IFOS-2013

Q1 Prove that $\text{Curl}(\text{Curl } F) = \text{grad}(\text{div}\vec{F}) - \nabla^2 \vec{F}$

sol: Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\text{Curl }\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}$$

$$\text{Curl}(\text{Curl }\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) & \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) & \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum \frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{i}$$

$$= \sum \left(\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial x \partial z}\right)\hat{i}$$

$$= \sum \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial x \partial z}\right) - \left(\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$= \sum \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial x \partial x}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\hat{i} - \sum\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\hat{i}$$

$$= \sum \frac{\partial}{\partial x}(\nabla\cdot\vec{F})\hat{i} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\sum F_1\hat{i}$$

$$= \frac{\partial}{\partial x}\hat{i}(\nabla\cdot\vec{F}) + \frac{\partial}{\partial y}\hat{j}(\nabla\cdot F) + \frac{\partial}{\partial z}\hat{k}(\nabla\cdot\vec{F})$$

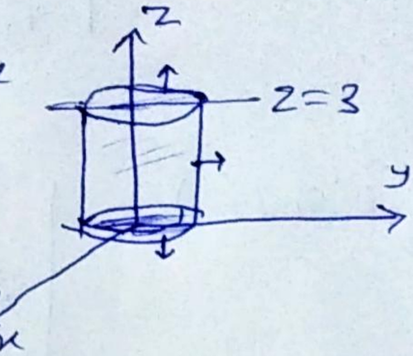
$$+ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \nabla(\nabla\cdot\vec{F}) - \nabla^2(\vec{F})$$

Hence $\boxed{\text{Curl}(\text{Curl }\vec{F}) = \text{grad}(\text{div }\vec{F}) - \nabla^2\vec{F}}$

Q Evaluate $\int_S \vec{F}\cdot d\vec{S}$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$ & $z=3$.

Surface S is closed & let us assume the volume enclosed by it is V.



Then By Gauss divergence theorem

$$\int_S \vec{F}\cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{F})\, dV$$ where V = Volume enclosed by the surface.

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$
$$= 4 - 4y + 2z \quad = 2(2-2y+z)$$

$$\therefore \iiint_V \operatorname{div}\vec{F}\, dV = \iiint_V 2(2-2y+z)\, dV$$

Converting integral to cylindrical co-ordinates

$z = z \qquad x^2+y^2 = r^2 \qquad x = r\cos\theta \quad y = r\sin\theta$

$r^2 = 4 \Rightarrow 0 \le r \le 2$

and $0 \le \theta \le 2\pi$, also $0 \le z \le 3$.

and $V = r\,dr\,d\theta\,dz$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2-2r\sin\theta+z)\, r\,dr\,d\theta\,dz$$

$$= 2\int_0^2\int_0^{2\pi} \left| 2z - 2r\sin\theta\, z + \frac{z^2}{2}\right|_0^3 r\,dr\,d\theta$$

$$= 2\int_0^2\int_0^{2\pi} \left(6 - 6r\sin\theta + \frac{9}{2}\right) r\,dr\,d\theta$$

$$= 2\int_0^2 \left| 6\theta + 6r\cos\theta + \frac{9}{2}\theta \right|_0^{2\pi} r\,dr$$

$$= 2\int_0^2 6(2\pi) + 6r(1-1) + \frac{9}{2}(2\pi)\, r\,dr$$

$$= 2\int_0^2 \frac{21}{2}(2\pi)\, r\,dr = 42\pi \int_0^2 r\,dr = 42\pi \left| \frac{r^2}{2} \right|_0^2$$

$$= 42\pi \left(\frac{4}{2} - 0 \right) = 84\pi$$

$$\therefore \boxed{\int_S \vec{F}\cdot d\vec{S} = \iiint_V \mathrm{div}\vec{F}\, dV = 84\pi}$$

Q3 Verify the divergence theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$ taken over rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

sol. Gauss divergence theorem states that for any closed surface S enclosing volume V.

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \mathrm{div}(\vec{F})\, dV$$

where $\vec{F}$ is any vector
$\hat{n}$ is normal to surface
$\mathrm{div}\vec{F}$ = Divergence of $\vec{F}$

Given $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - xz)\hat{j} + (z^2 - xy)\hat{k}$

$$\therefore \mathrm{div}\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - xz) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x+y+z)$$

$$\iiint \mathrm{div}\vec{F}\, dV = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 2(x+y+z)\, dx\, dy\, dz$$

$$= 2\int_0^a\int_0^b \left| xz+yz+\frac{z^2}{2} \right|_0^c dx\,dy$$

$$= 2\int_0^a\int_0^b cx+cy+\frac{c^2}{2}\, dx\,dy$$

$$= 2\int_0^a \left| cxy+\frac{cy^2}{2}+\frac{c^2y}{2} \right|_0^b dx$$

$$= 2\int_0^a bcx+\frac{b^2c}{2}+\frac{bc^2}{2}\, dx$$

$$= 2\left| bc\frac{x^2}{2}+\left(\frac{b^2c}{2}+\frac{bc^2}{2}\right)x \right|_0^a$$

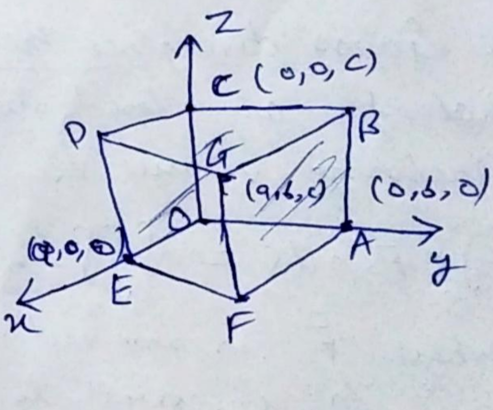
$$= 2\left(\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right) = 2\cdot\frac{abc}{2}(a+b+c)$$

$$= abc(a+b+c)$$

$$\therefore \boxed{\iiint_V \operatorname{div}\vec{F}\, dV = abc(a+b+c)}$$

Now,

$$\iint_S \vec{F}\cdot\hat{n}\, dS$$



For surface OABC: $x=0$, $\hat{n}=-\hat{i}$

$$\iint \vec{F}\cdot\hat{n}\, dS = \int_0^c\int_0^b (x^2-yz)\hat{i}+(y^2-xz)\hat{j} + (z^2-xy)\hat{k}\cdot(-\hat{i})\, dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (yz-x^2)\, dy\,dz = \int_{z=0}^{c}\int_{y=0}^{b} yz\, dy\,dz \quad (\because x=0 \text{ on OABC})$$

$$= \int_0^c \left| z\frac{y^2}{2} \right|_0^b dz = \int_0^c \frac{b^2}{2}z\, dz = \left| \frac{b^2}{4}z^2 \right|_0^c = \frac{b^2c^2}{4} \quad -(1)$$

For surface DEFG: $x=a$, $\hat{n}=\hat{i}$

$$\iint \vec{F}\cdot\hat{n}\, dS = \int_{z=0}^{c}\int_{y=0}^{b} (x^2-yz)\hat{i}+(y^2-xz)\hat{j}+(z^2-xy)\hat{k}\cdot\hat{i}\, dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (a^2-yz)\, dy\,dz = \int_{z=0}^{c} \left| a^2y-\frac{y^2}{2}z \right|_0^b dz = \int_0^c \left(a^2b-\frac{b^2}{2}z\right) dz$$

$= \left| a^2bz - \frac{b^2z^2}{4} \right|_0^b = a^2bc - \frac{b^2c^2}{4}$ — (ii)

For surface ABGF: $y = b$, $\hat{n} = \hat{j}$

$\iint \vec{F}\cdot\hat{n}\, ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k} \cdot \hat{j}\, dz\, dx$

$= \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - xz)\, dz\, dx = \int_0^c \left| b^2x - \frac{x^2}{2}z \right|_0^a dz = \int_0^c b^2a - \frac{a^2}{2}z\, dz$

$= \left| b^2az - \frac{a^2z^2}{4} \right|_0^c = ab^2c - \frac{a^2c^2}{4}$ — (iii)

For surface OCDE: $y = 0$ $\hat{n} = -\hat{j}$

$\iint F\cdot\hat{n}\, ds = \int_{z=0}^{c}\int_{x=0}^{a} (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k} \cdot (-\hat{j})\, dz\, dx$

$= \int_0^c\int_0^a (xz - 0^2)\, dz\, dx = \int_0^c \left| \frac{x^2}{2}z \right|_0^a = \int_0^c \frac{a^2}{2}z\, dz = \left| \frac{a^2z^2}{4} \right|_0^c$

$= \frac{a^2c^2}{4}$ — (iv)

For surface OAFE: $z = 0$, $\hat{n} = -\hat{k}$

$\iint \vec{F}\cdot\hat{n}\, ds = \int_{y=0}^{b}\int_{x=0}^{a} (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k} \cdot (-\hat{k})\, dx\, dy$

$= \int_{y=0}^{b}\int_{x=0}^{a} (xy - 0^2)\, dx\, dy = \int_0^b \left| \frac{x^2y}{2} \right|_0^a dy = \int_0^b \left(\frac{a^2y}{2}\right) dy$

$= \left| \frac{a^2y^2}{4} \right|_0^b = \frac{a^2b^2}{4}$ — (v)

For surface BCDG: $z = c$, $\hat{n} = \hat{k}$

$\iint F\cdot\hat{n}\, dS = \int_{y=0}^{b}\int_{x=0}^{a} (x^2-yz)\hat{i} + (y^2-xz)\hat{j} + (z^2-xy)\hat{k} \cdot \hat{k}\, dx\, dy$

$= \int_0^b\int_0^a (c^2 - xy)\, dx\, dy = \int_0^b \left| c^2x - \frac{x^2}{2}y \right|_0^a dy = \int_0^b c^2a - \frac{a^2}{2}y\, dy$

$= \left| c^2ay - \frac{a^2y^2}{4} \right|_0^b = abc^2 - \frac{a^2b^2}{4}$ — (vi)

Hence, $\iint_S \vec{F}\cdot\hat{n}\, ds =$ Sum of surface integral of all faces.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \left(\frac{b^2c^2}{4}\right) + \left(a^2bc - \frac{b^2c^2}{4}\right) + \left(ab^2c - \frac{a^2c^2}{4}\right) + \left(\frac{a^2c^2}{4}\right)$$

$$+ \left(\frac{a^2b^2}{4}\right) + \left(abc^2 - \frac{a^2b^2}{4}\right)$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc\,(a+b+c)$$

$$\boxed{\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV = abc\,(a+b+c)}$$

Hence, Gauss divergence theorem is verified.