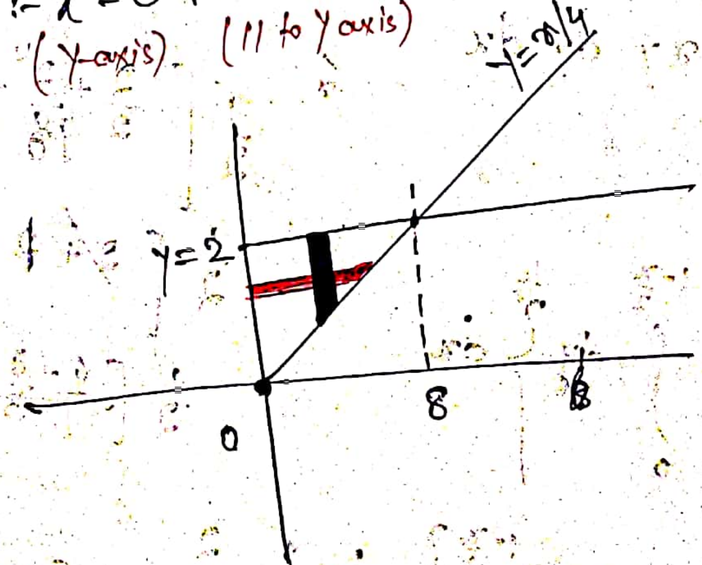
a) $4y$ b) $16y^2$ c) x d) 8

Soln

(1) Identification of limits

y Limits :- $y = x/4$ to 2 [Vertical strip]
(Line Passing through origin) (∥ to X axis)

x Limits :- $x = 0$ to 8
(Y-axis) (∥ to Y axis)



y limits :- $y = 0$ to 2

x limits :- $x = 0$ to $4y$

Q. The area enclosed between the curves $y^2=4x$ and $x^2=4y$ is a) $\frac{16}{3}$ b) 8 c) $\frac{32}{3}$ d) 16

Soln

$$\text{Area} = \iint_S dx\,dy$$

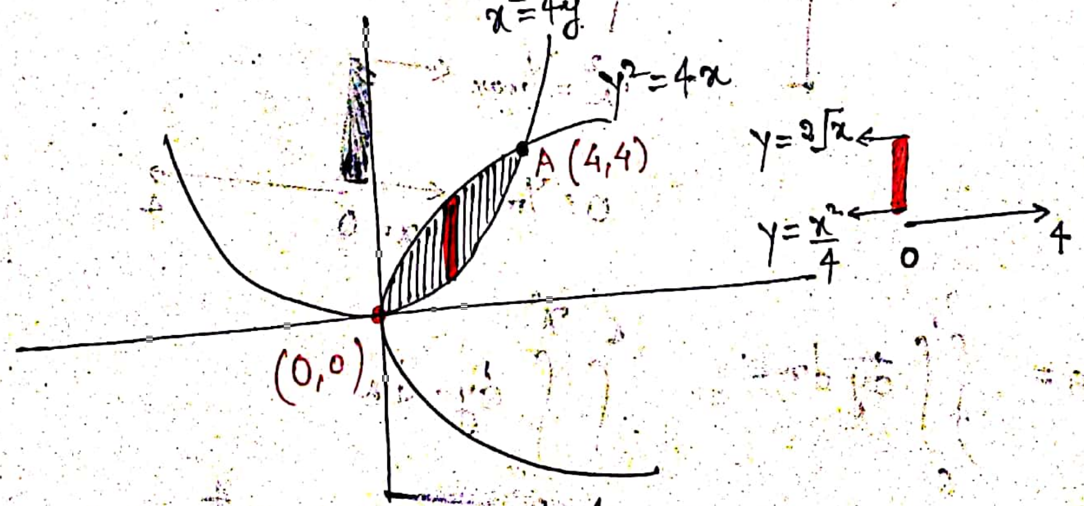
Limits → Tracing

$y^2 = 4x \Rightarrow$ Parabola

↳ Even [symmetric about x-axis]

$x^2 = 4y \Rightarrow$ Parabola

↳ Even [symmetric about y-axis]



$$\text{Area} = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} 1\,dy\,dx = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \left[2\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4 = \frac{16}{3}$$

Option (a)

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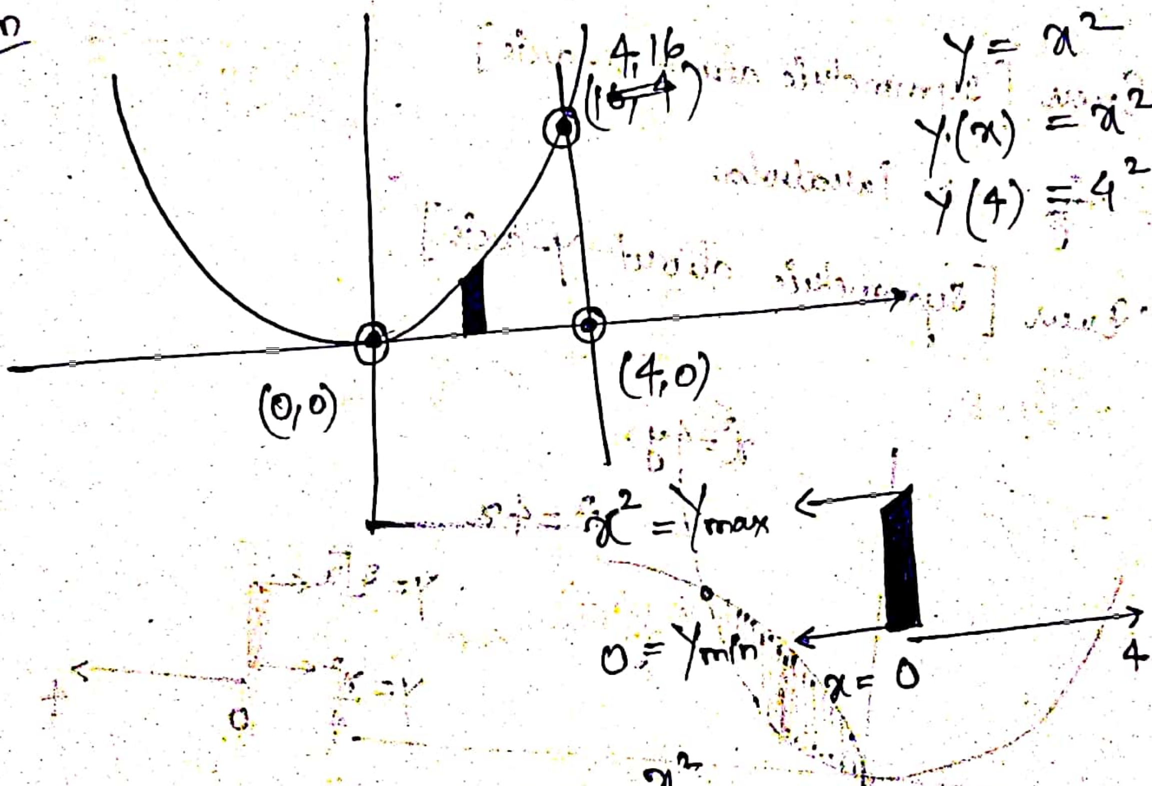
Q, Area bounded by the curve $y=x^2$ and lines $x=4$ and $y=0$ is given by

a) 64 b) $\frac{64}{3}$ c) $\frac{128}{3}$ d) $\frac{128}{4}$

Soln



$Y = x^2$

$Y(x) = x^2$

$Y(4) = 4^2 = 16$

$$\text{Area} = \iint_S dy\,dx = \int_0^4 \int_0^{x^2} dy \cdot dx$$

$$= \int_0^4 [y]_0^{x^2} dx = \int_0^4 [x^2] dx$$

$$= \left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{64}{3}$$ Ans

Q, A triangle in the x-y plane is bounded by the straight lines $2x = 3y$, $y = 0$ and $x = 3$. The volume above the triangle and under the plane $x+y+z=6$ is ______

Soln

$$V = \iiint_V dx\,dy\,dz$$

If ~~limits~~ Value of z is given then

$$\iint_S z\,dx\,dy$$

$$Vol = \iint_S f(x,y)\,dx\,dy$$

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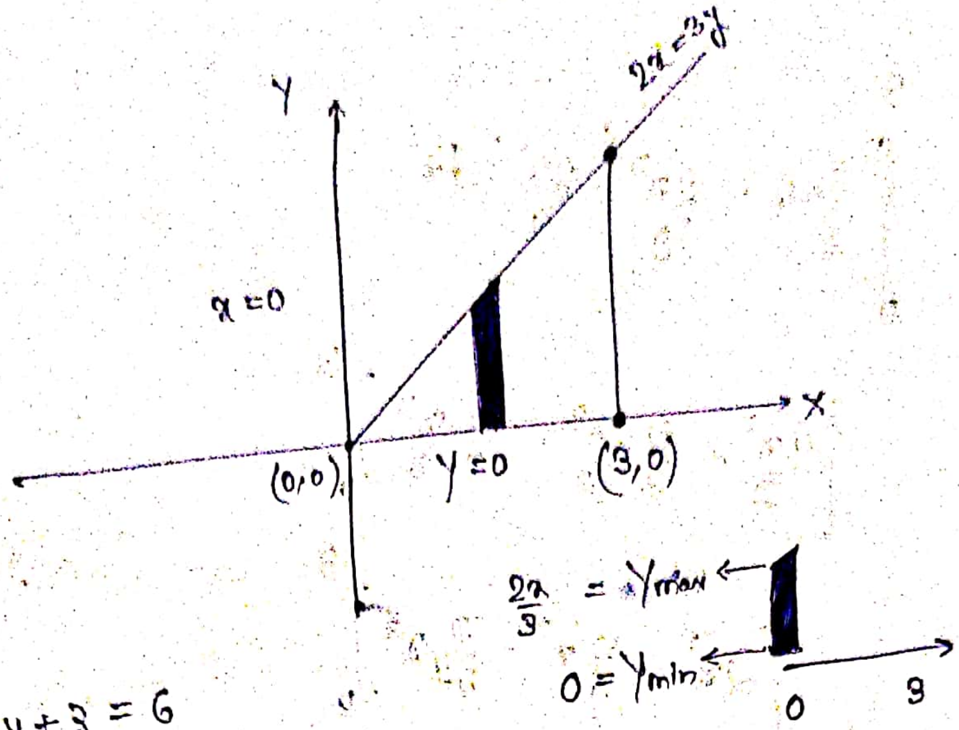

$$x + y + z = 6$$

$$z = 6 - x - y$$

$$V = \iint_S z \cdot dx\, dy$$

$$V = \iint_S (6 - x - y)\, dx\, dy$$

$$V = \int_0^3 \int_0^{2x/3} (6 - x - y)\, dy\, dx$$

$$= \int_0^3 \left[6y - xy - \frac{y^2}{2}\right]_0^{2x/3} dx$$

$$= \int_0^3 \left[\cancel{6}^{2} \cdot \frac{2x}{\cancel{3}} - x \cdot \frac{2x}{3} - \frac{1}{\cancel{2}} \cdot \frac{\cancel{4}^{2} x^2}{9}\right] dx$$

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$$\int_0^3 \left[4x - \frac{2x^2}{3} - \frac{2x^2}{9}\right] dx$$

$$\left[\frac{4x^2}{2} - \frac{2}{3}\frac{x^3}{3} - \frac{2}{9}\frac{x^3}{3}\right]_0^3$$

$$\left[2x^2 - \frac{2}{9}x^3 - \frac{2}{27}x^3\right]_0^3$$

$$2(3)^2 - \frac{2}{9}(3)^3 - \frac{2}{27}(3)^3$$

$$2(9) - \frac{2}{\cancel{9}}(\cancel{27})^{3} - \frac{2}{\cancel{27}}(\cancel{27})$$

$$18 - 6 - 2$$

$$12 - 2$$

$$= 10 \quad \underline{Ans}$$

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Q. If $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ and $|\vec{r}| = r$ then $div[r^2 \nabla(\ln r)]$ is ______

Soln $div[r^2 \nabla(\ln r)] = div\left[r^2 \frac{d}{dr}(\ln r)\right]$

$= div\left[r^{\cancel{2}} \times \frac{1}{\cancel{r}}\right]$

$= div[r]$

$\nabla \cdot \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$

$= 1 + 1 + 1$

$= 3$ Ans

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Q) A vector $\vec{P}$ is given by $\vec{P} = x^3 y \, \vec{a_x} - x^2 y^2 \, \vec{a_y} - x^2 y z \, \vec{a_z}$.

Which is true?

a) $\vec{P}$ is solenoidal but not irrotational

b) $\vec{P}$ is irrotational but not solenoidal

c) $\vec{P}$ is neither solenoidal nor irrotational

d) $\vec{P}$ is both solenoidal and irrotational

Soln

IRROTATIONAL $\Rightarrow \nabla \times \vec{A} = 0$

ROTATIONAL $\Rightarrow \nabla \times \vec{A} \neq 0$

$$\vec{\nabla} \times \vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 y & -x^2 y^2 & -x^2 y z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x^2 y z) - \frac{\partial}{\partial z}(-x^2 y^2)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x^2 y z) - \frac{\partial}{\partial z}(x^3 y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-x^2 y^2) - \frac{\partial}{\partial y}(x^3 y)\right]$$

Not Irrotational

$$= \hat{i}\left[(-x^2 z) - 0\right] - \hat{j}\left[(-2xyz) - 0\right] + \hat{k}\left[(-2xy) - x^3\right]$$

$$= -x^2 z \, \hat{i} + 2xyz \, \hat{j} + \left[(-2xy - x^3)\right]\hat{k} \neq 0$$

ROTATIONAL

SOLENOIDAL / INCOMPRESSIBLE / NO FLOW

$\Downarrow$

$$\nabla \cdot \vec{A} = 0$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x^3 y\,\hat{i} - x^2 y^2\,\hat{j} - x^2 y z\,\hat{k}\right)$$

$$\frac{\partial}{\partial x}(x^3 y) + \frac{\partial}{\partial y}(-x^2 y^2) + \frac{\partial}{\partial z}(-x^2 y z)$$

$$3x^2 y - x^2 \cdot 2y - x^2 y$$

$$3x^2 y - 3x^2 y, \cancel{\neq 0} = 0$$ ~~NOT~~ <u>SOLENOIDAL</u>

<u>$\vec{P}$ is solenoidal but not irrotational</u>

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Q. The directional derivative of $f(x,y) = \frac{xy}{\sqrt{2}}(x+y)$ at (1,1) in the direction of unit vector at an angle of $\frac{\pi}{4}$ with y axis is given by ——

VECTOR DERIVATIVE OF A SCALAR → Gradient

→ Max. rate of change → Normal vector of scalar quantity

→ Max. slope → Direction of Plane

DIRECTIONAL DERIVATIVE

$$D.D(\phi) = grad(\phi) \cdot \hat{d}$$

direction

$$|D.D| = |grad(\phi)|$$

$$\boxed{grad(\phi) = \nabla(\phi)}$$

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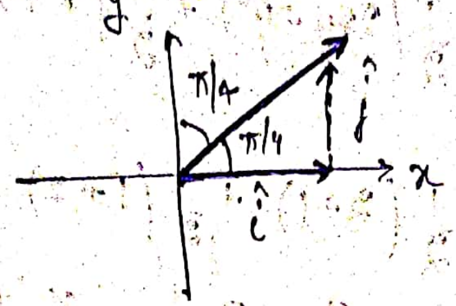
Soln

$$D.D = \text{grad}(f) \cdot \hat{d}$$



$$\hat{d} = \left(\sin\frac{\pi}{4}\right)\cdot\hat{i} + \cos\left(\frac{\pi}{4}\right)\hat{j}$$

$$\hat{d} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$$

$$D.D = \nabla(f)\cdot\hat{d}$$

$$= \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)\cdot\left(\hat{i}\frac{1}{\sqrt{2}} + \hat{j}\frac{1}{\sqrt{2}} + \hat{k}\cdot 0\right)$$

$$= \frac{1}{\sqrt{2}}\frac{\partial f}{\partial x} + \frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x}\left[x^2y + xy^2\right]$$

$$= \frac{1}{\sqrt{2}}\cdot\left[2xy + \cancel{2xy}\,y^2\right] = \frac{1}{\sqrt{2}}\left[2xy + y^2\right]$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial y}\left[x^2y + xy^2\right] = \frac{1}{\sqrt{2}}\left[x^2 + 2xy\right]$$

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$$DD = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(2xy+y^2)\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(x^2+2xy)\right]$$

$$= \frac{1}{2}(2xy+y^2) + \frac{1}{2}(x^2+2xy)$$

$$DD_{(1,1)} = \frac{1}{2}(2+1) + \frac{1}{2}(1+2)$$

$$= \frac{3}{2} + \frac{3}{2} = \cancel{\frac{6}{2}} \; 3 \; \text{Ans}$$

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Find Directional Derivative of $\phi = 2xy + z^2$ in the direction of $\vec{r} = \hat{i} + 2\hat{j} + 2\hat{k}$ at the point $(1, -1, 3)$

Soln

$$DD = \nabla\phi \cdot \hat{n}$$

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(2xy+z^2) + \hat{j}\frac{\partial}{\partial y}(2xy+z^2) + \hat{k}\frac{\partial}{\partial z}(2xy+z^2)$$

$$= \hat{i}(2y) + \hat{j}(2x) + \hat{k}(2z)$$

$$\nabla\phi\Big|_{1,-1,3} = \hat{i}(2(-1)) + \hat{j}(2(1)) + \hat{k}(2(3))$$

$$= -2\hat{i} + 2\hat{j} + 6\hat{k}$$

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$$\hat{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1^2+2^2+2^2}} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{9}}$$

$$\hat{n} = \frac{1}{3}\left(\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$DD = \nabla\phi \cdot \hat{n}$$

$$= \left(-2\hat{i}+2\hat{j}+6\hat{k}\right)\cdot\frac{1}{3}\left(\hat{i}+2\hat{j}+2\hat{k}\right)$$

$$= \frac{1}{3}\left[(-2\times1)+(2\times2)+(6\times2)\right]$$

$$= \frac{1}{3}\left[-2+4+12\right]$$

$$= \frac{14}{3} \text{ Ans}$$

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Q. $\vec{V} = x^2yz\,\hat{i} + xy^2z\,\hat{j} + xyz^2\,\hat{k}$ Find $\nabla \cdot \vec{V}$ at $(1,-1,1)$

$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xyz^2)$$

$$= 2xyz + 2xyz + 2xyz$$

$$= 6xyz$$

$$\nabla \cdot \vec{V}\Big|_{(1,-1,1)} = -6 \quad \text{Ans}$$

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Q> $\vec{F} = \nabla(2x^3y^2z^4)$ Find div $\vec{F}$, Curl $\vec{F}$

Soln

$$\text{div}\,\vec{F} = \text{div}(\nabla\phi) = \nabla^2\phi$$

$$= \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial y}\right] + \frac{\partial}{\partial z}\left[\frac{\partial\phi}{\partial z}\right]$$

$$= \frac{\partial}{\partial x}\left[6x^2y^2z^4\right] + \frac{\partial}{\partial y}\left[4x^3yz^4\right] + \frac{\partial}{\partial z}\left[8x^3y^2z^3\right]$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \quad \text{Ans}$$

$$\text{Curl}(\nabla\phi) = \bar{0} \quad \text{Ans}$$

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Q. Find a, b, c of $\vec{F} = (x+2y+az)\hat{i} + (bx-3y+z)\hat{j} + (4x+cy+2z)\hat{k}$ if $\vec{F}$ is irrotational

Soln $\text{Curl } \vec{F} = \bar{0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y+z) & (4x+cy+2z) \end{vmatrix}$

$$\hat{i}\left\{\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y+z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right\}$$

$$+\hat{k}\left\{\frac{\partial}{\partial x}(bx-3y+z) - \frac{\partial}{\partial y}(x+2y+az)\right\}$$

$$\hat{i}\{c-1\} - \hat{j}\{4-a\} + \hat{k}\{b-2\} = \bar{0}$$

$$\cancel{c\hat{i} - 4\hat{j} + (b-2)\hat{k} = 0}$$

$$(c-1)\hat{i} + \hat{j}(a-4) + \hat{k}(b-2) = 0$$

$c-1=0$
$a-4=0$
$b-2=0$

$c=1$
$a=4$
$b=2$

Ans

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Q> Find a if $\vec{F} = 2xy\,\hat{i} + 3x^2y\,\hat{j} - 3ayz\,\hat{k}$ is solenoidal at $(1,2,3)$

Soln

$$\nabla\cdot\vec{F} = 0 \quad \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\,2xy + \hat{j}\,3x^2y - 3ayz\hat{k}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(-3ayz) = 0$$

$$2y + 3x^2 - 3ay = 0$$

$$2(2) + 3(1) - 3a(2) = 0$$

$$4 + 3 - 6a = 0$$

$$6a = 7$$

$$a = 7/6 \quad \text{Ans}$$

Q> If the linear velocity $\vec{V}$ is given by $\vec{V} = x^2y\,\hat{i} + xyz\,\hat{j} - yz^2\,\hat{k}$ at the point $(1,1,-1)$ is then the angular velocity $\vec{W}$

$$\vec{\omega} = \frac{1}{2}\,\text{curl}\,\vec{V}$$

$$= -(\hat{i}+\hat{k})$$

3. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.
(a) Determine its velocity and acceleration at any time.
(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

(a) The position vector $\mathbf{r}$ of the particle is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = e^{-t}\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}$.

Then the velocity is $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{i} - 6\sin 3t\,\mathbf{j} + 6\cos 3t\,\mathbf{k}$

and the acceleration is $\mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = e^{-t}\mathbf{i} - 18\cos 3t\,\mathbf{j} - 18\sin 3t\,\mathbf{k}$

(b) At $t = 0$, $\dfrac{d\mathbf{r}}{dt} = -\mathbf{i} + 6\mathbf{k}$ and $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{i} - 18\mathbf{j}$. Then

magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (-18)^2} = \sqrt{325}$.

4. A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Velocity $= \dfrac{d\mathbf{r}}{dt} = \dfrac{d}{dt}[2t^2\mathbf{i} + (t^2 - 4t)\mathbf{j} + (3t - 5)\mathbf{k}]$

$= 4t\mathbf{i} + (2t - 4)\mathbf{j} + 3\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ at $t = 1$.

Unit vector in direction $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ is $\dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{(1)^2 + (-3)^2 + (2)^2}} = \dfrac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{14}}$.

Then the component of the velocity in the given direction is

$$\frac{(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (-2)(-3) + (3)(2)}{\sqrt{14}} = \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

Acceleration $= \dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right) = \dfrac{d}{dt}[4t\mathbf{i} + (2t-4)\mathbf{j} + 3\mathbf{k}] = 4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$.

Then the component of the acceleration in the given direction is

$$\frac{(4\mathbf{i} + 2\mathbf{j} + 0\mathbf{k})\cdot(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{14}} = \frac{(4)(1) + (2)(-3) + (0)(2)}{\sqrt{14}} = \frac{-2}{\sqrt{14}} = \frac{-\sqrt{14}}{7}$$

5. A curve C is defined by parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$, where s is the arc length of C measured from a fixed point on C. If $\mathbf{r}$ is the position vector of any point on C, show that $d\mathbf{r}/ds$ is a unit vector tangent to C.

The vector $\dfrac{d\mathbf{r}}{ds} = \dfrac{d}{ds}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \dfrac{dx}{ds}\mathbf{i} + \dfrac{dy}{ds}\mathbf{j} + \dfrac{dz}{ds}\mathbf{k}$ is tangent to the curve $x = x(s)$, $y = y(s)$, $z = z(s)$. To show that it has unit magnitude we note that

$$\left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{(dx)^2 + (dy)^2 + (dz)^2}{(ds)^2}} = 1$$

since $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$ from the calculus.

6. (a) Find the unit tangent vector to any point on the curve $x = t^2 + 1$, $y = 4t - 3$, $z = 2t^2 - 6t$.
(b) Determine the unit tangent at the point where $t = 2$.

(a) A tangent vector to the curve at any point is

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}[(t^2+1)\mathbf{i} + (4t-3)\mathbf{j} + (2t^2-6t)\mathbf{k}] = 2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}$$

The magnitude of the vector is $\left|\dfrac{d\mathbf{r}}{dt}\right| = \sqrt{(2t)^2 + (4)^2 + (4t-6)^2}$.

Then the required unit tangent vector is $\mathbf{T} = \dfrac{2t\mathbf{i} + 4\mathbf{j} + (4t-6)\mathbf{k}}{\sqrt{(2t)^2 + (4)^2 + (4t-6)^2}}$

Note that since $\left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt}$, $\mathbf{T} = \dfrac{d\mathbf{r}/dt}{ds/dt} = \dfrac{d\mathbf{r}}{ds}$.

(b) At $t = 2$, the unit tangent vector is $\mathbf{T} = \dfrac{4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (4)^2 + (2)^2}} = \dfrac{2}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$.

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1. If $\mathbf{R}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where x, y and z are differentiable functions of a scalar u, prove that $\dfrac{d\mathbf{R}}{du} = \dfrac{dx}{du}\mathbf{i} + \dfrac{dy}{du}\mathbf{j} + \dfrac{dz}{du}\mathbf{k}$.

$$\frac{d\mathbf{R}}{du} = \lim_{\Delta u \to 0} \frac{\mathbf{R}(u+\Delta u) - \mathbf{R}(u)}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{[x(u+\Delta u)\mathbf{i} + y(u+\Delta u)\mathbf{j} + z(u+\Delta u)\mathbf{k}] - [x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}]}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{x(u+\Delta u) - x(u)}{\Delta u}\mathbf{i} + \frac{y(u+\Delta u) - y(u)}{\Delta u}\mathbf{j} + \frac{z(u+\Delta u) - z(u)}{\Delta u}\mathbf{k}$$

$$= \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

2. Given $\mathbf{R} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, find (a) $\dfrac{d\mathbf{R}}{dt}$, (b) $\dfrac{d^2\mathbf{R}}{dt^2}$, (c) $\left|\dfrac{d\mathbf{R}}{dt}\right|$, (d) $\left|\dfrac{d^2\mathbf{R}}{dt^2}\right|$.

(a) $\dfrac{d\mathbf{R}}{dt} = \dfrac{d}{dt}(\sin t)\mathbf{i} + \dfrac{d}{dt}(\cos t)\mathbf{j} + \dfrac{d}{dt}(t)\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}$

(b) $\dfrac{d^2\mathbf{R}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{R}}{dt}\right) = \dfrac{d}{dt}(\cos t)\mathbf{i} - \dfrac{d}{dt}(\sin t)\mathbf{j} + \dfrac{d}{dt}(1)\mathbf{k} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}$

(c) $\left|\dfrac{d\mathbf{R}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{2}$

(d) $\left|\dfrac{d^2\mathbf{R}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

3. A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.
(a) Determine its velocity and acceleration at any time.
(b) Find the magnitudes of the velocity and acceleration at $t = 0$.

(a) The position vector $\mathbf{r}$ of the particle is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = e^{-t}\mathbf{i} + 2\cos 3t\,\mathbf{j} + 2\sin 3t\,\mathbf{k}$.

Then the velocity is $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{i} - 6\sin 3t\,\mathbf{j} + 6\cos 3t\,\mathbf{k}$

and the acceleration is $\mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = e^{-t}\mathbf{i} - 18\cos 3t\,\mathbf{j} - 18\sin 3t\,\mathbf{k}$

(b) At $t = 0$, $\dfrac{d\mathbf{r}}{dt} = -\mathbf{i} + 6\mathbf{k}$ and $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{i} - 18\mathbf{j}$. Then

magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (-18)^2} = \sqrt{325}$.

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21. Show that $\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \frac{\tau}{\rho^2}$.

$$\frac{d\mathbf{r}}{ds} = \mathbf{T}, \quad \frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d^3\mathbf{r}}{ds^3} = \kappa\frac{d\mathbf{N}}{ds} + \frac{d\kappa}{ds}\mathbf{N} = \kappa(\tau\mathbf{B} - \kappa\mathbf{T}) + \frac{d\kappa}{ds}\mathbf{N} = \kappa\tau\mathbf{B} - \kappa^2\mathbf{T} + \frac{d\kappa}{ds}\mathbf{N}$$

$$\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \mathbf{T}\cdot\kappa\mathbf{N}\times(\kappa\tau\mathbf{B} - \kappa^2\mathbf{T} + \frac{d\kappa}{ds}\mathbf{N})$$

$$= \mathbf{T}\cdot(\kappa^2\tau\mathbf{N}\times\mathbf{B} - \kappa^3\mathbf{N}\times\mathbf{T} + \kappa\frac{d\kappa}{ds}\mathbf{N}\times\mathbf{N}) = \mathbf{T}\cdot(\kappa^2\tau\mathbf{T} + \kappa^3\mathbf{B}) = \kappa^2\tau = \frac{\tau}{\rho^2}$$

The result can be written

$$\tau = [(x'')^2 + (y'')^2 + (z'')^2]^{-1} \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

where primes denote derivatives with respect to s, by using the result of Problem 20.

22. Given the space curve $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$, find (a) the curvature κ, (b) the torsion τ.

(a) The position vector is $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$.

Then $$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}} = \sqrt{(1)^2 + (2t)^2 + (2t^2)^2} = 1 + 2t^2$$

and $$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}}{1 + 2t^2}.$$

$$\frac{d\mathbf{T}}{dt} = \frac{(1 + 2t^2)(2\mathbf{j} + 4t\mathbf{k}) - (\mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k})(4t)}{(1 + 2t^2)^2} = \frac{-4t\mathbf{i} + (2 - 4t^2)\mathbf{j} + 4t\mathbf{k}}{(1 + 2t^2)^2}$$

Then $$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{-4t\mathbf{i} + (2 - 4t^2)\mathbf{j} + 4t\mathbf{k}}{(1 + 2t^2)^3}.$$

Since $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{\sqrt{(-4t)^2 + (2 - 4t^2)^2 + (4t)^2}}{(1 + 2t^2)^3} = \frac{2}{(1 + 2t^2)^2}$$

(b) From (a), $$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{-2t\mathbf{i} + (1 - 2t^2)\mathbf{j} + 2t\mathbf{k}}{1 + 2t^2}$$

Then $$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{1 + 2t^2} & \frac{2t}{1 + 2t^2} & \frac{2t^2}{1 + 2t^2} \\ \frac{-2t}{1 + 2t^2} & \frac{1 - 2t^2}{1 + 2t^2} & \frac{2t}{1 + 2t^2} \end{vmatrix} = \frac{2t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}}{1 + 2t^2}$$

Now $$\frac{d\mathbf{B}}{dt} = \frac{4t\mathbf{i} + (4t^2 - 2)\mathbf{j} - 4t\mathbf{k}}{(1 + 2t^2)^2} \quad \text{and} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{4t\mathbf{i} + (4t^2 - 2)\mathbf{j} - 4t\mathbf{k}}{(1 + 2t^2)^3}$$

Also, $$-\tau\mathbf{N} = -\tau\left[\frac{-2t\mathbf{i} + (1 - 2t^2)\mathbf{j} + 2t\mathbf{k}}{1 + 2t^2}\right].$$ Since $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, we find $\tau = \frac{2}{(1 + 2t^2)^2}$.

Note that $\kappa = \tau$ for this curve.

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38. Two rectangular xyz and $x'y'z'$ coordinate systems having the same origin are rotated with respect to each other. Derive the transformation equations between the coordinates of a point in the two systems.

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of any point P in the two systems (see figure on page 58). Then since $\mathbf{r} = \mathbf{r}'$,

(1) $$x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}' = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Now for any vector $\mathbf{A}$ we have (Problem 20, Chapter 2),

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{A}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{A}\cdot\mathbf{k}')\mathbf{k}'$$

Then letting $\mathbf{A} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ in succession,

(2) $$\begin{cases} \mathbf{i} = (\mathbf{i}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{i}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{i}\cdot\mathbf{k}')\mathbf{k}' = l_{11}\mathbf{i}' + l_{21}\mathbf{j}' + l_{31}\mathbf{k}' \\ \mathbf{j} = (\mathbf{j}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{j}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{j}\cdot\mathbf{k}')\mathbf{k}' = l_{12}\mathbf{i}' + l_{22}\mathbf{j}' + l_{32}\mathbf{k}' \\ \mathbf{k} = (\mathbf{k}\cdot\mathbf{i}')\mathbf{i}' + (\mathbf{k}\cdot\mathbf{j}')\mathbf{j}' + (\mathbf{k}\cdot\mathbf{k}')\mathbf{k}' = l_{13}\mathbf{i}' + l_{23}\mathbf{j}' + l_{33}\mathbf{k}' \end{cases}$$

Substituting equations (2) in (1) and equating coefficients of $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ we find

(3) $$x' = l_{11}x + l_{12}y + l_{13}z, \qquad y' = l_{21}x + l_{22}y + l_{23}z, \qquad z' = l_{31}x + l_{32}y + l_{33}z$$

the required transformation equations.

39. Prove
$$\begin{aligned} \mathbf{i}' &= l_{11}\mathbf{i} + l_{12}\mathbf{j} + l_{13}\mathbf{k} \\ \mathbf{j}' &= l_{21}\mathbf{i} + l_{22}\mathbf{j} + l_{23}\mathbf{k} \\ \mathbf{k}' &= l_{31}\mathbf{i} + l_{32}\mathbf{j} + l_{33}\mathbf{k} \end{aligned}$$

For any vector $\mathbf{A}$ we have $\mathbf{A} = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k}$.

Then letting $\mathbf{A} = \mathbf{i}', \mathbf{j}', \mathbf{k}'$ in succession,

$$\begin{aligned} \mathbf{i}' &= (\mathbf{i}'\cdot\mathbf{i})\mathbf{i} + (\mathbf{i}'\cdot\mathbf{j})\mathbf{j} + (\mathbf{i}'\cdot\mathbf{k})\mathbf{k} = l_{11}\mathbf{i} + l_{12}\mathbf{j} + l_{13}\mathbf{k} \\ \mathbf{j}' &= (\mathbf{j}'\cdot\mathbf{i})\mathbf{i} + (\mathbf{j}'\cdot\mathbf{j})\mathbf{j} + (\mathbf{j}'\cdot\mathbf{k})\mathbf{k} = l_{21}\mathbf{i} + l_{22}\mathbf{j} + l_{23}\mathbf{k} \\ \mathbf{k}' &= (\mathbf{k}'\cdot\mathbf{i})\mathbf{i} + (\mathbf{k}'\cdot\mathbf{j})\mathbf{j} + (\mathbf{k}'\cdot\mathbf{k})\mathbf{k} = l_{31}\mathbf{i} + l_{32}\mathbf{j} + l_{33}\mathbf{k} \end{aligned}$$

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