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(INSTITUTE OF MATHEMATICAL SCIENCES)

IAS MATHEMATICS (OPT.)-2013

PAPER - II : SOLUTIONS

IAS-2013

(1)(a) Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f : \mathbb{C} \to S$ defined by $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.)

Ans. TS: $(S, +, \cdot)$ is a field.

i) $(S, +)$ is an abelian group.

Closure: Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S$

then, clearly, $A+B = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$
$\Rightarrow a+c, -(b+d), b+d, a+c \in \mathbb{R}$

$\Rightarrow A+B \in S$

Associativity: $\forall A, B, C \in S$

$\Rightarrow (A+B)+C = A+(B+C)$

Since, matrix addition is associative in nature.

Identity: $\forall A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$,

such that $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity of S.

Inverse: $\forall A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element $\begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} \in S$

such that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} -a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Commutative: $\forall A, B \in S$ such that,

we have, $A+B = B+A$

as $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$\forall a, b, c, d \in \mathbb{R}$

ii) $(S,\cdot)$ is an abelian group.

Commutative: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S, \quad a,b,c,d \in \mathbb{R}$

we have, $A\cdot B = \begin{bmatrix} ac & -bd \\ -bd & ac \end{bmatrix} \in S \quad \therefore ac, bd \in \mathbb{R}$

Associative: $\forall\ A, B, C \in S$

we have, $(AB)C = A(BC)$

As, Matrix multiplication is Associative in nature.

Identity: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $\exists$ a unique element $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$

such that $AI = A = I\cdot A$.

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity of S.

Inverse: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element

$B = \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}$, for $a,b \neq 0$ (not both zero) such that

$AB = BA = I$.

$\therefore B = A^{-1}$ is the inverse element.

Commutative: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S, B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S$, we have

$AB = \begin{bmatrix} ac-bd & bc+ad \\ -bc-ad & -bd+ac \end{bmatrix}^T = BA = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & -bd+ac \end{bmatrix}$

$\therefore (S,\cdot)$ is an abelian group.

iii) Distributive laws:-

$\forall\ A, B, C \in S$

we have, $A\cdot(B+C) = A\cdot B + A\cdot C$

$(B+C)\cdot A = B\cdot A + C\cdot A$

Thus, distribution property is satisfied.

Hence $(S,+,\cdot)$ is a field i.e. a commutative division ring with unity.

The additive identity of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and multiplicative identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The inverse of $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

<u>T.S:</u> $f : \mathbb{C} \longrightarrow S$ defined by $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is an isomorphism.

<u>Well Defined:</u> Let $a+ib = c+id$

$$\Rightarrow a = c,\ b = d$$

$$\therefore \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$\Rightarrow f(a+ib) = f(c+id)$$

$\therefore$ f is well defined.

<u>1-1:</u> Let $f(a+ib) = f(c+id)$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$\Rightarrow a = c \text{ and } b = d$$

$$\therefore a+ib = c+id$$

$\Rightarrow$ f is 1-1.

<u>Onto:</u> $\forall \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in S$, $\exists$ a unique preimage $a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$ such that $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

<u>Homomorphism:</u> Let $a+ib$, $c+id \in \mathbb{C}$ be any two arbitrary elements.

$$f((a+ib)+(c+id)) = f(a+c+i(b+d))$$

$$= \begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix}$$

$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = f(a+ib) + f(c+id)$$

Now, $f((a+ib)\cdot(c+id)) = f((ac-bd) + i(ad+bc))$

$$= \begin{pmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$= f(a+ib)\cdot f(c+id)$$

$\Rightarrow$ f is ~~an~~ homomorphism map.

Thus, f is an isomorphism from $\mathbb{C}$ to S.

1(b) Give an example of an infinite group in which every element is of finite order.

Sol^n Let $\langle Z, + \rangle$ be the group of integers under addition.

Let $G = \left\{ Z + \frac{m}{p^n} \;\middle|\; \begin{array}{l} m, n \text{ are integers}, \\ n \geq 0 \end{array} \; p = \text{fixed prime} \right\}$

Then G is a subgroup of $\frac{Q}{Z}$ where $\langle Q, + \rangle$ is the group of rationals under addition.

Now $p^n\left(Z + \frac{m}{p^n}\right) = Z + \frac{m}{p^n} p^n = Z + m = Z$ $= \text{Zero of } G$

$\Rightarrow$ order of $Z + \frac{m}{p^n}$ divides p^n

$\Rightarrow$ order of $Z + \frac{m}{p^n}$ is p^r, $r < n$

$\Rightarrow$ order of every element in G is finite.

Here G is an infinite group.

IAS-2013 2(a) What are the orders of the following permutations in S_{10}?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$$ and $(12345)(67)$

Ans. Let $S = \{1, 2, 3, 4, --, 10\}$, then $S_{10} = \{f \mid f : S \to S$ is a permutation on $S\}$ is a group with respect to permutation multiplication of order $10!$

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$

$= (2\ 8)(3\ 7\ 4)(5\ 10\ 9\ 6)(1)$

Thus, Order of α

$= O(\alpha) =$ lcm of $\{2, 3, 4\} = 12$

Let $\beta = (1\ 2\ 3\ 4\ 5)(6\ 7)$

Order of β

$= O(\beta)$

$=$ lcm of $\{5, 2\}$

$= 10$

$\Rightarrow O(\beta) = 10$

and $O(\alpha) = 12$

2(b) IAS -2018 P-II → What is the maximal possible order of an element in S_{10}? why? Give an example of such an element. How many elements will there be in S_{10} of that order?

Sol let $S = \{1,2,3,4,5,6,7,8,9,10\}$ then

$S_{10} = \{f \mid f : S \to S$ is a permutation on $S\}$

is a group w.r.t permxn. of order 8!

∴ The Maximum possible order of one of the perm. in S_{10} is given by l.c.m of disjoint cycle lengths and which are relatively prime.

let $f \in S_{10}$ then $o(f) = l.c.m(2,3,5)$
$= 30$.

i.e let $f = (12)(345)(6789)$
then $o(f) = 15$

(or)

If $\sigma \in S_{10}$, we can express σ as a product of disjoint cycles $\sigma_1, \sigma_2, \cdots \sigma_t$ of lengths $k_1, k_2, \cdots k_t$ ($k_i \geq 2$) where $k_1 + k_2 + \cdots + k_t \leq 10$, The order of σ is then $l.c.m(k_1, k_2, \cdots k_t)$.

Thus we have to find the maximum value of $lcm(k_1, k_2, \cdots k_t)$

over all sets of integers $\{k_1, k_2 \cdots k_t\}$ satisfying $k_i \geq 2$ and $k_1 + k_2 + \cdots + k_t \leq 10$

We may as well assume the k_i's are distinct from each other (since repeating one of them does not alter the lcm). This narrows the search considerably. We list the possibilities for $k_1 + k_2 + \cdots + k_t$ and the corresponding lcm:

$2+3+4$, lcm$(2,3,4) = 12$
$2+3+5$, lcm$(2,3,5) = 12$
$2+4$, lcm$(2,4) = 4$
$2+5$, lcm$(2,5) = 10$
$2+6$, lcm$(2,6) = 6$
$2+7$, lcm$(2,7) = 14$
$2+8$, lcm$(2,8) = 8$
$3+4$, lcm$(3,4) = 12$
$3+5$, lcm$(3,5) = 15$
$3+6$, lcm$(3,6) = 6$
$3+5$, lcm$(3,5) = 15$
$4+6$, lcm$(4,6) = 12$.

We see that the highest possible order is 30 and this is achieved by the product of a 2-cycle, a 3-cycle and 5-cycle; example: $(12)(345)(678910)$.

IAS' 2013

(5a) Let $J = \{a+ib \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers (subring of $\mathbb{C}$)
Which of the following is J: Euclidean domain, PID, UFD? Justify your answer.

Ans. Let $\alpha = a_1 + ib_1$, $\beta = a_2 + ib_2$ where $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ be any two arbitrary elements.

Now, $d(x) \leq d(xy)$ for $x, y \in Z[i]$ and $y \neq 0$, then $xy^{-1} \in Q[i]$, the field of quotients of $Z[i]$.

Say, $xy^{-1} = s + ti$, where $s, t \in Q$.

Now let m be the integer nearest s and let n be the integer nearest t.

(These integers may not be uniquely determined, but that does not matter).

Thus, $|m-s| \leq 1/2$ and $|n-t| \leq 1/2$.

Then, $xy^{-1} = s+ti = (m-m+s) + (n-n+t)i$
$$= (m+ni) + [(s-m) + (t-n)i]$$

We claim that the division condition of the definition of a Euclidean domain is satisfied with $q = m+ni$ and
$$r = [(s-m) + (t-n)i]y$$

Clearly, $q \in Z[i]$ and since $r = x - qy$, so does r.

Finally, $d(r) = d([(s-m) + (t-n)i])d(y)$
$$= [(s-m)^2 + (t-n)^2]\, d(y)$$
$$\leq \left(\frac{1}{4} + \frac{1}{4}\right) d(y) < d(y)$$

Thus, J is Euclidean domain.

Since, $ED \Rightarrow PID \Rightarrow UFD$

Since, J is ED, thus J is PID and J is UFD also.

IAS 2013 3(b). Let R^c = Ring of all the real valued continuous functions on the interval $[0,1]$, under the operations

$$(f+g)(x) = f(x)+g(x)$$
$$(fg)(x) = f(x)\,g(x)$$

Let $M = \{ f \in R^c \mid f(\tfrac{1}{2}) = 0 \}$

Is M a maximal ideal of R? Justify your answer.

Ans. Given that R be the ring of all the real valued continuous functions on the closed unit interval

i.e $R = \{ f \mid f:[0,1] \to \mathbb{R} : f$ is cts on $[0,1]\}$

where $\mathbb{R}$ denote the set of all real numbers.

Here, R is a ring w.r.t compositions

$$(f+g)(x) = f(x)+g(x)$$
$$(fg)(x) = f(x)\,g(x) \quad \forall x \in [0,1] \text{ and } f,g \in R.$$

Now we shall show that

$M = \{ f \in R \mid f(\tfrac{1}{2}) = 0 \}$ is a maximal ideal.

First of all we shall show that M is non-empty

As, the real valued function 'e' on $[0,1]$ defined by $e(x) = 0 \quad \forall x \in [0,1]$

$\therefore e \in M$

$\therefore M$ is non-empty subset of R.

Now let $f, g \in M$, then $f(\tfrac{1}{2}) = 0$, $g(\tfrac{1}{2}) = 0$

We have, $(f-g)(1/2) = f(1/2) - g(1/2)$
$= 0 - 0$
$= 0$

$\therefore f-g \in M$

$\therefore (M,+)$ is a subgroup of $(R,+)$

Let $f \in M$ and $h \in R$, then $f(1/2) = 0$

Now, we have,

$(fh)(1/2) = f(1/2)\,h(1/2) = 0 \cdot h(1/2) = 0$
$\Rightarrow fh \in M$

IIIly, $h f \in M$

$\therefore M$ is an Ideal of R.

Finally, we shall show that M is a maximal ideal of R.

Let us define a function $\theta: [0,1] \to \mathbb{R}$ such that

$\theta(x) = 1 \quad \forall\, x \in [0,1]$

then θ is a continuous function.

$\therefore \theta \in R$ (ring)

But, $\theta \notin M$ as $\theta(\frac{1}{2}) = 1 \neq 0$

$\therefore M \neq R$

Let U be any other ideal of R such that

$M \subset U \subseteq R$ and $M \neq U$

We need to show that $U = R$

Since $M \subset U$ and $M \neq U$

We need to show that $U = R$

$\exists$ a function $\lambda \in U$ s.t. $\lambda \notin M$ i.e. $\lambda(\frac{1}{2}) \neq 0$

Let $\lambda(\frac{1}{2}) = c \neq 0$

Let us define a function $\beta: [0,1] \to \mathbb{R}$ st

$\beta(x) = c \quad \forall\, x \in [0,1]$

then, $\beta \in R$

Let $\psi = \lambda - \beta$, then $\psi(\frac{1}{2}) = \lambda(\frac{1}{2}) - \beta(\frac{1}{2})$

$= c - c = 0$

$\Rightarrow \psi \in M$

$\Rightarrow \psi \in U$ as $M \subset U$

i.e. $\beta = \lambda - \psi \in U \quad (\because \lambda, \psi \in U)$

If γ be a function from $[0,1]$ to $\mathbb{R}$

st $\gamma(x) = \frac{1}{c} \; (c \neq 0)$

then $\gamma \in R$

Now, we have, $(\gamma\beta)(x) = \gamma(x)\beta(x)$

$= \frac{1}{c} \cdot c = 1 = \theta(x) \; \forall x$

$\Rightarrow \gamma\beta = \theta$

Since $\beta \in U$, $\gamma \in R$, $\therefore \gamma\beta \in U$

$\Rightarrow \theta \in U$

Hence M is maximal ideal of the ring R.

4(b) → (IAS-2013 P-II) Using Cauchy's residue theorem, evaluate the integral $I = \int_0^{\pi} \sin^4\theta \, d\theta$.

Sol^n: $I = \int_0^{2\pi} \frac{(1-\cos t)(1-\cos t)}{4} \frac{dt}{2}$

$\because 1-\cos 2\theta = 2\sin^2\theta$

$\frac{(1-\cos 2\theta)}{2} \frac{(1-\cos 2\theta)}{2}$

take $2\theta = t$

$= \frac{1}{8}\int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt$

let $z = e^{it}$

$dz = ie^{it}dt$

$dz = iz\,dt$

$\therefore \oint_C \frac{(z^2-2z+1)^2}{1+4z^2} dz$ etc.

2013 - P-II

Q.4.C. Minimize $z = 5x_1 - 4x_2 + 6x_3 - 8x_4$

Subject to

$$x_1 + 2x_2 - 2x_3 + 4x_4 \le 40$$

$$2x_1 - x_2 + x_3 + 2x_4 \le 8$$

$$4x_1 - 2x_2 + x_3 - x_4 \le 10$$

$$x_i \geq 0$$

Solution:-

As the objective function should be in Maximization type

maximization W = minimize $-z$.

$$= -5x_1 + 4x_2 - 6x_3 + 8x_4 + 0\cdot s_1 + 0\cdot s_2 + 0\cdot s_3$$

Subject to

$$x_1 + 2x_2 - 2x_3 + 4x_4 + s_1 = 40$$

$$2x_1 - x_2 + x_3 + 2x_4 + s_2 = 8$$

$$4x_1 - 2x_2 + x_3 - x_4 + s_3 = 10$$

$$x_i \geq 0 ; s_i \geq 0$$

where s_1, s_2, s_3 are slack Variables

Initial basic feasible solution:-

$(x_1, x_2, x_3, x_4, s_1, s_2, s_3) = (0, 0, 0, 0, 40, 8, 10)$

$W = 0$

	C_j	-5	4	-6	8	0	0	0		
C_B	basis	x_1	x_2	x_3	x_4	s_1	s_2	s_3	B	θ
0	s_1	1	2	-2	4	1	0	0	40	10
0	s_2	2	-1	1	2	0	1	0	8	4 ←
0	s_3	4	-2	1	-1	0	0	1	10	-
z_j	$\Sigma C_B a_{ij}$	0	0	0	0	0	0	0	0	
C_j	$c_j - z_j$	-5	4	-6	8 ↑	0	0	0		

entering : x_4

exiting : s_2

key element : 2

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	C_j	-5	4	-6	8	0	0	0		
CB	basis	x1	x2	x3	x4	S1	S2	S3	B	θ
0	S1	-3	4	-4	0	1	-2	0	24	6
8	x4	1	-1/2	1/2	1	0	1/2	0	4	-
0	S3	5	-5/2	3/2	0	0	1/2	1	14	-
Wj	$\Sigma C_B a_{ij}$	8	-4	4	8	0	4	0	32	
Cj	Cj-Wj	-13	8	-10	0	0	-4	0		
4	x2	-3/4	1	-1	0	1/4	-1/2	0	6	
8	x4	5/8	0	0	1	1/8	1/4	0	7	
0	S3	25/8	0	-1	0	5/8	-3/4	1	29	
Wj	$\Sigma C_B a_{ij}$	2	4	-4	8	2	0	0	80	
Cj	Cj-Zj	-7	0	-2	0	-2	0	0		

entering : x2
exiting : s1
key element : 4
$R_2 \to R_2 + 1/8 R_1$,
$R_3 \to R_3 + \frac{5}{8} R_1$,

$\therefore$ all $C_j \le 0$; optimality is obtained with

$x_1 = x_3 = 0$; $x_2 = 4$; $x_4 = 8$ with minimize $z = -80$

2014

• Find all optimal solutions of the following LPP by Simplex method

Maximize $z = 30x_1 + 24x_2$

Subject to $5x_1 + 4x_2 \le 200$

$x_1 \le 32$

$x_2 \le 40$

$x_1, x_2 \ge 0$

Solution:

Standard form:

Maximize $z = 30x_1 + 24x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3$

Subject to $5x_1 + 4x_2 + s_1 = 200$

$x_1 + s_2 = 32$

$x_2 + s_3 = 40$

$x_1, x_2, s_1, s_3, s_2 \ge 0$

where s_1, s_2, s_3 are slack variables ... (0, 0, 200, 32, 40)

P-II 2013

5(a) Form a partial differential equation by eliminating the arbitrary functions 'f' and 'g' from $z = yf(x) + xg(y)$.

Sol:- Given $z = yf(x) + xg(y)$ —— ①

Differentiating ① partially w.r.t 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) \quad \text{—— ②}$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y). \quad \text{—— ③}$$

Differentiating (3) w.r.t x, we have.

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \quad \text{—— ④}$$

From (2) & (3).

$$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right], \quad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

Substituting these values in eqⁿ ④,

we have.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{xg(y) + yf(x)\}$$

or. $$\boxed{xy\frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z}$$ — from ①

5(b) Reduce the equation $yr+(x+y)s+xt=0$ to canonical form and hence find its general solution.

IAS-2013

Sol^n Given $yr+(x+y)s+xt=0 \rightarrow (1)$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=y, S=x+y$ and $T=x$ So that

$$S^2-4RT=(x+y)^2-4xy=(x-y)^2>0 \text{ for } x\neq y.$$

ans so (1) is hyperbolic. The λ-quadratic equation

$R\lambda^2+S\lambda+T=0$ reduces to $y\lambda^2+(x+y)\lambda+x=0$

(or) $(y\lambda+x)(\lambda+1)=0$

So that $\lambda=-1, -x/y$. Then the corresponding characteristic equations are given by

$$\frac{dy}{dx}-1=0 \text{ and } \frac{dy}{dx}-(x/y)=0$$

Integrating these $y-x=c_1$ and $\frac{y^2}{2}-x^2/2=c_2$

In order to reduce one (1) to its canonical form, we choose.

$$u=y-x \text{ and } v=y^2/2-x^2/2 \rightarrow (2)$$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right), \text{ using (2)} \rightarrow (3)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\Big), \text{ using (2)} \rightarrow (4)$$

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right)$ using (3)

$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\right] = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - \frac{\partial z}{\partial v}$

$\Rightarrow -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right) - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$

$= -\left(-\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u}\right) - x\left(-\frac{\partial^2 z}{\partial u \partial v} - x\frac{\partial^2 z}{\partial v^2}\right) - \frac{\partial z}{\partial v}$ using (2)

$r = \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}$

Now, $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$

$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial v}\right)$ by (4)

$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}$

$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} + \frac{\partial z}{\partial v}$

$\Rightarrow \frac{\partial^2 z}{\partial u^2} + y\frac{\partial^2 z}{\partial u \partial v} + y\left(\frac{\partial^2 z}{\partial u \partial v} + y\frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial v}$.

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$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}$$

Also $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right)$ using (4)

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial}\left(y\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$\Rightarrow -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial u \partial u} - y\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}. \text{ using (2)}$$

$$\therefore\ s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2} \quad \to (7)$$

using (5), (6) and (7) in (1) we get

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right) +$$

$$(x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\}$$

$$+ x\left\{\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right\} = 0$$

(or) $\left\{4xy - (x+y)^2\right\}\dfrac{\partial^2 z}{\partial u \partial v} - y\dfrac{\partial z}{\partial v} + x\dfrac{\partial z}{\partial v} = 0$

(or) $(y-x)^2\dfrac{\partial^2 z}{\partial u \partial v} + (y-x)\dfrac{\partial z}{\partial v} = 0$

(or) $u^2 \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial v} = 0$ (or) $u \frac{\partial^2 z}{\partial v \partial v} + \frac{\partial z}{\partial v} = 0 \rightarrow (8)$

[$\because$ $u \neq 0$ and $y - x = u$, by (2)]

(8) is the required Canonical form of (1).

Solution of (8) multiplying both sides of (8) by v we get

$uv \left(\frac{\partial^2 z}{\partial u \partial v}\right) + v\left(\partial z/\partial v\right) = 0$ (or) $(uvDD' + vD')z = 0 \rightarrow (9)$

where $D \equiv \frac{\partial}{\partial u}$ and $D' \equiv \partial/\partial v$. To reduce (9) in to linear equation with constant coefficients, we take new variables x and y as follows

let $u = e^X$ and $v = e^Y$ so that $X = \log u$, $Y = \log v$ $\rightarrow (10)$

let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$ then (9) reduces to

$(D_1 D_1' + D_1')z = 0$ (or) $D_1'(D+1)z = 0$

Its general solution is

$z = e^{-X}\phi_1(Y) + \phi_2(X) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$

(or) $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$,

where ψ_1 and ψ_2 are arbitrary functions.

Ques:- 5(c) In an examination the number of candidate whose obtained marks between certain limits were given in the following table:

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Using Newton forward interpolation formula, find the number of students whose marks lie b/w 45 and 50.

Sol:- First of all we find or construct the comulative frequency table for the given data.

upper limit of the Class Interval	40	50	60	70	80
Commulative frequency	31	73	124	159	190

The difference table is:-

x marks	y c.f	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

we have $x_0 = 40$; $x = 45$, $h = 10$

$$u = \frac{45-40}{10} = 0.5$$

From Newton forward Difference formula –

$$y_0 = 31 \ , \ \Delta y_0 = 42 \ , \ \Delta^2 y_0 = 9 \ , \ \Delta^3 y_0 = -25$$

$$\Delta^4 y_0 = 37$$

$$f(x) = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$f(45) = 31 + (0.5)(42) + (0.5)\frac{(-0.5)}{2}\times 9 + \frac{(0.5)(-0.5)(-1.5)}{6}(-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}\times 37$$

$f(45) = 47.8673$

$f(45) = 48$ (approximately)

The number of student who obtained marks less than $45 = 48$.

Hence, the no. of student whose marks between $45 \,\&\, 50 = 73 - 48 = 25$

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5(d) IAS-2013 P-II

Prove that the necessary and sufficient condition that vortex lines may be at right angles to the streamlines are $u, v, w = \mu\left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right)$, where μ and ϕ are functions of x, y, z, t.

Sol^n: The differential equations of streamlines and vortex lines are respectively.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{—} (1)$$

and $$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad \text{—} (2)$$

(1) and (2) will intersect orthogonally iff

$$u\xi + v\eta + w\zeta = 0$$

$$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

But this is the condition that

$u\,dx + v\,dy + w\,dz$ is perfect differential

$$\Rightarrow u\,dx + v\,dy + w\,dz = \mu\, d\varphi$$

$$= \mu\left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right)$$

This $\Rightarrow u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$.

IAS-2013 P-II 6(a)

Solve $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$.

Sol^n:- Given equation is $(D+3D')(D-2D')z = x^2\sin(x+y)$ ——(1)

Its A.E. is $(m+3)(m-2)=0$ So that $m=-3,2$.

$\therefore$ CF $= \phi_1(y-3x)+\phi_2(y+2x)$,

ϕ_1, ϕ_2 being arbitorary function.

$$PI = \frac{1}{(2+3D')(D-2D')}x^2\sin(x+y)$$

$$= \frac{1}{D+3D'}\left\{\frac{1}{D-2D'}, x^2\sin(x+y)\right\}$$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx \quad \text{where } c=y+2x$$

$$= \frac{1}{D+3D'}\int x^2\sin(c-x)\,dx$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)-\int 2x\cos(c-x)dx\right.$$

Integrating by parts.

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)-\left\{-2x\sin(c-x)+\int 2\sin(c-x)\,dx\right\}\right]$$

integrating by parts.

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)+2x\sin(c-x)-2\cos(c-x)\right]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y)+2x\sin(x+y)\right]$$

as $c=y+2x$.

$$= \int\left[(x^2-2)\cos(x+c'+3x)+2x\sin(x+c'+3x)dx\right.$$

where $c'=y-3x$

$$= \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\frac{\sin(4x+c')}{4}dx + 2\int x\sin(4x+c')dx$$

[integrating by parts 1st integral and keeping the second integral unchanged]

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x \sin(4x+c')\,dx$$

$$= \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x\cos(4x+c')}{4} + \int \frac{\cos(4x+c')}{4}\,dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x)$$

$$(\because c' = y-3x)$$

$$= \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y).$$

on simplification.

The required solution is -

$$z = CF + PI$$

$$= \phi_1(y-3x) + \phi_2(y+2x) + \left[\frac{x^2}{4} - \frac{13}{32}\right]\sin(x+y) - \left(\frac{3x}{8}\right)\cos(x+y).$$

2013 Q.6(b) Find the surface which intersects the surfaces of the system $z(x+y) = c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1$, $z=1$?

Sol:- The given system of surfaces is given by

$$f(x,y,z) = \frac{z(x+y)}{3z+1} = c \quad \text{——— (1)}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{z}{3z+1} \; ; \; \frac{\partial f}{\partial y} = \frac{z}{3z+1} \; ; \; \frac{\partial f}{\partial z} = \frac{x+y}{(3z+1)^2}$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

or $$\frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $$z(3z+1)p + z(3z+1)q = x+y \quad \text{——— (2)}$$

Lagrange's auxillary equations for (2) are.

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \quad \text{——— (3)}$$

Taking the first two fractions of (3), we get

$$dx - dy = 0$$

so that $$x - y = c_1 \quad \text{——— (4)}$$

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction of (3)

$$= \frac{x\,dx + y\,dy - z(3z+1)dz}{xz(3z+1) + yz(3z+1) - z(3z+1)(x+y)}$$

$$= \frac{x\,dx + y\,dy - z(3z+1)dz}{(x+y)z(3z+1) - (x+y)z(3z+1)}$$

$$\Rightarrow x\,dx + y\,dy - z(3z+1)dz = 0 \quad \text{— Integrate it}$$

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{\cancel{3}z^3}{\cancel{3}} - \frac{1}{2}z^2 = \frac{1}{2}c_2$$

or $x^2+y^2-2z^3-z^2=C_2$ ——— (5)

Hence, any surface which is orthogonal to (1) has equation of the form

$$x^2+y^2-2z^3-z^2=\phi(x-y);$$ ϕ being an arbitrary function ——— (6)

In order to get the desired surface passing through circle $x^2+y^2=1$, $z=1$.

we must choose $\phi(x-y)=-2$

Thus, the required particular surface is

$$\underline{x^2+y^2-2z^3-z^2=-2}$$

6(c) IAS 2013 P-II → A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating by giving each point a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.

Solⁿ → Let a tightly stretched string be in equilibrium position along x-axis with the ends fixed at $x=0$ and $x=l$.

The vibrations of the string are governed by one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (A)}$$

where $y(x,t)$ is the deflection of the point of the string at distance x from the end $x=0$ at any time t.

Here we have to solve (A) subject to the following initial and boundary conditions.

Boundary conditions: $y(0,t)=0$ and $y(l,t)=0$ —— (B)

Initial Conditions: $y(x,0)=0$ —— (C)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda(lx-x^2)$ —— (D)

Since the vibration of the string are periodic, so the solution of (A) will be of the form

$$y(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 \cos c\lambda t + C_4 \sin c\lambda t) \quad \text{—— (1)}$$

Now proceeding and we get

$$C_1 = 0,\ \lambda = n\pi/l,\ C_n = 0 \text{ and}$$

$$y(x,t) = \sum_{n=1}^{\infty} D_n \sin(n\pi ct/l) \cdot \sin(n\pi x/l)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi ct}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda(lx - x^2)$$

$$\Rightarrow \lambda(lx - x^2) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

Obviously the right hand side is the fourier sine series of L.H.S.

$$\therefore \left(\frac{n\pi c}{l}\right) D_n = \frac{2}{l}\int_0^l \lambda(lx - x^2) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

or $$D_n = \frac{2\lambda}{n\pi c}\left[\left\{(lx - x^2)\left(\frac{-l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right)\right\}_0^l + \int_0^l (-2)\left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right) dx\right]$$

$$= \frac{2\lambda}{n\pi c}\left[0 - \frac{2l^3}{n^3\pi^3}\left\{\cos\frac{n\pi x}{l}\right\}_0^l\right]$$

$$= \frac{4\lambda l^3}{c\pi^4 n^4}(1 - \cos n\pi)$$

$= 0$ if n is even and $\dfrac{8\lambda l^3}{c\pi^4 n^4}$ if n is odd.

$$\therefore y(x,t) = \frac{8\lambda l^3}{c\pi^4}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4} \sin\frac{(2m-1)\pi ct}{l} \cdot \sin\frac{(2m-1)\pi x}{l}.$$

7(b)

Qus) Use Euler's method with step size $h = 0.15$ to compute the approximation value of $y(0.6)$, correct up to five decimal places from the initial value problem

$$y' = x(y+x) - 2 \quad \& \quad y(0) = 2.$$

Sol.) Let, $f(x,y) = xy + x^2 - 2$

Given; $x_0 = 0$, $y_0 = 2$, $h = 0.15$

So; $x_1 = x_0 + h = 0 + 0.15 = 0.15$

$x_2 = x_1 + h = 0.15 + 0.15 = 0.3$

$x_3 = x_2 + h = 0.3 + 0.15 = 0.45$

$x_4 = x_3 + h = 0.45 + 0.15 = 0.6$

$y_1 = y_0 + hf(x_0, y_0)$

$y_1 = 2 + 0.15\, f(0, 2) = 1.7000$

$y_2 = y_1 + hf(x_1, y_1)$

$= 1.70 + 0.15\, f(0.15, 1.70) = 1.44163$

$y_3 = y_2 + hf(x_2, y_2)$

$y_3 = 1.44163 + 0.15\, f(0.30, 1.44163)$

$\boxed{y_3 = 1.22000}$

$y_4 = y_3 + hf(x_3 + y_3)$

$y_4 = 1.22000 + 0.15\, f(0.45, 1.22000)$

$\boxed{y_4 = 1.03273}$

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7(c)

Q. The velocity of a train which starts from rest is given in the table. The time is in minutes and velocity in km/hr.

t	2	4	6	8	10	12	14	16	18	20
V	16	28.8	40	46.4	51.2	32	17.6	8	3.2	0

Estimate the total distance run in 20 minutes by using composite Simphson' 1/3 rule.

Sol: Now $h=2$

if $S \to$ Km is the distance covered in time 't' (min)

then $\frac{ds}{dt} = V$

$$S = \int_0^{20} v\,dt$$

Given set of data

$V_0 = 16$, $V_1 = 28.8$, $V_2 = 40$, $V_3 = 46.4$; $V_4 = 51.2$

$V_5 = 32$, $V_6 = 17.6$, $V_7 = 8$, $V_8 = 3.2$; $V_9 = 0$

Applying Simphson 1/3 formula

$$\int_0^{20} v\,dt = \frac{h}{3}\Big[(V_0+V_9) + 4(V_1+V_3+V_5+V_7) + 2(V_2+V_4+V_6+V_8)\Big]$$

$$= \frac{h}{3}[700.8] = 467.2 \text{ mtr} \cancel{\text{sec}}.$$

Hence, the total distance run in 20 minutes is 467.2 mtrs.

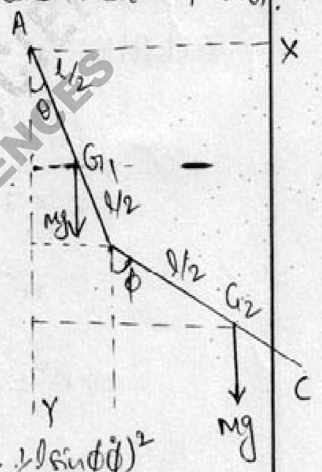
Q. Two equal rods AB and BC each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $2\pi/n$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

IAS-2013 P-I, 8(a)

Sol'n: Let AB and BC be the rods of equal length l and mass M.

At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by



$x_{G_1} = \frac{1}{2} l\sin\theta$; $y_{G_1} = \frac{1}{2} l\cos\theta$

$\therefore x_{G_2} = l\sin\theta = \frac{1}{2} l\sin\phi$, $y_{G_2} = l\cos\theta + \frac{1}{2} l\cos\phi$

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2, then

$$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\tfrac{1}{2} l\cos\theta\dot{\theta}\right)^2 + \left(-\tfrac{1}{2} l\sin\theta\dot{\theta}\right)^2$$

$$= -\tfrac{1}{4} l^2\dot{\theta}^2$$

$$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot{\theta} + \tfrac{1}{2} l\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \tfrac{1}{2} l\sin\phi\dot{\phi}\right)^2$$

$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right], \quad (\because \theta, \phi \text{ are small})$$

If T be the total K.E. and W the work function of the system, then

$$T = \text{K.E of rod AB} + \text{K.E of rod BC}$$

$$= \left[\tfrac{1}{2} M\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)^2\dot{\theta}^2 + \tfrac{1}{2}M\cdot v_{G_1}^2\right] + \left[\tfrac{1}{2}m\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)\dot{\phi}^2 + \tfrac{1}{2}M\cdot v_{G_2}^2\right]$$

$$= \tfrac{1}{2}Ml^2\left(\tfrac{4}{3}\dot{\theta}^2 + \tfrac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$$

and $W = Mg y_{G_1} + Mg y_{G_2} + c = Mg\left[\tfrac{1}{2}l\cos\theta + l\cos\theta + \tfrac{1}{2}\cos\phi\right] + c$

$$= \tfrac{1}{2}Mgl(3\cos\theta + \cos\phi)$$

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

i.e. $\dfrac{d}{dt}\left[\tfrac{1}{2}Ml^2\left(\tfrac{8}{3}\dot{\theta} + \dot{\phi}\right)\right] - 0 = \tfrac{1}{2}Mgl(-3\sin\theta) = -\tfrac{3}{2}Mgl\theta$ ($\because \theta$ is small)

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$\Rightarrow 8\ddot{\theta} + 3\ddot{\phi} = -9c\theta$, (where $c = g/l$) — (1).

equations (1) and (2) can be written as

$(8D^2+9c)\theta + 3D^2\phi = 0$ and $3D^2\theta + (2D^2+3c)\phi = 0$

Eliminating ϕ b/w these two equations, we get

$[(2D^2+3c)(8D^2+9c) - 9D^4] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$\theta = A\cos(nt+B) \quad \therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$

Substituting in (3) we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0 \quad \because \theta \neq 0.$

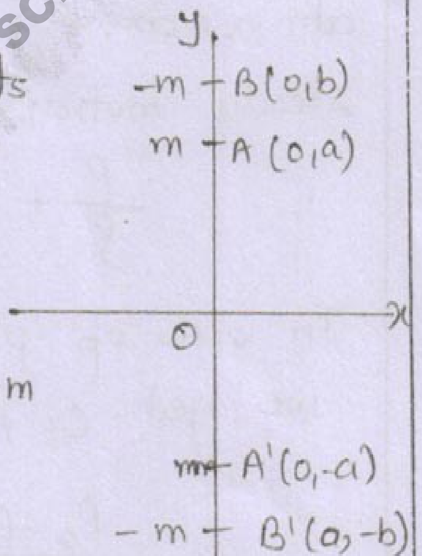
$$\therefore n^2 = \frac{42c \pm \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$$

$$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l} \quad (\because c = g/l).$$

Q (b) IAS-2013 P-II

If the fluid fills the region of space on the positive side of x-axis is a rigid boundry, and if there be a source $+m$ at the point $(0,a)$ and an equal sink at $(0,b)$ and if the pressure on the negative side of the boundry be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundry is $\pi\rho m^2(a-b)^2/ab(a+b)$, where ρ is the density of the fluid.

Sol^n The object system consists of source $+m$ at A $(0,a)$, i.e. at $z=ia$ and sink $-m$ at $z=ib$. The image system consist of source $+m$ at A' $(z=-ia)$ and sink $-m$ at B' $(z=-ib)$ w.r.t. the positive line OX which is rigid boundry.



The complex potential due to object system with rigid boundry is equivalent to the object system and its image system with no rigid boundry.

$$\therefore \quad w = -m\log(z-ia) + m\log(z-ib) - m\log(z+ia) + m\log(z+ib)$$

or $\quad w = -m\log(z^2+a^2) + m\log(z^2+b^2)$

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$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

for any point on x-axis, we have $z=x$ so that
$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x=\infty$. By Bernoulli's equation for steady motion.

$$\therefore \quad \frac{p}{\rho} + \frac{1}{2}q^2 = C$$

In view of $p=p_0$, $q=0$ when $x=\infty$, we get $C = p_0/\rho$.

$$\frac{p_0-p}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundary is given by,

$$P = \int_{-\infty}^{\infty}(p_0-p)\,dx = \int_{-\infty}^{\infty}\frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx.$$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

for any point on x-axis, we have $z = x$ so that
$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x = \infty$. By Bernoulli's equation for steady motion.

$$\therefore \quad \frac{p}{\rho} + \frac{1}{2}q^2 = C$$

In view of $p = p_0$, $q = 0$ when $x = \infty$, we get $C = p_0/\rho$.

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundary is given by,

$$P = \int_{-\infty}^{\infty} (p_0 - p)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty} \frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx.$$

$$= 4\rho m^2(a^2-b^2)^2\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

$$= 4m^2\rho\int_0^\infty \left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right\}-\frac{a^2}{(x^2+a^2)^2}-\frac{b^2}{(x^2+b^2)^2}\right]dx.$$

$$= 4m^2\rho\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b}-\frac{\pi}{2a}\right\}-\frac{\pi}{4a}-\frac{\pi}{4b}\right]$$

$$= \frac{\pi\rho m^2(a-b)^2}{ab(a+b)} \quad \text{(Ans.)}$$

for $\displaystyle\int_0^\infty \frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^\infty \cos^2\theta\,d\theta;\quad x = a\tan\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{\pi}{2}\cdot\frac{1}{2a^3}$$

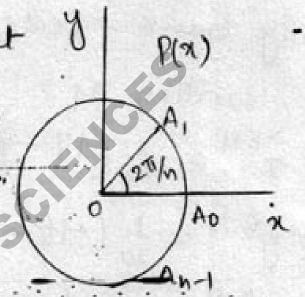
$$= \frac{\pi}{4a^3}$$

4(c) IFoS-2013 8(C) P-II

If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid.

Sol^n: from the fig., the n vortices are at $A_0, A_1, A_2 \ldots A_{n-1}$ such that

$\angle A_0OA_1 = \angle A_1OA_2 = \ldots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$



The coordinates of the points A_r are given by -

$$z = z_r = ae^{(2\pi/n)ir} \text{ where } r = 0, 1, 2 \ldots n-1.$$

These are n roots of the equation $z^n - a^n = 0$

$[\text{For } z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi ri}]$

Hence $z^n - a^n = (z-z_0)(z-z_1)\ldots(z-z_{n-1})$

The complex potential due to n vortices at P is given by

$$w = \frac{ik}{2\pi}\left[\log(z-z_0) + \log(z-z_1) + \ldots + \log(z-z_{n-1})\right]$$

$$= \frac{ik}{2\pi}\log(z-z_0)(z-z_1)\ldots(z-z_{n-1}) = \frac{ik}{2\pi}\log(z^n - a^n) \quad \text{--- (1)}$$

For the point A_0, $z = a$ so that $r = a, \theta = 0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\left[\log(z^n - a^n) - \log(z-a)\right]$$

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$$\phi' + i\psi' = \frac{ik}{2\pi}\left[\log(r^n e^{in\theta} - a^n) - \log(re^{i\theta} - a)\right]$$

$$\psi' = \frac{k}{4\pi}\left[\log(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta) - \log(r^2 + a^2 - 2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1} - 2nr^{n-1}a^n\cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta} - \frac{2r - 2a\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi}\left[\frac{2nr^n a^n \sin n\theta}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} - \frac{2ra\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a} = \frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right) - \left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right] = \frac{k}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta} - \frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\underset{x\to 0}{Lt}\ \frac{F(x)}{G(x)} = \underset{x\to 0}{Lt}\ \frac{F'(x)}{G'(x)} = \underset{x\to 0}{Lt}\ \frac{F''(x)}{G''(x)}$ $\left[\text{form } \frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta} - \frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta} - \frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}[0+0] = 0$$

finally $\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi a}(n-1)$, $\frac{\partial\psi'}{\partial\theta} = 0$ as $r\to a,\ \theta\to 0$.

Consequently, the velocity q_0 of the vortex A_0 is given by

$$q_0 = \left[\left(\frac{\partial\psi'}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2} = \frac{k(n-1)}{4\pi a}$$