Example 19.1. *Reduce $1 - \cos\alpha + i \sin\alpha$ to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$\therefore$ $$r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e., $$r = 2\sin\alpha/2$$

and $$\tan\theta = \frac{\sin\alpha}{1-\cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore \theta = \frac{\pi - \alpha}{2}.$$

Thus $1 - \cos\alpha + i\sin\alpha = 2\sin\dfrac{\alpha}{2}\left[\cos\dfrac{\pi-\alpha}{2} + i\sin\dfrac{\pi-\alpha}{2}\right].$

Example 19.2. *Find the complex number z if arg $(z + 1) = \pi/6$ and arg $(z - 1) = 2\pi/3$.* (*Mumbai*, 2009)

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $\quad arg\,(z + 1) = \pi/6, \quad \therefore \quad \tan^{-1}\left(\dfrac{y}{x+1}\right) = 30°$

i.e., $$\frac{y}{x+1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3y} = x + 1 \quad ...(i)$$

Also since $\quad \arg\,(z - 1) = 2\pi/3, \quad \therefore \tan^{-1}\left(\dfrac{y}{x+1}\right) = 120°$

i.e., $$\frac{y}{x-1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \quad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\quad \sqrt{3}\,y = 1/2 + 1$, *i.e.,* $\quad y = \sqrt{3}/2$

Hence $z = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i.$

Example 19.3. *Find the real values of x, y so that $-3 + ix^2y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that $$z = (x^2 + y) - 4i$$

$\therefore$ $$-3 + ix^2y = x^2 + y - 4i$$

Scanned with CamScanner

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, \; x^2 y = -4$$

Eliminating x, $(y + 3)y = -4$

or $y^2 + 3y - 4 = 0$ *i.e.*, $y = 1$ or -4

When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible

When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1$, $y - 4$ or $x = -1, y = -4$.

Scanned with CamScanner

Example 19.5. *Find the locus of P(z) when*

(*i*) $|z - a| = k$;

(*ii*) *amp* $(z - a) = \alpha$, where k and α are constants.

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(*i*) $|z - a| = k$ means that $AP = k$.

Scanned with CamScanner

Thus the locus of P(z) is a circle whose centre is A(a) and radius k.

(*ii*) amp $(z - a)$, *i.e.*, amp $(\vec{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of P(z) is a straight line through A(a) making an ∠α with OX.

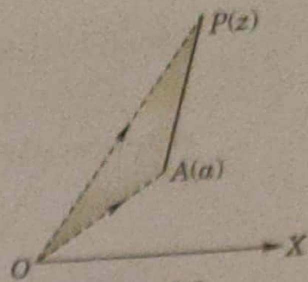

Fig. 19.6

Example 19.6. *Determine the region in the z-plane represented by*
(*i*) $1 < |z + 2i| \le 3$ (*ii*) $R(z) > 3$ (*iii*) $\pi/6 \le amp\ (z) \le \pi/3$.

Solution. (*i*) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

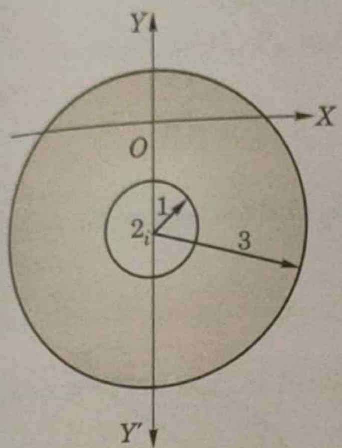

Fig. 19.7

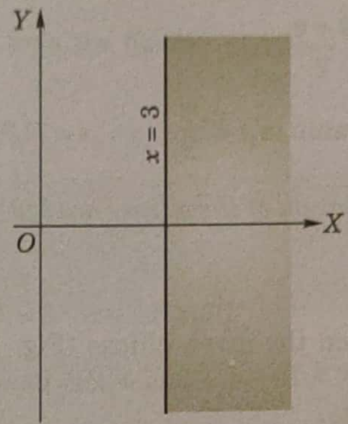

Fig. 19.8

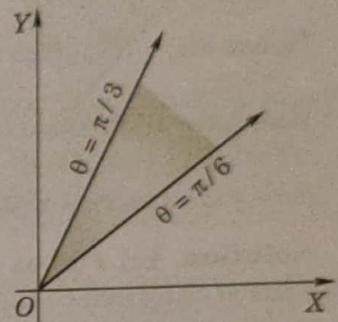

Fig. 19.9

(*ii*) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(*iii*) If $z = r(\cos\theta + i \sin\theta)$, then amp $(z) = \theta$.

$\therefore \pi/6 \le$ amp $(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If z_1, z_2 be any two complex numbers, prove that*
(*i*) $|z_1 + z_2| \le |z_1| + |z_2|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(*ii*) $|z_1 - z_2| \ge |z_1| - |z_2|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and

$$|z_1 + z_2| = OP.$$

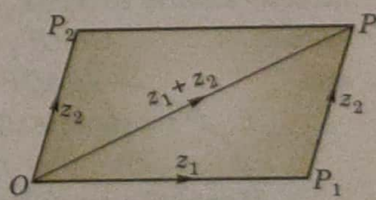

Fig. 19.10

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence $|z_1 + z_2| \le |z_1| + |z_2|$ ...(*i*)

Again $|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2|$ [By (*i*)]

Thus $|z_1 - z_2| \ge |z_1| - |z_2|$ ...(*ii*)

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Scanned with CamScanner

Example 19.8. *If* $|z_1+z_2| = |z_1-z_2|$, *prove that the difference of amplitudes of* z_1 *and* z_2 *is* $\pi/2$. (Mumbai, 2007)

Solution. Let $z_1+z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then
$$2z_1 = r[(\cos\theta+\cos\phi)+i(\sin\theta+\sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}+2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or
$$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2}+i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., } \operatorname{amp}(z_1)=\frac{\theta+\phi}{2} \quad ...(i)$$

Also
$$2z_2 = r(\cos\theta-\cos\phi)+i(\sin\theta-\sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2}+i\cos\frac{\theta+\phi}{2}\right)$$

or
$$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)+i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}$$

i.e.,
$$\operatorname{amp}(z_2)=\frac{\pi}{2}+\frac{\theta+\phi}{2} \quad ...(ii)$$

Hence [(*ii*) – (*i*)], gives $\operatorname{amp}(z_2)-\operatorname{amp}(z_1)=\dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at* z_1, z_2 *and major axis* $2a$, *is* $|z-z_1|+|z-z_2| = 2a$.

Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z-z_1|$ and $S'P = |z-z_2|$.

We know that $SP + S'P = AA'$ $(= 2a)$

i.e.,
$$|z-z_1|+|z-z_2| = 2a$$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or $\left|\vec{OS'}-\vec{OS}\right| = 2ae$ or $|z_2-z_1| = 2ae$

or $|z_1-z_2| = 2ae$ whence $e = |z_1-z_2|/2a$.

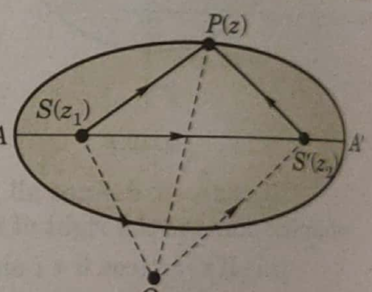

Fig. 19.11

(3) Geometric Representation of z_1z_2. Let P_1, P_2 represent the complex numbers

$$z_1 = x_1+iy_1 = r_1(\cos\theta_1+i\sin\theta_1)$$
and
$$z_2 = x_2+iy_2 = r_2(\cos\theta_2+i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that
$$OP/OP_1 = OP_2/OA \text{ i.e., } OP = OP_1 \cdot OP_2 = r_1r_2$$
and
$$\angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2+\theta_1$$

$\therefore$ P represents the number
$$r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)].$$

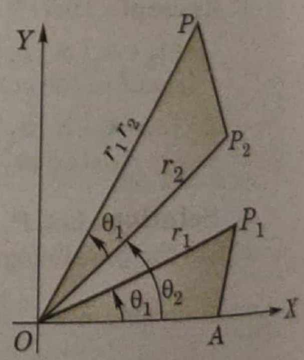

Fig. 19.12

Hence *the product of two complex numbers* z_1, z_2 *is represented by the point* P, *such that* (*i*) $|z_1z_2| = |z_1|\cdot|z_2|$.

(*ii*) $\operatorname{amp}(z_1z_2) = \operatorname{amp}(z_1)+\operatorname{amp}(z_2)$.

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta+i\sin\theta$ is unity.

(4) Geometric representation of z_1/z_2.

Let P_1, P_2 represent the complex numbers

Scanned with CamScanner

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

and $$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad i.e., \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and $$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2) [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

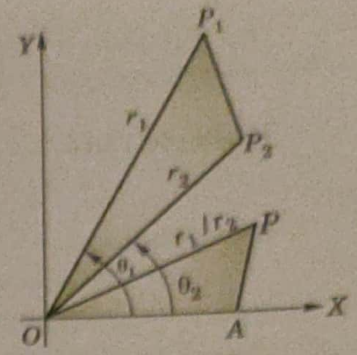

Fig. 19.13

Hence the complex number z_1/z_2 is represented by the point P, such that

(i) $|z_1/z_2| = |z_1|/|z_2|$

(ii) **amp (z_1/z_2) = amp (z_1) – amp (z_2).**

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

$$\overrightarrow{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \overrightarrow{P_2P_3} = z_3 - z_2$$

$$\therefore \quad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\overrightarrow{P_2P_3}}{\overrightarrow{P_2P_1}}\right]$$

$$= \text{amp}(\overrightarrow{P_2P_3}) - \text{amp}(\overrightarrow{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

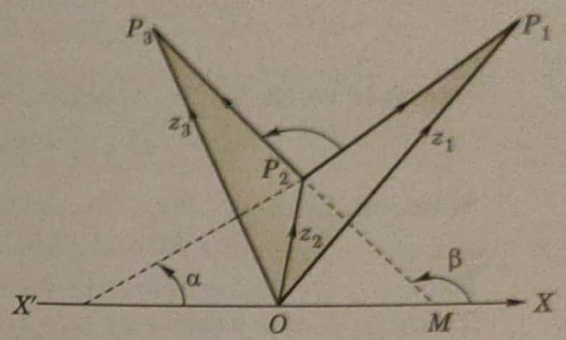

Fig. 19.14

Example 19.10. *Find the locus of the point z, when*

(i) $\left|\frac{z-a}{z-b}\right| = k$ (ii) *amp* $\left(\frac{z-a}{z-b}\right) = \alpha$ *where k and α are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(i) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\frac{z-a}{z-b}\right| = \left|\frac{z-a}{z-b}\right| = \frac{AP}{BP} = k$

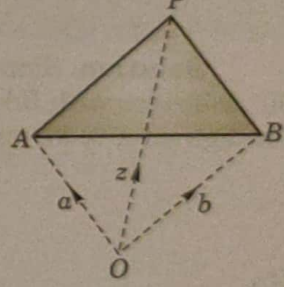

Fig. 19.15

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

Hence the locus of P(z) is a circle (unless k = 1, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

(ii) From the figure 19.16, we have amp $\left(\frac{z-a}{z-b}\right) = \angle APB = \alpha.$

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, $P'(z')$ be a point on the lower arc AB of this circle, then

amp $\left(\frac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

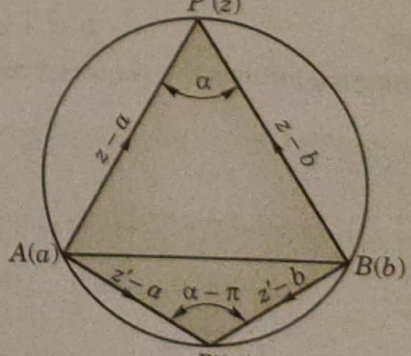

Fig. 19.16

Scanned with CamScanner

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[\,|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of ΔABC such that

$AB = BC$ and $\angle ABC = \pi/2$. (Fig. 19.17)

Then $\quad |z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i \sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i \sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i \sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_3(z_1 + z_2).$$

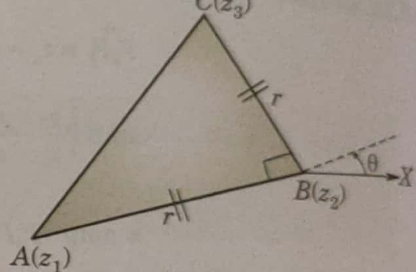

Fig. 19.17

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through $60°$ coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\,\theta$, we multiply it by $\cos\theta + i\sin\theta$.

$$\therefore \quad \vec{BC}\,(\cos \pi/3 + i \sin \pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}\,(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1) = 0$$

whence follows the required condition.

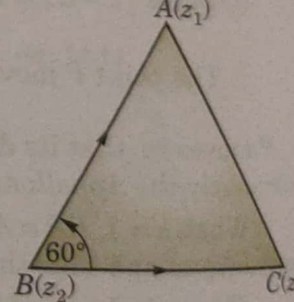

Fig. 19.18

Scanned with CamScanner

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$

$$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$$

$$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos \theta + i \sin \theta)^{12}$$

$$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos \theta + i \sin \theta)^{-20}$$

$\therefore$ The given expression $= \dfrac{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}}{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{-20}} = 1.$

Example 19.15. *Prove that*

$$(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n (\theta/2) . (\cos n\theta/2).$$

Solution. Put $1 + \cos \theta = r \cos \alpha$, $\sin \theta = r \sin \alpha$.

$\therefore$ $r^2 = (1 + \cos \theta)^2 + \sin^2 \theta = 2 + 2 \cos \theta = 4 \cos^2 \theta/2$ *i.e.*, $r = 2 \cos \theta/2$

and $\tan \alpha = \dfrac{\sin \theta}{1 + \cos \theta} = \dfrac{2 \sin \theta/2 . \cos \theta/2}{2 \cos^2 \theta/2} = \tan \theta/2$ *i.e.*, $\alpha = \theta/2$.

$\therefore$ L.H.S. $= [r(\cos \alpha + i \sin \alpha)]^n + [r (\cos \alpha - i \sin \alpha)]^n$

$= r^n[(\cos \alpha + i \sin \alpha)^n + (\cos \alpha - i \sin \alpha)^n] = r^n (\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha)$

$= r^n . 2 \cos n\alpha$ [Substituting the values of r and α]

$= 2^{n+1} \cos^n (\theta/2) \cos (n\theta/2)$.

Example 19.16. *If* $2 \cos \theta = x + \dfrac{1}{x}$, *prove that*

(*i*) $2 \cos r\theta = x^r + 1/x^r$, (*ii*) $\dfrac{x^{2n} + 1}{x^{2n-1} + x} = \dfrac{\cos n\theta}{\cos (n-1)\theta}$

Solution. Since $x + 1/x = 2 \cos \theta$ $\therefore$ $x^2 - 2x \cos \theta + 1 = 0$

whence $x = \dfrac{2 \cos \theta \pm \sqrt{(4 \cos^2 \theta - 4)}}{2} = \cos \theta \pm i \sin \theta.$

(*i*) Taking the + ve sign, $x^r = (\cos \theta + i \sin \theta)^r = \cos r\theta + i \sin r\theta$ (*C.S.V.T.U.*, 2009)

and $x^{-r} = (\cos \theta + i \sin \theta)^{-r} = \cos r\theta - i \sin r\theta$

Adding $x^r + 1/x^r = 2 \cos r\theta$. Similarly with the – ve sign, the same result follows.

Scanned with CamScanner

(ii)
$$\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{(\cos\theta + i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$
$$= \frac{\cos 2n\theta + i\sin 2n\theta + 1}{\cos(2n-1)\theta + i\sin(2n-1)\theta + \cos\theta + i\sin\theta}$$
$$= \frac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta+\cos\theta)+i(\sin\overline{2n-1}\,\theta+\sin\theta)}$$
$$= \frac{2\cos^2 n\theta + 2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta + 2i\sin n\theta\cos\overline{n-1}\,\theta}$$
$$= \frac{\cos n\theta\,(2\cos n\theta + 2i\sin n\theta)}{\cos\overline{n-1}\,\theta\,(2\cos n\theta+2i\sin n\theta)} = \frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, prove that* (i) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\Sigma\sin 2(\alpha+\beta)$

(iv) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$ *(Mumbai, 2009)*

Solution. Let $a=\text{cis}\,\alpha$, $b=\text{cis}\,\beta$ and $c=\text{cis}\,\gamma$.

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$ ...(1)

(i) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$

$$=\sum\frac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}\cdot\frac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$$

$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0$ (Given)

or $bc+ca+ab=0$ ...(2)

$\therefore\ a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$ [By (1) & (2) ...(3)]

or $(\text{cis}\,\alpha)^2+(\text{cis}\,\beta)^2+(\text{cis}\,\gamma)^2=\text{cis}\,2\alpha+\text{cis}\,2\beta+\text{cis}\,2\gamma=0$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$$

(ii) Since $a+b+c=0$, $\therefore\ a^3+b^3+c^3=3abc$

or $(\text{cis}\,\alpha)^3+(\text{cis}\,\beta)^3+(\text{cis}\,\gamma)^3=3\,\text{cis}\,\alpha\,\text{cis}\,\beta\,\text{cis}\,\gamma$

or $\text{cis}\,3\alpha+\text{cis}\,3\beta+\text{cis}\,3\gamma=3\,\text{cis}\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

(iii) From (1), $a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(\text{cis}\,\alpha)^4+(\text{cis}\,\beta)^4+(\text{cis}\,\gamma)^4=2\sum(\cos\alpha)^2(\text{cis}\,\beta)^2$

or $\text{cis}\,4\alpha+\text{cis}\,4\beta+\text{cis}\,4\gamma=2\sum\text{cis}\,2\alpha\,\text{cis}\,2\beta=2\sum\text{cis}\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$$

(iv) From (2), $ab+bc+ca=0$

or $\text{cis}\,\alpha\,\text{cis}\,\beta+\text{cis}\,\beta\,\text{cis}\,\gamma+\text{cis}\,\gamma\,\text{cis}\,\alpha=0$

or $\text{cis}\,(\alpha+\beta)+\text{cis}\,(\beta+\gamma)+\text{cis}\,(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

Scanned with CamScanner

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis } 0)^{1/3} = (\text{cis } 2n\pi)^{1/3} = \text{cis } 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are cis $0 = 1$,

$$\text{cis } 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and $$\text{cis } 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

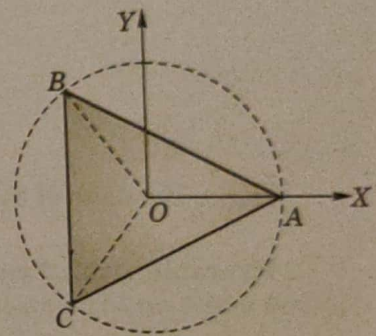

Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example. 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$. (*D.T.U.*, *2013*)

Also show that the continued product of these values is 1. (*Nagpur, 2009*)

Solution. Put $1/2 = r\cos\theta$ and $\sqrt{3}/2 = r\sin\theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis } \pi)^{1/4}$$

$$= [\text{cis } (2n + 1)\pi]^{1/4} = \text{cis } (2n + 1)\,\pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are cis $\pi/4$, cis $3\pi/4$, cis $5\pi/4$ and cis $7\pi/4$.

$$\therefore \quad \text{their continued product} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis } 4\pi = 1.$$

Scanned with CamScanner

Example 19.20. *Use De Moivre's theorem to solve the equation.*

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

(P.T.U., 2005)

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e.,
$$x^5 = -1 = \text{cis}\ \pi = \text{cis}\ (2n+1)\pi$$

$\therefore$
$$x = [\text{cis}\ (2n+1)\pi]^{1/5} = \text{cis}\ (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or
$$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos\pi/5 \pm i\sin\pi/5, \cos 3\pi/5 \pm i\sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i\cot r\pi/n)$, where r has the values 1, 2, 3, ..., n − 1.*

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1$ or $1 - \frac{1}{x} = (1)^{1/n}$

or
$$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis}\ \frac{2r\pi}{n}, r = 0, 1, 2, \dots (n-1). \qquad [\because\ 1 = \text{cis}\ 2r\pi]$$

or
$$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$
$$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \dots (n-1). \qquad [\because\ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i\cot r\pi/n)$ where $r = 1, 2, 3, \dots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $\quad (1)^{1/7} = (\cos 2r\pi + i\sin 2r\pi)^{1/7} = \text{cis}\ \frac{2r\pi}{7} = \left(\text{cis}\ \frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \dots + \rho^{6n}$ ...(i)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis}\ 2\pi)^n = 1$.

i.e.,
$$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of 7 $= 7p$ (say)

From (i), $\quad S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \dots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$

Example 19.23. *Find the equation whose roots are 2 cos π/7, 2 cos 3π/7, 2 cos 5π/7.*

Solution. Let $y = \cos\theta + i\sin\theta$, where $\theta = \pi/7, 3\pi/7, \dots, 13\pi/7$.

Scanned with CamScanner

Then $y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1$ or $y^7 + 1 = 0$

or $(y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0$ ...(*i*)

Its roots are $y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\{(y + 1/y)^3 - 3(y + 1/y)\} - \{(y + 1/y)^2 - 2\} - (y + 1/y) - 1 = 0$

or $x^3 - x^2 - 2x + 1 = 0$ ...(*ii*)

where $x = y + 1/y = 2\cos\theta$.

Now since $\cos 13\pi/7 = \cos \pi/7$, $\cos 11\pi/7 = \cos 3\pi/7$, $\cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\frac{\pi}{7}, 2\cos\frac{3\pi}{7}, 2\cos\frac{5\pi}{7}$.

Scanned with CamScanner

Example 19.24. *Express cos 6θ in terms of cos θ.* (Madras, 2002)

Solution. We know that $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6\theta - {}^6C_2 \cos^4\theta \sin^2\theta + {}^6C_4 \cos^2\theta \sin^4\theta - {}^6C_6 \sin^6\theta$

$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$

$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$

(2) Addition formulae for any number of angles

We have, $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1+i\tan\theta_1)$, $\cos\theta_2 + i\sin\theta_2 = \cos\theta_2(1+i\tan\theta_2)$ and so on.

$\therefore$ $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$

$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots(1+i\tan\theta_n)$

$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n[1 + i(\tan\theta_1+\tan\theta_2+\ldots+\tan\theta_n) + i^2(\tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3+\ldots) + i^3(\tan\theta_1\tan\theta_2\tan\theta_3+\ldots)+\ldots+\ldots]$

$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 + is_1 - s_2 - is_3 + s_4 + \ldots)$

where $s_1 = \tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_n$, $s_2 = \Sigma\tan\theta_1\tan\theta_2$, $s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3$ etc.

Equating real and imaginary parts, we have

$$\cos(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1 - s_2 + s_4 - \ldots)$$

$$\sin(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1 - s_3 + s_5 - \ldots)$$

and by division, we get $\mathbf{\tan(\theta_1+\theta_2+\ldots+\theta_n) = \dfrac{s_1 - s_3 + s_5 - \ldots}{1 - s_2 + s_4 - s_6 + \ldots}}$.

Example 19.25. *If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, show that $xy + yz + zx = 1$.* (P.T.U., 2003)

Solution. Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$, $\tan^{-1}z = \gamma$ so that $x = \tan\alpha$, $y = \tan\beta$, $z = \tan\gamma$

We know that $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$$

$\therefore$ $$\tan\pi/2 = \frac{x+y+z-xyz}{1-xy-yz-zx} \quad \text{or} \quad 1 - xy - yz - zx = 0$$

Hence $xy + yz + zx = 1$.

Example 19.26. *If $\theta_1, \theta_2, \theta_3$ be three values of θ which satisfy the equation $\tan 2\theta = \lambda\tan(\theta+\alpha)$ and such that no two of them differ by a multiple of π, show that $\theta_1+\theta_2+\theta_3+\alpha$ is a multiple of π.*

Solution. Given equation can be written as $\dfrac{2t}{1-t^2} = \lambda\dfrac{t+\tan\alpha}{1-t\,.\tan\alpha}$ where $t = \tan\theta$

or $$\lambda t^3 + (\lambda-2)\tan\alpha\,.\,t^2 + (2-\lambda)t - \lambda\tan\alpha = 0$$

$\therefore$ $\tan\theta_1, \tan\theta_2, \tan\theta_3$, being its roots, we have [By § 1.3]

$$s_1 = \Sigma\tan\theta_1 = -\frac{\lambda-2}{\lambda}\tan\alpha$$

$$s_2 = \Sigma\tan\theta_1\tan\theta_2 = \frac{2-\lambda}{\lambda} \quad \text{and} \quad s_3 = \tan\alpha$$

$\therefore$ $$\tan(\theta_1+\theta_2+\theta_3) = \frac{s_1 - s_3}{1 - s_2} = \frac{(-1+2/\lambda)\tan\alpha - \tan\alpha}{1-(2/\lambda-1)}$$

$$= -\tan\alpha = \tan(n\pi - \alpha)$$

Thus $\theta_1+\theta_2+\theta_3 = n\pi - \alpha$, whence follows the result.

(3) To expand $\sin^m\theta$, $\cos^n\theta$ or $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ

If $z = \cos\theta + i\sin\theta$ then $1/z = \cos\theta - i\sin\theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i\sin p\theta$ and $1/z^p = \cos p\theta - i\sin p\theta$

$\therefore$ $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$; $z^p + 1/z^p = 2\cos p\theta$, $z^p - 1/z^p = 2i\sin p\theta$

Scanned with CamScanner

These results are used to expand the powers of sin θ or cos θ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8\theta$ in a series of cosines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$, so that $z + 1/z = 2\cos\theta$ and $z^p + 1/z^p = 2\cos p\theta$.

$\therefore$ $(2\cos\theta)^8 = (z + 1/z)^8$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2\cos 8\theta) + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70.$$

Hence $$\cos^8\theta = \frac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35].$$

Example 19.28. *Expand $\sin^7\theta\cos^3\theta$ in a series of sines of multiples of θ.*

Solution. Let $z = \cos\theta + i\sin\theta$

so that $z + 1/z = 2\cos\theta,\ z - 1/z = 2i\sin\theta$ and $z^p - 1/z^p = 2i\sin p\theta$.

$\therefore$ $(2i\sin\theta)^7(2\cos\theta)^3 = (z - 1/z)^7(z + 1/z)^3$

$$= (z - 1/z)^4[(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4(z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i\sin 10\theta - 4(2i\sin 8\theta) + 3(2i\sin 6\theta) + 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$$

Since $i^7 = -i$,

$\therefore$ $$\sin^7\theta\cos^3\theta = -\frac{1}{2^9}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta].$$

Obs. The expansion of $\sin^m\theta\cos^n\theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

Scanned with CamScanner