Mains Test Series – 2020

Test-9 (Paper-I), Batch-I

Answer key (full syllabus)

1(a) M_{22} is the vector space of 2×2 matrices. Let S_{22} denote the set of all 2×2 symmetric matrices. That is

$S_{22} = \{A \in M_{22} \mid A^t = A\}$

(i) show that S_{22} is a subspace of M_{22}.

(ii) Exhibit a basis for S_{22} and prove that it has the required properties.

(iii) what is the dimension of S_{22}?

Sol'n: The zero vector of M_{22} is the zero matrix, O which is a symmetric matrix. so S_{22} is not empty,

since $O \in S_{22}$.

Suppose that A and B are two matrices in S_{22}.

Then we know that $A^t = A$ and $B^t = B$.

$$(A+B)^t = A^t + B^t$$

$$= A+B \qquad A, B \in S_{22}$$

So $A+B$ is symmetric and $A+B \in S_{22}$.

Suppose that $A \in S_{22}$ and $\alpha \in F$. Is $\alpha A \in S_{22}$?

we know that $A^t = A$.

Now check that, $\alpha A^t = \alpha A^t$

$$= \alpha A \qquad A \in S_{22}$$

So αA is also symmetric and $\alpha A \in S_{22}$.

$\therefore$ S_{22} is a subspace of M_{22}.

(2) An arbitrary matrix from S_{22} can be written as $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

we can express this matrix as

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

this equation says that the set

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } S_{22}.$$

Is it also linearly independent?

write a relation of a linear dependence on S,

$$0 = a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

The equality of these two matrices tells us that $a_1 = a_2 = a_3 = 0$, and the only relation of linear dependence on T is trivial. So T is linearly independent, and hence is a basis of S_{22}.

(3) and therefore $\dim(S_{22}) = 3$.

1(b)(ii) Sol'n : Since $\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}$

we have $T\left(\begin{bmatrix}4\\3\end{bmatrix}\right) = T\left(\begin{bmatrix}2\\1\end{bmatrix} + 2\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$= T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) + 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}3\\4\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}1\\8\end{bmatrix}}.$$

(or)

1(b)(i) The co-ordinate vectors given matrices relative to the usual basis are as follows.

$[-2\ 3\ 4\ -1\ 3\ -2]$, $[4\ -2\ 2\ 0\ -1\ 1]$, $[-1\ -2\ -2\ 2\ 2\ 2]$, $[-1\ 1\ 0\ -1\ 0\ -2]$, $[-1\ 2\ -2\ 0\ -1\ -2]$

Form the matrix M whose rows are the above co-ordinate vectors. Row reduce M to echelon form

$$M = \begin{pmatrix} -2 & 3 & 4 & -1 & 3 & -2 \\ 4 & -2 & 2 & 0 & -1 & 1 \\ -1 & -2 & -2 & 2 & 2 & 2 \\ -1 & 1 & 0 & -1 & 0 & -2 \\ -1 & 2 & -2 & 0 & -1 & -2 \end{pmatrix} \sim \begin{bmatrix} -1 & 2 & -2 & 0 & -1 & -2 \\ 4 & -2 & 2 & 0 & -1 & 1 \\ -1 & -2 & -2 & 2 & 2 & 2 \\ -1 & 1 & 0 & -1 & 0 & -2 \\ -2 & 3 & 4 & -1 & 3 & -2 \end{bmatrix} \quad R_5 \leftrightarrow R_1$$

$$\sim \begin{bmatrix} -1 & 2 & -2 & 0 & -1 & -2 \\ 0 & 6 & -6 & 0 & -3 & -7 \\ 0 & -4 & 0 & 2 & 3 & 4 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 8 & -1 & 5 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1 \end{matrix} \sim \begin{bmatrix} -1 & 2 & -2 & 0 & -1 & -2 \\ 0 & 6 & -6 & 0 & -3 & -7 \\ 0 & 0 & -4 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -1 \\ 0 & 0 & 7 & -1 & 9/2 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 + \frac{R_1}{2} \\ R_5 \to R_5 + \frac{R_1}{2} \end{matrix}$$

$$\sim \begin{bmatrix} -1 & 2 & -2 & 0 & -1 & -2 \\ 0 & 6 & -6 & 0 & -3 & -7 \\ 0 & 0 & -4 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1/2 & 3/4 & -1 \\ 0 & 0 & 0 & \text{[illegible]} & & \end{bmatrix} \begin{matrix} R_4 \to R_4 + \frac{R_3}{4} \\ R_5 \to R_5 + R_3\frac{7}{4} \end{matrix} \sim \begin{bmatrix} -1 & 2 & -2 & 0 & -1 & -2 \\ 0 & 6 & -6 & 0 & -3 & -7 \\ 0 & 0 & -4 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1/2 & 3/4 & -1 \\ 0 & 0 & 0 & 0 & 10 & -4 \end{bmatrix} R_5 \to R_5 + 5R_4$$

= U.T.

1(b) (i) Determine if the set S below is linearly independent in $M_{2,3}$.

$$\left\{\begin{bmatrix}-2 & 3 & 4\\ -1 & 3 & -2\end{bmatrix}, \begin{bmatrix}4 & -2 & 2\\ 0 & -1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -2 & -2\\ 2 & 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & -2\end{bmatrix}, \begin{bmatrix}-1 & 2 & -2\\ 0 & -1 & -2\end{bmatrix}\right\}$$

(ii) If $T : C^2 \to C^2$ satisfies $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix}$

find $\left(\begin{bmatrix}4\\3\end{bmatrix}\right)$.

Sol'n: Suppose that there exist constants a_1, a_2, a_3, a_4 and a_5 so that

$$a_1\begin{bmatrix}-2 & 3 & 4\\ -1 & 3 & -2\end{bmatrix} + a_2\begin{bmatrix}4 & -2 & 2\\ 0 & -1 & 1\end{bmatrix} + a_3\begin{bmatrix}-1 & -2 & -2\\ 2 & 2 & 2\end{bmatrix}$$

$$+ a_4\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & -2\end{bmatrix} + a_5\begin{bmatrix}-1 & 2 & -1\\ 0 & -1 & -2\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Then, we have the matrix equality.

$$\begin{bmatrix}-2a_1+4a_2-a_3-a_4-a_5 & 3a_1-2a_2-2a_3+a_4+2a_5 & 4a_1+2a_2-2a_3-2a_5\\ -a_1+2a_3-a_4 & 3a_1-a_2+2a_3-a_5 & -2a_1+a_2+2a_3+2a_4-2a_5\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

which yields the linear system of equations.

$$-2a_1+4a_2-a_3-a_4-a_5 = 0$$

$$3a_1-2a_2-2a_3+a_4+2a_5 = 0$$

$$4a_1+2a_2-2a_3-2a_5 = 0$$

$$-a_1+2a_3-a_4 = 0$$

$$3a_1-a_2+2a_3-a_5 = 0$$

$$-2a_1+a_2+2a_3+2a_4-2a_5 = 0$$

By row reducing the associated 6×5 homogeneous system, we see that the only solution is $a_1=a_2=a_3=a_4=a_5=0$, so these matrices are linearly independent subset of $M_{2,3}$.

1(c) Discuss the continuity of $f(x)$ in $[0,2]$ where

$$f(x)=\begin{cases}[\cos\pi x], & x\le 1\\ |(2x-3)|[x-2], & x>1,\end{cases}$$

where [.] denotes the greatest integral function.

Sol'n: Consider $x\in[0,1]$

then $f(x)=[\cos\pi x]$ is discontinuous where $\cos\pi x \in I$

In $[0,1]$, $\cos\pi x$ is an integer at $x=0,\frac{1}{2},1$.

$\Rightarrow x=0,\frac{1}{2}$ and 1 may be the points at which $f(x)$ may be discontinuous.

Again, consider $x\in[1,2]$

If we consider $f(x)=[x-2]|2x-3|$

then if $x\in(1,2)$; $[x-2]=-1$

and for $x=2$; $[x-2]=0$

Also, $|2x-3|=0$ at $x=3/2$

$\Rightarrow x=3/2$ and 2 may be the points at which $f(x)$ may be discontinuous. Hence, the possible points of discontinuity may be $x=0,\frac{1}{2},1,3/2,2$.

Now,

$$f(x)=\begin{cases}1; & x=0\\ 0; & 0<x\le\frac{1}{2}\\ -1; & \frac{1}{2}<x\le 1\\ -(3-2x); & 1<x\le 3/2\\ -(2x-3); & 3/2<x\le 2\\ 0; & x=2\end{cases}$$

Now, we will consider the cases one by one.

$\lim\limits_{x\to 0+0} f(x)=0$ and $f(0)=1$

$\therefore f(x)$ is discontinuous at $x=0$

$$\lim_{x\to \frac{1}{2}-0} f(x) = \lim_{x\to \frac{1}{2}-0} 0 = 0$$

$$\lim_{x\to \frac{1}{2}+0} f(x) = \lim_{x\to \frac{1}{2}+0} (-1) = -1$$

$\therefore f(x)$ is discontinuous at $x = \frac{1}{2}$

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} (-1) = -1$$

$$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0} -(3-2x) = -1$$

and $f(1) = -1$

$\therefore f(x)$ is continuous at $x = 1$

$$\lim_{x\to \frac{3}{2}-0} f(x) = \lim_{x\to \frac{3}{2}-0} (2x-3) = 0$$

$$\lim_{x\to \frac{3}{2}+0} f(x) = \lim_{x\to \frac{3}{2}+0} (3-2x) = 0$$

$$f(3/2) = 0$$

$\therefore f(x)$ is continuous at $x = 3/2$

$$\lim_{x\to 2-0} f(x) = \lim_{x\to 2-0} (3-2x) = -1$$

$f(x)$ is discontinuous at $x = 2$

$\therefore f(x)$ is continuous when

$x \in [0,2] - \{0, \frac{1}{2}, 2\}$.

1(d) → Use a double integral to determine the volume of the solid that is bounded by $z = 8 - x^2 - y^2$ —① and $z = 3x^2 + 3y^2 - 4$ —②

Sol^n: We have $z = 8 - x^2 - y^2$.

$$\Rightarrow x^2 + y^2 = 8 - z.$$

and from ②

$$z = 3(x^2+y^2) - 4$$
$$= 3(8-z) - 4$$
$$4z = 20$$
$$\Rightarrow z = 5$$

$\therefore$ from ①, $x^2 + y^2 = 3$.

Required volume $(V) = \iint\limits_{x^2+y^2 \le 3} \left[(8 - x^2 - y^2) - (3x^2 + 3y^2 - 4)\right] dx\,dy$

$$= \iint\limits_{x^2+y^2\le 3} \left[12 - 4(x^2+y^2)\right] dx\,dy$$

put $x = r\cos\theta$, $y = r\sin\theta$.

$$\Rightarrow dx\,dy = r\,dr\,d\theta$$

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}} (12 - 4r^2)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} (12r - 4r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(6r^2 - r^4\right)_0^{\sqrt{3}} d\theta$$

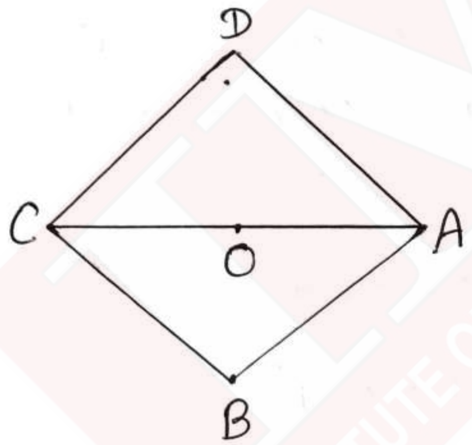
$$= \int_0^{2\pi} (18 - 9)\,d\theta = \int_0^{2\pi} 9\,d\theta = 18\pi$$

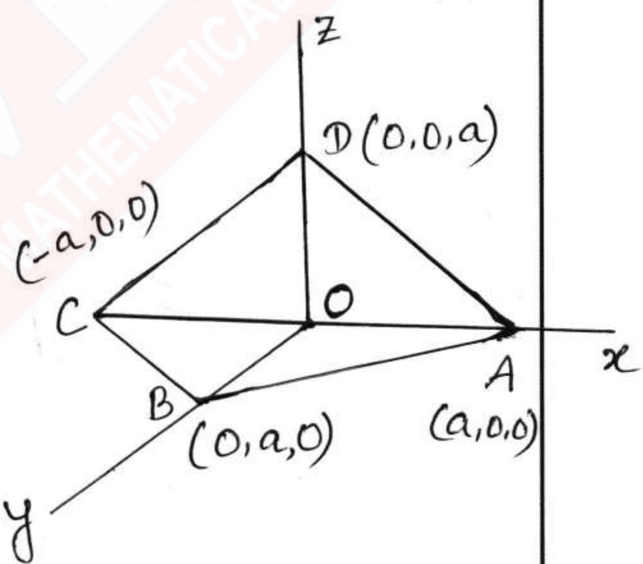
1(e) A Square ABCD of diagonal $2a$ is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the S.D. between DC and AB.

Sol: Let O, the centre of the square, be taken as the origin and OA, OB and OD be taken as x, y and z-axes respectively.

Therefore equations of AB are:

$$\frac{x-a}{a}=\frac{y-0}{-a}=\frac{z-0}{0} \quad \text{—— (1)}$$





Equations of DC are:

$$\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-a}{a} \quad \text{—— (2)}$$

Now any point on the line DC is $(0,0,a)$.

And the equation of the plane through the line AB and parallel to DC i.e. through ① and parallel to ② is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ a & -a & 0 \\ a & 0 & a \end{vmatrix} = 0$$

or $(x-a)(-a\cdot a) - y(a\cdot a) + z(a\cdot a) = 0$

or $x+y-z-a=0$ ——— ③

$\therefore$ Required S.D. = length of perpendicular from $(0,0,a)$ to the plane ③

$$= \frac{0+0-a-a}{\sqrt{[1^2+1^2+(-1)^2]}}$$

$$= \frac{2a}{\sqrt{3}} \text{ (numerically)}$$

2(a) (i) Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

(ii) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the characteristic roots of the n-square matrix A and K is a scalar, prove that characteristic roots of $A-KI$ are α_1-K, $\alpha_2-K, \cdots, \alpha_n-K$.

Solⁿ: (i) we have '0' is an eigen value of $A \Rightarrow \lambda=0$ satisfy the equation $|A-\lambda I|=0$

$\Rightarrow |A|=0 \Rightarrow A$ is singular.

conversely, A is singular $\Rightarrow |A|=0$

$\Rightarrow \lambda=0$ satisfy the equation $|A-\lambda I|=0$

$\Rightarrow$ 0 is an eigen value of A.

(iii) Since $\alpha_1, \alpha_2, \cdots \alpha_n$ are the characteristic roots of A, therefore the characteristic polynomial of A is

$$|A-\lambda I| = (\alpha_1-\lambda)(\alpha_2-\lambda)\cdots(\alpha_n-\lambda).$$

The characteristic polynomial of $A-KI$ is

$$|A-KI-\lambda I| = |A-(K+\lambda)I|$$

$$= \{\alpha_1-(\lambda+K)\}\{\alpha_2-(\lambda+K)\}\cdots\{\alpha_n-(\lambda+K)\}$$ from (i)

$$= \{(\alpha_1-K)-\lambda\}\{(\alpha_2-K)-\lambda\}\cdots\cdots\{(\alpha_n-K)-\lambda\}$$

which shows that the characteristic roots of $A-KI$ are

$\alpha_1-K, \alpha_2-K, \cdots, \alpha_n-K$.

2(a)(iii) Let $U = \text{Span}\{(1,3,-2,2,3), (1,4,-3,4,2), (2,3,-1,-2,9)\}$
$W = \text{Span}\{(1,3,0,2,1), (1,5,-6,6,3), (2,5,3,2,1)\}$
be the subspace of $\mathbb{R}^5$.
Find the basis and dimension of $U, W, U+W$ and $U \cap W$.

Sol^n: $U+W$ is the space spanned by all six vectors. Hence form the matrix whose rows are the given six vectors and then row reduce to echelon form.

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & -1 & 7 & -2 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - R_1 \\ R_6 \to R_6 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 6 & 0 & -6 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 3R_2 \\ R_5 \to R_5 - 2R_2 \\ R_6 \to R_6 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 6 & 0 & -6 \end{bmatrix} \begin{matrix} R_3 \leftrightarrow R_4 \\ R. \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_5 \to R_5 + R_3 \\ R_6 \to R_6 - 3R_3 \end{matrix}$$

The non-zero rows of the echelon matrix $(1,3,-2,2,3),(0,1,-1,2,-1)\ (0,0,2,0,-2)$ form a basis of $U+W$.

$\therefore \dim(U+3)=3.$

To find the basis and the dimension of U:-

Reduce to echelon form the matrix whose rows span U:

$$\begin{bmatrix}1&3&-2&2&3\\1&4&-3&4&2\\2&3&-1&-2&9\end{bmatrix}\sim\begin{bmatrix}1&3&-2&2&3\\0&1&-1&2&-1\\0&-3&3&-6&3\end{bmatrix}\begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1&3&-2&2&3\\0&1&-1&2&-1\\0&0&0&0&0\end{bmatrix}\ R_3\to R_3+3R_2$$

The two non-zero rows of the echelon matrix $(1,3,-2,2,3)\ (0,1,-1,2,-1)$ form a basis of U

and $\therefore$ therefore $\dim U=2.$

To find the basis and the dimension of W:-

Reduce to echelon form the matrix whose rows span W.

$$\begin{pmatrix}1&3&0&2&1\\1&5&-6&6&3\\2&5&3&2&1\end{pmatrix}\sim\begin{pmatrix}1&3&0&2&1\\0&2&-6&4&2\\0&-1&3&-2&-1\end{pmatrix}\begin{matrix}R_2\to R_2-R_1\\R_3\to R_3-2R_1\end{matrix}$$

$$\sim\begin{pmatrix}1&3&0&2&1\\0&1&-3&2&1\\0&0&0&0&0\end{pmatrix}\begin{matrix}R_2\to \frac{R_2}{2}\\R_3\to R_3-2R_2\end{matrix}$$

The two non-zero rows of the echelon matrix $(1,3,0,2,1)$, $(0,1,-3,2,1)$ form a basis of W. $\therefore \dim W=2$

$\therefore \dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2+2-3$

$= 4-3$

$= 1$

$\therefore \dim(U \cap W) = 1$

2(b) (i) Show that the function $f(x,y)=\begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2\neq 0 \\ 0, & \text{when } x^2+y^2=0 \end{cases}$

is continuous but not differentiable at $(0,0)$.

(ii) If $z=f\left(\frac{x-y}{y}\right)$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

Sol'n: Putting $x=r\cos\theta$, $y=r\sin\theta$; we get

$$|f(x,y)-f(0,0)| = \left|\frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2}-0\right|$$

$$= r|\cos\theta|\,|\cos\theta|\,|\sin\theta|$$

$$\leq r = \sqrt{x^2+y^2}$$

Let $\epsilon>0$ be given choose $\delta=\epsilon$. Then

$|f(x,y)-f(0,0)|<\epsilon$ if $\sqrt{x^2+y^2}<\delta$

Hence f is continuous at the origin

Now $f_x(0,0)=\lim_{h\to 0}[f(h,0)-f(0,0)]/h$

$$=\lim_{h\to 0}\frac{0-0}{h}=0.$$

Similarly $f_y(0,0)=0$

Let, if possible f be differentiable at $(0,0)$. Then

$f(h,k)-f(0,0)=Ah+Bk+\sqrt{h^2+k^2}\,g(h,k)$

where $A=f_x(0,0)$, $B=f_y(0,0)$ and $g(h,k)\to 0$ as

$(h,k)\to(0,0)$ —— ①

$$\therefore \frac{h^2k}{h^2+k^2}=\sqrt{h^2+k^2}\,g(h,k) \Rightarrow g(h,k)=\frac{h^2k}{(h^2+k^2)^{3/2}}$$

Now $\lim_{h\to 0} g(h,mh)=\frac{m}{(1+m^2)^{3/2}}$ $(k=mh)$

$\therefore \lim_{(h,k)\to(0,0)} g(h,k)=\frac{m}{(1+m^2)^{3/2}}$. which is depends on m

and so the limit does not exist. This contradicts ①. Hence f is not differentiable at $(0,0)$.

2(b)(ii) → Given $z = f\left(\frac{x-y}{y}\right)$ —— (1)

Differentiating (1) w.r.t x partially, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{x-y}{y}\right)\left(\frac{1}{y}\right)$$

Differentiating w.r.t y, partially we get

$$\frac{\partial z}{\partial y} = f'\left(\frac{x-y}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xf'\left(\frac{x-y}{y}\right)\cdot\frac{1}{y} + yf'\left(\frac{x-y}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$= \cancel{xf'\left(\frac{x-y}{y}\right)\cdot\frac{1}{y}} - \cancel{\frac{x}{y}f'\left(\frac{x-y}{y}\right)}$$

$$= 0$$

2(c)(ii) A variable plane is parallel to the given plane $x/a + y/b + z/c = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right) + zx\left(\frac{c}{a}+\frac{a}{c}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

Sol: The equation of any plane parallel to the given plane is

$$(x/a) + (y/b) + (z/c) = K \quad \text{——— (1)}$$

The plane (1) meets the axes in A, B and C, so the coordinates of A, B and C are $(aK, 0, 0)$; $(0, bK, 0)$ and $(0, 0, cK)$ respectively.

$\therefore$ The equation of the sphere O, ABC is

$$x^2 + y^2 + z^2 - aKx - bKy - cKz = 0$$

or $$(x^2+y^2+z^2) - K(ax+by+cz) = 0 \quad \text{——— (2)}$$

The equation of the circle ABC are given by (1) and (2) hence eliminating K between (1) and (2), we get the required locus of circle ABC as

$$(x^2+y^2+z^2)-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$$

or $\frac{x}{a}(by+cz)+\frac{y}{b}(ax+cz)+\frac{z}{c}(ax+by)=0$

or $yz\left(\frac{c}{b}+\frac{b}{c}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0$

2(c)(ii) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $1/a+1/b+1/c=0$.

Sol: Let the plane $ax+by+cz=0$ cut the cone $yz+zx+xy=0$ in a line

$$x/l = y/m = z/n. \text{ Then}$$

$$mn+nl+lm=0 \text{ and } al+bm+cn=0$$

Eliminating n between these relations, we get

$$(m+l)\left[-(al+bm)/c\right]+lm=0$$

or $al^2+(a+b-c)lm+bm^2=0$

or $a(l/m)^2+(a+b-c)(l/m)+b=0$

If the roots of this equation are l_1/m_1 and l_2/m_2, then

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \text{product of the roots} = \frac{b}{a}$$

or $\dfrac{l_1 l_2}{1/a} = \dfrac{m_1 m_2}{1/b} = \dfrac{n_1 n_2}{1/c}$, by symmetry.

If these lines are at right angles,

then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

or $(1/a) + (1/b) + (1/c) = 0.$

Hence proved.

3(a)(i) Let $T : \mathbb{C}^4 \to M_{2,2}$ be given by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix}$

Find a basis of $R(T)$. Is T surjective?

Sol'n. The range of T is

$$R(T) = \left\{ \begin{bmatrix} a+b & a+b+c \\ a+b+c & a+d \end{bmatrix} \middle| a,b,c,d \in \mathbb{C} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \middle| a,b,c,d \in \mathbb{C} \right\}$$

$$= \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

These three matrices are linearly independent, So a basis of $R(T)$ is $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Thus T is not Surjective, Since the range has dimension 3 which is shy of $\dim(M_{2,2}) = 4$. (Notice that the range is actually the subspace of symmetric 2×2 matrices in $M_{2,2}$).

3(a)(ii) Determine the values of k so that the following system in unknowns x, y, z has: (i) a unique solution, (ii) no solution, (iii) an infinite number of solutions:

$kx+y+z=1, \ x+ky+z=1, \ x+y+kz=1.$

Sol'n: write the matrix equation of the given system is

$$AX = \begin{bmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = B$$

The augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & k-1 & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - kR_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k \end{array}\right] \quad R_3 \to R_3 + R_2$$

If $2-k-k^2 \neq 0$
i.e, $k \neq 1, -2$
then $\rho(A|B) = \rho(A) = 3 =$ the no. of unknown variables.

$\therefore$ The equations are consistent.
and have unique solution.

If $k=1$, then $\rho(A|B) = \rho(A) = 2 <$ the no of unknown variables.
$\therefore$ The given equations are consistent and have infinite solutions.

If $k=-2$, then $\rho(A|B) = 3$, $\rho(A) = 2$.
$\therefore \rho(A|B) \neq \rho(A)$. $\therefore$ The given equations have no solution.

3(b) By using Lagrange's Multipliers method find the maximum and minimum values of $f(x,y,z) = 3x^2+y$ subject to the constraints $4x-3y=9$ and $x^2+z^2=9$.

Soln: Given that

$f(x,y,z) = 3x^2+y$ —— ①

Subject to the

$4x-3y=9$ —— ②

and $x^2+z^2=9$ —— ③

Let us consider a function F of independent variables x,y,z.

where $F = 3x^2+y+\lambda_1(4x-3y-9)+\lambda_2(x^2+z^2-9)$ —— ④

$dF = (6x+4\lambda_1+2\lambda_2 x)dx+(1-3\lambda_1)dy+2z\lambda_2 dz$

At stationary points, $dF=0$ ($\because dF = F_x dx + F_y dy + F_z dz$) —— ④

$6x+4\lambda_1+2\lambda_2 x = 0$

$\therefore F_x=0 \Rightarrow 6x+4\lambda_1+2\lambda_2 x=0$

$F_y=0 \Rightarrow 1-3\lambda_1=0 \Rightarrow \boxed{\lambda_1=\frac{1}{3}}$

$F_z=0 \Rightarrow 2\lambda_2 z=0 \Rightarrow \boxed{\lambda_2=0}$ (or) $\boxed{z=0}$

Let $z=0$

$\therefore$ from ③ $x=\pm 3$

from ② $y=-7$ (if $x=-3$)

and $y=1$ (if $x=3$)

From this we have two possible extreme $(-3,-7,0)$, $(3,1,0)$.

If $\lambda_2 = 0$ and $\lambda_1 = \frac{1}{3}$,
from (4), we get $6x = \frac{-4}{3}$

$\Rightarrow \boxed{x = \frac{-2}{9}}$

$\therefore$ from (2), $\cancel{3y}\ y = 9 - 4x = 9 - 4\left(-\frac{2}{9}\right)$

$\Rightarrow \boxed{y = -\frac{89}{27}}$ $\left(\because x = -\frac{2}{9}\right)$

from (3), $x^2 + z^2 = 9 \Rightarrow z^2 = 9 - \frac{4}{81}$

$\Rightarrow z^2 = \frac{725}{81}$

$\Rightarrow z = \frac{\pm 5\sqrt{29}}{9}$

$\therefore$ the two more potential absolute extrema

$\left(\frac{-2}{9}, \frac{-89}{27}, \frac{5\sqrt{29}}{9}\right), \left(\frac{-2}{9}, \frac{-89}{27}, \frac{5\sqrt{29}}{9}\right)$

$\therefore$ The maximum value of
$f(x,y,z) = 28$ at $(3,1,0)$

and minimum value of $f(x,y,z) = -\frac{85}{27}$

at $\left(-\frac{2}{9}, \frac{-89}{27}, \pm\frac{5\sqrt{29}}{9}\right)$

3(c) Show that the equation $x^2+y^2+z^2+yz+zx+xy+3x+y+4z+4=0$ represents a surface of revolution and determine the equations of its axis of rotation.

Sol: Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

or $$\begin{vmatrix} 1-\lambda & 1/2 & 1/2 \\ 1/2 & 1-\lambda & 1/2 \\ 1/2 & 1/2 & 1-\lambda \end{vmatrix} = 0 \quad \text{--- (1)}$$

or $(1-\lambda)\left[(1-\lambda)^2-(1/4)\right]-(1/2)\left[(1/2)(1-\lambda)-1/4\right]+(1/2)\left[(1/4)-(1/2)(1-\lambda)\right]=0$

or $(1-\lambda)^3-(3/4)(1-\lambda)+(1/4)=0$

or $4(1-\lambda)^3-3(1-\lambda)+1=0$

or $4\lambda^3-12\lambda^2+9\lambda-2=0$

or $(\lambda-2)(2\lambda-1)^2=0$ or $\lambda=2, 1/2, 1/2$

$\therefore$ We observe that two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents a surface of revolution [either ellipsoid or hyperboloid of revolution]

The central planes are given by

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0$$

i.e. $2x + y + z + 3 = 0$, $x + 2y + z + 1 = 0$,

$$x + y + 2z + 4 = 0.$$

Solving these we get

$$x = -1, \; y = 1, \; z = -2$$

$\therefore$ Centre of the given surface is $(-1, 1, -2)$.

$\therefore$ $d' = u\alpha + v\beta + w\gamma + d$

$$= (3/2)(-1) + (1/2)(1) + (2)(-2) + 4 = -1$$

$\therefore$ The reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

or $(1/2)x^2 + (1/2)y^2 + 2z^2 - 1 = 0$

or $\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{(1/2)} = 1$.

which is an ellipsoid (of revalution), the squares of whose semiaxes are 2, 2, 1/2.

Now putting $\lambda = 2$ in the determinant given by ① and associating each row with l, m, n the direction cosines of the principal axis (or axis of revalution), we have

$$-l + (1/2)m + (1/2)n = 0,$$
$$(1/2)l - m + (1/2)n = 0,$$
$$(1/2)l + (1/2)m - n = 0$$

i.e. $-2l + m + n = 0$, $l - 2m + n = 0$, $l + m - 2n = 0$

and these gives $l = m = n = 1/\sqrt{3}$

$\because l^2 + m^2 + n^2 = 1$.

Now the required axis of rotation (or principal axes) is a line through the centre $(-1, 1, -2)$ of the surface of revalution and direction cosines $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ or direction ratios 1, 1, 1.

$\therefore$ The required equations of the axis of rotation are

$$\frac{x-(-1)}{1} = \frac{y-1}{1} = \frac{z-(-2)}{1}$$

or $\quad x+1 = y-1 = z+2.$

4(a)(i) For the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$. find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A.

Sol^n: we write $A = IAI$ i.e $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

we shall affect every elementary row (column) transformation of the product by subjecting the prefactor (post factor) of A to the same.

operate $C_2 - C_1$, $C_3 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_2 - R_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $C_3 - C_2$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

operate $R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is of the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and

$\rho(A) = 2$.

4(a)(ii) → find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$ and hence, find the matrix

represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol^n: The characteristic equation of the matrix A is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(2-\lambda)\right] - 1(0-0) + 1\left[0 - (1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 4\lambda + 3\right] = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Caley-Hamilton theorem, the matrix A must satisfy its characteristic equation so we must have.

$$A^3 - 5A^2 + 7A - 3I = 0 \longrightarrow (1)$$

To Evaluate

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$\Rightarrow A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I.$$

$$= A^5(0) + A(0) + A^2 + A + I \quad (\because \text{ from } (1))$$

$$= A^2 + A + I.$$

$$\therefore A^2 + A + I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

which is the required matrix.

4(b)(i) Show that the function $f(x,y) = 2x^4 - 3x^2y + y^2$ has neither a maximum nor a minimum at $(0,0)$.

(ii) Evaluate $\int_0^1 (x\ln x)^3\, dx$

(iii) Show that if $a>1$, $\int_0^\infty \frac{x^a}{a^x}\, dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$.

Sol^n: Given $f(x,y) = 2x^4 - 3x^2y + y^2$

$$f_x(x,y) = 8x^3 - 6xy, \quad f_y(x,y) = -3x^2 + 2y$$

$$f_x(0,0) = 0 = f_y(0,0)$$

Now $f_{xx}(x,y)\Big|_{(0,0)} = 24x^2 - 6y = 0$

At $(0,0)$

$$f_{xy}(x,y) = -6x = 0.$$

At $(0,0)$

$$f_{yy}(x,y) = 0$$

$$f_{xx}f_{yy} - (f_{xy})^2 = 0$$

Thus it is a doubtful case and so requires further examination.

we have

$$f(x,y) = (x^2-y)(2x^2-y), \quad f(0,0) = 0$$

$$\therefore f(x,y) - f(0,0) = (x^2-y)(2x^2-y)$$

$$> 0 \text{ for } y<0 \text{ or } x^2>y>0$$

$$< 0 \text{ for } y > x^2 > y/2$$

Note that $f(x,y) - f(0,0)$ does not keep the same sign near the origin. Hence the given function has neither a maximum nor a minimum value at $(0,0)$

(ii) To evaluate $\int_0^1 (x\ln x)^3$

Put $\log x = -t$

$\Rightarrow x = e^{-t}$

$\Rightarrow dx = -e^{-t}\,dt$

Limits As $x \to 0$, $-t \to -\infty$ i.e. $t \to \infty$

As $x \to 1$, $t \to 0$

$$\int_0^1 (x\ln x)^3\,dx = \int_\infty^0 (-te^{-t})^3(-e^{-t})dt$$

$$= -\int_\infty^0 (-t)^3 e^{-3t} e^{-t}\,dt$$

$$= \int_\infty^0 t^3 e^{-4t}\,dt$$

$$= -\int_0^\infty t^3 e^{-4}dt \quad \text{——} ①$$

Now put $4t = z \Rightarrow t = z/4$

$dt = \frac{1}{4}dz$

Limits As $t \to 0$, $z \to 0$

As $t \to \infty$; $z \to \infty$

$$\therefore ① \equiv -\int_0^\infty t^3 e^{-4t}\,dt = -\int_0^\infty \left(\frac{z}{4}\right)^3 e^{-z}\left(\frac{1}{4}dz\right)$$

$$= -\left(\frac{1}{4}\right)^4 \int_0^\infty z^3 e^{-z}\,dz$$

$$= -\frac{1}{256}\int_0^\infty z^{4-1}e^{-z}\,dz$$

$$= -\frac{1}{256}\,3! \qquad \left(\because \Gamma n = \int_0^\infty x^{n-1}e^{-x}dx = n!\ \forall\ n>0.\right)$$

$$= -3/128$$

4(b)(iii) →

$$\int_0^\infty \frac{x^a}{a^x}\,dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$

$\because a = e^{\log a} \quad \therefore a^x = e^{x\log a}$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty \frac{x^a}{e^{x\log a}}\,dx$$

$$= \int_0^\infty e^{-x\log a}\cdot x^a\,dx$$

put $x\log a = z$, i.e $x = \dfrac{z}{\log a}$

So that $dx = \dfrac{dz}{\log a}$

when $x=0, z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx = \int_0^\infty e^{-z}\,\frac{z^a}{(\log a)^a}\cdot\frac{dz}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}\cdot(z)^{a+1-1}\,dz$$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$

4(c): Find the point of intersection P, Q of the generators of opposite system drawn through the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1.$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$(x^2/a^2)+(y^2/b^2)=2, \quad z=\pm c.$$

Sol: Let the coordinates of the point P be (x_1, y_1, z_1).

The equation of the tangent plane to the given hyperboloid at P is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$$

and its meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \quad z=0 \quad \text{——— (1)}$$

which is the same as the line joining the point A and B

i.e. $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right),\ z=0$ ——— (2)

comparing (1) and (2), we get

$$\frac{x_1/a^2}{\frac{1}{a}\cos\frac{\alpha+\beta}{2}}=\frac{y_1/b^2}{\frac{1}{b}\sin\frac{\alpha+\beta}{2}}=\frac{1}{\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}},\quad \frac{y_1}{b}=\frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \qquad \text{——— (3)}$$

Again $(x_1^2/a^2)+(y_1^2/b^2)-(z_1^2/c^2)=1$

or $\left[1/\cos^2\left(\frac{\alpha-\beta}{2}\right)\right]-\frac{z_1^2}{c^2}=1$, substituting values from (3).

or $\frac{z_1}{c^2}=\sec^2\left(\frac{\alpha-\beta}{2}\right)-1=\tan^2\left(\frac{\alpha-\beta}{2}\right)$

or $\frac{z_1}{c}=\pm\frac{\sin\{(\alpha-\beta)/2\}}{\cos\{(\alpha-\beta)/2\}}$ ——— (4)

From (3) and (4) we get the coordinates of $P(x_1,y_1,z_1)$ as

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{\pm c\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \right)$$

Again as A and B are extremities of two semi-conjugate diameters, we have

$$\alpha-\beta = \pi/2 \quad \text{——— (5)}$$

$\therefore$ from (3) we get

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = \frac{1}{\cos^2\{(\alpha-\beta)/2\}}$$

$$= \frac{1}{\cos^2(\pi/4)}, \quad \text{from (5)}$$

or $(x_1^2/a^2)+(y_1^2/b^2) = 2$

And from (4),

$$z_1 = \pm c\tan\left(\frac{\alpha-\beta}{2}\right)$$

$$= \pm c\tan\left(\frac{\pi}{4}\right), \quad \text{from (5)}$$

or $z_1 = \pm c$

$\therefore$ The locus of P and Q are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2, \quad z = \pm c$$

Hence proved.

5(a) (i) solve $x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$

(ii) Solve $(x^2 - 2x + 2y^2)dx + 2xy\,dy = 0$

Sol^n: (i) Re-writing the given equation

$$\left\{\frac{x-y}{(x^2+y^2)}\right\}dx + \left\{\frac{y+x}{x^2+y^2}\right\}dy = 0 \quad \text{--- (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$

$$M = \frac{x-y}{x^2+y^2} \quad \text{and} \quad N = \frac{y+x}{x^2+y^2}$$

$$\therefore \frac{\partial M}{\partial y} = 0 - \frac{1.(x^2+y^2) - y.2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and $$\frac{\partial N}{\partial x} = 0 + \frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence (1) is exact and therefore its solution is

$$\int M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = \tfrac{1}{2}c$$

(Treating y = constant)

$$\Rightarrow \int \left\{ x - y/(x^2+y^2) \right\} dx + \int y\,dy = \tfrac{1}{2}c$$

$$\frac{x^2}{2} - y \times \frac{1}{y}\tan^{-1}(x/y) + \frac{y^2}{2} = \frac{c}{2}$$

$$\Rightarrow x^2 + y^2 - 2\tan^{-1}(x/y) = c$$

5(a)(ii) Rewriting the given equation, we have

$$2xy\frac{dy}{dx}+x^2-2x+2y^2=0$$

$$\Rightarrow 2y\frac{dy}{dx}+\frac{x^2-2x}{x}+\frac{2y^2}{x}=0$$

$$\Rightarrow 2y\frac{dy}{dx}+\frac{2}{x}y^2=\frac{2x-x^2}{x} \quad \text{--- (1)}$$

putting $y^2=v$ so that $2y\left(dy/dx\right)=dv/dx$ — (2)

using (2), (1) gives $\frac{dv}{dx}+\frac{2}{x}v=\frac{2x-x^2}{x}$ — (3)

comparing (3) with $\frac{dv}{dx}+Pv=Q$,

we have $P=\frac{2}{x}$ and $Q=(2x-x^2)/x$ — (4)

$\therefore$ Since $\int P\,dx=\int\frac{2}{x}dx=2\log x=\log x^2$

hence I.F of (3) $=e^{\int P dx}=e^{\log x^2}=x^2$.

and solution of (3) is

$y(\text{I.F})=\int Q\,(\text{I.F})\,dx+c$, c being an arbitrary constant

$$\Rightarrow y^2x^2=\int\left(\frac{2x-x^2}{x}\right)x^2\,dx$$

$$\Rightarrow \int(2x^2-x^3)\,dx+c$$

$$\Rightarrow y^2x^2=\frac{2x^3}{3}-\frac{x^4}{4}+c$$

5(b) (I) Find the Laplace transform of $\frac{1}{\sqrt{\pi t}}$.

(II) show that (i) $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$ (ii) $\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \pi/4$.

Sol'n: (I) $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{\pi}} L\{t^{-1/2}\}$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}, \quad s>0$$

$$\left[\because L(t^n) = \Gamma(n+1)/s^{n+1},\ s>0\right]$$

$$= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{2})}{s^{1/2}} = \frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}}, \quad s>0.$$

$$\left[\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}\right]$$

(II) (i) Here $L\{\sin t\} = \frac{1}{(s^2+1)} = f(s)$, say ——— (1)

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds, \text{ using (1)}$$

$$\Rightarrow L\left\{\frac{\sin t}{t}\right\} = \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s = \frac{\pi}{2} - \tan^{-1}s$$

$$\Rightarrow \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}s. \quad \text{——— (2)}$$

(i) Taking limit of bothsides of (2) as $s\to 0$, we get

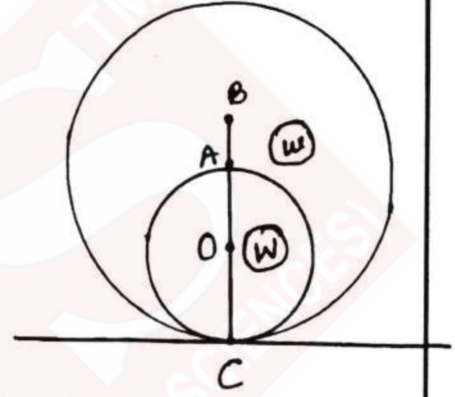
$$\int_0^t \frac{\sin t}{t}\,dt = \pi/2$$

(ii) Taking limit of bothsides of (2) as $s\to 1$, we get

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \pi/4.$$

5(c). A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$

Sol: C is the point of contact of the sphere and the spherical shell, O is the centre of the sphere, CA is the vertical diameter of the sphere and B is the centre of the spherical shell.



We have $OC = a$ and $BC = b$.
The weight W of the sphere acts at O and a particle of weight w is attached to A If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then

$$h = \frac{W\cdot a + w\cdot 2a}{W+w} = \frac{W+2w}{W+w}a$$

Here $\rho_1 = a$ and $\rho_2 = -b$
The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

i.e. $$\frac{1}{h} > \frac{1}{a} - \frac{1}{b}$$

i.e. $$\frac{W+w}{a(W+2w)} > \frac{b-a}{ab}$$

i.e. $(W+w)ab > a(b-a)(W+2w)$

i.e. $(W+w)b > (b-a)(W+2w)$

i.e. $W\{b-(b-a)\} > w\{2(b-a)-b\}$

i.e. $Wa > w(b-2a)$

i.e. $\dfrac{W}{w} > \dfrac{b-2a}{a}$

5(d). A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{gm^2}{\mu^2}\left[e^{-(\mu/m)t}-1+\frac{\mu t}{m}\right]$$

Sol: Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e., in the direction of x decreasing. The weight mg of the particle acts vertically downwards i.e., in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

or, $\dfrac{dv}{dt} = g - \dfrac{\mu}{m}v$, $\qquad \left[\because \dfrac{d^2x}{dt^2} = \dfrac{dv}{dt}\right]$

or, $dt = \dfrac{dv}{g-(\mu/m)v}$

Integrating, we have

$$t = -\frac{m}{\mu}\log\left(g-\frac{\mu}{m}v\right)+A,$$

where A is a constant.

But initially when $t=0$, $v=0$;

$\therefore \; A = (m/\mu) \log g$

$\therefore \; t = -\frac{m}{\mu} \log\left(g - \frac{\mu}{m} v\right) + \frac{m}{\mu} \log g$

or, $t = -\frac{m}{\mu} \log\left\{\frac{g - (\mu/m) v}{g}\right\}$

or, $-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm} v\right)$

or, $1 - \frac{\mu}{gm} v = e^{-\mu t/m}$

or, $v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)$

or, $dx = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right) dt.$

Integrating, we have

$$x = \frac{gm}{\mu}\left[t + \frac{m}{\mu} e^{-\mu t/m}\right] + B \quad \text{——— (1)}$$

where B is a constant.

But initially when $t=0$, $x=0$.

$$\therefore \; 0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B \quad \text{——— (2)}$$

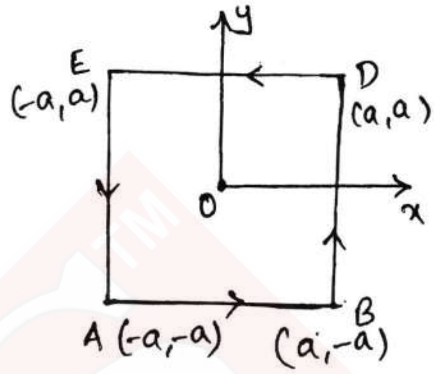
Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu} e^{-\mu t/m} - \frac{m}{\mu} + t\right]$$

$$= \frac{gm^2}{\mu^2}\left[e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right]$$

5(e) → Evaluate $\int_C \frac{-y^3 i + x^3 j}{(x^2+y^2)^2} dr$, where C is the boundary of the square $x = \pm a$, $y = \pm a$ in the counter clockwise sense.

Sol'n: Let $F = \frac{-y^3 i + x^3 j}{(x^2+y^2)^2}$



we have

$$F \cdot dr = \frac{-y^3 i + x^3 j}{(x^2+y^2)^2} \cdot dr$$

$$= \frac{-y^3 i + x^3 j}{(x^2+y^2)^2} \cdot (dx\, i + dy\, j)$$

$$= \frac{-y^3 dx + x^3 dy}{(x^2+y^2)^2}$$

The curve C consists of the four straight lines AB, BD, DE and EA.

Along the line AB, $y=-a$, $dy=0$ and x varies from $-a$ to a.

Along the line BD, $x=a$, $dx=0$ and y varies from $-a$ to a.

Along the line DE, $y=a$, $dy=0$ and x varies from a to $-a$

Along the line EA, $x=-a$, $dx=0$ and y varies from a to $-a$.

we have $\int_C F\cdot dr = \int_{AB} F\cdot dr + \int_{BD} F\cdot dr + \int_{DE} F\cdot dr + \int_{EA} F\cdot dr.$

$$= \int_{x=-a}^{a} \frac{a^3 dx}{(x^2+a^2)^2} + \int_{y=-a}^{a} \frac{a^3 dy}{(y^2+a^2)^2} + \int_{x=a}^{-a} \frac{-a^3 dx}{(x^2+a^2)^2} + \int_{y=a}^{-a} \frac{-a^3 dy}{(y^2+a^2)^2}$$

$$= 4\int_{-a}^{a} \frac{a^3 dx}{(x^2+a^2)^2}$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy = -\int_b^a f(x)\,dx\right]$$

$$= 8a^3 \int_0^a \frac{dx}{(x^2+a^2)^2}$$

$$= 8a^3 \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^4 \sec^4\theta}, \text{ putting } x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$$

$$= 8 \int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= 4 \int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$= 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= 4\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \pi + 2$$

6(a) (i) Find the Wronskian of the set of function $\{3x^3, |3x^3|\}$ on the interval $[-1,1]$ and determine whether the set is linearly dependent on $[-1,1]$.

(ii) Show that the differential equation $(3y^2-x)+2y(y^2-3x)y'=0$ admits an integrating factor which is a function of $(x+y^2)$. Hence solve the equation.

Sol'n: (i) Given functions are $\{3x^3, |3x^3|\}$

Let $f(x)=3x^3$
$g(x)=|3x^3|$

$$\Rightarrow f(x)=\begin{cases}3x^3 & \text{for } x\in[-1,0]\\ 3x^3 & \text{for } x\in[0,1]\end{cases}$$

$$\Rightarrow g(x)=\begin{cases}-3x^3 & \text{for } x\in[-1,0]\\ 3x^3 & \text{for } x\in[0,1]\end{cases}$$

Wronskian of

$$W(f(x),g(x))=\begin{vmatrix} f(x) & g(x)\\ f'(x) & g'(x)\end{vmatrix}$$

for $x\in[-1,0]$

$$W=\begin{vmatrix}3x^3 & -3x^3\\ 9x^2 & -9x^2\end{vmatrix}=0$$

for $x\in[0,1]$

$$W=\begin{vmatrix}3x^3 & 3x^3\\ 9x^2 & 9x^2\end{vmatrix}=0$$

$W=0$ for $x\in[-1,1]$ the given functions are linearly dependent on $[-1,1]$.

6(a)(ii) Here $(3y^2-x)+2y(y^2-3x)dy=0$ —— ①

$$M=3y^2-x \quad ; \quad N=2y^3-6xy$$

$$\frac{\partial M}{\partial y}=6y \neq \frac{\partial N}{\partial x}=-6y$$

$\therefore$ The equation is not an exact solution.

Let $f(x+y^2)$ be an I.F of equation ①, then we have,

$$f(x+y^2)(3y^2-x)dx+f(x+y^2)\times 2y(y^2-3x)dy=0$$

$$\therefore \frac{\partial M}{\partial y}=6y\,f(x+y^2)+(3y^2-x)\,f'(x+y^2)\times 2y$$

$$\frac{\partial N}{\partial x}=(-6y)\,f(x+y^2)+(2y^3-6xy)f'(x+y^2)$$

For the equation to be exact, we must have,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\Rightarrow 6y\,f(x+y^2)+(6y^3-2xy)f'(x+y^2)$$

$$=-6y\,f(x+y^2)+(2y^3-6xy)f'(x+y^2)$$

$$12y\,f(x+y^2)=(2y^3-6xy-6y^3+2xy)f'(x+y^2)$$

$$=(-4y^3-4xy)f'(x+y^2)$$

$$\Rightarrow 12y\,f(x+y^2)=-4y(y^2+x)f'(x+y^2)$$

$$\Rightarrow \frac{f(x+y^2)}{f'(x+y^2)}=\frac{-(x+y^2)}{3}.$$

$$\therefore \frac{f'(x+y^2)}{f(x+y^2)}=\frac{-3}{(x+y)^2}$$

$$\int\frac{f'(x+y^2)}{f(x+y^2)}\,d(x+y^2)=\int\frac{-3}{(x+y)^2}\,d(x+y^2)$$

$$\ln f(x+y^2) = (-3)\log(x+y^2)$$

$$\ln f(x+y^2) = \log(x+y^2)^{-3}$$

$$I.F = f(x+y^2) = \frac{1}{(x+y^2)^3}$$

$$(3y^2-x) \times \frac{1}{(x+y^2)^3}\,dx + \frac{2y(y^2-3x)}{(x+y^2)^3}\,dy = 0.$$

The general solution is:

$$\int M\,dx + \int N\,dy = 0$$

$$\int \frac{(3y^2-x)}{(x+y^2)^3}\,dx + \int 0 = c$$

$$\Rightarrow \int \frac{3y^2-x}{(x+y^2)^3}\,dx = c$$

$$\Rightarrow -\frac{1}{2}\frac{y^2}{(x+y^2)^2} - \frac{1}{(x+y^2)} = c$$

$$\Rightarrow \frac{1}{2}\frac{y^2}{(x+y^2)^2} + \frac{1}{(x+y^2)} = c.$$

6.b) A weight of 60 kg can just rest on a rough inclined plane of inclination 30° to the horizon. When inclination is increased to 60°, find the least horizontal force which will support it. Find also the least force along the plane that will drag it up.

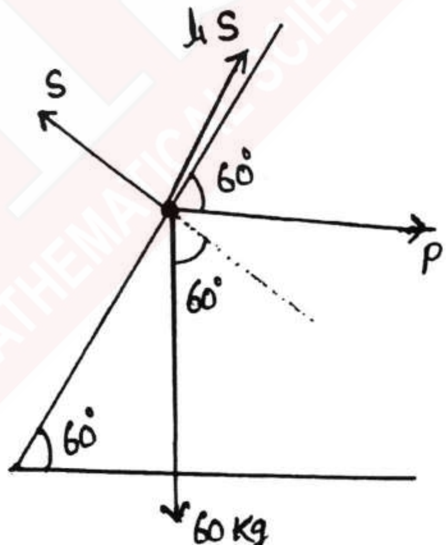
Sol: As the body rests in limiting equilibrium on a plane of inclination 30°.

$$\therefore \lambda = \alpha = 30^\circ$$

$\Rightarrow$ co-efficient of friction $= \mu = \tan\lambda$

$$= \tan 30^\circ = \frac{1}{\sqrt{3}}$$

(i) When the inclination of the plane is increased to 60°, Let S be the normal reaction and P, the least horizontal force which supports the body.



Resolving perpendicular to the plane.

$$S = 60\cos 60^\circ + P\sin 60^\circ$$

$$\Rightarrow S = 60 \times \frac{1}{2} + P \cdot \frac{\sqrt{3}}{2}$$

$$= 30 + \frac{\sqrt{3}}{2}P$$

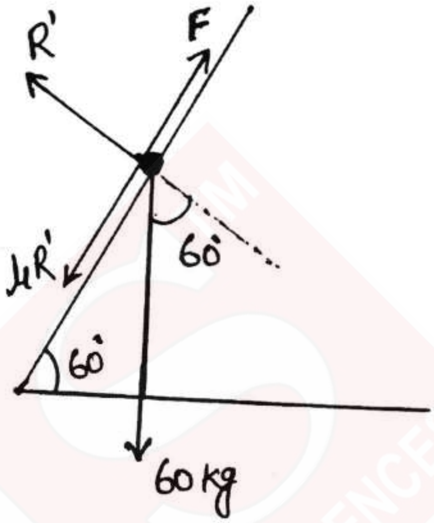
Resolving along the plane

$$\mu S + P\cos 60^\circ = 60\sin 60^\circ$$

$$\Rightarrow \frac{1}{\sqrt{3}}\left(30 + \frac{\sqrt{3}}{2}P\right) + P \cdot \frac{1}{2} = 60 \times \frac{\sqrt{3}}{2}$$

$$\Rightarrow 10\sqrt{3} + \frac{1}{2}P + \frac{1}{2}P = 30\sqrt{3} \qquad \therefore P = 20\sqrt{3}\text{ kg}.$$

(ii) Let F be the least force along the plane that drags the body up the plane so that the force of friction acts down the plane.



Resolving perpendicular to plane

$R' = 60 \cos 60°$

$= 60 \times \frac{1}{2}$

$= 30$ Kg.

Resolving along the plane

$F = \mu R' + 60 \sin 60°$

$= \frac{1}{\sqrt{3}} \times 30 + 60 \times \frac{\sqrt{3}}{2}$

$= 10\sqrt{3} + 30\sqrt{3}$

$= 40\sqrt{3}$ kg.

6(c) → A particle moves so that its position vector is given by $r = \cos\omega t\, i + \sin\omega t\, j$ where ω is a constant. Show that (i) the velocity v of the particle is ⊥ar to r, (ii) the acceleration a is directed toward the origin and has magnitude proportional to the distance from the origin. (iii) $r \times v = a$ constant vector.

Sol'n: (i) $v = \frac{dr}{dt} = -\omega \sin\omega t\, i + \omega\cos\omega t\, j$

then $r.v = [\cos\omega t\, i + \sin\omega t\, j] \cdot [-\omega\sin\omega t\, i + \omega\cos\omega t\, j]$

$= (\cos\omega t)(-\omega\sin\omega t) + (\sin\omega t)(\omega\cos\omega t) = 0$

and r and v are perpendicular.

(ii) $\frac{d^2 r}{dt^2} = \frac{dv}{dt} = -\omega^2\cos\omega t\, i - \omega^2 \sin\omega t\, j$

$= -\omega^2 [\cos\omega t\, i + \sin\omega t\, j] = -\omega^2 r$

Then the acceleration is opposite to the direction of r. i.e. it is directed toward the origin. Its magnitude is proportional to $|r|$ which is the distance from the origin.

(iii) $r \times v = [\cos\omega t\, i + \sin\omega t\, j] \times [-\omega\sin\omega t\, i + \omega\cos\omega t\, j]$

$$= \begin{vmatrix} i & j & k \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$$

$= \omega(\cos^2\omega t + \sin^2\omega t) k = \omega k$, a constant vector.

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration directed toward the centre of the circle, is the centripetal acceleration.

6(d) Show that the Frenet – Serret formula can be written the form

$$\frac{d\vec{T}}{ds} = \vec{\omega} \times \vec{T}, \quad \frac{d\vec{N}}{ds} = \vec{\omega} \times \vec{N} \text{ and } \frac{d\vec{B}}{ds} = \vec{\omega} \times \vec{B}$$

where $\vec{\omega} = \tau\vec{T} \times k\vec{B}$.

Sol'n:
$$\vec{\omega} \times \vec{T} = (\tau\vec{T} + k\vec{B}) \times \vec{T}$$
$$= \tau\vec{T} \times \vec{T} + k\vec{B} \times \vec{T}$$
$$= 0 + k\vec{N} \quad (\because \vec{B} \times \vec{T} = \vec{N})$$
$$= k\vec{N}$$
$$= \frac{d\vec{T}}{ds}$$

$$\vec{\omega} \times \vec{N} = (\tau\vec{T} + k\vec{B}) \times \vec{N}$$
$$= \tau\vec{T} \times \vec{N} + k\vec{B} \times \vec{N}$$
$$= \tau\vec{B} - k\vec{T}$$
$$= \frac{d\vec{N}}{ds}$$

$$\vec{\omega} \times \vec{N} = (\tau\vec{T} + k\vec{B}) \times \vec{B}$$
$$= \tau\vec{T} \times \vec{B} + k\vec{B} \times \vec{B}$$
$$= -\tau\vec{B} + 0$$
$$= -\tau B.$$

7(a)(i) solve $x^3 (d^3y/dx^3) + 2x(dy/dx) - 2y = x^2 \log x + 3x$.

Sol'n: Re-writing, the given equation is

$$(x^3D^3 + 2xD - 2)y = x^2\log x + 3x, \quad D \equiv \frac{d}{dx} \quad \text{—(1)}$$

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$ —(2)

Then $xD = D_1$ and $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$ and so (1) becomes.

$$\{D_1(D_1-1)(D_1-2) + 2D_1 - 2\}y = ze^{2z} + 3e^z$$

$$\Rightarrow (D_1^3 - 3D_1^2 + 4D_1 - 2)y = ze^{2z} + 3e^z$$

Its auxiliary equation is

$$D_1^3 - 3D_1^2 + 4D_1 - 2 = 0$$

$$\Rightarrow (D_1 - 1)(D_1^2 - 2D_1 + 2) = 0$$

$$\Rightarrow D_1 = 1, \; 1 \pm i$$

$$\therefore \text{C.F} = c_1e^z + e^z(c_2\cos z + c_3\sin z)$$

$$= x(c_1 + c_2\cos\log x + c_3\sin\log x)$$

where c_1, c_2 and c_3 are arbitrary constants.

P.I corresponding to ze^{2z}

$$= \frac{1}{D_1^3 - 3D_1^2 + 4D_1 - 2} ze^{2z} = e^{2z}\frac{1}{(D_1+2)^3 - 3(D_1+2)^2 + 4(D_1+2) - 2} z$$

$$= e^{2z}\frac{1}{D_1^3 + 3D_1^2 + 4D + 2} z = \frac{e^{2z}}{2}\left[1 + \frac{D_1^3 + 3D_1^2 + 4D_1}{2}\right]^{-1} z$$

$$= e^{2z}/2 \left\{1 - \tfrac{1}{2}(D_1^3 + 3D_1^2 + 4D_1 + \cdots\right\} z.$$

$$= e^{2z}/2 \left\{z - \tfrac{1}{2} \times 4\right\} = \frac{x^2}{2}(\log x - 2)$$

P.I corresponding to $3e^z$.

$$= 3\frac{1}{(D_1-2)(D_1^2 - 2D_1 + 2)} e^z = 3\frac{1}{D_1 - 1}\frac{1}{1^2 - 2\cdot 1 + 2} e^z = 3\frac{z}{1!}e^z = 3ze^z = 3x\log x$$

$\therefore$ Solution is $y = x(c_1 + c_2\cos\log x + c_2\sin\log x) + \frac{x^2}{2}(\log x - 2) + 3x\log x.$

7(a)(ii) Apply the method of variation of parameters to solve $(1-x)y_2 + xy_1 - y = (1-x)^2$.

Sol'n: Dividing by $(1-x)$ and re-writing, the given equation becomes

$$y_2 - \left\{\frac{x}{x-1}\right\}y_1 + \left\{\frac{1}{(x-1)}\right\}y = -(x-1) \quad \text{— (1)}$$

Consider $y_2 - \left\{\frac{x}{(x-1)}\right\}y_1 + \left\{\frac{1}{(x-1)}\right\}y = 0$ — (2)

Comparing (2) with $y_2 + Py_1 + Qy = 0$, $P = (-x)/(x-1)$ and $Q = \frac{1}{(x-1)}$.

$$\therefore P + Qx = (-x)/(x-1) + \frac{x}{(x-1)} = 0 \text{ and}$$

$$1 + P + Q = 1 + (-x)/(x-1) + \frac{1}{(x-1)} = 0$$

Hence by working rule,
we see that x and e^x are integrals of C.F of (1) (or) solutions of (2). Again the Wronskian W of x & e^x is given by

$$W = \begin{vmatrix} x & e^x \\ dx/dx & d(e^x)/dx \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0 \quad \text{— (3)}$$

showing that x and e^x are linearly independent solutions of (2). Hence the general solution of (2) is $y = ax + be^x$ and therefore C.F of (1) is $ax + be^x$, a & b being arbitrary constants.

Comparing (1) with $y_2 + Py_1 + Qy = R$ here

$$R = -(x-1) \quad \text{— (4)}.$$

Let $u = x$ and $v = e^x$ — (5).

Then, P.I of ① $= uf(x) + vg(x)$ —— ⑥

where $f(x) = -\int \frac{vR}{W}dx = \int \frac{e^x(x-1)}{e^x(x-1)}dx = \int dx = x$ by ③ ④ & ⑤

and $g(x) = \int \frac{uR}{W}dx = -\int \frac{x(x-1)}{e^x(x-1)}dx = -\int xe^{-x}dx$

$$= -\left\{x(-e^{-x}) - \int(-e^{-x})dx\right\}$$

$$= -(-xe^{-x} - e^{-x}) = e^{-x}(x+1)$$

Substituting the above values of $u, v, f(x)$ & $g(x)$ in ⑥

we have

P.I of ⑦ $= x \cdot x + e^x \cdot e^{-x}(x+1) = x^2 + x + 1$.

Hence the general solution of ① is

$y = C.F + P.I$,

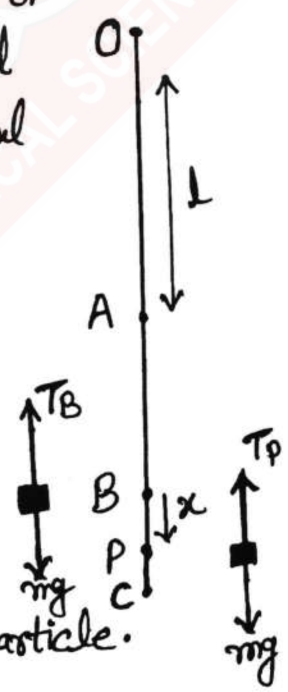
i.e. $y = ax + be^x + x^2 + x + 1$; a & b being arbitrary constants.

7(b) A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that

$$g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$$

Sol: One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m.

Let $AB = d$.



In the equilibrium position of the particle of mass m at B, the tension $T_B = \lambda(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l} = mg \quad \text{or} \quad \frac{\lambda}{lm} = \frac{g}{d} \quad \text{———} \ (1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP = x$. When the particle is at P, the tension T_P in the string P is $\lambda \frac{d+x}{l}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m \frac{d^2x}{dt^2} = \frac{\lambda(d+x)}{l} + mg,$$

[NOTE that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards i.e., in the direction of x-increasing.]

or
$$m \frac{d^2x}{dt^2} = \frac{\lambda d}{l} - \frac{\lambda x}{l} + mg$$

$$= -\frac{\lambda x}{l}, \quad \left[\because \frac{\lambda d}{l} = mg\right]$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{lm} \cdot x = -\frac{g}{d} x, \quad [\text{from } ①]$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\frac{2\pi}{\sqrt{(g/d)}}$ i.e., $2\pi\sqrt{\left(\frac{d}{g}\right)}$

But according to the question, the period is t_1 when $d = c_1$ and the period is t_2 when $d = c_2$

$$\therefore t_1 = 2\pi\sqrt{(c_1/g)} \text{ and } t_2 = 2\pi\sqrt{(c_2/g)},$$

so that $t_1^2 - t_2^2 = (4\pi^2/g)(c_1 - c_2)$

or $g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$.

7(c)(i)) Find the constants a and b so that the surface $ax^2-byz=(a+2)x$. will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$.

Sol^n: The given surfaces are

$$f_1 = ax^2 - byz - (a+2)x = 0 \quad \text{--- (1)}$$

and $f_2 = 4x^2y + z^3 - 4 = 0$ ——— (2)

The point $(1,-1,2)$ obviously lies on the surface (2). It will also lie on the surface (1) if

$$a + 2b - (a+2) = 0 \Rightarrow 2b - 2 = 0$$
$$\Rightarrow \boxed{b = 1}$$

Now $\text{grad } f_1 = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$.

and $\text{grad } f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$.

Then $n_1 = \text{grad} f_1$ at the point $(1,-1,2)$

$$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$$

and $n_2 = \text{grad } f_2$ at the point $(1,-1,2)$

$$= -8\hat{i} + 4\hat{j} + 12\hat{k}$$

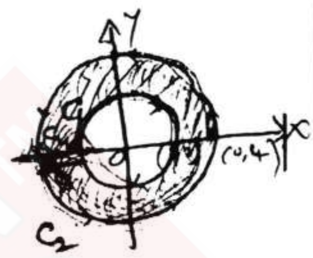
The vectors n_1 and n_2 are along the normals to the surfaces (1) and (2) at the point $(1,-1,2)$.

These surfaces will intersect orthogonally at the point $(1,-1,2)$ if the vectors n_1 and n_2 perpendicular i.e, if $n_1 \cdot n_2 = 0$

$$\Rightarrow -8(a-2) - 8b + 12b = 0$$
$$\Rightarrow b - 2a + 4 = 0$$
$$\Rightarrow 1 - 2a + 4 = 0 \quad (\because b = 1)$$
$$\Rightarrow \boxed{a = 5/2}$$

$\therefore a = 5/2, \ b = 1$.

7(c)(iii) verify Green's theorem in the plane for $\oint_C x^2y\,dx + (y^3 - xy^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2+y^2=4$, $x^2+y^2=16$



Sol^n. By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{--- (1)}$$

The boundary of the curve C is given by
$C: C_1 + C_2 + C_3 + C_4$ and R is the region bounded by the circles $x^2+y^2=4$ and $x^2+y^2=16$.

$$L.H.S. = \int_C M\,dx + N\,dy = \int_{C_1+C_2+C_3+C_4} M\,dx + N\,dy \quad \text{--- (2)}$$

Here note that along C_3 & C_4

$$\int_{C_3+C_4} M\,dx + N\,dy = 0 \qquad \left[\because C_3 \,\&\, C_4 \text{ are in opposite directions, i.e. } \int_{C_3} = -\int_{C_4}\right]$$

$\therefore$ from (1), we have

$$\int_C M\,dx + N\,dy = \int_{C_1+C_2} M\,dx + N\,dy$$

$$= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$

Let $\int_{C_1} M\,dx + N\,dy = \int_{C_1} x^2y\,dx + (y^3 - xy^2)\,dy$

putting $x = 2\cos\theta$, $y = 2\sin\theta$
$\Rightarrow dx = -2\sin\theta\,d\theta$ & $dy = 2\cos\theta\,d\theta$.

$$\therefore \int_{C_1} M\,dx + N\,dy = \int_{2\pi}^{0} 4\cos^2\theta \cdot (+2\sin\theta)(-2\sin\theta)\,d\theta + \int_{2\pi} \left(8\sin^3\theta - 2\cos\theta \cdot 4\sin^2\theta\right) 2\cos\theta\,d\theta$$

$$= \int_{2\pi}^{0} -32\cos^3\theta\sin^2\theta\, d\theta + \int_{2\pi}^{0} 16\sin^3\theta\cos\theta\, d\theta$$

$$= \int_{0}^{2\pi} 8\sin^2 2\theta\, d\theta + 16\left[\frac{\sin^4\theta}{\theta}\right]_{2\pi}^{0}$$

$$= 8\left[\frac{1-\cos 4\theta}{2}\right]_0^{2\pi} = 8\pi.$$

$$\int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} -256\cos^2\theta\sin^2\theta\, d\theta + \int_0^{2\pi}(64\sin^3\theta - 4\cos\theta\, 16\sin^2\theta)\, 4\cos\theta\, d\theta$$

by putting $x = 4\cos\theta$; $dx = -4\sin\theta\, d\theta$
$y = 4\sin\theta$; $dy = 4\cos\theta\, d\theta$

$$= -64\int_0^{2\pi} \sin^2 2\theta\, d\theta + 256\int_0^{2\pi} \sin^3\theta\cos\theta\, d\theta - 64\int_0^{2\pi}$$

$$= -128\int_0^{2\pi} \sin^2 2\theta + \left[\frac{\sin^4\theta}{4}\right]_0^{2\pi}(256)$$

$$= -128\int_0^{2\pi}\left[\frac{1-\cos 4\theta}{2}\right]$$

$$= -\frac{128}{2}[\theta]_0^{2\pi}$$

$$= -128\pi$$

$\therefore$ from (2)

$$\oint_C M\,dx + N\,dy = 8\pi - 128\pi = -120\pi$$

$M = x^2 y$ $\qquad N = y^3 - xy^2$

$\Rightarrow \dfrac{\partial M}{\partial y} = x^2$ $\qquad \dfrac{\partial N}{\partial x} = -y^2$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint -y^2 - x^2\, dx\,dy$$

$$= -\iint (x^2+y^2)\, dx\,dy$$

$x = r\cos\theta$; $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

$$= -\int_0^{2\pi}\int_{r=2}^{4} r^2 \cdot r\,d\theta\,dr$$

$$= -\int_0^{2\pi}\left[\frac{r^4}{4}\right]_2^4 d\theta$$

$$= -\int_0^{2\pi}\left[\frac{256-16}{4}\right] d\theta$$

$$= -\int_0^{2\pi} 60\,d\theta$$

$$= -60\,[\theta]_0^{2\pi}$$

$$= -60\,(2\pi)$$

$$= -120\pi.$$

$\therefore$ Hence Green's theorem is verified.

8(a)(i) → Find the general and singular solution of $y^2(y-xp)=x^4p^2$

Sol'n: Given that $y^2(y-xp)=x^4p^2$ —— (1)

Let $x=1/u$ and $y=1/v$

$\Rightarrow dx=-\frac{1}{u^2}du$ and $dy=-\frac{1}{v^2}dv$.

$\therefore \frac{dy}{dx}=\frac{u^2}{v^2}\frac{dv}{du}$

$\Rightarrow p=\frac{u^2}{v^2}P$ where $p=dy/dx$ & $P=\frac{dv}{du}$

Using the values of P, x and y, given equation becomes

$$\frac{1}{v^2}\left(\frac{1}{v}-\frac{1}{u}\frac{u^2}{v^2}P\right)=\frac{1}{u^4}\frac{u^4}{v^4}P^2$$

$$\Rightarrow \frac{1}{v}-\frac{u}{v^2}P=\frac{1}{v^2}P^2$$

$$\Rightarrow v^2\left(\frac{1}{v}-\frac{u}{v^2}P\right)=P^2$$

$$\Rightarrow v-uP=P^2$$

$\Rightarrow v=uP+P^2$ which is of Clairaut's form.

and hence its solution is

$$v=uc+c^2$$

i.e. $\frac{1}{y}=\frac{c}{x}+c^2$

$$\Rightarrow \boxed{x=cy+c^2xy}$$

$\Rightarrow xyc^2+yc-x=0$ —— (2)

which is a quadratic equation in c and so its c-discriminant relation is $y^2-4(xy)(-x)=0$

$\Rightarrow y(y+4x^2)=0$

Now, $y=0$ gives $p=dy/dx=0$. These values satisfy (1). So $y=0$ is a singular solution. Again $y=-4x^2$ gives

$p=dy/dx=-8x$.

These values satisfy (1). Hence $y+4x^2=0$ is also singular solution.

8 (a)(ii) Solve $(D^2+n^2)y = a\sin(nt+\alpha)$, if $y = Dy = 0$ when $t=0$.

sol'n: Given that $y'' + n^2 y = a\sin(nt+\alpha)$

i.e. $y'' + n^2 y = a(\sin nt\cos\alpha + \cos nt\sin\alpha)$ —— (1)

with initial conditions: $y(0) = 0$ and $y'(0)=0$. —— (2)

Taking Laplace transform of bothsides of (1), we have

$L\{y''\} + n^2 L\{y\} = a\cos\alpha\, L\{\sin nt\} + a\sin\alpha\, L\{\cos nt\}$

$\Rightarrow s^2 L\{y\} - sy(0) - y'(0) + n^2 L\{y\} = (an\cos\alpha)/(s^2+n^2)$
$+ (as\sin\alpha)/(s^2+n^2)$

$\Rightarrow (s^2+n^2)L\{y\} = (an\cos\alpha + as\sin\alpha)/(s^2+n^2)$, by (2)

$\Rightarrow L\{y\} = (an\cos\alpha)(s^2+n^2)^2 + (as\sin\alpha)/(s^2+n^2)^2$ —— (3)

Taking inverse Laplace transform of bothsides of (3), we get

$$y = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha\, L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} \quad \text{—— (4)}$$

Now,

$$L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = -\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+n^2}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1\, t\, L^{-1}\left\{\frac{1}{s^2+n^2}\right\}.$$

Thus $L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = \frac{t}{2n}\sin nt$ —— (5)

Let $f(s) = \frac{1}{(s^2+n^2)}$ and $g(s) = \frac{1}{(s^2+n^2)}$ —— (6)

Then $F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s^2+n^2)}\right\} = \frac{1}{n}\sin nt$

and $G(t) = L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{(s^2+n^2)}\right\} = \frac{1}{n}\sin nt$ —— (7)

Now, by the convolution theorem, we have

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} = \int_0^t \frac{\sin nu}{n}\cdot\frac{\sin n(t-u)}{n}\,du \text{ by } ⑥ \,\&\, ⑦$$

$$= \frac{1}{2n^2}\int_0^t [\cos n(t-2u) - \cos nt]\,du$$

$$= \frac{1}{2n^2}\int_0^t [\cos n(t-2u) - \cos nt]\,du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} - t\cos nt + \frac{\sin nt}{2n}\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right] \quad ⑧$$

Using ⑤ and ⑧, ④ reduces for

$$y = an\cos\alpha\cdot\frac{1}{2n^2}\left(\frac{\sin nt}{n} - t\cos nt\right) + a\sin\alpha\cdot\frac{1}{2n}t\sin nt$$

$$\Rightarrow y = \frac{a}{2n^2}\cos\alpha\sin nt - \frac{at}{2n}(\cos\alpha\cos nt - \sin\alpha\sin nt)$$

$$\Rightarrow y = \frac{a}{2n^2}[\cos\alpha\sin nt - nt\cos(\alpha+nt)]$$

8(b) → A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.

Sol^n: Take the centre of force O as origin. Suppose a particle starts from rest at A, where OA = a. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where OP = x. The acceleration of the particle at P is $\mu a^{5/3} x^{-2/3}$ directed towards O.

∴ the equation of the particle is

$$\frac{d^2x}{dt^2} = -\mu a^{5/3} x^{-2/3} \quad \text{——} \; (1)$$

Multiplying bothsides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu^{5/3} x^{1/3}}{1/3} + K = -6\mu a^{5/3} x^{1/3} + K$$

K = constant

At A, $x = a$ & $\frac{dx}{dt} = 0$, so that

$$-6\mu a^{5/3} a^{1/3} + K = 0 \Rightarrow K = 6\mu a^2.$$

$$\therefore (dx/dt)^2 = -6\mu a^{5/3} x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}(a^{1/3} - x^{1/3}) \quad \text{——} \; (2)$$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x = 0$ and $(dx/dt)^2 = v_1^2$. So from (2) we have

$$v_1^2 = 6\mu a^{5/3}(a^{1/3} - 0) = 6\mu a^2$$

$$\Rightarrow v_1 = a\sqrt{6\mu}$$

Now taking square root of ②, we get

$$\frac{dx}{dt} = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3}-x^{1/3})}$$

where the $-$ve sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}} \quad \text{——— ③}$$

Let t_1 be the time from A to O. Then integrating ③ from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}} \int_a^0 \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}$$

$$= \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^a \frac{dx}{\sqrt{(a^{1/3}-x^{1/3})}}$$

Put $x = a\sin^6\theta$, so that $dx = 6a\sin^5\theta\cos\theta\, d\theta$. when $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore t_1 = \frac{1}{\sqrt{(6\mu a^{5/3})}} \int_0^{\pi/2} \frac{6a\sin^5\theta\cos\theta\, d\theta}{a^{1/6}\cos\theta}$$

$$= \sqrt{\left(\frac{6}{\mu}\right)} \int_0^{\pi/2} \sin^5\theta\, d\theta$$

$$= \sqrt{\left(\frac{6}{\mu}\right)} \frac{4.2}{5.3.1}$$

$$= \frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$$

8(c) If $A = 2yz\,i - (x+3y-2)\,j + (x^2+z)\,k$, evaluate $\iint_S (\nabla\times A).n\,ds$ over the surface if intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant.

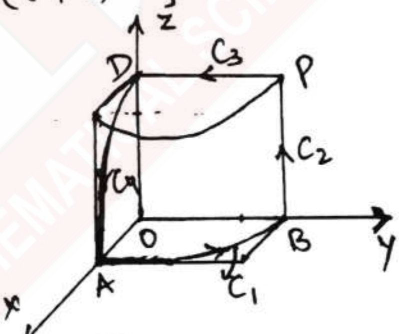
Sol'n: By Stoke's theorem

$$\iint_S (\nabla\times\vec{A}).\hat{n}\,ds = \int_C \vec{A}.d\vec{r} \quad \text{—— (1)}.$$

Here C is the curve consisting of four arcs namely $C_1: AB$, $C_2: BP$, $C_3: PD$, $C_4: DA$. Then we evaluate RHS of (1), along these four arcs one by one.

Along C_1: $z=0$, $x^2+y^2=a^2$, y varies from 0 to a.

$$\int_{C_1} \vec{A}.d\vec{r} = \int_{C_1} [2yz\,dx - (x+3y-2)\,dy + (x^2+z)\,dz]$$



$$= -\int_{C_1} (x+3y-2)\,dy$$

$$= -\int_0^a \left[\sqrt{a^2-y^2} + 3y - 2\right] dy$$

$$= -\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{2} + \frac{3}{2}y^2 - 2y\right]_0^a$$

$$\int_{C_1} \vec{A}.d\vec{r} = -\frac{\pi a^2}{2} - \frac{3a^2}{2} + 2a \quad \text{—— (2)}.$$

Along C_2: $x=0$, $y=a$; $dx=0$; $dz=0$ and y varies from 0 to a.

$$\therefore \int_{C_2} \vec{A}.d\vec{r} = \int_{C_2} z\,dz = \int_0^a z\,dz = \left[\frac{z^2}{2}\right]_0^a = \frac{a^2}{2} \quad \text{—— (3)}$$

Along C_3: $x=0$, $z=a$; $dx=0$; $dz=0$ and y varies from a to 0.

$$\therefore \int_{C_3} \vec{A}.d\vec{r} = \int_{C_3} (3y-2)\,dy = -\int_0^a (3y-2)\,dy$$

$$= \left[-\frac{3y^2}{2} + 2y\right]_a^0 = \frac{3a^2}{2} - 2a \quad \text{--- (4)}$$

Along C_4: $y=0$, $x^2+z^2=a^2$, z varies from a to 0.

$$\therefore \int_{C_4} \vec{A}\cdot d\vec{r} = \int_a^0 (x^2+z)\,dz = \int_a^0 (a^2-z^2+z)\,dz$$

$$= \left(a^2 z - \frac{1}{3}z^3 + \frac{z^2}{2}\right)_a^0 = \frac{-2a^3}{3} - \frac{a^2}{2} \quad \text{--- (5)}$$

Thus, the desired integral is sum of (2),(3),(4),(5)

i.e. $$\iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds = -\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2a^3}{3} - \frac{a^2}{2}$$

$$= -\frac{\pi a^2}{4} - \frac{2a^3}{3}$$

$$= -\frac{a^2}{12}(3\pi + 8a)$$