MAINSTORMING – 2019
MATHEMATICS
TEST- 1

Time Allowed: 3.00 Hrs ***Maximum: 250 Marks***

Units: Modern Algebra + Linear Algebra

Instructions

1. Question paper contains **8** questions out of this candidate need to answer **5** questions in the following pattern
2. Candidate should attempt question No's **1** and **5** compulsorily and any **three** of the remaining questions selecting **atleast one** question from each section.
3. The number of marks carried by each question is indicated at end of each question.
4. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/Notations carry their usual meanings, unless otherwise indicated.

Section- A

Q.1

a) Is the set of all rational numbers x such that $0 < x \leq 1$, a group w.r.t multiplication. (5 Marks)

b) Show that every finite group of prime order does not have any proper subgroup. (5 marks)

c) Let H be a subgroup of a group G and define $T = \{x \in G | xH = Hx\}$. Prove that T is a subgroup of G. (10 marks)

d) $(R, +)$ is an abelian group. Show that $(R, +, .)$ is a ring if multiplication (.) is defined as $a.b = 0 \ \forall a, b \in R$. (5 marks)

e) If R is a commutative ring prove that $(a+b)^2 = a^2 + 2ab + b^2$ $\forall a, b \in R$. (5 marks)

f) Show that the set R of all real valued continuous functions defined on [0,1] is a commutative ring with unity, w.r.t addition (+) and multiplication (*) of functions defined as

$(f+g)(x) = f(x) + g(x)$ and

$(f * g)(x) = f(x) * g(x)$

$\forall x \in [0,1]$ and $f, g \in R$. (10 marks)

g) The set N of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ $\forall a, b \in Z$ is a left ideal but not right ideal in the ring R of 2×2 matrices with elements as integers. Here N is the subset of R consisting of those elements whose second column contains only zeros. (10 marks)

Q.2

i. Do the following set form groups w.r.t binary operation * defined on them as follows $a * b = a + b + ab$ (10 marks)

ii. Show that the set of matrices A= $\begin{pmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{pmatrix}$ where α is a real number, forms a group under multiplication. (10 marks)

iii. If in a group G, $xy^2 = y^3x$ and $yx^2 = x^3y$, then show that $x = y = e$ where e is the identity in G. (15 marks)

iv. If H *and* K are two subgroups of a group G then show that HK is a subgroup of G iff $HK = KH$ (10 marks)

v. If H_1 *and* H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G. (5 marks)

Q.3

i. Show that two right cosets Ha, Hb of a group G are distinct iff two left cosets $a^{-1}H, b^{-1}H$ of G are distinct. (10 marks)

ii. Let H, K are two finite normal subgroups of co prime orders of a group G. Prove that $hk = kh, \forall h \in H, k \in K$. (20 marks

iii. Prove that those elements of a group G which commute with square of a given element b of G form a subgroup H of G and those which commute With b itself form a subgroup of H. (20 marks)

Q.4

i. Show that the set of Gaussian integers forms a ring under addition and multiplication of complex numbers. Is it an Integral domain? (15 marks)

ii. Show that set $\{a + b\omega, \omega^3 = 1\}\ \forall a, b \in R$ is a field w.r.t addition and multiplication. (10 marks)

iii. Construct a field of two elements. (10 marks)

iv. Prove that Z_7 is a field. (10 marks)

v. Find all solutions of $x^3 - 2x^2 - 3x = 0$ in Z_{12} (5 marks)

Section- B

Q.5

i. Show that the set W of elements of vector space $V_3(R)$ of the form $(x + 2y, y, -x + 3y)$ Where $x, y \in R$ is the subspace of $V_3(R)$. (10 marks)

ii. Express the vector $\alpha = (1, -2, 5)$ as a linear combination of the elements of the set $\{(1,1,1), (1,2,3), (2,-1,1)\} \subseteq R^3$ (10 marks)

iii. Show that the set $S = \{(1,2,1), (3,1,5)(3,-4,7)\} \subseteq R^3$ is linearly dependent. (10 marks)

iv. State Cayley Hamilton theorem and using it find inverse of $\begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix}$. (10 marks)

v. If A, B are two non singular matrices, then show that AB is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$. (10 marks)

Q.6

i. Find out for what values of λ *and* μ the equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have

a) No solution, b) unique solution and c) infinitely many solutions. (20 marks)

ii. Find a Basis and dimension of the solution space 'S' of linear equations $x + 2y - 2z + 2s - t = 0$

$x + 2y - z + 3s - 2t = 0$

$2x + 4y - 7z + s + t = 0$ (20 marks)

iii. Show that following matrices have same row space

$A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{bmatrix}$

(10 marks)

Q.7

i. Let V *and* W be following subspaces of R^4,

$V = \{(a, b, c, d)/b - 2c + d = 0\}$ $W = \{(a, b, c, d)/a = d, b = 2c\}$.

Fins Basis and dimension of V, W *and* $V \cap W$. Hence prove that $R^4 = V + W$. (20 marks)

ii. Find Eigen values and Eigen vectors of $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1 . \lambda_2 . \lambda_3$ equals determinant. Where λ_1, λ_2 *and* λ_3 are Eigen values of A.

(20 marks)

iii. If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1+i\sqrt{3}}{2} \end{bmatrix}$ the find the trace of A^{102}.

(10 marks)

Q.8

i. Show that $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is similar to a diagonal matrix. Also find transforming matrix and diagonal matrix

(20 marks)

ii. Show that every square matrix can be uniquely expressed as $P + iQ$, where P, Q are Hermition matrices. (10 marks)

iii. In the vector space R^3 let $\alpha = (1,2,1), \beta = (3,1,5), \gamma = (3,-4,7)$. Show that the subspace spanned by $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \gamma\}$ are same.

(10 marks)

iv. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then determine A^{50}. (10 marks)