Mains Test Series - 2018

Test - 16, Paper - II, Answer Key

1(a) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ in S_5

Find a permutation γ in S_5 such that $\alpha\gamma = \beta$.

Sol^n: Here $\alpha, \beta, \gamma \in S_5$

Given $\alpha\gamma = \beta$

$$\alpha^{-1}\alpha\gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 1 \end{pmatrix}$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

$$\therefore \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

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1(b) Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} / a \in \mathbb{R}, a \neq 0 \right\}$ show that G is a group under matrix multiplication. Explain why each element of G has an inverse even though the matrices have 0 determinants.

Sol'n: Hint: $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$

and $2ab \neq 0$, we have closure, matrix multiplication is associative.

from the product above we observe that the identity is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and the

inverse of $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{bmatrix}$

The group $GL(2, \mathbb{R})$ has a diffrent identity than the group G.

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1(c) Prove that the sequence $\{a_n\}$ recursively defined by $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5+a_n}$, $n \geq 1$ converges to the +ve root of the equation $x^2 - x - 5 = 0$.

Given that $a_1 = \sqrt{5}$, $a_{n+1} = \sqrt{5+x_n}$

$x_2 = \sqrt{5+x_1} = \sqrt{5+\sqrt{5}} > \sqrt{5} = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$

Suppose $x_{n+1} > x_n$ for some 'n'

$\Rightarrow 5+x_{n+1} > 5+x_n$

$\Rightarrow \sqrt{5+x_{n+1}} > \sqrt{5+x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$

$\therefore$ By mathematical induction $x_{n+1} > x_n \; \forall n$

$\therefore (x_n)$ is monotonically increasing

Now $x_1 = \sqrt{5} < 5$

$x_2 = \sqrt{5+\sqrt{5}} < 5$

similarly $x_3 < 5$

Suppose $x_n < 5$

$\Rightarrow 5+x_n < 10$

$\Rightarrow \sqrt{5+x_n} < \sqrt{10} < \sqrt{25} = 5$

$\therefore x_{n+1} < 5$

$\therefore$ By mathematical induction $x_n < 5 \; \forall n$

$\therefore (x_n)$ is bounded above.

Since (x_n) is monotonically increasing and bounded above.

$\therefore$ It is convergent.

$\lim_{n\to\infty} x_n = l$ & $\lim_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{5+x_n} \Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{5+\lim_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{5+l}$

$\Rightarrow l^2 - l - 5 = 0$

$\Rightarrow l = \frac{1 \pm \sqrt{21}}{2}$, but $l = \frac{1-\sqrt{21}}{2} < 0$ whereas $x_n > 0 \; \forall n$

$\therefore l \neq \frac{1-\sqrt{21}}{2}$, $\therefore x_n$ cgs to $\frac{1+\sqrt{21}}{2}$ which is +ve root of the eqn $x^2 - x - 5 = 0$

1(d) If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ interms of z.

Sol^n: Now $f(z)=u+iv$ so that $if(z)=iu-v$

$F(z)=(1+i)f(z)=(u-v)+i(u+v)=U+iV$, say

Here $U=u-v=(x-y)(x^2+4xy+y^2)$

$$\frac{\partial U}{\partial x}=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=x^2+4xy+y^2+(x-y)(2x+4y)$$
$$=3x^2+6xy-3y^2$$

$$\frac{\partial U}{\partial y}=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=-(x^2+4xy+y^2)+(x-y)(4x+2y)$$
$$=3x^2-6xy-3y^2.$$

Let $\frac{\partial U}{\partial x}=\phi_1(x,y)$ and $\frac{\partial U}{\partial y}=\phi_2(x,y)$.

By Milne's method we have

$$F'(z)=\phi_1(z,0)-i\phi_2(z,0).$$
$$F(z)=\int[\phi_1(z,0)-i\phi_2(z,0)]\,dz+C.$$
$$=\int(3z^2-3iz^2)\,dz+C$$
$$=3\int(1-i)z^2\,dz+C$$
$$\Rightarrow F(z)=(1-i)z^3+C$$
$$\Rightarrow (1+i)f(z)=\frac{1-i}{}z^3+\frac{C}{}.$$
$$\Rightarrow f(z)=\frac{1-i}{1+i}z^3+\frac{C}{1+i}.$$
$$\Rightarrow f(z)=-iz^3+C_1 \quad \text{where } C_1=\frac{C}{1+i}.$$

1(e) If $x_1=2,\ x_2=3,\ x_3=1$ is a feasible solution of the LPP

Maximize $Z = x_1+2x_2+4x_3$

subject to $2x_1+x_2+4x_3=11$

$3x_1+x_2+5x_3=14$

$x_1, x_2, x_3 \geq 0$.

find a basic feasible solution of the problem

Sol^n: The given system of equations may be put in matrix notations as $\begin{pmatrix}2 & 1 & 4\\ 3 & 1 & 5\end{pmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{pmatrix}11\\ 14\end{pmatrix}$

$\Rightarrow AX=B$.

where $\begin{pmatrix}2 & 1 & 4\\ 3 & 1 & 5\end{pmatrix}$; $B=\begin{pmatrix}11\\ 14\end{pmatrix}$; $X=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

Let the columns of A be denoted by

$A_1=\begin{pmatrix}2\\ 3\end{pmatrix}$, $A_2=\begin{pmatrix}1\\ 1\end{pmatrix}$, $A_3=\begin{pmatrix}4\\ 5\end{pmatrix}$

Here $\rho(A)=2$

∴ A basic solution to the given system of equations exist with not more than two variable different from zero

Also, the column vectors A_1, A_2, A_3 are linear dependent (we can easily verify)

$A_1\lambda_1+A_2\lambda_2+A_3\lambda_3=0$

$\Rightarrow 2\lambda_1+\lambda_2+4\lambda_3=0$

$3\lambda_1+\lambda_2+5\lambda_3=0$

Clearly this is a system of two equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$.

Let us choose one of the λ_i arbitrarily

say $\lambda_1 = 1$.

$\therefore \lambda_2 + 4\lambda_3 = -2$

$\lambda_2 + 5\lambda_3 = -3$

solving, we get $\lambda_2 = 2$, $\lambda_3 = -1$

To reduce the no. of +ve variables, the variable to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \text{Min}\left\{\frac{x_i}{\lambda_i} \,/\, \lambda_i > 0\right\}$$

$$= \text{Min}\left\{\frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3}\right\} = \text{Min}\left\{\frac{2}{1}, \frac{3}{2}, \frac{1}{-1}\right\} = \frac{3}{2}$$

Thus, we can remove vector A_2 for which $\frac{x_2}{\lambda_2} = \frac{3}{2}$ and obtain new solution with not more than two non-negative (non-zero) variables

The variables of new are given by

$\hat{x}_1 = x_1 - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2}$

$\hat{x}_2 = x_2 - \frac{3}{2}(2) = 3 - 3 = 0$

$\hat{x}_3 = x_3 - \frac{3}{2}(-1) = 1 + \frac{3}{2} = \frac{5}{2}$.

$\therefore$ The basic feasible solution is

$x_1 = \frac{1}{2}$, $x_3 = \frac{5}{2}$ with $x_2 = 0$ (non-basic)

2(a) let H be a subgroup of a group G such that $(G:H)=2$ then prove that H is a normal subgroup of G.
Is converse true? Justify your answer.

Sol let $(G,\times)$ be a group and $H \leq G$ such that $(G:H)=2$

To P.T $H \trianglelefteq G$.

$\because (G:H)=2$

There are two distinct left & right cosets of H in G.

let H & Ha ; H & aH be two distinct right cosets & left cosets of H in G; $a \in G$

then $G = H \cup Ha = H \cup aH$ ——— (1)

$\because a \in G \Rightarrow a \in H$ or $a \notin H$

Case(i) when $a \in H$

$\therefore Ha = H = aH$

$\Rightarrow Ha = aH$.

case(ii) when $a \notin H$

$\therefore Ha \neq H$ & $aH \neq H$

$\because G = H \cup Ha$ & $G = H \cup aH$

$\therefore H \cup Ha = H \cup aH$

$\Rightarrow Ha = aH$

$\therefore Ha = aH \;\; \forall \; a \in G$.

$\therefore H \trianglelefteq G$.

The converse of the above need not be true.

i.e If $H \trianglelefteq G$ then $(G:H) \neq 2$

For example:

let $G = \{1, -1, i, -i, j, -j, k, -k\}$ be a group

let $H = \{1, -1\}$

clearly $H \trianglelefteq G$

but $(G:H) = 4$

$\neq 2$.

2(b) Find all the subgroups of $\frac{Z}{21Z}$.

Sol let $(Z = \{ \cdots -3, -2, -1, 0, 1, 2, \cdots \}, +)$ be a group.

let $H = 21Z = \{ 21x \mid x \in I \}$

$= \{ \cdots -21, 0, 21, \cdots \}$

clearly $21Z \trianglelefteq Z$

$\therefore \frac{Z}{21Z} = \frac{Z}{H} = \{ H + a \mid a \in Z \}$ is a quotient group w.r.t ~~+~~ of cosets.

let $K \leq \frac{Z}{21Z}$ Then $K = \frac{H_1}{21Z}$ for some subgroup H_1 of Z such that $21Z \subseteq H_1$.

Again, if $H_1 \leq Z$ such that $21Z \subseteq H_1$ then $\frac{H_1}{21Z} \leq \frac{Z}{21Z}$

Now we have to determine all subgroups of Z such that $21Z$.

Now $1, 3, 7, 21$ are the only +ve divisors of 21

$\therefore Z, 3Z, 7Z,$ and are the only subgroups of Z that contain $21Z$.

Then $\frac{Z}{21Z}, \frac{3Z}{21Z}, \frac{7Z}{21Z}$ and $\frac{21Z}{21Z}$ are the only subgroups of $\frac{Z}{21Z}$.

2(c) → Let $R = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in M_2(\mathbb{C}) \,/\, \bar{\alpha}, \bar{\beta} \text{ denote the Conjugates of } \alpha, \beta \right\}$

Define addition + and multiplication . in R by usual matrix addition and matrix multiplication. Show that R is a division ring but not a field

Sol^n: Let $A = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ and $B = \begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$

Adding, A & B,

$$A+B = \begin{bmatrix} (a+r)+i(b+t) & (c+u)+i(d+v) \\ -(c+u)-i(d+v) & (a+r)-i(b+t) \end{bmatrix} \in R$$

and $AB = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$ (property of matrices and Complex numbers)

Here $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$ is zero element for any $A \in R$, we have $-A \in R$ such that

$$A+(-A) = 0$$

further, distributive properties hold

Hence R is a ring with identity

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R.$$

Now $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \in R$

and $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \in R$

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

$\therefore$ R is a non-commutative ring.

Let $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ be a non-zero element of R.

Then either $a+ib \neq 0$ or $c+id \neq 0$

i.e., $a^2+b^2 \neq 0$ or $c^2+d^2 \neq 0$.

Hence $a^2+b^2+c^2+d^2 \neq 0$.

Let $k = a^2+b^2+c^2+d^2$.

Note that $\frac{1}{k}\begin{bmatrix} a-ib & c-id \\ -(c+id) & a+ib \end{bmatrix} \in R$.

is the inverse of $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \in R$.

Hence each non-zero element of R has an inverse in R. Hence R is a division ring.

But R is non-commutative; clearly R is not a field.

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2(d) Every irreducible element in $R[x]$ is an irreducible polynomial, R being an integral domain with unity.

sol^n: let a be any irreducible element of R then we have to P.T a is also an irreducible element of $R[x]$.

If possible let a be not an irreducible element of $R[x]$

then we have

$a = f(x) \cdot g(x)$, where $f(x), g(x) \in R[x]$ and $f(x), g(x)$ are both non-unit in $R[x]$. — (1)

we know that the units of R and $R[x]$ are the same.

$\therefore$ $f(x)$ & $g(x)$ cannot be in R.

let $f(x) \notin R$, from (1), we have

$$\deg a = \deg (f(x) \cdot g(x)) = \deg f(x) + \deg g(x) \quad (\because R \text{ is I.D})$$

$$\Rightarrow 0 = \deg f(x) + \deg g(x) \quad (\because a \in R \Rightarrow \deg a = 0)$$

$$\Rightarrow \deg f(x) = 0 \text{ and } \deg g(x) = 0$$

If $\deg f(x) = 0$ then $f(x)$ is a constant polynomial of the form $f(x) = \alpha$ where $\alpha \neq 0 \in R$

$\Rightarrow f(x) \in R$ which is a contradiction.

Hence a is an irreducible element in $R[x]$

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3(a) Prove that the function f defined on $\mathbb{R}$ by $f(x)=\frac{1}{x^2+1}$, $x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Soln: Given that $f(x)=\frac{1}{x^2+1}$, $x\in\mathbb{R}$

$$f'(x)=\frac{-2x}{(x^2+1)^2}, \; x\in\mathbb{R}.$$

$$\therefore |f'(x)|<2 \quad \forall x\in\mathbb{R}$$

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $x_1<x_2$.

f is continuous on $[x_1,x_2]$ and f is differentiable on (x_1,x_2).

By the mean value theorem, $\exists$ a point $\xi\in(x_1,x_2)$ such that $\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(\xi)$

Since $|f'(x)|<2 \;\forall x\in\mathbb{R}$,

$$|f(x_1)-f(x_2)|<2|x_2-x_1|$$

Let us choose $\epsilon>0$, $\exists$ a positive $\delta(=\epsilon/2)$ such that $|f(x_1)-f(x_2)|<\epsilon \;\forall x_1,x_2$ in $\mathbb{R}$ satisfying $|x_2-x_1|<\delta$.

This proves that f is uniformly continuous on $\mathbb{R}$.

3(c) Discuss the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \frac{5^5x^5}{5!} + \cdots$$

Sol^n: Here $u_n = \frac{n^n x^n}{n!}$ and $u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1}} \frac{1}{x^{n+1}}$$

$$= \frac{n^n}{n!} \frac{(n+1)\, n!}{(n+1)^{n+1}} \cdot \frac{1}{x}$$

$$= \frac{n^n}{(n+1)^n} \cdot \frac{1}{x} = \frac{1}{(1+\frac{1}{n})^n} \frac{1}{x}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{ex}$$

$\therefore$ By D'Alembert's Ratio test, the series converges

if $\frac{1}{ex} > 1$, i.e. if $x < \frac{1}{e}$

and diverges if $\frac{1}{ex} < 1$ or $x > \frac{1}{e}$.

If $x = \frac{1}{e}$, then the ratio test fails.

Now, when $x = \frac{1}{e}$, we have

$$\frac{u_n}{u_{n+1}} = \frac{1}{(1+\frac{1}{n})^n} \cdot e$$

Since $\frac{u_n}{u_{n+1}}$ involves the number e, we prefer to apply log test rather than Raabe's test

$$\therefore \log \frac{u_n}{u_{n+1}} = \log e - n \log(1+\tfrac{1}{n}).$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} - \cdots$$

$\therefore \underset{n\to\infty}{Lt}\; n\log\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\left(\frac{1}{2}-\frac{1}{3n}+\cdots\right)$

$= \frac{1}{2} < 1$

$\therefore$ The series diverges by log test.

Hence the given series Σu_n converges if $x<\frac{1}{e}$ and diverges if $x\geq\frac{1}{e}$.

3(b) Given the series $\sum_{n=1}^{\infty} f_n$ for which $S_n(x)=\frac{1}{2n^2}\log(1+n^4x^2)$, $0\leq x\leq 1$. Show that the series $\sum_{n=1}^{\infty} f_n'$ does not converge uniformly, but the given series can be differentiated term by term.

Sol'n: Here $f(x) = \underset{n\to\infty}{Lt}\, S_n(x) = \underset{n\to\infty}{Lt}\,\frac{\log(1+n^4x^2)}{2n^2}$ (form $\frac{\infty}{\infty}$)

$$= \underset{n\to\infty}{Lt}\,\frac{\frac{4n^3x^2}{1+n^4x^2}}{4n}$$

$$= \underset{n\to\infty}{Lt}\,\frac{n^2x^2}{1+n^4x^2} = 0 \text{ for } 0\leq x\leq 1$$

$\therefore f'(x)=0$

Also $\underset{n\to\infty}{Lt}\, S_n'(x) = \underset{n\to\infty}{Lt}\left(\frac{1}{2n^2}\,\frac{2n^4x}{1+n^4x^2}\right)$

$$= \underset{n\to\infty}{Lt}\,\frac{n^2x}{1+n^4x^2} = 0 \text{ for } 0\leq x\leq 1$$

$\therefore f'(x) = Lt\, S_n'(x)$

Thus term by term differentiable.

However, the series $\sum_{n=1}^{\infty} f_n'$ is not uniformly convergent for $0\leq x\leq 1$. Since the sequence $\langle S_n'\rangle$ i.e. $\left\langle\frac{n^2x}{1+n^4x^2}\right\rangle$ has $x=0$ as a point of non-uniform convergence.

3(d)(i) Is the intersection of an arbitrary collection of open sets open? Justify your answer by a proof or by a counter example.

Sol^n: The intersection of an arbitrary family of open sets may or may not be an open set.

for example

(i) Let $I_n = (0,n)$, $n \in N$.

Then $\{I_n\}_{n\in N}$ is an infinite family of open sets.

$$\bigcap_{n\in N} I_n = I_1 \cap I_2 \cap I_3 \cap \cdots I_n, \cdots$$
$$= (0,1)\cap(0,2)\cap(0,3)\cap \cdots (0,n)\cap \cdots$$
$$= (0,1)$$

which is an open set.

(ii) Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in N$.

Then $\{I_n\}_{n\in N}$ is an infinite family of open sets.

$$\bigcap_{n\in N} I_n = I_1 \cap I_2 \cap I_3 \cap \cdots \cap I_n \cap \cdots$$
$$= (-1,1)\cap\left(-\tfrac{1}{2},\tfrac{1}{2}\right)\cap \cdots \cap \left(-\tfrac{1}{n},\tfrac{1}{n}\right)\cap \cdots$$
$$= \{0\}$$

which is not an open set.

3(d)(ii) Show that the union of an infinite number of closed sets in IR is not necessarily a closed set.

Sol'n - Let us consider the sets F_i where

$F_1 = \{x \in \mathbb{R} / -1 \leq x \leq 1\}$

$F_2 = \{x \in \mathbb{R} / -\frac{1}{2} \leq x \leq \frac{1}{2}\}$

$\vdots$

$F_n = \{x \in \mathbb{R} / -\frac{1}{n} \leq x \leq \frac{1}{n}\}$

Here each F_i is a closed set

as $\bigcup_{i=1} F_i = F_1$. ∴ clearly F_1 is a closed set.

Let us consider the sets F_i, where

$F_1 = \{x \in \mathbb{R} / 1 \leq x \leq 2\}$ $3-\frac{1}{1}$

$F_2 = \{x \in \mathbb{R} / \frac{1}{2} \leq x \leq 3-\frac{1}{2}\}$

$\vdots$

$F_n = \{x \in \mathbb{R} / \frac{1}{n} \leq x \leq 3-\frac{1}{n}\}$

Here each F_i is closed set.

$\bigcup_{i=1} F_i = \{x \in \mathbb{R} / 0 < x < 3\}$

clearly it is not a closed set.

∴ These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

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4(a) Show that the function of defined by
$$f(z) = u+iv = \begin{cases} \frac{\operatorname{Im}(z^2)}{\bar{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
satisfies the Cauchy-Riemann equations at the origin, yet it is not differentiable there.

Sol^n: The function f defined by
$$f(z) = u+iv = \begin{cases} \frac{\operatorname{Im}(z^2)}{\bar{z}} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

$$= \begin{cases} \frac{2xy}{x-iy} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

$z = x+iy$
$z^2 = (x+iy)^2$
$= x^2-y^2+2ixy$
$\operatorname{Im}(z^2) = 2xy$

$$= \begin{cases} \frac{2xy(x+iy)}{x^2+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

Here $u = \frac{2x^2y}{x^2+y^2}$; $v = \frac{2xy^2}{x^2+y^2}$ where $x \neq 0, y \neq 0$

To show that Cauchy-Riemann equations are satisfied at $z=0$:

Since $f(0) = 0$

$\Rightarrow u(0,0) + iv(0,0) = 0$

$\Rightarrow u(0,0) = v(0,0) = 0$

Now $\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

Similarly $\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = 0$ and $\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = 0$.

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z=0$

$\therefore$ Cauchy - Riemann equations are satisfied

To prove that $f(z)$ doesnot differentiable at $(0,0)$:

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z}$$

$$= \underset{(x,y)\to 0}{Lt} \frac{2xy(x+iy)}{(x^2+y^2)(x+iy)}$$

Let $(x,y) \longrightarrow (0,0)$ along $y=mx$

$$f'(0) = \underset{x\to 0}{Lt} \frac{2m^2x}{x^2+m^2x^2}$$

$$= \underset{x\to 0}{Lt} \frac{2m}{1+m^2}$$

clearly which depends on m.

$\therefore f'(0)$ doesnot exist

$\therefore f(z)$ doesnot differentiable at $(0,0)$

$\therefore$ The given function f satisfies C-R equations. although it is not differentiable at $(0,0)$

4(b) Use the method of contour integration to prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}, \quad a^2>b^2+c^2$$

Sol'n:
$$I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2} \qquad a^2>b^2+c^2$$

$$= \int_0^{2\pi} \frac{d\theta}{a+\frac{b}{2}(e^{i\theta}+e^{-i\theta})+\frac{c}{2i}(e^{i\theta}-e^{-i\theta})^2}$$

write $e^{i\theta} = z \Rightarrow d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{dz/iz}{\left[a + \frac{b}{2}\left(z+\frac{1}{z}\right) + \frac{c}{2i}\left(z-\frac{1}{z}\right)\right]^2}$$

$$= \int_C \frac{4iz\,dz}{\left[2iza + bi(z^2+1) + c(z^2-1)\right]^2}$$

$$= \int_C \frac{4iz\,dz}{\left[z^2(bi+c) + 2iza + (bi-c)\right]^2}$$

$$= \frac{4i}{(bi+c)^2}\int_C \frac{z\,dz}{\left[z^2 + \frac{2iaz}{bi+c} + \frac{bi-c}{bi+c}\right]^2} = \frac{4i}{(bi+c)^2}\int_C f(z)\,dz \quad \text{—— (1)}$$

where C is the unit circle.

f(z) has poles of order 2 given by

$$z^2 + \frac{2ia}{bi+c}z + \frac{bi-c}{bi+c} = 0$$

$$\Rightarrow z = \frac{-\frac{2ia}{bi+c} \pm \sqrt{\frac{-4a^2}{(bi+c)^2} - \frac{4(bi-c)}{bi+c}}}{2}$$

$$\Rightarrow z = \frac{1}{2}\left[\frac{-2ic}{bi+c} \pm \frac{2i\sqrt{a^2-b^2-c^2}}{bi+c}\right]$$

$$\Rightarrow z = \frac{i}{bi+c}\left[-a \pm \sqrt{a^2-b^2-c^2}\right]$$

$z = \frac{i}{bi+c}\left(-a+\sqrt{a^2-b^2-c^2}\right)$ & $z = \frac{i}{(bi+c)}\left(-a-\sqrt{a^2-b^2-c^2}\right)$

$= \alpha$ (say) $\qquad = \beta$ (say)

only $z=\alpha$ lies inside C of order 2.

Residue at $z=\alpha$ is

$$\lim_{z\to\alpha} \frac{d}{dz}\left\{(z-\alpha)^2 \frac{z}{(z-\alpha)^2(z-\beta)^2}\right\}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{(z-\beta)^2}\right]$$

$$= \lim_{z\to\alpha} \frac{-(z+\beta)}{(z-\beta)^3}$$

$$= \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{\frac{-i}{(bi+c)}(-2a)}{\left(\frac{2i}{bi+c}\right)^3\left(\sqrt{a^2-b^2-c^2}\right)^3} = \frac{a}{\frac{-4}{(bi+c)^2}\left(\sqrt{a^2-b^2-c^2}\right)^3}$$

Then from (1).

$$I = \frac{4i}{(bi+c)^2}[2\pi i](\text{Residue at } z=\alpha)$$

$$= \frac{4i}{(bi+c)^2}(2\pi i)\frac{-a}{\frac{4}{(bi+c)^2}(a^2-b^2-c^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2-b^2-c^2)^{3/2}}$$

$$\therefore \int_0^{2\pi}\frac{d\theta}{[a+b\cos\theta+c\sin\theta]^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}$$

4(c) Use simplex method, to solve Max $Z = 3x_1 + 2x_2$
subject to
$2x_1 + x_2 \le 2$
$3x_1 + 4x_2 \ge 12$
$x_1, x_2 \ge 0$.

Soln: write the given LPP in standard form.

$\text{Max } Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 - MA_1$

subject to

$2x_1 + x_2 + S_1 + 0S_2 + 0A_1 = 2$

$3x_1 + 4x_2 + 0S_1 - S_2 + A_1 = 12$

$x_1, x_2, S_1, S_2, A_1 \ge 0$

where S_1 is the slack variable, S_2 is the surplus variable and A_1 is the artificial variable.

Now IBFS is $S_2 = x_1 = x_2 = 0$ (non basic)
$S_1 = 2$, $A_1 = 12$ (basic)

for which $Z = -12M$.

Now we put the above information in the simplex table.

	c_j	3	2	0	0	-M		
C_B	Basis	x_1	x_2	S_1	S_2	A_1	b	θ
0	S_1	2	(1)	1	0	0	2	2 →
-M	A_1	3	4	0	-1	1	12	3
$Z_j = \Sigma C_B a_{ij}$		-3M	-4M	0	M	-M	-12M	
$C_j = c_j - Z_j$		3+3M	2+4M ↑	0	-M	0		

from the above table,

x_2 is the entering variable.

S_1 is the outgoing variable and (1) is the key element and all other elements in its column equal to zero.

Then the revised simplex table is

	C_j	3	2	0	0	$-M$	
CB	Basis	x_1	x_2	S_1	S_2	A_1	b
2	x_2	2	1	1	0	0	2
$-M$	A_1	-5	0	-4	-1	1	4
$Z_j = \Sigma C_B a_{ij}$		$4+5M$	2	$2+4M$	M	$-M$	$4-4M$
$\bar{C}_j = C_j - Z_j$		$-1-5M$	0	$-(2+4M)$	$-M$	0	

from the above table

all C_{ij}'s are ≤ 0 and artificial variable appears in the basis at non-zero level.

Thus there exists a pseudo optimal solution to the problem.

i.e, the given LPP doesnot possess any feasible solution.

4(d) A company has three plants at locations A, B and C, which supply to warehouses located at D, E, F, G and H. Monthly plant capacities are 800, 500 and 900 units respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units respectively. Unit transportation costs (in rupees) are given below:

	To				
	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	5
C	8	4	6	6	4

Determine an optimum distribution for the company in order to minimize the total transportation cost.

Sol: Given Transportation table is

	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	5
C	8	4	6	6	4

In this transportation problem
total capacity of the plant = 2200 units
total Requirement of the warehouse = 2500 units

Since the total Requirement is more than total capacity.

∴ The given transportation problem is unbalanced.

∴ we introduce an artificial source X with capacity 2500 − 2200 = 300 units.

Costs C_{ij}'s for all cells corresponding to artificial source X are taken as zeros.

By this we get the balance Transportation problems

	D	E	F	G	H	
A	5	8	6	6	3	800
B	4	7	7	6	5	500
C	8	4	6	6	4	900
X	0	0	0	0	0	300
	400	400	500	400	800	

Using the Vogel's Approximation method, the initial basic feasible solution is as shown below.

5	8	6 (500)	6	3 (300)	800
4 (400)	7	7	6 (100)	5	500
8	4 (400)	6	6	4 (500)	900
0	0	0	0 (300)	0	300
400	400	500	400	800	

No. of allocations (basic cells) which is less than above table

$= m+n-1 = 5+4-1 = 8$ (Here the no of allocations which is less than $m+n-1 = 8$)

The solution is feasible, but not basic feasible.

i.e, the solution is degenerate.

In order to complete the basis and there by remove degeneracy, we require only one more non-negative basic variable.

To break degeneracy, allocate a very small +ve quantity $\epsilon (\to 0)$ to occupied cell with minimum cost.

Minimum entry in unoccupied position is in cell $(2,5)$ and the no. of allocations will be independent because no closed loop is formed and also the net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for unoccupied cells are exhibited as below

					u_i
5 (-2)	8 (-5)	(500−θ) 6 (8)	6 (-2)	(300+θ) 3	-2
(400) 4	7 (-1)	7 (-1)	(400) 6	5 (ε−θ)	0
8 (-5)	(400) 4	6 (1)	6 (-4)	(500) 4	-1
0 (-2)	(-1) 0	θ (+2) 0	(300−θ) 0	(-1) 0	-6
v_j: 4	5	8	6	5	

Since the net evaluations in 2 cells are +ve.

∴ The current basic feasible solution is not optimal

Choose unoccupied cell with maximum Δ_{ij}

Clearly $\Delta_{43} = +2$ is the most +ve.

∴ The cell (4,3) enters the basis

- we allocate an unknown quantity θ to this cell (4,3) and identify a loop involving basic cells around this entering cell.

- Add and subtract θ alternately to and from the transition cells of the loop subject to the rim requirements as shown in the above table.

Now, assign maximum value θ so that one basic variable becomes zero and other basic variables ≥ 0.

Taking $\theta = \epsilon = (0)$

∴ $x_{25} = 0$

∴ The cell (2,5) leaves the basis.

The new basic feasible solution is shown below

					u_i
(-1) 5	(-5) 8	(500) 6	(-2) 6	3 (300)	0
(400) 4	(-4) 7	(-4) 7	(400) 6	(ε) 5 (-2)	0
(-3) 8	4	(-1) 6	6	(500) 4	1
(-2) 0	(-3) 0	(ε) 0	(300) 0	(-1) 0	-6
v_j: 4	3	6	6	3	

Proceeding in the same way, we get the new basic feasible solution.
(Here in the above table (3,4) enters the basis and (4,4) leaves the basis)

					u_i
(-2) 5	(-5) 8	200 6	(-1) 6	600 3	-1
400 4	(-3) 7	(0) 7	100 6	(-1) 5	0
(-4) 8	400 4	(1) 6	300 6	(0) 4	0
(-3) 0	(-3) 0	200 0	(-1) 0	(-3) 0	-7
v_j: 4	4	7	6	4	

In the above table (3,3) enters the basis and (3,5) leaves the basis and therefore the new basic feasible solution is

					u_i
(-1) 5	(-4) 8	(0) 6	(0) 6	800 3	0
400 4	(-3) 7	(-1) 7	100 6	(-2) 5	0
(-4) 8	400 4	200 6	300 6	(-1) 4	0
(-2) 0	(-2) 0	300 0	(0) 0	(-3) 0	-6
v_j: 4	4	6	6	3	

Since all the net evaluations ≤ 0.

$\therefore$ The current basic feasible solution is optimal.

Hence, the optimum solution is

$x_{15} = 800$, $x_{21} = 400$, $x_{24} = 100$, $x_{32} = 400$, $x_{33} = 200$

$x_{34} = 300$ and $x_{43} = 300$.

i.e, the optimum transportation Schedule is:

A → 800 units to H ;

B → 400 units to D and 100 units to G.

C → 400 units to E, 200 units to F & 300 units to G.

∴ The transportation cost according to this optimum route will be Rs 9200/-.

5(a) Find the surface which is orthogonal to the one parameter system $z = cxy(x^2+y^2)$ which passes through the hyperbola $x^2-y^2=a^2$, $z=0$.

Sol^n. The given system of surfaces

$$f(x,y,z) = \frac{z}{x^3y+xy^3} = c$$

$$\frac{\partial f}{\partial x} = \frac{-z(3x^2y+y^3)}{(x^3y+xy^3)^2}, \quad \frac{\partial f}{\partial y} = \frac{-z(3y^2x+x^3)}{(x^3y+xy^3)^2}$$

$$\frac{\partial f}{\partial z} = \frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

$$\frac{-z(3x^2y+y^3)}{(x^3y+xy^3)^2}p - \frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q = \frac{1}{(x^3y+xy^3)^2}$$

$$\left\{\frac{(3x^2+y^2)}{x}\right\}p + \left\{\frac{3y^2+x^2}{y}\right\}q = \frac{-(x^2+y^2)}{z} \quad \text{---} (2)$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{\left(\frac{3x^2+y^2}{x}\right)} = \frac{dy}{\left(\frac{3y^2+x^2}{y}\right)} = \frac{dz}{-\frac{(x^2+y^2)}{z}} \quad \text{---} (3)$$

Taking the first two fractions of (3)

$2xdx - 2ydy = 0$ so that $x^2-y^2 = c_1$

choosing $x, y, 4z$ as multipliers, each fraction of (3) $= \frac{xdx+ydy+4zdz}{0}$

$\therefore 2xdx + 2ydy + 8zdz = 0$

$\Rightarrow x^2+y^2+4z^2 = c_2$

Hence any surface which is orthogonal to (1) is of the form $x^2+y^2+4z^2 = \phi(x^2-y^2)$ --- (4), ϕ being an arbitrary function.

For the particular surface passing through the hyperbola $x^2-y^2=a^2$, $z=0$

we must take $\phi(x^2-y^2) = \dfrac{a^4(x^2+y^2)}{(x^2-y^2)^2}$

Hence the required surface is given by $(x^2+y^2+4z^2)^2(x^2-y^2)^2 = a^4(x^2+y^2)$

5(b) Solve $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$.

sol^n: Given equation is $(D+D')^2 z = 2\cos y - x\sin y$ —— ①

Its auxiliary equation is $(m+1)^2=0$ so that $m=-1,-1$

$\therefore C.F = \phi_1(y-x) = x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now $P.I = \dfrac{1}{D+D'}\dfrac{1}{D+D'}(2\cos y - x\sin y)$

$= \dfrac{1}{D+D'}\int[2\cos(x+c) - x\sin(x+c)]dx$, where $c=y-x$

$= \dfrac{1}{D+D'}\left[2\int\cos(x+c)dx - \int x\sin(x+c)dx\right]$

$= \dfrac{1}{D+D'}\left[2\int\cos(x+c)dx - \int x\sin(x+c)dx\right]$

$= \dfrac{1}{D+D'}\left[2\sin(x+c) - \left\{-x\cos(x+c) + \int\cos(x+c)dx\right\}\right]$

$= \dfrac{1}{D+D'}\left[2\sin(x+c) + x\cos(x+c) - \sin(x+c)\right]$

$= \dfrac{1}{D+D'}(\sin y + x\cos y)$, as $c=y-x$

$= \int[\sin(x+c') + x\cos(x+c')]dx$, where $c'=y-x$

$= -\cos(x+c') + x\sin(x+c') - \int\{1.\sin(x+c')\}dx$

$= -\cos(x+c') + x\sin(x+c') + \cos(x+c')$

$= x\sin y$, as $c'=y-x$

So the required solution is

$z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$.

5(c) Evaluate the integral $I = \int_1^2 \frac{2x\,dx}{1+x^4}$ using the Gauss-Legendre 1-point, 2-point, and 3-point quadrature rules. compare with the exact solution

$I = \tan^{-1}4 - \left(\frac{\pi}{4}\right)$

Solⁿ: To use the Gauss-Legendre rules, the interval $[1,2]$ is to be reduced to $[-1,1]$.

writing $x = at+b$,

we get $1 = -a+b$, $2 = a+b$

$\Rightarrow a = \frac{1}{2}, b = \frac{3}{2}$.

$\therefore x = \frac{t+3}{2}$ and $dx = \frac{1}{2}dt$

$$I = \int_{-1}^{1} \frac{8(t+3)}{[16+(t+3)^4]}\,dt$$

$$= \int_{-1}^{1} f(t)\,dt$$

where $f(t) = \frac{8(t+3)}{[16+(t+3)^4]}$

Using the 1-point rule, we get

$$I = 2f(0) = 2\left[\frac{24}{16+81}\right] = 0.4948$$

Using the 2-point rule, we get

$$I = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= 0.3842 + 0.1592$$

$$= 0.5434.$$

Using the 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\sqrt{3/5}\right) + 8f(0) + 5f\left(\sqrt{3/5}\right)\right]$$

$$= \frac{1}{9}\left[5(0.4393) + 8(0.2494) + 5(0.1379)\right]$$

$$= 0.5406$$

The exact solution is $I = 0.5404$.

5(d) (i) Convert the decimal number 15359 into hexadecimal.
(ii) Convert the hexadecimal number 8A3 into octal.

Sol^n: (i)

16	15359	
16	959	remainder F
16	59	remainder F
16	3	remainder B
	0	remainder 3

$$\therefore (15359)_{10} = (3BFF)_{16}$$

(ii) Hexadecimal number can be converted to equivalent octal number by converting the hex number to equivalent binary and then to octal

$$(8A3)_{16} = (1000\ 1010\ 0011)_2$$

$$= (\underset{4}{100}\ \underset{2}{010}\ \underset{4}{100}\ \underset{3}{011})_2$$

$$= (4243)_8$$

5(e) If velocity distribution of an incompressible fluid at point (x, y, z) is given by $\{3xz/r^5, 3yz/r^5, (kz^2-r^2)/r^5\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential.

Sol^n: Here $u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{kz^2-r^2}{r^5} = \dfrac{kz^2}{r^5} - \dfrac{1}{r^3}$ — (1)

where $r^2 = x^2+y^2+z^2$ — (2)

from (2), $\dfrac{\partial r}{\partial x} = x/r$, $\dfrac{\partial r}{\partial y} = y/r$ and $\dfrac{\partial r}{\partial z} = z/r$ — (3)

from (1) (2) and (3), we have

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2z}{r^7}, \quad \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2z}{r^7} \quad \text{— (4)}$$

and $\dfrac{\partial w}{\partial z} = \dfrac{2kz}{r^5} - 5kz^2r^{-6}\dfrac{\partial r}{\partial z} + 3r^{-4}\dfrac{\partial r}{\partial z}$

$$= \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7} \quad \text{— (5)}$$

Since (1) gives a possible liquid motion, the equation of continuity must be satisfied and so

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

$$\Rightarrow \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = 0 \Rightarrow \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}\cdot r^2 = 0$$

using (2), (4) & (5)

$\Rightarrow (2k-6)z/r^5 = 0$ so that $2k-6=0$ giving $k=3$

Substituting the above value of k in (1), we have

$$u = \frac{3xz}{r^5}, \quad v = \frac{3yz}{r^5}, \quad w = \frac{(3z^2-r^2)}{r^5} \quad \text{— (6)}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2)$$

$$= \frac{15z}{r^5} - \frac{15z}{r^7}r^2 = 0$$

Since the equation of continuity is satisfied by the given values of u, v, w, the motion is possible.

Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = -(u\,dx + v\,dy + w\,dz)$$

by definition of ϕ.

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right]$$

$$= \frac{r^2dz - 3z(x\,dx + y\,dy + z\,dz)}{r^5}$$

$$= \frac{r^3dz - 3r^2z\,dr}{(r^3)^2} = d\left(\frac{z}{r^3}\right), \text{ using (2)}$$

Integrating $\phi = z/r^3$

$\therefore$ the required velocity potential ϕ is given by $\frac{z}{r^3}$.

6(a) Find a complete integral of $p^2+q^2-2px-2qy+2xy=0$

Sol'n: Given equation is

$$f(x,y,z,p,q) = p^2+q^2-2px-2qy+2xy=0 \quad \text{--- (1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}$$

which gives $\dfrac{dp+dq}{2(x+y-p-q)} = \dfrac{dx+dy}{2(x+y-p-q)}$

$\Rightarrow dp+dq = dx+dy$ i.e, $dp-dx+dq-dy=0$

Integrating $(p-x)+(q-y)=a$ --- (2).

Rewriting (1) $(p-x)^2+(q-y)^2=(x-y)^2$ --- (3)

putting the value of $(q-y)$ from (2) in (3), we get

$(p-x)^2 + [a-(p-x)]^2 = (x-y)^2$

$\Rightarrow 2(p-x)^2 - 2a(p-x) + \{a^2-(x-y)^2\} = 0$

$\therefore \ p-x = \dfrac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2-(x-y)^2\}]}}{4}$

$\Rightarrow p = x + \frac{1}{2}\left[a \pm \sqrt{\{2(x-y)^2 - a^2\}}\right]$

$\therefore$ ② gives $q = a + y - p + x$

$\Rightarrow q = y + \frac{1}{2}\left[a \pm \sqrt{\{2(x-y)^2 - a^2\}}\right]$

putting these values of p and q in $dz = pdx + qdy$, we get

$dz = x\,dx + y\,dy + \frac{a}{2}(dx+dy) \pm \frac{1}{2}\sqrt{\{2(x-y)^2-a^2\}}\,(dx-dy)$

$\Rightarrow dz = x\,dx + y\,dy + \frac{a}{2}(dx+dy) \pm \frac{1}{\sqrt{2}}\sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}}(dx-dy)$

Integrating the desired complete integral is

$$z = \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2}(x+y) \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}} - \frac{a^2}{4}\log\left[(x-y) + \sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}}\right]\right)$$

$$\Rightarrow 2z = x^2 + y^2 + ax + ay \pm \frac{1}{\sqrt{2}}\left((x-y)\sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}} - \frac{a^2}{2}\log\left[(x-y) + \sqrt{\left\{(x-y)^2 - \frac{a^2}{2}\right\}}\right]\right).$$

6(b) solve $(x-y)p + (x+y)q = 2xz$

Sol'n: Here the Lagrange's subsidiary equations are

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \quad \text{— ①}$$

Taking first two fractions of ①.

$$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1+y/x}{1-y/x} \quad \text{— ②}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ from ②, we have

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{1+v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1+v^2} - \frac{2v}{1+v^2}\right)dv = \frac{2dx}{x}$$

Integrating, we get

$$2\tan^{-1}v - \log(1+v^2) = 2\log x - \log c_1$$

$$\Rightarrow \log\left(x^2 c_1^{-1}(1+v^2)\right) = 2\tan^{-1}v$$

$$\Rightarrow \frac{x^2(1+v^2)}{c_1} = e^{2\tan^{-1}v}$$

$$\Rightarrow \boxed{(x^2+y^2)e^{-2\tan^{-1}y/x} = c_1} \text{ (as } y = vx\text{)} \quad \text{—— ③}$$

choosing $1, 1, -\frac{1}{2}$ as multipliers

each fraction of ①

$$= \frac{dx + dy - \frac{1}{2}dz}{(x-y)+(x+y)-\frac{1}{2}(2x z)} = \frac{dx+dy-\frac{1}{2}dz}{0}$$

$$\Rightarrow dx + dy - \frac{1}{2}dz = 0$$

$$\Rightarrow \boxed{x + y - \log z = c_2} \quad \text{—— ④}$$

from ③ & ④ the required general solution is

$$\phi\left(x+y-\log z;\ (x^2+y^2)e^{-2\tan^{-1}(y/x)}\right) = 0$$

where ϕ is an arbitrary constant.

6(C) ⇒ Use the Runge Kutta method to solve

$10\frac{dy}{dx} = x^2+y^2$, $y(0)=1$ for the interval $0<x\le 0.4$ with $h=0.1$.

Sol^n: Given that $\frac{dy}{dx} = \frac{x^2+y^2}{10}$

To find $y(0.1)$: $h=0.1$; $x_0=0$, $y_0=1$

$$K_1 = hf(x_0,y_0) = (0.1)\frac{1}{10} = 0.01$$

$$K_2 = hf\left(x_0+h/2,\ y_0+\frac{K_1}{2}\right) = 0.01012$$

$$K_3 = h\left[f\left(x_0+h/2,\ y_0+\frac{K_2}{2}\right)\right] = 0.01012$$

$$K_4 = hf\left(x_0+h,\ y_0+K_3\right) = 0.0103$$

$$y(0.1) = y_0 + \frac{K_1+2K_2+2K_3+K_4}{6}$$

$$=1.0101$$

To find $y(0.2)$:

$$x_1 = x_0+h = 0+0.1 = 0.1,\ y_1 = 1.0101,\ h=0.1$$

$$K_1 = hf(x_1,y_1) = (0.1)f(0.1,\ 1.0101) = 0.0103$$

$$K_2 = 0.01053$$

$$K_3 = 0.01053$$

$$K_4 = 0.0108$$

$$y(0.2) = 1.0101 + \tfrac{1}{6}(K_1+2K_2+2K_3+K_4) = 1.0206.$$

To find $y(0.3)$:

$$x_2 = 0.2,\ y_2 = 1.0206,\ h=0.1$$

$$K_1 = hf(x_2,y_2) = 0.0108$$

$$K_2 = hf\left(x_2+h/2,\ y_2+\frac{K_1}{2}\right) = 0.0111$$

$$K_3 = hf\left(x_2+h/2,\ y_2+\frac{K_2}{2}\right) = 0.0111$$

$$K_4 = hf\left(x_2+h,\ y_2+K_3\right) = 0.0115$$

$$y(0.3) = 1.0317$$

To find $y(0.4)$:

$$x_3 = 0.3,\ y_3 = 1.0317,\ h=0.1$$

$K_1 = 0.0115$
$K_2 = 0.01198$
$K_3 = 0.01199$
$K_4 = 0.01249$
$y(0.4) = 1.04369.$

6(d). Write Hamilton's equations in polar coordinates for a particle of mass m moving in three dimensions in a force field of potential V.

Sol'n: At time t, let (r, θ, ϕ) be polar coordinates of the particle m at P. If (x,y,z) are the Cartesian coordinates of P, then $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\therefore K.E = T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi)^2 + (\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi)^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta) + r^2\dot{\theta}^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta) + r^2\dot{\phi}^2\sin^2\theta(\sin^2\phi + \cos^2\phi)\right]$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V$$

Here r, θ and ϕ are the generalised coordinates

$$\therefore p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r},\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \text{ and } p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta \quad \text{—(1)}$$

Since L does not contain, t explicity.

$$\therefore H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + V$$

Eliminating $\dot{r}, \dot{\theta}, \dot{\phi}$ with the help of relations (1)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V.$$

Hence the six Hamilton's equations are (note that V is function of r, θ, ϕ and t)

$$\dot{p}_r = -\frac{\partial H}{\partial r} \text{ i.e. } \dot{p}_r = -\frac{1}{2m}\left(-\frac{2p_\theta^2}{r^3} - \frac{2p_\phi^2}{r^3\sin^2\theta}\right) - \frac{\partial V}{\partial r}$$

$$\Rightarrow \dot{p}_r = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{\partial V}{\partial r} \quad \text{—} (H_1)$$

$$\dot{r} = \partial H/\partial p_r \text{ i.e. } \dot{r} = p_r/m \quad \text{—} (H_2)$$

$$\dot{p}_\theta = -\partial H/\partial\theta \text{ i.e. } \dot{p}_\theta = -\frac{1}{2m}\left(\frac{2\cos\theta}{r^2\sin^3\theta}p_\phi^2\right) - \frac{\partial V}{\partial\theta}$$

$$\Rightarrow \dot{p}_\theta = \cos\theta/mr^2\sin^3\theta \, p_\phi^2 - \frac{\partial V}{\partial\theta} \quad \text{—— } (H_3)$$

$$\dot{\theta} = \partial H/\partial p_\theta \text{ i.e. } \dot{\theta} = p_\theta/mr^2 \quad \text{—— } (H_4)$$

$$\dot{p}_\phi = -\partial H/\partial\phi = -\partial V/\partial\phi \quad \text{—— } (H_5)$$

$$\dot{\phi} = \partial H/\partial p_\phi = p_\phi/mr^2\sin^2\theta \quad \text{—— } (H_6)$$

7(a) Reduce $x^2r + 2xys + y^2t = 0$ to canonical form and hence solve

Soln: Given $x^2r + 2xys + y^2t = 0$ —— ①

comparing ① with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

here $R = x^2$, $S = 2xy$ and $T = y^2$ so that

$S^2 - 4RT = 0$ showing that ① is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$

reduces to $x^2\lambda^2 + 2xy\lambda + y^2 = 0$

$$\Rightarrow (x\lambda + y)^2 = 0$$

$$\Rightarrow \lambda = -y/x, \; -y/x.$$

The corresponding characteristic equation is

$$\frac{dy}{dx} - \frac{y}{x} = 0 \Rightarrow \frac{1}{y}dy - \frac{dx}{x} = 0$$

$$\Rightarrow \log y - \log x = \log c$$

$$\Rightarrow y/x = c.$$

Choose $u = y/x$ and $v = y$ —— ②

where we have chosen $v = y$ in such a manner that u and v are independent functions.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{\partial z}{\partial u}$ —— ③

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v} \quad \text{——— (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{-y}{x^2}\frac{\partial z}{\partial u}\right)$$

$$=\frac{2y}{x^3}\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$=\frac{2y}{x^3}\frac{\partial z}{\partial u}-\frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$=\frac{2y}{x^3}\frac{\partial z}{\partial u}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} \quad \text{——— (5)}$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=-\frac{1}{x^2}\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=-\frac{1}{x^2}\frac{\partial z}{\partial u}+\frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}$$

$$+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial x}\right)$$

$$=-\frac{1}{x^2}\frac{\partial z}{\partial u}-\frac{y}{x^3}\frac{\partial^2 z}{\partial u^2}-\frac{y}{x^2}\frac{\partial^2 z}{\partial u\partial v} \quad \text{——— (6)}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]$$

$$+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$=\frac{1}{x^2}\frac{\partial^2 z}{\partial u^2}+\frac{2}{x}\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2} \quad \text{——— (7)}$$

Using (5), (6) and (7) in (1), we get

$\frac{\partial^2 z}{\partial v^2}=0$ which is the required Canonical form. ——— (8)

Integrating (8) partially w.r.t v, $\frac{\partial z}{\partial v}=\phi(u)$ ——— (9)

Integrating (9) partially w.r.t v, $z=v\phi(u)+\psi(u)$

$z=y\phi(y/x)+\psi(y/x)$, ϕ, ψ being arbitrary functions

7(b) The following are the number of deaths in four successive ten year age groups. By using Newton's forward formula find the number of deaths at 45-50 and 50-55.

Age group	25-35	35-45	45-55	55-65
Deaths	13229	18139	24225	31496

Ans:- First we prepare the cumulative frequency table, as follows.

Age group Under	35	45	55	65
Deaths	13229	31368	55593	87089

The difference table is

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$
35	13229	18139	6086	1185
45	31368	24225	7271	
55	55593	31496		
65	87089			

we have to obtain $f(50)$ i.e. the no. of deaths above the age 25 and below 50.

Taking $x_0 = 35$, $x = 50$, $h = 10$

Here $p = \frac{50-35}{10} = \frac{15}{10} = 1.5$ $\quad p = \frac{x-x_0}{h}$

By Newton's forward formula

$$y_{50} = y_{35} + p\Delta y_{35} + \frac{p(p-1)}{2!}\Delta^2 y_{35} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{35} +$$

$$\Rightarrow f(50) = 13229 + (1.5)(18129) + \frac{(1.5)(0.5)}{2!}(6086) + \frac{(1.5)(0.5)(-0.5)}{3!}(1185)$$

$$= 13229 + 27208.5 + 2282.25 - 74.0625$$

$$\simeq 42646$$

Therefore the required no. of deaths between 45 and 50 $= 42646 - 31368 = 11278$

Hence the no. of deaths between 50 and 55

$$= 24225 - 11278$$

$$= \underline{\underline{12947}}$$

7(c) Solve the system of equations

$$x = \frac{1}{20}(17 - y + 2z)$$
$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{— (1)}$$
$$z = \frac{1}{20}(25 - 2x + 3y)$$

by Gauss-Seidel iterative method to and perform the first 3 iterations.

Sol^n: By Gauss-Seidel method, system (1) can be written as

$$x^{k+1} = \frac{1}{20}(17 - y^{k} + 2z^{k})$$
$$y^{k+1} = \frac{1}{20}(-18 - 3x^{k+1} + z^{k})$$
$$z^{k+1} = \frac{1}{20}(25 - 2x^{k+1} + 3y^{k+1})$$

Now taking $X^{(0)} = 0$ (i.e. $x = y = z = 0$), we obtain the following iterations.

First iteration

put $k = 0$

$$x^{(1)} = \frac{1}{20}(17 - y^{(0)} + 2z^{(0)}) = 0.8500$$
$$y^{(1)} = \frac{1}{20}(-18 - 3x^{(1)} + z^{(0)}) = -1.0275$$
$$z^{(1)} = \frac{1}{20}(25 - 2x^{(1)} + 3y^{(1)}) = 1.0109$$

Second iteration:

put $k = 1$

$$x^{(2)} = \frac{1}{20}(17 - y^{(1)} + 2z^{(1)}) = 1.0025$$
$$y^{(2)} = \frac{1}{20}(-18 - 3x^{(2)} + z^{(1)}) = -0.9998$$
$$z^{(2)} = \frac{1}{20}(25 - 2x^{(2)} + 3y^{(2)}) = 0.9998$$

Third iteration

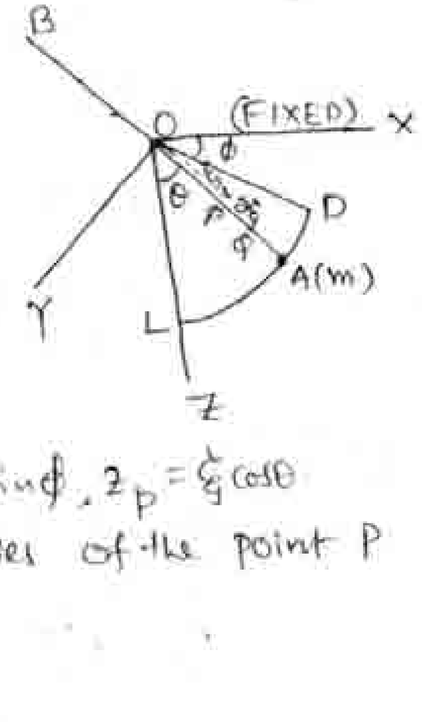
put $k = 2$

$$x^{(3)} = \frac{1}{20}(17 - y^{(2)} + 2z^{(2)}) = 1.0000$$
$$y^{(3)} = \frac{1}{20}(-18 - 3x^{(3)} + z^{(2)}) = -1.0000$$
$$z^{(3)} = \frac{1}{20}(25 - 2x^{(3)} + 3y^{(3)}) = 1.0000$$

$\therefore$ The solution is given by $x = 1$, $y = -1$ & $z = 1$.

7(d) → A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. the rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{(2ng)/l}$. show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}[\sqrt{n^2+1}-n]$, and will then rise again.

Sol'n: Let AB be the rod of mass 3m & length 2l. The middle point O of the rod is fixed and a mass m attached at the extremity A. Initially let the rod rest along OX in the plane of the paper. Let a line OY ⊥lar to OX in the plane of the paper and a line OZ ⊥lar to the plane of the paper be taken as axes of Y and Z respectively. At time t, let the rod turn through an angle ϕ to OX i.e. the plane OAL containing the rod and Z axis make an angle ϕ with X-Z plane. And let θ be the inclination of the rod with OZ at this time t. If P is a point of the rod at a distance $OP=\xi$, from O then coordinates of P are given by



$$x_p=\xi\sin\theta\cos\phi,\quad y_p=\xi\sin\theta\sin\phi,\quad z_p=\xi\cos\theta$$

∴ If v_p and v_A are the velocities of the point P and A respectively, then

$$v_p^2=\dot{x}_p^2+\dot{y}_p^2+\dot{z}_p^2$$

$$=(\xi\cos\theta\cos\phi\dot{\theta}-\xi\sin\theta\sin\phi\dot{\phi})^2$$
$$+(\xi\cos\theta\sin\phi\dot{\theta}+\xi\sin\theta\cos\phi\dot{\phi})^2+(-\xi\sin\theta\dot{\theta})^2$$

$= \xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

$\therefore$ At A, $\xi = OA = l$, $\therefore v_A^2 = l^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

Let $PQ = \delta\xi$ be an element of the rod at P, then mass of this element,

$$\delta m = \frac{3m}{2l} \cdot \delta\xi$$

$\therefore$ K.E of the element $PQ = \frac{1}{2}\delta m \cdot v_P^2 = \frac{1}{2}\xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \cdot \frac{3m}{2l}\delta\xi$

$\therefore$ K.E of the rod $AB = \frac{3m}{4l}\int_{-l}^{l} \xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\, d\xi$

$= \frac{1}{2}m(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\, l^2$

and K.E of mass m at $A = \frac{1}{2}mv_A^2 = \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

$\therefore$ The total K.E of the system

$T =$ K.E of the rod + K.E of the particle

$= ml^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

The work function $W = mg \cdot z_A = mgl\cos\theta$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}(2ml^2\dot{\theta}) - 2ml^2\dot{\phi}^2 \sin\theta\cos\theta = -mgl\sin\theta$

$\Rightarrow 2l\ddot{\theta} - 2l\dot{\phi}^2 \sin\theta\cos\theta = -g\sin\theta$ ——— (1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

$\Rightarrow \frac{d}{dt}(2ml^2\dot{\phi}\sin^2\theta) = 0 \Rightarrow \frac{d}{dt}(\dot{\phi}\sin^2\theta) = 0$ —— (2)

Integrating (2) we get $\dot{\phi}\sin^2\theta = C$ (Const)

($\because$ Rod was horizontal)

But initially when $\theta = \pi/2$

$\dot{\phi} = \sqrt{(2ng/l)}$ ——— (3)

$\therefore C = \sqrt{(2ng/l)}$

$\therefore \dot{\phi}\sin^2\theta = \sqrt{(2ng/l)}$

Substituting the value of d from ③ in ① we get

$$2l\ddot{\theta} - 2l \cdot \frac{2ng}{l\sin^4\theta} \sin\theta\cos\theta = -g\sin\theta$$

$$\Rightarrow 2l\ddot{\theta} - 4ng\cot\theta\,\text{cosec}^2\theta = -g\sin\theta \quad \text{—— ④}$$

Multiplying both sides by $\dot{\theta}$ & integrating, we get

$$l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta + D$$

But initially when $\theta = \pi/2$, $\dot{\theta} = 0$. $\therefore D = 0$

$$\therefore l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta \quad \text{—— ⑤}$$

The rod will fall till $\dot{\theta} = 0$

$$\Rightarrow 2ng\cot^2\theta = g\cos\theta \Rightarrow 2n\cos^2\theta - \cos\theta\sin^2\theta = 0$$

$$\Rightarrow \cos\theta\,(2n\cos\theta - \sin^2\theta) = 0.$$

$\therefore$ either $\cos\theta = 0$ i.e. $\theta = \pi/2$

$$\Rightarrow 2n\cos\theta - \sin^2\theta = 0 \Rightarrow 2n\cos\theta - (1-\cos^2\theta) = 0.$$

$$\Rightarrow \cos^2\theta + 2n\cos\theta - 1 = 0$$

$$\therefore \cos\theta = \frac{-2n \pm \sqrt{(4n^2+4)}}{2} \Rightarrow \cos\theta = -n + \sqrt{(n^2+1)}, \text{ Leaving -ve sign.}$$

$\because$ -ve value of $\cos\theta$ is inadmissible as θ cannot be obtuse.

$$\therefore \theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right].$$

from ④, we have $$2l\ddot{\theta} = \frac{g\,(4n\cos\theta - \sin^4\theta)}{\sin^3\theta} \quad \text{—— ⑥}$$

when $\cos\theta = -n + \sqrt{(n^2+1)}$, $\cos^2\theta = 2n^2 + 1 - 2n\sqrt{(n^2+1)}$

$$\begin{aligned}\therefore 4n\cos\theta - \sin^4\theta &= 4n\cos\theta - (1-\cos^2\theta)^2 \\ &= 4n\left[-n + \sqrt{(n^2+1)}\right] - \left[-2n^2 + 2n\sqrt{(n^2+1)}\right]^2 \\ &= -8n^2 - 8n^4 + 4n\sqrt{(n^2+1)} + 8n^3\sqrt{(n^2+1)} \\ &= 4n\sqrt{(n^2+1)}\left[-n + \sqrt{(n^2+1)}\right]^2, \text{ which is +ve}\end{aligned}$$

$\because \theta$ is acute angle $\therefore \sin^3\theta$ is also +ve

$\therefore$ when $\theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right]$, from ⑥, we see that $\ddot{\theta}$ is +ve. Hence from this position the rod will rise again.

8(a) A square plate is bounded by the lines $x=0, y=0, x=10$ & $y=10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,10) = x(10-x)$ while the other three faces are kept at $0°C$. Find the steady state temperature in the plate.

Sol'n: The steady state temperature $u(x,y)$ is the solution of Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— ①}.$$

subject to boundary conditions

$u(0,y) = u(10,y) = 0, \quad 0 \le y \le 10$ — (2)

$u(x,0) = 0, \quad 0 \le x \le 10$ — (3a)

and $u(x,10) = 10x - x^2, \quad 0 \le x \le 10$ — (3b)

Suppose (1) has a solution of the form

$u(x,y) = X(x)\,Y(y)$ — (4)

Substituting this value of u in (1), we get

$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$ — (5)

Since x and t are independent variables, each side of (5) must be equal to the constant, say μ.

Then (5) gives $X'' - \mu X = 0$ — (6)

and $Y'' + \mu Y = 0$ — (7)

Using (2), (4) gives $X(0)\,Y(y) = 0$ & $X(10)\,Y(y) = 0$

giving $X(0) = 0$ and $X(10) = 0$ — (8)

where $Y(y) \neq 0$, since otherwise $u \equiv 0$ which does not satisfy 3b)

We now solve (6) under B.C. (8). Three cases arise.

Case(1). Let $\mu = 0$. The solution of (6) is

$X(x) = Ax + B$ — (9)

Using B.C. (8), we get $A = B = 0$

So that $X(x) = 0$. This leads $u \equiv 0$ which doesnot satisfy 3(b). So we reject $\mu = 0$.

Case(2): Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of (6) is

$X(x) = A e^{x\lambda} + B e^{-\lambda x}$ — (10)

Using B.C (8), we get $A = B = 0$ so that $X(x) = 0$ and hence $u \equiv 0$.

So we reject $\mu = \lambda^2$.

Case (B): Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of ⑥ is

$X(x) = A\cos\lambda x + B\sin\lambda x$ — (11)

Using B.C. ⑧, (11) gives $0 = A$ and $0 = A\cos\lambda(10) + B\sin\lambda(10)$

$\Rightarrow \sin\lambda(10) = 0$, $B \neq 0$.

since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which doesnot satisfy 3(5).

Now, $\sin\lambda a = 0 \Rightarrow \lambda(10) = n\pi$; $n = 1, 2, 3 \ldots$

$\Rightarrow \lambda = \frac{n\pi}{10}$; $n = 1, 2, \ldots$ — (12)

Hence non-zero solutions $X_n(x)$ of ⑥ are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{10}\right)$ — (13)

Using $\mu = -\lambda^2 = -\frac{n^2\pi^2}{(10)^2}$, ⑦ becomes $Y'' - \left(\frac{n^2\pi^2}{10^2}\right)Y = 0$ — (14)

whose general solution is —

$Y_n(y) = C_n e^{n\pi y/10} + D_n e^{-n\pi y/10}$ — (15)

Using 3(4), ① gives $0 = X(x)\, Y(0)$ so that $Y(0) = 0$, where we have taken $X(x) \neq 0$.

for otherwise we will get $u = 0$ which doesnot satisfy 3(5).

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ — (16)

Putting $y = 0$ in (15) and using (16), we have

$0 = C_n + D_n \Rightarrow D_n = -C_n$. Then ⑮ reduces to

$Y_n(y) = C_n\left(e^{n\pi y/10} - e^{-n\pi y/10}\right)$

$= 2\sinh(n\pi y/10)$ — (17)

$\therefore u_n(x,y) = X_n(x)\, Y_n(y) = E_n \sin\left(\frac{n\pi x}{10}\right)\sinh\left(\frac{n\pi y}{10}\right)$ — (18)

are solutions of ①, satisfying ② and 3(4).

Here $E_n = 2B_nC_n$.

In order to satisfy condition 3(5), now consider more general solution given by

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) \sinh\left(\frac{n\pi y}{10}\right) \quad \text{--- (18)}$$

put $y=10$ in (18) and using 3(b), we get

$$u(x,10) = \underset{= f(x)}{10x - x^2} = \sum_{n=1}^{\infty} E_n \sinh(n\pi) \sin\left(\frac{n\pi x}{10}\right).$$

which is the half range fourier sine series of $f(x)$ in $(0,10)$

Hence, we have

$$E_n \sinh(n\pi) = \frac{2}{10}\int_0^{10} (10x - x^2) \sinh\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5\sinh n\pi}\left[(10x - x^2)\left(\frac{-10}{n\pi} \frac{\cos n\pi x}{10}\right) - (10-2x)\left(\frac{-100}{n^2\pi^2}\right)\sin\frac{n\pi x}{10} + 2\left(\frac{1000}{n^3\pi^3}\right)\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5\sinh n\pi}\left[\frac{-2000(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3}\right]$$

$$= \frac{400(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}$$

$$\therefore E_n = \begin{cases} 0, & \text{if } n = 2m \text{ and } m = 1,2,3\ldots \\ \dfrac{800\,\text{cosech}(2m-1)\pi}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1,\ m = 1,2,\ldots \end{cases}$$

$\therefore$ from (18), we have

$$u(x,y) = \frac{800}{\pi^3}\sum_{m=1}^{\infty} \frac{\text{cosech}(2m-1)\pi}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{10} \sinh\frac{(2m-1)\pi y}{10}$$

which is the required temperature.

8(c) when a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that path of each vortex is given by the equation,

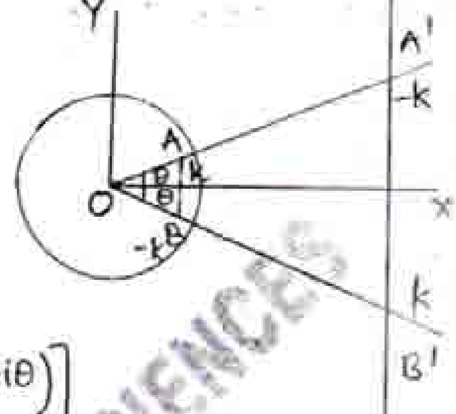
$$(r^2\sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2b^2r^2\sin^2\theta.$$

Sol'n: Let x-axis be the axis of the cylinder. Consider the vortices $+k$ at $A(r,\theta)$ & $-k$ at $B(-r,\theta)$ inside the cylinder s.t distances of A and B from the axis are equal. Evidently, AB is ⊥lar to x-axis. The image of vortex $+k$ at A w.r. to the cylinder is a vortex $-k$ at A', the inverse point of A. Similarly the image of vortex $-k$ at B is a vortex $+k$ at B'.

$$OB \cdot OB' = a^2 = OA \cdot OA',$$

where a is the radius of the cylinder. Then

$$OB' = \frac{a^2}{r} = OA' \text{ as } OB = OA = r$$



The complex potential due to this System at $P(z)$ is.

$$W = \frac{ik}{2\pi}\Big[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right)$$

$$-\log(z - re^{-i\theta}) + \frac{ik}{2\pi}\log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\Big]$$

The motion of the vortex at A is due to other Vortices. If W' be the complex potential for the motion of A, then

$$W' = W - \frac{ik}{2\pi}\log(z - re^{i\theta}) \text{ at } z = re^{i\theta}$$

$$= \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2 e^{i\theta}}{r}\right) - \log(z - re^{-i\theta}) - \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

at $z = re^{i\theta}$

$$W' = \frac{-ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta} - re^{-i\theta}) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$= \frac{-ik}{2\pi}\log(r^2 - a^2)e^{i\theta} - \log r + \log(2ir\sin\theta)$$

$$-\left[\log\{(r^2 - a^2)\cos\theta + i\sin\theta(r^2 + a^2)\} + \log r\right]$$

$$\psi = \frac{-k}{2\pi}\left[\log|(r^2 - a^2)e^{i\theta}| + \log(2ir\sin\theta) - \log|\{(r^2 - a^2)\cos\theta + i\sin\theta(r^2 + a^2)\}|\right]$$

$$= \frac{-k}{2\pi}\left[\log(r^2 - a^2) + \log 2r\sin\theta\right] - \frac{1}{2}\log\{(r^2 - a^2)^2\cos^2\theta + \sin^2\theta(r^2 + a^2)^2\}$$

Streamlines are given by $\psi = \text{const}$. i.e.

$$\log\left\{\frac{(r^2 - a^2)^2(2r\sin\theta)^2}{(r^2 - a^2)^2\cos^2\theta + (r^2 + a^2)^2\sin^2\theta}\right\} = \text{const} = \log 4b^2$$

$$\Rightarrow (r^2 - a^2)^2 r^2 \sin^2\theta = b^2\left[r^4 + a^4 - 2r^2 a^2\cos 2\theta\right]$$

$$\Rightarrow (r^2 - a^2)^2\left[r^2\sin^2\theta - b^2\right] = 4r^2 a^2 b^2 \sin^2\theta$$ This completes the proof.

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