

## IAS/IFoS MATHEMATICS by K. Venkanna

Set - III.

Duality in Linear Programming (73)

We were introduced to several real life problems that were formulated as L.P. models. Let us pick up one of those problems, say the diet problem with different data namely the following:

A mother wishes her children to obtain certain amounts of nutrients from their breakfast cereals. The children have the choice of eating Krunchies or crispies or a mixture of the two. From their breakfast, they should obtain 1 mg of Vitamin A, 0.05 mg of Vitamin B and 400 calories. One ounce of Krunchies contains 0.01 mg of Vitamin A, 1 mg of Vitamin B and 110 calories. One ounce of crispies contains 0.25 mg of Vitamin A, 0.5 mg of Vitamin B and 120 calories. One ounce of Krunchies cost 4 rupees and 1 ounce of crispies cost 5 rupees.

If we formulate a linear programming model for the above problem, assuming that any of the children eats  $x_1$  ounces of Krunchies and  $x_2$  ounces of crispies, then problem reduces to

Mimimize

$$Z = 4x_1 + 5x_2$$

Subject to

$$0.01x_1 + 0.25x_2 \geq 1$$

$$1x_1 + 0.5x_2 \geq 0.5$$

$$110w_1 + 120w_2 \geq 400$$

$$w_1, w_2 \geq 0.$$

Let us consider the same problem from a different angle.

Consider a salesman, who sells nutrients in the form of vitamin tablets and calories in the form of chocolate candy. Each milligram of vitamin A tablet costs Rs.  $w_1$ , Vitamin B tablet costs Rs.  $w_2$  and the amount of chocolate candy containing one calorie costs Rs.  $w_3$ .

To replace Krunchies, mother has to spend  $.01w_1 + 1w_2 + 110w_3$  for which this amount should be less than 4 cents.

Similarly, to replace Crispies mother has to spend  $.25w_1 + .5w_2 + 120w_3$  for which amount is less than 5 cents.

On the other hand the salesman tries to maximize his revenue which is the total cost requirement of the mother and again the mathematical formulation of the problem is given by

Maximize

$$Z = w_1 + .5w_2 + 400w_3$$

Sub. to

$$0.01w_1 + w_2 + 110w_3 \leq 4$$

$$0.25w_1 + 0.5w_2 + 120w_3 \leq 5$$

$$w_1, w_2, w_3 \geq 0.$$

These two problems have different mathematical formulations, one being a minimization problem whereas other being a maximization problem. Although they are expressed in terms of same basic data with different arrangements.

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### (1) DUALITY CONCEPT

One of the most interesting concepts in linear programming is the duality theory.

Every linear programming problem has associated with it, another L.P.P. revolving the same data and closely related optimal solutions. Such two problems are said to be duals of each other. While one of these is called the primal, the other the dual.

The importance of the duality concept is due to two main reasons. Firstly, if the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it into the dual problem and then solving it.

Secondly, the interpretation of the dual variables from the cost or economic point of view proves extremely useful in making future decisions for the

activities being programmed.

The notion of duality in linear programming was first introduced by von-Neumann and was later explicitly given by Gale, Kuhn and Tucker.

## ② formulation of dual problem:

Consider the following L.P.P.:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

sub. to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

$$\dots \dots \dots \dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1, x_2, \dots, x_n \geq 0.$$

To construct the dual problem, we adopt the following guidelines:

- (i) The maximization problem in the primal becomes the minimization problem in the dual and vice versa.
- (ii) ( $\leq$ ) type of constraints in the primal become ( $\geq$ ) type of constraints in the dual and vice versa
- (iii) The coefficients  $c_1, c_2, \dots, c_n$  in the objective function of the primal become  $b_1, b_2, \dots, b_m$  in the objective function of the dual and vice versa
- (iv) If the primal has 'n' variables and 'm' constraints,

the dual will have 'm' variables and 'n' constraints, i.e. the transpose of the Lody matrix of the primal problem gives the Lody matrix of the dual.

(v) The variables in both the primal and

dual are non-negative.

Then the dual problem will be

Minimize

$$N = b_1 w_1 + b_2 w_2 + \dots + b_m w_m$$

Sub. to constraints

$$a_{11} w_1 + a_{12} w_2 + \dots + a_{1n} w_n \geq c_1$$

$$a_{21} w_1 + a_{22} w_2 + \dots + a_{2n} w_n \geq c_2$$

$$\dots$$

$$a_{m1} w_1 + a_{m2} w_2 + \dots + a_{mn} w_n \geq c_n$$

$$w_1, w_2, \dots, w_m \geq 0$$

The primal-dual relationships can be conveniently displayed as below:

	Primal variables			
Dual variables	$w_1$	$w_2$	$w_m$	R.H.S of primal constraints.
	$a_{11}$	$a_{12}$	$\dots$	$b_1$
	$a_{21}$	$a_{22}$	$\dots$	$b_2$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$a_{m1}$	$a_{m2}$	$\dots$	$b_m$
	$c_1$	$c_2$	$\dots$	$c_n$
	R.H.S of the dual constraints.			

The information regarding the primal-dual objective, type of constraints and the sign of dual variables may be summarized in the following table:

Standard primal objective	Dual		
	Objective function type	constraints type	variable sign
Maximization	Minimization	$\geq$	unrestricted
Minimization	Maximization	$\leq$	unrestricted

Note!— All primal constraints are equations with non-negative RHS and all primal variables are non-negative.

problems

→ Write the dual of the following L.P.P.

$$\text{MAX } Z = x_1 + 2x_2$$

sub. to

$$2x_1 - 3x_2 \leq 3$$

$$4x_1 + x_2 \leq -4$$

$$x_1, x_2 \geq 0$$

Sol Dual to this L.P.P. is

$$\text{MIN } Z' = 3y_1 - 4y_2$$

sub. to

$$2y_1 + 4y_2 \geq 1$$

$$-3y_1 + y_2 \geq 2$$

$$y_1, y_2 \geq 0$$

→ Write the dual of the following L.P.P

MINIMIZE

$$Z = 3x_1 - 2x_2 + 4x_3$$

sub. to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2; \quad x_1, x_2, x_3 \geq 0$$

Since the problem is of minimization type,  
all constraints should be of ( $\geq$ ) type.  
we multiply 4th constraint throughout  
by -1, so that  $-7x_1 + 2x_2 + x_3 \geq -10$ .

Let  $y_1, y_2, y_3, y_4$  and  $y_5$  be the dual variables  
associated with the above five constraints.

then the dual problem is given by  
Maximize  $\text{D}_{\text{prob}}$

$$W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5$$

Sub. to

$$3y_1 + 6y_2 - 7y_3 + 4y_4 + 4y_5 \leq 3$$

$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$

$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

$\rightarrow$  write dual for the following L.P. problems:

\*  $\text{Max}$   
 $Z = 5x_1 + 3x_2$

Sub. to

$$3x_1 + 5x_2 \leq 5$$

$$x_1 + 3x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

\*  $\text{Max}$   
 $Z = 2x_1 + 3x_2 + 5x_3$   
Sub. to:  
 $5x_1 + 6x_2 - x_3 \leq 3$   
 $-2x_1 + x_2 + 2x_3 \leq 2$   
 $x_1 + 5x_2 - 3x_3 \leq 1$   
 $-3x_1 + 3x_2 - 7x_3 \leq 6$   
 $x_1, x_2, x_3 \geq 0$ .

(3) formulation of dual problem when the primal has equality constraints:

Consider the problem

$$\text{Max}$$
  
 $Z = c_1x_1 + c_2x_2$

Sub. to.

$$a_{11}x_1 + a_{12}x_2 = b_1,$$

$$a_{21}x_1 + a_{22}x_2 = b_2,$$

$$x_1, x_2 \geq 0.$$

Sol The equality constraint can be written as

$$a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 \geq b_1$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 \leq b_1 \text{ and } -a_{11}x_1 - a_{12}x_2 \leq -b_1$$

NOW we restate the problem

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MAX

$$Z = c_1 x_1 + c_2 x_2$$

Sub. to.

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$

$$-a_{11}x_1 - a_{12}x_2 \leq -b_1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

Now, this primal is in the proper form i.e a maximization problem subject to all constraints ' $\leq$ ' type

$\therefore$  The dual problem is

Minimize

$$W = b_1 y_1 - b_2 y_2 + b_2 y_3$$

Sub. to

$$a_{11}y_1 + a_{12}y_2 + a_{21}y_3 \geq c_1$$

$$a_{12}y_1 - a_{12}y_2 + a_{22}y_3 \geq c_2$$

$$y_1, y_2, y_3 \geq 0$$

This can be written as

Minimize

$$W = b_1(y_1 - y_2) + b_2 y_3$$

Sub. to

$$a_{11}(y_1 - y_2) + a_{21}y_3 \geq c_1$$

$$a_{12}(y_1 - y_2) + a_{22}y_3 \geq c_2$$

$$y_1, y_2, y_3 \geq 0$$

The term  $(y_1 - y_2)$  appears in both objective function and all the constraints of the dual. This will always happen whenever there is an equality constraint in the primal. Then the new variable  $(y_1 - y_2) = u_1$  becomes unrestricted in sign being the

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difference of two non-negative variables  
 $\therefore$  the above dual problem takes the form.

$$\min W = b_1 u_1 + b_2 y_2$$

Sub. to

$$a_{11} u_1 + a_{21} y_2 \geq c_1$$

$$a_{12} u_1 + a_{22} y_2 \geq c_2$$

$u_1$  unrestricted in sign,

$$y_2 \geq 0.$$

construct the dual of the L.P.P.

$$\max Z = 4x_1 + 9x_2 + 2x_3$$

subject to

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 5$$

$$x_1, x_2, x_3 \geq 0.$$

$\frac{1989}{1989}$  → Obtain the dual problem of the following L.P.P.

$$\max Z = x_1 - 2x_2 + 3x_3$$

Sub. to

$$-2x_1 + x_2 + 3x_3 = 2$$

$$2x_1 + 3x_2 + 4x_3 = 1,$$

$$x_1, x_2, x_3 \geq 0.$$

→  $\max$

$$Z = 3x_1 + 5x_2 + 6x_3$$

Sub. to

Primal  $\begin{cases} x_1 + x_2 \leq 5 \\ 2x_1 + x_2 - 2x_3 \leq 3 \\ 7x_2 + 2x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases}$

→  $\max$

$$Z = 3x_1 - 2x_2 + 4x_3 + 2x_4$$

Sub. to

$$\begin{cases} 2x_1 + x_2 + 4x_3 + 2x_4 = 2 \\ 4x_1 - 2x_2 + x_3 - 2x_4 \leq 7 \\ 3x_1 + x_2 + 2x_3 + 2x_4 \leq 3 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

(4) Dual of a problem with Unrestricted variables:

Let us now consider a case in which there is no restriction on the variables i.e. when the variables involved in the problem may or may not be non-negative.

for example:

MAX

$$Z = 3x_1 + 4x_2$$

Sub. to

primal 
$$\begin{cases} 4x_1 + 2x_2 \leq 7 \\ x_1 + 3x_2 \leq 5 \\ x_1, x_2 \text{ unrestricted} \end{cases}$$

$\Rightarrow$  put  $x_1 = x_1' - x_1''$

$$x_2 = x_2' - x_2''$$

so that  $x_1', x_1'', x_2', x_2'' \geq 0$ .

and the primal can be written as

MAXIMIZE

$$Z = 3x_1' - 3x_1'' + 4x_2' - 4x_2''$$

Sub. to.

$$4x_1' - 4x_1'' + 2x_2' - 2x_2'' \leq 7$$

$$x_1' - x_1'' + 3x_2' - 3x_2'' \leq 5.$$

$$x_1', x_1'', x_2', x_2'' \geq 0$$

i.e. Dual to this L.P.P.

MINIMIZE

$$N = 7y_1 + 5y_2$$

Sub. to

$$4y_1 + y_2 \geq 3 \quad |$$

$$-4y_1 - y_2 \geq -3 \quad |$$

$$2y_1 + 3y_2 \geq 4 \quad |$$

$$-2y_1 - 3y_2 \geq -4 \quad |$$

$$y_1, y_2 \geq 0.$$

This can be written as

$$\text{min } N = 7y_1 + 5y_2$$

$$\text{sub. to } 4y_1 + y_2 \geq 3$$

$$4y_1 + y_2 \leq 3$$

$$2y_1 + 3y_2 \geq 4$$

$$2y_1 + 3y_2 \leq 4$$

$$y_1, y_2 \geq 0.$$

$\Rightarrow$

$$\text{min } N = 7y_1 + 5y_2$$

$$\text{sub. to } 4y_1 + y_2 = 3$$

$$2y_1 + 3y_2 = 4$$

$$y_1, y_2 \geq 0.$$

which is the reqd dual form of  
given primal L.P.P.

\* Write the dual of the following:

Hence MAX

$$Z = 2x_1 + 3x_2 + 4x_3$$

sub. to

$$\left\{ \begin{array}{l} x_1 - 5x_2 + 2x_3 \leq 7 \\ 2x_1 - 3x_2 + x_3 \leq 3 \end{array} \right.$$

$$\left. \begin{array}{l} x_1 + 2x_2 - 4x_3 \leq 2 \\ x_1, x_2 \geq 0, x_3 \text{ unrestricted} \end{array} \right.$$

→ Max

$$Z = 3x_1 + 6x_2 + 7x_3$$

subject to

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 \leq 11 \\ 7x_1 + 2x_2 + 4x_3 \leq 7 \\ x_1 + 5x_2 + 4x_3 \leq 5 \\ x_1, x_2 \text{ unrestricted}, x_3 \geq 0 \end{cases}$$

→ Maximize

$$Z = 6x_1 - 5x_2 - 7x_3$$

Sub. to

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 \leq 3 \\ 7x_1 + 2x_2 + 4x_3 \leq 4 \\ 3x_1 + 5x_2 + 4x_3 \leq 2 \\ x_1 \geq 0, \\ x_2, x_3 \text{ unrestricted} \end{cases}$$

⑤ Dual of a problem with equality, Inequality constraints and Unrestricted variables :

for example:

$$\text{Max } Z = 8x_1 + 6x_2 + 7x_3 + 3x_4 - 2x_5$$

Sub. to

$$4x_1 + 2x_2 + 5x_3 + 6x_4 + 2x_5 \stackrel{\geq}{=} 11$$

$$4x_1 + 3x_2 - 2x_3 - 5x_4 + 3x_5 \leq 13$$

$$x_1, x_2, x_3 \geq 0,$$

$x_4$  and  $x_5$  unrestricted.

→ Write the dual of the following linear programming in a form such that dual variables are all non-negative.

Maximize

$$Z = 3x_1 + 2x_2 - 4x_3 + x_4$$

Sub. to

$$6x_1 + 4x_2 - 3x_3 + 2x_4 \leq 22$$

$$4x_1 - 3x_2 + 2x_3 + x_4 \leq 3$$

$x_1, x_2$  unrestricted,  $x_3, x_4 \geq 0$ .

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$$\underline{x_1} \quad \text{put } x_1 = x_1' - x_1'' \\ x_2 = x_2' - x_2''$$

We write the first equality constraint in the form of two inequalities of ( $\leq$ ) type.

we may rewrite the primal

$$\text{Max } Z = 3x_1' - 3x_1'' + 2x_2' - 2x_2'' - 4x_3 + x_4$$

Sub. to

$$6x_1' - 6x_1'' + 4x_2' - 4x_2'' - 3x_3 + 2x_4 \leq 2$$

$$-6x_1' + 6x_1'' - 4x_2' + 4x_2'' + 3x_3 - 2x_4 \leq -2$$

$$4x_1' - 4x_1'' - 3x_2' + 3x_2'' + 2x_3 + x_4 \leq 3$$

$$x_1', x_1'', x_2', x_2'', x_3, x_4 \geq 0$$

Therefore, dual is given by

$$\text{Minimize } W = 2y_1 - 2y_2 + 3y_3$$

Sub. to

$$6y_1 - 6y_2 + 4y_3 \geq 3$$

$$-6y_1 + 6y_2 - 4y_3 \geq -3$$

$$4y_1 - 4y_2 - 3y_3 \geq 2$$

$$-4y_1 + 4y_2 - 3y_3 \geq -2$$

$$-3y_1 + 2y_2 + 2y_3 \geq -4$$

$$2y_1 - 2y_2 + y_3 \geq 1$$

$$y_1, y_2, y_3 \geq 0$$

(OR)

$$\text{Min } W = 2y_1 - 2y_2 + 3y_3$$

Sub. to

$$6y_1 - 6y_2 + 4y_3 = 3, \quad 4y_1 - 4y_2 - 3y_3 = 2,$$

$$-3y_1 + 2y_2 + 2y_3 \geq -4, \quad 2y_1 - 2y_2 + y_3 \geq 1$$

$$\underline{y_1}, \underline{y_2}, \underline{y_3} \geq 0$$

→ Write the dual of the following primal

$$\text{Max } Z = 2x_1 - x_2 + 2x_3$$

Sub. to

$$4x_1 + 2x_2 + x_3 \leq 7$$

$$2x_1 - x_2 + 2x_3 \geq 8$$

$x_1$  unrestricted;  $x_2, x_3 \geq 0$

→ Write the dual of the following problem in a form such that dual variables are all non-negative

Max

$$Z = 6x_1 + 4x_2 + x_3 + 7x_4 + 5x_5$$

Sub. to

$$3x_1 + 7x_2 + 8x_3 + 5x_4 + x_5 = 2$$

$$2x_1 + x_2 + 3x_3 + 2x_4 + 9x_5 = 6$$

$x_1, x_2, x_3, x_4 \geq 0$ ;

$x_5$  unrestricted.

### ⑥ Dual of a problem in a standard form:

Let us now consider a general L.P.P in its standard form:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to.

$$\begin{aligned} \text{primal} \quad & \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ x_1, x_2, x_3, \dots, x_n \geq 0 \end{array} \right. \end{aligned}$$

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All constraints in the primal are equality constraints, therefore, all dual variables should be unrestricted.

Now we may write the dual as

Maximize

$$Z = b_1 y_1 + b_2 y_2 + b_3 y_3 + \dots + b_m y_m$$

Sub. to.

$$a_{11}y_1 + a_{12}y_2 + \dots + a_{1m}y_m \geq c_1$$

$$a_{21}y_1 + a_{22}y_2 + \dots + a_{2m}y_m \geq c_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{m1}y_1 + a_{m2}y_2 + \dots + a_{mm}y_m \geq c_m$$

$y_1, y_2, \dots, y_m$  are all unrestricted.

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Given the programme

$$\text{Max } u = 5x + 3y$$

Sub. to

$$x + 3y \leq 12$$

$$3x - 4y \leq 9$$

$$7x + 8y \leq 20$$

$$\text{and } x, y \geq 0.$$

Write its dual in the standard form.



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Find the dual of the following LPP

$$\text{Max. } Z = 2x_1 - x_2 + x_3$$

such that

$$x_1 + x_2 - 3x_3 \leq 8$$

$$4x_1 - x_2 + x_3 = 2$$

$$2x_1 + 3x_2 - x_3 \geq 5$$

$$x_1, x_2, x_3 \geq 0$$

→ Dual of the dual is the primal. (81)

For example:

$$\text{Max } Z = 6x_1 - 2x_2 + x_3$$

Sub. to

$$4x_1 - 2x_2 - 3x_3 \leq 7$$

$$2x_1 + x_2 - 4x_3 \geq 5$$

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0.$$

Sol. First, we shall write the given primal L.P.P. in canonical form.

$$\text{Max } Z = 6x_1 - 2x_2 + x_3$$

Sub. to

$$4x_1 - 2x_2 - 3x_3 \leq 7$$

$$-2x_1 - x_2 + 4x_3 \leq -5$$

$$x_1 + x_2 + x_3 \leq 1$$

$$-x_1 - x_2 - x_3 \leq -1; x_1, x_2, x_3 \geq 0.$$

Dual to this L.P.P. is

$$\text{Min } W = 7y_1 - 5y_2 + 4y_3 - y_4$$

Sub. to

$$4y_1 - 2y_2 + 4y_3 - y_4 \geq 6$$

$$-2y_1 - y_2 + 4y_3 - y_4 \geq -2$$

$$-3y_1 + 4y_2 + 4y_3 - y_4 \geq 1$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

Let us now write this dual in a canonical form, i.e. a minimization type with ' $\leq$ ' type of constraints.

∴ we have  $\text{Max } -W = -7y_1 + 5y_2 - y_3 + y_4$

Sub. to

$$-4y_1 + 2y_2 - y_3 + y_4 \leq -6$$

$$-2y_1 + y_2 - y_3 + y_4 \leq 2$$

$$3y_1 - 4y_2 - y_3 + y_4 \leq -1$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

Dual to this L.P.P is

-Maximize

$$-P = -6w_1 + 2w_2 + w_3$$

Sub. to

$$-4w_1 + w_2 + 3w_3 \geq -7$$

$$2w_1 + w_2 - 4w_3 \geq 5$$

$$-w_1 - w_2 - w_3 \geq -1$$

$$w_1 + w_2 + w_3 \geq 1$$

$$w_1, w_2, w_3 \geq 0$$

We can rewrite this L.P.P as

Maximize

$$P = 6w_1 - 2w_2 + w_3$$

$$\text{Sub. to } 4w_1 - 2w_2 - 3w_3 \leq 7$$

$$2w_1 + w_2 - 4w_3 \geq 5$$

$$w_1 + w_2 + w_3 = 1$$

$$w_1, w_2, w_3 \geq 0$$

which is precisely the given primal

HW verify that dual of the dual is the  
is the primal for the following L.P.P

Maximize

$$Z = 4x_1 + 3x_2$$

Sub. to.

$$5x_1 + 6x_2 = -7$$

$$x_1 + 6x_2 \leq 3$$

$$2x_1 - x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

**32.12. (1) DUALITY PRINCIPLE**

If the primal and the dual problems have feasible solutions then both have optimal solutions and the optimal value of the primal objective function is equal to the optimal value of the dual objective function i.e.

$$\text{Max. } Z = \text{Min. } W$$

This is the fundamental theorem of duality. It suggests that an optimal solution to the primal problem can directly be obtained from that of the dual problem and vice-versa.

**(2) Working rules for obtaining an optimal solution to the primal (dual) problem from that of the dual (primal) :**

Suppose we have already found an optimal solution to the dual (primal) problem by simplex method.

**Rule I.** If the primal variable corresponds to a slack starting variable in the dual problem, then its optimal value is directly given by the coefficient of the slack variable with changed sign, in the  $C_j$  row of the optimal dual simplex table and vice-versa.

**Rule II.** If the primal variable corresponds to an artificial variable in the dual problem, then its optimal value is directly given by the coefficient of the artificial variable, with changed sign, in the  $C_j$  row of the optimal dual simplex table, after deleting the constant  $M$  and vice-versa.

On the other hand, if the primal has an unbounded solution, then the dual problem will not have a feasible solution and vice-versa.

Now we shall workout two examples to demonstrate the primal dual relationships.

**Example 32.24.** Construct the dual of the following problem and solve both the primal and the dual :

$$\text{Maximize } Z = 2x_1 + x_2,$$

$$\text{subject to } -x_1 + 2x_2 \leq 2, x_1 + x_2 \leq 4, x_1 \leq 3, x_1, x_2 \geq 0.$$

**Solution using the primal problem.** Since only two variables are involved, it is convenient to solve the problem graphically.

In the  $x_1, x_2$ -plane, the five constraints show that the point  $(x_1, x_2)$  lies within the shaded region  $OABCD$  of Fig. 32.12. Values of the objective function  $Z = 2x_1 + x_2$  at these corners are  $Z(0) = 0$ ,  $Z(A) = 6$ ,  $Z(B) = 7$ ,  $Z(C) = 6$  and  $Z(D) = 1$ . Hence the optimal solution is  $x_1 = 3$ ,  $x_2 = 1$  and max.  $(Z) = 7$ .

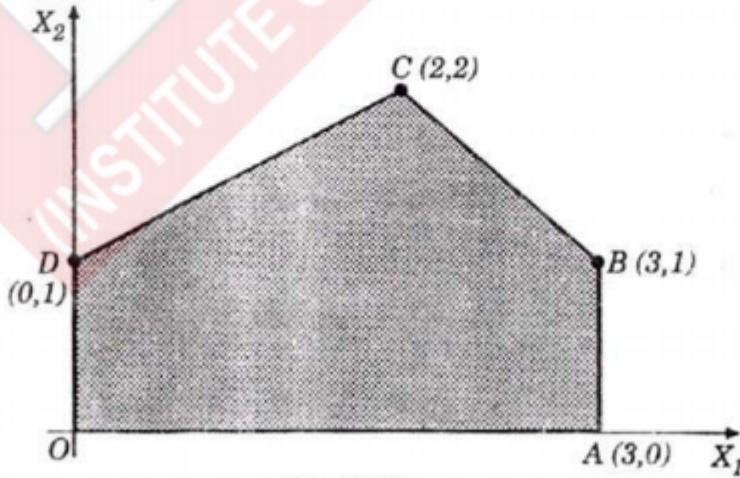


Fig. 32.12.

**Solution using the dual problem.** The dual problem of the given primal is :

$$\text{Minimize } W = 2y_1 + 4y_2 + 3y_3$$

$$\text{subject to } -y_1 + y_2 + y_3 \geq 2, 2y_1 + y_2 \geq 1, y_1, y_2 \geq 0.$$

*Step 1. Express the problem in the standard form.*

Introducing the slack and the artificial variables, the dual problem in the standard form is

$$\text{Max. } W' = -2y_1 - 4y_2 - 3y_3 + 0s_1 + 0s_2 - MA_1 - MA_2 \quad .$$

$$\text{subject to } -y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 + 0A_2 = 2,$$

$$2y_1 + y_2 + 0y_3 + 0s_1 - s_2 + 0A_1 + A_2 = 1$$

*Step 2. Find an initial basic feasible solution.*

Setting the non-basic variables  $y_1, y_2, y_3, s_1, s_2, A_1, A_2$ , each equal to zero, we get the initial basic feasible solution as

$$y_1 = y_2 = y_3 = s_1 = s_2 = 0 \text{ (non-basic)} ; A_1 = 2, A_2 = 1. \quad (\text{basic})$$

∴ Initial simplex table is

	$c_j$	-2	-4	-3	0	0	-M	-M		
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	0
-M	$A_1$	-1	1	1	-1	0	1	0	2	2/1
-M	$A_2$	2	(1)	0	0	-1	0	1	1	1/1←
	$Z_j$	-M	-2M	-M	M	M	-M	-M	-3M	
	$C_j$	$M - 2$	$2M - 4$	$M - 3$	$-M$	$-M$	0	0		

As  $C_j$  is positive under some columns, the initial solution is not optimal.

*Step 3. Iterate towards an optimal solution.*

(i) Introduce  $y_2$  and drop  $A_2$ . Then the new simplex table is

	$c_j$	-2	-4	-3	0	0	-M	-M		
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	0
-M	$A_1$	-3	0	(1)	-1	1	1	-1	1	1/1←
-4	$y_2$	2	1	0	0	-1	0	1	1	1/0
	$Z_j$	$3M - 8$	-4	-M	M	$4 - M$	-M	$M - 4$	$-M - 4$	
	$C_j$	$6 - 3M$	0	$M - 3$	$-M$	$M - 4$	0	$4 - 2M$		

As  $C_j$  is positive under some columns, this solution is not optimal.

(ii) Now introduce  $y_3$  and drop  $A_1$ . Then the revised simplex table is

	$c_j$	-2	-4	-3	0	0	-M	-M		
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$A_1$	$A_2$	$b$	
-3	$y_3$	-3	0	1	-1	1	1	-1	1	
-4	$y_2$	2	1	0	0	-1	0	1	1	1
	$Z_j$	1	-4	-3	3	1	-3	-1	-7	
	$C_j$	-3	0	0	-3	-1	$3 - M$	$1 - M$		

As all  $C_j \leq 0$ , the optimal solution is attained.

Thus an optimal solution to the dual problem is

$$y_1 = 0, y_2 = 1, y_3 = 1, \text{ Min. } W = -\text{Max. } (W') = 7.$$

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To derive the optimal basic feasible solution to the primal problem, we note that the primal variables  $x_1, x_2$  correspond to the artificial starting dual variables  $A_1, A_2$  respectively. In the final simplex table of the dual problem,  $C_j$  corresponding to  $A_1$ , and  $A_2$  are 3 and 1 respectively after ignoring  $M$ . Thus by rule II, we get opt.  $x_1 = 3$  and opt.  $x_2 = 1$ .

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 3, x_2 = 1; \text{ max. } Z = 7.$$

**Obs.** The validity of the duality theorem is therefore, checked since  $\text{max. } Z = \text{min. } W = 7$  from both the methods.

**Example 32.25.** Using duality solve the following problem :

$$\text{Minimize } Z = 0.7x_1 + 0.5x_2$$

$$\text{subject to } x_1 \geq 4, x_2 \geq 6, x_1 + 2x_2 \geq 20, 2x_1 + x_2 \geq 18, x_1, x_2 \geq 0.$$

(V.T.U., 2004)

**Sol.** The dual of the given problem is Max.  $W = 4y_1 + 6y_2 + 20y_3 + 18y_4$ ,

$$\text{subject to } y_1 + y_3 + 2y_4 \leq 0.7, y_2 + 2y_3 + y_4 \leq 0.5, y_1, y_2, y_3, y_4 \geq 0.$$

**Step 1.** Express the problem in the standard form.

Introducing slack variables, the dual problem in the standard form becomes

$$\text{Max. } W = 4y_1 + 6y_2 + 20y_3 + 18y_4 + 0s_1 + 0s_2,$$

$$\text{subject to } y_1 + 0y_2 + y_3 + 2y_4 + s_1 + 0s_2 = 0.7,$$

$$0y_1 + y_2 + 2y_3 + y_4 + 0s_1 + s_2 = 0.5, y_1, y_2, y_3, y_4, s_1, s_2 \geq 0.$$

**Step 2.** Find an initial basic feasible solution.

Setting non-basic variables  $y_1, y_2, y_3, y_4$  each equal to zero, the basic solution is

$$y_1 = y_2 = y_3 = y_4 = 0 \quad (\text{non-basic}) ; s_1 = 0.7, s_2 = 0.5 \quad (\text{basic})$$

Since the basic variables  $s_1, s_2 > 0$ , the initial basic solution is feasible and non-degenerate.

Initial simplex table is

	$c_j$	4	6	20	18	0	0	$b$	0
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$		
0	$s_1$	1	0	1	2	1	0	0.7	0.7/1
0	$s_2$	0	1	(2)	1	0	1	0.5	0.5/2 ←
	$Z_j$	0	0	0	0	0	0	0	
	$C_j$	4	6	20	18	0	0		

As  $C_j$  is positive in some columns, the initial basic solution is not optimal.

**Step 3.** Iterate towards an optimal solution.

(i) Introduce  $y_3$  and drop  $s_2$ . Then the new simplex table is

	$c_j$	4	6	20	18	0	0	$b$	0
$c_B$	Basis	$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$		
0	$s_1$	1	-1/2	0	(3/2)	1	-1/2	9/20	3/10 ←
20	$y_3$	0	1/2	1	$\frac{1}{2}$	0	1/2	1/4	1/2
	$Z_j$	0	10	20	10	0	10	5	
	$C_j$	4	-4	0	8	0	-10		

As  $C_j$  is positive under some of the columns, this solution is not optimal.

(ii) Introduce  $y_4$  and drop  $s_1$ . Then the revised simplex table is

$c_B$	$Basis$	$4$	$6$	$20$	$18$	$0$	$0$	$b$
$c_j$		$y_1$	$y_2$	$y_3$	$y_4$	$s_1$	$s_2$	
18	$y_4$	2/3	-1/3	0	1	2/3	-1/3	3/10
20	$y_3$	-1/3	2/3	1	0	-1/3	2/3	1/10
	$Z_j$	16/3	22/3	20	18	16/3	22/3	74/10
		-4/3	-4/3	0	0	-16/3	-22/3	

As all  $C_j \leq 0$ , this table gives the optimal solution.

Thus the optimal basic feasible solution is  $y_1 = 0, y_2 = 0, y_3 = 20, y_4 = 18$ ; max.  $W = 7.4$

Step 4. Derive optimal solution to the primal.

We note that the primal variables  $x_1, x_2$  correspond to the slack starting dual variables  $s_1, s_2$  respectively. In the final simplex table of the dual problem,  $C_j$  values corresponding to  $s_1$  and  $s_2$  are  $-16/3$  and  $-22/3$  respectively.

Thus, by rule I, we conclude that opt.  $x_1 = 16/3$  and opt.  $x_2 = 22/3$ .

Hence an optimal basic feasible solution to the given primal is

$$x_1 = 16/3, x_2 = 22/3; \text{ min. } Z = 7.4.$$

**Obs.** To check the validity of the duality theorem, the student is advised to solve the given L.P.P. directly by simplex method and see that min.  $Z = \text{max. } W = 7.4$ .

### Problems 32.7

Using duality solve the following problems (1 – 4) :

- Minimize  $Z = 2x_1 + 9x_2 + x_3$ ,  
subject to  $x_1 + 4x_2 + 2x_3 \geq 5, 3x_1 + x_2 + 2x_3 \geq 4$  and  $x_1, x_2 \geq 0$ .
- Maximize  $Z = 2x_1 + x_2$ ,  
subject to  $x_1 + 2x_2 \leq 10, x_1 + x_2 \leq 6, x_1 - x_2 \leq 2, x_1 - 2x_2 \leq 1, x_1, x_2 \geq 0$ .
- Maximize  $Z = 3x_1 + 2x_2$ ,  
subject to  $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \leq 10, x_2 \leq 3, x_1, x_2 \geq 0$ . (Madras, 1996)
- Maximize  $Z = 3x_1 + 2x_2 + 5x_3$   
subject to  $x_1 + 2x_2 + x_3 \leq 430, 3x_1 + 2x_3 \leq 460, x_1 + 4x_2 \leq 420, x_1, x_2, x_3 \geq 0$ .

### 32.13. (1) DUAL SIMPLEX METHOD

we have seen that a set of basic variables giving a feasible solution can be found by introducing artificial variables and using  $M$ -method or Two phase method. Using the primal-dual relationships for a problem, we have another method (known as *Dual simplex method*) for finding an initial feasible solution. Whereas the regular simplex method starts with a basic feasible (but non-optimal) solution and works towards optimality, the dual simplex method starts with a basic unfeasible (but optimal) solution and works towards feasibility. The dual simplex method is quite similar to the regular simplex method, the only difference lies in

the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable while in the case of regular simplex method reverse is done.

**(2) Working procedure for dual simplex method :**

*Step 1. (i) Convert the problem to maximization form, if it is not so.*

*(ii) Convert ( $\geq$ ) type constraints, if any to ( $\leq$ ) type by multiplying such constraints by -1.*

*(iii) Express the problem in standard form by introducing slack variables.*

*Step 2. Find the initial basic solution and express this information in the form of dual simplex table.*

*Step 3. Test the nature of  $C_j = c_j - Z_j$ :*

*(a) If all  $C_j \leq 0$  and all  $b_i \geq 0$ , then optimal basic feasible solution has been attained.*

*(b) If all  $C_j \leq 0$  and at least one  $b_i < 0$ , then go to step 4.*

*(c) If any  $C_j \geq 0$ , the method fails.*

*Step 4. Mark the outgoing variable.* Select the row that contains the most negative  $b_i$ . This will be the key row and the corresponding basic variable is the outgoing variable.

*Step 5. Test the nature of key row elements :*

*(a) If all these elements are  $\geq 0$ , the problem does not have a feasible solution.*

*(b) If at least one element  $< 0$ , find the ratios of the corresponding elements of  $C_j$ -row to these elements. Choose the smallest of these ratios. The corresponding column is the key column and the associated variable is the incoming variable.*

*Step 6. Iterate towards optimal feasible solution.* Make the key element unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

**Example 32.26. Using dual simplex method :**

maximize  $-3x_1 - 2x_2$ ,

subject to  $x_1 + x_2 \geq 1, x_1 + x_2 \leq 7, x_1 + 2x_2 \geq 10, x_2 \geq 3, x_1 \geq 0, x_2 \geq 0$ .

**Solution** consists of the following steps :

*Step 1. (i) Convert the first and third constraints into ( $\leq$ ) type. These constraints become*

$$-x_1 - x_2 \leq -1, -x_1 - 2x_2 \leq -10.$$

*(ii) Express the problem in standard form*

Introducing slack variables  $s_1, s_2, s_3, s_4$  the given problem takes the form

$$\text{Max. } Z = -3x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$\text{subject to } -x_1 - x_2 + s_1 = -1, x_1 + x_2 + s_2 = 7, -x_1 - 2x_2 + s_3 = -10,$$

$$x_2 + s_4 = 3, x_1, x_2, s_1, s_2, s_3, s_4 \geq 0.$$

*Step 2. Find the initial basic solution*

Setting the decision variables  $x_1, x_2$  each equal to zero, we get the basic solution

$$x_1 = x_2 = 0, s_1 = -1, s_2 = 7, s_3 = -10, s_4 = 3 \text{ and } Z = 0.$$

∴ Initial solution is given by the table below :

	$c_j$	-3	-2	0	0	0	0	
$c_B$	<i>Basis</i>	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$
0	$s_1$	-1	-1	1	0	0	0	-1
0	$s_2$	1	1	0	1	0	0	7
0	$s_3$	-1	(-2)	0	0	1	0	-10 ←
0	$s_4$	0	1	0	0	0	1	3
$Z_j = \Sigma c_B a_{ij}$		0	0	0	0	0	0	0
$C_j = c_j - Z_j$		-3	-2	0	0	0	0	

*Step 3. Test nature of  $C_j$ .*

Since all  $C_j$  values are  $\leq 0$  and  $b_1 = -1$ ,  $b_3 = -10$ , the initial solution is optimal but infeasible. We therefore, proceed further.

*Step 4. Mark the outgoing variable.*

Since  $b_3$  is negative and numerically largest, the third row is the key row and  $s_3$  is the outgoing variable.

*Step 5. Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key row.*

These ratios are  $-3/-1 = 3$ ,  $-2/-2 = 1$  (neglecting ratios corresponding to +ve or zero elements of key row). Since the smaller ratio is 1, therefore,  $x_2$ -column is the key column and (-2) is the key element.

*Step 6. Iterate towards optimal feasible solution.*

(i) Drop  $s_3$  and introduce  $x_2$  alongwith its associated value -2 under  $c_B$  column. Convert the key element to unity and make all other elements of the key column zero. Then the second solution is given by the table below :

	$c_j$	-3	-2	0	0	0	0	
$c_B$	<i>Basis</i>	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$
0	$s_1$	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	0	4
0	$s_2$	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	2
-2	$x_2$	$\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	0	5
0	$s_4$	$\left(-\frac{1}{2}\right)$	0	0	0	$\frac{1}{2}$	1	-2 ←
$Z_j = \Sigma c_B a_{ij}$		-1	-2	0	0	1	0	-10
$C_j = c_j - Z_j$		-2	0	0	0	-1	0	

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Since all  $C_j$  values are  $\leq 0$  and  $b_4 = -2$ , this solution is optimal but infeasible. We therefore proceed further.

(ii) *Mark the outgoing variable.*

Since  $b_4$  is negative, the fourth row is the key row and  $s_4$  is the outgoing variable.

(iii) *Calculate ratios of elements in  $C_j$ -row to the corresponding negative elements of the key row.*

This ratios is  $-2 / -\frac{1}{2} = 4$  (neglecting other ratios corresponding to +ve or 0 elements of key row).

$\therefore x_1$ -column is the key column and  $\left(-\frac{1}{2}\right)$  is the key element.

(iv) *Drop  $s_4$  and introduce  $x_1$  with its associated value  $-3$  under the  $c_B$  column. Convert the key element to unity and make all other elements of the key column zero. Then the third solution is given by the table below :*

	$c_j$	-3	-2	0	0	0	0	
$c_B$	Basis	$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$
0	$s_1$	0	0	1	0	-1	-1	6
0	$s_2$	0	0	0	1	1	1	0
-2	$x_2$	0	1	0	0	0	1	3
-3	$x_1$	1	0	0	0	-10	-2	4
$Z_j$		-3	-2	0	0	3	4	-18
$C_j$		0	0	0	0	-3	-4	

Since all  $C_j$  values are  $\leq 0$  and all  $b$ 's are  $\geq 0$ , therefore this solution is optimal and feasible. Thus the *optimal solution* is  $x_1 = 4$ ,  $x_2 = 3$  and  $Z_{max} = -18$ .

**Example 32.27.** Using dual simplex method, solve the following problem:

$$\text{Minimize } Z = 2x_1 + 2x_2 + 4x_3$$

$$\text{subject to } 2x_1 + 3x_2 + 5x_3 \geq 2, 3x_1 + x_2 + 7x_3 \leq 3, x_1 + 4x_2 + 6x_3 \leq 5, x_1, x_2, x_3 \geq 0$$

**Solution** consists of the following steps :

*Step 1. (i) Convert the given problem to maximization form by writing*

$$\text{Maximize } Z' = -2x_1 - 2x_2 - 4x_3.$$

*(ii) Convert the first constraint into ( $\leq$ ) type. Thus it is equivalent to*

$$-2x_1 - 3x_2 - 4x_3 \leq -2$$

*(iii) Express the problem in standard form.*

Introducing slack variables,  $s_1, s_2, s_3$ , the given problem becomes

$$\text{Max. } Z' = -2x_1 - 2x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } -2x_1 - 3x_2 - 5x_3 + s_1 + 0s_2 + 0s_3 = -2,$$

$$3x_1 + x_2 + 7x_3 + 0s_1 + s_2 + 0s_3 = 3,$$

$$x_1 + 4x_2 + 6x_3 + 0s_1 + 0s_2 + s_3 = 5,$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

**Step 2. Find the initial basic solution.**

Setting the decision variables  $x_1, x_2, x_3$  each equal to zero, we get the basic solution

$$x_1 = x_2 = x_3 = 0, s_1 = -2, s_2 = 3, s_3 = 5 \text{ and } Z' = 0.$$

∴ Initial solution is given by the table below :

		$c_j$	-2	-2	-4	0	0	0	
$c_B$	Basis	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	
0	$s_1$	-2	(-3)	-5	1	0	0	-2 ←	
0	$s_2$	3	1	7	0	1	0	3	
0	$s_3$	1	4	6	0	0	1	5	
	$Z_j$	0	0	0	0	0	0	0	
	$C_j$	-2	-2 ↑	-4	0	0	0		

**Step 3. Test nature of  $C_j$ .**

Since all  $C_j$  values are  $\leq 0$  and  $b_1 = -2$ , the initial solution is optimal but infeasible.

**Step 4. Mark the outgoing variable.**

Since  $b_1 < 0$ , the first row is the key row and  $s_1$  is the outgoing variable.

**Step 5. Calculate the ratio of elements of  $C_j$ -row to the corresponding negative elements of the key row.**

These ratios are  $-2/-2 = 1, -2/-3 = 0.67, -4/-5 = 0.8$ .

Since 0.67 is the smallest ratio,  $x_2$ -column is the key column and (-3) is the key element.

**Step 6. Iterate towards optimal feasible solution.**

Drop  $s_1$  and introduce  $x_2$  with its associated value -2 under  $c_B$  column. Then the revised dual simplex table is

		$c_j$	-2	-2	-4	0	0	0	
$c_B$	Basis	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	
-2	$x_2$	2/3	1	5/3	-1/3	0	0	2/3	
0	$s_2$	7/3	0	16/3	1/3	1	0	7/3	
0	$s_3$	-5/3	0	-2/3	4/3	0	1	7/3	
	$Z_j$	-4/3	-2	-10/3	2/3	0	0	-4/3	
	$C_j$	-2/3	0	-2/3	-2/3	0	0		

Since all  $C_j \leq 0$  and all  $b_i$ 's are  $> 0$ , this solution is optimal and feasible. Thus the optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 0 \text{ and } \max. Z' = -4/3 \text{ i.e. min. } Z = 4/3.$$

**Problems 32-8**

(86)

Using dual simplex method, solve the following problems :

1. Maximize  $Z = -3x_1 - x_2$   
subject to  $x_1 + x_2 \geq 1, 2x_1 + 3x_2 \geq 2; x_1, x_2 \geq 0.$
2. Minimize  $Z = 2x_1 + x_2,$   
subject to  $3x_1 + x_2 \geq 3, 4x_1 + 3x_2 \geq 6, x_1 + 2x_2 \leq 3, x_1, x_2 \geq 0$
3. Minimize  $Z = x_1 + 2x_2 + 3x_3,$   
subject to  $2x_1 - x_2 + x_3 \geq 4, x_1 + x_2 + 2x_3 \leq 8, x_2 - x_3 \geq 2; x_1, x_2, x_3 \geq 0.$
4. Minimize  $Z = x_1 + 2x_2 + x_3 + 4x_4$   
subject to  $2x_1 + 4x_2 + 5x_3 + x_4 \geq 10, 3x_1 - x_2 + 7x_3 - 2x_4 \geq 2$   
 $5x_1 + 2x_2 + x_3 + 6x_4 \geq 15, x_1, x_2, x_3, x_4 \geq 0.$

3000 15M Using duality or otherwise solve the linear programming problem

Maximize

$$Z = 18x_1 + 12x_2$$

Subject to

$$2x_1 - 2x_2 \geq -3$$

$$3x_1 + 2x_2 \geq 3$$

$$x_1, x_2 \geq 0.$$

