7/25

Date : 30/06/2019

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
INSTITUTE OF MATHEMATICAL SCIENCES

199/250

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

PAPER - II : PDE, NUMERICAL & COMP. PROG. AND MECHANICS & FD

TEST CODE: TEST-4: IAS(M)/30-JUNE.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has **30 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name YASH MESHRAM

Roll No. 115

Test Centre HOME

Medium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			08	38
	(c)			08	
	(d)			08	
	(e)			06	
2	(a)			13	
	(b)			14	40
	(c)			13	
	(d)				
3	(a)			13	
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5	(a)			08	
	(b)			08	
	(c)			08	40
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6	(a)				
	(b)				
	(c)				
	(d)				
7	(a)			08	
	(b)			10	39
	(c)			11	
	(d)			10	
8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					

199/250

DO NOT WRITE ON
THIS SPACE

SECTION – A

1. (a) Solve $x^2p^2 + y^2q^2 = z^2$. [10]

$x^2p^2 + y^2q^2 = z^2 \Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$ — ①

Let $dX = \frac{dx}{x}$, $dY = \frac{dy}{y}$ and $dZ = \frac{dz}{z}$ — ②

From ① & ②, $\left(\frac{dZ}{dX}\right)^2 + \left(\frac{dZ}{dY}\right)^2 = 1$ or $P^2 + Q^2 = 1$

where $P = \frac{dZ}{dX}$ and $Q = \frac{dZ}{dY}$

$P^2 + Q^2 = 1$ is of the form of $f(P, Q) = 0$ — ③

$\therefore$ Solution of ③ is $Z = aX + bY + c$ where $a^2 + b^2 = 1$ and a, b, c are arbitrary constants. — ④

From ②, $X = \log x$, $Y = \log y$ and $Z = \log z$ — ⑤

From ④ & ⑤, $\log z = a \log x + \sqrt{1-a^2} \log y + \log c'$ [Here $c = \log c'$]

$\therefore \log z = \log x^a y^{(1-a^2)^{1/2}} c'$

$\therefore z = c' x^a y^{(1-a^2)^{1/2}}$ where a & c' are arbitrary constants

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1. (b) Solve $(D^2 - 4D'^2)z = (4x/y^2) - (y/x^2)$. **[10]**

$$(D^2-4D'^2)z = \frac{4x}{y^2} - \frac{y}{x^2}$$

Auxiliary equation :- $m^2 - 4 = 0 \Rightarrow m = \pm 2$

$\therefore C.F = \phi_1(y-2x) + \phi_2(y+2x)$ where ϕ_1, ϕ_2 are arbitrary functions

$$P.I. = \frac{1}{(D^2-4D'^2)}\left[\frac{4x}{y^2} - \frac{y}{x^2}\right] = \frac{1}{(D+2D')(D-2D')}\left[\frac{4x}{y^2} - \frac{y}{x^2}\right]$$

$$= \frac{1}{(D+2D')}\int\left[\frac{4x}{(c-2x)^2} - \frac{(c-2x)}{x^2}\right]dx \quad \text{where } c = y+2x$$

$$= \frac{1}{(D+2D')}\int\left[\frac{-2}{c-2x} + \frac{2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x}\right]dx$$

$$= \frac{1}{(D+2D')}\left[\log(c-2x) + \frac{c}{c-2x} + \frac{c}{x} + \log x^2\right]$$

$$= \frac{1}{(D+2D')}\left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + \log x^2\right]$$

$$= \int\left[\log(c'+2x) + 1 + \frac{2x}{c'+2x} + \frac{c'+2x}{x} + 2 + 2\log x\right]dx \quad \text{where } y-2x = c'$$

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$$= x\log(c'+2x) + 5x + 2x\log x + c\log x - 2x$$

$$= x\log y + y\log x + 3x$$

$$\therefore z = C.F + P.I.$$
$$= \phi_1(y-2x) + \phi_2(y+2x) + x\log y + y\log x + 3x \;//$$

1. (c) Obtain the Newton-Raphson extended formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{\{f(x_0)\}^2 f''(x_0)}{\{f'(x_0)\}^3}$$

for the root of the equation f(x) = 0. **[10]**

$f(x) = 0$ or $f[x_0 + (x - x_0)] = 0$ — ①

Taylor series expansion of ① to first approximation,

i.e; $f(x) = f[x_0 + (x - x_0)] = 0$

$\therefore f(x_0) + (x - x_0) f'(x_0) = 0$

$\therefore x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ — ②

By Taylor series expansion to the second approximation,

$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \dfrac{(x_1 - x_0)^2}{2!} f''(x_0)$ — ③

But $f(x_1) \simeq 0$ [$\because$ it is approximation to root] — ④

$\therefore f(x_0) + (x_1 - x_0) f'(x_0) + \dfrac{(x_1 - x_0)^2}{2} f''(x_0) = 0 \rightarrow$ From ③ & ④

$\therefore x_1 - x_0 = -\dfrac{f(x_0)}{f'(x_0)} - \dfrac{1}{2}\left[-\dfrac{f(x_0)}{f'(x_0)}\right]^2 \dfrac{f''(x_0)}{f'(x_0)} \rightarrow$ From ②

$\therefore x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} - \dfrac{1}{2}\dfrac{\{f(x_0)\}^2}{\{f'(x_0)\}^3} f''(x_0)$ //

08

1. (d) (i) Simplify the expression $AB+\overline{AC}+A\bar{B}C(AB+C)$.

(ii) Simplify the given Boolean expression
$Y=\bar{A}\bar{B}\bar{C}+\bar{A}B\bar{C}+A\bar{B}\bar{C}+AB\bar{C}$.

[10]

i> Let $x = AB + \overline{AC} + A\bar{B}C(AB+C)$

$= AB + \bar{A} + \bar{C} + A\bar{B}CAB + A\bar{B}CC$

$= \bar{A} + B + \bar{C} + A\bar{B}CAB + A\bar{B}C$ $[\because \bar{A}+\bar{\bar{A}}B = \bar{A}+B]$

$= \bar{A} + B + \bar{C} + A\bar{B}C[AB+1]$

$[\because AB+1=1]$

$= \bar{A} + \bar{B} + \bar{C} + A\bar{B}C$

$= \bar{A} + B + \bar{C} + A\bar{B}C$

$= \overline{A\bar{B}C} + \overline{\overline{A\bar{B}C}}$ [de morgan's law]

$= \overline{A\bar{B}C} + \overline{\bar{A}+B+\bar{C}}$

$= \overline{(\bar{A}+B+\bar{C})A\bar{B}C}$

$= \overline{\bar{A}A\bar{B}C + BA\bar{B}C + \bar{C}A\bar{B}C}$

$= \overline{0+0+0}$

$= \bar{0} = 1$

ii> $Y = \underbrace{\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C}} + \underbrace{A\bar{B}\bar{C} + AB\bar{C}}$

$= \bar{A}.\bar{C}.(\bar{B}+B) + A.\bar{C}(\bar{B}+B)$

$= \bar{A}.\bar{C}.1 + A.\bar{C}.1$ $[\because \bar{B}+B=1]$

$= (\bar{A}+A).\bar{C}.1$

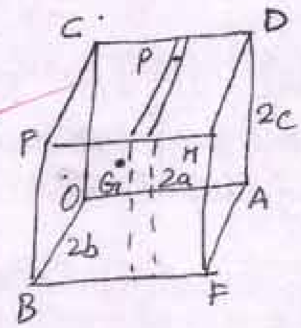
$= 1.\bar{C}.1$ $[\because \bar{A}+A=1]$

$= \bar{C}$ $[\because 1.\bar{C}=\bar{C}$ & $\bar{C}.1=\bar{C}]$

08

1. (e) Find the M.I. of a rectangular parallelopiped about an edge. **[10]**

Assume the lengths of the edges of rectangular parallelopiped of Mass M be 2a, 2b, 2c



$\therefore$ Moment of Inertia about an edge OA

$$= \frac{M}{3}(b^2+c^2) + M(b^2+c^2)$$

$$= \frac{4M}{3}(b^2+c^2) \quad //$$

where b^2+c^2 = perpendicular distance from G to OA

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2. (a) Reduce the equation

$u_{xx} + xu_{yy} = 0, \quad x \neq 0$

for all x, y to canonical form. **[16]**

$u_{xx} + xu_{yy} = 0 \Rightarrow r + xt = 0 \quad - (1)$

comparing it with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$,

$R = 1, S = 0, T = x$. $\quad S^2 - 4RT = -4x \begin{cases} \text{Elliptic}, & x > 0 \\ \text{Hyperbolic}, & x < 0 \end{cases}$

$\therefore R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda^2 + x = 0$

$\therefore \lambda = ix^{1/2}, -ix^{1/2}$

corresponding characteristic equations are :-

$\frac{dy}{dx} + ix^{1/2} = 0$ and $\frac{dy}{dx} - ix^{1/2} = 0$

$\therefore y + \frac{2i}{3}x^{3/2} = c_1$ and $y - \frac{2i}{3}x^{3/2} = c_2$

Let $u = y + \frac{2i}{3}x^{3/2} = \alpha + i\beta$ and $v = y - \frac{2i}{3}x^{3/2} = \alpha - i\beta$

where $\alpha = y$ & $\beta = \frac{2}{3}x^{3/2} \quad - (2)$

We have $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{1/2}\frac{\partial z}{\partial \beta} \quad - (3)$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \quad - (4)$

$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{1/2}\frac{\partial z}{\partial \beta}\right) = \frac{x^{-1/2}}{2}\frac{\partial z}{\partial \beta} + x^{1/2}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]$

$= \frac{x^{-1/2}}{2}\frac{\partial z}{\partial \beta} + x^{1/2}\left[0 + x^{1/2}\frac{\partial^2 z}{\partial \beta^2}\right] = \frac{x^{-1/2}}{2}\frac{\partial z}{\partial \beta} + x\frac{\partial^2 z}{\partial \beta^2} \quad - (5)$

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$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)$

$t = \frac{\partial^2 z}{\partial \alpha^2} + 0 = \frac{\partial^2 z}{\partial \alpha^2} \quad - (6)$

From (1), (5) & (6), $r + xt = 0$

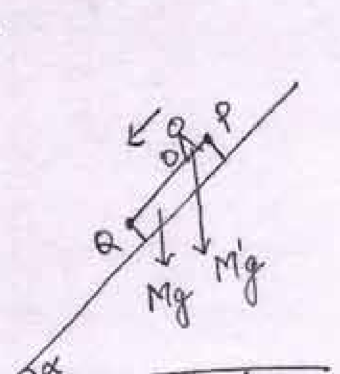
$\Rightarrow \frac{x^{-1/2}}{2}\frac{\partial z}{\partial \beta} + x\frac{\partial^2 z}{\partial \beta^2} + x\frac{\partial^2 z}{\partial \alpha^2} = 0$

$\therefore \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} = 0 \Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{3\beta}\frac{\partial z}{\partial \beta} = 0$

as $\beta = \frac{2}{3}x^{3/2}$

2. (b) A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M' starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time $\sqrt{\left\{\dfrac{2M'a}{(M+M')g\sin\alpha}\right\}}$, where a is the length of the plane.

[17]



Plank PQ has mass M.

Distance PO = x → travelled by a man of mass M'

Centre of gravity of block $= \bar{x}$ ~~Length = a/2~~

$$\therefore \bar{x} = \frac{M(a/2) + M'x}{M+M'} \quad -(1)$$

$$\cancel{\frac{d\bar{x}}{dx} \neq \frac{d\bar{x}}{dx} \neq M'(x)}$$

$$\frac{d\bar{x}}{dt} = \frac{M'}{M+M'}\frac{dx}{dt}$$

$$\therefore \frac{d^2\bar{x}}{dt^2} = \frac{M'}{M+M'}\frac{d^2x}{dt^2} \quad -(2)$$

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By balancing forces, we get

$$(M+M')\frac{d^2\bar{x}}{dt^2} = (M+M')\, g \sin\alpha \quad -③$$

$\therefore$ From ② & ③, $M'\dfrac{d^2x}{dt^2} = (M+M')\, g\sin\alpha \quad -④$

Integrating ④ 2 times, we get $M'x = (M+M')g\sin\alpha \dfrac{t^2}{2} + c_1$

Initial condition :- $t=0, x=0 \Rightarrow c_1 = 0$

$$\therefore \quad M'x = (M+M')g\sin\alpha\,\frac{t^2}{2}$$

$$\therefore \quad t^2 = \frac{2M'x}{(M+M')g\sin\alpha}$$

$$\therefore \quad t = \left[\frac{2M'x}{(M+M')g\sin\alpha}\right]^{1/2}$$

Put $x = PQ = a$, we get $t = \left[\dfrac{2M'a}{(M+M')g\sin\alpha}\right]^{1/2}$

2. (c) Four equidistant values u_{-1}, u_0, u_1 and u_2 being given, a value is interpolated by Lagrange's formula. Show that it may be written in the form

$$u_x = yu_0 + xu_1 + y\frac{(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0,$$

where $x + y = 1$. **[16]**

Difference table :-

x	u	Δu	$\Delta^2 u$
-1	u_{-1}		
		$u_0 - u_{-1}$	
0	u_0		$u_1 - 2u_0 + u_{-1}$
		$u_1 - u_0$	
1	u_1		$u_2 - 2u_1 + u_0$
		$u_2 - u_1$	
2	u_2		

Here, $\Delta^2 u_{-1} = u_1 - 2u_0 + u_{-1}$ & $\Delta^2 u_0 = u_2 - 2u_1 + u_0$

Consider RHS $= yu_0 + xu_1 + \frac{y}{3!}(y^2-1)\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$

$= u_0 - xu_0 + xu_1 + \frac{(1-x)}{6}(1+x^2-2x-1)[u_1 - 2u_0 + u_{-1}] + \frac{(x^3-x)}{6}[u_2 - 2u_1 + u_0]$

$= \frac{1}{6}(x^2 - 2x - x^3 + 2x^2)u_{-1} + u_0\left[1 - x - \frac{1}{3}(3x^2 - 2x - x^3) + \frac{x^3-x}{6}\right]$
$+ u_1\left[x - \frac{x^2}{2} - \frac{x^3}{2}\right] + u_2\left[\frac{(x^3-x)}{6}\right]$

$= -\frac{1}{6}(x^3 - 3x^2 + 2x)u_{-1} + \frac{1}{2}(x^3 - 2x^2 - x + 2)u_0 - \frac{1}{2}(x^3 - x^2 - 2x)u_1$
$+ \frac{1}{6}(x^3 - x)u_2 \quad - (1)$

We have $(x_0, x_1, x_2, x_3) \equiv (-1, 0, 1, 2)$ & $(f_0, f_1, f_2, f_3) \equiv (u_{-1}, u_0, u_1, u_2)$

By Lagrange Interpolation formula,

$u_x = \frac{(x)(x-1)(x-2)}{(-1)(-2)(-3)}u_{-1} + \frac{(x+1)(x-1)(x-2)}{(+1)(-1)(-2)}u_0 + \frac{(x+1)(x)(x-2)}{(2)(1)(-1)}u_1 + \frac{(x+1)x(x-1)}{(3)(2)(1)}u_2$

$= -\frac{1}{6}[x^3 - 3x^2 + 2x]u_{-1} + \frac{1}{2}[x^3 - 2x^2 - x + 2]u_0$
$+ \left(-\frac{1}{2}\right)[x^3 - x^2 - 2x]u_1 + \frac{1}{6}[x^3 - x]u_2 \quad - (2)$

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From (1) & (2),

$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$

3. (a) Find the characteristics of the PDE

$p^2 + q^2 = 2$

and determine the integral surface which passes through $x = 0$, $z = y$. **[15]**

Integral surface passes through $x=0$ and $y=z$ — ①

$\therefore$ We have $x=0$, $y=\lambda$ and $z=\lambda$ [λ being a parameter] ②

Let the initial values of x, y, z, p, q be

$x_0 = 0, y_0 = \lambda, z_0 = \lambda, p_0 = p_0$ & $q_0 = q_0$ — ③

From ③, and $p^2+q^2=2 \Rightarrow p_0^2+q_0^2 = 2$ — ④

Also, we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

$\therefore 1 = p_0(0) + q_0(1) \Rightarrow q_0 = 1$ — ⑤

From ④ & ⑤, $p_0^2 = 2-(1)^2 = 1 \Rightarrow p_0 = 1$ or -1

Choosing $p_0 = 1$, we get

$x_0 = 0$, $y_0 = \lambda$, $z_0 = \lambda$, $p_0 = 1$ and $q_0 = 1$ when $t = t_0 = 0$ ⑥

We have $f(x, y, z, p, q) \equiv p^2+q^2-2 = 0$

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Characteristic equations are :- $\frac{dx}{dt} = \frac{\partial f}{\partial p} = 2p$, $\frac{dy}{dt} = \frac{\partial f}{\partial q} = 2q$

$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = \underbrace{2p^2+2q^2}_{=4}$, $\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = 0$, $\frac{dq}{dt} = 0$ — (7)

solving (7), we get $p = p_0$, $q = q_0$. — (8)

$\therefore \frac{dx}{dt} = 2p_0$, $\frac{dy}{dt} = 2q_0$ and $\frac{dz}{dt} = 2p_0^2 + 2q_0^2 = 4$

$\therefore x = 2p_0 t + c_1$, $y = 2q_0 t + c_2$ and $z = 4t + c_3$ — (9)

From (5) & (9), $c_1 = 0$, $c_2 = \lambda$, $c_3 = \lambda$ — (10)

From (8), (9) & (10), $x = \underbrace{2p_0 t}_{=2t}$, $y = 2q_0 t + \lambda = 2t + \lambda$, $z = 4t + \lambda$,

$p = 1$, $q = 1$ — (11)

From (11), $x = 2t$, $y = 2t + \lambda$, $z = 4t + \lambda \rightarrow$ Required characteristics

$\therefore z - y = 4t + \lambda - 2t - \lambda = 2t = x$

$\therefore \boxed{z = x + y}$ is the integral surface which passes through $x = 0$ and $z = y$ [for $p_0 = 1$]

Also, $\boxed{z = -x + y}$ is the integral surface which passes through $z = y$ & $x = 0$ [for $p_0 = -1$]

3. (b) If the velocity of an incompressible fluid at the point (x,y,z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right), r^2 = x^2+y^2+z^2,$$

then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$.

Further, determine the streamlines. **[18]**

velocity $= \frac{3xz}{r^5}\hat{i} + \frac{3yz}{r^5}\hat{j} + \frac{3z^2-r^2}{r^5}\hat{k}$. Here, $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$ & $w = \frac{3z^2-r^2}{r^5}$ — ①

We have $r^2 = x^2+y^2+z^2$

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$ — ②

consider $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$$= \frac{\partial}{\partial x}\left[\frac{3xz}{r^5}\right] + \frac{\partial}{\partial y}\left[\frac{3yz}{r^5}\right] + \frac{\partial}{\partial z}\left[\frac{3z^2-r^2}{r^5}\right]$$

$$= \left[\frac{3z}{r^5} - \frac{15xz}{r^6}\frac{\partial r}{\partial x}\right] + \left[\frac{3z}{r^5} - \frac{15yz}{r^6}\frac{\partial r}{\partial y}\right] + \frac{r^5(6z-2z) - (3z^2-r^2)(5r^4\frac{\partial r}{\partial z})}{r^{10}}$$

$$= \left[\frac{3z}{r^5} - \frac{15x^2z}{r^7}\right] + \left[\frac{3z}{r^5} - \frac{15y^2z}{r^7}\right] + \frac{4z}{r^5} - \frac{(3z^2-r^2)(5r^3z)}{r^{10}}$$

$$= \frac{10z}{r^5} - \frac{15}{r^7}(x^2+y^2+z^2)z + \frac{5z}{r^5}$$

$$= \frac{15z}{r^5} - \frac{15}{r^7}zr^2 = \frac{15z}{r^5} - \frac{15z}{r^5} = 0 \qquad \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$\therefore$ Liquid motion is possible

Let ϕ be the velocity potential

$$\therefore d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = -u\,dx - v\,dy - w\,dz \quad \text{[By definition]}$$

$$\therefore d\phi = -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right]$$

$$= \frac{[r^2dz - 3z(x\,dx + y\,dy + z\,dz)]}{r^5} = \frac{[r^2dz - 3rz\,dr]}{r^5}$$

$$\therefore d\phi = \frac{[r^3dz - 3r^2dr.z]}{r^6} \Rightarrow d\phi = d\left(\frac{z}{r^3}\right) \Rightarrow \text{Integrating it, we get } \phi = \frac{z}{r^3}$$

$$\therefore \phi = \frac{z}{r^3}$$

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Streamlines :- $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{\omega} \Rightarrow \frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2-r^2)/r^5}$

or $\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{2z^2-x^2-y^2}$ — (3)

From (3), $\frac{dx}{3xz} = \frac{dy}{3yz} \Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0 \Rightarrow \log x - \log y = c_1 \Rightarrow x = c_1 y$

$c_1 \rightarrow$ arbitrary constant

Each fraction of (3) $= \frac{x\,dx + y\,dy + z\,dz}{3x^2z + 3y^2z + 2z^3 - x^2z - y^2z}$

$= \frac{x\,dx + y\,dy + z\,dz}{2z^3 + 2x^2z + 2y^2z} = \frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)}$

$\therefore \frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)} = \frac{dx}{3xz} \Rightarrow \frac{2}{3}\frac{dx}{x} = \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)}$

$\therefore \frac{2}{3}\log x = \frac{1}{2}\log(x^2+y^2+z^2) + \log c_2'$

$\therefore x = c_2 (x^2+y^2+z^2)^{3/4}$, $c_2 \rightarrow$ arbitrary constant

streamlines are given by $x = c_1 y$ and $x = c_2(x^2+y^2+z^2)^{3/4}$

3. (c) Solve the following system

$$\begin{bmatrix} 17 & 65 & -13 & 50 \\ 12 & 16 & 37 & 18 \\ 56 & 23 & 11 & -19 \\ 3 & -5 & 47 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 84 \\ 25 \\ 36 \\ 18 \end{bmatrix}$$

by Gauss-Seidel method and do computations to two decimal places and obtain upto 10 iterations. [17]

The above system can be rearranged as :-

$56x_1 + 23x_2 + 11x_3 - 19x_4 = 36$

$17x_1 + 65x_2 - 13x_3 + 50x_4 = 84$

$3x_1 - 5x_2 + 47x_3 + 10x_4 = 18$

$12x_1 + 16x_2 + 37x_3 + 18x_4 = 25$

By Gauss Seidel method, we have

$x_1^{(K+1)} = \frac{1}{56}\left[36 - 23x_2^K - 11x_3^K + 19x_4^K\right]$

$x_2^{(K+1)} = \frac{1}{65}\left[84 - 17x_1^{(K+1)} + 13x_3^K - 50x_4^K\right]$

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$$x_3^{(K+1)} = \frac{1}{47}\left[18 - 3x_1^{(K+1)} + 5x_2^{(K+1)} - 10x_4^{K}\right]$$

$$x_4^{(K+1)} = \frac{1}{18}\left[25 - 12x_1^{(K+1)} - 16x_2^{(K+1)} - 37x_3^{(K+1)}\right]$$

Take $(x_1^0, x_2^0, x_3^0, x_4^0) \equiv (0,0,0,0)$

K	$x_1^{(K+1)}$	$x_2^{(K+1)}$	$x_3^{(K+1)}$	$x_4^{(K+1)}$
0	0.64286	1.12418	0.46154	-0.98767
1	-0.24461	2.20834	0.84366	-2.14519
2	-1.15769	3.41397	1.27649	-3.49785
3	-2.19682	4.81281	1.77943	-5.08233
4	-3.40772	6.44893	2.36789	-6.93902
5	-4.82524	8.36558	3.05731	-9.11483
6	-6.48608	10.61154	3.86520	-11.66466
7	-8.43234	13.24354	4.81194	-14.65281
8	-10.71315	16.32798	5.92144	-18.15462
9	-13.38602	19.94265	7.22165	-22.25839

$x_1 = -13.39,\ x_2 = 19.94,\ x_3 = 7.22,\ x_4 = -22.26$ //

$x_1 = 4.84$
$x_2 = -4.70$
$x_3 = -1.64$
$x_4 = 5.72$

SECTION – B

5. (a) Find the integral surface of the equation

$(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$, through the curve $xz = a^3$, $y = 0$. **[10]**

$(x-y)y^2 p + (y-x)x^2 q = (x^2+y^2)z$ — ①

Lagrange equations :- $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Here $P = y^2(x-y)$, $Q = x^2(y-x)$, $R = (x^2+y^2)z$

$\therefore \frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{(x^2+y^2)}$ — ②

From ②, $\frac{dx}{y^2} = \frac{dy}{-x^2} \Rightarrow x^2dx + y^2dy = 0$

$\therefore x^3 + y^3 = 3c' = c_1$ where c_1, c' are arbitrary constants

From ②, ~~each fraction~~ $= \cancel{\frac{\frac{1}{y}dx + \frac{1}{x}dy}{yx - y^2 - xy}}$

Each fraction $= \frac{dx - dy}{y^2(x-y) - x^2(y-x)} = \frac{dx - dy}{-(x^2+y^2)(y-x)}$

08

$\therefore \dfrac{dx-dy}{(x-y)(x^2+y^2)} = \dfrac{dz}{z(z^2+y^2)} \Rightarrow \dfrac{d(x-y)}{(x-y)} = \dfrac{dz}{z}$

$\therefore \log(x-y) = \log z + \log c_2 \Rightarrow \dfrac{x-y}{z} = c_2$

where c_2 is an arbitrary constant

$\therefore$ Solution $\Rightarrow \phi\left(x^3+y^3, \dfrac{x-y}{z}\right) = 0$ where ϕ is an arbitrary function.

Surface passes through $xz = a^3$ and $y = 0$ — ③

$\therefore x^3+y^3 = c_1 \Rightarrow x^3 = c_1$ — ④

and $\dfrac{x-y}{z} = c_2 \Rightarrow \dfrac{x}{z} = c_2 \Rightarrow \dfrac{x}{a^3/x} = c_2$

$\therefore x^2 = a^3 c_2$ — ⑤

From ④ & ⑤, $c_1^{2/3} = a^3 c_2$ — ⑥

From ③, ④ & ⑥, $(x^3+y^3)^{2/3} = a^3 \dfrac{(x-y)}{z}$

$\Rightarrow (x^3+y^3)^2 = \dfrac{a^9}{z^3}(x-y)^3$

$\therefore z^3(x^3+y^3)^2 = a^9(x-y)^3$ is the required integral surface.

5. (b) Solve

$$\left.\begin{aligned} 10x-7y+3z+5u&=6,\\ -6x+8y-z-4u&=5,\\ 3x+y+4z+11u&=2,\\ 5x-9y-2z+4u&=7 \end{aligned}\right\}$$

by Gauss's elimination method. [10]

Here, augmented matrix $= [A \mid B]$

$$\therefore [A\mid B] = \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 38/10 & 8/10 & -1 & 86/10 \\ 0 & 31/10 & 31/10 & 95/10 & 2/10 \\ 0 & -55/10 & -35/10 & 15/10 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + \frac{6}{10}R_1 \\ R_3 \to R_3 - \frac{3}{10}R_1 \\ R_4 \to R_4 - \frac{5}{10}R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 38 & 8 & -10 & 86 \\ 0 & 31 & 31 & 95 & 2 \\ 0 & -55 & -35 & 15 & 40 \end{array}\right] \quad \begin{array}{l} R_2 \to 10R_2 \\ R_3 \to 10R_3 \\ R_4 \to 10R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 38 & 8 & -10 & 86 \\ 0 & 0 & 465/19 & 1960/19 & -1295/19 \\ 0 & 0 & -445/19 & 10/19 & 3125/19 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - \frac{31}{38} R_2 \\ R_4 \to R_4 + \frac{55}{38} R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 38 & 8 & -10 & 86 \\ 0 & 0 & 465/19 & 1960/19 & -1295/19 \\ 0 & 0 & 0 & \frac{9230}{93} & \frac{9230}{93} \end{array}\right] \quad R_4 \to R_4 + \frac{445}{465} R_3$$

Hence, we have $\frac{9230}{93} u = \frac{9230}{93} \Rightarrow u = 1$

$\therefore \frac{465z + 1960 \times 1}{19} = \frac{-1295}{19} \Rightarrow 465z = -1295 - 1960$
$\Rightarrow z = -7$

$\therefore 38y + 8 \times (-7) - 10(1) = 86 \Rightarrow 38y = 86 + 56 + 10 = 152$
$\Rightarrow y = 4$

$\therefore 10x - 7(4) + 3(-7) + 5(1) = 6 \Rightarrow 10x = 6 + 28 + 21 - 5$
$\Rightarrow x = 5$

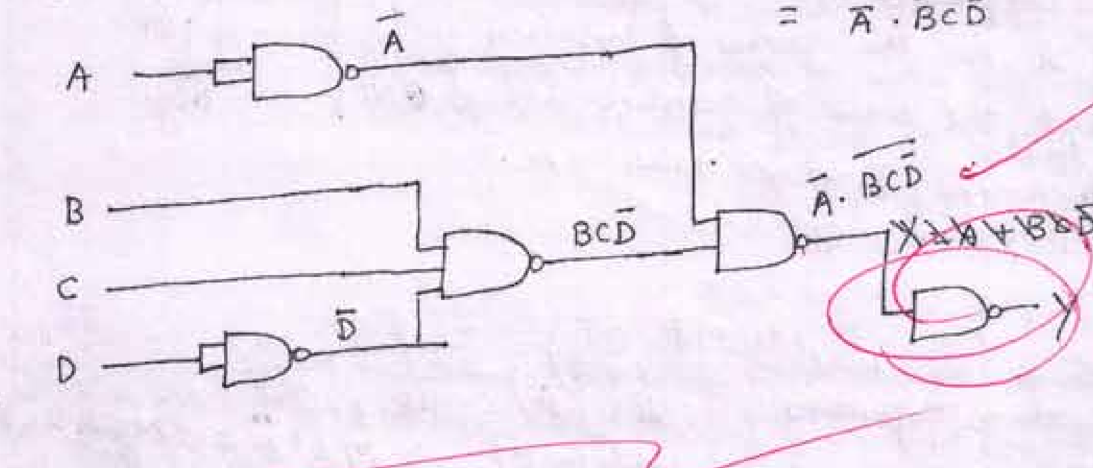
$\therefore (x, y, z, u) \equiv (5, 4, -7, 1)$ //

5. (c) Realise (a) $Y = A + BC\bar{D}$ using NAND gates and

(b) $Y = (A+C)(A+\bar{D})(A+B+\bar{C})$ using NOR gates. **[10]**

a)

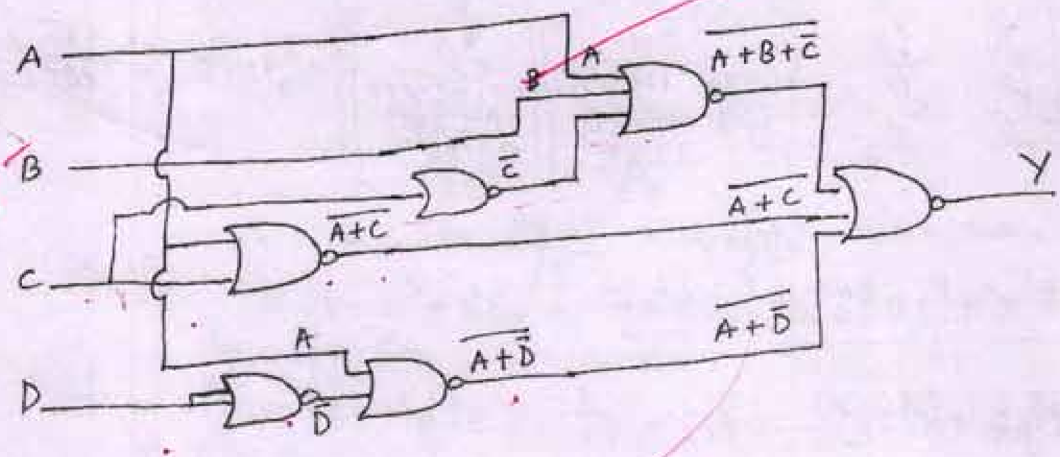
$Y = A + BC\bar{D}$
$= \overline{\bar{A} \cdot \overline{BC\bar{D}}}$



b) $Y = (A+C)(A+\bar{D})(A+B+\bar{C})$ using NOR gates

$= \overline{\overline{(A+C)(A+\bar{D})(A+B+\bar{C})}}$

$= \overline{\overline{(A+C)} + \overline{(A+\bar{D})} + \overline{(A+B+\bar{C})}}$



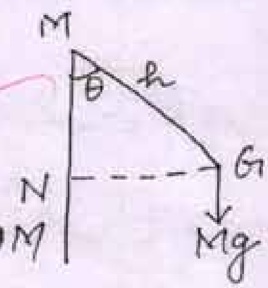
08

5. (d) Use Lagrange's equations to find the equation of motion of the compound pendulum which oscillates in a vertical plane about a fixed horizontal axis. **[10]**

Here, $l(MG) = h$

θ is the only generalised coordinate

Let k be the radius of gyration about the axis of rotation through M



MG makes θ angle with MN

$\therefore T = \frac{1}{2} M k^2 \dot{\theta}^2$

Also, $V = -Mgh\cos\theta$

Lagrange equation $\rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$

$\therefore \frac{d}{dt}(Mk^2\dot{\theta}) = -\frac{\partial}{\partial \theta}[-Mgh\cos\theta] \Rightarrow Mk^2\ddot{\theta} = -Mgh\sin\theta$

$\therefore \ddot{\theta} = -\left(\frac{gh}{k^2}\right)\sin\theta \simeq -\left(\frac{gh}{k^2}\right)\theta$ // [$\because \theta$ is very small]

08

5. (e) Find the stream lines and paths of the particles for the two dimensional velocity field :

$u = \frac{x}{1+t}, v = y, w = 0.$ **[10]**

Here, $u = \frac{x}{1+t}$, $v = y$, $\omega = 0$

i) Stream lines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{\omega}$

$\therefore (1+t)\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{0}$

$\Rightarrow dz = 0$ and $\frac{dy}{y} = (1+t)\frac{dx}{x}$

$\therefore z = C_1$ and $\log y = \log x^{(1+t)} + \log C_2$

$\therefore z = c_1$ and $y = c_2 x^{(t+1)}$ // [c_1 & c_2 are arbitrary constants]

ii) Path lines are given by :- $\frac{dx}{dt} = \frac{x}{1+t}$, $\frac{dy}{dt} = y$, $\frac{dz}{dt} = 0$

$\Rightarrow \log x = \log(1+t) + \log C_3$, $\log y = t + \log C_4$, $z = C_5$

[Integrating the above equations]

08

$\therefore x = c_3(1+t)$, $y = c_4 e^t$, $z = c_5$

$\therefore y = c_4 e^{\left[\frac{x}{c_3} - 1\right]}$ and $z = c_5$ represent path lines where c_3, c_4, c_5 are arbitrary constants.

6. (a) Form a partial differential equation by eliminating arbitrary function f and g from $z = f(x^2 - y) + g(x^2 + y)$. **[06]**

7. (a) Use Euler's modified method to compute y for x = 0.05 and x = 0.1. Given that $\frac{dy}{dx} = x + y$ with the initial condition $x_0 = 0, y_0 = 1$. Give the correct result upto four decimal places. [13]

We have $\frac{dy}{dx} = x+y$, $x_0 = 0$, $y_0 = 1$

Here , $f(x,y) = x+y$, $h = 0.05$, $x_0 = 0$, $y_0 = 1$

$x_1 = x_0 + h = 0 + 0.05 = 0.05$, $x_2 = x_0 + 2h = 0 + 2(0.05) = 0.1$

~~By Modified Euler's method~~ $y_1^{(0)} = y_0 + h f(x_0, y_0)$

$= 1 + 0.05[0+1] = 1.05$

By Modified Euler's method, $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(0)})]$

$\therefore y_1^{(1)} = 1 + \frac{0.05}{2}[1 + 0.05 + 1.05] = 1.0525$

$\therefore y(0.05) = 1.05250 \approx 1.0525$ //

We have $y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1, y_1^{(1)})]$

$= 1 + \frac{0.05}{2}[1 + 0.05 + 1.0525]$

08 $\therefore y_1^{(2)} = 1.05256$

$$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.0525 + 0.01\,[0.05 + 1.0525]$$
$$= 1.06353$$

$$\therefore y_2^{(1)} = y_1 + \frac{h}{2}\,[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$
$$= 1.0525 + \frac{0.05}{2}\,[0.05 + 1.0525 + 0.1 + 1.06353]$$

$$= 1.1092$$

$$\therefore y(0.1) = 1.1092 \,//$$

7. (b) Evaluate the integrals

(i) $I=\int_0^2 \frac{dx}{3+4x}$, (ii) $\int_0^2 \frac{dx}{x^2+2x+10}$

by Gauss-Legendre two-point and three-point formulas. **[12]**

Two point Gauss Legendre formula :- $\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$

Three point Gauss Legendre formula :- $\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right]$

i> $I_1 = \int_0^2 \frac{dx}{3+4x}$, let $x = t+1$, $dx = dt \Rightarrow I_1 = \int_{-1}^{1} \frac{dt}{3+4t+4}$

$\therefore I_1 = \int_{-1}^{1} \frac{dt}{7+4t} = \frac{1}{7-\frac{4}{\sqrt{3}}} + \frac{1}{7+\frac{4}{\sqrt{3}}}$ [By 2point Gauss formula]

$= \frac{14}{49-\frac{16}{3}} = \frac{42}{147-4\cancel{8}6} = \cancel{\frac{42}{131}} \cancel{\frac{14}{33}} \cancel{= 0.4242} = \frac{42}{131} = \underline{\underline{0.3206}}$

Using 3-point Gauss formula :- $I_1 = \int_{-1}^{1} \frac{dt}{7+4t}$

$= \frac{1}{9}\left[5\left(\frac{1}{7-4\sqrt{\frac{3}{5}}}\right) + 8\left(\frac{1}{7+0}\right) + 5\left(\frac{1}{7+4\sqrt{\frac{3}{5}}}\right)\right]$

$= \frac{1}{9}\left[\frac{8}{7} + \frac{70}{49-\frac{48}{5}}\right] = \frac{1}{9}\left[\frac{8}{7}+\frac{350}{197}\right] = \underline{\underline{0.3244}}$

ii> $I_2 = \int_0^2 \frac{dx}{x^2+2x+10}$, let $x = t+1$, $dx = dt$, $-1 \le t \le 1$

Using 2-point Gauss formula :-

$I_2 = \int_{-1}^{1} \frac{dt}{t^2+2t+1+2t+2+10} = \int_{-1}^{1} \frac{dt}{t^2+4t+13}$

$= \frac{1}{\left(\frac{-1}{\sqrt{3}}\right)^2 + 4\left(\frac{-1}{\sqrt{3}}\right)+13} + \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2+4\left(\frac{1}{\sqrt{3}}\right)+13} = 0.0907 + 0.0639$

$= \underline{\underline{0.1546}}$

Using 3-point Gauss formula :-

$I_2 = \int_{-1}^{1} \frac{dt}{t^2+4t+13} = \frac{1}{9}\left[5\left\{\frac{1}{\left(-\sqrt{\frac{3}{5}}\right)^2+4\left(-\sqrt{\frac{3}{5}}\right)+13} + \frac{1}{\left(\sqrt{\frac{3}{5}}\right)^2+4\left(\sqrt{\frac{3}{5}}\right)+13}\right\} + 8\left(0^2+4\times 0+13\right)^{-1}\right]$

$= \frac{1}{9}\left[5\left(0.0952+0.0599\right)+0.6154\right] = \underline{\underline{0.1545}}$

10

7. (c) The following are the numbers of deaths in four successive ten year age groups. Find the number of deaths at 45 – 50 and 50 – 55.

Age group :	25 – 35	35 – 45	45 – 55	55 – 65
Deaths :	13229	18139	24225	31496

[13]

Preparing the table according to the cumulative frequency

Age under	35	45	55	65
Deaths	13229	31368	55593	87089

Difference table :-

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

y_{50} is the number of deaths above the age 35 & below 50

Here, $x_0 = 35$, $x = 50$, $h = 10$, $p = x - x_0/h = 50-35/10 = 1.5$

By Newton's forward formula,

$$y_{50} = y_{35} + p\Delta y_{35} + \frac{p(p-1)}{2!}\Delta^2 y_{35} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{35}$$

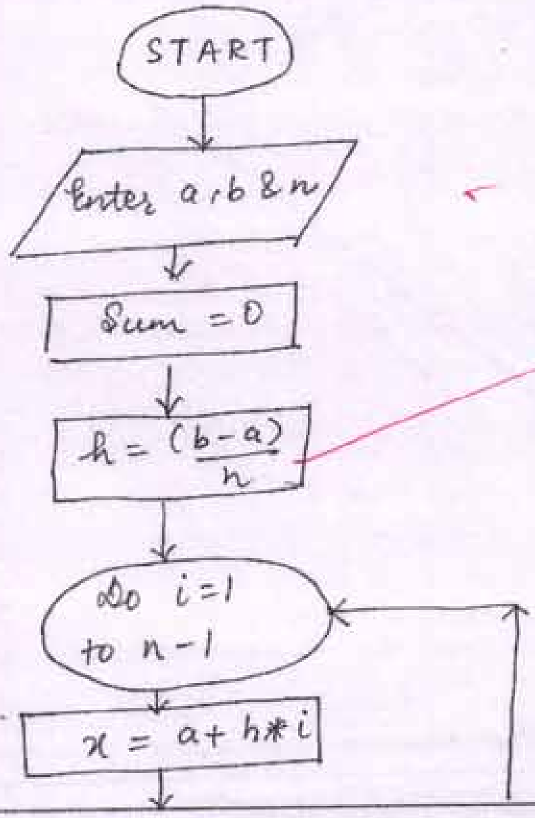
$$= 13229 + 1.5(18139) + \frac{1.5(0.5)}{2}(6086) + \frac{1.5(0.5)(-0.5)(1185)}{6}$$

≈ 42646

$\therefore$ Number of deaths between 45 & 50 $= 42646 - 31368 = \underline{\underline{11278}}$

$\therefore$ Number of deaths between 50 & 55 $= 24225 - 11278$
$= \underline{\underline{12947}}$

7. (d) Draw a flow chart for Simpson's $\frac{3}{8}$th rule **[12]**



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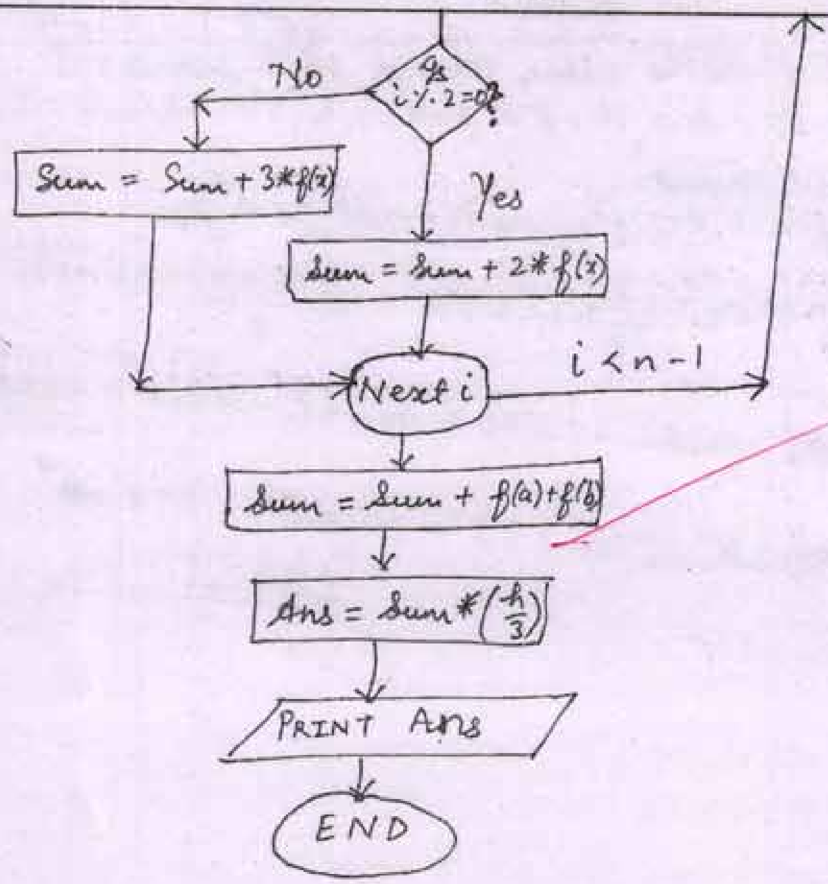

8. (a) Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$ **[17]**