Date : 14/07/2019

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

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MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

$\frac{213}{250}$

PAPER - II : FULL SYLLABUS

TEST CODE: TEST-6: IAS(M)/14-JULY.-2019

Time: 3 Hours | **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has **33 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name YASH MESHRAM

Roll No. 115

Test Centre HOME

Medium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			09	
	(c)			08	40
	(d)			09	
	(e)			06	
2	(a)				
	(b)				
	(c)				
	(d)				
3	(a)				
	(b)				
	(c)				
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4	(a)			11	
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	(c)			10	
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5	(a)			09	
	(b)			09	
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	(e)			09	
6	(a)			05	
	(b)			12	
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7	(a)			11	
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8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					

$\frac{213}{250}$

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SECTION – A

1. (a) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some a, b, $\in$ G, find o(b). **[10]**

We have $aba^{-1} = b^2$ ⓵ and $a^5 = e$ ⓶, $a, b \in G$

From ⓵, $(aba^{-1})(aba^{-1}) = b^2 . b^2$

$\Rightarrow ab(a^{-1}a)ba^{-1} = b^4$

$\Rightarrow abeba^{-1} = b^4$ [Inverse property of group]

$\Rightarrow ab^2a^{-1} = b^4$ — ⓷ [$\because beb = b(eb) = b.b = b^2$]

From ⓵ & ⓷, $(aba^{-1})(ab^2a^{-1}) = b^2 . b^4$

$\Rightarrow ab(a^{-1}a)b^2a^{-1} = b^6$

$\Rightarrow ab^3a^{-1} = b^6$ — ⓸

From ⓷, $ab^2a^{-1} = b^4 \Rightarrow a(aba^{-1})a^{-1} = b^4$ [$\because aba^{-1} = b^2$]

$\Rightarrow a^2ba^{-2} = b^4$ — ⓹

From ⓷, $(ab^2a^{-1})(ab^2a^{-1}) = b^4 . b^4$

$\Rightarrow ab^4a^{-1} = b^8$ — ⓺

$\Rightarrow a(a^2ba^{-2})a^{-1} = b^8$ [From ⓹]

$\Rightarrow a^3ba^{-3} = b^8$ — ⓻

$\cancel{\text{Hence, } (a^3ba^{-3})(a^3ba^{-3}) = b^8 . b^8}$

$\cancel{\Rightarrow a^3b(a^{-3}a^3)ba^{-3} = b^{16}}$

$\cancel{\Rightarrow a^3b^2a^{-3}}$

From ⓺, $(ab^4a^{-1})(ab^4a^{-1}) = b^8 . b^8$

$\Rightarrow ab^8a^{-1} = b^{16}$ — ⓼

From ⓻ & ⓼,

$a(a^3ba^{-3})a^{-1} = b^{16}$, i.e., $a^4ba^{-4} = b^{16}$ — ⓽

From ⓼, $(ab^8a^{-1})(ab^8a^{-1}) = b^{16} . b^{16}$

$\Rightarrow ab^{16}a^{-1} = b^{32}$ — ⓾

From ⓽ & ⓾, $a(a^4ba^{-4})a^{-1} = b^{32}$

$\Rightarrow a^5ba^{-5} = b^{32}$

$\Rightarrow ebe = b^{32}$ [$\because a^5 = e$ and $a^{-5} = (a^5)^{-1} = e^{-1} = e$]

$\Rightarrow b = b^{32}$

$\Rightarrow b^{31} = e$

08

As 31 is a prime number, $O(b) = 31$ //

1. (b) Show that $Z[\sqrt{-3}]$ is not a UFD. **[10]**

Consider $4 = 4 + 0\sqrt{-3} \in Z[\sqrt{-3}]$

We have $4 = 2.2 = (1+\sqrt{-3})(1-\sqrt{-3})$ — ①

We need to show 2 and $1 \pm \sqrt{-3}$ are irreducible elements of $Z[\sqrt{-3}]$

~~Let $2 = (a+\sqrt{3}bi)(c+\sqrt{3}di)$ where $a,b,c,d \in Z$~~
~~$\therefore \bar{2} = 2 = (a-\sqrt{3}bi)(c-d\sqrt{3}i)$~~

Let $2 = (a+\sqrt{3}bi)(c+\sqrt{3}di)$ where $a,b,c,d \in Z$

$\therefore \bar{2} = 2 = (a-\sqrt{3}bi)(c-d\sqrt{3}i)$

$\therefore 4 = (a^2+3b^2)(c^2+3d^2)$

Consider the following cases :-

i> $a^2+3b^2 = 1$ and $c^2+3d^2 = 4$

ii> $a^2+3b^2 = 4$ and $c^2+3d^2 = 1$

iii> $a^2+3b^2 = 2$ and $c^2+3d^2 = 2$

Case iii> is not possible in Z

Case i> is possible where $a = \pm 1$ and $b = 0$, i.e., $a+b\sqrt{3}i = \pm 1$.

But ± 1 are units in $Z[\sqrt{-3}]$

Case ii> is possible when $c = \pm 1$ and $d = 0$, i.e., $c+d\sqrt{3}i = \pm 1$.

Here, ± 1 are units in $Z[\sqrt{-3}]$

Hence, 2 is an irreducible element of $Z[\sqrt{-3}]$. — ②

Let $1+\sqrt{-3} = (a+\sqrt{3}bi)(c+d\sqrt{3}i)$ where $a,b,c,d \in Z$

$\therefore 1-\sqrt{-3} = (a-\sqrt{3}bi)(c-d\sqrt{3}i)$

$\therefore 4 = (a^2+3b^2)(c^2+3d^2)$

Cases :- i> $a^2+3b^2 = 1$ and $c^2+3d^2 = 4$

ii> $a^2+3b^2 = 4$ and $c^2+3d^2 = 1$

iii> $a^2+3b^2 = 2$ and $c^2+3d^2 = 2$

Cases i> & ii> gives us $a = \pm 1, b = 0$ and $c = \pm 1, d = 0$ respectively. But these are units in $Z[\sqrt{-3}]$.

Case iii> is not possible in Z.

Hence, $1+\sqrt{-3}$ is irreducible in $Z[\sqrt{-3}]$. Similarly, $1-\sqrt{-3}$ is also irreducible in $Z[\sqrt{-3}]$. — ③

From ①, ② & ③, $4 \in \mathbb{Z}[\sqrt{-3}]$ has two distinct expressions as product of irreducible elements of $\mathbb{Z}[\sqrt{-3}]$.

$\therefore \mathbb{Z}[\sqrt{-3}]$ is not a UFD.

09

1. (c) Prove that between any two real roots of the equation $e^x \cos x + 1 = 0$ there is at least one real root of the equation $e^x \sin x + 1 = 0$. **[10]**

Let the two roots of the equation $e^x \cos x + 1 = 0$ be α & β — ①

$\therefore e^{\alpha} \cos\alpha + 1 = 0 \Rightarrow \cos\alpha = -e^{-\alpha}$ — ②

and $e^{\beta}\cos\beta + 1 = 0 \Rightarrow \cos\beta = -e^{-\beta}$ — ③

Here, we consider $\alpha < \beta$.

Consider $f(x) = -\cos x - e^{-x}$ — ④

As $\cos x$ and e^{-x} are continuous on $\mathbb{R}$, they are continuous on $[\alpha, \beta]$.

Hence, $f(x)$ is continuous on $[\alpha, \beta]$. — ⑤

Differentiating ④ wrt x, we get

$f'(x) = \sin x + e^{-x}$ — ⑥

We have $f(\alpha) = -\cos\alpha - e^{-\alpha} = 0$ and $f(\beta) = -\cos\beta - e^{-\beta} = 0$ — ⑦

$[\because \cos\alpha = -e^{-\alpha} \text{ \& } \cos\beta = -e^{-\beta}]$

From ⑤, ⑥ & ⑦, $f(x)$ is continuous over $[\alpha, \beta]$ and $f(\alpha) = f(\beta) = 0$.

Hence, by Rolle's Theorem, $\exists\, \gamma$, $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$, i.e., $\sin\gamma + e^{-\gamma} = 0$

$$\Rightarrow e^{\gamma}\sin\gamma + 1 = 0 \text{ //}$$

Hence, we conclude that between any two real roots of $e^x\cos x + 1 = 0$, there exists atleast one real root of the equation $e^x \sin x + 1 = 0$.

08

1. (d) Prove that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} (z \neq 0), f(0) = 0;$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there **[10]**

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3-y^3}{x^2+y^2} + i\frac{(x^3+y^3)}{x^2+y^2} = u + iv$$

Here, $u = \dfrac{x^3 - y^3}{x^2+y^2}$ and $v = \dfrac{x^3+y^3}{x^2+y^2}$.

$$\therefore \left.\frac{\partial u}{\partial x}\right|_{(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{\frac{x^3}{x^2} - 0}{x} = 1$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,0)} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y^3/y^2 - 0}{y} = -1$$

$$\left.\frac{\partial v}{\partial x}\right|_{(0,0)} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x^3/x^2 - 0}{x} = 1$$

$$\left.\frac{\partial v}{\partial y}\right|_{(0,0)} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y^3/y^2 - 0}{y} = 1$$

We know that u & v are rational & finite for all values of $z \neq 0$
So, u & v are continuous at all those points where $z \neq 0$.
At origin, $f(z) = 0 \Rightarrow u = 0, v = 0$
$\therefore f(z)$ is continuous at origin and everywhere else.
Also, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at origin
Hence, Cauchy Riemann equations are satisfied. //
We have $f'(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{(x,y)\to(0,0)} \left[\frac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{(x+iy)}\right]$
We move along $y = 0$ towards $(0,0)$, then
$f'(z) = \lim_{x \to 0} \frac{x^3(1+i)}{x^2 \cdot x} = (1+i)$
Again, we move along $y = x$ towards $(0,0)$, then
$f'(z) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3+x^3)}{x^2+x^2} \frac{1}{(x+ix)} = \lim_{x \to 0} \frac{2ix^3}{2x^2 \cdot x(1+i)} = \frac{1}{2}(1-i)$
We see that $f'(z)$ is not unique
Therefore, $f'(z)$ doesn't exist at origin. //

1. (e) Obtain the dual of the LP problem :
Min. $z = x_1 + x_2 + x_3$ subject to the constraints :
$x_1 - 3x_2 + 4x_3 = 5$, $x_1 - 2x_2 \leq 3$, $2x_2 - x_3 \geq 4$; $x_1, x_2 \geq 0$ and x_3 is unrestricted. [10]

09

Primal :- Maximize $z = -x_1 - x_2 - x_3' + x_3''$ $\left[\because x_3 \text{ is unrestricted}, x_3 = x_3' - x_3''\right]$

subject to $x_1 - 3x_2 + 4x_3' - 4x_3'' \leq 5$

$$-x_1 + 3x_2 - 4x_3' + 4x_3'' \leq -5$$

$$x_1 - 2x_2 + 0x_3' + 0x_3'' \leq 3$$

$$0x_1 - 2x_2 + x_3' - x_3'' \leq -4$$

$$x_1, x_2, x_3', x_3'' \geq 0$$

Dual of LP problem is given by :-

Minimize $z = 5\omega_1 - 5\omega_2 + 3\omega_3 - 4\omega_4$

subject to $\omega_1 - \omega_2 + \omega_3 \geq -1$

$$-3\omega_1 + 3\omega_2 - 2\omega_3 - 2\omega_4 \geq -1$$

$$4\omega_1 - 4\omega_2 + \omega_4 \geq -1$$
$$-4\omega_1 + 4\omega_2 - \omega_4 \geq 1$$
$$\omega_1, \omega_2, \omega_3, \omega_4 \geq 0$$

Rewriting dual complex and taking $\omega^* = \omega_1 - \omega_2$

Minimize $z = 5\omega^* + \omega_3 - 4\omega_4$

subject to $\omega^* + \omega_3 \geq -1$

$$-3\omega^* + 2\omega_3 + 2\omega_4 \leq 1$$
$$4\omega^* + \omega_4 = -1$$
$\omega_3, \omega_4 \geq 0$ and ω^* unrestricted

06

2. (a) Let R be a commutative ring with unity. An ideal M of R is maximal ideal of R iff $\frac{R}{M}$ is a field. [15]

4. (a) Let G be the set of all real 2 × 2 matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where ad ≠ 0, under matrix multiplication. Let $N = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$. Prove that

(a) N is a normal subgroup of G.

(b) G/N is abelian. **[13]**

a) $G = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} / a, b, d \in \mathbb{R} \text{ and } ad \neq 0 \right\}$

Let $g \in G$ where $g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$\therefore g^{-1} = \frac{1}{ad}\begin{bmatrix} d & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix}$ {Property of matrices}

$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} / b \in \mathbb{R} \right\}$

Let $h \in N$, such that $h = \begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix}$

Consider $ghg^{-1} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 & b_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix} = \begin{bmatrix} a & ab_1+b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix}$

$= \begin{bmatrix} a \cdot 1/a & a\left(\frac{-b}{ad}\right) + \frac{ab_1+b}{d} \\ 0 & d \cdot 1/d \end{bmatrix}$

$\therefore ghg^{-1} = \begin{bmatrix} 1 & K \\ 0 & 1 \end{bmatrix} \in N \quad \left[\because ab \in \mathbb{R} \text{ and since } ad \neq 0, \text{ and } d \neq 0\right]$ $K = \frac{ab_1+b}{d} - \frac{b}{d}$

$ghg^{-1} \in N \Rightarrow N$ is a normal subgroup of G.

b) Let Nx, Ny be any 2 elements belonging to G/N and $x, y \in G$

We know that $\frac{G}{N}$ will be abelian iff $Nx \cdot Ny = Ny \cdot Nx$

i.e., $Nxy = Nyx$, i.e., $xy(yx)^{-1} \in N$, i.e., $xyx^{-1}y^{-1} \in N$

Let, $x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $y = \begin{bmatrix} l & m \\ 0 & n \end{bmatrix}$

$\therefore xyx^{-1}y^{-1} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} l & m \\ 0 & n \end{bmatrix} \begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix} \begin{bmatrix} 1/l & -m/ln \\ 0 & 1/n \end{bmatrix}$

$= \begin{bmatrix} al & am+bn \\ 0 & dn \end{bmatrix} \begin{bmatrix} 1/al & -m/aln - b/adn \\ 0 & 1/dn \end{bmatrix}$

$= \begin{bmatrix} 1 & (am+bn)\left(-\frac{m}{aln} - \frac{b}{adn}\right) \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}$ where $p = (am+bn)\left(\frac{-m}{aln} - \frac{b}{adn}\right)$

$\in \mathbb{R}$ as $a, b, d, l, m, n \in \mathbb{R}$ and $a, d, l, n \neq 0$

$\in N$

$\therefore xyx^{-1}y^{-1} \in N$

$\therefore \frac{G}{N}$ is an abelian group.

4. (b) Show that the function $f(x) = \sin\frac{1}{x}$ when x irrational,

$= 0$ Otherwise,

is not Riemann integrable on [0, 1]. **[10]**

Clearly, $f(x)$ is bounded on $[0,1]$ as $0 \le f(x) \le 1 \ \forall x \in [0,1]$

Let $P = \{0 = x_0, x_1, x_2 \ldots\ldots x_n = 1\}$ be a partition of $[a,b]$ and $I_r = [x_{r-1}, x_r]$

$r \in \mathbb{Z}^+$ be its sub-interval. Length of $I_r = \delta_r = x_r - x_{r-1}$

I_r contains rational as well as irrational numbers.

Let M_r and m_r be supremum & infimum of function $f(x)$ in I_r respectively $\Rightarrow m_r = 0$ and $M_r = \sin\frac{1}{x}$ for $r \in \mathbb{Z}^+$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n}\left[\sin\frac{1}{x}\right](x_r - x_{r-1}) = \int_0^1 \sin\frac{1}{x}\,dx$ — ① $\rightarrow$ By definition

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \int_0^1 0\,dx = 0$ — ②

From ①, $U(P,f) = \int_0^1 \sin\frac{1}{x}\,dx$

$\sin\frac{1}{x} > 0$ when $x \in [0,1]$

$\therefore \int_0^1 \sin\frac{1}{x}\,dx > \int_0^1 0\,dx$ — ③ [Property of Integration]

$\therefore$ From ①, ② & ③, $U(P,f) > L(P,f)$, i.e.,

$$U(P,f) \neq L(P,f)$$

Hence, f is not Riemann Integrable on $[0,1]$ //

08

4. (c) Using Cauchy's/Cauchy's integral formula evaluate.

(i) $\int_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1| = 1$.

(ii) $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3}dz$. If C is the circle $|z| = 1$. **[12]**

Cauchy's Integral formula :- $f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

i> $z^2+2z+5=0 \Rightarrow z=-1-2i$ and $z=-1+2i$ [Here, f(z) is not analytic]

C: circle $|z+1|=1$

For $z=-1-2i$, LHS $= |z+1| = |-1-2i+1| = 2 >$ RHS

$\therefore$ It lies outside C

For $z=-1+2i$, LHS $= |z+1| = |-1+2i+1| = 2 >$ RHS

$\therefore$ It also lies outside C

Hence, by Cauchy's Theorem, $\boxed{\int_C \frac{z+4}{z^2+2z+5}dz = 0}$

ii> $I = \oint_C \frac{\sin^6 z\,dz}{(z-\pi/6)^3}$, C: $|z|=1$

$f(z) = \frac{\sin^6 z}{(z-\pi/6)^3}$ is not analytic at $z=\frac{\pi}{6}$ and $z=\frac{\pi}{6}$ lies inside C

$\therefore \frac{d^2}{dz^2}[\sin^6 z]_{z=\frac{\pi}{6}} = \frac{2!}{2\pi i}\int_C \frac{\sin^6 z\,dz}{(z-\pi/6)^{2+1}}$

$\therefore \int_C \frac{\sin^6 z\,dz}{(z-\frac{\pi}{6})^{2+1}} = \frac{2\pi i}{2}\frac{d}{dz}\left[6\sin^5 z\cos z\right]_{z=\frac{\pi}{6}}$

$= 6\pi i\left[-\sin^6 z + 5\sin^4 z\cos^2 z\right]_{z=\frac{\pi}{6}}$

$= 6\pi i\left[-\left(\frac{1}{2}\right)^6 + 5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2\right]$

$= 6\pi i\left[\frac{1}{2^6}(14)\right] = \frac{84\pi i}{64} = \frac{21\pi i}{16}$

$\therefore \boxed{\oint_C \frac{\sin^6 z\,dz}{(z-\frac{\pi}{6})^3} = \frac{21\pi i}{16}}$

10

4. (d) Maximize $z = 4x_1 + 5x_2 - 3x_3 + 50$, subject to the constraints :

$x_1 + x_2 + x_3 = 10$

$x_1 - x_2 \geq 1$

$2x_1 + 3x_2 + x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$

[15]

The above LPP can be written as :-

Max. $z^* = 4x_1 + 5x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$ [Leave +50 as it can be added afterwards]

subject

$x_1 + x_2 + x_3 + A_1 = 10$

$x_1 - x_2 - S_1 + A_2 = 1$

$2x_1 + 3x_2 + x_3 + S_2 = 40$

and $x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

SIMPLEX METHOD (BIG-M)

1st Iteration :-

Basis	c_j	4	5	-3	0	0	-M	-M		
	C_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	b	θ (Ratio)
A_1	-M	1	1	1	0	0	1	0	10	$\frac{10}{1} = 10$
A_2	-M	(1)	-1	0	-1	0	0	1	1	$\frac{1}{1} = 1 \rightarrow$
S_2	0	2	3	1	0	1	0	0	40	$40/2 = 20$
z_j		-2M	0	-M	M	0	-M	-M	-11M	
$c_j - z_j$		2M+4	5	M-3	M	0	0	0		
		↑								

Optimal solution isn't arrived as all $c_j - z_j \nleq 0$

2nd iteration :-

Basis	c_j	4	5	-3	0	0	-M		
	C_B	x_1	x_2	x_3	S_1	S_2	A_1	b	θ (Ratio)
A_1	-M	0	(2)	1	1	0	1	9	$9/2 = 4.5 \rightarrow$
x_1	4	1	-1	0	-1	0	0	1	—
S_2	0	0	5	1	2	1	0	38	$\frac{38}{5} = 7.6$
z_j		4	-2M-4	-M	-M-4	0	-M	4-9M	
$c_j - z_j$		0	2M+9	M-3	M+4	0	0		
			↑						

Optimal solution isn't arrived as all $c_j - z_j \nleq 0$

3^{rd} Iteration :-

Basis	C_j	4	5	-3	0	0		
	C_B	x_1	x_2	x_3	s_1	s_2	b	θ (Ratio)
x_2	5	0	1	1/2	1/2	0	9/2	
x_1	4	1	0	1/2	-1/2	0	11/2	
s_2	0	0	0	-3/2	-1/2	1	31/2	
z_j		4	5	9/2	1/2	0	89/2	
$c_j - z_j$		0	0	-15/2	-1/2	0		

Here, all $c_j - z_j \le 0$. Hence, optimality is arrived.

$\therefore z^*_{max} = \frac{89}{2}$ where $x_1 = \frac{11}{2}$, $x_2 = \frac{9}{2}$, $x_3 = 0$

$\therefore z_{max} = z^*_{max} + 50 = \frac{89}{2} + 50 = \frac{189}{2}$ //

14

SECTION – B

5. (a) Find a complete integral of $z^2(p^2 + q^2) = x^2 + y^2$, i.e.,

$z^2\left[(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right] = x^2 + y^2$. **[10]**

$z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2 \Rightarrow \left[\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2$ – ①

Let $dZ = z\,dz$ so that $Z = z^2/2$ – ②

$\therefore \left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2+y^2$ or $P^2 + Q^2 = x^2+y^2$ – ③

where $P = \frac{\partial Z}{\partial x}$ & $Q = \frac{\partial Z}{\partial y}$

From ③, $P^2 - x^2 = -Q^2 + y^2$ [Separation of variables]

$\therefore$ Equating each side to c^2 (an arbitrary constant)

$\therefore P^2 - x^2 = c^2$ and $-Q^2 + y^2 = c^2$

$\Rightarrow P = \sqrt{x^2+c^2}$ and $Q = \sqrt{y^2-c^2}$

$\Rightarrow dZ = P\,dx + Q\,dy = \sqrt{x^2+c^2}\,dx + \sqrt{y^2-c^2}\,dy$ – ④

Integrating (4), we get

$$z = \int\sqrt{x^2+c^2}\,dx + \int\sqrt{y^2-c^2}\,dy$$

$$= \frac{x}{2}\sqrt{x^2+c^2} + \frac{c^2}{2}\log\{x+\sqrt{x^2+c^2}\} + \frac{y}{2}\sqrt{y^2-c^2} + \frac{c^2}{2}\log\{y+\sqrt{y^2-c^2}\} + b \quad - (5)$$

From (2) & (5),

$$\frac{z^2}{2} = \frac{x}{2}\sqrt{x^2+c^2} + \frac{y}{2}\sqrt{y^2-c^2} + \frac{c^2}{2}\log\{x+\sqrt{x^2+c^2}\}\{y+\sqrt{y^2-c^2}\} + b$$

$$\therefore z^2 = x\sqrt{x^2+c^2} + y\sqrt{y^2-c^2} + c^2\log\{x+\sqrt{x^2+c^2}\}\{y+\sqrt{y^2-c^2}\}^{-1} + b$$

where b & c are arbitrary constants.

i.e. $$z^2 = x\sqrt{x^2+c^2} + y\sqrt{y^2-c^2} + c^2\log\frac{x+\sqrt{x^2+c^2}}{y+\sqrt{y^2-c^2}} + b$$ //

09

5. (b) Solve $\left(D^3 - 4D^2D' + 5DD'^2 - 2D'^3\right)z = e^{y+2x} + (y+x)^{1/2}$ **[10]**

Auxiliary Equation :- $m^3 - 4m^2 + 5m - 2 = 0$

$\Rightarrow (m-1)(m^2-3m+2) = 0 \Rightarrow (m-1)^2(m-2) = 0$

1	1	−4	5	−2
	↓	1	−3	2
	1	−3	2	0

$\therefore CF = \phi_1(y+2x) + x\phi_2(y+x) + \phi_3(y+x)$

P.I. corresponding to $e^{y+2x} = \dfrac{1}{(D-D')^2(D-2D')}e^{y+2x}$

$= \dfrac{1}{(2-1)^2(D-2D')}e^{y+2x} = \dfrac{1}{(D-2D')}e^{y+2x}$

$= e^{y+2x}\cdot\dfrac{1}{D+2-2D'-2}\cdot 1 = e^{y+2x}\cdot\dfrac{1}{D-2D'}\cdot 1$

$= e^{y+2x}\cdot\dfrac{1}{D}\left(1-\dfrac{2D'}{D}\right)^{-1}[1] = e^{y+2x}\cdot\dfrac{1}{D}\cdot[1] = xe^{y+2x}$

P.I. corresponding to $(y+x)^{1/2} = \dfrac{1}{(D-D')^2(D-2D')}\sqrt{y+x}$

$= \dfrac{1}{(D-D')^2}\left\{\dfrac{1}{D-2D'}(y+x)^{1/2}\right\} = \dfrac{1}{(D-D')^2}\dfrac{1}{(1-2)}\int u^{1/2}du$ where $u = y+x$

$$= -\frac{1}{(D-D')^2}\cdot\frac{2}{3}(y+x)^{3/2} = -\frac{2}{3}\cdot\frac{x^2}{2!}(y+x)^{3/2}$$
$$= -\frac{x^2}{3}(y+x)^{3/2}$$

$\therefore$ Required general solution is

$$z = \phi_1(y+2x) + x\phi_2(y+x) + \phi_3(y+x) + xe^{y+2x} - \frac{x^2}{3}(y+x)^{3/2}$$

where ϕ_1, ϕ_2 & ϕ_3 are arbitrary functions.

09

5. (c) The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5. **[10]**

As per the given conditions, $0.5 = 4e^{-2t} + e^{-0.1t}$

Consider $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$

To find out the time required for the bacteria concentration to be 0.5, we need to find the roots of $f(t)$.

By Newton Raphson method, $t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$

We have $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$

$\therefore f'(t) = -8e^{-2t} - 0.1e^{-0.1t} \neq 0$ for any value of t

Hence, we can proceed. Let the initial time be $t_0 = 0$

$\therefore t_1 = t_0 - \frac{f(t_0)}{f'(t_0)} = 0 - \frac{4.5}{-8.1} = 0.55555$

$$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = 0.55555 - \frac{1.76273}{-2.72843} = 1.20168$$

$$t_3 = t_2 - \frac{f(t_2)}{f'(t_2)} = \cancel{1.20168} \approx 2.12340$$

$$t_4 = t_3 - \frac{f(t_3)}{f'(t_3)} = 3.99661$$

$$t_5 = t_4 - \frac{f(t_4)}{f'(t_4)} = 6.46087$$

$$t_6 = t_5 - \frac{f(t_5)}{f'(t_5)} = 6.92059$$

$$t_7 = t_6 - \frac{f(t_6)}{f'(t_6)} = 6.93154$$

$$t_8 = t_7 - \frac{f(t_7)}{f'(t_7)} = 6.93155$$

$$t_9 = t_8 - \frac{f(t_8)}{f'(t_8)} = 6.93155, \quad t_{10} = t_9 - \frac{f(t_9)}{f'(t_9)} = 6.93155$$

$\therefore$ Time required is $\underline{\underline{6.93155 \text{ units}}}$

0

5. (d) Convert the following

(i) 7765_8 to decimal

(ii) 199.3 to octal and then to binary

(iii) $FB17_{16}$ to binary

(iv) $641A_{16}$ to octal. **[10]**

$$i)\ (7765)_8 = (\underset{7}{111}\ \underset{7}{111}\ \underset{6}{110}\ \underset{5}{101})_2$$

$$\therefore (111111110101)_2 = 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 0 + 2^2 + 0 + 1$$
$$= (4085)_{10} //$$

$$ii)\ (199.3)_{10} = (307.23146314631463 14\ldots\ldots)_8 \quad 4$$
$$= (011\ 000\ 111.010011001100110 0)_2$$
$$= (11000111.0100\ 11001100 1100)_2 \quad //$$

iii) $(FB17)_{16} = (F\ B\ 1\ 7)_{16}$
$= (1111\ 1011\ 0001\ 0111)_{16}$
$= (1111101100010111)_{16}$ //

iv) $(641A)_{16} = (6\ 4\ 1\ A)_{16} = (0110\ 0100\ 0001\ 1010)_2$
$= (000\ 110\ 010\ 000\ 011\ 010)_2$
$= (0\ 6\ 2\ 0\ 3\ 2)_8$
$= (62032)_8$ //

09

5. (e) Show that u = 2cxy, v = c(a² + x² − y²) are the velocity components of a possible fluid motion. Determine the stream function. [10]

We have $u = 2cxy$ and $v = c(a^2+x^2-y^2)$

Fluid motion is possible if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ [in 2-D]

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[2cxy] = 2cy$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}[c(a^2+x^2-y^2)] = 0 + 0 - 2cy = -2cy$

$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2cy - 2cy = 0 \Rightarrow$ Fluid motion is possible //

Let ψ be the stream function.

We know that $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$

$\Rightarrow 2cxy = -\frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial x} = c(a^2+x^2-y^2)$ — ①

$\Rightarrow \psi = -cxy^2 + f(x,t)$ — ②

From ②, differentiating it wrt x

$$\frac{\partial\psi}{\partial x} = -cy^2 + \frac{\partial f(x,t)}{\partial x} \quad - ③$$

From ① & ③, $c(a^2+x^2-y^2) = -cy^2 + \frac{\partial f}{\partial x}(x,t)$

$\therefore \frac{\partial f}{\partial x}(x,t) = c(a^2+x^2) \Rightarrow f(x,t) = c(a^2x+\frac{x^3}{3})+g(y,t)$

$\therefore f(x,t) = c(a^2x+\frac{x^3}{3})+g(y,t)$

where f & g are arbitrary function of x & t and y & t respectively

Hence, $\psi = c(a^2x+\frac{x^3}{3}-xy^2)+g(y,t)$ is the required stream function.

09

6. (a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x^2+y^2+z^2, z^2-2xy)=0$. **[06]**

$\phi(x^2+y^2+z^2, z^2-2xy)=0 = \phi(u,v)$ where

$u=x^2+y^2+z^2$ and $v=z^2-2xy$

Partial differential equation for the above the given conditions will be of the form $Pp+Qq=R$ where

$$P = \frac{\partial(u,v)}{\partial(y,z)}, \quad Q = \frac{\partial(u,v)}{\partial(z,x)}, \quad R = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\therefore P = \frac{\partial(u,v)}{\partial(y,z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 2y & 2z \\ -2x & 2z \end{vmatrix} = 4z(x+y)$$

$$Q = \frac{\partial(u,v)}{\partial(z,x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 2z & 2x \\ 2z & -2y \end{vmatrix} = -4z(x+y)$$

$$R = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -2y & -2x \end{vmatrix} = -4(x^2 - y^2)$$

$$\therefore \quad Pp + Qq = R$$

$$\Rightarrow 4z(x+y)p + [-4z(x+y)]q = -4(x^2 - y^2)$$

$$\Rightarrow z(p-q) = -\frac{(x^2-y^2)}{(x+y)} \Rightarrow \boxed{z(p-q) = y-x}$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

05

6. (b) Reduce $x(\partial^2 z/\partial x^2) + \partial^2 z/\partial y^2 = x^2$ $(x > 0)$ to canonical form. **[13]**

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2,\ x>0 \Rightarrow xr + t = x^2 \quad — (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, we get

$R = x,\ S = 0,\ T = 1,\ f(x,y,z,p,q) = -x^2$

$S^2 - 4RT = 0^2 - 4(x)(1) = -4x <$ as $x > 0$

$\Rightarrow$ (1) is elliptic

Consider, $R\lambda^2 + S\lambda + T = 0 \Rightarrow x\lambda^2 + 0 + 1 = 0$

$\Rightarrow \lambda^2 = -1/x \quad \Rightarrow \quad \lambda = ix^{-1/2},\ -ix^{-1/2}$

Corresponding characteristic equations :- $\frac{dy}{dx} \pm ix^{-1/2} = 0$

$\Rightarrow y + i(2x^{1/2}) = c_1$ and $y - i(2x^{1/2}) = c_2$

choose $\alpha = y$ and $\beta = 2x^{1/2}$ as independent variables — (2)

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta} \quad — (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \quad — (4)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right) = -\frac{x^{-3/2}}{2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\frac{x^{-3/2}}{2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{0 + x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right\} = \frac{-x^{-3/2}}{2}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \quad - (5)$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\cdot\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \beta}{\partial y} = \frac{\partial^2 z}{\partial \alpha^2} \quad - (6)$$

From (1), (3), (4), (5) & (6), $x\left(\frac{\partial^2 z}{\partial x^2}\right) + \left(\frac{\partial^2 z}{\partial y^2}\right) = x^2$

$$\Rightarrow \frac{\beta^2}{4}\left[\frac{4}{\beta^2}\frac{\partial^2 z}{\partial \beta^2} - \frac{8}{2\beta^3}\frac{\partial z}{\partial \beta}\right] + \frac{\partial^2 z}{\partial \alpha^2} = \frac{\beta^4}{16}$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} - \frac{1}{\beta}\frac{\partial z}{\partial \beta} = \frac{\beta^4}{16}}$$ is the required canonical form

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6. (c) Find the characteristics of the equation

$pq = xy$

and determine the integral surface which passes through the curve $z = x$, $y = 0$. **[15]**

$pq = xy$ is the given equation $- (1)$

Integral surface passes through $z = x$ and $y = 0$, $- (2)$

Consider, parametric form $\equiv$ $x = \lambda$, $y = 0$, $z = \lambda$ $- (3)$

where λ is a parameter

Let the initial values of p, q, x, y, z be p_0, q_0, x_0, y_0, z_0 respectively $- (4)$

$x_0 = \lambda$, $y_0 = 0$, $z_0 = \lambda$ and $p_0 q_0 = x_0 y_0 = 0$ $- (5)$

We have $z_0' = x_0' p_0 + y_0' q_0 \Rightarrow 1 = 1(p_0) + 0 \Rightarrow p_0 = 1$

$\therefore q_0 = \frac{0}{p_0} = \frac{0}{1} = 0$

$\therefore x_0 = \lambda$, $y_0 = 0$, $z_0 = \lambda$, $p_0 = 1$, $q_0 = 0$ $- (6)$

Consider $f(x, y, z, p, q) = pq - xy = 0$ $- (7)$

Hence, the characteristic equations are :-

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = q \quad — \quad (8)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = p \quad — \quad (9)$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = pq + qp = 2pq \; - \; (10)$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -(-y) - p(0) = y \; - \; (11)$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -(-x) - q(0) = x \; - \; (12)$$

From (8) & (12), $\frac{dx/dt}{dq/dt} = \frac{q}{x} \Rightarrow \frac{dx}{dq} = \frac{q}{x}$

$\Rightarrow x\,dx - q\,dq = 0 \Rightarrow x^2 - q^2 = c_1^2 \; - \; (13)$

From (9) & (11), $\frac{dy/dt}{dp/dt} = \frac{p}{y} \Rightarrow y\,dy - p\,dp = 0$

$\Rightarrow y^2 - p^2 = c_2^2 \; - \; (14)$

From (6) & (13), $\lambda^2 - 0^2 = c_1^2 \Rightarrow c_1^2 = \lambda^2 \Rightarrow x^2 = \lambda^2 + q^2 - (15)$

From (6) & (14), $0 - 1^2 = c_2^2 \Rightarrow c_2^2 = 1 \Rightarrow y^2 = 1 + p^2 - (16)$

Also, from (8), $q = \frac{dx}{dt} \Rightarrow \frac{dq}{dt} = \frac{d^2x}{dt^2} = x$ as $\frac{dq}{dt} = x$

$\therefore \frac{d^2x}{dt^2} - x = 0 \Rightarrow x = Ae^t + Be^{-t} \; - \; (17)$

From (9) & (11) $\frac{d^2y}{dt^2} = \frac{dp}{dt} = y$

$\therefore \frac{d^2y}{dt^2} = y \Rightarrow y = Ce^t + De^{-t} \; - \; (18)$

From (12) & (17), $\frac{dq}{dt} = Ae^t + Be^{-t} \Rightarrow q = Ae^t - Be^{-t} \; - \; (19)$

From (11) & (18), $\frac{dp}{dt} = y \Rightarrow \frac{dp}{dt} = Ce^t + De^{-t} \Rightarrow p = Ce^t - De^{-t} \; (20)$

$\therefore z = 2pq = 2[Ae^t - Be^{-t}][Ce^t - De^{-t}] = 2ACe^{2t} + 2BDe^{-2t} - 2(AD+BC) \; (21)$

Initial condition :- At $t = 0$, $q_0 = 0$ & $p_0 = 1$

$\therefore 0 = A - B$, $C - D = 1$, $\lambda = A + B$,

$0 = C + D$ and $p_0q_0 - x_0y_0 = 0$ $pq - xy = 0$

$\therefore (C-D)(A-B) = (A+B)(C+D)$

$\therefore C = D = \frac{1}{2}$ and $A = B = \frac{\lambda}{2}$

Characteristics :-

$x = Ae^t + Be^{-t}$

$y = Ce^t + De^{-t}$

$p = Ce^t - De^{-t}$

$q = Ae^t - Be^{-t}$

$z = 2ACe^{2t} + 2BDe^{-2t} - 2(AD+BC)$

$\therefore x = \lambda \frac{(e^t+e^{-t})}{2}, y = \frac{e^t - e^{-t}}{2}, z = \frac{\lambda}{2}e^{2t} + \frac{\lambda}{2}e^{-2t} - \lambda$

$\therefore \frac{x^2}{\lambda^2} + y^2 = \frac{1}{4}[e^{2t}+e^{-2t}+2+e^{2t}+e^{-2t}-2] = \frac{e^{2t}+e^{-2t}}{2}$

$\therefore \cancel{\frac{x^2}{\lambda^2} + y^2 = \frac{z+\lambda}{\lambda}}$ and $\frac{x^2}{\lambda^2} - y^2 = \frac{1}{4}[4] = 1$

$\Rightarrow \lambda^2 = x^2/y^2+1 \quad \Rightarrow \lambda = \pm\, x/\sqrt{1+y^2}$

We have $z = \frac{\lambda}{2}(e^{2t}+e^{-2t}-2) = \frac{\lambda}{2}(e^t+e^{-t})^2$

$\therefore z = \frac{\lambda}{2} \cdot \frac{4x^2}{2\lambda^2} = \frac{2x^2}{2\lambda} = \cancel{2x^2} \frac{x^2}{x/\sqrt{1+y^2}} = x\sqrt{1+y^2}$ [+ve sign is taken as $z = x$ at $y = 0$]

$\therefore z = x\sqrt{1+y^2}$ //

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6. (d) Obtain temperature distribution y(x, t) in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Further it is given that y(x, 0) = 1 – x, 0 < x < 1. **[16]**

Bar is placed along the x-axis with its one-end at origin (at 10°C) and other (at 0°C) at $x = 1$. K is the thermal conductivity

Heat Equation :- $\frac{\partial y}{\partial t} = K \frac{\partial^2 y}{\partial x^2}$ — ①

Initial conditions :- $y(x,0) = 1-x$ — ②

Boundary conditions :- $y_x(1,t) = 0, y(0,t) = 10$ — ③

Let $y(x,t) = u(x,t) + 10$, i.e., $u(x,t) = y(x,t) - 10$ — ④

$\therefore$ New Initial conditions :- $u(x,0) = -x-9$ — ⑤

and Boundary conditions :- $u_x(1,t) = 0, u(0,t) = 0$ — ⑥

$\therefore \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$ — ⑦

Let the solution of ⑦ be of the form $u(x,t) = X(x)\,T(t)$ ⑧

From ⑦ & ⑧, $XT' = KX''T \Rightarrow \frac{X''}{X} = \frac{T'}{KT} = \mu$ (say) — ⑨

Separation of variables :- $X'' - \mu X = 0$ — (10)

$T' - \mu K T = 0$ — (11)

From (6) & (8), $X'(1)T(t) = 0 = X(0)T(t)$. But, as $T(t) = 0$ means $u = 0$, so we discard $T(t) \neq 0$

Hence, $X'(1) = 0$ and $X(0) = 0$ — (11)

- Case I :- Let $\mu = 0 \Rightarrow X(x) = Ax + B$ — (12)

From (11) & (12), $0 = A = B$ which means $X(x) = 0$, i.e., $u = 0$.

Discard $\mu = 0$

- Case II :- Let $\mu = \lambda^2, \lambda \neq 0 \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x} \Rightarrow X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ (13)

From (11) & (13), $0 = A + B$ & $0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda} \Rightarrow A = B = 0$

which gives $X(x) = 0$, i.e., $u = 0$. Discard $\mu = \lambda^2$

- Case III :- Let $\mu = -\lambda^2, \lambda \neq 0 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$ — (14)

$X'(x) = B\lambda\cos\lambda x - A\lambda\sin\lambda x$. From (11) & (14), $B\lambda\cos\lambda - A\lambda\sin\lambda = 0$ and $A = 0 \Rightarrow B\lambda\cos\lambda = 0 \Rightarrow \cos\lambda = 0$ as B cannot be zero.

Now, $\cos\lambda = 0 \Rightarrow \lambda = (2n-1)\frac{\pi}{2}, n \in Z^+$ — (15)

$\therefore$ Non-zero solutions of (14) are $X_n(x) = B_n \sin\{(2n-1)\frac{\pi}{2}x\}$ (16)

From (11) & (15), $T' = -(2n-1)^2\frac{\pi^2}{4}KT \Rightarrow \frac{dT}{T} = -c_n^2 dt$

$\Rightarrow T_n(t) = D_n e^{-c_n^2 t}$ where $c_n^2 = (2n-1)^2\frac{\pi^2}{4}K$ — (17)

From (16) & (17), $u_n(x,t) = X_n T_n = E_n \sin\{(2n-1)\frac{\pi}{2}x\}e^{-c_n^2 t}$ where $E_n = B_n D_n$

More general solution $\equiv u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\{(2n-1)\frac{\pi}{2}x\}e^{-c_n^2 t}$ (18)

From (5) & (18), $-x-9 = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$ — (19)

Multiplying both sides of (19) by $\sin\{(2m-1)\frac{\pi}{2}x\}$ and integrating, we get

$$-\int_0^1 (x+9)\sin\{(2m-1)\frac{\pi}{2}x\}dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}dx$$

We know that $\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}dx = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$

Hence, for $m = n$, $E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}dx$

$$= -2\left[(x+9)\frac{2}{\pi(2n-1)}\left\{-\cos\frac{(2n-1)\pi}{2}x\right\} + \frac{4}{\pi^2(2n-1)^2}\sin(2n-1)\frac{\pi}{2}x\right]_{x=0}^{x=1}$$

$$= \frac{(-1)^n 8}{\pi^2(2n-1)^2} - \frac{36}{\pi(2n-1)}$$

$\therefore$ Solution is ~~given by~~ $y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin(2n-1)\frac{\pi x}{2} e^{-C_n^2 t}$

where $C_n = (2n-1)^2 \frac{\pi^2}{4} K$ and $E_n = \frac{(-1)^n 8}{\pi^2(2n-1)^2} - \frac{36}{\pi(2n-1)}$ //

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7. (a) Solve the equations :

$10x_1 - 2x_2 - x_3 - x_4 = 3$

$-2x_1 + 10x_2 - x_3 - x_4 = 15$

$-x_1 - x_2 + 10x_3 - 2x_4 = 27$

$-x_1 - x_2 - 2x_3 + 10x_4 = -9$

by Gauss-Seidal iteration method. **[13]**

Rewriting each equation after dividing by 10.

$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$

$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$

$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$

$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$

By Gauss Siedel Method,

$x_1^{K+1} = 0.3 + 0.2x_2^K + 0.1x_3^K + 0.1x_4^K$

$x_2^{K+1} = 1.5 + 0.2x_1^{K+1} + 0.1x_3^K + 0.1x_4^K$

$x_3^{K+1} = 2.7 + 0.1x_1^{K+1} + 0.1x_2^{K+1} + 0.2x_4^K$ where $K = 0,1,2 \dots$

$x_4^{K+1} = -0.9 + 0.1x_1^{K+1} + 0.1x_2^{K+1} + 0.2x_3^{K+1}$

Taking $X^0 = (x_1^0, x_2^0, x_3^0, x_4^0) \equiv (0, 0, 0, 0)$

K	x_1^{K+1}	x_2^{K+1}	x_3^{K+1}	x_4^{K+1}
0	0.3	1.56	2.886	-0.1368
1	0.8869	1.9523	2.9566	-0.0248
2	0.9836	1.9899	2.9924	-0.0042
3	0.9968	1.9982	2.9987	-0.0008
4	0.9994	1.9997	2.9997	-0.0001
5	0.9999	1.9999	2.9999	-0.0001

∴ The solution for the above system of equations is

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$x_4 = 0$

7. (b) The velocity v of a particle at distance s from a point on its path is given by the table :

s ft :	0	10	20	30	40	50	60
v ft / sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule. **[12]**

We know that $v = \frac{ds}{dt} \Rightarrow \int dt = t = \int \frac{1}{v}\, ds$

s	0	10	20	30	40	50	60	-
$\frac{1}{v}$	$\frac{1}{47}$	$\frac{1}{58}$	$\frac{1}{64}$	$\frac{1}{65}$	$\frac{1}{61}$	$\frac{1}{52}$	$\frac{1}{38}$	-

Simpson's $\frac{1}{3}^{rd}$ Rule :- $h = \frac{b-a}{n} = \frac{60-0}{6} = 10$

$$t = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{10}{3}\left[\left(\frac{1}{47}+\frac{1}{38}\right) + 2\left(\frac{1}{64}+\frac{1}{61}\right) + 4\left(\frac{1}{58}+\frac{1}{65}+\frac{1}{52}\right)\right]$$

$$\cancel{= 1.0635238} = 1.06352 \text{ sec} //$$

~~∴ Time taken to travel is 1.0635238 seconds~~

Simpson's $\frac{3}{8}^{th}$ Rule :- $h = \frac{b-a}{n} = \frac{60-0}{6} = 10$

$$t = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2y_3\right]$$

$$= \frac{30}{8}\left[\left(\frac{1}{47}+\frac{1}{38}\right) + 3\left(\frac{1}{58}+\frac{1}{64}+\frac{1}{61}+\frac{1}{52}\right) + 2\left(\frac{1}{65}\right)\right]$$

$$\cancel{= 1.0643825} = 1.06438 \text{ sec} //$$

~~∴ Time required to travel is 1.06382~~

Time required $\begin{cases} \text{By Simpson's } 1/3^{rd} \text{ Rule :- } 1.06352 \text{ sec} \\ \text{By Simpsons } 3/8^{th} \text{ Rule :- } 1.06438 \text{ sec} \end{cases}$

Difference between these two methods $= |1.06352 - 1.06438|$
$= 0.00086$ sec //

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7. (c) Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$. **[12]**

Consider $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

$x_0 = 0,\ y_0 = 1,\ h = 0.2$

By Runge Kutta method, $y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

where $K_1 = hf(x_n, y_n)$, $K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$ and $K_4 = hf(x_n + h, y_n + K_3)$

To find $y(0.2)$:- $K_1 = 0.2 f(0, 1) = 0.2$

$K_2 = 0.2 f(0.1, 1.1) = 0.1967$

$K_3 = 0.2 f(0.1, 1.09836) = 0.1967$

$K_4 = 0.2 f(0.2, 1.1967) = 0.1891$

$\therefore K = \frac{1}{6}[0.2 + 2(0.1967) + 2(0.1967) + 0.1891] = 0.1960$

$\therefore y(0.2) = y_0 + K = 1 + 0.1960 = \underline{\underline{1.1960}}$

To find $y(0.4)$:- $x_1 = 0.2,\ y_1 = 1.1960,\ h = 0.2$

$K_1 = 0.2 f(0.2, 1.1960) = 0.1891$

$K_2 = 0.2 f(0.3, 1.2906) = 0.1795$

$K_3 = 0.2 f(0.3, 1.2858) = 0.1793$

$K_4 = 0.2 f(0.4, 1.3753) = 0.1688$

$\therefore K = \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$

$\therefore y(0.4) = y_1 + K = 1.1960 + 0.1792 = 1.3752$ //

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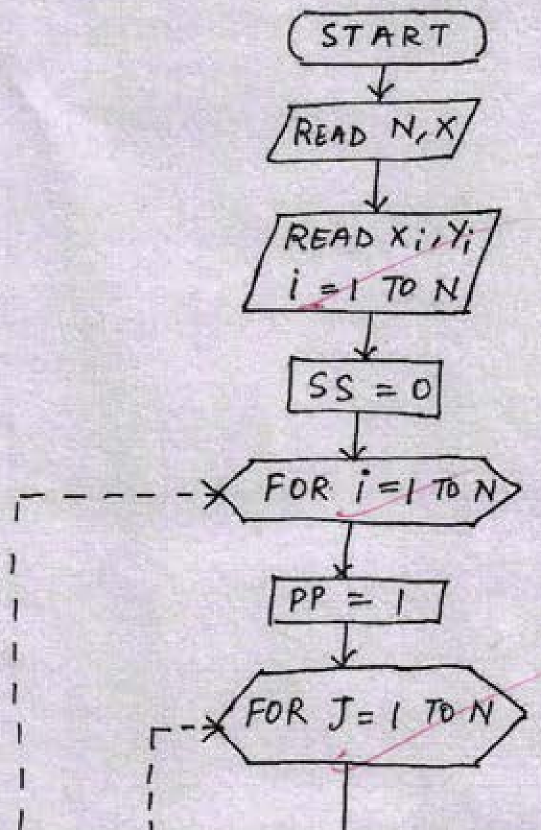
7. (d) Draw a flow chart for Lagrange's interpolation formula. **[13]**



FLOWCHART FOR LAGRANGE'S INTERPOLATION FORMULA

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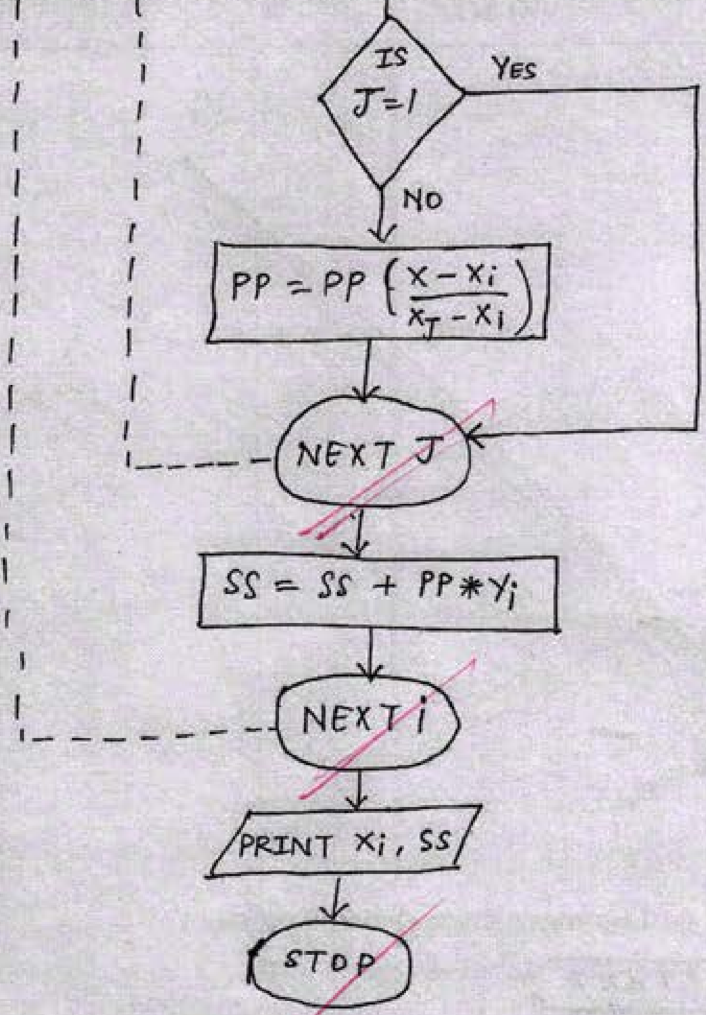

IS
J=1
YES
NO
PP = PP ((x - xi)/(xJ - xi))
NEXT J
SS = SS + PP*yi
NEXT i
PRINT xi, SS
STOP