# IFS MATHEMATICS MAIN EXAM., 2014

## PAPER-I

**Instructions:** *Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols and notations carry their usual meaning.*

## Section-A

**1.** (*a*) Show that $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for $R^3$. Express $(5, 3, 4)$ in terms of $u_1$, $u_2$ and $u_3$. **8**

(*b*) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that

$A^n = A^{n-2} + A^2 - 1, n \geq 3.$ **8**

(*c*) Show that the function given by

$$f(x) = \begin{cases} \dfrac{x\left(e^{1/x} - 1\right)}{\left(e^{1/x} + 1\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. **8**

(*d*) Evaluate $\iint\limits_R y \dfrac{\sin x}{x} dx\, dy$ over R, where

$R = \{(x, y) : y \leq x \leq \pi/2, 0 \leq y \leq \pi/2\}$. **8**

(*e*) Prove that the locus of a variable line which intersects the three lines:

$y = mx, z = c; y = -mx, z = -c; y = z, mx = -c$

is the surface $y^2 - m^2x^2 = z^2 - c^2$. **8**

**2.** (*a*) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over:

(*i*) the real field $\mathcal{R}$

(*ii*) the complex field C. **10**

(*b*) If $xyz = a^3$, then show that the minimum value of $x^2 + y^2 + z^2$ is $3a^2$. **10**

(*c*) Prove that every sphere passing through the circle $x^2 + y^2 - 2ax + r^2 = 0, z = 0$ cut orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$. **10**

(*d*) Show that the mapping $T : V_2(\bar{R}) \to V_3(\bar{R})$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation. Find the range, rank and nullity of T. **10**

**3.** (*a*) Examine whether the matrix

$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find

all eigen values. Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix. **10**

(*b*) A moving plane passes through a fixed point (2, 2, 2) and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C. **10**

(*c*) Evaluate the integral

$$I = \int_0^{\infty} 2^{-ax^2} dx$$

using Gamma function. **10**

(*d*) Prove that the equation:
$4x^2 - y^2 + z^2 - 3yz + 2xy + 12x - 11y + 6z + 4 = 0$ represents a cone with vertex at $(-1, -2, -3)$. 10

4. (*a*) Let $f$ be a real valued function defined on $[0, 1]$ as follows:

$$f(x) = \begin{cases} \frac{1}{a^{r-1}}, & \frac{1}{a^r} < x \le \frac{1}{a^{r-1}}, r = 1,2,3,... \\ 0 & x = 0 \end{cases}$$

where $a$ is an integer greater than 2. Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$. 10

(*b*) Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. 10

(*c*) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0 \le x \le \frac{y^2}{2}$ and $0 \le y \le 2$. 10

(*d*) Consider the linear mapping $F : R^2 \to R^2$ given as $F(x, y) = (3x + 4y, 2x - 5y)$ with usual basis.

Find the matrix associated with the linear transformation relative to the basis $S = \{u_1, u_2\}$ where $u_1 = (1, 2)$, $u_2 = (2, 3)$. 10

## Section-B

5. (*a*) Solve the differential equation:

$$y = 2px + p^2 y,\ p = \frac{dy}{dx}$$

and obtain the non-singular solution. 8

(*b*) Solve:

$$\frac{d^4y}{dx^4} - 16y = x^4 + \sin x.$$ 8

(*c*) A particle whose mass is $m$, is acted upon by a force $m\mu\left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance '$a$' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$. 8

(*d*) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12 : \sqrt{73}$. 8

(*e*) For three vectors show that:

$$\bar{a}\times(\bar{b}\times\bar{c}) + \bar{b}\times(\bar{c}\times\bar{a}) + \bar{c}\times(\bar{a}\times\bar{b}) = 0.$$ 8

6. (*a*) Solve the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x\tan\frac{y}{x^2}.$$ 10

(*b*) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2 - \frac{1}{2}\right)$. 10

(*c*) Solve by the method of variation of parameters:

$y'' + 3y' + 2y = x + \cos x$. 10

(*d*) For the vector $\bar{A} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine $\phi$ such that $\bar{A} = \nabla\phi$. 10

7. (*a*) A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$ radius of hemisphere. 15

(d) Write a BASIC program to sum the series $S = 1 + x + x^2 + ... + x^n$, for $n = 30, 60$ and $90$ for the values of $x = 0.1\ (0.1)\ 0.3$. **10**

7. (a) Solve:

$$(D - 3D' - 2)^2 z = 2e^{2x}\cot(y + 3x)$$ **10**

(b) Solve the following system of equations:

$$2x_1 + x_2 + x_3 - 2x_4 = -10$$
$$4x_1 + 2x_3 + x_4 = 8$$
$$3x_1 + 2x_2 + 2x_3 = 7$$
$$x_1 + 3x_2 + 2x_3 - x_4 = -5$$

(c) A uniform rod OA of length $2a$ is free to turn about its end O, revolves with uniform angular velocity $\omega$ about a vertical axis OZ through O and is inclined at a constant angle $\alpha$ to OZ. Show that the value of $\alpha$ is either zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$ **15**

8. (a) Using Runge-Kutta 4th order method, find $y$ from

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

with $y(0) = 1$ at $x = 0.2, 0.4$. **10**

(b) A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle $\alpha$ to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M + M')g\sin\alpha}}$$

where $\alpha$ is the length of the plank. **15**

(c) Prove that

$$\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$$

is a possible form for the bounding surface of a liquid and find the velocity components. **15**

# ANSWERS

## PAPER-I

### Section-A

1. (a) $x \cdot u_1 + y \cdot u_2 + z \cdot u_3 = (5, 3, 4)$

$x(1, -1, 0) + y(1, 1, 0) + z(0, 1, 1) = (5, 3, 4)$

$x + y = 5, \quad -x + y + z = 3$ and $z = 4$.

$\therefore x + y = 5$ and $-x + y + 4 = 3$

so, $x - y = 1$

From, $x + y = 5$ and $x - y = 1$

$x = 3, y = 2$ and $z = 4$

$\therefore u_1 = 3(1, -1, 0) = (3, -3, 0)$

$u_2 = 2(1, 1, 0) = (2, 2, 0)$

$u_3 = 4(0, 1, 1) = (0, 4, 4)$

1. (b)
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^5 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

for $n$ = even number

$$A^n = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{2} & 1 & 0 \\ \frac{n}{2} & 0 & 1 \end{bmatrix}$$

and, for $n$ = odd number

$$A^n = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ \frac{n+1}{2} & 0 & 1 \\ \frac{n-1}{2} & 1 & 0 \end{bmatrix}$$

$$A^n = A^{n-2} + A^2 - I$$

For $\quad n$ = even

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{2} & 1 & 0 \\ \frac{n}{2} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{n-2}{2} & 1 & 0 \\ \frac{n-2}{2} & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{n}{2} & 1 & 0 \\ \frac{n}{2} & 0 & 1 \end{bmatrix}$$

Again for $n$ = odd

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{n+1}{2} & 0 & 1 \\ \frac{n-1}{2} & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{n-1}{2} & 0 & 1 \\ \frac{n-3}{2} & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{n+1}{2} & 0 & 1 \\ \frac{n-1}{2} & 1 & 0 \end{bmatrix}$$

Hence, it is **proved.**

**1.** (*c*) Check for continuity at $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\lim_{x\to 0^-} \frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)} = \lim_{x\to 0^+} \frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)}$$

$$\frac{-x\left(e^{-\frac{1}{x}}-1\right)}{\left(e^{-\frac{1}{x}}+1\right)} = \frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)}$$

$$\frac{-x\left(1-e^{\frac{1}{x}}\right)}{\left(1+e^{\frac{1}{x}}\right)} = \frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)}$$

$$\frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)} = \frac{x\left(e^{\frac{1}{x}}-1\right)}{\left(e^{\frac{1}{x}}+1\right)}$$

So, $\quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.
But at $x = 0$, $f(x)$ is not definded.
Hence, it is not differentiable at $x = 0$.

1. (*d*) $\iint_R y \cdot \frac{\sin x}{x} \cdot dx \cdot dy$

$$= \int_0^{\pi/2} y \cdot dy \int_y^{\pi/2} \frac{\sin x}{x} dx$$

$$= \int_0^{\pi/2} y \cdot dy \cdot [\sin x]_y^{\pi/2}$$

$$= \int_0^{\pi/2} y \cdot dy \cdot \left[\sin\frac{\pi}{2} - \sin y\right]_y$$

$$= \int_0^{\pi/2} y \cdot dy \cdot (1 - \sin y)$$

$$= \int_0^{\pi/2} y(1 - \sin y) \cdot dy$$

$$= \left[\frac{y^2}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} y \cdot \sin y \cdot dy$$

$$= \frac{\pi^2}{8} + [y.\cos y - \sin y]_0^{\pi/2}$$

$$= \frac{\pi^2}{8} - 1$$

1. (*e*) The given lines are

$$y - mx = 0,\ z - c = 0 \quad \text{...}(i)$$

$$y + mx = 0,\ z + c = 0 \quad \text{...}(ii)$$

and $\quad y - z = 0,\ mx + c = 0 \quad \text{...}(iii)$

Any line intersecting (*i*) and (*ii*) is

$$y - mx - k_1 (z - c) = 0,\ y + mx - k_2 (z + c) = 0 \quad \text{...}(iv)$$

If it intersects (*iii*) also, we have to eliminate $x, y, z$ from (*iii*) and (*iv*).

Now putting $y = z$,

and $mx = -c$ from (*iii*) in (*iv*), we get

$$z + c - k_1(z - c) = 0,$$

$$z - c - k_2(z + c) = 0$$

or $\quad z(1 - k_1) + c(1 + k_1) = 0$

$$z(1 - k_2) - c(1 + k_2) = 0.$$

Equating the two values of $z$, we get

$$\frac{c(1+k_1)}{k_1 - 1} = \frac{c(1+k_2)}{1-k_2} (= z)$$

or $\quad (1 + k_1)(1 - k_2) = (1 + k_2)(k_1 - 1)$

or $\quad 1 + k_1 - k_2 - k_1k_2 = k_1 + k_1k_2 - 1 - k_2$

or $\quad 2k_1k_2 - 2 = 0$

or $\quad k_1k_2 = 1 \quad \text{...}(v)$

To find the locus, we have to eliminate $k_1$, $k_2$ from (*iv*) and (*v*).

From (*iv*)

$$k_1 = \frac{y - mx}{z - c}$$

$$k_2 = \frac{y + mx}{z + c}$$

Putting these values in (*v*), we get

$$\left(\frac{y - mx}{z - c}\right)\left(\frac{y + mx}{z + c}\right) = 1$$

or $\quad \frac{y^2 - m^2x^2}{z^2 - c^2} = 1$

or $\quad y^2 - m^2x^2 = z^2 - c^2$

which is the required locus.

2. (*b*) Here, we want to find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ with the constraint function $g(x, y, z) = xyz - a^3 = 0$. Using Lagrange's multiplier, the auxiliary equations are

$$f_x = \lambda g_x,\ f_y = \lambda g_y,\ f_z = \lambda g_z,\ g(x, y, z) = 0$$

$\Rightarrow \quad 2x = \lambda yz \quad$ or $\quad \lambda xyz = 2x^2 \quad \text{...}(i)$

$2y = \lambda xz \quad$ or $\quad \lambda xyz = 2y^2 \quad \text{...}(ii)$

$2z = \lambda xy \quad$ or $\quad \lambda xyz = 2z^2 \quad \text{...}(iii)$

$xyz - a^3 = 0 \quad$ or $\quad xyz = a^3 \quad \text{...}(iv)$

$x^2 = y^2 = z^2$ and $\quad xyz = a^3 \quad \text{...}(v)$

Therefore, the solutions of equation (*v*) are A($a, a, a$), B($a, -a, -a$), C($-a, -a, a$), D($-a, a, -a$), *i.e.*, the points A, B, C, D are those of minima and the value of $f(x, y, z)$ at these points is $3a^2$. Hence the minimum value is $3a^2$.

**2.** (*d*) $T(a, b) = (a+b, a-b, b), \forall\ a, b \in R$

Let $\alpha_1 = (a_1, b_1)$ and $\alpha_2 = (a_2, b_2)$ be any two elements of $V_2$ (R) then

$$\left.\begin{array}{l} T(\alpha_1) = T(a_1, b_1) = (a_1 + b_1, a_1 - b_1, b_1) \\ \text{and } T(\alpha_2) = T(a_2, b_2) = (a_2 + b_2, a_2 - b_2, b_2) \end{array}\right\} \quad ...(i)$$

Now $a, b \in R \Rightarrow a\alpha_1 + b\alpha_2 \in V_2$ (R)

$\therefore T(a\,\alpha_1 + b\,\alpha_2) = T[a(a_1, b_1) + b(a_2, b_2)]$

$= T[aa_1 + ba_2, ab_1 + bb_2]$

$= (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2)$

(by def. of T)

$= [a(a_1 + b_1) + b(a_2 + b_2), a(a_1 - b_1) + b(a_2 - b_2), ab_1 + bb_2]$

$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$

$= a\,T(\alpha_1) + b\,T(\alpha_2),$ [from (*i*)]

showing that $T : V_2 (R) \to V_3 (R)$ as defined in (*i*) is a linear transformation.

Null space of T.

If $\alpha = (a, b)$, then

$N(T) = \{\alpha \in V_2 (R); T(\alpha) = 0 \in V_3 (R)\}$

Now $T(\alpha) = T(a, b) = (a+b, a-b, b) = (0, 0, 0)$

$\Rightarrow a + b = 0, a - b = 0, b = 0$

$\Rightarrow a = 0, b = 0.$

$\therefore \alpha = (a, b) = (0, 0) \in N(T),$

showing that null space consists of only zero vector of $V_2$ (R) *i.e.* domain or in other words null space of T is the zero subspace of $V_2$ (R) *i.e.* nullity of T = dim [N (T)] = 0.

Range space of T.

We have

$R(T) = \{\beta \in V_3 (R) : \beta = T(\alpha), \alpha \in V_2 (R)\}$

Now $\{(1, 0), (0, 1)\}$ is the basis of $V_2$(R). Also $T(1, 0) = (1 + 0, 1 - 0, 0) = (1, 1, 0)$, and $T(0, 1) = (0 + 1, 0 - 1, 1) = (1, -1, 1)$.

Hence the range space of T is a sub-space of $V_3$ (R) generated by (1, 1, 0) and (1, –1, 0).

Now $a(1, 1, 0) + b(1, -1, 1) = (0, 0, 0) \ \forall\ a, b \in R$

$\Rightarrow (a + b, a - b, b) = (0, 0, 0)$

$\Rightarrow a + b = 0, a - b = 0, b = 0 \Rightarrow a = 0, b = 0.$

Therefore (1, 1, 0), (1, –1, 1); elements of R (T) are L.I. and generates R (T).

Hence,

$\{(1, 1, 0), (1, -1, 1)\}$ is the basis of R(T)

Therefore dim (R(T)) = rank (T) = 2.

**3.** (*a*) Let the variable plane passing through (*a*, *b*, *c*) meets X-axis at A, Y-axis at B and Z-axis at C.

Let OA = *p*, OB = *q*, OC = *r*

Coordinates of A are (*p*, 0, 0), B are (0, *q*, 0) and C are (0, 0, *r*).

Centre of the sphere OABC is $\left(\frac{p}{2}, \frac{q}{2}, \frac{r}{2}\right)$

Equation of the plane ABC is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$

Thus plane passes through P (*a*, *b*, *c*)

$$\therefore \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 1$$

$$\Rightarrow \frac{a}{2\left(\frac{p}{2}\right)} + \frac{b}{2\left(\frac{q}{2}\right)} + \frac{c}{2\left(\frac{r}{2}\right)} = 1$$

$$\Rightarrow \frac{a}{\left(\frac{p}{2}\right)} + \frac{b}{\left(\frac{q}{2}\right)} + \frac{c}{\left(\frac{r}{2}\right)} = 2$$

The centre of the sphere $\left(\frac{p}{2}, \frac{q}{2}, \frac{r}{2}\right)$ lies on the surface $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$

Here (*a*, *b*, *c*) = (2, 2, 2)

Hence, centre of the sphere lies on the surface

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1.$$

3.(*d*) The given equation is

$$4x^2 - y^2 + z^2 - 3yz + 2xy + 12x - 11y + 6z + 4 = 0 \quad ...(i)$$

Making it homogeneous in $x, y, z, t$ by introducing proper powers of $t$, we have,

$$F(x, y, z, t) = 4x^2 - y^2 + z^2 + 2xy - 3yz + 12x.t - 11y.t + 6z.t + 4t^2 = 0$$

$$\therefore \quad \frac{\partial F}{\partial x} = 8x + 2y + 12t$$

$$= 8x + 2y + 12 \quad [\because t = 1]$$

$$\frac{\partial F}{\partial y} = -2y + 2x - 3z - 11t$$

$$= -2y + 2x - 3z - 11 \quad [\because t = 1]$$

$$\frac{\partial F}{\partial z} = 2z - 3y + 6t$$

$$= 2z - 3y + 6 \quad [\because t = 1]$$

and $$\frac{\partial F}{\partial t} = 12x - 11y + 6z + 8t$$

$$= 12x - 11y + 6z + 8 \quad [\because t = 1]$$

Putting $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial t}$ each equal to zero. We have,

$$8x + 2y + 12 = 0$$

$$\Rightarrow \quad 4x + y + 6 = 0 \quad ...(ii)$$

$$-2y + 2x - 3z - 11 = 0 \quad ...(iii)$$

$$4z - 3y + 6 = 0 \quad ...(iv)$$

and $$12x - 11y + 6z + 8 = 0 \quad ...(v)$$

Multiply (*ii*) by 2 and add with (*iii*)

$$10x - 3z + 1 = 0 \quad ...(vi)$$

From (*ii*) and (*iv*)

$$\begin{array}{r} 12x + 3y + 18 = 0 \\ 4z - 3y + 6 = 0 \\ \hline 12x + 4z + 24 = 0 \end{array}$$

$$3x + z + 6 = 0 \quad ...(vii)$$

From (*vi*) and (*vii*), we get

$$19x + 19 = 0 \quad \Rightarrow x = -1.$$

and $$3(-1) + z + 6 = 0 \quad \Rightarrow z = -3$$

and from $4z - 3y + 6 = 0$

$$4(-3) - 3y + 6 = 0 \quad \Rightarrow y = -2$$

Now, putting $(x, y, z) = (-1, -2, -3)$ into (*v*), we get

$$12(-1) - 11(-2) + 6(-3) + 8 = 0$$

$$30 - 30 = 0$$

Hence, vertex is at $(-1, -2, -3)$

4.(*b*) Let $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$ be the line of section.

$\therefore$ $l, m, n$ satisfies equation of cone and plane.

$$\therefore \quad lm + mn + ln = 0$$

and $$al + bm + cn = 0 \quad ...(i)$$

$$\Rightarrow \quad n = \frac{al + bm}{-c}$$

Substituting we get

$$lm - \frac{m}{c}(al + bm) - \frac{l}{c}(al + bm) = 0$$

$$\therefore \quad al^2 + lm\,(a + b - c) + bm^2 = 0$$

$$\therefore \quad a\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + b = 0$$

which is a quadratic in $\frac{l}{m}$

Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ be the two roots of this equation.

$\therefore$ Product of the roots $= \frac{b}{a}$

i.e. $$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

$$\therefore \quad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

by symmetry

$\therefore \perp$ if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$

DATE

2(a) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors of $B$ viewed as a matrix over:

i) the real field $R$

ii) the complex field $C$. (10)

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda+1) + 2 = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = i, -i$$

Let $v$ is the eigen-vector

$$Bv = \lambda v \Rightarrow (B - \lambda I)v = 0$$

$$\lambda = i \Rightarrow \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)x - y = 0$$
$$2x - (1+i)y = 0 \Rightarrow y = (1-i)x$$

$$\therefore v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1-i)x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

$\lambda = -i$

$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1+i)x - y = 0$$
$$2x - y(1-i) = 0 \quad \Rightarrow \quad y = (1+i)x$$

$$v = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1+i)x \end{bmatrix} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} x$$

When B is viewed as matrix ~~over complex~~ ~~ov~~ over the complex field, then eigen vectors are

$$\begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

When B is viewed as matrix over the real field the eigen-vectors can be taken as

$$v = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{for } \lambda = i \text{ ie } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\left( a+ib \rightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right)$$

$$v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{for } \lambda = -i \text{ ie } \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

verification

AS $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

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2(c) Prove that every sphere passing through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ cut orthogonally every sphere through the circle $x^2+z^2=r^2,\ y=0$. (10)

Equations of two spheres can be taken as

$$S_1 : x^2+y^2+z^2-2ax+r^2+\lambda z=0$$

$$S_2 : x^2+y^2+z^2-r^2+\mu y=0$$

Condition of orthogonality –

$$2(u_1u_2+v_1v_2+w_1w_2)=d_1+d_2$$

$$2\left[a\cdot 0+0\cdot\left(-\frac{\mu}{2}\right)+\left(-\frac{\lambda}{2}\right)\cdot 0\right]=r^2+(-r)^2$$

$$2(0+0+0)=0$$

$$0=0$$

which is true for all values of parameters $\lambda$ and $\mu$.

Hence proved.

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3(b) A moving plane passes through a fixed point (2, 2, 2) and meets the coordinate axes at the points A, B, C, all away from origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C. (10).

Let eqn of plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\therefore$ $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

and $\frac{2}{a}+\frac{2}{b}+\frac{2}{c}=1$ —(1)

Let general eqn of sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$O(0,0,0)$ lies on it $\Rightarrow$ $d=0$

$A(a,0,0)$ lies on it $\Rightarrow$ $a^2+2ua=0$

$\Rightarrow$ $2u=-a$

$B(0,b,0)$ gives $2v=b$

$C(0,0,c)$ gives $2w=c$

$\therefore$ $x^2+y^2+z^2-ax-by-cz=0$

Centre $x_1=\frac{a}{2}$, $y_1=\frac{b}{2}$, $z_1=\frac{c}{2}$

Using (1) $\frac{2}{2x_1}+\frac{2}{2y_1}+\frac{2}{2z_1}=\cancel{a}\,1$

$\Rightarrow$ $\boxed{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1}$ is the required locus.

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3(c) Evaluate the integral

$$I = \int_0^\infty 2^{-ax^2}\, dx$$

using gamma function. (10).

$$I = \int_0^\infty 2^{-ax^2}\, dx$$

$$= \int_0^\infty e^{-ax^2(\log 2)}\, dx \qquad \left[t = e^{\log t}\right]$$

Let $a(\log 2)x^2 = y$

$a(\log 2)\, 2x\, dx = dy$

$$I = \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)\, x}$$

$$= \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)\sqrt{y}} \times \sqrt{a\log 2}$$

$$= \frac{1}{2\sqrt{a\log 2}} \int_0^\infty e^{-y} \cdot y^{-1/2}\, dy$$

$$= \frac{1}{2\sqrt{a\log 2}} \cdot \Gamma\left(\frac{-1}{2}+1\right) = \frac{\sqrt{\pi}}{2\sqrt{a\log 2}}.$$

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4(a) Let $f$ be a real valued function defined on $[0,1]$ as follows:

$$f(x) = \begin{cases} \frac{1}{a^{r-1}}, & \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}},\ r = 1,2,3,\ldots \\ 0, & x = 0. \end{cases}$$

where 'a' is an integer greater than 2. Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$. (10).

$$f(x) = \begin{cases} \frac{1}{a^{1-1}} = 1, & \frac{1}{a} < x \leq 1 \\ \frac{1}{a}, & \frac{1}{a^2} < x \leq \frac{1}{a} \\ \frac{1}{a^2}, & \frac{1}{a^3} < x \leq \frac{1}{a^2} \\ \vdots \\ \frac{1}{a^{r-1}}, & \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \\ \vdots \\ 0, & x = 0. \end{cases}$$

Clearly $f(x) \in [0,1]$ for all $x \in [0,1]$ as $a > 2$

$\Rightarrow f$ is bounded on $[0,1]$

Also, it is continuous on $[0,1]$ except at points $0, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \ldots$

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4(b)

The set of points of discontinuities has only one limit point 0, hence $f$ is Integrable on $[0,1]$.

$$\int_{\frac{1}{a^r}}^{1} f(x)\,dx = \int_{1/a}^{1} f\,dx + \int_{1/a^2}^{1/a} f\,dx + \dots + \int_{1/a^r}^{1/a^{r-1}} f\,dx$$

$$= \int_{1/a}^{1} 1\cdot dx + \int_{1/a^2}^{1/a} \frac{1}{a}\,dx + \int_{1/a^3}^{1/a^2} \frac{1}{a^2}\,dx + \dots$$

$$\dots + \int_{1/a^r}^{1/a^{r-1}} \frac{1}{a^{r-1}}\,dx$$

$$= \left(1-\frac{1}{a}\right) + \left(\frac{1}{a}-\frac{1}{a^2}\right)\frac{1}{a} + \left(\frac{1}{a^2}-\frac{1}{a^3}\right)\frac{1}{a^2} + \dots$$

$$\dots + \left(\frac{1}{a^{r-1}}-\frac{1}{a^r}\right)\cdot\frac{1}{a^{r-1}}$$

$$= \left(1-\frac{1}{a}\right)\left[1+\frac{1}{a^2}+\frac{1}{a^4}+\dots+\frac{1}{a^{2(r-1)}}\right]$$

$$= \frac{a-1}{a} \times \frac{1\left[1-\left(1/a^2\right)^r\right]}{1-\frac{1}{a^2}} \qquad \left(S_n = \frac{a(1-r^n)}{1-r}\right)$$

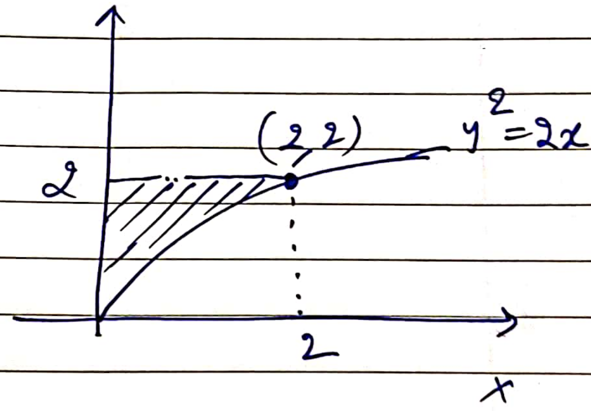
$$= \frac{a}{a+1}\cdot\left(1-\frac{1}{a^{2r}}\right)$$

Taking limit $r \to \infty$

$$\boxed{\int_0^1 f(x)\,dx = \frac{a}{a+1}}$$

4(b) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0 \le x \le \frac{y^2}{2}$ and $0 \le y \le 2$. (10).

$$I = \iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$$




$$= \int_{x=0}^{2}\int_{y=\sqrt{2x}}^{2} \frac{2y}{2\sqrt{1+x^2+y^2}}\,dy\,dx$$

$$= \int_{x=0}^{2} \left(1+x^2+y^2\right)^{1/2}\Big|_{y=\sqrt{2x}}^{y=2}\,dx$$

$$= \int_{x=0}^{2}\left(\sqrt{1+4+x^2} - \sqrt{1+x^2+2x}\right)dx$$

$$= \int_{0}^{2} \sqrt{5+x^2} - (1+x)\,dx$$

$$= \left[\frac{1}{2}x\sqrt{5+x^2} + \frac{5}{2}\log\left|x+\sqrt{5+x^2}\right|\right]_0^2 - \left[x+\frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{\cancel{2}\times\sqrt{9}}{\cancel{2}} + \frac{5}{2}\log(2+\sqrt{9}) - \frac{5}{2}\log\sqrt{5}\right) - \left(2+\frac{4}{2}\right)$$

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$$\boxed{I = \frac{5}{4}\log 5 - 1}$$

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