Set-VI

Convolution

→ Another important general Property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms $f(p)$ and $g(p)$ whose inverses $F(t)$ and $G(t)$. (i.e. $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$).

we would like to calculate the inverse of the product $h(p) = f(p) \cdot g(p)$ (i.e. $L^{-1}\{h(p)\} = L^{-1}\{f(p)\,g(p)\}$). From those known inverses $F(t)$ and $G(t)$. This inverse $H(t)$ is denoted by $(F*G)(t)$ and is called Convolution of $F(t)$ and $G(t)$.

→ Let $F(t)$ and $G(t)$ be two functions of class A then the convolution of the two functions $F(t)$ and $G(t)$ denoted by $F*G$ and is defined as

$$F*G = \int_0^t F(x)\,G(t-x)\,dx.$$

→ Properties of Convolution:

(i) $F*G = G*F$
(i.e, Commutative)

Sol'n: $F*G = \int_0^t F(x)\,G(t-x)\,dx$ Putting $t-x = y$ $\Rightarrow x = t-y$, $dx = -dy$

$$= \int_t^0 F(t-y)\,G(y)\,(-dy)$$

$$= -\int_t^0 F(t-y)\,G(y)\,dy$$

$$= \int_0^t G(y)\,F(t-y)\,dy$$

$$= G*F$$

H.W → (ii) $(F*G)*H = F*(G*H)$

(iii) $F*(G+H) = (F*G)+(F*H)$

Sol'n: $F*(G+H) = \int_0^t F(x)\,(G+H)(t-x)\,dx$

$$= \int_0^t F(x)\left[G(t-x)+H(t-x)\right]dx$$

$$= \int_0^t F(x)\,G(t-x)\,dx + \int_0^t F(x)\,H(t-x)\,dx$$

$$= (F*G)+(F*H)$$

→ Convolution theorem: (Convolution Property):—

Let $F(t)$ and $G(t)$ be two functions of Class A and let $L^{-1}\{f(p)\} = F(t)$ and $L^{-1}\{g(p)\} = G(t)$ then

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\,G(t-x)\,dx = F*G$$

Proof: We have to show that

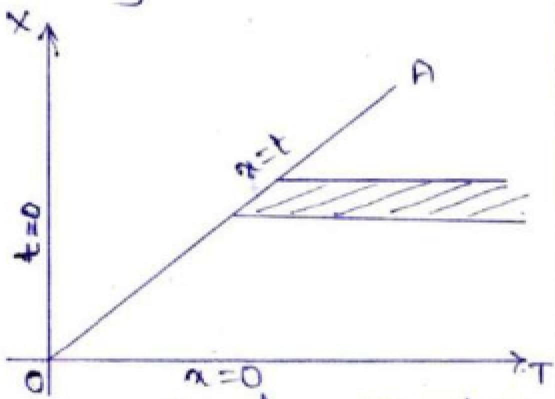
$$L\left\{\int_0^t F(x)\,G(t-x)\,dx\right\} = f(p)\cdot g(p)$$

Let $H(t) = \int_{x=0}^{t} F(x)\,G(t-x)\,dx = F*G.$

Since $L\{H(t)\} = \int_{t=0}^{\infty} e^{-pt}\,H(t)\,dx$ (by definition of L.T)

$$\therefore L\{H(t)\} = \int_{t=0}^{\infty} e^{-pt}\left[\int_{x=0}^{t} F(x)\,G(t-x)\,dx\right]dt \quad \text{—— (1)}$$

In equation (1), the region of integration in the double integral is the infinite area below the line OA (with equation $x=t$) and above the line OT (with equation $x=0$). Here t & x are measured along OT and OX respectively.



To change the order of Integration:

Draw an elementary strip parallel to T-axis. One end lies on $x=t$ and the other end on $t=\infty$.

For this strip x varies from 0 to ∞. Hence changing the order of integration ① reduces to.

$$L\{H(t)\} = \int_{x=0}^{\infty} F(x)\left\{\int_{t=x}^{\infty} e^{-pt} G(t-x)\,dt\right\}dx$$

$$= \int_{x=0}^{\infty} F(x)\left\{\int_{y=0}^{\infty} e^{-p(x+y)} G(y)\,dy\right\}dx$$

Putting $t-x=y$
$t=y+x$
$dt=dy$
limits for y: 0 to ∞.

$$= \int_{x=0}^{\infty} F(x)\left\{e^{-px}\int_{y=0}^{\infty} e^{-py} G(y)\,dy\right\}dx$$

$$= \int_{x=0}^{\infty} e^{-px}F(x)\,dx\left[\int_{y=0}^{\infty} e^{-py} G(y)\,dy\right]$$

$$= \left[\int_{t=0}^{\infty} e^{-pt}F(t)\,dt\right]\left[\int_{t=0}^{\infty} e^{-pt} G(t)\,dt\right]$$ (By the Property of definite integral)

$$= L\{F(t)\}\cdot L\{G(t)\}$$

$$= f(p)\cdot g(p)$$

$$\therefore H(t) = L^{-1}\{f(p)\cdot G(p)\}$$

i.e. $L^{-1}\{f(p)\cdot g(p)\} = H(t)$

$$\Rightarrow L^{-1}\{f(p)g(p)\} = \int_0^t F(x)\,G(x-t)\,dx$$

$$= \underline{F * G}$$

<u>Note</u>: (1). The convolution theorem can be re-written as

$$L\left\{\int_0^t F(x)G(x-t)\,dx\right\} = L\{F(t)*G(t)\}$$

$$= L\{F(t)\}\cdot L\{G(t)\}$$

(2). while using the convolution theorem, we use one of the following two forms

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(x)\,G(t-x)\,dx \quad \text{(or)}$$

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t G(x)\,F(t-x)\,dx.$$

→ Use the convolution theorem to find

(i) $L^{-1}\left\{\frac{1}{(p+a)(p+b)}\right\}$ (ii) $L^{-1}\left\{\frac{1}{(p+1)(p-2)}\right\}$ (iii) $L^{-1}\left\{\frac{1}{(p+1)(p-1)}\right\}$ (iv) $L^{-1}\left\{\frac{1}{(p-1)(p+2)}\right\}$

(v) $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$ (vi) $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\}$ (vii) $L^{-1}\left\{\frac{1}{(p-2)(p^2+1)}\right\}$

Sol'n: (i) $L^{-1}\left\{\frac{1}{(p+a)(p+b)}\right\}$

Let $f(p) = \frac{1}{p+a}$; $g(p) = \frac{1}{p+b}$

then, $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p+a}\right\} = e^{-at}$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p+b}\right\} = e^{-bt}$

Now using the convolution theorem, we have

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$\Rightarrow L^{-1}\left\{\frac{1}{(p+a)}\,\frac{1}{(p+b)}\right\} = \int_0^t e^{-au}\,e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\,du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t = \frac{e^{-bt}}{b-a}\left[e^{(b-a)t}-1\right] = \frac{1}{(b-a)}\left[e^{-at}-e^{-bt}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{p+a}\,\frac{1}{p+b}\right\} = \frac{1}{(b-a)}\left\{e^{-at}-e^{-bt}\right\}$$

→ $L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\}$

Sol'n: Let $f(p) = \frac{p}{p^2+a^2}$ and $g(p) = \frac{1}{p^2+a^2}$

Then $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

Now, using the convolution theorem, we have

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$L^{-1}\left\{\frac{p}{(p^2+a^2)^2}\right\} = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \cos au\left[\sin at\cos au - \cos at\sin au\right]du.$$

$$= \frac{1}{a}\int_0^t \cos^2 au \ \sin at \, du - \frac{1}{a}\int_0^t \sin au \cos au \cdot \cos at \, du.$$

$$= \frac{1}{2a}\sin at \int_0^t (1+\cos 2au)\, du - \frac{1}{2a}\cos at \int_0^t \sin 2au \, du$$

$$= \frac{1}{2a}\sin at \left[u + \frac{\sin 2au}{2a}\right]_0^t - \frac{1}{2a}\cos at \left[-\frac{\cos 2au}{2a}\right]_0^t$$

$$= \frac{1}{2a}\sin at \left[t + \frac{1}{2a}\sin 2at - 0\right] + \frac{1}{2a}\cos at \cdot \frac{1}{2a}\left(\cos 2at - 1\right)$$

$$= \frac{t}{2a}\sin at + \frac{1}{4a^2}\sin at \underbrace{\sin 2at}_{2\sin at \cos at} + \frac{1}{2a^2}\cos at\left(-\sin^2 at\right)$$

$$\left(\because \frac{1-\cos 2at}{2} = \sin^2 at\right)$$

$$= \frac{t}{2a}\sin at + \frac{1}{2a^2}\cancel{2\sin^2 at \cos at} - \cancel{\frac{1}{2a^2}\cos at \sin^2 at}$$

$$= \frac{t \sin at}{2a} \quad \text{Ans.}$$

(vi) $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\}$.

Sol: $L^{-1}\left\{\frac{1}{p(p^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{p^2}\cdot\frac{p}{(p^2+4)^2}\right\}$.

Let $f(p) = \frac{1}{p^2}$ and $g(p) = \frac{p}{(p^2+4)^2}$

Then $F(t) = L^{-1}(f(p)) = L^{-1}\left(\frac{1}{p^2}\right) = t$.

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{p}{(p^2+4)^2}\right\}$.

$$= L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{(p^2+4)}\right)\right\}$$

$$= -\frac{1}{2}L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{(p^2+4)}\right)\right\}$$

$$= -\frac{1}{2}(-1)^1 t \, L^{-1}\left\{\frac{1}{(p^2+4)}\right\}$$

$$= \frac{1}{2}t \frac{1}{2}\sin 2t$$

$$= \frac{t}{4}\sin 2t.$$

Now, using the convolution theorem, we have

$$L^{-1}\{f(p)\, g(p)\} = \int_0^t G(u)\, F(t-u)\, du$$

$$= \int_0^t \frac{u}{4}\sin 2u \,(t-u)\, du.$$

$$= \frac{1}{4}\int_0^t (tu - u^2)\sin 2u \, du.$$

$$= \frac{1}{4}\left\{\left[(tu-u^2)\left(-\frac{\cos 2u}{2}\right)\right]_0^t - \int_0^t (t-2u)\left(-\frac{\cos 2u}{2}\right)du\right\}$$ Integrating by parts

$$= \frac{1}{4}\left\{[0-0] + \left[(t-2u)\left(\frac{\sin 2u}{4}\right)\right]_0^t + \int_0^t \cancel{2}\left(\frac{\sin 2u}{\cancel{4}_2}\right)du\right.$$

$$= +\frac{1}{4}\left[\left(\frac{-t}{4}\right)\sin 2t - 0\right] + \frac{1}{8}\left[\frac{-\cos 2u}{2}\right]_0^t$$

$$= -\frac{t}{16}\sin 2t + \frac{1}{8}\left[\frac{-\cos 2t+1}{2}\right]$$

$$= -\frac{t}{16}\sin 2t - \frac{1}{16}\cos 2t + \frac{1}{16}.$$

→ Use the convolution theorem to find

(i) $L^{-1}\left\{\frac{p^2}{(p^2+4)^2}\right\}$ (ii) $L^{-1}\left\{\frac{p}{(p^2+4)^3}\right\}$.

[Hint (i) $L^{-1}\left\{\frac{p^2}{(p^2+4)^2}\right\} = L^{-1}\left\{\underset{f(p)}{\frac{p}{(p^2+4)}}\ \underset{g(p)}{\frac{p}{p^2+4}}\right\}$

(ii) $L^{-1}\left\{\frac{p}{(p^2+4)^3}\right\} = L^{-1}\left\{\underset{f(p)}{\frac{p}{(p^2+4)^2}}\cdot\underset{g(p)}{\frac{1}{(p^2+4)}}\right\}$]

→ Show that $1*1*1*\cdots*1$ (n times) $= \frac{t^{n-1}}{(n-1)!}$ where $n=1,2,3,\ldots$

Sol^n: Since $F*G = \int_0^t F(u)\,G(t-u)\,du$

$\left[\begin{array}{l} F(t)=1\ \&\ G(t)=1 \\ \Rightarrow F(u)=1\ \&\ G(t-u)=1 \end{array}\right]$

$\therefore 1*1 = \int_0^t 1\cdot 1\,du = [u]_0^t = t$

$\therefore 1*1*1 = t*1 = \int_0^t u\cdot 1\,du$

$= \left[\frac{u^2}{2}\right]_0^t = \frac{t^2}{2}.$

$\left[\begin{array}{l} \because F(t)=t \\ F(u)=u. \\ \&\ G(t)=1 \\ G(t-u)=1 \end{array}\right]$

and $1*1*1*1 = \int_0^t \frac{u^2}{2}\cdot 1\,du = \left[\frac{u^3}{6}\right]_0^t = \frac{t^3}{3!}.$

proceeding similarly, we have

$1*1*1*\cdots*1$ (n times) $= \frac{t^{n-1}}{(n-1)!}$ where $n=1,2,3,\ldots$

→ Show that $L^{-1}\left\{\frac{1}{p\sqrt{p+4}}\right\} = \frac{1}{2}\,\text{erf}(2\sqrt{t})$.

Sol^n: Let $f(p) = \frac{1}{p}$ and $g(p) = \frac{1}{\sqrt{p+4}}$

Then $F(t) = L^{-1}\{f(p)\} = L^{-1}\left\{\frac{1}{p}\right\} = 1$

and $G(t) = L^{-1}\{g(p)\} = L^{-1}\left\{\frac{1}{(p+4)^{1/2}}\right\} = e^{-4t} L^{-1}\left\{\frac{1}{p^{1/2}}\right\}$

$= e^{-4t}\frac{t^{-1/2}}{\Gamma\frac{1}{2}} = \frac{e^{-4t}}{\sqrt{\pi t}}$ $(\because \Gamma\tfrac{1}{2} = \sqrt{\pi})$

i.e, $G(t) = \frac{e^{-4t}}{\sqrt{\pi t}}$.

Now, using the convolution theorem.

$$L^{-1}\{f(p)\,g(p)\} = \int_0^t G(u)\,F(t-u)\,du$$

$$= \int_0^t e^{-4u}\frac{1}{\sqrt{\pi}\sqrt{u}}\cdot 1\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}\frac{1}{\sqrt{u}}\,du.$$

put $4u = y^2 \Rightarrow y = 2\sqrt{u}$

$\Rightarrow du = \frac{1}{2}y\,dy$

$\Rightarrow du = \frac{1}{2}2\sqrt{u}\,dy$

$\Rightarrow \frac{du}{\sqrt{u}} = dy$.

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-y^2}\,dy$$

$$= \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-y^2}\,dy$$

$$= \underline{\underline{\frac{1}{2}\,\text{erf}(2\sqrt{t})}}.$$ $\left(\because \text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx\right)$

→ Apply convolution theorem to show that

$\int_0^t \sin u\cos(t-u)\,du = \frac{t}{2}\sin t$.

Sol^n: ~~By~~ convolution theorem, we have

$$L\left\{\int_0^t F(u)\,G(t-u)\,du\right\} = L\{F(t)\}\,L\{G(t)\} \quad \text{①}$$

Take $F(t) = \sin t$ and $G(t) = \cos t$.

Then eqn ① reduces to

$$L\left\{\int_0^t \sin u\cos(t-u)\,du\right\} = L\{\sin t\}\,L\{\cos t\}$$

$$= \frac{1}{p^2+1}\,\frac{p}{p^2+1} = \frac{p}{(p^2+1)^2}$$

$$\therefore \int_0^t \sin u\cos(t-u)\,du = L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}.$$

$= \frac{1}{2} t \sin t.$ $\quad \left[\text{we know } L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2} t \sin t \right]$

$\therefore \int_0^t \sin u \cos(t-u)\, du = \frac{t}{2} \sin t.$

→ Evaluate $\int_0^t J_0(u)\, J_0(t-u)\, du.$

Soln. By convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\}\, L\{G(t)\}. \quad \text{---} (1)$$

Take $F(t) = J_0(t)$ and $G(t) = J_0(t)$.

Then eqn (1) reduces to

$$L\left\{\int_0^t J_0(u)\, J_0(t-u)\, du\right\} = L\{J_0(t)\}\, L\{J_0(t)\}.$$

$$= \left(\frac{1}{\sqrt{p^2+1}}\right)^2 \quad \left(\because L\{J_0(t)\} = \frac{1}{\sqrt{p^2+1}}\right)$$

$$= \frac{1}{p^2+1}$$

$$\therefore \int_0^t J_0(u)\, J_0(t-u)\, du = L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \sin t.$$

→ Apply convolution theorem to prove that

$$B(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad m>0, n>0 \quad \text{(Beta function)}$$

Hence deduce that

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2} B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)}.$$

Soln. By the convolution theorem,

$$L\left\{\int_0^t F(u)\, G(t-u)\, du\right\} = L\{F(t)\} \cdot G\{F(t)\}. \quad \text{---} (1)$$

Take $F(t) = t^{m-1}$ and $G(t) = t^{n-1}$.

Then eqn (1) reduces to

$$L\left\{\int_0^t u^{m-1}(t-u)^{n-1}\, du\right\} = L\{t^{m-1}\}\, L\{t^{n-1}\}.$$

$$= \frac{\Gamma(m)}{p^m} \frac{\Gamma(n)}{p^n} = \frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}$$

$$\therefore \int_0^t u^{m-1}(t-u)^{n-1}\, du = L^{-1}\left\{\frac{\Gamma(m)\,\Gamma(n)}{p^{m+n}}\right\}$$

$$= \Gamma(m)\,\Gamma(n)\, L^{-1}\left\{\frac{1}{p^{m+n}}\right\}$$

$$= \Gamma(m)\Gamma(n)\frac{t^{m+n-1}}{\Gamma(m+n)}$$

putting $t=1$ in the above result, we get

$$B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ where } m>0, n>0. \quad \text{--- (2)}$$

Deduction:

Taking $u=\sin^2\theta$. $\quad u=0 \Rightarrow \theta=0$

$du = 2\sin\theta\cos\theta\, d\theta$. $\quad u=1 \Rightarrow \theta=\pi/2$.

$\therefore$ from (2), we have

$$\int_0^{\pi/2}\sin^{2m-2}\theta\,(\underbrace{1-\sin^2\theta}_{\cos^2\theta})^{n-1}\,2\sin\theta\cos\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$$

$$\Rightarrow \int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2}B(m,n)$$

→ find $L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\}$ by the convolution theorem and deduce the value of $L^{-1}\left\{\frac{1}{p\sqrt{p+a}}\right\}$.

Solⁿ: Let $f(p)=\frac{1}{\sqrt{p}}$ and $g(p)=\frac{1}{p}$

Then, $F(t)=L^{-1}\left\{\frac{1}{\sqrt{p}}\right\} = L^{-1}\left\{\frac{1}{p^{1/2}}\right\} = \frac{t^{-1/2}}{\Gamma\frac{1}{2}} = \frac{1}{\sqrt{\pi t}}$.

and $G(t)=L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$.

Now, using the convolution theorem, we have

$$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\} = \int_0^t \frac{1}{\sqrt{\pi}\sqrt{u}}e^{a(t-u)}du$$

$$= \frac{e^{at}}{\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{u}}e^{-au}du$$

put $au = x^2 \Rightarrow x=\sqrt{au}$ $\quad$ if $u=0$ then $x=0$; $u=t$ then $x=\sqrt{at}$

$$a\,du = 2x\,dx \Rightarrow du = \frac{2x\,dx}{a}$$

$$\Rightarrow \frac{du}{\sqrt{au}} = \frac{2dx}{a} \Rightarrow \frac{du}{\sqrt{u}} = \frac{2}{\sqrt{a}}dx$$

$$= \frac{e^{at}}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-x^2}\frac{2}{\sqrt{a}}dx$$

$$= \frac{e^{at}}{\sqrt{a}}\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{at}} e^{-x^2}dx = \frac{e^{at}}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$$ (by defn of error function)

Deduction :

we have proved that

$$L^{-1}\left\{\frac{1}{\sqrt{p}(p-a)}\right\} = \frac{e^{at}}{\sqrt{a}}\operatorname{erf}(\sqrt{at}) \quad \text{---} ①$$

Now, $$L^{-1}\left\{\frac{1}{p\sqrt{p+a}}\right\} = L^{-1}\left\{\frac{1}{(p+a-a)\sqrt{p+a}}\right\}$$

$$= e^{-at}L^{-1}\left\{\frac{1}{(p-a)\sqrt{p}}\right\}$$

$$= e^{-at}\frac{e^{at}}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$$

$$= \frac{1}{\sqrt{a}}\operatorname{erf}(\sqrt{at})$$

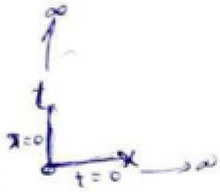
→ Show that (i) $\int_0^\infty \cos tx^2\,dx = \frac{1}{2}\sqrt{\pi/2t}$ (ii) $\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

(iii) $\int_0^\infty \sin tx^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$ (iv) $\int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

(iv) $\int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ (vi) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Sol: (i) Let $F(t) = \int_0^\infty (\cos tx^2)\,dx$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt}F(t)\,dt \quad \text{(By defn of LT)}$$



$$= \int_0^\infty e^{-pt}\left\{\int_0^\infty \cos tx^2\,dx\right\}dt$$

$$= \int_0^\infty\left[\int_0^\infty e^{-pt}\cos tx^2\,dt\right]dx \text{, changing the order of integration.}$$

$$= \int_0^\infty L(\cos t x^2)\,dx \text{, by defn of LT.}$$

$$= \int_0^\infty \frac{p}{p^2+x^4}\,dx \qquad \left[\because L\{\cos at\} = \frac{p}{p^2+a^2}\right]$$

Put $x^2 = p\tan\theta \Rightarrow x = \sqrt{p\tan\theta}$. Limits: if $x=0 \Rightarrow \theta=0$; if $x=\infty \Rightarrow \theta=\pi/2$

and $dx = \frac{1}{2\sqrt{p\tan\theta}}\,p\sec^2\theta\,d\theta$.

$$= \int_0^{\pi/2}\frac{p}{p^2+p^2\tan^2\theta}\,\frac{1}{2\sqrt{p\tan\theta}}\,p\sec^2\theta\,d\theta.$$

$$= \int_0^{\pi/2}\frac{p^2}{p^2(\sec^2\theta)}\,\frac{\sec^2\theta}{2\sqrt{p\tan\theta}}\,d\theta = \frac{1}{2\sqrt{p}}\int_0^{\pi/2}\frac{1}{\sqrt{\tan\theta}}\,d\theta.$$

$$= \frac{1}{2\sqrt{p}}\int_0^{\pi/2}\sin^{-1/2}\theta\,.\cos^{1/2}\theta\,d\theta.$$

$$= \frac{1}{2\sqrt{p}}\left(\frac{1}{2}\right)\frac{\Gamma\frac{1}{4}\,\Gamma 3/4}{\Gamma\frac{1}{4}+\frac{3}{4}}$$

(Here $2m-1=-\frac{1}{2}$; $2n-1=\frac{1}{2}$; $2m=\frac{1}{2}$; $2n=3/2$; $m=\frac{1}{4}$; $n=3/4$

$\because \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\frac{\Gamma m\,\Gamma n}{\Gamma m+n}$ ...)

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$$= \frac{1}{4\sqrt{p}}\frac{\Gamma\frac{1}{4}\,\Gamma 3/4}{\Gamma 1}$$

$(\because \Gamma 1 = 1)$

$$= \frac{1}{4\sqrt{p}}\Gamma(1/4)\,\Gamma(3/4) = \frac{1}{4\sqrt{p}}\Gamma\tfrac{1}{4}\,\Gamma 1-\tfrac{1}{4} \quad (\text{Here } p_1 = \tfrac{1}{4})$$

$\left(\because \Gamma(p_1)\,\Gamma(1-p_1) = \frac{\pi}{\sin p_1\pi}, \text{ where } 0<p_1<1\right)$

$$= \frac{1}{4\sqrt{p}}\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

$$= \frac{\pi}{4\times\frac{1}{\sqrt{2}}\sqrt{p}} = \frac{\pi}{2\sqrt{2}\sqrt{p}}$$

$$\therefore L\{F(t)\} = \frac{\pi}{2\sqrt{2}\sqrt{p}}$$

$$\Rightarrow F(t) = L^{-1}\left\{\frac{\pi}{2\sqrt{2}\sqrt{p}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}} L^{-1}\left\{\frac{1}{\sqrt{p}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{\pi t}} = \frac{\sqrt{\pi}}{2\sqrt{2t}} = \frac{1}{2}\sqrt{\frac{\pi}{2t}}.$$

$$\therefore \int_0^\infty \cos t x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2t}} \quad \text{——— (1)}$$

(ii) Taking $t=1$ in eqn (1)

$$\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(v) $\int_0^\infty e^{-tx^2} dx$.

Let $F(t) = \int_0^\infty e^{-tx^2} dx$.

$$\therefore L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\, dt$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^\infty e^{-tx^2} dx\right\} dt$$

$$= \int_0^\infty\left[\int_0^\infty e^{-pt} e^{-tx^2} dt\right] dx., \text{ changing the order of integration}$$

$$= \int_0^\infty L(e^{-tx^2})\,dx$$

$$= \int_0^\infty \frac{1}{p+x^2}\,dx \qquad \left[\because L^{-1}\{e^{-at}\} = \frac{1}{p+a}\right]$$

$$= \int_0^\infty \frac{1}{(\sqrt{p})^2+x^2}\,dx$$

$$= \left[\frac{1}{\sqrt{p}}\tan^{-1}\left(\frac{x}{\sqrt{p}}\right)\right]_0^\infty$$

$$= \left[\frac{1}{\sqrt{p}}\frac{\pi}{2} - 0\right] = \frac{1}{\sqrt{p}}\frac{\pi}{2}.$$

$$\therefore L\{F(t)\} = \frac{1}{\sqrt{p}}\left(\frac{\pi}{2}\right)$$

$$F(t) = L^{-1}\left\{\frac{1}{\sqrt{p}}\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{p}}\right)$$

$$F(t) = \frac{\pi}{2}\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{t}}$$

$$\therefore \int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}} \quad \text{—— (2)}$$

(vi) putting $t=1$ in eqn (2)

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

→ we know that $\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$.

The complementary error function is defined

as $\text{erf}\,c(t) = 1 - \text{erf}(t)$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx - \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx + \frac{2}{\sqrt{\pi}}\int_t^0 e^{-x^2}\,dx \qquad \left[\because \text{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx. = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \quad \left(\because \int_0^\infty e^{-x^2}dx = 1\right) \Rightarrow \text{erf}(\infty) = 1\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_t^0 e^{-x^2}\,dx + \int_0^\infty e^{-x^2}\,dx\right]$$

$$\boxed{\text{erf}\,c(t) = \frac{2}{\sqrt{\pi}}\int_t^\infty e^{-x^2}\,dx.}$$

* → Find $L^{-1}\left\{\frac{e^{-\sqrt{p}}}{p}\right\}$. and hence deduce that

$$L^{-1}\left\{\frac{e^{-x\sqrt{p}}}{p}\right\} = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right).$$

Sol^n: Let $f(p) = e^{-\sqrt{p}}$.

$$\Rightarrow L\{F(t)\} = e^{-\sqrt{p}}$$

$$\Rightarrow F(t) = L^{-1}\{e^{-\sqrt{p}}\}$$

$$= L^{-1}\left\{1-\sqrt{p}+\frac{(\sqrt{p})^2}{2!}-\frac{(\sqrt{p})^3}{3!}+\frac{(\sqrt{p})^4}{4!}-\cdots\right\}$$

$$= L^{-1}\left[1-\sqrt{p}+\frac{p}{2!}-\frac{p^{3/2}}{3!}+\frac{p^2}{4!}-\cdots\right]$$

$$= L^{-1}\{1\}-L^{-1}\{p^{1/2}\}+\frac{1}{2!}L^{-1}\{p\}-\frac{1}{3!}L^{-1}\{p^{3/2}\}$$
$$+\frac{1}{4!}L^{-1}\{p^2\}+\cdots$$

$$= L^{-1}\{1\}-L^{-1}\left\{\frac{1}{p^{-1/2}}\right\}+\frac{1}{2!}L^{-1}\left\{\frac{1}{p^{-1}}\right\}-\frac{1}{3!}L^{-1}\left\{\frac{1}{p^{-3/2}}\right\}$$
$$+\frac{1}{4!}L^{-1}\left\{\frac{1}{p^{-2}}\right\}-\cdots$$

→ Heaviside's expansion theorem (or formula).

(3)

Let $F(p)$ and $G(p)$ be two polynomials in p where $F(p)$ has degree less than that of $G(p)$. If $G(p)$ has 'n' distinct zeros α_r, $(r = 1, 2, \ldots, n)$.

i.e, $G(p) = (p-\alpha_1)(p-\alpha_2)\ldots\ldots(p-\alpha_n)$. then

$$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}.$$

proof: Since $F(p)$ is a polynomial of degree less than that of $G(p)$ and $G(p)$ has n distinct zeros α_r, $r = 1, 2, \ldots n$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(p)}{(p-\alpha_1)(p-\alpha_2)\ldots\ldots(p-\alpha_n)}$$

$$= \frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \ldots\ldots + \frac{A_r}{p-\alpha_r} + \ldots\ldots + \frac{A_n}{p-\alpha_n}$$

To compute A_r, multiplying bothsides by $p-\alpha_r$.

i.e, $$\frac{F(p)}{G(p)}\cdot(p-\alpha_r) = (p-\alpha_r)\left[\frac{A_1}{p-\alpha_1} + \frac{A_2}{p-\alpha_2} + \ldots\ldots + \frac{A_r}{p-\alpha_r} + \ldots\ldots + \frac{A_n}{p-\alpha_n}\right]$$

Taking limit as $p \to \alpha_r$, we get

$$\Rightarrow \lim_{p\to\alpha_r} \frac{F(p)}{G(p)}(p-\alpha_r) = A_r.$$

$$\Rightarrow A_r = \lim_{p\to\alpha_r} \frac{F(p)}{G(p)}(p-\alpha_r)$$

$$= F(\alpha_r) \lim_{p\to\alpha_r} \frac{(p-\alpha_r)}{G(p)} \quad \left(\text{Form } \tfrac{0}{0}\right)$$

where $G(p) = (p-\alpha_1)\ldots(p-\alpha_r)\ldots(p-\alpha_n)$

$$= F(\alpha_r) \lim_{p\to\alpha_r} \frac{1}{G'(p)}$$

$$= F(\alpha_r)\frac{1}{G'(\alpha_r)}$$

$$\therefore \frac{F(p)}{G(p)} = \frac{F(\alpha_1)}{G'(\alpha_1)}\frac{1}{(p-\alpha_1)} + \frac{F(\alpha_2)}{G'(\alpha_2)}\frac{1}{(p-\alpha_2)} + \frac{F(\alpha_3)}{G'(\alpha_3)}\frac{1}{(p-\alpha_3)} + \ldots\ldots + \frac{F(\alpha_r)}{G'(\alpha_r)}\frac{1}{(p-\alpha_r)} + \ldots\ldots + \frac{F(\alpha_n)}{G'(\alpha_n)}\frac{1}{(p-\alpha_n)}$$

Hence $$L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = \frac{F(\alpha_1)}{G'(\alpha_1)}L^{-1}\left(\frac{1}{p-\alpha_1}\right) + \frac{F(\alpha_2)}{G'(\alpha_2)}L^{-1}\left(\frac{1}{p-\alpha_2}\right) + \ldots\ldots + \frac{F(\alpha_r)}{G'(\alpha_r)}L^{-1}\left(\frac{1}{p-\alpha_r}\right) + \ldots\ldots + \frac{F(\alpha_n)}{G'(\alpha_n)}L^{-1}\left(\frac{1}{p-\alpha_n}\right)$$

$$= \frac{F(\alpha_1)}{G'(\alpha_1)} e^{\alpha_1 t} + \frac{F(\alpha_2)}{G'(\alpha_2)} e^{\alpha_2 t} + \cdots + \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} + \cdots + \frac{F(\alpha_n)}{G'(\alpha_n)} e^{\alpha_n t}$$

$$= \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}.$$

→ Apply Heaviside expansion theorem to obtain

(i) $L^{-1}\left\{\frac{2p^2+5p-4}{p^3+p^2-2p}\right\}$ (ii) $L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\}$

Solⁿ: (i) $F(p) = 2p^2+5p-4$,

and $G(p) = p^3+p^2-2p$

$= p(p^2+p-2) \Rightarrow \alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = -2$

$G'(p) = 3p^2+2p-2$

Using Heaviside's expansion formula, we have

$$\therefore L^{-1}\left\{\frac{2p^2+5p-4}{p^3+p^2-2p}\right\} = \frac{F(0)}{G'(0)} e^{0\cdot t} + \frac{F(1)}{G'(1)} e^{t} + \frac{F(-2)}{G'(-2)} e^{-2t}$$

$$= \frac{-4}{-2} + \frac{3}{3}e^{t} + \frac{-6}{6}e^{-2t} = 2+e^{t}-e^{-2t}.$$

(ii) $F(p) = 3p+1$ and $G(p) = (p-1)(p^2+1) = (p-1)(p+i)(p-i)$

$\Rightarrow \alpha_1 = 1,\ \alpha_2 = -i\ \ \alpha_3 = i$

$G(p) = p^3-p^2+p-1$

$G'(p) = 3p^2-2p+1$

$$\therefore L^{-1}\left\{\frac{F(p)}{G(p)}\right\} = L^{-1}\left\{\frac{3p+1}{(p-1)(p^2+1)}\right\} = \frac{F(1)e^{t}}{G'(1)} + \frac{F(-i)}{G'(-i)}e^{-it} + \frac{F(i)}{G'(i)}e^{it}$$

$$= \frac{4e^{t}}{2} + \frac{-3i+1}{-2+2i}e^{-it} + \frac{3i+1}{-(2+2i)}e^{it}$$

$$= 2e^{t} + \frac{(3i-1)(1+i)}{2(1-i)(1+i)}e^{-it} - \frac{3i+1}{2(1+i)}\frac{(1-i)}{(1-i)}e^{it}$$

$$= 2e^{t} + \frac{1}{2}(i-2)e^{-it} - \left(\frac{1}{2}\right)(i+2)e^{it}$$

$$= 2e^{t} - \frac{i}{2}(e^{it}-e^{-it}) - (e^{it}+e^{-it})$$

$$= 2e^{t} + \sin t - 2\cos t$$

→ Apply Heaviside expansion theorem to obtain

(i) $L^{-1}\left\{\frac{p+5}{(p+1)(p^2+1)}\right\}$ (ii) $L^{-1}\left\{\frac{2p^2-6p+5}{p^3-6p^2+11p-6}\right\}$.

→ Solution of ordinary Differential equations with constant coefficients: (32)

The Laplace transform is very useful in solving ordinary linear differential eqns. with constant coefficients.

Suppose we wish to solve the n^{th} order ordinary linear differential eqn with constant coefficients

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \cdots + a_n y = F(t) \quad \text{—— (1)}$$

where $F(t)$ is a function of the independent variable t. and $a_0, a_1, a_2, \ldots a_n$ are constants, subject to the initial conditions

$$y(0) = k_0,\; y'(0) = k_1,\; y''(0) = k_2, \ldots\; y^{n-1}(0) = k_{n-1} \quad \text{—— (2)}$$

where $k_0, k_1, \ldots k_{n-1}$ are constants.

On taking the Laplace transform of bothsides of eqn (1) and using conditions (2), we obtain an algebraic equation known as "subsidiary eqn" for determination of $L\{y(t)\}$. The required solution is then obtained by finding the inverse Laplace transform of

$L\{y(t)\} = f(p)$

Notations

→ $\frac{dy}{dt} = Dy = y'(t) = y^{(1)}(t)$;

$\frac{d^2y}{dt^2} = D^2y = y''(t) = y^{(2)}(t)$; ..... $\frac{d^ny}{dt^n} = D^ny = y^n(t)$ etc.

→ At $t=0$, we have

$y(0) = y_0,\; y'(0) = y_1,\; y''(0) = y_2, \ldots\; y^{(n)}(0) = y_n.$

→ Solve $\frac{d^2y}{dt^2}+y=0$ under the conditions that $y=1$, $\frac{dy}{dt}=0$ when $t=0$.

Sol^n: Given that $\frac{d^2y}{dt^2}+y=0$, i.e. $y''+y=0$ —— ①

Taking Laplace transform of both sides of eqn ①, we get

$L(y'')+L(y)=L(0)$

$p^2L\{y(t)\}-py(0)-y'(0)+L\{y(t)\}=0$

$p^2L\{y(t)\}-p(1)-0+L\{y(t)\}=0$ $\left[\because y(0)=1 \;\&\; y'(0)=0\right]$.

$(p^2+1)L\{y(t)\}=p$

$\Rightarrow L\{y(t)\}=\frac{p}{p^2+1}$

Taking inverse Laplace transform, we get

$\Rightarrow y(t)=L^{-1}\left\{\frac{p}{p^2+1}\right\}$

$=\cos t$.

$\therefore y(t)=\cos t$.

which is the required solution.

→ Solve $(D^2+m^2)x=a\cos nt$, $t>0$.

$x=x_0$ and $Dx=x_1$, when $t=0$, $n\neq m$

Sol^n: Given eqn is

$(D^2+m^2)x=a\cos nt$.

$x''+m^2x=a\cos nt$ —— ①

Taking Laplace transform of both sides of ①

we get

$L\{x''\}+L\{m^2x\}=L\{a\cos nt\}$

$p^2L\{x(t)\}-p\{x(0)\}-x'(0)+m^2Lx(t)=\frac{ap}{p^2+n^2}$ —— ②

Using the given conditions $x=x_0$ and $Dx=x_1$ when $t=0$

eqn ② reduces to

$p^2L\{x(t)\}-px_0-x_1+m^2L\,x(t)=\frac{ap}{p^2+n^2}$

$\Rightarrow (p^2+m^2)L\{x(t)\}=\frac{ap}{p^2+n^2}+px_0+x_1$

$$\Rightarrow L\{x(t)\} = \frac{p}{p^2+m^2}x_0 + \frac{1}{p^2+m^2}x_1 + \frac{a.p}{(p^2+m^2)(p^2+n^2)}$$ (33)

$$= x_0\frac{p}{p^2+m^2} + x_1\frac{1}{p^2+m^2} + \frac{a}{m^2-n^2}\left[\frac{-p}{p^2+m^2}+\frac{p}{p^2+n^2}\right]$$

Taking the inverse Laplace transform, we get

$$x(t) = x_0 L^{-1}\left\{\frac{p}{p^2+m^2}\right\} + x_1 L^{-1}\left\{\frac{1}{p^2+m^2}\right\} + \frac{a}{m^2-n^2}\left[L^{-1}\left\{\frac{-p}{p^2+m^2}\right\} + L^{-1}\left\{\frac{p}{p^2+n^2}\right\}\right].$$

$$= x_0\cos mt + x_1\frac{1}{m}\sin mt + \frac{a}{m^2-n^2}(-\cos mt + \cos nt)$$

$$x(t) = x_0\cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}\left[-\cos mt + \cos nt\right]$$

→ Solve $(D+2)^2 y = 4e^{-2t}$, $y(0)=-1$ & $y'(0)=4$
$(D^2+4D+4)y = 4e^{-2t}$

→ Solve $(D^2-2D+2)y=0$, $y = Dy = 1$, when $t=0$

→ Solve $(D^2+4D+4)x = \sin\omega t$, $t>0$ with x_0 and x_1 for values of x and Dx when $t=0$ (i.e., $x=x_0$ & $Dx=x_1$, when $t=0$)

→ Solve $(D^2+9)y = \cos 2t$ if $y(0)=1$, $y(\pi/2) = -1$

Sol^n: Given eqn $(D^2+9)y = \cos 2t$ ——— (1)

Taking Laplace transform of both sides of eqn (1), we get

$$L\{y''\} + 9L\{y\} = L\{\cos 2t\}$$

$$\Rightarrow p^2 L\{y(t)\} - p\,y(0) - y'(0) + 9L\{y(t)\} = \frac{p}{p^2+4} \quad \text{——— (2)}$$

Using the given condition $y(0)=1$ eqn (2) reduces to

$$p^2 L\{y(t)\} - p(1) - A + 9L\{y(t)\} = \frac{p}{p^2+4}$$

(Taking $y'(0) = A$ constant)

$$\Rightarrow (p^2+9)L\{y(t)\} = \frac{p}{p^2+4} + p + A.$$

$$\Rightarrow L\{y(t)\} = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + A.\frac{1}{p^2+9}.$$

Taking inverse Laplace transform: we get

$$y(t) = L^{-1}\left\{\frac{p}{(p^2+4)(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + A\, L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5} L^{-1}\left\{\frac{p}{(p^2+4)} - \frac{p}{(p^2+9)}\right\} + L^{-1}\left\{\frac{p}{p^2+9}\right\} + \frac{A}{3} L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$= \frac{1}{5} L^{-1}\left\{\frac{p}{p^2+4}\right\} - \frac{1}{5} L^{-1}\left\{\frac{p}{p^2+9}\right\} + \cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{A}{3}\sin 3t.$$

$$\Rightarrow y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{A}{3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos \pi + \frac{4}{5}\cos\left(3\frac{\pi}{2}\right) + \frac{A}{3}\sin\left(\frac{3\pi}{2}\right) \quad \left(\because y\left(\frac{\pi}{2}\right) = -1\right)$$

$$-1 = \frac{1}{5}(-1) - \frac{1}{5}(0) + 0 + \frac{A}{3}(-1)$$

$$\Rightarrow -1 = -\frac{1}{5} - \frac{A}{3} \Rightarrow -\frac{4}{5} = -\frac{A}{3}$$

$$\Rightarrow A = \frac{12}{5}$$

$\therefore$ The required solution is

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5\times 3}\sin 3t$$

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t.$$

→ $(D^2+3D+2)y = 0$; $y = y_0$ and $Dy = y_1$ at $t=0$

→ solve $(D^2-D-2)y = 20\sin 2t$; $y=-1$, $Dy = 2$ when $t=0$

→ Solve $(D+1)^2 y = t$; $y=-3$ when $t=0$ and $y=-1$ when $t=1$

→ solve $(D^2+2D+1)y = 3te^{-t}$, $t>0$ subject to the conditions $y=4$, $Dy=2$ when $t=0$

→ solve $(D^2+1)y = t\cos 2t$, $y=0$, $\frac{dy}{dt}=0$ when $t=0$

→ solve $(D^2-3D+2)y = 1-e^{2t}$, $y=1$, $Dy=0$ when $t=0$

→ solve $(D^2+1)y = \sin t \sin 2t$, $t>0$ if $y=1$, $Dy=0$ when $t=0$

→ solve. $(D^3-D)y = 2\cos t$, $y=3$, $Dy=2$, $D^2y=1$ when $t=0$

→ Solve $(D^3 - D^2 - D + 1)y = 8te^{-t}$
if $y = D^2y = 0$, $Dy = 1$ when $t = 0$

→ Solve $(D^4 - 1)y = 1$, $y = Dy = D^2y = D^3y = 0$ at $t = 0$

→ Solve $(D^2 + D)y = t^2 + 2t$. where $y(0) = 4$, $y'(0) = -2$

→ Solve $(D^4 + 2D^2 + 1)y = 0$ where $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$
and $y'''(0) = -3$.

→ Solve $(D^3 + 1)y = 1$, $t > 0$
$y = Dy = D^2y = 0$ when $t = 0$

→ Solve $(D^2 + n^2)y = a\sin(nt + \alpha)$, $y = 0$, $Dy = 0$ when $t = 0$

Independence of Solutions of Linear Differential Equations

Wronskian and Its Properties

SOLVED EXAMPLES

Ex. 1. *If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of differential equation $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.* **[Delhi B.Sc. (Hons) 1996]**

Sol. The Wronskian of $y_1(x)$ and $y_2(x)$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} = -3\sin^2 3x - 3\cos^2 3x$$

$$= -3(\sin^2 3x + \cos^2 3x) = -3 \neq 0.$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

Ex. 2. *Prove that $\sin 2x$ and $\cos 2x$ are solutions of the differential equation $y'' + 4y = 0$ and these solutions are linearly independent.*

[Delhi (B.Sc.) (G) 1998]

Sol. Given equation is $\qquad y'' + 4y = 0. \qquad$ ...(1)

Let $\qquad y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x. \qquad$ ...(2)

Now, $\qquad y_1' = 2\cos 2x$ and $y_1'' = -4\sin 2x. \qquad$ ...(3)

$\therefore \qquad y_1''(x) + 4y_1(x) = -4\sin 2x + 4\sin 2x = 0$, by (2) and (3)

Hence, $y_1(x) = \sin 2x$ is a solution of (1). Similarly we can prove that $y_2(x)$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2\sin^2 2x - 2\cos^2 2x$$

$$= -2(\sin^2 2x + \cos^2 2x) = -2 \neq 0.$$

Since $W(x) \neq 0$, $\sin 2x$ and $\cos 2x$ are linearly independent solutions of (1).

Ex. 3. *Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$.* **[Delhi B.Sc. (P) 96, Delhi B.Sc. (H) 2002]**

Sol. Given equation is $\qquad y'' - 2y' + 2y = 0. \qquad$ ...(1)

Let $\qquad y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x. \qquad$ ...(2)

From (2), $\quad y_1'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x) \qquad$ ...(3)

From (3), $\quad y_1''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x. \qquad$ ...(4)

Now, $y_1''(x) - 2y_1'(x) + 2y_1(x) = 2e^x \cos x - 2e^x (\sin x + \cos x) + 2e^x \sin x = 0$ showing that $y_1(x) = e^x \sin x$ is a solution of (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x (\sin x + \cos x) & e^x (\cos x - \sin x) \end{vmatrix}$$

$$= e^{2x} (\sin x \cos x - \sin^2 x) - e^{2x} (\sin x \cos x + \cos^2 x) = -e^{2x} \neq 0,$$

showing that $W(x) \neq 0$, and hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

The general solution of (1) is given by [Refer Art. 2.11]

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = e^x (c_1 \sin x + c_2 \cos x), \quad \ldots(5)$$

where c_1 and c_2 are arbitrary constants.

From (5), $y'(x) = e^x (c_1 \sin x + c_2 \cos x) + e^x (c_1 \cos x - c_2 \sin x)$....(6)

Putting $x = 0$ in (5) and using the given result $y(0) = 2$, we get

$$y(0) = c_2 \text{ or } c_2 = 2.$$

Putting $x = 0$ in (6) and using the given result $y'(0) = -3$, we get

$$y'(0) = c_2 + c_1 \text{ or } -3 = 2 + c_2 \text{ or } c_1 = -5.$$

$\therefore$ from (5), the solution of given equation satisfying the given properties is

$$y = e^x (2 \cos x - 5 \sin x).$$

Ex. 4. *Show that e^{2x} and e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with the property that $y(0) = 0$ and $y'(0) = 1$.* **[Delhi B.Sc. (G) 1998]**

Sol. Given equation is $\quad y'' - 5y' + 6y = 0. \quad \ldots(1)$

Let $\quad y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}. \quad \ldots(2)$

From (2) $\quad y_1'(x) = 2e^{2x}$ and $y_1''(x) = 4e^{2x}. \quad \ldots(3)$

$\therefore y_1''(x) - 5y_1'(x) + 6y_1(x) = 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = 0,$

showing that $y_1(x)$ is a solution of (1).

Similarly, we find that $y_2(x) = e^{3x}$ is a solution of (1),

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0,$$

showing that that $y_1(x) = e^{2x}$ and $y_2(x) = e^{3x}$ are linearly independent solutions of (1).

The general solution of (1) is given by

$$y(x) = c_1 e^{2x} + c_2 e^{3x}, \; c_1 \text{ and } c_2 \text{ being arbitrary constants.} \quad \ldots(4)$$

From (4), $\quad y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}. \quad \ldots(5)$

Putting $x = 0$ in (4) and using $y(0) = 0$, we get $c_1 + c_2 = 0$. ...(6)

Putting $x = 0$ in (5) and using $y'(0) = 1$, we get $2c_1 + 3c_2 = 1$. ...(7)

Solving (6) and (7), $c_1 = -1$ and $c_2 = 1$ and so from (4), we have

$$y(x) = e^{3x} - e^{2x} \text{ as the required solution.}$$

Ex. 5. *Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants c_1 and c_2 so that the solution $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$.* **[Delhi B.A. (P) 2002]**

Sol. Given equation is $\quad y'' + y = 0.$ ...(1)

Here $y_1(x) = \sin x$ so that $y_1'(x) = \cos x$ and $y_1''(x) = -\sin x$. ...(2)

Hence $\quad y_1''(x) + y_1(x) = -\sin x + \sin x = 0$, showing that $y_1(x)$ is a solution of (1). Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x(\cos x + \sin x) - \cos x(\sin x - \cos x) = 1 \neq 0,$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$

or $\quad \sin x + 3\cos x \equiv c_1 \sin x + c_2(\sin x - \cos x)$. ...(3)

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides of (3), we get

$$c_1 + c_2 = 1 \text{ and } -c_2 = 3 \text{ so that } c_1 = 4 \text{ and } c_2 = -3.$$

Ex. 6. *Show that x and $x\,e^x$ are linearly independent on the x-axis.*

Sol. The Wronskian $W(x)$ of x and $x\,e^x$ is given by

$$W(x) = \begin{vmatrix} x & xe^x \\ \dfrac{dx}{dx} & \dfrac{d(x e^x)}{dx} \end{vmatrix} = \begin{vmatrix} x & x e^x \\ 1 & e^x + x e^x \end{vmatrix}$$

$$= x(e^x + x e^x) - x e^x = x^2 e^x.$$

We observe that $W(x) \neq 0$ for $x \neq 0$ on the x-axis. Hence x and $x\,e^x$ are linearly independent on the x-axis [Refer corollary to theorem III of Art 2.6]

Ex. 7. *Show that the Wronskian of the functions x^2 and $x^2 \log x$ is non-zero. Can these functions be indpendent solutions of an ordinary differential equation. If so, determine this differential equation.* **[Meerut 1988, 98]**

Sol. Let $\quad y_1(x) = x^2$ and $y_2(x) = x^2 \log x$.

The Wronskian $W(x)$ of y_1 and y_2 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \log x \\ 2x & 2x\log x + x \end{vmatrix} = x^2(2x\log x + x) - 2x^3 \log x.$$

$\therefore \quad W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$. Hence solution y_1 and y_2 can be linearly independent solutions of an ordinary differential equation.

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2 \log x. \qquad \ldots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $\quad y' = 2Ax + B(2x \log x + x). \qquad \ldots(2)$

Differentiating (2), $\quad y'' = 2A + B(2 \log x + 2 + 1). \qquad \ldots(3)$

We now eliminate A and B from (1), (2) and (3). To this end, we first solve (2) and (3) for A and B. Multiplying both sides of (3) by x, we get

$$xy'' = 2Ax + B(3x + 2x \log x). \qquad \ldots(4)$$

Subtracting (2) from (4), $\quad xy'' - y' = 2Bx \quad$ or $\quad B = (xy'' - y')/2x.$

Substituting this value of B in (3), we have

$$2A = y'' - (1/2x)(xy'' - y')(3 + 2 \log x)$$

or $\quad A = (1/4x)[2xy'' - (xy'' - y')(3 + 2 \log x)].$

Substituting the above values A and B in (1), we have

$$y = (x/4)[2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \log x (xy'' - y')$$

or $\quad 4y = x(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x \log x (xy'' - y')$

or $\quad x^2y'' - 3xy' + 4y = 0$, which is the required equation.

Ex. 8. *Evaluate the Wronskian of the functions x and xe^x. Hence conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.*

[Meerut 97]

Sol. Let $y_1 = x$ and $y_2 = xe^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence y_1 and y_2 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax + Bxe^x, \qquad \ldots(1)$$

where A and B are arbitrary constants.

Differentiating (1), $\quad y' = A + B(e^x + xe^x) = A + B(1 + x)e^x. \qquad \ldots(2)$

Differentiating (2), $\quad y'' = B[e^x + (1 + x)e^x] = Be^x(2 + x). \qquad \ldots(3)$

We now eliminate A and B from (1), (2) and (3). To this we first solve (2) and (3) for A and B.

From (3), $\quad B = y''/[e^x(2 + x)].$

Substituting this value of B in (2), we have

$$A = y' - B(1 + x)e^x = y' - \frac{1+x}{2+x}y'' = \frac{(2+x)y' - (1+x)y''}{2+x}.$$

Substituting the above values of A and B in (1), we get

$$y = \left[\frac{(2+x)y' - (1+x)y''}{2+x}\right]x + \left[\frac{y''}{e^x(2+x)}\right]x e^x$$

or $$(2+x)y = x(2+x)y' - x(1+x)y'' + xy''$$

or $x^2y'' - x(2+x)y' + (2+x)y = 0$, which is required equation.

Ex. 9.(a) *Show that the solutions* e^x, e^{-x}, e^{2x} of $(d^3y/dx^3) - 2(d^2y/dx^2) - (dy/dx) + 2y = 0$ *are linearly independent and hence or otherwise solve the given equation.* **[Delhi B.Sc. (G) 1993, 98 ; Meerut 87, 98]**

Sol. Given equation is $\quad y''' - 2y'' - y' + 2y = 0.$ ...(1)

Let $\quad y_1 = e^x, y_2 = e^{-x}$ and $y_3 = e^{2x}$. ...(2)

Here $\quad y_1' = e^x, y_1'' = e^x$ and $y_1''' = e^x$. ...(3)

$\therefore \quad y_1''' - 2y_1'' - y_1' + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0$, by (2) and (3)

Hence $y_1 = e^x$ in a solution of (1). Similarly, we can show that e^{-x} and e^{2x} are also solutions of (1).

Now, the Wronskian $W(x)$ of y_1, y_2, y_3 in given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= (e^x \cdot e^{-x} \cdot e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \text{ by } \begin{matrix} C_2 \to & C_2 - C_1 \\ C_3 \to & C_3 - C_1 \end{matrix}$$

$= -6e^{2x}$, which is not identically zero on $(-\infty, \infty)$

Hence y_1, y_2, y_3 are linearly independent solutions of (1) [Refer corollary of theorem III of Art 2.6]. Since the order of the given equation (1) is three, it follows that the general solution of (1) will contain three arbitrary constants c_1, c_2, c_3 and is given by [Refer Art. 2.11]

$$y = c_1y_1 + c_2y_2 + c_3y_3 \text{ i.e., } y = c_1e^x + c_2e^{-x} + c_3e^{2x}.$$

Ex. 9. (b) *Show that the solutions* e^x, e^{2x}, e^{-2x} of $y''' - y'' - 4y' + 4 = 0$ *are linearly independent and hence or otherwise solve the given equation.*

Hint. Try yourself as in Ex. 9. (a) **Ans.** $y = c_1e^x + c_2e^{2x} + c_3e^{-2x}$.

Ex. 10. *Prove that the functions* $1, x, x^2$ *are llinearly independent. Hence from the differential equation whose roots are* $1, x, x^2$. **[Meerut 1996, 97]**

Sol. Let $\quad y_1(x) = 1, y_2(x) = x$ and $y_3(x) = x^2$. ...(1)

Then the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}, \text{ using (1)}$$

or $\quad W(x) = 2 \neq 0$ for any $x \in (-\infty, \infty)$.

Hence, y_1, y_2, and y_3 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 + Cy_3 = A + Bx + Cx^2, \qquad ...(1)$$

where A, B, C are arbitrary constants.

Differentiating (1), $\quad y' = B + 2Cx. \qquad ...(2)$

Differentiating (2), $\quad y'' = 2C. \qquad ...(3)$

Differentiating (3), $\quad y''' = 0$, *i.e.*, $d^3y/dx^3 = 0. \qquad ...(4)$

Since (4) is free from arbitrary constants A, B, C, hence (4) is the required differential equation.

Ex. 11. *Use Wronskian to show that the functions x, x^2, x^3 are independent. Determine the differential equation with these as independent solutions.*

[Meerut 86, 95]

Sol. Let $\quad y_1(x) = x, y_2(x) = x^2$ and $y_3 = x^3. \qquad ...(1)$

The Wronskian $W(x)$ of y_1, y_2 and y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}, \text{ using (1)}$$

or $\quad W(x) = x(12x^2 - 6x^2) - (1)(6x^3 - 2x^3) = 2x^3,$

which is not identically equal to zero. Hence the functions y_1, y_2 and y_3 are linearly independent.

To form the differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 + Cy_3 = Ax + Bx^2 + Cx^3. \qquad ...(2)$$

Differentiating (2), $\quad y' = A + 2Bx + 3Cx^2. \qquad ...(3)$

Differentiating (3), $\quad y'' = 2B + 6Cx \qquad ...(4)$

Differentiating (4), $\quad y''' = 6C. \qquad ...(5)$

To find the required equation, we now eliminate A, B, C, from (2), (3), (4) and (5).

From (5), $C = (1/6)y'''$. Then, from (4) $B = (1/2)(y'' - xy'''). \qquad ...(6)$

Multiplying both sides of (3) by x, $xy' = Ax + 2Bx^2 + 3Cx^3$. $\qquad ...(7)$

Substraticng (7) from (2), $y - xy' = -Bx^2 - 2Cx^3$

or $\quad y - xy' = -(1/2)x^2(y'' - xy''') - (2x^3)(1/6)y'''$, using (6)

or $\quad 6y - 6xy' = -3x^2y'' + 3x^3y''' - 2x^3y'''$ or $x^3y''' - 3x^2y'' + 6xy' - 6y = 0,$

which is the required differential equation.

EXERCISE

1. Prove that the Wronskian of the functions $e^{m_1x}, e^{m_2x}, e^{m_3x}$, is equal to $(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)\, e^{(m_1+m_2+m_3)x}$. Are these functions linearly independent.

[**Ans.** Given functions are linearly independent if $m_1 \neq m_2 \neq m_3$.]

2. Test the linear independence of the following sets of functions :

(*i*) $\sin x$, $\cos x$. [**Ans.** Linearly independent]

(*ii*) $1 + x$, $1 + 2x$, x^2. [**Ans.** Linearly independent]

(*iii*) $x^2 - 1$, $x^2 - x + 1$, $3x^2 - x - 1$. [**Ans.** Linearly dependent]

(*iv*) $\sin x$, $\cos x$, $\sin 2x$. [**Ans.** Linearly independent]

(*v*) e^x, e^{-x}, $\sin ax$. [**Ans.** Linearly independent]

(*vi*) e^x, $x e^x$, $\sinh x$. [**Ans.** Linearly independent]

(*vii*) $\sin 3x$, $\sin x$, $\sin^3 x$. [**Ans.** Linearly dependent]

3. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form the differential equation of second order having these two functions as independent solutions. [**Ans.** $y'' - 2y' + 2y = 0$]

4. Evaluate the Wronskian of the functions e^x and $x e^x$. Hence conclude whether or not they are linearly independent. If they are independent set up the differential equation having them as its independent solutions. [**Ans.** $y'' - 2y' + y = 0$]

5. Show that any two solutions of the equation $y'' + f(x)y' + g(x)y = 0$, $f(x)$ and $g(x)$ being continuous on an open interval I, are linearly independent, if and only if, their Wronskian is zero for some $x = x_0$ on I. **[Meerut 1992]**

[**Hint.** Proceed exactly as in theorem V of Art. 2.13 for $n = 2$.]

6. If the functions $p(x)$ and $q(x)$ are continuous on $\alpha < x < \beta$, and if the functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$, then prove that the Wronskian $W(y_1, y_2)$ in non- vanishing on $\alpha < x < \beta$.

[**Hint.** Proceed exactly as in theorem V of Art 2.13 for $n = 2$]

7. Show that linearly independent solutions of $y'' - 3y' + 2y = 0$ are e^x and e^{2x}. Find the solution $y(x)$ with the property that $y(0) = 0$, $y'(0) = 1$. [**Ans.** $y(x) = e^{2x} - e^x$]

(Delhi B.Sc. (G) 2000)

8. Show that the $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the real line, even though the Wronskian cannot be computed.

9. Show graphically that $y_1(x) = x^2$ and $y_2(x) = x\,|\,x\,|$ are linearly independent on $-\infty < x < \infty$, however Wronskian vanishes for every real value of x.

10. Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ on any interval.

[Nagpur 96]

11. Show that $y_1(x) = e^{-x/2} \sin(x\sqrt{3}/2)$ and $y_2(x) = e^{-x/2} \cos(x\sqrt{3}/2)$ are lineraly independent solutions of the differential equation $y'' + y' + y = 0$. **(Delhi B.Sc. (G) 1999, 2001)**

[**Hint:** Proceed as in solved Ex. 2 on page 35]