IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2019

Mains

MATHEMATICS

Optional Paper-II

Solutions

SECTION – A

1.(a) → Let G be a finite group, H and K subgroups of G such that $K \subset H$. Show that $(G:K) = (G:H)(H:K)$.

Solution:

Since $K \subset H \subseteq G$ and H, K are subgroups of G, therefore K is a subgroup of H.

By Lagrange's Theorem,

$$(G:K) = \frac{o(G)}{o(K)}$$

Similarly, $(G:H) = \frac{o(G)}{o(H)}$,

$$(H:K) = \frac{o(H)}{o(K)}$$

Hence,
$$(G:H)(H:K) = \frac{o(G)}{o(H)} \times \frac{o(H)}{o(K)}$$
$$= \frac{o(G)}{o(K)}$$
$$= (G:K)$$

Hence, the result.

1.(b) → Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x-y}, & (x,y) \neq (1,-1),(1,1) \\ 0, & (x,y) = (1,1),(1,-1) \end{cases}$$

is continuous and differentiable at $(1,-1)$.

Solution:

Given expression can be written as

$$f(x,y) = \begin{cases} x+y, & (x,y) \neq (1,-1),(1,1) \\ 0, & (x,y) = (1,1),(1,-1) \end{cases}$$

$$\because \lim_{(x,y)\to(1,-1)} f(x,y) = \lim_{(x,y)\to(1,-1)} (x+y) = 1+(-1) = 0 = f(1,-1)$$

$\Rightarrow$ $f(x,y)$ is continuous at $(1,-1)$

Since; $f_x(x,y) = 1$ and $f_y(x,y) = 1$ which are continuous everywhere including $(1,-1)$.

Therefore, f is differentiable everywhere including $(1,-1)$.

$\Rightarrow$ $f(x,y)$ is differentiable at $(1,-1)$.

QED.

Ques: 1(d) Suppose $f(z)$ is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation $\operatorname{Im} f(z) = (\operatorname{Re} f(z))^2$, $z \in D$. Show that $f(z)$ is a constant in D.

Solution:- Given $f(z)$ is an analytic function as Domain D in $\mathbb{C}$

Also $\operatorname{Im} f(z) = (\operatorname{Re} f(z))^2$; $z \in D$ — ①

If $f(z) = u + iv$

$\operatorname{Im} f(z) = v$ & $\operatorname{Re} f(z) = u$ — ②

$\because$ $f(z)$ is analytic function

$$\therefore \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{— ③}$$

from ① & ② ; $v = u^2$

$$\therefore \left.\begin{aligned} \frac{\partial v}{\partial x} &= 2u\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} &= 2u\frac{\partial u}{\partial y} \end{aligned}\right\} \text{— ④}$$

from ③ & ④ ; $\frac{\partial v}{\partial x} = 0$ & $\frac{\partial v}{\partial y} = 0$

$$\therefore \frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \Rightarrow \frac{\partial f}{\partial z} = 0$$

By Integrating $\Rightarrow$ $\boxed{f(z) = c \text{ [constant]}}$

Hence the result

Ques: 1(e) Use graphical method to solve the linear programming problem

Maximize $z = 3x_1 + 2x_2$

Subject to $x_1 - x_2 \geq 1$

$x_1 + x_3 \geq 3$

and $x_1, x_2, x_3 \geq 0$

Solution:- Given linear programming problem

Maximize $z = 3x_1 + 2x_2$

Subject to $x_1 - x_2 \geq 1$

$x_1 + x_3 \geq 3$

and $x_1, x_2, x_3 \geq 0$

Taking $-x_2 = y_2$

$\therefore z_{max} = 3x_1 - 2y_2$

$x_1 + y_2 \geq 1$

$x_1 + x_3 \geq 3$

$x_1, x_3 \geq 0, \ y_2 \leq 0$

From the graph, we have only three intersection points A, B and C.

A : (1, 0, 3)

B : (0, 1, 3)

for C : (solving planes P_1 & P_2)

$$\left.\begin{aligned} x_1 + y_2 = 1 \quad &\Rightarrow y_2 = 1 - x_1 \\ x_1 + x_3 = 3 \quad & \quad\ x_3 = 3 - x_1 \end{aligned}\right\} \begin{matrix} -(1) \\ -(2) \end{matrix}$$

as $x_1 \geq 0$

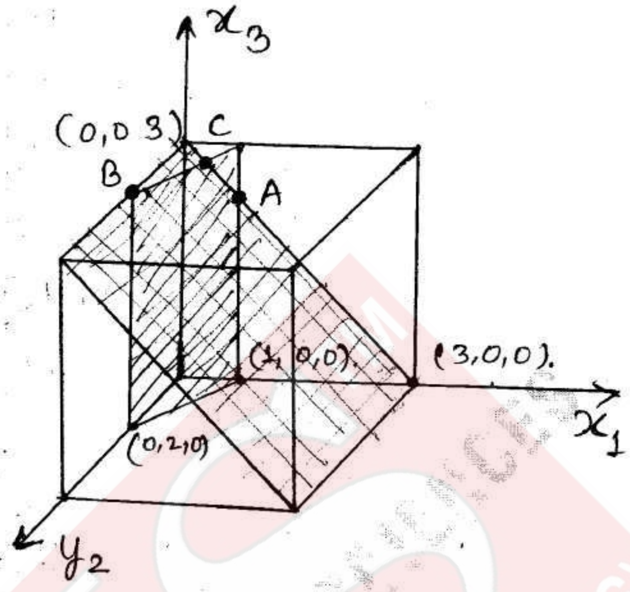
but by (1) $x_1 \geq 1$.

from (2) ; $x_3 \geq 0$

so $x_1 \leq 3$.

So; $1 \leq x_1 \leq 3$

$-2 \leq y_2 \leq 0$



$\therefore Z_{max} = 3x_1 - 2y_2$

$= 3\times3 - 2(-2)$

$= 9+4 = 13$.

$\therefore \boxed{Z_{max} = 13}$

Required solution.

2.(a) → If G and H are finite groups whose orders are relatively prime, then prove that there is only one homomorphism from G to H, the trivial one.

Solution:

Let G and H be finite groups such that orders of G and H are relatively prime to each other.

Consider $\phi : G \to H$ to be a homomorphism

To show that : ϕ must be trivial.

As $\phi(G)$ is a subgroup of H

$\Rightarrow$ order of $\phi(G)$ divides order of H.

Also, $\phi(G)$ is also a quotient of G.

$\Rightarrow$ order of $\phi(G)$ divides order of G.

$\because$ orders of G and H are coprimes.

$\Rightarrow$ $\phi(G)$ is trivial.

Hence, the result.

2(b) Write down all quotient groups of the group Z_{12}.

Sol

Let $(Z_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +_{12})$ be a group.

Clearly Z_{12} is a cyclic group and generated by '1'

i.e $Z = \langle 1 \rangle$ s.t $O(1) = 12 = O(Z_{12})$

$\therefore$ The subgroups of Z_{12} are precisely the subgroup generated by $m(1)$ where m divides 12

since $\frac{12}{m} \Rightarrow m = 1, 2, 3, 4, 6$ and 12

$\therefore \langle 1 \rangle, \langle 2(1) \rangle, \langle 3(1) \rangle, \langle 4(1) \rangle, \langle 6(1) \rangle$ and $\langle 12(1) \rangle$ are cyclic subgroups of Z_{12}

i.e $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle, \langle 6 \rangle$ and $\langle 12 \rangle$.

since Z_{12} is cyclic.

$\therefore$ its every subgroup is normal.

Thus the normal subgroups of Z_{12} are given by

$\langle 1 \rangle = Z_{12}$, $\langle 2 \rangle = \{2x / x \in I\}$

$= \{2, 4, 6, 8, 10, 0, \ldots\}$

$= \{0, 2, 4, 6, 8, 10\}$

$\langle 3 \rangle = \{0, 3, 6, 9\}$, $\langle 4 \rangle = \{0, 4, 8\}$

$\langle 6 \rangle = \{0, 6\}$ and $\langle 12 \rangle = \{0\} = \langle 0 \rangle$

Finally, the quotient groups of Z_{12}

are given by

$$\frac{Z_{12}}{\langle 1\rangle} = \{\langle 1\rangle + x \,/\, x \in Z_{12}\}$$
$$= \{\langle 1\rangle\},$$

$$\frac{Z_{12}}{\langle 2\rangle} = \{\langle 2\rangle + x \,/\, x \in Z_{12}\}$$
$$= \{\langle 2\rangle, \langle 2\rangle + 1\},$$

$$\frac{Z_{12}}{\langle 3\rangle} = \{\langle 3\rangle, \langle 3\rangle + 1, \langle 3\rangle + 2\},$$

$$\frac{Z_{12}}{\langle 4\rangle} = \{\langle 4\rangle, \langle 4\rangle + 1, \langle 4\rangle + 2, \langle 4\rangle + 3\}$$

$$\frac{Z_{12}}{\langle 6\rangle} = \{\langle 6\rangle, \langle 6\rangle + 1, \langle 6\rangle + 2, \langle 6\rangle + 3, \langle 6\rangle + 4, \langle 6\rangle + 5\} \text{ and}$$

$$\frac{Z_{12}}{\langle 12\rangle} = $$
$$= \{\langle 12\rangle, \langle 12\rangle + 1, \langle 12\rangle + 2, \langle 12\rangle + 3, \langle 12\rangle + 4, \langle 12\rangle + 5, \langle 12\rangle + 6, \langle 12\rangle + 7, \langle 12\rangle + 8, \langle 12\rangle + 9, \langle 12\rangle + 10, \langle 12\rangle + 11\}.$$

Hence, the result.

Ques: 2(c) Using differentials, find an approximate value of $f(4.1, 4.9)$, where

$$f(x,y) = (x^3 + x^2 y)^{1/2}$$

Solution:-

Given; $f(x,y) = (x^3 + x^2y)^{1/2}$ —— (1)

then; $f(4.1, 4.9) = \left[(4.1)^3 + (4.1)^2 \times 4.9\right]^{1/2}$

$f(4.1, 4.9) = [151.29]^{1/2}$

Let $f(4.1, 4.9) = f(X) = Y$

$\therefore Y = [X]^{1/2}$ —— (2)

$X = 151.29$.

Which can be break into two parts

$X = 144 + 7.29$

$X = x + \Delta x$

$\frac{dY}{dX} = \frac{1}{2\sqrt{X}}$ [by using (2)]. —— (3)

Also $\Delta Y = \frac{dY}{dX} \cdot \Delta X$

$\Delta Y = \frac{1}{2\sqrt{X}} \cdot \Delta X$

$\Delta Y = \frac{1}{2\sqrt{144}} \times 7.29 = \frac{1}{24} \times 7.29$.

$$\boxed{\Delta Y = 0.30375}$$

as $\Delta X = 7.29 \qquad \Delta Y = 0.30375$

$$\Delta Y = f(X+\Delta X) - f(X)$$

$$0.30375 = \sqrt{151.29} - \sqrt{144}$$

$$f(X+\Delta X) = \sqrt{151.29} = 12 + 0.30375$$

$$\boxed{f(X+\Delta X) = \sqrt{151.29} = 12.30375.}$$

$$\therefore f(x,y) = f(4.1, 4.9) = (x^3+x^2y)^{1/2} = 12.30375$$

Hence the approximate value of

$f(4.1, 4.9) = 12.304$ where

$f(x,y) = (x^3+x^2y)^{1/2}$.

Hence the result.

2.(d) → Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

Moreover $\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$.

Solution:

Since $f(z)$ has a pole of order m, then by definition, for $0 < |z-z_0| < R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}, \quad b_m \neq 0$$

$$\Rightarrow f(z) = \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \dots + b_m\right]$$

$$\Rightarrow f(z) = \frac{\phi(z)}{(z-z_0)^m}.$$

Clearly, $\phi(z_0) = b_m \neq 0$ and is analytic at z_0, as it has Taylor series expansion about z_0.

Conversely,

Suppose $f(z)$ can be written in the form:

$f(z) = \frac{\phi(z)}{(z-z_0)^m}$, then

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z-z_0) + \frac{\phi''(z_0)}{2!}.(z-z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \cdots$$

$\Rightarrow$ In $0 < |z-z_0| < R$,

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}\cdot\frac{1}{(z-z_0)}$$

$$+\frac{\phi^{(m)}(z_0)}{m!}+\frac{\phi^{(m+1)}(z_0)}{(m+1)!}(z-z_0)+\cdots\cdots$$

Since $\phi(z_0)\neq 0$,

$f(z)$ has a pole of order m

with residue,

$$\boxed{b_1=\frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$

In case of simple pole,

i.e., $m=1$, $\operatorname{Res}_{z=z_0} f(z_0)=\phi(z_0)$.

Hence, proved.

3.(a) → Discuss the uniform convergence of

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad \forall x \in \mathbb{R} \ (-\infty, \infty)$$
$$n = 1, 2, 3, \ldots$$

Solution:

Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

Suppose that $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$.

Also, the point-wise limit f is given as

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{nx^2 + 1/n}$$
$$= 0, \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}.$$

Now, from our assumption, $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$ so that we have the point-wise limit f is also the uniform limit.

Let $\varepsilon > 0$ be given. Then there exists m such that $\forall x \in (-\infty, \infty)$ and $\forall n \geq m$.

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \varepsilon.$$

We take $\varepsilon = 1/4$.

Now there exists an integer k such that

$$k \geq m \quad \text{and} \quad 1/k \in (-\infty, \infty).$$

Taking $n = k$ and $x = 1/k$, we have

$$\frac{nx}{1+n^2x^2} = \frac{1}{2}$$ which is not less than $1/4$.

Thus, we arrive at a contradiction and so, the sequence $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $\forall x \in \mathbb{R}(-\infty,\infty)$, $n = 1, 2, 3, \ldots$

is not uniformly convergent with 0, in the interval $(-\infty, \infty)$ even though it is point-wise convergent.

Hence, the result.

Ques: 3(b) Solve the linear programming problem using simplex method.

$$\text{Minimize } z = x_1 + 2x_2 - 3x_3 - 2x_4$$

Subject to

$$x_1 + 2x_2 - 3x_3 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 = 4$$

and $x_1, x_2, x_3, x_4 \geq 0$

Solution:-

$$\text{Min } Z = x_1 + 2x_2 - 3x_3 - 2x_4$$

subject to $x_1 + 2x_2 - 3x_3 + x_4 = 4$
$x_1 + 2x_2 + x_3 + 2x_4 = 4.$

and $x_1, x_2, x_3, x_4 \geq 0$;

$$\therefore \text{Max } z = -x_1 - 2x_2 + 3x_3 + 2x_4$$

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate

(1) As the constraint-1 is of type '=' we should add artificial variable A_1

(2) As the constraint-2 is of type '=' we should add artificial variable A_2

After introducing artificial variables

$$\text{Max } z = -x_1 - 2x_2 + 3x_3 + 2x_4 - MA_1 - MA_2$$

subject to

$$x_1 + 2x_2 - 3x_3 + x_4 + A_1 = 4$$
$$x_1 + 2x_2 + x_3 + 2x_4 + A_2 = 4$$

and $x_i, A_j \geq 0$.

	C_j	-1	-2	3	2	-M	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	A_2	b	θ
-M	A_1	1	(2)	-3	1	1	0	4	$4/2=2 \rightarrow$
-M	A_2	1	2	1	2	0	1	4	$4/2=2$
$Z_j=\Sigma a_{ij}C_B$		-2M	-4M	2M	-3M	-M	-M	-8M	
$C_j=C_j-Z_j$		-1+2M	4M-2 ↑	3-2M	3M+2	0	0		

$x_2 \rightarrow$ incoming variable (2) – key element
$A_1 \rightarrow$ Outgoing variable $A_1 \rightarrow$ column can be omitted.

-2	x_2	1/2	1	-3/2	1/2	1/2	0	2	(-ve)
-M	A_2	0	0	(4)	1	-1/2	1	0	$0/4=0 \rightarrow$
$Z_j=\Sigma a_{ij}C_B$		-1	-2	3-4M	-1-M	$-1+\frac{M}{2}$	-M	-4	
		0	0	4M ↑	M+3	omitted	0		

$A_2 \rightarrow$ outgoing variable A_2 column omitted
$x_3 \rightarrow$ incoming variable (4) key element.

	C_j	-1	-2	3	2		
C_B	Basis	x_1	x_2	x_3	x_4	b	θ
-2	x_2	1/2	1	0	7/8	2	$2/7/8=\frac{16}{7}=2.28$
3	x_3	0	0	1	(1/4)	0	$0/1/4=0 \rightarrow$
$Z_j=\Sigma a_{ij}C_B$		-1	-2	3	-1		
$C_j=C_j-Z_j$		0	0	0	3 ↑		

Positive Maximum $C_j = 3$ of x_4

hence $x_4 \rightarrow$ incoming $x_3 \rightarrow$ outgoing

(1/4) $\rightarrow$ Key Element

$\Rightarrow R_2 \longrightarrow R_2 \times 4$

$\Rightarrow R_1 \rightarrow R_1 - \frac{7}{8} R_2$

	C_j	-1	-2	3	2	
C_B	Basis	x_1	x_2	x_3	x_4	b
-2	x_2	1/2	1	-7/2	0	2
2	x_4	0	0	4	1	0
$Z_j = \Sigma C_B a_{ij}$		-1	-2	15	2	$Z = -4$.
$C_j = C_j - Z_j$		0	0	-12	0	

$\because C_j \leq 0$

Hence, optimal system is arrived with value of variables as;

$$x_1 = 0, \; x_2 = 2, \; x_3 = 0, \; x_4 = 0$$

$$\text{Max } z = -x_1 - 2x_2 + 3x_3 + 2x_4$$
$$= 0 - 2 \times 2 + 0 + 0 = -4.$$

$\boxed{\therefore \text{Min } z = 4}$

Required Solution

Ques: 3(c) Evaluate the integral $\int_C Re(z^2)dz$ from 0 to $2+4i$ along the curve C where C is a parabola $y=x^2$.

Solution:-

Given function $I = \int_C Re(z^2)dz$ —— (1)

from 0 to $2+4i$ along the curve C where C is parabola $y=x^2$.

$$y = x^2$$

$$dy = 2x\,dx. \quad \text{—— (2)}$$

$$z = x+iy$$

$$z^2 = x^2-y^2+2xyi$$

$$Re(z^2) = x^2-y^2 = x^2-x^4 \quad [\because y=x^2]$$

$$dz = dx+i\,dy. = dx+2xi\,dx \quad (\text{from (2)})$$

$$= (1+2xi)dx$$

$$\therefore \int_0^{2+4i} (x^2-y^2)\,dz = \int_0^2 (x^2-x^4)(1+2xi)\,dx$$

$$= \int_0^2 [x^2+2x^3i-x^4-2x^5i]\,dx.$$

$$= \left[\frac{x^3}{3}+\frac{x^4}{2}i-\frac{x^5}{5}-\frac{1}{3}x^6 i\right]_0^2$$

$$= \left[\frac{8}{3}+8i-\frac{32}{5}-\frac{64}{3}i\right]$$

$$\boxed{\int_C Re(z^2)dz = -\frac{56}{15} - i\,\frac{40}{3}}$$ Required Solution.

3.(d) → Let a be an irreducible element of the Euclidean ring R, then prove that $R|\langle a\rangle$ is a field.

Solution:

Let $A = (a)$, where a is an irreducible element of R.

We shall show that A is a maximal ideal of R.

Let I be any ideal of R such that

$$A \subseteq I \subseteq R.$$

Since R is an Euclidean ring and

$$E.D \Rightarrow P.I.D,$$

∴ We have, R is a PID;

Let $I = \langle d\rangle$, for some $d \in R$

Case (i): Let $d \in A = (a)$. Then $d = ax$ for some $x \in R$

for any $r \in I = (d)$, $r = dy$, for some $y \in R$

$$\Rightarrow r = (ax)y = a(xy)$$

$$\Rightarrow r \in A$$

$$\Rightarrow I \subseteq A$$

Also, $A \subseteq I$.

$\therefore A = I$.

Case (ii): Let $d \notin A$.

Since $a \in A$ and $A \subseteq I = (d)$,

so $a = dt$, for some $t \in R$

Since R is irreducible, either d or t is a unit.

If t is a unit, then $t^{-1} \in R$ and so

$d = at^{-1}$

Since $a \in A$ and $t^{-1} \in R$, $at^{-1} \in A$, as A is an ideal of R.

Thus, $d \in A$, which is a contradiction.

Consequently, d must be a unit.

i.e. $d^{-1} \in R$

Now, $d \in I$ and $d^{-1} \in R \Rightarrow 1 = dd^{-1} \in I$

$\Rightarrow I = R$.

Hence, $A \subseteq I \subseteq R$

$\Rightarrow A = I$ or $I = R$

$\Rightarrow$ A is a maximal ideal of R.

i.e. Since (a) is a maximal ideal of R, therefore $R/(a)$ is a field.

Hence, proved.

4.(a). → Find the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the subsidiary condition $x^2+y^2+z^2 = c^2$, $(x,y,z>0)$.

Solution:

On the spherical surface $x^2+y^2+z^2=c^2$, the function must assume the greatest value, since the surface is a bounded and closed set.

According to the Method of Undetermined Multipliers, we form the expression

$$F = x^2y^2z^2 + \lambda(x^2+y^2+z^2-c^2)$$

and by differentiation, we obtain

$$2xy^2z^2 + 2\lambda x = 0,$$
$$2x^2yz^2 + 2\lambda y = 0,$$
$$2x^2y^2z + 2\lambda z = 0.$$

The solutions with $x=0$, $y=0$, or $z=0$ can be excluded, for at these points the function takes on its least value, zero.

The other solutions of the equation are $x^2 = y^2 = z^2$, $\lambda = -x^4$. Using the subsidiary condition, we obtain the values

$$x = \pm \frac{c}{\sqrt{3}}, \quad y = \pm \frac{c}{\sqrt{3}}, \quad z = \pm \frac{c}{\sqrt{3}}$$

for the required coordinates.

At all these points, the function assumes the same value $\underline{c^6/27}$, which accordingly is the maximum.

Hence, any triad of numbers satisfies the relation

$$\sqrt[3]{x^2y^2z^2} \leq \frac{c^2}{3} = \frac{x^2+y^2+z^2}{3},$$

which states that the geometric mean of three nonnegative numbers x^2, y^2, z^2 is never greater than their arithmetic mean.

Ques: 4(b) Obtain the first three terms of Laurent series expansion of the function $f(z)=\frac{1}{(e^z-1)}$ about the point $z=0$ valid in the region $0<|z|<2\pi$.

Solution:- Given ; $f(z)=\frac{1}{(e^z-1)}$

about the point $z=0$ valid in region $0<|z|<2\pi$.

we have that

$$\frac{1}{(e^z-1)} = \frac{1}{z\left(1+\left[\left(z/2!\right)+\left(z^2/3!\right)+\cdots\right]\right)}; \text{ from where}$$

we get

$$P(z) = \left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)$$

If $0<|z|\leq 1$ so $|z|^2, |z|^3, \ldots < 1$

so $P(z) = \left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right) < 1$

[$\because$ at $z=1$. $\Rightarrow$ $P(z) = \left[\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots\right] < 1$

Then ;

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{P(z)}{z} + \frac{P(z)^2}{z} - \frac{P(z)^3}{z} + \cdots\cdots$$

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} - \frac{z}{3!} + \frac{z}{(2!)^2} - \frac{z^2}{4!} + \frac{2z^2}{(2!3!)} - \frac{z^2}{(2!)^3} + O(z^3)$$

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + z^2\left[\frac{1}{6}-\frac{1}{24}-\frac{1}{8}\right] + O(z^3)$$

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + z^2(0) + O(z^3)$$

$$\boxed{\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3)} \quad \text{——— (1)}$$

Also, first z terms asked coefficient $\left(\frac{1}{z}\right)$

is $\text{Res}\left[f(z)\right]_{z=0} \Rightarrow \lim_{z\to 0} \frac{z}{e^z-1} = 1$

so varifies.

If $1<|z|<2\pi$

$\because$ asked about $z=0$, so (1) holds as Laurent series is unique.

Hence; $\boxed{\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + O(z^3)}$

Required result.

4.(c) → Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$.

Solution:

Let $f(x) = \frac{\sqrt{x}}{\ln x}$

1 is the only point of infinite discontinuity of 'f' on $[1,2]$.

Take $g(x) = \frac{1}{(x-1)^n}$

$$\therefore \lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)^n\sqrt{x}}{\ln x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1^+} \frac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x}$$

$$= \lim_{x\to 1^+} (x-1)^{n-1}\left[nx^{3/2} + \left(\frac{x-1}{2}\right)\sqrt{x}\right]$$

$= 1$ if $n = 1$.

($\therefore$ a non-zero finite number)

$\therefore$ By Comparison test,

$\int_1^2 f(x)\,dx$ & $\int_1^2 g(x)\,dx$ are convergent (or) divergent together. But $\int_1^2 g(x)\,dx$ diverges ($\because n=1$).

$\therefore \int_1^2 f(x)\,dx$ diverges

i.e. $\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$ diverges.

Section-B

Ques:-5(a) Form a partial differential equation of the family of surfaces given by the following expression:

$$\psi(x^2+y^2+2z^2,\ y^2-2zx)=0$$

Solution:

Given $\psi(x^2+y^2+2z^2,\ y^2-2zx)=0$

$$u = x^2+y^2+z^2 \qquad v = y^2-2zx$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$\therefore\ \frac{\partial u}{\partial x}=2x\ ;\ \frac{\partial u}{\partial z}=4z\ ;\ \frac{\partial v}{\partial x}=-2z\ ;\ \frac{\partial v}{\partial z}=-2x$$

$$\frac{\partial\phi}{\partial u}(2x+4pz)+\frac{\partial\phi}{\partial v}(-2z+p(-2x))=0$$

$$\frac{\partial\phi}{\partial u}(2x+4pz)-\frac{\partial\phi}{\partial v}(2z+2xp)=0$$

$$\frac{\partial\phi}{\partial u}(x+2pz)-\frac{\partial\phi}{\partial v}(z+xp)=0$$

$$\boxed{\frac{\partial\phi}{\partial u}(x+2pz)=\frac{\partial\phi}{\partial v}(z+xp)} \qquad \text{---}(1)$$

$$\frac{\partial u}{\partial y}=2y\ ,\ \frac{\partial u}{\partial z}=4z\ ;\ \frac{\partial v}{\partial y}=2y\ ;\ \frac{\partial v}{\partial z}=-2x.$$

$$\frac{\partial\phi}{\partial u}(2y+4zq)+\frac{\partial\phi}{\partial v}(2y-2qx)=0$$

$$\frac{\partial \phi}{\partial u}(y+2zq)+\frac{\partial \phi}{\partial v}(y-xq)=0$$

$$\boxed{\frac{\partial \phi}{\partial u}(y+2zq)=(xq-y)\frac{\partial \phi}{\partial v}} \quad \text{——— } (2)$$

Divide eq^n (1) by (2), we get

$$\frac{(x+2pz)}{(y+2zq)}=\frac{(z+px)}{(xq-y)}$$

$$\therefore (xq-y)(x+2pz)=(y+2zq)(z+px)$$

$$\Rightarrow x^2q+\cancel{2xpqz}-xy-2pyz=zy+2z^2q+pxy+\cancel{2xpqz}$$

$$\Rightarrow x^2q-yx-2pyz-zy-2z^2q+(-pxy)=0$$

$$\Rightarrow (x^2-2z^2)q-(x+z)y-(xy+2yz)p=0$$

Hence;

$$\boxed{(x^2-2z^2)q-(x+z)y-(xy+2yz)p=0}$$

Required Solution

Ques: 5(b): Apply Newton-Raphson method, to find a real root of transcendental equation $x\log x = 1.2$, correct to three decimal places.

Solution:- Here; $x\log x = 1.2$

i.e $x\log x - 1.2 = 0$

Let, $f(x) = x\log x - 1.2$

$\therefore f'(x) = \log x + 1$

Here

x	1	2	3
$f(x)$	-1.2	-0.6	0.23

Here $f(2) = -0.6 < 0$ and $f(3) = 0.23 > 0$

$\therefore$ Root lies between 2 and 3.

$$x_0 = \frac{2+3}{2} = 2.5 \Rightarrow \boxed{x_0 = 2.5}$$

Ist iteration:-

$$f(x_0) = f(2.5) = (2.5)\log(2.5) - 1.2 = -0.21.$$

$$f'(x_0) = f'(2.5) = \log(2.5) + 1 = 1.4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{(-0.21)}{1.4} = 2.650$$

2nd Iteration

$$f(x_1) = f(2.65) = (2.65)\log(2.65) - 1.2 = -0.08$$

$$f'(x_1) = f'(2.65) = \log(2.65) + 1 = 1.42.$$

$$x_2 = 2.65 - \frac{(-0.08)}{1.42} = 2.65 + 0.056 = 2.706$$

3rd iteration

$f(x_2) = f(2.706) = 2.706 \log 2.706 - 1.2$
$= 1.16987 - 1.2 = -0.0301$

$f'(x_2) = f'(2.706) = \log(2.706) + 1 = 1.432.$

$x_3 = 2.706 - \frac{(-0.0301)}{1.432} = 2.706 + 0.021$

$\boxed{x_3 = 2.727}$

4th iteration

$f(x_3) = f(2.727) = 2.727 \log 2.727 - 1.2$
$= 1.188 - 1.2 = -0.012.$

$f'(x_3) = f'(2.727) = \log(2.727) + 1 = 1.436.$

$x_4 = 2.727 - \frac{(-0.012)}{1.436} = 2.727 + 0.008$

$\boxed{x_4 = 2.735}$

5th iteration

$f(x_4) = f(2.735) = 2.735 \log(2.735) - 1.2$
$= 1.195 - 1.2 = -0.005$

$f'(x_4) = f'(2.735) = \log(2.735) + 1 = 1.437.$

$x_5 = 2.735 - \frac{(-0.005)}{1.437} = 2.735 + 0.003$

$\boxed{x_5 = 2.738}$

6th iteration

$f(x_5) = f(2.738) = -0.0003$

$f'(x_5) = f'(2.738) = 1.438$

$x_6 = 2.738 - \frac{(-0.0003)}{1.438} = \underline{\underline{2.738}}$

Hence from fifth and sixth iterations

$x_5 = 2.738$ & $x_6 = 2.738$

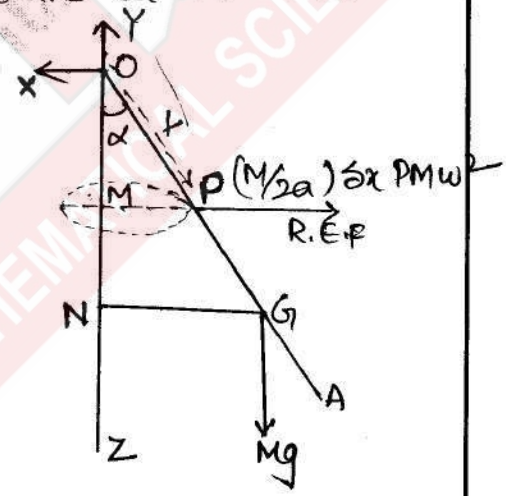
So, 2.738 is the real root of the given equation correct upto 3 decimal.

5(c) A uniform rod OA, of length $2a$, free to turn about its end O, revolves with angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ; find the value of α.

Sol'n: Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ.

Let $PQ = \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$.

This element PQ will make a circle in the horizontal plane with radius PM ($= x\sin\alpha$) and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\frac{M}{2a}\delta x \cdot PM \cdot \omega^2$ along PM.



Thus the reversed effective force on the element PQ is

$\frac{M}{2a}\delta x \cdot x\sin\alpha \cdot \omega^2$ along MP.

Now By D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and reaction at 'O' are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\sum\left(\frac{M}{2a}\delta x \cdot \omega^2 \cdot \sin\alpha\right) \cdot OM - Mg \cdot NG = 0$$

$$\Rightarrow \int_0^{2a} \frac{M}{2a}\omega^2 x^2 \sin\alpha \cos\alpha \, dx - Mg \cdot a\sin\alpha = 0, \quad (\because OM = x\cos\alpha)$$

$\Rightarrow \frac{M}{2a}\omega^2 \cdot \left\{\frac{1}{3}(2a)^3\right\} \cdot \sin\alpha\cos\alpha - Mg\,a\sin\alpha = 0$

$\Rightarrow Mga\sin\alpha\left(\frac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0$

$\therefore$ either $\sin\alpha = 0$ i.e. $\alpha = 0$

$\Rightarrow \frac{4a}{3g}\omega^2\cos\alpha - 1 = 0$, i.e. $\cos\alpha = \frac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero (or) $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

Ques-5(d)> Using Runge-Kutta method for fourth order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$. Use four decimal places for calculation and step length 0.2.

Solution:- Given;

$$y' = \frac{y^2-x^2}{y^2+x^2}$$

at $x=0$, $y(0)=1$; $h=0.2$; $y(0.2)=?$

using Runge-kutta method for fourth order

$$y(0.2) = y(0) + K$$

where $K = \frac{1}{6}[K_1+K_4+2(K_2+K_3)]$

$$K_1 = h f(x_0, y_0)$$
$$K_2 = h f(x_0+\tfrac{h}{2}, y_0+K_1/2)$$
$$K_3 = h f(x_0+\tfrac{h}{2}, y_0+\tfrac{K_2}{2})$$
$$K_4 = h f(x_0+h, y_0+K_3)$$

$\therefore$

$$K_1 = 0.2 f(0,1) = 0.2\times 1 = 0.2$$
$$K_2 = 0.2 f(0.1, 1.1) = 0.2\times 0.98361 = 0.19672$$
$$K_3 = 0.2 f(0.1, 1.09836) = 0.2\times 0.98356 = 0.19671$$
$$K_4 = 0.2 f(0.2, 1.19671) = 0.2\times 0.94566 = 0.18913$$

$$y(0.2) = 1+\frac{1}{6}\left(0.2+0.18913+2(0.19672+0.19671)\right)$$
$$y(0.2) = 1+\frac{1}{6}[0.38913 + 0.78686]$$
$$y(0.2) = 1+\frac{1}{6}[1.17599]$$
$$y(0.2) = 1+0.195998$$
$$\boxed{y(0.2) = 1.1960}$$

Required solution.

Ques: 5(c) Draw the flow chart and write a basic algorithm (in FORTRAN/C/C++) for evaluating $y = \int_0^6 \frac{dx}{1+x^2}$ using Trapezoidal rule.

Solution :- Given; $y = \int_0^6 \frac{dx}{1+x^2}$ —— (1)

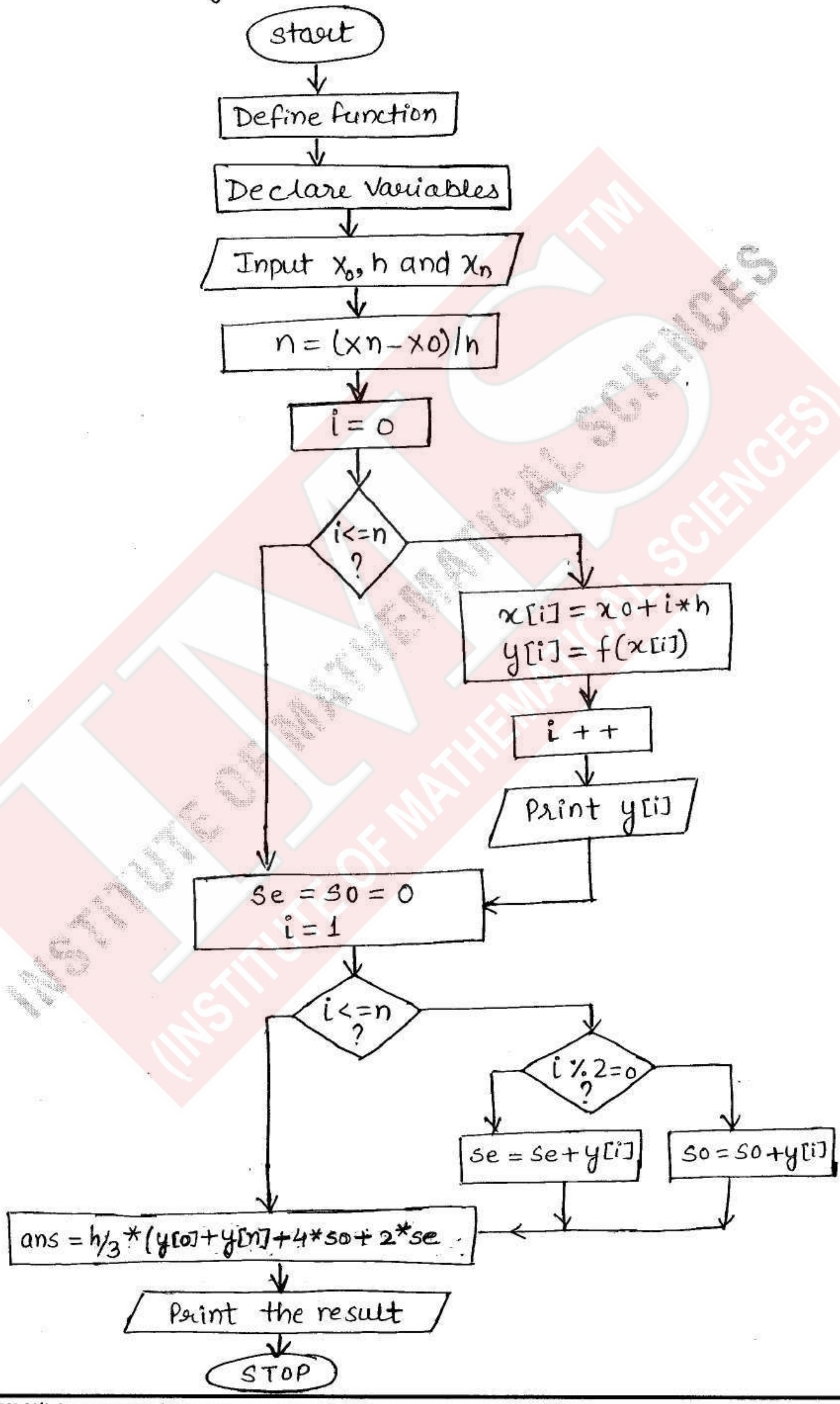
$f(x) = \frac{1}{1+x^2}$ $a = 0$, $b = 6$.

Hence; Algorithm for evaluating (1) using Trapezoidal rule in C++

```
// C++ program to implement Trapezoidal rule
# include <stdio.h>

// A sample function whose definite integral's
// approximate value is computed using Trapezoidal
// rule
float y (float x)
{
  // Declaring the function f(x) = 1/(1+x*x)
  return 1/(1+x*x);
}
// Function to evaluate the value of integral
float trapezoidal (float a, float b, float n)
{
  // Grid spacing
  float h = (b-a)/n;
```

```
// computing sum of first and last terms
// in above formula
 float s = y(a)+ y(b);

// Adding middle terms in above formula
 for ( int i = 1 ; i<n ;i++)
 s += 2*y(a+i*h);

// h/2 indicates (b-a)/2n. Multiplying h/2
// with s.
 return (h/2)*s ;
}
// Driver program to test above function
 int main()
{
 // Range of definite integral
    float x0 = 0;
    float xn = 1;

// Number of grids. Higher value means
// more accuracy
  int n=6;

printf ("Value of integral is %6.4f\n",
                  trapezoidal (x0, xn, n));
  return 0;
}
```

Flow chart of Trapezoidal Rule



Ques: 6(b) Find the equivalent numbers given in a specified number to the system mentioned against them:

(i) Integer 524 in binary system.

Solution: $(524)_{10} \longleftrightarrow (\quad)_2$.

2	524	
2	262	0
2	131	0
2	65	1
2	32	1
2	16	0
2	8	0
2	4	0
2	2	0
	1	0

$\therefore (524)_{10} \longrightarrow (1000001100)_2$.

(ii) 101 010 110 101 . 101 1010 . 11 to octal sysem.

Solution: $(\underbrace{101}_{5}\ \underbrace{010}_{2}\ \underbrace{110}_{6}\ \underbrace{101}_{5}\ .\ \underbrace{101}_{5}\ \underbrace{101}_{5}\ \underbrace{011}_{3})_2 \longleftrightarrow (\quad)_8$

$\therefore (101\ 010\ 110\ 101 . 101\ 101\ 011)_2 \longleftrightarrow (5265.553)_8$

(iii) Decimal number 5280 to hexadecimal system.

Solution: $(5280)_{10} \longleftrightarrow (\quad)_{16}$

16	5280	
16	330	0
16	20	A
	1	4

$\therefore (5280)_{10} \longleftrightarrow (14A0)_{16}$.

(iv) Find the unknown number $(1101.101)_8 \to (?)_{10}$.

Solution: $(1101.101)_8$

$1101 = 1\times 8^0 + 0\times 8^1 + 1\times 8^2 + 1\times 8^3$

$= 1+0+64+512 = 577.$

For Decimal (101)

$\Rightarrow 1\times 8^{-1} + 0\times 8^{-2} + 1\times 8^{-3}$

$\Rightarrow 0.125 + 0 + 0.001953125.$

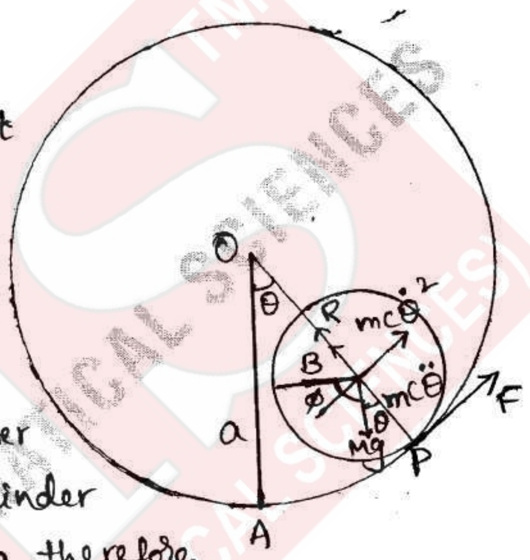
$\Rightarrow 0.126953125$

$\therefore (1101.101)_8 \longleftrightarrow (577.126953125)_{10}$

6(C). A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$.

Sol^n:

Let P be the point of contact of the two cylinders at time t such that $\angle AOP = \theta$. Let ϕ the angle which the line CB fixed in moving cylinder make with the vertical at time t.



Here radius of fixed cylinder is a and that of moving cylinder is a. Since there is pure rolling therefore

Arc AP = Arc BP

$\Rightarrow b\theta = a(\phi+\theta)$

$\Rightarrow a\phi = (b-a)\theta$

$\therefore \ddot{\phi} = c\ddot{\theta}$ ——— (1)

where $c = (b-a)$

Let R be the normal reaction and F the friction at the point P.

$\therefore$ The centre C describes a circle of radius $OC = b-a = c$

$\therefore$ Its accelerations along and perpendicular to CO are $c\dot{\theta}^2$ and $c\ddot{\theta}$ respectively

$\therefore$ The equations of motion of the moving cylinder are

$Mc\dot{\theta}^2 = R - Mg\cos\theta$ ——— (2)

and $Mc\ddot{\theta} = F - Mg\sin\theta$ ——— (3)

Also for the motion relative to the centre of Interia C,

$Mk^2\ddot{\phi}$ = Moment of the forces about $C = -Fa$ —— (4)

$$Mk^2 \frac{c}{a}\ddot{\theta} = -Fa$$

$$\Rightarrow F = -Mk^2 \frac{c}{a^2}\ddot{\theta}$$

Substituting in (3), we get

$$Mc\ddot{\theta} = -Mk^2 \frac{c}{a^2}\ddot{\theta} - Mg\sin\theta$$

$$\Rightarrow c\left(1+\frac{k^2}{a^2}\right)\ddot{\theta} = -g\sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{c\left(1+k^2/a^2\right)}\theta$$

$$= -\mu\theta$$

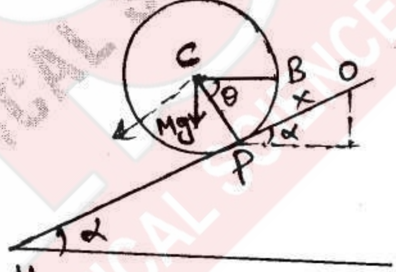
$\because$ θ is very small

$\therefore$ Length of the simple equivalent pendulum is

$$g/\mu = c\left(1+\frac{k^2}{a^2}\right) = (b-a)\left(1+k^2/a^2\right).$$

7(a) Using Hamilton's equation, find the acceleration for a sphere rolling down a rough inclined plane, if x be the distance of the point of contact of the sphere from a fixed point on the plane.

Sol^n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during this time let it turn through an angle θ.



Since there is no slipping

$\therefore x = OA = \text{arc } AB = a\theta,$

So that $\dot{x} = a\dot{\theta}$

If T and V are the kinetic & potential energies of the sphere, then

$T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2$

$= \frac{1}{2}M\,\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$

$\Rightarrow T = \frac{7}{10}M\dot{x}^2$

and $V = -MgOL = -Mg\,x\sin\alpha$, (Since the sphere move down the plane)

$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$

Here x is the only generalised coordinate.

$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x}$

Since L does not contain t explicitly.

$\therefore H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha$

$\Rightarrow H = \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx\sin\alpha$

$$= \frac{5}{14M} p_x^2 - Mgx\sin\alpha \quad \text{from (1)}.$$

Hence the two Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \quad \text{— }(H_1), \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M} p_x \quad \text{— }(H_2)$$

Differentiating (H_2) and using (H_1), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M} Mg\sin\alpha$$

$$\Rightarrow \ddot{x} = \frac{5}{7} g\sin\alpha.$$

which gives the required acceleration.

Ques: 7(b)) Apply Gauss – Seidel iteration method to solve the following system of equations:

$$2x + y - 2z = 7$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

correct to three decimal places.

Solution:- Given equations are.

$$2x + y - 2z = 7$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Which can be re-written as–

$$x = \frac{1}{2}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

$$(x_0, y_0, z_0) = (0, 0, 0)$$

Ist iteration:

$$x_1 = \frac{1}{2}[17 - (0) + 2(0)] = \frac{1}{2} \times 17 = 8.5$$

$$y_1 = \frac{1}{20}[-18 - 3(8.5) + 0] = \frac{1}{20}[-43.5] = -2.175$$

$$z_1 = \frac{1}{20}[25 - 2(8.5) + 3(-2.175)] = \frac{1}{20}[1.475]$$

$$z_1 = 0.07375.$$

$$(x_1, y_1, z_1) = (8.5, -2.175, 0.07375).$$

or

$$(x_1, y_1, z_1) = (8.5, -2.175, 0.074).$$

2nd Approximation

$$x_2 = \frac{1}{2}[17-(-2.175)+2(0.0734)] = \frac{1}{2}[19.323] = 9.661$$

$$y_2 = \frac{1}{20}[-18-3(9.661)+(0.0734)] = \frac{1}{20}[-46.91] = -2.345$$

$$z_2 = \frac{1}{20}[25-2(9.661)+3(-2.345)] = \frac{1}{20}[-1.359] = -0.068$$

$$(x_2, y_2, z_2) = (9.661, -2.345, -0.068)$$

3rd Approximation

$$x_3 = \frac{1}{2}[17-(-2.345)+2(-0.068)] = \frac{1}{2}[19.345-0.136]$$
$$= 9.6045$$

$$y_3 = \frac{1}{20}[-18-3(9.6045)+(-0.068)] = \frac{1}{20}[-46.8135-0.068]$$
$$= -\frac{1}{20}[46.8815] = -2.344.$$

$$z_3 = \frac{1}{20}[25-2(9.6045)+3(-2.344)]$$
$$= \frac{1}{20}[25-19.209-7.032] = \frac{1}{20}[25-26.241]$$
$$z_3 = -0.062$$

$$(x_3, y_3, z_3) = (9.6045, -2.344, -0.062)$$

4th Iteration

$$x_4 = \frac{1}{2}[17-(-2.344)+2(-0.062)] = \frac{1}{2}[19.344-0.124]$$
$$x_4 = \frac{19.22}{2} = 9.609$$

$$y_4 = \frac{1}{20}[-18-3(9.609)-0.062] = -\frac{1}{20}[46.891]$$

$$\boxed{y_4 = -2.344}$$

$$z_4 = \frac{1}{20}\left[25 - 2\times 9.609 + 3\times(-2.344)\right] = \frac{1}{20}\left[-1.253\right]$$

$$z_4 = -0.063$$

5th Iteration

$$(x_4, y_4, z_4) = (9.609, -2.344, -0.063).$$

$$x_5 = \frac{1}{2}\left[17 - (-2.344) + 2(-0.063)\right] = 9.609$$

$$y_5 = \frac{1}{20}\left[-18 - 3(9.609) + (-0.063)\right] = -2.344.$$

$$z_5 = \frac{1}{20}\left[25 - 2\times 9.609 + 3(-2.344)\right] = -0.063.$$

$\therefore$ Solution by Gauss - Seidel method

$$\boxed{x = 9.609, \; y = -2.344, \; z = -0.063}$$

Required solution.

Ques: 8(a) Given the Boolean expression

$$X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$$

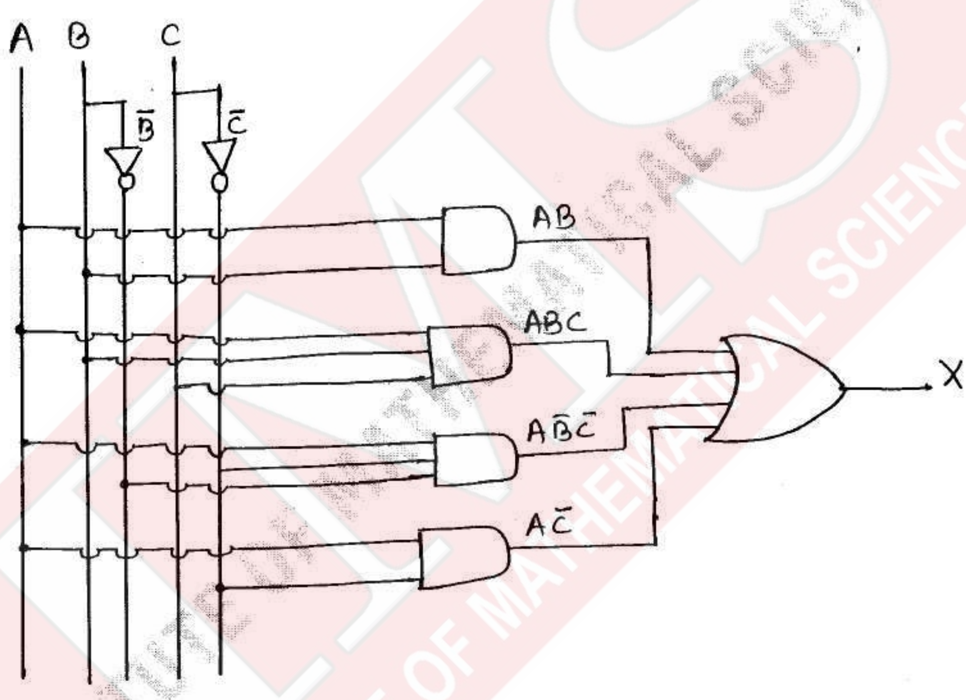
(i) Draw the logical diagram for the expression.

(ii) Minimize the expression

(iii) Draw the logical diagram for the reduced expression.

Solution:- Given $X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$

(i)



(ii) To minimize the expression.

$$X = AB + ABC + A\bar{B}\bar{C} + A\bar{C}$$

$$X = AB(1+C) + A\bar{C}(\bar{B}+1)$$

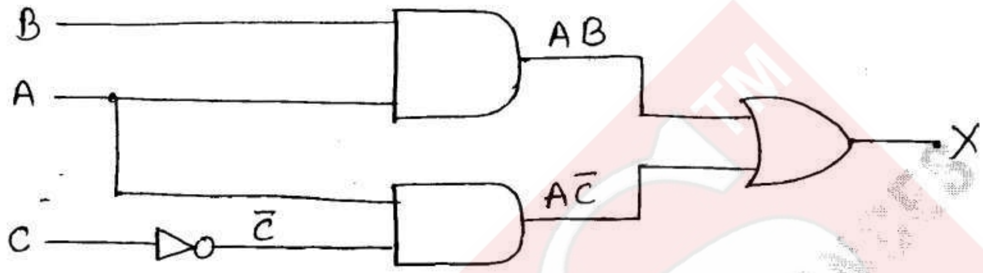
$$\boxed{X = AB + A\bar{C}} \qquad \left[\because \begin{array}{l} 1+C = 1 \\ \bar{B}+1 = 1 \end{array}\right]$$

or

$$\boxed{X = A(B+\bar{C})}$$

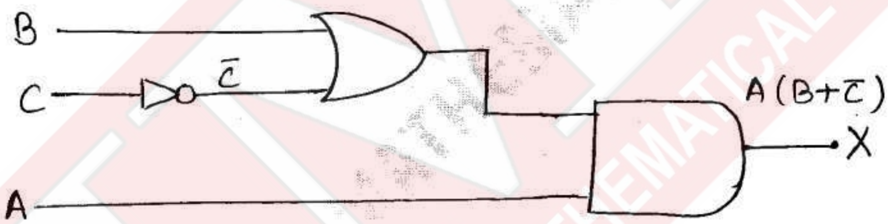
(iii) Logical diagram of reduced expression.

$$\boxed{X = AB + A\bar{C}}$$



(OR)

$$\boxed{X = A(B + \bar{C})}$$



8(b). A sphere of radius R, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, so that the pressure at the surface of the sphere at time t is $\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$.

Sol^n: Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin), and time t only. Let p be pressure at a distance r'. Let P be the pressure on the surface of the sphere of radius R and V be the velocity there. Then the equation of continuity is

$$r'^2 v' = R^2 V = F(t) \quad \text{—— (1)}$$

from (1) $$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \quad \text{—— (2)}$$

Again equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\Rightarrow \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}, \text{ using (2)}$$

Integrating w.r.t r', (3) reduces to

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + C,$$ C being an arbitrary constant.

when $r' = \infty$, then $v' = 0$ and $p = \Pi$ so that $C = \frac{\Pi}{\rho}$.

Then, we get

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho}$$

$$\Rightarrow p = \Pi + \frac{1}{2}\rho\left[2\frac{F'(t)}{r'} - v'^2\right]. \quad \text{—— (4)}$$

But $p = P$ and $v' = V$ when $r' = R$, Hence (4) gives

$$P = \Pi + \frac{1}{2}\rho\left[\frac{2}{R}\{F'(t)\}_{r'=R} - V^2\right] \quad \text{—— (5)}$$

Also $V = dR/dt$. Hence using (1), we have

$$\{F'(t)\}_{r'=R} = \frac{d}{dt}(R^2 V) = \frac{d}{dt}\left(R^2 \frac{dR}{dt}\right) = \frac{d}{dt}\left(\frac{R}{2}\cdot\frac{dR^2}{dt}\right)$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + \frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt}.$$

$$= \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

Using the above values of V and $\{F'(t)\}_{r'=R}$, (5) reduces to.

$$P = \Pi + \frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2\right\} - \left(\frac{dR}{dt}\right)^2\right]$$

$$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right].$$

8(C) Two sources, each of strength m are placed at the points $(-a,0)$, $(a,0)$ and a sink of strength $2m$ at origin. show that the streamlines are the curves $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$, where λ is a variable parameter.

show also that the fluid speed at any point is

$2ma^2/(r_1 r_2 r_3)$, where r_1, r_2 and r_3 are the distances of the points from the sources and the sink, respectively.

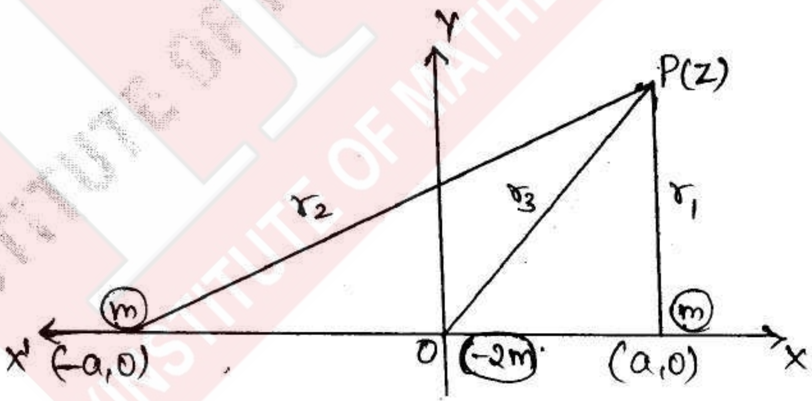
Sol'n: First Part:

The complex potential w at any point $P(z)$ given by

$$w = -m\log(z-a) - m\log(z+a) + 2m\log z \quad \text{—} ①$$

$$\Rightarrow w = m\left[\log z^2 - \log(z^2-a^2)\right]$$

$$\Rightarrow \phi + i\psi = m\left[\log(x^2-y^2+2ixy) - \log(x^2-y^2-a^2+2ixy)\right], \text{ as } z = x+iy$$



Equating the imaginary parts, we have

$$\psi = m\left[\tan^{-1}\{2xy/(x^2-y^2)\} - \tan^{-1}\{2xy/(x^2-y^2-a^2)\}\right]$$

$$\therefore \psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right], \text{ on simplification.}$$

The desired streamlines are given by $\psi =$ Constant $= m\tan^{-1}(-2/\lambda)$.

Then we obtain

$$(-2/\lambda) = (-2a^2xy)/[(x^2+y^2)^2 - a^2(x^2-y^2)]$$

$$\Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy).$$

Second Part: from (1), we have

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z}$$

$$= -\frac{2a^2m}{z(z-a)(z+a)}$$

$$\therefore \quad q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|}$$

$$= \frac{2a^2m}{r_1 r_2 r_3}$$

where $r_1 = |z-a|$, $r_2 = |z+a|$ and $r_3 = |z|$.