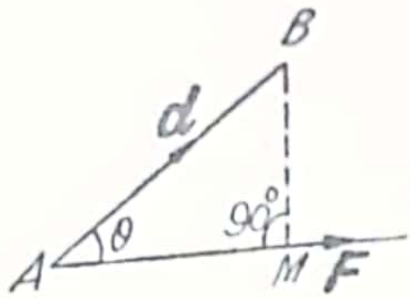
Work, Energy and Impulse

§ 1. The concept of work. We know that a force, when applied to a particle or body, often causes a change in its position. A force is said to do *work* when its point of application is displaced.

§ 2. Work done by a constant force. Definition. [Rohilkhand 1977]



Suppose a constant force represented by the vector **F** *acts at the point* A. *Let the point* A *be displaced to the point* B *where* $\overrightarrow{AB}=\mathbf{d}$. *Then the work* W *done by the constant force* **F** *during the displacement* **d** *of its point of application is defined as*

$$W=\mathbf{F}\cdot\mathbf{d}, \qquad ...(1)$$

where $\mathbf{F}\cdot\mathbf{d}$ *is the scalar product of the vectors* **F** *and* **d**.

Let θ be the angle between the vectors **F** and **d**. If $F=|\mathbf{F}|$ and $d=|\mathbf{d}|=AB$, then using the definition of the scalar product of two vectors, the equation (1) defining the work may be written as

$$W=Fd\cos\theta. \qquad ...(2)$$

Obviously $d\cos\theta$ is the displacement of the point of application of the force **F** in the direction of the force. *Hence the work done by a constant force is equal to the magnitude of the force multiplied by the displacement of the point of application of the force in the direction of the force.*

From the equation (2) we make the following observations :

(i) If $\theta=\frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force is perpendicular to the direction of the force, then $W=0$.

(ii) If $0\leqslant\theta<\frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is in the direction of the force, then W is positive.

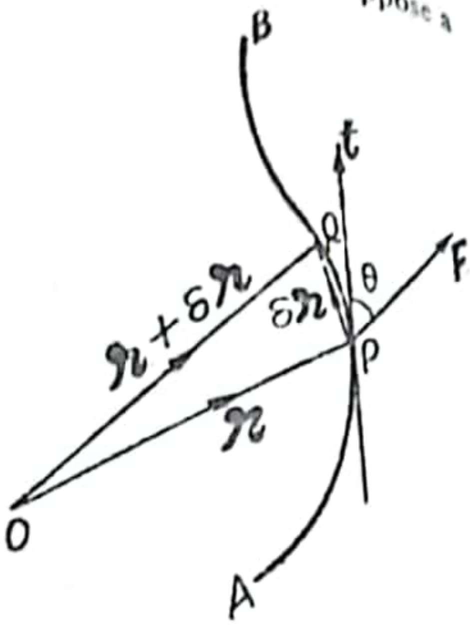
(iii) If $\frac{1}{2}\pi<\theta\leqslant\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is opposite to the direction of the force, then W is negative.

Example. If a particle of mass m is displaced on a horizontal plane through a distance h, then during this displacement the work done by the weight mg of the particle is zero.

If a particle of mass m is raised through a vertical height h, then during this displacement the work done by the weight mg of the particle is $-mgh$.

Again if a particle of mass m falls through a vertical depth h, then during this displacement the work done by the weight mg of the particle is mgh.

§ 3. **Work done by a variable force. Definition.** Suppose a variable force $\mathbf{F}$ acts on a particle which moves along an arc AB from A to B. Let P, Q be neighbouring points on this curve such that



$\overrightarrow{OP}=\mathbf{r}$, $\overrightarrow{OQ}=\mathbf{r}+\delta\mathbf{r}$,

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\delta\mathbf{r}$.

During the small displacement $\delta\mathbf{r}$ of its point of application the force $\mathbf{F}$ may be regarded as a constant force. So the work done by the force $\mathbf{F}$ during the displacement $\overrightarrow{PQ}=\delta\mathbf{r}$ of its point of application is equal to $\mathbf{F}\cdot\delta\mathbf{r}$. The work W done by the force $\mathbf{F}$ in displacing its point of application from A to B along the given arc AB is defined as the limiting sum of the elemental expressions $\mathbf{F}\cdot\delta\mathbf{r}$ as the point of application of the force moves from A to B along the given arc AB. Thus

$$W=\int_A^B \mathbf{F}\cdot d\mathbf{r}, \qquad \ldots(1)$$

where the integration is to be performed along the arc AB.

Referred to some frame of rectangular co-ordinate axes OX, OY and OZ let (x, y, z) be the co-ordinates of the point P. Then $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ so that $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$. Also let $\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ where X, Y, Z are the components of the force $\mathbf{F}$ along OX, OY, OZ respectively. We have

$$\mathbf{F}\cdot d\mathbf{r}=(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$
$$=Xdx+Ydy+Zdz.$$

$\therefore$ the equation (1) defining the work W done by the force $\mathbf{F}$ may be written as

$$W=\int_A^B (Xdx+Ydy+Zdz). \qquad \ldots(2)$$

Again if s denotes the arc length of the curve AB measured from some fixed point on the curve to any other point P whose position vector is $\mathbf{r}$, then $d\mathbf{r}/ds = \mathbf{t}$, where $\mathbf{t}$ is the unit vector along the tangent at P to the curve in the sense of s increasing. We may write the equation (1) as

$$W = \int_A^B \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \int_A^B \mathbf{F} \cdot \mathbf{t} \, ds$$

$$= \int_A^B F \cos \theta \, ds, \qquad \ldots(3)$$

where $F = |\mathbf{F}|$ and θ is the angle which the direction of the force $\mathbf{F}$ makes with the direction of the tangent of the curve in the sense of s increasing.

The integration on the right hand side of the above equations (1), (2) and (3) is to be performed along the arc AB of the given path of the particle.

§ 4. **Units of work.** In the Centimeter Gram Second (C. G. S.) system the absolute unit of work is called an *erg*. It is the work done by a force of one dyne in displacing its point of application through 1 centimeter in its direction. Also in this system the gravitational (or practical) unit of work is one *gram-cm*. It is the work done by a force of one gram weight as its point of application is displaced through 1 cm. in its direction. The two units are related as follows : *one gm.-cm. = g ergs = 981 ergs.*

In the Meter Kilogram Second (M. K. S.) system the absolute unit of work is called a *joule*. It is the work done by a force of one newton in displacing its point of application through 1 meter in its direction. Also in this system the gravitational (or practical) unit of work is one *kilogram-meter*. It is the work done by a force of one kg wt. as its point of application is displaced through 1 meter in its direction. We have

one kg.-m. = g joules = 9·8 joules.

In the Foot Pound Second (F. P. S.) system the absolute unit of work is called a *foot poundal*. It is the work done by a force of one poundal in displacing its point of application through 1 foot in its direction. Also in this system the gravitational unit of work is one *foot-pound*. It is the work done by a force of one pound weight as its point of application is displaced through 1 foot in its direction. We have

one ft.-lb. = g ft.-poundals = 32 ft.-poundals.

§ 5. **Power.** The amount of work done by a force depends upon the time as well. *The power of an agent supplying the force is defined as the rate of doing the work.* Thus the power of an agent is the amount of work done by the agent in a unit time. The units of power may be taken as the *units of work per second.*

In the British system *i.e.*, in the F. P. S. system the unit of power used in engineering practice is one *Horse power* while in the M.K.S. system the unit of power used in engineering practice is one *watt.* We have

$$\text{one Horse power } (H.\ P.) = 550\ ft.\text{-}lbs/sec.$$

and

$$\text{one Watt} = \text{one joule}/sec. = 10^7\ ergs/sec.$$

Thus an engine is said to be of one H. P. if the work done by it per second is 550 foot-pounds or 550×32 foot-poundals.

Also remember that $\quad 1$ H. P. $= 746$ watts.

Illustrative Examples

Ex. 1. *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.*

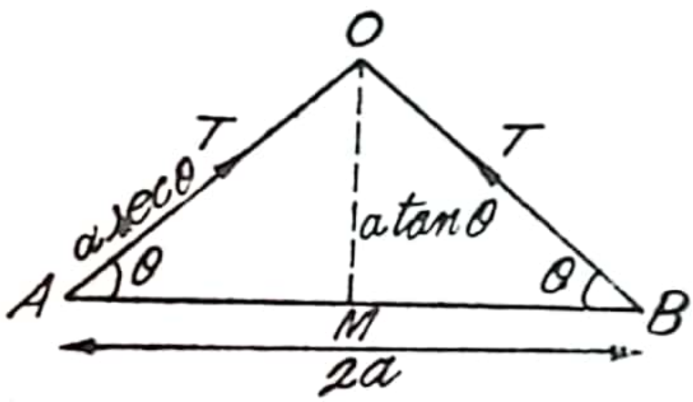
[Kanpur 1977; Rohilkhand 81]

Sol. For the complete solution of this problem refer § 8, chapter 2, page 90.

Ex. 2. *If a light elastic string, whose natural length is that of a uniform rod be attached to the rod at both the ends and suspended by the middle point, show that the rod will descend until each of the two portions of the string is inclined to the horizon at an angle θ, given by the equation*

$$\cot^3 \tfrac{1}{2}\theta - \cot \tfrac{1}{2}\theta = 2n,$$

the modulus of elasticity of the string being n times the weight of the rod.



Sol. Let $2a$ be the length of the rod AB, O the middle point of the string AOB whose natural length is also $2a$. The string is suspended at the fixed point O. Initially the rod is held at rest in the level of O and then released. Due to the weight of the rod the string is stretched and the rod moves down. Let θ be the inclination of each of the two portions of the string to the horizontal when the rod again comes to rest.

The vertical distance moved by the centre of gravity of the rod $= OM = a \tan \theta$.

$\therefore$ the work done by the weight of the rod

$$= mg\, a \tan \theta, \quad \ldots(1)$$

where m is the mass of the rod.

In the initial position the tension in the string is zero because then there is no extension.

In the final position the extension in the length of the string

$$= 2a \sec \theta - 2a.$$

$\therefore$ in the final position, by Hooke's law, the tension T in the string $= \dfrac{\lambda\,(2a \sec \theta - 2a)}{2a}$, where λ is the modulus of elasticity

$$= \frac{nmg\,(2a \sec \theta - 2a)}{2a} \qquad [\because \lambda = nmg]$$

$$= nmg\,(\sec \theta - 1).$$

We know that the work done in stretching an elastic string $=$(mean of the initial and final tensions)$\times$(the extension).

$\therefore$ the work done in stretching the string in question

$$= \tfrac{1}{2}\,(0+T)\times(2a \sec \theta - 2a)$$

$$= \tfrac{1}{2}.nmg\,(\sec \theta - 1).2a\,(\sec \theta - 1)$$

$$= nmga\,(\sec \theta - 1)^2. \quad \ldots(2)$$

Since the works (1) and (2) are equal, therefore

$$mga \tan \theta = nmga\,(\sec \theta - 1)^2$$

or $\quad \tan \theta = n\,(\sec \theta - 1)^2$

or $\quad \sin \theta \cos \theta = n\,(1 - \cos \theta)^2$

or $\quad 2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta\,(\cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta) = 4n \sin^4 \tfrac{1}{2}\theta$

or $\quad \cot^3 \tfrac{1}{2}\theta - \cot \tfrac{1}{2}\theta = 2n,$

which proves the required result.

Ex. 3. *A spider hangs from the ceiling by a thread of modulus of elasticity equal to its weight. Show that it can climb to the ceiling with an expenditure of work equal to only three quarters of what would be required if the thread were inelastic.*

Sol. Let l be the natural length of the thread and l_1 its length when the spider hangs in equilibrium. In this position of equilibrium we should have

the weight of the spider $=$ the tension in the thread.

$\therefore \quad mg = \lambda \dfrac{l_1 - l}{l}$, where m is the mass of the spider and λ is the modulus of elasticity of the thread.

But $\quad \lambda = mg$.

$$\therefore \quad mg = mg\,\frac{l_1 - l}{l}$$

i.e., $\quad l_1 - l = l \qquad$ *i.e.*, $\quad l_1 = 2l$.

Thus the spider hangs in equilibrium by the free end of the thread at a depth $2l$ below the ceiling.

If the length $2l$ were inelastic, the work that the spider does against its weight in climbing to the ceiling $= mg \cdot 2l = 2mgl$.

In case the thread is elastic the work done in stretching it to a length $2l$

$= \frac{1}{2}$ (initial tension + final tension) × extension

$= \frac{1}{2}(0+mg)(2l-l) = \frac{1}{2}mgl.$

In this case when the spider reaches the ceiling the thread reverts from its stretched to natural length, so the work done against the tension is the same as above but negative.

Therefore when the thread is elastic the total work done in climbing to the ceiling

$= 2mgl - \frac{1}{2}mgl = \frac{3}{2}mgl = \frac{3}{4}(2mgl)$

$= \frac{3}{4}$ of the work if the thread were inelastic.

Ex. 4 *A cylindrical cork of length l and radius r is slowly extracted from the neck of a bottle. If the normal pressure per unit of area between the bottle and unextracted part of the cork at any instant be constant and equal to P, show that the work done in extracting it is $\pi\mu r l^2 P$, where μ is the coefficient of friction.*

Sol. At any instant if x is the length of the cork in contact with the bottle, then the area of the surface of the cork in contact with the bottle is equal to $2\pi r x$.

The normal pressure on this surface $= 2\pi r x P$.

∴ the force of friction on the cork when it moves rubbing the bottle $= \mu 2\pi r x P$.

∴ work done against this friction in extracting a length δx

$= \mu 2\pi r x P \delta x$

$= 2\pi\mu r P x \delta x.$

Hence the total work done in extracting the whole length of the cork $= \int_0^l 2\pi\mu r P x\,dx = 2\pi\mu r P \int_0^l x\,dx$

$$= 2\pi\mu r P \left[\frac{x^2}{2}\right]_0^l = 2\pi\mu r P.\frac{l^2}{2} = \pi\mu r P l^2.$$

Ex. 5. *Prove that the work done in stretching an elastic string AB, of natural length l and modulus λ, from tension T_1 to tension T_2 is*

$$(l/2\lambda)(T_2^2 - T_1^2).$$

Sol. Let l_1 be the stretched length of the string in the state of tension T_1 and l_2 the stretched length in the state of tension T_2. Then by Hooke's law, we have

$$T_1 = \lambda \frac{l_1 - l}{l} \qquad (1)$$

and
$$T_2=\lambda\frac{l_2-l}{l}. \qquad \ldots(2)$$

Let W be the work done in stretching the string from tension T_1 to tension T_2. Then

$W=\frac{1}{2}$ (initial tension + final tension) × extension

$=\frac{1}{2}(T_1+T_2)(l_2-l_1)$. ...(3)

Subtracting (1) from (2), we have
$$T_2-T_1=\frac{\lambda}{l}(l_2-l_1). \qquad \ldots(4)$$

Substituting for l_2-l_1 from (4) in (3), we have
$$W=\frac{1}{2}(T_1+T_2).\frac{l}{\lambda}(T_2-T_1)=\frac{l}{2\lambda}(T_2^2-T_1^2).$$

Ex. 6. *A motor car weighing 10 quintals and travelling at 12 meters/sec. is brought to rest in 18 meters by the application of its brakes. Find the work done by the force of resistance due to brakes.*

Sol. Assuming that the resistance is uniform, let the retardation due to this resistance be r m /sec^2.

Here the initial velocity $u=12$ m./sec., final velocity $v=0$ m./sec. and the distance travelled $s=18$ metres. Therefore using the formula $v^2=u^2+2fs$, we have
$$0=12^2-2r\times 18 \quad i.e., \; r=\frac{144}{36}=4.$$

Now mass of the car = 1000 kg.

∴ using the formula $P=mf$, the force of resistance

$=1000\times 4$ newtons = 4000 newtons.

Hence the required work done = 4000×18 joules

$=72000$ joules $=\dfrac{72000}{9\cdot 8}$ kg.-meters.

$=7347$ kg.-meters (approx.).

Ex. 7. *A train of total mass 250 tons is drawn by an engine working at 560 H.P. If at a certain instant the total resistance is 16 lbs. wt. per ton the weight of the train, and the velocity 30 miles an hour, what is the train's acceleration, measured in miles per hour per second.*

Sol. The velocity of 30 miles per hour = 44 ft. per sec.

Let P lbs. wt. be the pull of the engine when the velocity is 44 ft./sec. Then the rate at which the engine works

$=P\times 44$ ft.-lbs/sec.

But the engine is working at 560 H.P.

i.e., at the rate of 560×550 ft.-lbs/sec.

$\therefore \quad P \times 44 = 560 \times 550 \quad$ or $\quad P = 7000.$

Total resistance $= (16 \times 250)$ lbs. wt. $= 4000$ lbs. wt.

$\therefore$ the net force in the direction of motion

$= (7000 - 4000)$ lbs. wt. $= 3000$ lbs. wt.

$= 3000 \times 32$ poundals $= 96000$ poundals.

If f ft./sec^2 be the acceleration of the train, we have by Newton's second law of motion

$$96000 = 250 \times 2240 f \qquad [1 \text{ ton} = 2240 \text{ lbs.}]$$

or
$$f = \frac{96000}{250 \times 2240} = \frac{6}{35}.$$

$\therefore$ the acceleration of the train $= \frac{6}{35}$ ft. per sec. per sec.

$= \frac{6 \times 60 \times 60}{35 \times 1760 \times 3}$ miles per hour per second

$= \frac{9}{77}$ miles per hour per second.

§ 6. **Kinetic energy.** The capacity of a body for doing work is known as energy of the body. The kinetic energy (K.E.) of a body is the energy which the body possesses on account of being in motion. We can define it precisely as follows :

Kinetic energy. Definition. *The kinetic energy of a body is the amount of work which the body can perform against some resistance till reduced to rest.* [Rohilkhand 1977]

Since the K.E. has been defined to be equal to work done in some way, the units for measuring K.E. are the same as those for work.

Calculation of kinetic energy. *If at any instant a body of mass m be moving with velocity u, then the kinetic energy of the body at that instant is equal to $\frac{1}{2} mu^2$.*

At any instant let a body of mass m be moving with velocity u. Then, by definition, the K.E. of the body at that instant is equal to the amount of work which the body can perform against some resistance, say P, till reduced to rest. Suppose the body moves from the point A to the point B while the resistance P reduces its velocity from u to 0. The direction of the force of resistance is against the direction of motion. If v be the velocity of the body at any point between the above two positions A and B, we should have

$$mv \frac{dv}{ds} = -P, \qquad \ldots(1)$$

assuming that the body is moving in the direction of s increasing so that the resistance P acts in the sense of s decreasing.

From (1), we have

$$Pds = -mvdv. \qquad \ldots(2)$$

Let $s=0$ at A and $s=b$ at B. Then integrating (2) from A to B, we have

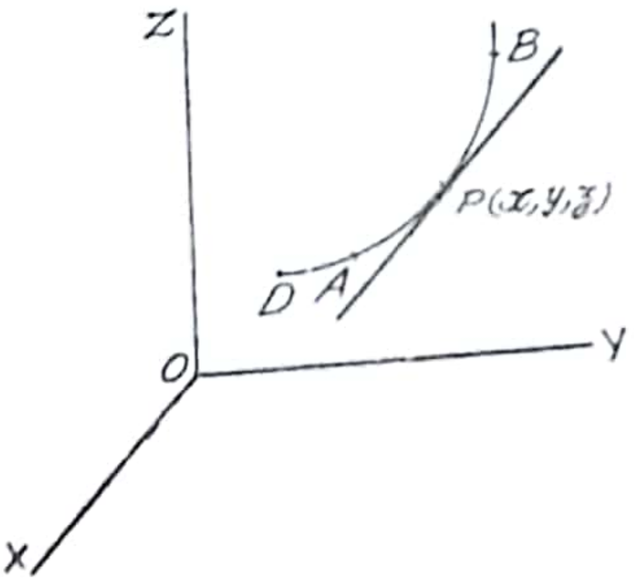
$$\int_0^b Pds = \int_u^0 -mvdv = -m\left[\tfrac{1}{2}v^2\right]_u^0 = \tfrac{1}{2}mu^2.$$

But $\int_0^b Pds$ is the work done against the resistance P while the body moves from A to B and so equal to the K.E. of the body at A.

Hence the K.E. of a body of mass m moving with velocity $u = \frac{1}{2}mu^2$.

Remark. If the mass m is measured in gms and the velocity u in cm./sec., then the K.E. is in ergs. If the mass m is measured in kgs. and the velocity u in m./sec., the K.E. is in joules.

§ 7. **The work-energy principle.** *The change in the kinetic energy of a particle during its motion from a position A to a position B is equal to the work done by the forces acting on the particle during that motion.*



Suppose a particle of mass m moves along any path under the action of any system of forces. Let X, Y, Z be the components of these forces along any three mutually perpendicular lines OX, OY, OZ taken as the co-ordinate axes. Suppose the velocity of the particle changes from v_1 to v_2 when it moves from A to B. Let $P(x, y, z)$ be the position of the particle at any time t where arc $DP = s$, D being some fixed point on the path. The direction cosines of the tangent at P to the path in the sense of s increasing are dx/ds, dy/ds, dz/ds. Let v be the velocity of the particle at P. Then the expression for the tangential acceleration of the particle at P is $v\ (dv/ds)$, +ive in the direction of s increasing. Resolving the forces acting on the particle at P along the tangent at P, the tangential equation of motion of P is

$$mv\frac{dv}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}.$$

[By Newton's second law of motion]

$\therefore\quad mvdv=Xdx+Ydy+Zdz.$ ...(1)

Integrating both sides of (1) from A to B, we have

$$\int_{v_1}^{v_2} mv\,dv=\int_A^B (Xdx+Ydy+Zdz).$$

Now $\int_{v_1}^{v_2} mv\,dv=m\left[\frac{v^2}{2}\right]_{v_1}^{v_2}=\frac{1}{2}mv_2^2-\frac{1}{2}mv_1^2$ ...(2

= K.E. of the particle at B − K.E. of the particle at A

= change in K.E. of the particle in moving from A to B.

Also $\int_A^B (Xdx+Ydy+Zdz)$ is the work done by the forces acting on the particle during its motion from A to B.

Hence from (2) we conclude that

the change in the K.E. = the work done by the forces.

This is known as the *principle of energy* or *the principle of work and energy.* (Lucknow 1977)

Remark. If a particle of mass m starts from rest and has velocity v after any time t, then by the principle of work and energy

$\frac{1}{2}mv^2-\frac{1}{2}m.0^2$ = the work done by the forces acting on the particle during that time t.

Thus we can say that *the kinetic energy of a moving particle is the amount of work done by the forces acting on it giving it that motion, starting from rest.*

§ 8. Conservative and non-conservative forces.

Conservative forces. Definition. *A force, is said to be conservative if the work done by it in displacing its point of application from one given point to another depends upon these points only and not upon the path followed.*

If a variable force $\mathbf{F}$ displaces its point of application from a point A to a point B, along a curve C, then the work W done by the force is given by

$$W=\int_A^B \mathbf{F}\cdot d\mathbf{r},$$

...ere the integration is to be performed ...ong the curve C. The force $\mathbf{F}$ is con...rvative if and only if the value of the ...ove integral does not depend upon the ...rve C.

Non-conservative forces. *The forces ...hich are not conservative are called non-conservative.*

We shall give below (without proof) two characteristic properties of conser...vative forces and any of these can be ...taken as an equivalent definition of a ...ystem of conservative forces.

(i) *A force is conservative if and only ...f the work done by it on a particle as it makes a complete circuit (i.e., comes to the position that it started from) is zero.*

(ii) *A force* $\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ *is conservative if and only if there exists a single valued function* $f(x, y, z)$ *such that*

$$\frac{\partial f}{\partial x}=X, \frac{\partial f}{\partial y}=Y, \frac{\partial f}{\partial z}=Z.$$

The function $f(x, y, z)$ *is called the* **potential function** *of the force* $\mathbf{F}$.

If a particle is displaced from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ under such a force $\mathbf{F}$ along any curve C, then the work W done by $\mathbf{F}$ is given by

$$W=\int_A^B (Xdx+Ydy+Zdz)$$

$$=\int_A^B \left(\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz\right)$$

$$=\int_A^B df=\Big[f(x, y, z)\Big]_A^B$$

$$=f(x_2, y_2, z_2)-f(x_1, y_1, z_1),$$

which obviously depends upon the points A and B and not upon the curve C.

Conservative forces do not change their character on account of any restraint while non-conservative forces change their character on account of extraneous circum... [illegible] few examples of the conservative forces are force of gravity, tension and normal reaction while a few examples of the non-conservative forces are force of friction and resistance of the air. Remember that a constant force **F** is always a conservative force and a central force **F** is also always a conservative force.

For instance suppose a particle is projected vertically upwards from a point O and after reaching a height h it comes back to the point of projection. Then the work done by gravity when the particle completes this circuit $= -mgh + mgh = 0$. Thus gravity is a conservative force.

Again consider a body put on a rough horizontal table. Let the frictional force be F. If the body is moved in a straight line from A to B, the work done by the force of friction F is $-F.AB$. Now if the body is moved back from B to A, the work done by the force of friction is $-F.AB$.

Thus the total work done in completing the circuit

$$= -F.AB + (-F.AB)$$

$$= -2F.AB \text{ which is not zero.}$$

Therefore frictional force is not conservative.

§ 9. **Potential Energy (P. E.).** The potential energy of a body acted upon by a conservative system of forces, is the capacity of the body for doing work on account of its position. We may define it precisely as follows :

If a body is acted upon by a conservative system of forces, then its potential energy in any position is the amount of the work done by these forces in bringing the body from that position to some standard position.

For example the potential energy of a body of mass m placed at a height h above the ground is the amount of the work which its weight mg does when the body moves from this position to the ground which is usually supposed to be the standard position. Thus *for a body of mass m placed at a height h, potential energy*

$$= mgh.$$

§ 10. **The principle of conservation of energy.** *If a particle acted upon by a conservative system of forces moves along any path, the sum of its kinetic and potential energies remains constant.*

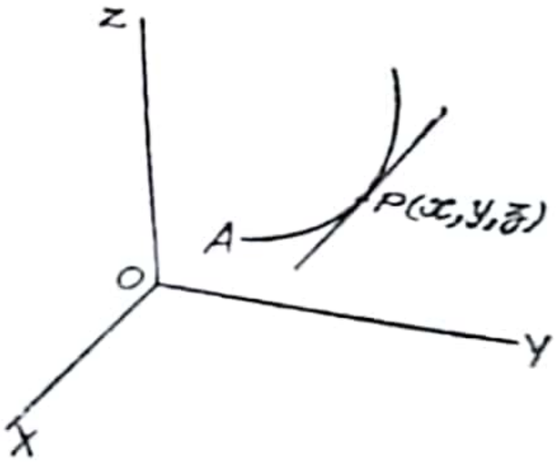
[Meerut 1979]

Suppose a particle of mass m moves along any path under the action of a system of conservative forces whose potential function is, say, $f(x, y, z)$.

Then $$\frac{\partial f}{\partial x}=X,\ \frac{\partial f}{\partial y}=Y,\ \frac{\partial f}{\partial z}=Z, \qquad \ldots(1)$$

where X, Y, Z are the components of the forces along the co-ordinate axes OX, OY, OZ respectively.

Let $P(x, y, z)$ be the position of the particle at any time t, where are $AP=s$, A being some fixed point on the path. The direction cosines of the tangent at P to the path in the sense of s increasing are dx/ds, dy/ds, dz/ds. Let v be the velocity of the particle at P. Then the tangential equation of motion of the particle at P is



$$mv\frac{dv}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}$$

or $$mv\,dv=X\,dx+Y\,dy+Z\,dz.$$

Integrating both sides, we get

$\frac{1}{2}mv^2=\int(X\,dx+Y\,dy+Z\,dz)+C$, where C is a constant

$$=\int\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz\right)+C \qquad \text{[from (1)]}$$

$$=\int df+C=f(x, y, z)+C.$$

[Note that if $f(x, y, z)$ is a function of x, y, z, then from partial differentiation, we have

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz.]$$

Now $\frac{1}{2}mv^2$ is the K.E. of the particle at the point P.

Thus the K.E. of the particle at $P=f(x, y, z)+C$. $\qquad \ldots(2)$

Again the potential energy of the particle at $P(x, y, z)$ is equal to the work done by the conservative forces in moving the particle from P to some standard position, say, (x_1, y_1, z_1).

$$\therefore \text{ P.E. at } P=\int_{(x,y,z)}^{(x_1,y_1,z_1)}(X\,dx+Y\,dy+Z\,dz)$$

$$=\int_{(x,y,z)}^{(x_1,y_1,z_1)}df=\Big[f(x, y, z)\Big]_{(x,y,z)}^{(x_1,y_1,z_1)}$$

$$=f(x_1, y_1, z_1)-f(x, y, z). \qquad \ldots(3)$$

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Adding (2) and (3), we have

K.E. at P+P.E. at $P=f(x_1, y_1, z_1)+C$,

which is constant because (x_1, y_1, z_1) is a fixed point.

This proves the principle of conservation of energy.

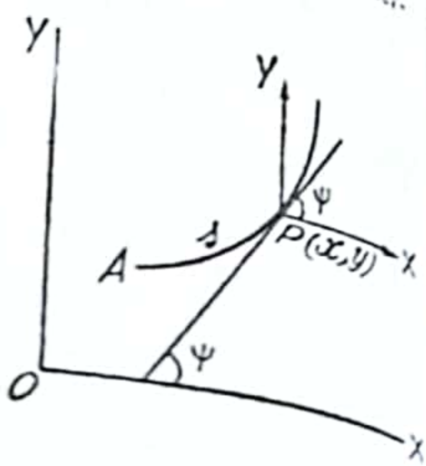
§ 11. The principle of conservation of energy for the motion in a plane. We have established the principle of work and energy and the principle of conservation of energy for the general motion in three dimensions. These principles can be similarly established, as special case, for the motion in two dimensions. We shall here establish the principle of conservation of energy for the motion in a plane.

If a particle acted upon by a conservative system of forces moves in a plane along any path, the sum of its kinetic and potential energies remains constant.

Suppose a particle of mass m moves in the plane XOY along any path under the action of a system of conservative forces whose potential function is, say, $f(x, y)$. Then

$$\frac{\partial f}{\partial x}=X, \quad \frac{\partial f}{\partial y}=Y, \qquad \ldots(1)$$

where X, Y are the components of the forces along the co-ordinate axes OX, OY respectively.



Let $P(x, y)$ be the position of the particle at any time t, where arc $AP=s$, A being some fixed point on the path. If the tangent at P to the path makes an angle ψ with OX, we have

$$\cos\psi=dx/ds \quad \text{and} \quad \sin\psi=dy/ds.$$

Let v be the velocity of the particle at P. The tangential equation of motion of P is

$$mv\frac{dv}{ds}=X\cos\psi+Y\sin\psi=X\frac{dx}{ds}+Y\frac{dy}{ds}.$$

$\therefore \quad mv\,dv=X\,dx+Y\,dy.$

Integrating both sides, we have

$\frac{1}{2}mv^2=\int(X\,dx+Y\,dy)+C$, where C is a constant

$$=\int\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\right)+C$$

$$=\int df+C=f(x, y)+C.$$

But $\frac{1}{2}mv^2$ is the kinetic energy of the particle at P.

$\therefore$ K.E. of the particle at $P = f(x, y) + C$. ...(2)

Again the potential energy of the particle at $P(x, y)$ is equal to the work done by the conservative forces acting on the particle in moving it from $P(x, y)$ to some standard position, say, (x_1, y_1).

$$\therefore \quad \text{P.E. at } P = \int_{(x, y)}^{(x_1, y_1)} (X dx + Y dy)$$

$$= \left[f(x, y) \right]_{(x, y)}^{(x_1, y_1)}$$

$$= f(x_1, y_1) - f(x, y). \quad \ldots(3)$$

Adding (2) and (3), we have

$$\text{K.E. at } P + \text{P.E. at } P = f(x_1, y_1) + C,$$

which is constant because (x_1, y_1) is a fixed point.

§ 12. The principle of conservation of linear momentum.

Momentum. Definition *If at any instant a particle of mass m moves with velocity* $\mathbf{v}$, *then the vector* $m\mathbf{v}$ *is called the* **momentum** *of the particle at that instant.* The direction of the momentum vector is obviously the same as that of the velocity vector.

If a particle of mass m grams moves in a straight line and its velocity at any instant is v cm./sec., then its momentum at that instant is mv gm.-cm./sec. and is in the direction of v.

The principle of conservation of linear momentum for a particle. *If the sum of the resolved parts of the forces acting on a particle in motion in any given direction is zero, then the resolved part of the momentum of the particle in that direction remains constant.*

[Meerut 1977]

Suppose a particle of mass m moves under the action of a force $\mathbf{F}$ whose resolved part in a given direction is zero. If $\mathbf{a}$ is the unit vector in the given direction, then the resolved part of $\mathbf{F}$ in the direction of $\mathbf{a}$ is $\mathbf{F} \cdot \mathbf{a}$. Thus it is given that $\mathbf{F} \cdot \mathbf{a} = 0$.

Let $\mathbf{v}$ be the velocity of the particle at any time t. Then the momentum of the particle at that instant $= m\mathbf{v}$. The resolved part of $m\mathbf{v}$ in the direction of $\mathbf{a}$ is $m\mathbf{v} \cdot \mathbf{a}$. We have

$$\frac{d}{dt}(m\mathbf{v} \cdot \mathbf{a}) = m \frac{d}{dt}(\mathbf{v} \cdot \mathbf{a}), \text{ if } m \text{ is constant}$$

$$= m \left(\frac{d\mathbf{v}}{dt} \cdot \mathbf{a} + \mathbf{v} \cdot \mathbf{0} \right) \quad [\because \ \mathbf{a} \text{ is a constant vector}]$$

$$= \left(m \frac{d\mathbf{v}}{dt} \right) \cdot \mathbf{a}$$

$$= \mathbf{F} \cdot \mathbf{a} \quad [\because \text{ by Newton's second law of motion, } m\,(d\mathbf{v}/dt) = \mathbf{F}]$$

$$= 0. \quad [\because \ \mathbf{F} \cdot \mathbf{a} = 0]$$

Thus $\frac{d}{dt}(m\mathbf{v}\cdot\mathbf{a})=0$ and so $m\mathbf{v}\cdot\mathbf{a}$ is constant.

Remark. The principle of conservation of linear momentum also holds good for a system of particles. *Thus if the sum of the resolved parts of the forces acting on a system of particles in any given direction is zero, then the resolved part of the total momentum of the system in that direction remains constant.*

§ 13. **Impulse. Definition. When the force is constant.** *If a constant force* **F** *acts on a particle during the time interval* (t_0, t_1), *the vector* $\mathbf{I}=(t_1-t_0)\,\mathbf{F}$ *is called the impulse of the force* **F** *during the interval* (t_0, t_1). Obviously, here direction of the impulse vector **I** is the same as that of the force **F**. [Meerut 1975]

When the force is variable. *If a variable force* $\mathbf{F}(t)$ *acts on a particle during the time interval* (t_0, t_1), *then the vector*

$$\mathbf{I}=\int_{t_0}^{t_1} \mathbf{F}(t)\, dt$$

is called the impulse of the force $\mathbf{F}(t)$ *during the interval* (t_0, t_1). Here the direction of the vector **I** is that of the time average of **F** over the interval (t_0, t_1).

Impulse-Momentum principle for a particle. *The change of momentum vector of a particle during a time interval is equal to the net impulse vector of the external forces during this interval.*

[Rohilkhand 1981]

Let a particle of mass m move under the action of an external force **F**. Let **v** be the velocity of the particle at the beginning of the time interval (t_0, t_1) and $\mathbf{v}_1$ be the velocity at the end of this time interval. If **v** is the velocity of the particle at any time t, then by Newton's second law of motion

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}. \qquad \ldots(1)$$

If **I** is the impulse vector of the force **F** during the time interval (t_0, t_1), then

$$\mathbf{I}=\int_{t_0}^{t_1} \mathbf{F}dt=\int_{t_0}^{t_1} m\frac{d\mathbf{v}}{dt}\,dt \qquad \text{[from (1)]}$$

$$=m\int_{t_0}^{t_1} d\mathbf{v}=m\Big[\,\mathbf{v}\,\Big]_{t_0}^{t_1}=m(\mathbf{v}_1-\mathbf{v}_0)$$

$$=m\mathbf{v}_1-m\mathbf{v}_0$$

= change in the momentum vector in the interval (t_0, t_1)

The equation $\mathbf{I}=m\mathbf{v}_1-m\mathbf{v}_0$ is known as the impulse-momentum principle. It gives us an exact relation between the impulse of force and the change in motion produced.

... a is constant.
...ation of linear momentum
... *Thus if the sum of the*
...stem of particles in any
...rt of the total momentum
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[Meerut 1975]
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Rectilinear motion with constant acceleration. Suppose a particle of mass m moves in a straight line under the action of a constant force F producing a constant acceleration f. Let u be the initial velocity of the particle and v be its velocity after time t.

The impulse of the force F during the time t

$=$ the product of the force F and the time t

$=Ft=mft$ [$\because$ $F=mf$, by Newton's second law of motion]

$=m(v-u)$ [$\because$ $v=u+ft$]

$=mv-mu$

$=$ the change of the momentum of the particle in time t.

If the interval t is indefinitely small, but u, v are finite *i.e.*, change in momentum is finite, then certainly the force F must be indefinitely large. Such a force is called an **impulsive force.**

Thus a very large force acting for a very short period of time is called an impulsive force. For example, the blow by a hammer on a peg is an impulsive force. An impulsive force is measured by the change in the momentum of the body produced by it. The students should distinguish carefully between *impulse* and *impulsive force.*

Units of Impulse. The equivalence of impulse and the change in momentum enables us to adopt the same units for impulse as those used for momentum. Thus the absolute units of impulse are :

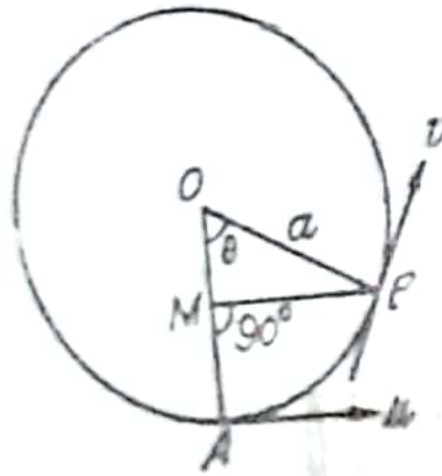
In C. G. S. system, gm. cm./sec.

In M. K. S. system, kg. m./sec.

In F. P. S. system, lb.-ft./sec.

Illustrative Examples

Ex. 8. *A bead of mass m is projected with velocity u along the inside of a smooth fixed vertical circle of radius a from the lowest point A. Use the principle of work and energy to find the velocity of the bead when it is at B, where* $\angle AOB=\theta$, *O being the centre of the circle.*



Sol. Let v be the velocity of the bead when it is at B. Then the change in the K. E. of the bead in moving from A to B

$=\frac{1}{2}mv^2-\frac{1}{2}mu^2.$

The only force that does work in this displacement is the weight mg. The work done by the weight mg of the bead during its displacement from A to $B = -mg.AM = -mg.(OA - OM)$

$$= -mg.(a - a\cos\theta)$$

$$= -mga\,(1-\cos\theta).$$

Now by the principle of work and energy, the change in the kinetic energy = work done by the forces.

$$\therefore \quad \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = -mga\,(1-\cos\theta)$$

or

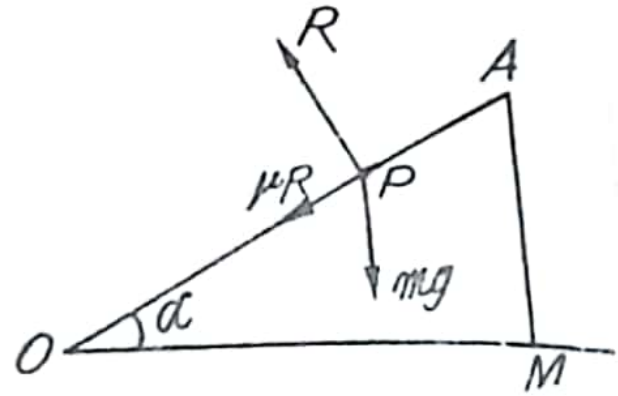
$$v^2 - u^2 = -2ga\,(1-\cos\theta)$$

or

$$v^2 = u^2 - 2ag\,(1-\cos\theta),$$

which gives the velocity of the bead at B.

Ex. 9. *A particle is set moving with kinetic energy E straight up an inclined plane of inclination α and coefficient of friction μ. Prove that the work done against friction before the particle comes to rest is* $E\mu\cos\alpha/(\sin\alpha + \mu\cos\alpha)$.



Sol. Suppose a particle of mass m starts moving from O with kinetic energy E up an inclined plane of inclination α to the horizontal. Let P be the position of the particle at any time t. The forces acting on the particle at P are (i) its weight mg, which has component $mg\sin\alpha$ down the plane and $mg\cos\alpha$ perpendicular to the plane, (ii) the normal reaction R of the plane and (iii) the force of friction μR acting down the plane because its direction is opposite to the direction of motion.

Since there is no motion of the particle perpendicular to the inclined plane, therefore $R = mg\cos\alpha$.

$\therefore$ the force of friction $= \mu R = \mu mg\cos\alpha$.

Suppose the particle comes to rest at A where $OA = x$.

The only forces which work during the displacement of the particle from O to A are its weight and the force of friction. The work done by the weight $= -mg.AM = -mgx\sin\alpha$.

The work done by the force of friction

$$= -(\mu mg\cos\alpha).x = -\mu mgx\cos\alpha. \qquad \ldots(1)$$

Since the kinetic energy of the particle at O is E and at A is zero, therefore by the principle of work and energy during the motion of the particle from O to A

change in K. E. = work done by the forces i.e., $\quad 0-E=-mgx \sin \alpha-\mu mgx \cos \alpha$

or $\quad E=xmg (\sin \alpha+\mu \cos \alpha)$

or $\quad x=\dfrac{E}{mg (\sin \alpha+\mu \cos \alpha)}$.

Putting this value of x in (1), the work done by the force of friction

$$=-\mu\, mg \cos \alpha . \frac{E}{mg (\sin \alpha+\mu \cos \alpha)}$$

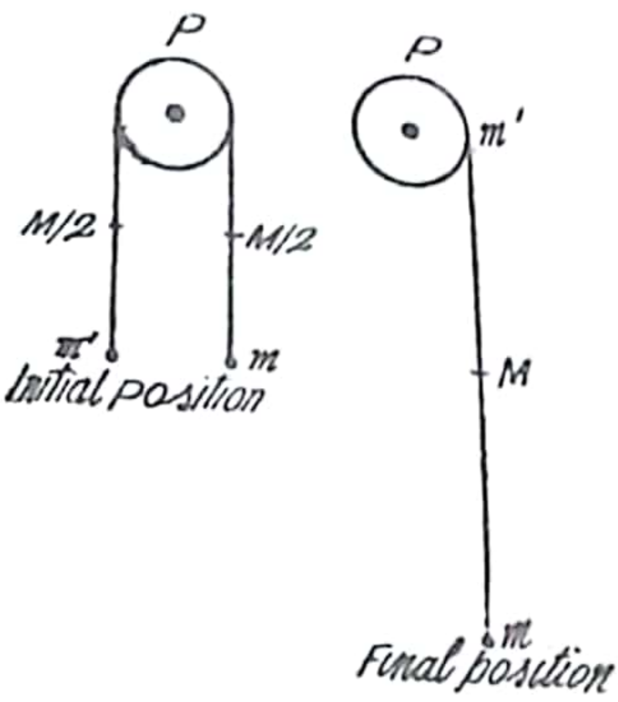
$$=\frac{-E\mu \cos \alpha}{\sin \alpha+\mu \cos \alpha}.$$

Hence the work done against friction $=\dfrac{E\mu \cos \alpha}{\sin \alpha+\mu \cos \alpha}$.

Ex. 10. *A uniform string of mass M and length 2a is placed symmetrically over a smooth peg and has particles of masses m and m' attached to its ends (m>m'). Show that the string runs off the peg when its velocity is*

$$\sqrt{\left[\frac{M+2(m-m')}{M+m+m'}\, ag\right]}.$$

Sol. In the initial position depth of the centre of gravity of the system from the peg P



$$=\frac{M.\tfrac{1}{2}a+ma+m'a}{M+m+m'}$$

$$=\frac{a}{2}\,\frac{M+2m+2m'}{M+m+m'}.$$

In the final position depth of the centre of gravity of the system from the peg P

$$=\frac{M\ a+m.2a+m'.0}{M+m+m'}$$

$$=a\,\frac{M+2m}{M+m+m'}.$$

$\therefore$ displacement in the position of centre of gravity

$$=a\,\frac{M+2m}{M+m+m'}-\frac{a}{2}\,\frac{M+2m+2m'}{M+m+m'}=\frac{a}{2}\,\frac{M+2m-2m'}{M+m+m'}.$$

The initial velocity of the system is zero; and let the final velocity be v.

By the principle of work and energy, we have

change in K. E. = work done by the forces.

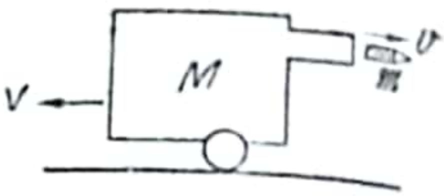
i.e., $\frac{1}{2}(M+m+m')v^2-0=(M+m+m')g.\frac{a}{2}\cdot\frac{M+2(m-m')}{M+m+m'}$

or $v^2=\frac{M+2(m-m')}{M+m+m'}ag$

or $v=\sqrt{\left[\frac{M+2(m-m')}{M+m+m'}ag\right]}$

This gives the velocity of the string when it runs off the peg.

Ex. 11. *A shot of mass m is fired horizontally from a gun of mass M with velocity u relative to the gun; show that the actual velocities of the shot and the gun are* $\frac{Mu}{M+m}$ *and* $\frac{mu}{M+m}$ *respectively, and that their kinetic energies are inversely proportional to their masses.*



Sol. Let v be the actual velocity of the shot and V be the actual velocity with which the gun recoils.

Then the velocity of the shot relative to the gun $=v+V$.

But according to the question the velocity of the shot relative to the gun is u.

$\therefore \quad u=v+V.$ ...(1)

Since in the horizontal direction no external force acts on the system, therefore by the principle of conservation of linear momentum applied in the horizontal direction

momentum before firing = momentum after firing

i.e., $0=mv-MV$

i.e., $mv=MV.$ ...(2)

From (2), $v=\frac{M}{m}V$. Substituting this value of v in (1), we have

$$u=\frac{M}{m}V+V=\left(\frac{M}{m}+1\right)V=\frac{M+m}{m}V.$$

$$\therefore \quad V=\frac{mu}{M+m} \quad \text{and so } v=\frac{M}{m}V=\frac{M}{m}\cdot\frac{mu}{M+m}=\frac{Mu}{M+m}.$$

$\therefore$ the actual velocity of the shot $=v=\frac{Mu}{M+m}$,

and the actual velocity of the gun $=V=\frac{mu}{M+m}$.

Again $\frac{\text{the K. E. of the shot}}{\text{the K. E. of the gun}}=\frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}=\frac{m}{M}\frac{v^2}{V^2}$

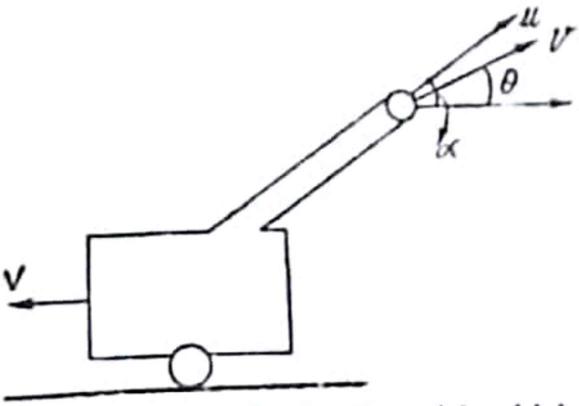
$$=\frac{m}{M}\cdot\frac{M^2}{m^2} \qquad \left[\because \text{ from (2), } \frac{v}{V}=\frac{M}{m}\right]$$

$$=\frac{M}{m}=\frac{\text{the mass of the gun}}{\text{the mass of the shot}}.$$

Hence their kinetic energies are inversely proportional to their masses.

Ex. 12. *A gun is mounted on a gun carriage, movable on a smooth horizontal plane, and the gun is elevated at an angle α to the horizon. A shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon. If the mass of the gun and its carriage be n times that of the shot, prove that*

$$\tan\theta=\left(1+\frac{1}{n}\right)\tan\alpha. \qquad \textbf{[Lucknow 1979, 81]}$$



Sol. Let v be the actual velocity with which the shot leaves the gun and V the actual velocity with which the gun carriage recoils horizontally. According to the question the direction of v makes an angle θ with the horizontal.

The velocity of the shot relative to the gun in the horizontal direction $=v\cos\theta+V$

and the velocity of the shot relative to the gun in the vertical direction $=v\sin\theta$.

If u be the velocity of the shot relative to the gun, then the direction of u makes an angle α with the horizontal.

$$\therefore \quad \tan\alpha=\frac{\text{vertical component of } u}{\text{horizontal component of } u}=\frac{v\sin\theta}{v\cos\theta+V}. \qquad ...(1)$$

Now if the mass of the shot is m, then the mass of the gun and the carriage is nm.

Since in the horizontal direction no external force acts on the system, therefore applying the principle of conservation of linear momentum in the horizontal direction, we have

$$mv\cos\theta-nmV=0$$

i.e.,

$$V=(v\cos\theta)/n.$$

Substituting this value of V in (1), we have

$$\tan\alpha=\frac{v\sin\theta}{v\cos\theta+(v\cos\theta)/n}=\frac{v\sin\theta}{v\cos\theta\,(1+1/n)}$$

$$=\frac{\tan\theta}{1+1/n}.$$

$\therefore\quad \tan\theta=\left(1+\frac{1}{n}\right)\tan\alpha.$

Ex. 13. *A shell of mass m is fired from a gun of mass M which can recoil freely on a horizontal base, and the elevation of the gun is α. Prove that the inclination of the path of the shell to the horizon at the time of projection is*

$$\tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\}.$$

Prove also that the energy of the shell on leaving the gun is to that of the gun as $[M^2+(m+M)^2\tan^2\alpha]:mM$, *assuming that none of the energy of the explosion is lost.*

Sol. Let v be the actual velocity and θ the actual elevation of the shell on leaving the gun. Suppose V is the actual velocity with which the gun recoils horizontally.

The velocity of the shell relative to the gun in the horizontal direction $=v\cos\theta+V$

and the velocity of the shell relative to the gun in the vertical direction $=v\sin\theta$.

Since the inclination of the velocity of the shell relative to the gun to the horizontal is equal to the elevation α of the gun,

therefore $\quad \tan\alpha=\dfrac{v\sin\theta}{v\cos\theta+V}.\qquad\ldots(1)$

Applying the principle of conservation of linear momentum in the horizontal direction, we have

momentum after firing = momentum before firing

i.e., $\quad mv\cos\theta-MV=0$

i.e., $\quad mv\cos\theta=MV.\qquad\ldots(2)$

Substituting the value of V from (2) in (1), we have

$$\tan\alpha=\frac{v\sin\theta}{v\cos\theta+(mv\cos\theta)/M}=\frac{v\sin\theta}{v\cos\theta\,(1+m/M)}$$

$$=\frac{\tan\theta}{1+m/M}.$$

$\therefore\quad \tan\theta=\left(1+\frac{m}{M}\right)\tan\alpha$

or $$\theta=\tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\}.$$

which proves the first result.

Squaring both sides of (2), we have

$$m^2v^2\cos^2\theta=M^2V^2.$$

$$\therefore\quad \frac{mv^2}{MV^2}=\frac{M}{m}\sec^2\theta$$

or $$\frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}=\frac{M}{m}(1+\tan^2\theta)=\frac{M}{m}\left\{1+\left(1+\frac{m}{M}\right)^2\tan^2\alpha\right\}.$$

Hence on leaving the gun, we have

$$\frac{\text{kinetic energy of the shell}}{\text{kinetic energy of the gun}}=\frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}$$

$$=\frac{M}{mM^2}\left\{M^2+(m+M)^2\tan^2\alpha\right\}=\frac{M^2+(m+M)^2\tan^2\alpha}{mM},$$

which proves the second result.

Ex. 14. *Assuming that in a canon the force on the ball depends only on the volume of gas generated by the gun powder, show that the ratio of the final velocity of the ball when the gun is free to recoil to its velocity when the gun is fixed is $\sqrt{\left(\frac{M}{M+m}\right)}$, where M and m are the masses of the gun and the ball respectively.* [Lucknow 1976]

Sol. Let E be the energy released by the explosion.

When the gun is free to recoil let v be the velocity of the ball and u the velocity with which the gun recoils. In this case the energy released is

$$E=\tfrac{1}{2}mv^2+\tfrac{1}{2}Mu^2. \quad ...(1)$$

Also by the principle of conservation of linear momentum, we have

$$mv-Mu=0 \quad i.e., \quad mv=Mu. \quad ...(2)$$

Again when the gun is fixed, let V be the velocity of the ball. The energy released is then

$$E=\tfrac{1}{2}mV^2. \quad ...(3)$$

From (1) and (3), on eliminating E, we get

$$mv^2+Mu^2=mV^2$$

or $$mv^2+M\frac{m^2}{M^2}v^2=mV^2 \quad \text{[substituting for } u \text{ from (2)]}$$

or $$mv^2\left(1+\frac{m}{M}\right)=mV^2$$

or $$v^2\left(\frac{M+m}{M}\right)=V^2$$

or $$\frac{v^2}{V^2}=\frac{M}{M+m} \quad \text{or} \quad \frac{v}{V}=\sqrt{\left(\frac{M}{M+m}\right)}.$$

Ex. 15. *A gun of mass M fires a shell of mass m horizontally, and the energy of explosion is such as would be sufficient to project the shot vertically to a height h. Show that the velocity of recoil of the gun is*

$$\left[\frac{2m^2gh}{M(M+m)}\right]^{1/2}.$$

[Rohilkhand 1979]

Sol. Let E be the energy of the explosion. Since E is just sufficient to project a mass m vertically to a height h, therefore $E=\frac{1}{2}mu^2$, where u is the vertical velocity of projection just sufficient to raise a particle to a height h.

But for such a velocity of projection u, we have

$$0=u^2-2gh \quad i.e., \quad u^2=2gh.$$

$$\therefore \quad E=\tfrac{1}{2}m.2gh=mgh. \qquad ...(1)$$

When the shell is fired horizontally from the gun, let v be the velocity of the shell and V the velocity with which the gun recoils. We then have

$$E=\tfrac{1}{2}mv^2+\tfrac{1}{2}MV^2. \qquad ...(2)$$

Also by the principle of conservation of linear momentum, we have

$$mv-MV=0 \quad i.e., \quad mv=MV. \qquad ...(3)$$

From (1) and (2), we have equating the two values of E

$$mgh=\tfrac{1}{2}mv^2+\tfrac{1}{2}MV^2$$

$$=\tfrac{1}{2}m.\frac{M^2V^2}{m^2}+\tfrac{1}{2}MV^2 \quad \text{[substituting for } v \text{ from (3)]}$$

$$=\tfrac{1}{2}MV^2\left(\frac{M}{m}+1\right)=\tfrac{1}{2}MV^2\,\frac{M+m}{m}.$$

$$\therefore \quad V^2=\frac{2m^2gh}{M(M+m)} \quad \text{or} \quad V=\left[\frac{2m^2gh}{M(M+m)}\right]^{1/2}.$$

Ex. 16. *A shell of mass m is projected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is* $\sqrt{\left[\frac{2EM}{m(M+m)}\right]}$, *it being assumed that at the instant of explosion the gun is free to recoil.*

Sol. Let u be the velocity of the shell while leaving the gun and v the velocity with which the gun recoils. Then we have

$$E=\tfrac{1}{2}mu^2+\tfrac{1}{2}Mv^2. \qquad ...(1)$$

Also by the principle of conservation of linear momentum, we have

$$mu-Mv=0 \quad i.e., \quad mu=Mv. \qquad ...(2)$$

To find u we have to eliminate v from (1) and (2).

From (2), we have $v=mu/M$. Putting this value of v in (1), we get

$$E=\tfrac{1}{2}mu^2+\tfrac{1}{2}M.\frac{m^2u^2}{M^2}=\tfrac{1}{2}mu^2\left(1+\frac{m}{M}\right)=\frac{mu^2(M+m)}{2M}$$

$\therefore \quad u^2 = \dfrac{2EM}{m(M+m)}$ or $u = \sqrt{\left[\dfrac{2EM}{m(M+m)}\right]}$.

Ex. 17. *A body of mass (m_1+m_2) moving in a straight line is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that if after the explosion the two parts move in the same line as before, their relative speed is*

$$\sqrt{\left[\frac{2E(m_1+m_2)}{m_1m_2}\right]}.$$

Sol. Let u be the velocity of the body of mass (m_1+m_2) before explosion and u_1 and u_2 the velocities of parts m_1 and m_2 after explosion. Then by the principle of conservation of linear momentum, we have

$$m_1u_1+m_2u_2=(m_1+m_2)\,u. \qquad \ldots(1)$$

Also K.E. after splitting = K.E. before splitting.

$\therefore \quad \frac{1}{2}m_1u_1^2+\frac{1}{2}m_2u_2^2=E+\frac{1}{2}(m_1+m_2)\,u^2$

or $\quad m_1u_1^2+m_2u_2^2=2E+(m_1+m_2)\,u^2. \qquad \ldots(2)$

We are to find the relative velocity which is equal to the difference of u_1 and u_2.

Multiplying (2) by (m_1+m_2) and then subtracting from it the square of (1), we get

$$(m_1+m_2)(m_1u_1^2+m_2u_2^2)-(m_1u_1+m_2u_2)^2=2E(m_1+m_2)$$

or $\quad m_1m_2(u_1^2+u_2^2-2u_1u_2)=2E(m_1+m_2)$

or $\quad m_1m_2(u_1-u_2)^2=2E(m_1+m_2)$

or $\quad (u_1-u_2)^2=\dfrac{2E(m_1+m_2)}{m_1m_2}$.

Hence $\quad u_1-u_2=\sqrt{\left[\dfrac{2E(m_1+m_2)}{m_1m_2}\right]}$.

It gives the relative velocity of m_1 with respect to m_2 after explosion.

Ex. 18. *A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s is the distance apart, in the tube, of the masses after a time t, show that the work done by the explosion is*

$$\frac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\cdot\frac{s^2}{t^2}.$$

Sol. Since the shell is lying in the tube, its velocity before explosion is zero. Let u_1 and u_2 be the velocities, of the masses m_1 and m_2 respectively after explosion. Then the relative velocity of the masses after explosion is u_1+u_2. Since the tube is smooth and horizontal, u_1+u_2 will remain constant.

$$\therefore \quad (u_1+u_2)\, t=s. \qquad \ldots(1)$$

Also by the principle of conservation of linear momentum, we have

$$m_1u_1-m_2u_2=0$$

i.e.,

$$m_1u_1=m_2u_2. \qquad \ldots(2)$$

Substituting for u_2 from (2) in (1), we get

$$\left(u_1+\frac{m_1u_1}{m_2}\right) t=s$$

or

$$u\left(\frac{m_1+m_2}{m_2}\right) t=s$$

or

$$u_1=\frac{m_2 s}{(m_1+m_2)\, t}.$$

$$\therefore \quad u_2=\frac{m_1}{m_2}\, u_1=\frac{m_1}{m_2}\cdot\frac{m_2 s}{(m_1+m_2)\, t}=\frac{m_1 s}{(m_1+m_2)\, t}.$$

Now the work done by the explosion

$$=\text{the kinetic energy released due to the explosion}$$

$$=\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2$$

$$=\tfrac{1}{2}\, m_1\cdot\frac{m_2^2 s^2}{(m_1+m_2)^2\, t^2}+\tfrac{1}{2}\, m_2\frac{m_1^2 s^2}{(m_1+m_2)\, t^2}$$

$$=\tfrac{1}{2}\,\frac{s^2}{t^2}\cdot\frac{1}{(m_1+m_2)^2}\,(m_1m_2^2+m_2m_1^2)$$

$$=\tfrac{1}{2}\,\frac{s^2}{t^2}\,\frac{m_1m_2\,(m_1+m_2)}{(m_1+m_2)^2}=\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\cdot\frac{s^2}{t^2}.$$

Ex. 19. *A shell is moving with velocity u in the line AB. An internal explosion, which generates an energy E, breaks it into two fragments of masses m_1 and m_2 which move in the line AB. Show that their velocities are*

$$u+\sqrt{\left[\frac{2Em_2}{m_1\,(m_1+m_2)}\right]} \quad \textit{and} \quad u-\sqrt{\left[\frac{2Em_1}{m_2\,(m_1+m_2)}\right]}.$$

[Lucknow 1978]

Sol. Let u_1 and u_2 be the velocities of the masses m_1 and m_2 respectively after the explosion. By the principle of conservation of linear momentum, we have

$$(m_1+m_2)\, u=m_1u_1+m_2u_2. \qquad \ldots(1)$$

Now the energy before explosion is $\frac{1}{2}\,(m_1+m_2)\, u^2$ and E is the energy due to explosion. Also the total energy after explosion is

$$(\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2).$$

Since there has been no dissipation of energy, therefore by the principle of conservation of mechanical energy, we have

$$\tfrac{1}{2}\,(m_1+m_2)\, u^2+E=\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2. \qquad \ldots(2)$$

It is easy to observe that for all values of x

$$u_1=u+\frac{x}{m_1} \quad \text{and} \quad u_2=u-\frac{x}{m_2} \qquad \ldots(3)$$

satisfy the equation (1). In order that these values of u_1 and u_2 may also satisfy the equation (2), we should have

$$\tfrac{1}{2}(m_1+m_2)\,u^2+E=\tfrac{1}{2}m_1\left(u+\frac{x}{m_1}\right)^2+\tfrac{1}{2}m_2\left(u-\frac{x}{m_2}\right)^2$$

or
$$(m_1+m_2)\,u^2+2E=m_1\left(u^2+\frac{2xu}{m_1}+\frac{x^2}{m_1^2}\right)+m_2\left(u^2-\frac{2xu}{m_2}+\frac{x^2}{m_2^2}\right)$$

or
$$2E=x^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right),$$ the other terms cancelling one another

or
$$2E=x^2\frac{(m_1+m_2)}{m_1m_2} \quad \text{or} \quad x^2=\frac{2Em_1m_2}{m_1+m_2}$$

or
$$x=\sqrt{\left(\frac{2Em_1m_2}{m_1+m_2}\right)}.$$

Putting this value of x in (3), we get

$$u_1=u+\sqrt{\left[\frac{2Em_2}{m_1(m_1+m_2)}\right]} \quad \text{and} \quad u_2=u-\sqrt{\left[\frac{2Em_1}{m_2(m_1+m_2)}\right]}.$$

Ex. 20. *A shell of mass M is moving with velocity V. An internal explosion generates an amount of energy E and breaks the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Show that their velocities are*

$$V+\sqrt{\left(\frac{2m_2E}{m_1M}\right)} \quad \text{and} \quad V-\sqrt{\left(\frac{2m_1E}{m_2M}\right)}.$$

[Lucknow 1980; Rohilkhand 80]

Sol. Since the whole mass M is divided in the ratio $m_1 : m_2$, therefore masses of the fragments are

$$\frac{m_1M}{m_1+m_2} \quad \text{and} \quad \frac{m_2M}{m_1+m_2}.$$

Now proceed as in Ex. 19.

Ex. 21. *A shot of mass m fired horizontally penetrates a thickness s of a fixed plate of mass M; prove that if M is free to move the thickness penetrated is $Ms/(M+m)$.*

Sol. Let u be the striking velocity of the shot and P be the force of resistance offered by the plate assumed to be uniform.

When the plate is fixed the velocity of the shot reduces to zero after penetrating a thickness s. During the motion of the shot the change in the K.E. of the shot$=0-\frac{1}{2}mu^2=-\frac{1}{2}mu^2$ and the work done by the force of resistance$=-Ps$. By the principle of work and energy, we have

change in K.E.$=$work done by the forces.

$$\therefore \quad -\tfrac{1}{2}mu^2=-Ps \quad \text{or} \quad \tfrac{1}{2}mu^2=Ps. \qquad \ldots(1)$$

Again consider the case when the plate is free to move. In this case let x be the thickness penetrated and V be the common velocity of the shot and the plate when the penetration ceases. By the principle of work and energy applied to the shot and the plate considered together as one system, we have

$$\tfrac{1}{2}(m+M)\,V^2-\tfrac{1}{2}mu^2=-Px$$

or
$$\tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)\,V^2=Px. \qquad \ldots(2)$$

Also in this case during the time of impact the resultant horizontal force on the whole system is zero because the mutual impulsive action and reaction between the shot and the plate are equal and opposite. Therefore by the principle of conservation of linear momentum, we have

momentum before impact$=$momentum after impact

i.e.,
$$mu=(m+M)\,V. \qquad \ldots(3)$$

Dividing (2) by (1), we get

$$\frac{x}{s}=\frac{mu^2-(m+M)\,V^2}{mu^2}$$

$$=\frac{mu^2-(m+M)\cdot\dfrac{m^2u^2}{(m+M)^2}}{mu^2}, \text{ substituting for } V \text{ from (3)}$$

$$=\frac{mu^2-\dfrac{m^2u^2}{m+M}}{mu^2}=1-\frac{m}{m+M}=\frac{M}{m+M}.$$

$$\therefore \quad x=\frac{Ms}{M+m}, \text{ which proves the required result.}$$

Ex. 22. *If a shot of mass m striking a fixed metal plate with velocity u, penetrates it through a distance a, show that it will completely pierce through a plate free to move, of mass M and thickness b, if* $b<\dfrac{Ma}{m+M}$, *the resistance being supposed uniform.*

Sol. When the plate is free to move let x be the distance penetrated. Then proceeding as in Ex. 21, we have

$$x=\frac{Ma}{M+m}.$$

Since the thickness of the plate is b, therefore the shot will completely pierce through if

$$b < \frac{Ma}{M+m}.$$

Ex. 23. *A block of mass M rests on a smooth horizontal table and a bullet of mass m is fired into it. The penetration of the bullet is opposed by a constant resisting force. If the experiment is repeated with the block firmly fixed, show that the depth of penetration of the bullet and the time which elapses before the bullet is at rest relatively to the block are in each case increased in the ratio* $\left(1+\frac{m}{M}\right) : 1$.

Sol. Let u be the striking velocity of the bullet and P be the force of resistance offered by the block assumed to be uniform.

Case I. When the block is fixed. In this case let s be the thickness penetrated and t the time that elapses when the penetration stops.

By the principle of work and energy, we have

$$0-\tfrac{1}{2}mu^2=-Ps \quad \text{i.e.,} \quad \tfrac{1}{2}mu^2=Ps. \qquad \ldots(1)$$

Also by the impulse-momentum principle, we have

$$0-mu=-Pt \quad \text{i.e.} \quad mu=Pt. \qquad \ldots(2)$$

Case II. When the block is free to move. In this case let s' be the thickness penetrated, t' the time taken when the penetration ceases and V the common velocity of the bullet and the block at the end of the penetration. In this case, we have

$$(m+M)\,V=mu, \qquad \ldots(3)$$

$$\tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)\,V^2=Ps', \qquad \ldots(4)$$

and

$$mu-mV=Pt'. \qquad \ldots(5)$$

The equation (3) has been written by applying the principle of conservation of momentum to the impact of the bullet and the block, the equation (4) has been obtained by applying the work-energy principle to the motion of the bullet and block considered together and the equation (5) has been obtained by applying the impulse-momentum principle to the motion of the bullet only.

Dividing (1) by (4), we get

$$\frac{s}{s'}=\frac{mu^2}{mu^2-(m+M)\,V^2}=\frac{mu^2}{mu^2-(m+M)\cdot\frac{m^2u^2}{(m+M)^2}} \quad \text{[from (3),}$$

$$\frac{1}{1-\dfrac{m}{m+M}}, \text{ dividing the Nr. and Dr. each by } mu^2$$

$$=\frac{m+M}{M}=1+\frac{m}{M}.$$

Thus $s : s' : : \left(1+\dfrac{m}{M}\right) : 1.$ This proves one result.

Again dividing (2) by (5), we have

$$\frac{t}{t'}=\frac{mu}{mu-mV}=\frac{mu}{mu-m.\dfrac{mu}{m+M}}, \text{ substituting for } V \text{ from (3)}$$

$$=\frac{1}{1-\dfrac{m}{m+M}}=\frac{m+M}{M}=1+\frac{m}{M}.$$

Thus $t : t' : : \left(1+\dfrac{m}{M}\right) : 1.$ This proves the other result.

Ex. 24. *A bullet of mass m moving with a velocity u strikes a block of mass M, which is free to move in the direction of the motion of the bullet, and is embedded in it. Show that a portion $M/(M+m)$ of the kinetic energy is lost. If the block is afterwards struck by an equal bullet moving in the same direction with the same velocity, show that there is a further loss of kinetic energy equal to*

$$\frac{M^2mu^2}{2\,(M+2m)\,(M+m)}.$$

Sol. Let v be the velocity of the block after the first bullet strikes it and is embedded in it. Then by the principle of conservation of momentum, we have

$$(m+M)\,v=mu.$$

Loss of K.E. $=\frac{1}{2}mu^2-\frac{1}{2}\,(m+M)\,v^2$ ...(1)

$$=\tfrac{1}{2}mu^2-\tfrac{1}{2}\,(m+M).\,\frac{m^2u^2}{(m+M)^2}, \text{ substituting for } v \text{ from (1)}$$

$$=\tfrac{1}{2}mu^2.\left[1-\frac{m}{m+M}\right]=\tfrac{1}{2}mu^2.\,\frac{M}{m+M}$$

$$=\frac{M}{m+M}.\ (\text{K.E. before striking}).$$

Thus the fraction of K.E. lost $=\dfrac{M}{m+M}.$

Again let V be the velocity of the block after the second bullet strikes it. Then

$$(2m+M)\,V=(m+M)\,v+mu=2mu. \quad \ldots(2)$$

$[\because$ from (1), $(m+M)\,v=mu]$

$\therefore$ further loss of K.E. $=\frac{1}{2}(m+M)\,v^2+\frac{1}{2}mu^2-\frac{1}{2}(2m+M)\,V^2$

$$=\frac{1}{2}(m+M).\frac{m^2u^2}{(m+M)^2}+\frac{1}{2}mu^2-\frac{1}{2}(2m+M).\frac{4m^2u^2}{(2m+M)^2}$$

[substituting for v and V from (1) and (2)]

$$=\frac{M^2mu^2}{2(M+2m)(M+m)}$$

Ex. 25. *A hammer of mass M lbs. falls freely from a height h feet on the top of an inelastic pile of mass m lbs. which is driven into the ground a distance a feet. Assuming that the resistance of the ground is constant, find its value and show that the time during which the pile is in motion is given by* $\frac{a(M+m)}{M}\left(\frac{2}{gh}\right)^{1/2}$ *Find also what fraction of kinetic energy is lost by impact.* [Rohilkhand 1978]

Sol. Let u ft./sec. be the velocity of the hammer just before impact with the pile. Then

$$u=\sqrt{(2gh)}. \quad \ldots(1)$$

Since the pile is inelastic, therefore after impact the hammer will not rebound and the hammer and the pile will begin to move together as one body, say, with velocity v ft./sec.

By the principle of conservation of momentum, we have

$$Mu+m.0=(M+m)\,v.$$

$$\therefore \quad v=\frac{Mu}{M+m}. \quad \ldots(2)$$

Suppose the resistance of the ground is R poundals and the retardation produced by it is f ft./sec^2.

Since the velocity v becomes zero after penetrating a distance a feet in the ground, therefore

$$0=v^2-2fa \quad \text{or} \quad f=\frac{v^2}{2a}. \quad \ldots(3)$$

By Newton's second law of motion, $P=mf$, we have

$$R-(m+M)\,g=(m+M)\,f.$$

$$\therefore \quad R=(m+M)\,g+(m+M)\,f$$

$$=(m+M)\,g+(m+M)\,\frac{v^2}{2a}\left[\because \text{ from (3)}, f=\frac{v^2}{2a}\right]$$

$$=(m+M)\,g+\frac{(m+M)}{2a}\cdot\frac{M^2u^2}{(m+M)^2} \text{ [from (2)]}$$

$$=(m+M)\,g+\frac{M^2}{2a}\,\frac{2gh}{m+M} \qquad [\because \text{ from (1), } u^2=2gh]$$

$$=(m+M)\,g+\frac{M^2gh}{a\,(m+M)}$$

Hence the resistance of the ground $=\left\{(m+M)+\frac{M^2h}{a\,(m+M)}\right\}$ lbs. wt.

Let t seconds be the time during which the pile is in motion. Then $\qquad 0=v-ft.$

$$\therefore \quad t=\frac{v}{f}=\frac{v}{v^2/2a} \qquad \text{[from (3)]}$$

$$=\frac{2a}{v}=\frac{2a}{Mu/(m+M)} \qquad \text{[from (2)]}$$

$$=\frac{2a\,(m+M)}{Mu}=\frac{2a\,(m+M)}{M\sqrt{(2gh)}}=\frac{a\,(m+M)}{M}\cdot\sqrt{\left(\frac{2}{gh}\right)}.$$

Loss of K.E. by impact $=\frac{1}{2}Mu^2-\frac{1}{2}\,(M+m)\,v^2$

$$=\frac{1}{2}Mu^2-\frac{1}{2}\,(M+m)\cdot\frac{M^2u^2}{(m+M)^2} \qquad \text{[from (2)]}$$

$$=\frac{1}{2}Mu^2\left[1-\frac{M}{m+M}\right]=\frac{1}{2}Mu^2\cdot\frac{m}{m+M} \text{ units of energy.}$$

$$\therefore \quad \text{fraction of K.E. lost}=\frac{\frac{1}{2}Mu^2\cdot\frac{m}{m+M}}{\frac{1}{2}Mu^2}=\frac{m}{m+M}.$$

Ex. 26. *Prove that if a hammer weighing W lbs. striking a nail weighing w lbs with velocity V feet per second, drives it a feet into a fixed block of wood, the average resistance of the wood in pounds to the penetration of the nail is*

$$\frac{W^2}{W+w}\cdot\frac{V^2}{2ga}.$$

If, however, the block is free to recoil and weighs M lbs., the resistance obtained would be

$$\frac{MW^2}{(M+W+w)\,(W+w)}\cdot\frac{V^2}{2ag}.$$

It is to be noted that motion in the case of a nail being driven is in the horizontal direction.

Sol. When the block is fixed. Let u be the common velocity of the nail and the hammer immediately after striking. By the principle of conservation of momentum, we have

$$(W+w)\,u=WV$$

or $\qquad u=WV/(W+w). \qquad \ldots(1)$

Let P pounds weight be the average resistance of the wood to the penetration of the nail. Then by the principle of work and energy, we have $\frac{1}{2}(W+w)u^2=Pg.a.$ ...(2)

Putting the value of u from (1) in (2), we have

$$\frac{1}{2}(W+w).\frac{W^2V^2}{(W+w)^2}=Pag$$

or $$P=\frac{W^2}{W+w}.\frac{V^2}{2ag}.$$ This proves the first result.

When the block is free to recoil. In this case let u_1 be the common velocity of the hammer, nail and the block when the penetration ceases. By the principle of conservation of momentum, we have $(M+W+w)u_1=WV$

or $u_1=WV/(M+W+w)$. ...(3)

If R pounds weight be the resistance in this case, then by the work-energy principle, we have

$$Rg.a=\frac{1}{2}(W+w)u^2-\frac{1}{2}(M+W+w)u_1^2. \quad ...(4)$$

Substituting the values of u and u_1 from (1) and (3) in (4), we get

$$R=\frac{MW^2}{(M+W+w)(W+w)}.\frac{V^2}{2ag},$$

which proves the second result.

Ex. 27. *A hammer head of mass W kg. moving horizontally with velocity u m/sec. strikes an inelastic nail of mass m kg. fixed in a block of mass M kg. which is free to move. Prove that if the mean resistance of the block to penetration by the nail is a force P kg. wt., then the nail will penetrate with each blow a distance*

$$\frac{MW^2u^2}{2gP(W+m)(W+m+M)} \text{ metres.}$$

Sol. First consider the impulsive action between the hammer and the nail. Since the nail is inelastic, therefore immediately after striking, the hammer and the nail will begin to move as one body, say, with velocity v m./sec. By the principle of conservation of momentum, we have $Wu=(W+m)v$. ...(1)

Now the nail penetrates the block and let V m./sec. be the common velocity when penetration ceases. Then again by the principle of conservation of momentum, we have

$$(W+m+M)V=Wu. \quad ...(2)$$

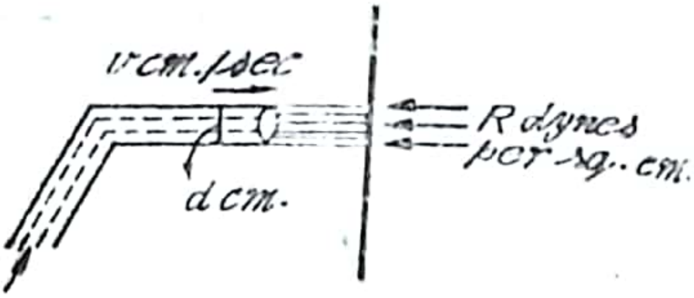
If x metres is the distance penetrated, then by the principle of work and energy, we have

$$-Pgx=\frac{1}{2}(W+m+M)V^2-\frac{1}{2}(W+m)v^2$$

or $$2Pgx=(W+m)v^2-(W+m+M)V^2.$$

Substituting the values of v and V found from (1) and (2) in (3), we get $\quad x=\dfrac{MW^2u^2}{2gP\,(W+m)\,(W+m+M)}.$

Ex. 28. *Water issuing from a nozzle of diameter d cms. with a velocity v cm./sec. impinges on a vertical wall, the jet being at right angles to the wall. If there is no splash, find the pressure exerted on the wall.*



Sol. As the jet strikes the wall, the wall exerts a force on it and destroys its momentum perpendicular to the wall. Let the force exerted by the wall on the jet be R dynes per sq. cm.

The impulse of the force exerted by the wall on the jet over the period of 1 second $=\frac{1}{4}\pi d^2R\times 1=\frac{1}{4}\pi d^2R$ gm.-cm./sec.

Mass of water that reaches the wall in 1 second
$=$ volume of water coming out of jet in 1 sec. $\times$ density of water
$=\frac{1}{4}\pi d^2v\,.\,1=\frac{1}{4}\pi d^2v$ gms.

[$\because$ density of water $=1$ gm. per cubic cm]

Change in the momentum of this mass of water on striking the wall $\quad=\frac{1}{4}\pi d^2v\,[0-(-v)]$

$=\frac{1}{4}\pi d^2v^2$ gm.-cm./sec

By the impulse-momentum principle, we have
impulse of the force for any time $=$ change in the momentum of the mass during that time.

$\therefore\quad \frac{1}{4}\pi d^2R=\frac{1}{4}\pi d^2v^2 \quad$ or $\quad R=v^2.$

By Newton's third law, action and reaction being equal in magnitude, pressure on the wall

$=v^2$ dynes per sq. cm.

Ex. 29. *A jet of water issues vertically at a speed of 30 feet per second from a nozzle of 0·1 square inch section. A ball weighing 1 lb. is balanced in the air by the impact of water on its underside. Show that the height of the ball above the level of the jet is 4·6 feet approximately.*

Sol. Let the height of the ball above the level of the jet be h feet. Suppose v ft./sec. is the velocity of the water at the time of striking the ball. Then

$$v^2 = 30^2 - 2gh \quad \text{or} \quad v = (900 - 2gh)^{1/2}. \qquad \ldots(1)$$

Since the ball is balanced in the air by the impact of the water on its underside, therefore the force exerted by the water on the ball is equal and opposite to the weight of the ball. Hence the force exerted by the ball on the water is equal to the weight of the ball. Thus the force exerted by the ball on the water is equal to $1 \times g$ *i.e.*, g poundals in the vertically downwards direction.

The impulse of this force over the period of 1 second $= g \times 1$ $= g$ lb.-ft./sec.

Cross-section of the nozzle $= 0.1$ square inch

$$= \frac{1}{10 \times 12 \times 12} \text{ sq. ft.}$$

Density of water $= 62{\cdot}5$ lbs. per cubic foot. Mass of water coming out of the nozzle per second

$=$ volume of water coming out of jet in 1 second $\times$ density of water

$$= \frac{1}{1440} \times 30 \times 62{\cdot}5 \text{ lbs.}$$

This mass of water strikes the ball with velocity v ft./sec. and is reduced to rest.

Change in the momentum of this mass of water on striking the ball $= \frac{1}{1440} \times 30 \times 62{\cdot}5 \times v$ lb.-ft./sec.

By the impulse-momentum principle, we have

impulse of the force for any time $=$ change in the momentum of the mass during that time.

$$g = \frac{1}{1440} \times 30 \times 62{\cdot}5 \times v \quad \text{or} \quad v = \frac{1440g \times 10}{30 \times 625} = \frac{96g}{125}. \qquad \ldots(2)$$

Equating the values of v from (1) and (2), we get

$$(900 - 2gh)^{1/2} = \tfrac{96}{125} g$$

or $$900 - 2gh = \left(\frac{96}{125}\right)^2 g^2 \quad \text{or} \quad h = \frac{900}{2g} - \left(\frac{96}{125}\right)^2 \cdot \frac{g}{2}$$

or $$h = \tfrac{900}{64} - \left(\tfrac{96}{125}\right)^2 \times 16 = 14{\cdot}006 - 9{\cdot}4 = 4{\cdot}6 \text{ approx.}$$

Ex. 30. *Two men, each of mass M, stand on two inelastic platforms each of mass m, hanging over a smooth pulley. One of the men leaping from the ground could raise his centre of gravity through a height h. Show that if he leaps with the same* [illegible] *from the platform, his centre of gravity will rise a height $\frac{M+2m}{2(M+m)} h$. Initially the platforms hang at rest.*

Sol. Let u be the velocity of the man at the time he leaps up from the platform and v the common velocity of the remaining system.

If I be the impulsive force on the man due to which he leaps up with velocity u, we have

I=change in the momentum of the man$=Mu$. ...(1)

Considering the motion of the platform from which the man leaps up and assuming that the impulsive tension is I' in the string, we have

$$I-I'=mv. \quad ...(2)$$

Also considering the motion of the man and the platform at the other end of the string, we have

$$I'=(M+m)\,v. \quad ...(3)$$

Now the energy with which the man jumps up is given equal to Mgh. Since an equal energy is imparted to the system by the sudden pressing of the platform due to the jumping of the man, therefore $\frac{1}{2}mv^2+\frac{1}{2}(m+M)\,v^2+\frac{1}{2}Mu^2=Mgh$

or $$(2m+M)\,v^2+Mu^2=Mgh. \quad ...(4)$$

Eliminating I, I' and v between (1), (2), (3) and (4), we get

$$\frac{u^2}{2g}=\frac{2m+M}{2\,(m+M)}\,h.$$

Now the height through which the man rises while leaping up from the platform with velocity u

$$=\frac{u^2}{2g}=\frac{2m+M}{2\,(m+M)}\,h.$$