Date : ....

NO.1 INSTITUTE FOR IAS/IFOS EXAMINATIONS

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MATHEMATICS CLASSROOM TEST
2022-23

Under the guidance *of* K. Venkanna

MATHEMATICS
COMPLEX ANALYSIS CLASS TEST

Date: 27 Jan. 2022

Time: 03:00 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. Write your Name & Name of the Test Centre in the appropriate space provided on the right side.
2. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
3. Candidates should attempt All Question.
4. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/notations carry their usual meanings, unless otherwise indicated.
6. All questions carry equal marks.
7. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. The candidate should respect the instructions given by the invigilator.
10. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name:

Mobile No.

Test Centre

Email.:

I have read all the instructions and shall abide by them

Signature of the Candidate

_ _

I have verified the information filled by the candidate above

Signature of the invigilator

_ _

P.T.O.

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Total Marks

1. Prove that $u(x, y) = 4xy - x^3 + 3xy^2$ is a harmonic function. Determine its harmonic Conjugate, hence find corresponding analytic function $f(z)$ interms of z. **[10]**

2. Prove that the function f defined by $f(z)=\begin{cases}\dfrac{z^3}{\overline{z}^2} & \text{for } z\neq 0\\ 0 & \text{for } z=0\end{cases}$

Satisfies the C-R equation at the origin, yet it is not differentiable there.

[12]

3. Use canchy's theorem and /or Cauchy integralformula

(i) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$

(ii) $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle |z| = 3.

(iii) $\oint_C \frac{e^{-z}}{z+1} dz,$ where C is the circle $|z| = \frac{1}{2}$ **[15]**

4. Prove that

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2} d\theta = \pi \frac{1-p+p^2}{1-p}, 0 < p < 1.$$ **[15]**

5. Let $f(z) = \dfrac{a_0 + a_1 z + \ldots\ldots. + a_{n-1} z^{n-1}}{b_0 + b_1 z + \ldots\ldots. + b_n z^n}, b_n \neq 0.$

Assume that the zeroes of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\dfrac{a_{n-1}}{b_n}$. **[13]**

6. Use the method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}, a^2 > b^2+c^2$$ **[15]**

7. (i) Let f = u + iv be an analytic function on the unit disc D = $\{z \in \mathbb{C} : |z| < 1\}$. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ at all points of D.

(ii) Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\frac{1}{z}\right)$. **[20]**

8. Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, z \neq 0; f(0) = 0$$

satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist. **[10]**

9. Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has n zeroes in the unit circle. **[10]**

10. Show that an isolated singular point z_0 of a function f(z) is a pole of order m if and only if f(z) can be written in the form $f(z)=\frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_o. Moreover $\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$. **[16]**

11. Expand the function $f(z)=\dfrac{2z^2+11z}{(z+1)(z+4)}$ in a Laurent's series valid for 2 < z < 3. **[10]**

12. If $f(z) = u + iv$ is analytic function and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z.

[10]

13. (i) Suppose f(z) is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation Im f(z) = (Re f(z))^2, Z $\in$ D. Show that f(z) is a constant in D.

(ii) Evaluate the integral $\int_C \operatorname{Re}\left(z^2\right) dz$ from 0 to 2 + 4i along the curve C where C is a parabola $y = x^2$. **[16]**

14. Using Cauchy's Integral formula, evaluate the integral

$$\oint_C \frac{dz}{\left(z^2+4\right)^2} \text{ where } C : |z-i| = 2.$$ **[07]**

15. If the function f(z) is analytic and one valued in $|z-a| < \mathbf{R}$, prove that for $0 < r < \mathbf{R}$

$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta) e^{-i\theta} d\theta,$ where $P(\theta)$ real part of $(a + r e^{i\theta})$. **[13]**

16. Let $c : [0, 1] \to \mathbb{C}$ be curve, where $c(t) = e^{4\pi it}$, $0 \le t \le 1$. Evaluate the contour integral $\int_c \frac{dz}{2z^2 - 5z + 2}$ **[10]**

17. Find the Laurent series expansion of $f(z)=\dfrac{z^2-z+1}{z(z^2-3z+2)}$ in the powers of (z + 1) in the region $|z+1|>3$. **[15]**

18. Let f be an entire function whose Taylor series expansion with centre $z = 0$ has infinitely many terms. Show that $z = 0$ is an essential singularity of $f\left(\frac{1}{z}\right)$. **[15]**

19. Using contour integration, evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin x dx}{x\left(x^2 + a^2\right)}, a > 0$. **[18]**

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ROUGH SPACE