REAL ANALYSIS

IFS PYQs

2020

1. 1b

(i) If $u = u(y-z,\ z-x,\ x-y)$, then find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

(ii) If $u(x, y, z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$. 8

2. 1c

Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)\,dx\,dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$. 8

3. 2b

Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x) = nx(1-x)^n$, does not converge uniformly on $[0, 1]$. 15

4. 3a

Find the extreme values of $f(x, y, z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$. 10

2019

1. 1b

Show that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$. *8*

2. 1c

Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on the interval $[0, 1]$. *8*

3. 2b 2019

(b) Show that the integral $\int_0^{\pi/2} \log \sin x \, dx$ is convergent and hence evaluate it. *15*

4. 3b

(b) Show that the sequence $\{\tan^{-1} nx\}$, $x \geq 0$ is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$. *15*

2018

5. 1b

(b) A function $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$. 10

6. 2b

(b) Consider the function f defined by

$$f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x^2 + y^2 \neq 0 \\ 0, & \text{where } x^2 + y^2 = 0 \end{cases}$$

Show that $f_{xy} \neq f_{yx}$ at $(0, 0)$. 10

7. 3a

(a) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. 10

8. 3d

(d) Show that the improper integral $\int_0^1 \dfrac{\sin \frac{1}{x}}{\sqrt{x}}\, dx$ is convergent. 10

9. 4a

Show that

$$\iint_R x^{m-1} y^{n-1} (1-x-y)^{l-1} dx\, dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)};\ l, m, n > 0$$

taken over R : the triangle bounded by $x=0,\ y=0,\ x+y=1$. 10

10. 4b

Let $f_n(x) = \dfrac{x}{n+x^2},\ x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$. 8

2017

11. 1b

1. (b) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as below :

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. 10

12. 3a

Evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ given that

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y} & \text{if } xy \neq 0 \\ 0 & , \text{otherwise} \end{cases}$$

10

13. 3b

Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the condition $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$. 10

14. 3c

Prove that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent but not absolutely convergent. 12

15. 3d

Find the volume of the region common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. 8

16. 4c

4.(c) Evaluate $\int_{x=0}^{\infty}\int_{y=0}^{x} x\, e^{-x^2/y}\, dy\, dx$ 8

2016

17. 1b

Examine the Uniform Convergence of

$f_n(x)=\dfrac{\sin(nx+n)}{n},\ \forall x\in\mathbb{R},\ n=1,\ 2,\ 3,\ \ldots$ 8

18. 1c

Find the maxima and minima of the function
$f(x,\ y)\ =\ x^3+y^3-3x-12y+20.$ 8

19. 3a

If $f_n(x)=\dfrac{3}{x+n},\ 0\leq x\leq 2$, state with reasons whether $\{f_n\}_n$ converges uniformly on $[0,\ 2]$ or not. 10

20. 3b

Examine the continuity of $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y)\neq(0,0)\\ \frac{1}{2}, & (x,y)=(0,0)\end{cases}$

at the point $(0,\ 0)$. 8

21. 3c

If $u(x,y)=\cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\},\ 0<x<1,\ 0<y<1$ then find the value of

$x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. 10

22. 3d

Evaluate the integral $\int_0^2 \int_0^{y^2/2} \frac{y}{(x^2+y^2+1)^{\frac{1}{2}}}\,dx\,dy$. 12

23. 4a

4.(a) Evaluate the integral $\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$. 8

2015

24. 1b

(b) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

Suppose $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8}$. What is $\sum_{n=1}^{\infty} a_n$?

Justify your answer. (Majority of marks is for the correct justification). 8

25. 2b

(b) Let X = (a, b]. Construct a continuous function $f : X \to \mathbb{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a + \varepsilon, b]$, $a + \varepsilon < b$? Why ? 14

26. 3b

(b) Let $f_n(x) = \dfrac{x}{1 + nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f ? Show that for $x \neq 0$, $f'_n(x) \to f'(x)$ but $f'_n(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\dfrac{1}{2\sqrt{n}}$. 13

27. 4a

(a) Compute the double integral which will give the area of the region between the y-axis, the circle $(x - 2)^2 + (y - 4)^2 = z^2$ and the parabola $2y = x^2$. Compute the integral and find the area. 15

2014

28. 1b

(b) Let f be defined on $[0, 1]$ as

$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } x \text{ is rational} \\ 1-x \quad, & \text{if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over $[0, 1]$. 8

29. 2b

(b) Show that the function $f(x) = \sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0, \pi)$. 15

30. 3b

(b) Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$. 15

31. 4a

(a) Show that the function $f(x) = \sin x$ is Riemann integrable in any interval $[0, t]$ by taking the partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \frac{3t}{n}, \ldots, \frac{nt}{n}\right\}$ and $\int_0^t \sin x\,dx = 1 - \cos t$. 10

2013

32. 1a

Evaluate :

$$\lim_{x \to 0} \left(\frac{e^{ax} - e^{bx} + \tan x}{x} \right)$$

33. 3a (13m)

Show that the function $f(x) = x^2$ is uniformly continuous in $(0, 1)$ but not in $\mathbb{R}$.

34. 4a (13m) 2013

Find the area of the region between the x-axis and $y = (x - 1)^3$ from $x = 0$ to $x = 2$.

2012

35. 1a

1. Answer the following :

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , \quad x \text{ is irrational} \\ -1 & , \quad x \text{ is rational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$. 10

36. 1d

(d) Show that the functions :

$$u = x^2 + y^2 + z^2$$

$$v = x + y + z$$

$$w = yz + zx + xy$$

are not independent of one another. 10

37. 2b

(b) If

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right),$$

show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$$ 13

38. 3a

3. (a) Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$. 13

39. 4b

(b) Examine the series

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

for uniform convergence. Also, with the help of this example, show that the condition of uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for the sum $S(x)$ of the series to be continuous. 13

2011

40. 1c

(c) Determine whether

$$f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is Riemann-integrable on [0, 1] and justify your answer. 10

41. 2b

(b) Let the function f be defined by

$$f(x) = \frac{1}{2^t}, \quad \text{when} \quad \frac{1}{2^{t+1}} < x \leq \frac{1}{2^t}$$

$$(t = 0, 1, 2, 3, \ldots)$$

$$f(0) = 0$$

Is f integrable on [0, 1] ? If f is integrable, then evaluate $\int_0^1 f\, dx$. 13

42. 3a

3. (a) Examine the convergence of

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$$

and evaluate, if possible. 13

43. 4a

4. (a) Evaluate

$$\iint \sqrt{4x^2 - y^2}\, dx\, dy$$

over the triangle formed by the straight lines $y = 0$, $x = 1$, $y = x$. 13

2010

44. 1c

(c) If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(x+y) = f(x)\,f(y)$$

for all x, y in $\mathbb{R}$ and $f(x) \neq 0$ for any x in $\mathbb{R}$, show that $f'(x) = f(x)$ for all x in $\mathbb{R}$ given that $f'(0) = f(0)$ and the function is differentiable for all x in $\mathbb{R}$. 10

45. 2a

2. (a) A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum. 13

46. 3b

(b) Evaluate

$$\iint_R (x - y + 1)\, dx\, dy$$

where R is the region inside the unit square in which $x + y \geq \frac{1}{2}$. 13