Chapter 2

2019

2.1 Section-A

Question-1(a) **Let $T : R^3 \to R^3$ be a linear operator on R^3 defined by**

$$T(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix of T in the basis $\{(1, 1, 1),\ (1, 1, 0),\ (1, 0, 0)\}$

[8 Marks]

Solution: Given $T : R^3 \to R^3$ such that Basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$$\begin{aligned}
T(1, 1, 1) &= (3, -3, 3) \\
&= 3(1, 1, 1) - 6, (1, 1, 0) + 6(1, 0, 0) \quad \text{(using calculator)} \\
T(1, 1, 0) &= (2, -3, 3) \\
&= 3(1, 1, 1) - 6(1, 1, 0) + 5(1, 0, 0) \\
T(1, 0, 0) &= (0, 1, 3) \\
&= 3(1, 1, 1) - 2(1, 1, 0) - 1(1, 0, 0)
\end{aligned}$$

$$\begin{aligned}
[T]_B &= \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}^\top \\
&= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}
\end{aligned}$$

Question-1(b) **The eigenvalues of a real symmetric matrix A are -1, 1 and -2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1, 1, 0)^T$, $(0, 0, 1)^T$ and $\frac{1}{\sqrt{2}}(-1, -1, 0)^T$ respectively. Find the matrix A^4.**

[8 Marks]

Solution: If a matrix A is diagonalizable, then

$$P^{-1}AP = D$$
$$\therefore \quad A = PDP^{-1}$$

$$\begin{aligned} A^4 &= \left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right)\left(PDP^{-1}\right) \\ &= PD^4P^{-1} \end{aligned}$$

Where P is diagonalizing matrix consisting of eigenvectors of A.
Also, D is diagonal matrix containing eigenvalues of A at diagonal entries.

$$P = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

$$|P| = -1.\left[\left(\frac{-1}{\sqrt{2}}\right)\cdot\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)\right]$$

[Expanding Along C_2]

$$= -\left(\frac{1}{2} + \frac{1}{2}\right) = -1$$

$$\text{Adj } P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{Adj}{|A|} = \frac{-1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} (-1)^4 & 0 & 0 \\ 0 & (1)^4 & 0 \\ 0 & 0 & (-2)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Hence, $A^4 = PD^4P^{-1}$

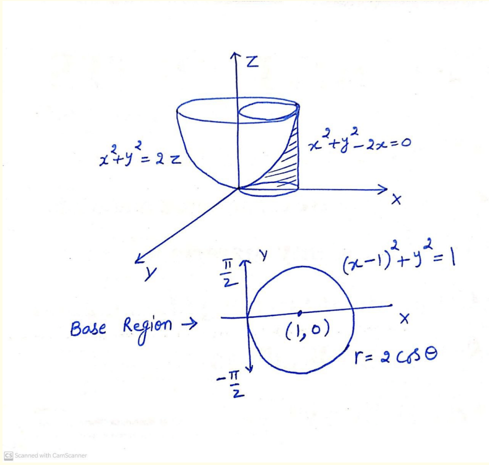
$$\begin{aligned} &= \frac{1}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -1 & 0 & -16 \\ 1 & 0 & -16 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -\sqrt{2} \\ 1 & 1 & 0 \end{bmatrix} \\ &= \frac{-1}{2} \begin{bmatrix} -17 & -15 & 0 \\ -15 & -17 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{17}{2} & \frac{15}{2} & 0 \\ \frac{15}{2} & \frac{17}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore \quad A^4 = PD^4P^{-1} = \frac{1}{2} \begin{bmatrix} 17 & 15 & 0 \\ 15 & 17 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Question-1(c) Find the volume lying inside the cylinder $x^2 + y^2 - 2x = 0$ and outside the paraboloid $x^2 + y^2 = 2z$, while bounded by xy-plane.

[8 Marks]

Solution: The required volume is found by integrating $z = \frac{1}{2}(x^2 + y^2)$ over the circle $x^2 + y^2 = 2x$



Changing to polar coordinates in the xy-plane,

$$x = r\cos\theta, y = r\sin\theta$$

$$\therefore z = \frac{1}{2}\left(x^2 + y^2\right) = \frac{r^2}{2}$$

$$r^2\, cos^2\theta + r^2 \sin^2\theta = 2rcos\theta$$

$$r = 2\cos\theta$$

To cover this circle, r varies from 0 to $2\cos\theta$ and θ varies from $\frac{-\pi}{2}$ to $\frac{\pi}{2}$
$\therefore$ Required volume

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} z \cdot r dr d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} \frac{r^3}{2} dr d\theta$$

$$V = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 dr d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left.\frac{r^4}{4}\right|_0^{2cos\theta} d\theta$$

$$= \frac{1}{8}\int_{-\pi/2}^{\pi/2} 16\cos^4\theta d\theta$$

$$= 2 \times 2 \cdot \int_0^{\pi/2} \cos^4\theta d\theta$$

$$\left(\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, if, f(x) = f(-x)\right)$$

$$= 4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \text{(Using Walli's formula)}$$
$$= \frac{3}{4}\pi$$

Question-1(d) Justify by using Rolle's theorem or mean value theorem that there is no number k for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1, 1]$.

[8 Marks]

Solution:

$$f(x) = x^3 - 3x + k$$

We will prove the result by using method of contradiction.
Let $f(x)$ has two distinct roots a and b in [-1,1] i.e.

$$f(a) = 0 \quad = f(b), -1 \le a, b \le 1, a \ne b$$

$f(x)$ is continuous and differentiable over the interval $[a, b]$.
Hence, by Rolle's theorem, there exist some $c \in (a, b)$ s.t.

$$f'(c) = 0$$

i.e.

$$3c^2 - 3 = 0 \quad \Rightarrow \quad c = \pm 1$$

which is contradiction to the fact that a and b lies within [-1,1].
Hence $f(x)$ cannot have two distinct roots in [-1,1] for any value of $'k'$.

Question-1(e) If the coordinates of the points A and B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the line joining A and B is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then show that $x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0$

[8 Marks]

Solution: Point $M(x, y)$ divides the line-segment AB in the ratio $b : a$ externally, We take it as $b : -a$ internally.

$$x = \frac{ab\cos\beta - ab\cos\alpha}{b - a}$$
$$= \frac{ab}{b-a}(\cos\beta - \cos\alpha)$$
$$= \frac{ab}{b-a}\left(-2\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \quad \ldots (1)$$
$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\begin{aligned}
y &= \frac{b\cdot(a\sin\beta)-a(b\sin\alpha)}{b-a} \\
&= \frac{ab}{b-a}(\sin\beta-\sin\alpha) \\
&= \frac{ab}{b-a}\left(2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right) \quad \ldots(2)
\end{aligned}$$

$$\left[\because \sin C-\sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$\frac{x}{y} = \frac{-\sin((\alpha+\beta)/2)}{\cos((\alpha+\beta)/2)}$$

$$\Rightarrow \quad x\cdot\cos\frac{\alpha+\beta}{2}+y\sin\frac{\alpha+\beta}{2}=0$$

Question-2(a) Determine the extreme values of the function $f(x,y)=3x^2-6x+2y^2-4y$ in the region $\{(x,y)\in\mathbb{R}^2 : 3x^2+2y^2\le 20\}$

[10 Marks]

Solution: **Method-1:**

First we find the critical points $f(x,y)=3x^2-6x+2y^2-4y$

$$f_x=0\Rightarrow 6x-6=0\Rightarrow x=1$$

$$f_y=0\Rightarrow 4y-4=0\Rightarrow y=1$$

$$f_{xy}=0$$

$\therefore P(1,1)$ is the only critical point. As

$$3(1)^2+2(1)^2=5<20$$

$\Rightarrow P(1,1)$ lies in the given elliptical region.

$$f(1,1)=3-6+2-4=-5 \quad \ldots(1)$$

$$f_{xx}f_{yy}-f_{xy}^2=(6)(4)-0^2=24>0$$

and $f_{xx}=6>0$ at $P(1,1)$

Hence point (1,1) is a point of local minima. Let us check at boundaries of the ellipse i.e. $3x^2+2y^2=20$

$$\begin{aligned}
\therefore f(x,y) &= 3x^2-6x+2y^2-4y \\
&= 20-6x-4y \\
&= 20-6x\pm 2\sqrt{2}\sqrt{20-3x^2}
\end{aligned}$$

Let

$$g(x)=20-6x+2\sqrt{2}\sqrt{20-3x^2}$$

$$g'(x)=-6+2\sqrt{2}\frac{(-6x)}{2\sqrt{20-3x^2}}$$

$g'(x) = 0$ gives $x = \pm 2 \Rightarrow y = \mp 2$

$$\begin{aligned} &\text{At } (2,-2) \\ f(x,y) =& 20 - 6(2) - 4(-2) \\ =& 20 - 12 + 8 = 16 \quad \ldots(2) \end{aligned}$$

$$\begin{aligned} &\text{At } (-2,2), \\ f(x,y) &= 20 - 6(-2) - 4(2) \\ &= 12 + 12 - 8 = 16 \quad \ldots(3) \end{aligned}$$

Again let

$$h(x) = 20 - 6x - 2\sqrt{2}\sqrt{20-3x^2}$$

$$\left(y = \frac{1}{\sqrt{2}}\sqrt{20-3x^2}\right)$$

$$h'(x) = -6 + 2\sqrt{2}\cdot\frac{6x}{2\sqrt{20-3x^2}}$$

$$h'(x) = 0 \quad \Rightarrow x = \pm 2 \quad \Rightarrow \quad y = \pm 2$$

At

$$(2,2) \Rightarrow f(x,y) = 20 - 12 - 8 = 0 \quad \ldots(4)$$

At

$$(-2,-2) \quad f(x,y) = 20 + 12 + 8 = 40 \quad \ldots(5)$$

From (1),(2),(3),(4) and (5) ,we get max at $(-2,-2)$,

$$f(x,y) = 40$$

min at $(1,1)$,

$$f(x,y) = -5$$

Method-2:

Using polar coordinates (elliptical)

$$3x^2 + 2y^2 = 20 \Rightarrow \quad \frac{x^2}{\frac{20}{3}} + \frac{y^2}{10} = 1$$

Let

$$x = 2\sqrt{\frac{5}{3}}r\cos\theta, \quad y = \sqrt{10}\,\mathrm{r}\sin\theta$$

for $0 \le r \le 1$, it gives elliptical region $\{3x^2 + 2y^2 \le 20\}$.

$$\begin{aligned} f(x,y) &= 3x^2 - 6x + 2y^2 - 4y \\ &= 20r^2 - 12\sqrt{\frac{5}{3}}r\cos\theta - 4\sqrt{10}\cdot r\sin\theta \\ &= 20r^2 - 4\sqrt{5}r(\sqrt{3}\cos\theta - \sqrt{2}\sin\theta) \\ &= 20r^2 - 20\,\mathrm{r}\left(\sqrt{\frac{3}{5}}\cos\theta - \sqrt{\frac{2}{5}}\sin\theta\right) \\ &= 20r^2 - 20r(\sin(A-\theta)) \end{aligned}$$

where,

$$\left(\sin A = \sqrt{\frac{3}{5}}, \quad \cos A = \sqrt{\frac{2}{5}}\right)$$

$$f(r,\theta) = 20r[r - \sin(A-\theta)]$$

Max value of $f(r,\theta)$ will occur where

$$\sin(A-\theta) = -1 \text{ and } r = 1$$

$$f(r,\theta) = 20(1)(1-(-1)) = 40$$

for minimum , $sin(A-\theta) = 1$

$$f(r,\theta) = 20r(r-1) = 20(r^2 - r)$$

$$f'(r,\theta) = 20r(r-1) \Rightarrow r = \frac{1}{2}$$

$$\text{min value} \quad 20 \times \frac{1}{2}(\frac{1}{2} - 1) = -5$$

Question-2(b) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1,1,0,0)^T$.
Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$.

[15 Marks]

Solution: Let $\lambda_1 = 4$, $v_2 = (1,1,0,0)^T$

$$Av_2 = \lambda_2 v_2$$

$$\begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(2,2,0,0) = (\lambda_2, \lambda_2, 0, 0)$$

$$\Rightarrow \quad \lambda_2 = 2$$

Let the other two eigenvalues be λ_3 and λ_4.

Trace (A) = sum of eigenvalues

$$4 + 2 + \lambda_3 + \lambda_4 = -1 + 5 + (-10) + 8$$

$$\lambda_3 + \lambda_4 = -4$$

Also, product of eigenvalues = Det(A)

$$4 \cdot 2 \cdot \lambda_3 \cdot \lambda_4 = 0 \quad \Rightarrow \lambda_3\lambda_4 = 0$$

i.e.

$$\lambda_3(-4-\lambda_3)=0$$
$$\Rightarrow \quad \lambda_3=0 \quad or \quad \lambda_3=-4$$
$$\therefore \quad \lambda_4=-4 \quad or \quad \lambda_4=0$$

Characteristic polynomial

$$\Pi(x-\lambda i)=0$$
$$(x-4)(x-2)(x+4)(x-0)=0$$
$$\left(x^2-16\right)(x-2)x=0$$
$$\left(x^3-16x-2x^2+32\right)x=0$$
$$x^4-16x^2-2x^3+32x=0$$

Since, every square matrix satisfies its characteristic equation (Cayley Hamilton Theorem)

$$\therefore \quad A^4-2A^3-16A^2+32A=0$$

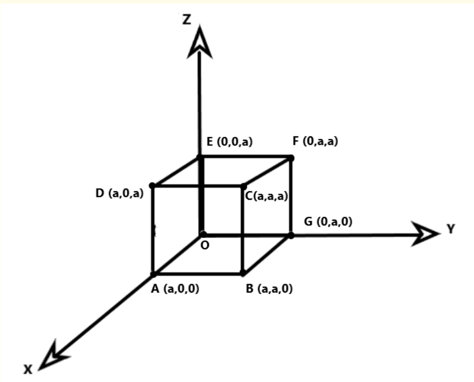
$$\therefore p=-2, q=-16, r=32$$

Question-2(c) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Show that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

[15 Marks]

Solution: The D.R. of four diagonals



$$AF=(-a,a,a)$$
$$=(-1,1,1)$$
$$BE=(-a,-a,a)$$
$$=(1,1,-1)$$
$$CO=(-a,-a,-a)$$
$$=(1,1,1)$$
$$DG=(-a,a-a)$$
$$=(1,-1,1)$$

Let the D.R.'s of line are $\langle l,m,n\rangle$

$$\cos\alpha=\frac{-l+m+n}{\sqrt{3}\cdot\sqrt{\ell^3+m^2+n^2}}; \quad cos\beta=\frac{l+m-n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}}$$

$$\cos\gamma = \frac{\ell+m+n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}} \quad \cos\delta = \frac{l-m+n}{\sqrt{3}\cdot\sqrt{l^2+m^2+n^2}}$$

$$\begin{aligned}\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta &= \frac{1}{3\left(\ell^2+m^2+n^2\right)}\left[(-l+m+n)^2+(l+m-n)^2\right.\\ &\qquad \left.+(l+m+n)^2+(l-m+n)^2\right]\\ &= \frac{4\left(l^2+m^2+n^2\right)}{3\left(l^2+m^2+n^2\right)} = \frac{4}{3}\end{aligned}$$

Question-3(a) Consider the vectors $x_1 = (1,2,1,-1)$, $x_2 = (2,4,1,1)$, $x_3 = (-1,-2,0,-2)$ and $x_4 = (3,6,2,0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1,x_2,x_3,x_4\}$ is a subspace of $\mathbb{R}^4$, defined as

$$\left\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbf{R}^4 : 2\xi_1-\xi_2=0, \quad 2\xi_1-3\xi_3-\xi_4=0\right\}$$

Can this subspace be written as $\{(\alpha,2\alpha,\beta,2\alpha-3\beta) : \alpha,\beta\in \mathrm{R}\}$? What is the dimension of this subspace?

[15 Marks]

Solution:

$$x_1=(1,2,1,-1), \quad x_2=(2,4,1,1)$$
$$x_3=(-1,-2,0,-2), \quad x_4=(3,6,2,0)$$

We find span $\{x_1,x_2,x_3,x_4\}$

$$\begin{bmatrix} 1 & 2 & 1 & -1\\ 2 & 4 & 1 & 1\\ -1 & -2 & 0 & -2\\ 3 & 6 & 2 & 0\end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1\\ 0 & 0 & -1 & 3\\ 0 & 0 & 1 & -3\\ 0 & 0 & -1 & 3\end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -1\\ 0 & 0 & -1 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2\\ 0 & 0 & 1 & -3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$= \operatorname{span}\{(1,2,0,2),(0,0,1,-3)\}$$
$$= \{a(1,2,0,2)+b(0,0,1,-3); a,b\in R\}$$
$$= \{(a,2a,b,2a-3b)\}$$
$$= \{(x,y,z,w) : x=a, y=2a, z=b, w=2a-3b\}$$
$$i.e. \quad y=2x, w=2x-3z\}$$

If we take $a=\alpha, b=\beta$ then above subspace can be written as

$$\{\alpha,2\alpha,\beta,2\alpha-3\beta\}, \quad Dim=2\ .$$

as α and β are linearly independent.

Question-3(b) The dimensions of a rectangular box are linear functions of time- $l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and the height is decreasing at the rate 3 cm/sec find the rates at which the volume V and with respect to time. If $l(0) = 10$, $w(0) = 8$ and the surface area S are changing $h(0) = 20$, is V increasing or decreasing, when $t = 5$ sec? What about S, when $t = 5$ sec?

[10 Marks]

Solution: Given

$$\frac{dl}{dt} = 2\text{cm/sec} \quad \frac{dw}{dt} = 2 \quad \frac{dh}{dt} = -3$$

$$l = 2t + \ell_0, \quad w = 2w + w_0, \quad h = -3t + h_0$$

Using $l(0) = 10, w(0) = 8, h(0) = 20$

$$l = 2t + 10$$

$$w = 2t + 8$$

$$h = -3t + 20$$

At $t = 5$ sec,

$$l = 20cm, \quad w = 18cm, \quad h = 5cm$$

$$\begin{aligned} V &= lwh \\ &= (2t+10)(2t+8)(-3t+20) \end{aligned}$$

$$\frac{dV}{dt} = 2(2t+8)(-3t+20) + 2(2t+10)(-3t+20)$$

$$(-3)(2t+10)(2t+8)$$

$$\begin{aligned} \left.\frac{dv}{dt}\right|_{t=5} &= 2(18)(5) + 2(20)(5) - 3(20)(18) \\ &= 180 + 200 - 1080 \\ &= -700 < 0 \quad (\text{ Decreasing } V) \end{aligned}$$

$$\text{Surface Area, } S = 2(lw + wh + hl)$$

$$\begin{aligned} \frac{dS}{dt} &= 2\left[(w+h)\frac{dl}{dt} + (l+h)\frac{dw}{dt} + (l+w)\frac{dh}{dt}\right] \\ &= 2[23(2) + 25(2) + 38(-3)] \\ &= 2(46 + 50 - 114) = -36 < 0 \end{aligned}$$

Therefore, S is decreasing.

Question-3(c) Show that the shortest distance between the straight lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

and

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

is $3\sqrt{30}$. Find also the equation of the line of shortest distance.

[15 Marks]

Solution: Let $A(3,8,3)$ and $B(-3,-7,6)$ are points lying on the lines L_1 and L_2.

$$L_1:\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$
$$L_2:\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

D.R. of line which is perpendicular to L_1 and L_2 both (i.e. shortest distance line)

$$\frac{l}{-4-2}=\frac{m}{-3-12}=\frac{n}{6-3}$$
$$\frac{l}{-6}=\frac{m}{-15}=\frac{n}{3}$$
$$<2,5,-1>$$

$$\text{D.R.'s of AB}=\langle 3+3,8+7,3-6\rangle$$
$$=\langle 6,15,-3\rangle$$

$$\therefore S.D.=\frac{1}{\sqrt{4+25+1}}(2\cdot 6+5\cdot 15+(-1)(-3))$$
$$=\frac{1}{\sqrt{30}}(12+75+3)=\frac{90}{\sqrt{30}}=3\sqrt{30}$$

Since, S. D. line is parallel to AB. Hence taking $A(3,8,3)$ as one point its equation is

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

[**Alternate Method**]: (When equation of Shortest Distance is asked)
Let

$$L_1:\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$
$$L_2:\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

Any general point on $L_1$is $P(3a+3,-a+8,a+3)$ and,
any general piint on L_2 is $Q(-3b-3,2b-7,4b+6)$.
$\therefore$ D.R.'s of PQ are $\langle P-Q\rangle$ i.e. $\langle 3a+3b+6,-a-2b+15,a-4b-3\rangle$.

If PQ is the shortest distance line, it will be perpendicular to both the lines L_1 and L_2.

$$\therefore \quad 3(3a+3b+6)-1(-a-2b+15)+1(a-4b-3)=0$$
$$\Rightarrow 11a+7b=0 \qquad \ldots(1)$$
$$Also, -3(3a+3b+6)+2(-a-2b+15)+4(a-4b-3)=0$$
$$\Rightarrow (-9a-9b-18)+(-2a-4b+30)+(4a-16b-12)=0$$
$$\Rightarrow -7a-29b=0 \qquad \ldots(2)$$

From (1) and (2) we get $a=0, b=0$.

$$\therefore \quad P \text{ is } (3,8,3) \text{ and } Q \text{ is } (-3,-7,6)$$

$$\text{Shortest Distance,} \quad PQ = \sqrt{(3+3)^2+(8+7)^2+(3-6)^2}$$
$$= \sqrt{36+225+9}$$
$$= \sqrt{270} = 3\sqrt{30}$$

D.R.'s of PQ $\langle 6, 15, -3\rangle$ i.e. $\langle 2, 5, -1\rangle$

$$\therefore \text{Equation of Shortest Distance is: } \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

.

Question-4(a) Using elementary row operations, reduce the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX=b$, where $X=(x,y,z,u)^T$ and $b=(2,1,0,4)^T$

[15 Marks]

Solution:

$$A = IA$$

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 3 & 0 \\ 2 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1, \quad R_4 - 1R_4 - 2R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & 1 & -2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} A$$

$$R_3 \to R_3 - 3R_2, \quad R_4 \to R_4 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & -4 & 8 \\ 0 & 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ -3 & 1 & 3 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A$$

$$R_4 \to R_4 - \frac{R_3}{2}, R_3 \to \frac{R_3}{-4}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ 1/2 & -1/2 & -3/2 & 1 \end{bmatrix} A$$

$$R_4 \to -R_4, \quad R_2 \to -R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 \\ +3/4 & -1/4 & -3/4 & 0 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A$$

$$R_3 \to R_3 + 2R_4, \quad R_2 \to R_2 - 2R_4, R_1 \to R_1 - R_4$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & -1/2 & 1 \\ 0 & -1 & -1 & 2 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix}$$

$$R_2 \to R_2 + R_3, \quad R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/4 & -5/4 & 11/4 & 3 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix}$$

$$R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -4 & 3 \\ -1/4 & -1/4 & 5/4 & 0 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix} A$$

$$AX = b$$

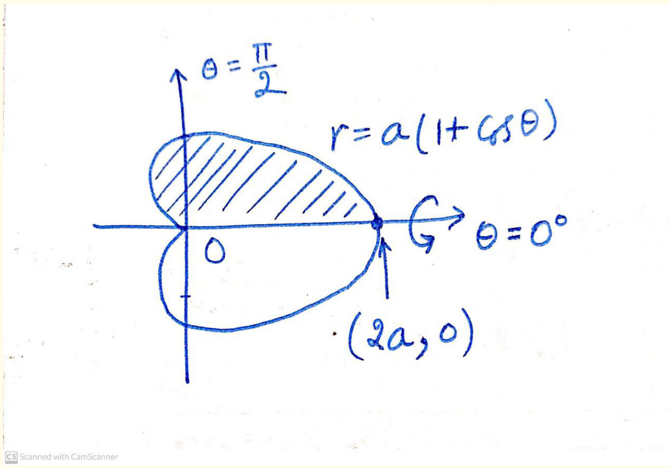
$$X = A^{-1}b$$

$$X = \frac{1}{4} \begin{bmatrix} 4 & -4 & -16 & 12 \\ -1 & -1 & 5 & 0 \\ -1 & 3 & 9 & -8 \\ -2 & 2 & 6 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8-4+48 \\ -2-1+0 \\ -2+3-32 \\ -4+2-16 \end{bmatrix} = \begin{bmatrix} 13 \\ -3/4 \\ -31/4 \\ -9/2 \end{bmatrix}$$

Question-4(b) Find the centroid of the solid generated by revolving the upper half of the cardioid $r = a(1+\cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform.

[10 Marks]



Solution:

As the solid of revolution is symmetric about initial line (x-axis), the centroid will lie on it. ie. y-coordinate will be zero. $\bar{y} = 0$
x-coordinate

$$\bar{x} = \frac{\int x dV}{\int dV}$$

[in polar-coordinates $x = r\cos\theta$

$$dV = 2\pi r^2 \sin\theta d\theta dr,$$

θ varies from 0 to π (upper part)]

$$V = \int_0^{\pi}\int_0^{a(1+cos\theta)} 2\pi r^2 \sin\theta dr d\theta$$

$$= 2\pi \int_0^{\pi} \left.\frac{r^3}{3}\right|_0^{a(1+cos\theta)} \sin\theta d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta d\theta$$

$$= \frac{2\pi a^3}{3} \cdot \left.\frac{(1+cos\theta)^4}{-4}\right|_0^{\pi}$$

$$= \frac{2\pi a^3}{3} \cdot \frac{16}{-4} = \frac{8\pi}{3}a^3$$

$$\begin{aligned}
\int xdV &= \int_0^\pi \int_0^{a(1+cos\theta)} (rcos\theta)\left(2\pi r^2\sin\theta\right) drd\theta \\
&= 2\pi \int_0^\pi \int_0^{a(1+cos\theta)} r^3 cos\theta sin\theta drd\theta \\
&= \frac{2\pi}{4}\int_0^\pi a^4(1+cos\theta)^4\cos\theta\sin\theta d\theta \\
&= \frac{2\pi}{4}\int_0^\pi a^4(1+cos\theta)^4(\cos\theta+1-1)\sin\theta d\theta \\
&= \frac{\pi}{2}a^4\int_0^\pi \left[(1+\cos\theta)^5\sin\theta-(1+cos\theta)^4\sin\theta\right] d\theta \\
&= \frac{\pi}{2}a^4\left[\frac{(1+\cos\theta)^6}{6}-\frac{(1+\cos\theta)^5}{5}\right]_0^\pi \\
&= \frac{\pi a^4}{2}\left(\frac{64}{6}-\frac{32}{5}\right)=\frac{\pi a^4\times 32}{15}
\end{aligned}$$

$$\therefore \bar{x}=\frac{32\pi a^4}{15}\times\frac{3}{8\pi a^3}=\frac{4a}{5}$$

$\therefore$ Centroid is $\left(\frac{4a}{5},0\right)$.

Question-4(c) A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A,B and C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

[15 Marks]

Solution: Let the equation of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=p \quad \ldots(1)$$

It meets the axis at points $A(ap,0,0)$, $B(0,bp,0)$, $c(0,0,cp)$ We find equation of sphere passing through origin $O(0,0,0)$ and A,B,C

$$x^2+y^2+z^2-apx-bpy-cpz=0 \quad \ldots(2)$$

Equation (1) and (2) together gives the equation of circle ABC. If we homogenize equation (2) with help of equation (1) , we will get the equation of cone with vertex at origin.

$$x^2+y^2+z^2-(apx+bpy+cpz)\left(\frac{x}{ap}+\frac{y}{bp}+\frac{z}{cp}\right)=0$$

$$\therefore yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

2.2 Section-B

Question-5(a) Solve the differential equation $(D^2+1)\,y = x^2 \sin 2x; \quad D \equiv \dfrac{d}{dx}.$

[8 Marks]

Solution: Auxiliary Equation:

$$D^2+1=0 \Rightarrow D=\pm i$$

$$C\cdot F \quad y = c_1\cos x + c_2 \sin x$$

$$\begin{aligned}
P.I. &= \frac{1}{D^2+1}x^2\sin 2x \\
&= Img \text{ part of } \frac{1}{D^2+1}x^2\cdot e^{i2x} \\
&= \operatorname{Im}\left[e^{i2x}\frac{1}{(D+2i)^2+1}x^2\right]
\end{aligned}$$

$$\left(\because \frac{1}{f(D)}Ve^{ax} = e^{ax}\frac{1}{f(D+a)}V\right)$$

$$\begin{aligned}
&= \operatorname{Im}\left(e^{i2x}\frac{1}{D^2+4iD-4+1}x^2\right) \\
&= \operatorname{Im}\left(\frac{-e^{i2x}}{3}\cdot\left(1-\left(\frac{D^2+4Di}{3}\right)\right)^{-1}x^2\right) \\
&= \operatorname{Im}\left[-\frac{e^{i2x}}{3}\left(1+\frac{D^2+4Di}{3}+\frac{16D^2i^2}{9}+\dots\right)x^2\right]
\end{aligned}$$

$\{$ using Binomial expansion and neglecting, higher powers of D. $\}$

$$\begin{aligned}
&= \operatorname{Im}\left(-\frac{e^{i2x}}{3}\left(x^2+\frac{(-26)}{9}+\frac{8xi}{3}\right)\right) \\
&= Im\left[-\frac{1}{3}\left(\cos^2 x + i\sin 2x\right)\left(x^2-\frac{26}{9}+i\frac{8x}{3}\right)\right] \\
&= \frac{-1}{3}\left[(\sin 2x)\left(x^2-\frac{26}{9}\right)+(\cos 2x)\frac{8x}{3}\right]
\end{aligned}$$

$\therefore$ complete solution

$$y = (C\cdot F + P\cdot I)$$

$$\Rightarrow y = c_1\cos x + c_2\sin x - \frac{1}{3}\left[(\sin 2x)\left(x^2-\frac{26}{9}\right)+(\cos 2x)\frac{8x}{3}\right]$$

Question-5(b) Solve the differential equation $(px-y)(py+x)=h^2p$, where $p=y'$.

[8 Marks]

Solution:

$$p^2xy+px^2-py^2-xy=h^2p$$

Put,

$$x^2=u, \quad y^2=v$$

$$P=\frac{dv}{du}=\frac{y}{x}p$$

$$(2xdx=du, \quad 2ydy=dv)$$

i.e.

$$p=\frac{x}{y}P=\sqrt{\frac{u}{v}}P$$

$\therefore$ The given $D\cdot E.$ transforms to

$$\frac{u}{v}P^2\sqrt{u}\sqrt{v}+\sqrt{\frac{u}{v}}Pu-\sqrt{\frac{u}{v}}Pv-\sqrt{u}\sqrt{v}$$

$$=h^2\sqrt{\frac{u}{v}}P$$

$$uP^2\sqrt{u}+\sqrt{u}\cdot uP-\sqrt{u}vP-\sqrt{u}v=h^2\sqrt{u}P$$

i.e.

$$uP^2+uP-vP-v=h^2P$$

$$u\left(P^2+P\right)-v(P+1)=h^2P$$

$$uP-v=\frac{h^2P}{P+1}$$

$$\left[v=uP-\frac{h^2P}{P+1}\right]$$

This is in Clairaut's form

$$y=px+f(p)$$

So, replacing P with c, we have general solution

$$v=cu-\frac{ch^2}{c+1}$$

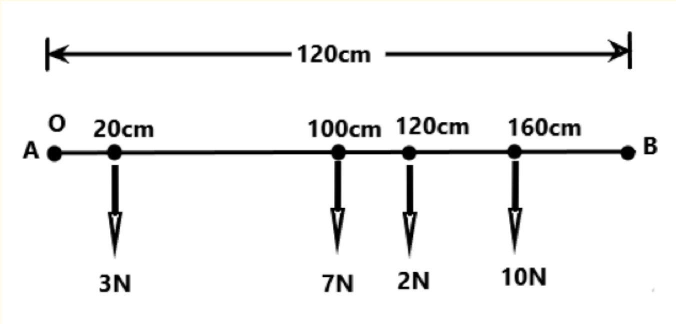
ie.

$$y^2=cx^2-\frac{ch^2}{c+1}$$

Question-5(c) A 2 meters rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hing loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal.

[8 Marks]

Solution: Varignon's Theorem: Moment of a force about a point is equal to the sum of the moments of the forces components about the point.



Let us take moments about point A. Resultant of all forces $= 3+7+2+10 = 22N$

$$\therefore (22 \times r) = 3 \times 20 + 7 \times 100 + 2 \times 120 + 10 \times 160 = 2600$$

$$\therefore \quad r = \frac{2600}{22} = \frac{1300}{11} = 118 \cdot 18cm$$

Hence rod must be supported at a point $118.18cm$ from end A.

Question-5(d) Let $\bar{r} = \bar{r}(s)$ represent a space curve. Find $\dfrac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T}, \bar{N}$ and $\bar{B}$ where $\bar{T}$, $\bar{N}$ and $\bar{B}$ represent tangent, principal normal and binormal respectively. Compute $\dfrac{d\bar{r}}{ds} \cdot \left(\dfrac{d^2\bar{r}}{ds^2} \times \dfrac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion.

[8 Marks]

Solution:

$$\vec{T} = \frac{d\vec{r}}{ds}$$

$$k\vec{N} = \frac{d\vec{T}}{ds} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right) = \frac{d^2\vec{r}}{ds^2}$$

i.e.

$$\frac{d^2\vec{r}}{ds^2} = k\vec{N}$$

$$\Rightarrow \frac{d^3\vec{r}}{ds^3} = k\frac{d\vec{N}}{ds} + \frac{dk}{ds}\vec{N} \ldots (1)$$

$$\frac{d^3\vec{r}}{ds^3} = k(\vec{B}\tau - k\vec{T}) + \frac{d}{ds}\left(\left|\frac{d\vec{T}}{ds}\right|\right) - \vec{N} \ldots (2)$$

(Serret-Frenet)

$$\Rightarrow \frac{d\vec{N}}{ds} = \tau\vec{B} - k\vec{T}$$

$$\begin{aligned}\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} &= k\vec{N} \times \left[k(\vec{B}\tau - k\vec{T}) + \frac{dk}{dS}\vec{N}\right] \qquad [\text{ using } (1)]\\ &= k^2\tau(\vec{N} \times \vec{B}) - k^3(\vec{N} \times \vec{T})\\ &= k^2\tau\vec{T} - k^3\vec{B}\end{aligned}$$

$$\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) = \vec{T} \cdot \left(k^2\tau\vec{T} - k^3\vec{B}\right)$$

$$= k^2\tau \qquad (\because \vec{T} \cdot \vec{B} = 0)$$

Question-5(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\, dx - 3x^2y^2 dy$ **along the path** $x^4 - 6xy^3 = 4y^2$**.**

[8 Marks]

Solution: The integral is of the form

$$\int_c M dx + N dy$$

where

$$M = 10x^4 - 2xy^3$$

$$N = -3x^2y^2$$

$$\frac{\partial M}{\partial y} = -6xy^2, \quad \frac{\partial N}{\partial x} = -6xy^2$$

Method-1: As

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given integral is path-independent. It means we can use any path.

Let the path consists of straight line L_1 : from $(0, 0)$ to $(2, 1)$ and then L_2 : from $(2, 0)$ to $(2, 1)$

Along $L_1 : y = 0 \Rightarrow dy = 0$

Along $L_2 : x = 2 \Rightarrow dx = 0$

Value of integral

$$\int_{x=0}^{2} 10x^4 dx + \int_{y=0}^{1} -3(2)^2 y^2 dy = 2x^5\Big|_0^2 - 4y^3\Big|_0^1$$
$$= 64 - 4$$
$$= 60$$

Method-2: As

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$
$$\therefore \left(10x^4 - 2xy^3\right) dx - \left(3x^2y^2\right) dy$$

is an exact differential of $(2x^5 - x^2y^3)$.

$$\therefore \int_{(0,0)}^{(2,1)} \left(10x^4 - 2xy^3\right) dx - 3x^2y^2 dy = \int_{(0,0)}^{(2,1)} d\left(2x^5 - x^2y^3\right)$$
$$= \left(2x^5 - x^2y^3\right)\Big|_{(0,0)}^{(2,1)}$$
$$= 64 - 4$$
$$= 60$$

Question-6(a) Solve by the method of variation of parameters the differential equation

$$x''(t) - \frac{2x(t)}{t^2} = t, \text{ where } 0 < t < \infty$$

[15 Marks]

Solution:

$$\left[D^2 - \frac{2}{t^2}\right] x(t) = t$$

i.e.

$$\left[t^2D^2 - 2\right] x(t) = t^3$$

Put

$$t = e^u \quad \therefore \quad u = \log t$$
$$D' = \frac{d}{du} = tD; \quad D'\left(D' - 1\right) = t^2D^2$$
$$\therefore \left[D'\left(D' - 1\right) - 2\right] x = e^{3u}$$
$$\left(D'^2 - D' - 2\right) x = e^{3u} - (1)$$
$$D' = 2, -1$$
$$C.F = c_1' e^{2u} + c_2' e^{-u}$$

Now, we use the variation of parameters to find complete integral of $D \cdot F$ (1) Replacing c_1', c_2' by unknown functions A and B, the complete solution is

$$y = Ae^{2u} + Be^{-u}$$
$$= Ay_1 + By_2$$

where,

$$y_1 = e^{2u}, \quad y_2 = e^{-u}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2u} & e^{-u} \\ 2e^{2u} & -e^{-u} \end{vmatrix}$$
$$= -e^u - 2e^u = -3e^u \neq 0$$

$\therefore y_1$ & y_2 are independent.

$$A = -\int \frac{y_2 R}{w} du \quad , \quad R = e^{3u}$$
$$= -\int \frac{e^{-u} \cdot e^{3u}}{-3e^u} du$$
$$= \frac{1}{3} \int e^u du = \frac{e^u}{3} + c_1$$

$$B = \int \frac{y_1 R}{w} du$$
$$= \int \frac{e^{2u} \cdot e^{3u}}{-3e^u} du$$
$$= -\frac{1}{3} \int e^{4u} du = \frac{-e^{4u}}{12} + c_2$$

$\therefore$ Complete Solution

$$y = Ay_1 + By_2$$
$$= \left(\frac{e^u}{3} + c_1 \right) e^{2u} + \left(\frac{-e^{4u}}{12} + c_2 \right) e^{-u}$$
$$y = \left(\frac{t}{3} + c_1 \right) t^2 + \left(-\frac{t^4}{12} + c_2 \right) \frac{1}{t}$$

Question-6(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force).

[15 Marks]

Solution: $r^2 = a^2 \cos 2\theta$ or $a^2 u^2 \cos 2\theta = 1$, $u = \frac{1}{r}$ – (1) Taking log,

$$2 \log a + 2 \log u + \log \cos 2\theta = 0$$

Differentiating w.r.t. θ

$$0 + \frac{2}{u} \cdot \frac{du}{d\theta} - \frac{2 \sin 2\theta}{\cos 2\theta} = 0$$
$$\frac{du}{d\theta} = u \tan 2\theta$$

$$\frac{d^2u}{d\theta^2} = 2u\sec^2 2\theta + \frac{du}{d\theta}\tan 2\theta$$
$$= 2u\sec^2 2\theta + u\tan^2 2\theta$$
$$\therefore \frac{d^2u}{d\theta^2} + u = 2u\sec^2 2\theta + u\tan^2 2\theta + u$$
$$= 3u\sec^2 2\theta = 3u\left(a^2u^2\right)^2 \text{[from (1)]}$$
$$= 3a^4u^5$$

WKT DE of the central orbit in polar form is

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$\therefore \frac{F}{h^2u^2} = 3a^4u^5 \Rightarrow F = 3h^2a^4u^7$$

from (1)

$$= k\frac{1}{r^7}$$

i.e.

$$F \propto \frac{1}{r^7}$$

Hence the force varies inversely as the 7^{th} power of the distance from the pole.

Question-6(c) Verify Stokes' theorem for $\bar{V} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

[10 Marks]

Solution: Stokes' Theorem:

$$\oint_C \vec{F}\cdot dr = \iint_S (\vec{\nabla}\times\vec{F})\cdot \hat{n}dS$$

Here, the boundary C of S is a circle in xy plane $x^2+y^2 = 1$
Let

$$x = \cos t, y = \sin t, \quad z = 0$$

$0 \le t \le 2\pi$ be parametric equation of C

$$\oint_c \vec{V}\cdot dr = \oint_c \left[(2x-y)i - yz^2j - y^2zk\right]\cdot[dxi + dyj + dzk]$$

$$= \oint (2x-y)dx - yz^2dy - y^2zdz$$

$$= \phi(2x-y)dx \quad [\because z = 0 \quad \& \quad dz = 0]$$

$$= \int_0^{2\pi}(2\cos t - \sin t)(-\sin t)dt$$

$$= -\int_0^{2\pi} \sin 2t - \left(\frac{1-cos2t}{2}\right)dt$$

$$= -\left[-\frac{cos2t}{2} - \frac{t}{2} + \frac{\sin^2 t}{4}\right]_0^{2\pi}$$
$$= -\left[\left(-\frac{1}{2} - \frac{2\pi}{2} + 0\right) - \left(\frac{-1}{2} - 0 + 0\right)\right]$$
$$= \pi$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} 1 & J & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$
$$= i(-2yz + 2yz) + j(0 - 0) + k(0 + 1)$$
$$= k - (1)$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} dS = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$
$$= \iint_D k \cdot (xi + yj + zk) \frac{dxdy}{z}$$
$$= \iint_D dxdy$$

$[D : x^2 + y^2 \leq 1$ unit circle in xy plane centered at origin] Area of circle D

$$\pi(1)^2 = \pi - (2)$$

From (1) and (2), we see that

$$\oint_C \vec{V} \cdot dr = \iint_S (\nabla \times \vec{V}) \cdot \hat{n} dS$$

Question-7(a) Find the general solution of the differential equation

$$\ddot{x} + 4x = \sin^2 2t$$

Hence find the particular solution satisfying the conditions

$$x\left(\frac{\pi}{8}\right) = 0 \quad \text{and} \quad \dot{x}\left(\frac{\pi}{8}\right) = 0$$

[15 Marks]

Solution: Let

$$D = \frac{d}{dt}, \quad D^2 = \frac{\partial^2}{dt^2}$$
$$\left(D^2 + 4\right) x = \sin^2 2t$$

Auxiliary Eqn: $D^2 + 4 = 0$

$$D = \pm 2i$$
$$C \cdot F = c_1 \cos 2t + c_2 \sin 2t$$

$$P \cdot I = \frac{1}{D^2+4}\sin^2 2t$$
$$= \frac{1}{D^2+4}\left(\frac{1-\cos 4t}{2}\right)$$
$$= \frac{1}{2}\frac{1}{D^2+4}\cdot 1 - \frac{1}{2}\frac{1}{D^2+4}\cos 4t$$
$$= \frac{1}{2}\frac{1}{D^2+4}e^{0t} - \frac{1}{2}\cdot\frac{\cos ut}{(-16)+4}$$
$$= \frac{1}{8} + \frac{1}{24}\cos 4t$$

General solution: $\quad x = C \cdot F + P \cdot I$

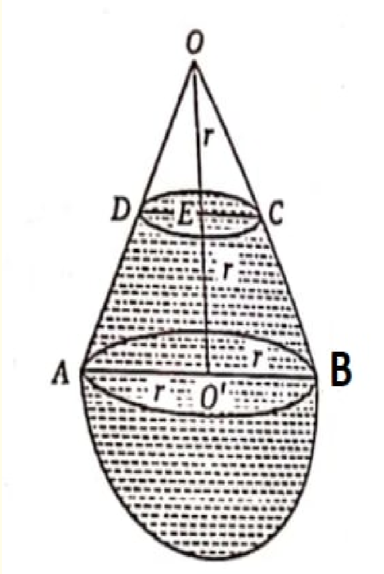
$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{24}(3 + \cos 4t)$$

$$x\left(\frac{\pi}{8}\right) = 0 \Rightarrow c_1 + c_2 = \frac{-\sqrt{2}}{8}$$
$$\dot{x}\left(\frac{\pi}{8}\right) = 0 \Rightarrow \quad c_2 - c_1 = \frac{\sqrt{2}}{12}$$
$$\therefore \quad c_1 = \frac{-5\sqrt{2}}{48}, \quad c_2 = \frac{-\sqrt{2}}{48}$$

Question-7(b) A vessel is in the shape of a hollow hemisphere surmounted by a cone held with the axis vertical and vertex uppermost. If it is filled with a liquid so as to submerge half the axis of the cone in the liquid and height of the cone be double the radius (r) of its base, find the resultant downward thrust of the liquid on the vessel in terms of the radius of the hemisphere and density (p) of the liquid.

[15 Marks]

Solution: Let r be the radius of the base of the hemisphere or cone so that the height of the surmounting cone is $2r$.



The vessel is filled upto CD so as to submerge half the axis of the cone in the liquid. From similar triangles OEC and $OO'B$, we have

$$\frac{EC}{O'B} = \frac{OE}{OO'} = \frac{r}{2r} = \frac{1}{2}$$

$$\therefore C = \frac{1}{2}OB' = \frac{1}{2}r$$

The resultant downward thrust of the liquid on the vessel = weight of the liquid contained in the vessel = wt. of the liquid in the hemisphere+ wt. of the liquid in the frustum

$$= \frac{2}{3}\pi r^3 w + \left[\frac{1}{3}\pi r^2 \cdot 2r - \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 \cdot r\right] w$$

$$= \frac{2}{3}\pi r^3 w + \frac{1}{3}\pi r^3 w\left(2 - \frac{1}{4}\right) = \frac{1}{3}\pi r^3 w\left(2 + \frac{7}{4}\right) = \frac{1}{3}\pi r^3 w \cdot \frac{15}{4}$$

$$= \frac{15}{8}(\frac{2}{3}\pi r^3 w)$$

Question-7(c) Derive the Frenet-Serret formulae. Verify the same for the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$

[10 Marks]

Solution: i) $\dfrac{dT}{ds} = kN$

ii) $\dfrac{dB}{ds} = -\tau N$

iii) $\dfrac{dN}{ds} = \tau B - kT$

where T, N, B are unit vectors along tangent principal normal and binormal directions.

$$|T| = 1 \quad \Rightarrow \quad T \cdot T = 1$$

$$\Rightarrow \quad 2T \cdot \frac{dT}{ds} = 0$$

$\Rightarrow \dfrac{dT}{ds}$ is $\perp$ to T.

Also, $\dfrac{dT}{ds}$ lies in oscillating plane.

$$\therefore \frac{dT}{ds} \text{ is parallel to } N$$

$$\therefore \frac{dT}{ds} = kN$$

ii) since, $|B| = 1$, unit vector

$$\therefore B \cdot B = 1 \quad \Rightarrow \quad 2B \cdot \frac{dB}{ds} = 0$$

$\Rightarrow dB/ds$ is $\perp$ to B $\quad \ldots \quad (1)$

W.K.T $\dfrac{dB}{dS}$ lies in oscillating plane. Also, since B and T are $\perp$

$$B \cdot T = 0$$

$$\Rightarrow \quad B \cdot \frac{dT}{ds} + T \cdot \frac{dB}{ds} = 0$$

$$B \cdot (kN) + T\frac{dB}{ds} = 0$$

$$(B \cdot N)k + \frac{dB}{ds} \cdot T = 0$$

$$\Rightarrow \frac{dB}{ds} \cdot T = 0$$

$$(\because B \perp N)$$

i.e. $\dfrac{dB}{ds}$ is $\perp$ to T $\quad \ldots \quad (2)$

From (1) and (2), $\dfrac{dB}{ds}$ is parallel to N

$$\Rightarrow \frac{dB}{ds} = -\tau N \quad (\tau = torsion)$$

iii) $B \times T = N$

$$B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$$

$$k(-T) - \tau(-B) = \frac{dN}{ds}$$

$$\therefore \quad \frac{dN}{ds} = \tau B - kT$$

Here, $\quad x = 3\cos t, \quad y = 3\sin t, \quad z = 4t$

$$\vec{R} = (3\cos t)i + (3\sin t)j + (4t)k$$

$$\frac{d\vec{r}}{dt} = (-3\sin t)i + (3\cos t)j + 4k$$

$$\left|\frac{dr}{dt}\right| = \sqrt{9\sin^2 t + 9cos^2 t + 16} = 5$$

Let S be length of arc from $t = 0$ to any point t on the curve, then

$$S = \int_0^t \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^t 5dt = 5t$$

$$\therefore \vec{r} = \left(3\cos\frac{s}{5}\right)i + \left(3\sin\frac{s}{5}\right)j + \left(\frac{4}{5}s\right)k$$

$$T = \frac{d\vec{r}}{ds} = \left(-\frac{3}{5}\sin\frac{s}{5}\right)i + \left(\frac{3}{5}\cdot\sin\frac{s}{5}\right]j + \frac{4}{5}k$$

$$\frac{dT}{ds} = \left(\frac{-3}{25}\cos\frac{s}{5}\right) - \frac{3}{25}\sin\frac{s}{5}j + 0$$

Principal Normal, N is parallel to $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix}$$

$$= i(0 + 12\sin t) + j(-12\cos t + 0) + k\left(+9\sin^2 t + 9\cos^2 t\right)$$

$$= (12\sin t)i - (12\cos t)j + 9k$$

$$\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}}) = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ 12\cos t & -12\sin t & 9 \end{vmatrix}$$

$$= i(27\cos t + 48\cos t) + j(48\sin t + 27\sin t) + k\left(36\sin t\cos t - 12\sin t\cos t\right)$$

$$= (75\cos t)i + (75\sin t)j$$

$$\left[\therefore \quad N = \pm\frac{1}{75}(75\cos ti + 75\sin tj) = -(\cos t)i + (-\sin t)j\right]$$

$$N = -(\cos\frac{s}{5})i + (-sin\frac{s}{5})j$$

(taking -ve sign)

$$\frac{dN}{ds} = -\frac{1}{5}\sin\frac{s}{5} + \frac{1}{5}\cos\frac{s}{5}j$$

Binormal vector B, is parallel to $\dot{\vec{r}} \times \ddot{\vec{r}}$

$$B = \frac{1}{\sqrt{144 + 81}}[12\sin ti - 12\cos tj + 9k]$$

$$B = \frac{12}{15}, \sin ti - \frac{12}{15}\cos tj + \frac{9}{15}k$$

$$B = \frac{4}{5}\sin\frac{s}{5}; -\frac{4}{5}\cdot\cos\frac{s}{5}j + \frac{3}{5}k$$

$$\frac{dB}{ds} = \frac{4}{25}\cos\frac{s}{5}i + \frac{4}{25}\sin\frac{s}{5}j$$

$$k = \left|\frac{dT}{ds}\right| = \frac{3}{25}, \quad \tau = \left|\frac{dB}{ds}\right| = \frac{4}{25}$$

Taking,

$$N = -\cos\frac{s}{5}; -\sin\frac{s}{5}j$$

i)

$$kN = \frac{3}{25}\left(-\cos\frac{s}{5}i - \sin\frac{s}{5}j\right) = \frac{dT}{ds}$$

ii)

$$-\tau N = \frac{-4}{25}\left(-\cos\frac{s}{5}i - \sin\frac{s}{5}j\right) = \frac{dB}{ds}$$

iii)

$$\begin{aligned}\tau B - kT &= \frac{4}{25}\left(\frac{4}{5}\sin\frac{s}{5}i - \frac{4}{5}\cos\frac{s}{5}j + \frac{3}{5}k\right) - \frac{3}{25}\left(\frac{-3}{5}\sin\frac{s}{5}i + \frac{3}{5}\cos\frac{s}{5}j + \frac{4}{5}k\right)\\ &= \frac{1}{5}\sin\frac{s}{5}; -\frac{1}{5}\cos\frac{s}{5}j\\ &= \frac{dN}{ds}\end{aligned}$$

Hence, we see that Frenet-Serret formulae are satisfied by the given curve in space.

Question-8(a) Find the general solution of the differential equation

$$(x-2)y'' - (4x-7)y' + (4x-6)y = 0$$

[10 Marks]

Solution:

$$y'' - \left(\frac{4x-7}{x-2}\right)y' + \left(\frac{4x-6}{x-2}\right)y = 0 - (1)$$

Comparing with:

$$y'' + Py' + Qy = 0$$

.

$$P = \frac{-(4x-7)}{x-2}, \quad Q = \frac{4x-6}{x-2}$$

Let e^{ax} be a solution, then

$$a^2 + aP + Q = 0$$

$$a^2 - \frac{a(4x-7)}{x-2} + \frac{4x-6}{x-2} = 0$$

$$\Rightarrow \quad a^2(x-2) - 4ax + 7a + 4x - 6 = 04$$

i.e.

$$x\left(a^2 - 4a + 4\right) - 2a^2 + 7a - 6 = 0$$

$$\Rightarrow \quad a^2 - 4a + 4 = 0 \quad \Rightarrow (a-2)^2 = 0 \Rightarrow a = 2$$

$$2a^2 - 7a + 6 = 0 \Rightarrow (2a-3)(a-2) = 0$$

i.e.

$$a = \frac{3}{2}, 2$$

$\therefore a = 2$ is common root $\Rightarrow e^{2x}$ is one solution.

$$\text{Consider } y = ve^{2x}$$
$$\frac{dy}{dx} = \frac{dv}{dx}e^{2x} + 2ve^{2x}$$
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^{2x} + 4\frac{dv}{dx}e^{2x} + 4ve^{2x}$$

Putting these values in (1)

$$\left(\frac{d^2v}{dx^2}e^{2x} + 4\frac{dv}{dx}e^{2x} + 4ve^{2x}\right) - \left(\frac{4x-7}{x-2}\right)\left(\frac{dv}{dx}e^{2x} + 2v^2x\right) + \frac{4x-6}{x-2}ve^{2x} = 0$$
$$\frac{d^2v}{dx^2} + \frac{dv}{dx}\left[4 - \frac{(4x-7)}{x-2}\right] + 4v - \frac{2(4x+7)}{x-2} + \frac{(4x-6)}{x-2} = 0$$
$$\frac{d^2v}{dx^2} - \frac{1}{x-2}\frac{dv}{dx} = 0$$

Let

$$\frac{dv}{dx} = p \Rightarrow \frac{dp}{dx} - \frac{1}{x-2} \cdot p = 0$$
$$\frac{dp}{p} = \frac{dx}{x-2}$$
$$\log p = \log(x-2) + \log C_1$$
$$\Rightarrow \quad p = c_1(x-2)$$

i.e.,

$$\frac{dv}{dx} = c_1(x-2)$$
$$v = c_1\left(\frac{x^2}{2} + 2x\right) + c_2\Big)$$
$$\therefore y = e^{2x} \cdot v = e^{2x}\left[c_1\left(\frac{x^2}{2} - 2x\right) + c_2\right]$$

Question-8(b) A shot projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h meters above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection.

[15 Marks]

Solution: We know that

$$x = (u\cos\theta)t$$
$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Equation of trajectory,

$$y = x\tan\theta - \frac{g}{2} \cdot \frac{x^2}{u^2\cos^2\theta}$$

When velocity is u, Range, $\quad R = \dfrac{u^2 \sin 2\theta}{g}$ With new velocity (say v), point $p(R, h)$ lies on the equation of trajectory

$$h = R\tan\theta - \frac{g}{2}\cdot\frac{R^2}{v^2\cos^2\theta}$$

$$= \frac{u^2\sin 2\theta}{g}\cdot\tan\theta - \frac{g}{2}\left(\frac{u^2\sin 2\theta}{g}\right)^2\frac{1}{v^2\cos^2\theta}$$

$$= \frac{2u^2\sin^2\theta}{g} - \frac{2u^4\sin^2\theta}{g\cdot v^2}$$

$$h = \frac{2u^2\sin^2\theta}{g}\left(1 - \frac{u^2}{v^2}\right)$$

$$1 - \frac{u^2}{v^2} = \frac{gh}{2u^2\sin^2\theta}$$

$$\frac{u}{v} = \left[1 - \frac{gh}{2u^2\sin^2\theta}\right]^{1/2}$$

i.e.

$$v = u\left(1 - \frac{gh}{2u^2\sin^2\theta}\right)^{-1/2}$$

$$\simeq u\left(1 + \frac{1}{2}\cdot\frac{gh}{2u^2\sin^2\theta}\right)$$

(Binomial Approximation)

$$\therefore v - u = \frac{gh}{4u\sin^2\theta}$$

Which is the required increase in the velocity of projection with same elevation θ.

Question-8(c) Derive $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right)$ in spherical coordinates.

[15 Marks]

Solution:

$$\nabla^2 F = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial F}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial F}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial F}{\partial u_3}\right)\right]$$

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$$

$$h_1 = h_R = 1 \quad , \quad h_2 = h_\theta = r$$

$$h_3 = h_\phi = r\sin\theta$$

$$\nabla^2 F = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r\cdot r\sin\theta}{1}\frac{\partial F}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta\cdot 1}{r}\frac{\partial F}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1\cdot r}{r\sin\theta}\cdot\frac{\partial F}{\partial\phi}\right)\right.$$
$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial F}{\partial\theta}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\frac{\partial F}{\partial\phi}\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial F}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 F}{\partial\phi^2}$$

$$F = \frac{x}{(x^2+y^2+z^2)^{3/2}}$$
$$= \frac{r\sin\theta cos\phi}{r^3}$$
$$= \frac{\sin\theta\cos\phi}{r^2}$$

$$\frac{\partial F}{\partial r} = \frac{-2\sin\theta cos\phi}{r^3}$$
$$\frac{\partial F}{\partial\theta} = \frac{\cos\theta cos\phi}{r^2}$$
$$\frac{\partial F}{\partial\phi} = \frac{-\sin\theta\sin\phi}{r^2}$$
$$\frac{\partial^2 F}{\partial\phi^2} = \frac{-\sin\theta cos\phi}{r^2}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{-2\cdot r^2\sin\theta\cos\phi}{r^3}\right)$$
$$= \frac{-2\sin\theta\cos\phi}{r^2}$$

$$\therefore \nabla^2 F = \frac{1}{r^2}\left(\frac{2\sin\theta cos\phi}{r^2}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta\cos\theta cos\phi}{r^2}-\frac{1}{r^2 sin^2\theta}\frac{sin\theta cos\phi}{r^2}$$
$$= \frac{2\sin\theta cos\phi}{24}+\frac{\cos 2\theta\cdot\cos\phi}{r^4\sin\theta}+\frac{cos\phi}{r^4\sin\theta}$$
$$= \frac{\cos\phi}{r^4}\left[2sin\theta+\frac{\cos 2\theta-1}{\sin\theta}\right]$$
$$= \frac{\cos\phi}{r^4}\left[\frac{2\sin^2\theta+\left(1-2\sin^2\theta\right)-1}{\sin\theta}\right]$$
$$= 0$$