11 Years
Solved Papers
2009-2019

MADE EASY
Publications

Civil Services
Main Examination

TOPICWISE PREVIOUS YEARS' SOLVED PAPERS

Mathematics
Paper - I

2 Calculus

1. Function of a Real Variable

1.1 **Suppose that f'' is continuous on [1, 2] and that f has three zeros in the interval (1, 2). Show that f'' has at least one zero in the interval (1, 2).**

(2009 : 12 Marks)

Solution:

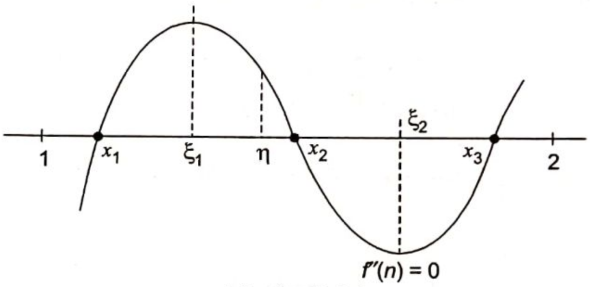
Insight : This question uses the fact that **continuity of any derivative of a function ensures continuity and differentiability of lower order derivatives** and the Rolle's theorem.

f'' is continuous on [1, 2]

$\Rightarrow f'$ is continuous and differentiable on [1, 2]

$\Rightarrow f$ is continuous and differentiable on [1, 2]

f has three zeros in (1, 2). Let them be x_1, x_2, x_3 with $x_1 < x_2 < x_3$.



Inflection Point

In the interval $[x_1, x_2]$, applying Rolle's theorem.

f is continuous on $[x_1, x_2]$.

f is differentiable on (x_1, x_2).

$$f(x_1) = f(x_2) = 0$$

$\Rightarrow \exists\ \xi_1 \in (x_1, x_2)$ such that $f'(\xi) = 0$ by Rolle's theorem.

Similarly, applying Rolle's theorem in interval $[x_2, x_3]$, $\exists\ \xi_2 \in (x_2, x_3)$ such that $f'(\xi_2) = 0$.

As $\xi_1 < x_2$ and $\xi_2 > x_2 \Rightarrow \xi_1 < \xi_2$.

Applying Rolle's theorem on f' in (ξ_1, ξ_2).

f' is continuous on $[\xi_1, \xi_2]$.

f' is differentiable on (ξ_1, ξ_2) as f'' is continuous on that interval.

$$f'(\xi_1) = f'(\xi_2) = 0$$

$\Rightarrow \exists\ \eta \in (\xi_1, \xi_2)$ so that $f''(\eta) = 0$ by Rolle's theorem.

Also, $(x_1, x_2) \subset (1, 2) \Rightarrow \eta \in (1, 2)$

1.2 **If f is the derivative of some function defined on $[a, b]$ prove that there exists a number $\eta \in [a, b]$ such that**

$$\int_a^b f(t)dt = f(\eta)(b-a)$$

(2009 : 12 Marks)

Solution:

Insight : This is the mean value theorem of integral calculus with the difference that f is not given as continuous but as derivative of some function. We use 2nd Fundamental Theorem of Calculus.

f is derivative on some function

$\Rightarrow$
$$f(x) = F(x) \text{ on } [a, b]$$

i.e., $f(x)$ has an anti-derivative $F(x)$ defined on $[a, b]$.

By 2nd Fundamental Theorem of Algebra, for any $x_1, x_2 \in [a, b]$

$$\int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Proof of 2nd Fundamental Theorem : $F(x)$ is an anti-derivative of $f(x)$.

Let
$$G(x) = \int_{x_1}^{x} f(x)dx \text{ where } x_1 \in (a, b)$$

Then $G(x)$ is an anti-derivative.

As **two anti-derivatives differ by a constant**

$$G(x) = F(x) + C$$

$\therefore$
$$G(x_1) = F(x_1) + C$$

and
$$\int_{x_1}^{x_1} f(x) = G(x_1) = 0$$

$\Rightarrow$
$$F(x_1) + C = 0 \Rightarrow C = -F(x_1)$$

$\therefore$
$$G(x) = F(x) - F(x_1)$$

$\therefore$
$$\int_{x_1}^{x} f(x)dx = F(x) - F(x_1)$$

$\Rightarrow$
$$\int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Now, $F'(x) = f(x)$ on $[a, b]$ so $F(x)$ is continuous on $[a, b]$ as **every differentiable function is continuous** and $F(x)$ is differentiable on (a, b).

So by Mean Value Theorem, $\exists \eta \in (a,b)$ such that

$$F'(\eta) = \frac{F(b) - F(a)}{b - a}$$

$\Rightarrow$
$$F(b) - F(a) = (b - a)f(\eta)$$

$\Rightarrow$
$$\int_a^b f(x)dx = (b - a)f(\eta)$$

1.3 A twice differentiable function $f(x)$ is such that $f(a) = 0 = f(b)$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one point x, $a < \xi < b$, for which $f''(\xi) < 0$.

(2010 : 12 Marks)

Solution:

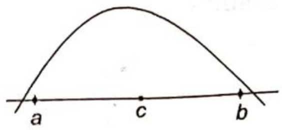
Given $f(a) = f(b) = 0$ and $c \in (a, b)$ such that $f(c) > 0$.

By Lagrange's Mean Value Theorem (LMVT), $\exists\ \alpha \in (a,c)$ and $\beta \in (c,b)$:

$$f'(\alpha) = \frac{f(c) - f(a)}{c - a} \text{ and } f'(\beta) = \frac{f(b) - f(c)}{b - c} \quad ...(1)$$

Now let $\xi \in (\alpha, \beta)$

$\therefore$ $\xi \in (a, b)$ or $a < \xi < b$



By Lagrange's Mean Value Theorem (LMVT)

$$f''(\xi) = \frac{f'(\beta) - f'(\alpha)}{\beta - \alpha}$$

Using vaues in eqn. (1), we get

$$f''(\xi) = \frac{\frac{f(b)-f(c)}{b-c} - \frac{f(c)-f(a)}{c-a}}{\beta - \alpha}$$

$$\Rightarrow \quad f''(\xi) = \frac{-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}}{\beta - \alpha} \qquad [f(a) = f(b) = 0]$$

$$\Rightarrow \quad f''(\xi) = -\left[\frac{\frac{f(c)}{b-c} + \frac{f(c)}{c-a}}{\beta - \alpha}\right] < 0 \text{ as } \beta > \alpha,\ b > c,\ c > a$$

$\therefore \exists\, \xi$ such that $f''(\xi) < 0$, where $a < \xi < b$.

1.4 Show that a box (rectangular parallelopiped) of maximum volume V with prescribed surface area is a cube.

(2010 : 20 Marks)

Solution:

Let x, y, z be length of edges of given rectangular parallelopiped. Its surface area be S, volume be V. (x, y, $z \neq 0$).

$$\therefore \quad S = 2xy + 2yz + 2zx,\ V = xyz$$

Let λ be the Lagrange's multiplier ($\lambda \neq 0$)

So, $$f = xyz + \lambda(2xy + 2yz + 2xz - S)$$

$\therefore$ At extremum,

$$f_x = 0 \Rightarrow yz + \lambda(2y + 2z) = 0 \qquad \text{...(1)}$$

$$f_y = 0 \Rightarrow xz + \lambda(2x + 2z) = 0 \qquad \text{...(2)}$$

$$f_z = 0 \Rightarrow xy + \lambda(2y + 2x) = 0 \qquad \text{...(3)}$$

Multiplying (1) by x and (2) by y and subtracting then, we get

$$xyz + \lambda(2xy + 2xz) - xyz - \lambda(2xy + 2yz) = 0$$

$$\Rightarrow \quad \lambda(2xz - 2yz) = 0$$

$$\Rightarrow \quad 2xz - 2yz = 0 \qquad (\lambda \neq 0)$$

$$\Rightarrow \quad x = y \text{ as } z \neq 0 \qquad \text{...(4)}$$

Similarly, multiplying (2) by y and (3) by z and subtracting, we get

$$xyz + \lambda(2xy + 2yz) - xyz - \lambda(2yz + 2xz) = 0$$

$$\Rightarrow \quad \lambda(2xy - 2xz) = 0$$

$$\Rightarrow \quad y = z\ (x \neq 0) \qquad \text{...(5)}$$

$\therefore$ from (4) and (5), it can be concluded $x = y = z$ which is a cube.

1d

1.5 Let f be a function defined on $\mathbb{R}$ such that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x in $\mathbb{R}$. How long can $f(2)$ possibly be?

(2011 : 10 Marks)

Solution :

$$f'(x) \le 5 \ \forall \ x \in R$$

Let $\quad f'(x) = 5 + g(x)$ where $g(x) \le 0 \ \forall \ x \in R$

$\Rightarrow \quad f(x) = 5x + G(x) + C_1,\ h(x) = \int g(x)dx$ is a decreasing function

$\Rightarrow \quad f(0) = G(0) + C_1 = -3$

$\Rightarrow \quad C_1 = -3 - G(0)$

$\Rightarrow \quad f(x) = 5x + G(x) - 3 - G(0)$

$\Rightarrow \quad f(2) = 10 + G(2) - 3 - G(0)$

$\quad = 7 + (G(2) - G(0))$

$\quad \le 7 \qquad (\because G$ is a decreasing function, $G(2) - G(0) < 0)$

$\therefore \quad f(2) \le 7$

1.6 Evaluate :

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$$\lim_{x\to 2} f(x), \text{ where } f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \ne 2 \\ \pi, & x = 2 \end{cases}$$

(2011 : 8 Marks)

Solution:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2) = 2 + 2 = 4$$

$$\therefore \quad \lim_{x\to 2} f(x) = 4$$

1.7 Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that for real numbers, $a, b \ge 0$

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

(2012 : 12 Marks)

Solution:

If $b^q = 0$, the inequality reduces to $0 \le \frac{a^p}{p}$ which is true for $a \ge 0$.

If $b^q > 0$, put $t = a^p b^{-q}$, the inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ or $\frac{ab}{b^q} \le \frac{a^p}{b^q} \cdot \frac{1}{p} + \frac{1}{q}$

or $\quad \dfrac{a}{b^{q-1}} \le \dfrac{a^p}{b^q} \cdot \dfrac{1}{p} + \dfrac{1}{q}$ reduces to

$$t^{1/p} \le \frac{t}{p} + \frac{1}{q} \qquad \left[\because t^{1/p} = \frac{a}{b^{q/p}} = \frac{a}{b^{q-1}} \text{ as } \frac{1}{p} + \frac{1}{q} = 1\right]$$

Let $\quad f(t) = t^{1/p} - \dfrac{t}{p} - \dfrac{1}{q}$

By the Mean Value Theorem, for fixed $t \geq 0$, there is θ between 1 and t such that

$$f(t) = f(t) - f(1) \qquad (\because f(1) = 0)$$
$$= f'(\theta)\,(t-1)$$
$$= \left(\frac{1}{p}\theta^{\frac{1}{p}-1} - \frac{1}{p}\right)(t-1) = \frac{1}{p}\left(\theta^{\frac{1}{p}-1} - 1\right)(t-1)$$
$$= \frac{1}{p}\left(\theta^{-\frac{1}{2}} - 1\right)(t-1) = \frac{1}{p}\theta^{-\frac{1}{q}}\left(1-\theta^{\frac{1}{2}}\right)(t-1)$$

Thus for $t \geq 0$, $\frac{1}{p}\theta^{-\frac{1}{q}}(1-\theta^{1/q})(t-1) \leq 0$

$\therefore$ for $t \geq 0$, $f(t) \leq 0$. Hence the result.

1.8 Define a sequence s_n of real numbers by

$$s_n = \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{n+i}$$

Does $\lim_{n\to\infty} s_n$ exist? If so, compute the value of this limit and justify your answer.

(2012 : 20 Marks)

Solution:

Given,
$$S_n = \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{n+i}$$
$$= \sum_{i=1}^{n} \frac{\left[\log\left(1+\frac{i}{n}\right)\right]^2}{n+i}$$
$$= \frac{\left[\log\left(1+\frac{1}{n}\right)\right]^2}{n+1} + \frac{\left[\log\left(1+\frac{1}{n}\right)\right]^2}{n+2} + \ldots\ldots + \frac{\left[\log\left(1+\frac{x}{n}\right)\right]^2}{x+n}$$

From Ist term,
$$a_1 = \frac{\left[\log\left(\frac{n+1}{n}\right)\right]^2}{n+1}$$
$$= \frac{[\log(n+1) - \log n]^2}{n+1}$$
$$= \frac{\log^2(n+1) + \log^2 n - 2\log n \log(n+1)}{n+1}$$

$$\therefore \quad \lim_{n\to\infty} a_1 = \lim_{n\to\infty}\left[\frac{\log^2(n+1)}{n+1} + \frac{\log^2 n}{n+1} - \frac{2\log n\log(n+1)}{n+1}\right]$$

$$\lim_{n\to\infty} a_1 = \lim_{n\to\infty}\left[\frac{2\log^2(n+1)}{n+1} + \frac{2\log n}{n} - \frac{2\log n}{n+1} - \frac{2\log(n+1)}{n}\right] \text{(L' Hospital's Rule)}$$

$$= \lim_{n\to\infty} \frac{2}{n+1} - \frac{2}{n} - \frac{2}{n} - \frac{2}{n+1} \quad \text{(L' Hospital's Rule)}$$

$$= \lim_{n\to\infty} -\frac{4}{n} = 0$$

Similarly, $\lim_{n\to\infty} a_2 = \lim_{n\to\infty} a_3 = \ldots.. = \lim_{n\to\infty} a_n = 0$

$\therefore$ $\lim_{n\to\infty} S_n = 0$

1.9 Find all the real values of p and q so that the integral $\int_0^1 x^p \left(\log\frac{1}{x}\right)^q dx$ converges.

(2012 : 20 Marks)

Solution:

The given integral is $\int_0^1 x^p \left(\log\frac{1}{x}\right)^q dx$

Put $\log\frac{1}{x} = t$ or $x = e^{-t}$,

$$\int_0^1 x^p \left(\log\frac{1}{x}\right)^q dx = \int_0^\infty t^q e^{-(p+1)t} dt$$

The integrand of the last integral has a point of infinite discontinuity 0. So, let us write

$$\int_0^\infty t^q e^{-(p+1)t} dt = \int_0^1 t^q e^{-(p+1)t} dt + \int_1^\infty t^q e^{-(p+1)t} dt$$

Convergence at 0.

Let $f(t) = t^q \cdot e^{-(p+1)t} = \dfrac{e^{-(p+1)t}}{t^{-q}}$

On comparing $\int_0^1 t^q \cdot e^{-(p+1)t} dt$ with $\int_0^1 \frac{dt}{t^{-q}}$, we can conclude that $\int_0^1 t^q e^{-(p+1)t} dt$ converges at 0 only if $-q < 1$ or $q > -1$, for all values of p.

[$\because$ the improper integral $\int_a^b \frac{dx}{(x-a)^n}$ converges if and only if $n < 1$.]

Convergence at ∞ :

Let $f(t) = t^q e^{-(p+1)t} = \dfrac{t^q}{e^{(p+1)t}}$

Let $f(t) = \dfrac{1}{t^2}$, so that

$$\frac{f(t)}{g(t)} = \frac{t^{q+2}}{e^{(p+1)t}} \to 0 \text{ as } t \to \infty, \text{ for all values of } q \text{ and } p+1 > 0.$$

But $\int_1^\infty \frac{dt}{t^2}$ converges. $\left[\because \int_a^\infty \frac{c\,dx}{x^n} \text{ where } a > 0, c > 0 \text{ converges iff } n > 1\right]$

Hence, $\int_1^\infty t^2 e^{-(p+1)t} dt$ converges for all q and only if $p + 1 > 0$.

[$\because$ If f and g are positive in $[a, x]$ and $\frac{f}{g} \to 0$, $\int_a^\infty g\,dx$ converges then $\int_a^\infty f\,dx$ converges].

Hence, $\int_0^1 x^p \left(\log\frac{1}{x}\right) dx$ is convergent for $p > -1$, $q > -1$ and divergent for all other values of p and q.

1.10 Prove that between two real roots of $e^x \cos x + 1 = 0$ a real root of $e^x \sin x + 1 = 0$ lies.

(2014 : 10 Marks)

Solution:

1C

Let $x = a$ and $x = b$ be live distinct roots of the given equation

$$e^x \cos x + 1 = 0$$

$\therefore$ $$e^a \cos a = -1$$

and $$e^b \cos b = -1$$

$\Rightarrow$ $$\cos a = -e^{-a}$$

and $$\cos b = -e^{-b} \qquad \text{...(i)}$$

Let $$f(x) = -\cos x - e^{-x} \ \forall\ x \in [a, b]$$

(i) Since $\cos x$ and e^{-x} are continuous in $[a, b]$

$\therefore$ $f(x)$ is continuous in $[a, b]$.

(ii) $$f'(x) = \sin x + e^{-x}$$

which exists for all $x \in (a, b)$

(iii) $$f(a) = -\cos a - e^{-a} = 0 \qquad \text{(by (i))}$$

$$f(b) = -\cos b - e^{-b} = 0 \qquad \text{(by (ii))}$$

$\therefore$ $$f(a) = f(b) = 0$$

$\therefore$ The conditions of Rolle's theorem are satisfied.

$\therefore$ At least one point $\in (a, b)$ such that $f'(c) = 0$.

$\Rightarrow$ $$f'(c) = \sin c + e^{-c} = 0$$

$\Rightarrow$ $$\sin c = -e^{-c}$$

$\Rightarrow$ $$e^c \sin c = -1$$

$\Rightarrow$ $$e^c \sin c + 1 = 0$$

$\Rightarrow$ $x = c \in (a, b)$ is a root of the equation $e^x \sin x + 1 = 0$

$\therefore$ $e^x \sin x + 1 = 0$ has a real root between any two roots of the equation $e^x \cos x + 1 = 0$

1.11 Find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a.

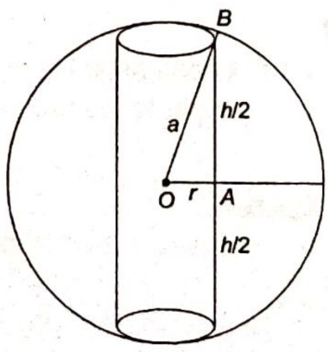
(2014 : 15 Marks)

Solution:

3a

Let O be the centre of sphere of radius 'a'.

Let 'h' be the height and 'r' be the radius of the cylinder.



Then $$OA = r = \sqrt{a^2 - \frac{h^2}{4}}$$

The volume V of the cylinder $$= \pi r^2 h$$

$$= \pi\left(a^2 - \frac{h^2}{4}\right)h$$

$\therefore$ $$\frac{dv}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right)$$

For v to be maximum or minimum, we must have

$$\frac{dv}{dh} = 0$$

i.e., $$\pi\left(a^2 - \frac{3}{4}h^2\right) = 0 \Rightarrow h = \frac{2a}{\sqrt{3}}$$

Also, $$\frac{d^2v}{dh^2} = \frac{-6h}{H} = \frac{-3}{2}h < 0 \text{ at } h = \frac{2a}{\sqrt{3}}$$

Hence, V is maximum when $h = \frac{2a}{\sqrt{3}}$

and maximum value $$= \pi\left(a^2 - \frac{h^2}{H}\right)h$$

$$= \pi\left[a^2 - \frac{1}{4}\left(\frac{2a}{\sqrt{3}}\right)\right]\frac{2a}{\sqrt{3}} = \frac{4\pi a^3}{\sqrt{3}}$$

1.12 Evaluate the following limit : $\lim_{x\to a}\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

(2015 : 10 Marks)

1c

Solution:

Let $$L = \lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi}{2a}\cdot x\right)}$$

$\Rightarrow$ $$\log L = \lim_{x\to a}\tan\left(\frac{\pi x}{2a}\right)\cdot\log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to a}\frac{\frac{-1}{a}}{\left(2-\frac{x}{a}\right)}\cdot\frac{1}{\left(-\text{cosec}^2\left(\frac{\pi x}{2a}\right)\cdot\frac{\pi}{2a}\right)} \qquad \text{(Using L-Hospital Rule)}$$

$$= \frac{2}{\pi} \qquad \left[\because \text{cosec}\left(\frac{\pi x}{2a}\right)\to 1 \text{ as } x\to a\right]$$

$\therefore$ $$L = e^{2/\pi}$$

2b

1.13 A conical tent is of given capacity. For the least amount of canvas required for it, find the ratio of its height to the radius of its base.

(2015 : 13 Marks)

Solution:

Volume is fixed here.

$$\frac{1}{3}\pi r^2 h = V$$

Surface Area (Lateral)

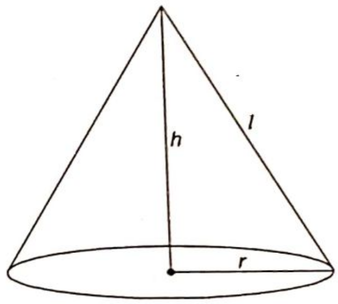
$$S = \pi r l = \pi r\sqrt{r^2 + h^2}$$

S is minimized whenever (S^2) is minimized (for non-negative values).
Hence, we take

$$S^2 = \pi^2 r^2(r^2 + h^2)$$

$$= \pi^2 \cdot \frac{3V}{\pi h}\left[\frac{3V}{\pi h} + h^2\right] \qquad \left[\text{from (i), } r^2 = \frac{3V}{\pi h}\right]$$

$$= 3\pi V\left[\left(\frac{3V}{\pi}\right)\cdot\frac{1}{h^2} + h\right]$$



Differentiating w.r.t. h

$$\frac{d(S^2)}{dh} = 3\pi V\left[\frac{-6V}{\pi}\cdot\frac{1}{h^3} + 1\right]$$

$$\frac{d^2(S^2)}{dh^2} = 3\pi V\left[\frac{18V}{\pi}\cdot\frac{1}{h^4}\right] > 0$$

For critical points, $$\frac{d(S^2)}{dh} = 0$$

$$3\pi V\left[\frac{-6V}{\pi h^3} + 1\right] = 0$$

$$\Rightarrow \qquad 6V = \pi h^3$$

$$6\cdot\frac{1}{3}\pi r^2 h = \pi h^3$$

$$\Rightarrow \qquad \frac{h}{r} = \sqrt{2}$$

and $$\frac{d^2(S^2)}{dh^2} > 0. \text{ Hence Minima.}$$

1.14 For the cardioid, $r = a(1 + \cos\theta)$, show that the square of the radius of curvature at any point (r, θ) is proportional to r. Also find the radius of curvature if $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

(2016 : 15 Marks)

Solution:

Cardioid: $$r = a(1 + \cos\theta) \qquad \text{...(i)}$$

Differentiating w.r.t. θ,

$$\frac{dr}{d\theta} = -a\sin\theta = r_1$$

$$\frac{d^2r}{d\theta^2} = -a\cos\theta = r_2$$

Radius of curvature (in Polar Form):

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{(r^2 + 2r_1^2 - rr_2)} \text{ (Standard Result)}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - a^2(1+\cos\theta)(-a\cos\theta)}$$

$$= \frac{a^3[1+2\cos\theta+\cos^2\theta+\sin^2\theta]^{3/2}}{a^2[1+\cos^2\theta+2\cos\theta+2\sin^2\theta+\cos\theta+\cos^2\theta]}$$

$$= \frac{a(2(1+\cos\theta))^{3/2}}{3(1+\cos\theta)} = \frac{2\sqrt{2}}{3}a\sqrt{1+\cos\theta} \quad \text{...(ii)}$$

$$\rho^2 = \frac{8}{9}a(1+\cos\theta)$$

$$= \frac{8}{9}r \quad \text{(from (i))}$$

$$\therefore \quad \rho^2 \propto r$$

$$\rho_{\theta=0} = \frac{2\sqrt{2}}{3}a\sqrt{1+\cos 0^\circ} = \frac{4}{3}a$$

$$\rho_{\theta=\frac{\pi}{4}} = \frac{2}{3}a\left(\sqrt{2+\sqrt{2}}\right)$$

$$\rho_{\theta=\frac{\pi}{2}} = \frac{2\sqrt{2}}{3}a \quad \text{...(from (ii))}$$

1.15 Determine if $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists or not. If the limit exists, then find its value.

(2018 : 10 Marks)

Solution:

If $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists, than the given function should have finite value as $z \to 1$.

Now,

$$\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2} = \lim_{z\to 1}\frac{(1-z)}{\cos\frac{nz}{2}}\times\sin\left(\cancel{\frac{\pi z}{2}}\right)^{1}$$

$$= \lim_{z\to 1}\frac{(1-z)}{\cos\left(\frac{nz}{2}\right)} \quad \left(\frac{0}{0}\text{ form}\right)$$

Applying L-pital rule, we get

$$\lim_{z\to 1}\frac{(-1)}{-\frac{\pi}{2}\sin\left(\cancel{\frac{\pi z}{2}}\right)^{1}} = \frac{2}{\pi}$$

$\therefore$ limit exists end is equal to $\frac{2}{\pi}$.

1.16 Find the limit $\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\sqrt{n^2-r^2}$.

(2018 : 10 Marks)

Solution:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\Sigma\sqrt{n^2-r^2} = \lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{r=0}^{n-1}\Sigma\sqrt{1-\left(\frac{r}{n}\right)^2}$$

Consider the function $\sqrt{1-x^2}$ in the interval [0, 1]. Divide it into partitions of length $\frac{1}{n}$.

Therefore, given sum can be written in integral as limit of sum.

i.e.,
$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\sqrt{1-\left(\frac{r}{n}\right)^2} = \lim_{\delta_r\to 0}\Sigma f(\xi^2)\delta_r$$

where
$$f(\xi) = \sqrt{1-\xi^2} \text{ and } \delta_r = \frac{1}{n}$$

∴ given integral becomes
$$\int_0^1\sqrt{1-x^2}\,dx = I$$

Let $x = \sin\theta$,
$$dx = \cos\theta\, d\theta$$

∴
$$I = \int_0^{\pi/2}\sqrt{1-\sin 2\theta}\cos\theta\, d\theta$$
$$= \int_0^{\pi/2}\cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)d\theta$$
$$= \frac{1}{2}\int_0^{\pi/2}d\theta + \frac{1}{2}\int_0^{\pi/2}\cos 2\theta\, d\theta$$
$$= \frac{\pi}{4}+0$$
$$= \frac{\pi}{4}$$

∴
$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{r=0}^{n-1}\sqrt{n^2-r^2} = \frac{\pi}{4}$$

1.17 Let $f: \left[0, \frac{\pi}{2}\right] \to R$ be a continuous function such that

$$f(x) = \frac{\cos^2 x}{4x^2-\pi^2},\quad 0 \le x < \frac{\pi}{2}$$

1a

Find the value of $f\left(\frac{\pi}{2}\right)$.

(2019 : 10 Marks)

Solution:

Given that $f: \left[0, \frac{\pi}{2}\right] \to R$ is a continuous function such that

$$f(x) = \frac{\cos^2 x}{4x^2-\pi^2}$$
$$0 \le x < \frac{\pi}{2}$$

Now, since f is continuous on $\left[0, \frac{\pi}{2}\right]$

We have

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right)$$

$$= f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos^2 x}{4x^2-\pi^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{2\cos x(-\sin x)}{8x}$$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-\sin 2x}{8x}$$

$$= \frac{-\sin 2\left(\frac{\pi}{2}\right)}{8\left(\frac{\pi}{2}\right)} = 0$$

$\therefore$ The value of $f\left(\frac{\pi}{2}\right) = 0$.

1.18 Is $f(x) = |\cos x| + |\sin x|$ differentiable at $x = \frac{\pi}{2}$? If yes, find its derivative at $x = \frac{\pi}{2}$. If no, then give a proof of it.

(2019 : 15 Marks)

Solution:

Given function is

$$f(x) = |\cos x| + |\sin x|$$

Clearly $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$\because$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right) = 1$$

Now, we inspect whether the given function is differentiable at $x = \frac{\pi}{2}$.

Consider,

$$\lim_{x\to\frac{\pi}{2}} \frac{f(x)-f\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}} = \lim_{n\to 0}\left[\frac{f\left(\frac{\pi}{2}-n\right)-1}{-h}\right]$$

$$= \lim\left[\frac{|\sin h|+|\cos h|-1}{-h}\right] \quad (\because \text{ Both mod values are positive})$$

So,

$$= \lim_{n\to 0}\frac{\sin h+\cos h-1}{-h} = \lim_{n\to 0}\frac{\cos h-\sin h}{-1}$$

$$= -1$$

Now, consider

$$\lim_{x\to\frac{\pi}{2}^+}\left[\frac{f(x)-f\left(\frac{\pi}{2}\right)}{x-\frac{\pi}{2}}\right] = \lim_{n\to 0}\left[\frac{f\left(\frac{\pi}{2}+h\right)-1}{h}\right]$$

$$= \lim_{n\to 0}\left[\frac{|\cos h|+|\sin h|-1}{h}\right] = \lim_{n\to 0}\frac{-\sin h+\cos h}{1}$$

$$= 1$$

$\because$ Left hand derivative $\neq$ Right hand derivative

The function $f(x) = |\cos x| + |\sin x|$ is not differentiable at $x = \frac{\pi}{2}$. Hence, the result.

3a

1.19 Find the maximum and the minimum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$ on the interval [2, 3].

(2019 : 13 Marks)

Solution:

Given :
$$f(x) = 2x^3 - 9x^2 + 12x + 6$$
$$f'(x) = 6x^2 - 18x + 12$$

Let
$$f'(x) = 0$$
$$6x^2 - 18x + 12 = 0$$
$$\Rightarrow \quad x^2 - 3x + 2 = 0$$
$$\Rightarrow \quad (x-1)(x-2) = 0$$
$$\Rightarrow \quad x = 1, 2$$

These are the critical points of the $f(x)$ which $\notin [2,3]$. Thus, $f(x)$ is monotonic on [2, 3].

Now to check whether it is monotonic increasing or decreasing :
$$f'(2) = 6(4) - 18(2) + 12 = 24 - 36 + 12 = 0$$
$$f'(3) = 6(9) - 18(2) + 12 = 54 + 12 - 54 = 12 > 0$$

Since, $f'(2) = 0$, $f'(3) > 0$ and $f'(3) > f'(2)$.

Hence, $f(x)$ is monotonically increasing on [2, 3].

Thus,
$$f_{\min} = f(x)\big|_{x=2} = 2\times(2)^3 - 9\times 4 + 12\times 2 + 6$$
$$= 16 - 36 + 24 + 6$$
$$= 10$$
$$f_{\max} = f(x)\big|_{x=3} = 2\times(3)^3 - 9\times 9 + 12\times 3 + 6$$
$$= 54 - 81 + 36 + 6$$
$$= 96 - 81$$
$$= 15$$

$\therefore \quad f_{\min} = 10$ and $f_{\max} = 15$. Hence, the result.

+
~~5.2~~° 2018
2(b)

2. Functions of Two and Three Variables

2.1 If $x = 3 \pm 0.01$ and $y = 4 \pm 0.01$ with approximately what accuracy can you calculate the polar co-ordinates r and θ of the point $P(x, y)$? Express your estimates as percentage changes of the value that r and θ have at point (3, 4).

(2009 : 20 Marks)

Solution:

Approach : r and θ are functions of two variables x and y. We write the total derivative of each to calculate error from the error in the variables.

We have $x = 3$, $dx = 0.01$, $y = 4$, $dy = 0.01$

By definition
$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

By total derivative

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

$$= \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$$

$$= \frac{3\times 0.01}{5} + \frac{4}{5}\times 0.01 = \frac{7}{5}\times 0.01$$

Percentage error in r

$$= \frac{\Delta r}{r}\times 100 = \frac{\frac{7}{5}\times 0.01}{\sqrt{3^2+4^2}}\times 100$$

$$= \frac{7}{25} = 0.28\%$$

Similarly,

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{3}$$

$$d\theta = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy$$

$$= \frac{1}{1+\frac{y^2}{x^2}} - \frac{y}{x^2}dx + \frac{1}{1+\frac{y^2}{x^2}} - \frac{1}{x}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$\therefore \quad d\theta = \frac{-4\times 0.01}{25} + \frac{3}{25}\times 0.01 = \frac{-1}{25}\times 0.01$$

$$\therefore \quad \text{Percentage error} = \frac{\frac{1}{2.5}}{\tan^{-1}\frac{4}{3}}\times 0.01\times 100 = 0.043\%$$

2.2 Let $f: R^2 \to R$ be defined as

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$$

Is f continuous at (0, 0)? Compute partial derivatives of f at any point (x, y) if exist.

(2009 : 20 Marks)

Solution:

Approach : Using definition of continuity and partial derivative.

Function being rational is continuous at all (x, y) except possibly at (0, 0).

Continuity at (0, 0) :

To check whether $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0.$

Converting to polar coordinates with $x = r\cos\theta$, $y = r\sin\theta$, $\lim_{(x,y)\to(0,0)} \sim \lim_{r\to 0}$

$$\lim_{r\to 0}\frac{r^2\sin\theta\cos\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} = \lim_{r\to 0}\frac{r^2\sin\theta\cos\theta}{r} \quad (r \text{ is always +ve}]$$

$$= \lim_{r\to 0} r\sin\theta\cos\theta = 0 \text{ for all values of } \theta.$$

$\therefore f(x, y)$ is cont. at (0, 0) as well.

Partial derivative at $(x, y) \neq (0, 0)$.

$$\frac{\partial f}{\partial x} = \frac{y\sqrt{x^2+y^2} - xy\cdot\frac{2x}{2}\sqrt{x^2+y^2}}{(x^2+y^2)} = \frac{y(x^2+y^2)-yx^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{y^3}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{x\sqrt{x^2+y^2} - xy\cdot\frac{2y}{2\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

(Simpy $x \leftrightarrow y$ as it is symmetrical function in x and y]

Partial derivative at (0, 0)

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

2.3 **A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth's atmosphere and its surface begins to heat. After one hour the temperature at the point (x, y, z) on the probe surface is given by**

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600$$

Find hottest point on the probe surface.

(2009 : 20 Marks)

Solution:

Approach : Use Lagrange's multipliers.

$$\text{Max. } T(x, y, z) = 8x^2 + 4yz - 16z + 600 \quad \text{...(i)}$$

subject to

$$4x^2 + y^2 + 4z^2 = 16 \quad \text{...(ii)}$$

Let

$$f = (8x^2 + 4yz - 16z + 600) + \lambda(4x^2 + y^2 + 4z^2 - 16)$$

Taking total derivative of f

$$df = (16 + 8\lambda)x\,dx + (4z + 2\lambda y)dy + (8\lambda z - 16)dz + 4y$$

For maximum/minimum

$$df = 0 \Rightarrow (16 + 8\lambda)x = 0;\ 4z + 2\lambda y = 0;\ 8\lambda z + 4y - 16 = 0 \quad \text{...(iii)}$$

Let $x \neq 0 \Rightarrow$

$$\lambda = -\frac{1}{2}$$

$$\left.\begin{aligned} 4z - y &= 0 \\ -4z + 4y &= 16 \end{aligned}\right\} y = \frac{16}{3};\ z = \frac{4}{3}$$

Putting this in (iii)

$$x^2 = 4 - \frac{1}{4}\left(\frac{256}{9} + \frac{64}{9}\right) = \frac{-44}{9} < 0$$

So this value is not possible.

$$\therefore \qquad x = 0$$

$$\begin{rcases} 4z + 2\lambda y = 0 \\ 8\lambda z + 4y = 16 \end{rcases} \qquad (4\lambda^2 - 4)y = -16$$

$$\Rightarrow \qquad y = \frac{4}{1-\lambda^2}$$

and
$$z = \frac{-\lambda y}{2} = \frac{2\lambda}{\lambda^2 - 1}$$

Substituting in (ii)

$$\frac{16}{(1-\lambda^2)^2} + \frac{16\lambda^2}{(1-\lambda^2)^2} = 16$$

$$\Rightarrow \qquad 1 + \lambda^2 = (1-\lambda^2)^2$$

$$\Rightarrow \qquad 1 + \lambda^2 = \lambda^4 - 2\lambda^2 + 1$$

$$\Rightarrow \qquad \lambda^2(\lambda^2 - 3) = 0$$

$$\Rightarrow \qquad \lambda = 0; \lambda^2 = 3$$

$$\lambda = 0 \Rightarrow x = 0, y = 4, z = 0$$

Let us consider x and y as independent and z as dependent variable.

(**Note :** We can do this because we have been given a relation between x, y and z)

From (ii) partially differentiating w.r.t. x.

$$8x + 8z\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$$

$$\therefore \qquad \frac{\partial T}{\partial x} = 16x + 4y\frac{\partial z}{\partial x} - 16$$

$$\frac{\partial z}{\partial x} = 16x - \frac{4xy}{z} + \frac{16x}{z}$$

$$\frac{\partial^2 T}{\partial x^2} = 16 + (16 - 4y)\left(\frac{1}{z} - \frac{x}{z^2}\cdot\frac{\partial z}{\partial x}\right)$$

$$= 16 + (16 - 4y)\left(\frac{1}{z} + \frac{x^2}{z^3}\right)$$

At $y = 4$, $\frac{\partial^2 T}{\partial x^2} > 0$, so this cannot be maxima.

$$\lambda^2 = 3, \lambda = \pm\sqrt{3}$$

$$\lambda = \sqrt{3}, z = \sqrt{3}, y = -2, x = 0$$

Again $\frac{\partial^2 T}{\partial x^2} > 0$ so this can not be a maxima.

$$\lambda = -\sqrt{3}, z = -\sqrt{3}, y = -2, x = 0$$

$$\frac{\partial^2 T}{\partial x^2} = 16 - \frac{24}{\sqrt{3}} < 0$$

So, this is a maxima.

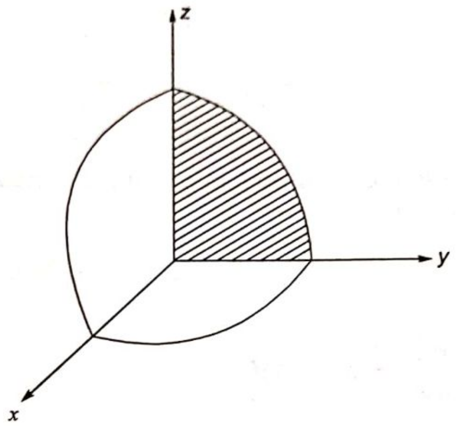
$\therefore$ Hottest point is $(0, -2, -\sqrt{3})$.

2.4 Evaluate : $I = \iint_S x\,dy\,dz + dz\,dx + xz^2\,dx\,dy$ where S is the outer surface of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first quadrant.

(2009 : 20 Marks)

Solution:

Approach : Surface integral with two changing variables can be calculated by taking the projection of the surface in the required plane. Use of polar coordinates can simplify integration.



$$I = I_1 + I_2 + I_3$$

$$I_1 = \iint_S x\,dy\,dz$$

$$I_2 = \iint_S dz\,dx$$

$$I_3 = \iint_S xz^2\,dx\,dy$$

For I_1 the area of integration is projection on YZ plane, i.e., $y^2 + z^2 \le 1$ in the first quadrant.

Also, $$x = (1 - y^2 - z^2)^{1/2}$$

Converting to polar coordinates

$$y = r\cos\theta,\ z = r\sin\theta$$

$$x = (1 - y^2 - z^2)^{1/2} = (1 - r^2)^{1/2}$$

$$I_1 = \iint_{dS} x\,dy\,dz = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (1-r^2)^{1/2} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (1-t)^{1/2} \frac{dt}{2} d\theta \qquad (r^2 = t)$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{-(1-t)^{3/2}}{3}\right]_{r=0}^{1} d\theta = \frac{1}{2}\int_{\theta=0}^{\pi/2} d\theta = \frac{\pi}{6}$$

For I_2 the area of integration is $x^2 + z^2 \le 1$ in first quadrant converting to polar coordinates.

$$I_2 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\,dr\,d\theta = \left[\frac{r^2}{2}\right]_0^1 [\theta]_0^{\pi/2} = \frac{\pi}{4}$$

For I_3 the area of integration is $x^2 + y^2 \le 1$ in the first quadrant.

Take $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$z^2 = 1 - x^2 - y^2 = (1 - r^2)$$

$$\therefore \quad I_3 = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r\cos\theta(1-r^2)rdrd\theta$$

$$= \int_{r=0}^{1} r^2 - r^4 dr \int_{\theta=0}^{\pi r} \cos\theta d\theta$$

Since r and θ are now explicitly separated so integration can now be done separately.

$$= \left[\frac{r^3}{3} - \frac{r^5}{5}\right]_0^1 \times 1 = \frac{2}{15}$$

$$\therefore \quad I = \frac{\pi}{6} + \frac{\pi}{4} + \frac{2}{15} = \frac{5\pi}{12} + \frac{2}{15}$$

2.5 If $f(x, y)$ is a homogenous function of degree n in x and y and has continuous first and second order partial derivatives then show that

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

(ii) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

(2010 : 20 Marks)

Solution:

(i) Given $f(x, y)$ is a homogenous function, of degree n.

$$\therefore \quad f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

Differentiating on both sides, we get

$$\frac{\partial f}{\partial(x\lambda)} \times \frac{\partial(\lambda x)}{\partial\lambda} + \frac{\partial f}{\partial(y\lambda)} \times \frac{\partial(\lambda y)}{\partial\lambda} = n\lambda^{n-1}f(x, y)$$

$$\Rightarrow \quad x\frac{\partial f}{\partial(\lambda x)} + y\frac{\partial f}{\partial(\lambda y)} = n\lambda^{n-1}f(x, y)$$

Putting $\lambda = 1$, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

(It is called as Euler's Theorem)

(ii)
$$x\frac{\partial f}{\partial(\lambda x)} + y\frac{\partial f}{\partial(\lambda y)} = n\lambda^{n-1}f(x, y)$$

$$\Rightarrow x\frac{\partial^2 f}{\partial(\lambda x)^2} \times \frac{\partial\lambda x}{\partial\lambda} + y\frac{\partial^2 f}{\partial(\lambda y)^2} \times \frac{\partial(\lambda y)}{\partial\lambda} + x\frac{\partial^2 f}{\partial(\lambda x)\partial(\lambda y)} \times \frac{\partial\lambda y}{\partial\lambda} + y\frac{\partial^2 f}{\partial(\lambda y)\partial(\lambda x)} \times \frac{\partial(\lambda x)}{\partial\lambda} = n(n-1)\lambda^{n-2}f(x, y)$$

$$\Rightarrow x^2\frac{\partial^2 f}{\partial(\lambda x)^2} + y^2\frac{\partial^2 f}{\partial(\lambda y)^2} + 2xy\frac{\partial^2 f}{\partial(\lambda x)\partial(\lambda y)} = n(n-1)\lambda^{n-2}f(x, y)$$

Putting $\lambda = 1$, we get

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} = n(n-1)f(x, y)$$

or
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

2.6 Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^3+y^3}$ if it exists.

(2011 : 10 Marks)

Solution:

Let $(x, y) \to (0, 0)$ along the line $y = mx$.

$$\therefore \quad \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3}$$

$$= \lim_{x\to 0} \frac{mx^3}{x^3+m^3x^3} = \frac{m}{1+m^3}$$

which is not unique as it assumes different values for different values of m.

$\therefore \quad \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2.7 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

(2011 : 20 Marks)

Solution:

Distance of the point $(3, 1, -1)$ from $x^2 + y^2 + z^2 = 4$ is given by

$$D = \sqrt{(x-3)^2+(y-1)^2+(z+1)^2}$$

Let $\quad f(x, y, z) = D^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$

and $\quad g(x, y, z) = x^2 + y^2 + z^2 = 4$

Then using $\quad \nabla f = \lambda \nabla g$

i.e., $\left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)\left[(x-3)^2+(y-1)^2+(z+1)^2\right] = \lambda\left(\hat{i}\dfrac{\partial}{\partial x}+\hat{j}\dfrac{\partial}{\partial y}+\hat{k}\dfrac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$$\Rightarrow \quad 2(x-3)\hat{i}+2(y-1)\hat{j}+2(z+1)\hat{k} = \lambda(2x\hat{i}+2y\hat{j}+2z\hat{k})$$

$$\Rightarrow \quad x = \frac{3}{1-\lambda},\ y = \frac{1}{1-\lambda},\ z = \frac{-1}{1-\lambda}$$

Putting these values in $x^2 + y^2 + z^2 = 4$, we get

$$\frac{9}{(1-\lambda)^2}+\frac{1}{(1-\lambda)^2}+\frac{1}{(1-\lambda)^2} = 4$$

$$\Rightarrow \quad 1-\lambda = \pm\frac{\sqrt{\pi}}{2}$$

For $1-\lambda = \dfrac{\sqrt{\pi}}{2}$, $\quad x = \dfrac{6}{\sqrt{\pi}},\ y = \dfrac{2}{\sqrt{\pi}},\ z = \dfrac{-2}{\sqrt{\pi}}$

For $1-\lambda = -\dfrac{\sqrt{\pi}}{2}$, $\quad x = -\dfrac{6}{\sqrt{\pi}},\ y = -\dfrac{2}{\sqrt{\pi}},\ z = \dfrac{2}{\sqrt{\pi}}$

$\therefore$ The point on the sphere that is closest to $(3, 1, -1)$ is $\left(\dfrac{6}{\sqrt{11}}, \dfrac{2}{\sqrt{11}}, \dfrac{-2}{\sqrt{11}}\right)$ and the point on the sphere that is farthest from $(3, 1, -1)$ is $\left(\dfrac{-6}{\sqrt{11}}, \dfrac{-2}{\sqrt{11}}, \dfrac{2}{\sqrt{11}}\right)$.

2.8 Define a function f of two real variables in the xy-plane by

$$f(x, y) = \begin{cases} \dfrac{x^3 \cos \frac{1}{y} + y^3 \cos \frac{1}{x}}{x^2 + y^2} & \text{for } x, y \neq 0 \\ \end{cases}$$

Check the continuity and differentiability of f at (0, 0).

(2012 : 12 Marks)

Solution:

$$f(x, y) = \begin{cases} \dfrac{x^3 \cos \frac{1}{y} + y^3 \cos \frac{1}{x}}{x^2 + y^2}, & x, y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $\in > 0$ be arbitrary, then

$$|x - 0 < \frac{\in}{2}, |y - 0| < \frac{\in}{2}, (x, y) \neq (0, 0)$$

$$\Rightarrow \quad f(x, y) - 0 = \left| \frac{x^3 \cos \frac{1}{y} + y^3 \cos \frac{1}{x}}{x^2 + y^2} \right|$$

$$\leq |x| \left| \frac{x^2}{x^2 + y^2} \right| \left| \cos \frac{1}{y} \right| + |y| \left| \frac{y^2}{x^2 + y^2} \right| \left| \sin \frac{1}{x} \right|$$

$[\because |a + b| \leq |a| + |b| \text{ and } |ab| = |a||b|]$

$$\leq |x| + |y| \quad \left[\because \left| \frac{x^2}{x^2 + y^2} \right| \leq 1, \left| \frac{y^2}{x^2 + y^2} \right| \leq 1 \text{ and } \left| \cos \frac{1}{y} \right|, \left| \sin \frac{1}{x} \right| \leq 1 \right]$$

$$< \frac{\in}{2} + \frac{\in}{2} = E$$

$$\therefore \quad |f(x, y) - f(0, 0)| < E$$

$$\Rightarrow \quad \lim_{(x,y)} f(x, y) = 0 = f(0, 0)$$

$\Rightarrow f$ is continuous at (0, 0].

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \text{ does not exist.}$$

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0,k) - f(0,0)}{k} \text{ does not exist.}$$

$\therefore f$ is not differential at the origin.

2.9 Find the points of local extrema and saddle points of the function f of two variables defined by

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$$

(2012 : 20 Marks)

Solution:

Here

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$$

Then,

$$\frac{\partial f}{\partial x} = 3x^2 - 63 + 12y$$

$$\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x$$

Solving $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$, we get

$$3x^2 - 63 + 12y = 0, 3y^2 - 63 + 12x = 0$$

or

$$x^2 + 4y - 2 = 0$$

$$y^2 + 4x - 21 = 0$$

...(i)

...(ii)

Subtracting (ii) from (i), we get

$$(x^2 - y^2) + 4(y - x) = 0$$

$$\Rightarrow \quad (x - y)(x + y - 4) = 0$$

$\Rightarrow$ Either $x - y = 0$ or $x + y - 4 = 0$

Solving $x - y = 0$ and $x^2 + 4y - 21 = 0$, we get

$$x = 3, -7 \text{ and } y = 3, -7$$

Again, solving $x + y - 4 = 0$ and $x^2 + 4y - 21 = 0$, we get

$$y = 5, -1 \text{ and } x = -1, 5$$

$\therefore$ There are four critical points namely (3, 3), (–7, –7), (–1, 5), (5, –1).

Let

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 12$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

At (3, 3), $rt - s^2 = 36(3)(3) - 144 = 180 > 0$

and $r = 6(3) = 18 > 0$

$\therefore f(x, y)$ is minimum at (3, 3)

and

$$\text{minimum value} = f(3, 3)$$

$$= 27 + 27 - 63(3 + 3) + 12(3)(3)$$

$$= -216$$

At (–7, –7), $rt - s^2 = 1620 > 0$

and $r = -42 < 0$

$\therefore f(x, y)$ is maximum at (–7, –7)

and maximum value $= f(-7, -7) = 784$

At (–1, 5), $rt - s^2 = -324 < 0$

$\therefore$ at (–1, 5), $f(x, y)$ is neither maximum nor minimum.

At (5, –1), $rt - s^2 = -324 < 0$

$\therefore$ at (5, –1), $f(x, y)$ is neither maximum nor minimum.

2.10 **Using Lagrange's multiplier method, find the shortest distance between the line y = 10 – 2x and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.**

(2013 : 20 Marks)

Solution:

Approach : This problem can be solved geometrically. But to use Lagrange's multipliers we can parametrize or use geometric condition of either the ellipse or the line.

e.g., if we took $(2 \cos \theta, 3 \sin \theta)$ as general point on ellipse the question could be solved without lagrange's multipliers.

Square of distance of any point from line $y = 10 - 2x$ is $\dfrac{(2x+y-10)^2}{5}$.

So, we have to minimise $(2x + y - 10)^2$ ...(i)

subject to $\dfrac{x^2}{4}+\dfrac{y^2}{9} = 1$ ...(ii)

Let
$$u = (2x+y-10)^2$$
$$f = (2x+y-10)^2 - \lambda\left(\frac{x^2}{4}+\frac{y^2}{9}-10\right) \quad \text{...(iii)}$$
$$df = \left(4(2x+y-10)-\frac{\lambda x}{2}\right)dx + \left(2(2x+y-10)-\frac{2\lambda y}{9}\right)dy$$

For stationary point, we chose λ such that
$$4(2x+y-10)-\frac{\lambda x}{2} = 0$$
and
$$2(2x+y-10)-\frac{2\lambda x}{9} = 0$$

i.e.,
$$\left.\begin{aligned}\left(8-\frac{\lambda}{2}\right)x+4y=40\\ 4x+\left(2-\frac{2\lambda}{9}\right)y=20\end{aligned}\right\} \Rightarrow \left[16-\left(8-\frac{\lambda}{2}\right)\left(2-\frac{2\lambda}{9}\right)\right]y$$
$$= 160-20\left(8-\frac{\lambda}{2}\right)$$

$$\Rightarrow \quad \left[\frac{25\lambda}{9}-\frac{\lambda^2}{9}\right]y = 10\lambda \Rightarrow y=\frac{90}{25-\lambda}$$

$$\therefore \quad x = \frac{80}{25-\lambda}$$

Putting this in (ii)
$$\frac{1600}{(25-\lambda)^2}+\frac{900}{(25-\lambda)^2} = 1 \Rightarrow 25-\lambda = \pm 50$$

$$\therefore \quad x = \pm\frac{8}{5},\ y = \pm\frac{9}{5}$$

Now
$$\frac{\partial u}{\partial x} = 4(2x+y-10); \quad \frac{\partial^2 u}{\partial x^2} = 8$$
$$\frac{\partial u}{\partial y} = 2(2x+y-10); \quad \frac{\partial^2 u}{\partial y^2} = 2$$
$$\frac{\partial^2 u}{\partial x \partial y} = 4$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 16-16 = 0 \geq 0$$

And
$$\frac{\partial^2 u}{\partial x^2} > 0$$

So, given value is a minima.

$$\text{Shortest distance} = \frac{\left|2\cdot\frac{8}{5}+\frac{9}{5}-10\right|}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{5}} = \sqrt{5}$$

2.11 Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at (0, 0).

(2013 : 15 Marks)

Solution:

We first compute f_x and f_y at all point.

f_x at $(x, y) \neq (0, 0)$

$$f_x = \frac{\partial}{\partial x}\left(\frac{xy^3}{x+y^2}\right) = \frac{y^3(x+y^2)-xy^3}{(x+y^2)^2} = \frac{y^5}{(x+y^2)^2}$$

$$f_x(0, 0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(\frac{xy^3}{x+y^2}\right) = \frac{3y^2x(x+y^2)-2y\cdot xy^3}{(x+y^2)^2}$$

$$= \frac{xy^4+3x^2y^2}{(x+y^2)^2} \text{ when } (x, y) \neq (0, 0)$$

$$f_y(0, 0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

Now, we compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$

$$f_{xy}(0, 0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{k-0}{k} = 1$$

$\therefore$ $f_{xy}(0, 0) = 1$

$$f_{yx}(0, 0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$\therefore$ $f_{yx}(0, 0) = 0$

Now for continuity we compute f_{xy} and f_{yx} at other points.

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{y^5}{(x+y^2)^2}\right)$$

$$= \frac{5y^4(x+y^2)^2 - y^5\cdot 2(x+y^2)\cdot 2y}{(x+y^2)^4}$$

$$= \frac{5y^4(x+y^2)-4y^6}{(x+y^2)^3} = \frac{5xy^4+y^6}{(x+y^2)^3}$$

$$\lim_{(x,y)\to(0,0)} f_{xy} = \lim_{(x,y)\to(0,0)}\frac{5xy^4+y^6}{(x+y^2)^3}$$

Let $(x, y) \to (0, 0)$ along $x = my^2$

Then, $$\lim_{(x,y)\to(0,0)} \frac{5xy^4 + y^6}{(x+y^2)^3} = \lim_{y\to 0} \frac{(5m+1)y^6}{(m+1)^3 y^6}$$

$$= \frac{5m+1}{(m+1)^3} \text{ which depends on } m.$$

So, limit does not exist and so f_{xy} is not continuous at (0, 0) continuity of f_{yx} at (0, 0).

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{xy^4 + 3x^2y^2}{(x+y^2)^2}\right)$$

$$= \frac{(y^6 + 6xy^2)(x+y^2)^2 - 2(x+y^2)(xy^4 + 3x^2y^2)}{(x+y^2)^4}$$

$$= \frac{y^6 + 5xy^4}{(x+y^2)^3}$$

Again by taking x = my2 we see that $\lim_{(x,y)\to(0,0)} f_{yx}$ does not exist.

$\therefore f_{yx}$ is not continuous at (0, 0).

2.12 Find the maximum or minimum values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically.

(2014 : 20 Marks)

Solution:

Given that

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{...(i)}$$

subject to the conditions

$$ax^2 + by^2 + cz^2 = 1 \quad \text{...(ii)}$$

and $$lx + my + nz = 0 \quad \text{...(iii)}$$

Let us consider a function F of independent variables x, y, z

where $$F = x^2 + y^2 + z^2 + l(ax^2 + by^2 + cz^2 - 1) + l^2 (lx + my + nz) \quad \text{...(iv)}$$

$\therefore$ $$dF = (2x + 2ax\lambda_1 + l\lambda_2)\,dx + (2y + 2by\lambda_1 + m\lambda_2)dy$$
$$+ (2z + 2l_1cz + n\lambda_2)dz \qquad (\because dF = F_x dx + F_y dy + F_z dz) \quad \text{...(v)}$$

At stationary points, $$dF = 0$$

$\therefore$ $$F_x = 0 \Rightarrow 2x + 2ax\lambda_1 + l\lambda_2 = 0$$
$$F_y = 0 \Rightarrow 2y + 2by\lambda_1 + m\lambda_2 = 0$$
$$F_z = 0 \Rightarrow 2z + 2ca\lambda_1 + n\lambda_2 = 0$$

Multiplying (vi) by x, y, z respectively and adding we get

$$2(x^2 + y^2 + z^2) + 2(ax^2 + by^2 + cz^2)\lambda_1 + (lx + my + nz)\lambda_2 = 0$$

$\Rightarrow$ $$2u + 2(1)\lambda_1 + 0(\lambda_2) = 0$$ where $u = x^2 + y^2 + z^2$

$\Rightarrow$ $$\lambda_1 = -u$$

From (vi), we have

$$2x + 2ax(-u) + l\lambda_2 = 0 \Rightarrow x = \frac{l\lambda_2}{2(1-au)}$$

$$2y + 2by(-u) + m\lambda_2 = 0 \Rightarrow y = \frac{-m\lambda_2}{2(1-bu)}$$

$$2z + 2cz(-u) + n\lambda_2 = 0 \Rightarrow z = \frac{n\lambda_2}{2(1-cu)}$$

$$\text{(iii)} \equiv l\left(\frac{-l\lambda_2}{2(1-au)}\right)+m\left(\frac{-m\lambda_2}{2(1-bu)}\right)+n\left(\frac{-n\lambda_2}{2(1-cu)}\right)=0$$

$$\Rightarrow \qquad = \lambda_2\left[\frac{l^2}{1-au}+\frac{m^2}{1-bu}+\frac{n^2}{1-cu}\right]=0 \qquad \text{...(vii)}$$

If $\lambda_2 = 0$ then we get $x = y = z = 0$ but $(x, y, z) = (0, 0, 0)$ does not satisfy one of the condition of the constraint (i).

From (vii), we have

$$\frac{l^2}{1-au}+\frac{m^2}{1-bu}+\frac{n^2}{1-cu} = 0$$

which gives the maxima and minima of u

i.e., $\qquad u = x^2 + y^2 + z^2$

2.13 Which point of the sphere $x^2 + y^2 + z^2 = 1$ is at the maximum distance from the point (2, 1, 3)?

(2015 : 13 Marks)

Solution:

Calculus Approach : Let (x, y, z) be such point.

Then maximize, $\qquad u = (x-2)^2 + (y-1)^2 + (z-3)^2 \qquad$ ...(i)

such that, $\qquad x^2 + y^2 + z^2 = 1 \qquad$ ...(ii)

Let $\qquad u = x^2 + y^2 + z^2 - 1$

Consider, $\qquad F = u + \lambda u$

$$F = (x-2)^2 + (y-1)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

For critical points, $\qquad dF = 0$

$$2[(x-2) + \lambda x]dx + 2[(y-1) + \lambda y]dy + 2[(z-3) + \lambda z]dz = 0$$

$$(\lambda+1)x = 2 \Rightarrow x = \frac{2}{\lambda+1},\ y = \frac{1}{\lambda+1},\ z = \frac{3}{\lambda+1}$$

$$(\lambda+1)y = 1$$

$$(\lambda+1)z = 3$$

From (ii), $\qquad \dfrac{4+1+9}{(\lambda+1)^2} = 1 \Rightarrow \lambda + 1 \pm\sqrt{14}$

Taking $\lambda + 1 \pm\sqrt{14}$, $\qquad (x, y, z) = \left(\dfrac{2}{\sqrt{14}}, \dfrac{1}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right)$

$$\therefore \qquad u = \left(\frac{2}{\sqrt{14}}-2\right)^2+\left(\frac{1}{\sqrt{14}}-1\right)^2+\left(\frac{3}{\sqrt{14}}-3\right)^2 = \left(\sqrt{14}-1\right)^2$$

Taking $\lambda + 1 = -\sqrt{14}$, $\qquad (x, y, z) = \left(\dfrac{-2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$

$$\therefore \qquad u = \left(\frac{-2}{\sqrt{14}}-2\right)^2+\left(\frac{-1}{\sqrt{14}}-1\right)^2+\left(\frac{-3}{\sqrt{14}}-3\right)^2 = \left(\sqrt{14}+1\right)^2$$

Hence, the point $\left(\dfrac{-2}{\sqrt{14}}, \dfrac{-1}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$ of the sphere is at the maximum distance from point (2, 1, 3).

$$\text{Max. distance} = \sqrt{u} = \left(\sqrt{14}+1\right)$$

Geometrical Approach : The equation of straight line through centre (0, 0, 0) and point (2, 1, 3) is $\dfrac{x}{2} = \dfrac{y}{1} = \dfrac{2}{3}$.

This line will cut the sphere in two points (one maximum, one minimum).

2.14 For the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

Examine the continuity and differentiability.

(2015 : 12 Marks)

Solution:

$$f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

for continuity (0, 0)

check $f(x, y)$ along $x = x = M\sqrt{y}$ in the neighborhood of origin

$$f(x, y) = f(M\sqrt{y}, y) = \frac{M^2y - M\sqrt{y}\sqrt{y}}{M^2y + y}$$

$$= y\frac{(M^2 - M)}{(M^2 + 1)} = \frac{M^2 - M}{M^2 + 1}$$

As the values of $f(x, y)$ depends on the values of m in the neighborhood of origin, so there will not be any unique values there. So, $f(x, y)$ is not continuous at origin.

For differentiability at (0, 0): As $f(x, y)$ is not continuous at (0, 0)

$\Rightarrow$ It is not differentiable there.

For values other than (0, 0): As the function is continuous and one values everywhere, so it is differentiable also.

2.15 Find the maximum and minimum value of $x^2 + y^2 + z^2$ subject to the conditions $\dfrac{x^2}{4} + \dfrac{y^2}{5} + \dfrac{z^2}{25} = 1$ and $x + y - z = 0$.

(2016 : 20 Marks)

Solution:

Let $$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \mu(x + y - z)$$

For stationary points, $$dF = 0$$

$$dx = 2x + \frac{2x}{4}\lambda + \mu = 0 \quad \text{...(i)}$$

$$dy = 2y + \frac{2y}{5}\lambda + \mu = 0 \quad \text{...(ii)}$$

$$dz = 2z + \frac{2z}{25}\lambda - \mu = 0 \quad \text{...(iii)}$$

Multiplying equations (i), (ii), (iii) with x, y, z and adding, we get

$$2(x^2 + y^2 + z^2) + 2\lambda\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25}\right) + \mu(x + y - z) = 0$$

$\Rightarrow$ $$2(x^2 + y^2 + z^2) + 2\lambda = 0$$

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25} = 1 \quad \text{...(iv)}$$

$$x + y - z = 0 \quad \text{...(v)}$$

∴ $$x^2 + y^2 + z^2 = -\lambda \quad \text{...(vi)}$$

From eqns. (i), (ii) and (iii)

$$x = \frac{-\frac{\mu}{2}}{1+\frac{\lambda}{4}}, \; y = \frac{-\frac{\mu}{2}}{1+\frac{\lambda}{5}}, \; z = \frac{\frac{\mu}{2}}{1+\frac{\lambda}{25}}$$

Eqn. (v) ⇒ $$\frac{1}{1+\frac{\lambda}{4}}+\frac{1}{1+\frac{\lambda}{5}}+\frac{1}{1+\frac{\lambda}{25}} = 0$$

⇒ $$17\lambda^2 + 245\lambda + 750 = 0$$

⇒ $$\lambda = -10, -\frac{75}{17}$$

∴ $$\text{Max}(x^2 + y^2 + z^2) = 10$$

$$\text{Min}(x^2 + y^2 + z^2) = \frac{75}{17} \quad \text{(from (vi))}$$

2.16 Let $f(x) = \begin{cases} \dfrac{2x^4y-5x^2y^2+y^5}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$. Find a $\delta > 0$ such that $|f(x, y) - f(0, 0)| < 0.01$, whenever $\sqrt{x^2+y^2} < \delta$.

(2016 : 15 Marks)

Solution:

Here $$f(0, 0) = 0$$

∴ $$|f(x) - f(0, 0)| = \left|\frac{2x^4y-5x^3y^2+y^5}{(x^2+y^2)^2}\right|$$

(Some printing mistake in the question, take either $5x^3y^2$ or $5x^2y^3$ in numerator)

$$|f(x) - f(0, 0)| \leq 2\left|\frac{x^4y}{(x^2+y^2)^2}\right|+5\left|\frac{x^3y^2}{(x^2+y^2)^2}\right|+\left|\frac{y^5}{(x^2+y^2)^2}\right|$$

(using triangle inequality)

Now, as $x \leq \sqrt{x^2+y^2}$, $y \leq \sqrt{x^2+y^2}$

∴ $$x^p.y^q \leq \left(\sqrt{x^2+y^2}\right)^{p+q}$$

∴ $$|f(x) - f(0, 0)| \leq 2\left|\frac{(x^2+y^2)^{5/2}}{(x^2+y^2)^2}\right|+5\left|\frac{(x^2+y^2)^{5/2}}{(x^2+y^2)^2}\right|+\left|\frac{(x^2+y^2)^{5/2}}{(x^2+y^2)^2}\right|$$

$$\leq 8(x^2 + y^2)^{1/2}$$

$$< 0.01 \text{ if } \sqrt{x^2+y^2} < \frac{0.01}{8}$$

∴ Take $$\delta = \frac{0.01}{8} = 1.25 \times 10^{-3} \text{ so that } |f(x, y) - f(0, 0)| < 0.01$$

2.17 If $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$, calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$ at (0, 0).

(2017 : 15 Marks)

Solution:

By defintion,

$$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(0, 0) = \lim_{h \to 0} \frac{f(0,K) - f(0,0)}{K} = \lim_{K \to 0} \frac{0}{K} = 0$$

$$f_y(h, 0) = \lim_{K \to 0} \frac{f(h,K) - f(h,0)}{K} = \lim_{K \to 0} \frac{hK(h^2 - K^2)}{K(h^2 + K^2)} = h$$

$\therefore$ $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

Again, $$f_{yx}(0, 0) = \lim_{K \to 0} \frac{f_x(0,K) - f_x(0,0)}{K}$$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_x(0, K) = \lim_{h \to 0} \frac{f(h,K) + -f(0,K)}{h} = \lim_{h \to 0} \frac{hK(h^2 - K^2)}{h(h^2 + K^2)} = -K$$

$\therefore$ $$f_{yx}(0, 0) = \lim_{K \to 0} \frac{-K - 0}{K} = -1$$

$\therefore$ $$f_{xy}(0, 0) \neq f_{yx}(0, 0)$$

2.18 Find the shortest distance from the point (1, 0) to the parabola $y^2 = 4x$.

(2018 : 13 Marks)

Solution:

Let any point on parabola be (x, y) square of distance from given point (1, 0) is $(x - 1)^2 + y^2$. (x, y) also lies on $y^2 = 4x$.

Let $$f = (x - 1)^2 + y^2 + \lambda(y^2 - 4x), \text{ where } \lambda \text{ is Lagregian multiplier.}$$

At extremum, $$f_x = f_y = 0$$

i.e., $$f_x = 2(x - 1) - 4\lambda = 0 \quad \text{...(i)}$$

$$f_y = 2y + 2\lambda y = 0 \quad \text{...(ii)}$$

From (ii), $$y(\lambda + 1) = 0$$

i.e., either $y = 0$ or $\lambda = -1$

if $\lambda = -1$, then $$2(x - 1) = -4 \quad \text{(from (ii))}$$

$\therefore$ $$x = -1$$

But $x = -1 \Rightarrow y^2 = -4$ which is not possible.

$\therefore$ $$y = 0 \Rightarrow x = 0$$

So, $(x, y) \equiv (0, 0)$ is point of extremum.

Also, $$f_{xx} = 2x,\ f_{yy} = 2 + 2\lambda,\ f_{xy} = 0$$

$$d^2f = f_{xx}dx + f_{yy}dy + f_x d^2x + f_y d^2y$$

$$= 2xdx + 2(1 + \lambda)dy$$

$$= 0 + (x + 1)dy = (x + 1)dy > 0$$

$\therefore$ Shortest distance from (1, 0) is $\sqrt{(0-1)^2 + (0-0)^2}$.

2.19 Let

$$f(x, y) = xy^2 \text{ if } y > 0$$
$$= -xy^2 \text{ if } y \le 0$$

Determine which of $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ exists and which does not exist.

(2018 : 12 Marks)

Solution:

We have,

$$\frac{\partial f}{\partial x}(0,1) = \lim_{h\to 0}\frac{f(h,1)-f(0,1)}{h} = \lim_{n\to 0}\frac{n\times 1^2 - 0}{n} = 1$$

$$\frac{\partial f}{\partial y}(0,1) = \lim_{k\to 0}\frac{f(0,1+k)-f(0,1)}{k} = \lim_{k\to 0}\frac{0-0}{k} = 0$$

Thus, $f_x(0, 1) = 1$ and $f_y(0, 1) = 0$

So both exists.

2.20 (i) If

$$u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-\frac{1}{6}$. Hence show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

(2019 : 12 Marks)

(ii) Using the Jacobian method, show that if $f'(x) = \frac{1}{1+x^2}$ and $f(0) = 0$, then

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

(2019 : 8 Marks)

Solution:

(i) Given that
$$u = \sin^{-1}\sqrt{\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$$

we can write
$$\sin u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2} = \frac{x^{1/6}}{x^{1/4}}\left[\frac{1+(y/x)^{1/3}}{1+(y+x)^{1/2}}\right]^{1/2}$$

$$= x^{-1/2}f\left(\frac{y}{x}\right)$$

$\Rightarrow$
$$\sin^2 u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right] = \frac{x^{1/3}}{x^{1/2}}\left[\frac{1+\left(\frac{y}{x}\right)^{1/3}}{1+\left(\frac{y}{x}\right)^{1/2}}\right]$$

$$= x^{-1/6}f\left(\frac{y}{x}\right)$$

Thus, $z = \sin^2 u$ is a homogeneous function of x and y of degree $-\frac{1}{6}$. ...(1)

Now, by Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{-1}{12}z \text{, where } z = \sin u$$

$$\Rightarrow \quad x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{-1}{12}\sin u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{12}\tan u \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. x and y, respectively

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^1 u\frac{\partial u}{\partial y} \qquad \text{...(2)}$$

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{-1}{12}\sec^1 u\frac{\partial u}{\partial y} \qquad \text{...(3)}$$

Multiply (2) by (x), (3) by y and add to get

$$\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{-1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \qquad \text{...(4)}$$

From (1) and (4), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(1 + \frac{1}{12}(\sec^2 u)\right)$$

$$= \frac{\tan u}{12}\left(\frac{12 + 1 + \sec^2 u}{12}\right)$$

$$= \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right) \text{. Hence Proved.}$$

(ii) Given that : $f'(x) = \frac{1}{1+x^2}$ and $f(0) = 0$

Let $u = f(x) + f(y)$ ...(1)

and $v = \frac{x+y}{1+xy}$ ...(2)

We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & f'(y) \\ \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} & \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Since, $\dfrac{\partial(u,v)}{\partial(x,y)} = 0$

$\therefore$ The given functions are not independent, i.e., the functions u and v are functionally related.

Let $u = \phi(v)$, then

$$f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right)$$

For $y = 0$ gives $f(x) = \phi(x)$ $[\because f(0) = 0]$

$\therefore$ $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$. Hence, the result.

3. Riemann's Definition of Definite Integrals

3.1 Show that the function

$$f(x) = [x^2] + |x - 1|$$

is a Riemann-integreble in the interval [0, 2], where $[\alpha]$ denotes the greates integer less than or equal to α. Can you give an example of a function that is not Riemann integrable on [0, 2]? Compute $\int_0^2 f(x)dx$, where $f(x)$ is as above.

(2010 : 12 Marks)

Solution:

Given :
$$f(x) = \begin{cases} 1-x & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x < \sqrt{2} \\ x+1 & \sqrt{2} \le x < \sqrt{3} \\ x+2 & \sqrt{3} \le x < 2 \end{cases}$$

Now, $f(x)$ is discontinuous at x = 1, $\sqrt{2}$, $\sqrt{3}$, i.e., finite number of discontinuity.

$\therefore$ $f(x)$ is Riemann integrable in [0, 2].

Now,
$$\int_0^2 f(x)dx = \int_0^1 (1-x)dx + \int_1^{\sqrt{2}} x dx + \int_{\sqrt{2}}^{\sqrt{3}} (x+1)dx + \int_{\sqrt{3}}^{2} (x+2)dx$$

$$= [x]_0^1 - \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[\frac{x^2}{2}\right]_{\sqrt{2}}^{\sqrt{3}} + [x]_{\sqrt{2}}^{\sqrt{3}} + \left[\frac{x^2}{2}\right]_{\sqrt{3}}^{2} + [2x]_{\sqrt{3}}^{2}$$

$$= 1 - \frac{1}{2} + \frac{2}{2} - \frac{1}{2} + \frac{3}{2} - \frac{2}{2} + \sqrt{3} - \sqrt{2} + \frac{4}{2} - \frac{3}{2} + 2(2-\sqrt{3})$$

$$= 6 - \sqrt{2} - \sqrt{3}$$

2a

3.2 Evaluate : $\int_0^1 \ln x\, dx$

(2011 : 12 Marks)

Solution:

We know that
$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\cdot\int v\,dx\right)dx$$

$\therefore$
$$\int \ln(x)\,dx = \ln x\int 1\,dx - \int\left(\left(\frac{d}{dx}\ln(x)\right)\int 1\,dx\right)dx$$
$$= \ln x\cdot x - \int\frac{1}{x}\cdot x\,dx$$
$$= x - \ln x - \int 1\,dx$$
$$= x\ln x - x + C, \text{ where } C \text{ is the constant of integration.}$$

1c

3.3 Evaluate : $\int_0^1\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx$

(2013 : 10 Marks)

Solution:

Given integral $\int_0^1\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx$

Let
$$\frac{1}{x} = t \Rightarrow x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$$

when $x = 0$, $t = \infty$; $x = 1$, $t = 1$.

$\therefore$
$$I = \int_\infty^1\left(\frac{2}{t}\sin t - \cos t\right)\left(-\frac{1}{t^2}dt\right)$$
$$= \int_\infty^1\left(\frac{2}{t^3}\sin t - \frac{1}{t^2}\cos t\right)dt$$

Using integration by parts on 2nd term
$$= \int_1^\infty\frac{2}{t^3}\sin t - \left\{\left[\frac{1}{t^2}\sin t\right]_1^\infty - \int_1^\infty -\frac{2}{t^3}\sin t\right\}$$
$$= -\left[\frac{1}{t^2}\sin t\right]_1^\infty = \sin 1$$

3.4 Evaluate : $\int_0^1\frac{\log_e(1+x)}{1+x^2}dx$

1d

(2014 : 10 Marks)

Solution:

Let
$$I = \int_0^1\frac{\log(1+x)}{1+x^2}dx$$

Put $x = \tan\theta \Rightarrow$
$$dx = \sec^2\theta\, d\theta$$

when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{C_1}$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)d\theta \quad \text{...(i)}$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] \quad \left(\text{by the property } \int_0^a f(x)dx = \int_0^a f(a-x)\right)$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1+\tan\theta\}d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \log(1+\tan\theta)d\theta$$

$$I = \log 2 \cdot [\theta]_0^{\pi/4} - I \quad \text{(from (i))}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

$$\therefore \quad \int_0^{\pi/4} \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2$$

3.5 Evaluate the following integral : $\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx$

1d

(2015 : 10 Marks)

Solution:

$$I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}dx \quad \text{...(i)}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3} dx}{\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3}+\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{(\cos x)^{1/3}}{(\cos x)^{1/3}+(\sin x)^{1/3}}dx \quad \text{...(ii)}$$

$$= \left[\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

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Adding (i) and (ii)

$$2I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3} + (\cos)^{1/3}}{(\sin x)^{1/3} + (\cos x)^{1/3}} dx$$

$$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$\therefore \quad I = \frac{1}{2} \times \frac{\pi}{6} = \frac{\pi}{12}$$

3.6 Evaluate : $I = \int_0^1 3\sqrt{x\log\left(\frac{1}{x}\right)}dx$.

(2016 : 10 Marks)

Solution:

Improper integral as integrand becomes undefined at lower limit, i.e., $x = 0$.

Let $\log\frac{1}{x} = y \Rightarrow \quad x = e^{-y} \Rightarrow dx = -e^{-y}\, dy$

limits, $x = 0^+ \to y \to \infty$

$x = 1 \to y = 0$

$$I = -\int_{\infty}^{0} (e^{-y}y)^{1/3} \cdot e^{-y} dy = \int_0^{\infty} y^{1/3} \cdot e^{-\frac{4}{3}y}\, dy$$

Putting $\frac{4}{3}y = t \Rightarrow \quad dy = \frac{3}{4}dt$

$$I = \int_0^{\infty}\left(\frac{3}{4}t\right)^{1/3} e^{-t} \cdot \frac{3}{4}dt$$

$$= \frac{3}{4}\cdot\left(\frac{3}{4}\right)^{1/3}\int_0^{\infty} t^{1/3}\cdot e^{-t}dt$$

$$= \left(\frac{3}{4}\right)^{4/3} \cdot \int_0^{\infty} t^{\frac{4}{3}-1} \cdot e^{-t}dt = \left(\frac{3}{4}\right)^{\frac{4}{3}} \cdot \left\lceil\left(\frac{4}{3}\right)\right.$$

$$= \left(\frac{3}{4}\right)^{\frac{4}{3}} \cdot \left\lceil\left(1+\frac{1}{3}\right)\right. = \frac{1}{3}\left(\frac{3}{4}\right)^{\frac{4}{3}} \cdot \left\lceil\frac{1}{3}\right.$$

$$\left[\Gamma(m) = \int_0^{\infty} e^{-x} \cdot x^{m-1}dx;\ \Gamma(m+1) = m\Gamma(m)\right]$$

4. Indefinite Integrals

4.1 Does the integral $\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}dx$ exist? If so, find its value.

(2010 : 12 Marks)

Solution:

Given the integral is $\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}dx$

Rationaling integrand, we get

$$\int_{-1}^{1} \frac{\sqrt{(1+x)^2}}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{1+x}{\sqrt{1-x^2}} dx$$

Let $x = \sin\theta$, ∴ $dx = \cos\theta \, d\theta$

∴ Integral becomes $\int_{-\pi}^{\pi} \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}} d\theta \times \cos\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{1+\sin\theta}{|\cos\theta|} \times \cos\theta \, d\theta = \int_{-\pi/2}^{\pi/2} (1+\sin\theta) d\theta$$

$$= [\theta]_{-\pi/2}^{\pi/2} + 0 = \pi$$

5. Infinite and Improper Integrals

5.1 Examine if the improper integral $\int_0^3 \frac{2x dx}{(1-x^2)^{2/3}}$ exists.

(2017 : 10 Marks)

Solution:

Given integrand, $\frac{2x}{(1-x^2)^{2/3}}$ is undefined at $x = 1$.

Hence, we split the limit at $x = 1$.

$$\int_0^3 \frac{2x}{(1-x^2)^{2/3}} dx = \int_0^1 \frac{2x}{(1-x^2)^{2/3}} dx + \int_1^3 \frac{2x}{(1-x^2)^{2/3}} dx$$

$$= \lim_{a\to 1} \int_0^a \frac{2x}{(1-x^2)^{2/3}} dx + \lim_{a\to 1} \int_a^3 \frac{2x}{(1-x^2)^{2/3}} dx$$

$$= \lim_{a\to 1^-} \left[-3(1-x^2)^{1/3}\right]_0^a + \lim_{a\to 1^+} \left[-3(1-x^2)^{1/3}\right]_a^3$$

$$= [0-(-3)] + [-3(-2)-0]$$

$$= 3 + 6 = 9$$

?

5.2 Find the maximum and minimum values of $x^4 - 5x^2 + 4$ on the interval [2, 3].

(2018 : 13 Marks)

Solution:

Let $f(x) = x^4 - 5x^2 + 4$ (given)

Now, $f'(x) = 4x^3 - 10x + 4$

$f'(x) = 0$ at $x_1 = 1.32, x_2 = -1.75, 0.43$

Also, $f''(x) = 12x^2 - 10$

In the interval [2, 3], $f(x)$ is monotonous increasing function as $f'(x) > 0$ for $x > 1.32$.

Therefore, in the interval [2, 3], minimum value occurs at $x = 2$ and maximum value occurs at $x = 3$.

$f(2) = 0$ = minimum value

$f(3) = 40$ = maximum value

Thus, $f(x)$ has minimum value of 0 and maximum value of 40 in the interval [2, 3].

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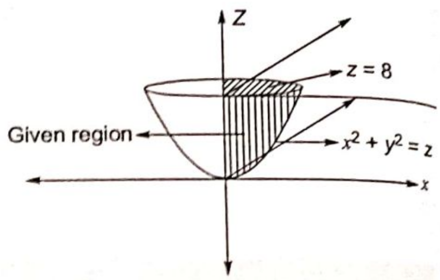
6. Double and Triple Integrals

6.1 Let D be the region determined by the inequalities $x > 0$, $y > 0$, $z < 8$ and $z > x^2 + y^2$. Compute $\iiint_D 2x\,dx\,dy\,dz$

(2010 : 20 Marks)

Solution:

The given region is $x, y > 0$ and $z < 8$.



$$z > x^2 + y^2$$

$\therefore$ Integral,
$$I = \iiint_D 2x\,dx\,dy\,dz$$

$\Rightarrow$
$$I = \iint \int_{z=x^2+y^2}^{8} 2x\,dx\,dy\,dz$$
$$= \iint [z]_{x^2+y^2}^{8} \cdot 2x\,dx\,dy = \iint (8 - x^2 - y^2)\cdot 2x\,dx\,dy$$
$$= 2\iint (8x - x^3 - y^2 x)\,dx\,dy$$

Let $x = r\cos\theta$, $y = r\sin\theta$

θ varies from 0 to $\dfrac{\pi}{2}$ as $x, y > 0$ in given region.

Also, r varies from 0 to $2\sqrt{2}$ as $x^2 + y^2 < z$.

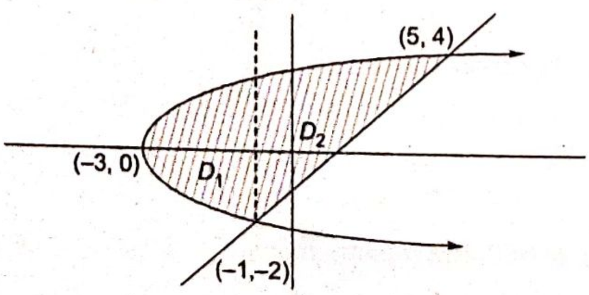
$\therefore$
$$I = 2\iint (8\cdot r\cos\theta - r^3\cos^3\theta - r^3\sin^2\theta\cdot\cos\theta)\,r\,dr\,d\theta$$
$$= 2\iint \left\{8r\cos\theta - r^3\cos\theta(\cos^2\theta + \sin^2\theta)\}r\,dr\,d\theta\right\}$$
$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\sqrt{2}} (8r^2\cos\theta - r^4\cos\theta)\,dr\,d\theta$$
$$= 2\left[8\left[\frac{r^3}{3}\right]_0^{2\sqrt{2}}[\sin\theta]_0^{\pi/2} - \left[\frac{r^5}{5}\right]_0^{2\sqrt{2}}[\sin\theta]_0^{\pi/2}\right]$$
$$= 2\left[8\times\frac{16\sqrt{2}}{3}\times 1 - \frac{128\sqrt{2}}{5}\times 1\right] = \frac{512\sqrt{2}}{15}$$

6.2 Evaluate $\iint_D xy\,dA$ where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

(2013 : 15 Marks)

Solution:

The required areas is shown shaded in the diagram.



The points of intersection of the curves are

$$(x-1)^2 = 2x+6 \Rightarrow x^2-4x-5=0$$
$$x = 5, -1$$

i.e., (–1, –2) and (5, 4)

$$I = \iint_D xy\,dA = \iint_{D_1} xy\,dA + \iint_{D_2} xy\,dA$$

$$= \int_{x=-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\,dy\,dx + \int_{x=-1}^{x=5}\int_{y=x-1}^{\sqrt{2x+6}} xy\,dy\,dx$$

$$= \int_{x=-3}^{-1}\left[\frac{xy^2}{2}\right]_{y=-\sqrt{2x+6}}^{\sqrt{2x+6}} dx + \int_{x=-1}^{5} x\left[\frac{y^2}{2}\right]_{x-1}^{\sqrt{2x+6}} dx$$

$$= \int_{x=-3}^{-1}\frac{x}{2}[(2x+6)-(2x+6)]dx + \int_{x=-1}^{5}\frac{x}{2}[(2x+6)-(x-1)^2]dx$$

$$= 0 + \int_{x=-1}^{5}\frac{x}{2}(4x+5-x^2)dx$$

$$= \int_{x=-1}^{5}\left(2x^2+\frac{5}{2}x-\frac{x^3}{2}\right)dx$$

$$= \left[\frac{2x^3}{3}+\frac{5}{4}x^2-\frac{x^4}{8}\right]_{x=-1}^{5}$$

$$= 36$$

6.3 By using the transformation $x + y = \mu$, $y = uv$, evaluate the integral $\iint\{xy(1-x-y)\}^{1/2}dxdy$ taken over the area enclosed by the straight lines $x = 0$, $y = 0$ and $x + y = 1$.

(2014 : 15 Marks)

Solution:

Given: $x + y = u,\ y = uv$

$\Rightarrow$ $x = u - av$

$x = u(1-v)$

Now
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} + \begin{vmatrix}-v & -u\\ v & u\end{vmatrix}$$

$$= u - vu + uv$$
$$= u$$

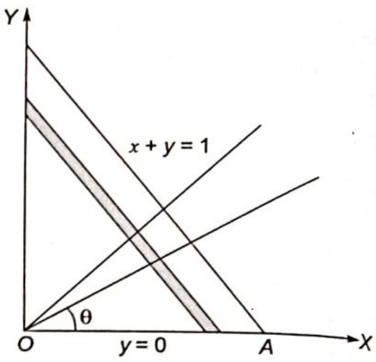
But
$$dxdy = \frac{\partial(x,y)}{\partial(u,v)}dud\theta$$

$$dxdy = ududv$$

and
$$\sqrt{xy(1-x-y} = \sqrt{u(1-v)uv(1-u)}$$
$$= uv^{1/2}\sqrt{(1-u)(1-v)}$$

Clearly, the region of integration is OAB.

The integration formulae are $x + y = u$, $y = uv = (x + y)v$



$$\Rightarrow \qquad y = \frac{v}{1-v}x$$

i.e., clearly the area for new variable is to be divided by the lines parallel to $x + y = 1$ and by lines $y = \frac{v}{1-v}x$,

i.e.,
$$y = x\tan\theta$$

where
$$\tan\theta = \frac{v}{1-v}$$

where θ varies from 0 to $\frac{\pi}{2}$ and so v varies from 0 to 1 and $u + x + y$ varies from 0 to 1.

i.e., limits of u are 0 to 1.

Hence, the given integral

$$= \int_0^1\int_0^1 uv^{1/2}\sqrt{(1-u)(1-v)}u\,du\,dv$$

$$= \int_0^1 u^2(1-u)^{1/2}du\int_0^1 v^{1/2}(1-v)^{1/2}dv$$

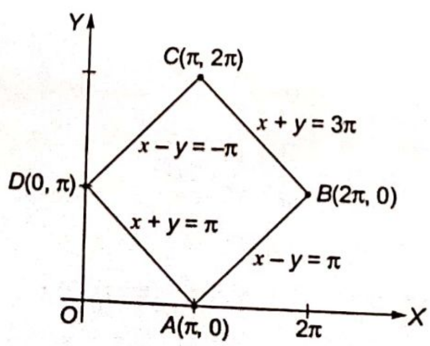
$$= \int_0^1 u^{3-1}(1-u)^{\frac{3}{2}-1}du\int_0^1 v^{\frac{3}{2}-1}(1-v)^{\frac{3}{2}-1}dv$$

$$= B\left(3,\frac{3}{2}\right).B\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\sqrt{3}\sqrt{\frac{3}{2}}}{\sqrt{\left(3+\frac{3}{2}\right)}}\cdot\frac{\sqrt{\frac{3}{2}}\cdot\sqrt{\frac{3}{2}}}{\sqrt{3}}$$

$$= \frac{2\pi}{105}.$$

6.4 Evaluate the integral : $\iint_R (x-y)^2 \cdot \cos^2(x+y)\,dx\,dy$ where R is the rhombus with successive vertices as $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$.

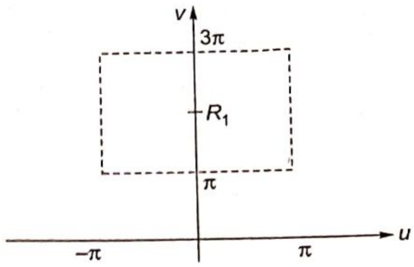


(2015 : 12 Marks)

Solution:

Using the transformation, $x - y = u$ and $x + y = v$

i.e., $$x = \frac{u+v}{2},\ y = \frac{-u+v}{2}$$



Our region of integration gets transformed to a square.

Jacobian,

$$\frac{J(x,y)}{J(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\iint_R f(x,y)dxdy = \iint_{R_1} f_1(u,v)|J(u,v)|dudv$$

$$= \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2 \cdot \cos^2 v \cdot \frac{1}{2} dudv$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} u^2 du \int_{\pi}^{3\pi} \frac{1}{2}(1+\cos 2v)dv$$

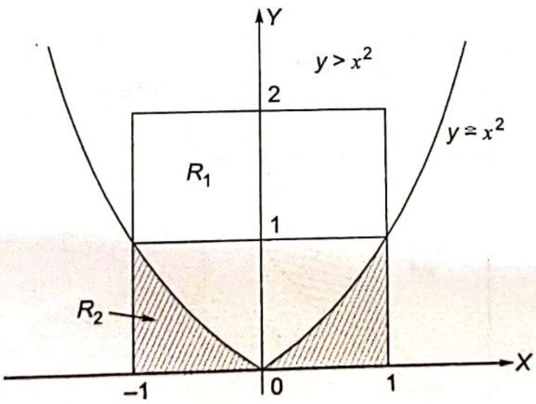
$$= \frac{1}{2}\cdot 2 \cdot \frac{\pi^3}{3}\cdot\frac{2\pi}{2} = \frac{\pi^4}{3}$$

6.5 Evaluate $\iint_R \sqrt{|y - x^2|}\,dxdy$ where $R = [-1, 1; 0, 2]$

(2015 : 13 Marks)

Solution:

We divide the domain R into two parts R_1 & R_2 in R_1, $y > x^2$



$$\therefore \quad |y - x^2| = (y - x^2)$$

In R_2, $$y < x^2$$

$$\therefore \quad |y - x^2| = (y - x^2)$$

In R_2, $$y < x^2$$

$$\therefore \quad |y - x^2| = -(y - x^2) = x^2 - y$$

$$\iint_R \sqrt{|y-x^2|}\,dxdy = \iint_{R_1}\sqrt{|y-x^2|}\,dxdy + \iint_{R_2}\sqrt{|y-x^2|}\,dxdy$$

$$= \int_{x=-1}^{1}\int_{y=x^2}^{2}\sqrt{y-x^2}\,dxdy + \int_{x=-1}^{1}\int_{y=0}^{x^{-2}}\sqrt{x^2-y}\,dxdy$$

$$= \int_{-1}^{1}\frac{2}{3}(y-x^2)^{3/2}\Bigg|_{y=x^2}^{y=2} dx + \int_{-1}^{1}-\frac{2}{3}(x^2-y)^{3/2}\Bigg|_{y=0}^{y=x^2} dx$$

$$= \frac{2}{3}\int_{-1}^{1}(2-x^2)^{3/2}dx + \left(-\frac{2}{3}\right)\int_{-1}^{1}(0-x^3)dx$$

$$= \frac{4}{3}\int_{0}^{1}(2-x^2)^{3/2}dx + \frac{2}{3}\left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$\left(\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ for even } f_x\right)$$

$$= \frac{16}{3}\int_{0}^{\pi/4}\cos^4\theta d\theta \text{, taking } x = \sqrt{2}\sin\theta$$

$$= \frac{4}{3}\int_{0}^{\pi/4}(1+\cos 2\theta)^2 d\theta = \frac{4}{3}+\frac{\pi}{2}$$

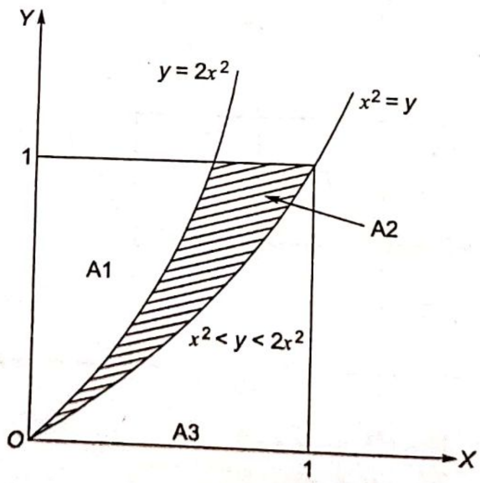
6.6 Evaluate $\iint_R f(x,y)dxdy$ over the rectangle $R = [0, 1; 0, 1]$ where

$$f(x, y) = \begin{cases} x+y, & \text{if } x^2 < y < 2x^2 \\ 0, & \text{elsewhere} \end{cases}$$

(2016 : 15 Marks)

Solution:

We split the region of integration R into three different regions A_1, A_2 and A_3 as per the definition of $f(x, y)$.



$A_2 : x^2 < y < 2x^2$

$$I = \iint_R f(x,y)dxdy$$

$$= \iint_{A_1} f(x,y)dxdy + \iint_{A_2} f(x,y)dxdy + \iint_{A_3} f(x,y)dxdy$$

$$= 0 + \int_{y=0}^{1} \int_{x=\sqrt{\frac{y}{2}}}^{\sqrt{y}} (x+y)\,dx\,dy + 0$$

$$= \int_{y=0}^{1} \left[\frac{x^2}{2} + yx\right]_{x=\sqrt{\frac{y}{2}}}^{\sqrt{y}} dy = \int_0^1 \frac{1}{2}\left(y - \frac{y}{2}\right) + y\left(\sqrt{y} - \sqrt{\frac{y}{2}}\right) dy$$

$$= \int_0^1 \frac{1}{4} \cdot y + y^{3/2} \cdot \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) dy$$

$$= \left[\frac{y^2}{8} + y^{5/2} \cdot \frac{2}{5}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\right]_0^1 = \frac{1}{8} + \left(\frac{2-\sqrt{2}}{5}\right)$$

$$= \frac{23 - 8\sqrt{2}}{40}$$

6.7 Integrate the function $f(x, y) = xy(x^2 + y^2)$ over the domain $R : \{-3 \le x^2 - y^2 \le 3,\ 1 \le xy \le 4\}$

(2017 : 10 Marks)

Solution:

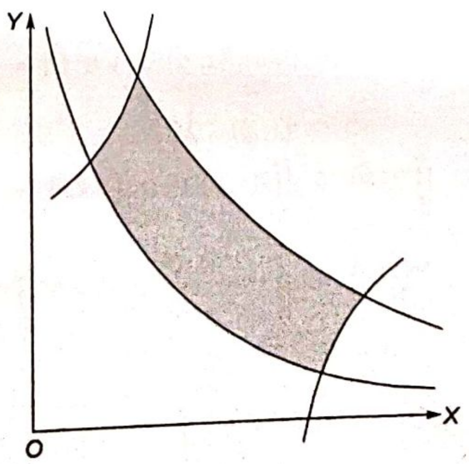
$$I = \iint_R xy(x^2 + y^2)\,dx\,dy$$

Put
$$x^2 - y^2 = u$$
$$xy = v$$

$\therefore$ Limits : $-3 \le u \le 3$

$1 \le v \le 4$ which bounds region R.

$$J = \frac{J(u,v)}{J(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}$$



$$= 2(x^2 + y^2)$$

$$\therefore \quad J(x, y) = \frac{1}{2(x^2 + y^2)} J(u,v)$$

$$\therefore \quad I = \iint_R xy \cdot \frac{(x^2+y^2)}{2(x^2+y^2)} du\,dv$$

$$= \frac{1}{2}\int_{u=-3}^{3}\int_{v=1}^{4} v\,dv\,du = \frac{1}{2}(3-(-3))\left(\frac{v^2}{2}\right)_1^4$$

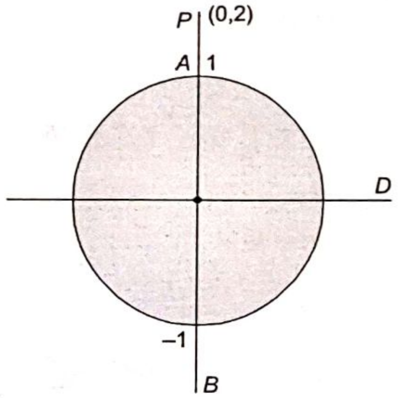
$$= \frac{6}{2}\left[\frac{16-1}{2}\right] = \frac{45}{2}$$

6.8 Prove that $\frac{\pi}{3} \le \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le \pi$ where D is the unit disc.

(2017 : 10 Marks)

Solution:

Here, integrand $\frac{1}{\sqrt{x^2+(y-2)^2}}[(= f(x,y)]$ represents the reciprocal of the distance of fixed point (0, 2), say P to another variable point (x, y) lying inside or on the disc D.



$D: x^2 + y^2 \le 1$

Let

$$M = \{\text{Max}\{f(x,y)\};\ x,\ y \in D\} = \frac{1}{|PA|} = 1$$

$$m = \{\text{Min}\{f(x,\ y);\ x,\ y \in D\} = \frac{1}{|PB|} = \frac{1}{3}$$

Hence,

$$m \le f(x,\ y) \le M$$

$$\therefore \quad \iint_D m\,dA \le \iint_D f(x,y)dA \le \iint_D M\,dA$$

$$\therefore \quad m\iint_D dA \le \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le M\iint_D dA$$

$$\Rightarrow \quad m(A) \le I \le M(A)$$

$$\Rightarrow \quad \frac{1}{3}(\pi \cdot 1^2) \le I \le 1(\pi \cdot 1^2)$$

i.e.,

$$\frac{\pi}{3} \le \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le \pi$$

6.9 Evaluate the integral $\int_0^a \int_{x/a}^x \frac{x\,dy\,dx}{x^2+y^2}$.

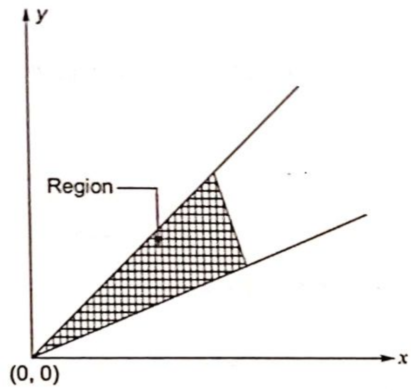
(2018 : 12 Marks)

Solution:

Given :
$$I = \int_0^a \int_{x/a}^x \frac{x}{x^2+y^2}\,dy\,dx$$

So,
$$I = \int_0^a \int_{x/a}^x \frac{x}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}\,dy\,dx$$

$$= \int_0^a \frac{1}{x}\int_{x/a}^x \frac{1}{1+\left(\frac{y}{x}\right)^2}\,dy\,dx = \int_0^a \frac{x}{x}\left[\tan^{-1}\frac{y}{x}\right]_{x/a}^x dx$$



$\Rightarrow$
$$I = \int_0^a \left(\tan^{-1}1 - \tan^{-1}\frac{1}{a}\right)dx$$

$$= \left(\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right)\int_0^a dx = a\left(\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right)$$

7. Area, Surface and Volume

7.1 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$.

(2011 : 20 Marks)

Solution:

The solid lies above the disk D whose boundary circle has the equation $x^2 + y^2 = 2x$ or $(x-1)^2 + y^2 = 1$.

Its equation in polar co-ordinates is given by $r = 2\cos\theta$ where θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\therefore$ The volume of the solid is

$$\iint_D (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta \qquad [\because x = r\cos\theta,\ y = r\sin\theta]$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{16\cos^4\theta}{4}\,d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = 4.2\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$\left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$= 8\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi}{2}$$

7.2 Compute the volume of the solid enclosed between the surfaces $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$.
(2012 : 20 Marks)

Solution:

The given surfaces are

$$x^2 + y^2 = 9$$

and $$x^2 + z^2 = 9$$

The volume of the required solid is

$$V = \int_{x=-3}^{3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{z=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dzdydx$$

$$= 8\int_{x=0}^{3} \int_{y=0}^{\sqrt{9-x^2}} \int_{z=0}^{\sqrt{9-x^2}} dzdydx \left[\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$= 8\int_{x=0}^{3} \int_{y=0}^{\sqrt{9-x^2}} \sqrt{9-x^2}\,dydx$$

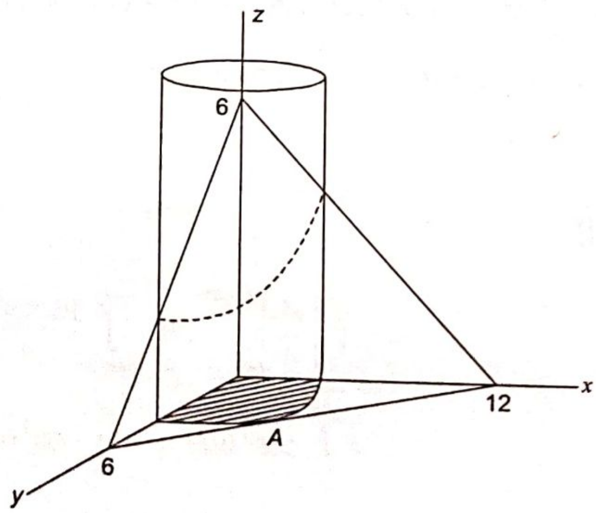
$$= 8\int_{x=0}^{3} \sqrt{9-x^2}\cdot\sqrt{9-x^2}\,dx = 8\int_{x=0}^{3} (9-x^2)dx$$

$$= 8\left(9x - \frac{x^3}{3}\right)_0^3 = 8(27-9) = 8 \times 18 = 144 \text{ cubic units}$$

7.3 Find the surface area of the plane $x + 2y + 2z = 12$ cut off by $x = 0$, $y = 0$ and $x^2 + y^2 = 16$.
(2016 : 15 Marks)

Solution:

Plane $x + 2y + 2z = 12$ or $\frac{x}{12} + \frac{y}{6} + \frac{z}{6} = 1$ cuts the co-ordinates at a distance of 12, 6 and 6 from origin.



Cylinder : $$x^2 + y^2 = 16$$

Planes : $$x = 0, y = 0$$

$$\text{Surface Area} = \iint_A \sqrt{1 + z_x^2 + z_y^2}\,dxdy$$

$$= \iint_A \sqrt{1+\left(-\frac{1}{2}\right)^2+(-1)^2}\,dxdy \qquad \left[\begin{array}{l} z=-\frac{x}{2}-y+6 \\ z_x=-\frac{1}{2}, z_y=-1 \end{array}\right]$$

$$= \frac{3}{2}\iint_A dxdy$$

(A : Projection of surface on xy-plane $x^2+y^2 \le 16, x \ge 0, y \ge 0$)

$$= \frac{3}{2}\cdot\left[\frac{1}{4}\pi(4)^2\right] = 6\pi$$

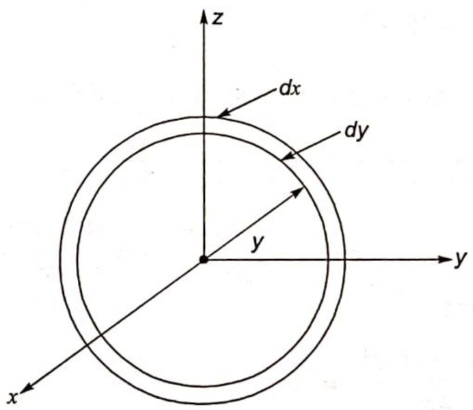
7.4 **The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about x-axis. Find the volume of solid of revolves.**

(2018 : 13 Marks)

Solution:

Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If it is revolved around x-axis, then each cross-section is circular disc of radius y. Taking an element of thickness dx and length dy, we get area of this cross-section as $2\pi y dy dx$.



$\therefore$ Volume,
$$V = \int_{x=-a}^{x=a}\int_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} 2\pi y dy dx$$

$\Rightarrow$
$$V = \int_{x=-a}^{a} 2\pi\left[\frac{y^2}{2}\right]_{y=0}^{y=b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$\Rightarrow$
$$V = \pi\int_{x=-a}^{a} b^2\left(1-\frac{x^2}{a^2}\right)dx = \pi b^2\left[2a-\frac{2a}{3}\right]$$

$\Rightarrow$
$$V = \frac{4\pi}{3}ab^2$$

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