Set - V

* The Laplace Transform *

Introduction:

Laplace transform or Laplace transformation is a method for solving linear differential equations and satisfying given boundary conditions without use of a general solution.

These particular solutions are the ones widely used in physics, mechanics, chemistry, medicine, national defence and many fields of pratical research. The knowledge of Laplace transforms in recent years has an essential part of mathematical background required for engineers and scientists.

* Integral Transform:

Let $K(P,t)$ be a function of two variables P and t, where P is a parameter (may be real (or) complex) independent of t. The function $f(P)$ defined by the integral (assume to be convergent)

$f(P) = \int_{-\infty}^{\infty} K(P,t) F(t)\,dt$, is called the integral transform of the function $F(t)$ and is denoted by $T\{F(t)\}$.

The function $K(P,t)$ is called the kernel of the transformation.

* Laplace Transform:

If the kernal $K(P,t)$ is defined as

$$K(P,t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-Pt} & \text{for } t \geq 0 \end{cases}$$

then $f(P) = \int_0^{\infty} e^{-Pt} F(t)\,dt$ ——— ①

$$f(P) = \int_{-\infty}^{\infty} K(P,t) F(t)\,dt = \int_{-\infty}^{0} (0) F(t)\,dt + \int_0^{\infty} e^{-Pt} \cdot F(t)\,dt = \int_0^{\infty} e^{-Pt} F(t)\,dt$$

the function $f(P)$ defined by the integral (1) is called the Laplace transform of the function $F(t)$ and is denoted by $L\{F(t)\}$ (or) $\bar{F}(P)$ (or) $L[F(t)]$

i.e. $L[F(t)] = \int_0^{\infty} e^{-Pt} F(t)\,dt$

Thus Laplace transform is a

function of a new variable (or Parameter) 'P' given by (1).

Note: The Laplace transform of $F(t)$ is said to exist if the integral (1) converges for some values of P, otherwise it does not exist.

* Linearity Property of Laplace Transformation:

A transformation T is said to be linear if for every pair of functions $F_1(t)$ and $F_2(t)$ and for every pair of constants a_1 and a_2.

we have

$$T\{a_1F_1(t)+a_2F_2(t)\}=a_1T\{F_1(t)\}+a_2T\{F_2(t)\}$$

→ The Laplace transformation is a linear transformation.

i.e, $L\{a_1F_1(t)+a_2F_2(t)\}$

$$=a_1L\{F_1(t)\}+a_2L\{F_2(t)\}$$

where a_1 & a_2 are constants.

Sol'n: we have

$$L\{F(t)\}=\int_0^\infty e^{-Pt}F(t)\,dt$$

$\therefore L\{a_1F_1(t)+a_2F_2(t)\}$

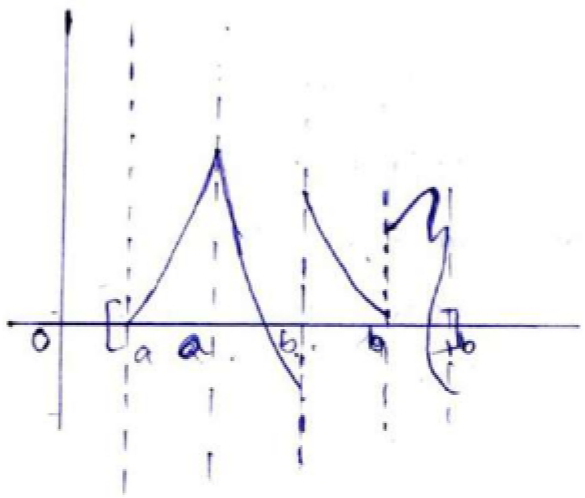
$$=\int_0^\infty e^{-Pt}\{a_1F_1(t)+a_2F_2(t)\}dt$$

$$=a_1\int_0^\infty e^{-Pt}F_1(t)dt+a_2\int_0^\infty e^{-Pt}F_2(t)dt$$

$$=a_1L\{F_1(t)\}+a_2L\{F_2(t)\}$$

→ Piecewise (or) Sectionally Continuous Function:

A function $F(t)$ is said to be piecewise (or sectionally) continuous on $t\in[a,b]$, if it defined on that interval and is such that the interval can be subdivided into finite number of intervals, in each of which $F(t)$ is continuous and has finite right and left hand limits.



Existence of Laplace Transform:

If $F(t)$ is a function which is piecewise continuous on every finite interval in the range $t \geq 0$ and satisfies

$|F(t)| \leq Me^{at}$ for all $t \geq 0$ and for some constants a and M, then the laplace transform of $F(t)$ exists for all $p > a$.

Proof: we have

$$L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$= \int_0^{t_0} e^{-pt} F(t)\,dt + \int_{t_0}^\infty e^{-pt} F(t)\,dt \quad \text{---} (1)$$

The integral $\int_0^{t_0} e^{-pt} F(t)\,dt$ exists since $F(t)$ is piecewise continuous on every finite interval $0 \leq t \leq t_0$.

Now $\left|\int_{t_0}^\infty e^{-pt} F(t)\,dt\right| \leq \int_{t_0}^\infty |e^{-pt} F(t)|\,dt$

$$\leq \int_{t_0}^\infty e^{-pt} Me^{at}\,dt$$

since $|F(t)| \leq Me^{at}$

$$= \int_{t_0}^\infty e^{-(p-a)t} M\,dt$$

$$= \left.\frac{-e^{(p-a)t} M}{(p-a)}\right|_{t_0}^\infty$$

$$= \frac{Me^{-(p-a)t_0}}{p-a}, \; p > a$$

$$\therefore \left|\int_{t_0}^\infty e^{-pt} F(t)\,dt\right| \leq \frac{Me^{-(p-a)t_0}}{p-a}; \; p > a$$

But $\frac{Me^{-(p-a)t_0}}{p-a}$ can be made as small as we please by taking t_0 sufficiently large.

Thus from (1), we conclude that $L\{F(t)\}$ exists for all $p > a$.

Note (1): Above theorem of existence of laplace transform can also be stated as:

"If $F(t)$ is a function which is piece-wise continuous on every finite interval in the range $t \geq 0$ and is of exponential order 'a' as $t \to \infty$, the laplace transform of $F(t)$ exists for all $p > a$".

(or)

"If $F(t)$ is a function of class A, the laplace transform of $F(t)$ exists for all $p > a$".

Note(2): Conditions in the above theorem are sufficient but not necessary for the existence of the Laplace transform. If these conditions are satisfied, the Laplace transform must exist.

If these conditions are not satisfied, the Laplace transform may or may not exist.

For eg: Consider the function

$$F(t) = \frac{1}{\sqrt{t}}$$

Here $F(t) \to \infty$ as $t \to 0$ from the right. Thus the function $F(t)$ is not piece wise continuous on every finite interval in the range $t \geq 0$.

But $F(t)$ is integrable from 0 to any positive value t_0.

Also $|F(t)| < Me^{at}$ for all $t > 1$ with $M = 1$ and $a = 0$.

Now $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$= \int_0^\infty e^{-pt} \frac{1}{\sqrt{t}}\,dt$$ which is converge for $p > 0$.

$$= \frac{2}{\sqrt{p}} \int_0^\infty e^{-x^2}\,dx$$

Putting $\sqrt{pt} = x \Rightarrow pt = x^2$

$\sqrt{p}\,\frac{1}{2\sqrt{t}}\,dt = dx$

$\Rightarrow \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{p}}\,dx$

Limits of x: if $t = 0 \Rightarrow x = 0$; $t = \infty \Rightarrow x = \infty$

$$= \frac{2}{\sqrt{p}} \cdot \frac{\sqrt{\pi}}{2}$$

$\left(\because \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\right)$

$$= \sqrt{\frac{\pi}{p}}, \quad p > 0$$

$\therefore L\left\{\frac{1}{\sqrt{t}}\right\}$ exists for $p > 0$ even if $F(t) = \frac{1}{\sqrt{t}}$ is not piece-wise continuous in the range $t \geq 0$.

* Functions of Exponential order:

A function $F(t)$ is said to be of exponential order α as $t \to \infty$ if there exists a +ve constant (real) M, a number α and a finite number 't_0' such that

$$|F(t)| < Me^{\alpha t} \quad \forall\ t \geq t_0.$$

(or) $|e^{-\alpha t} F(t)| < M \quad \forall\ t \geq t_0$.

If a function $F(t)$ is of exponential order α, it also of β, $\beta > \alpha$ (OR)

A function $F(t)$ is said to be exponential order α as $t \to \infty$ if $\lim_{t\to\infty} e^{-\alpha t} F(t) = $ finite quantity.

* Function of Class A:

A function $F(t)$ is said to be function of class A if (i) it is piecewise (or sectionally) continuous on every finite interval in the range $t \geq 0$.

(ii) $F(t)$ is of exponential order as $t \to \infty$

→ Prove that $F(t) = t^n$ is of exponential order as $t \to \infty$, 'n' being any +ve integer.

Sol^n: $\underset{t\to\infty}{Lt}\{e^{-at}F(t)\} = \underset{t\to\infty}{Lt}\, e^{-at}t^n,\ a>0$

$= \underset{t\to\infty}{Lt} \frac{t^n}{e^{at}}$, ($\frac{\infty}{\infty}$ form)

$= \underset{t\to\infty}{Lt} \frac{n!}{a^n e^{at}}$ (by L.Hospital rule)

$= \frac{n!}{\infty} = 0$

$\therefore \underset{t\to\infty}{Lt}\, e^{-at}t^n = 0 =$ finite number

$\therefore t^n$ is of exponential order as $t\to\infty$

Note: $|F(t)| = t^n < e^{nt}\ \forall\ t>0$.

$\therefore$ The given function is of exponential order n.

Ex: show that t^2 is of exponential order 3.

Sol^n: we have

$\underset{t\to\infty}{Lt}\{e^{-at}F(t)\} = \underset{t\to\infty}{Lt}\left(\frac{t^2}{e^{at}}\right)$

$= \underset{t\to\infty}{Lt} \frac{2t}{ae^{at}}$ (By L-Hospital's rule)

$= \underset{t\to\infty}{Lt} \frac{2}{a^2e^{at}}$ (By L-Hospital rule)

$= 0$, if $a>0$

$\therefore F(t) = t^2$ is of exponential order.

Now $|t^2| = t^2 < e^{2t} < e^{3t}\ \forall\ t>0$

$\therefore$ The given function is of exponential order 3. ($\because$ if F(t) is of exponential Order $\alpha=2$ it is also of $\beta=3$, $3>2$)

→ show that the function e^{t^2} is not of exponential order as $t\to\infty$

Sol^n: we have

$\underset{t\to\infty}{Lt}\{e^{-at}F(t)\} = \underset{t\to\infty}{Lt}\{e^{-at}e^{t^2}\}$

$= \underset{t\to\infty}{Lt}\, e^{t(t-a)}$

$= \infty\ \forall a$.

Hence whatever be the value of a, we cannot find a number M such that $e^{t^2} < Me^{at}$.

$\therefore$ The given function is not of exponential order as $t\to\infty$.

→ Find the Laplace transform of the function $F(t) = 1$

Sol^n: we have $L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$

$$\therefore L\{1\} = \int_0^\infty e^{-pt} \cdot 1\, dt$$

$$= \left[\frac{-e^{-pt}}{p}\right]_0^\infty \quad (\because e^{-pt} \to 0 \text{ as } t\to\infty)$$

$$= \frac{1}{p}, \; p>0$$

Note: Here the condition $p>0$ is necessary. Since the integral is convergent for $p>0$ and divergent for $p\le 0$.

→ Find $L\{t^n\}$, n is +ve integer.

Sol^n: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$$\therefore L\{t^n\} = \int_0^\infty e^{-pt}\, t^n\, dt$$

Integrating by parts

$$= \left[-\frac{1}{p} t^n e^{-pt}\right]_0^\infty + \frac{1}{p}\int_0^\infty n t^{n-1} e^{-pt}\, dt$$

$$= -\frac{1}{p} \lim_{t\to\infty} \frac{t^n}{e^{pt}} + 0 + \frac{n}{p}\int_0^\infty e^{-pt} t^{n-1}\, dt$$

$$= 0 + \frac{n}{p}\int_0^\infty e^{-pt}\, t^{n-1}\, dt$$

$\left(\because \lim_{t\to\infty} \frac{t^n}{e^{pt}} = 0\right.$ by L-Hospital's rule$\left.\right)$

$$= \frac{n}{p}\int_0^\infty e^{-pt}\, t^{n-1}\, dt$$

Proceeding similarly, we get

$$L\{t^n\} = \frac{n!}{p^n}\int_0^\infty e^{-pt}\, dt$$

$$= \frac{n!}{p^n}\left[-\frac{e^{-pt}}{p}\right]_0^\infty$$

$$= -\frac{n!}{p^n}\left[0 - \frac{1}{p}\right]$$

$$= \frac{n!}{p^{n+1}}, \; p>0.$$

→ Show that the Laplace transform of the function $F(t) = t^n$, $-1<n<0$, exists, although it is not a function of class A.

Sol^n: Given $F(t) = t^n$, $-1<n<0$ $\quad \left| \begin{array}{l} n=-\frac{1}{2} \\ F(t) = \frac{1}{\sqrt{t}} \end{array}\right.$

Here $F(t)\to\infty$ as $t\to 0$ (for $t\ge 0$)

i.e, the function is not piecewise continuous on every finite interval in the range $t\ge 0$.

we have $\lim_{t\to\infty}\{e^{-at}F(t)\} = \lim_{t\to\infty}\left(\frac{t^n}{e^{at}}\right)$

$$= \lim_{t\to\infty} \frac{1}{t^{-n} e^{at}}$$

$$= \lim_{t\to\infty} \frac{1}{t^m e^{at}}, \quad \text{where } 0<m<1$$

$= 0$, if $a>0$.

$-1<n<0$

$1>-n>0$

$1>m>0$

put $m=-n$

$\therefore F(t) = t^n$ is of exponential order.

Since $F(t) = t^n$ is not sectionally continuous over every finite interval in the range $t \geq 0$.

$\therefore$ It is not a function of class A.

But t^n is integrable from 0 to any +ve number t_0.

Now $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$

$= \int_0^\infty e^{-pt} t^n\, dt$

$= \int_0^\infty e^{-x} \left(\frac{x}{p}\right)^n \frac{1}{p} dx$; Putting $pt = x$, $p\,dt = dx$ $\Rightarrow dt = \frac{dx}{p}$ & taking $p>0$

$= \frac{1}{p} \cdot \frac{1}{p^n} \int_0^\infty e^{-x} x^n\, dx$

$= \frac{1}{p^{n+1}} \int_0^\infty e^{-x} x^{(n+1)-1} dx$

(By definition of Gamma function $\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ if $n>0$)

$= \frac{\Gamma(n+1)}{p^{n+1}}$, if $p>0$ and $n+1>0$ i.e., $n>-1$.

Hence Laplace transform of t^n, $0>n>-1$ exists, although it is not a function of class A.

→ Find $L\{e^{at}\}$.

Sol'n: Here $L\{e^{at}\} = \int_0^\infty e^{-pt} e^{at}\, dt$

$= \int_0^\infty e^{-(p-a)t} dt$

$= \left[\frac{-e^{-(p-a)t}}{p-a}\right]_0^\infty$, $p>a$

$= \frac{1}{p-a}$

→ Find $L\{\cos at\}$ and hence obtain $L(\sin^2 at)$

Sol'n: $L\{\cos at\} = \int_0^\infty e^{-pt} \cos at\, dt$

$$\boxed{\begin{array}{l}\int e^{ax}\sin bx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \\ \int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}\end{array}}$$

$= \left[\frac{e^{-pt}}{p^2+a^2}(-p\cos at + a\sin at)\right]_0^\infty$

$= 0 - \left(\frac{-p}{p^2+a^2}\right)$, $p>0$ $\quad (\because$ as $t\to\infty$, $e^{-pt}\to 0)$

$= \frac{p}{p^2+a^2}$, $p>0$ —— ①

Now $L(\sin^2 at) = L\left\{\frac{1-\cos 2at}{2}\right\}$

$= \frac{1}{2} L\{1 - \cos 2at\}$

$= \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos(2a)t\}$

$= \frac{1}{2p} - \frac{1}{2} \frac{p}{p^2+(2a)^2}$ $(\because$ by ①$)$

$$= \frac{1}{2p} - \frac{p}{2(p^2+4a^2)}$$

$$= \frac{2a^2}{p(p^2+4a^2)}$$

→ Find $L\{\cosh at\}$

Sol'n: $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\frac{1}{p-a} + \frac{1}{2}\frac{1}{p+a}$$

$$= \frac{p}{p^2-a^2}, \ p>a \ \& \ p>-a$$

i.e. $p>|a|$

i.e. $|a|<p$.

→ Find (1) $L\{\sin at\}$ (2) $L\{\sinh at\}$

→ Find $L\{\sin t \cos t\}$

Sol'n: Given $L\{\sin t\cos t\} = L\{\frac{1}{2}\sin 2t\}$

$$= \frac{1}{2}L\{\sin 2t\}$$

$$= \frac{1}{2}\frac{2}{p^2+4}, p>0$$

$$= \frac{1}{p^2+4}, p>0.$$

→ Find (1) $L\{\cosh^2 2t\}$

(2) $L\{7e^{2t}+9e^{-2t}+5\cos t+7t^3+5\sin 3t+2\}$

→ Find $L\{F(t)\}$, where $F(t)=\begin{cases}0, & 0<t<1\\ t, & 1<t<2\\ 0, & t>2\end{cases}$

Sol'n: Here $F(t)$ is not defined at $t=0$, $t=1$ & $t=2$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt$$

$$= \int_0^1 e^{-pt}\cdot 0\,dt + \int_1^2 e^{-pt}t\,dt + \int_2^\infty e^{-pt}\cdot 0\,dt$$

$$= \int_1^2 e^{-pt}t\,dt$$

$$= \left[-t\frac{e^{-pt}}{p}\right]_1^2 - \int_1^2 \frac{e^{-pt}}{-p}dt$$

$$= -\frac{2}{p}e^{-2p} + \frac{e^{-p}}{p} - \left[\frac{e^{-pt}}{p^2}\right]_1^2$$

$$= -\frac{2}{p}e^{-2p} + \frac{e^{-p}}{p} - \frac{e^{-2p}}{p^2} + \frac{e^{-p}}{p^2}$$

$$= \left(\frac{1}{p}+\frac{1}{p^2}\right)e^{-p} - \left(\frac{1}{p^2}+\frac{2}{p}\right)e^{-2p}$$

→ Find the L.T. of the function $F(t)$, where $F(t)=\begin{cases}4, & 0<t<1\\ 3, & t>1\end{cases}$

→ Find the L.T of the function $F(t)$, where $F(t)=\begin{cases}2t & 0\le t\le 5\\ 1 & t>5\end{cases}$

→ Find the L.T of the function $F(t)$, where $F(t)=\begin{cases}\sin t, & 0<t<\pi\\ 0, & t>\pi\end{cases}$

→ Find $L\{\sin\sqrt{t}\}$

Sol'n: we have

$$L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} \cdots\right\}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

$= L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \cdots$

$= \frac{\Gamma 3/2}{p^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{p^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{p^{7/2}} - \frac{1}{7!}\frac{\Gamma 9/2}{p^{9/2}} + \cdots$

$\left[\because L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}} \text{ if } n > -1\right]$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{p^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{7/2}} - \cdots$

$\left[\because \Gamma n+1 = n\Gamma n \right.$
$= n\Gamma(n-1)+1$
$= n(n-1)\Gamma(n-1)$
$\left.= n(n-1)(n-2)\Gamma(n-2) \text{ etc}\right]$

$= \frac{\frac{1}{2}\sqrt{\pi}}{p^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{5/2}} + \frac{1}{\cancel{120}\ 248}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{p^{7/2}} + \cdots$

$(\because \Gamma\frac{1}{2} = \sqrt{\pi})$

$= \frac{\sqrt{\pi}}{2p^{3/2}}\left[1 - \frac{1}{4p} + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \frac{1}{3!}\left(\frac{1}{4p}\right)^3 + \cdots\right]$

$= \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$

→ Find the L.T of the function $F(t) = (\sin t - \cos t)^2$.

→ Find the L.T of the function $F(t) = \frac{e^{at}-1}{a}$.

→ Evaluate $L\{F(t)\}$, if
$F(t) = \begin{cases}(t-1)^2, & t>1\\ 0, & 0<t<1\end{cases}$

→ Find $L\{F(t)\}$, if $F(t) = \begin{cases} e^t, & 0<t\le 1\\ 0, & t>1\end{cases}$

→ Prove that $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{p}}$.

* Laplace Transforms of some elementry Functions:

(1) $L\{1\} = \frac{1}{p}$, $p>0$

(2) $L\{t^n\} = \frac{n!}{p^{n+1}}$, $p>0$, where n is +ve integer.

(3) $L\{t^n\} = \frac{\Gamma n+1}{p^{n+1}}$, $p>0$, where $n>-1$

(4) $L\{e^{at}\} = \frac{1}{p-a}$, $p>a$

(5) $L\{\sin at\} = \frac{a}{p^2+a^2}$, $p>0$

(6) $L\{\cos at\} = \frac{p}{p^2+a^2}$, $p>0$

(7) $L\{\sinh at\} = \frac{a}{p^2-a^2}$, $p>|a|$ i.e. $|a|<p$

(8) $L\{\cosh at\} = \frac{p}{p^2-a^2}$, $p>|a|$ i.e. $|a|<p$

Note: If n is +ve integer, then $\Gamma n+1 = n!$

* First Translation (or) shifting Theorem:

If $L\{F(t)\} = f(p)$, then
$L(e^{at}F(t)) = f(p-a)$.

Proof: By definition of Laplace Transform, we have

$$L\{F(t)\} = f(P)$$
$$= \int_0^\infty e^{-Pt} F(t)\,dt$$

$$\therefore f(P-a) = \int_0^\infty e^{-(P-a)t} F(t)\,dt$$
$$= \int_0^\infty e^{-Pt}(e^{at}F(t))\,dt$$
$$= L\{e^{at} F(t)\}$$

→ * **Second translation (or) shifting theorem:**

If $L\{F(t)\} = f(P)$

and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$ then

$$L\{G(t)\} = e^{-aP} f(P).$$

Proof: By definition of Laplace transformation, we have

$$L(G(t)) = \int_0^\infty e^{-Pt} G(t)\,dt$$

$$= \int_0^a e^{-Pt} G(t)\,dt + \int_a^\infty e^{-Pt} G(t)\,dt$$

$$= \int_0^a e^{-Pt}\cdot 0\,dt + \int_a^\infty e^{-Pt} F(t-a)\,dt$$

$$= \int_a^\infty e^{-Pt} F(t-a)\,dt$$

Putting $t-a=x$ $\Rightarrow t = a+x$ $\Rightarrow dt = dx$

$$= \int_0^\infty e^{-P(a+x)} F(x)\,dx$$

$$= e^{-Pa}\int_0^\infty e^{-Px} F(x)\,dx$$

$$= e^{-Pa}\int_0^\infty e^{-Pt} F(t)\,dt$$

$\left(\because \int_0^\infty e^{-Px} F(x)\,dx = \int_0^\infty e^{-Pt} F(t)\,dt\right.$ By property of definite integrals, i.e., $\left.\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$

$$= e^{-Pa} L\{F(t)\}$$

$$= e^{-Pa} f(P)$$

→ * **Change of scale Property**

Theorem: If $L\{F(t)\} = f(P)$, then

$$L\{F(at)\} = \frac{1}{a} f(P/a)$$

Proof: By definition, we have

$$L\{F(at)\} = \int_0^\infty e^{-Pt} F(at)\,dt$$

Putting $at = x$ $\Rightarrow t = x/a$ $\Rightarrow dt = \frac{1}{a}dx$

$$= \int_0^\infty e^{-P(x/a)} F(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(P/a)x} F(x)\,dx$$

$$= \frac{1}{a}\int_0^\infty e^{-(P/a)t} F(t)\,dt$$

$\left(\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$

$$= \frac{1}{a} f(P/a).$$

→ Find $L\{t^3 e^{-3t}\}$ i.e. $L\{\underbrace{e^{-3t}}_{e^{at}}\underbrace{t^3}_{F(t)}\}$

clearly which is in the form $L\{e^{at}F(t)\}$

Sol'n: Now $L\{F(t)\} = L\{t^3\}$

$$= \frac{3!}{P^4} = \frac{6}{P^4} = f(P) \text{ (say)}$$

∴ from first shifting theorem

$$L\{e^{at}F(t)\} = f(P-a)$$

$$L\{t^3 e^{-3t}\} = \frac{6}{(P+3)^4} \qquad \boxed{a=-3}$$

→ Find $L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$

sol'n: we have

$$L\{3\cos 6t - 5\sin 6t\} = 3L\{\cos 6t\} - 5L\{\sin 6t\}$$

$$= 3\cdot\frac{P}{P^2+36} - 5\cdot\frac{6}{P^2+36}$$

$$= \frac{3P-30}{P^2+36} = f(P) \text{ say}$$

∴ from first shifting theorem, we have

$$L\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(P+2)$$

$$= \frac{3(P+2)-30}{(P+2)^2+36} \qquad \boxed{a=-2}$$

If $L\{F(t)\} = f(P)$ then $L\{e^{at}F(t)\} = f(P-a)$

$$= \frac{3P-24}{P^2+4P+40}$$

(OR)

By definition of L.T

$$L\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$$

$$= \int_0^\infty e^{-Pt}e^{-2t}(3\cos 6t - 5\sin 6t)\,dt$$

$$= 3\int_0^\infty e^{-(P+2)t}\cos 6t - 5\int_0^\infty e^{-(P+2)t}\sin 6t\,dt.$$

$$= \left\{3\cdot\frac{e^{-(P+2)t}}{(P+2)^2+6^2}\left[-(P+2)\cos 6t + 6\sin 6t\right]\right\}_0^\infty$$

$$- \left[5\,\frac{e^{-(P+2)t}}{(P+2)^2+6^2}\left\{-(P+2)\sin 6t - 6\cos 6t\right\}\right]_0^\infty$$

$$= 0 + \frac{3(P+2)}{(P+2)^2+6^2} - 5\left[0 - \frac{(-6)}{(P+2)^2+6^2}\right]$$

$$= \frac{3(P+2)}{(P+2)^2+36} - \frac{30}{(P+2)^2+36} = \frac{3P-24}{P^2+4P+40}$$

→ Find $L\{e^t(3\sinh 2t - 5\cosh 2t)\}$

→ Find (i) $L\{e^t\sin^2 t\}$ (ii) $L(e^t(t+3)^2)$

→ Find $L\left\{\frac{e^{-at}\,t^{n-1}}{(n-1)!}\right\}$, where n is +ve integer.

sol'n: we have

$$L\left\{\frac{t^{n-1}}{(n-1)!}\right\} = \frac{(n-1)!}{P^n(n-1)!} \qquad \left(\because L(t^n) = \frac{n!}{P^{n+1}}\right)$$

$$= \frac{1}{P^n} = f(P), \text{ say}$$

$$\therefore L\left\{e^{-at}\frac{t^{n-1}}{(n-1)!}\right\} = \frac{1}{(P+a)^n}$$

→ Applying change of scale Property, Obtain the Laplace transform of (i) $\sinh 3t$ (ii) $\cos 5t$

sol'n: (i) we have $L\{\sinh t\} = \frac{1}{P^2-1}$

$$= f(P), \text{ say}$$

$$\therefore L\{\sinh 3t\} = \tfrac{1}{3}f(P/3) \qquad \because L(F(t)) = f(P),\ L\{F(at)\} = \frac{1}{a}f\left(\frac{P}{a}\right)$$

$$= \frac{1}{3}\,\frac{1}{(p/3)^2-1}$$

$$= \frac{3}{p^2-9}$$

→ Given $L\{F(t)\} = \dfrac{p^2-p+1}{(2p+1)^2(p-1)}$; applying the change of scale property show that

$$L\{F(2t)\} = \frac{p^2-2p+4}{4(p+1)^2(p-2)}$$

→ Find $L\{G(t)\}$, where $G(t) = \begin{cases} e^{t-a}, & t>a \\ 0, & t<a \end{cases}$

Sol'n: From second shifting theorem we know that if $L\{F(t)\} = f(p)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

then $L\{G(t)\} = e^{-ap} f(p)$.

Here let $F(t) = e^t$

$$\therefore L\{F(t)\} = L\{e^t\} = \int_0^\infty e^{-pt} e^t\, dt$$

$$= \int_0^\infty e^{-(p-1)t}\, dt$$

$$= \left[\frac{e^{-(p-1)t}}{-(p-1)}\right]_0^\infty$$

$$= \frac{1}{p-1}, \quad p>1$$

$$= f(p), \text{ say}$$

and $G(t) = \begin{cases} F(t-a) = e^{t-a}, & t>a \\ 0, & t<a \end{cases}$

$$\therefore L\{G(t)\} = e^{-ap} f(p)$$

$$= \frac{e^{-ap}}{p-1}, \quad p>1$$

(OR)

we have $L\{G(t)\} = \int_0^\infty e^{-pt} G(t)\, dt$

$$= \int_0^a e^{-pt} G(t)\, dt + \int_a^\infty e^{-pt} G(t)\, dt$$

$$= \int_0^a e^{-pt}(0)\, dt + \int_a^\infty e^{-pt} e^{t-a}\, dt$$

$$= e^{-a}\int_a^\infty e^{-(p-1)t}\, dt$$

$$= e^{-a}\left[\frac{e^{-(p-1)t}}{-(p-1)}\right]_a^\infty$$

$$= e^{-a}\left[0 - \frac{e^{-(p-1)a}}{-(p-1)}\right]$$

$$= \frac{e^{-ap}}{p-1}$$

→ Find $L\{F(t)\}$, where

$$F(t) = \begin{cases} \cos\left(t-\frac{2}{3}\pi\right), & t>\frac{2\pi}{3} \\ 0, & t<\frac{2\pi}{3} \end{cases}$$

→ Find $L\{G(t)\}$, where

$$F(t) = \begin{cases} \sin\left(t-\frac{\pi}{3}\right), & t>\frac{\pi}{3} \\ 0, & t<\frac{\pi}{3} \end{cases}$$

*Laplace Transform of Derivatives:

Theorem: Let $F(t)$ be continuous for all $t\geq 0$ and be of exponential order 'a' as $t\to\infty$ and if $F'(t)$ is of class A, then Laplace transform of the derivative $F'(t)$ exists when $p>a$, and $L\{F'(t)\} = pL\{F(t)\} - F(0)$.

Proof:

Case(i) $F'(t)$ is continuous for all $t\geq 0$, then

$$L\{F'(t)\} = \int_0^\infty e^{-pt} F'(t)\,dt \quad \text{——} (1)$$

$$= \left[e^{-pt}F(t)\right]_0^\infty + \int_0^\infty pe^{-pt}F(t)\,dt$$

(Integrating by parts)

$$= \lim_{t\to\infty} e^{-pt}F(t) - F(0) + p\int_0^\infty e^{-pt}F(t)\,dt$$

$$L\{F'(t)\} = \lim_{t\to\infty} e^{-pt}F(t) - F(0) + pL\{F(t)\} \quad \text{——} (2)$$

Since $F(t)$ is continuous for all $t\geq 0$ and is of exponential order 'a' as $t\to\infty$.

$|F(t)| \leq Me^{at}$ $\forall$ $t\geq 0$ and for some constants a and M,

we have
$$|e^{-pt}F(t)| = e^{-pt}|F(t)|$$
$$\leq e^{-pt}Me^{at}$$
$$= Me^{-(p-a)t} \to 0 \text{ as } t\to\infty \text{ if } p>a$$

$\therefore \lim_{t\to\infty} e^{-pt}F(t) = 0$ for $p>a$.

$\therefore$ from (2) we conclude that $L\{F'(t)\}$ exists and $L\{F'(t)\} = pL\{F(t)\} - F(0)$.

Case-2: $F'(t)$ is merely piecewise continuous, the integral (1) may be broken as the sum of integrals in different ranges from 0 to ∞ such that $F'(t)$ is continuous in each of such parts.

Then proceeding as in case(1), we get

$$L\{F'(t)\} = pL\{F(t)\} - F(0)$$

$$\Big[\, L\{F'(t)\} = \int_0^\infty e^{-pt}F'(t)$$

$$= \int_0^{t_0} e^{-pt}F'(t) + \int_{t_0}^{t_1} e^{-pt}F'(t)\,dt + \cdots$$

$$+ \int_{t_0}^{\infty} e^{-pt}F'(t)\,dt, \left[\text{In particular } LF'(t) = \int_0^{t_0} + \int_{t_0}^{\infty}\right]$$

$$= \left[e^{-pt}F(t)\right]_0^{t_0} + p\int_0^{t_0} e^{-pt}F(t)\,dt + \cdots$$

$$+ \left[e^{-pt}F(t)\right]_{t_0}^{\infty} + p\int_{t_0}^{\infty} e^{-pt}F(t)\,dt$$

$$= e^{-pt_0}\cancel{F(t_0)} - F(0) + \lim_{t\to\infty} e^{-pt}F(t) - e^{-pt_0}\cancel{F(t_0)}$$

$$+ p\int_0^{t_0} e^{-pt}F(t)\,dt + p\int_{t_0}^{\infty} e^{-pt}F(t)\,dt$$

$$= -F(0) + p\int_0^\infty e^{-p(t)}F(t)\,dt$$

$$= pL\{F(t)\} - F(0) \,\Big]$$

Note 1: If $F(t)$ fails to be continuous at $t=0$ but $\lim_{t\to 0+} F(t) = F(0+0)$ exists $= F(0+)$

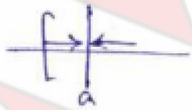
[i.e, $F(0+0)$ is not equal to $F(0)$, which may or may not exist]

then $L\{F'(t)\} = PL\{F(t)\} - F(0+0)$

Note 2: If $F(t)$ fails to be continuous at $t=a$, then

$$L\{F'(t)\} = PL\{F(t)\} - F(0) - e^{-ap}[F(a+0) - F(a-0)]$$

where $F(a+0)$ and $F(a-0)$ are the limits of F at $t=a$, as t approaches 'a' from the right and from the left respectively.



The quantity $F(a+0) - F(a-0)$ is called the jump discontinuity at $t=a$.

Proof: $L\{F'(t)\} = \int_0^\infty e^{-Pt} F'(t)\,dt$

$$= \int_0^a e^{-Pt} F'(t)\,dt + \int_a^\infty e^{-Pt} F'(t)\,dt$$

$$= \left[e^{-Pt} F(t)\right]_0^a + P\int_0^a e^{-Pt} F(t)\,dt + \left[e^{-Pt} F(t)\right]_a^\infty + P\int_a^\infty e^{-Pt} F(t)\,dt$$

$$= e^{-Pa} F(a-0) - F(0) + P\int_0^a e^{-Pt} F(t)\,dt + \lim_{t\to\infty} e^{-Pt} F(t) - e^{-Pa} F(a+0) + P\int_a^\infty e^{-Pt} F(t)\,dt$$

$\left(\because \lim_{t\to\infty} e^{-Pt} F(t) = 0 \text{ by case (1)}\right)$

$$= e^{-aP} F(a-0) - e^{-Pa} F(a+0) + 0 - F(0) + P\int_0^\infty e^{-Pt} F(t)\,dt$$

$$= e^{-ap}\left[F(a-0) - F(a+0)\right] - F(0) + PL\{F(t)\}$$

$$= PL\{F(t)\} - F(0) + e^{-ap}[F(a-0) - F(a+0)]$$

Note (3): For more than one discontinuity of the function $F(t)$, appropriate modification can be made.

* Laplace Transform of the n^{th} order derivative of $F(t)$:

Let $F(t)$ and its derivatives $F'(t), F''(t), \ldots, F^{n-1}(t)$ be continuous functions for all $t \geq 0$ and be of exponential orders as $t\to\infty$ and if $F^n(t)$ is of class A, then laplace transform of $F^n(t)$ exists when $P > a$, and is given by

$$L\{F^n(t)\} = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

Proof: From the above theorem, we have

$$L\{F'(t)\} = PL\{F(t)\} - F(0) \quad \text{--- (1)}$$

Applying the result (1) to the 2^{nd} order derivative $F''(t)$.

$$L\{F''(t)\} = PL\{F'(t)\} - F'(0)$$
$$= P[PL\{F(t)\} - F(0)] - F'(0) \quad (\because \text{by}①)$$
$$= P^2L\{F(t)\} - PF(0) - F'(0) \quad \text{——} ②$$

Again applying (1) to the 3rd order derivative $F'''(t)$, we have

$$L\{F'''(t)\} = PL\{F''(t)\} - F''(0)$$
$$= P[P^2L\{F(t)\} - PF(0) - F'(0)] - F''(0) \quad (\because \text{by}②)$$
$$= P^3L\{F(t)\} - P^2F(0) - PF'(0) - F''(0)$$

proceeding similarly, we get

$$L\{F^{n}(t)\} = P^nL\{F(t)\} - P^{n-1}F(0) - P^{n-2}F'(0) - \cdots - F^{n-1}(0).$$

*** Initial-Value Theorem:**

Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim\limits_{t\to 0} F(t) = \lim\limits_{P\to\infty} PL\{F(t)\}$.

Proof: we know that

$$L\{F'(t)\} = PL\{F(t)\} - F(0)$$

$$\Rightarrow \int_0^\infty e^{-Pt} F'(t)\,dt = PL\{F(t)\} - F(0) \quad \text{——(1)}$$

Taking limit as $P \to \infty$ in (1)

$$\lim_{P\to\infty} \int_0^\infty e^{-Pt} F'(t)\,dt = \lim_{P\to\infty} [PL\{F(t)\} - F(0)] \quad \text{——} ②$$

[Since $F'(t)$ is of class A, i.e, $F'(t)$ is sectionally continuous and of exponential order.

$$\therefore \lim_{P\to\infty} \int_0^\infty e^{-Pt} F'(t)\,dt = 0$$

$\therefore$ from ②

$$0 = \lim_{P\to\infty} PL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{P\to\infty} PL\{F(t)\} = F(0)$$

$$\Rightarrow \lim_{P\to\infty} PL\{F(t)\} = \lim_{t\to 0} F(t)$$

$$(\text{or}) \quad \lim_{t\to 0} F(t) = \lim_{P\to\infty} PL\{F(t)\}$$

Final-Value Theorem:

Let $F(t)$ be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim\limits_{t\to\infty} F(t) = \lim\limits_{P\to 0} PL\{F(t)\}$

Proof: we know that

$$L\{F'(t)\} = PL\{F(t)\} - F(0)$$

$$\Rightarrow \int_0^\infty e^{-Pt} F'(t)\,dt = PL\{F(t)\} - F(0) \quad \text{——} ①$$

Taking limit as $P \to 0$ in (1), we get

$$\lim_{P\to 0} \int_0^\infty e^{-Pt} F'(t)\,dt = \lim_{P\to 0} [PL\{F(t)\} - F(0)]$$

$$\Rightarrow \int_0^\infty F'(t)dt = \lim_{p\to 0} pL\{F(t)\} - F(0)$$

$$\Rightarrow [F(t)]_0^\infty = \lim_{p\to 0} pL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t\to\infty} F(t) - F(0) = \lim_{p\to 0} pL\{F(t)\} - F(0)$$

$$\Rightarrow \lim_{t\to\infty} F(t) = \lim_{p\to 0} pL\{F(t)\}$$

* → Laplace-Transform of Integrals:

Theorem: If $F(t)$ is piecewise continuous and satisfies $|F(t)| \le Me^{at}$ for all $t \ge 0$ for some constants 'a' and M, then

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{p}L\{F(t)\} = \frac{1}{p}f(p) \quad (p>0,\ p>a)$$

Proof: Let $F(t)$ be piecewise continuous such that

$|F(t)| \le Me^{at}$ for some constants a & M. —①

If ① holds for some negative value of 'a' then it also holds for positive value of 'a'

$\therefore$ Suppose that a is +ve.

Let $G(t) = \int_0^t F(x)\,dx$

$G(t)$ is continuous and

$$|G(t)| = \left|\int_0^t F(x)\,dx\right|$$

$$\le \int_0^t |F(x)|\,dx$$

$$\le \int_0^t Me^{ax}\,dx \quad (\because \text{by ①})$$

$$= \frac{M(e^{at}-1)}{a},\ a>0$$

$$\therefore |G(t)| \le \frac{M}{a}(e^{at}-1),\ a>0.$$

Also $G'(t) = F(t)$, except for points at which $F(t)$ is discontinuous.

$G'(t) = \frac{d}{dt}\int_0^t F(x)dx$ by Leibnitz rule

$= \int_0^t \frac{\partial}{\partial t}F(x)dx + F(x)\Big|_{x=t}\frac{dt}{dt} - F(x)\Big|_{x=0}\frac{d0}{dt}$

$= 0 + F(t) - 0 = F(t)$

$\therefore G'(t)$ is piecewise continuous on each finite interval and is of exponential order.

$\therefore$ By existence theorem, laplace transform of $G'(t)$ exists.

$$\therefore L(G'(t)) = pL\{G(t)\} - G(0)$$

$$= pL\{G(t)\} \quad \left(\because G(0) = \int_0^0 F(x)dx \Rightarrow G(0) = 0\right)$$

$$\Rightarrow L\{G(t)\} = \frac{1}{p}L(G'(t))$$

$$\Rightarrow L\left\{\int_0^t F(x)dx\right\} = \frac{1}{p}L\{F(t)\}$$

* Multiplication by powers of t:

Multiplication by t

Theorem: If $F(t)$ is a function of class A and if $L\{F(t)\}=f(P)$ then $L\{tF(t)\}=-f'(P)$.

Proof: we have

$$f(P)=L\{F(t)\}=\int_0^\infty e^{-Pt}F(t)\,dt$$

$$\therefore f'(P)=\frac{d}{dP}\int_0^\infty e^{-Pt}F(t)\,dt$$

$$=\int_0^\infty \frac{\partial}{\partial P}\{e^{-Pt}F(t)dt\}+0-0 \quad \text{(By Leibnitz's rule)}$$

$$=-\int_0^\infty te^{-Pt}F(t)\,dt$$

$$=-\int_0^\infty e^{-Pt}\{tF(t)\}dt$$

$$=-L\{tF(t)\}$$

$$\therefore L\{tF(t)\}=-f'(P)$$

(or)

$$L\{tF(t)\}=(-1)^1\frac{d}{dP}f(P)$$

Differentiation of a function under an integral sign.

$$\frac{d}{dx}\int_A^B F(x,t)dt=\int_A^B\frac{\partial}{\partial x}\{F(x,t)\}dt+F(x,t)\Big|_{x=B}\frac{dB}{dx}-F(x,t)\Big|_{x=A}\frac{dA}{dx}$$

where A & B are functions of x or constants.

Multiplication by t^n:

If $F(t)$ is a function of class A and if $L\{F(t)\}=f(P)$ then

$$L\{t^nF(t)\}=(-1)^n\frac{d^n}{dP^n}f(P),$$

where $n=1,2,3\ldots$

Division by t:

Theorem: If $L\{F(t)\}=f(P)$, then $L\left\{\frac{1}{t}F(t)\right\}=\int_P^\infty f(x)dx$ provided $\lim_{t\to0}\left\{\frac{1}{t}F(t)\right\}$ exists.

proof: Let $G(t)=\frac{1}{t}F(t)$

$$\Rightarrow F(t)=tG(t)$$

Apply L.T. on bothsides

$$L\{F(t)\}=L\{tG(t)\}$$

$$\Rightarrow f(P)=-\frac{d}{dP}L\{G(t)\} \qquad \boxed{\because L\{tF(t)\}=-f'(P)}$$

Integrating bothsides w.r.t P from P to ∞, we get

$$\int_P^\infty f(P)dP=-\Big[L\{G(t)\}\Big]_P^\infty=-\left[\int_0^\infty e^{-Pt}G(t)dt\right]_{P=P}^{\infty}$$

$$\Rightarrow\int_P^\infty f(P)dP=-\lim_{P\to\infty}\left[\int_0^\infty e^{-Pt}G(t)dt\right]+\lim_{P\to P}\left[\int_0^\infty e^{-Pt}G(t)dt\right]$$

$$=0+\int_0^\infty e^{-Pt}G(t)dt \qquad \left[\because \lim_{P\to\infty}L\{G(t)\}=\lim_{P\to\infty}\int_0^\infty e^{-Pt}G(t)dt=0\right]$$

$$=L\{G(t)\}$$

$\Rightarrow L(G(t)) = \int_p^\infty f(p)\,dp = \int_p^\infty f(x)\,dx$

i.e,

$$\boxed{L\left\{\frac{1}{t}F(t)\right\} = \int_p^\infty f(x)\,dx.}$$

$\because$ By property of definite integral $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Problems:

→ Using the laplace transform of Derivative of $F(t)$ i.e,

$L(F^n(t)) = p^n L\{F(t)\} - p^{n-1}F(0) - p^{n-2}F'(0) - \cdots - F^{n-1}(0).$

show that (1) $L\{t\} = \frac{1}{p^2}$

(2) $L\{e^{at}\} = \frac{1}{p-a}$ (3) $L\{-a\sin at\} = \frac{-a^2}{p^2+a^2}$

Sol'n: (1) we have

$L\{F'(t)\} = pL\{F(t)\} - F(0)$ —— ①

Here let $F(t) = t$ then $F'(t) = 1$ and $F(0) = 0$

$\therefore$ from ①

$L\{1\} = pL\{t\} - 0$

$L\{t\} = \frac{1}{p}L\{1\}$

$= \frac{1}{p}\cdot\frac{1}{p}$ $\quad (\because L\{1\} = \frac{1}{p})$

$= \frac{1}{p^2}$, $p>0$

(2) Let $F(t) = e^{at}$

then $F'(t) = ae^{at}$ and $F(0) = 1$

$\therefore$ from (1)

$L\{ae^{at}\} = pL\{e^{at}\} - 1$

$\Rightarrow aL\{e^{at}\} = pL\{e^{at}\} - 1$

$\Rightarrow (p-a)L\{e^{at}\} = 1$

$\Rightarrow L\{e^{at}\} = \frac{1}{p-a}$

(3) Let $F(t) = -a\sin at$

$F'(t) = -a^2\cos at$ and $F''(t) = a^3\sin at$

Also $F(0) = 0$ and $F'(0) = -a^2$

we know that

$L\{F''(t)\} = p^2L\{F(t)\} - pF(0) - F'(0)$

$\Rightarrow L\{a^3\sin at\} = p^2L\{-a\sin at\} - p(0) - (-a^2)$

$\Rightarrow a^2L\{a\sin at\} = p^2L\{-a\sin at\} + a^2$

$\Rightarrow p^2L\{-a\sin at\} - a^2L\{a\sin at\} = -a^2$

$\Rightarrow p^2L\{-a\sin at\} + a^2L\{-a\sin at\} = -a^2$

$\Rightarrow (p^2+a^2)L\{-a\sin at\} = -a^2$

$\Rightarrow L\{-a\sin at\} = \frac{-a^2}{p^2+a^2}$ —— ②

Note: from equation ②

$$\boxed{L\{\sin at\} = \frac{a}{p^2+a^2}}$$

→ Find $L\{t\cos at\}$, It is in the form of $L\{t\,F(t)\}$.

Sol'n: Since $L\{\cos at\} = \frac{p}{p^2+a^2} = (\text{say } f(p))$, $p>0$

$\therefore L\{t\cos at\} = -\frac{d}{dp} L\{\cos at\}$

$= -\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)$ [$\because$ if $L\{F(t)\} = f(p)$ then $L\{tF(t)\} = -\frac{d}{dp}f(p)$]

$= -\frac{p^2+a^2-p(2p)}{(p^2+a^2)^2}$

$= -\frac{a^2-p^2}{(p^2+a^2)^2} = \frac{p^2-a^2}{(p^2+a^2)^2}$

→ Find (i) $L\{t^2\sin at\}$ (ii) $L\{t^2e^{2t}\}$

(iii) $L\{t^3\cos t\}$ (iv) $L\{t(3\sin 2t - 2\cos 2t)\}$

(v) $L\{\sin at - at\cos at\}$

→ show that

$L\{t^n e^{at}\} = \frac{n!}{(p-a)^{n+1}}$, $p>a$

Sol'n: Since $L\{e^{at}\} = \frac{1}{p-a}$ (say $f(p)$), $p>a$

$\therefore L\{t^n e^{at}\} = (-1)^n \frac{d^n}{dp^n} f(p)$

$= (-1)^n \frac{d^n}{dp^n}\left(\frac{1}{p-a}\right)$

$= (-1)^n \frac{(-1)^n n!}{(p-a)^{n+1}}$

$= \frac{n!}{(p-a)^{n+1}}$, $p>a$

→ show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

Sol'n: Let $F(t) = \sin\sqrt{t}$

then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(0)=0$

$\therefore$ From $L\{F'(t)\} = pL\{F(t)\} - F(0)$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = pL\{\sin\sqrt{t}\} - 0$

$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2p\, L\{\sin\sqrt{t}\}$

$= 2p\frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$ [w.k.t $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2p^{3/2}} e^{-1/4p}$]

$= \frac{\sqrt{\pi}}{\sqrt{p}} e^{-1/4p}$

$= \sqrt{\frac{\pi}{p}}\, e^{-1/4p}$

→ Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{p}\right)$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist?

Sol'n: Let $F(t) = \sin t$ [Given problem is in the form of $L\left\{\frac{F(t)}{t}\right\}$ · Check $\lim_{t\to 0}\frac{F(t)}{t}$ exist or not.]

Now $\lim_{t\to 0}\frac{F(t)}{t} = \lim_{t\to 0}\frac{\sin t}{t}$

$= 1$

Since $L\{\sin t\} = \frac{1}{p^2+1} = f(p)$, say

$\therefore$ from $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$, we have

$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{x^2+1}\,dx$ [$\because f(p) = \frac{1}{p^2+1} \Rightarrow f(x) = \frac{1}{x^2+1}$]

$= \left[\tan^{-1}x\right]_p^\infty$ [$\because$ If $x\in\mathbb{R}$, $x\neq 0$ then $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $\Rightarrow \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$]

$= \left[\frac{\pi}{2} - \tan^{-1}p\right]$

$= \cot^{-1}p$

$= \tan^{-1}\frac{1}{p}$ $\left[\because \text{Let } \cot^{-1}p = x \Rightarrow p = \cot x \Rightarrow \tan x = \frac{1}{p} \Rightarrow x = \tan^{-1}(1/p)\right]$

$= f_1(p)$ say

Now $L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin(at)}{at}\right\}$

$= a\frac{1}{a}\tan^{-1}\frac{1}{(p/a)}$

$= \tan^{-1}a/p$ $\left[\because L\{F(at)\} = \frac{1}{a}f_1(p/a)\right]$

Since $L\{\cos at\} = \frac{p}{p^2+a^2} = f(p)$, say

we have $L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{x}{x^2+a^2}dx$

$= \left[\frac{1}{2}\log(x^2+a^2)\right]_p^\infty$

$= \frac{1}{2}\lim_{x\to\infty}\log(x^2+a^2) - \frac{1}{2}\log(p^2+a^2)$

which does not exist.

$\left(\because \lim_{x\to\infty}\log(x^2+a^2) \text{ is infinite}\right)$

Hence $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

→ If $L\{F(t), t\to p\} = f(p)$, Show that

$L\left\{\int_0^t \frac{F(u)}{u}du, t\to p\right\} = \frac{1}{p}\int_p^\infty f(y)dy$

Hence show that

$L\left\{\int_0^t \frac{\sin u}{u}du, t\to p\right\} = \frac{\cot^{-1}p}{p}$

Sol'n: From the laplace transform of integrals

we know that $L\left\{\int_0^t F(u)du\right\} = \frac{1}{p}f(p)$ —— ①

where $f(p) = L\{F(t)\}$.

Let $G(t) = \frac{F(t)}{t}$

$\therefore L\{G(t)\} = L\left\{\frac{F(t)}{t}\right\}$ $\left[\because \text{Division by } t: L\left(\frac{F(t)}{t}\right) = \int_p^\infty f(x)dx \text{ if } L(F(t)) = f(p)\right]$

$= \int_p^\infty f(y)dy$

$= g(p)$, say

$\therefore$ from ①

$L\left\{\int_0^t G(u)du\right\} = \frac{1}{p}g(p)$

$\Rightarrow L\left\{\int_0^t \frac{F(u)}{u}du\right\} = \frac{1}{p}\int_p^\infty f(y)dy$ —— ②

Deduction:

Let $F(t) = \sin t$

so that $f(p) = L\{\sin t\} = \frac{1}{p^2+1}$

$\therefore$ from ②

$L\left\{\int_0^t \frac{\sin u}{u}du\right\} = \frac{1}{p}\int_p^\infty \frac{1}{y^2+1}dy$

$= \frac{1}{p}\left[\tan^{-1}y\right]_p^\infty$

$= \frac{1}{p}\left[\frac{\pi}{2} - \tan^{-1}p\right]$

$= \frac{1}{p}\cot^{-1}p$

→ Prove that $\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)dx$,

provided that the integral converges.

Sol'n: we have $L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)dx$

(Division by t)

$$\Rightarrow \int_0^\infty e^{-pt}\frac{F(t)}{t}dt = \int_p^0 f(x)dx + \int_0^\infty f(x)dx$$

$$= -\int_0^p f(x)dx + \int_0^\infty f(x)dx$$

Taking limit as $p \to 0+$, (assuming the integral converges)

i.e. $\lim_{p\to 0+}\int e^{-pt}\frac{F(t)}{t}dt$

$= \lim_{p\to 0^+}\left[-\int_0^p f(x)dx + \int_0^\infty f(x)dx\right]$

i.e. $-\int_0^p f(x)dx + \int_0^\infty f(x)dx$

we have

$$\int_0^\infty \frac{F(t)}{t}dt = \int_0^\infty f(x)dx$$

→ Show that $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

Sol'n: Let $F(t) = \sin t$

$f(p) = L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1}$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-pt}\frac{\sin t}{t}dt$$

$$\left[\because L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)dx\right]$$

$$= \int_p^\infty f(x)dx$$

$$= \int_p^\infty \frac{1}{1+x^2}dx$$

$$= \left[\tan^{-1}(x)\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}p$$

Taking limit as $p \to 0$,

we have $\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2}$

Evaluation of Integrals:

If $L\{F(t)\} = f(p)$

i.e, $\int_0^\infty e^{-pt}F(t)dt = f(p)$

taking limit as $p \to 0$, we have

$\int_0^\infty F(t)dt = f(0)$

assuming that the integral is convergent.

→ Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt$

Sol'n: Let $F(t) = e^{-at} - e^{-bt}$

$$\therefore f(p) = L\{F(t)\} = L\{e^{-at}\} - L\{e^{-bt}\}$$

$$= \frac{1}{p+a} - \frac{1}{p+b}$$

$$\therefore L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)dx$$

$$\Rightarrow \int_0^\infty e^{-pt}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \int_p^\infty\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$$

$$= \left[\log(x+a) - \log(x+b)\right]_p^\infty$$

$$= \log\left(\frac{1+\frac{a}{x}}{1+\frac{b}{x}}\right)_p^\infty$$

$$= 0 - \left[\log(p+a) - \log(p+b)\right]$$

$$= \log\left(\frac{p+b}{p+a}\right)$$

Taking limit as $p \to 0$, we have

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}dt = \log\frac{b}{a}.$$

→ show that $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt = \log 2$

→ Evaluate $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

→ Show that $\int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}$

Sol'n: Given $\int_0^\infty t e^{-3t} \sin t \, dt =$

$$= \int_0^\infty e^{-3t} (t \sin t) dt.$$

which is in the form of

$\int_0^\infty e^{-pt} (t \sin t) dt = L(t \sin t)$ where $p = 3$

Since $L\{\underbrace{\sin t}_{F(t)}\} = \frac{1}{p^2+1} = f(p)$ say

and $L\{t \sin t\} = (-1) \frac{d}{dp} \frac{1}{p^2+1}$

$(\because L\{t F(t)\} = (-1) f'(p))$

(or) $\int_0^\infty e^{-pt} t \sin t \, dt = (-1) \frac{-2p}{(p^2+1)^2}$

putting $p = 3$, we have

$$\int_0^\infty e^{-3t} t \sin t \, dt = \frac{3}{50}$$

$$\Rightarrow \int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}$$

→ show that $\int_0^\infty t e^{-2t} \cos t \, dt = \frac{3}{25}$

Sol'n: we have $L\{t \cos t\} = -\frac{d}{dp} L\{\cos t\}$

$$\Rightarrow \int_0^\infty e^{-pt} t \cos t \, dt = -\frac{d}{dp}\left(\frac{p}{p^2+1}\right)$$

$$= -\frac{[p^2+1 - p(2p)]}{(p^2+1)^2}$$

$$= \frac{p^2-1}{(p^2+1)^2}$$

Putting $p = 2$, we get

$$\int_0^\infty t e^{-2t} \cos t \, dt = \frac{3}{25}$$

→ Prove that $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$.

*Periodic Functions:

Let F be a periodic function with period $T > 0$, that is $F(u+T) = F(u)$, $F(u+2T) = F(u)$ etc. then

$$L\{F(t)\} = \frac{\int_0^T e^{-pt} F(t) dt}{1 - e^{-pT}}$$

Proof: we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t) dt$

$$= \int_0^T e^{-pt} F(t) dt + \int_T^{2T} e^{-pt} F(t) dt + \int_{2T}^{3T} e^{-pt} F(t) dt + \cdots \quad \text{—— (1)}$$

Putting $t = u+T$ in 2nd integral

$\Rightarrow dt = du$

limits: if $t = T \Rightarrow u = 0$

$t = 2T \Rightarrow u = T$

$t = u+2T$ in 3rd integral & so on.

$\Rightarrow dt = du$

if $t = 2T \Rightarrow u = 0$

if $t = 3T \Rightarrow u = T$

$\therefore$ from (1)

$$L\{F(t)\} = \int_0^T e^{-pt} F(t) dt + \int_0^T e^{-p(u+T)} F(u+T) du + \int_0^T e^{-p(u+2T)} F(u+2T) du + \cdots$$

In the 1st integral $\because \int_0^T e^{-pt} F(t) dt = \int_0^T e^{-pu} F(u) du$

$$= \int_0^T e^{-pu} F(u) du + e^{-pT} \int_0^T e^{-pu} F(u) du + e^{-2pT} \int_0^T e^{-pu} F(u) du + \cdots$$

$$= \underbrace{[1 + e^{-pT} + e^{-2pT} + \cdots]}_{[(1-x)^{-1}]} \int_0^T e^{-pu} F(u) du$$

* A real function $f: A \to R$ is said to be a periodic function if $\exists$ a +ve real no. P such that $f(x+P)=f(x), \forall x \in A$. The least +ve real number P such that $f(x+P)=f(x), \forall x \in A$ is called period of f.

Ex: $f(x)=\sin x$ is a periodic function with Period 2π.

→ for $\cos x$ is 2π, → for $\tan x$ is π.

* Some Special Functions:

→ The Sine and Cosine Integrals:

The sine and cosine integrals, denoted by $S_i(t)$ and $C_i(t)$ respectively are defined by

$$S_i(t) = \int_0^t \frac{\sin u}{u}\,du$$

and $$C_i(t) = \int_t^\infty \frac{\cos u}{u}\,du$$

→ The Error Function:

The Error function, denoted by $erf(t)$, is defined by

$$erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$$

→ The Gamma Function:

If $n>0$, the Gamma function is defined by

$$\Gamma(n) = \int_0^\infty u^{n-1}e^{-u}\,du$$

→ The unit step Function (also called Heaviside's Unit Function):

The unit step function, denoted by $H(t-a)$ is defined by

$$H(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a. \end{cases}$$

→ The Exponential Integral:

The exponential integral is defined by $E_i(t) = \int_t^\infty \frac{e^{-u}}{u}\,du$

* → The Bessel Function:

Bessel Function of order n is defined by

$$J_n(t) = \frac{t^n}{2^n\Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot 4(2n+2)(2n+4)} \cdots\right]$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$$

$$\left\{ J_n(t) = \frac{(-1)^0}{\Gamma n+1}\left(\frac{t}{2}\right)^n + \frac{(-1)^1}{1!\,\Gamma n+2}\left(\frac{t}{2}\right)^{n+2} + \frac{(-1)^2}{2!\,\Gamma n+3}\left(\frac{t}{2}\right)^{n+4} + \cdots \right.$$

$$= \frac{1}{\Gamma n+1}\cdot\frac{t^n}{2^n} - \frac{1}{(n+1)\Gamma n+1}\frac{t^2}{2^2\cdot 2^n} + \frac{1}{2(n+2)(n+1)\Gamma n+1}\frac{t^n}{2^n 2^4} + \cdots$$

$$= \frac{t^n}{2^n\Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot 2^2(2n+2)(2n+4)} \cdots\right].$$

* → Laguerre Polynomial

Laguerre polynomial is defined by

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}t^n), \quad n=0,1,2\cdots$$

→ show that

(i) $L\{\sinh at\cos at\} = \frac{a(p^2-2a^2)}{p^4+4a^4}$

and (ii) $L\{\sinh at \sin at\} = \frac{2a^2 p}{p^4+4a^4}$

Sol'n: Since $L\{\sinh at\} = \frac{a}{p^2-a^2} = f(p)$, say

$\therefore L\{e^{iat}\sinh at\} = f(p-ia)$ (by first shifting property)

$= \frac{a}{(p-ia)^2-a^2}$

$= \frac{a}{(p^2-2a^2)-2iap}$ $(p-ia)^2-a^2 = p^2-2iap-a^2-a^2 = p^2-2a^2-2iap$

$= \frac{a\{(p^2-2a^2)+2iap\}}{(p^2-2a^2)^2+4a^2p^2}$ ($\because$ by dividing and multiplying by $(p^2-2a^2)+2iap$)

$= \frac{a(p^2-2a^2)+2ia^2p}{p^4-\cancel{4a^2p^2}+4a^4+\cancel{4a^2p^2}}$

$= \frac{a(p^2-2a^2)+2ia^2p}{p^4+4a^4}$

$\Rightarrow L\{\sinh at(\cos at+i\sin at)\}$

$= \frac{a(p^2-2a^2)}{p^4+4a^4} + i\frac{2a^2p}{p^4+4a^4}$

$\Rightarrow L\{\sinh at\cos at\} + iL\{\sinh at\sin at\}$

$= \frac{a(p^2-2a^2)}{p^4+2a^4} + i\frac{2a^2p}{p^4+4a^4}$

Comparing real and imaginary terms on bothsides, we get

$L\{\sinh at\cos at\} = \frac{a(p^2-2a^2)}{p^4+4a^4}$

and $L\{\sinh at\sin at\} = \frac{2a^2p}{p^4+4a^4}$

→ Prove that $L\{J_0(t)\} = \frac{1}{\sqrt{1+p^2}}$

and hence deduce that (i) $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

(ii) $L\{t\,J_0(at)\} = \frac{p}{(p^2+a^2)^{3/2}}$

(iii) $L\{e^{-at}J_0(at)\} = \frac{1}{\sqrt{p^2+2ap+2a^2}}$

(iv) $\int_0^\infty J_0(t)\,dt = 1$

Sol'n: we have

$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$

if $n=0$

$J_0(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{t}{2}\right)^{2r}$

$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,r!}\left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$

$= 1 - \frac{t^2}{2^2} + \frac{t^4}{(2!)^2 2^4} - \frac{t^6}{(3!)^2\cdot 2^6} + \cdots$

$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$

$$L\{J_0(t)\} = L\{1\} - \frac{1}{2^2}L\{t^2\} + \frac{1}{2^2.4^2}L\{t^4\} - \frac{1}{2^2.4^2.6^2}L\{t^6\} + \cdots$$

$$= \frac{1}{p} - \frac{1}{2^2}\cdot\frac{2!}{p^3} + \frac{1}{2^2.4^2}\cdot\frac{4!}{p^5} - \frac{1}{2^2.4^2.6^2}\frac{6!}{p^7} + \cdots$$

$$= \frac{1}{p}\left[1 - \frac{1}{2p^2} + \frac{1}{2^2.4^2}\frac{4\times3\times2\times1}{p^4} - \frac{1}{2^2.4^2.6^2}\frac{6\times5\times4\times3\times2\times1}{p^6} + \cdots\right]$$

$$= \frac{1}{p}\left[1 - \frac{1}{2}\left(\frac{1}{p^2}\right) + \frac{1.3}{2.4}\left(\frac{1}{p^2}\right)^2 - \frac{1.3.5}{2.4.6}\left(\frac{1}{p^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{p}\left[1 + \frac{1}{p^2}\right]^{-1/2}$$

$$= \frac{1}{p\left(1+\frac{1}{p^2}\right)^{1/2}} = \frac{1}{\frac{p\sqrt{1+p^2}}{p}}$$

$$= \frac{1}{\sqrt{1+p^2}}$$

(i) we know that by change of scale property if $L\{F(t)\} = f(p)$, then

$$L\{F(at)\} = \frac{1}{a}f\left(p/a\right)$$

$$\therefore J_0(at) = \frac{1}{a}\frac{1}{\sqrt{1+\left(\frac{p}{a}\right)^2}} = \frac{1}{\sqrt{p^2+a^2}}$$

(ii) $L\{t\,J_0(at)\} = \frac{-d}{dp}L\{J_0(at)\}$.

$$\left[\because L\{t\,F(t)\} = \frac{-d}{dp}L\{F(t)\}\right]$$

$$= \frac{-d}{dp}\left(\frac{1}{\sqrt{p^2+a^2}}\right) = \frac{p}{(p^2+a^2)^{3/2}}$$

(iii) By first shifting theorem

$L\{e^{at}F(t)\} = f(p-a)$ if $L\{F(t)\} = f(p)$

$$\Rightarrow L\{e^{-at}F(t)\} = f(p+a)$$

$$\therefore L\{e^{-at}J_0(at)\} = \frac{p+a}{\sqrt{(p+a)^2+a^2}}$$

$$= \frac{p+a}{\sqrt{p^2+2ap+2a^2}} \quad \left(\because L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}\right)$$

(iv) we have $L\{J_0(at)\} = \frac{1}{\sqrt{p^2+a^2}}$

$$\Rightarrow \int_0^\infty e^{-pt}J_0(at)\,dt = \frac{1}{\sqrt{p^2+a^2}}$$

$$\Rightarrow \int_0^\infty e^{-pt}J_0(t)\,dt = \frac{1}{\sqrt{p^2+1}} \text{ (Putting } a=1)$$

$\therefore$ putting $p=0$, we get

$$\int_0^\infty J_0(t)\,dt = 1$$

→ Prove that $L\{J_1(t)\} = 1 - \frac{p}{\sqrt{p^2+1}}$ and hence deduce that $L\{t\,J_1(t)\} = \frac{1}{(p^2+1)^{3/2}}$

Sol'n: we know that

Hint $\quad J_n(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$

Put $n=1$

$$\therefore J_1(t)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma{r+2}}\left(\frac{t}{2}\right)^{2r+1}$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r+1)!}\cdot\left(\frac{t}{2}\right)^{2r+1}$$

$$=\frac{t}{2}-\frac{1}{2^2}\cdot\frac{t^3}{4}+\frac{1}{2^2\cdot4^2}\cdot\frac{t^5}{6}\cdots$$

Imp → Find $L\{\text{erf}\sqrt{t}\}$ and hence prove that $L\{t.\text{ erf }2\sqrt{t}\}=\frac{3P+8}{P^2(P+4)^{3/2}}$

Sol^n: We know that

$$\text{erf}\sqrt{t}=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}e^{-u^2}du$$

$$\left(\because \text{erf}(t)=\frac{2}{\sqrt{\pi}}\int_0^{t}e^{-u^2}du\right)$$

$$=\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left(1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\cdots\right)du$$

$$=\frac{2}{\sqrt{\pi}}\left[u-\frac{u^3}{3}+\frac{u^5}{5(2!)}-\frac{u^7}{7(3!)}+\cdots\right]_0^{\sqrt{t}}$$

$$=\frac{2}{\sqrt{\pi}}\left[t^{1/2}-\frac{t^{3/2}}{3}+\frac{t^{5/2}}{5(2!)}-\frac{t^{7/2}}{7(3!)}+\cdots\right]$$

$$\therefore L(\text{erf}\sqrt{t})=\frac{2}{\sqrt{\pi}}\left[L\{t^{1/2}\}-\frac{1}{3}L\{t^{3/2}\}+\frac{1}{5(2!)}L\{t^{5/2}\}-\frac{1}{7(3!)}L\{t^{7/2}\}+\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{p^{3/2}}-\frac{1}{3}\frac{\Gamma 5/2}{p^{5/2}}+\frac{1}{5(2)}\frac{\Gamma 7/2}{p^{7/2}}-\frac{1}{7.6}\frac{\Gamma 9/2}{p^{9/2}}+\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}}{p^{3/2}}-\frac{1}{3}\frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{5/2}}+\frac{1}{5(2)}\frac{\frac{5}{2}\,\frac{3}{2}\,\Gamma\frac{1}{2}}{p^{7/2}}-\frac{1}{7.6}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{p^{9/2}}+\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2p^{3/2}}-\frac{\sqrt{\pi}}{2}\cdot\frac{1}{2}\frac{1}{p^{5/2}}-\frac{\sqrt{\pi}}{2}\frac{1.3}{2.4}\frac{1}{p^{7/2}}-\frac{\sqrt{\pi}}{2}\frac{1.3.5}{2.4.6}\frac{1}{p^{9/2}}+\cdots\right]$$

$$=\frac{2}{\sqrt{\pi}\,p^{3/2}}\frac{\sqrt{\pi}}{2}\left[1-\frac{1}{2}\left(\frac{1}{p}\right)+\frac{1.3}{2.4}\left(\frac{1}{p^2}\right)-\frac{1.3.5}{2.4.6}\left(\frac{1}{p^3}\right)+\cdots\right]$$

$$=\frac{1}{p^{3/2}}\left(1+\frac{1}{p}\right)^{-1/2}=\frac{1}{p^{3/2}\sqrt{1+\frac{1}{p}}}$$

$$=\frac{1}{p^{3/2}\frac{\sqrt{1+p}}{p^{1/2}}}=\frac{1}{p\sqrt{1+p}}$$

since $L\{F(at)\}=\frac{1}{a}f(p/a)$ by change of scale property

$$\therefore L\{\text{erf}(2\sqrt{t})\}=L\{\text{erf}(\sqrt{4t})\}$$

$$=\frac{1}{4}\frac{1}{\frac{p}{4}\sqrt{\frac{p}{4}+1}}=\frac{2}{p\sqrt{p+4}}$$

Hence $L\{t.\text{ erf }2\sqrt{t}\}=\frac{-d}{dp}L\{\text{erf}.2\sqrt{t}\}$

$$=-\frac{d}{dp}\left(\frac{2}{p\sqrt{p+4}}\right)\quad\left[\because L\{tF(t)\}=\frac{-d}{dp}L\{F(t)\}\right]$$

$$=-\frac{\left[0-\left\{p\left[\frac{1}{2\sqrt{p+4}}\right]+\sqrt{p+4}\right\}\right]}{p(\sqrt{p+4})^2}$$

$$=\frac{p+2(p+4)}{p^2(\sqrt{p+4})(p+4)}=\frac{3p+8}{p^2(p+4)^{3/2}}$$

→ Find the Laplace Transform of $S_i(t)$

<u>Sol'n</u>: we know that

$$S_i(t) = \int_0^t \frac{\sin u}{u}\,du$$

$$= \int_0^t \left(1 - \frac{u^3}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \cdots\right) du$$

$$= t - \frac{t^3}{3(3!)} + \frac{t^5}{5(5!)} - \frac{t^7}{7(7!)} + \cdots$$

$$\therefore L\{S_i(t)\} = L(t) - \frac{1}{3(3!)} L\{t^3\} + \frac{1}{5(5!)} L\{t^5\} - \frac{1}{7(7!)} L\{t^7\} + \cdots$$

$$= \frac{1!}{p^2} - \frac{1}{3(3!)} \cdot \frac{3!}{p^4} + \frac{1}{5(5!)} \frac{5!}{p^6} - \frac{1}{7(7!)} \frac{7!}{p^8} + \cdots$$

$$= \frac{1}{p}\left[\frac{1}{p} - \frac{1}{3}\frac{1}{p^3} + \frac{1}{5}\frac{1}{p^5} - \frac{1}{7}\cdot\frac{1}{p^7} + \cdots\right]$$

$$= \frac{1}{p}\tan^{-1} p, \text{ by Gregrey's series}$$

→ Find $L\{C_i(t)\}$

<u>Sol'n</u>: $L\{C_i(t)\} = L\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\}$

Let $F(t) = \int_t^\infty \frac{\cos u}{u}\,du = -\int_\infty^t \frac{\cos u}{u}\,du$

$$\Rightarrow F'(t) = -\frac{d}{dt}\int_\infty^t \frac{\cos u}{u}\,du$$

By Leibnitz's

So that $F'(t) = -\frac{\cos t}{t}$

(or) $tF'(t) = -\cos t$

$\therefore L\{tF'(t)\} = L\{-\cos t\}$

(or) $\frac{-d}{dp} L\{F'(t)\} = -\frac{p}{p^2+1}$

(or) $\frac{d}{dp}\left[pf(p) - F(0)\right] = \frac{p}{p^2+1}$ where $f(p) = L\{F(t)\}$

(or) $\frac{d}{dp}\left[pf(p)\right] = \frac{p}{p^2+1}$, since $F(0)$ is Constant.

Integrating $p f(p) = \frac{1}{2}\log(p^2+1) + c$ (Constant) ——(i)

But from the final value theorem

$$\lim_{p\to 0} pf(p) = \lim_{t\to\infty} F(t) = 0$$

$\therefore$ from (i) as $p \to 0$, we have

$0 = 0 + c$ or $c = 0$

$\therefore$ from (i), $pf(p) = \frac{1}{2}\log(p^2+1)$

(or) $f(p) = L\{F(t)\} = L\{C_i(t)\} = \frac{\log(p^2+1)}{2p}$

→ If $F(t) = t^2$, $0 < t < 2$ and $F(t+2) = F(t)$, find $L\{F(t)\}$

<u>Sol'n</u>: Here $F(t)$ is a periodic function with period $T = 2$.

$\therefore$ from fundamental theorem (Periodic function) we have

$$L\{F(t)\} = \frac{\int_0^T e^{-pt} F(t)dt}{1-e^{-pT}}$$

$$= \frac{\int_0^2 t^2 e^{-pt} dt}{1-e^{-2p}}$$

$$= \frac{\left(-\frac{t^2}{p} e^{-pt}\right)_0^2 + \frac{2}{p}\int_0^2 te^{-pt}dt}{1-e^{-2p}}$$

$$= \frac{-\frac{4}{p} e^{-2p} + \frac{2}{p}\left\{\left(-\frac{t}{p} e^{-pt}\right)_0^2 + \frac{1}{p}\int_0^2 e^{-pt}dt\right\}}{1-e^{-2p}}$$

$$= \frac{-\frac{4}{p}e^{-2p} - \frac{4}{p^2}e^{-2p} - \frac{2}{p^2}\left(\frac{-e^{-pt}}{p}\right)_0^2}{1-e^{-2p}}$$

$$= \frac{-(4p^2+4p+2)e^{-2p}+2}{p^3(1-e^{-2p})}$$

→ Find the Laplace transform of Heaviside's unit function $H(t-a)$

Sol'n: we have $H(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$

$$\therefore L\{H(t-a)\} = \int_0^\infty e^{-pt} H(t-a)dt$$

$$= \int_a^\infty e^{-pt}.1\,dt$$

$$= \frac{e^{-ap}}{p}.$$

→ show that $L\{E_i(t)\} = \frac{\log(p+1)}{p}$

Sol'n: we have $E_i(t) = \int_t^\infty \frac{e^{-u}}{u}du$

$$\therefore L\{E_i(t)\} = L\left\{\int_t^\infty \frac{e^{-u}}{u}du\right\}$$

$$= L\left\{\int_1^\infty \frac{e^{-tv}}{v}dv\right\}dt,$$

Putting $u = tv$ so that $du = t\,dv$

$$= \int_0^\infty e^{-pt}\left\{\int_1^\infty \frac{e^{-tv}}{v}dv\right\}dt$$

(By definition of L.T)

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-pt}e^{-tv}dt\right\}dv,$$

Changing the order of integration.

$$= \int_1^\infty \frac{1}{v}\frac{1}{p+v}dv$$

$$= \int_1^\infty \frac{1}{p}\left(\frac{1}{v} - \frac{1}{p+v}\right)dv$$

$$= \frac{1}{p}\Big[\log v - \log(p+v)\Big]_1^\infty$$

$$= \frac{1}{p}\left[-\log\left(\frac{p}{v}+1\right)\right]_1^\infty$$

$$= \frac{1}{p}\log(p+1)$$

→ Find $L\{L_n(t)\}$

we have $L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}.t^n)$

$$\therefore L\{L_n(t)\} = \int_0^\infty e^{-pt}.\frac{e^t}{n!}\frac{d^n}{dt^n}(e^{-t}.t^n)dt$$

$$= \frac{1}{n!}\int_0^\infty e^{-(p-1)t}\frac{d^n}{dt^n}(e^{-t}.t^n)dt$$

$$= \frac{1}{n!}\left[e^{-(p-1)t} \cdot \frac{d^{n-1}}{dt^{n-1}} e^{-t} \cdot t^n\right]_0^\infty$$

$$- \frac{1}{n!}\left[-(p-1)\int_0^\infty e^{-(p-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t}t^n)dt\right]$$

$$= 0 + \frac{p-1}{n!}\int_0^\infty e^{-(p-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} \cdot t^n)dt, \quad p>1$$

[$\because$ each term in $\frac{d^{n-1}}{dt^{n-1}}(e^{-t} \cdot t^n)$ Contains some integral Power of t].

Proceeding similarly again and again, we have

$$L\{L_n(t)\} = \frac{(p-1)^n}{n!}\int_0^\infty e^{-(p-1)t} (e^{-t} t^n) dt$$

$$= \frac{(p-1)^n}{n!}\int_0^\infty e^{-pt} t^n dt$$

$$= \frac{(p-1)^n}{n!} L\{t^n\}$$

$$= \frac{(p-1)^n}{n!} \frac{n!}{p^{n+1}}$$

$$= \frac{(p-1)^n}{p^{n+1}}$$

→ show that $L\{(1+te^{-t})^3\} =$

$$\frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

Sol'n: we have

$$L\{(1+te^{-t})^3\} = L\{1 + 3te^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}\}$$

$$= \frac{1}{p} + 3\frac{1!}{(p+1)^2} + 3 \cdot \frac{2!}{(p+2)^3} + \frac{3!}{(p+3)^4}$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}$$

Aliter

$$L\{(1+te^{-t})^3\} = \int_0^\infty (1+te^{-t})^3 \cdot e^{-pt} dt$$

$$= \int_0^\infty [e^{-pt} + 3te^{-(p+1)t} + 3t^2 e^{-(p+2)t} + t^3 e^{-(p+3)t}]dt$$

$$= \int_0^\infty e^{-pt}dt + 3\int_0^\infty te^{-(p+1)t}dt + 3\int_0^\infty t^2 e^{-(p+2)t}dt + \int_0^\infty t^3 e^{-(p+3)t}dt$$

$$= \frac{1}{p} + \frac{3}{(p+1)^2} + \frac{6}{(p+2)^3} + \frac{6}{(p+3)^4}, \quad p>0$$

[$\because \int_0^\infty e^{-at} t^{n-1} dt = \frac{\Gamma(n)}{a^n}$, if $a>0$ and $n>0$.]

* The Inverse Laplace Transform

Definition: If $f(P)$ is the Laplace transform of a function $F(t)$, i.e, $L\{F(t)\} = f(P)$, then $F(t)$ is called the Inverse Laplace transform of $f(P)$ and is written as

$F(t) = L^{-1}\{f(P)\}$.

Ex: $L\{t^n\} = \dfrac{n!}{p^{n+1}}$

$$\therefore t^n = L^{-1}\left\{\frac{n!}{p^{n+1}}\right\}$$

Linearity Properties:

Let $f_1(P)$ and $f_2(P)$ be the Laplace transforms of functions $F_1(t)$ and $F_2(t)$ respectively and C_1, C_2 be two constants, then

$$L^{-1}\{c_1 f_1(P) + c_2 f_2(P)\}$$
$$= c_1 L^{-1}\{f_1(P)\} + c_2 L^{-1}\{f_2(P)\}$$
$$= c_1 F_1(t) + c_2 F_2(t)$$

Proof: Given $L\{F_1(t)\} = f_1(P)$
$\Rightarrow F_1(t) = L^{-1}\{f_1(P)\}$

and $L\{F_2(t)\} = f_2(P) \Rightarrow F_2(t) = L^{-1}\{f_2(P)\}$

we have

$L\{C_1F_1(t) + C_2F_2(t)\} = C_1 L\{F_1(t)\} + C_2 L\{F_2(t)\}$

$$= c_1 f_1(P) + c_2 f_2(P)$$

$$\therefore L^{-1}\{C_1 f_1(P) + C_2 f_2(P)\} = C_1F_1(t) + C_2F_2(t)$$
$$= C_1 L^{-1}\{f_1(P)\} + C_2 L^{-1}\{f_2(P)\}$$

→ Find (i) $L^{-1}\left\{\frac{1}{P}\right\}$, $P>0$

Sol^n: Since $L\{1\} = \frac{1}{P}$

$\therefore L^{-1}\left\{\frac{1}{P}\right\} = 1$

→ (ii) $L^{-1}\left\{\frac{1}{p^{n+1}}\right\}$, n is any real number such that $n > -1$

Sol^n: Since $L\{t^n\} = \dfrac{\Gamma(n+1)}{p^{n+1}}$, $P>0, n>-1$

$$\therefore L^{-1}\left\{\frac{\Gamma(n+1)}{p^{n+1}}\right\} = t^n$$

$$\Rightarrow L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}, \; n>-1, P>0$$

If n is +ve integer, then $\Gamma(n+1) = n!$

$\therefore$ from ① $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$; $P>0$.

→ (iii) $L^{-1}\left\{\frac{1}{P-a}\right\}$

Sol^n: Since $L\{e^{at}\} = \frac{1}{P-a}$

$\therefore L^{-1}\left\{\frac{1}{P-a}\right\} = e^{at}$

→ Find (i) $L^{-1}\left\{\frac{1}{P^2+a^2}\right\}_{P>0}$ (ii) $L^{-1}\left\{\frac{P}{P^2+a^2}\right\}_{P>0}$

(iii) $L^{-1}\left\{\frac{1}{P^2-a^2}\right\}_{P>|a|}$ (iv) $L^{-1}\left\{\frac{P}{P^2-a^2}\right\}_{P>|a|}$

Sol'n: (i) Since $L\{\sin at\} = \frac{a}{p^2+a^2}$

$\therefore L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

$\Rightarrow L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

(ii) Since $L\{\cos at\} = \frac{p}{p^2+a^2}$

$\therefore L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

(iii) Since $L\{\sinh at\} = \frac{a}{p^2-a^2}$

$\therefore L^{-1}\left\{\frac{a}{p^2-a^2}\right\} = \sinh at$

$\Rightarrow L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}\sinh at$

(iv) Since $L\{\cosh at\} = \frac{p}{p^2-a^2}$

$\therefore L^{-1}\left\{\frac{p}{p^2-a^2}\right\} = \cosh at$

→ Find (i) $L^{-1}\left\{\frac{1}{p^4}\right\}$ (ii) $L^{-1}\left\{\frac{1}{p^2+4}\right\}$

(iii) $L^{-1}\left\{\frac{4}{(p^2-2)}\right\}$ (iv) $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{1}{p^4}\right\} = L^{-1}\left\{\frac{1}{p^{3+1}}\right\} = \frac{t^3}{3!}$

$\left[\because L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}\right.$ if n is +ve integer$\left.\right]$

(iv) $L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{1/2}}\right\}$

$= L^{-1}\left\{\frac{1}{p^{-1/2+1}}\right\}$

$= \frac{t^{-1/2}}{\Gamma(-\frac{1}{2}+1)}$ $(n=-\frac{1}{2})$

$= \frac{t^{-1/2}}{\Gamma\frac{1}{2}}$ $\left[\because L\{t^n\} = \frac{\Gamma(n+1)}{p^{n+1}}, n>-1 \Rightarrow L^{-1}\left(\frac{1}{p^{n+1}}\right) = \frac{t^n}{\Gamma(n+1)}\right]$

$= \frac{t^{-1/2}}{\sqrt{\pi}}$

$(\because \Gamma\frac{1}{2} = \sqrt{\pi})$

→ Find (i) $L^{-1}\left\{\frac{1}{p^{7/2}}\right\}$

(ii) $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{1}{p^{7/2}}\right\} = L^{-1}\left\{\frac{1}{p^{5/2+1}}\right\}$

$= \frac{t^{5/2}}{\Gamma(\frac{5}{2}+1)} = \frac{t^{5/2}}{\Gamma\frac{7}{2}}$

$= \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$

$= \frac{t^{5/2}}{\frac{15}{8}\sqrt{\pi}}$

$= \frac{t^{1/2}\cdot t^2}{\frac{15}{8}\sqrt{\pi}}$

$= \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}$

(ii) $L^{-1}\left\{\frac{p}{p^2+2} + \frac{6p}{p^2-16} + \frac{3}{p-3}\right\}$

$= L^{-1}\left\{\frac{p}{p^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{p}{p^2-4^2}\right\} + 3L^{-1}\left\{\frac{1}{p-3}\right\}$

$= \cos\sqrt{2}t + 6\cosh 4t + 3e^{3t}.$

→ Find (i) $L^{-1}\left\{\frac{6}{2p-3}-\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\}$

(ii) $L^{-1}\left\{\frac{3}{p^2-3}+\frac{3p+2}{p^3}-\frac{3p-27}{p^2+9}+\frac{6-30\sqrt{p}}{p^4}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{6}{2p-3}-\frac{3+4p}{9p^2-16}+\frac{8-6p}{16p^2+9}\right\}$

$$=L^{-1}\left\{\frac{6}{2\left[p-\frac{3}{2}\right]}\right\}-L^{-1}\left\{\frac{3}{9p^2-16}\right\}+4L^{-1}\left\{\frac{p}{9p^2-16}\right\}$$

$$+8L^{-1}\left\{\frac{1}{16p^2+9}\right\}-6L^{-1}\left\{\frac{p}{16p^2+9}\right\}$$

$$=3L^{-1}\left\{\frac{1}{p-3/2}\right\}-\frac{3}{9}L^{-1}\left\{\frac{1}{p^2-(4/3)^2}\right\}+$$

$$\frac{4}{9}L^{-1}\left\{\frac{p}{p^2-(4/3)^2}\right\}+\frac{8}{16}L^{-1}\left\{\frac{1}{p^2+(3/4)^2}\right\}-\frac{6}{16}L^{-1}\left\{\frac{p}{p^2+(3/4)^2}\right\}$$

$$=3e^{3/2t}-\frac{1}{3}\frac{1}{4/3}\sinh\frac{4}{3}t+\frac{4}{9}\cdot\cosh\frac{4}{3}t+$$

$$\frac{1}{2}\cdot\frac{1}{(3/4)}\sin 3/4t-\frac{3}{8}\cos 3/4t$$

$$=3e^{3/2t}-\frac{1}{4}\sinh 4/3t+4/9\cosh\frac{4t}{3}$$

$$+\frac{2}{3}\sin 3/4t-3/8\cos 3/4t.$$

→ Prove that

$$L^{-1}\left\{\frac{5}{p^2}+\left(\frac{\sqrt{p}-1}{p}\right)^2-\frac{7}{3p+2}\right\}$$

$$=1+6t-4\sqrt{\frac{t}{\pi}}-\frac{7}{3}e^{-2/3t}$$

→ show that

(i) $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\}=1-\frac{t^2}{(2!)^2}+\frac{t^4}{(4!)^2}-\frac{t^6}{(6!)^2}+\cdots$

(ii) $L^{-1}\left\{\frac{1}{p}\sin\frac{1}{p}\right\}=t-\frac{t^3}{(3!)^2}+\frac{t^5}{(5!)^2}-\frac{t^7}{(7!)^2}+\cdots$

(iii) $L^{-1}\left\{\frac{1}{p^3+1}\right\}=\frac{t^2}{2!}-\frac{t^5}{5!}+\frac{t^8}{8!}-\frac{t^{11}}{11!}+\cdots$

$\left(\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)$

Sol'n: (i) $L^{-1}\left\{\frac{1}{p}\cos\frac{1}{p}\right\}$

$$=L^{-1}\left\{\frac{1}{p}\left[1-\frac{(1/p)^2}{2!}+\frac{(1/p)^4}{4!}-\frac{(1/p)^6}{6!}\cdots\right]\right\}$$

$$=L^{-1}\left\{\frac{1}{p}-\frac{1}{p^3\,2!}+\frac{1}{4!\,p^5}-\frac{1}{6!\,p^7}\cdots\right\}$$

$$=L^{-1}\left\{\frac{1}{p}\right\}-\frac{1}{2!}L^{-1}\left\{\frac{1}{p^3}\right\}+\frac{1}{4!}L^{-1}\left\{\frac{1}{p^5}\right\}$$

$$-\frac{1}{6!}L^{-1}\left\{\frac{1}{p^7}\right\}+\cdots$$

$$=1-\frac{1}{2!}\frac{t^2}{2!}+\frac{1}{4!}\frac{t^4}{4!}-\frac{1}{6!}\frac{t^6}{6!}+\cdots$$

$$=1-\frac{t^2}{(2!)^2}+\frac{t^4}{(4!)^2}-\frac{t^6}{(6!)^2}+\cdots$$

(iii) $L^{-1}\left\{\frac{1}{p^3+1}\right\}=L^{-1}\left\{\frac{1}{p^3\left[1+\frac{1}{p^3}\right]}\right\}$

$\left[(1+x)^{-1}=1-x+x^2-x^3+x^4+\cdots\right]$

$$=L^{-1}\left\{\frac{1}{p^3}\left(1+\frac{1}{p^3}\right)^{-1}\right\}$$

$$=L^{-1}\left\{\frac{1}{p^3}\left[1-\frac{1}{p^3}+\left(\frac{1}{p^3}\right)^2-\left(\frac{1}{p^3}\right)^3+\left(\frac{1}{p^3}\right)^4-\cdots\right]\right\}$$

$$=L^{-1}\left\{\frac{1}{p^3}\right\}-L^{-1}\left\{\frac{1}{p^6}\right\}+L^{-1}\left\{\frac{1}{p^9}\right\}-L^{-1}\left\{\frac{1}{p^{12}}\right\}+\cdots$$

$$=\frac{t^2}{2!}-\frac{t^5}{5!}+\frac{t^8}{8!}-\frac{t^{11}}{11!}+\cdots$$

→ show that $L^{-1}\left\{\frac{e^{-1/p}}{p}\right\}=J_0\,2\sqrt{t}$

where

$$\boxed{J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4^2}-\frac{x^6}{2^2\cdot4^2\cdot6^2}\cdots}$$

First translation (or) shifting Theorem:

If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(p-a)\} = e^{at}F(t) = e^{at}L^{-1}\{f(p)\}$$

Proof: By definition of L.T. we have $f(p) = \int_0^\infty e^{-pt}F(t)\,dt$ — (1)

Replacing p by p−a on both sides of (1), we have

$$f(p-a) = \int_0^\infty e^{-(p-a)t}F(t)\,dt = \int_0^\infty e^{-pt}\{e^{at}F(t)\}\,dt$$

$\therefore f(p-a) = L\{e^{at}F(t)\}$ ($\because$ by def'n of L.T)

Hence, by def'n of inverse L.T, we get

$$L^{-1}\{f(p-a)\} = e^{at}F(t) = e^{at}L^{-1}\{f(p)\} \quad \text{— (2)}$$

Note: Replacing 'a' by −a in (2), we get

$$L^{-1}\{f(p+a)\} = e^{-at}F(t) = e^{-at}L^{-1}\{f(p)\}$$

Second translation (or) shifting theorem:

If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{e^{-ap}f(p)\} = G(t)$ where

$$G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a. \end{cases}$$

Proof: By definition of L.T., we have

$$L\{G(t)\} = \int_0^\infty e^{-pt}G(t)\,dt$$

$$= \int_0^a e^{-pt}G(t)\,dt + \int_a^\infty e^{-pt}G(t)\,dt$$

$$= \int_0^a e^{-pt}(0)\,dt + \int_a^\infty e^{-pt}F(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-pt}F(t-a)\,dt$$

Put $t-a = x \Rightarrow dt = dx$ and $t = x+a$

Limits: if $t=a \Rightarrow x=0$; $t=\infty \Rightarrow x=\infty$

$$= \int_0^\infty e^{-p(x+a)}F(x)\,dx$$

$$= \int_0^\infty e^{-ap}e^{-px}F(x)\,dx$$

$$= e^{-ap}\int_0^\infty e^{-pt}F(t)\,dt$$

by property of definite integrals $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$$= e^{-ap}L\{F(t)\}$$

$$\therefore L\{G(t)\} = e^{-ap}L\{F(t)\}$$

$$= e^{-ap}f(p)$$

Given $L^{-1}\{f(p)\} = F(t) \Rightarrow L(F(t)) = f(p)$

Hence by the definition of inverse L.T we get

$$L^{-1}\{e^{-ap}f(p)\} = G(t)$$

Note: The result of this theorem can also be expressed in the following two ways

1. $L^{-1}\{f(p)\} = F(t)$, $L^{-1}\{e^{-pa}f(p)\} = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

2. $L^{-1}\{f(p)\} = F(t)$, $L^{-1}\{e^{-pa}f(p)\} = F(t-a)H(t-a)$

where $H(t-a)$ is Heaviside unit step function which is defined as follows:

$$H(t-a) = \begin{cases} 1, & \text{when } t>a \\ 0, & \text{when } t<a. \end{cases}$$

→ * **Change of Scale Property:**

Theorem: If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\{f(ap)\} = \frac{1}{a}F(t/a)$$

Sol'n: Given $L^{-1}\{f(p)\} = F(t)$

$$\Rightarrow f(p) = L\{F(t)\}$$

$$\therefore f(p) = \int_0^\infty e^{-pt}F(t)\,dt$$

$$\therefore f(ap) = \int_0^\infty e^{-apt}F(t)\,dt$$

Put $at = x \Rightarrow dt = \frac{dx}{a}$

$t = x/a$

$$= \frac{1}{a}\int_0^\infty e^{-px}F\left(\frac{x}{a}\right)dx$$

$$= \frac{1}{a}\int_0^\infty e^{-pt}F(t/a)\,dt$$

$$\left(\because \int_a^b f(x)dx = \int_a^b F(t)dt\right)$$

$$= \frac{1}{a}L\{F(t/a)\}$$

$$= L\left\{\frac{1}{a}F(t/a)\right\}$$

$$\therefore f(ap) = L\left\{\frac{1}{a}F(t/a)\right\}$$

$$\Rightarrow L^{-1}\{f(ap)\} = \frac{1}{a}F(t/a)$$

→ Find $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\}$

Sol'n: $L^{-1}\left\{\frac{1}{p^2-6p+10}\right\} = L^{-1}\left\{\frac{1}{(p-3)^2+1}\right\}$

$$= e^{3t}L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$= e^{3t}\sin t$ (using first shifting theorem)

→ Find (i) $L^{-1}\left\{\frac{1}{p^2+8p+16}\right\}$.

(ii) $L^{-1}\left\{\frac{p-1}{(p+3)(p^2+2p+2)}\right\}$

Sol'n (i) $L^{-1}\left\{\frac{1}{p^2+8p+16}\right\} = L^{-1}\left\{\frac{1}{(p+4)^2}\right\}$

$= e^{-4t}\left\{\frac{1}{p^2}\right\}$ (by using first shifting theorem)

$$= e^{-4t}\frac{t}{1!} = te^{-4t}.$$

(ii) $L^{-1}\left\{\frac{P-1}{(P+3)(P^2+2P+2)}\right\}$

Now $\frac{P-1}{(P+3)(P^2+2P+2)} = \frac{A}{P+3} + \frac{BP+C}{P^2+2P+2}$

$P-1 = A(P^2+2P+2)+(BP+C)(P+3)$

$P-1 = (A+B)P^2+(2A+3B+C)P+2A+3C$

Comparing on bothsides

$A+B=0 \Rightarrow A=-B$

$2A+3B+C=1 \Rightarrow 2(-B)+3B+C=1$

$\Rightarrow B+C=1$

$2A+3C=-1$

$\Rightarrow 2(-B)+3C=-1$

$\Rightarrow -2B+3C=-1$

$$\begin{array}{r} 2B+2C=2 \\ -2B+3C=-1 \\ \hline 5C=1 \end{array}$$

$\boxed{C=1/5}$

$B+C=1$

$B=1-1/5$

$\boxed{B=4/5}$

$A+B=0$

$\boxed{A=-\frac{4}{5}}$

$\therefore \frac{P-1}{(P+3)(P^2+2P+2)} = -\frac{4}{5(P+3)} + \frac{\frac{4}{5}P+\frac{1}{5}}{P^2+2P+2}$

$= \frac{-4}{5(P+3)} + \frac{4P+1}{5(P^2+2P+2)}$

$L^{-1}\left\{\frac{P-1}{(P+3)(P^2+2P+2)}\right\}$

$= -\frac{4}{5}L^{-1}\left\{\frac{1}{P+3}\right\} + \frac{1}{5}L^{-1}\left\{\frac{4P+1}{(P^2+2P+2)}\right\}$

$= \frac{-4}{5}e^{-3t}L^{-1}\left\{\frac{1}{P}\right\} + \frac{1}{5}L^{-1}\left\{\frac{4(P+1)-3}{(P+1)^2+1}\right\}$

$= -\frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}L^{-1}\left\{\frac{4P-3}{P^2+1}\right\} \quad \left(\because L^{-1}\left\{\frac{1}{P}\right\}=1\right)$

$= -4/5\, e^{-3t} + \frac{e^{-5}}{5}\left[L^{-1}\left\{\frac{4P}{P^2+1}\right\} + L^{-1}\left\{\frac{-3}{P^2+1}\right\}\right]$

$= -4/5\, e^{-3t} + \frac{e^{-5}}{5}(4\cos t - 3\sin t)$

$L^{-1}\left\{\frac{3P-2}{P^2-4P+20}\right\} = L^{-1}\left\{\frac{3P-2}{(P-2)^2+16}\right\}$

$= L^{-1}\left\{\frac{3(P-2)+4}{(P-2)^2+4^2}\right\}$

$= e^{2t}L^{-1}\left\{\frac{3P+4}{P^2+4^2}\right\}$

$= e^{2t}\left[L^{-1}\left\{\frac{3P}{P^2+4^2}\right\} + L^{-1}\left\{\frac{4}{P^2+4^2}\right\}\right]$

$= e^{2t}\left[3\cos 2t + 4\sin 4t\right]$

$L^{-1}\left\{\frac{3P+7}{P^2-2P-3}\right\} = L^{-1}\left\{\frac{3P+7}{(P-1)^2-4^2}\right\}$

$= L^{-1}\left\{\frac{3(P-1)+10}{(P-1)^2-4}\right\}$

$= e^t L^{-1}\left\{\frac{3P+10}{P^2-2^2}\right\}$

$= e^t\left[3L^{-1}\left\{\frac{P}{P^2-2^2}\right\} + 10L^{-1}\left\{\frac{1}{P^2-2^2}\right\}\right]$

$= e^t\left[3\cosh 2t + \frac{10}{2}\sinh 2t\right]$

$= e^t\left[3\cosh 2t + 5\sinh 2t\right]$

$\boxed{\because L^{-1}\left\{\frac{1}{P^2-a^2}\right\} = \frac{1}{a}\sinh at}$

→ Find (i) $L^{-1}\left\{\frac{1}{(p+a)^n}\right\}$ (ii) $L^{-1}\left\{\frac{p}{(p+1)^{5/2}}\right\}$

(iii) $L^{-1}\left\{\frac{p}{(p+1)^5}\right\}$ (iv) $L^{-1}\left\{\frac{3p+2}{4p^2+12p+9}\right\}$

→ Evaluate (i) $L^{-1}\left\{\frac{1}{\sqrt{2p+3}}\right\}$ (ii) $L^{-1}\left\{\frac{1}{(8p-27)^{1/3}}\right\}$

Sol'n (ii): $L^{-1}\left\{\frac{1}{(8p-27)^{1/3}}\right\} = L^{-1}\left\{\frac{1}{8^{1/3}\left(p-\frac{27}{8}\right)^{1/3}}\right\}$

$$= \frac{1}{2} L^{-1}\left\{\frac{1}{\left(p-\frac{27}{8}\right)^{1/3}}\right\}$$

$$= \frac{1}{2} e^{27/8\,t} L^{-1}\left\{\frac{1}{p^{1/3}}\right\}$$

$$= \frac{e^{27/8\,t}}{2} \frac{t^{1/3-1}}{\Gamma 1/3}$$

$\boxed{L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1}}$

$$= \frac{e^{27/8\,t}}{2} \frac{t^{-2/3}}{\Gamma 1/3}$$

→ If $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$, find $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$ where $a>0$.

Sol'n: Since $L^{-1}\{f(ap)\} = \frac{1}{a}F(t/a)$

Hence $L^{-1}\left\{\frac{e^{-1/p}}{\sqrt{p}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ gives

$$L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{pk}}\right\} = \frac{1}{k}\frac{\cos 2\sqrt{t/k}}{\sqrt{\frac{\pi t}{k}}}$$

$$\Rightarrow \frac{1}{\sqrt{k}} L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{p}}\right\} = \frac{1}{k}\frac{\cos\left(2\sqrt{t/k}\right)}{\sqrt{\frac{\pi t}{k}}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/pk}}{\sqrt{p}}\right\} = \frac{\sqrt{k}}{k}\frac{\cos 2(t/k)^{1/2}}{\frac{1}{\sqrt{k}}\sqrt{\pi t}}$$

Putting $k = 1/a$

$$\therefore L^{-1}\left\{\frac{e^{-a/p}}{\sqrt{p}}\right\} = \frac{\cos 2(at)^{1/2}}{\sqrt{\pi t}}$$

$$= \frac{\cos 2\sqrt{at}}{\sqrt{\pi t}}$$

Imp → $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find $L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}$

Sol'n: Given $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$

Since $f(ap) = \frac{1}{a}F(t/a)$

$$\therefore L^{-1}\left\{\frac{ap}{((ap)^2+1)^2}\right\} = \frac{1}{2}\left(\frac{1}{a}\right)\frac{t}{a}\sin\frac{t}{a}$$

Putting $a=4$

$$L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{t}{4}\sin\frac{t}{4}$$

$$L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{1}{8}\frac{t}{4}\sin\frac{t}{4}$$

$$\Rightarrow 8L^{-1}\left\{\frac{4p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}.$$

$$\Rightarrow L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}$$

→ If $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$, Prove that

$$L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = \frac{1}{9}t\cos t/3$$

Sol^n: Given $L^{-1}\left\{\frac{p^2-1}{(p^2+1)^2}\right\} = t\cos t$

then $L^{-1}\left\{\frac{9p^2-1}{(9p^2+1)^2}\right\} = L^{-1}\left\{\frac{(3p)^2-1}{((3p)^2+1)^2}\right\}$

$\left[\because L^{-1}(f(ap)) = \frac{1}{a}F(t/a)\right]$

$$= \frac{1}{3}\,\frac{t}{3}\cos\frac{t}{3}$$

$$= \frac{t}{9}\cos t/3$$

→ If $L^{-1}\left\{\frac{p}{p^2-16}\right\} = \cosh 4t$, then Prove

$$L^{-1}\left\{\frac{p}{2p^2-8}\right\} = \frac{1}{2}\cosh 2t$$

→ Find $L^{-1}\left\{\frac{e^{-5p}}{(p-2)^4}\right\}$ It is in the form of $L^{-1}\left\{e^{-ap}f(p)\right\}$

Sol^n: Let $f(p) = \frac{1}{(p-2)^4}$

i.e. $L\{F(t)\} = \frac{1}{(p-2)^4}$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{(p-2)^4}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{1}{p^4}\right\}$$

$$= e^{2t}\frac{t^3}{3!} = \frac{1}{6}t^3e^{2t}$$

Hence by second shifting theorem, we have

$$L^{-1}\left\{e^{-ap}f(p)\right\} = \begin{cases}F(t-a), & t>a\\ 0, & t<a\end{cases}$$

$$\therefore L^{-1}\left\{e^{-5p}f(p)\right\} = \begin{cases}F(t-5), & t>a\\ 0, & t<a\end{cases}$$

$$\therefore L^{-1}\left\{e^{-5p}\frac{1}{(p-2)^4}\right\} = \begin{cases}\frac{1}{6}(t-5)^3e^{2(t-5)}, & t>5\\ 0, & t<5\end{cases}$$

$$= \frac{1}{6}(t-5)^3e^{2(t-5)}$$

interms of Heaviside unit step-function.

→ Find $L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\}$

Sol^n: Let $f(p) = \frac{1}{(p+4)^{5/2}}$

$$F(t) = L^{-1}\left\{\frac{1}{(p+4)^{5/2}}\right\} = e^{-4t}L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$$

$$= e^{-4t}\frac{t^{5/2-1}}{\Gamma 5/2}$$

$\left| L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\Gamma n+1}, \; n>-1 \right.$

$$= e^{-4t}\frac{t^{3/2}}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{4}{3}e^{-4t}\frac{t^{3/2}}{\sqrt{\pi}}$$

$$\therefore L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^4L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{3/2}}\right\}$$

∴ by second shifting theorem, we have

$$L^{-1}\left\{e^{-ap}f(p)\right\} = \begin{cases}F(t-a) & t>a\\ 0 & t<a\end{cases}$$

$$\text{i.e, } e^4L^{-1}\left\{e^{-3p}f(p)\right\} = \begin{cases}e^4F(t-3) & t>a\\ 0 & t<a\end{cases}$$

$$e^4 L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{3/2}}\right\}$$

$$= \begin{cases} e^4 \frac{4}{3\sqrt{\pi}} e^{-4(t-3)} (t-3)^{3/2}; & t>3 \\ 0 & , t<3 \end{cases}$$

$$= \begin{cases} \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2}, & t>3 \\ 0 & , t<3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}} e^{-4(t-4)} (t-3)^{3/2} H(t-3)$$

interms of Heaviside unit step function.

→ For $a>0$, Prove that $L^{-1}\{f(p)\} = F(t)$ implies $L^{-1}\{f(ap+b)\} = \frac{1}{a} e^{-bt/a} F(t/a)$

Sol^n: we have $L^{-1}\{f(p)\} = F(t)$

$$\Rightarrow f(p) = L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$$

$$\Rightarrow f(ap+b) = \int_0^\infty e^{-(ap+b)t} F(t)\,dt$$

$$= \int_0^\infty e^{-apt} e^{-bt} F(t)\,dt$$

$$= \int_0^\infty e^{-px} e^{-bx/a} F(x/a) \frac{dx}{a}$$

Putting $at = x$
$t = x/a$
$\Rightarrow dt = \frac{1}{a} dx$

$$= \int_0^\infty e^{-pt} \left\{\frac{1}{a} e^{-bt/a} F(t/a)\right\} dt$$

by property of definite integrals $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

$$= L\left\{\frac{1}{a} e^{-bt/a} F(t/a)\right\}$$

$$\therefore f(ap+b) = L\left\{\frac{1}{a} e^{-bt/a} F(t/a)\right\}$$

$$\Rightarrow L^{-1}\{f(ap+b)\} = \frac{1}{a} e^{-bt/a} F(t/a)$$

→ Evaluate $L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\}$

Sol^n: $L^{-1}\left\{\frac{p+1}{p^2+6p+25}\right\} = L^{-1}\left\{\frac{p+1}{(p+3)^2+16}\right\}$

$$= L^{-1}\left\{\frac{(p+3)-2}{(p+3)^2+4^2}\right\}$$

$$= e^{-3t} L^{-1}\left\{\frac{p-2}{p^2+4^2}\right\}$$

$$= e^{-3t}\left[L^{-1}\left\{\frac{p}{p^2+4^2}\right\} - L^{-1}\left\{\frac{2}{p^2+4^2}\right\}\right]$$

$$= e^{-3t}\left[\cos 2t - 2\cdot\frac{1}{4}\sin 4t\right]$$

$$= e^{-3t}\left[\cos 2t - \tfrac{1}{2}\sin 4t\right]$$

→ Evaluate $L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\}$

Sol^n:

$$L^{-1}\left\{\frac{6p^2+22p+18}{p^3+6p^2+11p+6}\right\} = L^{-1}\left\{\frac{6p^2+22p+18}{(p+1)(p+2)(p+3)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1} + \frac{2}{p+2} + \frac{3}{p+3}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+1}\right\} + L^{-1}\left\{\frac{2}{p+2}\right\} + L^{-1}\left\{\frac{3}{p+3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{p}\right\} + 2e^{-2t} L^{-1}\left\{\frac{1}{p}\right\} + 3e^{-3t} L^{-1}\left\{\frac{1}{p}\right\}$$

$$= e^{-t} + 2e^{-2t} + 3e^{-3t}$$

→ Evaluate $L^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

sol'n: $L^{-1}\left\{\frac{3(p^2+2p+3)}{(p^2+2p+2)(p^2+2p+5)}\right\}$

$$= L^{-1}\left\{\frac{1}{p^2+2p+2}+\frac{2}{p^2+2p+5}\right\}$$

$$= L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}+2L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$$

$$= e^{-t}\sin t + 2e^{-t}\sin 2t$$

→ Prove that $L^{-1}\left\{\frac{4p+5}{(p-1)^2(p+2)}\right\}$

$$= 3te^{t}+\tfrac{1}{3}e^{t}-\tfrac{1}{3}e^{-2t}$$

→ Evaluate $L^{-1}\left\{\frac{4p+5}{(p-4)^2(p+3)}\right\}$

→ Find $L^{-1}\left\{\frac{5p^2-15p-11}{(p+1)(p-2)^3}\right\}$

sol'n: $\frac{5p^2-15p-11}{(p+1)(p-2)^3}$

$$= \frac{-1}{3(p+1)}+\frac{1}{3(p-2)}+\frac{4}{(p-2)^2}-\frac{7}{(p-2)^3}$$

→ Prove that

$$L^{-1}\left\{\frac{2p+1}{(p+2)^2(p-1)}\right\} = \tfrac{1}{3}t\left[e^{t}-e^{-2t}\right]$$

→ Find $L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$

→ Prove that

$$L^{-1}\left\{\frac{p}{(p^2-2p+2)(p^2+2p+2)}\right\} = \tfrac{1}{2}\sin t \cdot \sinh t$$

→ Prove that

$$L^{-1}\left\{\frac{p}{p^4+p^2+1}\right\} = \frac{2}{\sqrt{3}}\sinh\tfrac{1}{2}t\,\sin\tfrac{1}{2}\sqrt{3}t$$

$$\left[\frac{p}{p^4+p^2+1} = \frac{p}{(p^2+1)^2-p^2} = \frac{p}{(p^2+1+p)(p^2+1-p)}\right]$$

→ Find $L^{-1}\left\{\frac{2p^3+2p^2+4p+1}{(p^2+1)(p^2+p+1)}\right\}$

→ Show that

$$L^{-1}\left\{\frac{p^2}{p^4+4a^4}\right\} = \frac{1}{2a}\left[\cosh at\,\sin at+\sinh at\,\cos at\right]$$

sol'n: $L.H.S = L^{-1}\left\{\frac{p^2}{(p^2+2a^2)^2-4a^2p^2}\right\}$

$$= L^{-1}\left\{\frac{p^2}{(p^2+2a^2-2ap)(p^2+2a^2+2ap)}\right\}$$

$$= L^{-1}\left\{\frac{p}{4a(p^2-2ap+2a^2)}-\frac{p}{4a(p^2+2ap+2a^2)}\right\}$$

$$= \frac{1}{4a}L^{-1}\left\{\frac{(p-a)+a}{(p-a)^2+a^2}-\frac{(p+a)-a}{(p+a)^2+a^2}\right\}$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left\{\frac{p+a}{p^2+a^2}\right\}-e^{-at}L^{-1}\left\{\frac{p-a}{p^2+a^2}\right\}\right]$$

$$= \frac{1}{4a}\left[e^{at}\left(L^{-1}\left\{\frac{p}{p^2+a^2}\right\}+L^{-1}\left\{\frac{a}{p^2+a^2}\right\}\right)\right.$$
$$\left.-e^{-at}\left(L^{-1}\left\{\frac{p}{p^2+a^2}\right\}-L^{-1}\left\{\frac{a}{p^2+a^2}\right\}\right)\right]$$

Inverse Laplace Transform of derivatives:

Theorem: If $L^{-1}\{f(p)\} = F(t)$, then

$L^{-1}\{f^n(p)\} = (-1)^n t^n F(t)$ i.e,

$$L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t),\ n = 1,2,3\ldots$$

Proof: we know that

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p)$$
$$= (-1)^n f^n(p)$$

$$\therefore L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n F(t)$$

$$\text{(or) } L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n F(t)$$

Note: The result of this theorem can also be written as

$$L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = L^{-1}\{f^{(n)}(p)\} = (-1)^n t^n L^{-1}\{f(p)\}$$

→ Find $L^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\}$

Sol'n: Let $f(p) = \frac{1}{p^2-a^2}$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-2p}{(p^2-a^2)^2}$$

$$\Rightarrow \frac{p}{(p^2-a^2)^2} = -\frac{1}{2}\frac{d}{dp} f(p)$$

$$= -\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)$$

$f(p) = \frac{1}{p^2-a^2}$

$L\{F(t)\} = f(p)$

$F(t) = L^{-1}\{f(p)\}$

$F(t) = L^{-1}\left\{\frac{1}{p^2-a^2}\right\}$

$= \frac{1}{a}\sinh at$

$$\therefore L^{-1}\left\{\frac{p}{(p^2-a^2)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)\right\}$$

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2-a^2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{p^2-a^2}\right\} \text{ by formula}$$

$$= \frac{1}{2}t\left(\frac{1}{a}\right)\sinh at$$

$$= \frac{t}{2a}\sinh at$$

$L^{-1}\left\{\frac{1}{p^2-a^2}\right\} = \frac{1}{a}$

→ Find (i) $L^{-1}\left\{\frac{p}{(p^2-16)^2}\right\}$ (ii) $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$ (iii) $L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$

→ Evaluate

(i) $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$ (ii) $L^{-1}\left\{\frac{p+2}{(p^2+4p+5)^2}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{p+1}{(p^2+2p+2)^2}\right\}$

Let $f(p) = \frac{1}{p^2+2p+2} = (p^2+2p+2)^{-1}$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{-(2p+2)}{(p^2+2p+2)^2} = \frac{-2(p+1)}{(p^2+2p+2)^2}$$

$$\Rightarrow \frac{p+1}{(p^2+2p+2)^2} = -\frac{1}{2}\frac{d}{dp} f(p)$$

$$= -\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+2p+2}\right)$$

$$\left(\because f(p) = \frac{1}{p^2+2p+2}\right)$$

$$\therefore L^{-1}\left(\frac{p+1}{(p^2+2p+2)^2}\right) = -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+2p+2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{p^2+2p+2}\right\}$$

$$= \frac{1}{2} t L^{-1}\left\{\frac{1}{(p+1)^2+1}\right\}$$

$$= \frac{1}{2} t e^{-t} L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{1}{2} t e^{-t} \sin t$$

→ Evaluate $L^{-1}\left\{\log \frac{p+3}{p+2}\right\}$

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Sol'n: Let $f(p) = \log\left(\frac{p+3}{p+2}\right)$

$$= \log(p+3) - \log(p+2)$$

$$\Rightarrow \frac{d}{dp} f(p) = \frac{1}{p+3} - \frac{1}{p+2}$$

$$\Rightarrow L^{-1}\left\{\frac{d}{dp} f(p)\right\} = L^{-1}\left\{\frac{1}{p+3} - \frac{1}{p+2}\right\}$$

$$= L^{-1}\left\{\frac{1}{p+3}\right\} - L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$\Rightarrow (-1)^1 t L^{-1}\{f(p)\} = e^{-3t} - e^{-2t}$$

$$\Rightarrow L^{-1}\{f(p)\} = \frac{1}{t}(e^{-2t} - e^{-3t})$$

$$\Rightarrow L^{-1} \log\left\{\frac{p+3}{p+2}\right\} = \frac{1}{t}(e^{-2t} - e^{-3t})$$

$\because L^{-1}\left(\frac{1}{p+3}\right) = e^{-3t} L^{-1}\left\{\frac{1}{p}\right\}$
$= e^{-3t}(1)$
$= e^{-3t}$

→ Find (i) $L^{-1}\left\{\log\left(\frac{1+p}{p}\right)\right\}$

(ii) $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$

Hint: $L^{-1} \log\left(1+\frac{1}{p^2}\right) = L^{-1} \log\left(\frac{p^2+1}{p^2}\right)$

(iii) $L^{-1}\left\{\log\left(1-\frac{1}{p^2}\right)\right\}$ (iv) $L^{-1}\left\{\log\left(1+\frac{\omega^2}{p^2}\right)\right\}$

→ Find $L^{-1}\left\{\frac{1}{(p+a)^3}\right\}$

Sol'n: Let $f(p) = \frac{1}{p+a}$

$$\Rightarrow \frac{d^2}{dp^2} f(p) = \frac{2}{(p+a)^3}$$

$$\Rightarrow \frac{1}{(p+a)^3} = \frac{1}{2}\frac{d^2}{dp^2} f(p)$$

$$= \frac{1}{2}\frac{d^2}{dp^2}\left(\frac{1}{p+a}\right)$$

$$\therefore L^{-1}\left\{\frac{1}{(p+a)^3}\right\} = \frac{1}{2} L^{-1}\left\{\frac{d^2}{dp^2}\left(\frac{1}{p+a}\right)\right\}$$

$$= \frac{1}{2}(-1)^2 t^2 L^{-1}\left(\frac{1}{p+a}\right)$$

$$= \frac{1}{2} t^2 e^{-at}$$

→ **Inverse Laplace Transform of Integrals:**

Theorem: If $L^{-1}\{f(p)\} = F(t)$, then

$$L^{-1}\left\{\int_p^\infty f(x)\,dx\right\} = \frac{F(t)}{t}$$

Proof: we know that

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\,dx$$

provided $\lim_{t\to 0} \frac{F(t)}{t}$ exists.

$$\therefore L^{-1}\left\{\int_p^\infty f(x)\,dx\right\} = \frac{F(t)}{t}$$

→ Prove that $L^{-1}\left\{\int_p^\infty \frac{1}{p(p+1)}\,dp\right\} = \frac{1-e^{-t}}{t}$

Sol'n: Let $f(p) = \frac{1}{p(p+1)}$ and

$$L^{-1}\{f(p)\} = F(t)$$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{P(P+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{P} - \frac{1}{P+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{P}\right\} - L^{-1}\left\{\frac{1}{P+1}\right\} = 1 - e^{-t}$$

Now,

$$L^{-1}\left\{\int_P^\infty f(P)\,dP\right\} = \frac{F(t)}{t}$$

$$\Rightarrow L^{-1}\left\{\int_P^\infty \frac{1}{P(P+1)}\,dP\right\} = \frac{1-e^{-t}}{t}$$

* Multiplication by powers of P:

Theorem: If $L^{-1}\{f(P)\} = F(t)$ and $F(0) = 0$ then $L^{-1}\{Pf(P)\} = F'(t)$.

Proof: we know that

$$L\{F'(t)\} = PL\{F(t)\} - F(0)$$

$$= Pf(P) - F(0) \qquad \because L\{F(t)\} = f(P)$$

Since $F(0) = 0$

$$L\{F'(t)\} = Pf(P)$$

$$\Rightarrow L^{-1}\{Pf(P)\} = F'(t)$$

Note: If $F(0) = F'(0) = F''(0) = \cdots$ $\cdots F^{n-1}(0) = 0$,

then $L^{-1}\{P^n f(P)\} = F^{(n)}(t)$

→ Find $L^{-1}\left\{\frac{P^2}{(P^2+4)^2}\right\} = L^{-1}\left\{P\cdot\frac{P}{(P^2+4)^2}\right\}$

$$= L^{-1}\{Pg(P)\}$$

Say where $g(P) = \frac{P}{(P^2+4)^2}$

Sol'n: Let $f(P) = \frac{1}{P^2+4}$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{1}{P^2+4}\right\} = \frac{1}{2}\sin 2t \quad \text{--- (1)}$$

and $\frac{d}{dP}f(P) = \frac{-2P}{(P^2+4)^2}$

$$\Rightarrow L^{-1}\left\{\frac{d}{dP}f(P)\right\} = L^{-1}\left\{\frac{-2P}{(P^2+4)^2}\right\}$$

$$= -2L^{-1}\left\{\frac{P}{(P^2+4)^2}\right\}$$

$$\Rightarrow (-1)^1 tF(t) = -2L^{-1}\left\{\frac{P}{(P^2+4)^2}\right\}$$

$$\Rightarrow L^{-1}\left\{\frac{P}{(P^2+4)^2}\right\} = \frac{1}{2}tF(t)$$

$$= \frac{1}{2}t\,\frac{1}{2}\sin 2t \quad \text{by (1)}$$

$$= \frac{1}{4}t\sin 2t = G(t) \text{ say}$$

Let $g(P) = \frac{P}{(P^2+4)^2}$ and $G(t) = \frac{1}{4}t\sin 2t$

Here $G(0) = 0$

Hence we have $L^{-1}\{Pg(P)\} = G'(t)$.

$$\Rightarrow L^{-1}\left\{P\cdot\frac{P}{(P^2+4)^2}\right\} = \frac{d}{dt}\left[\frac{t}{4}\sin 2t\right]$$

$$\Rightarrow L^{-1}\left\{\frac{P^2}{(P^2+4)^2}\right\} = \frac{1}{4}\left[\sin 2t + 2t\cos 2t\right]$$

* Division by Powers of P:

Theorem I: Let $L^{-1}\{f(P)\} = F(t)$. If $F(t)$ is sectionally continuous and of exponential order 'a' such that $\lim_{t\to 0} \frac{F(t)}{t}$ exists, then for $P>a$

$$L^{-1}\left\{\frac{f(P)}{P}\right\} = \int_0^t F(x)\,dx$$

Proof: Let $G(t) = \int_0^t F(x)\,dx$

Then $G'(t) = \frac{d}{dt}\int_0^t F(x)\,dx$

$= F(t)$ (by Leibnitz rule)

and $G(0)=0$

W.K.T. $L\{G'(t)\} = PL\{G(t)\} - G(0)$

$L\{F(t)\} = PLG\{t\} - 0$

$\Rightarrow f(P) = PL\{G(t)\}$

$\Rightarrow L\{G(t)\} = \frac{f(P)}{P}$

Hence $L^{-1}\left\{\frac{f(P)}{P}\right\} = G(t)$

$= \int_0^t F(x)\,dx$

$\therefore L^{-1}\left\{\frac{f(P)}{P}\right\} = \int_0^t F(x)\,dx$

Theorem-II: Let $L^{-1}\{f(P)\} = F(t)$, then

$$L^{-1}\left\{\frac{f(P)}{P^2}\right\} = \int_0^t\int_0^y F(x)\,dx\,dy = \int_0^t\int_0^t F(t)\,dt$$

Proof: Let $G(t) = \int_0^t\int_0^y F(x)\,dx\,dy$

$G'(t) = \frac{d}{dt}\int_0^t\left[\int_0^y F(x)\,dx\right]dy$

$= \int_0^t\left(\frac{\partial}{\partial t}\int_0^y F(x)\,dx\right)dy + \left[\int_0^y F(x)\,dx\right]_{y=t}\frac{dt}{dt} + 0$

$= 0 + \int_0^t F(t)\,dt$

$G'(t) = \int_0^t F(t)\,dt$

$\Rightarrow G'(t) = \int_0^t F(x)\,dx$ (By the Prop. definite integral)

$G''(t) = F(t)$ (by Leebnitz rule)

Also $G(0) = 0 = G'(0)$

Now $L\{G''(t)\} = P^2 L\{G(t) - PG(0) - G'(0)\}$

$\Rightarrow L\{F(t)\} = P^2 L\{G(t)\}$

$\Rightarrow f(P) = P^2 L\{G(t)\}$

$\Rightarrow \frac{1}{P^2} f(P) = L\{G(t)\}$

$\Rightarrow L^{-1}\left\{\frac{1}{P^2} f(P)\right\} = G(t) = \int_0^t\int_0^y F(x)\,dx\,dy$

In general

$$L^{-1}\left\{\frac{f(P)}{P^n}\right\} = \int_0^t \cdots \int_0^t\int_0^t F(t)\,dt^n.$$

→ Find $L^{-1}\left\{\frac{1}{P^3(P^2+1)}\right\}$.

It is the form of $L^{-1}\left\{\frac{f(P)}{P^3}\right\}$

Sol'n: Since $L^{-1}\left\{\frac{1}{P^2+1}\right\} = \sin t = F(t)$

(Here $F(t) = \sin t$)

$\left[\because L^{-1}\left\{\frac{f(P)}{P}\right\} = \int_0^t F(x)\,dx\right]$

$\therefore L^{-1}\left\{\frac{1}{P(P^2+1)}\right\} = \int_0^t \sin x\,dx$

$= \left[-\cos x\right]_0^t$

$= \cos t + 1 = 1 - \cos t = F_1(t)$, say

Let $L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\}$.

Clearly it is in the form of $L^{-1}\left\{\frac{f_1(p)}{p}\right\}$

[where $f_1(p) = \frac{1}{p(p^2+1)}$

and $L^{-1} f_1(p) = L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} = 1 - \cos t$
$= F_1(t)$]

$$\therefore L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\} = \int_0^t F_1(x)\,dx$$
$$= \int_0^t (1-\cos x)\,dx$$
$$= [x - \sin x]_0^t$$
$$= t - \sin t$$

Now let $L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\}$

clearly it is in the form of $L^{-1}\left\{\frac{f_2(p)}{p}\right\}$

where $f_2(p) = \frac{1}{p^2(p^2+1)}$

and $L^{-1}(f_2(p)) = t - \sin t = F_2(t)$, say

$$\therefore L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} = \int_0^t F_2(x)\,dx$$
$$= \int_0^t (x - \sin x)\,dx$$
$$= \left[\frac{x^2}{2} + \cos x\right]_0^t$$
$$= \frac{t^2}{2} + \cos t - 1$$

→ Find $L^{-1}\left\{\frac{1}{p(p^2+1)}\right\}$

→ Find $L^{-1}\left\{\frac{1}{p}\log\frac{p+2}{p+1}\right\}$

Sol'n: $f(p) = \log\frac{p+2}{p+1}$
$$= \log(p+2) - \log(p+1)$$
$$f'(p) = \frac{1}{p+2} - \frac{1}{p+1}$$
$$\therefore L^{-1}\{f'(p)\} = L^{-1}\left\{\frac{1}{p+2} - \frac{1}{p+1}\right\}$$
$$-t L^{-1}\{f(p)\} = L^{-1}\left(\frac{1}{p+2}\right) - L^{-1}\left(\frac{1}{p+1}\right)$$
$$= e^{-2t} - e^{-t} \quad \left[\because L^{-1} f'(p) = (-1)^1 t\, L^{-1} f(p)\right]$$
$$\Rightarrow L^{-1}\{f(p)\} = \frac{e^{-t} - e^{-2t}}{t} = F(t), \text{ say}$$
$$\therefore L^{-1}\left\{\frac{1}{p} f(p)\right\} = \int_0^t F(x)\,dx$$
$$= \int_0^t \frac{e^{-x} - e^{-2x}}{x}\,dx$$
$$\therefore L^{-1}\left\{\frac{1}{p} f(p)\right\} = \int_0^t \left[\frac{e^{-x} - e^{-2x}}{x}\right] dx$$

→ Find $L^{-1}\left\{\frac{1}{p}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}\right\}$

→ Find $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}$

Sol'n: Given $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\}$

Clearly it is in the form of $L^{-1}\left\{\frac{f(p)}{p}\right\}$

Let $f(p) = \frac{1}{(p+1)^3}$

$$\Rightarrow L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{(p+1)^3}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$= e^{-t}\frac{t^2}{2} = F(t), \text{ say}$$

Now $L^{-1}\left\{\frac{1}{p(p+1)^3}\right\} = \int_0^t F(x)\,dx$

$$= \int_0^t e^{-x}\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^t e^{-x} x^2\,dx$$

$$= \frac{1}{2}\left\{\left(-e^{-x}x^2\right)_0^t + 2\int_0^t e^{-x} x\,dx\right]$$

$$= \frac{1}{2}\left\{\left[-e^{-t}t^2+0\right] + 2\left[-e^{-x}x\right]_0^t - 2\left[e^{-x}\right]_0^t\right\}$$

$$= \frac{1}{2}e^{-t}t^2 + (-e^{-t}t+0) - (e^{-t}-1)$$

$$= -\frac{1}{2}e^{-t}t^2 - e^{-t}t - e^{-t} + 1$$

$$= 1 - e^{-t}\left[\frac{t^2}{2}+t+1\right]$$

→ If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t}{2}\sin t$,

find $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$

Sol'n: $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{1}{p}\cdot\frac{p}{(p^2+1)^2}\right\}$

which is in the form of $L^{-1}\left\{\frac{f(p)}{p}\right\}$

Here $f(p) = \dfrac{p}{(p^2+1)^2}$

and we know that

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{t}{2}\sin t = F(t), \text{ say}$$

$$L^{-1}\left\{\frac{1}{p}\frac{p}{(p^2+1)^2}\right\} = \int_0^t F(x)\,dx$$

$$= \int_0^t \frac{x}{2}\sin x\,dx$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} = \frac{1}{2}\left[-x\cos x+\sin x\right]_0^t$$

$$= \frac{1}{2}\left(\sin t - t\cos t\right)$$