Mains Test Series - 2020

Test-7, Paper-I, full Syllabus

Answer Key

1(a)(i) Define a finite dimensional vector space and prove that every finite dimensional vectorspace has a basis. Is $F[x]$ finite dimensional? Justify.

Sol'n: Definition: The vectorspace $V(F)$ is said to finite dimensional vectorspace or finitely generated if there exists a finite subset S of V such that $V = L(S)$.

Proof: Let $V(F)$ be a finite dimensional vectorspace.

then $\exists$ a finite subset S of V such that $L(S)=V$.

i.e., Let $S=\{\alpha_1, \alpha_2, \dots \alpha_m\} \subseteq V$ such that $L(S)=V$.

If S is LI then S itself is a basis of V.

If S is LD then thereexists a vector $\alpha_i \in S$ is a linear combination of its preceding vectors $\alpha_1, \alpha_2 \dots \alpha_{i-1}$

i.e, $\alpha_i = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{i-1}\alpha_{i-1}$ — (I)

where $a_1, a_2 \dots a_{i-1} \in F$

Now if we omit this vector α_i from the set S then the remaining set S' having $m-1$ vectors $\alpha_1, \alpha_2, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_m$.

i.e, $S' = \{\alpha_1, \alpha_2, \dots \alpha_{i-1}, \alpha_{i+1}, \dots \alpha_m\} \subset S$

Now we show that $L(S')=V$

Clearly $S'\subset S \Rightarrow L(S')\subset L(S)$

$\Rightarrow L(S')\subset V$ — ① $(\because L(S)\subset V)$

Let $\alpha\in V$ then α is l.c of elts of S.

$\therefore \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_i\alpha_i + b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

where $b_1, b_2, \cdots b_{i-1}, b_i, b_{i+1}, \cdots b_m \in R$

(I) $\equiv \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1} + b_i(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1})$

$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m.$

$= (b_1 + b_i a_1)\alpha_1 + (b_2 + b_i a_2)\alpha_2 + \cdots + (b_{i-1} + b_i a_{i-1})\alpha_{i-1}$

$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

$=$ l.c of $\alpha_1, \alpha_2 \cdots \alpha_{i-1}, \alpha_{i+1}, \cdots \alpha_m$.

$=$ l.c of elts of the set S'.

$\in L(S')$

$\therefore \alpha\in L(S')$

$\therefore V\subseteq L(S')$ — ②.

from ① & ② $V = L(S')$

If S' is LI then S' is a basis of $V(F)$.

If S' is LD then proceeding as above we get new set S'' of $m-2$ vectors which generates V.

i.e. $L(S'')=V$.

Continuing in this way, after finite no of steps, obtain a LI subset of S which generates V and therefore it is a basis of V.

At the most repeating the procedure we left with a subset having a single non-zero vector which generates V and we know that a set containing a single non-zero vector is LI.

$\therefore$ It forms a basis of V.

→ Clearly it is not a finite dimensional vectorspace because $\dim F[x] =$ infinite.

The set $S = \{1, x, x^2, \ldots x^n, \ldots\}$ is a basis of the vectorspace $F[x]$ of all polynomials over the field F.

1(b). Let $U = \text{Span}\{(1,3,-2,2,3), (1,4,-3,4,2), (2,3,-1,-2,9)\}$
$W = \text{Span}\{(1,3,0,2,1), (1,5,-6,6,3), (2,5,3,2,1)\}$ be the subspace of $\mathbb{R}^5$.
Find the basis and dimension of $U, W, U+W$ and $U \cap W$.

Sol'n. Let us construct matrices A and B

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 3R_2 \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix} R_2 \to \tfrac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 3R_2 \quad \text{——— (1)} \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 9 & -4 & -2 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 3R_2 \quad \text{——— (2)}$$

Clearly the above forms are row reduced echelon forms.
The bases of U and W are
$S = \{(1,0,1,-4,0), (0,1,-1,2,-1)\}$ and
$T = \{(1,0,9,-4,-2), (0,1,-3,2,1)\}$.
$\therefore \dim U = 2, \ \dim W = 2.$

$$\text{let } C = \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 1 & 0 & 9 & -4 & -2 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 8 & 0 & -2 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 8 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix} \quad R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 8 & 0 & -2 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad R_4 \to R_4 + 4R_3.$$

Clearly it is in echelon form. and the number of non-zero rows are 4.

$\therefore$ Basis of $U+W$ is

$S_1 = \{(1,0,1,-4,0), (0,1,-1,2,-1), (0,0,-2,0,2), (0,0,0,0,2)\}$

and $\dim(U+W) = 4$.

from ① & ②, we have

$U = \{(x, y, x-y, -4x+2y, -y) / x, y \in \mathbb{R}\}$

$W = \{(a, b, 9a-3b, -4a+2b, -2a+b) / a, b \in \mathbb{R}\}$

We have

$x = a$ —(i), $y = b$ —(ii) ; $x - y = 9a - 3b$, —(iii)

$-4x + 2y = -4a + 2b$, —(iv)

$-y = -2a + b$. —(v)

(iii) $\equiv a - b = 9a - 3b$

$\Rightarrow 8a - 2b = 0$ —(vi)

(v) $\equiv -b = -2a + b$

$\Rightarrow -2a + 2b = 0$. —(vii)

(vi) + (vii) $\equiv 6a = 0$

$\Rightarrow \boxed{a = 0}$.

and $\boxed{b = 0}$.

$\therefore U \cap W = \{(0, 0, 0, 0, 0)\}$

$\dim(U \cap W) = 0$.

$\therefore \dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

is verified

1(c) Show that $f(xy, z-2x)=0$ satisfies, under suitable conditions, the equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$. what are these conditions?

Sol^n: Let $u = xy$, $v = z-2x$; then $f(u,v)=0$, and

$$df = f_u du + f_v dv = f_u(x\,dy + y\,dx) + f_v(dz - 2dx) = 0$$

Taking z as dependent variable and x & y as independent variables, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\therefore\ df = f_u(x\,dy + y\,dx) + f_v\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy - 2dx\right) = 0.$$

$$\Rightarrow \left\{y f_u + f_v\left(\frac{\partial z}{\partial x} - 2\right)\right\}dx + \left\{x f_u + f_v\frac{\partial z}{\partial y}\right\}dy = 0$$

But since x and y are independent, we have

$$y f_u + f_v\left(\frac{\partial z}{\partial x} - 2\right) = 0, \text{ and } x f_u + f_v\frac{\partial z}{\partial y} = 0$$

Finding f_u from one equation and putting in the other, we get

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x, \text{ Provided } f_v \neq 0.$$

Thus, the result holds when f is differentiable and $f_v \neq 0$ (and then $f_u \neq 0$).

1(d). Obtain the volume bounded by the elliptic paraboloids given by the equations $z = x^2+9y^2$ and $z = 18-x^2-9y^2$.

Sol^n: The elliptic paraboloids intersect on the elliptic cylinder $x^2+9y^2 = 18-x^2-9y^2$

$$\Rightarrow x^2+9y^2 = 9$$

The volume projects into the region R (in the xy-plane) that is enclosed by the elliptic $x^2+9y^2=9$.

In the double integral w.r.t x and y over R, if we integrate w.r.t x, holding y fixed, x varies from $-\sqrt{9-9y^2}$ to $\sqrt{9-9y^2}$, Then y varies from -1 to 1.

Thus we have

$$V = \int_{y=-1}^{1}\int_{x=-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}}\int_{z=x^2+9y^2}^{18-x^2-9y^2} dz\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} (18-2x^2-18y^2)\,dx\,dy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{9-9y^2}} \left[18(1-y^2)-2x^2\right]dx\,dy$$

$$= 2\int_{-1}^{1}\left[18(1-y^2)x-\frac{2}{3}x^3\right]_0^{3\sqrt{1-y^2}} dy = 2\int_{-1}^{1} 36(1-y^2)^{3/2}\,dy$$

$$= 72\times 2\int_0^1 (1-y^2)^{3/2}\,dy$$

put $y = \sin\theta \Rightarrow dy = \cos\theta\, d\theta$

$$= 72\times 2\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 144\times\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 27\pi.$$

1(e). P is the variable point on the given line and A, B, C are its projections on the axes. Show that the sphere OABC passes through a fixed circle.

Sol'n: Let the equations of the given line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Any variable point on this line may be taken as $P(\alpha+l\gamma, \beta+m\gamma, \gamma+n\gamma)$. Given that the points A, B, C are the projections of P on the axes, so the coordinates of A, B and C are $(\alpha+l\gamma, 0, 0)$, $(0, \beta+m\gamma, 0)$ and $(0, 0, \gamma+n\gamma)$ respectively.

Now the equation of the sphere through $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ is

$$x^2+y^2+z^2-ax-by-cz=0$$

∴ In this problem the equation of the sphere O, ABC is

$$x^2+y^2+z^2-(\alpha+l\gamma)x-(\beta+m\gamma)y-(\gamma+n\gamma)z=0$$

$$(x^2+y^2+z^2-\alpha x-\beta y-\gamma z)-\gamma(lx+my+nz)=0$$

which is of the form $S+\lambda P=0$, where $\lambda=-\gamma$

∴ The sphere for all values of γ passes through the fixed circle.

$$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0,\quad lx+my+nz=0$$

Hence proved.

2(b) → Let $T: R^3 \to R^2$ be the linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$
If $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$, $\beta' = \{(1,0), (0,1)\}$ be ordered basis of R^3 and R^2, respectively, then find the matrix of T relative to β, β'. Also find rank and nullity.

Sol^n: Let $T: R^3 \to R^2$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1) \quad \text{—(1)}$$

$$T(1,0,-1) = (1,-3)$$
$$T(1,1,1) = (2,1)$$
$$T(1,0,0) = (1,-1)$$

further
$$T(1,0,-1) = (1,-3) = 1(1,0) + -3(0,1)$$
$$T(1,1,1) = (2,1) = 2(1,0) + 1(0,1)$$
$$T(1,0,0) = (1,-1) = 1(1,0) + (-1)(0,1)$$

Hence the matrix of T relative to β, β' is

$$[T]_{\beta,\beta'} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Let $(x_1, x_2, x_3) \in \ker T$ be arbitrary. Then

$$T(x_1, x_2, x_3) = (0,0)$$
$$\Rightarrow (x_1 + x_2, 2x_3 - x_1) = 0$$
$$\Rightarrow x_1 + x_2 = 0 \text{ and } 2x_3 - x_1 = 0$$
$$\Rightarrow x_1 = 2x_3, \; x_2 = -x_1 = -2x_3$$
$$\therefore (x_1, x_2, x_3) = (2x_3, -2x_3, x_3)$$
$$= x_3(2,-2,1)$$

Hence $\ker T = \{x_3(2,-2,1) / x_3 \in R\}$

This shows that $\ker T$ is spanned by $(2,-2,1) \neq (0,0,0)$ and so $\{(2,-2,1)\}$ is a basis of $\ker T$.

$\therefore \dim \ker T = 1$

i.e, nullity$(T) = 1$

w.k.T rank(T) + nullity$(T) = \dim R^3 = 3$

Hence rank $T = 3 - 1 = 2$.

2(c)(i) Show that $\int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \pi\log 2$

Sol: Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore I = \int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_0^{\pi/2}\log\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2}\log\left(\frac{\tan^2\theta+1}{\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta$$

$$= -\int_0^{\pi/2}\log\sin\theta\, d\theta - \int_0^{\pi/2}\log\cos\theta\, d\theta$$

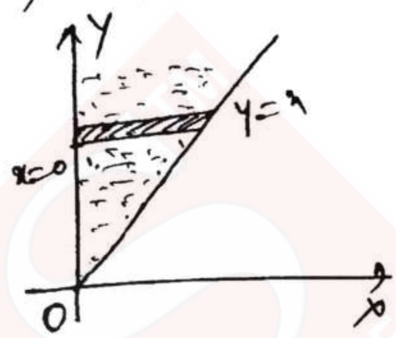
$$= -\left(-\frac{\pi}{2}\log 2\right) - \left(-\frac{\pi}{2}\log 2\right)$$

$$= \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2 \qquad \left(\because \int_0^{\pi/2}\log\sin x = \int_0^{\pi/2}\log\cos x = -\frac{\pi}{2}\log 2\right)$$

$$= \pi\log 2$$

2.(c)(ii) Let $E = \{(x,y) \in \mathbb{R}^2 / 0 < x < y\}$. Then evaluate $\iint_E y e^{-(x+y)}\,dx\,dy$.

Sol: Let $I = \iint_E y e^{-(x+y)}\,dx\,dy$ where $E = \{(x,y) \in \mathbb{R}^2 / 0 < x < y\}$



$$= \int_{y=0}^{\infty}\int_{x=0}^{y} y e^{-(x+y)}\,dx\,dy$$

$$= \int_{y=0}^{\infty} y\left(\frac{e^{-x}}{-1}\right)_0^y e^{-y}\,dy$$

$$= \int_{y=0}^{\infty} y e^{-y}\left(e^{-x}\right)_y^0\,dy$$

$$= \int_0^{\infty} y e^{-y}\left(1-e^{-y}\right)dy$$

$$= \int_0^{\infty}\left(y e^{-y} - y e^{-2y}\right)dy$$

$$= \int_0^{\infty} e^{-y} y^{2-1}\,dy - \int_0^{\infty} e^{-2y} y^{2-1}\,dy \qquad \left(\because \int_0^{\infty} e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}\right)$$

$$= \frac{\Gamma 2}{1^2} - \frac{\Gamma 2}{2^2}$$

$$= 1 - \frac{1}{4} \qquad \left(\because \Gamma n+1 = n!\right)$$

$$= \frac{3}{4}$$

2(d) (i) Prove that the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie.

Sol'n: Given lines are coplanar, if

$$\begin{vmatrix} (a-d)-(b-c) & a-b & (a+d)-(b+c) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Adding 3rd column to first we get

$$\begin{vmatrix} 2(a-b) & a-b & (a+d)-(b+c) \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

The first column being twice the second column, the determinant on the left vanishes, hence the given lines are coplanar.

Also the equation of the plane in which the two given lines lie is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$ adding 3rd column to the first.

$$\Rightarrow \begin{vmatrix} (x+z-2a)-2(y-a) & y-a & z-a-d \\ 2\alpha-2\alpha & \alpha & \alpha+\delta \\ 2\beta-2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$ substracting twice second column from first.

$$\Rightarrow \begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix} = 0 \Rightarrow (x+z-2y)\left[\alpha(\beta+\gamma)-\beta(\alpha+\delta)\right]=0$$

$$\Rightarrow x+z-2y=0.$$

2(d)(ii) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.

Sol^n: If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one of the three mutually perpendicular generators, then it is normal to the plane through the vertex cutting the cone in two $\perp$lar generators and therefore the equation of the plane is

$$x+2y+3z=0 \quad \text{——— (1)}$$

Now we are to find the lines of intersection of this plane and the given cone and let one of the lines of intersection be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Then we have $l+2m+3n=0$ & $5mn-8nl-3lm=0$ ——— (2)

Eliminating l between these we get

$$5mn-(8n+3m)[-(2m+3n)]=0$$

$$\Rightarrow 24n^2+30mn+6m^2=0$$

$$\Rightarrow m^2+5mn+4n^2=0 \Rightarrow (m+n)(m+4n)=0$$

when $m=-n$, from (2) we get $l+n=0 \Rightarrow l=-n$

$$\therefore \frac{l}{1}=\frac{m}{1}=\frac{n}{(-1)} \quad \text{——— (3)}$$

when $m=-4n$, from (2) we get $l-5n=0 \Rightarrow l=5n$

$$\therefore \frac{l}{5}=\frac{m}{(-4)}=\frac{n}{1} \quad \text{——— (4)}$$

Hence from (3) and (4) the other two generators are

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{(-1)} \quad \text{and} \quad \frac{x}{5}=\frac{y}{(-4)}=\frac{z}{1}$$

and evidently these are $\perp$lar as $1\cdot5+1\cdot(-4)+(-1)1=0$ and also each one of them is $\perp$lar to the given generator.

3.(a) Let $H = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H \bar{P}$ is diagonal.

Sol'n. Let us form the block matrix

$$[H|I] = \left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 1-i & 4 & 2-3i & 0 & 1 & 0 \\ -2i & 2+3i & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5 & 3 & 2i & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to (-1+i)R_1 + R_2 \\ R_3 \to 2iR_1 + R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5i & 3 & 2i & 0 & 1 \end{array}\right] \begin{array}{l} C_2 \to C_2 + (-1-i)C_1 \\ C_3 \to C_3 + 2iC_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 0 & -19 & 5+9i & -5i & 2 \end{array}\right] R_3 \to 2R_3 - 5iR_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1+i & 1 & 0 \\ 0 & 0 & -38 & 5+9i & -5i & 2 \end{array}\right] C_3 \to 2C_3 + 5iC_2$$

Clearly H has been diagonalized

Set $P = \begin{bmatrix} 1 & -1+i & 5+9i \\ 0 & 1 & -5i \\ 0 & 0 & 2 \end{bmatrix}$ and then $P^T H \bar{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -38 \end{bmatrix}$

3.(b)(i) Test the convergence of the integral $\int_1^2 \frac{dx}{\sqrt{x^4-1}}$.

Sol'n: In the given integral the integrand $f(x) = \frac{1}{\sqrt{x^4-1}}$ is unbounded at the lower limit of integration $x=1$.

Take $\phi(x) = \frac{1}{\sqrt{x^2-1}}$

then $\lim_{x\to 1} \frac{f(x)}{\phi(x)} = \lim_{x\to 1}\left[\frac{1}{\sqrt{x^4-1}}\sqrt{x^2-1}\right]$

$= \lim_{x\to 1} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{2}}$

which is finite and non-zero.

$\therefore$ by Comparison test,

$\int_1^2 f(x)\,dx$ and $\int_1^2 \phi(x)\,dx$

are either both convergent or both divergent.

But $\int_1^2 \phi(x)\,dx = \int_1^2 \frac{dx}{\sqrt{x^2-1}} = \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x^2-1}}$

$= \lim_{\varepsilon\to 0}\left[\log\{x+\sqrt{x^2-1}\}\right]_{1+\varepsilon}^2$

$= \lim_{\varepsilon\to 0}\left[\log(2+\sqrt{3}) - \log\{1+\varepsilon+\sqrt{\varepsilon^2+\varepsilon}\}\right]$

$= \log(2+\sqrt{3})$

which is a definite real number

$\therefore \int_1^2 \phi(x)\,dx$ is Convergent.

Hence $\int_1^2 \frac{1}{\sqrt{x^4-1}}\,dx$ is also Convergent.

3.(b)(ii) Show that the function f, where

$$f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

is continuous possesses partial derivations but is not differentiable at the origin.

Sol'n: put $x = r\cos\theta,\ y = r\sin\theta$

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2}\right| = |r(\cos^3\theta - \sin^3\theta)| \le 2|r| = 2\sqrt{x^2+y^2} < \varepsilon,$$

$$x^2 < \frac{\varepsilon^2}{8},\quad y^2 < \frac{\varepsilon^2}{8}$$

(or), if

$$|x| < \frac{\epsilon}{2\sqrt{2}},\quad |y| < \frac{\epsilon}{2\sqrt{2}}$$

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2} - 0\right| < \epsilon, \text{ when } |x| < \frac{\epsilon}{2\sqrt{2}},\ |y| < \frac{\epsilon}{2\sqrt{2}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence the function is continuous at $(0,0)$

Again

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{-k}{k} = -1$$

Thus, the function possesses partial derivatives at $(0,0)$.

If the function is differentiable at $(0,0)$, then by definition

$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi \quad \text{——— (1)}$$

when A and B are constants ($A = f_x(0,0) = 1$, $B = f_y(0,0) = -1$)

and ϕ, ψ tend to zero as $(h,k) \to (0,0)$

Putting $h = \rho\cos\theta$, $k = \rho\sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta \quad \text{——— (2)}$$

For arbitrary $\theta = \tan^{-1}(h/k)$, $\rho \to 0$ implies that $(h,k)\to(0,0)$.

Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

which is plainly impossible for arbitrary θ.

Thus, the function is not differentiable at the origin.

3(C) If Q is a point on the normal to the ellipsoid $\sum\left(\frac{x^2}{a^2}\right)=1$ at the point P, such that $3PQ = PG_1 + PG_2 + PG_3$, where G_1, G_2, G_3 are the points where the normal at P meets the yz, zx and xy planes, then the locus of Q is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2} + \frac{b^2y^2}{(2b^2-c^2-a^2)^2} + \frac{c^2z^2}{(2c^2-a^2-b^2)^2} = \frac{1}{9}.$$

Sol^n: Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)} = \frac{y-\beta}{(p\beta/b^2)} = \frac{z-\gamma}{(p\gamma/c^2)} = r \text{ (say)} \quad \text{——} (1)$$

$$\text{where } \frac{1}{p^2} = \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} \quad \text{——} (2)$$

$\therefore$ The coordinates of any point Q on the normal (1) are

$$\left(\alpha + \frac{p\alpha}{a^2}r,\ \beta + \frac{p\beta}{b^2}r,\ \gamma + \frac{p\gamma}{c^2}r\right), \quad \text{——} (3)$$

where r is the distance of Q from P.

If Q lies on the given ellipsoid i.e. PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha + \frac{p\alpha}{a^2}r\right)^2 + \frac{1}{b^2}\left(\beta + \frac{p\beta}{b^2}r\right)^2 + \frac{1}{c^2}\left(\gamma + \frac{p\gamma}{c^2}r\right)^2 = 1$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right) + 2rp\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right) + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right) = 1$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right) + 2rp\left(\frac{1}{p^2}\right) = 0, \text{ from (2) \& } \sum\frac{\alpha^2}{a^2} = 1$$

as $P(\alpha,\beta,\gamma)$ lies on the given conicoid.

$$r = \frac{-2}{p^3\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right)} = \text{length of normal chord PQ} \quad \text{——} (4)$$

Given $PQ = \frac{1}{3}(PG_1 + PG_2 + PG_3)$

$\Rightarrow r = \frac{1}{3}\left(-\frac{a^2}{p} - \frac{b^2}{p} - \frac{c^2}{p}\right)$, from The normal at any point $P(\alpha,\beta,\gamma)$ to the conicoid meets the three principal planes at G_1, G_2, G_3: then $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.

$$\Rightarrow pr = -\frac{1}{3}(a^2+b^2+c^2) \quad \text{—— (5)}$$

$\therefore$ from (3) & (5), we have

$$x_1 = \alpha + \frac{\alpha}{a^2}\left[-\frac{1}{3}(a^2+b^2+c^2)\right] = \frac{\alpha(2a^2-b^2-c^2)}{3a^2}$$

$$\frac{\alpha}{a} = \frac{3ax_1}{2a^2-b^2-c^2} \quad \text{—— (6)}$$

Similarly from (3) and (5), we can get

$$\frac{\beta}{b} = \frac{3by_1}{2b^2-c^2-a^2}, \quad \frac{\gamma}{c} = \frac{3cz_1}{2c^2-a^2-b^2} \quad \text{— (7)}$$

Also as $P(\alpha,\beta,\gamma)$ lies on the ellipsoid $\Sigma(x^2/a^2)=1$, so we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1 \Rightarrow \left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2 + \left(\frac{\gamma}{c}\right)^2 = 1$$

$$\Rightarrow \left(\frac{3ax_1}{2a^2-b^2-c^2}\right)^2 + \left(\frac{3by_1}{2b^2-c^2-a^2}\right)^2 + \left(\frac{3cz_1}{2c^2-a^2-b^2}\right)^2 = 1$$

$\therefore$ The locus of $Q(x_1,y_1,z_1)$ is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)} + \frac{b^2y^2}{(2b^2-c^2-a^2)^2} + \frac{c^2z^2}{(2c^2-a^2-b^2)^2} = \frac{1}{9}.$$

4(a)(i) Find the condition on a, b and c so that the following system in unknowns x, y, and z has a solution?

$$x+2y-3z=a$$
$$2x+6y-11z=b$$
$$x-2y+7z=c.$$

Sol^n: The matrix form of the given system of equations is

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 6 & -11\\ 1 & -2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & -4 & +10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c-a\end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c+2b-5a\end{bmatrix}$$

The system will have no solution if $c+2b-5a \neq 0$.

Thus the system will have atleast one solution if $c+2b-5a=0$ i.e. $5a=2b+c$.
which is the required condition.

Note: In this case the system ~~will~~ have infinitely many solutions. In otherwords, the system cannot have a unique solution.

4.(b) → A flat circular plate has the shape of the region $x^2+y^2 \le 1$. The plate, including the boundary where $x^2+y^2=1$, is heated so that the temperature at any point (x,y) is

$$T(x,y) = x^2+2y^2-x.$$

Find the hottest and coldest points on the plate, and the temperature at each of these points.

Sol'n: $T(x,y) = x^2+2y^2-x$ — ①.

Given that $x^2+y^2 \le 1$

Let $x^2+y^2=k$

where k is some +ve number which is <1 (i.e. $0<k\le 1$)

At the boundary $x^2+y^2=1$ —②

putting ② in ①, we get $T(x) = -x^2-x+2$

$\frac{dT(x)}{dx} = -2x-1$

For max or minimum $\frac{dT}{dx}=0$

$\Rightarrow -2x-1=0$

$\Rightarrow x=-\frac{1}{2}$.

from② $y = \pm\sqrt{3}/2$.

also $\frac{d^2T}{dx^2} = -2 < 0$.

$\therefore$ Maxima at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

In the interior region of plate

$\frac{\partial T}{\partial x} = 2x-1$, $\frac{\partial T}{\partial y} = 4y$

$\frac{\partial T}{\partial x}=0$ and $\frac{\partial T}{\partial y}=0$

$\Rightarrow x=\frac{1}{2}, y=0$

This is the point of minimum value.

from① $T\left(\frac{1}{2},0\right) = \frac{1}{4}+0-\frac{1}{2} = -\frac{1}{4}$

and $T\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$ and $T\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)=\frac{9}{4}$.

$\therefore$ Hottest points are $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $T=\frac{9}{4}$ units

Coldest point is $\left(\frac{1}{2},0\right)$ and $T=-\frac{1}{4}$ units

4.(c) Prove that in general two generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse $(x^2/a^6)+(y^2/b^6)=c^4/(a^4b^4)$.

Sol^n We know for the given hyperboloid, the generator belonging to λ-system is given by –

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \text{ and } \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right) \quad \text{—(i)}$$

or $\frac{x}{a}+\frac{\lambda y}{b}-\frac{z}{c}=\lambda$ and $\lambda\frac{x}{a}-\frac{y}{b}+\lambda\frac{z}{c}=1$

$\therefore$ If l_1, m_1, n_1 be the dr.'s of the generator (i)

then $\frac{l_1}{a}+\frac{\lambda m_1}{b}-\frac{n_1}{c}=0$ and $\frac{\lambda l_1}{a}-\frac{m_1}{b}+\frac{\lambda n_1}{c}=0$

Solving these simultaneously, we get

$$\frac{l_1/a}{\lambda^2-1}=\frac{m_1/b}{-\lambda-\lambda}=\frac{n_1/c}{-1-\lambda^2}$$

or $$\frac{l_1}{-a(\lambda^2-1)}=\frac{m_1}{2\lambda b}=\frac{n_1}{c(1+\lambda^2)} \quad \text{—— (ii)}$$

Similarly the direction ratio l_2, m_2, n_2 of the generator belonging to μ-system viz.

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right) \text{ and } \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad \text{—(iii)}$$

are given by, $$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)} \quad \text{—(iv)}$$

If these two generators given by (i) and (iii) are perpendicular then, $-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(\mu^2+1)=0$ —(v)

Now if λ-generator is given, then λ is constant and (v) will be a quadratic equation in μ which gives two values of μ and this shows that there will be two generators of μ-system which will be perpendicular to a generator of λ-system.

Now let the generators of μ-system meet the plane $z=0$ in the points $P(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$

$\therefore$ The generator of the μ-system through these points are given by

$$\frac{x-a\cos\alpha}{a\sin\alpha} = \frac{y-b\sin\alpha}{-b\cos\alpha} = \frac{z}{c} \quad \text{—— (vi)}$$

and $$\frac{x-a\cos\beta}{a\sin\beta} = \frac{y-b\sin\beta}{-b\cos\beta} = \frac{z}{c} \quad \text{—— (vii)}$$

These two generators intersect at right angles a generator of λ-system through any point $(a\cos\theta, b\sin\theta, 0)$ say, whose equations are

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{-c} \quad \text{—— (viii)}$$

As (vi) and (vii) are both perpendicular to (viii), so

$$a^2\sin\alpha\sin\theta + b^2\cos\alpha\cos\theta - c^2 = 0$$

and $$a^2\sin\beta\sin\theta + b^2\cos\beta\cos\theta - c^2 = 0$$

Solving these simultaneously for $a^2\sin\theta$, $b^2\cos\theta$ and $-c^2$, we get

$$\frac{a^2\sin\theta}{\cos\alpha-\cos\beta} = \frac{b^2\cos\theta}{\sin\beta-\sin\alpha} = \frac{-c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

or $\dfrac{a\sin\theta}{2\sin\frac{\alpha+\beta}{2}.\sin\frac{\beta-\alpha}{2}} = \dfrac{b^2\cos\theta}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}} = \dfrac{-c^2}{\sin(\alpha-\beta)}$

$= \dfrac{-c^2}{2\sin\frac{\alpha-\beta}{2}.\cos\frac{\alpha-\beta}{2}}$

$\Rightarrow \dfrac{a^2\sin\theta}{c^2} = \dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \qquad \dfrac{b^2\cos\theta}{c^2} = \dfrac{\cos\frac{(\alpha+\beta)}{2}}{\cos\frac{\alpha-\beta}{2}}$ ——— (ix)

Also equation of the joining P and Q is

$\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}, \; z=0$

or $\dfrac{x}{a}\left(\dfrac{b^2\cos\theta}{c^2}\right) + \dfrac{y}{b}\left(\dfrac{a^2\sin\theta}{c^2}\right) = 1, \; z=0$ —(x)

using the results of (ix).

Now in order to find its envelope, we should differentiate (x) with respect to θ and then eliminate θ.

Differentiating (x) w.r.to θ, we get

$-\dfrac{xb^2}{ac^2}\sin\theta + \dfrac{ya^2}{bc^2}\cos\theta = 0, \; z=0$ —(xi)

Squaring and adding (x) and (xi), θ is eliminated and we get the required envelope of PQ as

$\dfrac{x^2b^4}{a^2c^4} + \dfrac{y^2a^4}{b^2c^4} = 1, \; z=0$ or $\dfrac{x^2}{a^6} + \dfrac{y^2}{b^6} = \dfrac{c^4}{a^4b^4}, \; z=0$

which represents an ellipse on the plane $z=0$

Hence proved.

5.(a) Solve $\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1-\cot x)y = e^x \sin x$.

Sol^n Comparing the given equation with $y''+Py'+Qy=R$, we get, $P=-\cot x$, $Q=-1+\cot x$, $R=e^x\sin x$ —(i)

Here $1+P+Q=0$, showing that $u=e^x$ —(ii) is a part of C.F. of the given Equation.

Let the required general solution be

$$y = uv \quad \text{———(3)}$$

Then v is given by $\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

or $\frac{d^2v}{dx^2}+\left(-\cot x+\frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{e^x\sin x}{e^x}$

or $\frac{d^2v}{dx^2}+(2-\cot x)\frac{dv}{dx} = \sin x$

Let $\frac{dv}{dx}=q$ so that $\frac{d^2v}{dx^2}=\frac{dq}{dx}$ —(4)

Hence we get,

$$\frac{dq}{dx}+(2-\cot x)q = \sin x,$$

which linear in q and x

Its I.F $= e^{\int(2-\cot x)dx} = e^{2x-\log \sin x} = e^{2x}\cdot e^{\log(\sin x)^{-1}}$

$= e^{2x}(\sin x)^{-1}$ and solution is

$q\frac{e^{2x}}{\sin x} = \int\left(\sin x\cdot\frac{e^{2x}}{\sin x}\right)dx + C_1 = \frac{1}{2}e^{2x}+C_1$. C_1 being an arbitrary constant.

or $q = \left(\frac{1}{2}\right)x\sin x + C_1 e^{-2x}\sin x$

or $\frac{dv}{dx} = \left(\frac{1}{2}\right)x\sin x + C_1 e^{-2x}\sin x$

or $dv = \left[\left(\frac{1}{2}\right)x\sin x + c_1 e^{-2x}\sin x\right]dx$.

Integrating,

$$v = \left(\frac{-1}{2}\right)x\cos x + c_1\int e^{-2x}\sin x\,dx + c_2$$

where c_2 being an arbitrary constant.

or $$v = -\frac{\cos x}{2} + \frac{c_1}{1^2+(-2)^2}(-2\sin x - \cos x) + c_2$$

or $$v = -\frac{\cos x}{2} - \frac{c_1}{5}(2\sin x + \cos x) + c_2 \quad \text{——(5)}$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

Hence from (2), (3) and (5), the required general solution is

$$y = uv$$

$$y = e^{x}\left[-\left(\frac{1}{2}\right)x\cos x - \left(\frac{c_1}{5}\right)\times e^{-2x}(2\sin x + \cos x) + c_2\right]$$

or $$y = c_1' e^{-2x}(2\sin x + \cos x) + c_2 e^{x} - \left(\frac{1}{2}\right)x e^{x}\cos x,$$

where $c_1' = -\left(\frac{c_1}{5}\right)$.

5.(b)(i) Evaluate $L^{-1}\{3(1+e^{-s\pi})/(s^2+9)\}$.

Sol^n

$$L^{-1}\left\{\frac{3(1+e^{-s\pi})}{s^2+9}\right\} = L^{-1}\left\{\frac{3}{s^2+9}\right\} + L^{-1}\left\{\frac{3e^{-s\pi}}{s^2+9}\right\} \quad \text{——(1)}$$

Let $f(s) = 3/(s^2+9)$ ——— (2)

$$\therefore\ F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{3}{s^2+3^2}\right\} = 3\cdot\frac{1}{3}\sin 3t$$

$$F(t) = \sin 3t \quad \text{——— (3)}$$

By second shifting theorem, we have

$$L^{-1}\{e^{-s\pi}f(s)\} = \begin{cases} F(t-\pi) & , t>\pi \\ 0 & , t<\pi \end{cases}$$

or $$L^{-1}\left\{e^{-s\pi}\frac{3}{s^2+9}\right\} = \begin{cases} \sin 3(t-\pi), & t>\pi \\ 0 & , t<\pi \end{cases}$$

[using (1) and (2)]

$$= \begin{cases} -\sin 3t, & t>\pi \\ 0 & , t<\pi \end{cases}$$

$$= -\sin 3t\, H(t-\pi) \quad \text{——— (4)}$$

where $H(t-\pi)$ is the Heaviside unit step function.

Using (3) and (4), (1) to reduces

$$L^{-1}\left\{\frac{3(1+e^{-s\pi})}{s^2+9}\right\} = \sin 3t - \sin 3t\, H(t-\pi).$$

5.(b)(ii) Evaluate $L^{-1}\left\{\dfrac{1}{s^4(s^2+1)}\right\}$

Sol^n

Here $L^{-1}\left\{\dfrac{1}{(s^2+1)}\right\} = \sin t$

$$\therefore\ L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+1)}\right\} = \int_0^t \sin x\,dx = \left[-\cos x\right]_0^t$$
$$= 1-\cos t$$

$$\therefore\ L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos x)\,dx = t-\sin t$$

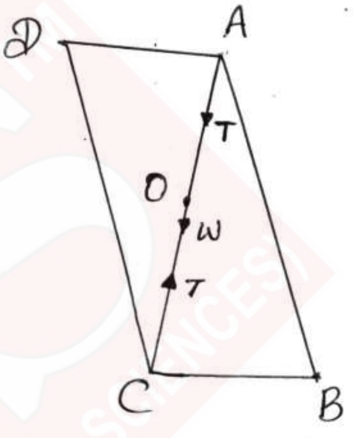
$$\therefore\ L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (x-\sin x)\,dx = \frac{1}{2}t^2+\cos t-1$$

$$\therefore\ L^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = \int_0^t \left(\frac{1}{2}x^2+\cos x-1\right)dx$$

$$= \left[\frac{x^3}{3}+\sin x-x\right]_0^t = \left[\frac{t^3}{3}+\sin t-t\right]$$

$$\therefore\ L^{-1}\left\{\frac{1}{s^4(s^2+1)}\right\} = \left[\frac{t^3}{3}+\sin t-t\right]$$

Q. 5.C) → Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.



Sol: ABCD is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC. Let T be the tension in the string AC. The total weight W of all the four rods, AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC = 2x$.

Give the system a small displacement in which x changes to $x+\delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes.

We have, $AO = x$

By the principle of virtual work, we have

$$-T\delta(AC) + W\delta(AO) = 0$$

or $$-T\delta(2x) + W\delta(x) = 0$$

or $$-2T\delta x + W\delta x = 0$$

or $[-2T+W]\delta x = 0$ or $-2T+W=0$ $[\because \delta x \neq 0]$

or $T = \frac{1}{2}W = \frac{1}{2}$ (total weight of all the four rods).

Q. 5.d) → A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$$

where a is the radius of the earth.

Sol: Let O be the centre of the earth and A the point of projection on the surface.

If P is the position of the particle at any time t, such that $OP = x$, then the acceleration at $P = \mu/x^2$ directed towards O.

∴ the equation of motion of the particle P is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$ ——— (1)

But at the point A, on the surface of the earth, $x = a$ and $\frac{d^2x}{dt^2} = -g$

$\therefore -g = -\mu/a^2$ or $\mu = a^2 g$.

$$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}$$

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + c,$$ where c is constant.

But when $x \to \infty$, $dx/dt \to 0$. $\therefore c = 0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} \text{ or } \frac{dx}{dt} = \frac{a\sqrt{(2g)}}{\sqrt{g}}$$ ——— (2)

[Here +ive sign is taken because the particle is moving in the direction of x increasing.]

Seperating the variables, we have

$$dt = \frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx$$

Integrating between the limits $x=a$ to $x=a+h$, the required time t to reach a height h is given by

$$t = \frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx$$

$$= \frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$= \frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2}-a^{3/2}\right]$$

$$= \frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$$

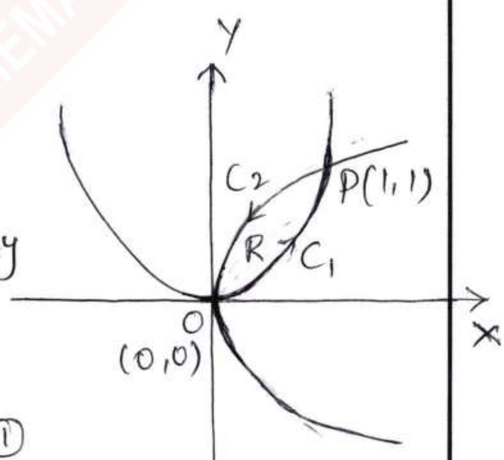
5(e) → Verify Green's theorem in the plane for $\int_C [(2xy-x^2)dx + (x^2+y^2)dy]$, where C is boundary of the region enclosed by $y=x^2$ and $y^2=x$ described in the positive sense.

Sol'n: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = 2xy - x^2$ $\quad N = x^2 + y^2$

The parabolas $y^2 = x$ and $x^2 = y$ intersect at the points (0,0) and (1,1). The closed curve C consists of the arc C_1 of the parabola $y=x^2$ and the arc C_2 of the parabola $y^2=x$. Also R is the region bounded by the closed curve C.



we have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right]dx\,dy$$

$$= \iint_R (2x - 2x)\,dx\,dy = \iint_R 0\,dx\,dy = 0 \quad \text{--- (1)}$$

Now the line integral along the closed curve C.

$$= \oint_C (M\,dx + N\,dy) = \int_{C_1} (M\,dx + N\,dy) + \int_{C_2} (M\,dx + N\,dy)$$

Along C_1, $x^2 = y$, $dy = 2x\,dx$ and x varies from 0 to 1.

$\therefore$ the line integral along C_1

$$= \int_{x=0}^{1} \left[(2x^3 - x^2)\,dx + (x^2 + x^4)\,2x\,dx\right]$$

$$= \int_0^1 (4x^3 - x^2 + 2x^5)\,dx = \left[x^4 - \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{3} = 1.$$

Along C_2, $y^2 = x$, $dx = 2y\,dy$ and y varies from 1 to 0.

$\therefore$ the line integral along C_2

$$= \int_{y=1}^{0} \left[(2y^3 - y^4)\,2y\,dy + (y^4 + y^2)\,dy\right]$$

$$= \int_1^0 (5y^4 - 2y^5 + y^2)\,dy.$$

$$= \left[y^5 - \frac{y^6}{3} + \frac{y^3}{3}\right]_1^0$$

$$= -1 + \frac{1}{3} - \frac{1}{3} = -1$$

$\therefore$ Total line integral along the closed curve C

$$= 1 - 1 = 0 \quad \text{——— (2).}$$

from (1) and (2), we see that the two integrals are equal and hence Green's theorem is verified.

6(a)(i) Solve $(4y+3x)dy + (y-2x)dx = 0$

Sol^n: Rewriting the given equation

$$\frac{dy}{dx} = -\frac{y-2x}{4y+3x} = \frac{2-(y/x)}{3+4(y/x)} \quad \text{—— (1)}$$

Let $y/x = v$ so that $y = xv$ —— (2)

from (2), $dy/dx = v + x\left(\frac{dv}{dx}\right)$ —— (3)

Using (2) and (3), (1) reduces to

$$v + x\frac{dv}{dx} = \frac{2-v}{3+3v} \Rightarrow x\frac{dv}{dx} = \frac{2-v}{3+4v} - v$$

$$\Rightarrow \frac{2dx}{x} = \frac{3+4v}{1-2v-2v^2}$$

Integrating,

$$2\log x = \int \frac{(3+4v)dv}{1-2v-2v^2} = \int \frac{-(-2-4v)}{1-2v-2v^2}dv$$

$$\Rightarrow \log x^2 + \log c = -\log(1-2v-2v^2) + \frac{1}{2}\int \frac{dv}{\frac{1}{2}-v-v^2}$$

$$\Rightarrow \log\{cx^2(1-2v-2v^2)\} = \frac{1}{2}\int \frac{dv}{(3/4)-(v^2+v+1/4)}$$

$$= \frac{1}{2}\int \frac{dv}{(\sqrt{3}/2)^2 - (v+1/2)^2}$$

$$= \frac{1}{2} \times \frac{1}{2(\sqrt{3}/2)} \log \frac{\sqrt{3}/2 + (v+1/2)}{(\sqrt{3}/2) - (v+1/2)}$$

$$= \frac{1}{2\sqrt{3}} \log \frac{\sqrt{3}+2v+1}{\sqrt{3}-v-1}$$

$$\log\left[cx^2\left(1 - \frac{2y}{x} - \frac{2y^2}{x^2}\right)\right] = \frac{1}{2\sqrt{3}} \log \frac{(\sqrt{3}+1)+2(y/x)}{(\sqrt{3}-1)-2(y/x)}, \text{ as } v = y/x$$

$$c(x^2-2xy-2y^2) = \left\{\frac{(\sqrt{3}+1)x+2y}{(\sqrt{3}-1)x-2y}\right\}^{1/2\sqrt{3}}$$, c being an arbitrary Constant.

6(a)(ii) Solve $(xy^2+e^{-1/x^3})dx - x^2y\,dy = 0$

Sol'n: Rewriting the given equation, we have

$$x^2y\frac{dy}{dx} = xy^2 + e^{-1/x^3} \Rightarrow 2y\frac{dy}{dx} - \frac{2}{x}y^2 = \frac{2}{x^2}e^{-1/x^3} \quad \text{—} (1)$$

Putting $y^2 = v$ and $2y(dy/dx) = \frac{dv}{dx}$, (1) reduces to

$$\frac{dv}{dx} - \left(\frac{2}{x}\right)v = \left(\frac{2}{x^2}\right)e^{-1/x^3}$$, which is linear equation.

Its I.F $= e^{\int(-2/x)dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$ and

solution is

$$vx^{-2} = \int(x^{-2}) \times (2x^{-2}e^{-x^{-3}})dx + C$$

$$vx^{-2} = 2\int x^{-4}e^{-x^{-3}}dx + C \quad \text{———} (3)$$

Putting $-x^{-3} = u$ so that $3x^{-4}dx = du$

$$\Rightarrow x^{-4}dx = \tfrac{1}{3}du$$

$\therefore$ (3) reduces to $vx^{-2} = \frac{2}{3} \times \int e^u du + C$

$$\Rightarrow vx^{-2} = \tfrac{2}{3}e^u + C$$

$$y^2x^{-2} = \tfrac{2}{3}(-x^{-3}) + C, \text{ as } v = y^2 \text{ \& } u = -x^{-3}$$

$$y^2/x^2 = \tfrac{2}{3}e^{-1/x^3} + C$$, C being arbitrary constant.

6(b) Find the orthogonal trajectories of cardioids $r = a(1-\cos\theta)$, a being parameter.

Sol'n: The given family of cardioids is

$$r = a(1-\cos\theta) \quad \text{—— (1)}$$

Taking logarithm of bothsides of (2), we get

$$\log r = \log a + \log(1-\cos\theta) \quad \text{—— (2)}$$

Differentiating (2) with respect to 'θ', we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \quad \text{—— (3)}$$

Hence (3) is the differential equation of the given family (1)

Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is $\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta}$

$$= \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$\Rightarrow \frac{1}{r}dr = -\tan(\theta/2)\,d\theta$, on separating variables

Integrating, $\log r = 2\log\cos\theta/2 + \log c$

$$\log r = \log(c\cos^2\theta/2)$$

$$\Rightarrow r = c/2\,(1+\cos\theta)$$

$$\Rightarrow r = b(1+\cos\theta) \quad \text{—— (4)}$$

where $b(=c/2)$ is arbitrary constant.

(4) gives another family of cardioids.

6. (c) Solve $y = 2px + y^2p^3$.

Sol^n Given,

$$y = 2px + y^2p^3 \quad \text{where } p = \frac{dy}{dx}. \quad \text{---(1)}$$

Solving (1) for x,

$$x = y\left(\frac{1}{2p}\right) - y^2\left(\frac{p^2}{2}\right). \quad \text{---(2)}$$

Differentiating (2) w.r.t 'y' and writing $\frac{1}{p}$ for $\frac{dx}{dy}$, we get

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{2yp^2}{2} - \frac{y^2}{2}.2p\frac{dp}{dy}$$

Or $\frac{1}{2p} + yp^2 + \frac{dp}{dy}\left(\frac{y}{2p^2} + yp^2\right) = 0$

Or $\quad p\left(py + \frac{1}{2p^2}\right) + y\frac{dp}{dy}\left(py + \frac{1}{2p^2}\right) = 0$

Or $\quad \left[py + \left(\frac{1}{2}p^2\right)\right]\left[p + y\left(\frac{dp}{dy}\right)\right] = 0.$

$p + y\left(\frac{dp}{dy}\right) = 0 \quad$ Or $\quad \left(\frac{1}{p}\right)dp + \left(\frac{1}{y}\right)dy = 0$

Integrating,

$$\log p + \log y = \log c \quad \text{Or} \quad py = c$$

Or $p = c/y$

Substituting this value of p in (1), we get the required solution.

$y = (2x)(c/y) + y^2(c/y)^3$ Or $\boxed{y^2 = 2cx + c^3}$

6(d)(i) → Find $L\{(2/t)(1-\cosh 2t)\}$

Sol^n: Here $L(1-\cosh 2t) = L\{1\} - L\{\cosh 2t\}$

$$= \frac{1}{s} - \frac{s}{s^2-2^2} = -\frac{4}{s^2-4} = f(s) \text{ say}$$

$$\therefore L\{(2/t)(1-\cosh 2t)\} = 2L\left\{\frac{1-\cosh 2t}{t}\right\}$$

$$= 2\int_s^\infty f(s)\,ds = 2\int_s^\infty \left(\frac{-4}{s^2-4}\right)ds, \text{ by (1)}$$

$$= -8\,\frac{1}{(2\times 2)}\left[\log\frac{s-2}{s+2}\right]_s^\infty$$

$$= -2\left[\lim_{s\to\infty}\log\frac{s-2}{s+2} - \log\frac{s-2}{s+2}\right]$$

$$= -2\left[\lim_{s\to\infty}\log\frac{1-2/s}{1+2/s} - \log\frac{s-2}{s+2}\right]$$

$$= 2\log\frac{s-2}{s+2}$$

6.(d)ii) By using Laplace transformation solve $(D^2+m^2)x = a\cos nt$, $t>0$, if x, Dx equal to x_0 and x_1, when $t=0$, $n \neq m$.

Sol'n: Rewriting the given equation and conditions we get.

$$x'' + m^2 x = a\cos nt \quad \text{——} (1)$$

with the initial conditions: $x(0) = x_0$ and $x'(0) = x_1$ —— (2)

Taking Laplace transform of both sides of (1), we get

$$L\{x''\} + m^2 L\{x\} = a L\{\cos nt\}$$

$$\Rightarrow s^2 L\{x\} - s x\{0\} - x'(0) + m^2 L\{x\} = as/(s^2+n^2)$$

$$\Rightarrow (s^2+m^2) L\{x\} - s x_0 - x_1 = as/(s^2+n^2), \text{ using } (2)$$

$$\Rightarrow L\{x\} = \frac{s x_0}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{as}{(s^2+m^2)(s^2+n^2)}$$

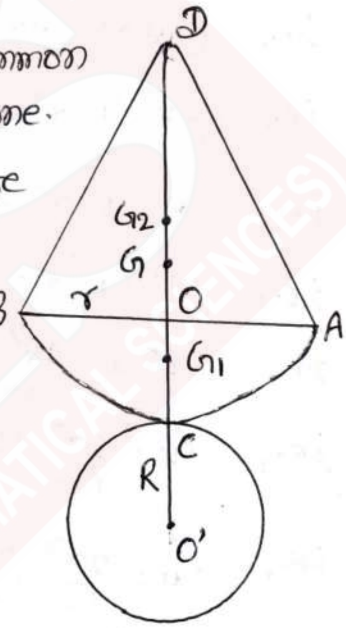
$$\Rightarrow L\{x\} = \frac{s x_0}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{a}{m^2-n^2}\left[\frac{s}{s^2+n^2} - \frac{s}{s^2+m^2}\right].$$

Taking Inverse Laplace transform of bothsides, we get

$$x = x_0 \cos mt + \frac{x_1}{m} \sin mt + \frac{a}{m^2-n^2}\left[\cos nt - \cos mt\right].$$

Q. 7a) → A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the bases; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right]$$



Sol: Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of radius R and centre O', their point of contact being C. For equilibrium the line O'COD must be vertical. Let H be the length of the axis OD of the cone. It is given that OB = OC = r the radius of the hemisphere.

If G_1 and G_2 are the centres of gravity of the hemisphere and the cone respectively, then

$$OG_1 = 3r/8 \quad \text{and} \quad OG_2 = H/4$$

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C, then

$$h = \frac{\frac{2}{3}\pi r^3 \cdot \frac{5}{8}r + \frac{1}{3}\pi r^2 H \cdot (r+\frac{1}{4}H)}{\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 H}$$

$$= \frac{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body $= r$.

and ρ_2 = the radius of curvature at C of the lower body = R

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{i.e.,} \quad \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

i.e., $$\frac{H+2r}{H(r+\frac{1}{4}H)+\frac{5}{4}r^2} > \frac{R+r}{rR}$$

i.e., $$(R+r)\{Hr+\tfrac{1}{4}H^2+\tfrac{5}{4}r^2\} - rR(H+2r) < 0$$

i.e., $$\tfrac{1}{4}H^2(R+r) + H\{(R+r)r - rR\} + \tfrac{5}{4}r^2(R+r) - 2r^2R < 0$$

i.e., $$H^2(R+r) + 4r^2H + 5r^3 - 3r^2R < 0$$

i.e., $$H^2(R+r) + 4r^2H - r^2(3R-5r) < 0$$

i.e., $$H^2 + \frac{4r^2}{R+r}H - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4}{(R+r)^2} - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4+r^2(3R-5r)(R+r)}{(R+r)^2} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2[4r^2+3R^2-2rR-5r^2]}{(R+r)^2} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2(3R^2-2rR-r^2)}{(R+r)^2} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 < \frac{r^2(3R+r)(R-r)}{(R+r)^2}$$

i.e., $$H+\frac{2r^2}{R+r} < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$$

i.e., $$H < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}} - \frac{2r^2}{R+r}$$

i.e., $$H < \frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right]$$

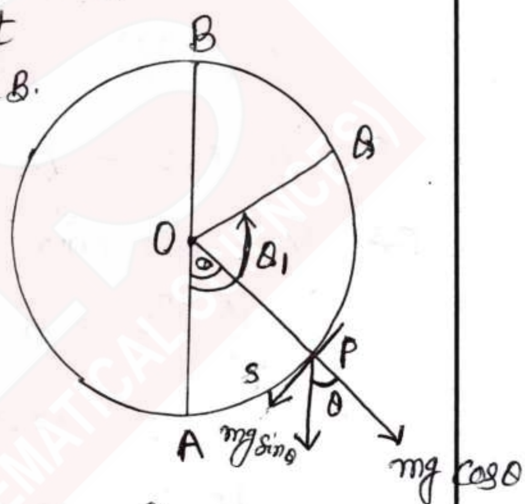
therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right].$$

Q.7b.) → A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. show that the reaction between the particle and the wire is zero after a time

$$\sqrt{(a/g)}\cdot\log(\sqrt{5}+\sqrt{6}).$$

Sol: Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the heighest point B.



If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{——} (1)$$

and $$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{——} (2)$$

Also $$s = a\theta \quad \text{——} (3)$$

From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying both sides by $2a(d\theta/dt)$ and Integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But according to the question $v=0$ at the heighest point B, where $\theta = \pi$

$$\therefore\ 0 = 2ag\cos\pi + A \quad \text{or} \quad A = 2ag.$$

$$\therefore \quad v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag \quad \text{——— (4)}$$

From (2) and (4), we have

$$R = \frac{m}{a}(v^2 + ag\cos\theta)$$

$$= \frac{m}{a}(2ag + 3ag\cos\theta) \quad \text{——— (5)}$$

If the reaction $R=0$ at the point Q where $\theta = \theta_1$, then from (5), we have

$$0 = \frac{m}{a}(2ag + 3ag\cos\theta_1)$$

or $\cos\theta_1 = -2/3$ ——— (6)

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta + 1)$$

$$= 2ag \cdot 2\cos^2\tfrac{1}{2}\theta = 4ag\cos^2\tfrac{1}{2}\theta$$

$$\therefore \quad \frac{d\theta}{dt} = 2\sqrt{(g/a)}\cos\tfrac{1}{2}\theta,$$

the positive sign being taken before the radical sign because θ increases as t increases.

or $dt = \frac{1}{2}\sqrt{(a/g)}\sec\frac{1}{2}\theta\, d\theta$.

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t = \frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1} \sec\tfrac{1}{2}\theta\, d\theta$$

or $$t = \sqrt{(a/g)}\left[\log\left(\sec\tfrac{1}{2}\theta + \tan\tfrac{1}{2}\theta\right)\right]_0^{\theta_1}$$

or $$t = \sqrt{(a/g)}\log\left(\sec\tfrac{1}{2}\theta_1 + \tan\tfrac{1}{2}\theta_1\right). \quad \text{——— (7)}$$

From (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1 - 1 = -\frac{2}{3}$$

or $$2\cos^2\tfrac{1}{2}\theta_1 = 1-\frac{2}{3} = \frac{1}{3}$$

or $$\cos^2\tfrac{1}{2}\theta_1 = \frac{1}{6}$$

or $$\sec^2\tfrac{1}{2}\theta_1 = 6$$

$\therefore$ $$\sec\tfrac{1}{2}\theta_1 = \sqrt{6}$$

and $$\tan\tfrac{1}{2}\theta_1 = \sqrt{(\sec^2\tfrac{1}{2}\theta_1 - 1)}$$

$$= \sqrt{(6-1)} = \sqrt{5}$$

Substituting in (7), the required time is given by

$$t = \sqrt{(a/g)}\,\log(\sqrt{6}+\sqrt{5})$$

Q.7.C) → A particle moves with a central acceleration $\mu/(\text{distance})^2$, it is projected with velocity v at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is

$$\sin^{-1}\left[\frac{\mu}{VR\left(v^2-\frac{2\mu}{R}\right)^{1/2}}\right]$$

Sol: If the particle describes a hyperbola under the central acceleration $\mu/(\text{distance})^2$, then the velocity v of the particle at a distance r from the centre of force is given by

$$v^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right) \quad \text{——— (1)}$$

where $2a$ is the transverse axis of the hyperbola. Since, the particle is projected with velocity V at a distance R, therefore from (1), we have

$$v^2=\mu\left(\frac{2}{R}+\frac{1}{a}\right)$$

or $$\frac{\mu}{a}=v^2-\frac{2\mu}{R} \quad \text{——— (2)}$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h=vp$, we have

$$h=VP=VR\sin\alpha \quad \text{——— (3)}$$

$[\because p=r\sin\phi$ and initially $r=R,\ \phi=\alpha]$

Also,
$$h=\sqrt{(\mu l)}=\sqrt{\{\mu\cdot(b^2/a)\}}=\sqrt{(\mu a)} \quad \text{——— (4)}$$

[$\because$ $b = a$ for a rectangular hyperbola]

From (3) and (4), we have

$$VR \sin\alpha = \sqrt{(\mu a)}$$

or $$\sin\alpha = \frac{\sqrt{(\mu a)}}{VR}$$

$$= \frac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \frac{\mu}{VR\sqrt{(\mu/a)}}.$$

Substituting for μ/a from (2), we have

$$\sin\alpha = \frac{\mu}{VR\sqrt{(V^2 - 2\mu/R)}}$$

or $$\alpha = \sin^{-1}\left[\frac{\mu}{VR\sqrt{(V^2 - 2\mu/R)}}\right]$$

which is the required angle of projection.

8.(a)(ii) A particle moves along the curve $x=4\cos t, y=4\sin t, z=6t$. Find the velocity and acceleration at time $t=0$ and $t=\frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.

Sol^n: Let $\vec{r}$ be the position vector of the particle at time t.

Then, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= 4\cos t\,\hat{i} + 4\sin t\,\hat{j} + 6t\,\hat{k}$

If $\vec{v}$ is the velocity of the particle at time t and $\vec{a}$ its acceleration at that time then,

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= -4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 6\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -4\cos t\,\hat{i} - 4\sin t\,\hat{j}$$

Magnitude of the velocity at time

$t = |\vec{v}|$

$= \sqrt{16\sin^2 t + 16\cos^2 t + 36}$

$= \sqrt{52}$

$= 2\sqrt{13}$

Magnitude of acceleration

$|\vec{a}| = \sqrt{16\cos^2 t + 16\sin^2 t}$

$= \sqrt{16} = 4$

At $t=0$, $\vec{v} = 4\hat{j} + 6\hat{k}$, $\vec{a} = -4\hat{i}$

At $t = \frac{1}{2}\pi$, $\vec{v} = -4\hat{i} + 6\hat{k}$, $\vec{a} = -4\hat{j}$

8·(b) → Find the curvature and torsion of the curve.

$$\vec{r} = a(u-\sin u)\vec{i} + a(1-\cos u)\vec{j} + bu\,k$$

Sol:- Given, $\vec{r} = a(u-\sin u)\hat{i} + a(1-\cos u)\hat{j} + bu\,\hat{k}$

$$\frac{d\vec{r}}{du} = (a - a\cos u)\hat{i} + a\sin u\,\hat{j} + bu\,\hat{k} \quad \text{——— (1)}$$

$$\therefore \left|\frac{d\vec{r}}{du}\right| = \sqrt{a^2 - 2a^2\cos^2 u + a^2\cos^2 u + a^2\sin^2 u + b^2}$$

$$= \sqrt{2a^2(1-\cos u) + b^2}$$

$$= \sqrt{\frac{2a^2}{b^2}[1-\cos u + 1]}$$

$$\left|\frac{d\vec{r}}{du}\right| = \frac{a}{b}\sqrt{2(2-\cos u)} \quad \text{——— (2)}$$

Also: $\dfrac{d^2r}{du^2} = a\sin u\,\hat{i} + a\cos u\,\hat{j}$ ——— (3)

& $\dfrac{d^3r}{du^3} = a\cos u\,\hat{i} - a\sin u\,\hat{j}$ ——— (4)

Now:
$$\frac{d\vec{r}}{du} \times \frac{d^2r}{du^2} = \begin{vmatrix} i & j & k \\ a - a\cos u & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix}$$

$$= i\,[(a\sin u \times 0) - ab\cos u] - j\,[-ab\sin u] + k\,(a^2\cos u - a^2\cos^2 u - a^2\sin^2 u]$$

$$\frac{d\vec{r}}{du} \times \frac{d^2r}{du^2} = -ab\cos u\,\hat{i} + ab\sin u\,\hat{j} + a^2(\cos u - 1)\hat{k}$$

$$\left|\frac{d\vec{r}}{du} \times \frac{d^2r}{du^2}\right| = \sqrt{a^2b^2\cos^2 u + a^2b^2\sin^2 u + a^4[\cos u - 1]^2}$$

$$= \sqrt{(ab)^2[\cos^2 u + \sin^2 u] + a^4[\cos^2 u + 1 - 2\cos u]}$$

$$= \sqrt{a^2b^2 + a^4[\cos u - 1]^2}$$

$$\therefore \left|\frac{d\vec{r}}{du} \times \frac{d^2 r}{du^2}\right| = \sqrt{a^2 b^2 + a^4 (\cos u - 1)^2}$$

$$= a\sqrt{b^2 + a^2(\cos u - 1)^2}$$

$$\kappa = \frac{\left|\frac{d\vec{r}}{du} \times \frac{d^2 u}{du^2}\right|}{\left|\frac{dr}{du}\right|^3} = \frac{\sqrt{a^2 b^2 + a^4 (\cos u - 1)^2}}{\left(2a^2(1-\cos u) + b^2\right)^{3/2}}$$

$$T = \frac{\left[\frac{d\vec{r}}{du} \cdot \frac{d^2\vec{r}}{du^2} \cdot \frac{d^3\vec{r}}{du^3}\right]}{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|} = \frac{[-ab\cos u\,\hat{i} + ab\sin a\,\hat{j} + a^2(\cos u - 1)\hat{k} - (a\cos u\,\hat{i} - a\sin u\,\hat{j})}{[a^2 b^2 + a^4(\cos u - 1)^2}$$

$$T = \frac{-a^2 b\cos^2 u - a^2 b\sin^2 u}{(a^2 b^2 + a^4(\cos u - 1)^2}$$

$$T = \frac{-a^2 b(\cos^2 u + \sin^2 u)}{a^2 b^2 + a^4(\cos u - 1)^2}$$

$$\boxed{T = \frac{a^2 b}{a^2 b^2 + a^4(\cos u - 1)^2}}$$

8(C) Find the workdone in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field F is given by $F = (2x - y + 2z)i + (x + y - z)j + (3x - 2y - 5z)k$.

Sol'n: In the xy-plane, we have $z = 0$.

$$\therefore F = (2x - y)i + (x + y)j + (3x - 2y)k$$

The circle C is given by $x^2 + y^2 = 4$ (or) $x = 2\cos t,\ y = 2\sin t$

$$\therefore r = xi + yj = 2\cos t\, i + 2\sin t\, j$$

$$\therefore \frac{dr}{dt} = -2\sin t\, i + 2\cos t\, j$$

Also $F = (4\cos t - 2\sin t)i + (2\cos t + 2\sin t)j + (6\cos t - 4\sin t)k$

In moving round the circle once t will vary from 0 to 2π.

The required workdone is $= \int_C F \cdot dr = \int_0^{2\pi} F \cdot \frac{dr}{dt}\, dt$

$$= \int_0^{2\pi} [-2\sin t(4\cos t - 2\sin t) + 2\cos t(2\cos t + 2\sin t)]\, dt$$

$$= \int_0^{2\pi} [4(\sin^2 t + \cos^2 t) - 4\sin t \cos t]\, dt$$

$$= \int_0^{2\pi} (4 - 4\sin t \cos t)\, dt$$

$$= \left[4t - 2\sin 2t\right]_0^{2\pi}$$

$$= 8\pi$$

8(d) By using Gauss divergence theorem evaluate $\iint_S (x^2+y^2)\,ds$, where S is the surface of the cone $z^2 = 3(x^2+y^2)$ bounded by $z=0$ and $z=3$.

Sol'n.: Let S be the surface of the cone $z^2=3(x^2+y^2)$ bounded by the planes $z=0$ and $z=3$. The plane $z=3$ cuts the surface $z^2=3(x^2+y^2)$ in the circle $x^2+y^2=3$, $z=3$. Let S_1 be the plane region bounded by this circle. Let S' be the closed surface consisting of the surface S & S_1. Let us first put the integral

$\iint_S (x^2+y^2)\,ds$ in the form $\iint_S F\cdot\hat{n}\,ds$

where $\hat{n}$ is a unit vector along the outward drawn normal to the surface S whose equation is

$\phi(x,y,z) \equiv 3(x^2+y^2)-z^2=0$

we have
$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{6x\hat{i}+6y\hat{j}-2z\hat{k}}{\sqrt{36x^2+36y^2+4z^2}}$$
$$= \frac{3x\hat{i}+3y\hat{j}-z\hat{k}}{\sqrt{9(x^2+y^2)+z^2}}$$
$$= \frac{3x\hat{i}+3y\hat{j}-z\hat{k}}{\sqrt{3z^2+z^2}}$$
$$= \frac{3xi+3yj-zk}{2z} \quad (\because \text{ on } S\ \ 3(x^2+y^2)=z^2)$$

Now take $F = \frac{2z}{3}(xi+yj)$. Then on S, $F\cdot\hat{n} = x^2+y^2$

By Gauss divergence theorem, we have
$$\iint_{S'} F\cdot\hat{n}\,ds = \iiint_V \operatorname{div}F\,dv \quad \text{——} \ (1)$$

where V is the volume enclosed by the closed surface S'.

we have $\operatorname{div}F = \operatorname{div}\left(\frac{2}{3}zxi + \frac{2}{3}zyj\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2}{3}zx\right) + \frac{\partial}{\partial y}\left(\frac{2}{3}zy\right)$$

$$= \frac{2}{3}z + \frac{2}{3}z = \frac{4}{3}z.$$

$\therefore \iiint_V \operatorname{div} F\, dV = \iiint_V \frac{4}{3} z\, dv$, where V is the volume bounded by $z=0$, $z=3$ & $x^2+y^2=\frac{z^2}{3}$

$$= \frac{4}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=-\sqrt{\frac{z^2}{3}-y^2}}^{\sqrt{z^2/3-y^2}} z\, dz\,dy\,dx$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=0}^{\sqrt{\frac{z^2}{3}-y^2}} z\, dz\,dy\,dx$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-\frac{z}{\sqrt{3}}}^{z/\sqrt{3}} z\,[x]_{x=0}^{\sqrt{\frac{z^2}{3}-y^2}}\, dz\,dy$$

$$= 2\cdot\frac{8}{3}\int_{z=0}^{3}\int_{y=0}^{z/\sqrt{3}} z\left[\sqrt{\frac{z^2}{3}-y^2}\right] dz\,dy$$

$$= \frac{16}{3}\int_{z=0}^{3}\left[\frac{y}{2}\sqrt{\left(\frac{z^2}{3}-y^2\right)} + \frac{z^2}{6}\sin^{-1}\left(\frac{y}{z/\sqrt{3}}\right)\right]_{y=0}^{z/\sqrt{3}} dz$$

$$= \frac{16}{3}\int_{z=0}^{3}\left[0+\frac{z^2}{6}\sin^{-1}(1)\right]dz$$

$$= \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\int_0^3 z^3\,dz = \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\left[\frac{z^4}{4}\right]_0^3$$

$$= \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\cdot\frac{81}{4} = 9\pi$$