Mains Test Series – 2018.

Test – 10, Paper – II, Answer Key.

1(c) A function $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(0) = 0$ and

$f(x) = 0$, if x is irrational

$= \frac{1}{q}$, if $x = p/q$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\gcd(p, q) = 1$.

Show that f is not differentiable at 0.

Sol^n: $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \frac{f(x)}{x}$

Let $\phi(x) = \frac{f(x)}{x}$.

Let $\{x_n\}$ be a sequence of rational points converging to 0 where $x_n = \frac{1}{n}$, $n \in \mathbb{N}$

Then $\lim\limits_{n \to \infty} \phi(x_n) = \lim\limits_{n \to \infty} n \cdot \frac{1}{n} = 1$

Let $\{y_n\}$ be a sequence of irrational points converging to 0.

$\lim\limits_{n \to \infty} \phi\{y_n\} = \lim\limits_{n \to \infty} \frac{f(y_n)}{y_n} = 0$

Since $f(y_n) = 0$ for all $n \in \mathbb{N}$

Therefore $\lim\limits_{x \to 0} \phi(x)$ does not exist,

Since for two sequences $\{x_n\}$ and $\{y_n\}$ both converging to 0, the sequences $\{\phi(x_n)\}$ and $\{\phi(y_n)\}$ converge to two different limits.

Hence f is not differentiable at 0.

1(8). Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4}\,dz$, where C is the circle $|z|=2$.

Sol'n: Comparing the given integral with

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz,$$

we get

$$f(z)=e^{3z}, \quad z=-1$$

Since $f(z)$ is analytic in $|z|=2$ and $z_0=-1$ is a point inside $|z|=2$

$\therefore$ we apply Cauchy integral formula

$$\int_C \frac{dz}{(z-z_0)^4} = \frac{2\pi i}{n!} f^3(z_0) \quad \text{--- (1)}$$

Now $f(z) = e^{3z}$

$\Rightarrow f'(z) = 3e^{3z} \Rightarrow f'(-1) = 3e^{-3} \quad (\because z_0=-1)$

$\Rightarrow f''(z) = 9e^{3z} \Rightarrow f''(-1) = 9e^{-3}$

$\Rightarrow f'''(z) = 27e^{3z} \Rightarrow f'''(-1) = 27e^{-3}$

$\therefore$ from (1), we have

$$\int_C \frac{dz}{(z+1)^4} = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{6}(27)e^{-3}$$

$$= 9\pi i e^{-3}.$$

1(e) A construction Company has to move four large cranes from old construction site to new construction site. The distance in kilometers between the old and new locations are as given in the adjoining table. The at O_3 cannot be used at N_2 but all the cranes can work equally well at any of the other new sites. Determine a plan for moving the cranes that will minimise the total distance involved in the move.

New Const

	N_1	N_2	N_3	N_4
O_1	15	20	13	40
O_2	38	42	15	20
O_3	25	17	30	18
O_4	18	30	40	35.

Sol Given O_3 cannot be assigned to N_2

So Replace corresponding value in table to very large value

	N_1	N_2	N_3	N_4
O_1	15	20	13	40
O_2	38	42	15	20
O_3	25	∞	30	18
O_4	18	30	40	35

$\Longrightarrow$ Substracting smallest number of each row with whole row values.

	N_1	N_2	N_3	N_4
O_1	2	7	0	27
O_2	23	27	0	5
O_3	7	∞	12	0
O_4	0	12	22	17

⇓ Substracting smallest Column value with the corresponding column values

	N_1	N_2	N_3	N_4
O_1	2	$\boxed{0}$	0	27
O_2	23	20	$\boxed{0}$	5
O_3	7	∞	12	$\boxed{0}$
O_4	$\boxed{0}$	5	22	17

Assigning, $O_1 \rightarrow N_2$

$O_2 \rightarrow N_3$

$O_3 \rightarrow N_4$

$O_4 \rightarrow N_1$

minimise the total distance involved in moving the cranes.

2(a) Let G be a group and H, K be two normal subgroups of G. If G is an internal direct product of H and K then, (i) $G \simeq H \times K$,
(ii) $G/H \simeq K$, and $G/K \simeq H$.

Sol'n: (i) Suppose that G is an internal direct product of the normal subgroups H and K.
Then $G = HK$ and $H \cap K = \{e\}$. Hence for every $g \in G$, there are unique elements $a \in H$ and $b \in K$ such that $g = ab$.

So we can define $f: G \to H \times K$ by $f(g) = (a, b)$ when $g = ab$, $a \in H$, $b \in K$. Let $g_1 = a_1 b_1$ and $g_2 = a_2 b_2$ be two elements of G, where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Now $g_1 g_2 = a_1 b_1 a_2 b_2 = a_1 a_2 b_1 b_2$.
Hence $f(g_1 g_2) = (a_1 a_2, b_1 b_2) = (a_1, b_1)(a_2, b_2) = f(g_1) f(g_2)$.

This shows that f is a homomorphism.
Since for every $g \in G$, there are unique elements $a \in H$ and $b \in K$ such that $g = ab$, it follows that f is an injective function. Again, if $(a, b) \in H \times K$, then $g = ab \in G$ and hence $f(g) = (a, b)$. Combining all these, we find that f is an isomorphism and so $G \simeq H \times K$.

(ii) Since for each $g \in G$, there are unique elements $a \in H$, $b \in K$, the function $\varphi: G \to H$ defined by $\varphi(g) = \varphi(ab) = a$, for all $g = ab \in G$ can be shown to be an epimorphism. Hence by the first isomorphism theorem $G/\ker\varphi \simeq H$.

Now, $\ker\varphi = \{g \in G \mid \varphi(g) = e\}$
$= \{g = ab \in G \mid a \in H, b \in K \text{ and } \varphi(g) = e\}$

$= \{ g = ab \in G \mid a \in H, b \in K \text{ and } a = e \}$

$= \{ g = eb \in G \mid b \in K \} = K$

Thus $G/K \cong H$. Similarly, we can show that $G/H \cong K$.

2(b) → Let a function f be continuous on an open bounded interval (a,b). Then f is uniformly continuous on (a,b) if and only if $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Sol^n: Let f be continuous on an open bounded interval (a,b) and let $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ both exist finitely.

Let us define a function g on $[a,b]$ by $g(x) = f(x)$, for all $x \in (a,b)$ and $g(a) = \lim\limits_{x \to a+} f(x)$, $g(b) = \lim\limits_{x \to b-} f(x)$.

g is continuous on (a,b), since f is continuous on (a,b).

$g(a) = \lim\limits_{x \to a+} f(x)$ (by definition) $= \lim\limits_{x \to a+} g(x)$ and

$g(b) = \lim\limits_{x \to b-} f(x)$ (by definition) $= \lim\limits_{x \to b-} g(x)$.

$\therefore$ g is right continuous at a and left continuous at b and consequently, g is continuous on $[a,b]$.

[By theorem: Let $I = [a,b]$ be a closed and bounded interval and a function $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I]

– g is uniformly continuous on $[a,b]$

[By theorem: Let I be an interval & a function $f: I \to \mathbb{R}$ be uniformly continuous on I. Then f is continuous on I], g is uniformly continuous on (a,b).

Since $g = f$ on (a,b), it follows that f is uniformly continuous on (a,b).

Conversely, let f be uniformly continuous on (a,b).

We prove that both the limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to b-} f(x)$ exist finitely.

Let $\{x_n\}$ be a sequence in (a,b) converging to a. Then $\{x_n\}$ is a Cauchy sequence in (a,b). Since f is uniformly continuous on (a,b), the sequence $\{f(x_n)\}$ is a cauchy sequence in $\mathbb{R}$ and therefore it is convergent. Let $\lim_{n\to\infty} f(x_n)=l$.

Let $\{y_n\}$ be another sequence in (a,b) converging to a. Then the sequence $\{x_n-y_n\}$ is a sequence in (a,b) converging to 0.

Let $\epsilon>0$. Since f is uniformly continuous on (a,b), there exists a positive δ such that for any two points $x_1, x_2 \in (a,b)$

$$|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\epsilon/2$$

Since $\{x_n-y_n\}$ is a sequence in (a,b) converging to 0, there exists a natural number k such that $|x_n-y_n|<\delta$ for all $n\geq k$.

$\therefore |f(x_n)-f(y_n)|<\epsilon/2$ for all $n\geq k$.

$|f(y_n)-l|\leq|f(y_n)-f(x_n)|+|f(x_n)-l|<\epsilon$ for all $n\geq k$. This proves that $\lim_{n\to\infty} f(y_n)=l$.

Thus for every sequence $\{x_n\}$ in (a,b) converging to a, the sequence $\{f(x_n)\}$ converges to the limit l. This implies $\lim_{x\to a+} f(x)=l$.

In a similar manner it can be proved that $\lim_{x\to b-} f(x)$ exists finitely.

2(c). Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$.

Assume that the zeroes of the denominator are simple. Show that the sum of residues of $f(z)$ at its poles is equal to $\dfrac{a_{n-1}}{b_n}$.

Sol^n: Let $f(z) = \dfrac{a_0}{b_0 + b_1 z}$, where $b_1 \neq 0$.

$$= \frac{a_0}{b_1\left[\frac{b_0}{b_1} + z\right]}.$$

$z = \frac{-b_0}{b_1}$ is a pole of order 1 i.e. simple pole.

The residue at $z = -b_0/b_1$ is

$$= \lim_{z \to b_0/b_1} \left(\frac{b_0}{b_1} + z\right) \frac{a_0}{b_1\left(\frac{b_0}{b_1} + z\right)}$$

$$= \frac{a_0}{b_1}$$

Now let us assume that $f(z) = \dfrac{a_0 + a_1 z}{b_0 + b_1 z + b_2 z^2}$

$$= \frac{a_0 + a_1 z}{b_2\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)}$$

Let $z = \alpha, \beta$ be the simple poles of

$$\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)$$

[Now since α, β are the roots of $z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2}$

$$z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2} = (z-\alpha)(z-\beta)$$

$$= z - (\alpha + \beta) + \alpha\beta \quad]$$

Residue at $z = \alpha$ is $= \lim_{z \to} (z-\alpha) \dfrac{a_0 + a_1 z}{b_2 (z-\alpha)(z-\beta)}$

$$= \frac{a_0 + a_1\alpha}{b_2(\alpha-\beta)}$$

Residue at $z=\beta$ is $= \lim_{z\to\beta} (z-\beta) \dfrac{a_0+a_1 z}{b_2(z-\alpha)(z-\beta)}$

$$= \frac{a_0+a_1\beta}{b_2(\beta-\alpha)}$$

$$= -\frac{a_0+a_1\beta}{b_2(\alpha-\beta)}$$

$\therefore$ Sum of the residues of $f(z) = \dfrac{a_0+a_1 z}{b_2\left(z^2+\frac{b_1}{b_2}z+\frac{a_0}{b_0}\right)}$

$$= \frac{1}{b_2}\left[\frac{a_0+a_1\alpha}{\alpha-\beta} - \frac{a_0+a_1\beta}{\alpha-\beta}\right]$$

$$= \frac{1}{b_2}\left[\frac{a_1(\cancel{\alpha-\beta})}{\cancel{\alpha-\beta}}\right] = \frac{a_1}{b_2}$$

Let $f(z) = \dfrac{a_0+a_1 z+a_2 z^2}{b_0+b_1 z+b_2 z^2+b_3 z^3}$

$$= \frac{a_0+a_1 z+a_2 z^2}{b_3\left[(z-\alpha)(z-\beta)(z-\gamma)\right.}$$

where α, β, γ are the simple poles.

Residue at $z=\alpha$ is $\lim_{z\to\alpha} \cancel{(z-\alpha)} \dfrac{a_0+a_1 z+a_2 z^2}{b_3\cancel{(z-\alpha)}(z-\beta)(z-\gamma)}$

$$= \frac{a_0+a_1\alpha+a_2\alpha^2}{b_3(\alpha-\beta)(\alpha-\gamma)}$$

Residue at $z=\beta$ is $\lim_{z\to\beta} \cancel{(z-\beta)} \dfrac{a_0+a_1 z+a_2 z^2}{b_3(z-\alpha)\cancel{(z-\beta)}(z-\gamma)}$

$$= \frac{a_0 + a_1\beta + a_2\beta^2}{b_3(\beta-\alpha)(\beta-\gamma)}$$

Residue at $z=\gamma$ is $= \lim_{z\to\gamma} \cancel{(z-\gamma)} \dfrac{a_0 + a_1 z + a_2 z^2}{b_3(z-\alpha)(z-\beta)\cancel{(z-\gamma)}}$

$$= \frac{a_0 + a_1\gamma + a_2\gamma^2}{b_3(\gamma-\alpha)(\gamma-\beta)}$$

$\therefore$ Sum of the residues of $f(z)$ at its poles α, β, γ

$$= \frac{1}{b_3}\left[\frac{a_0 + a_1\alpha + a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} + \frac{a_0 + a_1\beta + a_2\beta^2}{(\beta-\alpha)(\beta-\gamma)} + \frac{a_0 + a_1\gamma + a_2\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0 + a_1\alpha + a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} - \frac{a_0 + a_1\beta + a_2\beta^2}{(\alpha-\beta)(\beta-\gamma)} + \frac{a_0 + a_1\gamma + a_2\gamma^2}{(\alpha-\gamma)(\beta-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{(a_0 + a_1\alpha + a_2\alpha^2)(\beta-\gamma) - (a_0 + a_1\beta + a_2\beta^2)(\alpha-\gamma) + (a_0 + a_1\gamma + a_2\gamma^2)(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0(0) + a_1(0) + a_2\left(\alpha^2(\beta-\gamma) - \beta^2(\alpha-\gamma) + \gamma^2(\alpha-\beta)\right)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_2\left[\alpha^2\beta - \gamma\alpha^2 - \beta^2\alpha + \gamma\beta^2 + \gamma^2\alpha - \gamma^2\beta\right]}{\alpha^2\beta - \alpha\beta\gamma - \alpha^2\gamma + \alpha\gamma^2 - \beta^2\alpha + \beta^2\gamma + \alpha\beta\gamma - \gamma^2\beta}\right]$$

$$= \frac{a_2}{b_3} \quad (b_3 \neq 0)$$

$\therefore$ we conclude that

$$f(z) = \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + b_2 z^2 + \cdots + b_n z}$$

the sum of the residues of $f(z)$ at its poles

is $= \dfrac{a_{n-1}}{b_n}$ where $b_n \neq 0$.

4(b) → Let $f(x)$, $(x \in (-\pi, \pi))$ be defined by $f(x) = \sin|x|$. Is continuous on $(-\pi, \pi)$? If it is continuous, then is it differentiable on $(-\pi, \pi)$?

Sol^n: Given that $f(x) = \sin|x|$, $x \in (-\pi, \pi)$

$$\text{i.e. } f(x) = \begin{cases} \sin(-x) & \text{if } x \in (-\pi, 0) \\ \sin x & \text{if } x \in (0, \pi) \end{cases}$$

clearly $f(x)$ is continuous and differentiable over each subinterval. The only doubtful point is the breaking point $x = 0$.

At $x=0$, $f(x)=0$

Now LHL: $\underset{x\to 0-}{\text{Lt}} f(x) = \sin(-x) = 0$

RHL: $\underset{x\to 0+}{\text{Lt}} f(x) = \sin x = 0$

$\therefore \underset{x\to 0-}{\text{Lt}} f(x) = 0 = \underset{x\to 0+}{\text{Lt}} f(x)$

$\therefore f$ is continuous at $x=0$

Now RHD: $Rf'(0) = \underset{x\to 0+}{\text{Lt}} \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0+}{\text{Lt}} \dfrac{\sin x - 0}{x}$

$= \underset{x\to 0+}{\text{Lt}} \dfrac{\sin x}{x} = 1$

LHD: $Lf'(0) = \underset{x\to 0-}{\text{Lt}} \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0-}{\text{Lt}} \dfrac{\sin(-x)-0}{x-0}$

$$= \lim_{x \to 0-} \frac{-\sin x}{x} = -1$$

$\therefore Lf'(0) \neq Rf'(0)$

$\therefore f(x)$ is not differentiable at $x=0$

Hence f is continuous on $(-\pi, \pi)$

Also f is differentiable on $(-\pi,\pi)$ except at $x=0$.

4(c) (i) Find the expansion of $\frac{1}{(z^2+1)(z^2+2)}$ in powers of z when

(a) $|z|<1$ (b) $1<|z|<\sqrt{2}$ (c) $|z|>\sqrt{2}$.

Sol^n: Let $f(z) = \frac{1}{(z^2+1)(z^2+2)}$

Resolving $f(z)$ into partial fractions, we obtain

$$f(z) = \frac{1}{(z^2+1)(z^2+2)} = \frac{1}{z^2+1} - \frac{1}{z^2+2}$$

(i) For $|z|<1$, $f(z)$ is

$$f(z) = \frac{1}{z^2+1} - \frac{1}{z^2+2}$$

$\left[\because |z| < 1 < \sqrt{2} \Rightarrow \frac{|z|}{\sqrt{2}} < 1 \Rightarrow \frac{z^2}{2} < 1\right]$

$$= (1+z^2)^{-1} - \frac{1}{2}\frac{1}{\left(1+\frac{z^2}{2}\right)}$$

$$= (1+z^2)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)$$

$$= \left(1 - z^2 + z^4 - z^6 + \cdots\right) - \frac{1}{2}\left(1 - \frac{z^2}{2} + \frac{z^4}{2^2} - \frac{z^6}{2^3} \cdots\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z^2}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}}\right) z^{2n}$$

(ii) for $1 < |z| < \sqrt{2}$ $\left(\because 1 < |z| \Rightarrow \frac{1}{|z|} < 1 \ \& \ |z| < \sqrt{2} \Rightarrow \frac{|z|}{\sqrt{2}} < 1 \Rightarrow \frac{z^2}{2} < 1\right)$

$$\frac{1}{z^2+1} - \frac{1}{z^2+2} = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{2}\frac{1}{\left(1+\frac{z^2}{2}\right)}$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$$

$$= \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} + \cdots\right) - \frac{1}{2}\left(1 - \frac{z^2}{2} + \frac{z^4}{2^2} + \cdots\right)$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n}} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n+2}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n}}{2^{n+1}}$$

(iii) $|z| > \sqrt{2} \Rightarrow |z| > 1$

$\frac{1}{|z|} < 1$ and $\frac{\sqrt{2}}{|z|} < 1$

$$\therefore \frac{1}{z^2+1} - \frac{1}{z^2+2} = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{z^2}\left(1+\frac{2}{z^2}\right)^{-1}$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{z^2}\left(1+\frac{2}{z^2}\right)^{-1}$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{1}{z^{2n}} - \frac{1}{z^2}\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{z^{2n}}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(1-2^n)}{z^{2n+2}}$$

4(c)(ii) If $f(z)$ is a regular function z prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Sol^n. Let the analytic function $f(z)$ be

$f(z)=u+iv$, then $|f(z)|^2=u^2+v^2$

we have

$$\frac{\partial}{\partial x}u^2 = 2u\frac{\partial u}{\partial x}$$

$$\therefore \frac{\partial^2}{\partial x^2}(u^2) = 2\left(\frac{\partial u}{\partial x}\right)^2+2u\frac{\partial^2 u}{\partial x^2}$$

Similarly $\frac{\partial^2}{\partial y^2}u^2 = 2\left(\frac{\partial u}{\partial y}\right)^2+2u\frac{\partial^2 u}{\partial y^2}$

on adding, we have

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]+2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]$$

since u is harmonic

$$= 2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2\right]$$

$(\because u_y=-v_x)$

$$= 2|f'(z)|^2$$

$\left(\because f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)$

Similarly $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v^2 = 2|f'(z)|^2$

Hence $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2) = 4|f'(z)|^2$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

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4(d) Solve the following LPP by Simplex Method

Maximize $Z = 3x_1 + 9x_2$

subject to

$x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1, x_2 \geq 0$.

Sol'n: The objective function of the given LPP is of maximization type and RHS of all constraints are ≥ 0.

Now we write the given LPP in the standard form.

$\text{Max}\, Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$

subject to

$x_1 + 4x_2 + s_1 = 8$

$x_1 + 2x_2 + s_2 = 4$

$x_1, x_2, s_1, s_2 \geq 0$.

where s_1, s_2 are slack variables

Now the initial basic feasible solution is given by

letting $x_1 = x_2 = 0$ (non-basic)

$s_1 = 8, \; s_2 = 4$. (basic)

$\therefore$ The initial basic-feasible solution is $(0, 0, 8, 4)$ for which $Z = 0$.

put the above information in tableau form:

	C_j	3	9	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	1	4	1	0	8	2
0	S_2	1	2	0	1	4	2 →
	$Z_j = \Sigma C_B a_{ij}$	0	0	0	0	0	
	$C_j = C_j - Z_j$	3	9 ↑	0	0		

From the above table, we observe that the non basic variable x_2 enters into the basis. since the minimum ratio is 2 for both the slack variables S_1 and S_2, there is a tie for the variable to leave the basis. This is an indication of the existence of

first column of the unit matrix 1 and 0 in the tied rows. Dividing these by the corresponding elements of the key column, we get $1/4$ & $0/2$, S_2-row gives the smaller ratio and therefore S_2 is the first outgoing variable and (2) is the key element.

Thus the new simplex table is

	C_j	3	9	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	b
0	s_1	-1	0	1	-2	0
9	x_2	1/2	1	0	1/2	2
Z_j	$=\sum C_B a_{ij}$	9/2	9	0	9/2	18
C_j	$= C_j - Z_j$	-3/2	0	0	-9/2.	

As C_j is either zero or negative (i.e $C_j \le 0$) under all columns, the ~~above~~ table gives the optimal basic feasible solution.

$\therefore$ The optimal solution is $x_1 = 0$
$x_2 = 2$

and maximize $Z = 18$.

5(a) Solve $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y}\sin(2x+y)$. 10

Sol: $C.F = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x)$

ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

$$P.I = \frac{1}{(D+D'-1)(D+D'-3)(D+D')} e^{x+y}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+1+D'+1-1)(D+1+D'+1-3)(D+1+D'+1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+1)(D+D'-1)(D+D'+2)}\sin(2x+y)$$

$$= e^{x+y}\cdot\frac{1}{(D+D'+2)(D^2+2DD'+D'^2-1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+2)}\frac{1}{-2^2-2\times(2\times1)-1^2-1}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+2)}\frac{1}{-10}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}\frac{D+D'-2}{(D+D')^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{D^2+2DD'+D'^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{-2^2-2\times(2\times1)-1^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}\frac{1}{(-13)}(D+D'-2)\sin(2x+y)$$

$$= \frac{1}{130}e^{x+y}\,2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)$$

$\therefore$ The solution is given by

$$z = e^{x}\phi_1(y-x) + e^{3x}\phi_2(y-x) + \phi_3(y-x) + \frac{1}{130}\left[3\cos(2x+y)-2\sin(2x+y)\right]$$

5(b) → Reduce $x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 \, (x>0)$ to canonical form.

Sol'n: Rewriting the given equation, we get

$$xr + t^2 - x^2 = 0, \; x>0 \quad \text{——} \; (1)$$

comparing (1), with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$.

here $R = x$, $S = 0$, $T = 1$ so that

$S^2 - 4RT = -4x < 0$, showing that (1) is elliptic.

The λ- quadratic equation

$R\lambda^2 + S\lambda + T = 0$ reduces to

$x\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -\frac{1}{x}$

i.e., $\lambda = \frac{i}{x^{1/2}}, \frac{-i}{x^{1/2}}$

The corresponding characteristic equations are given by

$$\frac{dy}{dx} + ix^{-1/2} = 0 \quad \text{and} \quad \frac{dy}{dx} - ix^{-1/2} = 0$$

Integrating, we get

$$y + 2ix^{1/2} = C_1 \quad \text{and} \quad y - 2ix^{1/2} = C_2$$

Choose $u = y + 2ix^{1/2}$ and $v = y - 2ix^{1/2}$
$= \alpha + i\beta$ $\qquad = \alpha - i\beta$

where $\alpha = y$ and $\beta = 2x^{1/2}$ —— (2)

are now new independent variables.

NOW, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta}$ —— (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}$ —— (4)

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right)$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-\frac{1}{2}}\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)\right]$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-\frac{1}{2}}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{0 + x^{-\frac{1}{2}}\frac{\partial^2 z}{\partial \beta^2}\right\}$$

$$= -\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \quad \text{(5)}$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial \alpha}\left(\dfrac{\partial z}{\partial \alpha}\right) = \dfrac{\partial^2 z}{\partial \alpha^2}$ — (6)

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right) + \frac{\partial^2 z}{\partial \alpha^2} = x^2$$

$$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = x^2 + \frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

$$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\beta^4}{16} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}, \quad \text{as } \beta = 2x^{1/2}.$$

5(C) → The observed values of a function are respectively 168, 120, 72 and 63 at the four positions 3, 7, 9 and 10 of the independent variable. What is the best estimate for the value of the function at the position 6.

Sol: Given that

x	3	7	9	10
y	168	120	72	63

; To find the value at 6

using Lagrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(6) = \frac{(6-7)(6-9)(6-10)}{(3-7)(3-9)(3-10)}(168) + \frac{(6-3)(6-9)(6-10)}{(7-3)(7-9)(7-10)}(120)$$

$$+ \frac{(6-3)(6-7)(6-10)}{(9-3)(9-7)(9-10)}(72) + \frac{(6-3)(6-7)(6-9)}{(10-3)(10-7)(10-9)}(63)$$

$$= \frac{(-1)(-3)(-4)}{(-4)(-6)(-7)}(168) + \frac{(3)(-3)(-4)}{(4)(-2)(-3)}(120)$$

$$+ \frac{(3)(-1)(-4)}{(6)(2)(-1)}(72) + \frac{(3)(-1)(-3)}{(7)(3)(1)}(63)$$

$$= 12 + 180 - 72 + 27$$

$$f(6) = 147.$$

5(d) → A majority function is a digit circuit whose output is 1 iff the majority of the inputs are 1. The output is 0 otherwise. obtain the truth table of a three-input majority function can be obtained with 4 NAND gates.

Sol^n: Let A, B, C be three inputs. The design of this circuit can be done in the following steps.

→ Prepare a truth table. The output Y is 1 whenever two (or) more inputs are 1. otherwise output is 0.

A	B	C	Y	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{A}BC$
1	0	0	0	
1	0	1	1	$A\bar{B}C$
1	1	0	1	$AB\bar{C}$
1	1	1	1	ABC

→ Write AND terms when $Y=1$. These terms have each input variable in either non-inverted or inverted form. These terms are shown on ~~truth~~ table.

→ the output expression is

$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC.$$

→ Simplify the expression.

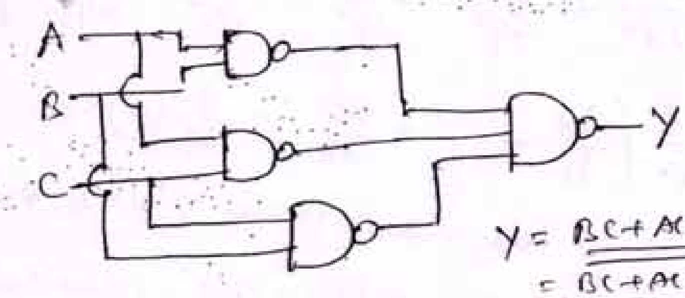
$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC + ABC + ABC$$

$$= \bar{A}BC + ABC + A\bar{B}C + ABC + AB\bar{C} + ABC$$

$$= BC(\bar{A}+A) + AC(\bar{B}+B) + AB(\bar{C}+C)$$

$$= BC + AC + AB$$

The logic circuit using 4 NAND gates be given below.



$$Y = BC + AC + AB$$

$$= \overline{\overline{BC + AC + AB}}$$

$$= \overline{\overline{BC} \cdot \overline{AC} \cdot \overline{AB}}.$$

5(e) $\rightarrow$ In an incompressible fluid the vorticity at every point is constant in magnitude and direction; Prove that the components of velocity u, v, w are the solutions of Laplace equation.

Sol^n: Let $W = \xi i + \eta j + \zeta k$,

$$q = ui + vj + wk$$

Vorticity is constant in magnitude and direction.

$\Rightarrow \xi, \eta, \zeta$ are constant.

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{const.}, \quad \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{const.}$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z}\right) = \zeta = \text{const.}$$

$$\therefore \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{Const.} \quad \text{——} (1) \qquad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{Const.} \quad \text{——} (2)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial z} = \text{Constant.} \quad \text{——} (3)$$

Differentiation of (2) and (3) w.r.t z and y gives

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \partial z}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial x}(0) = 0$$

$\therefore \nabla^2 u = 0$. Similarly we can prove $\nabla^2 v = 0, \nabla^2 w = 0$

It means that components of velocity are solutions of Laplace's equation.

6(a) Find a surface satisfying $r-2s+t=6$ and touching the hyperbolic paraboloid $z=xy$ along its section by the plane $y=x$.

Sol^n: Re-writing the given equation:

$$\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=6$$

i.e $(D^2-2DD'+D'^2)z=6$

$\Rightarrow (D-D')^2 z=6$ ——— (1)

$\therefore$ C.F $=\phi_1(y+x)+x\phi_2(y+x)$,

ϕ_1, ϕ_2 being arbitrary functions.

Now $P.I = \frac{1}{(D-D')^2}6 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}6$

$= \frac{1}{D^2}\left[1+\frac{2D'}{D}+\cdots\right](6)$

$= \frac{1}{D^2}(6) = 3x^2$.

$\therefore$ General solution of (1) is $z=CF+PI$

$z=\phi_1(y+x)+x\phi_2(y+x)+3x^2$ — (2)

Since the required surface (1) touches the given surface $z=xy$ ——— (3)

along the section $y=x$, the values of p and q ~~for~~ the two surfaces must be equal for any point on the plane $y=x$. ——— (4)

Now equating the values of p and q from (2) & (3),

we have $p=\phi_2(y+x)+x\phi_2'(y+x)+\phi_1'(y+x)+6x=y$ ——— (5)

and $q=x\phi_2'(y+x)+\phi_1'(y+x)=x$ ——— (6)

Subtracting (6) from (5) and using (4), we get

$\phi_2(2x)=-6x=-3(2x)$

which gives $\phi_2(y+x)=-3(y+x)$ ——— (7)

from (7), $\phi_2'(y+x)=-3$. Then (6) becomes

$-3x+\phi_1'(y+x)=x$ so that $\phi_1'(2x)=2\times 2x$ as $y=x$.

Now $\phi_1'(2x)=2(2x) \Rightarrow \phi_1'(x)=2x$ ——— (8)

Integrating (8), $\phi_1(x)=x^2+C$

which gives $\phi_1(y+x)=(y+x)^2+C$ ——— (9)

putting the values of $\phi_2(y+x)$ and $\phi_1(y+x)$ given by (7) and (9) in (2), we get

$z=x\{-3(y+x)\}+(y+x)^2+c+3x^2$

$\Rightarrow z=x^2-xy+y^2+c$ ——— (10)

Equating the values of z from (5) and (10), we get $xy=x^2-xy+y^2+c$

(or) $x^2=x^2-x^2+x^2+C \Rightarrow C=0$.

Hence the required surface is

$z=x^2-xy+y^2$.

6(b) Find the differential equation of the set of all right circular cones whose axes coincide with z-axis.

Sol: The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and the vertex at $(0,0,c)$ is given by

$$x^2+y^2=(z-c)^2\tan^2\alpha \quad \text{--- (1)}$$

in which both constants α and c are arbitrary.

Differentiating (1) partially w.r.t x & y, we get

$$2x=2(z-c)\left(\frac{\partial z}{\partial x}\right)\tan^2\alpha \quad \text{--- (2)}$$

and $$2y=2(z-c)\left(\frac{\partial z}{\partial y}\right)\tan^2\alpha. \quad \text{--- (3)}$$

multipling equations (2) and (3) by y and x respectively, we get

$$xy=y(z-c)\frac{\partial z}{\partial x}\tan^2\alpha$$

and $$xy=x(z-c)\frac{\partial z}{\partial y}\tan^2\alpha.$$

$$\Rightarrow y(z-c)\frac{\partial z}{\partial x}\tan^2\alpha=x(z-c)\frac{\partial z}{\partial y}\tan^2\alpha$$

$$\Rightarrow y\frac{\partial z}{\partial x}=x\frac{\partial z}{\partial y}$$

which is the required partial differential equation.

6(c) Write down and integrate completely the equations for the characteristics of $(1+q^2)z = px$. Expressing x, y, z and p in terms of ϕ, where $q = \tan\phi$ and determine the integral surface which passes through the parabola $x^2 = 2z,\ y = 0$

Sol: Given that $(1+q^2)z = px$.

Let $f(x, y, z, p, q) = (1+q^2)z - px$ — ①

we are to find the integral surface of ① which passes through parabola $x^2=2z,\ y=0$ whose parametric equations are

$x = \lambda,\ y = 0,\ z = \frac{\lambda^2}{2}$.

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = f_1(\lambda) = \lambda,\quad y_0 = f_2(\lambda) = 0,\quad z_0 = f_3(\lambda) = \frac{\lambda^2}{2}$.

To find p_0 and q_0.

we have $f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$

$\lambda = p_0(1) + q_0(0)$

$\Rightarrow \boxed{p_0 = \lambda}$

Also $(1+q_0^2) z_0 = p_0 x_0$

$\Rightarrow (1+q_0^2)\frac{\lambda^2}{2} = \lambda^2 \quad \Rightarrow \frac{\lambda^2}{2} q_0^2 = \frac{\lambda^2}{2}$

$\Rightarrow q_0^2 = 1 \Rightarrow \boxed{q_0 = 1}$

$\therefore x_0 = \lambda,\ y_0 = 0,\ z_0 = \lambda^2/2,\ p_0 = \lambda,\ q_0 = 1.$

The characteristic equations of (1) are

$\frac{dx}{d\phi} = f_p = -x$ —(2) ; $\frac{dy}{d\phi} = f_q = 2zq$ —(3)

$\frac{dz}{d\phi} = pf_p + qf_q = p$
$= p(-x) + q(2zq)$
$= -z - zq^2 + 2zq^2 \quad (\because px = (1+q^2)z)$
$= -z(1-q^2)$ —(4)

$\frac{dp}{d\phi} = -f_x - pf_z = p - p(1+q^2) = -pq^2$ —(5)

$\frac{dq}{d\phi} = -f_y - qf_z = 0 - q(1+q^2) = -q(1+q^2)$ —(6)

Dividing (4) by (6)

$\frac{dz}{dq} = \frac{-z(1-q^2)}{-q(1+q^2)}$

$\Rightarrow \frac{dz}{z} = \frac{1-q^2}{q(1+q^2)}\,dq$ —(7)

since $q = \tan\phi$ and $q_0 = 1$

$\Rightarrow \boxed{\phi_0 = \pi/4}$

and $dq = \sec^2\phi\, d\phi$.

from (10)

$\frac{dz}{z} = \frac{1-\tan^2\phi}{(1+\tan^2\phi)\tan\phi}\sec^2\phi\, d\phi$

$\Rightarrow \frac{dz}{z} = \frac{1-\tan^2\phi}{\sec^2\phi\tan\phi}\sec^2\phi\, d\phi$

$\Rightarrow \frac{dz}{z} = (\cot\phi - \tan\phi)\, d\phi.$

Integrating, we get

$\log z = \log(\sin\phi) + \log(\cos\phi) + \log c_1$

$\Rightarrow z = c_1 \sin\phi \cos\phi$. ——— (8)

Now, using the initial values in (8)

$z_0 = c_1 \sin\frac{\pi}{4} \cos\pi/4$

$\Rightarrow \frac{c_1}{2} = \frac{\lambda^2}{2} \Rightarrow c_1 = \lambda^2$

$\therefore z = \lambda^2 \sin\phi \cos\phi$.

$\Rightarrow \boxed{z = \frac{\lambda^2}{2} \sin 2\phi}$ ——— (9)

Now, dividing (2) by (6)

$\frac{dx}{dq} = \frac{x}{-q(1+q^2)}$

$\Rightarrow \frac{dx}{x} = \frac{dq}{q(1+q^2)} dq$ $\left(\because q = \tan\phi \atop dq = \sec^2\phi\, d\phi \right.$

$\Rightarrow \frac{dx}{x} = \frac{\sec^2\phi\, d\phi}{\tan\phi(1+\tan^2\phi)}$

$\Rightarrow \frac{dx}{x} = \frac{1}{\tan\phi} d\phi$

integrating

$\log x = \log\sin\phi + \log c_2$

$\Rightarrow x = c_2 \sin\phi$

Using the initial values

$x_0 = c_2 \sin\phi_0$

$\Rightarrow \lambda = c_2 \sin\pi/4$

$\Rightarrow c_2 = \lambda\sqrt{2}$.

$\therefore \boxed{x = \lambda\sqrt{2} \sin\phi.}$ ——— (10)

now, divide (5) by (6)

$\frac{dp}{dq} = \frac{pq}{1+q^2} \Rightarrow \frac{dp}{p} = \frac{q\,dq}{1+q^2}$

$\Rightarrow \frac{dp}{p} = \frac{\tan\phi \sec^2\phi \, d\phi}{1+\tan^2\phi} = \tan\phi \, d\phi$

$\Rightarrow \log p = \log\sec\phi + \log c_3$

$\Rightarrow p = c_3 \sec\phi.$

Using initial values

$p_0 = c_3 \sec\phi_0$

$\Rightarrow \lambda = c_3\sqrt{2}$

$\Rightarrow c_3 = \lambda/\sqrt{2}$

$\boxed{p = \frac{\lambda}{\sqrt{2}} \sec\phi.}$ —— (11)

multiplying (7) and (6), we get

$\frac{dy}{dq} = \frac{2zq}{-q(1+q^2)} = \frac{-2z}{1+q^2}$

$\Rightarrow -2 \cdot \frac{\lambda^2}{2} \frac{\sin 2\phi}{1+q^2} dq$ $\left(\because \text{from (9)} \; z = \frac{\lambda^2}{2}\sin 2\phi \right)$

$\Rightarrow dy = -\frac{\lambda^2 \sin 2\phi \sec^2\phi \, d\phi}{1+\tan^2\phi}$

$\Rightarrow dy = -\lambda^2 \sin 2\phi \, d\phi$

$\Rightarrow y = \frac{\lambda^2}{2}\cos 2\phi + c_4.$

Using initial values,

$y_0 = \frac{\lambda^2}{2}\cos 2\phi_0 + c_4$

$0 = \frac{\lambda^2}{2}(0) + c_4 \Rightarrow c_4 = 0.$

$\therefore \boxed{y = \frac{\lambda^2}{2}\cos 2\phi}$ —— (12)

from (9) & (12),

$$y^2+z^2 = \frac{\lambda^4}{4}\cos^2 2\phi + \frac{\lambda^4}{4}\sin^2 2\phi$$

$$= \frac{\lambda^4}{4}(1)$$

$$\Rightarrow y^2+z^2 = \lambda^4/4$$

$$\Rightarrow \lambda^4 = 4(y^2+z^2)$$

from (10), $x^4 = 4\lambda^4 \sin^4\phi$

$$\Rightarrow x^4 = 4(4)(y^2+z^2)\sin^4\phi$$

$$\Rightarrow x^4 = 16(y^2+z^2)\sin^4\phi.$$

Now $x = \lambda\sqrt{2}\sin\phi$

$$z = \lambda^2 \sin\phi\cos\phi = \frac{\lambda^2}{2}\sin 2\phi.$$

$$\frac{x^2}{z} = \frac{2\lambda^2\sin^2\phi}{\lambda^2\sin\phi\cos\phi} = 2\tan\phi.$$

$$\Rightarrow \tan\phi = \frac{x^2}{2z}$$

$$\therefore \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}} = \frac{1}{\sqrt{1+\left(\frac{x^2}{2z}\right)^2}}$$

Also $\sin\phi = \cos\phi\tan\phi$

$$\sin^4\phi = \cos^4\phi\tan^4\phi$$

$$\frac{x^4}{16(y^2+z^2)} = \frac{1}{\left(1+\frac{x^4}{4z^2}\right)^2}\cdot\frac{x^4}{4z^2}$$

$$\Rightarrow \frac{1}{16(y^2+z^2)} = \frac{4z^2\times 4z^2}{(4z^2+x^4)^2}\cdot\frac{1}{4z^2}$$

$$\Rightarrow \boxed{(4z^2+x^4)^2 = 64z^2(y^2+z^2)}$$

which is the required integral surface

6(d) → A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity v given by $v = kx$ if $0 \le x \le l/2$ and $v = k(l-x)$ if $l/2 \le x \le l$. Find the displacement function $y(x,t)$.

Sol^n: The required displacement function $y(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{—— (1)}$$

subject to boundary conditions

$$y(0,t) = y(l,t) = 0 \quad \forall t \ge 0 \quad \text{—— (2)}$$

and the given initial conditions,

Initial displacement $= y(x,0) = f(x) = 0$ —— (3)

and initial velocity

$$= \left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x) = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l-x), & l/2 \le x \le l \end{cases} \quad \text{—— (4)}$$

Suppose (1) has the solution of the form $y(x,t) = X(x)\,T(t)$ —— (5)

Substituting this value of y in ①, we have $XT'' = c^2X''T$.

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu$ (say)

$\Rightarrow X'' - \mu X = 0$ —⑥ and $T'' - \mu c^2 T = 0$ —⑦

using ②, ⑤ gives
$X(0)\,T(t) = 0$ and $X(l)\,T(t) = 0$

suppose that $T(t) \neq 0$ ($\because T(t)=0$ leads to $y \equiv 0\ \forall t$)

$\therefore \boxed{X(0)=0}$ and $\boxed{X(l)=0}$ —⑧

which are boundary conditions.

we now solve ⑥ under boundary conditions ⑧.

Three cases arise.

Case(1): Let $\mu = 0$
The solution of ⑥ is $X(x) = Ax + B$

using B.C. ⑧, we get $A=0, B=0$

$\Rightarrow X(x) = 0$

This leads to $y \equiv 0$
which doesnot satisfy I.C. ③ and ④.

so we reject $\mu = 0$

Case(2): Let $\mu = \lambda^2$, $\lambda \neq 0$. Then the solution of ⑥ is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$.

using B.C. ⑧, we get $A=0, B=0$

$\Rightarrow X(x) = 0$

This leads to $y \equiv 0$ which doesnot satisfy I.C. ③ & ④.

so reject $\mu = \lambda^2$.

Case(3): Let $\mu = -\lambda^2$, $\lambda \neq 0$.

The solution of (6) is

$X(x) = A\cos\lambda x + B\sin\lambda x$

Using B.C. (8), we get

$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$

and $X(l) = 0 = 0 + B\sin\lambda l \Rightarrow B = \sin\lambda l$

$\Rightarrow \sin\lambda l = 0 \quad (\because B \neq 0)$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}, \; n = 1, 2, 3, \ldots$

$\therefore X(x) = B\sin\frac{n\pi x}{l}, \; n = 1, 2, 3, \ldots$

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$ —— (9)

from (7), $T'' - \mu c^2 T = 0$

$\Rightarrow T'' + \lambda^2 c^2 T = 0 \quad (\because \mu = -\lambda^2)$

$\Rightarrow T'' + \frac{n^2\pi^2}{l^2} c^2 T = 0 \quad (\because \lambda = \frac{n\pi}{l})$

whose general solution is

$T_n(t) = C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)$

$\therefore y_n(x,t) = X_n(x)\, T_n(t)$

$= B_n \sin\left(\frac{n\pi x}{l}\right)\left[C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)\right]$

$= \left[E_n \cos\frac{n\pi c t}{l} + F_n \sin\frac{n\pi c t}{l}\right]\sin\frac{n\pi x}{l}$

are solutions of (1) satisfying (2)

Here $E_n = B_n C_n$ and $F_n = B_n D_n$.

In order to obtain a solution also satisfying (3) and (4), we consider more general solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

i.e. $$y(x,t) = \sum_{n=1}^{\infty}\left[E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad \text{—(10)}$$

putting $t=0$ in (10).

$$y(x,0) = 0 = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

where $$E_n = \frac{2}{l}\int_0^l (0)\cos\left(\frac{n\pi c t}{l}\right)\sin\frac{n\pi x}{l} = 0$$

From (10),

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \quad \text{—(11)}$$

$\frac{\partial y}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} \sin\frac{n\pi c}{l}\cdot\cos\frac{n\pi ct}{l}$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\cdot\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{n\pi c}\left[\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} g(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{n\pi c}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{2k}{n\pi c}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2k}{n\pi c}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$
$$+ \frac{2k}{n\pi c}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2k}{n\pi c}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4kl^2}{cn^3\pi^3}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & \text{if } n=2m \text{ and } m=1,2,3,\ldots \\ (-1)^{m+1}\left[\frac{4kl^2}{c\pi^3(2m-1)^3}\right], & \text{if } n=2m-1 \text{ and } m=1,2,3,\ldots \end{cases}$$

when $n=2m-1$, $\sin\frac{n\pi}{2} = \sin\frac{\pi}{2}(2m-1)$

$$= \sin\left(m\pi - \frac{\pi}{2}\right)$$

$$= \sin m\pi\cos\frac{\pi}{2} - \cos m\pi\sin\frac{\pi}{2}$$

$$= 0 - (-1)^m = (-1)^{m+1}.$$

$\therefore$ from (11), the required displacement function is given by

$$y(x,t) = \frac{4kl^2}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}.$$

9(a) Apply Newton-Raphson method to determine a root of the equation $\cos x - xe^{x} = 0$ Correct upto four decimal places.

Sol^n: Let $f(x) = \cos x - xe^{x}$.

$$f'(x) = -\sin x - xe^{x} - e^{x}$$

$f(0) = 1 - 0 = 1 > 0$ & $f(1) = \cos 1 - 1(e) = -ve < 0$.

so a root of $f(x) = 0$ lies between 0 and 1.

Let us take $x_0 = 0$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n + \frac{\cos x_n - x_n e^{x_n}}{(\sin x_n + x_n e^{x_n} + e^{x_n})}$$

Put $n = 0$, the first approximation is

$$x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 + x_0 e^{x_0} + e^{x_0}} = 1$$

$x_2 = 0.653071$

$x_3 = 0.5313$

$x_4 = 0.5179$

$x_5 = 0.51775$

$\therefore x = 0.517$ is root of $f(x)$ Correct upto three decimal places.

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7(d) Convert the following binary numbers to the base indicated.

(i) $(10111011001.101110)_2$ to octal
(ii) $(10111011001.1011100)_2$ to hexadecimal.
(iii) $(0.101)_2$ to decimal

Sol: (i) Binary number can be converted into equivalent octal number by making groups of 3 bits starting from LSB and moving towards MSB for integer part of the number and then replacing each group of three bits by its octal representation.
For fractional part, the grouping of 3 bits are made from the binary point.

Given $(10111011001.101110)_2 = (010111011001.101110)_2$

$$= \underbrace{010}_{2}\ \underbrace{111}_{7}\ \underbrace{011}_{3}\ \underbrace{001}_{1} \cdot \underbrace{101}_{5}\ \underbrace{110}_{6}$$

$$= (2731.56)_8.$$

(ii) Similarly, Binary number can be converted into equivalent hexadecimal number by making groups of 4 bits.

$$(10111011001.10111000)_2 = (\underbrace{0101}_{5}\underbrace{1101}_{D}\ \underbrace{1001}_{9} \cdot \underbrace{1011}_{B}\ \underbrace{1000}_{8})$$

$$= (5D9.B8)_{16}$$

(iii) $(0.101)_2 = 1\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3}$

$= \frac{1}{2} + 0 + \frac{1}{8}$

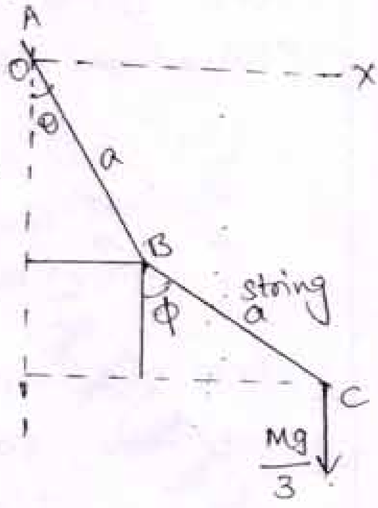
$= 0.5 + 0.125$

$= (0.625)_{10}$

8(a): A uniform straight rod of length $2a$ is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of length a to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{a/g}$.

Sol'n: Let M be the mass of the rod AB of length $2a$, BC the string and $M/3$ the mass at C.

At time t, let the rod and the string make angles θ & ϕ to the vertical respectively.



The middle point O of the rod AB as origin, horizontal and vertical lines OX & OY through O as axes, the coordinates of C are given by

$$x_C = a(\sin\theta + \sin\phi)$$
$$y_C = a(\cos\theta + \cos\phi)$$

$$\therefore v_C^2 = \dot{x}_C^2 + \dot{y}_C^2$$

$$= a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$$

$$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)] = a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

($\because \theta, \phi$ are small)

If T be the total kinetic energy and W the work function of the system, then

$$T = \text{K.E of the rod} + \text{K.E of the particle at } C.$$
$$= \left[\tfrac{1}{2}M\cdot\tfrac{1}{3}a^2\dot{\theta}^2 + \tfrac{1}{2}Mv_0^2\right] + \tfrac{1}{2}\left(\tfrac{1}{3}M\right)v_C^2$$

$= \frac{1}{2}Ma^2\dot\theta^2 + \frac{1}{6}Ma^2(\dot\theta^2+\dot\phi^2+2\dot\theta\dot\phi)$

$= \frac{1}{6}Ma^2(2\dot\theta^2+\dot\phi^2+2\dot\theta\dot\phi)$ $\quad (\because v_0=0)$

and $W = mg.0$

$= \frac{1}{2}Mg\cdot y_c + C = \frac{1}{3}Mga(\cos\theta+\cos\phi)+C$

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{6}Ma^2(4\dot\theta+2\dot\phi)\right] - 0 = \frac{1}{3}Mga(-\sin\theta) = -\frac{1}{3}Mga\theta$,

$(\because \theta \text{ is small})$

or $2\ddot\theta+\ddot\phi = -c\theta$, (where $c = g/a$) — ①

And Lagrange's ϕ equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{6}Ma^2(2\dot\phi+2\dot\theta)\right] - 0 = \frac{1}{3}Mga(-\cos\phi) = \frac{1}{3}Mga\phi$

$(\because \phi \text{ is small})$ — ②

(or) $\ddot\theta+\ddot\phi = -c\phi$, where $c = g/a$

Equations ① & ② can be written as

$(2D^2+c)\theta + D^2\phi = 0$ and $D^2\theta + (D^2+c)\phi = 0$

Eliminating ϕ between these two equations, we get

$[(D^2+c)(2D^2+c) - D^4]\theta = 0$

(or) $(D^4+3cD^2+c^2)\theta = 0$ —— ③

Let the solution of (3) be given by $\theta = A\cos(pt+B)$

$\therefore D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in (2), we get

$(p^4-3cp^2+c^2)\theta = 0$ or $p^4-3cp^2+c^2=0 \quad \because \theta\neq 0$.

$\therefore p^2 = \frac{3c\pm\sqrt{(9c^2-4c^2)}}{2} = \left(\frac{3\pm\sqrt5}{2}\right)c = \left(\frac{3\pm\sqrt5}{2}\right)\frac{g}{a}$

$\therefore$ one value of p^2 is $p_1^2 = \left(\frac{3-\sqrt5}{2}\right)\frac{g}{a}$.

$\therefore$ One period of principal oscillation $= \frac{2\pi}{p_1}$

$= 2\pi\sqrt{\frac{2}{3-\sqrt5}\cdot\frac{a}{g}} = 2\pi\sqrt{\frac{2(3+\sqrt5)}{(3-\sqrt5)(3+\sqrt5)}\cdot\frac{a}{g}}$

$= 2\pi\sqrt{\frac{6+2\sqrt5}{4}\cdot\frac{a}{g}} = 2\pi\sqrt{\left(\frac{\sqrt5+1}{2}\right)^2\frac{a}{g}} = (\sqrt5+1)\pi\sqrt{a/g}$

8(b) Test whether the motion specified by $q = \frac{k^2(xj - yi)}{x^2+y^2}$ (k = const) is a possible motion for an incompressible fluid. If so, determine the equations of streamlines. Also tell whether the motion is of the potential kind and if it determines the velocity potential.

Sol: Here $u = \frac{-k^2 y}{x^2+y^2}$, $v = \frac{k^2 x}{x^2+y^2}$, $w=0$

I. Equation of continuity for incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

But $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2k^2 xy}{(x^2+y^2)^2} - \frac{2k^2 xy}{(x^2+y^2)^2} + 0$

Hence equation of continuity is satisfied.

II. Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0}$$

$$\Rightarrow \frac{dx\,(x^2+y^2)}{-k^2 y} = \frac{(x^2+y^2)dy}{k^2 x} = \frac{dz}{0}$$

$\Rightarrow x\,dx + y\,dy = 0$, $dz = 0$

$\Rightarrow x^2+y^2 = a^2$, $z = b$

Hence streamlines are circles whose centres lie on z-axis.

III. To test the existence of velocity potential.

$$-d\phi = u\,dx + v\,dy + w\,dz$$

$$= -k^2 y \frac{dx}{x^2+y^2} + k^2 x \frac{dy}{x^2+y^2}$$

$$d\phi = k^2\left[\frac{y\,dx}{x^2+y^2} - \frac{x}{x^2+y^2}\,dx\right]$$

$$= k^2(M\,dx + N\,dy), \text{ say}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2} + y\left[\frac{-2y}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = -\left[\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\right] = \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\partial M}{\partial y}$$

Hence $M\,dx + N\,dy$ is exact. Therefore its solution given by

$$\phi = \int \frac{k^2 y}{x^2+y^2}dx + \int 0\,dy + C = \frac{k^2 y}{y}\tan^{-1}\left(\frac{x}{y}\right)+C$$

Hence ϕ exists and is given by

$$\phi = k^2\tan^{-1}\left(\frac{x}{y}\right)+C$$

8(c) When an infinite liquid contains two parallel equal and opposite vortices at a distance $2b$, Prove that the streamlines relative to the vortices are given by the equation $\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{y}{b} = C$; the origin being the middle point of the join which is taken for the axis of y.

Sol'n: Suppose there are two vortices of strengths $k, -k$ at A_1, A_2 respectively such that origin O is the middle point of $A_1A_2 = 2b$ and A_1, A_2 lie along y-axis. Both vortices will along a line parallel to x-axis with velocity.

$$q = \frac{k}{2\pi(A_1A_2)} = \frac{k}{2\pi.2b} = \frac{k}{4\pi b}$$

The complex potential w at P due to these two vortices given by

$$w = \frac{ki}{2\pi}\log(z-ib) - \frac{ki}{2\pi}\log(z+ib)$$

$$= \frac{ki}{2\pi}\log[x+i(y-b)] - \frac{ki}{4\pi}\log[x+i(y+b)]$$

Equating imaginary parts from both sides,

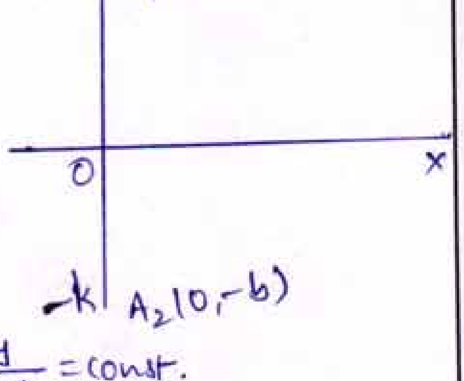
$$\psi = \frac{k}{4\pi}\log[x^2+(y-b)^2] - \frac{k}{4\pi}\log[x^2+(y+b)^2]$$

$$\Rightarrow \psi = \frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]$$

To reduce the vortex system to rest, we superimpose a velocity $\frac{k}{4\pi b}$ along x-axis to the system. Let ψ' be the stream function due to this addition, then

$$-\frac{\partial\psi'}{\partial y} = -\frac{\partial\phi'}{\partial x} = -\frac{k}{4\pi b}$$

$$\therefore \psi' = \frac{ky}{4\pi b}$$



Hence the streamlines relative to vortices are given by

$$\psi = \frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{ky}{4\pi b} = \text{const.}$$

$$\Rightarrow \log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{y}{b} = c$$

If we take $PA_1 = r_1 = [x^2+(y-b)^2]^{1/2}$

and $PA_2 = r_2 = [x^2+(y+b)^2]^{1/2}$, then the last gives.

$$\log\frac{r_1^2}{r_2^2} + \frac{y}{b} = \text{const.}$$

$$\Rightarrow \log\frac{r_1}{r_2} + \frac{y}{2b} = c.$$