(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. (14)

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}.$$ (13)

8. (a) Solve $x = y\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$. (10)

(b) Find the value of the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$, where the vector field,

$\vec{F} = (\sin y)\vec{i} + x(1+\cos y)\vec{j}$. (10)

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$ (10)

(d) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1-x)^{-2}$. (10)

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PAPER-II

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. Answer the following:

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \text{ is irrational} \\ -1, & x \text{ is rational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$. (10)

(b) Show that every field is without zero divisor. (10)

(c) Evaluate the integral

$$\int_{2-i}^{4+i} (x+y^2-ixy)\,dz$$

along the line segment AB joining the points A(2, –1) and B(4, 1). (10)

(d) Show that the functions:

$u = x^2 + y^2 + z^2$

$v = x + y + z$

$w = yz + zx + xy$

are not independent of one another. (10)

2. (a) Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. (13)

(b) If

$$u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right),$$

show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$$ (13)

(c) Solve the following problem by Simplex Method. How does the optimal table indicate that the optimal solution obtained is not unique?

Maximize $z = 8x_1 + 7x_2 - 2x_3$

subject to the constraints

$x_1 + 2x_2 + 2x_3 \le 12$

$2x_1 + x_2 - 2x_3 \le 12$

$x_1, x_2, x_3 \ge 0$ (14)

3. (a) Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$. (13)

(b) Show that the function

$u(x, y) = e^{-x}(x \cos y + y \sin y)$

is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$. (13)

(c) If R is an integral domain, show that the polynomial ring R[x] is also an integral domain. (14)

4. (a) Using the Residue Theorem, evaluate the integral $\int_C \frac{e^z - 1}{z(z-1)(z+i)^2} dz$, where C is the circle $|z| = 2$. (13)

(b) Examine the series

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \right]$$

for uniform convergence. Also, with the help of this example, show that the condition of uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for the sum S(x) of the series to be continuous. (13)

(c) Find the initial basic feasible solution of the following minimum cost transportation problem by Least Cost (Matrix Minima) Method and using it find the *optimal* transportation cost:

		Destinations				
		D_1	D_2	D_3	D_4	Supply
Sources	S_1	5	11	12	13	10
	S_2	8	12	7	8	30
	S_3	12	7	15	6	35
Requirement		15	15	20	25	

(14)

Section-B

5. Answer the following:

(a) Using Lagrange's interpolation formula, show that $32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4)$. (10)

(b) Solve

$(D^3 D'^2 + D^2 D'^3)z = 0$,

where D stands for $\frac{\partial}{\partial x}$ and D′ stands for $\frac{\partial}{\partial y}$. (10)

(c) Write a computer program to implement trapezoidal rule to evaluate

$$\int_0^{10} \left(1 - e^{-\frac{x}{2}}\right) dx.$$ (10)

(d) Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{q} + \nu \nabla^2 \vec{\Omega},$$

where ν is kinematic viscosity. (10)

6. (a) Using Method of Separation of Variables, solve Laplace Equation in three dimensions. (13)

(b) Derive the differential equation of motion for a spherical pendulum. (13

(c) A river is 80 metres wide. The depth d (in metres of the river at a distance x from one bank of the river is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section of the river. (14)

7. (a) Show that

$$u = \frac{A(x^2 - y^2)}{(x^2 + y^2)^2}, \; v = \frac{2Axy}{(x^2 + y^2)^2}, \; w = 0$$

are components of a possible velocity vector for inviscid incompressible fluid flow. Determine the pressure associated with this velocity field. (13)

(b) Solve the following system of equations using Gauss–Seidel Method:

$28x + 4y - z = 32$

$2x + 17y + 4z = 35$

$x + 3y + 10z = 24$

correct to three decimal places. (13)

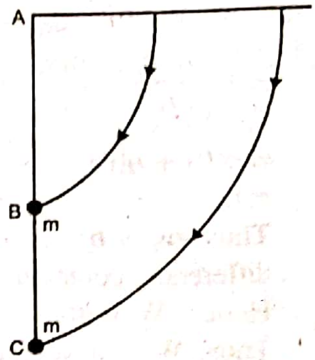
(c) Draw a flow chart for interpolation using Newton's forward difference formula. (14)

8. (a) Solve

$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

using Lagrange's Method. (13)

(b) A weightless rod ABC of length $2a$ is movable about the end A which is fixed and carries two particles of mass m each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$. (13)



(c) Using Euler's Modified Method, obtain the solution of

$$\frac{dy}{dx} = x + \left|\sqrt{y}\right|,\ y(0) = 1$$

for the range $0 \le x \le 0.6$ and step size 0.2. (14)

ANSWERS

PAPER-I

Section-A

1. (a) $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$, $\alpha_4 = (-1, 0, -1)$ are the elements of V.

Let a, b be two scalars such that the subspace generated is

$= a\,\alpha_1 + b\,\alpha_2 = a(1, 1, 2) + b(0, 1, 3)$

$= (a, a + b, 2a + 3b)$

Now similarly, subspace spanned by (α_3, α_4) is

$= c\,\alpha_3 + d\,\alpha_4 = c(2, 4, 5) + d(-1, 0, -1)$

$= (2c - d, 4c, 5c - d)$

According to the question,

intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$ is given by

$(a, a + b, 2a + 3b) = (2c - d, 4c, 5c - d)$

$\Rightarrow (a = 2c - d), (a + b = 4c),$

$(2a + 3b = 5c - d)$

$d = 2c - a,\ b = 4c - a,\ 3b = 5c - d - 2a$

1. (b) Let W be the set of all functions which satisfy the differential equation,

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$

$$\therefore\ W = \left\{ f : \frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0 \right\}$$

Let $y = f(x)$

Obviously $f(x) = 0$ or $y = 0$ satisfy the given differential equation and as such it belongs to W and thus $W \neq \phi$

Now let $y_1, y_2 \in W$, then

$$\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx} = 0 \quad ...(1)$$

and $$\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx} = 0 \quad ...(2)$$

Let $a, b \in R$. If W is to be a subspace then we should show that $ay_1 + by_2$ also belongs to W *i.e.*, it is a solution of the given differential equation.

We have

$$\frac{d^2}{dx^2}(ay_1 + by_2) + 3\frac{d}{dx}(ay_1 + by_2)$$

$$= a\frac{d^2 y_1}{dx^2} + b\frac{d^2 y_2}{dx^2} + 3a\frac{dy_1}{dx} + 3b\frac{dy_2}{dx}$$

PAPER-II

Section-A

1. (a) $f(x) = \begin{cases} 1, & x \text{ is irrational} \\ -1, & x \text{ is rational} \end{cases}$

First, Let a be any rational number so that $f(a) = -1$.

Since in any interval, there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that $|a_n - a| < \frac{1}{n}$.

Thus, the sequence $\{a_n\}$ converges to a. But $f(a_n) = 1$ for all n and $f(a) = -1$, so that

$$\lim_{n\to\infty} f(a_n) \neq f(a)$$

Thus, by theorem (*i.e.,* A function f defined on an interval I is continuous at a point $c \in$ I iff for every sequence $\{c_n\}$ in I converging to c, we have

$$\lim_{n\to\infty} f(c_n) = f(c)$$

applications, the function is discontinuous at any rational number a.

Hence, the function is discontinuous at all rational points. ...(1)

Next,

Let b be any irrational number. For each positive integer n, we can choose a rational number b_n such that

$|b_n - b| < \frac{1}{n}$, thus the sequence $\{b_n\}$ converges to b.

But $f(b_n) = -1$ for all n

and $f(b) = 1$

$$\therefore \lim_{n\to\infty} f(b_n) \neq f(b)$$

Hence, function is discontinuous at all irrational points. ...(2)

From (1) and (2) we conclude that the given function is discontinuous at every point in R.

1. (b) **To Prove:** Every field is without zero divisors —

Proof: Let a, b be elements of field F with $a \neq 0$ such that $ab = 0$

since $a \neq 0$ a^{-1} exists and we have

$$\begin{aligned} & ab = 0 \\ \Rightarrow\ & a^{-1}(ab) = a^{-1}(0) \\ \Rightarrow\ & (a^{-1}a)\,b = 0 \\ \Rightarrow\ & 1b = 0 && [\because a^{-1}a = 1] \\ \Rightarrow\ & b = 0 && [\because 1b = b] \end{aligned}$$

Similarly, let $ab = 0$ and $b \neq 0$.

Since $b \neq 0$; b^{-1} exists and we have

$$\begin{aligned} & ab = 0 \\ \Rightarrow\ & (ab)b^{-1} = 0b^{-1} \Rightarrow a(bb^{-1}) = 0 \\ \Rightarrow\ & a1 = 0 \quad \Rightarrow \quad a = 0 \end{aligned}$$

Thus in a field F, $ab = 0$

$\Rightarrow a^2 = 0$ or $b = 0$

Therefore, a field has no zero divisors.

1. (c) Evaluating $\int_C f(z)\,dz$

Here $f(z) = x + y^2 - ixy$

and C is the line segment AB joining the points A(2, –1) and B(4, 1).

In other words, C is a line segment from $z = 2 - i$ to $z = 4 + i$.

$\because\ z = x + iy$

$\therefore\ dz = dx + idy$

The equation of line AB is

$$y + 1 = \left[\frac{1-(-1)}{4-2}\right](x-2)$$

$$y + 1 = \frac{2}{2}(x-2)$$

$$y + 1 = x - 2$$

$$y = x - 3$$

$\therefore$ On the line, $y = x - 3$

$$dy = dx$$

Therefore, substituting these values, we get

Now, $\int_C f(z)\,dz = \int (x + y^2 - ixy)(dx + idy)$

$\because$ x varies from 2 to 4

$$= \int_2^4 \left[x+(x-3)^2 - ix(x-3)\right](dx+idx)$$

$$= \int_2^4 \left[x+x^2+9-6x-i(x^2-3x)\right](1+i)dx$$

$$= \int_2^4 \left[x^2-5x+9+i(3x-x^2)\right](1+i)dx$$

$$= \int_2^4 \left[x^2-5x+9-(3x-x^2)\right]dx +$$

$$i\int_2^4 \left[3x-x^2+x^2-5x+9\right]dx$$

$$= \int_2^4 \left[2x^2-8x+9\right]dx + i\int_2^4 (9-2x)dx$$

$$= \left[\frac{2x^3}{3} - \frac{8x^2}{2} + 9x\right]_2^4 + i\left[9x - \frac{2x^2}{2}\right]_2^4$$

$$= \left[\frac{2}{3}(4)^3 - 4(4)^2 + 9(4)\right] -$$

$$\left[\frac{2}{3}(2)^3 - 4(2)^2 + 9(2)\right]$$

$$+i\left\{\left[9(4)-(4)^2\right] - \left[9(2)-(2)^2\right]\right\}$$

$$= \left[\left(\frac{128}{3} - 28\right) - \left(\frac{16}{3} + 2\right) + i(20-14)\right]$$

$$= \left(\frac{112}{3} - 30\right) + 6i = \frac{22}{3} + 6i$$

1. (d) $u = x^2 + y^2 + z^2$ (given)

$v = x + y + z$

$w = yz + zx + xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial u}{\partial z} = 2z$$

$$\frac{\partial v}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 1, \quad \frac{\partial v}{\partial z} = 1$$

$$\frac{\partial w}{\partial x} = y + z, \quad \frac{\partial w}{\partial y} = x + z, \quad \frac{\partial w}{\partial z} = x + y$$

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix},$$

$R_1 \to R_1 + R_3$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$= 0$ $(\because R_1 = R_2)$

Hence, the functional relationship exists between u, v, w.

In other words, Functions u, v and w are not independent of one another.

2. (a) Elements of S_3

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

σ_1 = Identity

Now, $$\sigma_2^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \text{Identity}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

$$\sigma_4^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\sigma_4^3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

$$\sigma_5^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\sigma_5^3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

$$\sigma_6^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

So, $\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_6^2\} = I$

and $\{\sigma_1^3, \sigma_4^3, \sigma_5^3\} = I$.

2. (b) Here, $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$ (given)

From given u, it can be inferred that u is a homogeneous function of degree 2.

$\therefore$ From Euler theorem, we get

$$x\frac{du}{dx} + y\frac{du}{dy} = 2u \qquad \text{...(1)}$$

By partial differentiation of (1) with respect to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} \qquad \text{...(2)}$$

Now, by partial differentiation of (1) with respect to y, we get

$$\frac{x\partial^2 u}{\partial y \partial x} + \frac{y\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{2\partial u}{\partial y}$$

$$\Rightarrow x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} \qquad \text{...(3)}$$

(Using $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ if u is a continuous function having continuous first order partial derivatives.)

Multiplying equation (2) by x and equation (3) by y respectively, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = x\frac{\partial u}{\partial x} \qquad \text{...(4)}$$

and $$y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = y\frac{\partial u}{\partial y} \qquad \text{...(5)}$$

Adding (4) and (5), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u$$

[using relation (1)]

2. (c) Maximize $z = 8x_1 + 7x_2 - 2x_3$

subject to the constraints

$x_1 + 2x_2 + 2x_3 \leq 12$

$2x_1 + x_2 - 2x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

Introducing clock variables s_1 and s_2 then above LPP reduces to

$z = 8x_1 + 7x_2 - 2x_3$

Subject to $x_1 + 2x_2 + 2x_3 + s_1 = 12$

$x_1 + x_2 - 2x_3 + s_2 = 12$

Choose the initial basic solution

$x_1 = x_2 = x_3 = 0$

then $s_1 = s_2 = 12$.

Writing the above equations in tabular form we get

C_j →		8	7	−2	0	0.		
B	C_B	x_1	x_2	x_3	s_1	s_2	6	0 (Ratio)
s_1	0	1	2	2	1	0	12	12
s_2	0	2	1	−2	0	1	12	6
								→ Lowest +ve value
E_j →		0	0	0	0	0		
$D_j = C_j - E_j$		8 ↑	7	−2	0	0		

Choose the maximum D_j value which would indicate the incoming vector.

$\therefore$ x_1 is the incoming variable.

$\therefore$ s_2 is the outgoing variable.

(using minimum value rule)

Now, the next table is

[By dividing the second row by 2 and making other elements = 0 in that column (key).]

		8	7	−2	0	0		
		x_1	x_2	x_3	s_1	s_2	6	0
s_1	0	0	$\frac{3}{2}$	(3)	1	$\frac{-1}{2}$	6	$\frac{6}{2}=3\rightarrow$
s_2	8	1	$\frac{1}{2}$	−1	0	$\frac{1}{2}$	6	$\frac{6}{7}$ = negative
$E_j\rightarrow$		8	4	−8	0	4		
D_j=C_j−E_j			0	3	6	0	−4	
				↑				

Choosing maximum D_j value to indicate the incoming variable

$\therefore$ x_3 is the incoming variable.

and using minimum value rule

s_1 is the outgoing variable.

$\therefore$ Next table is:

	8	7	−2	0	0		
	x_1	x_2	x_3	s_1	s_2	6	0
x_3	6	0	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{-1}{6}$	2
x_1	8	1	1	0	$\frac{1}{3}$	$\frac{1}{3}$	7
$E_j\rightarrow$		8	11	6	$\frac{14}{3}$	$\frac{5}{3}$	
D_j=C_j − E_j		0	−4	−8	$\frac{-14}{3}$	$\frac{-5}{3}$	

The above table is the optimal table.

$\because$ all D_j are either negative or zero.

Thus, we get $x_1 = 7$, $x_3 = 2$

Thus, $z = 8(7) + 7x_2 - 2(2)$

$= 56 - 4 + 7x_2$

$$z = 52 + 7x_2 \quad \text{...(1)}$$

Using given equations we get

$7 + 2x_2 + 2(2) \le 12$

$\Rightarrow \quad 2x_2 \le 1$

$$x_2 \le \frac{1}{2}$$

and $2(7) + x_2 - 2(2) \le 12$

$\Rightarrow x_2 \le 2$

Also we know that $x_2 \ge 0$.

Taking these factors into account

$$0 \le x_2 \le \frac{1}{2} \quad \text{...(2)}$$

From (1) and (2) we get

$$z = 52 + 7\left(\frac{1}{2}\right) = 55.5$$

3. (a) Equation of the parabolic cylinder

$z = 4 - y^2$ (given)

and equation of the elliptic paraboloid is

$z = x^2 + 3y^2$

The required volume bounded

$$V = \int_A \int_{x^2+3y^2}^{4-y^2} dz\,dA$$

$$= 4\int_0^1 \int_0^{\sqrt{4-4y^2}} (4 - 4y^2 - x^2)\,dx\,dy$$

$$= 4\int_0^1 \left[(4-4y^2)x - \frac{x^3}{3}\right]_0^{\sqrt{4-4y^2}} dy$$

$$= 4\int_0^1 \left[\left(4-4y^2\right)^{3/2} - \frac{1}{3}\left(4-4y^2\right)^{3/2}\right] dy$$

$$= \frac{8}{3}\int_0^1 4^{3/2}(1-y^2)^{3/2}\,dy = \frac{64}{3}\int_0^1 \left(1-y^2\right)^{3/2} dy$$

Let $y = \sin\theta$ $dy = \cos\theta\,d\theta$

$$= \frac{64}{3}\int_0^{\pi/2} \cos^3\theta\cos\theta\,d\theta = \frac{64}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$V = \frac{64}{3} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$V = 4\pi$ cubic units.

3. (b) $u(x, y) = e^{-x}(x \cos y + y \sin y)$ [given]

Then $\dfrac{\partial u}{\partial x} = e^{-x}(\cos y) + (x \cos y + y \sin y)(-e^{-x})$

$= e^{-x}(\cos y - x \cos y - y \sin y)$

$\dfrac{\partial u}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y)$

Now,

$\dfrac{\partial^2 u}{\partial x^2} = e^{-x}(-\cos y) + (\cos y - x \cos y - y \sin y)(-e^{-x})$

$= e^{-x}(-\cos y - \cos y + x \cos y + y \sin y)$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2} = e^{-x}(x \cos y + y \sin y - 2\cos y)$

and $\dfrac{\partial^2 u}{\partial y^2} = e^{-x}(-x\cos y + \cos y - y \sin y + \cos y)$

$\Rightarrow \quad \dfrac{\partial^2 u}{\partial y^2} = e^{-x}(-x \cos y - y \sin y + 2\cos y)$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{x}(x\cos y + y \sin y - 2\cos y)$
$+ e^{-x}(-x \cos y - y \sin y + 2\cos y)$
$= e^{-x}[x \cos y + y \sin y - 2\cos y - x \cos y - y \sin y + 2\cos y]$

$\Rightarrow \quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic.

Let v be the harmonic conjugate of u. then $f(z) = u + iv$ is analytic.

$\therefore \quad f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

($\because$ By Cauchy-Riemann equations, $\dfrac{\partial v}{\partial x} = \dfrac{-\partial u}{\partial y}$)

$= e^{-x}(\cos y - x \cos y - y \sin y) - ie^{-x}(-x \sin y + y \cos y + \sin y)$

To apply **Milue-Thomson's method**, putting $x = z$ and $y = 0$, we get

$f'(z) = e^{-z}(1 - z - 0) - ie^{-z}(0)$

$\Rightarrow f'(z) = e^{-z}(1 - z)$

Now, integrating with respect to z, we get

$f(z) = (1 - z)(-e^{-z}) - \int -e^{-z}(-1)dz + \text{constant}$

$= -(1-z)e^{-z} - \int e^{-z}dz + \text{constant}$

$= -(1-z)e^{-z} - (-e^{-z}) + \text{constant}$

$= e^{-z}(-1 + z + 1) + \text{constant}$

$f(z) = e^{-x-iy}(x + iy) + \text{constant}$

$= e^{-x}(\cos y - i \sin y)(x + iy) + \text{constant}$

$= e^{-x}(x \cos y + y \sin y) + ie^{-x}(y \cos y - x \sin y) + iC$

$= u + i\left[e^{-x}(y\cos y - x \sin y) + C\right]$

$\therefore$ $v = e^{-x}(y \cos y - x \sin y) + C$

Hence, the harmonic conjugate of u is

$v = e^{-x}(y \cos y - x \sin y) + C$

Thus, the corresponding analytic function

$f(z) = u + iv$

$\Rightarrow \quad f(z) = e^{-x}(x\cos y + y \sin y)$
$+ i\left[e^{-x}(y\cos y - x\sin y) + C\right]$

3. (c) Let R be a commutative ring without zero divisors and with unity element 1.

Thus, R[x] is also a ring.

(Using, the set R[x] of all polynomials over an arbitrary ring R is a ring with respect to addition and multiplication of polynomials)

Now, to prove that R[x] is an integral domain, we should prove that

(*i*) R[x] is commutative

(*ii*) R[x] is without zero divisors

(*iii*) R[x] possesses the unity element

(*i*) R[x] is commutative

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots$ be any two elements of R[x].

If n is any non-negative integer, then the coefficient of x^n in

$$f(x)\,g(x) = \sum_{i+j=n} a_i b_j = \sum_{i+j=n} b_j a_i$$

(since R is commutative)

= coefficient of x^n in $g(x)\,f(x)$

$\therefore$ $f(x)g(x) = g(x)\,f(x)$

Hence, R[x] is a commutative ring.

(*ii*) R[x] is without zero divisors

Let $f(x) = a_0 + a_1x + b_2x^2 + \ldots + a_mx_m$, $a_m \neq 0$

$g(x) = b_0 + b_1x + b_2x^2 + \ldots\ldots\ldots + b_nx^n$, $b_n \neq 0$

be two non-zero elements of R[x].

Then $f(x)\, g(x)$ cannot be a zero polynomial *i.e.*, the zero element of R[x].

The reason is that at least one coefficient of $f(x)\, g(x)$ namely a_mb_n of x^{m+n} is $\neq 0$ because a_m, b_n are non-zero elements of R and R is without zero divisors.

(*iii*) R[x] possesses the unity element.

If 1 is the unity element of R, then the constant polynomial $1 + 0x + 0x^2 + 0x^3 + \ldots$ is the unity element of R[x]. We have

$[a_0 + a_1x + a_2x^2 + \ldots][1 + 0x + 0x^2 + 0x^3 + \ldots]$

$= (a_01) + (a_11)x + (a_21)x^2 + \ldots$

$= a_0 + a_1x + a_2x^2 + \ldots$

$\therefore$ Polynomial $1 + 0x + 0x^2 + \ldots$ or simply 1 is the unity element of R[x]

Hence, R[x] is an integral domain.

4. (a) We have $f(z) = \dfrac{e^z - 1}{z(z-1)(z+i)^2}$

The function $f(z)$ is analytic at every point within C except of the Poles $z = 0, 1, -i$

Note: All these poles lie inside C *i.e.*, $|z| = 2$

$\therefore$ Residues of all these poles have to be calculated.

Now, Residue at $z = 0$ is

$$\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{e^z-1}{(z-1)(z+i)^2}$$

$$= \frac{1-1}{(0-1)(0+i)^2} = \frac{0}{1}$$

$\therefore$ Residue at $z = 0$ is 0.

Now, residue at $z = 1$ is $\lim\limits_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1}\frac{e^z-1}{z(z+i)^2} = \frac{e-1}{1(1+i)^2}$$

$\therefore$ Residue at $z = 1$ is $\dfrac{e-1}{2i}$

($\because i^2 = -1$)

Now residue at $z = -i$ is

$$= \frac{1}{1!}\left[\frac{d}{dz}(z+i)^2 f(z)\right]_{z=-i}$$

($\because$ at $z = -i$, there is double pole of $f(z)$.)

$$= \left\{\frac{d}{dz}\left(\frac{e^z-1}{z(z-1)}\right)\right\}_{z=-i}$$

$$= \left[\frac{z(z-1)e^z - (e^z-1)(2z-1)}{z^2(z-1)^2}\right]_{z=-i}$$

$$= \left[\frac{z^2e^z - ze^z - (2ze^z - 2z - e^z + 1)}{z^2(z-1)^2}\right]_{z=-i}$$

$$= \left[\frac{e^z(z^2-3z+1) + 2z - 1}{z^2(z-1)^2}\right]_{z=-i}$$

$$= \left.\frac{-3ze^z + 2z - 1}{-(-i-1)^2}\right|_{\text{at } z=-i}$$

$$= \left.\frac{-1}{2i}(2z-1-3ze^z)\right|_{\text{at } z=-i}$$

Residue at $z = -i$

$$= -1\frac{(-2i-1-3ie^{-i})}{2i} = \frac{2i+1+3ie^{-i}}{2i}$$

$\therefore$ By residue theorem, we have

$$\int_C f(z)dz = 2\pi i$$

(sum of residues at $z = 0$, $z = 1$ and $z = -i$).

4. (b) Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$

Here, clearly $f'(x) = 0$

and $f_n'(x) = \dfrac{n(1-n^2x^2)}{(1+n^2x^2)^2}$

Therefore,

when $x \neq 0$, $f_n'(x) \to 0$ as $n \to \infty$ and so the formula $f'(x) = \lim\limits_{x\to\infty} f_n'(x)$ is true.

But at $x = 0$

$$f_n'(0) = \lim_{x \to 0} \frac{n}{(1+n^2x^2)^2}$$

$= n$, which tends to ∞ as $n \to \infty$

Thus, at $x = 0, f'(x) = \lim_{n \to 0} f_n'(x)$ is false.

It is $[f_n']$ that does not converge uniformly in an interval that contains zero.

4. (c) The transportation problem is given as

Destinations

		D_1	D_2	D_3	D_4	Supply
	S_1	5	11	12	13	10
Sources	S_2	8	12	7	8	30
	S_3	12	7	15	6	35
Requirement		15	15	20	25	

Note: This is a balanced transportation problem.

($\because$ Here total supply = total demand = 75 units.)

Now, using Least Cost Method or Matrix Minima Method we use the following approach.

The cheapest route is the one connecting the source S_1 to the destination D_1 where cost per unit is 5.

i.e., $C_{11} = 5$

Availability at S_1 is 10, demand at D_1 is 15.

Min [10, 15] = 10

$\therefore$ S_1 supplies, all the 10 units needed by D_1. Set $x_{11} = 10$ and encircle it.

Set $x_{12} = x_{13} = x_{14} = 0$ and put ×

	D_1	D_2	D_3	D_4	
S_1	5 (10)	11 (×)	12 (×)	13 (×)	10
S_2	8	12	7	8	30
S_3	12	7	15	6	35
	15 − 10 = 5	15	20	25	

Next cheapest route among uncrossed cells.

This route is (3, 4) connecting source S_3 to Destination D_4 with $C_{34} = 6$.

Availability at S_3 is 35 and demand D_4 is 25.

$\therefore$ Min (35, 25) = 25

Set $x_{34} = 25$ and encircle it. Set $x_{14} = x_{24} = 0$ and put ×.

	D_1	D_2	D_3	D_4	
S_1	5 (10)	11 (×)	12 (×)	13 (×)	
S_2	8	12	7	8 (×)	30
S_3	12	7	15	6 (25)	35 − 25 = 10
	5	15	20	25	

The next cheapest route among uncrossed cells is

C_{23} or $C_{32} = 7$

Let us choose $C_{23} = 7$

(**Note:** we can choose any of these two cells.)

Availability at S_2 = 30 and demand at $D_3 = 20$

Min (30, 20) = 20

$\therefore$ Set $x_{23} = 20$ and encircle it.

Set $x_{33} = 0$ and put ×.

	D_1	D_2	D_3	D_4	
S_1	5 (10)	11 (×)	12 (×)	13 (×)	
S_2	8	12	7 (20)	8 (×)	30 − 20 = 10
S_3	12	7	15 (×)	6 (25)	10
	5	15	20		

Now, the next cheapest route among uncrossed cells is

$C_{32} = 7$

Availability at S_3 = 10 and demand at $D_2 = 15$

$\therefore$ Min (10, 15) = 10

Set $x_{32} = 10$ and encircle it

Set $x_{31} = 0$ and put ×.

	D_1	D_2	D_3	D_4	
S_1	5 (10)	11 (×)	12 (×)	13 (×)	
S_2	8	12	7 (20)	8 (×)	10
S_3	12	7 (10)	15 (×)	6 (25)	10
	5	15 − 10 = 5			

Now, the next cheapest route among uncrossed cells is

$C_{21} = 8$ in the cell (2, 1)

Availability at S_2 = 10 while demand at $D_1 = 5$

$\therefore$ Set $x_{21} = 5$ and encircle it.

	D_1	D_2	D_3	D_4	
S_1	5 (10)	11 (×)	12 (×)	13 (×)	
S_2	8 (5)	12	7 (20)	8 (×)	10 − 5 = 5
S_3	12 (×)	7 (10)	15 (×)	6 (25)	10
	5	5			

Hence, the only uncrossed route is $C_{22} = 12$ in the cell (2, 2).

Thus, set $x_{22} = 5$ and encircle it.

	D_1	D_2	D_3	D_4
S_1	5 (10)	11 (×)	12 (×)	13 (×)
S_2	8 (5)	12 (5)	7 (20)	8 (×)
S_3	12 (×)	7 (10)	15 (×)	6 (25)

Now, all the minimum requirements have been satisfied and hence, the **Initial Basic Feasible Solution (IBFS)** is given as

$x_{11} = 10, x_{21} = 5, x_{22} = 5, x_{23} = 20, x_{32} = 10, x_{34} = 25$

and other variables, are non-basic.

Also the number of basic variables

$= (m + n) - 1 = (3 + 4) - 1 = 6.$

Thus, the total transportation cost

$= (10 \times 5) + (5 \times 8) + (5 \times 12) + (20 \times 7) + (10 \times 7) + (25 \times 6)$

$= 50 + 40 + 60 + 140 + 70 + 150 = 510$

Section-B

5. (a) To prove using Lagrange's interpolation formula

$32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4).$

Here we note that the values of x given are $-4, -2, 1, 2, 4$.

x	$f(x)$
$x_1 = -4$	$f(-4)$
$x_2 = -2$	$f(-2)$
$x_3 = 2$	$f(2)$
$x_4 = 4$	$f(4)$

By the help of these values, we can interpolate the function $f(x)$ using Lagrange's interpolation as:

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2) + \frac{(x-x_1)(x-x_2)(x-x_4)f(x_3)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$\Rightarrow f(x) = \frac{(x+2)(x-2)(x-4)}{(-4+2)(-4-2)(-4-4)}f(-4) + \frac{(x+4)(x-2)(x-4)}{(-2+4)(-2-2)(-2-4)}f(-2) + \frac{(x+4)(x+2)(x-4)}{(2+4)(2+2)(2-4)}f(2) + \frac{(x+4)(x+2)(x-2)}{(4+4)(4+2)(4-2)}f(4)$$

$$\Rightarrow f(x) = -\frac{(x^2-4)(x-4)}{96}f(-4) + \frac{(x^2-16)(x-2)}{48}f(-2) - \frac{(x^2-16)(x+2)}{48}f(2) + \frac{(x^2-4)(x+4)}{96}f(4)$$

Now, put $x = 1$ in the resulting polynomial we get

$$f(1) = \frac{-3}{32}f(-4) + \frac{10}{32}f(-2) + \frac{30}{32}f(2) - \frac{5}{32}f(4)$$

$$\Rightarrow 32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4)$$

5. (b) Solve $(D^3 D'^2 + D^2 D'^3)z = 0$

The given equation can be written as

$D^2 D'^2 (D + D')z = 0$

The auxiliary equation of the above equation is (by putting $D = m$ and $D' = 1$)

$m^2 (m + 1) = 0$

$m = 0, 0, -1$

Hence, the general solution of the given equation is

$z = \phi_1(y) + x\phi_2(y) + \phi_3(y - x)$

where, ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

5. (c) $I = \int_0^{10} \left(1 - e^{-x/2}\right) dx$

Trapezoidal rule,

$$I = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_{n-1} + y_n]$$

Here, $h = \frac{10-0}{n}$

$h = \frac{10}{n}$

n is our choice

Let $n = 10$ then $h = 1$.

$$I = \frac{1}{2}\left\{\left(1-e^{-0/2}\right)+2\left[\left(1-e^{-1/2}\right)+\left(1-e^{-2/2}\right)+\left(1-e^{-3/2}\right)+...+\left(1-e^{-10/2}\right)\right]\right\}$$

In this way, I is being calculated using trapezoidal rule.

5. (d) To prove $\left[\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}\cdot\nabla)\vec{q} + \nu\nabla^2\vec{\Omega}\right]$

where $\vec{\Omega}$ = vorticity vector = curl $\vec{q}$

$\vec{q}$ = velocity vector

ν = kinematic viscosity

The Navier-stokes equation of motion of an incompressible viscous fluid is given as

$$\frac{\partial q}{\partial t} - q\times(\nabla\times q) = -\nabla\left(\frac{p}{\rho}+\frac{1}{2}q^2\right)+\nabla^2 q \quad ...(1)$$

= Taking curl of the relation (1) we get

$$\frac{\partial}{\partial t}(\Omega) - \text{curl }(q\times\Omega) = \nu \text{ curl }(\nabla^2 q)$$

[where, vorticity vector $\vec{\Omega}$ = curl $\vec{q}$ = $\nabla \times q$ and using the property (curl grad $\phi = 0$)]

$$\therefore \text{curl}\left(-\nabla\left(\frac{p}{\rho}+\frac{1}{2}q^2\right)\right) = 0$$

$$\Rightarrow \frac{\partial\Omega}{\partial t} - [(\Omega\cdot\nabla)q - \Omega \text{ div } q - (q\cdot\nabla)\Omega + q \text{ div } \Omega] = u\,\nabla^2 \text{ curl } q$$

Using again two properties *i.e.*,

For incompressible fluids, div $q = 0$

and also always div (curl $\vec{F}$) = 0

$\therefore$ Here div (curl q) = 0

$\Rightarrow$ div (Ω) = 0

$\therefore$ The above relation becomes

$$\Rightarrow \frac{\partial\Omega}{\partial t} - [(\Omega\cdot\nabla)q - (q\cdot\nabla)\Omega] = \nu\nabla^2\Omega$$

$$\Rightarrow \left[\frac{\partial\Omega}{\partial t} + (q\cdot\nabla)\Omega\right] = (\Omega\cdot\nabla)q + \nu\nabla^2\Omega$$

$$\Rightarrow \frac{D\Omega}{Dt} = (\Omega\cdot\nabla)q + \nu\nabla^2\Omega$$

$\left[\text{using the fact that } \frac{D}{Dt} = \frac{\partial}{\partial t} + q\cdot\nabla\right]$

6. (a) Laplace equation in three dimensions is given as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ...(I)$$

or $\nabla^2 u = 0$

Let $u(x, y, z) = X(x)\,Y(y)\,Z(z)$

or simply $u = XYZ$

$$\therefore \frac{\partial u}{\partial x} = YZ\frac{dX}{dx}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = YZ\frac{d^2X}{dx^2} \quad ...(1)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = ZX\frac{\partial^2 Y}{\partial y^2} \quad ...(2)$$

and $$\frac{\partial^2 u}{\partial z^2} = XY\frac{\partial^2 Z}{\partial z^2} \quad ...(3)$$

Substituting the above values from (1), (2), (3) in I we get

$$YZ\frac{d^2X}{dx^2}+ZX\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}=0$$

Divide by XYZ to both sides, we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \quad ...(4)$$

which is of the form

$F_1(x) + F_2(y) + F_3(z) = 0$

Since x, y, z are independent, this is possible only when F_1, F_2, F_3 are constants. Let them be $-m^2$, $-n^2$ and $m^2 + n^2$.

Then from (4), we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -m^2, \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = -n^2,$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = m^2+n^2$$

$$\Rightarrow \frac{d^2X}{dx^2}+m^2X = 0, \quad \frac{d^2Y}{dy^2}+n^2Y = 0,$$

$$\frac{d^2Z}{dz^2}-(m^2+n^2)Z = 0$$

Their solutions are

$$X = A\cos mx + B\sin mx$$

$$Y = C\cos ny + D\sin ny$$

and $\quad Z = Ee^{\sqrt{m^2+n^2}z}+Fe^{-\sqrt{m^2+n^2}z}$

Hence, the general solution of I is

$u=XYZ$

$u=(A\cos mx+B\sin mx)(C\cos ny+D\sin ny)$

$$\left[E\exp\left(\sqrt{m^2+n^2}\right)z+F\exp\left(-\sqrt{m^2+n^2}\right)z\right]$$

In case we choose constants $m^2, n^2, -(m^2+n^2)$ then the general solution of I is

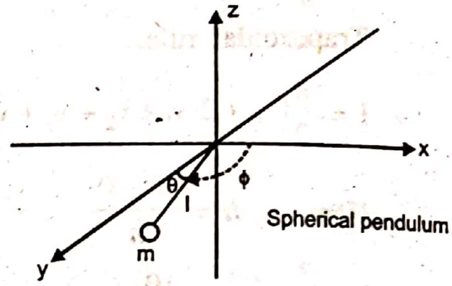
$$u = \left(Ae^{mx}+Be^{-mx}\right)\left(Ce^{ny}+De^{-ny}\right)$$

$$\times\left(E\cos\sqrt{m^2+n^2}\,z+F\sin\sqrt{m^2+n^2}\,z\right)$$

The choice of the constants and hence the general solution depends on the given initial and boundary conditions.

6. (b) Deriving the equation of motion for a spherical pendulum.

The spherical pendulum is similar to the simple pendulum, but moves in 3- dimensional space. This means, we need to introduce a new variable ϕ in order to describe the rotation of the pendulum around z-axis. We can then describe the position of the pendulum in reference to the variables θ and ϕ, and so the system has 2 degrees of freedom.



In order to describe the motion, we use spherical polar coordinates

$$x = l\sin\theta\cos\phi$$

$$y = l\sin\theta\sin\phi.$$

$$z = l\cos\theta$$

Now, we need to find the Lagrangian of the system.

Remember that

$$L = T - U \quad ...(1)$$

(T is kinetic energy and U is potential energy.)

where (in this case)

$$T = \frac{m}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2) \quad ...(2)$$

$$U = -mgz \quad ...(3)$$

Thus, in order to find T, we start calculations

$$\begin{cases}\dot{x}=l\dot{\theta}\cos\theta\cos\phi-l\dot{\phi}\sin\theta\sin\phi\\ \dot{y}=l\dot{\theta}\cos\theta\sin\phi+l\dot{\phi}\sin\theta\cos\phi\\ \dot{z}=-l\dot{\theta}\sin\theta\end{cases}$$

$$\left(\textbf{Note: } \dot{\theta}=\frac{d\theta}{dt}, \dot{\phi}=\frac{d\phi}{dt}\right)$$

So,

$$\dot{x}^2 = \left(l\dot{\theta}\cos\theta\cos\phi-l\dot{\phi}\sin\theta\sin\phi\right)^2$$

$$= l^2\dot{\theta}^2\cos^2\theta\cos^2\phi+l^2\dot{\phi}^2\sin^2\theta\sin^2\phi$$

$$-2l^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi$$

$$\dot{y}^2 = l^2\dot{\theta}^2\cos^2\theta\sin^2\phi+l^2\dot{\phi}^2\sin^2\theta\cos^2\phi$$

$$+2l^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi$$

and $\dot{z}^2 = l^2\dot{\theta}^2\sin^2\theta$

Therefore, substituting above values in equation (2) we get

$$T=\frac{m}{2}(l^2\dot{\theta}^2\cos^2\theta\cos^2\phi-2l^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\,.$$

$$\sin\phi\cos\phi+l^2\dot{\phi}^2\sin^2\theta\sin^2\phi+l^2\dot{\theta}^2\cos^2\theta$$

$$\sin^2\phi+2l^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\sin\phi\cos\phi$$

$$+l^2\dot{\phi}^2\sin^2\theta\cos^2\phi+l^2\dot{\theta}^2\sin^2\theta$$

Simplifying, we get

$$T=\frac{m}{2}(l^2\dot{\theta}^2+l^2\dot{\phi}^2\sin^2\theta)$$

(Using the relations $\sin^2\theta + \cos^2\theta = 1$ and $\sin^2\phi + \cos^2\phi = 1$)

And we know

$U = -mgz = -mg\ l\cos\theta$

$\therefore$ On substituting above values of T and U in (1) we get

$$L=\frac{m}{2}(l^2\dot{\theta}^2+l^2\dot{\phi}^2\sin^2\theta)+mgl\cos\theta$$

The resulting Lagrangian is

$$L(\theta,\phi)=\frac{1}{2}ml^2\left[\dot{\theta}^2+(1-\cos^2\theta)\dot{\phi}^2\right]$$

$$+mgl\cos\theta \quad ...(4)$$

The generalized coordinates are $q = (\theta, \phi)$ Substituting the Lagrangian into Lagrange's equation of motion we find the following individual terms:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0 \quad ...(5)$$

where $\dfrac{\partial L}{\partial\theta} = \phi^2\, ml^2\sin\theta\cos\theta - mgl\sin\theta$ [using (4)]

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$$

and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta}$ [using (4)]

On substituting above values in (5) we get

$$ml^2\ddot{\theta}-\phi^2ml^2\sin\theta\cos\theta+mgl\sin\theta=0$$

...(6)(*a*)

And rearranging for $\ddot{\theta}$ gives

$$\dot{\theta}=\frac{l\phi^2\sin\theta\cos\theta-g\sin\theta}{l} \quad ...(6)(b)$$

Similarly, in order to find the equation we need for ϕ, we use

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0 \quad ...(7)$$

where from (4), we get

$$\frac{\partial L}{\partial\phi}=0,\ \frac{\partial L}{\partial\dot{\phi}}=\dot{\phi}ml^2\sin^2\theta \quad \text{[using (4)]}$$

and

$$\therefore\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)=\ddot{\phi}ml^2\sin^2\theta+2\dot{\phi}ml^2\dot{\theta}\sin\theta\cos\theta$$

So, on putting above values in (7) we get

$$\ddot{\phi}ml^2\sin^2\theta+2\dot{\phi}ml^2\dot{\theta}\sin\theta\cos\theta=0$$

...(8)(*a*)

and rearranging for $\ddot{\phi}$ gives

$$\ddot{\phi}=\frac{-2\dot{\phi}\dot{\theta}\cos\theta}{\sin\theta} \quad ...(8)(b)$$

Thus, the dynamics of the system are given by (8)(*a*) and (6)(*a*)

$$\text{as}\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2\sin^2\theta\end{bmatrix}\begin{bmatrix}\ddot{\theta}\\ \ddot{\phi}\end{bmatrix}$$

$$+\begin{bmatrix}-ml^2\sin\theta\cos\theta\phi^2\\ 2ml^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}\end{bmatrix}+\begin{bmatrix}mgl\sin\theta\\ 0\end{bmatrix}=0$$

Given the initial position and velocity of the centre of the mass. This equation uniquely describes the motion of the pendulum.

6. (c)

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

We know that

$$\text{Area} = \int_0^{80} d(dx)$$

(when d = depth (in metres) at distance x from one bank.)

with $h = 10$, the trapezoidal rule is

$$\text{Area} = \frac{10}{2}[0 + 2(4 + 7 + 9 + 12 + 15 + 14 + 8) + 3]$$

$$= 5[2(69) + 3]$$

$$\text{Area} = 705 \text{ m}^2$$

Hence, area of the cross-section of the 80 m wide river is 705 m^2.

7. (a) Here $u(x, y) = \dfrac{A(x^2 - y^2)}{(x^2+y^2)^2}$,

$$v(x, y) = \frac{2Axy}{(x^2+y^2)^2},\ w = 0 \quad \text{...(1)}$$

$$\frac{\partial u}{\partial x} = \frac{A\left[2x(x^2+y^2)^2 - 4x(x^2-y^2)(x^2+y^2)\right]}{(x^2+y^2)^4}$$

$$\frac{\partial u}{\partial x} = \frac{2Ax(3y^2 - x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{A\left[-2y(x^2+y^2)^2 - 4y(x^2-y^2)(x^2+y^2)\right]}{(x^2+y^2)^4}$$

$$\frac{\partial u}{\partial y} = \frac{-2Ay(3x^2 - y^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial x} = \frac{2A\left[y(x^2+y^2)^2 - 4x^2y(x^2+y^2)^2\right]}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial x} = \frac{2Ay(y^2 - 3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{2A\left[x(x^2+y^2)^2 - 4xy^2(x^2+y^2)\right]}{(x^2+y^2)^4}$$

$$\frac{\partial v}{\partial y} = \frac{2Ax(x^2+y^2)}{(x^2+y^2)^3}$$

The equations of motion for inviscid and incompressible flow under no body force, in cartesian coordinates, are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial x} \quad \text{...(2)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial y} \quad \text{...(3)}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{-1}{\rho}\frac{\partial P}{\partial z} \quad \text{...(4)}$$

From (1) and (2), (3), (4) we have

$$\frac{2A^2x}{(x^2+y^2)^3} = +\frac{1}{\rho}\frac{\partial P}{\partial x} \quad \text{...(5)}$$

$$\frac{2A^2y}{(x^2+y^2)^3} = +\frac{1}{\rho}\frac{\partial P}{\partial y} \quad \text{...(6)}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z} \quad \text{...(7)}$$

Note: The relations, (5) and (6) show that the velocity components satisfy the equations of motion.

Now, equation (7) shows that the pressure P is independent of z *i.e.*, $P = P(x, y)$

Therefore,

$$dP = \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy$$

or $$dP = \frac{2A^2\rho x}{(x^2+y^2)^3}dx + \frac{2A^2\rho y}{(x^2+y^2)^3}dy$$

or $$dP = 2A^2\rho\frac{x\,dx + y\,dy}{(x^2+y^2)^3}$$

By integrating, we have

$$P = \frac{1}{2}\frac{A^2\rho}{(x^2+y^2)^2}$$

7. (b) Solve by Gauss–Seidel Method

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Note: Since in each equation, the coefficient of one of the unknowns (*i.e.*, x, y and z) is greater numerically than the sum of the numerical values of the other coefficients of the other unknowns.

∴ It can be solved by Gauss–Seidel Method as follows:

[*i.e.*, In first equation, the coefficient of x *i.e.*, 28 is greater than the sum of the coefficients of y and z (*i.e.*, 4 + 1)].

(Similarly, in equation second, the coefficient of y is greater than the sum of the coefficients of x and z)

(Similarly, in third equation, the coefficient of z is greater than the sum of the coefficients of x and y.)

Now, rewriting the given equations in the form

$$x = \frac{1}{28}(32-4y+z)$$

$$y = \frac{1}{17}(35-2x-4z)$$

$$z = \frac{1}{10}(24-x-3y)$$

We can start with any values of y and z. Let us take $y = 0$ and $z = 0$. Then, first approximations are given by

$$x^{(1)} = \frac{1}{28}(32-0-0) = \frac{32}{28} = 1.143$$

$$y^{(1)} = \frac{1}{17}[35-2(1.143)-4(0)]$$

$$= \frac{1}{17}(35-2.286) = \frac{32.714}{17}$$

$$y^{(1)} = 1.924$$

and $$z^{(1)} = \frac{1}{10}(24-1.143-3(1.924))$$

$$= \frac{1}{10}(24-1.143-5.772)$$

$$z^{(1)} = 1.708$$

Note: We are using the most recent values of x, y, z. Now, to obtain the second approximations, we have

$$x^{(2)} = \frac{1}{28}[32-4(1.924)+1.708]$$

$$= \frac{26.012}{28}$$

$$x^{(2)} = 0.929$$

$$y^{(2)} = \frac{1}{17}[35-2(0.929)-4(1.708)]$$

$$= \frac{1}{17}(35-2.153-6.832) = \frac{26.015}{17}$$

$$y^{(2)} = 1.53$$

$$z^{(2)} = \frac{1}{10}[24-0.929-3(1.53)]$$

$$= \frac{1}{10}(24-5.519)$$

$$z^{(2)} = 1.848$$

Proceeding in this way, we get the third approximation as

$$x^{(3)} = \frac{1}{28}[32-4(1.53)+1.848]$$

$$x^{(3)} = 0.99$$

$$y^{(3)} = \frac{1}{17}[35-2(0.99)-4(1.848)]$$

$$y^{(3)} = 1.507$$

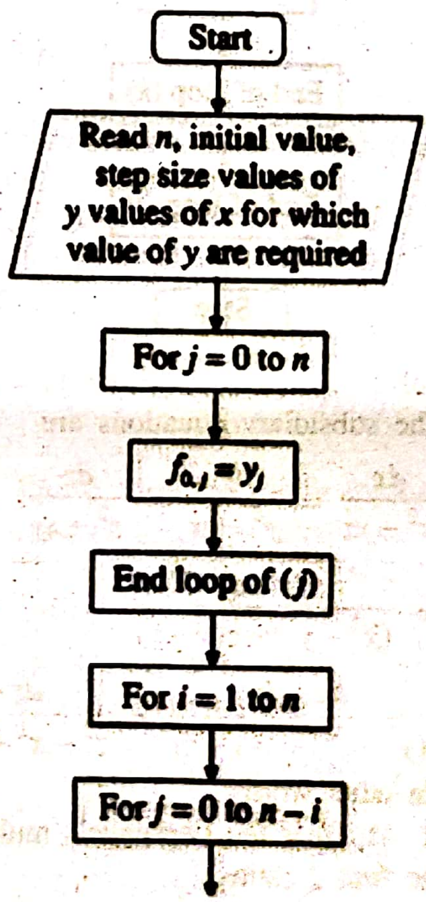
$$z^{(3)} = \frac{1}{10}[24-0.99-3(1.507)]$$

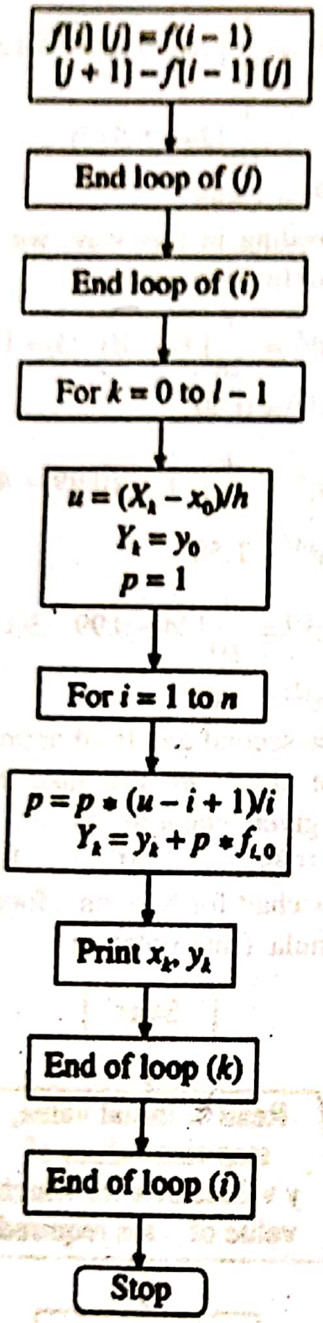
$$z^{(3)} = 1.849$$

Since second and third approximations are quite close, we can take the solution of the given equation as

$x = 0.99$, $y = 1.507$, $z = 1.849$

7. (c) Flow chart for Newton's forward difference formula (Interpolation)





8. (a) Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

The subsidiary equations are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\Rightarrow \frac{dx-dy}{(x^2-y^2)+z(x-y)}$$

$$= \frac{dy-dz}{(y^2-z^2)+x(y-z)} = \frac{dz-dx}{z^2-x^2+y(z-x)}$$

[In other words,

(1, –1, 0) is the Lagrange's multiplier for the first 2 ratios.

Similarly, (0, 1, –1) is the Lagrange's multiplier for the other ratios and (–1, 0, 1) is the Lagrange's multiplier for the other ratios.]

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$= \frac{dz-dx}{(z-x)(x+y+z)}$$

Now taking the first two members, we get

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$\Rightarrow \frac{d(x-y)}{(x-y)} = \frac{d(y-z)}{(y-z)}$$

Integrating and simplifying, we get

$$\ln(x-y) = \ln(y-z) + \ln C_1$$

(where $\ln C_1$ is an integration constant)

$$\text{or} \quad \ln\left(\frac{x-y}{y-z}\right) = \ln C_1$$

$$\Rightarrow \frac{x-y}{y-z} = C_1$$

Similarly, taking the last members, we get

$$\frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

$$\Rightarrow \frac{d(y-z)}{(y-z)} = \frac{d(z-x)}{(z-x)}$$

Integrating and simplifying, we get

$$\ln(y-z) = \ln(z-x) + \ln C_2$$

where $\ln C_2$ is an integration constant

$$\Rightarrow \ln\left(\frac{y-z}{z-x}\right) = \ln C_2$$

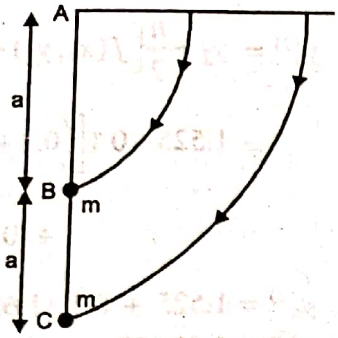
$$\Rightarrow \frac{y-z}{z-x} = C_2$$

Hence, the general solution can be given either as

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0$$

$$\text{or} \quad \frac{x-y}{y-z} = \phi\left(\frac{y-z}{z-x}\right)$$

8. (b) Since the rod ABC is weightless and the masses are symmetrical. (given)
Also, no external forces acting here.



$\therefore$ Total energy of the system remains constant.
Law of conservation of energy
Total P.E. = Total K.E.

$$\Rightarrow mg(a) + mg(2a) = \frac{1}{2}I_1w_1^2 + \frac{1}{2}I_2w_2^2$$

$$\Rightarrow \quad 3mga = \frac{w^2}{2}[I_1 + I_2] \quad (\because w_1 = w_2)$$

$$6mga = w^2\left[m(2a)^2 + ma^2\right]$$

$$\Rightarrow \quad w^2 = \frac{6mga}{5ma^2}$$

$$\Rightarrow \quad w = \sqrt{\frac{6g}{5a}}$$

8. (c) $\frac{dy}{dx} = x + \left|\sqrt{y}\right|,\ y(0) = 1$ (given)

Step size = 0.2 *i.e.,* $h = 0.2$

[Note: $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$]

Here it is given that $0 \le x \le 0.6$
It means we have to find
i.e., $y(0.2)$, $y(0.4)$ and $y(0.6)$
I I I
y_1, y_2 and y_3
Now, using Euler's Modified Method
First approximation, $y_1 = y_0 + hf(x_0, y_0)$

i.e., $y_1 = y(0.2) = y_0 + h\left(x_0 + \left|\sqrt{y_0}\right|\right)$

$$y_1 = y(0.2) = 1 + 0.2\left(0 + \left|\sqrt{1}\right|\right)$$
$$= 1 + 0.2(1)$$
$$y_1 = y(0.2) = 1.2$$

or $y_1^{(1)} = 1.2$

Now, second approximation of y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f(x_1, y_1^{(1)})\right]$$

$$= y_0 + \frac{h}{2}\left[\left(x_0 + \left|\sqrt{r_0}\right|\right) + \left(x_1 + \left|\sqrt{y_1}\right|\right)\right]$$

$$= 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \left|\sqrt{1.2}\right|\right)\right]$$

$$= 1 + 0.1\ (1 + 0.2 + 1.095)$$

$$y_1^{(2)} = 1.22954$$

Third approximation is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$\Rightarrow y_1^{(3)} = 1 + \frac{0.2}{2}\left[\left(0 + \left|\sqrt{1}\right|\right) + \left(0.2 + \sqrt{1.22954}\right)\right]$$

$$= 1 + 0.1\ [2.3088]$$
$$= 1 + 0.23088$$
$$y_1^{(3)} = 1.23088$$

Similarly, fourth approximation is

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + 0.2 + \left|\sqrt{1.23088}\right|\right]$$

$$= 1 + 0.1\ (2.30945)$$

$$y_1^{(4)} = 1.23094$$

Since, third and fourth approximation give almost similar results

$\therefore$ $y_1 = y(0.2) = 1.2309$

Now obtain $y_2 = y(0.4)$, we have first approximation

$$y_2^{(1)} = y_1 + h\,f(x_1, y_1)$$

$$= 1.2309 + 0.2\left(0.2 + \left|\sqrt{1.2309}\right|\right)$$

$$= 1.2309 + 0.26189$$

$$y_2^{(1)} = 1.4928$$

The second approximation is

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.2309 + \frac{0.2}{2}\left[\left(0.2 + \left|\sqrt{1.2309}\right|\right)\right.$$

$$\left. + \left(0.4 + \left|\sqrt{1.4928}\right|\right)\right]$$

$$= 1.2309 + 0.1 \ (2.93125)$$

$$y_2^{(2)} = 1.524$$

The third approximation is

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 1.2309 + 0.1\left[\left(0.2 + \left|\sqrt{1.2309}\right|\right)\right.$$

$$\left. + \left(0.4 + \left|\sqrt{1.524}\right|\right)\right]$$

$$= 1.2309 + 0.1 \ [2.94395]$$

$$y_2^{(3)} = 1.525$$

Since, second and third approximations are same,

$\therefore \ y_2 = y(0.4) = 1.525$

Now, obtain $y_3 = y(0.6)$, we have first approximation as

$$y_3^{(1)} = y_2 + hf(x_2, y_2)$$

$$= 1.525 + 0.2\left[0.4 + \left|\sqrt{1.525}\right|\right]$$

$$= 1.525 + 0.2 \ (1.6349)$$

$$y_3^{(1)} = 1.85198$$

Now second approximation is

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right]$$

$$= 1.525 + 0.1\left[\left(0.4 + \left|\sqrt{1.525}\right|\right)\right.$$

$$\left. + \left(0.6 + \left|\sqrt{1.85198}\right|\right)\right]$$

$$y_3^{(2)} = 1.525 + 0.1 \ [1.6349 + 1.96087]$$

$$y_3^{(2)} = 1.88458$$

Now, third approximation is

$$y_3^{(3)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(2)})\right]$$

$$= 1.525 + 0.1\left[1.6349 + 0.6 + \left|\sqrt{1.88458}\right|\right]$$

$$y_3^{(3)} = 1.8857$$

Since second and third approximations are similar,

$\therefore \ y_3 = 1.88$

Hence, $y_3 = y(0.6) = 1.88$

Thus finally, we get

$$\left[\begin{array}{ll} & y_1 = y(0.2) = 1.2309 \\ & y_2 = y(0.4) = 1.525 \\ \text{and} & y_3 = y(0.6) = 1.88 \end{array}\right]$$