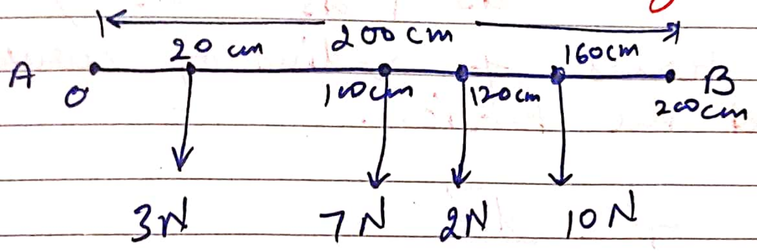
IFoS 2019 (S&D)

Date 14/6/20

5(c) A 2m rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal.



Varignon's Theorem: Moment of a force about a point is equal to the sum of the moments of the forces' components about the point.

Let us take moments about point A.

Resultant of all forces $= 3+7+2+10$

$= 22\,N$

$\therefore (22 \times r) = 3\times20 + 7\times100 + 2\times120 + 10\times160$

$= 2600$

$\therefore r = \frac{2600}{22} = \frac{1300}{11} = 118.18\,cm.$

Hence rod must be supported at a point 118.18 cm from end A.

6(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force).

$$r^2 = a^2 \cos 2\theta$$

or $\quad a^2 u^2 \cos 2\theta = 1$, where $u = \frac{1}{r}$ — (1)

Taking log on both sides, we have

$$2 \log a + 2 \log u + \log \cos 2\theta = 0 \quad \text{— (2)}$$

Differentiating w.r.t. θ

$$0 + \frac{2}{u} \cdot \frac{du}{d\theta} - \frac{2 \sin 2\theta}{\cos 2\theta} = 0$$

$$\Rightarrow \quad \frac{du}{d\theta} = u \tan 2\theta \quad \text{— (3)}$$

Again Differentiating w.r.t. θ,

$$\frac{d^2u}{d\theta^2} = 2u \sec^2 2\theta + \frac{du}{d\theta} \tan 2\theta$$

$$= 2u \sec^2 2\theta + u \tan^2 2\theta$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = 2u \sec^2 2\theta + u \tan^2 2\theta + u$$

$$= 2u \sec^2 2\theta + u \cdot \sec^2 2\theta$$

$$= 3u \sec^2 2\theta$$

$$= 3u \cdot a^4 \cdot u^4 \quad \text{[from (1)]}$$

$$= 3a^4 u^5 \quad \text{— (4)}$$

The DE of the central orbit in polar form is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2} \quad \text{— (5)}$$

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From (4) and (5)

$$3a^4u^5 = \frac{F}{h^2u^2}$$

$$F = 3h^2a^4 \cdot u^7$$

$$= \lambda u^7 \quad , \text{ where } \lambda = 3h^2a^4$$

i.e. $F = \frac{\lambda}{r^7}$

$$\therefore \boxed{F \propto \frac{1}{r^7}}$$

Hence the force varies inversely as the 7^{th} power of the distance from the pole.

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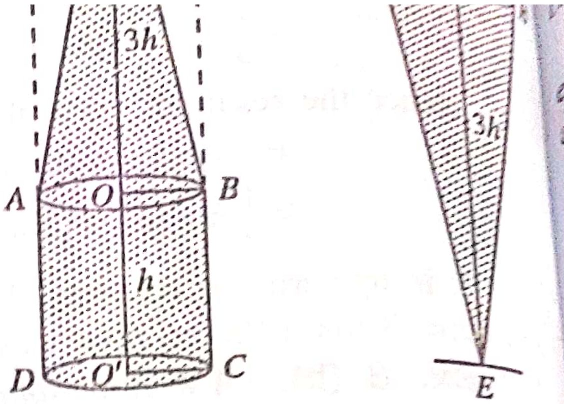
of the cone = wt. of the superincumbent liquid

$$= \pi r^2 . 3hw - \frac{1}{3}\pi r^2 . 3h . w = 2\pi r^2 hw \quad \ldots(1)$$



I POSITION

II POSITION

2nd position. It is the inverted position of the 1st case. Now the cone EAB is the downward portion of the vessel. The resultant vertical thrust on the surface of the cone

= wt. of the superincumbent liquid

$$= \pi r^2 hw + \frac{1}{3}\pi r^2 . 3hw = 2\pi r^2 hw \quad \ldots(2)$$

From (1) and (2), we get the required result.

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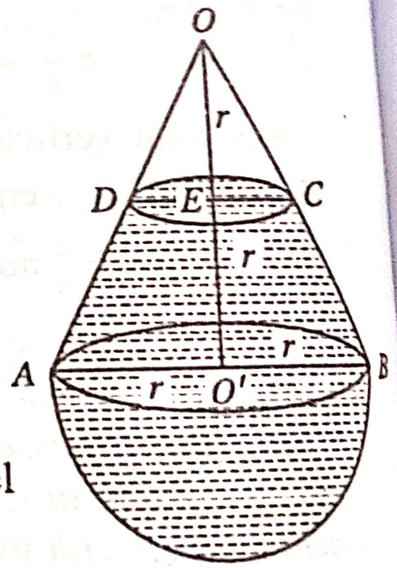
Ex. 11. *A vessel in the shape of a hollow hemisphere surmounted by a cone is held with the axis vertical and vertex uppermost. If it be filled with a liquid so as to submerge half the axis of the cone in the liquid, and height of the cone be double the radius of its base, show that the resultant downward thrust of the liquid on the vessel is $\frac{15}{8}$ times the weight of the liquid that the hemisphere can hold.*



Sol. Let r be the radius of the base of the hemisphere or cone so that the height of the surmounting cone is $2r$.

The vessel is filled upto CD so as to submerge half the axis of the cone in the liquid.

From similar triangles OEC and $OO'B$, we have

$$\frac{EC}{O'B} = \frac{OE}{OO'} = \frac{r}{2r} = \frac{1}{2}.$$

$$\therefore \qquad EC = \frac{1}{2} OB' = \frac{1}{2} r.$$

The resultant downward thrust of the liquid on the vessel

= weight of the liquid contained in the vessel

= wt. of the liquid in the hemisphere

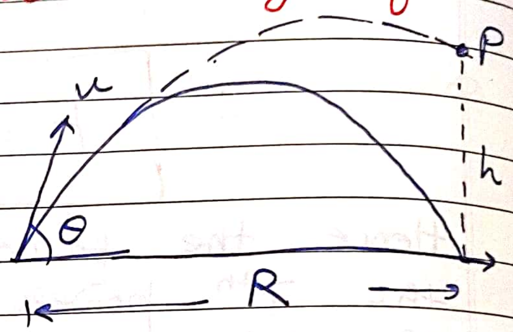
+ wt. of the liquid in the frustum

$$= \frac{2}{3}\pi r^3 w + \left[\frac{1}{3}\pi r^2 . 2r - \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 . r\right] w$$

$$= \frac{2}{3}\pi r^3 w + \frac{1}{3}\pi r^3 w\left(2 - \frac{1}{4}\right) = \frac{1}{3}\pi r^3 w\left(2 + \frac{7}{4}\right) = \frac{1}{3}\pi r^3 w \cdot \frac{15}{4}$$

$$= \frac{15}{8}\left(\frac{2}{3}\pi r^3 w\right)$$

8(b) A shot projected with a velocity 'u' can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h meter above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection.



We know that

$$x = (u\cos\theta)t$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Equation of trajectory,

$$y = x\tan\theta - \frac{g}{2}\cdot\frac{x^2}{u^2\cos^2\theta}$$

When velocity is u,

Range, $R = \dfrac{u^2\sin 2\theta}{g}$

With new velocity (say v), point P(R, h) lies on the eqn of trajectory

$$h = R\tan\theta - \frac{g}{2}\cdot\frac{R^2}{v^2\cos^2\theta}$$

$$= \frac{u^2\sin 2\theta}{g}\cdot\tan\theta - \frac{g}{2}\left(\frac{u^2\sin 2\theta}{g}\right)^2\frac{1}{v^2\cos^2\theta}$$

$$= \frac{2u^2\sin^2\theta}{g} - \frac{2u^4\sin^2\theta}{g\cdot v^2}$$

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$$h = \frac{2u^2 \sin^2\theta}{g}\left(1 - \frac{u^2}{v^2}\right)$$

$$\Rightarrow 1 - \frac{u^2}{v^2} = \frac{gh}{2u^2\sin^2\theta}$$

$$\frac{u}{v} = \left[1 - \frac{gh}{2u^2\sin^2\theta}\right]^{1/2}$$

i.e. $$v = u\left(1 - \frac{gh}{2u^2\sin^2\theta}\right)^{-1/2}$$

$$\simeq u\left(1 + \frac{1}{2}\cdot\frac{gh}{2u^2\sin^2\theta}\right)$$ (Binomial Approximation)

$$\therefore \boxed{v - u = \frac{gh}{4u\sin^2\theta}}$$

Which is the required increase in the velocity of projection with same elevation θ^0.