IFoS-2012 → Paper II

5) a) Using Lagrange's interpolation formula, show that,
$$32 f(1) = -3f(-4) + 10 f(-2) + 30 f(2) - 5 f(4)$$

⇒ here,

x	-4	-2	2	4
y	$f(-4)$	$f(-2)$	$f(2)$	$f(4)$

here $x = 1 \Rightarrow y = f(x) = f(1)$

				D_r	y_r	y_r/D_r
$x-x_0=\underline{5}$	$x_0-x_1=-2$	$x_0-x_2=-6$	$x_0-x_3=-8$	-480	$f(-4)$	$f(-4)/-480$
$x_1-x_0=2$	$x-x_1=\underline{3}$	$x_1-x_2=-4$	$x_1-x_3=-6$	144	$f(-2)$	$f(-2)/144$
$x_2-x_0=6$	$x_2-x_1=4$	$x-x_2=\underline{-1}$	$x_2-x_3=-2$	48	$f(2)$	$f(2)/48$
$x_3-x_0=8$	$x_3-x_1=6$	$x_3-x_2=2$	$x-x_3=\underline{-3}$	-288	$f(4)$	$f(4)/-288$.

and $\omega(x) = 5 \times 3 \times -1 \times -3 = 45$

$$\therefore f(1) = 45\left[-\frac{f(-4)}{480} + \frac{f(-2)}{144} + \frac{f(2)}{48} - \frac{f(4)}{288}\right]$$

$$\Rightarrow f(1) = \cancel{45}^{\cancel{3}\cancel{9}}\left[\frac{-3f(-4) + 10f(-2) + 30f(2) - 5f(4)}{\underset{16}{\cancel{48}} \times \cancel{3} \times \cancel{5} \times 2}\right]$$

$$\Rightarrow 32 f(1) = -3f(-4) + 10 f(-2) + 30 f(2) - 5 f(4)$$

6) (c) A river 80 meters wide. The depth 'd' (in meters) of the river at a distance x from one bank of the river is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section of the river.

⇒ Let 'A' be the area of the Cross-section, then,

$$A = \int_0^{80} (xd)\,dx = \int_0^{80} y\,dx$$

here, $h = 10$

x_i ($i=0$ to 8)	y_i ($i=0$ to 8)	y_i ($i=0,8$)	y_i ($i=1,3,5,7$)	y_i ($i=2,4,6$)
$x_0 = 0$	0	0	—	—
$x_1 = 10$	40	—	40	—
$x_2 = 20$	140	—	—	140
$x_3 = 30$	270	—	270	—
$x_4 = 40$	480	—	—	480
$x_5 = 50$	750	—	750	—
$x_6 = 60$	840	—	—	840
$x_7 = 70$	560	—	560	—
$x_8 = 80$	240	240	—	—
		$\Sigma y_i = 240 (= Y_0)$	$\Sigma y_i = 1620 (= Y_1)$	$\Sigma y_i = 1460 (= Y_2)$

Now, by simpson $1/3$rd rule,

$$A = \frac{h}{3}\left[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{h}{3}\left[Y_0 + 4Y_1 + 2Y_2\right]$$

$$= \frac{10}{3}\left[240 + 4 \times 1620 + 2 \times 1460\right]$$

$$= \frac{10}{3} \times 9640 = 10711.11 \text{ sq. meters.}$$

7) (b) Solve the following system of equation using Gauss-Seidel Method:

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Correct to three decimal places.

⇒ The given equations are diagonally dominant.
Now, we write iteration formula as,

$$x^{(k+1)} = \frac{1}{28}\left[32 - 4y^{(k)} + z^{(k)}\right]$$
$$y^{(k+1)} = \frac{1}{17}\left[35 - 2x^{(k+1)} - 4z^{(k)}\right]$$
$$z^{(k+1)} = \frac{1}{10}\left[24 - x^{(k+1)} - 3y^{(k+1)}\right]$$

we take the initial guess values are,

$$x^{(0)} = 0, \quad y^{(0)} = 0, \quad z^{(0)} = 0$$

$$\therefore \quad x^{(1)} = \frac{1}{28}\left[32 - 4\times 0 + 0\right] = 1.1429$$
$$y^{(1)} = \frac{1}{17}\left[35 - 2\times 1.1429 - 4\times 0\right] = 1.9244$$
$$z^{(1)} = \frac{1}{10}\left[24 - 1.1429 - 3\times 1.9244\right] = 1.7084$$

$$x^{(2)} = \frac{1}{28}\left[32 - 4\times 1.9244 + 1.7084\right] = 0.9989$$
$$y^{(2)} = \frac{1}{17}\left[35 - 2\times 0.9989 - 4\times 1.7084\right] = 1.5476$$
$$z^{(2)} = \frac{1}{10}\left[24 - 0.9989 - 3\times 1.5476\right] = 1.8428$$

$$x^{(3)} = \frac{1}{28}\left[32 - 4\times 1.5476 + 1.8428\right] = 0.9876$$
$$y^{(3)} = \frac{1}{17}\left[35 - 2\times 0.9876 - 4\times 1.8428\right] = 1.5090$$
$$z^{(3)} = \frac{1}{10}\left[24 - 0.9876 - 3\times 1.5090\right] = 1.8485$$

$$x^{(4)} = \frac{1}{28}\left[32 - 4\times 1.5090 + 1.8485\right] = 0.9933$$
$$y^{(4)} = \frac{1}{17}\left[35 - 2\times 0.9933 - 4\times 1.8485\right] = 1.5070$$
$$z^{(4)} = \frac{1}{10}\left[24 - 0.9933 - 3\times 1.5070\right] = 1.8486$$

$x^{(5)} = \frac{1}{28}[32 - 4\times 1.5070 + 1.8486] = 0.9935$

$y^{(5)} = \frac{1}{17}[35 - 2\times 0.9935 - 4\times 1.8486] = 1.5070$

$z^{(5)} = \frac{1}{10}[24 - 0.9935 - 3\times 1.5070] = 1.8486$

$\therefore x = 0.993$; $y = 1.507$; $z = 1.849$ is the solution. Correct upto 3-decimal places.

8) (c) Using Euler's Modified Method, obtain the solution of $\frac{dy}{dx} = x + |\sqrt{y}|$, $y(0) = 1$ for the range $0 \le x \le 0.6$ and step size 0.2

$\Rightarrow$ Here, $f(x,y) = x + |\sqrt{y}|$; $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$\therefore f(x_0, y_0) = x_0 + |\sqrt{y_0}| = 0 + 1 = 1$

we have $y_1^{(1)} = y_0 + hf(x_0, y_0)$

$= 1 + 0.2 \times 1 = 1.2$

$\therefore f(x_1, y_1^{(1)}) = x_1 + |\sqrt{y_1^{(1)}}|$

$= 0.2 + |\sqrt{1.2}| = 1.2954$

The second approximation to y_1 is,

$$y_1^{(2)} = y_0 + h\left[\frac{f(x_0,y_0) + f(x_1, y_1^{(1)})}{2}\right]$$

$$= 1 + 0.2 \times \left(\frac{1 + 1.2954}{2}\right) = 1.2295$$

Again, $f(x_1, y_1^{(2)}) = x_1 + |\sqrt{y_1^{(2)}}|$

$= 0.2 + |\sqrt{1.2295}| = 1.3088$

So, $y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(2)})]$

$= 1 + \frac{0.2}{2}(1 + 1.3088)$

$= 1.2309$

we have, $f(x_1, y_1^{(3)}) = 0.2 + |\sqrt{1.2309}| = 1.3090$

Then, $y_1^{(4)} = 1 + \frac{0.2}{2}(1 + 1.3090)$

$= 1.2309$

Since, $y_1^{(4)} = y_1^{(3)}$ hence $\boxed{y_1 = 1.2309}$

Now, $y_2^{(1)} = y_1 + h\, f(x_1, y_1)$

$= 1.2309 + 0.2\{0.2 + |\sqrt{1.2309}|\}$

$= 1.4927$

$f(x_2, y_2^{(1)}) = x_2 + |\sqrt{y_2^{(1)}}|$

$= 0.4 + \sqrt{1.4927} = 1.6220$

Then, $y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.2309 + \frac{0.2}{2}[(0.2 + \sqrt{1.2309}) + 1.6220]$

$= 1.5240$

Now, $y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.2309 + \frac{0.2}{2}[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5240})]$

$= 1.5253$

$y_2^{(4)} = 1.2309 + \frac{0.2}{2}[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.5253})]$

$= 1.5253$

$\therefore y_2^{(3)} = y_2^{(4)}$ hence $\boxed{y_2 = 1.5253}$

Now, $y_3^{(1)} = y_2 + h f(x_2, y_2) = 1.8523$

$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(1)})]$

$= 1.5253 + \frac{0.2}{2}[(0.4 + \sqrt{1.5253}) + (0.6 + \sqrt{1.8523})]$

$= 1.8849$

Similarly, $y_3^{(3)} = y_3^{(4)} = 1.8861$

Hence $y_3 = 1.8861$