Ex. 9. *Show that in a two-dimensional incompressible steady flow field the equation of continuity is satisfied with the velocity components in* rectangular coordinates given by

$$u(x, y) = \frac{k(x^2 - y^2)}{(x^2 + y^2)^2}, \qquad v(x, y) = \frac{2kxy}{(x^2 + y^2)^2},$$

where k is an arbitrary constant. **[Meerut 1994; Rolhilkhand 2001, 03, 04]**

Sol. The equation of continuity for incompressible steady flow in cartesian coordinates is

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0 \qquad ...(1)$$

For a two dimensional flow in xy-plane, $w = 0$ so that (1) reduce to

$$\partial u / \partial x + \partial v / \partial y = 0 \qquad ...(2)$$

Differentiating the given values of u and v partially *w.r.t* 'x' and 'y' respectively, we get

$$\frac{\partial u}{\partial x} = k(x^2 - y^2)\frac{(-2) \times (2x)}{(x^2 + y^2)^3} + \frac{k \times 2x}{(x^2 + y^2)^2} = -4kx\frac{(x^2 - y^2)}{(x^2 + y^2)^3} + \frac{2kx}{(x^2 + y^2)^2} \qquad ...(3)$$

$$\frac{\partial v}{\partial y} = 2kxy\frac{(-2) \times (2y)}{(x^2 + y^2)^3} + \frac{2kx}{(x^2 + y^2)^2} = -\frac{8kxy^2}{(x^2 + y^2)^3} + \frac{2kx}{(x^2 + y^2)^2} \qquad ...(4)$$

From (3) and (4), we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{-4kx^3 + 4kxy^2 - 8kxy^2}{(x^2 + y^2)^3} + \frac{4kx}{(x^2 + y^2)^2} = \frac{-4kx^3 + 4kxy^2 - 8kxy^2 + 4kx(x^2 + y^2)}{(x^2 + y^2)^3} = 0$$

Hence, the equation of continuity (2) is satisfied.

Ex. 1. *Show that the surface* $\dfrac{x^2}{a^2k^2t^4} + kt^2\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right) = 1$ *is a possible form of boundary surface of a liquid at time t.* **[I.A.S. 1992; Punjab 2002; Rohilkhand 2001]**

Sol. The given surface

$$F(x, y, z, t) = \frac{x^2}{a^2k^2t^4} + kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - 1 = 0 \qquad ...(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F / \partial t + u(\partial F / \partial x) + v(\partial F / \partial y) + w(\partial F / \partial z) = 0 \qquad ...(2)$$

and the same values of u, v, w satisfy the equation of continuity

$$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0 \qquad ...(3)$$

From (1), $\dfrac{\partial F}{\partial t} = -\dfrac{4x^2}{a^2k^2t^5} + 2kt\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)$, $\dfrac{\partial F}{\partial x} = \dfrac{2x}{a^2k^2t^4}$, $\dfrac{\partial F}{\partial y} = \dfrac{2kt^2 y}{b^2}$, $\dfrac{\partial F}{\partial z} = \dfrac{2kt^2 z}{c^2}$

With these values, (2) reduces to

$$-\frac{4x^2}{a^2k^2t^5} + 2kt\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + \frac{2xu}{a^2k^2t^4} + \frac{2kt^2 yv}{b^2} + \frac{2kt^2 zw}{c^2} = 0,$$

or

$$\frac{2x}{a^2k^2t^4}\left(u - \frac{2x}{t}\right) + \frac{2kyt}{b^2}(y + vt) + \frac{2ktz}{c^2}(z + wt) = 0,$$

which is identically satisfied if we take

$$u = 2x/t, \qquad v = -y/t, \qquad w = -z/t \qquad ...(4)$$

From (4), $$\frac{\partial u}{\partial x} = \frac{2}{t}, \qquad \frac{\partial v}{\partial y} = -\frac{1}{t}, \qquad \frac{\partial w}{\partial z} = -\frac{1}{t} \qquad ...(5)$$

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4).

Ex. 2. (*i*) *Determine the restrictions on* f_1, f_2, f_3 *if* $(x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) = 1$ *is a possible boundary surface of a liquid.*

[Agra 2005; I.A.S.1995; Kanpur 2011; Meerut 2000]

(*ii*) Show that $(x^2/a^2) f(t) + (y^2/b^2)\,\phi(t) + (z^2/c^2)\,\psi(t) = 1$ *is a possible from of the boundary surface if* $f(t)\,\phi(t)\,\psi(t) = 1$.

Sol. (*i*) The given surface

$$F(x, y, z, t) = (x^2/a^2) f_1(t) + (y^2/b^2) f_2(t) + (z^2/c^2) f_3(t) - 1 = 0 \qquad ...(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$(\partial F/\partial t) + u(\partial F/\partial x) + v(\partial F/\partial y) + w(\partial F/\partial z) = 0 \qquad ....(2)$$

and the same values of u, v w satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(3)$$

Using dashes for differentiation with respect to t, (1) gives

$$\frac{\partial F}{\partial t} = \frac{x^2}{a^2} f_1'(t) + \frac{y^2}{b^2} f_2'(t) + \frac{z^2}{c^2} f_3'(t), \qquad \frac{\partial F}{\partial x} = \frac{2x}{a^2} f_1(t), \qquad \frac{\partial F}{\partial y} = \frac{2y}{b^2} f_2(t), \qquad \frac{\partial F}{\partial z} = \frac{2z}{c^2} f_3(t)$$

With these values, (2) reduces to

$$\frac{x^2 f_1'}{a^2} + \frac{y^2 f_2'}{b^2} + \frac{z^2 f_3'}{c^2} + \frac{2xf_1u}{a^2} + \frac{2yf_2v}{b^2} + \frac{2zf_3w}{c^2} = 0$$

or $$\frac{2xf_1}{a^2}\left(u + \frac{xf_1'}{2f_1}\right) + \frac{2yf_2}{b^2}\left(v + \frac{yf_2'}{2f_2}\right) + \frac{2zf_3}{c^2}\left(w + \frac{zf_3'}{2f_3}\right) = 0$$

which is identically satisfied if we take

$$u = -\frac{xf_1'}{2f_1}, \qquad v = -\frac{yf_2'}{2f_2}, \qquad w = -\frac{zf_3'}{2f_3} \qquad ...(4)$$

From (4), $$\frac{\partial u}{\partial x} = -\frac{f_1'}{2f_1}; \qquad \frac{\partial v}{\partial y} = -\frac{f_2'}{2f_2}; \qquad \frac{\partial w}{\partial z} = -\frac{f_3'}{2f_3} \qquad ...(5)$$

Now the required restriction will be obtained if the above velocity components satisfy (3). Hence, we get

$$-\frac{f_1'}{2f_1}-\frac{f_2'}{2f_2}-\frac{f_3'}{2f_3}=0 \qquad \text{or} \qquad \frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}=0$$

Integrating, $\log f_1 + \log f_2 + \log f_3 = \log c$

or $\log (f_1 f_2 f_3) = \log c$ or $f_1 f_2 f_3 = c$, where c is an arbitrary constant.

(*ii*) Proceed as in the above example. There is no loss of generality if c is taken as unity.

Ex. 3. *Show that $(x^2/a^2)\tan^2 t + (y^2/b^2)\cot^2 t = 1$ is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.*

[Garhwal 2005; I.A.S. 1997; Kanpur 1999, 2004, 08; Rajasthan 2004; Meerut 2003, 05; Rohilkhand 2002; 05; Purvanchel 2004]

Sol. For the present two dimensional motion $(\partial F/\partial z = 0$ and $\partial w/\partial z = 0)$, the surface

$$F(x, y, t) = (x^2/a^2)\tan^2 t + (y^2/b^2)\cot^2 t - 1 = 0 \qquad \text{...(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\partial F/\partial t + u(\partial F/\partial x) + v(\partial F/\partial y) = 0 \qquad \text{...(2)}$$

and the same values of u and v satisfy the equation of continuity

$$\partial u/\partial x + \partial v/\partial y = 0 \qquad \text{...(3)}$$

From (1), $\dfrac{\partial F}{\partial t}=\dfrac{x^2}{a^2}\cdot 2\tan t\sec^2 t-\dfrac{y^2}{b^2}\cdot 2\cot t\,\mathrm{cosec}^2 t$, $\dfrac{\partial F}{\partial x}=\dfrac{2x}{a^2}\tan^2 t$, $\dfrac{\partial F}{\partial y}=\dfrac{2y}{b^2}\cot^2 t$

With these values, (2) reduces to

$$\frac{x\tan t}{a^2}(x\sec^2 t+u\tan t)+\frac{y\cot t}{b^2}(-y\,\mathrm{cosec}^2 t+v\cot t)=0,$$

which is identically satisfied if we take

$$x\sec^2 t + u\tan t = 0 \qquad \text{and} \qquad -y\,\mathrm{cosec}^2 t + v\cot t = 0$$

i.e. $$u=-\frac{x}{\sin t\cos t} \qquad \text{and} \qquad v=\frac{y}{\sin t\cos t} \qquad \text{...(4)}$$

From (4), $$\frac{\partial u}{\partial x}=-\frac{1}{\sin t\cos t} \qquad \text{and} \qquad \frac{\partial v}{\partial y}=\frac{1}{\sin t\cos t} \qquad \text{...(5)}$$

Using (5), we find that (3) is also satisfied by the above values of u, v. Hence (1) is a possible bounding surface with velocity components given by (4).

Using remark 2 of Art. 2.18 (with $\partial F/\partial z = 0$ here), the normal velocity

$$=\frac{u(\partial F/\partial x)+v(\partial F/\partial y)}{\sqrt{\{(\partial F/\partial x)^2+(\partial F/\partial y)^2\}}}=\frac{-\dfrac{x}{\sin t\cos t}\cdot\dfrac{2x\tan^2 t}{a^2}+\dfrac{y}{\sin t\cos t}\cdot\dfrac{2y\cot^2 t}{b^2}}{\left\{\left(\dfrac{2x\tan^2 t}{a^2}\right)^2+\left(\dfrac{2y\cot^2 t}{b^2}\right)^2\right\}^{1/2}}$$

$$=\frac{a^2y^2\cot t\,\mathrm{cosec}^2 t-b^2x^2\tan t\sec^2 t}{\sqrt{(x^2b^4\tan^4 t+y^2a^4\cot^4 t)}}$$

Ex. 4. **(*a*)** *Show that the ellipsoid* $x^2/\left(a^2k^2t^{2n}\right)+kt^n\{(y/b)^2+(z/c)^2\}=1$ *is a possible form of the boundary surface of a liquid. Derive also velocity components.*

(Kanpur 2009; 2010; Meerut 2007)

(*b*) Show that the variable ellipsoid $x^2/\left(a^2k^2t^4\right)+kt^2\{(y/b)^2+(z/c)^2\}=1$ is a possible form for the boundary surface at any time t. **(Kanpur 2007)**

Sol. (a) The given surface

$$F(x,y,z,t)=x^2/\left(a^2k^2t^{2n}\right)+kt^n\{(y/b)^2+(z/c)^2\}-1=0 \qquad \text{... (1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condtion

$$\partial F/\partial t+u(\partial F/\partial x)+v(\partial F/\partial y)+w(\partial F/\partial z)=0 \qquad \text{...(2)}$$

and the same values of u, v w stisfy the equation of continuity

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z=0. \qquad \text{...(3)}$$

From (1), $$\frac{\partial F}{\partial t}=-\frac{x^2}{a^2k^2}\cdot\frac{2n}{t^{2n+1}}+nkt^{n-1}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right),$$

$$\frac{\partial F}{\partial t}=\frac{2x}{a^2k^2t^{2n}}, \qquad \frac{\partial F}{\partial y}=\frac{2kt^n y}{b^2} \qquad \text{and} \qquad \frac{\partial F}{\partial z}=\frac{2kt^n z}{c^2}.$$

With these values, (2) reduces to

$$-\frac{x^2}{a^2k^2}\frac{2n}{t^{2n+1}}+nkt^{n-1}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\frac{2xu}{a^2k^2t^{2n}}+\frac{2kt^n vy}{b^2}+\frac{2kt^n zw}{c^2}=0$$

or $$\left(u-\frac{nx}{t}\right)\frac{2x}{a^2k^2t^{2n}}+\left(v+\frac{ny}{2t}\right)\frac{2kyt^n}{b^2}+\left(w+\frac{nz}{2t}\right)\frac{2kzt^n}{c^2}=0,$$

which is identically satisfied if we take

$u-(nx/t)=0,\qquad v+(ny/2t)=0 \qquad$ and $\qquad w+(nz/2t)=0$

or $\qquad u=nx/t,\qquad v=-ny/2t \qquad$ and $\qquad w=-nz/2t. \qquad$ ...(4)

From (4), $\quad \partial u/\partial x=n/t, \qquad \partial v/\partial y=-n/2t \qquad$ and $\quad \partial w/\partial z=-n/2t \qquad$ ...(5)

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given by (4)

(b) Proceed as in part (a) by taking $n=2$

2.20. Streamline or line of flow.
[I.A.S. 1995; Kurkshetra 1998; U. P. P. C. S. 2000, Agra 2004, 2009 Kanpur 2000, 04, Meerut 2001, 02, 05, 12; G. N. D. U. Amritsar 1999]

A streamline is a curve drawn in the fluid so that its tangent at each point is the direction of motion (*i.e.* fluid velocity) at that point.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a straight line and let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ be the fluid velocity at P. Then $\mathbf{q}$ is parallel to $d\mathbf{r}$ at P on the steamline. Thus, the equation of streamlines is given by

$$\mathbf{q} \times d\mathbf{r} = \mathbf{0} \qquad ...(1)$$

i.e., $$(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \times (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) = \mathbf{0}$$

or $$(vdz - wdy)\,\mathbf{i} + (wdx - udz)\,\mathbf{j} + (udy - vdx)\,\mathbf{k} = \mathbf{0}$$

whence $$vdz - wdy = 0, \qquad wdx - udz = 0, \qquad udy - vdx = 0$$

so that $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \qquad ...(2)$$

The equations (2) have a double infinite set of solutions. Through each point of the flow field where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ do not all vanish, there passes one and only one streamline at a given instant. This fact can be verified by employing the well known existence theorem for the system of equations (2). If the velocity vanishes at a given point, various singularities occur there. Such a point is known as a *critical point* or *stagnation point*.

2.21. Path line or path of a particle.
[Meerut 2012; Kanpur 2000, 02]

A path line is the curve or trajectory along which a particular fluid particle travels during its motion.

The differential equation of a path line is $$d\mathbf{r}/dt = \mathbf{q} \qquad ...(1)$$

so that $$dx/dt = u, \qquad dy/dt = v \qquad \text{and} \qquad dz/dt = w \qquad ...(2)$$

where $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} \qquad \text{and} \qquad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + w\mathbf{k}.$$

Remark. Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then the path line is determined from equations

$$\left.\begin{aligned} dx/dt &= u(x, y, z, t) \\ dy/dt &= v(x, y, z, t) \\ dz/dt &= w(x, y, z, t) \end{aligned}\right\} \qquad ...(3)$$

with initial conditions $$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0 \qquad ...(4)$$

Ex. 4. *Find the streamlines and paths of the particles when*

$$u = x/(1 + t), \qquad v = y/(1 + t), \qquad w = z/(1 + t). \qquad \textbf{[I.A.S. 1994]}$$

Sol. Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e. \qquad \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

i.e. $$(dx)/x = (dy)/y = (dz)/z \qquad ...(1)$$

Taking the first two members of (1), we get $x/y = c_1$ ...(2)

Taking the last two members of (1), we get $y/z = c_2$ ...(3)

The desired streamlines are given by the intersection of (2) and (3).

The paths of the particle are given by

$$dx/dt = u, \qquad dy/dt = v, \qquad dz/dt = w$$

i.e. $$dx/dt = x/(1 + t), \qquad dy/dt = y/(1 + t), \qquad dz/dt = z/(1 + t)$$

giving $\dfrac{dx}{x} = \dfrac{dt}{1+t}$, $\dfrac{dy}{y} = \dfrac{dt}{1+t}$, $\dfrac{dz}{z} = \dfrac{dt}{1+t}$

Integrating, $x = c_3 (1 + t)$, $y = c_4 (1 + t)$, $z = c_5 (1 + t)$
which give the desired paths of the particles, c_3, c_4 and c_5 being arbitrary constants.

Ex.10. *Determine the streamlines and the path lines of the particle when the components of the velocity field are given by $u = x / (1 + t)$, $v = y/(2 + t)$ and $w = z/(3 + t)$. Also state the condition for which the steamlines are identical with path lines.* **[I.A.S. 2000]**

Sol. Streamlines are given by $dx/u = dy/v = dz/w$

or $(1 + t) (1/x)dx = (2 + t) (1/y)dy = (3 + t) (1/z)dz$. ...(1)

Taking the first two members of (1), we have

$$(1/x)dx + (t/x)dx = (2/y)dy + (t/y)dy$$

or $(1/x)dx - (2/y)dy = t \{(1/y)dy - (1/x)dx\}$.

Integrating, $\log x - 2 \log y = t (\log y - \log x) + \log c_1$, c_1 being an arbitrary constant

or $\log (x/y^2) = \log \{c (y/x)^t\}$ so that $(y/x)^t = x/c_1y^2$. ...(2)

Similarly, taking the last two members of (1), we have

or $\log (y^2/z^3) = \log\{ c_2 (y/z)^t\}$ or $(y/z)^t = y^2/c_2z^3$. ...(3)

The desired stremlines at a given instant $t = t_0$ are given by the intersection of the surfaces (2) and (3) by substituting t_0 for t.

Again, the path lines are given by

$dx/dt = u$, $dy/dt = v$ $dz/dt = w$

or $dx/dt = x/(1 + t)$, $dy/dt = y/(2 + t)$, $dz/dt = z/(3 + t)$,

giving $dx/x = dt/(1 + t)$, $dy/y = dt/(2 + t)$, $dz/z = dt/(3 + t)$.

Integraing, $x = c_3 (1 + t)$, $y = c_4 (2 + t)$, $z = c_5 (3 + t)$, c_3, c_4, c_5 being arbitrary constants which gives the desired paths of the given particle in terms of the parameter t.

Condition under which the stremlines and path linear are indentical.

In the case of steady motion the steamlines remain unchanged as time progresses and hence they are identical with the path lines.

2.26. The velocity potential or velocity function.

[Meerut 2005, 09; Rohilkhand 2004, 05]

Suppose that the fluid velocity at time t is $\mathbf{q} = (u, v, w)$. Further suppose that at the considered instant t, there exists a scalar function ϕ (x, y, z, t), uniform throughout the entire field of flow and such that

$$-d\phi = u\, dx + v\, dy + w\, dz \qquad \text{...(1)}$$

i.e.

$$-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = u\, dx + v\, dy + w\, dz \qquad \text{...(2)}$$

Then the expression on the R.H.S. of (1) is an exact differential and we have

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(3)}$$

$$\therefore \qquad \mathbf{q} = -\nabla\phi = -\text{grad}\,\phi. \qquad \text{...(4)}$$

ϕ is called the *velocity potential*. The negative sign in (4) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \mathbf{q} = 0, \qquad i.e. \qquad \text{curl}\ \mathbf{q} = \mathbf{0} \qquad \text{...(5)}$$

or

$$\mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \mathbf{0} \qquad \text{...(6)}$$

Remark 1. The surfaces $\qquad \phi\,(x, y, z, t) = \text{const.} \qquad$ ...(7)

are called the *equipotenials*. The streamlines

$$dx/u = dy/v = dz/w \qquad \text{...(8)}$$

are cut at right angles by the surfaces given by the differential equation

$$udx + vdy + wdz = 0 \qquad \text{...(9)}$$

and the condition for the existence of such orthogonal surfaces is the condition that (9) may posses a solution of the form (7) at the considered instant t, the analytical condition being

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad \text{...(10)}$$

When the velocity potential exists, (3) holds. Then

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\phi}{\partial z\partial y} = 0, \qquad i.e., \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \qquad \text{...(11)}$$

Similarly, $\qquad \partial u/\partial z = \partial w/\partial x \qquad$ and $\qquad \partial v/\partial x = \partial u/\partial y \qquad$ ...(12)

Using (11) and (12), we find that the condition (10) is satisfied. Hence surfaces exist which cut the streamlines orthogonallly. We also conclude that at all points of field of flow the equipotentials are cut orthogonally by the streamlines.

Remark 2. When (5) holds, the flow is known as the *potential kind*. It is also known as *irrototional*. For such flow the field of $\mathbf{q}$ is *conservative*.

Remark 3. The equation of continuity of an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(13)}$$

Suppose that the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y, \qquad w = -\partial\phi/\partial z \qquad \text{...(14)}$$

Using (14), (13) reduces to

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2 = 0, \qquad ...(15)$$

showing that ϕ is a harmonic function satisfying the Laplace euation $\nabla^2\phi = 0$, where

$$\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad ...(16)$$

2.27. The Vorticity Vector. [Kanpur 2004, Garhwal 2005]

Let $\mathbf{q} = ui + vi + wk$ be the fluid velocity such that curl $\mathbf{q} \neq \mathbf{0}$. Then the vector

$$\mathbf{\Omega} = \text{curl}\,\mathbf{q} \qquad ...(1)$$

is called the *vorticity vector*.

Let $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ be the components of $\mathbf{\Omega}$ in cartesian coordinates. Then (1) reduces to

$$\mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so that $\qquad \mathbf{\Omega}_x = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}, \qquad \mathbf{\Omega}_y = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}, \qquad \mathbf{\Omega}_z = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}.$

Note. Some authors use ξ, η, ζ, for $\mathbf{\Omega}_x, \mathbf{\Omega}_y, \mathbf{\Omega}_z$ and define $\mathbf{\Omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = (1/2) \times \text{curl}\,\mathbf{q}$. Thus, we have

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

2.28. Vortex line [Agra 2004, 2009; Garhwal 2005]

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector $\mathbf{\Omega}$.

Let $\mathbf{\Omega} = \mathbf{\Omega}_x\mathbf{i} + \mathbf{\Omega}_y\mathbf{j} + \mathbf{\Omega}_z\mathbf{k}$ and let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of a point P on a vortex line. Then $\mathbf{\Omega}$ is parallel to $d\mathbf{r}$ at P on the vortex line. Hence the equation of vortex lines is given by $\qquad \mathbf{\Omega} \times d\mathbf{r} = \mathbf{0}$,

i.e. $\qquad (\Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}) \times (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \mathbf{0}$

or $\qquad (\Omega_y dz - \Omega_z dy)\mathbf{i} + (\Omega_z dx - \Omega_x dz)\mathbf{j} + (\Omega_x dy - \Omega_y dx)\mathbf{k} = \mathbf{0}$

whence $\qquad \Omega_y dz - \Omega_z dy = 0, \qquad \Omega_z dx - \Omega_x dz = 0, \qquad \Omega_x dy - \Omega_y dx = 0$

so that

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \qquad ...(1)$$

(1) gives the disired equations of vortex lines.

2.30. Rotational and irrotationl motion.

[Agra 2011; Garhwal 2005; I.A.S. 2000; G.N.D.U. Amritser 2003; Meerut 2002, 09, 10]

The motion of a fluid is said to be *irrotational* when the vorticity vector Ω of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be *rotational*.

Since $\quad \mathbf{\Omega} = \text{curl}\,\mathbf{q} \quad$ and $\quad \mathbf{\Omega} = \left(\dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}\right)\mathbf{k}$,

we conclude that the motion is irrotation if $\quad \text{curl}\,\mathbf{q} = \mathbf{0}$

or $\quad \partial w/\partial y = \partial v/\partial z, \quad \partial u/\partial z = \partial w/\partial x, \quad \partial v/\partial x = \partial u/\partial y,$

When the motion is irrotational i.e. when $\text{curl}\,\mathbf{q} = \mathbf{0}$, then $\mathbf{q}$ must be of the form ($-$ grad ϕ) for some scalar point function ϕ (say) because curl grad $\phi = 0$. Thus velocity potential exists whenever the fluid motion is irrotational. Again notice that when velocity potential exists, the motion is irrotational because $\mathbf{q} = -\,\text{grad}\,\phi \Rightarrow \text{curl}\,\mathbf{q} = -\,\text{curl grad}\,\phi = \mathbf{0}$.

Thus, the fluid motion is irrotational if and only if the velocity potential exists. **(Meerut 2009, 10)**

Rotational motion is also said to be *vortex motion*. Again by definition it follows that there are no vortex lines in an irrotational fluid motion.

Ex. 3. (*a*) *Test whether the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2+y^2}$ (k = const), *is a possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.*

[Kanpur 2006; I.A.S. 1996, Rohilkhand 2003, 04]

(*b*) *Determine the velocity potential for the motion specified by* $\mathbf{q} = \dfrac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2+y^2}$, ($k$ = const).

[Agra 2007]

Sol. (*a*) Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$. Then here

$$u = -\frac{k^2 y}{x^2+y^2}, \qquad v = \frac{k^2 x}{x^2+y^2}, \qquad w = 0 \qquad \text{...(1)}$$

The equation of continuity for an incompressible fluid is

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad \text{...(2)}$$

Form (1), $\quad \dfrac{\partial u}{\partial x} = \dfrac{2k^2xy}{(x^2+y^2)^2}, \qquad \dfrac{\partial v}{\partial y} = -\dfrac{2k^2xy}{(x^2+y^2)^2}, \qquad \dfrac{\partial w}{\partial z} = 0$

Hence (2) is satisfied and so the motion specified by given $\mathbf{q}$ is possible.

The equation of the streamlines are $\quad dx/u = dy/v = dz/w$

i.e.
$$\frac{dx}{-k^2y/(x^2+y^2)} = \frac{dy}{k^2x/(x^2+y^2)} = \frac{dz}{0} \qquad \text{...(3)}$$

Taking the last fraction, $\quad dz = 0 \quad$ so that $\quad z = c_1$. $\qquad$ ...(4)

Taking the first two fractions in (3) and simplifying, we get

$$dx/(-y) = dy/x \qquad \text{or} \qquad 2xdx + 2ydy = 0$$

Integrating, $\quad x^2 + y^2 = c_2$, c_2 being an arbitrary constant $\qquad$ ...(5)

(4) and (5) together give the streamlines. Clearly, the streamlines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

Again $\quad \operatorname{curl}\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{k^2 y}{x^2+y^2} & \dfrac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} = k^2\left\{\dfrac{y^2-x^2}{(x^2+y^2)^2} + \dfrac{x^2-y^2}{(x^2+y^2)^2}\right\}\mathbf{k} = \mathbf{0}.$

Hence the flow is of the potential kind and we can find velocity potential $\phi\ (x, y, z)$ such that $\mathbf{q} = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \qquad \text{...(6)}$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2} \qquad \text{...(7)}$$

$$\partial\phi/\partial z = -w = 0 \qquad \text{...(8)}$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x, y)$.

Integrating (6), $\phi\ (x, y) = k^2 \tan^{-1}(x/y) + f(y)$, where $f(y)$ is an arbitrary function, ...(9)

From (9), $$\frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2} \qquad \text{...(10)}$$

Comparing (7) and (10), we have $\quad f'(y) = 0 \quad$ so that

$f(y)$ = constant.

Since the constant can be omitted while writting velocity potential, the required velocity potential can be taken as [refer equation (9)]

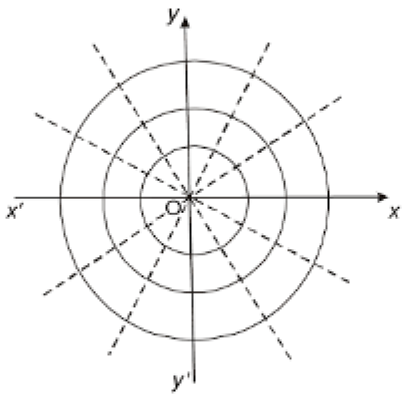
$$\phi(x, y) = k^2 \tan^{-1}(x/y) \qquad \text{...(11)}$$

The equipotentials are given by

$$k^2 \tan^{-1}(x/y) = \text{constant} = k^2 \tan^{-1} c$$

or $\quad x = cy$, c being a constant

which are planes through the z-axis. They are intersected by the streamlines as shown in the figure. Dotted lines represent equipotentials and ordinary lines represent streamlines.



(*b*) Proceed as in part (*a*) upto equation (11). Then the required velocity potential is given by (11).

Ex. 6. (*a*) *Show that* $u = -\dfrac{2xyz}{(x^2+y^2)^2}$, $v = \dfrac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w = \dfrac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrotational.

[Garhwal 2004; Agra 2004; Kerala 2001; I.A.S. 2000, 2002 Meerut 2002, 04]

(*b*) *Show that a fluid of constant density can have a velocity* **q** *given by*

$$\mathbf{q} = \left[-\frac{2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{x^2+y^2} \right]$$

Find the vorticity vector. **[Kanpur 2007; I.A.S. 1988, 98, 2000]**

Sol. Part (a). Here, we have

$$\frac{\partial u}{\partial x} = -2yz \frac{1 \cdot (x^2+y^2)^2 - x \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4} = -2yz \frac{x^2+y^2-4x^2}{(x^2+y^2)^3} = -2yz \frac{y^2-3x^2}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = z \frac{-2y(x^2+y^2)^2 - 2(x^2+y^2) \cdot 2y(x^2-y^2)}{(x^2+y^2)^4} = -2yz \frac{x^2+y^2+2(x^2-y^2)}{(x^2+y^2)^3} = -2yz \frac{3x^2-y^2}{(x^2+y^2)^3}$$

and $$\partial w / \partial z = 0$$

Hence the equation of continuity $$\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$$

is satisfied and so the liquid motion is possible.

Furthermore, we have

$$\mathbf{\Omega}_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\mathbf{\Omega}_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

and $$\mathbf{\Omega}_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} = 0$$

$\therefore \quad \partial w / \partial y = \partial v / \partial z, \quad \partial u / \partial z = \partial w / \partial x, \quad \partial v / \partial x = \partial u / \partial y$

and hence the motion is irrotational.

Part (b). Let **q** = (u, v, w). Then we have the same values of u, v, w as in part (a). By definition, the vorticity vector $\mathbf{\Omega}$ is given by

$$\mathbf{\Omega} = \mathbf{\Omega}_x \mathbf{i} + \mathbf{\Omega}_y \mathbf{j} + \mathbf{\Omega}_z \mathbf{k} = \mathbf{0}, \text{ using part (a)}$$

Ex. 7. *Show that* $\phi = (x-t)(y-t)$ *represents the velocity potential of an incompressible two dimensional fluid. Show that the streamlines at time 't' are the curves*

$$(x-t)^2 - (y-t)^2 = \text{constant},$$

and that the paths of the fluid particles have the equations

$$\log (x-y) = (1/2) \times \{(x+y) - a(x-y)^{-1}\} + b, \text{ where } a, b \text{ are constants.}$$

Sol. Given $\qquad \phi = (x-t)(y-t) \qquad \ldots (1)$

From (1), we have

$u = -\partial\phi/\partial x = -(y-t), \quad v = -\partial\phi/\partial y = -(x-t)$ and so $\quad \partial u/\partial x = 0 = \partial v/\partial y$

Thus the equation of continuity $\partial u/\partial x + \partial v/\partial y = 0$ is satisfied. Hence ϕ given by (1) represents the velocity potential of an incompressible two-dimensional flow.

Again, the equaiton of streamlines are given by

$$\frac{dx}{u} = \frac{dv}{v} \qquad \text{or} \qquad \frac{dx}{-(y-t)} = \frac{dy}{-(x-t)}$$

or $\qquad (x-t)\,dx - (y-t)\,dy = 0$

Integrating, $\qquad (x-t)^2 - (y-t)^2 = \text{constant}$

Finally, the paths of particles are given by

$$u = dx/dt = -(y-t) \qquad \text{and} \qquad v = dy/dt = -(x-t)$$

$\therefore \qquad dx/dt = t - y \qquad \ldots(2)$

and $\qquad dy/dt = t - x \qquad \ldots(3)$

From (2) and (3), $\qquad dx/dt + dy/dt = 2t - (x+y) \qquad \ldots(4)$

Let $x + y = z$ $\qquad$ so that $\qquad dx/dt + dy/dt = dz/dt \qquad \ldots(5)$

Then (4) gives $\quad dz/dt = 2t - z \quad$ or $\quad dz/dt + z = 2t \qquad \ldots(6)$

which is a linear differential equation.

Its integrating factor $= e^{\int dt} = e^t$. Here solution of (6) is

$$ze^t = \int 2te^t dt + c_1 = 2t \cdot e^t - \int (2)e^t dt + c_1 = 2te^t - 2e^t + c_1$$

$\therefore \qquad z = 2t - 2 + c_1 e^{-t} \qquad$ or $\qquad x + y = 2t - 2 + c_1 e^{-t}$, by (5) $\qquad \ldots(7)$

Again from (2) and (3), $\qquad dx/dt - (dy/dt) = x - y$

or $\qquad \dfrac{dx - dy}{dt} = x - y \qquad$ or $\qquad \dfrac{dx - dy}{x - y} = dt$

Integrating, $\qquad \log (x-y) - \log c_2 = t \qquad$ or $\qquad x - y = c_2\, e^t. \qquad \ldots(8)$

Using (7) and (8), we have

$\therefore \qquad x + y - a(x-y)^{-1} = 2t - 2 + c_1 e^{-t} - \dfrac{a}{c_2} e^{-t} = 2t - 2, \qquad$ taking $\quad c_1 = \dfrac{a}{c_2}$

$\therefore \qquad (1/2) \times \{(x+y) - a(x-y)^{-1}\} = t - 1 \qquad \ldots(9)$

But from (8), $\qquad e^t = (x-y)/c_2 \qquad$ so that $\qquad t = \log (x-y) - \log c_2$

$\therefore \qquad t - 1 = \log (x-y) - (\log c_2 + 1)$

$\therefore \qquad t - 1 = \log (x-y) - b, \qquad$ taking $\qquad b = -(\log c_2 + 1) \qquad \ldots(10)$

Using (10), (9) reduces to the required equations

$$(1/2) \times \{(x+y) - a(x-y)^{-1}\} = \log (x-y) + b.$$

Ex. 8(*a*). *If the velocity of an incompressible fluid at the point (x, y, z) is given by*

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$$

prove that the liquid motion is possible and that the velcity potential is $(\cos\theta)/r^2$. *Also determine the streamlines.*

Sol. Here $u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{3z^2-r^2}{r^5} = \dfrac{3z^2}{r^5} - \dfrac{1}{r^3}$ ...(1)

where $r^2 = x^2 + y^2 + z^2$ ...(2)

From (2), $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$, $\partial r/\partial z = z/r$ ...(3)

From (1), (2) and (3), we have

$$\frac{\partial u}{\partial x} = 3z\left[\frac{1}{r^5} + (-5x)r^{-6}\frac{\partial r}{\partial x}\right] = \frac{3z}{r^5} - \frac{15x^2 z}{r^7}$$

$$\frac{\partial v}{\partial y} = 3z\left[\frac{1}{r^5} + (-5y)r^{-6}\frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2 z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - 15z^2\, r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{9z}{r^5} - \frac{15z^3}{r^7}$$

$$\therefore \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = \frac{15z}{r^5} - \frac{15z}{r^7}\times r^2 = 0.$$

Since the equation of continuity is satisfied by the given values of u, v and w, the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(udx + vdy + wdz), \text{ by definition of } \phi$$

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right] = \frac{r^2dz - 3z\,(xdx + ydy + zdz)}{r^5}$$

Thus, $$d\phi = \frac{r^3dz - 3r^2zdr}{(r^3)^2} = d\left(\frac{z}{r^3}\right), \text{ using (2)}$$

Intergrating , $\phi = z/r^3$

[Omitting constant of integration, for it has no significance in ϕ]

In spherical polar coordinates (r, θ, ϕ), we know that $z = r \cos\theta$. Hence the required potential is given by $\phi = (r\cos\theta)/r^3 = (\cos\theta)/r^2$

We now obtain the streamlines. The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad i.e., \qquad \frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2 - r^2)/r^5}$$

or
$$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - r^2} \qquad ...(4)$$

Taking the first two members of (4) and simplifying, we get

$dx/x = dy/y$ or $dx/x - dy/y = 0$

Integrating, $\log x - \log y = \log c_1$ *i.e.* $x/y = c_1$, c_1 being a constant ...(5)

Now, each member in (4) $= \dfrac{xdx + ydy + zdz}{3x^2z + 3y^2z + 3z^3 - r^2z} = \dfrac{xdx + ydy + zdz}{3z(x^2 + y^2 + z^2) - r^2z}$

$$= \frac{xdx + ydy + zdz}{3z(x^2 + y^2 + z^2) - z(x^2 + y^2 + z^2)} = \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)}, \text{ by (2)} \qquad ...(6)$$

Taking the first member in (4) and (6), we get

$$\frac{dx}{3xz} = \frac{xdx + ydy + zdz}{2z(x^2 + y^2 + z^2)} \qquad \text{or} \qquad \frac{2}{3}\frac{dx}{x} = \frac{1}{2}\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2}$$

Integrating, $(2/3) \times \log x = (1/2) \times \log (x^2 + y^2 + z^2) + \log c_2$

or $x^{2/3} = c_2 (x^2 + y^2 + z^2)^{1/2}$, c_2 being an arbitrary constant ...(7)

The required streamlines are the curves of intersction of (5) and (7).

Ex. 8(*b*). If velocity distributon of an incompressible fluid at point (x, y, z) is gives by $\{3xz/r^5, 3yz/r^5\ (kz^2 - r^2)/r^5\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential. **[I.A.S. 2001]**

Sol. Here $u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{kz^2 - r^2}{r^5} = \dfrac{kz^2}{r^5} - \dfrac{1}{r^3}$, ...(1)

where $r^2 = x^2 + y^2 + z^2$ ...(2)

From (2), $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$ ...(3)

Now proceed as in solved Ex. 8(*a*) and obtain

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2z}{r^7}, \qquad \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2z}{r^7} \qquad ...(4)$$

and
$$\frac{\partial w}{\partial z} = \frac{2kz}{r^5} - 5kz^2r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7} \qquad ...(5)$$

Since (1) gives a possible liquid motion, the equation of continuity must be satisfied and so

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

or $\dfrac{(2k+9)z}{r^5} - \dfrac{15z}{r^7}(x^2 + y^2 + z^2) = 0$ or $\dfrac{(2k+9)z}{r^5} - \dfrac{15z}{r^7}\cdot r^2 = 0$, using (2),(4) and (5)

or $(2k - 6)\ z/r^5 = 0$ so that $2k - 6 = 0$ giving $k = 3$.

Substituting the above value of k in (1), we have

$u = (3xz)/r^5$, $v = (3yz)/r^5$, $w = (3z^2 - r^2)/r^5$. ...(6)

Using (6) and proceeding as in Ex. 8(*a*), the required velocity potential ϕ is given by $\phi = z/r^3$.

Ex. 9. (*a*) *Show that if the velocity potential of an irroatioanl fluid motion is equal to*

$$A\,(x^2+y^2+z^2)^{-3/2}\,z\,\tan^{-1}(y/x)$$

the lines of flow will be on the series of the surfaces $x^2+y^2+z^2 = c^{2/3}\,(x^2+y^2)^{2/3}$.

[Agra 2004, 06; Kanpur 2002, 11; Meerut 2004]

(*b*) *If the velocity potential of a fluid is* $\phi = (z/r^3)\tan^{-1}(y/x)$ *where* $r^2 = x^2+y^2+z^2$, then *show that the streamlines lie on the surfaces* $x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}$, *c being an arbitrary constant.* **[I.A.S. 2008]**

Sol. (*a*) The veloctiy potential ϕ is given by

$$\phi(x,y,z) = A(x^2+y^2+z^2)^{-3/2}z\tan^{-1}(y/x) = Ar^{-3}z\tan^{-1}(y/x) \qquad \text{...(1)}$$

where $$r^2 = x^2+y^2+z^2 \qquad \text{...(2)}$$

so that $$\partial r/\partial x = x/r, \qquad \partial r/\partial y = y/r, \qquad \partial z/\partial r = z/r \qquad \text{...(3)}$$

$\therefore$
$$u = -\frac{\partial\phi}{\partial x} = 3Azxr^{-5}\tan^{-1}\frac{y}{x} + \frac{Azyr^{-3}}{x^2+y^2}$$

$$v = -\frac{\partial\phi}{\partial y} = 3Azyr^{-5}\tan^{-1}\frac{y}{x} - \frac{Azxr^{-3}}{x^2+y^2}$$

$$w = -\frac{\partial\phi}{\partial z} = 3Az^2r^{-5}\tan^{-1}\frac{y}{x} - Ar^{-3}\tan^{-1}\frac{y}{x}$$

The equation of lines of flow are given by $dx/u = dy/v = dz/w$

i.e.
$$\frac{dx}{3Azxr^{-5}\tan^{-1}\dfrac{y}{x}+\dfrac{Azyr^{-3}}{x^2+y^2}} = \frac{dy}{3Azyr^{-5}\tan^{-1}\dfrac{y}{x}-\dfrac{Azxr^{-3}}{x^2+y^2}} = \frac{dz}{A(3z^2r^{-5}-r^{-3})\tan^{-1}\dfrac{y}{x}} \qquad \text{...(4)}$$

Each member of (4) is $= \dfrac{xdx+ydy+zdz}{(3x^2+3y^2+3z^2)r^{-2}-1} = \dfrac{xdx+ydy}{(3x^2+3y^2)/r^2}$ (on simplification)

or
$$\frac{xdx+ydy+zdz}{2} = \frac{r^2(xdx+ydy)}{3(x^2+y^2)}$$

or
$$\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{2}{3}\cdot\frac{2xdx+2ydy}{x^2+y^2} \qquad \text{...(5)}$$

Integrating (5), $\log(x^2+y^2+z^2) = (2/3)\times\log(x^2+y^2) + (2/3)\times\log c$

or $x^2+y^2+z^2 = c^{2/3}\,(x^2+y^2)^{2/3}$, c being an arbitrary constant ...(6)

(6) gives the required series of the surfaces on which the desired lines of flow will lie.

(*b*). Proceed like part (*a*) by taking $A = 1$. Thus obtain (5). Integrating (5), $\log(x^2+y^2+z^2) = (2/3)\times\log(x^2+y^2)+\log c$ giving $x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}$, c being an arbitrary constant.

Ex. 10. *Given $u = -Wy$, $v = Wx$, $w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of flow.*

Sol. Given $\qquad u = -Wy, \qquad v = Wx, \qquad w = 0 \qquad \ldots(1)$

$\therefore \qquad \partial u/\partial x = 0, \qquad \partial v/\partial y = 0, \qquad \partial w/\partial z = 0$

so that $\qquad \partial u/\partial x + \partial y/\partial y + \partial w/\partial z = 0. \qquad \ldots(2)$

(2) shows that the equation of continuity is satisfied and so the motion specified by (1) is possible. The equations of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w} \qquad i.e., \qquad \frac{dx}{-Wy} = \frac{dy}{Wx} = \frac{dz}{0}$$

giving $\qquad xdx + ydy = 0 \qquad$ and $\qquad dz = 0$

Integrating, $\quad x^2 + y^2 = c_1$ and $z = c_2$, c_1 and c_2 being arbitrary constants $\quad \ldots(3)$

Hence the streamlines are circles given by the intersection of surfaces (3).

The surfaces which cut the stream lines orthogonally are

$$udx + vdy + wdz = 0$$

i.e. $\qquad -Wydx + Wxdy = 0 \qquad$ or $\qquad dx/x - dy/y = 0$

Integrating, $\quad x/y = c \quad$ or $\quad x = cy$, c being an arbitrary constant, $\quad \ldots(4)$

which repressents a plane through z-axis and cuts the stream lines (3) orthogonally

Now $\qquad udx + vdy + wdz = -Wydx + Wxdy \qquad \ldots(5)$

Here $\qquad \frac{\partial}{\partial y}(-Wy) = -W \qquad$ and $\qquad \frac{\partial}{\partial x}(Wx) = W. \qquad \ldots(6)$

Hence $udx + vdy + wdz$ is not a perfect differential and so the velocity potential does not exist. Again, we have

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -Wy & Wx & 0 \end{vmatrix} = 2W\mathbf{k}.$$

Since curl $\mathbf{q} \neq \mathbf{0}$, the motion is rotational. Notice that a rigid body rotating about z-axis with constant vector angular velocity $2W\mathbf{k}$ will produce the above type of motion.

Ex. 11. *Prove that the liquid motion is possible when velocity at (x, y, z) is given by*

$$u = (3x^2 - r^2)/r^5, \qquad v = 3xy/r^5, \qquad w = 3xz/r^5,$$

where $r^2 = x^2 + y^2 + z^2$, and the streamlines lines are the intersection of the surfaces $(x^2 + y^2 + z^2)^3 = c\,(y^2 + z^2)^2$ by the planes passing through OX. State if the motion is irrotational giving reasons for your answer. **[Kanpur 2011; Agra 2008]**

Sol. Given $\quad u = (3x^2 - r^2)/r^5, \qquad v = 3xy/r^5, \qquad w = 3xz/r^5 \quad$ ...(1)

For the motion to be possible, we must show that the equation of continuity

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \qquad ...(2)$$

must be satisfied.

From (1),
$$\frac{\partial u}{\partial x} = \frac{[6x - 2r(\partial r/\partial x)]r^5 - 5r^4(\partial r/\partial x)(3x^2 - r^2)}{r^{10}} \qquad ...(3)$$

But
$$r^2 = x^2 + y^2 + z^2 \qquad ...(4)$$

From (4), $\quad \partial r/\partial x = x/r, \qquad \partial r/\partial y = y/r \quad$ and $\quad \partial r/\partial z = z/r \quad$ ...(5)

Using (5), (3) gives

$$\frac{\partial u}{\partial x} = \frac{(6x - 2x)r^5 - 5r^3x\,(3x^2 - r^2)}{r^{10}} = \frac{3x(3r^2 - 5x^2)}{r^7}$$

Similarly, $\quad \dfrac{\partial v}{\partial y} = \dfrac{3x(r^2 - 5y^2)}{r^7} \quad$ and $\quad \dfrac{\partial w}{\partial z} = \dfrac{3x(r^2 - 5z^2)}{r^7}.$

$\therefore$ L.H.S. of (2) $= \dfrac{3x[5r^2 - 5(x^2 + y^2 + z^2)]}{r^7} = 0$, using (4)

(2) is satisfied. So the liquid motion is possible. The equation of streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \text{or} \qquad \frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} \qquad ...(6)$$

Taking the last two members (6), we get

$dy/y = dz/z \qquad$ giving $\qquad y = az$, a being an arbitrary constant $\quad$ ...(7)

which is a plane passsing through OX.

Now each member of (6) $= \dfrac{xdx + ydy + zdz}{x(3r^2 - r^2)} = \dfrac{ydy + zdz}{3x(y^2 + z^2)}$

Thus,
$$\frac{3(2xdx + 2ydy + 2zdz)}{x^2 + y^2 + z^2} = \frac{2(2ydy + 2zdz)}{y^2 + z^2}$$

Integrating, $\qquad 3 \log (x^2 + y^2 + z^2) = 2 \log (y^2 + z^2) + \log c$

or $\qquad (x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, c being an arbitrary constant $\quad$ ...(8)

The required streamlines are given by the intersection of surfaces (8) by the planes (7) passing through OX.

Finally, to show that the motion is irrotational, we should verify the conditions:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad ...(9)$$

From (11), we have

$$\frac{\partial u}{\partial y} = -\frac{3y(5x^2 - r^2)}{r^7}, \qquad \frac{\partial u}{\partial z} = -\frac{3z(5x^2 - r^2)}{r^7}, \qquad \frac{\partial v}{\partial x} = \frac{3y(r^2 - 5x^2)}{r^7},$$

$$\frac{\partial v}{\partial z} = -\frac{15xyz}{r^7}, \qquad \frac{\partial w}{\partial x} = \frac{3z(r^2 - 5x^2)}{r^7}, \qquad \frac{\partial w}{\partial y} = -\frac{15xyz}{r^7}.$$

With these values, conditions (9) are all satisfied. Hence the motion is irrotational.

Ex. 14. *Show that the velocity poremtial* $\phi = (a/2)\times(x^2+y^2-2z^2)$ *satisfies the Laplace equation. Also determine the streamlines.* **[Nagpur 2003, I.A.S. 2002]**

Sol. We know that the velocity $\mathbf{q}$ of the fluid is given by

$$\mathbf{q} = -\nabla\phi = -\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left\{\frac{a}{2}(x^2+y^2-2z^2)\right\}$$

or $$\mathbf{q} = -(a/2)\times(2x\mathbf{i}+2y\mathbf{j}-4z\mathbf{k}). \qquad \ldots(1)$$

But $$\mathbf{q} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k}. \qquad \ldots(2)$$

Comparing (1) and (2), $u = -ax$, $v = -ay$, $w = 2az$.

The equaitons of streamlines are given by $dx/u = dy/v = dz/w$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \quad \text{or} \quad \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z}. \qquad \ldots(3)$$

Taking the first two fractions of (3), $(1/x)dx = (1/y)\,dy$.

Integrating, $\log x = \log y + \log c_1$ or $x = c_1 y$. ...(4)

Taking the last two fractions of (3), $(2/y)dy + (1/z)dz = 0$

Integrating, $2\log y + \log z = \log c_2$ or $y^2 z = c_2$. ...(5)

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that $$\phi = (a/2)\times(x^2+y^2-2z^2). \qquad \ldots(6)$$

From (6), $\partial\phi/\partial x = ax$, $\partial\phi/\partial y = ay$ and $\partial\phi/\partial z = -2az$

$\Rightarrow$ $\partial^2\phi/\partial x^2 = a$, $\partial^2\phi/\partial y^2 = a$ and $\partial^2\phi/\partial z^2 = -2a$

$\therefore$ $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2+\partial^2\phi/\partial z^2 = a+a-2a$ or $\nabla^2\phi = 0$,

showing that ϕ satisfies the Laplace equation.

Ex. 15. *Show that* $\phi = x f(r)$ *is a possible form for the velocity potential of an incompressible liquid motion. Given that the liquid speed* $q \to 0$ *as* $r \to \infty$, *deduce that the surfaces of constant speed are* $(r^2+3x^2)r^{-8}$ *= constant.*

Sol. Given $$\phi = xf(r). \qquad \ldots(1)$$

$\therefore$ $$\mathbf{q} = -\nabla\phi = -\nabla[xf(r)] = -[f(r)\nabla x + x\nabla f(r)]. \qquad \ldots(2)$$

Now, $r^2 = x^2+y^2+z^2 \Rightarrow 2r(\partial r/\partial x) = 2x \Rightarrow \partial r/\partial x = x/r$. ...(3)

Similarly, $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$. ...(4)

Also, $$\nabla x = [\mathbf{i}(\partial/\partial x)+\mathbf{j}(\partial/\partial y)+\mathbf{k}(\partial/\partial z)]x = \mathbf{i}$$

and $$\nabla f(r) = [\mathbf{i}(\partial/\partial x)+\mathbf{j}(\partial/\partial y)+\mathbf{k}(\partial/\partial z)]f(r)$$

$$= \mathbf{i}\, f'(r)\,(\partial r/\partial x) + \mathbf{j}\, f'(r)\,(\partial r/\partial y) + \mathbf{k}\, f'(r)\,(\partial r/\partial z)$$

$$= \mathbf{i}\, f'(r)\,(x/r) + \mathbf{j}\, f'(r)\,(y/r) + \mathbf{k}\, f'(r)\,(z/r), \text{ by (3) and (4)}$$

$$= (1/r) f'(r)\,(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = (1/r) f'(r)\mathbf{r}.$$

$\therefore$ (2) $\Rightarrow$ $$\mathbf{q} = -f(r)\mathbf{i} - (x/r) f'(r)\mathbf{r}. \qquad ...(5)$$

For a possible motion of an incompressible fluid, we have

$$\nabla \cdot \mathbf{q} = 0 \qquad \text{or} \qquad \nabla \cdot (-\nabla\phi) = 0 \qquad \text{or} \qquad \nabla^2\phi = 0$$

or $$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial/\partial z^2)[x\, f(r)] = 0, \text{ using (1)} \qquad ...(6)$$

Now, $$\frac{\partial^2}{\partial x^2}[x\, f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x\, f(r)\}\right] = \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$$

Also $$\frac{\partial^2}{\partial y^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2}{\partial z^2}[x\, f(r)] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ (6) becomes $$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0. \qquad ...(7)$$

Now, $$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'\frac{x}{r}, \text{ using (3).} \qquad ...(8)$$

and $$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$$

$$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x} = \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}.$$

$\therefore$ $$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \qquad ...(9)$$

Similarly, $$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f' \qquad ...(10)$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f' \qquad ...(11)$$

Adding (9), (10) and (11), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{x^2 + y^2 + z^2}{r^2}f'' - \frac{x^2 + y^2 + z^2}{r^3}f'$$

$$= \frac{3f'}{r} + f'' - \frac{f'}{r}, \text{ as } x^2 + y^2 + z^2 = r^2.$$

$\therefore$ $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 2f'/r + f'' \qquad ...(12)$$

Using (8) and (12), (7) reduces to

$$\frac{2f'x}{r} + x\left(\frac{2f'}{r} + f''\right) = 0 \qquad \text{or} \qquad f'' + \frac{4f'}{r} = 0$$

or

$$f''/f' + 4/r = 0.$$

Integrating $\quad \log f' + 4\log r = \log c_1 \quad$ so that $\quad f' = c_1 r^{-4}, \quad$ ...(13)

Integrating (13), $\quad f = -(c_1/3) \times r^{-3} + c_2$, c_2 being an arbitrary constant ...(14)

Substituting the values of f' and f from (13) and (14) in (5), we get

$$\mathbf{q} = -\{(c_1/3r^2) - c_2\}\mathbf{i} - (c_1 x/r^5)\,\mathbf{r} \qquad ...(15)$$

Given that $\mathbf{q} \to 0$ as $r \to \infty$, hence (15) shows that $c_2 = 0$.

$\therefore$ from (15),

$$\mathbf{q} = \frac{c_1}{3r^3}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) \qquad ...(16)$$

Now,

$$q^2 = \mathbf{q}\cdot\mathbf{q} = \frac{c_1^2}{9r^6}\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right)\cdot\left(\mathbf{i} - \frac{3x\mathbf{r}}{r^2}\right) = \frac{c_1^2}{9r^6}\left[\mathbf{i}\cdot\mathbf{i} - \frac{6x}{r^2}\mathbf{r}\cdot\mathbf{i} + \frac{9x^2}{r^4}\mathbf{r}\cdot\mathbf{r}\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2r^2}{r^4}\right), \qquad \text{as} \qquad \mathbf{r}\cdot\mathbf{r} = r^2 \qquad \text{and} \qquad \mathbf{r}\cdot\mathbf{i} = x$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right) = \frac{c_1^2}{9r^8}(r^2 + 3x^2).$$

Hence the required surfaces of constant speed are

$q^2 = \text{constant} \quad$ or $\quad (c_1^2/9r^8)\,(r^2 + 3x^2) = \text{constant} \quad$ or $\quad (r^2 + 3x^2)\,r^{-8} = \text{constant}.$

Ex. 4. *Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated ; prove that if* Π *be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is*

$$\Pi \frac{\log r - \log b}{\log a - \log b}.$$

[Kanpur 2000; Meerut 2000; Agra 1995; I.A.S. 2006]

Sol. Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder $|z| = b$. Hence the free surface would be cylindrical. Thus the liquid velocity v' will be radial and v' will be function of r' (the radial distance from the centre of the cylinder $|z| = b$ which is taken as origin) and time t only. Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad \ldots(1)$$

From (1), $$\partial v'/\partial t' = F'(t)/r' \qquad \ldots(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating, $$F'(t)\log r' + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C, \; C \text{ being an arbitrary constant} \qquad \ldots(3)$$

Initially when $t = 0$, $v' = 0$, $p = P$. So (3) $\Rightarrow$ $F'(0)\log r' = -(P/\rho) + C$ $\qquad \ldots(4)$

Again, $P = \Pi$ when $r' = a$ and $P = 0$ when $r' = b$. So (3) yields

$\therefore$ $$F'(0)\log a = -(\Pi/\rho) + C \quad \text{and} \quad F'(0)\log b = C \qquad \ldots(5)$$

Solving (5) for $F'(0)$ and C, we have

$$C = -\log b \frac{\Pi}{\rho \log(a/b)}, \qquad F'(0) = -\frac{\Pi}{\rho\log(a/b)}.$$

Putting these values in (4), we get

$$\frac{P}{\rho} = \frac{\Pi}{\rho\log(a/b)}\log r' - \frac{\Pi}{\rho\log(a/b)} \log b$$

or $$P = \Pi\frac{\log r' - \log b}{\log(a/b)} = \Pi\frac{\log r' - \log b}{\log a - \log b}. \qquad \ldots(6)$$

For the required result, replace r' by r in (6).

Ex. 12(*a*). *A steady inviscid incompressible fluid flow has a velocity field $u = fx$, $v = -fy$, $w = 0$, where f is a constant. Derive an expression for the pressure field $p(x, y, z)$ if the pressure $p(0, 0, 0) = p_0$ and $\mathbf{F} = -g\,\mathbf{i}\,z$.* **[I.A.S. 2006]**

Sol. Given $u = fx$, $v = -fy$, $w = 0$, f being a constant ... (1)

Also, given that $p = p_0$, when $x = 0$, $y = 0$, $z = 0$ ... (2)

Again, $\mathbf{F} = -g\mathbf{i}z \Rightarrow X = 0$, $Y = 0$ and $Z = -gz$ ... (3)

Equations of motion for steady motion $(\partial/\partial t) = 0$ of an incompressible fluid flow (see Art 3.1) are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = X - (1/\rho)\times(\partial p/\partial x) \quad \text{... (4)}$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = Y - (1/\rho)\times(\partial p/\partial y) \quad \text{... (5)}$$

$$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = Z - (1/\rho)\times(\partial p/\partial z) \quad \text{... (6)}$$

Using (1) and (3), (4), (5) and (6) reduce to

$$f^2x = -(1/\rho)\times(\partial p/\partial x), \qquad -f^2y = -(1/\rho)\times(\partial p/\partial y), \qquad 0 = -gz - (1/\rho)\times(\partial p/\partial z)$$

$$\Rightarrow \quad \partial p/\partial x = -f^2\rho x, \qquad \partial p/\partial y = f^2\rho y \quad \text{and} \quad \partial p/\partial z = -\rho g z \quad \text{... (7)}$$

Now, $$dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy + (\partial p/\partial z)\,dz$$

$\therefore$ $$dp = -(f^2\rho x)\,dx + (f^2\rho y)dy - (\rho g z)\,dz, \quad \text{using (7)}$$

Integrating, $p = -(f^2\rho x^2)/2 + (f^2\rho y^2)/2 - (\rho g z^2)/2 + C$, C being a constant ... (8)

Putting $x = y = z = 0$ and $p = p_0$ (see condition (2)), in (8), we get $C = p_0$

Thus, the required expression for the pressure field is given by

$$p(x, y, z) = p_0 - \rho\,(f^2x^2 - f^2y^2 + g z^2)/2$$

Ex. 12(b). *For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the vorticity* $\mathbf{w}$ *and velocity* $\mathbf{q}$ *satisfies*

$$(\mathbf{q}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\,\mathbf{q}.$$ **[I.A.S. 1989]**

Sol. Vector equation of motion for invicid incompressible fluid is (refer Art. 3.1A)

$$\partial\mathbf{q}/\partial t + \nabla(\mathbf{q}^2/2) - \mathbf{q}\times curl\ \mathbf{q} = \mathbf{F} - (1/\rho)\,\nabla p \quad \text{... (1)}$$

Since the motion is steady, $\partial\mathbf{q}/\partial t = \mathbf{0}$ ... (2)

Since ρ is uniform, $$(1/\rho)\,\nabla p = \nabla(p/\rho) \qquad \dots (3)$$

Since $\mathbf{F}$ is conservative, $\mathbf{F} = -\nabla\Omega$, where Ω is some scalar function. ... (4)

Again, by definition, vorticity vector $= \boldsymbol{w} = \text{curl}\ \boldsymbol{q}$.

Using (2), (3), (4) and (5) in (1), we obtain

$$\nabla(\boldsymbol{q}^2/2) - \boldsymbol{q}\times\boldsymbol{w} = -\nabla\Omega - \nabla(p/\rho) \qquad \text{or} \qquad \boldsymbol{q}\times\boldsymbol{w} = \nabla(\boldsymbol{q}^2/2 + \Omega + p/\rho)$$

Taking the curl of both sides of the above equation and using the vector identity curl gral $\phi = \mathbf{0}$, we have

$$\text{curl}\ (\boldsymbol{q}\times\boldsymbol{w}) = \mathbf{0} \qquad \text{or} \qquad (\nabla\cdot\boldsymbol{w})\,\boldsymbol{q} - (\boldsymbol{q}\cdot\nabla)\,\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q} - (\nabla\cdot\boldsymbol{q})\boldsymbol{w} = \mathbf{0}$$

$$\text{or} \qquad -(\boldsymbol{q}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{q} = \mathbf{0} \qquad \text{or} \qquad (\boldsymbol{q}\cdot\nabla)\boldsymbol{w} = (\boldsymbol{w}\cdot\nabla)\,\boldsymbol{q}.$$

where we have used the following two results

$$\nabla\cdot\boldsymbol{w} = \nabla\cdot\nabla\times\boldsymbol{q} = 0 \qquad \text{and} \qquad \nabla\cdot\boldsymbol{q} = 0 \ \ \text{(continuity equation).}$$

Ex. 13. *Show that if the velocity field*

$$u(x,y) = \frac{B(x^2-y^2)}{(x^2+y^2)^2}, \qquad v(x,y) = \frac{2Bxy}{(x^2+y^2)^2}, \qquad w(x,y) = 0$$

satisfies the equations of motion for inviscid incompressible flow, then determine the pressure associated with this velocity field, B being a constant.

[Kanpur 2002, 03, 05; Rohilkhand 2000, 05]

Sol. The equations of motion for steady inviscid incompressible flow are given by

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = -(1/\rho)\,(\partial p/\partial x), \qquad \dots (1)$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = -(1/\rho)\,(\partial p/\partial y) \qquad \dots (2)$$

and $$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = -(1/\rho)\,(\partial p/\partial z). \qquad \dots (3)$$

From the given values of u, v and w, we have

$$\frac{\partial u}{\partial x} = B\,\frac{2x(x^2+y^2)^2 - 4x(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial u}{\partial y} = B\,\frac{-2y(x^2+y^2)^2 - 4y(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^4} = -\frac{2By(3x^2-y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial v}{\partial x} = 2B\,\frac{y(x^2+y^2)^2 - 4x^2y(x^2+y^2)}{(x^2+y^2)^4} = \frac{2By(y^2-3x^2)}{(x^2+y^2)^3},$$

$$\frac{\partial v}{\partial y} = 2B\,\frac{x(x^2+y^2)^2 - 4xy^2(x^2+y^2)}{(x^2+y^2)^4} = \frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}, \qquad \frac{\partial v}{\partial z} = 0,$$

$$\partial w/\partial x = 0, \qquad \partial w/\partial y = 0 \qquad \text{and} \qquad \partial w/\partial z = 0.$$

Substituting the given values of u, v and w and also using the above relations, (1), (2) and (3) reduce to

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2Bx(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2By(3x^2-y^2)}{(x^2+y^2)^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{B(x^2-y^2)}{(x^2+y^2)^2}\cdot\frac{2By(y^2-3x^2)}{(x^2+y^2)^3}+\frac{2Bxy}{(x^2+y^2)^2}\cdot\frac{2Bx(x^2-3y^2)}{(x^2+y^2)^3}=-\frac{1}{\rho}\frac{\partial p}{\partial y}.$$

and $$0=-(1/\rho)(\partial p/\partial z)$$

Simplifying the above equations, we have

$$\frac{2B^2x}{(x^2+y^2)^5}[(x^2-y^2)(3y^2-x^2)-2y^2(3x^2-y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{2B^2y}{(x^2+y^2)^5}[(x^2-y^2)(y^2-3x^2)+2x^2(x^2-3y^2)]=-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$0=\partial p/\partial z$$

Again simplifying the above equations, we have

or $$\frac{2B^2x}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad i.e., \quad \frac{2B^2x\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial x} \quad \ldots(1)$$

$$\frac{2B^2y}{(x^2+y^2)^5}(-x^4-2x^2y^2-y^4)=-\frac{1}{\rho}\frac{\partial p}{\partial z} \quad i.e., \quad \frac{2B^2y\rho}{(x^2+y^2)^3}=\frac{\partial p}{\partial y} \quad \ldots(2)$$

and $$0=\partial p/dz. \quad \ldots(3)$$

Relation (3) shows that the pressure p is independent of z, *i.e.*, $p = p\ (x, y)$. Hence, we have

$$dp=(\partial p/\partial x)\,dx+(\partial p/\partial y)\,dy$$

or $$dp=\frac{2B^2x\rho}{(x^2+y^2)^3}dx+\frac{2B^2y\rho}{(x^2+y^2)^3}dy=B^2\rho(x^2+y^2)^{-3}(2xdx+2ydy)$$

or $$dp=B^2\rho(x^2+y^2)^{-3}d(x^2+y^2).$$

Integrating, $$p=C-(1/2)\times B^2\rho(x^2+y^2)^{-2}=C-\{B^2\rho/2(x^2+y^2)^2\},$$

where C is a constant of integratioon. It gives the required pressure distribution.

Ex. 14. *The particle velocity for a fluid motion referred to rectangular axes is given by the components* $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$, *where A is a constant. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube, $-a \le x \le a$, $0 \le z \le 2a$. Also, find the pressure associated with this velocity field.* **[I.A.S. 1994; Meerut 2003]**

Sol. Given $u = A\cos(\pi x/2a)\cos(\pi z/2a)$, $v = 0$, $w = A\sin(\pi x/2a)\sin(\pi z/2a)$. ...(1)

From (1), $\partial u/\partial x=-(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a)$, $\quad \partial v/\partial y=0$

and $\partial w/\partial z=(A\pi/2a)\sin(\pi x/2a)\cos(\pi z/2a)$. $\Big\}$ ...(2)

$$\therefore \quad \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0,$$

showing that the given velocity components represent a physically possible flow.

The equations of motion for steady inviscid incompressible flow under no body forces are

$$u(\partial u/\partial x) + v(\partial u/\partial y) + w(\partial u/\partial z) = -(1/\rho)\ (\partial p/\partial x), \quad \ldots(3)$$

$$u(\partial v/\partial x) + v(\partial v/\partial y) + w(\partial v/\partial z) = -(1/\rho)\ (\partial p/\partial y) \quad \ldots(4)$$

and $$u(\partial w/\partial x) + v(\partial w/\partial y) + w(\partial w/\partial z) = -(1/\rho)\ (\partial p/\partial z). \quad \ldots(5)$$

$$\left.\begin{aligned}&\text{From (1)}\ \partial u/\partial y=0, && \partial u/\partial z=-(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)\\ &\partial v/\partial x=\partial v/\partial z=0, && \partial w/\partial x=(A\pi/2a)\cos(\pi x/2a)\sin(\pi z/2a)\\ &\text{and} && \partial w/\partial y=0.\end{aligned}\right\} \quad \ldots(6)$$

Using (1), (2) and (6), the equations of motion (3), (4) and (5) become

$$-A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}-A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$0=-(1/\rho)\,(\partial p/\partial y)$$

$$A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}+A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\cdot\frac{A\pi}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Simplifying the above equations, we have

$$\partial p/\partial x=(\pi\rho A^2/2a)\cos(\pi x/2a)\sin(\pi x/2a), \qquad \text{...(7)}$$

$$\partial p/\partial y=0 \qquad \text{...(8)}$$

and $$\partial p/\partial z=-(\pi\rho A^2/2a)\cos(\pi z/2a)\sin(\pi z/2a). \qquad \text{...(9)}$$

Equation (8) shows that the pressure p is independent of y so that $p = p(x, z)$. Then

$$dp=(\partial p/\partial x)dx+(\partial p/\partial z)dz$$

or $dp = (\pi\rho A^2/2a)\,[\cos(\pi x/2a)\sin(\pi x/2a)\,dx - \cos(\pi z/2a)\sin(\pi z/2a)dz]$, using (7) and (9)

Integrating, $$p=(\pi\rho A^2/2a)\,[(a/\pi)\sin^2(\pi x/2a)-(a/\pi)\sin^2(\pi z/2a)]+C$$

or $p=(\rho A^2/2)\,[\sin^2(\pi x/2a)-\sin^2(\pi z/2a)]+C$, C being a constant of integration. ...(10)

(10) gives the required pressure associated with the velocity field given by (1).

Ex. 1. *An infinite mass of fluid is acted on by a force* $\mu/r^{3/2}$ *per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere* $r = c$ *in it, show that the cavity will be filled up after an interval of time* $(2/5\mu)^{1/2}c^{5/4}$.

[Kanpur 1999, 2009; Meerut 2005; I.A.S. 2003]

Sol. Method I At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2v'=r^2v \qquad \text{...(1)}$$

When the radius of the cavity is r, then

Kinetic energy $=\int_r^\infty \frac{1}{2}(4\pi r'^2\rho dr')\cdot v'^2$ $\qquad \left[\because \text{Kinetic energy} = \frac{1}{2}\times\text{mass}\times(\text{velocity})^2\right]$

$$=2\pi\rho r^4v^2\int_r^\infty\frac{dr'}{r'^2}, \text{ using (1)}$$

$$=2\pi\rho r^3v^2.$$

The initial kinetic energy is zero.

Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'}=\frac{\mu}{r'^{3/2}} \qquad \text{so that} \qquad V=\frac{2\mu}{r'^{1/2}}$$

$\therefore$ the work done $=\int_r^c V\,dm$, dm being the elementary mass

$$=\int_r^c\left(\frac{2\mu}{r'^{1/2}}\right)\cdot 4\pi r'^2\,dr'\rho=8\pi\mu\rho\int_r^c r'^{3/2}\,dr'=\frac{16}{5}\pi\rho\mu\,(c^{5/2}-r^{5/2})$$

We now use energy equation, namely, Increase in kinetic energy = work done

This $\Rightarrow$ $2\pi\rho r^3 v^2 - 0 = (16/5) \times \pi \rho \mu (c^{5/2} - r^{52})$

$$\therefore \qquad v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2} \frac{(c^{5/2} - r^{5/2})^{1/2}}{r^{3/2}} \qquad \text{...(2)}$$

wherein negative sign is taken because r decreases as t increases.

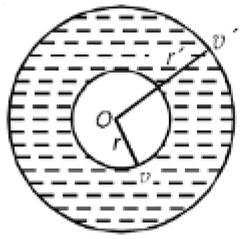
Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2} - r^{5/2})}} \quad \text{or} \quad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\,dr}{\sqrt{(c^{5/2} - r^{5/2})}}$$

Put $r^{5/2} = c^{5/2}\sin^2\theta$ so that $(5/2) \times r^{3/2}\,dr = 2c^{5/2}\sin\theta\cos\theta\,d\theta$.

$$\therefore \qquad T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/4} \sin\theta\,d\theta = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

Second Method. Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin) and time t. Also, let v be the velocity at a distance r.



Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v. \qquad \text{...(1)}$$

From (1), $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}. \qquad \text{...(2)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)} \qquad \text{...(3)}$$

Integrating (3) with respect to r', we have

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{2\mu}{r'^{1/2}}-\frac{p}{\rho}+C, \text{ } C \text{ being an arbitrary constant} \qquad ...(4)$$

When $r' = \infty$, $v' = 0$, $p = 0$. So from (4), $C = 0$. Then (4) becomes

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{2\mu}{r'^{1/2}}-\frac{p}{\rho}. \qquad ...(5)$$

Now when $r' = r$, $v' = v$ and $p = 0$. So (5) reduces to

$$-\frac{F'(t)}{r}+\frac{1}{2}v'^2=\frac{2\mu}{r^{1/2}}. \qquad ...(6)$$

Now, (1) $\Rightarrow$ $F(t) = r^2 v$ $\Rightarrow$ $F'(t) = 2rv\,(dr/dt) + r^2(dv/dt)$

or
$$F'(t)=2rv\frac{dr}{dt}+r^2\frac{dv}{dr}\frac{dr}{dt}=2rv^2+r^2v\frac{dv}{dr}, \qquad \text{as} \qquad \frac{dr}{dt}=v.$$

Hence (6) gives

$$-\frac{1}{r}\left[2rv^2+r^2v\frac{dv}{dr}\right]+\frac{v^2}{2}=\frac{2\mu}{r^{1/2}} \qquad \text{or} \qquad rv\frac{dv}{dr}+\frac{3}{2}v^2=-\frac{2\mu}{r^{1/2}}.$$

Multiplying both sides by $2r^2$, the above equation can be written as

$$2r^3v\,dv+3r^2v^2\,dr=-4\mu r^{3/2}\,dr \qquad \text{or} \qquad d\,(r^3v^2)=-4\mu r^{3/2}\,dr.$$

Integrating, $\quad r^3v^2 = -(8\mu/5)r^{5/2} + D$, D being an arbitrary constant $\qquad ...(7)$

When $r = c$, $v = 0$. So (7) gives $D = (8\mu/5)c^{5/2}$. Hence (7) reduces to

$$r^3v^2 = (8\mu/5)\times(c^{5/2}-r^{5/2})$$

or
$$v=\frac{dr}{dt}=-\left(\frac{8\mu}{5}\right)^{1/2}\left(\frac{c^{5/2}-r^{5/2}}{r^3}\right)^{1/2},$$

taking negative sign for dr/dt since velocity increases as r decreases.

Let T be the time of filling up the cavity, then

$$T=-\left(\frac{5}{8\mu}\right)^{1/2}\int_c^0\frac{r^{3/2}\,dr}{(c^{5/2}-r^{5/2})^{1/2}}. \qquad ...(8)$$

Let $\quad r^{5/2}=c^{5/2}\sin^2\theta \quad$ so that $\quad (5/2)\times r^{3/2}\,dr=c^{5/2}\sin\theta\cos\theta\,d\theta.$

$$\therefore \qquad T=\frac{4}{5}\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\frac{c^{5/2}\sin\theta\cos\theta}{c^{5/4}\cos\theta}\,d\theta=\frac{4c^{5/4}}{5}\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\sin\theta\,d\theta$$

or
$$T=(2/5\mu)^{1/2}\times c^{5/4}.$$

5.2. Stream function or current function.

[Agra 2005; Rohilkhand 2002, 03; Meerut 1999, 2010; Kanpur 2010, 09]

Let u and be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow or streamline is

$$dx/u = dy/ \qquad \text{or} \qquad dx - udy = 0 \qquad \text{...(1)}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y} = \frac{\partial(-u)}{\partial x} \qquad \text{...(2)}$$

(2) shows that *L.H.S.* of (1) must be an exact differential, $d\psi$ (say). Thus, we have

$$dx - udy = d\psi = (d\psi/\partial x)dx + (\partial\psi/\partial y)dy \qquad \text{...(3)}$$

so that $\qquad u = -\partial\psi/\partial y \qquad$ and $\qquad = \partial\psi/\partial x \qquad$ ...(4)

This function ψ is known as the ***stream function***. Then using (1) and (3), the streamlines are given by $d\psi = 0$ *i.e.*, by the equation $\psi = c$, where c is an arbitrary constant. Thus the stream function is constant along a streamline. Clearly the current function exists by virtue of the equation of continuity and incompressibility of the fluid. Hence the current function exists in all types of two-dimensional motion whether rotational or irrotational.

5.5. Some aspects of elementary theory of functions of a complex variables.

Suppose that $\quad z = x + iy \quad$ and that $\quad w = f(z) = \phi(x, y) + i\psi(x, y)$,

where x, y, ϕ, ψ are all real and $i = \sqrt{-1}$. Also, suppose that ϕ and ψ and their first derivatives are everywhere continuous within a given region. If at any point of the region specified by z the derivative $dw/dz\ (= f'(z))$ is unique, then w is said to be *analytic* or *regular* at that point. If the derivative is unique throughout the region, then w is said to be analytic or regular throughout the region. It can be shown that the necessary and sufficient conditions for w to be analytic at z are

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x,$$

which are known as the *Cauchy-Riemann equations*. The functions ϕ, ψ are known as *conjugate functions*.

5.6. Irrotational motion in two-dimensions. **[Meerut 2007; Purvanchal 2004, 05]**

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = -\partial\phi/\partial x \qquad \text{and} \qquad = -\partial\phi/\partial y \qquad \text{....(1)}$$

In two-dimensional flow the stream function ψ always exists such that

$$u = -\partial\psi/\partial y \qquad \text{and} \qquad = \partial\psi/\partial x \qquad \text{...(2)}$$

From (1) and (2), we have

$$\partial\phi / \partial x = \partial\psi / \partial y \qquad \text{and} \qquad \partial\phi / \partial y = -\partial\psi / \partial x \qquad ...(3)$$

which are well known *Cauchy-Riemann's equations*. Hence $\phi + i\psi$ is an analytic function of $z = x + i\,y$. Moreover ϕ and ψ are known as *conjugate functions*.

On multiplying and re-writing, (3) gives

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y}=0\,, \qquad ...(4)$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally.

Thus the curves of equi-velocity potential and the stream lines intersect orthogonally.

Differentiating the equations given in (3) with respect to x and y respectively, we get

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\psi}{\partial x\partial y} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial y^2}=-\frac{\partial^2\psi}{\partial x\partial y}. \qquad ...(5)$$

Since $\dfrac{\partial^2\psi}{\partial x\partial y}=\dfrac{\partial^2\psi}{\partial y\partial x}$, adding (5) gives

$$\partial^2\phi / \partial x^2 + \partial^2\phi / \partial y^2 = 0. \qquad ...(6)$$

Again, differentiating the equations given in (3) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2\phi}{\partial x\partial y}=-\frac{\partial^2\psi}{\partial x^2}$$

Since $\partial^2\phi / \partial x\partial y = \partial^2\phi / \partial y\,\partial x$, subtracting these, we get $\partial^2\psi / \partial x^2 + \partial^2\psi / \partial y^2 = 0$ ...(7)

Equations (6) and (7) show that *ϕ and ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.* **[Meerut 2010]**

5.7. Complex potential. [Meerut 2011]

[G.N.D.U. Amritsar 2003; Rohilkhand 2001; Kanpur 2001, 05; Agra 2005]

Let $w = \phi + i\psi$ be taken as a function of $x + iy$ *i.e.*, z. Thus, suppose that $w = f(z)$ *i.e.*

$$\phi + i\psi = f(x+iy) \qquad ...(1)$$

Differentiating (1) w.r.t x and y respectively, we get

$$\partial\phi / \partial x + i(\partial\psi / \partial x) = f'(x+iy) \qquad ...(2)$$

and $\qquad \partial\phi / \partial y + i(\partial\psi / \partial y) = if'(x+iy)$

or $\qquad \partial\phi / \partial y + i(\partial\psi / \partial y) = i\{\partial\phi / \partial x + i(\partial\psi / \partial x)\}$, by (2)

Equating real and imaginary parts, we get

$$\partial\phi / \partial x = \partial\psi / \partial y \qquad \text{and} \qquad \partial\phi / \partial y = -\partial\psi / \partial x \qquad ...(3)$$

which are *Cauchy-Riemann equations*. Then w is an analytic function of z and w is known as the *complex potential*.

Conversely, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion.

Remarks. If $\qquad \phi + i\psi = f(x+iy)$, $\qquad$ then $\qquad i\phi - \psi = if(x+iy)$

Thus, $\qquad \psi - i\phi = -if(x+iy) = g(x+iy)$, say

Hence proceeding as before, we get (3). Hence another irrotational motion is also possible in which lines of equi - velocity potential are given by ψ = constant and the streamlines by ϕ = constant.

5.8. Magnitude of velocity. **[G.N.D.U. Amritsar 2002; Kanpur 1997]**

Let $w = f(z)$ be the complex potential. Then

$$w = \phi + i\psi \qquad \text{and} \qquad z = x + iy \qquad \text{...(1)}$$

Also $\quad \partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\,\partial\psi/\partial x \qquad$ ...(2)

For two-diensional irrotational motion, we have (see Art. 5.1.)

$$u = -\partial\phi/\partial x \qquad \text{and} \qquad = -\partial\phi/\partial y \qquad \text{...(3)}$$

From (1), $\quad \dfrac{dw}{dz}.\dfrac{\partial z}{\partial x} = \dfrac{\partial\phi}{\partial x} + i\dfrac{\partial\psi}{\partial x} \qquad \text{and} \qquad \dfrac{\partial z}{\partial x} = 1$

$$\therefore \qquad \frac{dw}{dz} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}, \text{ using (2)} \qquad \text{...(4)}$$

or $\qquad dw/dz = -u + i \quad$ using (3) $\qquad$ ...(5)

which is called the *complex velocity*.

From (4) and (5), we see that the magnitude of velocity q at any point in a two-dimensional irrotational motion is given by $|\, dw/dz \,|$, where

$$|\, dw/dz \,| = \{(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2\}^{1/2} = (u^2 + \ \ ^2)^{1/2} = q \qquad \text{...(5)}$$

Remarks. The points where velocity is zero are known as *stagnation points*.

Ex.15 *Show that $u = 2cxy$, $\ = c(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.* **[Rohilkhand 1999]**

Sol. Given $\qquad u = 2cxy, \qquad = c(a^2 + x^2 - y^2) \qquad$ ...(1)

Equation of continuity in xy-plane is given by

$$\partial u/\partial x + \partial \ \ /\partial y = 0 \qquad \text{...(2)}$$

From (1), $\partial u/\partial x = 2cy$ and $\partial \ \ /\partial y = -2cy$. Putting these values in (2) we get $0 = 0$, showing (2) is satisfied by u, given by (1). Hence u and constitute a possible fluid motion.

Let ψ be the required stream function. Then, we have

$$u = -(\partial\psi/\partial y) \qquad \text{or} \qquad \partial\psi/\partial y = -2cxy \qquad \text{...(3)}$$

and $\qquad = \partial\psi/\partial x \qquad$ or $\qquad \partial\psi/\partial x = c(a^2 + x^2 - y^2) \qquad$ ...(4)

Integrating (3) partially w.r.t. 'y' $\qquad \psi = -cxy^2 + \phi(x, t), \qquad$ ...(5)

where $\phi(x, t)$ is an arbitrary function of x and t.

Differentiating (5) partially w.r.t. 'x', $\qquad \partial\psi/\partial x = -cy^2 + \partial\phi/\partial x \qquad$ ...(6)

(4) and (6) $\Rightarrow \qquad -cy^2 + \partial\phi/\partial x = c(a^2 + x^2 - y^2) \qquad$ or $\qquad \partial\phi/\partial x = c(a^2 + x^2) \qquad$ ...(7)

Integrating (7) partially w.r.t. 'x', $\qquad \phi(x, t) = c(a^2 x + x^3/3) + \psi(y, t)$,

where $\psi(y, t)$ is an arbitrary function of y and t.

Substituting the above value of $\phi(x, t)$ in (5), we get

$\psi = c(ax^2 + x^3/3 - xy^2) + \psi(y, t)$, which is the required stream function.

Ex. 16. *Show that $u = -\omega y$, $\quad = \omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between this motion and one represented by the potential* $\phi = A \log r$, where $r = (x^2 + y^2)^{1/2}$.

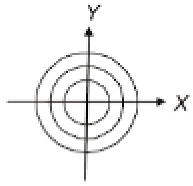
Sol. Given $\quad u = -\omega y, \quad\quad = \omega x \quad$ and $\quad w = 0 \qquad \ldots(1)$

(1) $\Rightarrow$ $\partial u / \partial x = 0 = \partial \quad / \partial y$. Hence the equation of continuity $\partial u / \partial x + \partial \quad / \partial y = 0$ is satisfied. Hence these exist a two dimensional motion defined by (1).

Now, $$\partial\psi = (\partial\psi / \partial x) dx + (\partial\psi / \partial y) dy \qquad \ldots(2)$$

But $$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} = \quad = \omega x \quad \text{and} \quad \frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x} = -u = \omega y$$

$\therefore$ (3) reduces to $$d\psi = \omega x\, dx + \omega y\, dy = d\{\omega (x^2 + y^2)/2\}$$



Integrating, $\psi = \omega(x^2 + y^2)/2 + c$, where c is an arbitrary constant.

The required streamlines are given by ψ = constant = c', say

i.e. $c' = \omega (x^2 + y^2)/2 + c \quad$ or $\quad x^2 + y^2 = 2(c' - c)/w = a^2$, say

Hence the required streamlines are concentric circles with centres at origin as shown in the adjoining figure.

Second part: Given
$$\phi = A \log r = A \log (x^2 + y^2)^{1/2} = (A/2) \times \log (x^2 + y^2) \qquad \ldots(3)$$

$$\therefore \quad u = -\frac{\partial\phi}{\partial x} = -\frac{Ax}{x^2 + y^2} \quad \text{and} \quad = -\frac{\partial\phi}{\partial y} = -\frac{Ay}{x^2 + y^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -A\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = A\frac{x^2 - y^2}{(x^2 + y^2)^2}, \qquad \frac{\partial \quad}{\partial y} = -A\frac{x^2 - y^2 - 2y^2}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$\therefore$ $\partial u / \partial x + \partial \quad / \partial y = 0$ so that the equation of continuity is satisfied.

Hence there exists exists a motion for the given value of ϕ.

Third part. Difference between the two given motions.

For the fluid motion given by (1), we have

$$\text{curl } \boldsymbol{q} = \boldsymbol{i} (\partial w / \partial y - \partial \quad / \partial z) + \boldsymbol{j} (\partial u / \partial z - \partial w / \partial x) + \boldsymbol{k} (\partial \quad / \partial x - \partial u / \partial y)$$
$$= \boldsymbol{i} (0 - 0) + \boldsymbol{j} (0 - 0) + \boldsymbol{k} (\omega + \omega) \neq \boldsymbol{0},$$

showing that curl $\boldsymbol{q} \neq 0$. Hence velocity potential does not exist for the fluid motion defined by (1) (refer Art. 2.26), whereas velocity potential exist for the second fluid motion.

Ex. 19. *In two-dimensional motion show that, if the streamlines are confocal ellipses*

$$x^2/(a^2+\lambda)+y^2/b^2+\lambda)=1, \quad \text{then} \quad \psi = A\log(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda})+B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point. **[Rajasthan 1998]**

Sol. Take $\quad z = C\cos w \quad$ then $\quad x+iy = C\cos(\phi+i\psi) \quad$ ...(1)

or $$x+iy = C(\cos\phi\cos i\psi - \sin\phi\sin i\psi)$$

or $$x+iy = C\cos\phi\cosh\psi - iC\sin\phi\sinh\psi \qquad ...(2)$$

Equating real and imaginary parts, (2) gives

$$x = C\cos\phi\cosh\psi \qquad \text{and} \qquad y = -C\sin\phi\sinh\psi$$

so that $$\cos\phi = \frac{x}{C\cosh\psi} \qquad \text{and} \qquad \sin\phi = -\frac{y}{C\sinh\psi}$$

Squaring and adding these, we obtain

$$\frac{x^2}{C^2\cosh^2\psi}+\frac{y^2}{C^2\sinh^2\psi}=1 \qquad ...(3)$$

which give the streamlines in two-dimensions.

Again, given that the streamlines are confocal ellipses

$$x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1 \qquad ...(4)$$

Since (3) and (4) must be identical, we have

$$C^2\cosh^2\psi = a^2+\lambda \qquad \text{and} \qquad C^2\sinh^2\psi = b^2+\lambda$$

$$\therefore \quad C(\cosh\psi+\sinh\psi) = \sqrt{a^2+\lambda}+\sqrt{b^2+\lambda} \qquad \text{or} \qquad Ce^{\psi} = \sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}$$

$[\because \cosh\psi = (e^{\psi}+e^{-\psi})/2 \quad \text{and} \quad \sinh\psi = (e^{\psi}-e^{-\psi})/2\ ]$

or $$\psi = \log(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda})-\log C \qquad ...(5)$$

If ϕ, ψ are velocity potential and stream function, so also will be $A\phi$ and $A\psi$ where A is a constant. Hence (5) may be re-written as

$$\psi = A\log(\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda})+B$$

From (1), $$\frac{dz}{dw} = -C\sin w = -C\sqrt{1-\cos^2 w} = -C(1-z^2/C^2)^{1/2}$$

$$= -\sqrt{C^2-z^2} = -\sqrt{(C+z)(C-z)} = -\sqrt{r_1 r_2}$$

where r_1 and r_2 are the focal distances (radii) of any point $P(z)$ from the foci $S(C, 0)$ and $S'(-C, 0)$ of the ellipses.

Thus $$q = |dw/dz| = 1/\sqrt{r_1 r_2}.$$

Ex. 20. *Show that the velocity potential* $\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$

gives a possible motion. Determine the streamlines and show also that the curves of equal speed are the ovals of Cassini given by rr' = const. **[Rajasthan 2000; I.A.S. 1990]**

Sol. Given $\phi = (1/2)\times\log[(x+a)^2+y^2]-(1/2)\times\log[x-a)^2+y^2]$

$$\therefore \qquad u = -\frac{\partial\phi}{\partial x} = -\frac{x+a}{(x+a)^2+y^2}+\frac{x-a}{(x-a)^2+y^2} \qquad ...(1)$$

and

$$= -\frac{\partial\phi}{\partial y} = -\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2} \qquad ...(2)$$

From (1)

$$\frac{\partial u}{\partial x} = -\frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2}+\frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \qquad ...(3)$$

From (2)

$$\frac{\partial}{\partial y} = -\frac{(x+a)^2-y^2}{[(x+a)^2+y^2]^2}+\frac{(x-a)^2-y^2}{[(x-a)^2+y^2]^2} \qquad ...(4)$$

Adding (3) and (4), we see that the equation of continuity $\partial u/\partial x+\partial \ /\partial y=0$ is satisfied. Hence there exists a motion for the given ϕ.

To determine the streamlines, we use the fact that velocity potential ϕ and the stream function ψ satisfy the Cauchy-Riemann equations, namely,

$$\partial\phi/\partial x = \partial\psi/\partial y \qquad \text{and} \qquad \partial\phi/\partial y = -\partial\psi/\partial x \qquad ...(5)$$

From (1) and (5), we have $\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2}$

Integrating it w.r.t. y, we get

$$\psi = \tan^{-1}\frac{y}{x+a}\tan^{1}\frac{y}{x-a}+f(x),\ f(x) \text{ being an arbitrary function of } x \qquad ...(6)$$

$$\therefore \qquad \frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2}+f'(x) \qquad ...(7)$$

Again from (5) and (2), we get

$$\frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2} \qquad ...(8)$$

Comparing (7) and (8) $f'(x) = 0$ so that $f(x)$ = constant. Omitting the additive constant, (6) gives

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} = \tan^{-1}\frac{[y/(x+a)]-[y/(x-a)]}{1-[y/(x+a)][y/(x-a)]}$$

$\therefore$
$$\psi = \tan^{-1}\frac{(-2ay)}{x^2+y^2-a^2}$$

Hence the streamlines are given by ψ = const. = $\tan^{-1}$ $(-2a/C)$, that is,

$$x^2 + y^2 - Cy = a^2 \qquad \text{....(9)}$$

which are circles. When $C = 0$, the stream line is the circle passing through $(a, 0)$ and $(-a, 0)$. Again, if C is infinite then stream line $y = 0$ [divide (9) by C and then let $C \to \infty$]

Now, $w = \phi + i\psi = \dfrac{1}{2}\log\left[(x+a)^2 + y^2\right] - \dfrac{1}{2}\log\left[(x-a)^2 + y^2\right] + i\tan^{-1}\dfrac{y}{x+a} - i\tan^{-1}\dfrac{y}{x-a}$

$$= \log[(x+a)+iy] - \log[(x-a)+iy] = \log(z+a) - \log(z-a), \text{ as } z = x+iy$$

$\therefore$
$$q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a|\cdot|z-a|} = \frac{2a}{rr'},$$

where r, r' are the distances of the point from the points P (x, y) from the points $(a, 0)$ and $(-a, 0)$. The curves of equal speed are given by q = constant or rr' = constant, which are Cassini ovals.

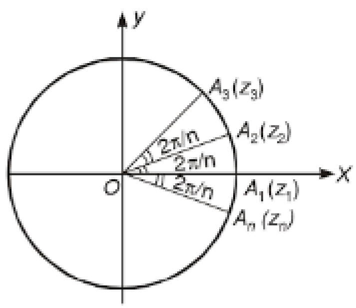
Sol. Let $A_1 A_2 A_3$ be the circle of radius a. Suppose that n rectilinear vortices each of strength k be situated at points $z_m = ae^{2\pi im/n}$, $m = 0, 1, 2, \ldots, n-1$ of the circle. Then the complex potential due to these n vortices is given by

$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(z - ae^{2\pi im/n}) = \frac{ik}{2\pi}\log\prod_{m=0}^{n-1}(z - ae^{2\pi im/n}) = \frac{ik}{2\pi}\log(z^n - a^n).$$

[using a well known result of algebra]

Now the fluid velocity q at any point out of all the n vortices is given by

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{nz^{n-1}}{z^n - a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n - a^n}\right|$$



Again the velocity induced at A_1 $(z = a)$, by others is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n - a^n) - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\log\frac{z^n - a^n}{z-a}$$

$$\therefore \quad w' = (ik/2\pi)\log(z^{n-1} + z^{n-2}a + \ldots + za^{n-2} + a^{n-1})$$

so that $$\frac{dw'}{dz}=\frac{ik}{2\pi}\;\frac{(n-1)z^{n-2}+(n-2)z^{n-2}a+\ldots+a^{n-2}}{z^{n-1}+z^{n-2}a+\ldots+za^{n-2}+a^{n-1}}$$

$$\therefore \qquad \left(\frac{dw'}{dz}\right)_{z=a}=\frac{ik}{2\pi}\;\frac{(n-1)+(n-2)+\ldots+2+1}{na}=\frac{ik\,(n-1)}{4\pi a}$$

[$\because$ By algebra $(n-1)+(n-2)+\ldots+2+1=\{(n-1)/2\}\times\{(n-1)+1]=n(n+1)/2$

or $$u_1-iv_1=\left(-\frac{dw'}{dz}\right)_{z=a}=-\frac{ik\,(n-1)}{4\pi a},$$

so that $u_1=0$ and $v_1=k\,(n-1)/4\pi a$. If q_r and q_0 be the radial and transverse velocity components of the velocity at $z=a$, then we have $q_r=0$ and $q_0=\mathrm{k}(n-1)/4\pi a$. Due to symmetry of the problem, it follows that each vortex moves with the same transverse velocity $k\,(n-1)/4\pi a$. Hence

the required time T is given by $$T=\frac{2\pi a}{k\,(n-1)/4\pi a}=\frac{8\pi^2a^2}{(n-1)\,k}.$$

Ex. 3. ***What an infinite liquid contains two parallel equal and opposite rectilinear vortices at a distance 2b, prove that the stream lines relative to the vortices are by the equation***

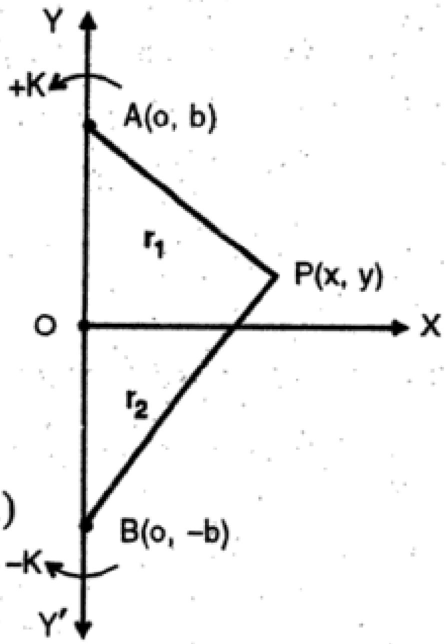
$$\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]+\frac{y}{b}=C,$$

the origin being the middle point of the join, which is taken for axis of Y. Show that for a vortex pair the relative streamlines are given by

$$k\,\{\log(r_1/r_2)+(y/2b)\}=constant,$$

where r_1, r_2 are the distances of any point from them.

Solution. The vortex system is moving with uniform velocity $(k/2\pi)AB$ perpendicular to the line AB. Now, superposing a velocity $(-k/4\pi b)$ to redue the vortex system to rest. Let ψ' be the stream function of the superposed system then



$$-\frac{\partial\psi'}{\partial y}=-\frac{k}{4\pi b}\Rightarrow\psi'=\frac{ky}{4\pi b}$$

If w_1 be the complex potential then

$$w_1=kz/4\pi b. \qquad \ldots(1)$$

The complex potential w_2 at any point $P\,(x,y)$ due to fixed vortices of strength k and $-k$ is given by

$$w_2=(ik/2\pi)\log(z-ib)-(ik/2\pi)\log(z+ib). \qquad \ldots(2)$$

Thus the total complex potential w at a point is given by

$$w = (ik/2\pi) \log (z - ib) - (ik/2\pi) \log (z + ib) + (k/4\pi b) z$$

or
$$w = (ik/2\pi) \log \{x + i (y - b)\} - (ik/2\pi) \log \{x + i (y + b)\} + (k/4\pi b) (x + iy)$$

Equating the imaginiary parts, the stream function of the relative motion becomes

$$\psi = (k/2\pi) \log \{x^2 + (y - b)^2\}^{1/2} - (k/2\pi) \log \{x^2 + (y + b)^2\}^{1/2} + (k/4\pi b) y$$

$$\psi = \frac{k}{4\pi} \log \left[\frac{x^2 + (y - b)^2}{x^2 + (y + b)^2}\right] + \frac{k}{4\pi} \frac{y}{b} . \quad ...(4)$$

Thus the relative stream lines are given by $\psi = \text{const.}$

$$\frac{k}{4\pi} \log \left[\frac{x^2 + (y - b)^2}{x^2 + (y + b)^2}\right] + \frac{k}{4\pi} \frac{y}{b} \text{const.}$$

or
$$\log \left[\frac{x^2 + (y - b)^2}{x^2 + (y + b)^2}\right] = C.$$
Proved.

From (3), we have

$$\psi = \frac{k}{4\pi} \left[\log \frac{r_1}{r_2} + \frac{y}{b}\right],$$

where
$$r_1^2 = x^2 + (y - b)^2 \text{ and } r_2^2 = x^2 + (y + b)^2.$$

The relative stream lines are given by $\psi = \text{constant}$.

$$\Rightarrow \quad k \left\{\log \left(\frac{r_1}{r_2}\right) + \frac{y}{b}\right\} = \text{const.}$$
Proved.

Ex. 11. *If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time*

$$8\pi^2 a^2/(n - 1) k,$$

and find the velocity at any point of the liquid.

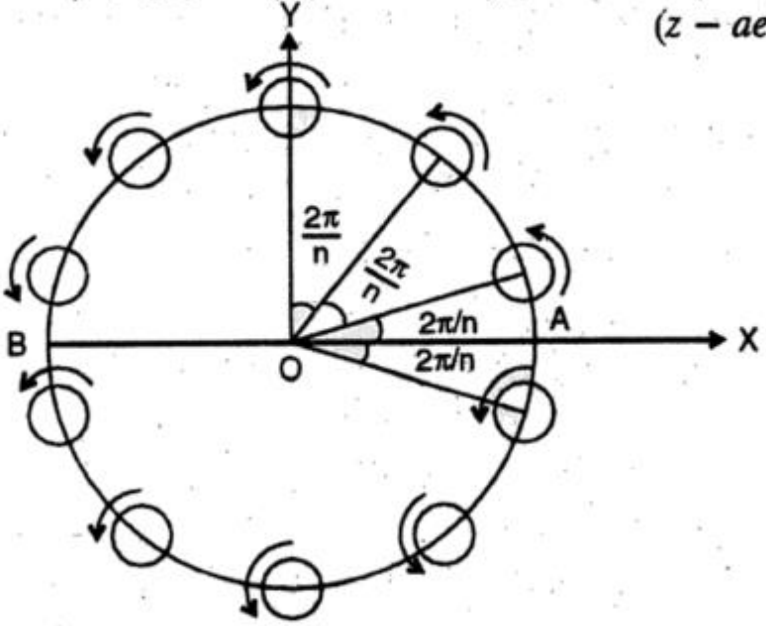
Solution. Since the n rectilinear vortices of strength k are symmetrically distributed round the circular cylinder then the angular distance between any two consecutive vortices will be $2\pi/n$. Let the line through centre of the cylinder and one of vortices be taken as X-axis. Thus the vertices are situated at the point

$$z = ae^0, ae^{2\pi i/n}, ae^{4\pi i/n}, ... ae^{2\pi ri/n}, ... ae^{2\pi (n - 1) i/n},$$

which are the n roots of the equation $z^n - a^n = 0$.

The complex potential due to n vortices becomes

$$w = (ik/2\pi)\,\{\log (z - a)(z - ae^{2\pi i/n})(z - ae^{4\pi i/n}) \ldots (z - ae^{2\pi (n-1) i/n})\}$$



or $\quad w = (ik/2\pi) \log (z^n - a^n)$

or $\quad \phi + i\psi = (ik/2\pi) \log \{(r^n \cos n\theta - a^n) + ir^n \sin n\theta\}$.

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi} \log \{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta\}.$$

Again $\qquad q^2 = \left(\frac{\partial \psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \psi}{\partial \theta}\right)^2$

or $\quad q^2 = \frac{k^2}{16\pi^2}\left\{\left(\frac{2nr^{2n-1} - 2nr^{n-1} a^n \cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta}\right)^2 + \frac{1}{r^2}\left(\frac{2nr^n a^n \sin n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta}\right)^2\right\}$

or $\quad q^2 = \frac{n^2k^2}{4\pi^2}\left\{\frac{(r^{2n-1} - r^{n-1} a^n \cos n\theta)^2 + r^{2n-2} a^{2n} \sin^2 n\theta}{(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)^2}\right\}$

or $\quad q^2 = \frac{n^2k^2}{4\pi^2}\left\{\frac{r^{4n-2} + r^{2n-2} a^{2n} - 2r^{2n-2} a^n \cos n\theta}{(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)^2}\right\}$

or $\quad q^2 = \frac{n^2k^2}{4\pi^2}\left\{\frac{r^{2n-2}(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)}{(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)^2}\right\}$

or $$q^2 = \frac{n^2k^2}{4\pi^2} \frac{r^{2n-2}}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta},$$

which determines the velocity at any point of the liquid.

The complex potential of any one of the vortices at $z = a$ reduces to

$$w' = (ik/2\pi) \log (z^n - a^n) - (ik/2\pi) \log (z - a).$$

or $$\phi' + i\psi' = (ik/2\pi) [\log \{(r^n \cos n\theta - a^n) + ir^n \sin n\theta\} - \log \{(r \cos \theta - a) + ir \sin \theta\}].$$

or $$\psi' = (k/4\pi) [\log (r^{2n} + a^{2n} - 2r^n a^n \cos n\theta) - \log (r^2 + a^2 - 2ar \cos \theta)].$$

or $$\frac{\partial \psi'}{\partial r} = \frac{k}{4\pi} \left[\frac{2nr^{2n-1} - 2nr^{n-1} a^n \cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta} - \frac{2r - 2a \cos \theta}{r^2 + a^2 - 2ar \cos \theta} \right]$$

Let ψ_0 be the stream function of the vortex $(a, 0)$ lying on X-axis.

$$\left(\frac{\partial \psi_0}{\partial r} \right)_{r=a} = \frac{k}{4\pi} \left[\frac{na^{2n-1} (1 - \cos n\theta)}{2a^{2n} (1 - \cos n\theta)} - \frac{2a (1 - \cos \theta)}{2a^2 (1 - \cos \theta)} \right]$$

$$\left(\frac{\partial \psi_0}{\partial r} \right)_{r=a} = \frac{k}{4\pi} \left(\frac{n}{a} - \frac{1}{a} \right) = \frac{k (n-1)}{4\pi a}$$

and $$\frac{1}{a} \frac{\partial \psi_0}{\partial \theta} = \left(\frac{1}{r} \frac{\partial \psi'}{\partial \theta} \right)_{r=a} \Rightarrow \frac{1}{a} \frac{\partial \psi_0}{\partial \theta} = 0.$$

It follows that the velocity is only along the tangent, there being no velocity along the normal to the circle. Due to symmetry, each vortices will move round the cylinder uniformly with the same transverse velocity

$$\frac{k(n-1)}{4\pi a}.$$

The required time of making one round becomes

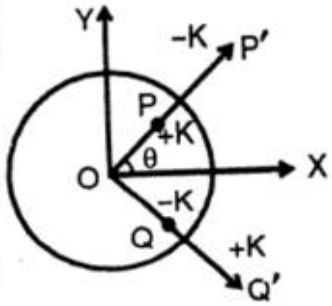
$$= \frac{2\pi a}{\{k(n-1)/4\pi a\}} = \frac{8\pi^2 a^2}{k(n-1)}.$$ **Proved.**

Ex. 12. *When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis. Show that the path of each vortex is given by the equation*

$$(r^2 \sin^2 \theta - b^2)(r^2 - a^2)^2 = 4a^2 b^2 r^2 \sin^2 \theta,$$

θ being measured from the line through the centre perpendicular to the join of the vortices.

Solution. Let a pair of equal and opposite vortices of strength $+k$ and $-k$ are placed at the point $P(r, \theta)$ and $Q(r, -\theta)$, respectively.



The image system consists of

(i) a vortex of strength $-k$ at an inverse point $P'(a^2/r, \theta)$,

(ii) a vortex of strength $+k$ at an inverse point $Q'(a^2/r, -\theta)$.

The complex potential at any point z becomes

$$w = (ik/2\pi) \log (z - re^{\theta i}) - (ik/2\pi) \log \{z - (a^2/r) e^{\theta i}\}.$$
$$- (ik/2\pi) \log (z - re^{-\theta i}) + (ik/2\pi) \log \{z - (a^2/r) e^{-\theta i}\}.$$

The motion of P is due to other vortices, thus for the motion of P, the complex potential at any point z $(z = re^{\theta i})$ becomes

$$w_1 = [-(ik/2\pi) \log \{z - (a^2/r) e^{\theta i}\} - (ik/2\pi) \log \{z - re^{-\theta i}\}.$$
$$- (ik/2\pi) \log \{z - (a^2/r) e^{-\theta i}\}]_{z = re^{\theta i}}$$

or $$w_1 = -\frac{ik}{2\pi} [\log \{re^{\theta i} - (a^2/r) e^{\theta i}\} \log (re^{\theta i} - re^{-\theta i})$$
$$- \log \{re^{\theta i} - (a^2/r) e^{-\theta i}\}]$$

or $$w_1 = -\frac{ik}{2\pi} [\log \{(r \cos \theta - (a^2/r) \cos \theta)$$
$$+ i (r \sin \theta - (a^2/r) \sin \theta)\} + \log (2ir \sin \theta)$$
$$- \log \{r \cos \theta - (a^2/r) \cos \theta\} + i (r \sin \theta + (a^2/r)) \sin \theta\}]$$

or $$\psi = -(k/2\pi) \left[\log \{r - (a^2/r)\} + \log (2r \sin \theta) \right.$$
$$\left. - \tfrac{1}{2} \log \{r^2 + (a^4/r^2) - 2a^2 \cos 2\theta\}\right]$$

or $$\psi = -(k/4\pi) [\log \{r - (a^2/r).2r \sin \theta\}^2$$
$$- \log \{r^2 + (a^4/r^2) - 2a^2 \cos 2\theta\}]$$

The stream lines are given by $\psi =$ Const.

i.e. $\quad -\dfrac{k}{4\pi}\left[\log\{4(r^2-a^2)^2\sin^2\theta\} - \log\left\{\dfrac{r^4+a^4-2a^2r^2\cos 2\theta}{r^2}\right\}\right] = \text{Const.}$

or $\quad \dfrac{r^2(r^2-a^2)^2\sin^2\theta}{r^4+a^4-2a^2r^2\cos 2\theta} = b^2$ (let)

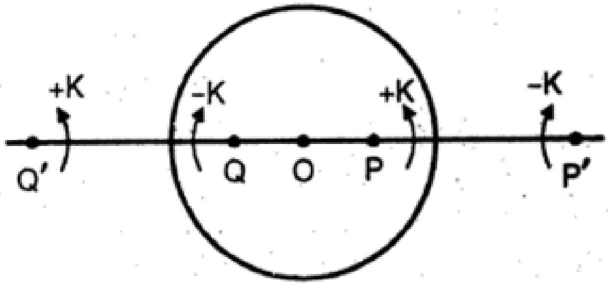
or $\quad b^2\{(r^2-a^2)^2+2a^2r^2(1-\cos 2\theta)\} = r^2(r^2-a^2)^2\sin^2\theta$

or $\quad 2a^2b^2r^2(1-\cos 2\theta) = (r^2-a^2)^2(r^2\sin^2\theta-b^2)$

or $\quad 4a^2b^2r^2\sin^2\theta = (r^2-a^2)^2(r^2\sin^2\theta-b^2).$ **Proved.**

Ex. 2. *If a vortex pair is situated within a cylinder, show that it will remain at rest if the distance of either from the centre is given by $(\sqrt{5}-2)^{1/2}a$, where a is the radius of the cylinder.*

Solution. Let P, Q be the vortex-pair situated with in the cylinder at an equidistant r ($OP = r = OQ$) from the centre. The image system give rise to a vortex of strength $-k$ at an inverse point P' and a vortex of strength $+k$ at an inverse point Q', such that



$$OP.OP' = a^2 = OQ.OQ' \Rightarrow OP' = a^2/r = OQ'.$$

The vortex will remain at rest if its velocity due to other three vortices be zero,

i.e., $$\frac{k}{2\pi}\left\{\frac{1}{PP'} - \frac{1}{PQ} + \frac{1}{Q'P}\right\} = 0$$

or $$\frac{k}{2\pi}\left\{\frac{1}{(a^2/r)-r} - \frac{1}{2r} + \frac{1}{(a^2/r)+r}\right\} = 0$$

or $2r\left(\dfrac{a^2}{r}+r\right) - \left(\dfrac{a^2}{r}+r\right)\left(\dfrac{a^2}{r}-r\right) + 2r\left(\dfrac{a^2}{r}-r\right) = 0.$

or $$(r^2/a^2)^2 + 4(r^2/a^2) - 1 = 0$$

or $$r^2/a^2 = (\sqrt{5}-2) \Rightarrow r = a(\sqrt{5}-2)^{1/2}.$$

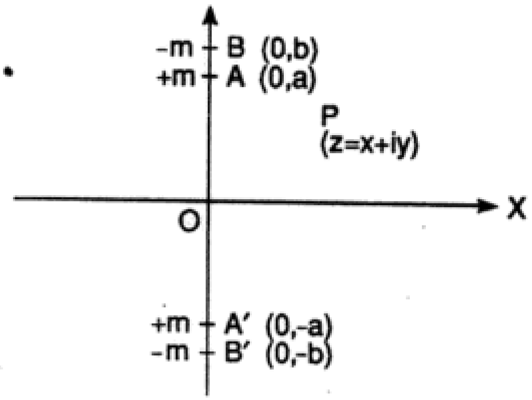
Proved.

Ex. 20. *If the fluid fill the region of spaces on the positive side of X-axis, which is a rigid boundary and if there be a source $+m$ at the point $(0, a)$ and an equal sink at $(0, b)$, and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is*

$$\pi\rho m^2 \frac{(a-b)^2}{ab\,(a+b)},$$

where ρ is the density of the fluid.

Solution. The image system due to a source of strength $+m$ placed at the point $A\,(0, a)$ and a sink of strength $-m$ placed at the point $B\,(0, b)$ on the same side consists of



(i) a source of strength $+m$ at the point $A\,(0, a)$ *i.e.*, at a distance $z = ai$ from O

(ii) a source of strength $+m$ at the other side of the surface at $A'\,(0, -a)$ *i.e.*, at a distance $z = -ai$ from O

(iii) a sink of strength $-m$ at the point $B\,(0, b)$ *i.e.*, at a distance $z = bi$ from O

(iv) a sink of strength $-m$ at the other side of the surface at $B'\,(0, -b)$ *i.e.*, at a distance $z = -bi$ from O.

Thus the complex potential is given by

$$w = -m\log(z - ai) - m\log(z + ai) + m\log(z - bi) + m\log(z + bi)$$

or $$w = -m\log(z^2 + a^2) + m\log(z^2 + b^2)$$

or $$\frac{dw}{dz} = -m\frac{2z}{z^2 + a^2} + m\frac{2z}{z^2 + b^2}$$

or $$q = \left|\frac{dw}{dz}\right| = 2m\left|\frac{z\,(a^2 - b^2)}{(z^2 + a^2)(z^2 + b^2)}\right|. \qquad \ldots (1)$$

Consider a point $R(x, 0)$ on the X-axis. Let q be the velocity at the point R, then substituting $z = x$ in (1), we have

$$q = 2m\,(a^2 - b^2)\left|\frac{x}{(x^2 + a^2)(x^2 + b^2)}\right|$$

$$q = 2m\,(a^2 - b^2)\frac{x}{(x^2 + a^2)(x^2 + b^2)}.$$

Let p be the pressure at any point, then by Bernoulli's theorem, we have

$$\frac{p}{\rho} = C - \tfrac{1}{2}q^2$$

Since $P_\infty = p_0$, $q = 0$; $C = p/\rho$

or $$\frac{p}{\rho} = \frac{p_0}{\rho} - \tfrac{1}{2}q^2 \Rightarrow p_0 - p = \tfrac{1}{2}\rho q^2.$$

Thus the resultant pressure on the boundary is

$$P = \int_{-\infty}^{\infty} (p_0 - p)\,dx = \tfrac{1}{2}\rho\int_{-\infty}^{\infty} q^2\,dx$$

$$\Rightarrow \quad P = 4m^2\rho\,(a^2 - b^2)^2\int_0^{\infty}\frac{x^2\,dx}{(x^2 + a^2)^2\,(x^2 + b^2)^2}$$

$$\Rightarrow \quad P = 4m^2\rho\int_0^{\infty}\left[\frac{a^2 + b^2}{b^2 - a^2}\left\{\frac{1}{x^2 + a^2} - \frac{1}{x^2 + b^2}\right\} - \frac{a^2}{(x^2 + a^2)^2} - \frac{b^2}{(x^2 + b^2)^2}\right]dx$$

$$\Rightarrow \quad P = 4m^2\rho\left[\frac{a^2 + b^2}{b^2 - a^2}\left(\frac{\pi}{2a} - \frac{\pi}{2b}\right) - \frac{\pi}{4a} - \frac{\pi}{4b}\right]$$

$$\Rightarrow \quad P = \pi\rho m^2\,\frac{(a - b)^2}{ab\,(a + b)}.$$

Proved.

Ex. 19. *Show that the velocity potential*

$$\phi = \tfrac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2},$$

gives a possible motion. Determine the form of streamlines and curves of equal speed.

Solution. $$\phi = \tfrac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2} \quad \ldots (1)$$

or $$\frac{\partial\phi}{\partial x} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2} \quad \ldots (2)$$

and $$\frac{\partial\phi}{\partial y} = \frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2} \quad \ldots (3)$$

Also $$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{-\frac{\partial\phi}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{\frac{x-a}{(x-a)^2+y^2} - \frac{x+a}{(x+a)^2+y^2}\right\}$$

or $$\frac{\partial u}{\partial x} = \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} - \frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2}$$

or $$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{y}{(x-a)^2+y^2} - \frac{y}{(x+a)^2+y^2}\right\}$$

or $$\frac{\partial v}{\partial y} = \frac{(x-a)^2-y^2}{[(x-a)^2+y^2]^2} - \frac{(x+a)^2-y^2}{[(x+a)^2+y^2]^2}$$

$\Rightarrow \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ which satisfies the equation of continuity. Hence the relation (1) gives a possible liquid motion.

Since the velocity potential ϕ and the stream function ψ satisfies the property of conjugate functions, so we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \quad \ldots (4)$$

or $$\frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

or $$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + f(x). \quad \ldots (5)$$

To determine $f(x)$, we shall differentiate (5) with regard to x.

$$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2 + y^2} + \frac{y}{(x-a)^2 + y^2} + f'(x)$$

$$\therefore \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \Rightarrow f'(x) = 0 \Rightarrow f(x) = \text{const.}$$

or $$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} = \tan^{-1}\left(-\frac{2ay}{x^2+y^2-a^2}\right).$$

The lines of flow can be obtained by $\psi = \text{const}$,

i.e., $$\tan^{-1}\left\{-\frac{2ay}{x^2+y^2-a^2}\right\} = \text{const.}$$

or $$\frac{2ay}{a^2-x^2-y^2} = \tan\mu = A = \text{const.}$$

The constant $A = 0$ gives the streamline $y = 0$ *i.e.*, a real axis and $A = \infty$ gives the streamline $x^2 + y^2 = a^2$, *i.e.*, a circle. **Ans.**

Again $w = \phi + i\psi = \frac{1}{2}\log\{(x+a)^2 - y^2\} - \frac{1}{2}\log\{(x-a)^2 + y^2\}$

$$- i\tan^{-1}\frac{y}{x+a} - i\tan^{-1}\frac{y}{x-a}$$

or $$w = \left[\tfrac{1}{2}\log\{(x+a)^2 + y^2\} + i\tan^{-1}\frac{y}{x-a}\right]$$

$$- \left[\tfrac{1}{2}\log\{(x-a)^2 + y^2\} + \tan^{-1}\frac{y}{x-a}\right]$$

or $$w = \log\{(x+a) + iy\} - \log\{(x-a) + iy\}$$

or $$w = \log(z+a) - \log(z-a);\ z = x + iy.$$

So $$q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{z+a} - \frac{1}{z-a}\right| = \frac{2a}{|z+a|\,|z-a|} = \frac{2a}{rr'},$$

where $|z+a| = r'$, $|z-a| = r$, r' and r are the distances from $(-a, 0)$ and $(a, 0)$.

Hence the curves of equal speeds are given by

$\frac{2a}{rr'} = \text{Const.} \Rightarrow rr' = \text{Const}$, which are known as *Cassini Ovals*.

Ans.

Ex. 12. *What arrangement of sources and sinks will give rise to the function*

$$w = \log\left(z - \frac{a^2}{z}\right)\ ?$$

Draw a rough sketch of the stream lines to this curve and prove that two of them sub-divide into the circle $r = a$ and the axis of Y.

Solution. The complex potential is given by

$$w = \log\left(z - \frac{a^2}{z}\right) = \log\left\{\frac{(z-a)(z+a)}{z}\right\}$$

or $$w = \log(z-a) + \log(z+a) - \log z, \qquad \text{...(1)}$$

which shows that there are two sinks of unit strength at distance $z = a$ and $z = -a$ and a source of unit strength at an origin.

The relation (1) can be expressed as

$$\phi + i\psi = \log\{(x-a) + iy\} + \log\{(x+a) + iy\} - \log(x + iy).$$

Equating the imaginary parts, we have

$$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

or $$\psi = \tan^{-1}\left(\frac{\dfrac{y}{(x-a)} + \dfrac{y}{(x+a)}}{1 - \dfrac{y}{(x-a)}\dfrac{y}{(x+a)}}\right) - \tan^{-1}\frac{y}{x}$$

or $$\psi = \tan^{-1}\left(\frac{2xy}{x^2 - y^2 - a^2}\right) - \tan^{-1}\frac{y}{x}$$

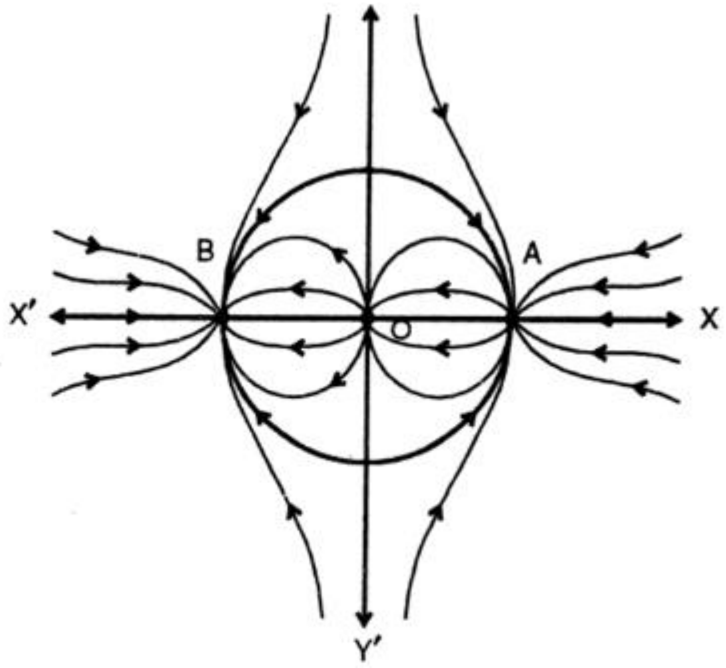
or $$\psi = \tan^{-1}\left(\frac{\dfrac{2xy}{x^2 - y^2 - a^2} - \dfrac{y}{x}}{1 + \dfrac{2xy}{x^2 - y^2 - a^2}\dfrac{y}{x}}\right),$$

or $$\psi = \tan^{-1}\frac{y}{x}\frac{(x^2 + y^2 + a^2)}{(x^2 + y^2 - a^2)}.$$

The streamlines are given by $\psi = \text{const.}$

i.e., $$\tan^{-1}\frac{y(x^2 + y^2 + a^2)}{x(x^2 + y^2 - a^2)} = \text{const.}$$

or $$\frac{y(x^2 + y^2 + a^2)}{x(x^2 + y^2 - a^2)} = k. \qquad \text{...(2)}$$



I. If the constant k is infinite then

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}=\infty \Rightarrow x(x^2+y^2-a^2)=0,$$

$$\Rightarrow x=0 \text{ and } x^2+y^2=a^2.$$

Thus $x=0$ shows that Y-axis is a streamline and the equation $x^2+y^2=a^2$ (*i.e.*, $r=a$) shows that the circle is a streamline with centre as origin.

II. If the constant k is zero, then $y=0$ implies that axis of X is a streamline. Therefore the rough sketch of the lines is with a source of unit strength at origin $O(0,0)$ and two sinks of unit strength at $A(a,0)$ and $B(-a,0)$.

Problem 6.1 : Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 k Pa, and with a mean velocity of 2 m/s. Neglecting friction, find the total head, if the pipe is 7 m above the datum line.

Solution: Given Data:

Diameter of pipe : d = 80 mm = 0.08 m

Gauge pressure of water : p = 60 kPa = 60×10^3 Pa or N/m^2

Mean velocity of water : V = 2 m/s

Datum head : z = 7 m

According to Bernoulli's equation:

$$\text{Total head of water : } H = \frac{p}{\rho g} + \frac{V^2}{2g} + z$$

$$= \frac{60\times10^3}{1000\times9.81} + \frac{(2)^2}{2\times9.81} + 7$$

$$= 6.11 + 0.20 + 7$$

= **13.31 m of water.**

Ex. 12. *A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being* **Π**. *Show that if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time t is*

$$\Pi + \tfrac{1}{2}\rho \left\{ \frac{d^2}{dt^2}(R)^2 + \left(\frac{dR}{dt}\right)^2 \right\}.$$

Solution. The only possible motion which can take place is one in which each element of liquid moves towards the centre, whence the free surface will remain shperical. In an incompressible fluid, the fluid velocity will be radial outside the sphere, so it will be a function of r and the time t only. The equation of continuity reduces to

$$\frac{1}{r^2}\frac{d}{dr}(r^2 v) = 0$$

or $$r^2 v = \text{constant} = F(t), \text{(let)} \qquad \dots (1)$$

where v is the velocity of a fluid particle at a distance r from the centre at any time t.

Equation of motion is

$$\frac{\partial v^*}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{F'(t)}{r^2} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

Integrating with regard to r, we have

$$-\frac{F'(t)}{r} + \tfrac{1}{2}v^2 = -\frac{p}{\rho} + B \qquad \dots (2)$$

where B is an arbitrary constant.

Initially $r \to \infty$, $v = 0$, $p = \Pi$; $B = \frac{\Pi}{\rho}$.

or $$-\frac{F'(t)}{r} + \tfrac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{p}{\rho}$$

or $$p = \Pi + \tfrac{1}{2}\rho\left[\frac{2F'(t)}{r} - v^2\right].$$

Let P be the pressure on the surface of the sphere of radius R and V be the velocity, then

$$P = \Pi + \tfrac{1}{2}\rho\left\{\frac{2F'(t)}{R} - V^2\right\}. \qquad \dots (3)$$

Again from (1), we have

$$R^2 V = F(t),\ F(t) = R^2\frac{dR}{dt}.$$

Differentiating with regard to t, we have

$$F'(t) = R^2 \frac{d^2R}{dt^2} + 2R \left[\frac{dR}{dt}\right]^2$$

From the equation (3), we get

$$P = \Pi + \tfrac{1}{2}\rho \left\{2R \frac{d^2R}{dt^2} + 4\left(\frac{dR}{dt}\right)^2 - V^2\right\}$$

---

*Since $r^2 v = F(t)$, $\dfrac{\partial v}{\partial t} = F'(t)/r^2$.

or $$P = \Pi + \tfrac{1}{2}\rho \left\{2R \frac{d^2R}{dt^2} + 4\left(\frac{dR}{dt}\right)^2 - \left(\frac{dR}{dt}\right)^2\right\}$$

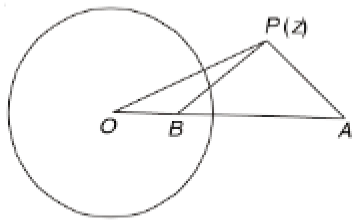
or $$P = \Pi + \tfrac{1}{2}\rho \left\{2\left[R \frac{d^2R}{dt^2} + \left(\frac{dR}{dt}\right)^2\right] + \left(\frac{dR}{dt}\right)^2\right\}$$

or $$P = \Pi + \tfrac{1}{2}\rho \left\{\frac{d^2}{dt^2}(R^2) + \left(\frac{dR}{dt}\right)^2\right\}.$$

Proved.

Ex. 7. *A source of fluid situated in space of two dimensions, is of such strength that* $2\pi\rho\mu$ *represents the mass of fluid of density* ρ *emitted pet unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is* $2\pi\rho\mu^2 a^2/r(r^2-a^2)$, *where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ?* **[Kanpur 2005, 06; Meerut 2005, 11; Rohilkhand 2002]**

Sol. Since the mass of fluid emitted is $2\pi\rho\mu$ per unit of time, by definition the strength of the given source is μ. Let this source be situated at A such that $OA = r$ and let B be the inverse point of A. Then, $OA \cdot OB = a^2$ so that $OB = a^2/r$. Here the equivalent image system consists of (taking OA as x-axis and using Art. 5.21)



(*i*) a source of strength μ at A $(r, 0)$

(*ii*) a source of strength μ at B $(a^2/r, 0)$

(*iii*) a sink of strength μ at O $(0, 0)$

Hence the complex potentioal at any point P $(z = x + iy)$ is given by

$$w = -\mu \log(z-r) - \mu \log(z - a^2/r) + \mu \log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{\mu}{z-r} - \frac{\mu}{z-a^2/r} + \frac{\mu}{z} \qquad \ldots(1)$$

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots(2)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}], \qquad \ldots(3)$$

wherein the indicated sum of the residues is calcualated at poles of $(dw/dz)^2$ lying within the circular boundary. Using (3), (2) reduces to

$$X - iY = -\pi\rho \times [\text{sum of the residues}] \qquad \ldots(4)$$

We proceed to find the residues of $(dw/dz)^2$. From (1), we have

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} - \frac{2}{z(z-a^2/r)} + \frac{2}{(z-r)(z-a^2/r)}\right]$$

$$= \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} + \frac{2}{rz} - \frac{2}{(a^2/r)(z-a^2/r)} \right.$$

$$\left. + \frac{2}{(a^2/r)z} + \frac{2}{(r-a^2/r)(z-r)} + \frac{2}{(a^2/r-r)(z-a^2/r)}\right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fractions]

From (5), we find that the poles inside the circular contour C are $z = 0$ and $z = a^2/r$.

$\therefore$ The required sum of the residues

= the sum of the coffecients of z^{-1} and $(z - a^2/r)^{-1}$ in R.H.S. of (5)

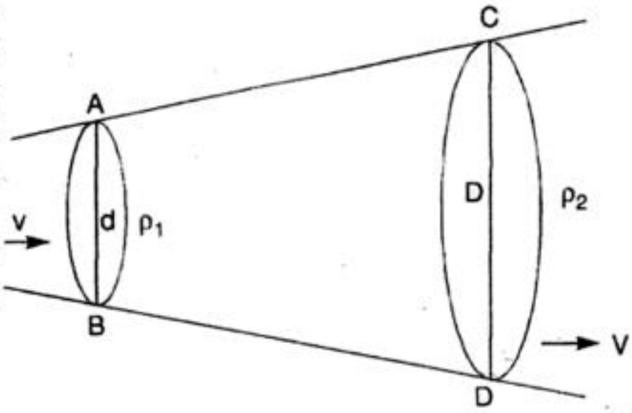
$$= \frac{2\mu^2}{r} + \frac{2\mu^2}{a^2/r} - \frac{2\mu^2}{a^2/r} + \frac{2\mu^2}{a^2/r - r} = \frac{2\mu^2 a^2}{r(a^2-r^2)} \qquad \ldots(6)$$

Using (6) in (4) and then equating real and imaginary parts, we have

$$X = 2\pi\rho\mu^2 a^2/r(r^2-a^2) \qquad \text{and} \qquad Y = 0.$$

Thus the disc is attracted towards the source along OA. Hence the disc will be urged to move along OA.

Ex. 8. *Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. If V and v be the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone, prove that*



$$\frac{v}{V} = \frac{D^2}{d^2} \exp.\left(\frac{v^2 - V^2}{2k}\right),$$

where k is the pressure divided by the density, and supposed constant.

Solution. Let ρ_1 and ρ_2 be the densities of steam at the ends of the conical pipe AB and CD. By the principle of conservation of mass, the mass of the steam that enters and leaves at the ends AB and CD are the same. Thus we have

$$\pi\left(\tfrac{1}{2}d\right)^2 v\rho_1 = \pi\left(\tfrac{1}{2}D\right)^2 V\rho_2$$

or $$\frac{v}{V} = \frac{D^2}{d^2}\frac{\rho_2}{\rho_1}. \qquad \ldots (1)$$

Let p be the pressure, ρ the density and u the velocity at distance r from AB, then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \; p = k\rho$$

or $$u\frac{\partial u}{\partial r} = -\frac{k}{\rho}\frac{\partial \rho}{\partial r}.$$

By integrating, we have

$$\tfrac{1}{2}u^2 = -k\log\rho + k\log E,$$

where E is an arbitrary constant.

or $\quad \log(\rho/E) = -\dfrac{u^2}{2k}$

or $\quad \rho = E \text{ exp. } (-u^2/2k).$

Again $\rho = \rho_1$ when $u = v$ then $\rho_1 = E \text{ exp. } (-v^2/2k),$

and $\quad \rho = \rho_2$ when $u = V$ then $\rho_2 = E \text{ exp. } (-V^2/2k).$

or $\quad \dfrac{\rho_1}{\rho_2} = \dfrac{\text{exp. } (-v^2/2k)}{\text{exp. } (-V^2/2k)} \Rightarrow \dfrac{\rho_2}{\rho_1} = \text{exp. } (v^2 - V^2)/2k. \qquad \ldots (2)$

From (1) and (2), we have

$$\frac{v}{V} = \frac{D^2}{d^2} \text{ exp. } \left(\frac{v^2 - V^2}{2k} \right)$$

Proved.

Ex. 1. *A sphere of radius a is surrounded by infinite liquid of density* ρ, *the pressure at infinity being* Π. *The sphere is suddenly annihilated. Show that the pressure at a distance r from the centre immediately falls to* $\Pi(1 - a/r)$. **[Purvanchel 2004, I.A.S. 1996]**

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius a/2, the impulsive pressure sustained by the surface of this sphere is $(7\Pi\rho^2/6)^{1/2}$.

Sol. Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad \ldots(1)$$

From (1), $\quad \partial v'/\partial t = F'(t)/r'^2 \qquad \ldots(2)$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial P}{\partial r'} \quad \text{or} \quad \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial P}{\partial r'}, \text{ using (2)}$$

Integrating, $\quad -\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = -\dfrac{p}{r'} + C$, C being an arbitrary constant.

When $r' = \infty$, then $p = \Pi$ and $v' = 0$ so that $C = \Pi/\rho$.

$$\therefore \quad -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho} \qquad \ldots(3)$$

When the sphere is suddenly annihilated, we have

$t = 0, \qquad r' = a, \qquad v' = 0 \qquad$ and $\qquad p = 0$

$\therefore$ From (3), $\quad -\dfrac{F'(0)}{a} = \dfrac{\Pi}{\rho} \quad$ so that $\quad F'(0) = -\dfrac{a\Pi}{\rho}$

Hence immediately after the annihilation of the sphere (with $t = 0,\ v' = 0$), (3) reduces to

$$\frac{a\Pi}{\rho r'} + 0 = \frac{\Pi - p}{\rho} \qquad \text{or} \qquad p = \Pi\left(1 - \frac{a}{r'}\right) \qquad ...(4)$$

Thus at the time of annihilation, when $r' = r$, the pressure is given by

$$p = \Pi\left(1 - a/r'\right). \qquad ...(5)$$

Second Part. If $\bar{\omega}$ be the impulsive pressure at distance r', then we have

$$d\tilde{\omega} = -\rho v' dr' \qquad ...(6)$$

Let r be the radius of the inner surface and v the velocity there. Then by the equation of continuity, we have

$$F(t) = r^2 v = r'^2 v' \qquad \text{so that} \qquad v' = (r^2 v)/r'^2 \qquad ...(7)$$

$\therefore$ (6) gives $\qquad d\omega' = \rho v(r^2/r'^2)dr'$

Integrating with respect to r', we get $\qquad \tilde{\omega} = \rho v(r^2/r') + C' \qquad ...(8)$

When $\quad r' = \infty, \quad \bar{\omega} = 0 \quad$ so that $\quad C' = 0.$

$$\therefore \qquad \tilde{\omega} = \rho v(r^2/r'), \qquad ...(9)$$

which gives the impulsive pressure $\tilde{\omega}$ at a distance r'. Since $r = a/2$, (9) reduces to

$$\tilde{\omega} = \frac{1}{4}\rho v a^2 \cdot \frac{1}{r'} \qquad ...(10)$$

We now determine velocity v at the inner surface of the sphere. Setting $r' = r,\ v' = v$ and $p = 0$ in (3), we get

$$-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} \qquad ... (11)$$

From (7), $F'(t) = \frac{d}{dt}(r^2 v) = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt} = 2r\frac{dr}{dt}v + r^2\frac{dv}{dt}\frac{dr}{dr}$

Thus, $F'(t) = 2rv^2 + r^2 v\frac{dv}{dr}$, as $v = \frac{dr}{dt}$

$\therefore$ (11) gives
$$-\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = \frac{\Pi}{\rho}$$

Multiplying both sides by $(-2r^2\, dr)$, we get

$$2r^3 v\,dv + 3r^2 v^2\, dr = -\frac{2\Pi r^2}{\rho}dr \qquad \text{or} \qquad d(r^3 v^2) = -\frac{2\Pi r^2}{\rho}dr$$

Integrating, $r^3 v^2 = -\frac{2\Pi r^3}{3\rho} + C''$, C'' being an arbitrary constant

When $r = a$, $v = 0$ so that $C'' = -\frac{2\Pi a^3}{3\rho}$.

$$\therefore \qquad r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) \qquad \text{...(12)}$$

The velocity v on the surface of the sphere of radius $a/2$ (which would be the inner surface on which the liquid impinges) is given by (12) by replacing r by $a/2$

$$\therefore \qquad v^2 = \frac{2\Pi}{3\rho} \times \frac{a^3 - a^3/8}{a^3/8} = \frac{14}{3} \times \frac{\Pi}{\rho}$$

Putting this value of v in (10), the impulsive pressure at a distance r' is given by

$$\tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \frac{a^2}{r'} \qquad \text{...(13)}$$

Hence the desired impulsive pressure on the surface of the sphere of radius $a/2$ is given by setting $r' = a/2$ in (13).

$$\therefore \qquad \tilde{\omega} = \frac{\rho}{4}\left(\frac{14}{3} \times \frac{\Pi}{\rho}\right)^{1/2} \times \frac{a^2}{(a/2)} = \left(\frac{7\Pi\rho a^2}{6}\right)^{1/2}$$

Ex. 3. *Two sources, each of strength m are placed at the points* $(-a, 0)$, $(a, 0)$ *and a sink of strength* $2m$ *at the origin. Show that the streamlines are the curves* $(x^2 + y^2)^2 = a^2 (x^2 - y^2 + \lambda xy)$ *where* λ *is a variable parameter.* **[U.P. P.C.S. 1999; I.A.S. 1999, 2003]**

Show *also that the fluid speed at any point is* $(2ma^2)/(r_1 r_2 r_3)$ *where* r_1, r_2, r_3, *are the distances of the points from the sources and the sink.*

[I.A.S. 1999, 2003; Meerut 2000; Garhwal 2005; Rohilkhand 2002]

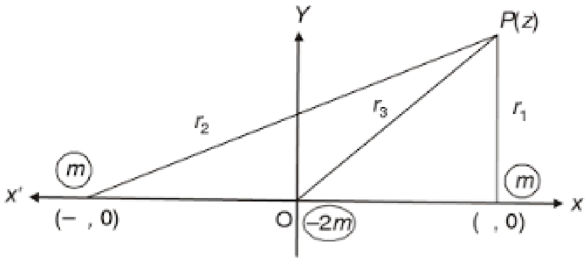
Sol. First Part.

The complex potential w at any point $P(z)$ is given by

$$w = -m \log (z - a) - m \log (z + a) + 2m \log z \qquad \ldots(1)$$

or $$w = m [\log z^2 - \log (z^2 - a^2)]$$

or $$\phi + i\psi = m[\log (x^2 - y^2 + 2ixy) - \log (x^2 - y^2 - a^2 + 2ixy)], \text{ as } z = x + iy$$



Equating the imaginary parts, we have

$$\psi = m\left[\tan^{-1}\left\{2xy/(x^2 - y^2)\right\} - \tan^{-1}\left\{2xy/(x^2 - y^2 - a^2)\right\}\right]$$

$\therefore$ $$\psi = m \tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2 - y^2)}\right], \text{ on simplification.}$$

The desired streamlines are given by ψ = constant = $m \tan^{-1} (-2/\lambda)$. Then we obtain

$$(-2/\lambda) = (-2a^2xy)/[(x^2+y^2)^2 - a^2(x^2-y^2)] \quad \text{or} \quad (x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy).$$

Second Part. From (1), we have

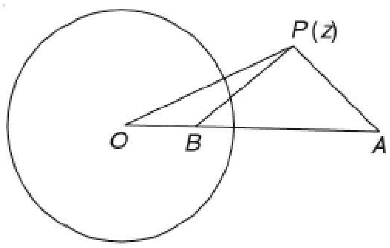
$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)(z+a)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where $r_1 = |z - a|$, $r_2 = |z + a|$ and $r_3 = |z|$.

Ex. 7. *A source of fluid situated in space of two dimensions, is of such strength that* $2\pi\rho\mu$ *represents the mass of fluid of density* ρ *emitted pet unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is* $2\pi\rho\mu^2 a^2 / r(r^2 - a^2)$, *where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ?* **[Kanpur 2005, 06; Meerut 2005, 11; Rohilkhand 2002]**

Sol. Since the mass of fluid emitted is $2\pi\rho\mu$ per unit of time, by definition the strength of the given source is μ. Let this source be situated at A such that $OA = r$ and let B be the inverse point of A. Then, $OA \cdot OB = a^2$ so that $OB = a^2/r$. Here the equivalent image system consists of (taking OA as x-axis and using Art. 5.21)



(*i*) a source of strength μ at A $(r, 0)$

(*ii*) a source of strength μ at B $(a^2/r, 0)$

(*iii*) a sink of strength μ at O $(0, 0)$

Hence the complex potentioal at any point P $(z = x + iy)$ is given by

$$w = -\mu \log (z - r) - \mu \log (z - a^2/r) + \mu \log z$$

$$\therefore \qquad \frac{dw}{dz} = -\frac{\mu}{z - r} - \frac{\mu}{z - a^2/r} + \frac{\mu}{z} \qquad \ldots (1)$$

If the pressure thrusts on the given circular disc are represented by (X, Y), then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad \ldots (2)$$

where C is the boundary of the disc. Again, by Cauchy's residue theorem, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times [\text{sum of the residues}], \qquad \ldots (3)$$

wherein the indicated sum of the residues is calcualated at poles of $(dw/dz)^2$ lying within the circular boundary. Using (3), (2) reduces to

$$X - iY = -\pi\rho \times [\text{sum of the residues}] \qquad \ldots (4)$$

We proceed to find the residues of $(dw/dz)^2$. From (1), we have

$$\left(\frac{dw}{dz}\right)^2 = \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z - a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} - \frac{2}{z(z - a^2/r)} + \frac{2}{(z-r)(z - a^2/r)}\right]$$

$$= \mu^2 \left[\frac{1}{(z-r)^2} + \frac{1}{(z-a^2/r)^2} + \frac{1}{z^2} - \frac{2}{z(z-r)} + \frac{2}{rz} - \frac{2}{(a^2/r)(z-a^2/r)} \right.$$

$$\left. + \frac{2}{(a^2/r)z} + \frac{2}{(r-a^2/r)(z-r)} + \frac{2}{(a^2/r-r)(z-a^2/r)} \right] \qquad \ldots(5)$$

[Resolving R.H.S. into partial fractions]

From (5), we find that the poles inside the circular contour C are $z = 0$ and $z = a^2/r$.

$\therefore$ The required sum of the residues

= the sum of the coffecients of z^{-1} and $(z-a^2/r)^{-1}$ in R.H.S. of (5)

$$= \frac{2\mu^2}{r} + \frac{2\mu^2}{a^2/r} - \frac{2\mu^2}{a^2/r} + \frac{2\mu^2}{a^2/r-r} = \frac{2\mu^2 a^2}{r(a^2-r^2)} \qquad \ldots(6)$$

Using (6) in (4) and then equating real and imaginary parts, we have

$$X = 2\pi\rho\mu^2 a^2 / r(r^2 - a^2) \qquad \text{and} \qquad Y = 0.$$

Thus the disc is attracted towards the source along OA. Hence the disc will be urged to move along OA.

Ex. 2. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure* Π *and contains a spherical cavity of radius a, filled with a gas at pressure* $m\Pi$; *prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na, where n is determined by the equation* $1 + 3m \log n - n^3 = 0$. **(Kanpur 2010)**

If m be nearly equal to 1, *the time of an oscillation will be* $2\pi\sqrt{(a^2\rho/3\pi)}$, ρ *being the density of the fluid.* **[Kanpur 2008; Agra 1998; I.A.S. 1994; Meernt 1999]**

Sol. As in Ex. 1, let at any time t, v' be the velocity at a distance t' and p' the pressure there. Also let v be the velocity at a distance r and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \qquad \ldots(1)$$

From (1),

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \qquad \ldots(2)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

i.e.

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \text{ using (2)} \qquad \ldots(3)$$

Integrating with respect to r', (3) gives

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=C-\frac{p'}{\rho}, \ C \text{ being an arbitrary constant}$$

When $r'=\infty$, then $v'=0, p'=\Pi$ so that $C=\Pi/\rho$. Hence, the above equation yields

$$\therefore \qquad -\frac{F'(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p'}{\rho} \qquad \ldots(4)$$

Since gas inside cavity obeys Boyle's law, we get

$(4/3)\times \pi a^3 m\Pi=(4/3)\times \pi r^3 p$ so that $p=(a^3 m\Pi)/r^3$

When $r'=r$ then $v'=v, p'=p=(a^3 m\Pi)/r^3$. So (4) gives

$$-\frac{F'(t)}{r}+\frac{1}{2}v^2=\frac{\Pi}{\rho}-\frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3} \qquad \ldots(5)$$

From (1), $F'(t)=2r\frac{dr}{dt}\cdot v+r^2\cdot\frac{dv}{dt}=2rv^2+r^2\frac{dv}{dr}\frac{dr}{dt}$, as $v=\frac{dr}{dt}$

or $$F'(t)=2rv^2+r^2v\,(dv/dr)$$

Hence (5) reduces to

$$-\frac{1}{r}\left(2rv^2+r^2v\frac{dv}{dr}\right)+\frac{1}{2}v^2=\frac{\Pi}{\rho}-\frac{a^3 m\Pi}{\rho}\cdot\frac{1}{r^3}$$

or $$rv\frac{dv}{dr}+\frac{3}{2}v^2=-\frac{\Pi}{\rho}+\frac{a^3 m\Pi}{\rho r^3} \qquad \ldots(6)$$

Multiplying both sides of (6) by $2r^2dr$, we get

$$2r^3v\,dv+3r^2v^2dr=\left(-\frac{2\Pi r^2}{\rho}+\frac{2a^3 m\Pi}{\rho r}\right)dr \quad \text{or} \quad d(r^3v^2)=\left(-\frac{2\Pi r^2}{\rho}+\frac{2a^3 m\Pi}{\rho r}\right)dr$$

Integrating, $r^3v^2=-\frac{2\Pi}{3\rho}r^3+\frac{2a^3 m\Pi}{\rho}\log r+C'$, C' being an arbitrary constant ...(7)

Initially, when $r=a$, then $v=0$. Hence (7) $\Rightarrow$ $C'=\frac{2\Pi a^3}{3\rho}-\frac{2a^3 m\Pi}{\rho}\log a$

$$\therefore \text{ From (7)}, \qquad r^3v^3=\frac{2\Pi}{3\rho}(a^3-r^3)+\frac{2a^3 m\Pi}{\rho}\log\left(\frac{r}{a}\right) \qquad \ldots(8)$$

Since the radius of the sphere oscillates between a and na, we have $v=0$, when $r=a$ and $r=na$. Putting $v=0$ and $r=na$ in (8), we have

$$0=\frac{2\Pi}{3\rho}\left\{a^3-n^3a^3+3ma^3\log\left(\frac{na}{a}\right)\right\}$$

so that $1+3m\log n-n^3=0$, as $a\neq 0$

Second Part. Let m be nearly equal to 1. Then, we take $r = a + x$ where x is small. Again, $v = dr/dt = dx/dt = \dot{x}$. Hence, taking $m = 1$, (8) reduces to

$$(a+x)^3 \dot{x}^2 = \frac{2\Pi}{3\rho}\left\{a^3 - (a+x)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(\frac{a+x}{a}\right)$$

or
$$a^3\left(1+\frac{x}{a}\right)^3 \dot{x}^2 = \frac{2\Pi a^3}{3\rho}\left\{1-\left(1+\frac{x}{a}\right)^3\right\} + \frac{2a^3\Pi}{\rho}\log\left(1+\frac{x}{a}\right)$$

or
$$\left(1+\frac{x}{a}\right)^3 \dot{x}^2 = \frac{2\Pi}{3\rho}\left\{1-\left(1+\frac{3x}{a}+\frac{3x^2}{a^2}+\ldots\right)\right\} + \frac{2a^3\Pi}{\rho}\left\{\frac{x}{a}-\frac{1}{2}\frac{x^2}{a^2}+\ldots\right\}$$

or
$$\dot{x}^2 = \frac{2\Pi}{3\rho}\left(1+\frac{x}{a}\right)^{-3}\left[-\frac{9}{2}\frac{x^2}{a^2}+\ldots\right] = \frac{2\Pi}{3\rho}\left(1-\frac{3x}{a}+\frac{6x^2}{a^2}-\ldots\right)\left[-\frac{9}{2}\frac{x^2}{a^2}+\ldots\right]$$

or
$$\dot{x}^2 = -[3\Pi x^2/\rho a^2], \text{ neglecting higher powers of } x$$

Differentiating the above relation with respect to t, we get

$$2\dot{x}\ddot{x} = -\frac{3\Pi}{\rho a}\cdot 2x\dot{x} \qquad \text{or} \qquad \ddot{x} = -\frac{3\Pi}{\rho a^2}x,$$

which represents the standard equation of simple harmonic motion and hence the required time of oscillation (*i.e.* periodic time) is given by

$$2\pi/\sqrt{(3\Pi/\rho a^2)} \qquad i.e. \qquad 2\Pi\sqrt{(\rho a^2/3\Pi)}.$$

Ex. 18. *λ denoting a variable parameter, and f a given function, find the condition that $f(x,y,\lambda)=0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.* **[Kurukshetra 1998]**

Sol. If ψ is the stream function, then streamlines are given by

$$\psi = C \text{ (constant)} \qquad ...(1)$$

Given that $$f(x,y,\lambda)=0 \qquad ...(2)$$

represents a system of streamlines, λ being parameter. Then for $\lambda=\lambda'$ (say), (2) must give a streamline which corresponds with (1) for $C=C'$. Hence ψ is a function of λ alone. Moreover λ is a function of x and y from (2). Hence, we obtain

$$\frac{\partial\psi}{\partial x}=\frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial x} \quad \text{and} \quad \frac{\partial\psi}{\partial y}=\frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial y}$$

Again, $$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial}{dx}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\cdot\frac{\partial\lambda}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\right)\frac{\partial\lambda}{\partial x}+\frac{d\psi}{d\lambda}\frac{\partial}{\partial x}\left(\frac{\partial\lambda}{\partial x}\right)$$

so that $$\frac{\partial^2\psi}{\partial x^2}=\left\{\frac{d}{d\lambda}\left(\frac{d\psi}{d\lambda}\right)\right\}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial x}+\frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2}$$

Thus, $$\frac{\partial^2\psi}{\partial x^2}=\frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial x}\right)^2+\frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2} \qquad ...(3)$$

Similarly, $$\frac{\partial^2\psi}{\partial y^2}=\frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial y}\right)^2+\frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial y^2} \qquad ...(4)$$

For the irrotational motion, $$\partial^2\psi/\partial x^2+\partial^2\psi/\partial y^2=0. \qquad ...(5)$$

Adding (3) and (4) and using (5), we get

$$\frac{d^2\psi}{d\lambda^2}\left[\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(\frac{\partial\lambda}{\partial y}\right)^2\right]+\frac{d\psi}{d\lambda}\left(\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}\right)=0$$

or $$\left[\left(\frac{\partial\lambda}{\partial x}\right)^2+\left(\frac{\partial\lambda}{\partial y}\right)^2\right]\Bigg/\left[\frac{\partial^2\lambda}{\partial x^2}+\frac{\partial^2\lambda}{\partial y^2}\right]=-\frac{d\psi/d\lambda}{d^2\psi/d\lambda^2} \qquad ...(6)$$

Since the R.H.S. of (6) is a function of λ alone, the required condition is that the L.H.S. of (6) should be a function of λ alone.

Theorem. *Show that the vorticity vector* Ω *of an incompressible viscous fluid moving under no external forces satisfies the differential equation*

$D\Omega/Dt = (\Omega\cdot\nabla)q + \nu\nabla^2\Omega$ where ν is the kinematic coefficient of viscosity.

[Agra 2000, 05, 06; Kolkata 2006; Himanchal 2000, 02, 03, 09; Meerut 2011]

Proof. Navier-Stokes equation for incompressible viscous fluid with constant viscosity (refer equation (17) in Art. 14.1) is

$$\partial\mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q}. \qquad \text{...(1)}$$

Let the forces be conservative. Then there exists a force potential V such that $\mathbf{B} = -\nabla V$.

Again, by vector calculus $\qquad \nabla\mathbf{q}^2 = \nabla(\mathbf{q}\cdot\mathbf{q}) = 2[(\mathbf{q}\cdot\nabla)\mathbf{q} + \mathbf{q}\times\text{curl }\mathbf{q}.]$

or $\qquad (\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - \mathbf{q}\times\text{curl }\mathbf{q}$

or $\qquad (\mathbf{q}\cdot\nabla)\mathbf{q} = \nabla(q^2/2) - 2\mathbf{q}\times\Omega \qquad \left[\text{Taking } \Omega = (1/2)\times\text{curl }\mathbf{q}\right]$

Then (1) reduces to $\qquad \partial\mathbf{q}/dt + \nabla(q^2/2) - 2\mathbf{q}\times\Omega = -\nabla V - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q}$

or $\qquad \partial\mathbf{q}/\partial t - 2\mathbf{q}\times\Omega = -\nabla(V + p/\rho + q^2/2) + \nu\,\nabla^2\mathbf{q}$

Taking curl of both sides and using the results curl grad $\equiv 0$ and curl $(\partial\mathbf{q}/\partial t) = \partial(\text{curl }\mathbf{q})/\partial t$ $= 2(\partial\Omega/\partial t)$ and curl $\nabla^2\mathbf{q} \equiv \nabla^2\text{curl }\mathbf{q} = 2\nabla^2\Omega$, we obtain

$$\partial\Omega/\partial t - \text{curl }(\mathbf{q}\times\Omega) = \nu\,\nabla^2\Omega$$

or $\qquad \partial\Omega/\partial t - [\mathbf{q}\,div\,\Omega - \Omega\,div\,\mathbf{q} + (\Omega\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\Omega] = \nu\,\nabla^2\Omega$

or $\qquad \partial\Omega/\partial t + (\mathbf{q}\cdot\nabla)\Omega = (\Omega\cdot\nabla)\mathbf{q} + \nu\nabla^2\Omega$

[$\because$ Equation of continuity is $div\ \mathbf{q} = 0$ Also $div\ \Omega = div$ curl $\mathbf{q} = 0$]

or $\qquad D\Omega/Dt = (\Omega\cdot\nabla)\mathbf{q} + \nu\,\nabla^2\Omega, \qquad \text{...(2)}$

which is known as *vorticity equation or vorticity transport equation.*

2. Prove that the equation of motion of a homogeneous inviscid liquid moving under conservative forces may be written in the form $\qquad \dfrac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\text{curl }\mathbf{q} = -\,\text{grad}\left(\dfrac{p}{\rho} + \dfrac{1}{2}q^2 + \Omega\right)$

[**Hint.** From Art. 3.1, we have $\qquad \dfrac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\text{curl }\mathbf{q} = \mathbf{F} - \dfrac{1}{\rho}\nabla p - \dfrac{1}{2}\nabla q^2 \qquad \text{...(1)}$

Since the forces form a conservative system, there exists a force potential Ω such that $\mathbf{F} = -\nabla\Omega$. Moreover, the fluid being homogeneous, we may write $\qquad (1/\rho)\,\nabla p = \nabla(p/\rho)$.

Hence (1) reduces to

$$\frac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\text{curl }\mathbf{q} = -\nabla\Omega - \nabla\left(\frac{p}{\rho}\right) - \nabla\left(\frac{1}{2}q^2\right) \quad \text{or} \quad \frac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\text{curl }\mathbf{q} = -\,\text{grad}\left(\frac{\text{p}}{\rho} + \frac{1}{2}q^2 + \Omega\right)$$

Ex. 1. *Show that for an incompressible steady flow with constant viscosity, the velocity components* $u(y) = y\dfrac{U}{h} + \dfrac{h^2}{2\mu}\left(-\dfrac{dp}{dx}\right)\dfrac{y}{h}\left(1-\dfrac{y}{h}\right)$, $\hat{v} = w = 0$ *satisfy the equation of motion, when the body force is neglected. h, U, dp/dx are constants and* $p = p(x)$. **(Meerut 2007, 11)**

Sol. Given $$u(y) = (yU/h) + (h^2/2\mu)(-dp/dx)(y/h)(1-y/h) \qquad \ldots(1)$$

$$v = 0 \quad \text{and} \quad w = 0 \qquad \ldots(2)$$

The equation of motion for viscous incompressible fluid is given by

$$\partial \mathbf{q}/\partial t + (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \mathbf{B} - (1/\rho)\times\nabla p + \nu\,\nabla^2\mathbf{q} \qquad \ldots(3)$$

Here $\partial \boldsymbol{q}/\partial t = \mathbf{0}$, the motion being steady. ...(4)

and $\mathbf{B} = \mathbf{0}$, as the body force is neglected. ...(5)

Since $v = w = 0$, we have $\mathbf{q} = \mathbf{i}u$

$$\therefore \qquad \nabla^2\mathbf{q} = \mathbf{i}\nabla^2 u \qquad \ldots(6)$$

Given that $p = p(x)$ so that $$\nabla p = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)p = \mathbf{i}\frac{dp}{dx} \qquad \ldots(7)$$

Also $$\mathbf{q}\cdot\nabla = (\mathbf{i}u)\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x}$$

$$\therefore \qquad (\mathbf{q}\cdot\nabla)\,\mathbf{q} = \left(u\frac{\partial}{\partial x}\right)(\mathbf{i}u) = \mathbf{i}u\frac{\partial u}{\partial x} = 0, \text{ as } u = u(y), \text{ given} \qquad \ldots(8)$$

Substituting (4), (5), (6), (7) and (8) into (3), we have

$$0 = -\frac{1}{p}\frac{dp}{dx} + \nu\nabla^2 u \quad \text{or} \quad \frac{1}{p}\frac{dp}{dx} = \frac{\mu}{\rho}\frac{d^2u}{dy^2} \quad \text{or} \quad \frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}. \qquad \ldots(9)$$

$$\left[\because \nu = \mu/\rho \text{ and } \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\right]$$

Now from (1), $$\frac{du}{dy} = \frac{U}{h} - \frac{h}{2\mu}\frac{dp}{dx}\left(1-\frac{2y}{h}\right), \text{ as } dp/dx \text{ is given to be constant}$$

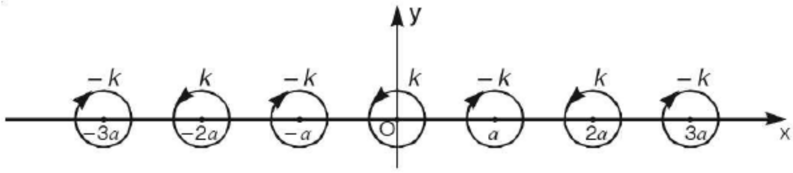
$$\therefore \qquad \frac{d^2u}{dy^2} = 0 - \frac{h}{2\mu}\frac{dp}{dx}\left(-\frac{2}{h}\right) = \frac{1}{\mu}\frac{dp}{dx},$$

which is the same as (9). This proves that the equation of motion is satisfied.

11.19. Illustrative solved examples.

Ex. 1. *An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show also that, if* α *be the radius of a vortex, the amount of flow between any vortex and the next is* $(k/\pi)\log\cot(\pi\alpha/2a)$. **[Bhopal 2000, Kolkata 2005; Agra 2003]**

Sol. Let the row of vortices be taken along the *x*-axis. Let there be vortices of strength k each at the points (0, 0), ($\pm 2a$, 0), ($\pm 4a$, 0), ... and those of strength $-$ k each at the points ($\pm a$, 0), ($\pm 3a$, 0), ($\pm 5a$, 0), ...



The complex potential of the entire system is given by

$$w = (ik/2\pi)\,[\{\log z + \log(z-2a) + \log(z+2a) + \log(z-4a) + \log(z+4a) + \ldots\}$$
$$- \{\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + \ldots\}]$$

$$= \frac{ik}{2\pi}\log\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\ldots}{(z^2-a^2)(z^2-3^2a^2)\ldots} = \frac{ik}{2\pi}\log\frac{\dfrac{z}{2a}\left[1-\left(\dfrac{z}{2a}\right)^2\right]\left[1-\left(\dfrac{z}{4a}\right)^2\right]\ldots}{\left[1-\left(\dfrac{z}{a}\right)^2\right]\left[1-\left(\dfrac{z}{3a}\right)^2\right]\ldots} + \text{a const.}$$

Thus, $$w = \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right), \qquad ..(1)$$

[Using well known expansions of $\sin(\pi z/2a)$ and $\cos(\pi z/2a)$ given in Art. 1.10]

which is the desired potential function that determines the velocity potential and stream function.

From (1), we get

$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x+iy) \quad \text{so that} \quad \phi - i\psi = -\frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x-iy)$$

$\sin(\pi z/2a)$ and $\cos(\pi z/2a)$]

Subtracting, these give $$2i\psi = \frac{ik}{2\pi}\left[\log\tan\frac{\pi}{2a}(x+iy) + \log\tan\frac{\pi}{2a}(x-iy)\right]$$

$$\therefore \quad \psi = \frac{k}{4\pi}\log\frac{\sin\dfrac{\pi}{2a}(x+iy)\sin\dfrac{\pi}{2a}(x-iy)}{\cos\dfrac{\pi}{2a}(x+iy)\cos\dfrac{\pi}{2a}(x-iy)} = \frac{k}{4\pi}\log\frac{\cosh(\pi y/a) - \cos(\pi x/a)}{\cosh(\pi y/a) + \cos(\pi x/a)} \qquad ...(2)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity, q_0 of vortex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a} - \frac{ik}{2a}\log z\right]\right\}_{z=0} = -\frac{ik}{2\pi}\left[\frac{\sec^2(\pi z/2a)}{\tan(\pi z/2a)}\times\frac{\pi}{2a} - \frac{1}{z}\right]_{z=0} = 0.$$

[On simplifying the indeterminate form with help of L'Hospital's rule]

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus we find that the vortex row induces no velocity on itself.

We now determine the required flow. For any point on the x-axis, $y = 0$ and hence ψ' at any point on the x-axis is given by [putting $y = 0$ in (2)]

$$\psi' = \frac{k}{4\pi}\log\frac{1-\cos(\pi x/a)}{1+\cos(\pi x/a)} = \frac{k}{4a}\log\frac{2\sin^2(\pi x/2a)}{2\cos^2(\pi x/2a)} = \frac{k}{4\pi}\log\tan\frac{\pi x}{2a} \quad \text{...(3)}$$

$\therefore$ The required flow between two consecutive vortices

$$= (\psi')_{a-\alpha} - (\psi')_{\alpha} = \frac{k}{2\pi}\left[\log\tan\frac{\pi(a-\alpha)}{2a} - \log\tan\frac{\pi\alpha}{2a}\right]$$

$$= \frac{k}{2\pi}\log\frac{\tan(\pi/2-\pi\alpha/2a)}{\tan(\pi\alpha/2a)} = \frac{k}{2\pi}\log\cot^2\frac{\pi\alpha}{2a} = \frac{k}{\pi}\log\cot\frac{\pi\alpha}{2a}.$$