3 Real Sequences

In this chapter we shall study a special class of functions whose domain is the set **N** of natural numbers and range a set of real numbers—the *Real Sequences*.

1. SEQUENCES

A function whose domain is the set **N** of natural numbers and range, a set of real numbers is called a *real sequence*. Thus a real sequence is denoted symbolically as $S: \mathbf{N} \to \mathbf{R}$.

Since we shall be dealing with real sequences only, we shall use the term *sequence* to denote a *Real sequence*.

NOTATION: Since the domain for a sequence is always **N**, a sequence is specified by the values S_n, $n \in \mathbf{N}$. Thus a sequence may be denoted as

$$\{S_n\}, n \in \mathbf{N} \text{ or } \{S_1, S_2, S_3, \ldots S_n, \ldots\}$$

The values $S_1, S_2, S_3, \ldots$ are called the first, second, third ... terms of the sequence. The mth and nth terms S_m and S_n for $m \neq n$ are treated as distinct terms even if $S_m = S_n$. Thus the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occurring at different positions are treated as distinct terms even if they have the same value. The number of terms in a sequence is always infinite.

In other words, we define a sequence as an ordered set of real numbers whose members can be put in an one-one correspondence with the set of natural numbers. However, a sequence may have only a finite number of distinct elements.

For example:

1. $\{S_n\} = \{(-1)^n\}, n \in \mathbf{N}$.

   Here $S_1 = -1$, $S_2 = 1$, $S_3 = -1$, $S_4 = 1$, ... so that there are only two, $1, -1$ distinct elements.

2. $\{S_n\} = \left\{\frac{1}{n}\right\}, n \in \mathbf{N}$.

   Here $S_1 = 1$, $S_2 = \frac{1}{2}, S_3 = \frac{1}{3}, \ldots$

   All the elements are distinct.

ILLUSTRATIONS

1. $\{S_n\}$, where $S_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbf{N}$.
2. $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbf{N}$.
3. $\{S_n\}$, where $S_n = 1$, $\forall\, n \in \mathbf{N}$.
4. $\left\{\frac{(-1)^{n-1}}{n!}\right\}$, $n \in \mathbf{N}$.

1.1 The Range

The Range or the Range Set is the set consisting of all distinct elements of a sequence, without repetition and without regard to the position of a term. Thus the range may be a finite or an infinite set, without ever being the null set.

1.2 Bounds of a Sequence

Bounded above sequences

A sequence $\{S_n\}$ is said to be *bounded above* if there exists a real number K such that

$$S_n \leq K \quad \forall\, n \in \mathbf{N}$$

Bounded below sequences

A sequence $\{S_n\}$ is said to be *bounded below* if there exists a real number k such that

$$S_n \geq k \quad \forall\, n \in \mathbf{N}.$$

Bounded sequences

A sequence is said to be *bounded* when it is bounded both above and below. K and k are respectively the upper and the lower bounds of the sequence.

Evidently a sequence is bounded iff its range is bounded. Also the bounds of the range are the bounds of the sequence.

1.3 Convergence of Sequences

Definition 1. *A sequence $\{S_n\}$ is said to converge to a real number l (or to have the real number l as its limit) if for each $\varepsilon > 0$, there exists a positive integer m (depending on ε) such that $|S_n - l| < \varepsilon$, for all $n \geq m$.*

The fact is expressed by saying that the terms approach the value l or tend to l as n becomes larger and larger. The same thing expressed in symbols is

$$S_n \to l \text{ as } n \to \infty \text{ or } \lim_{n\to\infty} S_n = l.$$

The definition ensures that

(*i*) From some stage onwards the difference between S_n and l can be made less than any preassigned positive number ε, however small, *i.e.*, given any positive real number ε, no matter

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however small, $\exists$ a positive integer m (finite) such that mth term onwards, S_n becomes and remains arbitrarily close to l, *i.e.*, l is a limit point of the sequence.

(*ii*) For any $\varepsilon > 0$, at the most a finite number of terms (depending on the choice of ε) of the sequence can lie outside $]l - \varepsilon, l + \varepsilon[$, *i.e.*, there is at the most a finite number of n's for which

$$S_n \le l - \varepsilon, \text{ and } S_n \ge l + \varepsilon.$$

(*iii*) Since $l - \varepsilon < S_n < l + \varepsilon$ for all $n \ge m$, therefore $S_n < l + \varepsilon$, for infinite number of terms, *i.e.*, infinite number of terms lie to the left of $l + \varepsilon$, or to the right of $l - \varepsilon$.

It may, therefore, be observed that if we can find even one $\varepsilon > 0$ for which infinitely many terms of the sequence lie outside $]l - \varepsilon, l + \varepsilon[$, then the sequence cannot converge to l.

1.4 Some Theorems

Theorem 1. *Every convergent sequence is bounded.*

Let a sequence $\{S_n\}$ converge to the limit l.

Let $\varepsilon > 0$ be a given number, so that $\exists$ a positive integer m such that

$$|S_n - 1| < \varepsilon \quad \forall\, n \ge m$$

$$\Leftrightarrow \qquad l - \varepsilon < S_n < l + \varepsilon \quad \forall\, n \ge m.$$

Let
$$g = \min\{l - \varepsilon, S_1, S_2, \ldots, S_{m-1}\}.$$
$$G = \max\{l + \varepsilon, S_1, S_2, \ldots, S_{m-1}\}.$$

Thus, we have

$$g \le S_n \le G \quad \forall\, n.$$

Hence, $\{S_n\}$ is a bounded sequence.

Remark: The converse of the above theorem may not be true. For example the sequence $\{S_n\}$, where $S_n = (-1)^n$, $n \in \mathbf{N}$, is bounded but it is not convergent. For, if possible, $\lim\limits_{n\to\infty} S_n = l$ then for $\varepsilon = 1$, $\exists\, m \in \mathbf{N}$ such that

$$|S_n - l| < 1, \quad \forall\, n \ge m,$$

i.e.,
$$\left|(-1)^{2m} - l\right| < 1 \text{ and } \left|(-1)^{2m+1} - l\right| < 1$$

or
$$|1 - l| < 1 \text{ and } |1 + l| < 1$$

$$\Rightarrow \qquad 2 = |1 - l + 1 + l| < 1 + 1 = 2$$

which is absurd.

Theorem 2. *A sequence cannot converge to more than one limit.*

Let, if possible, a sequence $\{S_n\}$ converges to two real numbers l and $l'\,(l \ne l')$. Let us select $\varepsilon = \dfrac{1}{3}|l - l'| > 0$

Since the sequence $\{S_n\}$ converges to l and l'; therefore, there exist positive integers m_1 and m_2 such that

$$|S_n - l| < \varepsilon, \quad \forall\, n \ge m_1 \qquad \ldots(1)$$

and $$|S_n - l'| < \varepsilon, \ \forall\, n \geq m_2 \qquad \text{...(2)}$$

Now from (1) and (2), for $n \geq \max(m_1, m_2)$

$$|l - l'| = |l - S_n + S_n - l'| \leq |l - s_n| + |S_n - l'| < 2\varepsilon$$

i.e., $|l - l'| < \frac{2}{3}|l - l'|$, which is not possible.

Hence, the sequence cannot converge to two limits.

It may be seen from the definition that the number to which a sequence converges is a limit point of the sequence. Consequently, the unique limit to which the sequence converges is called *the limit point* or the **limit** of the sequence. Symbolically, we write

$$\lim_{n\to\infty} S_n = l \ \text{ or } \ S_n \to l \text{ as } n \to \infty,$$

or simply

$$\lim S_n = l.$$

Thus, *Theorems* 1 *and* 2 may be stated in a combined form as:

Theorem 3. *Every convergent sequence is bounded and has a unique limit.*

2. LIMIT POINTS OF A SEQUENCE

A real number ξ is said to be a **limit point** of a sequence $\{S_n\}$, if every neighbourhood of ξ contains an infinite number of members of the sequence.

Thus ξ is a limit point of a sequence if given any positive number ε, however small, $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinite number of values of n, *i.e.*,

$$|S_n - \xi| < \varepsilon, \text{ for infinitely many values of } n.$$

Intuitively, it means that S_n is arbitrarily close to ξ for an infinite number of values of n or that infinitely many terms of the sequence occur very close to ξ.

As in a set, a *limit point* is also called a *cluster point* or a *point of condensation.*

Thus a number ξ is *not a limit point* of the sequence $\{S_n\}$ if $\exists$ a number $\varepsilon > 0$ such that $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for at the most a finite number of values of n.

Note: *A* more intuitive but rigorous way of finding a limit point l is to see if the terms get 'closer and closer' to l. This will provide a 'guess' as to the limit point, after which the definition is applied to see if the guess is correct.

It is clear from the definition that a limit point of the range set of a sequence is also a limit point of the sequence. But the converse may not always be true. It may happen that the limit point ξ of a sequence is such that $S_n = \xi$ for an infinitely many values of n, so that automatically $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinity of values of n. In such a situation ξ is just one element in the range set and as such fails to be a limit point thereof. However, if $S_n \in]\xi - \varepsilon, \xi + \varepsilon[$ for an infinite number of values of n and $S_n = \xi$ for at the most a finite number of values of n, then a limit point ξ of the sequence is as well a limit point of the range.

ILLUSTRATIONS

1. The constant sequence $\{S_n\}$, where $S_n = 1, \ \forall n \in \mathbf{N}$, has the only limit point 1. The range is $\{1\}$ and has no limit point.
2. The sequence $\{S_n\}$, where $S_n = \frac{1}{n}$, $n \in \mathbf{N}$, has 0 as a limit point which is as well a limit point of the range $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$
3. 0 and 2 are the only limit points of the sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbf{N}$. The range set $\{0, 2\}$ has no limit point.
4. 1 and -1 are the two limit points of the sequence $\{S_n\}$, where $S_n = (-1)^n$, $n \in \mathbf{N}$. The range set $\{1, -1\}$ has no limit point.
5. 1 and -1 are the two limit points of the sequence $\{S_n\}$, where $S_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $n \in \mathbf{N}$, which are also the limit points of the range.

2.1 Existence of Limit Points

Since members of a sequence form a set (the range), all the theorems relating to bounds and limit points of sets also hold for sequences of members with suitable modifications. It is not necessary, therefore, to list these again. However, we give a proof of one of the most fundamental theorems for sequences.

Bolzano-Weierstrass Theorem (for sequences)

Every bounded sequence has a limit point.

Let $\{S_n\}$ be a bounded sequence and $S = \{S_n : n \in \mathbf{N}\}$ be its range. Since the sequence is bounded, therefore its range set S is also bounded.

There are two possibilities:

(*i*) S is finite, (*ii*) S is infinite.

(*i*) If S is finite, then there must exist at least one member $\xi \in S$ such that $S_n = \xi$ for an infinite number of values of n. This means that every neighbourhood $]\xi - \varepsilon, \xi + \varepsilon[$ of ξ, contains $S_n (= \xi)$ for an infinite number of values of n.

Thus ξ is a limit point of the sequence $\{S_n\}$.

(*ii*) When S is infinite, since it is bounded, it has by Bolzano-Weierstrass theorem (for sets), at least one limit point, say ζ.

Again, since ζ is a limit point of S, therefore every neighbourhood $]\xi - \varepsilon, \xi + \varepsilon[, \varepsilon > 0$ of ζ contains an infinity of members of S, *i.e.*, $S_n \in]\zeta - \varepsilon, \zeta + \varepsilon[$ for an infinity of values of n. Hence ζ is a limit point of the sequence.

Note: The converse of the theorem is not always true, for there do exist unbounded sequences having only one real limit point.

For example $\{1, 2, 1, 4, 1, 6, \ldots\}$ has a unique limit point 1, but is not bounded above.

2.2 We have seen that a bounded sequence has at least one limit point. It may have one limit point, a finite or an infinite number of limit points. A number of questions arise. It may be asked, 'what is the greatest or the least limit point or whether the greatest or the least limit point exists at all?' In an attempt to answer such questions we now proceed to prove that such points do exist for bounded sequences.

Theorem 4. *The set of the limit points of a bounded sequence has the greatest and the least members.*

Let $\{S_n\}$ be a bounded sequence and S its range set. The set S is bounded and consequently its derived set S' is also bounded (Theorem 11, Ch. 2).

Let T be the set of limit points of the sequence. T is non-empty, for, by Bolzano-Weierstrass theorem the sequence has at least one limit point. Again, since T consists of the limit points of S (*i.e.*, the derived set S') and *those points of* S which are not the limit points of S but are limit points of the sequence. Therefore, T is bounded.

T may, however, be finite or infinite.

When $T(\neq \phi)$ is finite, it evidently has the greatest and the least members.

When T is infinite, being bounded set of real numbers, by the order-completeness property of real numbers, it has the Supremum M and the Infimum m, say.

It will now be shown that M and m are the limit points of the sequence, *i.e.*, $M \in T, m \in T$.

Let us first consider M.

Let $]M-\varepsilon, M+\varepsilon[, \varepsilon > 0$ be any *nbd* of M.

Since M is the supremum of T, therefore, $\exists$ at least one member ξ of T such that $M - \varepsilon < \xi \leq M$.

Thus $]M-\varepsilon, M+\varepsilon[$ is a *nbd* of ξ.

But ξ is a limit point of the sequence, so that $]M-\varepsilon, M+\varepsilon[$ contains an infinite members of the sequence.

$\Rightarrow$ and *nbd* $]M-\varepsilon, M+\varepsilon[$ of M contains an infinite number of members of the sequence

$\Rightarrow$ M is a limit point of the sequence

$\Rightarrow$ $M \in T$

Similarly it may be shown that $M \in T$.

Thus $M \in T, m \in T$ and being the supremum and infimum of T are the greatest and the least members of T respectively.

Thus *a bounded sequence has the greatest and the least limit points* (one farthest to the right and other farthest to the left).

2.3 The greatest and the smallest of the limit points of a (bounded) sequence are respectively called the **upper limit** and the **lower limit.**

ILLUSTRATIONS

1. The sequence $\{S_n\}$, where $S_n = (-1)^n$, $n \in \mathbf{N}$, is bounded, for $-1 \leq S_n \leq 1, \forall\, n \in \mathbf{N}$.

   Also -1, 1 are the only limit points.

   $\therefore$ Upper limit $= 1$, lower limit $= -1$.

2. The sequence $\{S_n\}$, where $S_n = 1 + (-1)^n$, $n \in \mathbf{N}$, is bounded, for $0 \leq S_n \leq 2, \forall\, n \in \mathbf{N}$.

   0, 2 are the only limit points.

   $\therefore$ Upper limit $= 2$, lower limit $= 0$.

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3. The sequence $\{S_n\}$, where $S_n = \dfrac{(-1)^{n-1}}{n!}$, $n \in \mathbf{N}$, *i.e.*, $\left\{1, \dfrac{-1}{2!}, \dfrac{-1}{3!}, \dfrac{-1}{4!}, \ldots\right\}$ is bounded, for $-\frac{1}{2} \leq S_n \leq 1, \forall\, n$; 0 being the only limit point, the upper and the lower limits coincide with 0, and so $\lim\limits_{n \to \infty} S_n = 0$.
4. The sequence $\{S_n\}$, where $S_n = n^2$, is $\{1, 4, 9, 16, 25, \ldots\}$. The sequence is bounded below but not above. There is no real limit point.

3. LIMITS — INFERIOR AND SUPERIOR

From the definition of limit in Section 1.4, it follows that the limiting behaviour of any sequence $\{a_n\}$ of real numbers, depends only on sets of the form $\{a_n : n \geq m\}$, *i.e.*, $\{a_m, a_{m+1}, a_{m+2}, \ldots\}$. In this regard we make the following definition.

Definition. Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\lim_{n\to\infty} \inf a_n = \sup_n \inf \left\{a_n, a_{n+1}, a_{n+2}, \ldots\right\}$$

and

$$\lim_{n\to\infty} \sup a_n = \inf_n \sup \left\{a_n, a_{n+1}, a_{n+2}, \ldots\right\}$$

as the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

We shall denote limit inferior and limit superior of $\{a_n\}$ by $\underline{\lim}_{n\to\infty}\, a_n$ and $\overline{\lim}_{n\to\infty}\, a_n$ or simply by $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ respectively.

We shall use the following notations for the sequence $\{a_n\}$, for each $n \in \mathbf{N}$.

$$\underline{A}_n = \inf \left\{a_n, a_{n+1}, a_{n+2}, \ldots\right\},$$

and

$$\overline{A}_n = \sup \left\{a_n, a_{n+1}, a_{n+2}, \ldots\right\}.$$

Therefore, we have

$$\underline{\lim}\, a_n = \sup_n \underline{A}_n$$

and

$$\overline{\lim}\, a_n = \inf_n \overline{A}_n$$

Now $\{a_{n+1}, a_{n+2}, \ldots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \ldots\}$ therefore by taking infimum and supremum respectively, it follows that

$$\underline{A}_{n+1} \geq \underline{A}_n \text{ and } \overline{A}_{n+1} \leq \overline{A}_n$$

This is true for each $n \in \mathbf{N}$.

The above inequalities show that the associated sequences $\{\underline{A}_n\}$ and $\{\overline{A}_n\}$ monotonically increase and decrease respectively with n.

Remark: It should be noted that both limits, inferior and superior, exist uniquely (finite or infinite) for all real sequences.

Theorem 5. *If $\{a_n\}$ is any sequence, then*

$$\inf a_n \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le \sup a_n.$$

Let
$$\underline{A}_k = \inf_{n \ge k} a_n \text{ and } \overline{A}_k = \sup_{n \ge k} a_n,\ k \in \mathbf{N}.$$

Then, for all $k,\ n \in \mathbf{N}$, we have

$$\underline{A}_k \le \underline{A}_{k+n} \le \overline{A}_{k+n} \le \overline{A}_n$$

$$\therefore \quad \underline{A}_k \le \overline{A}_n, \text{ for all } k, n \in \mathbf{N}$$

This implies that each $\underline{A}_k$ is a lower bound of the sequence $\{\overline{A}_n\}$, therefore

$$\underline{A}_k \le \inf_n \overline{A}_n = \overline{\lim}\, a_n, \text{ for each } k \in \mathbf{N}$$

This gives $\overline{\lim}\, a_n$ is an upper-bound of the sequence $\{\underline{A}_k\}$. Hence,

$$\underline{\lim}\, a_n = \sup_k \underline{A}_k \le \overline{\lim}\, a_n.$$

Other inequalities follow from

$$\inf a_n = \underline{A}_1 \le \underline{A}_1 \le \overline{A}_n \le \overline{A}_1 = \sup a_n$$

and the definition of $\underline{\lim}\, a_n$ and $\overline{\lim}\, a_n$ respectively.

Corollary. *If* a sequence $\{a_n\}$ is bounded then limit inferior and limit superior of $\{a_n\}$ are both finite. In fact

$$-\infty < \underline{\lim}\, a_n \le \overline{\lim}\, a_n < +\infty.$$

Theorem 6. *If $\{a_n\}$ is any sequence, then*

$$\underline{\lim}\,(-a_n) = -\overline{\lim}\, a_n, \text{ and } \overline{\lim}\,(-a_n) = -\underline{\lim}\, a_n.$$

Let $b_n = -a_n, n \in \mathbf{N}$, then we have

$$\begin{aligned}\underline{B}_n &= \inf \{b_n, b_{n+1}, \ldots\}\\ &= -\sup \{a_n, a_{n+1}, \ldots\} = -\overline{A}_n\end{aligned}$$

and so,

$$\begin{aligned}\underline{\lim}\,(-a_n) = \underline{\lim}\, b_n &= \sup (\underline{B}_1, \underline{B}_2, \ldots)\\ &= \sup \{-\overline{A}_1, -\overline{A}_2, \ldots\}\\ &= -\inf \{\overline{A}_1, \overline{A}_2, \ldots\}\\ &= -\inf \overline{A}_n = -\overline{\lim}\, a_n.\end{aligned}$$

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Also,

$$\underline{\lim}\, a_n = \underline{\lim}\,(-(-a_n)) = -\overline{\lim}\,(-a_n).$$

ILLUSTRATIONS

1. If $a_n = (-1)^n, n \in \mathbf{N}$, then

$$\underline{A}_n = -1 \text{ and } \overline{A}_n = 1, \text{ for each } n \in \mathbf{N}$$

$$\underline{\lim}\, a_n = \sup \underline{A}_n = -1 \text{ and } \overline{\lim}\, a_n = \inf \overline{A}_n = 1.$$

2. If $a_n = 1 + (-1)^n, n \in \mathbf{N}$, then

$$\underline{A}_n = \inf \left\{1 + (-1)^n, 1 + (-1)^{n+1}, \ldots\right\} = 0$$

and $$\overline{A}_n = \sup \left\{1 + (-1)^n, 1 + (-1)^{n+1}, \ldots\right\} = 2, \text{ for each } n \in \mathbf{N}$$

$$\underline{\lim}\, a_n = 0 \text{ and } \overline{\lim}\, a_n = 2.$$

3. If $a_n = n, n \in \mathbf{N}$, then

$$\underline{A}_n = n, \text{ and } \overline{A}_n = +\infty$$

$$\underline{\lim}\, a_n = \sup \{1, 2, 3, \ldots\} = +\infty, \text{ and } \overline{\lim}\, a_n = +\infty.$$

4. If $a_n = (-1)^n n, n \in \mathbf{N}$, then

$$\begin{aligned}\underline{\lim}\, a_n &= \sup_n \inf \left\{(-1)^n n, (-1)^{n+1}(n+1), \ldots\right\} \\ &= \sup \{-\infty, -\infty, \ldots\} = -\infty\end{aligned}$$

and $$\overline{\lim}\, a_n = +\infty.$$

5. If $a_n = \dfrac{(-1)^n}{n^2}, n \in \mathbf{N}$, then

$$\begin{aligned}\underline{\lim}\, a_n &= \sup_n \inf \left\{\frac{(-1)^n}{n^2}, \frac{(-1)^{n+1}}{(n+1)^2}, \ldots\right\} \\ &= \sup_n \begin{cases} \dfrac{-1}{n^2}, & \text{if } n \text{ is odd} \\ \dfrac{-1}{(n+1)^2}, & \text{if } n \text{ is even} \end{cases} \\ &= 0\end{aligned}$$

and $$\overline{\lim}\, a_n = 0.$$

6. If $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $n \in \mathbf{N}$, then

$$\underline{A}_n = \inf \left\{(-1)^n \left(1 + \frac{1}{n}\right), (-1)^{n+1} \left(1 + \frac{1}{n+1}\right), \ldots\right\}$$

$$= \begin{cases} -\left(1 + \frac{1}{n}\right), \text{ if } n \text{ is odd} \\ -\left(1 + \frac{1}{n+1}\right), \text{ if } n \text{ is even} \end{cases}$$

$$\underline{\lim}\, a_n = \sup \underline{A}_n = \sup \left\{-2, \frac{-4}{3}, \frac{-6}{5}, \ldots\right\} = -1$$

and $\overline{\lim}\, a_n = 1$.

7. The sequence

$$\left\{-2, 2, \frac{-3}{2}, \frac{3}{2}, \frac{-4}{3}, \frac{4}{3}, \ldots\right\}$$

has limit inferior -1 and limit superior 1. Note that -1 is not lower bound nor 1 is an upper bound of the sequence.

8. If $a_n = n\,(1 + (-1)^n)$, then $\underline{\lim}\, a_n = 0$ and $\overline{\lim}\, a_n = \infty$.

9. If $a_n = \sin \frac{n\pi}{3}$, $n \in \mathbf{N}$, then the sequence $\{a_n\}$ is

$$\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \frac{-\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \ldots\right\}$$

$$\underline{A}_n = \frac{-\sqrt{3}}{2} \text{ and } \overline{A}_n = \frac{\sqrt{3}}{2}, \text{ for each } n \in \mathbf{N}$$

$$\underline{\lim}\, a_n = \frac{-\sqrt{3}}{2} \text{ and } \overline{\lim}\, a_n = \frac{\sqrt{3}}{2}$$

10. If $\{r_n\}$ be an enumeration of all the rational numbers between 0 and 1, then

$\underline{\lim}\, r_n = 0$ and $\overline{\lim}\, r_n = 1$

Example 1. If $a_n = \sin\frac{n\pi}{2} + \frac{(-1)^n}{n}$, $n \in \mathbf{N}$, then show that

$$\underline{\lim}\, a_n = -1 \text{ and } \overline{\lim}\, a_n = 1$$

■ The sequence $\{a_n\}$ is bounded, for $\frac{-4}{3} \le a_n \le 1$, $\forall\, n$, the terms of the sequence are given by

$$a_{2n} = \frac{1}{2n}, \text{ for all } n,$$

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$$a_{4n+1} = 1 - \frac{1}{4n+1}, \text{ for all } n,$$

$$a_{4n+3} = -1 - \frac{1}{4n+3}, \text{ for all } n.$$

Therefore, $\overline{A}_n = 1$ for all n, and so

$$\overline{\lim}\, a_n = 1$$

Also $\underline{A}_n$ is given by

$$\underline{A}_{4n} = -1 - \frac{1}{4n+3}$$

$$\underline{A}_{4n+1} = -1 - \frac{1}{4n+3}$$

$$\underline{A}_{4n+2} = -1 - \frac{1}{4n+3}$$

and

$$\underline{A}_{4n+3} = -1 - \frac{1}{4n+3}, \text{for each } n \in \mathrm{N}.$$

Hence $\underline{\lim}\, a_n = -1$

Theorem 7. *If $\{a_n\}$ is any sequence, then*

$\underline{\lim}\, a_n = -\infty$ *if and only if $\{a_n\}$ is not bounded below,*

and $\overline{\lim}\, a_n = +\infty$ *if and only if $\{a_n\}$ is not bounded above.*

Let $\underline{A}_n = \inf \{a_n, a_{n+1}, ...\}$,

and $\overline{A}_n = \sup \{a_n, a_{n+1}, ...\}$, $n \in \mathbf{N}$

By definition, we have

$$\underline{\lim}\, a_n = -\infty \Leftrightarrow \sup \{\underline{A}_1, \underline{A}_2, ...\} = -\infty$$

$\Leftrightarrow$ $\underline{A}_n = -\infty, \forall\, n \in \mathbf{N} \Leftrightarrow \underline{A}_n = -\infty, \forall\, n \in \mathbf{N}$

$\Leftrightarrow$ $\inf \{a_n, a_{n+1}, ...\} = -\infty, \forall\, n \in \mathbf{N}$

$\Leftrightarrow$ $\{a_n\}$ is not bounded below.

The proof for limit superior is similar.

Corollary. If $\{a_n\}$ is any sequence, then

(*i*) $-\infty < \underline{\lim}\, a_n \leq +\infty$ iff $\{a_n\}$ is bounded below.

and (*ii*) $-\infty \leq \overline{\lim}\, a_n < +\infty$ iff $\{a_n\}$ is bounded above.

For bounded sequences, we have the following useful criteria for limits inferior and superior respectively.

Theorem 8. *A real number $\underline{a}$ is the limit inferior of a bounded sequence $\{a_n\}$ iff for each $\varepsilon > 0$, the following results hold:*

(*i*) $a_n < \underline{a} + \varepsilon$, *for infinitely many values of n, and*

(*ii*) *there exists a positive integer m such that*

$$a_n > \underline{a} - \varepsilon, \text{ for all } n \geq m.$$

Since $\{a_n\}$ is bounded, therefore $\underline{\lim}\, a_n$ is finite. Let $\varepsilon > 0$ be given. Then, we have

$$\underline{a} = \underline{\lim}\, a_n = \sup_n \underline{A}_n \Leftrightarrow \begin{cases} \underline{A}_n \leq \underline{a}, \text{ for all } n, \\ \exists\, m \in \mathbf{N} \text{ such that} \\ \underline{A}_m > \underline{a} - \varepsilon \end{cases}$$

$$\Leftrightarrow \begin{cases} \inf\{a_n, a_{n+1}, \ldots\} \leq \underline{a}, \forall n, \\ \exists\, m \in \mathbf{N} \text{ such that} \\ \inf\{a_m, a_{m+1}, \ldots\} > \underline{a} - \varepsilon \end{cases}$$

$$\Leftrightarrow \begin{cases} a_n < \underline{a} + \varepsilon, \text{ for infinitely many values of } n, \\ \exists\, m \in \mathbf{N} \text{ such that} \\ a_n > \underline{a} - \varepsilon, \forall\, n \geq m. \end{cases}$$

Hence the result follows.

Corollary 1. If $\{a_n\}$ is a bounded sequence and $\underline{a} = (\underline{\lim}\, a_n,)$, then

(*i*) for each real number $\alpha < \underline{a}$, $\exists$ a positive integer m such that

$$a_n > \alpha, \quad \forall\, n \geq m, \text{ and}$$

(*ii*) if $\alpha \in \mathbf{R}$ and $\exists\, m \in \mathbf{N}$ such that $a_n > \alpha$, $\forall\, n \geq m$, then $\underline{\lim}\, a_n \geq \alpha$.

For (*i*), take $\varepsilon = a - \alpha$ and use first part of the above theorem, and for (*ii*), such an α is then a lower bound of the set $\{a_n : n \geq m\}$

and so $$\underline{\lim}\, a_n = \sup_n \underline{A}_n \geq \underline{A}_m = \inf\{a_n : n \geq m\} \geq \alpha.$$

Corollary 2. Let $\{a_n\}$ be a bounded sequence, then

(*i*) if $\alpha \in \mathbf{R}$ and $\underline{\lim}\, a_n < \alpha$, then $a_n < \alpha$, for infinitely many values of n, and

(*ii*) if $\alpha \in \mathbf{R}$ is such that $\{n : a_n < \alpha\}$ is infinite, then $\underline{\lim}\, a_n \leq \alpha$.

The following theorem follows by applying the above theorem to the sequence $\{-a_n\}$ and using theorem (6 of section 3).

Theorem 9. *A real number is $\bar{a}$ the limit superior of a bounded sequence $\{a_n\}$ iff for each $\varepsilon > 0$, the following results hold:*

(*i*) $a_n > \bar{a} - \varepsilon$, *for infinitely many values of n,*

and (*ii*) $\exists$ *a positive integer m such that*

$$a_n < \bar{a} + \varepsilon, \quad \forall\, n \geq m.$$

Corollary 1. If $\{a_n\}$ is a bounded sequence and $\bar{a} = \overline{\lim}\, a_n$, then

(*i*) for each real number $\beta > \bar{a}, \exists\, m \in \mathbf{N}$, such that $a_n < \beta,\ \forall\, n \geq m$, and

(*ii*) if $\beta \in \mathbf{R}$ and $\exists\, m \in \mathbf{N}$ such that $a_n < \beta,\ \forall\, n \geq m$, then $\overline{\lim}\, a_n \leq \beta$.

Corollary 2. Let $\{a_n\}$ be a bounded sequence and $\beta \in \mathbf{R}$.

(*i*) if $\overline{\lim}\, a_n > \beta$, then $\{n : a_n > \beta\}$ is infinite, and

(*ii*) if $\{n : a_n > \beta\}$ is infinite, then $\overline{\lim}\, a_n \geq \beta$.

The following theorem shows that the limits–inferior and superior are the *smallest* and the *greatest* limit points, respectively, of a bounded sequence and hence the *lower* and *upper* limits of the bounded sequence.

Theorem 10. If $\{a_n\}$ *is a bounded sequence, then*

(*i*) $\underline{\lim}\, a_n$ = *smallest limit point of* $\{a_n\}$, *and*

(*ii*) $\overline{\lim}\, a_n$ = *greatest limit point of* $\{a_n\}$.

We shall prove (*i*) and leave (*ii*) as an exercise to the reader.

Let $\underline{\lim}\, a_n = \underline{a}$. Then $\underline{a}$ is finite, since $\{a_n\}$ is bounded.

Now by theorem (8), it follows that for each $\varepsilon > 0$,

$$\underline{a} - \varepsilon < a_n < \underline{a} + \varepsilon, \text{ for infinitely many values of } n.$$

Thus $\underline{a}$ is a limit point of $\{a_n\}$. Moreover, by the second part of the theorem (8), we have, for any $\varepsilon > 0$, there are at the most a finite number of terms of the bounded sequence $\{a_n\}$, for which $a_n \leq \underline{a} - \varepsilon$, and consequently any number smaller than $\underline{a}$ is not a limit point of $\{a_n\}$. Hence, $a = \underline{\lim}\, a_n$ is the smallest limit point of the bounded sequence $\{a_n\}$.

4. CONVERGENT SEQUENCES

A sequence may have no limit point, a unique limit point or a finite or an infinite number of limit points. Our interest lies chiefly in a bounded sequence with a unique limit point. Evidently such a sequence can have at the most a finite number of terms outside the interval $]l - \varepsilon, l + \varepsilon[,\ \varepsilon > 0$, however, small ε may be. For otherwise, by the *Bolzano-Weierstrass Theorem* the infinite number of outside terms will have another limit point. Further, the condition automatically ensures the existence of an infinite number of terms of the sequence within the interval. Such sequences are called *convergent sequences*.

Let us proceed to show that the converse of theorem 3 also holds.

4.1 Theorem 11. *Every bounded sequence with a unique limit point is convergent.*

Let l be the only limit point of a bounded sequence $\{S_n\}$, which surely exists by *Bolzano-Weierstrass Theorem*. Thus, for $\varepsilon > 0$, $S_n \in\,]l - \varepsilon, l + \varepsilon[$ for an infinite number of values of $n \in \mathbf{N}$.

l being the only limit point of the sequence there can exist only a finite number of values say m_1, m_2, ..., m_r of n such that the corresponding terms of the sequence do not belong to $]l - \varepsilon, l + \varepsilon[$. For otherwise, the infinitely many outside terms will have a limit point other than l.

Let $(m-1)$ be the greatest of such exceptional values of n. Thus, we have

$$S_n \in \,]l-\varepsilon, l+\varepsilon[, \ \forall\, n \ge m,$$

i.e.,

$$|S_n - l| < \varepsilon, \ \forall\, n \ge m.$$

Thus the sequence $\{S_n\}$ converges to its unique limit point l.

Theorems 3 and 11 may be stated in the combined form as:

Theorem 12. *A necessary and sufficient condition for the convergence of a sequence is that it is bounded and has a unique limit point.*

4.2 In view of the above discussion, we give below another equivalent definition of a convergent sequence.

Definition 2. A sequence is said to be convergent if it is bounded and has a unique limit point.

Starting with the definition 2 for the convergence of a sequence the reader is advised to prove the following theorem.

Theorem 13. *A necessary and sufficient condition for a sequence $\{S_n\}$ to converge to l is that to each $\varepsilon > 0$, there corresponds a positive integer m such that*

$$|S_n - l| < \varepsilon, \ \forall\, n \ge m.$$

The following theorem shows that a bounded sequence converges iff its limits inferior and superior coincide and the common value is the limit of the sequence.

Theorem 14. *A bounded sequence $\{a_n\}$ converges to a real number a if and only if*

$$\underline{\lim}\, a_n = \overline{\lim}\, a_n = a.$$

The condition is necessary: If the sequence $\{a_n\}$ converges to a, then, by theorem (12), a is the unique limit point of $\{a_n\}$. Hence the limits inferior and limit superior both are equal to a (using Theorem (10)).

The condition is sufficient: Let $\{a_n\}$ be a bounded sequence such that

$$\underline{\lim}\, a_n = \overline{\lim}\, a_n = a.$$

This shows that a is the unique limit point of the bounded sequence $\{a_n\}$. Since limit inferior and limit superior are the smallest and greatest limit points. Hence by theorem 12, it follows that $\{a_n\}$ converges to a.

5. NON-CONVERGENT SEQUENCES (DEFINITIONS)

(*a*) *Bounded Sequences*

A bounded sequence which does not converge, and has at least two limit points, is said to **oscillate finitely.**

(*b*) *Unbounded Sequences*

(*i*) If a sequence $\{S_n\}$ is unbounded on the left (below), then we say that $-\infty$ is a limit point of the sequence, and to each positive number G, however large, there corresponds a positive integer m, such that

$$S_n < -G, \ \forall\, n \ge m,$$

i.e., the sequence has an infinity of terms below $-G$.

Also, then $-\infty$ is the least limit point so that

$$\underline{\lim}\, S_n = -\infty.$$

(*ii*) If a sequence $\{S_n\}$ is unbounded on the right (above), then we say that $+\infty$ is a limit point of the sequence, and to each positive number G, however large, there corresponds a positive integer m, such that

$$S_n > G, \; \forall\, n \geq m.$$

i.e., the sequence has an infinite number of terms above G. Also, then $+\infty$ is the greatest limit point, so that $\overline{\lim}\, S_n = +\infty$.

(*iii*) If a sequence $\{S_n\}$ is bounded above (on the right) but not below and besides $-\infty$, has *no other* limit point, then $-\infty$ is not only its least but also its greatest limit point, and so we equate the upper limit also to $-\infty$, *i.e.*,

$$\underline{\lim}\, S_n = \overline{\lim}\, S_n = \lim S_n = -\infty.$$

The sequence is then said to **diverge to** $-\infty$.

(*iv*) If, finally, the sequence is bounded on the left (below) but not on the right (above) and besides $+\infty$, has *no other* limit point, then $+\infty$ is not only its greatest but also its least limit point. So we have

$$\overline{\lim}\, S_n = \underline{\lim}\, S_n = \lim S_n = +\infty.$$

The sequence is, then, said to **diverge to** $+\infty$.

(*v*) An unbounded sequence is said to oscillate infinitely if it diverges neither to $+\infty$ nor to $-\infty$. Thus a bounded sequence either *converges* or else *oscillates finitely*, but an unbounded sequence either *diverges* to $+\infty$ or $-\infty$ or *oscillates infinitely*.

Note: ∞ is not considered here as a real number because we are not dealing with the extended real number system. In the latter case, the definitions need lot of modifications.

ILLUSTRATIONS

1. $\{1 + (-1)^n\}$ oscillates finitely.
2. $\left\{(-1)^n\left(1 + \frac{1}{n}\right)\right\}$ oscillates finitely.
3. $\{n^2\}$ diverges to $+\infty$.
4. $\{-2^n\}$ diverges to $-\infty$.
5. $\{n(-1)^n\}$ oscillates infinitely.
6. $\left\{\frac{(-1)^{n-1}}{n!}\right\}$ converges to the limit 0.

$\left\{1+\frac{1}{n}\right\}$ converges to the limit 1.

8. $\left\{1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots\right\}$ is bounded below but unbounded above, and has a limit point 0 besides $+\infty$.

$$\underline{\lim}\, S_n = 0,\ \overline{\lim}\, S_n = +\infty.$$

The sequence oscillates infinitely.

9. $\{1, 2, 3, 2, 5, 2, 7, 2, 3, 2, 11, 2, 13, \ldots\}$,

where, $S_n = \begin{cases} 2, \text{ when } n \text{ is even,} \\ \text{lowest prime factor } (\neq 1) \text{ of } n, \text{ when } n \text{ is odd,} \end{cases}$

is bounded on the left but not on the right. It has infinite number of limit points 2, 3, 5, 7, 11, ..., so that

$$\underline{\lim}\, S_n = 2,\ \overline{\lim}\, S_n = +\infty.$$

The sequence oscillates infinitely.

10. The sequence $\left\{m+\frac{1}{n}\right\}$ where m, n are natural numbers, also oscillates infinitely 1, 2, 3, ... being its limit points.

Solved Examples

Example 2. Show that $\lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$

■ Let ε be any positive number.

$$\therefore \quad \left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| < \varepsilon, \text{ if } \left|\frac{3}{\sqrt{n}}\right| < \varepsilon \text{ or } n > \frac{9}{\varepsilon^2}.$$

Let m be a positive integer greater than $9/\varepsilon^2$.

Thus to $\varepsilon > 0, \exists$ a positive m, such that

$$\left|\frac{3+2\sqrt{n}}{\sqrt{n}} - 2\right| < \varepsilon,\ \forall\, n \geq m$$

$$\therefore \quad \lim_{n\to\infty} \frac{3+2\sqrt{n}}{\sqrt{n}} = 2.$$

Example 3. Show that

(i) $\lim_{n\to\infty} \sqrt[n]{a} = 1$, if $a > 0$ and

(ii) $\lim_{n\to\infty} \sqrt[n]{n} = 1$

■ (i) If $a > 1$, let $\sqrt[n]{a} = 1 + h_n$, where $h_n > 0$

$$\therefore \quad a = (1 + h_n)^n = 1 + n\, h_n + \frac{n\,(n-1)}{1.2} h_n{}^2 + \ldots + h_n{}^n$$

$$> 1 + n\, h_n, \ \forall\, n \qquad (\because h_n > 0)$$

$$\therefore \quad h_n < \frac{a-1}{n}, \ \forall\, n$$

Let ε be any positive number, then

$$|h_n| = h_n < \frac{a-1}{n} < \varepsilon, \text{ where } n > \frac{a-1}{\varepsilon}$$

Let m be any positive integer $> \dfrac{a-1}{\varepsilon}$, then

$$\left|\sqrt[n]{a} - 1\right| = |h_n| < \varepsilon, \ \forall\, n \geq m.$$

If $a = 1$, the result is trivial and if $0 < a < 1$ the result is obtained by taking reciprocals.

(ii) Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$

$$\therefore \quad n = (1 + h_n)^n = 1 + nh_n + \frac{1}{2}n(n-1)h_n^2 + \ldots + h_n^n$$

$$> \frac{1}{2}n(n-1)h_n^2, \ \forall n \qquad (\because h_n \geq 0)$$

$$\therefore \quad h_n^2 < \frac{2}{n-1}, \text{ for } n \geq 2$$

or $$|h_n| < \sqrt{\frac{2}{n-1}}, \text{ for } n \geq 2.$$

Let ε be any positive number, then

$$|h_n| < \sqrt{\frac{2}{n-1}} < \varepsilon, \text{ when } n > 1 + 2/\varepsilon^2.$$

Let m be any positive integer $> 1 + 2/\varepsilon^2$.

Thus for $\varepsilon > 0, \exists$ a positive integer m such that

$$\left|\sqrt[n]{n} - 1\right| = |h_n| < \varepsilon, \ \forall\, n \geq m.$$

Hence, $$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Example 4. Show that the sequence $\{r^n\}$ converges iff $-1 < r \leq 1$.

■ (i) When $r > 1$, let $r = 1 + h$, $h > 0$.

$\therefore$
$$r^n = (1+h)^n$$
$$> 1 + nh, \quad \forall\, n \in \mathbf{N}$$

If $G > 0$ be any number however large, we have

$$1 + nh > G, \text{ if } n > \frac{G-1}{h}.$$

Let m be a positive integer greater than $\frac{G-1}{h}$,

$\therefore$ for $G > 0$, $\exists$ a positive integer m such that

$$r^n > G, \quad \forall\, n \geq m.$$

Hence, the sequence diverges to ∞.

(*ii*) When $r = 1$, evidently $\lim r^n = 1$.

$\therefore$ The sequence converges to 1.

(*iii*) When $|r| < 1$, let $|r| = \frac{1}{1+h}$, where $h > 0$.

$$\therefore \qquad |r^n| = |r|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}, \quad \forall\, n \in \mathbf{N}.$$

Let ε be any positive number, then

$$\frac{1}{1+nh} < \varepsilon, \text{ when } n > \left(\frac{1}{\varepsilon} - 1\right)\Big/h$$

Let m be a positive integer greater than $\left(\frac{1}{\varepsilon} - 1\right)\Big/h$,

$\therefore$ for $\varepsilon > 0$, $\exists$ a positive integer m such that

$$|r^n| < \varepsilon, \quad \forall\, n \geq m.$$

Hence, $\{r^n\}$ converges to 0, *i.e.*,

$$\lim r^n = 0, \text{ when } |r| < 1.$$

(*iv*) When $r = -1$, the sequence $\{(-1)^n\}$ is bounded and has two limit points.

$\therefore$ the sequence oscillates finitely.

(*v*) When $r < -1$, let $r = -t$ so that $t > 1$.

Thus we get the sequence $\{(-1)^n t^n\}$, which has both positive and negative terms. The sequence is unbounded and the numerical values of the terms can be made greater than any number however large. Thus, it oscillates infinitely.

Hence, the sequence $\{r^n\}$ converges only when $-1 < r \leq 1$.

Note: The sequence $\{r^n\}$ converges to zero iff $|r| < 1$.

CHAPTER 3

Example 5. If $a > 0$ and p is real, then $\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$.

■ Let $r > p$ be any positive integer. For $n > 2r$

$$(1+a)^n > {}^nc_r a^r = \frac{n(n-1)\ldots(n-r+1)}{\lambda_r} a^r > \left(\frac{n}{2}\right)^r \frac{a^r}{\lambda_r}$$

Hence $$0 < \frac{n^p}{(1+a)^n} < \frac{2^r \lambda_r n^p}{a^r n^r} = \frac{2^r \lambda_r n^{p-r}}{a^r}$$

Since $p - r < 0$, therefore given $\varepsilon > 0$, $\exists$ a positive integer m such that

$$\left|n^{p-r} - 0\right| < \varepsilon, \text{ whenever } n \geq m \left[m > \frac{1}{\varepsilon^{1/r-p}}\right]$$

Thus $\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$.

Example 4 follows from this if we take $p = 0$.

EXERCISE

Show that:

1. $\lim \frac{2n-3}{n+1} = 2$
2. The sequence $\{(-1)^n\}$ oscillates finitely.
3. The sequence $\{S_n\}$, where $S_n = 1 + \frac{(-1)^n}{n}$ converges.
4. $\lim \frac{1+2+3+\ldots+n}{n^2} = \frac{1}{2}$
5. $\lim \frac{1+3+5+\ldots+(2n-1)}{n^2} = 1$.
6. The sequence $\{n+(-1)^n n\}$ oscillates infinitely.

6. CAUCHY'S GENERAL PRINCIPLE OF CONVERGENCE

Theorem 13 can be used to test the convergence of a sequence to a given limit l, but in cases where limit l is not known, nor can any guess be made of the same, the following theorem which involves only the terms of the sequence, is useful for determining whether a sequence converges or not.

Theorem 15. *A necessary and sufficient condition for the convergence of a sequence $\{S_n\}$ is that, for each $\varepsilon > 0$ there exists a positive integer m such that*

$$\left|S_{n+p} - S_n\right| < \varepsilon, \ \forall\, n \geq m \wedge p \geq 1.$$

Necessary: Let the sequence be convergent and let l be its limit, so that for a given $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$|S_n - l| < \frac{1}{2}\varepsilon, \ \forall n \ge m$$

If $p \ge 1$, then $n + p > n \ge m$, and so

$$|S_{n+p} - l| < \frac{1}{2}\varepsilon, \ \forall n \ge m \wedge p \ge 1$$

$$\Rightarrow \quad |S_{n+p} - S_n| = |S_{n+p} - l + l - S_n|$$

$$\le |S_{n+p} - l| + |l - S_n|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \forall n \ge m \wedge p \ge 1.$$

Sufficient: To establish the convergence of the sequence as a consequence of the given conditions, we first show that the sequence is bounded and then, that it converges to a limit.

Now by the given condition for $\varepsilon = 1$, $\exists$ a positive integer m_0 such that

$$|S_{n+p} - S_n| < 1, \ \forall n \ge m_0 \wedge p \ge 1$$

In particular, for $n = m_0$

$$|S_{m_0+p} - S_{m_0}| < 1, \ \forall p \ge 1$$

i.e.,

$$S_{m_0} - 1 < S_{m_0+p} < S_{m_0} + 1, \ \forall p \ge 1.$$

Let $g = \min\{S_1, S_2, \ldots, S_{m_0-1}, S_{m_0} - 1\}$

$G = \max\{S_1, S_2, \ldots, S_{m_0-1}, S_{m_0} + 1\}$.

Then, $g \le S_n \le G, \ \forall n$

Hence, the sequence is bounded and therefore by *Bolzano-Weierstrass Theorem* for sequences, it has at least one limit point, say l. We shall now show that the sequence converges to l, *i.e.*, $\lim S_n = l$.

Now by the given condition, for $\varepsilon > 0$, $\exists$ a positive integer m such that

$$|S_{n+p} - S_n| < \frac{1}{3}\varepsilon, \ \forall n \ge m \wedge p \ge 1$$

In particular, for $n = m$

$$|S_{m+p} - S_m| < \frac{1}{3}\varepsilon, \ \forall p \ge 1 \qquad \ldots(1)$$

As l is a limit point, $\exists$ an integer $m_1 > m$ such that

$$|S_{m_1} - l| < \frac{1}{3}\varepsilon \qquad \ldots(2)$$

Also, since $m_1 > m$ therefore from (1), we have

$$|S_{m_1} - S_m| < \frac{1}{3}\varepsilon \qquad \ldots(3)$$

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$$\therefore \quad |S_{m+p} - l| = |S_{m+p} - S_m + S_m - S_{m_1} + S_{m_1} - l|$$

$$\leq |S_{m+p} - S_m| + |S_m - S_{m_1}| + |S_{m_1} - l|$$

$$< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \forall\, p \geq 1$$

$$\Rightarrow \quad |S_n - l| < \varepsilon, \ \forall\, n \geq m$$

Hence, the sequence $\{S_n\}$ converges to l.

6.1 Cauchy Sequence

A sequence $\{S_n\}$ is called a *Cauchy sequence* or a *fundamental sequence* if for each $\varepsilon > 0, \exists$ a positive integer m, such that

$$|S_{n+p} - S_n| < \varepsilon, \ \forall\, n \geq m \wedge p \geq 1$$

or

$$|S_{n_1} - S_{n_2}| < \varepsilon, \ \forall\, n_1, n_2 \geq m$$

Thus in the field of real numbers, a sequence is convergent iff it is a Cauchy sequence.

Note: A sequence cannot converge if even one $\varepsilon > 0$ can be found such that for every positive integer m,

$$|S_{n+p} - S_n| \nless \varepsilon, \ \forall\, n \geq m \wedge p \geq 1$$

Ex. 1. If $\{a_n\}$ and $\{b_n\}$ are two Cauchy sequences, then the sequences $\{a_n \pm b_n\}$, $\{a_n \cdot b_n\}$ and $\{a_n/b_n\}$ (if $b_n \neq 0$ for all n) are also Cauchy sequences.

Ex. 2. Show that every Cauchy sequence is bounded. Is the converse true?

Example 6. Show that the sequence $\{S_n\}$, where $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ cannot converge.

■ Suppose, if possible, the sequence $\{S_n\}$ is convergent.

Let us take $\varepsilon = \frac{1}{2}$, and $n = m$ and $p = m$ in Cauchy's General Principle of Convergence, so that

$$|S_{2m} - S_m| < \frac{1}{2}$$

But $S_{2m} - S_m = \dfrac{1}{m-1} + \dfrac{1}{m-2} + \ldots + \dfrac{1}{2m}$,

$$> \frac{1}{2m} + \frac{1}{2m} + \ldots + \frac{1}{2m} = \frac{m}{2m} = \frac{1}{2}$$

i.e., $|S_{2m} - S_m| > \frac{1}{2}$, which contradicts Cauchy's general principle of convergence.

Hence, the sequence cannot converge.

Ex. Show that the sequence $\{S_n\}$, where

(i) $S_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}$

(ii) $S_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n-2}$

cannot converge.

7. ALGEBRA OF SEQUENCES

It may not always be easy to prove the convergence of a sequence by direct application of the definition of convergence or by Cauchy's General Principle of Convergence. But it may enable us to evaluate the limit of a sequence whose terms can be expressed as a sum, difference, product or quotient of the corresponding terms of two convergent sequences. The sequence $\{S_n\}$ whose nth term is $a_n + b_n$, $a_n - b_n$, a_nb_n or $\frac{a_n}{b_n}$ $(b_n \neq 0)$ is called the sum, difference, product or quotient of the sequences $\{a_n\}$ and $\{b_n\}$. We shall now discuss the convergence of such sequences.

Theorem 16. *If $\{a_n\}$, $\{b_n\}$ be two sequences such that* $\lim a_n = a$, $\lim b_n = b$, *then*

(i) $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n = a \pm b$

(ii) $\lim (a_nb_n) = (\lim a_n)(\lim b_n) = ab$

(iii) $\lim (a_n/b_n) = (\lim a_n/\lim b_n)$, if $b \neq 0, b_n \neq 0, \ \forall\, n$.

(i) Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$ and $\lim b_n = b$,

$\therefore$ $\exists$ positive integers m_1 and m_2, respectively, such that

$$|a_n - a| < \frac{1}{2}\varepsilon, \ \forall\, n \geq m_1, \text{ and}$$

$$|b_n - b| < \frac{1}{2}\varepsilon, \ \forall\, n \geq m_2.$$

Thus, for $m = \max(m_1, m_2)$, we have

$$|a_n - a| < \frac{1}{2}\varepsilon, \ |b_n - b| < \frac{1}{2}\varepsilon, \ \forall\, n \geq m$$

$$\therefore \quad |(a_n \pm b_n) - (a \pm b)| = |(a_n - a) \pm (b_n - b)|$$
$$\leq |a_n - a| \pm |b_n - b|$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon - \varepsilon, \ \forall\, n \geq m$$

Hence, $\quad \lim (a_n \pm b_n) = a \pm b - \lim a_n \pm \lim b_n$.

(ii) The two sequences $\{a_n\}$, $\{b_n\}$ being convergent, are bounded so that $\exists$ positive real numbers k, K such that

$$|a_n| \leq k, |b_n| \leq K, \ \forall\, n$$

Now

$$|a_nb_n - ab| = |a_n(b_n - b) + b(a_n - a)|$$

$$\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a|$$

$$\leq k\,|b_n - b| + |b| \cdot |a_n - a| \quad \text{...(1)}$$

Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$, $\lim b_n = b$, therefore there exist positive integers m_1 and m_2, respectively, such that

$$|a_n - a| < \frac{\frac{1}{2}\varepsilon}{|b| + 1}, \quad \forall\, n \geq m_1$$

$$|b_n - b| < \frac{\frac{1}{2}\varepsilon}{k}, \quad \forall\, n \geq m_2$$

Then, for $m = \max(m_1, m_2)$, we have from (1)

$$|a_nb_n - ab| < \frac{1}{2}\varepsilon + \frac{|b| \cdot \frac{1}{2}\varepsilon}{|b| + 1} < \varepsilon, \quad \forall\, n \geq m.$$

Hence, $\lim (a_nb_n) = ab = (\lim a_n)(\lim b_n)$.

(*iii*) *Lemma.* To show that if $\lim b_n = b \neq 0$, then $\exists$ a positive number λ and a positive integer m_3 such that

$$|b_n| > \lambda, \quad \forall\, n \geq m_3$$

Let us take $\varepsilon = \frac{1}{2}|b|$, so that there exists a positive integer m_3 such that

$$|b_n - b| < \frac{1}{2}|b|, \quad \forall\, n \geq m_3,$$

Thus,
$$|b| - |b_n| \geq |b_n - b| < \frac{1}{2}|b|.$$

$\Rightarrow$
$$|b_n| \geq \frac{1}{2}|b| \text{ (say)}, \quad \forall\, n \geq m_3. \quad \text{...(2)}$$

Let us apply the Lemma to prove the main theorem.

Now

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{ba_n - ab_n}{bb_n}\right| = \left|\frac{b(a_n - a) - a(b_n - b)}{bb_n}\right|$$

$$\leq \frac{|b||a_n - a| + |a||b_n - b|}{|b||b_n|}$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b|, \quad \forall\, n \geq m_3 \quad \text{[using (2)]}$$

Let $\varepsilon > 0$ be given.

Since $\lim a_n = a$, $\lim b_n = b$, therefore, $\exists$ positive integers m_1, m_2 such that

$$|a_n - a| < \frac{1}{4}|b|\varepsilon, \ \forall\, n \geq m_1$$

and
$$|b_n - b| < \frac{1}{4}\frac{|b|^2\varepsilon}{|a|+1}, \ \forall\, n \geq m_2.$$

Thus, for $m = \max(m_1, m_2, m_3)$, we have

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \ \forall\, n \geq m$$

Hence,
$$\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b} = \frac{\lim a_n}{\lim b_n}.$$

Note: It may be noted that while the sum, difference, product and quotient under certain conditions, of two convergent sequences is convergent, the converse may not be true, *i.e.*, if the sequence $\{a_n \pm b_n\}$, $\{a_n b_n\}$ or $\left\{\frac{a_n}{b_n}\right\}$ is convergent, the component sequences $\{a_n\}$ and $\{b_n\}$ may not be convergent, however, it is not difficult to see that both shall behave alike.

For example, consider the sequences $\{a_n\}$ and $\{b_n\}$.

(1) When $a_n = n^2$ and $b_n = -n^2$.

The sequence $\{a_n + b_n\}$ converges to zero and the sequence $\left\{\frac{a_n}{b_n}\right\}$ converges to -1, but the two sequences $\{a_n\}$, $\{b_n\}$ are divergent.

(2) If $a_n = b_n = (-1)^n$.

The sequence $\{a_n - b_n\}$ converges to zero, $\left\{\frac{a_n}{b_n}\right\}$ converges to 1 and $\{a_n b_n\}$ converges to 1, while both the sequences $\{a_n\}$ and $\{b_n\}$ oscillate finitely.

(3) If $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$.

The sequence $\{a_n + b_n\}$ converges to zero, $\{a_n b_n\}$ converges to -1 and $\left\{\frac{a_n}{b_n}\right\}$ converges to -1, but $\{a_n\}$ and $\{b_n\}$ are not convergent.

There are many inequalities relating the inferior and superior limits to the arithmetic operations of sequences. We shall now prove some of them, as a sample of the technique involved, for bounded sequences only. The results are, however, true for unbounded sequences as well with well-defined arithmetic operations.

Theorem 17. *If $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then*

(*i*) $\underline{\lim}\, a_n + \underline{\lim}\, b_n \leq \underline{\lim}\,(a_n + b_n)$,

(*ii*) $\underline{\lim}\,(a_n + b_n) \leq \underline{\lim}\, a_n + \overline{\lim}\, b_n$,

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(*iii*) $\underline{\lim}\, a_n + \overline{\lim}\, b_n \le \overline{\lim}\,(a_n + b_n)$.

(*iv*) $\overline{\lim}\,(a_n + b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$.

We shall prove only (*i*) and (*iii*). The other inequalities follow from (*i*) and (*iii*) applying to the sequences $\{-a_n\}$ and $\{-b_n\}$ respectively and using theorem (6 of section 3).

(*i*) Let

$$\underline{\lim}\, a_n = \underline{a} \text{ and } \underline{\lim}\, b_n = \underline{b},$$

then both $\underline{a}$ and $\underline{b}$ are real numbers, since $\{a_n\}$ and $\{b_n\}$ are bounded sequences. Let $\varepsilon > 0$ be given. Then by theorem (8 (*ii*) of section 3) there exist positive integers m_1 and m_2, respectively, such that

$$a_n > \underline{a} - \varepsilon/2, \ \forall\, n \ge m_1 \qquad ...(1)$$

and

$$b_n > \underline{b} - \varepsilon/2, \ \forall\, n \ge m_2 \qquad ...(2)$$

Put $m = \max\,(m_1, m_2)$ then from (1) and (2), we have

$$a_n + b_n > \underline{a} + \underline{b} - \varepsilon, \ \forall\, n \ge m$$

This implies that

$$\underline{a} + \underline{b} - \varepsilon \le \inf_{n \ge m}\,(a_n + b_n) \le \sup_{m}\, \inf_{n \ge m}\,(a_n + b_n) = \underline{\lim}\,(a_n + b_n)$$

But $\varepsilon > 0$ was chosen arbitrarily, hence

$$\underline{\lim}\, a_n + \underline{\lim}\, b_n = \underline{a} + \underline{b} \le \underline{\lim}\,(a_n + b_n)$$

(*iii*) Let

$$\underline{\lim}\, a_n = \underline{a} \text{ and } \overline{\lim}\, b_n = \bar{b},$$

then $\underline{a}, \bar{b} \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Then by theorem (3.8 (*ii*) of section 3), there exists a positive integer m such that

$$a_n > \underline{a} - \varepsilon/2, \ \forall\, n \ge m$$

Also by Theorem (9 (*i*) of Section 3), we have

$$b_n > \bar{b} - \varepsilon/2,$$

for infinitely many values of n. Thus

$$a_n + b_n > \underline{a} + \bar{b} - \varepsilon,$$

for infinitely many values of n.

Therefore, for given $k \in \mathbf{N}, \exists\, n_0 \ge k$ such that

$$a_{n_0} + b_{n_0} > \underline{a} + \bar{b} - \varepsilon$$

This implies that

$$\sup_{n \ge k}\,(a_n + b_n) \ge a_{n_0} + b_{n_0} > \underline{a} + \bar{b} - \varepsilon,$$

for each $k \in \mathbf{N}$.

Hence, $\underline{a} + \overline{b} - \varepsilon \le \inf\limits_{k} \sup\limits_{n \ge k} (a_n + b_n) = \overline{\lim}\,(a_n + b_n)$.

The inequality (*iii*) follows, since $\varepsilon > 0$ was arbitrary.

Corollary. If $\{a_n\}$ and $\{b_n\}$ are bounded sequences, then

$$\underline{\lim}\, a_n - \overline{\lim}\, b_n \le \underline{\lim}\,(a_n - b_n) \le \begin{Bmatrix} \overline{\lim}\, a_n - \overline{\lim}\, b_n \\ \underline{\lim}\, a_n - \underline{\lim}\, b_n \end{Bmatrix}$$

$$\le \overline{\lim}\,(a_n - b_n) \le \overline{\lim}\, a_n - \underline{\lim}\, b_n$$

This follows immediately by the above theorem and using theorem (6 of section 3).

Remark: Strict inequalities may hold in the above theorem as can be seen by the following example.

Take $\{a_n\} = \{-1, 0, 1, -1, 0, 1, \ldots\}$

and $\{b_n\} = \{0, 1, -1, 0, 1, -1, \ldots\}$

Then,

$$-2 = \underline{\lim}\, a_n + \underline{\lim}\, b_n < -1 = \underline{\lim}\,(a_n + b_n) < 0$$

$$= \underline{\lim}\, a_n + \overline{\lim}\, b_n < 1 = \overline{\lim}\,(a_n + b_n) < 2 = \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

Theorem 18. *If $\{a_n\}$ and $\{b_n\}$ are two bounded sequences of non-negative real numbers, then*

(*i*) $(\underline{\lim}\, a_n)(\underline{\lim}\, b_n) \le \underline{\lim}\,(a_n b_n)$

(*ii*) $\underline{\lim}\,(a_n b_n) \le (\underline{\lim}\, a_n)(\overline{\lim}\, b_n)$

(*iii*) $(\underline{\lim}\, a_n)(\overline{\lim}\, b_n) \le \overline{\lim}\,(a_n b_n)$.

(*iv*) $\overline{\lim}\,(a_n b_n) \le (\overline{\lim}\, a_n)(\overline{\lim}\, b_n)$.

We shall prove (*i*) and (*iii*) only, and leave the proof of (*ii*) and (*iv*) to the reader.

(*i*) We note that if $\underline{\lim}\, a_n = 0$ or $\underline{\lim}\, b_n = 0$, then the inequality (*i*) follows immediately. Therefore, we assume that

$$\underline{\lim}\, a_n = \underline{a} > 0 \text{ and } \underline{\lim}\, b_n = \underline{b} > 0,$$

Let $\varepsilon > 0$ be given. Then, by Theorem 8 (*ii*), there exist positive integers m_1 and m_2, respectively, such that

$$a_n > \underline{a} - \frac{\varepsilon}{2\underline{b}}, \quad \forall\, n \ge m_1$$

and

$$b_n > \underline{b} - \frac{\varepsilon}{2\underline{a}} \quad \forall\, n \ge m_2.$$

Therefore, for all $n \geq \max(m_1, m_2)$, we have

$$a_n b_n > \left(\underline{a} - \frac{\varepsilon}{2\underline{b}}\right)\left(\underline{b} - \frac{\varepsilon}{2\underline{a}}\right) = \underline{a}\,\underline{b} - \varepsilon + \frac{\varepsilon^2}{4\underline{a}\,\underline{b}} > \underline{a}\,\underline{b} - \varepsilon$$

This implies that, $\underline{\lim}(a_n b_n) \geq \underline{a}\,\underline{b} - \varepsilon$

But $\varepsilon > 0$ was arbitrary, hence the result follows.

$$\underline{\lim}(a_n b_n) \geq (\underline{\lim}\, a_n)(\underline{\lim}\, b_n)$$

(iii) We again suppose that

$$\underline{\lim}\ a_n = \underline{a} > 0, \text{ and } \overline{\lim}\, b_n = \overline{b} > 0$$

Then, (by Theorems 3.8 (*ii*) and 3.9 (*i*) of Section 3), for given $\varepsilon > 0$, there exists a positive integer m such that

$$a_n > \underline{a} - \frac{\varepsilon}{2\overline{b}}, \text{ for all } n \geq m,$$

and

$$b_n > \overline{b} - \frac{\varepsilon}{2\underline{a}}, \text{ for infinitely many values of } n.$$

Therefore, we have

$$a_n b_n > \underline{a}\,\overline{b} - \varepsilon, \text{ for infinitely many values of } n.$$

Thus, using corollary 2 of Theorem 3.9, it follows that

$$\overline{\lim}(a_n b_n) \geq \underline{a}\,\overline{b} - \varepsilon$$

Hence,

$$\overline{\lim}(a_n b_n) \geq (\underline{\lim}\, a_n)(\overline{\lim}\, b_n),$$

since $\varepsilon > 0$ was arbitrary.

Remark: Strict inequalities may hold here as well.

Take

$$\{a_n\} = \{1, 2, 2, 1, 2, 2, \ldots\}$$

and

$$\{b_n\} = \{3, 1, 2, 3, 1, 2, \ldots\}$$

$\therefore$

$$\{a_n b_n\} = \{3, 2, 4, 3, 2, 4 \ldots\}$$

Then,

$$1 = (\underline{\lim}\, a_n)(\underline{\lim}\, b_n) < 2 = \underline{\lim}\, a_n b_n < 3 = (\underline{\lim}\, a_n)$$

$$(\overline{\lim}\, b_n) < 4 = \overline{\lim}(a_n b_n) < 6 = (\overline{\lim}\, a_n)(\overline{\lim}\, b_n).$$

Theorem 19. *If $\{a_n\}$ is a bounded sequence such that $a_n > 0$ for all $n \in \mathbf{N}$, then*

(i) $\underline{\lim}\left(\dfrac{1}{a_n}\right) = \dfrac{1}{\overline{\lim}\, a_n}$, if $\overline{\lim}\, a_n > 0$

and (ii) $\overline{\lim}\left(\dfrac{1}{a_n}\right) = \dfrac{1}{\underline{\lim}\, a_n}$, if $\underline{\lim}\, a_n > 0$.

(i) Since $a_n > 0$, $\forall\, n \in \mathbf{N}$, therefore

$$1 = \frac{1}{a_n} \cdot a_n, \quad \forall\, n \in \mathbf{N}$$

By using the inequality (*iii*) of the above theorem, we have

$$1 = \overline{\lim}\left(\frac{1}{a_n} a_n\right) \geq \left(\underline{\lim}\left(\frac{1}{a_n}\right)\right)\left(\overline{\lim}\, a_n\right)$$

i.e., $$\underline{\lim}\frac{1}{a_n} \leq \frac{1}{\overline{\lim}\, a_n} \qquad \left(\because\ \overline{\lim}\, a_n > 0\right) \qquad \ldots(1)$$

Similarly, using the inequality (*ii*) of the above theorem, we obtain

$$1 = \underline{\lim}\left(\frac{1}{a_n} \cdot a_n\right) \leq \left(\underline{\lim}\frac{1}{a_n}\right)\left(\overline{\lim}\, a_n\right)$$

i.e., $$\underline{\lim}\frac{1}{a_n} \geq \frac{1}{\overline{\lim}\, a_n} \qquad \ldots(2)$$

Hence from (1) and (2), we get

$$\underline{\lim}\frac{1}{a_n} = \frac{1}{\overline{\lim}\, a_n}, \text{ if } \overline{\lim}\, a_n > 0.$$

The proof of (*ii*) is similar.

Corollary. If $\{a_n\}$ and $\{b_n\}$ are bounded sequences, $a_n \geq 0, b_n > 0$ for all $n \in \mathbf{N}$, then

(*i*) $\underline{\lim}\left(\dfrac{a_n}{b_n}\right) \geq \dfrac{\underline{\lim}\, a_n}{\overline{\lim}\, b_n}$, if $\overline{\lim}\, b_n > 0$,

and (*ii*) $\overline{\lim}\left(\dfrac{a_n}{b_n}\right) \leq \dfrac{\overline{\lim}\, a_n}{\underline{\lim}\, b_n}$, if $\underline{\lim}\, b_n > 0$.

Follows directly from the above two theorems.

EXERCISE

1. Construct $\{a_n\}$, $\{b_n\}$ such that $a_{n+3} = a_n$, $b_{n+3} = b_n$, $a_n \neq 0$, $b_n \neq 0$

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$\underline{\lim}\, a_n \neq 0$, $\{a_n b_n\}$ and $\{b_n\}$ do not converge but

$$\underline{\lim}(a_n b_n) = \underline{\lim}\, a_n \cdot \overline{\lim}\, b_n = \overline{\lim}\, a_n\ \underline{\lim}\, b_n$$

[*Hint*: $a_{3n} = 1, a_{3n-1} = 2, a_{3n-2} = -2, n = 1, 2, \ldots$
$b_{3n} = 2, b_{3n-1} = -2, b_{3n-2} = 1, n = 1, 2\ldots$]

2. Construct $\{a_n\}$ such that $\overline{\lim}\, a_n > \underline{\lim}\ a_n > 0$

   and either (*a*) $\overline{\lim}(a_n a_{n+1}) = \underline{\lim}(a_n a_{n+1})$,

   or (*b*) $\overline{\lim}(a_n a_{n+1} a_{n+2}) = \underline{\lim}(a_n a_{n+1} a_{n+2})$.

   Show that such a sequence cannot satisfy both (*a*) and (*b*).

   [*Hint*: (*a*) Take $a_{2n} = 1, a_{2n-1} = 2, n = 1, 2, \ldots$
   (*b*) Take $a_{3n} = 1, a_{3n-1} = 2, a_{3n-2} = 2, n = 1, 2\ldots$]

3. The real bounded sequence $\{a_n\}$ is such that

   $|a_n| \to l$, as $n \to \infty$, where l is a real number and

   $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$. Show that $l \neq 0$ and $\overline{\lim}\, a_n = -\underline{\lim}\, a_n$

   [*Hint*: If $|a_n| \to 0$, then $a_n \to 0$ and thus $\overline{\lim}\, a_n = \underline{\lim}\, a_n$.

   Hence $l \neq 0$, show that $\overline{\lim}\, a_n = |l|$ and $\underline{\lim}\, a_n = -|l|$]

4. Let $\{a_n\}$ be any bounded positive sequence and $\{b_n\}$ is a convergent positive sequence. Show that

   $$\overline{\lim}(a_n b_n) = \lim b_n\ \overline{\lim}\, a_n.$$

   and

   $$\underline{\lim}(a_n b_n) = \lim b_n\ \underline{\lim}\, a_n.$$

5. Let $\{a_n\}$ be a real bounded sequence and $\{b_n\}$ is a real convergent sequence. Show that

   $$\overline{\lim}(a_n + b_n) = \overline{\lim}\, a_n + \lim b_n \text{ and}$$

   $$\underline{\lim}(a_n + b_n) = \underline{\lim}\, a_n + \lim b_n$$

6. Let $\{b_n\}$ be a real bounded sequence such that for any real bounded sequence $\{a_n\}$, $\overline{\lim}(a_n + b_n) = \overline{\lim}\, a_n + \overline{\lim}\, b_n$. Show that $\{b_n\}$ is convergent.

7. Let $\{b_n\}$ be a positive real bounded sequence such that for any positive real bounded sequence $\{a_n\}$, $\overline{\lim}(a_n b_n) = (\overline{\lim}\, a_n)(\overline{\lim}\, b_n)$. Show that $\{b_n\}$ is convergent.

8. Let $\{a_n\}$ be a bounded sequence and z, ω be any given numbers (a_n, z, ω may be complex), show that:

   $$\overline{\lim}\, |a_n - z| \leq |z - \omega| + \overline{\lim}\, |a_n - \omega|$$

   $$\underline{\lim}\, |a_n - z| \geq |z - \omega| - \overline{\lim}\, |a_n - \omega|$$

Example 7. Show that $\lim \dfrac{(3n+1)(n-2)}{n(n+3)} = 3.$

■ We know that the sequence $\left\{\dfrac{1}{n}\right\}$ converges to zero, *i.e.*, $\lim \dfrac{1}{n} = 0$.

Now $$\lim \frac{(3n+1)(n-2)}{n(n+3)} = \lim \frac{\left(3+\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)}$$

$$= \frac{\lim\left(3+\frac{1}{n}\right)\lim\left(1-\frac{2}{n}\right)}{\lim\left(1+\frac{3}{n}\right)} = 3.$$

EXERCISE

1. Show that

(i) $\lim \frac{3+2\sqrt{n+1}}{\sqrt{n+1}} = 2$

(ii) $\lim \frac{2n^2-5}{3n^2+7n} = \frac{2}{3}$

(iii) $\lim \frac{1+2+3+\ldots+n}{n^2} = \frac{1}{2}$ $\frac{n(n+1)}{2}$

(iv) $\lim \frac{1+3+3+\ldots+(2n-1)}{n^2} = 1$ $\frac{n^2+2n}{n^2}$ $\frac{1+\frac{2}{n}}{1} = 1$

(v) $\lim [\sqrt{n+1}-\sqrt{n}] = 0.$

2. Show that $\lim a_n = a \Rightarrow \lim|a_n| = |a|.$

Also by considering $a_n = (-1)^n$ or $(-1)^n\left(1+\frac{1}{n}\right)$,

show that the converse is not always true.

Is the converse true if $a = 0$?

3. If $\{a_n\}$ converges and $\{b_n\}$ diverges, show that

(i) $\lim \frac{a_n}{b_n} = 0.$

(ii) $\{a_n + b_n\}$ is divergent.

4. Given that $\lim a_n = a$, $\lim b_n = b$, and $\{S_n\}$ and $\{T_n\}$ are two sequences, where

$$S_n = \max(a_n, b_n)$$
$$T_n = \min(a_n, b_n).$$

Show that the sequences $\{S_n\}$ and $\{T_n\}$ are convergent and that

$$\lim S_n = \max(a, b),$$
$$\lim T_n = \min(a, b).$$

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[Hint: $\max(a_n, b_n) = \frac{1}{2}(a_n + b_n) + \frac{1}{2}|a_n - b_n|$

$\min(a_n, b_n) = \frac{1}{2}(a_n + b_n) - \frac{1}{2}|a_n - b_n|$.]

5. Use Cauchy's General Principle of Convergence to show that the following sequences are convergent:

(i) $\left\{\frac{n}{n+1}\right\}$,

(ii) $\left\{\frac{(-1)^n}{n}\right\}$,

(iii) $\left\{1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right\}$,

(iv) $\left\{1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n-1}}{n}\right\}$.

8. SOME IMPORTANT THEOREMS

Theorem 20. *If* $\lim a_n = a$ *and* $a_n \geq 0$, *for all* n, *then* $a \geq 0$.

Let, if possible, $a < 0$.

Since $\lim a_n = a$, therefore for a given $\varepsilon > 0$, $\exists$ a positive integer m, such that

$$|a_n - a| < \varepsilon, \ \forall\, n \geq m$$

$$\Rightarrow \qquad a - \varepsilon < a_n < a + \varepsilon, \ \forall\, n \geq m$$

Let us select $\varepsilon = -\frac{1}{2}a, \lim\limits_{n\to\infty}$ so that corresponding to $\varepsilon = -\frac{1}{2}a > 0$, $\exists$ a positive integer m_1 such that

$$a + \frac{1}{2}a < a_n < a - \frac{1}{2}a, \ \forall\, n \geq m_1$$

$$\Rightarrow \qquad \frac{3}{2}a < a_n < \frac{a}{2}, \ \forall\, n \geq m_1$$

or

$$a_n < \frac{a}{2}, \ \forall\, n \geq m_1$$

Since $a < 0$, therefore it follows that $a_n < 0$, for $n \geq m_1$. This contradicts the fact that $a_n > 0$, for all n. Therefore, the supposition is wrong.

Hence

$$a \nless 0 \Rightarrow a \geq 0.$$

Theorem 21. *If* $\{a_n\}, \{b_n\}$ *are two sequences such that*

(i) $a_n \leq b_n$, $\forall\, n$ *and*

(ii) $\lim a_n = a$, $\lim b_n = b$, *then* $a \leq b$.

Let, if possible, $a > b$.

Let $a - b = 3\varepsilon$, so that the neighbourhoods $]b-\varepsilon, b+\varepsilon[$, $]a-\varepsilon, a+\varepsilon[$ of b and a, respectively, are disjoint.

Since $\{a_n\}$, $\{b_n\}$ converge to a and b, respectively, therefore corresponding to $\varepsilon > 0$, $\exists$ positive integers m_1 and m_2, respectively, such that

$$a - \varepsilon < a_n < a + \varepsilon, \ \forall\, n \geq m_1$$

$$b - \varepsilon < b_n < b + \varepsilon, \ \forall\, n \geq m_2.$$

Let $m = \max\,(m_1, m_2)$

$$\therefore \qquad a_n \in\,]a-\varepsilon, a+\varepsilon[, \ \forall\, n \geq m$$

$$b_n \in\,]b-\varepsilon, b+\varepsilon[, \ \forall\, n \geq m$$

Consequently

$$b_n < a_n, \ \forall\, n \geq m$$

which contradicts the fact that

$$a_n \leq b_n, \ \forall\, n.$$

Hence, our supposition is wrong and therefore $a \leq b$.

Ex. 1. Deduce theorem 21 from theorem 20 by considering the sequence $\{c_n\}$, where $c_n = b_n - a_n$.

Ex. 2. If $\{a_n\}$, $\{b_n\}$ are two sequences, such that $a_n \leq b_n$, $\forall\, n$, then show that $\underline{\lim}\, a_n \leq \underline{\lim}\, b_n$, and $\overline{\lim}\, a_n \leq \overline{\lim}\, b_n$.

Theorem 22. ***Sandwich theorem.*** *If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences such that*

(i) $a_n \leq b_n \leq c_n$, $\forall\, n$, ...(1)

and *(ii)* $\lim a_n = \lim c_n = l$

then

$$\lim b_n = l.$$

Let $\varepsilon > 0$ be given.

Now since $\{a_n\}$, $\{c_n\}$ both converge to l, therefore $\exists$ positive integers m_1, m_2, such that

$$|a_n - l| < \varepsilon, \ \forall\, n \geq m_1 \qquad ...(2)$$

and

$$|c_n - l| < \varepsilon, \ \forall\, n \geq m_2 \qquad ...(3)$$

Let $m = \max\,(m_1, m_2)$.

then, for $n \geq m$, we have from (1), (2) and (3)

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon$$

$$\Rightarrow \qquad l - \varepsilon < b_n < l + \varepsilon, \ \forall\, n \geq m$$

$$\Rightarrow \qquad |b_n - l| < \varepsilon, \ \forall\, n \geq m.$$

Hence

$$\lim b_n = l.$$

Ex. If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences such that

$$a_n \le b_n \le c_n \quad \forall\, n$$

Then

(*i*) $\overline{\lim}\, a_n = \overline{\lim}\, c_n = \bar{l}$ implies $\overline{\lim}\, b_n = \bar{l}$, and

(*ii*) $\underline{\lim}\, a_n = \underline{\lim}\, c_n = \underline{l}$ implies $\underline{\lim}\, b_n = \underline{l}$

Example 8. Show that the sequence $\{b_n\}$, where

$$b_n = \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}\right],$$

converges to zero.

▪ Evidently,

$$\frac{n}{(2n)^2} \le b_n \le \frac{n}{n^2}, \quad \forall\, n$$

$$\Rightarrow \qquad \frac{1}{4n} \le b_n \le \frac{1}{n}, \quad \forall\, n.$$

Now the sequences $\{a_n\}$, $\{c_n\}$, where $a_n = \dfrac{1}{4n}$ and $c_n = \dfrac{1}{n}$ are such that

(*i*) $a_n \le b_n \le c_n$, $\forall\, n$, and

(*ii*) $\lim a_n = \lim c_n = 0$

$\therefore\ \lim b_n = 0.$

Ex. 1. Show that the sequence $\{b_n\}$, where

$$b_n = \left\{\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}}\right\} = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$$

converges to 1.

Ex. 2. Show that:

(*i*) $\displaystyle \lim \sum_{k=1}^{n} \frac{1}{n^2+k} = 0,$

(*ii*) $\displaystyle \lim \sum_{k=1}^{n} \frac{1}{\sqrt{n+k}} = \infty.$

Theorem 23. ***Cauchy's first theorem on limits.*** *If* $\lim\limits_{n\to\infty} a_n = l$, then

$$\lim_{n\to\infty}\left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) = l.$$

Let $b_n = a_n - l$.

Now, since $\lim a_n = l$, therefore

$$\lim b_n = 0.$$

Also $$\frac{a_1 + a_2 + \dots + a_n}{n} = l + \frac{b_1 + b_2 + \dots + b_n}{n}, \quad \forall\, n$$

so that we have to show that

$$\lim \frac{b_1 + b_2 + \dots + b_n}{n} = 0, \text{ when } \lim b_n = 0.$$

Now, since $\{b_n\}$ is convergent, therefore it is bounded and hence $\exists$ a number $K > 0$ such that

$$|b_n| \le K, \quad \forall\, n.$$

Let $\varepsilon > 0$ be given. Since $\{b_n\}$ converges to zero, therefore $\exists$ a positive m such that

$$|b_n| < \tfrac{1}{2}\varepsilon, \text{ for } n \ge m$$

Also,

$$\left|\frac{b_1 + b_2 + \dots + b_n}{n}\right| = \left|\frac{b_1 + b_2 + \dots + b_m}{n} + \frac{b_{m+1} + \dots + b_m}{n}\right|$$

$$\le \frac{|b_1| + |b_2| + \dots + |b_m|}{n} + \frac{|b_{m+1}| + \dots + |b_n|}{n}$$

$$< \frac{mK}{n} + \frac{\varepsilon}{2}\frac{(n-m)}{n}, \quad \forall\, n \ge m$$

$$< \frac{mK}{n} + \frac{\varepsilon}{2}.$$

Let m_1 be a positive integer greater than $\dfrac{2mK}{\varepsilon}$, so that

$$\frac{mK}{n} < \frac{\varepsilon}{2}, \text{ where } n \ge m_1.$$

Thus, for $n \ge \max(m, m_1)$, we have

$$\left|\frac{b_1 + b_2 + \dots + b_n}{n}\right| < \varepsilon$$

$\Rightarrow$ $$\lim \frac{b_1 + b_2 + \dots + b_n}{n} = 0.$$

Hence, $$\lim \frac{a_1 + a_2 + \dots + a_n}{n} = l.$$

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Ex. Let $\{a_n\}$ be a bounded sequence.
Prove that

$$\underline{\lim}\, a_n \le \underline{\lim}\left((a_1 + a_2 + \ldots + a_n)/n\right) \le \overline{\lim}\left((a_1 + a_2 + \ldots + a_n)/n\right) \le \overline{\lim}\, a_n .$$

[*Hint*: For $\varepsilon > 0$ $\exists m$ such that $a_n > \underline{\lim}\, a_n - \varepsilon$, $\forall\, n > m$

Thus

$$\frac{a_1 + a_2 + \ldots + a_n}{n} > (\underline{\lim}\, a_n - \varepsilon)\frac{n - m}{n} + \frac{a_1 + a_2 + \ldots + a_m}{n}, \text{ for } n > m$$

and

$$\underline{\lim}\frac{(a_1 + a_2 + \ldots a_n)}{n} \ge (\underline{\lim}\, a_n - \varepsilon) \lim \frac{n - m}{n} = \underline{\lim}\, a_n - \varepsilon .$$

This is true for all $\varepsilon > 0$. Hence the result].

Note: The converse of the theorem is not true.
Let $a_n = (-1)^n$, so that

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = 0, \text{ if } n \text{ is even,}$$

$$= -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

$$\Rightarrow \quad \lim \frac{a_1 + a_2 + \ldots + a_n}{n} = 0.$$

But $\{a_n\}$ is not convergent.

Example 9. Show that

$$\lim_{n \to \infty}\left[\frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \ldots + \frac{1}{\sqrt{n^2 + n}}\right] = 1.$$

■ Let

$$a_k = \frac{n}{\sqrt{n^2 + k}}, \; k = 1, 2 \ldots n$$

$$\therefore \qquad \lim a_n = \lim \frac{1}{\sqrt{1 + \frac{1}{n}}} = 1.$$

Thus by *Cauchy's first theorem on limits*, we have

$$\lim \frac{a_1 + a_2 + \ldots + a_n}{n} = 1.$$

or

$$\lim \frac{1}{n}\left[\frac{n}{\sqrt{n^2+1}}+\frac{n}{\sqrt{n^2+2}}+\ldots+\frac{n}{\sqrt{n^2+n}}\right]=1$$

or

$$\lim\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right]=1.$$

Example 10. Show that

$$\lim_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}\right]=1.$$

■ Consider $a_n = n^{1/n}$.

Now $\lim a_n = \lim n^{1/n} = 1$

$\therefore$ *By Cauchy's first theorem on limits,*

$$\lim \frac{a_1+a_2+\ldots+a_n}{n}=1$$

$$\Rightarrow \qquad \lim \frac{1}{n}\left[1+2^{1/2}+3^{1/3}+\ldots+n^{1/n}\right]=1.$$

Ex. Show that:

(*i*) $\lim\left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+\ldots+\frac{1}{\sqrt{2n}}\right]=\infty$

(*ii*) $\lim \frac{1}{n}\left[1+\frac{1}{2}+\ldots+\frac{1}{n}\right]=0$

(*iii*) $\lim\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2}\right]=0$

Theorem 24. *If a sequence $\{a_n\}$ of positive terms converges to a positive limit l, then so does the sequences $\{(a_1a_2\ldots a_n)^{1/n}\}$ of its geometric means, i.e., if $\lim_{n\to\infty} a_n = l$, then*

$$\lim_{n\to\infty} (a_1 a_2 \ldots a_n)^{1/n} = l.$$

Let $\lim_{n\to\infty} a_n = l$.

Since all terms are positive, the sequence $\{\log a_n\}$ of logarithms converges to $\log l$, *i.e.*, $\lim_{n\to\infty} \log a_n = \log l$.

Hence by *Cauchy's first theorem on limits,*

$$\lim \frac{\log a_1 + \log a_2 + \ldots + \log a_n}{n} = \log l$$

$$\Rightarrow \qquad \lim \log (a_1 a_2 \ldots a_n)^{1/n} = \log l$$

$$\Rightarrow \qquad \lim (a_1 a_2 \ldots a_n)^{1/n} = l.$$

Note: The converse, however, is not true.

ILLUSTRATIONS

1. $\left(1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}\right)^{1/n} \to 1$ as $n \to \infty$

because $\lim \frac{n}{n-1} = 1.$

But $\left(1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1}\right)^{1/n} = (n)^{1/n}$

$$\therefore \qquad \lim_{n \to \infty} n^{1/n} = 1.$$

2. We know

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

or

$$\lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n = e$$

Hence, by theorem 3.24, as $n \to \infty$

$$\left[\left(\frac{2}{1}\right)^1 \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n} \to e$$

or

$$\frac{n+1}{(n!)^{1/n}} \to e$$

or

$$\frac{(n!)^{1/n}}{n+1} \to \frac{1}{e}$$

Note: $(n!)^{1/n} \to \infty$ as $n \to \infty$.

3. Let a sequence of positive monotonic decreasing terms

$$u_1, \frac{u_2}{u_1}, \frac{u_3}{u_2}, \ldots, \frac{u_{n+1}}{u_n}, \ldots \text{ converge to } l,$$

i.e.,
$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l.$$

Hence by theorem 3.24, the sequence

$$\left\{\left(u_1 \cdot \frac{u_2}{u_1} \cdot \frac{u_3}{u_2} \cdots \frac{u_{n+1}}{u_n}\right)^{1/n+1}\right\} \text{ converges to } l,$$

i.e.,
$$\left\{(u_{n+1})^{1/n+1}\right\} \text{ converges to } l$$

or

$$\lim_{n\to\infty} (u_n)^{1/n} = l$$

Thus,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l \quad \Rightarrow \quad \lim_{n\to\infty} (u_n)^{1/n} = l$$

The converse is not true.

A result that will be used while comparing the relative strength of the Ratio and the Root test for positive term infinite series. Also see Theorem 3.21.

Theorem 25. ***Cesaro's theorem.*** *If the sequences $\{a_n\}$ and $\{b_n\}$ converge to finite limits a and b respectively, then*

$$\lim_{n\to\infty} \frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} = ab$$

Let $a_n = a + \alpha_n$, where $|\alpha_n| \to 0$ as $n \to \infty$. On substituting for $a_1, a_2, \ldots, a_n$, we get

$$\frac{a_1 b_n + a_2 b_{n-1} + \ldots a_n b_1}{n} = \frac{a(b_1 + b_2 + \ldots + b_n)}{n} + \frac{\alpha_1 b_n + \alpha_2 b_{n-1} + \ldots + \alpha_n b_1}{n}$$

$$\leq \frac{a(b_1 + b_2 + \ldots + b_n)}{n} + \frac{B(|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|)}{n}$$

where B is the upper bound of the numbers $|b_1|, |b_2|, \ldots$

Also by *Cauchy's first theorem on limits*

$$\lim \frac{b_1 + b_2 + \ldots + b_n}{n} = b$$

and

$$\lim \frac{|\alpha_1|+|\alpha_2|+...|\alpha_n|}{n} = 0$$

$\therefore$

$$\lim \frac{a_1 b_n + a_2 b_{n-1} + ... a_n b_1}{n} = ab$$

Theorem 26. ***Cauchy's second theorem on limits.*** *If all the terms of a sequence $\{u_n\}$ are positive and if $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists then so does $\lim\limits_{n\to\infty} (u_n)^{1/n}$, and the two limits are equal, i.e., $\lim (u_n)^{1/n} = \lim \frac{u_{n+1}}{u_n}$, provided the latter limit exists.*

Let $\lim\limits_{n\to\infty} \frac{u_{n-1}}{u_n} = l$, a finite real number.

Hence for any $\varepsilon > 0$ there exists a positive integer m, such that

$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \ \forall\, n \geq m.$$

Putting $n = m, m+1, ..., n-1$ and multiplying, we get

$$(l-\varepsilon)^{n-m} < \frac{u_n}{u_m} < (l+\varepsilon)^{n-m}, \ \forall\, n \geq m$$

or

$$(l-\varepsilon)^{n-m} u_m < u_n < (l+\varepsilon)^{n-m} u_m, \ \forall\, n \geq m$$

or

$$(l-\varepsilon)^{1-m/n} u_m^{1/n} < u_n^{1/n} < (l+\varepsilon)^{1-m/n} u_m^{1/n}, \ \forall n \geq m.$$

If the first and the last expression in the above inequality are called A_n and B_n, then

$$\lim A_n = l - \varepsilon \text{ and } \lim B_n = l + \varepsilon$$

because u_m is a finite positive quantity.

It is, therefore, possible to choose a positive integer m_o such that for $n \geq m_o$,

$$A_n > l - 2\varepsilon \text{ and } B_n < l + 2\varepsilon$$

$\therefore$

$$l - 2\varepsilon < u_n^{1/n} < l + 2\varepsilon, \ \forall\, n \geq \max(m, m_0)$$

i.e.,

$$\lim u_n^{1/n} = l$$

Corollary. More generally, if $\overline{\lim} \frac{u_{n+1}}{u_n} = U$ and $\underline{\lim} \frac{u_{n+1}}{u_n} = L$, we can modify the inequalities by replacing $l - \varepsilon$ by $L - \varepsilon$ and $l + \varepsilon$ by $U + \varepsilon$. Thus

$$L - 2\varepsilon < u_n^{1/n} < U + 2\varepsilon, \ \forall\, n \geq \max(m, m_0)$$

$\Rightarrow$

$$L \leq \underline{\lim}\, u_n^{1/n} \text{ and } \overline{\lim}\, u_n^{1/n} \leq U$$

$$\therefore \qquad \underline{\lim}\frac{u_{n+1}}{u_n} \le \underline{\lim}\, u_n^{1/n} \le \overline{\lim}\, u_n^{1/n} \le \overline{\lim}\frac{u_{n+1}}{u_n},$$

a very useful result.

Note: Taking $u_{2n-1} = \dfrac{1}{2^n}$ and $u_{2n} = \dfrac{1}{3^n}$, for $n = 1, 2, 3, \ldots$

$$0 = \underline{\lim}\frac{u_{n+1}}{u_n} < \frac{1}{\sqrt{3}} = \underline{\lim}\, u_n^{1/n} < \frac{1}{\sqrt{2}} = \overline{\lim}\, u_n^{1/n} < \infty = \overline{\lim}\frac{u_{n+1}}{u_n}.$$

Example 11. Show that the sequences $\{a_n^{1/n}\}$ and $\{b_n^{1/n}\}$, where

(*i*) $a_n = \dfrac{\lfloor 3n}{(\lfloor n)^3}$ and

(*ii*) $b_n = \dfrac{n^n}{(n+1)(n+2)\ldots(n+n)}$

converge and find their limits.

■ (*i*)
$$a_n = \frac{\lfloor 3n}{(\lfloor n)^3}$$

$$a_{n+1} = \frac{\lfloor 3n+3}{(\lfloor n+1)^3}$$

$$\lim\frac{a_{n+1}}{a_n} = \lim\frac{\lfloor 3n+3}{(\lfloor n+1)^3}\,\frac{(\lfloor n)^3}{\lfloor 3n}$$

$$= \lim\frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = 27$$

By *Cauchy's second theorem on limits*,

$$\lim a_n^{1/n} = \lim\frac{a_{n+1}}{a_n} = 27$$

(*ii*) $b_n = \dfrac{n^n}{(n+1)(n+2)\ldots(n+n)}$

$$b_{n+1} = \frac{(n+1)^{n+1}}{(n+2)(n+3)\ldots 2n(2n+1)(2n+2)}$$

$$\therefore \quad \lim \frac{b_{n+1}}{b_n} = \lim \frac{(n+1)^{n+1}(n+1)(n+2)\ldots(2n)}{(n+2)(n+3)\ldots(2n+1)(2n+2)n^n}$$

$$= \lim \frac{\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^2}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} = \frac{e}{4}$$

Hence by *Cauchy's second theorem on limits,*

$$\lim b_n^{1/n} = \lim \frac{b_{n+1}}{b_n} = \frac{e}{4}$$

Ex. Find $\lim\limits_{n\to\infty} \frac{1}{n}((m+1)(m+2)\ldots(m+n))^{1/n}$.

Theorem 27. *If $\{a_n\}$ be a sequence such that $\lim \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$, then $\lim a_n = 0$.*

Since $|l| < 1$, we can choose a positive number ε, so small such that $|l| + \varepsilon < 1$.

Now since $\lim \frac{a_{n+1}}{a_n} = l$, therefore $\exists$ a positive integer m, such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \forall\, n \geq m$$

$$\Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| - |l| \leq \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon, \forall\, n \geq m$$

$$\Rightarrow \quad \left|\frac{a_{n+1}}{a_n}\right| < |l| + \varepsilon = k \text{ (say), where } k < 1, \ \forall\, n \geq m$$

Putting $n = m, m+1, \ldots, (n-1)$ in turn and multiplying, we get

$$\left|\frac{a_n}{a_m}\right| < k^{n-m}, \ \forall\, n \geq m$$

$$\Rightarrow \quad |a_n| < \frac{|a_m|}{k^m} \cdot k^n, \forall\, n \geq m$$

But as $k < 1$, $k^n \to 0$, hence

$$\lim a_n = 0.$$

Theorem 28. *If $\{a_n\}$ be a sequence, such that $\lim \frac{a_{n+1}}{a_n} = l > 1$, then*

$$\lim a_n = \infty.$$

Since $l > 1$, we can choose a positive number ε, such that

$$l - \varepsilon > 1.$$

Now, since $\lim \frac{a_{n+1}}{a_n} = l$, therefore $\exists$ a positive integer m, such that

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon, \ \forall \ n \geq m$$

$$\Rightarrow \quad l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon, \ \forall \ n \geq m$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} > l - \varepsilon = k \text{ (say)}, \ \forall \ n \geq m \text{ where } k > 1$$

Putting $n = m, m + 1, ..., (n - 1)$ in turn and multiplying, we get

$$\left| \frac{a_n}{a_m} \right| \geq \frac{a_n}{a_m} > k^{n-m}, \ \forall \ n \geq m$$

$$\Rightarrow \quad |a_n| > \frac{|a_m|}{k^m} k^n, \ \forall \ n \geq m$$

But as $k > 1$, $k^n \to \infty$

$$\therefore \quad \lim a_n = \infty$$

Example 12. Show that for any real number x, $\lim \frac{x^n}{n!} = 0.$

■ Let $a_n = \frac{x^n}{n!}$

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1},$$

so that $\lim \frac{a_{n+1}}{a_n} = 0 < 1.$

Hence by theorem 27, $\lim a_n = 0.$

Example 13. Show that

$$\lim \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!} x^n = 0, |x| < 1$$

■ Let

$$a_n = \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!} x^n$$

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$$\therefore \qquad \lim \frac{a_{n+1}}{a_n} = \lim \left(\frac{m-n}{n+1}\right) x = \lim \frac{\frac{m}{n} - 1}{1 + \frac{1}{n}} x = -x$$

But $|-x| = |x| < 1$.

$\therefore$ By theorem 3.27, $\lim a_n = 0$.

Ex. Show that $\lim nr^n = 0$, if $|r| < 1$.

9. MONOTONIC SEQUENCES

A sequence $\{S_n\}$ is said to be *monotonic increasing*, if $S_{n+1} \geq S_n \ \forall \ n$, and *monotonic decreasing* if $S_{n+1} \leq S_n \ \forall \ n$. It is said to be monotonic if it is either monotonic increasing or monotonic decreasing.

A sequence (S_n) is *strictly increasing* if $\forall \ n, S_{n+1} > S_n$ and *strictly decreasing* if $S_{n+1} < S_n$.

The importance of monotonic sequences lies in the fact that they cannot oscillate. They either converge or diverge. Following theorem will make the point clear.

Theorem 29. *A necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

The condition is necessary for we know that every convergent sequence is bounded.

The condition is sufficient. Let a bounded sequence $\{S_n\}$ be monotonic increasing. Let S denotes its range, which is evidently bounded. By the completeness property, S has the least upper bound (the supremum), say M.

We shall show that $\{S_n\}$ converges to M.

Let ε be any pre-assigned positive number.

Now since $M - \varepsilon$ is a number less than the supremum M, there exists at least one member say S_m such that $S_m > M - \varepsilon$.

As $\{S_n\}$ is a monotonic increasing sequence,

$$\therefore \qquad S_n \geq S_m > M - \varepsilon, \ \forall \ n \geq m$$

Again, since M is the supremum,

$$S_n \leq M < M + \varepsilon, \ \forall \ n$$

Thus

$$M - \varepsilon < S_n < M + \varepsilon, \ \forall \ n \geq m$$

$$\Rightarrow \qquad |S_n - M| < \varepsilon, \ \forall \ n \geq m$$

$$\Rightarrow \qquad \{S_n\} \text{ converges and } \lim S_n = M.$$

We may similarly consider the case of a bounded monotonic decreasing sequence.

Corollary 1. A monotonic increasing bounded above sequence converges to its least upper bound and a monotonic decreasing bounded below to the greatest lower bound.

Corollary 2. Every monotonic increasing sequence which is not bounded above, diverges to $+\infty$.

Let $\{S_n\}$ be a monotonic increasing sequence, not bounded above. Let G be any real number however large.

Since the sequence $\{S_n\}$ is unbounded above the monotonic increasing, $\exists$ a positive integer m such that

$$S_m > G \text{ and } S_n \geq S_m, \quad \forall\, n \geq m$$

$$\Rightarrow \qquad S_n > G, \quad \forall\, n \geq m$$

Hence, $\lim S_n = +\infty$.

Corollary 3. Every monotonic decreasing sequence which is not bounded below, diverges to $-\infty$.

9.1 Subsequences

If $\{S_n\} = \{S_1, S_2, S_3, ...\}$ be a sequence, then any infinite succession of its terms, picked out in any way (but preserving the original order), is called a *subsequence* of $\{S_n\}$, or, in other words if $\{n_k\}$ be a strictly monotonic increasing sequence of natural numbers, *i.e.*, $n_1 < n_2 < n_3 < ...$, then $\{S_{n_k}\}$ is a *subsequence* of the sequence $\{S_n\}$.

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ILLUSTRATIONS

1. $\{S_2, S_4, S_6, ..., S_{2n}, ...\}$ is a subsequence of $\{S_n\}$.
2. $\{S_1, S_4, S_9, ..., S_{n^2}, ...\}$ is a subsequence of $\{S_n\}$.
3. $\{S_7, S_8, S_9, ...\}$ is a subsequence of $\{S_n\}$, which is obtained by removing a finite number of terms from the beginning of $\{S_n\}$.

Without going into a formal proof we state :

1. *A sequence $\{S_n\}$ converges to s if and only if its every subsequence converges to s. Similarly* $\lim S_n = \infty(-\infty)$ *if and only if every subsequence of $\{S_n\}$ tends to* $\infty(-\infty)$.
2. *If ξ is a limit point of a sequence $\{S_n\}$, then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ which converges to ξ, i.e.,* $\lim_{k\to\infty} S_{n_k} = \xi$.

Ex. The subsequence $\{x, x^4, x^9, x^{16}, ..., x^{n^2}, ...\}$ of $\{x^n\}$ converges to zero if $|x| < 1$, for the sequence $\{x^n\}$ converges to zero for $|x| < 1$.

Example 14. Show that the sequence $\{S_n\}$, where $S_n = \left(1 + \frac{1}{n}\right)^n$, is convergent and that $\lim \left(1 + \frac{1}{n}\right)^n$ lies between 2 and 3.

■ Expanding by Binomial theorem, since n is a positive integer, we get

$$S_n = \left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots + \frac{n(n-1)\dots 3\cdot 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

Since each term beyond the first two terms on the R.H.S. is an increasing function of n, it follows that $\{S_n\}$ is a monotonic increasing sequence. Again since each bracket on the R.H.S. is positive, therefore, we have

$$2 < S_n < 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 3$$

$\Rightarrow$ $$2 < S_n < 3$$

Thus $\{S_n\}$ is a bounded and monotonic increasing sequence and so has a limit, which is generally denoted by e.

$\therefore$ $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e, \text{ where } 2 < e < 3$$

Example 15. Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}, \quad \forall\, n \in \mathbf{N}$$

is convergent.

■ Now

$$S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0, \ \forall\, n$$

$\therefore$ The sequence $\{S_n\}$ is monotonic increasing.

Again

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} = 1$$

i.e., $$0 < S_n < 1$$

$\therefore$ The sequence is bounded.

Hence, the sequence being bounded and monotonic increasing, is convergent.

Example 16. Show that the sequence, $\{S_n\}$, where

$$S_n = \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}, \quad \forall\, n \in \mathbf{N}$$

is convergent.

■ Now

$$S_{n+1} - S_n = \frac{1}{n+1!} > 0, \forall\, n$$

$\therefore$ The sequence $\{S_n\}$ is monotonic increasing.

Again,

$$S_n = \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$= \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} < 2$$

$\Rightarrow$ $$0 < S_n < 2$$

$\therefore$ The sequence is bounded.

Thus, the sequence $\{S_n\}$, being bounded and monotonic increasing, is convergent.

Example 17. Show that the sequence $\{S_n\}$, defined by the recursion formula $S_{n+1} = \sqrt{3S_n}$, $S_1 = 1$, converges to 3.

■ The terms of the sequence $\{S_n\}$ are

$$1, \sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \dots$$

Clearly

$$S_2 > S_1$$
$$S_3 > S_2$$

Also

$$S_{m+1} > S_m \Rightarrow \sqrt{3S_{m+1}} > \sqrt{3S_m} \Rightarrow S_{m+2} > S_{m+1}.$$

Thus by *Mathematical Induction*, the sequence $\{S_n\}$ is monotonic increasing.

Again,

$$S_1 < 3$$
$$S_2 < 3$$
$$S_3 = \sqrt{3S_2} < 3$$

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and
$$S_m < 3 \Rightarrow \sqrt{3S_m} < \sqrt{3.3} \Rightarrow S_{m+1} < 3$$

So, again, by *Mathematical Induction*,
$$0 < S_n < 3, \ \forall n$$

Hence, the sequence $\{S_n\}$, being bounded and monotonic increasing, is convergent.

Let $\lim S_n = l$.

Since $\lim S_{n+1} = \lim \sqrt{3S_n}$

$\therefore$
$$l = \sqrt{3l} \Rightarrow l = 0 \text{ or } 3$$

But $l \neq 0$, since $S_n \geq 1, \ \forall n$

Hence, $\lim S_n = 3$.

Example 18. If $\{S_n\}$ is a sequence, such that
$$S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}}, \ b > a, \ \forall n \geq 1 \text{ and } S_1 = a > 0$$

then show that the sequence $\{S_n\}$ is an increasing bounded above sequence and $\lim\limits_{n\to\infty} S_n = b$.

■ Given that
$$S_{n+1} = \sqrt{\frac{ab^2 + S_n^2}{a+1}}, \ b > a, \ \forall n \geq 1$$

For $n = 1$
$$S_2 = \sqrt{\frac{ab^2 + S_1^2}{a+1}} = \sqrt{\frac{ab^2 + a^2}{a+1}},$$

and
$$S_2^2 - S_1^2 = \frac{ab^2 + a^2}{a+1} - a^2 = \frac{a(b^2 - a^2)}{a+1} > 0$$

$\therefore$
$$S_2 > S_1 \qquad (\because b > a > 0)$$

For $n \geq 1$, we have
$$S_{n+1}^2 - S_n^2 = \frac{ab^2 + S_n^2}{a+1} - S_n^2 = \frac{a(b^2 - S_n^2)}{a+1} \quad ...(1)$$

Now,
$$S_1 = a < b, \ b^2 - S_2^2 = \frac{b^2 - a^2}{a+1} > 0$$

$\therefore$
$$S_2^2 < b^2$$

Assume $S_m < b$, for some $m \in \mathbf{N}$, then

$$S_{m+1}^2 - b^2 = \frac{ab^2 + S_m^2}{a+1} - b^2 = \frac{S_m^2 - b^2}{a+1} < 0$$

So by the Principle of Mathematical Induction,

$$0 < S_n < b, \ \forall\, n \in \mathbf{N}$$

$\therefore$ The sequence $\{S_n\}$ is bounded.

Also (1) implies that the sequence $\{S_n\}$ is monotonic increasing. Hence, the sequence $\{S_n\}$, being bounded and monotonically increasing, is convergent.

Let $$\lim_{n\to\infty} S_n = l$$

Since $$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} \sqrt{\frac{ab^2 + S_n^2}{a+1}}$$

$\therefore$ $$l^2 = \frac{ab^2 + l^2}{a+1} \Rightarrow l = \pm b \qquad (\because\ a \neq 0)$$

But $l \neq -b$ as $S_n \geq a > 0, \ \forall\, n$

Hence $l = b$.

Example 19. If $\{S_n\}$ be a sequence of positive real numbers such that $S_n = \frac{1}{2}(S_{n-1} + S_{n-2}), \forall\, n > 2$, then show that $\{S_n\}$ converges. Also find $\lim S_n$.

■ If $S_1 = S_2$, then evidently $S_n = S_1, \ \forall\, n$

so that the sequence converges to S_1.

When $S_1 \neq S_2$, let $S_1 < S_2$.

Putting $n = 3, 4, 5, \ldots, m$, in the relation $S_n = \frac{1}{2}(S_{n-1} + S_{n-2})$, we find that

$$\left.\begin{aligned} S_3 &= \frac{1}{2}(S_2 + S_1) \\ S_4 &= \frac{1}{2}(S_3 + S_2) \\ S_5 &= \frac{1}{2}(S_4 + S_3) \\ &\ldots\ldots\ldots\ldots \\ S_m &= \frac{1}{2}(S_{m-1} + S_{m-2}) \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \qquad \ldots(1)$$

$\Rightarrow$
$$\begin{aligned} S_1 &< S_3 < S_2 \\ S_3 &< S_4 < S_2 \\ S_3 &< S_5 < S_4 \\ S_5 &< S_6 < S_4 \\ &\ldots\ldots\ldots \end{aligned}$$

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Thus it appears that

$$\left.\begin{aligned} S_1 < S_3 < S_5 < \ldots \\ S_2 > S_4 > S_6 > \ldots \end{aligned}\right\} \qquad \ldots(2)$$

and

Now

$$S_{n+2} - S_n = \frac{1}{2}(S_{n+1} + S_n) - S_n$$

$$= \frac{1}{2}(S_{n+1} - S_n) \qquad \ldots(3)$$

$$= \frac{1}{4}(S_n - S_{n-2}) \qquad \ldots(4)$$

From (4), we easily see that

(*i*) The subsequence of odd terms is monotonic increasing, *i.e.*,

$$S_1 < S_3 < S_5 < \ldots$$

(*ii*) The subsequence of even terms is monotonic decreasing, *i.e.*,

$$\ldots < S_6 < S_4 < S_2.$$

Again from (3) when n is even, putting $n = 2m$, we get

$$S_{2m+2} - S_{2m} = \frac{1}{2}(S_{2m+1} - S_{2m})$$

$$\therefore \qquad S_{2m+2} < S_{2m} \Rightarrow S_{2m+1} < S_{2m}$$

but

$$S_{2m} < S_{2m-2} < \ldots < S_4 < S_2$$

$\therefore$ Every odd term is less than every even term, *i.e.*,

$$S_1 < S_3 < S_5 < \ldots < S_{2m+1} < S_{2m} < S_{2m-2} < \ldots < S_6 < S_4 < S_2$$

Thus, the odd term subsequence $\{S_{2n+1}\}$ is monotonic increasing and is bounded above (by S_2) and is, therefore, convergent.

Similarly the even term subsequence $\{S_{2n}\}$ is convergent.

Let us now show that the two subsequences converge to the same number.

Let $\{S_{2n+1}\} \to l$ and $\{S_{2n}\} \to l'$.

From the recursion formula,

$$S_{2m} = \frac{1}{2}(S_{2m-1} + S_{2m-2})$$

On taking limits as $m \to \infty$, we get

$$l' = \frac{1}{2}(l + l') \;\Rightarrow\; l = l'.$$

Thus both the subsequences of $\{S_n\}$ converge to l.

$\Rightarrow$ The sequence $\{S_n\}$ converges to l.

Again from (1) by adding,

$$S_k + \frac{1}{2} S_{k-1} = \frac{1}{2}(S_1 + 2S_2)$$

Taking limits when $k \to \infty$, we get

$$l + \frac{1}{2} l = \frac{1}{2}(S_1 + 2S_2)$$

$\therefore$
$$l = \frac{1}{3}(S_1 + 2S_2)$$

Thus the sequence $\{S_n\}$ converges to $\frac{1}{3}(S_1 + 2S_2)$.

It may similarly be shown that the case $S_1 > S_2$ leads to the same result.

Example 20. Show that the sequence $\{a_n\}$ defined by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right), \; n \geq 1 \text{ and } a_1 > 0$$

converges to 3.

■ Now
$$a_2 - a_1 = \frac{1}{2}\left(a_1 + \frac{9}{a_1}\right) - a_1 = \frac{9 - a_1^2}{2a_1} \geq 0, \text{ if } a_1 \leq 3$$

$\Rightarrow$
$$a_2 \geq a_1, \text{ if } a_1 \leq 3$$

Also
$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right) - a_n = \frac{9 - a_n^2}{2a_n} \geq 0, \text{ if } a_n \leq 3.$$

Thus the sequence $\{a_n\}$ is monotonically increasing if $a_n \leq 3$, $\forall\, n$ and decreasing if $a_n \geq 3$, $\forall\, n$. In either case the sequence is monotonic and bounded and therefore it is convergent.

let
$$\lim_{n\to\infty} a_n = l.$$

Now,
$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{9}{a_n}\right) = \frac{1}{2}\left(l + \frac{9}{l}\right)$$

or
$$l^2 = 9$$

or
$$l = \pm 3.$$

But
$$l \neq -3 \text{ as } a_n > 0, \; \forall\, n.$$

Hence $l = 3$.

Example 21. If a sequence $\{S_n\}$ is defined by

$$S_n = \frac{S}{1 + S_{n-1}}, \text{ where } S > 0, S_1 > 0, \; n \geq 2$$

then show that the sequence converges to the positive root of the equation

$$x^2 + x - S = 0.$$

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- Now

$$S_n = \frac{S}{1 + S_{n-1}} \qquad \text{...(1)}$$

$$S_n - S_{n-2} = \frac{S}{1 + S_{n-1}} - \frac{S}{1 + S_{n-3}}$$

$$= \frac{-S\left(S_{n-1} - S_{n-3}\right)}{\left(1 + S_{n-1}\right)\left(1 + S_{n-3}\right)} \qquad \text{...(2)}$$

$$= \frac{S^2\left(S_{n-2} - S_{n-4}\right)}{\left(1 + S_{n-1}\right)\left(1 + S_{n-2}\right)\left(1 + S_{n-3}\right)\left(1 + S_{n-4}\right)} \qquad \text{...(3)}$$

(3) shows that the even and odd terms form separate monotonic subsequences.

Again (2) shows that if odd terms form a monotonic decreasing subsequence, even terms will form a monotonic increasing subsequence and vice-versa.

Since every term of the sequence is positive,

$$\therefore \qquad S - S_n = \frac{SS_{n-1}}{1 + S_{n-1}} > 0$$

$$\Rightarrow \qquad 0 < S_n < S, \ \forall n$$

a result which could have been written directly from (1).

Thus the monotonic increasing subsequence is bounded above by S and the monotonic decreasing subsequence is bounded below by 0.

Hence the two subsequences converge.

Let, if possible, the even term subsequence converge to l and the odd term subsequence to l'.

$\therefore$ Taking the limit, we get from (1)

(*i*) for n even, $l = \dfrac{S}{1 + l'}$ or $ll' + l = S$

(*ii*) for n odd, $l' = \dfrac{S}{1 + l}$ or $ll' + l' = S$

$$\Rightarrow \qquad l = l'$$

Thus both the subsequences converge to the same number l,

$\Rightarrow$ The given sequence $\{S_n\}$ converges to l.

Again from (1), on proceeding to limits we get

$$l = \frac{S}{1 + l} \Rightarrow l^2 + l - S = 0$$

$\therefore$ l is a root of $x^2 + x - S = 0$

where l is positive, for, every term of the sequence is positive.

Example 22. Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by

$$x_1 = \frac{1}{2} \text{ and } y_1 = 1$$

and

$$x_n = \sqrt{x_{n-1}\, y_{n-1}}, \qquad n = 2, 3, 4, \ldots$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right), \qquad n = 2, 3, 4, \ldots$$

Prove that

$$x_{n-1} < x_n < y_n < y_{n-1}, \quad n = 2, 3, \ldots$$

and deduce that both the sequences converge to the same limit l, where $\frac{1}{2} < l < 1$.

- If $0 < a < b$, then geometric mean $G = \sqrt{ab}$ and the harmonic mean

$$H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1}$$

Also

$$a < H < G < b$$

we are given that

$$\frac{1}{2} = x_1 < y_1 = 1$$

On the assumption $x_{n-1} < y_{n-1}$, we have

$$x_{n-1} < x_n < y_{n-1} \qquad \left(\because\ x_n = \sqrt{x_{n-1}\, y_{n-1}}\right)$$

Further

$$x_n < y_n < y_{n-1}$$

because y_n is the harmonic means of x_n and y_{n-1}. It follows by induction that

$$x_{n-1} < x_n < y_n < y_{n-1}, \quad n = 2, 3, \ldots$$

The sequence $\{x_n\}$ increases and is bounded above by $y_1 = 1$. The sequence $\{y_n\}$ decreases and is bounded by $x_1 = \frac{1}{2}$. Hence, both sequences converge. Suppose $x_n \to l$ as $n \to \infty$ and $y_n \to m$ as $n \to \infty$, then

$$l^2 = lm$$

and

$$\frac{1}{m} = \frac{1}{2}\left(\frac{l + m}{lm}\right)$$

Both the sequences yield $l = m$.

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Theorem 30. ***Nested-intervals.*** *If a sequence of closed intervals* $[a_n, b_n]$ *is such that each member* $[a_{n+1}, b_{n+1}]$ *is contained in the preceding one* $[a_n, b_n]$ *and* $\lim (b_n - a_n) = 0$, *then there is one and only one point common to all the intervals of the sequence.*

Since each interval member of the sequence is contained in the preceding one, therefore, we have

$$a_1 \le a_2 \le \ldots \le a_n \le \ldots$$

$$b_1 \ge b_2 \ge \ldots \ge b_n \ge \ldots,$$

so that sequence $\{a_n\}$ is monotonic increasing, and $\{b_n\}$ is monotonic decreasing. Also $a_n \le b_1$ and $b_n \ge a_1$, for all n, so that sequence $\{a_n\}$ is bounded above by b_1 and $\{b_n\}$ bounded below by a_1.

Thus both the sequences are convergent.

Let $\lim a_n = \xi$ and $\lim b_n = \eta$.

Again

$$0 = \lim (b_n - a_n) = \lim b_n - \lim a_n = \xi - \eta$$

$$\Rightarrow \qquad \xi = \eta.$$

Obviously ξ is the upper bound of the sequence $\{a_n\}$ and lower bound of the sequence $\{b_n\}$, and hence

$$a_n \le \xi \le b_n, \ \forall\, n$$

so that ξ belongs to all the intervals.

Let, if possible, ξ_1, ξ_2 be two different points common to all the intervals, and let $\xi_1 < \xi_2$. Then

$$a_n \le \xi_1 < \xi_2 \le b_n, \ \forall\, n$$

i.e.,

$$b_n - a_n \ge \xi_2 - \xi_1 \ne 0, \text{ for all } n,$$

which is a contradiction to the fact that $\xi_1 < \xi_2$ $\quad (\because \lim (b_n - a_n) = 0)$

Hence the result.

The following result is an important generalization of the theorem on nested intervals and is due to G. Cantor.

Theorem 31. ***Cantor's intersection theorem for real line.*** *If* $F = \{F_n\}$ *is a countable class of non-empty closed and bounded sets such that*

$$F_1 \supset F_2 \supset F_3 \ldots \supset F_n, \text{ then } \bigcap_{n=1}^{\infty} F_n \text{ is non-empty.}$$

Since each F_n is a non-empty closed and bounded set, therefore, there exist sequences of real numbers M_n and m_n belonging to F_n, such that

$$M_n = \sup F_n,\ m_n = \inf F_n,$$

then $M_n \ge M_{n+1}$ and $m_n \le m_{n+1}$, for each $n \in \mathbf{N}$. Now the lower bound for the set $\bigcap_{n=1}^{\infty} F_n$ is the lower bound for the sequence $\{M_n\}$ of upper bounds. Thus $\{M_n\}$ is a non-increasing sequence which is bounded below and therefore convergent.

Let $\lim_{n\to\infty} M_n = M$

we shall show that $M \in \bigcap_{n=1}^{\infty} F_n$. Let, if possible, $M \notin \bigcap_{n=1}^{\infty} F_n$. Then there will be at least one neighbourhood, say, $]M-\varepsilon, M+\varepsilon[, \varepsilon > 0$ which contains no point of $\bigcap_{n=1}^{\infty} F_n$

$\Rightarrow$ $]M-\varepsilon, M+\varepsilon[$ contains no point of F_n for some value of n, say, m.

$\Rightarrow$ $]M-\varepsilon, M+\varepsilon[$ contains no point of F_n, for $n \geq m$

$\Rightarrow$ $M_n \notin\,]M-\varepsilon, M+\varepsilon[, \ \forall\, n \geq m$,

contradicting the fact that $\{M_n\}$ converges to M.

Hence, $M \in \bigcap_{n=1}^{\infty} F_n$.

EXERCISE

1. Show that the sequence $\{S_n\}$, where

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}$$

is convergent.

2. Prove that the sequence $\{S_n\}$ defined by the recursion formula

$$S_{n+1} = \sqrt{7 + S_n}, \ S_1 = \sqrt{7},$$

converges to the positive root of

$$x^2 - x - 7 = 0.$$

3. If $\{S_a\}$ be a sequence such that

$$S_{n+1} = 2 - \frac{1}{S_n}, \ n \geq 1 \text{ and } S_1 = \frac{3}{2},$$

then show that the sequence $\{S_n\}$ is bounded and monotonic and converges to 1.

4. Given that $\{a_n\}$ is a sequence such that

$$a_2 \leq a_4 \leq a_6 \leq \ldots \leq a_5 \leq a_3 \leq a_1$$

and a sequence $\{b_n\}$, where $b_n = a_{2n-1} - a_{2n}$, converges to 0, then show that the sequence $\{a_n\}$ is convergent.

5. Let $\{a_n\}$ be a sequence, defined by

$$a_{n+1} = \frac{4 + 3a_n}{3 + 2a_n}, \ n \geq 1 \ \ a_1 = 1,$$

show that $\{a_n\}$ converges to $\sqrt{2}$.

If $\{a_n\}$ be a sequence of positive real numbers such that

$$a_n = \sqrt{a_{n-1}\, a_{n-2}}, \ n > 2,$$

then show that the sequence converges to $(a_1 a_2^2)^{1/3}$.

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7. Let $\{a_n\}$ be a sequence defined by

$$a_{n+1} = \frac{1}{k}\left(a_n + \frac{k}{a_n}\right),\ k > 1 \text{ and } a_1 > 0,$$

show that $\{a_n\}$ converges to $\sqrt{\frac{k}{k-1}}$.

8. Show that the sequence $\{a_n\}$ defined by

$$a_{n+1} = 1 - \sqrt{1 - a_n},\ \forall\, n \geq 1 \text{ and } a_1 < 1$$

converges to '0'.

9. Show that the sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n) \text{ and } b_{n+1} = \sqrt{a_n b_n},$$

converge to the common limit, where

$$a > b > 0 \text{ and } a_1 = \tfrac{1}{2}(a + b),\ b_1 = \sqrt{ab}.$$

(*Hint:* A.M. $\geq$ G.M., the sequences $\{a_n\}$ and $\{b_n\}$ are monotonically decreasing and increasing respectively).

10. If a_1 and b_1 are positive and if for all $n \geq 1$,

$$a_{n+1} = \frac{a_n + b_n}{2},\ \frac{2}{b_{n+1}} = \frac{1}{a_n} + \frac{1}{b_n}$$

then show that $\{a_n\}$ and $\{b_n\}$ are monotonic sequences which converge to the common limit l, where $l^2 = a_1 b_1$.

11. If $\{a_{n_k}\}$ is a subsequence of the sequence $\{a_n\}$, then show that

$$\underline{\lim}\ a_n \leq \underline{\lim}\ a_{n_k} \leq \overline{\lim}\ a_{n_k} \leq \overline{\lim}\ a_n.$$

Deduce that, if the sequence $\{a_n\}$ converges to a, then all its subsequences will converge to the same limit a.

12. If $\{a_n\}$ is a bounded sequence, then show that there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$, such that

$$\lim_{k\to\infty} a_{n_k} = \lim a_n$$