Virtual Work

§ 1. **Displacement.** Suppose a particle moves from a position P to any other position Q by whatever path. Then the vector $\overrightarrow{PQ}$ is called the displacement of the particle with regard to P. If $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of the points P and Q referred to some origin O, then the displacement of the particle from P to Q is the vector

$$\overrightarrow{PQ}=\mathbf{r}'-\mathbf{r}.$$

§ 2. **A rigid body.** A rigid body is a collection of particles such that for any displacement of the body the distance between any two particles of the body remains the same in magnitude. Thus in the case of the rigid body, referred to some origin O, if $\mathbf{r}_1$, $\mathbf{r}_2$ are respectively the position vectors of the two particles before displacement and $\mathbf{r}_1'$, $\mathbf{r}_2'$ are their respective position vectors after displacement, then the condition of rigidity of the body requires that their mutual distance must remain the same before and after the displacement *i.e.*,

$$|\mathbf{r}_2-\mathbf{r}_1|=|\mathbf{r}_2'-\mathbf{r}_1'|, \text{ or } (\mathbf{r}_2-\mathbf{r}_1)^2=(\mathbf{r}_2'-\mathbf{r}_1')^2.$$

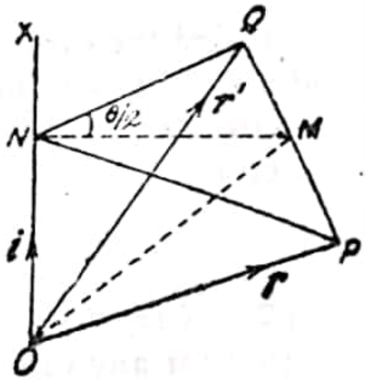
§ 3. **Kinds of displacement of a rigid body.** (*Translation, Rotation and General*).

One way of displacing a particle of a rigid body from one position to any other position is what we call *pure translation*. In this case the displacement is brought about without rotating the body. Thus if $\mathbf{r}$ be the position vector of a particle P referred to some origin O and if the particle is displaced from P to Q by giving a displacement $\mathbf{u}$ in the direction of OP only, then this displacement is called *translation* and we say that the displacement $\overrightarrow{PQ}=\mathbf{u}$ is a translation.

The other way of displacement of a particle is called *pure rotation*. In this case the displacement of the particle is brought about only by rotating the body about a fixed point, say O, so that the distance of the particle from the fixed point O does not change in the two positions P and Q before and after the displacement. Thus in the case of pure rotation, we have $OP=OQ$ in length but their directions are generally different.

If both the displacements translation and rotation take place simultaneously we call it a *general displacement* of the particle or of the body.

§ 4. **Rotation of a rigid body about a point.**



Suppose a rigid body rotates about a fixed point O. On account of this rotation suppose a particle is displaced from P to Q where

$$\overrightarrow{OP}=\mathbf{r} \text{ and } \overrightarrow{OQ}=\mathbf{r}'.$$

Since the displacement of the particle is that of rotation only about O, therefore

the length OP=the length OQ.

Let M be the middle point of PQ, so that

$$\overrightarrow{OM}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}').$$

Draw a line OX, through O, perpendicular to PQ and let $\mathbf{i}$ be a unit vector in the direction OX. Also let N be the foot of the perpendicular from M to OX.

Since $OP=OQ$ and M is the middle point of PQ, therefore from the $\triangle OPQ$ we observe that PQ is perpendicular to OM.

Thus PQ, being perpendicular to OM and ON both, is perpendicular to the plane OMN. Consequently PQ is perpendicular to MN because MN lies in the plane OMN. Thus NM is the perpendicular bisector of PQ and so we have

$$NP=NQ.$$

If $\angle PNQ=\theta$, then $\angle PNM=\angle MNQ=\tfrac{1}{2}\theta$.

The vector $\overrightarrow{PQ}$ is perpendicular to the vectors $\mathbf{i}$ and $\overrightarrow{OM}$ which implies that $\overrightarrow{PQ}$ is parallel to the vector $\mathbf{i}\times\overrightarrow{OM}$.

Also $|\overrightarrow{PQ}|=PQ=2PM=2NM\tan\tfrac{1}{2}\theta$

$$=2\,(OM\sin\angle MON)\tan\tfrac{1}{2}\theta=2\,|\mathbf{i}\times\overrightarrow{OM}|\tan\tfrac{1}{2}\theta.$$

$$[\because\ |\mathbf{i}\times\overrightarrow{OM}|=1.OM.\sin\angle MON]$$

Thus $\overrightarrow{PQ}$ is parallel to the vector $\mathbf{i}\times\overrightarrow{OM}$ and

$$|\overrightarrow{PQ}|=(2\tan\tfrac{1}{2}\theta)\,|\mathbf{i}\times\overrightarrow{OM}|.$$

Therefore by the definition of the multiplication of a vector by a scalar, we have

$$\overrightarrow{PQ}=(2\tan\tfrac{1}{2}\theta)\,\mathbf{i}\times\overrightarrow{OM}=(2\tan\tfrac{1}{2}\theta\,\mathbf{i})\times\overrightarrow{OM}.$$

Thus if $\mathbf{q}$ is the displacement of the particle from P to Q due to this rotation of the rigid body, we have

$$\mathbf{q}=\overrightarrow{PQ}=\mathbf{e}\times\mathbf{h}, \text{ where}$$

$$\mathbf{e}=(2\tan\tfrac{1}{2}\theta)\,\mathbf{i} \text{ and } \mathbf{h}=\overrightarrow{OM}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}').$$

The vector $\mathbf{e}$ is called the finite rotation about O which brings the particle from $\mathbf{r}$ to $\mathbf{r}'$, the direction of the vector $\mathbf{e}$ is called the axis of rotation and θ is called the angle of rotation.

When the rotation is small, Q tends to P *i.e.*, $\mathbf{r}'$ tends to $\mathbf{r}$, and then we have

$$\mathbf{h}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}')=\tfrac{1}{2}(\mathbf{r}+\mathbf{r})=\mathbf{r}$$

which leads to

$$\mathbf{q}=\mathbf{e}\times\mathbf{r}.$$

Remark. It can be easily seen that the displacement about a point is always a rotation. Also it can be easily shown that any displacement of a rigid body can be reduced to a translation together with a rotation.

§ 5. Position vector of a point after a general displacement.

Let $\mathbf{r}$ be the position vector of a point P referred to some origin O. If the particle is displaced from P to Q by giving only a displacement $\mathbf{u}$ in the direction of OP (*i.e.*, translation), then

$$\overrightarrow{PQ}=\mathbf{u}.$$

Also if P is displaced to Q by giving only a rotation $\mathbf{e}$ about O, then

$$\overrightarrow{PQ}=\mathbf{e}\times\tfrac{1}{2}(\mathbf{r}+\mathbf{r}')$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of P and Q respectively.

Now if the particle is displaced from P to Q by giving both the displacements translation $\mathbf{u}$ and rotation $\mathbf{e}$ simultaneously, then it is called a general displacement of the point. In this case, combining the above results, we have

$$\vec{PQ}=\mathbf{u}+\mathbf{e}\times\tfrac{1}{2}(\mathbf{r}+\mathbf{r}'). \qquad \ldots(1)$$

If this displacement is small, then writing $\mathbf{r}'=\mathbf{r}+d\mathbf{r}$ in the above result (1), we have

$$\vec{PQ}=\mathbf{u}+\mathbf{e}\times\tfrac{1}{2}(\mathbf{r}+\mathbf{r}+d\mathbf{r})$$
$$=\mathbf{u}+\mathbf{e}\times\mathbf{r}+\tfrac{1}{2}\mathbf{e}\times d\mathbf{r}$$
$$=\mathbf{u}+\mathbf{e}\times\mathbf{r},$$ because both the vectors $\mathbf{e}$ and $d\mathbf{r}$ are small.

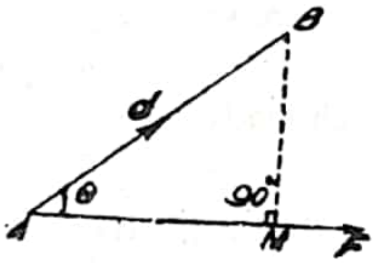
But $\vec{PQ}=\vec{OQ}-\vec{OP}=(\mathbf{r}+d\mathbf{r})-\mathbf{r}=d\mathbf{r}$.

Therefore if $\mathbf{r}$ is the position vector of a point P and if it is given a small displacement $d\mathbf{r}$ consisting of a small translation $\mathbf{u}$ and a small rotation $\mathbf{e}$, then we have

$$d\mathbf{r}=\mathbf{u}+\mathbf{e}\times\mathbf{r}. \qquad \text{[Remember]}$$

§ 6. Work done by a force. **[Meerut 74]**

Suppose a force represented by the vector $\mathbf{F}$ acts at the point A. Let the point A be displaced to the point B where $\vec{AB}=\mathbf{d}$.



Then the work W done by the force $\mathbf{F}$ during the displacement $\mathbf{d}$ of its point of application is defined as

$$W=\mathbf{F}\cdot\mathbf{d}, \qquad \ldots(1)$$

where $\mathbf{F}\cdot\mathbf{d}$ is the scalar product of the vectors $\mathbf{F}$ and $\mathbf{d}$.

Let θ be the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$ If $F=|\mathbf{F}|$ and $d=|\mathbf{d}|=AB$, then using the definition of the scalar product of two vectors, the equation (1) defining the work may be written as

$$W=F\,d\cos\theta. \qquad \ldots(2)$$

Obviously $d\cos\theta$ is the displacement of the point of application of the force $\mathbf{F}$ in the direction of the force. *Hence the work done by a force is equal to the magnitude of the force multiplied by the displacement of the point of application of the force in the direction of the force.*

From the equation (2) we make the following observations.

(i) If $\theta=\frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force is perpendicular to the direction of the force, then $W=0$.

(ii) If $0 \leqslant \theta < \frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is in the direction of the force, then W is positive.

(iii) If $\frac{1}{2}\pi < \theta \leqslant \pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is opposite to the direction of the force, then W is negative.

Remark. The work done by a force $\mathbf{F}$ acting at the point $\mathbf{r}$ during a small displacement $d\mathbf{r}$ of its point of application is

$$=\mathbf{F} \cdot d\mathbf{r}.$$

§ 7. Work done by a system of concurrent forces.

Theorem. *The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the separate forces.*

Proof. Let there be n forces represented by the vectors $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at a point P. Then during any displacement of P represented by the vector $\mathbf{d}$, the works done by the separate forces are respectively equal to

$$\mathbf{F}_1 \cdot \mathbf{d}, \ \mathbf{F}_2 \cdot \mathbf{d}, \ldots, \ \mathbf{F}_n \cdot \mathbf{d}.$$

The total work done is therefore

$$=\mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_2 \cdot \mathbf{d} + \ldots + \mathbf{F}_n \cdot \mathbf{d}$$

$$=(\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n) \cdot \mathbf{d}$$

[$\because$ scalar product is distributive]

$=\mathbf{R} \cdot \mathbf{d}$, where $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$ is the vector representing the resultant of these n concurrent forces.

But $\mathbf{R} \cdot \mathbf{d}$ is the work done by the resultant $\mathbf{R}$ during the displacement $\mathbf{d}$ of the point P.

Hence we have the result.

Example. *A particle acted on by constant forces* $4\mathbf{i}+\mathbf{j}-3\mathbf{k}$ *and* $3\mathbf{i}+\mathbf{j}-\mathbf{k}$ *is displaced from the point* $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ *to the point* $5\mathbf{i}+4\mathbf{j}+\mathbf{k}$. *Find the total work done by the forces.* **[Kanpur 76]**

Solution. Let $\mathbf{R}$ be the resultant of the two concurrent forces and $\mathbf{d}$ be the displacement of their point of application. Then, we have

$$\mathbf{R} = (4\mathbf{i}+\mathbf{j}-3\mathbf{k}) + (3\mathbf{i}+\mathbf{j}-\mathbf{k}) = 7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$$

and

$$\mathbf{d} = (5\mathbf{i}+4\mathbf{j}+\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 4\mathbf{i}+2\mathbf{j}-2\mathbf{k}.$$

$\therefore$ the total work done$=\mathbf{R}\cdot\mathbf{d}$

$$=(7\mathbf{i}+2\mathbf{j}-4\mathbf{k})\cdot(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})$$
$$=28+4+8=40 \text{ units of work.}$$

§ 8. Work done by a couple during a small displacement.

Let the two forces $\mathbf{F}$ and $-\mathbf{F}$ acting on a rigid body at the points whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$, be equivalent to a couple of moment $\mathbf{G}$, then

$$\mathbf{G}=\mathbf{r}_1\times\mathbf{F}+\mathbf{r}_2\times(-\mathbf{F})=(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{F}.$$

Suppose the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$. Then

$$d\mathbf{r}_1=\mathbf{u}+\mathbf{e}\times\mathbf{r}_1$$

and $$d\mathbf{r}_2=\mathbf{u}+\mathbf{e}\times\mathbf{r}_2. \qquad \text{[Refer § 5]}$$

$\therefore$ the work done by the couple

$$=\mathbf{F}\cdot d\mathbf{r}_1+(-\mathbf{F})\cdot d\mathbf{r}_2$$
$$=\mathbf{F}\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_1)+(-\mathbf{F})\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_2)$$
$$=\mathbf{F}\cdot(\mathbf{e}\times\mathbf{r}_1)-\mathbf{F}\cdot(\mathbf{e}\times\mathbf{r}_2)$$
$$=\mathbf{e}\cdot(\mathbf{r}_1\times\mathbf{F})-\mathbf{e}\cdot(\mathbf{r}_2\times\mathbf{F}), \text{ by a property of scalar triple product}$$
$$=\mathbf{e}\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{F}=\mathbf{e}\cdot\mathbf{G},$$

which is independent of the translation and depends upon rotation only.

§ 9. Work done by a system of forces during a small displacement.

Let a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. Suppose this system of forces is equivalent to a single force $\mathbf{R}$ acting at O, together with a couple of moment $\mathbf{G}$. Then

$$\mathbf{R}=\sum_{p=1}^{n}\mathbf{F}_p \text{ and } \mathbf{G}=\sum_{p=1}^{n}\mathbf{r}_p\times\mathbf{F}_p. \qquad \ldots(1)$$

If the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, then for a typical particle displaced from $\mathbf{r}_p$ to $\mathbf{r}_p+d\mathbf{r}_p$, the general displacement $d\mathbf{r}_p$ is given by

$$d\mathbf{r}_p=\mathbf{u}+\mathbf{e}\times\mathbf{r}_p. \qquad \ldots(2)$$

$\therefore$ the work done by the system of forces during this small displacement

$$=\sum_{p=1}^{n}\mathbf{F}_p\cdot d\mathbf{r}_p$$

$$=\sum_{p=1}^{n} \mathbf{F}_p \cdot (\mathbf{u}+\mathbf{e}\times\mathbf{r}_p) \qquad \text{[by (2)]}$$

$$=\mathbf{u}\cdot\sum_{p=1}^{n} \mathbf{F}_p+\mathbf{e}\cdot\sum_{p=1}^{n} \mathbf{r}_p\times\mathbf{F}_p, \quad \text{as } \mathbf{u} \text{ and } \mathbf{e} \text{ are constant vectors}$$

$$=\mathbf{u}\cdot\mathbf{R}+\mathbf{e}\cdot\mathbf{G}, \text{ by (1).}$$

§ 10. Virtual displacement and Virtual Work.

[Meerut 73, 74]

If a number of forces act on a body and displace it, these forces do some work actually. But if the forces are in equilibrium, then they do not displace their points of application and so there is actually no work done by these forces. However, if we imagine that the forces in equilibrium undergo some small displacement and find out the work done by the forces during that displacement, then such a displacement is called **virtual displacement** and such a work is called **virtual work.**

§ 11. The principle of virtual work.

The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation.

[Meerut 80, 81, 82, 83, 84, 85P, 85S, 89, 89P, 90, 90P; Lucknow 75, 76; Allahabad 75, 78; Kanpur 76, 78, 79, 80, 81, 82, 83, 87, 88; Jiwaji 80; Gorakhpur 78, 82; Rohilkhand 79, 80, 83, 86, 89]

Proof. Let a system of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Suppose this system of forces is equivalent to a single force $\mathbf{R}=\Sigma\,\mathbf{F}_1$ acting at O, together with a couple of moment $\mathbf{G}=\Sigma\mathbf{r}_1\times\mathbf{F}_1$. Then during any small displacement of the body consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, the sum of the works done by these forces

$$=\Sigma\,\mathbf{F}_1\cdot d\mathbf{r}_1=\Sigma\,\mathbf{F}_1\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_1)$$

$$=\mathbf{u}\cdot\Sigma\,\mathbf{F}_1+\mathbf{e}\cdot\Sigma\,\mathbf{r}_1\times\mathbf{F}_1$$

$$=\mathbf{u}\cdot\mathbf{R}+\mathbf{e}\cdot\mathbf{G}. \qquad \ldots(1)$$

The condition is necessary. Suppose the given system of forces is in equilibrium. Then $\mathbf{R}=\mathbf{0}$ and $\mathbf{G}=\mathbf{0}$. Therfore, from (1), the sum of the works done by the forces is zero. Hence the condition is necessary.

The condition is sufficient. Suppose the sum of the works done by the forces during **any small displacement** is zero. Then to prove that the forces are in equilibrium. We have, from (1)

$$\mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G} = 0, \quad \ldots(2)$$

for any small displacement consisting of a uniform translation **u** and a small rotation **e** about O.

Since the result (2) holds for any small displacement, therefore taking $\mathbf{e}=\mathbf{0}$ and $\mathbf{u} \neq \mathbf{0}$, we get from (2)

$$\mathbf{u} \cdot \mathbf{R} = 0. \quad \ldots(3)$$

Again taking **u** not perpendicular to **R**, we get from (3)

$$\mathbf{R} = \mathbf{0}.$$

Now taking $\mathbf{e} \neq \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$, we get from (2)

$$\mathbf{e} \cdot \mathbf{G} = 0. \quad \ldots(4)$$

Again taking **e** not perpendicular to **G**, we get from (4)

$$\mathbf{G} = \mathbf{0}.$$

Thus if the result (2) holds for any small displacement **u** and **e**, we must have $\mathbf{R}=\mathbf{0}$ and $\mathbf{G}=\mathbf{0}$. Hence the forces are in equilibrium and this proves that the condition is sufficient.

Remark 1. The equation (2) formed by equating to zero the sum of the virtual works done by the forces is called **the equation of virtual work.**

Remark 2. The above principle of virtual work and its proof equally holds whether the forces are coplanar or not and whether the forces act upon a particle or upon a rigid body.

§ 12. Forces which are omitted in forming the equation of virtual work.

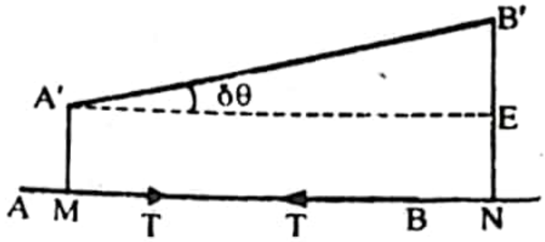
[Meerut 76, 85; Kanpur 79, 82, 87; Lucknow 75, 77; Allahabad 77; Gorakhpur 78, 80; Agra 75, 76; Jiwaji 81; Rokilkhand 78, 81]

The principle of virtual work gives us a very powerful method of attacking problems on equilibrium of forces. The mechanical advantage of this principle over other methods is that there are certain forces which are omitted in forming the equation of virtual work and consequently the solution of the problem becomes easy by this method. We now mention with proof the forces which are omitted in forming the equation of virtual work.

(*i*) *The work done by the tension of an inextensible string is zero during a small displacement.*

[Meerut 90P; Rohilkhand 77; Kanpur 83]

Let AB be an inextensible string of length l joining two points A and B of a rigid body. Let T be the tension in the string AB. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the



small angle between AB and $A'B'$. Since the string is inextensible, therefore $A'B'=AB=l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.

On account of the tension in the string AB, there are two forces each equal to T acting on A and B in opposite directions as shown in the figure. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during this displacement

$=T.AM-T.BN$ [Note that the displacement of B is in a direction opposite to that of the force T]

$=T.(AB-MB)-T.(MN-MB)$

$=T.(AB-MN)$

$=T.(AB-A'E)=T.(AB-A'B'\cos\delta\theta)$

$=T.(l-l\cos\delta\theta)$ $\qquad$ [$\because$ $AB=A'B'=l$]

$=T.l\,(1-\cos\delta\theta)$

$=T.l\left[1-\left\{1-\frac{(\delta\theta)^2}{2\,!}+\ldots\ldots\right\}\right]$, expanding $\cos\delta\theta$ in powers of $\delta\theta$

$=T.l.0$, to the first order of small quantities

$=0.$

Alternative Method.

Let T be the tension in an inextensible string connecting two points A and B whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. Then a force $\mathbf{T}$ acts at $\mathbf{r}_1$ and $-\mathbf{T}$ acts at $\mathbf{r}_2$. Since the string is inextensible, therefore for any displacement of A and B, we have

$$(\mathbf{r}_1-\mathbf{r}_2)^2=\text{constant}.$$

Differentiating,

$$2\,(\mathbf{r}_1-\mathbf{r}_2)\cdot(d\mathbf{r}_1-d\mathbf{r}_2)=0$$

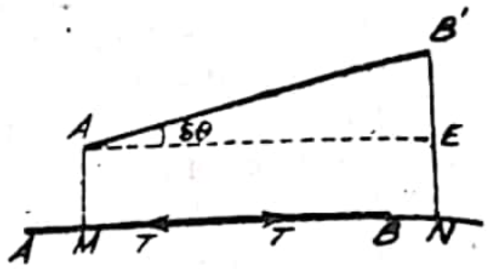
i.e., $\quad 2\mathbf{T}\cdot(d\mathbf{r}_1-d\mathbf{r}_2)=0 \qquad$ [$\because$ $\mathbf{T}$ is parallel to $\mathbf{r}_1-\mathbf{r}_2$]

or $\quad \mathbf{T}\cdot d\mathbf{r}_1+(-\mathbf{T})\cdot d\mathbf{r}_2=0,$

showing that the total work done by $\mathbf{T}$ at $\mathbf{r}_1$ and $-\mathbf{T}$ at $\mathbf{r}_2$ during a small displacement is zero. Hence the work done by the tension of an inextensible string is zero during a small displacement.

(*ii*) *The work done by the thrust of an intextensible rod is zero during a small displacement.*

Let T be the thrust in an inextensible rod AB joining two points A and B of a rigid body. Proceed as in part (i). Here the work done by the thrust in the rod AB during a small displacement



$$= -T.AM + T.BN = 0.$$

Remark. The forces of tension act inward and the forces of thrust act outwards. A common name for tension and thrust is stress. From (i) and (ii) we conclude that *if the distance between two particles of a system is invariable, the work done by the mutual action and reaction between the two particles is zero.*

(*iii*) *The reaction R of any smooth surface with which the body is in contact does no work.* For, if the surface is smooth, the reaction R on the point of contact A is along the normal to the surface. If A moves to a neighbouring point B, then the displacement AB is right angles to the direction of the force and so the work done by R is zero. If however, the surface is rough, the work done by the frictional force F *i.e.*, $F.(-AB)$ will come into the equation of virtual work.

(*iv*) *If a body rolls without sliding on any fixed surface, the work done in a small displacement by the reaction of the surface on the rolling body is zero.* For, the point of contact of the body is for the moment at rest, and so the normal reaction and the force of friction at the point of contact have zero displacements.

(*v*) *The work done by the mutual reaction between two bodies of a system is zero in any virtual displacement of the system.* For action and reaction are equal and opposite and so the work done by the action balances that done by the reaction.

(vi) *If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.* For in this case the displacement of the point of application of the force is zero.

§ **13.** (i) *To show that the work done by the tension T of an extensible string of length l during a small displacement is $-T\delta l$.*

[Allahabad 78; Jiwaji 82]·

Refer figure on page 9.

Let T be the tension in an extensible string AB of length l joining two points A and B of a rigid body. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is extensible, therefore let $A'B'=l+\delta l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.

On account of the tension in the string AB, there are two forces each equal to T acting on A and B in the opposite directions AB and BA respectively. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during the displacement

$$=T.AM-T.BN$$
$$=T.(AB-MB)-T.(MN-MB)$$
$$=T.(AB-MN)=T.(AB-A'E)$$
$$=T.(AB-A'B'\cos\delta\theta)$$
$$=T.[l-(l+\delta l)\cos\delta\theta]$$
$$=T.\left[l-(l+\delta l)\left\{1-\frac{(\delta\theta)^2}{2\,!}+\ldots\right\}\right], \text{ expanding } \cos\delta\theta \text{ in powers of } \delta\theta$$
$$=T.[l-l-\delta l], \text{ to the first order of small quantities}$$
$$=-T.\delta l.$$

(ii) Similarly it can be shown that *the work done by the thrust T of an extensible rod of length l during a small displacement is $T\delta l$.*

[Allahabad 79]

§ **14. Application of the principle of virtual work.** While applying the principle of virtual work we can give any small displacement to the system provided it is consistent with the geometrical conditions of the system. This displacement should be such as to exclude the forces which are not required and to include those which are required in the final result. After giving the displacement we must note the points and the lengths that change and that do not change during the displacement. If any length or angle etc. is to change during the displacement, we should first find its value in terms of some variable symbol and then after solving the problem we should put its value in the position of equilibrium.

In many cases we are required to find the tension of an inextensible string or the thrust or tension of an inextensible rod In order to find such a tension or thrust we must give the system a displacement in which the length of the string or the rod changes because otherwise the tension or thrust will not come in the equation of virtual work. But according to the geometrical conditions of the system we cannot give such a displacement to the body. So to get over this difficulty we replace the string or the rod by two equal and opposite forces T which are equivalent to the tension or the thirst in it. By doing so evidently the equation of virtual work is not affected while we become free to give the system a displacement in which the length of the string or the rod changes and consequently T will occur in the equation of virtual work and will thus be determined.

In any problem the virtual work done by the tension T of an extensible string of length l is $-T\delta l$ and the virtual work done by the thrust T of an extensible rod of length l is $+T\delta l$. In order to find the virtual work done by a force other than a tension or a thrust we first mark a fixed point or a fixed straight line. Then we measure the distance of the point of application of the force from this fixed point or line while moving along the line of action of the force. If this distance is x and the force is P, then the virtual work done by the force P during a small displacement is $P\delta x$ in magnitude. If the distance x is measured in the direction of the force P, the virtual work done by P is taken with positive sign and if the distance x is measured in the direction opposite to that of the force P, the virtual work done by P is taken with negative sign.

Equating to zero the total sum of the virtual works done by the forces, we get the equation of the virtual work. Solving this equation we get the value of the required thing to be determined.

If $f(x)$ is a function of x, then during a small displacement in which x changes to $x+\delta x$, we have

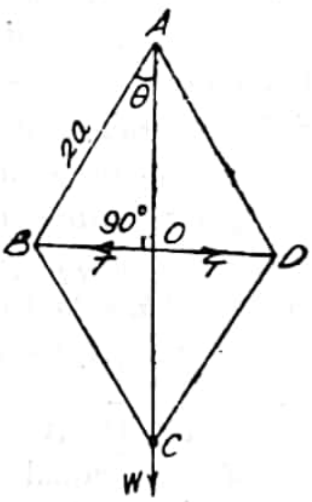
$$\delta f(x)=f(x+\delta x)-f(x)$$
$$=f(x)+\frac{\delta x}{1!}f'(x)+\ldots-f(x),$$
expanding $f(x+\delta x)$ by Taylor's theorem
$$=f'(x)\,\delta x, \text{ to the first order of small quantities.}$$

In many cases, the only forces that remain in the equation of virtual work are those due to gravity. In such cases if W is the total weight and $\bar{z}$ the height or depth of its point of application (*i.e.*, the centre of gravity of the system) above or below a fixed horizontal level, then by the principle of virtual work for the equilibrium of the body we must have $W\delta\bar{z}=0$ *i.e.*, $d\bar{z}=0$, which shows that $\bar{z}$ is a *maximum or minimum* in the position of equilibrium.

Illustrative Examples

Ex. 1. *Five weightless rods of equal length are jointed together so as to form a rhombus ABCD with one diagonal BD. If a weight W be attached to C and the system be suspended from A, show that there is a thrust in BD equal to* $W/\sqrt{3}$.

[Lucknow 76, 79; Kanpur 86]



Sol. Five equal weightless rods AB, BC, CD, DA and BD form the rhombus $ABCD$ and the diagonal BD. The system is suspended from A and a weight W is attached to C. Since the force of reaction at the point of suspension A balances the weight W at C, therefore the line AC must be vertical and so BD is horizontal. [Note that the diagonals of a rhombus bisect each other at right angles].

The rod BD prevents the points B and D from moving towards O and so it pushes them outwards, showing that there is a thrust in BD.

Let T be the thrust in the rod BD. Let $2a$ be the length of each of the rods AB, BC, CD and DA and let $\angle BAC=\theta$. In the position of equilibrium BD is also equal to $2a$ and so in the position of equilibrium ABD is an equilateral triangle and $\theta=\pi/6$.

To find the thrust T in the rod BD we shall have to give the system a displacement in which BD must change. So we replace the rod BD by two equal and opposite forces T as shown in the figure and then the distance BD can be changed.

Now we give the system a small symmetrical virtual displacement in which θ changes to $\theta+\delta\theta$. The point A remains fixed and so the distances will be measured from A. The points B, C and D change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes. The angle BOA will remain $90°$ because even after the displacement the figure remains a rhombus.

We have $\qquad BD=2BO=2AB\sin\theta=4a\sin\theta,$

and $\qquad AC=2AO=2.2a\cos\theta=4a\cos\theta.$

By the principle of virtual work, we have

$$T\delta(4a\sin\theta)+W\delta(4a\cos\theta)=0$$

or $$4aT\cos\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$$

or $$4a(T\cos\theta-W\sin\theta)\,\delta\theta=0$$

or $$T\cos\theta-W\sin\theta=0 \qquad [\because \delta\theta\neq 0]$$

or $$T\cos\theta=W\sin\theta$$

or $$T=W\tan\theta.$$

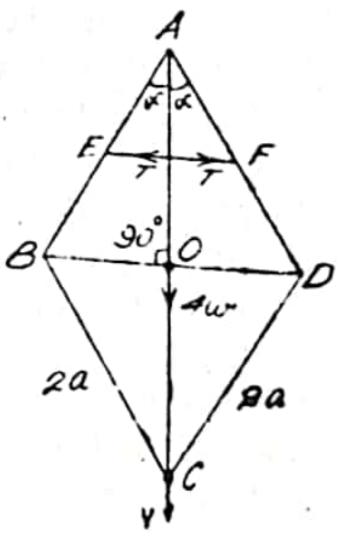
But in the position of equilibrium,

$$BD=2a \text{ and } \theta=\pi/6.$$

Therefore $$T=W\tan\tfrac{1}{6}\pi=W/\sqrt{3}.$$

Ex. 2. *Four rods of equal weights w form a rhombus ABCD, with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that the thrust in this stiffening rod is*

$$(2W+4w)\tan\alpha.$$



Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended by the point A and a weight W is attached to C. To keep the system in the form of a rhombus a light rod EF joins the middle points E and F of AB and AD respectively. Obviously the line AC must be vertical and so BD is horizontal.

We have $\angle BAC = \angle DAC = \alpha$.

Let T be the thrust in the rod EF. The total weight $4w$ of all the four rods can be taken acting at the point of intersection O of the diagonals AC and BD.

Replace the rod EF by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AC in which α changes to $\alpha+\delta\alpha$. The point A remains fixed and so the distances of the points of application of the weights $4w$ and W will be measured from A. The lengths of the rods AB, BC, CD and DA do not change, the length EF changes; the $\angle AOB$ remains 90° and the points O and C change.

We have

$$EF = 2 . AE \sin\alpha = 2a \sin\alpha,$$

$AO =$ depth of O below the fixed point A

$= AB \cos\alpha = 2a \cos\alpha$ and $AC = 2AO = 4a \cos\alpha$.

By the principle of virtual work, we have

$$T\delta\,(2a \sin\alpha) + 4w\delta\,(2a \cos\alpha) + W\delta\,(4a \cos\alpha) = 0$$

or $$2aT \cos\alpha\,\delta\alpha - 8aw \sin\alpha\,\delta\alpha - 4aW \sin\alpha\,\delta\alpha = 0$$

or $$2a\,[T \cos\alpha - 4w \sin\alpha - 2W \sin\alpha]\,\delta\alpha = 0$$

or $$T \cos\alpha - 4w \sin\alpha - 2W \sin\alpha = 0 \qquad [\because \delta\alpha \neq 0]$$

or $$T \cos\alpha = (4w + 2W) \sin\alpha$$

or $$T = (2W + 4w) \tan\alpha.$$

Ex. 3. *Four equal heavy uniform rods are freely jointed so as to form a rhombus which is freely suspended by one angular point, and the middle points of the two upper rods are connected by a light rod so that the rhombus cannot collapse. Prove that the thrust of this light rod is*

$$4W \tan\alpha,$$

where W is the weight of each rod and 2α is the angle of the rhombus at the point of suspension. **[Meerut 76; Kanpur 81]**

Sol. Proceed as in Ex. 2. Here a weight W is not attached at C. The equation of virtual work is

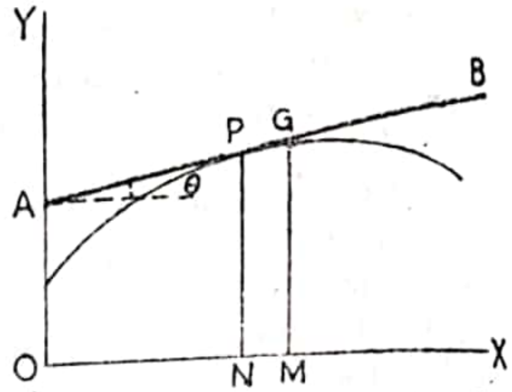
$$T\delta\,(2a \sin\alpha) + 4W\delta\,(2a \cos\alpha) = 0,$$

giving $$T = 4W \tan\alpha.$$

Ex. 4. *Four equal uniform rods, each of weight w, are freely jointed to form a rhombus $ABCD$. The frame work is suspended freely from A and a weight W is attached to each of the joints B, C, D. If two horizontal forces each of magnitude P acting at B and D*

Ex. 69. *A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any position, find the equation of the curve.* **[Kanpur 80]**

Sol. Let AB be the beam of length $2a$ touching the curve at P and resting with its end A in contact with the vertical wall OY. Take the wall OY as the y-axis and a fixed horizontal line OX as the x-axis. The weight W of the beam acts at its middle point G. Let z be the height of G above the fixed horizontal line OX i.e., $MG=z$. Suppose the beam makes an angle θ with the horizontal. The beam is in equilibrium in all positions. If we give the beam a small displacement in which θ changes to $\theta+\delta\theta$, then the equation of virtual work is



$$-W\delta\,(MG)=0,\ i.e.,\ \delta\,(z)=0,$$

for the reactions of the wall and the curve do not work.

$\therefore\quad z=\text{constant}=h$ (say).

Hence the coordinates of G are $(a\cos\theta, h)$.

Now the straight line AB passes through the point $G\,(a\cos\theta, h)$ and makes an angle θ with the x-axis. Therefore the equation of AB is $\quad y-h=\tan\theta\,(x-a\cos\theta)$

$i.e.,\quad x\tan\theta-y=a\sin\theta-h, \qquad \ldots(1)$

where θ is the parameter.

Since AB touches the curve, therefore the curve is the envelope of AB for varying values of θ.

Differentiating (1) partially with respect to θ, we have

$$x\sec^2\theta=a\cos\theta,\ i.e.,\ x=a\cos^3\theta. \qquad \ldots(2)$$

If we now eliminate θ between (1) and (2), we get the envelope of (1) i.e., the curve upon which the beam rests.

Putting $x=a\cos^3\theta$ in (1), we get

$$y-h=a\cos^3\theta\tan\theta-a\sin\theta=-a\sin\theta\,(1-\cos^2\theta)=-a\sin^3\theta \qquad \ldots(3)$$

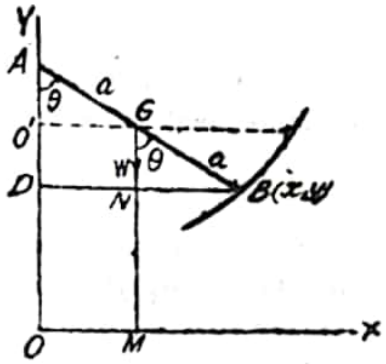
From (2) and (3), $x^{2/3}+(y-h)^{2/3}=(a\cos^3\theta)^{2/3}+(-a\sin^3\theta)^{2/3}$

$$=a^{2/3}(\cos^2\theta+\sin^2\theta)=a^{2/3}.$$

Hence the equation of the curve is $x^{2/3}+(y-a)^{2/3}=a^{2/3}$, which is an astroid.

to the wall; if the beam rests in all positions, show that the curve is an ellipse whose major axis lies along the horizontal line described by the centre of gravity of the beam. **[Kanpur 79]**

Sol. Let AB be the beam of weight W and length $2a$ which rests with its one end A against a smooth vertical wall OY and the other end B on a smooth curve which lies in a vertical plane perpendicular to the wall. The weight W of the beam acts at its middle point G. Let θ be the inclination of the beam to the vertical and z be the height of G above the fixed horizontal plane OX i.e. $MG=z$.



The beam rests in all positions. Give the beam a small displacement in which θ changes to $\theta+\delta\theta$, the end B remains on the curve and the end A remains on the wall. The only force which does the virtual work is the weight W of the beam. Hence the equation of virtual work is

$$-W\delta\,(MG)=0, \quad \text{or} \quad \delta(z)=0.$$

$$\therefore \quad z=\text{constant}=h \text{ (say)}.$$

Thus, in all positions of rest, the centre of gravity of the beam remains at a constant height h from the fixed horizontal plane OX and so it describes a horizontal line at a height h from the fixed horizontal plane OX.

Referred to OX and OY as the coordinate axes, let (x, y) be the coordinates of the end B of the beam. Then

$$x=DB=AB\sin\theta=2a\sin\theta \qquad \ldots(1)$$

and

$$y=OD=MN=MG-NG=h-a\cos\theta. \qquad \ldots(2)$$

From (2), $y-h=-a\cos\theta$

or

$$2(y-h)=-2a\cos\theta. \qquad \ldots(3)$$

Squaring and adding (1) and (3), we get

$$x^2+4(y-h)^2=4a^2$$

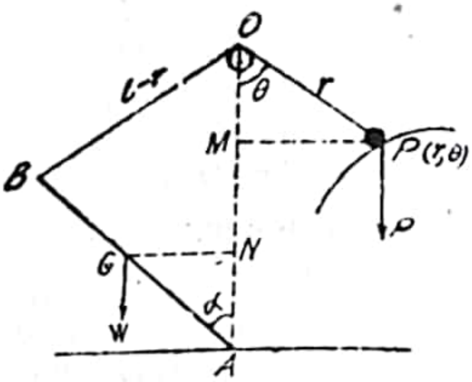
or

$$\frac{x^2}{4a^2}+\frac{(y-h)^2}{a^2}=1. \qquad \ldots(4)$$

The equation (4) is the locus of the point B *i.e.*, the equation of the curve on which the end B of the beam lies. We see that (4) is the equation of an ellipse whose centre is the point $(0, h)$ *i.e.*, the point O' where the line described by G meets the wall and whose major axis is the line $y-h=0$ *i.e.*, $y=h$ *i.e.*, the horizontal line $O'G$ described by the centre of gravity of the beam.

Ex. 67. *AB is a heavy beam which can turn about a horizontal axis at A. A cord fastened to B passes over a smooth pulley, at O, vertically above A, and is tied at the other end to a given weight P which moves on a smooth curve. Find the form of the curve if there is equilibrium in all positions.*

Sol. The beam AB of length $2a$ can turn about a horizontal axis at A. There is a smooth pulley at O which is vertically above A. One end of a cord of length l is fastened to B and the cord passes over the pulley at O and to the other end of the cord a weight P is tied which moves on a smooth curve. The beam is in equilibrium in all positions. The weight W of the beam acts at its middle point G.



Since both the points A and O are fixed, therefore the distance $AO=\text{constant}=c$ (say).

Take the fixed point O as pole and the fixed straight line OA as the initial line. Let the coordinates of P be (r, θ) so that $OP=r$ and $\angle POA=\theta$. We have $BO=l-r$. If $\angle BAO=\alpha$, then from $\triangle BAO$, by using the cosine formula, we have

$$\cos\alpha=\frac{4a^2+c^2-(l-r)^2}{4ac} \qquad ...(1)$$

The depth of P below the fixed point $O=OM=r\cos\theta$, and the depth of G below $O=ON=OA-AN=c-a\cos\alpha$.

Give the system a small displacement in which the length of the string does not change. The equation of virtual work is

$$P\delta(OM)+W\delta(ON)=0$$

or $$P\delta(r\cos\theta)+W\delta(c-a\cos\alpha)=0$$

or $$\delta[Pr\cos\theta+W(c-a\cos\alpha)]=0.$$

Integrating it, we get

$Pr\cos\theta+W(c-a\cos\alpha)=k$, where k is an arbitrary constant

or $Pr\cos\theta+W\left\{c-\dfrac{4a^2+c^2-(l-r)^2}{4c}\right\}=k$, substituting for $\cos\alpha$ from (1)

or $$4cPr\cos\theta+W\{3c^2-4a^2+(l-r)^2\}=4ck.$$

This is the equation of the curve on which the weight P rests.

Ex. 68. *One end of a beam rests against a smooth vertical wall and the other end on a smooth curve in a vertical plane perpendicular* and (ii) the weight W acting at A whose depth below S is $a-r$.

Since the ring is in equilibrium in any position on the wire, therefore by the principle of virtual work, we have

$w\delta\,(r\cos\theta)+W\delta\,(a-r)=0$ or $\delta\,[wr\cos\theta+W\,(a-r)]=0.$

Integrating it, we get

$$wr\cos\theta+W\,(a-r)=c, \text{ where } c \text{ is an arbitrary constant}$$

or $$Wa-c=Wr-wr\cos\theta$$

or $$Wa-c=Wr\,\{1-(w/W)\cos\theta\}$$

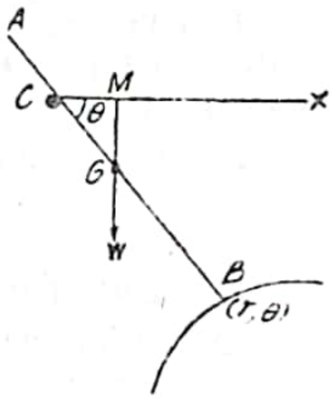
or $$\frac{a-(c/W)}{r}=1-\frac{w}{W}\cos\theta$$

which is of the form $l/r=1-e\cos\theta$.

This is the polar equation of a conic section whose focus is at S.

Ex. 66. *A heavy rod, of length 2l, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions, show that the curve is a conicoid whose polar equation, with C as origin, is* $r=l+(a/\sin\theta)$.

Sol. The end B of the rod AB of length $2l$ rests upon a smooth curve and the rod rests against a smooth peg at C. Take the fixed point C as pole and the fixed horizontal line CX as the initial line. The weight W of the rod acts at its middle point G. Let the coordinates of B be (r, θ) so that $CB=r$ and $\angle BCX=\theta$. We have



$$CG=CB-GB=r-l.$$

It is given that the rod is in equilibrium in all positions. Give the rod a small displacement in which r changes to $r+\delta r$ and θ changes to $\theta+\delta\theta$. The length of the rod does not change and the line CX remains fixed. The only force that does virtual work is the weight W of the rod acting at G whose depth below the fixed line CX is

$$=MG=CG\sin\theta=(r-l)\sin\theta.$$

The equation of virtual work is

$$W\delta\,[(r-l)\sin\theta]=0, \quad \text{or} \quad \delta[(r-l)\sin\theta]=0.$$

Integrating it, we get $\quad (r-l)\sin\theta=a,$

where a is an arbitrary constant

or $$r-l=\frac{a}{\sin\theta}, \quad \text{or} \quad r=l+\frac{a}{\sin\theta}$$

This is the locus of B *i.e.*, the equation of the curve on which the end B of the rod rests.

$$SP = r \quad \text{and} \quad \angle PSA' = \theta.$$

Since $SP + SQ = c$, therefore $SQ = c - SP = c - r$.

The weight W of the ring P acts at P and the weight W of the ring Q acts at Q.

Here S is the fixed point. The depth of P below S

$$= SM = SP \cos \theta = r \cos \theta = \frac{r - l}{e}, \qquad \text{from (1).}$$

Since the radius vector of the point Q is $c - r$, therefore the depth of Q below $S = \dfrac{c - r - l}{e}$.

Let the system be imagined to undergo a small displacement in which the length of the string remains unchanged and only the rings P and Q slightly slide on the wire. The sum of the virtual works done by the forces during this small displacement

$$= W\delta\,(SM) + W\delta\,(SN)$$

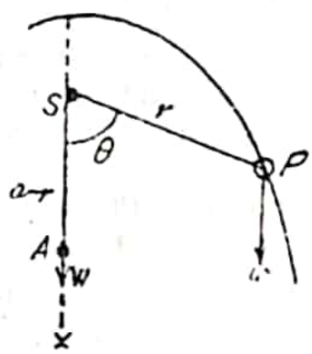
$$= W\delta\left(\frac{r-l}{e}\right) + W\delta\left(\frac{c-r-l}{e}\right)$$

$$= \frac{W}{e}\,\delta r - \frac{W}{e}\,\delta r = 0 \text{ (always).}$$

Thus the sum of virtual works is zero and is independent of r. Therefore the rings will be in equilibrium wherever they are placed.

Ex. 65. *A small heavy ring slides on a smooth wire whose plane is vertical, and is connected by a string passing over a small pulley in the plane of the curve with a weight which hangs freely. If the ring is in equilibrium in any position on the wire, show that the form of latter is a conic section whose focus is at the pulley.*

Sol. Let a string ASP of length a pass over a fixed pulley S. Let w be the weight of the ring P which slides on a smooth wire whose plane is vertical and W the weight attached to the other end A of the string.



Take the fixed point S as pole and the vertical line SAX as the initial line. Let the polar coordinates of the point P be (r, θ) so that $SP = r$ and $\angle PSX = \theta$.

We have $\qquad SA = a - r.$

Give the system a small displacement in which r changes to $r + \delta r$, θ changes to $\theta + \delta\theta$ and the length of the string remains unaltered so that the work done by its tension during this small displacement is zero. The only two forces contributing to the sum of virtual works are :

(i) the weight w of the ring acting at P whose depth below the fixed point S is $r \cos \theta$;

or $$\left[\left(\frac{l}{1-e\cos\theta}-c\right)\cos\theta-\frac{le\sin\theta}{(1-e\cos\theta)^2}\sin\theta\right]\delta\theta=0$$

or $$\left[\left(\frac{l}{1-e\cos\theta}-c\right)\cos\theta-\frac{le\sin^2\theta}{(1-e\cos\theta)^2}\right]=0\ [\because\ \delta\theta\neq0]\quad ...(2)$$

From (1), $e\cos\theta=1-\frac{l}{r}=\frac{r-l}{r}$.

$$\therefore\quad \cos\theta=\frac{r-l}{er}. \quad ...(3)$$

With the help of (1) and (3), the equation (2) becomes

$$(r-c)\frac{r-l}{er}-le\left\{1-\frac{(r-l)^2}{r^2e^2}\right\}\cdot\frac{r^2}{l^2}=0$$

or $$(r-c)\frac{r-l}{er}-\frac{1}{le}\{r^2e^2-(r-l)^2\}=0. \quad ...(4)$$

The equation (2) or (4) gives the position of equilibrium. When the length of rod is least, its upper end Q is just at the focus S so that $$r=2c.$$

Putting $r=2c$ in (4), we have

$$(2c-c)\frac{2c-l}{2ce}-\frac{1}{le}\{4c^2e^2-(2c-l)^2\}=0$$

or $$l(2c-l)-2\{4c^2e^2-(4c^2-4cl+l^2)\}=0$$

or $$8(1-e^2)c^2-6cl+l^2=0.$$

Solving it, we get

$$c=\frac{6l\pm\sqrt{[36l^2-32l^2(1-e^2)]}}{16(1-e^2)}=\frac{3l\pm l\sqrt{(1+8e^2)}}{8(1-e^2)}.$$

But we know that semi-latus rectum $l=b^2/a=a(1-e^2)$.

Hence $$c=\frac{3a\pm a\sqrt{(1+8e^2)}}{8}$$

or $$2c=\tfrac{3}{4}a\pm\tfrac{1}{4}a\sqrt{(1+8e^2)}.$$

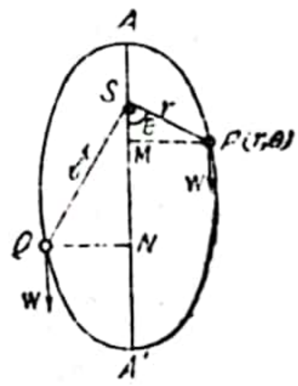
But $2c$ is definitely greater than SB' *i.e.* $2c>a$. Therefore neglecting the $-$ive sign, we get

$$2c=\tfrac{3}{4}a+\tfrac{1}{4}a\sqrt{(1+8e^2)},$$

giving the least length of the rod.

Ex. 64. *Two small smooth rings of equal weight slide on a fixed elliptical wire, whose major axis is vertical and they are connected by a string which passes over a small smooth peg at the upper focus; show that the weights will be in equilibrium wherever they are placed.*

Sol. The major axis AA' of the elliptical wire is vertical. The two rings are P and Q connected by the string PSQ which passes over a smooth peg at the focus S. Let c be the length of the string PSQ.



Referred to the focus S as pole and SA' as the initial line, the polar equation of the ellipse is $l/r=1-e\cos\theta$. ...(1)

Let the coordinates of the point P be (r,θ) so that or $\quad 2w\,\delta x+2W\,\delta x+Wl\cos\theta\,\delta\theta=0$

or $\quad 2(w+W)\,\delta x+Wl\cos\theta\,\delta\theta=0$

or $\quad -Wl\cos\theta\,\delta\theta=2(w+W)\,\delta x.$ ...(1)

Now we should find a relation between the two parameters x and θ.

From $\triangle QMP$, we have $MP=PQ\cos\theta=l\cos\theta$, giving the y-coordinate y of the point P of the parabola $y^2=4ax$.

Putting $y=l\cos\theta$ in the equation $y^2=4ax$, we have

$$l^2\cos\theta=4ax. \qquad ...(2)$$

Differentiating (2), we have

$$-2l^2\cos\theta\sin\delta\theta=4a\delta x$$

or
$$-l^2\cos\theta\sin\theta\,\delta\theta=2a\delta x. \qquad ...(3)$$

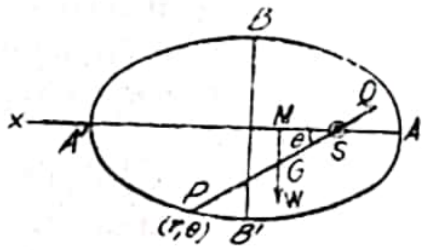
Dividing (3) by (1), we have

$$\frac{l\sin\theta}{W}=\frac{a}{w+W}\text{ or }\sin\theta=\frac{aW}{l(w+W)}$$

or
$$\theta=\sin^{-1}\left\{\frac{aW}{l(w+W)}\right\}.$$

Ex. 63. *A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal, and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find its position of equilibrium and show that its length must at least be $\frac{3}{4}a+\frac{1}{4}a\sqrt{(1+8e^2)}$ where $2a$ is the major axis and e is the eccentricity.*

Sol. There is a ring at the focus S of the ellipse whose major axis AA' is horizontal. The rod PQ, say of length $2c$, passes through the ring at S and the end P of the rod rests on the quadrant $A'B'$ of the ellipse which is farthest from the focus S. The weight W of the rod PQ acts at its middle point G.



Referred to the focus S as pole and SA' as the initial line, the polar equation of the ellipse is

$$l/r=1-e\cos\theta. \qquad ...(1)$$

Let the coordinates of P be (r,θ) so that $SP=r$ and $\angle PSX=\theta$.

Here the point S is fixed. The depth of G below the fixed line $SX=MG=SG\sin\theta$

$$=(SP-PG)\sin\theta=(r-c)\sin\theta$$

$$=\left(\frac{l}{1-e\cos\theta}-c\right)\sin\theta. \quad \text{[Substituting for } r \text{ from (1)]}$$

Give the rod a small displacement in which the end P of the rod remains on the ellipse and θ changes to $\theta+\delta\theta$. The equation of virtual work is $\quad W\delta(MG)=0$, or $\delta(MG)=0$

or
$$\delta\left[\left(\frac{l}{1-e\cos\theta}-c\right)\sin\theta\right]=0$$

$$\delta x = \tfrac{4}{3}a\,(1+2\sin\theta)\cos\theta\,\delta\theta. \qquad \ldots(2)$$

or Here OY is the fixed line. The depth of the middle point of AB or CD below $OY = x + a\cos\theta$ and the depth of the middle point of BC below $OY = x + 2a\cos\theta$.

Let the rods be given a small symmetrical displacement about the axis OX in which θ changes to $\theta+\delta\theta$. Then the equation of virtual work is

$$2W\delta\,(x+a\cos\theta) + W\delta\,(x+2a\cos\theta) = 0$$

or $$2W\delta x - 2aW\sin\theta\,\delta\theta + W\delta x - 2a\,W\sin\theta\,\delta\theta = 0$$

or $$3W\delta x - 4a\,W\sin\theta\,\delta\theta = 0$$

or $$3W.\tfrac{4}{3}a\,(1+2\sin\theta)\cos\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta = 0,$$

substituting for δx from (2)

or $$4aW\,[\cos\theta + 2\sin\theta\cos\theta - \sin\theta]\,\delta\theta = 0$$

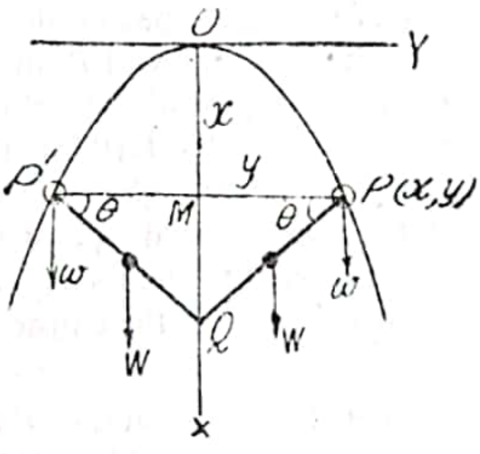
or $$\cos\theta + \sin 2\theta - \sin\theta = 0, \qquad [\because\ W \neq 0 \text{ and } \delta\theta \neq 0]$$

which gives the required result.

Ex. 62. *Two uniform straight rods PQ, $P'Q$, in all respects alike are smoothly jointed at Q and at P, P' carry small rings which slide on a smooth fixed parabolic wire whose axis is vertical and vertex upwards. Prove that in the symmetrical position of equilibrium the angle either rod makes with the horizontal is*

$$\sin^{-1}\left\{\frac{aW}{l\,(W+w)}\right\},$$

where W is the weight of either rod, w weight of either ring, l the length of either rod and $4a$ the latus rectum of the parabola.



Sol. Let the equation of the parabola be $y^2 = 4ax$. We have

$$PQ = P'Q = l.$$

The weights W of the rods PQ and $P'Q$ act at their respective middle points. The weight w of the ring P acts at P and the weight w of the ring P' acts at P'. We have $\angle QPP' = \theta = \angle QP'P$.

Let the co-ordinates of the point P be (x, y). Then $OM = x$ and $MP = y$.

Here OY is a fixed line. The depth of P or P' below OY

$$= OM = x,$$

and the depth of the middle point of PQ or $P'Q$ below OY

$$= x + \tfrac{1}{2}l\sin\theta.$$

Give the system (*i.e.*, rods and rings) a small symmetrical displacement about the axis OX of the parabola such that θ changes to $\theta+\delta\theta$, x changes to $x+\delta x$ and the lengths of the rods do not change. The equation of virtual work is

$$2w\,\delta(x) + 2W\,\delta(x + \tfrac{1}{2}l\sin\theta) = 0$$

$$= at^2 + l\sin\theta = a\left(\tan\theta - \frac{l}{2a}\cos\theta\right)^2 + l\sin\theta$$

$$= a\tan^2\theta + \frac{l^2}{4a}\cos^2\theta.$$

Now give the rod a small displacement in which θ changes to $\theta + \delta\theta$, the ends of the rod remaining in contact with the wire. If W be the weight of the rod, then the equation of virtual work is

$$-W\delta z = 0, \quad \text{or} \quad \delta z = 0$$

or $$\delta\left(a\tan^2\theta + \frac{l^2}{4a}\cos^2\theta\right) = 0$$

or $$\left(2a\tan\theta\sec^2\theta - \frac{l^2}{2a}\cos\theta\sin\theta\right)\delta\theta = 0$$

or $$2a\tan\theta\sec^2\theta - \frac{l^2}{2a}\cos\theta\sin\theta = 0 \qquad [\because \ \delta\theta \neq 0]$$

or $$\sin\theta\left(2a\sec^3\theta - \frac{l^2}{2a}\cos\theta\right) = 0.$$

$\therefore$ either $\sin\theta = 0$ *i.e.*, the rod is horizontal

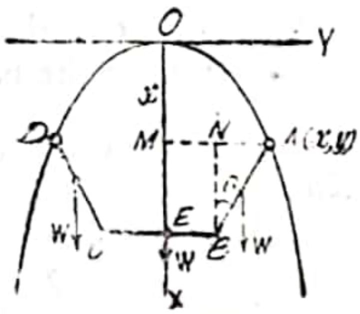
or $2a\sec^3\theta - \frac{l^2}{2a}\cos\theta = 0$ *i.e.*, $2a\sec^3\theta = \frac{l^2}{2a}\cos\theta$

i.e., $\cos^4\theta = 4a^2/l^2$ *i.e.*, $\cos^2\theta = 2a/l$, giving the inclined position of the rod.

Ex. 61. *Three equal and similar rods AB, BC, CD freely jointed at B and C have small weightless rings attached to them at A and D The rings slide on a smooth parabolic wire, whose axis is vertical and vertex upwards and whose latus rectum is half the sum of the lengths of the three rods. Prove that in the position of equilibrium, the inclination θ of AB or CD to the vertical is given by*

$$\cos\theta - \sin\theta + \sin 2\theta = 0. \qquad \textbf{[Lucknow 80]}$$

Sol. Let $AB = BC = CD = 2a$, so that the sum of their lengths $= 6a$.



Then the latus rectum of the parabola $= 3a$.

Hence the equation of the parabola is $y^2 = 3ax$.

In the position of equilibrium let θ be the inclination of AB or CD to the vertical. The weight W of each of the rods AB, BC and CD acts at their respective middle points.

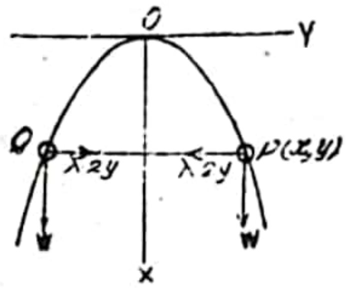
Let the coordinates of the point A be (x, y). Then $x = OM$ and $y = MA = MN + NA = EB + NA = a + 2a\sin\theta$. Since the point (x, y) lies on the parabola $y^2 = 3ax$, therefore

$$(a + 2a\sin\theta)^2 = 3ax. \qquad \ldots(1)$$

Differentiating (1), we get

$$2(a + 2a\sin\theta)\,2a\cos\theta\,\delta\theta = 3a\,\delta x$$

Suppose the two rings each of weight W are in equilibrium in any one symmetrical position P and Q. If (x, y) are the coordinates of P, then $PQ=2y$.



$\therefore$ force of attraction on each ring (P or Q)$=\lambda.2y$ and this force acts at P in the direction PQ and at Q in the direction QP.

A weight W acts at P and a weight W acts at Q. Give the rings a small displacement in which y changes to $y+\delta y$ and x changes to $x+\delta x$. The line OY remains fixed.

The depth of P or Q below OY is x and $PQ=2y$. The equation of virtual work is $\quad -\lambda.2y\ \delta\ (2y)+2W\delta x=0$

or $\quad 2\lambda y\ \delta y=W\delta x.$ ...(1)

But $\quad y^2=4ax.$ Therefore $2y\ \delta y=4a\ \delta x$

or $\quad y\ \delta y=2a\ \delta x.$ ...(2)

Dividing (1) by (2), we get

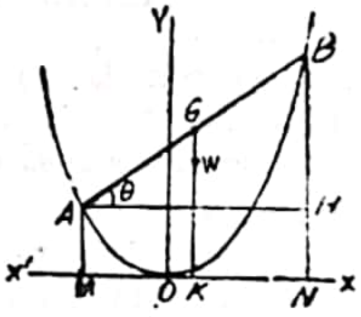
$$2\lambda=W/2a \quad \text{or} \quad \lambda=W/2a, \qquad ...(3)$$

as the condition for the equilibrium of the rings. The condition (3) is independent of the position of P *i.e.*, of (x, y). Hence if the rings rest in any one symmetrical position, they will rest in all symmetrical positions.

Ex. 60. *A smooth parabolic wire is fixed with its axis vertical and vertex downwards, and in it is placed a uniform rod of length $2l$ with its ends resting on the wire. Show that, for equilibrium, the rod is either horizontal, or makes with the horizontal an angle θ given by $\cos^2\theta=2a/l$, $4a$ being the latus rectum of the parabola.*

[Rohilkhand 89; Lucknow 74; Jiwaji 79; Kanpur 77]

Sol. Let AB be the rod of length $2l$ whose ends A and B rest on a smooth parabolic wire whose axis is vertical and vertex downwards. Referred to OX and OY as the coordinate axes, let the equation of the parabola be $x^2=4ay$.



Let the coordinates of the point A be $(2at, at^2)$ and let $\angle BAH=\theta$. Then the coordinates of B are

$$(2at+2l\cos\theta,\ at^2+2l\sin\theta).$$

Since the point B lies on the parabola, therefore

$$(2at+2l\cos\theta)^2=4a\ (at^2+2l\sin\theta)$$

or $\quad 4a^2t^2+8alt\cos\theta+4l^2\cos^2\theta=4a^2t^2+8al\sin\theta$

or $\quad 8alt\cos\theta=8al\sin\theta-4l^2\cos^2\theta$

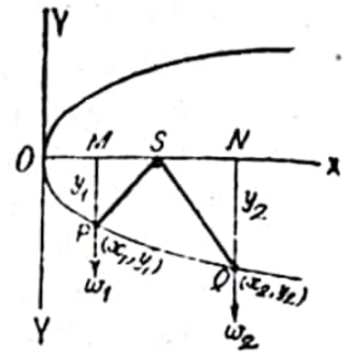
or $\quad t=\tan\theta-\dfrac{l}{2a}\cos\theta.$

If z be the height of the centre of gravity G of the rod above the fixed line OX, then

$$z=KG=\tfrac{1}{2}\ [MA+NB]=\tfrac{1}{2}\ [at^2+(at^2+2l\sin\theta)]$$

sting passing round a smooth peg at the focus. Prove that in the position of equilibrium their weights are proportional to their vertical depths below the axis. **[Kanpur 82]**

Sol. Let the equation of the parabola be $y^2=4ax$. For the sake of convenience the vertically downwards direction has been taken as the positive direction of y-axis.



There is a smooth peg at the focus S of the parabola. Let PSQ be the string of length l which passes over the peg at S. To the ends of the string two rings P and Q are attached which can slide on the parabolic wire. Let w_1 and w_2 be the weights of the rings P and Q respectively. Then a weight w_1 acts at P and a weight w_2 acts at Q. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q respectively.

Give the rings a small displacement in which y_1 changes to $y_1+\delta y_1$ and y_2 changes to $y_2+\delta y_2$. The line OX remains fixed and the length of the string PSQ remains unaltered so that the work done by the tension in this string during this small displacement is zero. The equation of virtual work is

$$w_1\ \delta\ (y_1)+w_2\ \delta\ (y_2)=0 \qquad \ldots(1)$$

or $$w_1\ \delta y_1=-w_2\ \delta y_2.$$

Now let us find a relation between the parameters y_1 and y_2 from the figure.

We have

the focal distance $SP=a+x_1$;
and the focal distance $SQ=a+x_2$.

$$\therefore \quad SP+SQ=2a+x_1+x_2$$

or $$l=2a+\frac{y_1^2}{4a}+\frac{y_2^2}{4a}. \qquad \ldots(2)$$

Differentiating (2), we get

$$0=\frac{2y_1}{4a}\ \delta y_1+\frac{2y_2}{4a}\ \delta y_2$$

or $$y_1\delta y_1=-y_2\ \delta y_2. \qquad \ldots(3)$$

Dividing (1) by (3), we get

$\dfrac{w_1}{y_1}=\dfrac{w_2}{y_2}$ *i.e.*, weights are proportional to their depths.

Ex. 59. *Two small rings of equal weights, slide on a smooth wire in the shape of a parabola whose axis is vertical and vertex upwards, and attract one another with a force which varies as the distance. If they can rest in any symmetrical position on the curve show that they will rest in all symmetrical positions.*

Sol. Here the vertex O is fixed. Taking the vertical line OX as the axis of x, let the equation of the parabola be $y^2=4ax$.

Let us assume that the stress in the rod AC is tension and let it be T. Let $\angle BAC=\theta=\angle CAD$.

Replace the rod AC by two equal and opposite forces T as shown in the figure. Take the centre O of the rhombus as origin, the line OA as the axis of x and the perpendicular line OB as the axis of y. Now give the rhombus a small symmetrical displacement about the centre O in which the centre O and the lines OX and OY remain fixed, θ changes to $\theta+\delta\theta$, the points A and C move on the axis of x and the points B and D move on the axis of y. The length AC changes, the lengths of the rods AB, BC, CD, DA do not change but their middle points are slightly displaced.

We have $AC=4a\cos\theta$ so that the work done by the tension T in the rod AC during this small displacement $=-T\delta\,(4a\cos\theta)$ $=4aT\sin\theta\,\delta\theta$.

By symmetry the forces P acting at the middle points of AB, BC, CD, DA contribute equal works, so that the sum of the works done by all of them is four times the work done by P acting at M.

The components (X, Y) of the force P acting at M along the fixed co-ordinate axes OX and OY are given by

$$X=-P\sin\theta,\ Y=-P\cos\theta.$$

Also the co-ordinates of the point M are $(a\cos\theta,\ a\sin\theta)$.

$\therefore$ the virtual work of the force P acting at M during this small displacement

$$=X\delta\,(a\cos\theta)+Y\delta\,(a\sin\theta)$$
$$=-P\sin\theta\delta\,(a\cos\theta)-P\cos\theta\delta\,(a\sin\theta)$$
$$=a\,P\sin^2\theta\,\delta\theta-aP\cos^2\theta\,\delta\theta=-aP\cos 2\theta\,\delta\theta.$$

Hence the total virtual work done by all the four forces P

$$=-4aP\cos 2\theta\,\delta\theta.$$

Now the equation of virtual work is

$$4aT\sin\theta\,\delta\theta-4aP\cos 2\theta\,\delta\theta=0$$

or $$4a\,(T\sin\theta-P\cos 2\theta)\,\delta\theta=0$$

or $$T\sin\theta-P\cos 2\theta=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$T=(P\cos 2\theta/\sin\theta).$$

Note. It may be seen that if 2θ is acute, then $\cos 2\theta$ is positive and so the value of T is positive which means that there is tension in the rod AC as we have assumed while solving the problem. But if 2θ is obtuse, then $\cos 2\theta$ is negative and so the value of T is negative which means that there is not tension but thrust in the rod AC.

Problems involving curves

Ex. 58. *Two heavy rings slide on a smooth parabolic wire, whose axis is horizontal and plane vertical, and are connected by a*

We have $AD=2a+4a\cos\theta$. Therefore the work done by the thrust T in the rod AD during this small displacement

$$=T\delta\,(2a+4a\cos\theta)=-4a\,T\sin\theta\,\delta\theta.$$

By symmetry the forces P acting at the middle points of AB, CD, DE and FA contribute equal works, so that the sum of the works done by all of them is four times the work done by P acting at M.

The components (X, Y) of the force P acting at M along the fixed coordinate axes OX and OY are given by

$$X=-P\sin\theta,\ Y=-P\cos\theta.$$

Also the coordinates of the point M are $(a+a\cos\theta,\ a\sin\theta)$.

$\therefore$ the virtual work of the force P acting at M during this small displacement

$$=X\delta\,(a+a\cos\theta)+Y\delta\,(a\sin\theta)$$
$$=-P\sin\theta\delta(a+a\cos\theta)-P\cos\theta\delta\,(a\sin\theta)$$
$$=aP\sin^2\theta\delta\theta-aP\cos^2\theta\delta\theta=-aP\,(\cos^2\theta-\sin^2\theta)\,\delta\theta$$
$$=-aP\cos 2\theta\,\delta\theta.$$

Hence the total virtual work done by all the forces P

$$=-4aP\cos 2\theta\,\delta\theta.$$

Now the equation of virtual work is

$$-4a\,T\sin\theta\,\delta\theta-4a\,P\cos 2\theta\,\delta\theta=0$$

or $\quad -4a\,(T\sin\theta+P\cos 2\theta)\,\delta\theta=0$

or $\quad T\sin\theta+P\cos 2\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=-\dfrac{P\cos 2\theta}{\sin\theta}.$

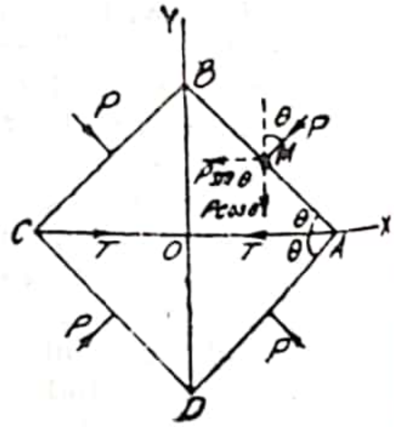
But in the position of equilibrium the hexagon is regular, so that $\theta=60°$.

$$\therefore\quad T=-\frac{P\cos 120°}{\sin 60°}=-\frac{P.(-\frac{1}{2})}{(\sqrt{3}/2)}=\frac{P}{\sqrt{3}}.$$

The positive value of T means that our assumption that there is thrust in AD is correct. Hence there is a thrust $P/\sqrt{3}$ in the rod AD.

Ex. 57. *A frame consists of five bars forming the sides of a rhombus ABCD with diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in AC is* $(P\cos 2\theta/\sin\theta)$ *where* θ *denotes the angle BAC.*

Sol. Let $2a$ be the length of each side of the rhombus $ABCD$ which we shall assume as placed on a smooth horizontal table. The four forces, each equal to P, act inwards at the middle points of AB, BC, CD, DA and at right angles to the respective sides. Let M be the middle point of AB where the force P acts.



Equating the two values of T, we get

$$\frac{W\cot\alpha}{2\pi}=\lambda\,\frac{x-a}{a}$$

or
$$x-a=\frac{a}{2\pi\lambda}\,W\cot\alpha$$

or
$$x=a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right),$$

which gives the radius of the string in equilibrium.

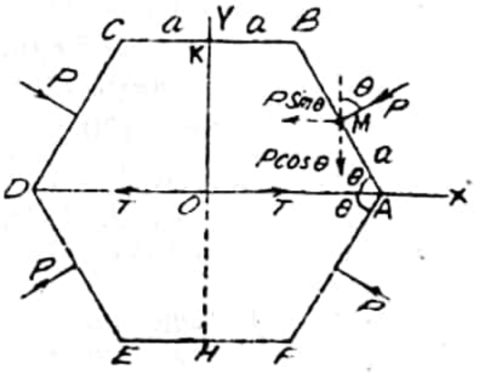
Ex. 55. *An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the verticle angle of the cone to be 2α.*

Sol. Proceed as in Ex. 54 Here in place of a heavy elastic string of weight W we have a heavy endless chain of weight W. If T is the tension in this chain, then proceeding as in Ex. 54, we get $\qquad T=(W\cot\alpha)/2\pi$.

Problems in which the nature of stress is to be found out.

Ex. 56. *$ABCDEF$ is a regular hexagon formed of light rods smoothly jointed at their ends with a diagonal rod AD. Four equal forces P act inwards at the middle points of AB, CD, DE, FA and at right angles to the respective sides. Find the stress in the diagonal AD and state whether it is a tension or thrust.*

Sol. Let $2a$ be the length of each side of the hexagon. The four forces, each equal to P, act inwards at the middle points of AB, CD, DE, FA and at right angles to the respective sides. Let M be the middle point of AB where the force P acts.



Let us assume that the stress in the rod AD is thrust and let it be T.

In the beginning we should not assume the hexagon to be a regular one, for we would give a displacement which will alter the length of AD, without altering the lengths of the sides of the hexagon. Let $\angle BAD=\theta=\angle FAD$.

Replace the rod AD by two equal and opposite forces T as shown in the figure. Take the centre O of the hexagon as origin, the line OA as the axis of x and the perpendicular line OK joining O to the middle point K of BC as the axis of y. Now give the hexagon a small symmetrical displacement about the centre O in which the centre O remains fixed and the lines OX and OY also remain fixed, θ changes to $\theta+\delta\theta$, the points A and D move on the axis of x and the points K and H move on the axis of y. The length AD changes and the middle points of AB, CD, DE and FA are slightly displaced.

Equating the two values of T, we have

$$2W=4W(2\sin\theta-1)$$

or $$1=2(2\sin\theta-1), \text{ or } 1=4\sin\theta-2$$

or $$4\sin\theta=3, \text{ or } \sin\theta=3/4, \text{ or } \theta=\sin^{-1}(3/4).$$

$\therefore$ in equilibrium the whole angle between AB and BC

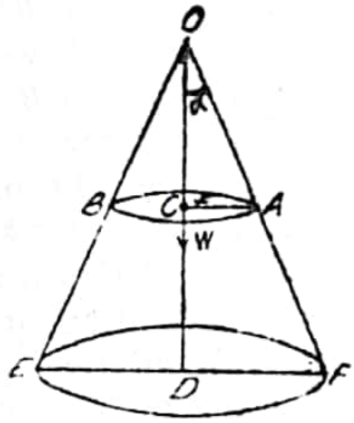
$$=2\theta=2\sin^{-1}(3/4).$$

Ex. 54. *A heavy elastic string, whose natural length is* $2\pi a$, *is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is* α. *If* W *be the weight and* λ *the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is*

$$a\left(1+\frac{W}{2\lambda\pi}\cot\alpha\right).$$

(Meerut 77; Lucknow 79; Gorakhpur 80; Kanpur 88; P.C.S. 79]

Sol. OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical. A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x. The weight W of the string acts at its centre of gravity C. Let T be the tension in this string.



Give the string a small displacement in which x changes to $x+\delta x$. The point O remains fixed, the point C is slightly displaced, $\angle\alpha$ is fixed and the length of the string slightly changes.

We have the length of the string AB in the form of a circle of radius $x=2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.

Also the depth of the point of application C of the weight W below the fixed point O

$$=OC=AC\cot\alpha=x\cot\alpha$$

and so the work done by the weight W during this small displacement $$=W\delta(x\cot\alpha).$$

Since the reactions at the various points of contact do no work, we have, by the principle of virtual work

$$-T\delta(2\pi x)+W\delta(x\cot\alpha)=0$$

or $$-2\pi T\,\delta x+W\cot\alpha\,\delta x=0 \quad \text{or} \quad (-2\pi T+W\cot\alpha)\delta x=0$$

or $$-2\pi T+W\cot\alpha=0$$

or $$T=(W\cot\alpha)/2\pi. \qquad [\because \delta x\neq 0]$$

By Hooke's law the tension T in the elastic string AB is given by

$$T=\lambda\frac{2\pi x-2\pi a}{2\pi a}=\lambda\frac{x-a}{a}.$$

$$=W\frac{a\sqrt{2}-l}{l}. \qquad [\because \lambda=W]$$

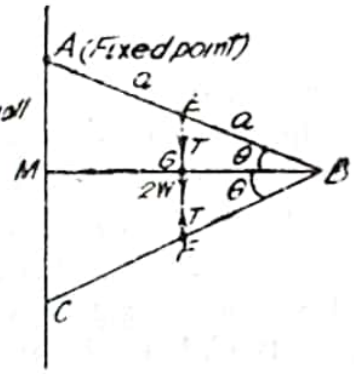
Equating the two values of T, we get

$$2W=W\frac{a\sqrt{2}-l}{l},$$

or $$2l=a\sqrt{2}-l, \text{ or } 3l=a\sqrt{2}$$

or $$l=a\sqrt{2}/3.$$

Ex. 53. *One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2\sin^{-1}(3/4)$.*



Sol. AB and BC are two rods each of length $2a$ and weight W smoothly joined together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a.

Let T be the tension in the string EF.

The total weight $2W$ of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M. Let $\angle ABM=\theta=\angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes to $\theta+\delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF=2EG=2EB\sin\theta=2a\sin\theta$.

Also the depth of G below the fixed point A

$$=AM=AB\sin\theta=2a\sin\theta.$$

The equation of virtual work is

$$-T\delta(2a\sin\theta)+2W\delta(2a\sin\theta)=0$$

or $$(-2aT\cos\theta+4aW\cos\theta)\,\delta\theta=0$$

or $$2a\cos\theta\,(-T+2W)\,\delta\theta=0$$

or $$-T+2W=0 \qquad [\because \delta\theta\neq 0 \text{ and } \cos\theta\neq 0]$$

or $$T=2W.$$

Also by Hooke's law the tension T in the elastic string EF is given by

$$T=\lambda\frac{2a\sin\theta-a}{a},$$

where λ is the modulus of elasticity of the string

$$=4W(2\sin\theta-1). \qquad [\because \lambda=4W]$$

$$-2cT\cos\alpha\,\delta\alpha+Wc\sin\alpha\,\delta\alpha+aW\operatorname{cosec}\alpha\cot\alpha\,\delta\alpha=0$$

or $\quad(-2cT\cos\alpha+Wc\sin\alpha+aW\operatorname{cosec}\alpha\cot\alpha)\,\delta\alpha=0$

or $\quad-2cT\cos\alpha+Wc\sin\alpha+aW\operatorname{cosec}\alpha\cot\alpha=0 \qquad [\because\ \delta\alpha\neq 0]$

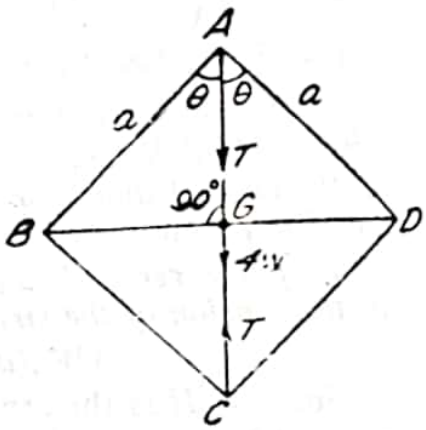
or $\quad 2cT\cos\alpha=W(c\sin\alpha+a\operatorname{cosec}\alpha\cot\alpha)$

or $\quad T=\dfrac{W}{2c\cos\alpha}(c\sin\alpha+a\operatorname{cosec}\alpha\cot\alpha)$

$\quad=\frac{1}{2}W\{\tan\alpha+(a/c)\operatorname{cosec}^2\alpha\}.$

Problems involving elastic strings

Ex. 52. *Four equal jointed rods, each of length a are hung from an angular point; which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is a* $\sqrt{2}/3$. **[Meerut 82, 85P]**



Sol. $ABCD$ is a framework formed of four equal rods each of length a and say of weight W. It is suspended from the point A. A and C are connected by an elastic string and in equilibrium $ABCD$ is square. The diagonal AC is vertical and so BD is horizontal. Let T be the tension in the string AC. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. Let $\angle BAC=\angle DAC$.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed, the length AC changes, the point G is slightly displaced, the lengths of the rods AB, BC, CD, DA do not change, and the $\angle BGA$ remains 90°. We have $\quad AC=2AG=2a\cos\theta$.

Also the depth of G below $A=AG=a\cos\theta$.

The equation of virtual work is

$$-T\delta(2a\cos\theta)+4W\delta(a\cos\theta)=0$$

or $\quad 2aT\sin\theta\ \ \theta-4aW\sin\theta\,\delta\theta=0$

or $\quad 2a\sin\theta\,(T-2W)\,\delta\theta=0$

or $\quad T-2W=0 \qquad [\because\ \delta\theta\neq 0 \text{ and } \sin\theta\neq 0]$

or $\quad T=2W.$

Let l be the natural length of the elastic string AC. In the position of equilibrium, $\angle BAC=45°$ and so the extended length AC of the elastic string $=2AG=2a\cos 45°=2a/\sqrt{2}=a\sqrt{2}$.

By Hooke's law, the tension T in the elastic string AC is given by $\quad T=\lambda\dfrac{AC-l}{l}$, where λ is the modulus of elasticity of the string

The equation of virtual work is

$$-T\delta\,(2l\cos\phi)+2lw\delta\,(\tfrac{1}{2}l\sin\phi-r\sec\phi)=0$$

or $2l\,T\sin\phi\,\delta\phi+2lw.\tfrac{1}{2}l\cos\phi\delta\phi-2lwr\sec\phi\tan\phi\,\delta\phi=0$

or $(2l\,T\sin\phi+l^2w\cos\phi-2lrw\sec\phi\tan\phi)\,\delta\phi=0$

or $2l\,T\sin\phi+l^2w\cos\phi-2lrw\sec\phi\tan\phi=0 \qquad [\because\ \delta\phi\neq 0]$

or $2lT\sin\phi=2lrw\sec\phi\tan\phi-l^2w\cos\phi$

or $T=\dfrac{1}{2l\sin\phi}(2lrw\sec\phi\tan\phi-l^2w\cos\phi)$

$\qquad=w\,(r\sec^2\phi-\tfrac{1}{2}l\cot\phi).$

If the string is slack, $T=0$. So putting $T=0$ in (1), we get

$$0=w\,(r\sec^2\phi-\tfrac{1}{2}l\cot\phi)$$

or $r\sec^2\phi-\tfrac{1}{2}l\cot\phi=0$

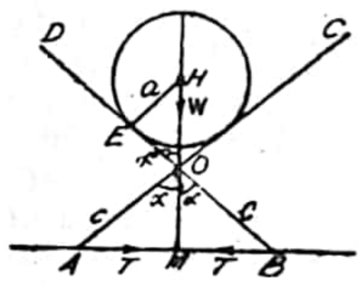
or $r\,(1+\tan^2\phi)=\tfrac{1}{2}l\cot\phi$

or $\dfrac{l}{2r}=\dfrac{1+\tan^2\phi}{\cot\phi}=\tan\phi+\tan^3\phi.$

Ex. 51. *Two light rods AOC, BOD are smoothly hinged at O, a point at a distance c from each of the ends A, B which are connected by a string of length 2c sin α. The rods rest in a vertical plane with the ends A and B on a smooth horizontal table. A smooth circular disc of radius a and weight W is placed on the rods above O with its plane vertical so that rods are tangents to the disc. Prove that the tension of the string is*

$$\tfrac{1}{2}W\,\{(a/c)\operatorname{cosec}^2\alpha+\tan\alpha\}.$$

Sol. If H is the centre of disc, then the weight W of the disc acts at H. The ends A and B of the rods AOC and BOD hinged at O are placed on a smooth horizontal table. Let T be the tension in the string AB. The line HO is vertical and meets AB at its middle point M. We have



$$AO=BO=c \text{ and } AB=2c\sin\alpha.$$

Therefore $\angle AOM=\angle BOM=\alpha=\angle HOD.$

If the rod BOD touches the disc at E, then $\angle OEH=90°$ and $HE=$ the radius of the disc $=a$.

Give the system a small symmetrical displacement about the vertical line HOM in which α changes to $\alpha+\delta\alpha$. The level of the line AB lying on the table remains fixed. The point H is slightly displaced, the length AB changes and the lengths of the rods do not change.

We have $AB=2c\sin\alpha$.

Also the height of H above $AB=MH=MO+OH$

$$=AO\cos\alpha+HE\operatorname{cosec}\alpha=c\cos\alpha+a\operatorname{cosec}\alpha.$$

The equation of virtual work is

$$-T\delta\,(2c\sin\alpha)-W\delta\,(c\cos\alpha+a\operatorname{cosec}\alpha)=0$$

the circle at M, then $\angle OMA = 90°$ and OM = the radius of the circle $= a$.

Give the rods a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90°. We have, the height of G above the fixed point O

$$= OG = OA - GA = OM \operatorname{cosec} \theta - AE \cos \theta$$
$$= a \operatorname{cosec} \theta - b \cos \theta.$$

The equation of virtual work is

$$-2W\delta\,(OG)=0, \quad \text{or} \quad \delta\,(OG)=0$$

or $$\delta\,(a \operatorname{cosec} \theta - b \cos \theta)=0$$

or $$(-a \operatorname{cosec} \theta \cot \theta + b \sin \theta)\,\delta\theta=0$$

or $$-a \operatorname{cosec} \theta \cot \theta + b \sin \theta = 0 \qquad [\because \delta\theta \neq 0]$$

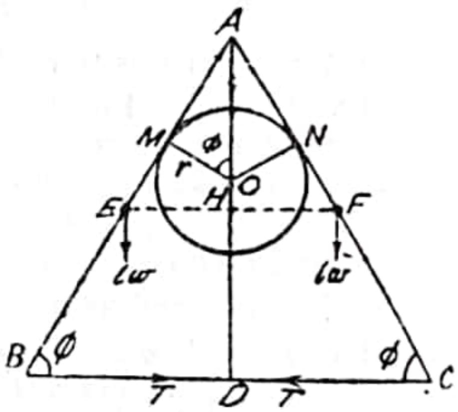
or $$a \operatorname{cosec} \theta \cot \theta = b \sin \theta$$

or $$a \cos \theta = b \sin^3 \theta.$$

Ex. 50. *Two equal rods, each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string and if the string is slack, show that ϕ satisfies the equation*

$$\tan^3 \phi + \tan \phi = l/2r. \qquad \text{[Gorakhpur 81; P.C.S. 78]}$$

Sol. Let AB and AC be the equal rods which are placed on a fixed horizontal cylindrical peg. In the figure we have shown a cross-section of the cylinder by a vertical plane passing through the points of contact of the rods. This cross section is a circle whose centre is O and radius is r. We have $AB=AC=l$. The line AO is vertical and meets EC at its middle point D. The weights lw and lw of the rods AB and AC act at their respective middle points E and F. Let T be the tension in the string BC. We have



$$\angle ABC = \angle ACB = \phi.$$

If the rod AB touches the circle at the point M, then

$$\angle OMA = 90°, \ \angle MOA = 90° - \angle BAD$$
$$= 90° - (90° - \phi) = \phi,$$

$OM = r$. Also $AE = \frac{1}{2} l$.

Give the rods a small symmetrical displacement in which ϕ changes to $\phi+\delta\phi$. The point O remains fixed, the length BC changes and the points E and F are slightly displaced. We have

$$BC = 2BD = 2AB \cos \phi = 2l \cos \phi.$$

Also the depth of E or F below the fixed point O

$$= OH = AH - AO = AE \sin \phi - OM \sec \phi = \tfrac{1}{2} l \sin \phi - r \sec \phi.$$

$= a\sqrt{2} \sin(\frac{1}{4}\pi + \theta) - PQ \cos\theta \sin\theta$

$[\because \quad AG = \frac{1}{2}AC = \frac{1}{2}\, 2a\sqrt{2} = a\sqrt{2},$ and $AP = PQ \cos\theta]$

$= a\sqrt{2} (\sin\frac{1}{4}\pi \cos\theta + \cos\frac{1}{4}\pi \sin\theta) - c\cos\theta\sin\theta$

$= a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta.$

The equation of virtual work is

$-W\delta(LG) = 0, \quad \text{or} \quad \delta(LG) = 0$

or $\quad \delta[a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta] = 0$

or $\quad [a(-\sin\theta + \cos\theta) - c(\cos^2\theta - \sin^2\theta)]\,\delta\theta = 0$

or $\quad a(\cos\theta - \sin\theta) - c(\cos^2\theta - \sin^2\theta) = 0 \qquad [\because \delta\theta \neq 0]$

or $\quad (\cos\theta - \sin\theta)[a - c(\cos\theta + \sin\theta)] = 0.$

$\therefore$ either $\quad \cos\theta - \sin\theta = 0$

i.e., $\quad \sin\theta = \cos\theta$ i.e., $\tan\theta = 1$ i.e., $\theta = \frac{1}{4}\pi,$

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $\quad a - c(\cos\theta + \sin\theta) = 0$

i.e., $\quad c^2(\cos\theta + \sin\theta)^2 = a^2$

i.e., $\quad c^2(1 + \sin 2\theta) = a^2$

i.e., $$\sin 2\theta = \frac{a^2}{c^2} - 1 = \frac{a^2 - c^2}{c^2}$$

i.e., $$\theta = \frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right),$$

giving the other position of equilibrium.

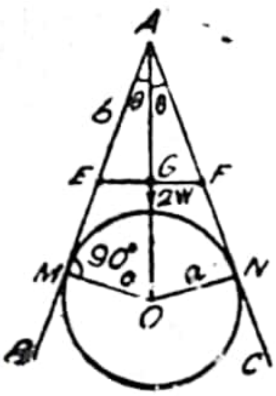
Ex. 48 (b). *A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove by the principle of virtual work that the side of length a makes with the vertical an angle θ given by* $2c\cos 2\theta = b\cos\theta - a\sin\theta$. **[Meerut 83, 87P]**

Sol. Proceed as in part (a).

Ex. 49. *Two equal rods, AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then*

$$b\sin^3\theta = a\cos\theta.$$

[Meerut 92; Lucknow 79; Kanpur 83, 87, 88; Jiwaji 80; Rohilkhand 78]



Sol. Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC. If E and F are the middle points of AB and AC, then the total weight $2W$ of the two rods can be taken acting at G, the middle point of EF. The line AO is vertical. We have

$$\angle BAO = \angle CAO = \theta.$$

Also $AB = 2b$, $AE = b$. If the rod AB touches

The equation of virtual work is

$$-W\delta\,(MG)=0, \quad \text{or} \quad \delta\,(MG)=0$$

or $$\delta\left[a \cot \alpha \sin \theta-\frac{a \cos 2\alpha}{4 \sin \alpha \cos \alpha}+\frac{a \cos 2\theta}{4 \sin \alpha \cos \alpha}\right]=0$$

or $$\left[a \cot \alpha \cos \theta-\frac{2a \sin 2\theta}{4 \sin \alpha \cos \alpha}\right] \delta\theta=0$$

or $$a \cot \alpha \cos \theta-\frac{4a \sin \theta \cos \theta}{4 \sin \alpha \cos \alpha}=0 \qquad [\because \ \delta\theta \neq 0]$$

or $$a \cos \theta\left(\cot \alpha-\frac{\sin \theta}{\sin \alpha \cos \alpha}\right)=0.$$

$\therefore$ $\cos \theta=0$ *i.e.*, $\theta=\frac{\pi}{2}$, giving one position of equilibrium in which the lamina rests symmetrically on the pegs

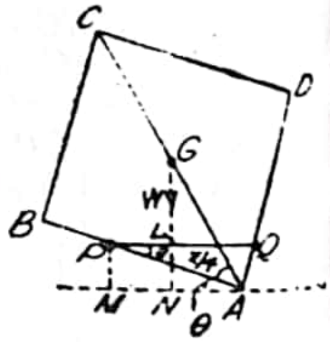
or $\cot \alpha-\dfrac{\sin \theta}{\sin \alpha \cos \alpha}=0$ *i.e.*, $\sin \theta=\cos^2 \alpha$ *i.e.*, $\theta=\sin^{-1}(\cos^2 \alpha)$, giving the other position of equilibrium.

Ex. 48. (a) *A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4} \ \text{or} \ \frac{1}{2} \sin^{-1}\left(\frac{a^2-c^2}{c^2}\right).$$

[Meerut 80, 88P; Gorakhpur 81; Jiwaji 82; P.C.S. 81]

Sol. The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ=c$ and $AB=2a$.



The weight W of the lamina acts at G, the middle point of the diaganal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

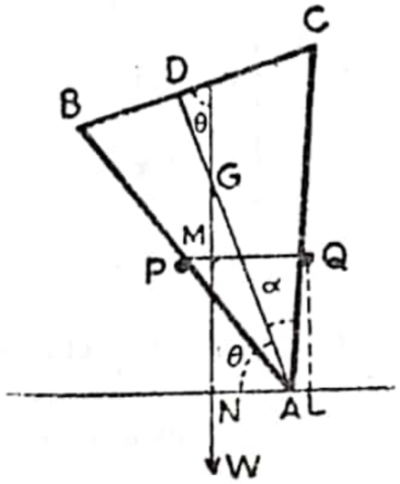
$$\angle PAM=\theta=\angle QPA.$$

We have $\angle BAC=\frac{1}{4}\pi=$constant.

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=LG=NG-NL=NG-MP$$
$$=AG \sin\left(\tfrac{1}{4}\pi+\theta\right)-AP \sin \theta$$

Sol. ABC is an isosceles triangular lamina in which $AB=AC$. The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.



Let $PQ=a$ so that $BC=3a$.

If D is the middle point of BC, then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG=\tfrac{2}{3}\,AD.$$

The weight W of the lamina acts vertically downwards at G. We have

$$\angle BAD=\angle CAD=\alpha.$$

Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendicular lines, therefore $\angle DAN=\theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line NMG].

Now $\quad\angle QPA=\angle PAN=\theta-\alpha,$

and $\quad\angle QAL=\pi-(\theta+\alpha).$

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=MG=NG-NM=NG-LQ$$
$$=AG\sin\theta-AQ\sin\{\pi-(\theta+\alpha)\}$$
$$=\tfrac{2}{3}\,AD\sin\theta-AQ\sin(\theta+\alpha).$$

Now $AD=CD\cot\alpha=\frac{3}{2}\,a\cot\alpha$. Also from the $\triangle AQP$, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ}=\frac{PQ}{\sin PAQ}\quad i.e.,\quad \frac{AQ}{\sin(\theta-\alpha)}=\frac{a}{\sin 2\alpha}.$$

$$\therefore\quad AQ=\frac{a}{\sin 2\alpha}\sin(\theta-\alpha).$$

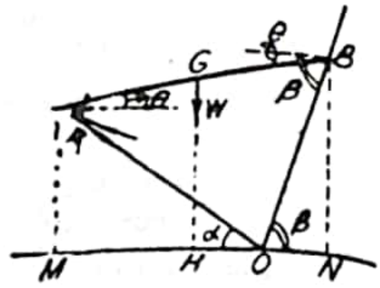
$$\therefore\quad MG=\tfrac{2}{3}\cdot\tfrac{3}{2}\,a\cot\alpha\sin\theta-\frac{a}{\sin 2\alpha}\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{2\sin 2\alpha}\,2\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{4\sin\alpha\cos\alpha}(\cos 2\alpha-\cos 2\theta)$$

$$=a\cot\alpha\sin\theta-\frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha}+\frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}.$$

Sol. Let AB be the rod of length $2a$ and G its middle point. Let AM, BN, and GH be the perpendiculars from A, B and G on the horizontal line through O, the point of intersection of the inclined planes OA and OB. The weight W of the rod acts at G and in equilibrium the rod makes an angle θ with the horizontal.



Give the rod a small displacement in which θ changes to $\theta+\delta\theta$. The horizontal line MON through O is the fixed line from which the distances will be measured. The angles α and β remain fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and B do no work. We have

the height of G above the fixed line MON

$$=HG=\tfrac{1}{2}(AM+BN)=\tfrac{1}{2}(OA\sin\alpha+OB\sin\beta).$$

From the $\triangle AOB$ by the sine theorem of trigonometry, we have

$$\frac{OA}{\sin(\beta-\theta)}=\frac{OB}{\sin(\theta+\alpha)}=\frac{AB}{\sin\{\pi-(\alpha+\beta)\}}=\frac{2a}{\sin(\alpha+\beta)}.$$

$$\therefore\quad OA=2a\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)},\; OB=\frac{2a\sin(\theta+\alpha)}{\sin(\alpha+\beta)}.$$

$$\therefore\quad HG=\tfrac{1}{2}\cdot\frac{2a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}.$$

The equation of virtual work is

$$-W\delta(HG)=0,\text{ or }\delta(HG)=0$$

or $$\delta\left[\frac{a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}\right]=0$$

or $$\frac{a}{\sin(\alpha+\beta)}[-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta]\,\delta\theta=0$$

or $$-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta=0 \qquad [\because\ \delta\theta\neq0]$$

or $$-(\cos\beta\cos\theta+\sin\beta\sin\theta)\sin\alpha+(\cos\theta\cos\alpha-\sin\theta\sin\alpha)\sin\beta=0$$

or $$2\sin\alpha\sin\beta\sin\theta=\cos\theta(\cos\alpha\sin\beta-\cos\beta\sin\alpha)$$

or $\tan\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta)$, giving the inclination of the rod to the horizontal in the position of equilibrium.

Ex. 47. *An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle* $\sin^{-1}(\cos^2\alpha)$ *with the vertical,* 2α *being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.*

[Meerut 81, 84P]

Differentiating (2),

$$-a \sin \theta \, \delta\theta = -b \sin \phi \, \delta\phi$$

or $$a \sin \theta \, \delta\theta = b \sin \phi \, \delta\phi. \qquad \ldots(3)$$

Dividing (1) by (3), we get

$$\frac{\frac{1}{2}l \sec^2 \theta - a \cos \theta}{a \sin \theta} = \frac{b \cos \phi}{b \sin \phi}$$

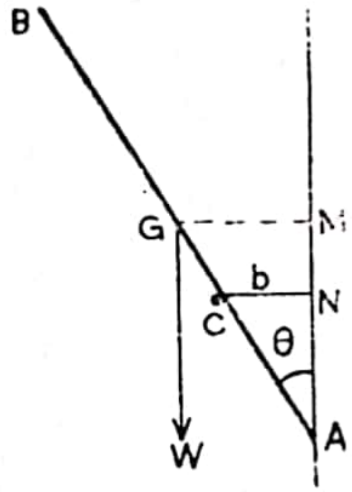
or $$\tfrac{1}{2}l \sec^2 \theta \sin \phi = a (\sin \theta \cos \phi + \cos \theta \sin \phi)$$

or $$\tfrac{1}{2}l \sec^2 \theta \sin \phi = a \sin (\theta + \phi). \qquad \ldots(4)$$

Thus θ and ϕ are given by the equations (2) and (4).

Ex. 45. *A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle* $\sin^{-1} (b/a)^{1/3}$. **[Gorakhpur 80, 82]**

Sol. A uniform beam AB of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b. Suppose the rod makes an angle θ with the wall *i.e.*, $\angle BAM = \theta$. The weight W of the rod acts at its middle point G.



Give the rod a small displacement in which θ changes to $\theta + \delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and C do not work.

We have, the height of G above the fixed point C

$$= NM = AM - AN = AG \cos \theta - CN \cot \theta$$
$$= a \cos \theta - b \cot \theta.$$

The equation of virtual work is

$$-W\delta (a \cos \theta - b \cot \theta) = 0,$$

or $$\delta (a \cos \theta - b \cot \theta) = 0$$

or $$-a \sin \theta \, \delta\theta + b \operatorname{cosec}^2 \theta \, \delta\theta = 0$$

or $$(-a \sin \theta + b \operatorname{cosec}^2 \theta) \, \delta\theta = 0$$

or $$-a \sin \theta + b \operatorname{cosec}^2 \theta = 0 \qquad [\because \delta\theta \neq 0]$$

or $$a \sin \theta = b \operatorname{cosec}^2 \theta \text{ or } \sin^3 \theta = b/a$$

or $$\theta = \sin^{-1} (b/a)^{1/3},$$

giving the inclination of the rod to the vertical in the position of equilibrium.

Ex. 46. *A heavy uniform rod, of length 2a, rests with its ends in contact with two smooth inclined planes, of inclination* α *and* β *to the horizon. If* θ *be the inclination of the rod to the horizon, prove, by the principle of virtual work, that*

$$\tan \theta = \tfrac{1}{2} (\cot \alpha - \cot \beta).$$

[Meerut 83, 83P, 84S, 87; Jiwaji 81]

The equation of virtual work is

$$-T\delta\,(4a\cos\alpha+2a)-2W\delta\,[(c-a)\tan\alpha-a\sin\alpha]-W\delta\,[(c-a)\tan\alpha]=0$$

or $\quad 4aT\sin\alpha\,\delta\alpha-2\,(c-a)\,W\sec^2\alpha\,\delta\alpha+2aW\cos\alpha\,\delta\alpha-(c-a)\,W\sec^2\alpha\,\delta\alpha=0$

or $\quad [4aT\sin\alpha-3\,(c-a)\,W\sec^2\alpha+2aW\cos\alpha]\,\delta\alpha=0$

or $\quad 4aT\sin\alpha-3\,(c-a)\,W\sec^2\alpha+2aW\cos\alpha=0 \qquad [\because\ \delta\alpha\neq 0]$

or $\quad 4aT\sin\alpha=3\,(c-a)\,W\sec^2\alpha-2aW\cos\alpha$

or $\quad T=\dfrac{1}{4a\sin\alpha}\,W\,[3c\sec^2\alpha-3a\sec^2\alpha-2a\cos\alpha]$

$\quad =\frac{1}{4}\,W\operatorname{cosec}\alpha\sec^2\alpha\,[(3c/a)-(3+2\cos^3\alpha)].$

Ex. 44. *Four light rods are jointed together to form a quadrilateral OABC. The lengths are such that*

$$OA=OC=a,\ AB=CB=b.$$

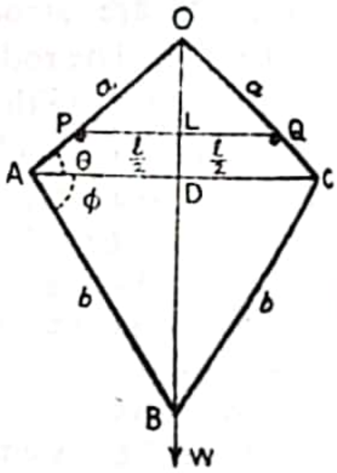
The framework hangs in a vertical plane OA and OC resting in contact with two smooth pegs distant l apart and on the same horizontal level. A weight hangs at B. If θ, ϕ are the inclinations of OA, AB to the horizontal, prove that these values are given by the equations

$$a\cos\theta=b\cos\phi$$

and $\qquad \frac{1}{2}l\sec^2\theta\sin\phi=a\sin(\theta+\phi).$ **[Jiwaji 82]**

Sol. $OABC$ is a framework formed of four light rods such that

$$OA=OC=a \text{ and } AB=CB=b.$$



The rods OA and OC are in contact with two smooth pegs P and Q which are in the same horizontal line and $PQ=l$. A weight W hangs at B. We have

$$\angle OAC=\theta \text{ and } \angle BAC=\phi.$$

Give the system a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W acting at B.

We have, the depth of B below $PQ=LB=OB-OL$

$$=OD+DB-OL=a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta.$$

The equation of virtual work is

$$W\delta\,(a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta)=0$$

or $\quad a\cos\theta\,\delta\theta+b\cos\phi\,\delta\phi-\frac{1}{2}l\sec^2\theta\,\delta\theta=0$

or $\quad (\frac{1}{2}l\sec^2\theta-a\cos\theta)\,\delta\theta=b\cos\phi\,\delta\phi. \qquad \ldots(1)$

Now let us find a relation between the parameters θ and ϕ from the figure. From the $\triangle OAD$, we have $AD=a\cos\theta$ and from the $\triangle BAD$, we have $AD=b\cos\phi$.

$$\therefore\quad a\cos\theta=b\cos\phi. \qquad \ldots(2)$$

$\therefore$ from $\triangle AOB$, we have

$$\sin\theta = \frac{BO}{AB} = \frac{\frac{1}{2}b}{a} = \frac{b}{2a}.$$

$$\therefore \quad \text{cosec}\,\theta = \frac{2a}{b} \text{ and } \cos\theta = \sqrt{(1-\sin^2\theta)}$$

$$= \sqrt{\left(1-\frac{b^2}{4a^2}\right)} = \frac{\sqrt{(4a^2-b^2)}}{2a}.$$

Substituting in (i), we have

$$T = W\,\frac{\frac{1}{2}c.(4a^2/b^2) - 2a.(b/2a)}{2a.\{\sqrt{(4a^2-b^2)}/2a\}} = W\,\frac{2a^2c - b^3}{b^2\,(4a^2-b^2)^{1/2}}.$$

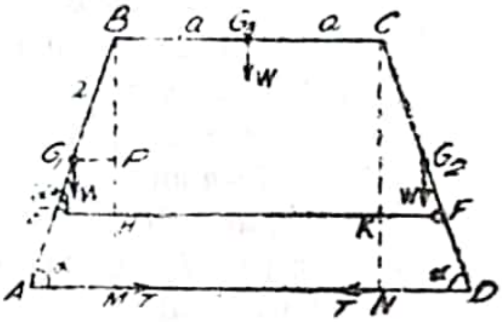
Ex. 43. *Three rigid rods AB, BC, CD each of length $2a$, are smoothly jointed at B and C. The system is placed in a vertical plane so that rods AB, CD are in contact with two smooth pegs distant $2c$ apart in the same horizontal line, the rods AB, CD make equal angle α with the horizon. Prove that the tension of the string AD which will maintain this configuration is*

$$\tfrac{1}{4} W\ \text{cosec}\,\alpha\ \sec^2\alpha\ \{(3c/a) - (3 + 2\cos^3\alpha)\},$$

where W is the weight of either rod.

Sol. Three rigid rods AB, BC, CD, each of length $2a$ and weight W are smoothly jointed at B and C. The rods AB and CD are in contact with two smooth pegs E and F which are in the same horizontal line and

$$EF = 2c.$$



Let T be the tension in the string AD joining A and D. The weights W of the rods AB, BC and CD act at their respective middle points.

We have $\qquad \angle BAD = \alpha = \angle CDA.$

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD do not change and the length AD changes.

We have,

$$AD = AM + MN + ND$$
$$= 2a\cos\alpha + 2a + 2a\cos\alpha$$
$$= 4a\cos\alpha + 2a.$$

The height of G_1 or G_2 above the fixed line EF

$$= HP = HB - PB = EH\tan\alpha - BG_1\sin\alpha$$
$$= \tfrac{1}{2}(2c - 2a)\tan\alpha - a\sin\alpha$$
$$= (c-a)\tan\alpha - a\sin\alpha,$$

and the height of G_3 above EF

$$= HB = (c-a)\tan\alpha.$$

and $$\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{(1-\tfrac{9}{16})}=\tfrac{1}{4}\sqrt{7}.$$

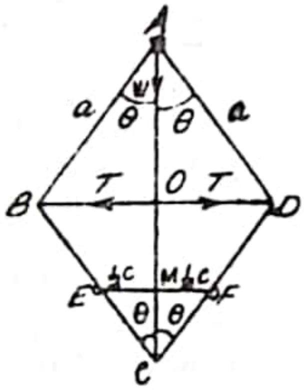
$$\therefore \quad T=\frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}}=\frac{32}{9\sqrt{7}}\frac{b}{a}W.$$

Ex. 42. *A rhomboidal framework $ABCD$ is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude*

$$W(2a^2c-b^3)/b^2(4a^2-b^2)^{1/2}.$$

Sol. The rods BC and CD of a rhomboidal framework $ABCD$ are in contact with two smooth pegs E and F which are in the same horizontal line and $EF=c$. The rods forming the rhombus are light and the length of each rod forming the rhombus is a. Let T be the thrust in the light rod BD joining B and D. A weight W is placed at the highest joint A. In the position of equilibrium, $BD=b$. The diagonal AC is vertical and BD is horizontal. Let

$$\angle BAC=\theta=\angle CAD.$$



Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The only forces contributing to the sum of virtual works are : (i) the weight W placed at A, and (ii) the thrust T in the rod BD. The reactions of the pegs E and F do not work. We have

$$BD=2BO=2AB\sin\theta=2a\sin\theta.$$

Also the height of A above the fixed line EF

$$=MA=CA-CM$$
$$=2OA-CM=2a\cos\theta-\tfrac{1}{2}c\cot\theta.$$

The equation of virtual work is

$$T\delta(2a\sin\theta)-W\delta(2a\cos\theta-\tfrac{1}{2}c\cot\theta)=0$$

or $$2aT\cos\theta\,\delta\theta+2aW\sin\theta\,\delta\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\delta\theta=0$$

or $$(2aT\cos\theta+2aW\sin\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\theta)\cdot\delta\theta=0$$

or $$2aT\cos\theta+2aW\sin\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\theta=0 \qquad [\because \delta\theta\neq 0]$$

or $$2aT\cos\theta=\tfrac{1}{2}cW\operatorname{cosec}^2\theta-2aW\sin\theta$$

or $$T=W.\frac{\tfrac{1}{2}c\operatorname{cosec}^2\theta-2a\sin\theta}{2a\cos\theta}. \qquad \text{...(1)}$$

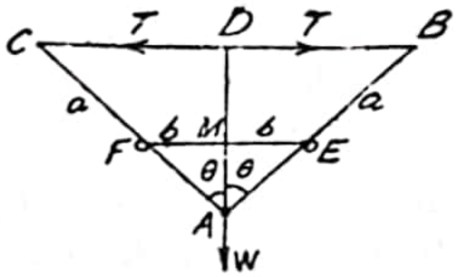
But in the position of equilibrium, we have

$$BD=b \text{ so that } BO=\tfrac{1}{2}b.$$

or $$T=\frac{1}{2l\cos\alpha}\,[d\,(W+4w)\,\text{cosec}^2\,\alpha-2l\sin\alpha\,(W+2w)]$$

or $$T=\tan\alpha\left[\frac{d}{2l}\,(W+4w)\,\text{cosec}^3\,\alpha-(W+2w)\right].$$

Ex. 41. *A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant 2b apart. A weight W is suspended from A, find the thrust in the rod BC.*



Sol. ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF=2b$. Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let

$$\angle BAD=\theta=\angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are : (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

We have,

$$BC=2BD=2AB\sin\theta=2a\sin\theta.$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$=MA=ME\cot\theta=b\cot\theta.$$

The equation of virtual work is

$$T\delta\,(2a\sin\theta)+W\delta\,(b\cot\theta)=0$$

or $$2a\,T\cos\theta\,\delta\theta-bW\,\text{cosec}^2\,\theta\,\delta\theta=0$$

or $$(2a\,T\cos\theta-bW\,\text{cosec}^2\,\theta)\,\delta\theta=0$$

or $$2a\,T\cos\theta-bW\,\text{cosec}^2\,\theta=0 \qquad [\because\ \delta\theta\neq0]$$

or $$2a\,T\cos\theta=bW\,\text{cosec}^2\,\theta$$

or $$T=\frac{Wb}{2a}\,\text{cosec}^2\,\theta\sec\theta.$$

But in the position of equilibrium,

$$BC=\tfrac{3}{2}a \text{ and so } BD=\tfrac{3}{4}a.$$

Therefore $$\sin\theta=\frac{BD}{AB}=\frac{\frac{3}{4}a}{a}=\frac{3}{4}$$

or $\quad -2aT\cos\alpha-5aW\sin\alpha+3Wc\operatorname{cosec}^2\alpha=0 \quad [\because\ \delta\alpha\neq 0]$

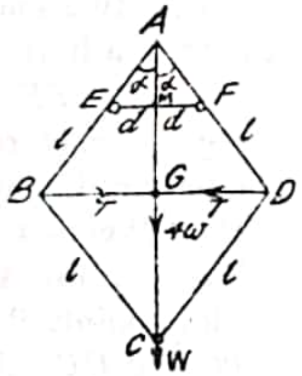
or $\quad 2aT\cos\alpha=3Wc\operatorname{cooec}^2\alpha-5aW\sin\alpha$

or $$T=\frac{W(3c\operatorname{cosec}^2\alpha-5a\sin\alpha)}{2a\cos\alpha}.$$

Ex. **40.** *ABCD is a rhombus formed with four rods each of length l and of weight w joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is 2d and the angle at A is 2α, show that the tension in the string is*

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and $EF=2d$. The length of each rod forming the rhombus is l. The total weight $4w$ of the rods forming the rhombus can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W is attached to the lowest point C. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the string BD. We have



$$\angle BAC=\alpha=\angle DAC.$$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remain fixed and the distances will be measured from this line. The $\angle AGB$ remains $90°$

We have the length of the string BD

$$=2BG=2AB\sin\alpha=2l\sin\alpha$$

The depth of G below EF

$$=MG=AG-AM=l\cos\alpha-d\cot\alpha,$$

and the depth of C below EF

$$=MC=AC-AM=2l\cos\alpha-d\cot\alpha.$$

The equation of virtual work is

$$-T\delta(2l\sin\alpha)+4w\delta(l\cos\alpha-d\cot\alpha)$$
$$+W\delta(2l\cos\alpha-d\cot\alpha)=0$$

or $$-2l\,T\cos\alpha\,\delta\alpha-4lw\sin\alpha\,\delta\alpha+4dw\operatorname{cosec}^2\alpha\,\delta\alpha-2l\,W\sin\alpha\,\delta\alpha$$
$$+dW\operatorname{cosec}^2\alpha\,\delta\alpha=0$$

or $$[-2lT\cos\alpha-2l\sin\alpha(2w+W)+d\operatorname{cosec}^2\alpha(4w+W)]\,\delta\alpha=0$$

or $$-2lT\cos\alpha-2l\sin\alpha(2w+W)+d\operatorname{cosec}^2\alpha(4w+W)=0$$
$$[\because\ \delta\alpha\neq 0]$$

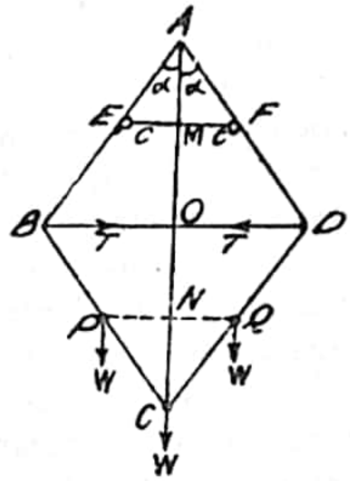
or $$2lT\cos\alpha=d(W+4w)\operatorname{cosec}^2\alpha-2l\sin\alpha(W+2w)$$

or $$2l\,T\cos\alpha = Wc\operatorname{cosec}^2\alpha - 2l\,W\sin\alpha$$

or $$T = \frac{1}{2l\cos\alpha}\,[Wc\operatorname{cosec}^2\alpha - 2lW\sin\alpha]$$

$$= W\tan\alpha\left[\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right]$$

Ex. 39. *A rhombus ABCD formed of four weightless rods each of length a freely jointed at the extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant 2c apart and in contact with AB and AD. Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If 2α be the angle of the rhombus at A, apply the principle of virtual work to find the tension of the string.*



Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and

$$EF = 2c.$$

The length of each rod of the rhombus is a and the rods forming the rhombus are light. Weights each equal to W are hung from the lowest corner C and from the middle points P and Q of the lower sides BC and CD. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the inextensible string joining B and D. We have

$$\angle BAC = \alpha = \angle DAC.$$

Replace the string BD by two equal and opposite forces T as shown in the figure so that the distance BD can be changed. Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$.

We have

$$BD = 2BO = 2AB\sin\alpha = 2a\sin\alpha.$$

The depth of C below $EF = MC = AC - AM$

$$= 2AO - AM = 2AB\cos\alpha - EM\cot\alpha$$
$$= 2a\cos\alpha - c\cot\alpha,$$

and the depth of P or Q below EF

$$= AN - AM = \tfrac{3}{2}AO - AM$$
$$= \tfrac{3}{2}a\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta\,(2a\sin\alpha) + W\delta\,(2a\cos\alpha - c\cot\alpha)$$
$$+ 2W\delta\,(\tfrac{3}{2}a\cos\alpha - c\cot\alpha) = 0$$

or $$(-2aT\cos\alpha - 2aW\sin\alpha + Wc\operatorname{cosec}^2\alpha - 3aW\sin\alpha$$
$$+ 2cW\operatorname{cosec}^2\alpha]\,\delta\alpha = 0$$

The equation of virtual work is

$$4W\,\delta\,(l\cos\theta - a\cot\theta) + W'\delta\,(2l\cos\theta - a\cot\theta) = 0$$

or $$-4lW\sin\theta\,\delta\theta + 4a\,W\operatorname{cosec}^2\theta\,\delta\theta - 2l\,W'\sin\theta\,\delta\theta + aW'\operatorname{cosec}^2\theta\,\delta\theta = 0$$

or $$[a\,(4W+W')\operatorname{cosec}^2\theta - l\,(4W+2W')\sin\theta]\,\delta\theta = 0$$

or $$a\,(4W+W')\operatorname{cosec}^2\theta - l\,(4W+2W')\sin\theta = 0 \qquad [\because\ \delta\theta \neq 0]$$

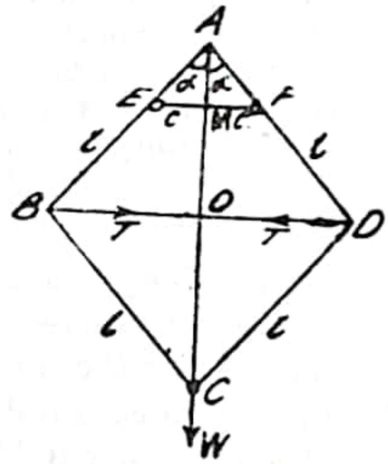
or $$a\,(4W+W')\operatorname{cosec}^2\theta = l\,(4W+2W')\sin\theta$$

or $$\sin^3\theta = \frac{a\,(4W+W')}{l\,(4W+2W')}.$$

Ex. 38. *ABCD is a rhombus with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD, B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is 2c and the angle at A is 2α, show that the tension in the string is*

$$W\tan\alpha\left(\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right)$$

Sol. The rods AB and AD of the frame rest on two smooth pegs E and F which are in the same horizontal line and $EF=2c$. The length of each rod of the rhombus is l and the rods forming the rhombus are weightless. A weight W is attached to the lowest point C. Let T be the tension in the string BD. We have



$$\angle BAC = \alpha = \angle CAD.$$

The diagonal AC is vertical and BD is horizontal.

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$. We have

$$BD = 2BO = 2AB\sin\alpha = 2l\sin\alpha.$$

Also the depth of the point C below EF

$$= MC = AC - AM = 2AO - AM$$

$$= 2AB\cos\alpha - EM\cot\alpha = 2l\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta\,(2l\sin\alpha) + W\delta\,(2l\cos\alpha - c\cot\alpha) = 0$$

or $$-2l\,T\cos\alpha\,\delta\alpha - 2lW\sin\alpha\,\delta\alpha + Wc\operatorname{cosec}^2\alpha\,\delta\alpha = 0$$

or $$(-2l\,T\cos\alpha - 2l\,W\sin\alpha + Wc\operatorname{cosec}^2\alpha)\,\delta\alpha = 0$$

or $$-2l\,T\cos\alpha - 2l\,W\sin\alpha + Wc\operatorname{cosec}^2\alpha = 0 \qquad [\because\ \delta\alpha \neq 0]$$

The equation of virtual work is

$$T\delta\,(4a\sin\theta)+4W\delta\,(2a\cos\theta-b\cot\theta)=0$$

$$4aT\cos\theta\,\delta\theta-8aW\sin\theta\,\delta\theta+4bW\operatorname{cosec}^2\theta\,\delta\theta=0$$

or $$4\,(aT\cos\theta-2aW\sin\theta+bW\operatorname{cosec}^2\theta)\,\delta\theta=0$$

or $$aT\cos\theta-2aW\sin\theta+bW\operatorname{cosec}^2\theta=0 \qquad [\because\ \delta\theta\neq 0$$

or $$aT\cos\theta=2aW\sin\theta-bW\operatorname{cosec}^2\theta$$

or $$T=\frac{W}{a\cos\theta}\,(2a\sin\theta-b\operatorname{cosec}^2\theta)$$

or $$=\frac{W}{a}\tan\theta\,(2a-b\operatorname{cosec}^3\theta).$$

But in the position of equilibrium, $\theta=45°$.

$$\therefore\quad T=\frac{W}{a}\tan 45°\,(2a-b\operatorname{cosec}^3 45°)$$

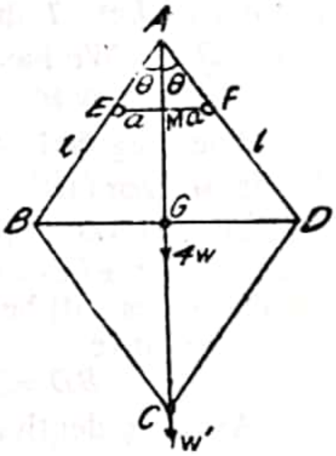
$$=\frac{W}{a}\,[2a-b\,(\sqrt{2})^3]=\frac{2W}{a}\,(a-b\sqrt{2}).$$

Remark. The pegs E and F may also be taken below the middle points of the rods AB and AD.

Ex. 37. *A rhombus is formed of rods each of weight W and length l with smooth joints. It rests symmetrically with its two upper sides in contact with two smooth pegs at the same level and at a distance 2a apart. A weight W' is hung at the lowest point. If the sides of the rhombus make an angle θ with the vertical, prove that*

$$\sin^3\theta=\frac{a\,(4W+W')}{l\,(4W+2W')}.$$

Sol. The rods AB and AD of the framework rest on two smooth pages E and F which are at the same level and $EF=2a$. The length of each rod of the rhombus is l. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W' is hung at the lowest point C. The diagonal AC is vertical and BD is horizontal. We have



$$\angle BAC=\theta=\angle CAD.$$

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AGB$ remains 90°.

We have, the depth of G below EF

$$=MG=AG-AM=l\cos\theta-a\cot\theta,$$

and the depth of C below EF

$$=MC=AC-AM=2l\cos\theta-a\cot\theta.$$

or $$\frac{\delta\,(AC)}{\delta\,(BD)}=-\frac{BD}{AC}. \qquad \ldots(3)$$

From (1) and (3), we get

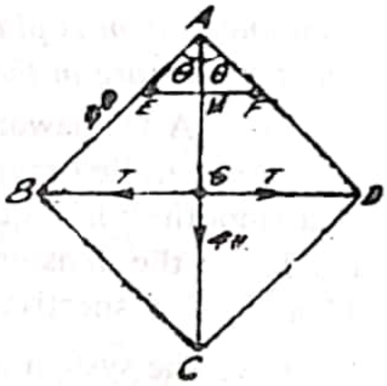
$$-\frac{T_2}{T_1}=-\frac{BD}{AC}$$

or $$\frac{T_1}{AC}=\frac{T_2}{BD},$$

i.e., tensions are in the ratio of the lengths of the strings.

Problems relating to bodies or frameworks resting on pegs or on inclined planes.

Ex. 36. *Four equal rods, each of length 2a and weight W, are freely jointed to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance 2b apart on the same level. Find the stress in the rod BD.*



Sol. The rods AB and AD of the framework rest on two fixed smooth pegs E and F which are at the same level and $EF=2b$. Let $2a$ be the length of each of the rods AB, BC, CD and DA. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the middle point of AC.

Let T be the thrust in the rod BD and let $\angle BAC=\theta=\angle CAD$.

Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed and the distance will be measured from this line. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The $\angle AGB$ remains $90°$.

The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BD, and (ii) the weight $4W$ acting at G. The reactions at the pegs do no work.

We have

$$BD=2BG=2.2a\sin\theta=4a\sin\theta.$$

Also the depth of G below the fixed line EF

$$=MG=AG-AM=AB\cos\theta-EM\cot\theta$$
$$=2a\cos\theta-b\cot\theta.$$

Sol. Proceed as in Ex. 33. Here the rod RS is also in a state of tension T'. The equation of virtual work is

$$-T\delta\,(PQ)-T'\delta\,(RS)=0$$

or
$$\frac{\delta\,(PQ)}{\delta\,(RS)}=-\frac{T'}{T}.$$

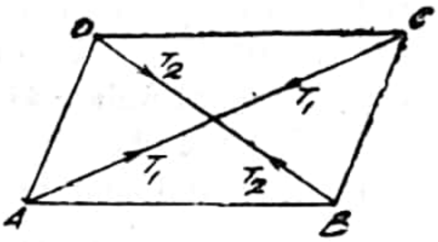
Also
$$\frac{\delta\,(PQ)}{\delta\,(RS)}=\frac{RS}{PQ},\text{ as found in Ex. 33.}$$

$$\therefore\quad -\frac{T'}{T}=\frac{RS}{PQ}$$

or
$$\frac{T}{PQ}+\frac{T'}{RS}=0$$

or
$$\frac{T}{l}+\frac{T'}{l'}=0.\qquad [\because \text{ in the position of equilibrium } PQ=l \text{ and } RS=l'].$$

Ex. 35. *Four rods are jointed together to form a parallelogram, the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.* **[Rohilkhand 82]**



Sol. A framework $ABCD$ is in the form of a parallelogram and is placed on a smooth horizontal table. Let T_1 and T_2 be the tensions in the strings AC and BD respectively.

Give the system a small displacement in the plane of the table in which AC changes to $AC+\delta(AC)$ and BD changes to $BD+\delta\,(BD)$. The lengths of the rods AB, BC, CD, DA do not change. During this displacement the weights of the rods do no work because the displacement of their points of application in the vertical direction is zero. The equation of virtual work is

$$-T_1\,\delta\,(AC)-T_2\,\delta\,(BD)=0$$

or
$$\frac{\delta\,(AC)}{\delta\,(BD)}=-\frac{T_2}{T_1}.\qquad \ldots(1)$$

Now let us find a relation between the parameters AC and BD from the figure. Since in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of its sides, therefore

$$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2=\text{constant}.\qquad \ldots(2)$$

Differentiating (2), we get

$$2AC\,\delta\,(AC)+2BD\,\delta\,(BD)=0$$

of the sides of the quadrilateral $ABCD$, therefore $PSQR$ is a parallelogram. Consequently the diagonals PQ and RS of this parallelogram bisect each other at O.

Replace the string PQ by two equal and opposite forces T as shown in the figure and replace the rod RS by two equal and opposite forces X as shown in the figure. Now give the system a small displacement in which PQ changes to $PQ+\delta(PQ)$ and RS changes to $RS+\delta(RS)$. The lengths of the rods AB, BC, CD, DA do not change. The equation of virtual work is

$$-T\delta(PQ)+X\delta(RS)=0$$

or
$$T\delta(PQ)=X\delta(RS)$$

or
$$\frac{\delta(PQ)}{\delta(RS)}=\frac{X}{T}. \quad \ldots(1)$$

Now let us find a relation between the parameters PQ and RS from the figure. Since OP is a median of the $\triangle OAB$, therefore

$$OA^2+OB^2=2OP^2+2AP^2=2\left(\tfrac{1}{2}PQ\right)^2+2\left(\tfrac{1}{2}AB\right)^2$$
$$=\tfrac{1}{2}(PQ^2+AB^2). \quad \ldots(2)$$

Similarly from $\triangle OCD$, we have

$$OC^2+OD^2=\tfrac{1}{2}(PQ^2+CD^2). \quad \ldots(3)$$

Adding (2) and (3), we get

$$OA^2+OB^2+OC^2+OD^2=\tfrac{1}{2}(2PQ^2+AB^2+CD^2) \quad \ldots(4)$$

Doing the same thing with $\triangle OAD$ and $\triangle OBC$, we get

$$OA^2+OB^2+OC^2+OD^2=\tfrac{1}{2}(2RS^2+BC^2+DA^2). \quad \ldots(5)$$

From (4) and (5), we get

$$\tfrac{1}{2}(2PQ^2+AB^2+CD^2)=\tfrac{1}{2}(2RS^2+BC^2+DA^2)$$

or
$$2(PQ^2-RS^2)=BC^2+DA^2-AB^2-CD^2$$

or
$$PQ^2-RS^2=\text{constant}, \quad \ldots(6)$$

since AB, BC, CD, DA are all of fixed lengths.

Differentiating (6), we get

$$2PQ\,\delta(PQ)-2RS\,\delta(RS)=0$$

or
$$\frac{\delta(PQ)}{\delta(RS)}=\frac{RS}{PQ}. \quad \ldots(7)$$

Equating the values of $\dfrac{\delta(PQ)}{\delta(RS)}$ from (1) and (7), we get

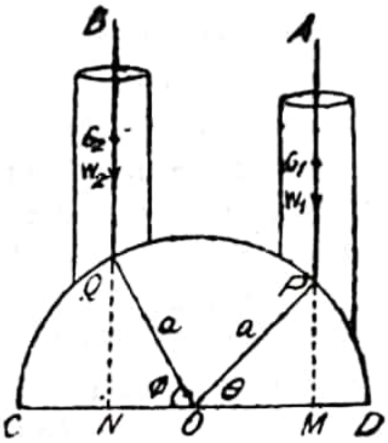
$$\frac{X}{T}=\frac{RS}{PQ} \text{ or } \frac{X}{RS}=\frac{T}{PQ}.$$

Ex. 34. *The middle points of the opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T' be the tensions in these rods, prove that*

$$\frac{T}{l}+\frac{T'}{l'}=0.$$

[Rohilkhand 79; Kanpur 78, 82]

whose lengths are say $2l_1$ and $2l_2$ respectively. Let θ and ϕ be the angles which the radii OP and OQ make with the horizontal diameter COD of the board. Let a be the radius of the board.



Here COD is a fixed horizontal line. The weight W_1 of the beam AP acts at its centre of gravity G_1 whose height above

$$CD = MG_1 = l_1 + a\sin\theta.$$

The weight W_2 of the beam BQ acts at G_2 whose height above CD is

$$l_2 + a\sin\phi.$$

Let the beams be imagined to undergo a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The equation of virtual work is $-W_1\delta\,(l_1+a\sin\theta)-W_2\delta\,(l_2+a\sin\phi)=0$

or $$-aW_1\cos\theta\,\delta\theta - a\,W_2\cos\phi\,\delta\phi = 0$$

or $$-W_1\cos\theta\,\delta\theta = W_2\cos\phi\,\delta\phi. \quad \ldots(1)$$

If b be the distance between the tubes in which the beams slide, then from the figure

$$a\cos\theta + a\cos\phi = b = \text{constant}$$

so that, $$-a\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi = 0$$

or $$-\sin\theta\,\delta\theta = \sin\phi\,\delta\phi. \quad \ldots(2)$$

Dividing (1) by (2), we have

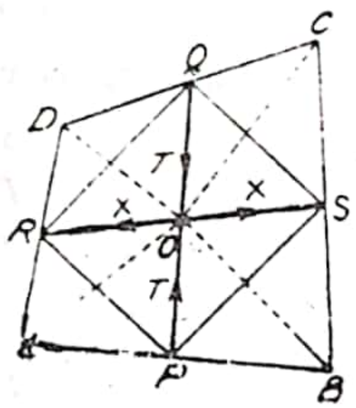
$$W_1\cot\theta = W_2\cot\phi$$

or $$\frac{W_1}{W_2} = \frac{\cot\phi}{\cot\theta} = \frac{\tan\theta}{\tan\phi},$$ which gives the required ratio.

Ex. 33. *A smoothly jointed framework of light rods forms a quadrilateral ABCD. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other pair by a light rod in a state of thrust X: show, by the method of virtual work, that*

$$T/PQ = X/RS.$$

Sol. $ABCD$ is a framework in the form of a quadrilateral formed of four light rods. The middle points P and Q of the rods AB and DC are joined by a string in a state of tension T and the middle points R and S of the rods AD and BC are joined by a light rod in a state of thrust X. [The framework is to be taken as placed on some smooth horizontal plane].



Since P, S, Q, R are the middle points

Sol. Proceed as in part (a).

Ex. 31. *A freely jointed framework is formed of five equal uniform rods each of weight W. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles θ and φ respectively with the vertical, prove that the tension of the string is to the weight of the rod as*

$$(4 \tan \theta + 2 \tan \phi) : (\tan \theta + \tan \phi).$$ **[Lucknow 80]**

Sol. Draw figure as in Ex. 30 (a). This question differs from the preceding one in having the string AM instead of the rod BE.

Let T be the tension in the string AM. The string AM is given to be inextensible, therefore before giving the displacement replace the string by two equal and opposite forces T so that the length AM may be changed.

Here $AM = 2a \cos \theta + 2a \cos \phi$.

The equation of virtual work is

$$-T\delta(2a \cos \theta + 2a \cos \phi) + 2W\delta(a \cos \theta) + 2W\delta(2a \cos \theta + a \cos \phi) + W\delta(2a \cos \theta + 2a \cos \phi) = 0$$

or $$2aT \sin \theta\, \delta\theta + 2aT \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 4aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi = 0$$

or $$2a \sin \theta\, (T - 4W)\, \delta\theta = 2a \sin \phi\, (2W - T)\, \delta\phi$$

or $$\sin \theta\, (T - 4W)\, \delta\theta = \sin \phi\, (2W - T)\, \delta\phi. \quad \ldots(1)$$

Also from the figure, we have

$$4a \sin \theta = 2a + 4a \sin \phi,$$

so that $$4a \cos \theta\, \delta\theta = 4a \cos \phi\, \delta\phi$$

or $$\cos \theta\, \delta\theta = \cos \phi\, \delta\phi. \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\tan \theta\, (T - 4W) = \tan \phi\, (2W - T)$$

or $$T(\tan \theta + \tan \phi) = W(2 \tan \phi + 4 \tan \theta)$$

or $$\frac{T}{W} = \frac{4 \tan \theta + 2 \tan \phi}{\tan \theta + \tan \phi},$$ which proves the required result.

Ex. 32. *A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.*

Sol. Let W_1 and W_2 be the weights of the beams AP and BQ

We have
$$BE=2BN=2.2a\sin\theta=4a\sin\theta,$$
the depth of the middle point of AB or AE below $A=a\cos\theta$,
the depth of the middle point of BC or ED below A
$$=2a\cos\theta+a\cos\phi,$$
and the depth of the middle point M of CD below A
$$=2a\cos\theta+2a\cos\phi.$$

The equation of virtual work is
$$T\delta(4a\sin\theta)+2W\delta(a\cos\theta)+2W\delta(2a\cos\theta+a\cos\phi)$$
$$+W\delta(2a\cos\theta+2a\cos\phi)=0$$
or $$4aT\cos\theta\,\delta\theta-2aW\sin\delta\theta-4aW\sin\theta\,\delta\theta-2aW\sin\phi\,\delta\phi$$
$$-2aW\sin\theta\,\delta\theta-2aW\sin\phi\,\delta\phi=0$$
or $$4a(T\cos\theta-2W\sin\theta)\,\delta\theta=4aW\sin\phi\,\delta\phi$$
or $$(T\cos\theta-2W\sin\theta)\,\delta\theta=W\sin\phi\,\delta\phi. \qquad ...(1)$$

From the figure, finding the length BE in two ways *i.e.*, from the upper portion ABE and from the lower portion $BCDE$, we have
$$4a\sin\theta=2a+4a\sin\phi.$$

Differentiating, we get $4a\cos\theta\,\delta\theta=4a\cos\phi\,\delta\phi$
or $$\cos\theta\,\delta\theta=\cos\phi\,\delta\phi. \qquad ...(2)$$

Dividing (1) by (2), we get
$$\frac{T\cos\theta-2W\sin\theta}{\cos\theta}=\frac{W\sin\phi}{\cos\phi}$$
or $$T-2W\tan\theta=2W\tan\phi$$
or $$T=W(\tan\phi+2\tan\theta).$$

But in the position of equilibrium,
$$\theta=\frac{1}{2}\cdot\frac{3}{5}\pi=\frac{3}{10}\pi,\ \phi=\frac{3}{5}\pi-\frac{1}{2}\pi=\frac{1}{10}\pi.$$
$$\therefore\quad T=W\left(\tan\frac{1}{10}\pi+2\tan\frac{3}{10}\pi\right)=W\left[\tan\frac{1}{10}\pi+2\cot\frac{2}{10}\pi\right]$$
$$\left[\because\ \tan\frac{3}{10}\pi=\cot\left(\frac{1}{2}\pi-\frac{3}{10}\pi\right)=\cot\frac{2}{10}\pi\right]$$
$$=W\left[\tan\frac{1}{10}\pi+2\cdot\frac{1-\tan^2(\pi/10)}{2\tan(\pi/10)}\right]$$
$$\left[\because\ \cot 2\alpha=\frac{1}{\tan 2\alpha}=\frac{1-\tan^2\alpha}{2\tan\alpha}\right]$$
$$=W\cot(\pi/10).$$

Ex. 30. (b) *A regular pentagon ABCDE formed of equal uniform rods each of weight W, is suspended from the point A and is maintained in shape by a light rod joining the middle points of BC and DE. Prove that the stress in the light rod is* $2W\cot(\pi/10)$.

(Rohilkhand 83)

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above. The equation of virtual work is

$$W\delta\,(l\cos\theta + a\cos\phi + \tfrac{3}{8}a\sin\phi) = 0$$

or $\quad -l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$ ...(1)

or $\quad l\sin\theta\,\delta\theta = a\,(\frac{3}{8}\cos\phi - \sin\phi)\,\delta\phi.$

From the figure, $EC = a$.

Also $EC = EM + MC = FA + MC = l\sin\theta + a\sin\phi$.

$\therefore \quad a = l\sin\theta + a\sin\phi.$

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$ ...(2)

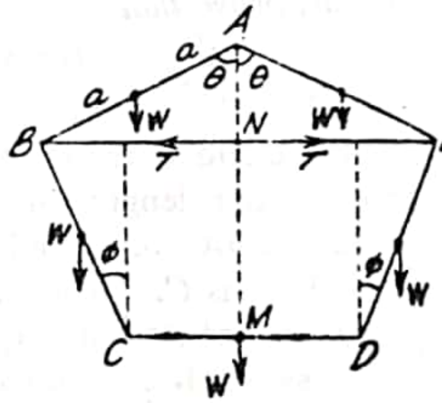
or $\quad -l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi.$

Dividing (1) by (2), we get

$$-\tan\theta = \tfrac{3}{8} - \tan\phi \quad \text{or} \quad \tan\phi = \tfrac{3}{8} + \tan\theta.$$

Ex. 30. (a) *Five equal uniform rods, freely jointed at their ends, form a regular pentagon ABCDE and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is W cot $\frac{1}{10}\pi$, where W is the weight of the rod.*

Sol. $ABCDE$ is a pentagon formed of five equal rods each of weight W and say of length $2a$. It is suspended from A and BF is jointed by a weightless bar. Let T be the thrust in the bar BE. The line AM joining A to the middle point M of CD is vertical and the line BE is horizontal. The weights of the rods AB, BC, CD, DE and EA act at their respective middle points. In the position of equilibrium the pentagon is a regular one so that each of its interior angles is $180° - 72°$ *i.e.*, $108°$ or $\frac{3}{5}\pi$ radians.



Let θ be the angle which the two upper slant rods AB and AE make with the vertical and ϕ be the angle which the two lower slant rods CB and DE make with the vertical.

Replace the rod BE by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AM in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$.

The point A remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length BE changes and the middle points of the rods AB, BC etc. are slightly displaced. The $\angle ANB$ remains $90°$.

Now consider a displacement when only θ changes and ϕ does not change so that $\delta\phi=0$. Then putting $\delta\phi=0$ in (1), we have

$$a\,[P\cos\theta-(W_1+W_2)\sin\theta]\,\delta\theta=0$$

or $$P\cos\theta-(W_1+W_2)\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$P=(W_1+W_2)\tan\theta. \qquad ...(2)$$

Again consider a displacement when only ϕ changes and θ does not change so that $\delta\theta=0$. Thus putting $\delta\theta=0$ in (1), we have

$$b\,[W_2\sin\phi-P\cos\phi]\,\delta\phi=0$$

or $$W_2\sin\phi-P\cos\phi=0 \qquad [\because\ \delta\phi\neq 0]$$

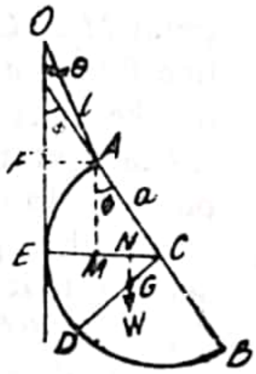
or $$P=W_2\tan\phi. \qquad ...(3)$$

From (2) and (3), we have

$$P=(W_1+W_2)\tan\theta=W_2\tan\phi.$$

Ex. 29. *A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that*

$$\tan\phi=\tfrac{3}{8}+\tan\theta.$$



Sol. O is a fixed point in the wall to which one end of the string has been attached. Let l be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C. The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that

$$CG=\tfrac{3}{8}a.$$

The hemisphere touches the wall at E. We have $\angle OEC=90^\circ$ so that EC is horizontal.

The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below $O=OF+AM+NG$

$$=l\cos\theta+a\cos\phi+\tfrac{3}{8}a\sin\phi.$$

[Note that $\angle NCG=90^\circ-\angle ACM=90^\circ-(90^\circ-\phi)=\phi$].

Give the system a small displacement in which θ changes to $\theta+\delta\theta$, ϕ changes to $\phi+\delta\phi$, the point O remains fixed, the length of the string AO does not change so that the work done by its tension is zero and the point G is slightly displaced. The $\angle OEC$ remains 90°.

$$a \sin \alpha = l \sin \beta, \quad \ldots(2)$$

so that $$a \cos \alpha \, \delta\alpha = l \cos \beta \, \delta\beta.$$

Dividing (1) by (2), we get

$$\frac{\frac{1}{2} W \cos \alpha - P \sin \alpha}{\cos \alpha} = \frac{P \sin \beta}{\cos \beta}$$

or $$\tfrac{1}{2} W \cos \alpha \cos \beta - P \sin \alpha \cos \beta = P \cos \alpha \sin \beta$$

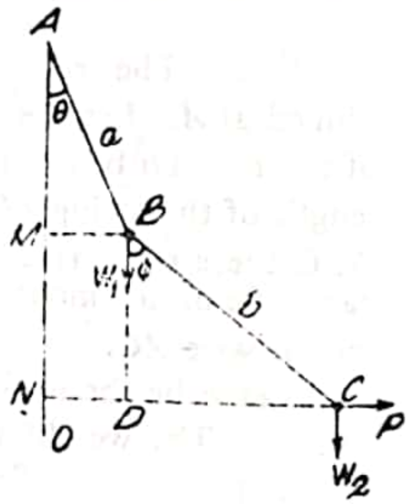
or $$P (\sin \beta \cos \alpha + \cos \beta \sin \alpha) = \tfrac{1}{2} W \cos \alpha \cos \beta$$

or $$P = \frac{W \cos \alpha \cos \beta}{2 \sin (\alpha + \beta)}.$$

Ex. 28. *Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then*

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi. \quad \text{[Meerut 81, 82, 85]}$$

Sol. Let the length of the portion AB of the string be a and that of BC be b. The point A is fixed and the vertical line AO through A is a fixed line.



From the fixed point A, the depth of B

$$= AM = a \cos \theta,$$

and the depth of C

$$= AN = AM + MN$$
$$= AM + BD = a \cos \theta + b \cos \phi.$$

Also the horizontal distance of the point C from the fixed line $AO = NC$

$$= ND + DC = MB + DC = a \sin \theta + b \sin \phi.$$

Now give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point A remains fixed, the length of the string remains unaltered and the points B and C are slightly displaced. The equation of virtual work is

$$W_1 \delta (a \cos \theta) + W_2 \delta (a \cos \theta + b \cos \phi) + P \delta (a \sin \theta + b \sin \phi) = 0$$

or $$-aW_1 \sin \theta \, \delta\theta - aW_2 \sin \theta \, \delta\theta - bW_2 \sin \phi \, \delta\phi + aP \cos \theta \, \delta\theta + bP \cos \phi \, \delta\phi = 0$$

or $$a [P \cos \theta - (W_1 + W_2) \sin \theta] \, \delta\theta = b [W_2 \sin \phi - P \cos \phi] \, \delta\phi,$$

where θ and ϕ are independent of each other. ...(1)

$$\therefore \quad T=\tfrac{1}{2}\,mg\left\{1+\tan 45^\circ.\frac{a}{\sqrt{(2l^2-a^2)}}\right\}$$

$$=\tfrac{1}{2}\,mg\left\{1+\frac{a}{\sqrt{(2l^2-a^2)}}\right\}.$$

When $l=a\sqrt{5}$, the tension T

$$=\tfrac{1}{2}\,mg\left\{1+\frac{a}{\sqrt{(2a^2.5-a^2)}}\right\}$$

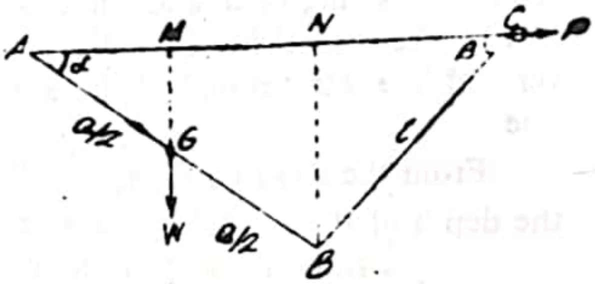
$$=\tfrac{1}{2}\,mg\,(1+\tfrac{1}{3})=\tfrac{2}{3}\,mg.$$

Ex. 27. *A rod is movable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is*

$$\frac{W\cos\alpha\cos\beta}{2\sin(\alpha+\beta)},$$

where W is the weight of the rod, and α, β the inclinations of the rod and the string to the horizontal.

[Lucknow 76; Allahabac Rohilkhand 87]



Sol. The rod AB is hinged at A. Let the length of the rod AB be a and the length of the string BC be l. At C there is a ring which can slide on a smooth horizontal wire AC.

Let P be the horizontal force applied at the ring C to keep it at rest. The weight W of the rod AB acts at its middle point G.

Let $\quad \angle BAC=\alpha$ and $\angle BCA=\beta$.

Give the system a small displacement in which α changes to $\alpha+\delta\alpha$ and β changes to $\beta+\delta\beta$. The point A remains fixed. The length of the rod AB remains fixed and the length of the string BC also remains fixed so that the work done by its tension is zero. The points G and C are slightly displaced. We have

the depth of G below $A=MG$

$=AG\sin\alpha=\tfrac{1}{2}a\sin\alpha$,

and the horizontal distance of C from $A=AC$

$=AN+NC=a\cos\alpha+l\cos\beta$.

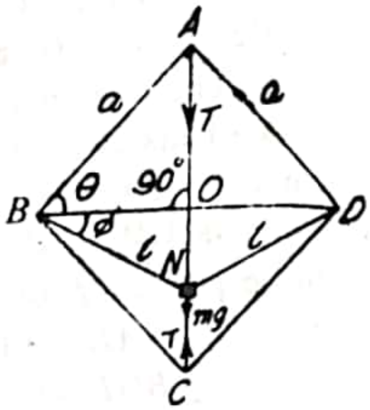
The equation of virtual work is

$$W\delta\,(\tfrac{1}{2}a\sin\alpha)+P\delta\,(a\cos\alpha+l\cos\beta)=0$$

or $\quad \tfrac{1}{2}a\,W\cos\alpha\,\delta\alpha-aP\sin\alpha\,\delta\alpha-lP\sin\beta\,\delta\beta=0$

or $\quad a\,(\tfrac{1}{2}W\cos\alpha-P\sin\alpha)\,\delta\alpha=lP\sin\beta\,\delta\beta. \quad \ldots(1)$

From the figure, equating the values of BN found from the triangles ANB and CNB, we get suspended from B and D by means of two strings BN and DN each of length l. Thus a weight mg acts at N. Let T be the tension in the string AC. In the position of equilibrium the figure is a square.



Let $\angle ABD=\theta$ and $\angle NBO=\phi$.

Give the system a small symmetrical displacement about the vertical AC in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA remain fixed and the length AC changes. The lengths of the strings BN and DN remain fixed so that the work done by their tensions is zero. The point N is slightly displaced.

We have

$$AC=2AO=2a\sin\theta,$$

and the depth of N below A

$$=AN=AO+ON=a\sin\theta+l\sin\phi.$$

The equation of virtual work is

$$-T\delta\,(2a\sin\theta)+mg\,\delta\,(a\sin\theta+l\sin\phi)=0$$

or $$-2aT\cos\theta\,\delta\theta+a\,mg\cos\theta\,\delta\theta+l\,mg\cos\phi\,\delta\phi=0$$

or $$a\cos\theta\,(2T-mg)\,\delta\theta=l\,mg\cos\phi\,\delta\phi. \qquad \ldots(1)$$

Now from the $\triangle AOB$, $BO=a\cos\theta$

and from the $\triangle BON$, $\quad BO=l\cos\phi$.

$$\therefore\quad a\cos\theta=l\cos\phi$$

so that $$-a\sin\theta\,\delta\theta=-l\sin\phi\,\delta\phi$$

or $$a\sin\theta\,\delta\theta=l\sin\phi\,\delta\phi. \qquad \ldots(2)$$

Dividing (1) by (2), we have

$$\frac{\cos\theta\,(2T-mg)}{\sin\theta}=\frac{mg\cos\phi}{\sin\phi}$$

or $$\cot\theta\,(2T-mg)=mg\cot\phi$$

or $$2T-mg=mg\tan\theta\cot\phi$$

or $$T=\tfrac{1}{2}mg\,(1+\tan\theta\cot\phi).$$

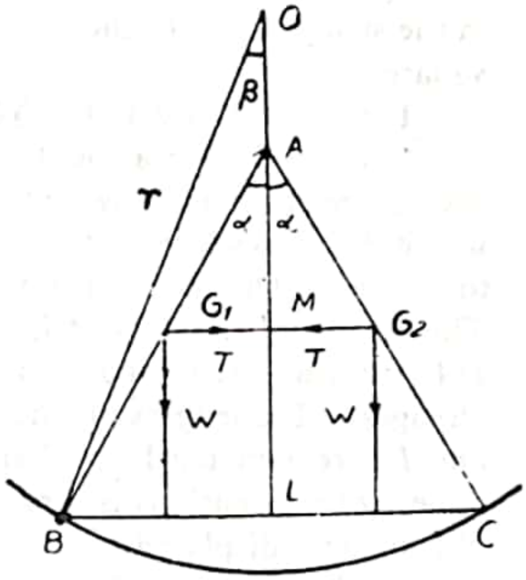
In the position of equilibrium $\theta=45°$,

$$BO=a\cos 45°=a/\sqrt{2},$$
$$ON=\sqrt{(BN^2-BO^2)}=\sqrt{\{l^2-(a^2/2)\}}$$
$$=\{\sqrt{(2l^2-a^2)}\}/\sqrt{2},$$

so that $$\cot\phi=\frac{BO}{ON}=\frac{a/\sqrt{2}}{\{\sqrt{(2l^2-a^2)}\}/\sqrt{2}}$$

$$=\frac{a}{\sqrt{(2l^2-a^2)}}.$$

smooth circular hoop. The radius $OB=r$ of the circular hoop is greater than the length $2a$ of either rod. Let T be the tension in the string connecting the middle points G_1 and G_2 of the rods. The weights W and W of the two rods act at their middle points G_1 and G_2 and the total weight $W+W=2W$ will act at the middle point M of G_1G_2. Given that $\angle BAL=\angle CAL=\alpha$ and $\angle BOL=\beta$.



Give the system a small displacement in which the angle α changes to $\alpha+\delta\alpha$ and β changes to $\beta+\delta\beta$. The smooth circular hoop remains fixed and hence its centre O can be taken as the fixed point. The lengths of the rods AB and AC do not change while the length of the string G_1G_2 changes.

The equation of virtual work is

$$-T\delta\,(G_1G_2)+2W\delta\,(OM)=0$$

or $$-T\delta\,(2a\sin\alpha)+2W\delta\,(r\cos\beta-a\cos\alpha)=0$$

or $$a\,(-T\cos\alpha+W\sin\alpha)\,\delta\alpha=Wr\sin\beta\,\delta\beta. \quad \text{...(1)}$$

In triangle OBL, $BL=r\sin\beta$,

and in triangle ABL, $BL=2a\sin\alpha$.

$$\therefore \quad 2a\sin\alpha=r\sin\beta$$

$$\therefore \quad \delta\,(2a\sin\alpha)=\delta\,(r\sin\beta)$$

or $$2a\cos\alpha\,\delta\alpha=r\cos\beta\,\delta\beta. \quad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{a\,(-T\cos\alpha+W\sin\alpha)}{2a\cos\alpha}=\frac{Wr\sin\beta}{r\cos\beta}$$

or $$-T+W\tan\alpha=2W\tan\beta$$

or $$T=W\,(\tan\alpha-2\tan\beta).$$

Ex. 26. *A frame, formed of four light rods, each of length a, freely jointed at A, B, C, D suspended at A. A mass m is suspended from B and D by two strings of length l ($l>a/\sqrt{2}$). The frame is kept in the form of a square by a string AC. Apply the method of virtual work to find the tension T in AC and show that when*

$l=a\sqrt{5}$, $T=2mg/3$. **[Ravishankar 86; Meerut 75]**

Sol. The framework is suspended from A and so A is a fixed point from which the distances are to be measured. A mass m is

Dividing (1) by (2), we have

$$\frac{W \cos B - T \sin B}{\cos B} = \frac{T \sin C}{\cos C}$$

or $$W - T \tan B = T \tan C$$

or $$T (\tan B + \tan C) = W$$

or $$T = \frac{W}{\tan B + \tan C}.$$

Ex. 24. *Two uniform rods AB, BC of weights W and W′ are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is*

$$(W + W') \cos A \cos C / \sin B.$$

Find the additional tension in the cord caused by suspending a weight W″ from B.

Sol. Draw figure and proceed as in Ex. 23.

In the first case, we shall get

$$T = \frac{(W+W')}{\tan A + \tan C} = \frac{(W+W') \cos A \cos C}{\sin A \cos C + \cos A \sin C}$$

$$= \frac{(W+W') \cos A \cos C}{\sin (A+C)} = \frac{(W+W') \cos A \cos C}{\sin (180° - B)}$$

$$= \frac{(W+W') \cos A \cos C}{\sin B}.$$

In the second case when a weight W'' is also suspended from B, let T' be the tension in the cord. Write the new equation of virtual work and find T'.

The required additional tension in the cord

$$= T' - T = (2W'' \cos A \cos C) / \sin B.$$

Ex. 25. *Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that if the string is stretched, its tension is* $W (\tan \alpha - 2 \tan \beta)$, *where* 2α *is the angle between the rods, and* β *the angle either rod subtends at the centre.*

Sol. AB and AC are two uniform rods freely jointed at A and resting with their extremeties B and C on the inside of a

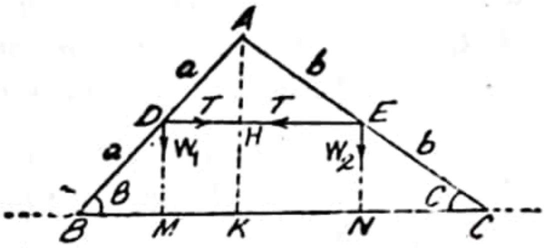
Ex. 23. *Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is*

$$\frac{W}{\tan B+\tan C}$$

where W is the total weight of the rods AB and AC.

[Gorakhpur 79; Jiwaji 78]

Sol. AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let



$$AB=2a \text{ and } AC=2b.$$

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore the total weight $W=W_1+W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B+\delta B$ and C changes to $C+\delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE=DH+HE=a\cos B+b\cos C,$$

the height of any point of the line DE above BC

$$=DM=a\sin B.$$

The equation of virtual work is

$$-T\delta\,(a\cos B+b\cos C)-W\delta\,(a\sin B)=0$$

or $$aT\sin B\,\delta B+bT\sin C\,\delta C-aW\cos B\,\delta B=0$$

or $$a\,(W\cos B-T\sin B)\,\delta B=bT\sin C\,\delta C. \quad \text{...(1)}$$

From the figure,

$$DM=a\sin B \text{ and } EN=b\sin C.$$

Since $DM=EN$, therefore $a\sin B=b\sin C$.

$$\therefore\quad \delta\,(a\sin B)=\delta\,(b\sin C)$$

or $$a\cos B\,\delta B=b\cos C\,\delta C. \quad \text{...(2)}$$

and the depth of the middle point of CB or CD below A

$$=2a\cos\theta+b\cos\phi.$$

The equation of virtual work is

$$-T\delta\,(2a\cos\theta+2b\cos\phi)+2W\delta\,(a\cos\theta)+2W'\delta\,(2a\cos\theta+b\cos\phi)=0$$

or $\quad 2aT\sin\theta\,\delta\theta+2b\,T\sin\phi\,\delta\phi-2aW\sin\theta\,\delta\theta-4aW'\sin\theta\,\delta\theta-2bW'\sin\phi\,\delta\phi=0$

or $\quad 2a\sin\theta\,(T-W-2W')\,\delta\theta=2b\sin\phi\,(W'-T)\,\delta\phi.$ ...(1)

But the parameters θ and ϕ are not independent of each other. From the figure, we can find a relation between θ and ϕ. From the $\triangle AMB$, we have $BM=2a\sin\theta$ and from the $\triangle CMB$, we have

$$BM=2b\sin\phi.$$

Equating the two values of BM, we have

$$2a\sin\theta=2b\sin\phi.$$

$\therefore \quad \delta\,(2a\sin\theta)=\delta\,(2b\sin\phi)$

or $\quad 2a\cos\theta\,\delta\theta=2b\cos\phi\,\delta\phi.$ ...(2)

Dividing (1) by (2), we get

$$\frac{\sin\theta\,(T-W-2W')}{\cos\theta}=\frac{\sin\phi\,(W'-T)}{\cos\phi} \qquad ...(3)$$

[Note that $\delta\theta$ and $\delta\phi$ cancel because $\delta\theta\neq 0$ and $\delta\phi\neq 0$]

or $\quad \sin\theta\cos\phi\,(T-W-2W')=\cos\theta\sin\phi\,(W'-T)$

or $\quad T(\sin\theta\cos\phi+\cos\theta\sin\phi)=W\sin\theta\cos\phi+W'\sin\theta\cos\phi+W'(\sin\theta\cos\phi+\cos\theta\sin\phi)$

or $\quad T\sin(\theta+\phi)=(W+W')\sin\theta\cos\phi+W'\sin(\theta+\phi).$

But in the position of equilibrium

$$\theta+\phi=90^\circ \quad \text{or} \quad \phi=90^\circ-\theta.$$

$\therefore \quad T\sin 90^\circ=(W+W')\sin\theta\cos(90^\circ-\theta)+W'\sin 90^\circ$

or $\quad T=(W+W')\sin^2\theta+W'.$

Ex. 22. *ABCD is a quadrilateral formed of four uniform freely jointed rods, of which $AB=AD$ and each of weight W, and $BC=CD$ each of weight W'. A string joins A to C. It is freely suspended from A. If $\angle BAD=2\theta$ and $\angle BCD=2\phi$, show that the tension in the string is*

$$\frac{W\tan\theta+W'(2\tan\theta+\tan\phi)}{\tan\theta+\tan\phi}.$$

[Meerut 86S]

Sol. Proceed as in Ex. 21.

From the result (3) of Ex. 21, we have

$$\tan\theta\,(T-W-2W')=\tan\phi\,(W'-T)$$

or $\quad T(\tan\theta+\tan\phi)=W\tan\theta+W'(2\tan\theta+\tan\phi)$

or $$T=\frac{W\tan\theta+W'(2\tan\theta+\tan\phi)}{\tan\theta+\tan\phi}$$

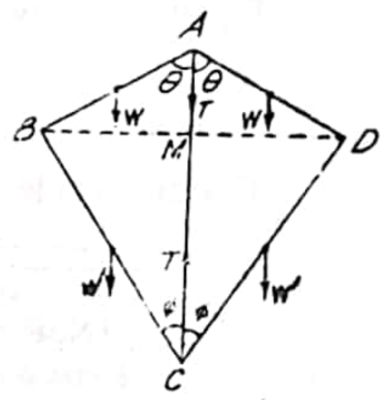
Problems involving two parameters.

Now we shall give the method to solve the problems involving two parameters.

Ex. 21. *A quadrilateral ABCD, formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is*

$$(W+W')\sin^2\theta+W',$$

where W is the weight of an upper rod and W' of a lower rod and 2θ is equal to the angle BAD.



Sol. The quadrilateral is suspended from the point A.

Let $\qquad AB=AD=2a$

and $\qquad BC=DC=2b.$

The diagonal AC must be vertical and BD horizontal. Let T be the tension in the string AC. The weights of the rods AB, BC, CD and DA act at their respective middle points. We have

$$\angle BAC=\theta=\angle DAC.$$

Let $\qquad \angle BCA=\phi=\angle DCA.$

Since in the position of equilibrium $\angle ABC=90°$, therefore in the position of equilibrium

$$\theta+\phi=90° \quad \text{or} \quad \phi=90°-\theta.$$

For initial calculation we cannot take $\angle ABC$ equal to 90° because during a displacement in which AC is to change this angle will not remain 90°.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA do not change while the length AC changes. The $\angle AMB$ remains 90°. We have

$$AC=AM+MC=AB\cos\theta+BC\cos\phi$$
$$=2a\cos\theta+2b\cos\phi,$$

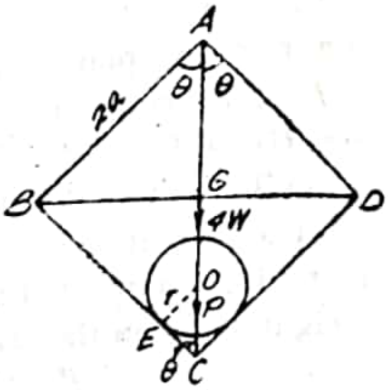
the depth of the middle point of AB or AD below A

$$=a\cos\theta,$$

one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing. Show that if 2θ be the angle at the fixed joint in the figure of equilibrium, then

$$\frac{\sin^3\theta}{\cos\theta}=\frac{Pr}{4(P+2W)a},$$

where r is the radius of the sphere and 2a is the length of each bar.



Sol. A rhombus $ABCD$ formed of four rods each of weight W and length $2a$ is suspended from the point A. A sphere of weight P and radius r is balanced inside the rhombus so as to is balanced inside the rhombus so as to keep it from collapsing. The diagonal AC of the rhombus must be vertical and the centre O of the sphere lies on it. The diagonal BD is horizontal.

The total weight $4W$ of all the four rods can be taken acting at G, the middle point of the diagonal AC and the weight P of the sphere acts at O. If the sphere touches the rod BC at E, then $\angle OEC=90°$, being the angle between the radius OE and the tangent BC. Also $\angle AGB=90°$.

We have

$$\angle BAC=\angle DAC=\theta=\angle BCA.$$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and the points G and O slightly displace. The angles AGB and OEC remain 90°. We have

the depth of G below $A=AG=2a\cos\theta$,

and the depth of O below A

$$=AO=AC-OC=4a\cos\theta-r\operatorname{cosec}\theta.$$

[Note that from the right angled triangle OEC,

$OC=OE\operatorname{cosec}\theta$]

The equation of virtual work is

$$4W\delta(2a\cos\theta)+P\delta(4a\cos\theta-r\operatorname{cosec}\theta)=0$$

or $$-8aW\sin\theta\,\delta\theta-4aP\sin\theta\,\delta\theta+Pr\operatorname{cosec}\theta\cot\theta\,\delta\theta=0$$

or $$[-8aW\sin\theta-4aP\sin\theta+Pr\operatorname{cosec}\theta\cot\theta]\,\delta\theta=0$$

or $$-8aW\sin\theta-4aP\sin\theta+Pr\operatorname{cosec}\theta\cot\theta=0\quad[\because\ \delta\theta\neq 0]$$

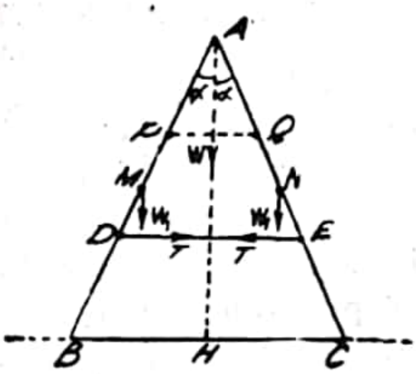
or $$4a\sin\theta\,(2W+P)=Pr\operatorname{cosec}\theta\cot\theta$$

or $$\frac{Pr}{4a(P+2W)}=\frac{\sin\theta}{\operatorname{cosec}\theta\cot\theta}=\frac{\sin^3\theta}{\cos\theta}$$

leg be W_1 and acts at their middle points and if a man of weight W is two-thirds the way up the ladder, show by the principle of virtual work, that the tension in the cord is

$$\tfrac{1}{2}\,(W+\tfrac{3}{2}W_1)\tan\alpha,$$

α being the inclination of each leg to the vertical.



Sol. Let AB and AC be the two legs each of weight W_1 and say of length $2a$. The points B and C rest on a horizontal plane. The points D and E are connected by a cord where $BD=CE=\frac{1}{3}AB$.

Let T be the tension in the cord DE. The weight W_1 of each leg acts at its middle point M or N. A man of weight W is on the level PQ where

$$BP=CQ=\tfrac{2}{3}\,AB.$$

The line AH is vertical and

$$\angle BAH=\angle CAH=\alpha.$$

Here the fixed level is the horizontal line BC.

The height of M or N above BC

$$=\tfrac{1}{2}AH=\tfrac{1}{2}.AB\cos\alpha=a\cos\alpha,$$

the height of the man (who is on PQ) above BC

$$=\tfrac{2}{3}AH=\tfrac{2}{3}.2a\cos\alpha=\tfrac{4}{3}a\cos\alpha.$$

The length of the cord (*i.e.*, string) DE

$$=2.AD\sin\alpha$$

$$=2.\tfrac{2}{3}AB\sin\alpha=2.\tfrac{2}{3}2a\sin\alpha=\tfrac{8}{3}a\sin\alpha.$$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The level of the line BC lying on the horizontal plane remains fixed, the lengths of the legs AB and AC do not change and the length DE changes.

The equation of virtual work is

$$-T\delta\,(\tfrac{8}{3}a\sin\alpha)-2W_1\delta\,(a\cos\alpha)-W\delta\,(\tfrac{4}{3}a\cos\alpha)=0$$

or $$-\tfrac{8}{3}a\,T\cos\alpha\,\delta\alpha+2aW_1\sin\alpha\,\delta\alpha+\tfrac{4}{3}aW\sin\alpha\,\delta\alpha=0$$

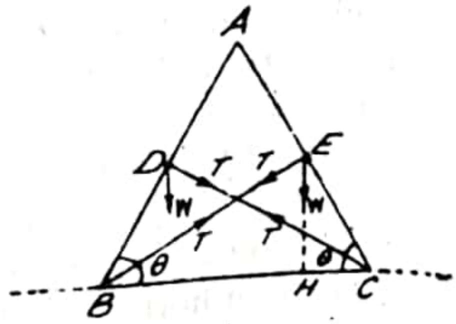
or $$2a\,[-\tfrac{4}{3}T\cos\alpha+W_1\sin\alpha+\tfrac{2}{3}W\sin\alpha]\,\delta\alpha=0$$

or $$-\tfrac{4}{3}T\cos\alpha+W_1\sin\alpha+\tfrac{2}{3}W\sin\alpha=0 \qquad [\because\ \delta\alpha\neq 0]$$

or $$\tfrac{4}{3}T\cos\alpha=(W_1+\tfrac{2}{3}W)\sin\alpha$$

or $$T=(\tfrac{3}{4}W_1+\tfrac{1}{2}W)\tan\alpha=\tfrac{1}{2}\,(W+\tfrac{3}{2}W_1)\tan\alpha.$$

Ex. 20. *Four equal uniform bars, each of weight W, are jointed together so as to form a rhombus. This is suspended vertically from*



Sol. AB and AC are equal beams each of weight W and say of length $2a$. They are hinged at A and are placed in a vertical plane with their ends B and C resting on a smooth horizontal plane. BE and CD are strings where D and E are the middle points of AB and AC respectively. Let T be the tension in each of the strings BE and CD. We have

$$\angle ABC = \angle ACB = \theta.$$

The weights of the rods AB and AC act at their middle points. Draw EH perpendicular to BC.

Here the fixed level is the horizontal line BC.

The height of the point D or E above BC

$$= EH = EC \sin \theta = a \sin \theta.$$

We have

$$BH = \tfrac{3}{4} BC = \tfrac{3}{4} . 2 . 2a \cos \theta = 3a \cos \theta.$$

$\therefore$ the length of the string $BE = \sqrt{(BH^2 + EH^2)}$

$$= \sqrt{(9a^2 \cos^2 \theta + a^2 \sin^2 \theta)} = a\sqrt{(1 + 8 \cos^2 \theta)}$$

$=$ the length of the string CD.

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on the line. The lengths of the rods AB and AC do not change while the lengths of the strings BE and CD change. The points D and E are slightly displaced.

The equation of virtual work is

$$-2T\delta\, [a\sqrt{(1+8 \cos^2 \theta)}] - 2W\delta\, (a \sin \theta) = 0$$

or $$-2aT\, \tfrac{1}{2}\, (1+8 \cos^2 \theta)^{-1/2}\, (-16 \cos \theta \sin \theta)\, \delta\theta - aW \cos \theta\, \delta\theta = 0$$

or $$2a\, [8 \sin \theta\, (1+8 \cos^2 \theta)^{-1/2}\, T - W] \cos \theta\, \delta\theta = 0$$

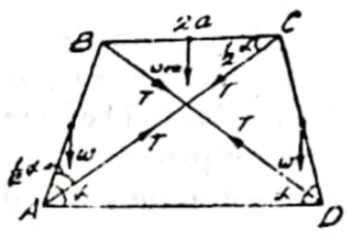
or $$8 \sin \theta\, (1+8 \cos^2 \theta)^{-1/2}\, T - W = 0$$

$$[\because \ \delta\theta \neq 0 \text{ and } \cos \theta \neq 0]$$

or $$T = \frac{W\sqrt{(1+8 \cos^2 \theta)}}{8 \sin \theta} = \frac{W\sqrt{(\text{cosec}^2\, \theta + 8 \cot^2 \theta)}}{8}$$

$$= \tfrac{1}{8} W\sqrt{(1 + 9 \cot^2 \theta)}.$$

Ex. 19. *A step ladder has a pair of legs which are jointed by a hinge at the top, and are connected by a cord attached at one-third of the distance from the lower end to the top. If the weight of each*



Sol. AB, BC and CD are three equal uniform rods each of weight w and say of length $2a$. The rods are freely jointed together at B and C and the frame rests in a vertical plane with the points A and D in contact with a smooth horizontal table. AC and BD are two equal light strings and $\angle BAD = \angle CDA = \alpha$. Obviously BC is horizontal. Let T be the tension in each of the strings AC and BD.

A mass of weight W is placed on BC at its middle point. The weights of the rods may be taken as acting at their middle points so that the total weight on the middle point of BC is $w+W$.

Since $AB = BC$, therefore $\angle BAC = \angle BCA = \angle CAD = \frac{1}{2}\alpha$.

Here the fixed horizontal level is AD. We have,

the height of the middle point of the rod AB or DC above AD

$$= a \sin \alpha,$$

and the height of the middle point of the rod BC above AD

$$= 2a \sin \alpha.$$

The length of the string AC or $BD = 4a \cos \frac{1}{2}\alpha$.

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The level of the line AD lying on the table remains fixed and the points A and D move on this line. The lengths of the rods AB, BC and CD do not change while the lengths of the strings AC and BD change. The middle points of the rods AB, BC and CD are slightly displaced.

The equation of virtual work is

$$-2T\delta\,(4a \cos \tfrac{1}{2}\alpha) - 2w\delta\,(a \sin \alpha) - (w+W)\,\delta\,(2a \sin \alpha) = 0$$

or $$-2T.4a.(-\sin \tfrac{1}{2}\alpha).\tfrac{1}{2}\delta\alpha - 2aw \cos \alpha\, \delta\alpha - 2a\,(w+W) \cos \alpha\, \delta\alpha = 0$$

or $$2a\,[2T \sin \tfrac{1}{2}\alpha - w \cos \alpha - (w+W) \cos \alpha]\, \delta\alpha = 0$$

or $$2T \sin \tfrac{1}{2}\alpha - (2w+W) \cos \alpha = 0 \qquad [\because \delta\alpha \neq 0]$$

or $$2T \sin \tfrac{1}{2}\alpha = (2w+W) \cos \alpha$$

or $$T = (w + \tfrac{1}{2}W) \cos \alpha \operatorname{cosec} \tfrac{1}{2}\alpha.$$

Ex. 18. *Two equal beams AC and AB, each of weight W, are connected by a hinge at A and are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. They are prevented from falling by strings connecting B and C with the middle points of the opposite beams. Show that the tension of each string is*

$$\tfrac{1}{4}W \sqrt{(1+9 \cot^2 \theta)},$$

where θ is the inclination of each beam to the horizon.

points H and K below A is $2a \cos \theta + 2a + \frac{1}{2}SD$ where in this case SD is fixed.

By the principle of virtual work, we have

$$T_1 \delta (4a \sin \theta) + 2W\delta (a \cos \theta) + 2W\delta (2a \cos \theta + a)$$
$$+ 2W\delta (2a \cos \theta + 2a + \tfrac{1}{2}SD) = 0$$

or $$4a\, T_1 \cos \theta\, \delta\theta - 10aW \sin \theta\, \delta\theta = 0$$

or $$2a\, (2T_1 \cos \theta - 5W \sin \theta)\, \delta\theta = 0$$

or $$2T_1 \cos \theta - 5W \sin \theta = 0 \qquad [\because \ \delta\theta \neq 0]$$

or $$T_1 = \tfrac{5}{2} W \tan \theta.$$

But in the position of equilibrium, the hexagon is a regular one and so $\theta = \pi/3$.

Therefore $T_1 = \frac{5}{2} W \tan \frac{1}{3}\pi = \frac{5}{2} W \sqrt{3}$.

Now let us proceed to find the thrust T_2.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and so replace the rod CE by two equal and opposite forces T_2 as shown in the figure. Give the system a small symmetrical displacement about the vertical line AD in which θ at both the ends A and D changes to $\theta + \delta\theta$ so that both the lengths BF and CE change. In this case the total weight $6W$ of all the six rods AB, BC etc. can be taken acting at the middle point O of AD.

We have

$$BF = 4a \sin \theta,\ CE = 4a \sin \theta \text{ and } AO = 2a \cos \theta + a.$$

By the principle of virtual work, we have

$$T_1 \delta (4a \sin \theta) + T_2 \delta (4a \sin \theta) + 6W\delta (2a \cos \theta + a) = 0$$

or $$4a\, T_1 \cos \theta\, \delta\theta + 4a\, T_2 \cos \theta\, \delta\theta - 12aW \sin \theta\, \delta\theta = 0$$

or $$4a\, [(T_1 + T_2) \cos \theta - 3W \sin \theta]\, \delta\theta = 0$$

or $$(T_1 + T_2) \cos \theta - 3W \sin \theta = 0 \qquad [\because \ \delta\theta \neq 0]$$

or $$T_1 + T_2 = 3W \tan \theta.$$

But in the position of equilibrium $\theta = \pi/3$.

$$\therefore \quad T_1 + T_2 = 3W \tan \tfrac{1}{3}\pi = 3W\sqrt{3}.$$

$$\therefore \quad T_2 = 3W\sqrt{3} - T_1 = 3W\sqrt{3} - \frac{5W\sqrt{3}}{2} = \frac{W\sqrt{3}}{2}.$$

Ex. 17. *Three equal uniform rods AB, BC, CD each of weight w, are freely jointed together at B and C, and rest in a vertical plane, A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework, so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then tension of each string will be of magnitude*

$$(w + \tfrac{1}{2}W) \cos \alpha \operatorname{cosec} \tfrac{1}{2}\alpha.$$

$$-4aT \sin\theta\,\delta\theta + 4aW' \cos\theta\,\delta\theta + 8aW \cos\theta\,\delta\theta = 0$$

or $$4a\,(-T \sin\theta + W' \cos\theta + 2W \cos\theta)\,\delta\theta = 0$$

or $$-T \sin\theta + 3W \cos\theta = 0 \qquad [\because \delta\theta \neq 0]$$

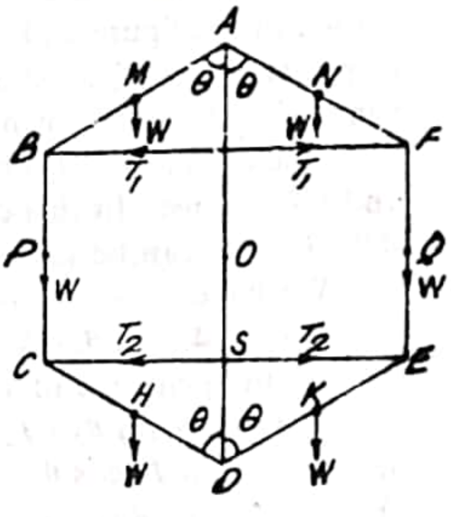
or $$T = 3W \cot\theta.$$

or

But in the position of equilibrium each of the rods AB, BC, EF and ED makes an angle 45° with the vertical and so also with the horizontal BE. Therefore in the position of equilibrium, $\theta = 45°$

and $$T = 3W \cot 45° = 3W.$$

Ex. 16. *Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are* $(5\sqrt{3}/2)\,W$ *and* $(\sqrt{3}/2)\,W$, *where W is the weight of each rod.* **(Meerut 81, 84R, 85S)**



Sol. Let the length of each of the rods AB, BC etc. be $2a$ and let θ be the angle which each of the slant rods AB, AF, DC and DE makes with the vertical AD.

Let T_1 and T_2 be the thrusts in the rods BF and CE respectively. Here A is the fixed point. The weights of the rods AB, BC etc. act at their respective middle points as shown in the figure.

Let us first find the thrust T_1.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and keep the rod CF intact so that during any displacement the length CE does not change. Now give the system a small symmetrical displacement about the vertical line AD in which θ at the end A changes to $\theta + \delta\theta$ while θ at the end D does not change. The portion $BCDEF$ moves as it is. The length BF changes while the length CE does not change so that during this small displacement the work done by the thrust T_2 of the rod CE is zero. The centres of gravity of all the six rods AB, BC etc. are slightly displaced.

We have $$BF = 4a \sin\theta.$$

In this case we cannot take the total weight of the rods AB, BC etc. act at the middle point O of AD. The depth of each of the points M and N below A is $a\cos\theta$, the depth of each of the points P and Q below A is $2a\cos\theta + a$, and the depth of each of the

By the principle of virtual work, we have

$$-2T\delta\,(2a\sin\theta)+6W\delta\,(2a\sin\theta)=0$$

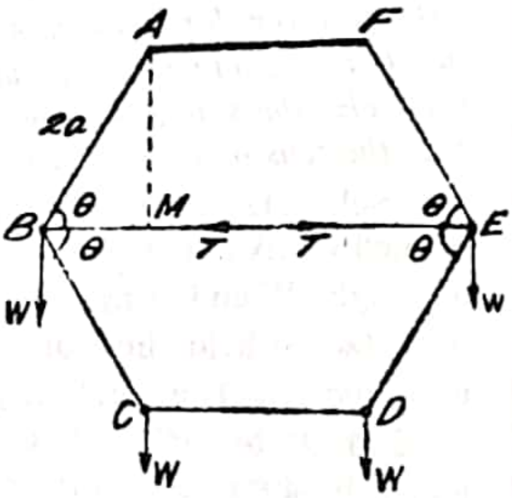
or $$-4aT\cos\theta\,\delta\theta+12aW\cos\theta\,\delta\theta=0$$

or $$4a\cos\theta\,(-T+3W)\,\delta\theta=0$$

or $$-T+3W=0 \qquad [\because\ \delta\theta\neq 0 \text{ and } \cos\theta\neq 0]$$

or $T=3W$ *i.e.*, the tension of each string is three times the weight of a bar.

Ex. 15. *Six equal light rods are joined to form a hexagon ABCDEF which is suspended at A and F so that AF is horizontal. A rod BE, also light, keeps the figure from collapsing and is of such a length that the rods ending in the points B, E are inclined at an angle of 45° to the vertical. Equal weights are suspended from B, C, D, E. Find the stress in BE.*



Sol. *ABCDEF* is a hexagon formed of six equal light rods say each of length $2a$. It is suspended at *A* and *F* so that *AF* is horizontal. Equal weights *W* are suspended from each of the points *B*, *C*, *D* and *E*. A light rod joining *B* and *E* saves the system from collapsing. Let *T* be the stress in the rod *BE*. Since the rod *BE* prevents the points *B* and *E* from moving inwards, therefore the stress in the rod *BE* is a thrust.

Let $\angle ABE=\theta=\angle FEB=\angle CBE=\angle DEB$.

Replace the rod *BE* by two equal and opposite forces *T* as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line *AF* remains fixed. The points *B*, *C*, *D* and *E* change. The lengths of the rods *AB*, *BC* etc. do not change while the length *BE* changes.

We have

$$BE=AF+2BM=2a+2.2a\cos\theta=2a+4a\cos\theta,$$

the depth of each of the points *B* and *E* below *AF*

$$=AM=2a\sin\theta,$$

and the depth of each of the points *C* and *D* below *AF*

$$=2AM=4a\sin\theta.$$

By the principle of virtual work, we have

$$T\delta\,(2a+4a\cos\theta)+2W\delta\,(2a\sin\theta)+2W\delta\,(4a\sin\theta)=0$$

vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length MN changes and the point O also changes.

We have

$$MN=2MO=2KF=2AF \sin\theta=4a \sin\theta.$$

Also the depth of O below the fixed line AB

$$=MO=2a \sin\theta.$$

By the principle of virtual work, we have

$$-T\delta\,(4a \sin\theta)+6W\delta\,(2a \sin\theta)=0$$

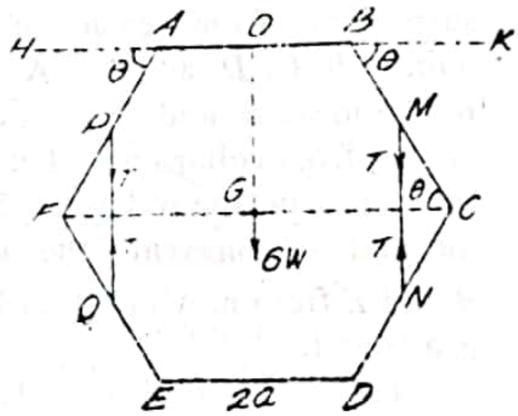
or $$-4aT \cos\theta\,\delta\theta+12aW \cos\theta\,\delta\theta=0$$

or $$4a\,[-T+3W] \cos\theta\,\delta\theta=0$$

or $$-T+3W=0 \qquad [\because \delta\theta\neq 0 \text{ and } \cos\theta\neq 0]$$

or $$T=3W.$$

Ex. 14. *Six equal bars are freely jointed at their extremities forming a regular hexagon ABCDEF which is kept in shape by vertical strings joining the middle points of BC, CD and AF, FE respectively, the side AB being held horizontal and uppermost. Prove that the tension of each string is three times the weight of a bar.*



Sol. $ABCDEF$ is a hexagon formed of six equal bars say each of weight W and length $2a$. The rod AB is held horizontal and uppermost. The middle points M and N of BC and CD are joined by a string and the middle points P and Q of AF and FE are also joined by a string. Let T be the tension in each of the strings PQ and MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at G, the middle point of FC.

Let $\qquad \angle HAF=\theta=\angle KBC.$

Give the system a small symmetrical displacement about the vertical line OG in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the lengths MN and PQ change and the point G also changes.

We have

$$PQ=MN=2MC \sin\theta=2a \sin\theta.$$

Also the depth of G below AB

$$=OG=BC \sin\theta=2a \sin\theta.$$

By the principle of virtual work, we have

$-T\delta\,(2a+4a\cos\theta)-6W\delta\,(2a\sin\theta)-W'\delta\,(4a\sin\theta)=0$

or $\quad 4aT\sin\theta\,\delta\theta-12aW\cos\theta\,\delta\theta-4aW'\cos\theta\,\delta\theta=0$

or $\quad 4a\,[T\sin\theta-3W\cos\theta-W'\cos\theta]\,\delta\theta=0$

or $\quad T\sin\theta-(3W+W')\cos\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=(3W+W')\cot\theta.$

But in the position of equilibrium, $\theta=60°$. Therefore

$$T=(3W+W')\cot 60°=(3W+W')/\sqrt{3}.$$

Ex. 12. *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.* **[Agra 76; Kanpur 79]**

Sol. Proceed as in Ex. 11. Here a weight W' is not placed at the middle point M of DE otherwise the question is the same.

The equation of virtual work is

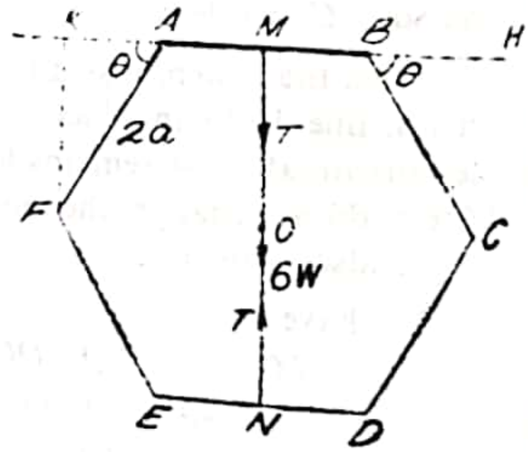
$$-T\delta\,(2a+4a\cos\theta)-6W\delta\,(2a\sin\theta)=0,$$

giving $\qquad T=3W\cot\theta.$

But in the position of equilibrium $\theta=60°$. Therefore

$$T=3W\cot 60°=3W/\sqrt{3}=W\sqrt{3}.$$

Ex. 13. *Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is 3W.* **[Kanpur 76, 78]**



Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let

$$\angle FAK=\theta=\angle CBH.$$

Give the system a small symmetrical displacement about the or $\quad 2a\,(T \sin \theta - 6W \cos \theta)\,\delta\theta = 0$

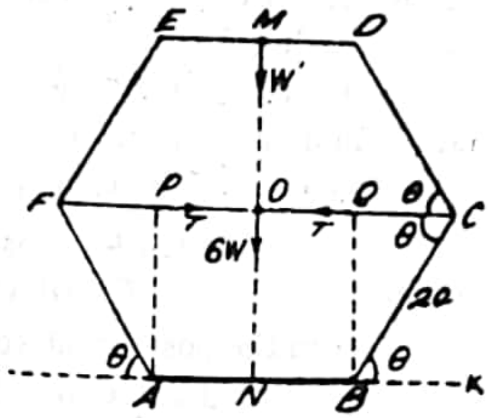
or $\quad T \sin \theta - 6W \cos \theta = 0 \qquad [\because \ \delta\theta \neq 0]$

or $\quad T = 6W \cot \theta.$

Ex. 11. *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely joined together. The two opposite angles C and F are connected by a string, which is horizontal, AB being in contact with a horizontal plane. A weight W′ is placed at the middle point of DE. Show that the tension of the string is* $(3W + W')/\sqrt{3}$.

[Meerut 79, 83, 86P, 80P; Rohilkhand 80; Kanpur 77]

Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The hexagon rests in a vertical plane with AB in contact with a horizontal plane. The opposite points C and F are connected by a string and a weight W' is placed at the middle point M of DE. Let T be the tension in the string FC.



The total weight $6W$ of the six rods AB, BC etc. can be taken acting at O, the middle point of FC.

Suppose BC and AF are inclined at an angle θ to the horizontal. Then in the position of equilibrium $\theta = \pi/3$ because in the position of equilibrium the hexagon is given to be a regular one and so $\angle CBK = 60°$.

Give the system a small symmetrical displacement about the vertical line MON in which θ changes to $\theta + \delta\theta$. The line AB on the horizontal plane remains fixed. The lengths of the rods AB, BC etc. do not change, the length FC changes and the points O and M also change.

We have

$$FC = FP + PQ + QC = 2a \cos \theta + 2a + 2a \cos \theta$$
$$= 2a + 4a \cos \theta,$$

the height of O above AB

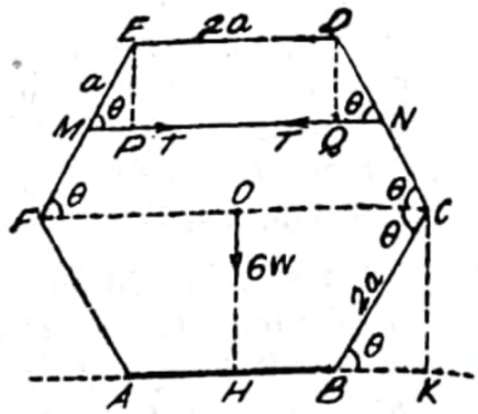
$$= NO = BQ = 2a \sin \theta,$$

and the height of M above AB

$$= NM = 2NO = 4a \sin \theta.$$

Ex. 10. *Six equal heavy beams are freely jointed at their ends to form a hexagon, and are placed in a vertical plane with one beam resting an a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Find its tension in terms of W and θ, where W is the weight of each beam.* **[Meerut 82]**

Sol. $ABCDEF$ is a hexagon formed of six equal heavy beams each of weight W and say of length $2a$. The frame is placed in a vertical plane with the beam AB resting on a horizontal plane. To save the system from collapsing the middle points M and N of the beams FE and CD are connected by a light cord. Let T be the tension in the cord MN.



The line FC is horizontal. We have $\angle EFC=\theta=\angle DCF$.

Draw EP and DQ perpendiculars to MN.

The total weight $6W$ of all the six rods can be taken, acting at O, the middle point of FC. Draw OH and CK perpendiculars to AB. We have $\angle CBK=\theta$.

Give the system a small symmetrical displacement about the vertical line OH in which θ changes to $\theta+\delta\theta$. The line AB on the horizontal plane remains fixed, and the distance of the point of application O of the weight $6W$ will be measured from AB. The lengths of the rods AB, BC etc. remain fixed while the length MN changes. The point O also changes.

We have $\quad MN=MP+PQ+QN$

$$=a\cos\theta+2a+a\cos\theta=2a+2a\cos\theta.$$

[Note that $PQ=ED=2a$, because ED remains fixed].

Also the height of O above the fixed line AB

$$=HO=KC=2a\sin\theta.$$

By the principle of virtual work, we have

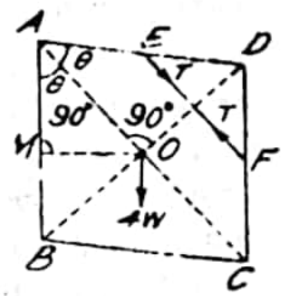
$$-T\delta(2a+2a\cos\theta)-6W\delta(2a\sin\theta)=0$$

[The work done by $6W$ is taken with $-$ive sign because the direction of HO is opposite to that of $6W$]

or $\quad 2aT\sin\theta\,\delta\theta-12aW\cos\theta\,\delta\theta=0$

Ex. 9. *Four equal uniform rods, each of weight W, are smoothly jointed so as to form a square ABCD; the side AB is fixed (clamped) in a vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC. Find the tension of the string.*

Sol. $ABCD$ is a framework formed of four equal uniform rods each of weight W and say of length $2a$. The side AB is fixed in a vertical position with A uppermost. A string joins the middle points E and F of AD and DC respectively and in equilibrium $ABCD$ is a square.



Let T be the tension in the string EF. The total weight $4W$ of all the rods AB, BC, CD and DA acts at O, the point of intersection of the diagonals AC and BD. We have, $\angle AOD=90°$. Let $\angle BAC=\theta=\angle DAC$. Draw OM perpendicular to AB.

[Note that we have drawn $ABCD$ as a rhombus and not as a square because in a displacement in which EF is to change the figure will not remain a square. After finding the value of the tension T we shall use the fact that in the position of equilibrium the figure is a square].

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AB will remain fixed and so A is a fixed point. The points C, D and O will change. The lengths of the rods AB, BC, CD and DA do not change while the length EF changes. The $\angle AOD$ remains $90°$.

We have $EF=\frac{1}{2}AC=AO=AD\cos\theta=2a\cos\theta$. Also the depth of O below the fixed point A *i.e.*, the distance of O from the fixed point A in the direction of the force $4W$

$$=AM=AO\cos\theta=(2a\cos\theta)\cos\theta=2a\cos^2\theta.$$

By the principle of virtual work, we have

$$-T\delta(2a\cos\theta)+4W\delta(2a\cos^2\theta)=0$$

or $\quad 2aT\sin\theta\,\delta\theta-16aW\cos\theta\sin\theta\,\delta\theta=0$

or $\quad 2a\sin\theta\,(T-8W\cos\theta)\,\delta\theta=0$

or $\quad T-8W\cos\theta=0 \qquad [\because \delta\theta\neq 0 \text{ and } \sin\theta\neq 0]$

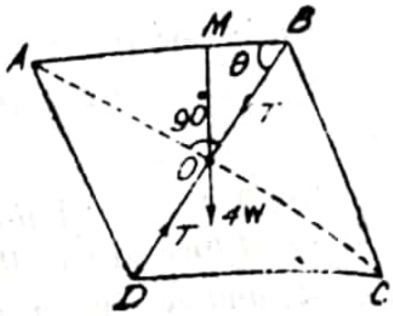
or $\quad T=8W\cos\theta.$

But in the position of equilibrium, $\theta=45°$.

$$\therefore \quad T=8W\cos 45°=8W(1/\sqrt{2})=4W\sqrt{2}.$$

$$\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

[Meerut 86, 88S, 90; Lucknow 77, 78, 79, 80; Rohilkhand 81, 85, 87; Kanpur, 86, P. C. S. 76]



Sol. $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have $\angle AOB=90°$.

Let $\angle ABO=\theta$. Draw OM perpendicular to AB.

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The $\angle AOB$ will remain $90°$.

We have $BD=2BO=2AB\cos\theta=2b\cos\theta$.

[Note that in the position of equilibrium $BD=a$. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line $AB=MO$.

$=BO\sin\theta=(AB\cos\theta)\sin\theta=b\sin\theta\cos\theta$.

By the principle of virtual work, we have

$$-T\delta(2b\cos\theta)+4W\delta(b\sin\theta\cos\theta)=0$$

or $$2bT\sin\theta\,\delta\theta+4bW(\cos^2\theta-\sin^2\theta)\,\delta\theta=0$$

or $$2b[T\sin\theta-2W(\sin^2\theta-\cos^2\theta)]\,\delta\theta=0$$

or $$T\sin\theta-2W(\sin^2\theta-\cos^2\theta)=0 \qquad [\because \delta\theta\neq 0]$$

or $$T=\frac{2W(\sin^2\theta-\cos^2\theta)}{\sin\theta}=\frac{2W(1-2\cos^2\theta)}{\sqrt{(1-\cos^2\theta)}}.$$

In the position of equilibrium, $BD=a$ or $BO=\frac{1}{2}a$. So in the position of equilibrium, $\cos\theta=\dfrac{BO}{AB}=\dfrac{\frac{1}{2}a}{b}=\dfrac{a}{2b}$.

$$\therefore\quad T=\frac{2W\{1-2(a^2/4b^2)\}}{\sqrt{\{1-(a^2/4b^2)\}}}=\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

or $\quad 4a\,T\cos\theta\,\delta\theta-8a\,W\sin\theta\delta\theta-4a\,W\sin\theta\,\delta\theta-4a\,W\sin\theta\delta\theta=0$

or $\quad 4a\,[T\cos\theta-4W\sin\theta]\,\delta\theta=0$

or $\quad T\cos\theta-4W\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$

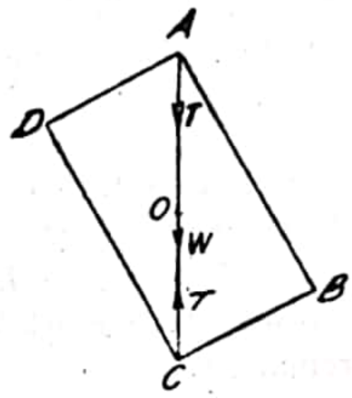
or $\quad T=4W\tan\theta.$

But in the position of equilibrium $\theta=45°$.

$\therefore\quad T=4W\tan 45°=4W=$ the total weight of the four rods.

Ex. 7. *Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.*

[Rohilkhand 88; Meerut 80, 88]



Sol. $ABCD$ is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC. Let T be the tension in the string AC. The total weight W of all the four rods AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC=2x$.

Give the system a small displacement in which x changes to $x+\delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes. We have, $AO=x$.

By the principle of virtual work, we have

$$-T\,\delta\,(AC)+W\delta\,(AO)=0$$

or $\quad -T\delta\,(2x)+W\delta\,(x)=0$

or $\quad -2T\,\delta x+W\,\delta x=0$

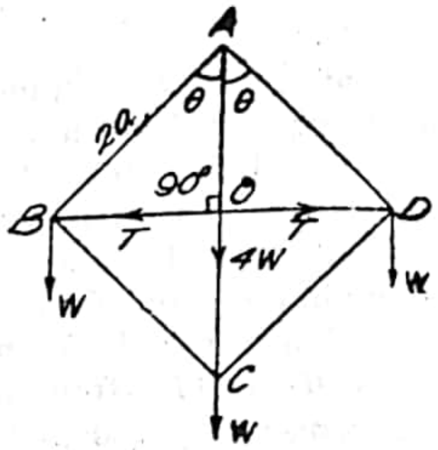
or $\quad [-2T+W]\,\delta x=0$

or $\quad -2T+W=0 \qquad [\because\ \delta x\neq 0]$

or $\quad T=\dfrac{1}{2}W=\dfrac{1}{2}$ (total weight of all the four rods).

Ex 8. *A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is*

Ex. 6. *A square framework, formed of uniform heavy rods of equal weight W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.*



Sol. $ABCD$ is a square framework formed of four rods each of weight W and say of length $2a$. It is suspended from the point A and a weight W is suspended from each of the three lower corners B, C and D. A light rod along the horizontal diagonal BD prevents the system from collapsing. Let T be the thrust in the rod BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at O.

To find T we shall have to give the system a displacement in which BD must change. So replace the rod BD by two equal and opposite forces T as shown in the figure and assume that $\angle BAC=\theta=\angle CAD$. [Note that the angle BAC will change during a displacement in which BD is to change.]

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and the points B, O, D and C change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes.

We have $BD=2BO=2AB \sin \theta=4a \sin \theta$,

the depth of each of the points B, C and D below the fixed point A

$$=AO=2a \cos \theta,$$

and the depth of C below $A=2AO=4a \cos \theta$.

$\therefore$ By the principle of virtual work, we have

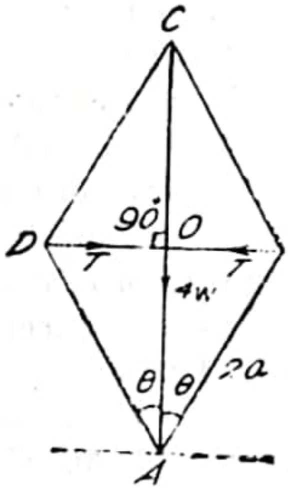
$$T\,\delta\,(4a \sin \theta)+4W\,\delta\,(2a \cos \theta)+2W\,\delta\,(2a \cos \theta)$$

$$+W\,\delta\,(4a \cos \theta)=0$$

$\therefore \quad P=2(W+w)\tan 60^\circ=2(W+w)\sqrt{3}=(W+w)\,2\sqrt{3}.$

Ex. 5. *Four equal uniform rods, each of weight W, are jointed to form a rhombus ABCD, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC=\theta$ by a light string joining B and D. Find the tension of the string.*

(Jiwaji 79; Kanpur 80, 88; P.C.S. 74)



Sol. $ABCD$ is a framework formed of four equal rods each of weight W and say of length $2a$. It is placed in a vertical plane with AC vertical and A resting on a horizontal plane. To keep the system in the form of a rhombus a light string joins B and D and prevents the points B and D from moving in the directions OB and OD respectively. Let T be the tension in the string BD. The total weight $4W$ of all the four rods may be taken acting at the point of intersection O of the diagonals AC and BD.

Let $\qquad \angle DAC=\theta=\angle BAC.$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A resting on the horizontal plane remains fixed. The points B, C, D and O will change. The lengths of the rods AB, BC, CD and DA will remain fixed while the length BD will change. The angle DOC will remain 90°.

We have $\qquad BD=2BO=2AB\sin\theta=4a\sin\theta,$

and the height of O above the fixed point A

$$=AO=2a\cos\theta.$$

By the principle of virtual work, we have

$$-T\delta(4a\sin\theta)-4W\delta(2a\cos\theta)=0. \qquad \ldots(1)$$

[Note that in the equation (1) the work done by the weight $4W$ has been taken with negative sign because the distance AO of its point of application O from the fixed point A is in a direction opposite to the direction of $4W$.]

From the equation (1), we have

$$-4aT\cos\theta\,\delta\theta+8aW\sin\theta\,\delta\theta=0$$

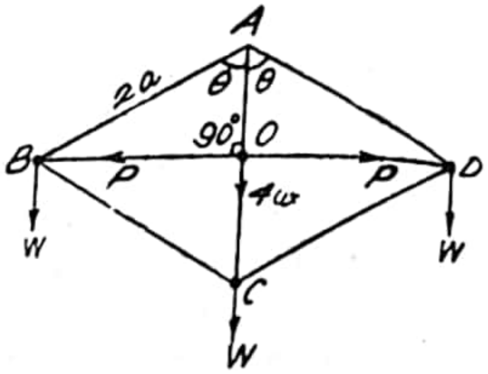
or $\qquad 4a[-T\cos\theta+2W\sin\theta]\,\delta\theta=0$

or $\qquad -T\cos\theta+2W\sin\theta=0 \qquad [\because \delta\theta\neq 0]$

or $\qquad T=2W\tan\theta.$

keep the angle BAD equal to 120°, *prove that*

$$P=(W+w)\,2\sqrt{3}.$$



Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$ It is suspended from the point A and a weight W is attached to each of the points B, C and D. To save the system from collapsing two horizontal forces each of magnitude P act at B and D and in equilibrium $\angle BAD=120°$. Obviously the nature of the forces P is like that of thrust.

The total weight $4w$ of all the four rods AB, BC, CD and DA can be taken acting at the point of intersection O of the diagonals AC and BD. Obviously the line AC must be vertical and so BD is horizontal.

To find P we have to give the system a displacement in which the length BD must change and consequently the angle BAD will change so let us assume that $\angle BAC=\theta=\angle DAC$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and we shall measure the distances of the points of application of various forces from the point A. The points B, C, D and O change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes. The angle AOB will remain 90°.

We have

$$BD=2BO=2AB\sin\theta=4a\sin\theta,$$

the depth of B or D or O below A

$$=AO=2a\cos\theta,$$

and the depth of C below A

$$=AC=2AO=4a\cos\theta.$$

By the principle of virtual work, we have

$$P\delta(4a\sin\theta)+4w\delta(2a\cos\theta)+2W\delta(2a\cos\theta)$$
$$+W\delta(4a\cos\theta)=0$$

or $$4aP\cos\theta\,\delta\theta-8aw\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$$

or $$4a\,[P\cos\theta-2w\sin\theta-W\sin\theta-W\sin\theta]\,\delta\theta=0$$

or $$P\cos\theta-2(W+w)\sin\theta=0$$

or $$P=2(W+w)\tan\theta.$$

But in the position of equilibrium, $\theta=60°$.