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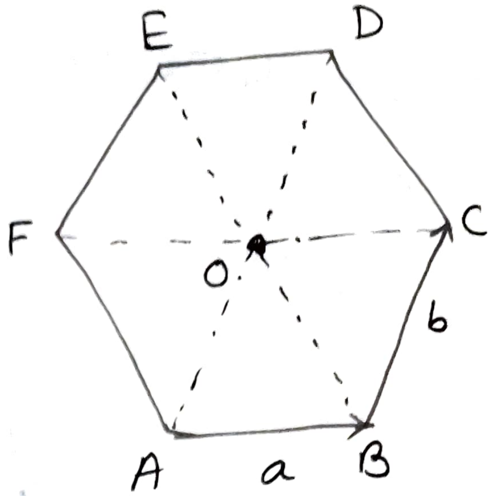
Q1.



$\vec{CD} = \vec{CO} + \vec{OD}$

(Triangle law of vector addition)

$\vec{CD} = -a + b$

$\vec{DE} = -a$

$\vec{EF} = \vec{EO} + \vec{OF} = -\vec{CD} + (-a)$

$= -(-a+b) - a$

$= -b$

$\vec{FA} = \vec{FO} + \vec{OA} = a + (-b) = a - b$

$\vec{AC} = \vec{AB} + \vec{BC} = a + b$

$\vec{AD} = 2(\vec{BC}) = 2b$

$\vec{AE} = \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE}$

$= \cancel{a+b}$

$\cancel{a} + b + (-\cancel{a}+b) + (-a)$

$= 2b - a$

$\vec{AF} = \vec{CD} = -a + b$

$\vec{CE} = \vec{CD} + \vec{DE} = -a + b + (-a)$

$= -2a + b$

Q2. Let

$$v_1 = 5a + 6b + 7c$$

$$v_2 = 7a - 8b + 9c$$

$$v_3 = 3a + 20b + 5c$$

We want to check whether vectors v_1, v_2 and v_3 are linearly independent or dependent.

Let us from the determinant of coefficients.

$$D = \begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$$

$$= 5(-40-180) - 6(35-27) + 7(140+24)$$

$$= 0$$

As $D = 0$, given vectors v_1, v_2, v_3 are linearly dependent,

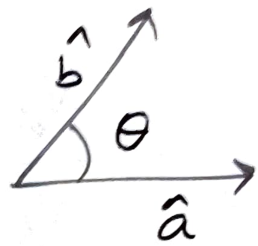
We also note that

$$v_3 = 2v_1 + (-1)v_2$$

ie. $2(5a+6b+7c) - (7a-8b+9c)$

$$= 3a + 20b + 5c = v_3.$$

Q3. $\hat{a}$ and $\hat{b}$ are unit vectors.



To show : $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$

$$|\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$$

$$= \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$$

$$= \cancel{1} - 2\,\hat{a}\cdot\hat{b} + \cancel{1} \qquad \left[\because \hat{x}\cdot\hat{x} = 1,\ \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}\right]$$

$$= 2 - 2\cos\theta \qquad \left[\vec{x}\cdot\vec{y} = |x||y|\cos\theta\right]$$

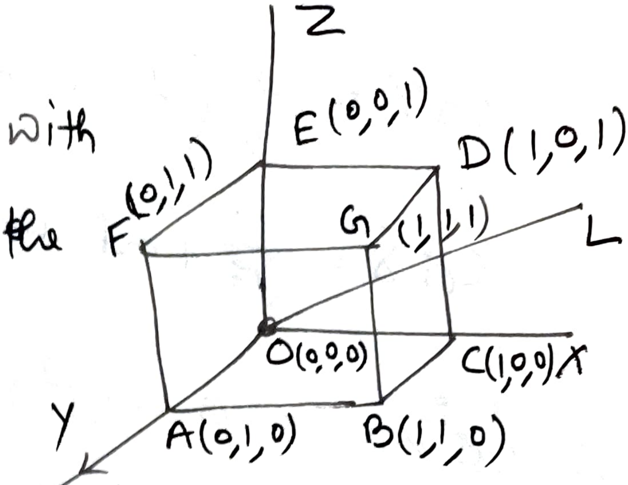
$$= +2(1-\cos\theta)$$

$$= 2\left(2\sin^2\frac{\theta}{2}\right)$$

$$\Rightarrow \quad |\hat{a}-\hat{b}| = 2\sin\frac{\theta}{2}$$

$$\Rightarrow \quad \boxed{\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|}$$

Q4 - Line L makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of the cube.



To show:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

D.R. of Diagonal OG $= \langle 1-0, 1-0, 1-0 \rangle$

D.C $= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

D.R of BE $= \langle 1-0, 1-0, 0-1 \rangle$

D.C $= \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$

D.R. of AD $= \langle 1-0, 0-1, 1-0 \rangle$

D.C $= \left\langle \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

D.R. of FC $= \langle 0-1, 1-0, 1-0 \rangle$

D.C $= \left\langle \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

Let l, m, n be d.c. of the given line L, then

$$\cos\alpha = \frac{1}{\sqrt{3}} \cdot l + \frac{1}{\sqrt{3}} \cdot m + \frac{1}{\sqrt{3}} \cdot n = \frac{1}{\sqrt{3}}(l+m+n)$$

$$\cos\beta = \frac{1}{\sqrt{3}}(l+m-n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(-l + m + n)$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\Big[(l+m+n)^2 + (l+m-n)^2 + (l-m+n)^2 + (-l+m+n)^2\Big]$$

$$= \frac{1}{3}\Big[(l^2+m^2+n^2 + \cancel{2lm} + \cancel{2mn} + \cancel{2nl})$$
$$+ (l^2+m^2+n^2 + \cancel{2lm} - \cancel{2mn} - \cancel{2nl})$$
$$+ (l^2+m^2+n^2 - \cancel{2lm} - \cancel{2mn} + \cancel{2nl})$$
$$+ (l^2+m^2+n^2 - \cancel{2lm} + \cancel{2mn} - \cancel{2nl})$$

$$= \frac{1}{3}\Big[4(l^2+m^2+n^2)\Big]$$

$$= \frac{4}{3} \qquad [\because l^2+m^2+n^2 = 1]$$

Q5. Let $v_1 = 2i - j + k$

$v_2 = i + 2j - 3k$

$v_3 = 3i + pj + 5k$

Let v_1, v_2 and v_3 are coplanar

then $[v_1, v_2, v_3] = 0$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 0$$

$$2(10 + 3p) + 1(5 + 9) + 1(p - 6) = 0$$

$$(20 + 6p) + 14 + p - 6 = 0.$$

$$7p = -28$$

$$\Rightarrow \boxed{p = -4}$$

Q6. To prove

$$a \times b = [(i \times a) \cdot b]\, i + [(j \times a) \cdot b]\, j + [(k \times a) \cdot b]\, k.$$

Let $a = a_1 i + a_2 j + a_3 k$

$b = b_1 i + b_2 j + b_3 k.$

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

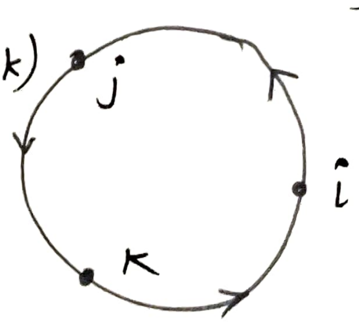
$$= i(a_2 b_3 - a_3 b_2) + j(b_1 a_3 - a_1 b_3) + k(a_1 b_2 - a_2 b_1) \quad \text{— (A)}$$

$$(i \times a) = i \times (a_1 i + a_2 j + a_3 k)$$

$$= a_1 (i \times i) + a_2 (i \times j) + a_3 (i \times k)$$

$$= a_2 k + a_3(-j) \qquad (\because \vec{v} \times \vec{v} = 0)$$

$$(i \times a) \cdot b = (a_2 k - a_3 j) \cdot (b_1 i + b_2 j + b_3 k)$$



$$= a_2 b_3 - a_3 b_2$$

$$[(i \times a) \cdot b]\, i = (a_2 b_3 - a_3 b_2)\, i \quad \text{— (1)}$$

$$(\hat{j}\times a)\cdot b = [\hat{j}\times(a_1\hat{i}+a_2\hat{j}+a_3k)]\cdot(b_1\hat{i}+b_2\hat{j}+b_3k)$$

$$= (-a_1 k + a_3\hat{i})\cdot(b_1\hat{i}+b_2\hat{j}+b_3k)$$

$$= a_3 b_1 - a_1 b_3 \qquad \text{--- (2)}$$

$$(k\times a)\cdot b = [k\times(a_1\hat{i}+a_2\hat{j}+a_3k)]\cdot(b_1\hat{i}+b_2\hat{j}+b_3k)$$

$$= [a_1\hat{j} + a_2(-\hat{i})]\cdot(b_1\hat{i}+b_2\hat{j}+b_3k)$$

$$= a_1 b_2 - a_2 b_1 \qquad \text{--- (3)}$$

Hence, ~~from~~ (1), (2) and (3)

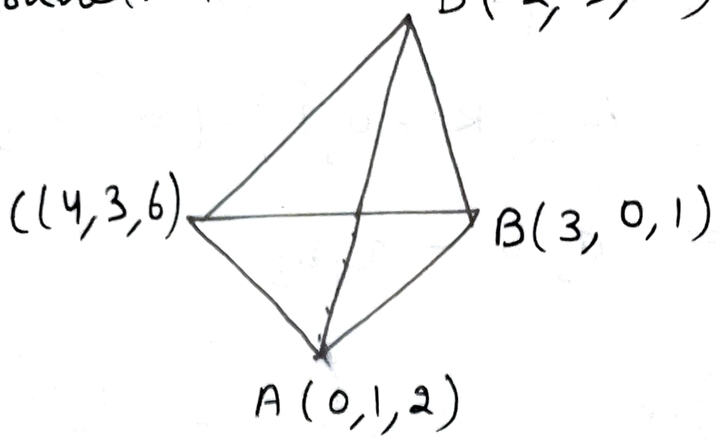
$$[(\hat{i}\times a)\cdot b]\hat{i} + [(\hat{j}\times a)\cdot b]\hat{j} + [(k\times a)\cdot b]\hat{j}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j}$$

$$+ (a_1b_2 - a_2b_1)k$$

$$= a\times b \qquad [\text{using } (\star)]$$

Q7. Tetrahedron



Volume of Tetrahedron

$$= \frac{1}{6}\left[\vec{AB}\ \vec{AC}\ \vec{AD}\right]$$

$$\vec{AB} = (3-0)i + (0-1)j + (1-2)k$$

$$= 3i - j - k$$

$$\vec{AC} = (4-0)i + (3-1)j + (6-2)k$$

$$= 4i + 2j + 4k$$

$$\vec{AD} = (2-0)i + (3-1)j + (2-2)k$$

$$= 2i + 2j + 0 \cdot k.$$

$$V = \frac{1}{6}\left\| \begin{matrix} 3 & -1 & -1 \\ 4 & 2 & 4 \\ 2 & 2 & 0 \end{matrix} \right\|$$

$$= \frac{1}{6}\left|\left[3(0-8) + 1(0-8) - 1(8-4)\right]\right|$$

$$= \frac{1}{6}\left|\left[-24-8-4\right]\right| = \left|-\frac{36}{6}\right| = 6$$

Q9. $\phi = xy^2z$

$A = xz\hat{i} - xy\hat{j} + yz^2 k$

To find $\dfrac{\partial^3 \phi A}{\partial x^2 \partial z}$ at $(2,-1,1)$

$$\phi A = (x^2y^2z^2)\hat{i} - (x^2y^3z)\hat{j} + (xy^3z^3)k$$

$$\frac{\partial \phi A}{\partial z} = (2x^2y^2z)\hat{i} - (x^2y^3)\hat{j} + (3xy^3z^2)k$$

$$\frac{\partial^2 \phi A}{\partial x \partial z} = (4xy^2z)\hat{i} - (2xy^3)\hat{j} + (3y^3z^2)k$$

$$\frac{\partial^3 \phi A}{\partial x^2 \partial z} = (4y^2z)\hat{i} + (2y^3)\hat{j} + 0.k$$

$$\left.\frac{\partial^3 \phi A}{\partial x^2 \partial z}\right|_{(2,-1,1)} = (4)\hat{i} - 2\hat{j} + 0.k.$$

Q10.

$$\vec{r} = xi + yj + zk$$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{grad}(\log|r|) = \nabla \log|r|$$

$$= i\frac{\partial}{\partial x}\log r + j\frac{\partial}{\partial y}\log r + k\frac{\partial}{\partial z}\log r$$

$$= i\frac{1}{r}\cdot\frac{\partial r}{\partial x} + j\frac{1}{r}\cdot\frac{\partial r}{\partial y} + k\frac{1}{r}\cdot\frac{\partial r}{\partial z}$$

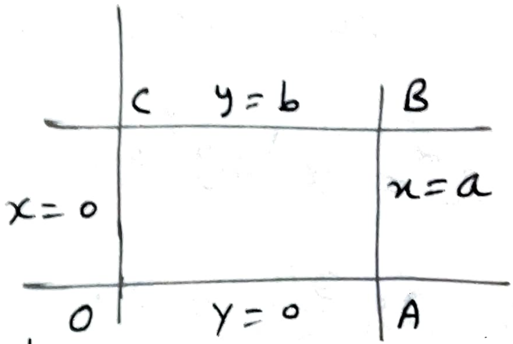
$$= \frac{i}{r}\cdot\frac{x}{r} + j\cdot\frac{1}{r}\cdot\frac{y}{r} + \frac{k}{r}\cdot\frac{z}{r}$$

$$= \frac{1}{r^2}(xi + yj + zk)$$

$$= \frac{\vec{r}}{r^2} = \frac{\vec{r}}{|\vec{r}|^2}$$

Q11. $\vec{F} = (x^2+y^2)\hat{i} - 2xy\,\hat{j}$

$$I = \oint_C F \cdot dr$$



$$= \int_C (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_{OA} F\cdot dr + \int_{AB} F\cdot dr + \int_{BC} F\cdot dr + \int_{CO} F\cdot dr$$

Along OA : $y=0$, x from 0 to a, $dy=0$

AB : $x=a$, y from 0 to b, $dx=0$

BC : $y=b$, x from a to 0, $dy=0$

CO : $x=0$, y from b to 0, $dx=0$

$$I = \int_0^a x^2\,dx + \int_0^b (-2a)y\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 0\,dy$$

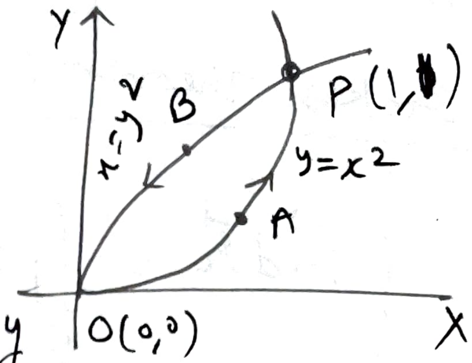
$$= \frac{x^3}{3}\Big|_0^a - a\cdot y^2\Big|_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^0$$

$$= \cancel{\frac{a^3}{3}} - ab^2 + \left(0 - \cancel{\frac{a^3}{3}} - b^2a\right)$$

$$= -2ab^2.$$

Q12.

$$\vec{F} = (2x+y^2)\,\hat{i} + (3y-4x)\,\hat{j}$$



$$I = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C (2x+y^2)\,dx + (3y-4x)\,dy$$

$$= \int_{OAP} \vec{F}\cdot d\vec{r} + \int_{P.BO} \vec{F}\cdot d\vec{r}$$

Along OAP: $y = x^2$, $dy = 2x\,dx$

x from 0 to 1

Along PBO: $x = y^2$, $dx = 2y\,dy$

y from 1 to 0.

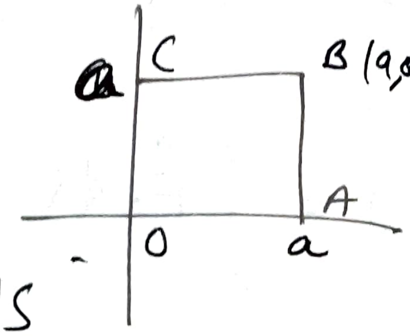
$$I = \int_0^1 (2x+x^2)\,dx + (3x^2-4x)\,2x\,dx + \int_1^0 (2y^2+y^2)\,2y\,dy + (3y-4y^2)\,dy$$

$$= \int_0^1 (6x^3-7x^2+2x)\,dx + \int_1^0 (6y^3-4y^2+3y)\,dy$$

$$= \left[\frac{6}{4}x^4 - \frac{7}{3}x^3 + x^2\right]_0^1 + \left[\frac{6}{4}y^4 - \frac{4}{3}y^3 + \frac{3}{2}y^2\right]_1^0$$

$$= \left(\cancel{\frac{3}{2}} - \frac{7}{3} + 1\right) + \left(0 - \cancel{\frac{3}{2}} + \frac{4}{3} - \frac{3}{2}\right) = \frac{-3}{2}.$$

Q15. $\vec{F} = x^2 \hat{i} + xy \hat{j}$ $\qquad z=0$



Stokes Theorem:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } F)\cdot \hat{n}\, dS$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2 dx + xy\, dy)$$

$$= \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}$$

Along OA : $y=0$, $dy=0$, $x: 0$ to a

AB : $x=a$, $dx=0$, $y=0$ to a

BC : $y=a$, $dy=0$, $x=a$, to 0

CO : $x=0$, $dx=0$, $y=a$ to 0.

$$I = \int_0^a x^2 dx + \int_0^a a\cdot y\, dy + \int_a^0 x^2 dx + \int_a^0 0\cdot dy$$

$$= \left.\frac{x^3}{3}\right|_0^a + a\cdot\left.\frac{y^2}{2}\right|_0^a + \left.\frac{x^3}{3}\right|_a^0$$

$$= \cancel{\frac{a^3}{3}} + \frac{a^3}{2} + \left(0 - \cancel{\frac{a^3}{3}}\right) = \frac{a^3}{2}.$$

1

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$

$$= i(0-0) + j(0-0) + k(y-0)$$

$$= y \cdot \hat{k}$$

Normal vector to given surface

$\hat{n} = k$ (plane $z = 0$ given)

$$(\text{curl } \vec{F}) \cdot \hat{n} = y.$$

$$dS = \frac{dx\,dy}{|\hat{n} \cdot k|} = dx\,dy$$

$$\therefore \iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, dS = \int_0^a \int_0^a y\, dy\, dx$$

$$= \int_0^a \left. \frac{y^2}{2} \right|_0^a dx$$

$$= \int_0^a \frac{a^2}{2}\, dx = \frac{a^2}{2} [x]_0^a = \frac{a^3}{2}. \quad (2)$$

Hence from (1) and (2)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, dS.$$

Stoke's Theorem is verified.

Q16. $\vec{F} = (y^2+z^2-x^2)\hat{i} + (z^2+x^2-y^2)\hat{j} + (x^2+y^2-z^2)k$

S: $x^2+y^2-2ax+az=0$

$(x-a)^2+y^2 = -a(z-a)$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}$$

$= i(2y-2z) + j(2z-2x) + k(2x-2y)$

S: $f = x^2+y^2-2ax+az=0$

$\nabla f = (2x-2a)\hat{i} + 2y\hat{j} + ak$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2x-2a)\hat{i} + 2y\hat{j} + ak}{\sqrt{(2x-2a)^2+4y^2+a^2}}$$

Q 17. $\vec{F} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)k$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}$$

$$= \hat{i}[-1-(-1)] - j[1-1] + k[\cos y - \cos y]$$

$$= 0$$

As $\boxed{\text{curl } \vec{F} = 0}$ $\Rightarrow$ $\boxed{\vec{F} \text{ is irrotational}}$

Let $\nabla\phi = \vec{F}$

i.e. $\frac{\partial \phi}{\partial x} = \sin y + z \Rightarrow \phi = x.\sin y + zx + f_1(y,z)$

$\frac{\partial \phi}{\partial y} = x\cos y - z \Rightarrow \phi = +x\sin y - yz + f_2(x,z)$

$\frac{\partial \phi}{\partial z} = x - y \Rightarrow \phi = xz - yz + f_3(x,y)$

Hence, $\boxed{\phi(x,y,z) = x.\sin y + zx - yz}$