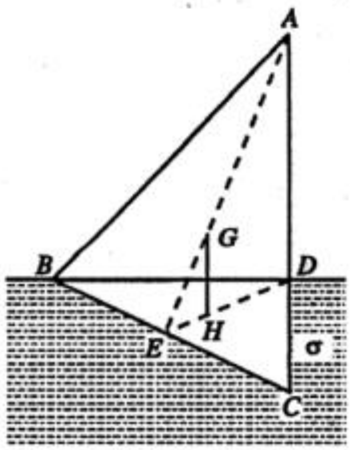
Ex. 17. *A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical, the angle B being in the surface of the liquid and the angle A not immersed. Show that*

$$\frac{\rho}{\sigma}=\frac{\sin A\cos C}{\sin B}=\frac{a^2+b^2-c^2}{2b^2},$$

a, b, c being the lengths of the sides of the triangle. **(Rohilkhand 1998, 99, 2000, 2004; Garhwal 2004)**

Sol. The portion BCD of the ΔABC is immersed in the liquid with BD in contact with the surface. Let G and H be the centres of gravity and buoyancy respectively.



E is the mid-point of BC.

The conditions of equilibrium are :

(i) The line GH must be vertical.

(ii) The weight of the lamina must be equal to the weight of the liquid displaced.

Since $EG=\frac{1}{3}EA$, $EH=\frac{1}{3}ED$, GH is parallel to AD.

But GH is vertical from the first condition so AC must be vertical.

From the second condition of equilibrium, we have

$$\Delta ABC\,\rho g=\Delta BDC\,\sigma\,g.$$

$$\therefore \qquad \frac{\rho}{\sigma}=\frac{\Delta BDC}{\Delta ABC}=\frac{\frac{1}{2}BD\,.\,DC}{\frac{1}{2}BD\,.\,AC}=\frac{DC}{AC}=\frac{BC\cos C}{AC}.$$

But $\qquad \dfrac{AC}{\sin B}=\dfrac{BC}{\sin A}$ or $BC=\dfrac{AC\sin A}{\sin B}$.

Hence $\qquad \dfrac{\rho}{\sigma}=\dfrac{AC\sin A\cos C}{AC\sin B}=\dfrac{\sin A\cos C}{\sin B}=\dfrac{a}{b}\cdot\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{a^2+b^2-c^2}{2b^2}.$

Ex. 14. *A body floating in water has volumes V_1, V_2, V_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Prove that*

$$\frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3}=0. \qquad \textbf{(Rohilkhand 1991, 93)}$$

Sol. Let V be the volume and W the weight of the body. Then the volumes immersed in water in the three cases are

$$(V-V_1), (V-V_2) \text{ and } (V-V_3).$$

Let ρ be the density of water.

For equilibrium, wt. of the body = wt. of water displaced + wt. of air displaced

$$\therefore \quad W=(V-V_1)\rho g+V_1\rho_1 g \quad \text{or} \quad W-V\rho g=V_1 g(\rho_1-\rho)$$

or $$\frac{W-V\rho g}{V_1}=g(\rho_1-\rho) \qquad \ldots(1)$$

Similarly $$\frac{W-V\rho g}{V_2}=g(\rho_2-\rho) \qquad \ldots(2)$$

and $$\frac{W-V\rho g}{V_3}=g(\rho_3-\rho) \qquad \ldots(3)$$

Multiplying (1) by $(\rho_2-\rho_3)$, (2) by $(\rho_3-\rho_1)$ and (3) by $(\rho_1-\rho_2)$ and adding, we get

$$(W-V\rho g)\left[\frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3}\right]=0$$

or $$\frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3}=0.$$

Note. The above result can be put in the form

$$V_2V_3(\rho_2-\rho_3)+V_3V_1(\rho_3-\rho_1)+V_1V_2(\rho_1-\rho_2)=0$$

or $$\rho_1V_1(V_2-V_3)+\rho_2V_2(V_3-V_1)+\rho_3V_3(V_1-V_2)=0.$$

Ex. 15. *If a body floats in a liquid with volumes V_1, V_2 and V_3 above the surface when the barometric heights are h_1, h_2 and h_3, prove that*

$$h_1 V_1 (V_2 - V_3) + h_2 V_2 (V_3 - V_1) + h_3 V_3 (V_1 - V_2) = 0.$$

Sol. Let V be the volume of the body and σ its density. Let ρ be the density of the liquid. Let the densities of the surrounding air be ρ_1, ρ_2 and ρ_3 corresponding to barometric heights, so we have

$$\rho_1 = \lambda h_1, \ \rho_2 = \lambda h_2, \ \rho_3 = \lambda h_3,$$

where λ is a constant.

For equilibrium,

wt. of the body = wt. of liquid displaced + wt. of air displaced.

$$\therefore \quad V\sigma g = (V - V_1)\rho g + V_1 \lambda h_1 g \quad \text{or} \quad V_1 \rho = \lambda V_1 h_1 + V(\rho - \sigma)$$

or
$$V_1 = \frac{\lambda}{\rho} V_1 h_1 + \frac{V}{\rho}(\rho - \sigma) \qquad \ldots(1)$$

Similarly,
$$V_2 = \frac{\lambda}{\rho} V_2 h_2 + \frac{V}{\rho}(\rho - \sigma) \qquad \ldots(2)$$

and
$$V_3 = \frac{\lambda}{\rho} V_3 h_3 + \frac{V}{\rho}(\rho - \sigma) \qquad \ldots(3)$$

Subtracting (2) from (3), we get

$$V_2 - V_3 = \frac{\lambda}{\rho}(V_2 h_2 - V_3 h_3).$$

$$\therefore \quad h_1 V_1 (V_2 - V_3) = \frac{\lambda}{\rho} h_1 V_1 (V_2 h_2 - V_3 h_3).$$

Similarly,
$$h_2 V_2 (V_3 - V_1) = \frac{\lambda}{\rho} h_2 V_2 (V_3 h_3 - V_1 h_1)$$

and
$$h_3 V_3 (V_1 - V_2) = \frac{\lambda}{\rho} h_3 V_3 (V_1 h_1 - V_2 h_2).$$

Adding the above three results, we get

$$h_1 V_1 (V_2 - V_3) + h_2 V_2 (V_3 - V_1) + h_3 V_3 (V_1 - V_2) = 0.$$

Ex. 3. *A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by a weight 2P. Prove that the densities of the body and liquid are as* $3:2$.

Sol. Let V be the volume of the body. Let ρ and σ be the densities of the body and the liquid respectively.

In the first case, let T be the tension in the string. Then

$P = T =$ wt. of the body − wt. of the liquid displaced $= V\rho g - V\sigma g$ ...(1)

In the second case, let T' be the tension in the string. Then

$$2P = T' = V\rho g - \frac{1}{2} V\sigma g \quad \text{...(2)}$$

Dividing (1) by (2), we get

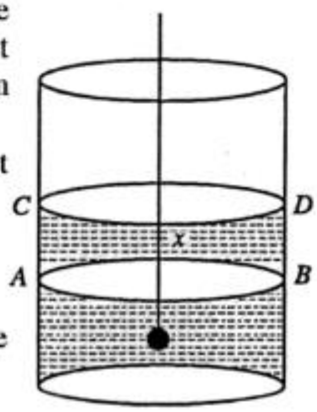
$$\frac{1}{2} = \frac{\rho - \sigma}{\rho - \frac{1}{2}\sigma} \quad \text{or} \quad \rho - \frac{1}{2}\sigma = 2\rho - 2\sigma$$

or $$\rho = \frac{3}{2}\sigma \quad \text{or} \quad \frac{\rho}{\sigma} = \frac{3}{2}.$$

Ex. 44. *A cylindrical vessel on a horizontal circular base of radius a, is filled with a liquid of density w to a height h. If now a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, prove that the increase of the whole pressure on the curved surface is*

$$\frac{8\pi}{3a} wc^3 \left(h + \frac{2c^3}{3a^2}\right) g.$$

Sol. Let the level of the liquid in the vessel be AB before the immersion of the sphere. After the sphere is immersed, let the level of the liquid be CD. If x be the increased height when the level is raised then $AC = BD = x$.



Since the volume of the liquid displaced by the sphere must be equal to the volume of the sphere, so we have

$$\pi a^2 x = \frac{4}{3}\pi c^3 \quad \Rightarrow \quad x = \frac{4}{3}\left(\frac{c^3}{a^2}\right).$$

Now the whole pressure on the curved surface before immersion $= P_1 = 2\pi ah \cdot \frac{1}{2}h \cdot wg = \pi ah^2 wg$.

Whole pressure on the curved surface after immersion $= P_2$

$$= 2\pi a(h+x) \cdot \frac{1}{2}(h+x)\, wg = \pi a(h+x)^2 wg.$$

$\therefore$ Increase of the whole pressure on the curved surface $= P_2 - P_1$

$$= \pi awg\,[(h+x)^2 - h^2] = \pi awg\,(x^2 + 2hx)$$

$$= \pi awg\, x(x+2h) = \pi awg \frac{4}{3} \cdot \left(\frac{c^3}{a^2}\right)\left(\frac{4}{3} \cdot \frac{c^3}{a^2} + 2h\right)$$

$$= \frac{8\pi}{3a} wgc^3 \left(h + \frac{2c^3}{3a^2}\right).$$

Ex. 1. *An ellipse is just immersed in water with its major axis vertical. Show that if the centre of pressure coincides with the focus, the eccentricity of the ellipse must be* $\frac{1}{4}$.

(Garhwal 2004; Bundelkhand 2001; Rohilkhand 1996)

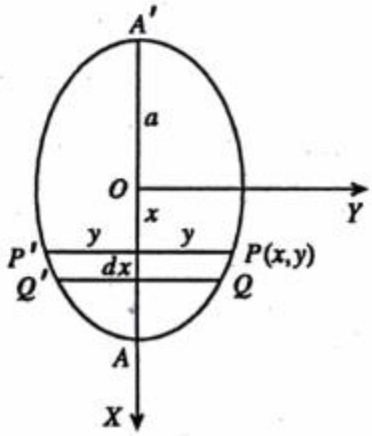
Sol. Take the major axis and minor axis respectively as the axes of x and y. Then the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(1)$$

By symmetry it is clear that the C.P. $(\bar{x}, \bar{y})$ will lie on the line AOA' *i.e.*, x-axis.

$\therefore \qquad \bar{y}=0.$

Take an elementary strip $PQQ'P'$ at a depth x below O, the centre of the ellipse, and of width dx. Then dS = area of the elementary strip $=2y\,dx$, p = intensity of pressure at any point of the strip $=\rho g\,(a+x)$, where ρ is the density of the liquid.



$\therefore \quad \bar{x}$ = depth of the C.P. of the ellipse below O

$$=\frac{\int x\,p\,dS}{\int p\,dS}, \text{ between suitable limits}$$

$$=\frac{\int_{-a}^{a} x\,\rho g\,(a+x)\,2y\,dx}{\int_{-a}^{a} \rho g\,(a+x)\,2y\,dx}=\frac{\int_{-a}^{a} xy\,(a+x)\,dx}{\int_{-a}^{a} y\,(a+x)\,dx}.$$

The parametric equations of the ellipse (1) are

$$x=a\cos t,\ \ y=b\sin t.$$

$\therefore \qquad dx=-\,a\sin t\,dt.$

Also when $x=a$, $\cos t=1 \Rightarrow t=0$ and when $x=-\,a$, $\cos t=-1 \Rightarrow t=\pi$.

$$\therefore \qquad \bar{x}=\frac{\int_{\pi}^{0} a\cos t\,.\,b\sin t\,(a+a\cos t)\,(-\,a\sin t\,dt)}{\int_{\pi}^{0} b\sin t\,(a+a\cos t)\,(-a\sin t\,dt)}$$

$$=\frac{a\int_{0}^{\pi}(\cos t\sin^2 t+\cos^2 t\sin^2 t)\,dt}{\int_{0}^{\pi}(\sin^2 t+\cos t\sin^2 t)\,dt}$$

$$= \frac{a\left[\int_0^{\pi} \cos t \sin^2 t\, dt + \int_0^{\pi} \cos^2 t \sin^2 t\, dt\right]}{\int_0^{\pi} \sin^2 t\, dt + \int_0^{\pi} \cos t \sin^2 t\, dt}$$

$$= \frac{a\left[0 + 2\int_0^{\pi/2} \cos^2 t \sin^2 t\, dt\right]}{2\int_0^{\pi/2} \sin^2 t\, dt + 0} = \frac{a\left(\frac{1\,.\,1}{4\,.\,2} \cdot \frac{\pi}{2}\right)}{\frac{1}{2} \cdot \frac{\pi}{2}} = \frac{a}{4}.$$

Now the C.P. of the ellipse will coincide with the focus, if $\bar{x} = ae$ *i.e.*, if $\frac{a}{4} = ae$ or $e = \frac{1}{4}$.

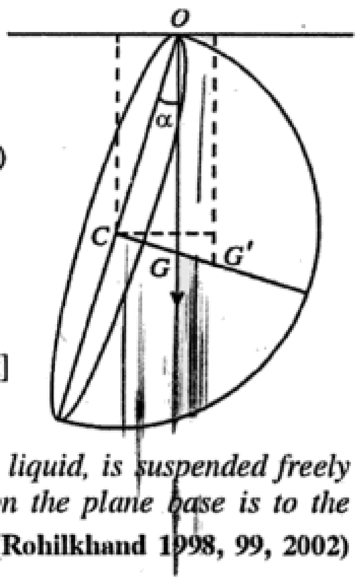
Ex. 29. *A hollow weightless hemisphere, filled with liquid, is suspended freely from a point in the rim of its base ; prove that the whole pressure on the curved surface and that on the base are in the ratio* 19 : 8. **(Garhwal 20002)**

Sol. Let a be the radius of the hemisphere and O the point of the rim from which it is suspended. If G be the C.G. of the hemisphere, then $CG = \frac{3a}{8}$ and OG must be vertical. If α be the inclination of the base to the vertical, then

$$\tan \alpha = \frac{3}{8}. \qquad \ldots(1)$$

The whole pressure on the base $= w\,.\,\pi a^2\,.\,a \cos \alpha$. ...(2)

Let G' be the C.G. of the curved surface of the hemisphere, then $CG' = \frac{1}{2}a$.



The depth of G' below $O = a \cos \alpha + \frac{1}{2} a \sin \alpha$.

$\therefore$ the whole pressure on the curved surface

$$= w\,.\,2\pi a^2\,.\left(a \cos \alpha + \frac{1}{2} a \sin \alpha\right) \qquad \ldots(3)$$

From (2) and (3), the required ratio is

$$2\left(\cos \alpha + \frac{1}{2} \sin \alpha\right) : \cos \alpha$$

or $\quad 2\left(1 + \frac{1}{2} \tan \alpha\right) : 1$ or $2\left(1 + \frac{3}{16}\right) : 1$ [From (1)]

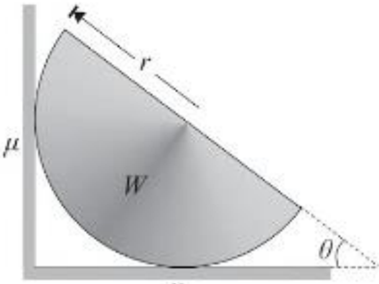
or $\quad 19 : 8.$

Ex. 30. *A hollow weightless hemisphere, filled with liquid, is suspended freely from a point in the rim of its base ; show that the thrust on the plane base is to the weight of the contained liquid as* $12 : \sqrt{(73)}$. **(Rohilkhand 1998, 99, 2002)**

Sol. Proceed as in Ex. 29.

The weight of the contained liquid $= \frac{2}{3}\pi a^3\,.\,w$.

A semi-circular disc of weight W rests in a vertical plane with its curved edge on a rough horizontal plane and an equally rough vertical plane as shown in Figure 3.2(a). Find the greatest angle θ that the diameter can make with the horizontal plane.



Solution. Fgure 3.2(b) shows a free body diagram of semi-circular disc. The condition of equilibrium is at the instant before sliding occurs. Thus, the sum of the horizontal forces, vertical forces and moments is zero, which gives

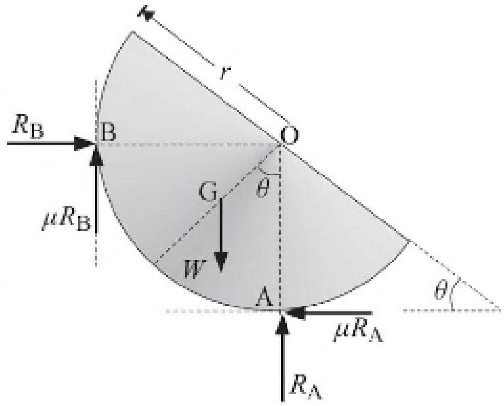


FIGURE 3.2(b)

$$\sum F_x = 0 \Rightarrow R_B - \mu R_A = 0 \qquad \text{(i)}$$

$$\sum F_y = 0 \Rightarrow R_A + \mu R_B - W = 0 \qquad \text{(ii)}$$

$$\sum M_A = 0 \Rightarrow R_B \times r + \mu R_B \times r - W \times \text{OG} \sin\theta = 0 \qquad \left[\text{OG} = \frac{4r}{3\pi}\right]$$

or $$(1+\mu) R_B \times r = W \frac{4r}{3\pi} \sin\theta \qquad \text{(iii)}$$

Solving for R_B from Equations (i) and (ii), we have

$$R_B = \frac{\mu}{1+\mu^2} W$$

Substituting the value of R_B in the Equation (iii), we have

$$\frac{\mu(1+\mu)}{1+\mu^2} = \frac{4}{3\pi} \sin\theta$$

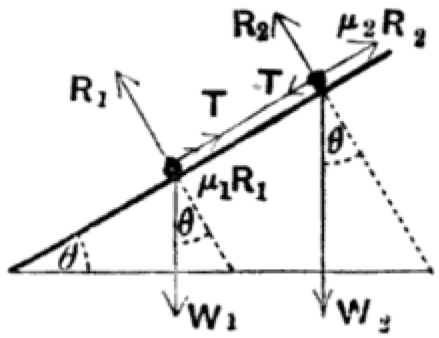
or $$\sin\theta = \frac{3\pi}{4} \times \left(\frac{\mu+\mu^2}{1+\mu^2}\right)$$

From which,

$$\boxed{\theta = \sin^{-1}\left(\frac{3\pi}{4} \times \frac{\mu+\mu^2}{1+\mu^2}\right)}$$

210. (193) **Ex.** *Two bodies, of weights W_1 and W_2, are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficients of friction between the bodies and the plane be respectively μ_1 and μ_2, find the inclination of the plane to the horizon when both bodies are on the point of motion, it being assumed that the smoother body is below the other.*

The lower body would slip when the inclination is $\tan^{-1}\mu_1$, but the upper would not do so till the inclination had the value $\tan^{-1}\mu_2$. When the two are tied together the inclination for slipping would be between these two values. Let it be θ and let R_1 and R_2 be the normal reactions of the bodies; also let T be the tension of the string.



The frictions $\mu_1 R_1$ and $\mu_2 R_2$ both act up the plane.

For the equilibrium of W_1, we have

$$W_1 \sin\theta = T + \mu_1 R_1,$$

and

$$W_1 \cos\theta = R_1.$$

$$\therefore \quad T = W_1(\sin\theta - \mu_1\cos\theta) \quad \text{.....................(1).}$$

For the equilibrium of W_2, we have

$$W_2 \sin\theta + T = \mu_2 R_2,$$

and

$$W_2 \cos\theta \quad = R_2.$$

$$\therefore \quad T = \mu_2 R_2 - W_2\sin\theta = W_2(\mu_2\cos\theta - \sin\theta) \quad \text{.........(2).}$$

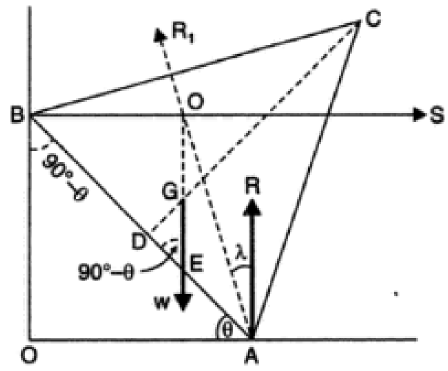
Hence, from (1) and (2),

$$W_1(\sin\theta - \mu_1\cos\theta) = W_2(\mu_2\cos\theta - \sin\theta).$$

$$\therefore \quad (W_1 + W_2)\sin\theta = (W_1\mu_1 + W_2\mu_2)\cos\theta.$$

$$\therefore \quad \tan\theta = \frac{W_1\mu_1 + W_2\mu_2}{W_1 + W_2}.$$

Example 4. *An equilateral triangle of uniform thickness and density, rests with one end of its base on a rough horizontal plane and the other against a smooth vertical wall. Show that the least angle that its base can make with the horizontal plane is given by* $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$, *'$\mu$' being the co-efficient of friction.*



Sol. The base AB will make the least angle with the horizontal plane when the triangle is in limiting equilibrium. Let $2a$ be a side of the triangle ABC.

Let the normal reaction S at B and the resultant reaction R_1 at A meet in O. Then OG is vertical, where G is the C.G. of ΔABC.

Let D be the mid-point of AB, then $GD \perp AB$.

$$GD = \frac{1}{3}CD = \frac{1}{3}\sqrt{BC^2 - BD^2} = \frac{1}{3}\sqrt{4a^2 - a^2}$$

$$= \frac{1}{3} \cdot a\sqrt{3} = \frac{a}{\sqrt{3}}$$

$$\angle DEG = 90° - \theta$$

$$\therefore \quad DE = GD\cot(90° - \theta) = \frac{a}{\sqrt{3}}\tan\theta$$

$$\therefore \quad AE = AD - DE = a - \frac{a}{\sqrt{3}}\tan\theta$$

$$BE = BD + DE = a + \frac{a}{\sqrt{3}}\tan\theta$$

$$\angle BOE = 90°, \angle AOE = \lambda$$

By "m–n theorem" in ΔAOB, we have

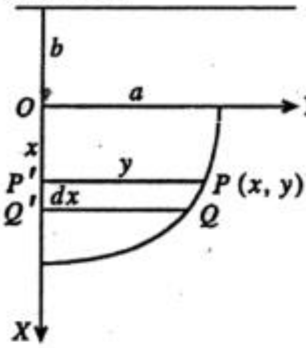
$$(AE + EB)\cot(90° - \theta) = AE\cot\lambda - EB\cot 90°$$

or
$$2a\tan\theta = \left(a - \frac{a}{\sqrt{3}}\tan\theta\right)\cot\lambda \quad \text{or} \quad \left(2 + \frac{1}{\sqrt{3}}\cot\lambda\right)\tan\theta = \cot\lambda$$

$$\therefore \quad \cot\theta = \left(2 + \frac{1}{\sqrt{3}}\cot\lambda\right) \cdot \frac{1}{\cot\lambda} = 2\tan\lambda + \frac{1}{\sqrt{3}} = 2\mu + \frac{1}{\sqrt{3}}.$$

Ex. 9. *A quadrant of a circle of radius a is immersed vertically with its bounding radius horizontal at a depth b. Find the centre of pressure.*

Sol. As in Ex.5, if $(\bar{x}, \bar{y})$ be the coordinates of the C.P., then



$$\bar{x} = \frac{\int x\, p\, dS}{\int p\, dS}, \quad \bar{y} = \frac{\int \frac{1}{2} y\, p\, dS}{\int p\, dS},$$

between suitable limits where $dS = y\, dx$, $p = \rho g\,(b + x)$ and $x^2 + y^2 = a^2$.

$$\therefore \qquad \bar{x} = \frac{\int_0^a x \rho g\,(b+x)\,.\,y\,dx}{\int_0^a \rho g\,(b+x)\,.\,y\,dx} = \frac{\int_0^a xy\,(b+x)\,dx}{\int_0^a y\,(b+x)\,dx}$$

$$= \frac{b\int_0^a x\sqrt{(a^2-x^2)}\,dx + \int_0^a x^2\sqrt{(a^2-x^2)}\,dx}{b\int_0^a \sqrt{(a^2-x^2)}\,dx + \int_0^a x\sqrt{(a^2-x^2)}\,dx}$$

$$= \frac{ba^3\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta + a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta}{ba^2\int_0^{\pi/2}\cos^2\theta\,d\theta + a^3\int_0^{\pi/2}\sin\theta\cos^2\theta\,d\theta},$$

putting $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$= \frac{ba^3\left(\frac{1}{3\,.\,1}\right) + a^4\left(\frac{1\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}\right)}{ba^2\left(\frac{1}{2}\cdot\frac{\pi}{2}\right) + a^3\,\frac{1}{3\,.\,1}} = \frac{1}{4}\cdot\frac{16ab + 3\pi a^2}{3b\pi + 4a}.$$

Similarly calculate

$$\bar{y} = \frac{3a^2 + 8ab}{2\,(3b\pi + 4a)}.$$

Ex. 10. *A semi-circular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of the centre of pressure below the surface is*

$$\frac{1}{4} \cdot \frac{3\pi (a^2 + 4b^2) + 32ab}{4a + 3\pi b}.$$

Sol. By symmetry, the C.P. lies on the vertical radius. If $\bar{x}$ be the depth of the C.P. below the centre, then as in Ex. 9, we get

$$\bar{x} = \frac{3\pi a^2 + 16ab}{4(4a + 3\pi b)}.$$

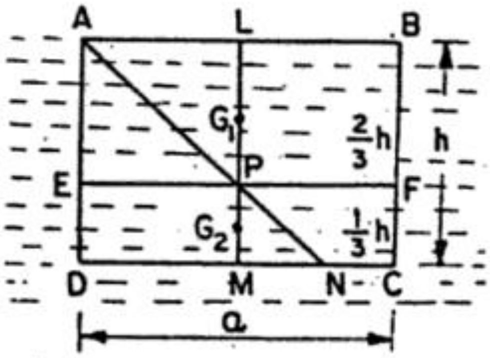
$\therefore$ Depth of the C.P. below the surface $= \bar{x} + b$

$$= \frac{3\pi a^2 + 16ab}{4(4a + 3\pi b)} + b = \frac{3\pi (a^2 + 4b^2) + 32ab}{4(4a + 3\pi b)}.$$

Example 2. (*a*) ***Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which are in the ratio 4 : 5.***

(*b*) ***P is the C.P. of rectangle ABCD, the side AB being in the surface. Prove that the line through A and P divides the area into two portions, the pressure on which are equal.***

Hint. (*a*) See Art. 3.7 (*i*)



$$\bar{x} = \frac{2}{3} h = \text{PL}$$

P_1 = Pressure on rectangle ABFE

$$= \rho g \left(a \cdot \frac{2}{3} h \right) \frac{1}{2} \cdot \frac{2}{3} h$$

$$= \frac{2}{9} \rho g a h^2$$

P_2 = Pressure on rectangle EFCD

$$= \rho g \left(a \cdot \frac{1}{3} h \right) \left(\frac{2}{3} h + \frac{1}{2} \cdot \frac{h}{3} \right) = \frac{5}{18} \rho g a h^2$$

$\therefore \qquad P_1 : P_2 = 4 : 5.$

Hint. (*b*) As in part (*a*), $\bar{x} = \text{LP} = \frac{2}{3} h.$

31. *A smooth wedge, of mass M, is placed on a horizontal plane, and a particle, of mass m, slides down its slant face, which is inclined at an angle α to the horizon; prove that the acceleration of the wedge is*

$$\frac{mg \sin \alpha \cos \alpha}{M + m \sin^2 \alpha}.$$

[Let f_1 be the acceleration of the particle in a direction perpendicular to, and towards, the slant face; f_2 the horizontal acceleration of the wedge; and R the normal reaction between the particle and the slant face, so that R acts in one direction on the particle and in the opposite direction on the wedge. Then

$$mf_1 = mg \cos \alpha - R \qquad \text{.........................(1)},$$

and

$$Mf_2 = R \sin \alpha \qquad \text{...............................(2)}.$$

Also, since the particle remains in contact with the slant face, the acceleration f_1 must be the same as the acceleration of the wedge resolved in a direction perpendicular to the slant face.

$$\therefore\ f_1 = f_2 \sin \alpha \qquad \text{..........................(3)}.$$

Solving (1), (2), and (3), we have f_2.]

Example 1. *Show that in a conic, the semi-latus rectum is the harmonic mean between the segments of any focal chord.*

Or

If PSP′ is a focal chord of a conic, prove that $\dfrac{1}{SP} + \dfrac{1}{SP'} = \dfrac{2}{l}$ *where 'l' is the semi-latus rectum.*

Or

Show that in a conic, the sum of the reciprocals of the segments of any focal chord is constant.

Sol. Let the equation of the conic be $\dfrac{l}{r} = 1 + e \cos \theta$ ...(1)

Let PSP′ be any focal chord. Let the vectorial angle of P be α, then the vectorial angle of P′ is $\pi + \alpha$.

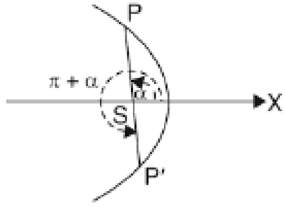
$\therefore$ Co-ordinates of P and P′ are (SP, α) and (SP′, $\pi + \alpha$).

Since P and P′ both lie on (1)

$$\therefore \quad \frac{l}{SP} = 1 + e \cos \alpha \qquad ...(2)$$

$$\frac{l}{S'P} = 1 + e \cos (\pi + \alpha)$$

$$= 1 - e \cos \alpha \qquad ...(3)$$



Adding (2) and (3),

$$\frac{l}{SP} + \frac{l}{SP'} = 2 \quad \therefore \quad \frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$$

Hence 'l' is the harmonic mean between SP and SP′.

Example 16. *A triangle ABC is immersed in a liquid with vertex C in the surface, and the sides BC and AC equally inclined to the surface, show that the vertical through C divides the triangle into two others, the fluid pressure upon which are*

$$(b^3 + 3ab^2) : (a^3 + 3a^2b).$$

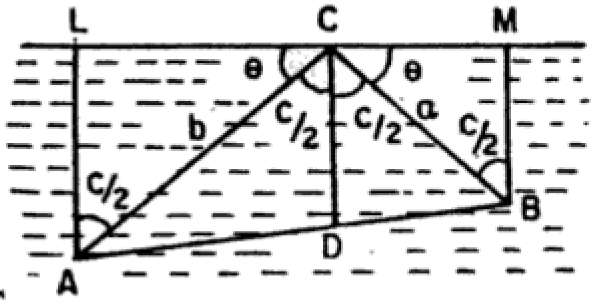
Sol. The vertex C of ΔABC is in free surface of liquid. Let

$\angle BCM = \angle LCA = \theta$

$\angle MCD = \angle LCD = \dfrac{\pi}{2}$

$\therefore \quad \angle BCD = \angle ACD$

$= \dfrac{\pi}{2} - \theta$



But $\angle BCD + \angle ACD = \angle C$

$\Rightarrow \quad \angle BCD = \angle ACD = \dfrac{\angle C}{2}$

(*i*) Depth of C.G. of ΔACB $= \dfrac{0 + LA + CD}{3}$

$$= \frac{0 + b\cos\frac{C}{2} + CD}{3}$$

In ΔALC
$\cos\dfrac{C}{2} = \dfrac{LA}{AC}$

Area of ΔACD $= \dfrac{1}{2}\,.\,AC\,.\,CD\sin\dfrac{C}{2}$

$= \dfrac{1}{2}b\,.\,CD\,.\,\sin\dfrac{C}{2}$

$$\therefore \quad \text{Pressure on } \Delta ACD = \rho g\,.\left(\frac{b\cos\frac{C}{2} + CD}{3}\right).\frac{1}{2}b\,.\,CD\,.\,\sin\frac{C}{2}$$

$$= \frac{1}{6} b \left(b \cos \frac{C}{2} + CD \right) . CD . \sin \frac{C}{2} . \rho g.$$

(*ii*) Depth of C.G. of $\Delta DCB = \frac{0 + CD + MB}{3} = \frac{1}{3} \left(CD + a \cos \frac{C}{2} \right)$

$$\left(\because \text{ In CBM, } \cos \frac{C}{2} = \frac{MB}{CB} \quad \therefore \; MB = a \cos \frac{C}{2} \right).$$

$$\text{Area of } \Delta DCB = \frac{1}{2} . BC . CD \sin \frac{C}{2}$$

$$= \frac{1}{2} . a . CD \sin \frac{C}{2}$$

$$\therefore \quad \text{Pressure on } \Delta DCB = \rho g . \frac{1}{3} \left(a \cos \frac{C}{2} + CD \right) . \frac{1}{2} . a . CD \sin \frac{C}{2}$$

$$= \frac{1}{6} a \left(a \cos \frac{C}{2} + CD \right) . CD . \sin \frac{C}{2} . \rho g$$

$$\therefore \quad \frac{\text{Pressure on } \Delta ACD}{\text{Pressure on } \Delta BCD} = \frac{\frac{1}{6} b \left(b \cos \frac{C}{2} + CD \right) . CD . \sin \frac{C}{2} . \rho g}{\frac{1}{6} a \left(a \cos \frac{C}{2} + CD \right) . CD \sin \frac{C}{2} \rho g}$$

$$= \frac{b \left(b \cos \frac{C}{2} + CD \right)}{a \left(a \cos \frac{C}{2} + CD \right)} \qquad ...(1)$$

Now $\Delta ACD + \Delta BCD = \Delta ABC$

or $\frac{1}{2} . b . CD . \sin \frac{C}{2} + \frac{1}{2} . a . CD . \sin \frac{C}{2} = \frac{1}{2} ab \sin C$

$$= \frac{1}{2} ab . 2 \sin \frac{C}{2} \cos \frac{C}{2}$$

$$\frac{1}{2} \cdot \text{CD}\,(b+a) \sin\frac{C}{2} = \frac{1}{2} ab \cdot 2 \sin\frac{C}{2} \cos\frac{C}{2}$$

$$\therefore \qquad \text{CD} = \frac{2\,ab \cos\dfrac{C}{2}}{a+b}$$

Putting the value of CD in (1)

$$\therefore \quad \frac{\text{Pressure on } \Delta\text{ACD}}{\text{Pressure on } \Delta\text{BCD}} = \frac{b\left(b \cos\dfrac{C}{2} + \dfrac{2ab}{a+b} \cos\dfrac{C}{2}\right)}{a\left(a \cos\dfrac{C}{2} + \dfrac{2ab}{a+b} \cos\dfrac{C}{2}\right)}$$

$$= \frac{b\left(b + \dfrac{2ab}{a+b}\right)}{a\left(a + \dfrac{2ab}{a+b}\right)} = \frac{b[b(a+b) + 2ab]}{a[a(a+b) + 2ab]}$$

$$= \frac{b[b^2 + 3ab]}{a(a^2 + 3ab)} = \frac{b^3 + 3ab^2}{a^3 + 3a^2b}.$$

Example 3 If the three thermodynamic variables P, V, T are connected by a relation $F(P,V,T)=0$ show that $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$

Solution Taking differential, we get

$$\frac{\partial F}{\partial P}dP + \frac{\partial F}{\partial V}dV + \frac{\partial F}{\partial T}dT = 0$$

If V is a constant $dV=0$, then

$$\frac{\partial F}{\partial P}dP + \frac{\partial F}{\partial T}dT = 0$$

$$i.e. \quad \frac{\partial F}{\partial P}\frac{\partial P}{\partial T} + \frac{\partial F}{\partial T} = 0$$

Since V remains constant $\frac{dP}{dT}$ is $\left(\frac{\partial P}{\partial T}\right)_V$

$$\therefore \quad \left(\frac{\partial P}{\partial T}\right)_V = -\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial P}} \qquad \text{...(i)}$$

Similarly, keeping P constant

$$\left(\frac{\partial T}{\partial V}\right)_P = -\frac{\frac{\partial F}{\partial V}}{\frac{\partial F}{\partial T}} \qquad \text{...(ii)}$$

Similarly, keeping T constant

$$\left(\frac{\partial V}{\partial P}\right)_T = -\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial V}} \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) and simplifying, we get

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1.$$