E.g. 1 Free particle Yup this is boring, but let's make sure it works for a particle moving in 1D with no forces, therefore no potential energy. There is only one generalized coordinate, x. The Lagrangian is simply $L = T = \frac{1}{2}m\dot{x}^2$.

Then the partial derivatives are $\frac{\partial L}{\partial x_i} = 0$ and $\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}$ and the equation of motion is

$$0 = \frac{d}{dt}(m\dot{x}) \tag{40}$$

i.e. linear momentum is constant, just a restatement of Newton's First Law.

E.g. 2 Projectile Motion in Uniform g Imagine motion in the $x - z$ plane. The generalized coordinates are just (x, z) and we can quickly write the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{z}^2) - mgz \tag{41}$$

The partials are

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial z} = -mg \qquad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{z}_i} = m\dot{z} \tag{42}$$

so the equations of motion are

$$0 = \frac{d}{dt}(m\dot{x}) \quad \text{i.e. horizontal momentum is constant} \tag{43}$$

$$-mg = \frac{d}{dt}(m\dot{z}) = m\ddot{z} \tag{44}$$

and these are just what we get from Newton's Laws

E.g. 3 Harmonic Oscillator in 1D We have a single generalized coordinate x representing the displacement of the mass from the unstretched location of the spring.

The Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ and the partials are

$$\frac{\partial L}{\partial x} = -kx \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{45}$$

and our equation of motion is simply

$$-kx = m\ddot{x} \tag{46}$$

which is just our usual differential equation.

E.g. 4 Central Force in 2D Here we have a constraint, $z = 0$ so we need 3(1)-1 = 2 generalized coordinates. We choose the polar coordinates (r, θ) and have transformation equations (First Path to Lagrangian)

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \tag{47}$$

Since the potential energy comes from a central force, it only depends on r so we write the Lagrangian

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{48}$$

The text illustrates getting the Lagrangian using the Third Path.

The partials are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = 0 \tag{49}$$

Recognizing that the force is $f(r) = -\partial V/\partial r$, we write the equations of motion as

$$m\ddot{r} = mr\dot{\theta}^2 + f(r) \qquad \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{50}$$

These are familiar from Chapter 6.

Notice that anytime a generalized coordinate (like θ for central forces) does not appear in the Lagrangian, then the conjugate momentum (yet to be discussed), $\partial L/\partial \dot{q}_i$ is constant. In this example the angular momentum, $mr^2\dot{\theta}$ is constant.

Problem 7.7.1 (Section 7.7)

Determine the moment of inertia of a right circular solid cone of uniform density about an axis through the centre of its base and its vertex. The cone has mass M, height h and the radius of its base is R.

Solution

The density of the cone $\quad \rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{M}{\frac{1}{3}\pi R^2 h} = \dfrac{3M}{\pi R^2 h}$

As illustrated in Figure 7.7.1a, the cone may be divided into thin disc segments of radius r and thickness Δx, each a distance x from the vertex, O, of the cone.

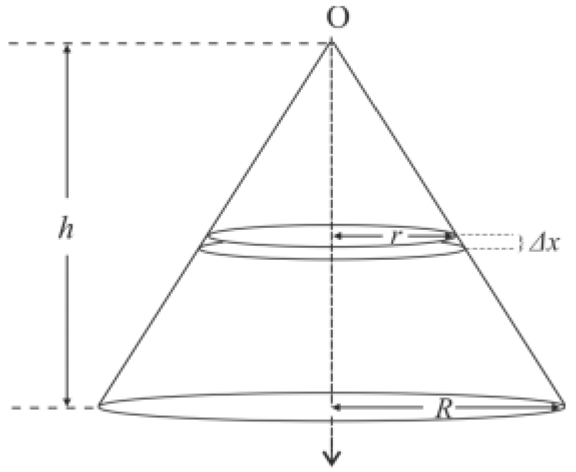


Figure 7.7.1a

From inspection of this figure $\quad \dfrac{x}{r} = \dfrac{h}{R} \quad \rightarrow \quad r = \dfrac{Rx}{h}$

The moment of inertia of the thin disc (using the Case 1 result of Section 7.7 of *Understanding Physics*)

$$\Delta I = \tfrac{1}{2}(\Delta M)r^2 = \tfrac{1}{2}(\rho\pi r^2 \Delta x)r^2 = \frac{1}{2}\frac{3M}{\pi R^2 h}\pi\left(\frac{Rx}{h}\right)^2(\Delta x)\left(\frac{Rx}{h}\right)^2 = \frac{3MR^2}{2h^5}x^4\Delta x$$

The moment of inertia of the cone is then

$$I = \int dI = \frac{3MR^2}{2h^5}\int_0^h x^4 dx = \frac{3MR^2}{2h^5}\left[\frac{x^5}{5}\right]_0^h = \frac{3MR^2}{10}$$

3). A bead slides without friction on a wire in the shape of a cycloid, with equations

$$x = a(\theta - \sin\theta), \qquad y = a(1 + \cos\theta),$$

where $0 \le \theta \le 2\pi$. Find the Lagrangian function and obtain the equations of motion.
(Answer):

$$(1 - \cos\theta)\ddot{\theta} + \frac{1}{2}\sin\theta\dot{\theta}^2 - \frac{g}{2a}\sin\theta = 0$$

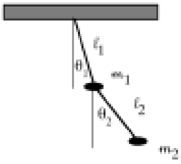
4). Consider the same problem as in (3) but with the cycloid replaced by a circular wire whose equation is

$$x = a\cos\theta, \qquad y = a(1 - \sin\theta), \qquad \text{for} \quad 0 \le \theta \le \pi$$

(Answer):

$$\dot{\theta} = \sqrt{\frac{2g\sin\theta}{a}}$$

5). A double pendulum consists of two particles suspended by massless rods. Assuming that all motion is in a vertical plane, find Lagrange's equations of motion. Assuming $m_1 = m_2$ and $l_1 = l_2$, linearize these equations, assuming small oscillations.



(Answer):

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -(m_1 + m_2) g l_1 \sin\theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = -m_2 g l_2 \sin\theta_2$$

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 = -2g\theta_1 \qquad l\ddot{\theta}_1 + l\ddot{\theta}_2 = -g\theta_2$$

6). A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed to be frictionless. Obtain the equations of motion for the particle. Eliminate the multiplier to obtain separate equations of motion for each of the variables (r, ϕ, z).

(Answer): Using cylindrical coordinates (r, ϕ, z)

$$m(\ddot{r} - r\dot{\phi}^2) = 2\lambda r, \qquad m\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0, \qquad m\ddot{z} = -mg - \lambda a, \qquad 2r\dot{r} - a\dot{z} = 0$$

The *Hamiltonian* H is defined to be the *sum* of the kinetic and potential energies:

$$H \equiv K + U \tag{28}$$

Here the Hamiltonian should be expressed as a function of position x and momentum p (rather than x and v, as in the Lagrangian), so that $H = H(x, p)$. This means that the kinetic energy should be written as $K = p^2/2m$, rather than $K = mv^2/2$.

Hamilton's equations in one dimension have the elegant nearly-symmetrical form

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p} && (29) \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial x} && (30) \end{aligned}}$$

Example: Simple Harmonic Oscillator

As an example, we may again solve the simple harmonic oscillator problem, this time using Hamiltonian mechanics. We first write down the kinetic energy K, expressed in terms of momentum p:

$$K = \frac{p^2}{2m} \tag{31}$$

As before, the potential energy of a simple harmonic oscillator is

$$U = \tfrac{1}{2}kx^2 \tag{32}$$

The Hamiltonian in this case is then

$$\begin{aligned} H(x, p) &= K + U && (33) \\ &= \frac{p^2}{2m} + \tfrac{1}{2}kx^2 && (34) \end{aligned}$$

Substituting this expression for H into the first of Hamilton's equations, we find

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial p} && (35) \\ &= \frac{\partial}{\partial p}\left(\frac{p^2}{2m} + \tfrac{1}{2}kx^2\right) && (36) \\ &= \frac{p}{m} && (37) \end{aligned}$$

Substituting for H into the second of Hamilton's equations, we get

$$\begin{aligned} \frac{dp}{dt} &= -\frac{\partial H}{\partial x} && (38) \\ &= -\frac{\partial}{\partial x}\left(\frac{p^2}{2m} + \tfrac{1}{2}kx^2\right) && (39) \\ &= -kx && (40) \end{aligned}$$

Equations (35) and (38) are two coupled first-order ordinary differential equations, which may be solved simultaneously to find $x(t)$ and $p(t)$. Note that for this example, Eq. (35) is equivalent to $p = mv$, and Eq. (38) is just Hooke's Law, $F = -kx$.

Example: Plane Pendulum

As with Lagrangian mechanics, more general coordinates (and their corresponding momenta) may be used in place of x and p. For example, in finding the motion of the simple plane pendulum, we may replace the position x with angle θ from the vertical, and the linear momentum p with the angular momentum $\mathcal{L}$.

To solve the plane pendulum problem using Hamiltonian mechanics, we first write down the kinetic energy K, expressed in terms of angular momentum $\mathcal{L}$:

$$K = \frac{\mathcal{L}^2}{2I} = \frac{\mathcal{L}^2}{2m\ell^2}, \tag{41}$$

where $I = m\ell^2$ is the moment of inertia of the pendulum. As before, the gravitational potential energy of a plane pendulum is

$$U = mg\ell(1 - \cos\theta). \tag{42}$$

The Hamiltonian in this case is then

$$\begin{aligned} H(\theta, \mathcal{L}) &= K + U && (43)\\ &= \frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos\theta) && (44) \end{aligned}$$

Substituting this expression for H into the first of Hamilton's equations, we find

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{\partial H}{\partial \mathcal{L}} && (45)\\ &= \frac{\partial}{\partial \mathcal{L}}\left[\frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos\theta)\right] && (46)\\ &= \frac{\mathcal{L}}{m\ell^2} && (47) \end{aligned}$$

Substituting for H into the second of Hamilton's equations, we get

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= -\frac{\partial H}{\partial \theta} && (48)\\ &= -\frac{\partial}{\partial \theta}\left[\frac{\mathcal{L}^2}{2m\ell^2} + mg\ell(1 - \cos\theta)\right] && (49)\\ &= -mg\ell\sin\theta && (50) \end{aligned}$$

Equations (45) and (48) are two coupled first-order ordinary differential equations, which may be solved simultaneously to find $\theta(t)$ and $\mathcal{L}(t)$. Note that for this example, Eq. (45) is equivalent to $\mathcal{L} = I\omega$, and Eq. (48) is the torque $\tau = -mg\ell\sin\theta$.

Example 1:

A sphere rolls down a rough included plane; if x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration.

Solution:

We have $T = \frac{1}{2}m(\dot{x}^2 + k^2\dot{\theta}^2) = \frac{1}{2}m\left(\dot{x}^2 + \frac{2}{5}a^2\dot{\theta}^2\right)\left(\because k^2 = \frac{2}{5}a^2\right)$

$$= \frac{1}{2}m\left(\dot{x}^2 \frac{2}{5}\dot{x}^2\right) = \frac{7}{10}m\dot{x}^2; \qquad ...(1)$$

$$V = -\,mgx\sin\alpha \qquad ...(2)$$

$$\therefore\ L = T - V = \frac{7}{10}m\dot{x}^2 + mgx\sin\alpha \qquad ...(3)$$

Now $p_x = (\partial L/\partial\dot{x}) = \frac{7}{10}m\dot{x} \Rightarrow \dot{x} = (5p_x/7m)$

Thus $H = -L + p_x\dot{x} = -\frac{7}{10}m\dot{x} - mgx\sin\alpha + p_x.(5p_x/7m)$

$$= \frac{5}{14}(p_x^2/m) - mgx\sin\alpha$$

$\therefore$ One of Hamilton's equations gives

$$p_x = -(\partial H/\partial x) = mg\sin\alpha \Rightarrow \frac{7}{5}m\ddot{x} = mg\sin\alpha \Rightarrow \ddot{x} = \frac{5}{7}g\sin\alpha$$

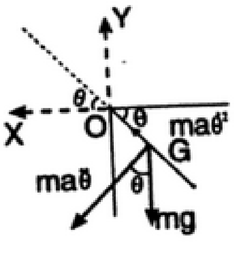
Ex.27. ***A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.***

[Meerut 1995]

Sol. Let $OA = 2a$, and let the rod make an angle θ with the horizontal after time t. Equations of motion of G along and perpendicular to GO are

$$ma\dot{\theta}^2 = Y\sin\theta + X\cos\theta - mg\sin\theta \qquad ...(1)$$

$$ma\ddot{\theta} = -Y\cos\theta + X\sin\theta + mg\cos\theta \qquad ...(2)$$



Since $k^2 = a^2 + \frac{a^2}{3} = \frac{4}{3}a^2$,

$\therefore$ moment equation about O is

$$m.\frac{4}{3}a^2\ddot{\theta} = mg.a\cos\theta \Rightarrow \ddot{\theta} = \frac{3g}{4a}\cos\theta. \qquad ...(3)$$

Intergrating (3), we get $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta + C$

when $\theta = 0,\ \dot{\theta} = 0\ \ \therefore C = 0,\ \therefore\ \dot{\theta}^2 = \frac{3g}{2a}\sin\theta.$

Putting this value of $\dot{\theta}^2$ in (1), we get

$$\frac{3}{2}mg\sin\theta = Y\sin\theta + X\cos\theta - mg\sin\theta$$

$\Rightarrow Y\sin\theta + X\cos\theta = \frac{5}{2}mg\sin\theta$ ...(4)

With the help of (3), the equation (2) becomes as

$-Y\cos\theta + X\sin\theta + mg\cos\theta = \frac{3mg}{4}\cos\theta$

$\Rightarrow -Y\cos\theta + X\sin\theta = -\frac{1}{4}mg\cos\theta$ ...(5)

Multiplying (4) by $\cos\theta$ and (5) by $\sin\theta$ and adding, we get

$X = (\frac{5}{2} - \frac{1}{4})\,mg\sin\theta\cos\theta = \frac{9}{4}mg\sin\theta\cos\theta = \frac{9}{8}mg\sin 2\theta.$

Similarly, we have $Y = mg\,(\frac{5}{2}\sin^2\theta + \frac{1}{4}\cos^2\theta).$

We observe that X is maximum when $\sin 2\theta = 1$

i.e. when $2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}.$

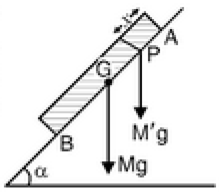
when $\theta = (\pi/4)$, we have $Y = mg\,[\frac{5}{2}\sin^2(\pi/4) + \frac{1}{4}\cos^2(\pi/4)]$

$= mg\,[\frac{5}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}] = \frac{11}{8}mg = \frac{11}{8}$ times the weight of the rod.

Ex. 4. *A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and man of mass M', starting from the upper end, walks down the plank so that it does not move; show that he gets to other end in time $\{2M'a/(M+M')g\sin\alpha\}^{1/2}$, where a is the length of the rod.*

[Agra 2006; Calcutta 2003; Kanpur 2010; Meerut 2000, 05; Purvanchel 2007; Guwahati 2007]

Sol. The plank and the man constitute a material system. The figure shows the given plank AB resting along the line of greatest slope of a given inclined plane. A man of mass M' starts moving down the plank from its upper end A. Let the man move down the plank through a distance $x(= AP)$ in time t. Since the plank does not move, hence if $\bar{x}$ is the distance of the C.G of the system from A in this position, we have



$$\bar{x} = \frac{M\cdot AG + M'\cdot AP}{M + M'} = \frac{M\times(a/2) + M'x}{M + M'}$$

Differentiating the above above relation twice w.r.t. 'x', we get

$$\frac{d^2\bar{x}}{dt^2} = \frac{M'}{M + M'}\frac{d^2x}{dt^2} \quad ...(1)$$

Now for the motion of translation of the system, we should consider only the motion of its C.G., supposing as if all the external forces act on it. Now the total weight $(M + M')g$ will act vertically downward at the C.G. of the system. Hence the equation of motion of the C.G. of the system along the plane is $$(M + M')(d^2\bar{x}/dt^2) = (M + M')g \sin\alpha$$

or $$M'(d^2x/dt^2) = (M + M')g \sin\alpha, \text{ using (1)} \quad \ldots(2)$$

Integrating (2), $$M'(dx/dt) = (M + M')g\, t \sin\alpha + C \quad \ldots(3)$$

But initially, when $t = 0$, $dx/dt = 0$. So (3) gives $C = 0$.

Then, (3) becomes $$M'(dx/dt) = (M + M')gt \sin\alpha \quad \ldots(4)$$

Integrating (4), $$M'x = (1/2) \times (M + M')g\, t^2 \sin\alpha + C' \quad \ldots(5)$$

But initially, when $t = 0$, $x = 0$. So (5) gives $C' = 0$

$\therefore$ (5) becomes $$M'x = (1/2) \times (M + M')gt^2 \sin\alpha$$

so that $$t = \{2M'x/(M + M')g \sin\alpha\}^{1/2} \quad \ldots(6)$$

Putting $x = a$ in (6), the required time $= \{2M'a/(M + M')g \sin\alpha\}^{1/2}$

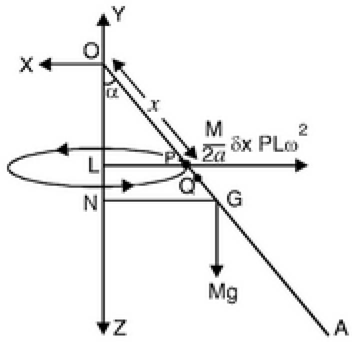
Ex. 5. **(*a*)** *A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or* $\cos^{-1}(3g/4a\omega^2)$.

[Agra 2009, 11; Guwahati 2007; Kanpur 2006, 2008]

(*b*) *A rod, of length 2a, revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that* $\omega^2 = 3g/(4a\cos\alpha)$. *Prove also that the direction of reaction at the hinge makes with the vertical an angle* $\tan^{-1}\{(3/4)\tan\alpha\}$.

[Meerut 2000, 2011; Kanpur 2011]

Sol. (*a*) Take an element $PQ(=\delta x)$ at a distance x from O, such that $OP = x$. The mass of the element PQ is $(M/2a)\,\delta x$. Draw PL perpendicular to OZ. Then element PQ will describe a circle of radius $PL(= x\sin\alpha)$ about L. Hence the effective force on this element PQ is $(M/2a)\,\delta x \cdot PL\omega^2$ along PL. So the reversed effective force on the element PQ is $(M/2a)\,\delta x \cdot x\sin\alpha \cdot \omega^2$ along LP as shown in the figure.



Now by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, namely weight Mg and reaction at O are in equilibrium. To avoid reaction at O, taking moments about O, we have

$$Mg \cdot a\sin\alpha - \{\Sigma(M/2a)\,\delta x \cdot \omega^2 x \sin\alpha\} \cdot x\cos\alpha = 0$$

or $$Mg\, a\sin\alpha - \frac{M\omega^2 \sin\alpha\cos\alpha}{2a}\int_0^{2a} x^2\, dx = 0$$

or $$Mg\, a\sin\alpha - (M/2a)\omega^2 \sin\alpha\cos\alpha\,\{(2a)^3/3\} = 0$$

or $$Mg\, a\sin\alpha\,\{1 - (4a/3g)\omega^2\cos\alpha\} = 0$$

giving either $\sin\alpha = 0$ *i.e.*, $\alpha = 0$ or $\cos\alpha = 3g/4a\omega^2$ ...(1)

Hence, the rod is inclined at an angle zero or $\cos^{-1}(3g/4a\omega^2)$

Remark. If $\omega^2 < 3g/4a$, then $\cos\alpha > 1$ and so in this case second value of α is not possible and hence $\alpha = 0$ is the only possible value.

(***b***) For first part refer part (*a*). To find the direction of reaction at the hinge O, let X, Y be the horizontal and vertical components of reaction at O. Then resolving the forces horizontally and vertically, we get

$$X = \Sigma\frac{M}{2a}\delta x\,\omega^2 x\sin\alpha = \int_0^{2a}\frac{M\omega^2\sin\alpha}{2a}x\, dx = Ma\omega^2\sin\alpha$$

and $$Y = Mg$$

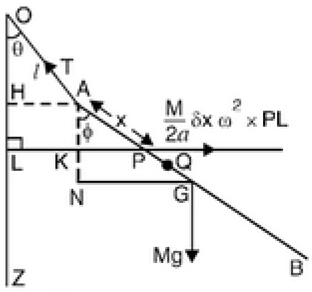
Let the reaction at O make an angle θ with vertical. Then

$$\tan\theta = \frac{X}{Y} = \frac{Maw^2\sin\alpha}{Mg} = \frac{a\sin\alpha}{g}\cdot\frac{3g}{4\cos\alpha}, \text{ using (1)}$$

or $\tan\theta = (3/4)\tan\alpha$ so that $\theta = \tan^{-1}\{(3/4)\tan\alpha\}$

Ex. 6. *A rod of length 2a, is suspended by a string of length l, attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, and their inclination to the vertical be θ and φ respectively, show that 3l/a = (4 tan θ − 3 tan φ) sin φ/(tan φ − tan θ) sin θ.*

[Kanpur 2011, I.F.S. 2003; Meerut 2002, 07; Purvanchel 2005]



Sol. Let AB be the given rod of length $2a$ and mass M. Suppose the rod is suspended by a string OA of length l. Let the string and the rod revolve about the vertical line OZ with uniform angular velocity ω and their inclination to the vertical be θ and ϕ respectively as in the figure.

Take an element $PQ(= \delta x)$ at a distance x from A. Then mass of this element is $(M/2a)\delta x$. Drop perpendicular PL to the vertical line OZ through O. Then $PL = PK + KL = PK + AH = x \sin\phi + l \sin\theta$

As the rod revolves about OZ, the element δx will describe a circle of radius PL in the horizontal plane. As shown in the figure, the reversed effective force on element δx along LP will be $(M/2a)\, \delta x\, \omega^2\, PL$ *i.e.*, $(M/2a)\, \delta x\, \omega^2\, (x \sin\phi + l \sin\theta)$. Again the external forces acting on the rod are (*i*) tension T at A along AO and (*ii*) its weight Mg acting vertically downwards at its middle point G.

For equilibrium of the rod resolving horizontally and vertically the forces acting on the rod, we get

$$T \sin\theta = \Sigma \frac{m}{2a} \delta x\, \omega^2 (x \sin\phi + l \sin\theta) = \int_0^{2a} \frac{ma}{2a} \omega^2 (x \sin\phi + l \sin\theta)\, dx$$

or $T \sin\theta = (M/2a)\omega^2[(l/2)x^2 \sin\phi + lx \sin\theta]_0^{2a} = M\omega^2(a \sin\phi + l \sin\theta)$ ...(1)

and $T \cos\theta = Mg$ ...(2)

Finally, taking moments about A of all forces acting on the rod, we get

$$-Mg \cdot NG + \Sigma (M/2a)\, \delta x\, \omega^2\, (x \sin\phi + l \sin\theta) \cdot AK = 0$$

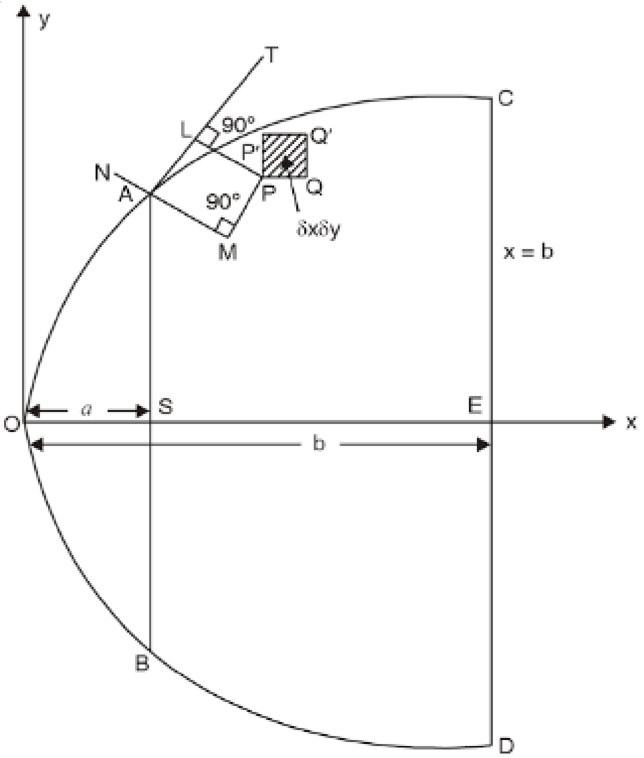
or $$Mg\, a \sin\phi = \int_0^{2a} \frac{M\omega^2}{2a} (x \sin\phi + l \sin\theta) \cdot x \cos\phi\, dx$$

Ex. 63. *A uniform lamina is bounded by a parabolic arc, of latusrectum 4a, and a double ordinate at a distance b from the vertex. If* $b = a(7+4\sqrt{7})/3$, *show that two of the principle axes at the end of a latus rectum are the tangent and normal there.*

Sol. Let the equation of given parabolic arc COD be given by

$$y^2 = 4ax \qquad \text{... (1)}$$

Let the given lamina COD be bounded by double ordinate CD such that $OE = b$. Let AB be the latusrectum of the parabola (1) so that $OS = a$ and $AS = BS = 2a$. Hence, the coordinates of the end A of the latus rectum are $(a, 2a)$. Let coordinates of any point P of the lamina be (x, y) and let $PQQ'P'$ be an elementary area $\delta x \delta y$ of the lamina COD.



13. *Show that the moment of inertia of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is the perpendicular from the centre on the tangent.*

Sol. Equation to any tangent to the ellipse is

$$y = mx + \sqrt{(a^2m^2+b^2)} \text{ where } m = \tan\theta.$$

$\therefore$ p = perpendicular from (0, 0) on the tangent

$$= \frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(1+m^2)}} = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}, \text{ putting } m = \tan\theta.$$

Now M.I. of the ellipse about x-axis $= \frac{1}{4}Mb^2$ and M.I. of the ellipse about y-axis $= \frac{1}{4}Ma^2$.

Also P.I. of the ellipse about these axes = 0.

$\therefore$ M.I. of the ellipse about the diameter parallel to the tangent

$$= \frac{1}{4}Mb^2 . \cos^2\theta + \frac{1}{4}Ma^2\sin^2\theta = \frac{1}{4}M . p^2.$$

$\therefore$ The required M.I. about the tangent

$$= \frac{M}{4}p^2 + Mp^2 = \frac{5}{4}Mp^2.$$

14. *Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same.*

Sol. M.I. about a tangent inclined at an angle θ with the major axis $= \frac{5}{4}Mp^2$ (See Ex. 13 above)

$$= \frac{5}{4}M(a^2\sin^2\theta + b^2\cos^2\theta)$$

and replacing θ by $\frac{\pi}{\theta} - \theta$, we get M. I. about a perpendicular tangent $= \frac{5}{4}M(a^2\cos^2\theta + b^2\sin^2\theta)$.

Hence the sum of the moments of inertia about two perpendicular tangents

$$\frac{5}{4}M[a^2+b^2],$$

which is always the same, being independent of θ.

Example 7.39. *A solid body of density ρ is in the shape of the solid formed by revolution of the centroid $r = a(1 + \cos\theta)$ about the initial line. Show that its moment of inertia about a straight line through the pole perpendicular to the initial line is* $\frac{352}{105}\pi\rho a^5$. (U.P.T.U., 2001)

Solution. An elementary area $rd\theta dr$, when revolved about OX generates a circular ring of radius $LP = r\sin\theta$ (Fig. 7.41).

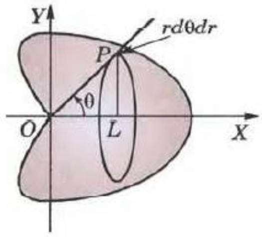


Fig. 7.41

M.I. of this ring about a diameter parallel to OY

$$= (2\pi r\sin\theta)(rd\theta dr)\rho\cdot\frac{(r\sin\theta)^2}{2}.$$

[$\because$ M.I. of a ring about a diameter $= Ma^2/2$.]

Now using Steiner's theorem, we have M.I. of the ring about OY = M.I. of the ring about a diameter LP parallel to OY + Mass of the ring $(OL)^2$ $(r\cos\theta)^2$

$$= 2\pi\rho r^4\sin^3\theta\, d\theta dr + 2\pi r\sin\theta\,(rd\theta dr)(r\cos\theta)^2$$

Hence M.I. of the solid generated by revolution about OY

$$= \pi\rho\int_0^{\pi}\int_0^{r=a(1+\cos\theta)}(r^4\sin^3\theta + 2r^4\sin\theta\cos^2\theta)\,d\theta dr$$

$$= \pi\rho\int_0^{\pi}(\sin^3\theta + 2\sin\theta\cos^2\theta)\,d\theta\int_0^{a(1+\cos\theta)} r^4\,dr$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi}\sin\theta\,(1+\cos^2\theta)(1+\cos\theta)^5\,d\theta \qquad [\text{Put } \theta = 2\phi]$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi/2}\sin 2\phi\,(1+\cos^2 2\phi)(1+\cos 2\phi)^5\,2d\phi$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi/2} 2\sin\phi\cos\phi\,\{1+(2\cos^2\phi-1)^2\}\,(2\cos^2\phi)^5\,2d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\int_0^{\pi/2}(\cos^{11}\phi - 2\cos^{13}\phi + 2\cos^{15}\phi)\sin\phi\,d\phi$$

$$= \frac{256\,\pi\rho a^5}{5}\left|-\frac{\cos^{12}\phi}{12}+\frac{2\cos^{14}\phi}{14}-\frac{2\cos^{16}\phi}{16}\right|_0^{\pi/2} = \frac{352\,\pi\rho a^5}{105}.$$

7.13 (1) PRODUCT OF INERTIA

If a particle of mass m of a body be at distances x and y from two given perpendicular lines, then $\sum mxy$ is called the *product of inertia* of the body about the given lines.

Consider an elementary mass $\delta x\delta y\delta z$ enclosing the point $P(x, y, z)$ of solid of volume V. Then the product of inertia (P.I.) of this element about the axes of x and $y = \rho\delta x\delta y\delta z\; xy$.

$\therefore$ P.I. of the solid about x and y-axes, *i.e.*, $P_{xy} = \iiint_V \rho xy\,dxdydz$

Similarly, $P_{yz} = \iiint_V \rho yz\,dxdydz$ and $P_{zx} = \iiint_V \rho zx\,dxdydz$.

In particular, for a plane lamina of surface density ρ and covering a region A in the xy-plane,

$$P_{xy} = \iint_A \rho xy\,dxdy \text{ whereas } P_{yz} = P_{zx} = 0. \qquad [\because\; z = 0]$$