(1) Prove that a non-commutative group of order $2n$ where n is odd prime, must have subgroup of order n. (8)

Proof: G is non-abelian group

suppose $\forall a \in G$

$o(a) = 2$

$\Rightarrow (ab)^2 = e$

$\Rightarrow ab = (ab)^{-1}$

$= b^{-1}a^{-1}$

$ab = ba \quad \because a^{-1} = a \;\forall a \in G$

$\Rightarrow G$ is abelian

hence contradiction

$\Rightarrow \exists$ a element of G of order other than 2.

$\therefore$ possible order are n and $2n$

if $a \in G \ni o(a) = 2n$

$\Rightarrow G = \langle a \rangle$

hence cyclic $\Rightarrow$ abelian

$\therefore$ contradiction

using lagrange's theorem

$\exists$ an element $b \in G$

$o(b) = n$

$\therefore H = \langle b \rangle$ is of order n.

79) Find all the homomorphism from the Group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$ -- (10)

Soln:- $\phi : \mathbb{Z} \to \mathbb{Z}_4$

$\cancel{\phi(x) = 0 \bmod 4 \quad \forall x \in \mathbb{Z}}$

A homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_4$ is determined by $\phi(1)$ since

$$\phi(n) = \phi(1+1+1+1 \cdots 1) \quad \{n \text{ times}\}$$
$$= \phi(1) + \phi(1) + \cdots + \phi(1)$$
$$\phi(n) = n\phi(1) \quad \forall n \in \mathbb{Z}$$

Also, for any $a \in \mathbb{Z}_4$ we can get a homomorphism $\mathbb{Z} \to \mathbb{Z}_4$ taking 1 to a by sending n to the reduction mod 4 of an.

So, there are 4 homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_4$, one for each value in $\mathbb{Z}_4$.

if $\phi(1) = 0$ we get trivial map.

$\ker\phi = \mathbb{Z}$ and $\operatorname{img}\phi$ is $\{0\}$

if $\phi(1) = 1 \bmod 4$

ϕ is onto. image of $\phi = \mathbb{Z}_4$.

$\ker\phi = 4\mathbb{Z}$

if $\phi(1) = 2 \bmod 4$

image $\phi = \{0, 2\}$

$\ker\phi = 2\mathbb{Z}$

if $\phi(1) = 3 \bmod 4$

$\operatorname{img}\phi = \mathbb{Z}_4$

$\ker\phi = 4\mathbb{Z}$

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2(d)) Let R be a commutative ring with unity. Prove that an ideal P of R is prime iff $\frac{R}{P}$ is an integral domain. (10)

Solⁿ:- **Defⁿ:- Prime ideal:-**
A prime ideal A of a commutative Ring R is a proper ideal of R $\ni$ $a, b \in R$ and $a \cdot b \in A$ imply $a \in A$ or $b \in A$.

Solⁿ:- let $\frac{R}{P}$ be an integral domain and $a \cdot b \in P$

$\therefore (a+P) \cdot (b+P) = ab + P = P$

where P is zero element of $\frac{R}{P}$

$\therefore$ either $a+P = P$ or $b+P = P$

$\Rightarrow a \in P$ or $b \in P$

$\therefore$ P is prime ideal.

conversely,

let P is prime ideal.

$\frac{R}{P}$ is commutative ring with unity where $1+P$ is unity element.

we just have to show that $\frac{R}{P}$ has no-zero divisior.

$\because$ P is prime ideal.

consider,

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$(a+P)\cdot(b+P) = 0+P = P$

$\therefore ab+P = P$

$\Rightarrow ab \in P$

$\Rightarrow a \in P$ or $b \in P$

$\Rightarrow$ either $a+P=P$ or $b+P=P$

$\therefore \frac{R}{P}$ is integral domain.

1b) show by an eg that in a finite commutative Ring, every maximial ideal need not be prime. 10

Let $R = \{0, 2, 4, 6\}$ under binary operation $+_8$ and $\times_8$.

then R is commutative ring.

$I = \{0, 4\}$ is maximal ided in R.

but $\bar{2} \times_8 \bar{6} = \bar{4} \in I$

but both $\bar{2}$ and $6 \notin I$.

hence I is not prime ideal.

4] c] let H be a cyclic subgroup of a group G. if H be a normal subgroup of G prove that every subgroup of H is a normal subgroup of G. (18)

Sol^n: let $H = \langle a \rangle$

H is normal subgroup of G

$\therefore \forall g \in G$

$$g^{-1}Hg = H$$

let K is subgroup of H.

any subgroup of cyclic group is cyclic.

$\therefore$ let $K = \langle a^m \rangle$

we have to show that K is normal subgroup of G.

$\therefore$ let $g \in G$ be arbitry and $k \in K$

$\therefore k = (a^m)^n$ for some $n \in \mathbb{Z}$

$$\therefore g^{-1}kg = g^{-1}(a^m)^n g$$

$$= g^{-1}(a^n)^m g$$

$$= (g^{-1}a^n g)^m$$

$\because H = \langle a \rangle$

$\therefore a^n \in H$

$\because H \trianglelefteq G$

$$g^{-1}a^n g \in H$$

$\therefore g^{-1}a^n g = a^t$ for $t \in \mathbb{Z}$

$$\therefore g^{-1}kg = (a^t)^m$$

$$= (a^m)^t \in K$$

$\therefore K \trianglelefteq G$.