EXADEMY

ONLINE NATIONAL TESTS SOLUTIONS

Course: Ordinary Differential Equations

Mathematics Optional

1.

Example 1: Find the differential equation corresponding to the equation $y = ae^{x} + be^{2x} + ce^{-3x}$ where a, b, c are arbitrary constants.

Solution: Given: $y = ae^{x} + be^{2x} + ce^{-3x}$

$y' = ae^{x} + 2be^{2x} – 3\ ce^{-3x}$

$y'' = ae^{x} + 4be^{2x} + 9\ ce^{-3x}$

$y''' = ae^{x} + 8be^{2x} – 27\ ce^{-3x}$

$y' – y = be^{2x} – 4ce^{-3x}$——————- (1)

$y'' – y' = 2be^{2x} + 12ce^{-3x}$——————- (2)

$y''' – y'' = 4be^{2x} – 36ce^{-3x}$——————- (3)

Now,with the help of matrices and by using elimination technique for eliminating b and c we get,

$$\begin{bmatrix} y'-y & 1 & -4 \\ y"-y' & 2 & 12 \\ y"'-y" & 4 & -36 \end{bmatrix} = 0$$

This matrix can be simplified as,

$$\begin{bmatrix} y'-y & 1 & -1 \\ y"-y' & 2 & 3 \\ y"'-y" & 4 & -9 \end{bmatrix} = 0$$

Expanding we get

$(y'-y)(-18-12) -1 (-9(y''-y') – 3(y''' – y'')) – 1(4(y'' – y') – 2(y''' – y'')) = 0$

$-30(y'-y) + 5(y'' – y') + 5(y''' – y'') = 0$

$(y''' – y'') + (y'' – y') – 6(y' – y) = 0$

$y''' – 7y' + 6y = 0$

$d^3y/dx^3 – 7\ \ dy/dx + 6y = 0$

This is the required differential equation.

2.

Example 2: Find the differential equation of all the hyperbolas whose axes are along both the axes.

Solution: The standard equation of a hyperbola is

$x^2/a^2 - y^2/b^2 = 1$, whose transverse and conjugate axes are along the coordinate axes.

As there are two arbitrary constants to eliminate them we need to differentiate the standard equation of hyperbola twice.

Differentiating the above equation with respect to x, we get,

$2x/a^2 - 2yy'/b^2 = 0$ .....(1)

Differentiating w. r. t. x again, we get,

$2/a^2 - 2/b^2 ((y')^2 + yy'') = 0$ .... (2)

Equating the values of b^2/a^2 from (1) and (2)

$yy'/x = y'^2 + yy''$

Or, $y (dy/dx) = x(dy/dx)^2 + xy (d^2y/dx^2)$

This represents a differential equation of second order obtained by eliminating two parameters.

3.

Example 3: Find the differential equation of the family of circles of radius 5cm and their centres lying on the x-axis.

Solution: Let the centre of the circle on x-axis be (a,0).

The equation of such a circle can be given as:

$(x-a)^2 + y^2 = 5^2$ ......(1)

$(x-a)^2 + y^2 = 25$

Differentiating this equation with respect to x, we get,

$2(x-a) + 2y\, dy/dx = 0$

Taking 2 as common and eliminating,

$(x-a) = -y\,(dy/dx)$

Substituting the value of (x – a) in equation (1), we get

$y^2(dy/dx)^2 + y^2 = 25$

This is the required differential equation.

4.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. (

Solution. Given equation is $x\,(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2 \int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or $$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \,.\, \frac{x^2}{2}\, dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

5.

Q 2 ⇒ Obtain the Equation of the orthogonal Trajectory of the family of curves represented by $r^n = a\sin n\theta$, (r,θ) being the Polar co-ordinates.

Sol^n Given $r^n = a\sin n\theta$

$$n\log r = \log a + \log\sin n\theta$$

diff w.r.t θ

$$\frac{n}{r}\frac{dr}{d\theta} = n\cot n\theta.$$

for finding orthogonal trajectories replace.

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

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$$n\left(-\frac{r^2 d\theta}{dr}\right) = n\cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = -\frac{dr}{r}$$

$$\frac{\log(\sec n\theta)}{n} = -\log r + c$$

$\cancel{\log n\theta =}$

$$\log\sec n\theta = -n\log r + k$$

$$\boxed{r^n = k\cos n\theta}$$ $k \to$ Constant

6.

Example 11.7. *Solve* $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$. (V.T.U., 2005)

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad [\text{Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\, d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan\frac{1}{2}(x+y)\right] + c.$

7.

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Q Find the differential eqn representing all the circles in xy plane.

sol: Eqn of circle in xy-plane

$$x^2+y^2+2gx+2fy+C=0$$

On differentiating w.r.t. to x

$$2x+2yy'+2g+2fy'=0$$

$$\Rightarrow \quad yy'+fy'=-(x+g)$$

Again differentiating

$$(y')^2+yy''+fy''=-1$$

$$\Rightarrow \quad (y')^2+yy''+1=-fy''$$

$$\Rightarrow \quad \frac{(y')^2+yy''+1}{y''}=-f$$

Again differentiating

$$\frac{(2y'y''+y''y'+yy''')y''-y'''((y')^2+yy''+1)}{(y'')^2}=0$$

$$\Rightarrow \quad \frac{(3y'y''+yy''')y''-y'''((y')^2+yy''+1)}{(y'')^2}=0$$

$$\Rightarrow \quad 3y'(y'')^2\cancel{+yy''y'''}-y'''(y')^2\cancel{-yy''y'''}-y'''=0$$

$$\cancel{3y'(y'')^2+yy''y'''-y'''(y')^2}$$

$$\Rightarrow \quad 3y'(y'')^2=y'''(1+(y')^2)$$

$$y'''=\frac{3y'(y'')^2}{1+(y')^2}$$

$$\Rightarrow \quad \boxed{\frac{d^3y}{dx^3}=\frac{3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2}{1+(dy/dx)^2}}$$ is the required equation

8.

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y \text{ const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 . 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

9.

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x$.

$$\therefore \qquad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y \text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y \text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

10.

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y}(1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\therefore \quad \text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y}(x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \frac{\partial}{\partial y}(\log y) = \frac{\partial}{\partial x}\left(\frac{x - \log y}{y}\right)\right.$

$$\therefore \quad \text{its solution is } \int_{(y\ \text{const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

or $$\log y \int dx + \int \left(\frac{-\log y}{y}\right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2}(\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m'ydx + n'xdy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a + h + 1}{m} = \frac{b + k + 1}{n}, \frac{a' + h + 1}{m'} = \frac{b' + k + 1}{n'}.$$

11.

Example 11.36. *Solve* $y(xy + 2x^2y^3)\, dx + x\,(xy - x^2y^2)\, dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m'ydx + n'xdy) = 0,$$

we have $a = b = 1, m = n = 1; a' = b' = 2, m' = 2, n' = -1.$

$$\therefore \quad \text{I.F.} = x^h y^k.$$

where $$\frac{a + h + 1}{m} = \frac{b + k + 1}{n}, \frac{a' + h + 1}{m'} = \frac{b' + k + 1}{n'}$$

i.e. $$\frac{1 + h + 1}{1} = \frac{1 + k + 1}{1}, \frac{2 + h + 1}{2} = \frac{2 + k + 1}{-1}$$

or $$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2 y} + \frac{2}{x}\right) dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right) dx = 0, \text{ which is an exact equation.}$$

$$\therefore \quad \text{The solution is } \int_{(y\ \text{const})} Mdx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

or $$\frac{1}{y}\left(-\frac{1}{x}\right) + 2 \log x - \log y = c \quad \text{or} \quad 2 \log x - \log y - 1/xy = c.$$

12.

Example 11.15. *Solve* $(x + 1)\dfrac{dy}{dx} - y\, e^{3x}(x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad \text{...}(i)$$

Here $\quad P = -\dfrac{1}{x+1}$ and $\displaystyle\int P dx = -\int \frac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \dfrac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \displaystyle\int [e^{3x}(x + 1)](\text{I.F.})\, dx + c$

or $\quad \dfrac{y}{x+1} = \displaystyle\int e^{3x}\, dx + c = \frac{1}{3}e^{3x} + c \quad$ or $\quad y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1)$.

13.

Example 11.17. *Solve* $3x(1 - x^2)\, y^2 \dfrac{dy}{dx} + (2x^2 - 1)y^3 = ax^3$ *(Rajasthan, 2006)*

Solution. Putting $y^3 = z$ and $3y^2 \dfrac{dy}{dx} = \dfrac{dz}{dx}$, the given equation becomes

$$x(1 - x^2)\frac{dz}{dx} + (2x^2 - 1)z = ax^3, \quad \text{or} \quad \frac{dz}{dx} + \frac{2x^2 - 1}{x - x^3}z = \frac{ax^3}{x - x^3} \qquad \text{...}(i)$$

which is Leibnitz's equation in z

$\therefore \quad$ I.F. $= \exp\left(\displaystyle\int \frac{2x^2 - 1}{x - x^3}dx\right)$

Now $\displaystyle\int \frac{2x^2 - 1}{x - x^3}\, dx = \int \left(-\frac{1}{x} - \frac{1}{2}\frac{1}{1+x} + \frac{1}{2}\cdot\frac{1}{1-x}\right) dx = -\log x - \frac{1}{2}\log(1 + x) - \frac{1}{2}\log(1 - x)$

$$= -\log[x\sqrt{(1 - x^2)}]$$

$\therefore \quad$ I.F. $= e^{-\log[x\sqrt{(1-x^2)}]} = [x\sqrt{(1 - x^2)}]^{-1}$

Thus the solution of (i) is

$$z(\text{I.F.}) = \int \frac{ax^3}{x - x^3}(\text{I.F.})\, dx + c$$

or $\quad \dfrac{z}{[x\sqrt{(1 - x^2)}]} = a\displaystyle\int \frac{x^3}{x(1 - x^2)}\cdot\frac{1}{x\sqrt{(1 - x^2)}}\, dx + c = a\int x(1 - x^2)^{-3/2}\, dx$

$$= -\frac{a}{2}\int(-2x)(1 - x^2)^{-3/2}\, dx + c = a(1 - x^2)^{-1/2} + c$$

Hence the solution of the given equation is

$$y^3 = ax + cx\sqrt{(1 - x^2)}. \qquad [\because z = y^3]$$

14.

Example 11.20. *Solve* $r \sin\theta\, d\theta + (r^3 - 2r^2 \cos\theta + \cos\theta)\, dr = 0.$

Solution. Given equation can be rewritten as

$$\sin\theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2)\cos\theta = -r^2 \qquad ...(i$$

Put $\cos\theta = y$ so that $-\sin\theta\, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = -r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r) dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y\left(\dfrac{1}{r} e^{r^2}\right) = \int r^2 . e^{r^2} . \dfrac{1}{r}\, dr + c$

or $$y e^{r^2}/r = \frac{1}{2}\int e^{r^2} 2r\, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2 e^{r^2} \cos\theta = r e^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2\cos\theta.$$

15.

Example 11.43. *Solve* $(px - y)(py + x) = a^2 p.$ (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2x dx = du$ and $2y dy = dv$

$\therefore$ $$p = \frac{dy}{dx} = \frac{dv}{y} \Big/ \frac{du}{x} = \frac{x}{y} P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

[redacted]

Then the given equation becomes $\left(\dfrac{xP}{y} . x - y\right)\left(\dfrac{xP}{y} . y + x\right) = a^2 \dfrac{xP}{y}$

or $$(uP - v)(P + 1) = a^2 P \text{ or } uP - v = \frac{a^2 P}{P + 1}$$

or $$v = uP - a^2P/(P + 1), \text{ which is Clairaut's form.}$$

$\therefore$ its solution is $$v = uc - a^2c/(c + 1), \text{ i.e., } y^2 = cx^2 - a^2c/(c + 1).$$

16.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$$\sin^{-1} p = y - xp \text{ or } y = px + \sin^{-1} p \text{ which is the } \textit{Clairaut's equation.}$$

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1 - c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = N(x^2 - 1)/x$$

Now substituting this value of c in (*i*), we get

$$y = N(x^2 - 1) + \sin^{-1} \{N(x^2 - 1)/x\}$$

which is the desired singular solution.

17.

Example 11.37. *Solve* $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution. Given equation is $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$ where $p = \frac{dy}{dx}$ or $p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0$.

Factorising $(p + y/x)(p - x/y) = 0$.

Thus we have $\quad p + y/x = 0 \quad ...(i) \quad$ and $\quad p - x/y = 0 \quad ...(ii)$

From (*i*), $\quad \frac{dy}{dx} + \frac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \frac{dy}{dx} - \frac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

18.

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $$p + y \cot x = \pm y \operatorname{cosec} x$$

i.e., $$p = y(-\cot x + \operatorname{cosec} x) \quad \ldots(i)$$

or $$p = y(-\cot x - \operatorname{cosec} x) \quad \ldots(ii)$$

From (*i*), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $$y = \frac{c}{2 \cos x^2/2} \text{ or } y(1 + \cos x) = c \quad \ldots(iii)$$

From (*ii*), $\dfrac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $$y = \frac{c}{2 \sin^2 \dfrac{x}{2}} \quad \text{or} \quad y(1 - \cos x) = c \quad \ldots(iv)$$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y(1 \pm \cos x) = c.$$

19.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right) \cdot \frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\displaystyle\int \frac{dx}{x} + 2\int \frac{dp}{p} = \text{a constant}$

or $$\log x + 2 \log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \quad \text{or} \quad p = \sqrt{(c/x)} \quad \ldots(ii)$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

20.

Example 11.40. *Solve* $y = 2px + p^n$. (Bhopal, 2009)

Solution. Given equation is $\quad y = 2px + p^n$ ...(i)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or
$$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (ii) is

$$x(\text{I.F.}) = \int(-np^{n-2}) . (\text{I.F.})\, dp + c \quad \text{or} \quad xp^2 = -n\int p^n\, dp + c = -\frac{np^{n+1}}{n+1} + c$$

or
$$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (i), we get $y = \frac{2c}{p} + \frac{1-n}{1+n}p^n$ ...(iv)

The equations (iii) and (iv) taken together, with parameter p, constitute the general solution (i).

21.

Example 11.41. *Solve* $y = 2px + y^2p^3$. (Bhopal, 2008)

Solution. Given equation, on solving for x, takes the form $x = \frac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\frac{dx}{dy}\left(=\frac{1}{p}\right) = \frac{1}{2} . \frac{p\left(1 - 2y . p^3 - y^2 3p^2\frac{dp}{dy}\right) - (y - y^2p^3)\frac{dp}{dy}}{p^2}$

or
$$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or
$$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p\,(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or
$$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) - 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\quad py = c.$ ...(i)

Thus eliminating from the given equation and (i), we get $y = 2\frac{c}{y}x + \frac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

22.

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics* $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, *where* λ *is the parameter.* (V.T.U., 2009 S)

Solution. Differentiating the given equation, we get $\frac{2x}{a^2} + \frac{2y}{a^2 + \lambda}\frac{dy}{dx} = 0$

or
$$\frac{y}{a^2 + \lambda} = -\frac{x}{a^2 (dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2 + \lambda} = \frac{-xy}{a^2 (dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2 (dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \qquad \ldots(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\, dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int y\,dy = \int \frac{a^2 - x^2}{x}\,dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

or
$$x^2 + y^2 = 2a^2 \log x + c' \qquad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

23.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \quad \ldots(i)$$

Differentiating, $$y\frac{dy}{dx} = 2a \quad \ldots(ii)$$

Substituting the value of a from (ii) in (i), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$ as the differential equation of the family. ...(iii)

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (iii), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$ which is the same as (iii).

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.*, each member of the family (i) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(i) Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.

(ii) Replace in this differential equation,

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[$\because$ for the given curve through $P(r, \theta)$ $\tan\phi = rd\theta/dr$

and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

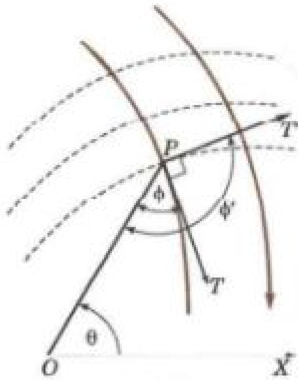

Fig. 12.8

(iii) Solve the differential equation of the orthogonal trajectories

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

24.

Example 12.7. *Find the orthogonal trajectory of the cardioids $r = a(1 - \cos\theta)$.* (*Kurukshetra, 2005*)

Solution. Differentiating $\quad r = a(1 - \cos\theta)$. ...(*i*)

with respect to θ, we get $\quad \dfrac{dr}{d\theta} = a \sin\theta$ ...(*ii*)

Eliminating a from (*i*) and (*ii*), we obtain

$\dfrac{dr}{d\theta} \cdot \dfrac{1}{r} = \dfrac{\sin\theta}{1 - \cos\theta} = \cot\dfrac{\theta}{2}$ which is the differential equation of the given family.

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\, d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)\,d\theta}{\cos\theta/2}$$

Integrating, $\quad \log r = 2 \log \cos\theta/2 + \log c$

or $\quad r = c\cos^2\theta/2 = \dfrac{1}{2}c(1 + \cos\theta) \quad$ or $\quad r = a'(1 + \cos\theta)$

which is the required orthogonal trajectory.

25.

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1 e^x + c_2 x e^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1 e^x + c_2 (e^x + xe^x)$$

$$\therefore \quad y - y_1 = c_2 e^x \quad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2 e^x \quad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx ($= m$) when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring

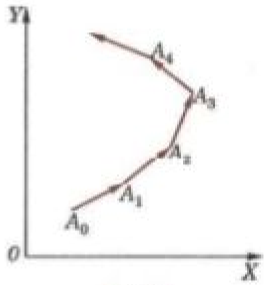

Fig. 11.1

point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point $A_0(x_0, y_0)$. Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.