MAINS TEST SERIES-2021

TEST-15 (BATCH-I)

FULL SYLLABUS (PAPER-I)

Answer Key

1.(a) Let V be the vector space of functions from R into R. Show that $f, g, h \in V$ are linearly independent where $f(t) = e^{2t}$, $g(t) = t^2$, $h(t) = t$.

Soln: Set a linear combination of the functions equal to the zero function 0 using unknown scalars x, y and z.

$$xf + yg + zh = 0$$

and then show that $x=0,\ y=0,\ z=0$.

We emphasize that $xf + yg + zh = 0$ means that, for every value of t,

$$xf(t) + yg(t) + zh(t) = 0.$$

In the equation

$$xe^{2t} + yt^2 + zt = 0, \quad \text{substitue}$$

$t=0$ to obtain $xe^0 + y0 + z0 = 0$ or $x = 0$

$t=1$ to obtain $xe^2 + y + z = 0$

$t=2$ to obtain $xe^4 + 4y + 2z = 0$

Solve the system $\begin{cases} x = 0 \\ xe^2 + y + z = 0 \\ xe^4 + 4y + 2z = 0 \end{cases}$

to obtain only the zero solution: $x=0, y=0, z=0$.

Hence f, g and h are independent

1.(b) Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$. Determine whether b is in the column space of A.

Solⁿ: The vector b is a linear combination of the columns of A if and only if b can be written as Ax for some x, that is, if and only if the equation $Ax = b$ has a solution.

Row reducing the augmented matrix $[A\ b]$,

$$\begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that $Ax = b$ is consistent and b is in col A.

The above solution shows that when a system of linear equations is written in the form $Ax = b$, the column space of A is the set of all b for which the system has a solution.

1.(C) Find the asymptotes of the curve:

$$x^3+2x^2y+xy^2-x^2-xy+2=0.$$

Solⁿ: $\phi_3(x,y)=x^3+2x^2y+xy^2$, $\phi_2(x,y)=-x^2-xy$, and $\phi_1(x,y)=0$.

Let the asymptotes be given by $y=mx+c$.

Step 1. Putting $x=1$ and $y=m$, we obtain

$$\phi_3(m)=1+2m+m^2,\ \phi_2(m)=-1-m,\ \phi_1(m)=0.$$

Step 2. The slopes of asymptotes are given by $\phi_3(m)=0$.

i.e., $1+2m+m^2=0$

$\Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1.$

Step 3. We have $c=-\dfrac{\phi_2(m)}{\phi_3'(m)}=\dfrac{1+m}{2+2m}$

We notice that c is $\frac{0}{0}$ form for $m=-1$.

Now c is determined by

$$\frac{c^2}{2}\phi_3''(m)+c\,\phi_2'(m)+\phi_1(m)=0.$$

or $\dfrac{c^2}{2}\times 2+c(-1)+0=0$

or $c^2-c=0 \Rightarrow c=0,1.$

Hence the two asymptotes corresponding to $m=-1,-1$ are

$$y=-x+0 \quad\text{and}\quad y=-x+1$$

or $x+y=0$ and $x+y=1$.

1.(d) Evaluate $\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\, dy$

by applying the transformation $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and integrating over an appropriate region in the uv-plane.

Soln: We know that

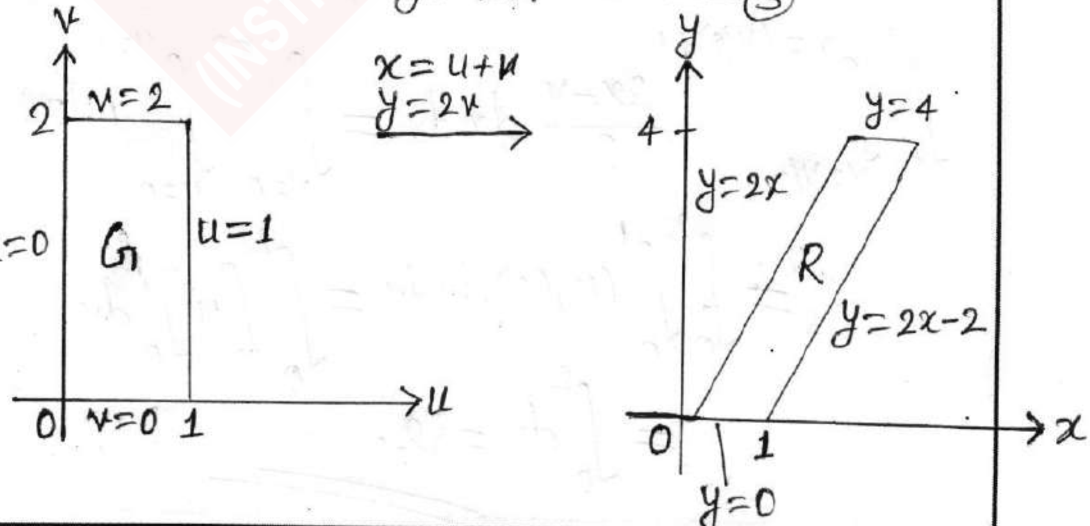
$$\iint_R f(x,y)\, dx\, dy = \iint_G f(g(u,v), h(u,v))\, |J(u,v)|\, du\, dv. \quad \text{—— (1)}$$

and given that $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ —— (2)

Now we sketch the region R of integration in the xy-plane and identify its boundaries.

To apply eqs. (1), we need to find the corresponding uv-region G and the jacobian of the transformation. To find them, we first solve eqs. (2) for x and y in terms of u and v. Routine algebra gives

$$x = u+v, \quad y = 2v. \quad \text{—— (3)}$$



We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R.

xy-equations for the boundary of R	corresponding uv-equations for the boundary of G	simplified uv-equations.
$x = y/2$	$u+v = 2v/2 = v$	$u=0$
$x = (y/2)+1$	$u+v = (2v/2)+1 = v+1$	$u=1$
$y = 0$	$2v = 0$	$v=0$
$y = 4$	$2v = 4$	$v=2$

The Jacobian of the transformation is

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply eq. (1):

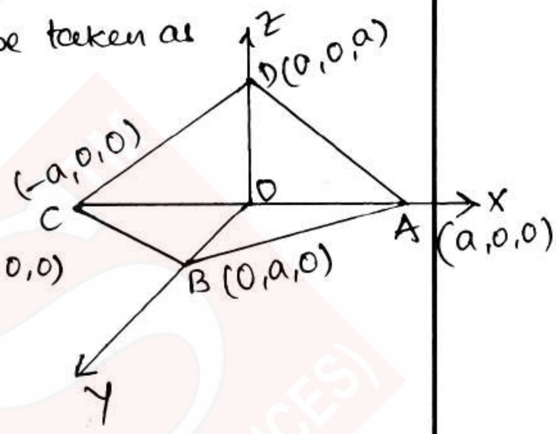
$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\, dx\,dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\,|J(u,v)|\, du\,dv$$

$$= \int_0^2 \int_0^1 (u)(2)\,du\,dv = \int_0^2 \left[u^2\right]_0^1 dv$$

$$= \int_0^2 dv = 2.$$

1(e) A Square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the S.D between DC and AB.

Sol^n: Let O the Centre of the Square be taken as the origin and OA, OB and OD be taken as x, y and z-axis respectively.



Then the coordinates of A, B, C and D are $(a,0,0)$, $(0,a,0)$, $(-a,0,0)$ and $(0,0,a)$ respectively.

Therefore equations of AB are

$$\frac{x-a}{a} = \frac{y-0}{-a} = \frac{z-0}{0} \quad \text{—} ①$$

Equations of DC are $\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-a}{a}$ — ②

Now any point on the line DC is $(0,0,a)$ and the equation of the plane through the line AB and parallel to DC.

i.e, through ① and parallel to ② is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ a & -a & 0 \\ a & 0 & a \end{vmatrix} = 0$$

$$\Rightarrow (x-a)(-a)(a) - y(a.a) + z(a.a) = 0$$

$$\Rightarrow -(x-a) - y + z = 0 \Rightarrow x+y-z-a=0 \quad \text{—} ③$$

$=$ length of $\perp$ar from $(0,0,a)$ to the plane ③

$$= \frac{|0+0-a-a|}{\sqrt{1^2+1^2+(-1)^2}}$$

$$= \frac{2a}{\sqrt{3}}.$$

2.a(i) Find the dimension and a basis of the solution space W of the system

$$x+2y-4z+3r-s=0$$
$$x+2y-2z+2r+s=0$$
$$2x+4y-2z+3r+4s=0$$

Sol^n: Reduce the system to echelon form:

$$x+2y-4z+3r-s=0$$
$$2z-r+2s=0$$
$$6z-3r+6s=0$$

and then $x+2y-4z+3r-s=0$
$\qquad 2z-r+2s=0$

There are five unknowns and two (nonzero) equations in echelon form.

Hence there are $5-2=3$ free variables. y, r and s. Thus $\dim W=3$.

Set (i) $y=1, r=0, s=0$ to obtain the solution

$$v_1=(-2,1,0,0,0)$$

(ii) $y=0, r=2, s=0$ to obtain the solution

$$v_2=(-2,0,1,2,0)$$

(iii) $y=0, r=0, s=1$ to obtain the solution

$$v_3=(-3,0,-1,0,1)$$

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space W.

2.b(i) For what value of a does $\frac{\sin 2x + a\sin x}{x^3}$ tends to a finite limit as $x \to 0$?

Sol^n: $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} \left(\frac{2+a}{0}\right)$$

To get the finite limit, we take

$$a + 2 = 0 \Rightarrow a = -2.$$

Thus $\lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$

$$= \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6} = -1.$$

Hence for $a = -2$, $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = -1.$

2.b(ii) If $I_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x \, dx$, prove that

$$I_{p,q} = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \frac{p-5}{p+q-4} \cdots \frac{2}{3+q} \cdot \frac{1}{1+q},$$

Where p is an odd positive integer and q is a positive integer, even or odd.

Solⁿ: $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{p-1}{p+q} \int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx$$

$$\therefore I_{p,q} = \frac{p-1}{p+q} I_{p-2,q}.$$

Replacing p by $p-2, p-4, \ldots, 3$; we obtain

$$I_{p-2,q} = \frac{p-3}{p+q-2} I_{p-4,q}$$

$$I_{p-4,q} = \frac{p-5}{p+q-4} I_{p-6,q}$$

$$\vdots$$

$$I_{3,q} = \frac{2}{3+q} I_{1,q}; \text{ if } p \text{ is odd.}$$

Now $$I_{1,q} = \int_0^{\pi/2} \sin x \cos^q x \, dx = -\left| \frac{\cos^{q+1} x}{q+1} \right|_0^{\pi/2} = \frac{1}{q+1}.$$

Hence from these relations, we obtain

$$I_{p,q} = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \frac{p-5}{p+q-4} \cdots \frac{2}{3+q} \cdot \frac{1}{1+q}.$$

2.(c)(1) Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{-z}{6}$.

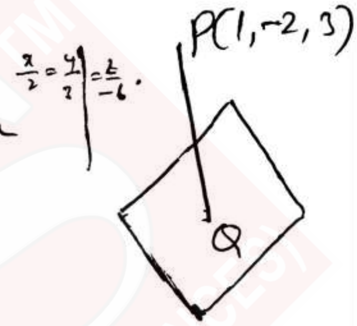
Sol: The given line is $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ —— ①

Its direction ratios are proportional to $2, 3, -6$.

Dividing each by $\sqrt{4+9+36}=\sqrt{49}=7$

the actual d.c.'s of the line ① are

$\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$.



Now equations of the line through $(1,-2,3)$ and parallel to ① are

$$\frac{x-1}{2/7}=\frac{y+2}{3/7}=\frac{z-3}{-6/7}=r \text{ (say)} \quad \text{—— ②}$$

Now any point on the line ②

$$Q\left(\frac{2}{7}r+1, \frac{3}{7}r-2, -\frac{6}{7}r+3\right) \quad \text{—— ③}$$

If this lies on the plane $x-y+z=5$.

then $\frac{2}{7}r+1-\frac{3}{7}r+2-\frac{6}{7}r+3=5$

$\Rightarrow -r+6=5$

$\Rightarrow r=1$

Substituting this value of r in ①, the point is $Q\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$ and therefore the required distance of this point from the given point $(1,-2,3)$

$$= \sqrt{\left(\frac{9}{7}-1\right)^2+\left(-\frac{11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = 1$$

2.(c)(ii) Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circle.

Sol: The centre of the given sphere is $\left(\frac{1}{2},0,-\frac{1}{2}\right)$

and its radius $=\sqrt{\left(\frac{1}{2}\right)^2+(0)^2+\left(-\frac{1}{2}\right)^2-(-2)}=\sqrt{\frac{5}{2}}=R$ (say)

Also length of perpendicular from $\left(\frac{1}{2},0,-\frac{1}{2}\right)$ to

$x+2y-z-4=0$ is $\dfrac{\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4}{\sqrt{2^2+2^2+(-1)^2}}=\frac{3}{\sqrt{6}}=p$ (say)

Then radius of the circle $=\sqrt{R^2-p^2}=\sqrt{\frac{5}{2}-\frac{9}{6}}=1$

The equations of the circle are

$$x^2+y^2+z^2-x+z-2=0,\quad x+2y-z-4=0.$$

$\therefore$ The equation of a sphere through this circle

is $(x^2+y^2+z^2-x+z-2)+\lambda(x+2y-z-4)=0$

$$\Rightarrow x^2+y^2+z^2+(\lambda-1)x+2\lambda y+(1-\lambda)z-(2+4\lambda)=0 \quad (1)$$

Its centre is $\left[-\frac{1}{2}(\lambda-1),\ -\lambda,\ -\frac{1}{2}(1-\lambda)\right]$.

If this circle is a great circle of the sphere (1), then the centre of (1) should lie on the plane of the circle. i.e., the plane $x+2y-z-4=0$

$$-\frac{1}{2}(\lambda-1)+2(-\lambda)+\frac{1}{2}(1-\lambda)-4=0$$

$$\Rightarrow -3\lambda-3=0$$

$$\Rightarrow \lambda=-1$$

$\therefore$ from (1), the equation of the required sphere is $x^2+y^2+z^2-2x-2y+2z+2=0$

3.a(i) → Find a basis for the column space of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solⁿ: Denote the columns of B by b_1, b_2, b_3, b_4, b_5 and note that $b_3 = -3b_1 + 2b_2$ and $b_4 = 5b_1 - b_2$. The fact that b_3 and b_4 are combinations of the pivot columns means that any combination of $b_1, b_2 \cdots b_5$ is actually just a combination of b_1, b_2 and b_5. Indeed, if v is any vector in col B, say,

$$v = c_1 b_1 + c_2 b_2 + c_3 b_3 + c_4 b_4 + c_5 b_5$$

then, substituting for b_3 and b_4, we can write v in the form of

$$v = c_1 b_1 + c_2 b_2 + c_3(-3b_1 + 2b_2) + c_4(5b_1 - b_2) + c_5 b_5$$

which is a linear combination of b_1, b_2 and b_5. So $\{b_1, b_2, b_5\}$ spans col B.

Also, b_1, b_2 and b_5 are linearly independent, because they are columns from an identity matrix. So the pivot columns of B form a basis for col B.

The matrix B is in reduced echelon form. To handle a general matrix A, recall that linear dependence relations among the columns of A can be expressed in the form $Ax = 0$ for some x. (If some columns are not involved in a particular dependence relation, then the corresponding entries in x are zero.)

When A is row reduced to echelon form B, the columns are drastically changed, but the equations $Ax=0$ and $Bx=0$ have the same set of solutions. That is, the columns of A have exactly the same linear dependence relationships as the columns of B.

3.b(i) State Euler's theorem for homogeneous functions and verify it for the function $z = \sin u$, where $u = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$.

Solⁿ: We have $z = \sin u = \frac{\sqrt{x^2+y^2}}{x+y} = \frac{\sqrt{1+y^2/x^2}}{(1+y/x)}$.

Thus z is a homogeneous function of x and y of degree 0. We shall verify that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

We have $\frac{\partial z}{\partial x} = \frac{(x+y)\frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}}{(x+y)^2}$,

$$\frac{\partial z}{\partial y} = \frac{(x+y)\frac{y}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}}{(x+y)^2}.$$

$$\therefore\ x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{(x^2+y^2)\left(\frac{x+y}{\sqrt{x^2+y^2}}\right) - (x+y)\sqrt{x^2+y^2}}{(x+y)^2} = 0.$$

3.b(ii) The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$. in an ellipse. Find the points on the ellipse that lie closest to and farthest from the origin.

Solⁿ: We find the extreme values of

$$f(x,y,z) = x^2+y^2+z^2$$

(the square of the distance from (x,y,z) to the origin)

subject to the constraints.

$$g_1(x,y,z) = x^2+y^2-1=0 \quad \text{———} (1)$$

$$g_2(x,y,z) = x+y+z-1=0. \quad \text{———} (2)$$

We know that gradient equation

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2,$$

where $g_1(x,y,z)=0$, $g_2(x,y,z)=0$.

then gives

$$2xi+2yj+2zK = \lambda(2xi+2yj)+\mu(i+j+K).$$

$$2xi+2yj+2zK = (2\lambda x+\mu)i+(2\lambda y+\mu)j+\mu K$$

or $2x = 2\lambda x+\mu$, $2y = 2\lambda y+\mu$, $2z=\mu$ ——— (3)

The scalar equations in (3) yield

$$\left.\begin{aligned} 2x &= 2\lambda x+2z \Rightarrow (1-\lambda)x = z, \\ 2y &= 2\lambda y+2z \Rightarrow (1-\lambda)y = z, \end{aligned}\right\} \text{———} (4)$$

equations (4) are satisfied simultaneously if either $\lambda=1$ and $z=0$ or $\lambda\neq 1$ and $x=y=\frac{z}{1-\lambda}$.

If $z=0$, then solving eqs. (1) and (2) simultaneously to find the corresponding points on the ellipse gives the two points $(1,0,0)$ and $(0,1,0)$.

If $x=y$, then eqs. ① and ② give

$$x^2+x^2-1=0 \Rightarrow 2x^2=1 \Rightarrow x=\pm\frac{\sqrt{2}}{2}$$

$$x+x+z-1=0 \Rightarrow z=1-2x \Rightarrow z=1\pm\sqrt{2}.$$

The corresponding points on the ellipse are

$$P_1=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1-\sqrt{2}\right) \text{ and}$$

$$P_2=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},1+\sqrt{2}\right).$$

But here we need to be careful. while P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are $(1,0,0)$ and $(0,1,0)$. The point on the ellipse farthest from the origin is P_2.

3.(c) A tangent plane to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ meets the coordinate axes in the points P, Q and R. Find the locus of the centroid of the triangle PQR.

Sol^n: Any tangent plane to the given ellipsoid is

$$lx+my+nz=\sqrt{(a^2l^2+b^2m^2+c^2n^2)} \quad \text{——} (1)$$

This plane meets the x-axis at P, so the coordinates P are $[(1/l)\sqrt{(a^2l^2+b^2m^2+c^2n^2)}, 0, 0]$ Putting $y=0=z$ in (1).

Similarly Q and R are

$$\left[0, \frac{1}{m}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}, 0\right] \text{ and } \left[0, 0, \frac{1}{n}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}\right]$$

$\therefore$ If (x_1, y_1, z_1) be the centroid of ΔPQR, then

$$x_1=\frac{1}{3}\left[\frac{1}{l}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}+0+0\right]$$

$$=\frac{1}{3l}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$$

Similarly,

$$y_1=\frac{1}{3m}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$$

and $z_1=\frac{1}{3n}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$.

$$\therefore (3lx_1)^2=a^2l^2+b^2m^2+c^2n^2$$

or $9l^2x_1^2=a^2l^2+b^2m^2+c^2n^2$

or $9a^2l^2/(a^2l^2+b^2m^2+c^2n^2) = a^2/x_1^2$

Similarly,

$$9b^2m^2/(a^2l^2+b^2m^2+c^2n^2) = b^2/y_1^2$$

$$9c^2n^2/(a^2l^2+b^2m^2+c^2n^2) = c^2/z_1^2$$

Adding these we get

$$\frac{9(a^2l^2+b^2m^2+c^2n^2)}{(a^2l^2+b^2m^2+c^2n^2)} = (a^2/x_1^2)+(b^2/y_1^2)+(c^2/z_1^2)$$

or $9 = (a^2/x_1^2)+(b^2/y_1^2)+(c^2/z_1^2)$

$\therefore$ The required locus of (x_1, y_1, z_1) is

$$(a^2/x_1^2)+(b^2/y_1^2)+(c^2/z_1^2) = 9.$$

4.(a) Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$, a symmetric matrix. Find the non-singular matrix P such that $P^T A P$ is diagonal and find $P^T A P$.

Solⁿ: First form the block matrix (A, I):

$$(A, I) = \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{array}\right)$$

Apply the operations $R_2 \to -2R_1 + R_2$ and $R_3 \to 3R_1 + R_3$ to (A, I) and then the corresponding operations $C_2 \to -2C_1 + C_2$ and $C_3 \to 3C_1 + C_3$ to A to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right)$$

and then $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right)$

Next apply the operation $R_3 \to -2R_2 + R_3$ and then the corresponding operation $C_3 \to -2C_2 + C_3$ to obtain

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right)$$

and then $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right)$

Now A has been diagonalized. set

$$P = \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and then $P^T A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{pmatrix}$

4.(b) $$I = \iiint \frac{dx\,dy\,dz}{\sqrt{x^2+z^2+(z-1)^2}} = \frac{\pi}{6}.$$

Where the integration is taken over the region $4(x^2+y^2+z^2) \leq 1$.

Sol^n: With change of variables to spherical coordinates

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

We have $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

Where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq r \leq 1/2$.

So that $$I = \int_{r=0}^{1/2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{r^2\sin\theta\,d\phi\,d\theta}{\sqrt{r^2-2r\cos\theta+1}}$$

$$= 2\pi\int_{r=0}^{1/2}\left(r\sqrt{r^2-2r\cos\theta+1}\right)_{\theta=0}^{\pi} dr$$

$$= 2\pi\int_0^{1/2}\{r(r+1)-r(1-r)\}\,dr$$

$$= 2\pi\,\frac{1}{3}\cdot\frac{2}{2^3}$$

$$= \frac{\pi}{6}.$$

4.(C) → Prove that the equations of the generating lines, through the point (θ, ϕ) on the hyperboloid of one sheet are

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)} = \frac{z - c\tan\phi}{\pm c}.$$

Soln: The point $P(\theta, \phi)$ on the hyperboloid of one sheet $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ ——— (1)

is $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi\ \ c\tan\phi)$.

Now we know that the generating lines through P are the lines of intersection of the hyperboloid (1) with the tangent plane at P whose equation is

$$(x/a)\cos\theta\sec\phi + (y/b)\sin\theta\sec\phi - (z/c)\tan\phi = 1 \quad \text{——— (2)}$$

This plane (2) meets the plane $z=0$ in the line given by

$$(x/a)\cos\theta\sec\phi + (y/b)\sin\theta\sec\phi = 1,\ z=0$$

or $(x/a)\cos\theta + (y/b)\sin\theta = \cos\phi,\ z=0$ ——— (3)

Also the section of the hyperboloid (1) by the plane $z=0$ is

$$(x^2/a^2) + (y^2/b^2) = 1,\ z=0 \quad \text{——— (4)}$$

Let the line (3) meet the section (4) of the hyperboloid (1) in the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$. Then the equation of AB is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right),\ z=0. \quad \text{——— (5)}$$

comparing (3) and (5), we get

$$\theta = \frac{\alpha+\beta}{2} \text{ and } \phi = \frac{\alpha-\beta}{2}$$

Adding and substracting, these, we get

$$\alpha = \theta+\phi \text{ and } \beta = \theta-\phi \quad \text{——— (6)}$$

Hence the two generators through P are AP and BP.
Now the direction ratios of AP are

$$a(\cos\alpha - \cos\theta\sec\phi),\ b(\sin\alpha - \sin\theta\sec\phi),\ c(0-\tan\phi).$$

$$\text{or } a\left[\frac{\cos(\theta+\phi)\cos\phi - \cos\theta}{\cos\phi}\right],\ b\left[\frac{\sin(\theta+\phi)\cos\phi - \sin\theta}{\cos\phi}\right],$$

$$\frac{-c\sin\phi}{\cos\phi} \quad \text{from (6)}$$

$$\text{or } a\left[\frac{\cos\theta\cos^2\phi - \sin\theta\sin\phi\cos\phi - \cos\theta}{\cos\phi}\right],$$

$$b\left[\frac{\sin\theta\cos^2\phi + \cos\theta\sin\phi\cos\phi - \sin\theta}{\cos\phi}\right],\ \frac{-c\sin\phi}{\cos\phi}$$

$$\text{or } a\left[\frac{-\cos\theta(1-\cos^2\phi) - \sin\theta\sin\phi\cos\phi}{\sin\phi}\right],$$

$$b\left[\frac{\cos\theta\sin\phi\cos\phi - \sin\theta(1-\cos^2\theta)}{\sin\phi}\right],\ -c,$$

multiplying each term by $(\cos\phi)/\sin\phi$.

$$\text{or } a(-\cos\theta\sin\phi - \sin\theta\cos\phi),\ b(\cos\theta\cos\phi - \sin\theta\sin\phi),\ -c$$

$$\text{or } a\sin(\theta+\phi),\ -b\cos(\theta+\phi),\ c,$$

Where $\theta+\phi$ is constant from (6) for all points on the generator AP, whose equations therefore are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c}\ \text{———} (7)$$

Similarly we can show that the equations of the generator BP are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta-\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta-\phi)}=\frac{z-c\tan\phi}{c}, \text{———} (8)$$

where $\theta-\phi$ is constant from (6) for all points on the generator.

Combining (7) and (8) we get the required result.

5.(a) Find the orthogonal trajectories of $r = a(1+\cos n\theta)$.

Sol'n: Given family is $r = a(1+\cos n\theta)$, where a is parameter. —(1)

Taking logarithm of bothsides,

$$\log r = \log a + \log(1+\cos n\theta) \quad \text{—(2)}$$

Differentiating (2) w.r.t θ

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = -(n\sin n\theta)\Big/(1+\cos n\theta) \quad \text{—(3)}$$

which is differential equation of the family of curves (1). Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3) the differential equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta}$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}\,d\theta$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{2\cos^2\left(\frac{n\theta}{2}\right)d\theta}{2\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)}$$

$$\Rightarrow n\frac{dr}{r} = \cot\left(\frac{n\theta}{2}\right)d\theta.$$

Integrating, $n\log r = \frac{2}{n}\times\log\sin\left(\frac{n\theta}{2}\right) + \frac{1}{n}\log c$, c being arbitrary constant.

$$n^2\log r = \log\sin^2\left(\frac{n\theta}{2}\right) + \log c$$

$$\Rightarrow r^{n^2} = c\sin^2\left(\frac{n\theta}{2}\right)$$

$$\Rightarrow r^{n^2} = \frac{c}{2}(1-\cos n\theta)$$

$$\Rightarrow r^{n^2} = b(1-\cos n\theta), \text{ taking } b = c/2$$

which is the equation of required orthogonal trajectories with b as parameter.

5.(b)(i)

If $L^{-1}\left\{\dfrac{e^{-1/p}}{p^{1/2}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, find $L^{-1}\left\{\dfrac{e^{-a/p}}{p^{1/2}}\right\}$

where $a>0$.

Solution:

(i) Since $L^{-1}\left\{\dfrac{e^{-1/p}}{p^{1/2}}\right\} = \dfrac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$

$$\therefore L^{-1}\left\{\frac{e^{-1/pk}}{(pk)^{1/2}}\right\} = \frac{1}{k}\cdot\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)/k}}$$

$$\Rightarrow L^{-1}\left\{\frac{e^{-1/pk}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)}}$$

Taking $k = 1/a$, we have,

$$\boxed{L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\} = \frac{\cos 2\sqrt{at}}{\sqrt{(\pi t)}}} \quad \text{---} (1)$$

5 b(ii) Find $L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\}$

Sol^n: We have $L^{-1}\left\{\frac{1}{(p+4)^{5/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{p^{5/2}}\right\}$

$$= e^{-4t}\frac{t^{5/2-1}}{\Gamma(5/2)} = \frac{4t^{3/2}e^{-4t}}{3\sqrt{\pi}}.$$

$$\therefore L^{-1}\left\{\frac{e^{4-3p}}{(p+4)^{5/2}}\right\} = e^{4} L^{-1}\left\{\frac{e^{-3p}}{(p+4)^{5/2}}\right\}$$

$$= \begin{cases} e^{4}\frac{4}{3\sqrt{\pi}}(t-3)^{3/2} e^{-4(t-3)}, & t>3 \\ 0, & t<3 \end{cases}$$

$$= \begin{cases} \frac{4}{3\sqrt{\pi}}(t-3)^{3/2} e^{-4(t-3)}, & t>3 \\ 0, & t<3 \end{cases}$$

$$= \frac{4}{3\sqrt{\pi}}(t-3)^{3/2} e^{-4(t-3)} H(t-3).$$

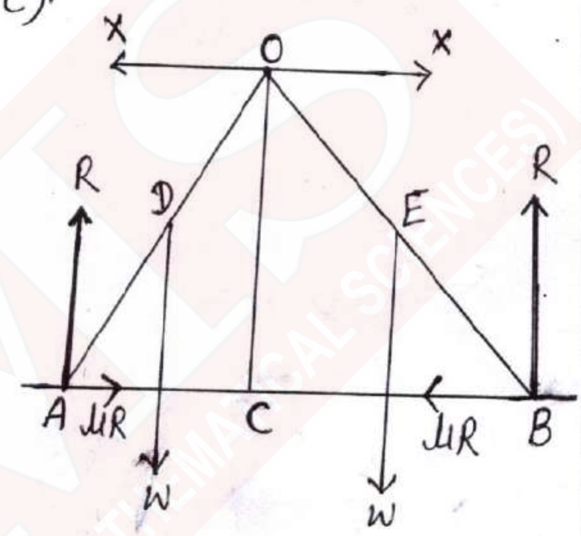
in terms of the Heaviside unit step function.

5.(C) → Two equal uniform rods AB, BC, each of weight W, lean against each other and rest in vertical plane with ends A and B on a rough horizontal plane. The angle ACB is 2α and the co-efficient of friction μ. Find what weight placed at C would cause them to slip.

Soln: Let w be the weight placed at C. (w acts vertically downwards at C).

Resolving vertically for the whole system

$$2W + w = 2R \quad \text{——} (1)$$



Taking moments about C for the equilibrium of rod AC only,

We have

$$W \cdot CD \sin\alpha + \mu R \cdot AC \cos\alpha = R \cdot AC \sin\alpha$$

$$\Rightarrow W \cdot CD \sin\alpha + \mu R \cdot 2CD \cos\alpha = R \cdot 2CD \sin\alpha$$

$$\Rightarrow W \sin\alpha + 2\mu R \cos\alpha = 2R \sin\alpha$$

$$\Rightarrow W \sin\alpha + \mu(2W + w)\cos\alpha = (2W + w)\sin\alpha \quad [\text{by using } (1)]$$

$$\Rightarrow \mu(2W + w)\cos\alpha = (W + w)\sin\alpha$$

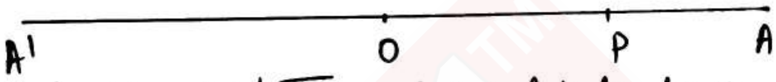
$$\Rightarrow \mu(2W + w) = (W + w)\tan\alpha$$

$$\Rightarrow 2\mu W - W\tan\alpha = w(\tan\alpha - \mu)$$

$$\therefore \quad w = \frac{W(2\mu - \tan\alpha)}{\tan\alpha - \mu}.$$

Q 5(d) A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP=b$ with velocity v in the direction OP; Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$.

Sol'n Let the eq^n of the S.H.M with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$



The time period $T = 2\pi/\sqrt{\mu}$, Let aplitude be a

Then $(dx/dt)^2 = \mu(a^2-x^2)$ ——— (1)

When particle passes through P its velocity is given to be v in the direction OP. Also $OP=b$. So putting $x=b$ and $dx/dt = v$ in (1)

We get $v^2 = \mu(a^2-b^2)$.

In S.H.M the time from P to A is equal to the time from A to P.

$\therefore$ the required time $= 2 \cdot$ time from A to P.

Now Motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2-x^2)} \Rightarrow dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}$$

Let t_1 be the time from A to P. Then at A, $t=0, x=a$ and at P, $t=t_1$ and $x=b$, Therefore integrating (3) we get

$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{(a^2-x^2)}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

Hence required time $= 2t_1 = \frac{2}{\sqrt{\mu}}\cos^{-1}\left(\frac{b}{a}\right)$

$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{(a^2-b^2)}}{b}\right\} = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right) \quad [\text{from }(2)$$

$$= \frac{2}{2\pi/T}\tan^{-1}\left\{\frac{v}{b(2\pi/T)}\right\} \quad \left[\because T = 2\pi/\sqrt{\mu} \text{ so that } \sqrt{\mu} = 2\pi/T\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$$

5.(e) If $f = yi + (x - 2xz)j - xyk$, evaluate $\int_S (\nabla \times f) \cdot n\, dS$, where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

Solⁿ: Let $F = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$

$$= xi + yj - 2zk$$

$\phi = c$ is given by $x^2+y^2+z^2=a^2$

grad ϕ is along the normal and

$$\text{grad } \phi = \sum i \frac{\partial \phi}{\partial x} = 2xi + 2yj + 2zk$$

$$\therefore n = \text{a unit normal} = \frac{2xi+2yj+2zk}{\sqrt{(4x^2+4y^2+4z^2)}}$$

$$= \frac{xi+yj+zk}{a}$$

$$\therefore F \cdot n = (xi+yj-2zk) \cdot \frac{(xi+yj+zk)}{a}$$

$$= \frac{x^2+y^2+2z^2}{a}$$

Also, we know that

$$\int_S F \cdot n\, dS = \iint_{S_3} F \cdot n \frac{dx\,dy}{n \cdot k}$$

Where S_3 is the projection of S on xy-plane.

$$n \cdot k = \frac{xi+yj+zk}{a} \cdot k = \frac{z}{a} = \frac{\sqrt{(a^2-x^2-y^2)}}{a}.$$

Also $F.n = \frac{x^2+y^2-2z^2}{a} = \frac{x^2+y^2-2(a^2-x^2-y^2)}{a}$

$$= \frac{3(x^2+y^2)-2a^2}{a}$$

$$\therefore \int_S F.n\,dS = \iint_{S_3} \frac{3(x^2+y^2)-2a^2}{a} \cdot \frac{dx\,dy}{\sqrt{(a^2-x^2-y^2)}}\,a \quad \text{---} (1)$$

Now, S_3 is the projection of $x^2+y^2+z^2=a^2$ in the xy-plane and is given by

$$x^2+y^2=a^2.$$

In order to integrate (1) put $x=r\cos\theta, y=r\sin\theta$.

$\therefore dx\,dy = r\,d\theta\,dr$ and limits of θ are from 0 to 2π and that of r from 0 to a.

$$\therefore \int_S F.n\,dS = \int_0^{2\pi}\int_0^a \frac{3r^2-2a^2}{\sqrt{(a^2-r^2)}}\,r\,d\theta\,dr$$

$$= 2\pi\int_0^a \frac{3r^2-2a^2}{\sqrt{(a^2-r^2)}}\,r\,dr.$$

Put $a^2-r^2=t$, $\therefore -2r\,dr = 2t\,dt$

$$\therefore \int_S F.n\,dS = 2\pi\int_0^a \frac{3(a^2-t^2)-2a^2}{t}(-t)\,dt$$

$$= 2\pi\int_0^a (a^2-3t^2)\,dt$$

$$= 2\pi(a^3-a^3) = 0.$$

6.a(i) Solve $y(1+xy)dx + x(1-xy)dy = 0$.

Soln: Given $(1+xy)y\,dx + (1-xy)x\,dy = 0$. —— ①

comparing ① with $M dx + N dy = 0$,

$$M = (1+xy)y \text{ and } N = (1-xy)x,$$

Showing that ① is of the form

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0.$$

Again, $Mx - Ny = xy(1+xy) - xy(1-xy) = 2x^2y^2 \neq 0$,

Showing that I.F. of ① $= \dfrac{1}{Mx-Ny} = \dfrac{1}{2x^2y^2}$.

on multiplying ① by $1/(2x^2y^2)$, we have

$$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0, \quad \text{—— ②}$$

Which must be exact and so by the usual rule, solution of ② is

$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(-\frac{1}{2y}\right)dy = \frac{1}{2}\log c$$

[Treating y as constant.]

or $\dfrac{1}{-2xy} + \dfrac{1}{2}\log x - \dfrac{1}{2}\log y = \dfrac{1}{2}\log c$

or $\log(x/y) - \log c = 1/(xy)$

or $\log(x/cy) = 1/(xy)$

or $x = cy e^{1/(xy)}$.

6(a)(ii) Solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

Sol^n: Given that $(D^4+D^2+1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$

The auxiliary equation is $D^4+D^2+1=0$

$\Rightarrow (D^2+1)^2-D^2=0$

$\Rightarrow (D^2+D+1)(D^2-D+1)=0$

$\Rightarrow D^2+D+1=0,$ (or) $D^2-D-1=0$

$\Rightarrow D = \frac{-1\pm i\sqrt{3}}{2}, \frac{1\pm i\sqrt{3}}{2}$

$\therefore C\cdot F = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$

$+e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right)+c_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

$P\cdot I = \frac{1}{D^4+D^2+1}\, e^{-x/2}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\left(\frac{\sqrt{3}}{2}x\right)$

$= e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})(D^2-2D+\frac{7}{4})}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{(-\frac{3}{4}-2D+\frac{7}{4})}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D^2+3/4)}\frac{1}{(1-2D)}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1+2D}{(1-4D^2)}\cos\frac{\sqrt{3}}{2}x$

$= e^{-x/2}\frac{1}{(D^2+3/4)}\frac{1}{4}(1+2D)\cos\frac{\sqrt{3}}{2}x$

$= \frac{e^{-x/2}}{4}\frac{1}{(D^2+3/4)}\left(\cos\frac{\sqrt{3}}{2}x - \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$

$$= \frac{1}{4} e^{-x/2} \left[\frac{1}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2} \cos\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2} \sin\frac{\sqrt{3}}{2}x \right.$$

$$= \frac{1}{4} e^{-x/2} \left[\frac{x}{2(\sqrt{3}/2)} \sin\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}x}{2(\sqrt{3}/2)} \left(-\cos\frac{\sqrt{3}}{2}x\right)\right]$$

$$= \frac{x}{4\sqrt{3}} e^{-x/2} \left[\sin\frac{\sqrt{3}}{2}x + \sqrt{3}\cos\frac{\sqrt{3}}{2}x \right]$$

$$\therefore y = C.F + P.I$$

$$= e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x \right]$$

$$+ e^{x/2}\left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x \right]$$

$$+ \frac{x}{4\sqrt{3}} e^{-x/2}\left[\sin\frac{\sqrt{3}}{2}x + \sqrt{3}\cos\frac{\sqrt{3}}{2}x \right]$$

6.(b) → Solve the differential equation:

$$y'' + (\tan x - 3\cos x)y' + 2y\cos^2 x = \cos^4 x.$$

Sol^n: Please do yourself.

ANS: $\boxed{y = c_1 e^{\sin x} + c_2 e^{-\sin x} - (5/4) - (3/2)\times \sin x - (1/2)\times \sin^2 x.}$

Ques: 6(c)) Solve $y'' + 3y' + 2y = x + \cos x$ by the method of variation of parameters.

Solution:- Given;

$$y'' + 3y' + 2y = x + \cos x \quad \text{--- (1)}$$

Compare (1) with $y'' + Py' + Qy = R$

here; $R = x + \cos x$ --- (2)

$P = 3, \; Q = 2$

consider $y'' + 3y' + 2y = 0$

$$(D^2 + 3D + 2)y = 0 \qquad D \equiv d/dx \quad \text{--- (3)}$$

Its auxillary equation

$$m^2 + 3m + 2 = 0$$
$$(m+1)(m+2) = 0$$
$$m = -1, -2.$$

$\therefore \boxed{\text{C.F of (1)} = c_1 e^{-x} + c_2 e^{-2x}}$

c_1 & c_2 being arbitrary constants

Let; $u = e^{-x}$ and $v = e^{-2x}$ --- (4)

Here; $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$

$$W = -2e^{-x} \cdot e^{-2x} + e^{-x} \cdot e^{-2x}$$

$$\omega = -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0 \quad \text{--- (5)}$$

Then, P.I of (1) $= uf(x) + vg(x)$, where --- (6)

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{e^{-2x}(x + \cos x)}{(-e^{-3x})} dx$$

$$f(x) = \int e^{x}(x + \cos x)\,dx \quad \text{— by (2), (4) \& (5)}$$

$$= \int x e^{x} dx + \int e^{x} \cos x \, dx$$

$$= x e^{x} - \int (1 \times e^{x}) dx + \frac{1}{2} \times e^{x} (\cos x + \sin x)$$

$$\left[\because \int e^{ax} \cos bx \, dx = \left\{ \frac{1}{a^2+b^2} \right\} \times e^{ax} (a \cos bx + b \sin bx) \right]$$

Thus; $f(x) = x e^{x} - e^{x} + \frac{1}{2} e^{x} (\cos x + \sin x)$ ——— (7)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{e^{-x}(x + \cos x)}{(-e^{-3x})} dx$

$$g(x) = -\int e^{2x}(x + \cos x) dx = -\int x e^{2x} dx - \int e^{2x} \cos x \, dx$$

$$g(x) = -\left[x \cdot \frac{1}{2} \cdot e^{2x} - \int \left(1 \times \frac{1}{2} \right) \times e^{2x} dx \right] - \frac{1}{5} e^{2x} (2 \cos x + \sin x)$$

$$\therefore g(x) = -\frac{x}{2} \cdot e^{2x} + \frac{1}{4} e^{2x} - \frac{1}{5} e^{2x} (2 \cos x + \sin x) \quad \text{——— (8)}$$

from (4), (6), (7) and (8), we have

$$\text{P.I of (1)} = e^{-x} \left\{ x e^{x} - e^{x} + \frac{1}{2} \cdot e^{x} (\cos x + \sin x) \right\}$$
$$+ e^{-2x} \left\{ -(x/2) \times e^{2x} + \frac{1}{4} \times e^{2x} - \frac{1}{5} \times e^{2x} (2 \cos x + \sin x) \right\}$$

$$\text{P.I of (1)} = x - 1 + \frac{(\cos x + \sin x)}{2} - \frac{x}{2} + \frac{1}{4} - \frac{(2 \cos x + \sin x)}{5}.$$

$$\text{P.I of (1)} = \frac{x}{2} - \frac{3}{4} + \frac{1}{10} (3 \sin x + \cos x)$$

$\therefore$ Required solution; $y = C.F + P.I$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x}{2} - \frac{3}{4} + \frac{1}{10} (3 \sin x + \cos x)}$$

which is required solution

6.(d) → By using Laplace transform method solve $(D^2+2D+1)\,y = 3te^{-t}$, $t>0$, subject to the conditions, $y=4$, $Dy=2$ when $t=0$.

Soln: Taking the Laplace transform of both sides of the given equation, we have

$$L\{y''\} + 2L\{y'\} + L\{y\} = 3L\{te^{-t}\}$$

or $p^2L\{y\} - py(0) - y'(0) + 2[pL\{y\} - y(0)] + L\{y\}$
$$= 3L\{te^{-t}\}$$

or $(p^2+2p+1)\,L\{y\} - 4p - 10 = -3\frac{d}{dp}[L\{e^{-t}\}]$

or $(p+1)^2 L\{y\} - 4p - 10 = -3\frac{d}{dp}\left(\frac{1}{p+1}\right)$

or $(p+1)^2 L\{y\} = \dfrac{3}{(p+1)^2} + 4p + 10$

or $L\{y\} = \dfrac{3}{(p+1)^4} + \dfrac{4(p+1)+6}{(p+1)^2}$

$$= \frac{3}{(p+1)^4} + \frac{4}{(p+1)} + \frac{6}{(p+1)^2}$$

$$\therefore y = 3L^{-1}\left\{\frac{1}{(p+1)^4}\right\} + 4L^{-1}\left\{\frac{1}{p+1}\right\} + 6L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$$

$$= 3e^{-t}L^{-1}\left\{\frac{1}{p^4}\right\} + 4e^{-t} + 6e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\}$$

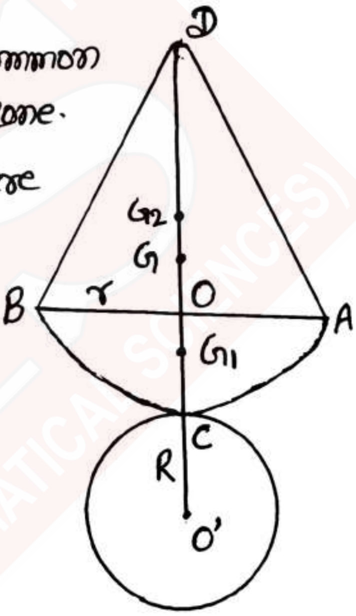
$$= 3e^{-t}\frac{t^3}{3!} + 4e^{-t} + 6e^{-t}\cdot t = \frac{1}{2}e^{-t}\cdot t^3 + 4e^{-t} + 6te^{-t},$$

which is the required solution.

Q.7a) → A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the bases; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right]$$

Sol: Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of radius R and centre O', their point of contact being C. For equilibrium the line O'COD must be vertical. Let H be the length of the axis OD of the cone. It is given that OB−OC = r the radius of the hemisphere.



If G_1 and G_2 are the centres of gravity of the hemisphere and the cone respectively, then

$$OG_1 = 3r/8 \quad \text{and} \quad OG_2 = H/4$$

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C, then

$$h = \frac{\frac{2}{3}\pi r^3 \cdot \frac{5}{8}r + \frac{1}{3}\pi r^2 H \cdot (r+\frac{1}{4}H)}{\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 H}$$

$$= \frac{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body $= r$.

and ρ_2 = the radius of curvature at C of the lower body $= R$

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{i.e.,} \quad \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

i.e, $$\frac{H+2r}{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2} > \frac{R+r}{rR}$$

i.e, $$(R+r)\left\{Hr+\frac{1}{4}H^2+\frac{5}{4}r^2\right\} - rR(H+2r) < 0$$

i.e, $$\frac{1}{4}H^2(R+r) + H\{(R+r)r - rR\} + \frac{5}{4}r^2(R+r) - 2r^2R < 0$$

i.e, $$H^2(R+r) + 4r^2H + 5r^3 - 3r^2R < 0$$

i.e, $$H^2(R+r) + 4r^2H - r^2(3R-5r) < 0$$

i.e, $$H^2 + \frac{4r^2}{R+r}H - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e, $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4}{(R+r)^2} - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e, $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4+r^2(3R-5r)(R+r)}{(R+r)^2} < 0$$

i.e, $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2[4r^2+3R^2-2rR-5r^2]}{(R+r)^2} < 0$$

i.e, $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2(3R^2-2rR-r^2)}{(R+r)^2} < 0$$

i.e, $$\left(H+\frac{2r^2}{R+r}\right)^2 < \frac{r^2(3R+r)(R-r)}{(R+r)^2}$$

i.e, $$H+\frac{2r^2}{R+r} < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$$

i.e, $$H < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}} - \frac{2r^2}{R+r}$$

i.e, $$H < \frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right]$$

therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right].$$

7(b). A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by

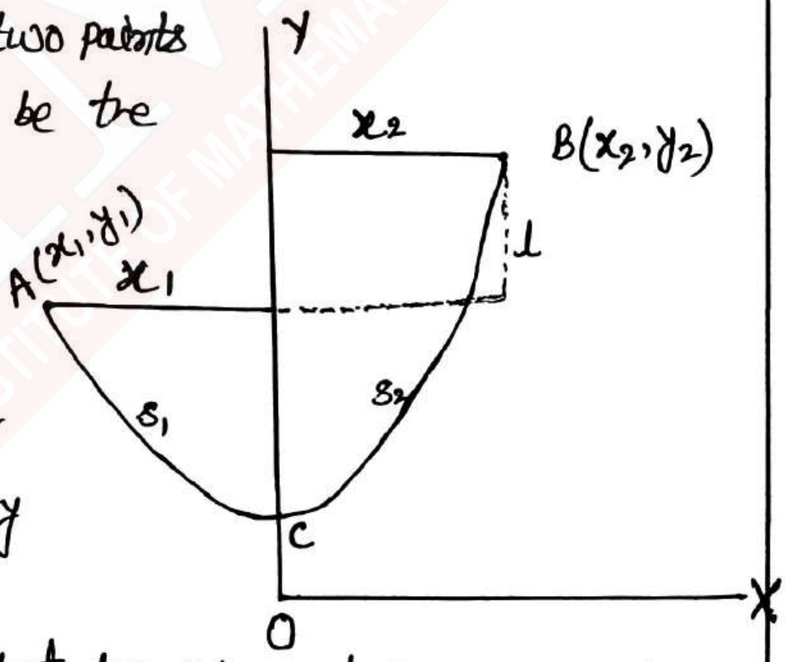
$$2c \sinh(a/2c) = \sqrt{(l^2 - b^2)}.$$

Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1 + T_2 = W\sqrt{\left(1 + \frac{4c^2}{l^2 - b^2}\right)} \text{ and } T_2 - T_1 = Wb/l,$$

where W is the weight of the chain.

Sol: A uniform chain of length l and weight W hangs between two points A and B. Let C be the vertex, OX the directrix, OY the axis and c the parameter of the catenary in which the chain hangs.



Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points A and B respectively and let arc $CA = s_1$ and arc $CB = s_2$.

We have $s_1 + s_2 = l$.

Since the horizontal distance between A and B is a, therefore

$$x_1 + x_2 = a.$$

Again since the vertical distance of B above A is b, therefore

$$y_2 - y_1 = b.$$

Let w be the weight per unit length of the chain. Then

$$W = lw, \quad \text{or} \quad w = W/l.$$

By the formula $s = c\sinh(x/c)$, we have

$$s_1 = c\sinh(x_1/c) \quad \text{and}$$

$$s_2 = c\sinh(x_2/c).$$

$$\therefore l = s_1 + s_2 = c\left[\sinh(x_1/c) + \sinh(x_2/c)\right] \quad \text{--- (1)}$$

Again by the formula $y = c\cosh(x/c)$, we have

$$y_1 = c\cosh(x_1/c) \quad \text{and}$$

$$y_2 = c\cosh(x_2/c)$$

$$\therefore b = y_2 - y_1 = c\left[\cosh(x_2/c) - \cosh(x_1/c)\right] \quad \text{--- (2)}$$

Squaring and subtracting (1) and (2), we have

$$l^2 - b^2 = c^2\Big[-\{\cosh^2(x_1/c) - \sinh^2(x_1/c)\} - \{\cosh^2(x_2/c) - \sinh^2(x_2/c)\} + 2\{\cosh(x_1/c)\cosh(x_2/c) + \sinh(x_1/c)\sinh(x_2/c)\}\Big]$$

$$= c^2\left[-1-1+2\cosh(x_1/c+x_2/c)\right]$$

$$= c^2\left[-2+2\cosh\{(x_1+x_2)/c\}\right]$$

$$= 2c^2\left\{\cosh\frac{a}{c}-1\right\}$$

$$= c^2\left\{1+2\sinh^2\frac{a}{2c}-1\right\}$$

$$= 4c^2\sinh^2\frac{a}{2c} \quad \text{———} \quad (3)$$

$\therefore$ c is given by $2c\sinh(a/2c)=\sqrt{(l^2-b^2)}$.

[Remember that

$$\cosh(\alpha+\beta)=\cosh\alpha\cdot\cosh\beta+\sinh\alpha\cdot\sinh\beta$$

and $\cosh 2\alpha = 1+2\sinh^2\alpha$]

Now let T_1 and T_2 be the tensions at the points A and B respectively. Then by the formula $T=wy$, we have

$$T_1=wy_1, \quad T_2=wy_2.$$

$$\therefore T_2-T_1 = w(y_2-y_1)=wb=(W/l)b=Wb/l.$$

Also, $T_1+T_2=w(y_1+y_2)=\frac{W}{l}(y_1+y_2)=W\frac{y_1+y_2}{s_1+s_2}$

$$= W\frac{c\cosh(x_1/c)+c\cosh(x_2/c)}{c\sinh(x_1/c)+c\sinh(x_2/c)}$$

$$= W \frac{\cosh(x_1/c) + \cosh(x_2/c)}{\sinh(x_1/c) + \sinh(x_2/c)}$$

$$= W \frac{2\cosh\frac{1}{2}(x_1/c + x_2/c)\cosh\frac{1}{2}(x_1/c - x_2/c)}{2\sinh\frac{1}{2}(x_1/c + x_2/c)\sinh\frac{1}{2}(x_1/c - x_2/c)}$$

$$= W \coth\left(\frac{x_1 + x_2}{2c}\right)$$

$$= W \coth\frac{a}{2c}$$

$$= W\sqrt{\left[1 + \operatorname{cosech}^2\frac{a}{2c}\right]} \quad \left[\because \coth^2\alpha = 1 + \operatorname{cosech}^2\alpha\right]$$

$$= W\sqrt{\left[1 + \frac{4c^2}{l^2 - b^2}\right]}$$

Substituting for $\operatorname{cosech}^2(a/2c)$ from (3).

7(C) → A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is

$$r \cos(\theta/\sqrt{2}) = a.$$

Soln → Here a central acceleration varies inversely as the cube of the distance

i.e., $P = \mu/r^3 = \mu u^3$, where μ is contant.

If V is the velocity for a circle of radius a, then,

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

or $V = \sqrt{(\mu/a^2)}$

∴ the velocity of projection

$$v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2(du/d\theta)$ and integrating we have,

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$v^2 = \mu u^2 + A \text{ ——— } (1)$$

Where A is a constant.

But initialy when $r = a$

i.e., $u = 1/a$, $\frac{du}{d\theta} = 0$ (at an apse)

and $v = v_1 = \sqrt{(2\mu/a^2)}$

from (1), we have,

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$\therefore$ $h^2 = 2\mu$ and $A = \mu/a^2$

Substituting the values of h^2 and A in (1), we have,

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

or, $$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1-a^2u^2}{a^2}$$

or, $$\sqrt{2}\,a\,\frac{du}{d\theta} = \sqrt{(1-a^2u^2)}$$

or, $$\frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1-a^2u^2)}}$$

Integrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$,

where B is constant.

But initially, when $u = 1/a$, $\theta = 0$.

$$\therefore B = \sin^{-1} 1 = \frac{1}{2}\pi$$

$$\therefore (\theta/\sqrt{2}) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$$

or, $$au = \frac{a}{r} = \sin\left\{\frac{1}{2}\pi + (\theta/\sqrt{2})\right\}$$

or, $$a = r\cos(\theta/\sqrt{2})$$

Which is the required equation of the path.

8.(b) → If u_1, u_2, u_3 are orthogonal curvilinear co-ordinates, Show that, $\partial r/\partial u_1$, $\partial r/\partial u_2$, $\partial r/\partial u_3$ and $\nabla u_1, \nabla u_2, \nabla u_3$ are reciprocal systems of vectors.

Sol^n: We know that

$$\nabla u_1 = e_1/h_1,\ \nabla u_2 = e_2/h_2,\ \nabla u_3 = e_3/h_3.$$

$$\therefore [\nabla u_1 + \nabla u_2 + \nabla u_3] = (\nabla u_1 \times \nabla u_2)\cdot \nabla u_3$$

$$= \left(\frac{1}{h_1} e_1 \times \frac{1}{h_2} e_2\right) \times \frac{1}{h_3} e_3$$

$$= \frac{1}{h_1 h_2 h_3} e_3 \cdot e_3 = \frac{1}{h_1 h_2 h_3} \quad \text{--- (1)}$$

Also, $\dfrac{\partial r}{\partial u_1} = h_1 e_1,\ \dfrac{\partial r}{\partial u_2} = h_2 e_2,\ \dfrac{\partial r}{\partial u_3} = h_3 e_3$ --- (2)

Now, $$\frac{\nabla u_2 \times \nabla u_3}{[\nabla u_1\ \nabla u_2\ \nabla u_3]} = \frac{(1/h_2)e_2 \times (1/h_3)e_3}{1/h_1 h_2 h_3}$$

$$= h_1 e_1 = \frac{\partial r}{\partial u_1} \quad (\text{by } (2))$$

Proceeding similarly, we get

$$\frac{\partial r}{\partial u_2} = \frac{\nabla u_3 \times \nabla u_1}{[\nabla u_1\ \nabla u_2\ \nabla u_3]},\quad \frac{\partial r}{\partial u_3} = \frac{\nabla u_1 \times \nabla u_2}{[\nabla u_1\ \nabla u_2\ \nabla u_3]}$$

Hence, $\partial r/\partial u_1$, $\partial r/\partial u_2$, $\partial r/\partial u_3$ and $\nabla u_1, \nabla u_2, \nabla u_3$ form reciprocal system of vectors.

8C Q) Find the value of

$$\int \text{curl } F \cdot da.$$

taken over the portion of the surface

$$x^2+y^2-2ax+az=0 \text{ for which}$$

$z \geq 0$ when

$$F=(y^2+z^2-x^2)i+(z^2+x^2-y^2)j+(x^2+y^2-z^2)k$$

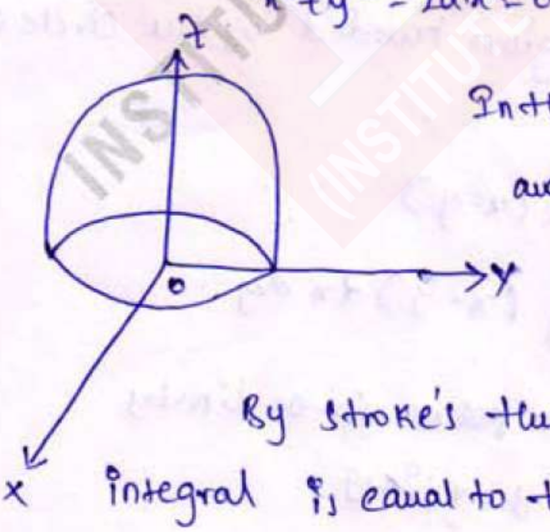
Sol^n Re writing the equation

$$x^2+y^2-2ax+az=0$$

or $\quad (x-a)^2+y^2=-a(z-a).$

we see that the surface is a paraboloid with its vertex at $(a,0,a)$ and axis parallel to z-axis and turned towards the negative direction of the same it meets the plane $z=0$ in the circle C given by

$$x^2+y^2-2ax=0 \quad , z=0$$



In the figure o is the point $(a,0,0)$ and ox, oy, oz are the lines parallel to the co-ordinate axes).

By Stroke's theorem, the given surface integral is equal to the line integral

$$\int_C F \cdot dx.$$

The circle 'c' is given by

$x = a(1+\cos\theta),\ y = a\sin\theta,\ z = 0$

Along c.

$$F = \left[a^2\sin^2\theta - a^2(1+\cos\theta)^2\right] i + \left[a^2(1+\cos\theta)^2 - a^2\sin^2\theta\right] j + \left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right] K$$

$$\therefore F\cdot\left(\frac{dx}{d\theta} i + \frac{dy}{d\theta} j + \frac{dz}{d\theta} K\right)$$

$$= \left[a^2\sin^2\theta - a^2(1+\cos\theta)^2\right]\left[-a\sin\theta - a\cos\theta\right].$$

$$\int_C F\cdot dr = \int_0^{2\pi} F\cdot\frac{dr}{d\theta}\, d\theta = 2a^3\pi$$

Another method: By a further application of Stoke's theorem, we see that the given integral

$$= \int_{S_1} \operatorname{curl} F\cdot n\, dS.$$

where S_1 is the plane region bounded by the circle c.

Here $n = K$

Thus $\operatorname{curl} F\cdot n = 2(x-y)$

$\therefore$ The integral $= 2\iint (x-y)\,dx\,dy$.

taken over S_1, changing to polar co-ordinates

so that $x = a + r\cos\theta,\ y = r\sin\theta$

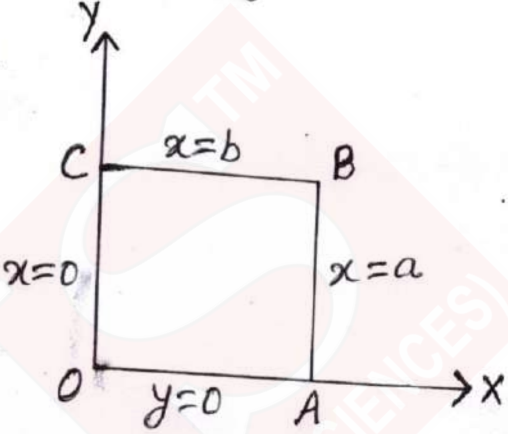
we see that the integral

$$= 2\int_0^a\int_0^{2\pi}(a + r\cos\theta - r\sin\theta)\, r\, d\theta\, dr = \underline{2a^3\pi}.\ \text{Ans.}$$

8.(d) → Verify Stoke's theorem for the function $F = x(ix+jy)$, integrated round the square in the plane $z=0$. Whose sides are along the lines $x=0, y=0, x=a, y=a$.

Solⁿ: We have $\text{curl}\; x(ix+jy) = ky$.

$\therefore \int_S \text{curl}\; x(ix+jy)\cdot da$



$= \int_0^a \int_0^a ky\cdot k\, dx\, dy$

$= \int_0^a \int_0^a y\, dx\, dy = \frac{a^3}{2}$

Again,

$+\int_{BC} F\cdot dr + \int_{CO} F\cdot dr + \int_{BC} F\cdot dr + \int_{CO} F\cdot dr.$

Now $\int_{OA} F\cdot dr = \int_0^a x(ix+jy)\cdot i\,dx = \int_0 x^2 dx = \frac{1}{3}a^3,$

$\int_{AB} F\cdot dr = \int_0^a x(ix+jy)\cdot j\,dy = \int_0^a ay\, dy = \frac{1}{2}a^3,$

$\int_{BC} F\cdot dr = \int_a^0 x(ix+jy)\cdot i\,dx = -\int_0^a x^2 dx = -\frac{1}{3}a^3,$

$\int_{CO} F\cdot dr = \int_a^0 x(ix+jy)\cdot j\,dy = 0.$

$\therefore \int_C F\cdot dr = \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3.$

Hence verified.