CSE-2015 → Paper II

6) (c) Find the Lagrange interpolating polynomial that fits the following data :

x : -1 $\quad 2 \quad 3 \quad 4$

$f(x)$: $-1 \quad 11 \quad 31 \quad 69$, find $f(1.5)$.

⇒ Lagrange interpolating formula is,

$$L(x) = \omega(x) \sum \frac{f(x_r)}{(x-x_r)\omega'(x_r)} = \omega(x) \sum_{r=0}^{n} \frac{y_r}{D_r}$$

where $\omega(x) = (x-x_0) \cdots (x-x_r) \cdots (x-x_n)$

$$D_r = (x-x_r)(x_r-x_0) \cdots (x_r-x_{r-1})(x_r-x_{r+1}) \cdots (x_r-x_n)$$

				D_r	y_r	y_r/D_r
$x+1$	-3	-4	-5	$-60(x+1)$	-1	$\frac{1}{60(x+1)}$
3	$x-2$	-1	-2	$6(x-2)$	11	$\frac{11}{6(x-2)}$
4	1	$x-3$	-1	$-4(x-3)$	31	$-\frac{31}{4(x-3)}$
5	2	1	$x-4$	$10(x-4)$	69	$\frac{69}{10(x-4)}$

$\omega(x) = (x+1)(x-2)(x-3)(x-4)$

$$\therefore f(x) = (x+1)(x-2)(x-3)(x-4)\left[\frac{1}{60(x+1)} + \frac{11}{6(x-2)} - \frac{31}{4(x-3)} + \frac{69}{10(x-4)}\right]$$

$$= \frac{1}{60}\Big[(x-2)(x-3)(x-4) + 110(x+1)(x-3)(x-4) - 465(x+1)(x-2)(x-4) + 414(x+1)(x-2)(x-3)\Big]$$

$$= \frac{1}{60}\Big[(x^3-9x^2+26x-24) + 110(x^3-6x^2+5x+12) - 465(x^3-5x^2+2x+8) + 414(x^3-4x^2+x+6)\Big]$$

$$= \frac{1}{60}\Big[(1+110-465+414)x^3 + (-9-660+2325-1656)x^2 + (26+550-930+414)x + (-24+1320-3720+2484)\Big]$$

$$= \frac{1}{60}\left[60x^3 + 0x^2 + 60x + 60\right] = x^3 + x + 1$$

$$\therefore f(1.5) = (1.5)^3 + 1.5 + 1 = 5.875$$

7) (b) Solve the initial value problem $\frac{dy}{dx} = x(y-x)$, $y(2)=3$ in the interval $(2, 2.4)$ using the Runge-kutta four-order method with step size, $h=0.2$.

$\Rightarrow$ For $y(2.2) \Rightarrow x_0 = 2,\ y_0 = 3,\ f(x,y) = xy - x^2,\ h = 0.2$

$k_1 = hf(x_0, y_0) = 0.2 f(2,3) = 0.4$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2 f(2.1, 3.2) = 0.462$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2 f(2.1, 3.231) = 0.47502$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(2.2, 3.47502) = 0.56101$

$\therefore y_1 = y(2.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 3 + \frac{1}{6} \times 2.83505$

$= 3.472508 \simeq 3.47251$

For $y(2.4) \Rightarrow x_1 = 2.2,\ y_1 = 3.47251,\ h = 0.2$

$k_1 = hf(x_1, y_1) = 0.2 f(2.2, 3.47251) = 0.5599$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.2 f(2.3, 3.75246) = 0.66813$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.2 f(2.3, 3.80658) = 0.69303$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 f(2.4, 4.16554) = 0.84746$

$\therefore y_2 = y(2.4) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 4.16079$

8) (b) Find the solution of the system,

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

using Guass-Seidal method (make four iteration)

$\Rightarrow$ The given system is clearly diagonally dominant. Now we write the iteration formula as:

$$x_1^{(k+1)} = \frac{1}{10}[3 + 3x_2^{(k)} + x_3^{(k)} + x_4^{(k)}]$$

$$x_2^{(k+1)} = \frac{1}{10}\left[15 + 2x_1^{(k+1)} + x_3^{(k)} + x_4^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{10}\left[27 + x_1^{(k+1)} + x_2^{(k+1)} + 2x_4^{(k)}\right]$$

$$x_4^{(k+1)} = \frac{1}{10}\left[-9 + x_1^{(k+1)} + x_2^{(k+1)} + 2x_3^{(k+1)}\right]$$

$\therefore$ $\cancel{x_1}$ Initially putting $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$

<u>first iteration</u> (when $k=0$)

$x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = 0.1368$

<u>2nd iteration</u> (when $k=1$)

$x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$

<u>3rd iteration</u> (when $k=2$)

$x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$, $x_4 = -0.0042$

<u>4th iteration</u> (when $k=3$)

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$

$\therefore$ The solution is,

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$