3

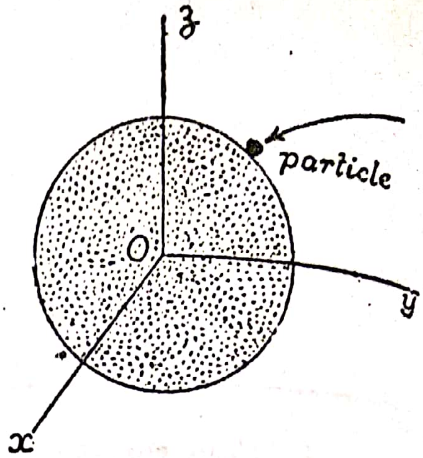
Lagrangian-Dynamics

3·1 Constraints and Generalised Co-ordinates.

(*a*) *Constraints.* If a particle moves in space, it requires three independent co-ordinates to specify its location in space and then, we say that the particle has *three* degree of freedom. On the other hand, if a particle moves on table-top, its motion is confined on a plane only implying that the particle requires two independent co-ordinates to find its position on the table-top. When the particle is not allowed to move freely in three-dimensions. we say that it is subjected to constraints In case, the constraints exist, the number of degree of freedom is reduced. For exampie, when particle moves on a table-tcp, one constraint exists which can be expressed in the form of an equation $z=0$ for all time. In general, when particle is constrained to move on a surface S "$f(x, y, z)=0$", the co-ordinates are not independent to each other as z can always be expressed in terms of x and y. In turn this implies that the particle has two degrees of freedom and not three. The constraints as discussed in the equation $f(x, y, z)=0$, is said to be an **integrable constraint*. Such constraints are also known as *holonomic constraints* and the system subjected to such constraints is known, as a holonomic system. Further there also exist another type of constraints being called as *non-integrable constraints*. Obviously such constraints are such that the differential equation expressing these cannot be integrated. Hence the constraint in which the differential from is non-integrable is called *non-holonomic* and the system subjected to such constraints is known as *non-holonomic system.*

For example (*for a holonomic constraint*), consider themotion of a single particle which is constrained to move on the sphere as shown in the adjoining diagram. The equation of the constraint

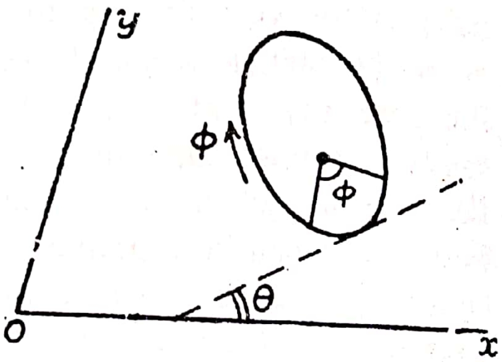
*The different al relation expressing the conduction that z does not changes is $dz=0$ which on the integration gives z=constant. This is the reason, such constraints are termed as *integrable constraints*.



is $x^2+y^2+z^2=a^2$. The displacements of the particle on the surface of the sphere in the three directions are connected by $xdx+ydy+zdz=0$. ...(1)

Obviously this expresses the equation of constraint in the differential form. Integrating (1), we get $x^2+y^2+z^2=a^2$ (same as the eq. of the constraint). Hence we say that the above constraint is a holonomic constraint. In spherical polar co-ordinates, the equation of the constraint is given by $r^2=a^2$ *i.e.* $r=a=$const. ...(2)

As an example of non-holonomic constraints; we consider the motion of a circular disc (of radius a) relling on a perfectly rough horizontal plane such that the plane of the circular disc always remains vertical and there exist on slipping. Now, let ψ be the angle between the plane of the disc and the $z=0$ plane then the equations of constraints are



$$z=a,\ \psi=\frac{\pi}{2}. \qquad (3)$$

The equation $z=a$ implies that there is no motion in the z-direction and $\psi=\frac{\pi}{2}$ implies that the plane of the disc always remains vertical. At any time t, the motion is exhibited in the above figure, where θ is the angle of the instantaneous direction of motion with the x-axis, and ϕ the angle of rotation about the centre of the circular disc. Further, the plane is perfectly rough hence no slipping occurs and only pure rolling ensues. If v is the velocity of a point on the periphery, then we get.

$$v=a\dot{\phi} \text{ i.e. } \dot{x}=v\cos\theta=a\dot{\phi}\cos\theta$$

and
$$\dot{y}=v\sin\theta=a\dot{\phi}\sin\theta.$$

Thus the infinitesimal displacement dx and dy are given by

$$dx=a\cos\theta\, d\phi \text{ and } dy=a\sin\theta\, d\phi \qquad ...(4)$$

Equations given by (4) which fix up the constraints cannot be integrated and hence there does not exist a relation of the type $f(x, y, \theta, \phi)=0$, compatible with the two equations given by (4).

In order to show that this is true, we assume that such a condition exists *i.e.* $f(x, y, \theta, \phi)=0$ exists implying

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial \theta}d\theta+\frac{\partial f}{\partial \phi}d\phi=0. \qquad ...(5)$$

Whence using $\left.\begin{matrix} dy=a\sin\theta\, d\phi \\ dx=a\cos\theta\, d\phi \end{matrix}\right\}$ in (5), we get

and

$$\frac{\partial f}{\partial x}a\cos\theta\, d\phi+\frac{\partial f}{\partial y}a\sin\theta\, d\phi+\frac{\partial f}{\partial \theta}d\theta+\frac{\partial f}{\partial \phi}d\phi=0$$

$$\Rightarrow d\phi\left(\frac{\partial f}{\partial x}a\cos\theta+\frac{\partial f}{\partial y}a\sin\theta+\frac{\partial f}{\partial \theta}\right)+\frac{\partial f}{\partial \theta}d\theta=0 \qquad ...(6)$$

$$\Rightarrow \frac{\partial f}{\partial x}a\cos\theta+\frac{\partial f}{\partial y}a\sin\theta+\frac{\partial f}{\partial \phi}=0 \text{ and } \frac{\partial f}{\partial \theta}=0. \qquad ...(7)$$

Differentiating the first relation of (7) partially w.r.t. "θ", we get

$$-a\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}a\cos\theta+\frac{\partial f}{\partial \phi}\cdot\frac{\partial f}{\partial \theta}=0$$

$$\Rightarrow -\frac{\partial f}{\partial x}\sin\theta+\frac{\partial f}{\partial y}\cos\theta=0. \qquad \left(\because \frac{\partial f}{\partial \theta}=0\right)$$

Again differentiating w.r.t. "θ", we get

$$-\frac{\partial f}{\partial x}\cos\theta+\frac{\partial f}{\partial y}\sin\theta\,(-1)=0. \qquad ...(8)$$

This is true for all values of θ, so we have $\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}$.

Substituting these values in the first relation of (7), we get $\frac{\partial f}{\partial \phi}=0$ and hence to sum up, we can say that

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial \theta}=\frac{\partial f}{\partial \phi} \qquad ...(9)$$

This implies that f does not involve θ, ϕ, x and y, *i.e.* a relation of the form $f(x, y, \theta, \phi)=0$ is not consistent with the constraint equations. Thus the equations governing the constraints are non-integrable.

(b) Generalised Co-ordinates.

Suppose that a particle or a system of N particles move subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates needed to specify the motion. These co-ordinates denoted *by $q_1, q_2, \ldots q_n$ are called generalised co-ordinates and may be distances, angles or quantities relating to them.*

However, generalised co-ordinates need not necessarily have the dimensions of length, leter we shall define *generalised force*, which again need not have the dimensions of forces, but the two are such that the scalar product of generalised co-ordinates and generalised force has the dimensions of work, The differential coefficients of q_α ($\alpha=1, 2, \ldots n$) w.r.t, "t" are termed as generalised velocities and are denoted by $\dot{q}_\alpha$. The number n of generalised co-ordinates is the number of degrees of freedom (to be defined in 3·2).

3·2 Degrees of freedom.

The number of co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

Example 1. *A particle moving freely in space requires* 3 *co-ordinates, e.g.* (x, y, z), *to specify its position. Thus the number of degress of freedom is* 3.

2. *A system consisting of N particles moving freely in space requires,* 3N *co-ordinates to specify the position. Thus number of degrees of freedom is* 3N.

A rigid body which can move freely in space has 6 degree of freedom i.e. 6 co-ordinates are required to specify the position.

Let 3 non-collinear points of a rigid body be fixed in space, then the rigid body is also fixed in space. Let these points have co-ordinates (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) respectively, a total of 9 since the body is rigid we must have

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=\text{constant},$$
$$(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2=\text{constant},$$
$$(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2=\text{constant}.$$

Hence 3 co-ordinates can be expressed in term of the remaining six. Thus six independent co-ordinates are needed to describe the motion i.e. there exist six degrees of freedom.

3·3. Transformation equations.

Let $\mathbf{r}_v=x_v\,\mathbf{i}+y_v\,\mathbf{j}+z_v\,\mathbf{k}$ be the position vector of v^{th} particle with respect to xyz co-ordinate system. The relationship of the generalised co-ordinates $q_1, q_2, \ldots, q_n$ to the position co-ordinates are given by the *transformation equations.*

$$\left.\begin{aligned} x_v &= x_v\,(q_1, q_2, \ldots, q_n;\ t) \\ y_v &= y_v\,(q_1, q_2, \ldots, q_n;\ t) \\ z_v &= z_v\,(q_1, q_2, \ldots, q_n;\ t) \end{aligned}\right\}$$

where t denotes the time. In vector, the above equations can be written as $\mathbf{r}_v = \mathbf{r}_v\,(q_1, q_2, \ldots q_n;\ t)$ ...(10)

where the functions in (10) are continuous and have continuous derivatives.

3·4. Classification of Mechanical System.

(a) Scleronomic system.

The mechanical system in which t, the time, does not enter explicitly in equations (1) or (2) is called a scleronomic system.

(b) Rheonomic system.

The mechanical system in which the moving constraints are involved and the time t does enter explicitly is called a Rheonomic system.

(c) Holonomic system and Non-Holonomic system.

Let $q_1, q_2, \ldots, q_n$ denote the generalised co-ordinates describing a system and let t denote the generalised co-ordinates describing system can be expressed as equations having the form $f(q_1, q_2 \ldots, q_n;\ t) = 0$ or their equivalent, then the system is said to be Holonomic otherwise it is said to be Non Holonomic system.

(d) Conservative and non-conservative systems.

If the forces acting on the system are derivable from a potential function [or potential energy] V, then the system is called conservative otherwise it is non-conservative.

3·5. Kinetic energy and generalised velocities.

The total K. E. of the system is $T = \frac{1}{2} \sum_{v=1}^{N} m_v \dot{\mathbf{r}}^2_v$. ...(11)

The K E. of the system can also be written as a quadratic form in the generalised velocities $\dot{q}_\alpha$. If the system is indepeudent of the explicitly i.e. Scleronomic, then quadratic form has only terms of the type $a_{\alpha\beta}$. In case the system is Rheonomic, linear terms in q_v are also present.

3·6. Generalised forces.

If W is total work bone on a system of particles by forces $\mathbf{F}_v$ acting on the v^{th} particle, then

$$dW = \sum_{\alpha=1}^{n} \phi_\alpha\, dq_\alpha \text{ where } \phi_\alpha = \sum_{v=1}^{N} \mathbf{F}_v \cdot \frac{\partial \mathbf{r}_v}{\partial q_\alpha}$$

is called the **generalised force** *associated with generalised coordinates* p_α.

Suppose that a system undergoes increments $dq_1, dq_2, \ldots, dq_n$ of the generalisd co-ordinates $q_1, q_2, \ldots, q_n$, then ν^{th} particle undergoes a displacement

$$d\mathbf{r}_\nu = \sum_{\nu=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} dq_\alpha. \quad \ldots(12)$$

$\therefore$ total work done is given by

$$dW = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot d\mathbf{r}_\nu = \sum_{\nu=1}^{N} \left\{ \sum_{\alpha=1}^{n} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \right\} dq_\alpha \quad \ldots(13)$$

Now, let $\qquad \phi_\alpha = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$

then (13) $\Rightarrow$ $dW = \sum_{\alpha=1}^{n} \left(\sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \right) dq_\alpha = \sum_{\alpha=1}^{n} \phi_\alpha \, dq_\alpha \quad \ldots(14)$

We have $dW = \sum_{\alpha=1}^{n} \frac{\partial W}{\partial q_\alpha} dq_\alpha$;

$$\therefore \quad \frac{\partial W}{\partial q_\alpha} = \phi_\alpha. \quad \ldots(15)$$

Note. (i) α-varies from 1 to n, the number of degree of freedom.

(ii) ν-varies from 1 to N, the number of particles in the system.

3·7. Lagrange's equations.

Let $\mathbf{F}_\nu$ be the net external force acting on the ν^{th} particle of a system; then by Newton's second law applied to ν^{th} particle, we have

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu \quad \ldots(16)$$

$$m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} = \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_x} \quad \ldots(17)$$

We also have

$$\mathbf{r}_\nu = \mathbf{r}_\nu (q_1, q_2, \ldots\ldots, q_n, t).$$

$$\dot{\mathbf{r}}_\nu = \frac{\partial \mathbf{r}_\nu}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}_\nu}{\partial q_2} \dot{q}_2 \ldots + \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \dot{q}_\alpha + \ldots + \frac{\partial \mathbf{r}_\nu}{\partial q_n} \dot{q}_n + \frac{\partial \mathbf{r}_\nu}{\partial q_t} \quad \ldots(18)$$

$$\therefore \quad \frac{\partial \dot{\mathbf{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \quad \ldots(19)$$

[Cancellation law of the dots]

and $\frac{\partial}{\partial q_\alpha} (\dot{\mathbf{r}}_\nu) = \frac{\partial}{\partial q_\alpha} \left(\frac{\partial \mathbf{r}_\nu}{\partial q_1} \dot{q}_1 + \frac{\partial \mathbf{r}_\nu}{\partial q_2} \dot{q}_2 + \ldots + \frac{\partial \mathbf{r}_\nu}{\partial q_n} \dot{q}_n + \frac{\partial \mathbf{r}_\nu}{\partial t} \right)$

$$=\frac{\partial^2 \mathbf{r}_v}{\partial q_\alpha \partial q_1}\dot{q}_1+\frac{\partial^2 \mathbf{r}_v}{\partial q_\alpha \partial q_2}\dot{q}_2+\ldots+\frac{\partial^2 \mathbf{r}_v}{\partial q_\alpha \partial q_n}\dot{q}_n+\frac{\partial}{\partial q_\alpha}\left(\frac{\partial \mathbf{r}_v}{\partial t}\right)$$

$$=\frac{\partial}{\partial q_1}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)\dot{q}_1+\frac{\partial}{\partial q_2}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)\dot{q}_2+\ldots+\frac{\partial}{\partial q_n}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)\dot{q}_n+\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)$$

or
$$\frac{\partial}{\partial q_\alpha}\left(\frac{d\mathbf{r}_v}{dt}\right)=\frac{d}{dt}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right) \qquad \ldots(20)$$

which shows that $\frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right)\equiv\frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right)$ ...(21)

[interchange of the order of operators]

Now
$$\frac{d}{dt}\left(\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)=\ddot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}+\dot{\mathbf{r}}_v\cdot\frac{d}{dt}\left(\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)$$
$$=\ddot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}+\dot{\mathbf{r}}_v\cdot\frac{\partial}{\partial q_v}\left(\frac{dr_v}{dt}\right) \qquad \text{[using (20)]}$$

or
$$\ddot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}=\frac{d}{dt}\left(\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)-\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial q_\alpha} \qquad \ldots(22)$$

$$\therefore \quad (17) \Rightarrow m_v\ddot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}=m_v\frac{d}{dt}\left(\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)-m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial q_\alpha}$$

$$\Rightarrow \frac{d}{dt}\left(m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right)-m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial q_\alpha}=\mathbf{F}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha} \qquad \ldots(23)$$

[using (17)]

Summing both sides w.r.t. v over all particles, we have

$$\frac{d}{dt}\left\{\sum_{v=1}^{N} m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}\right\}-\sum_v m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial q_\alpha}=\Sigma\,\mathbf{F}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha} \qquad \ldots(24)$$

Now, let T be the K.E. of a system of particles, then we have

$$T=\tfrac{1}{2}\sum_v m_v\dot{\mathbf{r}}_v^2=\tfrac{1}{2}\sum_v m_v\dot{\mathbf{r}}_v\cdot\dot{\mathbf{r}}_v \qquad \ldots(25)$$

$$\therefore \quad \frac{\partial T}{\partial q_\alpha}=\sum_v m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial q_\alpha} \qquad \ldots(26)$$

and
$$\frac{\partial T}{\partial \dot{q}_\alpha}=\sum_v m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \dot{\mathbf{r}}_v}{\partial \dot{q}_\alpha}=\sum_v m_v\dot{\mathbf{r}}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha} \qquad \ldots(27)$$

[using (19)]

$$\therefore \quad (24) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=\sum_v \mathbf{F}_v\cdot\frac{\partial \mathbf{r}_v}{\partial q_\alpha}$$

or
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=\phi_\alpha=\frac{\partial W}{\partial q_\alpha}. \qquad \ldots(28)$$

[using (16)]

For $\alpha=1, 2, \ldots, n$, we have n different equations which are called as Lagrange's equations.

Note. The quantity $p_\alpha=\dfrac{\partial T}{\partial \dot{q}_\alpha}$ is called the generalised momentum associated with the generalised co-ordinates q_α.

3·8. Lagrangian function.

If the forces are derivable from a potential V then

$$\phi_\alpha=\frac{\partial W}{\partial q_\alpha}=-\frac{\partial V}{\partial q_\alpha},$$

since the potential, or potential energy is a function of q's only (and possibly the time t) then, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=-\frac{\partial V}{\partial q_\alpha} \Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}(T-V)\right]-\left(\frac{\partial T}{\partial q_\alpha}-\frac{\partial V}{\partial q_\alpha}\right)=0$$

$$\because \frac{\partial V}{\partial \dot{q}_\alpha}=0.$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0, \text{ where } L=T-V \quad ...(29)$$

The function L defined by $L=T-V$ is said to be Lagrangian function.

3·9. Generalised momentum.

We define $p_\alpha=\dfrac{\partial T}{\partial \dot{q}_\alpha}$ to be the generalised momentum associated with generalised co-ordinate q_α. We usually refer p_α as the momentum conjugate to q_α, or the conjugate momentum.

In case the system is conservative, we have

$$T=L+V \Rightarrow \frac{\partial T}{\partial \dot{q}_\alpha}=\frac{\partial L}{\partial \dot{q}_\alpha}+\frac{\partial V}{\partial \dot{q}_\alpha}=\frac{\partial L}{\partial \dot{q}_\alpha} \quad ...(30)$$

because, V the P.E. of the system does not depend upon $\dot{q}_\alpha$.

$$\therefore \quad p_\alpha=\frac{\partial L}{\partial \dot{q}_\alpha}.$$

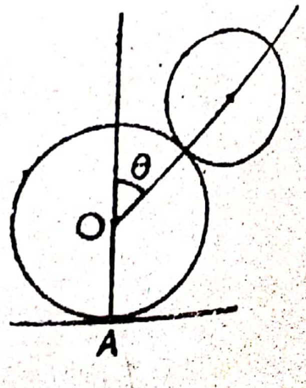
3·10. Solved Examples.

Ex. 1. Classification of Mechanical Systems.

Classify each of the following according as they are

(i) Scleronomic or Rheonomic, (ii) Holonomic or Nonoholonomic and (iii) Conservative or non-conservative.

(a) A sphere rolling down from the top of a fixed sphere.



In this case. *(i) The equations do not involve time "t" explicitly.*

(ii) Rolling sphere leaves the fixed sphere at some point.

(iii) Gravitational force acting on the body is derivable from a potential.

Hence the mechanical system is

Scleronomic non-holonomic conservative.

(*b*) *A cylinder rolling without slipping down a rough inclined plane of angle* α.

Scleronomic.

Holonomic equation of constraint is that of a line or plane.

Conservative.

(*c*) *A particle sliding down the inner surface, with coefficient of friction* μ, *of a paraboloid of revolution having its axis vertical and vertex downward.*

Scleronomic.
Holonomic.
Non-conserative [$\because$ force of friction is not derivable from a potential]

(*d*) *A particle moving on a very long frictionless wire which rotates with constant angular speed about a horizontal axis.*

Rheonomic (constraint involves time "t" explicity).

Holonomic (equation of constraint is that of a time which involves t explicitly).

Conservative.

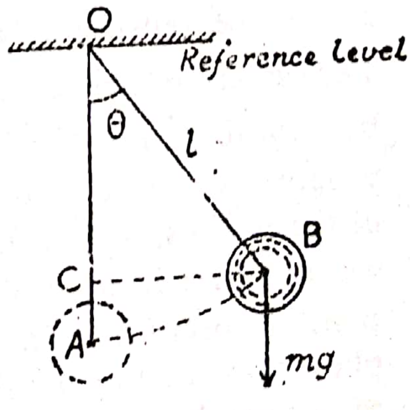
Ex. 2. (*i*) *Set up the Lagrangian for a simple pendulum, and*

(*ii*) *Obtain an equation describing its motion.*

Solution. (i) Choose as generalised co-ordinate the angle θ made by the string OB of the pendulum and the vertical OA. Let l be the length of OA, then K.E. is given by

$$T=\tfrac{1}{2}mv^2=\tfrac{1}{2}m\,(l\dot{\theta})^2=\tfrac{1}{2}ml^2\dot{\theta}^2 \quad \ldots(1)$$

where m is the mass of the job.



The potential energy of mass m is given by

$$V=mg\,(OA-OC)=mg\,(l-l\cos\theta)$$
$$=mgl\,(1-\cos\theta). \quad \ldots(2)$$

$$\therefore\quad L=T-V=\tfrac{1}{2}ml^2\dot{\theta}^2-mgl\,(1-\cos\theta).$$

(*ii*) Hence Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0 \Rightarrow \frac{d}{dt}(ml^2\dot{\theta})-(-mgl\sin\theta)=0$$

$$\Rightarrow l\ddot{\theta}=-g\sin\theta \Rightarrow \ddot{\theta}=-\frac{g}{l}\sin\theta$$

which is the required equation of motion.

Ex. 3. *Write down the Lagrange's equations, when the Lagrangian function has the form* $L=\dot{q}_k q_k-\sqrt{(1-\dot{q}_k^2)}$.

Sol. $L=\dot{q}_k q_k-\sqrt{(1-\dot{q}_k^2)} \Rightarrow \dfrac{\partial L}{\partial q_k}=\dot{q}_k$

and $\dfrac{\partial L}{\partial \dot{q}_k}=q_k-\frac{1}{2}(1-\dot{q}_k^2)^{-1/2}(-2\dot{q}_k)=q_k+\dfrac{\dot{q}_k}{\sqrt{(1-\dot{q}_k^2)}}$. ...(1)

Now, Lagrangian equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0. \qquad ...(2)$$

Whence putting the values of $\dfrac{\partial L}{\partial \dot{q}_k}$ and $\dfrac{\partial L}{\partial q_k}$ in (2), we get

$$\frac{d}{dt}\left\{q_k+\frac{\dot{q}_k}{\sqrt{(1-\dot{q}_k^2)}}\right\}-\dot{q}_k=0$$

$$\Rightarrow \quad \dot{q}_k+\frac{\ddot{q}_k\sqrt{(1-\dot{q}_k^2)}-\frac{1}{2}(1-\dot{q}_k^2)^{1/2}(-2\dot{q}_k)\,\dot{q}_k\ddot{q}_k}{(1-\dot{q}_k^2)}-\dot{q}_k=0$$

$$\Rightarrow \quad \ddot{q}_k\sqrt{(1-\dot{q}_k^2)}+\frac{\dot{q}_k^2\ddot{q}_k}{\sqrt{(1-\dot{q}_k^2)}}=0$$

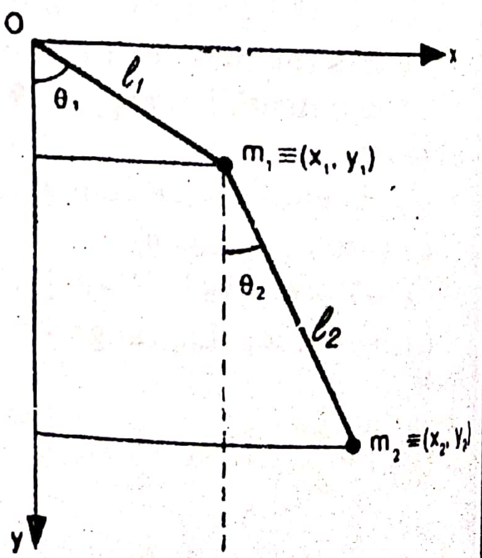
$$\Rightarrow \quad \ddot{q}_k(1-\dot{q}_k^2+\dot{q}_k^2)=0 \quad \Rightarrow \quad \ddot{q}_k=0$$

i.e., generalised acceleration is zero.

Ex. 4. *Obtain the Lagrangian equations of motion for a double pendulum vibrating in a vertical plane.*

(Rohilkhand 1981, 79, 78, 87)

Sol. The double pendulum consists of two masses connected by an inextensible light rod. The system is being suspended by another inextensible and weightless rod fastended to one of the masses, or in other words we, say that it is formed of two pendula oscillating in the same plane ; one having a bob of mass m_1 suspended from a flat hinge (here) at O by an inextensible and weightless rod of length l_1, while the other end has a bob of mass m_2 suspended from another hinge in mass m_1 by a similar rod of length l_2.



Now, $x_1=l_1\sin\theta_1,\ y_1=l_1\cos\theta_1,$

$x_2=l_1\sin\theta_1+l_2\sin\theta_2\,,\ y_2=l_1\cos\theta_1+l_2\cos\theta_2.$ ...(1)

$$\Rightarrow\quad \left.\begin{aligned}\dot{x}_1&=l_1\cos\theta_1\,\dot\theta_1,\\ \dot{y}_1&=-l_1\cos\theta_1\,\dot\theta_1,\\ \dot{x}_2&=l_1\cos\theta_1\,\dot\theta_1+l_2\cos\theta_2\,\dot\theta_2,\\ \dot{y}_2&=-l_1\sin\theta_1\,\dot\theta_1-l_2\sin\theta_2\,\dot\theta_2,\end{aligned}\right\}\qquad ...(2)$$

This kinetic energy of the system is given by

$$T=\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$$

$$=\tfrac{1}{2}m_1\{(l_1\cos\theta_1\,\dot\theta_1)^2+(-l_1\sin\theta_1\dot\theta_1)^2\}$$

$$+\tfrac{1}{2}m_2\{(l_1\cos\theta_1\dot\theta_1+l_2\cos\theta_2\dot\theta_2)^2+(-l_1\sin\theta_1\dot\theta_1-l_2\sin\theta_2\dot\theta_2)^2\}$$

$$=\tfrac{1}{2}m_1l_1^2\dot\theta_1^2+\tfrac{1}{2}m_2\{l_1^2\dot\theta_1^2+l_2^2\dot\theta_2^2+2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\}.$$

Whence taking the reference level as a horizontal plane at a distance (l_1+l_2) below the point of suspension O, the potential energy V, of the system is given by

$$V=m_1g\,(l_1+l_2-l_2\cos\theta_1)+m_2g\,\{l_1+l_2-(l_1\cos\theta_1+l_2\cos\theta_2)\}.$$

Also, Lagrangian of the system is given by $L=T-V$

$$=\tfrac{1}{2}m_1l_1^2\dot\theta_1^2+\tfrac{1}{2}m_2\{l_1^2\dot\theta_1^2+l_2^2\dot\theta_2^2+2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)\}$$

$$-m_1g\,(l_1+l_2-l_1\cos\theta_1)-m_2g\,(l_1+l_2-l_1\cos\theta_1-l_2\cos\theta_2)$$

...(3)

Now, Lagrange's$-\theta_1$ equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_1}\right)-\frac{\partial L}{\partial\theta_1}=0.\qquad ...(4)$$

$$\Rightarrow\quad \frac{d}{dt}\{m_1l_1\dot\theta_1+m_2l_1^2\dot\theta_1+m_2l_1l_2\dot\theta_2\cos(\theta_1-\theta_2)\}+m_2l_2l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)$$

$$+m_1gl_1\sin\theta_1+m_2gl_1\sin\theta_1=0$$

or $(m_1+m_2)l_1^2\ddot\theta_1+m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2)-m_2l_1l\,\dot\lambda_2\sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$

$$-m_2l_1l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)+m_1gl_1\sin\theta_1+m_2gl_1\sin\theta_1=0$$

$$\Rightarrow\quad (m_1+m_2)\,l_1^2\ddot\theta_1+m_2l_1l_2\ddot\theta_2\cos(\theta_1-\theta_2)+m_2l_1l_2\dot\theta_2^2\sin(\theta_1-\theta_1)$$

$$+(m_1-m_2)\,gl_1\sin\theta_1=0\qquad ...(5)$$

Also, Lagrange's θ_2 equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta_2}\right)-\frac{\partial L}{\partial\theta_2}=0.\qquad ...(6)$$

$$\Rightarrow\quad \frac{d}{dt}\{m_2l_2^2\dot\theta_2+m_2l_1l_2\dot\theta_1\cos(\theta_1-\theta_2)\}-m_2l_1l_2\dot\theta_1\dot\theta_3\sin(\theta_1-\theta_2)$$

$$+m_2gl_2\sin\theta_2=0$$

$$\Rightarrow\quad m_2l_2^2\ddot\theta_2+m_2l_1l_2\ddot\theta_1\cos(\theta_1-\theta_2)-m_2l_1\dot\theta_1\sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)$$

$$-m_2l_1l_2\dot\theta_1\dot\theta_2\sin(\theta_1-\theta_2)+m_2gl_2\sin\theta_2=0$$

$$\Rightarrow\quad m_2l_1^2\ddot\theta-m_2l_2i_2\dot\theta_2\cos(\theta_1-\theta_2)-m_2l_1l_2\dot\theta_1^2\sin(\theta_1-\theta_2)$$

$$+m_2gl_2\sin\theta_2=0.\qquad ...(7)$$

Thus equations (5) and (7) are required equations of motion.

Ex. 5. *A particle is constrained to move in a plane under the influence of an attraction towards the origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction. Find (a) the Lagrangian and (b) the equations of motion.*

Sol. Let (r, θ) be the co-ordinates of the particle of mass m at any instant t. The kinetic energy of the particle is then given by

$$T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2), \quad \ldots(1)$$

But, here the force acting on the particle is given by

$F=-kr$, where k is constant of proportionality. ...(2)

$\Rightarrow \quad F=-\dfrac{dV}{dr}$, where V is the potential energy

$\Rightarrow \quad dV=-F dr.$

$\Rightarrow$ P.E of the particle $=V=\int -F\,dr+A$,

where A is constant of integration

$$=\int kr\,dr+A=k(r^2/2)+A$$

But at the origin, potential energy is zero,

i.e. $V=0$ at $r=0$. $\Rightarrow \quad A=0. \quad \Rightarrow \quad V=(kr^2/2)$

$\therefore$ Lagrangian$=L=T-V=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)-\dfrac{kr^2}{2}$ ...(3)

$$\Rightarrow \frac{\partial L}{\partial \dot{r}}=m\dot{r}, \; \frac{\partial L}{\partial r}=mr\dot{\theta}^2-kr, \; \frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}; \; \frac{\partial L}{\partial \theta}=0. \quad \ldots(3')$$

Hence, Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=Q_r', \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=Q_\theta',$$

where Q_r' and Q_θ' are non-conservative generalised forces. ...(4)

Here non-conservative force is perpendicular to the radius vector **r**, so R_r' reduces to zero,

and
$$Q_\theta'=F'\cdot\frac{\partial r}{\partial \theta}=rF'$$

$=\dfrac{r\lambda}{r}\Big[\because$ non-conservative force is inversely proportional to r,

i.e., $F=\dfrac{\lambda}{r}$ where λ is some constant$\Big]=\lambda.$

Thus Lagrang's equations are reduced to

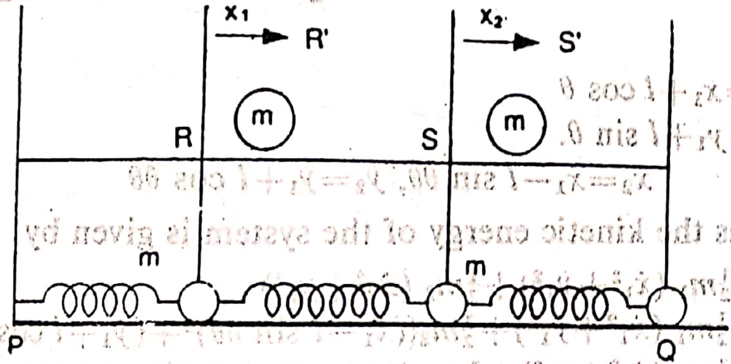
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0,\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=\lambda.$$

Now Making substitutions from (3′), these equations

$$\Rightarrow\ m\ddot{r}-mr\dot{\theta}^2+kr=0 \quad \text{and} \quad \frac{d}{dt}(mr^2\dot{\theta})=\lambda.$$

Ex. 6. *Two equal masses m are connected by springs having equal spring constants as shown in fig. below, so that the masses are free to sliae on a frictionless table PQ. The points P and Q to which the ends of the springs are attached are fixed. Use Lagrange's equations to obtain the equations of motion of the vibrating masses.*

Sol. Let x_1 and x_2 be the displacements of the masses from their initial positions, say R and S at any instant t.



From the kinetic energy of the system is given by

$$T=\tfrac{1}{2}m\dot{x}_1^2+\tfrac{1}{2}m\dot{x}_2^2,$$

Now let k represent the spring constant of the spring, then the potential energy of the system is given by

$$V=\tfrac{1}{2}kx_1^2+\tfrac{1}{2}k\,(x_2-x_1)^2+\tfrac{1}{2}kx_2^2,$$

[$\because$ extensions of the springs PR, RS and SQ are x_1, (x_2-p_1) and x_2 respectively].

The Lagrangian of the system is given by

$$L=T-V=\tfrac{1}{2}m\dot{x}_1^2+\tfrac{1}{2}m\dot{x}_2^2-\tfrac{1}{2}k(x_2-x_1)^2-\tfrac{1}{2}kx_2^2-\tfrac{1}{2}k\,x_1^2,$$

$$\Rightarrow\ \frac{\partial L}{\partial \dot{x}_1}=m\dot{x}_1,\ \frac{\partial L}{\partial x_1}=-k\dot{x}_1+k\,(x_2-x_1)=k\,(x_2-2x_1)$$

$$\frac{\partial L}{\partial \dot{x}_2}=mx_2\ \frac{\partial L}{\partial x_2}=-k\,(x_2-x_1)-kx_2=k(x_1-2x_2)$$

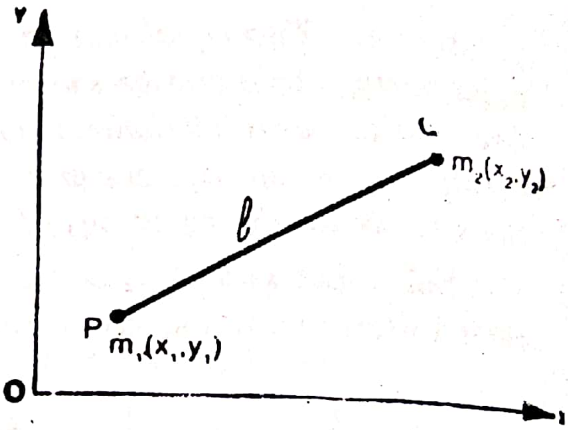
Whence, Lagrange's equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}=0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right)-\frac{\partial L}{\partial x_2}=0$$

$$\Rightarrow\ m\ddot{x}_1-k\,(x_2-2x_1)=0 \quad \text{and} \quad m\ddot{x}_2-k(x_1-2x_2)=0.$$

Ex. 7. *Use Lagrange's equations to set up the equations of motion of a dumb-bell in a vertical plane.*

Sol. By a dumb-bell we mean a system consisting of two particles of masses m_1 and m_2; rigidly fastened to a rod of length l and negligible mass. The system is free to move in a verticle plane (say x-y plane) under the action of gravity alone.



Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of the position of two masses respectively. Let θ be the inclination of the rod of length l with x-axis, then we have

$$x_2=x_1+l\cos\theta$$
$$y_2=y_1+l\sin\theta. \qquad \ldots(1)$$

$$\Rightarrow \qquad \dot{x}_2=\dot{x}_1-l\sin\theta\dot{\theta},\ \dot{y}_2=\dot{y}_1+l\cos\theta\dot{\theta} \qquad \ldots(2)$$

Thus the kinetic energy of the system is given by

$$T=\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)$$
$$=\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2\{(\dot{x}_1-l\sin\theta\dot{\theta})^2+(\dot{y}_1+l\cos\theta\dot{\theta})^2\}$$
$$=\tfrac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2\{(\dot{x}_1^2+l^2\sin^2\theta\dot{\theta}^2-2\dot{x}_1 l\sin\theta\dot{\theta}$$
$$+\dot{y}_1^2+l^2+\cos^2\theta\dot{\theta}^2+2\dot{y}_1 l\cos\theta\dot{\theta}\}$$
$$=\tfrac{1}{2}(m_1+m_2)(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2\{l^2\dot{\theta}^2-2l\dot{x}_2\dot{\theta}\sin\theta$$
$$+2l\dot{y}_1\dot{\theta}\cos\theta\} \qquad \ldots(3)$$

Further, potential energy of the system, (x-axis as the reference level), is given by

$$V=m_1gy_1+m_2gy_2=m_1gy_1+m_2g(y_1+l\sin\theta)$$
$$=(m_1+m_2)gy_1+m_2gl\sin\theta.$$

Then the Lagrangian of the system is given by

$$L=T-V=\tfrac{1}{2}(m_1+m_2)(\dot{x}_1^2+\dot{y}_1^2)+\tfrac{1}{2}m_2\{l^2\dot{\theta}-2l\dot{x}_1\dot{\theta}\sin\theta$$
$$+2l\dot{y}_1\dot{\theta}\cos\theta\}-(m_1+m_2)gy_1-m_2gl\sin\theta \qquad \ldots(4)$$

$$\Rightarrow \quad \left.\begin{aligned}
&\frac{\partial L}{\partial \dot{x}_1}=(m_1+m_2)\dot{x}_r-m_2l\dot{\theta}\sin\theta, && \frac{\partial L}{\partial x_1}=0\\
&\frac{\partial L}{\partial \dot{y}_1}=(m_1+m_2)\dot{y}_1+m_2l\dot{\theta}\cos\theta, && \frac{\partial L}{\partial y_1}=-(m_1+m_2)g\\
&\frac{\partial L}{\partial \dot{\theta}}=m_2l^2\dot{\theta}-m_2l\dot{x}_1\sin\theta+m_2l\dot{y}_1\cos\theta, && \frac{\partial L}{\partial \theta}=-m_1gl\cos\theta
\end{aligned}\right\} \ldots(5)$$

Also, Lagrange's equations in the variables x_1, y_1 and θ are given by

$$\left.\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right)-\frac{\partial L}{\partial x_1}&=\frac{d}{dt}\{(m_1+m_2)\,\dot{x}_1-m_2l\dot{\theta}\sin\theta\}=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right)-\frac{\partial L}{\partial y_1}&=\frac{d}{dt}\{(m_1+m_2)\,\dot{y}_1+m_2l\dot{\theta}\cos\theta\}+(m_1-m_2)\,g=0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}&=\frac{d}{dt}\{m_2l^2\dot{\theta}-m_2l\dot{x}_1\sin\theta+m_2l\dot{y}_1\cos\theta\}+m_2gl\cos\theta=0.\end{aligned}\right\}$$

Hence, we can say that the [using (5)] the equations of motion of the dumb bell are given by

$$(m_1+m_2)\,\ddot{x}_1-m_2l\ddot{\theta}\sin\theta-m_2l\dot{\theta}^2\cos\theta=0,$$

$$(m_1=m_1)\,\ddot{y}_1+m_2l\ddot{\theta}\cos\theta-m_2l\dot{\theta}^2\sin\theta+(m_1+m_2)\,g=0,$$

$$m_2l^2\ddot{\theta}-m_2l\ddot{x}_1\sin\theta-m_2l\dot{x}_1\dot{\theta}\cos\theta+m_2l\ddot{y}_1\cos\theta-m_2l\dot{y}_1\dot{\theta}\sin\theta$$
$$+m_2gl\cos\theta=0.$$

Ex. 8. *A particle moves in a plane under the influence of a force, acting towards a centre of force, whose magnitude is* $f=\frac{1}{r^2}\left(1-\frac{\dot{\theta}r^2-2\ddot{r}r}{c^2}\right)$, *where r is the distance of the particle to the centre of force. Obtain the Lagrangian for the motion in a plane.*

(*Rohilkhand* 1986)

Sol. $F_r=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2\ddot{r}r}{c^2}\right)=\frac{c^2-\dot{r}^2+2\ddot{r}r}{r^2c^2}=\frac{1}{r^2}+\frac{1}{c^2}\left(\frac{2\ddot{r}r-\dot{r}^2}{r^2}\right)$

Now taking $U=q\phi-q(\bar{A}\,.\,\bar{v})$ and $F_r=-\frac{\partial U}{\partial r}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}^2}\right)$ in usual notations, we get

$$F_r=-\frac{\partial U}{\partial r}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right)$$

$$\Rightarrow\quad \frac{\partial U}{\partial r}=-\frac{1}{r^2}\quad\Rightarrow\quad U=r^{-1}+\phi(\dot{r})$$

Choosing $\quad U=\frac{1}{r}+\frac{1}{c^2}\left(\frac{\dot{r}^2}{r}\right)$, we obtain

$$\frac{\partial U}{\partial r}=-\frac{1}{r^2}+\frac{\dot{r}^2}{c^2}\left(-\frac{1}{r^2}\right)=-\frac{1}{r^2}-\frac{\dot{r}^2}{c^2r^2}\text{ and }\frac{\partial U}{\partial \dot{r}}=\frac{1}{c^2}\cdot\frac{2\dot{r}}{r}$$

$$\Rightarrow\quad \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right)=\frac{2}{c^2}\frac{d}{dt}\left(\frac{\dot{r}}{r}\right)=\frac{2}{c^2}\left(\frac{\ddot{r}r-\dot{r}^2}{r^2}\right)$$

$$\therefore\quad -\frac{\partial U}{\partial r}+\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right)=\frac{1}{r^2}+\frac{\dot{r}^2}{c^2r^2}+\frac{2\ddot{r}r}{c^2r^2}-\frac{2\dot{r}^2}{c^2r^2}$$

$$=\frac{1}{r^2}\left(1-\frac{\dot{r}^2-2\ddot{r}r}{c^2}\right)$$

and hence justifies $F_r=\frac{1}{r^2}\left(1-\frac{\dot r^2-2\ddot r r}{c^2}\right)$, as given above. Thus the generalised potential, $U=\frac{1}{r}+\frac{1}{c^2}\frac{\dot r^2}{r}$ and $T=\frac{1}{2}[\dot r^2+r^2\dot\theta^2]$.

$\Rightarrow$ Lagrangian $L=T-U=\frac{\dot r^2}{2}+\frac{r^2\dot\theta^2}{2}-\frac{1}{r}-\frac{1}{c^2}\frac{\dot r^2}{r}$

3·11 Kinetic energy as a Quadratic function of velocities.

If at time t, the position of the ν^{th} particle (mass m_ν) of a holonomic system is defined by $\mathbf{r}_\nu$; then the K.E. is

$$T=\tfrac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{\mathbf{r}}_\nu^2, \quad \ldots(31)$$

where $\mathbf{r}_\nu=\mathbf{r}_\nu(q_1, q_2, \ldots q_n, t)$

$$\Rightarrow \dot{\mathbf{r}}_\nu=\dot q_1\frac{\partial \mathbf{r}_\nu}{\partial q_1}+\dot q_2\frac{\partial \mathbf{r}_\nu}{\partial q_2}+\ldots+\dot q_n\frac{\partial \mathbf{r}_\nu}{\partial q_n}+\frac{\partial \mathbf{r}_\nu}{\partial t}$$

Thus $$T=\tfrac{1}{2}\sum_{\nu=1}^{N} m_\nu\left\{\dot q_1\frac{\partial \mathbf{r}_\nu}{\partial q_1}+\dot q_2\frac{\partial \mathbf{r}_\nu}{\partial q_2}+\ldots+\dot q_n\frac{\partial \mathbf{r}_\nu}{\partial q_n}+\frac{\partial \mathbf{r}_\nu}{\partial t}\right\}^2$$

$$=\tfrac{1}{2}\{(a_{11}\dot q_1^2+a_{22}\dot q_2^2+\ldots+a_{nn}\dot q_n^2+2a_1a_2\dot q_1\dot q_2+\ldots)\}$$
$$+2(a_1\dot q_1+a_2\dot q_2+\ldots+a_n\dot q_n)+a\} \quad \ldots(32)$$

where
$$\left.\begin{aligned} a_{rs}&=\sum_{\nu=1}^{N} m_\nu(\partial \mathbf{r}_\nu/\partial q_r)\cdot(\partial \mathbf{r}_\nu/\partial q_s) \quad (s\geqslant r)\\ a_r&=\sum_{\nu=1}^{N} m_\nu(\partial \mathbf{r}_\nu/\partial q_r)\cdot(\partial \mathbf{r}_\nu/\partial t)\\ a&=\sum_{\nu=1}^{N} m_\nu(\partial \mathbf{r}_\nu/\partial t)^2. \end{aligned}\right\}$$

From (32), we see that T is a quadratic function of the generalised velocities.

The case when t is not explicity involved is of considerable importance. Here we have $\frac{\partial \mathbf{r}_\nu}{\partial t}=0$ and therefore (32)

$$\Rightarrow T=\tfrac{1}{2}(a_{11}\dot q_1^2+a_{22}\dot q_2^2+\ldots+a_{nn}\dot q_n^2+2a_1a_2\dot q_1\dot q_2+\ldots) \quad \ldots(33)$$

$$=\tfrac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} a_{rs}\dot q_r\dot q_s, \text{ where } a_{rs}=a_{sr}. \quad \ldots(34)$$

Now using Euler's theorem for homogeneous functions, we get

$$\dot q_1\frac{\partial T}{\partial \dot q_1}+\dot q_2\frac{\partial T}{\partial \dot q_2}+\ldots+\dot q_n\frac{\partial T}{\partial \dot q_n}=2T$$

$$\Rightarrow 2T=\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}=\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha=p_1\dot{q}_1+p_2\dot{q}_2+\ldots+p_n\dot{q}_n.$$

3·12. Equilibrium Configurations for Conservative Holonomic Dynamical systems.

For a conservative holonomic dynamical system specified by generalised co-ordinates $q_1, q_2, \ldots q_n$ with time explcitly absent the K.E. has been shown in the last section to be a homogeneous quadratic function of velocities of the form

$$T=\tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{q}_r\dot{q}_s, \text{ where } a_{rs}=a_{sr} \qquad \ldots(35)$$

and these coefficients are each function of $q_1, q_2, \ldots q_n$ for all r, s.

Also the P.E. is of the form $V=V(q_1, q_2, \ldots, q_n)$.

$\therefore$ Lagrange's equation are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=-\frac{\partial V}{\partial q_\alpha} \quad (\alpha=1, 2, \ldots, n) \qquad \ldots(36)$$

Now $\dfrac{\partial T}{\partial q_\alpha}=\tfrac{1}{2}\displaystyle\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{\partial a_{rs}}{\partial q_\alpha}\right)\dot{q}_r\dot{q}_s\Big\}.$

$$\frac{\partial T}{\partial \dot{q}_\alpha}=\tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\left(\dot{q}_r\frac{\partial \dot{q}_s}{\partial \dot{q}_\alpha}+\dot{q}_s\frac{\partial \dot{q}_r}{\partial \dot{q}_\alpha}\right)$$

$$=\tfrac{1}{2}\sum_{r=1}^{n} a_{r\alpha}\dot{q}_r+\tfrac{1}{2}\sum_{s=1}^{n} a_{\alpha s}\dot{q}_s=\sum_{r=1}^{n} a_{r\alpha}a_r \qquad (\because \; a_{\alpha r}=a_{r\alpha})$$

$$\text{Thus } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)=\sum_{r=1}^{n}\left\{\frac{d}{dt}(a_{r\alpha}\dot{q}_r)\right\}=\sum_{r=1}^{n}\left\{a_{r\alpha}\ddot{q}_r+\dot{q}_r\frac{d}{dt}(a_{r\alpha})\right\}$$

$$=\sum_{r=1}^{n}\left\{a_{r\alpha}\ddot{q}_r+\dot{q}_r{}^*\left(\dot{q}_r\frac{\partial a_\alpha}{\partial q_1}+\dot{q}_2\frac{\partial a_{r\alpha}}{\partial q_2}+\ldots+\dot{q}_n\frac{\partial a_{r\alpha}}{\partial q_2}\right)\right\}$$

Now, let us assume that the system be in equilibrium for a confiuration specified by the co-ordinates $q_\alpha=\beta_\alpha$ $(\alpha=1, 2, \ldots n)$.

For these values $\dot{q}_\alpha=0, \ddot{q}_\alpha=0$ $(\alpha=1, 2, \ldots, n)$.

That it $\Rightarrow$ that $\dfrac{\partial T}{\partial q_\alpha}=0, \; \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{q}_\alpha}\right)=0, \; (\alpha=1, 2, \ldots, n).$

---

*$a_{r\alpha}$ is a function of $q_1, q_2, \ldots, q_n$

$$\therefore \frac{d}{d}(a_{r\alpha})=\dot{q}_1\frac{\partial a_{r\alpha}}{\partial q_1}+\dot{q}_2\frac{\partial a_{r\alpha}}{\partial q_2}+\ldots+\dot{q}_n\frac{\partial a_{r\alpha}}{\partial q_n}.$$

$$\therefore \quad (36) \Rightarrow \frac{\partial V}{\partial q_\alpha}=0 \;(\alpha=1, 2, ..., n). \quad ...(37)$$

Thus we can say that :

In order to find the equilibrium configurations for system specified by n generalised co-ordinates q_α ($\alpha=1, 2, ..., n$), we express the potential energy V as a function of these co-ordinates and then solve the n equations (37) for the co-ordinates specifying the equilibrium configuration or configurations.

After obtaining the configuration of equilibrium according to the above rule it is important to investigate whether these positions are stable or unstable.

For a conservative system, $T+V=$const., we know that if small purturbations are made from an equilibrium position, T must de rease or increase according to whether the position is stable or unstable *i.e.* V must increase or decrease, respectively. Hence for the stable position, V should be minimum and for unstable position V may be maximum.

Example 9. *The ends of a uniform rod AB of length $2a \cos 15°$ and weight W, are constrained to slide on a smooth circular wire of radius a fixed with its plane vertical. The end A is connected by an elastic string, of natural length a and modulus of elasticity (W/2), to the highest point of the wire. If θ is the angle which the perpendicular bisector of the rod makes with the downward vertical show that the potential energy V is given by*

$$V=-\frac{Wa}{2}\{\cos(\theta-75°)+2\cos\tfrac{1}{2}(\theta+75°)\}+const.$$

Verify that $\theta=25°$ defines a position of equilibrium and investigate its stability.

Solution. Let H be the highest point of the circle.

Then $\angle AOB=\angle BOA=15°$, $\angle AOB=150°$.

$$\therefore \quad \angle OHA=\angle OAH=\tfrac{1}{2}\angle KOA=\frac{\theta+75°}{2} \text{ where } \angle KOG=\theta.$$

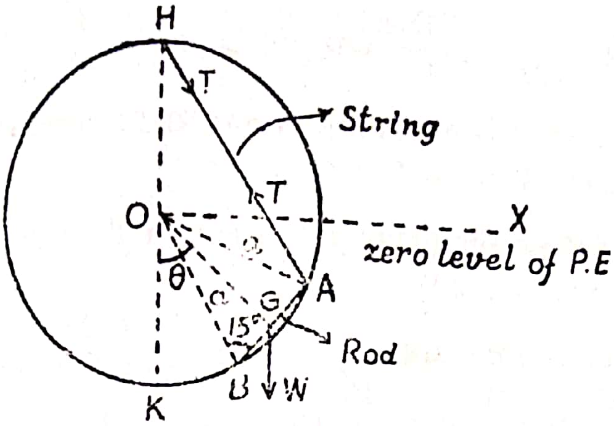
Taking OX as the zero level of P.E., we see that :

(i) *the gravitational potential energy is* $-Wa \sin 15° \cos\theta$.

[$\because$ depth of G of AB below OX is $OG \cos\theta$ or $\sin 15° \cos\theta$],

and (ii) *the elastic potential energy is*

$$\frac{Wa}{4}\left\{2\cos\left(\frac{\theta+75°}{2}\right)-1\right\}^2$$



$$AH=2a\cos \widehat{OAH}=2a\cos\frac{\theta+75^\circ}{2}$$

$$\Rightarrow \text{ Extension of the string}=2a\cos\frac{\theta+75^\circ}{2}-a;$$

$\therefore$ Elastic potential energy of the string is

$$=\frac{W}{2}a^2\,\frac{\left\{2\cos\left(\frac{\theta+75^\circ}{2}\right)-a\right\}^2}{a}\quad \text{as } T=\frac{W}{2}\cdot\frac{2a\cos\left(\frac{\theta+75^\circ}{2}\right)-a}{a}$$

Hence the P E. of the system is given by

$$V=\frac{W}{2}a\left\{2\cos\left(\frac{\theta+75^\circ}{2}\right)-1\right\}^2-Wa\sin 15^\circ\cos\theta \qquad \ldots(1)$$

$$\Rightarrow V=Wa\left\{\cos^2\left(\frac{\theta+75^\circ}{2}\right)-\cos\left(\frac{\theta+75^\circ}{2}\right)\right\}-Wa\sin 15^\circ\cos\theta+\text{const.}$$

$$=\frac{Wa}{2}\left\{1+\cos(\theta+75^\circ)-2\cos\left(\frac{\theta+75^\circ}{2}\right)-\sin(15^\circ+\theta)-\sin(15^\circ-\theta)\right\}+\text{const.}$$

$$=\frac{Wa}{2}\{\cos(\theta+75^\circ)-2\cos\tfrac{1}{2}(\theta+75^\circ)-\cos(75^\circ-\theta)-\cos(75^\circ+\theta)\}+\text{const.}$$

$$=-\frac{Wa}{2}\{\cos(\theta-75^\circ)+2\cos\tfrac{1}{2}(\theta+75^\circ)\}+\text{const.} \qquad \ldots(2)$$

$$\therefore \frac{\partial V}{\partial\theta}=\frac{Wa}{2}\{\sin(\theta-75^\circ)+\sin\tfrac{1}{2}(\theta+75^\circ)\}.$$

and $$\frac{\partial^2 V}{\partial\theta^2}=\frac{Wa}{2}\{\cos(\theta-75^\circ)+\tfrac{1}{2}\cos\tfrac{1}{2}(\theta+75^\circ)\}.$$

When $\theta=25^\circ$, we have

$$\frac{\partial V}{\partial\theta}=\frac{Wa}{2}\{\sin(-50^\circ)+\sin 50^\circ\}=0 \Rightarrow \theta=25^\circ$$

is an equilibrium configuration.

But $\left(\frac{\partial^2 V}{\partial \theta^2}\right)_{\theta=25^\circ}=\frac{3Wa}{4}\cos 50^\circ>0.$

This shows that V is a minimum and the configuration $\theta=25^\circ$ is one of stable equilibrium.

3·13. The deduce the principal of energy from the Lagrange's equations (conservative field).

Lagrange's equations are ;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=-\frac{\partial V}{\partial q_\alpha};\ (\alpha=1, 2, \ldots, n). \qquad \ldots(38)$$

Also by 3·11, equation 33 we know that

$$T=\tfrac{1}{2}(a_{11}\dot{q}_1^2+a_{22}\dot{q}_2^2+\ldots+a_{nn}\dot{q}_n^2+2a_1a_2\dot{q}_1\dot{q}_2+\ \ )$$

i.e., T can be expressed as a quadratic expression in generalised velocities. Hence applying Euler's Theorem, we get

$$\sum_{\alpha=2}^{n}\dot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}=2T. \qquad \ldots(39)$$

Also, $$\frac{dT}{dt}=\sum_{\alpha=1}^{n}\frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{n}\ddot{q}_\alpha\cdot\frac{\partial T}{\partial \dot{q}_\alpha} \qquad \ldots(40)$$

Now multiplying the n equations of (38) by $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$, respectively and then adding, we get

$$\left\{\dot{q}_1\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right)+\ldots+\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right)\dot{q}_n\right\}-\left\{\dot{q}_1\frac{\partial T}{\partial q_1}+\ldots+\dot{q}_a\frac{\partial T}{\partial q_n}\right\}$$

$$=-\left\{\dot{q}_1\frac{\partial V}{\partial q_1}+\ldots+\dot{q}_n\frac{\partial V}{\partial q_n}\right\}$$

$$\Rightarrow \frac{d}{dt}\left\{\sum_{\alpha=1}^{n}\dot{q}_n\frac{\partial T}{\partial \dot{q}_\alpha}\right\}-\left(\sum_{\nu=1}^{n}\ddot{q}_\nu\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\left(\sum_{\alpha=1}^{n}\dot{q}_\alpha\frac{\partial T}{\partial q_\nu}\right)=-\left(\sum_{\alpha=1}^{n}\dot{q}_\alpha\frac{\partial V}{\partial q_\alpha}\right)$$

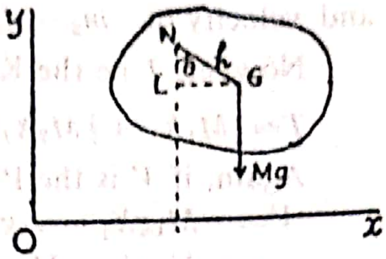
$$\Rightarrow \frac{d}{dt}(2T)-\frac{dT}{dt}=-\frac{dV}{dt}\Rightarrow\frac{dT}{dt}+\frac{dV}{dt}=0$$

$$\Rightarrow \frac{d}{dt}(T+V)=0\Rightarrow T+V=\text{constant}. \qquad \text{[Using (40)]} \quad \ldots(41)$$

Ex. 10. *Use Lagrange's equations to find differential equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.* **(Agra 1973)**

Solution. Let the plane of oscillation be represented by xy plane, where N is its intersection with the axis of rotation and G

is the centre of gravity. Let the mass of the pendulum be M and let its moment of intertia about the axis of rotation be Mk^2. Then potential energy relative to the horizontal plane through N is $V=-Mgh \cos \theta$.



Also $\quad T=\frac{1}{2} Mk^2\dot{\theta}^2$.

$$\therefore \quad L=T-V=\tfrac{1}{2}Mk^2\dot{\theta}^2+Mgh \cos \theta. \quad \ldots(1)$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{\theta}}=Mk^2\dot{\theta} \text{ and } \frac{\partial L}{\partial \theta}=-Mgh \sin \theta. \quad \ldots(2)$$

Now Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \Rightarrow \frac{d}{dt}(Mk^2\dot{\theta})+Mgh \sin \theta=0$$

i.e. $$Mk^2\ddot{\theta}+Mgh \sin \theta=0 \Rightarrow \ddot{\theta}=-\frac{gh}{k^2} \sin \theta.$$

When θ is small, we have $D^2\theta=-\dfrac{gh}{k^2}\theta \Rightarrow \left(D^2+\dfrac{gh}{k^2}\right)\theta=0.$

$(\because \sin \theta=\theta)$

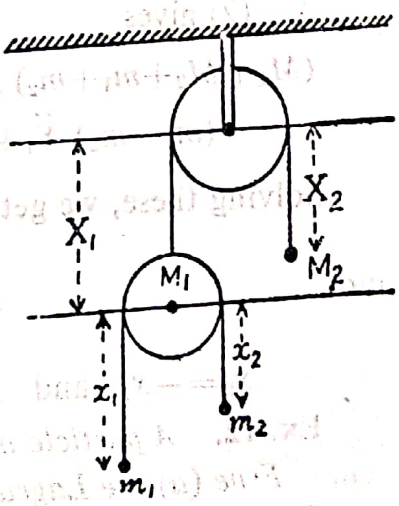
This is the Differential equation of the compound pendulum.

Ex. 11. *A mass M_2 hangs at one end of a string which passes over a fixed frictionless non rotating pulley. At the other end of this string there is a non-rotating pulley of mass M_1 over which there is a string carrying masses m_1 and m_2.*

(*a*) *Set up the Lagrangian of the system,*

(*b*) *Find the acceleration of mass M_2.*

Solution. Let X_1 and X_2 be the distance of masses M_1 and M_2 respectively. Let x_1 and x_2 be the distances of masses m_1 and m_2 respectively below the centre of the movable pulley M_1. But the lengths of the strings are constant, so we have $X_1+X_2=\text{constant}=k_1$ say, $x_1+x_2=\text{constant}=k_2$ say. So we have $\quad \dot{X}+\dot{X}_2=0$ and $\dot{x}_1+\dot{x}_2=0$



$$\Rightarrow \dot{x}_1=-\dot{x}_2 \text{ and } \dot{X}_1=-\dot{X}_2.$$

Hence :

Velocity of $\quad M_1=\dot{X}_1$; Velocity of $\quad M_2=\dot{X}_2=-\dot{X}_1$

Velocity of $\quad m_1=(d/dt)\,(X_1+x_1)=\dot{X}_1+\dot{x}_1$

and velocity of $m_2 = (d/dt)(X_1 + x_2) = \dot{X}_1 - \dot{x}_1$ $\quad (\because \dot{x}_2 = -\dot{x}_1)$

Now let, T be the K.E. of the system then, we have

$$T = \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2\dot{X}_1^2 + \tfrac{1}{2}m_1(\dot{X}_1 + \dot{x}_1)^2 + \tfrac{1}{2}m_2(\dot{X}_1 - \dot{x}_1)^2 \quad \ldots(1)$$

Again, if V is the P.E. of the system then we have

$$V = -M_1 g X_1 - M_2 g X_2 - m_1 g (X_1 + x_1) - m_2 g (X_1 + x_2)$$
$$= -M_1 g X_1 - M_2 g (k_1 - X_1) - m_1 g (X_1 + x_1) - m_2 g (X_1 + k_2 - x_1)$$

Hence the Lagrangian is given by

$$L = T - V = \tfrac{1}{2}M_1\dot{X}_1^2 + \tfrac{1}{2}M_2\dot{X}_1^2 + \tfrac{1}{2}m_1(\dot{X}_1 + \dot{x}_1)^2$$
$$+ \tfrac{1}{2}m_2(\dot{X}_1 - \dot{x}_1)^2 + M_1 g X_1 + M_2 g (k_1 - X_1) + m_1 g (X_1 + x_1)$$
$$+ m_2 g (X_1 + k_2 - x_1)$$

Thus Lagrangian equation (corresponding to X_1 and x_1) are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0. \quad \ldots(2)$$

But $\dfrac{\partial L}{\partial \dot{X}_1} = M_1\dot{X}_1 + M_2\dot{X}_1 + m_1(\dot{X}_1 + \dot{x}_1) + m_2(\dot{X}_1 - \dot{x}_1)$

$$= (M_1 + M_2 + m_1 + m_2)\dot{X}_1 + (m_1 - m_2)\dot{x}_1$$

$$\frac{\partial L}{\partial X_1} = M_1 g - M_2 g + m_1 g + m_2 g = (M_1 - M_2 + m_1 + m_2) g$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1(\dot{X}_1 + \dot{x}_1) - m_2(\dot{X}_1 - \dot{x}_1)$$

$$= (m_1 - m_2)\dot{X}_1 + (m_1 + m_2)\dot{x}_1$$

$$\frac{\partial L}{\partial x_1} = m_1 g - m_2 g = (m_1 - m_2) g.$$

$\therefore$ (2) gives

$$(M_1 + M_2 + m_1 + m_2)\ddot{X}_1 + (m_1 - m_2)\ddot{x}_1 = (M_1 - M_2 + m_1 + m_2) g$$
$$(m_1 - m_2)\ddot{X}_1 + (m_1 + m_2)\ddot{x}_1 = (m_1 - m_2) g$$

Solving these, we get $\ddot{x}_1 = \dfrac{2M_2(m_1 - m_2)}{(M_1 + M_2)(m_1 + m_2) + 4m_1m_2}\, g$

and $\ddot{X}_1 = \dfrac{(M_1 - M_2)(m_1 + m_2) + 4m_1m_2}{(M_1 + M_2)(m_1 + m_2) + 4m_1m_2}\, g$

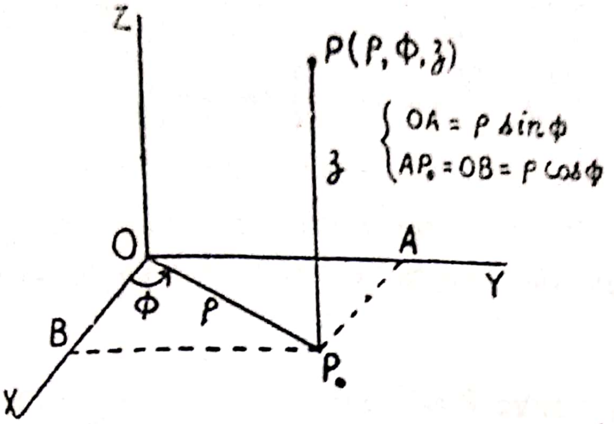
$\therefore$ $\ddot{x}_2 = -\ddot{x}_1$ and $\ddot{X}_2 = -\ddot{X}_1$ etc.

Ex. 12. *A particle of mass m moves in a conservative force field. Find (a) the Lagrangian function, (b) the equations of motion in cylindrical co-ordinates (ρ, ϕ, z).*

Solution. We have

$$\mathbf{OP} = \mathbf{OP_0} + \mathbf{P_0P} = \mathbf{OA} + \mathbf{AP_0} + \mathbf{P_0P} = \vec{\rho} \text{ say}$$

$\therefore \quad \vec{\rho} = \rho \sin\phi\, \mathbf{j} + \rho\cos\phi\, \mathbf{i} + z\mathbf{k}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along OX, OY and OZ respectively.



*Hence the unit vector along the direction of ρ increasing is given by

$$\vec{\rho}_1 = \frac{\partial \vec{\rho}}{\partial \rho} \bigg/ \left|\frac{\partial \vec{\rho}}{\partial \rho}\right| = \sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{i}.$$

$$\vec{\phi}_1 = \frac{\partial \vec{\rho}}{\partial \phi} \bigg/ \left|\frac{\partial \vec{\rho}}{\partial \phi}\right| = \frac{\rho\cos\phi\,\mathbf{j} - \rho\sin\phi\,\mathbf{i}}{\rho} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.$$

Now $\quad \mathbf{v} = \dfrac{d\vec{\rho}}{dt} = \dfrac{d}{dt}(\rho\sin\phi\,\mathbf{j} + \rho\cos\phi\,\mathbf{i} + z\mathbf{k})$

$$= \rho\cos\phi\,\dot{\phi}\,\mathbf{j} + \dot{\rho}\sin\phi\,\mathbf{j} - \rho\sin\phi\,\dot{\phi}\,\mathbf{i} + \dot{\rho}\cos\phi\,\mathbf{i} + \dot{z}\mathbf{k}$$

$$= \dot{\rho}\cos\phi\,\mathbf{i} + \dot{\rho}\sin\phi\,\mathbf{j} + \rho\dot{\phi}(\cos\phi\,\mathbf{j} - \sin\phi\,\mathbf{i}) + \dot{z}\mathbf{k} = \dot{\rho}\,\vec{\rho} + \rho\,\dot{\phi}\,\vec{\phi}_1 + \dot{z}\mathbf{k}.$$

$\therefore \quad T = \frac{1}{2}m\{\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\}$ and $V = V(\rho, \phi, z)$

(a) Hence the Lagrangian function is

$$L = T - V = \tfrac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2] - V(\rho, \phi, z)$$

(b) Lagrange's equation are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0 \text{ i.e. } \frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$$

---

*If $\mathbf{r}$ is the position vector of the particle at any time t, then $\partial\mathbf{r}/\partial r$ is a vector tangent to the curve θ=constant *i.e.*, a vector in the direction of $\mathbf{r}$ (increasing r). A unit vector in this direction is thus given by

$$\mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial r} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial r}\right|.$$

Similarly, $\partial\mathbf{r}/\partial\theta$ is a vector tangent to the curve r=constant. A unit vector in this direction is given by

$$\theta_1 = \frac{\partial \mathbf{r}}{\partial \theta} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right|.$$

$$m\,(\ddot{\rho}-\rho\dot{\phi}^2)=-\frac{\partial V}{\partial \rho}$$

$\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0 \text{ i.e. } \frac{d}{dt}\,(m\rho\dot{\phi})+\frac{\partial V}{\partial \phi}=0 \Rightarrow \frac{d}{dt}\,(\rho^2\dot{\phi})=-\frac{\partial V}{\partial \phi},$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=0 \text{ i.e. } \frac{d}{dt}\,(m\dot{z})+\frac{\partial V}{\partial z}=0 \Rightarrow m\ddot{z}=-\frac{\partial V}{\partial z}. \quad ...(3)$$

Ex. 13. *Work out the previous problem, if the particle moves in xy plane and the potential depends only on the distance from the origin.*

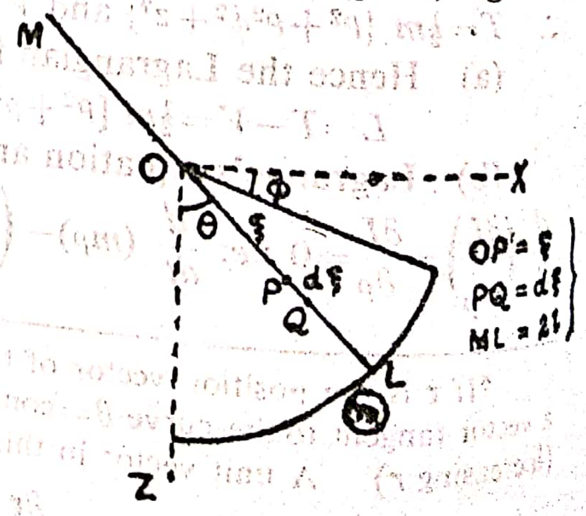
Solution. Here $V=V(\rho)$; $\Rightarrow \dfrac{\partial V}{\partial z}=0, \dfrac{\partial V}{\partial \phi}=0.$

$\therefore$ (1), (2), (3) of Ex. 12 $\Rightarrow m\,(\ddot{\rho}-\rho\dot{\phi}^2)=-\dfrac{\partial V}{\partial \rho}$

and $$\frac{d}{dt}\,(\rho^2\dot{\phi})=0.$$

Ex. 14. *A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to* $\sqrt{\left(\dfrac{2ng}{l}\right)}$. *Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is* $\cos^{-1}[\sqrt{(n^2+1)}-n]$ *and will then rise again.* **[Raj. 88, Agra 88]**

Solution. The mass m is attached at L. On the rod ML take a point P such that $OP=\xi$, the element $PQ=d\xi$. Further at any time t, let the plane through it and the vertical have turned through an angle ϕ from its initial position and let the rod be inclined at an angle θ to the vertical.



Taking O, the mid point of the rod, as the origin and OX, OY (a line perpendicular to the plane of the paper) and OZ as axis of reference, then co-ordinates of the point P on the rod are:

$$x=\xi\sin\theta\cos\phi,\; y=\xi\sin\theta\sin\phi,\; z=\xi\cos\theta$$

$\therefore \quad \dot{x}=\xi \cos\theta \cos\phi\,\dot{\theta}-\xi \sin\theta \sin\phi\dot{\phi},$

$\dot{y}=\xi \cos\theta \sin\phi\dot{\theta}+\xi \sin\theta \cos\phi\dot{\phi},\ \dot{z}=-\xi \sin\theta\dot{\theta}.$

Thus, $v_P^2=(\text{velocity})^2$ of $P=\dot{x}^2+\dot{y}^2+\dot{z}^2=\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$.

$\therefore\ v_L^2=l^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)=(\text{velocity})^2$ of mass m.

Now, mass of the element $PQ=\dfrac{3m}{2l}\,d\xi=dm$, say.

$\therefore$ Its kinetic energy

$$=\tfrac{1}{2}\,dm.v_P^2=\tfrac{1}{2}.\frac{3m}{2l}\,d\xi\,(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,\xi^2=\frac{3m}{4l}(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,\xi^2\,d\xi,$$

and K.E. of the rod$=\dfrac{3m}{4l}(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\displaystyle\int_{-l}^{l}\xi^2\,d\xi$

$$=\tfrac{1}{2}m\,(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\,l^2.$$

Again, $(\text{Velocity})^2$ of the particle $m=l^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$.

$\therefore$ Kinetic energy of the particle of mass m

$$=\tfrac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta).$$

$\therefore$ Total K.E.$=T=$K.E. of the rod$+$K.E. of the particle

$$=\tfrac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)+\tfrac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)=ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta).$$

Also the work function is given by $W=mgl\cos\theta+C$.

Lagrange's ϕ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\phi}}\right)-\dfrac{\partial T}{\partial\phi}=\dfrac{\partial W}{\partial\phi}$

which gives $\quad \dfrac{d}{dt}(2ml^2\dot{\phi}\sin^2\theta)=0.$

Integrating it we get $\dot{\phi}\sin^2\theta=K$ (constant).

Initially, $\quad \theta=\dfrac{\pi}{2}$ and $\dot{\phi}=\sqrt{\left(\dfrac{2ng}{l}\right)}$. $\quad\therefore\quad K=\sqrt{\left(\dfrac{2ng}{l}\right)}$

$$\Rightarrow\quad \dot{\phi}\sin^2\theta=\sqrt{\left(\frac{2ng}{l}\right)} \qquad \text{...(1)}$$

and Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right)-\dfrac{\partial T}{\partial\theta}=\dfrac{\partial W}{\partial\theta}$

i.e. $\quad \dfrac{d}{dt}(ml^2\dot{\theta})-2ml^2\dot{\phi}^2\sin\theta\cos\theta=-mgl\sin\theta.$

or

$$2l\ddot{\theta}-2l\dot{\phi}^2\sin\theta\cos\theta=-g\sin\theta. \qquad \text{...(2)}$$

Substituting value of $\dot{\phi}$ from (1) in (2), we have

$$2l\ddot{\theta}+4ng\cot\theta\operatorname{cosec}^2\theta=-g\sin\theta. \qquad \text{...(3)}$$

Integration provides us

$$2l\dot{\theta}^2+4ng\cot^2\theta=2g\cos\theta+k.$$

Initially $\quad \theta=\frac{\pi}{2},\ \dot{\theta}=0, \quad \therefore \quad k=0.$

$$\therefore \quad 2l\dot{\theta}^2+4ng\cot^2\theta=2g\cos\theta. \qquad \ldots(4)$$

The rod will fall till $\dot{\theta}=0$,

i.e. $\qquad 4ng\cot^2\theta=2g\cos\theta$

or $\qquad 2n\cos^2\theta-\cos\theta=\cos\theta\sin^2\theta=0.$

$\therefore$ either $\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$ which gives initial position,

or $\qquad 2n\cos\theta-\sin^2\theta=0 \Rightarrow \cos^2\theta=2n\cos\theta-1=0.$

Solving it, $\quad \cos\theta=\dfrac{-2n+\sqrt{(4n^2+4)}}{2}=-n+\sqrt{(n^2+1)}$

(the other value being inadmissible because θ cannot be obtuse)

or $\qquad \theta=\cos^{-1}[-n+\sqrt{(n^2+1)}].$

This proves the required result.

If we substitute this value of θ in equation (3), then we find that $\ddot{\theta}$ comes out to be positive.

Hence at that time the rod begins to rise.

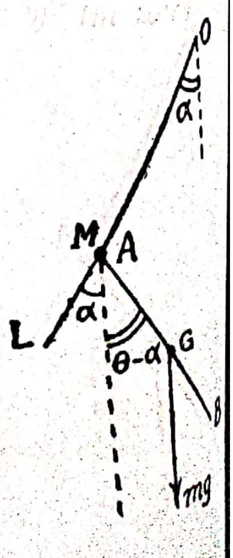
Ex. 15. *A bead, of mass M, slides on a smooth fixed wire, whose inclination to the vertical is α, and has hinged to it a rod, of mass m and length 21, which can move freely in the vertical plane through the wire. If the system starts from rest with the rod hanging vertically show that*

$$\{4M+m(1+3\cos^2\theta)\}\,l\dot{\theta}^2=6(M+m)\,g\sin\alpha\,(\sin\theta-\sin\alpha),$$

where θ is the angle between the rod and the lower part of the wire.

(Agra 1992)

Sol. Let OL be the fixed wire. At any time t, let the bead of mass M be at A where $OA=x$, also let θ be the angle which the rod AB makes with the lower part of the fixed wire.



Take O as origin and the fixed wire OL as x-axis ; then co-ordinates of G, the C.G. of the rod AB, are $\{x+l\cos\theta,\ l\sin\theta\}$,

i.e. $\qquad x_G=x+l\cos\theta$ and $y_G=l\sin\theta$.

$\therefore$ (Velocity)^2 of $G=v_G^2=\dot{x}_G^2+\dot{y}_G^2$

$$=(\dot{x}-l\sin\theta\,\dot{\theta})^2+(l\cos\theta\,\dot{\theta})^2.$$

Now let T be be kinetic energy and W the work function of the system. Then we easily get,

T=K.E. of the bead+K.E. of the rod,

$$=\tfrac{1}{2}M\dot{x}^2+\tfrac{1}{2}m\left[\frac{l^2}{3}\dot{\theta}^2+(\dot{x}-l\sin\theta\dot{\theta})^2+(l\cos\theta\dot{\theta})^2\right]$$

$$=\tfrac{1}{2}(M+m)\dot{x}^2-2ml\dot{x}\dot{\theta}\sin\theta+\tfrac{4}{3}ml^2\dot{\theta}^2.$$

Also the work function is given by

$$W=Mgx\cos\alpha+mg\{x\cos\alpha+l\cos(\theta-\alpha)\}$$
$$=(M+m)gx\cos\alpha+mgl\cos(\theta-\alpha).$$

$\therefore$ Lagrange's x-equation gives $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{x}}\right)-\dfrac{\partial T}{\partial x}=\dfrac{\partial W}{\partial x}$

i.e. $$\frac{d}{dt}[(M+m)\dot{x}-ml\dot{\theta}\sin\theta]=(M+m)g\cos\alpha$$

or $$(M+m)\ddot{x}-ml\ddot{\theta}\sin\theta-ml\dot{\theta}^2\cos\theta=(M+m)g\cos\alpha. \quad \text{...(i)}$$

Again Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial W}{\partial\theta}$$

i.e. $$\frac{d}{dt}[-ml\dot{x}\sin\theta+\tfrac{4}{3}ml^2\dot{\theta}]+ml\dot{x}\dot{\theta}\cos\theta=-mgl\sin(\theta-\alpha),$$

or $$-\ddot{x}ml\sin\theta-\dot{x}\dot{\theta}ml\cos\theta+\tfrac{4}{3}ml^2\ddot{\theta}+\dot{x}\dot{\theta}\,ml\cos\theta=-mgl\sin(\theta-\alpha),$$

or $$-\ddot{x}\sin\theta+\tfrac{4}{3}l\ddot{\theta}=-g\sin(\theta-\alpha) \quad \text{...(ii)}$$

Eliminating $\ddot{x}$ between (i) and (ii), we get

$$\ddot{\theta}\,[-ml\sin^2\theta+\tfrac{4}{3}(M+m)l]-ml\dot{\theta}^2\sin\theta\cos\theta$$
$$=(M+m)g[\cos\alpha\sin\theta-\sin(\theta-\alpha)],$$

or $$l\ddot{\theta}\,[4M+m+3m\cos^2\theta]-3ml\dot{\theta}^2\sin\theta\cos\theta$$
$$=3(M+m)g\cos\theta\sin\alpha.$$

Whence on integrating, we get

$$l\dot{\theta}^2[4M+m+3m\cos^2\theta]=6(M+m)g\sin\alpha\sin\theta+C. \quad \text{...(iii)}$$

When $\theta=\alpha$, $\dot{\theta}=0$. $\therefore$ $C=-6(M+m)g\sin^2\alpha$.

Putting the value of C in (ii), we get

$$l\dot{\theta}^2(4M+m+3m\cos^2\theta)=6(M+m)g\sin\alpha(\sin\theta-\sin\alpha).$$

3·14. Theory of small oscillations of conservative Holonomic Dynamical Systems.

Let us assume $q_\alpha=\beta_\alpha$ $(\alpha=1, 2, ..., n)$, where

$$V_0=V(\beta_1, \beta_2, ..., \beta_n).$$

Now let the system under go a small displacement from equilibrium so that at time t, $q_\alpha=\beta_\alpha+\xi_\alpha$ $(\alpha=1, ..., n)$ where ξ_α and ξ_α are very small.

But $$\frac{\partial V}{\partial q_\alpha}=0 \text{ for } q_\alpha=\beta_\alpha \ (\alpha=1,\ldots, n),$$

hence by Taylor's theorem, we get

$$V=V_0+\tfrac{1}{2}\left\{\left(\xi_1 \frac{\partial}{\partial q_1}+\ldots+\xi_n \frac{\partial}{\partial q_n}\right)^2 V\right\}$$

and second order, the zero signifying that we put

$$q_\alpha=\beta_\alpha \ (\alpha=1, 2,\ldots, n)$$

after performing differentiations.

So V takes the form as a quadratic function of the ξ's (to the second order)

i.e. $V=V_0+\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} C_{rs}\xi_r\xi_s$ where $C_{rs}=C_{sr}=\left(\frac{\partial^2 V}{\partial q_r\, \partial q_s}\right)=\text{constant}$ ...(42)

we also have $T=\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{q}_r\dot{q}_s=\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}\dot{\xi}_r\dot{\xi}_s.$ ...(42′)

Now let $\overset{*}{a}_{rs}$ be the value of each a_{rs} in the equilibrium configuration $q_\alpha=\beta_\alpha$.

Then $T=\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} \overset{*}{a}_{rs}\,\dot{\xi}_r\dot{\xi}_s$ where $\overset{*}{a}_{rs}=\overset{*}{a}_{sr}$

(to the 2nd order)

$\therefore$ $L=T-V=\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}(\overset{*}{a}_{rs}\dot{\xi}_r\dot{\xi}_s-C_r\xi_r\xi_s)-V_0.$ ...(42″)

where each a_{rs} and each C_{rs} is constant.

Thus Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_\alpha}\right)-\frac{\partial L}{\partial \xi_\alpha}=0 \ (\alpha=1, 2,\ldots, n). \qquad \ldots(43)$$

From (42″), we have $\frac{\partial L}{\partial \dot{\xi}_\alpha}=\sum_{s=1}^{n}\overset{*}{a}_{\alpha s}\dot{\xi}_s$;

and $$\frac{\partial L}{\partial \xi_\alpha}=-\tfrac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} C_{rs}\left(\xi_r\frac{\partial \xi_s}{\partial \xi_\alpha}+\xi_s\frac{\partial \xi}{\partial \xi_\alpha}\right)$$

$$=-\tfrac{1}{2}\sum_{r=1}^{n} C_{r\alpha}\xi_r-\tfrac{1}{2}\sum_{s=1}^{n} C_{\alpha s}\xi_s=-\sum_{s=1}^{n} C_{\alpha s}\xi_s.$$

$\therefore$ (43) $\Rightarrow \sum_{r=1}^{n}(\overset{*}{a}_{\alpha s}\ddot{\xi}_s+C_{\alpha s}\xi_s)=0$ $(\alpha=1, 2, \ldots, n)$

$$= \sum_{s=1}^{n} (\overset{*}{a}_{\alpha s} D_s^2 + C_{\alpha s})\ \xi_s = 0 \text{ where } D\xi_s = \frac{\partial \xi_s}{\partial t} \qquad \ldots(44)$$

In order to solve (44), we try to find constants M_s $(s=1, 2, \ldots, n)$ such that $\xi_r = M_s \sin(\omega t+\epsilon)$ $[(s=1, \ldots, n)]$ is a solution.

We have $\ddot{\xi}_s = -\omega^2 M_s \sin(\omega t+\epsilon)$,

i.e. $\quad D^2\xi_s = -\omega^2 M_s \sin(\omega t+\epsilon)$.

Substituting these values in (44), we get

$$\sum_{s=1}^{n} (-\overset{*}{a}_{\alpha s}\, \omega^2 M_s + C_{\alpha s} M_s) \sin(\omega t+\epsilon) = 0 \quad (\alpha = 1, 2, \ldots, n)$$

$$\Rightarrow \sum_{s=1}^{n} (\overset{*}{a}_{\alpha s} \omega^2 - C_{\alpha s})\, M = 0 \qquad (\alpha = 1, 2, \ldots, n) \qquad \ldots(45)$$

Eliminating M_s from (45), we get

$$\det \,|\, \overset{*}{a}_{\alpha s}\omega^2 - C_{\alpha s} \,| = 0 \qquad (\alpha, s = 1, \ldots, n), \qquad \ldots(46)$$

which is the equation of n^{th} degree for ω^2.

Let the n roots be $\omega^2 = \omega_s^2$. $\quad (s=1, \ldots, n)$.

If the n roots of the above equation are positive, we obtain n periodic expression for the ξ_s implying that each corresponding configuration is stable. A negative root would imply a periodic solution showing the corresponding configuration to be unstable. When n roots of (46) are all positive than we have seen that the equilibrium is stable. This will lead to the n periods of oscillation $\frac{2\pi}{\omega_s}$ $(s=1, \quad, n)$, which are called as *normal periods of oscillation.*

From (45), we can derive the ratios of the amplitudes

$$M_1 : M_2 : \ldots : M_n.$$

For if, when $\omega = \omega_s$, the cofactors of any row or column in the determinant on the L.H.S. of (46), are

$$C_1(\omega_s),\ C_2(\omega_s),\ \ldots,\ C_n(\omega_s).$$

Then we shall have

$$\frac{M_1}{C_1(\omega_s)} = \frac{M_2}{C_2(\omega_s)} = \ldots = \frac{M_n}{C_n(\omega_s)} = a_s \text{ (say)}. \qquad \ldots(47)$$

The solutions for $\omega = \omega_s$ are given by

$\xi_1 = C_1(\omega_s)\, a_s \sin(\omega_2 t+\epsilon_s)$, $\xi_2 = C_2(\omega_s)\, a_s \sin(\omega_s t+\epsilon_s)$. $\qquad \ldots(48)$

But the differential equations (44) are linear so the general solutions are obtained by the method of superposition giving

$$\xi_1 = \sum_{s=1}^{n} a_s C_1(\omega_s) \sin(\omega_s t + \epsilon_s), \quad \xi_2 = \sum_{s=1}^{n} a_s C_2(\omega_s) \sin(\omega_s t + \epsilon_s) \ldots,$$

$$\xi_n = \sum_{s=1}^{n} a_s C_n(\omega_s) \sin(\omega_s t + \epsilon_s). \qquad \ldots(49)$$

Alternative approach.

We have

$$T = \tfrac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} {}^{*}a_{rs} \dot{\xi}_r \dot{\xi}_s; \; V - V_0 = \tfrac{1}{2} \sum_{r=1}^{n} \sum_{s=1}^{n} C_{rs} \xi_r \xi_s.$$

Obvious T and $V - V_0$ both are homogeneous quadratic forms, the first being in $\dot{\xi}_r$ and the second in the co-ordinates ξ_r.

Now by linear algebra, we know that in general one can transform the co-ordinates ξ_r $(r = 1, 2, \ldots, n)$ to a new system of co-ordinates $\eta_1, \eta_2, \ldots, \eta_n$ by making use of

$$\xi_r = b_{r1} \eta_1 + b_{r2} \eta_2 + \ldots + b_{rn} \eta_n \; (r = 1, \ldots, n). \qquad \ldots(50)$$

Now with suitable choice of b's, T and $V - V_0$ take the following forms :

$$T = \tfrac{1}{2} (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \ldots + \dot{\eta}_n^2), \; V - V_0 = \tfrac{1}{2} (\gamma_1 \eta_1^2 + \gamma_2 \eta_2^2 + \ldots + \gamma_n \eta_n^2). \qquad \ldots(51)$$

Thus $L = T - V$

$$= \tfrac{1}{2} (\dot{\eta}_1^2 + \ldots + \dot{\eta}_n^2) - \tfrac{1}{2} (\gamma_1 \eta_1^2 + \gamma_2 \eta_2^2 + \ldots + \gamma_n \eta_n^2) - V_0$$

$$= \tfrac{1}{2} \sum_{r=1}^{n} (\dot{\eta}_r^2 - \gamma_r \eta_r^2) - V_0.$$

Hence Lagrange's equation for the new generalised co-ordinates $\eta_1, \ldots, \eta_n$ are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}_r}\right) - \frac{\partial L}{\partial \eta_r} = 0 \; (r = 1, 2, \ldots, n)$$

i.e. by

$$\ddot{\eta}_r + \mu_r \eta_r = 0 \qquad (r = 1, 2, \ldots, n). \qquad \ldots(52)$$

Now let $\mu_r = \omega_r^2$ so (52) $\Rightarrow$ $(D^2 + \omega_r^2)\, \eta_r = 0 \qquad \ldots(53)$

$(r = 1, \ldots, n)$.

$\therefore \quad \eta_r = A_r \sin(\omega_r t + \epsilon_r) \quad (r = 1, 2, \ldots, n)$.

The co-ordinates $\eta_1, \eta_2, \ldots, \eta_n$ are called the **normal co-ordinates** of the system.

Having determined the values of $\eta_1, \eta_2, \ldots, \eta_n$ we can obtain the solution of the original co-ordinates ξ_r $(r=1, \ldots, n)$.

Normal modes of vibration.

We have $\xi_r = b_{r_1} \eta_1 + b_{r_2} \eta_2 + \ldots + b_{r_n} \eta_n$ $(r=1, 2, \ldots, n)$,

which is linear super-position of the normal co-ordinates.

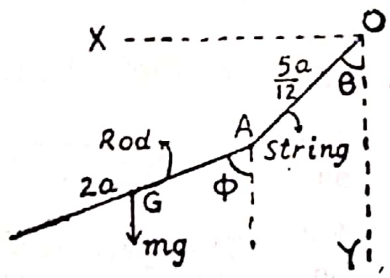
$(\eta_1, \eta_2, \ldots, \eta_n)$.

If the $(n-1)$ normal co-ordinates $\eta_1, \eta_2, \ldots, \eta_{r-1}, \eta_{r+1}, \ldots, \eta_n$ are all permanently zero for a system vibrating about a configuration of stable equilibrium. But on the other hand if $\eta_r \neq 0$, then the system is said to execute a *normal mode of vibration.*

Ex. 16. *A uniform rod of length 2a, which has one end attached to a fixed point by a light inextensible string, of length $\frac{5}{12}a$, is performing small oscillations in a vertical plane about its position of equilibrium. Find the position at any time, and show that the period of its principal oscillations are*

$$2\pi \sqrt{\left(\frac{5a}{3g}\right)} \quad \textit{and} \quad \pi \sqrt{\left(\frac{a}{3g}\right)}.$$

Solution. Figure is self explanatory.



At any time t, let the string and the rod be inclined at θ and ϕ to the vertical OY.

Co-ordinates of G are given by

$x_G = \frac{5}{12} a \sin\theta + a \sin\phi$,

$y_G = \frac{5}{12} a \cos\theta + \cos\phi$.

$\therefore \quad \dot{x}_G^2 + \dot{y}_G^2 = (\text{velocity})^2$ of G

$$= \frac{25}{144} a^2 \dot{\theta}^2 + a^2 \dot{\phi}^2 + \frac{5a^2}{6} \dot{\theta}\dot{\phi} \cos(\theta - \phi)$$

$$= \frac{25}{144} a^2 \dot{\theta}^2 + a^2 \dot{\phi}^2 + \frac{5}{6} a^2 \dot{\theta}\dot{\phi} \quad [\because \ \theta \text{ and } \phi \text{ are small}]$$

Again let T, be the kinetic energy and W, the work function of the system, then we easily get

$$T = \tfrac{1}{2} m \left[\frac{a^2}{3} \dot{\phi}^2 + \left(\frac{25}{144} a^2 \dot{\theta}^2 + a^2 \dot{\phi}^2 + \tfrac{5}{6} a^2 \dot{\theta}\dot{\phi} \right) \right]$$

$$= \frac{ma^2}{288} \left[25\dot{\theta}^2 + 192\dot{\phi}^2 + 120\dot{\theta}\dot{\phi} \right],$$

and $$W = mg \left[\frac{5}{12} a \cos\theta + a \cos\phi \right].$$

$\therefore$ Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right)-\frac{\partial T}{\partial \theta}=\frac{\partial W}{\partial \theta} \Rightarrow \frac{d}{dt}\left\{\frac{ma^2}{144}(25\dot\theta+50\dot\phi)\right\}=-\frac{5mga}{12}\theta$$

$\{\sin\theta=\theta$ as θ is small$\}$

$$\Rightarrow 5\ddot\theta+12\ddot\phi=-\frac{12g}{a}\theta \qquad \text{...(1)}$$

and Lagrange's ϕ equation gives

$$\frac{d}{dt}\left\{\frac{ma^2}{144}(192\dot\phi+60\dot\theta)\right\}=-mga\phi \Rightarrow 5\ddot\theta+16\ddot\phi=-12\frac{g}{a}\phi. \qquad \text{...(2)}$$

Equations (1) and (2) $\Rightarrow$ $(5D^2+12c)\,\theta+12D^2\,\phi=0$ ...(3)

and $(5D^2\theta+16D^2+12c)\,\phi=0$ where $(g/a)=c.$ ...(4)

Now eliminating ϕ between two equations, we get

$$[(5D^2+12c)\,(16D^2+12c)-60D^4]\,\theta=0 \qquad \text{...(5)}$$

or $$(5D^4+63cD^2+36c^2)\,\theta=0.$$

Let $\theta=A\cos(pt+B) \Rightarrow \therefore\ D\theta=-pA\sin(pt+B),$

$D^2\theta=-p^2A\cos(pt+B)=-p^2\theta$ and $D^4\theta=p^4\theta.$

Substituting these values in (5), we get

$(5p^4-63cp^2+36c^2)\,\theta=0 \Rightarrow (5p^4-63cp^2+36c^2)=0$ $(\because\ \theta\neq 0)$

$$\Rightarrow (5p^2-3c)\,(p^2-12c)=0 \Rightarrow \left(5p^2-\frac{3g}{a}\right)\left(p^2-\frac{12g}{a}\right)=0$$

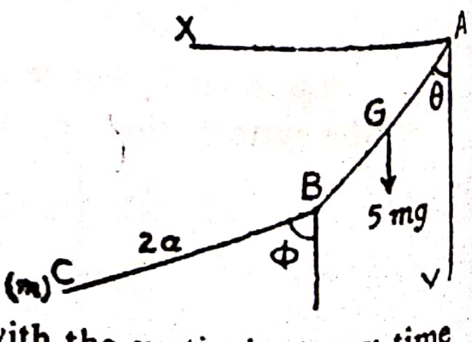
$$\therefore\ p_1^2=\frac{3g}{5a} \text{ and } p_2^2=\frac{12g}{a}$$

Thus periods of oscillations are $\dfrac{2\pi}{p_1}$ and $\dfrac{2\pi}{p_2}$

i.e. $2\pi\sqrt{\left(\dfrac{5a}{3g}\right)}$ and $2\pi\sqrt{\left(\dfrac{a}{12g}\right)}$ *i.e.* $2\pi\sqrt{\left(\dfrac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\dfrac{a}{3g}\right)}$

Ex. 17. *A uniform rod, of mass 5m and length 2a turns freely about one end which is fixed, to its other extremity is attached one end of a light string, of length 2a, which carries at its other end a particle of mass m, show that the periods of the small oscillations in a vertical plane are the same as those of simple pendulums of lengths* $\dfrac{2a}{3}$ *and* $\dfrac{20a}{7}$.

(Ranchi 93)



Solution. Let the string BC and the rod AB make angles ϕ and θ with the vertical at any time t. The particle of mass m is tied to the end C of the string.

Now $x_c=2a\sin\theta+2a\sin\phi,\ y_c=2a\cos\theta+2a\cos\phi.$

$\therefore$ (velocity)^2 of $m = \dot{x}_c^2 + \dot{y}_c^2$
$$= 4a^2 (\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}).$$

Again co-ordinates of G are $(a \sin \theta, a \cos \theta)$.
$\therefore$ (velocity)^2 of $G = a^2\dot{\theta}^2$.

Now let T be the kinetic energy and W the work function of the system, then we have

$$T = \tfrac{1}{2}.5m\left(\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \tfrac{1}{2}m.4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$
$$= ma^2\left(\tfrac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\right)$$

and $\quad W = 5mga \cos \theta + mg.2a(\cos \theta + \cos \phi)$

$$= 7mg \cos \theta + 2mga \cos \phi = 7mag\left(1 - \frac{\theta^2}{2}\right) + 2mag\left(1 - \frac{\phi^2}{2}\right)$$

$\therefore$ Lagrange's θ equation is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta} \Rightarrow \frac{d}{dt}\left(\frac{32}{3}\dot{\theta} + 4\dot{\phi}\right) = -\frac{7g}{a}\theta$$

$$\Rightarrow 32\ddot{\theta} + 12\ddot{\phi} = -21\frac{g}{a}\theta. \qquad \ldots(1)$$

Lagrange's ϕ-equation is given by

$$\frac{d}{dt}(4\dot{\phi} + 4\dot{\theta}) = -\frac{2g}{a}\phi \Rightarrow 2\ddot{\theta} + 2\ddot{\phi} = -\frac{g}{a}\phi \qquad \ldots(2)$$

$\therefore$ (1) and (2) $\Rightarrow (32D^2 + 21c)\,\theta + 12D^2\phi = 0 \qquad \ldots(3)$

and $\quad 2D^2\theta + (2D^2 + c)\,\phi = 0$ where $\dfrac{g}{a} = c. \qquad \ldots(4)$

Now eliminating "ϕ" between (3) and (4), we get

$$[(32D^2 + 21c)(2D^2 + c) - 24D^2]\,\theta = 0$$
$$\Rightarrow (40D^4 + 74cD^2 + 21c^2)\,\theta = 0. \qquad \ldots(5)$$

Now let $\theta = A \cos (pt + B) \Rightarrow D\theta = -pA \sin (pt + B)$.
$D^2\theta = -p^2A \cos (pt + B) = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting these in (5), we get

$(40p^4 - 74cp^2 + 21c^2)\,\theta = 0$ *i.e.* $40p^4 - 74cp^2 + 21c^2 = 0$ as $\theta \neq 0$

or $\quad (2p^2 - 3c)(20p^2 - 7c) = 0$ *i.e.* $\left(2p^2 - \dfrac{3g}{a}\right)\left(20p^2 - 7\dfrac{g}{a}\right) = 0.$

$$\Rightarrow p_1^2 = \frac{3g}{2a} \text{ and } p_2^2 = \frac{7g}{20a}.$$

*Hence length of equivalent pendulums are $\dfrac{g}{p_1^2}$ and $\dfrac{g}{p_2^2}$.

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*When $\theta = A \cos (pt + B)$, the period of motion is given by $T = \dfrac{2\pi}{p}$; if l is the length of the simple equivalent pendulum, we have $T = 2\pi\sqrt{\left(\dfrac{l}{g}\right)}$

Hence $\quad l = \dfrac{g}{p^2}$

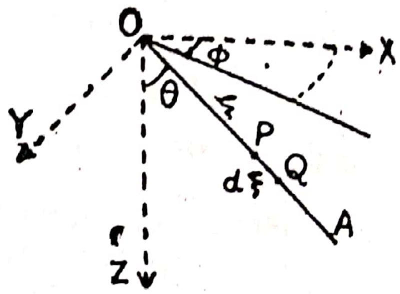
i.e. $\frac{2a}{3}$ and $\frac{20}{7}a$.

Ex. 18. *A uniform rod, of length 2a can turn freely about one end, which is fixed. Initially it is inclined at an angle α, to the downward drawn vertical and it is set rotating about a vertical axis through its fixed end with angular velocity ω. Show that, during the motion, the rod is always inclined to the vertical at an angle which is $>$ or $<$ α, according as $\omega^2 > \text{or} < \frac{3g}{4a\cos\alpha}$ and that in each case its motion is included between the inclinations α and $\cos^{-1}[-n+\sqrt{(1-2n\cos\alpha+n^2)}]$, where $n=\frac{a\omega^2\sin^2\alpha}{3g}$.*

If it be slightly disturbed when revolving steadily at a constant angle α, show that the time of a small oscillation is

$$2\pi\sqrt{\left[\frac{4a\cos\alpha}{3g(1+3\cos^2\alpha)}\right]}.$$

Solution. The rod OA is turning about the end O. Take a point P on the rod such that $OP=\xi$, and the element $PQ=d\xi$.



$\therefore$ mass of element $PQ=\frac{m}{2a}d\xi$.

where m is the mass of the rod.

Further at any time t, let the rod be inclined at an angle θ to the vertical, and let the plane through the rod and the vertical have turned through on angle ϕ from its initial position OX, then co-ordinates of the point P are

$$x_P=\xi\sin\theta\cos\phi,\ y_P=\xi\sin\theta\sin\phi,\ z_P=\xi\cos\theta.$$

$\therefore$ v_P^2 (Velocity)^2 of $P=\dot{x}_P^2+\dot{y}_P^2+\dot{z}_P^2=\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$.

and kinetic energy of the element PQ

$$=\tfrac{1}{2}\frac{m}{2a}v_P^2=\tfrac{1}{2}\frac{m}{2a}d\xi\,(\dot{\theta}^2+\dot{\phi}^2\sin\theta)\,\xi^2.$$

Now let, T, be the K.E. of the rod OA, then we have

$$T=\tfrac{1}{2}\frac{m}{2a}(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\int_0^{2a}\xi\,d\xi=\frac{2ma^2}{3}(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$$

$$=\frac{2ma^2}{3}.(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta).$$

Also the work function $W=mga\cos\theta+C$.

Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\phi}\sin^2\theta\right)=0 \text{ i.e., } \frac{d}{dt}[\dot{\phi}\sin^2\theta]=0 \quad ...(1)$$

$$\Rightarrow \dot{\phi}\sin^2\theta=K \text{ (constant).} \quad ...(2)$$

Initially $\theta=\alpha$, $\dot{\phi}=\omega$, $\therefore \quad K=\omega\sin^2\alpha$ ...(3)

Thus (2) gives $\qquad \dot{\phi}\sin^2\theta=\omega\sin^2\alpha$

and Lagrange' θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right)-\dfrac{\partial T}{\partial\theta}=0,$

$$\Rightarrow \frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\theta}\right)-\frac{2ma^2}{3}\dot{\phi}^2.2\sin\theta\cos\theta=-mga\sin\theta,$$

$$\Rightarrow \ddot{\theta}-\dot{\phi}^2\sin\theta\cos\theta=-\frac{g}{4a}\sin\theta. \quad ...(4)$$

Eliminating $\dot{\phi}$ between (3) and (4), we have

$$\ddot{\theta}-\frac{\omega^2\sin^4\alpha}{\sin^3\theta}\cos\theta=-\frac{3g}{4a}\sin\theta. \quad ...(5)$$

$$\Rightarrow \dot{\theta}^2+\frac{\omega^2\sin^4\alpha}{\sin^2\theta}=\frac{3g}{2a}\cos\theta+A. \quad ...(6)$$

Initial $\theta=\alpha$, $\dot{\theta}=0$, $\therefore \quad A=\omega^2\sin^2\alpha-\dfrac{3g}{2a}\cos\alpha,$

Substituting this value of A in (6), we get

$$\dot{\theta}^2+\frac{\omega^2\sin^4\alpha}{\sin^2\theta}=\frac{3g}{2a}\cos\theta+\omega^2\sin^2\alpha-\frac{3g}{2a}\cos\alpha.$$

or

$$\dot{\theta}^2=\omega^2\sin^2\alpha\left(1-\frac{\sin^2\alpha}{\sin^2\theta}\right)+\frac{3g}{2a}(\cos\theta-\cos\alpha)$$

$$=\frac{3ng}{a}\left(1-\frac{\sin^2\alpha}{\sin^2\theta}\right)+\frac{3g}{2a}(\cos\theta-\cos\alpha) \quad \left[\because n=\frac{a\omega^2\sin^2\alpha}{3g}\right]$$

$$=\frac{3g}{2a}\frac{\cos\alpha-\cos\theta}{\sin^2\theta}[2n(\cos\alpha+\cos\theta)-\sin^2\theta]$$

i.e., $\dot{\theta}^2=\dfrac{3g}{2a}\cdot\dfrac{\cos\alpha-\cos\theta}{\sin^2\theta}[\cos^2\theta+2n\cos\theta+2n\cos\alpha-1].$ ...(7)

From (7), we see that $\dot{\theta}=0$ when

$$(\cos\alpha-\cos\theta)[\cos^2\theta+2n\cos\theta+2n\cos\alpha-1]=0,$$

i.e., if either $\quad \cos\alpha-\cos\theta=0 \quad$ i.e., $\quad \theta=\alpha$ (the initial position)

or $\quad \cos^2\theta+2n\cos\theta+2n\cos\alpha-1=0,$

i.e., $\quad \cos\theta=\dfrac{-2n\pm\sqrt{[4n^2+4(1-2n\cos\alpha)]}}{2}$ ...(8)

or $\quad \cos\theta=-n+\sqrt{(1-2n\cos\alpha+n^2)},$

(the other value being inadmissible because that gives values of $\cos\theta$ numerically greater than unity).

Hence the motion is included between $\theta=\alpha$ and $\theta=\theta_1$ where
$$\cos\theta_1=\{\sqrt{(1-2n\cos\alpha+n^2)}-n\}.$$

The rod will move above or below its initial position if
$$\theta_1 > \text{or} < \alpha \quad \text{or} \quad \text{if } \cos\theta_1 < \text{or} > \cos\alpha$$
i.e., if
$$1-2n\cos\alpha+n^2 < \text{or} > (n+\cos\alpha)^2$$
i.e., if $\dfrac{3ng}{a\omega^2} < \text{or} > 4n\cos\alpha$, *i.e.*, if $\omega^2 > \text{or} < \dfrac{3g}{4a\cos\alpha}$.

Second Part.

Small oscillations about the steady motion. The motion will be steady if the rod goes round inclined at the same angle α with the vertical or mathematically if $\theta=\alpha$ (throughout the motion), then $\ddot{\theta}=0$.

Making these substitutions in (5), we get
$$=\frac{\omega^2\sin^4\alpha}{\sin^3\theta}\cos\theta=-\frac{3g}{4a}\sin\theta \ \ i.e.\ \ \omega^2=\frac{3g}{4a\cos\alpha}.$$

When ω^2 has this value and there are small oscillations about the position $\theta=\alpha$, then putting $\theta=\alpha+\psi$ in equation (5), we get
$$\ddot{\psi}=\frac{3g}{4a\cos\alpha}\,\frac{\sin^4\alpha}{\sin^3(\alpha+\psi)}\cos(\alpha+\psi)-\frac{3g}{4a}\sin(\alpha+\psi)$$
$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha\,(\cos\alpha\cos\psi-\sin\alpha\sin\psi)}{\cos\alpha\,(\sin\alpha\cos\psi-\cos\alpha\sin\psi)^3}-(\sin\alpha\cos\psi+\cos\alpha\sin\psi)\right]$$
$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha\,(\cos\alpha-\psi\sin\alpha)}{\cos\alpha\,(\sin\alpha+\psi\cos\alpha)^3}-(\sin\alpha+\psi\cos\alpha)\right]$$
approximately
$$=\frac{3g\sin\alpha}{4a}\,[(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3}-(1+\cot\alpha)]$$
approximately
$$=-\frac{3g\sin\alpha}{4a}(4\cot\alpha+\tan\alpha)\,\psi \text{ approximately}$$
$$=-[3g\,(1+3\cos^2\alpha)/4a\cos\alpha]\,\psi=-\mu\psi \text{ say}$$
$\Rightarrow$ Time of small oscillation$=[2\pi/\sqrt{\mu}]$.

3·15. Lagrange's Equations with Impulsive Forces.
When the forces are finite, we have by 3·7 eq. (28)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\frac{\partial T}{\partial q_\alpha}=\phi_\alpha \text{ where } \phi_\alpha=\sum_\nu \mathbf{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \qquad \ldots(54)$$

Integrating both sides of (1) w.r.t. "t" from $t=0$, $t=\tau$, we get

$$\left[\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)\right]_{t=0}^{t=\tau}-\int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt=\int_{t=0}^{\tau}\phi_\alpha\,dt$$

$$=\sum_\nu\left\{\left(\int_{t=0}^{\tau}\mathbf{F}_{\nu\alpha}dt\right)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\}$$

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau}-\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0}-\int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt=\sum_\nu\left\{\left(\int_0^\tau \mathbf{F}_\nu\cdot dt\right)\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\}. \qquad \ldots(55)$$

Taking the limit as $\tau\to 0$, we get

$$\lim_{\tau\to 0}\left\{\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau}-\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0}\right\}-\lim_{\tau\to 0}\int_0^\tau\frac{\partial T}{\partial q_\alpha}\,dt$$

$$=\sum_\nu\left\{\left(\lim_{\tau\to 0}\int_0^\tau \mathbf{F}_\nu\,dt\right)\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right\}$$

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_2-\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1=\Sigma^*\,\mathbf{I}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}=\dagger\,P_\nu,\ \text{(say)} \qquad \ldots(56)$$

where subscripts 1 and 2 denote respectively quantities before and after the application of the Impulsive forces.

These equations are known as Lagrange's equations under impulsive forces.

Ex. 19. *A square ABCD formed by four rods of length 2l and mass m hinged at their ends, rests on horizontal frictionless table. An impulse of magnitude I is applied to the vertex A in the direction AD. Find the equations of motion, and prove that the K.E.*

---

*Suppose that the force $\mathbf{F}_\nu$ acting on a system are such that

$$\lim_{\tau\to 0}\int_0^\tau \mathbf{F}_\nu\,dt=\mathbf{I}_\nu$$

where τ represents a time interval. Then we call $\mathbf{F}_\nu$ impulsive forces and $\mathbf{I}_\nu$ are called impulses.

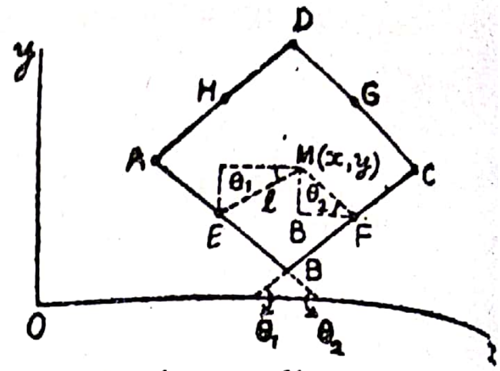
† If we call P_α to be the generalised impulse, (56) can be written as

Generalised impulse=Change in generalised momentum.

developed immediately after the application of the impulsive forces is

$$T = I^2/2m.$$

Solution. When the square is struck, its shape will in general be a rhombus. Suppose that at any time t, the angles made by side AD (or BC) and AB or (CD) with the x-axis are θ_1 and θ_2 respectively, while the co-ordinates of the centre M are (x, y). Hence α, y, θ_1, θ_2 are the generalised co-ordinates.



From the adjoining diagram, we see that the position vectors of the centres E, F, G, H of the rods are given respectively by

$$\mathbf{r}_E = (x - l\cos\theta_1)\,\mathbf{i} + (y - l\sin\theta_1)\,\mathbf{j},$$

$$\mathbf{r}_F = (x + l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_2)\,\mathbf{j}$$

$$\mathbf{r}_G = (x + l\cos\theta_1)\,\mathbf{i} + (y + l\sin\theta_1)\,\mathbf{j}$$

and $$\mathbf{r}_H = (x - l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_2)\,\mathbf{j}.$$

$\therefore$ $$\mathbf{v}_E = \dot{\mathbf{r}}_E = (\dot{x} + l\sin\theta_1\dot{\theta}_1)\,\mathbf{i} + (y - l\cos\theta_1\dot{\theta}_1)\,\mathbf{j}$$

$$\mathbf{v}_E = \mathbf{r}_F = (\dot{x} - l\sin\theta_2\dot{\theta}_2)\,\mathbf{i} + (y + l\cos\theta_2\dot{\theta}_2)\,\mathbf{j}$$

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = (\dot{x} - l\sin\theta_1\dot{\theta}_1)\,\mathbf{i} + (y + l\cos\theta_1\dot{\theta}_1)\,\mathbf{j}$$

$$\mathbf{v}_H = \mathbf{r}_H = (\dot{x} + l\sin\theta_2\dot{\theta}_2)\,\mathbf{i} + (y + l\cos\theta_1\,\dot{\theta}_2)\,\mathbf{j}.$$

Now,

K.E. of the rod $AB = \frac{1}{2}m\,\dot{\mathbf{r}}_E^2 + \frac{1}{3}ml^2\dot{\theta}_2^2\cdot\frac{1}{2} = T_{AB}$ say

K.E. of the rod $CB = \frac{1}{2}m\,\dot{\mathbf{r}}_F^2 + \frac{1}{3}ml^2\dot{\theta}_1^2\cdot\frac{1}{2} = T_{CB}$ say

K.E. of the rod $CD = \frac{1}{2}m\,\mathbf{r}_G^2 + \frac{1}{3}ml^2\theta_2^2\cdot\frac{1}{2} = T_{CD}$ say

and K.E. of the rod $DA = \frac{1}{2}m\,\dot{\mathbf{r}}_H^2 + \frac{1}{3}ml^2\dot{\theta}_1^2\cdot\frac{1}{2} = T_{DA}$ say

$\therefore$ K.E. of the system

$$= \tfrac{1}{2}m\,(\dot{\mathbf{r}}_E^2 + \dot{\mathbf{r}}_F^2 + \dot{\mathbf{r}}_G^2 + \mathbf{r}_H^2) + \tfrac{2}{3}ml^2\,(\dot{\theta}_1^2 + \dot{\theta}_2^2)\cdot\tfrac{1}{2} = T \text{ say}$$

or $$T = \tfrac{1}{2}m\,(4\dot{x}^2 + 4\dot{y}^2 + 2l^2\dot{\theta}_1^2 + 2l\dot{\theta}_2^2) + \frac{ml^2}{3}\,(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

$$= 2m\,(\dot{x}^2 + \dot{y}^2) + \tfrac{4}{3}ml^2\,(\dot{\theta}^2{}_1 + \dot{\theta}_2^2). \qquad \ldots(1)$$

Let us now assume that initially the rhombas is a square at rest with its sides parallel to the co-ordinate axes and its centre located at the origin. Then we get

$$x=0=y,\ \theta_1=\frac{\pi}{2},\ \theta_2=0,\ \dot{x}=0,\ \dot{y}=0,\ \dot{\theta}_1=0,\ \dot{\theta}_2=0.$$

Now if we use the notation $(\ )_1$ and $(\ )_2$ to denotes quantities before and after the impulse is applied, we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1=(4m\dot{x})_1=0,\ \left(\frac{\partial T}{\partial \dot{y}}\right)_1=(4m\dot{y})_1=0,$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1=\left(\frac{8}{3}\,ml^2\dot{\theta}_1\right)_1=0,\ \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_1=\left(\frac{8}{3}\,ml^2\dot{\theta}_2\right)_1=0$$

and

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2=(4m\dot{x})_2=4m\dot{x}.\ \left(\frac{\partial T}{\partial \dot{y}_2}\right)_2=(4m\dot{y})_2=4m\dot{y},$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2=\left(\frac{8}{3}\,ml^2\dot{\theta}_1\right)_2=\frac{8}{3}\,ml^2\dot{\theta}_1,\ \left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2=\frac{8}{3}\,ml^2\dot{\theta}_2$$

Hence Lagrange's equations are

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2-\left(\frac{\partial T}{\partial \dot{x}}\right)_1=P_x \Rightarrow 4m\dot{x}=P_x \qquad ...(2)$$

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2-\left(\frac{\partial T}{\partial \dot{y}}\right)_1=P_y \Rightarrow 4m\dot{y}=P_y \qquad ...(3)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_2-\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)_1=P_{\theta_1} \Rightarrow \frac{8}{3}\,ml^2\dot{\theta}_2=P_{\theta_1} \qquad ...(4)$$

$$\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)_2-\left(\frac{\partial T}{\partial \ddot{\theta}_1}\right)_1=P_{\theta_2} \Rightarrow \frac{8}{3}\,ml^2\dot{\theta}_2=P_{\theta_2}. \qquad ...(5)$$

Now we shall find the value of P_x, P_y, P_{θ_1} and P_{θ_2}.

We have $P_\alpha=\sum_\nu \mathbf{I}_y \cdot \dfrac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$ where $\mathbf{I}_\nu$ are the impulsive forces.

$$\therefore\ P_x=\mathbf{I}_A\cdot\frac{\partial \mathbf{r}_A}{\partial x}+\mathbf{I}_B\cdot\frac{\partial \mathbf{r}_B}{\partial x}+\mathbf{I}_C\cdot\frac{\partial \mathbf{r}_C}{\partial x}+\mathbf{I}_D\cdot\frac{\partial \mathbf{r}_D}{\partial x} \qquad ...(6)$$

$$P_y=\mathbf{I}_A\cdot\frac{\partial \mathbf{r}_A}{\partial y}+\mathbf{I}_B\cdot\frac{\partial \mathbf{r}_B}{\partial y}+\mathbf{I}_C\cdot\frac{\partial \mathbf{r}_C}{\partial y}+\mathbf{I}_D\cdot\frac{\partial \mathbf{r}_D}{\partial y} \qquad ...(7)$$

$$P_{\theta_1}=\mathbf{I}_A\cdot\frac{\partial \mathbf{r}_A}{\partial \theta_1}+\mathbf{I}_B\cdot\frac{\partial \mathbf{r}_B}{\partial \theta_1}+\mathbf{I}_C\cdot\frac{\partial \mathbf{r}_C}{\partial \theta_1}+\mathbf{I}_D\cdot\frac{\partial \mathbf{r}_D}{\partial \theta_1} \qquad ...(8)$$

$$P_{\theta_2}=\mathbf{I}_A\cdot\frac{\partial \mathbf{r}_A}{\partial \theta_2}+\mathbf{I}_B\cdot\frac{\partial \mathbf{r}_C}{\partial \theta_2}+\mathbf{I}_C\cdot\frac{\partial \mathbf{r}_C}{\partial \theta_2}+\mathbf{I}_D\cdot\frac{\partial \mathbf{r}_D}{\partial \theta_2} \qquad ...(9)$$

where $\mathbf{r}_A=(x-l\cos\theta_1-l\cos\theta_1)\,\mathbf{i}+(y-l\sin\theta_1+l\sin\theta_2)\,\mathbf{j}$ ...(10)

$\mathbf{r}_B=(x-l\cos\theta_1+l\cos\theta_2)\,\mathbf{i}+(y-l\sin\theta_1-l\sin\theta_2)\,\mathbf{j}$ ...(11)

$\mathbf{r}_C=(x+l\cos\theta_1+l\cos\theta_2)\,\mathbf{i}+(y+l\sin\theta_1-l\sin\theta_2)\,\mathbf{j}$ ...(12)

$\mathbf{r}_D=(x+l\cos\theta_1-l\cos\theta_2)\,\mathbf{i}+(y+l\sin\theta_1+l\sin\theta_2)\,\mathbf{j}$. ...(13)

But initially the impulsive force at A is in the direction of the positive y-axis, so we have $\mathbf{r}_A=l\mathbf{j}$. ...(14)

$\therefore$ Equations (6), (7), (8), (9)

$\Rightarrow$ $P_x=0,\ P_y=I,\ P_{\theta_1}=-Il\cos\theta_1$ and $P_{\theta_2}=Il\cos\theta_2$. ...(15)

Thus equations (2), (3), (4) and (5) give

$$4m\dot{x}=0,\ 4m\dot{y}=I,\ \frac{8}{3}ml^2\dot{\theta}_1=-Il\cos\theta_1,\ \frac{8}{3}ml^2\dot{\theta}_2=Il\cos\theta_2. \quad ...(16)$$

Second Part :

We have $\dot{x}=0,\ \dot{y}=\frac{I}{4m},\ \dot{\theta}_1=-\frac{3I}{8ml}\cos\theta_1$ and $\dot{\theta}_2=\frac{3I}{8ml}\cos\theta_2$.

$$\therefore\quad T=2m(\dot{x}^2+\dot{y}^2)+\frac{4}{3}ml^2(\dot{\theta}_1{}^2+\dot{\theta}_2{}^2)$$

$$=2m\left(0+\frac{I^2}{16m^2}\right)+\frac{4}{3}ml^2\left(\frac{9I^2}{54m^2l^2}\cos^2\theta_1+\frac{9I^2}{64m^2l^2}\cos^2\theta_2\right). \quad ...(17)$$

But immediately after the application of the impulsive forces, $\theta_1=\frac{\pi}{2}$ and $\theta_2=0$ approximately, so (17) gives $T=\frac{I^2}{2m}$.

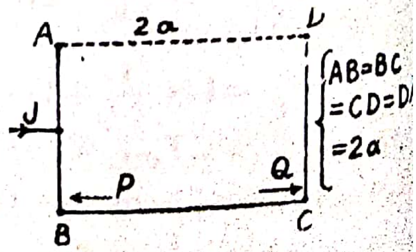
Ex. 20. *Three equal uniform rods AB, BC, CD are freely jointed at B and C and the ends A and D are fastend to smooth fixed points whose distance apart is equal to the length of either rod. The frame being at rest in the form of the square, a blow J is given perpendicular to AB at its middle point and in the plane of square. Show that the energy set up is* $\frac{3J^2}{40m}$ *where m is the mass of each rod.*

Find also the blows at the joints B and C.

Solution. The blow J is given at G_1, the C.G. of AB, at right angle to AB.

The rods AB and CD will turn through the same angle θ but the rod BC will remain parallel to AD.

Now let T be the K.E. of the system then we have



$$T=2\tfrac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2+a^2\dot{\theta}^2\right]+\tfrac{1}{2}m(2a\dot{\theta})^2$$

$$=\frac{10ma^2}{3}\dot{\theta}^2.$$

$\therefore$ Lagrange's θ-equation given

$$\left(\frac{\partial T}{\partial \dot\theta}\right)_2 - \left(\frac{\partial T}{\partial \dot\theta}\right)_1 = J \Rightarrow \frac{20ma^2}{3}\dot\theta = Ja \Rightarrow \dot\theta = \frac{3J}{20ma};$$

and $$T = \frac{10ma^2}{3} \cdot \frac{9J^2}{400m^2a^2} = \frac{3J^2}{40m}.$$

Second Part.

Let P and Q be the impulses at B and C respectively, with directions, as shown in the figure, then

Considering the motion of AB and taking moments about A, we get

Change in the angular momentum about the axis through A = moments of the impulses about this axis

$$\Rightarrow m\frac{4a^2}{3}\dot\theta = Ja - P.2a \Rightarrow P = \tfrac{1}{2}J - \frac{2ma}{3}\dot\theta$$

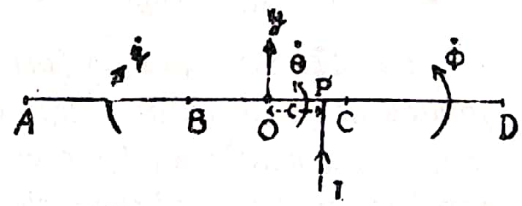
$$\Rightarrow P = \tfrac{1}{2}J - \frac{2ma}{3} \cdot \frac{3J}{20ma} = \frac{2J}{5}.$$

Again considering the motion of CD and taking moments bout D, we obtain

$$m\frac{4a^2}{3}\dot\theta = Q.2a \Rightarrow Q = \frac{2ma}{3}\dot\theta = \frac{2ma}{3} \cdot \frac{3J}{20ma} = \frac{1}{10}J.$$

Ex. 21. *Three equal uniform rods AB, BC, CD, each of mass and length $2a$, are at rest in a straight line smoothly joined at and C. A blow I is given to the middle rod at a distance c from e centre O in a direction peraendicular to it, show that the initial locity of O is $(2I/m)$ and that the initial angular velocities of the ds are $\frac{(5a+9c)I}{10ma^2}$, $\frac{6cI}{5ma^2}$ and $\frac{(5a-9c)I}{10mn^2}$.* **[Nagpur 1978]**

Solution. Just after the below, y by the linear velocity of O, d let θ, ϕ, ψ be the inclinations BC, CD and AB to their initial sitions. Now let T be K. E. of system, then we have



$$T = \tfrac{1}{2}m\left(\frac{a^2}{3}\dot\theta^2 + \dot y^2\right) + \tfrac{1}{2}m\left\{\frac{a^2}{3}\dot\psi^2 + (\dot y - a\dot\theta + a\dot\psi)^2\right\}$$
$$+ \tfrac{1}{2}m\left\{\frac{a^2}{3}\dot\phi^2 + (\dot y + a\dot\theta + a\dot\phi)^2\right\}$$

$$=\tfrac{1}{2}m\left[3\dot{y}^2+\frac{7a^2}{3}\dot{\theta}^2+\frac{4b^2}{3}\dot{\psi}^2+\frac{4}{3}a^2\dot{\phi}^2+2a\dot{y}\dot{\psi}+2a\dot{\phi}\right.$$

$$\left.+2a\dot{\theta}\dot{\phi}-2a\dot{\theta}\dot{\psi}\right]$$

Also just before the action of the impulse, we have $T=0$;

$\therefore$ Lagrange's equation for below gives

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2-\left(\frac{\partial T}{\partial \dot{y}}\right)_1=I \Rightarrow 3m\dot{y}+ma\dot{\phi}+ma\dot{\psi}=I$$

$$\Rightarrow 3\dot{y}+a\dot{\phi}+a\dot{\psi}=\frac{I}{m}.$$

Similarly Lagrange's θ, ϕ, ψ equations give

$$\tfrac{7}{3}a\dot{\theta}+a\dot{\phi}-a\dot{\psi}=\frac{Ic}{ma},\ \dot{y}+\tfrac{4}{3}a\dot{\phi}+a\dot{\theta}=0,\ \dot{y}+a\dot{\theta}+\tfrac{4}{3}a\dot{\psi}=0.$$

Last two equations $\Rightarrow 3\dot{y}+2a\dot{\phi}+2a\dot{\psi}=0$

$$3\dot{y}=\frac{2I}{m} \quad \text{or} \quad \dot{y}=\frac{2I}{3m}.$$

Also $\qquad 2a\dot{\theta}=\frac{4a}{3}(\dot{\psi}-\dot{\phi}),$ *i.e.* $\tfrac{3}{2}a\dot{\theta}+a\dot{\phi}-a\dot{\psi}=0;$

$$\therefore \quad \frac{5a}{6}\dot{\theta}=\frac{Ic}{ma} \quad i.e. \quad \dot{\theta}=\frac{6Ic}{5ma^2}.$$

Substituting now the values of $\dot{y}$ and $\dot{\theta}$ in the equation

$$\dot{y}+\tfrac{4}{3}a\dot{\phi}+a\dot{y}=0,$$

we get $\qquad \frac{2I}{3m}+\tfrac{4}{3}a\dot{\phi}+\frac{6Ic}{5ma}=0 \quad \text{or} \quad \dot{\phi}=\frac{(5a+9c)}{10ma^2}.$

Also substituting values of $\dot{y}$ and $\dot{\theta}$ in the equation

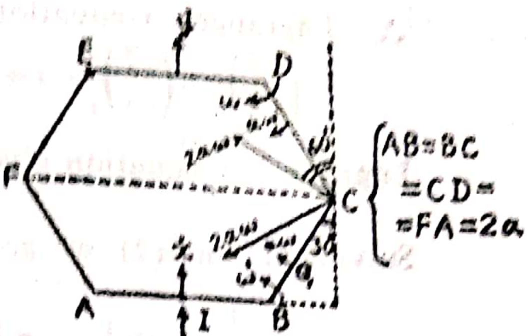
$$\dot{y}-a\dot{\theta}+\tfrac{4}{3}a\dot{\psi}=0,$$

we get $\qquad \frac{2I}{3m}-\frac{6Ic}{5ma}+\tfrac{4}{3}a\dot{\psi}=0 \quad \text{or} \quad \dot{\psi}=\frac{(5a-9c)\,I}{10ma^2}.$

Ex. 22. *Six equal uniform rods form a regular hexagon, loosely jointed at the angular points and rest on a smooth table; a blow is given perpendicularly to one of them at its middle point; find the resulting motion and show that the opposite rod begins to move with one tenth of the velocity of the rod that is struck.*

[Raj. 92; I. A. S. 92; Agra 80]

Solution. The impulse is given at the middle point of AB in the direction as shown in the figure, so the ensuring motion of BC, AF and CD, EF will be symmetrical.



Therefore, they have after impulse same angular velocity say ω.

Also the rods AB and DE will begin to move at right angles to themselves. Now let $\dot{x}$ and $\dot{y}$ be their velocities just after the action of impulse.

$\therefore$ Velocity of C in the direction of $\dot{x}$

$$=\text{Velocity of } B+\text{velocity of } C \text{ relative to } B$$

$$=\dot{x}+2a\omega \sin 30=\dot{x}+a\omega \qquad \ldots(1)$$

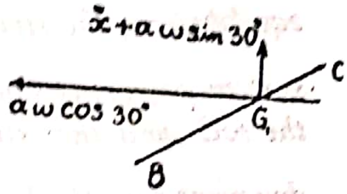
and velocity of C in the same direction (with respect to the rod CD)

$$=\text{Velocity of } D+\text{velocity of } C \text{ relative to } D,$$

$$=\dot{y}-2a\omega \sin 30=\dot{y}-a\omega. \qquad \ldots(2)$$

$\therefore$ (1) and (2) $\Rightarrow \dot{x}+a\omega=\dot{y}-a\omega \Rightarrow a\omega=\frac{1}{2}(\dot{y}-\dot{x})$.

Now actual velocity of the C.G. of the rod BC is $a\omega \cos 30°$ parallel to AB and $\dot{x}+a\omega \sin 30°$ at right angles to AB.



$\therefore$ (Velocity)^2 of G_1

$$=(a\omega \cos 30°)^2+(\dot{x}+a\omega \sin 30)^2.$$

Thus K.E. of BC

$$=\frac{1}{2}m\left[\frac{a^2}{3}\omega^2+\{(a\omega \cos 30°)^2+(\dot{x}+a\omega \sin 30)^2\}\right]$$

$$=\frac{1}{2}m\left[\frac{4}{3}a^2\omega^2+\dot{x}^2+a\omega\dot{x}\right]=\frac{m}{12}\left[5\dot{x}^2+2\dot{y}^2-\dot{x}\dot{y}\right]=T_1 \text{ say}$$

$$\left(\because\quad a\omega=\frac{\dot{y}-\dot{x}}{2}\right).$$

Also, Kinetic energy of CD

$$=\frac{1}{2}m\left[\frac{a^2}{3}\omega^2+\{(a\omega \cos 30)^2+(\dot{y}-a\omega \sin 30)^2\}\right]$$

$$=\frac{1}{2}m\left[\frac{4}{3}a^2\omega^2+\dot{y}^2-a\omega\dot{y}\right]=\frac{m}{12}\left[2\dot{x}^2+5\dot{y}^2-\dot{x}\dot{y}\right]=T_2 \text{ say.}$$

$\therefore$ Total Kinetic energy T is given by

$$T=2T_1+2T_2+T_3+T_4,$$

where T_3 and T_4 are the K.E's. of AB and DE respectively.

$$\therefore\quad T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2+2.\frac{m}{12}(5\dot{x}^2+2\dot{y}^2-\dot{x}\dot{y})$$

$$+2.\frac{m}{12}(5\dot{x}^2+2y^2-\dot{x}\dot{y})$$

or

$$T=\frac{m}{3}(5\dot{x}^2+2\dot{y}^2+\dot{x}\dot{y}).$$

$\therefore$ Lagrange's x-equation is

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2-\left(\frac{\partial T}{\partial \dot{x}}\right)_1=I \Rightarrow \frac{m}{3}(10\dot{x}-\dot{y})=I. \qquad ...(1)$$

Lagrange's y-equation gives $\frac{m}{3}(10\dot{y}-\dot{x})=0. \qquad ...(2)$

Solving (1) and (2), we get $\dot{x}=\frac{10I}{33m}$ and $\dot{y}=\frac{I}{33m}$

$$\dot{y}=\frac{1}{10}\dot{x} \text{ and } a\omega=\tfrac{1}{2}(\dot{y}-\dot{x}) \Rightarrow \omega=-\frac{3I}{22ma},$$

negative sign simply indicates that ω will be in the opposite direction.

Ex. 23. *A uniform rod AB, of mass 2m and length 2a, swings freely about a horizontal axis through the end A. One end of a light elastic string is attached to the end B of rod and carries a particle of mass m at its other end. When the system is in stable equilibrium, the string is of length $\frac{4a}{3}$, its extension being ϵ. If the system performs small oscillations in the vertical plane through AB, the rod and the string making angle θ, ϕ respectively with the downward vertical, and the length of the string being $x+\frac{4a}{3}$, show that x, $\phi+2\theta$, $2\phi-3\theta$ are normal co-ordinates for the system, and the length of the corresponding simple pendulums are $\epsilon, \frac{8a}{3}, \frac{a}{3}$.*

Solution. We have $x_G=a\sin\theta$, $y_G=a\cos\theta$,

$$\because \quad \dot{x}^2_G+\dot{y}^2_G=a^2\dot{\theta}^2=v^2_G. \qquad ...(1)$$

Thus K.E. of the rod

$$=\frac{1}{2}.2m\left\{a^2\dot{\theta}^2+\frac{a^2}{3}\dot{\theta}^2\right\}=\frac{1}{2}.2m.\frac{4a^2}{3}\dot{\theta}^2=\frac{4a^2}{3}m\dot{\theta}^2. \qquad ...(2)$$

Also we have $\quad y_c=2a\cos\theta+\left(x+\frac{4a}{3}\right)\cos\phi$

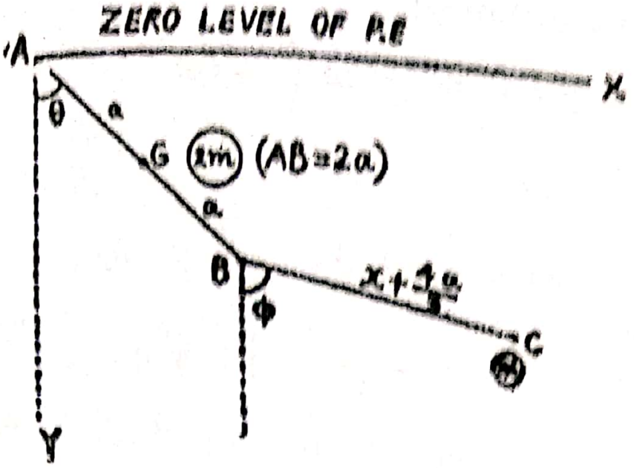
and $\quad x_c=2a\sin\theta+\left(x+\frac{4a}{3}\right)\sin\phi$

$$\Rightarrow v_\theta^2=(2a)^2\dot{\theta}^2+\left(x+\frac{4a}{3}\right)^2\dot{\phi}^2+2.2a\cos\theta\,.\,\dot{\theta}\,.\left(x+\frac{4a}{3}\right)\dot{\phi}\cos\phi$$

$$+\dot{x}^2+4a\,\dot{\theta}\dot{x}\sin(\phi-\theta)+2.2a\sin\theta\,.\,\dot{\theta}\,.\left(x+\frac{4a}{3}\right)\dot{\phi}\sin\phi$$

$$=4a^2\dot{\theta}^2+\left(x+\frac{4a}{3}\right)^2\dot{\phi}^2+4a\,\dot{\theta}\dot{\phi}\left(x+\frac{4a}{3}\right)\cos(\theta-\phi)$$

$$+\dot{x}^2+4a\dot{\theta}\,\dot{x}\sin(\phi-\theta)$$



Thus K.E. of mass m

$$=\tfrac{1}{2}m\left\{4a^2\dot{\theta}^2+\dot{x}^2+\left(x+\frac{4a}{3}\right)^2\dot{\phi}^2+4a\,\dot{\theta}\dot{\phi}\left(x+\frac{4a}{3}\right)\cos(\theta-\phi)\right.$$
$$\left.+4a\,\dot{\theta}\dot{x}\sin(\phi-\theta)\right\}$$

$\therefore$ K.E. of the system is, ...(3)

$$T=\frac{4a^2}{3}m\dot{\theta}^2+\tfrac{1}{2}m\left\{4a^2\dot{\theta}^2+\dot{x}^2+\left(x+\frac{4a}{3}\right)^2\dot{\phi}^2\right.$$
$$\left.+4a\,\dot{\theta}\dot{\phi}\left(x+\frac{4a}{3}\right)\cos(\theta-\phi)+4a\,\dot{\theta}\dot{x}\sin(\phi-\theta)\right\}$$
$$=\frac{4a^2}{3}m\dot{\theta}^2+\tfrac{1}{2}m\left[\dot{x}^2+4a^2\dot{\theta}^2+\left(x+\frac{4a^2}{3}\right)\dot{\phi}^2+4a\,\dot{\theta}\dot{\phi}\left(x+\frac{4a}{3}\right)\right.$$
$$\left.+4a\,\dot{\theta}\dot{x}(\phi-\theta)\right]\text{ approx.} \quad ...(4)$$
$$=\frac{4a^2}{3}m\dot{\theta}^2+\tfrac{1}{2}m\left\{\dot{x}^2+4a^2\dot{\theta}^2+\frac{16a^2}{9}\dot{\phi}^2+\frac{16a^2}{3}\dot{\theta}\dot{\phi}\right\}$$

[correct to second order]

$$=\frac{10}{3}a^2m\dot{\theta}^2+\tfrac{1}{2}m\dot{x}^2+\frac{8}{9}ma^2\dot{\phi}^2+\frac{8ma^2}{3}\dot{\theta}\dot{\phi}$$

Also $W=2mg\,a\cos\theta+mg\left\{2a\cos\theta+\left(x+\frac{4a}{3}\right)\cos\phi\right\}^*$
$$-\frac{mg}{2\epsilon}(x+\epsilon)^2-D$$

$$^*T=\lambda\,\frac{\text{Final length}-\text{Initial length}}{\text{Initial length}}$$
$$\Rightarrow mg=\frac{\lambda\epsilon}{[(4a)/3]-\epsilon}\quad\Rightarrow\quad\lambda=\frac{4a-3\epsilon}{3\epsilon}mg.$$

But the stretch of string is $x+\epsilon$, hence the workdone is

$$-\int_0^x\lambda\,\frac{x+\{(4a)/3\}-\{(4a)/3\}+\epsilon}{\{(4a)/3\}-\epsilon}\,dx$$
$$=-\int_0^x\frac{3\lambda}{4a-3\epsilon}(x+\epsilon)\,dx=-\frac{mg}{\epsilon}\int_0^x(x+\epsilon)\,dx=-\frac{mg}{2\epsilon}(x+\epsilon)^2-D$$

But $W+V=$ constant.

$$\therefore \quad V=-2mg\, a\cos\theta-mg\left(2a\cos\theta+x\cos\phi+\frac{4a}{3}\cos\phi\right)$$
$$+\frac{mg}{2\epsilon}(x+\epsilon)^2+\text{constant}$$
$$=-2mg\, a-mg\left(2a+x+\frac{4a}{3}\right)+mg\, a\theta^2+mg\, a\theta^2$$
$$-\frac{4a}{3}mg\left(-\frac{\phi^2}{2}\right)+\frac{mg}{2}\left(2x+\frac{x^2}{\epsilon}\right)+\text{constant}$$
$$=mg\, a\theta^2+mg\, a\theta^2-4mg\, a+\frac{2a}{3}mg\,\phi^2+\frac{mg}{2\epsilon}x^2+\text{const.}$$
$$=mg\frac{x^2}{2\epsilon}+2mg\, a\theta^2+\frac{2}{3}mg\, a\phi^2+\text{constant.} \qquad \ldots(5)$$

Thus $L=T-V=\frac{1}{2}m\dot{x}^2+\frac{10}{3}ma^2\dot{\theta}^2+\frac{8}{9}ma^2\dot{\phi}^2+\frac{8}{3}ma^2\dot{\theta}\dot{\phi}$

$$-mg\frac{x^2}{2\epsilon}-2mg\, a\theta^2-\frac{2}{3}mg\, a\phi^2=\text{const.}$$

$\therefore$ Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \Rightarrow \ddot{x}+\frac{g}{\epsilon}x=0 \qquad \ldots(6)$$
$$\frac{dt}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \Rightarrow \frac{20}{3}a\ddot{\theta}+\frac{8}{3}a\ddot{\phi}+4g\theta=0 \qquad \ldots(7)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0 \Rightarrow \frac{8}{3}a\ddot{\theta}+\frac{16}{9}a\ddot{\phi}+\frac{4}{3}g\phi=0. \qquad \ldots(8)$$

Now (6) shows that x is the normal co-ordinate and the corresponding length of the simple equivalent pendulum is ϵ. In (7), (8) put $\theta=A\sin(\omega t+\epsilon)$, $\phi=B\sin(\omega t+\epsilon)$.

Then simplifying, we get

$$(5a\omega^2-3g)\,A+2a\omega^2\,B=0 \qquad \ldots(9)$$
$$6a\omega^2\,A+(4a\omega^2-3g)\,B=0. \qquad \ldots(10)$$

Eliminating A, B we get

$$(5a\omega^2-3g)+(4a\omega^2-3g)-12a^2\omega^4=0$$
$$\Rightarrow \quad \omega_1{}^2=\frac{3g}{a},\ \omega_2{}^2=\frac{3g}{8a}.$$

From (9), $\dfrac{A}{\left(\frac{2a\omega^2}{g}\right)}=-\dfrac{B}{\left(\frac{5a\omega^2}{g}-3\right)}.$

When $\omega=\omega_1$, these are satisfied by $A=C_1$, $B=-2C_1$,
When $\omega=\omega_2$, solutions are $A=2C_2$, $B=3C_2$.

$\therefore$ General solution is

$$\theta=C_1 \sin(\omega_1 t+\epsilon_1)+2C_2 \sin(\omega_2 t+\epsilon_2) \quad ...(11)$$

and

$$\phi=-2C_1 \sin(\omega_1 t+\epsilon_1)+3C_2 \sin(\omega_2 t+\epsilon_2) \quad ...(12)$$

(11) and (12)

$$\Rightarrow \quad 2\theta+\phi=7C_2 \sin(\omega_2 t+\epsilon_2),\ 2\phi-3\theta=-7C_1 \sin(\omega_1 t+\epsilon_1)$$

which shows that $2\theta+\phi$, $2\phi-3\theta$ are expressible as single sinusoidal terms and are normal co-ordinates. Their periods being

$$\frac{2\pi}{\omega_2}=2\pi\left(\frac{8a}{3g}\right)^{1/2} \text{ and } \frac{2\pi}{\omega_1}=2\pi\left(\frac{a}{3g}\right)^{1/2}.$$

Alternative approach.

Equations (7), (8) can be written as

$$5a\ddot{\theta}+3g\,\theta+2a\,\ddot{\phi}=0 \quad ...(13)$$

and

$$6a\ddot{\theta}+4a\ddot{\phi}+3g\,\phi=0; \quad ...(14)$$

$\therefore$ (13) multiplied by λ+(14) multiplied by μ $\Rightarrow$

$$a(5\lambda+6\mu)\,\ddot{\theta}+3g\,\lambda\theta+2a\,(\lambda+2\mu)\ddot{\phi}+3g\,\mu\phi=0. \quad ...(15)$$

Now we choose λ and μ in such a manner that

$$\frac{\text{coefficient of } \ddot{\theta}}{\text{coefficient of } \theta}=\frac{\text{coefficient of } \ddot{\phi}}{\text{coefficient of } \phi}$$

$$\Rightarrow \quad \frac{a\,(5\lambda+6\mu)}{3g\lambda}=\frac{2a\,(\lambda+2\mu)}{3g\mu} \Rightarrow \frac{\lambda}{\mu}=-\frac{3}{2} \text{ or } \frac{\lambda}{\mu}=2.$$

When $\lambda=-\frac{3}{2}$, equation (15) gives

$$-3a\ddot{\theta}-9g\theta+2a\ddot{\phi}+6g\phi=0$$

$$\Rightarrow \quad \left(D^2+\frac{3g}{a}\right)(2\phi-3\theta)=0.$$

$$\therefore \quad 2\phi-3\theta=A_1 \sin\left\{\sqrt{\left(\frac{3g}{a}\right)}\,t+\epsilon_1\right\}. \quad ...(16)$$

Further when $\frac{\lambda}{\mu}=2$, equation (15) gives

$$16a\ddot{\theta}+6g\theta+8a\ddot{\phi}+3g\phi=0$$

$$\Rightarrow \quad \left(D^2+\frac{3g}{8a}\right)(2\theta+\phi)=0.$$

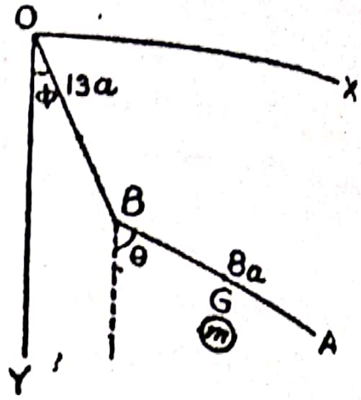
$$\therefore \quad 2\theta+\phi=A_2 \sin\left\{\sqrt{\left(\frac{3g}{8a}\right)}\,t+\epsilon_2\right\}$$

Thus x, $2\theta+\phi$, $2\phi-3\theta$ are normal co-ordinates.

Ex. 24. *A uniform rod AB of length 8a is suspended from a fixed point O by means of light inextensible string, of length 13a, attached to B. If the system is slighty displaced in a vertical plane, show that $(\theta+3\phi)$ and $(12\theta-13\phi)$ are principal co-ordinates, where θ and ϕ are the angles which the rod and string respectively* *make with the vertical. Also show that periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{52a}{3g}\right)}.$$

Solution. We have



$$x_G = 13a \sin\phi + 4a \sin\theta$$

and $$y_G = 13a \cos\phi + 4a \cos\theta$$

$$\therefore \quad \dot{x}^2_G + \dot{y}^2_G = 169a^2\dot{\phi}^2 + 16a^2\dot{\theta}^2 + 104a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi).$$

Thus, $T = \frac{1}{2}m\,[k^2\dot{\theta}^2 + (\dot{x}^2_G + \dot{y}^2_G)]$

$$= \tfrac{1}{2}ma^2\left[\frac{64}{3}\dot{\theta}^2 + 169\dot{\phi}^2 + 104\dot{\theta}\dot{\phi}\right]$$

and the work function

$$W = mg\,(13a\cos\phi + 4a\cos\theta);$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta} \Rightarrow 16\ddot{\theta} + 39\ddot{\phi} = -\frac{3g}{a}\theta. \quad \ldots(1)$$

While Lagrange's ϕ-equation gives

$$4\ddot{\theta} + 13\ddot{\phi} = -\frac{g}{a}\phi. \quad \ldots(2)$$

Equations (1) and (2)

$$\Rightarrow D^2(\theta+3\phi) = -\frac{3g}{52a}(\theta+3\phi) \Rightarrow D^2(12\theta-13\phi) = -\frac{g}{a}(12\theta-13\phi).$$

Now putting $\theta+3\phi = X$ and $12\theta-13\phi = Y$ in these equations, we get

$$D^2X = -\frac{3g}{52a}X \text{ and } D^2Y = -\frac{g}{a}Y.$$

which obviously represent two independent simple harmonic montions. Hence X and Y are principal co-ordinates, that is $(\theta+3\phi)$ and $(12\theta-13\phi)$ are principal co-ordinates.

Also, periods of small oscillations are given by

$$2\pi\sqrt{\left(\frac{52a}{3g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{g}\right)}.$$

3·26. Lagrange's equations for Non-Holonomic system with moving constraints.

Consider a system of N particles with masses $(m_1, m_2, \ldots, m_N)$, position vectors $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)$.

Let $\mathbf{F}_\nu$ be the total force acting on the typical particle, the reactions of constraints being included in $\mathbf{F}_\nu$.

$\therefore$ Equation of motion of the typical particle is

$$m_\nu \ddot{\mathbf{r}}_\nu = \mathbf{F}_\nu \quad \text{...(1)}$$

These N vector equations are precisely equivalent to the single scalar equation

$$\sum_{\nu=1}^{N} (m_\nu \ddot{\mathbf{r}}_\nu - \mathbf{F}_\nu) \cdot \mathbf{P}_\nu = 0, \quad \text{...(2)}$$

where $\mathbf{P}_\nu$ indicates a set of N arbitrary vectors.

Now if we put $\mathbf{P}_\nu = \delta \mathbf{r}_\nu$ and regard $\delta \mathbf{r}_\nu$ as arbitrary vertical displacement, we have

$$\sum_{\nu=1}^{N} (m_\nu \ddot{\mathbf{r}}_\nu - \mathbf{F}_\nu) \cdot \delta \mathbf{r}_\nu = 0 \Rightarrow \sum_{\nu=1}^{N} m_\nu \ddot{\mathbf{r}}_\nu = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \delta \mathbf{r}_\nu = \delta W,$$

where δW is the virtual work done by the force in the virtual displacement. The equation,

$$\sum_{\nu=1}^{N} m_\nu \ddot{\mathbf{r}}_\nu = \delta W$$

is known as D' Alemberts equation and is closely related to D' Alemberts principle.

Now let us assume that the system is subject to constraints, in general non-holonomic.

Let $q_1, q_2, \ldots, q_n$ be the generalised co-ordinates such that these co-ordinates together with the time t, determine the positions of the particles

i.e. $\quad \mathbf{r}_\nu = \mathbf{r}_\nu (q_1, q_2, \ldots, q_n, t) \quad \text{...(3)}$

Now assume that there are m constraint conditions of the form

$$\sum_{\alpha=1}^{n} A_\alpha \, dq_\alpha + A \, dt = 0 \; ; \; \sum_{\alpha=1}^{m} B_\alpha \, dq_\alpha + B \, dt = 0, \text{ where } m < n \quad \text{...(4)}$$

The K.E. of the system is given by

$$T = \tfrac{1}{2} \sum_{\nu=1}^{N} m_\nu \dot{\mathbf{r}}_\nu \cdot \dot{\mathbf{r}} \text{ and } \mathbf{r}_\nu = \mathbf{r}_\nu (q_1, q_2, \ldots, q_n, t)$$

Hence T may be written as $\quad \text{...(5)}$

$$T = T(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)$$

$$\left\{ \because \; \dot{\mathbf{r}}_\nu = \sum_{\alpha=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \mathbf{r}_\nu}{\partial t} \right\}$$

Let us now define

$$\ldots \frac{d}{dt}\left(\frac{\partial T}{\ldots}\right) \ldots \quad Y_\alpha = \Sigma \, \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}.$$

If $\delta\mathbf{r}_\nu$ are virtual displacements which satisfy the instantaneous constraints (taking t to be constant at that time), we have

$$\delta\mathbf{r}_\nu=\sum_\alpha \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\,\delta q_\alpha. \quad \ldots(6)$$

$\therefore$ virtual work done is $\delta W=\sum_\nu m_\nu \ddot{\mathbf{r}}_\nu\cdot\delta\mathbf{r}_\nu$

$$=\sum_\nu\sum_\alpha m_\nu \ddot{\mathbf{r}}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\,\partial q_\alpha \quad \ldots(7)$$

or
$$\delta W=\sum_\alpha\left(\sum_\nu \mathbf{F}_\nu\cdot\frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}\right)\delta q_\alpha=\sum_\alpha Y_\alpha\,\delta q_\alpha. \quad \ldots(8)$$

But the virtual work done can also be written as (in terms of the generalised forces ϕ_α) as

$$\delta W=\sum_\alpha \phi_\alpha\,\delta q_\alpha \quad \ldots(9)$$

$\therefore$ (8) and (9) give $\quad \Sigma\,(Y_\alpha-\phi_\alpha)\,\delta q_\alpha=0. \quad \ldots(10)$

But δq_α are not all independent, so we cannot conclude from (10) that $Y_\alpha=\phi_\alpha$.

From (4), since t is constant for instantaneous constraints, we have the m equations

$$\sum_\alpha A_\alpha\,\delta_\alpha q_\alpha=0,\ \sum_\alpha B_\alpha\,\delta q_\alpha=0 \quad \ldots(11)$$

Multiplying these by the m Lagrange's multipliers $\lambda_1, \lambda_2,\ldots\lambda_m$ and adding, we get

$$\Sigma\,(\lambda_1 A_\alpha+\lambda_2 B_\alpha+\ldots)\,\delta q_\alpha=0. \quad \ldots(12)$$

Subtracting (68) from (66), we obtain

$$\sum_\alpha\,[Y_\alpha-\phi_\alpha-\lambda_1 A_\alpha-\lambda_2 B_\alpha-\ldots]\,\delta q_\alpha=0. \quad \ldots(13)$$

Now because of equations (11), we can solve form of the quantities δq_α (say $\delta q_1, \delta q_2, \ldots, \delta q_m$) in terms of the remaining δq_α (say $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$).

Hence in (13), we consider $\delta q_1, \delta q_2, \ldots, \delta q_m$ as dependent and $\delta q_{m+1}, \delta q_{m+2}, \ldots, \delta q_n$ as independent.

Let us arbitrary set the coefficients of the dependent variables equal to zero

i.e. $\quad Y_\alpha-\phi_\alpha-\lambda A_\alpha-\lambda_2 B_\alpha-\ldots=0 \quad (\alpha=1, 2, \ldots m). \quad \ldots(14)$

Then the independent quantities δq_α are left in (69) and since these are arbitrary, it follows that their coefficients must vanish

i.e. $\quad Y_\alpha-\phi_\alpha-\lambda_1 A_\alpha-\lambda_\alpha-\ldots=0 \quad (\alpha=1, 2\ldots m). \quad \ldots(15)$

If $\delta \mathbf{r}_\nu$ are virtual displacements which satisfy the instantaneous constraints (taking t to be constant at that time), we have

$$\delta \mathbf{r}_\nu = \sum_\alpha \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \delta q_\alpha. \qquad ...(6)$$

$\therefore$ virtual work done is $\delta W = \sum_\nu m_\nu \ddot{\mathbf{r}}_\nu \cdot \delta \mathbf{r}_\nu$

$$= \sum_\nu \sum_\alpha m_\nu \ddot{\mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \partial q_\alpha \qquad ...(7)$$

or
$$\delta W = \sum_\alpha \left(\sum_\nu \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} \right) \delta q_\alpha = \sum_\alpha Y_\alpha \, \delta q_\alpha. \qquad ...(8)$$

But the virtual work done can also be written as (in terms of the generalised forces ϕ_α) as

$$\delta W = \sum_\alpha \phi_\alpha \, \delta q_\alpha \qquad ...(9)$$

$\therefore$ (8) and (9) give $\quad \Sigma (Y_\alpha - \phi_\alpha) \, \delta q_\alpha = 0. \qquad ...(10)$

But δq_α are not all independent, so we cannot conclude from (10) that $Y_\alpha = \phi_\alpha$.

From (4), since t is constant for instantaneous constraints, we have the m equations

$$\sum_\alpha A_\alpha \, \delta_\alpha q_\alpha = 0, \ \sum_\alpha B_\alpha \, \delta q_\alpha = 0 \qquad ...(11)$$

Multiplying these by the m Lagrange's multipliers $\lambda_1, \lambda_2, ... \lambda_m$ and adding, we get

$$\Sigma (\lambda_1 A_\alpha + \lambda_2 B_\alpha + ...) \, \delta q_\alpha = 0. \qquad ...(12)$$

Subtracting (68) from (66), we obtain

$$\sum_\alpha [Y_\alpha - \phi_\alpha - \lambda_1 A_\alpha - \lambda_2 B_\alpha - ...] \, \delta q_\alpha = 0. \qquad ...(13)$$

Now because of equations (11), we can solve form of the quantities δq_α (say $\delta q_1, \delta q_2, ..., \delta q_m$) in terms of the remaining δq_α (say $\delta q_{m+1}, \delta q_{m+2}, ..., \delta q_n$).

Hence in (13), we consider $\delta q_1, \delta q_2, ..., \delta q_m$ as dependent and $\delta q_{m+1}, \delta q_{m+2}, ..., \delta q_n$ as independent.

Let us arbitrary set the coefficients of the dependent variables equal to zero

i.e. $\quad Y_\alpha - \phi_\alpha - \lambda A_\alpha - \lambda_2 B_\alpha - ... = 0 \quad (\alpha = 1, 2, ... m). \qquad ...(14)$

Then the independent quantities δq_α are left in (69) and since these are arbitrary, it follows that their coefficients must vanish

i.e. $\quad Y_\alpha - \phi_\alpha - \lambda_1 A_\alpha - \lambda_\alpha - ... = 0 \quad (\alpha = 1, 2 ... m). \qquad ...(15)$

Thus equations (14) and (15)

$$\Rightarrow \quad Y_\alpha - \phi_\alpha + \lambda_1 A_\alpha + \lambda_2 B_\alpha \ldots ; \quad \alpha = 1, 2, \ldots, n,$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha + \lambda_1 A_1 A_\alpha + \lambda_2 B_\alpha + \ldots ; \alpha = 1, 2, \ldots n \qquad \ldots(16)$$

Obviously (16) denotes n equations. These n equations together with m equations [denoted by 4] lead to $n+m$ equations in $n+m$ unknowns.

Note. If forces are derivable from a potential *viz.*

$$\phi_\alpha = -\frac{\partial V}{\partial q_\alpha}$$

where V does not bayond on $\dot{q}_\alpha$, then (16) can be written as

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial V}{\partial \dot{q}_\alpha}\right] - \frac{\partial}{\partial q_\alpha}(T - V) = \lambda_1 A_\alpha + \lambda_2 B_\alpha + \ldots$$

or

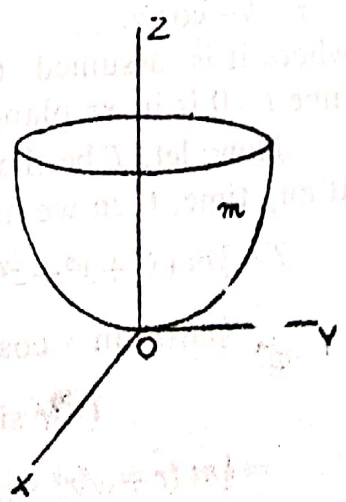
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 A_\alpha + \lambda_2 B_\alpha + \qquad \ldots(17)$$

where L is said to be the Lagrangian function or the kinetic potential of the system.

Ex. 25. *A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless [see the adjoining diagram], Obtain the equations of motion.*

Solution. We have,

$$L = \tfrac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2) - mgz \qquad \ldots(1)$$



But $x^2 + y^2 = \rho^2$ so that constraint condition is

$$\rho^2 - az = 0 \Rightarrow 2\rho\delta\rho - a\delta z = 0. \qquad \ldots(2)$$

Let us now assume

$$q_1 = \rho, \; q_2 = \phi, \; q_3 = z.$$

$$\therefore \; (2) \Rightarrow 2\rho\delta q_1 - a\delta q_3 + 0.\delta q_2 = 0. \qquad \ldots(3)$$

Comparing it with

$$A_1 \, \delta q_1 + A_2 \, \delta q_2 + A_3 \, \delta q_3 = 0 \qquad \ldots(4)$$

we get $\quad A_1 = 2\rho, \; A_2 = 0, \; A_3 = -a.$

Now Lagrange's equation are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = \lambda_1 A \quad [\because \text{ only the constraint is given}]$$

$$\alpha = 1, 2, 3$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial \rho} = 2\lambda_1 \rho \quad \ldots(5) ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \qquad \ldots(6)$$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=-\lambda_1 a$ ...(7) $\Rightarrow m(\ddot{\rho}-\rho\,\dot{\phi}^2)=2\lambda_1\,\rho$; ...(8)

$m\frac{d}{dt}(\rho^2\dot{\phi})=0$ ...(9) and $m\ddot{z}=-mg-\lambda_1 a$ ...(10)

[using (1)]

Note. The constraint condition is

$$2\rho\,\delta q_1+0\,\delta q_2-a\,\delta q_3=0,$$

i.e. $2\rho\frac{\delta q_1}{\delta t}-a\frac{\delta q_3}{\delta t}=0$ i.e. $2\rho\frac{\partial q_1}{\partial t}-a\frac{\partial q_3}{\partial t}=0$ i.e. $2\rho\dot{\rho}-a\,\dot{z}=0$...(11)

For equations, *viz*. (8), (9), (10) and (11) enable us to find the four unknown ρ, ϕ, z, λ_1.

Ex. 26. *In the adjoining diagram AB is a straight frictionless wire fixed at point A on a vertical axis OA such that AB rotates about OA with contant angular velocity ω. A bead of mass m is constrained to move on the wire.*

(*a*) *Set up the Lagrangian.*

(*b*) *Write Lagrange's equations.*

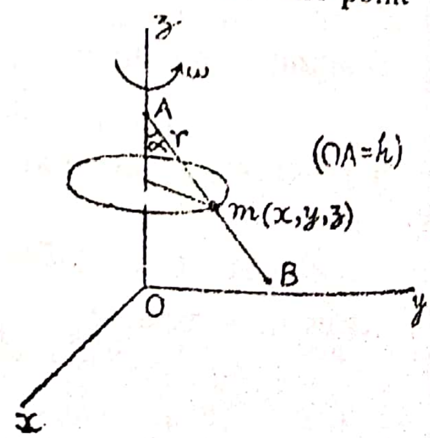
(*c*) *Determine the motion at any time.*

(*d*) *If the bead starts from rest at A, how long will if take to reach the end B of the wire assuming that the length of the wire is l.*

Solution. (*a*) Let r be the distance of the bead from the point A of the wire at time t. Cartesian co-ordinates of the bead are then given by

$x=r\sin\alpha\cos\omega t,\ y=r\sin\alpha\sin\omega t,$
$z=h-\cos\alpha,$

where it is assumed that the wire at time $t=0$ is in xz plane.



Now, let, T be the K.E. of the bead at any time, then we have

$T=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$=\frac{1}{2}m\{(\dot{r}\sin\alpha\cos\omega t-wr\sin\omega t)^2$

$(+\dot{r}\sin\alpha\sin\omega t+\omega r\sin\alpha\cos\omega t)^2+(-\dot{r}\cos\alpha)^2\}$

$=\frac{1}{2}m(\dot{r}+\omega^2 r^2\sin^2\alpha).$

Also, taking xy plane as reference level, we have

$T=mgz=mg(h-r\cos\alpha)$

$\therefore$ Lagrangian is given by

$L=T-V=\frac{1}{2}m(\dot{r}^2+\omega^2 r^2\sin^2\alpha)-mg(h-r\cos\alpha)$

(b) $\frac{\partial L}{\partial \dot{r}}=m\dot{r}$ and $\frac{\partial L}{\partial r}=m\omega^2 r \sin^2 \alpha+mg \cos \alpha.$

$\therefore$ Lagrange's r-equation gives

$$(d/dt)\,(\partial L/\partial \dot{r})-(\partial L/\partial r)=0 \Rightarrow m\ddot{r}-(m\omega^2 r \sin \alpha+mg \cos \alpha)=0$$
$$\Rightarrow \ddot{r}-(\omega^2 \sin^2 \alpha)\, r=g \cos \alpha.$$
$$\Rightarrow (D^2-\omega^2 \sin^2 \alpha)\, r=g \cos \alpha. \quad ...(1)$$

(c) Solving (1), we get $r=$C.F,+P.I.

where $\text{C.F.}=c_1 e^{(\omega \sin \alpha)t}+c_2\, e^{-(\omega \sin \alpha)t}$ and $\text{P.I.}=\frac{-g \cos \alpha}{\omega^3 \sin^2 \alpha}$

(d) The bead starts from rest at $t=0$, we have

$$r=0,\ \dot{r}=0 \text{ at } t=0.$$

$$\therefore \quad 0=c_1+c_2-\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \text{ and } c_1-c_2=0$$

Thus $r=\frac{g \cos \alpha}{2\omega^2 \sin^2 \alpha}\{e^{-\sin \alpha t}+e^{-\omega \sin \alpha t}\}-\frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}$

$$=\frac{g \cos \lambda}{\omega^2 \sin^2 \alpha}(\cosh (\omega \sin \alpha)\, t-1\}.$$

Hence the required time is given by

$$T=\frac{1}{\omega \sin \alpha}\cosh^{-1}\left(1+\frac{l\omega^2 \sin^2 \alpha}{g \cos \alpha}\right) \quad [\text{Putting } t=T \text{ and } r=l].$$

Ex. 27. *A particle of mass m is moving in a plane under an inverse square law attractive force. Set up the Lagrangian and hence obtain the equation describing its motion.* **(Meerut 1971)**

Sol. Let (r, θ) be the plane polar co-ordinates of the particle of mass m, then kinetic energy of the particle T is given by

$$T=\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2).$$

Let F be the force experienced by the particle, then we have

$F \propto \frac{1}{r^2} \Rightarrow F=-\frac{k}{r^2}$, where k is the constant of proportionality.

But $\qquad F=-\frac{dV}{dr} \Rightarrow dV=-F\,dr.$

$\Rightarrow$ Potential energy is given by

$$V=-\int_{\infty}^{r} F\,dr=\int_{\infty}^{r}\frac{k}{r^2}\,dr=\left[-\frac{k}{r}\right]_{\infty}^{r}=-\frac{k}{r}.$$

$$\therefore \quad L=T-V=\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2)+\frac{k}{r},$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{r}}=m\dot{r},\ \frac{\partial L}{\partial r}=mr\dot{\theta}^2-\frac{k}{r^2};\ \frac{\partial L}{\partial \theta}=0,\ \frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}.$$

Now, Lagrangian equation or equation of motion in the variable r is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0, \quad \Rightarrow \quad \frac{d}{dt}(m\dot{r})-\left(mr\dot{\theta}^2-\frac{k}{r^2}\right)=0,$$

$$\Rightarrow \quad m\ddot{r}-mr\dot{\theta}^2+\frac{k}{r^2}=0.$$

Again, Lagrangian equation in the variable θ is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0 \quad \Rightarrow \quad \frac{d}{dt}(mr^2\dot{\theta})=0. \quad \Rightarrow \quad mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}=0.$$

Ex. 28. *Set up the Lagrange's equation of a particle moving on the surface of earth using spherical polar co-ordinates.*

Sol. Let at any instant (r, θ, ϕ) be the spherical polar co-ordinates of a particle moving on the surface of earth. Assuming the centre of earth as origin, let (x, y, z) be the cartesian co-ordinates of the particle at the instant. Then we have

$$x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\theta,\ z=r\cos\theta. \qquad \ldots(1)$$

$$\Rightarrow \quad \dot{x}=r\cos\phi\cos\theta\dot{\theta}-r\sin\theta\sin\phi\dot{\phi},$$

$$\dot{y}=r\sin\theta\cos\phi\dot{\phi}+r\sin\phi\cos\theta\dot{\theta},\quad \dot{z}=-r\sin\theta\dot{\theta}.$$

Whence, kinetic energy of the particle is given by

$$\begin{aligned} T &= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)\\ &= \tfrac{1}{2}m\,[(r\cos\phi\cos\theta\dot{\phi}-r\sin\theta\sin\phi\dot{\phi})^2\\ &\qquad +(r\sin\theta\cos\phi\dot{\phi}+r\sin\phi\cos\theta\dot{\theta})^2+(-r\sin\theta\dot{\theta})^2]\\ &= \tfrac{1}{2}mr^2\,[\dot{\theta}^2+\sin^2\theta\dot{\phi}^2]. \end{aligned} \qquad \ldots(2)$$

Now, let potential energy of the particle be $V(r, \theta, \phi)$, then the Lagrangian of the system is given by

$$L=T-V=\tfrac{1}{2}mr^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)-V(r, \theta, \phi) \qquad \ldots(3)$$

Now Lagrange's θ equations

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \theta}=0.$$

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta}=mr^2\sin\theta\cos\theta\dot{\phi}^2-\frac{\partial V}{\partial \theta}. \qquad \ldots(4)$$

Making these substitutions, in equation (4) we get

$$\frac{d}{dt}(mr^2\dot{\theta})-mr^2\sin\theta\cos\theta\dot{\phi}^2+\frac{\partial V}{\partial \theta}=0. \qquad \ldots(5)$$

Also, the Lagrangian equation for conservative system in the variable ϕ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)-\frac{\partial L}{\partial \phi}=0.$$

Further, eqn. (3) $\Rightarrow \dfrac{\partial L}{\partial \phi}=-\dfrac{\partial V}{\partial \phi}$ and $\dfrac{\partial L}{\partial \dot{\phi}}=mr^2 \sin \theta\, \dot{\phi}$.

$$\Rightarrow \frac{d}{dt}(mr^2 \sin^2 \theta \dot{\phi})+\frac{\partial V}{\partial \phi}=0.$$

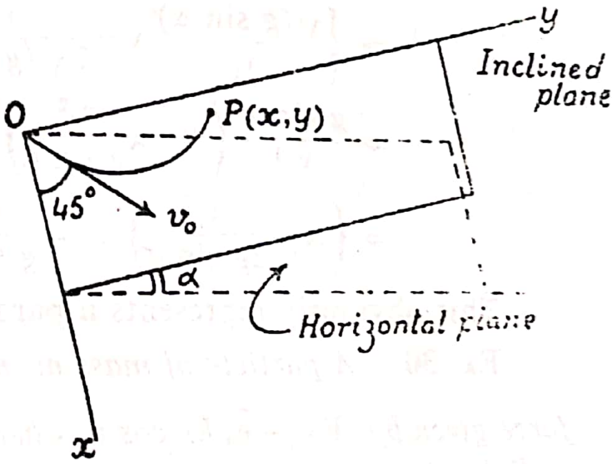
$$\Rightarrow mr^2 \sin^2 \theta \ddot{\phi}+2mr^2 \sin \theta \cos \theta\, \dot{\theta} \dot{\phi}+\frac{\partial V}{\partial \phi}=0. \qquad \ldots(6)$$

Whence equations (5) and (6) are required Lagrange's equations for a particle moving on the surface of earth.

Ex. 29. *A particle is constrained to move two-dimensionally on a smooth inclined plane making an angle α ($\alpha<90°$) to the horizontal. Assume that the line of intersection of the inclined plane with the horizontal plane is the x-axis and a line on the inclined plane drawn perpendicular to the x-axis is the y-axis.*

(*a*) *Obtain an expression for the Lagrangian of the particle in terms of the x and y co-ordinates.*

(*b*) *Solve the Lagrangian equations of the motion and show that the trajectory of the particle on the inclined plane will be a parabola, if the particle was initially at $x=0$, $y=0$ with an initial velocity v_0 along the line $x=y$.* (Agra 1967)



Solution. (*a*) Figure is self explanatory. K.E. of the particle is given by

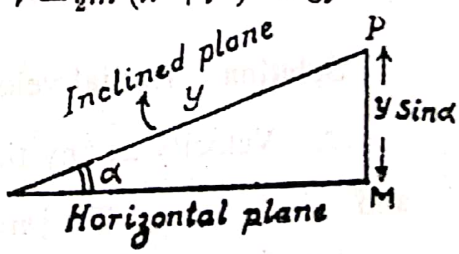
$$T=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2). \qquad \ldots(1)$$

The potential energy of the particle is

$$V=mg \cdot PM=mgy \sin \alpha.$$

$\therefore$ Lagrangian function $(L)=T-V=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)-mgy \sin \alpha$.

(*b*) Hence Lagrange's x and y equations are :



$(d/dt)\,(\partial L/\partial \dot{x})-(\partial L/\partial x)=0$

and $(d/dt)\,(\partial L/\partial \dot{y})-(\partial L/\partial y)=0$

$\Rightarrow (d/dt)\,(m\dot{x})-0=0$

and $(d/dt)\,(m\dot{y})+mg \sin \alpha=0$

$\Rightarrow m\ddot{x}=0$ and $m\ddot{y}+mg \sin \alpha=0$ $\qquad \ldots(2)$

$\Rightarrow \ddot{x}=0$ and $\ddot{y}=-g \sin \alpha$.

Now $\ddot{x}=0 \Rightarrow \dot{x}=$ constant $=C$ say $=v_0\cos 45°=(v_0/\sqrt{2})$. ...(3)

Again integrating, we get $x=(v_0/\sqrt{2})\,t+D$, where D is the constant of integration. Initially $t=0$, $x=0 \Rightarrow D=0$ and thus

$$x=(v_0/\sqrt{2})\,t. \quad ...(4)$$

Further $\ddot{y}=-g\sin\alpha$, $\therefore\ \dot{y}=-g\sin\alpha t+E$, where D is any constant of integration; when $t=0$, we have $\dot{y}=(v_0/\sqrt{2})$,

$\therefore\ E=(v_0/\sqrt{2})$ and thus $\dot{y}=(v_0/\sqrt{2})-g\sin\alpha t$.

This implies $y=(v_0/\sqrt{2})\,t-g\,(\sin\alpha/2)\,t^2-F$ where F is the constant of integration. When $t=0$, we have $y=0$.

$$\therefore\ F=0 \Rightarrow y=(v_0 t/\sqrt{2})-\tfrac{1}{2}g\sin\alpha t^2. \quad ...(5)$$

Now eliminating t between (4) and (5), we get

$$y=\frac{v_0}{\sqrt{2}}\cdot\frac{\sqrt{2}x}{v_0}-\tfrac{1}{2}g\sin\alpha\cdot\left(\frac{\sqrt{2}x}{v_0}\right)^2=x-\tfrac{1}{2}g\sin\alpha\cdot\frac{2x^2}{v_0^2}=x-\frac{g\sin\alpha}{v_0^2}x^2$$

$$\Rightarrow y=\frac{v_0^2}{4g\sin\alpha}-\left\{\frac{\sqrt{(g\sin\alpha)}}{v_0}x-\frac{v_0}{2\sqrt{(g\sin\alpha)}}\right\}^2$$

$$\Rightarrow \left\{\frac{\sqrt{(g\sin\alpha)}}{v_0}x-\frac{v_0}{2\sqrt{(g\sin\alpha)}}\right\}^2=\left(\frac{v_0^2}{4g\sin\alpha}-y\right)$$

$$\Rightarrow \frac{g\sin\alpha}{v_0^2}\left(x-\frac{v_0^2}{2g\sin\alpha}\right)^2=-\left(y-\frac{v_0^2}{4g\sin\alpha}\right)$$

$$\Rightarrow \left(x-\frac{v_0^2}{2g\sin\alpha}\right)^2=-\frac{v_0^2}{g\sin\alpha}\left(y-\frac{v_0^2}{4g\sin\alpha}\right).$$

This obviously represents a parabola.

Ex. 30. *A particle of mass m moves in a plane in the field of force given by* $\mathbf{F}=-\hat{\mathbf{e}}_r\,kr\cos\theta$, *where k is a constant and* $\hat{\mathbf{e}}_r$ *is the radial unit vector.*

(*a*) *Will the angular momentum of the particle about the origin be conserved ? Justify your statement.*

(*b*) *Obtain the differential equation of the orbit of the particle.*

(Agra M.Sc. Physics 1988)

Solution. Radial velocity $=\dot{r}$, transverse velocity $=r\dot{\theta}$.

$\therefore$ Velocity at any time $=\sqrt{\{\dot{r}^2+r^2\dot{\theta}^2\}}$

and

$$T=\tfrac{1}{2}mv^2=\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2). \quad ...(1)$$

(*a*) Lagrange's θ-equation gives $(d/dt)\,(\partial T/\partial\dot{\theta})-(\partial T/\partial\theta)=Q_\theta$, *i.e.* $(d/dt)\,(mr^2\dot{\theta})-0=0$ ($\because$ $Q_\theta=0$ as there is no force in the θ-direction) *i.e.*, $mr^2\dot{\theta}=$constant *which implies that the angular momentum about the origin is conserved.*

(*b*) Lagrange's *r*-equation gives

$$(d/dt)\,(\partial T/\partial \dot{r})-(\partial T/\partial r)=Q_r \Rightarrow (d/dt)\,(m\dot{r})-mr\dot{\theta}^2=-kr\cos\theta$$

$$(\because\ Q_r=-kr\cos\theta) \Rightarrow \ddot{r}-r\dot{\theta}^2=-(k/m)\,r\cos\theta.$$

This is the required differential equation of the orbit of the particle.

3·17. First Integrals of Motion. The most important technique of finding solutions of the Lagrangian equation consists in obtaining the first integrals of motion. It is evident that the first integral of a set of differential equations is a function consisting of constants, unknowns and differentials of one order less than the order of the original differential equations. Such methods reduce the order of the system of differential equations whose solutions are to be obtained. Now we shall discuss the following two types of integrals.

(a) **Jacobian Integral.** When the potential energy of a system does not depend on the time, explicity (*i.e.* it does not contain terms involving time), then we come across with an important theorem stated below :

Theorem. *If the Lagrangian of a system does not contain time explicity, the quanity* $E=\Sigma\dfrac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-L$, *sum extending over* α, *known as the Jacobian integral, is a constant of motion, where E represents the total energy of the system.*

We have $$E=\Sigma\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-L \qquad \ldots(1)$$

$$\Rightarrow \frac{dE}{dt}=\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}-L\right\}=\Sigma\frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha+\Sigma\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha-\frac{dL}{dt}$$

$$=\Sigma\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha+\Sigma\frac{\partial S}{\partial \dot{q}_\alpha}\ddot{q}_\alpha-\left\{\frac{\partial L}{\partial t}+\Sigma\frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha+\Sigma\frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha\right\}$$

$$(\because\ L=L\,(q_\alpha, \dot{q}_\alpha, t))$$

$$=\left\{\Sigma\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\Sigma\frac{\partial L}{\partial q_\alpha}\right\}-\frac{\partial L}{\partial t}=-\frac{\partial L}{\partial L}$$

$$\left[\text{Using Lagrange's eq., } viz.\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0\right]. \qquad \ldots(2)$$

But, when L does not depend on time explicity we have $\dfrac{\partial L}{\partial t}=0$ implying $\dfrac{dE}{dt}=0 \Rightarrow S=\text{constant} \Rightarrow \Sigma\dfrac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-L=\text{constant}.$

Now we must find to that physical entity E refers. To do this, we proceed as follows :

By Euler's theorem on homogeneous functions, we have

$$\Sigma \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha = 2T, \text{ where } T \text{ is the K.E. of system.}$$

$$\therefore \ \Sigma \frac{\partial L}{\partial \dot{q}_\alpha} \dot{q}_\alpha - L = \Sigma \left\{ \frac{\partial}{\partial \dot{q}_\alpha} (T-V)\, \dot{q}_\alpha \right\} - (T-V)$$

$$= \Sigma \left\{ \left(\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial V}{\partial \dot{q}_\alpha} \right) \dot{q}_\alpha \right\} - T + V = \Sigma \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha - T + V$$

[$\because$ V the P.E. does not contain $\dot{q}_\alpha$]

$$= 2T - T + V = T + V$$

= Kinetic energy + Potential energy

= Total energy of the system $= E$. ...(3)

(b) Momentum Integrals. When the Lagrangian does not involve a certain co-ordinate, there exists another kind of integral which is known as *momentum integeal.*

The co-ordinate which is missing, is termed as *cyclic* or *ignorable co-ordinate.*

Lagrange's equations are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0.$

Now, let L does not contain q_α explicity so that $\frac{\partial L}{\partial q_\alpha} = 0$....(5)

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_\alpha} = \text{constant} = p_\alpha \text{ say.} \quad ...(6)$$

Hence if a co-ordinate q_α is cyclic, the quantity $(\partial L/\partial \dot{q}_\alpha)$ represents an integral of motion and is denoted by p_α. The constant p_α as defined above is known as the *conjugate momentum* to *momentum conjugate* to the variable q_α.

3·18. Velocity-dependent Potentials.

When the force system is conservative, the generalised forces can be derived from a scalar potential function independent of velocities the equations of motion can be exdressed in Lagrangian form. For non-conservative systems in which the potential depends on the velocities, the equations of motion can be expressed in Lagrangian from provided the generalised forces Q_α given by

$$Q_\alpha = -\frac{\partial V}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right).$$

The quatity V is known as generalised potential or the velocity dependent potential.

3·19. Lagrangian for a charged particle in an electro-magnetic field.

When a charged particle moves in an electro-magnetic field, the force acting on it given by

$$\mathbf{F}=e\,[\mathbf{E}+(1/c)\,\mathbf{v}\times\mathbf{B}] \qquad ...(1)$$

where E, B are the components of the electro-magnetic field and v represents for the velocity of the charged particle. Also further if ϕ denotes the scalar potential and A a vector potential then we have

$$\mathbf{B}=\nabla\times\mathbf{A} \text{ and } \mathbf{E}=-\nabla\phi-(1/c)\,\partial\mathbf{A}\,\partial t). \qquad ...(2)$$

The objective of this section is to obtain the generalised potential V from which the *Lorentz force* (F) can be derived. To achieve this, we have

$$(d/dt)\,(m\mathbf{v})=\mathbf{F}=e\,[\mathbf{E}+(1/c)\,\mathbf{v}\times\mathbf{B}]$$
$$=e\,[-\nabla\phi-(1/c)\,(\partial\mathbf{A}/\partial t)+(1/c)\,\dot{\mathbf{v}}\times(\nabla\times\mathbf{A})] \qquad ...(3)$$

But $(d\mathbf{A}/dt)=(\partial\mathbf{A}/\partial t)+\partial\mathbf{A}/\partial x)\,\dot{x}+(\partial\mathbf{A}/\partial y)\,\dot{y}+(\partial\mathbf{A}/\partial z)\,\dot{z}$

$$=(\partial\mathbf{A}/\partial t)+(\partial\mathbf{A}/\partial x)v_x+(\partial\mathbf{A}/\partial y)\,v_y+(\partial\mathbf{A}/\partial z)\,v_z$$
$$=(\partial\mathbf{A}/\partial t)+\mathbf{v}\cdot\nabla\mathbf{A} \qquad ...(4)$$

and $\mathbf{v}\times(\nabla\times\mathbf{A})=-(\mathbf{v}\cdot\nabla)\,\mathbf{A}+\nabla\,(\mathbf{v}\cdot\mathbf{A}). \qquad ...(5)$

$\therefore$ Equation (3)

$\Rightarrow (d/dt)\,(m\mathbf{r})=e\,[-\nabla\phi-(1/c)\,\{\partial\mathbf{A}/\partial t)+\mathbf{v}\nabla\mathbf{A}\}+(1/c)\,\nabla\,(\mathbf{v}\cdot\mathbf{A})]$

whence making use of (4), we again get

$$(d/dt)\,m\mathbf{v}=e\,[-\nabla\phi-(1/c)\,d\mathbf{A}/dt)+(1/c)\,\nabla\,(\mathbf{v}\cdot\mathbf{A})]$$

$$=e\left[-\nabla\left(\phi-\frac{\mathbf{v}\cdot\mathbf{A}}{c}\right)-(1/c)\,d\mathbf{A}/dt)\right]$$

$$\Rightarrow \frac{d}{dt}\left(m\mathbf{v}+\frac{e}{c}\,\mathbf{A}\right)=-e\nabla\left(\phi-\frac{\mathbf{v}\cdot\mathbf{A}}{c}\right). \qquad ...(6)$$

Equation (6) is having analogy with Lagrengian from of the type

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right)=\frac{\partial L}{\partial q_a}. \qquad ...(7)$$

In carterian system, equation (5) can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x},\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)=\frac{\partial L}{\partial y} \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)=\frac{\partial L}{\partial z} \qquad ...(8)$$

Comparing (8) with (6), we get

$$\frac{\partial L}{\partial \dot{x}}=mv_x+\frac{e}{c}\,A_x \text{ etc. and } \frac{\partial L}{\partial x}=\frac{\partial}{\partial x}\left[-e\left(\phi-\frac{\mathbf{v}\cdot\mathbf{A}}{c}\right)\right] \text{ etc.} \qquad ...(9)$$

or in vector notations,

$$\frac{\partial L}{\partial \mathbf{v}}=m\mathbf{v}+(c/e)\,\mathbf{A} \text{ and } \frac{\partial L}{\partial q_\alpha}=\frac{\partial}{\partial q_\alpha}\left[-e\left(\phi-\frac{\mathbf{v}\cdot\mathbf{A}}{c}\right)\right] \qquad ...(10)$$

where $q_\alpha,=x,\ y$ or z. ...(11)

This gives $K=\frac{1}{2}m\mathbf{v}^2+e\phi+(e/c)\,(\mathbf{v}\cdot\mathbf{A})=T-V.$ ...(12)

$\therefore$ Generalised potential, $V=e\phi-(e/c)\,(\mathbf{v}\cdot\mathbf{A})$.

Also the momentum $\mathbf{p}$ in the elecro-magnetic field is given by ...(13)

$$\mathbf{p}=(\partial L/\partial\mathbf{v})=m\mathbf{v}+(e/c)\,\mathbf{A}.$$

Obviously p_α is a constant of motion, if $(\partial L/\partial q_\alpha)=0$, and thus, p_α is the conjugate momentum to the cyclic variable q_α.

3·20. Lagrange's equation for electrical circusts.

For electrical circuits containing finite number of inductonces, condensers and resistances, Lagrange's equation may be written down in the following manner

$$(d/dt)\,(\partial L_{e.c.}/\partial \dot{q}_k)-(\partial L_{e.c.}/\partial q_k)=Q_k, \quad ...(1)$$

where $L_{e.c.}$ is the Lagrangian for the electrical circuit and equals to $T_m-V_{e.c.}$ i.e., $L_{e.c.}=T_m-V_{e.c.}$, ...(2)

where T_m is the magntic energy of the electrical circuit and is similar to the K.E, of the mechanical system, $V_{e.c.}$ is the electrical energy of the electrical circuit and is similar to the P.E. of the mechanical system and Q_k is the generalised force due to friction,

Note. The conservative forces are included in the Lagrangian L. If the system does not contain dissipative or frictional forces *i.e.* if $Q_k=0$. then equation (1).

$$\Rightarrow \quad (d/dt)\,(\partial L_{e.e.}/\partial \dot{q}_k)-(\partial L_{e.c.}/\partial q_k)=0 \quad ...(3)$$

Ex. 31. *An electrical circuit contains an inductance L and capacitance. Find the Lagrangian equation of motion when the charge of the condenser is q.*

Solution. Let T_m be the magnetic energy of the electrical circuit, then we have

$$T_m=\tfrac{1}{2}L\dot{q}^2=\tfrac{1}{2}Li^2. \quad ...(1)$$

[$\therefore$ current$=i=\dot{q}$]

Also the electrical energy of the electrical circuit is given by $V_{e.c.}=\frac{1}{2}Cq^2$ where C is the capacitance of the circuit, ...(2)

$\therefore$ Lagrangian function for the electrical circuit is given by

$$L_{e.c.}=T_m-V_{e.c.}=\tfrac{1}{2}L\dot{q}^2-\tfrac{1}{2}q^2. \quad ...(A)$$

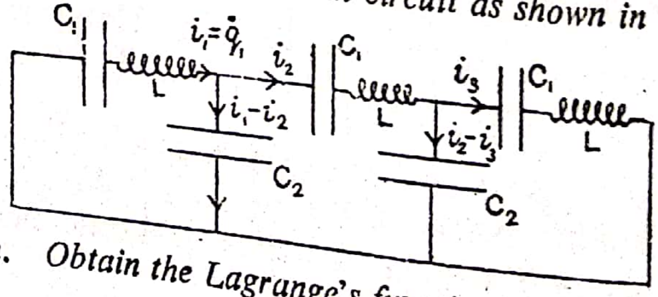
Now Lagrange's q-equation gives.

$$(d/dt)\,(\partial L_{e.c.}/\partial \dot{q})-(\partial L_{e,c.}/\partial q)=0 \quad ...(3)$$

$$\Rightarrow \quad (d/dt)\,(L\dot{q})+(q/C)=0 \quad \text{[using (A)]}$$

$$\Rightarrow \quad L\ddot{q}+(q/C)=0.$$

Ex. 32. *Consider an electrical circuit as shown in the adjoin-*



ing diagram. Obtain the Lagrange's function for the system and *hence obtain the equations of motion where C and L denote that capacitance and inductance respectively.*

Solution. The figure is self-explanatory. Now magnetic energy of the electrical circuit is given by

$$T_m = \tfrac{1}{2}L\,(i_1^2 + i_2^2 + i_3^2)$$
$$= \tfrac{1}{2}L\,(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2),$$

while electrical energy of the electrical circuit is given by ...(1)

$$V_{e.c.} = \tfrac{1}{2}\frac{(q_1^2 + q_2^2 + q_3^2)}{C_1} + \tfrac{1}{2}\frac{(q_1 - q_2)^2}{C_2} + \tfrac{1}{2}.\frac{(q_2 - q_3)^2}{C_3}$$

...(2)

$$\therefore\; L_{e.c.} = T_m - V_{e.c.} = \tfrac{1}{2}L\,(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \frac{q_1^2 + q_2^2 + q_3^2}{2C_1} - \tfrac{1}{2}\frac{(q_1 - q_2)^2}{C_2} - \tfrac{1}{2}\frac{(q_2 - q_3)^2}{C_3}$$

$$\left.\begin{aligned}
&\Rightarrow \frac{\partial L_{e.c.}}{\partial \dot{q}_1} = L\dot{q}_1,\; \frac{\partial L_{e.c.}}{\partial \dot{q}_2} = L\dot{q}_2,\; \frac{\partial L_{e.c.}}{\partial \dot{q}_3} = L\dot{q}_3,\; \frac{\partial L_{e.c.}}{\partial q_1} = \frac{-q_1}{C_1} - \frac{(q_1 - q_2)}{C_2} \\
&\frac{\partial L_{e.c.}}{\partial q_2} = -\frac{-q_2}{C_1} + \frac{q_1 - q_2}{C_2} - \frac{q_2 - q_3}{C_2},\; \frac{\partial L_{e.c.}}{\partial q_3} = \frac{-q_3}{C_1} + \frac{q_2 - q_3}{C_2}
\end{aligned}\right\}$$

...(3)

Hence Lagrange's q_1-equation gives

$$\frac{d}{dt}\left(\frac{\partial L_{e.c.}}{\partial \dot{q}_1}\right) - \frac{\partial L_{e.c.}}{\partial q_1} = 0 \;\Rightarrow\; L\ddot{q}_1 + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0.$$

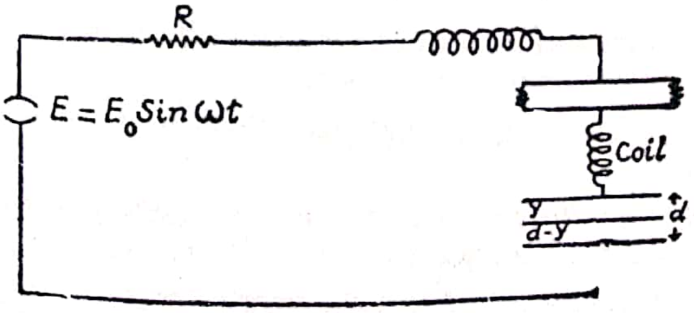
...(4)

Similarly Lagrange's q_2 and q_3 equations

$$\Rightarrow\; L\ddot{q}_2 + \frac{q_2}{C_1} - \frac{q_1 + q_3 - 2q_2}{C_2} = 0 \text{ and } L\ddot{q}_3 + \frac{q_3}{C_1} - \frac{q_2 - q_3}{C_2} = 0.$$

Ex. 33. *Consider an electro-magnetic system as shown in the adjoining figure. The upper plate of the condenser of capacity C, having mass m, is suspended from a coil of spring constant k while its lower plate is fixed. The upper plate is free to move vertically under the action of the spring and electric field between the plates, i.e., the consider is of variable capacity. Find (a) the Lagrangian and (b) the equation of motion.*

Solution. In the position of equilibrium, when consider is



uncharged, let d be the distance between the plates. When the charge on the condenser is q, let $d-y$ be the distance between the plates resulting its capacity as

$$C=\frac{\lambda A}{4\pi (a-y)}=\frac{A}{4\pi (d-y)} \quad (\because \quad \lambda=1 \text{ as the medium is air}),$$

$\therefore$ Magnetic or kinetic energy of the electrical system

$$=\tfrac{1}{2}L\dot{q}^2=T_{e.c.}$$

and kinetic energy of the mechanical system

$$=\tfrac{1}{2}m\dot{y}^2=T_{m.s.}$$

Thus potential energy of the electrical system is

$$V_{e.c.}=\tfrac{1}{2}(q^2/C)-qE_0 \sin \omega t,$$

potential energy of the mechanical system is $V_{m.s.}=\frac{1}{2}\kappa y^2$,

$$\Rightarrow L_{e.c.}=T_{e.c.}+T_{m.s.}-V_{e.c.}-V_{m.s.}$$

$$=\tfrac{1}{2}L\dot{q}^2+\tfrac{1}{2}m\dot{y}^2-\tfrac{1}{2}(q^2/C)+qE_0 \sin \omega t-\tfrac{1}{2}\kappa y^2$$

$$=\tfrac{1}{2}L\dot{q}^2+\tfrac{1}{2}m\dot{y}^2+qE_0 \sin \omega t-\tfrac{1}{2}\kappa y^2-\frac{4\pi (d-y)}{2A}\cdot q^2 \quad ...(1)$$

Also the frictional or dissipative force is given by

$$Q_q=-R\dot{q}.$$

Now Lagrange's q and x-equations are

$$\frac{d}{dt}\left(\frac{\partial L_{e.m.}}{\partial \dot{q}}\right)-\frac{\partial L_{e.m.}}{\partial q}=Q_k \text{ and } \frac{d}{dt}\left(\frac{\partial L_{e.m.}}{\partial \dot{x}}\right)-\frac{\partial L_{e.c.}}{\partial x}=Q_x. \quad ...(3)$$

Now equation (1)

$$\Rightarrow \frac{\partial L_{e.m.}}{\partial \dot{q}}=L\dot{q}, \quad \frac{\partial L_{e.m.}}{\partial q}=E_0 \sin \omega t-\frac{4\pi (d-y)}{A} q,$$

$$\frac{\partial L_{e.m.}}{\partial \dot{x}}=m\dot{y}, \quad \frac{\partial L_{e.m.}}{\partial x}=-\kappa y+\frac{2\pi q^2}{A}.$$

$\therefore$ (3) gives $L\ddot{q}+(4\pi/A)(d-y)\, q-E_0 \sin \omega t=-R\dot{q}$

and $\quad m\ddot{y}+\kappa y-(2\pi/A)\, q^2=0.$

Supplementary Problems

1. Classify each of the following according as they are (i) Scleronomic or Rheonomic, (ii) Holonomic or Non-Holonomic, and (iii) Conservative or Non-Conservative.

   (*a*) a horizontal cylinder of radius a rolling inside a perfectly rough hollow horizontal cylinder of radius $b > a$;

   (*b*) a cylinder rolling (and possibly sliding) down an inclined plane of angle α;

   (*c*) a sphere rolling down another sphere is rolling with uniform, speed along a horizontal plane;

   (*d*) a particle constrained to move along a line under the influence of a force which is inversely proportional to the square of its distance from a fixed point and a damping force proportional to the square of the instantanous speed.

Ans. (*a*) Scleronomic, holonomic, conservative,
(*b*) Scleronomic, non-holonomic, conservative,
(*c*) Rheonomic, non-holonomic, conservative,
(*d*) Scleronomic holonomic, non-conservative.

2. If $F(\lambda x, \lambda y, \lambda z) = \lambda_n F(x, y, z)$ where λ is a parameter, then F is said to be a homogeneous function of other n. Determine which (if any) of the following functions are homogeneous, giving the order in each case :

(*a*) $x^2+y^2+z^2+xy+yz+zx$,

(*b*) $3x-2y+4z$,

(*c*) $xyz+2xy+2xz+2yz$.

(*d*) $(x+y+z)/x$,

(*e*) $x^2 \tan^{-1}(y/x)$,

(*f*) $(x+y+z)/(x^2+y^2+z^2)$.

Ans. (*a*) homogeneous of order 2, (*b*) homogeneous of order 1, (*c*) non-homogeneous, (*d*) homogeneous of order zero, (*e*) homogeneous of order 3, (*f*) homogeneous of order -1.

3. (*a*) Set up the Lagrangian for a particle of mass m falling freely in a uniform gravitational field and (*b*) write Lagrange's equations.

4. Use Lagrange's equations to describe the motion of a projectile of mass m down a frictionless inclined plane of angle α.

5. Use Lagrange's equations to describe the motion of a particle launched with speed v_0 at angle α to the horizontal.

6. A particle of mass m is connected to a fixed point P on a horizontal plane by a string of length l. The plane rotates with constant angular speed ω about a vertical axis through a point O of the plane where $OP=a$, (*a*) Set up the Lagrangian of the system, (*b*) Write the equations of motion of the particle.

7. The rectangular co-ordinates (x, y, z) defining the position of a particle of mass m moving in a force field having potential V are given in terms of spherical co-ordinates (r, θ, ϕ) by the transformation equations

$$x = r \sin\theta \cos\phi,\ y = r \sin\phi \sin\theta,\ z = r\cos\theta.$$

Use Lagrange's equation to set up the equation of motion.

Ans.
$$m[\ddot{r} - r\dot{\phi}^2 \cos^2\phi] = -\frac{\partial V}{\partial r}$$

$$m\left[\frac{d}{dt}(r^2\dot{\phi}) + r^2\dot{\theta}^2 \sin\phi\cos\phi\right] = -\frac{1}{r}\frac{\partial V}{\partial \theta}$$

$$m\frac{d}{dt}(r^2\dot{\theta}\sin^2\phi) = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}.$$

8. Use the method of Lagrange's equations for the non-holonomic system to solve the problem of a particle of mass m sliding down a frictionless inclined plane of angle α.

9. A uniform rod AB of mass m and length $2a$, attached to a fixed point O by a light elastic string OA of natural length l and modulus of elasticity λ, moves in a vertical plane through O with the string taught. At time t, OA, AB make angle θ, ϕ with the downward vertical and the length of OA is $l+x$. Prove that the kinetic energy of the rod is

$$\tfrac{1}{2}m\,[\dot{x}^2+(l+x)^2\,\dot{\theta}^2+\tfrac{4}{3}a^2\dot{\phi}^2]$$
$$+ma\dot{\phi}\,[(l+x)\,\dot{\theta}\cos(\phi-\theta)-\dot{x}\sin(\phi-\theta)]$$

Derive Lagrange's equations of motion of the system, if in addition to gravity an external force with horizontal and vertical component X, Y acts on the rod at B.

Ans. $m\,\{\ddot{x}-a\ddot{\phi}\sin(\phi-\theta)-a\dot{\phi}^2\cos(\phi-\theta)-(l+x)\,\dot{\theta}^2\}$
$$=-\lambda x/l+mg\cos\theta+X\sin\theta+Y\cos\theta;$$
$$m\,\{(l+x)^2\,\ddot{\theta}+2\,(l+x)\,\dot{x}\dot{\theta}+a\ddot{\phi}\,(l+x)\cos(\phi-\theta)$$
$$-a\dot{\phi}^2\,(l+x)\sin(\phi-\theta)\}$$
$$=-mg\,(l+x)\sin\theta+X\,(l+x)\cos\theta-Y\,(l+x)\sin\theta;$$
$$m\,\{\tfrac{4}{3}\,a\ddot{\phi}+2\dot{x}\dot{\theta}\cos(\phi-\theta)+(l+x)\,\ddot{\theta}\cos(\phi-\theta)$$
$$+(l+x)\,\dot{\theta}^2\sin(\phi-\theta)-\ddot{x}\sin(\phi-\theta)\}$$
$$=-mg\sin\phi+2X\cos\phi-2Y\sin\phi.$$

10. The ends A, B of a light rod of length $2a$ are constrained to move along two fixed smooth wires OC, OZ respectively; OC is horizontal and OZ vertically downwards. A particle of mass m is attached to each end of the rod and a bead of mass M can slide freely along the rod. The whole system is made to rotate about OZ with constant angular velocity ω and at time t the bead is distant $a+x$ from B, while the rod makes an angle θ with the upward, prove that the kinetic energy of the system is

$$2ma^2\,[\dot{\theta}^2+\omega^2\sin^2\theta]+\tfrac{1}{2}M\,[\dot{x}^2+(a^2+2ax\cos 2\theta+x^2)\,\dot{\theta}^2$$
$$+2a\,\dot{x}\dot{\theta}\sin 2\theta+(a+x)^2\,\omega^2\sin^2\theta].$$

Show, by using Lagrange's equations, that a steady motion in which $x=0$, $\theta=\alpha$ ($=$constant), is possible only if $a\omega^2=g\cot\alpha\,\mathrm{cosec}\,\alpha$

and $$(M+2m)\tan^2\alpha=M+4m.$$

11. A uniform rod of length l and mass M is at rest on a horizontal functionless table. An impulse of magnitude I is applied to one end A of the rod and perpendicular to it Prove that (a) the velocity given to end A is $4I/M$, (b) the velocity of the centre of mass is I/M and (c) the rod rotates about the centre of mass with angular velocity of magnitude $6I/Ml$.

12. A square of side a and mass M, formed from 4 uniform rods which are smoothly hinged at their edges, rests on a horizontal frictionless plane. An impulse is applied at a vertex in a direction of the diagonal through the vertex so that the vertex is given a velocity of magnitude v_0. Prove, that the rods move about their centres of mass with angular speed $3v_0/4a$.

13 Two uniform rods AB, BC of masses m_1, m_2 and lengths $2a$. $2b$ are smoothly hinged at B and initially they lie at rest on a smooth table and in a straight line. AB receives a blow of impulse I at A perpendicular to AB. Construct the equations of motions of the system just after impulse.

14. AB, BC, CD are three equal rods freely joined together to form the three sides of a square, the rods lying on a smooth horizontal table. The end A is pivoted to the table. If the end D receives a blow in the direction AD, show that the initial angular velocity of the rods are as $1 : 6 : 11$.

15. The ends of a uniform rigid rod of mass m are moving with velocity u, v. Prove that the K. E. of the rod is $1/6m$ (u^2+u, v+v^2). Three uniform rods OA, AB, BC each of mass m, are smoothly jointed together at A, B and hang in equilibrium under gravity from fixed smooth joint at O. An impulsive couple G is applied to the rod AB in a vertical plane through O. Prove that the initial kinetic energy of the system is $\dfrac{57G^2}{26\,Ml^2}$, where l is the length of AB.

16. A square plate has mass M and side $2a$. It is freely suspended from a corner by a string of length $2a\sqrt{2}$ and receives a blow F in the plant of the as its lowest point. Show that the plate will begin to rotates with angular velocity $\dfrac{3F}{Ma\sqrt{2}}$.

17. The end points of a uniform rod of mass m are moving perpendicular to the length of the rod and in the same direction with velocities u, v ; prove that the kinetic energy of the rod is $1/6m\,(u^2+uv+v^2)$.
Three rods A_1A_2, A_2A_3, A_3A_1, each of mass m, smoothly hinged together at A_2, A_3, hang in equilibrium under gravity from a fixed smooth hinge at A_1. A horizontal impulse is epplied to a point P of A_1A_3. If there is no impulsive reaction on the hinge A_1, show that $\dfrac{A_1P}{PA_2}=\dfrac{26}{7}$.

18. Explain in detail the Lagrangian method for finding the periods of the normal modes of small oscillation about a position of stable equilibrium for a holonomic system with n degrees of freedom.

A bead of mass $3m$ is free to slide on a smooth thin uniform wire of mass $2m$ bent into the shape of a circle of a radius a. One point of the wire is fixed and the system is performing small oscillations in a vertical plane about its position of stable equilibrium. Show that the periods of the normal modes are $2\pi\,(2a/g)^{1/2}$, $2\pi\,(2a/5g)^{1/2}$.

19. A uniform rod BC of mass m and length $2a$ is attached to a fixed point A by a light elastic string, AB of natural length b. The modulus of elasticity of the string is such that its stretched length is a when the rod hangs from A in equilibrium under gravity. The system makes small oscillations, about the configuration of equilibrium in vertical plane through A. Find the kinetic and potential energies of the system if at time t the length of AB is $a+x$ and AB, BC makes angle θ, ϕ respectively, with the vertical. Show that one normal period of vibration is $2\pi\,[(a-b)/g]^{1/2}$ and find the other two periods. Ans. Periods are $2\pi/p$, where $p^2=\frac{1}{2}\,(7\pm\sqrt{7})\,g/a$.

20. Two double pendulums ABC, $A'B'C'$, smoothly jointed at B and B' are freely suspended from the fixed points A, A' at the same level and smoothly connected by a rod BB'. The lengths AB, BC, BB' $B'C'$, $B'A'$, AA' are

each equal to $2a$. The rods AB, BB' $B'A'$ are each uniform and of mass m and the rods BC, BC' are each uniform and of mass $4m$, the rods AB, BC $B'C'$ makes angle θ, ϕ, ψ with the downward vertical. Prove that $\phi-\psi$ is a normal co-ordinates for small oscillations of the system in a verticle plane through AA' about the equilibrium configuration, the length of the equivalent simple pendulum in the corresponding normal mode of oscillation being $4a/3$.

Show that the remaining normal co-ordinates are

$$5\theta-2\phi-2\psi \quad \text{and} \quad 4\theta+\phi+\psi$$

and that the corresponding equivalent simple pendulum of lengths

$$a/3, 44a/15.$$