5(a). Solve the PDE $(D^2 - D'^2 + D + 3D' - 2)z = e^{(x-y)} - x^2 y$

SOLUTION

Given equation can be re written as

$$[(D+D_1')(D-D_1') + 2(D+D_1') - (D-D') - 2)z = e^{x-y} - x^2 y$$

$$(D - D' + 2)(D + D' + 1)z = e^{x-y} - x^2 y$$

C.F. $= e^{-2x}\phi_1(y+x) + e^x \phi_2(y-x)$

ϕ_1, ϕ_2 being arbitrary constants.

P.I. corresponding to e^{x-y}

$$\Rightarrow \quad \frac{1}{(D^2 - D'^2 + D + 3D' - 2)} e^{x-y} = \frac{1}{1-1+1+(3)-2} e^{x-y}$$

$$= \frac{e^{x-y}}{2}$$

P.I. corresponding to $(-x^2y)$

$$\Rightarrow \quad \frac{1}{D^2 - D_3^2 + D + 3D' - 2}(-x^2 y) = \frac{1}{2}\left(1 + \frac{(D-D')}{2}\right)^{-1} (1-(D+D'))^{-1} x^2 y$$

$$= \frac{1}{2}\left(1 + \frac{(D-D')}{2}\right)^{-1} \left(1 + (D+D') + (D+D')^2 \ldots\right) x^2 y$$

$$= \frac{1}{2}\left(1 + \left(\frac{D-D'}{2}\right)^{-1} \left(x^2 y + 2xy + x^2 + xy + 4x + 6\right)\right)$$

$$= \frac{1}{2}\left(1 - \left(\frac{D-D'}{2}\right) + \left(\frac{D-D'}{2}\right)^2 - \left(\frac{D-D'}{2}\right)^3 \left(x^2 y + 2xy + x^2 + 2y + 4x + 6\right)\right)$$

$$= \frac{1}{2}\left(1 - \frac{D}{2} + \frac{D'}{2} + \frac{D^2}{4} + \frac{D'^2}{4} - \frac{DD'}{2} + \frac{3D'D^2}{8} \ldots\right)\left(x^2 y + 2xy + x^2 + 2y + 4x + 6\right)$$

$$= \frac{1}{2}\left(x^2 y + 2xy + x^2 + 2y + 4x + 6 - \frac{1}{2}(2xy + 2y + 2x + 4)\right)$$

$$+ \frac{1}{2}\left(x^2 + 2x + 2\right) + \frac{1}{4}(2y + 2) + \frac{1}{4}(0) - (2x+2)\frac{3}{8}(2)$$

$$\text{P.I.} = \frac{1}{2}\left(x^2 y + xy + 3x^2/2 + 3y/2 + 3x + 21/4\right)$$

$$\therefore \text{ General solution } = e^{-2x}\phi_1(y+x) + e^x \phi_2(y-x) \frac{+1}{2} e^{x-y} + \frac{1}{2}\left(x^2 y + xy + \frac{3x^2}{2} + \frac{3y}{2} + 3x + 21/4\right)$$

5(b). Solve the PDE $(x+2z)\frac{\partial z}{\partial x}+(4zx-y)\frac{\partial z}{\partial y}=2x^2+y$.

SOLUTION

Lagranges auxiliary equations are given by $\frac{dx}{x+2z}=\frac{dy}{4xz-y}=\frac{dz}{2x^2+y}$ .....(1)

Choose y,x,–2z as multiplirs

$$(1) = \frac{ydx+xdy-2zdz}{y(x+2z)+x(4xz-y)-2z(2x^2+y4)}=\frac{d(xy-z^2)}{0}$$

$$= \boxed{xy-z^2=c_1}$$ .....(2)

Choosing 2x, –1, –1 as multipliers

$$\frac{2xdx-dy-dz}{2x(x+2z)-(4xz-y)-(2x^2+y4)}=\frac{d(x^2-y-z)}{0}$$

$x^2-y-z=c_2$ .....(3)

from (2) and (3). solution is given by

$\phi(xy-z^2, x^2-y-z)=0$

ϕ being arbitrary functions.

6(a). Find surface satisfying $\frac{\partial^2 z}{\partial x^2}=6x+2$ and touching $z = x^3+y^3$ along its section by the plane $x+y+1=0$.

SOLUTION

$$\frac{\partial^2 z}{\partial x^2}=6x+2$$

$$\frac{\partial p}{\partial x}=6x+2$$

$$p=3x^2+2x+\phi(y)$$

$$z=x^3+x^2+x\phi(y)+\psi(y) \qquad \text{....(1)}$$

Given surface satisfies $z=x^3+y^3$ and $(x+y+1) = 0$. (1) & (2) touch along their common section by $x+y+1=0$.

The values of $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ from (1) & (2) must be the same.

$\therefore$ from (1) $\frac{\partial z}{\partial x}=3x^2+2x+\phi(y)$ .....(4)

from the given surface $\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(x^3+y^3)=3x^2$ .....5)

Equating, (4) = (5) $\Rightarrow$

$$3x^2 = 3x^2+2x+\phi(y)$$

$$\phi(y) = -2x \text{ as } x+y+1=0.$$

$$\phi(y) = 2y+2 \qquad \text{.....(6)}$$

Similarly

$$\frac{\partial z}{\partial y} = x\phi'(y)+\psi'(y)$$

$$\frac{\partial z}{\partial y} = x(2)+\psi'(y) \qquad \text{from (6) } \phi'(y)=2$$

from (2)

$$\frac{\partial z}{\partial y} = 3y^2$$

$$3y^2 = 2x+\psi'(y)$$

$$\psi'(y) = 3y^2-2x$$

$$\psi'(y) = 3y^2+2(y+1)$$

$$\psi(y) = y^3+y^2+2y+c$$

$$\therefore \quad Z = x^3+x^2+x(2y+2)+y^3+y^2+2y+c$$

$$Z = x^3+y^3+x^2+2xy+y^2+2x+2y+c$$

Putting $z = x^3+y^3$, $x + y = -1$
Given $c = 1$

$\therefore$ $\boxed{Z=x^3+y^3+(x+y+1)^2}$ Required solution

6(c). Obtain the temperature distribution y(x, t) in a uniform bar of unit length whose one end is kept at 10^0C and the other end is insulated. also it is given that y(x,0) = 1–x, 0<x<1.

SOLUTION

Let the bar be placed along x axis with its one end at origin and other end at x = 1.

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2} \quad \text{....(1) heat equations}$$

Boundary conditions $y_x(1, t) = 0 \quad y(0,t) = 10.$

Initial conditions $y(x,0) = 1-x \qquad 0<x<1$

Taking y(x, t) = u (x, t) + 10

Boundary condition changes to

$$u_x(1,t) = 0 \quad \text{....(2)}$$

$$u(0,t) = 0 \quad \text{....(3)}$$

$$u(x,0) = -(x+9) \quad \text{....(4)}$$

Suppose $u(x,t) = X(x)\ T(t)$

$\therefore \quad XT' = kX''T$

$$\frac{T'}{\kappa T} = \frac{X''}{X} = \mu$$

$\Rightarrow \quad X'' - \mu X = 0 \quad \text{.....(5)}$

$T' - \mu kT = 0 \quad \text{.....(6)}$

From (2) and (3) $X'(1)T(t)=0;\ X(0)T(t)=0$

T(t) depends on t and for some t , $T(t)\neq 0$.

$\Rightarrow X'(1) = 0,\ X(0) = 0$

From (5)

Case(i): $\mu = 0$

$$X(x) = Ax + B$$

$$X'(1) = 0 \Rightarrow A = 0$$

$$X(0) = 0 \Rightarrow B = 0$$

$\therefore\ X(x) = 0\ \therefore$ we reject $\mu = 0$.

Case(ii) $\mu = \lambda^2,\ \lambda \neq 0$.

$$X'' - \mu x = 0$$

$$X'' - \lambda^2 X = 0$$

$\Rightarrow \quad X = A e^{\lambda x} + B e^{-\lambda x}$

$$X'(1) = A\lambda e^{\lambda} + Be^{-\lambda}$$

$$X(0) = A+B$$

$$X'(1) = 0;\ X(0) = 0$$

$\Rightarrow A = 0,\ B = 0$

$\therefore$ we reject $\mu = \lambda^2$

Case(iii) $\mu = -\lambda^2,\ \lambda \neq 0$

$$X'' + \lambda^2 X = 0$$

$\therefore \quad X(x) = A\cos\lambda x + B\sin\lambda x$

$$X'(1) = -A\sin\lambda + B\lambda\cos\lambda = 0$$

$$X(0) = A = 0$$

$$\cos\lambda = 0 \Rightarrow \lambda = (2n-1)\frac{\pi}{2}$$

$$X_n(n) = B_n \sin\left(\frac{2n-1}{2}\pi x\right)$$

From(6) $\quad T - \mu k T = 0$

$$T' + \left[\left(\frac{2n-1}{2}\right)\pi\right]^2 \kappa T = 0$$

$$\Rightarrow T_n(t) = D_n e^{-c_n^2 t}$$

$$\boxed{c_n^2 = \frac{(2n-1)^2}{4}\pi^2 k}$$

$$\therefore \quad u_n(x,t) = X(x)T(t)$$

$$\Rightarrow \quad u_n(x,t) = E_n\left[\sin.\frac{(2n-1)\pi x}{2}\right]e^{-c_n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} E_n\left[\sin\frac{(2n-1)\pi x}{2}\right]e^{-c_n^2 t}$$

Putting t = 0

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$$

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$$

$$\therefore \quad E_n = \frac{2}{2}\int_0^1 (-x-9)\sin\frac{(2n-1)\pi x}{2}.dx$$

$$\therefore \quad E_n = -2\left[\frac{(x+9)\left(-\cos\left(\frac{2n-1}{2}\right)\pi x\right)}{(2n-1)\pi/2} - \frac{(-1)\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right]_0^1$$

$$\boxed{E_n = \frac{8(-1)^n}{(2n-1)\pi^2} - \frac{36}{(2n-1)\pi}}$$

$$\boxed{y(x,t) = 10 + E_n\left[\sin(2n-1)\frac{\pi x}{2}\right]e^{-c_n^2 t}}$$

Required solutions.