2018

Prove that the function

$$u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$$

is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Let, $f(z) = u + iv$

$$u = (x-1)^3 - 3xy^2 + 3y^2$$

$$u_x = 3(x-1)^2 - 3y^2$$

$$u_{xx} = 6(x-1) \quad \text{— (1)}$$

$$u_y = -6xy + 6y$$

$$u_{yy} = -6x + 6 = -6(x-1) \quad \text{— (2)}$$

$\therefore$ $u_{xx} + u_{yy} = 0$

Hence function u is harmonic.

Using Milne's Method to find Harmonic Conjugate

$$\phi_1(z,0) = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=z\\y=0}} = 3(z-1)^2$$

$$\phi_2(z,0) = \left.\frac{\partial u}{\partial y}\right|_{\substack{x=z\\y=0}} = 0$$

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$$

$$= \int 3(z-1)^2\,dz + C$$

$$= (z-1)^3 + C.$$

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$$\therefore \quad f(z) = (z-1)^3 + C$$

$$f(z) = [(x-1)+iy]^3 + C$$

$$= (x-1)^3 - iy^3 + 3(x-1)\, iy\, [(x-1)+iy]$$

$$= (x-1)^3 - 3(x-1)y^2 - iy^3 + 3i(x-1)^2 y$$

$$= (x-1)^3 - 3(x-1)y^2 + i\left[3(x-1)^2 y - y^3\right]$$

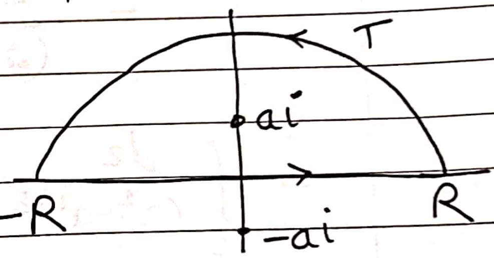
$$\therefore \quad v(x,y) = 3(x-1)^2 y - y^3$$

CSS/2018

3) Show by applying the residue Theorem that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a>0.$$

Let $f(z) = \dfrac{1}{(z^2+a^2)^2}$ over a contour ~~containing~~ consisting of a large semi-circle, T of radius R, together with part of real axis from $-R$ to R.



Poles of $f(z)$ are given by

$(z^2+a^2)^2 = 0$

$z = ai, -ai$ (Twice)

out of which only $z=ai$ (order = 2) lies incide C.

By Cauchy's residue Theorem

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_T \frac{dz}{(z^2+a^2)^2} \quad \text{—— (1)}$$

$$\lim_{z\to\infty} z\cdot f(z) = \lim_{z\to\infty} \frac{z}{(z^2+a^2)^2} = \lim_{z\to\infty} \frac{z}{z^4\left(1+\frac{a^2}{z^2}\right)^2}$$

$$\longrightarrow 0$$

$$\therefore \lim_{R\to\infty} \int_T \frac{dz}{(z^2+a^2)^2} = 0 \quad \text{—— (2)}$$

and $$\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = 2\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$$

—— (3)

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$$f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z+ai)^2} \cdot \frac{1}{(z-ai)^2}$$

Residue at $ai = \frac{d}{dz} (z-ai)^2 \cdot f(z) \Big|_{z=a\hat{i}}$

$$= \frac{d}{dz} \cdot \frac{1}{(z+ai)^2}\Big|_{z=ai} = \frac{-2}{(z+ai)^3}\Big|_{z=ai}$$

$$= \frac{-2}{(2ai)^3} = \frac{1}{4a^3 i}$$

$$\therefore \int_C \frac{dz}{(z^2+a^2)^2} = 2\pi i \left(\frac{1}{4a^3 i}\right)$$ (Cauchy Residue Theorem)

$$= \frac{\pi}{2a^3} \quad \text{———} \quad (4)$$

using (2), (3), (4) in (1) and taking $R \to \infty$

$$\frac{\pi}{2a^3} = 2\int_0^\infty \frac{dx}{(x^2+a^2)^2} + 0$$

ie $\boxed{\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}}$, $a>0$.

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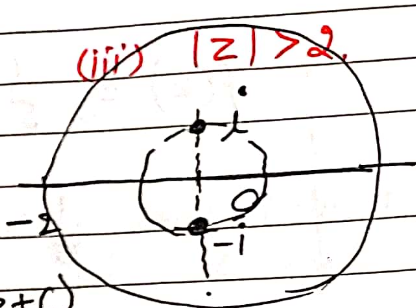
2018/4b/15m/CSE

4(b) Find the Laurent's series which represents the function $\frac{1}{(1+z^2)(z+2)}$ when

i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$



$$f(z)=\frac{1}{(z^2+1)(z+2)}=\frac{A}{z+2}+\frac{Bz+C}{z^2+1}$$

$$1 = A(z^2+1)+(z+2)(Bz+C)$$

Let, $z=-2 \Rightarrow 1=5A \Rightarrow A=1/5$

Coeff of $z^2 \Rightarrow 0=A+B \Rightarrow B=-1/5$

Constant term $\Rightarrow 1=A+2C \Rightarrow C=\frac{1}{2}\left(1-\frac{1}{5}\right)=\frac{2}{5}$

$$f(z)=\frac{1}{(z^2+1)(z+2)}=\frac{1}{5(z+2)}+\frac{-z+2}{5(z^2+1)}$$

for $|z|<1$

$$f(z)=\frac{1}{5\cdot 2\left(1+\frac{z}{2}\right)}-\frac{(z-2)}{5(1+z^2)}$$

$$=\frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}-\frac{1}{5}(z-2)(1+z^2)^{-1}$$

$$=\frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\dots\right)-\frac{(z-2)}{5}\left(1-z^2+z^4-\dots\right)$$

$$=\frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n-\frac{(z-2)}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

$\left(\text{as } |z|<1,\ \therefore \left|\frac{z}{2}\right|<1 \ \&\ |z^2|<1\right)$

It represents Taylor series, in the region $|z|<1$.

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(ii) $1 < |z| < 2$

$1 < |z|$, $\therefore \frac{1}{|z|} < 1$ | $|z| < 2$ $\therefore \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{5} \cdot \frac{1}{(z+2)} - \frac{(z-2)}{5} \cdot \frac{1}{(z^2+1)}$$

$$= \frac{1}{5} \cdot \frac{1}{2\left(1+\frac{z}{2}\right)} - \frac{(z-2)}{5} \times \frac{1}{z^2\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1} - \frac{(z-2)}{5z^2} \cdot \left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \dots\right) - \frac{(z-2)}{5z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots\right)$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{(z-2)}{5z^2} \cdot \sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n}}$$

(iii) $|z| > 2$, $\therefore \frac{2}{|z|} < 1$ & $\frac{1}{|z|} < 1$

$$f(z) = \frac{1}{5 \times z\left(1+\frac{2}{z}\right)} - \frac{z-2}{5} \cdot \frac{1}{z^2\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1} - \frac{z-2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{5z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right) - \frac{(z-2)}{5z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots\right)$$

$$= \frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{(z-2)}{5z^2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^{2n}}$$

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