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1.c If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u+iv$ is a regular function of $x+iy$.

$$u = (x-1)^3 - 3xy^2 + 3y^2$$

We find $f(z) = u+iv$ using Milne's Method

$$\phi_1(z,0) = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=z\\y=0}} = 3(z-1)^2$$

$$\phi_2(z,0) = \left.\frac{\partial u}{\partial y}\right|_{\substack{x=z\\y=0}} = 0$$

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz + C$$

$$= \int 3(z-1)^2\,dz + C$$

$$= (z-1)^3 + C.$$

Being a polynomial function in z, $f(z)$ is regular.

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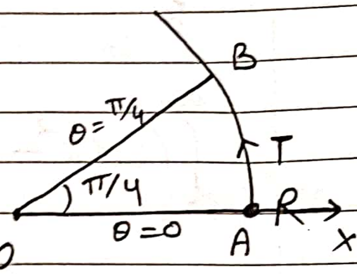
2(c) Prove that

$$\int_0^\infty \cos x^2\, dx = \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Consider $\int_C f(z)dz = \int_C e^{-z^2} dz$



where C is the contour as shown in the diagram.
Since $f(z)$ is regular within and on the boundary of C therefore by Cauchy's residue theorem,

$$0 = \int_C f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BO} f(z)dz$$

or $$\int_0^R e^{-x^2} dx + \int_T e^{-z^2} dz + \int_R^0 e^{-r^2 e^{i\pi/2}} . e^{i\pi/4} dr = 0 \quad \text{—(1)}$$

Now $$\left| \int_T e^{-z^2} dz \right| = \left| \int_0^{\pi/4} e^{-R^2 e^{i2\theta}} . iRe^{i\theta} d\theta \right|$$

$$\le \int_0^{\pi/4} R e^{-R^2 \cos 2\theta} . d\theta$$

$$\le \frac{1}{2} \int_0^{\pi/2} R e^{-R^2 \sin\phi} . d\phi$$

$$\left(\text{Putting } 2\theta = \frac{\pi}{2} - \phi \right)$$

$$\le \frac{1}{2} \int_0^{\pi/2} R e^{-\left(\frac{2\phi}{\pi}\right) R^2} . d\phi, \quad \text{(Jordan's Inequality)}$$

$$= \frac{\pi}{4\cancel{R}} \left(1 - e^{-R^2}\right) \to 0 \text{ as } R \to \infty$$

[Jordan's inequality: $\frac{2\theta}{\pi} \le \sin\theta \le \theta$, where $0 \le \theta \le \frac{\pi}{2}$]

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Hence as $R \to \infty$, we have from (1)

$$\int_0^\infty e^{-x^2} dx + 0 + (-1)\int_0^\infty e^{-ir^2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) dr = 0$$

$$\int_0^\infty (\cos x^2 - i \sin x^2)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) dx = \int_0^\infty e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2}$$

Equating real and imaginary parts on both sides

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{\sqrt{\pi}}{2} \quad \text{and}$$

$$\int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Adding, we get the result

$$\int_0^\infty \cos x^2 dx = \frac{1}{2} \frac{\sqrt{\pi}}{2}$$

Subtracting, $\displaystyle\int_0^\infty \sin x^2 dx = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2}.$

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3.c Evaluate the integral $\int_0^{2\pi} \cos^{2n}\theta\, d\theta$, where n is a positive integer.

$$I = \int_0^{2\pi} \cos^{2n}\theta\, d\theta = \int_0^{2\pi} (\cos^2\theta)^n\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right)^n d\theta = \frac{1}{2^n}\int_0^{2\pi} (1+\cos 2\theta)^n\, d\theta$$

Let $2\theta = t$, $2\,d\theta = dt$

$$I = \frac{1}{2^n}\int_0^{4\pi} (1+\cos t)^n \frac{dt}{2}$$

$$= \frac{1}{2^n}\int_C \left[1+\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^n \frac{dz}{iz},$$

Put $z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

C : unit circle

$|z| = 1$.

$$= \frac{1}{2^n}\int_C \left(\frac{z^2+2z+1}{2z}\right)^n \frac{dz}{iz}$$

$$= \frac{1}{2^{2n}i}\int_C \frac{(z+1)^{2n}}{z^{n+1}}\, dz$$

$$= \frac{1}{2^{2n}i}\int_C \frac{f(z)}{z^{n+1}}\, dz \qquad \left[\text{Let } f(z) = (z+1)^{2n}\right.$$

$z=0$ is the pole of order $n+1$ of $\frac{f(z)}{z^{n+1}}$.

Residue at $z=0$ is $\frac{1}{n!} D^n\left[(z+1)^{2n}\right]_{z=0}$

$$= \frac{1}{n!}\,\frac{2n(2n-1)\ldots(2n-n+1)}{} = \frac{(2n)!}{(n!)^2}$$

$$\therefore I = \frac{1}{2^{2n}\cancel{i}}\left[2\pi\cancel{i} \times \frac{(2n)!}{(n!)^2}\right] = \frac{(2n)!\,\pi}{2^{(2n-1)}(n!)^2}$$