5(a) Find the partial differential equation of the family of all tangent planes to the ellipsoid: $x^2+4y^2+4z^2=4$, which are not perpendicular to the xy-plane.

Sol^n: Given that $x^2+4y^2+4z^2=4$ —— ①

Its tangent plane at a point $P(x_1, y_1, z_1)$ is

$$xx_1+4yy_1+4zz_1=4 \quad \text{—— ②}$$

Let the plane be $lx+my+nz=p$ —— ③

then $\dfrac{x_1}{l}=\dfrac{4y_1}{m}=\dfrac{4z_1}{n}=\dfrac{4}{p}$

$$\Rightarrow x_1=\frac{4l}{p}, \quad y_1=\frac{m}{p}, \quad z_1=\frac{n}{p}$$

$$① \equiv \frac{16l^2}{p^2}+\frac{4m^2}{p^2}+\frac{4n^2}{p^2}=4$$

$$\Rightarrow 4l^2+m^2+n^2=p^2$$

$$\therefore ③ \equiv lx+my+nz=\pm\sqrt{4l^2+m^2+n^2} \quad \text{—— ④}$$

Since it is not perpendicular to xy-plane,

$$\therefore n\neq 0$$

$$\therefore ④ \equiv \frac{l}{n}x+\frac{m}{n}y+z=\pm\sqrt{4\left(\frac{l}{n}\right)^2+\left(\frac{m}{n}\right)^2+1}$$

$$\Rightarrow \alpha x+\beta y+z=\pm\sqrt{4\alpha^2+\beta^2+1} \quad \text{—— ⑤}$$

which is the required tangent plane

Differentiating ④ partially w.r.t 'x' & 'y'

we get

$$\alpha+\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=-\alpha \Rightarrow p=-\alpha \Rightarrow \alpha=-p$$

$$\beta+\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial z}{\partial y}=-\beta \Rightarrow q=-\beta \Rightarrow \beta=-q$$

$\therefore$ ⑤ $\equiv$

$$-px-qy+z=\pm\sqrt{4p^2+q^2+1}$$

$$\Rightarrow (px+qy-z)^2=(4p^2+q^2+1)$$

which is the required partial Differential Equation.

Q.6(a) Solve $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$.

Sol:- Here Lagrange's Auxillary equations are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3-y^3)} \quad \text{—} (1)$$

Taking first two fractions of (1), we have

$$(2y^4 - x^3y)dx = (y^3x - 2x^4)dy.$$

$$\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left[\frac{1}{x^2} - \frac{2x}{y^3}\right]dy$$

[$\because$ By diving it by x^3y^3]

or $\left[\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$

or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$

Integrating, we get

$$(y/x^2) + (x/y^2) = c_1$$

$$\boxed{x^3 + y^3 = x^2y^2c_1} \quad \text{—} (2)$$

Choosing $(1/x), (1/y), (1/3z)$ as multiplier of each fraction of (1).

$$= \frac{1/x\,dx + 1/y\,dy + 1/3z\,dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)}$$

$$= \frac{1/x\,dx + 1/y\,dy + 1/3z\,dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz = 0$$

so that $\log x + \log y + \frac{1}{3}\log z = \log c_2$

$$\log x + \log y + \log z^{1/3} = \log c_2$$

$$\log(x y z^{1/3}) = \log c_2$$

$$\boxed{xyz^{1/3} = c_2} \quad \text{— (3)}$$

from (2) & (3), the required general solution is

$$\boxed{\phi\left(xyz^{1/3},\ y/x^2 + x/y^2\right) = 0}$$

ϕ being the arbitrary function

7(a) Solve the partial differential Equation

$$(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

where; $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Sol: Given P.D.E is

$$(2D^2 - 5DD' + 5D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

The auxillary of the given equation is

$$2m^2 - 5m + 2 = 0$$

$$(2m-1)(m-1) = 0$$

$$m = 1/2, 1$$

$$\therefore C.F = \phi_1(y + x/2) + \phi_2(y+2x)$$

$$= \phi_1[1/2(2y+x)] + \phi_2(y+2x).$$

$$\boxed{C.F = \phi_1(2y+x) + \phi_2(y+2x)}$$

where, ϕ_1, ϕ_2 being arbitrary functions.

Now $P.I = \frac{1}{2D^2 - 5DD' + 2D'^2}[5\sin(2x+y) + 24(y-x) + e^{3x+4y}]$

$$P.I = \underbrace{\frac{1}{2D^2-5DD'+2D'^2}[5\sin(2x+y)]}_{P.I.1} + \underbrace{\frac{1}{2D^2-5DD'+2D'^2}24(y-x)}_{PI.2} + \underbrace{\frac{1}{2D^2-5DD'+2D'^2}\cdot e^{3x+4y}}_{P.I.3}.$$

$$P.I_2 = \frac{1}{2D^2-5DD'+2D'^2}[24(y-x)]$$

$$= 24 \cdot \frac{1}{2D^2-5DD'+2D'^2}(y-x)$$

$$= 24 \cdot \frac{1}{2(-1)^2 - 5\times 1\times -1 + 2\times(1)^2}\iint v\,dv\,dv \quad [\because v = y-x]$$

$$= \cancel{\frac{24}{20}}\ \frac{24}{20}\int \frac{v^2}{2}\,dv$$

8(c) A thin annulus occupies the region $0 \leq a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0° while along the outer edge the temperature is held at $T = K\cos\theta/2$, where K is a constant. Determine the temperature distribution in the annulus.

Soln:

We are given a circular annulus whose inner and outer radii are a and b respectively.

The steady state temperature $T(r,\theta)$ at any point $P(r,\theta)$ of the annulus is the solution of the Laplace's equation in polar co-ordinates (r,θ) namely

$$r^2\left(\frac{\partial^2 T}{\partial r^2}\right) + r\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{—}(1)$$

Since the temperatures along the inner $(r=a)$ and outer boundary $(r=b)$ are maintained at 0° and $K\cos\theta/2$ respectively

i.e. $T(a,\theta) = 0$ and $T(b,\theta) = K\cos\theta/2$. ②

Clearly the temperature function $T(r,\theta)$ must be periodic in θ of period 2π. Accordingly, we now proceed to solve (1)

Suppose (1) has a solution of the form

$$T(r,\theta) = R(r)\,H(\theta) \quad \text{--- (3)}$$

where R and H are functions of r and θ respectively

Using (3), (1) reduces to

$$r^2R''H + rR'H + RH'' = 0$$

$$\Rightarrow (r^2R'' + rR')H = -RH''$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{H''}{H} \quad \text{--- (4)}$$

Since L.H.S of (4) is a function of r only and the R.H.S is a function of θ only, the two sides of (4) must be equal to the same constant say μ.

Then (4) gives

$$r^2R'' + rR' - \mu R = 0 \quad \text{--- (5)}$$

$$\text{and} \quad H'' + \mu H = 0 \quad \text{--- (6)}$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Re-writing (5), $(r^2D^2 + rD - \mu)R = 0$ --- (7)

where $D \equiv \frac{d}{dr}$

Let $r = e^z \Rightarrow z = \log r$. and $D_1 = \frac{d}{dz}$

Then w.r.t $rD_0 = D_1$ and $r^2D^2 = D_1(D_1-1)$.

$$(7) \equiv \left(D_1(D_1-1)+D_1-\mu\right)R = 0$$

$$\Rightarrow (D_1^2-\mu)R = 0 \quad \text{--- (8)}$$

Again, let $D_2 = \frac{d}{d\theta}$. Then (6) may be re-written as $(D_2^2+\mu)H = 0$ --- (9)

The solutions of (8) and (9) depend on μ.

Consider following cases:

Case(i): Let $\mu = 0$. Then (8) and (9) reduces to

$$\frac{d^2R}{dz^2} = 0 \quad \text{and} \quad \frac{d^2H}{d\theta^2} = 0.$$

solving these,

$$R(r) = c_1 z + c_2 = c_1 \log r + c_2$$

$$\text{and } H = c_3\theta + c_4.$$

Hence, from (3), solution of (1) is of the form

$$T(r,\theta) = (c_1\log r + c_2)(c_3\theta + c_4) \quad \text{--- (10)}$$

Since $T(r,\theta)$ is periodic in θ, we must take $c_3 = 0$. Then equation (10) becomes

$$T(r,\theta) = (c_1\log r + c_2)\, c_4$$

$$= \frac{1}{2}(a_0\log r + b_0) \quad \text{--- (11)}$$

where $a_0 = 2c_1c_4$ and $b_0 = c_2c_4$. are new arbitrary constants.

Case (ii): Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then ⑧ and ⑨ become

$$(D_1^2 - \lambda^2)R = 0 \quad \text{and} \quad (D_2^2 + \lambda^2)H = 0. \quad (12)$$

[Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $(D_2^2 - \lambda^2)H = 0$ whose solution will not contain trigonometric functions and hence periodic nature of $T(r,\theta)$ will not be attained]

Solving (12)

$$R(r) = c_5 e^{\lambda z} + c_6 e^{-\lambda z}$$
$$= c_5 (e^z)^\lambda + c_6 (e^z)^{-\lambda} = c_5 r^\lambda + c_6 r^{-\lambda}$$

and $H(\theta) = c_7 \cos\lambda\theta + c_8 \sin\lambda\theta$.

Hence, from (3), a solution of (1) is of the form

$$T(r,\theta) = (c_5 r^\lambda + c_6 r^{-\lambda})(c_7 \cos\lambda\theta + c_8 \sin\lambda\theta) \quad (13)$$

Since $T(r,\theta)$ is periodic in θ with period 2π.

∴ we must take $\lambda = n$, where $n = 1, 2, 3, \ldots$

Hence (13) takes the form

$$T(r,\theta) = (c_5 r^n + c_6 r^{-n})(c_7 \cos n\theta + c_8 \sin n\theta) \quad (14)$$

$n = 1, 2, 3, \ldots$

with the help of (11) and (14), the most general solution of (1) is

$$T(r,\theta) = \frac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos n\theta + (c_n r^n + d_n r^{-n}) \sin n\theta \quad \text{—(15)}$$

which holds for $a \le r \le b$.

Here $a_n = c_5 c_7$, $b_n = c_6 c_7$; $c_n = c_5 c_8$. $d_n = c_6 c_8$. are new arbitrary constants.

putting $r=a$, and $r=b$ by turn in (15) and B.C (2) we have

$$0 = \frac{a_0 \log a + b_0}{2} + \sum_{n=1}^{\infty} (a_n a^n + b_n a^{-n}) \cos n\theta + \sum (c_n a^n + d_n a^{-n}) \sin n\theta \quad \text{—(16)}$$

$$k \cos\frac{\theta}{2} = \frac{a_0 \log b + b_0}{2} + \sum_{n=1}^{\infty} (a_n b^n + b_n b^{-n}) \cos n\theta + (c_n b^n + d_n b^{-n}) \sin n\theta \quad \text{—(17)}$$

(16) and (17) are usual expansions of $f_1(\theta) = 0$ and $f_2(\theta) = k\cos\theta/2$ as Fourier series in $(0, 2\pi)$.

Hence we have

$$a_0 \log a + b_0 = \frac{1}{\pi} \int_0^{2\pi} (0)\, d\theta \;\&\; a_0 \log b + b_0 = \frac{1}{\pi} \int_0^{2\pi} (0)\, d\theta.$$

solving these we get

$a_0 \log a + b_0 = 0$ and $a_0 \log b + b_0 = 0$ —(18)

$$\Rightarrow \boxed{a_0 = 0 \text{ and } b_0 = 0}$$

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$\because\ a_n a^n + b_n a^{-n} = \frac{1}{\pi}\int_0^{2\pi}(0)\cos n\theta, = 0 \Rightarrow b_n = -a_n a^{2n}$

& $a_n b^n + b_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} k\cos\frac{\theta}{2}\cos n\theta\, d\theta$

$$a_n b^n + b_n b^{-n} = \frac{k}{2\pi}\int_0^{2\pi}\left[\cos\left(n+\tfrac{1}{2}\right)\theta + \cos\left(n-\tfrac{1}{2}\right)\theta\right]d\theta$$

$$= \frac{k}{2\pi}\left[\frac{\sin(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\sin(n-\frac{1}{2})\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= 0$$

$\Rightarrow a_n b^n - a_n a^{2n} b^{-n} = 0$

$\Rightarrow a_n\left[b^n - a^{2n} b^{-n}\right] = 0$

$\Rightarrow a_n = 0$ and $b_n = 0$

$c_n a^n + d_n a^{-n} = \frac{1}{\pi}\int_0^{2\pi}(0)\sin n\theta\, d\theta = 0$

& $c_n b^n + d_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} k\cos\frac{\theta}{2}\sin n\theta\, d\theta$

$$c_n b^n + d_n b^{-n} = \frac{k}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\tfrac{1}{2}\right)\theta + \sin\left(n-\tfrac{1}{2}\right)\theta\right]d\theta$$

$$= -\frac{k}{2\pi}\left[\frac{\cos(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\cos(n-\frac{1}{2})\theta}{(n-\frac{1}{2})}\right]_0^{2\pi}$$

$$= -\frac{k}{2\pi}\left[-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$$

$$= \frac{k}{\pi}\left[\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right]$$

$$= \frac{k}{\pi}\frac{2n}{(n^2-\frac{1}{4})}$$

$$C_n b^n + d_n b^{-n} = \frac{8kn}{\pi(4n^2-1)} \quad \text{--- (20)}$$

Now we have

$$C_n a^n + d_n a^{-n} = 0$$

$$\Rightarrow d_n = -C_n a^{2n}$$

$\therefore$ from (20),

$$C_n b^n - C_n a^{2n} b^{-n} = \frac{8kn}{\pi(4n^2-1)}$$

$$C_n\left[b^n - b^{-n} a^{2n}\right] = \frac{8kn}{\pi(4n^2-1)}$$

$$C_n = \frac{8kn}{\pi(4n^2-1)} \frac{1}{(b^n - b^{-n}a^{2n})}$$

$\therefore$ from (I)

$$T(r,\theta) = \sum_{n=1}^{\infty}\left(C_n r^n + d_n r^{-n}\right) \sin n\theta$$

$$= \sum_{n=1}^{\infty}\left(C_n r^n - C_n a^{2n} r^{-n}\right) \sin n\theta$$

$$(\because d_n = -C_n a^{2n})$$

$$= \sum_{n=1}^{\infty} C_n\left[r^n - a^{2n} r^{-n}\right] \sin n\theta$$

$$= \frac{8k}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1} \frac{\left[r^n - a^{2n} r^{-n}\right]}{\left(b^n - b^{-n}a^{2n}\right)}$$

$$T(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\left[\frac{(r/a)^n - (a/r)^n}{(b/a)^n - \left(\frac{a}{b}\right)^n}\right]\sin n\theta$$

which is the required temperature distribution in the given annulus.