5

Indeterminate Forms

§ 1. Definition.

If $f(x)$ and $\phi(x)$ be any two functions of x, then we know that

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)}, \text{ provided } \lim_{x \to a} \phi(x) \neq 0.$$

In case $\phi(x) \to 0$ but $\lim f(x) \neq 0$, the fraction tends to $+\infty$ or $-\infty$ or it may not have any limit.

When $f(x)$ and $\phi(x)$ both tend to zero as $x \to a$, then

$\lim_{x \to a} \dfrac{f(x)}{\phi(x)}$, when written in the form $\dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)}$ reduces to the form

0/0, which is meaningless.

A fraction whose numerator and denominator both tend to zero as $x \to a$ is called the **indeterminate form 0/0**. It has no definite value. The other indeterminate forms are ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0 and ∞^0.

If a fraction $f(x)/\phi(x)$ takes the indeterminate form 0/0 when $x \to a$ it does not mean that $\lim_{x \to a} \dfrac{f(x)}{\phi(x)}$ will not exist. For example the fraction $\dfrac{x^2 - a^2}{x - a}$ takes the indeterminate form $\dfrac{0}{0}$ when $x \to a$. But

$$\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a)}{x - a} = \lim_{x \to a} (x + a) = 2a,$$ showing

that the limit exists in this case. Thus $\lim_{x \to a} \dfrac{f(x)}{\phi(x)}$ may exist even if the fraction $\dfrac{f(x)}{\phi(x)}$ takes the indeterminate form 0/0. In this chapter we shall give methods to find the limits of the functions which take indeterminate forms.

§ 2. The form 0/0. L'Hospital's Rule.

If $f(x)$ and $\phi(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = \phi(a) = 0$, then

§ 3. Method of expansion (Algebraic Methods).

In many cases the limit of an indeterminate form can be easily obtained by using some well known algebraic and trigonometrical expansions. We can also make use of some well-known limits in order to solve the problems or to shorten the work. The following expansions should be remembered :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, \ |x| < 1$$

$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \ |x| < 1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right), \ |x| < 1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots; \ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots; \ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

Also **remember** that

$$\log 1 = 0, \ \log e = 1, \ \log \infty = \infty; \ \log 0 = -\infty.$$

Sometimes the use of the following limits shortens the work :

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1,$ (ii) $\lim_{x \to 0} \cos x = 1,$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1,$ (iv) $\lim_{x \to 0} (1+x)^{1/x} = e,$

(v) $\lim_{x \to 0} (1+nx)^{1/x} = e^n.$ (vi) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e,$

(vii) $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$

Solved Examples

Ex. 1. *Evaluate* (a) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x},$

(b) $\lim_{x \to 0} \frac{1 - \sqrt{(1-x^2)}}{x^2},$

(e) We have

$$\lim_{x \to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}, \quad \text{[form 0/0 so we shall apply L'Hospital's rule]}$$

$$= \lim_{x \to 1} \frac{5x^4 - 6x^2 - 8x + 9}{4x^3 - 6x^2 + 2}, \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 1} \frac{20x^3 - 12x - 8}{12x^2 - 12x}, \quad \text{[form 0/0, } \therefore \text{ again apply L'Hospital's rule]}$$

$$= \lim_{x \to 1} \frac{60x^2 - 12}{24x - 12} = \frac{60 - 12}{24 - 12} = \frac{48}{12} = 4.$$

Ex. 2. *Evaluate*

(a) $\lim_{x \to 0} \frac{\sin x}{x}$, (b) $\lim_{x \to 0} \frac{\tan x}{x}$, (c) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

Sol. (a) $\lim_{x \to 0} \frac{\sin x}{x}$ [form 0/0]

$$= \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

(b) Proceeding as in part (a), we get

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

(c) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$, [form 0/0]

$$= \lim_{x \to 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}.$$

Ex. 3. *Evaluate* $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$.

Sol. We have $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$ [form 0/0]

$$= \lim_{x \to 0} \frac{a^x \log a}{b^x \log b} = \frac{\log a}{\log b} = \log_b a.$$

Ex. 4. *Evaluate* $\lim_{x \to 0} \frac{x - \sin x}{x^3}$. **(Rohilkhand 1983; Meerut 94)**

Sol. Here $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ [form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Aliter. $$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - [x - (x^3/3\,!) + (x^5/5\,!) - \ldots]}{x^3}$$

$$= \lim_{x \to 0} \frac{(x^3/6) - (x^5/120) + \ldots}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{120} + \ldots\right) = \frac{1}{6}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \dfrac{x - \sin x}{\tan^3 x}$.

Sol. $$\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x} = = \lim_{x \to 0} \left\{\frac{x - \sin x}{x^3} \cdot \left(\frac{x}{\tan x}\right)^3\right\} \quad \text{(Note)}$$

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^3$$

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3}, \qquad \left[\because \lim_{x \to 0} \frac{x}{\tan x} = 1\right]$$

[Proceeding as in Ex. 4.].

$$= \tfrac{1}{6}$$

Ex. 6. *Evaluate* $\lim_{x \to 0} \dfrac{x - \tan x}{x^3}$.

(Delhi 1975; Kanpur 72; Kashmir 72; Meerut 73)

Sol. $$\lim_{x \to 0} \frac{x - \tan x}{x^3}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{-2 \sec x \,.\, \sec x \tan x}{6x}$$

$$= \lim_{x \to 0} - \frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = -\frac{1}{3} \times 1 = -\frac{1}{3}.$$

Aliter. $$\lim_{x \to 0} \frac{x - \tan x}{x^3}$$

$$= \lim_{x \to 0} \frac{x - [x + (x^3/3) + (2x^5/15) + \ldots]}{x^3}$$

$$= \lim_{x \to 0} \left(-\tfrac{1}{3} - \tfrac{2}{15}x^2 - \ldots\right) = -\tfrac{1}{3}.$$

Ex. 7. (a). *Evaluate* $\lim_{x \to 0} \dfrac{1 - \cos x}{3x^2}$. **(Meerut 1988)**

Sol. We have $$\lim_{x \to 0} \frac{1 - \cos x}{3x^2}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{\sin x}{6x}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Ex. 7 (b). *Evaluate* $\lim_{x \to 0} \dfrac{x^2 + 2 \cos x - 2}{x \sin^3 x}$. **(Meerut 1991)**

Sol. $$\lim_{x \to 0} \frac{x^2 + 2 \cos x - 2}{x \sin^3 x}$$

$$= \lim_{x \to 0} \left[\frac{x^2 + 2 \cos x - 2}{x^4} \cdot \frac{x^3}{\sin^3 x}\right]$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^3 \cdot \lim_{x \to 0} \frac{x^2 + 2 \cos x - 2}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 + 2 \cos x - 2}{x^4}, \qquad \left[\because \lim_{x \to 0} \frac{x}{\sin x} = 1\right]$$

$$= \lim_{x \to 0} \frac{2x - 2 \sin x}{4x^3},$$

by L'Hospital's rule for the form 0/0

$$= \lim_{x \to 0} \frac{2 - 2 \cos x}{12x^2} \qquad [\text{form again } 0/0]$$

$$= \lim_{x \to 0} \frac{2 \sin x}{24x} = \frac{1}{12} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{12} \cdot 1 = \frac{1}{12}.$$

Ex. 8. *Evaluate* $\lim_{x \to 0} \dfrac{1 - \cos x}{x \log (1 + x)}$.

Sol. $$\lim_{x \to 0} \frac{1 - \cos x}{x \log (1 + x)}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{\sin x}{\log (1 + x) + \{x/(1 + x)\}}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{\cos x}{\dfrac{1}{1 + x} + \dfrac{1 + x - x}{(1 + x)^2}} = \lim_{x \to 0} \frac{\cos x}{\dfrac{1}{1 + x} + \dfrac{1}{(1 + x)^2}} = \frac{1}{2}.$$

Ex. 9. *Evaluate* $\lim_{x \to 1} \dfrac{\log x}{x - 1}$. (Agra 1983; Meerut 95, 98)

Sol. $$\lim_{x \to 1} \frac{\log x}{x - 1} \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 1} \frac{1/x}{1} = 1.$$

Ex. 10. *Evaluate* $\lim_{x \to 0} \dfrac{a \sin x - \sin ax}{x (\cos x - \cos ax)}$. [form 0/0]

Sol. $$\lim_{x \to 0} \frac{a \sin x - \sin ax}{x (\cos x - \cos ax)}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{a \cos x - a \cos ax}{(\cos x - \cos ax) + x(-\sin x + a \sin ax)}.$$

$$= \lim_{x \to 0} \frac{-a \sin x + a^2 \sin ax}{(-\sin x + a \sin ax) + (-\sin x + a \sin ax) + x(-\cos x + a \cos ax)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-a \sin x + a^2 \sin ax}{2(a \sin ax - \sin x) + x(a^2 \cos ax - \cos x)}$$

$$= \lim_{x \to 0} \frac{-a \cos x + a^3 \cos ax}{2(a^2 \cos ax - \cos x) + (a^2 \cos ax - \cos x) + x(-a^3 \sin ax + \sin x)}$$

$$= \lim_{x \to 0} \frac{-a \cos x + a^3 \cos ax}{3(a^2 \cos ax - \cos x) + x(\sin x - a^3 \sin ax)}$$

$$= \frac{-a + a^3}{3(a^2 - 1)} = \frac{a}{3} \frac{(a^2 - 1)}{(a^2 - 1)} = \frac{a}{3}.$$

Ex. 11. *Evaluate* $\lim_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1 + bx)}$. [form 0/0]

Sol. $\lim_{x \to 0} \dfrac{e^{ax} - e^{-ax}}{\log(1 + bx)}$,

$$= \lim_{x \to 0} \frac{ae^{ax} + ae^{-ax}}{b/(1 + bx)} = \frac{a + a}{b} = \frac{2a}{b}.$$

Ex. 12. *Evaluate* $\lim_{x \to 0} \dfrac{\log(1 + x^3)}{\sin^3 x}$.

Sol. $$\lim_{x \to 0} \frac{\log(1 + x^3)}{\sin^3 x} = \lim_{x \to 0} \left\{ \frac{\log(1 + x^3)}{x^3} \cdot \left(\frac{x}{\sin x} \right)^3 \right\}$$

$$= \lim_{x \to 0} \frac{\log(1 + x^3)}{x^3}, \text{ since } \lim_{x \to 0} \frac{x}{\sin x} = 1, \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{3x^2/(1 + x^3)}{3x^2} = \lim_{x \to 0} \frac{1}{1 + x^3} = 1.$$

Ex. 13. *Evaluate* $\lim_{x \to 0} \dfrac{\sin x \sin^{-1} x}{x^2}$. **(Rohilkhand 1979; Raj. 79)**

Sol. $$\lim_{x \to 0} \frac{\sin x \sin^{-1} x}{x^2} = \lim_{x \to 0} \left(\frac{\sin^{-1} x}{x} \cdot \frac{\sin x}{x} \right) \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\sin^{-1} x}{x},$$

$$= \lim_{x \to 0} \frac{1/\sqrt{(1 - x^2)}}{1} = 1.$$

Ex. 14. *Evaluate* $\lim_{x \to 0} \dfrac{\log(1 + kx^2)}{1 - \cos x}$. [form 0/0]

Sol. $\lim_{x \to 0} \dfrac{\log(1 + kx^2)}{1 - \cos x}$,

$$= \lim_{x \to 0} \frac{2kx/(1 + kx^2)}{\sin x} = \lim_{x \to 0} \left(\frac{2k}{1 + kx^2} \cdot \frac{x}{\sin x} \right)$$

$$= 2k \times 1 = 2k.$$

Ex. 15. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x - \sin x}$.

Sol. We have $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x - \sin x}$, [Form 0/0]

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)}{1 - \cos x}, \text{ by Hospital's Rule}$$

[Form again 0/0]

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)^2 - e^{x \cos x}(-2 \sin x - x \cos x)}{\sin x},$$

by Hospital's rule

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)^2 + e^{x \cos x}(2 \sin x + x \cos x)}{\sin x},$$

[form 0/0]

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)^3 - 2e^{x \cos x}(\cos x - x \sin x) \times (-2 \sin x - x \cos x) + e^{x \cos x}(\cos x - x \sin x) . (2 \sin x + x \cos x) + e^{x \cos x}(3 \cos x - x \sin x)}{\cos x}$$

$$= \frac{1 - 1(1 - 0)^3 - 2 \times 1 \times 1 \times 0 + 1 \times 1 \times 0 + 1 \times (3 - 0)}{1}$$

$$= \frac{1 - 1 + 3}{1} = 3.$$

Ex. 16. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$. **(Kanpur 1979; Meerut 94P, 96P)**

Sol. Here $\text{Nr.} = e^x - e^{\sin x} = e^x - e^{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$

$$= e^x - e^x . e^{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$$

$$= e^x(1 - e^z), \text{ where } z = -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= e^x \left[1 - \left(1 + z + \frac{z^2}{2!} + \dots \right) \right] = -e^x \left[z + \frac{z^2}{2!} + \dots \right]$$

$$= -e^x \left[\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) + \frac{1}{2!} \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)^2 + \dots \right]$$

$$= -e^x \left[-\frac{x^3}{6} + \frac{x^5}{120} - \dots \right] = x^3 e^x \left(\frac{1}{6} - \frac{x^2}{120} + \dots \right).$$

Also Denom. $= x - \sin x = x - \left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \dots\right)$

$$= \frac{x^3}{6} - \frac{x^5}{120} + \dots = x^3\left(\frac{1}{6} - \frac{x^2}{120} + \dots\right).$$

$$\therefore \quad \lim_{x\to 0}\frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x\to 0}\frac{x^3 . e^x\,[(1/6) - (x^2/120) + \dots]}{x^3\,[(1/6) - (x^2/120) + \dots]}$$

$$= \lim_{x\to 0}\frac{e^x\,[(1/6) - (x^2/120) + \dots]}{(1/6) - (x^2/120) + \dots} = \frac{1/6}{1/6} = 1.$$

Note. The students can also solve Ex. 15 by the method we adopted in Ex. 16.

Ex. 17. *Evaluate* $\lim_{x\to b}\frac{x^b - b^x}{x^x - b^b}$. **(Meerut 1983, Rohilkhand 87)**

Sol. Here the form is 0/0. To differentiate the Dr. we shall require the diff. coeff. of x^x. Let $y = x^x$; then $\log y = x\log x$.

Differentiating, $\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + \log x$

or $\frac{dy}{dx} = y\,(1 + \log x) = x^x\,(1 + \log x)$. $[\because y = x^x]$.

$$\therefore \quad \frac{d}{dx}(x^x) = x^x\,(1 + \log x).$$

$$\therefore \quad \lim_{x\to b}\frac{x^b - b^x}{x^x - b^b} = \lim_{x\to b}\frac{bx^{b-1} - b^x\log b}{x^x\,(1 + \log x) - 0}, \text{ by Hospital's rule}$$

$$= \frac{b\,.\,b^{b-1} - b^b\log b}{b^b\,(1 + \log b)} = \frac{b^b\,(1 - \log b)}{b^b\,(1 + \log b)} = \frac{1 - \log b}{1 + \log b}.$$

****Ex. 18.** *Evaluate* $\lim_{a\to b}\frac{a^b - b^a}{a^a - b^b}$.

Sol. Here $a \to b$. Therefore, while differentiating we shall regard a as variable and b as constant. **(Note)**

We have $\lim_{a\to b}\frac{a^b - b^a}{a^a - b^b}$, [form 0/0]

$$= \lim_{a\to b}\frac{ba^{b-1} - b^a\log b}{a^a\,(1 + \log a) - 0}, \quad \left[\because \frac{d}{da}(a^a) = a^a\,(1 + \log a)\right]$$

$$= \frac{b\,.\,b^{b-1} - b^b\log b}{b^b\,(1 + \log b)} = \frac{b^b\,(1 - \log b)}{b^b\,(1 + \log b)} = \frac{1 - \log b}{1 + \log b}.$$

Ex. 19. *Evaluate* $\lim_{x\to 0}\frac{[e^x + \log\{(1-x)/e\}]}{\tan x - x}$.

(Rohilkhand 1986, Gorakhpur 82; Kanpur 79; Lucknow 75)

Sol. $\lim_{x\to 0}\frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x} = \lim_{x\to 0}\frac{e^x + \log(1-x) - \log e}{\tan x - x}$ **(Note)**

$$= \lim_{x\to 0}\frac{e^x + \log(1-x) - 1}{\tan x - x}, \quad \text{[form 0/0]}$$

$$= \lim_{x\to 0}\frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right) - 1}{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\right) - x}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{6}x^3\,(1 + \text{terms containing } x \text{ and its higher powers})}{\frac{1}{3}x^3\,(1 + \text{terms containing } x \text{ and its higher powers})}$$

$$= -\tfrac{1}{2}.$$

Ex. 20. *Evaluate* $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}$.

(Agra 1982, Meerut 82, 96 BP, 98; Vikram 77)

Sol. $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2}$ $\left[\text{form }\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\cos x - x\sin x - 1/(1+x)}{2x} \quad \left[\text{form }\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{-\sin x - \sin x - x\cos x + 1/(1+x)^2}{2} = \tfrac{1}{2}.$$

Ex. 21. *Evaluate* $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x^2\sin x}$.

Sol. $\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x^2\sin x}$, $\left[\text{form }\frac{0}{0}\right]$

(Note)

$$= \lim_{x\to 0}\left\{\frac{e^x - e^{-x} - 2x}{x^3}\cdot\left(\frac{x}{\sin x}\right)\right\},$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x^3}, \quad \left[\text{form }\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x} - 2}{3x^2}, \quad \left[\text{form }\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x}}{6x}, \quad \left[\text{form }\frac{0}{0}\right]$$

$$= \lim_{x\to 0}\frac{e^x + e^{-x}}{6} = \frac{1+1}{6} = \frac{1}{3}.$$

Ex. 22. *Evaluate* $\lim_{x\to 0}\frac{\cosh x - \cos x}{x\sin x}$.

(Meerut 1990, 97; Agra 85; Delhi 86; Kashmir 74; Punjab 72)

Sol. $\lim_{x\to 0}\frac{\cosh x - \cos x}{x\sin x}$, $\left[\text{form }\frac{0}{0}\right]$

(Note)

$$= \lim_{x\to 0}\left\{\frac{\cosh x - \cos x}{x^2}\cdot\left(\frac{x}{\sin x}\right)\right\},$$

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{2x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{2} = \frac{1+1}{2} = 1.$$

Ex. 23. *Evaluate* $\lim_{x \to 0} \frac{e^x - \log(e + ex)}{x^2}$.

Sol. $$\lim_{x \to 0} \frac{e^x - \log(e + ex)}{x^2} = \lim_{x \to 0} \frac{e^x - \log\{e(1+x)\}}{x^2}$$

$$= \lim_{x \to 0} \frac{e^x - \log e - \log(1+x)}{x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - 1 - \log(1+x)}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - \{1/(1+x)\}}{2x},$$

$$= \lim_{x \to 0} \frac{e^x + \{1/(1+x)^2\}}{2} = \frac{1+1}{2} = 1.$$

Ex. 24. *Evaluate* $\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$.

Sol. $$\lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} \qquad \text{(Note)}$$

$$= \lim_{x \to 0} \left\{\frac{e^x - 2\cos x + e^{-x}}{x^2} \cdot \frac{x}{\sin x}\right\}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + 2\sin x - e^{-x}}{2x},$$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{2} = \frac{1+2+1}{2} = 2. \qquad \left[\text{form } \frac{0}{0}\right]$$

Ex. 25. *Evaluate* $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^3}$,

Sol. Required limit,

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - x - x^2}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3}x^3 + \text{terms containing powers of } x \text{ higher than } 3}{x^3}$$

$$= \lim_{x \to 0} \left(\tfrac{1}{3} + \text{terms containing } x \text{ and its higher powers}\right) = \tfrac{1}{3}.$$

Ex. 26. *Evaluate* $\lim_{x \to 1} \frac{x^x - x}{1 - x + \log x}$. **(Kanpur 1979)**

Sol. $$\lim_{x \to 1} \frac{x^x - x}{1 - x + \log x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{x^x(1 + \log x) - 1}{-1 + (1/x)}, \qquad \left\{\because \frac{d}{dx}(x^x) = x^x(1 + \log x)\right\}$$

[form again 0/0]

$$= \lim_{x \to 1} \frac{x^x(1/x) + x^x(1 + \log x) \cdot (1 + \log x)}{(-1/x^2)}$$

$$= \frac{1 + 1(1+0)(1+0)}{-1} = -2.$$

Ex. 27. *Evaluate* $\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$. **(Indore 1974)**

Sol. $$\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to \frac{1}{2}} \frac{2\cos \pi x \cdot (-\pi \sin \pi x)}{2e^{2x} - 2e}$$

$$= \lim_{x \to \frac{1}{2}} \frac{-\pi \sin 2\pi x}{2e^{2x} - 2e}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to \frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{4e^{2x}} = \frac{-2\pi^2(-1)}{4e} = \frac{\pi^2}{2e}.$$

Ex. 28. *Evaluate* $\lim_{\theta \to 0} \frac{\sin\theta - \theta\cos\theta}{\sin\theta - \theta}$.

Sol. $$\lim_{\theta \to 0} \frac{\sin\theta - \theta\cos\theta}{\sin\theta - \theta}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{\theta \to 0} \frac{\cos\theta - \cos\theta + \theta\sin\theta}{\cos\theta - 1} = \lim_{\theta \to 0} \frac{\theta \sin\theta}{\cos\theta - 1}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{\theta \to 0} \frac{\theta\cos\theta + \sin\theta}{-\sin\theta}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{\theta \to 0} \frac{\cos\theta - \theta\sin\theta + \cos\theta}{-\cos\theta} = \frac{1 - 0 + 1}{-1} = -2.$$

Ex. 29. *Evaluate* $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$. **(Gorakhpur 1989)**

Sol. $$\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)}$$

$$= \lim_{x \to 0} \frac{\{x + x^2 - \frac{1}{6}x^3 + \frac{1}{2}x^3 + \text{terms containing powers of } x \text{ higher than } 3\} - x - x^2}{-\frac{1}{2}x^3 + \text{terms containing powers of } x \text{ higher than } 3}$$

$$= \lim_{x \to 0} \frac{x^3\,[\frac{1}{3} + \text{terms containing } x \text{ and its higher powers}]}{x^3\,[-\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}]}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{2}} = -\frac{2}{3}.$$

Ex. 30. *Evaluate* $\lim_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$.

(Meerut 1977, 96 BP; Rohilkhand 88)

Sol. $\lim_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$, $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x \to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \dots}{x^5} = \lim_{x \to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right) = \frac{1}{5!} = \frac{1}{120}.$$

Ex. 31. *Evaluate* $\lim_{x \to 0} \dfrac{a^x - 1 - x \log a}{x^2}$.

Sol. $\lim_{x \to 0} \dfrac{a^x - 1 - x \log a}{x^2}$, $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{a^x \log a - \log a}{2x}, \quad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{a^x (\log a)^2}{2} = \frac{1}{2} (\log a)^2.$$

****Ex. 32.** *Evaluate* $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$.

(Rohilkhand 1990; Kanpur 86; Gorakhpur 87; Allahabad 8; Bihar 7; Delhi 76; G.N.U. 74; Agra 88; U.P. P.C.S. 97; Magadh 71; Vikram 7

Sol. Here $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$ is of the form $\dfrac{0}{0}$ becaus $\lim_{x \to 0} (1+x)^{1/x} = e$. First we shall obtain an expansion for $(1+x)^{1/x}$ i ascending powers of x. Let $y = (1+x)^{1/x}$. Then

$$\log y = \frac{1}{x} \log (1+x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

$$= 1 + z, \quad \text{where } z = -(x/2) + (x^2/3) - \dots$$

$$\therefore \quad y = e^{1+z} = e \,.\, e^z = e \cdot \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)^2 + \dots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8}x^2 + \text{terms containing powers of } x \text{ higher than } 3\right]$$

$$= e\left[1 - \tfrac{1}{2}x + \tfrac{11}{24}x^2 + \dots\dots\right].$$

Now $$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \dots\dots\right] - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left[-\frac{1}{2}x + \frac{11}{24}x^2 + \dots\dots\right]}{x}$$

$$= \lim_{x \to 0} e\left[-\tfrac{1}{2} + \tfrac{11}{24}x + \dots\dots\right] = -\tfrac{1}{2}e.$$

Ex. 33. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$.

Sol. $\lim_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$

$$= \lim_{x \to 0} \left\{\frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \frac{x}{\sin x}\right\} \quad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2}, \quad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2/(1+x)}{2x}, \quad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + 2/(1+x)^2}{2} = \frac{1 - 1 + 2}{2} = 1.$$

Ex. 34. *Evaluate* $\lim_{x \to 0} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

Sol. $\lim_{x \to 0} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$, $[\text{form } 0/0]$

$$= \lim_{x \to 0} \frac{2\cos 2x + 4 \sin x \cos x - 2\cos x}{-\sin x - 2\cos x\,(-\sin x)}$$

$$= \lim_{x \to 0} \frac{2\cos 2x + 2\sin 2x - 2\cos x}{-\sin x + \sin 2x}, \quad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 4\cos 2x + 2\sin x}{-\cos x + 2\cos 2x} = \frac{-0 + 4 + 0}{-1 + 2} = 4.$$

**Ex. 35. Evaluate $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$.

(Meerut 1982, 83, 85, 88, 94P)**

Sol. Here $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$, [form 0/0]

$= \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right] - e + \frac{1}{2}ex}{x^2}$ (See Ex. 32)

$= \lim_{x \to 0} \frac{e\,[(11/24)\,x^2 + \text{terms containing higher powers of } x]}{x^2}$

$= \lim_{x \to 0} e\,[(11/24) + \text{terms containing } x \text{ and its higher powers}]$

$= (11/24)\,e.$

Ex. 36. *Evaluate* $\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$.

(Kumayun 1983; Agra 81; Bundelkhand 78; U.P. P.C.S. 94)

Sol. $\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$, [form 0/0]

$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[\{1 + x + (x^2/2\,!) + \ldots\ldots\} - 1]^{3/2}}$

$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[x + (x^2/2) + \ldots\ldots]^{3/2}}$

$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{x^{3/2}\,[1 + (x/2) + \ldots\ldots]^{3/2}}$

$= \lim_{x \to 0} \frac{\tan x}{x\,[1 + (x/2) + \ldots\ldots]^{3/2}}$

$= \lim_{x \to 0} \left[\frac{1}{[1 + (x/2) + \ldots\ldots]^{3/2}} \cdot \frac{\tan x}{x}\right]$

$= 1 \cdot 1 = 1.$

Ex. 37. *Evaluate* $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$. **(Meerut 1990P, 91S)**

Sol. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x \to 0} \left[\frac{\tan x - x}{x^3} \cdot \frac{x}{\tan x}\right]$, (Note)

[form 0/0]

$= \lim_{x \to 0} \frac{\tan x - x}{x^3}$, [form 0/0]

$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}$,

$= \lim_{x \to 0} \frac{2 \sec x \,.\, \sec x \tan x}{6x}$

$= \lim_{x \to 0} \left(\frac{1}{3} \cdot \sec^2 x \cdot \frac{\tan x}{x}\right) = \frac{1}{3} \cdot 1.\,1 = \frac{1}{3}.$

Ex. 38. *Evaluate* $\lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log\{x/(x-1)\}}$.

Sol. We have $\lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log\{x/(x-1)\}}$

$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log[x/x\,\{1 - (1/x)\}]} = \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log[1/\{1 - (1/x)\}]}$

$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log 1 - \log\{1 - (1/x)\}}$

$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{-\log\{1 - (1/x)\}}$, [form 0/0]

$= \lim_{x \to +\infty} \frac{(a^{1/x} \log a)(-1/x^2) - (b^{1/x} \log b)(-1/x^2)}{-1/\{1 - (1/x)\}\,.\,(1/x^2)}$,

by Hospital's rule

$= \lim_{x \to +\infty} \frac{b^{1/x} \log b - a^{1/x} \log a}{-1/\{1 - (1/x)\}}$,

cancelling $1/x^2$ from the Nr. and the Dr.

$= \frac{b^0 \log b - a^0 \log a}{-1/(1-0)} = \frac{\log b - \log a}{-1} = \log a - \log b = \log(a/b).$

Ex. 39. *Evaluate* $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$.

Sol. We have $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$, [form 0/0]

$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots\right) - \left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots\ldots\right)}{x^3}$

$= \lim_{x \to 0} \frac{\left(\frac{1}{3} + \frac{1}{3\,!}\right)x^3 + \left(\frac{2}{15} - \frac{1}{5\,!}\right)x^5 + \ldots\ldots}{x^3}$

$= \lim_{x \to 0} \frac{\frac{1}{2}x^3 + (1/8)\,x^5 + \ldots\ldots}{x^3} = \lim_{x \to 0} \left(\frac{1}{2} + \frac{1}{8}x^2 + \ldots\ldots\right) = \frac{1}{2}.$

Ex. 40. *Evaluate* $\lim_{x \to 0} \left\{\frac{a^x - b^x}{x}\right\}$.

(Rohilkhand 1991; Kashmir 84; Ranchi 74; Vikram 78; Meerut 94)

Sol. We have $\lim_{x \to 0} \left\{\frac{(a^x - b^x)}{x}\right\}$, [form 0/0]

$= \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1}$, by Hospital's rule

$= \log a - \log b = \log(a/b).$

Ex. 41. *Evaluate* $\lim_{x \to 0} \frac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$.

(Meerut 1983S; Lucknow 74)

Sol. $\lim_{x \to 0} \dfrac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$, [form 0/0]

$$= \lim_{x \to 0} \frac{5\left(x - \frac{x^3}{3!} + \ldots\right) - 7\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} + 3\left\{(3x) - \frac{(3x)^3}{3!} + \ldots\right\}}{(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots) - x}$$

$$= \lim_{x \to 0} \frac{x^3 (-\frac{5}{6} + \frac{28}{3} - \frac{27}{2} + \text{higher powers of } x)}{(\frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots)}$$

$$= \lim_{x \to 0} \frac{-5 + \text{terms containing } x \text{ and its higher powers}}{(\frac{1}{3} + \frac{2}{15}x^2 + \ldots)}$$

$$= -\frac{5}{1/3} = -15.$$

****Ex. 42.** *Find the values of a and b in order that*

$\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$ *may be equal to* 1.

(Agra 1980; Meerut 1981, 83, 85, 90S, 92; Garhwal 77; G.N.U. 72; Kashmir 71; Kanpur 87; Raj. 73)

Sol. We have $\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$,

[form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{1 + a \cos x - ax \sin x - b \cos x}{3x^2}. \quad \ldots(1)$$

Now the denominator of (1) $\to 0$ as $x \to 0$. Therefore if the numerator of (1) does not tend to 0 as $x \to 0$, then the given limit cannot be equal to 1. Hence for the given limit to be equal to 1 the numerator of (1) must also $\to 0$ as $x \to 0$.

$\therefore \quad 1 + a - b = 0$ or $a - b = -1$. ...(2)

Now if $1 + a - b = 0$, then (1) takes the form 0/0. Hence by applying L' Hospital's rule to (1), the given limit is equal to

$$\lim_{x \to 0} \frac{-a \sin x - a \sin x - ax \cos x + b \sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-ax \cos x + (b - 2a) \sin x}{6x}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-a \cos x + ax \sin x + (b - 2a) \cos x}{6}, \quad \text{[by L'Hospital's rule]}$$

$$= \frac{-a + b - 2a}{6} = \frac{b - 3a}{6} = 1, \text{ (as given).} \quad \ldots(3)$$

$\therefore \quad b - 3a = 6.$

Adding (2) and (3), we have $-2a = 5$ or $a = -5/2$.

$\therefore \quad b = a + 1 = (-5/2) + 1 = -3/2.$

Hence $a = -5/2, \quad b = -3/2.$

Ex. 43. *Find the values of a and b in order that*

$\lim_{x \to 0} \dfrac{x(1 - a \cos x) + b \sin x}{x^3}$ *may be equal to* $\frac{1}{3}$. **(Delhi 1977)**

Sol. Proceed as in Ex. 42. The answer is $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

Ex. 44. *Find the values of a, b, c so that*

$$\lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2. \quad \textbf{(Meerut 1982, 96)}$$

Sol. Here the given limit

$$= \lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x}, \quad \left[\text{form } \frac{a - b + c}{0}\right].$$

$\therefore$ for the given limit to be equal to 2, we must have

$a - b + c = 0.$ ...(1)

Now applying L'Hospital's rule for the form 0/0, we have the given limit

$$= \lim_{x \to 0} \frac{ae^x + b \sin x - ce^{-x}}{\sin x + x \cos x}, \quad \left[\text{form } \frac{a - c}{0}\right].$$

$\therefore$ for the given limit to be equal to 2, we must have

$a - c = 0.$ ...(2)

Now again applying L' Hospital's rule for the form 0/0, we have the given limit

$$= \lim_{x \to 0} \frac{ae^x + b \cos x + ce^{-x}}{\cos x + \cos x - x \sin x} = \frac{a + b + c}{2}.$$

$\therefore$ for the given limit to be equal to 2, we must have

$(a + b + c)/2 = 2$ *i.e.*, $a + b + c = 4.$ ...(3)

Solving (1), (2) and (3), we get $a = 1, b = 2, c = 1$.

Ex. 45. *If* $\lim_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ *be finite, find the value of 'a' and the limit.*

Sol. We have $\lim_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$, [form 0/0]

$$= \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} \quad \ldots(1)$$

Now the denominator of (1) $\to 0$ as $x \to 0$.

But if the numerator of (1) does not tend to zero as $x \to 0$, then the given limit becomes infinite. Therefore for the given limit to be finite the numerator of (1) must $\to 0$ as $x \to 0$.

Hence $\quad 2 + a = 0$ or $a = -2$.

With this value of a, we get from (1), the given limit

[form 0/0]

$$= \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2},$$ [form 0/0]

$$= \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x},$$

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -\frac{6}{6} = -1.$$

Form II : ∞/∞.

If $f(x)$ and $\phi(x)$ be two functions such that

$\lim_{x \to a} f(x) = \infty$ **and** $\lim_{x \to a} \phi(x) = \infty$, **then**

$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$, **provided the limit exists.**

Proof. We have, $\lim_{x \to a} \frac{f(x)}{\phi(x)}$ [form 0/0]

$$= \lim_{x \to a} \left[\frac{1}{\phi(x)}\right] \Big/ \left[\frac{1}{f(x)}\right],$$

$$= \lim_{x \to a} \left[-\frac{1}{\{\phi(x)\}^2}\phi'(x)\right] \Big/ \left[-\frac{1}{\{f(x)\}^2} f'(x)\right],$$ [By L' Hospital's rule]

$$= \left[\lim_{x \to a} \frac{\phi'(x)}{f'(x)}\right]\left[\lim_{x \to a} \frac{f(x)}{\phi(x)}\right]^2 \qquad ...(1)$$

Now three cases arise :

Case I. *When* $\lim_{x \to a} f(x)/\phi(x)$ *is neither zero nor infinite..*

In this case dividing both sides of (1) by

$\left[\lim_{x \to a} \{f(x)/\phi(x)\}\right]^2$, we get

$$\left[\lim_{x \to a} \frac{f(x)}{\phi(x)}\right]^{-1} = \lim_{x \to a} \frac{\phi'(x)}{f'(x)}$$

or $$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Case II. *When* $\lim_{x \to a} \left\{f(x)/\phi(x)\right\} = 0.$

In this case, we have

$$1 + \lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \left[1 + \frac{f(x)}{\phi(x)}\right]$$ [form ∞/∞]

$$= \lim_{x \to a} \frac{\phi(x) + f(x)}{\phi(x)},$$

$$= \lim_{x \to a} \frac{\phi'(x) + f'(x)}{\phi'(x)},$$

[by case I, since the form is ∞/∞ and the limit is equal to $1 + 0$ *i.e.*, 1 which is neither zero nor infinite]

$$= \lim_{x \to a} \left[1 + \frac{f'(x)}{\phi'(x)}\right] = 1 + \lim_{x \to a} \frac{f'(x)}{\phi'(x)}. \qquad ...(2)$$

Subtracting 1 from both sides of (2), we get

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Case III. When $\lim_{x \to a} \{f(x)/\phi(x)\}$ is infinite. In this case,

$$\lim_{x \to a} \frac{\phi(x)}{f(x)} = 0 = \lim_{x \to a} \frac{\phi'(x)}{f'(x)},$$ [by case II].

$$\therefore \quad \lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Note. The above proposition is also true when $x \to \infty$ or $-\infty$ in place of a.

Important. It is interesting to note that in both cases when the form is ∞/∞ or $0/0$ the rule of evaluating the limit by differentiating the numerator and denominator separately holds good. But while evaluating $\lim_{x \to a} \{f(x)/\phi(x)\}$, when it is of the form ∞/∞, it is sometimes necessary to change it into the form $0/0$, otherwise the process of differentiating the numerator and the denominator will never end.

Remember. $\log 0 = -\infty$, and $\log \infty = \infty$.

Solved Examples

Ex. 1. *Evaluate* $\lim_{x \to \infty} \frac{x^2 + 2x}{5 - 3x^2}$.

Sol. We have, $\lim_{x \to \infty} \frac{x^2 + 2x}{5 - 3x^2}$, [form ∞/∞]

$$= \lim_{x \to \infty} \frac{2x + 2}{-6x},$$ [form ∞/∞]

$$= \lim_{x \to \infty} \frac{2}{-6} = -\frac{1}{3}.$$

Aliter. $$\lim_{x \to \infty} \frac{x^2 + 2x}{5 - 3x^2} = \lim_{x \to \infty} \frac{x^2\{1 + (2/x)\}}{x^2\{(5/x^2) - 3\}}$$

$$= \lim_{x \to \infty} \frac{1 + (2/x)}{(5/x^2) - 3} = -\tfrac{1}{3}.$$

Ex. 2. *Evaluate* $\lim_{x \to 0} \frac{\log x}{\cot x}$. (Kanpur 1988; Meerut 98)

Sol. We have, $\lim_{x \to 0} \frac{\log x}{\cot x}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{-\sin^2 x}{x}$, [form 0/0]

$= \lim_{x \to 0} \frac{-2 \sin x \cos x}{1} = \frac{-2 \times 0 \times 1}{1} = 0.$

Ex. 3. *Evaluate* $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$. **(Meerut 1994)**

Sol. We have $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{(1/\sin 2x)(2 \cos 2x)}{(1/\sin x) \cos x} = \lim_{x \to 0} \frac{2 \cot 2x}{\cot x}$, $\left[\text{form } \frac{\infty}{\infty}\right]$

$= \lim_{x \to 0} \frac{2 \tan x}{\tan 2x}$, [form 0/0]

(By L' Hospital's rule)

$= \lim_{x \to 0} \frac{2 \sec^2 x}{2 \sec^2 2x}$,

$= 1.$

Ex. 4. *Evaluate* $\lim_{x \to \infty} \left(\frac{\log x}{x}\right)$.

Sol. We have $\lim_{x \to \infty} \frac{\log x}{x}$, [form ∞/∞]

$= \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \left(\frac{1}{x}\right) = 0.$

Ex. 5. *Evaluate* $\lim_{x \to 0} \frac{\log \sin x}{\cot x}$. [form ∞/∞]

Sol. $\lim_{x \to 0} \frac{\log \sin x}{\cot x}$,

$= \lim_{x \to 0} \frac{\cos x/\sin x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \left(-\frac{\cos x}{\sin x} \sin^2 x\right)$

$= \lim_{x \to 0} (-\sin x \cos x) = 0.$

Ex. 6. *Evaluate* $\lim_{x \to \pi/2} \frac{\log (x - \frac{1}{2}\pi)}{\tan x}$. **(Meerut 1989P, 97)**

Sol. We have $\lim_{x \to \frac{1}{2}\pi} \frac{\log (x - \frac{1}{2}\pi)}{\tan x}$, [form ∞/∞]

$= \lim_{x \to \pi/2} \frac{1/(x - \frac{1}{2}\pi)}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\cos^2 x}{x - \frac{1}{2}\pi}$, [form 0/0]

$= \lim_{x \to \pi/2} \frac{-2 \cos x \sin x}{1} = \lim_{x \to \pi/2} (-\sin 2x) = 0.$

Ex. 7. *Evaluate* $\lim_{x \to 0} \frac{\log \sin x}{\log x}$.

Sol. We have $\lim_{x \to 0} \frac{\log \sin x}{\log x}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{\cos x/\sin x}{1/x} = \lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x}{\tan x} = 1.$

Ex. 8. *Evaluate* $\lim_{x \to 0} \frac{\log x^2}{\cot x^2}$.

Sol. We have $\lim_{x \to 0} \frac{\log x^2}{\cot x^2}$ [form ∞/∞]

$= \lim_{x \to 0} \frac{(1/x^2) \cdot 2x}{(-\operatorname{cosec}^2 x^2)(2x)} = \lim_{x \to 0} \left(-\frac{\sin^2 x^2}{x^2}\right)$

$= \lim_{x \to 0} \left\{\left(\frac{\sin x^2}{x^2}\right)(-\sin x^2)\right\}$ **(Note)**

$= \lim_{x \to 0} (-\sin x^2) = 0.$ $\left[\because \lim_{x \to 0} \frac{\sin x^2}{x^2} = 1\right]$

Ex. 9. *Evaluate* $\lim_{x \to 0} \frac{\log \tan x}{\log x}$.

Sol. We have $\lim_{x \to 0} \frac{\log \tan x}{\log x}$ [form ∞/∞]

$= \lim_{x \to 0} \frac{\sec^2 x/\tan x}{1/x} = \lim_{x \to 0} \left\{\left(\frac{x}{\tan x}\right)\left(\frac{1}{\cos^2 x}\right)\right\} = 1 \times 1 = 1.$

Ex. 10. *Evaluate* $\lim_{x \to \infty} x^m e^{-x}$. **(K.U. 1977)**

Sol. We have $\lim_{x \to \infty} x^m e^{-x} = \lim_{x \to \infty} \frac{x^m}{e^x}$, [form ∞/∞]

$= \lim_{x \to \infty} \frac{m x^{m-1}}{e^x}$, [form ∞/∞]

$= \lim_{x \to \infty} \frac{m(m-1) x^{m-2}}{e^x}$, [form ∞/∞]

....

$= \lim_{x \to \infty} \frac{m(m-1)(m-2) \ldots 3 \cdot 2 \cdot 1}{e^x} = \lim_{x \to \infty} \frac{m!}{e^x}$

$[\because e^x \to \infty \text{ when } x \to \infty]$.

$= 0,$

Ex. 11. *Evaluate* $\lim_{x \to 0+} \frac{\operatorname{cosec} x}{\log x}$. **(Agra 1978)**

Sol. $\lim_{x \to 0+} \frac{\operatorname{cosec} x}{\log x}$, [form ∞/∞]

$= \lim_{x \to 0+} \frac{-\operatorname{cosec} x \cot x}{1/x} = \lim_{x \to 0+} -\left[\left(\frac{x}{\tan x}\right) \operatorname{cosec} x\right]$

$$= \lim_{x \to 0+} (-\operatorname{cosec} x) = -\infty.$$

Ex. 12. *Evaluate* $\lim_{x \to \infty} \dfrac{3x+4}{\sqrt{(2x^2+5)}}$. (K.U. 1975)

[form ∞/∞]

Sol. We have $\lim_{x \to \infty} \dfrac{3x+4}{\sqrt{(2x^2+5)}}$,

$$= \lim_{x \to \infty} \frac{x\,[3+(4/x)]}{x\,[\sqrt{\{2+(5/x^2)\}}]} = \lim_{x \to \infty} \frac{3+(4/x)}{\sqrt{\{2+(5/x^2)\}}} = \frac{3}{\sqrt{2}}.$$

Ex. 13. *Evaluate* $\lim_{x \to \infty} \dfrac{\log (x-a)}{\log (e^x - e^a)}$.

[form ∞/∞]

Sol. $\lim_{x \to a} \dfrac{\log (x-a)}{\log (e^x - e^a)}$, [form 0/0]

$$= \lim_{x \to a} \frac{1/(x-a)}{\{1/(e^x - e^a)\}\, e^x} = \lim_{x \to a} \frac{e^x - e^a}{e^x (x-a)},$$

$$= \lim_{x \to a} \frac{e^x}{e^x (x-a) + e^x} = \lim_{x \to a} \frac{e^x}{e^x\,[(x-a)+1]}$$

$$= \lim_{x \to a} \frac{1}{x-a+1} = 1.$$

Ex. 14. *Evaluate* $\lim_{x \to 0} \dfrac{\log (\tan^2 2x)}{\log (\tan^2 x)}$. (Magadh 1972)

Sol. We have $\lim_{x \to 0} \dfrac{\log (\tan^2 2x)}{\log (\tan^2 x)}$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{2 \log (\tan 2x)}{2 \log (\tan x)},$$

$$= \lim_{x \to 0} \frac{(1/\tan 2x) \cdot 2 \sec^2 2x}{(1/\tan x) \sec^2 x} = \lim_{x \to 0} \frac{2 \tan x \cos^2 x}{\tan 2x \cos^2 2x}$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin 2x \cos 2x} = \lim_{x \to 0} \frac{\sin 2x}{\sin 2x \cos 2x}$$

$$= \lim_{x \to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1.$$

Ex. 15. *Evaluate* $\lim_{x \to 0} \dfrac{\log \log (1-x^2)}{\log \log \cos x}$. (Meerut 1984)

[form ∞/∞]

Sol. We have, $\lim_{x \to 0} \dfrac{\log \log (1-x^2)}{\log \log \cos x}$,

$$= \lim_{x \to 0} \frac{\dfrac{1}{\log (1-x^2)} \cdot \dfrac{1}{1-x^2} \cdot (-2x)}{\dfrac{1}{\log \cos x} \cdot \dfrac{1}{\cos x} \cdot (-\sin x)}$$

$$= 2 \lim_{x \to 0} \frac{x \cos x \log \cos x}{\sin x \,.\, (1-x^2) \log (1-x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1-x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log (1-x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log (1-x^2)},$$ [form 0/0]

$$= 2 \lim_{x \to 0} \frac{\dfrac{1}{\cos x} \cdot (-\sin x)}{\dfrac{1}{1-x^2} \cdot (-2x)} = 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1-x^2}{\cos x} \right)$$

$$= 1.$$

***Ex. 16.** *Evaluate* $\lim_{x \to \pi/2} \dfrac{\tan 5x}{\tan x}$. (Agra 1988)

[form ∞/∞]

Sol. $\lim_{x \to \pi/2} \dfrac{\tan 5x}{\tan x}$

$$= \lim_{x \to \pi/2} \left(\frac{\sin 5x}{\sin x} \cdot \frac{\cos x}{\cos 5x} \right)$$

$$= \lim_{x \to \pi/2} \frac{\sin 5x}{\sin x} \cdot \lim_{x \to \pi/2} \frac{\cos x}{\cos 5x}$$ [form 0/0]

$$= 1 \cdot \lim_{x \to \pi/2} \frac{\cos x}{\cos 5x},$$

$$= \lim_{x \to \pi/2} \frac{-\sin x}{-5 \sin 5x},$$ by L'Hospital's rule

$$= \frac{-1}{-5} = \frac{1}{5}.$$

§ 5. Form III. $0 \times \infty$.

This form can be easily reduced to the form 0/0 or to the form ∞/∞.

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} \phi(x) = \infty$.

Then we can write $\lim_{x \to a} f(x) \,.\, \phi(x)$

$$= \lim_{x \to a} \frac{f(x)}{1/\phi(x)},$$ [form 0/0]

or $$= \lim_{x \to a} \frac{\phi(x)}{1/f(x)}.$$ [form ∞/∞]

Thus $\lim_{x \to a} f(x) \,.\, \phi(x)$ is reduced to the form 0/0 or ∞/∞ which can now be evaluated by L' Hospital's Rule or otherwise.

Solved Examples

Ex. 1. *Evaluate* $\lim_{x \to \infty} \left\{ x \tan \left(\dfrac{1}{x} \right) \right\}$. (Meerut 1982S)

Sol. We have $\lim_{x \to \infty} [\tan(1/x)] \cdot x$, [form $0 \times \infty$]

$$= \lim_{x \to \infty} [\tan(1/x)]/[1/x], \quad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{[(-1/x^2)\sec^2(1/x)]}{[-1/x^2]} = \lim_{x \to \infty} \sec^2(1/x) = 1.$$

Ex. 2. *Evaluate* $\lim_{x \to 1} (1 - x) \cdot \tan \frac{\pi x}{2}$. **(Meerut 1983, 97; Agra 85)**

Sol. We have $\lim_{x \to 1} (1 - x) \cdot \tan \frac{\pi x}{2}$, [form $0 \times \infty$]

$$= \lim_{x \to 1} \frac{1 - x}{\cot(\pi x/2)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{-1}{-\frac{\pi}{2} \cdot \operatorname{cosec}^2 \frac{\pi x}{2}} = \lim_{x \to 1} \frac{2}{\pi} \sin^2 \frac{\pi x}{2} = \frac{2}{\pi}.$$

Ex. 3. *Evaluate* $\lim_{x \to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x$.

Sol. We have $\lim_{x \to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x$ [form $\infty \times 0$]

$$= \lim_{x \to 1} \frac{\log x}{\cos(\pi/2x)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{1/x}{-[\sin(\pi/2x)] \cdot (-\pi/2x^2)} = \lim_{x \to 1} \frac{2x}{\pi} \operatorname{cosec} \frac{\pi}{2x} = \frac{2}{\pi}.$$

Ex. 4. *Evaluate* $\lim_{x \to \infty} (a^{1/x} - 1) x$. **(Meerut 1980, 88P)**

Sol. We have $\lim_{x \to \infty} (a^{1/x} - 1) x$ [form $0 \times \infty$]

$$= \lim_{x \to \infty} \frac{a^{1/x} - 1}{1/x}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{a^{1/x} \cdot \log a \cdot (-1/x^2)}{-1/x^2}$$

$$= \lim_{x \to \infty} a^{1/x} \log a = a^0 \log a = \log a.$$

Ex. 5. (a) *Evaluate* $\lim_{x \to 0} x \log x$.

(Kanpur 1985, Agra 86; Meerut 79, 95)

Sol. $\lim_{x \to 0} x \log x$, [form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log x}{1/x}, \quad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{1/x}{-(1/x^2)} = \lim_{x \to 0} (-x) = 0.$$

Ex. 5. (b) *Evaluate* $\lim_{x \to 0} x^m (\log x)^n$, *where* m, n *are positive integers.* **(Allahabad 1971)**

Sol. We have $\lim_{x \to 0} x^m (\log x)^n$, [form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{(\log x)^n}{x^{-m}}, \quad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{n(\log x)^{n-1}(1/x)}{-m x^{-m-1}} = \lim_{x \to 0} \left[-\frac{n}{m} \cdot \frac{(\log x)^{n-1}}{x^{-m}}\right],$$

[form ∞/∞ if $n > 1$]

$$= \lim_{x \to 0} \left(-\frac{n}{m}\right) \cdot \frac{(n-1)(\log x)^{n-2} \cdot (1/x)}{-m x^{-m-1}}$$

$$= \lim_{x \to 0} (-1)^2 \frac{n(n-1)}{m^2} \cdot \frac{(\log x)^{n-2}}{x^{-m}}, \quad \text{[form } \infty/\infty \text{ if } n > 2]$$

$$= \lim_{x \to 0} (-1)^n \cdot \frac{n(n-1)(n-2)\ldots \text{upto } n \text{ factors}}{m^n} \cdot \frac{(\log x)^{n-n}}{x^{-m}},$$

[by repeated application of the above process]

$$= \lim_{x \to 0} (-1)^n \frac{n!}{m^n} \cdot x^m = (-1)^n \frac{n!}{m^n} \cdot \lim_{x \to 0} x^m = 0.$$

Ex. 6. *Evaluate* $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$.

Sol. We have $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$, [form $\infty \times 0$]

$$= \lim_{x \to \infty} \frac{\sin(a \cdot 2^{-x})}{2^{-x}}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{\{\cos(a \cdot 2^{-x})\} \cdot a \cdot 2^{-x}(\log 2) \cdot (-1)}{2^{-x}(\log 2) \cdot (-1)}$$

$$= \lim_{x \to \infty} a \cos\left(\frac{a}{2^x}\right) = a \cos 0 = a \cdot 1 = a.$$

Ex. 6. (a) *Evaluate* $\lim_{x \to \infty} a^x \sin\left(\frac{b}{a^x}\right)$, $a > 1$. **(Gorakhpur 1988)**

Sol. Proceed as in Ex. 6. **Ans.** b.

§ 6. Form IV : $[\infty - \infty]$.

When $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} \phi(x) = \infty$, we have

$$= \lim_{x \to a} [f(x) - \phi(x)], \quad \text{[form } \infty - \infty]$$

$$= \lim_{x \to a} \left[\frac{1}{1/f(x)} - \frac{1}{1/\phi(x)}\right]$$

$$= \lim_{x \to a} \frac{\{1/\phi(x)\} - \{1/f(x)\}}{\{1/f(x)\} \cdot \{1/\phi(x)\}}. \qquad \text{[form 0/0]}$$

Now this can be evaluated by applying L' Hospital's Rule.

Solved Examples

Ex. 1. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$.

(Agra 1987; Raj. 77; Gurunanak 73; Ranchi 76; Meerut 86, 88P, 96; Rohilkhand 87; Vikram 75; Delhi 75)

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$ [form $\infty - \infty$]

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}, \qquad \text{[form 0/0]}$$

$$= \left(\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4} \right) \cdot \left(\lim_{x \to 0} \frac{x^2}{\sin^2 x} \right), \qquad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}, \qquad \left[\because \lim_{x \to 0} \frac{x^2}{\sin^2 x} = 1 \right]$$

$$= \lim_{x \to 0} \frac{1}{x^4} \left[\left(x - \frac{x^3}{3!} + \dots \right)^2 - x^2 \right]$$

$$= \lim_{x \to 0} \frac{1}{x^4} \left[\left\{ x^2 - \frac{1}{3} x^4 + \dots \right\} - x^2 \right]$$

$$= \lim_{x \to 0} \left[-\tfrac{1}{3} + \text{terms containing } x \text{ and its higher powers} \right]$$

$$= -\tfrac{1}{3}.$$

Ex. 2. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$. **(Agra 1979; Utkal 72)**

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \frac{\sin x - x \cos x}{x \sin x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x - x \cos x}{x^2} \cdot \frac{x}{\sin x} \right)$$

$$= \lim_{x \to 0} \frac{\sin x - x \cos x}{x^2}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x - \cos x + x \sin x}{2x} = \lim_{x \to 0} \left(\frac{1}{2} \sin x \right) = 0.$$

Ex. 3. (a) *Evaluate* $\lim_{x \to \pi/2} (\sec x - \tan x)$. **(Vikram 1970; Indore 70)**

[form $\infty - \infty$]

Sol. We have $\lim_{x \to \pi/2} (\sec x - \tan x)$,

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \lim_{x \to \pi/2} \cot x = 0.$$

(b) $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x = 0.$$

Ex. 4. *Evaluate* $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x} \right]$.

Sol. We have $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x} \right]$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \frac{x - e^x + 1}{x(e^x - 1)}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - e^x}{e^x - 1 + xe^x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x \right)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x} \right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \left(\frac{\sin^2 x - x^2 \cos^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x} \right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4}$$

$$= \lim_{x \to 0} \frac{[x - (x^3/3!) + \dots]^2 - x^2 [1 - (x^2/2!) + (x^4/4!) - \dots]^2}{x^4}$$

$$= \lim_{x \to 0} \frac{[x^2 - (2x^4/3!) + \dots] - x^2 (1 - x^2 + \dots)}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 - (x^4/3) + \dots - x^2 + x^4 + \dots}{x^4}$$

$$= \lim_{x \to 0} \frac{(\frac{2}{3}) x^4 + \text{terms containing higher powers of } x}{x^4} = \frac{2}{3}.$$

Ex. 6. *Evaluate* $\lim\limits_{x \to 0} \dfrac{\cot x - (1/x)}{x}$. **(Kanpur 1986; Gorakhpur 89)**

Sol. We have $\lim\limits_{x \to 0} \dfrac{\cot x - (1/x)}{x}$ [form 0/0]

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x},$$

$$= \lim_{x \to 0} \left(\frac{x \cos x - \sin x}{x^3} \cdot \frac{x}{\sin x} \right)$$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x^3}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{3x^2}$$

$$= \lim_{x \to 0} \left(-\frac{1}{3} \cdot \frac{\sin x}{x} \right) = -\frac{1}{3} \cdot 1 = -\frac{1}{3}.$$

Ex. 7. *Evaluate* $\lim\limits_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\log x} \right]$.

Sol. We have $\lim\limits_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\log x} \right]$, [form $\infty - \infty$]

$$= \lim_{x \to 1} \left[\frac{x \log x - x + 1}{(x-1) \log x} \right], \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \left[\frac{x \cdot (1/x) + \log x - 1}{(x-1)(1/x) + \log x} \right] \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{\log x}{1 - (1/x) + \log x},$$

$$= \lim_{x \to 1} \frac{1/x}{(1/x^2) + (1/x)} = \frac{1}{1+1} = \frac{1}{2}.$$

Ex. 8. *Evaluate* $\lim\limits_{x \to 0} \left[\dfrac{1}{x(1+x)} - \dfrac{\log(1+x)}{x^2} \right]$.

Sol. $\lim\limits_{x \to 0} \left[\dfrac{1}{x(1+x)} - \dfrac{\log(1+x)}{x^2} \right]$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left[\frac{x - (1+x) \log(1+x)}{x^2 (1+x)} \right], \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - (1+x) \cdot \{1/(1+x)\} - \log(1+x)}{2x + 3x^2} \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-\log(1+x)}{2x + 3x^2},$$

$$= \lim_{x \to 0} \frac{-\{1/(1+x)\}}{2 + 6x} = -\frac{1}{2}.$$

Ex. 9. *Evaluate* $\lim\limits_{x \to 0} \left(\operatorname{cosec}^3 x - \dfrac{1}{x^3} \right)$.

Sol. We have $\lim\limits_{x \to 0} \left(\operatorname{cosec}^3 x - \dfrac{1}{x^3} \right)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{\sin^3 x} - \frac{1}{x^3} \right) = \lim_{x \to 0} \frac{x^3 - \sin^3 x}{x^3 \sin^3 x}$$

$$= \lim_{x \to 0} \left\{ \left(\frac{x^3 - \sin^3 x}{x^6} \right) \cdot \left(\frac{x}{\sin x} \right)^3 \right\}$$

$$= \lim_{x \to 0} \frac{x^3 - \sin^3 x}{x^6} = \lim_{x \to 0} \frac{x^3 - \{x - (x^3/3!) + (x^5/5!) - \ldots\}^3}{x^6}$$

$$= \lim_{x \to 0} \frac{x^3 - [x + \{-(x^3/3!) + (x^5/5!) - \ldots\}]^3}{x^6}$$

$$= \lim_{x \to 0} \frac{x^3 - [x^3 + 3x^2 \{-(x^3/3!) + (x^5/5!)\} + \ldots]}{x^6}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} x^5 + \text{terms containing powers of } x \text{ higher than } 5}{x^6}$$

$$= \lim_{x \to 0} \frac{x^5 [\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}]}{x^6}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}}{x} = \infty.$$

Ex. 10. *Evaluate* $\lim\limits_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{x^2} \log(1+x) \right]$ **(Gorakhpur 1981)**

Sol. We have $\lim\limits_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{x^2} \log(1+x) \right]$

$$= \lim_{x \to 0} \frac{x - \log(1+x)}{x^2}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - \{1/(1+x)\}}{2x} = \lim_{x \to 0} \frac{1 + x - 1}{2x(1+x)}$$

$$= \lim_{x \to 0} \frac{1}{2(1+x)} = \frac{1}{2}.$$

Ex. 11. *Evaluate* $\lim\limits_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$. **(Vikram 1978; Meerut 89S)**

Sol. We have $\lim\limits_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$, [form $\infty - \infty$]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{1}{1 - \sin x} \right) = \lim_{x \to \pi/2} \left[\frac{1 - \sin x - \cos x}{\cos x (1 - \sin x)} \right], \quad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \left[\frac{-\cos x + \sin x}{-\sin x (1 - \sin x) + \cos x (-\cos x)} \right]$$

$$= \lim_{x \to \pi/2} \left[\frac{\sin x - \cos x}{- \sin x + \sin^2 x - \cos^2 x} \right] = \frac{1}{-1+1} = \infty.$$

§ 7. The forms 0^0, 1^∞, ∞^0.

Suppose $\lim_{x \to a} [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then let $y = \lim_{x \to a} [f(x)]^{\phi(x)}$.

Taking logarithm of both sides, we get

$$\log y = \lim_{x \to a} \phi(x) . \log f(x).$$

Now in any of the above three cases $\log y$ takes the form $0 \times \infty$ which is changed to the form $[0/0]$ or $[\infty/\infty]$ whichever is convenient and then its limit is evaluated by L' Hospital's Rule or by using standard Expansions.

Solved Examples

Ex. 1. (a) *Evaluate* $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$.

(Kanpur 1988; Rohilkhand 82; Allahabad 72; Gorakhpur 74; Meerut 77, 79 S)

Sol. Let $y = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$. [form 1^∞]

$\therefore \quad \log y = \lim_{x \to \infty} \left\{ x \log \left(1 + \frac{a}{x}\right) \right\}$, [form $\infty \times 0$]

$= \lim_{x \to \infty} \frac{\log \{1 + (a/x)\}}{1/x}$, [form 0/0]

$= \lim_{x \to \infty} \frac{[1/\{1 + (a/x)\}] . (-a/x^2)}{-(1/x^2)} = \lim_{x \to \infty} \frac{a}{1 + (a/x)} = a.$

$\therefore \quad y = e^a$ or $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$.

Ex. 1. (b) *Evaluate* $\lim_{x \to 0} (1 + x)^{1/x}$.

(Kanpur 1987; Gorakhpur 87)

[form 1^∞].

Sol. Let $y = \lim_{x \to 0} (1 + x)^{1/x}$, [form 0/0]

$\therefore \quad \log y = \lim_{x \to 0} \frac{\log (1 + x)}{x}$,

$= \lim_{x \to 0} \frac{1/(1 + x)}{1} = \lim_{x \to 0} \frac{1}{1 + x} = 1.$

$\therefore \quad y = e^1 = e$ or $\lim_{x \to 0} (1 + x)^{1/x} = e.$

Ex. 2. *Evaluate* $\lim_{x \to 0} (\cos x)^{\cot^2 x}$.

(Allahabad 1987; Rohilkhand 88, 89; Gorakhpur 86)

Sol. Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$. [form 1^∞]

$\therefore \quad \log y = \lim_{x \to 0} (\cot^2 x) . (\log \cos x)$, [form $\infty \times 0$]

$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x}$, [form 0/0]

$= \lim_{x \to 0} \frac{\{(1/\cos x) . (-\sin x)\}}{2 \tan x . \sec^2 x}$, [by L' Hospital's rule]

$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x . \sec^2 x} = \lim_{x \to 0} \frac{-1}{2 \sec^2 x} = -\frac{1}{2}.$

$\therefore \quad y = e^{-1/2}$ i.e., $\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$.

Ex. 3. *Evaluate* $\lim_{x \to \pi/2} (\sin x)^{\tan x}$.

(Kanpur 1989; Agra 84, 87; Gorakhpur 79; Meerut 88 S)

Sol. Let $y = \lim_{x \to \pi/2} (\sin x)^{\tan x}$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to \pi/2} \tan x . \log \sin x$, [form $\infty \times 0$]

$= \lim_{x \to \pi/2} \frac{\log \sin x}{\cot x}$, [form 0/0]

$= \lim_{x \to \pi/2} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x}$, [by L' Hospital's rule]

$= \lim_{x \to \pi/2} (- \sin x \cos x) = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 4. *Evaluate* $\lim_{x \to \infty} (1 + x^{-2})^x$.

Sol. Let $y = \lim_{x \to \infty} \left(1 + \frac{1}{x^2}\right)^x$, [form 1^∞].

$\therefore \quad \log y = \lim_{x \to \infty} x \log \left(1 + \frac{1}{x^2}\right)$, [form $\infty \times 0$]

$= \lim_{x \to \infty} x \left[\frac{1}{x^2} - \frac{1}{2x^4} + \ldots\right]$, [Expanding $\log \{1 + (1/x^2)\}$]

$= \lim_{x \to \infty} \left[\frac{1}{x} - \frac{1}{2x^3} + \ldots\right] = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 5. *Evaluate* $\lim_{x \to 0} (\cos x)^{1/x^2}$ **(Meerut 1991 P; Kanpur 76)**

[form 1^∞].

Sol. Let $y = \lim_{x \to 0} (\cos x)^{1/x^2}$ [form $\infty \times 0$]

$\therefore \quad \log y = \lim_{x \to 0} \left(\frac{1}{x^2}\right) (\log \cos x),$ [form 0/0]

$= \lim_{x \to 0} \frac{\log \cos x}{x^2},$ [by L' Hospital's rule]

$= \lim_{x \to 0} \frac{(1/\cos x)(-\sin x)}{2x},$ [form 0/0]

$= \lim_{x \to 0} \left(\frac{-\tan x}{2x}\right),$ [by L' Hospital's Rule]

$= \lim_{x \to 0} \left(\frac{-\sec^2 x}{2}\right),$

$= -\frac{1}{2}. \qquad \therefore\ y = e^{-1/2}.$

Ex. 6. *Evaluate* $\lim_{x \to 0} (\cos x)^{1/x}$. (**Magadh 1970**)

[form 1^∞]

Sol. Let $y = \lim_{x \to 0} (\cos x)^{1/x}$. [form $\infty \times 0$]

$\therefore \quad \log y = \lim_{x \to 0} \left(\frac{1}{x}\right) (\log \cos x),$ [form 0/0]

$= \lim_{x \to 0} \frac{\log \cos x}{x},$

$= \lim_{x \to 0} \frac{(1/\cos x)(-\sin x)}{1} = \lim_{x \to 0} (-\tan x) = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 7. *Evaluate* $\lim_{x \to 1} (x)^{1/(x-1)}$. (**Vikram 1977**)

[form 1^∞]

Sol. Let $y = \lim_{x \to 1} x^{1/(x-1)}$. $\left[\text{form } \frac{0}{0}\right]$

$\therefore \quad \log y = \lim_{x \to 1} \frac{1}{(x-1)} \log x = \lim_{x \to 1} \frac{\log x}{(x-1)},$

$= \lim_{x \to 1} \frac{1/x}{1} = 1.$ Hence $y = e^1 = e$.

****Ex. 8.** *Evaluate* $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

(**Allahabad 1987; Gurunanak 73; Magadh 76; Bihar 72; Kanpur 85; Meerut 94P, 96P**)

[form 1^∞]

Sol. Let $y = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$.

$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x} \log \frac{\tan x}{x}$

$= \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{x + (x^3/3) + (2x^5/15) + \dots}{x}\right\}\right],$

(writing the expansion for $\tan x$)

$= \lim_{x \to 0} \frac{1}{x} \log \left[1 + \frac{x^2}{3} + \frac{2x^4}{15} + \dots\right]$

$= \lim_{x \to 0} \frac{1}{x} \log (1 + z)$, where $z = \frac{x^2}{3} + \frac{2x^4}{15} + \dots$

$= \lim_{x \to 0} \frac{1}{x} \left[z - \frac{z^2}{2} + \dots\right],$ [Expanding $\log (1 + z)$]

$= \lim_{x \to 0} \frac{1}{x} \left[\left(\frac{x^2}{3} + \frac{2x^4}{15} + \dots\right) - \frac{1}{2}\left(\frac{x^2}{13} + \frac{2x^4}{15} + \dots\right)^2 + \dots\right]$

$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{3} + \left(\frac{2}{15} - \frac{1}{18}\right) x^4 + \dots\right]$

$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$

$= \lim_{x \to 0} \left[\frac{x}{3} + \frac{7}{90} x^3 + \dots\right] = 0.$

$\therefore \quad y = e^0 = 1.$

****Ex. 9.** *Evaluate* $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

(**Meerut 1981, 85, 87; Avadh 87; Kanpur 77; Rohilkhand 81; Gorakhpur 80**)

Sol. Proceeding as in Ex. 8, we have

$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$

$= \lim_{x \to 0} \left[\frac{1}{3} + \frac{7}{90} x^2 + \dots\right] = \frac{1}{3}.$

$\therefore \quad y = e^{1/3}.$

Ex. 10. *Evaluate* $\lim_{x \to 0+} \left(\frac{\tan x}{x}\right)^{1/x^3}$. (**U.P. P.C.S. 1995; Meerut 79**)

Sol. Proceeding as in Ex. 8, we have

$\log y = \lim_{x \to 0+} \frac{1}{x^3} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$

$= \lim_{x \to 0+} \left[\frac{1}{3x} + \frac{7}{90} x + \dots\right] = +\infty.$

$\therefore \quad y = e^\infty = \infty.$

Ex. 11. *Evaluate* $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$ (**K.U. 1977; Pbi 75**)

Sol. Let $y = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$, $\left[\text{form } 1^\infty, \text{since } \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$.

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \frac{\sin x}{x}\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \log \left\{\frac{x - (x^3/3\,!) + (x^5/5\,!) - \ldots}{x}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \log \left(1 - \frac{x^2}{3\,!} + \frac{x^4}{5\,!} - \ldots\right)$$

$$= \lim_{x \to 0} \frac{1}{x} \log \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \log (1 - z), \text{ where } z = \frac{x^2}{6} - \frac{x^4}{120} + \ldots$$

$$= \lim_{x \to 0} \frac{1}{x} \left(-z - \frac{z^2}{2} - \ldots\right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left[-\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)^2 - \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[-\frac{x^2}{6} + \left(\frac{x^4}{120} - \frac{x^4}{72}\right) + \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[-\frac{x^2}{6} - \frac{x^4}{180} + \ldots\right]$$

$$= \lim_{x \to 0} \left[-\frac{x}{6} - \frac{x^3}{180} + \ldots\right] = 0.$$

$$\therefore \quad y = e^0 = 1.$$

****Ex. 12.** *Evaluate* $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

(Kashmir 1975; Ranchi 75; Meerut 89; Andhra 7…)

Sol. Proceeding as in Ex. 11, we have

$$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[-\frac{x^2}{6} - \frac{x^4}{180} + \ldots\right]$$

$$= \lim_{x \to 0} \left[-\frac{1}{6} - \frac{x^2}{180} + \ldots\right] = -\frac{1}{6}.$$

$$\therefore \quad y = e^{-1/6}.$$

Ex. 13. *Evaluate* $\lim_{x \to 0+} \left(\frac{\sin x}{x}\right)^{1/x^3}$

Sol. Proceeding as in Ex. 11, we have

$$\log y = \lim_{x \to 0+} \frac{1}{x^3} \cdot \left[-\frac{x^2}{6} - \frac{x^4}{180} + \ldots\right] = -\infty.$$

$$\therefore \quad y = e^{-\infty} = 0.$$

Ex. 14. *Evaluate* $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x}$ [form 1…

Sol. Let $y = \lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x}$,

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{\sinh x}{x}\right\}\right].$$

Now $\quad \sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \ldots$

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x} \log \left\{\frac{x + (x^3/3\,!) + (x^5/5\,!) + \ldots}{x}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x} \log \left[1 + \frac{x^2}{6} + \frac{x^4}{120} + \ldots.\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \log (1 + z), \text{ where } z = \frac{x^2}{6} + \frac{x^4}{120} + \ldots$$

$$= \lim_{x \to 0} \frac{1}{x} \left[z - \frac{z^2}{2} + \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\left(\frac{x^2}{6} + \frac{x^4}{120} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{6} + \frac{x^4}{120} + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{6} + \left(\frac{1}{120} - \frac{1}{72}\right)x^4 + \ldots.\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{6} - \frac{1}{180}x^4 + \ldots\right]$$

$$= \lim_{x \to 0} \left[\frac{x}{6} - \frac{1}{180}x^3 + \ldots\right] = 0.$$

$$\therefore \quad y = e^0 = 1.$$

Ex. 15. *Evaluate* $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$

(Garhwal 1983; Gorakhpur 73)

Sol. Proceeding as in Ex. 14, we have

$$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{6} - \frac{1}{180}x^4 + \ldots\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{6} - \frac{1}{180}x^2 + \ldots\right] = \frac{1}{6}.$$

$$\therefore \quad y = e^{1/6}.$$

Ex. 16. *Evaluate* $\lim_{x \to 0} \left[\frac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$ **(Meerut 1984)**

[form 1^∞].

Sol. Let $y = \lim_{x \to 0} \left[\frac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$,

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2(\cosh x - 1)}{x^2}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2}{x^2}\left(1 + \frac{x^2}{2\,!} + \frac{x^4}{4\,!} + \ldots - 1\right)\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2}{x^2}\left(\frac{x^2}{2} + \frac{x^4}{24} + \ldots\right)\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left[1 + \left(\frac{x^2}{12} + \ldots\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \left[\left(\frac{x^2}{12} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{12} + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{12} + \text{higher powers of } x\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{12} + \text{terms containing } x \text{ and its higher powers}\right] = \frac{1}{12}.$$

$\therefore \quad y = e^{1/12}.$

Ex. 17. *Evaluate* $\lim_{x \to 0} (a^x + x)^{1/x}$. **(Kanpur 1980)**

Sol. Let $y = \lim_{x \to 0} (a^x + x)^{1/x}$, [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log (a^x + x)\right] \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\log (a^x + x)}{x},$$

$$= \lim_{x \to 0} \frac{\{1/(a^x + x)\}\,(a^x \log a + 1)}{1}$$

$$= \lim_{x \to 0} \frac{a^x \log a + 1}{a^x + x} = \frac{\log a + 1}{1 + 0} = \log a + 1 \quad [\because 1 = \log e]$$

$$= \log a + \log e = \log (ae),$$

$\therefore \quad y = ae.$

Ex. 18. *Evaluate* $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ **(Rohilkhand 1982, 76)**

Sol. Let $y = \lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$, [form 1^∞].

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{a^x + b^x}{2}\right\}\right] \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\log \{(a^x + b^x)/2\}}{x}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\log (a^x + b^x) - \log 2}{x},$$

$$= \lim_{x \to 0} \frac{\dfrac{a^x \log a + b^x \log b}{a^x + b^x}}{1} = \lim_{x \to 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

$$= \frac{\log a + \log b}{1 + 1} = \tfrac{1}{2} \log (ab) = \log \sqrt{(ab)}.$$

$\therefore \quad y = \sqrt{(ab)}.$

Ex. 19 (a). *Evaluate* $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan (\pi x/2a)}$ **(G.N.U. 1977; Delhi 70)**

Sol. Let $y = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan (\pi x/2a)}$ [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to a} \tan \left(\frac{\pi x}{2a}\right) \left[\log \left(2 - \frac{x}{a}\right)\right], \quad [\text{form } \infty \times 0]$$

$$= \lim_{x \to a} \left[\left\{\log \left(2 - \frac{x}{a}\right)\right\} \Big/ \cot \left(\frac{\pi x}{2a}\right)\right], \quad \text{[form 0/0]}$$

$$= \lim_{x \to a} \left[\left(-\frac{1}{a}\right)\left(2 - \frac{x}{a}\right)^{-1}\right] \Big/ \left[\left\{-\operatorname{cosec}^2 \left(\frac{\pi x}{2a}\right)\right\} \frac{\pi}{2a}\right]$$

$$= \lim_{x \to a} \frac{1}{a} \cdot \frac{2a}{\pi} \cdot \frac{1}{\{2 - (x/a)\}} \sin^2 \left(\frac{\pi x}{2a}\right) = \frac{2}{\pi}.$$

$\therefore \quad y = e^{2/\pi}.$

Ex. 19 (b). *Evaluate* $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$. **(Rohilkhand 1978; Raj. 78)**

Sol. Let $y = \lim_{x \to \pi/4} [(\tan x)^{\tan 2x}]$ [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to \pi/4} [(\tan 2x) . (\log \tan x)] \quad [\text{form } \infty \times 0]$$

$$= \lim_{x \to \pi/4} \left[\frac{\log \tan x}{\cot 2x}\right], \quad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/4} \left[\frac{(1/\tan x) . \sec^2 x}{-2 \operatorname{cosec}^2 (2x)}\right] = \lim_{x \to \pi/4} \left[\frac{\sec^2 x . \sin^2 (2x)}{-2 \tan x}\right]$$

$$= \frac{(\sqrt{2})^2 . (1)^2}{-2 . 1} = -1.$$

$\therefore \quad y = e^{-1} = 1/e.$

Ex. 20. *Evaluate* $\lim_{x \to \infty} \frac{Ax^n + Bx^{n-1} + Cx^{n-2} + \ldots}{ax^m + bx^{m-1} + cx^{m-2} + \ldots}$.

Sol. Given limit $= \lim_{x \to \infty} \frac{x^n [A + (B/x) + (C/x^2) + \ldots]}{x^m [a + (b/x) + (c/x^2) + \ldots]}$

$$= \lim_{x \to \infty} \left(x^{n-m} . \frac{A}{a}\right).$$

Now, if $n > m$, $\lim_{x \to \infty} \left(\frac{A}{a} x^{n-m}\right) = \infty$;

if $n = m$, $\lim_{x \to \infty} \frac{A}{a} x^{n-m} = \lim_{x \to \infty} \frac{A}{a} = \frac{A}{a}$;

if $n < m$, $\lim_{x \to \infty} \frac{A}{a} x^{n-m} = \lim_{x \to \infty} \left(\frac{A}{a} \cdot \frac{1}{x^{m-n}}\right) = 0.$

Ex. 21. *Evaluate* $\lim_{x \to 0} x^x$. **(Meerut 1979, 93)**

[form 0^0]

Sol. Let $y = \lim_{x \to 0} x^x$. [form $0 \times \infty$]

$\therefore \log y = \lim_{x \to 0} x \log x$, [form ∞/∞]

$= \lim_{x \to 0} \dfrac{\log x}{(1/x)}$,

$= \lim_{x \to 0} \dfrac{1/x}{(-1/x^2)} = \lim_{x \to 0} (-x) = 0.$

$\therefore \; y = e^0 = 1.$

Ex. 22. *Evaluate* $\lim_{x \to 0} \dfrac{x^x - 1}{x}$.

$\left[\text{form } 0/0, \; \because \lim_{x \to 0} x^x = 1\right]$

Sol. $\lim_{x \to 0} \dfrac{x^x - 1}{x}$, [by L' Hospital's Rule]

$= \lim_{x \to 0} \dfrac{d/dx\,(x^x)}{1}$,

$= \lim_{x \to 0} \left\{\dfrac{d}{dx}(x^x)\right\}$

Now let $y = x^x$; then $\log y = x \log x$. Now differentiating,

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1 \,.\, \log x = (1 + \log x).$$

$\therefore \; \dfrac{dy}{dx} = y\,(1 + \log x)$. Thus $\dfrac{d}{dx}(x^x) = x^x(1 + \log x)$.

Hence $\lim_{x \to 0} \dfrac{x^x - 1}{x} = \lim_{x \to 0} [x^x(1 + \log x)]$ $\left[\because \lim_{x \to 0} x^x = 1\right]$

$= 1\,(1 - \infty) = -\infty.$

Ex. 23. *Evaluate* $\lim_{x \to 0} (\sin x)^{\tan x}$ [form 0^0]

Sol. Let $y = \lim_{x \to 0} (\sin x)^{\tan x}$ [form $0 \times \infty$]

$\therefore \log y = \lim_{x \to 0} (\tan x) \,.\, (\log \sin x)$, [form ∞/∞]

$= \lim_{x \to 0} \dfrac{\log \sin x}{\cot x}$, [form ∞/∞]

$= \lim_{x \to 0} \dfrac{(1/\sin x)\cos x}{-\operatorname{cosec}^2 x}$, [form $0/0$]

$= \lim_{x \to 0} \left[-\left(\dfrac{\sin^2 x}{\tan x}\right)\right]$,

$= \lim_{x \to 0} \left[\dfrac{-2 \sin x \cos x}{\sec^2 x}\right] = 0.$

$\therefore \; y = e^0 = 1.$

Ex. 24. *Evaluate* $\lim_{x \to \infty} \left(\dfrac{1}{x}\right)^{1/x}$ (Agra 1974)

Sol. Let $y = \lim_{x \to \infty} \left(\dfrac{1}{x}\right)^{1/x}$ [form 0^0]

$\therefore \log y = \lim_{x \to \infty} \dfrac{1}{x} \log\left(\dfrac{1}{x}\right)$, [form $0 \times \infty$]

$= \lim_{x \to \infty} \dfrac{-\log x}{x}$,

$[\because \log(1/x) = \log 1 - \log x = -\log x]$

$= -\lim_{x \to \infty} \dfrac{\log x}{x}$, [form ∞/∞]

$= -\lim_{x \to \infty} \dfrac{(1/x)}{1} = 0.$

$\therefore \; y = e^0 = 1.$

Ex. 25. *Evaluate* $\lim_{x \to \infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$.

Sol. Let $y = \lim_{x \to \infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$. [form 0^0]

$\therefore \log y = \lim_{x \to \infty} \left(\dfrac{1}{x}\right) \log(\frac{1}{2}\pi - \tan^{-1} x)$

$= \lim_{x \to \infty} \dfrac{\log(\frac{1}{2}\pi - \tan^{-1} x)}{x}$, [form ∞/∞]

$= \lim_{x \to \infty} \dfrac{\{1/(\frac{1}{2}\pi - \tan^{-1} x)\} \,.\, \{-1/(1 + x^2)\}}{1}$

$= \lim_{x \to \infty} \left\{\dfrac{-1/(1 + x^2)}{(\frac{1}{2}\pi - \tan^{-1} x)}\right\}$, [form $0/0$]

$= \lim_{x \to \infty} \dfrac{\{1/(1 + x^2)^2\} \,.\, 2x}{-1/(1 + x^2)}$, [by L' Hospital's rule]

$= \lim_{x \to \infty} \left\{\dfrac{-2x}{1 + x^2}\right\}$, [form ∞/∞]

$= \lim_{x \to \infty} \left\{\dfrac{-2}{2x}\right\} = 0.$

$\therefore \; y = e^0 = 1.$

Ex. 26. *Evaluate* $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$.

(Agra 1978, 75; Rajasthan 77; Indore 72)

Sol. Let $y = \lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$. [form ∞^0]

$\therefore \log y = \lim_{x \to 0} \dfrac{1}{\log x} (\log \operatorname{cosec} x)$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/\operatorname{cosec} x)\,(-\operatorname{cosec} x \cot x)}{1/x}$$ [form 0/0]

$$= \lim_{x \to 0} \left(\frac{-x}{\tan x}\right),$$

$$= \lim_{x \to 0} \left(\frac{-1}{\sec^2 x}\right) = -1.$$

$\therefore \quad y = e^{-1} = 1/e.$

Ex. 27. *Evaluate* $\lim_{x \to 0} (\cot x)^{\sin x}$. **(Allahabad 1974; Andhra 71)**

[form ∞^0]

Sol. Let $y = \lim_{x \to 0} (\cot x)^{\sin x}$.

$$\therefore \quad \log y = \lim_{x \to 0} (\sin x) \cdot (\log \cot x),$$ [form $0 \times \infty$]

$$= \lim_{x \to 0} \left(\frac{\log \cot x}{\operatorname{cosec} x}\right),$$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/\cot x) \cdot (-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} = \lim_{x \to 0} \left(\frac{\operatorname{cosec} x}{\cot^2 x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{\cos^2 x}\right) = 0.$$

$\therefore \quad y = e^0 = 1.$

Ex. 28. *Evaluate* $\lim_{x \to \frac{1}{2}\pi} (\sec x)^{\cot x}$. **(Gorakhpur 1971)**

[form ∞^0]

Sol. Let $y = \lim_{x \to \frac{1}{2}\pi} (\sec x)^{\cot x}$.

$$\therefore \quad \log y = \lim_{x \to \pi/2} (\cot x) \cdot (\log \sec x),$$ [form $0 \times \infty$]

$$= \lim_{x \to \pi/2} \frac{\log \sec x}{\tan x},$$ [form ∞/∞]

$$= \lim_{x \to \pi/2} \frac{(1/\sec x) \cdot \sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\tan x}{\sec^2 x} = \lim_{x \to \pi/2} (\sin x \cos x) = 0.$$

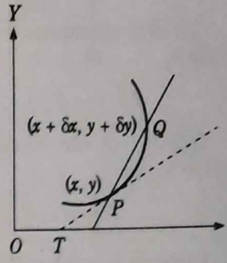
$\therefore \quad y = e^0 = 1.$ □

6

Tangents and Normals

§ 1. Tangent.

Definition. *The tangent to a plane curve at the point $P(x, y)$ on it is defined as the limiting position of the chord PQ as the point $Q(x + \delta x, y + \delta y)$ approaches the point P, provided such a limiting position exists.*



Equation of the tangent. Let P be any point (x, y) on the curve $y = f(x)$, and Q a neighbouring point $(x + \delta x, y + \delta y)$ on it. The point Q may be taken on either side of P. The equation of the chord PQ is

$$Y - y = \frac{(y + \delta y) - y}{(x + \delta x) - x}(X - x),$$ [Here (X, Y) are the current coordinates of any point on the chord]

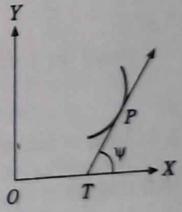
or
$$Y - y = (\delta y / \delta x)(X - x). \qquad \text{...(1)}$$

Now as $Q \to P$, $\delta x \to 0$, $\delta y/\delta x \to dy/dx$, and the chord PQ tends to the tangent at P. Therefore taking limit of (1) as $Q \to P$, we see that the equation of the tangent to the curve $y = f(x)$ at the point (x, y) is

$$\mathbf{Y - y = (dy/dx)\,(X - x)} \qquad \text{...(2)}$$

Geometrical meaning of dy/dx. The equation (2) of the tangent at any point $P(x, y)$ on the curve $y = f(x)$ may be written as

$$Y = \left(\frac{dy}{dx}\right) X + \left(y - x\frac{dy}{dx}\right),$$



which is of the form $Y = mX + c$. On comparing the two equations, we have $m = dy/dx$. Therefore if ψ is the angle which the tangent to the curve $y = f(x)$ at the point (x, y) on it makes with the positive direction of x-axis then

$$\mathbf{dy/dx = \tan \psi}.$$

Remember. *The differential coefficient dy/dx at any point (x, y) on the curve $y = f(x)$ is equal to $\tan \psi$, where ψ is the angle which the positive direction of the tangent at P to the curve makes with the positive direction of the axis of x.*