Mains Test Series - 2020

Test-3 (Paper-I)

Answer key

ODE, VECTOR ANALYSIS AND DYNAMICS & STATICS

1(a) → solve $(D^4+D^2+1)y = ax^2+be^x \sin 2x$.

Sol^n: Given $(D^4+D^2+1)y = ax^2+be^x \sin 2x$ ——— (1)

Auxiliary Equation is $(D^4+D^2+1)=0$

$$\Rightarrow (D^2+1)^2 - D^2 = 0$$

$$\Rightarrow (D^2+D+1)(D^2-D+1)=0$$

So that $D^2+D+1=0$ (or) $D^2-D+1=0$

$$\Rightarrow D = \frac{-1\pm i\sqrt{3}}{2},\ \frac{1\pm i\sqrt{3}}{2}.$$

$$\therefore\ C.F = e^{-x/2}\left[c_1\cos\left(\frac{x\sqrt{3}}{2}\right)+c_2\sin\left(\frac{x\sqrt{3}}{2}\right)\right]+e^{x/2}\left[c_3\cos\left(\frac{x\sqrt{3}}{2}\right)+c_4\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

Now P.I corresponding to ax^2

$$= a\frac{1}{D^4+D^2+1}x^2 = a[1+(D^4+D^2)]^{-1}x^2$$

$$= a[1-D^2+\cdots]x^2 = a(x^2-2).$$

Next P.I corresponding to $be^{-x}\sin 2x$.

$$= b\frac{1}{D^4+D^2+1}e^{+x}\sin 2x = be^{+x}\frac{1}{(D+1)^4+(D+1)^2+1}\sin 2x$$

$$= be^{-x}\frac{1}{D^4+4D^3+6D^2+4D+1+D^2+2D+1+1}\sin 2x$$

$$= be^{+x}\frac{1}{(D^2)^2+4D(D^2)+7D^2+6D+3}\sin 2x$$

$$= be^{+x}\frac{1}{(-2^2)^2+4D(-2^2)+7(-2^2)+6D+3}\sin 2x.$$

$$= be^{+x}\frac{1}{10D+9}\sin 2x$$

$$= be^{+x}(10D-9)\cdot\frac{1}{100D^2-81}\sin 2x$$

$$= be^{+x}(10D-9)\frac{1}{100(-2^2)-81}\sin 2x$$

$$= be^{+x}(10D-9)\frac{1}{481}\sin 2x$$

$$= \frac{be^{+x}}{481}(20\cos 2x-9\sin 2x).$$

Required solution is

$$y = e^{-x/2}\left[c_1\cos(x\sqrt{3}/2)+c_2\sin(x\sqrt{3}/2)\right]+e^{x/2}\left[c_3\cos(x\sqrt{3}/2)+c_4\sin(x\sqrt{3}/2)\right]$$
$$+a(x^2-2)+(b/481)e^{x}(20\cos 2x-9\sin 2x).$$

1(b) (i) Prove $L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\frac{s^2+4}{s^2}$

(ii) Evaluate $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\}$.

Sol^n: (i) $L\{\sin^2 t\} = L\{\frac{1}{2}(1-\cos 2t)\}$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] = f(s), \text{ say } \quad \text{—— (1)}$$

$$\therefore L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty f(s)\,ds = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right]ds, \text{ using (1)}$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\lim_{s\to\infty}\log\frac{s^2}{s^2+4} - \frac{1}{4}\log\frac{s^2}{s^2+4}$$

$$= \frac{1}{4}\lim_{s\to\infty}\log\frac{1}{1+(4/s^2)} + \frac{1}{4}\log\frac{s^2+4}{s^2}$$

$$= 0 + \frac{1}{4}\log\left\{\frac{(s^2+4)}{s^2}\right\}$$

$$= \underline{\frac{1}{4}\log\left\{\frac{(s^2+4)}{s^2}\right\}}$$

(ii) $L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+1)-1](s+1)^3}\right\}$

$$= e^{-t}L^{-1}\left\{\frac{1}{(s-1)s^3}\right\} \quad \text{—— (1)}$$

[Using first shifting theorem]

Now, $L^{-1}\left\{\frac{1}{s-1}\right\} = e^t \Rightarrow L^{-1}\left\{\frac{1}{(s-1)s}\right\} = \int_0^t e^x\,dx = e^t - 1$.

$$\therefore L^{-1}\left\{\frac{1}{(s-1)s^2}\right\} = \int_0^t (e^x-1)\,dx = e^t - t - 1$$

$$\therefore L^{-1}\left\{\frac{1}{(s-1)s^3}\right\} = \int_0^t (e^x - x - 1)\,dx = e^t - \tfrac{1}{2}t^2 - t - 1$$

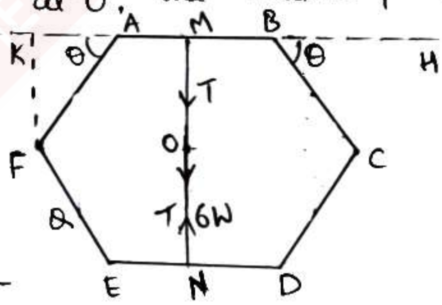
Putting this value in ①, we get

$$L^{-1}\left\{\frac{1}{s(s+1)^3}\right\} = e^{-t}\left(e^t - \tfrac{1}{2}t^2 - t - 1\right) = 1 - e^{-t}\left(1 + t + \tfrac{1}{2}t^2\right).$$

1(c) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; Prove that its tension is 3W.

Sol^n: ABCDEF is a hexagon formed of six equal rods each of weight W and say of length $2a$. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN. The total weight 6W of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let

$\angle FAK = \theta = \angle CBH$.



Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remains fixed, the length MN changes and the point O also changes.

we have $MN = 2MO = 2KF = 2AF\sin\theta = 4a\sin\theta$

Also the depth of O below the fixed line $AB = MO = 2a\sin\theta$.

By the principle of virtual work, we have

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

$$\Rightarrow -4aT\cos\theta\,\delta\theta + 12aW\cos\theta\,\delta\theta = 0$$

$$\Rightarrow 4a[-T+3W]\cos\theta\,\delta\theta = 0$$

$$\Rightarrow -T + 3W = 0 \quad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$\Rightarrow T = 3W$$

1(d) If $\frac{d^2\vec{A}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\hat{k}$, find $\vec{A}$ given that $\vec{A} = 2i + j$ and $\frac{dA}{dt} = -i - 3k$ at $t=0$.

Sol'n: Given that

$$\frac{d^2\vec{A}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\hat{k}$$

Integrating, we get

$$\frac{d\vec{A}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\hat{k} + b,$$ b is an arbitrary constant vector.

But it is given that when $t=0$,

$$\frac{dA}{dt} = -\hat{i} - 3\hat{k}$$

$$\therefore -i - 3k = -4k + b$$

$$\Rightarrow b = -i + k$$

$$\therefore \frac{d\vec{A}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\hat{k} - \hat{i} + \hat{k}$$

$$= (3t^2 - 1)\hat{i} - 8t^3\hat{j} + (-4\cos t + 1)\hat{k}$$

Integrating again, w.r.t t, we get

$$\vec{A} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + c.$$

when c is an arbitrary constant vector. But it is given that when $t=0$, $\vec{A} = 2\hat{i} + \hat{j}$

$$\therefore 2\hat{i} + \hat{j} = 0 + c = c$$

$$\therefore \vec{A} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + 2\hat{i} + \hat{j}$$

$$\Rightarrow \vec{A} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}$$

is the required solution of the given differential equation.

1(e) → Find the curvature and the torsion of the space curve $x = a(3u-u^3)$, $y = 3au^2$, $z = a(3u+u^3)$.

Sol^n: we know that

$$\vec{r} = (x, y, z)$$

i.e. $\vec{r} = (3au - au^3, 3au^2, 3au + au^3)$ —— ①

$$\frac{d\vec{r}}{du} = (3a - 3au^2, 6au, 3a + 3au^2) \text{ —— ②}$$

$$\left|\frac{d\vec{r}}{du}\right| = 3a\sqrt{2}\,(1+u^2) \text{ —— ③}$$

Also $\frac{d^2\vec{r}}{du^2} = (-6au, 6a, 6au)$ on diff ② w.r.t u —— ④

$$\frac{d^3\vec{r}}{du^3} = (-6a, 0, 6a) \text{ —— ⑤}$$

Now
$$\frac{d\vec{r}}{du} \times \frac{d^2 r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3a-3au^2 & 6au & 3a+3au^2 \\ -6au & 6a & 6au \end{vmatrix}$$

$$= \hat{i}\,(36a^2u^2 - 18a^2 - 18a^2u^2) - \hat{j}\,(18a^2u - 18a^2u^3 + 18a^2u + 18a^2u^3) + \hat{k}\,(18a^2 - 18a^2u^2 + 36a^2u^2)$$

$$= \hat{i}\,(18a^2u^2 - 18a^2) - \hat{j}\,(36a^2u) + \hat{k}\,(18a^2u^2 + 18a^2)$$

$$\therefore \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} = (18a^2u^2 - 18a^2, -36a^2u, 18a^2 + 18a^2u^2) \text{ —— ⑥}$$

$$\therefore \left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right| = 18a^2\sqrt{(u^2-1)^2 + (-2u)^2 + (1+u^2)^2} = 18a^2\sqrt{2}\,(u^2+1) \text{ —— ⑦}$$

Now
$$\left(\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right) \cdot \frac{d^3 r}{du^3} = (-\cancel{108a^3u^2} + 108a^3 + 0 + 108a^3 + \cancel{108a^3u^2})$$
$$= 216a^3 \text{ —— ⑧}$$

By using the formula,

$$K = \frac{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|}{\left|\frac{d\vec{r}}{du}\right|^3}$$

$$= \frac{18a^2\sqrt{2}(u^2+1)}{[3a\sqrt{2}(1+u^2)]^3} = \frac{18a^2\sqrt{2}(u^2+1)}{27a^3\, 2\sqrt{2}\,(u^2+1)^3}$$

$$\therefore \boxed{K = \frac{1}{3a(1+u^2)^2}} \quad \text{——— (A)}$$

Similarly, we obtain

$$\tau = \frac{\left[\frac{d\vec{r}}{du} \;\; \frac{d^2\vec{r}}{du^2} \;\; \frac{d^3\vec{r}}{du^3}\right]}{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|^2}$$

$$= \frac{216a^3}{[18a^2\sqrt{2}(u^2+1)]^2}$$

$$\therefore \boxed{\tau = \frac{1}{3a(1+u^2)}} \quad \text{——— (B)}$$

Hence, Proved.

2(a) Find the orthogonal trajectories of $r = a(1-\cos n\theta)$.

Sol'n: Given that $r = a(1-\cos n\theta)$, where a is parameter — (1)

Take logarithm of both sides

$$\log r = \log a + \log(1-\cos n\theta) \quad \text{— (2)}$$

Differentiating (2) w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{n\sin n\theta}{1-\cos n\theta} \quad \text{— (3)}$$

which is diff equation of the family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3), the diff. equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{n\sin n\theta}{1-\cos n\theta}$$

$$\Rightarrow -n\frac{dr}{r} = \frac{1-\cos n\theta}{\sin n\theta}$$

$$\Rightarrow -n\frac{dr}{r} = \frac{2\sin^2(n\theta/2)}{2\sin n\theta/2 \cos n\theta/2}$$

$$\Rightarrow -n\frac{dr}{r} = \tan n\theta/2.$$

Integrating.

$$\Rightarrow -n\log r = -\frac{2}{n}\log(\cos n\theta/2) + \frac{1}{n}\log c, \quad c \text{ being parameter}$$

$$\Rightarrow n^2\log r = \log\cos^2\frac{n\theta}{2} - \log c$$

$$\Rightarrow r^{n^2} = \frac{\cos^2 n\theta/2}{c}$$

$$\Rightarrow r^{n^2} = \frac{1}{2c}(1+\cos n\theta)$$

$$\Rightarrow r^{n^2} = b(1+\cos n\theta) \quad \text{taking } b = \frac{1}{2c}$$

which is the equation of required orthogonal trajectories with b as parameter

2(b) Solve $y = yp^2 + 2px$

Sol^n: Given $y = yp^2 + 2px$, where $p = dy/dx$ —— (1)

solving (1) for y, $y(1-p^2) = 2px$

$\Rightarrow y = 2px/(1-p^2)$ —— (2)

Differentiating (2) w.r.t 'x' and writing p for dy/dx, we get

$p = [(1-p^2)\{2p + 2x(dp/dx)\} - 2px(-2p)(dp/dx)]/(1-p^2)^2$

$\Rightarrow p(1-p^2)^2 = 2p(1-p^2) + 2x(1-p^2)(dp/dx) + 4p^2x(dp/dx)$

$\Rightarrow p(1-p^2)[(1-p^2)-2] - 2x(dp/dx)(1-p^2+2p^2) = 0$

$\Rightarrow p(p^2-1)(1+p^2) - 2x(dp/dx)(1+p^2) = 0$

$\Rightarrow (1+p^2)[p(p^2-1) - 2x(dp/dx)] = 0$ —— (3)

Neglecting the first factor which does not involve dp/dx, (3) reduces to

$$p(p^2-1) - 2x\frac{dp}{dx} = 0$$

$$\Rightarrow \left(\frac{-1}{p-1} + \frac{1}{p+1} - \frac{2}{p}\right)dp = \frac{dx}{x}$$

Integration $\log(p-1) + \log(p+1) - 2\log p = \log x + \log c$

$\Rightarrow [(p-1)(p+1)/p^2] = cx$

$\Rightarrow (p^2-1)/p^2 = cx$

$\Rightarrow p^2 - 1 = cxp^2$

$$\Rightarrow p = \left[\frac{1}{(1+cx)}\right]^{1/2}$$

putting this value of p in (1), the required solution is

$$y = \frac{y}{1+cx} + \frac{2x}{(1+cx)^{1/2}}$$

$$\Rightarrow y\left[1 - \frac{1}{1+cx}\right] = \frac{2x}{(1+cx)^{1/2}}$$

$$\Rightarrow \frac{cy}{(1+cx)^{1/2}} = 2$$

Squaring bothsides of above equation, $c^2y^2 = 4(1+cx)$

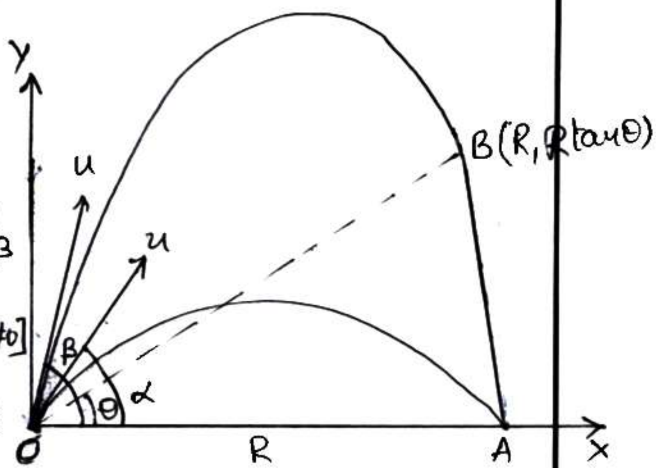
c being an arbitrary constant.

2(C) A shot fired at an elevation α is observed to strike the foot of a tower which raise above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is $\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha \cos\theta)\right]$.

Sol^n: Let AB be the tower and O the point of projection.
It is given that $\angle AOB = \theta$.
Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB, let $OA = R$.

Then $R = \dfrac{u^2 \sin 2\alpha}{g}$.

Referred to the horizontal & vertical lines OX & OY lying in the plane of motion as the coordinate axes, the coordinates of the top B of the tower are $(R, R\tan\theta)$.
If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is



$$y = x\tan\beta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\beta}$$

$$\therefore R\tan\theta = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{R}{u^2\cos^2\beta} \quad [\because R \neq 0]$$

Substituting the value of R from (1), we get

$$\tan\theta = \tan\beta - \frac{1}{2}g\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

$$\Rightarrow \tan\theta = \tan\beta - \frac{\sin 2\alpha}{2\cos^2\beta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{\sin 2\alpha}{2\cos^2\beta}$$

multiplying bothsides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta = 2\sin\beta\cos\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow (1+\cos 2\beta)\sin\theta = \sin 2\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin 2\beta\cos\theta - \cos 2\beta\sin\theta = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin(2\beta-\theta) = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow 2\beta - \theta = \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow 2\beta = \theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow \beta = \tfrac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)\right]$$

2(d) (i) Prove the identity

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$$

(ii) Derive the identity

$$\iiint_V (\phi\nabla^2\varphi - \varphi\nabla^2\phi)\,dv = \iint_S (\phi\nabla\varphi - \varphi\nabla\phi)\,\hat{n}\cdot ds$$

Sol'n: (i) we have

$$\text{grad}(\vec{A}\cdot\vec{B}) = \nabla(\vec{A}\cdot\vec{B}) = \sum i\frac{\partial}{\partial x}(\vec{A}\cdot\vec{B})$$

$$= \sum i\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right)$$

$$= \sum\left\{\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}\right\} + \sum\left\{\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}\right\} \quad \text{--- (1)}$$

Now we know that

$$a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$$

$$\therefore (a\cdot b)c = (a\cdot c)b - a\times(b\times c)$$

$$\therefore \left(A\cdot\frac{\partial B}{\partial x}\right)\hat{i} = (\vec{A}\cdot i)\frac{\partial B}{\partial x} - A\times\left(\frac{\partial B}{\partial x}\times i\right)$$

$$= (A\cdot i)\frac{\partial B}{\partial x} + A\times\left(i\times\frac{\partial B}{\partial x}\right)$$

Thus $\sum\left\{\left(A\cdot\frac{\partial B}{\partial x}\right)i\right\} = \sum\left\{(A\cdot i)\frac{\partial B}{\partial x}\right\} + \sum\left\{A\times\left(i\times\frac{\partial B}{\partial x}\right)\right\}$

$$= \left\{A\cdot\sum i\frac{\partial}{\partial x}\right\}B + A\times\sum\left(i\times\frac{\partial B}{\partial x}\right)$$

$$= (A\cdot\nabla)B + A\times(\nabla\times B) \quad \text{--- (2)}$$

Similarly $\sum\left\{\left(B\cdot\frac{\partial A}{\partial x}\right)i\right\} = (B\cdot\nabla)A + B\times(\nabla\times A) \quad \text{--- (3)}$

Putting the values from (2) & (3) in (1), we get

$$\text{grad}(\vec{A}\cdot\vec{B}) = (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) + (\vec{B}\cdot\nabla)\vec{A} + \vec{B}\times(\nabla\times\vec{A})$$

2(d)(ii) Sol'n: By divergence theorem, we have

$$\iiint_V \nabla\cdot F\, dV = \iint_S F\cdot n\, ds$$

Putting $F = \phi\nabla\psi$, we get

$$\nabla\cdot F = \nabla\cdot(\phi\nabla\psi)$$
$$= \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi)$$
$$= \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)$$

Also $F\cdot n = (\phi\nabla\psi)\cdot n$

$\therefore$ divergence theorem gives

$$\iiint_V (\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]dV = \iint_S (\phi\nabla\psi)\cdot n\, ds \quad \text{--- (1)}$$

This is called Green's first identity.

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]dV = \iint_S [\psi\nabla\phi]\cdot n\, ds$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot n\, ds \quad \text{--- (3)}$$

3(a) → Reduce the equation $x^2(\log x)^2(d^2y/dx^2) - 2x\log x(dy/dx) + [2+\log x - 2(\log x)^2]y = x^2(\log x)^3$.

Sol'n: Dividing by $x^2(\log x)^2$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2}{x\log x}\frac{dy}{dx} + \frac{2+\log x - 2(\log x)^2}{x^2(\log x)^2}y = \log x \quad \text{—— (1)}$$

comparing (1) with $y''+Py'+Qy=R$,

$$P = -\frac{2}{x\log x}, \quad Q = \frac{2}{x^2(\log x)^2} + \frac{1}{x^2\log x} - \frac{2}{x^2} \text{ and } R = \log x \quad \text{—— (2)}$$

we choose

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{\int \frac{(1/x)}{\log x}dx} = e^{\log(\log x)} = \log x \quad \text{—— (3)}$$

Let the required general solution be $y = uv$ —— (4).

Then v is given by the normal form

$$\frac{d^2v}{dx^2} + Iv = S \quad \text{—— (5)}$$

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$$= \frac{2}{x^2(\cos x)^2} + \frac{1}{x^2\cos x} - \frac{2}{x^2} - \frac{1}{x^2(\cos x)^2} + \left[-\frac{1}{x^2\cos x} - \frac{1}{x^2(\cos x)^2}\right]$$

$\Rightarrow I = -2/x^2$ and $S = R/u = (\log x)/(\log x) = 1$.

Then (5) reduce to $d^2v/dx^2 - (2/x^2)v = 1$

$$\Rightarrow (x^2D^2 - 2)v = x^2 \quad \text{—— (6)}$$

Let $x = e^z$ (or $z = \log x$), $D \equiv d/dx$ and $D = d/dz$

So that $x^2D^2 = D_1(D_1-1)$.

Then (6) reduces to $[D_1(D_1-1)-2]v = e^{2z}$

$$\Rightarrow (D_1^2 - D_1 - 2)v = e^{2z} \quad \text{—— (7)}$$

Its auxiliary equation is $D_1^2 - D_1 - 2 = 0$ so that $D_1 = 2, -1$

$\therefore$ C.F $= c_1 e^{2z} + c_2 e^{-z} = c_1 (e^z)^2 + c_2 (e^z)^{-1} = c_1 x^2 + c_2 x^{-1}$

and $\text{P.I} = \dfrac{1}{D_1^2 - D_1 - 2} e^{2z} = \dfrac{1}{(D_1-2)(D_1+1)} e^{2z}$

$$= \frac{1}{D_1-2} \frac{1}{2+1} e^{2z}$$

$$= \frac{1}{3} \frac{1}{(D_1-2)^1} e^{2z}$$

$$= \frac{1}{3} \frac{z}{1!} e^{2z} \quad \left[\because \frac{1}{(D_1-a)^n} e^{az} = \frac{z^n}{n!} e^{az}\right]$$

$$= \frac{1}{3} z (e^z)^2$$

$$= \frac{1}{3} x^2 \log x \quad \text{as } z = \log x \text{ \& } e^z = x$$

$\therefore$ solution of (7) is $v = \text{C.F} + \text{P.I}$

$$= c_1 x^2 + c_2 x^{-1} + \frac{1}{3} x^2 \log x \quad \text{---} \quad (8)$$

from (3), (4) and (8), the required general solution is

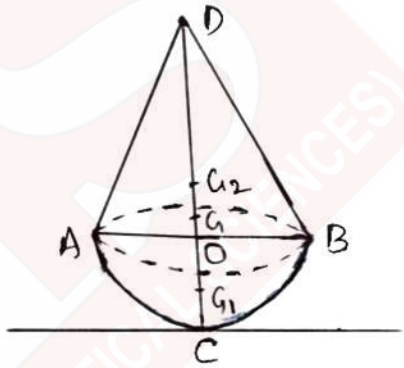
$y = uv$

$\Rightarrow y = \log x \left[c_1 x^2 + c_2 x^{-1} + \frac{1}{3} x^2 \log x\right]$

3(b) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.

Sol'n: AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C.

Let H be the height OD of the cone and r be the radius OA (or) OC of the hemisphere. Let G_1 and G_2 be the Centres of Gravity of the hemisphere and the cone respectively. Then



$OG_1 = 3r/8$ and $OG_2 = H/4$

If h be the height of the Centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact C, then using the formula $x = \dfrac{w_1x_1 + w_2x_2}{w_1+w_2}$,

we have $$h = \frac{\frac{1}{3}\pi r^2 H \cdot CG_2 + \frac{2}{3}\pi r^3 \cdot CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{\frac{1}{3}\pi r^2 H(r+\frac{1}{4}H) + \frac{2}{3}\pi r^3 \cdot \frac{5}{8}r}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{H(r+\frac{1}{4}H) + \frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of Curvature at the point of contact C of the upper body which is spherical $= r$.

and ρ_2 = the radius of Curvature of the lower body at the point of contact $= \infty$

$\therefore$ the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$\Rightarrow \frac{1}{h} > \frac{1}{r} + \frac{1}{\infty}$$

$$\Rightarrow \frac{1}{h} > \frac{1}{r}$$

$$\Rightarrow h < r$$

$$\Rightarrow \frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r} < r$$

$$\Rightarrow Hr + \frac{1}{4}H^2 + \frac{5}{4}r^2 < Hr + 2r^2$$

$$\Rightarrow \frac{1}{4}H^2 < \frac{3}{4}r^2.$$

$$\Rightarrow H^2 < 3r^2$$

$$\Rightarrow H < r\sqrt{3}$$

Hence the greatest height of the Cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

3(C) (i) If f and g are irrotational then show that $f\times g$ is a solenoidal vector.

(ii) If $f = (\vec{a}\times\vec{r})r^n$,

show that $\text{div} f = 0$, $\text{curl} f = (n+2)r^n\vec{a} - nr^{n-2}(\vec{a}\cdot\vec{r})\vec{r}$.

Sol^n: (i) If f and g are irrotational, then

$$\text{curl} f = 0, \quad \text{curl}\, g = 0$$

Now $\text{div}(f\times g) = g\cdot \text{curl} f - f\cdot \text{curl}\, g$

$$= g\cdot 0 - f\cdot 0$$
$$= 0$$

Since $\text{div}(f\times g)$ is zero, therefore $f\times g$ is solenoidal.

(ii) We know that $\text{div}(\phi A) = \phi\, \text{div}\vec{A} + \vec{A}\cdot \text{grad}\,\phi$

$\therefore \text{div}\{r^n(\vec{a}\times\vec{r})\} = r^n \text{div}(\vec{a}\times\vec{r}) + (\vec{a}\times\vec{r})\cdot \text{grad}\, r^n$

$$= r^n \text{div}(\vec{a}\times\vec{r}) + (\vec{a}\times\vec{r})\cdot(nr^{n-1}\,\text{grad}\, r)$$

$$= r^n[\vec{r}\cdot \text{curl}\,\vec{a} - \vec{a}\cdot \text{curl}\,\vec{r}] + (\vec{a}\times\vec{r})\cdot\left(nr^{n-1}\frac{1}{r}\vec{r}\right)$$

$$= r^n(\vec{r}\cdot 0 - \vec{a}\cdot 0) + nr^{n-2}(\vec{a}\times\vec{r})\cdot\vec{r}$$

[$\because$ curl of constant vector is zero and $\text{curl}\,\vec{r} = 0$]

$$= nr^{n-2}[\vec{a}, \vec{r}, \vec{r}]$$

$= 0$ ($\because$ scalar triple product having two equal vectors is zero)

We know that $\text{curl}(\phi\vec{A}) = (\text{grad}\,\phi)\vec{A} + \phi\, \text{curl}\,\vec{A}$

Putting $\phi = r^n$ and $\vec{A} = \vec{a}\times\vec{r}$

$\therefore \text{curl}[r^n(\vec{a}\times\vec{r})] = \nabla r^n \times (\vec{a}\times\vec{r}) + r^n\,\text{curl}(\vec{a}\times\vec{r})$ ——— (1)

Now $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\left(\frac{1}{r}\right)\vec{r} = nr^{n-2}\vec{r}$

$$\therefore \nabla r^n \times (\vec{a}\times\vec{r}) = (n r^{n-2}\vec{r}) \times (\vec{a}\times\vec{r})$$
$$= n r^{n-2}\{\vec{r}\times(\vec{a}\times\vec{r})\}$$
$$= n r^{n-2}\left[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\right]$$
$$= n r^{n-2}\left[r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\right]$$
$$= n r^n\vec{a} - n r^{n-2}(\vec{r}\cdot\vec{a})\vec{r} \quad \text{—— (2)}$$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

where the scalars a_1, a_2, a_3 are all constants.

Then $\vec{a}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= i(a_2 z - a_3 y) + j(a_3 x - a_1 z) + k(a_1 y - a_2 x)$$

$$\therefore \text{Curl}(\vec{a}\times\vec{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= (a_1 + a_1)\hat{i} + \hat{j}(a_2 + a_2) + (a_3 + a_3)\hat{k}$$

$$= 2(a_1 i + a_2 j + a_3 k) = 2\vec{a} \quad \text{—— (3)}$$

Substituting (2) & (3) in (1), we get

$$\text{Curl}\left[r^n(\vec{a}\times\vec{r})\right] = n r^n\vec{a} - n r^{n-2}(\vec{r}\cdot\vec{a})\vec{r} + r^n(2\vec{a})$$

$$= (n+2) r^n - n r^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

4(a) Solve the equation $d^2y/dx^2 + (2\cos x + \tan x)\frac{dy}{dx} + y\cos^2 x = \cos^4 x$.

Sol'n: Comparing the given equation with

$$y'' + Py' + Qy = R, \text{ we have}$$

$$P = 2\cos x + \tan x, \quad Q = \cos^2 x \text{ and } R = \cos^4 x \quad \text{—— (1)}$$

Choose z such that $(dz/dx)^2 = \cos^2 x$

$$\Rightarrow \frac{dz}{dx} = \cos x \quad \text{—— (2)}$$

from (2), $dz = \cos x\, dx$

$$\Rightarrow z = \sin x \quad \text{—— (3)}$$

With this value of z, the given equation transforms to

$$d^2y/dz^2 + P_1(dy/dz) + Q_1 y = R_1 \quad \text{—— (4)}$$

where $P_1 = \dfrac{d^2z/dx^2 + P(dz/dx)}{(dz/dx)^2} = \dfrac{-\sin x + (2\cos x + \tan x)\cos x}{\cos^2 x} = 2$

$Q_1 = \dfrac{Q}{(dz/dx)^2} = 1$ and $R_1 = \dfrac{R}{(dz/dx)^2} = \dfrac{\cos^4 x}{\cos^2 x} = \cos^2 x$

$= 1 - \sin^2 x$

$= 1 - z^2$

Hence (4) yields $\dfrac{d^2y}{dz^2} + 2\left(\dfrac{dy}{dz}\right) + y = 1 - z^2$

$$\Rightarrow (D_1^2 + 2D_1 + 1)y = 1 - z^2$$

$$\Rightarrow (D_1 + 1)^2 y = 1 - z^2 \quad \text{where } D_1 \equiv d/dz \quad \text{—— (5)}$$

The auxiliary equation of (5) is $(D_1 + 1)^2 = 0$

$$\Rightarrow D_1 = -1, -1$$

$\therefore$ C.F $= (C_1 + C_2 z)e^{-z}$, C_1 & C_2 being arbitrary constants

and P.I $= \dfrac{1}{(D_1+1)^2}(1-z^2) = (1+D_1)^{-2}(1-z^2)$

$= (1 - 2D_1 + 3D_1^2 + \cdots)(1-z^2)$

$= 1 - z^2 - 2D_1(1-z^2) + 3D_1^2(1-z^2) + \cdots = 1 - z^2 - 2(-2z) + 3(-2)$

$= -z^2 + 4z - 5$

Hence the required solution is $y = \text{C.F} + \text{P.I}$

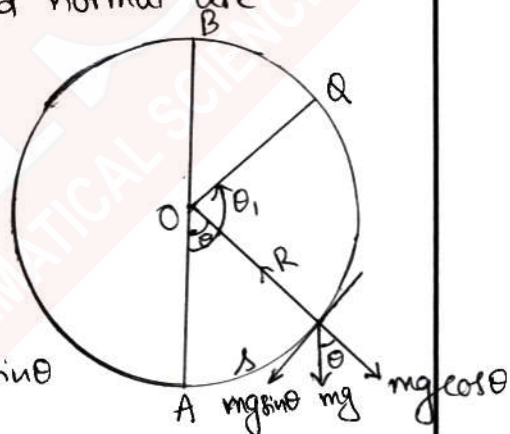
$$\Rightarrow y = (C_1 + C_2 z)e^{-z} - z^2 + 4z - 5$$

$$\Rightarrow y = (C_1 + C_2 \sin x)e^{-\sin x} - \sin^2 x + 4\sin x - 5, \text{ as } z = \sin x.$$

4(b) → A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{(a/g)}\log(\sqrt{5}+\sqrt{6})$.

Sol'n: Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$\& \quad m\frac{v^2}{a} = R - mg\cos\theta$$

Also $s = a\theta$

from ① & ③, we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying bothsides by $2a\,(d\theta/dt)$ & integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But according to the question $v=0$ at the highest point B, where $\theta = \pi$.

$$\therefore 0 = 2ag\cos\pi + A \Rightarrow A = 2ag$$

$$\therefore v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag$$

from ② and ④, we have

$$R = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(2ag + 3ag\cos\theta) \text{ — ⑤}$$

If the reaction $R=0$ at the point Q where $\theta = \theta_1$, then from ⑤, we have

$$0 = \frac{m}{a}(2ag + 3ag\cos\theta_1)$$

$$\Rightarrow \cos\theta_1 = -2/3 \quad \text{——— (6)}$$

from (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta + 1) = 2ag \cdot 2\cos^2\tfrac{1}{2}\theta$$

$$= 4ag\cos^2\tfrac{1}{2}\theta$$

$\therefore \frac{d\theta}{dt} = 2\sqrt{g/a}\cos\frac{1}{2}\theta$, the +ve sign being taken before the radical sign because θ increases as t increases

$$\Rightarrow dt = \tfrac{1}{2}\sqrt{a/g}\sec\tfrac{1}{2}\theta\, d\theta.$$

Integrating from $\theta = 0$ to $\theta = \theta_1$, the required time t is given by $t = \frac{1}{2}\sqrt{a/g}\int_{\theta=0}^{\theta_1}\sec\frac{1}{2}\theta\, d\theta$

$$\Rightarrow t = \sqrt{a/g}\left[\log(\sec\tfrac{1}{2}\theta + \tan\tfrac{1}{2}\theta)\right]_0^{\theta_1}$$

$$\Rightarrow t = \sqrt{a/g}\log(\sec\tfrac{1}{2}\theta_1 + \tan\tfrac{1}{2}\theta_1) \text{ ——— (7)}$$

from (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1 - 1 = -2/3$$

$$\Rightarrow 2\cos^2\tfrac{1}{2}\theta_1 = 1 - 2/3 = 1/3$$

$$\Rightarrow \cos^2\tfrac{1}{2}\theta_1 = 1/6$$

$$\Rightarrow \sec^2\tfrac{1}{2}\theta_1 = 6.$$

$$\therefore \sec\tfrac{1}{2}\theta_1 = \sqrt{6}$$

and $\tan\frac{1}{2}\theta_1 = \sqrt{(\sec^2\frac{1}{2}\theta_1 - 1)} = \sqrt{6-1} = \sqrt{5}$

Substituting in (7), the required time is given by

$$t = \sqrt{(a/g)}\log(\sqrt{6} + \sqrt{5})$$

4(c) Show that $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential for $\vec{F}$ and the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

Sol'n: We have $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$

$$\nabla \times \vec{F} = \hat{i}(0-0) - \hat{j}(3z^2-3z^2) + \hat{k}(2x-2x)$$

$$\therefore \nabla \times \vec{F} = 0$$

$\Rightarrow \vec{F}$ is a conservative force field scalar potential for $\vec{F}$ is given by

$$\vec{F} = \nabla\phi$$

$$\Rightarrow (2xy+z^3)i + x^2 j + 3xz^2 k = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 2xy + z^3$$

$$\Rightarrow \phi = yx^2 + xz^3 + f_1(y,z) \quad \text{——} (1)$$

$$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2 y + f_2(x,z) \quad \text{——} (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f_3(x,y) \quad \text{——} (3)$$

from (1), (2) & (3)

we get $\phi = x^2 y + xz^3$ is the scalar potential for $\vec{F}$.

$$\text{Work done} = \int_{(1,-2,1)}^{(3,1,4)} F \cdot dr$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi = \left[\phi\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= 202$$

4(d) Verify Stoke's theorem for the vector $F = z\hat{i} + x\hat{j} + y\hat{k}$ taken over the half of the sphere $x^2+y^2+z^2=a^2$ lying above the xy-plane.

Sol^n: Here let S be surface of the sphere $x^2+y^2+z^2=a^2$ lying above the xy-plane and let the curve C be the boundary of this surface. Obviously the curve C is a circle in the xy-plane of radius 'a' and centre origin and its equations are $x^2+y^2=a^2,\ z=0$.

Suppose $x = a\cos t,\ y = a\sin t,\ z=0,\ 0 \le t \le 2\pi$ are parametric equations of C.

By Stoke's theorem, we have

$$\oint_C F\cdot d\vec{r} = \iint_S \text{curl}\, F\cdot \hat{n}\, ds$$

Now

$$\oint_C F\cdot d\vec{r} = \oint_C (z\hat{i}+x\hat{j}+y\hat{k})\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint (z\,dx + x\,dy + y\,dz)$$

$$= \oint_C x\,dy \quad (\because \text{ on } C,\ z=0 \text{ and } dz=0)$$

$$= \int_0^{2\pi} a\cos t\ \frac{dy}{dt}\,dt$$

$$= \int_0^{2\pi} a\cos t\cdot a\cos t\, dt = a^2\int_0^{2\pi}\cos^2 t\, dt$$

$$= a^2\int_0^{2\pi}\frac{1+\cos 2t}{2}dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \pi a^2 \quad \text{—— (1)}$$

Now let us find $\iint_S \text{curl}\, F\cdot\hat{n}\, ds$

we have $\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = i+j+k$

If $\hat{n}$ is a unit vector along outward drawn normal at any point (x,y,z) on the surface S. i.e., the surface $\phi(x,y,z) = x^2+y^2+z^2-a^2$.

then $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$

$\therefore \iint_S \text{curl } F\cdot\hat{n}\,ds = \iint_S (\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)ds$

$= \frac{1}{a}\iint_S (x+y+z)\,ds$

Taking the polar spherical coordinates (r,θ,ϕ). we have $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$. Here $r = a$,

$\therefore x = a\sin\theta\cos\phi,\quad y = a\sin\theta\sin\phi,\quad z = a\cos\theta.$

Also ds = an elementary area on the surface of the sphere at the point (a,θ,ϕ).

$= a\,d\theta\; a\sin\theta\, d\phi = a^2\sin\theta\, d\theta\, d\phi.$

$\therefore \iint_S \text{curl } F\cdot\hat{n}\,dS = \frac{1}{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(a\sin\theta\cos\phi + a\sin\theta\sin\phi + a\cos\theta)\,a^2\sin\theta\, d\theta\, d\phi$

$= a^2\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(\sin^2\theta\cos\phi + \sin^2\theta\sin\phi + \cos\theta\sin\theta)\,d\theta\, d\phi$

$= a^2\int_{\theta=0}^{\pi/2}\left(\sin^2\theta\sin\phi - \sin^2\theta\cos\phi + \phi\cos\theta\sin\theta\right)_0^{2\pi} d\theta$

$= \int_{\theta=0}^{\pi/2} 2\pi\cos\theta\sin\theta\, d\theta = \pi a^2\int_0^{\pi/2}\sin 2\theta\, d\theta$

$= \pi a^2\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$

$= -\frac{\pi a^2}{2}[\cos\pi - \cos 0]$

$= -\frac{\pi a^2}{2}(-2) = \pi a^2$ ——— (2)

$\therefore$ from (1) & (2) $\int_C F\cdot d\vec{r} = \iint_S \text{curl } F\cdot\hat{n}\,ds$

5(a) → solve $dy/dx + (x-y-2)/(x-2y-3) = 0$.

Sol'n: Given equation is $dy/dx = -(x-y-2)/(x-2y-3)$

Take $x = X+h$, $y = Y+k$ so that $\frac{dy}{dx} = \frac{dY}{dX}$ —— ①

The given equation becomes

$$\frac{dY}{dX} = -\frac{X-Y+h-k-2}{X-2Y+h-2k-3} \quad \text{—— ②}$$

choose h, k so that $h-k-2=0$ and $h-2k-3=0$ —— ③

solving ③, we get $h=1$, $k=-1$ so that from ①, we have

$X = x-1$ & $Y = y+1$ —— ④

and ② becomes $\frac{dY}{dX} = -\frac{X-Y}{X-2Y} = -\frac{1-\left(\frac{Y}{X}\right)}{1-2\left(\frac{Y}{X}\right)}$ —— ⑤

Take $Y/X = v$ i.e., $Y = vX$. so that $\frac{dY}{dX} = v + X\frac{dv}{dX}$ —— ⑥

from ⑤ & ⑥, $v + X\frac{dv}{dX} = -\frac{1-v}{1-2v} \Rightarrow X\frac{dv}{dX} = \frac{1-2v^2}{2v-1}dv$

$$\Rightarrow \frac{dX}{X} = \frac{2v-1}{1-2v^2}dv$$

$$\Rightarrow \frac{dX}{X} = \left[-\frac{1}{2}\frac{(-4v)}{1-2v^2} - \frac{1}{1-\left(\frac{v}{\sqrt{2}}\right)^2}\right]dv$$

Integrating, $\log X = -\frac{1}{2}\log(1-2v^2) - \frac{1}{2\sqrt{2}}\log\frac{1+v\sqrt{2}}{1-v\sqrt{2}} - \frac{1}{2}\log c$

$\Rightarrow 2\log X + \log(1-2v^2) + \log c = -\frac{1}{\sqrt{2}}\log\left[\frac{1+v\sqrt{2}}{1-v\sqrt{2}}\right]$

$\Rightarrow \log\{X^2(1-2v^2)\} = \log\left[\frac{1-v\sqrt{2}}{1+v\sqrt{2}}\right]^{1/\sqrt{2}}$

$\Rightarrow X^2\left(1-2\frac{Y^2}{X^2}\right) = \left[\frac{1-\left(\frac{Y}{X}\right)\sqrt{2}}{1+\left(\frac{Y}{X}\right)\sqrt{2}}\right]^{1/\sqrt{2}} \Rightarrow c(X^2-2Y^2) = \left[\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right]^{1/\sqrt{2}}$

$\Rightarrow c\{(x-1)^2 - 2(y+1)^2\} = \left[\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}\right]^{1/\sqrt{2}}$

$\Rightarrow c(x^2-2y^2-2x-4y-1) = \left\{\frac{x-y\sqrt{2}-\sqrt{2}-1}{x+y\sqrt{2}+\sqrt{2}-1}\right\}^{1/\sqrt{2}}$, c being arbitrary Constant.

5(b) (i) Prove that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\left(\frac{2}{3}\right)$

(ii) If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$.

Sol'n: (i) Here $L\{\cos 6t - \cos 4t\} = L\{\cos 6t\} - L\{\cos 4t\}$

$$= \frac{s}{(s^2+6^2)} - \frac{s}{(s^2+4^2)} = f(s) \text{ say}$$

$$\therefore L\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$= \int_s^\infty \left[\frac{s}{s^2+36} - \frac{s}{s^2+16}\right] ds$$

$$= \left[\frac{1}{2}\log(s^2+36) - \frac{1}{2}\log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\log\left[\frac{s^2+36}{s^2+16}\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+36}{s^2+16} - \frac{1}{2}\log\frac{s^2+36}{s^2+16}$$

$$= \frac{1}{2}\lim_{s\to\infty}\frac{1+\left(\frac{36}{s^2}\right)}{1+\left(\frac{16}{s^2}\right)} + \frac{1}{2}\log\frac{s^2+16}{s^2+36}$$

$$\Rightarrow \int_0^\infty e^{-st}\frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\frac{s^2+16}{s^2+36} \quad \text{—— (1)}$$

Taking limit of bothsides of (1) as $s\to 0$, we get

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\frac{16}{36} = \frac{1}{2}\log\left(\frac{2}{3}\right)^2$$

$$= \log\frac{2}{3}.$$

(iii) $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{s}{(s^2+1)^2}\right\}$

$= \int_0^t \frac{1}{2} x \sin x \, dx$ (using the given result)

$= \frac{1}{2}\left\{\left[x(-\cos x)\right]_0^t - \int_0^t 1\cdot(-\cos x)\,dx\right\}$

$= \frac{1}{2}(-t\cos t + \sin t)$

5(C) → A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $W/2$, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is lW/c.

Sol'n: Let a rod AB of length $2a$ (say) be movable in a vertical plane about a smooth hinge at the end A.
A weight $W/2$ is attached at the other end B of the rod and this end is fastened by a string BC of length l to a point C at a height $AC = c$ vertically over the hinge at A. The rod is in equilibrium under the action of the following forces:

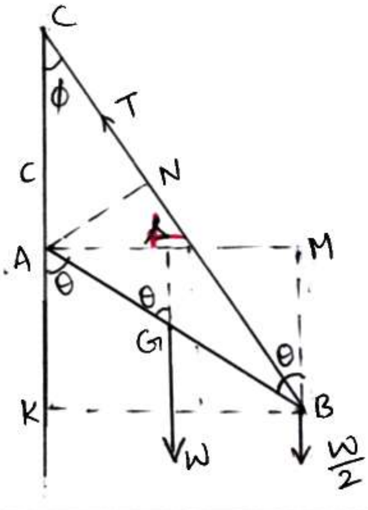
(i) W, weight of the rod at its mid point G, acting vertically downwards.

(ii) $W/2$, weight attached at the end B, acting vertically downwards,

(iii) T, tension in the string along BC, and

(iv) the reaction at the hinge at A.

Let θ & ϕ be the angles of inclination of the rod & the string respectively to the vertical.
To avoid reaction at A, taking moments about the point A, we have $T.AN = W\,AL + \frac{1}{2}W\,AM$



$\Rightarrow T.AC\sin\phi = W.AG\sin\theta + \frac{1}{2}W.AB\sin\theta$

$\Rightarrow T.c\sin\phi = W.a\sin\theta + \frac{1}{2}W.2a\sin\theta$ $[\because AB = 2a]$

$\Rightarrow T = W\dfrac{2a\sin\theta}{c\sin\phi}$ —— (1)

Now from the ΔCBK, $BK = BC\sin\phi = l\sin\phi$
& from the ΔABK, $BK = AB\sin\theta = 2a\sin\theta$

$\therefore l\sin\phi = 2a\sin\theta$ —— (2)

$\therefore$ from (1) & (2), we get $\boxed{T = Wl/c}$

5(d) A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is $2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$.

sol'n: Let the equation of the S.H.M with centre O as origin be $d^2x/dt^2=-\mu x$. Then the time period

$$T=2\pi/\sqrt{\mu}.$$

If a be the amplitude of the motion, we have

$$v^2=\mu(a^2-x^2), \quad \text{①}$$

where v is the velocity at a distance x from the centre.

But when $x=x_1$, $v=v_1$ and when $x=x_2$, $v=v_2$

Therefore from ①, we have

$$v_1^2=\mu(a^2-x_1^2) \text{ and } v_2^2=\mu(a^2-x_2^2).$$

These give $v_2^2-v_1^2=\mu\{(a^2-x_2^2)-(a^2-x_1^2)\}$

$$=\mu(x_1^2-x_2^2)$$

i.e, $\mu=\dfrac{(v_2^2-v_1^2)}{(x_1^2-x_2^2)}$.

Hence the time period $T=\dfrac{2\pi}{\sqrt{\mu}}$

$$=2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}.$$

5(e) Find the workdone in moving the particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, $z=0$ under the field of force given by $\vec{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$.

Soln: Given that $z=0$.

$\therefore F=(2x-y)\hat{i}+(x+y)\hat{j}+(3x-2y)\hat{k}$

The ellipse 'C' is given by $\frac{x^2}{25}+\frac{y^2}{16}=1$

where $a=5, b=4$.

put $x=5\cos t$, $y=4\sin t$

$\therefore \vec{r}=x\hat{i}+y\hat{j}=5\cos t\,\hat{i}+4\sin t\,\hat{j}$

$\therefore \frac{d\vec{r}}{dt}=-5\sin t\,\hat{i}+4\cos t\,\hat{j}$

and $\vec{F}=(10\cos t-4\sin t)\hat{i}+(5\cos t+4\sin t)\hat{j}+(15\cos t-8\sin t)\hat{k}$.

In moving round the ellipse once t will vary from 0 to 2π.

The required workdone is $=\int_C F\cdot dr=\int_0^{2\pi} F\cdot\frac{dr}{dt}\,dt$

$=\int_0^{2\pi}\left[(10\cos t-4\sin t)(-5\sin t)+(5\cos t+4\sin t)(4\cos t)\right]dt$

$=\int_0^{2\pi}\left[-34\cos t\sin t+20\sin^2 t+20\cos^2 t\right]dt$

$=\int_0^{2\pi}\left[-17\sin 2t+20(1)\right]dt$

$=\left[-17\frac{\cos 2t}{2}+20t\right]_0^{2\pi}$

$=20(2\pi)=40\pi$.

6(a) Justify that a differential equation of the form: $[y+xf(x^2+y^2)]dx+[yf(x^2+y^2)-x]dy=0$ where $f(x^2+y^2)$ is an arbitrary function of (x^2+y^2), is not an exact differential equation for $f(x^2+y^2)=(x^2+y^2)^2$.

Sol^n: Given that

$$[y+xf(x^2+y^2)]dx+[yf(x^2+y^2)-x]dy=0 \quad \text{——} ①$$

which is in the form of $Mdx+Ndy=0$

where $M=y+xf(x^2+y^2)$, $N=yf(x^2+y^2)-x$.

$$\frac{\partial M}{\partial y}=1+f'(x^2+y^2).2yx \quad \& \quad \frac{\partial N}{\partial x}=yf'(x^2+y^2)2x-1$$

$$=1+2xyf'(x^2+y^2) \qquad\qquad = -1+2yxf'(x^2+y^2)$$

Clearly $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

$\therefore$ Given equation is not an exact differential equation.

To show $\frac{1}{x^2+y^2}$ is an integrating factor:

Multiplying equation ① by $\frac{1}{x^2+y^2}$.

Then

$$M=\frac{y+xf(x^2+y^2)}{x^2+y^2} \quad \& \quad N=\frac{yf(x^2+y^2)-x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y}=\frac{[1+2xyf'(x^2+y^2)](x^2+y^2)-[y+xf(x^2+y^2)](2y)}{(x^2+y^2)^2}$$

$$=\frac{x^2-y^2+2xy[(x^2+y^2)f'(x^2+y^2)-f(x^2+y^2)]}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial y}=\frac{[-1+2xyf'(x^2+y^2)](x^2+y^2)-[yf(x^2+y^2)-x](2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2 - y^2 + 2xy\left[(x^2+y^2) f'(x^2+y^2) - f(x^2+y^2)\right]}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\Rightarrow \frac{1}{x^2+y^2}$ is an integrating factor for equation (1)

Given $f(x^2+y^2) = (x^2+y^2)^2$

Then equation (1) becomes

$$\left[y + x^2(x^2+y^2)^2\right]dx + \left[y(x^2+y^2)^2 - x\right]dy = 0$$

This can be written as

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\Rightarrow \frac{y\,dx - x\,dy}{x^2+y^2} + (x\,dx + y\,dy)(x^2+y^2) = 0$$

$$d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) + \frac{x^2+y^2}{2}\,d(x^2+y^2) = 0$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\frac{(x^2+y^2)^2}{2} = C$$

which is the required solution.

6(b) Solve by the method of variation of parameters
$$\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x.$$

Sol^n: Given that
$$y'' + (1-\cot x)y' - y\cot x = \sin^2 x \quad \text{--- (1)}$$

Consider
$$y'' + (1-\cot x)y' - \cot x . y = 0 \quad \text{--- (2)}$$

Comparing (2) with $y'' + Py' + Qy = 0$

Here $P = 1-\cot x \quad Q = -\cot x \quad R = 0.$

$\therefore 1 - P + Q = 1-(1-\cot x) - \cot x = 0$ showing that $u = e^{-x}$ is a part of C.F of (2).

Let the complete solution of (2) be $y_c = uv$.

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

i.e, $\frac{d^2v}{dx^2} + \left[1-\cot x + \frac{2}{e^{-x}}(-e^{-x})\right]\frac{dv}{dx} = 0$

$$\Rightarrow \frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = 0 \quad \text{--- (3)}$$

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = \frac{dq}{dx}$

$\therefore$ (3) becomes
$$\frac{dq}{dx} - (1+\cot x)q = 0$$
$$\Rightarrow \frac{1}{q}dq = (1+\cot x)dx$$
Integrating it,
$$\log q - \log c_1 = x + \log\sin x$$
$$\Rightarrow q = c_1 e^x \sin x.$$
$$\Rightarrow \frac{dv}{dx} = c_1\int e^x \sin x\, dx + c_2$$

$v = \frac{1}{2}c_1 e^x(\sin x - \cos x) + c_2$

(or) $v = c_1' e^x(\sin x - \cos x) + c_2$ where $c_1' = \frac{c_1}{2}$.

$\therefore$ C.F. of (2) is given by

$y_c = e^{-x}\{c_1' e^x(\sin x - \cos x) + c_2\}$

$y_c = c_1'(\sin x - \cos x) + c_2 e^{-x}$ ——— (4)

P.I. of (1) $y_p = u f(x) + v g(x)$

$u = \sin x - \cos x$, $v = e^{-x}$

and $W = \begin{vmatrix} \sin x - \cos x & e^{-x} \\ \cos x + \sin x & -e^{-x} \end{vmatrix} = -2e^{-x}\sin x$

where

$f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{-x}\sin^2 x}{-2e^{-x}\sin x}dx$ ($\because$ from (1) $R = \sin^2 x$)

$= \frac{1}{2}\int \sin x\, dx = -\frac{1}{2}\cos x$

$g(x) = \int \frac{uR}{W} = \int \frac{(\sin x - \cos x)\sin^2 x}{-2e^{-x}\sin x}dx$

$= -\frac{1}{2}\int e^x(\sin x - \cos x)\sin x . dx$

$= -\frac{1}{2}\int e^x(\sin^2 x - \cos x \sin x)dx$

$= -\frac{1}{2}\left[\int e^x\left(\frac{1-\cos 2x}{2}\right)dx - \int e^x \frac{\sin 2x}{2}dx\right]$

$= -\frac{1}{4}\left[e^x - \int e^x \cos 2x\, dx - \int e^x \sin 2x\, dx\right]$

$= -\frac{1}{4}\left[e^x - \frac{e^x}{5}(\cos 2x + 2\sin 2x) - \frac{e^x}{5}(\sin 2x - 2\cos 2x)\right]$

$= -\frac{1}{4}\left[e^x - \frac{e^x}{5}(\cos 2x - 3\sin 2x)\right]$

$= -\frac{1}{4}e^x\left[1 + \frac{1}{5}(\cos 2x - 3\sin 2x)\right.$

$\therefore$ (4) $\Rightarrow$

$$y_p = -\frac{1}{2}\cos x(\sin x - \cos x) + e^{-x}\left[-\frac{1}{4}e^{x}\left(1+\frac{1}{5}(\cos 2x - 3\sin 2x)\right)\right]$$

$$= -\frac{1}{2}\sin x\cos x + \frac{1}{2}\cos^2 x - \frac{1}{4} - \frac{1}{20}(\cos 2x - 3\sin 2x)$$

$$= -\frac{1}{4}\sin 2x + \frac{1}{4}(1+\cos 2x) - \frac{1}{4} - \frac{1}{20}(\cos 2x - 3\sin 2x)$$

$$= \frac{1}{20}(4\cos 2x - 2\sin 2x)$$

$$= \frac{1}{10}(2\cos 2x - \sin 2x)$$

$\therefore$ $y = y_c + y_p$

$$y = c_1'(\sin x - \cos x) + c_2 e^{-x} + \frac{1}{10}(2\cos 2x - \sin 2x)$$

which is the solution of (1).

6(c) → Show that the Wronskian of the functions x^2 and $x^2\log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation. If so, determine this differential equation.

Sol^n: Let $y_1(x) = x^2$ and $y_2(x) = x^2\log x$.

The Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\log x \\ 2x & 2x\log x + x \end{vmatrix}$$

$$= x^2(2x\log x + x) - 2x^3\log x$$

$\therefore W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$. Hence, functions $y_1(x)$ and $y_2(x)$, i.e, x^2 and $x\log x$ can be linearly independent solutions of an ordinary differential equation.

To form the required differential equation.

The general solution of the required differential equation may be written as

$$y = Ay_1(x) + By_2(x) = Ax^2 + Bx^2\log x \quad \text{——} (1)$$

where A & B are arbitrary constants.

Diff. (1), we get

$$y' = 2Ax + B(2x\log x + x) \quad \text{——} (2)$$

Diff. (2), we get

$$y'' = 2A + B(2\log x + 2 + 1) \quad \text{——} (3)$$

We now eliminate A & B from (1), (2) & (3),

multiplying both sides of (3) by x, we get

$$xy'' = 2Ax + B(3x + 2x\log x) \quad \text{——} (4)$$

Subtracting (2) from (4), $xy'' - y' = 2Bx \Rightarrow B = (xy'' - y')/2x$

Substituting this value of B in (3), we have

$$2A = y'' - \frac{1}{2x}(xy''-y')(3+2\log x)$$

$$\Rightarrow A = \frac{1}{4x}\left[2xy'' - (xy''-y')(3+2\log x)\right].$$

Substituting the above values A & B in (1), we have

$$y = \left(\frac{x}{4}\right)\left[2xy'' - 3xy'' + 3y' - 2xy''\log x + 2y'\log x\right] + \frac{x}{2}(xy''-y')\log x$$

$$\Rightarrow 4y = x(-xy'' + 3y' - 2xy''\log x + 2y'\log x) + 2x(xy''-y')\log x$$

$$\Rightarrow x^2y'' - 3xy' + 4y = 0.$$

which is the required equation.

6(d) → By using Laplace transform method, solve the differential equation $(D^2+n^2)x = a\sin(nt+\alpha)$, $D^2=\frac{d^2}{dt^2}$ subject to the initial conditions $x=0$ and $\frac{dx}{dt}=0$ at $t=0$. in which a, n and α are constants.

Sol'n: The given equation can be written as

$$(D^2+n^2)x = a(\sin nt\cos\alpha + \cos nt\sin\alpha)$$

$$\therefore L(x'') + n^2 L(x) = a\cos\alpha\, L(\sin nt) + a\sin\alpha\, L(\cos nt)$$

$$\Rightarrow p^2L(x) - px(0) - x'(0) + n^2L(x) = a\cos\alpha\cdot\frac{n}{p^2+n^2} + a\cos\alpha\cdot\frac{p}{p^2+n^2}$$

$$\Rightarrow (p^2+n^2)L(x) = \frac{na\cos\alpha}{p^2+n^2} + \frac{ap\sin\alpha}{p^2+n^2}$$

$$\Rightarrow L(x) = \frac{an\cos\alpha}{(p^2+n^2)^2} + \frac{ap\sin\alpha}{(p^2+n^2)^2}$$

$$\therefore x = an\cos\alpha\; L^{-1}\left\{\frac{1}{(p^2+n^2)^2}\right\} + a\sin\alpha\cdot L^{-1}\left\{\frac{p}{(p^2+n^2)^2}\right\} \quad \text{—— (1)}$$

Now, $$L^{-1}\left\{\frac{p}{(p^2+n^2)^2}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{n^2+p^2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1\, t\; L^{-1}\left\{\frac{1}{p^2+n^2}\right\}$$

$$\therefore L^{-1}\left\{\frac{p}{(p^2+n^2)^2}\right\} = \frac{t}{2n}\sin nt \quad \text{—— (2)}$$

Let $f(p) = \frac{1}{p^2+n^2}$ and $g(p) = \frac{1}{p^2+n^2}$

Then $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p^2+n^2}\right\} = \frac{\sin nt}{n}$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+n^2}\right\} = \frac{\sin nt}{n}$.

Now, by the Convolution theorem

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(G)\,G(t-u)\,du$$

$$\therefore L^{-1}\left\{\frac{1}{(p^2+n^2)^2}\right\} = \int_0^t \left[\frac{\sin nu}{n}\;\frac{\sin n(t-u)}{n}\right]du$$

$$= \frac{1}{2n^2}\int_0^t \left[\cos n(t-2u) - \cos nt\right]du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} - t\cos nt + \frac{\sin nt}{2n}\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right] \longleftarrow (3)$$

using (2) & (3), (1) reduces to

$$y = an\cos\alpha\,\frac{1}{2n^2}\left(\frac{\sin nt}{n} - t\cos nt\right) + a\sin\alpha\,\frac{1}{2n}\,t\sin nt$$

$$= \frac{a}{2n^2}\cos\alpha\sin nt - \frac{at}{2n}\left[\cos\alpha\cos nt - \sin\alpha\sin nt\right]$$

$$= \frac{a}{2n^2}\cos\alpha\sin nt - \frac{at}{2n}\left(\cos(\alpha+nt)\right)$$

$$= \frac{a}{2n^2}\left[\cos\alpha\sin nt - nt\cos(\alpha+nt)\right]$$

which is the required solution.

7(a) → The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is $\mu \log\left[\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right]$.

where μ is the coefficient of friction.

Sol'n: Let the end links A and B of a uniform chain slide along a fixed rough horizontal rod. If AB is the maximum span, then A and B are in the state of limiting equilibrium.

Let R be the reaction of the rod at A acting ⊥lar to the rod. Then the frictional force μR will act at A along the rod in the outward direction BA as shown in the fig.

The resultant F of the forces R and μR at A will make an angle λ (where $\tan\lambda=\mu$) with the direction of R.

For the equilibrium of A the resultant F of R & μR at A will be equal and opposite to the tension T at A.

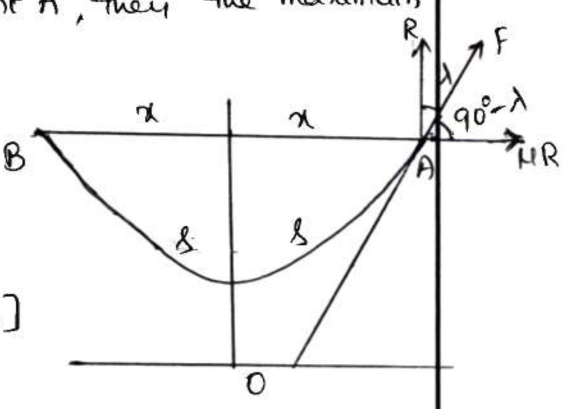
Since the tension at A acts along the tangent to the chain at A, therefore the tangent to the catenary at A makes an angle $\psi_A = \frac{1}{2}\pi-\lambda$ to the horizontal.

Thus for the point A of the catenary, we have $\psi=\psi_A=\frac{1}{2}\pi-\lambda$.

∴ the length of the chain $2s = 2c\tan\psi_A = 2c\tan(\frac{1}{2}\pi-\lambda)$

$= 2c\cot\lambda = \frac{2c}{\mu}$. [∵ $\tan\lambda=\mu$]

If (x_A, y_A) are the coordinates of the point A, then the maximum span $AB = 2x_A$



$= 2c\log(\tan\psi_A + \sec\psi_A)$

$= 2c\log\left\{\tan\psi_A + \sqrt{1+\tan^2\psi}\right\}$

$= 2c\log\left\{\cot\lambda + \sqrt{(1+\cot^2\lambda)}\right\}$

[∵ $\psi_A = \frac{1}{2}\pi-\lambda$]

$= 2c\log\left\{\frac{1}{\mu} + \sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\}$

$$= 2c \log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$$

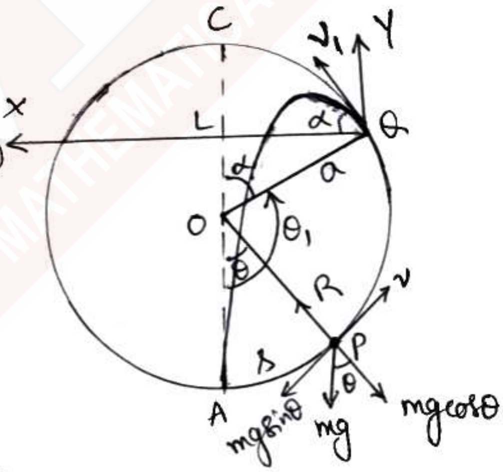
Hence the required ratio

$$= \frac{2x}{2s} = \frac{2c\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}}{(2c/\mu)}$$

$$= \mu\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$$

7(b) → A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{(\frac{7}{2}ag)}$. show that it will leave the sphere at a height $\frac{3}{4}a$ above the lowest point and its subsequent path meets the sphere again at the point of projection.

Sol'n: A particle is projected from the lowest point A of a sphere with velocity $u=\sqrt{(\frac{7}{2}ag)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where arc $AP = s$ and $\angle AOP = \theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{——} \ (1)$$

and $$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{——} \ (2)$$

Also $$s = a\theta \quad \text{——} \ (3)$$

from (1) & (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

multiplying bothsides by $2a\frac{d\theta}{dt}$ and then integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But at the point A, $\theta = 0$ and $v = u = \sqrt{(\frac{7}{2}ag)}$

$$\therefore A = \tfrac{7}{2}ag - 2ag = \tfrac{3}{2}ag$$

$$\therefore v^2 = \tfrac{3}{2}ag + 2ag\cos\theta \quad \text{——} \ (4)$$

Now from (2) and (4), we have

$$R = \frac{m}{a}\left[v^2 + ag\cos\theta\right]$$

$$= \frac{m}{a}\left[\frac{3}{2}ag + 2ag\cos\theta + ag\cos\theta\right]$$

$$= 3mg\left(\tfrac{1}{2} + \cos\theta\right)$$

If the particle leaves the sphere at the point Q, where $\theta = \theta_1$, then $0 = 3mg\left(\tfrac{1}{2} + \cos\theta_1\right)$ (or) $\cos\theta_1 = -\tfrac{1}{2}$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$

$$\therefore \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \tfrac{1}{2} \quad \text{—— (5)}$$

$$\therefore AL = AO + OL = a + a\cos\alpha = a + \tfrac{a}{2} = \tfrac{3a}{2}.$$

i.e, the particle leaves the sphere at a height $\tfrac{3}{2}a$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v = v_1$, $R = 0$ and $\theta = \theta_1$ in (2), we get

$$v_1^2 = -ag\cos\theta_1 = -ag\left(-\tfrac{1}{2}\right) = \tfrac{1}{2}ag.$$

$\therefore$ the particle leaves the sphere at the point Q with velocity $v_1 = \sqrt{\left(\tfrac{1}{2}ag\right)}$ making an angle α with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t QX & QY as coordinate axes is $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_1^2\cos^2\alpha}$

$$\Rightarrow y = x\sqrt{3} - \frac{gx^2}{2\cdot\frac{1}{2}ag\cdot\frac{1}{4}}$$

$$\Rightarrow y = \sqrt{3}x - \frac{4x^2}{a}. \quad \text{—— (6)}$$

from the fig (6), for the point A, $x = QL = a\sin\alpha = a\sqrt{3}/2$ & $y = -LA = -\tfrac{3}{2}a$.

If we put $x = a\sqrt{3}/2$ in the equation (6), we get

$$y = a\tfrac{\sqrt{3}}{2}\cdot\sqrt{3} - \tfrac{4}{a}\cdot\tfrac{3a^2}{4} = \tfrac{3a}{2} - 3a = -\tfrac{3}{2}a.$$

Thus the coordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

7(c) A particle moves with a central acceleration $\mu(r + a^4/r^3)$ being projected from an apse at a distance 'a' with a velocity $2a/\sqrt{\mu}$. Prove that it describes the curve $r^2(2 + \cos\sqrt{3}\theta) = 3a^2$.

Sol'n: Here, the central acceleration,

$P = \mu(r + a^4/r^3) = \mu\{\frac{1}{u} + a^4u^3\}$, where $u = 1/r$.

∴ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u} + a^4u^3\right)$$

$$\Rightarrow h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\left(\frac{1}{u^3} + a^4u\right).$$

Multiplying bothsides by $2(du/d\theta)$ and integrating w.r.t 'θ' we have,

$$h^2\left[2\cdot\frac{u^2}{2} + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(-\frac{1}{2u^2} + \frac{a^4u^2}{2}\right) + A$$

$$\Rightarrow v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left[-\frac{1}{u^2} + a^4u^2\right] + A \quad \text{--- (1)}$$

where A is a constant.

Now initially the particle has been projected from an apse (say, the point A) at a distance a with velocity $2\sqrt{\mu}a$.

∴ when $r = a$ i.e, $u = 1/a$, $du/d\theta = 0$ (at an apse) & $v = 2\sqrt{\mu}a$.

∴ from (1), we have

$$\underset{(i)}{4\mu a^2} = \underset{(ii)}{h^2\left[\frac{1}{a^2}\right]} = \underset{(iii)}{\mu\left(-a^2 + a^4\cdot\frac{1}{a^2}\right)} + A$$

from (i) & (ii), we have $h^2 = 4\mu a^4$ and from (i) & (iii) we have

$$4\mu a^2 = 0 + A \quad \text{i.e. } A = 4\mu a^2$$

Substituting the values of h^2 & A in (1), we have

$$4\mu a^4\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + 4\mu a^2$$

$$4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$$

$$4a^4u^2\left(\frac{du}{d\theta}\right)^2 = (-1 - 3a^4u^4 + 4a^2u^2) \quad \text{——} \quad (2)$$

$$2a^2u\frac{du}{d\theta} = \sqrt{(-1 - 3a^4u^4 + 4a^2u^2)}$$

$$\Rightarrow d\theta = \frac{2a^2u\,du}{\sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}} = \frac{2a^2u\,du}{\sqrt{3}.\sqrt{[-\frac{1}{3} - (a^4u^4 - \frac{4}{3}a^2u^2)]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}.\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2 + \frac{4}{9}]}}$$

$$\Rightarrow \sqrt{3}\,d\theta = \frac{2a^2u\,du}{\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2]}}$$

Substituting $a^2u^2 - \frac{2}{3} = z$, so that $2a^2u\,du = dz$, we have

$$\sqrt{3}\,d\theta = \frac{dz}{\sqrt{(\frac{1}{3})^2 - z^2}}$$

Integrating $\sqrt{3}\theta + B = \sin^{-1}(3z)$ where B is a constant.

$$\Rightarrow \sqrt{3}\theta + B = \sin^{-1}(3a^2u^2 - 2). \quad \text{——} \quad (3)$$

Now take the aspe-line OA as the initial line.

Then initially $r = a$, $u = \frac{1}{a}$ and $\theta = 0$.

$\therefore$ from (3), $0 + B = \sin^{-1} 1 \Rightarrow B = \frac{1}{2}\pi$

Putting $B = \frac{1}{2}\pi$ in (3), we have

$$\sqrt{3}\theta + \frac{1}{2}\pi = \sin^{-1}(3a^2u^2 - 2)$$

$$\Rightarrow 3a^2u^2 - 2 = \sin\left(\frac{1}{2}\pi + \sqrt{3}\theta\right) = \cos(\sqrt{3}\theta)$$

$$\Rightarrow \frac{3a^2}{r^2} - 2 = \cos(\sqrt{3}\theta)$$

$$\Rightarrow 3a^2 - 2r^2 = r^2\cos\sqrt{3}\theta$$

$$\therefore 3a^2 = r^2[2 + \cos\sqrt{3}\theta]$$

which is the equation of the required curve.

8(a) If $F\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\hat{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\hat{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\hat{k}$

Prove that (i) $F=r\times\nabla f$, (ii) $F\cdot r=0$, (iii) $F\cdot\nabla f=0$.

Sol^n: we have $\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$

and $r=x\hat{i}+y\hat{j}+z\hat{k}$

(i) $$r\times\nabla f=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{vmatrix}$$

$$=\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\hat{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\hat{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\hat{k}=F$$

(ii) $F\cdot r=(r\times\nabla f)\cdot r$

$=0$ [because the value of a scalar triple product having two vectors equal is zero].

(iii) $F\cdot\nabla f=(r\times\nabla f)\cdot\nabla f=[r,\nabla f,\nabla f]$

$=0$, because the value of a scalar triple product having two vectors equal is zero.

8(b). Find div grad r^m and verify that $\nabla \times \nabla r^m = 0$.

Sol^n: $\text{div grad } r^m = \text{div}\left(m r^{m-1} \text{grad } r\right)$

$$= \text{div}\left(m r^{m-1} \frac{1}{r} \vec{r}\right)$$

$$= \text{div}\left(m r^{m-2} \vec{r}\right)$$

$$= m r^{m-2} \text{div} \vec{r} + \vec{r} \cdot \left(\text{grad } m r^{m-2}\right)$$

$$= 3 m r^{m-2} + \vec{r} \cdot \left[m(m-2) r^{m-3} \text{grad } r\right]$$

$$= 3m r^{m-2} + \vec{r} \cdot \left[m(m-2) r^{m-3} \frac{1}{r} \vec{r}\right]$$

$$= 3m r^{m-2} + \vec{r} \cdot \left[m(m-2) r^{m-4} \vec{r}\right]$$

$$= 3m r^{m-2} + m(m-2) r^{m-4} (\vec{r} \cdot \vec{r})$$

$$= 3m r^{m-2} + m(m-2) r^{m-2}$$

$$= m r^{m-2} (3 + m - 2)$$

$$= m(m+1) r^{m-2}.$$

$\therefore \text{div}(\text{grad } r^m) = m(m+1) r^{m-2}$

To find $\nabla \times \nabla r^m$:

we have $\nabla r^m = \frac{\partial r^m}{\partial x} \hat{i} + \frac{\partial r^m}{\partial y} \cdot \hat{j} + \frac{\partial r^m}{\partial z} \hat{k}$

$$= m r^{m-2} x \hat{i} + m r^{m-2} y \hat{j} + m r^{m-2} z \hat{k}$$

$$\nabla \times \nabla r^m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ m r^{n-2} x & m r^{n-2} y & m r^{n-2} z \end{vmatrix}$$

$$= \hat{i}\left[(m(m-2) r^{m-3})(yz - yz)\right] + \hat{j}\left[m(m-2) r^{m-3} (xz - xz)\right] + \hat{k}\left[m(m-2) r^{m-3} (xy - yx)\right]$$

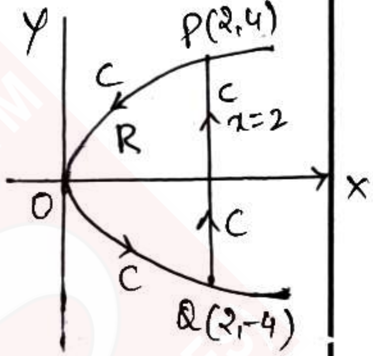
$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$= 0$$

8(C) → Verify Green's theorem in a plane for $\oint_C [(x^2-2xy)dx+(x^2y+3)dy]$ where C is the boundary of the region defined by $y^2=8x$ and $x=2$.

Sol'n: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy = \oint_C M\,dx+N\,dy$$



Here $M=x^2-2xy$; $N=x^2y+3$

The parabola $y^2=8x$ and the straight line $x=2$ intersect the points $P(2,4)$ & $Q(2,-4)$.

The positive direction in traversing C is as shown in the figure and R is the region bounded by the curve C.

We have $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

$$=\iint_R\left[\frac{\partial}{\partial x}(x^2y+3)-\frac{\partial}{\partial y}(x^2-2xy)\right]dxdy$$

$$=\iint_R (2xy+2x)\,dxdy$$

$$=\int_{x=0}^{2}\int_{y=-\sqrt{8x}}^{\sqrt{8x}}(2xy+2x)\,dxdy$$

[∵ for the region R, x varies from 0 to 2 and y varies from $-\sqrt{8x}$ to $\sqrt{8x}$].

$$=\int_0^2 2x\left[\frac{y^2}{2}+y\right]_{y=-\sqrt{8x}}^{\sqrt{8x}}dx$$

$$=\int_0^2 2x\left[0+2\sqrt{8x}\right]dx$$

$$=4\sqrt{8}\int_0^2 x^{3/2}\,dx$$

$$=8\sqrt{2}\cdot\tfrac{2}{5}\left[x^{5/2}\right]_0^2=\frac{128}{5}.$$

Now let us evaluate the line integral along C. Along $y^2=8x$, we have $x=y^2/8$, $dx=\frac{1}{4}y\,dy$ and y varies from 4 to -4.
Therefore along $y^2=8x$, the line integral equals

$$= \int_{y=4}^{-4}\left[\left[\frac{y^4}{64}-2\frac{y^2}{8}\cdot y\right]\cdot\frac{1}{4}y\,dy+\left(\frac{y^4}{64}\cdot y+3\right)dy\right]$$

$$= \int_{4}^{-4}\left[\frac{y^5}{256}+\frac{y^5}{64}-\frac{1}{16}y^4+3\right]dy$$

$$= -\int_{4}^{-4}\left[\frac{y^5}{256}+\frac{y^5}{64}-\frac{1}{16}y^4+3\right]dy =$$

$$= -2\int_{0}^{4}\left[-\frac{1}{16}y^4+3\right]dy \qquad \left[\because \int_{-a}^{a}f(x)dx=0 \text{ (or) } 2\int_0^a f(x)dx \text{ according as } f(x)=-f(x) \text{ (or) } f(-x)=f(x)\right]$$

$$= -2\left[-\frac{1}{16\cdot5}y^5+3y\right]_0^4$$

$$= -2\left[-\frac{1}{80}\cdot4^5+3.4\right] = \frac{128}{5}-24.$$

Along the straight line $x=2$, we have $dx=0$ and y varies from -4 to 4. Therefore along $x=2$, the line integral equals.

$$\int_{y=-4}^{4}\left[0+(2^2\cdot y+3)dy\right] = \int_{-4}^{4}(4y+3)\,dy$$

$$= 3\int_{-4}^{4}dy \qquad \left[\because \int_{-4}^{4}4y\,dy=0\right]$$

$$= 6\int_0^4 dy = 6\left[y\right]_0^4 = 6.4 = 24.$$

$\therefore$ the total line integral along the curve C i.e,

$$\oint_C (M\,dx+N\,dy) = \frac{128}{5}-24+24=\frac{128}{5}. \quad \text{——} (2)$$

from (1) & (2), we see that

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C (M\,dx+N\,dy).$$

which verifies Green's theorem in plane.

8(d) → verify the divergence theorem for $F = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by the surfaces $x^2+y^2=4$, $z=0$, $z=3$.

Sol'n: Let S denote the closed surface bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$, $z=3$.

Also let V be the volume bounded by the surface S.

By Gauss divergence theorem, we have

$$\iint_S F\cdot\hat{n}\,ds = \iiint_V \operatorname{div} F\,dv$$

volume integral $= \iiint_V \operatorname{div} F\,dv$

$$= \iiint_V \nabla\cdot F\,dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right]dv$$

$$= \iiint_V (4-4y+2z)\,dv$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2-2y+z)\,dz\,dx\,dy$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[2y-y^2+zy\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz\,dx$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[(2+z)\,2\sqrt{4-x^2}-0\right]dz\,dx$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\left[2\sqrt{4-x^2}+z\sqrt{4-x^2}\right]dz\,dx$$

$$= 4\int_{x=-2}^{2}\left[2z\sqrt{4-x^2}+\frac{z^2}{2}\sqrt{4-x^2}\right]_{z=0}^{3} dx$$

$$= 4\int_{x=-2}^{2}\left[6\sqrt{4-x^2} + \tfrac{9}{2}\sqrt{4-x^2}\right]dx$$

$$= 4\times\frac{21}{2}\int_{x=-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{84}{2}\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2\times\frac{\pi}{2}\right] = 84\pi$$

Now we shall evaluate the surface integral $\iint_S F.\hat{n}\,ds$

Here the surface S consists of three surfaces

(i) the surface S_1 of the base i.e, the plane face $z=0$ of the cylinder.

(ii) the surface S_2 of the top i.e, the plane $z=3$ of the cylinder and

(iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 i.e. $z=0$, $F = 4xi - 2y^2j$, putting $z=0$ in F.

A unit vector $\hat{n}$ along the outword drawn normal to S_1 is obviously $-\hat{k}$.

$$\therefore \iint_{S_1} F.n\,ds = \iint_{S_1}(4xi - 2y^2j).(-k)\,ds$$

$$= \iint_{S_1} 0\,ds = 0$$

For the surface S_2 i.e, $z=3$, $F = 4xi - 2y^2j + 9k$ putting $z=3$ in F.

A unit vector n along the outward drawn normal to S_2 is obviously k.

$$\therefore \iint_{S_2} F\cdot n\,ds = \iint_{S_2} (4xi - 2y^2 j + 9k)\cdot k\,ds$$

$$= \iint_{S_2} 9\,ds = 9(2\pi)(2)$$

$$= 36\pi \quad (\because \text{area of } S_2 = 2\pi r \text{ here } r=2)$$

For the convex portion S_3 i.e. $x^2+y^2=4$, a vector normal to S_3 is given by

$$\nabla(x^2+y^2) = 2xi + 2yj$$

$\therefore$ n = a unit vector along outward drawn normal at any point of S_3.

$$= \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{xi+yj}{2} \quad (\because x^2+y^2=4 \text{ on } S_3)$$

$\therefore$ on S_3 $\quad F\cdot\hat{n}\,ds = (4xi - 2y^2 j + z^2 k)\cdot\frac{1}{2}(xi+yj)$

$$= 2x^2 - y^3$$

Also ds = elementary area on the surface S_3

$= 2\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} F\cdot\hat{n}\,ds = \iint_{S_3} (2x^2 - y^3)\,2\,d\theta\,dz \quad \text{where } x = 2\cos\theta,\ y = 2\sin\theta$$

$$= \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi} [\cos^2\theta - \sin^3\theta]\,(z)_{z=0}^{3}\,d\theta$$

$$= 48 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48 \left[4 \int_0^{\pi/2} \cos^2\theta\, d\theta - \int_0^{2\pi} \sin^3\theta\, d\theta \right]$$

$$= 48 \left[4 \cdot \tfrac{1}{2} \cdot \tfrac{\pi}{2} - 0 \right] \quad \left(\because \sin^3(2\pi - \theta) = -\sin^3\theta \text{ odd function } \int_0^{2\pi} \sin^3\theta\, d\theta = 0 \right)$$

$$= 48\pi$$

Hence the required surface integral

$$= \iint_S F \cdot \hat{n}\, ds = \iint_{S_1} F \cdot \hat{n}\, ds + \iint_{S_2} F \cdot \hat{n}\, ds + \iint_{S_3} F \cdot \hat{n}\, ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$$\therefore \iint_S F \cdot \hat{n}\, ds = 84\pi$$

$$\therefore \iint_S F \cdot \hat{n}\, ds = \iiint_V \operatorname{div} F\, dv$$