* Applications of Cauchy's Theorem *

In this lesson we consider two most important applications of Cauchy's theorem. They are

(1) Cauchy's Integral formula

(2) Taylors Theorem

In Cauchy's Integral formula we establish that when $f(z)$ is analytic in a domain D, then $f(z)$ will have derivatives of all orders in D. This formula also helps us in evaluating integrals in certain cases. Let us suppose that $\int_C f(z)dz$ is to be evaluated. then this formula can be applied if $F(z)$ is of the form $\frac{f(z)}{(z-z_0)^n}$, $n=0,1,2,\ldots$ where z_0 is a point inside 'c' and $f(z)$ is analytic inside and on a simple closed contour c.

In Taylor's theorem we prove that if $f(z)$ is analytic in a domain D, then $f(z)$ can be expressed as infinite convergent series at every point of a circle of convergence.

Here we also prove a theorem called Morera's theorem, which is a partial converse of Cauchy's theorem

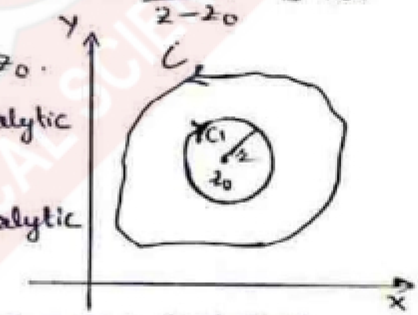
* Cauchy's First Integral Formula :-

(I) Let $f(z)$ be analytic in a simply-connected domain containing a simply closed curve 'c'. If z_0 is inside 'c' then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$

Proof: Given function $\frac{f(z)}{z-z_0}$ is not defined at $z=z_0$.

$\therefore$ It is not analytic at $z=z_0$.

Since $f(z)$ is analytic at $z=z_0$.



$\therefore$ given $\epsilon>0$, $\exists$ a $\delta>0$ such that

$|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$

Now choose a $\gamma>0$ and less than δ.

Let it be so small that the positively oriented circle $|z-z_0|=\gamma$ and is denoted by C_1 is interior to C. then

$|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|=\gamma$

Observe that the contour $C-C_1$ is a boundary of a multiply connected domain in which $\frac{f(z)}{z-z_0}$ is analytic.

$\therefore$ By Cauchy's theorem for multiply-connected domain, we get

$$\int_{C-C_1} \frac{f(z)}{z-z_0}\,dz = 0.$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz$$

$$= \int_{C_1} \frac{f(z_0) + f(z) - f(z_0)}{z-z_0}\,dz$$

$$= \int_{C_1} \frac{f(z_0)}{z-z_0}\,dz + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\int_{C_1} \frac{1}{z-z_0}\,dz + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)(2\pi i) + \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz - (2\pi i) f(z_0) = \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \quad \text{---} (1)$$

Consider

$$\left| \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le \int_{C_1} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$

$$\le \frac{\epsilon}{\delta}\int_{C_1} |dz|$$

$$= \frac{\epsilon}{\delta}(2\pi\delta) \quad (\because \text{length of the circle} = 2\pi r)$$

$$= 2\pi\epsilon$$

$\therefore$ from (1)

$$\left| \int_C \frac{f(z)}{z-z_0}\,dz - (2\pi i) f(z_0) \right| = \left| \int_{C_1} \frac{f(z)-f(z_0)}{z-z_0}\,dz \right| \le 2\pi\epsilon$$

Since this relation must be satisfied for every $\epsilon > 0$, we get

$$\left| \int_C \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0) \right| = 0 \quad (\because \epsilon \to 0)$$

$$\Rightarrow \int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

$$\boxed{\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$

Note: If $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$ for all points z_0 inside 'C', is $f(z)$ an analytic inside and on C?

Sol^n: Yes.

If $f(z)$ is not analytic inside and on C, we cannot obtain integral formula.

Theorem (II) * Cauchy's General Integral Formula :-

Let $f(z)$ be analytic in a simply connected domain containing the simple contour 'C'. Then $f(z)$ has derivatives of all orders at each point z_0 inside 'C' with $f^n(z_0) = \frac{\lfloor n}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$.

Proof: Since $f(z)$ is analytic in a simply connected domain D containing the simple closed contour 'C' and z_0 is any point inside 'C'

$$\therefore f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz \quad \text{---} (1)$$

Now choose h such that z_0+h lies inside 'c'.

Then $f(z_0+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+h)}\,dz$ ——②

Now $\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0-h)(z-z_0)}\,dz$

(from ①&②) ——③

On taking the limit as $h\to 0$, the LHS reduces to $f'(z_0)$.

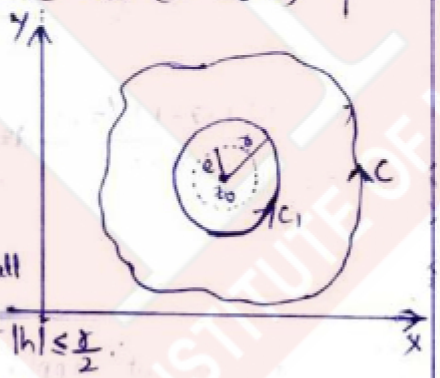
The integrand in RHS reduces to $\frac{f(z)}{(z-z_0)^2}$

For Proving the above, consider

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz\right|$$

$$= \left|\frac{h}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2(z-z_0-h)}\,dz\right| \text{ ——④}$$

Given a circle $C_1 : |z-z_0| = r$ contained in C.



Choose 'h' small enough so that $|h| \le \frac{r}{2}$.

Now by Cauchy's theorem for multiply-connected region we have.

$$\frac{h}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2(z-z_0-h)}\,dz = \frac{h}{2\pi i}\int_{C_1} \frac{f(z)}{(z-z_0)^2(z-z_0-h)}\,dz$$

Since $f(z)$ is continuous on C_1,

$\therefore$ it is bounded on C_1

$\therefore |f(z)| \le M$ (say)

Then we have

$$\frac{|h|}{2\pi}\int_{C_1} \frac{|f(z)|}{|z-z_0|^2|z-z_0-h|}\,|dz| \le \frac{|h|M}{2\pi r^2}\int_{C_1} \frac{1}{|z-z_0-h|}\,|dz|$$

$$\le \frac{|h|M}{2\pi r^2}\int_{C_1} \frac{|dz|}{|z-z_0|-|h|} \le \frac{|h|M}{2\pi r^2\left(\frac{r}{2}\right)}\int_{C_1} |dz|$$

$$\le \frac{|h|M}{\pi r^3}\cdot 2\pi r = |h|\left(\frac{2M}{r^2}\right).$$

$\therefore$ As $h\to 0$

④ tends to zero.

On using ③ we get

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = f'(z_0)$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_0-h)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

By continuing the above process, we get

$$\frac{f'(z_0+h)-f'(z_0)}{h} = \frac{1}{2\pi i}\int_C \frac{1}{h}\left[\frac{f(z)}{(z-z_0-h)^2} - \frac{f(z)}{(z-z_0)^2}\right]dz$$

$$= \frac{1}{2\pi i}\int_C \frac{(z-z_0)^2-(z-z_0-h)^2}{h(z-z_0)^2(z-z_0-h)^2}f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2(z-z_0)-h}{(z-z_0)^2(z-z_0-h)^2}f(z)\,dz$$

As $\lim_{h\to 0}$, we get

$$f''(z_0) = \frac{1.2}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}\,dz$$

Hence by using induction, we can

Prove that $f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

Hence the theorem.

Note:- The above two theorems tells us that the values of $f(z), f'(z), ----$ at any point z_0 inside C, can be expressed interms of the values of the function $f(z)$ on the boundary 'C'.

Note:- Here it is Proved that, if $f(z)$ is analytic in a simply connected domain D, then all order derivatives of $f(z)$ exist in D.

Note:- Cauchy's integral formula helps us to evaluate certain complex integrals along a contour.

Ex-①: Evaluate $\int_C \frac{e^z \sin z}{(z-2)^2} dz$ where $C: |z| = 3$.

Sol^n: Comparing the given integral with

$$\int_C \frac{f(z)}{(z-z_0)^2} dz$$

we get $f(z) = e^z \sin z$; $z_0 = 2$

Since $e^z \sin z$ is analytic in $|z| = 3$. and $z_0 = 2$ is a point inside $|z| = 3$,

∴ we can apply Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0) \quad \text{——— ①}$$

Now $f(z) = e^z \sin z$

$$\Rightarrow f'(z) = e^z(\sin z + \cos z)$$

$$\Rightarrow f'(2) = e^2(\sin 2 + \cos 2) \quad (\because z_0 = 2)$$

from ①, we have

$$\int_C \frac{f(z)}{(z-2)^2} dz = 2\pi i e^2 (\sin 2 + \cos 2)$$

Ex-②: Evaluate $\int_{|z|=2} \frac{z^3+3z-1}{(z-1)(z+3)} dz$.

Sol^n: Comparing the given integral with $\int_C \frac{f(z)}{z-z_0} dz$, we get

$$f(z) = \frac{z^3+3z-1}{z+3}, \quad z_0 = 1$$

Since $f(z)$ is analytic in $|z| = 2$ and $z_0 = 1$ is a point inside $|z| = 2$. we apply Cauchy's integral formula

$$\int \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0) \quad \text{——— ①}$$

Since $f(z_0) = f(1)$

$$= \frac{(1)^3 + 3(1) - 1}{1+3} = 3/4$$

$$\therefore \int_{|z|=2} \frac{(z^3+3z-1)/z+3}{z-1} dz = 2\pi i f(1)$$

$$= 2\pi i (3/4)$$

$$= 3/2 \pi i$$

Note: we cannot apply Cauchy's integral formula by taking $f(z) = \frac{z^3+3z-1}{z-1}$; $z_0 = -3$.

because $z_0 = -3$ is not inside $|z| = 2$.

H.W. Evaluate $\int_{|z|=3} \frac{z^3+3z-1}{(z-1)(z+3)} dz$

(by using partial fractions)

HW Evaluate (a) $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

(b) $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where 'c' is the circle $|z|=3$.

HW Evaluate the following integrals, where C is the circle $|z|=3$.

(a) $\int_C \frac{e^z}{z-2} dz$ Ans: $2\pi i e^2$

(b) $\int_C \frac{e^{2z}}{(z-2)^2} dz$ Ans: $8\pi i e^4$

(c) $\int_C \frac{3z^4+2z-6}{(z-2)^3}$ Ans: $144\pi i$.

→ Evaluate

(a) $\int_{|z|=2} \frac{1}{z^4-1} dz$ Ans: 0 $\left[\frac{1}{z^4-1} = \frac{1}{(z^2-1)(z^2+1)} = \frac{1}{(z-1)(z+1)(z-i)(z+i)}\right]$

(b) $\int_{|z|=2} \frac{1}{z^2+1} dz$ Ans: 0

→ Evaluate the integral $\int_C \frac{z}{(16-z^2)(z+i)} dz$ where C is circle.

(a) $|z|=2$ Ans: $2\pi/17$

(b) $|z+4|=2$ Ans: $\pi/17 - \frac{4\pi i}{17}$

(c) $|z|=5$ Ans: $\frac{2\pi}{17} - \frac{8\pi i}{17}$

Theorem (III) * **Cauchy's Integral Formula for multiply Connected Regions:-**

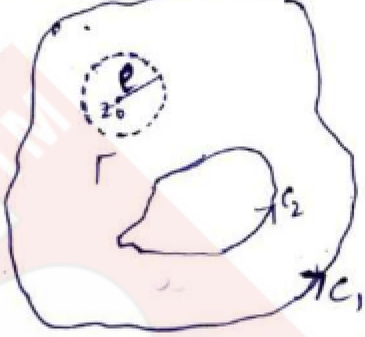
Let $f(z)$ be analytic in a multiply connected domain $C = C_1 \cup C_2$.

If z_0 is inside 'c' then

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z-z_0} dz$$

Proof: Let $f(z)$ be analytic in a multiply (doubly) connected region R whose boundary is

$C = C_1 \cup C_2$



Let z_0 be inside 'c'

Construct a circle Γ contained in 'c' and whose centre is z_0. Then in the region $C-\Gamma$, the function $\frac{f(z)}{z-z_0}$ is analytic. then by Cauchy's theorem

$$\int_{C-\Gamma} \frac{f(z)}{z-z_0} dz = 0$$

$$\Rightarrow \int_{C_1} \frac{f(z)}{z-z_0} dz - \int_{C_2} \frac{f(z)}{z-z_0} dz - \int_{\Gamma} \frac{f(z)}{z-z_0} dz = 0 \quad \text{—(1)}$$

But $\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-z_0} dz = f(z_0)$

(by Cauchy's integral formula)

∴ from (1), we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z-z_0} dz - \oint_{C_2} \frac{f(z)}{z-z_0} dz.$$

Hence the result.

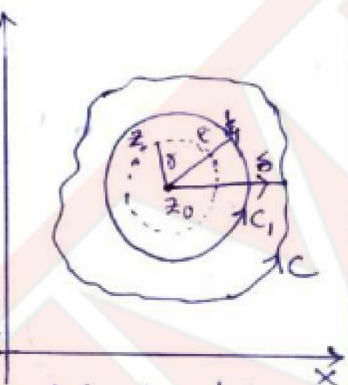
Note:- similarly we can show that the conclusion of the theorem (II) remains valid for the multiply-connected region R.

* Taylor's Theorem:-

Let $f(z)$ be analytic in a domain D whose boundary is 'C'. z_0 is any point in C. Then $f(z)$ can be expressed as

$$f(z) = f(z_0) + \frac{(z-z_0)f'(z_0)}{1!} + (z-z_0)^2\frac{f''(z_0)}{2!} + \cdots + \frac{f^n(z_0)}{n!}(z-z_0)^n + \cdots$$

The series converges for $|z-z_0| < \delta$, where δ is the distance of z_0 to the nearest point on 'C'.



Proof: construct a circle C_1 with radius ρ and centre z_0.

Let $\rho < \delta$

Let z be any point in C_1. Then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi)}{\xi - z}\, d\xi \quad \text{——} (1)$$

Let $|z-z_0| = r$. Then $r = |z-z_0| < |\xi - z_0| = \rho$

$$\text{Now } \frac{1}{\xi - z} = \frac{1}{(\xi-z_0)-(z-z_0)}$$

$$= \frac{1}{(\xi-z_0)\left(1-\frac{z-z_0}{\xi-z_0}\right)}$$

$$= \frac{1}{(\xi-z_0)}\left(1-\frac{z-z_0}{\xi-z_0}\right)^{-1}$$

$$= \frac{1}{(\xi-z_0)}\left[1+\frac{z-z_0}{\xi-z_0} + \left(\frac{z-z_0}{\xi-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{\xi-z_0}\right)^{n-1} + \sum_{k=n}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^k\right]$$

$$\boxed{\begin{aligned} &\sum_{k=n}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^k = \left(\frac{z-z_0}{\xi-z_0}\right)^n + \left(\frac{z-z_0}{\xi-z_0}\right)^{n+1} + \cdots \\ &= \left(\frac{z-z_0}{\xi-z_0}\right)^n\left[1+\left(\frac{z-z_0}{\xi-z_0}\right) + \cdots\right] \\ &= \left(\frac{z-z_0}{\xi-z_0}\right)^n\left(1-\frac{z-z_0}{\xi-z_0}\right)^{-1} \end{aligned}}$$

$$= \frac{1}{(\xi-z_0)}\left[1+\left(\frac{z-z_0}{\xi-z_0}\right)+\left(\frac{z-z_0}{\xi-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{\xi-z_0}\right)^{n-1} + \frac{\left[(z-z_0)/(\xi-z_0)\right]^n}{1-\frac{z-z_0}{\xi-z_0}}\right]$$

Substituting this in (1), we get

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi-z_0)}d\xi + \frac{z-z_0}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi-z_0)^2}d\xi + \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi-z_0)^n}d\xi + R_n \quad \text{——} (2)$$

$$\text{where } R_n = \frac{1}{2\pi i}\int_{C_1}\left(\frac{z-z_0}{\xi-z_0}\right)^n\frac{f(\xi)}{(\xi-z)}d\xi$$

By using the general form of Cauchy's integral formula in (2), we get

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!}f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \frac{(z-z_0)^{n-1}}{(n-1)!}f^{n-1}(z_0) + R_n \quad \text{——} (\mathrm{I})$$

The result follows if we can show that $\lim_{n\to\infty} R_n = 0$.

Since $f(z)$ is continuous on C_1, $\exists$ a constant M such that $|f(z)| \le M$ on C_1.

$$\therefore |R_n| = \left|\frac{1}{2\pi i}\int_{C_1}\frac{(z-z_0)^n}{(\xi-z_0)^n}\frac{f(\xi)}{\xi-z}d\xi\right|$$

$$\le \frac{1}{2\pi}\int_{C_1}\left|\frac{z-z_0}{\xi-z_0}\right|^n\left|\frac{f(\xi)}{\xi-z}\right||d\xi|$$

$$\le \frac{M}{2\pi}\frac{r^n}{\rho^n}\int_{C_1}\frac{1}{|\xi-z|}|d\xi| \quad \text{——} (3)$$

Now consider

$$\frac{1}{|\xi-z|} = \frac{1}{|\xi-z_0-(z-z_0)|} \le \frac{1}{|\xi-z_0|-|z-z_0|} = \frac{1}{\rho-r}$$

Substituting this in ③, we get

$$|R_n| \le \frac{M}{2\pi}\frac{r^n}{\rho^n}\frac{1}{\rho-r}\int_{C_1}|d\xi|$$

$$= \frac{M}{2\pi}\left(\frac{r}{\rho}\right)^n \frac{2\pi\rho}{\rho-r}$$

Since $\frac{r}{\rho}<1$, $\left(\frac{r}{\rho}\right)^n \to 0$ as $n\to\infty$

$\therefore \lim_{n\to\infty}|R_n| \le \lim \frac{M\rho}{\rho-r}\left(\frac{r}{\rho}\right)^n = 0$

$\therefore$ from ①, we have

$$f(z) = \sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n \quad \text{④}$$

Ex: Expand $f(z)=\sin z$ in a Taylor's series about $z=\pi/4$.

Sol'n: $f(z)=\sin(z) \Rightarrow f(\pi/4)=\sin\pi/4=\frac{1}{\sqrt{2}}$

$f'(z)=\cos z \Rightarrow f'(\pi/4)=\cos\pi/4=\frac{1}{\sqrt{2}}$

$f''(z)=-\sin z \Rightarrow f''(\pi/4)=-1/\sqrt{2}$

$f'''(z)=-\cos z \Rightarrow f'''(\pi/4)=-1/\sqrt{2}$

$f^{iv}(z)=\sin z \Rightarrow f^{iv}(\pi/4)=1/\sqrt{2}$

⋮

Substituting these values in ④, we get

$$\sin(z)=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}(z-\pi/4)-\frac{1}{\sqrt{2}.2!}(z-\pi/4)^2-\frac{1}{\sqrt{2}.3!}(z-\pi/4)^3+\cdots$$

$$=\frac{1}{\sqrt{2}}\left[1+(z-\pi/4)-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\cdots\right]$$

HW Expand $f(z)=\log(1+z)$ in a Taylor's series about $z=0$.

* **Note:-** we know that a power series represents an analytic function inside its circle is of convergence. The Taylor's theorem is converse to the above. Thus a function $f(z)$ is analytic at a point z_0 iff

$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in some disk $|z-z_0|\le r$

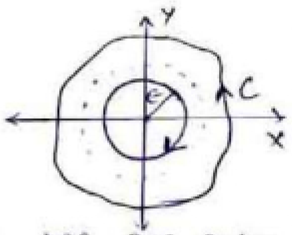
where $a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{n+1}}dz$

Note: we know that if $f(z)$ is analytic inside and on a simple closed contour C and z_0 is any point inside 'c', then the Cauchy's theorem says theorem $\int_C f(z)dz=0$. But the converse of this theorem is not true. That is even if $\int_C f(z)dz=0$, where C is closed contour containing a point z_0 inside it, $f(z)$ need not be analytic inside and on C.

For example $\int_C \frac{1}{z^2}dz=0$ along every simple closed curve 'c' having the origin as the interior point. This is because $f(z)$ is analytic in the region between 'C' and some circle $|z|=\epsilon$ contained in C.

$\therefore$ By Cauchy's theorem for multiply-connected regions

$$\int_C \frac{1}{z^2}dz=\int_{|z|=\epsilon}\frac{1}{z^2}dz=\int_0^{2\pi}\frac{i\epsilon e^{i\theta}}{\epsilon^2 e^{i2\theta}}d\theta=0.$$



But $\frac{1}{z^2}$ is not analytic at $z=0$.

Therefore in C, now we are going to state the theorem called Morera's theorem which is a partial converse to Cauchy's theorem.

→ **Morera's Theorem:**

Let $f(z)$ be continuous in a domain D and $\int_C f(z)dz = 0$ along every simple closed contour 'C' contained in D. then $f(z)$ is analytic in D.

Note:- In view of Morera's theorem we can say that a necessary and sufficient condition for a continuous function to be analytic in a simply connected domain is that the integral of it is independent of the path.

*LAURENT SERIES:

Earlier we have observed that $f(z)$ can be expressed in Taylor's series given by $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, if $f(z)$ is analytic in a disc $|z-z_0|<R$.

Now we consider that $f(z)$ is analytic in annulus given by $R_1 < |z-z_0| < R_2$, $R_1 < R_2$ instead of a disc.

In this lesson we show that such a function $f(z)$ can be expressed in a power series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$, which is known as Laurent Series. This Laurent series will have two parts named as
(i) Principal part (ii) Analytic part.

The analytic part is equalent to Taylor's series. The principal part helps us in determining the nature of singularities.

***Laurent Series:**

we know that when $f(z)$ is analytic at a point z_0 then $f(z)$ can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

that is valid in some neighbourhood of z_0. Now let us consider a function $f_1(z)$ defined as $f_1(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$

This can be viewed as a series in the variable $\frac{1}{z-z_0}$ then let this series be convergent in $\frac{1}{|z-z_0|}<R$.

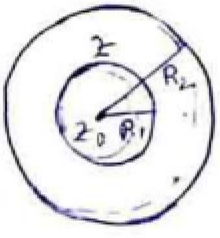
$$\therefore \; |z-z_0| > \frac{1}{R} = R_1 \text{ (say)}$$

i.e. $|z-z_0| > R_1$

Now suppose that the series

$\sum_{n=0}^{\infty} a_n (z-z_0)^n$

has radius of convergence R_2.



Then $f_2(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is analytic for $|z-z_0| < R_2$. If $R_2 > R_1$, then $f_1(z)$ and $f_2(z)$ are both analytic in the annulus $R_1 < |z-z_0| < R_2$

Hence the function

$$f(z) = f_1(z) + f_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

is analytic in the annulus $R_1 < |z-z_0| < R_2$

Let us take $a_{-n} = b_n$

then $f(z) = \sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$= \sum_{n=-1}^{-\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{--- (1)}$$

A series of the above form is known as Laurent's series.

* LAURENT'S THEOREM :-

Suppose $f(z)$ is analytic in the annulus $R_1 < |z-z_0| < R_2$. Then the representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is valid throughout the annulus.

Further more the coefficients are given by $a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$

where 'C' is any simple closed contour contained in the annulus that makes a clockwise revolution about the point z_0

→ **Principal Part** :- the series of negative powers of $(z-z_0)$ in $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is called the principal part of the Laurent Series of $f(z)$

This part is convergent every where outside the circle $|z-z_0| = R_1$

→ **Analytic Part** :-

The series of positive powers of $(z-z_0)$ in $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is called the analytic part of Laurent Series of $f(z)$.

this part is convergent every where inside the circle $|z-z_0| = R_2$.

Problems

2005 → Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a laurent series valid for (a) $1 < |z| < 3$ (b) $|z| > 3$ (c) $0 < |z+1| < 2$ (d) $|z| < 1$

Sol^n: The given function resolving into partial fractions,

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right) \quad \text{--- (1)}$$

(a) $1 < |z| < 3$

Now consider $\frac{1}{z+1}$

$$\frac{1}{z+1} = \frac{1}{z(1+1/z)} = \frac{1}{z}\left(1+1/z\right)^{-1}$$

Here we have taken z common because the required range is $|z|>1$

$$\therefore \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} = \frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right)$$

$$= \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots$$ (By using binomial expansion)

this expansion is possible only when

$\left|\frac{1}{z}\right|<1 \Rightarrow |z|>1$

which is required range.

Now consider $\frac{1}{z+3}$

$$\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

Here we have taken 3 common because the required range is $|z|<3$.

$$\therefore \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1} = \frac{1}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right)$$

$$= \frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\cdots$$

This expansion is possible only when $\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$.

which is the required range.

Substituting these in ①, we get

$$f(z) = \frac{1}{2}\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right) - \frac{1}{2}\left(\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\cdots\right)$$

$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}\cdots\right)+\left(-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\cdots\right)$$

$$= \cdots+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}+\frac{z^2}{54}+\cdots$$

which is required Laurent expansion valid for $1<|z|<3$.

Principal part : $\frac{1}{2z}-\frac{1}{2z^2}+\cdots$

Analytic part : $-\frac{1}{6}+\frac{z}{18}+\frac{z^2}{54}+\cdots$

(b) $|z|>3$:

Now consider $\frac{1}{z+1}$

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots$$

This expansion is valid for $\left|\frac{1}{z}\right|<1$

$\Rightarrow |z|>1$

which is valid for $|z|>3$ ($\because |z|>3>1$)

which is required range.

Now consider $\frac{1}{z+3}$

$$\frac{1}{z+3} = \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\cdots\right)$$

$$= \frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\cdots$$

This expansion is valid for $\left|\frac{3}{z}\right|<1$

$\left|\frac{3}{z}\right|<1 \Rightarrow |z|>3$

which is the required range.

$\therefore$ from ①,

$$f(z) = \frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\cdots\right]-\frac{1}{2}\left[\frac{1}{z}-\frac{3}{z^2}+\frac{9}{z^3}-\cdots\right]$$

$$= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\cdots$$

which is the required Laurent expansion valid for $|z|>3$.

(c) $0<|z+1|<2$:

Let $z+1=u$ then $0<|u|<2$

Now from ①,

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)}$$

$$= \frac{1}{2u(1+\frac{u}{2})}$$

$$= \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left[1-\frac{u}{2}+\frac{u^2}{4}-\cdots\right]$$

$$= \frac{1}{2u}-\frac{1}{4}+\frac{u}{8}-\cdots$$

This expansion is possible only when

$$0<\left|\frac{u}{2}\right|<1$$

$$\Rightarrow 0<|u|<2$$

which is the required range.

$\therefore$ The required laurent series valid for

$$0<|u|<2$$

i.e $0<|z+1|<2$ is

$$\frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\cdots$$

(d) $|z|<1$:

Now consider $\frac{1}{z+1}$

$$\therefore \frac{1}{z+1} = (1+z)^{-1} = (1-z+z^2-\cdots)$$

This expansion is valid only when $|z|<1$ which is required range.

Now Consider $\frac{1}{z+3}$

$$\therefore \frac{1}{z+3} = \frac{1}{3(1+\frac{z}{3})} = \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right]$$

This expansion is valid for

$$\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$$

which is also valid for $|z|<1$.

$\therefore$ from ①,

$$f(z) = \frac{1}{2}\left[1-z+z^2-\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right]$$

$$= \frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\cdots$$

which is the required laurent expansion valid for $|z|<1$.

This is a Taylor's Series.

2002 Show that when $0<|z-1|<2$, the function $f(z) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of $(z-1)$ as

$$\frac{-1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n}$$

Sol'n: Let $z-1=u$ then $0<|z-1|<2$

$$\Rightarrow 0<|u|<2$$

$$\therefore \frac{z}{(z-1)(z-3)} = \frac{u+1}{u(u-2)} = \frac{-1}{2u}+\frac{3}{2(u-2)}$$

Hence for $|u|<2$,

we have
$$f(z) = -\frac{1}{2u}-\frac{1}{4(1-\frac{u}{2})}$$

$$= \frac{-1}{2u}-\frac{3}{4}\left(1-\frac{u}{2}\right)^{-1}$$

$$= -\frac{1}{2u}-\frac{3}{4}\left[1+\frac{u}{2}+\frac{u^2}{2^2}+\cdots\right]$$

$$= \frac{-1}{2(z-1)}-\frac{3}{4}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\cdots\right]$$

$$= \frac{-1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

H.W. show that when $0<|z|<4$,

$$\frac{1}{4z-z^2}=\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$

H.W. Expand $\frac{1}{z(z^2-3z+2)}$ for the regions

(i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$.

→ Find the Laurent's expansion of $\frac{z^2}{z^4-1}$ is valid for $0<|z-i|<\sqrt{2}$.

Soln: Here $f(z)=\frac{z^2}{z^4-1}$

$$=\frac{z^2}{(z^2+1)(z^2-1)}$$

$$=\frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$$

consider $\frac{1}{z+i}=\frac{1}{(z-i+2i)}$

$$=\frac{1}{2i\left[1+\frac{z-i}{2i}\right]}$$

$$=\frac{1}{2i}\left[1+\frac{z-i}{2i}\right]^{-1}$$

This expansion is possible if $\left|\frac{z-i}{2i}\right|<1$.

$\therefore \left|\frac{z-i}{2i}\right|<1 \Rightarrow \frac{|z-i|}{2}<1$

$$\Rightarrow |z-i|<2$$

$$\therefore \frac{1}{z+i}=\frac{1}{2i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n \quad \text{--- (1)}$$

Now $\frac{1}{z+1}=\frac{1}{z-i+1+i}$

$$=\frac{1}{(1+i)\left[1+\frac{z-i}{1+i}\right]}$$

$$=\frac{1}{1+i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{1+i}\right)^n$$

This expansion is possible only when $\left|\frac{z-i}{1+i}\right|<1$.

$\Rightarrow |z-i|<|1+i|=\sqrt{1+1}=\sqrt{2}$

$\Rightarrow |z-i|<\sqrt{2}$

$$\therefore \frac{1}{z+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-i)^n \quad \text{--- (2)}$$

Now Consider

$$\frac{1}{z-1}=\frac{1}{z-i-(1-i)}=\frac{1}{(1-i)\left[1-\frac{z-i}{1-i}\right]}$$

$$=\frac{1}{1-i}\left[1-\frac{z-i}{1-1}\right]^{-1}$$

The expansion is possible if $\left|\frac{z-i}{i-1}\right|<1$

i.e $|z-i|<|i-1|=\sqrt{2}$

$\Rightarrow |z-i|<\sqrt{2}$

$$\therefore \frac{1}{z-1}=\frac{1}{i-1}\sum_{n=0}^{\infty}\left(\frac{z-i}{i-1}\right)^n=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(i-1)^{n+1}}$$

∴ The Laurent expansion of the given function is $\frac{z^2}{z^4-1}=\frac{z^2}{(z+i)(z-i)(z+1)(z-1)}$

$$=\frac{1}{4}\left[\frac{i}{z+i}-\frac{i}{z-i}-\frac{1}{z+1}+\frac{1}{z-1}\right]$$

$$=\frac{i}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2i)^{n+1}}(z-i)^n-\frac{i}{4(z-i)}$$

$$-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(i-1)^{n+1}}(z-i)^n+$$

$$\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+i)^{n+1}}(z-i)^n$$

Principal Part: $\frac{-i}{4(z-i)}$

→ Express $\sin z \sin\left(\frac{1}{z}\right)$ in a Laurent series valid for $|z|>0$.

Sol'n: we know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

and $\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \cdots$

$$\therefore \sin z \cdot \sin\left(\frac{1}{z}\right) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \cdots\right)$$

$$= \cdots + \left(\frac{1}{5!} + \frac{1}{7!}\cdot\frac{1}{3!} + \frac{1}{9!}\cdot\frac{1}{5!} + \cdots\right)\frac{1}{z^4} + \left(-\frac{1}{3!} - \frac{1}{5!}\cdot\frac{1}{3!} - \frac{1}{7!}\cdot\frac{1}{5!} + \cdots\right)\frac{1}{z^2} +$$

$$\left(1 + \left(\frac{1}{3!}\right)^2 + \left(\frac{1}{7!}\right)^2 + \cdots\right) - \left(\frac{1}{3!} + \frac{1}{3!}\cdot\frac{1}{5!} + \frac{1}{5!}\cdot\frac{1}{7!} + \cdots\right)z^2 + \cdots$$

H.W. Find the Laurent series for the following functions valid for the given region (a) $e^{z^2} + e^{1/z^2}$; $|z|>0$.

(b) $\frac{1}{(z-a)(z-b)}$; $0<|z-a|<|a+b|$

Here $0<|a|<|b|$

(c) $\frac{\sin z}{z^2}$; $|z|>0$.

H.W. Find the principal part for the following Laurent series:

$\frac{\sin z}{z^4}$; $|z|>0$ Ans: $\frac{1}{z^3} - \frac{1}{6z}$

→ For the function $f(z) = \frac{2z^3+1}{z^2+z}$

find (i) a taylor's series valid in the neighbourhood of the point $z=i$.

(ii) a Laurent series valid within the annulus of which centre is the origin.

Sol'n: (i) we have $f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}$ — ①

Let $f_1(z) = 2(z-1)$, $f_2(z) = \frac{1}{z}$, $f_3(z) = \frac{1}{z+1}$

Taylor's expansion for $f_1(z)$ about $z=i$ is given by $f_1(z) = \sum_{n=0}^{\infty} \frac{f_1^n(i)}{n!}(z-i)^n$ — ②

$f_1(z) = 2(z-1) \Rightarrow f_1(i) = 2(i-1)$

$f_1'(z) = 2 \Rightarrow f_1'(i) = 2$

$f_1''(z) = 0 \Rightarrow f_1''(i) = 0$

$\therefore f_1^n(i) = 0$ for $n \geq 2$.

$\therefore$ from ②,

$$\boxed{f_1(z) = 2(i-1) + 2(z-i)}$$

Similarly we can find

$$f_2(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}}(z-i)^n \ \&$$

$$f_3(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$$

$\therefore$ from ①, we have

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n$$

which is the required Taylor's expansion.

(ii) For $|z|<1$, Laurent series for $f(z)$ is given by

$$f(z) = 2(z-1) + \frac{2}{z} + (1+z)^{-1}$$

$$= 2(z-1) + \frac{2}{z} + (1 - z + z^2 - z^3 + \cdots)$$

Principal part: $\frac{2}{z}$.

→ Obtain the Taylor's (or) Laurent's Series which represents the function $f(z) = \frac{1}{(1+z^2)(z+2)}$

when (i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

* Classification of Singularities:

Upto now we have considered the functions which are analytic in and on a closed contour 'C'. Now we consider the domains that contain the points, where $f(z)$ is not analytic.

A point is said to be singular point of a function $f(z)$, if the function is not analytic at that point. These singularities are classified into three types, called (i) Removable singularity (ii) Pole (iii) Essential singularity. The behaviour of the function at these points is shown to be as:

(i) At removable singularity the function can be redefined such that the function is analytic at that point.

(ii) At a pole $f(z)$ tends to ∞ as z approaches the singular point.

(iii) At an essential singularity the function comes arbitrarily close to every complex number in the deleted neighbourhood of that point.

The method of finding singularities for a given function is considered.

* Singularities :-

Definition: A single valued function $f(z)$ is said to have a singularity at a point if the function is not analytic at that point.

* Isolated Singular Point :

If a function is analytic in some deleted neighbourhood of a singular point then that point is said to be an isolated singular point.

Ex ①: Consider $f(z) = \frac{\sin z}{z}$, $z=0$ is an isolated singular point of $f(z)$, since $f(z)$ is analytic in any deleted neighbourhood of $z=0$.

Ex ②: $f(z) = \frac{z+3}{z^2(z^2+1)}$ possesses three isolated singular points $z=0$, $z=i$ & $z=-i$

Ex ③: $f(z) = \frac{1}{\sin(\pi/z)}$ has an infinite number of isolated singularities all of which lie on the real axis from $z=-1$ to $z=1$.

These isolated singularities are at $z = \pm\frac{1}{n}$, $n=1,2,3,\ldots$

$z=1$
$0<|z-1|<\delta$
$z\in(1-\delta,1+\delta)$

The origin $z=0$ is also a singularity but it is not isolated. Since every neighbourhood of '0' contains other singularities of the function.

Ex ④: The function $\log z$ has a singularity at the origin which is not isolated.

Since every neighbourhood of '0' contains points on the negative real axis where $\log z$ ceases to be analytic

Def: Let $z = z_0$ be an isolated singularity of a function $f(z)$.

Since singularity isolated there exists a deleted neighbourhood $0 < |z - z_0| < \delta$ in which $f(z)$ is analytic then $f(z)$ has a laurent's expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$

The part $\sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$ of the laurent series is called the principal part of $f(z)$ at $z = z_0$.

→ **Riemann's Theorem**:-

If a function $f(z)$ has an isolated singularity at $z = z_0$ and is bounded in some deleted neighbourhood of z_0, then $f(z)$ can be defined at z_0 in such a way as to be analytic at z_0.

→ If $f(z)$ has an isolated singularity at $z = z_0$ and is bounded in some deleted neighbourhood of z_0, then $\lim_{z \to z_0} f(z)$ exists.

→ **Removable Singularity**:

Let $f(z)$ has an isolated singularity at z_0. Then z_0 is said to be removable singularity if $\lim_{z \to z_0} f(z)$ exists.

Example: $f(z) = \frac{\sin z}{z}$;

$z = 0$ is an isolated singular point

Also $\lim_{z \to 0} \frac{\sin z}{z} = 1$ (exists)

$\therefore z = 0$ is a removable singular point.

Theorem ①: If $f(z)$ has an isolated singularity at z_0 and $f(z) \to \infty$ as $z \to z_0$, then $f(z)$ has a pole at z_0.

Definition: Let $f(z)$ be analytic in a deleted neighbourhood of z_0 and 'n' be a +ve integer such that $\lim_{z \to z_0} (z-z_0)^n f(z)$

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0, \infty. \quad \text{--- ①}$$

Then $f(z)$ is said to have a pole of order 'n' at z_0.

When 'n' = 1, the pole is said to be a simple pole.

Note(1): From the nature of the condition that is satisfied by $f(z)$ when it has a pole of order 'n' at z_0 we conclude that it must be of the form $\frac{F(z)}{(z-z_0)^n} = f(z)$.

where $F(z)$ is analytic at z_0 and $F(z_0) \neq 0$.

$\therefore f(z)$ is not defined at z_0.

Ex ①: $f(z) = \frac{1}{(z-2)^3}$ has a pole of order 3 at $z=2$.

② $f(z) = \frac{3z-2}{(z-1)^2(z+1)(z-4)}$ has a pole of order '2' at $z=1$ and simple poles at $z=-1$ & $z=4$.

Note ②: The necessary and sufficient condition for an isolated singularity to be a pole is:

Necessary Condition:

If $f(z)$ has an isolated singularity at $z=z_0$ and $f(z) \to \infty$ as $z \to z_0$ then $f(z)$ has a pole at $z=z_0$.

Sufficient Condition:

Given that at $z=z_0$ is a pole of order k, then we have to prove that

$$\lim_{z\to z_0} f(z) = \infty.$$

By the definition of a pole

$$\lim_{z\to z_0} (z-z_0)^k f(z) \neq 0$$
$$= A \text{ (say)}$$

Now we consider

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \left[(z-z_0)^k \cdot f(z)\right] \frac{1}{(z-z_0)^k}$$
$$= A \times \infty$$
$$= \infty$$

$\therefore \lim_{z\to z_0} f(z) = \infty$

i.e. $f(z) \longrightarrow \infty$ as $z \to z_0$.

Theorem: If $f(z)$ has a pole at $z=z_0$, then $f(z)$ may be expressed as

$$f(z) = \sum_{n=-k}^{\infty} b_n (z-z_0)^n \text{ where } k \text{ is}$$

the order of the pole.

→ Isolated Essential Singularity:

An isolated singularity that is neither a removable singularity nor a pole is said to an isolated essential singularity.

Theorem: Cosorati – Weierstrass theorem:

If $f(z)$ has an isolated essential singularity at $z=z_0$ then $f(z)$ comes arbitrarily close to every complex value in each deleted neighbourhood of z_0.

For example: $f(z) = e^{\frac{1}{z-2}}$ has an essential singularity at $z=2$.

— If a function is single valued and has singularity, then the singularity is either a pole or an essential singularity.

For this reason a pole is sometimes called non-essential singularity.

Equivalently, $z=z_0$ is an essential singularity if we cannot find any +ve integer 'n' such that

$$\lim_{z\to z_0} (z-z_0)^n f(z) = A \neq 0.$$

Example ②: $f(z) = e^{1/z}$

$z=0$ is an essential singularity of $f(z)$.

Note: If $z = z_0$ is an essential singularity of $f(z)$ then principal part of the laurent expansion has an infinitely many terms.

From the above example

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots,$$

$z=0$ is an essential singularity.

*** Singularity at $z = \infty$:**

A singularity of $f(z)$ at $z=\infty$ is a removable, a pole (or) essential according as the singularity of $f(1/z)$ at $z=0$ is removable, a pole or essential.

Ex: ① $f(z) = z^2+1$ has a pole of order '2' at $z=\infty$

Since $f(1/z) = \frac{1}{z^2}+1$

$$\Rightarrow z^2 f(1/z) = 1+z^2$$

$$\lim_{z\to 0} z^2 f(1/z) = \lim_{z\to 0}(1+z^2)$$

$$= 1$$

$$\neq 0.$$

$\therefore f(1/z)$ has a pole of second order at $z=0$.

Ex ②: $f(z) = e^z$ has an isolated essential singularity at $z=\infty$ because $f(1/z) = e^{1/z}$ has an isolated essential singularity at $z=0$.

→ **Method of determining the nature of isolated singularities of a given function:**

Step 1: observe at what points the given function is not analytic i.e., not defined.

Step 2: At those points determine the value of the limit of $f(z)$ as z tends to those points.

Step 3: (i) If the limit exists then that point is a removable singularity.

(ii) If the value of the limit is ∞, then that point is a pole.

(iii) If the point is neither removable nor a pole, then say that it is a essential singularity.

1999 → Find all the finite isolated singularities of $\frac{1}{\sin z - \cos z}$

Sol'n: If the denominator is equal to zero, the given function is not defined.

$\therefore \sin z - \cos z = 0 \Rightarrow z = \pi/4$

$$\therefore \lim_{z\to\pi/4} f(z) = \lim_{z\to\pi/4}\frac{1}{\sin z - \cos z} = \infty$$

Hence $z = \pi/4$ is a simple pole.

$$= \lim_{z\to\pi/4} \frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}}\sin z - \frac{1}{\sqrt{2}}\cos z}$$

$$= \lim_{z\to\pi/4} \frac{1/\sqrt{2}}{\cos\frac{\pi}{4}\sin z - \sin\frac{\pi}{4}\cos z}$$

$$= \lim_{z\to\pi/4} \frac{1/\sqrt{2}}{\sin(z-\pi/4)}$$

$$\Rightarrow \lim_{z\to\pi/4} \frac{1/\sqrt{2}\,(z-\pi/4)}{\sin(z-\pi/4)}$$

$$= 1/\sqrt{2} \neq 0$$

$$\lim_{z\to\pi/4} (z-\pi/4)\cdot f(z) = 1/\sqrt{2} \neq 0$$

where $f(z) = \frac{\frac{1}{\sqrt{2}}}{\sin(z-\pi/4)}$

→ Describe the singularity of $f(z) = \frac{z^2}{z+1}$ at $z = \infty$.

Sol'n: $f(z) = \frac{z^2}{z+1}$

$$\therefore f(1/z) = \frac{1/z^2}{\frac{1}{z}+1}$$

$$= \frac{1}{z(z+1)}$$

$\therefore z = 0,\ z = -1$ are simple poles for $f(1/z)$.

$\therefore f(z)$ has pole at $z = \infty$.

Method of determining the nature of singularities using the principal part:

With the help of the no. of terms that are present in the principal part of Laurent series for a function $f(z)$, the nature of the singularity can be determined. For that consider the following cases.

$0<|z-z_0|<R_2$

Let us consider the special case when $R_1=0$. Then $f(z)$ is analytic in the deleted nbd of z_0. So z_0 is an isolated singularity.

Case(i): This isolated singularity will be removable singularity iff the principal part is zero. For proving this first let us suppose that the principal part is zero and then prove that the singularity is removable. Now the Laurent series becomes $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. This is equivalent to Taylor's series of $f(z)$ at z_0, which is cgt in $|z-z_0|<\delta$, $\delta>0$. Thus $f(z)$ can be defined such that it is analytic at z_0. Hence z_0 is removable singularity.

In vice versa let us suppose that z_0 is removable singularity and then prove that the principal part is zero.

Since z_0 is removable singularity, $f(z)$ can be defined at z_0 such that it is analytic at z_0 also. Hence $f(z)$ is analytic in $|z-z_0|\leq\delta$. Hence by Taylor's theorem $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Hence principal part is zero

Case(ii): In this case the principal part has finitely many terms. This is the situation iff the singularity at z_0 is a pole. To prove this, first let us suppose that the principal part has finite no. of terms i.e., say K terms, then we prove that z_0 is a pole of order K. For that let us consider the Laurent series for $f(z)$ at z_0.

$$f(z) = \sum_{n=-K}^{\infty} a_n (z-z_0)^n = \frac{a_{-K}}{(z-z_0)^K} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0$$
$$+ a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\therefore (z-z_0)^K f(z) = a_{-K} + a_{-K+1}(z-z_0) + \cdots + a_{-2}(z-z_0)^{K-2}$$
$$+ a_{-1}(z-z_0)^{K-1} + a_0(z-z_0)^K + a_1(z-z_0)^{K+1} +$$
$$a_2(z-z_0)^{K+2} + \cdots$$

$$\therefore \lim_{z\to z_0} (z-z_0)^K f(z) = a_{-K} \neq 0, \infty$$

$\therefore z_0$ is a pole of $f(z)$ of order K.

In converse let z_0 be a pole of order K for $f(z)$, then we have to prove that the Laurent series will have finitely K terms.

Now we have $\lim_{z\to z_0} (z-z_0)^K f(z) \neq 0, \infty$.

$\therefore$ Let $(z-z_0)^K f(z) = F(z)$.

Then $F(z)$ is analytic in $|z-z_0| < R$

$\therefore$ By Taylor's theorem $F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$.

$$\therefore F(z) = (z-z_0)^K f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$\therefore f(z) = \frac{a_0}{(z-z_0)^K} + \frac{a_1}{(z-z_0)^{K-1}} + \cdots + a_K + a_{K+1}(z-z_0) + \cdots$$

Let $a_{k+n} = b_n$, then $a_0 = b_{-k}$, $a_1 = b_{-k+1}$, - - -

$$\therefore f(z) = \frac{b_{-k}}{(z-z_0)^k} + \frac{b_{-k+1}}{(z-z_0)^{k-1}} + \cdots\cdots + b_0 + b_1(z-z_0) + \cdots$$

$$= \sum_{n=-k}^{\infty} b_n (z-z_0)^n$$

Hence the principal part of $f(z)$ is having finite no. of terms, i.e, k terms.

Case(iii) In this let the principal part has infinitely many terms. Then z_0 cannot be removable or a pole. Hence by a process of elimination z_0 can be an essential singularity.

Notes: The coefficients of a Laurent series are usually not found by evaluating the integrals in terms of which they are defined. Infact determining 'a_n' by other means, will enable us to evaluate the integral by which they are defined.

In general if the given function $f(z)$ consists trignometric, (or) exponential or logarithmic function, we make the known standard expression and determine the Laurent series.

But in the case of algebraic functions we make use of binomial expansion and find the Laurent series.

Example6: Find the nature and location of the singularities of the function $f(z) = \frac{1}{z(e^z-1)}$

Prove that it can be expanded in the form $\frac{1}{z^2} - \frac{1}{2z} + a_0 + a_2 z^2 + a_4 z^4 + \cdots\cdots$ where $0 < |z| < 2\pi$.

Solⁿ The given function is $f(z) = \frac{1}{z(e^z-1)}$

$\therefore$ The singularities of $f(z)$ are given by

$z(e^z-1) = 0$

$\Rightarrow z=0$ and $e^z-1=0 \Rightarrow e^z = 1 = e^{2n\pi i} \Rightarrow z = 2n\pi i$
$(n = 0, \pm 1, \pm 2, \ldots)$

Hence the singularities of $f(z)$ are at $z=0$ and
$z = 2n\pi i \quad (n = \pm 1, \pm 2, \ldots)$

$z=0$ also occurs as a factor of e^z-1.

Hence $z=0$ is a double pole of $f(z)$.

The other singularities, namely $\pm 2\pi i, \pm 4\pi i, \pm 6\pi i, \ldots$
are simple poles.

It follows that $f(z)$ can be expanded as a Laurent series in the region $0 < |z| < 2\pi$ in powers of z. Since $z=0$ is a double pole, the principal part of $f(z)$ will consist of two terms only

Now consider $f(z) = \frac{1}{z(e^z-1)}$

$$= \frac{1}{z\left[1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots-1\right]}$$

$$= \frac{1}{z^2\left[1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right]}$$

$$= \frac{1}{z^2}\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)^2 - \left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)^3+\left(\frac{z}{2}+\frac{z^2}{3!}+\cdots\right)^4+\cdots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2!}+z^2\left(-\frac{1}{3!}+\frac{1}{4}\right)+z^3\left(-\frac{1}{4!}+\frac{1}{6}-\frac{1}{8}\right)+\left(-\frac{1}{120}+\frac{1}{3}+\frac{1}{24}-\frac{1}{8}+\frac{1}{16}\right)z^4+\cdots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2}+\frac{1}{12}z^2+0+\frac{1}{360}z^4+\cdots\right]$$

$$= \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}+\frac{1}{360}z^2+\cdots$$

Q. Find Laurent series about the indicated singularity for each of the following conditions. Name the singularity in each case and give the region of convergence of each series. 42

a) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$ (b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$ (c) $\dfrac{z-\sin z}{z^3}$; $z=0$

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$

(e) $\dfrac{1}{z^2(z-3)^2}$; $z=3$

Sol: (a) $\dfrac{e^{2z}}{(z-1)^3}$; $z=1$

Let $z-1=u$, then $z=1+u$.

and $\dfrac{e^{2z}}{(z-1)^3} = \dfrac{e^{2+2u}}{u^3} = \dfrac{e^2 \cdot e^{2u}}{u^3}$

$= \dfrac{e^2}{u^3}\left[1+2u+\dfrac{(2u)^2}{2!}+\dfrac{(2u)^3}{3!}+\dfrac{(2u)^4}{4!}\cdots\right]$

$= \dfrac{e^2}{u^3}+\dfrac{2e^2}{u^2}+\dfrac{2e^2}{u}+\dfrac{4e^2}{3}+\dfrac{2e^2}{3}u+\cdots$

$= \dfrac{e^2}{(z-1)^3}+\dfrac{2e^2}{(z-1)^2}+\dfrac{2e^2}{(z-1)}+\dfrac{4}{3}e^2+\dfrac{2}{3}e^2(z-1)+\cdots$

$z=1$ is a pole of order 3 (or) triple pole.

The series cgs for all values of $z \neq 1$

(b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$.

Let $z+2=u$

then $(z-3)\sin\dfrac{1}{z+2} = (u-5)\sin\left(\dfrac{1}{u}\right)$

$= (u-5)\left\{\dfrac{1}{u}-\dfrac{1}{3!u^3}+\dfrac{1}{5!u^5}-\cdots\right\}$

$= 1-\dfrac{5}{u}-\dfrac{1}{3!u^2}+\dfrac{5}{3!u^3}+\dfrac{1}{5!u^4}+\cdots$

$= 1-\dfrac{5}{z+2}-\dfrac{1}{3!(z+2)^2}+\dfrac{5}{6(z+2)^3}-\cdots$

Since the principal part of $f(z)$ consists of an infinite no. of terms,

$\therefore$ $z=-2$ is an essential singularity

The series cgs for all values of $z \neq -2$

(c) $\dfrac{z-\sin z}{z^3}$; $z=0$

$$\frac{z-\sin z}{z^3} = \frac{1}{z^3}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!}-\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots\right\}$$

$$= \frac{1}{3!}-\frac{z^2}{3!}+\frac{z^4}{7!}-\cdots$$

Since this expansion contains no negative powers of z;

$\therefore$ $z=0$ is a removable singularity.

The series cgs for all values of z.

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$

Let $z+2=u$

then $\dfrac{z}{(z+1)(z+2)} = \dfrac{u-2}{u(u-1)} = \dfrac{2-u}{u(1-u)}$

$$= \frac{2-u}{u}(1-u)^{-1}$$

$$= \frac{2-u}{u}\left[1+u+u^2+u^3+\cdots\right]$$ ($\because$ By the binomial expansion it is possible when $|u|<1$ i.e. $|z+2|<1$)

$$= \left(\frac{2}{u}+2+2u+2u^2+\cdots\right)-\left(1+u+u^2+u^3+\cdots\right)$$

$$= \frac{2}{u}+1+u+u^2+u^3+\cdots$$

$$= \frac{2}{(z+2)}+1+(z+2)+(z+2)^2+\cdots$$

$z=-2$ is a pole of order 1.

(or) simple pole.

The series cgs for all values of z s.t $0<|z+2|<1$

$\dfrac{1}{z^2(z-3)^2}$; $z=3$

Let $z-3=u$.

Then $\dfrac{1}{z^2(z-3)^2} = \dfrac{1}{(u+3)^2u^2} = \dfrac{1}{9u^2\left[1+\frac{u}{3}\right]^2}$

$$= \frac{1}{9u^2}\left[1+\frac{u}{3}\right]^{-2}$$ by the binomial theorem when $\left|\frac{u}{3}\right|<1$

$$= \frac{1}{9u^2}\left[1+(-2)\left(\frac{u}{3}\right)+\frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3+\cdots\right]$$

$$= \frac{1}{9u^2}-\frac{2}{27u}+\frac{1}{27}-\frac{4u}{243}+\cdots = \frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4(z-3)}{243}+\cdots$$

$z=3$ is a pole of order 2 (or) double pole.

The series cgs for all values of z s.t $0<|z-3|<3$

RESIDUES

In the previous discussion, we considered the evaluation of complex integrals by using parametrization, Cauchy's theorem, Cauchy's integral formula etc.

At present, we consider another important method of evaluating certain integrals by using residues. We define the residue of $f(z)$ as the coefficient of $\frac{1}{z-z_0}$ in the Laurent expansion of $f(z)$. We also consider the different methods of evaluating residues, which help us in evaluating integrals.

* **Residue:** If $f(z)$ is analytic in a deleted nbd of z_0 then by Laurent theorem we may write

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + \cdots$$

$$\cdots + \frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-3}}{(z-z_0)^3} + \cdots$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$ —— ①

$n = 0, \pm 1, \pm 2, \ldots$

In the special case $n=-1$,

we have from (1)

$$a_{-1} = \frac{1}{2\pi i}\int_C f(z)\,dz \Rightarrow \int_C f(z)\,dz = 2\pi i\, a_{-1}$$

where C is a simple closed contour enclosing z_0 and contained in the nbd of z_0. Then the coefficient a_{-1} is called the Residue of $f(z)$ at z_0.

The above relationship helps in evaluating the integral $\int_C f(z)\,dz$. Therefore evaluation of an integral is akin to (similar to) evaluating a coefficient in the Laurent expansion of the function.

Example: ① Evaluate $\int_C e^{1/z}\,dz$.

since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

The residue at $z=0$ is $1 = a_{-1}$.

$\therefore \int_C e^{1/z}\,dz = 2\pi i\cdot 1$

$= 2\pi i$

Example ②. ~~Evaluate~~ $\int_C \sin\left(\frac{1}{z^2}\right) dz$.

Sol for $z \neq 0$, $\sin\left(\frac{1}{z^2}\right) = \frac{1}{z^2} - \frac{1}{3!}\left(\frac{1}{z^2}\right)^3 + \frac{1}{5!}\left(\frac{1}{z^2}\right)^5 + \cdots$

since there is no term of z^{-1}, $a_{-1} = 0$.

Hence $\int_C \sin\left(\frac{1}{z^2}\right) dz = 0$.

→ since $\int_C \sin\left(\frac{1}{z^2}\right) dz = 0$ along any simple closed contour contains the origin, why is not $\sin\left(\frac{1}{z^2}\right)$ analytic?

Sol At origin $\sin(1/z^2)$ is not defined.

so it is not continuous.

so we can not apply Morera's theorem to conclude that $\sin(1/z^2)$ is analytic.

Residue Theorem :-

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve 'C' except at the singularities $a, b, c, \ldots$ inside. which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$

Then $\int_C f(z)\,dz = 2\pi i(a_{-1}+b_{-1}+c_{-1}+\ldots)$

i.e the integral of $f(z)$ around 'C' is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C.

OR

→ suppose $f(z)$ is analytic inside and on a simple closed contour 'C' except for isolated singularities at $z_1, z_2, \ldots z_n$ inside 'C'. Let the residues at $z_1, z_2, \ldots z_n$ respectively be $\alpha_1, \alpha_2, \ldots \alpha_n$. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \alpha_k$$

proof About each singularity z_k, construct a circle C_k contained inside 'C' s.t $C_k \cap C_L = 0$ when $K \neq L$. Then by Cauchy's theorem,

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz$$

where the integration along each interior is counter clock wise.

$$\therefore \frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz + \frac{1}{2\pi i}\int_{C_2} f(z)\,dz$$

$$+\frac{1}{2\pi i}\int_{C_n} f(z)\,dz.$$

$$= (\alpha_1+\alpha_2+\cdots+\alpha_n)$$

$$= \sum_{k=1}^{n}\alpha_k.$$

$$\therefore \int_C f(z)\,dz = 2\pi i\sum_{k=1}^{n}\alpha_k.$$

→ Evaluate $\int_C \frac{1}{(z-1)(z-2)}dz$ along different simple closed contours C.

Sol: The given function $f(z) = \frac{1}{(z-1)(z-2)}$ has simple poles at $z=1$ and $z=2$

$\therefore$ the residue of $f(z)$ at $z=1$ is

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{1}{(z-1)(z-2)} = -1$$

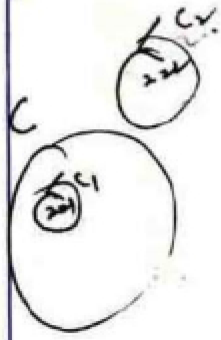
and the residue of $f(z)$ at $z=2$ is

$$\lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{1}{(z-1)(z-2)} = 1$$

Hence if $z=1$ is inside C and $z=2$ is outside C,

then

$$\int_C \frac{1}{(z-1)(z-2)}dz = 2\pi i(-1) = -2\pi i$$



if $z=1$ is outside C and $z=2$ is inside C

then $\int_C \frac{1}{(z-1)(z-2)}dz = 2\pi i(1) = 2\pi i$

if both $z=1$ and $z=2$ are inside C.

then $\int_C \frac{1}{(z-1)(z-2)}dz = 2\pi i(-1+1) = 0$

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Note ① Here C is taken as any simple closed contour C because the shape of C won't effect the value of the integral

Note②: If a function $f(z)$ has a simple pole at $z=z_0$ then the residue by using the formula

$$\lim_{z\to z_0}(z-z_0)\,f(z).$$

#w → Evaluate $\int_C \frac{dz}{z(z-3)}$ along different simple closed contours C

→ **Formula for calculating Residues:**

Let a function $f(z)$ has a pole of order k at $z=z_0$. Then the residue is calculated by using the formula $a_{-1}=\frac{1}{(k-1)!}\cdot\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]$

This formula can be obtained as below:

Let $f(z)$ has a pole of order k at $z=z_0$, then $f(z)$ can be expanded in Laurent series as

$$f(z)=\sum_{n=-k}^{\infty}a_n(z-z_0)^n$$

$$=a_{-k}\frac{1}{(z-z_0)^k}+a_{-k+1}\frac{1}{(z-z_0)^{k-1}}+\cdots+a_{-1}\frac{1}{(z-z_0)}+$$
$$a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

$$=\frac{1}{(z-z_0)^k}\left[a_{-k}+a_{-k+1}(z-z_0)+\cdots\cdots+a_{-1}(z-z_0)^{k-1}+a_0(z-z_0)^k+a_1(z-z_0)^{k+1}+\cdots\right]$$

$$\Rightarrow f(z)\cdot(z-z_0)^k=\left[a_{-k}+a_{-k+1}(z-z_0)+a_{-k+2}(z-z_0)^2+\cdots+a_{-1}(z-z_0)^{k-1}+a_0(z-z_0)^k+\cdots\right]$$

Differentiating w.r.t z we get

$$\frac{d}{dz}(z-z_0)^k f(z)=a_{-k+1}+2a_{-k+2}(z-z_0)+\cdots+a_{-1}(k-1)(z-z_0)^{k-2}+a_0k(z-z_0)^{k-1}+\cdots$$

repeating this process in total $(k-1)$ times we get

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = a_{-k}(k-1)! + k(k-1)(k-2)\cdots 3\cdot 2\cdot a_0(z-z_0)$$
$$+ (k+1)k(k-1)\cdots\cdots 4\cdot 3\cdot a_1(z-z_0)^2 + \cdots$$

$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = a_{-1}(k-1)! + 0 + 0 + \cdots$$

$$\therefore\ a_{-1} = \frac{1}{(k-1)!}\lim_{z\to z_0}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right]$$

Example.

Evaluate $\int_C \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\,dz$, where C is a simple closed contour containing $z=1$

Solⁿ. Let $f(z) = \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}$; $z_0 = 1$

$f(z)$ has a pole of order 3 at $z=1$

$$\therefore\ a_{-1} = \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left((z-1)^3 f(z)\right)$$
$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[(z-1)^3\frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\right]$$
$$= \frac{1}{2!}\lim_{z\to 1}\frac{d^2}{dz^2}\left[z^4 - z^3 - 17z + 3\right]$$
$$= \frac{1}{2!}\lim_{z\to 1}\left(12z^2 - 6z - 0\right)$$
$$= \frac{1}{2!}\left[(12)(1)^2 - 6(1)\right]$$
$$= \frac{1}{2!}(12-6)$$
$$= \frac{1}{2}(6) = 3$$

$$\therefore\ \int_C \frac{z^4 - z^3 - 17z + 3}{(z-1)^3}\,dz = 2\pi i\, a_{-1} = 2\pi i(3)$$
$$= 6\pi i$$

→ Find the residues of (a) $f(z) = \dfrac{z}{(z-1)(z+1)^2}$

(b) $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+4)}$

(c) $f(z) = e^z \operatorname{cosec}^2 z$

at all its poles in the finite plane.

Sol^n → (a) $f(z) = \dfrac{z}{(z-1)(z+1)^2}$

~~Here~~ $z=1$ and $z=-1$ are poles of orders one and two respectively

∴ Residue at $z=1$ is

$$a_{-1} = \lim_{z\to1} (z-1) f(z) = \lim_{z\to1} (z-1)\frac{z}{(z-1)(z+1)^2}$$

$$= \lim_{z\to1} \frac{z}{(z+1)^2} = \frac{1}{4}$$

Residue at $z=-1$ is

$$a_{1} = \lim_{z\to-1} \frac{1}{1!} \frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \lim_{z\to-1} \frac{d}{dz}\left[(z+1)^2 \frac{z}{(z-1)(z+1)^2}\right]$$

$$= \lim_{z\to-1} \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= \lim_{z\to-1} \frac{-1}{(z-1)^2} = -\frac{1}{4}$$

Sol^n → (c) It is given that $f(z) = e^z \operatorname{cosec}^2 z$

$$= \frac{e^z}{\sin^2 z} = \frac{e^z}{(\sin z)^2}$$

∴ the function has double poles at $z=0, \pm\pi, \pm2\pi, \ldots$

i.e, $z = m\pi$, where $m = 0, \pm1, \pm2, \ldots$

Using the general formula for calculating the residues we get

$$a_{-1} = \frac{1}{(1)!} \lim_{z\to m\pi} \frac{d}{dz}\left[(z-m\pi)^2 \frac{e^z}{(\sin z)^2}\right]$$

$$= \lim_{z\to m\pi} \frac{(\sin z)^2\left[2(z-m\pi)e^z + (z-m\pi)^2 e^z\right] - e^z (z-m\pi)^2\, 2\sin z\cos z}{(\sin z)^4}$$

$$= \lim_{z\to m\pi} \frac{e^z (z-m\pi)^2\left[\sin z - 2\cos z\right] + 2e^z (z-m\pi)\sin z}{(\sin z)^3}$$

Let $z - m\pi = t \Rightarrow z = m\pi + t$.

when $z \to m\pi$, $t \to 0$

then this limit can be written as

$$\lim_{t\to 0} e^{m\pi+t}\left\{\frac{t^2\left[\sin(m\pi+t) - 2\cos(m\pi+t)\right] + 2t\sin(m\pi+t)}{[\sin(m\pi+t)]^3}\right\}$$

either m is even or odd the limit becomes

$$\lim_{t\to 0} e^{m\pi+t}\left\{\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{(\sin t)^3}\right\}$$

$$= \lim_{t\to 0} e^{m\pi+t}\left\{\lim_{t\to 0}\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{(\sin t)^3}\right\}$$

$$= e^{m\pi}\left\{\lim_{t\to 0}\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{\sin^3 t}\right\}$$

This limit can be easily evaluated by applying L' Hospital's rule many times after making the following adjustment.

$$= e^{m\pi}\lim_{t\to 0}\left\{\frac{t^2(\sin t - 2\cos t) + 2t\sin t}{t^3}\cdot\frac{t^3}{\sin^3 t}\right\}$$

$$= e^{m\pi}\lim_{t\to 0}\left(\frac{t^2 \sin t - 2\cos t + 2t\sin t}{t^3}\right)\lim_{t\to 0}\frac{t^3}{\sin^3 t}$$

$$= e^{m\pi}\lim_{t\to 0}\left(\frac{t^2 \sin t - 2\cos t + 2t\sin t}{t^3}\right) 1$$

$= e^{m\pi}$ (by using L' Hospital's Rule many times)

$\therefore$ The residues are $e^{m\pi}$, $m = 0, \pm 1, \pm 2, ----$

Evaluation of real definite integrals:

Some types of real definite integrals can be evaluated by using the Residue theorem.

Let us first recollect what an improper integral is:
Let $f(x)$ be continuous over a semi-infinite interval $x \geq 0$ i.e. $[0,\infty)$. If $\underset{R\to\infty}{Lt} \int_0^R f(x)\,dx$ exists, the improper integral $\int_0^\infty f(x)\,dx$ is said to converge and its value is the value of the limit.

$$\therefore \int_0^\infty f(x)\,dx = \underset{R\to\infty}{Lt} \int_0^R f(x)\,dx.$$

Next if $f(x)$ is continuous for all x then the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as — (1)

$$\int_{-\infty}^{\infty} f(x)\,dx = \underset{R_1\to\infty}{Lt} \int_{-R_1}^{0} f(x)\,dx + \underset{R_2\to\infty}{Lt} \int_0^{R_2} f(x)\,dx. \quad \text{—— (2)}$$

When both the individual limits exist. The improper integral is said to converge also. Its value is defined as the sum of those two limits.

To the same integral another type of value called Cauchy's principal value is also defined as:

$$\text{principal value of } \int_{-\infty}^{\infty} f(x)\,dx = \underset{R\to\infty}{Lt} \int_{-R}^{R} f(x)\,dx. \quad \text{—— (3)}$$

provided this single limit exists.

If integral (2) cgs, the value obtained is the same as the cauchy principal value.

On the other hand, when $f(x)=x$ (for example), the cauchy principal value of integral (1) is zero, whereas that integral does not converge according to definition (2). In most of the cases these two values of the improper integral will be the same.

In problems, we consider in this lesson the cauchy's principal value is only obtained.

Some real integrals are evaluated using the Residue theorem together with a suitable function $f(z)$ and a suitable closed path or contour 'C'.

The following type of functions are considered.

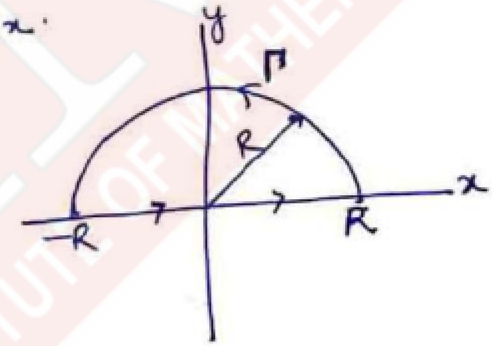
Type ①

⇒ $\int_{-\infty}^{\infty} F(x)\,dx$; $F(x)$ is a rational function.

For evaluating this type of integrals we consider $\int_C F(z)\,dz$ along a contour 'C' consisting of a line along the x-axis from $-R$ to R. and a semi-circle 'Γ' having this line as diameter as shown in the figure.

Then let $R \to \infty$.

If $F(x)$ is even function this can be used to evaluate $\int_0^\infty F(x)\,dx$.

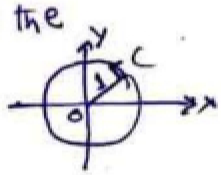


Type (2)

⇒ $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$;

$F(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta, \cos\theta$.

In this case we take $z = e^{i\theta}$. Then $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$. Then the given integral will be equivalent to $\int_C F(z)\,dz$, where C is the unit circle $|z|=1$ (as shown in the figure) with centre at the origin



$z = x+iy = e^{i\theta}$

$\Rightarrow z = \frac{1}{2}(z+\frac{1}{z}) + \frac{1}{2i}i(z-\frac{1}{z})$

$= \cos\theta + i\sin\theta$

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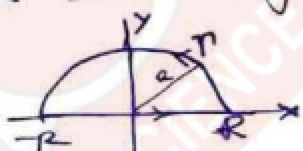
Type (3) $\int_{-\infty}^{\infty} F(x) \cdot \left\{ \begin{matrix} \cos mx \\ \sin mx \end{matrix} \right\} dx$; $F(x)$ is a rational function.

Here we consider $\int_C F(z) e^{imz} dz$, where C is a contour as that of type (1).

4) Miscellaneous integrals involving particular contours will be discussed.

→ If $|F(z)| \le M/R^k$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that $\underset{R\to\infty}{Lt} \int_\Gamma F(z)\,dz = 0$ where 'Γ' is the semi-circular arc of radius R shown in figure.



Sol: W.K.T if $f(z)$ is continuous on a contour C having length L, with $|f(z)| \le M$ on C. Then $\left| \int_C f(z)\,dz \right| \le ML$

$\therefore$ we have

$$\left| \int_\Gamma F(z)\,dz \right| \le \frac{M}{R^k} \cdot \pi R \quad (\because \text{length of arc } L = \pi R)$$

$$= \frac{\pi M}{R^{k-1}}$$

$\frac{2\pi R}{2} = \pi R$

Letting $R \to \infty$

$$\underset{R\to\infty}{Lt} \left| \int_\Gamma f(z)\,dz \right| = 0$$

$$\therefore \underset{R\to\infty}{Lt} \int_\Gamma F(z)\,dz = 0$$

→ S.T for $z = Re^{i\theta}$, $|f(z)| \le \frac{M}{R^k}$, $k > 1$; if $f(z) = \frac{1}{z^6+1}$

Sol: if $z = Re^{i\theta}$.

$$|f(z)| = |f(Re^{i\theta})|$$

$$= \left| \frac{1}{R^6 e^{6i\theta} + 1} \right| \le \frac{1}{|R^6 e^{6i\theta}| - 1}$$

$\left[\because |z_1 + z_2| \ge |z_1| - |z_2| \Rightarrow \frac{1}{|z_1+z_2|} \le \frac{1}{|z_1| - |z_2|} \right.$

$$= \frac{1}{R^6 - 1} \leq \frac{2}{R^6} \quad \text{if } R \text{ is large enough}$$
(say $R > 2$).

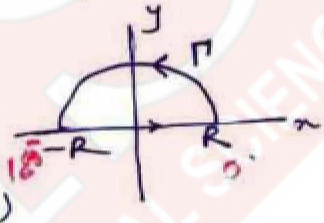
$\therefore |f(z)| \leq \frac{2}{R^6}$

problems on Type (1)

Here $M = 2$, $K = 6$

→ Evaluate $\int_0^\infty \frac{dx}{x^6+1}$.

Sol^n: for evaluating the given integral first let us consider $\int_C \frac{dz}{1+z^6}$, where C is the closed contour as shown in the figure consisting of the line from $-R$ to R and the semicircle Γ, traversed in the +ve (counter clockwise) sense.



The function $\frac{1}{z^6+1}$ will have poles at

$z^6 = e^{i(2n+1)\pi}$, where $n = 0, 1, 2, 3, 4, 5$.

i.e., $z = e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$

are the simple poles.

Only the poles $e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}$ lie within C.

$\therefore \int_C \frac{1}{z^6+1} dz = 2\pi i \left[\text{Residue at } e^{\pi i/6} + \text{Residue at } e^{3\pi i/6} + \text{Residue at } e^{5\pi i/6}\right]$

i.e., $$\int_C \frac{1}{z^6+1} dz = 2\pi i \left[\lim_{z \to e^{i\pi/6}} (z - e^{i\pi/6}) f(z) + \lim_{z \to e^{3\pi i/6}} (z - e^{3\pi i/6}) f(z) + \lim_{z \to e^{5\pi i/6}} (z - e^{5\pi i/6}) f(z)\right] \quad (1)$$

Now consider
$$\lim_{z \to e^{i\pi/6}} (z - e^{i\pi/6}) f(z) = \lim_{z \to e^{i\pi/6}} (z - e^{i\pi/6}) \frac{1}{z^6+1}$$
$= \frac{0}{0}$ form.

$$= \lim_{z \to e^{i\pi/6}} \frac{1}{6z^5}$$ (on using L' Hospital's Rule)

$$= \frac{1}{6} e^{-5\pi i/6} \quad \text{——(2)}$$

$$\lim_{z \to e^{3\pi i/6}} \left\{ (z - e^{3\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{3\pi i/6}} \frac{1}{6z^5}$$ (by using L-Hospital's Rule) $\frac{0}{0}$ form

$$= \frac{1}{6} e^{-5\pi i/2} \quad \text{——(3)}$$

$$\lim_{z \to e^{5\pi i/6}} \left\{ (z - e^{5\pi i/6}) \frac{1}{z^6+1} \right\} = \lim_{z \to e^{5\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6} e^{-25\pi i/6} \quad \text{——(4)}$$

Substituting (2), (3) & (4) in (1)

$$\int_C \frac{dz}{z^6+1} = 2\pi i \left\{ \frac{1}{6} e^{-5\pi i/6} + \frac{1}{6} e^{-5\pi i/2} + \frac{1}{6} e^{-25\pi i/6} \right\}$$

$$= \frac{2\pi i}{6} \left\{ \left(\cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6} \right) + \left(\cos \frac{5\pi}{2} - i \sin \frac{5\pi}{2} \right) + \left(\cos \frac{25\pi}{6} - i \sin \frac{25\pi}{6} \right) \right\}$$

$$= \frac{\pi i}{3} \left\{ \cos\left(\pi - \frac{\pi}{6}\right) - i \sin\left(\pi - \frac{\pi}{6}\right) + \cos\left(2\pi + \frac{\pi}{2}\right) - i \sin\left(2\pi + \frac{\pi}{2}\right) + \cos\left(4\pi + \frac{\pi}{6}\right) - i \sin\left(4\pi + \frac{\pi}{6}\right) \right\}$$

$$= \frac{\pi}{3} i \left\{ \cancel{-\cos \frac{\pi}{6}} - i \sin \frac{\pi}{6} + \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} + \cancel{\cos \frac{\pi}{6}} - i \sin \frac{\pi}{6} \right\}$$

$$= \frac{\pi i}{3} \left\{ -2i \sin \frac{\pi}{6} + 0 - i \right\} = \frac{\pi i}{3} (-i) \left\{ 2\left(\tfrac{1}{2}\right) + 1 \right\}$$

$$= \frac{\pi}{3}(2) = \frac{2\pi}{3}$$

$$\Rightarrow \boxed{\int_C \frac{dz}{z^6+1} dz = \frac{2\pi}{3}}$$

i.e, $\int_C \frac{1}{z^6+1} dz = \int_{-R}^{R} \frac{1}{x^6+1} dx + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3}$

$$\Rightarrow \int_{-R}^{R} \frac{1}{x^6+1} dx + \int_\Gamma \frac{dz}{z^6+1} = \frac{2\pi}{3} \quad \text{——(I)}$$

(in x-axis y=0)

first let us consider

$$\left|\int_0^{\pi}\frac{dz}{1+z^6}\right| \le \int_0^{\pi}\frac{|dz|}{|z^6+1|}$$

let $z = Re^{i\theta}$ then $|dz| = R\,d\theta$

$$|z^6+1| = |z^6-(-1)| \geq |z^6|-|1| = |z|^6-1 = R^6-1$$

$$\therefore \frac{1}{|z^6+1|} \le \frac{1}{R^6-1}$$

$$\therefore \left|\int_0^{\pi}\frac{dz}{1+z^6}\right| \le \int_0^{\pi}\frac{R\,d\theta}{R^6-1} = \frac{R}{R^6-1}\int_0^{\pi}d\theta$$

$$= \frac{R\pi}{R^6-1}$$

As $R\to\infty$, $\int_0^{\pi}\frac{dz}{1+z^6} = 0$ —— (II)

Taking the limit of both sides of (I) as $R\to\infty$ and using (II)

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{x^6+1} + 0 = \frac{2\pi}{3}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\therefore \int_0^{\infty}\frac{1}{x^6+1}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^6+1}$$ $\left(\because \int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx \text{ if } f(x) \text{ is even function}\right)$

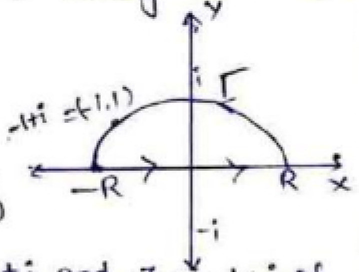
$$= \frac{1}{2}\left(\frac{2\pi}{3}\right)$$

$$\int_0^{\infty}\frac{1}{x^6+1}dx = \pi/3.$$

→ Show that $\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$.

Sol: Consider $\int_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)}$, where '$C$' is the closed contour consisting of the line from $-R$ to R (along x-axis) and semicircle Γ having this line as diameter.

The function



$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$$

has poles at $z = \pm i$ and $z = -1 \pm i$ of orders 2 and 1 respectively.

But only the poles are $z = i$ and $z = -1+i$ lie with in 'C'.

$$\therefore \int \frac{z^2}{(z^2+1)^2(z^2+2z+2)} dz = 2\pi i \left[\begin{matrix} (\text{Residue at } z = i) + \\ \text{Residue at } z = (-1+i) \end{matrix} \right] \quad \text{———} (1)$$

Now residue at $z = i$ of order 2 is

$$\frac{1}{(2-1)!} \lim_{z \to i} \frac{d^{2-1}}{dz^{2-1}} \left[(z-i)^2 f(z) \right]$$

$$= \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \frac{z^2}{(z^2+1)^2(z^2+2z+2)} \right]$$

$$= \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \frac{z^2}{(z-i)^2(z+i)^2(z^2+2z+2)} \right]$$

$$= \frac{9i-12}{100}$$

Now residue at $z = -1+i$ of order 1 is

$$= \lim_{z \to (-1+i)} \left[(z-(-1+i)) \cdot \frac{z^2}{(z^2+1)^2(z^2+2z+2)} \right]$$

$$= \lim_{z \to (-1+i)} \frac{z^2}{(z-(-1-i)) \cdot (z^2+1)^2}$$

$$= \frac{3-4i}{25}$$

$$\therefore (1) \equiv \int_C f(z) dz = 2\pi i \left[\frac{9i-12}{100} + \frac{3-4i}{25} \right]$$

$$= \frac{7\pi}{50} \quad \text{———} (2)$$

Now $\displaystyle\int_C \frac{z^2}{(z^2+1)^2(z^2+2z+2)} dz =$

$$\int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50}$$

(from (2))

$$\therefore \int_{-R}^{R} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} + \int_0^\pi \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \frac{7\pi}{50} \quad \text{———} (3)$$

Taking the limit as $R \to \infty$ and noting that the second integral approaches zero.

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$$

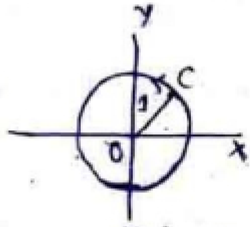
$\boxed{2001}$ → show that $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$

Problems on Type (2):

→ show that $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta = \frac{-\pi}{3}$

Sol'n: The given function is of the type (2).

Let the contour 'c' be a unit circle $|z| = 1$ as shown in the figure.



Let $z = e^{i\theta}$ then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= \frac{z + \frac{1}{z}}{2}$$

$$= \frac{z^2+1}{2z}$$

$$\therefore \frac{\cos\theta}{5+4\cos\theta} = \frac{z^2+1}{2z} \cdot \frac{1}{5 + 4\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{z^2+1}{2(2z^2+5z+2)}$$

$$= \frac{z^2+1}{2(2z+1)(z+2)}$$

Since $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\,d\theta$

$$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$$

$$\therefore \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \int_C \frac{(z^2+1)}{2(z+2)(2z+1)}\frac{dz}{iz} \quad \text{———} (I)$$

where c is a circle of unit radius with centre at origin.

$\therefore f(z) = \dfrac{z^2+1}{(2i)(z)(z+2)(2z+1)}$ has

Simple poles at $z=0, z=-2, z=-\frac{1}{2}$

But only the poles are $z=0, z=-\frac{1}{2}$ of order 1 lie with in 'c'

$$\therefore \int_C f(z)dz = 2\pi i \begin{bmatrix} (\text{Residue at } z=0) + \\ (\text{Residue at } z=-\tfrac{1}{2}) \end{bmatrix} \quad \text{———} (1)$$

Now residue at $z=0$ is

$$\lim_{z\to 0} \frac{z.(z^2+1)}{(2i)z(z+2)(2z+1)} = \frac{1}{4i}$$

and residue at $z=-\frac{1}{2}$ is

$$\lim_{z\to -\frac{1}{2}} \frac{(z+\frac{1}{2})(z^2+1)}{(2i)(z)(z+2)(2z+1)} =$$

$$\lim_{z\to -\frac{1}{2}} \frac{(z+\frac{1}{2})(z^2+1)}{(2i)(z)(z+2)2(z+\frac{1}{2})}$$

$$= -\frac{5}{12i}$$

$$\therefore (1) \equiv \int_C f(z)dz = 2\pi i\left[\frac{1}{4i} - \frac{5}{12i}\right]$$

$$= 2\pi i\left(\frac{3-5}{12i}\right)$$

$$= -\frac{\pi}{3}.$$

$\therefore$ from (I)

$$\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \frac{-\pi}{3}$$

2003 → Prove that $\displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$, $(a>0)$

Sol'n: Let $I = \displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+(1-\cos 2\theta)}$$

$$= \int_{\phi=0}^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi} \quad \text{Putting } 2\theta=\phi \quad \text{———} (i)$$

which is in the form of the type (2)

Let the contour c be the unit circle $|z|=1$ with centre at origin.

Let $z = e^{i\phi}$ then $\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$= \frac{1}{2z}(z^2+1)$$

$$\therefore \frac{a}{2a^2+1-\cos\phi} = \frac{a}{2a^2+1-\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{2az}{4a^2z+2z-z^2-1}$$

$$= \frac{-2az}{z^2-(2a^2+1)2z+1}$$

Since $z = e^{i\phi} \Rightarrow dz = ie^{i\phi}d\phi$

$$\Rightarrow dz = iz\,d\phi$$

$$\Rightarrow \boxed{d\phi = \frac{dz}{iz}}$$

$$\therefore (1) = \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{2a}{i}\int_C \frac{dz}{[2z(2a^2+1)-z^2-1]}$$

where 'c' is unit circle of radius with centre at the origin.

$$= 2ai\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$= 2ai\int_C f(z)\,dz \quad \text{———} (ii)$$

where $f(z) = \dfrac{1}{z^2 - 2(2a^2+1)z+1}$

Now the poles of $f(z)$ are given by

$z^2 - 2(2a^2+1)z + 1 = 0$

$$\Rightarrow z = \frac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2 - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm \sqrt{16a^4+16a^2}}{2}$$

$$\boxed{z = (2a^2+1) \pm 2a\sqrt{a^2+1}}$$

Let $\alpha = (2a^2+1) + 2a\sqrt{a^2+1}$

$\beta = (2a^2+1) - 2a\sqrt{(a^2+1)}$

Clearly $|\alpha| > 1 \quad (\because a > 0)$

Since $|\alpha\beta| = 1$

we have $|\beta| < 1$.

Hence the only pole inside 'c' is at $z = \beta$

$\therefore$ Residue at $z = \beta$ is

$$\lim_{z\to\beta} \cancel{(z-\beta)} \frac{1}{(z-\alpha)\cancel{(z-\beta)}} = \frac{1}{\beta - \alpha}$$

$$= \frac{1}{-4a\sqrt{a^2+1}}$$

Now from (ii),

$$\int_0^{\pi} \frac{a^2\, d\theta}{a^2 + \sin^2\theta} = 2ai \int_C f(z)\, dz$$

$$= (2ai)\, 2\pi i\, [\text{residue at } z = \beta]$$

$$= (2ai)(2\pi i)\left[\frac{1}{-4a\sqrt{a^2+1}}\right]$$

$$= \frac{\pi}{\sqrt{a^2+1}}$$

Note: (1) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ if $f(2a-x) = f(x)$

$= 0$ if $f(2a-x) = -f(x)$

HW → Prove that $\displaystyle\int_0^{2\pi} \frac{a\, d\theta}{a^2 + \sin^2\theta} = \frac{1}{2}\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta}$

$$= \frac{\pi}{2\sqrt{1+a^2}}$$

2007 → Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + 8\cos^2\theta}$

HW → Prove that $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta$

$$= \frac{2\pi}{b^2}\left[a - \sqrt{(a^2-b^2)}\right]$$

2006 → Prove that $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} = \frac{\pi a^2}{1-a^2}$; $-1 < a < 1$ i.e. $|a| < 1$.

Sol^n: Let $I = \displaystyle\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2}$$

$\left(\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(a-x) = f(x)\right)$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} \quad \text{--- (i)}$$

which is in the form of the type (2)

Let the contour 'c' be the unit circle $|z| = 1$ with centre at the origin.

Let $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$= \frac{z^2+1}{2z}$$

and $\cos 2\theta = \dfrac{e^{2i\theta} + e^{-2i\theta}}{2}$

$$= \frac{z^2 + \frac{1}{z^2}}{2}$$

$$= \frac{z^4+1}{2z^2}$$

$$\therefore \frac{\cos 2\theta}{1-2a\cos\theta+a^2} = \frac{\frac{z^4+1}{2z^2}}{1-2a\left(\frac{z^2+1}{2z}\right)+a^2}$$

$$= \frac{z^4+1}{z\left(2z-2a(z^2+1)+2a^2z\right)}$$

$$= \frac{z^4+1}{z\left[2z(1-az)+2a^2z-2a\right]}$$

Since $z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$

$$\Rightarrow dz = iz\,d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

$$\therefore (1) \equiv \int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{2}\int_C \frac{(z^4+1)\frac{dz}{iz}}{2z\left[z(1-az)+a^2z-a\right]}$$

where 'C' is unit circle of radius with centre at the origin.

$$= \frac{1}{4i}\int_C \frac{(z^4+1)\,dz}{z^2\left[z(1-az)+a^2z-a\right]}$$

$$= \frac{1}{4i}\int_C \frac{(z^4+1)\,dz}{z^2\left[z(1-az)-a(1-az)\right]}$$

$$= \frac{1}{4i}\int_C \frac{z^4+1}{z^2\left[(z-a)(1-az)\right]}\,dz$$

$$= \frac{1}{4i}\int_C f(z)\,dz \quad \text{--- (ii)}$$

where $f(z) = \dfrac{z^4+1}{\left[z^2(z-a)(1-az)\right]}$

$\therefore f(z)$ has poles at $z=0, z=a, z=1/a$ of orders 2, 1 and 1 respectively.

But only the poles are $z=0$ and $z=a$ inside 'C'.

$$\therefore \int_C f(z)\,dz = 2\pi i\left[(\text{Residue at } z=0) + (\text{Residue at } z=a)\right] \quad \text{--- (iii)}$$

Now residue at $z=0$ of order 2 is

$$\lim_{z\to 0}\frac{d}{dz}\left(z^2 f(z)\right) = \lim_{z\to 0}\frac{d}{dz}\left[\frac{z^4+1}{(z-az^2-a+a^2z)}\right]$$

$$= -\left(\frac{1+a^2}{a^2}\right)$$

and Residue at $z=a$ is

$$\lim_{z\to a}\left[(z-a)f(z)\right] = \frac{a^4+1}{a^2(1-a^2)}$$

$\therefore$ from (iii)

$$\int_C f(z)\,dz = 2\pi i\left[\frac{-(1+a^2)}{a^2} + \frac{a^4+1}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left[\frac{-(1-a^4)+a^4+1}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left[\frac{2a^4}{a^2(1-a^2)}\right]$$

$$= 2\pi i\left(\frac{2a^2}{1-a^2}\right) = \frac{4\pi i a^2}{1-a^2}$$

$\therefore$ from (ii), we have

$$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \frac{1}{4i}\left(\frac{4\pi i a^2}{1-a^2}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

HW → Apply Calculus of residues to Prove that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta\,d\theta}{5+4\cos\theta} = \pi/6$.

2004 → Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p}$, $0<p<1$.

Sol^n: Clearly the given function is of the type (2).

Let the contour 'c' be a unit circle $|z|=1$ with centre at origin.

Let $z=e^{i\theta}$ then $\cos 3\theta = \frac{e^{i3\theta}+e^{-i3\theta}}{2}$

$$= \frac{z^3+\frac{1}{z^3}}{2}$$

$$= \frac{z^6+1}{2z^3}$$

$$\therefore \cos^2 3\theta = \left(\frac{z^6+1}{2z^3}\right)^2$$

$$\boxed{\cos^2 3\theta = \frac{(z^6+1)^2}{4z^6}}$$

and $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2}$

$$= \frac{\left(z^2+\frac{1}{z^2}\right)}{2}$$

$$\boxed{\cos 2\theta = \frac{1}{2z^2}(z^4+1)}$$

Since $z=e^{i\theta} \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\,d\theta = \int_C \frac{\frac{(z^6+1)^2}{4z^6}\left(\frac{dz}{iz}\right)}{1-2p\left(\frac{z^4+1}{2z^2}\right)+p^2}$$

$$= \frac{1}{4i}\int_C \frac{(z^6+1)^2\,dz}{z^7\left[\frac{z^2-p(z^4+1)+p^2z^2}{z^2}\right]}$$

$$= \frac{1}{4i}\int_C \frac{(z^6+1)^2\,dz}{z^5\left[z^2-p(z^4+1)+p^2z^2\right]}$$

$$= \frac{1}{4i}\int_C f(z)\,dz \quad \text{———} \quad (i)$$

where $f(z) = \frac{(z^6+1)^2}{z^5\left[z^2-p(z^4+1)+p^2z^2\right]}$

$$= \frac{-(z^6+1)^2}{z^5\left[p(z^2)^2-(1+p^2)z^2+p\right]}$$

$$= \frac{-(z^6+1)^2}{z^5(pz^2-1)(z^2-p)}$$

Now $f(z)$ poles at $z=0$, $z=\pm\frac{1}{\sqrt{p}}$, $z=\pm\sqrt{p}$ of orders 5, 1 and 1 respectively.

Now residue at $z=0$ of order 5 is

$$\frac{1}{(5-1)!}\lim_{z\to 0}\frac{d^4}{dz^4}\left[z^5\cdot\frac{(z^6+1)^2}{z^5(pz^2-1)(z^2-p)}\right]$$

but this method is much more laborious to get the solution. so that we better to stop this method in particular problems.

<u>Now we follow the following Procedure:</u>

Let $I = \int_0^{2\pi}\frac{\cos^2 3\theta\,d\theta}{1-2p\cos 2\theta+p^2} = \frac{1}{2}\int_0^{2\pi}\frac{1+\cos 6\theta}{1-2p\cos 2\theta+p^2}\,d\theta$

$$= \text{real part of } \frac{1}{2}\int_0^{2\pi}\frac{1+e^{i6\theta}}{1-2p\cos 2\theta+p^2}\,d\theta \quad (1)$$

Let the contour 'c' be a unit circle $|z|=1$ with centre at the origin.

Let $z=e^{i\theta} \Rightarrow \cos 2\theta = \frac{e^{i2\theta}+e^{-i2\theta}}{2}$

$$= \frac{z^2+z^{-2}}{2} = \frac{z^4+1}{2z^2}$$

$$\therefore \frac{1+e^{6i\theta}}{1-2p\cos 2\theta+p^2} = \frac{1+z^6}{1-2p\left(\frac{z^4+1}{2z^2}\right)+p^2}$$

$$= \frac{(1+z^6)z^2}{z^2-p(z^4+1)+p^2z^2}$$

$$= \frac{z^2(1+z^6)}{z^2(1-pz^2)-p(1-pz^2)}$$

$$= \frac{z^2(1+z^6)}{(z^2-p)(1-pz^2)}$$

Since $z = e^{i\theta}$

$\Rightarrow d\theta = \frac{dz}{iz}$

Now consider

$$\int_0^{2\pi} \frac{1+e^{6i\theta}}{1-2p\cos 2\theta+p^2}\, d\theta = \int_C \frac{z^2(1+z^6)\frac{dz}{iz}}{(z^2-p)(1-pz^2)}$$

$$= \frac{1}{i}\int_C \frac{z(1+z^6)dz}{(z^2-p)(1-pz^2)}$$

$$= \frac{1}{i}\int_C f(z)dz \quad \text{--- (2)}$$

where C is unit circle of radius with Centre at origin

where $f(z) = \frac{z(1+z^6)}{(z^2-p)(1-pz^2)}$

The function $f(z)$ has the poles at $z = \pm\frac{1}{\sqrt{p}}$, $z = \pm\sqrt{p}$ of orders one.

But only the poles are $z = \pm\sqrt{p}$ are lie within C.

$$\therefore \int_C f(z)dz = \int_C \frac{z(1+z^6)dz}{(z^2-p)(1-pz^2)}$$

$$= 2\pi i\left[(\text{Residue at } z=\sqrt{p}) + (\text{Residue at } z=-\sqrt{p})\right] \quad \text{--- (3)}$$

Now residue at $z=\sqrt{p}$ is of order 1 is

$$\lim_{z\to\sqrt{p}} \frac{\cancel{(z-\sqrt{p})}z(1+z^6)}{\cancel{(z-\sqrt{p})}(z+\sqrt{p})(1-pz^2)}$$

$$= \frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)} = \frac{1+p^3}{2(1-p^2)}$$

Residue at $z=-\sqrt{p}$ is of order 1 is

$$\lim_{z\to-\sqrt{p}} \frac{\cancel{(z+\sqrt{p})}z(1+z^6)}{\cancel{(z+\sqrt{p})}(z-\sqrt{p})(1-pz^2)}$$

$$= \frac{-\sqrt{p}(1+p^3)}{-2\sqrt{p}(1-p^2)} = \frac{1+p^3}{2(1-p^2)}$$

$$\therefore (3) \equiv \int_C f(z)dz = 2\pi i\left[\frac{1+p^3}{2(1-p^2)} + \frac{1+p^3}{2(1-p^2)}\right]$$

$$= \frac{2\pi i}{2(1-p^2)}(2)(1+p^3)$$

$$= \frac{2\pi i(1+p^3)}{1-p^2}$$

$$= \frac{2\pi i(1+p)(1-p+p^2)}{(1-p)(1+p)} \quad \left[\because a^3+b^3=(a+b)(a^2-ab+b^2)\right]$$

$$= \frac{2\pi i(1-p+p^2)}{1-p}$$

$$\therefore (2) \equiv \int_0^{2\pi}\frac{1+e^{6i\theta}}{1-2p\cos\theta+p^2}d\theta = \frac{1}{i}\int_C f(z)dz$$

$$= \frac{1}{i}(2\pi i)\frac{(1-p+p^2)}{1-p}$$

$$= \frac{2\pi(1-p+p^2)}{(1-p)} \quad \text{--- (4)}$$

$\therefore$ from (1) and (4)

$$I = \int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos\ldots+p^2} = \text{real part of } \frac{1}{2}\left[\frac{2\pi(1-p+p^2)}{1-p}\right]$$

where $0<p<1$.

$$\therefore \int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p} \text{ where } 0<p<1.$$

→ Prove that $\int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{3+2\cos\theta}d\theta = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$ (n being +ve integer)

Sol^n: Let $I = \int_0^{2\pi}\frac{(1+2\cos\theta)^n\cos n\theta}{(3+2\cos\theta)}d\theta$

$$= \text{real part of } \int_0^{2\pi}\frac{(1+2\cos\theta)^n e^{in\theta}}{(3+2\cos\theta)}d\theta \quad \text{--- (1)}$$

Let the contour 'C' be a unit circle $|z|=1$ with centre at the origin.

Let $z = e^{i\theta} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and $e^{in\theta} = z^n$.

$$\frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} = \frac{\left(1+z+\frac{1}{z}\right)^n z^n}{\left(3+z+\frac{1}{z}\right)}$$

$$= \frac{(z^2+z+1)^n z^n}{z^n(z^2+3z+1)/z}$$

$$= \frac{z(z^2+z+1)^n}{(z^2+3z+1)}$$

Since $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

Consider $\int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} d\theta = \int_C \frac{z(z^2+z+1)^n}{(z^2+3z+1)} \frac{dz}{iz}$

$$= \frac{1}{i}\int_C \frac{(z^2+z+1)^n}{z^2+3z+1} dz$$

where c is unit circle of radius with centre at origin

$$= \frac{1}{i}\int_C f(z)\,dz \quad \text{——— (2)}$$

where $f(z) = \frac{(z^2+z+1)^n}{(z^2+3z+1)}$

Now the poles of $f(z)$ are given by

$$z^2+3z+1=0$$

$$\Rightarrow z = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$$

Let $\alpha = \frac{-3+\sqrt{5}}{2}$; $\beta = \frac{-3-\sqrt{5}}{2}$

clearly $|\beta|>1$

Since $|\alpha\beta| = 1 \Rightarrow |\alpha|<1$ ($\because |\beta|>1$)

Hence the only pole inside C is at $z=\alpha$ of order 1.

$$\therefore \int_C f(z)\,dz = \int_C \frac{(z^2+z+1)^n}{z^2+3z+1} dz$$

$$= 2\pi i \,(\text{Residue at } z=\alpha) \quad \text{——— (3)}$$

Now the residue at $z=\alpha$ is

$$\lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} \frac{(z-\alpha)(z^2+z+1)^n}{(z-\alpha)(z-\beta)}$$

$$= \frac{(\alpha^2+\alpha+1)^n}{\alpha-\beta}$$

$$= \frac{\left(1-\frac{3}{2}+\frac{\sqrt{5}}{2}+\frac{7-3\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$= \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$\therefore$ from (3)

$$\int_C f(z)\,dz = 2\pi i \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$\therefore$ from (2)

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{3+2\cos\theta} d\theta = \frac{1}{i} 2\pi i \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n \quad \text{——— (4)}$$

from (1) & (4)

$$I = \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta = \text{real part of } \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

$$= \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

$$\therefore \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} = \frac{2\pi(3-\sqrt{5})^n}{\sqrt{5}}$$

→ Prove that $\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)d\theta = \frac{2\pi}{n!}$

Sol'n: Let $I = \int_0^{2\pi} e^{\cos\theta} e^{-(n\theta-\sin\theta)i}\, d\theta$

$= \int_0^{2\pi} e^{\cos\theta} e^{-in\theta} e^{i\sin\theta}\, d\theta$

$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta} e^{-in\theta}\, d\theta$

$= \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\, d\theta$

Putting $z = e^{i\theta}$ and let the contour 'C' be a unit circle $|z|=1$ with Centre at the origin.

$I = \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\, d\theta = \int_C e^z z^{-n} \frac{dz}{iz}$

$\left(\because z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}\right)$

$= \frac{1}{i}\int_C \frac{e^z}{z^{n+1}}\, dz$

$= \frac{1}{i}\int_C f(z)\, dz$ —— ① where C is unit circle of radius with Centre at origin.

where $f(z) = \frac{e^z}{z^{n+1}}$

Clearly $f(z)$ has a pole at $z=0$ of order $n+1$.

$\therefore \int_C f(z)\, dz = 2\pi i\,(\text{Residue at } z=0)$

Now the residue at $z=0$ of order $(n+1)$ is

$\lim_{z\to 0}(z-0)^{n+1} f(z) = \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}\left(z^{n+1}\frac{e^z}{z^{n+1}}\right)$

$= \frac{1}{n!}\lim_{z\to 0}\frac{d^n}{dz^n}(e^z)$

$= \frac{1}{n!}\lim_{z\to 0} e^z = \frac{1}{n!}$

$\therefore \int_C f(z)\, dz = 2\pi i \frac{1}{n!}$

$\therefore$ from ①

$I = \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta}\, d\theta = \frac{1}{i}(2\pi i)\frac{1}{n!}$

$= \frac{2\pi}{n!}$

$\therefore \int_0^{2\pi} e^{\cos\theta + i\sin\theta} e^{-in\theta}\, d\theta = \frac{2\pi}{n!}$

$\Rightarrow \int_0^{2\pi} e^{\cos\theta} e^{-i(n\theta-\sin\theta)}\, d\theta = \frac{2\pi}{n!}$

Equating real & imaginary parts

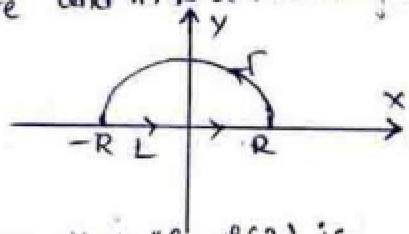
$\int_0^{2\pi} e^{\cos\theta}\cdot\cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$

and $\int_0^{2\pi} e^{\cos\theta}\cdot\sin(n\theta - \sin\theta)\, d\theta = 0$

$\therefore \int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$

Problems on Type (3): $\int_{-\infty}^{\infty} F(x)\begin{bmatrix}\cos mx\\ \sin mx\end{bmatrix}dx$

→ If $|F(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k>0$ and M are constants, Prove that $\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$, where Γ is the semicircular arc of radius R, shown in figure and m is a +ve constant.



Sol'n: we know that if $f(z)$ is continuous on a contour 'C' having length L, with $|f(z)| \le M$. Then $\left|\int_C f(z)dz\right| \le ML$

we have

$$\left|\int_\Gamma F(z)\,dz\right| \le \frac{M}{R^k}\cdot \pi R \quad (\because \text{length of arc } L = \pi R)$$

$$= \frac{\pi M}{R^{k-1}}$$

Now since $z = Re^{i\theta}$,

$$\int_\Gamma e^{imz}F(z)\,dz = \int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})\,iRe^{i\theta}\,d\theta$$

then

$$\left|\int_0^\pi e^{imRe^{i\theta}}F(Re^{i\theta})\,iRe^{i\theta}d\theta\right| \le \int_0^\pi \left|e^{imRe^{i\theta}}F(Re^{i\theta})\,iRe^{i\theta}\right|d\theta$$

$$= \int_0^\pi \left|e^{imR\cos\theta - mR\sin\theta}F(Re^{i\theta})\,iRe^{i\theta}\right|d\theta$$

$$= \int_0^\pi e^{-mR\sin\theta}\,|F(Re^{i\theta})|\,R\,d\theta \qquad (\because |e^{imR\cos\theta}| = 1)$$

$$\le \frac{M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

$$= \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \quad \text{——} \ (1)$$

For $0 \le \theta \le \pi/2$, $\sin\theta \ge \frac{2\theta}{\pi}$

[Since, consider $f(\theta) = \frac{\sin\theta}{\theta}$. Then

$$f(\pi/2) = \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

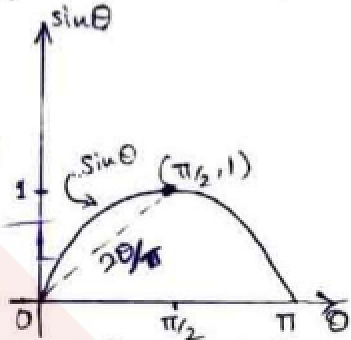
Similarly $f(0) = \frac{\sin 0}{0} \Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

But $1 > \frac{2}{\pi}$.

The value of $f(\theta)$ varies from 1 to $\frac{2}{\pi}$ as θ varies from 0 to $\pi/2$.

Now let us show that $f(\theta)$ is decreasing function in the given interval.

For that



Consider

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} = \frac{\cos\theta - \frac{\sin\theta}{\theta}}{\theta}$$

$\because$ cosine is a decreasing function in the interval $(0, \pi/2)$.

$\therefore f'(\theta) < 0$.

$\therefore f(\theta) = \frac{\sin\theta}{\theta} \ge \frac{2}{\pi}$ for all values of θ in $(0, \pi/2)$]

from (1), $$\frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \le \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{\frac{-2mR\theta}{\pi}}\,d\theta$$

$$= \frac{\pi M}{mR^k}\left(1 - e^{-mR}\right)$$

As $R \to \infty$, this approaches zero. Since m & k are +ve.

i.e. $$\lim_{R\to\infty}\int_\Gamma e^{imz}F(z)\,dz = 0$$

[2002] Show that $\int_0^\infty \frac{\cos ax}{x^2+1}dx = \frac{\pi}{2}e^{-a}, a > 0$

Sol'n: Clearly which is in the form of type(3).

Now consider $\int_0^\infty \frac{\cos ax}{x^2+1}dx = \int_C \frac{1}{z^2+1}e^{iaz}dz$

where C is the contour consisting of the semi-circle Γ of radius R together with the part of the real axis from $-R$ to R.

$$= \int_C f(z)\,dz \quad \text{——} \ (1)$$

where $f(z) = \frac{e^{iaz}}{1+z^2}$

The function $f(z)$ has simple poles at $z=+i$ and $z=-i$, of which $z=i$ only inside 'c'

$$\therefore \int_C f(z)dz = 2\pi i \{\text{Residue at } z=i\} \quad (2)$$

Now Residue at $z=i$ is

$$\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{e^{iaz}}{z+i} = \frac{e^{-a}}{2i}$$

$$(2) \equiv \int_C f(z)dz = 2\pi i\left(\frac{e^{-a}}{2i}\right)$$

$$= \pi e^{-a}$$

$$\text{i.e. } \int_C \frac{e^{iaz}}{1+z^2}dz = \pi e^{-a} \quad (3)$$

But $\displaystyle\int_C \frac{e^{iaz}}{1+z^2}dz = \int_{-R}^{R}\frac{e^{iax}}{1+x^2}dx + \int_\Gamma \frac{e^{iaz}}{1+z^2}dz$

$$\Rightarrow \int_{-R}^{R}\frac{\cos ax}{1+x^2}dx + i\int_{-R}^{R}\frac{\sin ax}{1+x^2}dx + \int_\Gamma \frac{e^{iaz}}{1+z^2}dz = \pi e^{-a} \quad (\text{from } (3)) \quad (4)$$

Now consider

$$\left|\int_\Gamma \frac{e^{iaz}}{1+z^2}dz\right| = \left|\int_0^\pi e^{iaz}\frac{1}{1+z^2}dz\right|$$

Let $z=Re^{i\theta} \Rightarrow dz = iRe^{i\theta}d\theta$

$$\left|\int_0^\pi e^{iaz}\frac{1}{1+z^2}dz\right| \le \int_0^\pi \left|\frac{e^{iaRe^{i\theta}}}{1+R^2e^{i2\theta}}iRe^{i\theta}\right|d\theta$$

$$\le \int_0^\pi \frac{|e^{iaR\cos\theta}||e^{-aR\sin\theta}|}{|1+R^2e^{i2\theta}|}Rd\theta$$

$$\le \int_0^\pi \frac{|e^{-aR\sin\theta}|}{R^2-1}Rd\theta \quad (\because |a+b| = |a-(-b)| \ge |a|-|b|)$$

$$= \frac{R}{R^2-1}\int_0^\pi e^{-aR\sin\theta}d\theta$$

$$= \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-aR\sin\theta}d\theta$$

But we know that $\sin\theta \ge \frac{2\theta}{\pi}$ when $0<\theta<\pi/2$

$$\therefore aR\sin\theta \ge \frac{2aR\theta}{\pi}, a>0.$$

$$\Rightarrow e^{aR\sin\theta} \ge e^{2aR\theta/\pi}$$

$$\Rightarrow e^{-aR\sin\theta} \le e^{-2aR\theta/\pi}$$

$$\therefore \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-aR\sin\theta}d\theta \le \frac{2R}{R^2-1}\int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta$$

$$= \frac{2R}{R^2-1}\left[\frac{e^{-2aR\theta/\pi}}{\frac{-2aR}{\pi}}\right]_0^{\pi/2}$$

$$= \frac{-\pi}{a(R^2-1)}\left[e^{-aR}-1\right]$$

$$= \frac{\pi}{a(R^2-1)}\left[1-e^{-aR}\right]$$

This tends to '0' as $R\to\infty$. Now as $R\to\infty$ from (4), we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}dx + i\int_{-\infty}^{\infty}\frac{\sin x}{1+x^2}dx = \pi e^{-a}$$

Comparing the real part, we get

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}dx = \pi/e^a$$

$$\Rightarrow 2\int_0^\infty \frac{\cos x}{1+x^2}dx = \pi e^{-a} \quad \left(\because f(x) = \frac{\cos x}{1+x^2} \text{ is evenfunction}\right)$$

$$= 2\int_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2}e^{-a}$$

Hence the result.

HW → Apply the Calculus of residues to evaluate $\displaystyle\int_0^\infty \frac{x\sin x}{x^2+a^2}dx; (a>0)$ $\left[\text{Hint } \int_C \frac{ze^{iz}}{z^2+a^2}dz\right]$

HW → Find $\displaystyle\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx$