$$\left.\begin{aligned} \therefore \quad \frac{\partial \mathbf{V}}{\partial x} &= l_1 \frac{\partial \mathbf{V}'}{\partial x'} + l_2 \frac{\partial \mathbf{V}'}{\partial y'} + l_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{Similarly} \quad \frac{\partial \mathbf{V}}{\partial y} &= m_1 \frac{\partial \mathbf{V}'}{\partial x'} + m_2 \frac{\partial \mathbf{V}'}{\partial y'} + m_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{and} \quad \frac{\partial \mathbf{V}}{\partial z} &= n_1 \frac{\partial \mathbf{V}'}{\partial x'} + n_2 \frac{\partial \mathbf{V}'}{\partial y'} + n_3 \frac{\partial \mathbf{V}'}{\partial z'} \end{aligned}\right\} \quad \ldots(3)$$

Taking dot product of these three equations by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{div } \mathbf{V} = \text{div } \mathbf{V}'.$$

Theorem 4. *If* $\mathbf{V}(x, y, z)$ *is a vector function invariant under a rotation of axes, then curl* **V** *is a vector invariant under this rotation.* **[Punjab 1966]**

Proof. Proceed exactly in the same manner as in theorem 3. In place of taking dot product of equations (3), take cross product. We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

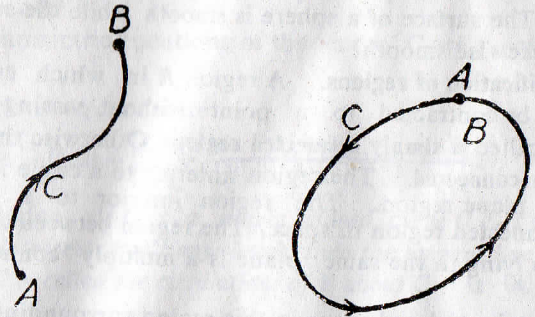
or $$\text{curl } \mathbf{V} = \text{curl } \mathbf{V}'.$$

3

Green's, Gauss's and Stoke's Theorems

§ 1. Some preliminary concepts.

Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction;* the opposite direction along C is then called the *negative direction.* Suppose A is the initial point and B the terminal point

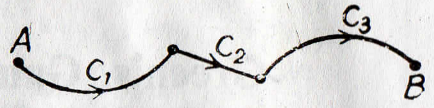


Oriented closed curve

of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve.*

Smooth curve. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t = t_1$ and $t = t_2$ respectively. We know that $\frac{d\mathbf{r}}{dt}$ is a tangent vector to this curve at the point **r**. Suppose the function $\mathbf{r}(t)$ is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a smooth curve.

A curve C is said to be piecewise smooth if it is composed of a finite number of smooth curves. The curve C in the adjoining figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3. The circle is a smooth closed curve while the curve



Piecewise smooth curve

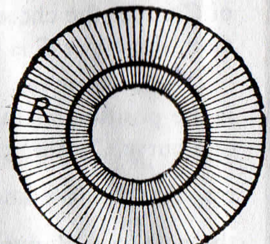
consisting of the four sides of a rectangle is a piecewise smooth closed curve.

Smooth surface. Suppose S is a surface which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S. Then S is called a **smooth surface.**

If a surface S is not smooth but can be subdivided into a finite number of smooth surfaces, then it is called a **piecewise smooth surface.** The surface of a sphere is smooth while the surface of a cube is piecewise smooth.

Classification of regions. A region R in which every closed curve can be contracted to a point without passing out of the region is called **a simply connected region.** Otherwise the region R is **multiply-connected.** The region interior to a circle is a simply-connected plane region. The region interior to a sphere is a simply-connected region in space. The region between two concentric circles lying in the same plane is a multiply connected plane region.

If we take a closed curve in this region surrounding the inner circle, then it cannot be contracted to a point without passing out of the region. Therefore the region is not simply-connected. However the region between two concentric spheres is a simply-connected region in space. The region between two infinitely long coaxial cylinders is a multiply-connected region in space.



§ 2. Line Integrals. *Any integral which is to be evaluated along a curve is called a line integral.*

Suppose $\mathbf{r}\ (t)=x\ (t)\ \mathbf{i}+y\ (t)\ \mathbf{j}+z\ (t)\ \mathbf{k}$, where $\mathbf{r}\ (t)$ is the position vector of (x, y, z) *i.e.* $\mathbf{r}\ (t)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, defines a piecewise smooth curve joining two points A and B. Let $t=t_1$ at A and $t=t_2$ at B. Suppose $\mathbf{F}\ (x, y, z)=F_1\ \mathbf{i}+F_2\ \mathbf{j}+F_3\ \mathbf{k}$ is a vector point function defined and continuous along C. If s denotes the arc length

of the curve C, then $\dfrac{d\mathbf{r}}{ds}=\mathbf{t}$ is a unit vector along the tangent to the curve C at the point $\mathbf{r}$. The component of the vector $\mathbf{F}$ along this tangent is $\mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$. The integral of $\mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$ along C from A to B written as

$$\int_A^B \left[\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right] ds=\int_A^B \mathbf{F}\cdot d\mathbf{r}=\int_C \mathbf{F}\cdot d\mathbf{r}$$

is an example of a *line integral.* It is called *the tangent line integral of* **F** *along C.*

Since $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, therefore, $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$.

$$\therefore\quad \mathbf{F}\cdot d\mathbf{r}=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$
$$=F_1\,dx+F_2\,dy+F_3\,dz.$$

Therefore in components form the above line integral is written as

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (F_1\,dx+F_2\,dy+F_3\,dz).$$

The parametric equations of the curve C are $x=x(t)$, $y=y(t)$ and $z=z(t)$.

Therefore we may write

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{t_1}^{t_2}\left[F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right] dt.$$

Circulation. *If C is a simple closed curve (i.e. a curve which does not intersect itself anywhere), then the tangent line integral of* **F** *around C is called the circulation of* **F** *about C.* It is often denoted by

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C (F_1\,dx+F_2\,dy+F_3\,dz).$$

Work done by a Force. Suppose a force **F** acts upon a particle. Let the particle be displaced along a given path C in space. If $\mathbf{r}$ denotes the position vector of a point on C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to C at the point $\mathbf{r}$ in the direction of s increasing. The component of force **F** along tangent to C is $\mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}$. Therefore the work done by **F** during a small displacement ds of the particle along C is $\left[\mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}\right] ds$ *i.e.*, $\mathbf{F}\cdot d\mathbf{r}$. The total work

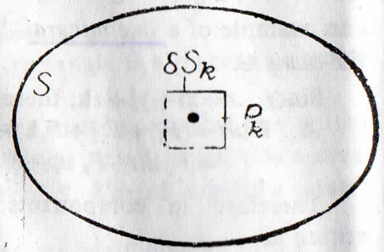
W done by **F** in this displacement along C is given by the line integral

$$W=\int_C \mathbf{F}\cdot d\mathbf{r},$$

the integration being taken in the sense of the displacement.

§3. Surface Integrals.

Any integral which is to be evaluated over a surface is called a surface integral. Suppose S is a surface of finite area. Suppose $f(x, y, z)$ is a single valued function of position defined over S. Sub divide the area S into n elements of areas $\delta S_1, \delta S_2, .., \delta S_n$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k). We define $f(P_k)=f(x_k, y_k, z_k)$. Form the sum



$$\sum_{k=1}^{n} f(P_k)\, \delta S_k.$$

Now take the limit of this sum as $n\to\infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the *surface integral of* $f(x, y, z)$ over S and is dedoted by

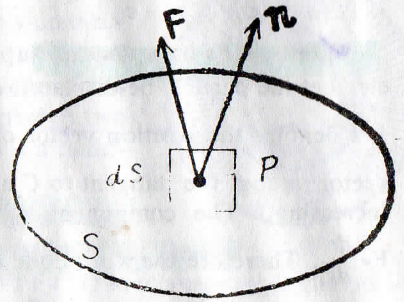
$$\iint_S f(x, y, z)\; dS.$$

It can be shown that if the surface S is piecewise smooth and the function $f(x, y, z)$ is continuous over S, then the above limit exists *i.e.,* is independent of the choice of sub-division and points P_k.

Flux. Suppose S is a piecewise smooth surface and

$$\mathbf{F}(x, y, z)$$

is a vector function of position defined and continuous over S. Let P be any point on the surface S and let **n** be the unit vector at P in the direction of **outward drawn normal** to the surface S at P. Then $\mathbf{F}\cdot\mathbf{n}$ is the normal component of **F** at P. The integral of $\mathbf{F}\cdot\mathbf{n}$ over S is



$$\iint_S \mathbf{F}\cdot\mathbf{n}\; dS.$$

It is called *the flux* of **F** over S.

Let us associate with the differential of surface area dS a vector $d\mathbf{S}$ (called *vector area*) whose magnitude is dS and whose direction is that of **n**. Then $d\mathbf{S}=\mathbf{n}dS$. Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\; dS=\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Suppose the outward drawn normal to the surface S at P makes angles α, β, γ with the positive directions of x, y and z axes respectively. If l, m, n are the direction cosines of this outward drawn normal, then

$$l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma.$$

Also $\mathbf{n}=\cos\alpha\mathbf{i}+\cos\beta\mathbf{j}+\cos\gamma\mathbf{k}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}.$

Let $\mathbf{F}(x, y, z)=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$. Then

$$\mathbf{F}\cdot\mathbf{n}=F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma=F_1 l+F_2 m+F_3 n.$$

Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\; dS=\iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)\; dS$$

$$=\iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy),$$ if we define

$$\iint_S F_1\cos\alpha\; dS=\iint_S F_1\, dy\, dz,\quad \iint_S F_2\cos\beta\; dS=\iint_S F_2\, dz\, dx,$$

$$\iint_S F_3\cos\gamma\; dS=\iint_S F_3\, dx\, dy.$$

Note 1. Other examples of surface integrals are

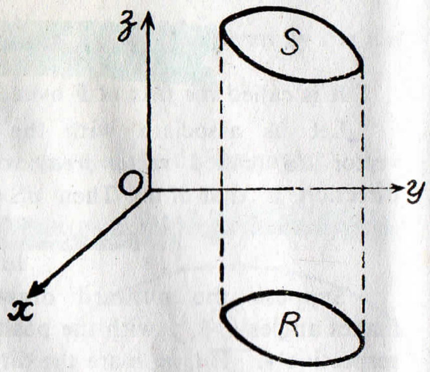
$$\iint_S f\,\mathbf{n}\; dS,\quad \iint_S \mathbf{F}\times d\mathbf{S}$$

where $f(x, y, z)$ is a scalar function of position.

Note 2. Important. In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the co-ordinate plane chosen meets the surface S in no more than one point. If the surface S does not satisfy this condition, then it can be sub-divided into surfaces which do satisfy this condition.

Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form

$$z=h\,(x, y).$$



Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle which the undirected normal $\mathbf{n}$ at $P\,(x, y, z)$ to the surface S makes with z-axis, then it can be shown that

$$\cos\gamma\ dS=dx\ dy,$$

where dS is the small element of area of surface S at the point P.

Therefore $dS=\dfrac{dx\ dy}{\cos\gamma}=\dfrac{dx\ dy}{|\,\mathbf{n}\cdot\mathbf{k}\,|}$, where $\mathbf{k}$ is the unit vector along z-axis.

Hence $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\ dS=\iint_R \mathbf{F}\cdot\mathbf{n}\ \frac{dx\ dy}{|\,\mathbf{n}\cdot\mathbf{k}\,|}.$

Thus the surface integral on S can be evaluated with the help of a double integral integrated over R.

§ 4 Volume Integrals.

Suppose V is a volume bounded by a surface S. Suppose $f\,(x, y, z)$ is a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define $f(P_k)=f(x_k, y_k, z_k)$.

Form the sum

$$\sum_{k=1}^{n} f\,(P_k)\ \delta V_k.$$

Now take the limit of this sum as $n\to\infty$ in such a way that the largest of the volumes δV_k approaches zero. This limit, if it exists, is called the volume integral of $f\,(x, y, z)$ over V and is denoted by

$$\iiint_V f\,(x, y, z)\ dV.$$

It can be shown that if the surface is piecewise smooth and the function $f\,(x, y, z)$ is continuous over V, then the above limit exists *i.e.* is independent of the choice of sub-divisions and points P_k.

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinate axes, then $dV=dx\,dy\,dz$ and the above volume integral becomes

$$\iiint_V f\,(x, y, z)\ dx\ dy\ dz.$$

If $\mathbf{F}\,(x, y, z)$ is a vector function, then

$$\iiint_V \mathbf{F}\ dV$$

is also an example of a volume integral.

SOLVED EXAMPLES

Ex. 1. *Evaluate* $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=x^2\mathbf{i}+y^3\mathbf{j}$ *and curve* C *is the arc of the parabola* $y=x^2$ *in the* x-y *plane from* $(0, 0)$ *to* $(1, 1)$.

Solution. We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y=x^2$ from $(0, 0)$ to $(1, 1)$.

Let $x=t$; then $y=t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then $\mathbf{r}\,(t)=x\mathbf{i}+y\mathbf{j}=t\mathbf{i}+t^2\mathbf{j}$.

$$\therefore\quad \frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}.$$

Also in terms of t, $\mathbf{F}=t^2\mathbf{i}+t^6\mathbf{j}$.

At the point $(0, 0)$, $t=x=0$. At the point $(1, 1)$, $t=1$.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_C\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\int_0^1 (t^2\mathbf{i}+t^6\mathbf{j})\cdot(\mathbf{i}+2t\mathbf{j})\ dt$$

$$=\int_0^1 (t^2+2t^7)\ dt=\left[\frac{t^3}{3}+\frac{2t^8}{8}\right]_0^1=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}.$$

Method 2. In the xy-plane we have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$.

$$\therefore\quad d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}.$$

Therefore $\mathbf{F}\cdot d\mathbf{r}=(x^2\mathbf{i}+y^3\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})=x^2\ dx+y^3\ dy.$

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (x^2\ dx+y^3\ dy).$$

Now along the curve C, $y=x^2$. Therefore $dy=2x\ dx$.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{x=0}^{1} [x^2\ dx+x^6\,(2x)\ dx]$$

$$=\int_0^1 (x^2+2x^7)\ dx=\left[\frac{x^3}{3}+\frac{2x^8}{8}\right]_0^1=\frac{7}{12}.$$

Ex. 2. *Evaluate* $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=(x^2-y^2)\,\mathbf{i}+xy\mathbf{j}$ *and curve* C *is the arc of the curve* $y=x^3$ *from* $(0, 0)$ *to* $(2, 8)$.

Solution. The curve C is the curve $y=x^3$ from $(0, 0)$ to $(2, 8)$. Let $x=t$, then $y=t^3$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\mathbf{r}(t)=x\mathbf{i}+y\mathbf{j}=t\mathbf{i}+t^3\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\mathbf{i}+3t^2\mathbf{j}.$$

Also in terms of t, $\mathbf{F}=(t^2-t^6)\,\mathbf{i}+t^4\mathbf{j}$.

At the point $(0, 0)$, $t=x=0$. At the point $(2, 8)$, $t=2$.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt=\int_0^2 [(t^2-t^6)\,\mathbf{i}+t^4\mathbf{j}]\cdot(\mathbf{i}+3t^2\mathbf{j})\, dt$$

$$=\int_0^2 [(t^2-t^6)+3t^6]\, dt=\int_0^2 [t^2+2t^6]\, dt$$

$$=\left[\frac{t^3}{3}+\frac{2t^7}{7}\right]_0^2=\left[\frac{8}{3}+\frac{256}{7}\right]=\frac{824}{21}.$$

Ex. 3. *If* $\mathbf{F}=3xy\,\mathbf{i}-y^2\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where C is the curve in the xy-plane,* $y=2x^2$, *from* $(0, 0)$ *to* $(1, 2)$.

[Kanpur 1978; Agra 76]

Solution. The parametric equations of the parabola $y=2x^2$ can be taken as

$$x=t,\ y=2t^2.$$

At the point $(0, 0)$, $x=0$ and so $t=0$. Again at the point $(1, 2)$, $x=1$ and so $t=1$.

Now $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (3xy\,\mathbf{i}-y^2\mathbf{j})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

[$\because$ $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, so that $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}$]

$$=\int_C (3xy\, dx-y^2\, dy)=\int_{t=0}^1 \left(3xy\frac{dx}{dt}-y^2\frac{dy}{dt}\right) dt$$

$$=\int_0^1 (3.t.2t^2.1-4t^4.4t)\, dt$$

[$\because$ $x=t$, $y=2t^2$ so that $dx/dt=1$ and $dy/dt=4t$]

$$=\int_0^1 (6t^3-16t^5)\, dt=\left[6.\frac{t^4}{4}-16.\frac{t^6}{6}\right]_0^1$$

$$=\frac{6}{4}-\frac{16}{6}=\frac{3}{2}-\frac{8}{3}=-\frac{7}{6}.$$

Ex. 4. *Find the work done when a force*

$$\mathbf{F}=(x^2-y^2+x)\,\mathbf{i}-(2xy+y)\,\mathbf{j}$$

moves a particle in xy-plane from $(0, 0)$ *to* $(1, 1)$ *along the parabola* $y^2=x$. **[Kanpur 1980]**

Solution. Let C denote the arc of the parabola $y^2=x$ from the point $(0, 0)$ to the point $(1, 1)$. The parametric equations of the parabola $y^2=x$ can be taken as $x=t^2$, $y=t$. At the point $(0, 0)$, $t=0$ and at the point $(1, 1)$, $t=1$. The required work done

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \{(x^2-y^2+x)\,\mathbf{i}-(2xy+y)\,\mathbf{j}\}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$=\int_C [(x^2-y^2+x)\, dx-(2xy+y)\, dy]$$

$$=\int_{t=0}^1 \left[(x^2-y^2+x)\frac{dx}{dt}-(2xy+y)\frac{dy}{dt}\right] dt$$

$$=\int_0^1 [(t^4-t^2+t^2).2t-(2t^3+t).1]\, dt$$

$$=\int_0^1 [2t^5-2t^3-t]\, dt=\left[2.\frac{t^6}{6}-2.\frac{t^4}{4}-\frac{t^2}{2}\right]_0^1$$

$$=\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}=-\tfrac{2}{3}.$$

Ex. 5. *Evaluate* $\int (x\, dy-y\, dx)$ *around the circle* $x^2+y^2=1$.

Solution. Let C denote the circle $x^2+y^2=1$. The parametric equations of this circle are $x=\cos t$, $y=\sin t$.

To integrate around the circle C we should vary t from 0 to 2π.

$$\therefore \quad \oint_C (x\, dy-y\, dx)=\int_0^{2\pi} \left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) dt$$

$$=\int_0^{2\pi} (\cos^2 t+\sin^2 t)\, dt=\int_0^{2\pi} dt=2\pi.$$

Ex. 6. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\mathbf{F}=\mathbf{i}\cos y-\mathbf{j}x\sin y$

and C is the curve $y=\sqrt{(1-x^2)}$ *in xy-plane from* $(1, 0)$ *to* $(0, 1)$.

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C (\mathbf{i}\cos y-\mathbf{j}x\sin y)\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy)=\int_C (\cos y\, dx-x\sin y\, dy)$$

$$=\int_C d\,(x\cos y)=\Big[x\cos y\Big]_{(1,0)}^{(0,1)}=0-1=-1.$$

Ex. 7. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=xy\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and curve C is the arc of* $y=x^2-4$ *from* $(2, 0)$ *to* $(4, 12)$.

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C [xy\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\mathbf{i}+dy\mathbf{j})$$

$$=\int_C [xy\,dx+(x^2+y^2)\,dy]=\int_C xy\,dx+\int_C (x^2+y^2)\,dy.$$

Along C, $y=x^2-4$ and $x^2=y+4$.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{x=2}^{4} x\,(x^2-4)\,dx+\int_{y=0}^{12} (y+4+y^2)\,dy$$

$$=\left[\frac{x^4}{4}-2x^2\right]_2^4+\left[\frac{y^2}{2}+4y+\frac{y^3}{3}\right]_0^{12}=732.$$

Ex. 8. *Evaluate* $\int_C xy^3\,ds$, *where C is the segment of the line $y=2x$ in the xy-plane from $(-1, -2)$ to $(1, 2)$.*

Solution. The parametric form of the curve C can be taken as $\mathbf{r}(t)=t\mathbf{i}+2t\mathbf{j}$ $(-1 \leqslant t \leqslant 1)$.

We have $\dfrac{d\mathbf{r}}{dt}=\mathbf{i}+2\mathbf{j}$.

Now $\dfrac{d\mathbf{r}}{dt}=\dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}$.

$\therefore \quad \left|\dfrac{d\mathbf{r}}{dt}\right|=\left|\dfrac{d\mathbf{r}}{ds}\right|\dfrac{ds}{dt}=\dfrac{ds}{dt}$, because $\dfrac{d\mathbf{r}}{ds}$ is unit vector.

$$\therefore \quad \frac{ds}{dt}=|\,\mathbf{i}+2\mathbf{j}\,|=\sqrt{5}.$$

$$\therefore \quad \int_C xy^3\,ds=\int_C \left(xy^3\,\frac{ds}{dt}\right)dt=\int_{-1}^{1} t\,(2t)^3\sqrt{5}\,dt$$

$$=8\sqrt{5}\int_{-1}^{1} t^4\,dt=\frac{16}{\sqrt{5}}.$$

Ex. 9. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\mathbf{F}=xy\mathbf{i}+yz\mathbf{j}+zx\mathbf{k}$ *and curve C is* $\mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$, *t varying from* -1 *to* $+1$.

Solution. Along the curve C,

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}.$$

$\therefore \quad x=t,\ y=t^2,\ z=t^3$ and $\dfrac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}$.

$\therefore$ Along the curve C, we have

$$\mathbf{F}=(t\times t^2)\,\mathbf{i}+(t^2\times t^3)\,\mathbf{j}+(t^3\times t)\,\mathbf{k}=t^3\mathbf{i}+t^5\mathbf{j}+t^4\mathbf{k}.$$

Hence $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}=\int_{-1}^{1}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$

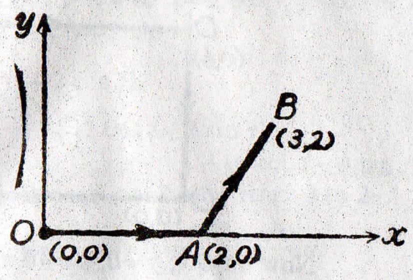
$$=\int_{-1}^{1} (t^3\mathbf{i}+t^5\mathbf{j}+t^4\mathbf{k})\cdot(\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k})\,dt=\int_{-1}^{1} (t^3+2t^6+3t^6)\,dt$$

$$=\int_{-1}^{1} (t^3+5t^6)\,dt=\int_{-1}^{1} t^3\,dt+5\int_{-1}^{1} t^6\,dt$$

$$=0+5\,(2)\int_0^1 t^6\,dt=10\left[\frac{t^7}{7}\right]_0^1=\frac{10}{7}.$$

Ex. 10. *If* $\mathbf{F}=(2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is the curve in the xy-plane consisting of the straight lines from* $(0, 0)$ *to* $(2, 0)$ *and then to* $(3, 2)$.

Solution. The path of integration C has been shown in the figure. It consists of the straight lines OA and AB,



We have $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C [(2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$=\int_C [(2x+y)\,dx+(3y-x)\,dy].$$

Now along the straight line OA, $y=0$, $dy=0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y-0=\frac{2-0}{3-2}\,(x-2)\ \text{i.e.,}\ y=2x-4.$$

$\therefore$ along AB, $y=2x-4$, $dy=2dx$ and x varies from 2 to 3.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^2 [(2x+0)\,dx+0]+\int_2^3 [(2x+2x-4)\,dx+(6x-12-x)\,2dx]$$

$$=\left[x^2\right]_0^2+\int_2^3 (14x-28)\,dx=4+14\int_2^3 (x-2)\,dx$$

$$=4+14\left[\frac{(x-2)^2}{2}\right]_2^3=4+7=11.$$

Ex. 11. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F}=(x^2+y^2)\,\mathbf{i}-2xy\mathbf{j}$, *curve C is the rectangle in the xy-plane bounded by* $y=0,\ x=a,\ y=b,\ x=0$.

[Meerut 1981; Kanpur 79]

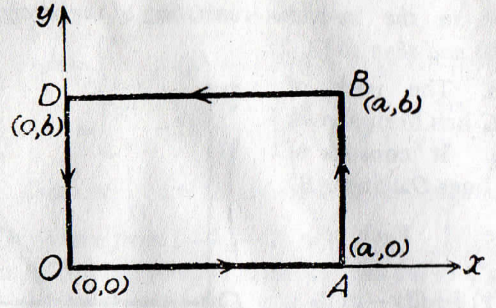
Solution. In the x-y plane $z=0$. Therefore

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j} \text{ and } d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}.$$

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [(x^2+y^2)\,\mathbf{i}-2xy\mathbf{j}]\cdot(dx\mathbf{i}+dy\mathbf{j})$

$$=\int_C [(x^2+y^2)\,dx-2xy\,dy]$$



Now on OA, $y=0$, $dy=0$ and x varies from 0 to a,
on AB, $x=a$, $dx=0$ and y varies from 0 to b,
on BD, $y=b$, $dy=0$ and x varies from a to 0,
on DO, $x=0$, $dx=0$ and y varies from b to 0.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^a x^2\,dx-\int_0^b 2ay\,dy+\int_a^0 (x^2+b^2)\,dx+\int_b^0 0\,dy$$

$$=\left[\frac{x^3}{3}\right]_0^a-2a\left[\frac{y^2}{2}\right]_0^b+\left[\frac{x^3}{3}+b^2x\right]_a^0+0=-2ab^2.$$

Ex. 12. *Find the total work done in moving a particle in a force field given by* $\mathbf{F}=3xy\mathbf{i}-5z\mathbf{j}+10x\mathbf{k}$ *along the curve* $x=t^2+1$, $y=2t^2$, $z=t^3$ *from* $t=1$ *to* $t=2$. **[Kanpur 1978]**

Solution. Let C denote the arc of the given curve from $t=1$ to $t=2$. Then the total work done

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (3xy\mathbf{i}-5z\mathbf{j}+10x\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\int_C (3xy\,dx-5z\,dy+10x\,dz)$$

$$=\int_1^2 \left(3xy\frac{dx}{dt}-5z\frac{dy}{dt}+10x\frac{dz}{dt}\right)dt$$

$$=\int_1^2 [3(t^2+1)(2t)^2(2t)-(5t^3)(4t)+10(t^2+1)(3t^2)]\,dt$$

$$=\int_1^2 (12t^5+12t^3-20t^4+30t^4+30t^2)\,dt$$

$$=\int_1^2 (12t^5+10t^4+12t^3+30t^2)\,dt=303.$$

Ex. 13. *Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field* $\mathbf{F}$ *is given by*

$$\mathbf{F}=(2x-y+2z)\,\mathbf{i}+(x+y-z)\,\mathbf{j}+(3x-2y-5z)\,\mathbf{k}.$$

[Kanpur 1979]

Solution. In the xy-plane, we have $z=0$. Therefore

$$\mathbf{F}=(2x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}+(3x-2y)\,\mathbf{k}.$$

The circle C is given by $x^2+y^2=4$ or $x=2\cos t$, $y=2\sin t$.

$\therefore\quad \mathbf{r}=x\mathbf{i}+y\mathbf{j}=2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}.$

$\therefore\quad \dfrac{d\mathbf{r}}{dt}=-2\sin t\,\mathbf{i}+2\cos t\,\mathbf{j}.$

Also $\mathbf{F}=(4\cos t-2\sin t)\,\mathbf{i}+(2\cos t+2\sin t)\,\mathbf{j}+(6\cos t-4\sin t)\,\mathbf{k}.$

In moving round the circle once t will vary from 0 to 2π.

The required work done is $=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}\mathbf{F}\cdot\dfrac{d\mathbf{r}}{dt}\,dt$

$$=\int_0^{2\pi}[-2\sin t\,(4\cos t-2\sin t)+2\cos t\,(2\cos t+2\sin t)]\,dt$$

$$=\int_0^{2\pi}[4(\sin^2 t+\cos^2 t)-4\sin t\cos t]\,dt$$

$$=\int_0^{2\pi}(4-4\sin t\cos t)\,dt=\Big[4t-2\sin 2t\Big]_0^{2\pi}=8\pi.$$

Ex. 14. *If* $\mathbf{F}=(3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is a straight line joining (0, 0, 0) to (1, 1, 1).*

[Meerut B. Sc. Physics 1983]

Solution. The equations of the straight line joining (0, 0, 0) and (1, 1, 1) are

$$\frac{x-0}{1-0}=\frac{y-0}{1-0}=\frac{z-0}{1-0}=t\ \text{(say)}.$$

Then along C, $x=t$, $y=t$, $z=t$.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=t\mathbf{i}+t\mathbf{j}+t\mathbf{k}$. $\therefore\ d\mathbf{r}=(\mathbf{i}+\mathbf{j}+\mathbf{k})\,dt.$

Also along C, $\mathbf{F}=(3t^2+6t)\,\mathbf{i}-14t^2\,\mathbf{j}+20t^3\,\mathbf{k}.$

At (0, 0, 0), $t=0$ and at (1, 1, 1), $t=1$.

$$\therefore\quad \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{t=0}^1 [(3t^2+6t)-14t^2+20t^3]\,dt=\frac{13}{3}.$$

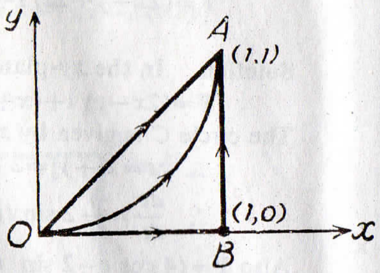
Ex. 15. *If* $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *from (0, 0) to (1, 1) along the following paths C :*

(a) *the parabola* $y=x^2$, [Agra 1973]

(b) *the straight lines from* $(0, 0)$ *to* $(1, 0)$ *and then to* $(1, 1)$.

(c) *the straight line joining* $(0, 0)$ *and* $(1, 1)$.

Solution. The three paths of integration have been shown in the figure. We have



$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_C (y\mathbf{i}-x\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})$$

$$=\int_C (y\,dx-x\,dy).$$

(a) C is the arc of parabola $y=x^2$ from $(0, 0)$ to $(1, 1)$. Here $dy=2x dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 [x^2\,dx-x\,(2x)\,dx]=\int_0^1 -x^2dx=-\tfrac{1}{3}.$$

(b) C is the curve consisting of straight lines OB and BA. Along OB, $y=0$, $dy=0$ and x varies from 0 to 1. Along BA, $x=1$, $dx=0$ and y varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 0\,dx+\int_0^1 -1\,dy=-1.$$

(c) C is the straight line OA. The equation of OA is

$$y-0=\frac{1-0}{1-0}(x-0) \text{ i.e. } y=x.$$

$\therefore$ $dy=dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^1 (x dx-x dx)=0.$$

Note. We observe here that $\mathbf{F}$ is a vector field such that its line integral depends not only on the end points but also on the geometric shape of the path of integration. We shall discuss this topic in depth in the latter portion of this chapter.

Ex. 16. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and* C *is the portion of the curve* $\mathbf{r}=a\cos t\,\mathbf{i}+b\sin t\,\mathbf{j}+ct\,\mathbf{k}$, *from* $t=0$ *to* $t=\pi/2$. [Agra 1975]

Solution. Along the curve C,

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=a\cos t\,\mathbf{i}+b\sin t\,\mathbf{j}+ct\,\mathbf{k}.$$

$$\therefore \quad x=a\cos t,\ y=b\sin t,\ z=ct.$$

Now $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$

$$=\int_C (yz\,dx+zx\,dy+xy\,dz)=\int_C d\,(xyz)$$

$$=\Big[xyz\Big]_{t=0}^{t=\pi/2}=\Big[(a\cos t).(b\sin t).(ct)\Big]_0^{\pi/2}$$

$$=abc\Big[t\cos t\sin t\Big]_0^{\pi/2}=abc\,(0-0)=0.$$

Ex 17. *Evaluate*

$$\int_C \{(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy\}$$

where C *is the arc of the parabola* $2x=\pi y^2$ *from* $(0, 0)$ *to* $(\frac{1}{2}\pi, 1)$. [Meerut 1977]

Solution. We know that $Mdx+Ndy$ is an exact differential if $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

Here $M=2xy^3-y^2\cos x$; $\therefore$ $\dfrac{\partial M}{\partial y}=6xy^2-2y\cos x$.

Also $N=1-2y\sin x+3x^2y^2$; $\therefore$ $\dfrac{\partial N}{\partial x}=-2y\cos x+6xy^2$.

Thus $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$. Therefore $Mdx+Ndy$ is an exact differential.

Let $\phi\,(x, y)$ be such that

$$d\phi=(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy.$$

Then $\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy=(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy.$

$\therefore$ $\dfrac{\partial\phi}{\partial x}=2xy^3-y^2\cos x$ which gives $\phi=x^2y^3-y^2\sin x+f_1(y)$ ...(1)

Also $\dfrac{\partial\phi}{\partial y}=(1-2y\sin x+3x^2y^2)$ which gives $\phi=y-y^2\sin x+x^2y^3+f_2(x)$. ...(2)

The values of ϕ given by (1) and (2) agree if we take $f_1(y)=y$ and $f_2(x)=0$. Then $\phi=y-y^2\sin x-x^2y^3$.

$\therefore$ The given integral

$$=\int_C d\phi=\int_C d\,(y-y^2\sin x+x^2y^3)$$

$$=\Big[y-y^2\sin x+x^2y^3\Big]_{(0,\,0)}^{(\pi/2,\,1)}$$

$$=\Big[\Big\{1-1\times\sin\frac{\pi}{2}+\frac{\pi^2}{4}\times 1\Big\}-0\Big]=\frac{\pi^2}{4}.$$

Ex. 18. *Find the circulation of* **F** *round the curve C where*

$$\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$$

and C is the circle $x^2+y^2=1$, $z=0$.

Solution. By definition, the circulation of **F** along the curve C is

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

$$=\oint_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\oint_C (y\,dx+z\,dy+x\,dz)$$

$$=\oint_C y\,dx \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi} \sin\theta\,(-\sin\theta)\,d\theta \quad [\because \text{ on } C,\ x=\cos\theta,\ y=\sin\theta]$$

$$=-\int_0^{2\pi} \sin^2\theta\,d\theta=-\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$=-\tfrac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi}=-\pi.$$

Ex. 19. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and S is that part of the surface of the sphere* $x^2+y^2+z^2=1$ *which lies in the first octant.* **[Agra 1974; Kanpur 79; Meerut 84 (P)]**

Solution. A vector normal to the surface S is given by

$$\nabla\,(x^2+y^2+z^2)=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

Therefore $\mathbf{n}$=a unit normal to any point (x, y, z) of S

$$=\frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

since $x^2+y^2+z^2=1$ on the surface S.

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1$, $z=0$.

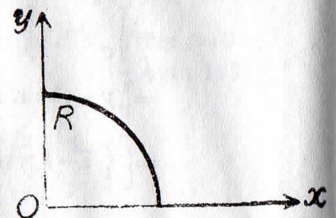
We have

$$\mathbf{F}\cdot\mathbf{n}=(yz\mathbf{i}+zx\mathbf{j}+xy\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$
$$=3xyz.$$

Also $\mathbf{n}\cdot\mathbf{k}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{k}=z.$

$\therefore\quad |\mathbf{n}\cdot\mathbf{k}|=z.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$



$$=\iint_R \frac{3xyz}{z}\,dx\,dy=3\iint_R xy\,dx\,dy$$

$$=3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\,r\,d\theta\,dr, \text{ on changing to polars}$$

$$=3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\,d\theta=\tfrac{3}{4}\,(\tfrac{1}{2})=\tfrac{3}{8}.$$

Ex. 20. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=z\mathbf{i}+x\mathbf{j}-3y^2z\mathbf{k}$ *and S is the surface of the cylinder* $x^2+y^2=16$ *included in the first octant between* $z=0$ *and* $z=5$.

Solution. A vector normal to the surface S is given by

$$\nabla\,(x^2+y^2)=2x\mathbf{i}+2y\mathbf{j}.$$

Therefore $\mathbf{n}$=a unit normal to any point of S

$$=\frac{2x\mathbf{i}+2y\mathbf{j}}{\sqrt{(4x^2+4y^2)}}=\frac{x\mathbf{i}+y\mathbf{j}}{4}, \text{ since } x^2+y^2=16, \text{ on the surface } S.$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now $\mathbf{F}\cdot\mathbf{n}=(z\mathbf{i}+x\mathbf{j}-3y^2z\mathbf{k})\cdot\left(\dfrac{x\mathbf{i}+y\mathbf{j}}{4}\right)=\tfrac{1}{4}\,(xz+xy),$

$$\mathbf{n}\cdot\mathbf{j}=\left(\frac{x\mathbf{i}+y\mathbf{j}}{4}\right)\cdot\mathbf{j}=\frac{y}{4}.$$

Therefore the required surface integral is

$$=\iint_R \frac{xz+xy}{4}\,\frac{dx\,dz}{y/4}$$

$$=\int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{(16-x^2)}}+x\right)dx\,dz, \text{ since } y=\sqrt{(16-x^2)} \text{ on } S$$

$$=\int_0^5 (4z+8)\,dz=90.$$

Ex. 21. *Evaluate* $\iiint_V \phi\,dV$, *where* $\phi=45x^2y$ *and V is the closed region bounded by the planes* $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Solution. We have

$$\iiint_V \phi\,dV=\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\,dx\,dy\,dz$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\left[\,z\,\right]_0^{8-4x-2y} dx\,dy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)\,dx\,dy$$

$$=45\int_{x=0}^{2}\left[x^2(8-4x)\frac{y^2}{2}-2x^2\frac{y^3}{3}\right]_0^{4-2x} dx$$

$$=45\int_0^2 \frac{x^2}{3}(4-2x)^3\,dx=128.$$

Ex. 22. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=(x+y^2)\,\mathbf{i}-2x\mathbf{j}+2yz\mathbf{k}$ *and* S *is the surface of the plane* $2x+y+2z=6$ *in the first octant.* **[Kanpur 1970]**

Solution. A vector normal to the surface S is given by
$$\nabla\,(2x+y+2z)=2\mathbf{i}+\mathbf{j}+2\mathbf{k}.$$

∴ $\mathbf{n}=$ a unit normal vector at any point (x, y, z) of S
$$=\frac{2\mathbf{i}+\mathbf{j}+2\mathbf{k}}{\sqrt{(4+1+4)}}=(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$.

We have $\mathbf{F}\cdot\mathbf{n}=[(x+y^2)\,\mathbf{i}-2x\,\mathbf{j}+2yz\,\mathbf{k}]\cdot(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})$
$$=\tfrac{2}{3}(x+y^2)-\tfrac{2}{3}x+\tfrac{4}{3}yz=\tfrac{2}{3}y^2+\tfrac{4}{3}yz.$$

Also $\quad \mathbf{n}\cdot\mathbf{k}=(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})\cdot\mathbf{k}=\tfrac{2}{3}.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R [\tfrac{2}{3}y^2+\tfrac{4}{3}yz]\cdot\tfrac{3}{2}dx\,dy$

$$=\iint_R (y^2+2yz)\,dx\,dy$$

$$=\iint_R \left[y^2+2y\left(\frac{6-2x-y}{2}\right)\right]dx\,dy,\text{ using the fact that}$$

$$z=\frac{6-2x-y}{2}\text{ from the equation of } S$$

$$=\iint_R (y^2+6y-2xy-y^2)\,dx\,dy=2\iint_R y\,(3-x)\,dx\,dy$$

$$=2\int_{y=0}^{6}\int_{x=0}^{(6-y)/2} y\,(3-x)\,dx\,dy.$$

[Note that R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$. To evaluate the double integral over R, keep y fixed and integrate with respect to x from $x=0$ to $x=\dfrac{6-y}{2}$; then

integrate with respect to y from $y=0$ to $y=6$. In this way R is completely covered].

$$=2\int_{y=0}^{6} y\left[3x-\frac{x^2}{2}\right]_{x=0}^{(6-y)/2} dy$$

$$=2\int_0^6 y\left[\frac{3\,(6-y)}{2}-\frac{(6-y)^2}{8}\right]dy$$

$$=2\int_0^6 y\left[9-\frac{3y}{2}-\frac{36}{8}+\frac{12y}{8}-\frac{y^2}{8}\right]dy$$

$$=2\int_0^6 y\left[\frac{36}{8}-\frac{y^2}{8}\right]dy=\int_0^6\left[9y-\frac{y^3}{4}\right]dy$$

$$=\left[9\frac{y^2}{2}-\frac{y^4}{16}\right]_0^6=\left[9.\frac{36}{2}-\frac{36\times36}{16}\right]=[162-81]=81.$$

Ex. 23. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F}=y\,\mathbf{i}+2x\,\mathbf{j}-z\mathbf{k}$ *and* S *is the surface of the plane* $2x+y=6$ *in the first octant cut off by the plane* $z=4$.

Solution. A vector normal to the surface S is given by
$$\nabla\,(2x+y)=2\mathbf{i}+\mathbf{j}.$$

Therefore $\mathbf{n}=$ a unit normal vector at any point (x, y, z) of S
$$=\frac{2\mathbf{i}+\mathbf{j}}{\sqrt{(4+1)}}=\frac{1}{\sqrt5}(2\mathbf{i}+\mathbf{j}).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\dfrac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to xy-plane.

Now $\mathbf{F}\cdot\mathbf{n}=(y\mathbf{i}+2x\mathbf{j}-z\mathbf{k})\cdot\left(\dfrac{2}{\sqrt5}\mathbf{i}+\dfrac{1}{\sqrt5}\mathbf{j}\right)=\dfrac{2}{\sqrt5}y+\dfrac{2}{\sqrt5}x.$

Also $\quad \mathbf{n}\cdot\mathbf{j}=\dfrac{1}{\sqrt5}(2\mathbf{i}+\mathbf{j})\cdot\mathbf{j}=\dfrac{1}{\sqrt5}.$

∴ the required surface integral is

$$=\iint_R\left(\frac{2}{\sqrt5}y+\frac{2}{\sqrt5}x\right)\cdot\sqrt5\,dx\,dz=\iint_R 2\,(y+x)\,dx\,dz$$

$$=2\iint_R [6-2x+x]\,dx\,dz,\text{ since } y=6-2x\text{ on } S$$

$$=2\iint_R (6-x)\,dx\,dz=2\int_{z=0}^{4}\int_{x=0}^{3}(6-x)\,dx\,dz$$

$$=2\int_{x=0}^{3}(6-x)\Big[z\Big]_0^4 dx=8\left[6x-\frac{x^2}{2}\right]_0^3=8\left[18-\frac{9}{2}\right]=108.$$

Exercises

1. Find $\int_C \mathbf{t} \cdot d\mathbf{r}$

where $\mathbf{t}$ is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin.

Hint. For any curve, $\dfrac{d\mathbf{r}}{ds}=$unit tangent vector$=\mathbf{t}$.

$$\therefore \int_C \mathbf{t} \cdot d\mathbf{r}=\int_C \mathbf{t} \cdot \frac{d\mathbf{r}}{ds}\, ds=\int_C \mathbf{t} \cdot \mathbf{t}\, ds=\int_C ds$$

$=\int_0^{2\pi} ds$, since along the unit circle C, s goes from 0 to 2π

$=2\pi$.

2. If $\mathbf{F}=(3x^2+6y)\,\mathbf{i}-14\,yz\mathbf{j}+20xz^2\,\mathbf{k}$, then evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve
$$x=t,\ y=t^2,\ z=t^3.$$
Ans. 5.

3. Integrate the function $\mathbf{F}=x^2\,\mathbf{i}-xy\,\mathbf{j}$ from the point $(0, 0)$ to $(1, 1)$ along the parabola $y^2=x$. **[Rohilkhand 1978]**
Ans. $-\frac{1}{12}$.

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is $x^2y^2\,\mathbf{i}+y\mathbf{j}$ and C is $y^2=4x$ in the xy-plane from $(0, 0)$ to $(4, 4)$. **[Agra 1978; Kanpur 77]**
Ans. 264.

5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where,
$\mathbf{F}=c\,[-3a \sin^2 t \cos t\mathbf{i}+a\,(2 \sin t-3 \sin^3 t)\,\mathbf{j}+b \sin 2t\mathbf{k}]$
and C is given by $\mathbf{r}=a \cos t\mathbf{i}+a \sin t\mathbf{j}+bt\mathbf{k}$
from $t=\pi/4$ to $\pi/2$. **[Delhi 1970]**
Ans. $\frac{1}{2}c\,(a^2+b^2)$.
$$\left[\textbf{Hint.} \int_C \mathbf{F} \cdot d\mathbf{r}=\int_{\pi/4}^{\pi/2} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt\right].$$

6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}=z\,\mathbf{i}+x\,\mathbf{j}+y\mathbf{k}$ and C is the arc of the curve $\mathbf{r}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\mathbf{k}$ from $t=0$ to $t=2\pi$.
Ans. 3π. **[Agra 1974, 77]**

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where
$$\mathbf{F}=xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$$
and C is the x-axis from $x=2$ to $x=4$ and the line $x=4$ from $y=0$ to $y=12$. **Ans.** 768.

8. Find the work done in moving a particle in a force field
$$\mathbf{F}=3x^2\,\mathbf{i}+(2xz-y)\,\mathbf{j}+z\mathbf{k}$$
along the line joining $(0, 0, 0)$ to $(2, 1, 3)$. **[Delhi 1969]**
Ans. 16.

9. Calculate $\int_C [(x$ [illegible] $)\,\mathbf{i}+(x^2-y^2)\,\mathbf{j}] \cdot d\mathbf{r}$

where C is the curve :

(i) $y^2=x$ joining $(0, 0)$ to $(1, 1)$.
(ii) $x^7=y$ joining $(0, 0)$ to $(1, 1)$.
(iii) consisting of two straight lines joining $(0, 0)$ to $(1, 0)$ and $(1, 0)$ to $(1, 1)$.
(iv) consisting of three straight lines joining $(0, 0)$ to $(2, -2)$, $(2, -2)$ to $(0, -1)$ and $(0, -1)$ to $(1, 1)$.

Ans. (i) $\frac{7}{16}$, (ii) $\frac{38}{45}$, (iii) 1, (iv) $-\frac{7}{3}$.

10. Find the circulation of $\mathbf{F}$ round the curve C, where
$$\mathbf{F}=\mathbf{i}e^x \sin y+\mathbf{j}\, e^x \cos y$$
and C is the rectangle whose vertices are
$$(0, 0), (1, 0), (1, \tfrac{1}{2}\pi), (0, \tfrac{1}{2}\pi).$$
Ans. 0.

11. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}=18z\,\mathbf{i}-12\,\mathbf{j}+3y\,\mathbf{k}$ and S is the surface of the plane
$$2x+3y+6z=12 \text{ in the first octant.}$$
Ans. 24.

12. Evaluate $\iint_S \mathbf{A} \cdot \mathbf{n}\, dS$, where $\mathbf{A}=xy\,\mathbf{i}-x^2\,\mathbf{j}+(x+z)\,\mathbf{k}$, S is the portion of the plane $2x+2y+z=6$ included in the first octant and $\mathbf{n}$ is a unit normal to S. **[Meerut 1974]**

13. If $\mathbf{F}=2y\,\mathbf{i}-z\,\mathbf{j}+x^2\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4$ and $z=6$, then evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.
Ans. 132.
$$\left[\textbf{Hint.} \iint_S \mathbf{F} \cdot \mathbf{n}\, dS=\iint_R \mathbf{F} \cdot \mathbf{n} \frac{dy\, dz}{|\mathbf{n} \cdot \mathbf{i}|}, \text{ where } R \text{ is the projection of } S \text{ on the } yz\text{-plane}\right].$$

14. If $\mathbf{F}=(2x^2-3z)\,\mathbf{i}-2xy\,\mathbf{j}-4x\,\mathbf{k}$, then evaluate $\iiint_V \nabla \cdot \mathbf{F}\, dV$ where V is the closed region bounded by the planes
$$x=0,\ y=0,\ z=0 \text{ and } 2x+2y+z=4.$$
[Kanpur 1976]
Ans. $\frac{8}{3}$.

§ 5. Green's theorem in the plane. *Let R be a closed bounded region in the x-y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives* $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ *in R. Then*

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C (M dx+N dy),$$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

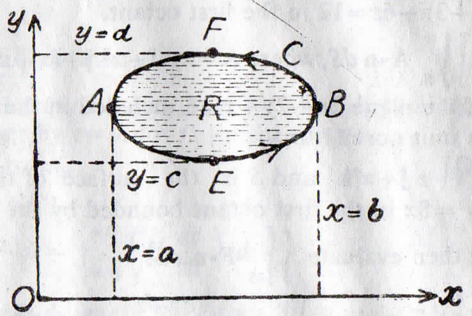
[Meerut 1978, 79, 81, 82, 84]

Proof. We shall first prove the theorem for a special region R bounded by a closed curve C and having the property that any straight line parallel to any one of the coordinate axes and intersecting R has only one segment (or a single point) in common with R. This means that R can be represented in both of the forms

$$a \leqslant x \leqslant b,\ f(x) \leqslant y \leqslant g(x)$$

and $$c \leqslant y \leqslant d,\ p(y) \leqslant x \leqslant q(y).$$

In the adjoining figure, the equations of the curves AEB and BFA are $y=f(x)$ and $y=g(x)$ respectively. Similarly the equations of the curves FAE and EBF are $x=p(y)$ and $x=q(y)$ respectively.



We have

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy=\int_{x=a}^{b}\left[\int_{y=f(x)}^{g(x)} \frac{\partial M}{\partial y}\, dy\right] dx$$

$$=\int_{x=a}^{b}\Big[M(x, y)\Big]_{y=f(x)}^{y=g(x)} dx$$

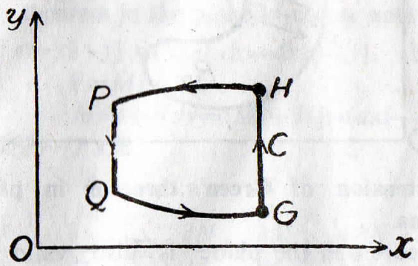
$$=\int_{x=a}^{b}\Big[M[x, g(x)]-M[x, f(x)]\Big]\, dx$$

$$=-\int_a^b M[x, f(x)]\, dx-\int_b^a M[x, g(x)]\, dx$$

$=-\oint_C M(x, y)\, dx$, since $y=f(x)$ represents the curve AEB and $y=g(x)$ represents the curve BFA.

If portions of C are segments parallel to y-axis such as GH and PQ in the figure on this page, then above result is not affected. The line integral $\int M\, dx$ over GH is zero because on GH, we have $x=$constant implies $dx=0$. Similarly the line integral over PQ is zero. The equations of QG and HP are $y=f(x)$ and $y=g(x)$ respectively. Hence we have

$$-\iint_R \frac{\partial M}{\partial y}\, dx\, dy=\oint_C M(x, y)\, dx. \qquad \text{...(1)}$$



Similarly,

$$\iint_R \frac{\partial N}{\partial x}\, dx\, dy=\int_{y=c}^{d}\left[\int_{x=p(y)}^{q(y)} \frac{\partial N}{\partial x}\, dx\right] dy$$

$$=\int_{y=c}^{d}\Big[N(x, y)\Big]_{x=p(y)}^{x=q(y)} dy$$

$$=\int_{y=c}^{d}\Big[N[q(y), y]-N[p(y), y]\Big]\, dy$$

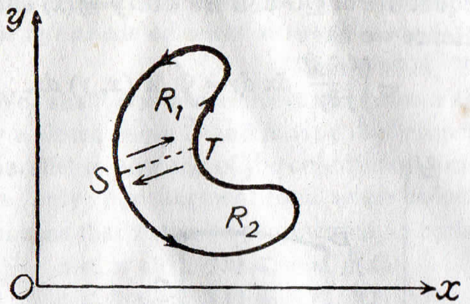
$$=\int_c^d N[q(y), y]\, dy+\int_d^c N[p(y), y]\, dy$$

$$=\oint_C N(x, y)\, dy. \qquad \text{...(2)}$$

From (1) and (2), we get on adding

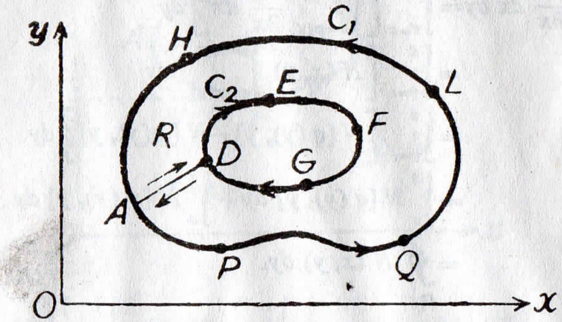
$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C (M dx+N dy). = \oint_C \vec{F}\cdot d\vec{r}$$

The proof of the theorem can now be extended to a region R which can be subdivided into finitely many special regions of the above type by drawing lines (TS in the figure on this page). In this case we apply the theorem to each subregion (R_1 and R_2 in the figure) and then add the results. The sum of the left hand members will be equal to the integral over R. The sum of the right hand members will be equal to the line integral over C plus the line integrals over the curves introduced for subdividing R. Each of the latter integrals comes twice, taken once in each direction (as ST and TS in the figure). Therefore these two integrals cancel each other and thus the sum of the right hand members will be equal to the line integral over C.



Note. Extension of Green's theorem in plane to multiply-connected regions.

Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the figure below. Here the boundary C of R consists of two parts ; the exterior boundary C_1



is traversed in the anticlockwise sense so that R is on the left, while the interior boundary C_2 is traversed in the clockwise sense so that R is on the left.

In order to establish the theorem, we construct a line such as AD (called a *cross cut*) connecting the exterior and interior bound-

aries. The region bounded by $ADEFGDAPQLHA$ is simply-connected and so Green's theorem is valid for it. Therefore

$$\oint_{ADEFGDAPQLHA} M\,dx+N\,dy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy.$$

The integral on the left hand side leaving out the integrand is equal to

$$\int_{AD}+\int_{C_2}+\int_{DA}+\int_{C_1}$$

$$=\int_{C_2}+\int_{C_1},\ \text{since}\int_{AD}=-\int_{DA}$$

$$=\oint_C (Mdx+Ndy).$$

Hence the theorem.

§ 6. Green's theorem in the plane in vector notation.

We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$ so that $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}$. Let

$$\mathbf{F}=M\mathbf{i}+N\mathbf{j}.$$

Then $\quad Mdx+Ndy=(M\mathbf{i}+N\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})=\mathbf{F}\cdot d\mathbf{r}.$

Also curl $\mathbf{F}=\nabla\times\mathbf{F}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ M & N & 0\end{vmatrix}=-\frac{\partial N}{\partial z}\mathbf{i}+\frac{\partial M}{\partial z}\mathbf{j}+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$$\therefore\quad (\nabla\times\mathbf{F})\cdot\mathbf{k}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}.$$

Hence Green's theorem in plane can be written as

$$\iint_R(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dR=\oint_C\mathbf{F}\cdot d\mathbf{r}$$

where $dR=dx\,dy$ and $\mathbf{k}$ is a unit vector perpendicular to the x-y plane.

If s denotes the arc length of C and $\mathbf{t}$ is the unit tangent vector to C, then

$$d\mathbf{r}=\frac{d\mathbf{r}}{ds}\,ds=\mathbf{t}\,ds.$$

Therefore the above result can also be written as

$$\iint_R(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dR=\oint_C\mathbf{F}\cdot\mathbf{t}\,ds.$$

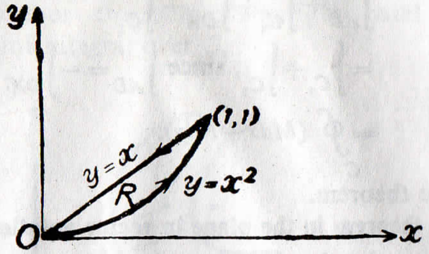
SOLVED EXAMPLES

Ex. 1. *Verify Green's theorem in the plane for*

$$\oint_C (xy+y^2)\,dx+x^2dy \text{ where } C \text{ is the closed curve of the region}$$

bounded by $y=x$ *and* $y=x^2$.

Solution. By Green's theorem in plane, we have



$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\oint_C (Mdx+Ndy).$$

Here $M=xy+y^2$, $N=x^2$.

The curves $y=x$ and $y=x^2$ intersect at (0, 0) and (1, 1). The positive direction in traversing C is as shown in the figure.

We have $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy$

$$=\iint_R \left[\frac{\partial}{\partial x}(x^2)-\frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$=\iint_R (2x-x-2y)\,dx\,dy=\iint_R (x-2y)\,dx\,dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x}(x-2y)\,dy\,dx=\int_{x=0}^{1}\left[xy-y^2\right]_{y=x^2}^{x}dx$$

$$=\int_0^1 [x^2-x^2-x^3+x^4]\,dx=\int_0^1 (x^4-x^3)\,dx$$

$$=\left[\frac{x^5}{5}-\frac{x^4}{4}\right]_0^1=\frac{1}{5}-\frac{1}{4}=-\frac{1}{20}.$$

Now let us evaluate the line integral along C. Along $y=x^2$, $dy=2x\,dx$. Therefore along $y=x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2)+x^4\}\,dx+x^2\,(2x)\,dx]$$

$$=\int_0^1 (3x^3+x^4)\,dx=\frac{19}{20}.$$

Along $y=x$, $dy=dx$. Therefore along $y=x$, the line integral uals

$$\int_1^0 [\{(x)\,(x)+x^2\}\,dx+x^2\,dx]=\int_1^0 3x^4\,dx=-1.$$

Therefore the required line integral$=\frac{19}{20}-1=-\frac{1}{20}$. Hence the theorem is verified.

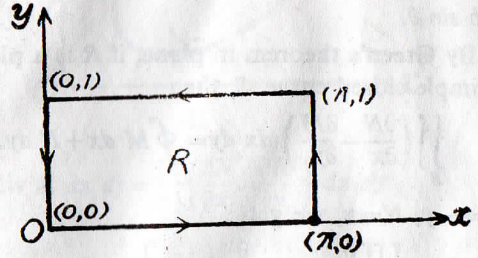
Ex. 2. *Evaluate by Green's theorem*

$$\oint_C (x^2-\cosh y)\,dx+(y+\sin x)\,dy,$$

where C is the rectangle with vertices (0, 0), (π, 0), (π, 1), (0, 1).

Solution. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\oint_C (Mdx+Ndy).$$



Here $M=x^2-\cosh y$, $N=y+\sin x$.

$\therefore$ $\frac{\partial N}{\partial x}=\cos x$, $\frac{\partial M}{\partial y}=-\sinh y$.

Hence the given line integral is equal to

$$\iint_R (\cos x+\sinh y)\,dx\,dy=\int_{x=0}^{\pi}\int_{y=0}^{1}(\cos x+\sinh y)\,dy\,dx$$

$$=\int_{x=0}^{\pi}\left[y\cos x+\cosh y\right]_{y=0}^{1}dx=\int_{x=0}^{\pi}[\cos x+\cosh 1-1]\,dx$$

$$=\left[\sin x+x\cosh 1-x\right]_0^{\pi}=\pi\,(\cosh 1-1).$$

Ex. 3. *Evaluate by Green's theorem*

$$\oint_C (\cos x\sin y-xy)\,dx+\sin x\cos y\,dy,$$

where C is the circle $x^2+y^2=1$.

Solution. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\,dy=\oint_C (M\,dx+N\,dy).$$

Here $M=\cos x \sin y-xy$, $N=\sin x \cos y$.

$$\therefore \quad \frac{\partial M}{\partial y}=\cos x \cos y-x, \quad \frac{\partial N}{\partial x}=\cos x \cos y.$$

Hence the given line integral is equal to

$$\iint_R x\, dx\, dy=\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta\; r\, d\theta\, dr, \text{ changing to polars}$$

$$=\int_{\theta=0}^{2\pi}\left[\frac{r^3}{3}\right]_0^1 \cos\theta\, d\theta=\tfrac{1}{3}\Big[\sin\theta\Big]_0^{2\pi}=\tfrac{1}{3}\,(0)=0.$$

Ex. 4. *Show that the area bounded by a simple closed curve C is given by* $\frac{1}{2}\oint (x\, dy-y\, dx)$. *Hence find the area of the ellipse* $x=a\cos\theta,\ y=b\sin\theta$. [Agra 1974]

Solution. By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\oint_C M\, dx+N\, dy.$$

Putting $M=-y$, $N=x$, we get

$$\oint_C (x\, dy-y\, dx)=\iint_R\left[\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right] dx\, dy$$

$$=2\iint_R dx\, dy=2A, \text{ where } A \text{ is the area bounded by } C.$$

Hence $\qquad A=\frac{1}{2}\oint_C (x\, dy-y\, dx).$

The area of the ellipse $=\frac{1}{2}\oint_C (x\, dy-y\, dx)$

$$=\frac{1}{2}\int_{\theta=0}^{2\pi}\left(a\cos\theta\,\frac{dy}{d\theta}-b\sin\theta\,\frac{dx}{d\theta}\right) d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}(ab\cos^2\theta+ab\sin^2\theta)\, d\theta=\tfrac{1}{2}ab\int_0^{2\pi} d\theta=\pi ab.$$

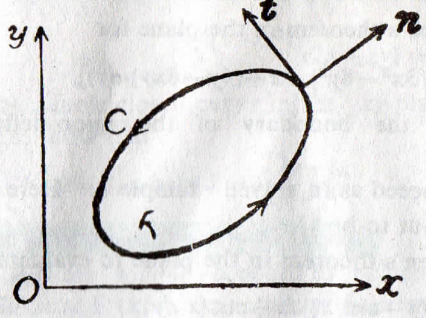
Ex. 5. *Introducing* $\mathbf{A}=N\mathbf{i}-M\mathbf{j}$, *show that the formula in Green's theorem may be written as*

$$\iint_R \operatorname{div}\mathbf{A}\, dx\, dy=\oint_C \mathbf{A}\cdot\mathbf{n}\, ds,$$

where $\mathbf{n}$ *is the outward unit normal vector to C and s is the arc length of C.*

Solution. We have $\mathbf{A}=N\mathbf{i}-M\mathbf{j}$.

$$\therefore \quad \operatorname{div}\mathbf{A}=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}.$$



$$\therefore \quad \iint_R \operatorname{div}\mathbf{A}\, dx\, dy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy$$

$$=\oint_C (M\, dx+N\, dy), \text{ by Green's theorem}$$

Now $\quad M\, dx+N\, dy=(M\mathbf{i}+N\mathbf{j})\cdot(dx\mathbf{i}+dy\mathbf{j})=(M\mathbf{i}+N\mathbf{j})\cdot d\mathbf{r}$

$$=\left\{(M\mathbf{i}+N\mathbf{j})\cdot\frac{d\mathbf{r}}{ds}\right\} ds.$$

Now if $\mathbf{t}$ is a unit tangent vector to C, then $\mathbf{t}=\dfrac{d\mathbf{r}}{ds}$. Also if $\mathbf{k}$ is a unit vector perpendicular to xy-plane, then $\mathbf{t}=\mathbf{k}\times\mathbf{n}$.

$\therefore \quad M\, dx+N\, dy=[(M\mathbf{i}+N\mathbf{j})\cdot\mathbf{t}]\, ds=[(M\mathbf{i}+N\mathbf{j})\cdot(\mathbf{k}\times\mathbf{n})]\, ds$

$$=[(M\mathbf{i}+N\mathbf{j})\times\mathbf{k}]\cdot\mathbf{n}\, ds=(M\mathbf{i}\times\mathbf{k}+N\mathbf{j}\times\mathbf{k})\cdot\mathbf{n}\, ds$$

$$=(N\mathbf{i}-M\mathbf{j})\cdot\mathbf{n}\, ds=\mathbf{A}\cdot\mathbf{n}\, ds.$$

Hence the result.

Note. Putting $\mathbf{A}=\nabla\phi$ in the above result, we get

$$\iint_R \operatorname{div}(\nabla\phi)\, dx\, dy=\oint_C (\nabla\phi)\cdot\mathbf{n}\, ds$$

or $$\iint \nabla^2\phi\, dx\, dy=\oint\frac{\partial\phi}{\partial n}\, ds, \quad \text{since } \nabla\phi=\frac{\partial\phi}{\partial n}\,\mathbf{n}.$$

Exercises

1. Verify Green's theorem in the plane for

$$\int_C (2xy-x^2)\, dx+(x^2+y^2)\, dy,$$

where C is the boundary of the region enclosed by $y=x^2$ and $y^2=x$ described in the positive sense. **[Meerut 1973]**

2. Verify Green's theorem in the plane for

$$\int_C [(3x^2-8y^2)\, dx+(4y-6xy)\, dy)],$$

where C is the boundary of the region defined by $y=\sqrt{x}$, $y=x^2$.

[Hint. Proceed as in solved example 1. Here each integral will come out to be $\frac{3}{2}$**]**.

3. Apply Green's theorem in the plane to evaluate

$$\int_C \{(y-\sin x)\, dx+\cos x\, dy\},$$

where C is the triangle enclosed by the lines

$y=0,\ x=\pi,\ \pi y=2x.$ **[Agra 1973]**

Ans. $-\frac{\pi}{4}-\frac{2}{\pi}$.

[Hint. Here $M=y-\sin x$, $N=\cos x$. Therefore the given

$$\text{integral}=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dx\, dy=\int_{x=0}^{\pi/2}\int_{y=0}^{(2/\pi)x} (-\sin x-1)\, dx\, dy\Big]$$

4. Evaluate by Green's theorem in plane

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$$\int_C (e^{-x}\sin y\, dx+e^{-x}\cos y\, dy),$$

where C is the rectangle with vertices

$(0, 0), (\pi, 0), (\pi, \frac{1}{2}\pi), (0, \frac{1}{2}\pi)$. **Ans.** $2\,(e^{-\pi}-1)$.

5. If $\mathbf{F}=(x^2-y^2)\,\mathbf{i}+2xy\mathbf{j}$ and $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$, find the value of $\int \mathbf{F}\cdot d\mathbf{r}$ around the rectangular boundary $x=0$, $x=a$, $y=0$, $y=b$. **[Gauhati 1973]**

Ans. $2ab^2$.

6. Verify Green's theorem in the plane for

$$\int_C (x^2-xy^3)\, dx+(y^2-2xy)\, dy,$$

where C is the square with vertices $(0, 0), (2, 0), (2, 2), (0, 2)$. **[Meerut 1974]**

7. Apply Green's theorem in the plane to evaluate

$$\int_C [(2x^2-y^2)\, dx+(x^2+y^2)\, dy],$$

where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y=(1-x^2)^{1/2}$. **Ans.** 4/3.

[Hint. By Green's theorem the given integral

$$=\int_{x=-1}^{1}\int_{y=0}^{\sqrt{(1-x^2)}} (2x+2y)\, dx\, dy\ \Big].$$

8. If C is the simple closed curve in the xy-plane not enclosing the origin, show that

$$\int_C \mathbf{F}\cdot d\mathbf{r}=0, \text{ where } \mathbf{F}=\frac{-\mathbf{i}y+\mathbf{j}x}{x^2+y^2}.$$

§ 7. The Divergence theorem of Gauss.

Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\mathbf{F}(x, y, z)$ is a vector function of position which is continuous and has continuous first partial derivatives in V. Then

$$\iiint_V \nabla\cdot\mathbf{F}\, dV=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is the outwards drawn unit normal vector to S.

[Kanpur 1977, 79; Agra 72; Allahabad 80; Rohilkhand 80; Madras 83; Kerala 75; Meerut B. Sc. Physics 83]

Since $\mathbf{F}\cdot\mathbf{n}$ is the normal component of vector $\mathbf{F}$, therefore divergence theorem may also be stated as follows :

The surface integral of the normal component of a vector $\mathbf{F}$ taken over a closed surface is equal to the integral of the divergence of $\mathbf{F}$ taken over the volume enclosed by the surface.

Cartesian equivalent of Divergence Theorem.

Let $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$. Then $\nabla\cdot\mathbf{F}=\text{div}\,\mathbf{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}$.

If α, β, γ are the angles which outward drawn unit normal $\mathbf{n}$ makes with positive directions of x, y, z-axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of $\mathbf{n}$ and we have

$$\mathbf{n}=\cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k}.$$

$$\therefore\ \mathbf{F}\cdot\mathbf{n}=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(\cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k})$$
$$=F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma.$$

Therefore the divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$=\iint_S (F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma)\, dS$$

$$=\iint (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

The significance of divergence theorem lies in the fact that a surface integral may be expressed as a volume integral and vice versa.

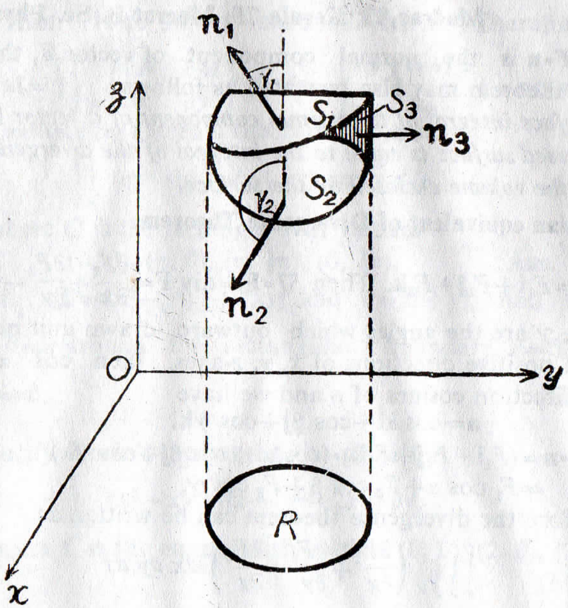
Proof of the divergence theorem.

We shall first prove the theorem for a special region V which is bounded by a piecewise smooth closed surface S and has the property that any straight line parallel to any one of the coordinate axes and intersecting V has only one segment (or a single point) in common with V. If R is the orthogonal projection of S on the xy-plane, then V can be represented in the form

$$f(x, y) \leqslant z \leqslant g(x, y)$$

where (x, y) varies in R.

Obviously $z=g(x, y)$ represents the upper portion S_1 of S, $z=f(x, y)$ represents the lower portion S_2 of S and there may be a remaining vertical portion S_3 of S.



We have

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz = \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz \right] dx\, dy$$

$$=\iint_R \Big[F_3(x, y, z) \Big]_{z=f(x, y)}^{g(x, y)} dx\, dy$$

$$=\iint_R \Big[F_3[x, y, g(x, y)] - F_3[x, y, f(x, y)] \Big] dx\, dy$$

$$=\iint_R F_3[x, y, g(x, y)]\, dx\, dy - \iint_R F_3[x, y, f(x, y)]\, dx\, dy \qquad \ldots(1)$$

Now for the vertical portion S_3 of S, the normal $\mathbf{n}_3$ to S_3 makes a right angle γ with $\mathbf{k}$. Therefore

$$\iint_{S_3} F_3\, \mathbf{k} \cdot \mathbf{n}_3\, dS_3 = 0, \text{ since } \mathbf{k} \cdot \mathbf{n}_3 = 0.$$

For the upper portion S_1 of S, the normal $\mathbf{n}_1$ to S_1 makes an acute angle γ_1 with $\mathbf{k}$. Therefore

$$\mathbf{k} \cdot \mathbf{n}_1\, dS_1 = \cos \gamma_1\, dS_1 = dx\, dy. \quad \text{Hence}$$

$$\iint_{S_1} F_3 \mathbf{k} \cdot \mathbf{n}_1\, dS_1 = \iint_R F_3[x, y, g(x, y)]\, dx\, dy.$$

For the lower portion S_2 of S, the normal $\mathbf{n}_2$ to S_2 makes an obtuse angle γ_2 with $\mathbf{k}$. Therefore

$$\mathbf{k} \cdot \mathbf{n}_2\, dS_2 = \cos \gamma_2\, dS_2 = -\,dx\, dy. \quad \text{Hence}$$

$$\iint_{S_2} F_3\, \mathbf{k} \cdot \mathbf{n}_2\, dS_2 = -\iint_R F_3[x, y, f(x, y)]\, dx\, dy.$$

$$\therefore \iint_{S_3} F_3\, \mathbf{k} \cdot \mathbf{n}_3\, dS_3 + \iint_{S_1} F_3\, \mathbf{k} \cdot \mathbf{n}_1\, dS_1 + \iint_{S_2} F_3\, \mathbf{k} \cdot \mathbf{n}_2\, dS_2$$

$$=0 + \iint_R F_3[x, y, g(x, y)]\, dx\, dy - \iint_R F_3[x, y, f(x, y)]\, dx\, dy$$

or with the help of (1), we get

$$\iint_S F_3 \mathbf{k} \cdot \mathbf{n}\, dS = \iiint_V \frac{\partial F_3}{\partial z}\, dV. \qquad \ldots(2)$$

Similarly, by projecting S on the other co-ordinate planes, we get

$$\iint_S F_2\, \mathbf{j} \cdot \mathbf{n}\, dS = \iiint_V \frac{\partial F_2}{\partial y}\, dV \qquad \ldots(3)$$

and
$$\iint_S F_1\, \mathbf{i} \cdot \mathbf{n}\, dS = \iiint_V \frac{\partial F_1}{\partial x}\, dV. \qquad \ldots(4)$$

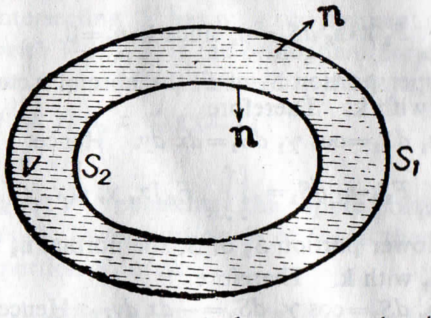
Adding (2), (3) and (4), we get

$$\iint_S (F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3 \mathbf{k}) \cdot \mathbf{n}\, dS = \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

or $$\iiint_V \nabla\cdot\mathbf{F}\, dV=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

The proof of the theorem can now be extended to a region V which can be subdivided into finitely many special regions of the above type by drawing auxiliary surfaces. In this case we apply the theorem to each sub-region and then add the results. The sum of the volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxiliary surfaces cancel in pairs, while the sum of the remaining surface integrals is equal to the surface integral over the whole boundary S of V.

Note. The divergence theorem is applicable for a region Y if it is bounded by two closed surfaces S_1 and S_2 one of which lies



within the other. Here outward drawn normals will have the directions as shown in the figure.

§ 8. Some deductions from divergence theorem.

1. Green's theorem. *Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\, \nabla^2\psi-\psi\, \nabla^2\phi)\, dV=\iint_S (\phi\, \nabla\psi-\psi\, \nabla\phi)\cdot\mathbf{n}\, dS.$$

[Agra 1971, Gauhati 72; M. U. 1979; Indore 1979]

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\, dV=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Putting $\mathbf{F}=\phi\, \nabla\psi$, we get

$$\nabla\cdot\mathbf{F}=\nabla\cdot(\phi\, \nabla\psi)$$
$$=\phi\, (\nabla\cdot\nabla\psi)+(\nabla\phi)\cdot(\nabla\psi)=\phi\, \nabla^2\psi+(\nabla\phi)\cdot(\nabla\psi).$$

Also $\mathbf{F}\cdot\mathbf{n}=(\phi\, \nabla\psi)\cdot\mathbf{n}.$

$\therefore$ divergence theorem gives

$$\iiint_V [\phi\nabla^2\psi+(\nabla\phi)\cdot(\nabla\psi)]\, dV$$
$$=\iint_S (\phi\nabla\psi)\cdot\mathbf{n}\, dS \quad \ldots(1)$$

[Meerut 1970]

This is called *Green's first identity or theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2\phi+(\nabla\psi)\cdot(\nabla\phi)]\, dV$$
$$=\iint_S [\psi\nabla\, \phi]\cdot\mathbf{n}\, dS \quad \ldots(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\, dV=\iint_S (\phi\nabla\, \psi-\psi\, \nabla\phi)\cdot\mathbf{n}\, dS \quad \ldots(3)$$

This is called *Green's second identity or Green's theorem in symmetrical form.*

Since $\nabla\psi=\dfrac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi=\dfrac{\partial\phi}{\partial n}\mathbf{n}$, therefore

$$(\phi\nabla\, \psi-\psi\, \nabla\, \phi)\cdot\mathbf{n}=\left(\phi\frac{\partial\psi}{\partial n}\mathbf{n}-\psi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}$$
$$=\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2\psi-\psi\, \nabla^2\, \phi)\, dV=\iint_S \left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right) dS.$$

[Meerut 1972, 80]

Note. **Harmonic function.** *If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi=0$, then ϕ is called harmonic function. If ϕ and ψ* are both harmonic functions, then $\nabla^2\phi=0,\ \nabla^2\psi=0$. Hence from Green's second identity, we get

$$\iint_S \left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right) dS=0.$$

2. *Prove that* $\displaystyle\iiint_V \nabla\phi\, dV=\iint_S \phi\mathbf{n}\, dS.$

[Agra 1972; Allahabad 77]

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\, dV=\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Taking $\mathbf{F}=\phi\mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla\cdot(\phi\mathbf{C})\, dV=\iint_S (\phi\mathbf{C})\cdot\mathbf{n}\, dS \qquad \ldots(1)$$

Now $\nabla\cdot(\phi\mathbf{C})=(\nabla\phi)\cdot\mathbf{C}+\phi\,(\nabla\cdot\mathbf{C})$

$=(\nabla\phi)\cdot\mathbf{C}$, since $\nabla\cdot\mathbf{C}=0$.

Also $(\phi\mathbf{C})\cdot\mathbf{n}=\mathbf{C}\cdot(\phi\mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C}\cdot(\nabla\phi)\, dV=\iint_S \mathbf{C}\cdot(\phi\mathbf{n})\, dS$$

or $$\mathbf{C}\cdot\iiint_V \nabla\phi\, dV=\mathbf{C}\cdot\iint_S \phi\mathbf{n}\, dS$$

or $$\mathbf{C}\cdot\left[\iiint_V \nabla\phi\, dV-\iint_S \phi\mathbf{n}\, dS\right]=0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla\phi\, dV=\iint_S \phi\mathbf{n}\, dS.$$

3. *Prove that* $\iiint_V \nabla\times\mathbf{B}\, dV=\iint_S \mathbf{n}\times\mathbf{B}\, dS.$

[Gauhati 1971, 74]

Proof. In divergence theorem taking $\mathbf{F}=\mathbf{B}\times\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla\cdot(\mathbf{B}\times\mathbf{C})\, dV=\iint_S (\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}\, dS. \qquad \ldots(1)$$

Now $\nabla\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{C}\cdot\text{curl }\mathbf{B}-\mathbf{B}\cdot\text{curl }\mathbf{C}$

$=\mathbf{C}\cdot\text{curl }\mathbf{B}$, since curl $\mathbf{C}=\mathbf{0}$.

Also $(\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}=[\mathbf{B},\ \mathbf{C},\ \mathbf{n}]=[\mathbf{C},\ \mathbf{n},\ \mathbf{B}]=\mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C}\cdot\text{curl }\mathbf{B})\, dV=\iint_S \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})\, dS$$

or $$\mathbf{C}\cdot\iiint_V (\nabla\times\mathbf{B})\, dV=\mathbf{C}\cdot\iint_S (\mathbf{n}\times\mathbf{B})\, dS$$

or $$\mathbf{C}\cdot\left[\iiint_V (\nabla\times\mathbf{B})\, dV-\iint_S (\mathbf{n}\times\mathbf{B})\, dS\right]=0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla\times\mathbf{B})\, dV-\iint_S (\mathbf{n}\times\mathbf{B})\, dS.$$

Hence we must have

$$\iiint_V (\nabla\times\mathbf{B})\, dV-\iint_S (\mathbf{n}\times\mathbf{B})\, dS=0$$

or $$\iiint_V (\nabla\times\mathbf{B})\, dV=\iint_S (\mathbf{n}\times\mathbf{B})\, dS.$$

SOLVED EXAMPLES

Ex. 1. *For any closed surface S, prove that*

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\, dS=0.$$

Solution. By divergence theorem, we have

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V (\text{div curl }\mathbf{F})\, dV,$$ where V is the volume enclosed by S

$=0$, since div curl $\mathbf{F}=0$.

Ex. 2. *Evaluate* $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$, *where S is a closed surface.*

[Madras 1983; Rohilkhand 76; Allahabad 75]

Solution. By the divergence theorem, we have

$$\iint_S \mathbf{r}\cdot\mathbf{n}\, dS=\iiint_V \nabla\cdot\mathbf{r}\, dV$$

$$=\iiint_V 3\, dV, \text{ since } \nabla\cdot\mathbf{r}=\text{div }\mathbf{r}=3$$

$=3V$, where V is the volume enclosed by S.

Ex. 3. *If* $\mathbf{F}=ax\mathbf{i}+by\mathbf{j}+cz\mathbf{k}$, *a, b, c are constants, show that* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\frac{4}{3}\pi\,(a+b+c)$, *where S is the surface of a unit sphere.*

[Kerala 1974; Agra 80; Rohilkhand 77; Allahabad 80, 82]

Solution. By the divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iiint_V (\nabla\cdot\mathbf{F})\, dV,$$

where V is the volume enclosed by S

$$=\iiint_V [\nabla\cdot(ax\mathbf{i}+by\mathbf{j}+cz\mathbf{k})]\, dV$$

$$=\iiint_V \left[\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right] dV$$

$$=\iiint_V (a+b+c)\, dV=(a+b+c)\, V=(a+b+c)\,\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi\,(1)^3$ *i.e.*, $\frac{4}{3}\pi$.

Ex. 4. *If* $\mathbf{n}$ *is the unit outward drawn normal to any closed surface S, show that* $\iiint_V \text{div }\mathbf{n}\, dV=S.$

Solution. We have by the divergence theorem,

$$\iiint_V \text{div } \mathbf{n}\, dV = \iint_S \mathbf{n}\cdot\mathbf{n}\, dS = \iint_S dS = S.$$

Ex. 5. *Prove that*

$$\iiint_V \nabla\phi\cdot\mathbf{A}\, dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}dS - \iiint_V \phi\nabla\cdot\mathbf{A}\, dV.$$

Solution. By divergence theorem, we have

$$\iiint_V \nabla\cdot(\phi\mathbf{A})\, dV = \iint_S (\phi\mathbf{A})\cdot\mathbf{n}\, dS. \qquad \ldots(1)$$

Now $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A} + \phi\,(\nabla\cdot\mathbf{A})$.
Also $(\phi\mathbf{A})\cdot\mathbf{n} = \phi\,(\mathbf{A}\cdot\mathbf{n})$.
Hence (1) gives

$$\iiint_V [(\nabla\phi)\cdot\mathbf{A} + \phi\,(\nabla\cdot\mathbf{A})]\, dV = \iint_S \phi\ \mathbf{A}\cdot\mathbf{n}\, dS$$

or $$\iiint_V (\nabla\phi)\cdot\mathbf{A}\, dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\, dS - \iiint_V \phi\ \nabla\cdot\mathbf{A}\, dV.$$

Ex. 6. *Prove that* $\int_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = 0.$

Solution. We have $\int_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = \int_S (\nabla\phi\times\nabla\psi)\cdot\mathbf{n}\, dS$

$= \int_V \nabla\cdot(\nabla\phi\times\nabla\psi)\, dV$, by divergence theorem

$= 0 \qquad [\because\ \nabla\cdot(\nabla\phi\times\nabla\psi) = 0.$ See Ex. 13 page 65]

Ex. 7. *Prove that*

$$\int_V \nabla\phi\cdot\text{curl } \mathbf{F}\, dV = \int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S}.$$

Solution. We have $\int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S} = \int_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\, dS$

$= \int_V \nabla\cdot(\mathbf{F}\times\nabla\phi)\, dV$, by divergence theorem applied to the vector function $\mathbf{F}\times\nabla\phi$

$= \int_V (\nabla\phi\cdot\text{curl } \mathbf{F} - \mathbf{F}\cdot\text{curl } \nabla\phi)\, dV$

[By vector identity 5 on page 57]

$= \int_V \nabla\phi\cdot\text{curl } \mathbf{F}\, dV. \qquad [\because\ \text{curl } \nabla\ \phi = 0]$

Ex. 8. *Prove that* $\iiint_V \dfrac{dV}{r^2} = \iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS.$

Solution. $\iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS = \iint_S \left(\dfrac{\mathbf{r}}{r^2}\right)\cdot\mathbf{n}\, dS$

$= \iiint_V \nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right) dV$, by divergence theorem.

Now $\nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right) = \dfrac{1}{r^2}(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\nabla\left(\dfrac{1}{r^2}\right)$

$= \dfrac{3}{r^2} + \mathbf{r}\cdot\left(-\dfrac{2}{r^3}\ \nabla r\right) = \dfrac{3}{r^3} - \dfrac{2}{r^3}\left(\mathbf{r}\cdot\dfrac{\mathbf{r}}{r}\right) = \dfrac{3}{r^2} - \dfrac{2}{r^4}\ r^2 = \dfrac{1}{r^2}.$

Hence $\iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS = \iiint_V \dfrac{dV}{r^2}.$

Ex. 9. *If* $\mathbf{F} = \nabla\ \phi$ *and* $\nabla^2\phi = 0$, *show that for a closed surface S*

$$\iiint_V \mathbf{F}^2\, dV = \iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS.$$ **[Rohilkhand 1978, 79]**

Solution. By divergence theorem, we have

$$\iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V [\nabla\cdot(\phi\mathbf{F})]\, dV.$$

Now $\nabla\cdot(\phi\mathbf{F}) = (\nabla\phi\cdot\mathbf{F}) + \phi\,(\nabla\cdot\mathbf{F}) = \mathbf{F}\cdot\mathbf{F} + \phi\,(\nabla\cdot\nabla\ \phi)$
$= \mathbf{F}^2 + \phi\nabla^2\phi = \mathbf{F}^2$, since $\nabla^2\phi = 0$.

Hence $\iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \mathbf{F}^2\, dV.$

Ex. 10. *If* $\mathbf{F} = \nabla\phi$, $\nabla^2\phi = -4\pi\rho$, *show that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = -4\pi \iiint_V \rho\, dV.$$

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V (\nabla\cdot\mathbf{F})\, dV.$$

Now $\nabla\cdot\mathbf{F} = \nabla\cdot(\nabla\phi) = \nabla^2\phi = -4\pi\rho$.

$$\therefore\ \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V (-4\pi\rho)\, dV = -4\pi\iiint_V \rho\, dV.$$

Ex 11. *If* $\mathbf{C} = \frac{1}{2}\nabla\times\mathbf{B}$, $\mathbf{B} = \nabla\times\mathbf{A}$, *show that*

$$\tfrac{1}{2}\iiint_V \mathbf{B}^2\, dV = \tfrac{1}{2}\iint_S \mathbf{A}\times\mathbf{F}\cdot\mathbf{n}\, dS + \iiint_V \mathbf{A}\cdot\mathbf{C}\, dV.$$

Solution. We have by divergence theorem

$$\tfrac{1}{2}\iint_S (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\, dS = \tfrac{1}{2}\iiint_V \nabla\cdot(\mathbf{A}\times\mathbf{B})\, dV.$$

Now $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl } \mathbf{A} - \mathbf{A}\cdot\text{curl } \mathbf{B}$
$= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot\mathbf{B} - \mathbf{A}\cdot(2\mathbf{C}) = \mathbf{B}^2 - 2\,(\mathbf{A}\cdot\mathbf{C})$.

Hence $\tfrac{1}{2}\iint_S (\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}\, dS = \tfrac{1}{2}\iiint_V [\mathbf{B}^2 - 2\,(\mathbf{A}\cdot\mathbf{C})]\, dV$

$= \tfrac{1}{2}\iiint_V \mathbf{B}^2\, dV - \iiint_V \mathbf{A}\cdot\mathbf{C}\, dV$

or $$\tfrac{1}{2}\iiint_V \mathbf{B}^2\, dV=\tfrac{1}{2}\iint_S \mathbf{A}\times\mathbf{B}\cdot\mathbf{n}\, dS+\iiint_V \mathbf{A}\cdot\mathbf{C}\, dV.$$

Ex. 12. *If ϕ is harmonic in V, then*

$$\iint_S \frac{\partial\phi}{\partial n}\, dS=0$$

where S is the surface enclosing V. **[Meerut 1972]**

Solution. We have $\displaystyle\iint_S \frac{\partial\phi}{\partial n}\, dS=\iint_S \left(\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}\, dS$

$$=\iint_S (\nabla\phi)\cdot\mathbf{n}\, dS$$

$$=\iiint_V \nabla\cdot(\nabla\phi)\, dV, \text{ by divergence theorem}$$

$$=\iiint_V \nabla^2\phi\, dV$$

$=0$, since $\nabla^2\phi=0$ in V because ϕ is harmonic in V.

Ex. 13. *If ϕ is harmonic in V, then*

$$\iint_S \phi\frac{\partial\phi}{\partial n}\, dS=\iiint_V |\nabla\phi|^2\, dV.$$

[Meerut 1969, Agra 70]

Solution. We have

$$\iint_S \phi\frac{\partial\phi}{\partial n}\, dS=\iint_S \left(\phi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}\, dS=\iint_S (\phi\nabla\phi)\cdot\mathbf{n}\, dS$$

$$=\iiint_V \nabla\cdot(\phi\nabla\phi)\, dV, \text{ by divergence theorem}$$

$$=\iiint_V [(\nabla\phi\cdot\nabla\phi)+\phi(\nabla\cdot\nabla\phi)]\, dV$$

$$=\iiint_V [(\nabla\phi)^2+\phi\nabla^2\phi]\, dV$$

$$=\iiint_V |\nabla\phi|^2\, dV, \text{ since } \nabla^2\phi=0 \text{ and } (\nabla\phi)^2=|\nabla\phi|^2.$$

Ex. 14. *If ϕ is harmonic in V and $\dfrac{\partial\phi}{\partial n}=0$ on S, then ϕ is constant in V.*

Solution. Since ϕ is harmonic in V, therefore as in exercise 13, we have

$$\iint_S \phi\frac{\partial\phi}{\partial n}\, dS=\iiint_V |\nabla\phi|^2\, dV.$$

But $\dfrac{\partial\phi}{\partial n}=0$ on S. Therefore $\displaystyle\iint_S \phi\frac{\partial\phi}{\partial n}\, dS=0.$

$$\therefore \quad \iiint_V |\nabla\phi|^2\, dV=0.$$

$\therefore \quad |\nabla\phi|^2=0$ in V.

$\therefore \quad \nabla\phi=0$ in V.

$\therefore \quad \phi=$constant in V.

Ex. 15. *If ϕ and ψ are harmonic in V and $\dfrac{\partial\phi}{\partial n}=\dfrac{\partial\psi}{\partial n}$ on S, then $\phi=\psi+c$ in V, where c is a constant.*

Solution. We have, $\nabla^2\phi=0$, $\nabla^2\psi=0$ in V.

$\therefore \quad \nabla^2(\phi-\psi)=\nabla^2\phi-\nabla^2\psi=0$ in V.

Therefore $\phi-\psi$ is harmonic in V.

Again on S, $\dfrac{\partial}{\partial n}(\phi-\psi)=\dfrac{\partial\phi}{\partial n}-\dfrac{\partial\psi}{\partial n}=0.$

Thus $\phi-\psi$ is harmonic in V and on S we have

$$\frac{\partial}{\partial n}(\phi-\psi)=0.$$

Hence as in exercise 14, we have

$\phi-\psi=c$, where c is a constant

or $\quad \phi=\psi+c$.

Ex. 16. *If div **F** denotes the divergence of a vector field **F** at a point P, show that*

$$\operatorname{div}\mathbf{F}=\lim_{\delta V\to 0}\frac{\iint_{\delta S}\mathbf{F}\cdot\mathbf{n}\, dS}{\delta V}$$

where δV is the volume enclosed by the surface δS and the limit is obtained by shrinking δV to the point P.

Solution. We have by the divergence theorem,

$$\iiint_{\delta V} \operatorname{div}\mathbf{F}\, dV=\iint_{\delta S}\mathbf{F}\cdot\mathbf{n}\, dS. \quad \ldots(1)$$

By the mean value theorem of integral calculus, the left hand side can be written as

$$\overline{\operatorname{div}\mathbf{F}}\iiint_{\delta V} dV=\overline{\operatorname{div}\mathbf{F}}\,\delta V,$$

where $\overline{\operatorname{div}\mathbf{F}}$ is some value intermediate between the maximum and minimum of div **F** throughout δV. Therefore (1) gives

$$\overline{\operatorname{div}\mathbf{F}}\,\delta V=\iint_{\delta S}\mathbf{F}\cdot\mathbf{n}\, dS$$

or
$$\overline{\text{div }\mathbf{F}}=\frac{\iint_{\delta S}\mathbf{F}\cdot\mathbf{n}\,dS}{\delta V}.$$

Taking the limit as $\delta V\to 0$ such that P is always interior to δV, $\overline{\text{div }\mathbf{F}}$ approaches the value div **F** at point P. Hence, we get
$$\text{div }\mathbf{F}=\lim_{\delta V\to 0}\frac{\iint_{\delta S}\mathbf{F}\cdot\mathbf{n}\,dS}{\delta V}.$$

Ex. 17. *Show that* $\iint_S \mathbf{n}\,dS=0$ *for any closed surface S.*

Solution. Let **C** be any arbitrary constant vector. Then
$$\mathbf{C}\cdot\iint_S \mathbf{n}\,dS=\iint_S \mathbf{C}\cdot\mathbf{n}\,dS$$
$$=\iiint_V (\nabla\cdot\mathbf{C})\,dV, \text{ by divergence theorem}$$
$$=0, \text{ since div }\mathbf{C}=0.$$

Thus $\mathbf{C}\cdot\iint_S \mathbf{n}\,dS=0$, where **C** is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n}\,dS=0$.

Ex. 18. *Prove that* $\iint_S \mathbf{r}\times\mathbf{n}\,dS=0$ *for any closed surface S.*

Solution. Let **C** be any arbitrary constant vector. Then
$$\mathbf{C}\cdot\iint_S \mathbf{r}\times\mathbf{n}\,dS=\iint_S \mathbf{C}\cdot[(\mathbf{r}\times\mathbf{n})]\,dS=\iint_S (\mathbf{C}\times\mathbf{r})\cdot\mathbf{n}\,dS$$
$$=\iiint_V [\nabla\cdot(\mathbf{C}\times\mathbf{r})]\,dV, \text{ by divergence theorem}$$
$$=\iiint_V [\mathbf{r}\cdot\text{curl }\mathbf{C}-\mathbf{C}\cdot\text{curl }\mathbf{r}]\,dV$$
$$=0, \text{ since curl }\mathbf{C}=0 \text{ and } \mathbf{r}=0.$$

Thus $\mathbf{C}\cdot\iint_S \mathbf{r}\times\mathbf{n}\,dS=0$, where **C** is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r}\times\mathbf{n}\,dS=0$.

Ex. 19. *Prove that* $\iint_S (\nabla\phi)\times\mathbf{n}\,dS=0$ *for a closed surface S.*

Solution Let **C** be an arbitrary constant vector. Then

$$\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\,dS=\iint_S \mathbf{C}\cdot[(\nabla\phi)\times\mathbf{n}]\,dS$$
$$=\iint_S [\mathbf{C}\times\nabla\phi]\cdot\mathbf{n}\,dS$$
$$=\iiint_V [\nabla\cdot(\mathbf{C}\times\nabla\phi)]\,dV, \text{ by div. theorem}$$
$$=\iiint_V [\nabla\phi\cdot\text{curl }\mathbf{C}-\mathbf{C}\cdot\text{curl }\nabla\phi]\,dV$$
$$=0, \text{ since curl }\mathbf{C}=0 \text{ and curl }\nabla\phi=0.$$

Thus $\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\,dS=0$, where **C** is an arbitrary vector.

Hence we must have $\iint_S (\nabla\phi)\times\mathbf{n}\,dS=0$.

Ex. 20. *Prove that* $\iint \mathbf{n}\times(\mathbf{a}\times\mathbf{r})\,dS=2V\mathbf{a}$, *where* **a** *is a constant vector and V is the volume enclosed by the closed surface S.*

Solution. We know that
$$\iiint_V \nabla\times\mathbf{B}\,dV=\iint_S \mathbf{n}\times\mathbf{B}\,dS.$$ [see page 110]

Putting $\mathbf{B}=\mathbf{a}\times\mathbf{r}$, we get
$$\iint_S \mathbf{n}\times(\mathbf{a}\times\mathbf{r})\,dS=\iiint_V \nabla\times(\mathbf{a}\times\mathbf{r})\,dV$$
$$=\iiint_V \text{curl }(\mathbf{a}\times\mathbf{r})\,dV$$
$$=\iiint_V 2\mathbf{a}\,dV, \text{ since curl }(\mathbf{a}\times\mathbf{r})=2\mathbf{a}$$
$$=2\mathbf{a}\iiint_V dV=2\mathbf{a}V.$$

Ex. 21. *A vector* **B** *is always normal to a given closed surface S. Show that* $\iiint_V \text{curl }\mathbf{B}\,dV=0$, *where V is the region bounded by S.*

Solution. We know that
$$\iiint_V \text{curl }\mathbf{B}\,dV=\iint_S \mathbf{n}\times\mathbf{B}\,dS.$$

Since **B** is normal to S, therefore **B** is parallel to **n**. Therefore $\mathbf{n}\times\mathbf{B}=0$.
$$\therefore\quad \iint_S \mathbf{n}\times\mathbf{B}\,dS=0.$$

$$\therefore \quad \iiint_V \text{curl } \mathbf{B} \, dV = 0.$$

Ex. 22. *Express* $\int_V \{(\text{grad } \rho)\cdot\mathbf{v} + \rho \text{ div } \mathbf{v}\} \, dV$, *as a surface integral.* **[Gauhati 1972, 77]**

Solution. We know that

$$\text{div } (\rho\mathbf{v}) = (\text{grad } \rho)\cdot\mathbf{v} + \rho \text{ div } \mathbf{v}.$$

[See vector identity 3 on page 56].

$$\therefore \quad \int_V \{(\text{grad } \rho)\cdot\mathbf{v} + \rho \text{ div } \mathbf{v}\} \, dV = \int_V \text{div } (\rho\mathbf{v}) \, dV$$

$$= \int_V \nabla\cdot(\rho\mathbf{v}) \, dV$$

$$= \int_S (\rho\mathbf{v})\cdot\mathbf{n} \, dS, \text{ by Gauss divergence theorem}$$

$$= \int_S \rho \, (\mathbf{v}\cdot\mathbf{n}) \, dS.$$

Ex. 23. *Using the divergence theorem, show that the volume V of a region T bounded by a surface S is*

$$V = \iint_S x \, dy \, dz = \iint_S y \, dz \, dx = \iint_S z \, dx \, dy$$

$$= \tfrac{1}{3} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy).$$

Solution. By divergence theorem, we have

$$\iint_S x \, dy \, dz = \iiint_V \left(\frac{\partial}{\partial x}(x) \right) dV = \iiint_V dV = V$$

$$\iint_S y \, dz \, dx = \iiint_V \left[\frac{\partial}{\partial y}(y) \right] dV = \iiint_V dV = V$$

$$\iint_S z \, dx \, dy = \iiint_V \left[\frac{\partial}{\partial z}(z) \right] dV = \iiint_V dV = V.$$

Adding these results, we get

$$3V = \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy)$$

or
$$V = \tfrac{1}{3} \iint_S (x \, dy \, dz + y \, dz \, dx + z \, dx \, dy).$$

Ex. 24. *Verify divergence theorem for*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped

$$0 \leqslant x \leqslant a, \; 0 \leqslant y \leqslant b, \; 0 \leqslant z \leqslant c.$$ **[Meerut 1976]**

Solution. We have div $\mathbf{F} = \nabla\cdot\mathbf{F}$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z.$$

$$\therefore \quad \text{volume integral} = \iiint_V \nabla\cdot\mathbf{F} \, dV = \iiint_V 2\,(x+y+z) \, dV$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x+y+z) \, dx \, dy \, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx \right]_{x=0}^{a} dy \, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{a^2}{2} + ay + az \right] dy \, dz = 2 \int_{z=0}^{c} \left[\frac{a^2}{2} y + a\frac{y^2}{2} + azy \right]_{y=0}^{b} dz$$

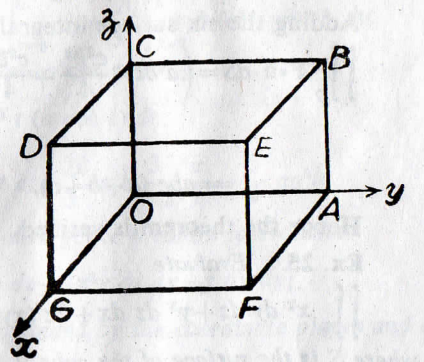
$$= 2 \int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right] dz = 2 \left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab\frac{z^2}{2} \right]_0^c$$

$$= [a^2bc + ab^2c + abc^2] = abc\,(a+b+c).$$

Surface Integral. We shall now calculate

$$\iint_S \mathbf{F}\cdot\mathbf{n} \, dS$$

over the six faces of the rectangular parallelopiped.



Over the face $DEFG$,

$\mathbf{n} = \mathbf{i}$, $x = a$.

Therefore

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n} \, dS$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} [(a^2 - yz)\,\mathbf{i} + (y^2 - za)\,\mathbf{j} + (z^2 - ay)\,\mathbf{k}]\cdot\mathbf{i} \, dy \, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} (a^2 - yz) \, dy \, dz = \int_{z=0}^{c} \left[a^2 y - z\frac{y^2}{2} \right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \left[a^2 b - \frac{zb^2}{2} \right] dz = \left[a^2 bz - \frac{z^2}{4} b^2 \right]_0^c$$

$$= a^2 bc - \frac{c^2 b^2}{4}.$$

Over the face $ABCO$, $\mathbf{n} = -\mathbf{i}$, $x = 0$. Therefore

$$\iint_{ABCO} \mathbf{F}\cdot\mathbf{n} \, dS = \iint [(0 - yz)\,\mathbf{i} + \ldots + \ldots]\cdot(-\mathbf{i}) \, dy \, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} yz \, dy \, dz = \int_{z=0}^{c} \left[\frac{y^2}{2} z \right]_{y=0}^{b} dz = \int_{z=0}^{c} \frac{b^2}{2} z \, dz = \frac{b^2 c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n}=\mathbf{j}$, $y=b$. Therefore

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{z=0}^{c}\int_{x=0}^{a} [(x^2-bz)\,\mathbf{i}+(b^2-zx)\,\mathbf{j}$$

$$+(z^2-bx)\,\mathbf{k}]\cdot\mathbf{j}\, dx\, dz$$

$$=\int_{z=0}^{c}\int_{x=0}^{a} (b^2-zx)\, dx\, dz=b^2\, ca-\frac{a^2c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n}=-\mathbf{j}$, $y=0$. Therefore

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{z=0}^{c}\int_{x=0}^{a} zx\, dx\, dz=\frac{c^2a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n}=\mathbf{k}$, $z=c$. Therefore

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{y=0}^{b}\int_{x=0}^{a} (c^2-xy)\, dx\, dy=c^2ab-\frac{a^2b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n}=-\mathbf{k}$, $z=0$. Therefore

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\, dS=\int_{y=0}^{b}\int_{x=0}^{a} xy\, dx\, dy=\frac{a^2b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_{S} \mathbf{F}\cdot\mathbf{n}\, dS=\left(a^2bc-\frac{c^2b^2}{4}+\frac{c^2b^2}{4}\right)+\left(b^2ca-\frac{a^2c^2}{4}+\frac{a^2c^2}{4}\right)$$

$$+\left(c^2ab-\frac{a^2b^2}{4}+\frac{a^2b^2}{4}\right)$$

$$=abc\,(a+b+c).$$

Hence the theorem is verified.

Ex. 25. *Evaluate*

$$\iint_S x^2\, dy\, dz+y^2\, dz\, dx+2z\,(xy-x-y)\, dx\, dy$$

where S is the surface of the cube

$$0\leqslant x\leqslant 1,\ 0,\leqslant y\leqslant 1,\ 0\leqslant z\leqslant 1.$$ **[Meerut 1968]**

Solution. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}\{2z\,(xy-x-y)\}\right] dV$$

$$=\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} [2x+2y+2xy-2x-2y]\, dx\, dy\, dz$$

$$=2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xy\, dx\, dy\, dz=2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}\, y\right]_{x=0}^{1} dy\, dz$$

$$=2\int_{z=0}^{1}\int_{y=0}^{1}\frac{y}{2}\, dy\, dz=\int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz$$

$$=\int_{z=0}^{1}\tfrac{1}{2}\, dz=\tfrac{1}{2}\Big[\, z\, \Big]_0^1=\tfrac{1}{2}.$$

Ex. 26. *By transforming to a triple integral evaluate*

$$I=\iint_S (x^3\, dy\, dz+x^2y\, dz\, dx+x^2z\, dx\, dy)$$

where S is the closed surface bounded by the planes $z=0$, $z=b$ and the cylinder $x^2+y^2=a^2$. **[Meerut 1969, 80]**

Solution. By divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3)+\frac{\partial}{\partial y}(x^2y)+\frac{\partial}{\partial z}(x^2z)\right] dV$$

$$=\int_{z=0}^{b}\int_{y=-a}^{a}\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (3x^2+x^2+x^2)\, dx\, dy\, dz$$

$$=4\times 5\int_{z=0}^{b}\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}} x^2\, dx\, dy\, dz$$

$$=20\int_{z=0}^{b}\int_{y=0}^{a}\left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy\, dz=\frac{20}{3}\int_{z=0}^{b}\int_{y=0}^{a}(a^2-y^2)^{3/2} dy\,dz$$

$$=\frac{20}{3}\int_{y=0}^{a}\Big[(a^2-y^2)^{3/2}\, z\Big]_{z=0}^{b} dy=\frac{20}{3}\int_{y=0}^{a} b\,(a^2-y^2)^{3/2}\, dy.$$

Put $y=a\sin t$ so that $dy=a\cos t\, dt$.

$$\therefore\quad I=\frac{20}{3}\, b\int_0^{\pi/2} a^3\cos^3 t\,(a\cos t)\, dt$$

$$=\frac{20}{3}\, a^4b\int_0^{\pi/2}\cos^4 t\, dt=\frac{20}{3}\, a^4b\,\frac{3}{4.2}\,\frac{\pi}{2}=\frac{5}{4}\,\pi a^4b.$$

Ex. 27. *Apply Gauss's divergence theorem to evaluate*

$$\iint_S [(x^3-yz)\, dy\, dz-2x^2y\, dz\, dx+z\, dx\, dy]$$

over the surface of a cube bounded by the coordinate planes and the planes $x=y=z=a$.

Solution. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy)$$

$$=\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

Here $F_1=x^3-yz$, $F_2=-2x^2y$, $F_3=z$.

$$\therefore\quad \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=3x^2-2x^2+1=x^2+1.$$

$\therefore$ the given surface integral is equal to the volume integral

$$\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} (x^2+1)\, dx\, dy\, dz$$

$$=\int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dy\, dz$$

$$=\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{3}+a\right)dy\,dz=a^2\left(\frac{a^3}{3}+a\right).$$

Ex. 28. *If* $\mathbf{F}=x\mathbf{i}-y\mathbf{j}+(z^2-1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *where S is the closed surface bounded by the planes* $z=0$, $z=1$ *and the cylinder* $x^2+y^2=4$. **[Kanpur 1978, 80]**

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{F}\,dV.$$

Here $\text{div}\,\mathbf{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(z^2-1)$

$=1-1+2z=2z.$

$$\therefore\ \iiint_V \text{div}\,\mathbf{F}\,dV=\int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}2z\,dx\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}\Big[2zx\Big]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}4z\sqrt{(4-y^2)}\,dy\,dz=\int_{y=-2}^{2}\left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1}dy$$

$$=2\int_{y=-2}^{2}\sqrt{(4-y^2)}\,dy=4\int_0^2\sqrt{(4-y^2)}\,dy$$

$$=4\left[\frac{y}{2}\sqrt{(4-y^2)}+2\sin^{-1}\frac{y}{2}\right]_0^2=4\,[2\sin^{-1}1]=4(2)\frac{\pi}{2}=4\pi.$$

Ex. 29. *Find* $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$,

where $\mathbf{A}=(2x+3z)\,\mathbf{i}-(xz+y)\,\mathbf{j}+(y^2+2z)\,\mathbf{k}$

and S is the surface of the sphere having centre at $(3,-1,2)$ *and radius* 3. **[Meerut 1974]**

Solution. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{A}\,dV.$$

Now $\text{div}\,\mathbf{A}=\frac{\partial}{\partial x}(2x+3z)+\frac{\partial}{\partial y}\{-(xz+y)\}+\frac{\partial}{\partial z}(y^2+2z)$

$=2-1+2=3.$

$$\therefore\ \iint_S \mathbf{A}\cdot\mathbf{n}\,dS=\iiint_V 3\,dV=3\iiint_V dV=3V.$$

But V is the volume of a sphere of radius 3. Therefore $V=\frac{4}{3}\pi\,(3)^3=36\pi.$

$$\therefore\ \iint_S \mathbf{A}\cdot\mathbf{n}\,dS=3V=3\times 36\pi=108\pi.$$

Ex. 30. *Apply divergence theorem to evaluate*

$$\iint_S [(x+z)\,dy\,dz+(y+z)\,dz\,dx+(x+y)\,dx\,dy]$$

where S is the surface of the sphere $x^2+y^2+z^2=4$.

Solution. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V\left[\frac{\partial}{\partial x}(x+z)+\frac{\partial}{\partial y}(y+z)+\frac{\partial}{\partial z}(x+y)\right]dV$$

$=\iiint_V 2dV=2\iiint_V dV=2V$, where V is the volume of the sphere $x^2+y^2+z^2=4$

$$=2\left[\frac{4}{3}\pi(2)^3\right]=\frac{64}{3}\pi.$$

Ex. 31. *If S is any closed surface enclosing a volume V and* $\mathbf{F}=x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k}$, *prove that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=6V.$$

[Kanpur 1979; Rohilkhand 80; Agra 78]

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iiint_V \text{div}\,\mathbf{F}\,dV=\iiint_V \text{div}\,(x\mathbf{i}+2y\mathbf{j}+3z\mathbf{k})\,dV$$

$$=\iiint_V\left[\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(2y)+\frac{\partial}{\partial z}(3z)\right]dV$$

$$=\iiint_V(1+2+3)\,dV=6\iiint_V dV=6V.$$

Ex. 32. *Evaluate*

$$\iint_S (y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\cdot\mathbf{n}\,dS$$

where S is the part of the sphere $x^2+y^2+z^2=1$ *above the xy-plane and bounded by this plane.* **[Agra 1969; Bombay 66]**

Solution. By divergence theorem, we have

$$\iint_S (y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\cdot\mathbf{n}\,dS$$

$$=\iiint_V \text{div}\,(y^2z^2\,\mathbf{i}+z^2x^2\,\mathbf{j}+z^2y^2\,\mathbf{k})\,dV,$$

where V is the volume enclosed by S

$$=\iiint_V\left[\frac{\partial}{\partial x}(y^2z^2)+\frac{\partial}{\partial y}(z^2x^2)+\frac{\partial}{\partial z}(z^2y^2)\right]dV$$

$$=\iiint_V 2zy^2\,dV=2\iiint_V zy^2\,dV.$$

We shall use spherical polar coordinates (r,θ,ϕ) to evaluate this triple integral. In polars $dV=(dr)\,(r d\theta)\,(r \sin\theta\, d\phi)=r^2 \sin\theta\, dr\, d\theta\, d\phi$. Also $z=r\cos\theta$, $y=r\sin\theta\sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π. The triple integral is

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(r\cos\theta)\,(r^2\sin^2\theta\sin^2\phi)\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi$$

$$=2\cdot\frac{1}{6}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\sin^3\theta\cos\theta\sin^2\phi\,d\theta\,d\phi,$$

on integrating with respect to r.

[Note that the order of integration is immaterial because the limits of r, θ and ϕ are all constants].

$$=\frac{1}{3}\cdot\frac{2}{4\cdot 2}\int_{0}^{2\pi}\sin^2\phi\,d\phi,\text{ on integrating with respect to }\theta$$

$$=\frac{1}{12}\cdot 4\int_{0}^{\pi/2}\sin^2\phi\,d\phi=\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{12}.$$

Ex. 33. *By converting the surface integral into a volume integral evaluate*

$$\iint_S (x^3\,dy\,dz+y^3\,dz\,dx+z^3\,dx\,dy),$$

where S is the surface of the sphere $x^2+y^2+z^2=1$. **[Bombay 1970]**

Solution. By divergence theorem, we have

$$\iint_S (F_1\,dy\,dz+F_2\,dz\,dx+F_3\,dx\,dy)$$

$$=\iiint_V\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)dx\,dy\,dz,$$

where V is the volume enclosed by S.

Here $F_1=x^3$, $F_2=y^3$, $F_3=z^3$.

$$\therefore\ \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=3\,(x^2+y^2+z^2).$$

$\therefore$ the given surface integral

$$=\iiint_V 3\,(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$=3\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}r^2r^2\sin\theta\,dr\,d\theta\,d\phi,$$ changing to polar spherical coordinates

$$=3\times 2\pi\times 2\times\frac{1}{5}=\frac{12\pi}{5}.$$

Ex. 34. *Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2=x^2+y^2$ and the plane $z=4$, if*

$$\mathbf{F}=4xz\,\mathbf{i}+xyz^2\,\mathbf{j}+3z\,\mathbf{k}.$$

Solution. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iiint_V \operatorname{div}\mathbf{F}\,dV,$$

where V is the volume enclosed by S.

Here $\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}(4xz)+\frac{\partial}{\partial y}(xyz^2)+\frac{\partial}{\partial z}(3z)=4z+xz^2+3.$

Also V is the region bounded by the surfaces

$$z=0,\ z=4,\ z^2=x^2+y^2.$$

Therefore $\iiint_V \operatorname{div}\mathbf{F}\,dV=\iiint_V (4z+xz^2+3)\,dx\,dy\,dz$

$$=\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}}(4z+xz^2+3)\,dx\,dy\,dz$$

$$=2\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=0}^{\sqrt{(z^2-y^2)}}(4z+3)\,dx\,dy\,dz,$$

since $\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}}x\,dx=0$

$$=2\int_{z=0}^{4}\int_{y=-z}^{z}(4z+3)\sqrt{(z^2-y^2)}\,dy\,dz,$$

on integrating with respect to x

$$=4\int_{z=0}^{4}\int_{y=0}^{z}(4z+3)\sqrt{(z^2-y^2)}\,dy\,dz$$

$$=4\int_{z=0}^{4}(4z+3)\left[\frac{y}{2}\sqrt{(z^2-y^2)}+\frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$=4\int_0^4(4z+3)\left[\frac{z^2}{2}\sin^{-1}1\right]dz=\pi\int_0^4(4z^3+3z^2)\,dz$$

$$=\pi\left[z^4+z^3\right]_0^4=\pi\,(256+64)=320\pi.$$

Ex. 35. *Show that $\iint_S (x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\cdot\mathbf{n}\,dS$ vanishes where S denotes the surface of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution. We have by divergence theorem

$$\iint_S (x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\cdot\mathbf{n}\,dS$$

$$=\iiint_V \operatorname{div}(x^2\,\mathbf{i}+y^2\,\mathbf{j}+z^2\,\mathbf{k})\,dV,$$ where V is the volume enclosed by S

$$=\iiint_V (2x+2y+2z)\,dx\,dy\,dz$$

$$=2\int_{z=-c}^{c}\int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}}\int_{x=-a\sqrt{\{1-y^2/b^2-z^2/c^2\}}}^{a\sqrt{\{1-(y^2/b^2)-(z^2/c^2)\}}}(x+y+z)\,dx\,dy\,dz$$

$$=4\int_{z=-c}^{c}\int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}}(y+z)\sqrt{\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)}\,dy\,dz,$$

on integrating with respect to x

$\left[\text{Note that } \int_{-a}^{a} f(x)\,dx=0 \text{ if } f(-x)=-f(x) \text{ and } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(-x)=f(x)\right]$

$$=8\int_{z=-c}^{c}\int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} z\sqrt{\left(1-\frac{z^2}{c^2}-\frac{y^2}{b^2}\right)}\,dy\,dz$$

$$=8\int_{z=-c}^{c}\int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} \frac{z}{b}\sqrt{\left\{b^2\left(1-\frac{z^2}{c^2}\right)-y^2\right\}}\,dy\,dz$$

$$=\frac{8}{b}\int_{z=-c}^{c} z\left[\frac{y}{2}\sqrt{\left\{b^2\left(1-\frac{z^2}{c^2}\right)-y^2\right\}}+\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\frac{y}{b\sqrt{\{1-(z^2/c^2)\}}}\right]_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}}dz$$

$$=\frac{8}{b}\int_{z=-c}^{c} z\left[\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}1\right]dz=\frac{8}{b}\int_{z=-c}^{c} z\,\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\frac{\pi}{2}\,dz=0$$

Ex. 36. *If* $\mathbf{F}=(x^2+y-4)\,\mathbf{i}+3xy\,\mathbf{j}+(2xz+z^2)\,\mathbf{k}$, *evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ *where* S *is the surface of the sphere* $x^2+y^2+z^2=16$ *above the xy-plane.*

Solution. The surface $x^2+y^2+z^2=16$ meets the plane $z=0$ in a circle C given by $x^2+y^2=16$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the region bounded by S'

If $\mathbf{n}$ denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n}=-\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

Now by an application of Gauss divergence theorem, we have

$$\iint_{S'} \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=0 \qquad \text{[See Ex. 1 page 111]}$$

or $$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS+\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS-\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dS=0 \qquad [\because \text{ on } S_1,\ \mathbf{n}=-\mathbf{k}]$$

or $$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot\mathbf{k}\,dS.$$

Now $$\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=0\mathbf{i}-z\,\mathbf{j}+(3y-1)\,\mathbf{k}=-z\,\mathbf{j}+(3y-1)\,\mathbf{k}.$$

$\therefore$ $\operatorname{curl}\mathbf{F}\cdot\mathbf{k}=\{-z\,\mathbf{j}+(3y-1)\,\mathbf{k}\}\cdot\mathbf{k}=3y-1.$

$\therefore$ $$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}(3y-1)\,dS$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}(3r\sin\theta-1)\,r\,d\theta\,dr,$$ changing to polars

[Note that S_1 is a circle in xy plane with centre origin and radius 4]

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}3r^2\sin\theta\,d\theta\,dr-\int_{\theta=0}^{2\pi}\int_{r=0}^{4}r\,d\theta\,dr$$

$$=0-\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^4 d\theta \qquad \left[\because \int_{\theta=0}^{2\pi}\sin\theta\,d\theta=0\right]$$

$$=-8\left[\theta\right]_0^{2\pi}=-16\pi.$$

Ex. 37. *Evaluate* $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, *where* $\mathbf{A}=(x-z)\,\mathbf{i}+(x^3+yz)\,\mathbf{j}-3xy^2\,\mathbf{k}$ *and* S *is the surface of the cone* $z=2-\sqrt{(x^2+y^2)}$ *above the xy-plane.* **[Meerut 1974]**

Solution. The surface $z=2-\sqrt{(x^2+y^2)}$ meets the xy-plane in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. By application of divergence theorem, we have

$$\iint_{S'}\operatorname{curl}\mathbf{A}\cdot\mathbf{n}\,dS=0 \qquad \text{[See Ex. 1 page 111]}$$

or $$\iint_S \operatorname{curl}\mathbf{A}\cdot\mathbf{n}\,dS+\iint_{S_1}\operatorname{curl}\mathbf{A}\cdot\mathbf{n}\,dS=0$$

or $$\iint_S \operatorname{curl}\mathbf{A}\cdot\mathbf{n}\,dS=\iint_{S_1}\operatorname{curl}\mathbf{A}\cdot\mathbf{k}\,dS \qquad [\because \text{ on } S_1,\ \mathbf{n}=-\mathbf{k}]$$

Now curl $\mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x-z & x^3+yz & -3xy^2 \end{vmatrix}$

$$=\mathbf{i}\,(-6xy-y)+\mathbf{j}\,(-1+3y^2)+\mathbf{k}\,(3x^2-0).$$

$\therefore$ curl $\mathbf{A}\cdot\mathbf{k}=3x^2$.

$$\therefore \quad \iint_S \text{curl } \mathbf{A}\cdot\mathbf{n}\; dS=\iint_{S_1} 3x^2\, dS$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\cos^2\theta\; r\, d\theta\, dr, \text{ changing to polars}$$

$$=3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\cos^2\theta\, d\theta\, dr=3\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2\cos^2\theta\, d\theta$$

$$=12\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$=12\times 4\int_0^{\pi/2}\cos^2\theta\, d\theta=48\times\frac{1}{2}\times\frac{\pi}{2}=12\pi.$$

Ex. 38. *Evaluate* $\iint_S (ax^2+by^2+cz^2)\, dS$ *over the sphere* $x^2+y^2+z^2=1$ *using the divergence theorem.*

Solution. Let us first put the integral

$$\iint_S (ax^2+by^2+cz^2)\, dS \text{ in the form}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\; dS,$$

where $\mathbf{n}$ is unit normal vector to S.

The normal vector to $\phi\,(x, y, z)\equiv x^2+y^2+z^2-1=0$ is

$$=\nabla\phi=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

$$\therefore \quad \mathbf{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{[4\,(x^2+y^2+z^2)]}}$$

$$=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \qquad [\because \quad x^2+y^2+z^2=1, \text{ on } S]$$

Now we are to choose $\mathbf{F}$ such that

$$\mathbf{F}\cdot\mathbf{n}=\mathbf{F}\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=ax^2+by^2+cz^2.$$

Obviously $\quad \mathbf{F}=ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$.

Now $\iint_S (ax^2+by^2+cz^2)\, dS$

$$=\iint_S \mathbf{F}\cdot\mathbf{n}\; dS, \text{ where } \mathbf{F}=ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$$

$$=\iiint_V \text{div } \mathbf{F}\; dV, \text{ by divergence theorem}$$

$$=\iiint_V (a+b+c)\, dV \qquad [\because \quad \text{div } \mathbf{F}=a+b+c]$$

$$=(a+b+c)\iiint_V dV=(a+b+c)\, V$$

$=(a+b+c)\,\frac{4}{3}\pi$, since the volume V enclosed by the sphere S of unit radius is $\frac{4}{3}\pi$.

Ex. 39. Gauss's theorem. *Let S be a closed surface and let $\mathbf{r}$ denote the position vector of any point (x, y, z) measured from an origin O. Then*

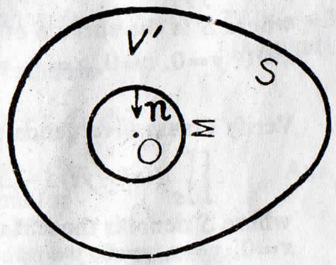
$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\, dS$$

is equal to (i) zero if O lies outside S; (ii) 4π if O lies inside S.

Proof. (*i*) *When origin O is outside S.* In this case $\mathbf{F}=\dfrac{\mathbf{r}}{r^3}$ is continuously differentiable throughout the region V enclosed by S. Hence by divergence theorem, we have

$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\; dS=\iiint_V \text{div}\left(\frac{\mathbf{r}}{r^3}\right) dV=0, \text{ since div}\left(\frac{\mathbf{r}}{r^3}\right)=0.$$

(*ii*) *When origin O is inside S.* In this case divergence theorem cannot be applied to the region V enclosed by S since $\mathbf{F}=\dfrac{\mathbf{r}}{r^3}$ has a point of discontinuity at the origin. To remove this difficulty let us enclose the origin by a small sphere Σ of radius ϵ.



The function $\mathbf{F}$ is continuously differentiable at the points of the region V' enclosed between S and Σ. Therefore applying divergence theorem for this region V', we have

$$\iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\; dS=\iint_\Sigma \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\; d\Sigma$$

$$=\iiint_{V'} \text{div}\left(\frac{\mathbf{r}}{r^3}\right) dV'=0, \text{ since div}\left(\frac{\mathbf{r}}{r^3}\right)=0.$$

$$\therefore \quad \iint_S \frac{\mathbf{r}}{r^3}\cdot\mathbf{n}\; dS=-\iint_\Sigma \left(\frac{\mathbf{r}}{r^3}\right)\cdot\mathbf{n}\; d\Sigma.$$

Now on the sphere Σ, the outward drawn normal $\mathbf{n}$ is directed towards the centre. Therefore on Σ, we have

$$\mathbf{n}=-\frac{\mathbf{r}}{\epsilon}.$$

$$\therefore \quad -\iint_{\Sigma} \frac{\mathbf{r}}{r^3} \cdot \mathbf{n}\, d\Sigma = -\iint_{\Sigma} \frac{\mathbf{r}}{\epsilon^3} \cdot \left(-\frac{\mathbf{r}}{\epsilon}\right) d\Sigma, \text{ since on } \Sigma, r=\epsilon$$

$$= \iint_{\Sigma} \frac{\mathbf{r}^2}{\epsilon^4}\, d\Sigma = \iint_{\Sigma} \frac{\epsilon^2}{\epsilon^4}\, d\Sigma = \frac{1}{\epsilon^2} \iint_{\Sigma} d\Sigma = \frac{1}{\epsilon^2}\, 4\pi\epsilon^2 = 4\pi.$$

Hence $\iint_S \frac{\mathbf{r}}{r^3} \cdot \mathbf{n}\, dS = 4\pi.$

Exercises

1. Verify divergence theorem for $\mathbf{F}=4xz\mathbf{i}-y^2\mathbf{j}+yz\mathbf{k}$ taken over the cube bounded by

$$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1.$$

[**Hint.** Proceed as in Ex. 24. Here we shall have

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \frac{3}{2}.$$

The six surface integrals will come out to be 2, 0, −1, 0, $\frac{1}{2}$ and 0. Their sum is $=\frac{3}{2}$.

Hence the theorem is verified].

2. Evaluate, by Green's theorem in space (*i.e.*, Gauss divergence theorem), the integral

$$\iint_S 4xz\,dy\,dz - y^2\,dz\,dx + yz\,dx\,dy,$$

where S is the surface of the cube bounded by

$x=0,\ y=0,\ z=0,\ x=1,\ y=1,\ z=1.$ **[Meerut 1974; Kanpur 77]**

Ans. $\frac{3}{2}$.

3. Verify Gauss divergence theorem to show that

$$\iint_S \{(x^3-yz)\,\mathbf{i}-2x^2y\mathbf{j}+2\mathbf{k}\} \cdot \mathbf{n}\, dS = \tfrac{1}{3}a^5,$$

where S denotes the surface of the cube bounded by the planes $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a.$

[Rohilkhand 1979; Agra 77]

4. Evaluate $\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \cdot \mathbf{n}\, dS$ where S denotes the surface of the cube bounded by the planes $x=0,\ x=a,\ y=0,\ y=a,\ z=0,\ z=a$ by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly.

[Agra 1979]

[**Hint.** Here $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \text{div } \mathbf{F}\, dV$$

$$= \iiint_V 3\,dV = 3V = 3a^3, \text{ as } V=a^3=\text{the volume of the cube}].$$

5. Evaluate by divergence theorem the integral

$$\iint_S xz^2\,dy\,dz + (x^2y-z^3)\,dz\,dx + (2xy+y^2z)\,dx\,dy,$$

where S is the entire surface of the hemispherical region bounded by $z=\sqrt{(a^2-x^2-y^2)}$ and $z=0.$ **[Meerut 1974]**

[**Hint.** Proceed as in Ex. 32]. **Ans.** $\frac{2\pi a^5}{5}$.

6. By using Gauss divergence theorem, evaluate

$$\iint_S (x\mathbf{i}+y\mathbf{j}+z^2\mathbf{k}) \cdot \mathbf{n}\, dS$$

where S is the closed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1.$ **[Agra 1973]**

[**Hint.** Proceed as in Ex. 34]. **Ans.** $7\pi/6$.

7. Use divergence theorem to find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ for the vector $\mathbf{F}=x\mathbf{i}-y\mathbf{j}+2z\mathbf{k}$ over the sphere $x^2+y^2+(z-1)^2=1.$ **Ans.** $8\pi/3$.

8. If $\mathbf{F}=ax\mathbf{i}+by\mathbf{j}+cz\mathbf{k}$, where a, b, c are constants, show that

$$\iint_S (\mathbf{n} \cdot \mathbf{F})\, dS = \frac{4\pi}{3}(a+b+c),$$

S being the surface of the sphere $(x-1)^2+(y-2)^2+(z-3)^2=1.$

[Gauhati 1971]

9. Use divergence theorem to evaluate

$$\iint_S [x\,dy\,dz + y\,dz\,dx + z\,dx\,dy],$$

where S is the surface $x^2+y^2+z^2=1.$ **Ans.** 4π.

19. Verify the divergence theorem for

$$\mathbf{F}=4x\mathbf{i}-2y^2\mathbf{j}+z^2\mathbf{k}$$

taken over the region bounded by the surfaces

$x^2+y^2=4,\ z=0,\ z=3.$ **[Allahabad 1978]**

[**Hint.** Show that each of the two integrals is $=84\pi$].

11. Verify divergence theorem for

$$\mathbf{F}=2x^2y\mathbf{i}-y^2\mathbf{j}+4xz^2\mathbf{k}$$

taken over the region in the first octant bounded by

$y^2+z^2=9$ and $x=2.$ **[Kanpur 1976]**

[**Hint.** Show that each of the two integrals is $=180$].

12. Verify divergence theorem for the function $\mathbf{F}=y\mathbf{i}+x\mathbf{j}+z^2\mathbf{k}$ over the cylindrical region bounded by

$x^2+y^2=a^2,\ z=0$ and $z=h.$ **[Kanpur 1975; Allahabad 79]**

13. If $\mathbf{F}=y\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **[Kanpur 1980]**

[**Hint.** Proceed as in Ex. 36. Here (curl $\mathbf{F}$)$\cdot\mathbf{k}=-z=0$ over the surface S_1 bounded by the circle $x^2+y^2=a^2$, $z=0$]. **Ans.** 0.

14. Evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, dS$, where

$\mathbf{A}=[xye^z+\log(z+1)-\sin x]\,\mathbf{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane. **Ans.** 0.

15. Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$, where

$\mathbf{F}=(x^2+y-4)\,\mathbf{i}+3xy\mathbf{j}+(2xz+z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z=4-(x^2+y^2)$ above the xy-plane. **Ans.** -4π.

16. Compute

(i) $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\, dS$, and

(ii) $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2+by^2+cz^2=1$.

Ans. (i) $\dfrac{4\pi\,(a+b+c)}{3\sqrt{(abc)}}$, (ii) $\dfrac{4\pi}{\sqrt{(abc)}}$.

17. Evaluate $\iint_S (x^2+y^2)\, dS$, where S is the surface of the cone $z^2=3\,(x^2+y^2)$ bounded by $z=0$ and $z=3$. **Ans.** 9π.

18. Prove that

$$\int_V \mathbf{f}\cdot\text{curl}\,\mathbf{F}\, dV=\int_S \mathbf{F}\times\mathbf{f}\cdot d\mathbf{S}+\int_V \mathbf{F}\cdot\text{curl}\,\mathbf{f}\, dV.$$

[**Hint.** Apply divergence theorem for the vector function $\mathbf{F}\times\mathbf{f}$].

19. Let $\mathbf{r}$ denote the position vector of any point (x, y, z) measured from an origin O and let $r=|\mathbf{r}|$.

Evaluate $\iint_S \dfrac{\mathbf{r}}{r^3}\cdot\mathbf{n}\, dS$ where S is the sphere $x^2+y^2+z^2=a^2$.

Ans. 4π. **[Calicut 1975]**

§ 9. Stoke's Theorem. *Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then*

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS=\iint_S (\text{curl}\,\mathbf{F})\cdot d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal $\mathbf{n}$ *to S, has the surface on the left.*

[**Agra 1970; Nagpur 70; Meerut 77, 80, 82, 84 (P), 85**]

Note. $\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C \left(\mathbf{F}\cdot\dfrac{d\mathbf{r}}{ds}\right) ds=\oint_C (\mathbf{F}\cdot\mathbf{t})\, ds$, where $\mathbf{t}$ is unit tangent vector to C. Therefore $\mathbf{F}\cdot\mathbf{t}$ is the component of $\mathbf{F}$ in the direction of the tangent vector of C. Also $(\nabla\times\mathbf{F})\cdot\mathbf{n}$ is the component of curl $\mathbf{F}$ in the direction of outward drawn normal vector $\mathbf{n}$ of S. Therefore in words Stoke's theorem may be stated as follows :

The line integral of the tangential component of vector $\mathbf{F}$ *taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of* $\mathbf{F}$ *taken over any surface S having C as its boundary.*

Cartesian equivalent of Stoke's theorem.

Let $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$. Let outward drawn normal vector $\mathbf{n}$ of S make angles α, β, γ with positive directions of x, y, z axes.

Then $\mathbf{n}=\cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k}$.

Also $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$

$$=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore\quad (\nabla\times\mathbf{F})\cdot\mathbf{n}=\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\cos\alpha+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\cos\beta+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\cos\gamma.$$

Also $\mathbf{F}\cdot d\mathbf{r}=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$
$=F_1dx+F_2dy+F_3dz.$

$\therefore$ Stoke's theorem can be written as

$$\oint_C F_1dx+F_2dy+F_3dz=\iint_S\left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\cos\alpha+\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\cos\beta+\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Proof of Stoke's theorem. Let S be a surface which is such that its projections on the xy, yz and xz planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in the forms

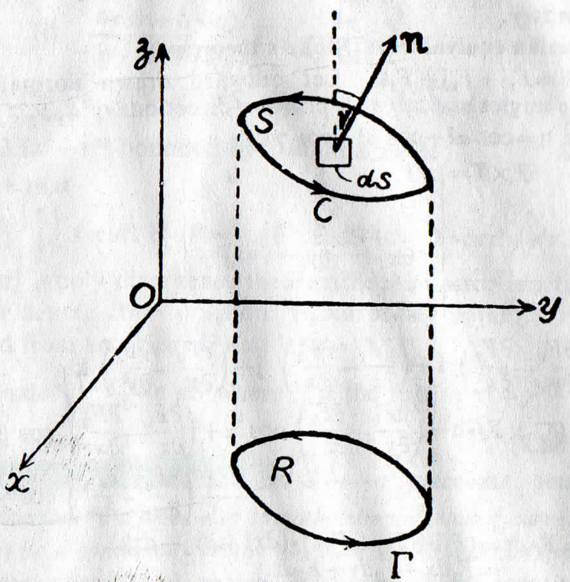
$$z=f(x, y),\ y=g(x, z),\ x=h(z, y)$$

where f, g, h are continuous functions and have continuous first partial derivatives.

Consider the integral $\iint_S [\nabla \times (F_1\mathbf{i})]\cdot\mathbf{n}\, dS$.

We have
$$\nabla \times (F_1\mathbf{i})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$$
$$=\frac{\partial F_1}{\partial z}\mathbf{j}-\frac{\partial F_1}{\partial y}\mathbf{k}.$$

$$\therefore\ [\nabla \times (F_1\mathbf{i})]\cdot\mathbf{n}=\left(\frac{\partial F_1}{\partial z}\mathbf{j}\cdot\mathbf{n}-\frac{\partial F_1}{\partial y}\mathbf{k}\cdot\mathbf{n}\right)=\frac{\partial F_1}{\partial z}\cos\beta-\frac{\partial F_1}{\partial y}\cos\gamma.$$



$$\therefore\ \iint_S [\nabla \times (F_1\mathbf{i})]\cdot\mathbf{n}\, dS=\iint_S\left(\frac{\partial F_1}{\partial z}\cos\beta-\frac{\partial F_1}{\partial y}\cos\gamma\right)dS.$$

We shall prove that

$$\iint_S\left(\frac{\partial F_1}{\partial z}\cos\beta-\frac{\partial F_1}{\partial y}\cos\gamma\right)dS=\oint_C F_1\, dx.$$

Let R be the orthogonal projection of S on the xy-plane and

let Γ be its boundary which is oriented as shown in the ugure. Using the representation $z=f(x, y)$ of S, we may write the line integral over C as a linc integral over Γ. Thus

$$\oint_C F_1(x, y, z)\, dx=\oint_\Gamma F_1[x, y, f(x, y)]\, dx$$
$$=\oint_\Gamma\{F_1[x, y, f(x, y)]\, dx+0\, dy\}$$
$$=-\iint_R\frac{\partial F_1}{\partial y}\, dx\, dy,$$ by Green's theorem in plane for the region R.

But
$$\frac{\partial F_1[x, y, f(x, y)]}{\partial y}=\frac{\partial F_1(x, y, z)}{\partial y}+\frac{\partial F_1(x, y, z)}{\partial z}\frac{\partial f}{\partial y}\qquad [\because\ z=f(x, y)]$$

$$\therefore\ \oint_C F_1(x, y, z)\, dx=-\iint_R\left(\frac{\partial F_1}{\partial y}+\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right)dx\, dy \qquad \text{...(1)}$$

Now the equation $z=f(x, y)$ of the surface S can be written as
$$\phi(x, y, z)\equiv z-f(x, y)=0.$$

We have $\text{grad}\ \phi=-\frac{\partial f}{\partial x}\mathbf{i}-\frac{\partial f}{\partial y}\mathbf{j}+\mathbf{k}$.

Let $|\text{grad}\ \phi|=a$.

Since grad ϕ is normal to S, therefore, we get
$$\mathbf{n}=\pm\frac{\text{grad}\ \phi}{a}.$$

But the components of both $\mathbf{n}$ and grad ϕ in positive direction of z-axis are positive. Therefore
$$\mathbf{n}=+\frac{\text{grad}\ \phi}{a}$$

or $$\cos\alpha\mathbf{i}+\cos\beta\mathbf{j}+\cos\gamma\mathbf{k}=-\frac{1}{a}\frac{\partial f}{\partial x}\mathbf{i}-\frac{1}{a}\frac{\partial f}{\partial y}\mathbf{j}+\frac{1}{a}\mathbf{k}.$$

$$\therefore\ \cos\alpha=-\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta=-\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma=\frac{1}{a}.$$

Now $$dS=\frac{dx\, dy}{\cos\gamma}=a\, dx\, dy.$$

$$\therefore\ \iint_S\left(\frac{\partial F_1}{\partial z}\cos\beta-\frac{\partial F_1}{\partial y}\cos\gamma\right)dS$$
$$=\iint_R\left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right)-\frac{\partial F_1}{\partial y}\frac{1}{a}\right]a\, dxdy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial f}{\partial y} \right) dx\, dy. \quad \ldots(2)$$

From (1) and (2), we get

$$\oint_C F_1\, dx = \iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) dS$$

$$= \iint_S [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n}\, dS. \quad \ldots(3)$$

Similarly, by projections on the other coordinate planes, we get

$$\oint_C F_2\, dy = \iint_S [\nabla \times (F_2 \mathbf{j})] \cdot \mathbf{n}\, dS \quad \ldots(4)$$

$$\oint_C F_3\, dz = \iint_S [\nabla \times (F_3 \mathbf{k})] \cdot \mathbf{n}\, dS \quad \ldots(5)$$

Adding (3), (4), (5), we get

$$\oint_C (F_1\, dx + F_2\, dy + F_3\, dz) = \iint_S [\nabla \times (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k})] \cdot \mathbf{n}\, dS$$

or $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS.$$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's theorem will be true provided S can be subdivided into surfaces $S_1, S_2, .., S_k$ with boundaries $C_1, C_2, .., C_k$ which do satisfy the restrictions. Stoke's theorem holds for each such surface. The sum of surface integrals over $S_1, S_2, .., S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, ..., C_k$ will give us line integral over C.

Note. Green's theorem in plane is a special case of Stoke's theorem. If R is a region in the xy-plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dR = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

This is nothing but a special case of Stoke's theorem because here $\mathbf{k} = \mathbf{n} =$ outward drawn unit normal to the surface of region R.

SOLVED EXAMPLES

Ex. 1. *Prove that* $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0.$

Solution. By Stoke's theorem

$$\oint_C \mathbf{r} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{r}) \cdot \mathbf{n}\, dS = 0, \text{ since curl } \mathbf{r} = \mathbf{0}.$$

Ex. 2. *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}.$

Solution. By Stoke's theorem, we have

$$\oint_C \nabla(\phi\psi) \cdot d\mathbf{r} = \iint_S [\text{curl grad } (\phi\, \psi)] \cdot \mathbf{n}\, dS$$

$$= 0, \text{ since curl grad } (\phi\, \psi) = \mathbf{0}.$$

But $\nabla\ (\phi\, \psi) = \phi \nabla \psi + \psi \nabla \phi.$

$$\therefore \quad \oint_C (\phi \nabla \psi + \psi \nabla \phi) \cdot d\mathbf{r} = 0$$

or $$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}.$$

Ex. 3. (a) *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n} dS.$

Solution. By Stoke's theorem, we have

$$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi \nabla \psi)] \cdot \mathbf{n}\, dS$$

$$= \iint_S [\nabla \phi \times \nabla \psi + \phi \text{ curl grad } \psi] \cdot \mathbf{n}\, dS$$

$$= \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n}\, dS, \text{ since curl grad } \psi = \mathbf{0}.$$

Ex. 3. (b) *Show that* $\int_C \phi \nabla \phi \cdot d\mathbf{r} = 0$, *$C$ being a closed curve.*

Solution. Applying Stoke's theorem to the vector function $\phi \nabla \phi$, we have

$$\int_C (\phi \nabla \phi) \cdot d\mathbf{r} = \iint_S [\text{curl } (\phi \nabla \phi)] \cdot \mathbf{n}\, dS$$

$$=\iint_S [\phi \text{ curl } \nabla\phi+\nabla\phi\times\nabla\phi]\cdot\mathbf{n}\, dS$$

$$=\iint_S \mathbf{0}\cdot\mathbf{n}\, dS \quad [\because \text{ curl } \nabla\phi=\mathbf{0} \text{ and } \nabla\phi\times\nabla\phi=\mathbf{0}]$$

$$=0.$$

Ex. 4. *Prove that* $\oint_C \phi\, d\mathbf{r}=\iint_S d\mathbf{S}\times\nabla\phi.$

[Kanpur 1977]

Solution. Let $\mathbf{A}$ be any arbitrary constant vector. Let $\mathbf{F}=\phi\mathbf{A}$. Applying Stoke's theorem for $\mathbf{F}$, we get

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S \{\nabla\times(\phi\mathbf{A})\}\cdot\mathbf{n}\, dS=\iint_S [\nabla\phi\times\mathbf{A}+\phi \text{ curl } \mathbf{A}]\cdot d\mathbf{S}$$

$$=\iint_S (\nabla\phi\times\mathbf{A})\cdot d\mathbf{S}, \text{ since curl } \mathbf{A}=\mathbf{0}.$$

$$\therefore \quad \oint_C (\phi\mathbf{A})\cdot d\mathbf{r}=\iint_S \mathbf{A}\cdot(d\mathbf{S}\times\nabla\phi)$$

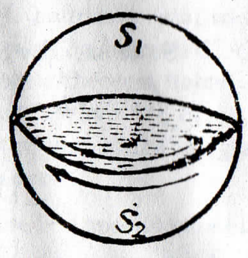
or $\quad \mathbf{A}\cdot\oint_C \phi\, d\mathbf{r}=\mathbf{A}\cdot\iint_S d\mathbf{S}\times\nabla\phi$ or $\mathbf{A}\cdot\left[\oint_C \phi\, d\mathbf{r}-\iint_S d\mathbf{S}\times\nabla\phi\right]=0.$

Since $\mathbf{A}$ is an arbitrary vector, therefore we must have

$$\oint_C \phi\, d\mathbf{r}=\iint_S d\mathbf{S}\times\nabla\phi.$$

Ex. 5. *By Stoke's theorem prove that div curl* $\mathbf{F}=0$.

Solution. Let V be any volume enclosed by a closed surfaces. Then by divergence theorem



$$\iiint_V \nabla\cdot(\text{curl } \mathbf{F})\, dV$$

$$=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS.$$

Divide the surface S into two portions S_1 and S_2 by a closed curve C. Then

$$\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS=\iint_{S_1} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS_1$$

$$+\iint_{S_2} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS_2. \quad \ldots(1)$$

By Stoke's theorem right hand side of (1) is

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}-\oint_C \mathbf{F}\cdot d\mathbf{r}=0.$$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \quad \iiint_V \nabla\cdot(\text{curl } \mathbf{F})\, dV=0.$$

Now this equation is true for all volume elements V. Therefore we have $\nabla\cdot(\text{curl } \mathbf{F})=0$

or $\quad$ div curl $\mathbf{F}=0$.

Ex. 6. *By Stoke's theorem prove that curl grad* $\phi=\mathbf{0}$.

Solution. Let S be any surface enclosed by a simple closed curve C. Then by stoke's theorem, we have

$$\iint_S (\text{curl grad } \phi)\cdot\mathbf{n}\, dS=\oint_C \text{grad } \phi\cdot d\mathbf{r}.$$

Now grad $\phi\cdot d\mathbf{r}=\left(\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$

$$=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi.$$

$\therefore \quad \oint_C \text{grad } \phi\cdot d\mathbf{r}=\oint_C d\phi=\Big[\phi\Big]_A^A$, where A is any point on C

$$=0.$$

Therefore we have $\iint_S (\text{curl grad } \phi)\cdot\mathbf{n}\, dS=0.$

Now this equation is true for all surface elements S.

Therefore we have, curl grad $\phi=\mathbf{0}$.

Ex. 7. *Verify Stoke's theorem for* $\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$ *where S is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and C is its boundary.* **[Bombay 1970; Meerut 81; Agra 79; Rohilkhand 77]**

Solution. The boundary C of S is a circle in the xy plane of radius unity and centre origin. The equations of the curve C are $x^2+y^2=1$, $z=0$. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C (y\,dx+z\,dy+x\,dz)=\oint_C y\,dx, \text{ since on } C,\ z=0 \text{ and } dz=0$$

$$=\int_0^{2\pi}\sin t\,\frac{dx}{dt}\,dt=\int_0^{2\pi}-\sin^2 t\,dt$$

$$=-\tfrac{1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt=-\tfrac{1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$=-\pi. \qquad \text{...(1)}$$

Now let us evaluate $\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS$. We have curl $\mathbf{F}$

$$=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & z & x\end{vmatrix}=-\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl } \mathbf{F}\cdot\mathbf{k}\,dS \text{ [See Ex. 36 Page 126]}$$

$$=\iint_{S_1}(-\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot\mathbf{k}\,dS=\iint_{S_1}(-1)\,dS=-\iint_{S_1}dS=-S_1.$$

But S_1=area of a circle of radius $1=\pi(1)^2=\pi$.

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS=-\pi. \qquad \text{...(2)}$$

Hence from (1) and (2), the theorem is verified.

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Ex. 8. *Verify Stoke's theorem for* $\mathbf{F}=(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}$, *where S is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and C is its boundary.*

[Kanpur 1970; Rohilkhand 78; Allahabad 78; Agra 73, 76, 80]

Solution. The boundary C of S is a circle in the xy plane of radius unity and centre origin. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C[(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}]\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C[(2x-y)\,dx-yz^2\,dy-y^2z\,dz]$$

$$=\oint_C(2x-y)\,dx, \text{ since } z=0 \text{ and } dz=0$$

$$=\int_0^{2\pi}(2\cos t-\sin t)\frac{dx}{dt}\,dt=-\int_0^{2\pi}(2\cos t-\sin t)\sin t\,dt$$

$$=-\int_0^{2\pi}[\sin 2t-\tfrac{1}{2}(1-\cos 2t)]\,dt=-\left[-\frac{\cos 2t}{2}-\tfrac{1}{2}t+\tfrac{1}{2}\,\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$=-[(-\tfrac{1}{2}+\tfrac{1}{2})-\tfrac{1}{2}(\pi-0)+\tfrac{1}{4}(0-0)]=\pi. \qquad \text{...(1)}$$

Also $(\nabla\times\mathbf{F})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}$

$$=(-2yz+2yz)\,\mathbf{i}-(0-0)\,\mathbf{j}+(0+1)\,\mathbf{k}=\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\iint_{S_1}(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dS$$

[by an application of divergence theorem, see Ex. 36, page 126]

$$=\iint_{S_1}\mathbf{k}\cdot\mathbf{k}\,dS=\iint_{S_1}dS=S_1=\pi. \qquad \text{...(2)}$$

Hence from (1) and (2), the theorem is verified.

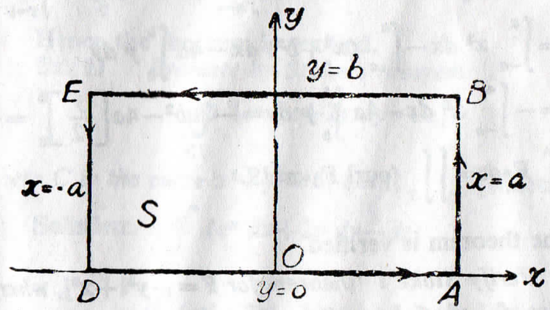
Ex. 9. *Verify Stoke's theorem for*

$$\mathbf{F}=(x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}$$

taken round the rectangle bounded by

$x=\pm a$, $y=0$, $y=b$. **[Meerut 1967]**

Solution. We have



$$\text{curl } \mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y^2 & -2xy & 0\end{vmatrix}$$

$$=(-2y-2y)\,\mathbf{k}=-4y\mathbf{k}.$$

Also $\mathbf{n}=\mathbf{k}$.

$$\therefore \quad \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS=\int_{y=0}^{b}\int_{x=-a}^{a}(-4y\mathbf{k})\cdot\mathbf{k}\, dx\, dy$$

$$=-4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy=-4\int_{y=0}^{b}\Big[xy\Big]_{x=-a}^{a} dy$$

$$=-4\int_{y=0}^{b} 2ay\, dy=-4\Big[ay^2\Big]_0^b=-4ab^2.$$

Also $\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C [(x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

$$=\oint_C [(x^2+y^2)\,dx-2xy\,dy]$$

$$=\int_{DA}[(x^2+y^2)\,dx-2xy\,dy]+\int_{AB}+\int_{BE}+\int_{ED}.$$

Along DA, $y=0$ and $dy=0$. Along AB, $x=a$ and $dx=0$.
Along BE, $y=b$ and $dy=0$. Along ED, $x=-a$ and $dx=0$.

$$\therefore \quad \oint_C \mathbf{F}\cdot d\mathbf{r}=\int_{x=-a}^{a} x^2\,dx+\int_{y=0}^{b}-2ay\,dy$$

$$+\int_{x=a}^{-a}(x^2+b^2)\,dx+\int_{y=b}^{0}2ay\,dy$$

$$=\int_{-a}^{a}x^2\,dx-\int_{-a}^{a}(x^2+b^2)\,dx-4a\int_0^b y\,dy$$

$$=-\int_{-a}^{a}x^2\,dx-4a\int_0^b y\,dy=-2ab^2-4a\left[\frac{y^2}{2}\right]_0^b=-4ab^2.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS.$

Hence the theorem is verified.

Ex. 10. *Verify Stoke's theorem for* $\mathbf{F}=-y^3\mathbf{i}+x^3\mathbf{j}$, *where* S *is the circular disc* $x^2+y^2\leqslant 1$, $z=0$.

Solution. The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C(-y^3\,\mathbf{i}+x^3\,\mathbf{j})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\oint_C(-y^3\,dx+x^3\,dy)=\int_{t=0}^{2\pi}\left\{-y^3\frac{dx}{dt}+x^3\frac{dy}{dt}\right\}dt$$

$$=\int_0^{2\pi}[-\sin^3 t\,(-\sin t)+\cos^3 t\,(\cos t)]\,dt$$

$$=\int_0^{2\pi}(\cos^4 t+\sin^4 t)\,dt=4\int_0^{\pi/2}(\cos^4 t+\sin^4 t)\,dt$$

$$=4\left\{\frac{3.1}{4.2}\frac{\pi}{2}+\frac{3.1}{4.2}\frac{\pi}{2}\right\}=\frac{3\pi}{2}.$$

Also $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ -y^3 & x^3 & 0\end{vmatrix}=(3x^2+3y^2)\,\mathbf{k}.$

Here $\mathbf{n}=\mathbf{k}$ because the surface S is the xy-plane.

$\therefore \quad (\nabla\times\mathbf{F})\cdot\mathbf{n}=(3x^2+3y^2)\,\mathbf{k}\cdot\mathbf{k}=3\,(x^2+y^2).$

$$\therefore \quad \iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=3\iint_S(x^2+y^2)\,dS$$

$$=3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2 r\,d\theta\,dr, \text{ changing to polars}$$

$$=\frac{3}{4}\int_0^{2\pi}d\theta=\frac{3}{4}(2\pi)=\frac{3\pi}{2}.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\frac{3\pi}{2}.$

Hence the theorem is verified.

Ex. 11. *Evaluate by Stoke's theorem*

$$\oint_C (e^x\,dx+2y\,dy-dz)$$

where C *is the curve* $x^2+y^2=4$, $z=2$. **[Meerut 1969; Agra 72]**

Solution. $\oint_C (e^x\,dx+2y\,dy-dz)$

$$=\oint_C(e^x\mathbf{i}+2y\mathbf{j}-\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{F}=e^x\mathbf{i}+2y\mathbf{j}-\mathbf{k}.$$

Now curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ e^x & 2y & -1\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS$$

$$=0, \text{ since curl } \mathbf{F}=0.$$

Ex. 12. *Evaluate by Stoke's theorem*

$$\oint_C (yz\, dx+xz\, dy+xy\, dz)$$

where C is the curve $x^2+y^2=1,\ z=y^2$. **[Kanpur 1980]**

Solution. Here $\mathbf{F}=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$.

$\therefore$ Curl $\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$

$$=(x-x)\,\mathbf{i}-(y-y)\,\mathbf{j}+(z-z)\,\mathbf{k}=0.$$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS$$

$$=0, \text{ since curl } \mathbf{F}=0.$$

Ex. 13. *Evaluate* $\oint_C (xy\, dx+xy^2\, dy)$ *by Stoke's theorem where C is the square in the xy-plane with vertices* $(1, 0), (-1, 0), (0, 1), (0, -1)$.

Solution. Here $\mathbf{F}=xy\mathbf{i}+xy^2\mathbf{j}$.

$\therefore$ curl $\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}=(y^2-x)\,\mathbf{k}.$

Also $\mathbf{n}=\mathbf{k}$.

$\therefore$ curl $\mathbf{F}\cdot\mathbf{n}=(y^2-x)\,\mathbf{k}\cdot\mathbf{k}=y^2-x.$

$\therefore$ The given line integral

$$=\iint_S (y^2-x)\, dS$$

$$=\int_{y=-1}^{1}\int_{x=-1}^{1} (y^2-x)\, dx\, dy=\int_{y=-1}^{1}\left[y^2x-\frac{x^2}{2}\right]_{x=-1}^{1} dy$$

$$=\int_{y=-1}^{1} 2y^2 dy=2\left[\frac{y^3}{3}\right]_{-1}^{1}=\frac{4}{3}.$$

Ex. 14. *Evaluate* $\oint_C \mathbf{F}\cdot d\mathbf{r}$ *by Stoke's theorem where* $\mathbf{F}=y^2\mathbf{i}+x^2\mathbf{j}-(x+z)\,\mathbf{k}$ *and C is the boundary of the triangle with vertices at* $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.

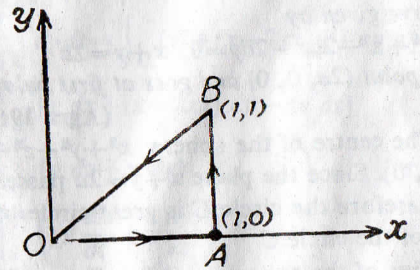
Solution. We have

Curl $\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}=0\mathbf{i}+\mathbf{j}+2\,(x-y)\,\mathbf{k}.$

Also we note that z-coordinate of each vertex of the triangle is zero. Therefore the triangle lies in the x-y plane. So $\mathbf{n}=\mathbf{k}$.

$\therefore$ Curl $\mathbf{F}\cdot\mathbf{n}=[\mathbf{j}+2\,(x-y)\,\mathbf{k}]\cdot\mathbf{k}=2\,(x-y).$

In the figure, we have only considered the x-y plane.



The equation of the line OB is $y=x$.

By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\, dS$$

$$=\int_{x=0}^{1}\int_{y=0}^{x} 2\,(x-y)\, dx\, dy=2\int_{x=0}^{1}\left[xy-\frac{y^2}{2}\right]_{y=0}^{x} dx$$

$$=2\int_{x=0}^{1}\left[x^2-\frac{x^2}{2}\right] dx=2\int_0^1 \frac{x^2}{2}\, dx=\int_0^1 x^2\, dx=\tfrac{1}{3}.$$

Ex. 15. *Evaluate by Stoke's theorem*

$$\oint_C (\sin z\, dx-\cos x\, dy+\sin y\, dz)$$

where C is the boundary of the rectangle

$0 \leqslant x \leqslant \pi,\ 0 \leqslant y \leqslant 1,\ z=3.$

Solution. Here $\mathbf{F}=\sin z\,\mathbf{i}-\cos x\,\mathbf{j}+\sin y\,\mathbf{k}$.

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}=\cos y\,\mathbf{i}+\cos z\,\mathbf{j}+\sin x\,\mathbf{k}.$$

Since the rectangle lies in the plane $z=3$, therefore $\mathbf{n}=\mathbf{k}$.

$\therefore$ curl $\mathbf{F}\cdot\mathbf{n}=(\cos y\mathbf{i}+\cos z\mathbf{j}+\sin x\mathbf{k})\cdot\mathbf{k}=\sin x$.

By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F}\cdot\mathbf{n})\,dS$$

$$=\int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\,dx\,dy=\int_{x=0}^{\pi} \sin x\,dx=2.$$

Ex. 16. *Apply Stoke's theorem to prove that*

$$\int_C (ydx+zdy+xdz)=-2\sqrt{2}\pi a^2$$

where C is the curve given by

$$x^2+y^2+z^2-2ax-2ay=0,\ x+y=2a$$

and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.

(Agra 1969; Meerut 82)

Solution. The centre of the sphere $x^2+y^2+z^2-2ax-2ay=0$ is the point $(a, a, 0)$. Since the plane $x+y=2a$ passes through the point $(a, a, 0)$, therefore the circle C is great circle of this sphere.

$\therefore$ Radius of the circle C

$=$radius of the sphere$=\sqrt{(a^2+a^2)}=a\sqrt{2}$.

Now $\int_C (ydx+zdy+xdz)=\int_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot d\mathbf{r}$

$$=\iint_S [\text{curl } (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})]\cdot\mathbf{n}\,dS,$$

where S is any surface of which circle C is boundary [Stoke's theorem].

$$\text{Now curl } (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$=-\mathbf{i}-\mathbf{j}-\mathbf{k}=-(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle C. Then a vector normal to S is grad $(x+y)=\mathbf{i}+\mathbf{j}$.

$\therefore$ $\mathbf{n}=$unit normal to $S=\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$.

$$\therefore \int_C (y\,dx+z\,dy+x\,dz)=\iint_S -(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\mathbf{j}\right)dS$$

$$=-\frac{2}{\sqrt{2}}\iint_S dS=-\frac{2}{\sqrt{2}}(\text{area of the circle of radius } a\sqrt{2})$$

$$=-\sqrt{2}\,(2\pi a^2).$$

Ex. 17. *Use Stoke's theorem to evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, *where $\mathbf{F}=y\mathbf{i}+(x-2xz)\mathbf{j}-xy\mathbf{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.*

Solution. The boundary C of the surface S is the circle $x^2+y^2=a^2$, $z=0$. Suppose $x=a\cos t$, $y=a\sin t$, $z=0$, $0 \leqslant t<2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [y\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}]\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)$$

$$=\int_C [y\,dx+(x-2xz)\,dy-xy\,dz]$$

$$=\int_C (y\,dx+x\,dy) \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi}\left(y\frac{dx}{dt}+x\frac{dy}{dt}\right)dt$$

$$=\int_0^{2\pi} [a\sin t\,(-a\sin t)+a\cos t\,(a\cos t)]\,dt$$

$$=a^2\int_0^{2\pi}(\cos^2 t-\sin^2 t)\,dt=a^2\int_0^{2\pi}\cos 2t\,dt=a^2\left[\frac{\sin 2t}{2}\right]_0^{2\pi}=0.$$

Ex. 18. *Evaluate the surface integral* $\iint_S$ *curl* $\mathbf{F}\cdot\mathbf{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid $z=1-x^2-y^2$ for which $z \geqslant 0$, and*

$\mathbf{F}=y\mathbf{i}+z\mathbf{j}+x\mathbf{k}$. **(Bombay 1970)**

Solution. The boundary C of the surface S is the circle $x^2+y^2=1$, $z=0$. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0 \leqslant t<2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS=\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_C (y\mathbf{i}+z\mathbf{j}+x\mathbf{k})\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz)=\int_C y\,dx+z\,dy+x\,dz$$

$$=\int_C y\,dx \quad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi} y\frac{dx}{dt}\,dt=\int_0^{2\pi}\sin t\,(-\sin t)\,dt=-\int_0^{2\pi}\sin^2 t\,dt$$

$$=-4\int_0^{\pi/2}\sin^2 t\,dt=-4\times\tfrac{1}{2}\times\frac{\pi}{2}=-\pi.$$

Ex. 19. *If* $\mathbf{F}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, *evaluate* $\iint \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$ *taken over the portion of the surface* $x^2+y^2+z^2-2ax+az=0$ *above the plane* $z=0$, *and verify Stoke's theorem.*

Solution. The surface $x^2+y^2+z^2-2ax+az=0$ meets the plane $z=0$ in the circle C given by $x^2+y^2-2ax=0,\ z=0$. The polar equation of the circle C lying in the xy-plane is $r=2a\cos\theta$, $0\leqslant\theta<\pi$. Also the equation $x^2+y^2-2ax=0$ can be written as $(x-a)^2+y^2=a^2$. Therefore the parametric equations of the circle C can be taken as

$$x=a+a\cos t,\ y=a\sin t,\ z=0,\ 0\leqslant t<2\pi.$$

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS.$$

Now curl $\mathbf{F}\cdot\mathbf{k}=$
$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}\cdot\mathbf{k}$$

$$=\left[\frac{\partial}{\partial x}(z^2+x^2-y^2)-\frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\mathbf{k}\cdot\mathbf{k}$$
$$[\because\ \mathbf{i}\cdot\mathbf{k}=\mathbf{j}\cdot\mathbf{k}=0]$$

$$=2(x-y).$$

$$\therefore\ \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS=\iint_{S_1}2(x-y)\,dS$$

$$=2\int_{\theta=0}^{\pi}\int_{r=0}^{2a\cos\theta}(r\cos\theta-r\sin\theta)\,r\,d\theta\,dr,$$

changing to polars

$$=2\int_{\theta=0}^{\pi}(\cos\theta-\sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta$$

$$=2\times\frac{8a^3}{3}\int_0^{\pi}(\cos\theta-\sin\theta)\cos^3\theta\,d\theta$$

$$=\frac{16a^3}{3}\int_0^{\pi}(\cos^4\theta-\cos^3\theta\sin\theta)\,d\theta$$

$$=\frac{16a^3}{3}\int_0^{\pi}\cos^4\theta\,d\theta\quad\left[\because\ \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta=0\right]$$

$$=2\times\frac{16a^3}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$=2\times\frac{16a^3}{3}\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}=2\pi a^3. \quad ...(1)$$

Also
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C (y^2+z^2-x^2)\,dx+(z^2+x^2-y^2)\,dy+(x^2+y^2-z^2)\,dz$$

$$=\int_C (y^2-x^2)\,dx+(x^2-y^2)\,dy\quad[\because\ \text{on } C,\ z=0 \text{ and } dz=0]$$

$$=\int_0^{2\pi}(x^2-y^2)\left(\frac{dy}{dt}-\frac{dx}{dt}\right)dt$$

$$=\int_0^{2\pi}[(a+a\cos t)^2-a^2\sin^2 t]\,(a\cos t+a\sin t)\,dt$$

$$=a^3\int_0^{2\pi}(1+\cos^2 t+2\cos t-\sin^2 t)(\cos t+\sin t)\,dt$$

$$=a^3\int_0^{2\pi}2\cos^2 t\,dt,\ \text{the other integrals vanish}$$

$$=2a^3\times4\int_0^{\pi/2}\cos^2 t\,dt=8a^3\times\tfrac{1}{2}\times\frac{\pi}{2}=2\pi a^3. \quad ...(2)$$

Comparing (1) and (2), we see that

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Hence Stoke's theorem is verified.

Ex. 20. *Prove that a necessary and sufficient condition that* $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ *for every closed curve* C *lying in a simply connected region* R *is that* $\nabla\times\mathbf{F}=\mathbf{0}$ *identically.*

Solution. Sufficiency. Suppose R is simply connected and curl $\mathbf{F}=\mathbf{0}$ everywhere in R. Let C be any closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS=0.$$

Necessity. Suppose $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ for every closed path C and assume that $\nabla\times\mathbf{F}\neq 0$ at some point A.

Then taking $\nabla\times\mathbf{F}$ as continuous, there must exist a region with A as an interior point, where $\nabla\times\mathbf{F}\neq\mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point is in the same direction as $\nabla\times\mathbf{F}$, *i.e.* $\nabla\times\mathbf{F}=\lambda\mathbf{n}$ where λ is a positive constant. Let C be the boundary of S. Then by Stoke's theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS=\iint_S \lambda\mathbf{n}\cdot\mathbf{n}\, dS$$

$$=\lambda S>0.$$

This contradicts the hypothesis that $\oint_C \mathbf{F}\cdot d\mathbf{r}=0$ for every closed path C. Therefore we must have $\nabla\times\mathbf{F}=\mathbf{0}$ everywhere in R.

Exercises

1. Verify Stoke's theorem for the function
$$\mathbf{F}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$$
where curve is the unit circle in the xy-plane bounding the hemisphere $z=\sqrt{(1-x^2-y^2)}$.
[Agra 1975; Rohilkhand 81; Kanpur 78]
[**Hint.** Proceed as in Ex. 7 Page 139. Show that
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\pi=\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS]$$

2. Verify Stoke's theorem for $\mathbf{A}=2y\mathbf{i}+3x\mathbf{j}-z^2\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2=9$ and C is its boundary. **[Meerut 1975]**

3. Verify Stoke's theorem for the vector $\mathbf{q}=z\mathbf{i}+x\mathbf{j}+y\mathbf{k}$ taken over the half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy-plane. **[Gauhati 1973]**

4. Verify Stoke's theorm for the function
$$\mathbf{F}=x^2\mathbf{i}+xy\mathbf{j}$$
integrated along the rectangle, in the plane $z=0$, whose sides are along the lines $x=0$, $y=0$, $x=a$ and $y=b$. **[Meerut 1976]**

5. Verify Stoke's theorem for a vector field defined by $\mathbf{F}=(x^2-y^2)\mathbf{i}+2xy\mathbf{j}$ in the rectangular region in the xy-plane bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.
[Kanpur 1975]

6. Verify Stoke's theorem for the function
$$\mathbf{F}=x^2\mathbf{i}+xy\mathbf{j},$$
integrated round the square, in the plane $z=0$, whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$. **[Bombay 1970]**
[**Hint.** Proceed as in Ex. 9 Page 141. Show that
$$\int_C \mathbf{F}\cdot d\mathbf{r}=\tfrac{1}{2}a^3=\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS].$$

7. Verify Stoke's theorem for the function
$$\mathbf{F}(x, y, z)=xy\mathbf{i}+xy^2\mathbf{j}$$
integrated round the square with vertices $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 0)$,
where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along x-axis and y-axis respectively. **[Meerut 1979]**

8. Verify Stoke's theorem for the vector $\mathbf{A}=3y\mathbf{i}-xz\mathbf{j}+yz^2\mathbf{k}$, where S is the surface of the paraboloid $2z=x^2+y^2$ bounded by $z=2$ and C is its boundary. **[Meerut 1973, 77]**

9. By converting into a line integral evaluate
$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\, dS, \text{ where } \mathbf{A}=(x-z)\mathbf{i}+(x^3+yz)\mathbf{j}-3xy^2\mathbf{k}$$
and S is the surface of the cone $z=2-\sqrt{(x^2+y^2)}$ above the xy-plane. **Ans.** 12π. **[Meerut 1974]**

10. By converting into a line integral evaluate
$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$$
where $\mathbf{F}=(x^2+y-4)\mathbf{i}+3xy\mathbf{j}+(2xz+z^2)\mathbf{k}$
and S is the surface of (i) the hemisphere $x^2+y^2+z^2=16$ above the xy-plane (ii) the paraboloid $z=4-(x^2+y^2)$ above the xy-plane. **Ans.** (i) -16π, (ii) -4π.

11. Evaluate $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\, dS$, where
$\mathbf{F}=(y-z+2)\mathbf{i}+(yz+4)\mathbf{j}-xz\mathbf{k}$ and S is the surface of the cube $x=y=z=0$, $x=y=z=2$ above the xy-plane. **Ans.** -4.
[**Hint.** The curve C bounding the surface S is the square, say $OABC$, in the xy-plane given by $x=0$, $x=2$, $y=0$, $y=2$].

12. Show that
$$\iint_S \phi \text{ curl } \mathbf{F}\cdot d\mathbf{S}=\int_C \phi\, \mathbf{F}\cdot d\mathbf{r}-\iint_S (\text{grad } \phi\times\mathbf{F})\cdot d\mathbf{S}.$$
[**Hint.** Apply Stoke's theorem to the vector $\phi\mathbf{F}$].

13. If $\mathbf{f}=\nabla\phi$ and $\mathbf{g}=\nabla\psi$ are two vector point functions, such that
$$\nabla^2\phi=0,\ \nabla^2\psi=0$$

show that

$$\iint_S (\mathbf{g}\cdot\nabla)\,\mathbf{f}\cdot d\mathbf{S}=\int_C (\mathbf{f}\times\mathbf{g})\cdot d\mathbf{r}+\iint_S (\mathbf{f}\cdot\nabla)\,\mathbf{g}\cdot d\mathbf{S}.$$

§ 10. Line integrals Independent of path.

Let $\mathbf{F}(x, y, z)=f(x, y, z)\,\mathbf{i}+g(x, y, z)\,\mathbf{j}+h(x, y, z)\,\mathbf{k}$ be a vector point function defined and continuous in a region R of space. Let P and Q be two points in R and let C be a path joining P to Q. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int (f dx+g dy+h dz) \qquad \ldots(1)$$

is called the line integral of $\mathbf{F}$ along C. In general the value of this line integral depends not only on the end points P and Q of the path C but also on C.

In other words, if we integrate from P to Q along different paths, we shall, in general, get different values of the integral. *The line integral* (1) *is said to be* **independent of path** *in R, if for every pair of end points P and Q in R the value of the integral is the same for all paths C in R starting from P and ending at Q.*

In this case the value of this line integral will depend on the choice of P and Q and not on the choice of the path joining P to Q.

Definition. *The expression $f\,dx+g\,dy+h\,dz$ is said to be an exact differential if there exists a single valued scalar point function $\phi(x, y, z)$, having continuous first partial derivatives such that $d\phi=f\,dx+g\,dy+h\,dz$.*

It can be easily seen that $f\,dx+g\,dy+h\,dz$ is an exact differential if and only if the vector function

$$\mathbf{F}=f\,\mathbf{i}+g\,\mathbf{j}+h\,\mathbf{k}$$

is the gradient of a single valued scalar function $\phi(x, y, z)$.

Because $\quad \mathbf{F}=\text{grad}\,\phi$

if, and only if $f\,\mathbf{i}+g\,\mathbf{j}+h\,\mathbf{k}=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$

if, and only if $f=\dfrac{\partial\phi}{\partial x},\ g=\dfrac{\partial\phi}{\partial y},\ h=\dfrac{\partial\phi}{\partial z}$

if, and only if $f\,dx+g\,dy+h\,dz=\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy+\dfrac{\partial\phi}{\partial z}dz$

if, and only if $f\,dx+g\,dy+h\,dz=d\phi$.

Thus $\mathbf{F}=\text{grad}\,\phi$ if, and only if $f\,dx+g\,dy+h\,dz$ is an exact differential $d\phi$.

Theorem 1. *Let $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ be continuous in a region R of space. Then the line integral*

$$\int (f\,dx+g\,dy+h\,dz)$$

is independent of path in R if and only if the differential form under the integral sign is exact in R. **[Meerut 1968]**

Or

Let $\mathbf{F}(x, y, z)$ be continuous in region R of space. Then the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

is independent of the path C in R joining P and Q if and only if $\mathbf{F}=\text{grad}\,\phi$ where (x, y, z) is a single-valued scalar function having continuous first partial derivatives in R. **[Kerala 1975]**

Proof. Suppose $\mathbf{F}=\text{grad}\,\phi$ in R. Let P and Q be any two points in R and let C be any path from P to Q in R.

Then

$$\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C \nabla\phi\cdot d\mathbf{r}$$

$$=\int_C \left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\int_C \left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)=\int_C d\phi$$

$$=\int_P^Q d\phi=\Big[\,\phi\,\Big]_P^Q=\phi(Q)-\phi(P).$$

Thus the line integral depends only on points P and Q and not the path joining them. This is true of course only if $\phi(x, y, z)$ is single valued at all points P and Q.

Conversely suppose the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path C joining any two points P and Q in R. Let P be a fixed point (x_0, y_0, z_0) in R and let Q be any point (x, y, z) in R. Let

$$\phi(x, y, z)=\int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F}\cdot d\mathbf{r}=\int_{(x_0, y_0, z_0)}^{(x, y, z)} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides, with respect to s, we get

$$\frac{d\phi}{ds}=\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}.$$

But $\quad \dfrac{d\phi}{ds}=\dfrac{\partial\phi}{\partial x}\dfrac{dx}{ds}+\dfrac{\partial\phi}{\partial y}\dfrac{dy}{ds}+\dfrac{\partial\phi}{\partial z}\dfrac{dz}{ds}$

$$=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}\right)$$

$$=\nabla\phi\cdot\frac{d\mathbf{r}}{ds}.$$

$$\therefore\quad \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}=\nabla\phi\cdot\frac{d\mathbf{r}}{ds}$$

or $$(\nabla\phi-\mathbf{F})\cdot\frac{d\mathbf{r}}{ds}=0.$$

Now this result is true irrespective of the path joining P to Q *i.e.* this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$$\nabla\phi-\mathbf{F}=\mathbf{0},$$

i.e., $$\nabla\phi=\mathbf{F}.$$

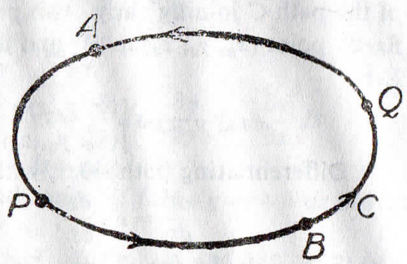
This completes the proof of the theorem.

Definition. *A vector field* $\mathbf{F}(x, y, z)$ *defined and continuous in a region R of space is said to be a conservative vector field if the line integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *is independent of the path C in R joining P and Q where P and Q are any two points in R.*

By theorem 1, the vector field $\mathbf{F}(x, y, z)$ is conservative if and only if $\mathbf{F}=\nabla\phi$ where $\phi(x, y, z)$ is a single valued scalar function having continuous first partial derivatives in R. The function $\phi(x, y, z)$ is called the **scalar potential** *of the vector field* $\mathbf{F}$.

Theorem 2. *Let* $\mathbf{F}(x, y, z)$ *be a vector function defined and continuous in a region R of space. Then the line integral* $\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ *is independent of the path joining any two points P and Q in R if and only if* $\oint \mathbf{F}\cdot d\mathbf{r}=0$ *for every simple closed path in R.*

Proof. Let C be any simple closed path in R and let the line integral be independent of path in R. Take two points P and Q on C and subdivide C into two arcs PBQ and QAP. Then



$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_{PBQAP}\mathbf{F}\cdot d\mathbf{r}$$

$$=\int_{PBQ}\mathbf{F}\cdot d\mathbf{r}+\int_{QAP}\mathbf{F}\cdot d\mathbf{r}=\int_{PBQ}\mathbf{F}\cdot d\mathbf{r}-\int_{PAQ}\mathbf{F}\cdot d\mathbf{r}$$

$=0$, since the integral from P to Q along a path through B is equal to the integral from P to Q along a path through A.

Conversely suppose that the integral under consideration is zero on every simple closed path in R. Let P and Q be any two points in R and let PBQ and PAQ be any two paths in R which join P to Q and do not cross. Then

$$\oint_{PBQAP}\mathbf{F}\cdot d\mathbf{r}=\int_{PBQ}\mathbf{F}\cdot d\mathbf{r}+\int_{QAP}\mathbf{F}\cdot d\mathbf{r}=\int_{PBQ}\mathbf{F}\cdot d\mathbf{r}-\int_{PAQ}\mathbf{F}\cdot d\mathbf{r}.$$

But as given, we have $\oint_{PBQAP}\mathbf{F}\cdot d\mathbf{r}=0$.

$$\therefore\quad \int_{PBQ}\mathbf{F}\cdot d\mathbf{r}-\int_{PAQ}\mathbf{F}\cdot d\mathbf{r}=0$$

or $$\int_{PBQ}\mathbf{F}\cdot d\mathbf{r}=\int_{PAQ}\mathbf{F}\cdot d\mathbf{r}.$$

This completes the proof of the theorem.

Theorem 3. *Let* $\mathbf{F}(x, y, z)=f\mathbf{i}+g\mathbf{j}+h\mathbf{k}$ *be a continuous vector function having continuous first partial derivatives in a region R of space. If* $\int fdx+gdy+hdz$ *is independent of path in R and consequently* $fdx+gdy+hdz$ *is an exact differential in R, then curl* $\mathbf{F}=\mathbf{0}$ *everywhere in R. Conversely, if R is simply connected and curl* $\mathbf{F}=\mathbf{0}$ *everywhere in R, then* $fdx+gdy+hdz$ *is an exact differential in R or* $\int fdx+gdy+hdz$ *is independent of path in R.* **[Allahabad 1979]**

Proof. Suppose $\int (fdx+g\,dy+h\,dz)$ is independent of path in R. Then $fdx+gdy+hdz$ is an exact differential in R. Therefore

$$\mathbf{F}=f\mathbf{i}+g\mathbf{j}+h\mathbf{k}=\text{grad}\,\phi.$$

$$\therefore\quad \text{curl}\,\mathbf{F}=\text{curl}\,(\text{grad}\,\phi)=\mathbf{0}.$$

Conversely suppose R is simply connected and curl $\mathbf{F}=\mathbf{0}$ everywhere in R. Let C be any simple closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\ dS=0.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is zero for every simple closed path C in R.

Therefore $\int \mathbf{F}\cdot d\mathbf{r}$ is independent of path in R.

Therefore $\mathbf{F}=\nabla\phi$ and consequently $fdx+gdy+hdz$ is an exact differential $d\phi$.

Note. The assumption that R be simply connected is essential and cannot be omitted. It is obvious from the following example.

Example. Let $\mathbf{F}=-\dfrac{y}{x^2+y^2}\,\mathbf{i}+\dfrac{x}{x^2+y^2}\,\mathbf{j}$.

Here $\mathbf{F}$ is not defined at origin. In every region R of the xy-plane not containing the origin, we have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$=0\mathbf{i}+0\mathbf{j}+\left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\}\mathbf{k}$$

$$=\left\{\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}+\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right\}\mathbf{k}=0\mathbf{k}=\mathbf{0}.$$

Suppose R is simply connected. For example let R be the region enclosed by a simple closed curve C not enclosing the origin. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C\left(-\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy\right)$$

$$=\iint_R\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)-\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]dx\,dy,$$

by Green's theorem in plane

$$=0.$$

Suppose R is not simply connected. Let R be the region of the xy-plane contained between concentric circles of radii $\frac{1}{2}$ and $\frac{3}{2}$ and having centre at origin. Obviously R is not simply connected. We have $\mathbf{F}=\mathbf{0}$, everywhere in R. Let C be a circle of radius one and centre at origin. Then C is a closed curve in R. The parametric

equations of C can be taken as $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$.

We have $$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int\left(-\frac{y}{x^2+y^2}\,dx+\frac{x}{x^2+y^2}\,dy\right)$$

$$=\int_{t=0}^{2\pi}\left[-\frac{\sin t}{\cos^2 t+\sin^2 t}\frac{dx}{dt}+\frac{\cos t}{\cos^2 t+\sin^2 t}\frac{dy}{dt}\right]dt$$

$$=\int_0^{2\pi}(\sin^2 t+\cos^2 t)\ dt=2\pi.$$

Thus we see that $\oint_C \mathbf{F}\cdot d\mathbf{r}\neq 0$.

Definition. Irrotational vector field. *A vector field* $\mathbf{F}$ *is said to be irrotational if curl* $\mathbf{F}=\mathbf{0}$. **(Calicut 1975; Allahabad 79)**

We see that an irrotational field $\mathbf{F}$ is characterised by any one of the three conditions :

(i) $\mathbf{F}=\nabla\phi$,

(ii) $\nabla\times\mathbf{F}=\mathbf{0}$,

(iii) $\oint \mathbf{F}\cdot d\mathbf{r}=0$ for every closed path.

Any one of these conditions implies the other two.

SOLVED EXAMPLES

Ex. 1. *Are the following forms exact ?*

(i) $xdx-ydy+zdz$. *(ii)* $e^ydx+e^xdy+e^zdz$.

(iii) $yzdx+xzdy+xydz$. *(iv)* $y^2z^3dx+2xyz^3dy+3xy^2z^2dz$.

Solution. (i) We have

$$xdx-ydy+zdz=(x\mathbf{i}-y\mathbf{j}+z\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$
$$=\mathbf{F}\cdot d\mathbf{r},\text{ where}$$
$$\mathbf{F}=x\mathbf{i}-y\mathbf{j}+z\mathbf{k}.$$

We have $$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & -y & z \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

$\therefore$ the given form is exact.

(ii) Here $\mathbf{F}=e^y\mathbf{i}+e^x\mathbf{j}+e^z\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+(e^x-e^y)\,\mathbf{k}.$$

Since curl $\mathbf{F}\neq 0$, therefore the given form is not exact.

(iii) Here $\mathbf{F}=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}=(x-x)\,\mathbf{i}-(y-y)\,\mathbf{j}+(z-z)\,\mathbf{k}=0.$$

Since curl $\mathbf{F}=0$, therefore the given form is exact.

(iv) Here $\mathbf{F}=y^2z^3\,\mathbf{i}+2xyz^3\,\mathbf{j}+3xy^2z^2\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$=(6xyz^2-6xyz^2)\,\mathbf{i}-(3y^2z^2-3y^2z^2)\,\mathbf{j}+(2yz^3-2yz^3)\,\mathbf{k}$$

$$=0.$$

$\therefore$ the given form is exact.

Ex. 2. *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(i) $x\,dx-y\,dy-z\,dz$. *(ii)* $dx+z\,dy+y\,dz$

(iii) $\cos x\,dx-2yz\,dy-y^2\,dz$. *(iv)* $(z^2-2xy)\,dx-x^2dy+2xz\,dz$.

Solution. (i) Here $\mathbf{F}=x\mathbf{i}-y\mathbf{j}-z\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & -z \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=0.$$

$\therefore$ the given form is exact.

Let $\mathbf{F}=\nabla\phi$,

or $x\mathbf{i}-y\mathbf{j}-z\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=x \text{ whence } \phi=\frac{x^2}{2}+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=-y \text{ whence } \phi=-\frac{y^2}{2}+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=-z \text{ whence } \phi=-\frac{z^2}{2}+f_3(x,y). \qquad \ldots(3)$$

The constants of integration are functions of the variables not involved in the integration because the derivatives are partial.

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y,z)=-\frac{y^2+z^2}{2},\ f_2(x,z)=\frac{x^2-z^2}{2},\ f_3(x,y)=\frac{x^2-y^2}{2}.$$

$\therefore$ $\phi=\frac{x^2-y^2-z^2}{2}$ to which may be added any constant.

Hence $\phi=\frac{x^2-y^2-z^2}{2}+C$, where C is a constant.

(ii) Here $\mathbf{F}=\mathbf{i}+z\mathbf{j}+y\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=0.$$

$\therefore$ the given form is exact.

Let $\mathbf{F}=\nabla\phi$

or $\mathbf{i}+z\mathbf{j}+y\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=1 \text{ whence } \phi=x+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=z \text{ whence } \phi=zy+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=y \text{ whence } \phi=yz+f_3(x,y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y,z)=zy,\ f_2(x,z)=x,\ f_3(x,y)=x.$$

$\therefore$ $\phi=x+yz$ to which may be added any constant.

$\therefore$ $\phi=x+yz+C$.

(iii) Here $\mathbf{F}=\cos x\,\mathbf{i}-2yz\mathbf{j}-y^2\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$$=(-2y+2y)\,\mathbf{i}+0\mathbf{j}+0\mathbf{k}=0.$$

$\therefore$ the given form is exact.

Let $\mathbf{F}=\nabla\phi$,

or $\cos x\,\mathbf{i}-2yz\mathbf{j}-y^2\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=\cos x \text{ whence } \phi=\sin x+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=-2yz \text{ whence } \phi=-y^2z+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=-y^2 \text{ whence } \phi=-y^2z+f_3(x,y). \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y,z)=-y^2z, f_2(x,z)=\sin x, f_3(x,y)=\sin x$.

$\therefore\quad \phi=\sin x-y^2z$ to which may be added any constant.

$\therefore\quad \phi=\sin x-y^2z+C$.

(iv) Here $\mathbf{F}=(z^2-2xy)\,\mathbf{i}-x^2\mathbf{j}+2xz\mathbf{k}$. We have

$$\text{Curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2-2xy & -x^2 & 2xz \end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F}=\nabla\phi$

or $\quad (z^2-2xy)\,\mathbf{i}-x^2\mathbf{j}+2xz\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=z^2-2xy \text{ whence } \phi=z^2x-x^2y+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=-x^2 \text{ whence } \phi=-x^2y+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=2xz \text{ whence } \phi=xz^2+f_3(x,y). \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y,z)=0, f_2(x,z)=xz^2, f_3(x,y)=-x^2y$.

$\therefore\quad \phi=z^2x-x^2y$ to which may be added any constant.

$\therefore\quad \phi=z^2x-x^2y+C$.

Ex. 3. *Show that* $\mathbf{F}=(2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\mathbf{k}$ *is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from*

$$(1,-2,1) \text{ to } (3,1,4).$$

Solution. The field $\mathbf{F}$ will be conservative if $\nabla\times\mathbf{F}=\mathbf{0}$.

We have

$$\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}=\mathbf{0}.$$

Therefore $\mathbf{F}$ is a conservative force field.

Let $\mathbf{F}=\nabla\phi$

or $\quad (2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=2xy+z^3 \text{ whence } \phi=x^2y+z^3x+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=x^2 \text{ whence } \phi=x^2y+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \text{ whence } \phi=xz^3+f_3(x,y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y,z)=0, f_2(x,z)=z^3x, f_3(x,y)=x^2y$.

$\therefore\quad \phi=x^2y+xz^3$ to which may be added any constant.

$\therefore\quad \phi=x^2y+xz^3+C$.

$$\text{Work done}=\int_{(1,-2,1)}^{(3,1,4)} \mathbf{F}\cdot d\mathbf{r}$$

$$=\int_{(1,-2,1)}^{(3,1,4)} d\phi=\Big[\phi\Big]_{(1,-2,1)}^{(3,1,4)}$$

$$=\Big[x^2y+xz^3\Big]_{(1,-2,1)}^{(3,1,4)}=202.$$

Ex. 4. *Show that the vector field* $\mathbf{F}$ *ginen by*

$$\mathbf{F}=(y+\sin z)\,\mathbf{i}+x\mathbf{j}+x\cos z\mathbf{k}$$

is conservative. Find its scalar potential.

Solution. We have

$$\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sin z & x & x\cos z \end{vmatrix}=\mathbf{0}.$$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Let $\mathbf{F}=\nabla\phi$

or $\quad (y+\sin z)\,\mathbf{i}+x\mathbf{j}+x\cos z\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=y+\sin z \text{ whence } \phi=xy+x\sin z+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=x \quad \text{ whence } \phi=xy+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=x\cos z \text{ whence } \phi=x\sin z+f_3(x,y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=0, f_2(x, z)=x \sin z, f_3(x, y)=xy.$$

$\therefore$ $\phi=xy+x \sin z$ to which may be added any constant.

$\therefore$ $\phi=xy+x \sin z+C$.

Ex. 5. *Evaluate*

$$\int_C 2xyz^2\, dx+(x^2z^2+z \cos yz)\, dy+(2x^2yz+y \cos yz)\, dz$$

where C is any path from $(0, 0, 1)$ *to* $(1, \frac{1}{4}\pi, 2)$. **(Meerut 1968)**

Solution. We have $\mathbf{F}=2xyz^2\, \mathbf{i}+(x^2z^2+z \cos yz)\, \mathbf{j}+(2x^2yz+y \cos yz)\, \mathbf{k}$.

$$\therefore \quad \nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2+z \cos yz & 2x^2yz+y \cos yz \end{vmatrix}$$

$$=(2x^2z+\cos yz-yz \sin yz-2x^2z-\cos yz+yz \sin yz)\, \mathbf{i}-(4xyz-4xyz)\, \mathbf{j}+(2xz^2-2xz^2)\, \mathbf{k}=0.$$

$\therefore$ the given line integral is independent of path in space.

Let $\mathbf{F}=\nabla\phi$. Then

$$\frac{\partial\phi}{\partial x}=2xyz^2 \text{ whence } \phi=x^2yz^2+f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=x^2z^2+z \cos yz \text{ whence } \phi=x^2z^2y+\sin yz+f_2(x, z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=2x^2yz+y \cos yz \text{ whence } \phi=x^2yz^2+\sin yz+f_3(x, y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=\sin yz, f_2(x, z)=0, f_3(x, y)=0.$$

$\therefore$ $\phi=x^2yz^2+\sin yz$ to which may be added any constant.

The given line integral is therefore

$$\int_C d(x^2yz^2+\sin yz)=\Big[x^2yz^2+\sin yz\Big]_{(0, 0, 1)}^{(1, \pi/4, 2)}$$

$$=\pi+\sin \tfrac{1}{2}\pi=\pi+1.$$

Ex. 6. *Evaluate*

$$\int_C yzdx+(xz+1)\, dy+xy\, dz,$$

where C is any path from $(1, 0, 0)$ *to* $(2, 1, 4)$.

[Meerut 1969; Agra 72]

Solution. We have $\mathbf{F}=yz\mathbf{i}+(xz+1)\, \mathbf{j}+xy\mathbf{k}$.

$$\therefore \quad \nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz+1 & xy \end{vmatrix}$$

$$=(x-x)\, \mathbf{i}-(y-y)\, \mathbf{j}+(z-z)\, \mathbf{k}=0.$$

$\therefore$ the differential form $yzdx+(xz+1)\, dy+xydz$ is exact and the given line integral is independent of path.

Let $\mathbf{F}=\nabla\phi$

or $yz\mathbf{i}+(xz+1)\, \mathbf{j}+xy\mathbf{k}=\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=yz \text{ whence } \phi=xyz+f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=xz+1 \text{ whence } \phi=xyz+y+f_2(x, z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=xy \text{ whence } \phi=xyz+f_3(x, y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z)=y, f_2(x, z)=0, f_3(x, y)=y.$$

$\therefore$ $\phi=xyz+y$ to which may be added any constant.

The given line integral is therefore

$$=\int_{(1, 0, 0)}^{(2, 1, 4)} d(xyz+y)=\Big[xyz+y\Big]_{(1, 0, 0)}^{(2, 1, 4)}$$

$$=[8+1-0-0]=9.$$

Ex. 7. *Show that the form under the integral sign is exact and evaluate*

$$\int_{(0, 2, 1)}^{(2, 0, 1)} [ze^x\, dx+2yz\, dy+(e^x+y^2)\, dz].$$

Solution. Here $\mathbf{F}=ze^x\, \mathbf{i}+2yz\, \mathbf{j}+(e^x+y^2)\, \mathbf{k}$.

$$\text{We have curl } \mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & 2yz & e^x+y^2 \end{vmatrix}$$

$$=(2y-2y)\, \mathbf{i}-(e^x-e^x)\, \mathbf{j}+0\mathbf{k}=0.$$

$\therefore$ the form under the integral sign is exact and consequently the line integral is independent of path in space.

Let $\mathbf{F}=\nabla\phi$

or $ze^x\,\mathbf{i}+2yz\,\mathbf{j}+(e^x+y^2)\,\mathbf{k}=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=ze^x \text{ whence } \phi=ze^x+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=2yz \text{ whence } \phi=y^2z+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=e^x+y^2 \text{ whence } \phi=e^xz+y^2z+f_3(x,y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y,z)=y^2z,\ f_2(x,z)=e^xz,\ f_3(x,y)=0.$$

$\therefore\quad \phi=ze^x+y^2z$ to which may be added any constant. The given line integral is therefore

$$=\int_{(0,2,1)}^{(2,0,1)} d\,(ze^x+y^2z)=\Big[ze^x+y^2z\Big]_{(0,2,1)}^{(2,0,1)}$$

$$=[e^2+0-1-4]=e^2-5.$$

Ex. 8. *If* $\mathbf{F}=\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* C *is the curve* $y=\sqrt{(1-x^2)}$ *in the x-y plane from* $(1,0)$ *to* $(0,1)$.

Solution. We have $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C(\cos y\,dx-x\sin y\,dy)$

$$=\int_1^0 \cos\sqrt{(1-x^2)}\,dx-\int_0^1\sqrt{(1-y^2)}\sin y\,dy.$$

It is difficult to evaluate the integrals directly. However we observe that

$$\text{curl }\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \cos y & -x\sin y & 0\end{vmatrix}$$

$$=0\,\mathbf{i}+0\,\mathbf{j}+(-\sin y+\sin y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ the given line integral is independent of path.

Let $\mathbf{F}=\nabla\phi$

or $\cos y\,\mathbf{i}-x\sin y\,\mathbf{j}=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=\cos y \text{ whence } \phi=x\cos y+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=-x\sin y, \text{ whence } \phi=x\cos y+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=0 \text{ whence } \phi=f_3(x,y). \qquad \ldots(3)$$

From (1), (2), (3), we see that $\phi=x\cos y$.

$\therefore$ The given line integral is equal to

$$\int_{(1,0)}^{(0,1)} d\,(x\cos y)=\Big[x\cos y\Big]_{(1,0)}^{(0,1)}=[0-1\cos 0]=-1.$$

Ex. 9. *Show that the vector field* $\mathbf{F}$ *given by*

$$\mathbf{F}=(x^2-yz)\,\mathbf{i}+(y^2-zx)\,\mathbf{j}+(z^2-xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ *such that* $\mathbf{F}=\nabla\phi$.

Solution. We have

$$\text{Curl }\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2-yz & y^2-zx & z^2-xy\end{vmatrix}$$

$$=(-x+x)\,\mathbf{i}-(-y+y)\,\mathbf{j}+(-z+z)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ The vector field $\mathbf{F}$ is irrotational.

Let $\mathbf{F}=\nabla\phi$

or $(x^2-yz)\,\mathbf{i}+(y^2-zx)\,\mathbf{j}+(z^2-xy)\,\mathbf{k}=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$.

Then

$$\frac{\partial\phi}{\partial x}=x^2-yz \text{ whence } \phi=\frac{x^3}{3}-xyz+f_1(y,z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y}=y^2-zx \text{ whence } \phi=\frac{y^3}{3}-xyz+f_2(x,z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z}=z^2-xy \text{ whence } \phi=\frac{z^3}{3}-xyz+f_3(x,y). \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y,z)=\dfrac{y^3}{3}+\dfrac{z^3}{3},\ f_2(x,z)=\dfrac{x^3+z^3}{3},\ f_3(x,y)=\dfrac{x^3+y^3}{3}$.

Therefore $\phi=\dfrac{x^3+y^3+z^3}{3}-xyz+C.$

Exercises

1. Show that $(y^2z^3\cos x-4x^3z)\,dx+2z^3y\sin x\,dy+(3y^2z^2\sin x-x^4)\,dz$ is an exact differential of some function ϕ and find this function.

Ans. $\phi=y^2z^3\sin x-x^4z+C.$

2. (i) Show that the vector field $\mathbf{F}=(2xy^2+yz)\,\mathbf{i}+(2x^2y+xz+2yz^2)\,\mathbf{j}+(2y^2z+xy)\,\mathbf{k}$ is conservative.

(ii) Show that $\mathbf{F}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is conservative and find ϕ such that $\mathbf{F}=\nabla\phi$. [Kanpur 1980]

Ans. $\phi=\frac{1}{2}(x^2+y^2+z^2)+C.$

3. Show that

$$\mathbf{F}=(\sin y+z)\,\mathbf{i}+(x\cos y-z)\,\mathbf{j}+(x-y)\,\mathbf{k}$$

is a conservative vector field and find a function ϕ such that

$$\mathbf{F}=\nabla\phi.$$ [Bombay 1966]

Ans. $\phi=x\sin y+xz-yz+C.$

4 Show that the vector field defined by

$$\mathbf{F}=(2xy-z^3)\,\mathbf{i}+(x^2+z)\,\mathbf{j}+(y-3xz^2)\,\mathbf{k}$$

is conservative, and find the scalar potential of **F**.

[Bombay 1970]

5. Show that the following vector functions **F** are irrotational and find the corresponding scalar ϕ such that

$$\mathbf{F}=\nabla\phi.$$

(i) $\mathbf{F}=x^3\,\mathbf{i}+y^3\,\mathbf{j}+z^3\,\mathbf{k}.$

(ii) $\mathbf{F}=(y\sin z-\sin x)\,\mathbf{i}+(x\sin z+2yz)\,\mathbf{j}+(xy\cos z+y^2)\,\mathbf{k}.$

(iii) $\mathbf{F}=(\sin y+z\cos x)\,\mathbf{i}+(x\cos y+\sin z)\,\mathbf{j}+(y\cos z+\sin x)\mathbf{k}.$

[Calcutta 1975]

Ans. (i) $\phi=\frac{1}{4}(x^4+y^4+z^4)+C.$

(ii) $\phi=xy\sin z+\cos x+y^2z+C.$

(iii) $\phi=x\sin y+y\sin z+z\sin x+C.$

6. Find a, b, c if $\mathbf{F}=(3x-3y+az)\,\mathbf{i}+(bx+2y-4z)\,\mathbf{j}+(2x+cy+z)\mathbf{k}$ is irrotational. [Calicut 1974]

Ans. $a=2,\ b=-3,\ c=-4.$

7. Show that

$$(2x\cos y+z\sin y)\,dx+(xz\cos y-x^2\sin y)\,dy+x\sin y\,dz=0$$

is an exact differential equation and hence solve it.

Ans. Solution is $x^2\cos y+xz\sin y=C.$

8. If **F** is irrotational in a simply connected region R, show that there exists a scalar field ϕ such that $\mathbf{F}=\text{grad}\,\phi$.

[Calicut 1975]

§ 11. Physical interpretation of divergence and curl.

[Meerut 1968]

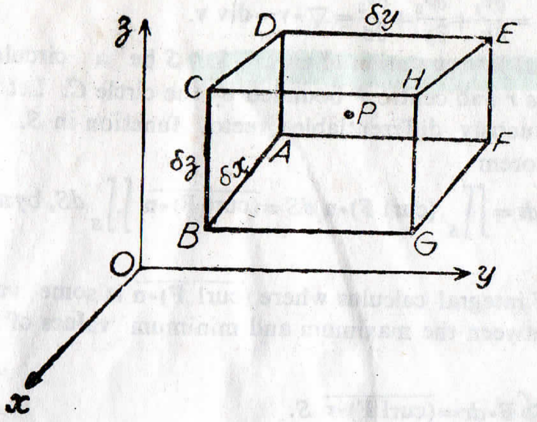
Physical interpolation of divergence. *Suppose that there is a fluid motion whose velocity at any point is* $\mathbf{v}(x, y, z)$. *Then the loss of fluid per unit unit volume per time in a small parallelopiped having centre at* $P(x, y, z)$ *and edges parallel to the co-ordinate axes and having lengths* $\delta x, \delta y, \delta z$ *respectively, is given approximately by*

$$\text{div}\,\mathbf{v}=\nabla\cdot\mathbf{v}.$$

Let $\mathbf{v}=v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}$.

x-component of velocity $\mathbf{v}$ at $P=v_1(x, y, z)$.

x-component of $\mathbf{v}$ at centre of face $AFED$ which is perpendicular to x-axis and is nearer to origin



$$=v_1\left(x-\frac{\delta x}{2}, y, z\right)$$

$$=v_1(x, y, z)-\frac{\delta x}{2}\frac{\partial v_1}{\partial x}+\ldots \text{ by Taylor s theorem}$$

$$=v_1(x, y, z)-\frac{\delta x}{2}\frac{\partial v_1}{\partial x} \text{ approximately.}$$

Similarly x-component of $\mathbf{v}$ at centre of opposite face

$$GHCB=v_1+\frac{\delta x}{2}\frac{\partial v_1}{\partial x} \text{ approximately.}$$

$\therefore$ volume of fluid entering the parallelopiped across $AFED$ per unit time$=\left(v_1-\frac{\delta x}{2}\frac{\partial v_1}{\partial x}\right)\delta y\,\delta z.$

Also volume of fluid going out the parallelopiped across $GHCB$ per unit time$=\left(v_1+\frac{\delta x}{2}\frac{\partial v_1}{\partial x}\right)\delta y\,\delta z.$

$\therefore$ loss in volume per unit time in the direction of x-axis

$$=\left(v_1+\frac{\delta x}{2}\frac{\partial v_1}{\partial x}\right)\delta y\,\delta z-\left(v_1-\frac{\delta x}{2}\frac{\partial v_1}{\partial x}\right)\delta y\,\delta z=\frac{\partial v_1}{\partial x}\,\delta x\,\delta y\,\delta z.$$

Similarly, loss in volume per unit time in y direction

$$=\frac{\partial v_2}{\partial y}\,\delta x\,\delta y\,\delta z,$$

and loss in volume per unit time in z direction$=\frac{\partial v_3}{\partial z}\,\delta x\,\delta y\,\delta z.$

$\therefore$ total loss of the fluid per unit volume per unit time

$$=\frac{\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)\delta x\,\delta y\,\delta z}{\delta x\,\delta y\,\delta z}$$

$$=\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}=\nabla\cdot\mathbf{v}=\text{div }\mathbf{v}.$$

Physical interpretation of curl. Let S be a circular disc of small radius r and centre P bounded by the circle C. Let $\mathbf{F}(x, y, z)$ be a continuously differentiable vector function in S. Then by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS=\overline{(\text{curl } \mathbf{F})\cdot\mathbf{n}}\iint_S dS,$$ by mean value

theorem of integral calculus where $\overline{(\text{curl } \mathbf{F})\cdot\mathbf{n}}$ is some value intermediate between the maximum and minimum values of $(\text{curl } \mathbf{F})\cdot\mathbf{n}$ over S.

$$\therefore \quad \oint_C \mathbf{F}\cdot d\mathbf{r}=\overline{(\text{curl } \mathbf{F})\cdot\mathbf{n}}\; S.$$

$$\therefore \quad \overline{(\text{curl } \mathbf{F})\cdot\mathbf{n}}=\frac{\left[\oint_C \mathbf{F}\cdot d\mathbf{r}\right]}{S}.$$

Taking limit as $r\to 0$, we get at P

$$(\text{curl } \mathbf{F})\cdot\mathbf{n}=\lim_{r\to 0}\frac{\oint_C \mathbf{F}\cdot d\mathbf{r}}{S}.$$

Now $(\text{curl } \mathbf{F})\cdot\mathbf{n}$ is normal component of curl $\mathbf{F}$ at P and $\oint \mathbf{F}\cdot d\mathbf{r}$ is circulation of $\mathbf{F}$ about C. Therefore the normal component of the curl can be interpreted physically as the limit of the circulation per unit area.