2011

1. $\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$, $\vec{b} = \sin t\,\hat{i} - \cos t\,\hat{j}$

(a) $\frac{d}{dt}(\vec{a}\cdot\vec{b})$ (ii) $\frac{d}{dt}(\vec{a}\times\vec{b})$

$\vec{a}\cdot\vec{b} = 5t^2\sin t - t\cos t$

(i) $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = \frac{d\vec{a}}{dt}\cdot\vec{b} + \vec{a}\cdot\frac{d\vec{b}}{dt}$

$= 10t\sin t - \cos t + 5t^2\cos t + t\sin$

$= 11t\sin t - \cos t + 5t^2\cos t$

(ii) $\left(\frac{d\vec{a}}{dt}\times\vec{b}\right) + \vec{a}\times\frac{d\vec{b}}{dt}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$= \hat{i}(-t^3\cos t) + \hat{j}(-t^3\sin t) + \hat{k}(-5t^2\cos t - t\sin t)$

$\frac{d}{dt}(\vec{a}\times\vec{b}) = (-3t^2\cos t + t^3\sin t)\hat{i}$

$+ \hat{j}(-3t^2\sin t - t^3\cos t) +$

$\hat{k}(-10t\cos t + 5t^2\sin t - t\cos t - \sin t)$

verification: $\frac{d}{dt}(\vec{a}\cdot\vec{b}) = 10t\sin t + 5t^2\cos t - \cos t + t\sin t$

$= 11t\sin t - \cos t + 5t^2\cos t$

proved.

2. u, v are scalar fields ; $\vec{f}$ = vector field

given $u\vec{f}$ = grad v Find $\vec{f} \cdot \text{curl} \vec{f}$

$$u\vec{f} = \nabla v$$

Taking curl both sides,

$$\nabla \times u\vec{f} = \nabla \times \nabla v$$

$$\nabla \times u\vec{f} = \sum i \times \frac{\partial}{\partial x} u\vec{f} - \sum i \times \frac{\partial u}{\partial x} \vec{f} + \sum i \times u \frac{\partial \vec{f}}{\partial x}$$

$$= \sum \frac{\partial u}{\partial x} i \times \vec{f} + u \sum i \times \frac{\partial f}{\partial x}$$

$$= (\nabla u) \times \vec{f} + u \sum \frac{\partial}{\partial x} i \times f$$

$$\because \sum \frac{\partial}{\partial x}(i \times \vec{f}) = \sum \frac{\partial i}{\partial x} \times \vec{f} + \sum i \times \frac{\partial f}{\partial x}$$

$$= 0 + \sum i \times \frac{\partial f}{\partial x}$$

Since curl $\vec{f} = \sum i \times \frac{\partial}{\partial x} \vec{f}$

$$\nabla \times u\vec{f} = \nabla u \times \vec{f} + u(\nabla \times \vec{f})$$

Multiply by $u\vec{f}$ both sides,

$$u\vec{f} \cdot (\nabla \times u\vec{f}) = u\vec{f} \cdot (\nabla u \times \vec{f}) + u^2 \vec{f} \cdot (\nabla \times \vec{f})$$

$$\Rightarrow u^2 \vec{f} \cdot (\nabla \times \vec{f}) = u\vec{f} \cdot (\nabla \times u\vec{f}) - u\vec{f} \cdot (\nabla u \times \vec{f})$$

$$= \nabla \vec{v} \cdot (\nabla \times \nabla \vec{v}) - u\vec{f} \cdot (\nabla u \times \vec{f})$$

$$= \nabla \vec{v} \cdot (\nabla \times \nabla \vec{v}) - 0$$

$$= \nabla v \cdot (\nabla \times \nabla v)$$

$$\left[\because u\vec{f} \cdot \nabla u \times \vec{f} = [u\vec{f} \; \nabla u \; \vec{f}] = \underbrace{[\nabla u \; u\vec{f} \; \vec{f}]}_{=0} \right]$$

$$\Rightarrow \vec{f} \cdot (\nabla \times \vec{f}) = \frac{\nabla v \cdot (\nabla \times \nabla \vec{v})}{u^2} = 0$$

3. $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$

Are $\nabla u, \nabla v, \nabla w$ coplanar.

$\nabla u = \hat{i} + \hat{j} + \hat{k}$

$\nabla v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla w = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$

They are coplanar if $\nabla u \cdot (\nabla v \times \nabla w) = 0$

ie $[\nabla u \ \nabla v \ \nabla w] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = 1\left(2xy+2y^2-2xz-2z^2\right)$$

$$-1\left(2x^2+2xy-2yz-2z^2\right)$$

$$+1\left(2x^2+2xz-2y^2-2yz\right)$$

$$= \cancel{2xy}+\cancel{2y^2}-\cancel{2xz}-\cancel{2z^2}-\cancel{2x^2}-\cancel{2xy}+\cancel{2yz}+\cancel{2z^2}$$

$$+\cancel{2x^2}+\cancel{2xz}-\cancel{2y^2}-\cancel{2yz}$$

$$= 0$$

Yes, they are coplanar.

4.

4. $\vec{u} = 4y\hat{i} + x\hat{j} - 2z\hat{k}$

Find $\iint (\nabla \times \vec{u}) \cdot \vec{ds}$ over the hemisphere given by $x^2+y^2+z^2 = a^2;\ z \geq 0$.

Using stokes Thm. $\iint \text{curl}\vec{F} \cdot \hat{n}\, ds$

$= \iint \text{curl}\vec{F} \cdot \vec{ds} = \oint_C \vec{F} \cdot d\vec{r}$

where C is the circle $x^2+y^2=a^2$ on xy-plane

$\oint_C \vec{F} \cdot d\vec{r} = \int (4y\hat{i} + x\hat{j} - 2z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

on $C: z=0, dz=0$

$I \Rightarrow \oint_C 4y\,dx + x\,dy$

let us parametrize C by $x = a\cos\theta,\ y = a\sin\theta$

$dx = -a\sin\theta \quad dy = a\cos\theta$

$I = \int_{\theta=0}^{2\pi} -4a^2\sin^2\theta\, d\theta + \int_{\theta=0}^{2\pi} a^2\cos^2\theta\, d\theta$

$= \int \left(-4a^2\left(\frac{1-\cos 2\theta}{2}\right) + a^2\left(\frac{1+\cos 2\theta}{2}\right)\right) d\theta$

$= -\frac{3}{\cancel{2}} a^2 \times \cancel{2}\pi = -3a^2\pi$

OR By using divergence theorem,

$\iint_S (\nabla \times \vec{u}) \cdot \vec{ds} + \iint_{S_1} (\nabla \times \vec{u}) \cdot \vec{ds} = \iiint \nabla \cdot (\nabla \times \vec{u})\, dV$

where S_1 is the enclosing plane $z=0,\ x^2+y^2=a^2$

so $\iint_S (\nabla \times \vec{u}) \cdot d\vec{r} = -\iint_{S_1} \nabla \times \vec{u} \cdot \vec{ds'}$

$= -\iint (-3\hat{k}) \cdot (-\hat{k})\, dx\,dy = -3\iint_S dx\,dy$

$= \boxed{-3\pi a^2}$

ROUGH

5. $\vec{r}$ position vector, find values of n for which vector $r^n\vec{r}$ is
(i.) irrotational (ii.) solenoidal

$$\nabla \times (r^n\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Note: IRROTATIONAL if $\boxed{\nabla \times r^n\vec{r} = 0}$

$$\nabla \times (r^n\vec{r}) = \hat{i}\left(z\frac{\partial}{\partial y}r^n - y\frac{\partial}{\partial z}r^n\right)$$

$$+\hat{j}\left(x\frac{\partial}{\partial z}r^n - z\frac{\partial}{\partial x}r^n\right) + \hat{k}\left(y\frac{\partial}{\partial x}r^n - x\frac{\partial}{\partial y}r^n\right)$$

$$\frac{\partial}{\partial x}r^n = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r}$$

$$\left[\because r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}\right]$$

So, $\frac{\partial}{\partial x}r^n = nr^{n-2}x$, $\frac{\partial}{\partial y}r^n = nr^{n-2}y$, $\frac{\partial}{\partial z}r^n = nr^{n-2}z$

So $\nabla \times (r^n\vec{r}) = nr^{n-2}(0\hat{i} + 0\hat{j} + 0\hat{k})$
$= 0$

$\therefore$ for any real n, $r^n\vec{r}$ is irrotational

(ii) $\nabla \cdot (r^n\vec{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$$= 3r^n + xnr^{n-1}\frac{\partial r}{\partial x} + ynr^{n-1}\frac{\partial r}{\partial y} + znr^{n-1}$$

$= 3r^n + nr^{n-1}\left(\frac{x^2+y^2+z^2}{r}\right)$

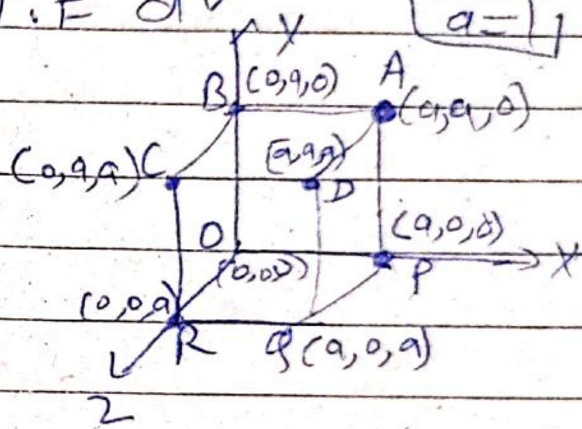
$= 3r^n + nr^n = (n+3)r^n$

so, solenoidal when $\nabla\cdot(r^n\vec{r}) = 0$

$\Rightarrow \boxed{n = -3}$

6. verify Gauss Divergence for $\vec{u} = x^2\hat{i} + y^2\hat{j} - z^2\hat{k}$ taken over cube $0 \le x, y, z \le 1$

$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV$ $\quad \boxed{a = 1}$



$\nabla\cdot\vec{F} = 2x + 2y - 2z$

$\iiint \nabla\cdot\vec{F}\, dV$

$= \iiint (2x + 2y - 2z)\, dx\, dy\, dz$

$= \iint \left(x^2 + 2yx - 2zx \Big|_{x=0}^{1}\right) dy\, dz$

$= \int_0^1 \left[\int_0^1 (1 + 2y - 2z)\, dy\right] dz = \int_0^1 \left(y + y^2 - 2zy \Big|_0^1\right) dz$

$= \int_0^1 (2 - 2z)\, dz = 2z - z^2 \Big|_0^1 = 1$

$\iint_S \vec{F}\cdot\vec{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

are 6 faces of cube

$= \iint_{APQD} + \iint_{OBCR} + \iint_{ABCD} + \iint_{OPQR} + \iint_{CDQR} + \iint_{ABOP}$

$= \iint x^2\hat{i}\cdot\hat{i}\,)\, dy\, dz + \iint_{x=0} -x^2\, dy\, dz - \iint_{z=1} z^2\, dx\, dy$

$+ \iint_{z=0} z^2\, dx\, dy + \iint_{y=a} y^2\, dx\, dz - \iint_{y=0} y^2\, dx\, dz$

$= 1 + 0 + 1 + 0 + 1 - 0 = 1$

$= RHS$

Hence verified

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