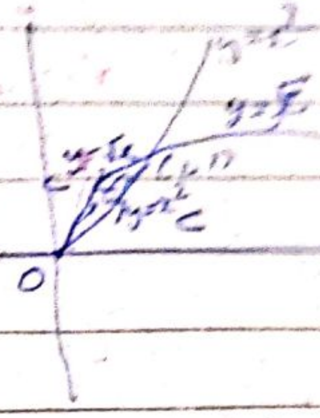
2011 IFoS

1. Verify Green's Theorem in plane to $\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, C = boundary of region enclosed by $y = \sqrt{x}$ and $y = x^2$.

The point of intersection of curve



$C = \; y = \sqrt{x} = x^2$

$\Rightarrow x^2 = \sqrt{x} \Rightarrow x^4 = x$

$\Rightarrow x(x^3 - 1) = 0 \Rightarrow \boxed{x = 0, x = 1}$

so, Region R is bounded by curve $C = C_1 + C_2$

C_1: $(0,0)$ to $(1,1)$ along $y = x^2$ and

C_2: $(1,1)$ to $(0,0)$ along $y = \sqrt{x}$.

so, By Green's Thm in plane -

$$\oint_C (P dx + Q dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint (-6y + 16y)\,dx\,dy = \iint 10y \,dx\,dy$$

$$= \int_{x=0}^{1} \left(\int_{y=x^2}^{\sqrt{x}} 10y \,dy \right) dx$$

$$= \int_0^1 \left(5y^2 \Big|_{x^2}^{\sqrt{x}} \right) dx$$

$$= \int_0^1 5(x - x^4)\,dx = 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[\frac{1}{2} - \frac{1}{5}\right] = \boxed{\frac{3}{2}}$$

Doubt.

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2. $m = 2$ unit, $\vec{r} = (t^2-2t)\hat{i} + \left(\frac{1}{2}t^2+1\right)\hat{j} + t^2\hat{k}$

At $t=1$, find K.E, angular momentum, time rate of change of angular momentum and moment of resultant force, acting at particle, about the origin.

$\vec{L} = \vec{r}\times\vec{p} = mr^2\omega = I\omega$

$\vec{r} = (t^2-2t)\hat{i} + \left(\frac{1}{2}t^2+1\right)\hat{j} + t^2\hat{k}$

$\vec{r}|_{t=1} = -\hat{i} + \frac{3}{2}\hat{j} + \hat{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = (2t-2)\hat{i} + (t)\hat{j} + 2t\hat{k}$

$\vec{v}|_{t=1} = 0\hat{i} + \hat{j} + 2\hat{k}$

$\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{i} + \hat{j} + 2\hat{k}$

$\boxed{m = 2 \text{ unit}}$

$KE = \frac{1}{2}mv^2 = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}\times 2\times(\sqrt{5})^2$

$= 5$ unit

angular momentum $= mrv = m\times|\vec{r}|\times|\vec{v}|$

$= m\times\frac{\sqrt{17}}{2}\times\sqrt{5} = \sqrt{85}$ unit

Time rate of change of angular momentum = moment of force moment of resultant force $= \vec{r} \times \vec{F}$

$$\vec{F} = m\frac{d\vec{v}}{dt} = 4\hat{i} + 2\hat{j} + 4\hat{k}$$

$$\vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & 1 \\ 4 & 2 & 4 \end{vmatrix} = 4\hat{i} + 8\hat{j} - 8\hat{k}$$

$$\text{moment} = |\vec{r} \times \vec{F}| = \sqrt{144} = 12\,\text{unit}$$

3. Find curvature, torsion and relation b/w arc length s and parameter u for the curve:-

$$\vec{r} = \vec{r}(u) = 2\log_e u\,\hat{i} + 4u\hat{j} + (2u^2+1)\hat{k}$$

$$\frac{d\vec{r}}{du} = \frac{2}{u}\hat{i} + 4\hat{j} + 4u\hat{k}$$

$$\left|\frac{d\vec{r}}{du}\right| = \sqrt{16 + 16u^2 + \frac{4}{u^2}} = \frac{\sqrt{16u^2 + 16u^4 + 4}}{u}$$

$$= \frac{2}{u}\sqrt{1 + 4(u^2 + u^4)} = \frac{2 + 4u^2}{u}$$

$$\text{Curvature } k = \frac{\left|\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2}\right|}{\left|\frac{d\vec{r}}{du}\right|^3}$$

$$\Rightarrow \frac{d^2r}{du^2} = -\frac{2}{u^2}\hat{i} + 0\hat{j} + 4\hat{k}$$

$$\frac{d^3r}{du^3} = +\frac{4}{u^3}\hat{i} + 0\hat{j} + 0\hat{k}$$

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$$\frac{d\vec{r}}{du}\times\frac{d^2r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{2}{u} & 4 & 4u \\ -\frac{2}{u^2} & 0 & 4 \end{vmatrix}$$

$$= \hat{i}(16) + \hat{j}\left(\frac{\cancel{16}}{u}\right) + \hat{k}\left(\frac{8}{u^2}\right)$$

$$\left|\frac{d\vec{r}}{du}\times\frac{d^2r}{du^2}\right| = \sqrt{\left(\frac{16}{u}\right)^2 + 16^2 + \left(\frac{8}{u^2}\right)^2} = \cancel{8\sqrt{4+\frac{1}{u^4}}}$$

$$\left(\frac{d\vec{r}}{du}\times\frac{d^2r}{du^2}\right)\cdot\frac{d^3r}{du^3} = \frac{16\times 4}{u^3} = \frac{64}{u^3}$$

$$\Rightarrow K = \frac{\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right|}{\left|\frac{dr}{du}\right|^3} = \cancel{\frac{\sqrt{16^2\times u^4+64}}{\frac{8}{u^3}(1+4(u^2+u^4))^3}}$$

$$\cancel{= \frac{8}{u}\sqrt{4u^2+1}}$$

$$\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right| = \sqrt{\frac{16\times16}{u\times u} + 16\times16 + \frac{8\times8}{u^2\times u^2}}$$

$$= \frac{8}{u^2}\sqrt{4u^2+4u^4+1} = \frac{8}{u^2}(2u^2+1)$$

$$\text{So } K = \frac{\frac{8}{u^2}(2u^2+1)}{\left(\frac{2}{u}+4u\right)^3} = \frac{\frac{8}{u^2}(2u^2+1)}{\frac{8}{u^3}(1+2u^2)^3}$$

$$= \frac{u}{(1+2u^2)^2}$$

$$\tau = \frac{\left[\frac{d\vec{r}}{du}\ \frac{d^2 r}{du^2}\ \frac{d^3 r}{du^3}\right]}{\left|\frac{dr}{du} \times \frac{d^2 r}{du^2}\right|^2} = \frac{\frac{64}{u^3}}{\frac{64}{u^4}(2u^2+1)^2}$$

$$= \frac{u}{(2u^2+1)^2} = k$$

Now $s = \int ds = \int \frac{ds}{du} du = \int \left|\frac{d\vec{r}}{du}\right| du$

$$\boxed{s = \int \left(\frac{2}{u} + 4u\right) du = 2\log u + 2u^2 + c \quad \text{Put } c = 0.}$$

Q-4 curl $(\vec{f} \times \vec{g}) = \vec{f}\,\text{div}\,\vec{g} - \vec{g}\,\text{div}\,\vec{f} + (\vec{g} \cdot \nabla)\vec{f} - (\vec{f} \cdot \nabla)\vec{g}$.

verify it for $\vec{f} = x\hat{i} + \cancel{z}\hat{j} + y\hat{k}$, $\vec{g} = y\hat{i} + z\hat{k}$.

curl $(\vec{f} \times \vec{g}) = \sum i \times \frac{\partial}{\partial x}(\vec{f} \times \vec{g})$

$$= \sum i \times \left(\frac{\partial \vec{f}}{\partial x} \times \vec{g}\right) + \sum i \times \left(\vec{f} \times \frac{\partial \vec{g}}{\partial x}\right)$$

$\left[\text{using } \frac{d}{dx}(\vec{f} \times \vec{g}) = \left(\frac{df}{dx} \times \vec{g}\right) + \left(\vec{f} \times \frac{dg}{dx}\right)\right]$

using $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$= \sum (i \cdot \vec{g})\frac{\partial f}{\partial x} - \sum \left(i \cdot \frac{\partial f}{\partial x}\right)\vec{g}$$

$$- \sum (i \cdot \vec{f})\frac{\partial g}{\partial x} + \sum \left(i \cdot \frac{\partial g}{\partial x}\right)\vec{f}$$

$$= \sum \left(\vec{g} \cdot i\frac{\partial}{\partial x}\right)\vec{f} - (\text{div}\,\vec{f})g - \sum \left(\vec{f} \cdot i\frac{\partial}{\partial x}\right)\vec{g}$$

$$+ (\text{div}\,\vec{g})\vec{f}$$

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$$= (\vec{g}\cdot\nabla)\vec{f} - (\mathrm{div}\,\vec{f})g - (\vec{f}\cdot\nabla)\vec{g} + (\mathrm{div}\,\vec{g})\vec{f}$$

$$\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$$
$$\vec{g} = y\hat{i} + 0\hat{j} + z\hat{k}$$
$$\mathrm{div}\,\vec{f} = 1 \;;\; \mathrm{div}\,\vec{g} = 1$$

$$(\vec{g}\cdot\nabla)\vec{f} = \sum\left(\vec{g}\cdot i\frac{\partial}{\partial x}\right)\vec{f} = \left(y\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\right)(\vec{f})$$
$$= y\hat{i} + z\hat{j}$$
$$(\vec{f}\cdot\nabla)\vec{g} = z\hat{i} + y\hat{k}$$
$$\vec{f}\times\vec{g} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ x & z & y \\ y & 0 & z \end{vmatrix}$$
$$= \hat{i}(z^2) + \hat{j}(y^2 - xz) + \hat{k}(-yz)$$
$$\nabla\times(\vec{f}\times\vec{g}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 - xz & -yz \end{vmatrix}$$
$$= \hat{i}(-z+x) + \hat{j}(2z) + \hat{k}(-z)$$

$$(g\cdot\nabla)\vec{f} - (\mathrm{div}\,F)g - (\vec{f}\cdot\nabla)\vec{g} + (\mathrm{div}\,\vec{g})f$$
$$= y\hat{i} + z\hat{j} - y\hat{i} - z\hat{k} - z\hat{i} - y\hat{k} + x\hat{i} + z\hat{j} + y\hat{k}$$
$$= \hat{i}(-z+x) + \hat{j}(2z) + \hat{k}(-z)$$

Hence proved

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Q-5 $\oint_C \sin x\,dx + y^2 dy - dz$ Stoke's Thm

C = Circle $x^2+y^2=16,\ z=3$.

Stoke's Thm

$$\oint \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, ds$$

$$\vec{F} = \sin x\,\hat{i} + y^2\,\hat{j} - \hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & y^2 & -1 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

so $\oint \sin x\,dx + y^2 dy - dz$

$= 0.$

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