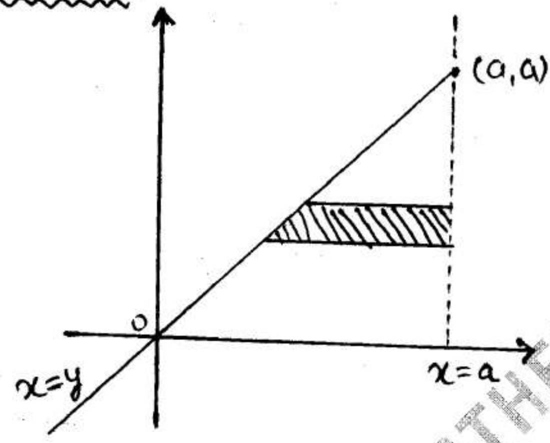
2008 - Paper - I :-

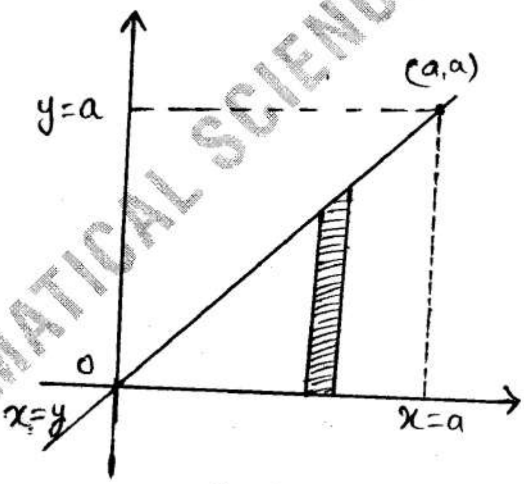
Ques: 3(b)) Evaluate the double integral

$$\int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2}$$

by changing the order of integration.

Solution:-




$$I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x\,dx\,dy}{x^2+y^2}$$

Integrating w.r.t $y$, keeping $x$ constant

$$I = \int_0^a x \cdot \left[\frac{1}{x} \cdot \tan^{-1}\frac{y}{x}\right]_0^x dx$$

$$I = \int_0^a x \left[\frac{1}{x} \tan^{-1} 1\right] dx = \int_0^a \cancel{x} \cdot \frac{1}{\cancel{x}} \cdot \frac{\pi}{4}\, dx$$

$$I = \frac{\pi}{4}\left[x\right]_0^a = \frac{\pi}{4} a.$$

$$\boxed{\therefore\ I = \int_0^a \int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \frac{\pi}{4} a}$$ Required result

Ques:-4(c)> Show that the enveloping cylinders of the ellipsoid $ax^2+by+cz^2=1$ with generators perpendicular to z axis meet the plane $z=0$ in parabolas.

Solution:-

Let generator be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Since, it is $\perp^r$ to z-axis so

$$l\cdot 0+m\cdot 0+n\cdot 1=0 \quad \Rightarrow \boxed{n=0}$$

Any general point on generator can be $(lr+x_1,\ mr+y,\ z_1)$ put this in equation of ellipsoid.

$$a(lr+x_1)^2+b(mr+y_1)^2+cz_1^2=1$$

$$r^2(al^2+bm^2)+2r(alx_1+bmy_1)+ax_1^2+by_1^2+cz_1^2-1=0 \quad \text{—(1)}$$

As generator touches ellipsoid so $r_1=r_2$ thus $B^2=4AC$ in equation (1)

$$4(alx_1+bmy_1)^2=4(al^2+bm^2)(ax_1^2+by_1^2+cz_1^2-1)$$

So equation of enveloping cylinder of ellipsoid is

$$[(alx+bmy)^2=(al^2+bm^2)(ax_1^2+by_1^2+cz_1^2-1)] \quad \text{—(2)}$$

Enveloping cylinder meeting $z=0$ in parabola if $h^2=ab$ where $h=\frac{\text{coefficient of } xy}{2}$

$a =$ coefficient of $x$ and
$b =$ coefficient of $y$

In ②

$a =$ coefficient of $x = a^2l^2 - a(al^2+bm^2) = -abm^2$

$b =$ coefficient of $y = b^2m^2 - b(al^2+bm^2)$

$= \cancel{b^2m^2} - bal^2 - \cancel{b^2m^2}$

$= -abl^2.$

$2h =$ coefficient of $xy = 2ablm$

$\boxed{\text{These values satisfy } h^2 = ab \text{ thus enveloping cylinder meet } z=0 \text{ in parabola}}$

required result.

2008 - Paper - II>

Ques: 2(d) Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod_{2}^{\infty}\left(1+\frac{1}{n^2-1}\right)$.

Solution:-

given; $f(n) = \sum \frac{1}{n(n+1)}$

Let; $P = \frac{1}{2}\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right)$

on solving.

$$P = \frac{1}{2}\prod_{n=2}^{\infty}\left(\frac{n^2}{n^2-1}\right) = \frac{1}{2}\prod_{n=2}^{\infty}\left[\left(\frac{n}{n-1}\right)\cdot\left(\frac{n}{n+1}\right)\right]$$

$$P = \lim_{n\to\infty}\frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{1}\cdot\frac{2}{\cancel{3}}\right]\left[\frac{\cancel{3}}{\cancel{2}}\cdot\frac{3}{\cancel{4}}\right]\left[\frac{\cancel{4}}{\cancel{3}}\cdot\frac{4}{\cancel{5}}\right]\cdots\left[\frac{\cancel{n}}{n-1}\cdot\frac{n}{n+1}\right]$$

$$P = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty} 1-\frac{1}{n+1}$$

$$\boxed{P = 1}$$

On solving $\sum \frac{1}{n(n+1)}$.

$$= \sum\left[\frac{1}{n}-\frac{1}{n+1}\right] = \left[1-\cancel{\frac{1}{2}}\right]+\left[\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right]+\left[\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right]+\left[\cancel{\frac{1}{4}}-\cancel{\frac{1}{5}}\right]$$
$$\cdots\left[\cancel{\frac{1}{n}}-\frac{1}{n+1}\right]$$

$$= \lim_{n\to\infty} 1-\frac{1}{n+1} = 1.$$

$$\therefore \boxed{\sum\frac{1}{n(n+1)} \text{ is equivalent to } \frac{1}{2}\prod_{n=2}^{\infty}\left[1+\frac{1}{n^2-1}\right]}$$

Hence proved.

Ques: 4(a)) Evaluate

$$\int_c \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2}\right] dz$$

where c is the circle $|z|=3$. State the theorems you use in evaluating above integral.

Solution:-

given; $I = \int_c \left[\frac{e^{2z}}{z^2(z^2+2z+2)} + \log(z-6) + \frac{1}{(z-4)^2}\right] dz$

where c is the circle $|z|=3$.

Since $\frac{1}{(z-4)^2}$ is analytic everywhere in $|z|=3$

$\therefore \int_c \frac{1}{(z-4)^2} dz = 0$ [by Cauchy's theorem] —(A)

Let; $z-6 = re^{i\theta}$

$(x-6)+iy = r\cos\theta + ir\sin\theta$

$\therefore x-6 = r\cos\theta \quad ; \quad y = r\sin\theta$

$\therefore r^2 = (x-6)^2+y^2$

$\tan\theta = \frac{y}{x-6} \Rightarrow \boxed{\theta = \tan^{-1}\frac{y}{x-6}}$

$\therefore \log(z-6) = \log|re^{i\theta}|$

$\log(z-6) = \log r + \log e^{i\theta}$

$\log(z-6) = \log r + i\theta$

$\log(z-6) = \log\sqrt{(x-6)^2+y^2} + i\tan^{-1}\left(\frac{y}{x-6}\right)$

$$\boxed{\log(z-6) = \frac{1}{2}\log|(x-6)^2+y^2| + i\tan^{-1}\left(\frac{y}{x-6}\right)} \text{ —(1)}$$

Clearly (1) is analytic everywhere except

$(x-6)^2+y^2=0$

i.e $\boxed{x=6\,,\; y=0}$

But $(6,0)$ lies Outside $|z|=3$

$\therefore \log|z-6|$ is analytic everywhere in $|z|=3$

$\boxed{\therefore \log|z-6| = 0}$ —— (B)

Now;

$$\int_c \frac{e^{2z}}{z^2(z^2+2z+2)}\,dz = \int_c f(z)\,dz. \quad (\text{say})$$

Where; $f(z) = \dfrac{e^{2z}}{z^2(z^2+2z+2)}$

Clearly; $f(z)$ is not an analytic function

when; $z^2(z^2+2z+2)=0$

$\Rightarrow z = 0, 0, -1 \pm i$

$\therefore f(z)$ has poles $z=0$ of order 2,

and $z = -1+i, -1-i$ of order 1, 1 respectively.

<u>By residue theorem;</u>

$$\text{Res} f(0) = \frac{1}{\lfloor 2-1} \lim_{z\to 0} \frac{d}{dz}\left[\cancel{(z-0)^2}\cdot \frac{e^{2z}}{\cancel{z^2}(z^2+2z+2)}\right]$$

$$= \lim_{z\to 0} \frac{(z^2+2z+2)\cdot 2e^{2z} - e^{2z}(2z+2)}{(z^2+2z+2)^2}$$

$$= \frac{(0+0+2)2\cdot e^0 - e^0(0+2)}{(0+0+2)^2}$$

$$\boxed{\text{Res} f(0) = \frac{4-2}{4} = \frac{1}{2}} \text{ —— (a)}$$

Residue at $f(-1+i) = \lim_{z\to -1+i} \cancel{(z+1-i)} \cdot \dfrac{e^{2z}}{z^2 \cancel{(z+1-i)}(z+1+i)}$

$$= \frac{e^{2(-1+i)}}{(-1+i)^2(\cancel{-1}+i+\cancel{1}+i)} = \frac{e^{2(-1+i)}}{2i\cdot(1+i^2-2i)}$$

$$\boxed{\text{Res } f(-1+i) = \frac{e^{2(-1+i)}}{4}} \quad \text{—— (b)}$$

Residue at $f(-1-i) = \lim_{z\to -1-i} \cancel{(z+1+i)} \dfrac{e^{2z}}{z^2(z+1-i)\cancel{(z+1+i)}}$

$$= \frac{e^{-2(1+i)}}{(-(1+i))^2\cdot(\cancel{-1}-i+\cancel{1}-i)} = \frac{e^{-2(1+i)}}{(1+i)^2\cdot(-2i)}$$

$$\boxed{\text{Residue at } f(-1-i) = \frac{e^{-(2+2i)}}{4} = \frac{e^{-2(1+i)}}{4}} \quad \text{—— (c)}$$

$$\therefore \int_C \frac{e^{2z}}{z^2(z^2+2z+2)}dz = 2\pi i\,(\text{Sum of all Residues})$$

$$= 2\pi i\left[\text{Res } f(0) + \text{Res } f(-1-i) + \text{Res } f(-1+i)\right]$$

$$= 2\pi i\left[\frac{1}{2} + \frac{e^{-2}}{2}\left[\frac{e^{2i}+e^{-2i}}{2}\right]\right] \text{— from a, b, c}$$

Hence,

$$\boxed{\int_C \frac{e^{2z}}{z^2(z^2+2z+2)}dz = \pi i\left(1 + \frac{e^{-2}}{2}\left[e^{2i}+e^{-2i}\right]\right)} \quad \text{—— (C)}$$

From (A), (B) and (C)

$$I = \int_C \left[\frac{1}{(z-4)^2} + \log(z-6) + \frac{e^{2z}}{z^2(z^2+2z+2)}\right]dz$$

$$I = 0 + 0 + \int_C \frac{e^{2z}}{z^2(z^2+2z+2)}dz$$

$$\boxed{I = \pi i\left(1 + \frac{e^{-2}}{2}\left[e^{2i}+e^{-2i}\right]\right)}$$ which is required result.

Ques:- 4(c) Solve the following transportation problem:

Destinations

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | Availability |
|---|---|---|---|---|---|---|---|
| $F_1$ | 2 | 1 | 3 | 3 | 2 | 5 | 50 |
| $F_2$ | 3 | 2 | 2 | 4 | 3 | 4 | 40 |
| $F_3$ | 3 | 5 | 4 | 2 | 4 | 1 | 60 |
| $F_4$ | 4 | 2 | 2 | 1 | 2 | 2 | 30 |
| Demand | 30 | 50 | 20 | 40 | 30 | 10 | |

FACTORIES

finding the initial solution by Matrix Minima Method.

Solution:-

From the given table

Total Demand $= 30+50+20+40+30+10$
$= 180$

Total Availability $= 50+40+60+30 = 180$

Hence; $\boxed{\text{Demand} = \text{Availability}}$

$\therefore$ Given Transportation problem is balanced.

Hence, for initial Solution - by Matrix Minima Mtd.

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | Availability |
|---|---|---|---|---|---|---|---|
| $F_1$ | × 2 | 50 1 | × 3 | × 3 | × 2 | × 5 | 50 ✓ |
| $F_2$ | 20 3 | 0 2 | 20 2 | × 4 | × 3 | × 4 | 40 ✓ |
| $F_3$ | 10 3 | × 5 | × 4 | 10 2 | 30 4 | 10 1 | 60 |
| $F_4$ | × 4 | × 2 | × 2 | 30 1 | × 2 | × 2 | 30 |
| Demand | 30 | 50 ✓ | 20 ✓ | 40 ✓ | 30 | 10 | |

Here ; number of positive allocations = 8

and $m+n-1 = 6+4-1 = 9$

$\therefore$ Solution degenerates.

Let, cell $a_{22} = 0$ be a zero basic cell in the initial Basic feasible solution.

$\therefore$ Initial feasible solution $= 2\times0 + 1\times50 + 3\times20 + 2\times20 + 3\times10 + 2\times10 + 4\times30 + 1\times10 + 1\times30$

IBFS $= 0+50+60+40+30+20+120+10+30$

$\boxed{\text{IBFS} = 360}$ using Matrix minima method.

Now, for OBFS

By using MODI method:

For Basic cells ; $\Delta_i = u_i + v_j - c_{ij} = 0$

$u_2 + v_2 - 2 = 0$

$\Rightarrow u_2 + v_2 = 2$

$\Rightarrow u_2 + v_1 = 3$

$u_3 + v_1 = 3$

$u_1 + v_2 = 1$

$u_2 + v_3 = 2$

$u_3 + v_4 = 2$

$u_4 + v_4 = 1$

$u_3 + v_5 = 4$

$u_3 + v_6 = 1$

Let $u_3 = 0$, $u_1 = -1$, $u_2 = 0$, $u_4 = -1$

$v_1 = 3$, $v_2 = 2$, $v_3 = 2$, $v_4 = 2$, $v_5 = 4$, $v_6 = 1$

Let us, calculate $\Delta_{ij}$ for Non-Basic Cells :-

$\Delta_{41} = u_4 + v_1 - 4 = -1+3-4 = -2$

$\Delta_{11} = u_1 + v_1 - 2 = -1+3-2 = 0$

$\Delta_{32} = u_3 + v_2 - 5 = 0+2-5 = -3$

$\Delta_{42} = u_4 + v_2 - 2 = -1+2-2 = -1.$

$\Delta_{13} = u_1 + v_3 - 3 = -1+2-3 = -2$

$\Delta_{33} = u_3 + v_3 - 4 = 0+2-4 = -2$

$\Delta_{43} = u_4 + v_3 - 2 = -1+2-2 = -1$

$\Delta_{15} = u_1 + v_5 - 2 = -1+4-2 = 1 > 0$

$\Delta_{25} = u_2 + v_5 - 3 = 0+4-3 = 1 > 0$

$\Delta_{45} = u_4 + v_5 - 2 = -1+4-2 = 1 > 0$

$\Delta_{16} = u_1 + v_6 - 5 = -1+1-5 = -5$

$\Delta_{26} = u_2 + v_6 - 4 = 0+1+(-4) = -3$

$\Delta_{46} = u_4 + v_6 - 2 = -1+1-2 = -2$

As we observe, there are some positive values in non-basic cells; hence,

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|---|---|
| $F_1$ | | 50 | | | | |
| $F_2$ | 20 | 0 | 20 | | | |
| $F_3$ | 10 | | | 10 $+\theta$ | 30 $-\theta$ | 10 |
| $F_4$ | | | | 30 $-\theta$ | $+\theta$ | |

| | |
|---|---|
| $+\theta$ 10 | 30 $-\theta$ |
| 30 $-\theta$ | $+\theta$ |

Here $\theta = 30$ $\Rightarrow$

| | |
|---|---|
| 40 | 0 |
| 0 | 30 |

$\therefore$ New feasible solution table is given by—

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|---|---|
| $F_1$ | | $50_1$ | | | | |
| $F_2$ | $20_3$ | $0_2$ | $20_2$ | | | |
| $F_3$ | $10_3$ | | | $40_2$ | 4 | $10_1$ |
| $F_4$ | | | | $0_1$ | $30_2$ | |

$\because$ No. of allocations $= 9 = m+n-1$

$\therefore$ Using MODI Method

For Basic cells

$u_1+v_2 = 1$ $\quad$ $u_3+v_4 = 2$

$u_2+v_1 = 3$ $\quad$ $u_4+v_4 = 1$

$u_3+v_1 = 3$ $\quad$ $u_3+v_6 = 1$

$u_2+v_2 = 2$ $\quad$ $u_4+v_5 = 2$

$u_2+v_3 = 2$

Put

$u_3 = 0$ $\quad$ $v_1 = 3$

$u_4 = -1$ $\quad$ $v_4 = 2$

$u_1 = -1$ $\quad$ $v_5 = 3$

$u_2 = 0$ $\quad$ $v_6 = 1$

$v_2 = 2$

$v_3 = 2$

for non basic cells :- $(\Delta_{ij})$

$\Delta_{11} = u_1+v_1 - 2 = -1+3-2 = 0$

$\Delta_{13} = u_1+v_3 - 3 = -1+2-3 = -2$

$\Delta_{14} = u_1+v_4 - 3 = -1+2-3 = -2$

$\Delta_{15} = u_1+v_5 - 2 = -1+3-2 = 0$

$\Delta_{16} = u_1+v_6 - 5 = -1+1-5 = -5$

$\Delta_{24} = u_2+v_4 - 4 = 0+2-4 = -2$

$\Delta_{25} = u_2+v_5 - 3 = 0+3-3 = 0$

$\Delta_{26} = u_2+v_6 - 4 = 0+1-4 = -3$

$\Delta_{32} = u_3+v_2 - 5 = 0+2-5 = -3$

$\Delta_{33} = u_3+v_3 - 4 = 0+2-4 = -2$

$\Delta_{41} = u_4+v_1 - 4 = -1+3-4 = -2$

$\Delta_{42} = u_4+v_2 - 2 = -1+2-2 = -1$

$\Delta_{43} = u_4+v_3 - 2 = -1+2-2 = -1$

$\Delta_{35} = u_3+v_5 - 4 = 0+3-4 = -1$

$\Delta_{46} = u_4+v_6 - 2 = -1+1-2 = -2$

By observing all the values of $\Delta_{ij}$ we get all

$\boxed{\Delta_{ij} \leq 0}$ Hence optimality obtained

$\therefore$ Optimal Transporation cost =

$$1\times 50 + 3\times 20 + 0\times 2 + 2\times 20 + 3\times 10 + 2\times 40 + 1\times 10 + 0\times 1 + 2\times 30$$

$$= 50 + 60 + 0 + 40 + 30 + 80 + 10 + 0 + 60$$

$$= 110 + 150 + 70 = \underline{\underline{330}}$$

$\boxed{\therefore \text{optimal cost for given Transportation} = \text{Rs: } 330}$

Ques:-5(d) State the principle of duality.

Solution :- A pair of expression is said to be Dual Expression if one can be obtained from the other by interchanging 1 with 0 and 0 with 1 in the expression and OR with AND and vice-versa.

(i) Give dual of the boolean expressions:

$$(x+y)\cdot(\bar{x}\cdot\bar{z})\cdot(y+z) \text{ and } x\cdot\bar{x}=0$$

Dual of $(x+y)(\bar{x}\cdot\bar{z})(y+z)$

$$=(x\cdot y)+(\bar{x}+\bar{z})+(y\cdot z)$$

Dual of $x\cdot\bar{x}=0$ be $x+\bar{x}=1$.

(ii) Represent

$(\bar{A}+\bar{B}+\bar{C})(A+\bar{B}+C)(A+B+\bar{C})$ in NOR to NOR logic circuit network.

Solution:-

given expression-

$$(\bar{A}+\bar{B}+\bar{C})(A+\bar{B}+C)(A+B+\bar{C})$$

$$=\left[\bar{A}[A+\bar{B}+C]+\bar{B}[A+\bar{B}+C]+\bar{C}[A+\bar{B}+C]\right](A+B+\bar{C})$$

$$=\left[A\bar{A}+\bar{A}\bar{B}+\bar{A}C+\bar{B}A+\bar{B}+\bar{B}C+\bar{C}A+\bar{C}\bar{B}+\bar{C}C\right](A+B+\bar{C})$$

$$=\left[0+\bar{A}\bar{B}+\bar{A}C+\bar{B}(A+1)+\bar{B}C+\bar{C}A+\bar{C}\bar{B}\right](A+B+\bar{C})$$

$$=\left[\bar{A}\bar{B}+\bar{A}C+\bar{B}[1+C]+\bar{C}A+\bar{C}\bar{B}\right](A+B+\bar{C})$$

$$=(\bar{A}\bar{B}+\bar{A}C+\bar{B}(1+\bar{C})+\bar{C}A)(A+B+\bar{C})$$

$$=(\bar{B}(1+\bar{A})+\bar{A}C+A\bar{C})(A+B+\bar{C}) \quad \left[\because \underline{1+K=1}\right]$$

$$y = (\bar{B} + \bar{A}C + A\bar{C})(A + B + \bar{C})$$

$$y = A(\bar{B} + \bar{A}C + A\bar{C}) + B(\bar{B} + \bar{A}C + A\bar{C}) + \bar{C}(\bar{B} + \bar{A}C + A\bar{C})$$

$$y = A\bar{B} + \underbrace{A\bar{A}\cdot C}_{0} + A\bar{C} + \underbrace{B\bar{B}}_{0} + B\bar{A}C + BA\bar{C} + \bar{C}\bar{B} + \underbrace{\bar{A}C\bar{C}}_{=0} + A\bar{C}$$

$$y = A\bar{B} + A\bar{C} + B\bar{A}C + BA\bar{C} + \bar{C}\bar{B}$$

$$y = A\bar{B} + \bar{B}\bar{C} + A\bar{C}(1 + B) + \bar{A}BC$$

$$y = A\bar{B} + \bar{B}\bar{C} + A\bar{C} + \bar{A}BC$$

$$y = A(\bar{B} + \bar{C}) + \bar{B}\bar{C} + \bar{A}BC$$

$$y = A(\overline{BC}) + (\overline{B+C}) + \bar{A}BC$$

$$y = \overline{\bar{A} + \overline{\overline{\bar{B}+\bar{C}}}} + \overline{A+\bar{B}+\bar{C}} + \overline{B+C}$$

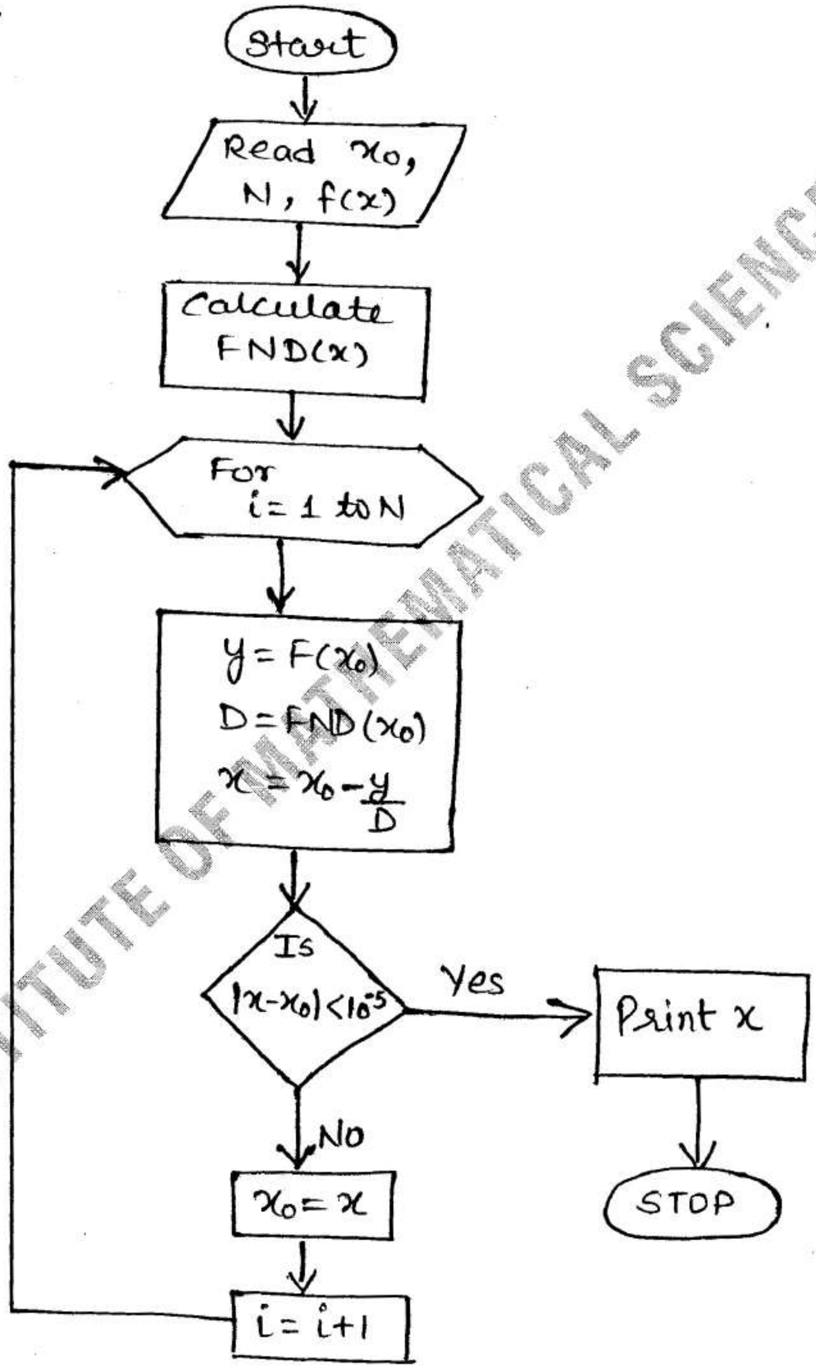
$$y = \overline{\bar{A}+\overline{\bar{B}+\bar{C}}} + \overline{A+\bar{B}+\bar{C}} + \overline{B+C}$$

$$\boxed{y = \overline{\bar{A} + \overline{\bar{B}+\bar{C}}} + \overline{A+\bar{B}+\bar{C}} + \overline{B+C}}$$

<u>Required result</u>

Ques:- 7(b)) Draw a flow chart for solving equation $F(x)=0$ correct to five decimal places by Newton - Raphson method.

Solution :-



2009 - Paper - II

Ques: 1(f) A paint factory produces both interior and exterior paint from two raw materials $M_1$ and $M_2$. The basic data is as follows:-

| | Tons of raw material per ton of | | Maximum Daily availability |
|---|---|---|---|
| | Exterior Paint | Interior Paint | |
| Raw material $M_1$ | 6 | 4 | 24 |
| Raw material $M_2$ | 1 | 2 | 6 |
| Profit per ton (Rs. 1,0000 | 5 | 4 | |

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The maximum daily demand of interior paint is 2 tons. The factory wants to determine the optimum product mix of interior paint and exterior paint that maximizes daily profits. Formulate the LP problem for this situation.

Solution:-

Let; $x$ = daily production of exterior paint (in tons)

$y$ = daily production of interior paint (in tons)

$\therefore$ Max $z = 5x + 4y$

Subject to;
$$6x + 4y \le 24$$
$$x + 2y \le 6$$
$$y \le x + 1 \quad \text{i.e } -x + y \le 1$$

$y \leq 2$

$x, y \geq 0$

Hence; the LPP is

$$\boxed{\begin{aligned} &\text{Max. } Z = 5x + 4y \\ &\text{subject to: } 6x + 4y \leq 24 \\ &\qquad\qquad x + 2y \leq 6 \\ &\qquad\qquad y - x \leq 1 \\ &\qquad\qquad y \leq 2 \\ &\qquad\qquad x, y \geq 0 \end{aligned}}$$

Required LPP

Ques: 2(c)) Show that the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{4}{6}\right)^2+\cdots+\left(\frac{1\cdot4\cdot7\ldots(3n-2)}{3\cdot6\cdot9\ldots3n}\right)^2+\cdots$$

converges.

Solution:-

given series; $\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\cdot\frac{4}{6}\right)^2+\cdots+\left(\frac{1\cdot4\cdot7\cdots(3n-2)}{3\cdot6\cdot9\cdots3n}\right)^2+\cdots$

Consider;

$$u_n=\left[\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}\cdots\cdots\frac{3n-2}{3n}\right]^2$$

$$u_{n+1}=\left[\frac{1}{3}\cdot\frac{4}{6}\cdot\frac{7}{9}\cdots\cdots\frac{3n-2}{3n}\cdot\frac{3n+1}{3n+3}\right]^2$$

Using ratio test

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left[\frac{3n+3}{3n+1}\right]^2$$

$$=\lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}}\left[\frac{3+3/n}{3+1/n}\right]^2$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\left[\frac{3}{3}\right]^2=1.$$

Thus ratio test fails

Modifying; $\frac{u_n}{u_{n+1}}=\frac{3n+3}{3n+1}$

$$\frac{u_n}{u_{n+1}}=\left(1+\frac{1}{n}\right)\left(1+\frac{1}{3n}\right)^{-1}$$

By Gauss's test

Expand $\frac{u_n}{u_{n+1}}$ using binomials expansions

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)\left(1-\frac{1}{3n}+\frac{1}{(3n)^2}-\cdots\right)$$

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{n}-\frac{1}{3n}+\frac{1}{(3n)^2}-\frac{1}{3n^2}+\frac{1}{9n^3}+\cdots\right]$$

$$\boxed{\frac{u_n}{u_{n+1}} = \left[1+\frac{2}{3n}-\frac{2}{9n^2}+\cdots\right]}$$

Since; $\lambda = \frac{2}{3} < 1$, that is coefficient of $1/n$ is less than 1, then by gauss's test given series is convergent.

Ques: 5(c) ii) Find the values of two valued boolean variables A, B, C, D by solving the following simultaneous equations :-

$$\bar{A} + AB = 0$$

$$AB = AC$$

$$AB + A\bar{C} + CD = \bar{C}D$$

where, $\bar{x}$ denotes the complement of $x$.

Solution:-

(i) $\bar{A} + AB = 0$

$(A+\bar{A})(\bar{A}+B) = 0 \quad [\because A+\bar{A} = 1]$

$\bar{A} + B = 0$

$\Rightarrow \boxed{B = -\bar{A}}$

(ii) $AB = AC$

$AB + \bar{A} = AC + \bar{A}$

$AB + \bar{A}(1+B) = AC + \bar{A}(1+C)$

$AB + \bar{A}B + \bar{A} = AC + \bar{A}C + \bar{A}$

$(A+\bar{A})B + \bar{A} = C(A+\bar{A}) + \bar{A}$

$B + \bar{A} = C + \bar{A}$

$\Rightarrow \boxed{B = C}$

(iii) $AB + A\bar{C} + CD = \bar{C}D$

$\Rightarrow A(B+\bar{C}) + CD = \bar{C}D$

$A(C+\bar{C}) + CD = \bar{C}D \quad [\because B = C]$

$A + CD = \bar{C}D$

$A = \bar{C}D - CD$

$A = D(\bar{C} - C) = D(-1) \Rightarrow \boxed{A = -D}$

Hence; $\boxed{A = -D}$ $\boxed{B = -\bar{A}}$ $\boxed{C = B}$ $\boxed{D = -A}$

Ques:- 5(d)) (i) Realize the following expression by using NAND gates only:

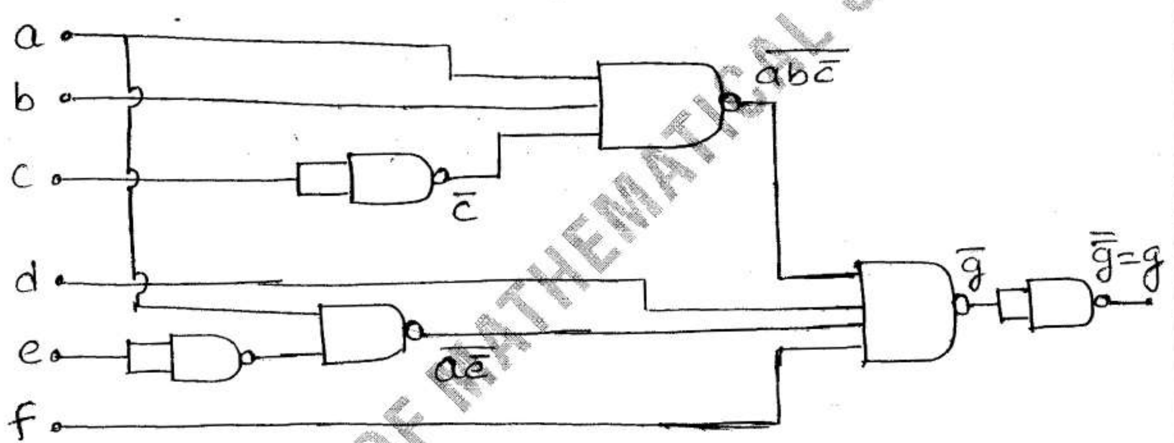
$$g = (\bar{a}+\bar{b}+c)\bar{d}\,(\bar{a}+e)f$$

where $\bar{x}$ denotes the complement of $x$.

Solution:-

given; $g = (\bar{a}+\bar{b}+c)\bar{d}\,(\bar{a}+e)f$

$$g = \overline{a\cdot b\bar{c}}\cdot \bar{d}\cdot(\overline{a\cdot \bar{e}})f$$




(ii) Find the decimal equivalent of $(357.32)_8$?

Solution:- $(357.32)_8 \longleftrightarrow (\,?\,)_{10}$

357

$7\times 8^0 = 7$

$5\times 8^1 = 40$

$3\times 8^2 = 192$

$(232)_{10}$

0.32

$2\times 8^{-2} = 0.03125$

$3\times 8^{-1} = 0.375$

$(0.40625)_{10}$

$$\boxed{\therefore\ (357.32)_8 \longleftrightarrow (239.40625)_{10}}$$

required solution.

2010-Paper-I:

Ques: 1.f) Show that the function

$$f(x) = [x^2] + |x-1|$$

is Riemann integrable in the interval $[0,2]$, where $[\alpha]$ denotes the greatest integer less than or equal to $\alpha$. Can you give an example of a function that is not Riemann integrable on $[0,2]$? Compute $\int_0^2 f(x)\,dx$, where $f(x)$ is as above.

Solution:-

given; $f(x) = [x^2] + |x-1|$

we have;

$$[x^2] = \begin{cases} 0 & : x \in [0,1) \\ 1 & : x \in [1,\sqrt{2}) \\ 2 & : x \in [\sqrt{2},\sqrt{3}) \\ 3 & : x \in [\sqrt{3},2) \end{cases}$$

and;

$$|x-1| = \begin{cases} -(x-1) & ; x \in [0,1) \\ (x-1) & ; x \in [1,2] \end{cases}$$

Clearly, $f(x)$ is bounded $\forall\ x \in [0,2]$ and it is having finite number of points of discontinuity namely $1, \sqrt{2}, \sqrt{3}, 2$ because $[x^2]$ is discontinuous at these points, while $|x-1|$ is continuous everywhere in $[0,2]$.

Hence; $f \in R[0,2]$

To find $\int_0^2 f(x)dx$ :-

$$\int_0^2 f(x)\,dx = \int_0^2 \{[x^2] + |x-1|\}\,dx$$

$$= \int_0^2 [x^2]dx + \int_0^2 |x-1|dx$$

$$= \int_0^1 0.dx + \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{\sqrt{3}} 2dx + \int_{\sqrt{3}}^{2} 3dx$$

$$+ \int_0^1 -(x-1)dx + \int_1^2 (x-1)dx$$

$$= 0 + (\sqrt{2}-1) + 2(\sqrt{3}-\sqrt{2}) + 3(2-\sqrt{3})$$

$$+ \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2$$

$$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} + \frac{1}{2}$$

$$+ \left[\cancel{2} - \cancel{2} - \left[\frac{1}{2} - 1\right]\right]$$

$$= -\sqrt{2} - \cancel{1} - \sqrt{3} + 6 + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}}$$

$$= 6 - \sqrt{3} - \sqrt{2}.$$

$$\therefore \boxed{\int_0^2 f(x)dx = 6 - \sqrt{3} - \sqrt{2}}$$ required Solution.

Ques: 4(b)) If $f(x,y)$ is a homogeneous function of degree 'n' in 'x' and 'y'. and has contirious first-order and second-order partial derivatives then show that,

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

(ii) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

Solution:-

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ —— ①

Let, f be a homogeneous function of degree 'n' in x and y.

$\therefore f = x^n f(y/x)$

$\therefore \frac{\partial f}{\partial x} = nx^{n-1}f(y/x) + x^n f'(y/x)(-y/x^2)$

$$\boxed{\frac{\partial f}{\partial x} = nx^{n-1}f(y/x) - x^{n-2}\cdot y f'(y/x)} \quad \text{—— ②}$$

$\frac{\partial f}{\partial y} = x^n f'(y/x)\cdot\left[\frac{1}{x}\right)$

$$\boxed{\frac{\partial f}{\partial y} = x^{n-1}\cdot f'(y/x)} \quad \text{—— ③}$$

$\therefore x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n f(y/x) - \cancel{x^{n-1}\cdot y f'(y/x)}$
$\qquad + \cancel{x^{n-1} f'(y/x)\cdot y}$

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n f(y/x)$

$$\boxed{\therefore x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf}$$ proved

(ii) $x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$

We have,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf \quad \text{——} (1)$$

Differentiating (1) partially w.r.t $x$, we get

$$\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\partial y}=n\frac{\partial f}{\partial x}$$

$$\Rightarrow \boxed{x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\partial y}=(n-1)\frac{\partial f}{\partial x}} \quad \text{——} (A)$$

Now, differentiating (1) partially, w.r.t $y$, we get

$$x\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}=n\frac{\partial f}{\partial y}$$

$$\boxed{x\frac{\partial^2 f}{\partial y\partial x}+y\frac{\partial^2 f}{\partial y^2}=(n-1)\frac{\partial f}{\partial y}} \quad \text{——} (B)$$

$A\times x+B\times y \Rightarrow$

$$x^2\frac{\partial^2 f}{\partial x^2}+xy\frac{\partial^2 f}{\partial x\partial y}+xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=(n-1)\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right]$$

$$\boxed{x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=(n-1)nf} \quad (\text{by } (1))$$

Hence proved

Ques: 5(c) Find K/T for the curve.

$$\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$$

Solution:-

Given; $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$ —— ①

$$\Rightarrow \frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} - a\sin t\,\hat{j} + 0\,\hat{k}$$

$$\therefore \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2+b^2} \quad \text{—— ②}$$

Unit tangent vector,

$$T = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\Big/\frac{ds}{dt} = \frac{d\vec{r}}{dt}\Big/\left|\frac{d\vec{r}}{dt}\right|$$

$$\boxed{T = \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\,\hat{k}}{\sqrt{a^2+b^2}}} \quad \text{—— ③}$$

$$\frac{dT}{dt} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{\sqrt{a^2+b^2}}$$

$$\therefore \frac{dT}{ds} = \frac{dT}{dt}\Big/\frac{ds}{dt} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{a^2+b^2} \quad \text{— ④}$$

$$\because \frac{dT}{ds} = KN \quad \Rightarrow \quad K = \left|\frac{dT}{ds}\right|$$

$$\therefore K = \frac{\sqrt{a^2\cos^2 t + a^2\sin^2 t}}{a^2+b^2} = \frac{\sqrt{a^2(\cos^2 t + \sin^2 t)}}{a^2+b^2}$$

$$\boxed{K = \frac{a}{a^2+b^2}} \quad \text{—— ⑤}$$

$$\therefore N = \frac{1}{K}\frac{dT}{ds} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{\cancel{(a^2+b^2)}} \cdot \frac{\cancel{(a^2+b^2)}}{a}$$

$$N = \frac{-\cancel{a}(\cos t\,\hat{i} + \sin t\,\hat{j})}{\cancel{a}}$$

$$\boxed{N = -\cos t\,\hat{i} - \sin t\,\hat{j}} \quad \text{——— (6)}$$

Binormal vector,

$$B = T \times N = \frac{1}{\sqrt{a^2+b^2}}\begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$B = \frac{1}{\sqrt{a^2+b^2}}\left[ i(b\sin t) - b\cos t\,\hat{j} + (a\sin^2 t + a\cos^2 t)\hat{k}\right]$$

$$B = \frac{1}{\sqrt{a^2+b^2}}\left[ b\sin t\,\hat{i} - b\cos t\,\hat{j} + a(\sin^2 t + \cos^2 t)\hat{k}\right]$$

$$B = \frac{[b\sin t\,\hat{i} - b\cos t\,\hat{j} + a\hat{k}]}{\sqrt{a^2+b^2}} \quad \text{——— (7)}$$

$$\therefore \frac{dB}{dt} = \frac{b\cos t\,\hat{i} + b\sin t\,\hat{j}}{\sqrt{a^2+b^2}}$$

$$\frac{dB}{ds} = \frac{dB}{dt}\Big/\frac{ds}{dt} = \frac{b}{(\sqrt{a^2+b^2})^2}[\cos t\,\hat{i} + \sin t\,\hat{j}] \text{ — (8)}$$

$$\therefore \frac{dB}{ds} = -\tau N$$

$$\Rightarrow \tau = \left|\frac{dB}{ds}\right| = \frac{b}{a^2+b^2}\left|\cos t\,\hat{i} + \sin t\,\hat{j}\right|$$

$$\boxed{\tau = \frac{b}{a^2+b^2}} \quad \text{——— (9)}$$

Hence; by dividing (5) by (5)

$$\frac{\kappa}{\tau} = \frac{a}{\cancel{a^2+b^2}} \cdot \frac{\cancel{a^2+b^2}}{b}$$

$$\boxed{\frac{\kappa}{\tau} = \frac{a}{b}}$$ required Solution.

2010 Paper-II

Ques: 1 (d)) Define $\{x_n\}$ by $x_1 = 5$ and

$$x_{n+1} = \sqrt{4 + x_n} \quad \text{for } n > 1.$$

Show that the sequence converges to $\frac{(1+\sqrt{17})}{2}$.

Solution:-

Given; $x_{n+1} = \sqrt{4 + x_n} \quad \forall\, n > 1$ —— (1)

and $x_1 = 5$

when; $n = 1$. $x_2 = \sqrt{4 + x_1} = \sqrt{4+5}$

$$x_2 = \sqrt{9} = 3 < x_1$$

$$\therefore x_2 < x_1$$

Let $x_{k+1} < x_k$

$\Rightarrow 4 + x_{k+1} < 4 + x_k$

$$\sqrt{4 + x_{k+1}} < \sqrt{4 + x_k}$$

$$x_{k+2} < x_{k+1} \quad \forall\, k > 2$$

$$\therefore x_{k+1} < x_k \quad \forall\, k > 1$$

$$\therefore x_{n+1} < x_n \quad \forall\, n > 1.$$

$\therefore \{x_n\}$ is monotonically decreasing.

Now; $\because x_1 = 5 < 12$.

$$x_2 = \sqrt{x_1 + 4} = 3 < 12$$

Let $x_k < 12$

$$4 + x_k < 16$$

$\sqrt{4+x_k} < \sqrt{16}$

$x_{k+1} < 4 < 12$

i.e $x_{k+1} < 12$

$\Rightarrow x_{n+1} < 12 \quad \forall\ n > 1$ [by mathematical Induction]

$\Rightarrow x_n < 12 \quad \forall\ n \geq 1$

$\therefore \{x_n\}$ is bounded above by 12.

Hence, $\{x_n\}$ is convergent.

Let, $\lim\limits_{n\to\infty} x_n = l$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = l$

$\because x_{n+1} = \sqrt{4+x_n}$

$\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \sqrt{4+x_n}$

$l = \sqrt{4+l}$

Squaring both side.

$l^2 = 4+l$

$l^2 - l - 4 = 0$

$\therefore l = \dfrac{+1 \pm \sqrt{1+4\times 4}}{2\times 1} = \dfrac{1 \pm \sqrt{1+16}}{2}$

$l = \dfrac{1 \pm \sqrt{17}}{2}$ ; But $l \neq \dfrac{1-\sqrt{17}}{2}$

because $x_1 = 5 > 0$

$\therefore \boxed{l = \dfrac{1+\sqrt{17}}{2}}$ Ans hence proved.

Ques: 2(d) Consider the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

Find the values of $x$ for which it is convergent and also the sum function.
Is the convergence uniform? Justify your answer.

Solution:- Given series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

$$S_n(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots\cdots + \frac{x^2}{(1+x^2)^n}$$

$$S_n(x) = x^2 \left\{ 1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots + \frac{1}{(1+x^2)^n} \right\}$$

$$S_n(x) = x^2 \cdot \frac{1\left(1 - \frac{1}{(1+x^2)^{n+1}}\right)}{1 - \frac{1}{1+x^2}}$$

$$S_n(x) = \frac{(1+x^2)^{n+1} - 1}{(1+x^2)^n}$$

$$\boxed{S_n(x) = (1+x^2) - \frac{1}{(1+x^2)^n}}$$

Sum function; $S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} (1+x^2) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$

$$|S_n(x) - S(x)| = \left| \frac{1}{(1+x^2)^n} \right| < \varepsilon$$

$$\Rightarrow (1+x^2)^n > 1/\varepsilon$$

$$\Rightarrow n \log(1+x^2) > \log 1/\varepsilon$$

$$\boxed{n > \frac{\log 1/\varepsilon}{\log(1+x^2)}}$$

At $x=0$ ; there exist no integer 'm' (large +ve number) such that

$|S_n(x) - S(x)| < \varepsilon \quad \forall \; n \geq m$

thus, the given series is not uniformly convergent at $x=0$, but it is convergent uniformly for all other 'x'.

Ques: 3(c)> Let $f_n(x) = x^n$ on $-1 < x \leq 1$ for $n = 1, 2, \dots$ Find the limit function. Is the convergence uniform? Justify your answer.

Solution:

given function; $f_n(x) = x^n$ on $-1 < x \leq +1$ for $n = 1, 2, \dots$

Limit function $f(x) = \lim_{n\to\infty} f_n(x)$

$= \lim_{n\to\infty} x^n$

$$f(x) = \begin{cases} 0 & : x \in (-1, 1) \\ 1 & : x = 1 \end{cases}$$

$|f_n(x) - f(x)| = |x^n| < \varepsilon$

$\Rightarrow \log x^n < \log \varepsilon$

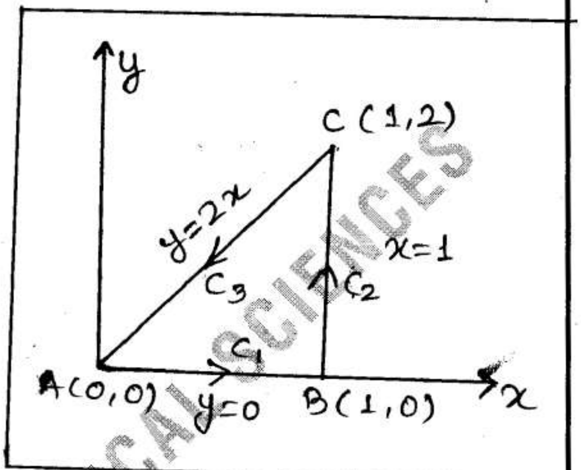
$n \log x < \log \varepsilon$

$n \log 1/x > \log 1/\varepsilon$

$$\boxed{n > \frac{\log 1/\varepsilon}{\log 1/x}}$$

At $x = 1$, there exist no integer $m$ (positive large number) such that $|f_n(x) - f(x)| < \varepsilon$ $\forall\ n \geq m$: Thus, at $x = 1$; $f_n(x)$ is not uniformly convergent. Although in interval $(-1, 1)$ it is uniformly convergent.

Ques: 4 (a)(i)) Evaluate the line integral $\int_C f(z)\,dz$ where $f(z) = z^2$, $C$ is the boundary of the triangle with vertices A (0,0), B(1,0), C (1,2) in that order.




Solution :-

Given; $f(z) = z^2$

C is the boundary of the triangle ABC A(0,0); B(1,0) & C(1,2).

Clearly,

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$

$$\int_{AB} z^2\,dz = \int_{x=0}^{1} (x+0)^2\,dx$$

$$= \int_{x=0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_0^1$$

$$\int_{AB} z^2\,dz = 1/3 \qquad \left[\begin{array}{l} \because z = x+iy \\ dx = dz \end{array}\right]$$

Along BC ; $x = 1 \Rightarrow dx = 0$

$$z = x+iy = 1+iy \qquad \therefore dz = i\,dy.$$

$$\int_{BC} f(z)\,dz = \int_{y=0}^{2} (1+iy)^2 i\,dy = i\int_{y=0}^{2} (1+iy)^2\,dy$$

$$\int_{BC} f(z)\,dz = i\int_{y=0}^{2} (1-y^2+2iy)\,dy$$

$$\Rightarrow \int_{BC} f(z)\,dz = i\left[y - \frac{y^3}{3} + iy^2\right]_0^2$$

$$= i\left[2 - \frac{8}{3} + i4\right]$$

$$\boxed{\int_{BC} f(z)\,dz = -\frac{2}{3}i - 4}$$

Along CA: $y = 2x \Rightarrow dy = 2dx$

$$z = x + iy = x + i2x = (1+2i)x.$$

$$dz = (1+2i)dx$$

$$\int_{CA} f(z)\,dz = \int_{x=1}^{0} (1+2i)x\cdot(1+2i)dx$$

$$= (1+2i)^2 \int_{x=1}^{0} x\,dx$$

$$= (1+2i)^2 \cdot \left[\frac{x^2}{2}\right]_1^0 = \frac{3-4i}{2}.$$

$$\boxed{\therefore \int_{CA} f(z)\,dz = \frac{3-4i}{2}}$$

Hence, $\int_C f(z)\,dz = \frac{1}{3} - \frac{2i}{3} - 4 + \frac{3}{2} - 2i$

$$\int_C f(z)\,dz = \frac{2-24+9}{6} - \frac{8i}{3}$$

$$\boxed{\int_C f(z)\,dz = -\frac{13}{6} - \frac{8}{3}i}$$ required solution.

Ques: 7(c)

(i) find the hexadecimal equivalent of the decimal number $(587632)_{10}$.

Solution:- $(587632)_{10} \longleftrightarrow (?)_{16}$

| | | |
|---|---|---|
| 16 | 587632 | |
| 16 | 36,727 | 0 |
| 16 | 2295 | 7 |
| 16 | 143 | 7 |
| | 8 | 15 |

$(8 \quad 15 \quad 7 \quad 7 \quad 0)_{16}$

$= (8 \; F \; 7 \; 7 \; 0)_{16}$

$\therefore \boxed{(587632)_{10} \longleftrightarrow (8F770)_{16}}$

(iii) Using Boolean algebra, simplify the following expressions

(i) $a + a'b + a'b'c + a'b'c'd + \cdots\cdots$

$y = (a + a'b) + a'b'(c + c'd) + \cdots\cdots$

$y = (a+b) + a'b'(c+d) + \cdots$

$[\because A + A'B = A + B]$

$y = (a + a'b'c) + (b + b'a'd)$

$y = (a + b'c) + (b + a'd) = (a + a'd) + (b + b'c)$

$y = (a+d) + (b+c)$

$\boxed{y = a+b+c+d}$ required solution.

(ii) $x'y'z + yz + xz$

where, $x'$ represents the complement of $x$.

Solution:- Given; $F = x'y'z + yz + xz$

$$F = (x'y' + y)z + xz$$

$$[\because A + A'B = A + B]$$

$$F = (y + x')z + xz$$

$$F = yz + x'z + xz$$

$$F = yz + z(x + x') \quad [\because x + x' = 1]$$

$$F = yz + z(1)$$

$$F = yz + z$$

$$F = z(1 + y) \quad [\because 1 + A = 1]$$

$\boxed{F = z}$ required solution.

Paper : 2011 – I

Ques: 1 (c) } Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^3+y^3}$ if it exists.

Solution:- given $f(x,y) = \dfrac{x^2 y}{x^3+y^3}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3+y^3}$$

Firstly; $y=0$ ; and $x$ approaching to 0, from x-axis

$$\lim_{x\to 0} \frac{x^2 . 0}{x^3+0} = 0$$

$\Rightarrow$ $x=0$ , $y \to 0$, from y axis.

$$\lim_{y\to 0} \frac{y . 0}{y^3+0} = 0$$

Let, $y=mx$, and $(x,y)\to(0,0)$, through this line

$$\lim_{x\to 0} \frac{x^2 . mx}{x^3+m^3x^3} = \lim_{x\to 0} \frac{\cancel{x^3} . m}{\cancel{x^3}(1+m^3)} = \lim_{x\to 0} \frac{m}{1+m^3}$$

Since; the value of limit depends on 'm' (i.e path) thus it is independent of x, y and hence limit does not exist

Ques: 1) d) 'f' be a function defined on $\mathbb{R}$ such that $f(0) = -3$ and $f'(x) \leq 5$ for all the values of 'x' in $\mathbb{R}$. How large can $f(2)$ possibly be?

Solution:-

Since; $f'(x) \leq 5 \quad \forall x \in \mathbb{R}$

thus $f(x)$ will be a linear polynomial

let; $f(x) = ax + b$ ——— (1)

$f(0) = a \times 0 + b = -3$

$\Rightarrow \boxed{b = -3}$ $\quad [\because f(0) = -3]$

$\therefore f(x) = ax - 3$

Differentiate w.r.t $x$.

$f'(x) = a \leq 5 \quad \forall x \in \mathbb{R}$

$\therefore$ In eqn (1)

$\boxed{f(2) = 2a - 3}$ ——— (2)

and value of $a \leq 5$

The largest possible value of $f(2)$ is attained only when 'a' is maximum, thus put $a = 5$ in eqn (2), we get

$f(2) = 2 \times 5 - 3 = 7$

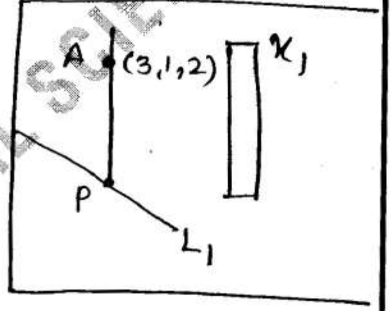
$\boxed{\therefore \text{largest value of } f(2) \text{ be } 7,}$ required Solution.

Ques: 1) e) Find the equations of the straight line through the point $(3,1,2)$ to intersect the straight line $x+4 = y+1 = 2(z-2)$ and parallel to the plane $4x+y+5z=0$.

Solution:-

Given, point $(3,1,2)$ to intersect the straight line $x+4 = y+1 = 2(z-2)$.

$\therefore L_1 \Rightarrow \frac{x+4}{2} = \frac{y+1}{2} = \frac{z-2}{1}$




and the general point on $L_1$, let P be the point

$$\frac{x+4}{2} = \frac{y+1}{2} = \frac{z-2}{1} = r$$

$$P\,(2r-4,\ 2r-1,\ r+2)$$

Let, straight line through $(3,1,2)$ meet the given line 'p', therefore Direction ratio's of line AP to be determined by.

$AP\,(2r-4-3,\ 2r-1-1,\ r+2-2] \Rightarrow AP(2r-7, 2r-2, r)$

The line is parallel to the plane $4x+y+5z=0$

$\therefore$ Dr's of normal to the plane is $\perp^r$ to Dr's of determined line thus

$$(2r-7)4 + 2r-2+5r = 0 \Rightarrow 15r = 30 \Rightarrow \boxed{r=2}$$

$\therefore$ Dr's of $AP = (-3, 2, 2)$

And Point $P = (0,3,4)$

Therefore; straight line AP, which intersect $(3,1,2)$.

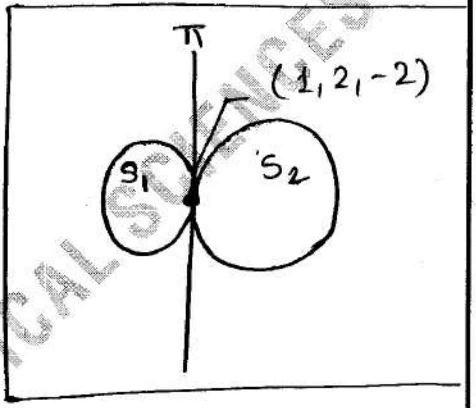
$$\left[\frac{x-3}{-3} = \frac{y-1}{2} = \frac{z-2}{2}\right]$$

Ques:17f) Show that the equation of the sphere which touches the sphere

$$4(x^2+y^2+z^2)+10x-25y-2z=0$$

at point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is $x^2+y^2+z^2+2x-6y+1=0$?

Solution:-




Given sphere $S_1$:

$$4(x^2+y^2+z^2)+10x-25y-2z=0$$

$$\Rightarrow x^2+y^2+z^2+\frac{5}{2}x-\frac{25}{4}y-\frac{z}{2}=0$$

Centre $C_1=\left(-\frac{5}{4},+\frac{25}{8},\frac{1}{4}\right)$

To evaluate sphere(s) that passes through $(-1,0,0)$ and touches $S_1$ at $(1,2,-2)$, this means that line through $C_1$ and $(1,2,-2)$ passes through centre of sphere 'S', line formed by joining $C_1$ and $P(1,2,-2)$ is

$$\frac{x-1}{9/4}=\frac{y-2}{-9/8}=\frac{z+2}{-9/4}$$

$$\Rightarrow \frac{x-1}{2}=\frac{y-2}{-1}=\frac{z+2}{-2}=\lambda$$

$$\Rightarrow x=2\lambda+1\ ,\ y=-\lambda+2\ ,\ z=-2\lambda-2$$

So any point on this line i.e.

$(2\lambda+1,-\lambda+2,-(2\lambda+2))$ be the centre of sphere S.

So the equation of sphere be -

$$x^2+y^2+z^2+2(-2\lambda-1)x+2(\lambda-2)y+2(2\lambda+2) + d = 0 \quad \text{—(1)}$$

as $(1,0,0)$ passes through it

i.e $1+0+0+2(-2\lambda-1)\cdot 1+0+0+d=0$

$$\Rightarrow 1+2(2\lambda+1)+d=0$$

$$\Rightarrow \boxed{d=-4\lambda-3}$$

Now put $(1,2,-2)$ in eqn (1)

$$9+2(-2\lambda-1)+4(\lambda-2)-8(\lambda+1)-4\lambda-3=0$$

$$9-\cancel{4\lambda}-2+\cancel{4\lambda}-8-8\lambda-8-4\lambda-3=0$$

$$12\lambda = 9-2-8-8-3$$

$$12\lambda=-12$$

$$\boxed{\lambda=-1}$$

Put $\lambda=-1$ in eq (1), we will get.

$$x^2+y^2+z^2+2(-2\times-1-1)x+2(-1-2)y+2(2\times-1+2)z + (-4\times-1)-3=0$$

$$x^2+y^2+z^2+2(1)x-6y+2(2-2)z+4-3=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2+2x-6y+1=0}$$

Which is required solution

Ques: 3 (a)) Evaluate

(i) $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ \pi & , x = 2 \end{cases}$

Solution:-

given; $f(x) = \begin{cases} \frac{x^2-4}{x-2} & ; x \neq 2 \\ \pi & ; x = 2 \end{cases}$

To check whether limit exist or not

$$\boxed{f(2+h) = f(2-h) = f(2)} \text{ - if exist.}$$

not

$$\therefore R.H.L = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} \frac{(2+h)^2-4}{(2+h)-2}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2-\cancel{4}}{\cancel{2}+h-\cancel{2}}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}} = 4.$$

$$L.H.L = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{(2-h)^2-4}{(2-h)-2}$$

$$= \lim_{h\to 0} \frac{\cancel{4}-4h+h^2-\cancel{4}}{\cancel{2}-h-\cancel{2}}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-4+h)}{-\cancel{h}} = +4.$$

since; $f(2) = \pi$.

$$\boxed{\therefore f(2+h) = f(2-h) = 4 \neq f(2)}$$ Thus, limit does exist, but it is not continuous.

Ques: 3(a) Evaluate:

(ii) $\int_0^1 \ln x \, dx$

Solution: $I = \int_0^1 \ln x \, dx$

Integration by parts.

$$I = \int_0^1 \ln x\,dx = \left[\ln x \cdot \int 1\,dx\right]_0^1 - \int_0^1 \frac{d}{dx}\ln x \cdot \int 1\,dx$$

$$= [x \cdot \ln x]_0^1 - \int_0^1 \frac{1}{x} \cdot x\,dx$$

$$= [x \ln x]_0^1 - \int_0^1 dx$$

$$= [\ln 1 - 0] - [x]_0^1$$

$$= 0 - [1 - 0] = -1.$$

$$\boxed{\therefore\ I = \int_0^1 \ln x\,dx = -1}$$

which is required solution.

Ques: 3(b)) find the points on the sphere $x^2+y^2+z^2=4$ that are closest and farthest from the point $(3,1,-1)$.

Solution:-

given sphere $\Rightarrow g(x,y) = x^2+y^2+z^2-4=0$ —— (1)

We need to find the closest and farthest from the point $(3,1,-1)$.

Let any point on sphere $(x,y,z)$ to determine using Lagrangian multiplier

$$d^2 = (x-3)^2+(y-1)^2+(z+2)^2 \quad \text{—— (2)}$$

Let; $f(x,y,z) = d^2+\lambda g(x,y)$

$$f(x,y,z) = (x-3)^2+(y-1)^2+(z+2)^2+\lambda(x^2+y^2+z^2-4) \quad \text{—— (3)}$$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= 2(x-3)+\lambda(2x)=0 \Rightarrow x = \frac{3}{1+\lambda}\\ \frac{\partial F}{\partial y} &= 2(y-1)+\lambda(2y)=0 \Rightarrow y = \frac{1}{1+\lambda}\\ \frac{\partial F}{\partial z} &= 2(z+1)+\lambda(2z)=0 \Rightarrow z = \frac{-1}{1+\lambda}\end{aligned}\right\} \text{—— (4)}$$

put the values of $x, y, z$ from (4) in eq (1)

$$\left(\frac{3}{1+\lambda}\right)^2+\left(\frac{1}{1+\lambda}\right)^2+\left(\frac{-1}{1+\lambda}\right)^2 = 4$$

$$3^2+1^2+1^2 = 4(1+\lambda)^2$$

$$(1+\lambda)^2 = \frac{11}{4}$$

$$\Rightarrow \boxed{1 + \lambda = \pm \frac{\sqrt{11}}{2}}.$$

$\therefore$ The stationary points are

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right) \text{ and } \left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

from 4

When $\lambda + 1 = \sqrt{11}/2$

$$x = \frac{6}{\sqrt{11}}, \quad y = \frac{2}{\sqrt{11}}, \quad z = \frac{-2}{\sqrt{11}}$$

$$(2) \equiv d^2 = 1.7325 \Rightarrow \boxed{d_{min} = 1.316}$$

When $\lambda + 1 = -\sqrt{11}/2$

$$(2) \equiv d^2 = 28.266 \Rightarrow \boxed{d_{max} = 5.316}$$

again partially differentiating (4) w.r.t $x, y, z$ respectively

$$\boxed{f_{xx} = 1 + \lambda = f_{yy} = f_{zz}}$$

Similarly :- $f_{yx} = 0 = f_{zx}$

$$f_{xz} = f_{xy} = 0$$

$$f_{yz} = f_{zy} = 0$$

$|F_{xx}|$, $\begin{vmatrix} F_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix}$, $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$ retain same +ve sign; thus it is a closest point.

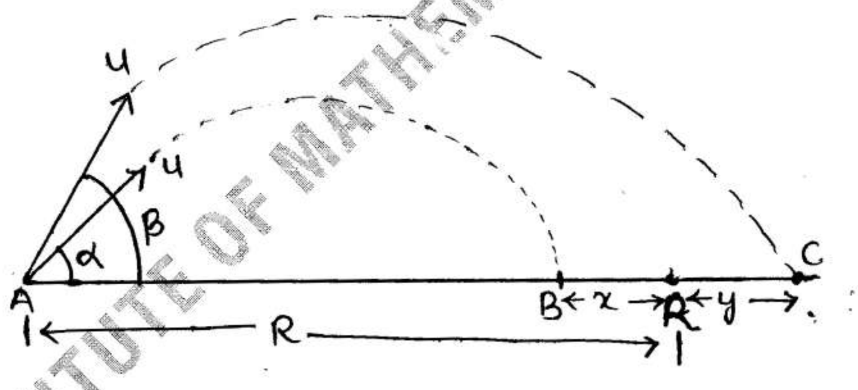
$$\therefore \text{Closest point} = d = 1.316 = \left[\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right]$$

$$\text{farthest point} = d = 5.316 = \left[\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right]$$

which is required solution.

Ques:- 5(d)) A projectile aimed at a mark which is in the horizontal plane through the point of projection, falls '$x$' meter short of it when the angle of projection is $\alpha$ and goes $y$ meter beyond when the angle of projection is '$\beta$'. If the velocity of projection is assumed same in all cases, find the correct angle of projection.

Solution:-




Let, correct angle of projection be $\theta$ such that, $\boxed{\alpha < \theta < \beta}$

Correct Range $= R$ at point R.

When projected at angle $\alpha$,

$$\text{Range} = AB = R - x.$$

When projected at angle $\beta$.

$$\text{Range} = AC = R + y$$

and 'u' is the initial velocity of projection which is same for all angles.

$\therefore$ Range at $\alpha$ projected at angle $\alpha$;

$$R - x = \frac{u^2 \sin 2\alpha}{g} \quad \text{— (1)}$$

Range when projected at angle $\beta$:

$$R + y = \frac{u^2 \sin 2\beta}{g} \quad \text{— (2)}$$

Correct Range when projected at angle $\theta$

$$R = \frac{u^2 \sin 2\theta}{g}$$

Put value of R in (1) & (2), we get

$$\frac{u^2 \sin 2\theta}{g} - x = \frac{u^2 \sin 2\alpha}{g}$$

$$\boxed{x = \frac{u^2}{g}(\sin 2\theta - \sin 2\alpha)} \quad \text{— (A)}$$

$$\frac{u^2 \sin 2\theta}{g} + y = \frac{u^2 \sin 2\beta}{g}$$

$$\boxed{y = \frac{u^2}{g}(\sin 2\beta - \sin 2\theta)} \quad \text{— (B)}$$

$\Rightarrow$ A ÷ B

$$\frac{x}{y} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$$

$$\Rightarrow x \sin 2\beta - x \sin 2\theta = y \sin 2\theta - y \sin 2\alpha$$

$$\Rightarrow x \sin 2\beta + y \sin 2\alpha = (y + x) \sin 2\theta$$

$$\sin 2\theta = \frac{x \sin 2\beta + y \sin 2\alpha}{x+y}$$

$$2\theta = \sin^{-1}\left[\frac{x \sin 2\beta + y \sin 2\alpha}{x+y}\right]$$

$$\boxed{\theta = \frac{1}{2}\sin^{-1}\left[\frac{x \sin 2\beta + y \sin 2\alpha}{x+y}\right]}$$

is required angle of projection.

2011 = Paper - II

Ques:- 2(d)) Find the shortest distance from the origin (0,0) to hyperbola.

$$x^2 + 8xy + 7y^2 = 225$$

Solution)

Let $(x, y)$ be the point on hyperbola which is at shortest distance from the origin (0,0)

So, evaluating minimum value of $x^2+y^2$ using Lagrange's Multiplier.

$$f(x, y) = x^2+y^2+\lambda(x^2+8xy+7y^2-225)$$

$$\frac{\partial f}{\partial x} = 2x+2x\lambda+8y\lambda = 0$$

$$\Rightarrow x+\lambda(x+4y) = 0 \quad \text{——} \; (1)$$

$$\frac{\partial f}{\partial y} = 2y+14y\lambda+8x\lambda = 0$$

$$y+\lambda(7y+4x) = 0 \quad \text{——} \; (2)$$

Multiplying (1) with $x$ and (2) with $y$ and adding both equation.

$$x^2+\lambda(x^2+4xy)+y^2+\lambda(7y^2+4xy) = 0$$

$$x^2+y^2+\lambda(x^2+7y^2+8xy) = 0$$

Let $x^2+y^2 = u$

$\therefore \lambda = \frac{-u}{225}$ $\quad [\because x^2+7y^2+8xy=225]$

from ① and ②

$$\frac{-x}{x+4y} = \frac{-y}{4x+7y} = \lambda = \frac{-u}{225}$$

$$\frac{x+4y}{x} = \frac{225}{u} \Rightarrow \frac{225}{u} - 1 = \frac{4y}{x}.$$

$$\frac{4x+7y}{y} = \frac{225}{u} \Rightarrow \frac{225}{u} - 7 = \frac{4x}{y}$$

Multiplying both

$$\frac{(225-u)}{u} \times \frac{(225-7u)}{u} = \frac{4\cancel{y}}{\cancel{x}} \times \frac{4\cancel{x}}{\cancel{y}}$$

$$\Rightarrow (225-u)(225-7u) = 16u^2$$

$$\Rightarrow 9u^2 + 8\times 225u - (225)^2 = 0$$

$$(9u-225)(u+225) = 0$$

$$\boxed{u = \frac{225}{9}, -225}$$

$u$ cannot be negative, as it is sum of squares i.e $u = x^2+y^2$.

$\therefore u = -225$ neglected.

so $\boxed{u = 25}$ i.e $x^2+y^2 = 25$

and hence $d = \sqrt{x^2+y^2} = \sqrt{25} = 5$

$\boxed{\text{Minimum Distance} = 5 \text{ units}}$ required solution

Ques: 3(b)) Show that the series for which the sum of first $n$ terms

$$f_n(x) = \frac{nx}{1+n^2x^2} ; \; 0 \leq x \leq 1$$

cannot be differentiated term by term at $x=0$. What happens at $x \neq 0$.

Solution:-

given ; $f_n(x) = \dfrac{nx}{1+n^2x^2}$ ; $0 \leq x \leq 1$

for $f_n(x)$ to be term by term differentiable $\lim\limits_{n\to\infty} f_n'(x)$ must be equal to $f'(x)$, where $f(x)$ is a limit function.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2}$$

$$= \lim_{n\to\infty} \frac{x}{2nx^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$f(x) = 0 \quad \Rightarrow \quad f'(x) = 0 \quad \forall x \in \mathbb{R}$$

Differentiating $f_n$ with respect to $x$:-

$$f_n'(x) = \frac{(1+n^2x^2)n - nx(2n^2x)}{(1+n^2x^2)^2}$$

$$= \frac{n + n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2}$$

$$\boxed{f_n'(x) = \frac{n - n^3x^2}{(1+n^2x^2)^2}}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{n - n^3x^2}{(1+n^2x^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 & ; x \neq 0 \\ \infty & ; x = 0 \end{cases}$$

Thus, $f_n(x)$ is not differentiable term by term at $x=0$ and differentiable term by term at $\forall\ x \neq 0$.

Ques: 4(b)) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$,

then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \text{; for all } x.$$

Solution:-

given function; $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$

$\therefore$ By Wiesstress M-test $\Sigma S_n$ is uniformly convergent, if there exist $\Sigma M_n$ convergent series of positive sequence $M_n$ such that

$|S_n| \leq M_n$.

$$S_n(x) = \frac{1}{n^3+n^4x^2}$$

thus; $|S_n| = \left|\frac{1}{n^3+n^4x^2}\right|$

$$|S_n(x)| = \left|\frac{1}{n^3+n^4x^2}\right| \leq \frac{1}{n^3} = M_n$$

$[\because n^4x^2 \geq 0] \ \forall\ x$

$\Sigma \frac{1}{n^3}$ is convergent series by p-test

$\left[\because \frac{1}{n^p} \ ; \ p > 1 - \text{convergent}\right]$

$\therefore \Sigma \frac{1}{n^3+n^4x^2}$ is uniformly convergent and can be differentiated term by term for all $x$, thus.

$$S'(x) = \sum_{n=1}^{\infty}\left(\frac{1}{n^3+n^4x^2}\right)' = \sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left[\frac{1}{1+nx^2}\right]'$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^3}\cdot\left[\frac{(1+nx^2)(0.) - 2x\,n(1)}{(1+nx^2)^2}\right]$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{\cancel{3}2}}\cdot\frac{-2x\cancel{n}}{(1+nx^2)^2}$$

$$\boxed{S'(x) = -2x\sum_{n=1}^{\infty}\frac{1}{(1+nx^2)^2}}$$

Which is required solution

Ques:- 5(d)

(i) Compute $(3205)_{10}$ to the base 8.

Solution:- To find $(3205)_{10} \longleftrightarrow (\_\_\_)_8$

$$\begin{array}{r|r|l} 8 & 3205 & \\ \hline 8 & 400 & 5 \\ \hline 8 & 50 & 0 \\ \hline & 6 & 2 \end{array}$$

$\Rightarrow (6205)_8$

$\therefore \boxed{(3205)_{10} \longleftrightarrow (6205)_8}$

(ii) Let 'A' be an arbitrary but fixed Boolean algebra with operations $\wedge, \vee$ and $'$ and the zero and the unit element denoted by 0 and 1 respectively. Let $x, y, z \ldots$ be elements of A.
If $x, y \in A$ be such that $x \wedge y = 0$ and $x \vee y = 1$ then prove that $y = x'$.

Solution:-

given that $x, y \in A$

and $x \wedge y = 0$ & $x \vee y = 1$ —— (1)

$(x \wedge y)' = 1$ $\left[\because \begin{array}{l} x \neq x' \\ y \neq y' \end{array}\right]$

$x' \vee y' = 1$ → [By De-Morgan law]

$x' \vee y' = x \vee y$ —— from (1)

Since; $x' \neq x$ & $y' \neq y$.

$\therefore \boxed{x' = y \text{ and } y' = x}$

Hence proved

Ques:- 7(b)) Find the logic circuit the represents the following Boolean function. Find also an equivalent simpler circuit.

| $x$ | $y$ | $z$ | $f(x,y,z)$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 0 |

Solution:-

| $x$ | $y$ | $z$ | $f(x,y,z)=m_i$ | | Minterm |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | $m_1$ | $xyz$ |
| 1 | 1 | 0 | 0 | $m_2$ | $xy\bar{z}$ |
| 1 | 0 | 1 | 0 | $m_3$ | $x\bar{y}z$ |
| 1 | 0 | 0 | 0 | $m_4$ | $x\bar{y}\bar{z}$ |
| 0 | 1 | 1 | 1 | $m_5$ | $\bar{x}yz$ |
| 0 | 1 | 0 | 0 | $m_6$ | $\bar{x}y\bar{z}$ |
| 0 | 0 | 1 | 0 | $m_7$ | $\bar{x}\bar{y}z$ |
| 0 | 0 | 0 | 0 | $m_8$ | $\bar{x}\bar{y}\bar{z}$ |

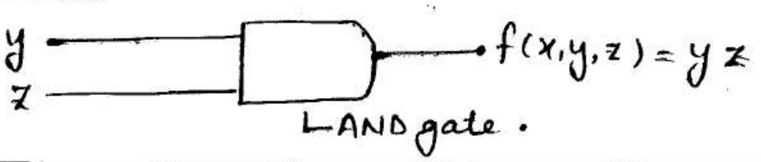
To get the Boolean function, let us add minterms corresponding to output '1'.

$$\therefore\ f(x,y,z) = m_1 + m_5 = xyz + \bar{x}yz$$

$$f(x,y,z) = (x+\bar{x})(yz) = 1\cdot yz = yz \quad [\because A+\bar{A}=1]$$

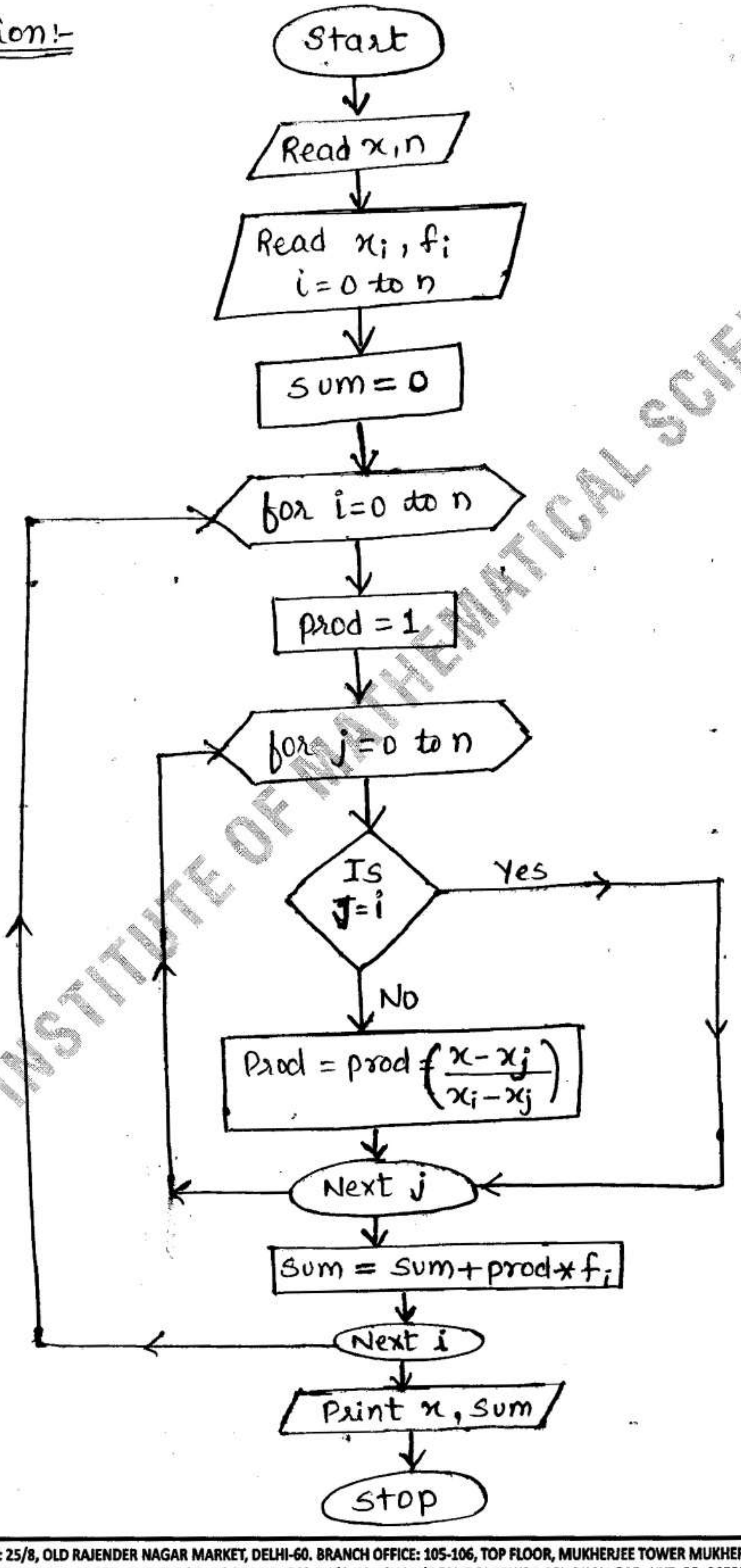
$$\therefore\ \boxed{f(x,y,z) = yz}$$

Simpler circuit is:-



Ques:- 7 (c)) Draw a flow chart for Lagrange's interpolation formula.

Solution:-



2012-Paper-1

Ques: 1(a)) Define a function 'f' of two real variables in the xy-plane by

$$f(x,y) = \begin{cases} \dfrac{x^3\cos\frac{1}{y} + y^3\cos\frac{1}{x}}{x^2+y^2} & ; \text{for } x, y \neq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Check the continuity and differentiability of f at (0, 0).

Solution:-

For continuity:- f(x, y) is continuous at (0,0), if $|f(x,y) - f(0,0)| < \epsilon$ ; whereas $\sqrt{x^2+y^2} < \delta$

considering $|f(x,y) - f(0,0)| = \left|\dfrac{x^3\cos\frac{1}{y} + y^3\cos\frac{1}{x}}{x^2+y^2}\right|$

$$\leq \left|\frac{x^3+y^3}{x^2+y^2}\right| \quad \text{—— (1)}$$

{Since ; $\cos\frac{1}{y} \leq 1$ ; $\cos\frac{1}{x} \leq 1$}

Let ; $x = r\cos\theta$ ; $y = r\sin\theta$ and put in (1)

$$\left|\frac{r^3\cos^3\theta + r^2\sin^3\theta}{r^2(\cos^2\theta + \sin^2\theta)}\right| \leq \left|\frac{r^{\cancel{3}}[\cos^3\theta + \sin^3\theta]}{\cancel{r^2}}\right|$$

$$\leq r|\cos^3\theta + \sin^3\theta| \leq 2r < \epsilon$$

i.e $r < \frac{\epsilon}{2}$ ; choosing $\delta = \epsilon/2$ ; $r^2 = x^2+y^2$

therefore; $|f(x,y) - f(0,0)| < \epsilon$

whenever $\sqrt{x^2+y^2} < \delta$

thus ; $f(x,y)$ is continuous at $(0,0)$

For Differentiability:

$$f(h,k) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k) \quad \text{--- (2)}$$

$A, B$ to be independent of $h, k$ and

$\phi(h,k)$ to tend to zero as $(h,k)$ tends to zero

$$A = f_x\Big|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$B = f_y\Big|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0.$$

In equation (2);

$$\frac{h^3\cos 1/k + k^3\cos 1/h}{h^2+k^2} = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\boxed{\phi(h,k) = \frac{h^3\cos 1/k + k^3\cos 1/h}{(h^2+k^2)^{3/2}}}$$

Taking limit

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{h^3\cos 1/k + k^3\cos 1/h}{(h^2+k^2)^{3/2}}$$

Suppose path of approaching is $k = mh$

$$\lim_{h\to 0} \frac{h^3 \cos\frac{1}{mh} + m^3h^3 \cos\frac{1}{h}}{(h^2+m^2h^2)^{3/2}}$$

$$= \lim_{h\to 0} \frac{\cancel{h^3}\left(\cos\frac{1}{mh} + m^3 \cos\frac{1}{h}\right)}{\cancel{h^3}(1+m^2)^{3/2}}$$

$$= \lim_{h\to 0} \frac{\cos\frac{1}{mh} + m^3\cos\frac{1}{h}}{(1+m^2)^{3/2}}$$

Since; $\cos 1/h$ lies between $-1$ and $1$ and limit depends on 'm', thus limit does not exist and the $f(x,y)$ is not differentiable at $(0,0)$.

Ques:- 4(a)) Compute the volume of the solid enclosed between the surfaces $x^2+y^2=9$ and $x^2+z^2=9$.

Solution:-

given solid surfaces are
$$x^2+y^2=9 \quad \text{and} \quad x^2+z^2=9$$

Volume enclosed $= \iiint_V dx\,dy\,dz$

$x$ varies from $-3$ to $3$.
$y$ varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$
$z$ varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$

$$V = \int_{-3}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dz$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [z]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dy$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 2[\sqrt{9-x^2}]\, dy$$

$$= \int_{-3}^{3} 2[\sqrt{9-x^2}]\,[y]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx$$

$$= \int_{-3}^{3} 4\,(9-x^2)\,dx \Rightarrow 4\left[9x-\frac{x^3}{3}\right]_{-3}^{3}$$

$$= 4\,[9\times 3 - 9 - (-27+9)]$$

$$V = 4\,(27-9+18) = 4\times 36 = 144.$$

$\boxed{\text{Volume enclosed is 144 units}}$ required Solution.

Ques:- 4 (b)) A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left[\frac{b}{c}+\frac{c}{b}\right]+zx\left[\frac{c}{a}+\frac{a}{c}\right]+xy\left[\frac{a}{b}+\frac{b}{a}\right]=0$$

Solution:-

Given; plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ —— (1)

Any parallel to above plane can be written

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=d \quad \text{—— (2)}$$

It meets axes in

A $(ad, 0, 0)$ ; B $(0, bd, 0)$ ; C $(0, 0, cd)$

Circle ABC constructed when sphere OABC is cut by the plane (2).

Sphere OABC can be given as,

$$x^2+y^2+z^2-adx-bdy-cdz=0 \quad \text{—— (3)}$$

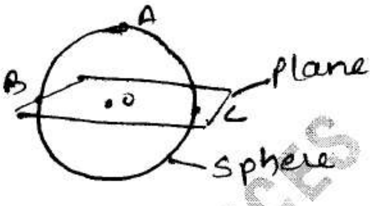
sphere OABC and plane (2), gives circle ABC thus, to construct cone in which circle lies can be attained by replacing 'd' from (2) and (3)

$$x^2+y^2+z^2-a\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)x-b\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)y-c\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)z=0$$

By solving →

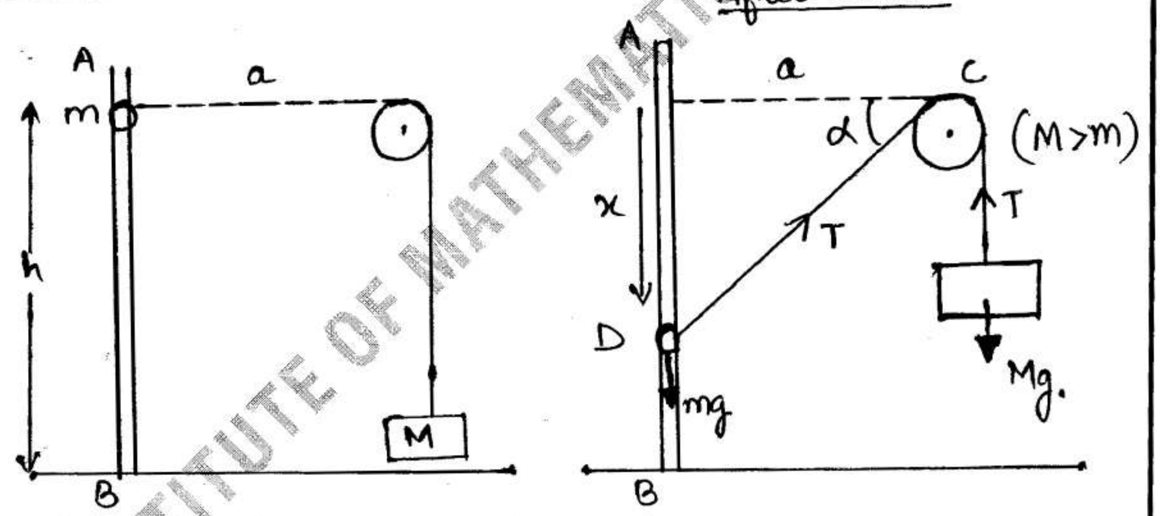
$$-ax\left(\frac{y}{b}+\frac{z}{c}\right)-by\left(\frac{x}{a}+\frac{z}{c}\right)-cz\left(\frac{x}{a}+\frac{y}{b}\right)=0$$

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$



Ques: 7(a)) A heavy ring of mass m, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant 'a' from the rod and has a mass M (>m) fastened to its other end. Show that if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2-m^2}$ before coming to rest.

Solution:-




Ring of mass 'm' slides over vertical rod AB pulley is at distant 'a' from the rod, string is attached with ring at one end of mass 'm' & with mass 'M' at other end.

Let, ring traverses distances 'x' before coming to rest along AB and $l$ be the length of string that crosses pulley, increasing the AC to CD i.e. 'a' to 'a+$l$'.

Applying work energy principle:

$$mgh = Mgl + mg(h-x)$$

$$\cancel{mgh} = Mgl + \cancel{mgh} - mgx$$

$$m\cancel{g}x = M\cancel{g}l$$

$$\boxed{l = \frac{mx}{M}} \quad \text{———} \quad (1)$$

In $\Delta ACD$

$$a^2 + x^2 = (a+l)^2$$

$$\cancel{a^2} + x^2 = \cancel{a^2} + l^2 + 2al$$

$$x^2 = l^2 + 2la$$

$$x^2 = \frac{m^2x^2}{M^2} + \frac{2amx}{M} \quad [\text{using } (1)]$$

$$\frac{2am\cancel{x}}{\cancel{M}} = x^{\cancel{2}}\left(\frac{M^2 - m^2}{M^{\cancel{2}}}\right)$$

$$\boxed{x = \frac{2aMm}{M^2 - m^2}}$$

or

i.e $\boxed{x = \frac{2Mma}{M^2 - m^2}}$

is the required solution.

2012 - Paper-II

Ques: 1(b)) Let

$$f_n(x) = \begin{cases} 0 & , \text{ if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x} & , \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & ; \text{ if } x > \frac{1}{n} \end{cases}$$

show that $f_n(x)$ converges to a continuous function but not uniformly.

Solution:-

$$\text{Given; } f_n(x) = \begin{cases} 0 & ; \text{ if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x} & ; \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & ; \text{ if } x > \frac{1}{n} \end{cases}$$

By Archemedi's theorem, there exist 'n' such that $xn > 1$ for $x \in R$

i.e. $x > \frac{1}{n}$; this implies $f_n(x) = 0$

So; $\lim\limits_{x \to 0} f_n(x) = f(x) = 0$

so, $f_n(x)$ converges to $f(x) = 0$.

For Uniform Convergence:

Consider;

$$|f_n(x) - f(x)| = \begin{cases} 0 & ; x < 1/n+1 \\ \left|\sin\frac{\pi}{x}\right| & ; \frac{1}{n+1} < x < \frac{1}{n} \\ 0 & ; x > \frac{1}{n} \end{cases}$$

let ; $M_n = \sup |f_n(x) - f(x)|$

$$= \sup \left|\sin\frac{\pi}{x}\right| \quad ; \frac{1}{n+1} < x < \frac{1}{n}$$

$$= 1.$$

Ques:1) e) Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^{n}\cdot n^{6}$ is convergent.

Solution:-

Consider; $u_n = \left(\frac{\pi}{\pi+1}\right)^n \cdot n^6$ ; therefore

$$u_{n+1} = \left(\frac{\pi}{\pi+1}\right)^{n+1}(n+1)^6$$

$$\frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1}\right)^6 \cdot \frac{\pi+1}{\pi}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^6\cdot\left(1+\frac{1}{\pi}\right)$$

$$= \lim_{n\to\infty}\frac{\cancel{n^6}}{\cancel{n^6}(1+1/n)^6}\left[1+\frac{1}{\pi}\right]$$

$$= \frac{1}{1}\left[1+\frac{1}{\pi}\right]$$

$$\Rightarrow \boxed{\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1+\frac{1}{\pi} > 1}$$

By ratio test,

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} > 1$$

$\therefore$ The series is convergent.

Ques: 2(b)> Let $f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 1 & ; \text{if } (x,y) = (0,0) \end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$.

Solution:-

Given; $f(x,y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2} & ; \text{if } (x,y) \neq (0,0) \\ 1 & ; \text{if } (x,y) = (0,0) \end{cases}$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\left[\frac{(h+0)^2}{h^2+0^2} - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{1-1}{h} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\left[\frac{k^2}{k^2} - 1\right]}{k}$$

$$= \lim_{k \to 0} \frac{1-1}{k} = 0$$

Therefore; $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$.

To check continuity

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$$

Assuming $(x,y)$ approaches $(0,0)$ along $y=mx$

$y = mx$

$$= \lim_{x \to 0} \frac{(x+mx)^2}{x^2+m^2x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}(1+m)^2}{\cancel{x^2}(1+m^2)}$$

$$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = \frac{(1+m)^2}{1+m^2}.}$$

Limit depends on 'm' thus limit does not exist and $f(x,y)$ is not continuous at $(0,0)$.

Ques: 2 (d)) Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ from the origin, by the method of Lagrange's multipliers.

Solution:-

Let $(x,y,z)$ be the point on line formed by intersection of $\pi_1 : 3x+4y+5z=7$ & $\pi_2 = x-z=9$

Distance from origin be $d^2 = x^2+y^2+z^2$

Using Lagrange's method to evaluate minimum value of $x^2+y^2+z^2$ subject to $\pi_1$ & $\pi_2$

$$F(x,y,z) = x^2+y^2+z^2+\lambda_1(3x+4y+5z-7)+\lambda_2(x-z-9)$$

$$\frac{\partial f}{\partial x} = 2x+\lambda_1 3+\lambda_2 = 0 \quad \text{——} (1)$$

$$\frac{\partial f}{\partial y} = 2y+4\lambda_1 = 0 \quad \text{——} (2) \Rightarrow \boxed{\lambda_1 = -\frac{y}{2}}$$

$$\frac{\partial f}{\partial z} = 2z+\lambda_1(5)-\lambda_2 = 0 \quad \text{——} (3)$$

Adding (1) and (3), we get

$$2(x+z)+8\lambda_1 = 0$$

$$\lambda_1 = -\frac{\cancel{2}(x+z)}{\cancel{8}4}$$

$$-\frac{y}{2} = -\frac{(x+z)}{4} \Rightarrow \boxed{2y = x+z} \text{——} (4)$$

Solving (4) & $\pi_2$

$$x = \frac{2y+9}{2} \; ; \; z = \frac{2y-9}{2}$$

Put these in $\pi_1$.

Ques: 3(b)) Let $f(x)$ be differentiable on $[0,1]$, such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x)\,dx = 1$. Prove that $\int_0^1 x f(x) f'(x)\,dx = -\frac{1}{2}$.

Solution:-

Since; $f(x)$ is differentiable on $[0,1]$

hence; $f(x)$ is continuous on $[0,1]$

$$\int_0^1 x f(x) f'(x)\,dx$$

integrating by using by parts integration

$$I = \int_0^1 x f(x) f'(x)\,dx = \left[x \int f(x) f'(x)\,dx\right]_0^1 - \int_0^1 \left[\int f(x) f'(x)\,dx\right]$$

$$I = \left[\frac{x (f(x))^2}{2}\right]_0^1 - \int_0^1 \frac{f^2(x)}{2}\,dx.$$

$$I = \left[\frac{(f(1))^2}{2} - 0\right] - \frac{1}{2} \quad \left[\because \int_0^1 f^2(x)\,dx = 1\right]$$

given.

$$I = [0 - 0] - \frac{1}{2}.$$

$$I = -1/2.$$

Thus; $\boxed{\int_0^1 x f(x) f'(x)\,dx = -1/2}$

Hence, required result

$$3\left(\frac{2y+9}{2}\right) + 4y + 5\left(\frac{2y-9}{2}\right) = 7$$

By solving, we get

$$\boxed{y = 4/3 \ ; \ x = 35/6 \ \& \ z = -19/6}$$

To check stationary points $(35/6, 4/3, -19/6)$ gives minimum distance or not

$$u = x^2+y^2+z^2$$

$$\frac{\partial u}{\partial x} = 2x + 2z\frac{\partial z}{\partial x}$$

$$\boxed{\frac{\partial u}{\partial x} = 2x - \frac{6}{5}z}$$

[ using $3x+4y+5z=7$

$3 + 5\frac{\partial z}{\partial x} = 0$

$\because$ $z$ is dependent on $x$ & $y$

$\partial z/\partial x = -3/5$ ]

$$r = \frac{\partial^2 u}{\partial x^2} = 2 - \frac{6}{5}\frac{\partial z}{\partial x} = 2 + \frac{6}{5}\times\frac{3}{5} = \frac{68}{25} > 0$$

$$\frac{\partial u}{\partial y} = 2y + 2z\frac{\partial z}{\partial y} = 2y - \frac{8z}{5} \qquad \left[\because \frac{\partial z}{\partial y} = \frac{-4}{5}\right]$$

$$t = \frac{\partial^2 u}{\partial y^2} = 2 - \frac{8}{5}\cdot\frac{\partial z}{\partial y} = 2 + \frac{32}{25} = \frac{82}{25} > 0$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\frac{6}{5}\frac{\partial z}{\partial y} = \frac{24}{25} > 0$$

Since, $rt - s^2 > 0$ & $r > 0$, $t > 0$, therefore it is a minimum, thus minimum distance is $d^2 = x^2+y^2+z^2 = \left(\frac{35}{6}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{-19}{6}\right)^2$

$$d = \sqrt{\left(\frac{35}{6}\right)^2 + \frac{16}{9} + \frac{369}{36}}$$

$\boxed{d = 6.77 \text{units}}$ which is required <u>result</u>

Ques: 3(d)) Evaluate by Contour Integration

$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}, \quad a^2<1$$

Solution:-

given; $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$ ; $a^2<1$

Let $z = e^{i\theta} \Rightarrow dz = e^{i\theta}.\, i\,d\theta$

$$dz = z\,i\,d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}.\frac{z^2+1}{z}.$$

$$I = \int_C \frac{1}{1-\cancel{2}a.\frac{1}{\cancel{2}}\left(\frac{z^2+1}{z}\right)+a^2}\cdot\frac{dz}{iz}$$

$$I = \frac{1}{i}\int_C \frac{\cancel{z}}{z-a(z^2+1)+a^2z}\cdot\frac{dz}{\cancel{z}}$$

$$I = \frac{1}{i}\int_C \frac{dz}{z-az^2-a+a^2z}$$

$$I = \frac{1}{i}\int_C \frac{dz}{z(1-az)-a(1-az)}$$

$$I = \frac{1}{i}\int_C \frac{dz}{(z-a)(1-az)}$$

$$I = \frac{1}{i}\int f(z)\,dz$$

$$f(z) = \frac{1}{(z-a)(1-az)}$$

The roots of $f(z)$ can be obtained from

$$(z-a)(1-az)$$

$$\Rightarrow z = a, 1/a \text{ with order } 1.$$

Now; Res at $z=a$

$$= \lim_{z\to a} \cancel{(z-a)} \frac{1}{\cancel{(z-a)}(1-az)}$$

$$= \frac{1}{(1-a.a)} = \frac{1}{1-a^2}$$

Res. at $z = 1/a$.

$$= \lim_{z\to \frac{1}{a}} \cancel{(1-az)} \frac{1}{(z-a)\cancel{(1-az)}}$$

$$= \frac{1}{\left(\frac{1}{a}-a\right)} = \frac{a}{1-a^2}$$

$$I = 2\pi i \left[\operatorname{Res} f(a) + \operatorname{Res} f(1/a)\right]$$

$$I = 2\pi \cancel{i} \cdot \frac{1}{\cancel{i}}\left[\frac{1}{1-a^2} + \frac{a}{1-a^2}\right]$$

$$I = 2\pi\left[\frac{1+a}{1-a^2}\right] = \frac{2\pi\cancel{(1+a)}}{\cancel{(1+a)}(1-a)}$$

$$\boxed{I = \frac{2\pi}{1-a}}$$

$$\therefore \boxed{\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2} = \frac{2\pi}{1-a}}$$ Required result

Ques:- 5(c)) Provide a computer algorithm to solve an ordinary differential equation $\frac{dy}{dx} = f(x,y)$ in the interval $[a,b]$ for $n$ number of discreate points, where the initial value is $y(\alpha) = \alpha$, using Euler's method.

Solve:-

1. Read $a, b, y_i$
2. Read number of discreate points '$n$'.
3. Evaluate $h = \frac{(b-a)}{n}$
4. Assign $x = a$
5. Assign $i = 1$
6. Check if $x >= (b+h)$, go to 10
   Otherwise go to 7.
7. Evaluate
   $$y_{i+1} = y_i + hf(x_i, y_i)$$
8. print $y_{i+1}$
9. Increament $i$ & go to 6
   and $x = x+h$
10. Stop.

Ques:- 8(a)) A pendulum consists of a rod of length 2a and mass m; to one end of which a spherical bob of radius a/3 and mass 15 m is attached. Find the moment of Inertia of the pendulum:

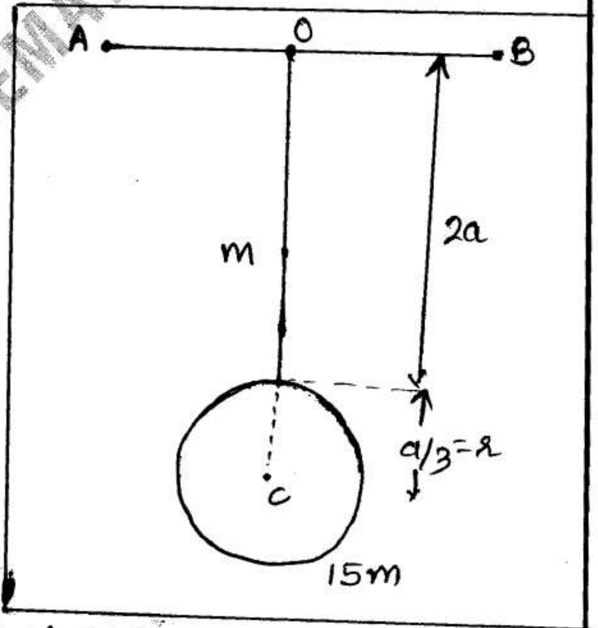
(i) about an axis through the other end of the rod and at right angles to the rod.

(ii) about a parallel axis through the centre of mass of the pendulum.

[Given: The centre of mass of the pendulum is a/12 above the centre of the sphere.].

Solution:-

A pendulum consists of a rod of length '2a' and mass 'm'.
a spherical bob of radius $= a/3$ and mass $= 15m$.




Now, the moment of Inertia about axis AB.

= MOI of rod OD + MOI of spherical bob about AB

$$= \frac{4}{3}ma^2 + \frac{2}{5}(15m)\cdot\left(\frac{a}{3}\right)^2 + 15m\left(2a+\frac{a}{3}\right)^2$$

$$= \frac{4}{3}ma^2 + \frac{2}{3}ma^2 + \frac{5m}{3}\cdot 49a^2$$

$$= \frac{251\,ma^2}{3} \Rightarrow \boxed{\text{MOI about axis AB} = \frac{251\,ma^2}{3}}$$

2013 - Paper-I)

Ques:- 8(d)) Use Stoke's theorem to evaluate the line integral $\int_C (-y^3 dx + x^3 dy - z^3 dz)$, where 'C' is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$.

Solution :- given

$$\vec{F} = -y^3\hat{i} + x^3\hat{j} - z^3\hat{k} \quad \text{--- (1)}$$

Stoke's theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times F)\cdot\hat{n}\, ds \quad \text{--- (A)}$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & -z^3 \end{vmatrix}$$

$$= \hat{i}(0+0) + (-\hat{j})(0+0) + \hat{k}(3x^2+3y^2) \quad \text{--- (2)}$$

$$= 0\hat{i} + 0\hat{j} + (3x^2+3y^2)\hat{k}$$

C, is intersection of $x^2+y^2=1$ & $x+y+z=1$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x+y+z-1)}{|\nabla\phi|}$$

$$\boxed{\hat{n} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}} \quad \text{--- (3)}$$

In (A) put (2) & (3)

$$\iint_S (3\hat{k}(x^2+y^2))\cdot\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\right) ds$$

$$= \iint_S \frac{3(x^2+y^2)}{\sqrt{3}}\, ds$$

Surface integral around $x^2+y^2=1$

2013-Paper-II

Ques: 1.(C) Let $f(x) = \begin{cases} \frac{x^2}{2}+4 & \text{if } x \geq 0 \\ -\frac{x^2}{2}+2 & \text{if } x < 0 \end{cases}$

Is 'f' Reimann integrable in the intervae $[-1,2]$? Why? Does there exist a function 'g' such that $g'(x) = f(x)$? Justify your answer?

Solution:-

given; $f(x) = \begin{cases} \frac{x^2}{2}+4 & ; x \geq 0 \\ -\frac{x^2}{2}+2 & ; x < 0 \end{cases}$

To check the continuity of $f(x)$ at $x=0$

$$\lim_{x\to 0^+} f(x+h) = \lim_{h\to 0^+} \left(\frac{0+h}{2}\right)^2 + 4 = 4$$

$$\lim_{x\to 0^-} f(x-h) = \lim_{h\to 0^-} -\left(\frac{0+h}{2}\right)^2 + 2 = 2$$

and $f(0) = 4$

$$\therefore f(x+h) = f(0) \neq f(x-h)$$

Hence, $f(x)$ is discontinuous at $x=0$ which is finite and countable. Therefore $f(x)$ is Reimann Integrable in $[-1,2]$.

Suppose; $g(x)$ exist such that $g'(x) = f(x)$

$$g(x) = \begin{cases} \int_0^x f(x)\,dx & ; \text{ if } x \geq 0 \\ \int_{-1}^x f(x)\,dx & ; \text{ if } x < 0 \end{cases}$$

$$g(x) = \begin{cases} \int_0^x \left(\frac{x^2}{2}+4\right)dx & ; \ 0 \le x < 2 \\ \int_{-1}^x \left(-\frac{x^2}{2}+2\right)dx & ; \ -1 \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ \left(-\frac{x^3}{6}+2x\right)_{-1}^{x} & ; \ -1 \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ -\frac{x^3}{6}+2x+\frac{11}{6} & ; \ -1 \le x < 0 \end{cases}$$

But, $\lim_{x\to 0} g(x)$ does not exist thus not continuous at $x=0$.

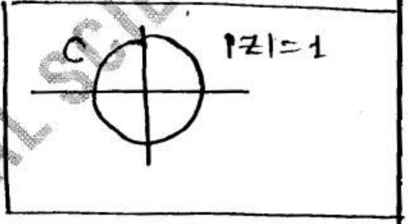
Therefore, there does not exist a function $g(x)$ such that $g'(x) = f(x)$.

Hence the result.

Ques: 1(d)) Prove that if $be^{a+1} < 1$, where a and b are positive real number, then the function $z^n e^{-a} - be^z$ has 'n' zeroes in the unit circle.

Solution:-

Using Rouche's theorem that if $f(z)$ & $g(z)$ are analytic function in 'C', then $f(z)+g(z)$ has same number of zeroes as $f(z)$ has if $|g(z)| < |f(z)|$ on C.




Consider;

$$f(z) = z^n e^{-a}$$
$$g(z) = -be^z$$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be^z}{z^n e^{-a}}\right| \quad \text{using } |z|=1$$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be}{1^n \cdot e^{-a}}\right| = \left|\frac{be}{e^{-a}}\right|$$

$$\left|\frac{g(z)}{f(z)}\right| = |be^{a+1}| < 1.$$

$$\Rightarrow \boxed{|g(z)| < |f(z)| \quad \text{on} \quad z = 1}$$

Then, function $z^n e^{-a} - be^z$ will have same number of zeros as $f(z) = z^n e^{-a}$ in $|z| = 1$.

Clearly, $f(z)$ has 'n' zeroes in unit circle.

So; $z^n e^{-a} - be^z$ will have 'n' zeroes in unit circle.

Ques:- 2(c) ) Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$, is uniformly convergent but not absolutely for all real values of $x$.

Solution:-

To show uniform convergence

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n+x^2} \qquad f(x) = (-1)^{n-1}$$

$$g(x) = \frac{1}{n+x^2}$$

Using dirichlet theorem

(1) Sum function; $S_n = \sum_{n=1}^{r} (-1)^{n-1} = \begin{cases} 0 & ; r \text{ is even.} \\ 1 & ; r \text{ is odd.} \end{cases}$

Thus; $S_n$ is bounded.

(2) To check sequence $\langle g(x) \rangle$ is positive, monotonic decreasing sequence converging to zero.

$$u_n - u_{n+1} = \frac{1}{n+x^2} - \frac{1}{n+1+x^2}$$

$$= \frac{1}{(n+x^2)(n+1+x^2)} > 0$$

$u_n > u_{n+1}$ ; thus $g(x)$ is positive monotonic decreasing sequence converging to zero.

$\therefore$ By Dirichlet theorem;

$\boxed{\sum (-1)^{n-1} \frac{1}{n+x^2} \text{ is uniformly convergent.}}$

Now, to check whether 'f' is absolutely convergent or not.

$\sum f(x)$ is absolutely convergent if $\sum|f(x)|$ is convergent.

$$\sum\left|\frac{(-1)^{n-1}}{n+x^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n+x^2}$$

Let $u_n = \frac{1}{n+x^2}$ ; $u_{n+1} = \frac{1}{n+1+x^2}$

$$\frac{u_n}{u_{n+1}} = \frac{n+1+x^2}{n+x^2} = \left(1+\frac{1}{n+x^2}\right)$$

$$= \left(1+\frac{1+x^2}{n}\right)\left(1+\frac{x^2}{n}\right)^{-1}$$

Doing binomial Expansion

$$= \left(1+\frac{1+x^2}{n}\right)\left(1-\frac{x^2}{n}+\frac{x^4}{2n^2}+\cdots\right)$$

$$= \left[1+\frac{1+x^2}{n}-\frac{x^2}{n}-\frac{x^2(1+x^2)}{n^2}+\frac{x^4}{2n^2}+\cdots\right]$$

$$= \left[1+\frac{1}{n}-\frac{x^2(1+x^2)}{n^2}+\frac{x^4}{2n^2}\cdots\right]$$

$\lambda = 1$ ; i.e coefficient of $1/n$ is 1, then by Gauss's test given series is divergent, thus.

$$\boxed{\sum\frac{(-1)^{n-1}}{n+x^2} \text{ is not absolutely convergent}}$$

Which is required result

Ques:-3(c)> Let $f(x,y) = y^2 + 4xy + 3x^2 + x^3 + 1$,
At what points will $f(x,y)$ have a maximum or minimum?

Solution:-

Given ; $f(x,y) = y^2 + 4xy + 3x^2 + x^3 + 1$

First to determine stationary points

$$f_x = 4y + 6x + 3x^2 = 0 \quad \text{---} (1)$$

$$f_y = 2y + 4x = 0 \Rightarrow \boxed{y = -2x}$$

Put $y = -2x$ in (1)

$$3x^2 + 6x - 8x = 0$$

$$3x^2 - 2x = 0$$

$$x(3x-2) = 0$$

$$\Rightarrow \boxed{x = 0, \ x = 2/3}$$

Stationary points are $(0,0)$, $(2/3, -4/3)$.

Check maximum or minimum at stationary points

$$f_{xx} = 6 + 6x$$

$$f_{xy} = 4$$

$$f_{yy} = 2$$

At $f(0,0)$ ; $f_{xx} = 6 > 0$

$f_{yy} = 2 > 0$

$f_{xy} = 4 > 0$

neither minimum or maximum at $f(0,0)$.

At $f(2/3, -4/3)$

$f_{xx} = 6 + 6 \times \frac{2}{3} = 10 > 0$

$f_{yy} = 2 \quad > 0$

$f_{xy} = 4 \quad > 0.$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 10 - 16 = 4 > 0$

$\boxed{\therefore \text{ Minimum at } f(2/3, -4/3) = 4}$

Now, value at $(2/3, -4/3)$.

$f(2/3, -4/3) = \frac{16}{9} - 4 \times \frac{2}{3} \times \frac{4}{3} + \frac{3 \times 4}{9} + \frac{8}{27} + 1$

$= \frac{16}{9} - \frac{32}{9} + \frac{12}{9} + \frac{35}{27}$

$= \frac{-4 \times 3}{9 \times 3} + \frac{35}{27} = \frac{35 - 12}{27}.$

$\boxed{f(2/3, -4/3) = 23/27}$

Ques: 3(d) ⟩ Let $[x]$ denote the integer part of the real number $x$, i.e., if $n \le x < n+1$, where $n$ is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$; Reimann integrable in $[-1, 2]$? If not, explain why. If it is integrable, compute $\int_{-1}^{2} ([x]^2+3]\,dx$.

Solution:-

given $f(x) = [x]^2 + 3$

where, $n \le x < n+1$, where $n$ is an integer, then $[x] = n$.

Let, we have,

$$[x]^2 = \begin{cases} 1 & : x \in [-1, 0) \\ 0 & : x \in [0, 1) \\ 1 & : x \in [1, \sqrt{2}) \\ 2 & : x \in [\sqrt{2}, \sqrt{3}) \\ 3 & : x \in (\sqrt{3}, 2) \end{cases}$$

$$\therefore f(x) = [x]^2 + 3 = \begin{cases} 4 & : x \in [-1, 0) \\ 3 & : x \in [0, 1) \\ 4 & : x \in [1, \sqrt{2}) \\ 5 & : x \in [\sqrt{2}, \sqrt{3}) \\ 6 & : x \in (\sqrt{3}, 2) \end{cases}$$

Clearly, $f(x)$ is bounded by $4 \le f(x) \le 6 \;\forall\; x \in [-1, 2]$ and it is having finite number of points of discontinuity namely $x = 0, 1, \sqrt{2}, \sqrt{3}, 2$

$\therefore f(x)$ is discontinuous at $0, 1, \sqrt{2}, \sqrt{3}, 2$ in the given interval $[-1, 2]$, which is countable and

Thus, f(x) is Riemann integrable

As integral, hence compute

$$\int_{-1}^{2} f(x)\,dx = \int_{-1} ([x]^2+3)\,dx$$

$$= \int_{-1}^{0} 4\cdot dx + \int_{0}^{1} 3\cdot dx + \int_{1}^{\sqrt{2}} 4\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 5\,dx + \int_{\sqrt{3}}^{2} 6\,dx$$

$$= 4\times1 + 3\times1 + 4(\sqrt{2}-1) + 5(\sqrt{3}-\sqrt{2}) + 6(2-\sqrt{3})$$

$$= \cancel{4} + 3 + 4\sqrt{2} - \cancel{4} + 5\sqrt{3} - 5\sqrt{2} + 12 - 6\sqrt{3}$$

$$= 15 - \sqrt{2} - \sqrt{3}.$$

$$\boxed{\therefore \int_{-1}^{2} f(x)\,dx = \int_{-1}^{2} ([x]^2+3)\,dx = 15 - \sqrt{2} - \sqrt{3}}$$

Ques: 4(a) Solve the minimum time assignment problem:

| | | Machines | | | |
|---|---|---|---|---|---|
| | | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
| Jobs | $J_1$ | 3 | 12 | 5 | 14 |
| | $J_2$ | 7 | 9 | 8 | 12 |
| | $J_3$ | 5 | 11 | 10 | 12 |
| | $J_4$ | 6 | 14 | 4 | 11 |

Solution:-

Subtract minimum element of each row from all other element of that row thus reduced matrix is -

$$\begin{bmatrix} 0 & 9 & 2 & 11 \\ 0 & 2 & 1 & 5 \\ 0 & 6 & 5 & 7 \\ 2 & 10 & 0 & 7 \end{bmatrix}$$

Subtract minimum element of each column from all elements of that column, we get

$$\begin{bmatrix} 0 & 7 & 2 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 5 & 2 \\ 2 & 8 & 0 & 2 \end{bmatrix}$$

Cover all the zeros by least number of horizontal and vertical lines. Exactly lines are required to cover all zeros should be 4.

$$\begin{bmatrix} 0 & 7 & 2 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 4 & 5 & 2 \\ 2 & 8 & 0 & 2 \end{bmatrix}$$

However lines are 3
3 < 4, thus to attain optimality.

Pick minimum element from uncovered elements i.e. 2 and subtract from all other uncovered element and also add to intersecting elements of two lines thus reduced matrix is –

$$\begin{bmatrix} 0 & 5 & 2 & 4 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 2 & 6 & 0 & 0 \end{bmatrix}$$

Now, again cover all zeros by least number of horizontal & vertical lines.

as $r = 4 = n$. Thus, optimality is attained and we can make the assignment.

Choose/Pick zero from rows containing only one zero and cancel all zeros in its respective column. Similarly doing for column, thus

$$\begin{bmatrix} \boxed{0} & 5 & 2 & 4 \\ 2 & \boxed{0} & 3 & \cancel{0} \\ \cancel{0} & 2 & 5 & \boxed{0} \\ 2 & 6 & \boxed{0} & \cancel{0} \end{bmatrix}$$

Hence;

$J_1 \to M_1$

$J_2 \to M_2$

$J_3 \to M_4$

$J_4 \to M_3$

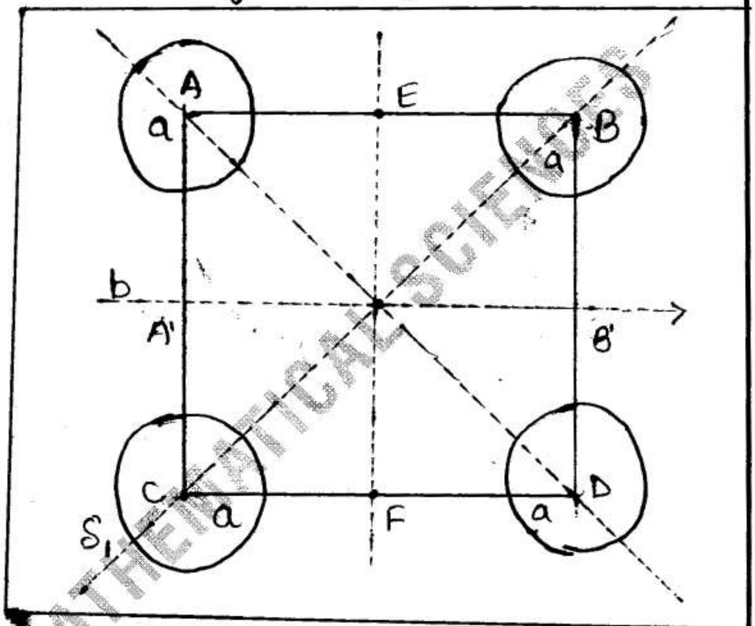
$$\boxed{\therefore \text{Minimum time} = 3 + 9 + 12 + 4 = 28.}$$ required solution.

Ques: 5(e)) Four solid spheres A, B, C and D, each of mass 'm' and radius 'a', are placed with their centres on the four corners of a square of side b. Calculate the moment of inertia of the system about a diagonal of the square?

Solve:-

Firstly evaluating moment of Inertia of system about AB.

Four solid sphere A, B, C and D each have mass M. and radius 'a'.




MOI of any solid sphere ($S_1$) about AB

= MOI of solid sphere about CD + MOI of centre C of $S_1$ about AB.

$$= \frac{2}{5}Ma^2 + \frac{Mb^2}{4}.$$

MOI of System about AB = 4 (MOI of $S_1$ about AB)

$$= 4\left(\frac{2}{5}Ma^2 + \frac{Mb^2}{4}\right)$$

$$= \frac{8}{5}Ma^2 + Mb^2$$

Similarly, MOI of System about EF is

$= \frac{8}{5}Ma^2 + Mb^2$

(where EF is axis perpendicular to AB)

Let; $A = B = \frac{8}{5}Ma^2 + Mb^2$

So, MOI of system about the diagonal

$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$

F : Product of inertial since the system is symmetric $F = 0$, therefore

$F = A\cos^2 45^\circ + A\sin^2 45^\circ$

$F = A(\cos^2 45^\circ + \sin^2 45^\circ)$

$F = A$

$\boxed{\therefore F = \frac{8}{5}Ma^2 + Mb^2}$ required result.