CSE - 2010

Fluid dynamics

Q5f For an incompressible fluid vorticity at every point is constant in direction and magnitude. Show that velocity components u, v, w are solutions of Laplace equation.

sol. Vorticity (Ω) = Curl q

$$\Omega = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$\because$ Ω is constant

$$\Rightarrow \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = c_1 \quad -(1)$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = c_2 \quad -(2)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = c_3 \quad -(3)$$

Differentiating (2) by z and (3) by y and substracting we get

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial z \partial x} - \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 w}{\partial x \partial z} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0 \quad -(4)$$

Acc. to equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\therefore -\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$ → Putting this in (4)

we get, $$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

$$\therefore \boxed{\nabla^2 u = 0}$$

Similarly we can show $\nabla^2 v = 0$ and $\nabla^2 w = 0$

Hence Proved.

Q8(b) When a pair of equal and opposite rectilinear vortices are situated in long circular cylinder at equal distance from its axis. Show that path of each vortex is given by equation

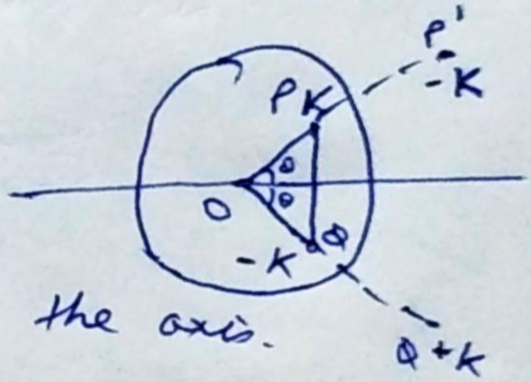
$$(r^2 \sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2 b^2 r^2 \sin^2\theta$$

θ being measured from the line through the centre $\perp$ to joint of the vortices.

sol. Let the axis of the cylinder be x-axis. and vortices are situated at $P(r, \theta)$ and $Q(r, -\theta)$ respectively.

Strength at $P = K$

" " $Q = -K$



Clearly line PQ is $\perp$ to the axis.

~~The complex pote~~

The image system due to P consists

→ Vortex of strength $-k$ at $\frac{a^2}{r}$ (P'), where a is radius of cylinder

→ Vortex at centre of cylinder of strength K

Image system due to Q consists

→ Vortex of strength k at $\frac{a^2}{r}$ (say Q')

→ Vortex of strength $-K$ at center of cylinder.

Vortices at the centre of cylinder cancel out.
At any point $S(z)$, the complex potential W is given by

$$W = \frac{iK}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

Complex Potential at $P(r, \theta)$ will be

$W' = W -$ Potential due to P

$$W' = \frac{iK}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - ie^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$W' = \frac{iK}{2\pi}\left[\log \frac{\left(z - \frac{a^2}{r}e^{-i\theta}\right)}{(z - re^{-i\theta})\left(z - \frac{a^2}{r}e^{i\theta}\right)}\right]$$

At P, $z = re^{i\theta}$

$$\therefore W' = \frac{ik}{2\pi}\left[\log \frac{\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)}{\left(re^{i\theta} - re^{-i\theta}\right)\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right)}\right]$$

As $w' = \phi + i\psi$

$$\therefore \psi = \frac{k}{2\pi}\log \frac{\left|re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right|}{\left|re^{i\theta} - re^{-i\theta}\right|\left|re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right)}$$

Streamlines are given by $\psi =$ Constant

$$\frac{K}{2\pi}\left[\log \frac{\left|\frac{r^2 e^{i\theta} - a^2 e^{-i\theta}}{r}\right|}{|2ir\sin\theta|\,\left|\frac{r^2 e^{i\theta} - a^2 e^{i\theta}}{r}\right|}\right] = C_1$$

$$\log \frac{|(r^2+a^2)\sin\theta\, i + (r^2-a^2)\cos\theta|}{|2ir\sin\theta|\,|(r^2-a^2)\cos\theta + i(r^2-a^2)\sin\theta|} = C_2$$

$$\frac{(r^2+a^2)^2\sin^2\theta + (r^2-a^2)^2\cos^2\theta}{4r^2\sin^2\theta\,((r^2-a^2)^2\cos^2\theta + (r^2-a^2)^2\sin^2\theta)} = C_3$$

$$\Rightarrow \frac{((r^2+a^2)^2 - (r^2-a^2)^2)\sin^2\theta + (r^2-a^2)^2}{4r^2\sin^2\theta\,(r^2-a^2)^2} = C_3$$

$$\Rightarrow \frac{4r^2a^2\sin^2\theta + (r^2-a^2)^2}{4r^2\sin^2\theta\,(r^2-a^2)^2} = C_3$$

$$\Rightarrow 4r^2a^2\sin^2\theta + (r^2-a^2)^2 = C_3\, 4r^2\sin^2\theta\,(r^2-a^2)^2$$

$$4r^2a^2\sin^2\theta = (r^2-a^2)^2\left[4r^2\sin^2\theta\, C_3 - 1\right]$$

If we take $C_3 = \frac{1}{4b^2}$, b is constant

$$4r^2a^2\sin^2\theta = (r^2-a^2)^2\left[4r^2\sin^2\theta\,\frac{1}{4b^2} - 1\right]$$

$$4r^2a^2\sin^2\theta = \frac{(r^2-a^2)^2}{b^2}\left(r^2\sin^2\theta - b^2\right)$$

$$\Rightarrow \boxed{4r^2a^2b^2\sin^2\theta = (r^2-a^2)^2\left(r^2\sin^2\theta - b^2\right)}$$

Hence Proved.

Mechanics (CSE-2010)

Q 5(e) Let the equation of the Parabola be

$$y^2 = 4ax$$

The ends of latus rectum LL' are $(a, 2a)$ and $(a, -2a)$

Here, $\frac{dy}{dx} = \frac{4a}{2y}$, $\therefore$ At L $\frac{dy}{dx} = \frac{4a}{2(2a)} = 1$

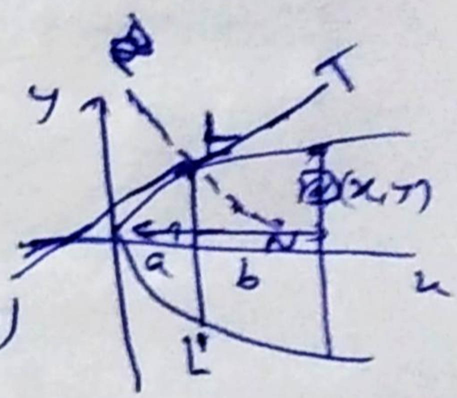
Equation of tangent at L is

$$y - 2a = 1.(x-a) \Rightarrow \boxed{y - x - a = 0} \rightarrow LT$$

Equation of Normal at L

$$y - 2a = -1\,(x-a)$$

$$\Rightarrow \boxed{y + x - 3a = 0} \rightarrow LN$$



Consider an element $dxdy$ at Point (x, y)

PM = length of $\perp$ from tangent LT

$$PM = \frac{y - x - a}{\sqrt{2}}$$

PN = length of $\perp$ from Normal LN

$$PN = \frac{y + x - 3a}{\sqrt{2}}$$

Product of Inertia about LT and $LN = PM.PN.\,dm$

$$= PM.PN.\,\rho\, dxdy$$

If LT and LN are principal axis then Product of inertia of the lamina must be 0.

$$\therefore \int_{x=0}^{b} \int_{y=-2\sqrt{ax}}^{2\sqrt{ax}} PM.PN\, \rho\, dx dy = 0$$

$$\Rightarrow \rho \int_{0}^{b} \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right) dx dy = 0$$

$$\Rightarrow \frac{\rho}{2} \int_{0}^{b} \int_{-2\sqrt{ax}}^{2\sqrt{ax}} (y^2 - 4ay + 3a^2 + 2ax - x^2)\, dx dy = 0$$

$$= \int_{0}^{b} \left| \frac{y^3}{3} - 2ay^2 + (3a^2 + 2ax - x^2)y \right|_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$= \int_{0}^{b} \frac{16}{3} ax\sqrt{ax} - 0 + 4\sqrt{ax}(3a^2 + 2ax - x^2)\, dx = 0$$

$$\Rightarrow \left| \frac{16}{15} a^{3/2} x^{5/2} + 4a^{5/2} x^{3/2} + \frac{8}{5} a^{3/2} x^{5/2} - \frac{4}{7} a^{1/2} x^{7/2} \right|_{0}^{b} = 0$$

$$\Rightarrow \frac{16}{15} a^{3/2} b^{5/2} + 4a^{5/2} b^{3/2} + \frac{8}{5} a^{3/2} b^{5/2} - \frac{4}{7} a^{1/2} b^{7/2} = 0$$

$$\Rightarrow \frac{16}{15} ab + 4a^2 + \frac{8}{5} ab - \frac{4}{7} b^2 = 0$$

$$\Rightarrow b^2 - \frac{14}{3} ab - 7a^2 = 0$$

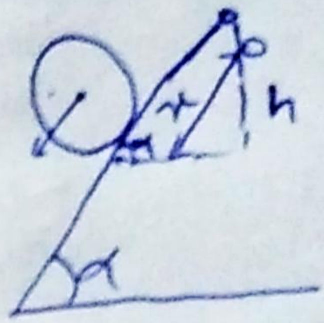
$$b = \frac{\frac{14}{3} a \pm \sqrt{\frac{196}{9} a^2 + 28a^2}}{2}$$

$\therefore b = \frac{a}{3}(7 \pm 4\sqrt{7})$

As b cannot be $-ve$, neglecting $-ve$ value

$$\therefore \boxed{b = \frac{a}{3}(7 + 4\sqrt{7})}$$

Q 8(a) As the inclined plane is rough and sphere is rolling down.



~~The dran~~ If at a distance x from the top of inclined plane the velocity is v then

Translational kinetic energy

$$= \frac{1}{2}Mv^2 = \frac{1}{2}m\dot{x}^2$$

Rotational kinetic energy

$$= \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{2}{5}Mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{M}{5}v^2 = \frac{m}{5}\dot{x}^2$$

$$\text{Total K.E.} = \frac{1}{2}m\dot{x}^2 + \frac{1}{5}m\dot{x}^2 = \frac{7}{10}m\dot{x}^2$$

$$\text{Potential energy} = -mgh = -mgx\sin\alpha$$

$\therefore$ Lagrangian $L = \text{Total K.E.} - \text{P.E}$

$$= \frac{7}{10}m\dot{x}^2 - (-mgx\sin\alpha)$$

$\therefore\ L = \frac{7}{10} m\dot{x}^2 + mgx \sin\alpha$

Now, $P_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5} m\dot{x} \quad \therefore\ \dot{x} = \frac{5P_x}{7m}$

Hence, $H = \dot{x}P_x - L$

$$H = \left(\frac{5P_x}{7m}\right)P_x - \left(\frac{7}{10} m \left(\frac{5P_x}{7m}\right)^2 + mgx \sin\alpha\right)$$

$$H = \frac{5P_x^2}{7m} - \left(\frac{5P_x^2}{14m} + mgx \sin\alpha\right)$$

$$\boxed{H = \frac{5P_x^2}{14m} - mgx \sin\alpha}$$ is the

Hamilton's equation:

The equation of motion is given by

$$m\frac{d}{dt}\left(\frac{\partial H}{\partial P}\right) - \frac{\partial H}{\partial x} = 0$$

$$\therefore\ m\frac{d}{dt}\left[\frac{\partial}{\partial P_x}\left(\frac{5P_x^2}{14m} - mgx\sin\alpha\right)\right] - \frac{\partial}{\partial x}\left(\frac{5P_x^2}{14m} - mgx\sin\alpha\right) = 0$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{5}{7}\frac{P_x}{m}\right) - (-mg\sin\alpha) = 0$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{5}{7m}\cdot\frac{7m\dot{x}}{5}\right) + mg\sin\alpha = 0$$

$$m\frac{d}{dt}(\dot{x}) + mg\sin\alpha = 0$$

$$\boxed{\ddot{x} = -g\sin\alpha} \rightarrow \text{Acceleration of the sphere.}$$