Krishna's

TEXT BOOK on

Calculus

(*For B.A. and B.Sc. Ist year students of All Colleges affiliated to universities in Uttar Pradesh*)

As per U.P. UNIFIED Syllabus

(*w.e.f.* 2011-2012)

Revised Edition

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Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **CALCULUS** has been specially written according to the latest **Unified Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-I Students** of all Universities in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *Managing Director,* **Mr. Sugam Rastogi**, *Executive Director,* **Mrs. Kanupriya Rastogi** *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd.**, **Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

Preface to the Revised Edition

The authors feel great pleasure in presenting the thoroughly revised edition of the book **CALCULUS** and wish to record thanks to the teachers and students for their warm reception to the previous edition.

The present edition has been specially designed, made up-to-date and well organised in a systematic order according to the latest syllabus.

The authors have always endeavoured to keep the text update in the best interests of the students community- a gesture which the authors hope would be appreciated by the students and teachers alike.

Suggestions for the improvement of the book will be thankfully received.

— Authors

Syllabus

Calculus

U.P. UNIFIED (*w.e.f.* 2011-12)

B.A./B.Sc. Paper-II M.M. : 33 / 65

Differential Calculus

Unit-1: ε-δ definition of the limit of a function, Continuous functions and classification of discontinuities, Differentiability, Chain rule of Differentiability, Rolle's theorem, First and second mean value theorems, Taylor's theorems with Lagrange's and Cauchy's forms of remainder, Successive differentiation and Leibnitz's theorem.

Unit-2: Expansion of functions (in Taylor's and Maclaurin's series), Indeterminate forms, Partial differentiation and Euler's theorem, Jacobians.

Unit-3: Maxima and Minima (for functions of two variables), Tangents and normals (polar form only), Curvature, Envelopes and evolutes.

Unit-4(a): Asymptotes, Tests for concavity and convexity, Points of inflexion, Multiple points, Tracing of curves in Cartesian and Polar coordinates.

Integral Calculus

Unit- 4(b): Reduction formulae, Beta and Gamma functions.

Unit-5: Quadrature. Rectification. Volumes and surfaces of solids of revolution, Pappus theorem, Double and triple integrals, Change of order of integration, Dirichlet's and Liouville's integral formulae.

Brief Contents

SECTION

A

DIFFERENTIAL CALCULUS

Chapters

Chapter 1

Limits and Continuity

1.1 Definitions

Constant : A symbol which retains the same value throughout a set of mathematical operations is called a constant.

A **variable** is a quantity, or a symbol representing a number, which is capable of assuming different values.

A **continuous variable** is one which can take all the numerical values between two given numbers.

An **independent variable** is one which may take up any arbitrary value that may be assigned to it.

A **dependent variable** is a symbol which can assume its value as a result of some other variable taking some assigned value.

Domain of a Variable : If we give the independent variable x only those values which lie between $x = a$ and $x = b$, then all these numerical values taken collectively will be called **domain** or **interval** of the variable. The domain is said to be **closed** if a and b are included in it and is denoted by the symbol $[a, b]$. An **open** domain is denoted by $]a, b[$ or by (a, b). Similarly the symbols $[a, b[$ and $]a, b]$ stand for semi-open domains. These semi-open domians are also denoted by $[a, b)$ and $(a, b]$ respectively.

Function : If y depends upon x in such a manner that for every value of x in its domain of variation there corresponds a definite (*i.e.*, a unique) value of y, then y is said to be a single-valued function of x and is denoted by $y = f(x), f$ denoting the kind of dependence or relationship that exists between x and y.

This relationship is often called functional relation and $f(x_1), f(x_2), \ldots, f(x_r)$ are called functional values of $f(x)$ for $x = x_1, x_2, \ldots, x_r$ respectively.

Note : The essential thing about the definition of a function is that for each value of x there must correspond a definite value of $f(x)$. We must be in possession of a set of rules which determine for each value of x in a certain interval, a definite value of the function. These rules may take the shape of a single compact formula such as $f(x) = \sin x$ or a number of such formulae that apply to different parts of the domain of x, for example

$$\left.\begin{array}{lll} f(x) = \sin x & \text{for} & 0 \le x \le \pi/2 \\ f(x) = x & \text{for} & \pi/2 < x < \pi \\ f(x) = \cos x & \text{for} & x \ge \pi. \end{array}\right\} . \qquad \ldots(1)$$

In the first case $f(x) = \sin x$ is defined for values of x in any interval. In the second case $f(x)$ given by (1) is defined in the interval $[0, \infty[$.

The above definition of a function of x brings about (1) idea of the dependence of the function on x (2) idea of definiteness of the values of the function for each value of x (3) idea of single valuedness of the function (4) idea of the domain of the variable x.

We are accustomed to think that every function is capable of graphical representation. Majority of functions are certainly capable of graphical representation but there are some functions which cannot be represented by a graph. The function defined as follows is such a function :

$$f(x) = 0 \text{ when } x \text{ is rational, } f(x) = 1 \text{ when } x \text{ is irrational.}$$

Set-theoretic definition of a function : Let A and B be two given sets. Suppose there exists a correspondence denoted by f, which associates to **each** member of A a **unique** member of B. Then f is called a **function** or a **mapping** from A to B.

The mapping f of A to B is denoted by $f : A \to B$. The set A is called the **domain** of the function f, and B is called the **co-domain** of f. The element $y \in B$ which the mapping f associates to an element $x \in A$ is denoted by $f(x)$ and is called the ***f*-image of** x or the **value** of the function f for x. Each element of A has a unique image and each element of B need not appear as the image of an element in A. We define the **range** of f to consist of those elements in B which appear as the image of at least one element in A.

Equality of two functions : Two functions f and g of $A \to B$ are said to be *equal* if and only if $f(x) = g(x) \ \forall x \in A$ and we write $f = g$. For two unequal mappings from A to B, there must exist at least one element $x \in A$ such that $f(x) \ne g(x)$.

Constant function : A function $f : A \to B$ is called a **constant function** if the same element $b \in B$ is assigned to every element in A.

Real valued function : If both A and B are the sets of real numbers, then $f : A \to B$ is called a real valued function of a real variable.

Single-valued and multiple-valued functions : If y has only one definite value when a definite value is given to x then y is called a **single-valued function of *x***. When y has more than one value for a value of x, it is called a **multiple-valued function of *x***.

Odd and Even functions : A function is said to be **odd** if it changes sign when the sign of the variable is changed *i.e.*, if $f(-x) = -f(x)$.

A function is said to be **even** if its sign does not change when the sign of the variable is changed *i.e.*, if $f(-x) = f(x)$.

Bounded and unbounded functions : If for all values of x in a given interval, $f(x)$ is never greater than some fixed number M, the number M is said to be an **upper bound** for f in that interval, whereas if $f(x)$ is never less than some number m then m is called a **lower bound** for f in that interval. If both upper and lower bounds of a function are finite, the function is said to be bounded otherwise it is said to be unbounded.

By a supremum of f in an interval we mean the least of all the upper bounds of f in that interval. Similarly an **infimum** of f is the greatest of all the lower bounds of f in the interval.

A rational integral function, or a **polynomial**, is a function of the form

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a positive integer or zero.

A rational function is defined as the quotient of one polynomial by another. For example,

$\dfrac{7x+4}{2x^2+3x+6}$ is a rational function.

An algebraical function is a function which can be expressed as the root of an equation of the form

$$y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_{n-1} y + A_n = 0$$

where $A_1, A_2, \ldots, A_n$ are rational functions of x. In particular a rational function is also algebraical.

A transcendental function is a function which is not algebraical. Trigonometrical, exponential and logarithmic functions are examples of transcendental functions.

Monotonic functions : The function $y = f(x)$ is said to be **monotonically increasing** if corresponding to an increase in the value of x in a certain interval I in which the function $f(x)$ is defined, the value of y never decreases *i.e.,*

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \quad \forall\, x_1, x_2 \in I.$$

Similarly the function $f(x)$ is **monotonically decreasing** if

$$x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \quad \forall\, x_1, x_2 \in I.$$

Also f is said to be **strictly increasing** iff $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ and **strictly decreasing** iff $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

The function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \leq x \leq \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \leq x \leq \pi$.

Explicit and implicit functions : A function is said to be *explicit* when expressed directly in terms of the independent variable or variables *e.g.*, $y = \sin^{-1} x + \log x$.

If the function cannot be expressed directly in terms of the independent variable or variables, the function is said to be *implicit e.g.*, the equation $x^y + y^x = a^b$ expresses y as an implicit function of x.

Sum, Difference, Product and Quotient of two functions : Let f, g be two functions with domains D_1 and D_2. If $D = D_1 \cap D_2$, then D is common to the domains of f and g.

The *sum function* $f + g$ is defined as $(f+g)(x) = f(x) + g(x) \ \forall\, x \in D$.

If $c \in \mathbf{R}$, the function cf is defined as $(cf)(x) = cf(x) \ \forall\, x \in D_1$.

The *difference function* $f - g$ is defined as $(f-g)(x) = f(x) - g(x) \ \forall\, x \in D$.

The *product function* fg is defined as $(fg)(x) = f(x)\, g(x) \ \forall\, x \in D$.

The *reciprocal function* $1/g$ of the function g is defined as

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} \ \forall\, x \in D_2 \text{ and } g(x) \neq 0.$$

The *quotient function* f/g is defined as $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} \ \forall\, x \in D$ and $g(x) \neq 0$.

1.2 Limits

Consider the function $y = (x^2 - 1)/(x - 1)$. The value of this function at $x = 1$ is of the form $0/0$ which is meaningless. In this case we cannot divide the numerator by the denominator since $x - 1$ is zero. Now suppose x is not actually equal to 1 but very nearly equal to 1. Then $x - 1$ is not equal to zero. Hence in this case we can divide the numerator by the denominator.

$$\therefore \qquad \frac{x^2 - 1}{x - 1} = x + 1.$$

If x is little greater than 1, then the value of y will be greater than 2 and as x gets nearer to 1, y comes nearer to 2. Now the difference between y and 2 is

$$\frac{x^2 - 1}{x - 1} - 2 = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x - 1)^2}{x - 1} = x - 1.$$

This difference $(x - 1)$ can be made as small as we please by letting x tend to 1.

Thus we see that when x has a fixed value 1, the value of y is meaningless but when x tends to 1, y tends to 2 and we say that the limit of y is 2 when x tends to 1. Thus we write as

$$\lim_{x \to 1} \left[(x^2 - 1)/(x - 1) \right] = 2.$$

Definition of limit : **(Bundelkhand 2006; Purvanchal 10; Kashi 14)**

Let f be a function defined on some neighbourhood of a point a except possibly at a itself. Then a real number l is said to be the **limit** *of f as x approaches a if for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number δ greater than zero such that*

$$| f(x) - l | < \varepsilon$$

for all values of x for which $0 < | x - a | < \delta$, where $| x - a |$ means the absolute value of $x - a$ without any regard to sign.

In symbols, we then write $\lim_{x \to a} f(x) = l$.

We have to negate the above definition in order to show that f does not approach l as x approaches a.

If it is not true that for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$0 < | x - a | < \delta \Rightarrow | f(x) - l | < \varepsilon,$$

then there must exist an $\varepsilon > 0$, such that for every $\delta > 0$, there is some x for which

$$0 < | x - a | < \delta \text{ but } | f(x) - l | \nless \varepsilon.$$

This means that in order to show that f does not approach l as x approaches a, it is sufficient to produce an $\varepsilon > 0$ such that for each $\delta > 0$ there is some x satisfying

$$0 < | x - a | < \delta \quad \text{and} \quad | f(x) - l | \geq \varepsilon.$$

Note 1 : It is not at all necessary for $\lim_{x \to a} f(x)$ to exist that f be defined at $x = a$. It is enough that for some $\delta > 0$, f be defined whenever $0 < |x - a| < \delta$.

Note 2 : If N be a neighbourhood of a, then $N \sim \{a\}$ is called a *deleted neighbourhood* of a.

Note 3 : If a function f has a finite limit at a point a, then by the definition of the limit of a function a deleted neighbourhood of a exists on which f is bounded.

Now we shall prove a theorem which is the foundation on which the definition of limit rests. If this theorem were not true, the definition of limit would have been useless.

Theorem : *If* $\lim_{x \to a} f(x) = l$, *and* $\lim_{x \to a} f(x) = m$, *then* $l = m$ *i.e., if* $\lim_{x \to a} f(x)$ *exists, then it is unique.*

Proof : Suppose, if possible, $l \neq m$.

Let us take $\varepsilon = \frac{1}{2} |l - m|$. Then $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = l$, for a given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1. \qquad \ldots(1)$$

Again since $\lim_{x \to a} f(x) = m$, for a given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(x) - m| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_2. \qquad \ldots(2)$$

If we choose $\delta = \min. \{\delta_1, \delta_2\}$, then $0 < |x - a| < \delta$ implies that both $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ hold, and hence, we have

$$|f(x) - l| < \varepsilon \text{ and } |f(x) - m| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

This implies that if $0 < |x - a| < \delta$, then

$$|l - m| = |\{f(x) - m\} - \{f(x) - l\}| \leq |f(x) - m| + |f(x) - l|$$
$$< \varepsilon + \varepsilon = 2\varepsilon = |l - m|$$

i.e., $|l - m| < |l - m|$, which is absurd and so our assumption is wrong.

Hence, $l = m$ *i.e.,* $\lim_{x \to a} f(x)$ is unique.

1.3 Algebra of Limits

Now we shall give some theorems on limits of functions which are similar to those of limits of sequences.

Theorem 1 : *If* $\lim_{x \to a} f(x) = l \neq 0$, *then there exist numbers* $k > 0$ *and* $\delta > 0$ *such that* $|f(x)| > k$ *whenever* $0 < |x - a| < \delta$.

Also then $\lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{l}$.

Proof : Let $\varepsilon = \frac{1}{2} |l|$. Then $\varepsilon > 0$, because $l \neq 0$.

Since $\lim_{x \to a} f(x) = l$, therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ whenever } 0 < |x - a| < \delta. \qquad \ldots(1)$$

Now $|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)| < \varepsilon + |f(x)|$,

whenever $0 < |x - a| < \delta$, from (1).

$\therefore$ Whenever $0 < |x - a| < \delta$, we have

$$|f(x)| > |l| - \varepsilon = |l| - \tfrac{1}{2}|l| = \tfrac{1}{2}|l| > 0. \qquad \text{...(2)}$$

Thus taking $k = \frac{1}{2}|l| > 0$, we get $|f(x)| > k$ whenever $0 < |x - a| < \delta$.

This proves the first part of the theorem.

Second part : Now to prove that $\lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{l}$.

We have $\left|\dfrac{1}{f(x)} - \dfrac{1}{l}\right| = \left|\dfrac{l - f(x)}{l \cdot f(x)}\right| = \dfrac{|l - f(x)|}{|l| \cdot |f(x)|}$. ...(3)

By first part of this theorem there exist numbers $k > 0$ and $\delta_1 > 0$ such that

$$|f(x)| > k \text{ i.e., } \frac{1}{|f(x)|} < \frac{1}{k} \text{ whenever } 0 < |x - a| < \delta_1. \qquad \text{...(4)}$$

Let $\varepsilon' > 0$ be given.

Since $\lim_{x \to a} f(x) = l$, therefore, given $\varepsilon' > 0$, there exists $\delta_2 > 0$ such that

$$|f(x) - l| < k\,|l|\,\varepsilon' \text{ whenever } 0 < |x - a| < \delta_2. \qquad \text{...(5)}$$

Let $\delta = \min. \{\delta_1, \delta_2\}$. Then from (3), (4) and (5), we have

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \frac{1}{|l|} \cdot k\,|l|\,\varepsilon' \cdot \frac{1}{k} \text{ whenever } 0 < |x - a| < \delta.$$
$$= \varepsilon'.$$

Thus for given $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \varepsilon' \text{ whenever } 0 < |x - a| < \delta.$$

Hence, $\lim_{x \to 0} \dfrac{1}{f(x)} = \dfrac{1}{l}$.

Theorem 2 : *The limit of a sum is equal to the sum of the limits.*

Proof : Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$.

We have to show that $\lim_{x \to a} \{(f + g)(x)\} = l + m$.

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to a} f(x) = l$, therefore, there exists $\delta_1 > 0$ such that

$$|f(x) - l| < \tfrac{1}{2}\varepsilon \text{ whenever } 0 < |x - a| < \delta_1.$$

Again since $\lim_{x \to a} g(x) = m$, therefore, there exists $\delta_2 > 0$ such that

$$|g(x) - m| < \tfrac{1}{2}\varepsilon \text{ whenever } 0 < |x - a| < \delta_2.$$

If we take $\delta = \min. \{\delta_1, \delta_2\}$, then $0 < |x - a| < \delta$

$$\Rightarrow \quad \text{both } 0 < |x - a| < \delta_1 \text{ and } 0 < |x - a| < \delta_2 \text{ hold,}$$

and consequently if $0 < |x - a| < \delta$, then both $|f(x) - l| < \frac{1}{2}\varepsilon$ and $|g(x) - m| < \frac{1}{2}\varepsilon$ are true.

Now if $0 < |x - a| < \delta$, then

$$|(f + g)(x) - (l + m)| = |f(x) - l + g(x) - m|$$

$$\leq |f(x) - l| + |g(x) - m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus $|(f + g)(x) - (l + m)| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\therefore \quad \lim_{x \to a} (f + g)(x)$ exists and $\lim_{x \to a} (f + g)(x) = l + m$.

The above result can be extended to any finite number of functions.

In the same way, we can prove that $\lim_{x \to a} (f - g)(x) = l - m$.

Theorem 3 : *The limit of a product is equal to the product of the limits.*

Proof : Using the notations of theorem 2, we have to prove that

$$\lim_{x \to a} (fg)(x) = lm.$$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} \text{Now } |(fg)(x) - lm| &= |f(x)\, g(x) - lg(x) + lg(x) - lm| \\ &\leq |f(x)\, g(x) - lg(x)| + |lg(x) - lm| \\ &= |g(x)|\ |f(x) - l| + |l|\ |g(x) - m|. \qquad \text{...(1)} \end{aligned}$$

Since $\lim_{x \to a} g(x) = m$, therefore $g(x)$ is bounded in some deleted neighbourhood of $x = a$. Hence there exists $k > 0$ and $\delta_1 > 0$ such that $|g(x)| \leq k$ whenever $0 < |x - a| < \delta_1$.

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, therefore, corresponding to any given $\varepsilon > 0$, we can find positive numbers δ_2 and δ_3 such that

$$|f(x) - l| < \frac{\varepsilon}{2k} \text{ whenever } 0 < |x - a| < \delta_2$$

and

$$|g(x) - m| < \frac{\varepsilon}{2(|l| + 1)} \text{ whenever } 0 < |x - a| < \delta_3.$$

If we take $\delta = \min. \{\delta_1, \delta_2, \delta_3\}$, then from (1), we get

$$\begin{aligned} |(fg)(x) - lm| &< k \cdot \frac{\varepsilon}{2k} + |l| \cdot \frac{\varepsilon}{2(|l| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \left[\because \frac{|l|}{|l| + 1} < 1\right] \\ &= \varepsilon \quad \text{whenever } 0 < |x - a| < \delta. \end{aligned}$$

Thus for $\varepsilon > 0$, we have $\delta > 0$ such that $|(fg)(x) - lm| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\therefore \quad \lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x)\, g(x)$ exists and $\lim_{x \to a} (fg)(x) = lm$.

The above theorem can evidently be extended to any finite number of functions.

Theorem 4 : *The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is not zero.*

Proof : Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m \neq 0$.

$$\begin{aligned} \text{Now} \quad \left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| &= \left|\left\{\frac{f(x)}{g(x)} - \frac{f(x)}{m}\right\} + \left\{\frac{f(x)}{m} - \frac{l}{m}\right\}\right| \\ &= \left|\frac{f(x)}{m\, g(x)}\{m - g(x)\} + \frac{1}{m}\{f(x) - l\}\right| \end{aligned}$$

$$\leq \frac{|f(x)|}{|m|\;|g(x)|}\;|m - g(x)| + \frac{1}{|m|}\;|f(x) - l| \qquad \ldots(1)$$

Since $\lim_{x \to a} f(x) = l$, therefore there exists a deleted neighbourhood $]a - \delta_1, a + \delta_1[- \{a\}$ of the point $x = a$ in which the function f is bounded. Let $K > 0$ be such that

$$|f(x)| \leq K \text{ whenever } 0 < |x - a| < \delta_1.$$

Again since $g(x) \neq 0$ for all x in the domain of g and $\lim_{x \to a} g(x) = m \neq 0$, therefore there exist numbers $k > 0$ and $\delta_2 > 0$ such that

$$|g(x)| > k \; i.e., \frac{1}{|g(x)|} < \frac{1}{k} \text{ whenever } 0 < |x - a| < \delta_2.$$

[See theorem 1 of article 1.3]

Let $\qquad \delta' = \min(\delta_1, \delta_2)$.

The inequality (1) can then be written as

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| \leq \frac{K}{k\,|m|}\;|m - g(x)| + \frac{1}{|m|}\,|f(x) - l|, \qquad \ldots(2)$$

for all x such that $0 < |x - a| < \delta'$.

Now take any given $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, we can find positive numbers δ_3 and δ_4 such that

$$|f(x) - l| < |m| \cdot \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_3$$

and $$|g(x) - m| < \frac{k\,|m|}{K} \cdot \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_4.$$

Take $\delta = \min\{\delta', \delta_3, \delta_4\}$. Then from (2), we get

$$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ whenever } 0 < |x - a| < \delta.$$

$\therefore \qquad \lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$, if $m \neq 0$.

Alternative Proof :

Since $m \neq 0$, therefore, by theorem 1 of § 3, $\lim_{x \to a} \frac{1}{g(x)}$ exists and equals $\frac{1}{m}$.

Now $$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \left\{f(x) \cdot \frac{1}{g(x)}\right\}$$

$$= \left\{\lim_{x \to a} f(x)\right\} \left\{\lim_{x \to a} \frac{1}{g(x)}\right\} \qquad \text{[By theorem 3 of article 1.3]}$$

$$= l \cdot \frac{1}{m} = \frac{l}{m}.$$

Theorem 5 : *Let f be defined on D and let $f(x) \geq 0$ for all $x \in D$. If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$.*

Proof : Suppose that $\lim_{x \to a} f(x) = l$ and l is negative.

Taking $\varepsilon = -\frac{1}{2}l$, we can find a positive number $\delta > 0$ such that

$$|f(x) - l| < -\tfrac{1}{2}l \text{ whenever } 0 < |x - a| < \delta.$$

It gives that

$$\frac{3l}{2} < f(x) < \frac{l}{2} < 0 \text{ whenever } 0 < |x - a| < \delta.$$

This is a contradiction since we are given that $f(x) \geq 0$ for all $x \in D$. Hence l cannot be negative.

Consequently $\lim_{x \to a} f(x) \geq 0$.

Corollary : *Let f be defined on D and let $f(x) > 0$ for all $x \in D$.*

If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$.

Proof : Since $f(x) > 0 \Rightarrow f(x) \geq 0$, therefore now we can apply theorem 5 of article 1.3.

Theorem 6 : *Let f and g be defined on D and let $f(x) \geq g(x)$ for all $x \in D$. Then*

$$\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x), \textit{ provided these limits exist.}$$

Proof : Let $\lim_{x \to a} f(x) = l, \lim_{x \to a} g(x) = m$.

Let us define a function h by $h(x) = f(x) - g(x) \ \forall x \in D$. Then, we have

(i) $h(x) \geq 0 \ \forall x \in D$.

(ii) $\lim_{x \to a} h(x)$ exsits and $\lim_{x \to a} h(x) = l - m$.

(iii) $\lim_{x \to a} h(x) \geq 0$, by theorem 5 of article 1.3.

Thus, from (ii) and (iii), we get $l - m \geq 0$ *i.e.,* $l \geq m$,

i.e., $$\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x).$$

Corollary : *Let $f(x) > g(x)$ for all $x \in D$. Then $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$, provided these limits exist.*

Theorem 7 : *Let f, g and h be defined on D and let $f(x) \geq g(x) \geq h(x)$ for all x. Let $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.*

Then $\lim_{x \to a} g(x)$ exists, and $\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Proof : Let $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$.

Then corresponding to any given $\varepsilon > 0$, we can find positive numbers δ_1 and δ_2 such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1$$

i.e., $$l - \varepsilon < f(x) < l + \varepsilon \text{ whenever } 0 < |x - a| < \delta_1 \qquad \ldots(1)$$

and $$l - \varepsilon < h(x) < l + \varepsilon \text{ whenever } 0 < |x - a| < \delta_2. \qquad \ldots(2)$$

Choosing δ to be smaller than δ_1 and δ_2, we see from (1) and (2) that

$$l - \varepsilon < h(x) \leq g(x) \leq f(x) < l + \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Thus $\quad l - \varepsilon < g(x) < l + \varepsilon$ whenever $0 < |x - a| < \delta$

or $\quad |g(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} g(x) = l$.

Theorem 8 : *If* $\lim_{x \to a} f(x) = l$, *then* $\lim_{x \to a} |f(x)| = |l|$.

Proof : We have $|f(x) - l| \geq \big| |f(x)| - |l| \big|$, for all x. ...(1)

$[\because \ |p - q| \geq \big| |p| - |q| \big|]$

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to a} f(x) = l$, therefore, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta. \qquad ...(2)$$

From (1) and (2), we get

$$\big| |f(x)| - |l| \big| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Consequently $\lim_{x \to a} |f(x)|$ exists and $\lim_{x \to a} |f(x)| = |l|$.

Theorem 9 : *If there is a number* $\delta > 0$ *such that* $h(x) = 0$ *whenever* $0 < |x - a| < \delta$, *then* $\lim_{x \to a} h(x) = 0$.

Proof : For any $\varepsilon > 0$, the number $\delta > 0$, given in the hypothesis of the theorem is such that $h(x) = 0$ whenever $0 < |x - a| < \delta$

or $\quad |h(x) - 0| = 0 < \varepsilon$ whenever $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} h(x) = 0$.

Corollary : *If there is a number* $\delta > 0$ *such that* $f(x) = g(x)$ *whenever* $0 < |x - a| < \delta$, *then* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

Proof : Let us define a function h by setting

$$h(x) = f(x) - g(x) \text{ for all } x.$$

Then $h(x) = 0$ whenever $0 < |x - a| < \delta$.

Now, apply theorem 9 of article 1.3.

Note : The above corollary has deep implications. It asserts that the concept of limit is a 'local' one. If two functions agree on some neighbourhood of a point a, then they cannot approach different limits as x approaches a.

Theorem 10 : *If* $\lim_{x \to a} f(x) = 0$ *and* $g(x)$ *is bounded in some deleted neighbourhood of* a, *then* $\lim_{x \to a} f(x)\, g(x) = 0$.

Proof : Since $g(x)$ is bounded in some deleted neighbourhood of a, therefore there exist numbers $k > 0$ and $\delta_1 > 0$ such that

$$|g(x)| \leq k \text{ whenever } 0 < |x - a| < \delta_1. \qquad ...(1)$$

Now take any given $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = 0$, therefore there exists $\delta_2 > 0$ such that

$$|f(x) - 0| = |f(x)| < \frac{\varepsilon}{k} \text{ whenever } 0 < |x - a| < \delta_2 \qquad \ldots(2)$$

Now take $\delta = \min(\delta_1, \delta_2)$. Then for all x such that $0 < |x - a| < \delta$, we have

$$|f(x)\, g(x) - 0| = |f(x)\, g(x)| = |f(x)| \,.\, |g(x)|$$

$$< \frac{\varepsilon}{k} \cdot k = \varepsilon \text{, using (1) and (2).}$$

Hence $\lim_{x \to a} f(x)\, g(x) = 0$.

Illustration : We have $\lim_{x \to 0} x \sin(1/x) = 0$

because $\lim_{x \to 0} x = 0$ and $|\sin(1/x)| \le 1$ for all $x \ne 0$ *i.e.*, $\sin(1/x)$ is bounded in some deleted neighbourhood of zero.

1.4 Right Hand and Left Hand Limits

Definition : (Right-hand limit) : *A function f is said to approach l as x approaches a from right (or from above) if corresponding to an arbitrary positive number ε, there exists a positive number δ such that* $|f(x) - l| < \varepsilon$ *whenever* $a < x < a + \delta$.

It is written as $\lim_{x \to a+0} f(x) = l$ or $f(a + 0) = l$.

The working rule for finding the right hand :

"Put $a + h$ for x in $f(x)$ where h is + ive and very very small and make h approach zero".

In short, we have $f(a + 0) = \lim_{h \to 0} f(a + h)$.

Definition : (Left-hand limit) : *A function f is said to approach l as x approaches a from the left (or from below) if corresponding to an arbitrary positive number ε, there exists a positive number δ such that*

$$|f(x) - l| < \varepsilon \textit{ whenever } a - \delta < x < a.$$

It is written as $\lim_{x \to a-0} f(x) = l$ or $f(a - 0) = l$.

The working rule for finding the left hand :

"Put $a - h$ for x in $f(x)$ where h is + ive and very very small and make h approach zero."

In this case, we have $f(a - 0) = \lim_{h \to 0} f(a - h)$.

Important Note : If both right hand limit and left hand limit of f as $x \to a$, exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

1.5 Limits as $x \to +\infty$ $(-\infty)$

Definition : *A function f is said to approach l as x becomes positively infinite, if corresponding to each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *whenever* $x \geq \delta$.

Then we write $\lim_{x \to \infty} f(x) = l$ *or* $f(x) \to l$ *as* $x \to \infty$.

Definition : *A function f is said to approach l as x becomes negatively infinite, if corresponding to each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *whenever* $x \leq -\delta$.

Then we write $\lim_{x \to -\infty} f(x) = l$

or $f(x) \to l$ *as* $x \to -\infty$.

Note 1 : The results on the limits of sum, product and quotient of functions also hold good here provided that in these cases $l + m$, lm, l/m are defined.

Note 2 : If $\lim_{x \to \infty} f(x) = l$ exists, $\lim_{x \to \infty} g(x)$ does not exist (as a finite real number), even then $\lim_{x \to \infty} f(x)\, g(x)$ can exist. Similar is the case as $x \to -\infty$.

1.6 Infinite Limits

Definition : *A function f is said to approach* $+\infty$ *as x approaches a, if corresponding to any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $f(x) > \varepsilon$ *whenever* $0 < |x - a| < \delta$.

Then we write $\lim_{x \to a} f(x) = \infty$ *or* $f(x)$ *tends to* ∞ *as x tends to a.*

Definition : *A function f is said to approach* $-\infty$ *as x approaches a, if corresponding to any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$f(x) < -\varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Then we write $\lim_{x \to a} f(x) = -\infty$ *or* $f(x)$ *tends to* $-\infty$ *as x tends to a.*

Illustrative Examples

Example 1 : *Let f be the function given by* $f(x) = \dfrac{x^2 - a^2}{x - a}$, $x \neq a$.

Using (ε, δ) *definition show that* $\lim_{x \to a} f(x) = 2a$.

Solution : Let $\varepsilon > 0$ be given. In order to show that

$$\lim_{x \to a} f(x) = 2a,$$

we have to show that for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

If $x \neq a$, then $|f(x) - 2a| = \left|\dfrac{x^2 - a^2}{x - a} - 2a\right| = |(x + a) - 2a|$ $[\because x \neq a]$

$$= |x - a|.$$

$\therefore \qquad |f(x) - 2a| < \varepsilon$, if $|x - a| < \varepsilon$.

Choosing a number δ such that $0 < \delta \le \varepsilon$, we have

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Hence $\lim_{x \to a} f(x) = 2a$.

Example 2 : *Using* (ε, δ) *definition show that* $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0 \cdot$

(Meerut 2012, 13; Rohilkhand 13B)

Solution : Let $\varepsilon > 0$ be given. In order to show that $\lim_{x \to 0} \left(x \sin \frac{1}{x} \right) = 0$, we have to show that for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\left| x \sin \frac{1}{x} - 0 \right| < \varepsilon \text{ whenever } 0 < |x - 0| < \delta.$$

Now $\qquad \left| x \sin \frac{1}{x} - 0 \right| = |x| \left| \sin \frac{1}{x} \right| \le |x|$, because $\left| \sin \frac{1}{x} \right| \le 1$.

$\therefore \qquad \left| x \sin \frac{1}{x} - 0 \right| < \varepsilon$ whenever $|x| < \varepsilon$.

Choosing a number δ such that $0 < \delta \le \varepsilon$, we have

$$\left| x \sin \frac{1}{x} - 0 \right| < \varepsilon \text{ whenever } 0 < |x| < \delta.$$

Hence $\lim_{x \to 0} x \sin \frac{1}{x} = 0$.

Example 3 : *Show by* (ε, δ) *method that the function f, defined on* $\mathbf{R} - \{0\}$ *by* $f(x) = \sin(1/x)$ *whenever* $x \ne 0$, *does not tend to* 0 *as x tends to* 0. **(Meerut 2013B)**

Solution : In order to show that $\sin(1/x)$ does not tend to 0 as x tends to 0, take $\varepsilon = \frac{1}{2}$. By Archimedean property of real numbers for any $\delta > 0$ there exists a positive integer n such that

$$n > \frac{1}{\pi\delta} \quad i.e., \ \delta > \frac{1}{n\pi} \cdot$$

$$\therefore \qquad 0 < \frac{2}{(4n + 1)\pi} < \frac{1}{2n\pi} < \frac{1}{n\pi} < \delta.$$

Take $\qquad x = \dfrac{2}{(4n + 1)\pi} \cdot$ Then $0 < |x - 0| < \delta$.

Also, $\qquad |\sin(1/x) - 0| = |\sin(2n\pi + \frac{1}{2}\pi)| = 1 > \varepsilon$.

Thus we have shown that there exists an $\varepsilon > 0$, namely $\frac{1}{2}$, such that for every $\delta > 0$ there is an $x \left[= \dfrac{2}{(4n + 1)\pi} \right.$, where n is a positive integer such that $\left. \dfrac{2}{(4n + 1)\pi} < \delta \right]$ such that

$$0 < |x - 0| < \delta \text{ and } |\sin(1/x) - 0| > \varepsilon.$$

Hence $\sin(1/x)$ does not tend to 0 as x tends to 0.

Example 4 : *Show that* $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$ *does not exist.*

Solution : Let $f(x) = |x-2|/(x-2)$.

We have the right hand limit *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{|2+h-2|}{(2+h-2)}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1;$$

and the left hand limit *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{|2-h-2|}{(2-h-2)}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Since $f(2+0) \neq f(2-0)$, hence $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

Example 5 : *Evaluate the following limits if they exist :*

(a) $\lim_{x \to 2} \frac{x^2+3x+2}{x-2}$.

Solution : Here the right hand limit *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{(2+h)^2+3(2+h)+2}{2+h-2}$$

$$= \lim_{h \to 0} \frac{12+7h+h^2}{h} = \lim_{h \to 0} \left(\frac{12}{h}+7+h\right) = \infty;$$

and the left hand limit *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{(2-h)^2+3(2-h)+2}{2-h-2}$$

$$= \lim_{h \to 0} \frac{12-7h+h^2}{-h} = \lim_{h \to 0} \left(-\frac{12}{h}+7-h\right) = -\infty.$$

Since $f(2+0) \neq f(2-0)$, hence $\lim_{x \to 2} f(x)$ does not exist.

(b) $\lim_{x \to 0} (1+x)^{1/x}$.

Solution : Here the right hand limit *i.e.*,

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1+h)^{1/h}$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{h} \cdot h + \frac{\frac{1}{h}\left(\frac{1}{h}-1\right)}{1.2} h^2 + \frac{\frac{1}{h}\left(\frac{1}{h}-1\right)\left(\frac{1}{h}-2\right)}{1.2.3} h^3 + \dots\right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{1!} + \frac{1.(1-h)}{2!} + \frac{1.(1-h)(1-2h)}{3!} + \dots\right]$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \infty = e.$$

Similarly, the left hand limit *i.e.*,

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (1-h)^{-1/h} = e.$$

Thus both $f(0+0)$ and $f(0-0)$ exist and are equal to e.

Hence $\lim_{x \to 0} (1+x)^{1/x} = e.$

(c) $\lim_{x \to 0} \frac{\sin x}{x}$. **(Bundelkhand 2008; Kanpur 09)**

Solution : Let $f(x) = \frac{\sin x}{x}$.

Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin h}{h}$

$$= \lim_{h \to 0} \frac{h - \frac{h^3}{3\,!} + \frac{h^5}{5\,!} - \dots}{h} = \lim_{h \to 0} \left(1 - \frac{h^2}{3\,!} + \frac{h^4}{5\,!} - \dots\right) = 1.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Since $f(0+0) = f(0-0) = 1$, hence $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(d) $\lim_{x \to \infty} \frac{\sin x}{x}$. **(Bundelkhand 2008)**

Solution : $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{y \to 0} y \sin(1/y)$, putting $x = 1/y$.

Let $f(y) = y \sin(1/y)$.

We have, right hand limit *i.e.*, $f(0+0) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin(1/h)$

$= 0 \times$ a finite quantity lying between -1 and 1

$= 0.$

Similarly, left hand limit *i.e.*, $f(0-0) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h) \sin(-1/h) = \lim_{h \to 0} h \sin(1/h) = 0.$

Since $f(0+0) = f(0-0) = 0$, therefore $\lim_{y \to 0} y \sin(1/y) = 0$

i.e., $\lim_{x \to \infty} \frac{\sin x}{x} = 0.$

(e) $\lim_{x \to 0} \sin \frac{1}{x}$.

Solution : Let $f(x) = \sin(1/x)$.

Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \sin \frac{1}{h}$.

As $h \to 0$, the value of $\sin(1/h)$ oscillates between $+1$ and -1, passing through zero and intermediate values an infinite number of times. Hence there is no definite number l to which $\sin(1/h)$ tends as h tends to zero. Therefore the right hand limit $f(0+0)$ does not exist.

Similarly the left hand limit $f(0-0)$ also does not exist. Thus $\lim_{x \to 0} \sin(1/x)$ does not exist.

(f) $\displaystyle \lim_{x \to 0} \frac{a^x - 1}{x}$. **(Meerut 2003; Kanpur 10)**

Solution : Let
$$f(x) = \frac{a^x - 1}{x}$$
$$= \frac{1 + x \log a + (x^2/2!)(\log a)^2 + \ldots - 1}{x}$$
$$= \frac{x[\log a + \frac{1}{2}x(\log a)^2 + \ldots]}{x}.$$

Here $\displaystyle f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h\left[\log a + \frac{h}{2}(\log a)^2 + \ldots\right]}{h} = \log a.$

Also $\displaystyle f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \log a.$

Since $\displaystyle f(0+0) = f(0-0) = \log a$, therefore $\displaystyle \lim_{x \to 0} \frac{a^x - 1}{x} = \log a.$

(g) $\displaystyle \lim_{x \to 0} \frac{1}{x} \cdot e^{1/x}$.

Solution : Let $\displaystyle f(x) = \frac{1}{x} \cdot e^{1/x}$.

Then $\displaystyle f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{h} e^{1/h}$

$= \infty$, since both $1/h$ and $e^{1/h}$ tend to ∞ as $h \to 0$.

Also $\displaystyle f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -\frac{1}{h} e^{-1/h}$

$$= \lim_{h \to 0} \frac{-1}{he^{1/h}} = \lim_{h \to 0} \frac{-1}{h\left(1 + \frac{1}{h} + \frac{1}{2!} \cdot \frac{1}{h^2} + \ldots\right)}$$

$$= \lim_{h \to 0} \frac{-1}{h + 1 + (1/2h) + \ldots} = 0.$$

Since $f(0+0) \neq f(0-0)$, therefore $\displaystyle \lim_{x \to 0} \frac{1}{x} e^{1/x}$ does not exist.

(h) $\displaystyle \lim_{x \to 0} \frac{(1+x)^n - 1}{x}$.

Let $\displaystyle f(x) = \frac{(1+x)^n - 1}{x}$.

Then $\displaystyle f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{(1+h)^n - 1}{h}$

$$= \lim_{h \to 0} \frac{1 + nh + \dfrac{n(n-1)}{2!} h^2 + \ldots - 1}{h}$$

$$= \lim_{h \to 0} \frac{h\left[n + \dfrac{n(n-1)}{2!} h + \ldots\right]}{h}$$

$$= \lim_{h \to 0} \left[n + \frac{n(n-1)}{2!} h + \ldots\right] = n.$$

Also $\quad f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{(1-h)^n - 1}{-h}$$

$$= \lim_{h \to 0} \frac{1 + n(-h) + \dfrac{n(n-1)}{2!}(-h)^2 + \ldots - 1}{-h} = n.$$

Since $f(0+0) = f(0-0) = n$, therefore $\lim_{x \to 0} f(x) = n$.

(i) $\quad \lim_{x \to a} \dfrac{x^m - a^m}{x - a}$.

Solution : Let $\quad f(x) = \dfrac{x^m - a^m}{x - a}$.

Then $\quad f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \dfrac{(a+h)^m - a^m}{a + h - a}$

$$= \lim_{h \to 0} \frac{a^m \left[\left(1 + \dfrac{h}{a}\right)^m - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{a^m}{h}\left[1 + m \cdot \frac{h}{a} + \frac{m(m-1)}{2!} \frac{h^2}{a^2} + \ldots - 1\right]$$

$$= \lim_{h \to 0} a^m \left[\frac{m}{a} + \frac{m(m-1)}{2} \cdot \frac{h}{a^2} + \ldots\right] = a^m \cdot \frac{m}{a} = ma^{m-1}.$$

Also $\quad f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \dfrac{(a-h)^m - a^m}{a - h - a} = ma^{m-1}$.

Since $\quad f(a+0) = f(a-0) = ma^{m-1}$, hence $\lim_{x \to a} f(x) = ma^{m-1}$.

Example 6 : *Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case :*

(i) $\lim_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$; $\qquad$ **(ii)** $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$;

(Meerut 2003; Kanpur 11; Rohilkhand 14)

(iii) $\lim_{x \to 0} f(x)$ *where* $f(x)$ *is defined as*

$f(x) = x$, *when* $x > 0$; $f(x) = 0$, *when* $x = 0$; $f(x) = -x$, *when* $x < 0$.

(Purvanchal 2008)

Solution : (i) Let $f(x) = \dfrac{2x^2 - 8}{x - 2}$.

We have $f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \dfrac{2(2+h)^2 - 8}{2+h-2}$

$$= \lim_{h \to 0} \frac{2(4+4h+h^2) - 8}{h} = \lim_{h \to 0} \frac{8h + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(8+2h)}{h} = \lim_{h \to 0} (8+2h) = 8.$$

Again $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \dfrac{2(2-h)^2 - 8}{2-h-2}$

$$= \lim_{h \to 0} \frac{2(4-4h+h^2) - 8}{-h} = \lim_{h \to 0} \frac{-8h + 2h^2}{-h}$$

$$= \lim_{h \to 0} \frac{-h(8-2h)}{-h} = \lim_{h \to 0} (8-2h) = 8$$

Since $f(2+0) = f(2-0) = 8$, therefore $\lim_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$ exists and is equal to 8.

(ii) Let $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$.

Here the right hand limit, *i.e.,*

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h \to 0} \frac{e^{1/h}[1 - (1/e^{1/h})]}{e^{1/h}[1 + (1/e^{1/h})]} = 1.$$

Again the left hand limit, *i.e.,*

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h \to 0} \frac{(1/e^{1/h}) - 1}{(1/e^{1/h}) + 1} = \frac{0-1}{0+1} = -1.$$

Since $f(0+0) \neq f(0-0)$, hence $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

(iii) We have the right hand limit *i.e.,* $f(0+0)$

$= \lim_{h \to 0} f(0+h)$, where h is + ive but sufficiently small

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h$, $\qquad [\because h > 0 \text{ and } f(x) = x \text{ if } x > 0]$

$= 0.$

Also, the left hand limit, *i.e.,* $f(0-0)$

$= \lim_{h \to 0} f(0-h)$, where h is + ive but sufficiently small

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h), \qquad [\because -h < 0 \text{ and } f(x) = -x \text{ if } x < 0]$$

$$= \lim_{h \to 0} h = 0.$$

Thus both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero.

Hence $\lim_{x \to 0} f(x)$ exists and is equal to zero.

Example 7 : *Let* $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if is irrational.} \end{cases}$

Show that $\lim_{x \to a} f(x)$ *exists only when* $a = 0$. **(Purvanchal 2007)**

Solution : Case I : If a is a non-zero rational number.

In this case $f(a-0) = \lim_{h \to 0} f(a-h)$

$$= \lim_{h \to 0} (a-h) \quad \text{or} \quad \lim_{h \to 0} -(a-h),$$

according as $(a-h)$ is rational or irrational

$= a$ or $-a$ *i.e.,* is not unique.

$\therefore$ $f(a-0)$ does not exist.

$\therefore$ $\lim_{x \to a} f(x)$ does not exist.

Case II : If $a = 0$. In this case $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$= \lim_{h \to 0} (-h)$ or $\lim_{h \to 0} h$, according as $-h$ is rational or irrational

$= 0.$

Again $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$= \lim_{h \to 0} h$ or $\lim_{h \to 0} (-h)$, according as h is rational or irrational

$= 0.$

Since $f(0+0) = f(0-0) = 0$, hence $\lim_{x \to 0} f(x)$ exists and is equal to zero.

Case III : If a is an irrational number.

In this case $f(a-0) = \lim_{h \to 0} f(a-h)$

$$= \lim_{h \to 0} (a-h) \quad \text{or} \quad \lim_{h \to 0} -(a-h),$$

according as $(a-h)$ is rational or irrational

$= a$ or $-a$ *i.e.,* is not unique.

$\therefore$ $f(a-0)$ does not exist.

$\therefore$ $\lim_{x \to a} f(x)$ does not exist.

Thus we see that $\lim_{x \to a} f(x)$ exists only when $a = 0$.

Example 8 : *Discuss the existence of the limit of the function f defined as*

$$f(x) = 1, \text{ if } x < 1; \; f(x) = 2 - x, \text{ if } 1 < x < 2; \; f(x) = 2, \text{ if } x \geq 2$$

at $x = 1$ *and* $x = 2$.

Solution : At $x = 1$. We have

$$f(1+0) = \lim_{h \to 0} f(1+h), \quad \text{where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [2 - (1+h)] = \lim_{h \to 0} (1-h) = 1;$$

and $$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1) = 1.$$

Since $f(1+0) = f(1-0) = 1$, hence $\lim_{x \to 1} f(x)$ exists and is equal to 1.

At $x = 2$. We have $f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2) = 2;$

and $$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2 - (2-h)] = \lim_{h \to 0} h = 0.$$

Since $f(2+0) \neq f(2-0)$, hence $\lim_{x \to 2} f(x)$ does not exist.

Example 9 : *If* $\lim_{x \to a} f(x) = \pm \infty$, *then* $\lim_{x \to a} \dfrac{1}{f(x)} = 0$.

Solution : Let $\lim_{x \to a} f(x) = +\infty$.

Let $\varepsilon > 0$ be given. If $\varepsilon_1 = 1/\varepsilon$, then $\varepsilon_1 > 0$.

Since $\lim_{x \to a} f(x) = \infty$, therefore for $\varepsilon_1 > 0$, there exists $\delta > 0$ such that

$$f(x) > \varepsilon_1 \qquad \text{whenever } 0 < |x - a| < \delta$$

i.e., $$\frac{1}{f(x)} < \frac{1}{\varepsilon_1} \qquad \text{whenever } 0 < |x - a| < \delta$$

i.e., $$0 < \frac{1}{f(x)} < \varepsilon \qquad \text{whenever } 0 < |x - a| < \delta \qquad [\because \varepsilon = 1/\varepsilon_1]$$

i.e., $$-\varepsilon < \frac{1}{f(x)} < \varepsilon \qquad \text{whenever } 0 < |x - a| < \delta$$

i.e, $$\left| \frac{1}{f(x)} - 0 \right| < \varepsilon \qquad \text{whenever } 0 < |x - a| < \delta.$$

$$\therefore \qquad \lim_{x \to a} \frac{1}{f(x)} = 0.$$

Similarly it can be proved that

$$\lim_{x \to a} \frac{1}{f(x)} = 0 \quad \text{when } \lim_{x \to a} f(x) = -\infty.$$

Example 10 : *If* $f(x) = \dfrac{\sin [x]}{[x]}$, $[x] \neq 0$ and $f(x) = 0$, $[x] = 0$,

where $[x]$ *denotes the greatest integer less than or equal to x, then find* $\lim_{x \to 0} f(x)$.

(Kanpur 2010)

Solution : Here $f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$$=\lim_{h\to 0} 0 \qquad [\because [h]=0]$$

$$=0.$$

Also $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0}\frac{\sin[-h]}{[-h]}, \qquad [\because [-h]=-1\neq 0]$$

$$=\lim_{h\to 0}\frac{\sin(-1)}{(-1)}=\frac{\sin(-1)}{(-1)}=\sin 1\neq 0.$$

Since $f(0+0)\neq f(0-0)$, therefore $\lim_{x\to 0} f(x)$ does not exist.

Comprehensive Exercise 1

1. Using definition of limit, show that $\lim_{x\to 0} f(x)=1$ where

$$f(x)=\begin{cases}1+x^2 & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$$

2. If f is defined on **R** as $f(x)=\begin{cases}2, & \text{if } x \text{ is irrational}\\ 1, & \text{if } x \text{ is rational,}\end{cases}$

prove that $\lim_{x\to a} f(x)$ does not exist for any $a\in \mathbf{R}$.

3. If f is defined on **R** as $f(x)=\begin{cases}0, & \text{if } x \text{ is irrational}\\ 1, & \text{if } x \text{ is rational,}\end{cases}$

prove that $\lim_{x\to a} f(x)$ does not exist for any $a\in \mathbf{R}$.

4. If $x\to 0$, then does the limit of the following function f exist or not ?

$$f(x)=x, \text{ when } x<0\,;\ f(x)=1, \text{ when } x=0\,;\ f(x)=x^2, \text{ when } x>0.$$

5. Use the formula $\lim_{x\to 0}\dfrac{a^x-1}{x}=\log a$ to find $\lim_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}$.

6. If $f(x)=e^{-1/x}$, show that at $x=0$, the right hand limit is zero while the left hand limit is $+\infty$, and thus there is no limit of the function at $x=0$.

7. Give an example to show that $\lim_{x\to a} f(x)$ may exist even when the function is not defined for $x=a$.

8. Let $f(x)=\begin{cases}x, & 0\leq x<1\\ 3-x, & 1\leq x\leq 2.\end{cases}$

Show that $\lim_{x\to 1+} f(x)=2$. Does the limit of $f(x)$ at $x=1$ exist ?

Give reasons for your answer.

9. Evaluate $\lim_{x\to 0}\dfrac{x-|x|}{x}$. **(Meerut 2001)**

10. Evaluate $\lim_{x \to 0} \dfrac{|\sin x|}{x}$.

11. Evaluate $\lim_{x \to 0} \dfrac{e^{1/x}}{e^{1/x} + 1}$. **(Avadh 2010)**

12. If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$, then prove that

$$\lim_{x \to a} f(x) = f(a).$$

(Garhwal 2011)

Answers 1

4. Yes; $\lim_{x \to 0} f(x) = 0$. **5.** $2 \log 2$. **8.** Does not exist.

9. Right hand limit is 0 and left hand limit is 2 and so the limit does not exist.

10. Does not exist because the right hand limit is 1 and the left hand limit is -1.

11. The limit does not exist because the right hand limit is 1 and the left hand limit is 0.

1.7 Continuity

(Purvanchal 2010, 11; Avadh 14)

The intuitive concept of continuity is derived from geometrical considerations. If the graph of the function $y = f(x)$ is a continuous curve, it is natural to call the function continuous. This requires that there should be no sudden changes in the value of the function. A small change in x should produce only a small change in y. Moreover for the graph to be a continuous running curve, it should possess a definite direction at each point.

But the continuity as defined in pure analysis is quite distinct from the intuitive or the geometrical concept of the term. Sometimes drawing a graph is difficult. We now give the arithmetical definition of continuity given by Cauchy.

Cauchy's definition of continuity : *A real valued function f defined on an open interval I is said to be continuous at $a \in I$ iff for any arbitrarily chosen positive number ε, however small, we can find a corresponding number $\delta > 0$ such that*

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta. \qquad \ldots(1)$$

(Bundelkhand 2010; Kanpur 11)

We say that f is a **continuous function** if it is continuous at every $x \in I$.

In other words, f is continuous at a if for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

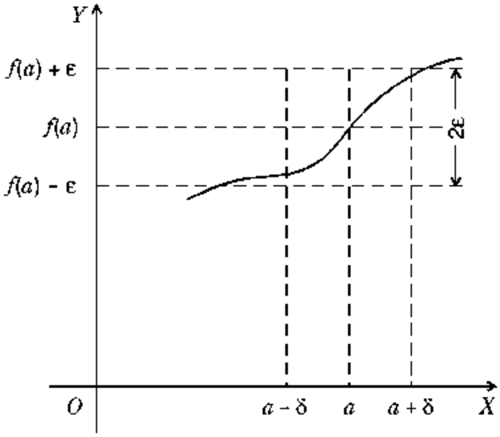
This means that the function f will be continuous at $x = a$ if the difference between $f(a)$ and the value of $f(x)$ at any point in the interval $]a - \delta, a + \delta[$ can be made less than a pre-assigned positive number ε. Note that we choose δ after we have chosen ε.

Geometrical Interpretation of Continuity :

A geometrical interpretation of the above definition is immediate. Corresponding to any pre-assigned positive number ε, we can determine an interval of width 2δ about the point $x = a$ (see the figure) such that for any point x lying in the interval $]a - \delta, a + \delta[$, $f(x)$ is confined to lie between $f(a) - \varepsilon$ and $f(a) + \varepsilon$.

The inequality (1) may be written in the form of an equality as

$$f(x) = f(a) + \eta, \text{ where } |\eta| < \varepsilon.$$



Note 1 : For a function $f(x)$ to be continuous at $x = a$, it is necessary that $\lim_{x \to a} f(x)$ must exist.

Note 2 : The function must be defined at the point of continuity.

Note 3 : The value of δ depends upon the values of ε and a.

Note 4 : The interval I may be of any one of the forms :

$$]a, b[\,,\;]-\infty, b[\,,\;]a, \infty[\,,\;]-\infty, \infty[.$$

An alternative definition of continuity : *A function f is said to be continuous at $a \in I$ iff $\lim_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$ otherwise the function is discontinuous at $x = a$.*

This definition of continuity follows immediately from the definition of limit and the definition of continuity. Thus a function f is said to be continuous at a, if $f(a + 0) = f(a - 0) = f(a)$. This is a working formula for testing the continuity of a function at a given point. **(Bundelkhand 2008, 10; Kashi 12)**

Important Remark : If $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ is a polynomial in x of degree n, then by the above definition it can be easily seen that $f(x)$ is continuous for all $x \in \mathbf{R}$.

If c be any real number, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \{a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n\}$$

$$= a_0 \lim_{x \to c} x^n + a_1 \lim_{x \to c} x^{n-1} + \ldots + a_{n-1} \lim_{x \to c} x + \lim_{x \to c} a_n$$

$$= a_0 c^n + a_1 c^{n-1} + \ldots + a_{n-1} c + a_n \qquad \left[\because \lim_{x \to c} x = c\right]$$

$$= f(c).$$

Since $\lim_{x \to c} f(x) = f(c)$, therefore $f(x)$ is continuous at $x = c$.

Thus $f(x)$ is continuous at every real number c and so $f(x)$ is continuous for all $x \in \mathbf{R}$.

Thus remember that a polynomial function $f(x)$ is always continuous at each point of its domain.

Continuity from left and continuity from right :

Let f be a function defined on an open interval I and let $a \in I$. We say that f is continuous from the left at a if $\lim_{x \to a-0} f(x)$ exists and is equal to $f(a)$. Similarly f is said to be continuous from the right at a if $\lim_{x \to a+0} f(x)$ exists and is equal to $f(a)$.

From the above definitions it is clear that for a function f to be continuous at a, it is necessary as well as sufficient that f be continuous from the left as well as from the right at a.

Continuous function : *A function f is said to be a continuous function if it is continuous at each point of its domain.*

Continuity in an open interval : A function f is said to be continuous in the open interval $]a, b[$ if it is continuous at each point of the interval. **(Bundelkhand 2009)**

Continuity in a closed interval : Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous at a if it is continuous from the right at a and also that f is continuous at b if it is continuous from the left at b. Further, f is said to be continuous on the closed interval $[a, b]$, if (i) it is continuous from the right at a, (ii) continuous from the left at b and (iii) continuous on the open interval $]a, b[$.

Thus if a function f is defined on the closed interval $[a, b]$, then

(i) it is continuous at the left end point a if $f(a) = f(a+0)$

i.e., $$f(a) = \lim_{x \to a+0} f(x)$$

(ii) it is continuous at the right end point b if $f(b) = f(b-0)$

i.e., $$f(b) = \lim_{x \to b-0} f(x)$$

and (iii) it is continuous at an interior point c of $[a, b]$ *i.e.*, at $c \in\ ]a, b[$ if

$$f(c-0) = f(c) = f(c+0) \text{ i.e., if } \lim_{x \to c-0} f(x) = f(c) = \lim_{x \to c+0} f(x).$$

1.8 Discontinuity

Definition : *If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of this function.*

Types of discontinuity : **(Avadh 2014)**

(i) Removable discontinuity : **(Meerut 2011)**

A function f is said to have a *removable discontinuity* at a point a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ *i.e.*, if

$$f(a+0) = f(a-0) \neq f(a).$$

The function can be made continuous by defining it in such a way that

$$\lim_{x \to a} f(x) = f(a).$$

(ii) Discontinuity of the first kind or ordinary discontinuity : **(Meerut 2010B)**

A function f is said to have a *discontinuity of the first kind* or *ordinary discontinuity* at a if $f(a+0)$ and $f(a-0)$ both exist but are not equal. The point a is said to be a point of discontinuity from the left or right according as $f(a-0) \neq f(a) = f(a+0)$ or $f(a-0) = f(a) \neq f(a+0)$.

(iii) Discontinuity of the second kind : A function f is said to have a *discontinuity of the second kind*, at a if none of the limits $f(a+0)$ and $f(a-0)$ exist. The point a is said to be a point of discontinuity of the second kind from the left or right according as $f(a-0)$ or $f(a+0)$ does not exist. **(Meerut 2003, 10B)**

(iv) Mixed discontinuity : **(Meerut 2012B)**

A function f is said to have a *mixed discontinuity* at a, if f has a discontinuity of second kind on one side of a and on the other side a discontinuity of first kind or may be continuous.

(v) Infinite discontinuity : A function f is said to have an *infinite discontinuity* at a if $f(a+0)$ or $f(a-0)$ is $+\infty$ or $-\infty$. Obviously, if f has a discontinuity at a and is unbounded in every neighbourhood of a, then f is said to have an infinite discontinuity at a.

1.9 Jump of a Function at a Point

If both $f(a+0)$ and $f(a-0)$ exist, then the **jump** in the function at a is defined as the non-negative difference $f(a+0) \sim f(a-0)$. A function having a finite number of jumps in a given interval is called **piecewise continuous** or **sectionally continuous**.

Illustrative Examples

Example 1 : *Test the following functions for continuity :*

(i) $f(x) = x \sin(1/x), x \neq 0, f(0) = 0$ *at* $x = 0$. **(Kanpur 2005; Avadh 08; Meerut 09B; Purvanchal 09; Kashi 12; Rohilkhand 14)**

Also draw the graph of the function.

(ii) $f(x) = 2^{1/x}$ *when* $x \neq 0, f(0) = 0$ *at* $x = 0$.

(iii) $f(x) = 1/(1 - e^{-1/x}), x \neq 0, f(0) = 0$ *at* $x = 0$.

Solution : (i) Here $f(0+0) = \lim_{h \to 0} f(0+h), h > 0$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin \frac{1}{h} = 0. \qquad \text{[See theorem 10 of article 1.3]}$$

$$\left[\because \lim_{h \to 0} h = 0 \text{ and } \left| \sin \frac{1}{h} \right| \leq 1 \text{ for all } h \neq 0 \text{ i.e., } \sin(1/h) \text{ is bounded in some deleted neighbourhood of zero} \right]$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h), h > 0$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h) \sin\left(\frac{1}{-h}\right) = \lim_{h \to 0} h \sin \frac{1}{h} = 0, \text{ as before.}$$

Also $f(0) = 0$.

Thus $f(0-0) = f(0) = f(0+0)$.

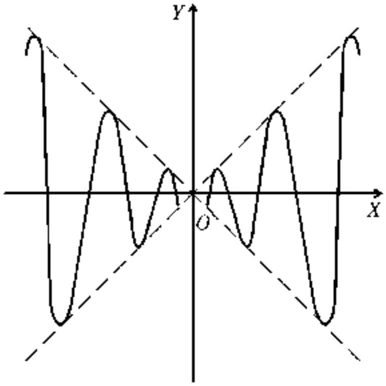
$\therefore$ the function $f(x)$ is continuous at $x = 0$.

To draw the graph of the function we put $y = f(x)$.

So the graph of the function is the curve

$$y = x \sin(1/x), x \neq 0$$

and $\qquad y = 0$ when $x = 0$.



If we put $-x$ in place of x, the equation of this curve does not change and so this curve is symmetrical about the y-axis and it is sufficient to draw the graph when $x > 0$.

Also

$$|f(x)| = |x \sin(1/x)| = |x| \cdot |\sin(1/x)|$$

$$\leq |x|. \qquad [\because |\sin(1/x)| \leq 1]$$

$\therefore$ for all x the curve $y = x \sin(1/x)$ lies between the lines $y = x$ and $y = -x$.

Excluding origin the curve meets the y-axis at the points where

$$\sin\frac{1}{x} = 0 \text{ i.e., where } \frac{1}{x} = \pi, 2\pi, 3\pi, \ldots \text{ i.e., where } x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots.$$

Also $y = x$ at the points where $\sin\frac{1}{x} = 1$ *i.e.*, $\frac{1}{x} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

i.e., $\qquad x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$

and $\qquad y = -x$ at the points where $\sin\frac{1}{x} = -1$ *i.e.*, $\frac{1}{x} = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

i.e., $\qquad x = \frac{2}{3\pi}, \frac{2}{7\pi}, \ldots.$

We have $\qquad \dfrac{dy}{dx} = \sin\dfrac{1}{x} + x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}.$

So at the points where $\sin(1/x) = 1$, we have $\cos(1/x) = 0$ and $dy/dx = 1$ *i.e.*, at these points the curve touches the straight line $y = x$. Similarly at the points where $\sin(1/x) = -1$, the curve touches the straight line $y = -x$.

Also $\qquad \lim\limits_{x \to \infty} x \sin\dfrac{1}{x} \qquad$ [Form $\infty \times 0$]

$$= \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta}, \text{ putting } \frac{1}{x} = \theta \text{ so that } \theta \to 0 \text{ as } x \to \infty$$

$$= 1.$$

Thus $y \to 1$ as $x \to \infty$ and so the straight line $y = 1$ is an asymptote of the curve.

Although the function is continuous at the origin, yet the graph of the function in the vicinity of the origin cannot be drawn, since the function oscillates infinitely often in any interval containing the origin.

From the graph it is clear that the function makes an infinite number of oscillations in the neighbourhood of $x = 0$. The oscillations, however, go on diminishing in length as $x \to 0$.

Note 1 : If we are to check the continuity of $f(x)$ at any point $x = c$, where $c \neq 0$, then we see that $\lim_{x \to c} f(x) = \lim_{x \to c} x \sin \frac{1}{x} = c \sin \frac{1}{c} = f(c)$ and so $f(x)$ is continuous at $x = c$.

Thus $f(x)$ is continuous for all $x \in \mathbf{R}$ *i.e.*, $f(x)$ is continuous on the whole real line.

Note 2 : If we take $f(0) = 2$, the function becomes discontinuous at $x = 0$ and has a **removable discontinuity** at $x = 0$.

(ii) Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} 2^{1/h} = 2^{\infty} = \infty$,

$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} 2^{-1/h} = 2^{-\infty} = 0$,

and $f(0) = 0$.

Since $f(0+0) \neq f(0-0)$, therefore the function is discontinuous at the origin.

It has an **infinite discontinuity** there.

(iii) Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \dfrac{1}{1-e^{-1/h}} = 1$,

$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \dfrac{1}{1-e^{1/h}} = 0$.

Since $f(0+0) \neq f(0-0)$, hence $f(x)$ is discontinuous at $x = 0$ and has discontinuity of the first kind. This function has a jump of one unit at 0 since $f(0+0) - f(0-0) = 1$.

Example 2 : *Consider the function f defined by* $f(x) = x - [x]$, *where x is a positive variable and* $[x]$ *deontes the integral part of x and show that it is discontinuous for integral values of x and continuous for all others. Draw its graph.*

Solution : From the definition of the function $f(x)$, we have

$f(x) = x - (n-1)$ for $n - 1 < x < n$,
$f(x) = 0$ for $x = n$,
$f(x) = x - n$ for $n < x < n + 1$, where n is an integer.

We shall test the function $f(x)$ for continuity at $x = n$.

We have $f(n) = 0$;

$$f(n+0) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} \{(n+h) - n\} \quad [\because n < n+h < n+1]$$

$$= \lim_{h \to 0} h = 0;$$

and

$$f(n-0) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} \{(n-h) - (n-1)\} \quad [\because n-1 < n-h < n]$$

$$= \lim_{h \to 0} (1-h) = 1;$$

Since $f(n+0) \neq f(n-0)$, the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. It is obviously continuous for all other values of x.

Since x is a positive variable, putting $n = 1, 2, 3, 4, 5, \ldots$ we see that the graph of $f(x)$ consists of the following straight lines :

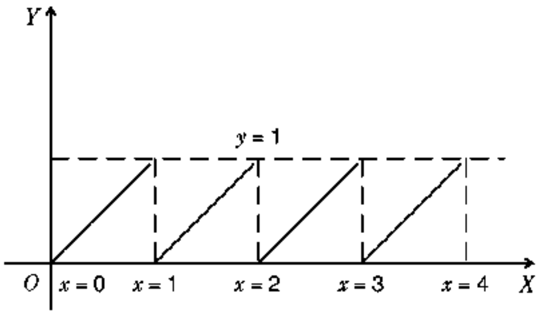
$y = x$ when $0 < x < 1$, $\quad y = 0$ when $x = 1$

$y = x - 1$ when $1 < x < 2$, $\quad y = 0$ when $x = 2$

$y = x - 2$ when $2 < x < 3$, $\quad y = 0$ when $x = 3$

$y = x - 3$ when $3 < x < 4$, $y = 0$ when $x = 4$ and so on.

The graph of the function thus obtained is shown by thick lines from $x = 0$ to $x = 4$. From the graph it is evident that :



(i) The function is discontinuous for all integral values of x but continuous for other values of x.

(ii) The function is bounded between 0 and 1 in every domain which includes an integer.

(iii) The lower bound 0 is attained but the upper bound 1 is not attained since $f(x) \neq 1$ for any value of x.

Example 3 : *Show that the function $f(x) = [x] + [-x]$ has removable discontinuity for integral values of x.* **(Kanpur 2009)**

Solution : We observe that $f(x) = 0$, when x is an integer and $f(x) = -1$, when x is not an integer. Hence if n is any integer, we have $f(n-0) = f(n+0) = -1$ and $f(n) = 0$. So the function $f(x)$ has a removable discontinuity at $x = n$, where n is an integer.

Example 4 : *Let $y = E(x)$, where $E(x)$ denotes the integral part of x. Prove that the function is discontinuous where x has an integral value. Also draw the graph.*

Solution : From the definition of $E(x)$, we have

$$E(x) = n - 1 \quad \text{for} \quad n - 1 \le x < n,$$
$$E(x) = n \quad \text{for} \quad n \le x < n + 1$$
$$E(x) = n + 1 \quad \text{for} \quad n + 1 \le x < n + 2,$$

and so on where n is an integer.

We consider $x = n$.

Then $E(n) = n$, $E(n-0) = n-1$ and $E(n+0) = n$.

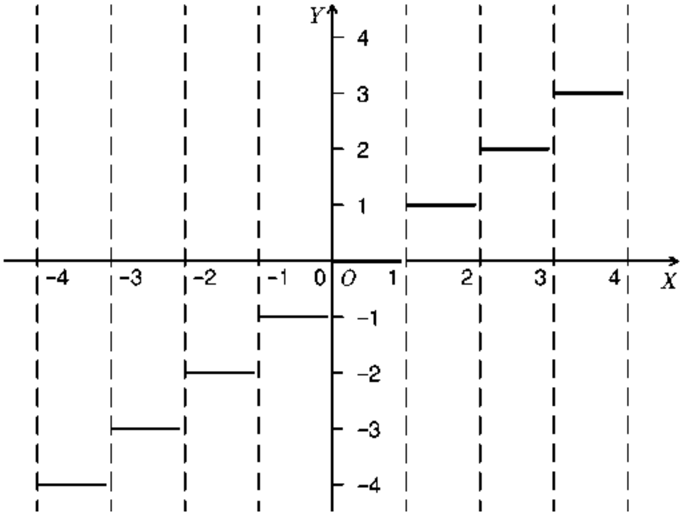
Since $E(n+0) \neq E(n-0)$, the function $E(x)$ is discontinuous at $x = n$ *i.e.,* when x has an integral value.

Evidently it is continuous for all other values of x.

To draw the graph, we put $n = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$, so that

$$\begin{aligned}
y &= -4, && \text{when} && -4 \le x < -3,\\
y &= -3, && \text{when} && -3 \le x < -2,\\
y &= -2, && \text{when} && -2 \le x < -1,\\
y &= -1, && \text{when} && -1 \le x < 0,\\
y &= 0, && \text{when} && 0 \le x < 1\\
y &= 1, && \text{when} && 1 \le x < 2\\
y &= 2, && \text{when} && 2 \le x < 3\\
y &= 3, && \text{when} && 3 \le x < 4\\
y &= 4, && \text{when} && 4 \le x < 5 \text{ and so on.}
\end{aligned}$$

The graph is shown by thick lines.



Example 5 : *Show that the function* ϕ *defined as*

$$\phi(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{2} - x & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{for } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{for } \frac{1}{2} < x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

has three points of discontinuity which your are required to find. Also draw the graph of the function. **(Rohilkhand 2009; Avadh 10, 13)**

Solution : Here the domain of the function $\phi(x)$ is the closed interval [0, 1].

When $0 < x < \frac{1}{2}$, $\phi(x) = \frac{1}{2} - x$ which is a polynomial in x of degree 1. We know that a polynomial function is continuous at each point of its domain and so $\phi(x)$ is continuous at each point of the open interval $0 < x < \frac{1}{2}$.

Again when $\frac{1}{2} < x < 1$, $\phi(x) = \frac{3}{2} - x$ which is also a polynomial in x and so $\phi(x)$ is also continuous at each point of the open interval $\frac{1}{2} < x < 1$.

Now it remains to test the function $\phi(x)$ for continuity at $x = 0, \frac{1}{2}$ and 1.

(i) For $x = 0$, we have $\phi(0) = 0$,

$$\phi(0+0) = \lim_{h \to 0} \phi(0+h) = \lim_{h \to 0} \phi(h) = \lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}.$$

Since $\phi(0) \neq \phi(0+0)$, the function $\phi(x)$ is discontinuous at $x = 0$ and the discontinuity is ordinary.

(ii) For $x = \frac{1}{2}$, we have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$,

$$\phi\left(\frac{1}{2} - 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} - h\right) = \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right], \left[\text{Note that } 0 < \frac{1}{2} - h < \frac{1}{2}\right]$$

$$= \lim_{h \to 0} h = 0.$$

Since $\phi\left(\frac{1}{2} - 0\right) \neq \phi\left(\frac{1}{2}\right)$, the function $\phi(x)$ is discontinuous from the left at $x = 1/2$.

Again $\quad \phi\left(\frac{1}{2} + 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} + h\right), h > 0$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right] \qquad \left[\because \frac{1}{2} < \frac{1}{2} + h < 1\right]$$

$$= \lim_{h \to 0} (1 - h) = 1 \neq \phi\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Thus the function $\phi(x)$ is discontinuous from the right also at $x = \frac{1}{2}$.

In this way $\phi(x)$ has discontinuity of the first kind *i.e.*, ordinary discontinuity at $x = \frac{1}{2}$ and the jump of the function at $x = 1/2$ is $\phi\left(\frac{1}{2} + 0\right) - \phi\left(\frac{1}{2} - 0\right)$ *i.e.*, $1 - 0$ *i.e.*, 1.

(iii) For $x = 1$, we have $\phi(1) = 1$,

$$\phi(1 - 0) = \lim_{h \to 0} \phi(1 - h)$$

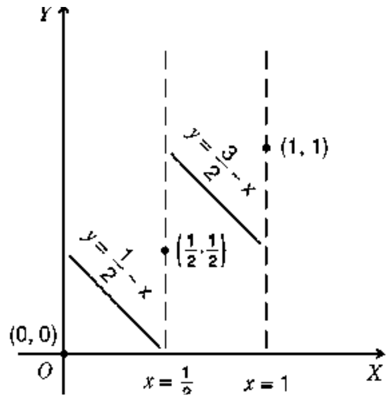
$$= \lim_{h \to 0} [(3/2) - (1 - h)], \qquad \left[\text{Note that } \frac{1}{2} < 1 - h < 1\right]$$

$$= \lim_{h \to 0} \left(\frac{1}{2} + h\right) = \frac{1}{2}.$$

Since $\phi(1) \neq \phi(1 - 0)$, $\phi(x)$ is discontinuous at $x = 1$ and the discontinuity is ordinary.

Hence the function $\phi(x)$ has three points of discontinuity at $x = 0, \frac{1}{2}$ and 1.

The graph of the function consists of the point $(0, 0)$; the segment of the line $y = \frac{1}{2} - x$, $0 < x < \frac{1}{2}$; the point $\left(\frac{1}{2}, \frac{1}{2}\right)$; the segment of the line $y = \frac{3}{2} - x$, $\frac{1}{2} < x < 1$; and the point $(1, 1)$.



Thus the graph is as shown in the figure. From the graph we observe that the function is discontinuous at $x = 0, \frac{1}{2}$ and 1.

Example 6 : *Determine the values of a, b, c for which the function*

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$$

is continuous at $x = 0$.

Solution : Here $f(0 + 0) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \dfrac{(h + bh^2)^{1/2} - h^{1/2}}{bh^{3/2}}$

$$= \lim_{h \to 0} \frac{(1 + bh)^{1/2} - 1}{bh} = \lim_{h \to 0} \frac{\{1 + \frac{1}{2}bh + \ldots\} - 1}{bh} = \frac{1}{2},$$

which is independent of b and so b may have any real value except 0.

Again $f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \dfrac{\sin(a+1)(-h) + \sin(-h)}{(-h)}$

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h} = \lim_{h \to 0} \frac{2\sin\left(\frac{1}{2}a + 1\right)h\cos(ah/2)}{h}$$

$$= \lim_{h \to 0} \frac{\sin \{(a+2)/2\} h}{\{(a+2)/2\} h} (a+2) \cos (ah/2) = a+2.$$

For continuity at $x = 0$, we have $f(0+0) = f(0-0) = f(0)$

i.e., $\frac{1}{2} = a + 2 = c.$ $\therefore$ $c = \frac{1}{2}$ and $a = -\frac{3}{2}$.

Example 7 : *A function $f(x)$ is defined as follows :*

$$f(x) = \begin{cases} (x^2/a) - a, & when \quad x < a \\ 0, & when \quad x = a \\ a - (a^2/x), & when \quad x > a. \end{cases}$$

Prove that the function $f(x)$ is continuous at $x = a$.

(Bundelkhand 2007; Avadh 09; Rohilkhand 13)

Solution : We have $f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left[a - \frac{a^2}{(a+h)} \right]$,

$[\because f(x) = a - (a^2/x) \text{ for } x > a]$

$$= [a - (a^2/a)] = a - a = 0 ;$$

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \left[\frac{(a-h)^2}{a} - a \right], [\because f(x) = (x^2/a) - a \text{ for } x < a]$$

$$= [(a^2/a) - a] = a - a = 0.$$

Also, we have $f(a) = 0$.

Since $f(a+0) = f(a-0) = f(a)$, therefore $f(x)$ is continuous at $x = a$.

Example 8 : *Examine the function defined below for continuity at $x = a$:*

$$f(x) = \frac{1}{x-a} \, cosec \left(\frac{1}{x-a} \right), x \neq a$$

$$f(x) = 0, x = a.$$

(Avadh 2004)

Solution : We have

$$f(a+0) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \frac{1}{a+h-a} \operatorname{cosec} \frac{1}{a+h-a} = \lim_{h \to 0} \frac{1}{h \sin (1/h)}$$

$$= +\infty \quad , \quad \text{since } h \sin (1/h) \to 0 \text{ as } h \to 0.$$

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{1}{a-h-a} \operatorname{cosec} \left(\frac{1}{a-h-a} \right)$$

$$= \lim_{h \to 0} - \left[\frac{1}{h} \cdot \frac{1}{\sin \{-(1/h)\}} \right] = \lim_{h \to 0} \frac{1}{h \sin (1/h)}$$

$$= +\infty, \quad \text{since } h \sin (1/h) \to 0 \text{ as } h \to 0.$$

Also, we have $f(a) = 0$.

Since $f(a+0) = f(a-0) \neq f(a)$, the function $f(x)$ is discontinuous at $x = a$, having an infinite discontinuity of the second kind.

Example 9 : *Examine the function defined below for continuity at $x = 0$:*

$$f(x) = \frac{sin^2 \, ax}{x^2} \; for \; x \neq 0, f(x) = 1 \; for \; x = 0.$$

(Meerut 2010)

Solution : We have $f(0) = 1$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah} \right)^2 . a^2 = 1 . a^2 = a^2 ;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin^2(-ah)}{(-h)^2}$$

$$= \lim_{h \to 0} \frac{\sin^2 ah}{h^2} = a^2.$$

Now $f(x)$ is continuous at $x = 0$ iff

$$f(0+0) = f(0-0) = f(0).$$

Hence $f(x)$ is discontinuous at $x = 0$ unless $a = 1$.

Example 10 : *A function $f(x)$ is defined as follows :*

$$f(x) = 1 + x \text{ if } x \le 2 \text{ and } f(x) = 5 - x \text{ if } x \ge 2.$$

Is the function continuous at $x = 2$? **(Meerut 2002, 06)**

Solution : Here $f(2) = 1 + 2$ or $5 - 2 = 3$;

$$f(2+0) = \lim_{h \to 0} f(2+h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [5 - (2+h)], \qquad [\because 2 + h > 2 \text{ and } f(x) = 5 - x \text{ if } x > 2]$$

$$= \lim_{h \to 0} (3 - h) = 3 ;$$

and $$f(2-0) = \lim_{h \to 0} f(2-h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [1 + (2-h)], \qquad [\because 2 - h < 2 \text{ and } f(x) = 1 + x \text{ if } x < 2]$$

$$= \lim_{h \to 0} (3 - h) = 3.$$

Thus $f(2+0) = f(2-0) = f(2)$. Hence the function $f(x)$ is continuous at $x = 2$.

Example 11 : *Discuss the continuity of the function $f(x)$ defined as follows:*

$$f(x) = x^2 \text{ for } x < -2, \quad f(x) = 4 \text{ for } -2 \le x \le 2, \quad f(x) = x^2 \text{ for } x > 2.$$

Solution : We shall test the continuity of $f(x)$ only at the points $x = -2$ and 2. Obviously it is continuous at all other points.

At $x = -2$. We have $f(-2) = 4$;

$$f(-2+0) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} 4 = 4 ;$$

$$f(-2-0) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} (-2-h)^2, \quad [\because -2 - h < -2]$$

$$= 4.$$

Since $f(-2+0) = f(-2-0) = f(-2)$, the function is continuous at $x = -2$.

At $x = 2$. We have $f(2) = 4$;

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4 ;$$

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4.$$

Since $f(2+0) = f(2-0) = f(2)$, the function is continuous at $x = 2$.

1.10 Algebra Of Continuous Functions

Theorem 1 : *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f + g$ is also continuous at a.*

Theorem 2 : *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Theorem 3 : *If f is continuous at a point a and $c \in \mathbf{R}$, then cf is continuous at a.*

Theorem 4 : *Let f and g be defined on an interval I, and let $g(a) \neq 0$. If f and g are continuous at $a \in I$, then f/g is continuous at a .*

Theorem 5 : *If f is continuous at a then $|f|$ is also continuous at a.*

Note : The converse is not true. For example, if

$f(x) = -1$, for $x < a$ and $f(x) = 1$ for $x \geq a$ then

$\lim_{x \to a} |f(x)| = 1 = |f(a)|$, but $\lim_{x \to a} f(x)$ does not exist.

Thus $|f|$ is continuous at a while f is not continuous at a.

Comprehensive Exercise 2

1. Discuss the continuity and discontinuity of the following functions :

(i) $f(x) = x^3 - 3x$.

(ii) $f(x) = x + x^{-1}$.

(iii) $f(x) = e^{-1/x}$.

(iv) $f(x) = \sin x$.

(v) $f(x) = \cos(1/x)$ when $x \neq 0, f(0) = 0$.

(vi) $f(x) = \sin(1/x)$ when $x \neq 0$, and $f(0) = 0$. **(Lucknow 2011)**

(vii) $f(x) = \dfrac{\sin x}{x}$ when $x \neq 0$ and $f(0) = 1$. **(Kanpur 2007; Avadh 08)**

(viii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ when $x \neq 0$ and $f(0) = 1$. **(Meerut 2004B)**

(ix) $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ when $x \neq 0, f(0) = 0$. **(Bundelkhand 2011)**

(x) $f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}} + \sin(1/x)$ when $x \neq 0, f(0) = 0$.

(xi) $f(x) = \sin x \cos(1/x)$ when $x \neq 0, f(0) = 0$.

2. (i) Discuss the continuity of $f(x)$ at $x = 0$, if $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$

(Meerut 2008)

(ii) If $f(x) = \dfrac{1}{x - a} \sin \dfrac{1}{x - a}$, find $f(a + 0)$ and $f(a - 0)$.

Is the function continuous at $x = a$?

3. Find out the points of discontinuity of the following functions :

(i) $f(x) = (2 + e^{1/x})^{-1} + \cos e^{1/x}$ for $x \neq 0, f(0) = 0$.

(ii) $f(x) = 1/2^n$ for $1/2^{n+1} < x \leq 1/2^n$, $n = 0, 1, 2, \ldots$ and $f(0) = 0$.

4. If $f(x) = \dfrac{1}{x} \sin \dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is finite for every value of x in the interval $[-1, 1]$ but is not bounded. Determine the points of discontinuity of the function if any.

5. A function f defined on $[0, 1]$ is given by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \frac{1}{2}$ · **(Rohilkhand 2012B)**

6. Prove that the function f defined by

$$f(x) = \begin{cases} \dfrac{1}{2}, & \text{if } x \text{ is rational} \\ \dfrac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere.

7. (i) Show that the function f defined by $f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 1$

is not continuous at $x = 0$ and also show how the discontinuity can be removed. **(Meerut 2011; Rohilkhand 06)**

(ii) Show that the function $f(x) = 3x^2 + 2x - 1$ is continuous for $x = 2$.

(iii) Show that the function $f(x) = (1 + 2x)^{1/x}$, $x \neq 0$, $f(x) = e^2$, $x = 0$ is continuous at $x = 0$.

8. Examine the continuity of the function

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$$

at $x = 0, 1$ and 2. **(Meerut 2004, 06B, 07B; Avadh 06; Purvanchal 06, 10)**

9. (i) Show that the function

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0 \text{ and } f(0) = 0$$

is discontinuous at $x = 0$.

Show that the following function is continuous at $x = 0$.

$$f(x) = \frac{\sin^{-1} x}{x}, x \neq 0, f(0) = 1.$$

10. Discuss the continuity of the function $f(x) = \dfrac{1}{1 - e^{1/x}}$, when $x \neq 0$ and $f(0) = 0$ for all values of x. **(Rohilkhand 2010B)**

11. Prove that the function $f(x) = \dfrac{|x|}{x}$ for $x \neq 0, f(0) = 0$ is continuous at all points except $x = 0$. **(Meerut 2009; Kanpur 08, 09)**

12. Test the continuity of the function $f(x)$ at $x = 0$ if $f(x) = \dfrac{e^{1/x} \sin(1/x)}{1 + e^{1/x}}, x \neq 0$ and $f(0) = 0$. **(Meerut 2005)**

13. Examine the following function for continuity at $x = 0$ and $x = 1$:

$$f(x) = \begin{cases} x^2 & \text{for } x \le 0 \\ 1 & \text{for } 0 < x \le 1 \\ \dfrac{1}{x} & \text{for } x > 1. \end{cases}$$

(Meerut 2001, 03, 04B, 05)

14. Discuss the continuity of the following function at $x = 0$:

$$f(x) = \begin{cases} \cos x, & x \ge 0 \\ -\cos x, & x < 0. \end{cases}$$

15. Test the continuity of the following functions at $x = 0$:

(i) $f(x) = x \cos(1/x)$, when $x \ne 0, f(0) = 0$. **(Meerut 2007)**

(ii) $f(x) = x \log x$, for $x > 0, f(0) = 0$.

16. Discuss the nature of discontinuity at $x = 0$ of the function $f(x) = [x] - [-x]$ where $[x]$ denotes the integral part of x.

17. Discuss the continuity of $f(x) = (1/x)\cos(1/x)$.

18. Give an example of each of the following types of functions :

(i) The function which possesses a limit at $x = 1$ but is not defined at $x = 1$.

(ii) The function which is neither defined at $x = 1$ nor has a limit at $x = 1$.

(iii) The function which is defined at two points but is nevertheless discontinuous at both the points.

19. In the closed interval $[-1, 1]$ let f be defined by

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \ne 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ?

Answers 2

1. (i) Continuous for all x. (ii) Discontinuous at $x = 0$.

(iii) Discontinuous at $x = 0$. (iv) Continuous for all x.

(v) Discontinuous at $x = 0$. (vi) Discontinuous at 0.

(vii) Continuous for all x. (viii) Discontinuous at 0.

(ix) Discontinuous at 0. (x) Discontinuous at 0.

(xi) Continuous for all x.

2. (ii) No, it has a discontinuity of second kind. Here both $f(a + 0)$ and $f(a - 0)$ do not exist.

3. (i) Discontinuous at $x = 0$.

(ii) Discontinuous at $x = 1/2^n, n = 1, 2, 3, \ldots$

4. Discontinuous at 0.

8. Continuous at $x = 1, 2$ and discontinuous at $x = 0$.

10. Discontinuous only at $x = 0$ and the discontinuity is ordinary.

12. Discontinuity of the second kind at $x = 0$.

13. Discontinuous at $x = 0$ and continuous at $x = 1$.

14. Discontinuous at $x = 0$.

15. (i) Continuous.

(ii) Continuous.

16. Discontinuity of the first kind.

17. Continuous for all x, except at $x = 0$ where it has discontinuity of the second kind.

18. (i) $f(x) = x^2$ for $x > 1, f(x) = x^3$ for $x < 1$.

(ii) $f(x) = -x^2$ for $x < 1, f(x) = x^2$ for $x > 1$.

(iii) $f(x) = 0$ for $x \leq 0, f(x) = \frac{3}{2} - x$ for $0 < x \leq \frac{1}{2}$, $f(x) = \frac{3}{2} + x$ for $x > \frac{1}{2}$.

19. (i) Yes; (ii) Yes.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = \ldots\ldots$.

(Bundelkhand 2008)

2. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \ldots\ldots$.

3. $\lim_{x \to 0} \frac{\sin(x/4)}{x} = \ldots\ldots$.

4. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \ldots\ldots$.

5. $\lim_{x \to 0} (1 + x)^{1/x} = \ldots\ldots$.

6. $\lim_{x \to 0} \frac{a^x - 1}{x} = \ldots\ldots$.

7. If $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x, then $f(x) = \ldots\ldots$, for $3 < x < 4$.

8. Let $f(x) = \begin{cases} x & , 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2. \end{cases}$

Then $\lim_{x \to 1-} f(x) = \ldots\ldots$.

9. Let $f(x) = \begin{cases} 1 & , \quad x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 2 & , \quad x \geq 2. \end{cases}$

Then (i) $f(\frac{3}{2}) = \ldots\ldots$ (ii) $\lim_{x \to 1+} f(x) = \ldots\ldots$ and (iii) $\lim_{x \to 2-} f(x) = \ldots\ldots$

(Meerut 2003)

10. $\lim_{x \to 0-} \frac{|\sin x|}{x} = \ldots\ldots$.

11. A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but is not equal to ......

12. The domain of the function $f(x) = \dfrac{\sin x}{x}$ is …… .

13. The domain of the function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1 & , x = 0 \end{cases}$

is …… .

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

14. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ is equal to

(a) 1 (b) –1 (c) 2 (d) The limit does not exist

15. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$ is equal to

(a) 0 (b) 1 (c) –1 (d) 2

16. $\lim\limits_{x \to 2+} \dfrac{|x - 2|}{x - 2}$ is equal to

(a) –1 (b) 1 (c) 2 (d) –2

17. $\lim\limits_{x \to 3-} \dfrac{|x - 3|}{x - 3}$ is equal to

(a) –1 (b) 3 (c) –3 (d) 1 **(Meerut 2003; Rohilkhand 14)**

18. $\lim\limits_{x \to 0+} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ is equal to

(a) –1 (b) 1 (c) 0 (d) 2

True or False:

Write 'T' for true and 'F' for false statement.

19. If $f(x) = \begin{cases} x, & \text{when } x < 0 \\ 1, & \text{when } x = 0 \\ x^2, & \text{when } x > 0, \end{cases}$

then $\lim\limits_{x \to 0} f(x) = 0.$

20. The function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 2 & , x = 0 \end{cases}$

is continuous at $x = 0$.

21. The function $f(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0 \end{cases}$

is continuous at $x = 0$.

22. For $\lim\limits_{x \to a} f(x)$ to exist, the function $f(x)$ must be defined at $x = a$.

23. The function $f(x) = \begin{cases} x\cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

is discontinuous at $x = 0$.

24. If a function f is continuous at a, then $|f|$ is also continuous at a.

25. The function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

is continuous at $x = 0$.

26. The function $f(x) = \begin{cases} 1, & x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 2, & x \geq 2 \end{cases}$

is discontinuous at $x = 1$.

27. $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 1$.

28. $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x} = 1$.

29. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = 2$.

30. $\lim\limits_{x \to 0} \dfrac{\sin x}{2x} = \dfrac{1}{2}$.

Answers

1.	$f(a)$.	**2.**	2.	**3.**	$\frac{1}{4}$.	**4.**	$\frac{3}{2}$.	**5.**	e.
6.	$\log_e a$.	**7.**	$x - 3$.	**8.**	1.				
9.	(i) $\frac{1}{2}$		(ii) 1		(iii) 0.	**10.**	-1.	**11.**	$f(a)$.
12.	$\mathbf{R} - \{0\}$.	**13.**	$\mathbf{R}$.	**14.**	(d).	**15.**	(b).	**16.**	(b).
17.	(a).	**18.**	(b).	**19.**	*T*.	**20.**	*F*.	**21.**	*T*.
22.	*F*.	**23.**	*F*.	**24.**	*T*.	**25.**	*T*.	**26.**	*F*.
27.	*F*.	**28.**	*F*.	**29.**	*T*.	**30.**	*T*.		

□

Chapter 2

Differentiability

2.1 Definitions

Derivative at a point : **(Bundelkhand 2010; Purvanchal 11)**

Let I denote the open interval $]a, b[$ in $\mathbf{R}$ and let $x_0 \in I$. Then a function $f : I \to \mathbf{R}$ is said to be **differentiable** (*or* **derivable**) *at x_0 iff*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ or equivalently } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists finitely and this limit, if it exists finitely, is called the **differential coefficient** *or* **derivative** *of f with respect to x at $x = x_0$.*

It is denoted by $f'(x_0)$ or by $Df(x_0)$.

Progressive and regressive derivatives :

The *progressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, h > 0.$$

It is also called the **right hand differential coefficient** of f at $x = x_0$ and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$.

The *regressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}, h > 0.$$

It is also called the **left hand differential coefficient** of f at $x = x_0$ and is denoted by $L f'(x_0)$ or by $f'(x_0 - 0)$.

It is obvious that *f is derivable at x_0 iff $L f'(x_0)$ and $Rf'(x_0)$ both exist and are equal.*

Remark : If $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ is a polynomial in x of degree n, then $f(x)$ is differentiable at every point a of $\mathbf{R}$.

Differentiability in an interval : **(Meerut 2003; Purvanchal 11)**

Open interval $]a, b[$: *A function $f :]a, b[\to \mathbf{R}$ is said to be differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.*

Closed interval $[a, b]$: *A function $f : [a, b] \to \mathbf{R}$ is said to be differentiable in $[a, b]$ iff $Rf'(a)$ exists, $L f'(b)$ exists and f is differentiable at every point of $]a, b[$.*

Derivative of a function : Let f be a function whose domain is an interval I. If I_1 be the set of all those points x of I at which f is differentiable *i.e.,* $f'(x)$ exists and if $I_1 \neq \varnothing$, we get another function f' with domain I_1. It is called the *first derivative* of f (or simply the derivative of f). Similarly 2nd, 3rd, …, nth derivatives of f are defined and are denoted by $f'', f''', \ldots, f^{(n)}$ respectively.

Note : The derivative of a function at a point and the derivative of a function are two different but related concepts. The derivative of f at a point a is a number while the derivative of f is a function. However, very often the term derivative of f is used to denote both number and function and it is left to the context to distinguish what is intended.

An alternate definition of differentiability :

Let f be a function defined on an interval I and let a be an interior point of I. Then, by the definition of $f'(a)$, assuming it to exist, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

i.e., $f'(a)$ exists if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

or equivalently

$$x \in]a - \delta, a + \delta[\Rightarrow f'(a) - \varepsilon < \frac{f(x) - f(a)}{x - a} < f'(a) + \varepsilon.$$

2.2 Geometrical Meaning of a Derivative

We take two neighbouring points $P[a, f(a)]$ and $Q[a + h, f(a + h)]$ on the curve $y = f(x)$.

Let the chord PQ and the tangent at P meet the x-axis in L and T respectively. Let $\angle QLX = \alpha$ and $\angle PTX = \psi$. Draw PN and $QM \perp$ to OX and $PH \perp$ to QM.

Then $PH = NM = OM - ON = a + h - a = h,$

and $QH = QM - MH = QM - PN = f(a + h) - f(a).$

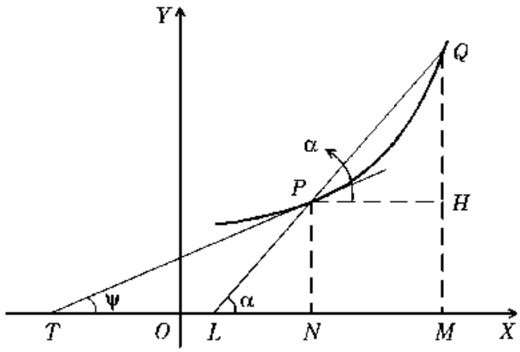
$$\therefore \quad \tan \alpha = \frac{QH}{PH} = \frac{f(a + h) - f(a)}{h}. \quad \ldots(1)$$

As $h \to 0$, the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position and the angle α approaches the angle ψ.

Hence taking limits as $h \to 0$, the equation (1) gives

$$\tan \psi = f'(a).$$

Hence $f'(a)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P[a, f(a)]$ makes with x-axis.



2.3 A Necessary Condition for the Existence of a Finite Derivative

Theorem : *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.* **(Meerut 2010, 10B, 11; Kanpur 07, 12; Avadh 10; Kashi 14)**

Proof : Let f be differentiable at x_0. Then $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists and equals $f'(x_0)$. Now, we can write $f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0)$, if $x \neq x_0$.

Taking limits as $x \to x_0$, we get

$$\lim_{x \to x_0} \left[f(x) - f(x_0)\right] = \lim_{x \to x_0} \left\{\frac{f(x) - f(x_0)}{x - x_0}(x - x_0)\right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \,.\, 0 = 0,$$

so that $$\lim_{x \to x_0} f(x) = f(x_0).$$

Hence f is continuous at x_0. Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative. The following example illustrates this fact :

Let $f(x) = x \sin(1/x), x \neq 0$ and $f(0) = 0$.

This function is continuous at $x = 0$ but not differentiable at $x = 0$.

Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$, therefore the function $f(x)$ is continuous at $x = 0$.

$$\text{Now } Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. Similarly $Lf'(0)$ does not exist.

Thus $f(x)$ is not differentiable at $x = 0$, though it is continuous there.

2.4 Algebra of Derivatives

Now we shall establish some fundamental theorems regarding the differentiability of the sum, product and quotient of differentiable functions.

Theorem 1 : *If a function f is differentiable at a point x_0 and c is any real number, then the function cf is also differentiable at x_0 and* $(cf)'(x_0) = cf'(x_0)$.

Proof : By the definition of $f'(x_0)$, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Now
$$\lim_{x \to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left\{ c \cdot \frac{f(x) - f(x_0)}{x - x_0} \right\}$$

$$= c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x_0).$$

Hence cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

Theorem 2 : *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is $f + g$ and*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

Proof : Since f and g are differentiable at x_0, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \qquad \ldots(1)$$

and
$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0). \qquad \ldots(2)$$

Now
$$\lim_{x \to x_0} \frac{(f + g)(x) - (f + g)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0},$$

as the limit of a sum is equal to the sum of the limits

$= f'(x_0) + g'(x_0)$, using (1) and (2).

Hence $f + g$ is differentiable at x_0 and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

Theorem 3 : *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is fg and* $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Proof : Since f and g are differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \qquad \ldots(1)$$

and

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) \qquad \ldots(2)$$

Now

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} g(x) + f(x_0) \cdot \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0),$$

using (1), (2) and the fact that $\lim_{x \to x_0} g(x) = g(x_0)$.

Note that $g(x)$ is differentiable at $x = x_0$ implies that $g(x)$ is continuous at x_0 and so

$$\lim_{x \to x_0} g(x) = g(x_0).$$

Hence fg is differentiable at x_0 and

$$(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0).$$

Theorem 4 : *If f is differentiable at x_0 and $f(x_0) \neq 0$, then the function $1/f$ is differentiable at x_0 and $(1/f)'(x_0) = -f'(x_0)/\{f(x_0)\}^2$.*

Proof : Since f is differentiable at x_0, therefore $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$.

$$\ldots(1)$$

Since f is differentiable at x_0, it is continuous at x_0, therefore

$$\lim_{x \to x_0} f(x) = f(x_0) \neq 0. \qquad \ldots(2)$$

Also, since $f(x_0) \neq 0$, hence, $f(x_0) \neq 0$ in some neighbourhood N of x_0. Now, we have for $x \in N$

$$\lim_{x \to x_0} \frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x - x_0} = \lim_{x \to x_0} \left\{ -\frac{f(x) - f(x_0)}{x - x_0} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(x_0)} \right\}$$

$$= -\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} \frac{1}{f(x)} \cdot \frac{1}{f(x_0)}$$

$$= - f'(x_0) \cdot \frac{1}{f(x_0)} \cdot \frac{1}{f(x_0)}, \text{ using (1) and (2)}$$

$$= - f'(x_0)/\{f(x_0)\}^2.$$

Hence $1/f$ is differentiable at x_0 and

$$(1/f)'(x_0) = - f'(x_0)/\{f(x_0)\}^2.$$

Theorem 5 : *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, and $g(x_0) \neq 0$, then the function f/g is differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{[g(x_0) f'(x_0) - f(x_0) g'(x_0)]}{[g(x_0)]^2}.$$

Proof : Use theorems 3 and 4 of article 2.4.

2.5 The Chain Rule of Differentiability

Theorem. *Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g o f$ is differentiable at x_0, and*

$$(g o f)'(x_0) = g'(f(x_0)) . f'(x_0).$$

Proof. Let $y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $$f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)] \qquad \ldots(1)$$

where $\lambda(x) \to 0$ as $x \to x_0$.

Further since g is differentiable at y_0, we have

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$$

or $$g(y) - g(y_0) = (y - y_0)[g'(y_0) + \mu(y)] \qquad \ldots(2)$$

where $\mu(y) \to 0$ as $y \to y_0$.

Now $(g o f)(x) - (g o f)(x_0) = g(f(x)) - g(f(x_0)) = g(y) - g(y_0)$

$$= (y - y_0)[g'(y_0) + \mu(y)], \text{ by (2)}$$

$$= [f(x) - f(x_0)][g'(y_0) + \mu(y)]$$

$$= (x - x_0)[f'(x_0) + \lambda(x)][g'(y_0) + \mu(y)], \text{ by (1).}$$

Thus if $x \neq x_0$, then

$$\frac{(g o f)(x) - (g o f)(x_0)}{x - x_0} = [g'(y_0) + \mu(y)] . [f'(x_0) + \lambda(x)]. \qquad \ldots(3)$$

Also f being differentiable at x_0, is continuous at x_0 and hence as $x \to x_0$, $f(x) \to f(x_0)$ *i.e.*, $y \to y_0$.

Consequently $\mu(y) \to 0$ as $x \to x_0$ and $\lambda(x) \to 0$ as $x \to x_0$.

Taking the limits as $x \to x_0$, we get from (3)

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(y_0) \cdot f'(x_0).$$

Hence the function gof is differentiable at x_0 and $(gof)'(x_0) = g'(f(x_0)) f'(x_0)$.

2.6 Derivative of the Inverse Function

Theorem : *If f be a continuous one-to-one function defined on an interval and let f be differentiable at x_0, with $f'(x_0) \neq 0$, then the inverse of the function f is differentiable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.*

Proof : Before proving the theorem we remind that if the domain of f be X and its range be Y, then the inverse function g of f usually denoted by f^{-1} is the function with domain Y and range X such that $f(x) = y \Leftrightarrow g(y) = x$. Also g exists if f is one-one.

Let $y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $$f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)] \qquad ...(1)$$

where $\lambda(x) \to 0$ as $x \to x_0$. Further, we have

$$g(y) - g(y_0) = x - x_0, \text{ by definition of } g.$$

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0) + \lambda(x)}, \text{ by (1).}$$

It can be easily seen that if $y \to y_0$, then $x \to x_0$.

In fact, f is continuous at x_0 implies that $g = f^{-1}$ is continuous at $f(x_0) = y_0$ and consequently

$$g(y) \to g(y_0) \text{ as } y \to y_0 \text{ i.e., } x \to x_0 \text{ as } y \to y_0, \text{ so that } \lambda(x) \to 0 \text{ as } y \to y_0.$$

$$\therefore \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(x_0) + \lambda(x)} = \frac{1}{f'(x_0)}$$

or $$g'(y_0) = \frac{1}{f'(x_0)} \quad \text{or} \quad g'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Illustrative Examples

Example 1 : *Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$ where $|x|$ means the numerical value or the absolute value of x.* **(Bundelkhand 2008; Rohilkhand 07; Avadh 11; Meerut 13B)**

Also draw the graph of the function.

Solution : We have $f(0) = |0| = 0$,

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$

and $\quad f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0}|-h|=\lim_{h\to 0} h=0.$

$\therefore \quad f(0)=f(0+0)=f(0-0).$

Hence $f(x)$ is continuous at $x=0$.

Also, we have $Rf'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\frac{|h|-0}{h}=\lim_{h\to 0}\frac{h}{h}, \text{ (}h\text{ being positive)}$$

$$=\lim_{h\to 0} 1=1,$$

and $\quad Lf'(0)=\lim_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\dfrac{f(-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\frac{|-h|-0}{-h}=\lim_{h\to 0}\frac{h}{-h}, \text{ (}h\text{ being positive)}$$

$$=\lim_{h\to 0}-1=-1.$$

Since $Rf'(0)\neq Lf'(0)$, the function $f(x)$ is not differentiable at $x=0$.

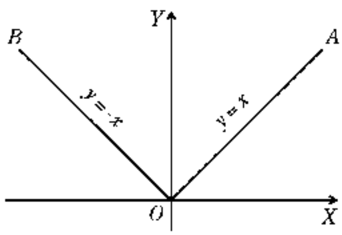
To draw the graph of the function $f(x)=|x|$.

We have $f(x)=\begin{cases} x, & x\geq 0\\ -x, & x\leq 0.\end{cases}$

Let $y=f(x)$. Then the graph of the function consists of the following straight lines :

$$y=x, \quad x\geq 0$$
$$y=-x, \quad x\leq 0.$$

The graph is as shown in the figure. From the graph we observe that the function is continuous at the point O *i.e.*, at the point $x=0$ but it is not differentiable at this point. The tangent to the curve at the point O from the right is the straight line OA and from the left is the straight line OB. Thus the tangent to the curve at O does not exist and so the function is not differentiable at O.



Example 2 : *Show that the function* $f(x)=|x|+|x-1|$ *is not differentiable at* $x=0$ *and* $x=1$. **(Meerut 2005B, 08; Kashi 14)**

Solution : We first observe that if $x<0$, then

$$|x|=-x \text{ and } |x-1|=|1-x|=1-x\ ;$$

if $0\leq x\leq 1$, then $|x|=x$ and $|x-1|=|1-x|=1-x$;

and if $x>1$, then $|x|=x$ and $|x-1|=x-1$.

$\therefore$ the function $f(x)$ is given by

$$f(x)=\begin{cases} 1-2x, & \text{if } x<0\\ 1, & \text{if } 0\leq x\leq 1\\ 2x-1, & \text{if } x>1.\end{cases}$$

At $x = 0$. We have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1-1}{h}, \text{ as } f(x) = 1 \text{ if } 0 \le x \le 1$$

$$= \lim_{h \to 0} 0 = 0,$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{[1 - 2(-h)] - 1}{-h}$$

$[\because f(x) = 1 - 2x, \text{ if } x < 0]$

$$= \lim_{h \to 0} \frac{2h}{-h} = \lim_{h \to 0} -2 = -2.$$

$\because$ $Rf'(0) \neq Lf'(0)$, so the given function is not differentiable at $x = 0$.

At $x = 1$. We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1+h) - 1] - 1}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 1 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2,$$

and $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} 0 = 0.$$

$\because$ $Rf'(1) \neq Lf'(1)$, so the given function $f(x)$ is not differentiable at $x = 1$.

Example 3 : *Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.*

Solution : $f(x)$ is an even function, so $f(-x) = f(x)$ $\forall x$.

$f'(0)$ exists $\Rightarrow Rf'(0) = Lf'(0) = f'(0)$.

Now $f'(0) = Rf'(0) = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}, h > 0$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{h} \qquad [\because f(-x) = f(x)]$$

$$= -\lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = -Lf'(0) = -f'(0).$$

$\therefore$ $2f'(0) = 0 \Rightarrow f'(0) = 0.$

Example 4 : *Let* $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2, \end{cases}$ *and*

$g(x) = f(|x|) + |f(x)|$. *Test the differentiability of $g(x)$ in* $]-2, 2[$.

Solution : When $-2 \le x \le 0$, $|x| = -x$ and when $0 < x \le 2$, $|x| = x$.

Now $-2 \le x \le 0 \Rightarrow |x| = -x$

$\Rightarrow f(|x|) = f(-x) = -x - 1$. $[\because 0 < -x \le 2]$

So we have $f(|x|) = \begin{cases} x - 1, & 0 < x \le 2 \\ -x - 1, & -2 \le x \le 0 \end{cases}$

and $$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ -x+1, & 0 < x \le 1 \\ x-1, & 1 < x \le 2. \end{cases}$$

$$\therefore \quad g(x) = f(|x|) + |f(x)| = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x \le 1 \\ 2x-2, & 1 < x \le 2. \end{cases}$$

We see that $g(x)$ is differentiable $\forall\, x \in\ ]-2, 2[$, except possibly at $x = 0$ and 1.

$$L g'(0) = \lim_{h \to 0} \frac{g(0-h) - g(0)}{-h} = \lim_{h \to 0} \frac{g(-h) - g(0)}{-h}$$

$$= \lim_{h \to 0} \frac{h - 0}{-h} = -1,$$

$$R g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

Since $L g'(0) \ne R g'(0)$, $g(x)$ is not differentiable at $x = 0$.

Again $$R g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{2(1+h) - 2 - 0}{h} = 2,$$

$$L g'(1) = \lim_{h \to 0} \frac{g(1-h) - g(1)}{-h} = \lim_{h \to 0} \frac{0-0}{-h} = 0 \ne R g'(1).$$

$\therefore$ g is not differentiable at $x = 1$.

Example 5 : *Suppose the function f satisfies the conditions :*

(*i*) $f(x+y) = f(x) f(y)\ \forall\, x, y$ (*ii*) $f(x) = 1 + x g(x)$ *where* $\lim_{x \to 0} g(x) = 1.$

Show that the derivative $f'(x)$ exists and $f'(x) = f(x)$ for all x.

Solution : Putting δx for y in the first condition, we have

$$f(x + \delta x) = f(x) f(\delta x).$$

Then $$f(x + \delta x) - f(x) = f(x) f(\delta x) - f(x)$$

or $$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{f(x)\,[f(\delta x) - 1]}{\delta x}$$

$$= \frac{f(x)\, \delta x\, g(\delta x)}{\delta x}, \text{ by given condition (ii)}$$

$$= f(x)\, g(\delta x).$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} f(x)\, g(\delta x) = f(x) \,.\, 1.$$

$$\left[\because \lim_{\delta x \to 0} g(\delta x) = 1\right]$$

$\therefore$ $f'(x) = f(x)$. Since $f(x)$ exists, $f'(x)$ also exists.

Example 6 : *Show that the function f given by $f(x) = x \tan^{-1}(1/x)$ for $x \ne 0$ and $f(0) = 0$ is continuous but not differentiable at $x = 0$.* **(Purvanchal 2008; Meerut 13)**

Solution : Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \tan^{-1} \frac{1}{x} = 0 = f(0)$, therefore the function f is continuous at $x = 0$.

Now $$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \tan^{-1}(1/h) - 0}{h} = \lim_{h \to 0} \tan^{-1}\left(\frac{1}{h}\right) = \tan^{-1} \infty = \frac{\pi}{2}$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \tan^{-1}(-1/h) - 0}{-h}$$

$$= \lim_{h \to 0} \tan^{-1}\left(-\frac{1}{h}\right) = -\tan^{-1} \infty = -\frac{\pi}{2}.$$

Since $Rf'(0) \neq Lf'(0)$, f is not differentiable at $x = 0$.

Example 7 : *Investigate the following function from the point of view of its differentiability. Does the differential coefficient of the function exist at $x = 0$ and $x = 1$?*

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ x^3 - x + 1 & \text{if } x > 1. \end{cases}$$

(Meerut 2006)

Solution : We check the function $f(x)$ for differentiability at $x = 0$ and $x = 1$ only. For other values of x, obviously $f(x)$ is differentiable because it is a polynomial function. It can be seen that $f(x)$ is continuous at $x = 0$ and $x = 1$.

Now $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-(0-h) - 0}{-h} = -1$$

and $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h)^2 - 0}{h} = 0.$$

$\because$ $Lf'(0) \neq Rf'(0)$, the function is not differentiable at $x = 0$.

Again $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{-2h + h^2}{-h} = \lim_{h \to 0} (2 - h) = 2$$

and $$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^3 - (1+h) + 1 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + 3h^2 + h^3}{h} = \lim_{h \to 0} (2 + 3h + h^2) = 2 = Lf'(1).$$

Hence $f'(1)$ exists *i.e.,* the function is differentiable at $x = 1$.

Example 8 : *Find $f'(1)$ if* $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & \text{when } x \neq 1 \\ -1/3, & \text{when } x = 1. \end{cases}$

Solution : We have $f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \left[\frac{1 + h - 1}{2(1+h)^2 - 7(1+h) + 5} - \left(-\frac{1}{3}\right) \right] \Big/ h$$

$$= \lim_{h \to 0} \frac{3h + 2(1+h)^2 - 7(1+h) + 5}{3h[2(1+h)^2 - 7(1+h) + 5]}$$

$$= \lim_{h \to 0} \frac{2h^2}{3h(-3h + 2h^2)} = \lim_{h \to 0} \frac{2}{-9 + 6h} = -\frac{2}{9}.$$

Example 9 : *Test the continuity and differentiability in* $-\infty < x < \infty$, *of the following function :*

$$f(x) = 1 \quad in \quad -\infty < x < 0$$
$$= 1 + \sin x \quad in \quad 0 \le x < \tfrac{1}{2}\pi$$
$$= 2 + \left(x - \tfrac{1}{2}\pi\right)^2 \quad in \quad \tfrac{1}{2}\pi \le x < \infty.$$

(Avadh 2009)

Solution : We shall test $f(x)$ for continuity and differentiability at $x = 0$ and $\pi/2$. It is obviously continuous as well as differentiable at all other points.

(i) Continuity and differentiability of $f(x)$ at $x = 0$.

We have $f(0) = 1 + \sin 0 = 1$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1 + \sin h) = 1;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} 1 = 1.$$

Since $f(0) = f(0+0) = f(0-0)$, $f(x)$ is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{(1 + \sin h) - (1 + \sin 0)}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1;$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$
$$= \lim_{h \to 0} \frac{1 - (1 + \sin 0)}{-h} = \lim_{h \to 0} \frac{0}{-h} = \lim_{h \to 0} 0 = 0.$$

Since $Rf'(0) \ne Lf'(0)$, $f(x)$ is not differentiable at $x = 0$.

(ii) Continuity and differentiability of $f(x)$ at $x = \frac{1}{2}\pi$.

We have $$f\left(\tfrac{1}{2}\pi\right) = 2 + \left(\tfrac{1}{2}\pi - \tfrac{1}{2}\pi\right)^2 = 2;$$

$$f\left(\tfrac{1}{2}\pi + 0\right) = \lim_{h \to 0} f\left(\tfrac{1}{2}\pi + h\right) = \lim_{h \to 0} \left[2 + \left\{\left(\tfrac{1}{2}\pi + h\right) - \tfrac{1}{2}\pi\right\}^2\right]$$
$$= \lim_{h \to 0} (2 + h^2) = 2;$$

and $$f\left(\tfrac{1}{2}\pi - 0\right) = \lim_{h \to 0} f\left(\tfrac{1}{2}\pi - h\right) = \lim_{h \to 0} \left[1 + \sin\left(\tfrac{1}{2}\pi - h\right)\right]$$
$$= \lim_{h \to 0} (1 + \cos h) = 1 + 1 = 2.$$

Since $f\left(\frac{1}{2}\pi\right) = f\left(\frac{1}{2}\pi + 0\right) = f\left(\frac{1}{2}\pi - 0\right)$, f is continuous at $x = \frac{1}{2}\pi$.

Now $$Rf'\left(\frac{1}{2}\pi\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}\pi + h\right) - f\left(\frac{1}{2}\pi\right)}{h}$$
$$= \lim_{h \to 0} \frac{\left[2 + \left\{\frac{1}{2}\pi + h - \frac{1}{2}\pi\right\}^2\right] - \left[2 + \left(\frac{1}{2}\pi - \frac{1}{2}\pi\right)^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{2 + h^2 - 2}{h} = \lim_{h \to 0} h = 0;$$

and $$Lf'\left(\frac{1}{2}\pi\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}\pi - h\right) - f\left(\frac{1}{2}\pi\right)}{-h} = \lim_{h \to 0} \frac{1 + \sin\left(\frac{1}{2}\pi - h\right) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-1 + \cos h}{-h} = \lim_{h \to 0} \frac{1 - \cos h}{h} = \lim_{h \to 0} \frac{2\sin^2(h/2)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin(h/2)}{h/2} . \sin(h/2)\right] = 1 \times 0 = 0.$$

Since $Rf'(0) = Lf'(0), f(x)$ is differentiable at $x = \frac{1}{2}\pi$.

Example 10 : *If $f(x) = x^2 \sin(1/x)$, for $x \neq 0$ and $f(0) = 0$, then show that $f(x)$ is continuous and differentiable everywhere and that $f'(0) = 0$. Also show that the function $f'(x)$ has a discontinuity of second kind at the origin.* **(Meerut 2006B; Kanpur 14)**

Solution : We have $f(0+0) = \lim_{h \to 0} (0+h)^2 \sin\frac{1}{0+h} = \lim_{h \to 0} h^2 \sin\frac{1}{h} = 0;$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h)^2 \sin(-1/h)$$

$$= -\lim_{h \to 0} h^2 \sin\frac{1}{h} = 0.$$

$\because$ $f(0+0) = f(0-0) = f(0)$, so the function is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin\frac{1}{h} = 0;$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h) - 0}{-h} = \lim_{h \to 0} h \sin\frac{1}{h} = 0.$$

Thus $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$. For all other values of $x, f(x)$ is easily seen to be continuous and differentiable.

Now $f'(x) = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ at $x \neq 0$ and $f'(0) = 0$.

$\therefore$ $$f'(0+0) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0} f'(h)$$

$$= \lim_{h \to 0} \left(2h \sin\frac{1}{h} - \cos\frac{1}{h}\right), \quad \text{which does not exist.}$$

Similarly it can be shown that $f'(0-0)$ does not exist.

Hence f' is discontinuous at the origin. Since both the limits $f'(0-0)$ and $f'(0+0)$ do not exist, therefore the discontinuity is of the second kind.

Example 11 : *A function f is defined by $f(x) = x^p \cos(1/x), x \neq 0; f(0) = 0$. What conditions should be imposed on p so that f may be*

(*i*) *continuous at $x = 0$* (*ii*) *differentiable at $x = 0$?*

Solution : We have

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\left[(0+h)^p\cos\{1/(0+h)\}\right]$$

$$=\lim_{h\to 0} h^p\cos(1/h) \qquad ...(1)$$

and $$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\left[(0-h)^p\cos\{1/(0-h)\}\right]$$

$$=\lim_{h\to 0}(-h)^p\cos(1/h). \qquad ...(2)$$

Now if the function $f(x)$ is to be continuous at $x=0$, then

$$f(0+0)=f(0)=0=f(0-0)$$

i.e., the limits given in (1) and (2) must both tend to zero.

This is possible only if $p>0$, which is the required condition.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^p\cos(1/h)-0}{h}=\lim_{h\to 0} h^{p-1}\cos\frac{1}{h} \qquad ...(3)$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{(-h)^p\cos(-1/h)-0}{-h}$$

$$=\lim_{h\to 0}-(-1)^p h^{p-1}\cos(1/h). \qquad ...(4)$$

Now if $f'(x)$ exists at $x=0$ then we must have $Rf'(0)=Lf'(0)$ and this is possible only if $p-1>0$ *i.e.,* $p>1$ which gives $Rf'(0)=0=Lf'(0)$. Hence in order that f is differentiable at $x=0$, p must be greater than 1.

Example 12 : *For a real number y, let [y] denote the greatest integer less than or equal to y. Then if* $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$, *show that $f'(x)$ exists for all x.*

Solution : From the definition of $[y]$, we see that $[x-\pi]$ is an integer for all values of x. Then $\pi(x-\pi)$ is an integral multiple of π and so $\tan(\pi[x-\pi])=0\ \forall x$. Since $[x]$ is an integer so $1+[x]^2\neq 0$ for any x. Thus $f(x)=0$ for all x *i.e.,* $f(x)$ is a constant function and so it is continuous and differentiable *i.e.,* $f'(x)$ exists $\forall x$ and is equal to zero.

Example 13 : *Determine the set of all points where the function* $f(x)=x/(1+|x|)$ *is differentiable.*

Solution : Since $|x|=x, x>0$, $|x|=-x, x<0$, $|x|=0, x=0$,

$$\therefore \quad f(x)=\frac{x}{1-x}, x<0; f(x)=0, x=0; f(x)=\frac{x}{1+x}, x>0.$$

We have $$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0}\frac{h}{1+h}=0;$$

$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0}\frac{-h}{1+h}=0.$$

Since $f(0+0)=f(0)=f(0-0)=0$ so the function is continuous at $x=0$.

Further $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{[-h/(1+h)] - 0}{-h} = \lim_{h \to 0} \frac{1}{1+h} = 1;$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{[h/(1+h)] - 0}{h} = 1.$$

Since $Lf'(0) = Rf'(0)$, so the function is differentiable at $x = 0$. It is obviously differentiable for all other real values of x. Hence it is differentiable in the interval $]-\infty, \infty[$.

Example 14 : *Let* $f(x) = \sqrt{(x)}\,\{1 + x \sin(1/x)\}$ *for* $x > 0, f(0) = 0$,

$$f(x) = -\sqrt{(-x)}\,\{1 + x \sin(1/x)\} \text{ for } x < 0.$$

Show that $f'(x)$ *exists everywhere and is finite except at* $x = 0$ *where its value is* $+\infty$.

Solution : We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{h})\,\{1 + h \sin(1/h)\} - 0}{h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin(1/h)\right] = \infty + 0 = \infty$$

and

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sqrt{[-(-h)]}\left[1 + (-h) \sin \dfrac{1}{-h}\right] - 0}{-h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin \frac{1}{h}\right] = .\infty + 0 = \infty$$

Since $Rf'(0) = Lf'(0) = \infty$, $\therefore f'(0) = \infty$.

We have $f'(x) = \dfrac{1}{2\sqrt{x}} + \dfrac{3}{2}\sqrt{x} \sin \dfrac{1}{x} - \dfrac{1}{\sqrt{x}} \cos \dfrac{1}{x}$ for $x > 0$

and $$f'(x) = \frac{1}{2\sqrt{(-x)}} + \frac{3}{2}\sqrt{(-x)} \sin \frac{1}{x} - \frac{1}{\sqrt{(-x)}} \cos \frac{1}{x} \text{ for } x < 0.$$

Hence $f'(a)$ is finite for all $a \neq 0$.

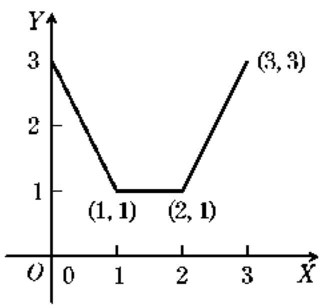
Example 15 : *Draw the graph of the function* $y = |x - 1| + |x - 2|$ *in the interval* $[0, 3]$ *and discuss the continuity and differentiability of the function in this interval.* **(Meerut 2007B, 09; Garhwal 08)**

Solution : From the given definition of the function, we have

$y = 1 - x + 2 - x = 3 - 2x$ when $x \leq 1$

$y = x - 1 + 2 - x = 1$ when $1 \leq x \leq 2$

$y = x - 1 + x - 2 = 2x - 3$ when $x \geq 2$.



Thus the graph consists of the segments of the three straight lines $y = 3 - 2x, y = 1$ and $y = 2x - 3$ corresponding to the intervals $[0, 1], [1, 2], [2, 3]$ respectively. The graph of the function for the interval $[0, 3]$ is as given in the figure.

The graph shows that the function is continuous throughout the interval but is not differentiable at

$x = 1, 2$ because the slopes at these points are different on the left and right hand sides.

To test it analytically, we write $y = f(x)$. Then

$$\begin{aligned} f(x) &= 3 - 2x \quad \text{when } x \le 1 \\ &= 1 \quad \text{when } 1 \le x \le 2 \\ &= 2x - 3 \quad \text{when } x \ge 2. \end{aligned}$$

This function is obviously continuous and differentiable at all points of the interval $[0, 3]$ except possibly at $x = 1$ and at $x = 2$.

At $x = 1$, we have $f(1) = 1$;

$$f(1-0) = \lim_{h \to 0} [3 - 2(1-h)] = 1; f(1+0) = \lim_{h \to 0} (1) = 1.$$

Since $f(1-0) = f(1+0) = f(1)$, f is continuous at $x = 1$.

Again $\quad Rf'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \dfrac{1-1}{h} = 0$

and $\quad Lf'(1) = \lim_{h \to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \dfrac{3 - 2(1-h) - 1}{-h} = -2.$

Since $Rf'(1) \neq Lf'(1)$, f is not differentiable at $x = 1$.

At $x = 2$, we have $f(2) = 1$;

$$f(2-0) = \lim_{h \to 0} (1) = 1; f(2+0) = \lim_{h \to 0} [2(2+h) - 3] = 1.$$

Since $f(2-0) = f(2+0) = f(2)$, f is continuous at $x = 2$.

Again $\quad Rf'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{2(2+h) - 3 - 1}{h} = 2$

and $\quad Lf'(2) = \lim_{h \to 0} \dfrac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \dfrac{1-1}{-h} = 0\cdot$

Since $Rf'(2) \neq Lf'(2)$, f is not differentiable at $x = 2$.

Example 16 : *Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = x\left[1 + \tfrac{1}{3} \sin \log x^2\right], x \neq 0 \text{ and } f(0) = 0$$

is everywhere continuous but has no differential coefficient at the origin.

(Garhwal 2009)

Solution : Obviously the function $f(x)$ is continuous at every point of $\mathbf{R}$ except possibly at $x = 0$. We test at $x = 0$. Given $f(0) = 0$.

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \left[(0+h)\left\{1 + \tfrac{1}{3} \sin \log (0+h)^2\right\}\right]$$

$$= \lim_{h \to 0} [h + (h/3) \sin \log h^2] = 0 + 0 \times \text{ a finite quantity} = 0.$$

[$\because \sin \log h^2$ oscillates between -1 and $+1$ as $h \to 0$]

Similarly we can show that $f(0-0) = 0$.

Hence f is continuous at $x = 0$.

$$\text{Now} \quad Rf'(0) = \lim_{h \to 0} \frac{(0+h)\left\{1 + \frac{1}{3} \sin \log (0+h)^2\right\} - 0}{h}$$

$$= \lim_{h \to 0} \left\{1 + \tfrac{1}{3} \sin \log h^2\right\}, \text{ which does not exist since } \sin \log h^2$$

oscillates between -1 and 1 as $h \to 0$.

$$Lf'(0) = \lim_{h \to 0} \frac{(0-h)\left\{1 + \frac{1}{3}\sin\log(0-h)^2\right\} - 0}{-h}$$

$$= \lim_{h \to 0} \left[1 + \tfrac{1}{3} \sin \log h^2\right], \text{ which does not exist as above.}$$

Hence f has no differential coefficient at $x = 0$.

Example 17 : *Let* $f(x) = x \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}, x \neq 0; f(0) = 0.$

Show that $f(x)$ *is continuous but not derivable at* $x = 0$.

(Meerut 2005; Purvanchal 07, 14; Kanpur 08; Bundelkhand 14)

Solution : We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \lim_{h \to 0} h \frac{1 - e^{-2/h}}{1 + e^{-2/h}}, \quad \text{dividing the Nr. and Dr. by } e^{1/h}$$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

and
$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -h \frac{e^{1/-h} - e^{-1/-h}}{e^{1/-h} + e^{-1/-h}} = \lim_{h \to 0} -h \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{h \to 0} -h \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = 0 \times \frac{0-1}{0+1} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

Now
$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \left[h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0\right] \Big/ h = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1-0}{1+0} = 1,$$

and
$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left[(-h) \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0\right] \Big/ (-h)$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0-1}{0+1} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function is not derivable at $x = 0$.

Example 18 : *Let* $f(x) = e^{-1/x^2} \sin(1/x)$ *when* $x \neq 0$ *and* $f(0) = 0$. *Show that at every point* f *has a differential coefficient and this is continuous at* $x = 0$.

Solution : We test differentiability at $x = 0$.

$$Rf'(0) = \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h) - 0}{h} = \lim_{h \to 0} \frac{\sin(1/h)}{he^{1/h^2}}$$

$$= \lim_{h \to 0} \frac{\sin(1/h)}{h\left\{1 + \frac{1}{h^2} + \frac{1}{2!\,h^4} + \ldots\right\}} = \lim_{h \to 0} \frac{\sin(1/h)}{h + \frac{1}{h} + \frac{1}{2!} \cdot \frac{1}{h^3} + \ldots}$$

$$= \frac{\text{a finite quantity lying between } -1 \text{ and } +1}{\infty} = 0.$$

Similarly $Lf'(0) = 0$.

Since $Rf'(0) = Lf'(0) = 0$, hence the function $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

If x is any point other than zero, then

$$f'(x) = (2/x^3)\, e^{-1/x^2} \sin(1/x) - (1/x^2)\, e^{-1/x^2} \cos(1/x)$$

$$= \{(2/x) \sin(1/x) - \cos(1/x)\}\, (1/x^2)\, (1/e^{1/x^2}) \qquad \ldots(1)$$

Now $\quad f'(0+0) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0} \left(\frac{2}{h} \sin\frac{1}{h} - \cos\frac{1}{h}\right) \cdot \frac{1}{h^2\, e^{1/h^2}}$

$$= \lim_{h \to 0} \left(\frac{2 \sin(1/h)}{h^3\, e^{1/h^2}} - \frac{\cos(1/h)}{h^2\, e^{1/h^2}}\right)$$

$$= \lim_{h \to 0} \left[\frac{2 \sin(1/h)}{h^3\left(1 + \frac{1}{h^2} + \frac{1}{2!.\,h^4} + \ldots\right)} - \frac{\cos(1/h)}{h^2\left(1 + \frac{1}{h^2} + \frac{1}{2!.\,h^4} + \ldots\right)}\right]$$

$$= \frac{\text{some finite quantity}}{\infty} - \frac{\text{some finite quantity}}{\infty} = 0.$$

Similarly $f'(0-0) = 0$. Hence f' is continuous at $x = 0$.

Comprehensive Exercise 1

1. Show that the function $f(x) = |x - 1|$ is not differentiable at $x = 1$.
2. (a) If $f(x) = x/(1 + e^{1/x}), x \neq 0, f(0) = 0$, show that f is continuous at $x = 0$, but $f'(0)$ does not exist. **(Purvanchal 2014)**

   (b) If $f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is continuous at $x = 0$, but $f'(0)$ does not exist.
3. A function ϕ is defined as follows :

   $$\phi(x) = -x \text{ for } x \leq 0, \; \phi(x) = x \text{ for } x \geq 0.$$

   Test the character of the function at $x = 0$ as regards continuity and differentiability.
4. Show that the function f defined on **R** by

   $$f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$$

   is continuous but not differentiable at the points 1, 2 and 3. **(Bundelkhand 2009)**
5. Show that the function

   $$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ x - 1, & 1 < x \leq 2 \end{cases}$$

   has no derivative at $x = 1$.

6. Show that the function

$$f(x) = x^2 - 1, x \geq 1 = 1 - x, x < 1$$

has no derivative at $x = 1$.

7. The following limits are derivatives of certain functions at a certain point. Determine these functions and the points.

(i) $\lim\limits_{x \to 2} \dfrac{\log x - \log 2}{x - 2}$. (ii) $\lim\limits_{h \to 0} \dfrac{\sqrt{(a+h)} - \sqrt{(a)}}{h}$.

8. Let $f(x) = x^2 \sin(x^{-4/3})$ except when $x = 0$ and $f(0) = 0$. Prove that $f(x)$ has zero as a derivative at $x = 0$.

9. A function $\phi(x)$ is defined as follows :

$$\phi(x) = 1 + x \text{ if } x \leq 2$$
$$\phi(x) = 5 - x \text{ if } x > 2.$$

Test the character of the function at $x = 2$ as regards its continuity and differen tiability.

10. Examine the following curve for continuity and differentiability at $x = 0$ and $x = 1$:

$$\begin{aligned} y &= x^2 && \text{for } x \leq 0 \\ y &= 1 && \text{for } 0 < x \leq 1 \\ y &= 1/x && \text{for } x > 1. \end{aligned}$$

Also draw the graph of the function. **(Meerut 2004B, 09B)**

11. A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x) &= 1 + x && \text{for } x \leq 0, \\ f(x) &= x && \text{for } 0 < x < 1, \\ f(x) &= 2 - x && \text{for } 1 \leq x \leq 2, \\ f(x) &= 3x - x^2 && \text{for } x > 2. \end{aligned}$$

Discuss the continuity of $f(x)$ and the existense of $f'(x)$ at $x = 0$, 1 and 2.

12. Discuss the continuity and differentiability of the following function :

$$\begin{aligned} f(x) &= x^2 && \text{for } x < -2 \\ f(x) &= 4 && \text{for } -2 \leq x \leq 2 \\ f(x) &= x^2 && \text{for } x > 2. \end{aligned}$$

Also draw the graph. **(Meerut 2007, 10B)**

13. A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x) &= x && \text{for } 0 \leq x \leq 1 \\ f(x) &= 2 - x && \text{for } x \geq 1. \end{aligned}$$

Test the character of the function at $x = 1$ as regards the continuity and differentiability. **(Meerut 2003)**

14. Examine the function defined by

$$f(x) = x^2 \cos(e^{1/x}), x \neq 0,$$
$$f(0) = 0$$

with regard to (i) continuity (ii) differentiability in the interval $]-1, 1[$.

15. (a) Define continuity and differentiability of a function at a given point. If a function possesses a finite differential coefficient at a point, show that it is continuous at this point. Is the converse true ? Give example in support of your answer.

(b) What do you understand by the derivative of a real valued function at a point ? Show that $f(x) = x \sin(1/x), x \neq 0, f(0) = 0$ is not derivable at $x = 0$.

(c) Prove that if a function $f(x)$ possesses a finite derivative in a closed interval $[a, b]$, then $f(x)$ is continuous in $[a, b]$.

Answers 1

3. Continuous at $x = 0$ but not differentiable at $x = 0$.

7. (i) The function is $\log x$ and the point is $x = 2$.
(ii) The function is $\sqrt{x}$ and the point is $x = a$.

9. Continuous but not differentiable at $x = 2$.

10. Discontinuous and non-differentiable at $x = 0$, continuous and non-differentiable at $x = 1$.

11. Discontinuous and non-differentiable at $x = 0, 2$ and continuous but not differentiable at $x = 1$.

12. Continuous but not differentiable at $x = -2, 2$.

13. Continuous but non-differentiable at $x = 1$.

14. Continuous and differentiable throughout **R**.

2.7 Rolle's Theorem

If a function $f(x)$ is such that

(*i*) *$f(x)$ is continuous in the closed interval* $[a, b]$,

(*ii*) *$f'(x)$ exists for every point in the open interval* $]a, b[$,

(*iii*) *$f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.* **(Lucknow 2007; Purvanchal 07; Kanpur 08; Meerut 12B; Avadh 14; Kashi 13,14)**

Proof : Since f is continuous on $[a, b]$, it is bounded on $[a, b]$. Let M and m be the supremum and infimum of f respectively in the closed interval $[a, b]$.

Now either $M = m$ or $M \neq m$.

If $M = m$, then f is a constant function over $[a, b]$ and consequently $f'(x) = 0$ for all x in $[a, b]$. Hence the theorem is proved in this case.

If $M \neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$. For the sake of definiteness, let $M \neq f(a)$.

Since every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a, b]$ such that $f(c) = M$. Also, since $f(a) \neq M \neq f(b)$, therefore, c is different from both a and b. This implies that $c \in]a, b[$.

Now $f(c)$ is the supremum of f on $[a, b]$, therefore,

$$f(x) \leq f(c) \quad \forall \quad x \text{ in } [a, b]. \qquad \ldots(1)$$

In particular, for all positive real numbers h such that $c - h$ lies in $[a, b]$,

$$f(c - h) \leq f(c)$$

$$\Rightarrow \quad \frac{f(c - h) - f(c)}{-h} \geq 0. \qquad \ldots(2)$$

Since $f'(x)$ exists at each point of $]a, b[$, and hence, in particular $f'(c)$ exists, so taking limit as $h \to 0$, (2) gives $\quad Lf'(c) \geq 0. \qquad \ldots(3)$

Similarly, from (1), for all positive real numbers h such that $c+h$ lies in $[a, b]$, we have

$$f(c+h) \leq f(c).$$

By the same argument as above, we get

$$Rf'(c) \leq 0. \qquad \ldots(4)$$

Since $f'(c)$ exists, hence, $Lf'(c) = f'(c) = Rf'(c)$. ...(5)

From (3), (4) and (5) we conclude that $f'(c) = 0$.

In the same manner we can consider the case $M = f(a) \neq m$.

Note 1 : There may be more than one point like c at which $f'(x)$ vanishes.

Note 2 : Rolle's theorem will not hold good

(i) if $f(x)$ is discontinuous at some point in the interval $a \leq x \leq b$,

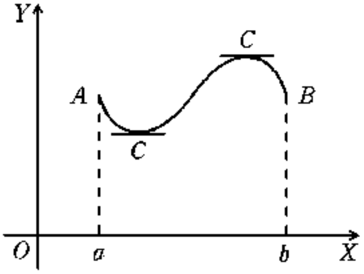
or **(ii)** if $f'(x)$ does not exist at some point in the interval $a < x < b$,

or **(iii)** if $f(a) \neq f(b)$.

Note 3 : It can be seen that the conditions of Rolle's theorem are not necessary for $f'(x)$ to vanish at some point in $]a, b[$. For example, $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ in the interval $[-1, 2]$ but $f'(x)$ vanishes at an infinite number of points in the interval.

Geometrical interpretation of Rolle's Theorem :

Suppose the function $f(x)$ is not constant and satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Then its geometrical interpretation is that the curve representing the graph of the function f must have a tangent parallel to x-axis, at least at one point between a and b.



Algebraical interpretation of Rolle's Theorem :

Rolle's theorem leads to a very important result in the theory of equations, when $f(a) = f(b) = 0$ and $f : [a, b] \to \mathbf{R}$ is a polynomial function $f(x)$. Here a and b are the roots of the equation $f(x) = 0$. Since a polynomial function $f(x)$ is continuous and differentiable at every point of its domain and we have taken $f(a) = f(b)$, therefore, all the three conditions of Rolle's theorem are satisfied and consequently there exists a point $c \in]a, b[$ such that $f'(c) = 0$ *i.e., if a and b are any two roots of the polynomial equation* $f(x) = 0$, *then there exists at least one root of the equation* $f'(x) = 0$ *which lies between a and b.*

Illustrative Examples

Example 1 : *Discuss the applicability of Rolle's theorem for* $f(x) = 2 + (x-1)^{2/3}$ *in the interval* $[0, 2]$. **(Meerut 2012)**

Solution : We have $f(x) = 2 + (x-1)^{2/3}$. Here $f(0) = 3 = f(2)$, which shows that the third condition of Rolle's theorem is satisfied.

Since $f(x)$ is an algebraic function of x, it is continuous in the closed interval $[0, 2]$. Thus the first condition of Rolle's theorem is satisfied.

Now $f'(x) = \frac{2}{3} \cdot [1/(x-1)^{1/3}]$. We see that for $x = 1$, $f'(x)$ is not finite while $x = 1$ is a point of the open interval $0 < x < 2$. Thus the second condition of Rolle's theorem is not satisfied.

Hence the Rolle's theorem is not applicable for the function $f(x) = 2 + (x-1)^{2/3}$ in the interval $[0, 2]$.

Example 2 : *Discuss the applicability of Rolle's theorem in the interval* $[-1, 1]$ *to the function* $f(x) = |x|$.

Solution : Given $f(x) = |x|$. Here $f(-1) = |-1| = 1$, $f(1) = |1| = 1$, so that $f(-1) = f(1)$.

Further the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but it is not differentiable at $x = 0$ which is a point of the open interval $]-1, 1[$. Thus the second condition of Rolle's theorem is not satisfied. Hence the Rolle's theorem is not applicable here.

Example 3 : *Are the conditions of Rolle's theorem satisfied in the case of the following functions ?*

(i) $f(x) = x^2$ *in* $2 \le x \le 3$, *(ii)* $f(x) = \cos(1/x)$ *in* $-1 \le x \le 1$,
(iii) $f(x) = \tan x$ *in* $0 \le x \le \pi$.

Solution : (i) The function $f(x) = x^2$ is continuous and differentiable in the interval $[2, 3]$. Also $f(2) = 4$ and $f(3) = 9$, so that $f(2) \ne f(3)$.

Thus the first two conditions of Rolle's theorem are satisfied and the third condition is not satisfied.

(ii) The function $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied.

Here $f(-1) = \cos(-1) = \cos 1$ and $f(1) = \cos 1$. Thus $f(-1) = f(1)$ i.e., the third condition is satisfied.

(iii) The function $f(x) = \tan x$ is not continuous at $x = \pi/2$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied here.

Further $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$. Thus $f(0) = f(\pi)$ *i.e.,* the third condition is satisfied.

Example 4 : *Discuss the applicability of Rolle's theorem to* $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$, *in the interval* $[a, b]$, $0 < a < b$. **(Rohilkhand 2014)**

Solution : Here $f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)a}\right] = \log 1 = 0,$

and $$f(b) = \log\left[\frac{b^2 + ab}{(a+b)b}\right] = \log 1 = 0.$$

Thus $f(a) = f(b) = 0$.

Also $$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log \frac{(x^2 + 2xh + h^2 + ab)(a+b)x}{(a+b)(x+h)(x^2+ab)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2 + 2xh + h^2 + ab)}{x^2 + ab} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{1 + \frac{2xh + h^2}{x^2 + ab}\right\} - \log\left\{1 + \frac{h}{x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2xh + h^2}{x^2 + ab} - \frac{h}{x} + \ldots\right], \qquad \ldots(1)$$

$$\left[\because \log(1+y) = y - \tfrac{1}{2}y^2 + \ldots\right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $Lf'(x) = \lim_{h \to 0}\left[\frac{f(x-h) - f(x)}{-h}\right]$

$$= \lim_{h \to 0} \frac{1}{(-h)}\left[\frac{-2hx + h^2}{x^2 + ab} - \frac{(-h)}{x} + \ldots\right], \text{ replacing } h \text{ by } -h \text{ in (1)}$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Since $Rf'(x) = Lf'(x)$, $f(x)$ is differentiable for all values of x in $[a, b]$. This implies that $f(x)$ is also continuous for all values of x in $[a, b]$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]a, b[$.

Now $\quad f'(x) = 0 \Rightarrow \frac{2x}{x^2 + ab} - \frac{1}{x} = 0$ or $2x^2 - (x^2 + ab) = 0$

or $\quad x^2 = ab$ or $x = \sqrt{(ab)}$,

which being the geometric mean of a and b lies in the open interval $]a, b[$. Hence the Rolle's theorem is verified.

Remark : In this question to find $f'(x)$, we can also proceed as follows :

We have $f(x) = \log(x^2 + ab) - \log(a + b) - \log x$.

$\therefore \quad f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}$.

Obviously $f'(x)$ exists for all values of x in $[a, b]$.

Example 5 : *Verify Rolle's theorem in the case of the functions*

(i) $f(x) = 2x^3 + x^2 - 4x - 2$,

(ii) $f(x) = \sin x$ *in* $[0, \pi]$,

(iii) $f(x) = (x - a)^m (x - b)^n$, *where* m *and* n *are +ive integers, and* x *lies in the interval* $[a, b]$.

Solution : (i) Since $f(x)$ is a rational integral function of x, therefore, it is continuous and differentiable for all real values of x. Thus the first two conditions of Rolle's theorem are satisfied in any interval.

Here $f(x) = 0$ gives $2x^3 + x^2 - 4x - 2 = 0$

or $\quad (x^2 - 2)(2x + 1) = 0$ *i.e.*, $x = \pm\sqrt{2}, -\frac{1}{2}$.

Thus $\quad f(\sqrt{2}) = f(-\sqrt{2}) = f\left(-\frac{1}{2}\right) = 0.$

If we take the interval $[-\sqrt{2}, \sqrt{2}]$, then all the three conditions of Rolle's theorem are satisfied in this interval. Consequently there is at least one value of x in the open interval $]-\sqrt{2}, \sqrt{2}[$ for which $f'(x) = 0$.

Now $\quad f'(x) = 0 \Rightarrow 6x^2 + 2x - 4 = 0 \Rightarrow 3x^2 + x - 2 = 0$

or $\quad (3x-2)(x+1) = 0 \quad$ or $\quad x = -1, 2/3 \quad$ *i.e.,* $f'(-1) = f'(2/3) = 0$.

Since both the points $x = -1$ and $x = 2/3$ lie in the open interval $]-\sqrt{2}, \sqrt{2}[$, Rolle's theorem is verified.

(ii) The function $f(x) = \sin x$ is continuous and differentiable in $[0, \pi]$.

Also $f(0) = 0 = f(\pi)$. Thus all the three condtions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]0, \pi[$.

Now $f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm \dfrac{\pi}{2}, \pm \dfrac{3\pi}{2}, \pm \dfrac{5\pi}{2}, \ldots$.

Since $x = \pi/2$ lies in the open interval $]0, \pi[$, the Rolle's theorem is verified.

(iii) We have $f(x) = (x-a)^m (x-b)^n$.

As m and n are positive integers, $(x-a)^m$ and $(x-b)^n$ are polynomials in x on being expanded by binomial theorem. Hence $f(x)$ is also a polynomial in x. Consequently $f(x)$ is continuous and differentiable in the closed interval $[a, b]$. Also $f(a) = f(b) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied so that there is at least one value of x in the open interval $]a, b[$ where $f'(x) = 0$.

Now $\quad f'(x) = (x-a)^m . n(x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n$.

Solving the equation $f'(x) = 0$, we get $x = a, b, (na + mb)/(m+n)$.

Out of these values the value $(na + mb)/(m+n)$ is a point which lies in the open interval $]a, b[$, since it divides the interval $]a, b[$ internally in the ratio $m : n$. Hence the Rolle's theorem is verified.

Example 6 : *Verify Rolle's theorem for*

$$f(x) = x(x+3) e^{-x/2} \text{ in } [-3, 0].$$

Solution : We have $f(x) = x(x+3) e^{-x/2}$.

$$\therefore \quad f'(x) = (2x+3) e^{-x/2} + (x^2 + 3x) e^{-x/2} . \left(-\frac{1}{2}\right)$$

$$= e^{-x/2}\left[2x + 3 - \frac{1}{2}(x^2 + 3x)\right] = -\frac{1}{2}\left(x^2 - x - 6\right) e^{-x/2},$$

which exists for every value of x in the interval $[-3, 0]$. Hence $f(x)$ is differentiable and so also continuous in the interval $[-3, 0]$. Also $f(-3) = f(0) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x) = 0$ for at least one value of x lying in the open interval $]-3, 0[$.

Now $\quad f'(x) = 0 \Rightarrow -\frac{1}{2}(x^2 - x - 6) e^{-x/2} = 0 \quad$ or $\quad x^2 - x - 6 = 0$

or $\quad (x-3)(x+2) = 0 \quad$ or $\quad x = 3, -2$.

Since the value $x = -2$ lies in the open interval $]-3, 0[$, the Rolle's theorem is verified.

Comprehensive Exercise 2

1. (i) State Rolle's theorem. **(Kanpur 2005)**

   (ii) Verify Rolle's theorem when $f(x) = e^x \sin x, a = 0, b = \pi$.

2. Verify Rolle's theorem for the following functions :

(i) $f(x) = (x-4)^5 (x-3)^4$ in the interval $[3, 4]$.

(ii) $f(x) = x^3 - 6x^2 + 11x - 6$.

(iii) $f(x) = x^3 - 4x$ in $[-2, 2]$.

(iv) $f(x) = e^x (\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$. **(Meerut 2013B)**

(v) $f(x) = 10x - x^2$ in $[0, 10]$. **(Kanpur 2006)**

3. Discuss the applicability of Rolle's theorem to the function

$$f(x) = x^2 + 1, \text{ when } 0 \le x \le 1$$
$$= 3 - x, \text{ when } 1 < x \le 2.$$

4. Show that between any two roots of $e^x \cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.

5. State and prove Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function $f(x) = x^2$ in $[-1, 1]$.

6. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ on the interval $[1, 2]$.

7. Does the function $f(x) = |x - 2|$ satisfy the conditions of Rolle's theorem in the interval $[1, 3]$. Justify your answer with correct reasoning.

8. The function f is defined in $[0, 1]$ as : $f(x) = 1$ for $0 \le x < \frac{1}{2}$

$= 2$ for $\frac{1}{2} \le x \le 1$.

Show that $f(x)$ satisfies none of the conditions of Rolle's theorem, yet $f'(x) = 0$ for many points in $[0, 1]$.

9. If $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $]0, 1[$.

10. Let $\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \dfrac{a_2}{n-1} + \ldots + \dfrac{a_{n-1}}{2} + a_n = 0$. Show that there exists at least one real x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$.

Answers 2

3. The given function is not differentiable at $x = 1$ and so Rolle's theorem is not applicable to the given function in the interval $[0, 2]$.

7. The function does not satisfy the third condition that $f(x)$ must be differentiable in the open interval $]1, 3[$.

2.8 Lagrange's Mean Value Theorem

Theorem : *If a function $f(x)$ is*

(*i*) *continuous in a closed interval* $[a, b]$,

and (*ii*) *differentiable in the open interval* $]a, b[$ *i.e.,* $a < x < b$, *then there exists at least one value 'c' of x lying in the open interval* $]a, b[$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

(Kanpur 2011; Rohilkhand 12, 12B; Meerut 12; Kashi 14; Avadh 07, 12, 14)

Proof : Consider the function $\phi(x)$ defined by $\phi(x) = f(x) + Ax$, ...(1)

where A is a constant to be chosen such that $\phi(a) = \phi(b)$

i.e., $$f(a) + Aa = f(b) + Ab \quad \text{or} \quad A = -\frac{f(b) - f(a)}{b - a}. \qquad \text{...(2)}$$

(i) Now the function f is given to be continuous on $[a, b]$ and the mapping $x \to Ax$ is continuous on $[a, b]$, therefore ϕ is continuous on $[a, b]$.

(ii) Also, since f is given to be differentiable on $]a, b[$ and the mapping $x \to Ax$ is differentiable on $]a, b[$, therefore, ϕ is differentiable on $]a, b[$.

(iii) By our choice of A, we have $\phi(a) = \phi(b)$.

From (i), (ii) and (iii), we find that ϕ satisfies all the conditions of Rolle's theorem on $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $]a, b[$, such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$, from (1).

$\therefore$ $$\phi'(c) = 0 \Rightarrow f'(c) + A = 0$$

or $$f'(c) = -A = \frac{f(b) - f(a)}{b - a}, \text{ from (2).}$$

This proves the theorem. It is usually known as the 'First Mean Value Theorem of Differential Calculus'.

Another form of Lagrange's mean value theorem.

If in the above theorem, we take $b = a + h$, then a number c, lying between a and b can be written as $c = a + \theta h$, where θ is some real number such that $0 < \theta < 1$.

Now Lagrange's theorem can be stated as follows :

If f be defined and continuous on $[a, a + h]$ *and differentiable on* $]a, a + h[$*, then there exists a point* $c = a + \theta h$ $(0 < \theta < 1)$ *in the open interval* $]a, a + h[$ *such that*

$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h)$$

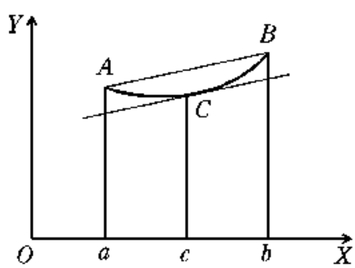
or $$\boldsymbol{f(a + h) - f(a) = hf'(a + \theta h).}$$

Geometrical interpretation of the mean value theorem.

Let $y = f(x)$ and let ACB be the graph of $y = f(x)$ in $[a, b]$. The coordinates of the point A are $(a, f(a))$ and those of B are $(b, f(b))$. If the chord AB makes an angle α with the x-axis, then

$$\tan \alpha = \frac{f(b) - f(a)}{b - a}$$
$$= f'(c),$$

by Lagrange's mean value theorem where $a < c < b$.



Thus Lagrange's mean value theorem says that there is some point c in $]a, b[$ such that the tangent to the curve at this point is parallel to the chord joining the points on the graph with abscissae a and b.

2.9 Important Deductions from the Mean Value Theorem

Theorem 1 : *If a function f is continuous on* $[a, b]$, *differentiable on* $]a, b[$ *and if* $f'(x) = 0$ *for all x in* $]a, b[$, *then* $f(x)$ *has a constant value throughout* $[a, b]$.

Proof : Let c be any point of $]a, b]$. Then the function f is continuous on $[a, c]$ and differentiable on $]a, c[$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a, c]$. Consequently there exists a real number d between a and c *i.e.*, $a < d < c$ such that

$$f(c) - f(a) = (c - a) f'(d).$$

But by hypothesis $f'(x) = 0$ throughout the interval $]a, b[$, therefore, in particular $f'(d) = 0$ and hence $f(c) - f(a) = 0$ or $f(c) = f(a)$. Since c is any point of $]a, b]$, therefore, it gives that $f(x) = f(a)\ \forall\ x$ in $]a, b]$. Thus $f(x)$ has a constant value throughout $[a, b]$.

Theorem 2 : *If $f(x)$ and $\phi(x)$ are functions continuous on $[a, b]$ and differentiable on $]a, b[$ and if $f'(x) = \phi'(x)$ throughout the interval $]a, b[$, then $f(x)$ and $\phi(x)$ differ only by a constant.*

Proof : Consider the function $F(x) = f(x) - \phi(x)$. Throughout the interval $]a, b[$, we have

$$F'(x) = f'(x) - \phi'(x) = 0, \text{ because } f'(x) = \phi'(x).$$

Consequently, from theorem 1, we get

$$F(x) = \text{constant or } f(x) - \phi(x) = \text{constant}.$$

Theorem 3 : *If $f'(x) = k$ for each point x of $[a, b]$, k being a constant, then*

$$f(x) = kx + C\ \forall\ x \in [a, b], \textit{ where C is a constant.}$$

Proof : Consider the interval $[a, x]$ such that $[a, x]$ lies in the interval $[a, b]$ *i.e.,* $[a, x] \subset [a, b]$. Since $f'(x)$ exists $\forall\ x \in [a, b]$, f is differentiable on $[a, b]$ and hence on $[a, x]$ and consequently continuous on $[a, x]$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a, x]$ and hence there is a point $c \in\ ]a, x[$ such that

$$f(x) - f(a) = (x - a) f'(c).$$

But by hypothesis $f'(x) = k\ \forall\ x \in [a, b]$, therefore, in particular $f'(c) = k$ as $a < c < x < b$ *i.e.,* $a < c < b$.

Hence $f(x) - f(a) = (x - a)k$ or $f(x) = kx + f(a) - ak$

or $\qquad f(x) = kx + C$ where $C = f(a) - ak$ is a constant.

Theorem 4 : *If f is continuous on $[a, b]$ and $f'(x) \geq 0$ in $]a, b[$, then f is increasing in $[a, b]$.*

Proof : Let x_1 and x_2 be any two distinct points of $[a, b]$ such that $x_1 < x_2$. Then f satisfies the conditions of the Lagrange's mean value theorem in $[x_1, x_2]$. Consequently there exists a number c such that $x_1 < c < x_2$, and $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$.

Now $x_2 - x_1 > 0$ and $f'(c) \geq 0$ (as $f'(x) \geq 0\ \forall\ x \in\ ]a, b[$ and c is a point of $]a, b[$), therefore

$$f(x_2) - f(x_1) \geq 0 \textit{ i.e., } f(x_1) \leq f(x_2).$$

Thus $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)\ \forall\ x_1, x_2 \in [a, b]$.

Hence f is an increasing function in the interval $[a, b]$.

Similarly, we can prove that if $f'(x) \leq 0$ in $]a, b[$, then f is decreasing in $[a, b]$.

Corollary : *If f is continuous on $[a, b]$, then f is strictly increasing or strictly decreasing on $[a, b]$ according as*

$$f'(x) > 0 \textit{ or } < 0 \textit{ in }]a, b[\cdot$$

2.10 Cauchy's Mean Value Theorem

If two functions $f(x)$ and $g(x)$ are

(*i*) *continuous in a closed interval* $[a, b]$,

(*ii*) *differentiable in the open interval* $]a, b[$,

and (*iii*) $g'(x) \neq 0$ *for any point of the open interval* $]a, b[$, *then there exists at least one value c of x in the open interval* $]a, b[$, *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, a < c < b.$$

(Kanpur 2007; Avadh 12; Rohilkhand 14)

Proof : First we observe that as a consequence of condition (iii), $g(b) - g(a) \neq 0$. For if $g(b) - g(a) = 0$ *i.e.*, $g(b) = g(a)$, then the function $g(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$ and consequently there is some x in $]a, b[$ for which $g'(x) = 0$, thus contradicting the hypothesis that $g'(x) \neq 0$ for any point of $]a, b[$.

Now consider the function $F(x)$ defined on $[a, b]$, by setting

$$F(x) = f(x) + Ag(x), \quad \text{...(1)}$$

where A is a constant to be chosen such that $F(a) = F(b)$

i.e., $$f(a) + Ag(a) = f(b) + Ag(b)$$

or $$-A = \frac{f(b) - f(a)}{g(b) - g(a)}. \quad \text{...(2)}$$

Since $g(b) - g(a) \neq 0$, therefore A is a definite real number.

(i) Now f and g are continuous on $[a, b]$, therefore, F is also continuous on $[a, b]$.

(ii) Again, since f and g are differentiable on $]a, b[$, therefore F is also differentiable on $]a, b[$.

(iii) By our choice of A, $F(a) = F(b)$.

Thus the function $F(x)$ satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Consequently there exists, at least one value, say c, of x in the open interval $]a, b[$ such that $F'(c) = 0$.

But $F'(x) = f'(x) + Ag'(x)$, from (1).

$\therefore$ $F'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$

or $$-A = \frac{f'(c)}{g'(c)}. \quad \text{...(3)}$$

From (2) and (3), we get $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$.

Another form : If $b = a + h$, then $a + \theta h = a$ when $\theta = 0$ and $a + \theta h = b$ when $\theta = 1$. Therefore, if $0 < \theta < 1$, then $a + \theta h$ means some value between a and b. So putting $b = a + h$ and $c = a + \theta h$, the result of the above theorem can be written as

$$\frac{f(a + h) - f(a)}{g(a + h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}, 0 < \theta < 1.$$

Note 1 : If we take $g(x) = x$ for all x in $[a, b]$, then Cauchy's mean value theorem gives Lagrange's mean value theorem as a particular case. For $g(x) = x$ means $g(b) = b$, $g(a) = a$, $g'(x) = 1$ and so $g'(c) = 1$. Putting these values in Cauchy's mean value theorem, we get Lagrange's mean value theorem. Thus Cauchy's mean value theorem is more general than Lagrange's mean value theorem.

Note 2 : Cauchy's mean value theorem cannot be obtained by applying Lagrange's mean value theorem to the functions f and g.

For applying Lagrange's mean value theorem to $f(x)$ and $g(x)$ separately, we get

$$f(b) - f(a) = (b - a) f'(c_1), \text{ where } a < c_1 < b$$

and $$g(b) - g(a) = (b - a) g'(c_2), \text{ where } a < c_2 < b.$$

Dividing, we have $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c_1)}{g'(c_2)}$.

Note that here c_1 is not necessarily equal to c_2.

Illustrative Examples

Example 1 : *If $f(x) = (x - 1)(x - 2)(x - 3)$ and $a = 0$, $b = 4$, find 'c' using Lagrange's mean value theorem.* **(Lucknow 2007; Rohilkhand 14)**

Solution : We have

$$f(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6.$$

$\therefore$ $$f(a) = f(0) = -6 \text{ and } f(b) = f(4) = 6.$$

$\therefore$ $$\frac{f(b) - f(a)}{b - a} = \frac{6 - (-6)}{4 - 0} = \frac{12}{4} = 3.$$

Also $f'(x) = 3x^2 - 12x + 11$ gives $f'(c) = 3c^2 - 12c + 11$.

Putting these values in Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c), \ (a < c < b), \text{ we get}$$

$$3 = 3c^2 - 12c + 11 \quad \text{or} \quad 3c^2 - 12c + 8 = 0$$

or $$c = \frac{12 \pm \sqrt{(144 - 96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

As both of these values of c lie in the open interval $]0, 4[$, hence both of these are the required values of c.

Example 2 : *Let $f : [0, 1] \to \mathbf{R}$ be defined by*

$$f(x) = (x - 1)^2 + 2 \ \forall x \in [0, 1].$$

Find the equation of the tangent to the graph of this curve which is parallel to the chord joining the points $(0, 3)$ and $(1, 2)$ of the curve.

Solution : Since $f(x)$ is a polynomial function, therefore it is continuous on $[0, 1]$ and differentiable in $]0, 1[$. Hence, by Lagrange's mean value theorem, there is some $c \in]0, 1[$ such that

$$\frac{f(1) - f(0)}{1 - 0} = f'(c) \text{ or } \frac{2 - 3}{1} = f'(c) \text{ or } -1 = f'(c).$$

Now $f'(x) = 2(x - 1)$ gives $f'(c) = 2(c - 1)$.

Thus $2(c - 1) = -1$ *i.e.,* $c = \frac{1}{2}$.

$\therefore f(c) = \frac{9}{4}$, so that the point of contact of the tangent is $\left(\frac{1}{2}, \frac{9}{4}\right)$ and its slope is $f'(c) = -1$. Hence the equation of the required tangent is

$$y - \frac{9}{4} = -1\left(x - \frac{1}{2}\right) \quad \text{or} \quad 4x + 4y = 11.$$

Example 3 : *Compute the value of* θ *in the first mean value theorem*

$$f(x+h)=f(x)+hf'(x+\theta h),\ \text{if } f(x)=ax^2+bx+c.$$

Solution : Here $f(x)=ax^2+bx+c$.

$\therefore$ $f(x+h)=a(x+h)^2+b(x+h)+c,$

$f'(x)=2ax+b, f'(x+\theta h)=2a(x+\theta h)+b.$

Substituting all these values in the Lagrange's mean value theorem, we get

$$a(x+h)^2+b(x+h)+c=ax^2+bx+c+h[2a(x+\theta h)+b] \quad \ldots(1)$$

The relation (1) being identically true for all values of x, hence when $x\to 0$, we have

$$ah^2+bh+c=c+h[2a\theta h+b]$$

or $ah^2=2a\theta h^2$ or $\theta=1/2$.

Example 4 : *A function* $f(x)$ *is continuous in the closed interval* $[0, 1]$ *and differentiable in the open interval* $]0, 1[$, *prove that*

$$f'(x_1)=f(1)-f(0), \text{ where } 0<x_1<1.$$

Solution : Here $a=0, b=1$ so that

$$\frac{f(b)-f(a)}{b-a}=\frac{f(1)-f(0)}{1-0}=f(1)-f(0).$$

If we take $c=x_1$, and substitute these values in the result of Lagrange's mean value theorem, we get

$$f(1)-f(0)=f'(x_1) \text{ where } 0<x_1<1.$$

This is a particular case of Lagrange's mean value theorem. Students can give an independent proof of this.

Example 5 : *Separate the intervals in which the polynomial*

$2x^3-15x^2+36x+1$ *is increasing or decreasing.*

Solution : We have $f(x)=2x^3-15x^2+36x+1$.

$\therefore$ $f'(x)=6x^2-30x+36=6(x-2)(x-3).$

Now $f'(x)>0$ for $x<2$ or for $x>3$,

$f'(x)<0$ for $2<x<3$, and $f'(x)=0$ for $x=2, 3$.

Thus $f'(x)$ is + ive in the intervals $]-\infty, 2[$ and $]3, \infty[$ and negative in the interval $]2, 3[$.

Hence f is monotonically increasing in the intervals $]-\infty, 2]$, $[3, \infty[$ and monotonically decreasing in the interval $[2, 3]$.

Example 6 : *Show that* $\dfrac{x}{1+x}<\log(1+x)<x$ *for* $x>0$. **(Bundelkhand 2011)**

Solution : Let $f(x)=\log(1+x)-\dfrac{x}{1+x}\cdot$ $\therefore f(0)=0$.

Then $f'(x)=\dfrac{1}{1+x}-\dfrac{1.(1+x)-x.1}{(1+x)^2}=\dfrac{1}{1+x}-\dfrac{1}{(1+x)^2}=\dfrac{x}{(1+x)^2}\cdot$

We observe that $f'(x)>0$ for $x>0$. Hence $f(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$f(x)>f(0) \text{ for } x>0 \text{ i.e., } \left[\log(1+x)-\frac{x}{1+x}\right]>0 \text{ for } x>0$$

i.e., $$\log(1+x)>\frac{x}{1+x} \text{ for } x>0. \quad \ldots(1)$$

Again, let $\phi(x) = x - \log(1+x)$.

$\therefore \quad \phi(0) = 0.$

Then $\phi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}$.

We observe that $\phi'(x) > 0$ for $x > 0$. Hence $\phi(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$\phi(x) > \phi(0) \text{ for } x > 0 \text{ } i.e., [x - \log(1+x)] > 0 \text{ for } x > 0$$

i.e., $\quad x > \log(1+x)$ for $x > 0$. ...(2)

From (1) and (2), we get

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.$$

Example 7 : *Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval* $[1, 2]$. **(Avadh 2013)**

Solution : Let $f(x) = x^2$ and $g(x) = x^3$. Then $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$ and differentiable in the open interval $]1, 2[$. Also $g'(x) = 3x^2 \neq 0$ for any point in the open interval $]1, 2[$. Hence by Cauchy's mean value theorem there exists at least one real number c in the open interval $]1, 2[$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}. \qquad ...(1)$$

Now $\quad \dfrac{f(2) - f(1)}{g(2) - g(1)} = \dfrac{4-1}{8-1} = \dfrac{3}{7}.$

Also $f'(x) = 2x, g'(x) = 3x^2$.

$\therefore \quad \dfrac{f'(c)}{g'(c)} = \dfrac{2c}{3c^2} = \dfrac{2}{3c}.$ Putting these values in (1), we get $\dfrac{3}{7} = \dfrac{2}{3c}$ or $c = \dfrac{14}{9}$ which lies in the open interval $]1, 2[$. Hence Cauchy's mean value theorem is verified.

Example 8 : *If in the Cauchy's mean value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

Solution : Here $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = -e^a e^b = -e^{a+b},$

and $\quad \dfrac{f'(x)}{g'(x)} = \dfrac{e^x}{-e^{-x}}$ so that $\dfrac{f'(c)}{g'(c)} = \dfrac{e^c}{-e^{-c}} = -e^{2c}.$

Putting these values in Cauchy's mean value theorem, we get

$$-e^{a+b} = -e^{2c} \quad \text{or} \quad 2c = a+b \quad \text{or} \quad c = \tfrac{1}{2}(a+b).$$

Thus c is the arithmetic mean between a and b.

Example 9 : *If in the Cauchy's mean value theorem, we write*

(i) $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$, then c is the geometric mean between a and b, and if **(Rohilkhand 2014)**

(ii) $f(x) = 1/x^2$ and $g(x) = 1/x$, then c is the harmonic mean between a and b. **(Bundelkhand 2005)**

Solution : (i) Here $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{\sqrt{b} - \sqrt{a}}{(1/\sqrt{b}) - (1/\sqrt{a})} = -\sqrt{(ab)},$

and $\quad \dfrac{f'(x)}{g'(x)} = \dfrac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}}$ so that $\dfrac{f'(c)}{g'(c)} = -\dfrac{c^{-1/2}}{c^{-3/2}} = -c.$

Putting these values in Cauchy's mean value theorem, we get

$$-\sqrt{(ab)} = -c \quad \text{or} \quad c = \sqrt{(ab)}$$

i.e., c is the geometric mean between a and b.

(ii) Here $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{(1/b^2) - (1/a^2)}{(1/b) - (1/a)} = \dfrac{a+b}{ab}$

and $\dfrac{f'(x)}{g'(x)} = \dfrac{-2x^{-3}}{-x^{-2}}$ so that $\dfrac{f'(c)}{g'(c)} = \dfrac{-2c^{-3}}{-c^{-2}} = \dfrac{2}{c}\cdot$

Putting these values in Cauchy's mean value theorem, we get

$$\frac{a+b}{ab} = \frac{2}{c} \quad \text{or} \quad c = \frac{2ab}{a+b}$$

i.e., c is the harmonic mean between a and b.

Comprehensive Exercise 3

1. State Lagrange's mean value theorem. Test if Lagrange's mean value theorem holds for the function $f(x) = |x|$ in the interval $[-1, 1]$. **(Kanpur 2010; Rohilkhand 13B)**
2. If $f(x) = 1/x$ in $[-1, 1]$, will the Lagrange's mean value theorem be applicable to $f(x)$? **(Meerut 2012B)**
3. Verify Lagrange's mean value theorem for the function
$$f : [-1, 1] \to \mathbf{R} \text{ given by } f(x) = x^3.$$
4. Find 'c' of the mean value theorem, if $f(x) = x(x-1)(x-2)$; $a = 0, b = \frac{1}{2}\cdot$
5. Find 'c' of mean value theorem when
(i) $f(x) = x^3 - 3x - 2$ in $[-2, 3]$ (ii) $f(x) = 2x^2 + 3x + 4$ in $[1, 2]$
(iii) $f(x) = x(x-1)$ in $[1, 2]$ **(Meerut 2013B)**
(iv) $f(x) = x^2 - 3x - 1$ in $\left(-\dfrac{11}{7}, \dfrac{13}{7}\right)\cdot$
6. State the conditions for the validity of the formula
$$f(x+h) = f(x) + hf'(x + \theta h)$$
and investigate how far these conditions are satisfied and whether the result is true, when
$$f(x) = x \sin(1/x) \text{ (being defined to be zero at } x = 0) \text{ and } x < 0 < x + h.$$
7. (a) Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.
(b) Show that $\log(1+x) - \dfrac{2x}{2+x}$ is increasing when $x > 0$.
8. Determine the intervals in which the function $(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$ is increasing or decreasing.
9. Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{e}.
10. Show that the set of all x for which $\log(1+x) \le x$ is equal to $[0, \infty[$.
11. Use Lagrange's mean value theorem to prove that
$$1 + x < e^x < 1 + xe^x \;\; \forall \, x > 0.$$

12. If $a = -1, b \geq 1$ and $f(x) = 1/|x|$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.

13. State Cauchy's mean value theorem. **(Kanpur 2007)**

Verify Cauchy's mean value theorem for $f(x) = \sin x, g(x) = \cos x$ in $[-\pi/2, 0]$.

14. If $f(x) = x^2, g(x) = \cos x$, then find the point $c \in]0, \pi/2[$ which gives the result of Cauchy's mean value theorem in the interval $[0, \pi/2]$ for the functions $f(x)$ and $g(x)$.

15. Use Cauchy's mean value theorem to show that

$$\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta, \quad \text{where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

Answers 3

1. The mean value theorem does not hold since the given function is not differentiable at $x = 0$.

2. not applicable. **4.** $1 - \dfrac{\sqrt{21}}{6}$.

5. (i) $\pm \sqrt{(7/3)}$. (ii) $3/2$. (iii) $3/2$. (iv) $1/7$.

6. Condition of differentiability is not satisfied in $x < 0 < x + h$ since $f(x)$ is non-differentiable at $x = 0$.

8. Increasing in the intervals $[-2, -1]$ and $[0, 1]$ and decreasing in the intervals $]-\infty, -2], [-1, 0]$ and $[1, \infty[$.

9. e^{π} is bigger than π^{e}.

14. Root of the equation $\sin c - (8c/\pi^2) = 0$ in the open interval $]\pi/6, \pi/2[$.

2.11 Taylor's Theorem with Lagrange's Form of Remainder after *n* Terms

If $f(x)$ is a single-valued function of x such that

(*i*) *all the derivatives of $f(x)$ upto $(n-1)$th are continuous in $a \leq x \leq a + h$,*

and (*ii*) *$f^{(n)}(x)$ exists in $a < x < a + h$, then*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h), \textit{ where } 0 < \theta < 1.$$

Proof : Consider the function ϕ defined by

$$\phi(x) = f(x) + (a + h - x) f'(x) + \frac{(a + h - x)^2}{2!} f''(x) + \dots + \frac{(a + h - x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{A}{n!}(a + h - x)^n,$$

where A is a constant to be suitably chosen.

We choose A such that $\phi(a) = \phi(a + h)$.

Now $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{A}{n!}h^n$,

and $\phi(a+h) = f(a+h)$.

Hence A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A. \quad \ldots(1)$$

Now, by hypothesis, all the functions

$$f(x), f'(x), f''(x), \ldots\ldots, f^{(n-1)}(x)$$

are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

Further $(a+h-x), (a+h-x)^2/2!, \ldots, (a+h-x)^n/n!$, all being polynomials, are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$. Also A is a constant.

$\therefore$ $\phi(x)$ is continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

By our choice of A, $\phi(a) = \phi(a+h)$. Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Consequently $\phi'(a+\theta h) = 0$, where $0 < \theta < 1$.

Now $\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x)$

$$+ \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{A}{(n-1)!}(a+h-x)^{n-1}$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A],$$

since other terms cancel in pairs.

$\therefore$ $\phi'(a+\theta h) = 0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or $$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or $$f^{(n)}(a+\theta h) - A = 0 \quad \text{or} \quad A = f^{(n)}(a+\theta h).$$

$[\because h \neq 0, (1-\theta) \neq 0 \text{ as } 0 < \theta < 1]$

Putting this value of A in (1), we get

$$\boldsymbol{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots}$$
$$\boldsymbol{+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h).}$$

This is **Taylor's development** of $f(a+h)$ in ascending integral powers of h. The $(n+1)$th term $\boldsymbol{\frac{h^n}{n!}f^{(n)}(a+\theta h)}$ is called **Lagrange's form of remainder** after n terms in Taylor's expansion of $f(a+h)$.

Note : If we take $n = 1$, we see that Lagrange's mean value theorem is a particular case of the above theorem.

Corollary : (Maclaurin's development) :

If we take the interval $[0, x]$ instead of $[a, a + h]$, so that changing a to 0 and h to x in Taylor's theorem, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x),$$

which is known as **Maclaurin's theorem** or **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Lagrange's form of remainder** $\frac{x^n}{n!} f^{(n)}(\theta x)$ after n terms.

2.12 Taylor's Theorem with Cauchy's Form of Remainder

If $f(x)$ is a single-valued function of x such that

(*i*) *all the derivatives of $f(x)$ upto $(n-1)$th are continuous in $a \le x \le a + h$,*

and (*ii*) *$f^{(n)}(x)$ exists in $a < x < a + h$, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h), \text{ where } 0 < \theta < 1.$$

Proof : Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x) A,$$

where A is a constant to be suitably chosen. We choose A such that $\phi(a) = \phi(a+h)$.

Now $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA,$

and $\phi(a+h) = f(a+h)$.

Hence A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA. \quad \ldots(1)$$

As explained earlier in article 2.11, it can be easily seen that $\phi(x)$ satisfies all the conditions of Rolle's theorem. Consequently

$$\phi'(a+\theta h) = 0, \text{ where } 0 < \theta < 1.$$

Now $\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - A$, since other terms cancel in pairs.

$\therefore$ $\phi'(a+\theta h) = 0$ gives $\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - A = 0$

or $A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$.

Putting this value of A in (1), we get

$$\boldsymbol{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)}$$

$$\boldsymbol{+ \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h).}$$

The $(n+1)$th term $\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ is called **Cauchy's form of remainder** after n terms in the **Taylor's expansion** of $f(a+h)$ in ascending integral powers of h.

Corollary : (Maclaurin's development with Cauchy's form of remainder) :
If we change a to 0 and h to x in the above result, we get

$$\boldsymbol{f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)}$$

$$\boldsymbol{+ \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x),}$$

which is **Maclaurin's theorem with Cauchy's form of remainder**. The $(n+1)$th term $\frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$ is known as Cauchy's form of remainder after n terms in Maclaurin's development of $f(x)$ in the interval $[0, x]$.

2.13 Expansions of Some Basic Functions

(i) Expansion of e^x :

Let $f(x) = e^x$.

Then $f^{(n)}(x) = e^x \ \forall\, n \in \mathbf{N}, \forall\, x \in \mathbf{R}$ so that

$$f^{(n)}(0) = e^0 = 1 \ \forall\, n \in \mathbf{N}.$$

Now Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $$R_n = \frac{x^n}{n!} f^{(n)}(\theta x),\ 0 < \theta < 1.$$

Now $$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} f^{(n)}(\theta x) = \lim_{n \to \infty} \frac{x^n}{n!} e^{\theta x}$$

$$= e^{\theta x} \lim_{n \to \infty} \frac{x^n}{n!} = e^{\theta x} \times 0 = 0. \qquad \left[\because \lim_{n \to \infty} \frac{x^n}{n!} = 0\right]$$

Thus $f^{(n)}(x)$ exists in $[0, x]$ for each $n \in \mathbf{N}$ and $R_n \to 0$ as $n \to \infty$ *i.e.*, all the conditions of Maclaurin's series expansion are satisfied.

Hence $\forall\, x \in \mathbf{R}$ the expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

(ii) Expansion of $\sin x$:

Let $f(x) = \sin x$.

Then $\quad f^{(n)}(x) = \sin\left(x + \frac{1}{2} n\pi\right) \ \forall\, n \in \mathbf{N}, \forall\, x \in \mathbf{R}$

so that $\quad f^{(n)}(0) = \sin\left(\frac{1}{2} n\pi\right) \ \forall\, n \in \mathbf{N}$

or $\quad f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ (-1)^{(n-1)/2}, & \text{if } n \text{ is odd}. \end{cases}$

The Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $\quad R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right), \ 0 < \theta < 1.$

Now $\quad |R_n| = \left| \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right) \right| = \left| \frac{x^n}{n!} \right| \left| \sin\left(\theta x + \frac{n\pi}{2}\right) \right| \le \left| \frac{x^n}{n!} \right|.$

$\therefore \quad \lim_{n \to \infty} |R_n| \le \lim_{n \to \infty} \left| \frac{x^n}{n!} \right| = 0 \qquad \left[\because \lim_{n \to \infty} \frac{x^n}{n!} = 0 \right]$

or $\quad \lim_{n \to \infty} R_n = 0.$

Thus all the conditions of Maclaurin's series expansion are satisfied. Hence the expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

(iii) Expansion of cos *x* :

Proceed as above. In this case we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(iv) Expansion of $\log_e(1 + x)$:

Let $\quad f(x) = \log_e(1 + x), \quad (-1 < x \le 1).$

Then $\quad f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \ \forall\, n \in \mathbf{N},$

so that $\quad f^{(n)}(0) = (-1)^{n-1}(n-1)! \ \forall\, n \in \mathbf{N}.$

The Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $\quad R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \ 0 < \theta < 1$

$$= \frac{x^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = (-1)^{n-1} \cdot \frac{1}{n} \left(\frac{x}{1+\theta x} \right)^n.$$

In order to show that $R_n \to 0$ as $n \to \infty$, we consider two cases :

Case I : $\quad 0 \le x \le 1$.

Since $0 \le x \le 1$ and $0 < \theta < 1$, therefore $x < 1 + \theta x$

and hence $\lim_{n \to \infty} \left(\frac{x}{1+\theta x} \right)^n = 0$. Also $\lim_{n \to \infty} \frac{(-1)^{n-1}}{n} = 0.$

Thus, in this case, $\lim_{n \to \infty} R_n = 0.$

Case II : $-1 < x < 0$.

In this case it will not be convenient to show that Lagrange's form of remainder $R_n \to 0$ as $n \to \infty$ because $x/(1+\theta x)$ may not be numerically less than unity. Therefore we use the Cauchy's form of remainder. We have

$$R_n = \frac{x^n}{(n-1)\,!}(1-\theta)^{n-1} f^{(n)}(\theta x)$$

$$= \frac{x^n}{(n-1)\,!}(1-\theta)^{n-1}(-1)^{n-1}(n-1)\,!\,(1+\theta x)^{-n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x}, \qquad (0 < \theta < 1).$$

Now as $0 < \theta < 1$ and $-1 < x < 0$, we have

$$0 < \frac{1-\theta}{1+\theta x} < 1, \text{ so that } \lim_{n \to \infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

Also, $\lim_{n \to \infty} x^n = 0$ as $-1 < x < 0$.

Thus, in this case also $R_n \to 0$ as $n \to \infty$.

Hence, $f(x)$ satisfies all the conditions of Maclaurin's series expansion for $-1 < x \leq 1$.

Therefore, for $-1 < x \leq 1$, we get

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

(v) Expansion of $(1+x)^m$:

Let $\quad f(x) = (1+x)^m, \forall\, x \in \mathbf{R}$.

Then $\quad f^{(n)}(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n} \quad \forall\, n \in \mathbf{N}$.

Now we consider two cases :

Case I : If m is a positive integer.

In this case, we notice that for $n > m$, $f^{(n)}(x) = 0$. So all the terms after the $(m+1)$th term vanish and so the expansion consists of finite number of terms in the form

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!}f''(0) + \ldots + \frac{x^m}{m\,!}f^{(m)}(0).$$

Case II : If m is a fraction or a negative integer.

In this case, let $|x| < 1$.

We have

$$f^{(n)}(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n}, \quad x \neq -1.$$

Here, we use Maclaurin's expansion with Cauchy's form of remainder. Thus, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)\,!}f^{(n-1)}(0) + R_n$$

where $R_n = \dfrac{x^n}{(n-1)\,!}(1-\theta)^{n-1} f^{(n)}(\theta x),\ 0 < \theta < 1$

$$= \frac{x^n}{(n-1)\,!}(1-\theta)^{n-1} \cdot m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$$

$$= \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot (1+\theta x)^{m-1} \cdot \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} \cdot x^n.$$

If $a_n = \dfrac{m(m-1)\ldots(m-n+1)}{(n-1)!} \cdot x^n$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{m-n}{n} \cdot x$

(on simplification)

$$\therefore \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty}\left(\frac{m}{n} - 1\right)x = (0-1)\,x = -\,x.$$

This gives $\lim_{n \to \infty} a_n = 0$, since $|-x| = |x| < 1$.

Further $0 < \theta < 1 \Rightarrow 0 < 1-\theta < 1+\theta x$

so that $\dfrac{1-\theta}{1+\theta x} < 1$ which gives $\lim_{n \to \infty}\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$

Hence in this case $\lim_{n \to \infty} R_n = 0$. Thus $f(x)$ satisfies the conditions of Maclaurin's series expansion.

Therefore for $-1 < x < 1$, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2\,!} \cdot x^2 + \frac{m(m-1)(m-2)}{3\,!} \cdot x^3 + \ldots$$

Illustrative Examples

Example 1 : *If* $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2\,!} f''(\theta x)$, ...(1)

find the value of θ *as* $x \to 1$, $f(x)$ *being* $(1-x)^{5/2}$.

Solution : Here $f(x) = (1-x)^{5/2}$.

$\therefore \quad f'(x) = -\frac{5}{2}(1-x)^{3/2}$ and $f''(x) = \dfrac{15}{4}(1-x)^{1/2}$.

Thus $f(0) = 1, f'(0) = -\frac{5}{2}, f''(\theta x) = \dfrac{15}{4}(1-\theta x)^{1/2}$.

Putting these values in (1), we get

$$(1-x)^{5/2} = 1 - \frac{5}{2}x + \frac{x^2}{2\,!} \times \frac{15}{4}(1-\theta x)^{1/2}.$$

Therefore as $x \to 1$, we have

$$0 = 1 - \frac{5}{2} + \frac{1}{2\,!} \cdot \frac{15}{4}(1-\theta)^{1/2}$$

or $(1-\theta)^{1/2} = \dfrac{4}{5}$ or $(1-\theta) = \dfrac{16}{25}$ or $\theta = \dfrac{9}{25}$.

Example 2 : *Show that the number* θ *which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit* $1/(n+1)$ *as* $h \to 0$ *provided that* $f^{(n+1)}(x)$ *is continuous and different from zero at* $x = a$.

Solution : Applying Taylor's theorem with Lagrange's form of remainder after n terms and $(n+1)$ terms successively, we get for $\theta, \theta' \in \,]0, 1[$,

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a + \theta h),$$

and
$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta' h).$$

Subtracting these, we have

$$\frac{h^n f^{(n)}(a)}{n!} + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta' h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$$

or
$$f^{(n)}(a + \theta h) - f^{(n)}(a) = \frac{h}{n+1} f^{(n+1)}(a + \theta' h) \qquad \ldots(1)$$

Applying Lagrange's mean value theorem to the function $f^{(n)}(x)$ in the interval $[a, a + \theta h]$, we get

$$f^{(n)}(a + \theta h) - f^{(n)}(a) = \theta h f^{(n+1)}(a + \theta\theta'' h), \; 0 < \theta'' < 1. \qquad \ldots(2)$$

From (1) and (2), we have

$$\theta h f^{(n+1)}(a + \theta\theta'' h) = \frac{h}{n+1} f^{(n+1)}(a + \theta' h)$$

or
$$\theta = \frac{1}{n+1} \frac{f^{(n+1)}(a + \theta' h)}{f^{(n+1)}(a + \theta\theta'' h)}.$$

$$\therefore \quad \lim_{h \to 0} \theta = \frac{1}{n+1} \cdot \frac{f^{(n+1)}(a)}{f^{(n+1)}(a)} = \frac{1}{n+1}, \text{ provided } f^{(n+1)}(a) \neq 0.$$

Example 3 : *Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that*

$$\phi'(x) = F'\{f(x)\} f'(x) \text{ where } \phi(x) = F\{f(x)\}.$$

Solution : Let $f(x) = t$ so that $\phi(x) = F(t)$.

$$\text{Now } \phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{h \to 0} \frac{F\{f(x+h)\} - F\{f(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{F\{f(x) + hf'(x + \theta_1 h)\} - F\{f(x)\}}{h}, \; (0 < \theta_1 < 1)$$

[$\because f(x+h) = f(x) + hf'(x + \theta_1 h)$, by mean value theorem]

$$= \lim_{h \to 0} \frac{F(t+H) - F(t)}{h}, \text{ where } H = hf'(x + \theta_1 h)$$

$$= \lim_{h \to 0} \frac{HF'(t + \theta_2 H)}{h}$$

[$\because F(t+H) = F(t) + HF'(t + \theta_2 H)$, by mean value theorem]

$$= \lim_{h \to 0} \frac{hf'(x + \theta_1 h) F'[t + \theta_2 hf'(x + \theta_1 h)]}{h}$$

$$= f'(x) F'(t) = F'\{f(x)\} f'(x).$$

Note : This example gives an alternative proof of the chain rule.

Comprehensive Exercise 4

1. If $f(x+h) = f(x) + h f'(x) + \dfrac{h^2}{2!} f''(x+\theta h)$,

find the value of θ as $x \to a$, $f(x)$ being $(x-a)^{5/2}$.

2. Find θ, if $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!} f''(x+\theta h)$, $0 < \theta < 1$, and

(i) $f(x) = ax^3 + bx^2 + cx + d$ (ii) $f(x) = x^3$.

3. Show that 'θ' (which occurs in the Lagrange's mean value theorem) approaches the limit $\frac{1}{2}$ as 'h' approaches zero provided that $f''(a)$ is not zero. It is assumed that $f''(x)$ is continuous.

4. Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n+1)$ as $h \to 0$ provided that $f^{(n+1)}(x)$ is continuous and different from zero at $x = a$.

Answers 4

1. $\theta = \dfrac{64}{225} \cdot$ **2.** (i) $\theta = \frac{1}{3} \cdot$ (ii) $\theta = \frac{1}{3} \cdot$

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ..." so that the following statements are complete and correct.

1. A function $f(x)$ is said to be differentiable at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - \ldots}{x - a} \text{ exists.}$$

2. The right hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{\ldots}, h > 0,$$

provided the limit exists.

3. The left hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{\ldots}, h > 0,$$

provided the limit exists.

4. A function $f :]a, b[\to \mathbf{R}$ is said to be differentiable in $]a, b[$ if and only if it is differentiable at every point in .... .

5. If a function $f(x)$ is differentiable at $x = a$, then $f'(a)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ makes with ....

6. Continuity is a necessary but not a ... condition for the existence of a finite derivative.

7. The function $f(x) = |x|$ is differentiable at every point of **R** except at $x = \dots$.

8. If a function $f(x)$ is such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$,

(ii) $f'(x)$ exists for every point in the open interval $]a, b[$,

(iii) $f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.

The above theorem is known as .... .

9. If a function $f(x)$ is

(i) continuous in the closed interval $[a, b]$, and

(ii) differentiable in the open interval $]a, b[$ *i.e.*, $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = \dots .$$

10. If two functions $f(x)$ and $g(x)$ are

(i) continuous in a closed interval $[a, b]$

(ii) differentiable in the open interval $]a, b[$, and

(iii) $g'(x) \neq 0$ for any point of the open interval $]a, b[$, then there exists at least one value c of x in the open interval $]a, b[$, such that

$$\frac{f(b) - f(a)}{\dots} = \frac{f'(c)}{g'(c)}.$$

11. If f is continuous in $[a, b]$ and $f'(x) \geq 0$ in $]a, b[$, then f is ... in $[a, b]$.

12. If $f(x) = \sin x$, then

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \dots .$$

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

13. The function $f(x) = |x - 1|$ is not differentiable at

(a) $x = 0$ (b) $x = -1$

(c) $x = 1$ (d) $x = 2$

14. The function $f(x) = |x + 3|$ is not differentiable at

(a) $x = 3$ (b) $x = -3$

(c) $x = 0$ (d) $x = 1$

15. A function $f(x)$ is differentiable at $x = a$ if

(a) $Rf'(a) = Lf'(a)$

(b) $Rf'(a) = 0$

(c) $Lf'(a) = 0$

(d) $Rf'(a) \neq Lf'(a)$

16. A function $\phi(x)$ is defined as follows :

$$\phi(x) = 1 + x \text{ if } x \leq 2$$
$$\phi(x) = 5 - x \text{ if } x > 2.$$

Then

(a) $\phi(x)$ is continuous but not differentiable at $x = 2$

(b) $\phi(x)$ is differentiable at every point of **R**

(c) $\phi(x)$ is neither continuous nor differentiable at $x = 2$

(d) $\phi(x)$ is differentiable at $x = 2$ but is not continuous at $x = 2$.

17. Out of the following four functions tell the function for which the conditions of Rolle's theorem are satisfied.

(a) $f(x) = |x|$ in $[-1, 1]$

(b) $f(x) = x^2$ in $2 \le x \le 3$

(c) $f(x) = \sin x$ in $[0, \pi]$

(d) $f(x) = \tan x$ in $0 \le x \le \pi$.

18. The function $f(x) = \sin x$ is increasing in the interval

(a) $[0, \pi]$ (b) $\left[0, \frac{\pi}{2}\right]$

(c) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (d) $\left[\frac{\pi}{2}, \pi\right]$

19. The value of 'c' of Lagrange's mean value theorem for $f(x) = x(x-1)$ in $[1, 2]$ is given by

(a) $c = \frac{5}{4}$ (b) $c = \frac{3}{2}$

(c) $c = \frac{7}{4}$ (d) $c = \frac{11}{6}$

20. The value of 'c' of Rolle's theorem for the function $f(x) = e^x \sin x$ in $[0, \pi]$ is given by

(a) $c = \frac{3\pi}{4}$

(b) $c = \frac{\pi}{4}$

(c) $c = \frac{\pi}{2}$

(d) $c = \frac{5\pi}{6}$

True or False:

Write 'T' for true and 'F' for false statement.

21. If a function $f(x)$ is continuous at $x = a$, it must also be differentiable at $x = a$.

22. If a function $f(x)$ is differentiable at $x = a$, it must be continuous at $x = a$.

23. If a function $f(x)$ is differentiable at $x = a$, it may or may not be continuous at $x = a$.

24. The function $f(x) = |x|$ is differentiable at every point of $\mathbf{R}$.

25. Rolle's theorem is applicable for $f(x) = \sin x$ in $[0, 2\pi]$.

26. Rolle's theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.

27. Lagrange's mean value theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.

28. The function $f(x) = \sin x$ is increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

29. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has no root in $]0, 1[$.

30. If f is continuous on $[a, b]$ and $f'(x) \le 0$ in $]a, b[$, then f is increasing in $[a, b]$.

31. The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is decreasing in the interval $[2, 3]$.

32. Let $f(x) = |x| + |x-1|$. Then $Rf'(0) = 0$.

33. Rolle's theorem is not applicable for the function $f(x) = x(x+2)e^{-x/2}$ in $[-2, 0]$.

34. The value of 'c' of Lagrange's mean value theorem for the function $f(x) = 2x^2 + 3x + 4$ in $[1, 2]$ is given by $c = \frac{5}{4}$.

35. If $f(x) = x^n$, then $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = nx^{n-1}$.

36. If $f(x) = \cos x$, then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = -\sin a$.

37. If $f(x) = e^x$, then $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = e^x$.

Answers

1.	$f(a)$.	**2.**	h.	**3.**	$-h$.	**4.**	$]a, b[$.	**5.**	the x-axis.
6.	sufficient.	**7.**	0.	**8.**	Rolle's theorem.			**9.**	$f'(c)$.
10.	$g(b) - g(a)$.	**11.**	increasing.	**12.**	$\cos x$.	**13.**	(c).	**14.**	(b).
15.	(a).	**16.**	(a).	**17.**	(c).	**18.**	(b).	**19.**	(b).
20.	(a).	**21.**	F.	**22.**	T.	**23.**	F.	**24.**	F.
25.	T.	**26.**	F.	**27.**	F.	**28.**	T.	**29.**	F.
30.	F.	**31.**	T.	**32.**	T.	**33.**	F.	**34.**	F.
35.	T.	**36.**	T.	**37.**	F.				

Chapter 3

Differentiation

3.1 Increments

In differential calculus we use the word *'Increment'* to denote a small change (increase or decrease) in the value of any variable. Thus if x be a variable, then a small change in the value of x is called the increment in x and we usually denote it by δx which is read as 'delta x'. It should be noted that δx does not mean δ multiplied by x. It represents a single quantity which stands for the increment in x. Sometimes we also use the single letters h, k etc. to denote increments.

Now suppose $y = f(x)$ is a function of the variable x. Let δy denote the increment in y corresponding to an increment δx in x.

Then $\quad y + \delta y = f(x + \delta x)$.

Therefore $\quad \delta y = f(x + \delta x) - f(x)$.

The quotient $\quad \dfrac{\delta y}{\delta x} = \dfrac{f(x + \delta x) - f(x)}{\delta x}$

is called the *average rate of change* of y with respect to x in the interval $(x, x + \delta x)$.

3.2 The Differential Coefficient

Definition : *The differential coefficient of a function $y = f(x)$ -with respect to x is defined as*

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

provided the limit exists. The differential coefficient is also called the **derivative**, or the **derived function**. The differential coefficient of $y = f(x)$ with respect to x may be denoted by any of the symbols

$$\frac{d}{dx} y, \frac{dy}{dx}, y', Dy, \frac{d}{dx} f(x), f'(x), Df(x).$$

The process of finding the differential coefficient is called *differentiation*. The differential coefficient (dy/dx) is also called the *instantaneous rate of change* (or simply, *the rate of change*) of y with respect to x.

The Differential Coefficient at a Point : If $y = f(x)$ is a function of x, then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is called the differential coefficient of $f(x)$ for $x = a$, provided the above limit exists. It is denoted by $\left(\frac{dy}{dx}\right)_{x=a}$, $(y')_a$, or $f'(a)$. It gives us the rate of change of y with respect to x at $x = a$.

3.3 Differential Coefficient of x^n (n being Real Number)

Let $y = x^n$. Then $y + \delta y = (x + \delta x)^n$.

Therefore $\delta y = (x + \delta x)^n - x^n$.

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}.$$

Taking limit when $\delta x \to 0$, we get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^n \left(1 + \frac{\delta x}{x}\right)^n - x^n}{\delta x} = \lim_{\delta x \to 0} \frac{x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right]}{\delta x}$$

[Expanding by binomial theorem since $\delta x/x$ is numerically less than unity, δx being numerically small]

$$= \lim_{\delta x \to 0} \frac{x^n \left[n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!}\frac{\delta x}{x^2} + \ldots\ldots\right] = x^n \cdot \frac{n}{x} = nx^{n-1}.$$

Thus $\quad \dfrac{d}{dx} x^n = nx^{n-1}.$

Illustration 1 : $\dfrac{d}{dx}(x^4) = 4x^{4-1} = 4x^3.$

Illustration 2 : $\dfrac{d}{dx}\left(\dfrac{1}{x^{1/3}}\right) = \dfrac{d}{dx}(x^{-1/3}) = -\dfrac{1}{3}x^{-4/3} = -\dfrac{1}{3x^{4/3}}.$

3.4 Differential Coefficient of sin *x*

We have $\dfrac{d}{dx}\sin x = \lim\limits_{\delta x \to 0} \dfrac{\sin(x+\delta x) - \sin x}{\delta x}$, by definition

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x} = \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}$$

$= \cos x$, since $\lim\limits_{\delta x \to 0} \dfrac{\sin(\delta x/2)}{\delta x/2} = 1.$

Thus $\quad \dfrac{d}{dx}\sin x = \cos x.$

Similarly, it can be shown that

$$\frac{d}{dx}\cos x = -\sin x.$$

Note : $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ is true only when x is expressed in terms of radians. In case x is given in terms of degrees, it should be first expressed in terms of radians before applying the above results.

3.5 Differential Coefficient of e^x

We have $\quad \dfrac{d}{dx}e^x = \lim\limits_{\delta x \to 0} \dfrac{e^{x+\delta x} - e^x}{\delta x}$, by definition

$$= \lim_{\delta x \to 0} \frac{e^x e^{\delta x} - e^x}{\delta x} = \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^x\left[1 + \frac{\delta x}{1!} + \frac{(\delta x)^2}{2!} + \frac{(\delta x)^3}{3!} + \ldots - 1\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^x\left[\frac{\delta x}{1!} + \frac{(\delta x)^2}{2!} + \frac{(\delta x)^3}{3!} + \ldots\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} e^x\left[1 + \frac{\delta x}{2!} + \frac{(\delta x)^2}{3!} + \ldots\right] = e^x.$$

Thus $\quad \dfrac{d}{dx}(e^x) = e^x.$

3.6 Differential Coefficient of $\log_e x$

We have $\dfrac{d}{dx}\log x = \lim\limits_{\delta x \to 0} \dfrac{\log(x+\delta x) - \log x}{\delta x}$, by definition

$$= \lim_{\delta x \to 0} \frac{\log\left(\dfrac{x+\delta x}{x}\right)}{\delta x} = \lim_{\delta x \to 0} \frac{\log\left(1+\dfrac{\delta x}{x}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{\delta x}{x} - \dfrac{(\delta x)^2}{2x^2} + \dfrac{(\delta x)^3}{3x^3} - \dots\dots}{\delta x}$$

(the expansion is justified since $\delta x / x$ is numerically less than unity, δx being numerically small)

$$= \lim_{\delta x \to 0} \left[\frac{1}{x} - \frac{\delta x}{2x^2} + \frac{(\delta x)^2}{3x^3} - \dots\dots\right] = \frac{1}{x}.$$

Thus $\boldsymbol{\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}}$.

3.7 Differential Coefficient of a Constant

Let $f(x) = c$, where c is a constant.

Then $\dfrac{d}{dx} f(x) = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$, by definition

$$= \lim_{\delta x \to 0} \frac{c - c}{\delta x}, \quad \text{since } f(x) = c \text{ for every value of } x$$

$$= \lim_{\delta x \to 0} \frac{0}{\delta x} = 0.$$

Thus, **the differential coefficient of a constant is zero.**

3.8 Differential Coefficient of the Product of a Constant and a Function

Let c be a constant and $f(x)$ be a function of x. Then by definition

$$\frac{d}{dx}\{c f(x)\} = \lim_{\delta x \to 0} \frac{c f(x+\delta x) - c f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} c \cdot \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= c \cdot \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = c \frac{d}{dx} f(x).$$

Thus the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.

Illustration 1 : $\dfrac{d}{dx}(4e^x) = 4 . \dfrac{d}{dx} e^x = 4e^x$.

Illustration 2 : $\dfrac{d}{dx}\left(\dfrac{1}{2x^{4/3}}\right) = \dfrac{1}{2}\dfrac{d}{dx}\left(\dfrac{1}{x^{4/3}}\right) = \dfrac{1}{2}\dfrac{d}{dx} x^{-4/3}$

$$= \frac{1}{2}\left(-\frac{4}{3}\right)x^{-7/3} = -\frac{2}{3x^{7/3}}.$$

3.9 Differential Coefficient of $\log_a x$

We have $\log_a x = \log_e x \log_a e = \log_a e \log_e x$, where $\log_a e$ is simply a constant.

$$\therefore \quad \frac{d}{dx}(\log_a x) = \log_a e \frac{d}{dx}(\log_e x) = (\log_a e) \cdot \frac{1}{x} = \frac{1}{x} \cdot \log_a e$$

$$= \frac{1}{x \log_e a}, \text{ since } \log_a e \,.\, \log_e a = 1.$$

Thus $\quad \boldsymbol{\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} = \frac{1}{x \log a}}.$

3.10 Differential Coefficient of the Sum of Two Functions

Let $\quad f(x) = f_1(x) + f_2(x)$.

Then $\quad f(x + \delta x) = f_1(x + \delta x) + f_2(x + \delta x)$.

Therefore, by definition

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{f_1(x + \delta x) + f_2(x + \delta x)\} - \{f_1(x) + f_2(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{f_1(x + \delta x) - f_1(x)\} + \{f_2(x + \delta x) - f_2(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{f_1(x + \delta x) - f_1(x)}{\delta x} + \frac{f_2(x + \delta x) - f_2(x)}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0} \frac{f_1(x + \delta x) - f_1(x)}{\delta x} + \lim_{\delta x \to 0} \frac{f_2(x + \delta x) - f_2(x)}{\delta x}$$

$$= \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x).$$

Thus, **the differential coefficient of the sum of two functions is equal to the sum of their differential coefficients.**

This theorem can be extended for the sum of any number of functions. Thus

$$\boldsymbol{\frac{d}{dx}\{f_1(x) + f_2(x) + \ldots + f_n(x)\} = \frac{d}{dx} f_1(x) + \frac{d}{dx} f_2(x) + \ldots + \frac{d}{dx} f_n(x)}.$$

Illustration 1 : $\frac{d}{dx}(7e^x + 4\,log\,x + x^3) = \frac{d}{dx}7e^x + \frac{d}{dx}4\,log\,x + \frac{d}{dx}x^3$

$$= 7e^x + 4 \,.\, (1/x) + 3x^2.$$

Illustration 2 : $\frac{d}{dx}(8x^3 - sin\,x) = \frac{d}{dx}8x^3 - \frac{d}{dx}sin\,x = 24x^2 - cos\,x.$

3.11 Differential Coefficient of the Product of Two Functions

Let $f(x) = f_1(x) f_2(x)$.

Then $f(x+\delta x) = f_1(x+\delta x) f_2(x+\delta x)$.

Therefore, by definition

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{f_1(x+\delta x) f_2(x+\delta x) - f_1(x) f_2(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{f_1(x+\delta x) f_2(x+\delta x) - f_1(x+\delta x) f_2(x) + f_1(x+\delta x) f_2(x) - f_1(x) f_2(x)}{\delta x}$$

(by adding and subtracting the term $f_1(x+\delta x) f_2(x)$ in the numerator)

$$= \lim_{\delta x \to 0} \frac{f_1(x+\delta x)\{f_2(x+\delta x) - f_2(x)\} + f_2(x)\{f_1(x+\delta x) - f_1(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} f_1(x+\delta x)\left[\frac{f_2(x+\delta x) - f_2(x)}{\delta x}\right] + \lim_{\delta x \to 0} f_2(x)\left[\frac{f_1(x+\delta x) - f_1(x)}{\delta x}\right]$$

$$= f_1(x) \frac{d}{dx} f_2(x) + f_2(x) \frac{d}{dx} f_1(x).$$

Thus, **the differential coefficient of the product of two functions**

= **first function × differential coefficient of the second**

+ **second function × differential coefficient of the first.**

Illustration 1 : $\frac{d}{dx}(e^x \cos x) = e^x \frac{d}{dx} \cos x + \cos x \frac{d}{dx} e^x$

$= -e^x . \sin x + \cos x . e^x = e^x(\cos x - \sin x)$.

Illustration 2 : $\frac{d}{dx}(x^3 \log x) = x^3 \frac{d}{dx}(\log x) + \log x \frac{d}{dx} x^3$

$= x^3 . (1/x) + (\log x) . 3x^2 = x^2(1 + 3 \log x)$.

3.12 Differential Coefficient of the Quotient of Two Functions

Let $f(x) = \frac{f_1(x)}{f_2(x)}$.

Then $f(x+\delta x) = \frac{f_1(x+\delta x)}{f_2(x+\delta x)}$.

Therefore, by definition

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\dfrac{f_1(x+\delta x)}{f_2(x+\delta x)} - \dfrac{f_1(x)}{f_2(x)}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{f_1(x+\delta x)}{f_2(x+\delta x)} - \dfrac{f_1(x)}{f_2(x+\delta x)} - \dfrac{f_1(x)}{f_2(x)} + \dfrac{f_1(x)}{f_2(x+\delta x)}}{\delta x}$$

$$\left(\text{by adding and subtracting the term } \frac{f_1(x)}{f_2(x+\delta x)} \text{ in the numerator}\right)$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{1}{f_2(x+\delta x)}\{f_1(x+\delta x) - f_1(x)\} - f_1(x)\left\{\dfrac{f_2(x+\delta x) - f_2(x)}{f_2(x) f_2(x+\delta x)}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{1}{f_2(x+\delta x)} \cdot \frac{f_1(x+\delta x) - f_1(x)}{\delta x}\right\}$$

$$- \lim_{\delta x \to 0} \left\{\frac{f_1(x)}{f_2(x) f_2(x+\delta x)} \cdot \frac{f_2(x+\delta x) - f_2(x)}{\delta x}\right\}$$

$$= \frac{1}{f_2(x)} \cdot \frac{d}{dx} f_1(x) - \frac{f_1(x)}{f_2(x) f_2(x)} \cdot \frac{d}{dx} f_2(x)$$

$$= \frac{f_2(x) \cdot \dfrac{d}{dx} f_1(x) - f_1(x) \cdot \dfrac{d}{dx} f_2(x)}{[f_2(x)]^2}.$$

Thus, $$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left\{\frac{\boldsymbol{f_1(x)}}{\boldsymbol{f_2(x)}}\right\} = \frac{\textbf{(Diff. coeff. of Numerator)} \times \textbf{(Denominator)} - \textbf{(Numerator)} \times \textbf{(Diff. Coeff. of Denominator)}}{\textbf{Square of the Denominator}}.$$

3.13 Differential Coefficient of tan x

We have, $\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \dfrac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x \frac{d}{dx}\cos x}{\cos^2 x}$, by article 3.12

$$= \frac{\cos x \,.\, \cos x - \sin x \,.\, (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

Thus, $\boldsymbol{\frac{d}{dx}\tan x = \sec^2 x}$.

Similarly, we can show that

$$\boldsymbol{\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x}.$$

3.14 Differential Coefficient of cosec x

We have, $\dfrac{d}{dx}\operatorname{cosec} x = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right)$

$$= \frac{\left(\dfrac{d}{dx}1\right).\sin x - 1 . \dfrac{d}{dx}(\sin x)}{\sin^2 x} = \frac{0 - \cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x.$$

Thus, $\quad \boldsymbol{\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x}.$

Similarly, we can show that $\boldsymbol{\dfrac{d}{dx}\sec x = \sec x \tan x}.$

3.15 Differential Coefficient of a Function of a Function

Consider the function $\log \sin x$. Here $\log(\sin x)$ is a function of $\sin x$ whereas $\sin x$ is itself a function of x. Thus we have case of a function of a function.

Let $\quad y = f\{\phi(x)\}$.

Put $\quad t = \phi(x)$.

Then $\quad t + \delta t = \phi(x + \delta x)$.

As $\quad \delta x \to 0, \delta t$ also $\to 0$.

We have

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta t}\cdot\frac{\delta t}{\delta x}\right)$$

$$= \left(\lim_{\delta x \to 0}\frac{\delta y}{\delta t}\right)\cdot\left(\lim_{\delta x \to 0}\frac{\delta t}{\delta x}\right)$$

$$= \left(\lim_{\delta t \to 0}\frac{\delta y}{\delta t}\right)\cdot\left(\lim_{\delta x \to 0}\frac{\delta t}{\delta x}\right), \text{ since } \delta t \to 0 \text{ when } \delta x \to 0$$

$$= \frac{dy}{dt}\cdot\frac{dt}{dx}.$$

Thus, if y is a function of t and t is a function of x, then y is also a function of x and we have

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}}.$$

Similarly, if y is a function of u, u is a function of v and v is a function of x,then y is also a function of x and we have,

$$\boldsymbol{\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}}.$$

Illustrative Examples

Example 1 : *Find the differential coefficient of sin 2x.*

Solution : Put $2x = t$.

Then $\quad \dfrac{d}{dx}\sin 2x = \dfrac{d}{dx}\sin t = \left(\dfrac{d}{dt}\sin t\right).\dfrac{dt}{dx} = \cos t . \dfrac{d}{dx}(2x)$

$$= (\cos t) . 2 = 2\cos t = 2\cos 2x.$$

Example 2 : *Find the differential coefficient of* $\tan^3 x$.

Solution : Put $\tan x = t$.

Then $\frac{d}{dx}\tan^3 x = \frac{d}{dx}t^3 = \left(\frac{d}{dt}t^3\right)\cdot\frac{dt}{dx} = 3t^2\cdot\frac{d}{dx}(\tan x)$

$$= 3t^2 . \sec^2 x = 3\tan^2 x \sec^2 x.$$

Example 3 : *Find the differential coefficient of* $\log \sin x$.

Solution : We have

$$\frac{d}{dx}\log\sin x = \left(\frac{d}{d(\sin x)}\log\sin x\right).\frac{d}{dx}(\sin x) = \frac{1}{\sin x}.\cos x = \cot x.$$

3.16 Differential Coefficient of a^x

We have $a^x = e^{\log a^x} = e^{x\log a}$.

$\therefore$ $\frac{d}{dx}a^x = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}.\frac{d}{dx}(x\log a)$

$$= (\log a).e^{x\log a} = a^x\log a.$$

Thus, $\boldsymbol{\frac{d}{dx}a^x = a^x\log a.}$

3.17 Differential Coefficient of $\sin^{-1} x$

Let $\sin^{-1} x = y$.

Then $x = \sin y$.

Differentiating both sides with respect to x, we get $1 = \frac{d}{dx}(\sin y)$

or $1 = \frac{d}{dy}\sin y.\frac{dy}{dx}$ or $1 = \cos y.\frac{dy}{dx}$.

$\therefore$ $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}$.

Thus, $\boldsymbol{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{(1-x^2)}}}$.

Similarly, we can prove the following other results for inverse circular functions :

$$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{(1-x^2)}};\quad \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2};\quad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2};$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{(x^2-1)}};\quad \frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{x\sqrt{(x^2-1)}}.$$

3.18 Inverse Functions

Let $y = f(x)$ be a function of x. If when solved for x, this relation can be written as $x = f^{-1}(y)$, then f^{-1} is called the inverse function of the function f.

Here f^{-1} should be regarded as one symbol like F, or ϕ, or g.

In the relation $y = f(x)$, x is **regarded** as the independent variable, while in the relation $x = f^{-1}(y)$, y is the independent variable.

By differentiation, we get $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ respectively.

The relation between these two differential coefficients can be obtained as follows :

Let δx and δy be the corresponding increments in x and y respectively.

Then we have

$$\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1.$$

Taking limit of both sides when $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} \right) = 1$$

or
$$\left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \right) \left(\lim_{\delta x \to 0} \frac{\delta x}{\delta y} \right) = 1$$

or
$$\left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \right) \cdot \left(\lim_{\delta y \to 0} \frac{\delta x}{\delta y} \right) = 1, \text{ since } \delta y \to 0 \text{ as } \delta x \to 0$$

or
$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{dx/dy}.$$

Thus
$$\boldsymbol{\frac{dy}{dx} \cdot \frac{dx}{dy} = 1} \quad \textit{i.e.,} \quad \boldsymbol{\frac{dy}{dx} = \frac{1}{dx/dy}}.$$

3.19 Logarithmic Differentiation

Whenever we are required to differentiate a function of x in which a function of x is raised to a power which itself is a function of x, neither the formula for a^x nor that for x^n is applicable. In such cases we first take logarithm of the function and then differentiate. This process is called *logarithmic differentiation*. It is also helpful in the cases where we are to differentiate a function which consists of the product or the quotient of a number of functions.

Illustrative Examples

Example 1 : *Find the differential coefficient of* $(\sin^{-1} x)^{\log x}$.

Solution : Let $y = (\sin^{-1} x)^{\log x}$.

Taking logarithm of both sides, we have $\log y = (\log x) . \log \sin^{-1} x$.

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = (\log x) \cdot \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{(1 - x^2)}} + \frac{1}{x} \cdot \log \sin^{-1} x.$$

$$\therefore \quad \frac{dy}{dx} = y \left[\frac{\log x}{(\sin^{-1} x)\sqrt{(1 - x^2)}} + \frac{\log \sin^{-1} x}{x} \right]$$

$$= (\sin^{-1} x)^{\log x} \left[\frac{\log x}{(\sin^{-1} x)\sqrt{(1 - x^2)}} + \frac{\log \sin^{-1} x}{x} \right].$$

3.20 Differential Coefficient of the Product of any Number of Functions

Let $y = f_1(x) f_2(x) f_3(x) \ldots f_n(x)$.

Then $\log y = \log f_1(x) + \log f_2(x) + \ldots + \log f_n(x)$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)}\right].$$

$$\therefore \quad \frac{dy}{dx} = y\left[\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)}\right]$$

or
$$\frac{dy}{dx} = f_1'(x) f_2(x) f_3(x) \ldots f_n(x) + f_1(x) f_2'(x) f_3(x) \ldots f_n(x) + \ldots + f_1(x) f_2(x) \ldots f_{n-1}(x) f_n'(x).$$

Thus *to differentiate the product of any number of functions multiply the differential coefficient of each function taken separately by the product of all the remaining functions and then add up the results.*

3.21 Implicit Functions

If y is a function of x given by a relation of the type $y = f(x)$, then y is said to be an *explicit function* of x. On the other hand, if the relation between x and y is given by an equation involving both x and y, then y is said to be an *implicit function* of x. If we are given y implicitly in terms of x, we can find dy/dx without first expressing y explicitly in terms of x. Thinking of y as a function of x, we differentiate both sides of the given equation with respect to x and then solve the resulting relation for dy/dx.

Illustrative Examples

Example 1 : *Find dy/dx when* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

Solution : Regarding y as a function of x, differentiating both sides of the given equation with respect to x, we get

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0.$$

Therefore $$\frac{dy}{dx}(2hx + 2by + 2f) = -(2ax + 2hy + 2g)$$

or $$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}.$$

Example 2 : *Find* $\dfrac{dy}{dx}$ *if* $y = (\cos x)^{(\cos x)^{(\cos x) \ldots \text{to inf}}}$.

Solution : From the given expression for y, it follows that

$$y = (\cos x)^y \quad \text{or} \quad \log y = y \log \cos x.$$

Now differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log\cos x + y \cdot \frac{1}{\cos x}(-\sin x)$$

or $$\frac{dy}{dx}\left[\frac{1}{y}-\log\cos x\right]=-y\tan x \quad \text{or} \quad \frac{dy}{dx}=-\frac{(y^2\tan x)}{1-y\log\cos x}.$$

3.22 Parametric Equations

If x and y are both expressed in terms of a third variable, say t, then t is usually called a *parameter.* In the case of parametric equations we can always find dy/dx, without first eliminating the parameter.

Thus, if the parametric equations are $x=\phi(t), y=\psi(t)$, then

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

or $$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}.$$

Example : *If* $x=a\left(\cos t+\log\tan\frac{1}{2}t\right)$, $y=a\sin t$, *find* dy/dx.

Solution : Here $$\frac{dx}{dt}=a\left\{-\sin t+\frac{1}{\tan(t/2)}\cdot\left(\sec^2\frac{t}{2}\right)\cdot\frac{1}{2}\right\}$$

$$=a\left\{-\sin t+\frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right\}=a\left(\frac{1-\sin^2 t}{\sin t}\right)=a\frac{\cos^2 t}{\sin t}.$$

Also $dy/dt=a\cos t$.

Now $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\cos t}{(a\cos^2 t)/\sin t}=\tan t.$$

3.23 Differentiation of a Function with Respect to a Function

Suppose we are to find the differential coefficient of the function $u=f(x)$ with respect to another function, say, $v=\phi(x)$.

It means we are to find du/dv, where u and v are both given in terms of a third variable x. Therefore, as in the case of parametric equations, we have $\dfrac{du}{dv}=\dfrac{du/dx}{dv/dx}$,

i.e., $$\frac{df(x)}{d\phi(x)}=\frac{df(x)}{dx}\Big/\frac{d\phi(x)}{dx}.$$

Example : *Differentiate* $x^{\sin^{-1}x}$ *with respect to* $\sin^{-1}x$.

Solution : Let $u=x^{\sin^{-1}x}$ and $v=\sin^{-1}x$.

Then $\log u=\sin^{-1}x.\log x$.

$$\therefore \quad \frac{1}{u}\frac{du}{dx}=\frac{1}{\sqrt{(1-x^2)}}\log x+\frac{1}{x}\sin^{-1}x$$

or $$\frac{du}{dx}=x^{\sin^{-1}x}\left[\frac{\log x}{\sqrt{(1-x^2)}}+\frac{\sin^{-1}x}{x}\right].$$

Again $$\frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}.$$

Now $$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{x^{\sin^{-1}x}\left[\dfrac{\log x}{\sqrt{(1-x^2)}}+\dfrac{\sin^{-1}x}{x}\right]}{\dfrac{1}{\sqrt{(1-x^2)}}}$$

$$=x^{\sin^{-1}x}\left[\frac{x\log x+\sqrt{(1-x^2)}\sin^{-1}x}{x}\right].$$

3.24 Trigonometrical Transformations

Sometimes a function can be easily differentiated after making some trigonometrical transformation. Following formulae of trigonometry are of frequent use in such cases :

(i) $1+\cos x=2\cos^2(x/2)$,

(ii) $1-\cos x=2\sin^2(x/2)$,

(iii) $\tan x=\dfrac{2\tan(x/2)}{1-\tan^2(x/2)}$,

(iv) $\sin x=\dfrac{2\tan(x/2)}{1+\tan^2(x/2)}$,

(v) $\cos x=\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$,

(vi) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\dfrac{x+y}{1-xy}$,

(vii) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}$,

(viii) $2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}$,

(ix) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$,

(x) $\sin 3x=3\sin x-4\sin^3 x$.

(xi) $\cos 3x=4\cos^3 x-3\cos x$.

Illustrative Examples

Example 1 : *Differentiate* $\tan^{-1}\dfrac{a-x}{1+ax}$ *with respect to x.*

Solution : Let $y=\tan^{-1}\dfrac{a-x}{1+ax}$.

Then $y=\tan^{-1}a-\tan^{-1}x$.

$\therefore$ $$\frac{dy}{dx}=0-\frac{1}{1+x^2}, \text{ since } \tan^{-1}a \text{ is constant}$$

$$=-1/(1+x^2).$$

Example 2 : *Differentiate* $\tan^{-1}[\{\sqrt{(1+x^2)}-1\}/x]$ *with respect to* $\tan^{-1}x$.

Solution : Let $u=\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$ and $v=\tan^{-1}x$.

Then to find du/dv.

Since $v=\tan^{-1}x$, therefore $x=\tan v$.

$\therefore$ $$u=\tan^{-1}\frac{\sqrt{(1+\tan^2 v)}-1}{\tan v}=\tan^{-1}\frac{\sec v-1}{\tan v}$$

$$= \tan^{-1} \frac{1 - \cos v}{\sin v} = \tan^{-1} \frac{2 \sin^2 \frac{1}{2} v}{2 \sin \frac{1}{2} v \cos \frac{1}{2} v} = \tan^{-1} \tan \tfrac{1}{2} v = \tfrac{1}{2} v.$$

Hence $du/dv = 1/2$.

Example 3 : *Differentiate* $\tan^{-1} \dfrac{\sqrt{(1 + x^2)} + \sqrt{(1 - x^2)}}{\sqrt{(1 + x^2)} - \sqrt{(1 - x^2)}}$.

Solution : Let $y = \tan^{-1} \dfrac{\sqrt{(1 + x^2)} + \sqrt{(1 - x^2)}}{\sqrt{(1 + x^2)} - \sqrt{(1 - x^2)}}$.

Put $x^2 = \cos 2\theta$.

Then $$y = \tan^{-1} \frac{\sqrt{(1 + \cos 2\theta)} + \sqrt{(1 - \cos 2\theta)}}{\sqrt{(1 + \cos 2\theta)} - \sqrt{(1 - \cos 2\theta)}}$$

$$= \tan^{-1} \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan^{-1} \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$= \tan^{-1} \tan\left(\frac{\pi}{4} + \theta\right) = \frac{\pi}{4} + \theta.$$

$\therefore$ $$y = \tfrac{1}{4}\pi + \tfrac{1}{2} \cos^{-1} x^2.$$

Hence $$\frac{dy}{dx} = -\frac{1}{2} \frac{1}{\sqrt{(1 - x^4)}} \cdot 2x = -\frac{x}{\sqrt{(1 - x^4)}}.$$

3.25 Hyperbolic Functions

We define the hyperbolic functions as follows :

$$\boldsymbol{\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2},}$$

$$\boldsymbol{\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x},}$$

$$\boldsymbol{\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}.}$$

Relations between different hyperbolic functions.

We have, $\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$

$$= \tfrac{1}{4}(e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4}(e^{2x} - 2 + e^{-2x}) = \tfrac{1}{4}(2 + 2) = 1.$$

Thus, $\boldsymbol{\cosh^2 x - \sinh^2 x = 1}$.

Similarly, we can establish the following other relations for hyperbolic functions :

$$\boldsymbol{\cosh 2x = \cosh^2 x + \sinh^2 x, \quad \sinh 2x = 2 \sinh x \cosh x}$$

$$\boldsymbol{\cosh 2x = 2\cosh^2 x - 1, \quad \cosh 2x = 1 + 2\sinh^2 x,}$$

$$\boldsymbol{\operatorname{sech}^2 x = 1 - \tanh^2 x, \quad \operatorname{cosech}^2 x = \coth^2 x - 1.}$$

Note : In order to remember the relations for hyperbolic functions it should be noted that they can be obtained from the corresponding relations for circular functions simply by changing them to hyperbolic functions and also by changing the sign of the term which contains the product of two sines.

Differential Coefficients of Hyperbolic Functions :

We have $\dfrac{d}{dx}(\cosh x) = \dfrac{d}{dx}\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{1}{2}(e^x - e^{-x}) = \sinh x.$

Thus, $\boldsymbol{\dfrac{d}{dx}\cosh x = \sinh x.}$

Similarly, $\boldsymbol{\dfrac{d}{dx}\sinh x = \cosh x.}$

Again $\dfrac{d}{dx}(\tanh x) = \dfrac{d}{dx}\left(\dfrac{\sinh x}{\cosh x}\right) = \dfrac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x.$$

Thus, $\boldsymbol{\dfrac{d}{dx}\tanh x = \text{sech}^2 x.}$

Similarly, $\boldsymbol{\dfrac{d}{dx}\coth x = -\text{cosech}^2 x,}$

$\boldsymbol{\dfrac{d}{dx}\text{cosech}\, x = -\text{cosech}\, x \coth x,}$

and $\boldsymbol{\dfrac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x.}$

3.26 Inverse Hyperbolic Functions and their Derivatives

If $\sinh y = x$, then we write $y = \sinh^{-1} x$. Similarly, we can define $\cosh^{-1} x$, $\text{sech}^{-1} x$ and other inverse hyperbolic functions.

Logarithmic values of inverse hyperbolic functions.

Let $y = \cosh^{-1} x$, then $\cosh y = x$.

$\therefore$ $\sinh y = \sqrt{(\cosh^2 y - 1)} = \sqrt{(x^2 - 1)}$.

But $e^y = \cosh y + \sinh y = x + \sqrt{(x^2 - 1)}$.

$\therefore$ $y = \log[x + \sqrt{(x^2 - 1)}]$ *i.e.,* $\boldsymbol{\cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]}$.

Similarly, $\boldsymbol{\sinh^{-1} x = \log[x + \sqrt{(x^2 + 1)}]}$ and $\boldsymbol{\tanh^{-1} x = \dfrac{1}{2}\log\dfrac{1 + x}{1 - x}}$.

Now, in order to find out the differential coefficient of $\cosh^{-1} x$, we have

$$\frac{d}{dx}\cosh^{-1} x = \frac{d}{dx}\log[x + \sqrt{(x^2 - 1)}] = \frac{1}{\sqrt{(x^2 - 1)}}.$$

Thus, $\boldsymbol{\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{(x^2 - 1)}}}$.

Similarly, $\boldsymbol{\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{(x^2 + 1)}}}$,

$\boldsymbol{\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1 - x^2}}$.

3.27 List of Standard Results to be Committed to Memory

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\log_e x = \frac{1}{x}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$$

$$\frac{d}{dx}a^x = a^x \log_e a$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x.$$

$$\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{(1-x^2)}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{(1-x^2)}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{(x^2+1)}}$$

$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}.$$

□

Chapter

4

Successive Differentiation

4.1 Successive Differential Coefficients

Let $y = f(x)$ be a differentiable function of x ; then its differential coefficient $\dfrac{dy}{dx}$ is called the *first differential coefficient* of y. If the first differential coefficient $\dfrac{dy}{dx}$ is differentiable, then its differential coefficient *i.e.,* $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is called the *second differential coefficient* of y and is denoted by $\dfrac{d^2 y}{dx^2}\cdot$ Similarly, the differential coefficient of $\dfrac{d^2 y}{dx^2}$ is called the *third differential coefficient* of y and is written as $\dfrac{d^3 y}{dx^3}\cdot$ In general, the n^{th} *differential coefficient* of y is denoted by $\dfrac{d^n y}{dx^n}\cdot$

If $y = f(x)$ be a function of x, then the various ways of writing the successive differential coefficients of y are as follows :

$$\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \ldots, \frac{d^n y}{dx^n}, \cdots; \ Dy, D^2 y, D^3 y, \ldots, D^n y, \ldots$$

$$y_1, y_2, y_3, \ldots, y_n, \ldots ; \; y', y'', y''', \ldots, y^{(n)}, \ldots$$
$$f'(x), f''(x), f'''(x), \ldots, f^{(n)}(x), \ldots.$$

If $y = f(x)$ be a function of x, then the n^{th} differential coefficient of y_r is the $(n + r)^{th}$ differential coefficient of y

i.e., $\quad D^n y_r = D^{n+r} y = y_{n+r}$. In particular, $D^n y_2 = D^{n+2} y = y_{n+2}$.

The value of the n^{th} differential coefficient of $y = f(x)$ at $x = a$ is denoted by $(y_n)_{x=a}$ or by $(y_n)_a$, or by $f^{(n)}(a)$. It should be noted that the differential coefficient of a given order at a point can exist only when the function and all derivatives of lower order are differentiable at the point.

Illustrative Examples

Example 1 : *Find the second differential coefficient of* $e^{3x} \sin 4x$.

Solution : Let $y = e^{3x} \sin 4x$.

Then $\dfrac{dy}{dx} = 3e^{3x} \sin 4x + 4e^{3x} \cos 4x = e^{3x}(3 \sin 4x + 4 \cos 4x)$.

$$\therefore \quad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\{e^{3x}(3 \sin 4x + 4 \cos 4x)\}$$
$$= 3e^{3x}(3 \sin 4x + 4 \cos 4x) + e^{3x}(12 \cos 4x - 16 \sin 4x)$$
$$= e^{3x}(24 \cos 4x - 7 \sin 4x).$$

Example 2 : *If* $y = (\sin^{-1} x)^2$, *prove that* $(1 - x^2) y_2 - x y_1 - 2 = 0$.

Solution : We have $y = (\sin^{-1} x)^2$.

Differentiating both sides with respect to x, we get $y_1 = \dfrac{2 \sin^{-1} x}{\sqrt{(1 - x^2)}}$.

Squaring both sides, we get

$$(1 - x^2) y_1^2 = 4 (\sin^{-1} x)^2$$

or $\quad (1 - x^2) y_1^2 - 4y = 0$, since $y = (\sin^{-1} x)^2$.

Differentiating again, we get $(1 - x^2) 2y_1 y_2 - 2xy_1^2 - 4y_1 = 0$.

Since $2y_1 \neq 0$, therefore cancelling $2y_1$, we get $(1 - x^2) y_2 - xy_1 - 2 = 0$.

Example 3 : *If* $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$, *find* $\dfrac{d^2 y}{dx^2}$.

Solution : We have $x = a(\cos \theta + \theta \sin \theta)$.

$$\therefore \quad \frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a \theta \cos \theta.$$

Also $\quad y = a(\sin \theta - \theta \cos \theta)$.

$$\therefore \quad \frac{dy}{d\theta} = a(\cos \theta - \cos \theta + \theta \sin \theta) = a \theta \sin \theta.$$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta.$

$$\therefore \qquad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta) = \left[\frac{d}{d\theta}(\tan\theta)\right]\frac{d\theta}{dx}$$

$$= \sec^2\theta \cdot \frac{1}{a\theta\cos\theta} = \frac{1}{a}\frac{\sec^3\theta}{\theta}.$$

Example 4 : *If* $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, *prove that* $p + \dfrac{d^2 p}{d\theta^2} = \dfrac{a^2 b^2}{p^3}$.

Solution : We have $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$. ...(1)

Differentiating both sides of (1) w.r.t. 'θ', we have

$$2p\frac{dp}{d\theta} = -2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta$$

or
$$p\frac{dp}{d\theta} = (b^2 - a^2)\sin\theta\cos\theta. \qquad ...(2)$$

Now differentiating both sides of (2) w.r.t. 'θ', we have

$$p\frac{d^2 p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)(\cos^2\theta - \sin^2\theta)$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - (a^2\cos^2\theta + b^2\sin^2\theta)$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - p^2. \qquad \text{[From (1)]}$$

$$\therefore \qquad p\frac{d^2 p}{d\theta^2} + p^2 = (b^2\cos^2\theta + a^2\sin^2\theta) - \left(\frac{dp}{d\theta}\right)^2$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - \frac{(b^2 - a^2)^2\sin^2\theta\cos^2\theta}{p^2},$$

substituting for $dp/d\theta$ from (2)

$$= \frac{1}{p^2}[p^2(b^2\cos^2\theta + a^2\sin^2\theta) - (b^2 - a^2)^2\sin^2\theta\cos^2\theta]$$

$$= \frac{1}{p^2}[(a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta)$$

$$- (b^2 - a^2)^2\sin^2\theta\cos^2\theta]$$

$$= \frac{1}{p^2}[a^2 b^2\cos^4\theta + a^2 b^2\sin^4\theta + 2a^2 b^2\sin^2\theta\cos^2\theta]$$

$$= \frac{a^2 b^2}{p^2}(\cos^2\theta + \sin^2\theta)^2 = \frac{a^2 b^2}{p^2}.$$

Thus $\quad p\dfrac{d^2 p}{d\theta^2} + p^2 = \dfrac{a^2 b^2}{p^2}.$

Dividing both sides by p, we have $\quad \dfrac{d^2 p}{d\theta^2} + p = \dfrac{a^2 b^2}{p^3}.$

Comprehensive Exercise 1

1. If $x = a(t - \sin t)$ and $y = a(1 + \cos t)$, prove that $\dfrac{d^2 y}{dx^2} = \dfrac{1}{4a}\operatorname{cosec}^4\left(\dfrac{t}{2}\right)$.

2. (i) If $y = A \sin mx + B \cos mx$, prove that $y_2 + m^2 y = 0$.

(ii) If $y = Ae^{ax} + Be^{-ax}$, show that $y_2 - a^2 y = 0$.

3. If $y = e^{ax} \cos bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2) y = 0$.

Also prove that $y_{n+1} = 2ay_n - (a^2 + b^2) y_{n-1}$.

4.2 *n*th Differential Coefficients of Some Standard Functions

(i) If $y = e^{ax+b}$, then $y_1 = ae^{ax+b}$, $y_2 = a^2 e^{ax+b}$, $y_3 = a^3 e^{ax+b}$, ...and so on.

In general $y_n = a^n e^{ax+b}$. **(Bundelkhand 2005; Agra 07; Rohilkhand 11B)**

Thus, $\boldsymbol{D^n e^{ax+b} = a^n e^{ax+b}}$.

(ii) If $y = a^x$, then $y_1 = (\log a)\, a^x$, $y_2 = (\log a)^2 a^x$, etc.

In general $y_n = (\log a)^n a^x$.

Thus $\boldsymbol{D^n a^x = (\log a)^n a^x}$. **(Meerut 2001; Bundelkhand 05; Agra 07)**

(iii) If $y = (ax + b)^m$, then $y_1 = ma\,(ax + b)^{m-1}$,

$$y_2 = m(m-1)\, a^2 (ax + b)^{m-2},$$

$$y_3 = m(m-1)(m-2)\, a^3 (ax + b)^{m-3}, \text{ etc.}$$

In general $y_n = m(m-1)(m-2) \ldots \{m - (n-1)\} a^n (ax + b)^{m-n}$.

Thus, $\boldsymbol{D^n (ax + b)^m = m(m-1)(m-2) \ldots (m - n + 1)\, a^n (ax + b)^{m-n}}$.

If m is a positive integer we can write the above result in a compact form by using the factorial notation. Thus, in this case $D^n (ax + b)^m$

$$= \frac{m(m-1)(m-2)\ldots(m-n+1)(m-n)(m-n-1)\ldots 2.1}{(m-n)(m-n-1)\ldots 2.1} \cdot a^n (ax + b)^{m-n}$$

$$= \frac{m\,!}{(m-n)\,!} a^n (ax + b)^{m-n}.$$

If m is a negative integer, say $m = -p$, where p is a positive integer, then

$$D^n (ax + b)^{-p} = (-p)(-p-1)(-p-2) \ldots \{-p - (n-1)\}\, a^n (ax + b)^{-p-n}$$

$$= (-1)^n p(p+1)(p+2) \ldots (p + n - 1)\, a^n (ax + b)^{-p-n}$$

$$= (-1)^n \frac{(p + n - 1)!}{(p-1)!} a^n (ax + b)^{-p-n}.$$

Note : If m is a positive integer, the m^{th} differential coefficient of $(ax + b)^m$ is constant. Therefore the $(m + 1)^{th}$ and all the higher differential coefficients of $(ax + b)^m$ will be zero.

(iv) If $y = (ax + b)^{-1}$, then **(Agra 2007)**

$$y_1 = (-1)\, a\, (ax + b)^{-2},\ y_2 = (-1)(-2)\, a^2 (ax + b)^{-3},$$

$$y_3 = (-1)(-2)(-3)\, a^3 (ax + b)^{-4}, \text{ etc.}$$

In general, $y_n = (-1)(-2)(-3)\ldots(-n)\, a^n (ax + b)^{-(n+1)}$.

Thus, $\boldsymbol{D^n (ax + b)^{-1} = (-1)^n n!\, a^n (ax + b)^{-n-1}}$.

(v) If $y = \log(ax + b)$, then $y_1 = \dfrac{a}{ax + b} = a\,(ax + b)^{-1}$,

$$y_2 = a^2 (-1)(ax + b)^{-2},\ y_3 = a^3 (-1)(-2)(ax + b)^{-3}, \text{ etc.}$$

In general, $y_n = a^n (-1)(-2)\ldots\{-(n-1)\}(ax + b)^{-n}$.

Thus, $$\boldsymbol{D^n \log(ax + b) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax + b)^n}}.$$ **(Meerut 2003)**

(vi) If $y = \cos(ax + b)$, then

$$y_1 = -a \sin(ax + b) = a \cos\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2 \cos\left(ax + b + 2\cdot\frac{\pi}{2}\right),\ y_3 = a^3 \cos\left(ax + b + 3\,.\frac{\pi}{2}\right), \text{ etc.}$$

In general, $y_n = a^n \cos\left(ax + b + \dfrac{n\pi}{2}\right)$.

Thus, $$\boldsymbol{D^n \cos(ax + b) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)}.$$

(vii) Similarly, $\boldsymbol{D^n \sin(ax + b) = a^n \sin\left(ax + b + \dfrac{n\pi}{2}\right)}$.

(Gorakhpur 2005; Bundelkhand 07; Kumaun 08)

(viii) If $y = e^{ax} \sin(bx + c)$, then

$$y_1 = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$$
$$= e^{ax}\{a \sin(bx + c) + b \cos(bx + c)\}.$$

Putting $a = r\cos\phi$, $b = r\sin\phi$, so that

$r^2 = a^2 + b^2$ and $\phi = \tan^{-1}(b/a)$, we get

$$y_1 = re^{ax} \sin(bx + c + \phi).$$

Similarly $y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$, etc.

In general, $y_n = r^n e^{ax} \sin(bx + c + n\phi)$.

Thus, $\boldsymbol{D^n \{e^{ax} \sin(bx + c)\} = r^n e^{ax} \sin(bx + c + n\phi)}$,

where, $\boldsymbol{r = (a^2 + b^2)^{1/2}}$, and $\boldsymbol{\phi = \tan^{-1}(b/a)}$.

(ix) Similarly,

$$\boldsymbol{D^n \{e^{ax} \cos(bx + c)\} = r^n e^{ax} \cos(bx + c + n\phi)}$$

where $\boldsymbol{r = (a^2 + b^2)^{1/2}}$, and $\boldsymbol{\phi = \tan^{-1}(b/a)}$.

4.3 Decomposition in a Sum

All the standard results obtained in article 4.2 should be committed to memory. In order to find the n^{th} differential coefficient of any other function, it will be often necessary to express that function as the sum or difference of suitable functions with the help of some algebraic or trigonometrical transformations as discussed below.

4.4 Use of Partial Fractions

In order to find the n^{th} differential coefficient of a fraction in which numerator and denominator are both rational, integral algebraic functions, we should resolve the fraction into partial fractions after breaking its denominator into linear factors, real or imaginary. In case we get imaginary factors in the denominator we shall make use of **De-Moivre's Theorem** of trigonometry in order to simplify the result.

Illustrative Examples

Example 1 : *Find the n^{th} differential coefficient of* $\dfrac{x^2}{(x-a)(x-b)}$.

(Rohilkhand 2014)

Solution : Let $y = \dfrac{x^2}{(x-a)(x-b)}$.

Since the given fraction is not a proper one, therefore we should first divide the numerator by the denominator before resolving it into partial fractions. Here we observe orally that the quotient will be 1. So let

$$\frac{x^2}{(x-a)(x-b)} \equiv 1 + \frac{A}{x-a} + \frac{B}{x-b}.$$

Clearing the fractions, we get

$$x^2 \equiv (x-a)(x-b) + A(x-b) + B(x-a).$$

Putting $x = a$, we get $A = a^2/(a-b)$ and putting $x = b$, we get $B = b^2/(b-a)$.

Hence $$y = 1 + \frac{a^2}{(a-b)(x-a)} + \frac{b^2}{(b-a)(x-b)}$$

$$= 1 + \frac{a^2}{(a-b)}(x-a)^{-1} - \frac{b^2}{(a-b)}(x-b)^{-1}.$$

Therefore differentiating both sides n times, we get

$$y_n = \frac{a^2}{(a-b)}(-1)^n n!\,(x-a)^{-n-1} - \frac{b^2}{(a-b)}(-1)^n n!\,(x-b)^{-n-1}$$

$$= \frac{(-1)^n n!}{(a-b)}\left[\frac{a^2}{(x-a)^{n+1}} - \frac{b^2}{(x-b)^{n+1}}\right].$$

Example 2 : *Find the n^{th} differential coefficient of*

(i) $\tan^{-1}(x/a)$. **(Meerut 2001, 05B, 09; Purvanchal 10, 14; Avadh 13)**

(ii) $\tan^{-1} x$.

Solution:(i) If $y = \tan^{-1}\dfrac{x}{a}$, then $y_1 = \dfrac{a}{x^2 + a^2} = \dfrac{a}{(x+ia)(x-ia)}$.

Now let $\dfrac{a}{(x+ia)(x-ia)} \equiv \dfrac{A}{x+ia} + \dfrac{B}{x-ia}$.

Clearing the fractions, we get $a \equiv A(x-ia) + B(x+ia)$.

Putting $x = ia$, we get $B = 1/2i$

and putting $x = -ia$, we get $A = -1/2i$.

$$\therefore \quad y_1 = \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right] = \frac{1}{2i}\left[(x-ia)^{-1} - (x+ia)^{-1}\right].$$

Now differentiating both sides $(n-1)$ times, we get

$$y_n = \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x-ia)^{-n} - (-1)^{n-1}(n-1)!\,(x+ia)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[(x-ia)^{-n} - (x+ia)^{-n}\right].$$

Put $x = r\cos\phi$ and $a = r\sin\phi$. Then

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[r^{-n}(\cos\phi - i\sin\phi)^{-n} - r^{-n}(\cos\phi + i\sin\phi)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\,r^{-n}\cdot\left[(\cos n\phi + i\sin n\phi) - (\cos n\phi - i\sin n\phi)\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\,r^{-n}.\,2i\sin n\phi = (-1)^{n-1}(n-1)!\,r^{-n}\sin n\phi$$

$$= (-1)^{n-1}(n-1)!\left(\frac{a}{\sin\phi}\right)^{-n}\sin n\phi \qquad \text{, since } r = \frac{a}{\sin\phi}$$

$$= (-1)^{n-1}(n-1)!\,a^{-n}\sin^n\phi\,\sin n\phi, \text{ where } \phi = \tan^{-1}(a/x).$$

(ii) Proceeding as in part (i), we get

$$D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\,\sin^n\phi\,\sin n\phi, \text{ where } \phi = \tan^{-1}(1/x).$$

4.5 Use of Trigonometrical Transformations

Suppose we are to find the n^{th} differential coefficient of the function $\sin^m x\cos^n x$, where m and n are positive integers. With the help of trigonometry, we express this function as the sum of sines or cosines of multiples of x and then we apply standard results.

Illustrative Examples

Example 1 : *Find the n^{th} differential coefficient of $\sin^2 x\sin 2x$.*

Solution : Let $y = \sin^2 x\sin 2x$.

Then $y = \frac{1}{2}(1-\cos 2x)\sin 2x$, since $2\sin^2 x = 1-\cos 2x$

$$= \frac{1}{2}\sin 2x - \frac{1}{2}\sin 2x\cos 2x = \frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x.$$

Now differentiating both sides n times, we have

$$y_n = \frac{1}{2}\cdot 2^n\sin\left(2x+\frac{n\pi}{2}\right) - \frac{1}{4}.\,4^n\sin\left(4x+\frac{n\pi}{2}\right)$$

$$= \frac{1}{4}\left[2.2^n\sin\left(2x+\frac{n\pi}{2}\right) - 4^n\sin\left(4x+\frac{n\pi}{2}\right)\right].$$

Example 2 : *Find the n^{th} differential coefficient of $e^{ax}\cos^2 x\sin x$.*

Solution : Let $y = e^{ax}\cos^2 x\sin x$.

Then $y = \frac{1}{2}e^{ax}(1+\cos 2x)\sin x = \frac{1}{2}e^{ax}\sin x + \frac{1}{2}e^{ax}\cos 2x\sin x$

$$= \frac{1}{2} e^{ax} \sin x + \frac{1}{2} e^{ax} [\sin 3x - \sin x]$$

as $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

$$= \frac{1}{2} e^{ax} \sin x + \frac{1}{4} e^{ax} \sin 3x - \frac{1}{4} e^{ax} \sin x$$

$$= \frac{1}{4} [e^{ax} \sin x + e^{ax} \sin 3x].$$

Now differentiating both sides n times, we have

$$y_n = \frac{1}{4}\left[(1 + a^2)^{n/2} e^{ax} \sin\left(x + n \tan^{-1} \frac{1}{a}\right) + (9 + a^2)^{n/2} e^{ax} \sin\left(3x + n \tan^{-1} \frac{3}{a}\right)\right]$$

$$= \frac{1}{4} e^{ax}\left[(1 + a^2)^{n/2} \sin\left(x + n \tan^{-1} \frac{1}{a}\right) + (9 + a^2)^{n/2} \sin\left(3x + n \tan^{-1} \frac{3}{a}\right)\right].$$

Example 3 : *Find the n^{th} differential coefficient of $\sin^5 x \cos^3 x$.*

Solution : Let $z = \cos x + i \sin x$, then

$$z^{-1} = (\cos x + i \sin x)^{-1} = \cos x - i \sin x.$$

Therefore $z + z^{-1} = 2 \cos x$ and $z - z^{-1} = 2i \sin x$.

Also by De-Moivre's theorem, $z^m = \cos mx + i \sin mx, z^{-m} = \cos mx - i \sin mx$.

Therefore $z^m + z^{-m} = 2 \cos mx$ and $z^m - z^{-m} = 2i \sin mx$.

Now $(2i \sin x)^5 (2 \cos x)^3 = (z - z^{-1})^5 (z + z^{-1})^3$

$$= (z^8 - z^{-8}) - 2 (z^6 - z^{-6}) - 2 (z^4 - z^{-4}) + 6 (z^2 - z^{-2})$$

$$= 2i \sin 8x - 2 (2i \sin 6x) - 2 (2i \sin 4x) + 6 (2i \sin 2x).$$

Therefore $\sin^5 x \cos^3 x = 2^{-7} [\sin 8x - 2 \sin 6x - 2 \sin 4x + 6 \sin 2x]$.

Hence $D^n (\sin^5 x \cos^3 x)$

$$= 2^{-7}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right) - 2 . 6^n \sin\left(6x + \frac{n\pi}{2}\right) - 2 . 4^n \sin\left(4x + \frac{n\pi}{2}\right) + 6.2^n \sin\left(2x + \frac{n\pi}{2}\right)\right].$$

Comprehensive Exercise 2

Find the n^{th} differential coefficients of :

1. (i) $\log [(ax + b) (cx + d)]$.

(ii) $\cos 2x \cos 3x$ **(Bundelkhand 2001; Kashi 11)**

(iii) $\cos x \cos 2x \cos 3x$.

(iv) $\cos^4 x$.

2. (i) $\cos^2 x \sin^3 x$.

(ii) $e^{ax} \cos^3 bx$.

(iii) $e^{ax} \sin bx \cos cx$.

(iv) $e^{2x} \sin^3 x$.

3. (i) $\dfrac{1}{1 - 5x + 6x^2}$

(ii) $\dfrac{1}{x^2 - a^2}$

(iii) $\dfrac{x^2}{(x+2)(2x+3)}$. (iv) $\dfrac{x}{(x-a)(x-b)(x-c)}$.

4. $\dfrac{x^4}{(x-1)(x-2)}$, $n \geq 3$. **(Agra 2014)**

5. (i) $\tan^{-1}\left\{\dfrac{1+x}{1-x}\right\}$. **(Purvanchal 2011)** (ii) $\tan^{-1}\left\{\dfrac{2x}{1-x^2}\right\}$.

6. If $y = \tan^{-1}\left\{\dfrac{\sqrt{(1+x^2)}-1}{x}\right\}$, show that $y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta$,

where $\theta = \cot^{-1} x$.

7. If $y = \dfrac{x}{(x^2+a^2)}$, prove that $y_n = (-1)^n n!\, a^{-n-1} \sin^{n+1}\phi \cos(n+1)\phi$,

where $\phi = \tan^{-1}(a/x)$. **(Kanpur 2009)**

8. If $y = \sin mx + \cos mx$, prove that $y_n = m^n [1 + (-1)^n \sin 2mx]^{1/2}$.

(Meerut 2000, 09B)

9. Prove that the value of the n^{th} differential coefficient of $x^3/(x^2-1)$ for $x = 0$ is zero if n is even, and is $-n!$ if n is odd and greater than 1.

10. If $y = (\tan^{-1} x)^2$, prove that $(x^2+1)^2 y_2 + 2x(x^2+1)y_1 - 2 = 0$.

Answers 2

1. (i) $(-1)^{n-1}(n-1)!\,\{a^n(ax+b)^{-n} + c^n(cx+d)^{-n}\}$.

(ii) $\dfrac{1}{2}\left\{5^n \cos\left(5x + \dfrac{1}{2}n\pi\right) + \cos\left(x + \dfrac{1}{2}n\pi\right)\right\}$.

(iii) $\dfrac{1}{4}\left\{2^n \cos\left(2x + \dfrac{1}{2}n\pi\right) + 4^n \cos\left(4x + \dfrac{1}{2}n\pi\right) + 6^n \cos\left(6x + \dfrac{1}{2}n\pi\right)\right\}$.

(iv) $\dfrac{1}{8}\left\{4^n \cos\left(4x + \dfrac{1}{2}n\pi\right) + 2^{n+2} \cos\left(2x + \dfrac{1}{2}n\pi\right)\right\}$.

2 (i) $\dfrac{1}{16}\left\{2\sin\left(x + \dfrac{1}{2}n\pi\right) + 3^n \sin\left(3x + \dfrac{1}{2}n\pi\right) - 5^n \sin\left(5x + \dfrac{1}{2}n\pi\right)\right\}$.

(ii) $(1/4)(a^2+9b^2)^{n/2}.\, e^{ax}\cos\{3bx + n\tan^{-1}(3b/a)\}$

$+ (3/4)(a^2+b^2)^{n/2} e^{ax} \cos\{bx + n\tan^{-1}(b/a)\}$.

(iii) $\dfrac{1}{2} r^n e^{ax} \sin\{(b+c)x + n\phi\} + \dfrac{1}{2} r_1{}^n e^{ax} \sin\{(b-c)x + n\psi\}$,

where $r^2 = a^2 + (b+c)^2$, $\phi = \tan^{-1}\{(b+c)/a\}$, $r_1{}^2 = a^2 + (b-c)^2$,

$\psi = \tan^{-1}\{(b-c)/a\}$.

(iv) $\dfrac{3}{4}\,.\,5^{n/2}\,.\,e^{2x}\sin[x + n\tan^{-1}(1/2)] - \dfrac{1}{4}\cdot(13)^{n/2}\,.\,e^{2x}[\sin 2x + n\tan^{-1}(3/2)]$.

3. (i) $(-1)^n n!\,[2^{n+1}(2x-1)^{-n-1} - 3^{n+1}(3x-1)^{-n-1}]$.

(ii) $(1/2a)\, n!\,(-1)^n \{(x-a)^{-n-1} - (x+a)^{-n-1}\}$.

(iii) $(-1)^n n!\left[\dfrac{9.2^{n-1}}{(2x+3)^{n+1}}-\dfrac{4}{(x+2)^{n+1}}\right]$.

(iv) $(-1)^n n!\left\{\dfrac{a}{(a-b)(a-c)(x-a)^{n+1}}+\dfrac{b}{(b-c)(b-a)(x-b)^{n+1}}\right.$

$$\left.+\frac{c}{(c-a)(c-b)(x-c)^{n+1}}\right\}.$$

4. $(-1)^n n!\{16(x-2)^{-n-1}-(x-1)^{-n-1}\}$.

5. (i) $(-1)^{n-1}(n-1)!\sin^n\phi\sin n\phi$, where $\phi=\tan^{-1}(1/x)$.

(ii) $2(-1)^{n-1}(n-1)!\sin^n\phi\sin n\phi$, where $\phi=\tan^{-1}(1/x)$.

4.6 Leibnitz's Theorem

This theorem is useful for finding the n^{th} differential coefficient of the product of two functions. The statement of this theorem is as follows :

If u and v are any two functions of x such that all their desired differential coefficients exist, then the n^{th} differential coefficient of their product is given by

$$D^n(uv)=(D^n u).v+{}^nC_1 D^{n-1}u.Dv+{}^nC_2 D^{n-2}u.D^2v+\ldots$$
$$\ldots+{}^nC_r D^{n-r}u.D^r v+\ldots+uD^n v.$$

(Meerut 2004, 05BP, 08, 09; Bundelkhand 05; Agra 07; Kumaun 08; Purvanchal 11; Kashi 13)

Proof : We shall prove this theorem by mathematical induction. By actual differentiation, we have

$$D(uv)=(Du).v+u.Dv. \qquad \ldots(1)$$

From (1) we see that the theorem is true for $n=1$.

Now suppose that the theorem is true for a particular value of n. Then we have

$$D^n(uv)=(D^n u)v+{}^nC_1 D^{n-1}uDv+{}^nC_2 D^{n-2}uD^2v+\ldots$$
$$\ldots+{}^nC_r D^{n-r}uD^r v+{}^nC_{r+1}D^{n-r-1}uD^{r+1}v+\ldots+u.D^n v. \qquad \ldots(2)$$

Differentiating both sides of (2) with respect to x, we have

$$D^{n+1}(uv)=\{(D^{n+1}u).v+D^n u\,Dv\}+\{{}^nC_1 D^n u.Dv+{}^nC_1 D^{n-1}u\,D^2v\}$$
$$+\{{}^nC_2 D^{n-1}u.D^2v+{}^nC_2 D^{n-2}u.D^3v\}+\ldots$$
$$\ldots+\{{}^nC_r D^{n-r+1}uD^r v+{}^nC_r D^{n-r}uD^{r+1}v\}$$
$$+\{{}^nC_{r+1}D^{n-r}u.D^{r+1}v+{}^nC_{r+1}D^{n-r-1}uD^{r+2}v\}+\ldots$$
$$\ldots+\{DuD^n v+uD^{n+1}v\}.$$

Rearranging the terms, we have

$$D^{n+1}(uv)=(D^{n+1}u).v+(1+{}^nC_1)(D^n u\,Dv)+({}^nC_1+{}^nC_2)(D^{n-1}u\,D^2v)$$
$$+\ldots+({}^nC_r+{}^nC_{r+1})(D^{n-r}u.D^{r+1}v)+\ldots+uD^{n+1}v. \qquad \ldots(3)$$

But we know that ${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$.

Therefore ${}^nC_0+{}^nC_1={}^{n+1}C_1$ where ${}^nC_0=1$, ${}^nC_1+{}^nC_2={}^{n+1}C_2$, etc.

Hence (3) becomes

$$D^{n+1}(uv) = (D^{n+1}u).v + {}^{n+1}C_1 D^n u.Dv + {}^{n+1}C_2 D^{n-1}u.D^2 v$$
$$+ \ldots + {}^{n+1}C_{r+1} D^{n-r}u.D^{r+1}v + \ldots + u.D^{n+1}v. \quad \ldots(4)$$

The result (4) shows that if the theorem is true for any particular value of n, it is also true for the next value of n. But we have already seen that the theorem is true for $n = 1$. Hence it must be true for $n = 2$ and so for $n = 3$; and so on. Therefore the theorem is true for every positive integral value of n.

Note : While applying Leibnitz's theorem if we see that one of the two functions is such that all its differential coefficients after a certain stage become zero then we should take that function as the second function.

Illustrative Examples

Example 1 : *Find the n^{th} differential coefficient of $x^3 \cos x$.* **(Meerut 2010)**

Solution : Since the fourth and higher derivatives of x^3 will become zero, therefore for the sake of convenience we should choose x^3 as the second function. Applying Leibnitz's theorem, we have

$$D^n[(\cos x).x^3] = (D^n \cos x).x^3 + {}^nC_1(D^{n-1}\cos x).(Dx^3)$$
$$+ {}^nC_2(D^{n-2}\cos x)(D^2x^3) + {}^nC_3(D^{n-3}\cos x)(D^3x^3),$$

since all other terms become zero

$$= \cos\left(x + \frac{n\pi}{2}\right).x^3 + n\cos\left\{x + (n-1)\frac{\pi}{2}\right\}.3x^2$$
$$+ \frac{n(n-1)}{1\cdot 2}\cos\left\{x + (n-2)\frac{\pi}{2}\right\}.6x + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cos\left\{x + (n-3)\frac{\pi}{2}\right\}.6$$
$$= x^3\cos\left(x + \frac{n\pi}{2}\right) + 3x^2.n\sin\left(x + \frac{n\pi}{2}\right)$$
$$- 3n(n-1)x\cos\left(x + \frac{n\pi}{2}\right) - n(n-1)(n-2)\sin\left(x + \frac{n\pi}{2}\right)$$
$$= \left[x^3 - 3n(n-1)x\right]\cos\left(x + \frac{n\pi}{2}\right) + \left[3x^2 n - n(n-1)(n-2)\right]\sin\left(x + \frac{n\pi}{2}\right).$$

Example 2 : *Find the n^{th} differential coefficient of $x^{n-1}\log x$.* **(Meerut 2010B)**

Solution : Let $y = x^{n-1}\log x$. $\quad\ldots(1)$

Then $y_1 = x^{n-1}.(1/x) + (n-1).x^{n-2}.\log x$.

Multiplying both sides by x, we have $xy_1 = x^{n-1} + (n-1)x^{n-1}\log x$

or $\quad xy_1 = x^{n-1} + (n-1)y. \quad\ldots(2)$

$[\because$ from (1), $y = x^{n-1}\log x]$

Differentiating both sides of (2), $(n-1)$ times, we have

$$D^{n-1}(y_1 x) = D^{n-1}x^{n-1} + (n-1)D^{n-1}y$$

or $\quad (D^{n-1}y_1).x + {}^{n-1}C_1(D^{n-2}y_1).1 = (n-1)! + (n-1)y_{n-1}$

or $\quad xy_n + (n-1)y_{n-1} = (n-1)! + (n-1)y_{n-1}$

or $\quad xy_n = (n-1)!$ or $y_n = (n-1)!/x$.

Hence $\quad D^n (x^{n-1} \log x) = (n-1)!/x$.

Example 3 : *If* $y = a \cos(\log x) + b \sin(\log x)$, *show that*

$$x^2 y_2 + xy_1 + y = 0,$$

(Bundelkhand 2006, 11, 12; Avadh 08; Kashi 12; Meerut 13)

and $$x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2+1) y_n = 0.$$

Solution : We have $\quad y = a \cos(\log x) + b \sin(\log x)$.

Differentiating both sides with respect to x, we have

$$y_1 = -\frac{a}{x} \sin(\log x) + \frac{b}{x} \cos(\log x)$$

or $$xy_1 = -a \sin(\log x) + b \cos(\log x).$$

Differentiating both sides again with respect to x, we have

$$xy_2 + y_1 = -\frac{a}{x} \cos(\log x) - \frac{b}{x} \sin(\log x)$$

or $$x^2 y_2 + xy_1 = -[a \cos(\log x) + b \sin(\log x)]$$

or $$x^2 y_2 + xy_1 = -y \quad \text{or} \quad x^2 y_2 + xy_1 + y = 0.$$

Differentiating both sides of this equation n times by Leibnitz's theorem, we get

$$D^n (x^2 y_2) + D^n (xy_1) + D^n (y) = 0$$

or $$(D^n y_2) x^2 + {}^nC_1 (D^{n-1} y_2) . (Dx^2) + {}^nC_2 . (D^{n-2} y_2) . (D^2 x^2)$$
$$+ (D^n y_1) . x + {}^nC_1 (D^{n-1} y_1) . (Dx) + D^n y = 0$$

or $$x^2 y_{n+2} + 2xn\, y_{n+1} + \frac{n(n-1)}{2} . 2y_n + xy_{n+1} + ny_n + y_n = 0$$

or $$x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2+1) y_n = 0.$$

Example 4 : *If* $y = e^{a \sin^{-1} x}$, *show that*

$$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2+a^2) y_n = 0.$$

(Garhwal 2000, 01; Gorakhpur 05; Rohilkhand 05, 08; Agra 06, 08; Purvanchal 07)

Solution : We have $\quad y = e^{a \sin^{-1} x}$.

Therefore $y_1 = e^{a \sin^{-1} x} . a/\sqrt{(1-x^2)}$

or $$y_1 . \sqrt{(1-x^2)} = ae^{a \sin^{-1} x} = ay, \qquad [\text{replacing } e^{a \sin^{-1} x} \text{ by } y]$$

or $$y_1^2 (1-x^2) = a^2 y^2. \qquad ...(1)$$

Differentiating (1) w.r.t. 'x', we have

$$2y_1 y_2 (1-x^2) + y_1^2 (-2x) = 2a^2 yy_1$$

or $$2y_1 [y_2 (1-x^2) - y_1 x - a^2 y] = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get

$$y_2 (1-x^2) - y_1 x - a^2 y = 0. \qquad ...(2)$$

(Bundelkhand 2007)

Differentiating (2) n times by Leibnitz's theorem, we have

$$D^n [y_2 (1-x^2)] - D^n (y_1 x) - a^2 D^n y = 0$$

or $$\left[y_{n+2}\cdot(1-x^2)+ny_{n+1}(-2x)+\frac{n(n-1)}{2!}y_n(-2)\right]$$
$$-[y_{n+1}x+ny_n\,.\,1]-a^2y_n=0$$

or $$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+a^2)\,y_n=0.$$

Example 5 : *If* $y^{1/m}+y^{-1/m}=2x$, *prove that*
$$(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0.$$

(Agra 2002; Meerut 04, 12B; Rohilkhand 06, 09B, 10B, 11; Purvanchal 06)

Solution : We have $y^{1/m}+y^{-1/m}=2x$.

Multiplying both sides by $y^{1/m}$, we get
$$y^{2/m}+1=2xy^{1/m} \quad\text{or}\quad y^{2/m}-2xy^{1/m}+1=0.$$
$$\therefore\quad y^{1/m}=\frac{2x\pm\sqrt{(4x^2-4)}}{2}=x\pm\sqrt{(x^2-1)}\ \text{ or }\ y=[x\pm\sqrt{(x^2-1)}]^m. \quad ...(1)$$
$$\therefore\quad y_1=m[x\pm\sqrt{(x^2-1)}]^{m-1}\left\{1\pm\frac{x}{\sqrt{(x^2-1)}}\right\}$$
$$=\pm\frac{my}{\sqrt{(x^2-1)}}[x\pm\sqrt{(x^2-1)}]^m=\pm\frac{my}{\sqrt{(x^2-1)}},\ \text{from (1)}.$$

Squaring both sides, we get

$y_1^2\,(x^2-1)=m^2y^2$. Differentiating again, we get
$$2y_1y_2\,(x^2-1)+2xy_1^2=2m^2yy_1\ \text{ or }\ 2y_1\,[y_2\,(x^2-1)+xy_1-m^2y]=0$$

or $$y_2\,(x^2-1)+xy_1-m^2y=0,\ \text{since } 2y_1\neq 0. \quad ...(2)$$

Differentiating (2) n times by Leibnitz's theorem, we get
$$D^n\{y_2\,(x^2-1)\}+D^n\,(y_1x)-m^2D^ny=0$$

or $$y_{n+2}\cdot(x^2-1)+{}^nC_1\,y_{n+1}\,2x+{}^nC_2\,y_n\,.\,2+y_{n+1}\,.\,x$$
$$+{}^nC_1\,y_n\,.\,1-m^2y_n=0$$

or $$(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0.$$

Example 6 : *If* $I_n=\dfrac{d^n}{dx^n}(x^n\log x)$, *prove that* $I_n=n\,I_{n-1}+(n-1)\,!$;

(Meerut 2004B; Agra 06; Gorakhpur 06; Rohilkhand 08; Avadh 11)

hence show that $I_n=n\,!\left(\log x+1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}\right)$.

Solution : We have, $$I_n=\frac{d^n}{dx^n}[x^n\log x]=\frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n\log x)\right]$$
$$=\frac{d^{n-1}}{dx^{n-1}}\left[nx^{n-1}\log x+x^n\cdot\frac{1}{x}\right]$$
$$=n\,\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x)+\frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$
$$=n\,I_{n-1}+(n-1)\ \textbf{Proved.} \quad ...(1)$$

We have just proved that $I_n=n\,I_{n-1}+(n-1)\,!$.

Dividing both sides by n !, we have $\dfrac{I_n}{n\,!}=\dfrac{I_{n-1}}{(n-1)\,!}+\dfrac{1}{n}\cdot$ ...(2)

Changing n to $n-1$ in the above relation (2), we have

$$\frac{I_{n-1}}{(n-1)\,!}=\frac{I_{n-2}}{(n-2)\,!}+\frac{1}{n-1}\cdot$$

Putting this value of $\dfrac{I_{n-1}}{(n-1)\,!}$ in (2), we have

$$\frac{I_n}{n\,!}=\frac{I_{n-2}}{(n-2)\,!}+\frac{1}{n-1}+\frac{1}{n}\cdot$$

Thus making repeated use of the reduction formula (2), we ultimately have

$$\frac{I_n}{n\,!}=\frac{I_1}{1\,!}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\cdot$$

But $I_1=\dfrac{d}{dx}(x\log x)=x\cdot\dfrac{1}{x}+\log x=\log x+1.$

$$\therefore\qquad \frac{I_n}{n\,!}=\log x+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

or $$I_n=n\,!\left(\log x+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\cdot$$

Comprehensive Exercise 3

1. State Leibnitz's theorem. **(Meerut 2005B, 08, 11; Bundelkhand 08; Agra 08)**

2. Find the 4^{th} differential coefficients of $x^3\log x$; $x^2\sin 3x$; $e^{2x}\sin 2x$.

Find the n^{th} differential coefficients of :

3. (i) $x^2 e^{-x}$. (ii) $x^3\log x$. (iii) $e^x\log x$. (iv) $x^2\tan^{-1}x$.

4. If $y=x^2e^x$, show that $y_n=\frac{1}{2}n(n-1)y_2-n(n-2)y_1+\frac{1}{2}(n-1)(n-2)y$.

(Bundelkhand 2008)

5. Prove that the n^{th} differential coefficient of $x^n(1-x)^n$ is equal to

$$n\,!\,(1-x)^n\left\{1-\frac{n^2}{1^2}\frac{x}{1-x}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}\frac{x^2}{(1-x)^2}-\ldots\right\}.$$

$\left[\textbf{Hint}.\ D^r x^n=\dfrac{n\,!}{(n-r)\,!}x^{n-r}\right].$ **(Rohilkhand 2007; Kanpur 08)**

6. Prove that $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right)=\dfrac{(-1)^n(n\,!)}{x^{n+1}}\left[\log x-1-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{n}\right].$

7. If $y=x^n\log x$, prove that $xy_{n+1}=n\,!$.

(Meerut 2001; Bundelkhand 09; Rohilkhand 11B)

Hence show that $I_n=n\,!\left(\log x+1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}\right).$

8. By forming in two different ways the n^{th} derivative of x^{2n}, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2}+\ldots=\frac{(2n)!}{(n!)^2}.$$

[**Hint.** Find the n^{th} derivative of $x^n.x^n$ and of x^{2n} and equate].

9. Prove that $D^n\left(\frac{\sin x}{x}\right)=\left\{P\sin\left(x+\frac{1}{2}n\pi\right)+Q\cos\left(x+\frac{1}{2}n\pi\right)\right\}\Big/x^{n+1}$,

where $P=x^n-n(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4}-\ldots$,
and $Q=nx^{n-1}-n(n-1)(n-2)x^{n-3}+\ldots$

10. If $y=e^{\tan^{-1}x}$, prove that

$$(1+x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0.$$

(Avadh 2010; Kanpur 14)

11. If $y=\cos(\log x)$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0.$

12. If $y=(\sin^{-1}x)^2$, prove that $(1-x^2)y_2-xy_1-2=0$,

and $(1-x^2)y_{n+2}-x(2n+1)y_{n+1}-n^2y_n=0.$ **(Meerut 2002; Agra 08)**

13. If $y=(x^2-1)^n$, prove that $(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0.$

(Meerut 2008; Rohilkhand 06, 11B; Kashi 13)

Hence if $P_n=\frac{d^n}{dx^n}(x^2-1)^n$, show that $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}+n(n+1)P_n=0.$

14. If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0.$

(Meerut 2006B; Rohilkhand 13; Purvanchal 14)

15. If $y=[x+\sqrt{(1+x^2)}]^m$, prove that $(1+x^2)y_2+xy_1-m^2y=0$

and $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$

(Kanpur 2006; Avadh 09; Bundelkhand 14)

16. If $y=[\log\{x+\sqrt{(1+x^2)}\}]^2$, prove that
$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0.$ **(Agra 2005; Purvanchal 09)**

17. If $y=\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}$, prove that $(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0.$

(Meerut 2007B; Kanpur 10)

Answers 3

2. $(6/x)$; $3^3(3x^2-4)\sin 3x-6^3x\cos 3x$; $-64e^{2x}\sin 2x$.

3. (i) $(-1)^n e^{-x}[x^2-2nx+n(n-1)]$.

(ii) $(-1)^n(n-4)!\,6x^{-n+3}$.

(iii) $e^x[\log x+{}^nC_1x^{-1}-{}^nC_2x^{-2}+{}^nC_3\,2!\,x^{-3}+\ldots+(-1)^{n-1}(n-1)!\,x^{-n}]$.

(iv) $(-1)^{n-1}(n-3)!\,\{(n-1)(n-2)x^2\sin^n\phi\sin n\phi$

$-{}^nC_1\,2x(n-2)\sin^{n-1}\phi\sin(n-1)\phi$

$+2.{}^nC_2\sin^{n-2}\phi\sin(n-2)\phi\}$, where $\phi=\tan^{-1}(1/x)$.

4.7 n^{th} Differential Coefficient for $x = 0$

Sometimes we are required to find the n^{th} differential coefficient of y for $x = 0$ *i.e.* $(y_n)_0$. This may be done even though we may not be able to find the nth differential coefficient in a compact form for the general value of x. The method will be clear from the following example :

Illustrative Examples

Example 1 : *If* $y = \sin(m \sin^{-1} x)$, *find* $(y_n)_0$. **(Meerut 2000, 03)**

Solution : We have $y = \sin(m \sin^{-1} x)$. ...(1)

Differentiating both sides with respect to x, we get

$$y_1 = \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{(1 - x^2)}}. \quad \ldots(2)$$

Squaring both sides of (2) and multiplying by $(1 - x^2)$, we get

$$(1 - x^2) y_1^2 = m^2 \cos^2 (m \sin^{-1} x)$$

or $$(1 - x^2) y_1^2 = m^2 [1 - \sin^2 (m \sin^{-1} x)]$$

or $$(1 - x^2) y_1^2 = m^2 (1 - y^2) \quad [\text{since } y = \sin(m \sin^{-1} x)]$$

or $$(1 - x^2) y_1^2 + m^2 y^2 - m^2 = 0. \quad \ldots(3)$$

Differentiating both sides of (3) with respect to x, we get

$$(1 - x^2) 2y_1 y_2 - 2x y_1^2 + 2m^2 yy_1 = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get $(1 - x^2) y_2 - xy_1 + m^2 y = 0$. ...(4)

Differentiating both sides of (4) n times by Leibnitz's theorem, we get

$$(1 - x^2) y_{n+2} + {}^nC_1 y_{n+1} (-2x) + {}^nC_2 y_n (-2) - xy_{n+1} - {}^nC_1 y_n + m^2 y_n = 0$$

or $$(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 - m^2) y_n = 0. \quad \ldots(5)$$

(Meerut 2000)

Putting $x = 0$ in (1), we get $(y)_0 = 0$.

Putting $x = 0$ in (2), we get $(y_1)_0 = m$.

Putting $x = 0$ in (4), we get $(y_2)_0 + m^2 (y)_0 = 0$ *i.e.*, $(y_2)_0 = 0$.

Also putting $x = 0$ in (5), we get $(y_{n+2})_0 = (n^2 - m^2)(y_n)_0 \ldots(6)$

Putting $n - 2$ in place of n in (6), we get

$$(y_n)_0 = \{(n-2)^2 - m^2\} (y_{n-2})_0$$

$$= \{(n-2)^2 - m^2\} \{(n-4)^2 - m^2\} (y_{n-4})_0.$$

[Since from (6), we have $(y_{n-2})_0 = \{(n-4)^2 - m^2\} (y_{n-4})_0$].

Now there arise two cases.

Case I : When n is even.

$$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \{(n-6)^2 - m^2\} \ldots \{4^2 - m^2\} \{2^2 - m^2\}(y_2)_0$$

$$= 0 \quad , \text{since } (y_2)_0 = 0.$$

Case II : When n is odd.

$$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}\{(n-6)^2 - m^2\}\dots \{3^2 - m^2\}\{1^2 - m^2\}(y_1)_0$$

$$= \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}\{(n-6)^2 - m^2\}\dots \{3^2 - m^2\}\{1^2 - m^2\}\, m.$$

Comprehensive Exercise 4

1. If $y = \sin^{-1} x$, prove that

$(1 - x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - n^2 y_n = 0,$ **(Agra 2005; Bundelkhand 11)**

and hence find the vlaue of $(y_n)_0$.

2. Find $(y_n)_0$, when $y = \log [x + \sqrt{(1 + x^2)}]$.

3. If $y = [\log \{x + \sqrt{(1 + x^2)}\}]^2$, prove that $(y_{n+2})_0 = -\, n^2 (y_n)_0$, hence find $(y_n)_0$.

(Meerut 2005, 09B)

4. If $y = (\sinh^{-1} x)^2$, prove that $(1 + x^2)\dfrac{d^{n+2} y}{dx^{n+2}} + (2n+1)\, x \dfrac{d^{n+1} y}{dx^{n+1}} + n^2 \dfrac{d^n y}{dx^n} = 0.$

Hence find, at $x = 0$, the value of $(d^n y / dx^n)$.

5. If $y = [x + \sqrt{(1 + x^2)}]^m$, find $(y_n)_{x=0}$.

(Meerut 2006, 07, 09; Bundelkhand 2001)

6. If $y = \cos (m \sin^{-1} x)$, find $(y_n)_0$.

7. If $x = \sin \left(\dfrac{1}{a} \log y \right)$ or if $y = e^{a \sin^{-1} x}$, prove that

$(1 - x^2)\, y_2 - x y_1 - a^2 y = 0,$ **(Bundelkhand 2007)**

$(1 - x^2)\, y_{n+2} - x (2n+1)\, y_{n+1} - (n^2 + a^2)\, y_n = 0,$

and hence find the value of $(y_n)_0$.

(Rohilkhand 2005, 08; Agra 06, 08; Gorakhpur 05)

8. If $y = e^{a \cos^{-1} x}$, prove that $(1 - x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2 + a^2)\, y_n = 0.$

Hence find the value of y_n for $x = 0$. **(Meerut 2001; Purvanchal 14)**

9. If $y = \tan^{-1} x$, prove that $(1 + x^2)\, y_2 + 2 x y_1 = 0$, and hence find the value of all the derivatives of y with respect to x, when $x = 0$.

Also show that $(y_n)_0$ is 0, $(n-1)\,!$ or $-\,(n-1)\,!$ according as n is of the form $2p$, $4p + 1$ or $4p + 3$ respectively.

Answers 4

1. 0 when n is even, and $(n-2)^2 (n-4)^2 \dots\dots 5^2 . 3^2 . 1^2$, when n is odd.
2. 0 when n is even, and $(-1)^{(n-1)/2} (n-2)^2 (n-4)^2 \dots 3^2 . 1^2$, when n is odd.

3. 0 when n is odd, and $(-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2\ldots 4^2 . 2^2 . 2$, when n is even.

4. 0 when n is odd, and
$(-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2\ldots 4^2.2^2.2$ when n is even.

5. $\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-1^2)\,m$, n odd;
$\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-2^2)\,m^2$, n even.

6. 0 when n is odd, and $-\{(n-2)^2-m^2\}\{(n-4)^2-m^2\}\ldots(2^2-m^2)\,m^2$, when n is even.

7. $\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(3^2+a^2)(1^2+a^2)\,a$, n odd;
$\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)\,a^2$, n even.

8. $-\{(n-2)^2+a^2\}\{(n-4^2)+a^2\}\ldots(3^2+a^2)(1^2+a^2)\,ae^{a\pi/2}$, n odd;
$\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)\,a^2e^{a\pi/2}$, n even.

9. 0 when n is even and $(-1)^{(n-1)/2}(n-1)!$ when n is odd.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. If $y=\sin(ax+b)$, then $D^n\sin(ax+b)=$ ....

2. If $y=(ax+b)^{-1}$, then $D^n(ax+b)^{-1}=$ ....

3. The n^{th} differential coefficient of $e^x\sin^2 x=$ ....

4. If $y=a\cos(\log x)+b\sin(\log x)$, then $x^2y_2+xy_1=$ ....

5. If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, then $(1-x^2)\,y_{n+1}-(2n+1)\,xy_n=$ ....

6. If $y=e^{-x}$, then $D^n e^{-x}=$ .... **(Agra 2006)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. If $y=\log x$, then $D^n\log x$ is

(a) $\dfrac{(-1)^n(n-1)!}{x^n}$

(b) $\dfrac{(-1)^{n-1}(n-1)!}{x^n}$

(c) $\dfrac{(-1)^{n-1}n!}{x^n}$

(d) $\dfrac{(-1)^{n-1}(n-1)!}{x^{n+1}}$

8. If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, then $\dfrac{d^2y}{dx^2}$ is

(a) $\dfrac{1}{a}\dfrac{\sec^3\theta}{\theta}$

(b) $\dfrac{a\sec^3\theta}{\theta}$

(c) $\dfrac{\theta\sec^3\theta}{a}$

(d) $a\theta\sec^3\theta$

9. By Leibnitz's theorem we find the n^{th} differential coefficient of the ...... of two functions.

(a) sum

(b) difference

(c) product

(d) quotient

10. If $y = e^{\tan^{-1}x}$, then $(1 + x^2)\, y_{n+2} + [2(n+1)x - 1]\, y_{n+1} = \ldots\ldots$

(a) $-n(n-1)\, y_n$

(b) $\dfrac{n}{2}(n+1)\, y_n$

(c) $-\dfrac{n}{2}(n-1)\, y_n$

(d) $-n(n+1)\, y_n$

True or False.

Write 'T' for true and 'F' for false statement.

11. If $y = f(x)$, then the n^{th} differential coefficient of y_r is the $(n + r)^{th}$ differential coefficient of y.

12. If $y = e^{ax}\sin(bx + c)$, then

$$D^n\{e^{ax}\sin(bx + c)\} = r^n e^{ax}\sin\{bx + c + (n+1)\phi\},$$

where $r = (a^2 + b^2)^{1/2}$ and $\phi = \tan^{-1}(b/a)$.

13. While applying Leibnitz's theorem if we observe that one of the two functions is such that all its differential coefficients after a certain stage become zero, then we should take that function as second function.

Answers

1. $a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.

2. $(-1)^n n!\, a^n (ax+b)^{-n-1}$.

3. $\frac{1}{2}[e^x - (5)^{n/2} e^x \cos(2x + n \tan^{-1} 2)]$.

4. $-y$.

5. $n^2 y_{n-1}$.

6. $(-1)^n e^{-x}$.

7. (b).

8. (a).

9. (c).

10. (d).

11. T.

12. F.

13. T.

Chapter 5

Expansions of Functions

5.1 Accurate Statement of Taylor's Theorem

If $f(x)$ is a single-valued function of x such that

(i) *all the derivatives of $f(x)$ upto $(n-1)^{th}$ are continuous in* $a \le x \le a+h$,

and (ii) *$f^{(n)}(x)$ exists in $a < x < a+h$, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h), \textit{ where } 0 < \theta < 1.$$

Taylor's Series : **(Meerut 2009B, 10B; Kashi 11, 13)**

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[a, a+h]$. Then for every positive integral value of n, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad \dots(1)$$

where $$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h), \ (0 < \theta < 1).$$

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots \qquad \ldots(2)$$

The series given in (2) is known as **Taylor's infinite series** for the expansion of $f(a+h)$ as a power series in h.

5.2 Maclaurin's Series

(Rohilkhand 2009B; Kashi 12)

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n, \ldots \qquad \ldots(1)$$

where $\qquad R_n=\frac{x^n}{n!}f^{(n)}(\theta x), (0<\theta<1).$

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$\boldsymbol{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots} \qquad \ldots(2)$$

The series given in (2) is known as **Maclaurin's infinite series** for the expansion of $f(x)$ as a **power series** in x. Maclaurin's series is a particular case of Taylor's series. If in Taylor's series we put $a=0$ and $h=x$, we get Maclaurin's series.

Maclaurin's expansion of $f(x)$ fails if any of the functions $f(x), f'(x), f''(x), \ldots$, becomes infinite or discontinuous at any point of the interval $[0, x]$ or if R_n does not tend to zero as $n \to \infty$.

5.3 Formal Expansions of Functions

We have seen that for the validity of the expansion of a function $f(x)$ as an infinite Maclaurin's series, it is necessary that $R_n \to 0$ as $n \to \infty$. But to examine the behaviour of R_n as $n \to \infty$ is not an easy job because in many cases it is not possible to find a general expression for the nth derivative of the function to be expanded. So in this chapter we shall simply obtain *formal expansion* of a function $f(x)$ without showing that $R_n \to 0$ as $n \to \infty$. Such an expansion will not give us any idea of the range of values of x for which the expansion is valid. To obtain such an expansion of $f(x)$ we have only to calculate the values of its derivatives for $x=0$ and substitute them in the infinite Maclaurin's series

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots$$

For the convenience of the students we shall now give formal proofs of Maclaurin's and Taylor's theorems without bothering about the nature of R_n as $n \to \infty$.

Maclaurin's Theorem : (Bundelkhand 2006; Kashi 12, 13; Purvanchal 14)

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$. Assuming that $f(x)$ can be expanded as an infinite power series in x, we have

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots$$

Proof : Suppose $f(x)=A_0+A_1x+A_2x^2+A_3x^3+\ldots \qquad \ldots(1)$

Let the expansion (1) be differentiable term by term any number of times. Then by successive differentiation, we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \ldots,$$
$$f''(x) = 2 \,.\, 1 A_2 + 3 \,.\, 2 A_3 x + 4 \,.\, 3 A_4 x^2 + \ldots,$$
$$f'''(x) = 3 \,.\, 2 \,.\, 1 A_3 + 4 \,.\, 3 \,.\, 2 A_4 x + \ldots, \text{ and so on.}$$

Putting $x = 0$ in each of these relations, we get

$$f(0) = A_0, f'(0) = A_1, f''(0) = 2!\, A_2, f'''(0) = 3!\, A_3, \ldots .$$

Substituting these values of $A_0, A_1, A_2, \ldots$ in (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n\,!} f^{(n)}(0) + \ldots$$

This is Maclaurin's Theorem. If we denote $f(x)$ by y, then Maclaurin's theorem can also be written in the following way :

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots .$$

Taylor's Theorem : **(Bundelkhand 2005; Avadh 09, 10, 14; Kashi 11, 13, 14)**

Let $f(x)$ be a function of x which possesses, continuous derivatives of all orders in the interval $[a, a+h]$. Assuming that $f(a+h)$ can be expanded as an infinite power series in h, we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots$$

Proof : Suppose $f(a+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots$ ...(1)

Let the expansion (1) be differentiable term by term any number of times w.r.t. 'h'. Then by successive differentiation w.r.t. 'h', we have

$$f'(a+h) = A_1 + 2A_2 h + 3A_3 h^2 + \ldots,$$
$$f''(a+h) = 2 \,.\, 1 A_2 + 3 \,.\, 2 A_3 h + \ldots,$$
$$f'''(a+h) = 3 \,.\, 2 \,.\, 1 A_3 + \ldots, \text{ and so on.}$$

Putting $h = 0$ in each of the above relations, we get

$$f(a) = A_0, f'(a) = A_1, f''(a) = 2!\, A_2, f'''(a) = 3!\, A_3, \text{ and so on.}$$

$$\therefore \quad A_0 = f(a), A_1 = f'(a), A_2 = \frac{1}{2!} f''(a), A_3 = \frac{1}{3\,!} f'''(a), \text{ and so on.}$$

Substituting these values of $A_0, A_1, A_2, A_3, \ldots$ in (1), we get

$$\boldsymbol{f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots}$$

This is **Taylor's theorem.** Another useful form is obtained on replacing h by $(x - a)$. Thus

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots \ldots + \frac{(x-a)^n}{n\,!} f^{(n)}(a) + \ldots,$$

which is an expansion of $f(x)$ as a power series in $(x - a)$.

Note : If we expand $f(x+h)$, by Taylor's theorem, as a power series in h, then the result is as follows :

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2\,!} f''(x) + \ldots + \frac{h^n}{n\,!} f^{(n)}(x) + \ldots$$

Illustrative Examples

Example 1 : *Expand e^x in ascending powers of x.* **(Bundelkhand 2008)**

Solution : Let $f(x) = e^x$. Then $f(0) = 1$, $f^{(n)}(x) = e^x$ so that $f^{(n)}(0) = 1$, where $n = 1, 2, 3, 4, \ldots$

Substituting these values in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots, \text{ we get}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots$$

This is known as **Exponential series**.

Example 2 : *Expand $(1 + x)^n$ in ascending powers of x.*

Solution : Let $f(x) = (1 + x)^n$, so that $f(0) = 1$.

We have $f^{(m)}(x) = n(n-1)\ldots(n-m+1)(1+x)^{n-m}$.

$$\therefore \qquad f^{(m)}(0) = n(n-1)\ldots(n-m+1).$$

Putting $m = 1, 2, 3, \ldots$, we have

$$f'(0) = n, f''(0) = n(n-1), f'''(0) = n(n-1)(n-2), \text{ and so on.}$$

Substituting these values in Maclaurin's series for $f(x)$, we get

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + \frac{n(n-1)\ldots(n-m+1)}{m!}x^m + \ldots$$

This is known as Binomial series. If n is a positive integer, the series will consist of $(n + 1)$ terms.

Example 3 : *Expand sin x.* **(Kashi 2012)**

Solution : Let $f(x) = \sin x$. Then $f(0) = 0$

$$f'(x) = \cos x, \quad f'(0) = 1$$
$$f''(x) = -\sin x, \quad f''(0) = 0$$
$$f'''(x) = -\cos x, \quad f'''(0) = -1.$$

In general $f^{(n)}(x) = \sin\left(x + \frac{1}{2}n\pi\right)$ so that

$$f^{(n)}(0) = \sin\tfrac{1}{2}n\pi = 0 \text{ when } n = 2m \quad \text{and} \quad = (-1)^m \text{ when } n = 2m+1.$$

Hence substituting these values in Maclaurin's series, we get

$$\sin x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!}(-1) + 0 + \ldots + 0 + (-1)^m\frac{x^{2m+1}}{(2m+1)!} + \ldots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m\frac{x^{2m+1}}{(2m+1)!} + \ldots.$$

This is known as **Sine series**.

Similarly we may obtain **Cosine series** :

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^m\frac{x^{2m}}{(2m)!} + \ldots.$$

(Kanpur 2006)

Example 4 : *Expand log (1 + x) by Maclaurin's theorem.*

(Meerut 2003, 11; Agra 05)

Solution : Let $f(x) = \log(1 + x)$.

Then $\quad f(0)=\log 1=0, f^{(n)}(x)=\dfrac{(-1)^{n-1}(n-1)!}{(x+1)^n}$

so that $\qquad f^{(n)}(0)=(-1)^{n-1}(n-1)!$, where $n=1, 2, 3, 4, \ldots$.

Now by Maclaurin's theorem,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots.$$

Substituting the values of $f(0), f'(0), f''(0)$, etc., we get

$$\log(1+x)=0+x-\frac{x^2}{2!}\cdot 1!+\frac{x^3}{3!}\cdot 2!-\frac{x^4}{4!}\cdot 3!+\ldots$$

$$\ldots+\frac{x^n}{n!}(-1)^{n-1}(n-1)!+\ldots$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

Example 5 : *Apply Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e(1+e^x)$ to the terms containing x^4.* **(Kanpur 11; Rohilkhand 12)**

Solution : Let $y=\log_e(1+e^x)$. Then $(y)_0=\log_e(1+e^0)=\log_e 2$.

Now $\quad y_1=\dfrac{e^x}{1+e^x}=\dfrac{(1+e^x)-1}{1+e^x}=1-\dfrac{1}{1+e^x}$ so that $(y_1)_0=1-\frac{1}{2}=\frac{1}{2}$,

$$y_2=0+\frac{e^x}{(1+e^x)^2}=\frac{e^x}{1+e^x}\cdot\frac{1}{1+e^x}=y_1(1-y_1)=y_1-y_1^2,$$

so that $\quad (y_2)_0=(y_1)_0-[(y_1)_0]^2=\frac{1}{2}-\left(\frac{1}{2}\right)^2=\frac{1}{4}$,

$y_3=y_2-2y_1y_2$ so that $(y_3)_0=\frac{1}{4}-2\cdot\frac{1}{2}\cdot\frac{1}{4}=0$,

$y_4=y_3-2y_2^2-2y_1y_3$ so that $(y_4)_0=0-2\cdot\left(\frac{1}{4}\right)^2-0=-\frac{1}{8}$, and so on.

Now by Maclaurin's theorem, we have

$$y=(y)_0+x(y_1)_0+\frac{x^2}{2!}(y_2)_0+\frac{x^3}{3!}(y_3)_0+\frac{x^4}{4!}(y_4)_0+\ldots.$$

$$\therefore \quad \log(1+e^x)=\log 2+x\cdot\frac{1}{2}+\frac{x^2}{2!}\cdot\frac{1}{4}+\frac{x^3}{3!}\cdot 0+\frac{x^4}{4!}\cdot\left(-\frac{1}{8}\right)+\ldots$$

$$=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots.$$

Example 6 : *Expand $\log\{x+\sqrt{(1+x^2)}\}$ in ascending powers of x and find the general term.*

Solution : Let $\quad y=\log\{x+\sqrt{(1+x^2)}\}$. $\quad\ldots(1)$

Then $\quad y_1=\dfrac{1}{x+\sqrt{(1+x^2)}}\cdot\left\{1+\dfrac{2x}{2\sqrt{(1+x^2)}}\right\}=\dfrac{1}{\sqrt{(1+x^2)}}$ $\quad\ldots(2)$

$\therefore \quad y_1^2(1+x^2)-1=0.$

Differentiating again, we get $(1+x^2)\,2y_1y_2+2xy_1^2=0$

or $\qquad 2y_1[(1+x^2)y_2+xy_1]=0$

or $\qquad (1+x^2)\,y_2+xy_1=0,$ $\qquad\qquad$ ...(3)

since $\qquad 2y_1 \neq 0.$

Now differentiating (3) n times by Leibnitz's theorem, we get

$$(1+x^2)\,y_{n+2}+n\,.\,y_{n+1}\,.\,2x+\frac{n\,(n-1)}{1\,.\,2}\,y_n\,.\,2+y_{n+1}\,.\,x+n\,.\,y_n\,.\,1=0$$

or $\qquad (1+x^2)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2\,y_n=0.$ $\qquad\qquad$ ...(4)

Putting $x=0$ in (1), (2), (3) and (4), we get

$$(y)_0=0,\,(y_1)_0=1,\,(y_2)_0=0,$$

and $\qquad (y_{n+2})_0=-\,n^2\,(y_n)_0.$ $\qquad\qquad$ ...(5)

Now putting $n=1, 3, 5, \ldots$ in (5), we get

$$(y_3)_0=-\,1^2\,(y_1)_0=-\,1^2,$$

$$(y_5)_0=(-\,3^2)\,(y_3)_0=(-\,3^2)\,(-\,1^2)=3^2.1^2,$$

$$(y_7)_0=(-\,5^2)\,(y_5)_0=(-\,5^2)\,(-\,3^2)\,(-\,1^2)=-\,5^2\,.\,3^2\,.\,1^2, \text{ and so on.}$$

Putting $n-2$ in place of n in (5), we get

$$(y_n)_0=\{-\,(n-2)^2\}\,(y_{n-2})_0 \qquad\qquad \ldots(6)$$

$$=\{-\,(n-2)^2\}\,\{-\,(n-4)^2\}\,(y_{n-4})_0\,.$$

[$\because$ replacing n by $n-2$ in (6), we have $(y_{n-2})_0=-\,(n-4)^2\,(y_{n-4})_0$]

Thus if n is odd, we have

$$(y_n)_0=\{-\,(n-2)^2\}\,\{-\,(n-4)^2\}\ldots(-\,5^2)\,(-\,3^2)\,(-\,1^2)\,.\,1$$

$$=(-\,1)^{(n-1)/2}\,(n-2)^2\,(n-4)^2\ldots5^2\,.\,3^2\,.\,1^2. \qquad\qquad \ldots(7)$$

Again, putting $n=2, 4, 6, \ldots$ in (5), we get

$$(y_4)_0=-\,2^2.\,(y_2)_0=0,\,(y_6)_0=-\,4^2\,.\,(y_4)_0=0, \text{ and so on.}$$

Thus, if n is even, we have $(y_n)_0=0.$

Now by Maclaurin's theorem, we have

$$y=(y)_0+\frac{x}{1\,!}\,(y_1)_0+\frac{x^2}{2\,!}\,(y_2)_0+\frac{x^3}{3\,!}\,(y_3)_0+\ldots\,.$$

$$\therefore \qquad \log\{x+\sqrt{(1+x^2)}\}=0+x\,.\,1+\frac{x^2}{2\,!}\,.\,0+\frac{x^3}{3\,!}\cdot(-\,1^2)+\frac{x^4}{4\,!}\cdot 0$$

$$+\frac{x^5}{5\,!}\,(3^2\cdot 1^2)+\frac{x^6}{6\,!}\cdot 0+\frac{x^7}{7\,!}\,(-\,5^2\cdot 3^2\cdot 1^2)+\ldots$$

$$=x-\frac{x^3}{3\,!}\cdot 1^2+\frac{x^5}{5\,!}\,(3^2\cdot 1^2)-\frac{x^7}{7\,!}\,(5^2\cdot 3^2\cdot 1^2)+\ldots$$

The general term $=\dfrac{x^n}{n!}\,(y_n)_0$, where $(y_n)_0$ is given by (7) when n is odd and $(y_n)_0=0$, when n is even.

Putting $2n-1$ in place of n in (7), we find that

$$(y_{2n-1})_0=(-\,1)^{n-1}\,(2n-3)^2\,(2n-5)^2\ldots5^2.\,3^2.\,1^2.$$

Hence $\log \{x + \sqrt{(1 + x^2)}\}$

$$= x - 1^2 \cdot \frac{x^3}{3!} + 1^2 \cdot 3^2 \cdot \frac{x^5}{5!} - 1^2 \cdot 3^2 \cdot 5^2 \cdot \frac{x^7}{7!} + \ldots$$

$$+ (-1)^{n-1} 1^2 \cdot 3^2 \cdot 5^2 \ldots (2n-3)^2 \cdot \frac{x^{2n-1}}{(2n-1)!} + \ldots .$$

Example 7 : *If* $y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots$, *prove that*

$(n+1)(n+2)\, a_{n+2} = n^2 a_n.$ **(Meerut 2010B; Kumaun 08)**

Solution : Let $y = \sin^{-1} x$ . ...(1)

Then $y_1 = \dfrac{1}{\sqrt{(1-x^2)}}$. ...(2)

$\therefore$ $y_1^2 (1 - x^2) - 1 = 0.$

Differentiating again, we get $(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = 0$

or $2y_1 [(1 - x^2) y_2 - xy_1] = 0$

or $(1 - x^2) y_2 - xy_1 = 0,$ ...(3)

since $2y_1 \neq 0.$

Now differentiating (3) n times by Leibnitz's theorem, we get

$$(1 - x^2) y_{n+2} + n \cdot y_{n+1} \cdot (-2x) + \frac{n(n-1)}{1 \cdot 2} y_n (-2) - y_{n+1} \cdot x - ny_n \cdot 1 = 0$$

or $(1 - x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$ ...(4)

Putting $x = 0$ in (4), we get $(y_{n+2})_0 = n^2 (y_n)_0.$ ...(5)

By Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1\,!}(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots + \frac{x^n}{n\,!}(y_n)_0 + \ldots$$

Also we are given that

$$y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots .$$

Equating the coefficients of x^n in the two expansions for y, we get $a_n = \dfrac{(y_n)_0}{n\,!}$.

$$\therefore \quad \frac{a_{n+2}}{a_n} = \frac{(y_{n+2})_0}{(n+2)!} \cdot \frac{n!}{(y_n)_0} = \frac{(y_{n+2})_0}{(y_n)_0} \cdot \frac{1}{(n+2)(n+1)}$$

$$= \frac{n^2}{(n+2)(n+1)}, \quad \text{substituting for } \frac{(y_{n+2})_0}{(y_n)_0} \text{ from (5).}$$

Hence $(n+1)(n+2)\, a_{n+2} = n^2 a_n.$

Example 8 : *Expand* $\sin x$ *in powers of* $\left(x - \frac{1}{2}\pi\right)$ *by using Taylor's series.*

(Meerut 2005; Rohilkhand 06, 10; Agra 06)

Solution : Let $f(x) = \sin x$. We want to expand $f(x)$ in powers of $x - \frac{1}{2}\pi$.

We can write $f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$.

Now expanding $f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right]$ by Taylor's theorem in powers of $\left(x - \frac{1}{2}\pi\right)$, we get

$$f(x) = f\left[\frac{1}{2}\pi + \left(x - \frac{1}{2}\pi\right)\right] = f\left(\frac{1}{2}\pi\right) + \left(x - \frac{1}{2}\pi\right) f'\left(\frac{1}{2}\pi\right)$$

$$+ \frac{1}{2!}\left(x - \frac{1}{2}\pi\right)^2 f''\left(\frac{1}{2}\pi\right) + \frac{1}{3!}\left(x - \frac{1}{2}\pi\right)^3 f'''\left(\frac{1}{2}\pi\right) + \dots \quad \dots(1)$$

Now $f(x) = \sin x$. Therefore $f\left(\frac{1}{2}\pi\right) = \sin\frac{1}{2}\pi = 1$,

$f'(x) = \cos x$ giving $f'\left(\frac{1}{2}\pi\right) = \cos\frac{1}{2}\pi = 0$,

$f''(x) = -\sin x$ so that $f''\left(\frac{1}{2}\pi\right) = -\sin\frac{1}{2}\pi = -1$,

$f'''(x) = -\cos x$ so that $f'''\left(\frac{1}{2}\pi\right) = -\cos\frac{1}{2}\pi = 0$,

$f^{iv}(x) = \sin x$ so that $f^{iv}\left(\frac{1}{2}\pi\right) = \sin\frac{1}{2}\pi = 1$, etc.

Substituting these values in (1), we get

$$\sin x = 1 + \left(x - \frac{1}{2}\pi\right) . 0 + \frac{1}{2!}\left(x - \frac{1}{2}\pi\right)^2 . (-1) + \frac{1}{3!}\left(x - \frac{1}{2}\pi\right)^3 . 0$$

$$+ \frac{1}{4!}\left(x - \frac{1}{2}\pi\right)^4 . 1 + \dots$$

$$= 1 - \frac{1}{2!}\left(x - \frac{1}{2}\pi\right)^2 + \frac{1}{4!}\left(x - \frac{1}{2}\pi\right)^4 - \dots .$$

Example 9 : *Expand* $\log \sin (x + h)$ *in powers of* h *by Taylor's theorem.*

(Purvanchal 2006; Meerut 10; Bundelkhand 09; Kashi 14)

Solution : Let $f(x + h) = \log \sin (x + h)$.

Then by Taylor's theorem, we have

$$\log \sin (x + h) = f(x + h)$$

$$= f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots .$$

Now $f(x + h) = \log \sin (x + h)$.

$\therefore$ $f(x) = \log \sin x$,

$f'(x) = (1/\sin x) . \cos x = \cot x$,

$f''(x) = -\operatorname{cosec}^2 x$,

$f'''(x) = 2 \operatorname{cosec} x \operatorname{cosec} x \cot x = 2 \operatorname{cosec}^2 x \cot x$.

… … … … … … … … …

Hence,

$$\log \sin (x + h) = \log \sin x + h \cot x - \frac{1}{2} h^2 \operatorname{cosec}^2 x + \frac{1}{3} h^3 \operatorname{cosec}^2 x \cot x + \dots$$

Ex. 10. *Use Taylor's theorem to prove that*

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\,\frac{\sin\theta}{1}-(h\sin\theta)^2\,\frac{\sin 2\theta}{2}$$
$$+(h\sin\theta)^3\,\frac{\sin 3\theta}{3}+\ldots+(-1)^{n-1}(h\sin\theta)^n\,\frac{\sin n\theta}{n}+\ldots$$

where $\theta=\cot^{-1}x$. **(Gorakhpur 2005; Agra 07; Rohilkhand 08B, 09B; Kashi 11; Avadh 13)**

Solution : Let $y=f(x)=\tan^{-1}x$.

Then $$y_1=\frac{1}{1+x^2}=\frac{1}{(x+i)(x-i)}=\frac{1}{2i}\left[\frac{1}{x-i}-\frac{1}{x+i}\right]$$

or $$y_1=\frac{1}{2i}[(x-i)^{-1}-(x+i)^{-1}]. \qquad \ldots(1)$$

Differentiating (1), $(n-1)$ times, we get

$$y_n=\frac{1}{2i}[(-1)^{n-1}(n-1)!\,(x-i)^{-n}-(-1)^{n-1}(n-1)!\,(x+i)^{-n}]$$

or $$y_n=\frac{(-1)^{n-1}(n-1)!}{2!}[(x-i)^{-n}-(x+i)^{-n}]. \qquad \ldots(2)$$

Now put $x=r\cos\theta,\ 1=r\sin\theta$ in (2). Then

$$y_n=\frac{(-1)^{n-1}(n-1)!}{2i}\,.\,r^{-n}[(\cos\theta-i\sin\theta)^{-n}-(\cos\theta+i\sin\theta)^{-n}]$$

$$=\frac{(-1)^{n-1}(n-1)!}{2i}\,r^{-n}[(\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)],$$

by De Moivre's theorem

$$=\frac{(-1)^{n-1}(n-1)!}{2i}\,r^{-n}.\,2i\sin n\theta$$

$$=(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta. \qquad [\because\ r^{-1}=1/r=\sin\theta]$$

Hence $f^{(n)}(x)=(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$,

where $\cot\theta=x$, *i.e.*, $\theta=\cot^{-1}x$.

Putting $n=1,2,3,\ldots$, we get

$$f'(x)=\sin\theta\,.\,\sin\theta,\ f''(x)=-\sin^2\theta\sin 2\theta,$$
$$f'''(x)=2!\,\sin^3\theta\sin 3\theta, \text{ and so on.}$$

Substituting these values in Taylor's series

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x)+\ldots, \text{ we get}$$

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\,.\,\sin\theta-\frac{h^2}{2!}\sin^2\theta\sin 2\theta$$
$$+\frac{h^3\,2!}{3!}\sin^3\theta\sin 3\theta-\ldots+\frac{h^n}{n!}(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta+\ldots$$

or $$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta.\,\frac{\sin\theta}{1}-(h\sin\theta)^2\,\frac{\sin 2\theta}{2}$$
$$+(h\sin\theta)^3\,.\,\frac{\sin 3\theta}{3}+\ldots+(-1)^{n-1}(h\sin\theta)^n.\,\frac{\sin n\theta}{n}+\ldots$$

Comprehensive Exercise 1

1. (i) State Maclaurin's theorem. **(Meerut 2000; Bundelkhand 01, 08, 11; Agra 07)**

(i) State Taylor's theorem. **(Bundelkhand 2006, 08, 11)**

Expand the following functions by Maclaurin's theorem :

2. (i) a^x **(Meerut 2012B)**

(ii) $\tan x$ **(Kanpur 2014)**

(iii) $e^{x \cos x}$

(iv) $\tan^{-1} x$ **(Bundelkhand 2001)**

(v) $\sec x$.

3. (i) Obtain by Maclaurin's theorem the first five terms in the expansion of $e^{\sin x}$.

(Bundelkhand 2007)

(ii) Expand by Maclaurin's theorem $\dfrac{e^x}{1+e^x}$ as far as the term x^3. **(Meerut 2006B)**

(iii) Obtain by Maclaurin's theorem the first five terms in the expansion of $\log(1+\sin x)$. **(Meerut 2007)**

(iv) Find the first three terms in the expansion in the powers of x of $\log(1+\tan x)$.

(Rohilkhand 2011B)

4. (i) Apply Maclaurin's theorem to prove that $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \dots$.

(Bundelkhand 2011; Rohilkhand 13)

(ii) Use Maclaurin's formula to show that $e^x \sec x = 1 + x + \dfrac{2x^2}{2!} + \dfrac{4x^3}{3!} + \dots$

(Meerut 2004; Rohilkhand 08B)

(iii) Expand $\sinh x \cos x$ to fifth powers of x.

5. Show that

(i) $$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \dots + \cos\left(\frac{1}{4}n\pi\right) \cdot \frac{2^{n/2}}{n!} x^n + \dots.$$

(Bundelkhand 2014; Agra 14)

(ii) $$e^x \sin x = x + x^2 + \frac{2}{3!}x^3 - \frac{2^2}{5!}x^5 - \dots + \sin\left(\frac{1}{4}n\pi\right)\frac{2^{n/2}}{n!}x^n + \dots$$

(Meerut 2003; Gorakhpur 06)

6. Apply Maclaurin's theorem to prove that

(i) $$e^{ax} \sin bx = bx + abx^2 + \frac{3a^2 b - b^3}{3!}x^3 + \dots + \frac{(a^2+b^2)^{n/2}}{n!}x^n \sin\left(n \tan^{-1}\frac{b}{a}\right) + \dots$$

(ii) $e^{ax}\cos bx = 1 + ax + \dfrac{a^2 - b^2}{2} x^2 + \dfrac{a(a^2 - 3b^2)}{3!} x^3 + \dots$

$$+ \frac{(a^2 + b^2)^{n/2}}{n!} x^n \cos\left(n \tan^{-1}\frac{b}{a}\right) + \dots .$$

7. Show that $e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \dfrac{x^2}{2!}\cos 2\alpha + \dfrac{x^3}{3!}\cos 3\alpha + \dots$

(Rohilkhand 2007; Avadh 11)

8. (i) Expand $\sin^{-1}(x + h)$ in powers of x as far as the term x^3.
[**Hint.** Use Taylor's series]

(ii) Prove that $\log(x + h) = \log h + \dfrac{x}{h} - \dfrac{x^2}{2h^2} + \dfrac{x^3}{3h^3} - \dots$

9. (i) Expand $\tan^{-1} x$ in powers of $\left(x - \frac{1}{4}\pi\right)$.

[**Hint.** Let $f(x) = \tan^{-1} x$. We can write
$f(x) = f\left[\frac{1}{4}\pi + \left(x - \frac{1}{4}\pi\right)\right]$. Now apply Taylor's theorem]

(ii) Expand $\sin\left(\frac{1}{4}\pi + \theta\right)$ in powers of θ.

(iii) Expand $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

(Meerut 2004B, 05B; Gorakhpur 06; Rohilkhand 09; Purvanchal 11; Kashi 11)

(iv) Write the value of α, if by Taylor's theorem

$$2x^3 + 7x^2 + x - 1 = \alpha + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3.$$

(Meerut 2001)

(v) Expand $\log\sin x$ in powers of $(x - a)$.

(Meerut 2001, 06; Rohilkhand 07B; Avadh 10)

10. (i) If $y = e^{a\sin^{-1}x}$, show that $(1 - x^2)\, y_{n+2} - (2n + 1)\, x y_{n+1} - (n^2 + a^2)\, y_n = 0$.
Hence by Maclaurin's theorem, show that

$$e^{a\sin^{-1}x} = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2 + a^2)}{3!}x^3 + \dots$$ **(Kumaun 2008)**

Also deduce that $e^{\theta} = 1 + \sin\theta + \dfrac{1}{2!}\sin^2\theta + \dfrac{2}{3!}\sin^3\theta + \dots$

(ii) If $y = \sin(m \sin^{-1} x)$, then show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$$

Hence or otherwise expand $\sin m\theta$ in powers of $\sin\theta$.

(iii) If $y = \sin\log(x^2 + 2x + 1)$, prove that

$$(x + 1)^2 y_{n+2} + (2n + 1)(x + 1)\, y_{n+1} + (n^2 + 4)\, y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as far as x^6.

11. By Maclaurin's theorem or otherwise find the expansion of $y = \sin(e^x - 1)$ upto and including the term in x^4. Find also the first two non-vanishing terms in the expansion of x as a series of ascending powers of y.

12. Expand $\log\{1 - \log(1 - x)\}$ in powers of x by Maclaurin's theorem as far as the term x^3. **(Avadh 2009)**

By substituting $\dfrac{x}{1 + x}$ for x deduce the expansion of $\log\{1 + \log(1 + x)\}$ as far as the term in x^3.

13. If $y = \sin^{-1} x / \sqrt{(1 - x^2)}$ when $-1 < x < 1$, and $-\frac{1}{2}\pi < \sin^{-1} x < \frac{1}{2}\pi$, prove that

$$(1 - x^2)\frac{d^{n+1} y}{dx^{n+1}} - (2n + 1)x\frac{d^n y}{dx^n} - n^2\frac{d^{n-1} y}{dx^{n-1}} = 0.$$ **(Meerut 2007B)**

Assuming that y can be expanded in ascending powers of x in the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

prove that $(n + 1) a_{n+1} = n a_{n-1}$, and hence obtain the general term of the expansion.

14. If $y = e^{m \tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$, prove that

$$(n + 1) a_{n+1} + (n - 1) a_{n-1} = m a_n .$$ **(Purvanchal 2007)**

15. Prove that

$$f(mx) = f(x) + (m - 1) x f'(x) + \frac{(m - 1)^2}{2!} x^2 f''(x) + \frac{(m - 1)^3}{3!} x^3 f'''(x) + \ldots$$

(Meerut 2001; Agra 07; Rohilkhand 13)

16. Prove that

(i) $f\left(\dfrac{x^2}{1 + x}\right) = f(x) - \dfrac{x}{1 + x} f'(x) + \dfrac{x^2}{(1 + x)^2}\dfrac{f''(x)}{2!} - \ldots$ **(Rohilkhand 2008)**

(ii) $f(x) = f(0) + x f'(x) - \dfrac{x^2}{2!} f''(x) + \dfrac{x^3}{3!} f'''(x) - \ldots$

[**Hint.** (i) Write $f\left(\dfrac{x^2}{1 + x}\right) = f\left(x - \dfrac{x}{1 + x}\right)$.

Now apply Taylor's theorem.

(ii) We have $f(0) = f(x - x)$.

Apply Taylor's theorem and transpose the terms to get the result.]

Answers 1

2. (i) $1 + x \log a + \dfrac{(x \log a)^2}{2!} + \ldots + \dfrac{(x \log a)^n}{n!} + \ldots$.

(ii) $x + \dfrac{x^3}{3} + \dfrac{2}{15} x^5 + \ldots$.

(iii) $1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{11 x^4}{24} - \dfrac{x^5}{5} + \ldots$.

(iv) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)} + \ldots$.

(v) $1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$.

3. (i) $1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$. (ii) $\frac{1}{2} + \frac{1}{4}x - \frac{x^3}{8}\frac{1}{3!} + \ldots$.

(iii) $x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \ldots$. (iv) $x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \ldots$.

4. (iii) $x - \frac{2x^3}{3!} - \frac{4x^5}{5!} + \ldots$.

8. $\sin^{-1} h + x(1-h^2)^{-1/2} + \frac{x^2}{2!} h(1-h^2)^{-3/2}$

$$+ \frac{x^3}{3!}\{(1-h^2)^{-5/2}(1+2h^2)\} + \ldots.$$

9. (i) $\tan^{-1}(\pi/4) + \left(x - \frac{1}{4}\pi\right) \Big/ (1 + \pi^2/16) - \pi\left(x - \frac{1}{4}\pi\right)^2 \Big/ \{4(1+\pi^2/16)^2\} + \ldots$.

(ii) $\frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \ldots\right)$.

(iii) $45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$.

(iv) 45.

(v) $\log \sin a + (x-a)\cot a - \frac{(x-a)^2}{2!}\operatorname{cosec}^2 a + \frac{(x-a)^3}{3!} 2\operatorname{cosec}^2 a \cot a + \ldots$.

10. (ii) $\sin m\theta = m\sin\theta + \frac{m(1^2 - m^2)}{3!}\sin^3\theta + \frac{m(1^2-m^2)(3^2-m^2)}{5!}\sin^5\theta + \ldots$.

(iii) $y = 2x - x^2 - \frac{2}{3}x^3 + \frac{3}{2}x^4 - \frac{5}{3}x^5 + \frac{3}{2}x^6 + \ldots$.

11. $x + \frac{x^2}{2!} - \frac{5x^4}{24} + \ldots,\ y - \frac{y^2}{2} + \ldots$

12. $x + \frac{x^3}{6} + \ldots,\ x - x^2 + \frac{7x^3}{6} + \ldots$

13. $a_{2m} = 0,\ a_{2m+1} = \frac{2m(2m-2)(2m-4)\ldots 2}{(2m+1)(2m-1)\ldots 3}$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. By Maclaurin's theorem expansion of $\sin^{-1} x$ is ....

2. By Maclaurin's theorem

$$y = \log(\sec x + \tan x) = x + \frac{x^3}{6} + \frac{x^5}{24} + \ldots\ldots, \text{ then } (y_3)_0 = \ldots\ldots.$$

3. By Maclaurin's theorem

$$y = e^x \sin x = x + x^2 + \frac{2}{3\,!} x^3 + \ldots\ldots + \frac{2^{n/2} \sin (n\,\pi/4)}{n\,!} x^n + \ldots\ldots,$$

then $(y_3)_0 = \ldots\ldots$. **(Meerut 2001)**

4. By Taylor's theorem the expansion of $\log (x + h)$ in ascending powers of x is …… .

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

5. By Taylor's theorem the second term in the expansion of $\log \sin x$ in the powers of $(x - a)$ is

(a) $(x - a) \cot a$ (b) $(x - b) \operatorname{cosec} a$

(c) $(x - a) \cot a \operatorname{cosec} a$ (d) $(x - a)^2 \cot a$

(Rohilkhand 2005)

6. By Maclaurin's theorem the second term in the expansion of $e^x/(1 + e^x)$ is

(a) $\dfrac{1}{48} x$ (b) $\dfrac{x}{4}$

(c) 0 (d) $-\dfrac{1}{48} x^3$ **(Rohilkhand 2007)**

True or False:

Write 'T' for true and 'F' for false statement.

7. The function $\log x$ does not possess Maclaurin's series expansion because it is not defined at $x = 0$.

8. If in Taylor's series we put $a = 0$ and $h = x$, we get Maclaurin's series.

9. Is this $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2\,!} f''(0) + \dfrac{x^3}{3\,!} f'''(0) + \ldots\ldots$ Taylor's Theorem.

(Agra 2005, 06)

Answers

1. $x + \dfrac{1^2 . x^3}{3\,!} + \dfrac{3^2 . 1^2 . x^5}{5\,!} + \dfrac{5^2 . 3^2 . 1^2 . x^7}{7\,!} + \ldots\ldots$

2. 1. **3.** 2. **4.** $\log h + \dfrac{x}{h} - \dfrac{x^2}{2h^2} + \dfrac{x^3}{3h^3} - \ldots\ldots$

5. (a). **6.** (b). **7.** *T*.

8. *T*. **9.** *F*.

Chapter

6

Indeterminate Forms

6.1 Indeterminate Forms

The form 0/0 has got no definite value. For if we write $0/0 = y$, then the equation $0y = 0$ reduces to an identity in y, *i.e.*, it is true for all values of y. We cannot cancel 0 from both sides. Therefore the form 0/0 is meaningless.

Now suppose $\lim_{x \to a} \phi(x) = 0$ and $\lim_{x \to a} \psi(x) = 0$.

Then we cannot write $\lim_{x \to a} \dfrac{\phi(x)}{\psi(x)} = \dfrac{\lim_{x \to a} \phi(x)}{\lim_{x \to a} \psi(x)}$

because in that case $\lim_{x \to a} \dfrac{\phi(x)}{\psi(x)}$ takes the form 0/0 which is meaningless. It, however, does not mean that if $\lim_{x \to a} \dfrac{\phi(x)}{\psi(x)}$ takes the form 0/0, then the limit itself does not exist.

For example, $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a}$ takes the form 0/0 if we write it as $\dfrac{\lim_{x \to a} (x^2 - a^2)}{\lim_{x \to a} (x - a)}$.

But, we have $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a} = \lim_{x \to a} \dfrac{(x - a)(x + a)}{(x - a)} = \lim_{x \to a} (x + a) = 2a$

and thus the limit exists.

Here it should not be confused that we have made an attempt to find the value of 0/0. We have simply evaluated the limit of a function which is the quotient of two functions such that if we take their limits separately, then the combination takes the form 0/0.

The form 0/0 is an **indeterminate form.** It has no definite value. The other indeterminate forms are ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0. In this chapter we shall discuss methods which enable us to evaluate the limits of indeterminate forms.

6.2 The Form 0/0

Suppose $\phi(x)$ and $\psi(x)$ are functions which can be expanded by Taylor's theorem in the neighbourhood of $x = a$. Also let $\phi(a) = 0$, and $\psi(a) = 0$. Then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

We have, by Taylor's theorem, $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)}$

$$= \lim_{x \to a} \frac{\phi(a) + (x-a)\phi'(a) + \dfrac{(x-a)^2}{2!}\phi''(a) + \ldots + R_1}{\psi(a) + (x-a)\psi'(a) + \dfrac{(x-a)^2}{2!}\psi''(a) + \ldots + R_2},$$

where $\quad R_1 = \dfrac{(x-a)^n}{n!}\phi^{(n)}\{a + \theta_1(x-a)\}, 0 < \theta_1 < 1,$

and $\quad R_2 = \dfrac{(x-a)^n}{n!}\psi^{(n)}\{a + \theta_2(x-a)\}, 0 < \theta_2 < 1.$

But, by hypothesis, $\phi(a) = 0$ and $\psi(a) = 0$.

Therefore, $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)} = \lim\limits_{x \to a} \dfrac{(x-a)\phi'(a) + \dfrac{(x-a)^2}{2!}\phi''(a) + \ldots + R_1}{(x-a)\psi'(a) + \dfrac{(x-a)^2}{2!}\psi''(a) + \ldots + R_2}.$

Dividing the numerator and denominator by $x - a$, we have

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(a) + (x-a)\left\{\dfrac{1}{2!}\phi''(a) + \dfrac{1}{3!}(x-a)\phi'''(a) + \ldots\right\}}{\psi'(a) + (x-a)\left\{\dfrac{1}{2!}\psi''(a) + \dfrac{1}{3!}(x-a)\psi'''(a) + \ldots\right\}}$$

$$= \frac{\phi'(a)}{\psi'(a)}, \quad \text{if } \phi'(a) \text{ and } \psi'(a) \text{ are not both zero}$$

$$= \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

This proves the theorem which is generally known as *L 'Hospital's Rule.*

It can be easily seen that if $\phi'(a)$, $\phi''(a)$,......, $\phi^{(n-1)}(a)$ and $\psi'(a)$, $\psi''(a)$,..., $\psi^{(n-1)}(a)$ are all zero, but $\phi^{(n)}(a)$ and $\psi^{(n)}(a)$ are not both zero, then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi^{(n)}(x)}{\psi^{(n)}(x)}.$$

The theorem of this article is true even if x tends to ∞ or $-\infty$ instead of a, *i.e.*, if

$$\lim_{x \to \infty} \phi(x) = 0, \quad \lim_{x \to \infty} \psi(x) = 0,$$

then
$$\lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lim_{x \to \infty} \frac{\phi'(x)}{\psi'(x)}.$$

Writing $x = 1/y$, we have as $x \to \infty$, $y \to 0$.

$$\therefore \quad \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lim_{y \to 0} \frac{\phi(1/y)}{\psi(1/y)} = \lim_{y \to 0} \frac{\phi'(1/y)\, y^{-2}}{\psi'(1/y)\, y^{-2}},$$

by L'Hospital's rule

$$= \lim_{y \to 0} \frac{\phi'(1/y)}{\psi'(1/y)} = \lim_{x \to \infty} \frac{\phi'(x)}{\psi'(x)}.$$

Note 1 : L'Hospital's rule implies

$$\lim_{x \to a+} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a-} \frac{\phi(x)}{\psi(x)}.$$

Note 2 : While applying L'Hospital's rule we are not to differentiate $\dfrac{\phi(x)}{\psi(x)}$ by the rule for finding the differential coefficient of the quotient of two functions. But we are to differentiate the numerator and denominator separately.

Note 3 : Important. Before applying L'Hospital's rule we must satisfy ourselves that the form is 0/0. Sometimes it happens that at some stage the resulting function is not indeterminate of the type 0/0 and we still apply L'Hospital's rule which is not justified in that case. This is a fairly common error.

Illustrative Examples

Example 1 : *Evaluate* $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$.

Solution : We have $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$ [form 0/0]

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - \dfrac{2}{(1+x)}}{\sin x + x \cos x} \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \dfrac{2}{(1+x)^2}}{\cos x + \cos x - x \sin x} = \frac{1 - 1 + 2}{1 + 1 - 0} = \frac{2}{2} = 1.$$

Example 2 : *Evaluate* $\displaystyle \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$.

Solution : We have $\displaystyle \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$ [form 0/0]

$$= \lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{5x^4} \quad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \frac{-\sin x + x}{20x^3} \qquad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \frac{-\cos x + 1}{60x^2} \qquad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \frac{\sin x}{120x} \qquad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \frac{\cos x}{120} = \frac{1}{120}.$$

6.3 Algebraic Methods

In many cases the limits are easily obtained by the use of well known algebraic and trigonometrical expansions. We can also make use of some well known limits in order to solve the problems or to shorten the work. The following expansions should be remembered :

(i) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\text{ad. inf.}, \ |x| < 1.$

(ii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\text{ad. inf.}, \ |x| < 1.$

(iii) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots\text{ad. inf.}$

(iv) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots\text{ad. inf.}$

(v) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \ldots\text{ad. inf.}$

(vi) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots\text{ad. inf.}$

(vii) $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots\text{ad. inf.}$

(viii) $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\text{ad. inf.}$

(ix) $\sin^{-1} x = x + 1^2 \cdot \dfrac{x^3}{3!} + 3^2 . 1^2 \cdot \dfrac{x^5}{5!} + 5^2 . 3^2 . 1^2 \cdot \dfrac{x^7}{7!} + \ldots\ldots$

The following values of logarithms to the base e should also be remembered :

$$\log 1 = 0, \log e = 1, \log \infty = \infty, \log 0 = -\infty.$$

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x\to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$. **(Meerut 2001)**

Solution : We have $\lim_{x\to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$

$$= \lim_{x\to 0} \frac{x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{2} - \frac{5}{6}x^3 + \ldots}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{1}{2} - \frac{5}{6}x + \text{ terms containing higher powers of } x \right) = \frac{1}{2}.$$

Example 2 : *Evaluate* $\lim_{x \to 0} \dfrac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$. **(Kanpur 2007; Rohilkhand 05)**

Solution : Here it should be noted that we cannot apply Hospital's rule since $x^{1/2}$ cannot be expanded by Taylor's theorem in the neighbourhood of $x = 0$. However, we can get the result by the use of algebraic methods. We have thus,

$$\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{\left(1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \ldots - 1\right)^{3/2}} = \lim_{x \to 0} \frac{x^{1/2} \tan x}{x^{3/2}\left(1 + \frac{x}{2\,!} + \ldots\right)^{3/2}}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\left(1 + \frac{x}{2\,!} + \ldots\right)^{3/2}}$$

$$= 1, \text{ since } \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \cos x = 1.$$

Example 3 : *Evaluate* $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e}{x}$. **(Rohilkhand 2013; Kashi 13)**

Solution : Here $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e}{x}$ is of the form $\dfrac{0}{0}$ because

$$\lim_{x \to 0} (1 + x)^{1/x} = e.$$

To evaluate the given limit first we shall obtain an expansion for $(1 + x)^{1/x}$ in ascending powers of x.

Let $y = (1 + x)^{1/x}$. Then

$$\log y = \frac{1}{x} \log (1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$= 1 + z \quad , \text{where } z = -(x/2) + (x^2/3) - \ldots$$

$$\therefore \quad y = e^{1 + z} = e \, . \, e^z = e \, . \left(1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \ldots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)^2 + \ldots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8}x^2 + \text{ terms containing powers of } x \text{ higher than } 3\right]$$

$$= e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right].$$

Now $$\lim_{x \to 0} \frac{(1 + x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right] - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left[-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\right]}{x} = \lim_{x \to 0} e\left[-\frac{1}{2} + \frac{11}{24}x + \ldots\right] = -\frac{1}{2}e.$$

Comprehensive Exercise 1

1. State L' Hospital's rule.

Evaluate the following limits :

2. (i) $\lim_{x \to 0} \frac{\sin x}{x}$.

(ii) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

(iii) $\lim_{x \to 0} \frac{e^x - 1}{x}$.

(iv) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$. **(Meerut 2012B)**

3. (i) $\lim_{x \to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$.

(ii) $\lim_{x \to 0} \frac{a^x - b^x}{x}$. **(Agra 2003)**

(iii) $\lim_{x \to 0} \frac{\log(1 - x^2)}{\log \cos x}$.

(iv) $\lim_{x \to 0} \frac{xe^x - \log(1 + x)}{x^2}$.

4. (i) $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x}$.

(ii) $\lim_{x \to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

(iii) $\lim_{x \to 0} \frac{(1 + x)^n - 1}{x}$.

(iv) $\lim_{x \to 1} \frac{\log x}{x - 1}$. **(Garhwal 2001)**

5. (i) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$.

(ii) $\lim_{x \to 0} \frac{\{1 - \sqrt{(1 - x^2)}\}}{x^2}$.

(iii) $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$. **(Avadh 2014)**

(iv) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$.

6. (i) $\lim_{x \to 0} \dfrac{\{\cosh x + \log(1 - x) - 1 + x\}}{x^2}$.

(ii) $\lim_{x \to 0} \dfrac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$. **(Meerut 2001)**

(iii) $\lim_{x \to 0} \dfrac{e^x + \log\left(\dfrac{1 - x}{e}\right)}{\tan x - x}$.

(iv) $\lim_{x \to 1} \dfrac{x \sqrt{(3x - 2x^4)} - x^{6/5}}{1 - x^{2/3}}$.

7. (i) $\lim_{x \to \pi/2} \dfrac{\cos x}{x - \frac{1}{2}\pi}$.

(ii) $\lim_{x \to a} \dfrac{a^x - x^a}{x^x - a^a}$.

(iii) $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$. **(Avadh 2006; Purvanchal 14)**

(iv) $\lim_{x \to 0} \dfrac{\sin x \,.\, \sin^{-1} x - x^2}{x^6}$.

8. (i) $\lim_{x \to 0} \dfrac{x^2 + 2 \cos x - 2}{x \sin^3 x}$.

(ii) $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x - \sin x}$.

(iii) $\lim_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$. **(Meerut 2012)**

(iv) $\lim_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^2}$.

9. (i) $\lim_{x \to 0} \dfrac{\sin^2 x - x^2}{x^4}$.

(ii) $\lim_{x \to 0} \dfrac{\sin x \sin^{-1} x}{x^2}$.

10. Find the values of a and b in order that $\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$, may be equal to 1. **(Meerut 2013B)**

11. Find the values of a, b, c so that $\lim_{x \to 0} \dfrac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$.

12. Find the value of a, b and c so that $\lim_{x \to 0} \dfrac{x(a + b \cos x) - c \sin x}{x^5} = 1$.

Answers 1

2. (i) 1. (ii) 1/6. (iii) 1. (iv) 1/2.

3. (i) 4. (ii) $\log(a/b)$. (iii) 2. (iv) 3/2.

4. (i) 1/6. (ii) 4. (iii) n. (iv) 1.
5. (i) 1. (ii) $\frac{1}{2}$. (iii) 1/3. (iv) 2.
6. (i) 0. (ii) -15. (iii) $-\frac{1}{2}$. (iv) 81/20.
7. (i) -1. (ii) $\dfrac{\log a-1}{\log a+1}$. (iii) $11e/24$. (iv) 1/18.
8. (i) 1/12. (ii) 3. (iii) 1.
   (iv) Infinite if $a \neq -2$ and 0 if $a=-2$.
9. (i) $-\frac{1}{3}$. (ii) 1.
10. $a=-5/2, b=-3/2$.
11. $a=1, b=2, c=1$.
12. $a=120,\ b=60,\ c=180$.

6.4 The Form $\frac{\infty}{\infty}$

Suppose $\lim_{x \to a} \phi(x)=\infty$ *and* $\lim_{x \to a} \psi(x)=\infty$.

Then $\displaystyle \lim_{x \to a} \frac{\phi(x)}{\psi(x)}=\lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}$.

We have
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)}=\lim_{x \to a} \frac{1/\psi(x)}{1/\phi(x)} \qquad \text{[Form 0/0]}$$
$$=\lim_{x \to a} \frac{\dfrac{-\psi'(x)}{[\psi(x)]^2}}{\dfrac{-\phi'(x)}{[\phi(x)]^2}} \qquad \text{[by L'Hospital's rule]}$$
$$=\lim_{x \to a} \left[\frac{\psi'(x)}{\phi'(x)} \cdot \left\{\frac{\phi(x)}{\psi(x)}\right\}^2\right].$$

Thus,
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)}=\lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \cdot \left\{\lim_{x \to a} \frac{\phi(x)}{\psi(x)}\right\}^2. \qquad ...(1)$$

Now suppose
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)}=\lambda. \qquad ...(2)$$

Then three cases arise.

Case I : λ is neither zero nor infinite. In this case dividing both sides of (1) by λ^2, we get
$$\lambda^{-1}=\lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \quad \text{or} \quad \lambda=\lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Case II : $\lambda=0$. In this case adding 1 to each side of equation (2), we get
$$\lambda+1=\lim_{x \to a} \frac{\phi(x)}{\psi(x)}+1=\lim_{x \to a} \left\{\frac{\phi(x)}{\psi(x)}+1\right\}$$
$$=\lim_{x \to a} \frac{\phi(x)+\psi(x)}{\psi(x)}=\lim_{x \to a} \frac{\phi'(x)+\psi'(x)}{\psi'(x)}$$
$$\left\{\text{by case I, since form is } \frac{\infty}{\infty} \text{ and } \lambda+1 \neq 0\right\}$$

$$= \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)} + 1.$$

Therefore $$\lambda = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Case III : $\lambda = \infty$. In this case, we have

$$\frac{1}{\lim_{x \to a} \frac{\phi(x)}{\psi(x)}} = \lim_{x \to a} \frac{\psi(x)}{\phi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}. \qquad \text{[by case II]}$$

Therefore $$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Hence in every case in which $\lim_{x \to a} \phi(x) = \infty$ and $\lim_{x \to a} \psi(x) = \infty$, we get

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Note 1 : By writing $x = 1/y$, we can show as in § 2 that the proposition of this article is also true when $x \to \infty$ or $-\infty$ in place of a.

Note 2 : Obviously the proposition of this article is true when one or both the limits are $-\infty$.

Important : We have seen that in both cases when the form is ∞/∞ or $0/0$ the rule of evaluating the limit by differentiating the numerator and denominator separately holds good. Also we can easily convert the form ∞/∞ to the form $0/0$ and vice-versa. Therefore at every stage we should note carefully that which form will be more suitable to evaluate the limit most quickly. Moreover in some cases it will be necessary to convert the form ∞/∞ to the form $0/0$, otherwise the process of differentiating the numerator and the denominator would never terminate.

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x \to 0} \frac{\log x}{\cot x}$. **(Agra 2014; Purvanchal 14)**

Solution : We have, $$\lim_{x \to 0} \frac{\log x}{\cot x} \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x} \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos x}{1} = \frac{-2 \times 0 \times 1}{1} = 0.$$

Example 2 : *Evaluate* $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$. **(Agra 2001)**

Solution : We have $$\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x} \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \cdot \frac{(2/\sin 2x) \cdot \cos 2x}{(1/\sin x) \cdot \cos x} = \lim_{x \to 0} \frac{2 \cot 2x}{\cot x} \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{-4 \operatorname{cosec}^2 2x}{-\operatorname{cosec}^2 x} \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{4 \sin^2 x}{\sin^2 2x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{4 \sin^2 x}{(2 \sin x \cos x)^2}$$

$$= \lim_{x \to 0} \frac{1}{\cos^2 x} = 1.$$

Example 3 : *Evaluate* $\lim_{x \to 0} \dfrac{\log \log (1 - x^2)}{\log \log \cos x}$. **(Kumaun 2000; Avadh 13)**

Solution : We have, $\lim_{x \to 0} \dfrac{\log \log (1 - x^2)}{\log \log \cos x}$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{\dfrac{1}{\log (1 - x^2)} \cdot \dfrac{1}{1 - x^2} \cdot (-2x)}{\dfrac{1}{\log \cos x} \cdot \dfrac{1}{\cos x} \cdot (-\sin x)}.$$

$$= 2 \lim_{x \to 0} \frac{x \cos x \log \cos x}{\sin x \,.\, (1 - x^2) \log (1 - x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1 - x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)} \qquad \text{[form 0/0]}$$

$$= 2 \lim_{x \to 0} \frac{\dfrac{1}{\cos x} \cdot (-\sin x)}{\dfrac{1}{1 - x^2} \cdot (-2x)}$$

$$= 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1 - x^2}{\cos x} \right)$$

$$= 1.$$

Comprehensive Exercise 2

Evaluate the following limits :

1. (i) $\lim_{x \to \infty} \dfrac{x}{e^x}$.

(ii) $\lim_{x \to \infty} \dfrac{e^x}{x^3}$.

2. (i) $\lim_{x \to 1} \dfrac{\log (1 - x)}{\cot \pi x}$.

(ii) $\lim_{x \to \infty} \dfrac{\log x}{a^x}$, $a > 1$.

3. (i) $\lim_{x \to a} \dfrac{\log (x - a)}{\log (e^x - e^a)}$.

(ii) $\lim_{x \to \frac{1}{2}} \dfrac{\sec \pi x}{\tan 3 \pi x}$.

4. (i) $\lim_{x \to \infty} \left\{ \dfrac{(\log x)^3}{x} \right\}$.

(ii) $\lim_{x \to \infty} \dfrac{x (\log x)^3}{1 + x + x^2}$.

5. (i) $\lim_{x \to \infty} x \tan \dfrac{1}{x}$. **(Bundelkhand 2001)**

(ii) $\lim_{x \to 0} \dfrac{\log \tan 2x}{\log \tan 3x}$.

6. (i) $\lim_{x \to \pi/2} \dfrac{\log \left(x - \frac{1}{2} \pi \right)}{\tan x}$.

(ii) $\lim_{x \to a} \dfrac{c \{ e^{1/(x - a)} - 1 \}}{\{ e^{1/(x - a)} + 1 \}}$.

7. (i) $\lim_{x \to 1} \left\{ \dfrac{2}{x^2 - 1} - \dfrac{1}{x - 1} \right\}$. **(Garhwal 2002)**

(ii) $\lim_{x \to \pi/2} (\sec x - \tan x)$.

Answers 2

1. (i) 0. (ii) ∞. **2.** (i) 0. (ii) 0.
3. (i) 1. (ii) 3 **4.** (i) 0 (ii) 0.
5. (i) 1 (ii) 1. **6.** (i) 0. (ii) c.
7. (i) $-\frac{1}{2}$. (ii) 0.

6.5 The Form $\infty - \infty$

This form can be easily reduced to the form $\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$.

Suppose $\lim_{x \to a} \phi (x) = \infty$ *and* $\lim_{x \to a} \psi (x) = \infty$.

Then $\lim_{x \to a} \{ \phi (x) - \psi (x) \}$ [form $\infty - \infty$]

$$= \lim_{x \to a} \left\{ \frac{1}{1/\phi (x)} - \frac{1}{1/\psi (x)} \right\}$$

$$= \lim_{x \to a} \frac{\dfrac{1}{\psi (x)} - \dfrac{1}{\phi (x)}}{\dfrac{1}{\phi (x) \,.\, \psi (x)}}, \text{ which is of the form } \frac{0}{0}.$$

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$.

Solution : We have, $\lim_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$ [form $\infty - \infty$]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{1}{1 - \sin x} \right) \qquad \text{[form } \infty - \infty]$$

$$= \lim_{x \to \pi/2} \frac{1 - \sin x - \cos x}{\cos x - \cos x \sin x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-\cos x + \sin x}{-\sin x + \sin^2 x - \cos^2 x} = \frac{-0 + 1}{-1 + 1 - 0} = \infty.$$

Example 2 : *Evaluate* $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$. **(Kanpur 2011)**

Solution : We have, $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$ [form $\infty - \infty$]

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\left(x - \dfrac{x^3}{3\,!} + \ldots \right)^2 - x^2}{x^2 \left(x - \dfrac{x^3}{3\,!} + \ldots \right)^2}$$

$$= \lim_{x \to 0} \frac{-\dfrac{2x^4}{3\,!} + \text{terms containing higher powers of } x}{x^4 + \text{terms containing higher powers of } x}$$

$$= \lim_{x \to 0} \frac{-\dfrac{2}{3\,!} + \text{terms containing powers of } x \text{ only in the numerator}}{1 + \text{terms containing powers of } x \text{ only in the numerator}}$$

$$= \frac{-2}{3\,!} = -\frac{1}{3}.$$

6.6 The Form $0 - \infty$

This form can be easily reduced to the form $\dfrac{0}{0}$ *or to the form* $\dfrac{\infty}{\infty}$.

Suppose $\lim_{x \to a} \phi(x) = 0$ *and* $\lim_{x \to a} \psi(x) = \infty$.

Then $\lim_{x \to a} \phi(x) . \psi(x)$ [form $0 \times \infty$]

$$= \lim_{x \to a} \frac{\phi(x)}{\dfrac{1}{\psi(x)}} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to a} \frac{\psi(x)}{\dfrac{1}{\phi(x)}} \qquad [\text{form } \infty/\infty]$$

We shall reduce the form $0 \times \infty$ to the form $0/0$ or ∞/∞ according to our convenience.

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x \to 0} x \log \sin x.$

Solution : We have, $\lim_{x \to 0} x \log \sin x$ [form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log \sin x}{1/x} \qquad [\text{form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{(1/\sin x) \,.\, \cos x}{-1/x^2} \qquad [\text{form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x} \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{x^2 \sin x - 2x \cos x}{\cos x} = 0.$$

Comprehensive Exercise 3

1. (i) $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{x}{\log x} \right).$

(ii) $\lim_{x \to 0} \left(\frac{\cot x - \frac{1}{x}}{x} \right).$

2. (i) $\lim_{x \to 0} \left(\frac{\operatorname{cosec} x - \cot x}{x} \right).$

(ii) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right).$

3. (i) $\lim_{x \to \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x \right).$

(ii) $\lim_{x \to 0} x \log x.$

4. (i) $\lim_{x \to 0} \sin x . \log x.$

(ii) $\lim_{x \to \infty} x (a^{1/x} - 1).$

5. (i) $\lim_{x \to \infty} 2^x \sin \frac{a}{2^x}.$ **(Agra 2003)**

(ii) $\lim_{x \to \pi/2} (1 - \sin x) \tan x.$

6. $\lim_{x \to 0} x^m (\log x)^n$, where m and n are positive integers.

Answers 3

1. (i) -1. (ii) $-\frac{1}{3}$. **2.** (i) $\frac{1}{2}$. (ii) $-\frac{1}{2}$.

3. (i) -1. (ii) 0. **4.** (i) 0. (ii) $\log a$.

5. (i) a. (ii) 0. **6.** 0.

6.7 The Forms 1^∞, 0^0, ∞^0

Suppose $\lim_{x \to a} \{\phi(x)\}^{\psi(x)}$ *takes any one of these three forms.*

Then let $$y = \lim_{x \to a} \{\phi(x)\}^{\psi(x)}.$$

Taking logarithm of both sides, we get $\log y = \lim_{x \to a} \psi(x) . \log \phi(x)$.

Now in any of the above three cases, $\log y$ takes the form $0 \times \infty$ which can be evaluated by the process of aricle 6.6.

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x \to 0} (\cos x)^{\cot^2 x}$.; **(Kumaun 2001; Bundelkhand 14)**

Solution : Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$. [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to 0} \cot^2 x . \log \cos x \qquad \text{[form } \infty \times 0]$$

$$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{(1/\cos x) . (-\sin x)}{2 \tan x \sec^2 x} \qquad \text{(by L'Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \sec^2 x} = \lim_{x \to 0} \frac{1}{-2 \sec^2 x} = -\frac{1}{2}.$$

$$\therefore \quad y = e^{-1/2}.$$

Example 2 : *Evaluate* $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$.

Solution : Let $y = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$. [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to \infty} \left\{x \log\left(1 + \frac{a}{x}\right)\right\} \qquad \text{[form } \infty \times 0]$$

$$= \lim_{x \to \infty} \frac{\log(1 + a/x)}{1/x} \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{1 + a/x} \cdot (-a/x^2)}{-1/x^2} = \lim_{x \to \infty} \frac{a}{1 + a/x} = a.$$

Therefore, $\quad y = e^a.$

6.8 Compound Forms

Suppose a function is the product of two or more factors the limit of each of which can be easily found. Then the limit of the entire function will be equal to the product of the limits of the factors provided that the product is not in itself an indeterminate form. A similar rule is applicable in the case of a sum, difference, quotient or power.

Illustrative Examples

Example 1 : *Evaluate* $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1 + x) \right].$

Solution : We have, $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1 + x) \right] = \lim_{x \to 0} \frac{x - \log(1 + x)}{x^2}$ [form 0/0]

$$= \lim_{x \to 0} \frac{1 - \frac{1}{1 + x}}{2x}, \text{ by L'Hospital's rule} \quad \text{[The form is again 0/0]}$$

$$= \lim_{x \to 0} \frac{\frac{1}{(1 + x)^2}}{2} = \frac{1}{2}.$$

Example 2 : *Evaluate* $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}.$

(Meerut 2000; Garhwal 01; Kumaun 02; Agra 03; Kashi 14; Purvanchal 14)

Solution : Let $y = \lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}.$

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \left(\frac{\tan x}{x} \right) = \lim_{x \to 0} \frac{1}{x^2} \log \left[\frac{1}{x} \left(x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots \right) \right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left[1 + \left(\frac{x^2}{3} + \frac{2}{15} x^4 + \ldots \right) \right]$$

$$= \lim_{x \to 0} \frac{\left(\frac{x^2}{3} + \frac{2}{15} x^4 + \ldots \right) - \frac{1}{2} \left(\frac{x^2}{3} + \frac{2}{15} x^4 + \ldots \right)^2 + \ldots}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{3} + \text{terms containing higher powers of } x}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{1}{3} + \text{terms containing powers of } x \text{ only in the numerator} \right]$$

$$= \frac{1}{3}.$$

$$\therefore \quad y = e^{1/3}.$$

Comprehensive Exercise 4

1. (i) $\lim_{x \to 0} x^x$. **(Agra 2002; Kanpur 04)**

(ii) $\lim_{x \to 0} (\cos x)^{1/x}$.

2. (i) $\lim_{x \to 0} (\cos x)^{1/x^2}$. **(Garhwal 2003)**

(ii) $\lim_{x \to \pi/2} (\sin x)^{\tan x}$.

3. (i) $\lim_{x \to \pi/2} (\sec x)^{\cot x}$.

(ii) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$.

4. (i) $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan (\pi x/2a)}$ **(Rohilkhand 2012)**

(ii) $\lim_{x \to 1} (1 - x^2)^{1/\log(1 - x)}$.

5. (i) $\lim_{x \to 1} x^{1/(1 - x)}$.

(ii) $\lim_{x \to \infty} (a_0 x^m + a_1 x^{m-1} + \ldots + a_m)^{1/x}$.

6. (i) $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$. **(Garhwal 2001, 03)**

(ii) $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^3}$.

7. (i) $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$. **(Kumaun 2008)**

(ii) $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$. **(Kumaun 2003)**

8. (i) $\lim_{x \to 0} \left\{\frac{2(\cosh x - 1)}{x^2}\right\}^{1/x^2}$.

(ii) $\lim_{x \to \infty} \left\{\frac{\log x}{x}\right\}^{1/x}$.

9. (i) $\lim_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x\right)^{1/x}$.

(ii) $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$.

10. $\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$.

Answers 4

1.	(i) 1.	(ii) 1.	**2.** (i) $e^{-1/2}$.	(ii) 1.
3.	(i) 1.	(ii) $1/e$.	**4.** (i) $e^{2/\pi}$.	(ii) e.
5.	(i) $1/e$.	(ii) 1.	**6.** (i) 1.	(ii) ∞.
7.	(i) $e^{1/6}$.	(ii) $e^{-1/6}$.	**8.** (i) $e^{1/12}$.	(ii) 1.
9.	(i) 1.	(ii) $1/e$.	**10.** $2/\pi$.	

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "……", so that the following statements are complete and correct.

1. $\lim_{x \to 0} \dfrac{\sin ax}{\sin bx} =$ …… .

2. $\lim_{x \to 0} \dfrac{\log(1 + kx^2)}{1 - \cos x} =$ …… .

3. $\lim_{x \to \infty} \dfrac{x^2 + 2x}{5 - 3x^2} =$ …… .

4. $\lim_{x \to 1} \left(\sec \dfrac{\pi}{2x}\right) \cdot \log x =$ …… .

5. The value of $\lim_{x \to 0} \dfrac{x - \tan x}{x^3} =$ …… . **(Agra 2002)**

6. The value of $\lim_{x \to 0} \dfrac{a^x - b^x}{x} =$ …… . **(Agra 2003)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. The value of $\lim_{x \to 0} \dfrac{1 - \cos x}{3x^2}$ is

(a) 0 (b) $\dfrac{2}{3}$

(c) $\dfrac{1}{6}$ (d) 1 **(Garhwal 2002)**

8. Which of the following is not an indeterminate form ?

(a) $\dfrac{\infty}{\infty}$ (b) $0 \times \infty$

(c) 1^0 (d) 0^0

9. The value of the $\lim_{x \to 0} \dfrac{\log \tan x}{\log x}$ is

(a) 0 (b) 1

(c) –1 (d) None of these

10. Which of the following is an indeterminate form ?

(a) $\infty + \infty$ (b) $\infty \times \infty$

(c) 1^{∞} (d) 0^{∞}

11. The value of $\lim\limits_{x \to 0} \dfrac{a^x - 1 - x \log_e a}{x^2}$ is

(a) $(\log_e a)^2$ (b) $\dfrac{(\log_e a)}{2}$

(c) $a - \log_e a$ (d) $\dfrac{(\log_e a)^2}{2}$ **(Garhwal 2001)**

12. The value of $\lim\limits_{x \to 1} \dfrac{\log x}{x - 1}$ is

(a) -1 (b) ∞ (c) 1 (d) 0

13. The value of $\lim\limits_{x \to \infty} \dfrac{\log_e x}{a^x}, a > 1$ is

(a) $\dfrac{1}{\log_e a}$ (b) a (c) 1 (d) 0

(Garhwal 2003)

True or False:

Write 'T' for true and 'F' for false statement.

14. While applying L' Hospital's rule to evaluate $\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)}$, if the form is $\dfrac{0}{0}$, we are to differentiate $f(x)/\phi(x)$ as a fraction.

15. The indeterminate form $\dfrac{\infty}{\infty}$ can be easily converted to the form $\dfrac{0}{0}$ and vice-versa.

16. If $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} \phi(x) = \infty$, then $\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{\phi'(x)}$.

17. $\lim\limits_{x \to a} \dfrac{(1 + x)^{1/x} - e + \dfrac{ex}{2}}{x^2}$ is of the form $\dfrac{\infty}{0}$. **(Meerut 2001)**

Answers

1. $\dfrac{a}{b}$.	**2.** $2k$.	**3.** $-\dfrac{1}{3}$.
4. $\dfrac{2}{\pi}$.	**5.** $-\dfrac{1}{3}$.	**6.** $\log \dfrac{a}{b}$.
7. (c).	**8.** (c).	**9.** (b).
10. (c).	**11.** (d).	**12.** (c).
13. (d)	**14.** *F*.	**15.** *T*.
16. *T*.	**17.** *F*.	

Chapter

7

Partial Differentiation

7.1 Functions of Several Independent Variables

So far we have considered functions of one independent variable only. However, in practice, we often come across functions of more than one independent variable. For example, the area of a rectangle depends upon two independent variables, namely the length and the breadth. Similarly, the volume of a rectangular parallelopiped depends upon three independent variables, namely the length, the breadth and the height.

There are a number of differences between the calculus of one and of two variables. Fortunately the calculus of functions of three or more variables differs only slightly from that of functions of two variables. The study here will be limited largely to functions of two variables.

Definition : *Let z be a symbol which has one definite value for every pair of values of x and y. Then z is called a function of the two independent variables x and y, and is usually written as $z = f(x, y)$. A function of x and y is also written as $\phi(x, y)$ or $\psi(x, y)$ etc.*

(Kanpur 2014)

A similar definition can be given for functions of more than two independent variables.

If to each point (x, y), of a part of the xy-plane is assigned a unique real number z, even then z is said to be given as a function, $z = f(x, y)$, of the independent variables x and y. The locus of all points (x, y, z) satisfying $z = f(x, y)$ is a surface in ordinary space.

7.2 Continuity of a Function of Two Variables

A function $f(x, y)$ is said to have a limit A as $x \to a$ and $y \to b$ if for any arbitrarily chosen positive number ε, however small (but not zero), there exists a corresponding number $\delta > 0$ such that

$$|f(x, y) - A| < \varepsilon,$$

for all values of x and y satisfying $0 < \sqrt{\{(x-a)^2 + (y-b)^2\}} < \delta$.

Here $0 < \sqrt{\{(x-a)^2 + (y-b)^2\}} < \delta$ defines a deleted neighbourhood of (a, b), namely all points except (a, b) lying within a circle of radius δ and centre (a, b).

A function $f(x, y)$ is said to be continuous at (a, b) provided $f(a, b)$ is defined and $\lim_{x \to a, y \to b} f(x, y) = f(a, b)$.

7.3 Partial Differential Coefficients

Suppose $z = f(x, y)$ is a function of two independent variables x and y. Since x and y are independent, we may (i) allow x to vary while y is kept fixed, (ii) allow y to vary while x is kept fixed, (iii) allow x and y to vary simultaneously. In the first two cases, z is practically a function of a single variable and we can differentiate it in accordance with the usual rules.

The partial differential coefficient of $z = f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ with respect to x when y is regarded as a constant. It is usually written as

$$\frac{\partial f}{\partial x} \text{ or } \frac{\partial z}{\partial x} \text{ or } f_x.$$

Thus, $\dfrac{\partial f}{\partial x} = \lim_{\delta x \to 0} \dfrac{f(x + \delta x, y) - f(x, y)}{\delta x}$, provided the limit exists.

Similarly, the partial differential coefficient of $z = f(x, y)$ with respect to y is the ordinary differential coefficient of $f(x, y)$ with respect to y when x is kept as constant. It is written as

$$\frac{\partial f}{\partial y} \quad \text{or} \quad \frac{\partial z}{\partial y} \quad \text{or} \quad f_y.$$

In a similar manner, if $z = f(x_1, x_2, \ldots, x_n)$ be a function of n independent variables $x_1, x_2, \ldots, x_n$, then the partial differential coefficient of z with respect to x_1, is the ordinary differential coefficient of z with respect to x_1, when all the variables except x_1 are regarded as constants. We shall write it as

$$\frac{\partial z}{\partial x_1} \quad \text{or} \quad \frac{\partial f}{\partial x_1}.$$

7.4 Partial Differential Coefficients of Higher Orders

The partial differential coefficient $\dfrac{\partial z}{\partial x}$ of $z = f(x, y)$ may again be differentiated partially with respect to x and to y, thus giving the second partial differential coefficients

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y\,\partial x} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

Similarly, from $\frac{\partial z}{\partial y}$ may be obtained

$$\frac{\partial^2 z}{\partial x\,\partial y} = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right).$$

If $z = f(x, y)$ and its partial derivatives are continuous (as is true in all ordinary cases), the order of differentiation is immaterial, that is,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}.$$

Illustrative Examples

Example 1 : *If* $u = ax^2 + 2hxy + by^2$, *find* $\frac{\partial^2 u}{\partial y\,\partial x}$ *and* $\frac{\partial^2 u}{\partial x\,\partial y}$.

Solution : We have, $u = ax^2 + 2hxy + by^2$.

$\therefore \quad \frac{\partial u}{\partial x} = 2ax + 2hy.$ (treating y as constant)

Hence $\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy)$

$= 2h.$ (treating x as constant)

Again $\frac{\partial u}{\partial y} = 2hx + 2by$ (treating x as constant)

$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by)$

$= 2h.$ (treating y as constant)

Here we note that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$.

Example 2 : *If* $u = f\left(\frac{y}{x}\right)$, *show that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$ **(Agra 2003)**

Solution : We have, $u = f\left(\frac{y}{x}\right)$.

$\therefore \quad \frac{\partial u}{\partial x} = \left\{f'\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right).$ (treating y as constant)

$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{y}{x}f'\left(\frac{y}{x}\right).$ ...(1)

Again $\frac{\partial u}{\partial y} = \left\{f'\left(\frac{y}{x}\right)\right\}\left(\frac{1}{x}\right).$ (treating x as constant)

$\therefore \quad y\frac{\partial u}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right).$ ...(2)

Adding (1) and (2), we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

Example 3 : *If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that*

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}.$$

(Kanpur 2007; Purvanchal 07; Rohilkhand 12; Avadh 13; Kashi 14)

Solution : We have, $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

Differentiating partially with respect to x, we have

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

or
$$\frac{\partial u}{\partial x} = \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}. \qquad \ldots(1)$$

Similarly, by symmetry, we have

$$\frac{\partial u}{\partial y} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz}, \qquad \ldots(2)$$

and
$$\frac{\partial u}{\partial z} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}. \qquad \ldots(3)$$

Adding (1), (2) and (3), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)} = \frac{3}{x + y + z}.$$

Now
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x + y + z}\right)$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x + y + z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x + y + z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x + y + z}\right)\right]$$

$$= 3\left[-\frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2} - \frac{1}{(x + y + z)^2}\right] = -\frac{9}{(x + y + z)^2}.$$

Example 4 : *If $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}, \quad \frac{\partial x}{r\partial\theta} = r\frac{\partial\theta}{\partial x},$$

and
$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Also find the value of $\dfrac{\partial\theta}{\partial x}$.

(Garhwal 2002)

Solution : We have $x = r\cos\theta$.

$\therefore \quad \dfrac{\partial x}{\partial r} = \cos\theta. \qquad$ (regarding θ as constant)

Also we have, $r^2 = x^2 + y^2$.

$\therefore \quad 2r\dfrac{\partial r}{\partial x} = 2x \qquad$ (regarding y as constant) or $$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

Thus $$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}\cdot$$

Again, $$\frac{\partial x}{\partial\theta} = -\,r\sin\theta. \qquad \text{(regarding } r \text{ as constant)}$$

$$\therefore \qquad \frac{1}{r}\frac{\partial x}{\partial\theta} = -\,\sin\theta.$$

Also we have, $\theta = \tan^{-1}(y/x)$.

$$\therefore \qquad \frac{\partial\theta}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}\cdot$$

$$\therefore \qquad r\frac{\partial\theta}{\partial x} = -\,\sin\theta.$$

Hence $$\frac{\partial x}{r\partial\theta} = r\frac{\partial\theta}{\partial x}\cdot$$

Finally, we have $\theta = \tan^{-1}(y/x)$.

$$\therefore \qquad \frac{\partial\theta}{\partial x} = -\frac{y}{(x^2+y^2)}\cdot \qquad \therefore \qquad \frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}\cdot \qquad \ldots(1)$$

Also $$\frac{\partial\theta}{\partial y} = \frac{1}{\left(1+\dfrac{y^2}{x^2}\right)}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}\cdot \qquad \therefore \qquad \frac{\partial^2\theta}{\partial y^2} = \frac{-\,2xy}{(x^2+y^2)^2}\cdot \qquad \ldots(2)$$

Adding (1) and (2), we get $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0.$

Example 5 : *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that*

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$$

Solution : Here $u = (1 - 2xy + y^2)^{-1/2}$.

$$\therefore \qquad \frac{\partial u}{\partial x} = -\,\tfrac{1}{2}(1-2xy+y^2)^{-3/2}(-\,2y) = yu^3,$$

and $$\frac{\partial u}{\partial y} = -\,\tfrac{1}{2}(1-2xy+y^2)^{-3/2}\,.\,(-\,2x+2y) = (x-y)\,u^3.$$

Now $$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} = \frac{\partial}{\partial x}\left\{(1-x^2)\,.\,yu^3\right\}$$

$$= y(-\,2x)\,u^3 + y(1-x^2)\,.\,3u^2\frac{\partial u}{\partial x}$$

$$= -\,2xy\,u^3 + 3y(1-x^2)\,u^2\,yu^3$$

$$= -\,2xy\,u^3 + 3y^2u^5(1-x^2). \qquad \ldots(1)$$

Also $$\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{y^2(x-y)\,u^3\right\} = \frac{\partial}{\partial y}\{(y^2x-y^3)\,u^3\}$$

$$= (2xy-3y^2)\,u^3 + (y^2x-y^3)\,.\,3u^2\frac{\partial u}{\partial y}$$

$$= (2xy-3y^2)\,u^3 + y^2(x-y)\,.\,3u^2\,.\,(x-y)\,u^3$$

$$= (2xy - 3y^2)\, u^3 + y^2 (x - y)^2 \,.\, 3u^5$$
$$= 2xyu^3 + 3y^2 u^5 [(x - y)^2 - u^{-2}]$$
$$= 2xyu^3 + 3y^2 u^5 [(x - y)^2 - (1 - 2xy + y^2)], \qquad [\because u^{-2} = 1 - 2xy + y^2]$$
$$= 2xyu^3 + 3y^2 u^5 [x^2 - 1] = 2xyu^3 - 3y^2 u^5 (1 - x^2). \qquad \ldots(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$

Example 6 : *If* $\theta = t^n e^{-r^2/4t}$, *what value of n will make* $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$ *?*

(Garhwal 2003; Lucknow 11)

Solution : We have $\dfrac{\partial \theta}{\partial r} = t^n \,.\, e^{-r^2/4t} \,.\left(-\dfrac{2r}{4t}\right) = -\dfrac{r}{2} t^{n-1} e^{-r^2/4t}.$

$$\therefore \qquad r^2 \frac{\partial \theta}{\partial r} = -\tfrac{1}{2} r^3 t^{n-1} e^{-r^2/4t}.$$

$$\therefore \qquad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{3r^2}{2} t^{n-1} e^{-r^2/4t} - \tfrac{1}{2} r^3 t^{n-1} e^{-r^2/4t} \,.\left(-\frac{2r}{4t}\right)$$
$$= -\tfrac{3}{2} r^2 t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^4 t^{n-2} e^{-r^2/4t}.$$

$$\therefore \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\tfrac{3}{2} t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$$

Also $$\frac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} \,.\, \frac{r^2}{4t^2}$$
$$= nt^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$$

Now $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$$

$$\Rightarrow \qquad -\tfrac{3}{2} t^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t} = nt^{n-1} e^{-r^2/4t} + \tfrac{1}{4} r^2 t^{n-2} e^{-r^2/4t}$$

$\Rightarrow \qquad -\tfrac{3}{2} t^{-n-1} e^{-r^2/4t} = nt^{n-1} e^{-r^2/4t}$, for all possible values of r and t

$$\Rightarrow \qquad n = -\tfrac{3}{2}.$$

Comprehensive Exercise 1

1. Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ when $u = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1.$

2. Prove that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$ in each of the following cases :

(i) $u = x^4 + x^2 y^2 + y^4.$ (ii) $u = \log \tan (y/x).$

(iii) $u = \log \left\{\dfrac{x^2 + y^2}{x + y}\right\}.$ (iv) $u = x^y.$

3. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

4. If $u = xyf\left(\dfrac{y}{x}\right)$, then write the value of the expression $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

(Meerut 2001; Kanpur 07)

5. If $z = f(x + ay) + \phi(x - ay)$, prove that $\dfrac{\partial^2 z}{\partial y^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$.

(Bundelkhand 2001; Kanpur 05)

6. If $u = f(r)$, where $r = \sqrt{(x^2 + y^2)}$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r}f'(r)$.

(Meerut 2001; Avadh 04)

7. If $z = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$, prove that $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{x^2 - y^2}{x^2 + y^2}$.

8. (i) If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(ii) If $u = e^x(x\cos y - y\sin y)$, then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. **(Garhwal 2003)**

9. If $z(x + y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

(Bundelkhand 2011; Kashi 13; Avadh 14)

10. If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$.

(Kanpur 2011; Kashi 12; Rohilkhand 13)

11. If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = -(x\log ex)^{-1}$.

(Rohilkhand 2013; Bundelkhand 14; Purvanchal 14)

12. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ when

(i) $u = e^{my}\cos mx$. **(Agra 2014)**

(ii) $u = \log(x^2 + y^2)$.

(iii) $u = \tan^{-1}(y/x)$.

(iv) $e^x(x\cos y - y\sin y)$ **(Garhwal 2003)**

13. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

(i) $x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} + z\dfrac{\partial V}{\partial z} = -V$.

(ii) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$. **(Kumaun 2008)**

14. If $u = \tan^{-1}\dfrac{xy}{\sqrt{(1 + x^2 + y^2)}}$, show that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{1}{(1 + x^2 + y^2)^{3/2}}$.

15. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, prove that

(Rohilkhand 2011B, 12B)

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

16. If $x=r\cos\theta$, $y=r\sin\theta$, prove that

(i) $(\partial r/\partial x)^2+(\partial r/\partial y)^2=1.$

(ii) $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}=\dfrac{1}{r}\left\{\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2\right\}.$

(Lucknow 2007, 11)

17. (i) If $u=x^2y+y^2z+z^2x$, show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=(x+y+z)^2.$

(ii) If $u=\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ x^2 & y^2 & z^2\end{vmatrix}$, show that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial y}{\partial z}=0.$

(Rohilkhand 2013B)

Answers 1

1. $\dfrac{2x}{a^2},\dfrac{2y}{b^2}.$

4. $2u.$

7.5 Homogeneous Functions

An expression in x and y in which every term is of the same degree is called a *homogeneous function* of x and y. Consider the function defined by

$$f(x,y)=a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots\ldots+a_{n-1}xy^{n-1}+a_ny^n. \quad \ldots(1)$$

In this function every term is of degree n. Therefore it is a homogeneous function of x and y of degree n. Moreover, (1) may be written as

$$f(x,y)=x^n\left\{a_0+a_1\left(\frac{y}{x}\right)+a_2\left(\frac{y}{x}\right)^2+\ldots+a_n\left(\frac{y}{x}\right)^n\right\}$$

or $\quad f(x,y)=x^nF(y/x),$

where $F(y/x)$ is some function of y/x. Thus a homogeneous function of x and y of degree n may be put in the form $x^nF(y/x)$. Therefore we give the general definition of a homogeneous function as follows :

$x^nF(y/x)$ is called a *homogeneous function of* x and y of degree n, whatever the function F may be. Similarly, $y^nF(x/y)$ is also a *homogeneous function* of x and y of degree n.

Thus $x^3\sin(y/x)$ is a homogeneous function of x and y of degree 3. Similarly, $y^2\cos(x/y)$ is a homogeneous function of x and y of degree 2.

In general, if the function $f(x_1,x_2,\ldots,x_p)$ of the p variables $x_1,x_2,\ldots,x_p$ can be put in the form

$$x_r^{\,n}F\left(\frac{x_1}{x_r},\frac{x_2}{x_r},\ldots,\frac{x_p}{x_r}\right),$$

then $f(x_1,x_2,\ldots,x_p)$ is called a homogeneous function of $x_1,x_2,\ldots,x_p$ of degree n.

7.6 Euler's Theorem on Homogeneous Functions

If u be a homogeneous function of x and y of degree n, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu.$

(Meerut 2000; Gorakhpur 06; Kashi 11, 13, 14)

Proof : Since u is a homogeneous function of x and y of degree n, therefore u may be put in the form

$$u=x^n F(y/x). \qquad \ldots(1)$$

Differentiating (1) partially with respect to x, we get

$$\frac{\partial u}{\partial x}=nx^{n-1}F\left(\frac{y}{x}\right)+x^n\left\{F'\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right).$$

$$\therefore \qquad x\frac{\partial u}{\partial x}=nx^n F\left(\frac{y}{x}\right)-yx^{n-1}F'\left(\frac{y}{x}\right). \qquad \ldots(2)$$

Again differentiating (1) partially with respect to y, we get

$$\frac{\partial u}{\partial y}=x^n\left\{F'\left(\frac{y}{x}\right)\right\}\left(\frac{1}{x}\right).$$

$$\therefore \qquad y\frac{\partial u}{\partial y}=yx^{n-1}F'\left(\frac{y}{x}\right). \qquad \ldots(3)$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n F\left(\frac{y}{x}\right)=nu.$$

Hence $\quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=nu.$

This proves the theorem.

In general, if u be a homogeneous function of $x_1, x_2, \ldots, x_m$ of degree n, then

$$\boldsymbol{x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+\ldots+x_m\frac{\partial u}{\partial x_m}=nu.}$$

The proof is similar to that of two variables.

Illustrative Examples

Example 1 : *Verify Euler's theorem for the function* $u=\dfrac{(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})}.$

(Rohilkhand 2014)

Solution : Here we see that u is a homogeneous function of x and y of degree $1/4-1/5$ *i.e.*, $1/20$. Therefore in order to verify Euler's theorem we are to show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{20}u.$$

We have, $\log u=\log(x^{1/4}+y^{1/4})-\log(x^{1/5}+y^{1/5}). \qquad \ldots(1)$

Differentiating (1) partially with respect to x, we have

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x^{1/4}+y^{1/4}}\cdot\left(\frac{1}{4}x^{-3/4}\right)-\frac{1}{x^{1/5}+y^{1/5}}\left(\frac{1}{5}x^{-4/5}\right).$$

$$\therefore \qquad \frac{\partial u}{\partial x}=u\left[\frac{1}{4}\frac{x^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{x^{-4/5}}{x^{1/5}+y^{1/5}}\right].$$

$$\therefore \quad x\frac{\partial u}{\partial x}=u\left[\frac{1}{4}\frac{x^{1/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{x^{1/5}}{x^{1/5}+y^{1/5}}\right]. \qquad \text{...(2)}$$

Again differentiating (1) partially with respect to y, we get

$$\frac{1}{u}\frac{\partial u}{\partial y}=\left[\frac{1}{4}\frac{y^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{y^{-4/5}}{x^{1/5}+y^{1/5}}\right].$$

$$\therefore \quad y\frac{\partial u}{\partial y}=u\left[\frac{1}{4}\frac{y^{1/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{y^{1/5}}{x^{1/5}+y^{1/5}}\right]. \qquad \text{...(3)}$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u\left[\frac{1}{4}\frac{x^{1/4}+y^{1/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{x^{1/5}+y^{1/5}}{x^{1/5}+y^{1/5}}\right]=u\left[\frac{1}{4}-\frac{1}{5}\right]=\frac{1}{20}u.$$

This verifies Euler's theorem.

Example 2 : *If* $u=\sin^{-1}\left\{\dfrac{x^2+y^2}{x+y}\right\}$, *show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u.$

(Garhwal 2002; Kumaun 08; Avadh 12)

Solution : We have, $\sin u=\dfrac{x^2+y^2}{x+y}$.

Let $v=\dfrac{x^2+y^2}{x+y}\cdot$ Then v is a homogeneous function of x and y of degree 1.

Therefore by Euler's theorem, we have $\quad x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}=v. \qquad$ ...(1)

Now $v=\sin u$.

$$\therefore \quad \frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}\cdot$$

Putting these values in (1), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=v$$

or

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{v}{\cos u}=\frac{\sin u}{\cos u} \qquad \left[\because \; v=\sin u=\frac{x^2+y^2}{x+y}\right]$$

$$=\tan u.$$

This proves the result.

Comprehensive Exercise 2

1. State Euler's theorem on homogeneous functions. **(Bundelkhand 2001)**
2. Verify Euler's theorem in the following cases :

(i) $u=ax^2+2hxy+by^2$.

(ii) $u=\dfrac{x(x^3-y^3)}{x^3+y^3}\cdot$

(iii) $u=axy+byz+czx$.

(iv) $u=x^n\sin(y/x)$.

(v) $u=x^n\log(y/x)$.

(vi) $u=1/\sqrt{(x^2+y^2)}$.

3. (i) If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$.

(Rohilkhand 2012B)

(ii) If $u = \sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

(Gorakhpur 2005; Garhwal 02)

4. (i) If $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

(ii) If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

(Rohilkhand 2013B)

5. (i) If $u = \log\dfrac{x^3+y^3}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2$.

(ii) If $u = \log\dfrac{x^4+y^4}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

(iii) If $u = \log\dfrac{x^2+y^2}{x+y}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$. **(Kanpur 2006; Avadh 13)**

6. Use Euler's theorem on homogeneous functions to show that if $u = \tan^{-1}(y/x)$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

7. If u be a homogeneous function of x and y of degree n, show that

(i) $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x\,\partial y} = (n-1)\dfrac{\partial u}{\partial x}\cdot$

(ii) $x\dfrac{\partial^2 u}{\partial x\,\partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}\cdot$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$.

8. If $u = \dfrac{xy}{x+y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

9. If $u = x\phi(y/x) + \psi(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

(Kanpur 2008)

7.7 Total Differential Coefficient

If $u = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$, then x and y are not independent variables. Substituting the values of x and y in u, we can express u as a function of the single variable t and we can find the ordinary differential coefficient du/dt.

To distinguish du/dt from the partial differential coefficients $\partial u/\partial x$ and $\partial u/\partial y$, we shall call du/dt as the *total differential coefficient*. We shall now obtain a formula which will enable us to find du/dt without first expressing u in terms of t only.

Suppose δx, δy and δu are the increments in x, y and u respectively corresponding to an increment δt in t.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y)$.

$\therefore \qquad \delta u = f(x+\delta x, y+\delta y) - f(x, y).$

$$\therefore \qquad \frac{\delta u}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \frac{\{f(x+\delta x, y+\delta y) - f(x, y+\delta y)\} + \{f(x, y+\delta y) - f(x, y)\}}{\delta t}$$

[adding and subtracting the term $f(x, y+\delta y)$ in the numerator]

$$= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta t} + \frac{f(x, y+\delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}.$$

Now $$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \lim_{\delta t \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}. \qquad \ldots(1)$$

Now δx and δy also tend to zero when δt tends to zero.

So we have

$$\lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x},$$

because while x becomes $x+\delta x$, $y+\delta y$ remains unchanged.

Similarly, $$\lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}.$$

Also $$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} \quad \text{and} \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Therefore (1) gives $$\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}.$$

Similarly, if $u = f(x_1, x_2, \ldots, x_m)$ and $x_1, x_2, \ldots, x_m$ are all functions of t, we can prove that

$$\boldsymbol{\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt} + \ldots + \frac{\partial u}{\partial x_m} \cdot \frac{dx_m}{dt}}.$$

7.8 First Differential Coefficient of an Implicit Function

Suppose $u = f(x, y)$, where $y = \phi(x)$. Then supposing t to be the same as x in the formula of article 7.7, we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \text{or} \quad \boldsymbol{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}.$$

Now suppose we are given an implicit relation between x and y of the form $u \equiv f(x, y) = c$, where c is a constant and y is a function of x.

Then, we have $du/dx = 0$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \quad \text{or} \quad \boldsymbol{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}.$$

Illustrative Examples

Example 1 : *If* $(\tan x)^y + (y)^{\cot x} = a$, *find the value of* dy/dx. **(Rohilkhand 2014)**

Solution : Let $f(x, y) \equiv (\tan x)^y + (y)^{\cot x} = a$.

Then, we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$. ...(1)

Now $\dfrac{\partial f}{\partial x} = y(\tan x)^{y-1}\sec^2 x + y^{\cot x} \cdot \log y \cdot (-\operatorname{cosec}^2 x)$

and $\dfrac{\partial f}{\partial y} = (\tan x)^y \log\tan x + (\cot x) \cdot y^{\cot x - 1}$.

Therefore (1) gives

$$\frac{dy}{dx} = -\frac{y(\tan x)^{y-1} \cdot \sec^2 x - y^{\cot x} \cdot \log y \cdot \operatorname{cosec}^2 x}{(\tan x)^y \log\tan x + \cot x \cdot y^{\cot x - 1}}.$$

Example 2 : *If* $u = x^2 y$, *where* $x^2 + xy + y^2 = 1$, *find* du/dx.

Solution : We have, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$. ...(1)

Now $\dfrac{\partial u}{\partial x} = 2xy$ and $\dfrac{\partial u}{\partial y} = x^2$.

Let $f(x, y) \equiv x^2 + xy + y^2 = 1$.

Then $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{2x + y}{x + 2y}$.

So putting the values in (1), we get

$$\frac{du}{dx} = 2xy + x^2 \cdot \left(-\frac{2x + y}{x + 2y}\right) = 2xy - \frac{x^2(2x + y)}{x + 2y}.$$

Example 3 : *If* $u = f(y - z, z - x, x - y)$, *prove that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

(Kanpur 2009; Avadh 10)

Solution : We have $u = f(y - z, z - x, x - y)$.

Let $y - z = A$, $z - x = B$, and $x - y = C$.

Then $u = f(A, B, C)$ where A, B and C are functions of x, y and z.

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial x} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial x} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial x}$$

$$= \frac{\partial u}{\partial A} \cdot (0) + \frac{\partial u}{\partial B} \cdot (-1) + \frac{\partial u}{\partial C} \cdot (1) = -\frac{\partial u}{\partial B} + \frac{\partial u}{\partial C}. \qquad ...(1)$$

Similarly $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial y} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial y} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial y}$$

$$= \frac{\partial u}{\partial A} \cdot (1) + \frac{\partial u}{\partial B} \cdot (0) + \frac{\partial u}{\partial C} \cdot (-1) = \frac{\partial u}{\partial A} - \frac{\partial u}{\partial C} \qquad ...(2)$$

and $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial z} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial z} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial z}$$

$$= \frac{\partial u}{\partial A} \cdot (-1) + \frac{\partial u}{\partial B} \cdot (1) + \frac{\partial u}{\partial C} \cdot (0) = -\frac{\partial u}{\partial A} + \frac{\partial u}{\partial B}. \qquad ...(3)$$

Adding (1), (2) and (3), we have $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0.$

Comprehensive Exercise 3

1. Find dy/dx in the following :
   (i) $x^y+y^x=a^b$. (ii) $ax^2+2hxy+by^2=1$. **(Kashi 2013)**
2. If $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$.
3. Find du/dx if $u=\sin(x^2+y^2)$, where $a^2x^2+b^2y^2=c^2$.
4. If $u=x^4y^5$, where $x=t^2$ and $y=t^3$, find du/dt.
5. If $f(x,y)=0$, $\phi(y,z)=0$, show that $\dfrac{\partial f}{\partial y}\cdot\dfrac{\partial \phi}{\partial z}\cdot\dfrac{dz}{dx}=\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial \phi}{\partial y}$. **(Lucknow 2009)**
6. If $u=\sqrt{(x^2+y^2)}$ and $x^3+y^3+3axy=5a^2$, find the value of du/dx when $x=a, y=a$.

Answers 3

1. (i) $-\dfrac{\{yx^{y-1}+y^x\log y\}}{\{x^y\log x+xy^{x-1}\}}$, (ii) $-\dfrac{(ax+hy)}{(hx+by)}$.
3. $2x\{\cos(x^2+y^2)\}\left(1-\dfrac{a^2}{b^2}\right)$.
4. $23t^{22}$.
6. 0.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. If $u=e^{my}\cos mx$, then $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=$ …….
2. If $u=\tan^{-1}\dfrac{x^2+y^2}{x+y}$, then $\dfrac{\partial u}{\partial x}=$ …….
3. An expression in which every term is of the same degree is called a …… function.
4. If $u=f(x,y)$, where $x=\phi(t)$ and $y=\psi(t)$, then $\dfrac{du}{dt}=$ …….
5. If $u(x,y)$ is a homogeneous function of x and y of degree n, then
$$x\frac{\partial}{\partial x}(u_x)+y\frac{\partial}{\partial y}(u_x)=\text{……},$$
where $u_x=\dfrac{\partial u}{\partial x}$.

6. If $u = f(y + ax) + \phi(y - ax)$, then $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{a^2 \partial^2 u}{\partial y^2} =$ .... **(Agra 2002)**

7. If $u = f(y/x)$, then, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} =$ .... **(Agra 2003)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

8. If $z = \tan(y + ax) + (y - ax)^{3/2}$, then $\left(\dfrac{\partial^2 z}{\partial x^2}\right) - a^2\left(\dfrac{\partial^2 z}{\partial y^2}\right)$ is equal to

(a) 0 (b) $(y - ax)$

(c) 1 (d) $\sec(y + ax)$

9. If u is a homogeneous function of x and y of degree n, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ is equal to

(a) n (b) nu

(c) $\dfrac{n}{u}$ (d) u

10. If $\dfrac{1}{u} = \sqrt{(x^2 + y^2 + z^2)}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z}$ is equal to

(a) u (b) $-u$

(c) u^2 (d) 0

11. If x and y are connected by an equation of the form $f(x, y) = 0$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{\partial f/\partial x}{\partial f/\partial y}$ (b) $(-1)^n \dfrac{\partial f/\partial x}{\partial f/\partial y}$

(c) $\dfrac{\partial f/\partial y}{\partial f/\partial x}$ (d) $-\dfrac{\partial f/\partial x}{\partial f/\partial y}$ **(Garhwal 2001)**

12. If $x = r\cos\theta$, $y = r\sin\theta$, then the value of $\dfrac{\partial \theta}{\partial x}$ is

(a) $-\dfrac{\sin\theta}{r}$ (b) $\dfrac{r}{\sin\theta}$

(c) $\dfrac{\sin\theta}{r}$ (d) $-\dfrac{1}{r\sin\theta}$ **(Garhwal 2002)**

13. If $f(x, y)$ be a homogeneous function of x and y of degree n, then

(a) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = f$ (b) $y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y} = f$

(c) $y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y} = nf$ (d) $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf$ **(Garhwal 2003)**

True or False:

Write 'T' for true and 'F' for false statement.

14. If $u = f(x, y)$ and its partial derivatives are continuous, then the order of differentiation is immaterial.

15. If u is a homogeneous function of x and y of degree n, then $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are also homogeneous functions of x and y each being of degree n.

16. If $f(x, y) = 0$ be an implicit function of x and y and

$$p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}, r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}, \text{ then}$$

$$\frac{d^2 y}{dx^2} = -(q^2 r - 2pqs + p^2 t)/q^3.$$

Answers

1. 0.

2. $\dfrac{(x^2 + 2xy - y^2)}{\{(x + y)^2 + (x^2 + y^2)^2\}}$.

3. homogeneous.

4. $\dfrac{\partial u}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dt}$.

5. $(n - 1)\, u_x$.

6. 0.

7. 0.

8. (a).

9. (b).

10. (b).

11. (d).

12. (a).

13. (d).

14. T.

15. F.

16. T.

□

Chapter

8

Jacobians

8.1 Jacobian

(Kanpur 2014)

Definition : If $u_1, u_2, \ldots, u_n$ are functions of n independent variables $x_1, x_2, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \ldots\ \ldots\ \ldots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \ldots\ \ldots\ \ldots & \dfrac{\partial u_2}{\partial x_n} \\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots\ \ldots & \ldots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \ldots\ \ldots\ \ldots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted either by $\dfrac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)}$ or by $J(u_1, u_2, \ldots, u_n)$. The second notation is used when there is no doubt as regards the independent variables.

Thus if u and v are functions of two independent variables x and y, we have

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = J(u, v).$$

Similarly if u, v and w are functions of three independent variables x, y and z, we have

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = J(u, v, w).$$

Note : If the functions $u_1, u_2, \ldots, u_n$ of n independent variables $x_1, x_2, \ldots, x_n$ are of the following forms,

$$u_1 = f_1 (x_1), u_2 = f_2 (x_1, x_2), \ldots, u_n = f_n (x_1, x_2, \ldots, x_n), \text{ then}$$

$$\frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \ldots \ldots \frac{\partial u_n}{\partial x_n},$$

i.e., in such cases the Jacobian reduces to the principal diagonal term of the determinant.

Illustrative Examples

Example 1 : *If* $x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$, *show that*

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = r^2 \sin \theta.$$

(Meerut 2003; Garhwal 02; Rohilkhand 12; Avadh 07, 12)

Solution : We have

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= \cos \theta (r^2 \sin \theta \cos \theta \cos^2 \phi + r^2 \sin \theta \cos \theta \sin^2 \phi) + r \sin \theta (r \sin^2 \theta \cos^2 \phi + r \sin^2 \theta \sin^2 \phi),$$

expanding the determinant along the third row

$$= r^2 \sin \theta \cos^2 \theta + r^2 \sin^3 \theta = r^2 \sin \theta (\cos^2 \theta + \sin^2 \theta) = r^2 \sin \theta.$$

Example 2 : *Find the Jacobian* $\dfrac{\partial (x, y, z)}{\partial (r, \theta, \phi)}$ *being given*

$$x = r \cos \theta \cos \phi, y = r \sin \theta \sqrt{(1 - m^2 \sin^2 \phi)},$$

$$z = r \sin \phi \sqrt{(1 - n^2 \sin^2 \theta)}, \text{ where } m^2 + n^2 = 1.$$

Solution : Here $x^2 + y^2 + z^2$

$$= r^2 \cos^2\theta \cos^2\phi + r^2 \sin^2\theta - r^2 m^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\phi - r^2 n^2 \sin^2\phi \sin^2\theta$$

$$= r^2 (\cos^2\theta \cos^2\phi + \sin^2\theta + \sin^2\phi - \sin^2\theta \sin^2\phi) \qquad [\because m^2 + n^2 = 1]$$

$$= r^2 (\cos^2\theta \cos^2\phi + \sin^2\theta + \sin^2\phi \cos^2\theta)$$

$$= r^2 (\sin^2\theta + \cos^2\theta) = r^2.$$

$$\left.\begin{aligned} \therefore \quad & x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r} + z\frac{\partial z}{\partial r} = r \,;\, x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} + z\frac{\partial z}{\partial \theta} = 0 \\ \text{and} \quad & x\frac{\partial x}{\partial \phi} + y\frac{\partial y}{\partial \phi} + z\frac{\partial z}{\partial \phi} = 0. \end{aligned}\right\} \qquad ...(1)$$

Now
$$J(x, y, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \frac{1}{x}\begin{vmatrix} x\frac{\partial x}{\partial r} & x\frac{\partial x}{\partial \theta} & x\frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \frac{1}{x}\begin{vmatrix} r & 0 & 0 \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}, \text{ by adding } yR_2 + zR_3 \text{ to } R_1 \text{ and using the relations (1)}$$

$$= \frac{r}{x}\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \frac{r}{x}\left\{\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi}\right\}$$

$$= \frac{r}{x}\left\{r\cos\theta\sqrt{(1 - m^2\sin^2\phi)}\,.\,r\cos\phi\sqrt{(1 - n^2\sin^2\theta)} - \frac{r\sin\phi\,.\,n^2\sin\theta\cos\theta}{\sqrt{(1 - n^2\sin^2\theta)}}\cdot\frac{r\sin\theta\,.\,m^2\sin\phi\cos\phi}{\sqrt{(1 - m^2\sin^2\phi)}}\right\}$$

$$= \frac{r^3\cos\theta\cos\phi}{x}\left[\frac{(1 - m^2\sin^2\phi)(1 - n^2\sin^2\theta) - n^2 m^2 \sin^2\theta\sin^2\phi}{\sqrt{[(1 - n^2\sin^2\theta)(1 - m^2\sin^2\phi)]}}\right]$$

$$= \frac{r^3\cos\theta\cos\phi}{r\cos\theta\cos\phi}\cdot\left[\frac{1 - m^2\sin^2\phi - n^2\sin^2\theta + m^2n^2\sin^2\phi\sin^2\theta - m^2n^2\sin^2\theta\sin^2\phi}{\sqrt{[(1 - n^2\sin^2\theta)(1 - m^2\sin^2\phi)]}}\right]$$

$$= \frac{r^2(m^2\cos^2\phi + n^2\cos^2\theta)}{\sqrt{[(1 - n^2\sin^2\theta)(1 - m^2\sin^2\phi)]}}. \qquad [\because m^2 + n^2 = 1]$$

Example 3 : *If* $y_1 = r\sin\theta_1\sin\theta_2$, $y_2 = r\sin\theta_1\cos\theta_2$, $y_3 = r\cos\theta_1\sin\theta_3$, $y_4 = r\cos\theta_1\cos\theta_3$, *find the value of the Jacobian* $\dfrac{\partial(y_1, y_2, y_3, y_4)}{\partial(r, \theta_1, \theta_2, \theta_3)}$.

Solution : Squaring and adding the given relations, we have

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2.$$

$$\left.\begin{aligned} \therefore \quad & y_1 \frac{\partial y_1}{\partial r} + y_2 \frac{\partial y_2}{\partial r} + y_3 \frac{\partial y_3}{\partial r} + y_4 \frac{\partial y_4}{\partial r} = r \\ \text{and} \quad & y_1 \frac{\partial y_1}{\partial \theta_r} + y_2 \frac{\partial y_2}{\partial \theta_r} + y_3 \frac{\partial y_3}{\partial \theta_r} + y_4 \frac{\partial y_4}{\partial \theta_r} = 0, \quad r = 1, 2, 3. \end{aligned}\right\} \quad \text{...(1)}$$

Also $y_3^2 + y_4^2 = r^2 \cos^2 \theta_1$, so that

$$\left.\begin{aligned} & y_3 \frac{\partial y_3}{\partial \theta_1} + y_4 \frac{\partial y_4}{\partial \theta_1} = - r^2 \cos \theta_1 \sin \theta_1 ; \\ & y_3 \frac{\partial y_3}{\partial \theta_r} + y_4 \frac{\partial y_4}{\partial \theta_r} = 0, \quad r = 2, 3. \end{aligned}\right\} \quad \text{...(2)}$$

Now the required Jacobian

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \frac{\partial y_1}{\partial \theta_3} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix}.$$

Operating $y_1 R_1 + (y_2 R_2 + y_3 R_3 + y_4 R_4)$, and using the results (1), we get

$$J = \frac{1}{y_1} \begin{vmatrix} r & 0 & 0 & 0 \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix} = \frac{r}{y_1} \begin{vmatrix} \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1 y_3} \begin{vmatrix} \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ - r^2 \cos \theta_1 \sin \theta_1 & 0 & 0 \\ \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix},$$

adding $y_4 R_3$ to $y_3 R_2$ and using the results (2)

$$= \frac{r}{y_1 y_3} \cdot r^2 \cos \theta_1 \sin \theta_1 \left[\frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_4}{\partial \theta_3} - \frac{\partial y_4}{\partial \theta_2} \cdot \frac{\partial y_2}{\partial \theta_3} \right]$$

$$= \frac{r^3 \cos \theta_1 \sin \theta_1}{y_1 y_3} [(- r \sin \theta_1 \sin \theta_2)(- r \cos \theta_1 \sin \theta_3) - 0]$$

$$= \frac{r^5 \sin^2 \theta_1 \cos^2 \theta_1 \sin \theta_2 \sin \theta_3}{r^2 \sin \theta_1 \cos \theta_1 \sin \theta_2 \sin \theta_3} = r^3 \sin \theta_1 \cos \theta_1 .$$

Comprehensive Exercise 1

1. If $x = r\cos\theta, y = r\sin\theta$, show that

(*i*) $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r,$ **(Kanpur 2005; Meerut 13B; Kashi 13)**

(*ii*) $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}\cdot$ **(Meerut 2003)**

2. If $x = u(1 + v), y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.

(Meerut 2013)

3. If $x = c\cos u\cosh v, y = c\sin u\sinh v$, prove that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2}c^2(\cos 2u - \cosh 2v).$$

(Rohilkhand 2013)

4. If $u = \dfrac{y^2}{2x}, v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}\cdot$ **(Meerut 2012)**

5. If $u_1 = x_2 x_3/x_1, u_2 = x_3 x_1/x_2, u_3 = x_1 x_2/x_3$, prove that $J(u_1, u_2, u_3) = 4$.

(Bundelkhand 2014; Purvanchal 14)

6. If $x = \sin\theta\sqrt{(1 - c^2\sin^2\phi)}, y = \cos\theta\cos\phi$, then show that

$$\frac{\partial(x, y)}{\partial(\theta, \phi)} = -\sin\phi\,\frac{[(1 - c^2)\cos^2\theta + c^2\cos^2\phi]}{\sqrt{(1 - c^2\sin^2\phi)}}\cdot$$

7. If $y_1 = 1 - x_1, y_2 = x_1(1 - x_2), y_3 = x_1 x_2(1 - x_3), \ldots, y_n = x_1 x_2 \ldots x_{n-1}(1 - x_n)$, prove that

$$J(y_1, y_2, \ldots, y_n) = (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

8. If $y_1 = \cos x_1, y_2 = \sin x_1\cos x_2, y_3 = \sin x_1\sin x_2\cos x_3, \ldots,$
$y_n = \sin x_1\sin x_2\sin x_3 \ldots \sin x_{n-1}\cos x_n$, find the Jacobian of $y_1, y_2, \ldots, y_n$ with respect to $x_1, x_2, \ldots, x_n$.

Answers 1

2. $1 + u + v$. **4.** $-y/2x$.

8. $(-1)^n\sin^n x_1\sin^{n-1}x_2 \ldots \sin x_n$.

8.2 Case of Functions of Functions (Chain Rule)

We shall establish the formula for two variables and the result can be easily extended to any number of variables.

Theorem : *If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2, then*

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)}\cdot\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

(Kumaun 2003)

Proof : We have

$$\left.\begin{aligned}\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1},\ \frac{\partial u_1}{\partial x_2}=\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}+\frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2},\\ \frac{\partial u_2}{\partial x_1}=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1},\ \frac{\partial u_2}{\partial x_2}=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2}+\frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2}.\end{aligned}\right\}\quad ...(1)$$

Now $\dfrac{\partial(u_1,u_2)}{\partial(y_1,y_2)}\cdot\dfrac{\partial(y_1,y_2)}{\partial(x_1,x_2)}=\begin{vmatrix}\dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial y_2}\\ \dfrac{\partial u_2}{\partial y_1} & \dfrac{\partial u_2}{\partial y_2}\end{vmatrix}\times\begin{vmatrix}\dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2}\\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2}\end{vmatrix}$

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_1}+\dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_2}+\dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_2}\\ \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_1}+\dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_2}+\dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_2}\end{vmatrix},$$

applying row-by-column multiplication rule

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2}\\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2}\end{vmatrix},\ \text{using the relations (1)}$$

$$=\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)}.$$

Note : The above formula resembles very much with the formula $\dfrac{df}{dx}=\dfrac{df}{dt}\cdot\dfrac{dt}{dx}$, for the derivative of the function of a function.

Generalization of the above formula : *If $u_1,u_2,...,u_n$ are functions of $y_1,y_2,\ldots,y_n$ and $y_1,y_2,...,y_n$ are functions of $x_1,x_2,...,x_n$, then*

$$\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,...,x_n)}=\frac{\partial(u_1,u_2,...,u_n)}{\partial(y_1,y_2,...,y_n)}\cdot\frac{\partial(y_1,y_2,...,y_n)}{\partial(x_1,x_2,...,x_n)}.$$

The proof may be easily extended as in the case of two variables and has been left as an exercise for the students.

8.3 Jacobian of Implicit Functions

Theorem : *Suppose $u_1,u_2,...,u_n$ instead of being given explicitly in terms of $x_1,x_2,...,x_n$ are connected with them by equations such as*

$$F_1(u_1,u_2,...,u_n,x_1,x_2,...,x_n)=0,$$

$$F_2(u_1,u_2,...,u_n,x_1,x_2,...,x_n)=0,$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$F_n(u_1,u_2,...,u_n,x_1,x_2,...,x_n)=0.$$

Then, we have $$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(x_1, x_2, ..., x_n)}}{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(u_1, u_2, ..., u_n)}}.$$

Proof : Here also we shall establish the result for two variables and the proof can be extended easily for n variables. The students should themselves write the proof for n variables on the basis of the proof given below for two variables.

In the case of two variables, the connecting relations are

$$\left.\begin{array}{l} F_1(u_1, u_2, x_1, x_2) = 0, \\ F_2(u_1, u_2, x_1, x_2) = 0. \end{array}\right\} \quad ...(1)$$

From relations (1), we have by differentiation

$$\left.\begin{array}{l} \dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} = 0, \\ \dfrac{\partial F_1}{\partial x_2} + \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} = 0, \\ \dfrac{\partial F_2}{\partial x_1} + \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} = 0, \\ \dfrac{\partial F_2}{\partial x_2} + \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} = 0. \end{array}\right\} \quad ...(2)$$

Now $$\frac{\partial(F_1, F_2)}{\partial(u_1, u_2)} \cdot \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \end{vmatrix},$$

applying row-by-column multiplication rule

$$= \begin{vmatrix} \dfrac{-\partial F_1}{\partial x_1} & \dfrac{-\partial F_1}{\partial x_2} \\ \dfrac{-\partial F_2}{\partial x_1} & \dfrac{-\partial F_2}{\partial x_2} \end{vmatrix}, \text{ using the relations (2)}$$

$$= (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)}.$$

Accordingly, we have

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\dfrac{\partial(F_1, F_2)}{\partial(x_1, x_2)}}{\dfrac{\partial(F_1, F_2)}{\partial(u_1, u_2)}}.$$

Illustrative Examples

Example 1 : *Prove that* $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1.$

(Bundelkhand 2011; Kanpur 08; Meerut 12; Avadh 13)

Solution : Let $u = f_1(x,y),\ v = f_2(x,y)$. ...(1)

Obviously x and y can also be expressed as functions of u and v. Differentiating relations (1) partially with respect to u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \quad 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \\ 0 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \quad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}. \end{aligned}\right\} \quad ...(2)$$

Now
$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix},$$

applying row-by-column multiplication

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \text{ using the relations (2)}$$

$$= 1.$$

Example 2 : *If* $u^3 + v + w = x + y^2 + z^2,\ u + v^3 + w = x^2 + y + z^2,$ $u + v + w^3 = x^2 + y^2 + z,$ *prove that*

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

Solution : The given relations can be written as

$$F_1 \equiv u^3 + v + w - x - y^2 - z^2 = 0,$$

$$F_2 \equiv u + v^3 + w - x^2 - y - z^2 = 0,$$

$$F_3 \equiv u + v + w^3 - x^2 - y^2 - z = 0.$$

Now
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}. \quad ...(1)$$

Here
$$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$

$$= -1(1 - 4yz) + 2x(2y - 4yz) - 2x(4yz - 2z)$$

$$= -1 + 4(yz + zx + xy) - 16xyz.$$

And
$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 3u^2 (9v^2 w^2 - 1) - 1 (3w^2 - 1) + 1 . (1 - 3v^2)$$
$$= 2 - 3 (u^2 + v^2 + w^2) + 27u^2 v^2 w^2.$$

$\therefore$ From (1), $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = \dfrac{1 - 4 (yz + zx + xy) + 16xyz}{2 - 3 (u^2 + v^2 + w^2) + 27u^2 v^2 w^2}$.

Comprehensive Exercise 2

1. If $u^3 + v^3 = x + y, u^2 + v^2 = x^3 + y^3$, show that
$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{1}{2} \frac{y^2 - x^2}{uv (u - v)}.$$

2. If $x + y + z = u, y + z = uv, z = uvw$, show that $\dfrac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v.$

(Rohilkhand 2005; Kashi 14)

3. If $u^3 = xyz, \dfrac{1}{v} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}, w^2 = x^2 + y^2 + z^2$, prove that
$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = - \frac{v (y - z) (z - x) (x - y) (x + y + z)}{3u^2 w (yz + zx + xy)}.$$
(Rohilkhand 2012B)

4. If $u^3 + v^3 + w^3 = x + y + z,\ u^2 + v^2 + w^2 = x^3 + y^3 + z^3,\ u + v + w = x^2 + y^2 + z^2$, then prove that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = \dfrac{(y - z) (z - x) (x - y)}{(u - v) (v - w) (w - u)}$.

(Purvanchal 2007; Kanpur 12)

5. Compute the Jacobian $\dfrac{\partial (u, v)}{\partial (r, \theta)}$ where

$u = 2xy, v = x^2 - y^2, x = r \cos \theta, y = r \sin \theta.$

6. If $u_1 = x_1 + x_2 + x_3 + x_4, u_1 u_2 = x_2 + x_3 + x_4,\ u_1 u_2 u_3 = x_3 + x_4, u_1 u_2 u_3 u_4 = x_4$, show that
$$\frac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_1, u_2, u_3, u_4)} = {u_1}^3 {u_2}^2 u_3 .$$

7. Given $y_1 (x_1 - x_2) = 0, y_2 ({x_1}^2 + x_1 x_2 + {x_2}^2) = 0$, show that
$$\frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = 3y_1 y_2 \frac{x_1 + x_2}{{x_1}^3 - {x_2}^3}.$$

8. If $u = x (1 - r^2)^{-1/2}, v = y (1 - r^2)^{-1/2}, w = z (1 - r^2)^{-1/2}$,

where $r^2 = x^2 + y^2 + z^2$, show that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = (1 - r^2)^{-5/2}$.

9. (a) Find the Jacobian of $y_1, y_2, y_3, ..., y_n$, being given
$$y_1 = x_1 (1 - x_2), y_2 = x_1 x_2 (1 - x_3), \ldots ,$$
$$y_{n-1} = x_1 x_2 \ldots x_{n-1} (1 - x_n), y_n = x_1 x_2 x_3 \ldots x_n .$$

(b) If $y_1 = r \cos \theta_1, y_2 = r \sin \theta_1 \cos \theta_2, y_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3$, ...

$y_{n-1} = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \cos \theta_{n-1}$ and $y_n = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1}$, prove that

$$\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (r, \theta_1, \ldots, \theta_{n-1})} = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2}.$$

10. If λ, μ, ν are the roots of the equation in k,

$$\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1,$$

prove that

$$\frac{\partial (x, y, z)}{\partial (\lambda, \mu, \nu)} = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

11. The roots of the equation in λ,

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w. Prove that

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$$ **(Kumaun 2008; Kanpur 10)**

12. If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that

$$\frac{\partial (y, z)}{\partial (x, z)} = \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}}$$

13. (i) Prove that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \times \dfrac{\partial (x, y, z)}{\partial (u, v, w)} = 1.$ **(Kanpur 2009, 11)**

(ii) Prove that $\dfrac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} \cdot \dfrac{\partial (x_1, x_2, \ldots, x_n)}{\partial (y_1, y_2, \ldots, y_n)} = 1.$

Answers 2

5. $-4r^3$.

9. (a) $x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$.

8.4 Necessary and Sufficient Condition for a Jacobian to Vanish

Theorem : *Let $u_1, u_2, \ldots, u_n$ be functions of n independent variables $x_1, x_2, \ldots, x_n$. In order that these n functions may not be independent, i.e., there may exist between these n functions a relation*

$$F(u_1, u_2, \ldots, u_n) = 0, \qquad \ldots(1)$$

it is necessary and sufficient that the Jacobian $\dfrac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)}$ should vanish identically.

Proof : The condition is necessary *i.e.*, if there exists between $u_1, u_2, \ldots, u_n$ a relation

$$F(u_1, u_2, \ldots, u_n) = 0 \qquad \ldots(1)$$

their Jacobian is necessarily zero.

Differentiating (1) partially with respect to $x_1, x_2, \ldots, x_n$, we get

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_1} = 0,$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_2}+\frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_2}+\cdots+\frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_2}=0,$$

....

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_n}+\frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_n}+\cdots+\frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_n}=0.$$

Eliminating $\frac{\partial F}{\partial u_1},\frac{\partial F}{\partial u_2},\ldots,\frac{\partial F}{\partial u_n}$ from these equations, we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}=0 \quad \text{or} \quad \frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}=0.$$

The condition is sufficient, *i.e.*, if the Jacobian $J(u_1, u_2, \ldots, u_n)$ is zero, then there must exist a relation between $u_1, u_2, \ldots, u_n$.

The equations connecting the functions $u_1, u_2, \ldots, u_n$ and the variables x_1, $x_2, \ldots, x_n$ are always capable of being put into the following form :

$$\phi_1(x_1, x_2, \ldots, x_n, u_1)=0$$

$$\phi_2(x_2, x_3, \ldots, x_n, u_1, u_2)=0$$

....

$$\phi_r(x_r, x_{r+1}, \ldots, x_n, u_1, u_2, \ldots, u_r)=0$$

....

$$\phi_n(x_n, u_1, u_2, \ldots, u_n)=0.$$

Then, we have

$$J=\frac{\partial(u_1,u_2,\ldots,u_n)}{\partial(x_1,x_2,\ldots,x_n)}=(-1)^n\frac{\dfrac{\partial(\phi_1,\phi_2,\ldots,\phi_n)}{\partial(x_1,x_2,\ldots,x_n)}}{\dfrac{\partial(\phi_1,\phi_2,\ldots,\phi_n)}{\partial(u_1,u_2,\ldots,u_n)}}$$

$$=(-1)^n\frac{\dfrac{\partial\phi_1}{\partial x_1}\dfrac{\partial\phi_2}{\partial x_2}\cdots\dfrac{\partial\phi_n}{\partial x_n}}{\dfrac{\partial\phi_1}{\partial u_1}\dfrac{\partial\phi_2}{\partial u_2}\cdots\dfrac{\partial\phi_n}{\partial u_n}}.$$ [See note after article 8.1]

Now, if $J=0$, we have

$$\frac{\partial\phi_1}{\partial x_1}\cdot\frac{\partial\phi_2}{\partial x_2}\cdots\frac{\partial\phi_r}{\partial x_r}\cdots\frac{\partial\phi_n}{\partial x_n}=0$$

i.e., $\dfrac{\partial \phi_r}{\partial x_r} = 0$ for *some value* of r between 1 and n.

Hence, for that particular value of r the function ϕ_r must not contain x_r; and accordingly the corresponding equation is of the form

$$\phi_r (x_{r+1}, \ldots, x_n, u_1, u_2, \ldots, u_r) = 0.$$

Consequently between this and the remaining equations $\phi_{r+1} = 0, \phi_{r+2} = 0, \ldots,$ $\phi_n = 0$, the variables $x_{r+1}, x_{r+2}, \ldots, x_n$ can be eliminated so as to give a final equation between $u_1, u_2, \ldots, u_n$ alone.

Hence the theorem is established.

Illustrative Examples

Example 1 : *Show that the functions*

$$u = x + y - z,$$
$$v = x - y + z,$$
$$w = x^2 + y^2 + z^2 - 2yz$$

are not independent of one another. Also find the relation between them.

(Garhwal 2000)

Solution : Here

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y - z) & 2(z - y) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y - z) & 0 \end{vmatrix}, \text{ adding } C_2 \text{ to } C_3$$

$$= 0.$$

Since the Jacobian is zero, the functions are not independent.

Now $u + v = 2x$ and $u - v = 2(y - z)$.

Therefore $(u + v)^2 + (u - v)^2 = 4(x^2 + y^2 + z^2 - 2yz) = 4w.$

This is the required relation between u, v, w.

Example 2 : *Show that $ax^2 + 2hxy + by^2$ and $Ax^2 + 2Hxy + By^2$ are independent unless*

$$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$

Solution : Let $u = ax^2 + 2hxy + by^2, v = Ax^2 + 2Hxy + By^2$.

If the functions u, v are not independent, then

$$\frac{\partial (u, v)}{\partial (x, y)} = 0$$

or $$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2(ax + hy) & 2(hx + by) \\ 2(Ax + Hy) & 2(Hx + By) \end{vmatrix} = 0$$

or $\quad (ax+hy)(Hx+By)-(hx+by)(Ax+Hy)=0$

or $\quad (aH-Ah)x^2+(aB-Ab)xy+(Bh-bH)y^2=0.$

Since the variables x, y are independent, the coefficients of x^2 and y^2 in the above equation must be separately zero. Hence, we have

$$aH-Ah=0 \text{ and } Bh-bH=0$$

whence $\dfrac{a}{A}=\dfrac{h}{H}=\dfrac{b}{B}\cdot$

$$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)}=-\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}\cdot$$

Comprehensive Exercise 3

1. If $u=x^2+y^2+z^2, v=x+y+z, w=xy+yz+zx$,

   show that the Jacobian $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ vanishes identically. Also find the relation between u, v and w. **(Avadh 2014)**

2. If $u=(x+y)/(1-xy)$ and $v=\tan^{-1}x+\tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}\cdot$

   Are u and v functionally related ? If so, find the relationship.

3. If the functions u, v, w of three independent variables x, y, z are not independent, prove that the Jacobian of u, v, w with respect to x, y, z vanishes.

4. Show that the functions $u=3x+2y-z, v=x-2y+z$ and $w=x(x+2y-z)$ are not independent and find the relation between them.

5. Show that the functions

$$u=x+y+z,$$
$$v=xy+yz+zx,$$
$$w=x^3+y^3+z^3-3xyz$$

   are not independent. Find the relation between them. **(Meerut 2013B)**

6. If $u=x+2y+z, v=x-2y+3z$ and $w=2xy-xz+4yz-2z^2$, show that they are not independent. Find the relation between u, v and w.

7. If $u=\dfrac{x+y}{z}, v=\dfrac{y+z}{x}, w=\dfrac{y(x+y+z)}{xz}$, show that u, v, w are not independent and find the relation between them.

8. If $u=x+y+z+t, v=x+y-z-t, w=xy-zt, r=x^2+y^2-z^2-t^2$, show that $\dfrac{\partial(u,v,w,r)}{\partial(x,y,z,t)}=0$ and hence find a relation between u, v, w and r.

9. If $f(0)=0$ and $f'(x)=\dfrac{1}{1+x^2}$, prove without using the method of integration, that

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

   (Meerut 2012B, 13)

Answers 3

1. $v^2 = u + 2w$. **2.** $u = \tan v$. **4.** $u^2 - v^2 = 8w$.
5. $u^3 = 3uv + w$. **6.** $u^2 - v^2 = 4w$. **7.** $uv = w + 1$.
8. $uv = r + 2w$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ..." so that the following statements are complete and correct.

1. If u and v are functions of two independent variables x and y, then

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dots \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

2. If $x = r \cos \theta, y = r \sin \theta$, then $\dfrac{\partial (r, \theta)}{\partial (x, y)} = \dots$.

3. $\dfrac{\partial (u, v)}{\partial (x, y)} \times \dfrac{\partial (x, y)}{\partial (u, v)} = \dots$.

4. If $x = u(1 + v), y = v(1 + u)$, then $\dfrac{\partial (x, y)}{\partial (u, v)} = \dots$.

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

5. If $x = r \cos \theta, y = r \sin \theta$, then

(a) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = r$

(b) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = \dfrac{1}{r}$

(c) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = r^2$

(d) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = \dfrac{1}{r^2}$

6. If $u = \dfrac{y^2}{2x}$, $v = \dfrac{x^2 + y^2}{2x}$, then

(a) $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{y}{2x}$

(b) $\dfrac{\partial (u, v)}{\partial (x, y)} = -\dfrac{y}{2x}$

(c) $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{2x}{y}$

(d) $\dfrac{\partial (u, v)}{\partial (x, y)} = -\dfrac{2x}{y}$

True or False:

Write 'T' for true and 'F' for false statement.

7. If $u_1 = \dfrac{x_2 x_3}{x_1}$, $u_2 = \dfrac{x_3 x_1}{x_2}$, $u_3 = \dfrac{x_1 x_2}{x_3}$, then $\dfrac{\partial (u_1, u_2, u_3)}{\partial (x_1, x_2, x_3)} = -4.$

8. If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2, then

$$\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} = \frac{\partial (u_1, u_2)}{\partial (y_1, y_2)} \cdot \frac{\partial (y_1, y_2)}{\partial (x_1, x_2)}.$$

9. If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, then $\dfrac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = r^2 \sin \theta.$

10. If u, v and w are functions of three independent variables x, y and z, then

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} \cdot \frac{\partial (x, y, z)}{\partial (u, v, w)} = 0.$$

11. If u_1, u_2 and u_3 are functions of three independent variables x_1, x_2 and x_3 connected by the equations

$F_1 (u_1, u_2, u_3, x_1, x_2, x_3) = 0,$

$F_2 (u_1, u_2, u_3, x_1, x_2, x_3) = 0,$

and $F_3 (u_1, u_2, u_3, x_1, x_2, x_3) = 0,$

then $\dfrac{\partial (u_1, u_2, u_3)}{\partial (x_1, x_2, x_3)} = \dfrac{\dfrac{\partial (F_1, F_2, F_3)}{\partial (x_1, x_2, x_3)}}{\dfrac{\partial (F_1, F_2, F_3)}{\partial (u_1, u_2, u_3)}}.$

12. The functions $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$ are not independent of each other.

13. If $x + y + z = u$, $y + z = v$, $z = uvw$,

then $\dfrac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v.$

14. If u, v, w are functions of three independent variables x, y, z, then u, v, w are not independent of each other if $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \neq 0$.

Answers

1. $\dfrac{\partial u}{\partial y}$.	**2.** $\dfrac{1}{r}$.	**3.** 1.	**4.** $1 + u + v$.
5. (a).	**6.** (b).	**7.** *F*.	**8.** *T*.
9. *T*.	**10.** *F*.	**11.** *F*.	**12.** *T*.
13. *T*.	**14.** *F*.		

Chapter

9

Maxima and Minima of Functions of Two Independent Variables

9.1 Maximum or Minimum

Definition : Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively. Then $f(a, b)$ is said to be a *maximum or a minimum* value of $f(x, y)$ according as $f(a+h, b+k)$ is *less* or *greater* than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

9.2 Necessary Conditions for the Existence of a Maximum or a Minimum of $f(x, y)$ at $x = a, y = b$

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y)$ at $x = a, y = b$ if the expression $f(a+h, b+k) - f(a, b)$ is of invariable sign for all sufficiently small independent values of h and k provided both of them are not equal to zero. If the sign of $f(a+h, b+k) - f(a, b)$ is negative, we shall have a maximum of $f(x, y)$ at $x = a, y = b$. If it is positive, $f(x, y)$ has a minimum at $x = a, y = b$.

By Taylor's theorem for two variables, we have

$$f(a+h, b+k) = f(a, b) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right)_{\substack{x = a,\\ y = b}}$$

$$+\frac{1}{2\,!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\,\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a,\\ y=b}}+\ldots$$

$$\therefore\quad f(a+h,b+k)-f(a,b)=h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}}+k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}}$$

$$+\ \text{terms of the second and higher orders in } h \text{ and } k. \qquad \ldots(1)$$

By taking h and k, sufficiently small, the first degree terms in h and k can be made to govern the sign of the right hand side and therefore of the left hand side of (1).

Thus the sign of $[f(a+h,b+k)-f(a,b)]$

$$=\text{the sign of}\left[h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}}+k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}}\right] \qquad \ldots(2)$$

Taking $k=0$, we find that if $\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}}\neq 0$, the right hand side of (2) changes sign when h changes sign. Therefore $f(x,y)$ cannot have a maximum or minimum at $x=a$, $y=b$ if $\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}}\neq 0.$

Similarly taking $h=0$, we can see that $f(x,y)$ cannot have a maximum or a minimum at $x=a, y=b$ if $\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}}\neq 0.$

Thus *a set of necessary conditions that $f(x,y)$ should have a maximum or a minimum at $x=a, y=b$ is that*

$$\left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)_{\substack{\boldsymbol{x}=\boldsymbol{a}\\ \boldsymbol{y}=\boldsymbol{b}}}=\mathbf{0}\ \textit{and}\ \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\right)_{\substack{\boldsymbol{x}=\boldsymbol{a}\\ \boldsymbol{y}=\boldsymbol{b}}}=\mathbf{0}.$$

The above conditions are necessary but not sufficient for the existence of maxima or minima.

9.3 Stationary , Extreme and Saddle Points (Kashi 2014)

A point (a,b) is called *a stationary point*, if both the first order partial derivatives of the function $f(x,y)$ vanish at that point. A stationary point which is either a maximum or a minimum is called an *extreme point* and the value of the function at the point is called an *extreme value*. A stationary point is not necessarily an extreme point. Thus a stationary point may be a maximum or a minimum or neither of these two. To decide whether a point is really an exreme point, a further investigation is necessary.

Saddle Point : A stationary point which is neither a maximum nor a minimum is called a *saddle point.*

9.4 Sufficient Condition for Maxima or Minima

(Lucknow 2008)

Let $r = \left(\dfrac{\partial^2 f}{\partial x^2}\right)_{\substack{x = a \\ y = b}}$, $s = \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x = a \\ y = b}}$ and $t = \left(\dfrac{\partial^2 f}{\partial y^2}\right)_{\substack{x = a \\ y = b}}$.

If $\left(\dfrac{\partial f}{\partial x}\right)_{\substack{x = a \\ y = b}} = 0$ and $\left(\dfrac{\partial f}{\partial y}\right)_{\substack{x = a \\ y = b}} = 0$, *i.e.* if the necessary conditions for the existence of maxima or minima are satisfied, we have

$$f(a + h, b + k) - f(a, b) = \frac{1}{2\,!}(rh^2 + 2shk + tk^2) + R_3,$$

where R_3 consists of terms of third and higher orders in h and k.

For sufficiently small values of h and k, the sign of

$$\frac{1}{2}(rh^2 + 2shk + tk^2) + R_3$$

is the same as that of $rh^2 + 2shk + tk^2$.

Now the following three different cases arise :

Case I : $rt - s^2 > 0$. In this case obviously neither r nor t can be zero. Therefore we write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[r^2 h^2 + 2srhk + rtk^2]$$

$$= \frac{1}{r}[(rh + sk)^2 + (rt - s^2) k^2].$$

Since $rt - s^2$ is positive, therefore $(rh + sk)^2 + (rt - s^2) k^2$ is positive for all values of h and k except when $rh + sk = 0, k = 0$ *i.e.* when $h = 0, k = 0$ which is obviously not possible.

Thus in this case the expression $rh^2 + 2shk + tk^2$ will have the *same* sign for all values of h and k. This sign is determined by the sign of r.

Thus $f(x, y)$ will have a maximum or a minimum at $x = a, y = b$ if $rt > s^2$. Further $f(x, y)$ is a maximum or a minimum according as r is negative or positive.

Case II : $rt - s^2 < 0$. In this case if $r \neq 0$, we can write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[(rh + sk)^2 + (rt - s^2) k^2].$$

If $k = 0, h \neq 0$, the sign of this expression will be the same as that of r. But if $k \neq 0$, $rh + sk = 0$, the sign of this expression will be opposite to that of r since $rt - s^2$ is negative. Thus in this case the expression $rh^2 + 2shk + tk^2$ is not of invariable sign.

A similar argument can be given if $t \neq 0$.

In case $r = 0$ as well as $t = 0$, we have

$$rh^2 + 2shk + tk^2 = 2shk,$$

which obviously does not keep the same sign for all values of h and k.

Thus $f(x, y)$ will have neither a maximum nor a minimum at $x = a$, $y = b$, if $rt < s^2$.

Case III : $rt - s^2 = 0$. If $r \neq 0$, we can write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[(rh + sk)^2 + (rt - s^2)k^2]$$

$$= \frac{1}{r}(rh + sk)^2. \qquad [\because rt - s^2 = 0]$$

This expression becomes zero when $rh + sk = 0$. Therefore the nature of the sign of

$$f(a + h, b + k) - f(a, b)$$

depends upon the consideration of R_3. The case is, therefore, doubtful.

In case $r = 0$, we must have $s = 0$, because of the condition $rt - s^2 = 0$.

$$\therefore \qquad rh^2 + 2shk + tk^2 = tk^2,$$

which is zero when $k = 0$ whatever h may be. The case is again doubtful.

Thus, if $rt - s^2 = 0$, the case is doubtful and further investigation is needed to determine whether $f(x, y)$ is a maximum or a minimum at $x = a$, $y = b$, or not.

9.5 Working Rule for Maxima and Minima

Suppose $f(x, y)$ is a given function of x and y. Find $\partial f/\partial x$ and $\partial f/\partial y$ and solve the simultaneous equations $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. In order to solve these equations we may either eliminate one of the variables, or factorise the equations. In the latter case each factor of the first equation must be solved in conjunction with each factor of the second equation. Suppose solving these equations we get the pairs of values of x and y as (a_1, b_1), (a_2, b_2) etc. Then all these pairs of roots will give stationary values of $f(x, y)$.

To discuss the maximum or minimum at $x = a_1, y = b_1$, we should find

$$r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\substack{x = a_1 \\ y = b_1}}, \quad s = \left(\frac{\partial^2 u}{\partial x \partial y}\right)_{\substack{x = a_1 \\ y = b_1}}, \quad t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{\substack{x = a_1 \\ y = b_1}}.$$

Then calculate $rt - s^2$.

If $rt - s^2 > 0$ and r is negative, $f(x, y)$ is maximum at $x = a_1, y = b_1$.

If $rt - s^2 > 0$ and r is positive, $f(x, y)$ is minimum at $x = a_1, y = b_1$.

If $rt - s^2 < 0$, $f(x, y)$ is neither maximum nor minimum at $x = a_1, y = b_1$. In this case the function $z = f(x, y)$ is stationary at $x = a, y = b$ but the stationary value $f(a, b)$ is neither maximum nor minimum. Hence, $(x = a, y = b, z = (a, b))$ is a **saddle point** of the surface $z = f(x, y)$.

If $rt - s^2 = 0$, the case is doubtful and further investigation will be required to decide it. We shall leave this case.

Illustrative Examples

Example 1 : *Discuss the maximum or minimum values of u where*

$$u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3. \qquad \textbf{(Avadh 2013)}$$

Solution : We have $\partial u/\partial x = 2a^2y - 6axy + 3x^2y + y^3$,

and $\qquad \partial u/\partial y = 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2.$

Also $\quad r = \partial^2 u/\partial x^2 = 6ay + 6xy,$

$$s = \partial^2 u/\partial x\,\partial y = 2a^2 - 6ax + 3x^2 + 3y^2,$$

and $\quad t = \partial^2 u/\partial y^2 = -6ay + 6xy.$

For a maximum or minimum of u, we have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0.$$

Thus, we have

$$\left.\begin{aligned} y(2a^2 - 6ax + 3x^2 + y^2) &= 0 \\ \text{and} \quad 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 &= 0 \end{aligned}\right\}.$$

Therefore we have to consider the pairs of equations, *viz.*,

$$\left.\begin{aligned} y &= 0 \\ 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 &= 0 \end{aligned}\right\} \qquad ...(1)$$

and

$$\left.\begin{aligned} 2a^2 - 6ax + 3x^2 + y^2 &= 0 \\ 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 &= 0 \end{aligned}\right\} \qquad ...(2)$$

Putting $y = 0$ in the second equation of the pair (1), we get

$$2a^2x - 3ax^2 + x^3 = 0 \quad i.e., \quad x(x^2 - 3ax + 2a^2) = 0$$

i.e., $\quad x(x - a)(x - 2a) = 0 \quad i.e., \quad x = 0, x = a, x = 2a.$

Thus the pair (1) gives the following values of x and y:

$$x = 0, y = 0; x = a, y = 0; x = 2a, y = 0.$$

Multiplying the first equation of the pair (2) by x and subtracting it from the second equation of the pair, we get

$$3ax^2 - 3ay^2 - 2x^3 + 2xy^2 = 0 \quad \text{or} \quad (x^2 - y^2)(3a - 2x) = 0.$$

$\therefore \quad x = \frac{3}{2}a \quad \text{and} \quad x = \pm y.$

When $x = \frac{3}{2}a$, the first equation of the pair (2) gives $y = \pm \frac{1}{2}a$.

When $x = y$, we have $2a^2 - 6ay + 4y^2 = 0 \quad i.e., \quad y = a, \frac{1}{2}a$.

Also when $x = -y$, we have $2a^2 + 6ay + 4y^2 = 0 \quad i.e., \quad y = -a, -\frac{1}{2}a$.

Thus in all we get the following pairs of values of x and y which make the function u stationary:

$$(0, 0), (a, 0), (2a, 0), \left(\frac{3}{2}a, \frac{1}{2}a\right), \left(\frac{3}{2}a, -\frac{1}{2}a\right)$$

$$(a, a), \left(\frac{1}{2}a, \frac{1}{2}a\right), (a, -a), \left(\frac{1}{2}a, -\frac{1}{2}a\right).$$

For (0, 0),

$$r = 0, s = 2a^2, t = 0 \text{ so that } rt - s^2 \text{ is negative.}$$

Therefore we have neither a maximum nor a minimum of u at $(0, 0)$.

Similarly, we can show that u has neither a maximum nor a minimum at $(a, 0)$, $(2a, 0)$, (a, a), $(a, -a)$.

For $(3a/2, a/2)$,

$r = \frac{3}{2}a^2, s = \frac{1}{2}a^2, t = \frac{3}{2}a^2$ so that $rt - s^2$ is positive. Since r is positive, therefore u has minimum at this point.

Similarly, we can show that u has a maximum at $\left(\frac{1}{2}a, -\frac{1}{2}a\right)$.

For $(3a/2, -a/2)$,

$r = -\frac{3}{2}a^2, s = -\frac{1}{2}a^2, t = -\frac{3}{2}a^2$ so that $rt - s^2$ is positive. Since r is negative, therefore u has a maximum at this point.

Similarly, we can show that u has a maximum at $(a/2, a/2)$.

Example 2 (a) : *Find the extreme values of* $xy(a - x - y)$.

Solution : Let $u = xy(a - x - y)$.

Then $$\frac{\partial u}{\partial x} = ay - 2xy - y^2$$

and $$\frac{\partial u}{\partial y} = ax - x^2 - 2xy.$$

Also $$r = \frac{\partial^2 u}{\partial x^2} = -2y, \; s = \frac{\partial^2 u}{\partial y \partial x} = a - 2x - 2y,$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -2x.$$

For a maximum or minimum of u, we have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0.$$

Thus, we have

$$\left.\begin{array}{l} ay - 2xy - y^2 = 0 \\ ax - x^2 - 2xy = 0 \end{array}\right\}.$$

These equations can be written as

$$y(a - 2x - y) = 0, x(a - x - 2y) = 0,$$

so that we have to consider the four pairs of equations, *viz.*,

$$y = 0, x = 0; a - 2x - y = 0, x = 0; y = 0, a - x - 2y = 0;$$

$$a - 2x - y = 0, a - x - 2y = 0.$$

Solving these, we get the following pairs of values of x and y which make the function stationary :

$$(0, 0), (0, a), (a, 0), \left(\frac{1}{3}a, \frac{1}{3}a\right).$$

For (0, 0),

$$r = 0, s = a, t = 0 \text{ so that } rt - s^2 \text{ is negative.}$$

Therefore we have neither a maximum nor a minimum of u at $(0, 0)$.

For $(0, a)$,

$$r = -2a, s = -a, t = 0 \text{ so that } rt - s^2 \text{ is negative.}$$

Therefore u has not an extreme value at $(0, a)$.

Similarly, we may show that u has not an extreme value at $(a, 0)$.

For $\left(\frac{1}{3}a, \frac{1}{3}a\right)$,

$$r = -\frac{2}{3}a, s = -\frac{1}{3}a, t = -\frac{2}{3}a \text{ so that } rt - s^2 \text{ is positive.}$$

Therefore u has an extreme value at $\left(\frac{1}{3}a, \frac{1}{3}a\right)$ and it will be a maximum or minimum according as, r is negative or positive *i.e.*, according as, a is positive or negative.

$\therefore$ the extreme value of u is $(1/27)\,a^3$.

Example 3 : *Show that the distance l of any point (x, y, z) on the plane $2x+3y-z=12$ from the origin is given by*

$$l=\sqrt{[x^2+y^2+(2x+3y-12)^2]}.$$

Hence find the point on the plane that is nearest to the origin.

Solution : If l is the distance from $(0,0,0)$ of any point (x, y, z), then $l=\sqrt{(x^2+y^2+z^2)}$. If the point (x, y, z) lies on the plane $2x+3y-z=12$, then $l=\sqrt{[x^2+y^2+(2x+3y-12)^2]}$.

[$\because z=2x+3y-12$, from the equation of the plane].

$$\therefore \quad l^2=x^2+y^2+(2x+3y-12)^2$$
$$=5x^2+10y^2+12xy-48x-72y+144=u, \text{ say.}$$

Now l is maximum or minimum according as l^2 *i.e.*, u is maximum or minimum.

For a maximum or minimum of u, we have

$$\frac{\partial u}{\partial x}=10x+12y-48=0,$$

and
$$\frac{\partial u}{\partial y}=20y+12x-72=0.$$

Solving these equations, we get

$$x=12/7, \text{ and } y=18/7.$$

Also
$$r=\frac{\partial^2 u}{\partial x^2}=10, s=\frac{\partial^2 u}{\partial x\,\partial y}=12,$$

and
$$t=\frac{\partial^2 u}{\partial y^2}=20.$$

$\therefore$ $rt-s^2=10\times 20-12^2=+$ ive. Since $rt-s^2>0$ and $r>0$, therefore u is minimum and hence l is minimum when $x=12/7$ and $y=18/7$. Putting these values of x and y in the equation of the plane, we get

$$z=2.(12/7)+3.(18/7)-12=-6/7.$$

Therefore the required point is $(12/7, 18/7, -6/7)$.

Example 4 : *Locate the stationary points of $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature.*

Solution : Let $u=x^4+y^4-2x^2+4xy-2y^2.$

Then
$$\frac{\partial u}{\partial x}=4x^3-4x+4y$$

and
$$\frac{\partial u}{\partial y}=4y^3+4x-4y.$$

The stationary points are given by

$$\frac{\partial u}{\partial x}=0 \text{ i.e., } 4x^3-4x+4y=0, \qquad \text{...(1)}$$

and
$$\frac{\partial u}{\partial y}=0 \text{ i.e., } 4y^3+4x-4y=0. \qquad \text{...(2)}$$

Now we shall find the points (x, y) satisfying the simultaneous equations (1) and (2).

Adding (1) and (2), we get

$$4x^3+4y^3=0 \quad \text{i.e.,} \quad x^3+y^3=0$$

i.e., $(x+y)(x^2-xy+y^2)=0$.

$\therefore$ either $x+y=0$, ...(3)

or $x^2-xy+y^2=0$. ...(4)

First we solve the simultaneous equations (1) and (3).

From (3), we have $y=-x$.

Putting $y=-x$ in (1), we get

$$4x^3-8x=0 \text{ i.e., } x^3-2x=0 \text{ i.e., } x(x^2-2)=0$$

i.e., $x=0, \sqrt{2}$ or $-\sqrt{2}$.

The corresponding values of y are $y=0, -\sqrt{2}, \sqrt{2}$.

Thus the points $(0,0)$, $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$ satisfy (1) and (2).

If we solve the equations (1) and (4), we get (0, 0) as the only real solution.

Hence the function u is stationary at the points

$$(0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2}).$$

We have $r=\dfrac{\partial^2 u}{\partial x^2}=12x^2-4$, $s=\dfrac{\partial^2 u}{\partial x\,\partial y}=4$, $t=\dfrac{\partial^2 u}{\partial y^2}=12y^2-4$.

At (0, 0), $r=-4, s=4, t=-4$, so that $rt-s^2=16-16=0$.

Thus at the point (0, 0), the case is doubtful and further investigation is needed.

At $(\sqrt{2},-\sqrt{2})$, $r=20, s=4, t=20$, so that $rt-s^2=400-16=+$ ive.

Therefore u has an extreme value at this point.

Since r is positive, therefore u has a minimum at this point.

At $(-\sqrt{2},\sqrt{2})$, $r=20, s=4, t=20$, so that $rt-s^2$ is positive. Since r is positive, therefore u has a minimum at this point also.

Note : We may tackle the doubtful case at the point (0, 0) by the following consideration :

We have $u=x^4+y^4-2(x-y)^2$.

At the point $(0,0)$, we have $u=0$.

At the points in the neighbourhood of the point (0, 0) where $x\neq y$, the value of u is approximately given by

$$u=-2(x-y)^2,$$

[Neglecting the terms x^4+y^4 because the numerical values of x and y are small].

Thus at such points u is –ive.

Again at the points in the neighbourhood of the point $(0,0)$, where $x=y$, we have $u=x^4+y^4$ which is positive.

Thus in the neighbourhood of the point $(0,0)$, there are points at which u takes values less than its value at the point (0, 0) and there are points at which u takes values greater than its value at the point $(0,0)$. Hence u cannot have a maximum or a minimum value at the point $(0,0)$.

Example 5 : *Find the minimum value of* $x^2+y^2+z^2$ *when* $ax+by+cz=p$.

(Kashi 2014)

Solution : Let $u=x^2+y^2+z^2$.

Here u is a function of three variables x, y and z. But we can eliminate one variable with the help of the given relation, *viz.*,

$$ax+by+cz=p.$$

From this relation, we have $z=\dfrac{p-ax-by}{c}$.

Putting this value of z in the value of u, we get

$$u = x^2 + y^2 + \frac{(p - ax - by)^2}{c^2},$$

where u has been expressed as a function of two independent variables x and y.

We have $$\frac{\partial u}{\partial x} = 2x - \frac{2a}{c^2}(p - ax - by),$$

and $$\frac{\partial u}{\partial y} = 2y - \frac{2b}{c^2}(p - ax - by).$$

Solving $\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0$, we get

$$x = \frac{ap}{a^2 + b^2 + c^2}, \text{ and } y = \frac{bp}{a^2 + b^2 + c^2}.$$

Again, we get $r = \dfrac{\partial^2 u}{\partial x^2} = 2 + \dfrac{2a^2}{c^2}$, $s = \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{2ab}{c^2}$,

and $$t = \frac{\partial^2 u}{\partial y^2} = 2 + \frac{2b^2}{c^2}.$$

$$\therefore \qquad rt - s^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4} = 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right).$$

Since $rt - s^2$ is positive and r is also positive, therefore u is minimum for the values of x and y found above.

The minimum value of u, therefore, is $\dfrac{p^2}{(a^2 + b^2 + c^2)}$.

Comprehensive Exercise 1

1. **Discuss the maxima and minima of the following functions :**

(i) $u = xy + a^3(1/x + 1/y)$. **(Meerut 2003, 13; Kashi 11; Avadh 11; Purvanchal 14)**

(ii) $u = x^3y^2(1 - x - y)$. **(Bundelkhand 2011, 14; Rohilkhand 12; Avadh 12)**

(iii) $u = x^3 + y^3 - 3axy$. **(Meerut 2003, 12; Kanpur 07, 08)**

(iv) $u = x^2 + y^2 + 6x + 12$. **(Kashi 2013)**

(v) $u = x^2 + y^2 + 2/x + 2/y$. **(Meerut 2012B)**

(vi) $u = y^2 + 4xy + 3x^2 + x^3$.

2. (i) $u = 3x^2 - y^2 + x^3$. **(Meerut 2013B)**

(ii) $u = 2x^2y + x^2 - y^2 + 2y$.

(iii) $u = 2\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y) + \cos(x + y)$.

(iv) $u = \sin x \sin y \sin(x + y)$. **(Bundelkhand 2011; Meerut 12B)**

[**Hint :** It is sufficient to consider the values of x and y between 0 and π since the function u is periodic with period π both for x and y]

(v) $u = x^2y^2 - 5x^2 - 8xy - 5y^2$. **(Avadh 2006)**

(vi) $u = x^4 + 2x^2 y - x^2 + 3y^2$.

3. If $u = ax^3 y^2 - x^4 y^2 - x^3 y^3$, prove that $x = a/2, y = a/3$ make u a maximum.

4. Investigate the maxima and minima of

$$2(x - y)^2 - x^4 - y^4,$$

leaving aside any doubtful case that may arise. **(Kanpur 2009)**

5. Examine the following surface for high and low points :

$$z = x^2 + xy + 3x + 2y + 5.$$

Find the saddle points of the surface if there are any.

6. Find all the stationary points of the function

$$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x,$$

examining whether they are maxima or minima.

7. Examine for maximum and minimum values of the function

$$z = x^2 - 3xy + y^2 + 2x.$$

8. Find the points (x, y) where the function $xy(1 - x - y)$ is maximum or minimum. What is the maximum value of the function ?

9. Examine the function $z = x^2 y - y^2 x - x + y$ for maxima and minima. Find the saddle points of the given surface.

10. Let $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$. Show that $f(x, y)$ has neither a maximum nor a minimum at $(0, 0)$.

11. Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum. **(Kanpur 2010; Kumaun 08)**

Answers 1

1. (i) Minimum at $x = y = a$. (ii) Maximum at $x = \frac{1}{2}, y = \frac{1}{3}$.

(iii) $x = y = a$ gives a maximum or a minimum according as a is negative or positive.

(iv) Minimum at $(-3, 0)$. (v) Minimum at $(1, 1)$.

(vi) Minimum at $x = \frac{2}{3}, y = -\frac{4}{3}$.

2. (i) Maximum at $(-2, 0)$. (ii) No exreme value.

(iii) Maximum when $x = y = n\pi + (-1)^n \pi/6$
and minimum when $x = y = 2n\pi - \pi/2$.

(iv) Maximum at $x = y = \pi/3$ and minimum at $x = y = 2\pi/3$.

(v) Maximum at $x = y = 0$.

(vi) Minimum when $x = \pm\sqrt{3}/2, y = -1/4$.

4. Maximum when $x = \pm\sqrt{2}, y = \mp\sqrt{2}$.

5. No high and low points. The point $(2, 1, 3)$ is a saddle point.

6. Maximum at $(4, 0)$ and minimum at $(6, 0)$.

7. The function z is stationary at the point $(4/5, 6/5)$. But it is neither maximum nor minimum at this point.

8. Maximum at the point $\left(\frac{1}{3}, \frac{1}{3}\right)$. Max. value $= 1/27$.

9. The function is stationary at the points (1, 1) and (− 1, − 1), but it has no exreme value. The points (1, 1, 0) and (− 1, − 1, 0) are the two saddle points of the surface $z = x^2 y - y^2 x - x + y$.

11. Centroid of the triangle.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …" so that the following statements are complete and correct.

1. Let $f(x, y)$ be a function of two independent variables x and y. The necessary conditions for the existence of a maximum or a minimum of $f(x, y)$ at $x = a, y = b$ are

$$\frac{\partial f}{\partial x} = 0, \text{ and } \frac{\partial f}{\partial y} = \ldots \text{ at } x = a, y = b.$$

2. Let $f(x, y)$ be a function of two independent variables x and y. If at the point (a, b), we have

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2 > 0$$

and $\dfrac{\partial^2 f}{\partial x^2} > 0$, then $f(x, y)$ is … at (a, b).

3. Let $f(x, y)$ be a function of two independent variables x and y. Let

$$r = \frac{\partial^2 f}{\partial x^2}, \; s = \frac{\partial^2 f}{\partial x \, \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}.$$

If at the point (a, b), we have

$$\frac{\partial f}{\partial x} = 0, \; \frac{\partial f}{\partial y} = 0, \; rt - s^2 > 0 \text{ and } r < 0, \text{ then } f(x, y) \text{ is … at } (a, b).$$

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

4. Let $f(x, y)$ be a function of two independent variables x and y. Let

$$r = \frac{\partial^2 f}{\partial x^2}, \; s = \frac{\partial^2 f}{\partial x \, \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}.$$

If at the point (a, b), we have $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$, then $f(x, y)$ is maximum at (a, b) if at (a, b) we have

(a) $rt - s^2 < 0$ (b) $rt - s^2 > 0$ and $r < 0$

(c) $rt - s^2 > 0$ and $r > 0$ (d) $rt - s^2 < 0$ and $r < 0$

5. The function $u = x^2 + y^2 + 6x + 12$ is minimum at

(a) (3, 0) (b) (0, 3)

(c) (− 3, 0) (d) (3, 3)

6. The function $u = 3x^2 - y^2 + x^3$ is maximum at

(a) (− 2, 0) (b) (2, 0)

(c) (0, − 2) (d) (0, 2)

True or False:

Write 'T' for true and 'F' for false statement.

7. Let $f(x, y)$ be a function of two independent variables x and y. Let

$$r = \frac{\partial^2 f}{\partial x^2},\ s = \frac{\partial^2 f}{\partial x\, \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}.$$

If at the point (a, b), we have $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $rt - s^2 < 0$, then $f(x, y)$ has an extreme value at (a, b).

8. Let $f(x, y)$ be a function of two independent variables x and y. If at (a, b), we have $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, then $f(x, y)$ must have a maximum or a minimum at (a, b).

9. The minimum value of $x^2 + y^2 + z^2$ when $ax + by + cz = p$ is $\frac{p^2}{(a^2 + b^2 + c^2)}$.

10. The function $xy(1 - x - y)$ has a maximum value at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

11. The function $x^3 + y^3 + 3xy$ has a maximum value at the point $(-1, -1)$.

12. The minimum value of the function $x^3 + y^3 - 6xy$ is -8.

Answers

1.	0.	**2.**	minimum.	**3.**	maximum.
4.	(b).	**5.**	(c).	**6.**	(a).
7.	*F*.	**8.**	*F*.	**9.**	*T*.
10.	*F*.	**11.**	*T*.	**12.**	*T*.

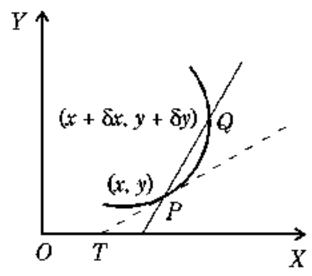
□

Chapter

10

Tangents and Normals

10.1 Tangent and Normal to a Curve



Tangent to a Curve : Let P be any given point on a curve and Q any other point on it in the neighbourhood of P. The point Q may be taken on either side of P. As Q tends to P, the straight line PQ, in general, tends to a definite straight line TP passing through P. This straight line is called the tangent to the curve at the point P.

(Kanpur 2014)

Normal to a Curve : The normal to a curve at any point P of it is the straight line which passes through that point and is perpendicular to the tangent to the curve at that point.

10.2 Equation of the Tangent (Cartesian Co-ordinates)

Let $y = f(x)$ be the cartesian equation of a curve. Let P be any given point (x, y) on this curve. Take a point Q $(x + \delta x, y + \delta y)$ on this curve in the neighbourhood of P.

If (X, Y) are current co-ordinates of any point on the chord PQ, then the equation of the chord PQ is

$$Y - y = \frac{(y + \delta y) - y}{(x + \delta x) - x}(X - x) \quad \text{or} \quad Y - y = \frac{\delta y}{\delta x}(X - x). \qquad \ldots(1)$$

Now, as Q tends to P, $\delta x \to 0$ and chord PQ tends to the tangent at P. Therefore the equation (1) tends to the equation

$$Y - y = \frac{dy}{dx}(X - x). \qquad \left[\because \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}\right].$$

Hence the equation of the tangent to the curve $y = f(x)$ at the point (x, y) is

$$\boldsymbol{Y - y = \frac{dy}{dx}(X - x).}$$

Note 1 : If we are to find the equation of the tangent to the curve $y = f(x)$ at the point (x_1, y_1) on it, we should first find the value of dy/dx of the curve at the point (x_1, y_1). The equation of the tangent at the point (x_1, y_1) will then be

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1),$$

where (x, y) are the current co-ordinates of any point on the tangent.

Note 2 : If the equations of the curve be given in parametric cartesian form say

$$x = f(t) \quad \text{and} \quad y = \phi(t), \text{ then}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}.$$

Hence the equation of the tangent at any point 't' on the curve is given by

$$Y - \phi(t) = \frac{\phi'(t)}{f'(t)}[X - f(t)].$$

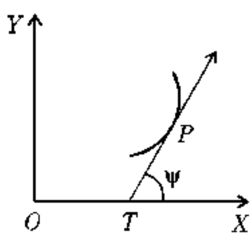
10.3 Geometrical Meaning of *dy/dx*

Let P be any given point (x, y) on the curve $y = f(x)$. Suppose the positive direction of the tangent at P is that in which x increases. Let ψ be the angle which the positive direction of the tangent at P makes with the positive direction of the axis of x. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

or

$$Y = \left(\frac{dy}{dx}\right)X + \left(y - x\frac{dy}{dx}\right). \qquad \ldots(1)$$



This equation is of the form $Y = mX + c$, $\qquad \ldots(2)$

which is the equation of the straight line whose **gradient** is m *i.e.*, the line makes an angle, with the positive direction of x-axis, whose tangent is m. Therefore comparing (1) and (2), we get

$$\boldsymbol{\frac{dy}{dx} = \tan \psi.}$$

Thus the differential coefficient dy/dx at any point (x, y) on the curve $y = f(x)$ is equal to the tangent of the angle which the positive direction of the tangent at P to the curve makes with the positive direction of the axis of x.

10.4 Tangents Parallel to the Co-ordinate Axes

If the tangent at any point is parallel to the axis of x, then $\psi = 0$ *i.e.*, $\tan\psi = 0$ and so we have $dy/dx = 0$ at that point.

On the other hand if a tangent is parallel to the axis of y or perpendicular to the axis of x, then

$$\psi = \pi/2 \text{ i.e., } \tan\psi = \tan(\pi/2) = \infty$$

and so we have $dy/dx = \infty$ or $dx/dy = 0$ at that point.

10.5 Equation of the Normal

Let P be any given point (x, y) on the curve $y = f(x)$. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \text{or} \qquad Y = \left(\frac{dy}{dx}\right)X + \left(y - x\frac{dy}{dx}\right).$$

Therefore the gradient of the tangent at P is dy/dx. If m be the gradient of the normal at P, then

$$m \cdot \frac{dy}{dx} = -1 \qquad \text{or} \qquad m = -\frac{1}{dy/dx} = -\frac{dx}{dy}.$$

Hence *the equation of the normal to the curve at P is*

$$\boldsymbol{Y - y = -\frac{dx}{dy}(X - x)} \quad \text{or} \quad \boldsymbol{\frac{dy}{dx}(Y - y) + (X - x) = 0}.$$

Important : If the equation of a curve is given in the form $f(x, y) = 0$, then

$$\boldsymbol{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}}.$$

Illustrative Examples

Example 1 : *Find the equations of the tangent and the normal at any point* (x, y) *of the curve* $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$.

Solution : Let $f(x, y) \equiv \dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} - 1 = 0$.

Then $\qquad \dfrac{\partial f}{\partial x} = \dfrac{mx^{m-1}}{a^m}$ and $\dfrac{\partial f}{\partial y} = \dfrac{my^{m-1}}{b^m}$.

$$\therefore \qquad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{b^m x^{m-1}}{a^m y^{m-1}}.$$

Hence the equation of the tangent at (x, y) is

$$Y - y = -\frac{b^m x^{m-1}}{a^m y^{m-1}}(X - x)$$

i.e.,
$$\frac{y^{m-1}}{b^m}(Y - y) = -\frac{x^{m-1}}{a^m}(X - x)$$

i.e.,
$$\frac{Y y^{m-1}}{b^m} + \frac{X x^{m-1}}{a^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m}.$$

But the point (x, y) lies on the given curve.

Therefore $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1.$

Hence the equation of the tangent at (x, y) is $\dfrac{X x^{m-1}}{a^m} + \dfrac{Y y^{m-1}}{b^m} = 1.$

Also, the equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$

i.e., $$Y - y = \frac{a^m y^{m-1}}{b^m x^{m-1}}(X - x) \quad i.e., \quad \frac{X - x}{b^m x^{m-1}} = \frac{Y - y}{a^m y^{m-1}}.$$

Example 2 : *Find the equation of the tangent at the point 't' to the cycloid*

$$x = a(t + \sin t), \quad y = a(1 - \cos t).$$ **(Agra 2014)**

Solution : We have $dx/dt = a(1 + \cos t)$, $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2 \sin t/2 \cos t/2}{2 \cos^2 t/2} = \tan \frac{t}{2}.$$

Hence the equation of the tangent at the point 't' is

$$y - a(1 - \cos t) = \tan \frac{t}{2}[x - a(t + \sin t)]$$

i.e., $$y - 2a \sin^2 \frac{t}{2} = (x - at) \tan \frac{t}{2} - a \sin t . \tan \frac{t}{2}$$

i.e., $$y - 2a \sin^2 \frac{t}{2} = (x - at) \tan \frac{t}{2} - 2a \sin^2 \frac{t}{2}$$

i.e., $$y = (x - at) \tan \frac{t}{2},$$

where (x, y) are the current co-ordinates of any point on the tangent.

Note : If the equations of a curve are given in the parametric form $x = f(t), y = \phi(t)$, then by the point t we mean the point whose co-ordinates are $x = f(t)$ and $y = \phi(t)$.

10.6 Angle of Intersection

The angle of intersection of two curves is defined as the angle between their tangents at their point of intersection.

In order to determine the angles of intersection of two given curves

$$f(x, y) = 0, \qquad \ldots(1)$$

and $$\phi(x, y) = 0 \qquad \ldots(2)$$

we should first solve the equations (1) and (2) simultaneously to get the points of intersection of (1) and (2).

If (x_1, y_1) is one of the points of intersection, then to find the angle of intersection at (x_1, y_1), we should find the slopes m_1 and m_2 of the tangents of the two curves at the point (x_1, y_1).

We have, $m_1 = \left(\dfrac{dy}{dx}\right)$ at (x_1, y_1) of the curve (1)

and $m_2 = \left(\dfrac{dy}{dx}\right)$ at (x_1, y_1) of the curve (2).

If $m_1 = m_2$, the angle of intersection is 0^o.

If $m_1 = \infty$, $m_2 = 0$, the angle of intersection is 90^o.

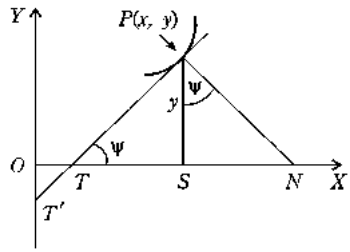
If $m_1 m_2 = -1$, again the angle of intersection is 90^o and we say that the *two curves intersect orthogonally.*

In all other cases, the acute angle between the tangents is equal to

$$\tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

10.7 Length of Cartesian Tangent, Normal, Subtangent and Subnormal

The lentgh of the tangent of a curve at one of its points is defined as the length of the portion of the tangent between its point of contact and the x-axis. The length of the projection of this segment on the x-axis is called the *length of the subtangent.*



Similarly the *length of the normal* is defined as the length of the portion of the normal between the point of contact of the tangent and the x-axis. The length of the projection of this segment on the x-axis is called the *length of the subnormal.*

Let P be any point (x, y) on the curve $y = f(x)$. Suppose tangent and the normal at P meet the x-axis in T and N respectively. Let PS be the ordinate of the point P.

Then $PS = y$.

If ψ be the angle which the tangent at P makes with x-axis, then $\angle PTS = \angle SPN = \psi$ and $\tan\psi = dy/dx$.

Length of tangent

$$= PT = y \operatorname{cosec}\psi = y\sqrt{(1 + \cot^2\psi)} = y\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

Length of sub-tangent

$$= TS = y\cot\psi = \boldsymbol{y}\frac{\boldsymbol{dx}}{\boldsymbol{dy}} = \frac{\boldsymbol{y}}{\boldsymbol{dy/dx}}.$$

Length of normal

$$= PN = y\sec\psi = y\sqrt{(1 + \tan^2\psi)} = \boldsymbol{y}\sqrt{\left\{\boldsymbol{1} + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\boldsymbol{2}}\right\}}.$$

Length of subnormal

$$= SN = y\tan\psi = \boldsymbol{y}\frac{\boldsymbol{dy}}{\boldsymbol{dx}}.$$

Intercepts made by the tangent on the coordinate axes.

The equation of the tangent at $P(x, y)$ is $Y - y = \dfrac{dy}{dx}(X - x)$.

This meets OX, where $Y = 0$ *i.e.*, where

$$0 - y = \frac{dy}{dx}(X - x) \quad \text{or} \quad X = x - \frac{y}{dy/dx}.$$

Hence the length of the intercept OT that the tangent cuts off from the x-axis is

$$\boldsymbol{x - \frac{y}{(dy/dx)}}.$$

Again the tangent meets y-axis, where $X = 0$, *i.e.*, where

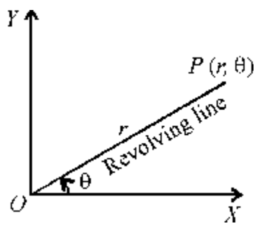
$$Y - y = \frac{dy}{dx}(0 - x) \quad \text{or} \quad Y = y - x\frac{dy}{dx}.$$

Hence the intercept OT', made by the tangent on y-axis is $\boldsymbol{y - x\frac{dy}{dx}}$.

10.8 Polar Co-ordinates

Besides the cartesian system of co-ordinates, there are other systems also for representing the position of a point in a plane. Polar system, which is one of them, will be described here.

In polar system, we start with a fixed half line OX, called the **initial line** and a fixed point O on it, called the **pole.** If P is any point in the plane, the distance $OP = r$ is called the **radius vector** of the point P and $\angle XOP = \theta$ is called the **vectorial angle** of P. Also (r, θ) are called polar coordinates of the point P. The line OP is called the **revolving line.** For any point $P(r, \theta)$ the angle θ is taken to be positive when measured in the anti-clockwise direction from the initial line and negative when measured in the clockwise direction from the initial line. The radius vector r is considered to be positive when measured away from O in the direction of the line governing the vectorial angle θ. If for any point $P(r, \theta)$, r is negative, we first draw through O a line making an angle θ with the initial line. Producing this line backwards through O, we mark a point P on it such that $OP = |r|$; P is then the required point (r, θ). Thus in polar coordinates both r and θ are capable of varying in the interval $(-\infty, \infty)$.

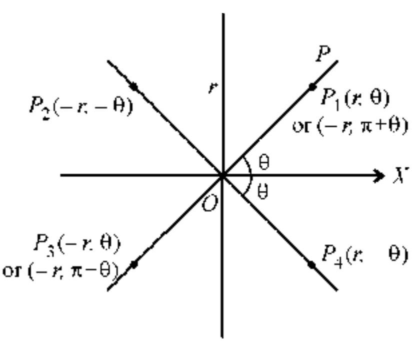


To each ordered pair (r, θ) of real numbers there corresponds one and only one point. But the converse is not true. For example, in the figure the point P whose polar coordinates are (r, θ) may also be described as $(r, \theta + 2\pi)$, $(-r, \theta + \pi)$ etc. In particular, the polar coordinates of pole may be given as $(0, \theta)$ where θ is perfectly arbitrary.

It is usual to regard θ as the independent variable and the curve whose equation is $r = f(\theta)$ or $F(r, \theta) = 0$ consists of the totality of distinct points (r, θ) which satisfy the equation.

Positive and Negative radii vectors :

If we measure the distance r along the revolving line in the direction in which the line projects from the pole O then the radius vector r is + ive and if it is measured in the opposite direction then the radius vector r is negative.



If the distance r is measured along OP in the direction of OP and $OP_1 = r$, then P_1 will be called (r, θ). But if r is measured in the opposite direction, then the point P_3 obtained is called $(-r, \theta)$.

Here the line OP_3 can be said to make an angle $\pi + \theta$ with the initial line. In this case the distance r measured along OP till P_3 will be called + ive and point P_3 will be said to be the point $(r, \pi + \theta)$. If now we measure a distance r in the opposite direction of OP_3, then we get the point P_1 which we shall call $(-r, \pi + \theta)$.

Positive and Negative Vectorial Angles :

If the revolving line makes an angle θ in the anticlockwise direction with the initial line, then the vectorial angle is said to be positive and if it makes an angle θ in the clockwise direction, then the vectorial angle is said to be negative. Thus, if $OP_4 = r$, then the point P_4 is said to be $(r, -\theta)$.

Similarly if $OP_2 = r$, then the point P_2 is said to be $(-r, -\theta)$.

Relation between Cartesian and polar coordinates :

Take the pole O as origin, the initial line as the positive direction of x-axis, and the line through O making angle $\frac{\pi}{2}$ with OX in the anti-clockwise direction as the positive direction of y-axis. Suppose (r, θ) are the polar and (x, y) are the cartesian coordinates of any point P. Draw PM perpendicular to OX. Then $OM = x$ and $MP = y$.

From $\Delta\ OPM$, we have

$$\boldsymbol{x = r\cos\theta}, \qquad \ldots(1) \qquad\qquad \boldsymbol{y = r\sin\theta}. \qquad \ldots(2)$$

Squaring and adding (1) and (2), we get $\boldsymbol{x^2 + y^2 = r^2}$

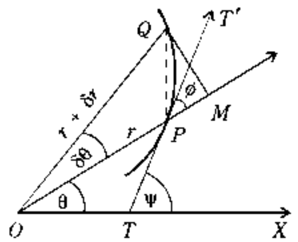
and dividing (2) by (1), we get

$$\boldsymbol{\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}}.$$

Exercise : *Plot the positions of the points whose polar coordinates are*

$$\left(-2, \frac{\pi}{2}\right), \left(2, -\frac{\pi}{2}\right), (3, \pi), (1, 0), \left(2, \frac{\pi}{4}\right), \left(2, -\frac{\pi}{4}\right), \left(1, \frac{3\pi}{4}\right), (1, -\pi), (1, \pi).$$

10.9 Angle Between Radius Vector and Tangent



Let P be any point (r, θ) on the curve $r = f(\theta)$. The line TPT' is tangent to this curve at P. We denote by ϕ the angle which the positive direction of the tangent at P (*the direction in which θ increases*) makes with the positive direction of the radius vector OP (*the direction in which r increases*).

Let Q be any other point $(r + \delta r, \theta + \delta\theta)$ on the curve in the neighbourhood of P. Draw QM perpendicular to OP. As $Q \to P$, we have $\delta\theta \to 0$, the chord $PQ \to$ tangent at P and the angle $QPM \to \phi$.

$$\text{Thus } \tan\phi = \lim_{\delta\theta \to 0} \tan\angle QPM = \lim_{\delta\theta \to 0} \frac{QM}{PM} = \lim_{\delta\theta \to 0} \frac{QM}{OM - OP}$$

$$= \lim_{\delta\theta \to 0} \frac{(r + \delta r) \sin \delta\theta}{(r + \delta r) \cos \delta\theta - r}$$

$$= \lim_{\delta\theta \to 0} \frac{(r + \delta r)\left(\delta\theta - \dfrac{\delta\theta^3}{3!} + \dots\right)}{(r + \delta r)\left(1 - \dfrac{\delta\theta^2}{2!} + \dots\right) - r} = \lim_{\delta\theta \to 0} \frac{r\,\delta\theta}{\delta r},$$

neglecting small quantities of the second and higher order

$$= r \frac{d\theta}{dr}.$$

Hence, $\tan \phi = r \dfrac{d\theta}{dr}$ **(Meerut 1990, 96)**

or $\cot \phi = \dfrac{1}{r} \dfrac{dr}{d\theta}$.

Note 1 : The angle ϕ is taken to be positive if it is measured in the anti-clockwise direction.

Note 2 : From the figure, we have an important relation $\psi = \theta + \phi$.

Note 3 : If the equation of a curve is given in the form $r = f(\theta)$ and we are to find the value of ϕ, then differentiating with respect to θ after taking logarithm of both sides, we shall at once get $\cot \phi$.

10.10 Angle of Intersection of Two Curves

Let $r = f(\theta)$ and $r = F(\theta)$ be the polar equations of two curves and P be one of their points of intersection. The two curves have a common radius vector at P. Suppose ϕ_1 is the angle which the tangent to the first curve at P makes with the radius vector of P and ϕ_2 is the angle which the tangent to the second curve at P makes with the radius vector of P. Then the acute angle of intersection of the two curves at P is obviously

$$= \phi_1 \sim \phi_2 \text{ i.e., } |\phi_1 - \phi_2|.$$

If $\tan \phi_1 = n_1$ and $\tan \phi_2 = n_2$, then the angle of intersection is

$$\phi_1 - \phi_2 = \tan^{-1} \tan (\phi_1 - \phi_2) = \tan^{-1} \left(\frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right) = \tan^{-1} \frac{n_1 - n_2}{1 + n_1 n_2}.$$

If $\dfrac{n_1 - n_2}{1 + n_1 n_2}$ is positive, we shall get acute angle of intersection at P and if $\dfrac{n_1 - n_2}{1 + n_1 n_2}$ is negative we shall get the obtuse angle of intersection at P.

In particular, the two curves intersect orthogonally if $n_1 n_2 = -1$, *i.e.*, $\tan \phi_1 . \tan \phi_2 = -1$.

Illustrative Examples

Example 1 : *Show that the parabolas* $r = a/(1 + \cos\theta)$ *and* $r = b/(1 - \cos\theta)$ *intersect orthogonally.* **(Meerut 2011; Avadh 13; Rohilkhand 14)**

Solution : The equations of the curves are

$r = a/(1 + \cos\theta)$ ...(1)

and $r = b/(1 - \cos\theta)$. ...(2)

Taking logarithm of both sides of (1), we get $\log r = \log a - \log(1 + \cos\theta)$.

Differentiating with respect to θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{(-\sin\theta)}{1 + \cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\frac{1}{2}\theta.$$

$$\therefore \quad \cot\phi = \tan\frac{1}{2}\theta = \cot\left(\frac{1}{2}\pi - \frac{1}{2}\theta\right) \quad \text{or} \quad \phi = \frac{1}{2}\pi - \frac{1}{2}\theta.$$

Hence $\phi_1 = \frac{1}{2}\pi - \frac{1}{2}\theta$.

Again taking logarithm of both sides of (2), we get

$\log r = \log b - \log(1 - \cos\theta)$.

Differentiating with respect to θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1 - \cos\theta} = -\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = -\cot\frac{1}{2}\theta.$$

$$\therefore \quad \cot\phi = -\cot\frac{1}{2}\theta = \cot\left(\pi - \frac{1}{2}\theta\right) \quad \text{or} \quad \phi = \pi - \frac{1}{2}\theta.$$

Hence $\phi_2 = \pi - \frac{1}{2}\theta$.

Therefore, angle of intersection $= \phi_1 \sim \phi_2 = \left(\pi - \frac{1}{2}\theta\right) - \left(\frac{1}{2}\pi - \frac{1}{2}\theta\right) = \frac{1}{2}\pi$.

Thus the two curves intersect orthogonally.

Example 2 : *Find the angle of intersection of the curves $r^2 = 16\sin 2\theta$ and $r^2\sin 2\theta = 4$.*

Solution : The given curves are

$r^2 = 16\sin 2\theta$, ...(1)

and $r^2\sin 2\theta = 4$. ...(2)

From (1), $2\log r = \log 16 + \log\sin 2\theta$.

Therefore $\frac{2}{r}\frac{dr}{d\theta} = 2\frac{\cos 2\theta}{\sin 2\theta}$ or $\cot\phi_1 = (1/r)(dr/d\theta) = \cot 2\theta$. Thus $\phi_1 = 2\theta$.

From (2), $2\log r + \log\sin 2\theta = \log 4$.

Therefore $\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$ or $\cot\phi_2 = \frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta = \cot(\pi - 2\theta)$.

Thus $\phi_2 = \pi - 2\theta$.

Now the angle of intersection of (1) and (2)

$= \phi_1 \sim \phi_2 = (\pi - 2\theta) - 2\theta = \pi - 4\theta$,

where θ is to be found at the point where (1) and (2) intersect.

Eliminating r between (1) and (2), we get $\sin^2 2\theta = \frac{1}{4}\cdot$ Therefore

$\sin 2\theta = \pm\frac{1}{2}\cdot$

But $\sin 2\theta = -\frac{1}{2}$ is inadmissible because it gives imaginary values of r from (1) and (2). Now $\sin 2\theta = \frac{1}{2}$ gives $2\theta = \frac{1}{6}\pi$ or $\theta = \pi/12$.

Hence, the angle of intersection of (1) and (2) at the point $\theta = \pi/12$ is $\pi - 4(\pi/12)$ *i.e.*, $2\pi/3$.

Comprehensive Exercise 1

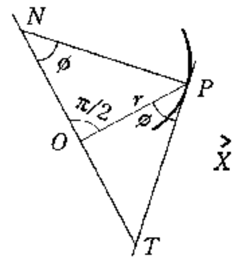
1. Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$ the tangent is inclined at a constant angle α to the radius vector. **(Avadh 2014)**
2. Find the angle at which the radius vector cuts the curve $l/r = 1 + e\cos\theta$.
3. Find the angle ϕ for the curve $a\theta = (r^2 - a^2)^{1/2} - a\cos^{-1}(a/r)$. **(Rohilkhand 2013)**
4. If ϕ be the angle between tangent to a curve and the radius vector drawn from the origin of coordinates to the point of contact, prove that
$$\tan\phi = \left(x\frac{dy}{dx} - y\right) \Big/ \left(x + y\frac{dy}{dx}\right).$$
$\left[\textbf{Hint}.\text{ We have } \psi = \theta + \phi, \tan\psi = \frac{dy}{dx} \text{ and } \tan\theta = \frac{y}{x}\right].$
5. Prove $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, where $u = \frac{1}{r}$ and p is the lenght of perpendicular form pole to the tangent of the curve at any point $p(r, \theta)$. **(Bundelkhand 2001)**
6. Show that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally. **(Kumaun 2008)**
7. Show that the cardioids $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ intersect at right angles. **(Meerut 2000; Kanpur 07, 11)**
8. Show that the curves $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$ cut orthogonally.
9. Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.
10. Show that the curves $r = 2\sin\theta$ and $r = 2\cos\theta$ intersect at right angles.
11. Find the angle between the tangent and the radius vector in the case of the curve $r^n = a^n \sec(n\theta + \alpha)$, and prove that this curve is intersected by the curve $r^n = b^n \sec(n\theta + \beta)$ at an angle which is independent of a and b.
12. Find the angle of intersection between the pair of curves $r = 6\cos\theta$ and $r = 2(1 + \cos\theta)$.

Answers 1

2. $\tan^{-1}\{(1 + e\cos\theta)/(e\sin\theta)\}$.
3. $\cos^{-1}(a/r)$.
9. $\pi/4$.
11. $(\pi/2) - (n\theta + \alpha)$.
12. $(\pi/6)$.

10.11 Lengths of Polar Sub-tangent and Polar Sub-normal

(Kashi 2013)



Let P be any point (r, θ) on a given curve. Suppose the tangent and normal at P meet the straight line through the pole O perpendicular to the radius vector OP in T and N respectively. Then OT and ON are respectively called the **polar subtangent** and the **polar sub-normal** at P.

We have $\angle OPT = \phi$ and $\angle ONP = \phi$.

From ΔPTO, $OT = OP \tan\phi = r \,.\, r\dfrac{d\theta}{dr} = r^2\dfrac{d\theta}{dr}$.

Hence **Polar sub-tangent** $= \boldsymbol{r^2\dfrac{d\theta}{dr}}$.

Again from ΔPON, $ON = OP\cot\phi = r\cdot\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{dr}{d\theta}$.

Hence **Polar sub-normal** $= \boldsymbol{\dfrac{dr}{d\theta}}$.

Example : *Show that in the curve $r = a\theta$ the polar sub-normal is constant and in the curve $r\theta = a$ the polar sub-tangent is constant.*

Solution : From the curve $r = a\theta$, we have $dr/d\theta = a$.

$\therefore$ Polar sub-normal $= dr/d\theta = a$, which is a constant

i.e., independent of r and θ.

Again, from the curve $r\theta = a$ or $r = a/\theta$, we have

$$\frac{dr}{d\theta} = -a/\theta^2 \quad \text{or} \quad \frac{d\theta}{dr} = -\frac{\theta^2}{a}.$$

$$\therefore \quad \text{Polar sub-tangent} = r^2\frac{d\theta}{dr} = \frac{a^2}{\theta^2}\cdot\left(\frac{-\theta^2}{a}\right) = -a, \quad \text{which is constant.}$$

10.12 Length of the Perpendicular from Pole to the Tangent

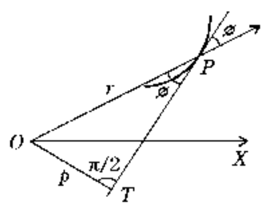
From the pole O draw OT perpendicular to the tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$.

Let $OT = p$. Thus, *p is the length of the perpendicular from pole to the tangent.*

We have $\angle OPT = \phi$.

From the right angle triangle OTP, we have

$$OT = OP\sin\phi \quad \text{or} \quad \boldsymbol{p = r\sin\phi}. \quad \ldots(1)$$



Often we require the value of p in terms of r and θ only.
For this we shall substitute the value of ϕ in (1) from the equation $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta}$.

Thus from (1), we have $\dfrac{1}{p^2} = \dfrac{1}{r^2}\operatorname{cosec}^2\phi = \dfrac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Hence $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. ...(2)

(Meerut 2003)

Sometimes, we write $u=\dfrac{1}{r}\cdot$

Then $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}\cdot$ $\therefore$ $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\left(\dfrac{du}{d\theta}\right)^2$

or $$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2. \qquad ...(3)$$

Note : The results (1), (2) and (3) are very important and should be committed to memory.

10.13 Pedal Equation **(Meerut 2009)**

The relation between p and r for a given curve is called its pedal equation where r is the radius vector of any point on the curve and p is the length of the perpendicular from pole to the tangent at that point.

Case I : To form the pedal equation of a curve whose cartesian equation is given.

Let $f(x, y) = 0,$ ...(1)

be the cartesian equation of the given curve.

The equation of the tangent at (x, y) to the curve (1) is

$$Y-y=\frac{dy}{dx}(X-x), \quad \text{or} \quad Y-X\frac{dy}{dx}+\left(x\frac{dy}{dx}-y\right)=0.$$

If p be the length of perpendicular from (0, 0) to this tangent, we have

$$p=\frac{x\dfrac{dy}{dx}-y}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}. \qquad ...(2)$$

Also $r^2=x^2+y^2.$ ...(3)

Eliminating x and y between (1), (2) and (3), we get a relation between p and r *i.e.*, the required pedal equation of the given curve.

Case II : To form the pedal equation of a curve whose polar equation is given.

Let $f(r, \theta) = 0$ ...(1)

be the polar equation of the given curve.

We have $p = r\sin\phi,$ ...(2)

and $\cot\phi=\dfrac{1}{r}\dfrac{dr}{d\theta}\cdot$ ...(3)

Eliminating θ and ϕ between (1), (2) and (3), we obtain the required pedal equation of the given curve.

Important : Sometimes we do not get the value of ϕ from equation (3) in a convenient form. In that case instead of using the relations (2) and (3), we can use the single relation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \quad \text{...(4)}$$

Eliminating θ between (1) and (4), we obtain the required pedal equation of the curve.

Illustrative Examples

Example 1 : *Show that the pedal equation of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *is*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

(Meerut 2010B; Avadh 13)

Solution : The equation of the curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. ...(1)

The co-ordinates (x, y) of any point P on (1) may be taken as $x = a \cos t$, $y = b \sin t$.

$$\therefore \quad \frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = b \cos t. \qquad \therefore \quad \frac{dy}{dx} = -\frac{b \cos t}{a \sin t}.$$

Hence the equation of the tangent to the ellipse at the point 't' is

$$Y - b \sin t = -\frac{b \cos t}{a \sin t}(X - a \cos t)$$

or $\quad ab - b \cos t \, . \, X - a \sin t \, . \, Y = 0.$ ...(2)

Therefore p = the length of perpendicular from (0, 0) to (2)

$$= \frac{ab}{\sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)}}$$

i.e., $\quad \frac{1}{p^2} = \frac{a^2 \sin^2 t + b^2 \cos^2 t}{a^2 b^2}.$ ...(3)

Also $\quad r^2 = x^2 + y^2 = a^2 \cos^2 t + b^2 \sin^2 t = a^2 (1 - \sin^2 t) + b^2 (1 - \cos^2 t)$

$= a^2 + b^2 - a^2 \sin^2 t - b^2 \cos^2 t.$ ...(4)

Eliminating t between (3) and (4), we obtain the pedal equation of the given curve.

From (4), we get $\quad a^2 \sin^2 t + b^2 \cos^2 t = (a^2 + b^2) - r^2.$

Hence (3) gives $\quad \frac{1}{p^2} = \frac{(a^2 + b^2) - r^2}{a^2 b^2} \quad$ or $\quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}$,

which is the required pedal equation of the ellipse.

Example 2 : *Show that the pedal equation of the parabola* $y^2 = 4a(x + a)$ *is* $p^2 = ar$.

(Meerut 2010)

Solution : Differentiating the equation of the curve, we get

$$2y \frac{dy}{dx} = 4a, \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}.$$

Therefore the equation of the tangent at (x, y) is

$$Y - y = (2a/y)(X - x) \qquad \text{or} \qquad (2a/y) X - Y + y - (2a/y) x = 0.$$

$$\therefore \quad p = \frac{y - (2ax/y)}{\sqrt{(1 + 4a^2/y^2)}} = \frac{y^2 - 2ax}{\sqrt{(y^2 + 4a^2)}}$$

$$= \frac{4a(x+a) - 2ax}{\sqrt{[4a(x+a) + 4a^2]}} = \frac{2ax + 4a^2}{\sqrt{[4a(x+2a)]}}$$

$$= \frac{2a(x+2a)}{\sqrt{[4a(x+2a)]}} = \sqrt{[a(x+2a)]}$$

Also $\quad r^2 = x^2 + y^2 = x^2 + 4a(x+a) = (x+2a)^2.$

$\therefore \quad r = (x+2a). \qquad \therefore \quad p^2 = a(x+2a)$ or $p^2 = ar$,

which is the required pedal equation of the given curve.

Example 3 : *Form the pedal equation of the sine spiral $r^n = a^n \sin n\theta$.*

Solution : The curve is $r^n = a^n \sin n\theta$. ...(1)

Taking logarithm of both sides, we get $n \log r = n \log a + \log \sin n\theta$.

Differentiating with respect to θ, we obtain

$$\frac{n}{r}\frac{dr}{d\theta} = n\frac{\cos n\theta}{\sin n\theta} = n \cot n\theta. \qquad \therefore \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta.$$

Therefore $\phi = n\theta$.

Now $\quad p = r\sin\phi. \qquad \therefore \quad p = r\sin n\theta.$ ...(2)

Eliminating θ between (1) and (2), we obtain the required pedal equation of the given curve.

From (2), we have $\sin n\theta = p/r$.

Putting this value in (i), we obtain

$$r^n = a^n(p/r) \quad \text{or} \quad pa^n = r^{n+1},$$

which is the required pedal equation.

Comprehensive Exercise 2

1. Find the polar subtangent of the ellipse $l/r = 1 + e\cos\theta$.
2. For the parabola $2a/r = 1 - \cos\theta$, prove that

   (i) $\phi = \pi - \frac{1}{2}\theta$, (ii) $p = a\operatorname{cosec}\frac{1}{2}\theta$,

   (iii) $p^2 = ar$,

   (iv) the polar subtangent $= 2a\operatorname{cosec}\theta$. **(Purvanchal 2014)**
3. For the cardioid $r = a(1 - \cos\theta)$, prove that

   (i) $\phi = \frac{1}{2}\theta$, **(Meerut 2001)**

   (ii) $p = 2a\sin^3\frac{1}{2}\theta$,

   (iii) the pedal equation is $2ap^2 = r^3$, **(Meerut 2001; Rohilkhand 12B)**

   (iv) the polar subtangent $= 2a\sin^2(\theta/2)\tan(\theta/2)$.
4. Show that the pedal equation

   (i) of the lemniscate $r^2 = a^2\cos 2\theta$ is $r^3 = a^2 p$,

   (ii) of the hyperbola $r^2\cos 2\theta = a^2$ is $pr = a^2$,

   (iii) of the cosine spiral $r^n = a^n\cos n\theta$ is $pa^n = r^{n+1}$,

   (iv) of the curve $r = a\theta$ is $p^2 = r^4/(r^2 + a^2)$.

5. Show that the pedal equation of the conic $\dfrac{l}{r} = 1 + e\cos\theta$ is

$$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right).$$

6. Show that the pedal equation of the spiral $r = a \operatorname{sech} n\theta$ is of the form

$$\frac{1}{p^2} = \frac{A}{r^2} + B.$$

7. Show that the pedal equation of the cardioid $r = a(1 + \cos\theta)$ is $r^3 = 2ap^2$.

8. Show that the pedal equation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 = a^2 - 3p^2$.

9. Show that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'\left(\theta - \frac{1}{2}\pi\right)$. **(Gorakhpur 2006)**

Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral.

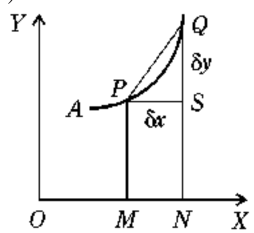
10. Prove that the normal at any point (r, θ) to the curve $r^n = a^n \cos n\theta$ makes an angle $(n + 1)\theta$ with the initial line.

Answers 2

1. $l/e \sin\theta$.

10.14 Differential Coefficient of Arc Length (Cartesian Formula)

Let s denote the length of the arc AP of the curve $y = f(x)$ measured from some fixed point A on it to any other point $P(x, y)$. Then s is obviously some function of x and we want to find ds/dx.



Take a point $Q(x + \delta x, y + \delta y)$ on the curve in the neighbourhood of P such that arc $AQ = s + \delta s$.

Then arc $PQ = \delta s$.

Also $\delta x \to 0$ as $Q \to P$.

From the right angled triangle PSQ, we have chord

$$(PQ)^2 = PS^2 + SQ^2 = (\delta x)^2 + (\delta y)^2. \quad \text{...(1)}$$

Dividing (1) throughout by $(\delta x)^2$, we get

$$\left(\frac{\text{chord } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2 \quad \text{or} \quad \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\text{arc } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

or $$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2.$$

Taking limit of both sides as $Q \to P$, we get

$$\lim_{Q \to P}\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \lim_{\delta x \to 0}\left(\frac{\delta s}{\delta x}\right)^2 = \lim_{\delta x \to 0}\left[1 + \left(\frac{\delta y}{\delta x}\right)^2\right]$$

or $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2. \qquad \left[\because \lim_{Q \to P}\frac{\text{chord } PQ}{\text{arc } PQ} = 1\right].$$

Thus $\dfrac{ds}{dx} = \pm\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}$,

where plus or minus sign is to be taken before the radical sign according as s increases as x increases or decreases.

Hence if s increases as x increases, we have

$$\boldsymbol{\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}.}$$

Corollary 1 : If $x = f(y)$ be the equation of the curve, then 's' is obviously a function of y. In this case dividing (1) throughout by $(\delta y)^2$ and proceeding to limits, we get

$$\boldsymbol{\frac{ds}{dy} = \pm\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}},}$$

where plus or minus sign is to be taken before the radical sign according as s increases as y increases or decreases.

Corollary 2 : If the equations of the curve be given in the parametric form $x = f_1(t)$ and $y = f_2(t)$, then 's' is evidently a function of t. In this case dividing (1) throughout by $(\delta t)^2$ and proceeding to limits, we have

$$\boldsymbol{\frac{ds}{dt} = \pm\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}},} \qquad (\textit{Parametric formula})$$

where plus or minus sign is to be taken before the radical sign according as s increases as t increases or decreases.

10.15 $\cos\psi = \dfrac{dx}{ds}$ and $\sin\psi = \dfrac{dy}{ds}$

We know that $\tan\psi = dy/dx$.

If s increases as x increases, we have

$$\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{(1+\tan^2\psi)} = \sec\psi.$$

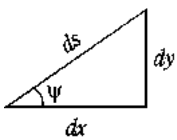
$$\therefore \quad \boldsymbol{\frac{dx}{ds} = \cos\psi.}$$

Again if s increases as y increases, we have

$$\frac{ds}{dy} = \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}} = \sqrt{(1+\cot^2\psi)} = \operatorname{cosec}\psi.$$

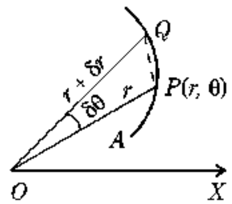
$$\therefore \quad \boldsymbol{\frac{dy}{ds} = \sin\psi.}$$

Important : We can remember these results very easily with the help of the adjoining hypothetical figure.



10.16 Differential Coefficient of Arc Length (Polar Formula)

To prove that $\dfrac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}$ *for the curve* $r = f(\theta)$.



Let s denote the length of the arc AP measured from some fixed point A on the curve $r = f(\theta)$ to any other point $P(r, \theta)$. Then s is a function of θ.

Take a point $Q(r + \delta r, \theta + \delta\theta)$ on the curve in the neighbourhood of P such that arc $AQ = s + \delta s$.

Then arc $PQ = \delta s$.

Also $\delta\theta \to 0$ and $\delta r \to 0$ as $Q \to P$.

From the triangle OPQ, we have

$$(\text{chord } PQ)^2 = OP^2 + OQ^2 - 2OP.\,OQ\cos\angle QOP$$

or $$(\text{chord } PQ)^2 = r^2 + (r + \delta r)^2 - 2r(r + \delta r)\cos\delta\theta$$

or $$(\text{chord } PQ)^2 = (\delta r)^2 + 2r\delta r(1 - \cos\delta\theta) + 2r^2(1 - \cos\delta\theta)$$

or $$(\text{chord } PQ)^2 = (\delta r)^2 + 2r\delta r(1 - \cos\delta\theta) + 2r^2(1 - \cos\delta\theta).$$

Dividing by $(\delta\theta)^2$, we get

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 . \left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r . \left(\frac{\sin\dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}\right)^2 . \delta r + r^2\left(\frac{\sin\dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}}\right)^2 .$$

Taking limits of both sides as $Q \to P$, we get

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r . 1 . 0 + r^2 . 1 . \left[\because \lim_{\delta\theta \to 0}\left(\sin\frac{\delta\theta}{2} \Big/ \frac{\delta\theta}{2}\right) = 1,\right.$$
$$\left.\lim_{\delta\theta \to 0}\frac{\delta r}{\delta\theta} = \frac{dr}{d\theta} \text{ and } \lim_{Q \to P}\frac{\text{chord } PQ}{\text{arc } PQ} = 1\right]$$

$$\therefore \quad \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 .$$

Thus $\dfrac{ds}{d\theta} = \pm\sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}$, where plus or minus sign is to be taken before the radical sign according as s increases or decreases as θ increases. Hence if s increases as θ increases, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}.$$

Corollary : If $\theta = f(r)$ be the equation of the curve, then 's' is a function of r and we have

$$\frac{ds}{dr} = \frac{ds}{d\theta}\frac{d\theta}{dr} = \frac{d\theta}{dr}\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

or $$\frac{ds}{dr} = \sqrt{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}}.$$

10.17 $\cos\phi = \dfrac{dr}{ds}$ and $\sin\phi = r\dfrac{d\theta}{ds}$

We know that $\tan\phi = r\,(d\theta/dr)$.

$$\therefore\quad \cos\phi = \frac{1}{\sqrt{(\sec^2\phi)}} = \frac{1}{\sqrt{(1+\tan^2\phi)}} = \frac{1}{\sqrt{\left\{1 + r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}}}$$

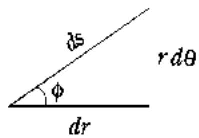
$$= \frac{1}{\left(\dfrac{d\theta}{dr}\right)\sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}} = \frac{dr/d\theta}{ds/d\theta} = \frac{dr}{ds},$$

s being measured in such a way that s increases as θ increases.

Thus $\quad \boldsymbol{\cos\phi = \dfrac{dr}{ds}}\cdot$

Also $\quad \sin\phi = \tan\phi\cos\phi = r\dfrac{d\theta}{dr}\cdot\dfrac{dr}{ds} = r\dfrac{d\theta}{ds}\cdot$

Hence $\quad \boldsymbol{\sin\phi = r\dfrac{d\theta}{ds}}\cdot$



Important : We can remember these results very easily with the help of the adjoining hypothetical figure.

Illustrative Examples

Example 1 : *For the curve $y = a \log\sec(x/a)$, prove that $\dfrac{ds}{dx} = \sec\left(\dfrac{x}{a}\right)$.*

Solution : We have $y = a\log\sec(x/a)$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = a\cdot\frac{1}{\sec(x/a)}\sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right)\cdot\frac{1}{a}$$

$$= \tan\left(\frac{x}{a}\right).$$

Now $$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

$$= \sqrt{\left\{1 + \tan^2\left(\frac{x}{a}\right)\right\}} = \sec\left(\frac{x}{a}\right).$$

Example 2 : *For the ellipse $x = a\cos t$, $y = b\sin t$, prove that*

$$ds/dt = a\,(1 - e^2\cos^2 t)^{1/2}.$$

Solution : Here $\dfrac{dx}{dt} = -a\sin t$ and $\dfrac{dy}{dt} = b\cos t$.

Now $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{(a^2\sin^2 t + b^2\cos^2 t)}$$

$$= \sqrt{\{a^2\sin^2 t + a^2(1 - e^2)\cos^2 t\}} \qquad [\because b^2 = a^2(1 - e^2)]$$

$$= a\sqrt{(\sin^2 t + \cos^2 t - e^2 \cos^2 t)}$$
$$= a\sqrt{(1 - e^2 \cos^2 t)}.$$

Example 3 : *Show that for the curve* $r^m = a^m \cos m\theta$, $\dfrac{ds}{d\theta} = \dfrac{a^m}{r^{m-1}}$.

Solution : We have $r^m = a^m \cos m\theta$.
Differentiating logarithmically, we obtain

$$\frac{m}{r}\frac{dr}{d\theta} = -\frac{m \sin m\theta}{\cos m\theta},$$

i.e.,
$$\frac{dr}{d\theta} = -r \tan m\theta.$$

Now
$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$
$$= \sqrt{(r^2 + r^2 \tan^2 m\theta)} = r \sec m\theta$$
$$= \frac{r}{\cos m\theta} = \frac{r a^m}{a^m \cos m\theta} = \frac{r a^m}{r^m} = \frac{a^m}{r^{m-1}}.$$

Comprehensive Exercise 3

1. Calculate ds/dx for the following curves:
(i) $y^2 = 4ax$;
(ii) $y = a \cosh (x/a)$;
(iii) $x^{2/3} + y^{2/3} = a^{2/3}$.

2. Calculate ds/dt for the following curves :
(i) $y = a(1 - \cos t), x = a(t + \sin t)$.
(ii) $x = a \cos^3 t, y = a \sin^3 t$.
(iii) $x = 2 \sin t, y = \cos 2t$.

3. Calculate $ds/d\theta$ for the following curves :
(i) $r = \log \sin 3\theta$;
(ii) $r = a(1 - \cos\theta)$.

4. For the curve $r = ae^{\theta \cot \alpha}$, prove that $s/r =$ constant, s being measured from the pole.

5. In any curve, prove that
(i) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$,
(ii) $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{(r^2 - p^2)}}$.

6. For the curve $r^n = a^n \cos n\theta$, prove that $a^{2n} \dfrac{d^2 r}{ds^2} + nr^{2n-1} = 0$.

7. For the cycloid $x = a(1 - \cos t)$, $y = a(t + \sin t)$, find

(i) $\dfrac{ds}{dt}$

(ii) $\dfrac{ds}{dx}$

(iii) $\dfrac{ds}{dy}$.

Answers 3

1. (i) $(1 + a/x)^{1/2}$,
(ii) $\cosh(x/a)$,
(iii) $(a/x)^{1/3}$.

2. (i) $2a \cos(t/2)$,
(ii) $3a \cos t \sin t$,
(iii) $2 \cos t \sqrt{(1 + 4 \sin^2 t)}$.

3. (i) $\sqrt{(r^2 + 9 \cot^2 3\theta)}$,
(ii) $2a \sin(\theta/2)$.

7. (i) $2a \cos \dfrac{t}{2}$,

(ii) $\operatorname{cosec} \dfrac{t}{2}$,

(iii) $\sec \dfrac{t}{2}$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. If ϕ is the angle between the radius vector and the tangent of a curve then $\tan \phi =$ …… .

2. For the curve $r = f(\theta)$, $\dfrac{ds}{d\theta} =$ …… .

3. For the parabola $\dfrac{2a}{r} = 1 - \cos\theta$, $\phi =$ …… .

4. For the cycloid $x = a(1 - \cos t)$, $y = a(t + \sin t)$, we have $\dfrac{ds}{dt} =$ …… .

5. For the curve $r^2 = a^2 \cos 2\theta$, the value of $\dfrac{ds}{d\theta}$ is …… .

6. If $\dfrac{d\theta}{dr} = \dfrac{7}{3}$, at a point on the curve $r = f(\theta)$, then at that point polar subnormal is .... **(Meerut 2001)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. For the cardioid $r = a(1 - \cos\theta)$, the value of ϕ is

(a) θ (b) $\dfrac{\theta}{2}$

(c) $-\dfrac{\theta}{2}$ (d) $-\theta$

8. Two curves cut orthogonally if $\tan\phi_1 . \tan\phi_2$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these

9. For the curve $r = f(\theta)$, the value of $\cos\phi$ is

(a) $r\dfrac{d\theta}{ds}$ (b) $r\dfrac{ds}{d\theta}$

(c) $\dfrac{ds}{dr}$ (d) $\dfrac{dr}{ds}$

10. For any curve $r = f(\theta)$, the value of $\dfrac{ds}{d\theta}$ is

(a) $\dfrac{r^2}{p}$ (b) $\dfrac{p}{r^2}$

(c) $\dfrac{r}{p}$ (d) $\dfrac{p}{r}$

True or False:

Write 'T' for true and 'F' for false statement.

11. If p be the length of perpendicular drawn from the pole O to tangent at any point $P(r, \theta)$ on the curve $r = f(\theta)$, then

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

12. The relation between p and r for a given curve is called its polar equation.

13. For the curve $r = f(\theta)$, we have $\left(\dfrac{dr}{ds}\right)^2 + \left(r\dfrac{d\theta}{ds}\right)^2 = 1.$

14. $p = r\sin\theta$ is the pedal equation of some curve.

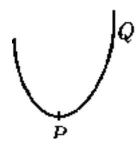
Answers

1. $r \dfrac{d\theta}{dr} \cdot$
2. $\sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}.$
3. $\pi - \dfrac{1}{2}\theta.$
4. $2a \cos \dfrac{1}{2} t.$
5. $\dfrac{a^2}{r}.$
6. $\dfrac{3}{7} \cdot$
7. (b).
8. (c).
9. (d).
10. (a).
11. T.
12. F.
13. T.
14. F.

Chapter 11

Curvature

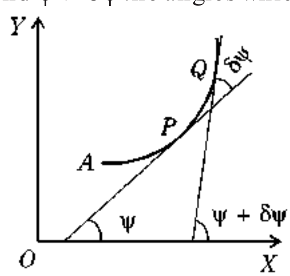
11.1 Meaning of Curvature



In the adjoining figure we see that the curve bends more sharply at the point P than at the point Q. We express this feeling by saying that the curve has a greater curvature at P than at Q. However, in order to get a quantitative estimate of curvature, we should give a mathematical definition of curvature which should be in agreement with our intuitive notion of curvature.

11.2 Definition of Curvature (Purvanchal 2010)

Let P and Q be two neighbouring points on a curve, ψ and $\psi + \delta\psi$ the angles which the tangents at P and Q make with the x-axis.



Let A be any fixed point on the curve.

Let arc $AP = s$, arc $AQ = s + \delta s$, so that arc $PQ = \delta s$.

The symbol, $\delta\psi$ denotes the angle through which the tangent turns as a point moves along the curve from P to Q through a distance δs. The angle $\delta\psi$ is called the *contingence* of the arc PQ. Obviously, $\delta\psi$ will be large or small, as compared with δs, depending on the degree of sharpness of the bend of the arc PQ. This suggests us to make the following definitions :

(*i*) $\delta\psi$ *is defined to be* **total curvature** *of the arc PQ ;*

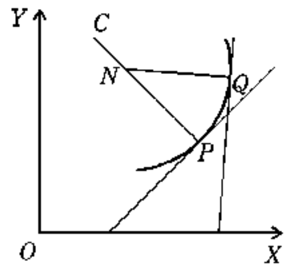
(*ii*) *the ratio* $\dfrac{\delta\psi}{\delta s}$ *is defined to be the* **average curvature** *of the arc PQ* ;

(*iii*) *the* **curvature** *of the curve at P is defined to be* $\lim\limits_{Q\to P}\dfrac{\delta\psi}{\delta s}$ *i.e.,* $\dfrac{d\psi}{ds}$.

Thus $\dfrac{d\psi}{ds}$ *is a mathematical measure for the curvature of curve at any point P.*

11.3 Radius of Curvature (Meerut 2010)

Let P be a given point on a given curve, and Q any other point on it in the neighbourhood of P. Let N be the point of intersection of the normals at P and Q. Suppose N tends to definite position C as Q tends to P, whether from the right or from the left.



Then C is called the **centre of curvature** of the curve at P.

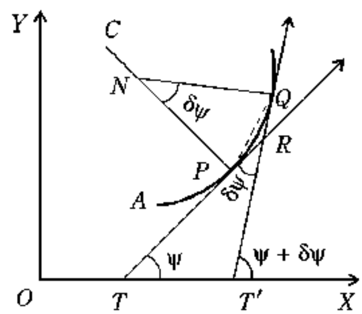
The distance CP is called the **radius of curvature** of the curve at P and is usually denoted by the Greek letter ρ.

The circle with its centre at C and radius CP is called the **circle of curvature** at P. Any chord drawn through P, of the circle of curvature at P, is called a **chord of curvature at** P.

11.4 Intrinsic Formula for the Radius of Curvature (Kashi 2011)

The relation between s and ψ for any curve is called its **intrinsic equation**. Let P be a given point on the curve $s = f(\psi)$, and Q a point on it in the neighbourhood of P. Let ψ and $\psi + \delta\psi$ be the angles which tangents at P and Q make with the x-axis. Let A be any fixed point on the curve. Let

arc $AP = s$, arc $AQ = s + \delta s$, so that arc $PQ = \delta s$.



Let R be the point of intersection of the tangents at P and Q and N be the point of intersection of the normals at these two points. Suppose $N \to C$ as $Q \to P$.

Then the radius of curvature at $P = \rho = \lim\limits_{Q\to P} PN$.

We have $\angle PNQ = \angle TRT' = \delta\psi$.

Now from the triangle PNQ, we have

$$\frac{PN}{\sin NQP} = \frac{\text{chord } PQ}{\sin PNQ} = \frac{\text{chord } PQ}{\sin \delta\psi}.$$

$$\therefore \qquad PN = \frac{\text{chord } PQ}{\sin \delta\psi}\sin NQP = \frac{\text{chord } PQ}{\delta s}\cdot\frac{\delta s}{\delta\psi}\cdot\frac{\delta\psi}{\sin\delta\psi}\cdot \sin NQP.$$

Now as $Q \to P$, we have $\delta\psi \to 0$, $\delta s \to 0$, chord $PQ \to$ tangent at P,

$QN \to$ normal at P and consequently $\angle NQP \to \dfrac{\pi}{2}$.

Therefore $\rho = \lim_{Q \to P} PN$

$$= \left(\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} \right) \cdot \left(\lim_{\delta\psi \to 0} \frac{\delta s}{\delta \psi} \right) \cdot \left(\lim_{\delta\psi \to 0} \frac{\delta\psi}{\sin \delta\psi} \right) \cdot \left(\lim_{Q \to P} \sin NQP \right)$$

$$= 1 \cdot \left(\frac{ds}{d\psi} \right) \cdot 1. \sin \frac{\pi}{2} = \frac{ds}{d\psi}.$$

Hence, $\rho = \dfrac{ds}{d\psi}$.

Corollary : The curvature of the curve at any point P is by definition, equal to $\dfrac{d\psi}{ds}$. Hence *the curvature of the curve at any point is equal to the reciprocal of the radius of curvature at that point i.e., curvature* $= \dfrac{1}{\rho}$.

Example : *Find the radius of curvature for the curve whose intrinsic equation is*

$$s = a \log \tan \left(\frac{\pi}{4} + \frac{\psi}{2} \right).$$

Solution : We have $\rho = \dfrac{ds}{d\psi} = a \dfrac{1}{\tan \left(\dfrac{\pi}{4} + \dfrac{\psi}{2} \right)} \sec^2 \left(\dfrac{\pi}{4} + \dfrac{\psi}{2} \right) \cdot \dfrac{1}{2}$

$$= \frac{a}{2 \sin \left(\frac{\pi}{4} + \frac{\psi}{2} \right) \cos \left(\frac{\pi}{4} + \frac{\psi}{2} \right)} = \frac{a}{\sin \left(\frac{\pi}{2} + \psi \right)} = \frac{a}{\cos \psi} = a \sec \psi.$$

11.5 Cartesian Formula for Radius of Curvature (Gorakhpur 2005)

Let the equation of the curve be $y = f(x)$.

We know that $\dfrac{dy}{dx} = \tan \psi$.

Differentiating with respect to x, we get

$$\frac{d^2 y}{dx^2} = \sec^2 \psi \cdot \frac{d\psi}{dx} = \sec^2 \psi \frac{d\psi}{ds} \cdot \frac{ds}{dx}.$$

$$\therefore \quad \frac{d\psi}{ds} = \frac{\dfrac{d^2 y}{dx^2}}{\sec^2 \psi \dfrac{ds}{dx}}. \qquad \therefore \quad \frac{ds}{d\psi} = \frac{(1 + \tan^2 \psi) \cdot \dfrac{ds}{dx}}{\dfrac{d^2 y}{dx^2}}.$$

But we have $\dfrac{ds}{dx} = \sqrt{\left\{ 1 + \left(\dfrac{dy}{dx} \right)^2 \right\}}$.

Hence $$\rho = \frac{ds}{d\psi} = \frac{\left\{ 1 + \left(\dfrac{dy}{dx} \right)^2 \right\}^{3/2}}{\dfrac{d^2 y}{dx^2}} = \frac{(1 + y_1^2)^{3/2}}{y_2}.$$ **(Bundelkhand 2008)**

Note 1 : The radius of curvature ρ can come out to be positive or negative. If in the relation

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

we take the positive sign before the radical, the value of ρ is positive or negative according as $\dfrac{d^2 y}{dx^2}$ is positive or negative. However, if we define the radius of curvature in such a way that it is to be always positive, then we should ignore the sign whenever we get a negative value for ρ.

Note 2 : Since the radius of curvature is a length therefore its value is independent of the choice of x-axis and y-axis. Hence interchanging x and y, we obtain

$$\rho = \frac{\left\{\mathbf{1} + \left(\dfrac{\mathbf{dx}}{\mathbf{dy}}\right)^2\right\}^{3/2}}{\dfrac{\mathbf{d^2x}}{\mathbf{dy^2}}}.$$

This formula is specially useful when (dy/dx) is infinite *i.e.*, when the tangent is perpendicular to x-axis.

Illustrative Examples

Example 1 : *Find the curvature at the point* $(3a/2, 3a/2)$ *of the curve* $x^3 + y^3 = 3axy$. **(Meerut 2010; Agra 05; Kashi 14; Avadh 14)**

Solution : The curve is $x^3 + y^3 = 3axy$. ...(1)

Differentiating with respect to x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx}$$

or $$x^2 + y^2 \frac{dy}{dx} = ay + ax \frac{dy}{dx}. \quad \text{...(2)}$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}. \qquad \therefore \quad \left[\dfrac{dy}{dx}\right]_{\left(\frac{3}{2}a,\, \frac{3}{2}a\right)} = -1.$

Again, differentiating (2), with respect to x, we get

$$2x + 2y\left[\frac{dy}{dx}\right]^2 + y^2 \frac{d^2 y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2 y}{dx^2}$$

or $$(ax - y^2)\frac{d^2 y}{dx^2} = 2x + 2y\left(\frac{dy}{dx}\right)^2 - 2a\frac{dy}{dx}. \quad \text{...(3)}$$

Putting $x = \dfrac{3a}{2}, y = \dfrac{3a}{2}$ and $\left[\dfrac{dy}{dx}\right]_{(3a/2,\, 3a/2)} = -1$ in (3), we get

$$\left[\frac{d^2 y}{dx^2}\right]_{(3a/2,\, 3a/2)} = -\frac{32}{3} \cdot \frac{1}{a}.$$

Hence the radius of curvature ρ at $\left(\dfrac{3a}{2}, \dfrac{3a}{2}\right)$

$$= \left[\frac{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{3/2}}{\frac{d^2 y}{dx^2}} \right]_{(3a/2\,,\,3a/2)} = \frac{(1+1)^{3/2}}{-\frac{32}{3} \cdot \frac{1}{a}} = -\frac{3a}{8\sqrt{2}}.$$

$\therefore$ Curvature at $\left(\frac{3a}{2}, \frac{3a}{2} \right) = \frac{1}{\rho} = -\frac{8\sqrt{2}}{3a}$.

If we ignore the negative sign, the value of curvature at $\left(\frac{3a}{2}, \frac{3a}{2} \right) = \frac{8\sqrt{2}}{3a}$.

Example 2 : *If a curve is defined by the equations $x = f(t)$ and $y = \phi(t)$, prove that the radius of curvature ρ is equal to $\frac{(x'^2 + y'^2)^{3/2}}{x' y'' - y' x''}$, where accents (i.e., dashes) denote differentiation with respect to t.*

Solution : We have $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)} = \frac{y'}{x'}$.

Also $\quad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'} \right) = \left\{ \frac{d}{dt}\left(\frac{y'}{x'} \right) \right\} \cdot \frac{dt}{dx}$.

$$= \frac{y'' x' - x'' y'}{x'^2} \cdot \frac{1}{x'} \qquad \left[\because \frac{dx}{dt} = x' \text{ and } \frac{dt}{dx} = \frac{1}{x'} \right]$$

$$= \frac{y'' x' - x'' y'}{x'^3}.$$

Hence $\quad \rho = \frac{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{3/2}}{\frac{d^2 y}{dx^2}} = \frac{\left\{ 1 + \frac{y'^2}{x'^2} \right\}^{3/2}}{\frac{y'' x' - x'' y'}{x'^3}} = \frac{(x'^2 + y'^2)^{3/2}}{y'' x' - x'' y'}.$

Example 3 : *In the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$, prove that $\rho = 4a \cos \frac{1}{2} t$.*

(Bundelkhand 2007; 12, 14)

Solution : Here $\frac{dx}{dt} = a(1 + \cos t)$ and $\frac{dy}{dt} = a \sin t$.

$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2 \sin t/2 \cos t/2}{2 \cos^2 t/2} = \tan \frac{t}{2}$.

Also $\quad \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx} \right) = \frac{d}{dx}\left(\tan \frac{t}{2} \right) = \frac{1}{2} \cdot \sec^2 \frac{t}{2} \cdot \frac{dt}{dx}$

$$= \frac{1}{2} \sec^2 \frac{t}{2} \cdot \frac{1}{a(1 + \cos t)} = \frac{1}{2} \sec^2 \frac{t}{2} \cdot \frac{1}{2a \cos^2 t/2} = \frac{1}{4a} \sec^4 \frac{t}{2}.$$

Hence $\quad \rho = \frac{\left\{ 1 + \tan^2 \frac{t}{2} \right\}^{3/2}}{\frac{1}{4a} \sec^4 \frac{t}{2}} = \frac{4a \sec^3 \frac{t}{2}}{\sec^4 \frac{t}{2}} = 4a \cos \frac{1}{2} t.$

Example 4 : *If CP, CD be a pair of conjugate semi-diameters of an ellipse, prove that the radius of curvature at P is CD^3/ab, a and b being the lengths of the semi-axes of the ellipse.* **(Meerut 2001, 05B; Rohilkhand 11)**

Solution : (Note. Two perpendicular diameters are called **conjugate diameters**)

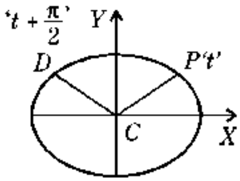
Let CP and CD be a pair of conjugate semi-diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1,$ the centre C is origin.

Let 't' be the eccentric angle of the point P. Then the co-ordinates of P are $x=a\cos t,\ y=b\sin t.$

The eccentric angle of D will be $t+\frac{1}{2}\pi$, so that the co-ordinates of D are

$$\left[a\cos\left(\tfrac{1}{2}\pi+t\right), b\sin\left(\tfrac{1}{2}\pi+t\right)\right]\ \textit{i.e.,}\ (-a\sin t, b\cos t).$$

Now for the point P we have $x=a\cos t$ and $y=b\sin t$.



$$\therefore\quad \frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=b\cos t.$$

Hence $\quad \dfrac{dy}{dx}=-\dfrac{b}{a}\cot t.$

Also $\quad \dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{dx}\left(-\dfrac{b}{a}\cot t\right)$

$$=\left\{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\right\}\cdot\frac{dt}{dx}$$

$$=\left(\frac{b}{a}\operatorname{cosec}^2 t\right)\cdot\left(-\frac{1}{a}\operatorname{cosec} t\right)=-\frac{b}{a^2}\operatorname{cosec}^3 t\cdot$$

$\therefore$ Radius of curvature of the point 't'

$$=\rho=\frac{\left\{1+\dfrac{b^2\cos^2 t}{a^2\sin^2 t}\right\}^{3/2}}{-\dfrac{b}{a^2}\operatorname{cosec}^3 t}=-\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}.$$

Neglecting the negative sign, we have $\rho=\dfrac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}.$

Now $\quad CD=\sqrt{\{(-a\sin t-0)^2+(b\cos t-0)^2\}}=(a^2\sin^2 t+b^2\cos^2 t)^{1/2}.$

$$\therefore\quad \frac{CD^3}{ab}=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}=\rho.$$

Example 5 : *For the curve* $y=\dfrac{ax}{a+x}$, *if* ρ *is the radius of curvature at any point* (x,y), *show that* $(2\rho/a)^{2/3}=(y/x)^2+(x/y)^2.$

(Kumaun 2008; Rohilkhand 10B, 13, 14; Avadh 10, 13)

Solution : We have $y=\dfrac{ax}{a+x}$ ...(1)

$$\therefore\quad \frac{dy}{dx}=a\frac{(a+x)-x}{(a+x)^2}=\frac{a^2}{(a+x)^2}=a^2(a+x)^{-2},$$

and $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -2a^2 (a+x)^{-3} = \frac{-2a^2}{(a+x)^3} = \frac{-2a^2}{(ax/y)^3} = -\frac{2y^3}{ax^3}.$$

Now $$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^4}{(a+x)^4} = 1 + \frac{a^4}{(ax/y)^4} = 1 + \frac{y^4}{x^4} = \frac{x^4+y^4}{x^4}.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2 y/dx^2} = \frac{[(x^4+y^4)/x^4]^{3/2}}{(2y^3/ax^3)},$$

(negative sign being neglected)

$$= \frac{a\,(x^4+y^4)^{3/2}}{2\,x^6\,(y^3/x^3)} = \frac{a}{2}\,\frac{(x^4+y^4)^{3/2}}{x^3\,y^3}.$$

Hence $$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^4+y^4}{x^2 y^2} = \frac{x^4}{x^2 y^2} + \frac{y^4}{x^2 y^2} = \frac{x^2}{y^2} + \frac{y^2}{x^2} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2.$$

11.6 Radius of Curvature at the Origin (Another Method)

Radius of curvature at the origin can be found by substituting $x = 0, y = 0$ in the value of ρ obtained from the formula of §11.5. Here we shall give an alternative method.

Since the curve passes through the origin, therefore $(y)_0 = 0$ *i.e.*, the value of y at $x = 0$ is 0.

Let $$\left(\frac{dy}{dx}\right)_{(0,0)} = (y_1)_0 = p \text{ and } \left(\frac{d^2 y}{dx^2}\right)_{(0,0)} = (y_2)_0 = q.$$

Then $$\rho \text{ (at origin)} = \frac{(1+p^2)^{3/2}}{q}. \qquad \text{...(1)}$$

To get the values of p and q, we know by Maclaurin's theorem that

$$y = (y)_0 + (y_1)_0\,x + \frac{(y_2)_0}{2!}\,x^2 + \ldots \qquad \text{...(2)}$$

Since the curve passes through origin, therefore (2) becomes

$$y = px + \tfrac{1}{2}\,q\,x^2 + \ldots\ldots$$

Thus to get the values of p and q, we should get from the equation of the curve an expansion for y in ascending powers of x by algebraic or trigonometric methods. The coefficient of x in this expansion will be equal to p and the coefficient of x^2 will be equal to $\frac{1}{2}q$. Putting the values of p and q in (1), we shall get ρ at origin.

11.7 Newton's Method for Radius of Curvature at the Origin

Suppose a curve passes through the origin and the x-axis is tangent to the curve at origin.

Then $(y)_0 = 0$ and $\left(\frac{dy}{dx}\right)_{(0,0)}$ *i.e,* $(y_1)_0 = 0$.

Therefore in this case by Maclaurin's expansion, we have

$$y = 0 + 0.x + \frac{q}{2}.x^2 + \frac{(y_3)_0}{3!}x^3 + \ldots \qquad \ldots(1)$$

where $(y_2)_0 = q$.

Multiplying (1) by $\frac{2}{x^2}$, we get $\frac{2y}{x^2} = q + \frac{2}{3!}(y_3)_0 x + \ldots$ ...(2)

Taking limit as $x \to 0$ of both sides of (2), we get $\lim_{x \to 0} \frac{2y}{x^2} = q$.

Also in this case ρ at origin $= \frac{(1+0)^{3/2}}{q} = \frac{1}{q} = \lim_{x \to 0} \frac{x^2}{2y}$.

Therefore *when x-axis is tangent to the curve at the origin,*

$$\rho\ (\textit{at origin}) = \lim_{x \to 0}\left(\frac{x^2}{2y}\right).$$

Similarly it can be shown that if y-axis is tangent to the curve at the origin, then

$$\rho\ (\textit{at origin}) = \lim_{x \to 0} \frac{y^2}{2x}.$$

These two formulae are known as **Newton's formulae.**

Illustrative Examples

Example 1 : *Find the radius of curvature at origin for the curve*

$$x^3 + y^3 - 2x^2 + 6y = 0.$$

Solution : The curve passes through origin. Equating to zero the lowest degree terms we get $y = 0$, *i.e.,* x-axis as tangent to the curve at origin

$\therefore$ By Newton's method ρ (at origin) $= \lim_{x \to 0} \frac{x^2}{2y}$.

Dividing by $2y$, the equation of the curve can be written as

$$x \cdot \frac{x^2}{2y} + \frac{1}{2}y^2 - 2.\frac{x^2}{2y} + 3 = 0.$$

Taking limit as $x \to 0$, $y \to 0$ and $\lim_{x \to 0} \frac{x^2}{2y} = \rho$, we get

$$0\rho + 0 - 2\rho + 3 = 0 \quad i.e., \quad \rho = 3/2.$$

Example 2 : *Show that the radii of curvature of the curve* $y^2 = x^2(a+x)/(a-x)$ *at the origin are* $\pm\, a\sqrt{2}$. **(Gorakhpur 2006)**

Solution : The curve passes through the origin and the tangents at origin are $y^2 = x^2$ *i.e.,* $y = \pm x$. Thus neither of the coordinate axes is tangent at the origin. Therefore we cannot apply Newton's method. But the equation of the curve can be written as

$$y = \frac{\pm x(a+x)^{1/2}}{(a-x)^{1/2}} \quad \text{or} \quad y = \pm x\left(1 + \frac{x}{a}\right)^{1/2}\left(1 - \frac{x}{a}\right)^{-1/2}$$

or $$y = \pm x\left\{1 + \frac{1}{2}\frac{x}{a} + \ldots\ldots\right\}\left\{1 + \frac{1}{2}\frac{x}{a} + \ldots\ldots\right\}$$

expanding by Binomial Theorem

or $$y = \pm x\left[1 + \frac{x}{a} + \ldots\ldots\right].$$

Comparing this equation with the equation

$$y = px + q\frac{x^2}{2} + \ldots, \text{ we get } p = 1, q = \frac{2}{a} \quad \text{or} \quad p = -1, q = -\frac{2}{a}.$$

But ρ at origin $= \dfrac{(1+p^2)^{3/2}}{q}$.

$\therefore$ When $p = 1, q = \dfrac{2}{a}$, ρ at origin $= \dfrac{(1+1)^{3/2}}{\frac{2}{a}} = a\sqrt{2}$.

Also when $p = -1, q = -\dfrac{2}{a}$, ρ at origin $= \dfrac{(1+1)^{3/2}}{-2/a} = -a\sqrt{2}$.

Example 3 : *Find the radii of curvature at the origin for the curve*

$$y^2 - 3xy - 4x^2 + x^3 + x^4 y + y^5 = 0.$$

Solution : The curve passes through the origin and the tangents at origin are $y^2 - 3xy - 4x^2 = 0$. Thus neither of the co-ordinate axes is tangent at the origin. Therefore Newton's method cannot be applied. Also we cannot put the equation of the curve in the form $y = px + \dfrac{qx^2}{2} + \ldots\ldots$

Hence substituting $px + \dfrac{qx^2}{2} + \ldots$ for y in the equation of the curve, we get the identity,

$$\left(px + \frac{qx^2}{2} + \ldots\ldots\right)^2 - 3x\left(px + \frac{qx^2}{2} + \ldots\ldots\right) - 4x^2 + x^3 + x^4\left(px + \frac{qx^2}{2} + \ldots\ldots\right) + \ldots = 0.$$

Equating to zero the coefficients of x^2 and x^3, we get

$$p^2 - 3p - 4 = 0 \quad \text{and} \quad pq - \frac{3q}{2} + 1 = 0.$$

Solving these we get $p = 4, -1$.

When $p = 4, q = -2/5$ and when $p = -1, q = 2/5$.

Now $\quad \rho$ (at origin) $= \dfrac{(1+p^2)^{3/2}}{q}$.

$\therefore$ When $p = 4, q = 2/5$, ρ at origin $= \dfrac{(1+16)^{3/2}}{-2/5} = -\dfrac{85\sqrt{17}}{2}$

and when $p = -1, q = 2/5$, ρ at origin $= \dfrac{(1+1)^{3/2}}{2/5} = 5\sqrt{2}$.

Comprehensive Exercise 1

1. Find the radius of curvature at the point (s, ψ) on the following curves :

(i) $s = c\tan\psi$ (Catenary)

(ii) $s = 8a\sin^2\frac{1}{6}\psi$ (Cardioid)

(iii) $s = 4a\sin\psi$ (Cycloid) **(Bundelkhand 2001; Rohilkhand 08; Kashi 11)**

(iv) $s = c \log \sec \psi$ (Tractrix). **(Kashi 2012)**

2. Find the radius of curvature at the point (x, y) on the following curves :

(i) $a^2 y = x^3 - a^3$ (ii) $y^2 = 4ax$

(iii) $xy = c^2$ (iv) $ay^2 = x^3$

(v) $y = \frac{1}{2} a (e^{x/a} + e^{-x/a})$ **(Agra 2007)**

(vi) $y = c \log \sec (x/c)$ **(Kanpur 2007; Purvanchal 09)**

(vii) $x^{1/2} + y^{1/2} = a^{1/2}$.

(viii) $x^{2/3} + y^{2/3} = a^{2/3}$ **(Rohilkhand 2009B; Kashi 12)**

(ix) $x^m + y^m = 1$.

3. (i) Find the radius of curvature of the curve $y = e^x$, at the point where it crosses the y-axis. **(Agra 2014)**

(ii) Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 1$ at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$.

4. (i) Prove that at the point $x = \frac{1}{2}\pi$ of the curve $y = 4 \sin x - \sin 2x$, $\rho = \dfrac{5\sqrt{5}}{4}$.

(ii) Prove that for the curve $s = a \log \cot \left(\dfrac{\pi}{4} - \dfrac{\psi}{2}\right) + a \sin \psi \sec^2 \psi$, $\rho = 2a \sec^3 \psi$;

and hence that $\dfrac{d^2 y}{dx^2} = \dfrac{1}{2a}$.

5. In the curve $y = ae^{x/a}$, prove that

$$\rho = a \sec^2 \theta \operatorname{cosec} \theta, \text{ where } \theta = \tan^{-1} (y/a).$$

6. Show that the radius of curvature at a point $(a \cos^3 \theta, a \sin^3 \theta)$ on the curve

$x^{2/3} + y^{2/3} = a^{2/3}$ is $3a \sin \theta \cos \theta$. **(Meerut 2000, 05; Kashi 13)**

7. Prove that for the ellipse $x^2/a^2 + y^2/b^2 = 1$,

$\rho = \dfrac{a^2 b^2}{p^3}$, p being the perpendicular from the centre upon the tangent at (x, y).

(Meerut 2002, 04B, 07; Avadh 05, 09)

8. In the ellipse $x^2/a^2 + y^2/b^2 = 1$, show that the radius of curvature at an end of the major axis is equal to the semi-latus rectum of the ellipse.

9. If ρ and ρ' be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that $(\rho^{2/3} + \rho'^{2/3})(ab)^{2/3} = a^2 + b^2$.

(Meerut 2001, 03, 04, 06, 11; Bundelkhand 06; Kanpur 11; Rohilkhand 13B; Kashi 14)

10. Prove that if ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then ρ^2 varies as $(SP)^3$.

11. If the co-ordinates of a point on a curve be given by the equations

$$x = c \sin 2\theta (1 + \cos 2\theta), y = c \cos 2\theta (1 - \cos 2\theta),$$

show that the radius of curvature at the point is $4c \cos 3\theta$.

12. If the co-ordinates of a point on a curve be given by the equations

$$x = a \sin t - b \sin (at/b), y = a \cos t - b \cos (at/b),$$

show that the radius of curvature at the point is $\dfrac{4ab}{a+b} \sin \dfrac{a-b}{2b} t$.

13. Prove that for the curve $s = ae^{x/a}$, $a\rho = s(s^2 - a^2)^{1/2}$.

Show that for the curve $s^2 = 8ay$, $\rho = 4a\sqrt{\left(1 - \dfrac{y}{2a}\right)}$. **(Kanpur 2009)**

14. Find the radius of curvature at the origin of the following curves :

(i) $y = x^4 - 4x^3 - 18x^2$. (ii) $y = x^3 + 5x^2 + 6x$.

15. Show that the radii of curvature of the curve $a(y^2 - x^2) = x^3$ at the origin are $\pm 2a\sqrt{2}$.

Answers 1

1. (i) $c \sec^2 \psi$ (ii) $\frac{4}{3} a \sin \frac{1}{3} \psi$ (iii) $4a \cos \psi$ (iv) $c \tan \psi$.

2. (i) $\dfrac{(a^4 + 9x^4)^{3/2}}{6a^4 x}$ (ii) $\left(\dfrac{2}{\sqrt{a}}\right)(x + a)^{3/2}$ (iii) $\dfrac{(x^2 + y^2)^{3/2}}{2c^2}$

(iv) $\dfrac{1}{6a}(4a + 9x)^{3/2} x^{1/2}$ (v) y^2/a (vi) $c \sec(x/c)$

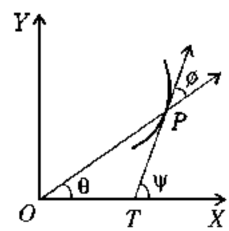
(vii) $\dfrac{2(x + y)^{3/2}}{\sqrt{a}}$ (viii) $3a^{1/3} x^{1/3} y^{1/3}$ (ix) $\dfrac{(x^{2m-2} + y^{2m-2})^{3/2}}{(1 - m) x^{m-2} y^{m-2}}$.

3. (i) $\sqrt{8}$. (ii) $\dfrac{1}{\sqrt{2}}$. **14.** (i) $1/36$, (ii) $37\sqrt{(37)}/10$.

11.8 Pedal Formula for Radius of Curvature

We have the relation $\psi = \theta + \phi$, ...(1)

as is obvious from the adjoining figure.



Differentiating (1) w.r.t. s, we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \quad \text{or} \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{dr} \cdot \frac{dr}{ds}$$

or $$\frac{1}{\rho} = \frac{1}{r} \sin \phi + \cos \phi \frac{d\phi}{dr}$$

$$\left[\because \ \rho = \frac{ds}{d\psi},\ \sin \phi = r\frac{d\theta}{ds} \text{ and } \cos \phi = \frac{dr}{ds}\right]$$

or $$\frac{1}{\rho} = \frac{1}{r}\left(\sin \phi + r \cos \phi \frac{d\phi}{dr}\right) = \frac{1}{r}\frac{d}{dr}(r \sin \phi) = \frac{1}{r}\frac{dp}{dr}. \quad [\because \ p = r \sin \phi]$$

Hence $\rho = \boldsymbol{r \dfrac{dr}{dp}}$. **(Meerut 2003, 07B)**

11.9 Polar Formula for Radius of Curvature

We know that $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. ...(1)

Differentiating (1) with respect to r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left\{\frac{d}{dr}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\left\{\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)^2\right\}\cdot\frac{d\theta}{dr}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\cdot 2\left(\frac{dr}{d\theta}\right)\cdot\frac{d^2 r}{d\theta^2}\cdot\frac{d\theta}{dr}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2 r}{d\theta^2}$$

$$\therefore \qquad \frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^5}\left\{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2 r}{d\theta^2}\right\}$$

Therefore $$\rho = r\frac{dr}{dp} = \frac{r\cdot\dfrac{1}{p^3}}{\dfrac{1}{r^5}\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}\right\}}.$$

But from (i), $$\frac{1}{p^3} = \left\{\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2} = \frac{1}{r^6}\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}.$$

Hence $$\rho = \frac{r^6\cdot\dfrac{1}{r^6}\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}}.$$

Therefore, $$\boldsymbol{\rho = \frac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}}}.$$

Corollary : If we put $u = \frac{1}{r}$ or $r = \frac{1}{u}$, then

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} \quad \text{and} \quad \frac{d^2 r}{d\theta^2} = \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2}\frac{d^2 u}{d\theta^2}.$$

Putting these values in the polar formula for ρ, we get

$$\rho = \frac{\left\{\dfrac{1}{u^2} + \dfrac{u'^2}{u^4}\right\}^{3/2}}{\dfrac{1}{u^2} + \dfrac{2u'^2}{u^4} - \dfrac{2u'^2}{u^4} + \dfrac{u''}{u^3}} = \boldsymbol{\frac{(u^2 + u'^2)^{3/2}}{u^3(u + u'')}},$$

where dashes denote differentiation with respect to θ.

Note : We see that the pedal formula for ρ is simpler than the polar formula. Therefore in case the equation of the curve is given in polar form, it is often convenient to change it first to pedal equation and then to find ρ with the help of the pedal formula.

Illustrative Examples

Example 1 : *Show that for the cardioid* $r = a(1+\cos\theta)$, $\rho = \frac{2}{3}\sqrt{(2ar)}$.

(Purvanchal 2006, 11; Rohilkhand 09B, 10; Agra 14)

Solution : The curve is $r = a(1+\cos\theta)$.

$$\therefore \qquad \frac{dr}{d\theta} = -a\sin\theta \quad \text{and} \quad \frac{d^2r}{d\theta^2} = -a\cos\theta.$$

Now
$$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}$$
$$= \frac{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}^{3/2}}{a^2(1+\cos\theta)^2 + 2(-a\sin\theta)^2 - a(1+\cos\theta)(-a\cos\theta)}$$
$$= \frac{\left(4a^2\cos^4\frac{1}{2}\theta + 4a^2\cos^2\frac{1}{2}\theta\sin^2\frac{1}{2}\theta\right)^{3/2}}{a^2 + 2a^2(\cos^2\theta + \sin^2\theta) + 3a^2\cos\theta}$$
$$= \frac{(4a^2\cos^2\frac{1}{2}\theta)^{3/2}[\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta]^{3/2}}{3a^2(1+\cos\theta)} = \frac{8a^3\cos^3\frac{1}{2}\theta}{6a^2\cos^2\frac{1}{2}\theta} = \left[\frac{4a}{3}\right]\cos\frac{1}{2}\theta.$$

But $\qquad r = a(1+\cos\theta) = 2a\cos^2\frac{1}{2}\theta$.

$$\therefore \qquad \cos\frac{1}{2}\theta = \sqrt{(r/2a)}.$$

Hence $\qquad \rho = \dfrac{4a}{3}\sqrt{\left[\dfrac{r}{2a}\right]} = (2/3)\sqrt{(2ar)}$.

Note : We could have solved this problem more easily by changing the equation of the curve to pedal form.

Example 2 : *Show that in the rectangular hyperbola* $r^2\cos 2\theta = a^2$, $\rho = r^3/a^2$.

Solution : The curve is $r^2\cos 2\theta = a^2$. ...(1)

Taking logarithm of both sides of (1), we get $2\log r + \log\cos 2\theta = 2\log a$.

Differentiating with respect to θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta}(-2\sin 2\theta) = 0$$

or
$$\frac{1}{r}\frac{dr}{d\theta} = \cot\phi = \tan 2\theta = \cot\left[\frac{\pi}{2} - 2\theta\right].$$

$$\therefore \qquad \phi = \frac{\pi}{2} - 2\theta.$$

Now $\qquad p = r\sin\phi = r\sin\left[\dfrac{\pi}{2} - 2\theta\right] = r\cos 2\theta. \qquad$ But $\cos 2\theta = \dfrac{a^2}{r^2}\cdot$

Hence the pedal equation of the curve is

$$p = r\cdot\frac{a^2}{r^2} \quad \text{or} \quad p = \frac{a^2}{r}\cdot$$

$$\therefore \qquad \frac{dp}{dr} = -\frac{a^2}{r^2}\cdot$$

Hence $\rho = r\dfrac{dr}{dp} = -\dfrac{r^3}{a^2}\cdot$

Neglecting the negative sign, we have $\rho = \dfrac{r^3}{a^2}\cdot$

Example 3 : *Find the radius of curvature at the point* (p, r) *on the ellipse*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

Solution : Differentiating the given equation with respect to r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2r}{a^2 b^2}. \qquad \therefore \quad \frac{dp}{dr} = \frac{rp^3}{a^2 b^2}.$$

Hence $\rho = r\dfrac{dr}{dp} = r\cdot\dfrac{a^2 b^2}{rp^3} = \dfrac{a^2 b^2}{p^3}\cdot$

11.10 Tangential Polar Formula for Radius of Curvature

A relation between p and ψ *holding for every point of a curve, is called tangential polar equation of the curve.* Thus the tangential polar equation of the curve is of the form $p = f(\psi)$.

We have $\dfrac{dp}{d\psi} = \dfrac{dp}{dr}\cdot\dfrac{dr}{ds}\cdot\dfrac{ds}{d\psi} = \dfrac{dp}{dr}\cos\phi\,.\,\rho$

$$= \frac{dp}{dr}\cos\phi\,.\,r\frac{dr}{dp} \qquad \left[\because \frac{dr}{ds} = \cos\phi \text{ and } \rho = \frac{ds}{d\psi} = r\frac{dr}{dp}\right]$$

$$= r\cos\phi.$$

Also $\quad p = r\sin\phi.$

$$\therefore \quad p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2(\sin^2\phi + \cos^2\phi) \text{ or } p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2. \qquad \ldots(1)$$

Differentiating (1) with respect to p, we get

$$2p + 2\left(\frac{dp}{d\psi}\right)\cdot\frac{d^2p}{d\psi^2}\cdot\frac{d\psi}{dp} = 2r\frac{dr}{dp} \qquad \text{or} \qquad r\frac{dr}{dp} = p + \frac{d^2p}{d\psi^2}.$$

Hence $\rho = \boldsymbol{p + \dfrac{d^2 p}{d\psi^2}}\cdot$

Illustrative Examples

Example 1 : *Show that for the epi-cycloid* $p = a\sin b\psi$, ρ *varies as p.*

Solution : We have $\dfrac{dp}{d\psi} = ab\cos b\psi$ and $\dfrac{d^2p}{d\psi^2} = -ab^2\sin b\psi = -b^2 p.$

$$\therefore \quad \rho = p + \frac{d^2p}{d\psi^2} = p - b^2 p = (1 - b^2)\,p.$$

Hence ρ varies as p.

Comprehensive Exercise 2

1. Find the radius of curvature at the point (p, r) on the following curves :

(i) $p^2 = ar$ (parabola). (ii) $2ap^2 = r^3$ (cardioid).

(iii) $a^2 p = r^3$ (Lemniscate). (iv) $p^2 = \dfrac{r^4}{(r^2 + a^2)}$.

2. Prove that for any curve $\dfrac{r}{\rho} = \sin \phi \left(1 + \dfrac{d\phi}{d\theta}\right)$, where ρ is the radius of curvature and $\tan \phi = r \dfrac{d\theta}{dr}$. **(Gorakhpur 2005)**

3. In the curve $p = r^{n+1}/a^n$, show that the radius of curvature varies inversely as the $(n-1)^{th}$ power of the radius vector.

4. Find the radius of curvature at the point (r, θ) on each of the following curves :

(i) $r = a \cos \theta$. **(Kanpur 2006)** (ii) $r(1 + \cos \theta) = 2a$.

(iii) $r^n = a^n \cos n\theta$. **(Rohilkhand 2005)**

(iv) $r^n = a^n \sin n\theta$ **(Agra 2006; Rohilkhand 12; Avadh 12)**

(v) $r = a(1 - \cos \theta)$. **(Avadh 2010)** (vi) $r^2 = a^2 \cos 2\theta$.

5. Forming the pedal equation of the curve

$$\theta = a^{-1} (r^2 - a^2)^{1/2} - \cos^{-1}\left(\frac{a}{r}\right), \text{ show that } \rho = \sqrt{(r^2 - a^2)}.$$

(Meerut 2006B, 08; Rohilkhand 06; Kashi 11)

6. For the rectangular hyperbola $xy = c^2$, prove that $\rho = \frac{1}{2} r^3/c^2$, r being the central radius vector of the point considered.

7. Show that at any point on the equiangular spiral $r = ae^{\theta \cot \alpha}$, $\rho = r \operatorname{cosec} \alpha$, and that it subtends a right angle at the pole.

8. If ρ_1, ρ_2 be radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos \theta)$, which passes through the pole, then show that

$$\rho_1^2 + \rho_2^2 = 16a^2/9.$$

(Kanpur 2008)

9. Show that the radius of curvature at any point on the curve $r = a(1 \pm \cos \theta)$ varies as square root of the radius vector.

10. Find the radius of curvature of the cardioid $r = a(1 - \cos \theta)$ at the pole (origin).

Answers 2

1. (i) $\dfrac{2p^3}{a^2}$ (ii) $\frac{2}{3}\sqrt{(2ar)}$ (iii) $a^2/3r$

(iv) $(r^2 + a^2)^{3/2} / (r^2 + 2a^2)$. **2.** $\dfrac{a^2 b^2}{p^3}$.

4. (i) $a/2$ (ii) $2\sqrt{(r^3/a)}$ (iii) $\dfrac{a^n}{(n+1) r^{n-1}}$

(iv) $\dfrac{a^n}{(n+1)\,r^{n-1}}$ (v) $\frac{2}{3}\sqrt{(2ar)}$ (vi) $\dfrac{a^2}{3r}$.

10. 0.

11.11 Co-ordinates of Centre of Curvature (Meerut 2008)

Let the equation of the curve be $y = f(x)$.

Let P be the given point (x, y) on this curve and Q the point $(x + \delta x, y + \delta y)$ in the neighbourhood of P. (See the fig. of article 11.3). Let N be the point of intersection of the normals at P and Q. As $Q \to P$, suppose $N \to C$. Then C is the centre of curvature of P.

Suppose co-ordinates of C are (α, β).

From the equation of the curve, we have $\dfrac{dy}{dx} = f'(x) = \phi(x)$, say.

The equation of normal at P is

$$(Y - y)\,\phi(x) + (X - x) = 0. \qquad \text{...(1)}$$

The equation of normal at Q is

$$\{Y - (y + \delta y)\}\,\phi(x + \delta x) + \{X - (x + \delta x)\} = 0. \qquad \text{...(2)}$$

Subtracting (1) from (2), we get

$$(Y - y)\,\{\phi(x + \delta x) - \phi(x)\} - \phi(x + \delta x)\,\delta y - \delta x = 0.$$

Dividing by δx, we get

$$(Y - y)\left\{\frac{\phi(x + \delta x) - \phi(x)}{\delta x}\right\} - \phi(x + \delta x)\,\frac{\delta y}{\delta x} - 1 = 0. \qquad \text{...(3)}$$

The value of Y obtained from this equation will give us the y co-ordinate of the point of intersection of (1) and (2).

Now as $Q \to P$, $\delta x \to 0$ and Y obtained from (3)$\to \beta$.

Therefore taking limit of (3) as $\delta x \to 0$, we get

$$(\beta - y)\lim_{\delta x \to 0}\left\{\frac{\phi(x + \delta x) - \phi(x)}{\delta x}\right\} - \lim_{\delta x \to 0}\phi(x + \delta x)\,.\,\lim_{\delta x \to 0}\frac{\delta y}{\delta x} - 1 = 0$$

or $$(\beta - y)\,\frac{d}{dx}\,\phi(x) - \phi(x)\cdot\frac{dy}{dx} - 1 = 0$$

or $$(\beta - y)\,\frac{d}{dx}\left(\frac{dy}{dx}\right) - \frac{dy}{dx}\cdot\frac{dy}{dx} - 1 = 0. \qquad \left[\because\ \phi(x) = \frac{dy}{dx}\right]$$

or $$(\beta - y)\,\frac{d^2 y}{dx^2} - \left\{\left(\frac{dy}{dx}\right)^2 + 1\right\} = 0.$$

$\therefore$ $$\boldsymbol{\beta = y + \dfrac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2 y}{dx^2}}}.$$

Also (α, β) lies on (1). Therefore, we get

$$(\beta - y)\frac{dy}{dx} + (\alpha - x) = 0$$

i.e.,
$$(\alpha - x) = -(\beta - y)\frac{dy}{dx} = -\frac{dy}{dx}\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

$$\therefore \quad \boldsymbol{\alpha = x - \frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}}.$$

11.12 Evolute of a Curve

The locus of the centre of curvature of a curve is called its evolute.

11.13 Equation of the Circle of Curvature

If (α, β) be the co-ordinates of the centre of curvature and ρ the radius of curvature at any point (x, y) on a curve, then the equation of the circle of curvature at that point is $\quad (X - \alpha)^2 + (Y - \beta)^2 = \rho^2.$

Illustrative Examples

Example 1 : *Find the co-ordinates of the centre of curvature for the point (x, y) on the parabola $y^2 = 4ax$.*

Also find the equation of the evolute of the parabola.

Solution : Here $2y\frac{dy}{dx} = 4a$, *i.e.,* $\frac{dy}{dx} = \frac{2a}{y} = a^{1/2}x^{-1/2} = \sqrt{\left(\frac{a}{x}\right)}$.

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{2}a^{1/2}x^{-3/2}.$$

If (α, β) be the centre of curvature for the point (x, y), then

$$\alpha = x - \frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}} = x - \frac{\sqrt{\left(\frac{a}{x}\right)}\left\{1+\frac{a}{x}\right\}}{-\frac{1}{2}\frac{1}{x}\sqrt{\left(\frac{a}{x}\right)}} = x + 2x(1 + a/x)\cdot$$

$$\therefore \quad \alpha = 3x + 2a, \qquad \text{...(1)}$$

and
$$\beta = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = y + \frac{1+\left(\frac{a}{x}\right)}{-\frac{1}{2}\frac{1}{x}\sqrt{\left(\frac{a}{x}\right)}}$$

$$= 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{3/2}(1 + a/x)$$
$$= 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{3/2} - 2a^{1/2}x^{1/2}.$$

$$\therefore \qquad \beta = -2x\sqrt{\left(\frac{x}{a}\right)}. \qquad \ldots(2)$$

Therefore the required centre of curvature is $\left((3x + 2a), -2x\sqrt{\left(\frac{x}{a}\right)}\right)$.

Evolute of the parabola : Let us eliminate x between (1) and (2). From (2), we get

$$\beta^2 = \frac{4x^3}{a} \quad \text{or} \quad x^3 = \frac{a\beta^2}{4}.$$

From (i), we get $x = \dfrac{\alpha - 2a}{3}$.

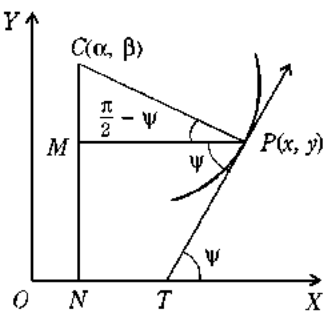
$$\therefore \qquad \left(\frac{\alpha - 2a}{3}\right)^3 = \frac{a\beta^2}{4} \quad \text{or} \quad 27\,a\beta^2 = 4\,(\alpha - 2a)^3.$$

Hence the locus of (α, β) is $27\,ay^2 = 4\,(x - 2a)^3$, which is the evolute of parabola.

Example 2 : *Prove that the co-ordinates* (α, β) *of the centre of curvature at any point* (x, y) *can be expressed in the form*

$$\alpha = x - \frac{dy}{d\psi} \text{ and } \beta = y + \frac{dx}{d\psi}.$$

Solution : Let $C(\alpha, \beta)$ be the centre of curvature of the point $P(x, y)$ on the curve $y = f(x)$. The line TP is tangent to the curve at P and obviously PC is normal at P to the curve, since the centre of curvature is a point on the normal.



ρ = radius of curvature at $P = PC$ and CN is the ordinate of C. Draw PM perpendicular to CN. Obviously

$$\angle\, CPM = \tfrac{1}{2}\pi - \psi.$$

Then $\quad \alpha = x - PM = x - \rho \sin\psi$

$$= x - \frac{dy}{ds}\cdot\frac{ds}{d\psi} \qquad \left[\because\ \rho = \frac{ds}{d\psi} \text{ and } \frac{dy}{ds} = \sin\psi\right]$$

$$= x - \frac{dy}{d\psi}.$$

Also $\quad \beta = y + CM = y + \rho\cos\psi$

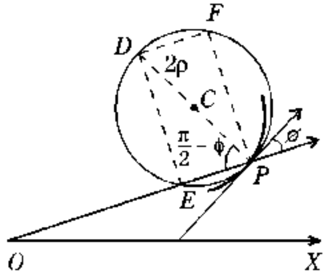
$$= y + \frac{dx}{ds}\cdot\frac{ds}{d\psi} \qquad \left[\because\ \cos\psi = \frac{dx}{ds}\right]$$

$$= y + \frac{dx}{d\psi}.$$

11.14 Chord of Curvature through the Origin (Pole)

Let C be the centre of curvature at the point P on any given curve.

Then $CP = \rho$ = radius of curvature at P. O is the pole. Join OP to meet the circle of curvature in E. Then PE is the chord of curvature through the origin.



PD is the diameter of the circle of curvature. We have $PD = 2\rho$

and $\angle PED = 90^o$, being an angle in a semi-circle.

Also PD is normal to the curve at P.

$\therefore \quad \angle EPD = \frac{1}{2}\pi - \phi.$

Hence from the right-angled triangle PED, we have

$$PE = PD\cos\left(\frac{1}{2}\pi - \phi\right) = 2\rho\sin\phi.$$

$\therefore$ *chord of curvature through pole* $=$ **2ρ sin ϕ**.

Corollary : Chord of curvature perpendicular to the Radius Vector :

Suppose a line through P, perpendicular to the radius vector OP, meets the circle of curvature in F.

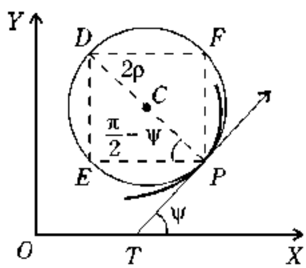
Then PF is the chord of curvature perpendicular to the radius vector.

We have $PF = ED = 2\rho\sin\left(\frac{1}{2}\pi - \phi\right) = 2\rho\cos\phi.$

$\therefore$ *Chord of curvature perpendicular to the radius vector* $= 2\rho$ **cos** ϕ.

11.15 Chord of Curvature Parallel to the Axes

(i) Chord of Curvature Parallel to the *x*-axis :



Let C be the centre of curvature at the point P on any given curve. Suppose a line through P, drawn parallel to the axis of x, meets the circle of curvature in E. Then PE is the chord of curvature parallel to x-axis.

PD is the diameter of the circle of curvature. We have $PC = 2\rho$ and $\angle PED = 90^o$.

Obviously $\angle EPD = \frac{1}{2}\pi - \psi.$

Hence from the right angled triangle PED, we have

$$PE = 2\rho\cos\left(\frac{1}{2}\pi - \psi\right) = 2\rho\sin\psi.$$

$\therefore$ *Chord of curvature parallel to x-axis* $=$ **2ρ sin ψ**.

(ii) Chord of Curvature Parallel to *y*-axis :

Draw a line through P, parallel to y-axis, to meet the circle of curvature in F.

Then $PF =$ chord of curvature parallel to y-axis

$$= ED = 2\rho\sin\left(\frac{\pi}{2} - \psi\right) = \mathbf{2\rho\cos\psi}.$$

Illustrative Examples

Example 1 : *Show that the chord of curvature through the pole of the curve $r^n = a^n\cos n\theta$ is $2r/(n+1)$.* **(Purvanchal 2014)**

Solution : The curve is $r^n = a^n\cos n\theta$. ...(1)

Taking logarithm of both sides, we get $n\log r = n\log a + \log\cos n\theta$.

Differentiating with respect to θ, we get $\dfrac{n}{r}\dfrac{dr}{d\theta} = -\dfrac{n}{\cos n\theta}\sin n\theta$

i.e., $\cot\phi = -\tan n\theta = \cot\left(\frac{1}{2}\pi + n\theta\right)$ ·

$\therefore \quad \phi = \frac{1}{2}\pi + n\theta.$

Now $p = r\sin\phi = r\sin\left(\frac{1}{2}\pi + n\theta\right) = r\cos n\theta.$

Therefore the pedal equation of the given curve is $p = r^{n+1}/a^n$.

$$\therefore \quad \frac{dp}{dr} = \frac{(n+1)\,r^n}{a^n}.$$

Also $$\rho = r\frac{dr}{dp} = \frac{a^n}{(n+1)\,r^{n-1}}.$$

Hence the chord of curvature through the pole

$$= 2\rho\sin\phi = 2\rho\sin\left(\frac{1}{2}\pi + n\theta\right) = 2\rho\cos n\theta$$

$$= 2\cdot\frac{a^n}{(n+1)\,r^{n-1}}\cdot\frac{r^n}{a^n} = \frac{2r}{(n+1)}\cdot$$

Example 2 : *In the curve $y = a\log\sec(x/a)$, prove that the chord of curvature parallel to the axis of y is of constant length.* **(Rohilkhand 2009)**

Solution : Differentiating the equation of the curve with respect to x, we get

$$\frac{dy}{dx} = a\,.\,\frac{1}{\sec(x/a)}\,.\sec(x/a)\tan(x/a)\,(1/a) = \tan(x/a).$$

$$\therefore \quad \frac{d^2y}{dx^2} = (1/a)\sec^2(x/a).$$

Chord of curvature parallel to y-axis

$$= 2\rho\cos\psi = \frac{2\rho}{\sec\psi} = \frac{2\rho}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}$$

$$= 2\cdot\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}\cdot\frac{1}{\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}}$$

$$= 2\cdot\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}}$$

$$= 2\cdot\frac{\left(1+\tan^2\dfrac{x}{a}\right)}{\dfrac{1}{a}\sec^2\left(\dfrac{x}{a}\right)} = 2a, \text{ which is constant.}$$

Comprehensive Exercise 3

1. In the parabola $x^2 = 4ay$, prove that the co-ordinates of the centre of curvature are $\left(-\dfrac{x^3}{4a^2}, 2a + \dfrac{3x^2}{4a}\right)$.
2. In the catenary $y = c \cosh(x/c)$, show that the centre of curvature (α, β) is given by $\alpha = x - y\{(y^2/c^2) - 1)\}^{1/2}, \beta = 2y$.
3. For the curve $a^2 y = x^3$, show that the centre of curvature (α, β) is given by
$$\alpha = \frac{x}{2}\left\{1 - \frac{9x^4}{a^4}\right\}, \quad \beta = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$
4. Show that the centre of curvature (α, β) at the point determined by t on the ellipse $x = a\cos t, y = b\sin t$, is given by $\alpha = \dfrac{a^2 - b^2}{a}\cos^3 t, \beta = -\dfrac{a^2 - b^2}{b}\sin^3 t$.
   Also show that the evolute of the ellipse is $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.
5. Prove that the centre of curvature (α, β) for the curve $x = 3t, y = t^2 - 6$ is $\alpha = -\frac{4}{3}t^3, \beta = 3t^2 - \frac{3}{2}$.
6. Show that in any curve the chord of curvature perpendicular to the radius vector is $2\rho\sqrt{(r^2 - p^2)}/r$.
7. Show that the chord of curvature through the pole of the equiangular spiral $r = ae^{m\theta}$ is $2r$.
8. Show that the chord of curvature, through the pole, for the cardioid
$$r = a(1 + \cos\theta) \text{ is } \frac{4}{3}r.$$
9. Show that the circle of curvature at the point $(am^2, 2am)$ of the parabola $y^2 = 4ax$ has for its equation
$$x^2 + y^2 - 6am^2 x - 4ax + 4am^3 y - 3a^2 m^4 = 0.$$
10. If C_x and C_y be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$, prove that $\dfrac{1}{C_x^2} + \dfrac{1}{C_y^2} = \dfrac{1}{2aC_x}$.

(Agra 2007; Rohilkhand 07; Purvanchal 07)

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. The relation between s and ψ for any curve is called its ...... equation.
2. By definition the curvature of the curve at any point P is equal to ....
3. For a curve $y = f(x)$, we have $\rho =$ ....
4. Intrinsic formula for the radius of curvature is ....
5. The radius of curvature at any point of the cycloid
$$x = a(t + \sin t), y = a(1 - \cos t) \text{ is ....}$$

6. For a curve defined by the equations $x = f(t)$ and $y = \phi(t)$ the radius of curvature is ....

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. If y-axis is the tangent to the given curve at the origin, then radius of curvature at the origin is equal to

(a) $\lim\limits_{x \to 0} \dfrac{x^2}{2y}$ (b) $\lim\limits_{x \to 0} \dfrac{y^2}{2x}$ (c) $\lim\limits_{x \to 0} \dfrac{x^2}{y}$ (d) $\lim\limits_{x \to 0} \dfrac{y^2}{x}$

8. Pedal formula for radius of curvature is

(a) $\dfrac{1}{r}\dfrac{dr}{dp}$ (b) $r\dfrac{dr}{dp}$ (c) $\dfrac{1}{r}\dfrac{dp}{dr}$ (d) $r\dfrac{dp}{dr}$

9. Chord of curvature parallel to y-axis is

(a) $2\rho \sin \phi$ (b) $2\rho \cos \phi$ (c) $2\rho \sin \psi$ (d) $2\rho \cos \psi$

True or False:

Write 'T' for true and 'F' for false statement.

10. There is no difference between curvature of the circle and circle of curvature. **(Meerut 2003)**

11. The curvature of the curve at any point is equal to the reciprocal of the radius of curvature at that point.

12. The polar formula for radius of curvature is : $\rho = \dfrac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$.

13. The tangential polar formula for radius of curvature is $\rho = p + \dfrac{d^2p}{d\psi^2}$.

14. Pedal formula for radius of curvature is $\rho = r\dfrac{dr}{dp}$. **(Agra 2006)**

Answers

1. intrinsic.
2. $\dfrac{d\psi}{ds}$.
3. $\dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$.
4. $\dfrac{ds}{d\psi}$.
5. $4a \cos \dfrac{1}{2} t$.
6. $\dfrac{(x'^2 + y'^2)^{3/2}}{x' y'' - y' x''}$, where $x' y'' - y' x'' \neq 0$.
7. (b).
8. (b).
9. (d).
10. *F*.
11. *T*.
12. *T*.
13. *T*.
14. *T*.

Chapter

12

Envelopes, Evolutes and Involutes

12.1 One Parameter Family of Curves

An equation of the form

$$F(x, y, \alpha) = 0 \qquad \text{...(1)}$$

in which α is a constant, represents a curve. If α is a parameter *i.e.*, if α can take all real values, then (1) is the equation of a *one parameter family of curves with parameter* α. If we give different values to α we get different members of this family. On any particular curve belonging to this family the value of α is constant but it changes from one curve to another.

An equation of the type $F(x, y, \alpha, \beta) = 0$ also defines a family of curves but in this case we have two parameters α and β.

Illustrations :

(i) The equation $x \cos \alpha + y \sin \alpha = a$ determines a family of straight lines, and α is the parameter of this family.

(ii) The equation $y = mx - 2am - am^3$ determines a family of straight lines which are normals to the parabola $y^2 = 4ax$. Here m is the parameter.

12.2 Envelope of a One Parameter Family of Curves (Kashi 2011)

Definition : Let $F(x, y, \alpha) = 0$ be a family of curves, the parameter being α.

Suppose P is a point of intersection of two members $F(x, y, \alpha) = 0$ and $F(x, y, \alpha + \delta\alpha) = 0$ of this family corresponding to the parameter values α and $\alpha + \delta\alpha$. As $\delta\alpha \to 0$, let P tend to a definite point Q on the member α. The locus of Q (for varying values of α) is called the envelope of the family.

The above definition can be given in concise form as below :

The envelope of a one parameter family of curves is the locus of the limiting positions of the points of intersection of any two members of the family when one of them tends to coincide with the other which is kept fixed.

Thus the envelope of a family of curves is the locus of the points of intersection of consecutive members of the family.

12.3 Method of Finding the Envelope

Suppose $\quad F(x, y, \alpha) = 0 \qquad \ldots(1)$

is the equation of a family of curves with parameter α.

Consider the two members

$$F(x, y, \alpha) = 0 \text{ and } F(x, y, \alpha + \delta\alpha) = 0 \qquad \ldots(2)$$

of this family corresponding to the parameter values α and $\alpha + \delta\alpha$.

The co-ordinates of the points of intersection of the curves (2) satisfy the equations

$$F(x, y, \alpha) = 0, \; F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha) = 0$$

and therefore the equations

$$F(x, y, \alpha) = 0, \; \frac{F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha)}{\delta\alpha} = 0.$$

Taking limits as $\delta\alpha \to 0$, we see that the co-ordinates of the limiting positions of the points of intersection of the curves (2) satisfy the equations

$$F(x, y, \alpha) = 0 \quad \text{and} \quad \frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0. \qquad \ldots(3)$$

Thus, for all values of α, the co-ordinates of the points on the envelope satisfy the equations (3). Therefore eliminating α between the equations (3), we shall get the envelope of the family of curves (1).

Working Rule : *The equation of the envelope of the family of curves* $F(x, y, \alpha) = 0$ *where* α *is the parameter, is obtained by eliminating* α *between the equations*

$$\boldsymbol{F(x, y, \alpha) = 0}$$

and
$$\boldsymbol{\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0.}$$

Here $\dfrac{\partial F(x, y, \alpha)}{\partial \alpha}$ is the partial derivative of $F(x, y, \alpha)$ with respect to the parameter α while x and y have been regarded as constants.

Note : The equations $x = \phi(\alpha), y = \psi(\alpha)$ obtained on solving $F(x, y, \alpha) = 0$ and $\dfrac{\partial F(x, y, \alpha)}{\partial \alpha} = 0$ are the parametric equations of the envelope, α being the parameter.

Illustrative Examples

Example 1 : *Find the envelope of the family of straight lines* $y = mx + (a/m)$, *the parameter being* m. **(Kanpur 2005)**

Solution : The equation of the given family of straight lines is

$$y = mx + (a/m) \text{, the parameter being } m. \quad ...(1)$$

Differentiating (1) partially with respect to m, we get

$$0 = x - (a/m^2) \quad \text{or} \quad m = (a/x)^{1/2}. \quad ...(2)$$

Eliminating m between (1) and (2), we get the required envelope.

Putting $m = (a/x)^{1/2}$ in (1), we get

$$y = x \cdot \frac{a^{1/2}}{x^{1/2}} + a \cdot \frac{x^{1/2}}{a^{1/2}} = 2a^{1/2} \,.\, x^{1/2}.$$

Squaring, we get $y^2 = 4ax$, which is the required envelope.

Example 2 : *Find the envelope of the family of straight lines*

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2,$$

where the parameter is θ. **(Kashi 2013; Avadh 13)**

Solution : The equation of the given family of straight lines is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2, \theta \text{ being the parameter.} \quad ...(1)$$

Differentiating (1) partially with respect to θ, we get

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0. \quad ...(2)$$

Eliminating θ between (1) and (2), we get the required envelope.

From (2), we get $\tan^3\theta = -(by/ax)$.

$\therefore$ $\tan\theta = -(by)^{1/3}/(ax)^{1/3}$.

$\therefore$ $\sin\theta = \dfrac{(by)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}, \cos\theta = -\dfrac{(ax)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}$

or $\sin\theta = -\dfrac{(by)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}, \cos\theta = \dfrac{(ax)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}$.

Substituting these values in (1), we get

$$\pm [(ax)^{2/3} + (by)^{2/3}] [(ax)^{2/3} + (by)^{2/3}]^{1/2} = a^2 - b^2$$

or $\pm [(ax)^{2/3} + (by)^{2/3}]^{3/2} = a^2 - b^2$

i.e., $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$,

which is the equation of the required envelope.

12.4 Envelope in Case the Equation of the Family of Curves is a Quadratic in the Parameter

(Garhwal 2002)

Let the equation of the family of curves be $F(x, y, \alpha) = 0$, the parameter being α. Suppose this equation can be arranged as a quadratic in α. Let this quadratic be

$$A\alpha^2 + B\alpha + C = 0, \quad ...(1)$$

where A, B and C are some functions of x and y.

Differentiating (1) partially with respect to α, we get

$$2A\alpha + B = 0. \quad ...(2)$$

Eliminating α between (1) and (2), we get the envelope.

From (2), we have $\alpha = (-B/2A)$.

Substituting this value of α in (1), we get

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0 \quad \text{or} \quad B^2 - 4AC = 0,$$

which is the required equation of the envelope.

Remember : *The envelope of the family of curves*

$$A\alpha^2 + B\alpha + C = 0,$$

where A, B, C are functions of x and y, is

$$\boldsymbol{B^2 - 4AC = 0.}$$

Illustrative Examples

Example 1 : *Find the envelope of the family of straight lines*

$y = mx + \sqrt{(a^2 m^2 + b^2)}$, *the parameter being m.* **(Garhwal 2002; Rohilkhand 14; Purvanchal 14)**

Solution : The equation of the given family of straight lines is

$$y = mx + \sqrt{(a^2 m^2 + b^2)} \quad \text{or} \quad y - mx = \sqrt{(a^2 m^2 + b^2)}$$

or $$(y - mx)^2 = a^2 m^2 + b^2 \quad \text{or} \quad y^2 - 2mxy + m^2 x^2 - a^2 m^2 - b^2 = 0$$

or $$m^2 (x^2 - a^2) - 2xym + y^2 - b^2 = 0. \qquad \text{...(1)}$$

The equation (1) is a quadratic in the parameter m. So the required envelope is obtained by equating to zero the discriminant of (1). Hence the required envelope is

$$(-2xy)^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

or $$x^2 y^2 - (x^2 - a^2)(y^2 - b^2) = 0$$

or $$x^2 y^2 - x^2 y^2 + x^2 b^2 + a^2 y^2 - a^2 b^2 = 0$$

or $$x^2 b^2 + a^2 y^2 = a^2 b^2$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$ which is an ellipse.

Example 2 : *Find the envelope of the family of straight lines $x/a + y/b = 1$, where the two parameters a, b are connected by the relation $ab = c^2$, c being a constant.* **(Kumaun 2002; Meerut 12)**

Solution : The equation of the given family of straight lines is

$$x/a + y/b = 1, \qquad \text{...(1)}$$

where the parameters a, b are connected by the relation

$$ab = c^2. \qquad \text{...(2)}$$

We shall eliminate one parameter, say b.

From (2), we have $b = c^2/a$. Putting the value of b in (1), we get

$$\frac{x}{a} + \frac{y}{c^2/a} = 1 \quad \text{or} \quad \frac{x}{a} + \frac{ay}{c^2} = 1, \qquad \text{...(3)}$$

which is the equation of the given family and it contains only one parameter a.

We can arrange (3) as a quadratic in a. Thus (3) can be written as

$$c^2 x + a^2 y = ac^2 \quad \text{or} \quad a^2 y - ac^2 + c^2 x = 0. \qquad \text{...(4)}$$

The equation (4) is a quadratic in the parameter a. So the required envelope is

$$(-c^2)^2 - 4yc^2x = 0 \quad \text{or} \quad c^4 - 4xyc^2 = 0$$

or $$xy = c^2/4.$$

12.5 Geometrical Significance of the Envelope

In general the envelope of a family of curves touches each member of the family.

Let the equation of the family of curves be

$$F(x, y, \alpha) = 0, \qquad \text{...(1)}$$

α being the parameter.

The envelope of (1) is obtained by eliminating α between (1) and

$$\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0. \qquad \text{...(2)}$$

Obviously we can take (1) as the equation of the envelope provided we regard α as a function of x and y given by (2).

Let (x, y) be a point common to the member 'α' of the family and the envelope. If at the point (x, y) we do not have $\dfrac{\partial F}{\partial x} = 0 = \dfrac{\partial F}{\partial y}$, then at this point the slope of the tangent to the member 'α' of the family is

$$-\frac{\partial F/\partial x}{\partial F/\partial y}. \qquad \text{...(3)}$$

[Refer the chapter on partial differentiation]

Also the slope of the tangent to the envelope at the point (x, y) is

$$-\frac{\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial \alpha}\dfrac{\partial \alpha}{\partial x}}{\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial \alpha}\dfrac{\partial \alpha}{\partial y}}. \qquad \text{...(4)}$$

Note that $F(x, y, \alpha) = 0$ is also the equation of the envelope provided α is not a constant but is a function of x and y given by $\partial F/\partial \alpha = 0$.

Since at every point of the envelope we have $\partial F/\partial \alpha = 0$, therefore the two slopes given by (3) and (4) are the same.

Hence the slopes of the tangents to the member of the family and the envelope at the common points are equal. This means that the curves of the family and the envelope have the same tangent at the points in common *i.e.*, they touch each other at these points.

Each point on the envelope is a point on some curve of the family and each curve of the family has some point which is on the envelope. At these common points both touch each other. Hence, *in general, the envelope of a family of curves touches each curve of the family and at each point is touched by some member of the family.*

Note : If $\partial F/\partial x$ and $\partial F/\partial y$ are both zero for any point on the curve, the slopes of the tangents cannot be found from (3) and (4). In this case the above argument breaks down and the envelope may not touch a curve at the points where $\partial F/\partial x = 0 = \partial F/\partial y$ *i.e.*, at the singular points.

If the given family of curves is a family of straight lines or a family of conics we have no singular points. Hence the envelope of a family of straight lines or of conics touches each member of the family at all their common points without exception.

Illustrative Examples

Example 1 : *Find the envelope of the family of straight lines*

$$x \cos \alpha + y \sin \alpha = a,$$

the parameter being α, *and interpret the result geometrically.*

Solution : The equation of the given family of straight lines is

$$x \cos \alpha + y \sin \alpha = a, \qquad ...(1)$$

the parameter being α.

Differentiating (1) partially with respect to α, we get

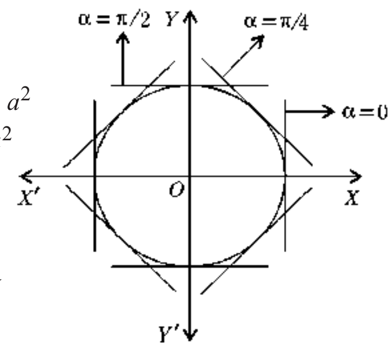
$$- x \sin \alpha + y \cos \alpha = 0. \qquad ...(2)$$

Eliminating α between (1) and (2), we get the envelope. So squaring and adding (1) and (2), we get

$$(x \cos \alpha + y \sin \alpha)^2 + (- x \sin \alpha + y \cos \alpha)^2 = a^2$$

or $\quad x^2 (\cos^2 \alpha + \sin^2 \alpha) + y^2 (\sin^2 \alpha + \cos^2 \alpha) = a^2$

or $\quad x^2 + y^2 = a^2$, which is the required envelope.



Geometrical interpretation : $x^2 + y^2 = a^2$ is the equation of a circle whose centre is origin and radius is a. This circle is the envelope of the family of straight lines $x \cos \alpha + y \sin \alpha = a$. So for each value of α, the straight line $x \cos \alpha + y \sin \alpha = a$ touches the circle $x^2 + y^2 = a^2$. Also the circle $x^2 + y^2 = a^2$ is touched at each point by some straight line belonging to the family $x \cos \alpha + y \sin \alpha = a$.

Example 2 : *Find the envelope of the family of circles*

$$x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = c^2,$$

where α *is the parameter, and interpret the result.* **(Bundelkhand 2014)**

Solution : The equation of the given family of circles can be written as

$$2ax \cos \alpha + 2ay \sin \alpha = x^2 + y^2 - c^2. \qquad ...(1)$$

[Note that we have brought the terms containing $\cos \alpha$ and $\sin \alpha$ to one side and the rest of the terms to the other side].

Differentiating (1) partially with respect to α, we get

$$- 2ax \sin \alpha + 2ay \cos \alpha = 0. \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$4a^2 x^2 + 4a^2 y^2 = (x^2 + y^2 - c^2)^2$$

or $$(x^2 + y^2 - c^2)^2 = 4a^2 (x^2 + y^2), \qquad ...(3)$$

which is the required envelope.

Interpretation : The equation (3) can be written as

$$(x^2 + y^2)^2 - 2 (2a^2 + c^2)(x^2 + y^2) + c^4 = 0.$$

Solving it as a quadratic in $(x^2 + y^2)$, we get

$$x^2 + y^2 = \frac{2 (2a^2 + c^2) \pm \sqrt{\{4 (2a^2 + c^2)^2 - 4c^4\}}}{2}$$

$$= 2a^2 + c^2 \pm 2a\sqrt{(c^2 + a^2)}$$
$$= (a^2 + c^2) \pm 2a\sqrt{(a^2 + c^2)} + a^2$$
$$= [\sqrt{(a^2 + c^2)} \pm a]^2.$$

Therefore the required envelope consists of the two circles

$$x^2 + y^2 = [\sqrt{(a^2 + c^2)} + a]^2$$

and
$$x^2 + y^2 = [\sqrt{(a^2 + c^2)} - a]^2.$$

These are the circles with centre at origin and radii

$$\sqrt{(a^2 + c^2)} \pm a.$$

Comprehensive Exercise 1

1. Find the envelope of the straight lines $(x/a)\cos\theta + (y/b)\sin\theta = 1$, the parameter being θ and interpret the result geometrically. **(Kashi 2012)**

Find the envelope of the following families of straight lines :

2. (i) $y = m^2 x + (1/m^2)$, the parameter being m.

(ii) $y = mx + a\sqrt{(1 + m^2)}$, the parameter being m. **(Kumaun 2000)**

(iii) $y = mx + am^3$, the parameter being m.

(iv) $y = mx + am^p$, the parameter being m.

(v) $x\operatorname{cosec}\theta - y\cot\theta = c$, the parameter being θ.

(vi) $x\cos^3\alpha + y\sin^3\alpha = a$, the parameter being α.

3. Find the envelope of the family of circles

$$x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha + c^2 = 0, \ (a^2 > c^2)$$

where α is the parameter, and interpret the result.

4. Find the envelope of the following families of circles :

(i) $(x - \alpha)^2 + y^2 = 4\alpha$, α being the parameter.

(ii) $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$, α being the parameter. **(Rohilkhand 2013)**

(iii) $(x - c)^2 + y^2 = R^2$, where c is the parameter.

(iv) $y^2 = m^2(x - m)$, m being the parameter.

(v) $tx^3 + t^2 y = a$, the parameter being t.

(vi) $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{k^2 - \alpha^2} = 1$, where α is the parameter. **(Kanpur 2008)**

5. Find the envelope of the family of curves

$$(a^2/x)\cos\theta - (b^2/y)\sin\theta = c^2/a, \ \theta \text{ being the parameter.}$$

6. Find the envelope of the family of straight lines

$$x\cos^n\theta + y\sin^n\theta = a, \text{ for different values of } \theta.$$

7. Find the envelope of the ellipse $x = a\sin(\theta - \alpha), y = b\cos\theta$, where α is the parameter. **(Kanpur 2009)**

8. Projectiles are fired from a gun with a constant initial velocity v_0. Supposing the gun can be given any elevation and is kept always in the same vertical plane, what is the envelope of all possible trajectories, assuming their equation to be

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha}\,?$$

9. Find the envelope of the straight lines $x\cos\alpha + y\sin\alpha = l\sin\alpha\cos\alpha$, where α is the parameter. Give the geometrical interpretation.
(Rohilkhand 2012; Avadh 07, 12)

10. Find the envelope of the family of straight lines
$$x\cos m\alpha + y\sin m\alpha = a(\cos n\alpha)^{m/n},$$
where α is the parameter. [**Hint.** Change the equation to polar co-ordinates by substituting $x = r\cos\theta, y = r\sin\theta$.]

11. Show that the radius of curvature of the envelope of the line
$$x\cos\alpha + y\sin\alpha = f(\alpha) \text{ is } f(\alpha) + f''(\alpha).$$

12. If $x^{2/3} + y^{2/3} = k^{2/3}$ is the envelope of the lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$, then find the necessary relation between a, b and k.

13. Find the envelope of the family of curves $x^2\sin\alpha + y^2\cos\alpha = a^2$, where α is the parameter.

14. Find the envelope of the family of curves $(y - c)^2 - \frac{2}{3}(x - c)^3 = 0$, where c is the parameter.

Answers 1

1. $x^2/a^2 + y^2/b^2 = 1$.
Each line of the given family is a tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

2. (i) $y^2 = 4x$. (ii) $x^2 + y^2 = a^2$.
(iii) $4x^3 + 27ay^2 = 0$. (iv) $(p-1)^{p-1}x^p + p^p ay^{p-1} = 0$.
(v) $x^2 - y^2 = c^2$. (vi) $a^2(x^2 + y^2) = x^2y^2$.

3. $(x^2 + y^2 + c^2)^2 = 4a^2(x^2 + y^2)$.
Circles with centre at origin and radii $a \pm \sqrt{(a^2 - c^2)}$.

4. (i) $y^2 - 4x - 4 = 0$. (ii) $(x + y + 1)^2 = 2(x^2 + y^2)$.
(iii) $y = \pm R$. (iv) $4x^3 = 27y^2$.
(v) $x^6 + 4ay = 0$. (vi) $x \pm y = \pm k$.

5. $(a^2/x)^2 + (b^2/y)^2 = (c^2/a)^2$.

6. $x^{2/(2-n)} + y^{2/(2-n)} = a^{2/(2-n)}$.

7. $x = \pm a$.

8. $y + (\frac{1}{2}gx^2)/v_0^2 = v_0^2/(2g)$.

9. $x^{2/3} + y^{2/3} = l^{2/3}$.

10. $r^{n/(m-n)} = a^{n/(m-n)}\cos\{n\theta/(m-n)\}$, where r, θ are the polar coordinates of (x, y).

12. $a^2 + b^2 = k^2$.

13. $x^4 + y^4 = a^4$.

14. $x - y = 0, x - y = \frac{2}{9}$.

12.6 Envelope of a Family of Curves whose Equation is not given in a Direct Form

Sometimes, we are to find the envelope of a family of curves whose equation is not given in a direct form, but we are given a law in accordance with which the equation of any member of the family can be obtained. In such cases we should first find the equation of the family of curves in a proper form and then we should find the envelope.

Illustrative Examples

Example 1 : *Find the envelope of the circles which pass through the origin and whose centres lie on the ellipse* $x^2/a^2+y^2/b^2=1$.

Solution : Any point on the ellipse $x^2/a^2+y^2/b^2=1$ is $(a\cos\theta, b\sin\theta)$.

Its distance from the origin is $\sqrt{(a^2\cos^2\theta+b^2\sin^2\theta)}$.

Therefore the equation of the given family of circles is

$$(x-a\cos\theta)^2+(y-b\sin\theta)^2=a^2\cos^2\theta+b^2\sin^2\theta$$

or $$x^2+y^2-2ax\cos\theta-2by\sin\theta=0. \qquad ...(1)$$

We are to find the envelope of the family of circles (1), where θ is the parameter. The equation (1) may be written as

$$2ax\cos\theta+2by\sin\theta=x^2+y^2. \qquad ...(2)$$

Differentiating (2) partially with respect to θ, we get

$$-2ax\sin\theta+2by\cos\theta=0. \qquad ...(3)$$

Squaring and adding (2) and (3), we get

$$4a^2x^2+4b^2y^2=(x^2+y^2)^2$$

or $$(x^2+y^2)^2=4(a^2x^2+b^2y^2),$$

which is the required envelope.

Example 2 : *Find the envelope of the circles drawn on the radii vectors of the parabola* $y^2=4ax$ *as diameter.*

Solution : Any point on the parabola $y^2=4ax$ is $(at^2, 2at)$. Equation of the circle drawn on the line joining the origin $(0,0)$ to the point $(at^2, 2at)$ as diameter is

$$(x-0)(x-at^2)+(y-0)(y-2at)=0$$

or $$x^2+y^2-axt^2-2aty=0. \qquad ...(1)$$

We are to find the envelope of the family of circles (1), where t is the parameter.

Differentiating (1) partially with respect to t, we get

$$0-2axt-2ay=0. \qquad ...(2)$$

Eliminating t between (1) and (2), we get the required envelope. From (2), we get $t=-y/x$.

Putting this value of t in (1), we get

$$x^2+y^2-ax.(y^2/x^2)+2ay.(y/x)=0$$

or $$x^2+y^2-(ay^2/x)+(2ay^2/x)=0 \quad \text{or} \quad x^2+y^2+(ay^2/x)=0$$

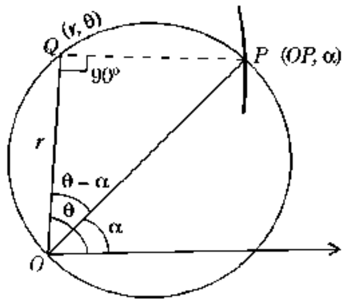
or $$ay^2+x(x^2+y^2)=0,$$ which is the required envelope.

Example 3 : *Find the envelope of the circles drawn on the radii-vectors of the curve* $r^n=a^n\cos n\theta$ *as diameter.*

Solution : Let P be any point on the curve $r^n=a^n\cos n\theta$. If α is the vectorial angle of P, then the radius vector OP is given by

$$(OP)^n=a^n\cos n\alpha.$$

$$\therefore \quad OP=a(\cos n\alpha)^{1/n}.$$



Let Q be any point (r, θ) on the circle drawn on OP as diameter. From the right angled triangle OQP, we have

$$OQ = OP \cos \angle POQ.$$

$$\therefore \quad r = a (\cos n\alpha)^{1/n} \cos (\theta - \alpha), \quad \ldots(1)$$

is the equation of the circle drawn on OP as diameter.

We are to find the envelope of the family of circles (1), where α is the parameter.

Taking logarithm of both sides of (1), we get

$$\log r = \log a + (1/n) \log \cos n\alpha + \log \cos (\theta - \alpha). \quad \ldots(2)$$

Differentiating (2) partially with respect to α, we get

$$0 = 0 + \frac{1}{n} \cdot \frac{n}{\cos n\alpha} (- \sin n\alpha) + \frac{\sin (\theta - \alpha)}{\cos (\theta - \alpha)}$$

or $$\tan n\alpha = \tan (\theta - \alpha).$$

$$\therefore \quad n\alpha = \theta - \alpha \text{ (taking principal value only)}$$

or $$\alpha = \theta/(n + 1).$$

Substituting this value of α in (1), we get the required envelope.

Thus from (1),

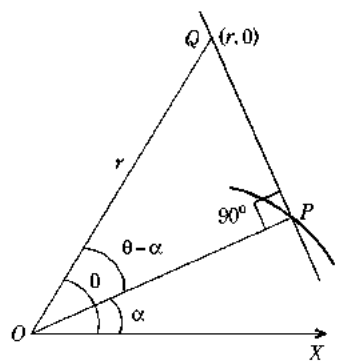
$$r = a (\cos n\alpha)^{1/n} \cos n\alpha \qquad [\because \theta - \alpha = n\alpha]$$

or $$r = a (\cos n\alpha)^{1 + 1/n} \quad \text{or} \quad r = a (\cos n\alpha)^{(n + 1)/n}$$

or $$r^{n/(n + 1)} = a^{n/(n + 1)} \cos \{n\theta/(n + 1)\},$$

which is the required envelope.

Example 4 : *Find the envelope of the straight lines drawn at right angles to the radii vectors of the cardioid $r = a(1 + \cos \theta)$ through their extremities.*



Solution : Let P be any point on the cardioid $r = a (1 + \cos \theta)$. If α is the vectorial angle of P, then the radius vector OP is given by

$$OP = a (1 + \cos \alpha).$$

Let Q be any point on the straight line drawn through P and at right angles to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ.$$

$$\therefore \quad a (1 + \cos \alpha) = r \cos (\theta - \alpha)$$

or $$r \cos (\theta - \alpha) = 2a \cos^2 \tfrac{1}{2} \alpha, \quad \ldots(1)$$

is the equation of the straight line drawn through P and at right angles to OP.

We are to find the envelope of the family of straight lines (1), where α is the parameter.

Taking logarithm of both sides of (1), we get

$$\log r + \log \cos (\theta - \alpha) = \log 2a + 2 \log \cos (\alpha/2). \quad \ldots(2)$$

Differentiating (2) partially with respect to α, we get

$$0 + \frac{1}{\cos (\theta - \alpha)} \sin (\theta - \alpha) = 0 - \frac{2}{\cos \frac{1}{2} \alpha} \left(\sin \frac{1}{2} \alpha \right) \cdot \frac{1}{2}$$

or $\tan(\theta - \alpha) = -\tan\frac{1}{2}\alpha = \tan(-\frac{1}{2}\alpha)$.

$\therefore$ $\theta - \alpha = n\pi - \frac{1}{2}\alpha$, where n is any integer.

$\therefore$ $\alpha = 2\theta - 2n\pi$.

Substituting this value of α in (1), we get

$$r\cos\{\theta - (2\theta - 2n\pi)\} = 2a\cos^2(\theta - n\pi)$$

or $r\cos(2n\pi - \theta) = 2a\cos^2\theta$

or $r\cos\theta = 2a\cos^2\theta$ or $r = 2a\cos\theta$.

Therefore the required envelope is $r = 2a\cos\theta$. It is a circle passing through the pole.

12.7 Two Parameters Connected by a Relation

Suppose the equation of the family of curves contains two parameters which are connected by a relation. We can find the envelope of this family by eliminating one parameter as we have done in one earlier example. But if the elimination of one parameter makes the subsequent process of finding the envelope difficult, we can proceed as in the following example.

Illustrative Examples

Example 1 : *Find the envelope of the family of curves* $\dfrac{x^m}{a^m} + \dfrac{y^m}{b^m} = 1$, *where the parameters a and b are connected by the relation* $a^p + b^p = c^p$.

Solution : The equation of the given family of curves is

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1, \qquad \text{...(1)}$$

where the parameters a and b are connected by the relation

$$a^p + b^p = c^p. \qquad \text{...(2)}$$

Since there is a relation between a and b, therefore we shall regard b as a function of a. Now we shall differentiate (1) and (2) with respect to a regarding x and y as constants and b as a function of a.

From (1), we get

$$-\frac{mx^m}{a^{m+1}} - \frac{my^m}{b^{m+1}}\frac{db}{da} = 0 \qquad i.e, \qquad \frac{db}{da} = -\frac{x^m/a^{m+1}}{y^m/b^{m+1}}. \qquad \text{...(3)}$$

Again from (2), we get

$$pa^{p-1} + pb^{p-1}(db/da) = 0 \qquad i.e., \qquad db/da = -a^{p-1}/b^{p-1}. \qquad \text{...(4)}$$

Equating the two values of (db/da), we get

$$\frac{x^m/a^{m+1}}{y^m/b^{m+1}} = \frac{a^{p-1}}{b^{p-1}} \quad \text{or} \quad \frac{x^m/a^m}{y^m/b^m} = \frac{a^p}{b^p}. \qquad \text{...(5)}$$

Eliminating a and b between (1), (2) and (5), we get the required envelope. From (5), we have

$$\frac{x^m/a^m}{a^p} = \frac{y^m/b^m}{b^p} = \frac{x^m/a^m + y^m/b^m}{a^p + b^p} = \frac{1}{c^p}. \qquad \textbf{[Note]}$$

$\therefore \qquad x^m/a^{p+m} = 1/c^p$

or $\qquad a^{p+m} = x^m c^p$

or $\qquad a = (x^m c^p)^{1/(p+m)}$

or $\qquad a^p = (x^m c^p)^{p/(p+m)} = x^{mp/(p+m)} c^{p^2/(p+m)}.$

Similarly $b^p = y^{mp/(p+m)} c^{p^2/(p+m)}$.

Substituting these values of a^p and b^p in (2), we get

$$c^{p^2/(p+m)} \{x^{mp/(p+m)} + y^{mp/(p+m)}\} = c^p$$

or $$x^{mp/(p+m)} + y^{mp/(p+m)} = c^{p - p^2/(p+m)}$$

or $$x^{mp/(p+m)} + y^{mp/(p+m)} = c^{mp/(p+m)},$$

which is the required envelope.

Comprehensive Exercise 2

1. Find the envelope of the circles drawn upon the radii vectors of the ellipse $x^2/a^2 + y^2/b^2 = 1$ as diameter.

2. Show that the envelope of the circles whose centres lie on the parabola $y^2 = 4ax$ and which pass through its vertex is the cissoid $y^2(2a + x) + x^3 = 0$.

3. Show that the envelope of the circles whose centres lie on the rectangular hyperbola $x^2 - y^2 = a^2$ and which pass through the origin is the lemniscate $r^2 = 4a^2 \cos 2\theta$.

4. Show that the envelope of the circles described on the central radii of a rectangular hyperbola as diameters is a lemniscate

$$r^2 = a^2 \cos 2\theta.$$

5. Show that the envelope of the polars of points on the ellipse $\dfrac{x^2}{h^2} + \dfrac{y^2}{k^2} = 1$ with respect to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{h^2 x^2}{a^4} + \dfrac{k^2 y^2}{b^4} = 1$. **(Bundelkhand 2011)**

6. Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of the ellipse

$x^2/a^2 + y^2/b^2 = 1$ is the ellipse $x^2/a^2 + y^2/b^2 = \frac{1}{2}$.

7. Find the envelopes of circles described on the radii vectors of the following curves as diameters

(i) $l/r = 1 + e \cos\theta$, (ii) $r^3 = a^3 \cos 3\theta$, (iii) $r \cos^n (\theta/n) = a$.

8. Find the envelopes of the straight lines drawn at right angles to the radii vectors of the following curves through their extremities :

(i) $r = ae^{\theta \cot \alpha}$,

(ii) $r^n = a^n \cos n\theta$,

(iii) $r = a + b \cos\theta$.

9. Find the envelope of the straight line $x/a + y/b = 1$, where the parameters a and b are connected by the following relations

(i) $a^n + b^n = c^n$, c being a constant.

(Garhwal 2003; Gorakhpur 06; Kashi 11; Meerut 13B)

(ii) $a^m b^n = c^{m+n}$, c being a constant.

(iii) $a + b = c$, **(Garhwal 2001, 03)**

(iv) $a^2 + b^2 = c^2$,

(v) $ab = c^2$, c is a constant

10. Prove that the envelope of the ellipses $x^2/a^2 + y^2/b^2 = 1$ having the sum of their semi-axes constant and equal to c, is the astroid $x^{2/3} + y^{2/3} = c^{2/3}$.

11. Find the envelope of the system of concentric and coaxial ellipses of constant area.

(Gorakhpur 2005; Kanpur 10)

[**Hint :** Taking the common centre as origin and the common axes as coordinate axes, let the equation of the family of ellipses be $x^2/a^2 + y^2/b^2 = 1$. The area of an ellipse is πab. Since the ellipses are of constant area, so let $ab = c^2$, where c is a constant. Now find the envelope.]

12. Show that the envelope of the family of parabolas

$$(x/a)^{1/2} + (y/b)^{1/2} = 1,$$

under the condition $ab = c^2$, is a hyperbola whose asymptotes coincide with the axes. **(Garhwal 2000)**

13. A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.

Answers 2

1. $(x^2 + y^2)^2 = a^2 x^2 + b^2 y^2$.

7. (i) $r^2 (e^2 - 1) - 2ler \cos\theta + l^2 = 0$.

(ii) $r^{3/4} = a^{3/4} \cos(3\theta/4)$.

(iii) $r \cos^{n-1} \{\theta/(n-1)\} = a$.

8. (i) $r \sin\alpha = ae^{(\alpha - \pi/2)\cot\alpha}\, e^{\theta \cot\alpha}$.

(ii) $r^{n/(1-n)} = a^{n/(1-n)} \cos\{n\theta/(1-n)\}$.

(iii) $r^2 - 2br\cos\theta + (b^2 - a^2) = 0$.

9. (i) $x^{n/(n+1)} + y^{n/(n+1)} = c^{n/(n+1)}$.

(ii) $\{(m+n)^{m+n} x^m y^n\}/m^m n^n = c^{m+n}$.

(iii) $x^{1/2} + y^{1/2} = c^{1/2}$. (iv) $x^{2/3} + y^{2/3} = c^{2/3}$. (v) $xy = c^2/4$.

11. $4x^2 y^2 = c^4$. **13.** A circle.

12.8 Evolute of a Curve **(Meerut 2012B)**

We define the **evolute** of a curve as the locus of the centre of curvature for that curve.

Evolute as the envelope of the normals : The centre of curvature of a curve for a given point P on it is the limiting position of the intersection of the normal at P with the normal at any other consecutive point Q as $Q \to P$. Therefore by the definition of envelope, the envelope of the normals to a curve is the **evolute** of that curve. Hence, the evolute of a curve is the envelope of the normals to that curve.

Theorem : *The normal at any point of a curve is a tangent to its evolute touching at the corresponding centre of curvature.*

Proof : The co-ordinates (α, β) of the centre of curvature for any point $P(x, y)$ on the given curve are given by $\alpha = x - \rho \sin \psi$; $\beta = y + \rho \cos \psi$.

Differentiating these with respect to x, we get

$$\frac{d\alpha}{dx} = 1 - \rho \cos \psi \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\psi} \cdot \frac{dx}{ds} \cdot \frac{d\psi}{dx} - \sin \psi \frac{d\rho}{dx} = - \sin \psi \frac{d\rho}{dx} \qquad \ldots(1)$$

and

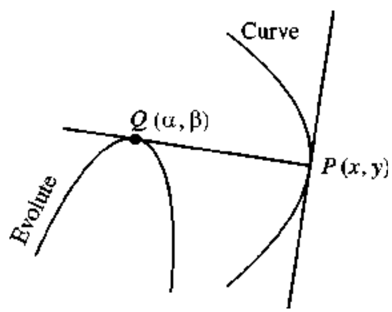
$$\frac{d\beta}{dx} = \frac{dy}{dx} - \rho \sin \psi \frac{d\psi}{dx} + \cos \psi \frac{d\rho}{dx}$$

$$= \frac{dy}{dx} - \frac{ds}{d\psi} \cdot \frac{dy}{ds} \cdot \frac{d\psi}{dx} + \cos \psi \frac{d\rho}{dx} = \cos \psi \frac{d\rho}{dx}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{d\beta}{d\alpha} = - \cot \psi \qquad \ldots(3)$$

But $\frac{d\beta}{d\alpha}$ is the slope of the tangent at Q to the evolute and $-\cot \psi$ is the slope of the normal PQ at P to the given curve. These two slopes are equal, and Q is a common point on both the lines. Hence the tangent at Q to the evolute and the normal at P to the given curve coincide *i.e.*, the normal at P to the given curve touches its evolute at the corresponding point.



12.9 Length of Arc of an Evolute

The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.

Let $C(\alpha, \beta)$ be the centre of curvature of the point $P(x, y)$ on the given curve. Then $CP = \rho$.

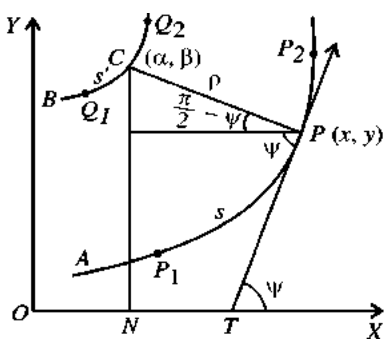
Also $\alpha = x - \rho \sin \psi$, ...(1)

and $\beta = y + \rho \cos \psi$. ...(2)

Let s be the arc length of the given curve measured from some fixed point A on it to the point (x, y) and s' the length of the evolute measured from some fixed point B on it to the point (α, β).

Differentiating both sides of (1) w.r.t. 's', we have

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - \frac{d\rho}{ds} \sin \psi - \rho \cos \psi \frac{d\psi}{ds}$$



$$= \cos\psi - \frac{d\rho}{ds}\sin\psi - \rho\cos\psi \cdot \frac{1}{\rho} \qquad \left[\because \frac{dx}{ds} = \cos\psi, \frac{d\psi}{ds} = \frac{1}{\rho}\right]$$

$$= -(d\rho/ds)\sin\psi. \qquad \text{...(3)}$$

Differentiating both sides of (2), w.r.t. 's', we have

$$\frac{d\beta}{ds} = \frac{dy}{ds} + \frac{d\rho}{ds}\cos\psi - \rho\sin\psi\frac{d\psi}{ds}$$

$$= \sin\psi + \frac{d\rho}{ds}\cos\psi - \rho\sin\psi \cdot \frac{1}{\rho} \qquad \left[\because \frac{dy}{ds} = \sin\psi\right]$$

$$= \frac{d\rho}{ds}\cos\psi. \qquad \text{...(4)}$$

Squaring and adding (3) and (4), we get

$$\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 = \left(\frac{d\rho}{ds}\right)^2.$$

$$\therefore \qquad \sqrt{\left[\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2\right]} = \frac{d\rho}{ds}. \qquad \text{...(5)}$$

Now s' denotes the arc length of the locus of the point (α, β). Regarding α, β as the functions of the parameter s, we have

$$\frac{ds'}{ds} = \sqrt{\left[\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2\right]} \qquad \text{...(6)}$$

From (5) and (6), we have $ds'/ds = d\rho/ds$.

$$\therefore \qquad ds' = d\rho \qquad \text{...(7)}$$

Let Q_1 and Q_2 be the points on the evolute corresponding to the points P_1 and P_2 on the given curve. Then integrating (7) between these points, we get

$$\left[s'\right]_{Q_1}^{Q_2} = \left[\rho\right]_{P_1}^{P_2}$$

i.e., $(s' \text{ at } Q_2) - (s' \text{ at } Q_1) = (\rho \text{ at } P_2) - (\rho \text{ at } P_1)$

i.e., the arc length of the evolute from Q_1 to $Q_2 =$ the difference between the radii of curvature of the given curve at the points P_1 and P_2.

Illustrative Examples

Example 1 : *Find the evolute of the parabola* $y^2 = 4ax$.

(Kanpur 2006; Avadh 08; Meerut 12B; Purvanchal 14)

Solution : We know that the evolute of a curve is the envelope of the normals to that curve.

Equation of any normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3, \qquad \text{...(1)}$$

where m is the parameter.

So the envelope of (1) is the evolute of $y^2 = 4ax$.

Differentiating (1) partially with respect to m, we get

$$0 = x - 2a - 3am^2$$

i.e., $m = \{(x - 2a)/3a\}^{1/2}$.

Substituting this value of m in (1), we get

$$y=\left(\frac{x-2a}{3a}\right)^{1/2}\left[x-2a-a\cdot\frac{x-2a}{3a}\right]$$

$$=\left(\frac{x-2a}{3a}\right)^{1/2}(x-2a)\cdot\frac{2}{3}=\frac{2\,(x-2a)^{3/2}}{3\sqrt{(3a)}}.$$

Squaring, we get $27ay^2=4\,(x-2a)^3$, which is the required evolute.

Example 2 : *Find the evolute of the hyperbola*

$$x^2/a^2-y^2/b^2=1.$$

Soution : The given hyperbola is

$$x^2/a^2-y^2/b^2=1. \qquad ...(1)$$

The evolute of the hyperbola (1) is the envelope of the family of normals to the hyperbola (1). The coordinates (x, y) of any point P on the hyperbola (1) may be taken as

$$x=a\sec\theta,\ y=b\tan\theta, \text{ where } \theta \text{ is the parameter.}$$

We have $dx/d\theta=a\sec\theta\tan\theta$, $dy/d\theta=b\sec^2\theta$.

$\therefore$ slope of the normal to the hyperbola (1) at the point $(a\sec\theta, b\tan\theta)$

$$=-\frac{dx}{dy}=-\frac{dx/d\theta}{dy/d\theta}$$

$$=-\frac{a\sec\theta\tan\theta}{b\sec^2\theta}=-\frac{a\tan\theta}{b\sec\theta}.$$

$\therefore$ Equation of the normal to the hyperbola (1) at the point $(a\sec\theta, b\tan\theta)$ is

$$y-b\tan\theta=-\frac{a\tan\theta}{b\sec\theta}(x-a\sec\theta)$$

or $$ax\tan\theta+by\sec\theta=(a^2+b^2)\sec\theta\tan\theta$$

or $$ax\cos\theta+by\cot\theta=a^2+b^2. \qquad ...(2)$$

Now the evolute of the hyperbola (1) is the envelope of the family of straight lines (2), where θ is the parameter.

Differentiating (2) partially with respect to θ, we get

$$-ax\sin\theta-by\operatorname{cosec}^2\theta=0$$

or $$ax\sin\theta=-by\operatorname{cosec}^2\theta$$

or $$\sin^3\theta=-\frac{by}{ax} \quad\text{or}\quad \sin\theta=-\frac{(by)^{1/3}}{(ax)^{1/3}}.$$

$\therefore$ $$\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{\left[1-\frac{(by)^{2/3}}{(ax)^{2/3}}\right]}$$

$$=\frac{[(ax)^{2/3}-(by)^{2/3}]^{1/2}}{(ax)^{1/3}}$$

and $$\cot\theta=\frac{\cos\theta}{\sin\theta}=-\frac{[(ax)^{2/3}-(by)^{2/3}]^{1/2}}{(by)^{1/3}}.$$

Substituting the values of $\cos\theta$ and $\cot\theta$ in (2), the envelope of the family of straight lines (2) is

$$ax\cdot\frac{[(ax)^{2/3}-(by)^{2/3}]^{1/2}}{(ax)^{1/3}}-by\cdot\frac{[(ax)^{2/3}-(by)^{2/3}]^{1/2}}{(by)^{1/3}}=a^2+b^2$$

or $\quad (ax)^{2/3}\,[(ax)^{2/3} - (by)^{2/3}]^{1/2} - (by)^{2/3}\,[(ax)^{2/3} - (by)^{2/3}]^{1/2} = a^2 + b^2$

or $\quad [(ax)^{2/3} - (by)^{2/3}]\,[(ax)^{2/3} - (by)^{2/3}]^{1/2} = a^2 + b^2$

or $\quad [(ax)^{2/3} - (by)^{2/3}]^{3/2} = (a^2 + b^2)$

or $\quad (ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$,

which is the required evolute of the hyperbola (1).

Example 3 : *Show that the whole length of the evolute of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *is* $4\,(a^2/b - b^2/a)$. **(Lucknow 2009, 10; Meerut 13)**

Solution : The given equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$. ...(1)

Now ρ at the point $(a\cos t, b\sin t)$ of (1) $= (a^2\sin^2 t + b^2\cos^2 t)^{3/2}/ab$.

[Refer the chapter on curvature]

But at the ends of major and minor axes t is equal to 0 and $\frac{1}{2}\pi$ respectively.

$\therefore \quad \rho_1 = \rho$ at the end of major axis $= (b^2)^{3/2}/ab = b^2/a$

and $\quad \rho_2 = \rho$ at the end of minor axis $= (a^2)^{3/2}/ab = a^2/b$.

Since the given ellipse is symmetrical about both the axes, therefore its evolute must also be symmetrical about both the axes. Hence the whole length of the evolute of the ellipse

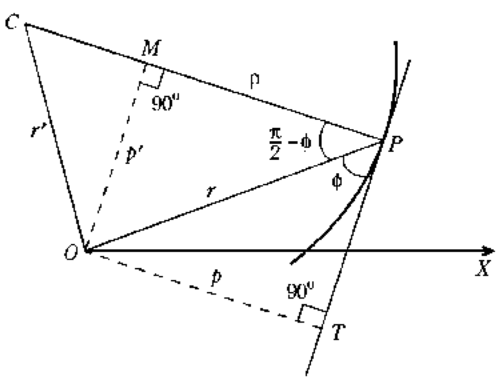
$$= 4\,(\rho_2 - \rho_1) = 4\,(a^2/b - b^2/a).$$

12.10 Evolute of Polar Curves

In the case of polar curves there is no standard method of finding the evolute of a curve. However, if a curve is given in pedal form, we can easily find the pedal equation of its evolute by the method given below.

Let the pedal equation of the given curve be

$$p = f(r). \qquad ...(1)$$



Let C be the centre of curvature corresponding to the point P on the given curve.

Then $PC = \rho$ and PC is normal to the given curve at the point P. Corresponding to the point P on the given curve the point on the evolute is C. Since the evolute of a curve is the envelope of the normals of that curve, therefore the normal PC of the given curve is tangent to the evolute at the point C.

If OT is the perpendicular from the pole O to the tangent to the given curve at the point P, then $OT = p$ and $OP = r$.

Draw OM perpendicular from the pole O to the tangent PC to the evolute at the point C. If $OC = r'$ and $OM = p'$, then the relation between p' and r' will be the pedal equation of the evolute.

From the $\Delta\, OPC$, we have

$$OC^2 = OP^2 + PC^2 - 2OP\,.\,PC\cos\angle\,OPC$$

i.e., $$r'^2 = r^2 + \rho^2 - 2r\rho \sin \phi$$

i.e., $$r'^2 = r^2 + \rho^2 - 2\rho p. \qquad [\because p = r \sin \phi]. \quad ...(2)$$

Now $OTPM$ is a rectangle and so $MP = OT = p$. From the right angled triangle OMP, we have

$$OM^2 = OP^2 - MP^2$$

i.e., $$p'^2 = r^2 - p^2. \quad ...(3)$$

Also $$\rho = r \frac{dr}{dp}. \quad ...(4)$$

Eliminating r, p and ρ between the equations (1), (2), (3) and (4), we get the pedal equation of the evolute.

Illustrative Examples

Example 1 : *Show that the evolute of an equiangular spiral is an equiangular spiral.* **(Rohilkhand 2012B)**

Solution : Let the pedal equation of the given equiangular spiral be

$$p = r \sin \alpha. \quad ...(1)$$

We have $dp/dr = \sin \alpha$.

$$\therefore \quad \rho = r \frac{dr}{dp} = r \cdot \frac{1}{\sin \alpha} = r \operatorname{cosec} \alpha. \quad ...(2)$$

Corresponding to the point (p, r) on the given curve (1), let the point on the evolute be (p', r'), the co-ordinates in each case being expressed in pedal form.

Then $$r'^2 = r^2 + \rho^2 - 2\rho p$$
$$= r^2 + r^2 \operatorname{cosec}^2 \alpha - 2r \operatorname{cosec} \alpha \,.\, r \sin \alpha,$$

[from (1) and (2)]

$$= r^2 \operatorname{cosec}^2 \alpha - r^2 = r^2 \cot^2 \alpha. \quad ...(3)$$

Also $$p'^2 = r^2 - p^2 = r^2 - r^2 \sin^2 \alpha = r^2 \cos^2 \alpha \quad ...(4)$$

Dividing (4) by (3), we get

$$\frac{p'^2}{r'^2} = \frac{r^2 \cos^2 \alpha}{r^2 \cot^2 \alpha} = \sin^2 \alpha.$$

$$\therefore \quad p'^2 = r'^2 \sin^2 \alpha \quad \text{or} \quad p' = r' \sin \alpha.$$

Hence the locus of the point (p', r') is $p = r \sin \alpha$. This is the pedal equation of the evolute and is an equiangular spiral.

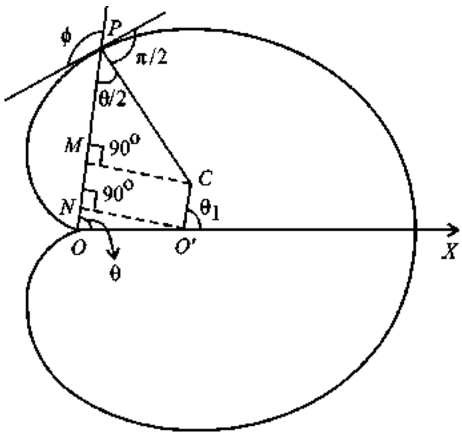
Example 2 : *Prove that the evolute of the cardioid* $r = a(1 + \cos\theta)$ *is the cardioid* $r = \frac{1}{3} a (1 - \cos \theta)$, *the pole of the latter equation being at the point* $(\frac{2}{3} a, 0)$.

Solution : The given cardioid is

$$r = a(1 + \cos \theta). \quad ...(1)$$

Let O be the pole and OX the initial line. The cardioid (1) has been drawn in the figure.

Take any point $P(r, \theta)$ on the given cardioid (1). Also let O' be the given point $(\frac{2}{3} a, 0)$ [this will be on the initial line because $\theta = 0$].



Let C be the centre of curvature of the curve (1) corresponding to the point P. Then $CP = \rho =$ radius of carvature of (1) at the point P.

We have to find the locus of the point C w.r.t. O' as pole. Let $O'C = r_1$ and $\angle CO'X = \theta_1$. Draw CM and $O'N$ perpendiculars from C and O' respectively to OP.

Let us first find the value of ρ. Taking logarithm of both sides of (1), we get

$$\log r = \log a + \log(1 + \cos\theta).$$

Differentiating w.r.t. 'θ', we get

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= -\tan\tfrac{1}{2}\theta = \cot(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta).$$

$\therefore \qquad \phi = \frac{1}{2}\pi + \frac{1}{2}\theta.$

Now $\qquad p = r\sin\phi = r\sin(\frac{1}{2}\pi + \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta.$

From (1), $\qquad r = 2a\cos^2\frac{1}{2}\theta = 2a\,(p/r)^2.$

$\therefore$ the pedal equation of the curve (1) is

$$r^3 = 2ap^2. \qquad \text{...(2)}$$

Differentiating (2) w.r.t. 'p', we get

$$3r^2\,(dr/dp) = 4ap.$$

$\therefore \qquad \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\left(\frac{r^3}{2a}\right)^{1/2}, \qquad \left[\because \text{ from (2), } p^2 = \frac{r^3}{2a}\right]$

$$= \frac{4a}{3}\left(\frac{r}{2a}\right)^{1/2} = \frac{4a}{3}\left\{\frac{a\,(1 + \cos\theta)}{2a}\right\}^{1/2}$$

$$= \frac{4a}{3}\left(\frac{2a\cos^2\frac{1}{2}\theta}{2a}\right)^{1/2}$$

$$= \frac{4a}{3}\cos\frac{1}{2}\theta. \qquad \text{...(3)}$$

Since $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$,

$\therefore \quad \angle CPM = \frac{1}{2}\theta$.

$\therefore \quad CM = PC \sin\frac{1}{2}\theta = \rho \sin\frac{1}{2}\theta = \frac{4}{3} a \cos\frac{1}{2}\theta \sin\frac{1}{2}\theta = \frac{2}{3} a \sin\theta$.

$\therefore \quad$ Also $O'N = OO' \sin\theta = \frac{2}{3} a \sin\theta$.

Thus $CM = O'N$ and consequently $O'NMC$ is a rectangle. Therefore $O'C$ is parallel to OP,

i.e., $\quad \theta_1 = \theta$. ...(4)

Now $\quad r_1 = O'C = NM = OP - ON - PM$

$$= OP - OO' \cos\theta - PC \cos\frac{1}{2}\theta$$

$$= r - \frac{2}{3} a \cos\theta - \frac{4}{3} a \cos\frac{1}{2}\theta \cos\frac{1}{2}\theta$$

$$= a(1 + \cos\theta) - \frac{2}{3} a \cos\theta - \frac{2}{3} a (1 + \cos\theta)$$

$$= \frac{1}{3} a (3 + 3\cos\theta - 2\cos\theta - 2 - 2\cos\theta)$$

$$= \frac{1}{3} a (1 - \cos\theta)$$

$$= \frac{1}{3} a (1 - \cos\theta_1). \qquad [\because \theta_1 = \theta]$$

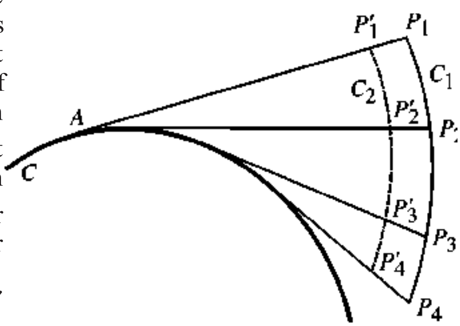
Evolute is the locus of the centre of curvature. Hence generalising $r_1 = \frac{1}{3} a(1 - \cos\theta_1)$, the required equation of the evolute is $r = \frac{1}{3} a (1 - \cos\theta)$ referred to O' as pole.

12.11 Involutes

Definition : *If one curve is the evolute of another, then the latter is called an* ***involute*** *of the former.* Thus a curve C_1 is an involute of a given curve C, if the curve C is the evolute of C_1.

Theorem : *Every curve has an infinite number of involutes.*

Let C be the given curve. Take an inextensible thin string and tie one end of this string to a fixed point, say A, of the curve C and then wrap the string round the convex side of C, keeping it taut all the while. Then any point on the string will describe an involute of C, since at each instant the free part of the string is a tangent to C whereas the direction of motion of the point on the string is at right angles to this tangent *i.e.,* this tangent is along the normal to the locus of that point on the string. In the figure the locus of the point P_1 on the string is the curve C_1 which is an involute of C and the locus of the point P_1' on the string is the curve C_2 which is also an involute of C. Similarly by taking some other point on the string we may obtain some other involute of C.



Since $\quad P_1P_1' = P_2P_2' = P_3P_3' = P_4P_4'$

etc., therefore the curves C_1 and C_2 are at a constant distance along their common normal at their corresponding points. We have thus shown that there are an infinite number of involutes to the given curve C and all of them form a system of ***parallel curves.***

Comprehensive Exercise 3

1. Show that the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2.$$

Hence find the evolute of the above ellipse. **(Avadh 2006, 10)**

2. Find the equation of the evolute of the parabola $y^2 = 2px$.

3. Show that the evolute of the tractrix

$$x = a\,(\cos t + \log \tan \tfrac{1}{2} t),\ y = a \sin t,$$

is the catenary $y = a \cosh (x/a)$.

4. Prove that the evolute of the hyperbola $2xy = a^2$ is

$$(x + y)^{2/3} - (x - y)^{2/3} = 2a^{2/3}.$$

5. Prove that the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is the envelope of the family of ellipses given by

$$a^2x^2 \sec^4 \alpha + b^2y^2 \operatorname{cosec}^4 \alpha = (a^2 - b^2)^2,$$

α being the variable parameter.

6. Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

7. Show that the evolute of the curve whose pedal equation is $r^2 - a^2 = mp^2$ is the curve whose pedal equation is $r^2 - (1 - m)\,a^2 = mp^2$.

8. Prove that normals to a given curve are always tangent lines to its evolute.

Answers 3

1. $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$. **2.** $27py^2 = 8\,(x - p)^3$.

6. $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "……" so that the following statements are complete and correct.

1. The equation of the envelope of the family of curves $F(x, y, \alpha) = 0$ where α is the parameter, is obtained by eliminating α between the equations $F(x, y, \alpha) = 0$ and $\dfrac{\partial F(x, y, \alpha)}{\partial \alpha} = \ldots\ldots$.

2. The envelope of the family of curves $A\alpha^2 + B\alpha + C = 0$, where A, B, C are functions of x and y is ....

3. The envelope of the family of curves $tx^2 + t^2 y = a$, the parameter being t is .... .

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

4. The envelope of the family of straight lines $y = mx + a/m$, the parameter being m is

(a) $y = 4ax$ (b) $y^2 = 4ax$ (c) $y^2 = ax$ (d) $y^2 = 2ax$.

5. The envelope of the family of straight lines $(x/a)\cos\theta + (y/b)\sin\theta = 1$, the parameter being θ is

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$ (c) $x^2 + y^2 = a^2 + b^2$ (d) $a^2x^2 + b^2y^2 = 0$.

6. The envelope of the straight line $y = m^2 x + 1/m^2$, the parameter being m is

(a) $y^2 = 4mx$ (b) $y^2 = 2mx$ (c) $y^2 = 4x$ (d) $y + x = m$.

7. The envelope of the family of curves $A\alpha^2 + B\alpha + C = 0$, where A, B, C are functions of x, y and α is a parameter is **(Garhwal 2002)**

(a) $A^2 = 4BC$ (b) $B^2 = 4AC$ (c) $A^2 + 4BC = 0$ (d) $A^2 - BC = 0$

8. The locus of the centres of curvatures of all points of a given plane curve is called

(a) radius of curvature (b) envelope

(c) evolute (d) normal **(Kumaun 2008)**

True or False:

Write 'T' for true and 'F' for false statement.

9. In general, the envelope of a family of curves touches each curve of the family and at each point is touched by some member of the family.

10. The envelope of $x^2 \sin\alpha + y^2 \cos\alpha = a^2$, the parameter being α is $x^2 + y^2 = a^2$.

11. The evolute of a curve is the envelope of the tangents of that curve.

12. If one curve is the evolute of another, then the latter is called an involute of the former.

13. The evolute of an equiangular spiral is not an equiangular spiral.

Answers

1. 0. **2.** $B^2 = 4AC$. **3.** $x^4 + 4ay = 0$.
4. (b). **5.** (a). **6.** (c).
7. (b). **8.** (c). **9.** *T*.
10. *F*. **11.** *F*. **12.** *T*.
13. *F*.

Chapter 13

Asymptotes

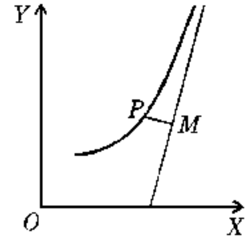
13.1 Asymptote

Suppose a curve is not limited in extent *i.e.* it has some branch or branches which extend to infinity. Parabola and Hyperbola are familiar curves of this type. Take a point on an infinite branch of such a curve and draw a tangent to the curve at this point. If the distance of the point of contact from the origin tends to infinity, the tangent itself, may or may not tend to a definite straight line. In case the tangent tends to a definite straight line, at a finite distance from the origin, it is called an **asymptote** of the curve. Thus we can define the asymptote of a curve as follows.

Definition : *A straight line at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.* **(Kanpur 2005, 07, 14; Kashi 11)**

The curve approaches the asymptote :

Roughly speaking an asymptote is a tangent with its point of contact at a great distance from the origin. Therefore, when a point P on a curve tends to infinity, its perpendicular distance from the corresponding asymptote tends to zero. This property of an asymptote enables us to draw more accurately those curves which have asymptotes. We draw the asymptotes first, and then the curve, making its branches approach the corresponding asymptotes.



Branch of a Curve : Suppose the equation of the curve is such that y has two or more values for every value of x. Corresponding to these distinct values of y we shall get *different branches* of the curve. It is just possible, that each branch may have its own separate asymptote. Therefore a *curve may have more than one asymptote.*

13.2 Determination of Asymptotes

Let the equation of the curve be $f(x, y) = 0$. ...(1)

We shall here consider the case of only those asymptotes which are not *parallel to y-axis*. We know that the equation of a straight line which is not parallel to y-axis is of the form

$$y = mx + c. \qquad ...(2)$$

The abscissa, x, must tend to infinity as the point $P(x, y)$ on the curve (1) tends to infinity along the line (2).

The equation of the tangent to the curve (1) at the point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x), \quad \text{or} \quad Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right). \qquad ...(3)$$

As $\quad x \to \infty, \qquad \dfrac{dy}{dx}$ and $y - x\dfrac{dy}{dx}$

must both tend to finite limits, in order that an asymptote might exist.

Suppose the tangent (3) tends to the straight line (2) as $x \to \infty$. Then (2) is an asymptote of the curve (1). Also we have

$$m = \lim_{x \to \infty} \frac{dy}{dx} \quad \text{and} \quad c = \lim_{x \to \infty} \left(y - x\frac{dy}{dx}\right).$$

Since c is finite, therefore $\displaystyle\lim_{x \to \infty} \frac{y - x\dfrac{dy}{dx}}{x} = 0$

i.e., $$\lim_{x \to \infty}\left(\frac{y}{x} - \frac{dy}{dx}\right) = 0 \quad \textit{i.e.,} \quad \lim_{x \to \infty}\frac{y}{x} = \lim_{x \to \infty}\frac{dy}{dx} = m.$$

Therefore $\displaystyle\lim_{x \to \infty}\frac{y}{x} = \lim_{x \to \infty}\frac{dy}{dx} = m.$

Also $$c = \lim_{x \to \infty}\left(y - x\frac{dy}{dx}\right) = \lim_{x \to \infty}(y - mx), \text{ since } \lim_{x \to \infty}\frac{dy}{dx} = m.$$

Hence, if *$y = mx + c$ is an asymptote to the curve $f(x, y) = 0$,*

$$m = \lim_{x \to \infty}\frac{y}{x} \textit{ and } c = \lim_{x \to \infty}(y - mx).$$

13.3 The Asymptotes of the General Rational Algebraic Curve

Let the equation to the curve be

$$\{a_0 y^n + a_1 y^{n-1}x + a_2 y^{n-2}x^2 + \ldots + a_{n-1} yx^{n-1} + a_n x^n\}$$
$$+ \{b_1 y^{n-1} + b_2 y^{n-2}x + \ldots + b_{n-1} yx^{n-2} + b_n x^{n-1}\}$$
$$+ \{c_2 y^{n-2} + \ldots\} + \ldots = 0, \qquad ...(1)$$

or
$$x^n\phi_n(y/x)+x^{n-1}\phi_{n-1}(y/x)+x^{n-2}\phi_{n-2}(y/x)+\ldots$$
$$\ldots+x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0, \quad \ldots(2)$$

where $\phi_r\left(\frac{y}{x}\right)$ is a polynomial in $\frac{y}{x}$ of degree r.

Dividing (2) by x^n, we get
$$\phi_n(y/x)+\frac{1}{x}\phi_{n-1}(y/x)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots\ldots$$
$$\ldots\ldots\ldots+\frac{1}{x^{n-1}}\phi_1\left(\frac{y}{x}\right)+\frac{1}{x^n}\phi_0\left(\frac{y}{x}\right)=0. \quad \ldots(3)$$

Excluding at present the case of asymptotes parallel to the y-axis (*i.e.* excluding the case in which $\lim\limits_{x\to\infty}(y/x)$ is equal to ∞), (3) gives, on taking limits as $x\to\infty$, the equation
$$\boldsymbol{\phi_n(m)=0}, \quad \ldots(4)$$
where
$$m=\lim_{x\to\infty}\left(\frac{y}{x}\right)=\text{slope of an asymptote.}$$

The equation (4) is, in general, of degree n in m. Solving this equation, we shall get the slopes of the asymptotes. This equation will give us n values of m, corresponding to the n branches of the curve (1). However, some of the values of m may be equal, and this will be the case of parallel asymptotes. Since we are concerned only with real asymptotes, therefore we shall reject the imaginary roots of (4) if there are any.

Now if $y=mx+c$ is an asymptote of (1), then we know that corresponding to a specified value of m, we have
$$c=\lim_{x\to\infty}(y-mx).$$

Therefore to determine the value of c corresponding to the value of m, we put $y-mx=p$ in the equation of the curve, where p is a variable which $\to c$ as $x\to\infty$.

So putting $y=mx+p$ *i.e.*,
$$\frac{y}{x}=m+\frac{p}{x} \text{ in (2), we get}$$
$$x^n\phi_n\left(m+\frac{p}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{p}{x}\right)+x^{n-2}\phi_{n-2}\left(m+\frac{p}{x}\right)+\ldots$$
$$\ldots+x\phi_1\left(m+\frac{p}{x}\right)+\phi_0\left(m+\frac{p}{x}\right)=0\cdot \quad \ldots(5)$$

Expanding each term of (5) by Taylor's theorem, we get
$$x^n\left[\phi_n(m)+\frac{p}{x}\phi_n'(m)+\frac{p^2}{x^2\,2\,!}\phi_n''(m)+\ldots\right]$$
$$+x^{n-1}\left[\phi_{n-1}(m)+\frac{p}{x}\phi'_{n-1}(m)+\ldots\right]$$
$$+x^{n-2}\left[\phi_{n-2}(m)+\frac{p}{x}\phi'_{n-2}(m)+\ldots\right]+\ldots\ldots=0. \quad \ldots(6)$$

Arranging terms in (6) according to descending powers of x, we get

$$x^n \phi_n(m) + x^{n-1}[p\phi_n'(m) + \phi_{n-1}(m)]$$
$$+ x^{n-2}\left[\frac{p^2}{2!}\phi_n''(m) + \frac{p}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m)\right] + \ldots\ldots = 0. \quad \ldots(7)$$

Putting $\phi_n(m) = 0$ in (7) and then dividing by x^{n-1}, we get

$$[p\,\phi_n'(m) + \phi_{n-1}(m)]$$
$$+ \frac{1}{x}\left[\frac{p^2}{2!}\phi_n''(m) + \frac{p}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m)\right] + \ldots = 0. \quad \ldots(8)$$

Taking limit as $x \to \infty$ and remembering that $\lim_{x \to \infty} p = c$, we get

$$\boldsymbol{c\,\phi_n'(m) + \phi_{n-1}(m) = 0}, \quad \ldots(9)$$

which determines one value of c for each value of m found from (4).

The asymptotes are then given by $y = mx + c$, where m is a root of (4) and the corresponding c is obtained from (9).

Important : The polynomial $\phi_n(m)$ is easily obtained by putting $y = m$ and $x = 1$ in $x^n \phi_n(y/x)$ *i.e.* the n^{th} degree terms in the equation of the curve. Similarly to obtain $\phi_{n-1}(m)$, we should put $y = m$ and $x = 1$ in the $(n-1)^{th}$ degree terms in the equation of the curve. In general, to obtain $\phi_r(m)$, we should put $y = m$ and $x = 1$ in the r^{th} degree terms in the equation of the curve.

Illustrative Examples

Example 1 : *Find the asymptotes of the curve* $\dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 1\cdot$

(Bundelkhand 2006; Agra 07, 08; Kashi 14)

Solution : The equation of the curve can be written as

$$a^2 y^2 - b^2 x^2 = x^2 y^2 \quad \text{or} \quad x^2 y^2 - a^2 y^2 + b^2 x^2 = 0.$$

Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.* of y^2), the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ *i.e.* $x = \pm a$.

Also equating to zero the coefficient of the highest power of x (*i.e.* of x^2), the asymptotes parallel to x-axis are given by $y^2 + b^2 = 0$, which gives two imaginary asymptotes.

Thus all the four possible asymptotes of the curve have been found and the only real asymptotes are $x = \pm a$.

Example 2 : *Find the asymptotes of the curve* $y^2(a^2 - x^2) = x^4$. **(Meerut 2010)**

Solution : The equation of the curve is $y^2 x^2 + x^4 - a^2 y^2 = 0$.

Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ *i.e.* $x = \pm a$.

The coefficient of the highest power x^4 of x is merely a constant. Hence there is no asymptote parallel to x-axis.

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.* four degree terms and we get $\phi_4(m) = m^2 + 1$.

The roots of the equation $\phi_4(m) = 0$ are imaginary and consequently the corresponding asymptotes are imaginary.

Hence the only real asymptotes of the curve are $x = \pm a$.

Comprehensive Exercise 1

Find all the asymptotes of the following curves:

1. $a^2/x^2 + b^2/y^2 = 1$. **(Meerut 2007B; Purvanchal 14)**
2. $y^2(x^2 - a^2) = x$. **(Rohilkhand 2014)**
3. $xy^2 = 4a^2(2a - x)$.
4. $x^2y^2 = a^2(x^2 + y^2)$. **(Bundelkhand 2001, 05, 08)**
5. $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$. **(Meerut 2003, 06; Agra 05; Rohilkhand 05, 06)**
6. $x^2/a^2 - y^2/b^2 = 1$. **(Gorakhpur 2005; Bundelkhand 11)**
7. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$. **(Meerut 2007)**
8. Find the asymptotes parallel to the axes of the curve
$$x^2y^2 - x^2 - y^2 - x - y + 1 = 0.$$ **(Bundelkhand 2001)**
9. Find the asymptotes of the curve
$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0.$$ **(Meerut 2001)**

Answers 1

1. $x = \pm a, y = \pm b$.
2. $x = \pm a, y = 0$.
3. $x = 0$.
4. $x = \pm a, y = \pm a$.
5. $x = \pm a, y = \pm x$.
6. $\dfrac{y}{b} = \pm \dfrac{x}{a}$.
7. $y = 0, y = 1, x = 0, x = 1$.
8. $x = \pm 1, y = \pm 1$.
9. $x = \pm a, y = \pm a$.

13.4 Asymptotes Might Not Exist

If one or more values of m obtained from $\phi_n(m) = 0$ are such that they make $\phi_n'(m) = 0$, but do not make $\phi_{n-1}(m)$ zero, then the equation for determining the corresponding values of c becomes

$$0 \, . \, c + \phi_{n-1}(m) = 0.$$

From this equation we get $c = +\infty$ or $-\infty$ and this corresponds to the case when the tangent goes farther and farther away from the origin as $x \to \infty$. Corresponding to such values of m, we shall get no asymptotes.

13.5 Two Parallel Asymptotes

Suppose the equation (iv) *i.e.* $\phi_n(m) = 0$ of § 3 gives us two equal values of m. This repeated value of m will make $\phi_n'(m) = 0$. In case it does not make $\phi_{n-1}(m)$ equal to zero, the asymptotes corresponding to it will not exist. If it also makes $\phi_{n-1}(m)$ equal to zero, the equation from which c is usually determined reduces to the identity $\quad 0 \cdot c + 0 = 0,$
and we cannot find the value of c in this way. To derermine c in this case, we put

$$\phi'_n(m) = \phi_{n-1}(m) = 0$$

in equation (vii) of § 3 and we get

$$x^{n-2}\left[\frac{p^2}{2\,!}\phi''_n(m) + \frac{p}{1\,!}\phi'_{n-1}(m) + \phi_{n-2}(m)\right] +$$
$$x^{n-3}\left[\frac{p^3}{3\,!}\phi_n'''(m) + \frac{p^2}{2\,!}\phi''_{n-1}(m) + \frac{p}{1\,!}\phi'_{n-2}(m) + \phi_{n-3}(m)\right] + \ldots\ldots = 0.$$

Dividing by x^{n-2}, taking limits as $x \to \infty$ and remembering that $\lim\limits_{x\to\infty} p = c$, we get

$$\frac{c^2}{2\,!}\phi''_n(m) + \frac{c}{1\,!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0.$$

This equation is quadratic in c. It will give us two values of c, say c_1 and c_2 corresponding to that repeated value of m. The two corresponding asymptotes will be $y = mx + c_1$ and $y = mx + c_2$, which are obviously parallel.

13.6 Three Parallel Asymptotes

If the equation $\phi_n(m) = 0$ gives us three equal values of m, then this repeated value of m will make $\phi'_n(m)$ and $\phi''_n(m)$ equal to zero. For the existence of corresponding asymptotes it must make $\phi_{n-1}(m)$ equal to zero. If it also makes $\phi'_{n-1}(m)$ and $\phi_{n-2}(m)$ equal to zero, then the equation to determine c reduces to the identity

$$0 \cdot c^2 + 0 \cdot c + 0 = 0,$$

and we shall not be able to find the value of c in this way.

So putting each of $\phi_n(m), \phi'_n(m), \phi''_n(m)$, $\phi_{n-1}(m)$, $\phi'_{n-1}(m)$ and $\phi_{n-2}(m)$ equal to zero in equation (vii) of § 3 and dividing by x^{n-3} and taking limit as $x \to \infty$, we get

$$\frac{\boldsymbol{c^3}}{\boldsymbol{3\,!}}\phi_n'''(\boldsymbol{m}) + \frac{\boldsymbol{c^2}}{\boldsymbol{2\,!}}\phi''_{n-1}(\boldsymbol{m}) + \frac{\boldsymbol{c}}{\boldsymbol{1\,!}}\phi'_{n-2}(\boldsymbol{m}) + \phi_{n-3}(\boldsymbol{m}) = \mathbf{0}.$$

This equation will give us three values of c corresponding to that repeated value of m and accordingly we shall get three parallel asymptotes.

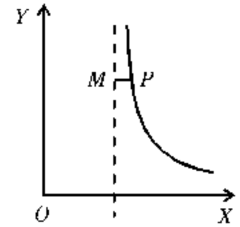
In a similar way we can discuss the case of more than three parallel asymptotes.

13.7 Asymptotes Parallel to the Co-ordinate Axes

(i) Asymptotes parallel to *y*-axis :

Let $x = k$ be an asymptote parallel to y-axis of the curve $f(x, y) = 0$. In this case, y, alone tends to infinity as a point $P(x, y)$ on the curve tends to infinity along the line $x = k$. Also

$$k = \lim_{y \to \infty} x, \text{ where } (x, y) \text{ lies on the curve.}$$



Therefore to find the asymptotes parallel to y-axis, we find the definite value or values k_1, k_2 etc. to which x tends as y tends to infinity.

Then the lines $x = k_1, x = k_2$, etc. are the required asymptotes.

Asymptotes parallel to *y*-axis of a rational algebraic curve. Let the equation of the curve, when arranged in descending powers of y, be

$$y^m \phi(x) + y^{m-1}\phi_1(x) + y^{m-2}\phi_2(x) + \ldots = 0, \qquad \ldots(1)$$

where $\phi(x), \phi_1(x), \phi_2(x)$ etc. are polynomials in x.

Dividing the equation (1) by y^m, we obtain

$$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots = 0. \qquad \ldots(2)$$

If $x = k$ is an asymptote parallel to y-axis of (1), then $k = \lim_{y \to \infty} x$. Therefore taking limit of (2) as $y \to \infty$ and remembering that $x \to k$ as $y \to \infty$, we get $\phi(k) = 0$.

Therefore k is a root of the equation $\phi(x) = 0$.

If k_1, k_2 etc., be the roots of $\phi(x) = 0$, then the asymptotes of (1) parallel to y-axis are $x = k_1, x = k_2$, etc.

From algebra, we know that if k_1 is a root of $\phi(x) = 0$, then $x - k_1$ must be a factor of $\phi(x)$. Also $\phi(x)$ is the coefficient of the highest power of y *i.e.* y^m in the equation of the curve. Hence we have the following simple rule :

The asymptotes parallel to the axis of y are obtained by equating to zero the coefficient of the highest power of y in the equation of the curve. In case the coefficient of the highest power of y, is a constant or if its linear factors are all imaginary, there will be no asymptotes parallel to y-axis.

(ii) Asymptotes parallel to *x*-axis : Proceeding as above, we have the following rule for finding asymptotes parallel to x-axis of a rational algebraic curve :

The asymptotes parallel to the axis of x are obtained by equating to zero the co-efficient of the highest power of x, in the equation of the curve. In case the coefficient of highest power of x, is a constant or if its factors are all imaginary, there will be no asymptotes parallel to x-axis.

13.8 Total Number of Asymptotes of a Curve

The number of asymptotes, real or imaginary, of an algebraic curve of the n^{th} degree cannot exceed n.

The slopes of the asymptotes which are not parallel to y-axis are given as the roots of the equation $\phi_n(m) = 0$ which is of degree n at the most. If the equation of the curve possesses some asymptotes parallel to y-axis, then we can easily see that the degree of $\phi_n(m) = 0$ will be smaller than n by at least the same number. In general one value of m gives only one value of c. In case the equation for determining c is a quadratic, the equation $\phi_n(m) = 0$ has two equal roots. Similarly if the equation for determining c is cubic, the equation $\phi_n(m) = 0$ has three equal roots.

Hence *a curve of degree n cannot have more than n asymptotes.* But the number of real asymptotes can be less than n. Some roots of the equation $\phi_n(m) = 0$ may come out to be imaginary or even corresponding to a real value of m the value of c may come out to be infinite.

13.9 Working Rule for Finding the Asymptotes of Rational Algebraic Curves

(i) A curve of degree n cannot have more than n asymptotes real or imaginary.

(ii) Equating to zero the coefficient of the highest power of y in the equation of the curve, we get asymptotes parallel to y-axis. Similarly equating to zero the coefficient of the highest power of x in the equation of the curve, we get asymptotes parallel to x-axis.

If $y = mx + c$ is an asymptote not parallel to y-axis, then the values of m and c are found as follows :

(iii) Putting $y = m$ and $x = 1$ in the highest *i.e.* nth degree terms in the equation of the curve, we get $\phi_n(m)$. Solving the equation $\phi_n(m) = 0$, we get the slopes of the asymptotes. If some values of m are imaginary, we reject them.

(iv) Corresponding to a value of m, the value of c is given by the equation

$$c\,\phi'_n(m) + \phi_{n-1}(m) = 0,$$

where $\phi_{n-1}(m)$ is obtained by putting $y = m$ and $x = 1$ in the $(n-1)^{th}$ degree terms in the equation of the curve. The asymptotes corresponding to $m = 0$ are already found in (ii). So we need not find the value of c corresponding to $m = 0$.

(v) If corresponding to two equal values of m, the equation for determining c, given in (iv) reduces to the identity $0 \cdot c + 0 = 0$, then the values of c are given by

$$\frac{c^2}{2!}\phi''_n(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0.$$

(vi) Similarly, if three values of m are equal and the equation for determining c, given in (v), reduces to the identity $0 \cdot c^2 + 0 \cdot c + 0 = 0$, then the corresponding values of c are given by

$$\frac{c^3}{3!}\phi'''_n(m) + \frac{c^2}{2!}\phi''_{n-1}(m) + \frac{c}{1!}\phi'_{n-2}(m) + \phi_{n-3}(m) = 0.$$

Illustrative Examples

Example 1 : *Find the asymptotes of the curve $y^2 = 4x$.*

(Meerut 2010B; Kumaun 08)

Solution : The equation of the curve is $y^2 - 4x = 0$.

Putting $y = m$ and $x = 1$ in the highest *i.e.* 2nd degree terms, we get $\phi_2(m) = m^2$.

Solving the equation $\phi_2(m) = 0$ *i.e.* $m^2 = 0$, we get $m = 0, 0$.

Also putting $y = m$ and $x = 1$ in the first degree terms, we get $\phi_1(m) = -4$.

Now c is given by the equation $c\phi'_2(m) + \phi_1(m) = 0$ *i.e.* $2mc - 4 = 0$.

If we put $m = 0$ in this equation, we get $c = \infty$. Hence no asymptote exists.

Example 2 : *Find the asymptotes of the curve* $x^3 + y^3 - 3axy = 0$. **(Agra 2005; Bundelkhand 09; Meerut 12; Kashi 12; Avadh 13; Rohilkhand 14)**

Solution : Obviously there are no asymptotes parallel to the co-ordinate axes.

Putting $y = m$ and $x = 1$ in the highest *i.e.* third degree terms in the equation of the curve, we get $\phi_3(m) = 1 + m^3$.

Solving the equation $\phi_3(m) = 0$,

i.e. $\quad (1 + m^3) = 0, \quad$ *i.e.* $\quad (m + 1)(m^2 - m + 1) = 0$,

we get $\quad m = -1$ as the only real root.

The other two roots are imaginary.

Again putting $y = m$ and $x = 1$ in the second degree terms in the equation of the curve, we get $\phi_2(m) = -3am$.

Now c is given by $c\,\phi'_3(m) + \phi_2(m) = 0$, *i.e.* $c(3m^2) - 3am = 0$.

Putting $m = -1$, we get $c = -a$.

Hence the only real asymptote of the curve is

$$y = -x - a \quad \text{or} \quad y + x + a = 0.$$

Example 3 : *Find all the asymptotes of the curve*

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Solution : Obviously there are no asymptotes parallel to the coordinate axes.

Putting $y = m$ and $x = 1$ in the highest *i.e.* third degree terms in the equation of the curve, we get

$$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3.$$

The slopes of the asymptotes are given by

$$\phi_3(m) = 2m^3 - 7m^2 + 2m + 3 = 0, \text{ or } (m - 1)(2m + 1)(m - 3) = 0.$$

$$\therefore \quad m = 1, 3, -\tfrac{1}{2}.$$

Again putting $y = m$ and $x = 1$ in the next highest *i.e.* second degree terms in the equation of the curve, we get $\phi_2(m) = -14m + 7m^2$.

Now c is given by $c\,\phi_3'(m) + \phi_2(m) = 0$,

i.e., $\quad c(6m^2 - 14m + 2) + (7m^2 - 14m) = 0$.

When $m = 1, c = -7/6$; when $m = 3, c = -3/2$ and when $m = -\frac{1}{2}, c = -5/6$.

$\therefore$ The required asymptotes are $y = x - 7/6$; $y = 3x - 3/2$ and $y = -\frac{1}{2}x - 5/6$

i.e., $\quad 6y - 6x + 7 = 0$; $2y - 6x + 3 = 0$ and $2y + x + 5/3 = 0$.

Example 4 : *Find all the asymptotes of the curve*

$$y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0.$$

Solution : Putting $y = m$ and $x = 1$ in the highest *i.e.* third degree terms of the equation of the curve, we get $\quad \phi_3(m) = m^3 - m^2 - m + 1$.

The slopes of the asymptotes are given by

$$\phi_3(m) = m^3 - m^2 - m + 1 = 0 \quad \text{or} \quad (m-1)^2(m+1) = 0.$$

$\therefore$ $m = 1, 1, -1.$

Now putting $y = m$ and $x = 1$ in the next highest *i.e.* second degree terms, we get

$$\phi_2(m) = 1 - m^2.$$

To determine c, we have $c\,\phi'_3(m) + \phi_2(m) = 0$,

i.e. $$c(3m^2 - 2m - 1) + (1 - m^2) = 0. \qquad \ldots(1)$$

When $m = -1$, we have $c = 0$ and the corresponding asymptote is $y = -x + 0$

i.e. $y + x = 0.$

When $m = 1$, the equation (1) reduces to the identity $c \cdot 0 + 0 = 0$ and we cannot determine c from it. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi''_3(m) + \frac{c}{1!}\phi'_2(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms in the equation of the curve, we get $\phi_1(m) = 0$, since there are no first degree terms.

Hence for $m = 1$, c is to be given by,

$$\frac{c^2}{2}(6m - 2) + c(-2m) = 0 \quad \text{i.e.} \quad (3m - 1)c^2 - 2mc = 0.$$

For $m = 1$, this becomes $2c^2 - 2c = 0$ *i.e.* $c = 0$ and 1.

Hence $y = x + 1$ and $y = x + 0$ are two parallel asymptotes corresponding to the slope $m = 1$.

$\therefore$ The required asymptotes are $y + x = 0, y - x = 0, y - x - 1 = 0.$

Example 5 : *Find all the asymptotes of the curve*

$$(x + y)^2(x + 2y + 2) = x + 9y + 2. \qquad \textbf{(Meerut 2011)}$$

Solution : The equation of the curve can be written as

$$(x + y)^2(x + 2y) + 2(x + y)^2 - (x + 9y) - 2 = 0.$$

Here $\phi_3(m) = (1 + m)^2(1 + 2m).$

The slopes of the asymptotes are given by

$$\phi_3(m) = (1 + m)^2(1 + 2m) = 0.$$

$\therefore$ $m = -1, -1, -\frac{1}{2}.$

Also $\phi_2(m) = 2(1 + m)^2.$

To determine c, we have $c\,\phi'_3(m) + \phi_2(m) = 0$,

i.e. $$c\{2(1 + m)(1 + 2m) + 2(1 + m)^2\} + 2(1 + m)^2 = 0. \qquad \ldots(1)$$

When $m = -\frac{1}{2}$, we have $c = -1$ and the corresponding asymptote is

$$y = -\frac{1}{2}x - 1 \quad \text{i.e.} \quad 2y + x + 2 = 0.$$

When $m = -1$, the equation (i) reduces to the identity $c \cdot 0 + 0 = 0$ and we cannot determine c from it. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi''_3(m) + \frac{c}{1!}\phi'_2(m) + \phi_1(m) = 0.$$

Now $\phi_1(m) = -(1 + 9m)$.

Hence for $m = -1$, c is given by

$$\frac{c^2}{2}\{2(1 + 2m) + 4(1 + m) + 4(1 + m)\}$$
$$+ c\{4(1 + m)\} - (1 + 9m) = 0$$

i.e. $$(6m + 5)c^2 + 4(1 + m)c - (1 + 9m) = 0.$$

For $m = -1$, this becomes $-c^2 + 8 = 0$, *i.e.* $c = \pm 2\sqrt{2}$.

Hence $y = -x + 2\sqrt{2}$ and $y = -x - 2\sqrt{2}$ are two parallel asymptotes corresponding to the slope $m = -1$.

$\therefore$ The required asymptotes are $2y + x + 2 = 0$ and $x + y \pm 2\sqrt{2} = 0$.

Example 6 : *Find the asymptotes of the curve* $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$.

(Meerut 2002; Kashi 12)

Solution : The given curve is of degree 3. So it cannot have more than three asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get $x = 0$ as an asymptote parallel to y-axis. Also there is no asymptote parallel to x-axis because the coefficient of x^2 is merely a constant.

Now we proceed to find the remaining oblique asymptotes.

Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = 1 + 2m + m^2, \text{ and } \phi_2(m) = -1 - m.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \text{ i.e., } 1 + 2m + m^2 = 0 \text{ i.e., } (1 + m)^2 = 0.$$

$\therefore$ $$m = -1, -1.$$

To determine c, we have the equation

$$c\phi_3'(m) + \phi_2(m) = 0 \quad \text{i.e.,} \quad c(2 + 2m) - 1 - m = 0. \qquad \ldots(1)$$

For $m = -1$, the equation (1) reduces to the identity $c \,.\, 0 + 0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Now $\phi_3''(m) = 2$, $\phi_2'(m) = -1$, and $\phi_1(m) = 0$ because there are no first degree terms in the equation of the curve. So for $m = -1$, c is to be given by

$$\frac{1}{2}c^2 . (2) + c . (-1) + 0 = 0 \quad \text{i.e.,} \quad c^2 - c = 0 \quad \text{i.e.,} \quad c(c - 1) = 0.$$

$\therefore$ $$c = 0, 1.$$

Hence $y = -x + 0$ and $y = -x + 1$ are two parallel asymptotes corresponding to the slope $m = -1$.

$\therefore$ the required asymptotes are $x = 0$, $x + y = 0$ and $x + y - 1 = 0$.

Comprehensive Exercise 2

Find all the asymptotes of the following curves :

1. $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.

2. $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$.

3. $x^3 + 2x^2 y - xy^2 - 2y^3 + xy - y^2 - 1 = 0.$ **(Bundelkhand 2006)**
4. $x^2 y + xy^2 + xy + y^2 + 3x = 0.$
5. $y^3 - 5xy^2 + 8x^2 y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0.$
6. $2x(y - 3)^2 = 3y(x - 1)^2.$
7. $y^2 (x - 2a) = x^3 - a^3.$ **(Bundelkhand 2008)**
8. $y^3 - 2y^2 x - yx^2 + 2x^3 + y^2 - 6xy + 5x^2 - 2y + 2x + 1 = 0.$
9. $(x^2 - y^2)^2 - 4y^2 + y = 0.$ **(Kanpur 2007)**
10. $x^3 + x^2 y - xy^2 - y^3 - 3x - y - 1 = 0.$ **(Meerut 2001, 04, 06B; Gorakhpur 06)**
11. $x^2 y^3 + x^3 y^2 = x^3 + y^3.$ **(Rohilkhand 2007)**
12. $x^3 - 2x^2 y + xy^2 + x^2 - xy + 2 = 0.$ **(Kanpur 2014)**
13. $x^3 - 5x^2 y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0.$
14. $(2x - 3y + 1)^2 (x + y) = 8x - 2y + 9.$

Answers 2

1. $y = x + 1, y = -x + 1$ and $x + 2y = 0.$
2. $y = -x + 2, y = x + 2, y = 2x - 4.$
3. $2y + x = 1, y = x, y + x + 1 = 0.$
4. $x = -1, y = 0, x + y = 0.$
5. $y = x, y = 2x + 2, y = 2x + 1.$
6. $x = 0, y = 0, 2y = 3x + 6.$
7. $x = 2a, y = x + a, y = -x - a.$
8. $y = x, y = 2x + 1, y = -x - 2.$
9. $x + y = \pm 1$ and $x - y = \pm 1.$
10. $y = x, y = -x + 1, y = -x - 1.$
11. $y = \pm 1, x = \pm 1, y = -x.$
12. $x = 0, y = x, y = x + 1.$
13. $y - x = 0, 2y - x = 0, 2y - x - 1 = 0.$
14. $y + x = 0, 3y - 2x - 3 = 0, 3y - 2x + 1 = 0.$

13.10 Asymptotes by Expansion **(Kumaun 2008)**

To show that $y = mx + c$ is an asymptote of the curve

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots$$

Let the equation of the curve be $y = mx + c + \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dots$ …(1)

where the series $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dots\dots$ is convergent for sufficiently large values of x. Differentiating (1), we have

$$\frac{dy}{dx} = m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots$$

$\therefore$ The equation of the tangent to (1) at (x, y) is

$$Y - y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\right)(X - x)$$

or
$$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\right)X + y - \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\right)x$$

or
$$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \ldots\right)X + c + \frac{2A}{x} + \frac{3B}{x^2} + \ldots, \qquad \ldots(2)$$

substituting the value of y from (1). Suppose now $x \to \infty$. The equation (2) then tends to the equation $Y = mX + c$.

Hence $\boldsymbol{y = mx + c}$ is the asymptote of the curve

$$\boldsymbol{y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \ldots}$$

Example : *Find the asymptotes of the hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

Solution : The equation of the hyperbola can be written as

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1, \quad \text{or} \quad y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

or
$$y^2 = \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right), \quad \text{or} \quad y = \pm\frac{b}{a}x\left(1 - \frac{a^2}{x^2}\right)^{1/2}$$

or
$$y = \pm\frac{b}{a}x\left\{1 - \frac{1}{2}\frac{a^2}{x^2} - \frac{1}{8}\frac{a^4}{x^4} + \ldots\right\}.$$

Hence by article 13.10, the asymptotes of the curve are $y = \pm\dfrac{b}{a}x$.

13.11 Alternative Methods of Finding Asymptotes of Algebraic Curves

Theorem : *The asymptotes of an algebraic curve are parallel to the lines obtained by equating to zero the linear factors of the highest degree terms in its equation.*

Let the equation of the curve be of degree n. Let $y - m_1 x$ be a factor of the nth degree terms in the equation of the curve. Then obviously $(m - m_1)$ is a factor of $\phi_n(m)$. Therefore m_1 is a root of the equation $\phi_n(m) = 0$ and there is an asymptote parallel to the line $y - m_1 x = 0$.

Conversely let m_1 be a root of the equation $\phi_n(m) = 0$, so that there is an asymptote parallel to the line $y - m_1 x = 0$. In this case $m - m_1$ must be a factor of $\phi_n(m)$. Therefore $(y/x - m_1)$ must be a factor of $\phi_n(y/x)$. Hence $(y - m_1 x)$ must be a factor of $x^n \phi_n(y/x)$ *i.e.* the highest degree terms in the equation of the curve.

If the highest degree terms contain, x, as a factor, then after a little consideration it will be obvious that the curve will possess asymptotes parallel to $x = 0$ *i.e.* y-axis.

Hence the theorem.

We know that if $y = mx + c$ is an oblique asymptote of the curve $f(x, y) = 0$, then $m = \lim_{x \to \infty} \frac{y}{x}$ and $c = \lim_{x \to \infty, y/x \to m} (y - mx)$. These facts together with the above theorem enable us to find the asymptotes of algebraic curves very easily. The first step for this purpose is that we should collect the highest degree terms in the equation of the curve and resolve them into real linear factors. Then the following different cases may arise :

Case I : Let $y - m_1 x$ be a non-repeated factor of the highest *i.e.* n^{th} degree terms in the equation of the curve. Then the equation to the curve can be written as

$$(y - m_1 x) F_{n-1} + P_{n-1} = 0, \qquad \text{...(1)}$$

where F_{n-1} contains only terms of degree $n - 1$, and P_{n-1} contains terms of various degrees, none of which is of a degree higher than $n - 1$.

Obviously $y - m_1 x = c$, where c is to be determined, is an asymptote of the curve.

Now $c = \lim_{x \to \infty, y/x \to m_1} (y - m_1 x)$ where (x, y) lies on (1).

But when (x, y) lies on (1), $y - m_1 x = - \frac{P_{n-1}}{F_{n-1}}$.

$$\therefore \qquad c = \lim_{x \to \infty, y/x \to m_1} \left(\frac{- P_{n-1}}{F_{n-1}} \right).$$

Hence $y - m_1 x = \lim_{x \to \infty, y/x \to m_1} \left(\frac{- P_{n-1}}{F_{n-1}} \right)$ is an asymptote of the curve.

Thus dividing (1) by F_{n-1} and taking limit as $x \to \infty, y/x \to m_1$ we shall get an asymptote of (1). Similarly we can find asymptotes corresponding to other non-repeated linear factors.

Example : *Find all the asymptotes of the curve*

$$(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0.$$

Solution : The equation of the curve can be written as

$$(x^2 - y^2)(x + 2y) + (x^2 - y^2) + x + y + 1 = 0$$

or
$$(x - y)(x + y)(x + 2y) = (y^2 - x^2) - x - y - 1.$$

The slope of the asymptote corresponding to the factor $x - y$ is 1. Hence the asymptote corresponding to this factor is

$$x - y = \lim_{x \to \infty, y/x \to 1} \frac{(y^2 - x^2) - x - y - 1}{(x + y)(x + 2y)}$$

$$= \lim_{x \to \infty, y/x \to 1} \frac{\left(\frac{y^2}{x^2} - 1 \right) - \frac{1}{x} - \frac{y}{x} \cdot \frac{1}{x} - \frac{1}{x^2}}{\left(1 + \frac{y}{x}\right)\left(1 + \frac{2y}{x}\right)},$$

on dividing the numerator and denominator by x^2

$$= \frac{(1 - 1)}{(1 + 1)(1 + 2)} = \frac{0}{6} = 0,$$

i.e., $x - y = 0$ is one asymptote of the curve.

Second asymptote corresponding to the factor $x + y$ is

$$x + y = \lim_{x\to\infty,\, y/x\to -1} \frac{(y^2 - x^2) - x - y - 1}{(x - y)(x + 2y)}$$

$$= \lim_{x\to\infty,\, y/x\to -1} \frac{\left[\dfrac{y^2}{x^2} - 1\right] - \dfrac{1}{x} - \dfrac{y}{x}\cdot\dfrac{1}{x} - \dfrac{1}{x^2}}{\left[1 - \dfrac{y}{x}\right]\left[1 + 2\dfrac{y}{x}\right]}$$

$$= \frac{(1 - 1)}{(1 + 1)(1 - 2)} = 0,$$

i.e. $x + y = 0$ is another asymptote of the curve.

The third asymptote of the curve is

$$x + 2y = \lim_{x\to\infty,\, y/x\to -\frac{1}{2}} \frac{(y^2 - x^2) + \text{terms of degree lower than 2}}{(x - y)(x + y)}$$

$$= \lim_{x\to\infty,\, y/x\to -\frac{1}{2}} \frac{(y^2/x^2 - 1) + \text{terms which} \to 0}{(1 - y/x)(1 + y/x)}$$

$$= \frac{\left(\frac{1}{4} - 1\right)}{\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)} = \frac{-\frac{3}{4}}{\frac{3}{4}} = -1.$$

Therefore $x + 2y + 1 = 0$ is the third asymptote of the curve.

Important : It should be noted that while taking limit we should reject all the terms in the numerator whose degree is lower than the degree of denominator. All such terms will tend to zero as $x\to\infty$.

Case II : If $(y - m_1 x)^2$ is a factor of the n^{th} degree terms but $(y - m_1 x)$ is not a factor of the $(n-1)^{th}$ degree terms, then $\phi_n'(m_1) = 0$ and $\phi_{n-1}(m_1) \neq 0$. Therefore as in § 4, the asymptotes corresponding to the factor $(y - m_1 x)^2$ will not exist. Therefore *there will be no asymptotes with slope m_1 if $(y - m_1 x)^2$ is a factor of the n^{th} degree terms and $y - m_1 x$ is not a factor of the $(n-1)^{th}$ degree terms.* In case the equation of the curve does not contain terms of degree $n - 1$, we can add them with zero coefficient and obviously $y - m_1 x$ can be taken as a factor of the $(n-1)^{th}$ degree terms.

Case III : Let the equation of the curve be of the form

$$(y - m_1 x)^2 F_{n-2} + (y - m_1 x)\, G_{n-2} + P_{n-2} = 0, \qquad \ldots(1)$$

where F_{n-2} and G_{n-2} contain only terms of degree $n - 2$, and P_{n-2} contains terms of various degrees, none of which is of a degree higher than $n - 2$.

Dividing (1) by F_{n-2} and taking limit as $x\to\infty$ and $y/x \to m_1$, we get an equation of the form $c^2 + Ac + B = 0$, giving the values of c corresponding to the slope m_1. If c_1 and c_2 are the roots of this equation, then $y - m_1 x = c_1$ and $y - m_1 x = c_2$ will be the corresponding asymptotes. After a little consideration it will be obvious that the asymptotes corresponding to the factor $(y - m_1 x)^2$ will be obtained by solving the quadratic

$$(y - m_1 x)^2 + A\,(y - m_1 x) + B = 0.$$

In a similar way we can discuss the case if the n^{th} degree *terms contain* $(y - m_1 x)^3$ or a higher power of $y - m_1 x$ as a factor.

Example : *Find the asymptotes of the curve*

$$x^2 (x^2 - y^2)(x - y) + 2x^3 (x - y) - 4y^3 = 0.$$

Solution : The equation of the curve can be written as

$$x^2 (x - y)^2 (x + y) + 2x^3 (x - y) - 4y^3 = 0.$$

The asymptotes corresponding to the factor $(x - y)^2$ are given by

$$(x - y)^2 + (x - y) \lim_{x \to \infty, y/x \to 1} \frac{2x^3}{x^2 (x + y)} - 4 \lim_{x \to \infty, y/x \to 1} \frac{y^3}{x^2 (x + y)} = 0$$

or
$$(x - y)^2 + (x - y) \lim_{x \to \infty, y/x \to 1} \frac{2}{(1 + y/x)} - 4 \lim_{x \to \infty, y/x \to 1} \frac{(y/x)^3}{(1 + y/x)} = 0$$

or
$$(x - y)^2 + 1 . (x - y) - \frac{1}{2} \cdot 4 = 0 \quad \text{or} \quad (x - y)^2 + (x - y) - 2 = 0$$

i.e.
$$(x - y) = \frac{-1 \pm \sqrt{(1 + 8)}}{2} \quad \textit{i.e.} \quad x - y = -2 \text{ and } x - y = 1.$$

The asymptote corresponding to the factor $x + y$ is

$$(x + y) = \lim_{x \to \infty, y/x \to -1} \frac{4y^3 - 2x^3 (x - y)}{x^2 (x - y)^2}$$

$$= \lim_{x \to \infty, y/x \to -1} \frac{4\left(\frac{y}{x}\right)^3 . \frac{1}{x} - 2\left(1 - \frac{y}{x}\right)}{\left(1 - \frac{y}{x}\right)^2}$$

$$= -4/4 = -1 \quad \textit{i.e.,} \quad x + y + 1 = 0.$$

Moreover the curve has two asymptotes parallel to y-axis and they can be obtained by equating to zero the coefficient of highest power of y *i.e.* y^3 in the equation of the curve.

So they are $x^2 - 4 = 0$ *i.e.* $x = \pm 2$.

Case IV : Let the equation of the curve be of the form

$$(ax + by + c) P_{n-1} + Q_{n-1} = 0, \qquad \text{...(1)}$$

where P_{n-1} and Q_{n-1} contain terms none of which is of a higher degree than $n - 1$, and P_{n-1} contains at least one term of degree $n - 1$ so as to ensure that the equation (1) is of degree n. Obviously $ax + by$ will be a factor of the n^{th} degree terms in the equation of the curve. Now (1) can be written as

$$(ax + by) P_{n-1} + cP_{n-1} + Q_{n-1} = 0.$$

The asymptote of (1) corresponding to the factor $ax + by$ (if it occurs as a non-repeated factor of the highest degree terms) is

$$(ax+by)+\lim_{x\to\infty,\, y/x\to -a/b}\left\{c+\frac{Q_{n-1}}{P_{n-1}}\right\}=0$$

or
$$(ax+by+c)+\lim_{x\to\infty,\, y/x\to -a/b}\left(\frac{Q_{n-1}}{P_{n-1}}\right)=0\cdot$$

Thus if the equation of a curve is given in the form (1), then there is no necessity of collecting separately the n^{th} degree terms. A similar modification can be made in case III.

Case V : Asymptotes by Inspection : *If the equation of a curve of the n^{th} degree can be put in the form $F_n + P = 0$, where F_n is of degree n (i.e., contains terms of degree n and may also contain terms of lower degrees), and P is of degree $n-2$, or lower, and if $F_n = 0$ can be broken up into n linear factors which represent n straight lines no two of which are parallel or coincident then all the asymptotes of the curve are given by equating to zero the linear factors of F_n.*

Let $ax+by+c=0$ be a non-repeated factor of F_n. Then the equation of the curve can be written as $(ax+by+c)\,F_{n-1}+P=0$, where F_{n-1} is of degree $n-1$.

The asymptote of the given curve parallel to the line $ax+by+c=0$ is

$$ax+by+c+\lim_{x\to\infty,\, y/x\to -a/b}\left(\frac{P}{F_{n-1}}\right)=0\cdot \qquad \ldots(1)$$

Since F_{n-1} contains at least one term of degree $n-1$ and P is of degree $n-2$, or lower, therefore we shall have

$$\lim_{x\to\infty,\, y/x\to -a/b}\left(\frac{P}{F_{n-1}}\right)=0\cdot$$

Thus $ax+by+c=0$ is an asymptote of the given curve.

Illustrative Examples

Example 1 : *Find all the asymptotes of the curve*

$$(x-y-1)^2(x^2+y^2+2)+6(x-y-1)(xy+7)-8x^2-2x-1=0.$$

Solution : The asymptotes parallel to the line $x-y-1=0$ are

$$(x-y-1)^2+6(x-y-1)\lim_{x\to\infty,\, y/x\to 1}\frac{xy+7}{x^2+y^2+2}$$
$$+\lim_{x\to\infty,\, y/x\to 1}\frac{-8x^2-2x-1}{x^2+y^2+2}=0,$$

or
$$(x-y-1)^2+6(x-y-1)\lim_{x\to\infty,\, y/x\to 1}\frac{\dfrac{y}{x}+\dfrac{7}{x^2}}{1+\left(\dfrac{y}{x}\right)^2+\dfrac{2}{x^2}}$$
$$+\lim_{x\to\infty,\, y/x\to 1}\frac{-8-\dfrac{2}{x}-\dfrac{1}{x^2}}{1+\dfrac{y^2}{x^2}+\dfrac{2}{x^2}}=0$$

or $$(x-y-1)^2+3(x-y-1)-4=0,$$

or $$(x-y-1)=\frac{-3\pm\sqrt{(9+16)}}{2}=1,-4.$$

Thus $x-y-2=0$ and $x-y+3=0$ are two parallel asymptotes of the given curve. Since the remaining linear factors of the fourth degree terms in the equation of the curve are imaginary, therefore the other two asymptotes are imaginary.

Example 2 : *Find all the asymptotes of the curve*

$$(x^2-y^2)(x+2y+1)+x+y+1=0.$$

Solution : The equation of the curve can be written as

$$(x-y)(x+y)(x+2y+1)+x+y+1=0.$$

Since no two of the straight lines $x-y=0, x+y=0$ and $x+2y+1=0$ are parallel and $x+y+1$ is of degree 1, therefore all the asymptotes of the curve are given by $$(x-y)(x+y)(x+2y+1)=0.$$

Comprehensive Exercise 3

Find all the asymptotes of the following curves :

1. $(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0.$
2. $x(y-x)^2-3y(y-x)+2x=0.$
3. $(y-x)(y-2x)^2+(y+3x)(y-2x)+2x+2y-1=0.$ **(Kanpur 2010; Meerut 12B; Bundelkhand 14; Purvanchal 14)**
4. $(x-2y)^2(x-y)-4y(x-2y)-(8x+7y)=0.$ **(Meerut 2005B; Bundelkhand 07)**
5. $(y-a)^2(x^2-a^2)=x^4+a^4.$
6. $x(y-3)^3=4y(x-1)^3.$
7. $(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0.$
8. $x^2(x+y)(x-y)^2+ax^3(x-y)-a^2y^3=0.$
9. $(x-y+1)(x-y-2)(x+y)=8x-1.$
10. $xy(x^2-y^2)(x^2-4y^2)+xy(x^2-y^2)+x^2+y^2-7=0.$

Answers 3

1. $y=x, y=2x, y+x+1=0, y+2x+1=0.$
2. $x=3, y-x=1, y-x=2.$
3. $y=2x-2, y=2x-3, y-x=4.$
4. $y=x+4, x-2y=2\pm3\sqrt{3}.$
5. $x=\pm a, y=x+a, y=-x+a.$
6. $x=0, y=0, y=2x+\frac{3}{2}, 2y+4x=15.$
7. $x-y-2=0, x-y-3=0.$
8. $x=\pm a, y=x+a, y=\pm x-\frac{1}{2}a.$
9. $y+x=0, y=x-3$ and $y=x+2.$
10. $x=0, y=0, x-y=0, x+y=0, x-2y=0$ and $x+2y=0.$

13.12 Intersection of a Curve and its Asymptotes

Let $y = mx + c$ ...(1)

be an asymptote of the curve

$$x^n \phi_n(y/x) + x^{n-1} \phi_{n-1}(y/x) + x^{n-2} \phi_{n-2}(y/x) + \ldots = 0, \quad \ldots(2)$$

which is of degree n.

To find the points of intersection of (1) and (2), we should solve the two equations simultaneously. So eliminating y between (1) and (2), we get

$$x^n \phi_n(m + c/x) + x^{n-1} \phi_{n-1}(m + c/x) + x^{n-2} \phi_{n-2}(m + c/x) + \ldots = 0.$$

Expanding each term by Taylor's theorem and arranging the terms in descending powers of x, we get

$$x^n \phi_n(m) + [c\phi'_n(m) + \phi_{n-1}(m)] x^{n-1}$$
$$+ \left[\frac{c^2}{2!} \phi''_n(m) + \frac{c}{1!} \phi'_{n-1}(m) + \phi_{n-2}(m)\right] x^{n-2} + \ldots = 0. \quad \ldots(3)$$

Since $y = mx + c$ is an asymptote of (2), therefore the coefficients of x^n and x^{n-1} are both zero in (3).

Hence (3) reduces to

$$\left\{\frac{c^2}{2!} \phi''_n(m) + \frac{c}{1!} \phi'_{n-1}(m) + \phi_{n-2}(m)\right\} x^{n-2} + \ldots = 0, \quad \ldots(4)$$

in which the coefficient of x^{n-2} will be non-zero provided there is no other asymptote of the given curve parallel to $y = mx + c$.

Now (4) gives us the abscissae of the points of intersection of (1) and (2). Since equation (4) is of degree $n - 2$ in x, therefore it will give $n - 2$ values of x.

Hence, *in general, any asymptote of a curve of the n^{th} degree cuts the curve in* $(n - 2)$ *points.*

Corollary 1 : *The n asymptotes of a curve of the n^{th} degree cut it in $n(n - 2)$ points.*

Corollary 2 : *If the equation of a curve of degree n can be put in the form $F_n + P = 0$, where P is of degree $n - 2$ at the most and F_n consists of n non-repeated linear factors, then the $n(n - 2)$ points of intersection of the curve with its asymptotes lie on the curve $P = 0$.*

The asymptotes of the curve $F_n + P = 0$, are given by the equation $F_n = 0$.

We know that if $S = 0$ and $S' = 0$ represent two curves, then $S - \lambda S' = 0$ represents some curve through the points of intersection of $S = 0$ and $S' = 0$.

If we take $\lambda = 1$, then we see that $(F_n + P) - F_n = 0$ *i.e.* $P = 0$ is a curve passing through the points of intersection of $F_n + P = 0$ and $F_n = 0$.

Thus, *a curve of degree $n - 2$, or less, can be made to pass through the $n(n - 2)$ points of intersection of a curve of degree n with its n asymptotes.*

Particular Cases :

(i) If the given curve is of degree 3, then the $3(3 - 2)$ *i.e.*, 3 points of intersection of the curve and its asymptotes lie on a curve of degree $3 - 2 = 1$ *i.e.*, on a straight line.

(ii) If the curve is of degree 4 then the $4(4 - 2)$ *i.e.*, 8 points of intersection of the curve and its asymptotes lie on a curve of degree $4 - 2 = 2$ *i.e.* on a conic.

Illustrative Examples

Example 1 : *Show that the asymptotes of the curve*

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

cut the curve in eight points which lie on the ellipse $x^2+4y^2=4$. **(Purvanchal 2007)**

Solution : The equation of the curve can be written as

$$(4x^4+4y^4-17x^2y^2)-4(4y^2x-x^3)+2x^2-4=0. \qquad \ldots(1)$$

Here $\phi_4(m)=4+4m^4-17m^2$.

The slopes of the asymptotes are given by the equation

$$\phi_4(m)=4m^4-17m^2+4=0 \qquad i.e., \quad (4m^2-1)(m^2-4)=0.$$

Therefore $m=\pm\frac{1}{2}, \pm 2$.

Also $\phi_3(m)=-4(4m^2-1)$ and $\phi'_4(m)=16m^3-34m$.

Now c is given by $c\phi'_4(m)+\phi_3(m)=0$

i.e., $c(16m^3-34m)-4(4m^2-1)=0$.

When $m=\frac{1}{2}$, $c=0$; when $m=-\frac{1}{2}$, $c=0$; when $m=2$, $c=1$; and when $m=-2$, $c=-1$.

Therefore the asymptotes are

$$y=\frac{1}{2}x, \; y=-\frac{1}{2}x, \; y=2x+1 \text{ and } y=-2x-1$$

i.e., $2y-x=0$, $2y+x=0$, $y-2x-1=0$ and $y+2x+1=0$.

The combined equation of the asymptotes is

$$(2y-x)(2y+x)(y-2x-1)(y+2x+1)=0$$

or $(4y^2-x^2)\{(y^2-4x^2)-4x-1\}=0$

or $(4y^2-x^2)(y^2-4x^2)-4x(4y^2-x^2)-4y^2+x^2=0$

or $4y^4-17x^2y^2+4x^4-4(4y^2x-x^3)-4y^2+x^2=0. \qquad \ldots(2)$

Now each asymptote of (1) will cut it in $4-2$ *i.e.* 2 points. Therefore the four asymptotes will cut it in 4×2 *i.e.* 8 points.

Subtracting (2) from (1), we get $2x^2-4+4y^2-x^2=0$ *i.e.* $x^2+4y^2=4$, which is the equation of an ellipse. Hence the eight points of intersection of (1) and (2) lie on the ellipse $x^2+4y^2=4$.

Example 2 : *Find the equation of the cubic which has the same asymptotes as the curve*

$$x^3-6x^2y+11xy^2-6y^3+x+y+1=0$$

and which passes through the points $(0,0)$, $(1,0)$ *and* $(0,1)$.

Solution : The equation of the given curve can be written as

$$(x-y)(x-2y)(x-3y)+x+y+1=0. \qquad \ldots(1)$$

By inspection, we find that $x-y=0$, $x-2y=0$, and $x-3y=0$ are the asymptotes of (1).

The combined equation of the asymptotes of (1) is

$$F_3 \equiv (x-y)(x-2y)(x-3y)=0.$$

Since the points of intersection of a cubic curve with its asymptotes lie on a straight line, therefore the most general equation of the curve having $F_3 = 0$ as its asymptotes is

$$(x - y)(x - 2y)(x - 3y) + ax + by + c = 0$$

or $$x^3 - 6x^2 y + 11xy^2 - 6y^3 + ax + by + c = 0. \qquad \ldots(2)$$

If (2) passes through the points (0, 0), (1, 0) and (0, 1), then

$$c = 0,\ 1 + a = 0 \ \textit{i.e.},\ a = -1 \text{ and } -6 + b = 0 \ \textit{i.e.}\ b = 6.$$

Hence the required curve is $x^3 - 6x^2 y + 11xy^2 - 6y^3 - x + 6y = 0.$

Comprehensive Exercise 4

1. Find the asymptotes of the curve $x^2 y - xy^2 + xy + y^2 + x - y = 0$ and show that they cut the curve again in three points which lie on the straight line $x + y = 0$.

2. Show that the asymptotes of the cubic $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$ cut the curve in three points which lie on the straight line $x - y + 1 = 0$.

(Kanpur 2009; Avadh 10)

3. Find the equation of the straight line on which lie the three points of intersection of the curve $(x + a) y^2 = (y + b) x^2$ and its asymptotes.

4. Show that the eight points of intersection of the curve

$$xy(x^2 - y^2) + x^2 + y^2 = a^2$$

and its asymptotes lie on a circle whose centre is at the origin.

5. Show that the four asymptotes of the curve

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2 y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in eight points which lie on the circle $x^2 + y^2 = 1$.

6. Show that the eight points of intersection of the curve

$$x^4 - 5x^2 y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

and its asymptotes lie on a rectangular hyperbola.

7. Find the equation of the quartic curve which has $x = 0, y = 0, y = x$ and $y = -x$ for asymptotes and which passes through (a, b) and which cuts its asymptotes again in eight points that lie on a circle whose centre is origin and radius a.

8. Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2 y + 11xy^2 - 6y^3 + x + y + 1 = 0$, and which touches the axis of y at the origin and passes through the point (3, 2). **(Agra 2007)**

Answers 4

1. $y = 0, x = 1, x - y + 2 = 0.$

3. $a^2 (y + b) = b^2 (x + a).$

7. $bxy(x^2 - y^2) + a(b^2 - a^2)(x^2 + y^2 - a^2) = 0.$

8. $x^3 - 6x^2 y + 11xy^2 - 6y^3 - x = 0.$

13.13 Asymptotes to Non-Algebraic Curves

The definition of the asymptotes helps us in finding the asymptotes of non-algebraic curves as is clear from the following example.

Example : *Find the asymptotes of the curve* $y = \sec x$.

Solution : Here $dy/dx = \sec x \tan x$.

Therefore the tangent at (x, y) to the given curve is $Y - \sec x = \sec x \tan x\,(X - x)$

i.e., $$Y \cos^2 x - \cos x = (X - x) \sin x. \qquad \ldots(1)$$

Now as $x \to \pi/2, y \to \infty$ and the distance of (x, y) from the origin tends to infinity. Therefore taking limit of (1) as $x \to \pi/2$, we get

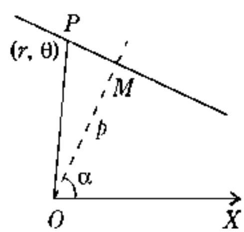
$$Y \,.\, 0 - 0 = \left(X - \frac{1}{2}\pi\right) . \, 1 \qquad i.e., \qquad X = \frac{1}{2}\pi.$$

This is one asymptote. The other asymptotes are

$$X = -\frac{1}{2}\pi, \pm \frac{3}{2}\pi, \ldots\ldots$$

13.14 Polar Curves

Lemma : *The polar equation of any line is* $p = r\cos(\theta - \alpha)$, *where, p is the length of the perpendicular from the pole to the line and* α *is the angle which the perpendicular makes with the initial line.*



Let O be the pole and OX the initial line. Let OM be the perpendicular from O to the given line.

Then it is given that $OM = p$ and $\angle XOM = \alpha$.

Let P be any point (r, θ) on the given line.

Then $OP = r, \angle XOP = \theta$ and $\angle MOP = \theta - \alpha$.

From the right angled triangle OMP, we have

$OM = OP \cos \angle POM$

i.e., $$p = r\cos(\theta - \alpha),$$

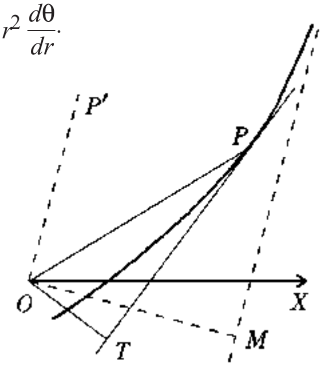
which is the required equation of the line.

Asymptotes of Polar Curves : *If* α *be a root of the equation* $f(\theta) = 0$, *then* $r\sin(\theta - \alpha) = 1/f'(\alpha)$ *is an asymptote of the curve* $1/r = f(\theta)$.

Take any point $P(r, \theta)$ on the curve

$$1/r = f(\theta). \qquad \ldots(1)$$

Draw a line through the pole O perpendicular to the radius vector OP and meeting the tangent at P in T.



Then $OT =$ polar subtangent of the curve at $P = r^2 \dfrac{d\theta}{dr}$.

But from (i), $$-\frac{1}{r^2}\frac{dr}{d\theta} = f'(\theta).$$

$$\therefore \quad r^2 \frac{d\theta}{dr} = -\frac{1}{f'(\theta)} = OT.$$

Now suppose θ tends to α. Since $f(\alpha) = 0$, therefore from (1), $r \to \infty$ *i.e.* the distance of P from the pole tends to infinity. Also PT tends to the asymptote and $OT \to \left[-\dfrac{1}{f'(\theta)}\right]_{\theta = \alpha}$

i.e. $OT \to -\dfrac{1}{f'(\alpha)}$, if $f'(\alpha) \neq 0$.

Also OP and PT will tend to become parallel as is obvious from the dotted lines in the figure. Therefore the angle OTP will tend to a right angle and OT will tend to OM where OM is perpendicular to the asymptote. Hence $OM = -\dfrac{1}{f'(\alpha)}$.

When $\theta = \alpha$, suppose OP tends to OP'.

Then $\angle XOP' = \alpha$.

$\therefore$ $\angle MOX = -\left(\dfrac{\pi}{2} - \alpha\right)$, negative sign indicating that it has been measured clockwise.

$\therefore$ The equation of the asymptote is

$$r\cos\left[\theta - \left\{-\left(\frac{\pi}{2} - \alpha\right)\right\}\right] = -\frac{1}{f'(\alpha)}$$

i.e.
$$r\cos\left(\theta + \frac{\pi}{2} - \alpha\right) = -\frac{1}{f'(\alpha)}$$

i.e.
$$-r\sin(\theta - \alpha) = -\frac{1}{f'(\alpha)} \qquad i.e.\ r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}.$$

13.15 Working Rule for Finding the Asymptotes of Polar Curves

(i) Put the equation of curve in the form $\dfrac{1}{r} = f(\theta)$.

(ii) Solve the equation $f(\theta) = 0$. Let $\alpha, \beta, \ldots$ be its roots.

(iii) The asymptote corresponding to $\theta = \alpha$ is

$$r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}, \text{ where } f'(\alpha) = \left[f'(\theta)\right]_{\theta = \alpha}.$$

Illustrative Examples

Example 1 : *Find the asymptotes of the curve $r = 2a/(1 - 2\cos\theta)$.*

Solution : The equation to the curve can be written as

$$\frac{1}{r} = \frac{1}{2a}(1 - 2\cos\theta) = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ if $1 - 2\cos\theta = 0$ *i.e* $2\cos\theta = 1$ *i.e.* $\cos\theta = \dfrac{1}{2}$

i.e. $\theta = 2n\pi \pm \dfrac{\pi}{3}$, where n is any integer

$= \alpha$, say.

Also $f'(\theta) = \dfrac{1}{2a}(2\sin\theta) = \dfrac{1}{a}(\sin\theta)$.

$$\therefore \quad f'(\alpha) = \frac{1}{a}\sin\left(2n\pi \pm \frac{\pi}{3}\right) = \pm\frac{1}{a}\sin\frac{\pi}{3} = \pm\frac{\sqrt{3}}{2a}.$$

$$\therefore \quad \frac{1}{f'(\alpha)} = \pm\frac{2a}{\sqrt{3}}.$$

Hence the asymptotes are given by

$$r\sin\left\{\theta-\left(2n\pi\pm\frac{\pi}{3}\right)\right\}=\pm\frac{2a}{\sqrt{3}}$$

or $$r\sin\theta\cos\left(2n\pi\pm\frac{\pi}{3}\right)-r\cos\theta\sin\left(2n\pi\pm\frac{\pi}{3}\right)=\pm\frac{2a}{\sqrt{3}}$$

or $$(r\sin\theta)\cos\frac{\pi}{3}\mp r\cos\theta\sin\frac{\pi}{3}=\pm\frac{2a}{\sqrt{3}}$$

i.e. $$r\sin\theta\cos\frac{\pi}{3}-r\cos\theta\sin\frac{\pi}{3}=\frac{2a}{\sqrt{3}}$$

and $$r\sin\theta\cos\frac{\pi}{3}+r\cos\theta\sin\frac{\pi}{3}=-\frac{2a}{\sqrt{3}}$$

i.e. $$r\sin\left(\theta-\frac{\pi}{3}\right)=\frac{2a}{\sqrt{3}}\text{ and } r\sin\left(\theta+\frac{\pi}{3}\right)=-\frac{2a}{\sqrt{3}}.$$

Example 2 : *Find the asymptotes of the curve $r=a\operatorname{cosec}\theta+b$.*

Solution : The equation of the curve can be written as

$$\frac{1}{r}=\frac{1}{a\operatorname{cosec}\theta+b}=\frac{\sin\theta}{a+b\sin\theta}=f(\theta),\text{ say.}$$

Now $f(\theta)=0$, if $\sin\theta=0$ *i.e.* $\theta=n\pi$, where n is any integer

$$=\alpha,\text{ say.}$$

Also $$f'(\theta)=\frac{\cos\theta.(a+b\sin\theta)-\sin\theta.b\cos\theta}{(a+b\sin\theta)^2}.$$

$\therefore$ $$f'(\alpha)=f'(n\pi)=\frac{\cos n\pi.(a+b\sin n\pi)-\sin n\pi.b\cos n\pi}{(a+b\sin n\pi)^2}$$

$$=\frac{a\cos n\pi}{a^2}=\frac{1}{a}\cos n\pi.$$

$\therefore$ $$\frac{1}{f'(\alpha)}=\frac{a}{\cos n\pi}.$$

Hence the asymptotes are given by $r\sin(\theta-n\pi)=\dfrac{a}{\cos n\pi}$

or $$(r\sin\theta\cos n\pi-r\cos\theta\sin n\pi)\cos n\pi=a$$

or $$r\sin\theta\cos^2 n\pi=a$$

or $$r\sin\theta=a.$$

13.16 Circular Asymptotes

Definition : *Let the equation of a curve be $r=f(\theta)$.*

If $\lim\limits_{\theta\to\infty} f(\theta)=l$, then the circle $r=l$ is called the circular asymptote of the curve $r=f(\theta)$.

Example 1 : *Find the circular asymptote of the curve $r=a\cdot\dfrac{\theta}{\theta-1}$.*

Solution : The circular asymptote is given by

$$r=a\lim_{\theta\to\infty}\frac{\theta}{\theta-1}=a.$$

Thus $r=a$ is the circular asymptote.

Comprehensive Exercise 5

1. If α is a root of the equation $f(\theta) = 0$, then write the equation of asymptote of the polar curve $\dfrac{1}{r} = f(\theta)$ corresponding to the root α. **(Meerut 2001)**

Find the asymptotes of the following curves :

2. $y = \tan x$.

3. $r \sin m\theta = a$. **(Meerut 2000)**

4. $r\theta = a$. **(Meerut 2008, 12B)**

5. $2/r = 1 + 2 \sin \theta$.

6. (i) $r \sin \theta = 2 \cos 2\theta$. (ii) $r \sin \theta = a \cos 2\theta$.

7. (i) $r \cos \theta = a \sin \theta$. **(Meerut 2004B)**

(ii) $r \sin \theta = 2 \cos \theta$.

8. $r = 4 (\sec \theta + \tan \theta)$.

9. $r \sin 2\theta = a$.

10. $r \cos \theta = 4 \sin^2 \theta$. **(Meerut 2005)**

11. $r\theta \cos \theta = a \cos 2\theta$.

12. $r (e^\theta - 1) = a (e^\theta + 1)$.

13. $r = \dfrac{2\theta}{\sin \theta}$.

Find the circular asymptotes of the following curves :

14. $r (e^\theta - 1) = a (e^\theta + 1)$.

15. $r = \dfrac{3\theta^2 + 2\theta + 1}{2\theta^2 + \theta + 1}$.

Answers 5

1. $r \sin (\theta - \alpha) = \dfrac{1}{[f'(\theta)]_{\theta = \alpha}}$.

2. $x = \pm \pi/2, \pm 3\pi/2, \ldots\ldots$.

3. $r \sin \left(\theta - \dfrac{k\pi}{m}\right) = \dfrac{a}{m \cos k\pi}$, where k is any integer.

4. $r \sin \theta = a$.

5. $r \sin \left[\theta \pm \dfrac{1}{6}\pi\right] = 2/\sqrt{3}$.

6. (i) $r \sin \theta = 2$. (ii) $r \sin \theta = a$.

7. (i) $r \cos \theta = \pm a$. (ii) $r \sin \theta = \pm 2$.

8. $r \cos \theta = 8$.

9. $r \sin \theta = \pm \frac{1}{2} a, r \cos \theta = \pm \frac{1}{2} a$.

10. $r \cos \theta = 4$.

11. $r \sin \theta = a, r \cos \theta = \dfrac{a}{\left(k + \dfrac{1}{2}\right)\pi}$, k is any integer.

12. $r \sin \theta = 2a$.

13. $r \sin \theta = 2k\pi, k = \pm 1, \pm 2, \ldots$.

14. $r = a$.

15. $r = \frac{3}{2}$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …", so that the following statements are complete and correct.

1. The asymptotes parallel to the axis of y are obtained by equating to zero the coefficients of the …… power of y in the equation of the curve.
2. A curve of degree n cannot have more than …… asymptotes. **(Agra 2008)**
3. The only asymptote of the curve $x^3 + y^3 - 3axy = 0$ is …… .
4. If α be a root of the equation $f(\theta) = 0$, then an asymptote of the curve $\frac{1}{r} = f(\theta)$ is …… .
5. Circular asymptote of the curve $r(\theta^2 + 1) = a\theta^2 - 1$ is …… . **(Meerut 2001)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

6. The number of asymptotes of the curve $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$ is
   (a) 2 (b) 3 (c) 4 (d) 1 **(Bundelkhand 2006)**
7. The asymptotes of the curve $y^2(x^2 - a^2) = x$, which are parallel to the x-axis are
   (a) $x = \pm a$ (b) $y = \pm a$ (c) $y = 0, y = 0$ (d) $x = 0$
8. The number of oblique asymptotes of the curve $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$ is
   (a) 4 (b) 3 (c) 2 (d) None of these

True or False:

Write 'T' for true and 'F' for false statement.

9. The number of asymptotes, real or imaginary, of an algebraic curve of the n^{th} degree cannot exceed n.
10. The curve $x^2(x - y)^2 + a^2(x^2 - y^2) - a^2xy = 0$ has no asymptotes parallel to y-axis.
11. The curve $r = \frac{a}{1 - \cos\theta}$ has no asymptotes.
12. An asymptote is a tangent of the curve at infinity.
13. An asymptote touch the curve at finite point. **(Agra 2007)**

Answers

1. highest. 2. n.
3. $x + y + a = 0$. 4. $r\sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$, $f'(\alpha) \neq 0$.
5. $r = a$. 6. (c). 7. (c). 8. (c).
9. T. 10. F. 11. T. 12. T. 13. F.

□

Chapter

14

Singular Points: Curve Tracing

14.1 Concavity and Convexity (Meerut 2009)

Let P be a given point on a curve and AB a given straight line which does not pass through P. Then the curve is said to be *concave* or *convex* at P with respect to AB, according as a sufficiently small arc of the curve containing P lies entirely *within* or *without* the acute angle formed by the tangent at P to the curve with the line AB. Thus in the figure 1 the curve at P is convex to AB, and in figure 2 it is concave to AB.

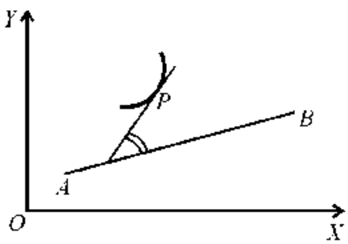


Fig. 1

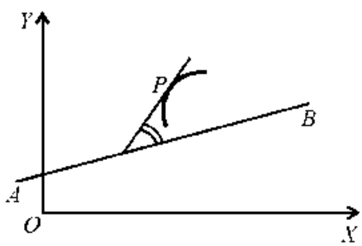


Fig. 2

The property of concavity or convexity of a curve at any point is not an *inherent* property of the curve. At a given point a curve may be concave with respect to some line, while at the same point it may be convex with respect to some other line.

Concavity upwards and Concavity downwards :

The curve shown in the Fig. 1 below is **concave upwards** and the curve shown in the Fig. 2 below is **concave downwards.**

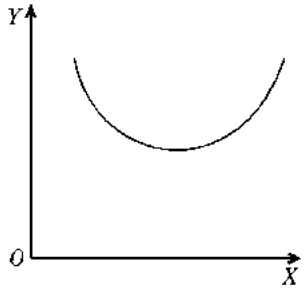


Fig. 1

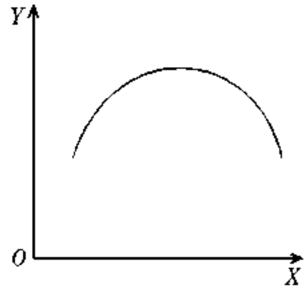


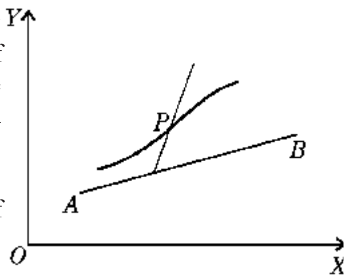
Fig. 2

14.2 Point of Inflexion

(Avadh 2014)

A point P on a curve is said to be a point of inflexion, if the curve is concave on one side and convex on the other side of P with respect to any line AB. Thus at a point of inflexion the curve changes its direction of bending from concavity to convexity or vice-versa. The two portions of the curve on the two sides of P lie on *different* sides of the tangent at P, *i.e.*, the curve *crosses* the tangent at P.

Thus *a point where the curve crosses the tangent is a point of inflexion*. Therefore the position of a point of inflexion of a curve will in no way depend on the choice of coordinate axes. In particular, the positions of x and y axes may be interchanged without affecting the positions of the points of inflexion on the curve.



Inflexional tangent. The tangent at a point of inflexion of a curve is called inflexional tangent.

14.3 Test of Concavity or Convexity

We shall consider concavity and convexity with respect to the axis of x.

Let the equation of the curve be $y = f(x)$ and let P be the point (x, y) on this curve. Suppose the tangent at P is not parallel to y-axis so that at P the value of $f'(x)$ is finite.

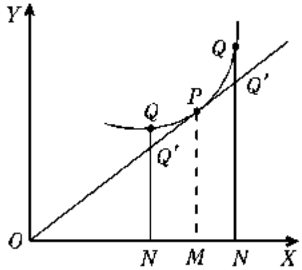


Fig. 1

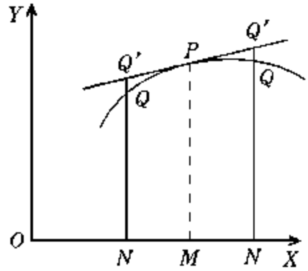


Fig. 2

Let Q be the point $(x+h, y+k)$ on the curve in the neighbourhood of P. The point Q may be taken on either side of P. Suppose the ordinate QN of Q meets the tangent to the curve at P in Q'.

The equation of the tangent at P is

$$Y-y=f'(x)(X-x), \qquad \text{...(1)}$$

where (X, Y) are the current coordinates.

Putting $X=x+h$ in (1), we get

$$NQ'-y=f'(x)\{x+h-x\}$$

or $$NQ'=f(x)+hf'(x). \qquad \text{...(2)}$$

$[\because \; y=f(x)]$

Also from the equation of the curve, we get

$$NQ=f(x+h)$$

$$=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{1}{(n-1)!}h^{n-1}f^{(n-1)}(x)$$

$$+\frac{1}{n!}h^n f^n(x+\theta h), \qquad \text{...(3)}$$

on expanding by Taylor's theorem, where $0<\theta<1$.

From (2) and (3), by subtraction, we get

$$NQ-NQ'=\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x+\theta h). \qquad \text{...(4)}$$

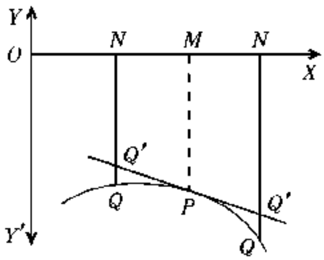
If $f''(x)\neq 0$, then by taking h sufficiently small, the second degree terms in h on the R.H.S. of (4) can be made to govern its sign. Therefore $(NQ-NQ')$ will be of the same sign as $\frac{h^2}{2!}f''(x)$.

Obviously $\frac{h^2}{2!}f''(x)$ will be of invariable sign whether h is positive or negative *i.e.*, whether Q lies to the right or the left of P.

The curve at P will be convex with respect to the axis of x if $NQ-NQ'$ is positive [See Fig. 1] and it will be concave at P with respect to the axis of x if $NQ-NQ'$ is negative [See Fig. 2].

Hence, the curve is convex at P to the axis of x if $f''(x)$ is positive, and concave if $f''(x)$ is negative.

We have drawn the above figures for the case when the curve is above the axis of x. If, however, the curve is below the axis of x, then NQ and NQ' are both negative, and $NQ-NQ'=-\{|NQ|-|NQ'|\}$.



Hence, in this case the curve at P is convex with respect to the axis of x if $|NQ|-|NQ'|$ is positive *i.e.*, if $NQ-NQ'$ is negative *i.e.*, if $f''(x)$ is negative.

Similarly the curve at P will be concave with respect to the axis of x if $f''(x)$ is positive.

From the above discussion we observe that whether the curve lies below or above the axis of x, we have the following criterion for concavity or convexity at P with respect to the axis of x.

A curve is convex or concave at P to the axis of x according as $y\dfrac{d^2y}{dx^2}$ is positive or negative at P.

Test for concavity upwards or Concavity downwards :

The curve $y = f(x)$ is **concave upwards** *in $[a, b]$ i.e., when $a \le x \le b$, if $f''(x) > 0 \ \forall \ x \in [a, b]$ and is concave downwards in $[a, b]$ if $f''(x) < 0 \ \forall \ x \in [a, b]$.*

Concavity and Convexity with respect to the axis of y :

By considering y as the independent variable, we can easily show that a curve $x = f(y)$, at a given point P on it, is convex or concave to the axis of y according as $x\dfrac{d^2x}{dy^2}$ is positive or negative at P.

Illustrative Examples

Example 1 : *Show that the curve $y = e^x$ is everywhere concave upwards and the curve $y = \log x$ is everywhere concave downwards.*

Solution : First consider the curve $y = e^x$.

We have $\quad \dfrac{dy}{dx} = e^x \quad$ and $\quad \dfrac{d^2y}{dx^2} = e^x$.

Obviously $\dfrac{d^2y}{dx^2} > 0, \forall \ x \in \mathbf{R}$, where $\mathbf{R}$ is the set of real numbers.

Hence , the curve $y = e^x$ is everywhere concave upwards.

Now consider the curve $y = \log x$, $0 < x < \infty$.

We have $\quad \dfrac{dy}{dx} = \dfrac{1}{x} \quad$ and $\quad \dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2}$.

Obviously, $\quad \dfrac{d^2y}{dx^2} < 0, \forall \ x \in \,]0, \infty[$.

Hence, the curve $y = \log x$ is everywhere concave downwards.

Example 2 : *Find the intervals in which the curve $y = e^x (\cos x + \sin x)$ is concave upwards or downwards; x varying in the interval $]0, 2\pi[$.*

Solution : The given curve is $y = e^x (\cos x + \sin x)$.

We have $\dfrac{dy}{dx} = e^x (\cos x + \sin x) + e^x (-\sin x + \cos x) = 2e^x \cos x$.

$$\therefore \quad \frac{d^2y}{dx^2} = 2e^x \cos x - 2e^x \sin x = 2e^x (\cos x - \sin x)$$

$$= 2\sqrt{2} \, . \, e^x \left(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4}\right) = 2\sqrt{2} \, . \, e^x \cos\left(x + \frac{\pi}{4}\right).$$

When $x \in \left]0, \dfrac{\pi}{4}\right[$ or when $x \in \left]\dfrac{5\pi}{4}, 2\pi\right[$, we have $\dfrac{d^2y}{dx^2} > 0$.

Hence, the given curve is concave upwards in $\left]0, \dfrac{\pi}{4}\right[$ and $\left]\dfrac{5\pi}{4}, 2\pi\right[$.

Again when $x \in \left]\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right[$, we have $\dfrac{d^2y}{dx^2} < 0$.

Hence, the given curve is concave downwards in $\left]\frac{\pi}{4}, \frac{5\pi}{4}\right[$.

Example 3 : *Show that the sine curve $y = \sin x$ is everywhere concave with respect to the axis of x excluding the points where it meets the axis of x.*

Solution : The given curve is $y = \sin x$.

We have $\quad \dfrac{dy}{dx} = \cos x \quad$ and $\quad \dfrac{d^2y}{dx^2} = -\sin x$.

The function $\sin x$ is a periodic function with period 2π. Hence, it is sufficient to consider the given curve in the interval $[0, 2\pi]$.

In the interval $[0, 2\pi]$, we have $y = 0$ when $x = 0$ or $x = \pi$ or $x = 2\pi$.

When $x \in \,]0, \pi[$, we have $y > 0$ and $\dfrac{d^2y}{dx^2} < 0$.

So $y\dfrac{d^2y}{dx^2} < 0$ when $x \in \,]0, \pi[$.

Hence, the curve $y = \sin x$ is concave to the axis of x in the interval $]0, \pi[$.

When $x \in \,]\pi, 2\pi[$, we have $y < 0$ and $\dfrac{d^2y}{dx^2} > 0$.

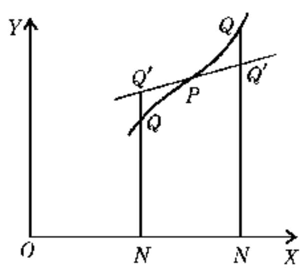
So $y\dfrac{d^2y}{dx^2} < 0$ when $x \in \,]\pi, 2\pi[$.

Hence, the curve $y = \sin x$ is concave to the axis of x in the interval $]\pi, 2\pi[$.

Thus the curve $y = \sin x$ is everywhere concave with respect to the axis of x excluding the points where it meets the axis of x.

14.4 Test for Point of Inflexion (Avadh 2014)

Let the equation of the curve be $y = f(x)$ and let P be the point (x, y) on this curve. Suppose the tangent at P is not parallel to y-axis so that at P the value of $f'(x)$ is finite. Let Q be the point $(x + h, y + k)$ on the curve in the neighbourhood of P. The point Q may be taken on either side of P. Suppose the ordinate QN of Q meets the tangent to the curve at P in Q'.



The equation of the tangent at P is

$$Y - y = f'(x)(X - x), \quad \text{...(1)}$$

where (X, Y) are the current coordinates.

Putting $X = x + h$ in (1), we get $NQ' - y = f'(x)\{x + h - x\}$

or $\quad NQ' = f(x) + hf'(x), \quad [\because y = f(x)] \quad \text{...(2)}$

Also from the equation of the curve, we get $NQ = f(x + h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots$$

$$+ \frac{1}{(n-1)!}h^{n-1}f^{(n-1)}(x) + \frac{1}{n!}h^n f^{(n)}(x + \theta h), \quad \text{...(3)}$$

on expanding by Taylor's theorem, if $0 < \theta < 1$.

From (2) and (3), by subtraction, we get

$$NQ - NQ' = \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x + \theta h). \quad \text{...(4)}$$

If $f''(x) \neq 0$, then by taking h sufficiently small, the second degree terms in h on the R.H.S. of (iv) can be made to govern its sign. Therefore $(NQ - NQ')$ will be of the same sign as $(h^2/2!) f''(x)$. But $(h^2/2!) f''(x)$ will be of invariable sign whether h is positive or negative *i.e.*, whether Q lies to the right or the left of P. Therefore on both sides of P the curve will be either concave or convex. Hence *the necessary condition for the existence of a point of inflexion at P is that*

$$\boldsymbol{f''(x) = 0.}$$

Now if $f''(x) = 0$, we have from (iv)

$$NQ - NQ' = \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{iv}(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x + \theta h). \qquad \ldots(5)$$

If $f'''(x) \neq 0$, then for sufficiently small values of h the sign of the right hand side of (5) is the same as that of $(h^3/3!) f'''(x)$, which changes sign when h changes sign. Thus with respect to x-axis, the curve will be concave on one side of P and convex on the other side of P. So there will be a point of inflexion at P.

Hence, *there will be a point of inflexion at P, if* $\boldsymbol{d^2 y/dx^2 = 0}$ *but* $\boldsymbol{d^3 y/dx^3 \neq 0}$.

Generalisation : If $f''(x) = f'''(x) = f^{iv}(x) = \ldots = f^{(n-1)}(x) = 0$, and $f^{(n)}(x) \neq 0$, it is easy to see from the value of $NQ - NQ'$, that there will be a point of inflexion if n is odd. If, however, n is even, the curve does not cross the tangent and so there will not be a point of inflexion at P. Such a point (if n is greater than 2) is called **a point of undulation**. To the eye a point of undulation appears just like an ordinary point.

Corollary : The position of a point of inflexion is independent of the choice of coordinate axes. Therefore on interchanging x and y in the above results, we can say that, *there will be a point of inflexion at P, if* $\boldsymbol{d^2 x/dy^2 = 0}$, *but* $\boldsymbol{d^3 x/dy^3 \neq 0}$.

(Bundelkhand 2007)

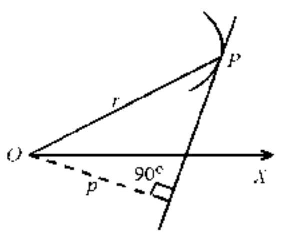
It will become necessary for us to use this criterion if the tangent at P is parallel to y-axis *i.e.*, if dy/dx is infinite at P. It will also be useful where the equation of the curve is of the form $x = f(y)$.

14.5 Concavity and Convexity for Polar Curves

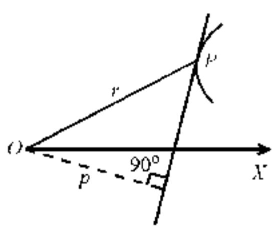
From the following figures it is obvious that if at any point P on the curve the perpendicular p drawn from the pole on the tangent increases as r increases, then the curve is concave at P to the pole. Thus, *a curve is concave at P to the pole if dp/dr is positive there.*

Similarly, *a curve is convex at P to the pole if dp/dr is negative there.*

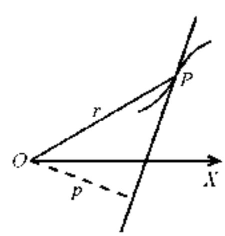
If dp/dr is zero at P, positive for points on one side of P and negative for the points on the other side of P, there must be a point of inflexion at P.



Curve concave at P to the pole O



Curve convex at P to the pole O



Point of inflexion at P

But $r\dfrac{dr}{dp} =$ radius of curvature at P

$$= \frac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}}$$

$$\therefore \qquad \frac{dp}{dr} = \frac{r\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}\right\}}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}$$

Hence, if $r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2} = 0$ at P, there is, in general, a point of inflexion at P.

Illustrative Examples

Example 1 : *Find the points of inflexion of the curve* $y = 3x^4 - 4x^3 + 1$.

(Rohilkhand 2010, 11B; Avadh 10; Meerut 12)

Solution : Differentiating the equation of the curve with respect to x, we get $dy/dx = 12x^3 - 12x^2$ and $d^2 y/dx^2 = 36x^2 - 24x$.

For the points of inflexion, we must have $d^2 y/dx^2 = 0$

i.e., $\qquad 36x^2 - 24x = 0, \quad$ *i.e.,* $\quad 12x(3x - 2) = 0,$

i.e., $\qquad x = 0 \quad$ or $\quad \dfrac{2}{3}$ for the points of inflexion.

Now $\qquad d^3 y/dx^3 = 72x - 24.$

When $x = 0$, $d^3 y/dx^3 \neq 0$, therefore $x = 0$ gives a point of inflexion.

Similarly, when $x = \dfrac{2}{3}$, $d^3 y/dx^3 \neq 0$; therefore $x = \dfrac{2}{3}$ also gives a point of inflexion.

From the equation of the curve, we have

$$y = 1, \text{ when } x = 0 \text{ and } y = \frac{11}{27}, \text{ when } x = \frac{2}{3}.$$

Hence $(0, 1)$ and $\left(\dfrac{2}{3}, \dfrac{11}{27}\right)$ are the required points of inflexion.

Important : Instead of finding $d^3 y/dx^3$, we can use another criterion for points of inflexion. If $d^2 y/dx^2 = 0$ at $x = a$ and the sign of $d^2 y/dx^2$ changes while x passes through a, then there will be a point of inflexion at $x = a$.

Example 2 : *Find the points of inflexion of the curve* $x = \log(y/x)$.

(Purvanchal 2011; Rohilkhand 14)

Solution : The given curve is $x = \log(y/x)$ or $y/x = e^x$

or $\qquad y = x e^x. \qquad$ ...(1)

Differentiating (1), we get

$$dy/dx = x e^x + e^x = (x + 1) e^x$$

and $\qquad d^2 y/dx^2 = e^x + (x + 1) e^x = e^x (x + 2).$

For points of inflexion $d^2 y/dx^2 = 0$.

$\therefore \quad e^x (x+2) = 0 \quad$ *i.e.,* $\quad x = -2, \qquad [\because \; e^x \neq 0]$

Now $\quad d^3 y/dx^3 = e^x + (x+2) e^x = e^x (x+3) \neq 0$ at $x = -2$.

$\therefore$ there is a point of inflexion at $x = -2$.

From (1), when $x = -2, y = -2e^{-2} = -2/e^2$.

Hence the point of inflexion is $(-2, -2/e^2)$.

Example 3 : *Find the points of inflexion on the curve*

$$x = a(2\theta - \sin\theta),\; y = a(2 - \cos\theta).$$ **(Avadh 2013)**

Solution : Differentiating w.r.t. θ, we have

$$dx/d\theta = a(2 - \cos\theta) \quad \text{and} \quad dy/d\theta = a\sin\theta.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{2 - \cos\theta}.$$

And
$$\frac{d^2 y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{2 - \cos\theta}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{(2 - \cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(2 - \cos\theta)^2} \cdot \frac{1}{a(2 - \cos\theta)}$$

$$= \frac{2\cos\theta - (\cos^2\theta + \sin^2\theta)}{a(2 - \cos\theta)^3} = \frac{2\cos\theta - 1}{a(2 - \cos\theta)^3}.$$

Now for the points of inflexion, we must have $d^2 y/dx^2 = 0$

i.e., $\quad 2\cos\theta - 1 = 0 \quad$ or $\quad \cos\theta = \frac{1}{2} = \cos\frac{1}{3}\pi$.

$\therefore \quad \theta = 2n\pi \pm \frac{1}{3}\pi$, where n is any integer.

Substituting the value of θ in the given equation of the curve we get the points of inflexion as

$$\left[a\left(4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right]. \quad \left[\because \sin\left(2n\pi \pm \frac{\pi}{3}\right) = (-1)^{2n}\sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}\right].$$

Example 4 : *Find the ranges of values of x for which the curve*

$$y = x^4 - 6x^3 + 12x^2 + 5x + 7$$

is concave upwards or downwards.

Also determine the points of inflexion. **(Purvanchal 2008)**

Solution : We have $\dfrac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$,

$$\frac{d^2 y}{dx^2} = 12x^2 - 36x + 24 = 12(x-1)(x-2)$$

and
$$\frac{d^3 y}{dx^3} = 24x - 36.$$

We have
$$\frac{d^2 y}{dx^2} > 0, \forall\, x \in\,]-\infty, 1[,$$

$$\frac{d^2 y}{dx^2} < 0, \forall\, x \in\,]1, 2[\text{ and } \frac{d^2 y}{dx^2} > 0, \forall\, x \in\,]2, \infty[.$$

Hence, the curve is concave upwards in the intervals $]-\infty, 1[$ and $]2, \infty[$ and concave downwards in the interval $]1, 2[$.

Now $\dfrac{d^2y}{dx^2} = 0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x = 1$ or $x = 2$.

At $x = 1$, $\dfrac{d^3y}{dx^3} = -12 \neq 0$ and at $x = 2$, $\dfrac{d^2y}{dx^2} = 12 \neq 0$. Thus, there are points of inflexion at $x = 1$ and at $x = 2$.

When $x = 1$, we have $y = 19$ and when $x = 2$, we have $y = 33$.

Hence, $(1, 19)$ and $(2, 33)$ are the two points of inflexion on the curve.

Example 5 : *Examine the curve $y = \sin x$ for concavity upwards, concavity downwards and for points of inflexion in the interval* $[-2\pi, 2\pi]$.

Solution : The given curve is $y = \sin x$.

We have $\dfrac{dy}{dx} = \cos x$, $\dfrac{d^2y}{dx^2} = -\sin x$ and $\dfrac{d^3y}{dx^3} = -\cos x$.

We have, $\dfrac{d^2y}{dx^2} < 0, \forall x \in\,]-2\pi, -\pi[,$

$$\frac{d^2y}{dx^2} > 0, \forall x \in\,]-\pi, 0[, \; \frac{d^2y}{dx^2} < 0, \forall x \in\,]0, \pi[$$

and $\dfrac{d^2y}{dx^2} > 0, \forall x \in\,]\pi, 2\pi[.$

Hence, the curve is concave downwards in the intervals $]-2\pi, -\pi[$ and $]0, \pi[$ and concave upwards in the intervals $]-\pi, 0[$ and $]\pi, 2\pi[$.

Now $\dfrac{d^2y}{dx^2} = 0 \Rightarrow \sin x = 0 \Rightarrow x = -2\pi$ or $x = -\pi$ or $x = 0$

or $x = \pi$ or $x = 2\pi$.

At each of the points $x = -2\pi, x = -\pi, x = 0, x = \pi$ and $x = 2\pi$, we have $\dfrac{d^3y}{dx^3} \neq 0$.

Thus there are points of inflexion at each of these points.

Also, $y = 0$ at each of these points.

Hence, the curve has points of inflexion at $(-2\pi, 0), (-\pi, 0), (0, 0), (\pi, 0)$ and $(2\pi, 0)$.

Example 6 : *Find the points of inflexion on the curve* $r(\theta^2 - 1) = a\theta^2$.

(Meerut 2013)

Solution : We have $r = a\theta^2/(\theta^2 - 1)$.

$$\therefore \quad \frac{dr}{d\theta} = a\,[(\theta^2 - 1).2\theta - \theta^2.2\theta]/(\theta^2 - 1)^2 = -2a\theta/(\theta^2 - 1)^2,$$

and
$$\frac{d^2r}{d\theta^2} = -2a\,[(\theta^2 - 1)^2.1 - \theta.2(\theta^2 - 1).2\theta]/(\theta^2 - 1)^4$$
$$= 2a(3\theta^2 + 1)/(\theta^2 - 1)^3.$$

We know that at the point of inflexion, the radius of curvature is infinite. Hence at the point of inflexion, we have

$$r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2) = 0$$

or
$$\frac{a^2\theta^4}{(\theta^2 - 1)^2} + \frac{8a^2\theta^2}{(\theta^2 - 1)^4} - \frac{2a^2\theta^2(3\theta^2 + 1)}{(\theta^2 - 1)^4} = 0$$

or $$\frac{a^2\theta^2(\theta^2-3)(\theta^2+2)}{(\theta^2-1)^3}=0$$

or $$\theta^2(\theta^2-3)(\theta^2+2)=0$$

$\therefore$ $$\theta^2=0, 3, -2.$$

Rejecting the values $\theta^2=-2$ and 0 we see that the points of inflexion are given by $\theta^2=3$ *i.e.*, $\theta=\pm\sqrt{3}$.

Comprehensive Exercise 1

1. Show that the points of inflexion upon the curve $x^2y=a^2(x-y)$ are given by $x=0, x=\pm a\sqrt{3}$. **(Meerut 2013B)**

2. Find the points of inflexion of the curve $y(a^2+x^2)=x^3$. **(Purvanchal 2010)**

3. Find the points of inflexion of the curve $xy=a^2\log(y/a)$.

4. Find the points of inflexion of the curve $x=(\log y)^3$. **(Purvanchal 2009)**

5. Investigate the points of inflexion of the curve $y=(x-1)^4(x-2)^3$. **(Agra 2014)**

6. Show that every point in which the sine curve $y=c\sin(x/a)$ meets the axis of x is a point of inflexion.

7. Show that points of inflexion of the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$; **(Agra 2006; Avadh 11; Kashi 12)**

8. Find the points of inflexion on the curve $y^2=x(x+1)^2$ and also obtain the equations of the inflexional tangents.

9. Show that origin is a point of inflexion of the curve $a^{m-1}.y=x^m$ if m is odd and greater than 2.

10. Show that the abscissae of the points of inflexion on the curve $y^2=f(x)$ satisfy the equation $[f'(x)]^2=2f(x).f''(x)$.

11. Show that the line joining the points of inflexion of the curve $y^2(x-a)=x^2(x+a)$ subtends an angle of $\pi/3$ at the origin.

12. Prove that the curve $y=\dfrac{1-x}{1+x^2}$, has three points of inflexion which lie in a straight line.

13. Show that the points of inflexion on the curve $y=be^{-(x/a)^2}$ are given by $x=\pm a/\sqrt{2}$. **(Agra 2005)**

14. Show that the points of inflexion of the curve $r=b\theta^n$ are given by $r=b\{-n(n+1)\}^{n/2}$.

Answers 1

2. $(0,0), \left(\sqrt{3}a, \dfrac{3\sqrt{3}}{4}a\right), \left(-\sqrt{3}a, \dfrac{-3\sqrt{3}a}{4}\right)$.

3. $\left(\frac{3}{2}ae^{-3/2}, ae^{3/2}\right)$.

4. $(0, 1), (8, e^2)$.

5. Points of inflexion at $x = 2, (11 \pm \sqrt{2})/7$; point of undulation at $x = 1$.

8. Points of inflexion are $(1/3, \pm 4/3\sqrt{3})$. Inflexional tangents are

$$9x \pm 3\sqrt{3}y + 1 = 0.$$

14.6 Multiple Points

(Meerut 2003)

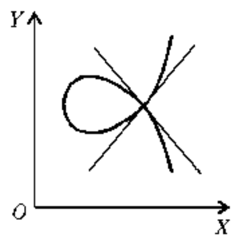
A point through which more than one branches of a curve pass is called a multiple point on the curve. A point on the curve is called a **double point** if two branches of the curve pass through it, a **triple point** if three branches pass through it. In general, if r branches pass through a point, it is called a multiple point of the r^{th} order.

14.7 Singular Points

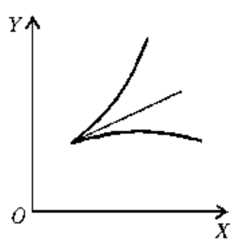
An unusual point on a curve is called a singular point. For example, at any point the tangent does not usually cross the curve. But at a point of inflexion the tangent crosses the curve and therefore it is a singular point. Similarly, through one point, usually one branch of the curve passes. But through a multiple point, more than one branches of the curve pass. Therefore multiple points are also singular points.

14.8 Classification of Double Points

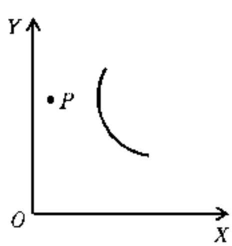
(i) Node : If the two branches through a double point on a curve are real and have *different* tangents there, then the double point is called a *node*. **(Kumaun 2008; Kashi 11)**



(ii) Cusp : If the two branches through a double point on a curve are real and have *coincident* tangents there, then the double point is called a *cusp*. **(Kashi 2011)**



(iii) Conjugate Point : If there are no real points on the curve in the neighbourhood of a point P on the curve, then P is called a *conjugate point* (or an *isolated point*). The process of finding the tangents usually gives imaginary tangents at such a point.



Since through a double point two branches of the curve pass, therefore in the process of finding tangents at a double point we must get *two* tangents there, one for each branch. If the two tangents are real and distinct, the double point will be a node. If the two tangents are imaginary, the double point will be a conjugate point. If the two tangents are real and coincident, the double point may be a cusp or a conjugate point. The possibility of the point being a conjugate point in this case arises on account of the fact that sometimes imaginary expressions $A \pm iB$ become real by chance when $B = 0$. In such cases the double point will be a cusp if there are other real points of the curve in its neighbourhood, otherwise it will be a conjugate point.

14.9 Species of Cusps

We know that two branches of a curve have a common tangent at a cusp. A cusp is said to be *single* or *double* according as the curve lies entirely on *one* side of the common normal or on *both* sides. Also it is of the *first* or *second species* according as the two branches lie on *opposite* sides or on the *same* side of the common tangents. We have the following *five* different types of cusps :

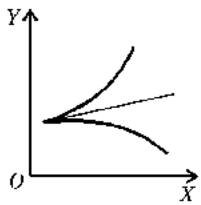


Fig. 1

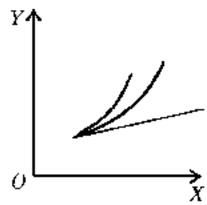


Fig. 2

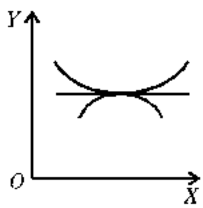


Fig. 3

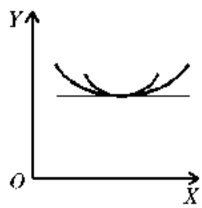


Fig. 4

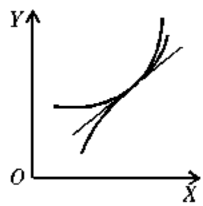


Fig. 5

Single cusp of the first species as shown in Fig. 1.

Single cusp of the second species as shown in Fig. 2.

Double cusp of the first species as shown in Fig. 3.

Double cusp of the second species as shown in Fig. 4.

Double cusp with change of species as shown in Fig. 5. Here the two branches lie on *both the sides* of the common normal but on one side they lie on the same and on the other on opposite sides of the common tangent. Such a point is called a point of **oscul-inflexion.**

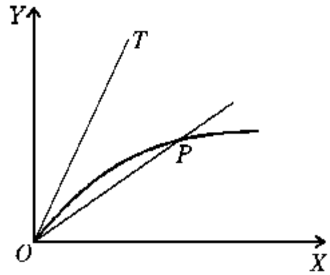
14.10 Tangents at Origin

In order to know the nature of a double point it is necessary to find the tangent or tangents there. Now we shall find a simple rule for writing down the *tangent* or *tangents* at the origin to rational algebraic curves.

If a curve passes through the origin and is given by a rational integral, algebraic equation, the equation to the tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

Let the equation of the curve when arranged according to ascending powers of x and y be

$$(a_1 x + a_2 y) + (b_1 x^2 + b_2 xy + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + \dots) + \dots = 0, \quad \dots(1)$$



where the constant term is absent since the curve passes through the origin.

Let $P(x, y)$ be any point on the curve. The slope of the chord OP is y/x. Therefore the equation to OP is $Y = (y/x)\,X$, where (X, Y) are current coordinates.

As $P \to O$ *i.e.*, as $x \to 0$ and $y \to 0$, the chord OP tends to the tangent at O.

Excluding for the present the case when the tangent is the y-axis *i.e.*, when $\lim_{x \to 0}\left(\frac{y}{x}\right) = \pm\infty$, we have the equation of the tangent at O as

$$Y = \left\{\lim_{x \to 0}\left(\frac{y}{x}\right)\right\} X. \quad \dots(2)$$

Case I : Let $a_2 \neq 0$. Dividing (1) by x and taking limit as $x \to 0$, we get

$$a_1 + a_2 \left\{\lim_{x \to 0}\left(\frac{y}{x}\right)\right\} = 0. \quad \dots(3)$$

Eliminating $\lim_{x \to 0}\left(\frac{y}{x}\right)$ between (2) and (3), we get $a_1 X + a_2 Y = 0$, as the equation of tangent at the origin to the curve (1).

Replacing the current coordinates X, Y by x, y this equation becomes

$$a_1 x + a_2 y = 0, \quad \dots(4)$$

which is obviously the equation obtained by equating to zero the lowest degree terms in (1).

If $a_2 = 0$, then a_1 is also zero from (3), and we get the next case.

Case II : Let $a_1 = 0$, $a_2 = 0$, but b_2 and b_3 are not both zero. Dividing (1) by x^2 and taking limit as $x \to 0$, we get

$$b_1 + b_2 \lim_{x \to 0}\left(\frac{y}{x}\right) + b_3 \lim_{x \to 0}\left(\frac{y}{x}\right)^2 = 0,$$

or $$b_1 + b_2 m + b_3 m^2 = 0, \quad \dots(5)$$

where $$\lim_{x \to 0}\left(\frac{y}{x}\right) = m.$$

Equation (5) is a quadratic in m, showing that there are two tangents at the origin in this case. Eliminating m between (2) and (5), we get

$$b_1 x^2 + b_2 xy + b_3 y^2 = 0, \quad \dots(6)$$

as the equation of the tangents at the origin to (1) in this case. In equation (6), we have taken x, y as current coordinates. Obviously the equation (6) is obtained by equating to zero the lowest degree terms in the equation of the curve (1), where

$$a_1 = a_2 = 0.$$

If $b_2 = b_3 = 0$, then by (v), $b_1 = 0$.

Case III : If $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, we can show by the same process that the rule still holds; and so on.

If tangent at the origin is the y-axis, we can easily show by supposing the axes of x and y to be interchanged for a moment, that the rule is still true.

Hence the equation of the tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

Corollary : If the origin is a double point on a curve, then the curve has two tangents at the origin. Therefore the equation of the curve should not contain the constant and the first degree terms and the second degree terms should be the lowest degree terms in the equation of the curve.

Example 1 : *Show that the origin is a node on the curve* $x^3 + y^3 - 3axy = 0$.

(Meerut 2003; Purvanchal 14)

Solution : The curve passes through the origin as its equation does not contain the constant term. Also equating to zero the lowest degree terms in the equation of the curve, we get the equation to the tangents at origin as $-3axy = 0$, *i.e.* $xy = 0$, *i.e.* $x = 0$, $y = 0$ are two real and distinct tangents at the origin. Therefore origin is a node.

Example 2 : Show that the origin is a conjugate point on the curve

$$a^2 x^2 + b^2 y^2 = (x^2 + y^2)^2.$$

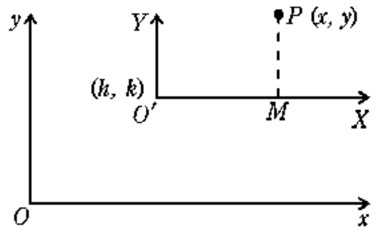
Solution : Obviously the curve passes through the origin. The equation to the tangent at origin is

$$a^2 x^2 + b^2 y^2 = 0, \quad i.e., \quad ax \pm iby = 0.$$

Thus there are two imaginary tangents at the origin. Therefore origin is a conjugate point.

14.11 Change of Origin

Let (x, y) be the coordinates of a point P with reference to Ox and Oy as coordinate axes. Referred to Ox and Oy as coordinate axes, let (h, k) be the coordinates of a point O'. Draw a line $O'X$ parallel to Ox and a line $O'Y$ parallel to Oy. Let (X, Y) be the coordinates of P with reference to $O'X$ and $O'Y$ as coordinate axes.



Obviously, we have

$$x = X + h \quad \text{and} \quad y = Y + k.$$

Thus to obtain the equation of the curve referred to the point (h, k) as origin, the coordinate axes remaining parallel to their original directions, we should put $X + h$ in place of x and $Y + k$ in place of y in the equation of the curve, where X, Y are the current coordinates in the new equation.

If in the new equation also we take x, y as the current coordinates, then in order to shift the origin to the point (h, k), we should replace x by $x + h$ and y by $y + k$ in the given equation of the curve.

14.12 Tangents at the Point (h, k) to a Curve

If we are to find the tangents at the point (h, k) to a curve, we should first shift the origin to the point (h, k) in the equation of the curve. Then the equation of the tangents at the new origin will be obtained by equating to zero the lowest degree terms in the new equation of the curve.

Example : *Show that the point* $(2, 1)$ *is a node on the curve* $(x-2)^2 = y(y-1)^2$.

Solution : Shifting the origin to the point $(2, 1)$, the equation of the curve becomes

$$\{(x+2)-2\}^2 = (y+1)\{(y+1)-1\}^2$$

i.e. $$x^2 = y^2(y+1). \qquad \text{...(1)}$$

Equating to zero the lowest degree terms in (1), the equation of the tangents at the new origin is

$$x^2 = y^2, \quad i.e. \quad y = \pm x.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(2, 1)$ on the given curve.

14.13 Position and Character of Double Points

Let $f(x, y) = 0$ be any curve and P be any point (x, y) on it. The slope of the tangent at P is equal to dy/dx and it is given by the equation

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad \text{...(1)}$$

At a multiple point of a curve, the curve has at least two tangents and accordingly dy/dx must have at least two values at a multiple point. The equation (1) is of first degree in dy/dx. It can be satisfied for more than one value of dy/dx, if and only if,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0.$$

Therefore the necessary and sufficient conditions for any point (x, y) *of the curve* $f(x, y) = 0$ *to be a multiple point are that*

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0.$$

Hence in order to find the multiple points of the curve $f(x, y) = 0$, we should simultaneously solve the equations,

$$\boldsymbol{\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, f(x, y) = 0.}$$

Differentiating (1) with respect to x again, we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) + \frac{d}{dx}\left\{\frac{\partial f}{\partial y}\frac{dy}{dx}\right\} = 0$$

or $$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) + \frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) \cdot \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dx^2} = 0$$

or $$\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \left\{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right\}\frac{dy}{dx}\right]$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \left\{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right\}\cdot\frac{dy}{dx}\right]\cdot\frac{dy}{dx} + \frac{\partial f}{\partial y}\cdot\frac{d^2 y}{dx^2} = 0$$

or $$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{dy}{dx} + \frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\cdot\frac{d^2 y}{dx^2} = 0$$

or $$\frac{\partial^2 f}{\partial x^2} + 2\cdot\frac{\partial^2 f}{\partial x\,\partial y}\cdot\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\cdot\left(\frac{dy}{dx}\right)^2 = 0,$$

since at a multiple point $\partial f/\partial y = 0$.

Therefore at the multiple point, the values of $\dfrac{dy}{dx}$ are given by the quadratic in $\dfrac{dy}{dx}$,

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 f}{\partial x^2} = 0. \qquad ...(2)$$

If $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x\,\partial y}, \dfrac{\partial^2 f}{\partial y^2}$ are not all zero, the equation (2) will be a quadratic in dy/dx and the multiple point will be a double point.

The two tangents will be real and distinct, coincident, or imaginary according as

$$4\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - 4\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} >, = \text{ or} < 0$$

i.e., in general, the double point will be a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 >, = \textbf{ or} < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$ **(Meerut 2003)**

If $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y^2} = 0$, then the point (x, y) will be a multiple point of order higher than the second.

14.14 Nature of a Cusp at the Origin

Suppose the origin is a cusp. Then the curve will have two coincident tangents at the origin. Therefore the equation of the curve must be of the form

$$(ax + by)^2 + \text{ terms of third and higher degrees} = 0. \qquad ...(1)$$

The common tangent at the origin to the two branches of the curve is

$$ax + by = 0. \qquad ...(2)$$

Let P be the perpendicular to (2) from any point (x, y) on (1) in the neighbourhood of the origin. Then

$$P = \frac{ax + by}{\sqrt{(a^2 + b^2)}},$$ which is proportional to $ax + by$. Let us put

$$p = ax + by. \qquad ...(3)$$

Eliminate x or y (whichever is convenient) between (1) and (3). Suppose we eliminate y. Then we shall get a relation between p and x. Since p is small and also there are only two branches of the curve (2) through the origin, therefore terms involving powers of p above the second will be neglected. Thus we shall get a quadratic in p of the form

$$Ap^2 + Bp + C = 0, \qquad ...(4)$$

where A, B and C are some functions of x. Solving (4), we get

$$p = \{-B \pm \sqrt{(B^2 - 4AC)}\}/2A. \qquad ...(5)$$

Also if p_1, p_2 are the roots of (4), we get

$$p_1 p_2 = C/A. \qquad ...(6)$$

The following different cases arise :

(i) If for all values of x, positive or negative, provided they are numerically small, the values of p given by (5) are imaginary, the origin will be a conjugate point.

(ii) If for all numerically small values of x, positive or negative, the values of p given by (5) are real, there will be a double cusp at the origin.

(iii) If the reality of the values of p given by (5) depends on the sign of x, there will be a single cusp at the origin.

(iv) If for numerically small values of x for which p is real, the sign of $p_1 p_2$ is positive, then p_1 and p_2 will be of the same sign. Therefore the two perpendiculars lie on the same side of the common tangent and there will be a cusp of the second species. If, on the other hand, the sign of $p_1 p_2$ is negative, then p_1 and p_2 are of opposite signs. Therefore the two perpendiculars lie on opposite sides of the common tangent and there will be a cusp of the first species.

Note : While investigating the sign of an expression for sufficiently small values of x, we should keep in mind only those terms which involve the lowest power of x.

14.15 Nature of a Cusp at any Point

If there is a cusp at the point (h, k), we should first shift the origin to (h, k) and then apply the methods given in article 14.14.

Illustrative Examples

Example 1 : *Examine the nature of the origin on the curve*

$$(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0.$$

Solution : The tangents at the origin are $(2x + y)^2 = 0$. Thus there are two coincident tangents at the origin. Therefore the origin may be a cusp or a conjugate point.

Let $p = 2x + y$.

Putting $y = p - 2x$ in the equation of the curve, we get

$$p^2 - 6xp(p - 2x) - 7x^3 = 0$$

or $$p^2(1 - 6x) + 12x^2 p - 7x^3 = 0. \qquad ...(1)$$

Let p_1, p_2 be the roots of (1). Then

$$p = \frac{-12x^2 \pm \sqrt{\{144x^4 + 28x^3(1 - 6x)\}}}{2(1 - 6x)}$$

i.e., $$p = \frac{-6x^2 \pm \sqrt{(7x^3 - 6x^4)}}{(1 - 6x)}, \qquad \text{...(2)}$$

and $$p_1 p_2 = -\frac{7x^3}{1 - 6x}. \qquad \text{...(3)}$$

From (2), we see that for sufficiently small positive values of x, p is real and for numerically small negative values of x, p is imaginary. Therefore, there is a single cusp at the origin.

Also when x is + ive and very small, then from (3) we notice that $p_1 p_2$ is –ive. Therefore p_1 and p_2 are of opposite signs. Hence there is a single cusp of the first species at the origin.

Example 2 : *Determine the existence and nature of the double points on the curve*

$$y^2 = (x - 2)^2 (x - 1).$$ **(Meerut 2003)**

Solution : The equation of the given curve is

$$f(x, y) \equiv y^2 - (x - 2)^2 (x - 1) = 0. \qquad \text{...(1)}$$

We have $\partial f/\partial x = -2(x - 2)(x - 1) - (x - 2)^2$

$$= -(x - 2)\{2(x - 1) + (x - 2)\} = -(x - 2)(3x - 4)$$

and $\partial f/\partial y = 2y$.

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x, y) = 0$.

Here $\partial f/\partial x = 0$ gives $(x - 2)(3x - 4) = 0$ *i.e.,* $x = 2, 4/3$

and $\partial f/\partial y = 0$ gives $y = 0$.

$\therefore$ the possible double points are $(2, 0), (4/3, 0)$.

Out of these only $(2, 0)$ satisfies the equation of the curve. Therefore $(2, 0)$ is the only double point on the given curve.

Nature of the double point at (2, 0) : Shifting the origin to the point $(2, 0)$, the equation of the curve becomes

$$y^2 = (x + 2 - 2)^2 (x + 2 - 1) \quad i.e., \quad y^2 = x^2 (x + 1) \qquad \text{...(2)}$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are $y^2 - x^2 = 0$ *i.e.,* $y^2 = x^2$ *i.e.,* $y = \pm x$.

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(2, 0)$ on the given curve.

Example 3 : *Examine the nature of the double points of the curve*

$$2(x^3 + y^3) - 3(3x^2 + y^2) + 12x = 4.$$

Solution : The equation of the given curve is

$$f(x, y) \equiv 2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0. \qquad \text{...(1)}$$

We have $\dfrac{\partial f}{\partial x} = 6x^2 - 18x + 12$ and $\dfrac{\partial f}{\partial y} = 6y^2 - 6y$.

For the double points, $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ and $f(x, y) = 0$.

Here $\dfrac{\partial f}{\partial x} = 0$ gives $6x^2 - 18x + 12 = 0$

i.e., $x^2 - 3x + 2 = 0$ *i.e.,* $(x - 1)(x - 2) = 0$ *i.e.,* $x = 1, 2$

and $\qquad \dfrac{\partial f}{\partial y} = 0$ gives $6y^2 - 6y = 0$ *i.e.,* $y(y-1) = 0$ *i.e.,* $y = 0, 1$.

$\therefore$ the possible double points are $(1, 0)$, $(1, 1)$, $(2, 0)$ and $(2, 1)$.

Out of these only $(1, 1)$ and $(2, 0)$ satisfy the equation of the curve. Therefore $(1, 1)$ and $(2, 0)$ are the only double points on the given curve.

Now $\qquad \dfrac{\partial^2 f}{\partial x^2} = 12x - 18, \dfrac{\partial^2 f}{\partial x \partial y} = 0, \dfrac{\partial^2 f}{\partial y^2} = 12y - 6.$

At the point $(1, 1)$, $\dfrac{\partial^2 f}{\partial x^2} = -6, \dfrac{\partial^2 f}{\partial x \partial y} = 0, \dfrac{\partial^2 f}{\partial y^2} = 6.$

$\therefore$ at the point $(1, 1)$, $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = -36.$

Thus at the point $(1, 1)$, $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}.$

Therefore there is a node at the point $(1, 1)$.

At the point $(2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = 6, \dfrac{\partial^2 f}{\partial x \partial y} = 0, \dfrac{\partial^2 f}{\partial y^2} = -6.$

$\therefore$ at the point $(2, 0)$, $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}.$

Thus there is a node at the point $(2, 0)$.

Example 4 : *Find the nature of the origin on the curve* $a^4 y^2 = x^4 (x^2 - a^2)$.

(Meerut 2006B)

Solution : The given curve is $a^4 y^2 = x^4 (x^2 - a^2)$. ...(1)

Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $a^4 y^2 = 0$ *i.e.,* $y = 0, y = 0$ are two real and coincident tangents at the origin.

Thus the origin may be a cusp or a conjugate point.

From (1), $\qquad y = \pm (x^2/a^2) \sqrt{(x^2 - a^2)}$.

For small values of $x \neq 0$, + ive or –ive, $(x^2 - a^2)$ is –ive *i.e.,* y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin. Hence origin is a conjugate point and not a cusp.

Example 5 : *Show that the origin is a conjugate point on the curve*

$$x^4 - a x^2 y + a x y^2 + a^2 y^2 = 0.$$

Solution : Equating to zero, the lowest degree terms in the given curve, the tangents at the origin are given by

$$a^2 y^2 = 0 \quad i.e., \; y^2 = 0 \quad i.e., \; y = 0, y = 0.$$

Thus there are two real and coincident tangents at the origin

$\therefore$ origin is either a cusp or a conjugate point.

Now the equation of the given curve is

$$ay^2 (x + a) - ax^2 y + x^4 = 0.$$

Solving it for y, we have

$$y = \frac{a x^2 \pm \sqrt{\{a^2 x^4 - 4ax^4 (x + a)\}}}{2a(x + a)} = \frac{a x^2 \pm x^2 \sqrt{(-4a x - 3a^2)}}{2a(x + a)}.$$

Now for small values of $x \neq 0$, $(-4ax - 3a^2)$ is $-$ ive. Thus y is imaginary in the neighbourhood of origin.

Hence origin is a conjugate point.

Comprehensive Exercise 2

1. Write down the equations to the tangents at the origin for the following curves :

(i) $y^2(a-x) = x^2(a+x)$, (ii) $x^4 + 3x^3y + 2xy - y^2 = 0$,

(iii) $(x^2+y^2)(2a-x) = b^2x$.

2. For the curve $y^2(a^2+x^2) = x^2(a^2-x^2)$, show that the origin is a node.

3. Show that the origin is a conjugate point on the curve $y^2 = 2x^2y + x^4y - 2x^4$.

4. Show that the curve $x^3 + x^2y = ay^2$ has a cusp at the origin.

5. Find the position and nature of double points of the following curves :

(i) $y^3 = x^3 + ax^2$. **(Meerut 2013B)**

(ii) $y^2 + 3ax^2 + x^3 = 0$

(iii) $x^3 + y^3 = 3axy$. **(Agra 2006; Rohilkhand 07; Kumaun 08; Meerut 12B, 13)**

(iv) $x^3 + y^3 = 3xy$. **(Meerut 2001, 05; Agra 14)**

(v) $a^4y^2 = x^4(2x^2 - 3a^2)$. **(Meerut 2007B)**

(vi) $x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0$.

(vii) $x^4 + y^3 + 2x^2 + 3y^2 = 0$. **(Bundelkhand 2001; Meerut 07; Avadh 13)**

6. Show that the curve $y^2 = bx\tan(x/a)$ has a node or a conjugate point at the origin according as a and b have like on unlike signs.

7. Prove that the curve $ay^2 = (x-a)^2(x-b)$ has at $x = a$, a conjugate point if $a < b$, a node if $a > b$, and a cusp if $a = b$.

8. Show that the curve $y^3 = (x-a)^2(2x-a)$ has a single cusp of the first species at the point $(a, 0)$.

9. Examine the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ for singular points and show that it has a cusp of the first kind at the point $(-1, -2)$.

10. Determine the position and character of the double points on :

(i) $y(y-6) = x^2(x-2)^3 - 9$. **(Rohilkhand 2008, 09)**

(ii) $y(y-1)^2 = (x-2)^2$. **(Meerut 2000, 02; Gorakhpur 05; Rohilkhand 12)**

(iii) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$.

(iv) $y^2 - x(x-a)^2 = 0, (a > 0)$.

(v) $y^2 - x^3 = 0$.

(vi) $a^4y^2 = x^4(a^2 - x^2)$.

(vii) $y^2 = x^2(9 - x^2)$.

11. Find the position and nature of the double points on the curve

$$x^2y^2 = (a+y)^2(b^2 - y^2) \text{ if}$$

(i) $b > a$, (ii) $b = a$, (iii) $b < a$.

12. Discuss the nature of double points of the curve $(x+y)^3-\sqrt{2}(x-y+2)^2=0$.

13. Show that the curve $(xy+1)^2+(x-1)^3(x-2)=0$ has a single cusp of the first species at the point $(1,-1)$.

Answers 2

1. (i) $y=\pm x$, (ii) $y=0, y=2x$, (iii) $x=0$.

5. (i) A cusp at the origin. (ii) A conjugate point at the origin.
(iii) A node at the origin. (iv) A node at the origin.
(v) A conjugate point at the origin.
(vi) Nodes at the points $(0,-1)$, $(1,0)$ and $(-1,0)$.
(vii) A conjugate point at the origin.

10. (i) $(0,3)$ is a conjugate point and at $(2,3)$ there is a single cusp of the first species.
(ii) $(2,1)$ is a node.
(iii) Node at $(3,2)$.
(iv) Node at $(a,0)$.
(v) Single cusp of the first kind at $(0,0)$.
(vi) Double cusp of the first species at $(0,0)$.
(vii) Node at $(0,0)$.

11. (i) When $b>a$, the point $(0,-a)$ is a node.
(ii) When $b=a$, the point $(0,-a)$ is a single cusp of first kind.
(iii) When $b<a$, the point $(0,-a)$ is a conjugate point.

12. There is a single cusp of the first species at $(-1,1)$.

14.16 Curve Tracing (Cartesian Equations)

To find the approximate shape of a curve whose cartesian equation is given, we should adopt the following procedure :

1. Symmetry : First we should find if the curve is symmetrical about any line. In this connection the following rules are helpful :

(i) If in the equation of a curve the powers of y are all even, the curve is symmetrical about the axis of x *i.e.*, the shape of the curve above and below the axis of x is symmetrical. The obvious reason is that the equation of the curve in this case remains unchanged if we replace y by $-y$. Thus the parabola $y^2=4ax$ is symmetrical about the axis of x.

(ii) If in the equation of a curve the powers of x are all even, the curve is symmetrical about the axis of y. For example, the parabola $x^2=4by$ is symmetrical about the axis of y.

(iii) If the equation of a curve remains unchanged when x is replaced by $-x$ and y is replaced by $-y$, then the curve is symmetrical in opposite quadrants. For example, the curve $xy=c^2$ is symmetrical in opposite quadrants.

(iv) If the equation of a curve remains unchanged when x and y are interchanged, the curve is symmetrical about the line $y=x$, (*i.e.*, the straight line passing through the origin and making an angle 45° with the positive direction of the axis of x). For example, the curve $x^3+y^3=3axy$ is symmetrical about the line $y=x$.

2. Nature of the Origin on the Curve : We should see whether the curve passes through the origin or not. If the point (0, 0) satisfies the equation of the curve, it passes through the origin. In order to know the shape of a curve at any point, we should draw the tangent or tangents to the curve at that point. Therefore if the curve passes through the origin, we should find the equation to the tangents at origin by equating to zero the lowest degree terms in the equation of the curve. If there are two tangents at the origin, then the origin will be a double point on the curve. We should also observe the nature of the double point.

3. Points of intersection of the curve with the co-ordinate axes :

We should find the points where the curve cuts the co-ordinate axes. To find the points where the curve cuts the x-axis we should put $y = 0$ in the equation of the curve and solve the resulting equation for x. Similarly the points of intersection with the y-axis are obtained by putting $x = 0$ and solving the resulting equation for y. **We should also obtain the tangents to the curve at the points where it meets the co-ordinate axes**. In order to find the tangent at the point (h, k), we should shift the origin to (h, k) and then the tangent or tangents at this new origin will be obtained by equating to zero the lowest degree terms. The value of dy/dx at the point (h, k) can also be used to find the slope of the tangent at that point.

4. We should solve the equation of the curve for y or x whichever is convenient. Suppose we solve for y. Starting from $x = 0$, we should see the nature of y as x increases and then tends to $+\infty$. Similarly we should see the nature of y as x decreases and then tends to $-\infty$. **We should pay special attention to those values of x for which y = 0 or $\rightarrow$ infinity.**

If we solve the equation of the curve for y and the curve is symmetrical about y-axis, then we should consider only positive values of x. The curve for negative values of x can be drawn from symmetry and there is no necessity of considering them afresh.

However, if we solve the equation for y and there is symmetry only about x-axis, then we are to consider both positive as well as negative values of x. If the curve is symmetrical in opposite quadrants, or if there is symmetry about the x-axis, then only positive values of y need be considered.

If $y \rightarrow$ infinity as $x \rightarrow a$, then the line $x = a$ will be an asymptote of the curve. Similarly if $x \rightarrow$ infinity as $y \rightarrow b$, then the line $y = b$ will be an asymptote of the curve.

5. Regions where the curve does not exist : We should find out if there is any region of the plane such that no part of the curve lies in it. Such a region is easily obtained on solving the equation for one variable in terms of the other. The curve will not exist for those values of one variable which make the other imaginary. For example, in the curve

$$a^2 y^2 = x^2 (x - a)(2a - x),$$

we find that for $0 < x < a$, y^2 is negative, *i.e.*, y is imaginary. Therefore the curve does not exist in the region bounded by the lines $x = 0$ and $x = a$. For $a < x < 2a$, y^2 is positive *i.e.*, y is real. Therefore the curve exists in the region bounded by the lines $x = a$ and $x = 2a$. Thus if y is imaginary when x lies between a and b, the curve does not exist in the region bounded by the lines $x = a$ and $x = b$.

6. Asymptotes : We should find all the asymptotes of the curve. If an infinte branch of the curve has an asymptote, then ultimately it must be drawn parallel to the asymptote. The asymptotes parallel to the x-axis can be obtained by equating to zero the coefficient of the highest power of x in the equation of the curve. Similarly the asymptotes parallel to the y-axis can be obtained by equating to zero the coefficient of the highest power of y in the equation of the curve.

7. The sign of dy/dx : We should calculate the value of dy/dx from the equation of the curve. Then we shall find the points at which dy/dx **vanishes** or **becomes infinite**. These will give us the points where the tangent is parallel or perpendicular to the x-axis.

If in any region $a < x < b$, dy/dx remains throughout positive, then in this region y increases continuously as x increases. If in any region $a < x < b$, dy/dx remains throughout negative, then in this region y decreases continuously as x increases.

8. Special Points : If necessary, we should find the co-ordinates of a few points on the curve.

9. Points of inflexion : While drawing the curve if it appears that the curve possesses some points of inflexion, then their positions can be accurately located by putting $d^2 y/dx^2$ or $d^2 x/dy^2$ equal to zero and solving the resulting equation.

Taking all the above isolated facts into consideration, we can draw the approximate shape of the curve.

Illustrative Examples

Example 1 (a) : *Trace the curve* $ay^2 = x^3$. **(semi-cubical parabola).**

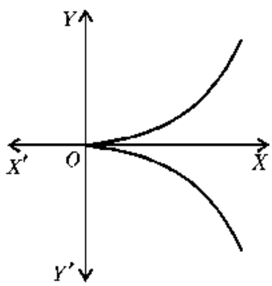
Solution : We note the following facts about this curve :

(i) Since in the equation of the curve the powers of y are all even, therefore the curve is *symmetrical about the axis of* x.

(ii) The curve passes through the origin.

(iii) Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin. Therefore the tangents at origin are

$$ay^2 = 0 \quad i.e., \quad y = 0, y = 0.$$



Thus the origin is a double point and it may be a cusp since there are two coincident tangents at the origin.

(iv) The curve does not intersect the coordinate axes anywhere except the origin.

(v) Solving the equation of the curve for y, we get $y^2 = \dfrac{x^3}{a}$.

When $x = 0, y^2 = 0$.

When $x > 0, y^2$ is positive *i.e.*, y is real. Therefore the curve exists in the region $x > 0$.

As x increases, y^2 also increases and when $x \to \infty, y^2 \to \infty$.

When $x < 0, y^2$ is negative *i.e.*, y is imaginary.

Therefore the curve does not exist in the region $x < 0$.

(vi) Obviously the curve has no asymptotes.

(vii) The curve exists in the neighbourhood of origin where $x > 0$. Also x-axis is a common tangent to the two branches of the curve passing through origin. Hence origin is a cusp.

Taking all these facts into consideration, the shape of the curve is as shown in the adjoining figure.

Example 1 (b) : *Trace the curve* $y^2 = x^3$. **(Bundelkhand 2006)**

Solution : Proceed as in part (a).

Example 2 : *Trace the curve* $y^2 (2a - x) = x^3$. **(Cissoid)**

(Meerut 2001, 11; Agra 06; Rohilkhand 06; Bundelkhand 08; Avadh 12; Kashi 12, 14)

Solution : We note the following particulars about the curve :

(i) It is symmetrical about the axis of x, since the powers of y that occur are all even.

(ii) The curve passes through the origin and the tangents at the origin are $2ay^2 = 0$ *i.e.*, $y = 0, y = 0$ are two coincident tangents at the origin. Therefore the origin may be a cusp.

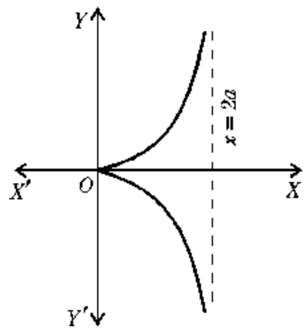
(iii) The curve meets the coordinate axes only at the origin.

(iv) Solving the equation of the curve for y, we get $y^2 = x^3/(2a - x)$.

When $x = 0, y^2 = 0$. When $x \to 2a, y^2 \to \infty$. Therefore $x = 2a$ is an asymptote of the curve.

When $0 < x < 2a, y^2$ is positive *i.e.*, y is real. Therefore the curve exists in this region.

When $x > 2a, y^2$ is negative *i.e.*, y is imaginary. Therefore the curve does not exist in the region $x > 2a$. When $x < 0, y^2$ is negative. Therefore the curve does not exist in the region $x < 0$. Since the curve exists in the neighbourhood of origin where $x > 0$, therefore there is a single cusp at the origin.



(v) Putting $y = m$ and $x = 1$ in the third degree terms in the equation of the curve, we get $\phi_3(m) = m^2 + 1$. The roots of the equation $m^2 + 1 = 0$ are imaginary, therefore $x = 2a$ is the only real asymptote of the curve.

(vi) For the branch of the curve lying above the x-axis, we have $y = \dfrac{x^{3/2}}{\sqrt{(2a - x)}}$.

$\therefore \dfrac{dy}{dx} = \dfrac{(3a - x)\sqrt{x}}{(2a - x)^{2/3}}$, which vanishes when $x = 0$, or $3a$.

But $x = 3a$ is outside the range of admissible values of x. Therefore dy/dx vanishes at no admissible value of x except $x = 0$.

When $0 < x < 2a$, dy/dx is positive. Therefore in this region y increases continuously as x increases.

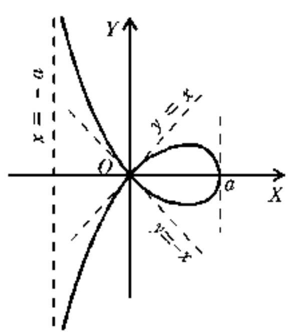
Combining all these facts, we see that the shape of the curve is as shown in the adjoining figure.

Example 3 (a) : *Trace the curve* $y^2(a + x) = x^2(a - x)$.

(Meerut 2000, 13; Kanpur 10; Bundelkhand 11; Avadh 13)

Solution : (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. The tangents at origin are $a(y^2 - x^2) = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve.



(iii) The curve intersects the x-axis where $y = 0$ *i.e.*, $x^2(a - x) = 0$.

Therefore the curve intersects the x-axis at $(0, 0), (a, 0)$.

The curve intersects the y-axis only at origin.

(iv) **Tangent at (a, 0).** Shifting the origin to $(a, 0)$ the equation of the curve becomes

$$y^2 (2a + x) = (x + a)^2 \{a - (x + a)\}$$

or
$$y^2 (2a + x) = - x (x^2 + 2ax + a^2).$$

Equating to zero the lowest degree terms, we get $x = 0$ (*i.e.*, the new y-axis) as the tangent at the new origin. Thus the tangent at $(a, 0)$ is perpendicular to x-axis.

(v) Solving the equation of the curve for y, we get

$$y^2 = x^2 (a - x)/(x + a).$$

When $x = 0, y^2 = 0$ and when $x = a$, $y^2 = 0$.

When $0 < x < a, y^2$ is positive. Therefore the curve exists in this region.

When $x > a, y^2$ is negative. Therefore the curve does not exist in the region $x > a$.

When $x \to - a, y^2 \to \infty$. Therefore $x = - a$ is an asymptote of the curve.

When $- a < x < 0, y^2$ is positive. Therefore the curve exists in this region.

When $x < - a, y^2$ is negative. Therefore the curve does not exist in the region $x < - a$.

(vi) Putting $y = m$ and $x = 1$ in the highest *i.e.*, third degree terms in the equation of the curve, we get $\phi_3 (m) = m^2 + 1$. The roots of the equation $\phi_3 (m) = 0$ are imaginary. Therefore $x = - a$ is the only real asymptote of the curve.

(vii) For the portion of the curve lying in the first quadrant, we have

$$y = x \sqrt{\left\{\frac{(a - x)}{(a + x)}\right\}} = x \frac{(1 - x/a)^{1/2}}{(1 + x/a)^{1/2}}.$$

When $0 < x < a, y$ is less than x. Therefore the curve lies below the line $y = x$ which is tangent at the origin.

For the portion of the curve lying in the second quadrant, we have

$$y = - x \frac{(1 - x/a)^{1/2}}{(1 + x/a)^{1/2}}, x < 0.$$

When $- a < x < 0, y$ is greater than the numerical value of x. Therefore the curve lies above the tangent $y = - x$.

Hence the shape of the curve is as shown in the figure.

Example 3 : (b) *Trace the curve* $y^2 (a + x) = x^2 (3a - x)$. **(Purvanchal 2011)**

Solution : Proceed as in part (a).

Example 4 : *Trace the curve* $y^2 (x^2 + y^2) + a^2 (x^2 - y^2) = 0$.

Solution : (i) The curve is symmetrical about both the axes.

(ii) It passes through the origin and $a^2 (x^2 - y^2) = 0$ *i.e.*, $y = \pm x$ are the two tangents at the origin. Therefore the origin is a node.

(iii) The curve intersects the x-axis only at origin. It intersects the y-axis at $(0, 0)$, $(0, a)$ and $(0, - a)$.

(iv) Shifting the origin to $(0, a)$, the equation of the curve becomes

$$(y + a)^2 \{x^2 + (y + a)^2\} + a^2 \{x^2 - (y + a)^2\} = 0$$

or
$$(y^2 + 2ay + a^2) \{x^2 + y^2 + 2ay + a^2\} + a^2 (x^2 - y^2 - 2ay - a^2) = 0.$$

Equating to zero the lowest degree terms, we get

$$2a^3 y + 2a^3 y - 2a^3 y = 0 \text{ } i.e., y = 0$$

as the tangent at the new origin. Thus the new x-axis is tangent at the new origin.

We need not find the tangent at $(0, - a)$ as the curve is symmetrical about x-axis.

(v) Solving the equation of the curve for x, we get

$$x^2 = y^2 (a^2 - y^2)/(a^2 + y^2).$$

When $y = 0, x^2 = 0$ and when $y = a, x^2 = 0$.

When $0 < y < a, x^2$ is positive. Therefore the curve exists in the region $0 < y < a$.

When $y > a, x^2$ is negative. Therefore the curve does not exist in the region $y > a$.

We need not consider the negative values of y as the curve is symmetrical about x-axis.

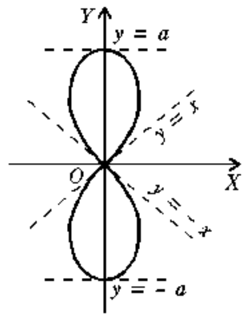
(vi) The asymptotes parallel to x-axis are given by $a^2 + y^2 = 0$ *i.e.*, $y = \pm ia$. Also $\phi_4 (m) = m^2 (1 + m^2)$. Its roots are $m = 0, 0, i, -i$. The asymptotes corresponding to $m = 0$ are imaginary. Hence all the four asymptotes are imaginary.

(vii) In the positive quadrant, we have

$$x = y (a^2 - y^2)^{1/2}/(a^2 + y^2)^{1/2}, \; y > 0$$

or

$$x = y \frac{\left(1 - \dfrac{y^2}{a^2}\right)^{1/2}}{\left(1 + \dfrac{y^2}{a^2}\right)^{1/2}}.$$



When $0 < y < a$, we see that x is less than y. Therefore the curve lies above the line $y = x$ which is tangent at the origin.

Combining all these facts, we see that the shape of the curve is as shown in the adjoining figure.

Example 5 : *Trace the curve* $x^2 (x^2 - 4a^2) = y^2 (x^2 - a^2)$.

Solution : (i) Symmetry about both the axes.

(ii) The curve passes through the origin and $a^2 y^2 - 4a^2 x^2 = 0$ *i.e.*, $y = \pm 2x$ are the tangents at the origin. Therefore origin is a node on the curve.

(iii) The curve cuts the x-axis at $(0, 0), (2a, 0), (-2a, 0)$. It cuts the y-axis only at the origin.

(iv) Shifting the origin to $(2a, 0)$, the equation of the curve becomes

$$(x + 2a)^2 (x^2 + 4ax) = y^2 (x^2 + 4ax + 3a^2).$$

The equation to the tangent at the new origin is $16a^3 x = 0$ *i.e.*, $x = 0$. Thus the new y-axis is tangent at the new origin.

(v) Solving the equation of the curve for y, we get $y^2 = \dfrac{x^2 (x^2 - 4a^2)}{(x^2 - a^2)}$.

When $x = 0, y^2 = 0$.

When $x \to a, y^2 \to \infty$ *i.e.*, $x = a$ is an asymptote of the curve.

When $0 < x < a, y^2$ is positive *i.e.*, the curve exists in this region.

When $x = 2a, y^2 = 0$.

When $a < x < 2a, y^2$ is negative *i.e.*, the curve does not exist in this region.

When $x > 2a, y^2$ is positive *i.e.*, the curve exists in this region.

When $x \to \infty, y^2 \to \infty$. We need not consider the negative values of x as the curve is symmetrical about the y-axis.

(vi) The asymptotes of the curve parallel to y-axis are given by $x^2 - a^2 = 0$. Thus $x = \pm a$ are two asymptotes of the curve.

Also the equation of the curve can be written as

$$x^2(y^2 - x^2) - a^2y^2 + 4a^2x^2 = 0.$$

$\therefore$ $\phi_4(m) \equiv m^2 - 1 = 0$ gives $m = \pm 1$.

Also $\phi_3(m) = 0$.

For $m = \pm 1$, c is given by $c\,\phi'_4(m) + \phi_3(m) = 0$.

When $m = 1, c = 0$. Also when $m = -1, c = 0$.

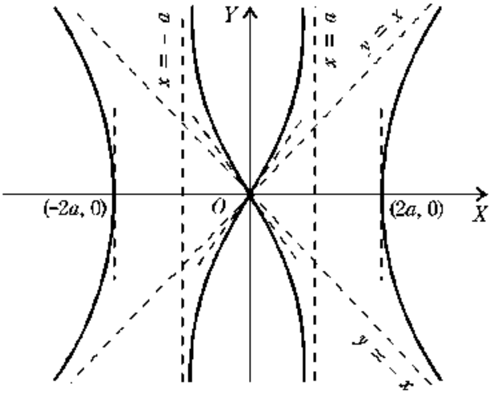
Therefore $y = \pm x$ are two oblique asymptotes of the curve.

(vii) In the positive quadrant, we have

$$y^2 = \frac{x^2(4a^2 - x^2)}{(a^2 - x^2)}, 0 < x < a$$

or
$$y = 2x\left(1 - \frac{x^2}{4a^2}\right)^{1/2} \Big/ \left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$

When $0 < x < a$, y is greater than $2x$. Therefore the curve lies above the line $y = 2x$ which is tangent at the origin.



Combining all these facts we see that the shape of the curve is as shown in the above figure.

Example 6 : *Trace the curve* $x^3 + y^3 = 3axy$. **(Folium of Descartes)**

(Meerut 2007, 08, 10B, 13B; Rohilkhand 08; Purvanchal 07)

Solution : (i) The curve is symmetrical about the line $y = x$, since its equation remains unchanged on interchanging x and y.

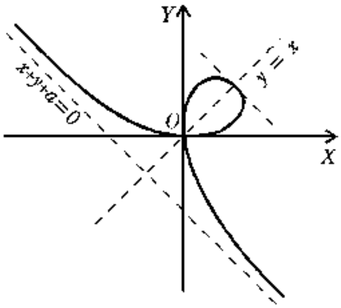
(ii) The curve passes through the origin and the tangents at origin are $3axy = 0$ *i.e.*, $x = 0, y = 0$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve.

(iii) The curve intersects the coordinate axes only at $(0, 0)$.

(iv) From the equation of the curve we see that x and y cannot be both negative because then the left hand side of the equation of the curve becomes negative while the right hand side becomes positive. Therefore the curve does not exist in the third quadrant.

(v) The curve meets the line $y = x$ at the point $(3a/2, 3a/2)$. From the equation of the curve, we have

$$\frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}.$$



At $\left(\frac{3a}{2}, \frac{3a}{2}\right), \frac{dy}{dx} = -1$. Therefore the tangent at this point makes an angle of 135° with the positive direction of x-axis.

(vi) **Asymptotes :** $\phi_3(m) = m^3 + 1$.

The only real root of the equation $\phi_3(m) = 0$

i.e., $m^3 + 1 = 0$, is $m = -1$.

Also $\phi_2(m) = -3am$.

For $m = -1$, c is given by $c(3m^2) - 3am = 0$.

$\therefore$ when $m = -1, c = -a$.

Hence $y = -x - a$ is the only real asymptote of the curve.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Example 7 : *Trace the curve* $y^3 + x^3 = a^2 x$.

(Meerut 2006, 10; Kanpur 08; Kashi 13)

Solution : (i) If we change the signs of x and y both, the equation of the curve does not change. Therefore the curve is symmetrical in opposite quadrants.

(ii) The curve passes through the origin and the tangent at origin is $x = 0$ *i.e.,* y-axis.

(iii) The curve cuts the x-axis where $y = 0$ *i.e.,* $x(x^2 - a^2) = 0$. Thus the curve cuts the x-axis at $(0, 0), (a, 0), (-a, 0)$.

The curve intersects the y-axis only at the origin.

(iv) From the equation of the curve, we have $\frac{dy}{dx} = \frac{a^2 - 3x^2}{3y^2}$.

At $(a, 0), \frac{dy}{dx} = \infty$ *i.e.,* the tangent is perpendicular to x-axis.

Also at $(-a, 0), \frac{dy}{dx} = -\infty$ *i.e.,* the tangent is perpendicular to x-axis.

(v) $\frac{dy}{dx} = 0$ at $x = \pm \frac{a}{\sqrt{3}}$. Therefore the tangents at these points are parallel to the x-axis.

(vi) Solving the equation of the curve for y, we get $y^3 = x(a^2 - x^2)$.

When $x = 0, y^3 = 0$ and when $x = a, y^3 = 0$.

When $0 < x < a$, y^3 is positive *i.e.,* y is positive in this region.

When $x > a$, y^3 is negative *i.e.,* y is negative in this region.

When $x \to \infty, y^3 \to -\infty$ *i.e.,* $y \to -\infty$.

We need not consider the negative values of x as there is symmetry in opposite quadrants.

Asymptotes : $\phi_3(m) = m^3 + 1, \phi_2(m) = 0$.

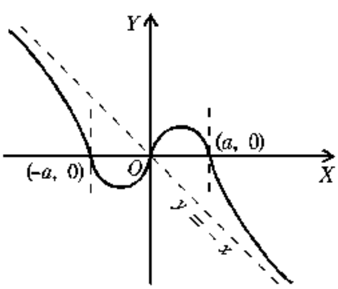
The only real root of $m^3 + 1 = 0$ is $m = -1$.

Also c is given by $c(3m^2) + 0 = 0$.

When $m = -1, c = 0$.

Hence $y = -x$ is the only real asymptote of the curve.

Combining all these facts the shape of the curve is as shown in the figure.



Comprehensive Exercise 3

Trace the following curves :

1. $x^3 y = x + 1$.

2. $x = (y - 1)(y - 2)(y - 3)$. **(Kanpur 2009; Purvanchal 06)**

3. $y = x(x^2 - 1)$.

4. $y^2 = 4ax$. **(parabola)**

5. $xy^2 = 4a^2 (2a - x)$. **(Witch of Agnesi)**

6. $x^2 y^2 = a^2 (x^2 + y^2)$. **(Gorakhpur 2006)**

7. $y(x^2 - 1) = (x^2 + 1)$. **(Bundelkhand 2001; Kashi 11)**

8. $y(x^2 + 4a^2) = 8a^3$. **(Agra 2008)**

9. $y^2 (1 - x^2) = x^2 (1 + x^2)$. **(Meerut 2007B; Bundelkhand 07, 10)**

10. $a^2 y^2 = x^2 (a^2 - x^2)$.

11. $a^2 y^2 = x^3 (2a - x)$.

12. $y^2 (a^2 + x^2) = x^2 (a^2 - x^2)$. **(Meerut 2001, 03, 12; Kumaun 08)**

13. $y^2 x = a^2 (x - a)$.

14. $9ay^2 = x(x - 3a)^2$.

15. $y^2 (x + a) = (x - a)^3$. **(Meerut 2004, 06B)**

16. $x^2 y^2 = (1 + y)^2 (4 - y^2)$.

17. $y^2 (x + 3a) = x(x - a)(x - 2a)$. **(Meerut 2005B)**

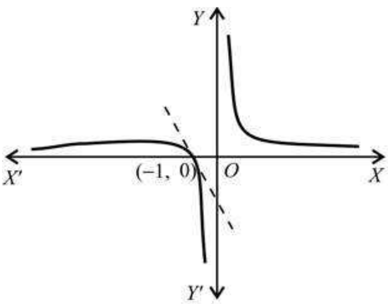
18. $a^3 y^2 = (x - a)^4 (x - b), a > b$.

19. $y^2 (x^2 - 1) = x$.

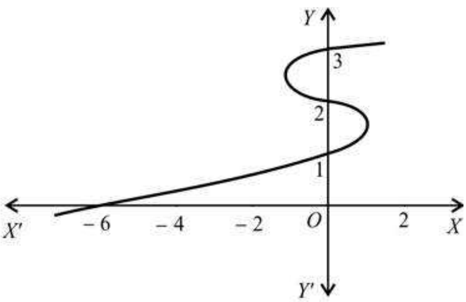
20. $x(x - 2a) y^2 = a^2 (x - a)(x - 3a)$.

21. $y^2 = (x - a)(x - b)(x - c), a > b > c$.

Answers 3



Ex. 1.



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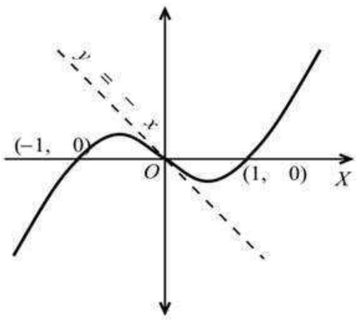

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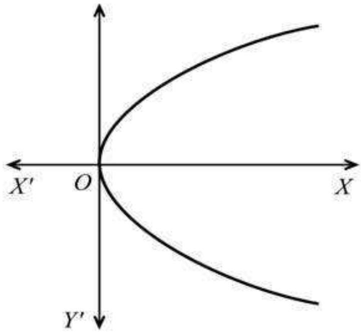

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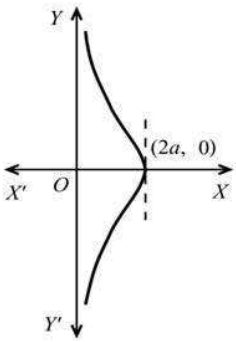

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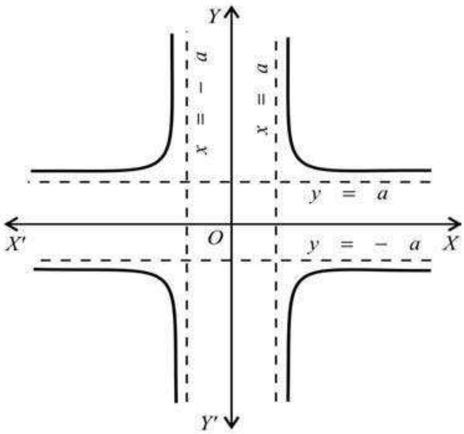

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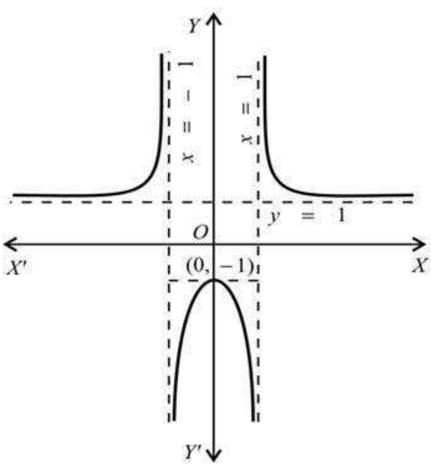

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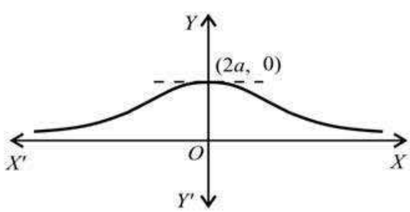

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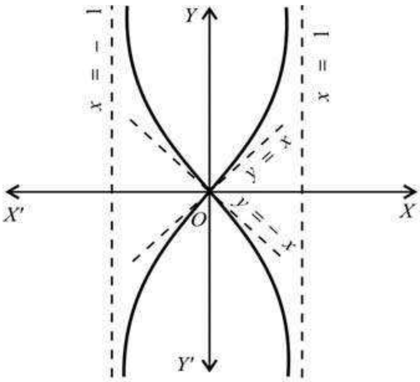

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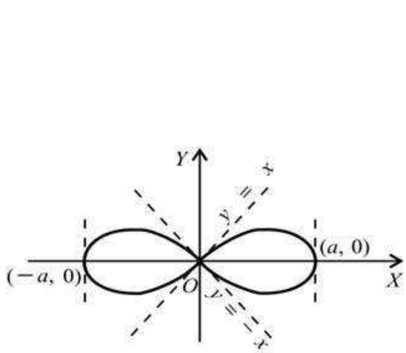

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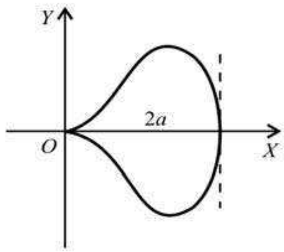

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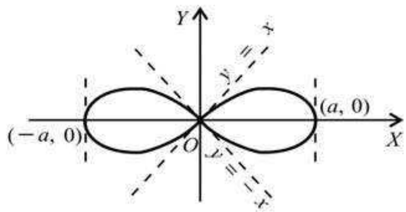

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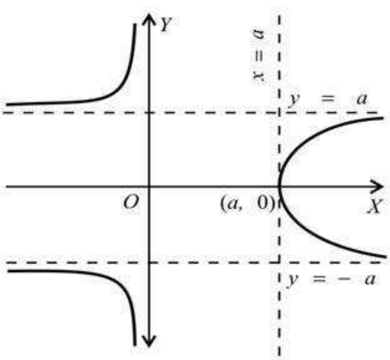

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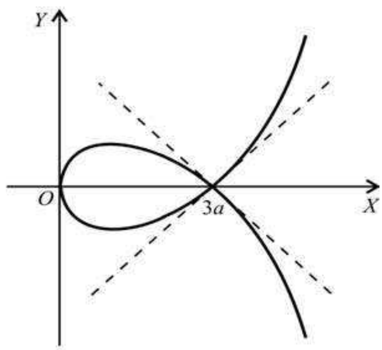

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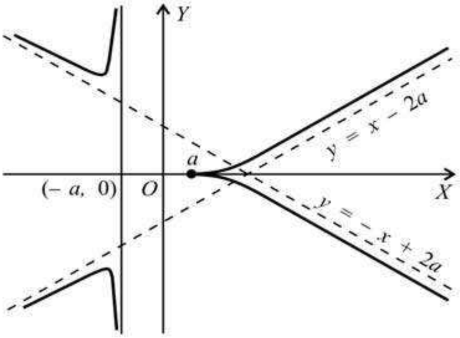

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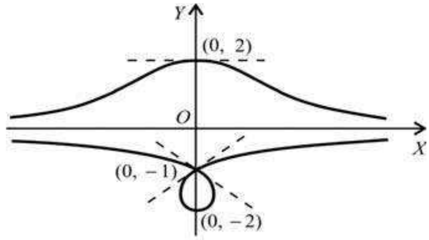

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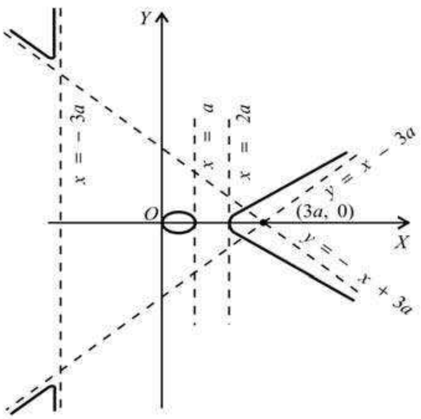

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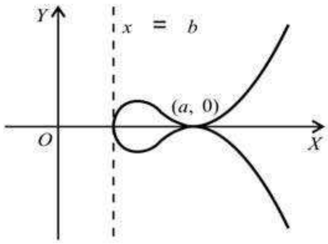

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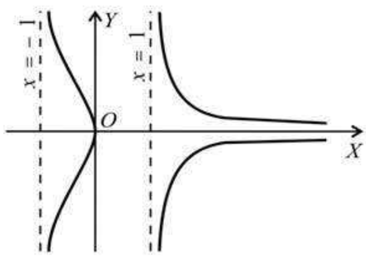

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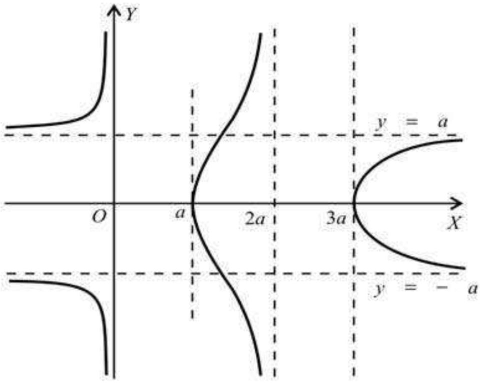

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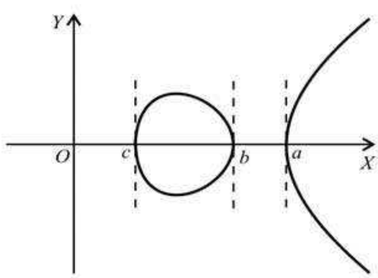


Ex. 21.

14.17 Polar Equations : Procedure for Tracing

1. Symmetry :

(i) If the equation of the curve does not change by changing the sign of θ, then the curve is **symmetrical about the initial line.**

(ii) If the equation of the curve remains unchanged by changing r into $-r$, then the curve is symmetrical about the pole and the pole is the centre of the curve.

2. Some Special points on the Curve : The curve will pass through the pole if for some value of θ the value of r comes out to be zero. Also if $r=0$ when $\theta=\alpha$, then usually the line $\theta=\alpha$ will be a tangent to the curve at the pole.

We should find the values of θ for which $r=0$, or r is maximum, or r is minimum, or $r \to \infty$.

3. Solve the equation of the curve for r and consider how r varies as θ increases from 0 to $+\infty$, and also as θ decreases from 0 to $-\infty$. We should pay special attention to the values of θ found in the paragraph 2.

We should form a table of corresponding values of r and θ which would give us a number of points on the curve. Plotting these points we shall find the shape of the curve.

In the polar equations in which only periodic functions ($\sin\theta$, $\cos\theta$, $\tan\theta$ etc.) occur, the values of θ from 0 to 2π (or sometimes some multiple or sub-multiple of 2π) need be considered, as the remaining values of θ do not give any new branch of the curve.

4. Regions where the curve does not exist : If r is imaginary when $\alpha<\theta<\beta$, then the curve does not exist in the region bounded by the lines $\theta=\alpha$ and $\theta=\beta$.

5. Asymptotes : Find the asymptotes if the curve possesses an infinite branch. If $r\to\infty$ as $\theta\to\alpha$, we should not assume that $\theta=\alpha$ is an asymptote. The asymptote might be parallel to the line $\theta=\alpha$ or even might not exist at all. The asymptotes should be found by the method given in the chapter on Asymptotes.

6. Find $\tan\phi$ *i.e.*, $r\,d\theta/dr$ which will indicate the direction of the tangent at any point. If for $\theta=\alpha$, ϕ comes out to be zero, then the line $\theta=\alpha$ will be a tangent to the curve at the point $\theta=\alpha$. If for $\theta=\alpha$, ϕ comes out to be $\pi/2$, then at the point $\theta=\alpha$, the tangent will be perpendicular to the radius vector $\theta=\alpha$.

7. Important : It is sometimes convenient to change the equation from the polar form to the cartesian form. Remember that the relations between the cartesian and polar coordinates are $x=r\cos\theta$, $y=r\sin\theta$.

Illustrative Examples

Example 1 : *Trace the curve* $r=a(1+\cos\theta)$. **(Cardioid)**

(Meerut 2009B; Bundelkhand 05; Rohilkhand 07)

Solution : (i) The curve is symmetrical about the initial line since its equation remains unchanged by writing $-\theta$ in place of θ.

(ii) $r=0$, when $\cos\theta=-1$ *i.e.*, $\theta=\pi$,
r is maximum when $\cos\theta=1$, *i.e.*, $\theta=0$. Then $r=2a$.

Also r is minimum when $\cos\theta=-1$ *i.e.*, $\theta=\pi$. Then $r=0$.

(iii) $\dfrac{dr}{d\theta}=-a\sin\theta$.

When $0<\theta<\pi$, $(dr/d\theta)$ is throughout negative.

Therefore r decreases continuously as θ increases from 0 to π.

(iv) Also $\tan\phi = r\dfrac{d\theta}{dr} = -\dfrac{a(1+\cos\theta)}{a\sin\theta} = -\cot\dfrac{\theta}{2}$.

$\phi = 0$ when $\theta = \pi$. Then $r = 0$.

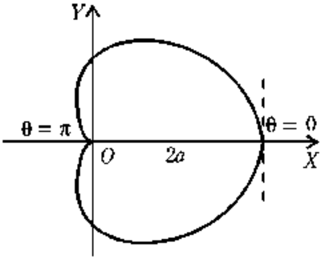
Therefore the line $\theta = \pi$ is tangent to the curve at the pole.

$\phi = 90^{o}$ when $\theta = 0$. Then $r = 2a$. Therefore the tangent at $\theta = 0$ is perpendicular to the radius vector $\theta = 0$.

(v) Since r is never greater than $2a$, therefore the curve will have no asymptotes.

(vi) The following table gives the corresponding values of θ and r.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	$2a$	$\frac{3}{2}a$	a	$\frac{a}{2}$	0



The portion of the curve lying in the region $\pi < \theta < 2\pi$ can be drawn by symmetry. Hence the shape of the curve is as shown in the figure.

Example 2 : *Trace the curve* $r = a\cos 2\theta$.

Solution : (i) The curve is symmetrical about the initial line.

(ii) $r = 0$, when $\cos 2\theta = 0$, *i.e.*, $2\theta = \pm\,\pi/2$ *i.e.*, $\theta = \pm\,\pi/4$.

Therefore the lines $\theta = \pm\,\pi/4$ are tangents to the curve at the pole.

r is maximum when $\cos 2\theta = 1$. Then $\theta = 0$ and $r = a$.

(iii) $$\tan\phi = r\frac{d\theta}{dr} = a\cos 2\theta\cdot\frac{1}{-2a\sin 2\theta}$$
$$= -\frac{1}{2}\cot 2\theta.$$

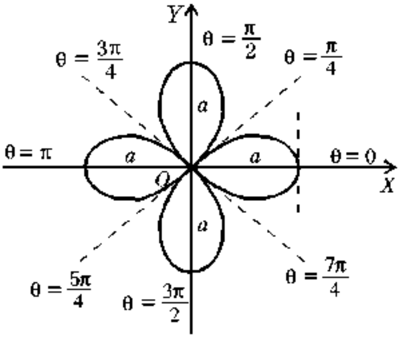
$\phi = 90^{o}$ when $2\theta = 0$ *i.e.*, $\theta = 0$. Therefore at the point $\theta = 0$, the tangent is perpendicular to the radius vector $\theta = 0$.

(iv) The following table gives the corresponding values of θ and r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5}{6}\pi$	π
r	a	$\frac{1}{2}a$	0	$-\frac{1}{2}a$	$-a$	$-\frac{1}{2}a$	0	$\frac{1}{2}a$	a

The variation of θ from π to 2π need not be considered because of symmetry about the initial line.

Hence the curve is as shown in the figure. The curve consists of four similar loops, all lying within a circle of radius a and centre at the pole.



Important : The above curve is a particular case of the curves of the type $r = a\cos n\theta$ which have n loops when n is odd and $2n$ loops when n is even.

Example 3 : *Trace the curve* $r = a \sin 3\theta$. **(Meerut 2003; Rohilkhand 12)**

Solution : (i) The curve is not symmetrical about the initial line.

(ii) $r = 0$ when $\sin 3\theta = 0$ *i.e.*, $3\theta = 0, \pi$, *i.e.*, $\theta = 0, \pi/3$.

Therefore the lines $\theta = 0$ and $\theta = \pi/3$ are tangents to the curve at the pole.

Also r is maximum when $\sin 3\theta = 1$ *i.e.*, $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$.

The maximum value of r is a.

(iii) $\tan \phi = r \dfrac{d\theta}{dr} = \dfrac{1}{3} \tan 3\theta$.

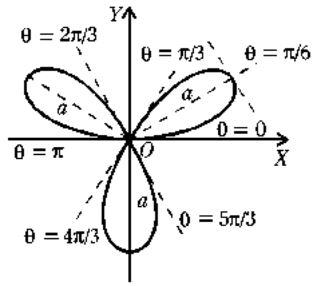
$\phi = 90^\circ$ when $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$.

Therefore at the point $\theta = \pi/6$, tangent is perpendicular to the radius vector $\theta = \pi/6$.

(iv) The following table gives the corresponding values of θ and r :

3θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π	$\frac{9\pi}{2}$	5π	$\frac{11\pi}{2}$	6π
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

Here one loop of the curve lies in the region $0 < \theta < \dfrac{\pi}{3}$, one loop lies in the region $\dfrac{\pi}{3} < \theta < \dfrac{2\pi}{3}$ and one loop lies in the region $\dfrac{2\pi}{3} < \theta < \pi$. If θ increases beyond π to 2π, the same branches of the curve are repeated and we do not get any new branch. Hence the complete curve is as shown in the adjoining figure.



Important Note : The above curve is a particular case of the curves of the type $r = a \sin n\theta$ which have n loops when n is odd and $2n$ loops when n is even.

Example 4 : *Trace the curve* $r = a + b \cos \theta$, *when* $a < b$. **(Limacon)**

Solution : (i) The curve is symmetrical about the initial line.

(ii) $r = 0$ when $a + b \cos \theta = 0$ *i.e.*, $\theta = \cos^{-1}\left(-\dfrac{a}{b}\right)$.

Since $\dfrac{a}{b} < 1$, therefore $\cos^{-1}\left(-\dfrac{a}{b}\right)$ is real.

Therefore the radius vector $\theta = \cos^{-1}\left(-\dfrac{a}{b}\right)$ is tangent to the curve at the pole.

r is maximum when $\cos \theta = 1$, *i.e.*, $\theta = 0$. Then $r = a + b$.

Also r is minimum when $\cos \theta = -1$, *i.e.*, $\theta = \pi$.

Then $r = a - b$, which is negative, $(\because a < b)$.

(iii) $\dfrac{dr}{d\theta} = -b \sin \theta$.

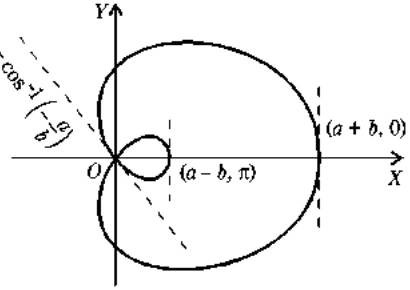
$\therefore \tan \phi = r \dfrac{d\theta}{dr} = -\dfrac{(a + b \cos \theta)}{b \sin \theta}$.

$\phi = 90^\circ$ when $\theta = 0$ and π. Therefore at the points $\theta = 0$ and $\theta = \pi$, the tangent is perpendicular to the radius vector.

(iv) The following table gives the corresponding values of r and θ.

θ	0	$\pi/2$	$\cos^{-1}\left(-\frac{a}{b}\right)$	$\cos^{-1}\left(-\frac{a}{b}\right) < \theta < \pi$	π
r	$a + b$	a	0	r is negative	$a - b$

The variation of θ from π to 2π need not be considered because of the symmetry about the initial line. Hence the curve is as shown in the adjoining figure.



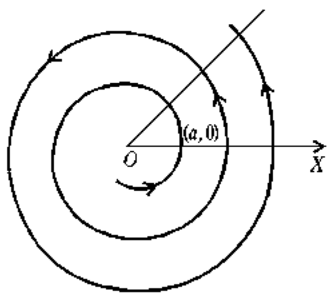
Example 5 : *Trace the curve* $r = ae^{m\theta}$. **(Equiangular Spiral)**

Solution : (i) The curve is not symmetrical about the initial line.

(ii) As $\theta \to \infty$, $r \to \infty$ and as $\theta \to -\infty$, $r \to 0$. Also r is always positive. When $\theta = 0, r = a$.

(iii) $\dfrac{dr}{d\theta} = ame^{m\theta}$.

When $-\infty < \theta < \infty$, $\dfrac{dr}{d\theta}$ is throughout positive. Therefore r increases continuously as θ increases from $-\infty$ to ∞.



(iv) $\tan\phi = r\dfrac{d\theta}{dr} = \dfrac{a\,e^{m\theta}}{am\,e^{m\theta}} = \dfrac{1}{m}$.

$\therefore \quad \phi = \tan^{-1}\left(\dfrac{1}{m}\right) = \text{constant}.$

Thus in this curve the angle between the radius vector and the tangent always remains constant.

Hence the shape of the curve is as shown in the adjoining diagram.

Comprehensive Exercise 4

Trace the following curves :

1. $r = 2a\cos\theta$. **(Circle)**
2. $r = a(1 - \cos\theta)$. **(Cardioid)** **(Meerut 2001)**
3. $r = a + b\cos\theta$, when $a > b$. **(Limacon)** **(Meerut 2000)**
4. $r^2 = a^2\cos 2\theta$. **(Lemniscate of Bernouli)** **(Meerut 2002, 08; Agra 07)**
5. $r^2 = a^2\sin 2\theta$. **(Lemniscate)**

6. $r = a \sin 2\theta$. **(Four leaved rose)** **(Bundelkhand 2009)**

7. $r = a \cos 3\theta$. **(Three leaved rose)** **(Avadh 2010; Karshi 13)**

8. $2a/r = 1 + \cos\theta$. **(Parabola)**

9. $r = \frac{1}{2} + \cos 2\theta$. **(Meerut 2004B)**

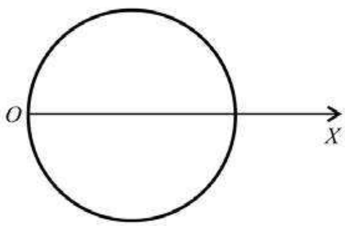
10. (i) $r = a(\sec\theta + \cos\theta)$.

[**Hint.** Changing to cartesian form, the equation becomes

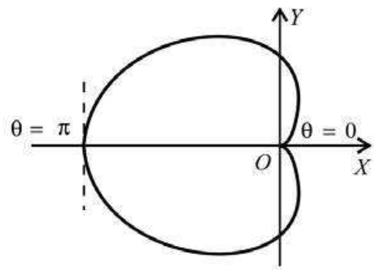
$$y^2 (x - a) = x^2 (2a - x)].$$

(ii) $r \cos\theta = 2a \sin^2\theta$. (**Cissoid.** For figure, see Example 2 after article 14.16)

Answers 4



Ex. 1.



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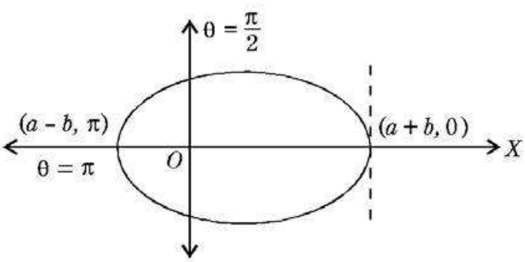

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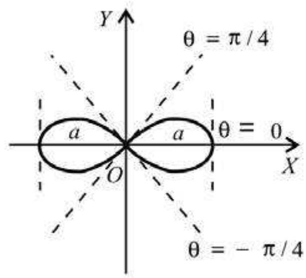

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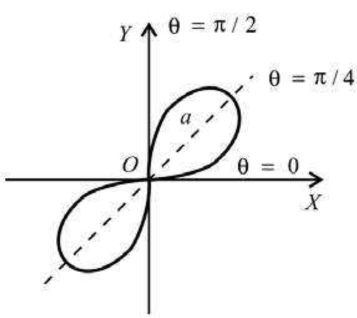

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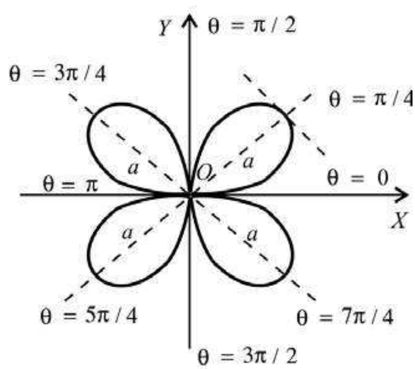

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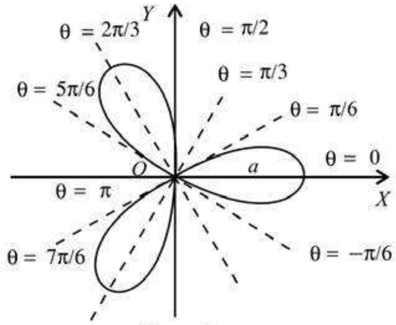

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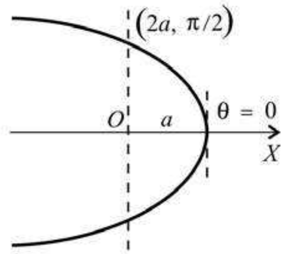

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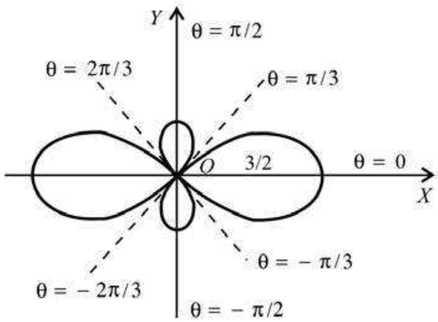
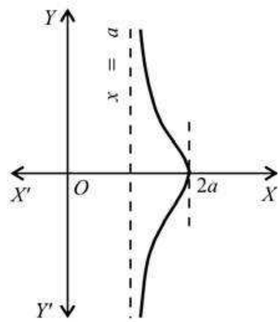


Ex. 9.



Ex. 10.

14.18 Parametric Equations

If the equation to a curve is given in a parametric form, $x = f(t)$, $y = \phi(t)$, then in some cases the curve can be easily traced by eliminating the parameter. But if it is not convenient to eliminate t, a series of values are given to t and the corresponding values of x, y and (dy/dx) are found. Then we plot the different points and observe the slopes of the tangents at these points given by the values of (dy/dx).

Illustrative Examples

Example 1 : *Trace the curve* $x = a(t + \sin t)$, $y = a(1 - \cos t)$, *when* $-\pi \leq t \leq \pi$. **(Cycloid)**

Solution : Here $\dfrac{dx}{dt} = a(1 + \cos t)$ and $\dfrac{dy}{dt} = a \sin t$.

Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a(1 + \cos t)} = \tan \dfrac{t}{2}$.

(i) $y = 0$, when $\cos t = 1$ *i.e.*, $t = 0$.

When $t = 0$, $x = 0$, $(dy/dx) = \tan 0 = 0$.

Therefore the curve passes through the origin and the axis of x is tangent at the origin.

(ii) y is maximum when $\cos t = -1$, *i.e.*, $t = \pi$ and $-\pi$. When $t = \pi$, $x = a\pi$, $y = 2a$ and $(dy/dx) = \infty$.

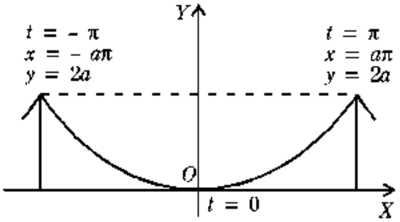
Therefore at the point $t = \pi$, whose cartesian coordinates are $(a\pi, 2a)$, the tangent is perpendicular to the x-axis. When $t = -\pi, x = -a\pi$, $y = 2a$, $(dy/dx) = -\infty$.

(iii) In this curve y cannot be negative. Therefore the curve lies entirely above the axis of x. Also no portion of the curve lies in the region $y > 2a$.

(iv) Corresponding values of x, y and (dy/dx) for different values of t are given in the following table :

t	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$-a\pi$	$-a(\frac{1}{2}\pi + 1)$	0	$a(\frac{1}{2}\pi + 1)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

If we put $-t$ in place of t in the equation of the curve, we get $x = -a(t + \sin t)$, and $y = a(1 - \cos t)$. Thus for every value of y, there are two equal and opposite values of x. Therefore the curve is symmetrical about the y-axis. Hence the shape of the curve is as shown in the diagram.



Example 2 : *Trace the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. **(Astroid)** **(Rohilkhand 2013B)**

Solution : The parametric equations of the curve are $x = a\cos^3 t, y = a\sin^3 t$.

We have $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\dfrac{3a\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\tan t$.

Also the equation of the curve can be written as

$$\left(\frac{x^2}{a^2}\right)^{1/3} + \left(\frac{y^2}{a^2}\right)^{1/3} = 1.$$

We observe the following facts about the curve.

(i) The curve is symmetrical about both the axes.

It is also symmetrical about the line $y = x$.

(ii) The curve does not pass through the origin.

(iii) The curve cuts the x-axis, where $y = 0$

i.e., $\left(\dfrac{x^2}{a^2}\right)^{1/3} = 1$ *i.e.,* $\dfrac{x^2}{a^2} = 1$ *i.e.,* $x = \pm a$.

Thus the curve cuts the x-axis at $(a, 0)$ and $(-a, 0)$.

Similarly the curve crosses the y-axis at $(0, a)$ and $(0, -a)$.

(iv) At the point $(a, 0)$, we have $x = a$.

Therefore $\cos^3 t = 1$ and thus $t = 0$.

When $t = 0, \dfrac{dy}{dx} = 0$.

Hence at the point $(a, 0)$, the x-axis is tangent to the curve.

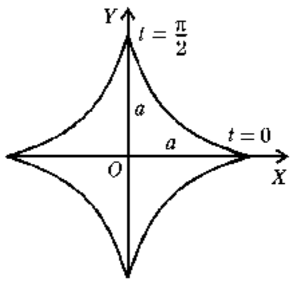
Again at the point $(0, a)$, we have $y = a$.

Therefore $\sin^3 t = 1$ and thus $t = \frac{\pi}{2}$.

When $t = \frac{\pi}{2}, \frac{dy}{dx} = -\infty$.

Hence at the point $(0, a)$, the y-axis is tangent to the curve.

(v) The values of $\sin t$ and $\cos t$ cannot numerically exceed 1. Therefore in this curve the values of x and y cannot numerically exceed a. Therefore the entire curve lies in the region bounded by the lines $x = a, x = -a, y = a$ and $y = -a$.



Hence the shape of the curve is as shown in the diagram.

Comprehensive Exercise 5

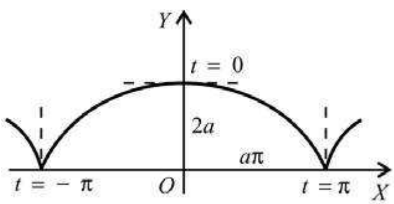
Trace the following curves :

1. $x = a(t + \sin t), y = a(1 + \cos t), -\pi \le t \le \pi$. **(Cycloid)**

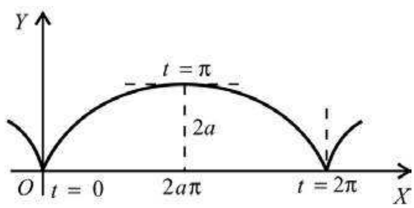
2. $x = a(t - \sin t), y = a(1 - \cos t)$. **(Meerut 2005)**

3. $x = a \cos t + \frac{1}{2} a \log \tan^2 (t/2), y = a \sin t$. **(Tractrix)**

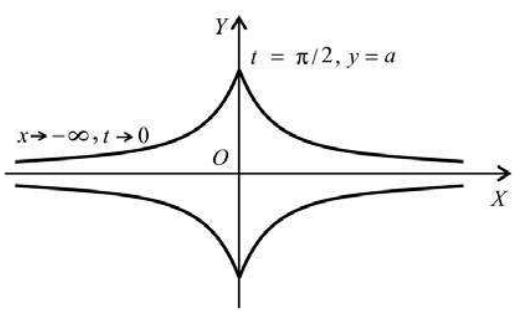
Answers 5



Ex. 1.



Ex. 2.



Ex. 3.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …", so that the following statements are complete and correct.

1. At the point of inflexion

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \ \ldots\ldots .$$ **(Agra 2007)**

2. If the two branches through a double point on a curve are real and have different tangents there, the double point is called a …… . **(Kumaun 2008)**

3. The double point on the curve $x^3 + y^3 = 3axy$ is …… .

4. The curve $y^2(1 - x^2) = x^2(1 + x^2)$ is symmetrical about …… .

5. If the equation of the curve $r = f(\theta)$ does not change by changing the sign of θ, then the curve is symmetrical about the …… .

6. The curve $x^3 + y^3 = 3axy$ is symmetrical about …… . **(Bundelkhand 2006)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. The tangents at origin to the curve $x^3 + y^3 = 3axy$ are

(a) $x = 0, y = 0$ (b) $x = 0, y = 1$
(c) $x = 1, y = 0$ (d) $x = 1, y = 1$

8. The curve $y = x^3$ is symmetrical about the

(a) x-axis (b) y-axis
(c) both the axes (d) opposite quadrants **(Bundelkhand 2008)**

9. The number of loops in the curve $r = a\cos 2\theta$ is

(a) 1 (b) 2 (c) 3 (d) 4

10. The curve $r = a\sin 3\theta$ is symmetrical about the

(a) initial line (b) pole
(c) the line $\theta = \frac{\pi}{2}$ (d) there is no symmetry

11. At the point of inflexion of the curve $x = f(y)$, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3}$ is not equal to

(a) 1 (b) 0
(c) -1 (d) 2 **(Bundelkhand 2007)**

12. Equation of Lemniscate is

(a) $r = a\cos\theta$ (b) $r = a\sin\theta$
(c) $r^2 = a^2\cos 2\theta$ (d) none **(Rohilkhand 2008)**

True or False:

Write 'T' for true and 'F' for false statement.

13. If the two branches through a double point on a curve are real and have coincident tangents there, then the double point is called a node.

14. If the equation of the curve $r = f(\theta)$ remains unchanged on changing the signs of r and θ both, the curve is symmetrical about the line $\theta = \frac{\pi}{2}\cdot$ **(Meerut 2001, 09)**

15. If the equation of a curve remains unchanged even when x and y are interchanged, the curve is symmetrical about the line $y = x$.

16. A point of inflexion is a point at which a curve is changing concave upward to concave downward, or vice-versa.

Answers

1. $\neq 0$.	**2.** node.	**3.** $(0, 0)$.	**4.** both the axes.	**5.** initial line.
6. $y = x$.	**7.** (a).	**8.** (d).	**9.** (d).	**10.** (c).
11. (b).	**12.** (c).	**13.** *F*.	**14.** *T*.	**15.** *T*.
16. *T*.				

□

SECTION

B

INTEGRAL CALCULUS

Chapters

Chapter

1

Reduction Formulae (For Trigonometric Functions)

1.1 Reduction Formulae

A reduction formula is a formula which connects an integral, which cannot otherwise be evaluated, with another integral of the same type but of lower degree. It is generally obtained by applying the rule of integration by parts.

1.2 Reduction Formulae for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$, n being a + ive Integer

(a) Let $\boldsymbol{I_n = \int \sin^n x\,dx}$ or $I_n = \int \sin^{n-1} x \sin x\,dx$. **(Note)**

Integrating by parts regarding $\sin x$ as the 2nd function, we have

$$I_n = \sin^{n-1} x.(-\cos x) - \int (n-1)\sin^{n-2} x.\cos x.(-\cos x)\,dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.\cos^2 x\,dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x.(1-\sin^2 x)\,dx \qquad \textbf{(Note)}$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$$

$$= -\sin^{n-1} x.\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\,I_n.$$

Transposing the last term to the left, we have

$$I_n(1+n-1) = -\sin^{n-1}x.\cos x + (n-1)I_{n-2},$$

$$[\because I_{n-2} = \int \sin^{n-2}x\,dx]$$

or $$nI_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2}$$

or $$I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}.$$

$\therefore$ $$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x.\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

(Bundelkhand 2008; Agra 14)

(b) Let $I_n = \int \cos^n x\,dx$ or $I_n = \int \cos^{n-1}x.\cos x\,dx$.

Integrating by parts regarding $\cos x$ as the 2nd function, we have

$$I_n = \cos^{n-1}x.\sin x - \int (n-1)\cos^{n-2}x.(\sin x).\sin x\,dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int \cos^{n-2}x.\sin^2 x\,dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x.\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)I_{n-2} - (n-1)I_n.$$

Transposing the last term to the left, we have

$$I_n(1+n-1) = \cos^{n-1}x.\sin x + (n-1)I_{n-2}$$

or $$nI_n = \cos^{n-1}x.\sin x + (n-1)I_{n-2}.$$

$\therefore$ $$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx.$$

1.3 Walli's Formula

To evaluate $\int_0^{\pi/2} \sin^n x\,dx$ **and** $\int_0^{\pi/2} \cos^n x\,dx$.

Proceeding as in the previous article, we have

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

$\therefore$ $$\int_0^{\pi/2} \sin^n x\,dx = -\left[\frac{\sin^{n-1}x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx. \quad ...(1)$$

Putting $(n-2)$ in place of n in (1), we have

$$\int_0^{\pi/2} \sin^{n-2}x\,dx = \frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4}x\,dx.$$

Substituting this value in (1), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\int_0^{\pi/2} \sin^{n-4} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\int_0^{\pi/2} \sin^{n-6} x\,dx. \qquad ...(2)$$

Now two cases arise *viz*, n is even or odd.

Case I : When *n* is odd : In this case by the repeated application of the reduction formula (1), the last integral of (2) is

$$\int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1.$$

Hence when n is odd, from (2), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\;\cdots\;\cdot\frac{2}{3}\cdot 1$$

$$= \frac{(n-1)(n-3)\ldots\ldots 4.2}{n(n-2)\ldots 3.1}\cdot 1.$$

Case II : When *n* is even :

In this case the last integral of (2) is

$$\int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} dx = \Big[x\Big]_0^{\pi/2} = \frac{\pi}{2}.$$

Hence when n is even, from (2), we have

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} \sin^0 x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2}\cdot\frac{\pi}{2}.$$

If we evaluate $\int_0^{\pi/2} \cos^n x\,dx$, we get the same results.

$$\therefore \qquad \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx. \qquad \textbf{(Note)}$$

Note : Walli's formula is applicable only when the limits are from 0 to $\frac{1}{2}\pi$.

Illustrative Examples

Example 1 : *Establish a reduction formula for* $\int \sin^n (2x)\,dx$.

Solution : Let $I_n = \int \sin^n (2x)\,dx$ or $I_n = \int \sin^{n-1}(2x)\sin(2x)\,dx$.

Integrating by parts regarding $\sin 2x$ as the 2nd function, we have

$$I_n = \sin^{n-1}(2x)\left[-\tfrac{1}{2}\cos 2x\right] - \int \{(n-1)\sin^{n-2} 2x \,.\, \cos 2x \,.\, 2\}\,.\,(-\tfrac{1}{2}\cos 2x)\,dx$$

$$= -\tfrac{1}{2}\sin^{n-1} 2x \,.\, \cos 2x + (n-1)\int \sin^{n-2} 2x \,.\, \cos^2 2x\,dx$$

$$= -\tfrac{1}{2}\sin^{n-1} 2x \,.\, \cos 2x + (n-1)\int \sin^{n-2} 2x \,.\, (1-\sin^2 2x)\,dx$$

$$= -\tfrac{1}{2}\sin^{n-1} 2x \,.\, \cos 2x + (n-1)\int \sin^{n-2} 2x\,dx - (n-1)\int \sin^{n} 2x\,dx$$

$$= -\tfrac{1}{2}\sin^{n-1} 2x \,.\, \cos 2x + (n-1)\,I_{n-2} - (n-1)\,I_n.$$

Transposing the last term to the left, we have

$$n\,I_n = -\tfrac{1}{2}\sin^{n-1} 2x \,.\, \cos 2x + (n-1)\,I_{n-2}$$

or $$I_n = -\frac{\sin^{n-1} 2x \,.\cos 2x}{2n} + \frac{n-1}{n}I_{n-2},$$ is the reduction formula.

***Example 2 :** *Prove that* $\displaystyle\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{(2m)\,!}{\{2^m \,.\, m\,!\}^2}\cdot\frac{\pi}{2}.$

Solution : Here $2m$ is even. Hence from article 1.3 (Case II), we get

$$\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{(2m-1)\,(2m-3)\,\ldots\,3\,.1}{(2m)\,(2m-2)\,\ldots\,4\,.\,2}\cdot\frac{\pi}{2} \qquad \textbf{(Walli's formula)}$$

$$= \frac{2m\,(2m-1)\,(2m-2)\,\ldots\,3\,.\,2\,.\,1}{\{2m\,(2m-2)\,\ldots.\,4\,.\,2\}^2}\cdot\frac{\pi}{2}.$$

[Multiplying Nr. & Dr. by $2m\,(2m-2)\,(2m-4)\,\ldots\,4\,.\,2$]

$$= \frac{(2m)\,!}{\{2^m\,.\,m\,(m-1)\,(m-2)\,\ldots\ldots 2\,.\,1\}^2}\cdot\frac{\pi}{2} = \frac{(2m)\,!}{\{2^m\,.\,m\,!\,\}^2}\cdot\frac{\pi}{2}.$$

Example 3 : *Evaluate* $\displaystyle\int_0^{2a}\frac{x^{9/2}\,dx}{\sqrt{(2a-x)}}.$

Solution : Put $x = 2a\sin^2\theta$, so that $dx = 2a\,.2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0$, $\sin^2\theta = 0$ *i.e.*, $\theta = 0$

and when $x = 2a$, $\sin^2\theta = 1$ *i.e.*, $\theta = \pi/2$.

Then $$\int_0^{2a}\frac{x^{9/2}\,dx}{\sqrt{(2a-x)}} = \int_0^{\pi/2}\frac{(2a\sin^2\theta)^{9/2}.\,4a\sin\theta\cos\theta\,d\theta}{\sqrt{(2a-2a\sin^2\theta)}}$$

$$= \int_0^{\pi/2}\frac{(2a)^{9/2}.\,4a\sin^{10}\theta.\,\cos\theta\,d\theta}{(2a)^{1/2}.\,\cos\theta}$$

$$= (2a)^4.\,4a\int_0^{\pi/2}\sin^{10}\theta\,d\theta$$

$$= 64\,a^5\cdot\frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{63\,a^5\,\pi}{8}.$$

1.4 Reduction Formula for $\int \tan^n x\,dx$ and $\int \cot^n x\,dx$

(a) We have $\int \tan^n x\,dx = \int \tan^{n-2} x \,.\, \tan^2 x\,dx$ **(Note)**

$$= \int \tan^{n-2} x \,.\, (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \,.\, \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{(\tan x)^{n-2+1}}{n-2+1} - \int \tan^{n-2} x\,dx$$

or $$\boldsymbol{\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx,}$$

which is the required reduction formula.

Application : *Evaluate* $\int \tan^4 x\,dx$.

Putting $n = 4$ in the above reduction formula, we have

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx$$

$$= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x.$$

(b) We have $\int \cot^n x\,dx = \int \cot^{n-2} x \,.\, \cot^2 x\,dx$

$$= \int \cot^{n-2} x \,.\, (\text{cosec}^2 x - 1)\,dx$$

$$= \int \cot^{n-2} x \,.\, \text{cosec}^2 x\,dx - \int \cot^{n-2} x\,dx$$

$$= -\frac{(\cot x)^{n-1}}{n-1} - \int \cot^{n-2} x\,dx$$

or $$\cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx,$$

which is the required reduction formula.

Application : Putting $n = 5$ in the above reduction formula and applying it repeatedly, we have

$$\int \cot^5 x\,dx = -\frac{1}{4}\cot^4 x - \int \cot^3 x\,dx$$

$$= -\frac{1}{4}\cot^4 x - [-\frac{1}{2}\cot^2 x - \int \cot x\,dx]$$

$$= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \int \cot x\,dx$$

$$= -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log \sin x.$$

1.5 Reduction Formulae for $\int \sec^n x\,dx$ and $\int \text{cosec}^n x\,dx$

(Bundelkhand 2011)

(a) We have $I_n = \int \sec^n x\,dx = \int \sec^{n-2} x \,.\, \sec^2 x\,dx.$ **(Note)**

Integrating by parts regarding $\sec^2 x$ as the 2nd function, we have

$$I_n = \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \qquad \textbf{(Note)}$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx.$$

Transposing the term containing $\int \sec^n x \, dx$ to the left, we have

$$(n-2+1) \int \sec^n x \, dx = \sec^{n-2} \tan x + (n-2) \int \sec^{n-2} x \, dx$$

or $$(n-1) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx.$$

Dividing both sides by $(n-1)$, we have

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx,$$

which is the required reduction formula.

(b) To find the reduction formula for $\int \operatorname{cosec}^n x \, dx$, proceed exactly in the same way as in part (a). Thus, we get

$$\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} \int \operatorname{cosec}^{n-2} x \, dx,$$

as the required reduction formula for $\int \operatorname{cosec}^n x \, dx$.

1.6 Reduction Formula for $\int \sin^m x \cos^n x \, dx$

(kanpur 2014)

Let $$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx$$

$$= \int \cos^{n-1} x \, . \, (\sin^m x \cos x) \, dx.$$

Integrating by parts taking $\sin^m x \cos x$ as the second function, we get

$$I_{m,n} = \frac{\sin^{m+1} x}{m+1} \cos^{n-1} x + \frac{n-1}{m+1} \int \sin^{m+1} x \cos^{n-2} x \sin x \, dx$$

$$= \frac{\sin^{m+1} x}{m+1} \cos^{n-1} x + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \sin^2 x \, dx$$

$$= \frac{\sin^{m+1} x}{m+1} \cdot \cos^{n-1} x + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, . \, (1 - \cos^2 x) \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, dx - \frac{n-1}{m+1} I_{m,n}.$$

Transposing the last term to the left, we have

$$I_{m,n} \left(1 + \frac{n-1}{m+1} \right) = \frac{\sin^{m+1} x . \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

or $$I_{m,n} \left(\frac{m+n}{m+1} \right) = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}.$$

Thus the required reduction formula is

$$I_{m,n} = \frac{\sin^{m+1} x \,.\, \cos^{n-1} x}{m+n} + \frac{(n-1)\, I_{m,n-2}}{m+n}.$$

Note : If we write $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^{m-1} x \,.\, (\cos^n x \sin x)\, dx,$$

then integrating by parts regarding $\cos^n x \sin x$ as the 2nd function, the reduction formula can be obtained as

$$\boldsymbol{I_{m,n} = -\frac{\sin^{m-1} x \,.\, \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}}.$$

Similarly other four reduction formulae for $\int \sin^m x \cos^n x \, dx$ may be obtained as

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} I_{m,n+2}.$$

[To obtain this reduction formula put $(n+2)$ in place of n in the reduction formula obtained in article 1.6 and adjust the result accordingly]

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}$$

$$I_{m,n} = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1} I_{m-2,n+2}$$

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m+2,n-2}.$$

[This reduction formula has been obtained in article 1.6 at the stage we applied integration by parts.]

Illustrative Examples

Example 1 : *Evaluate* $\displaystyle\int \frac{d\theta}{\sin^4 \frac{1}{2}\theta}$.

Solution : We have $\displaystyle\int \frac{d\theta}{\sin^4 \frac{1}{2}\theta} = \int \operatorname{cosec}^4 \frac{\theta}{2}\, d\theta = 2\int \operatorname{cosec}^4 x \, dx$, putting $\theta = 2x$.

But $$\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x \, dx.$$

[Derive this formula here]

Putting $n = 4$, we get

$$\int \operatorname{cosec}^4 x \, dx = -\frac{\operatorname{cosec}^2 x \cot x}{3} + \frac{2}{3}\int \operatorname{cosec}^2 x \, dx$$

$$= -\tfrac{1}{3}\operatorname{cosec}^2 x \cot x + \tfrac{2}{3}(-\cot x).$$

Hence the given integral

$$= 2\int \operatorname{cosec}^4 x\, dx = -\tfrac{2}{3}\operatorname{cosec}^2 x \cot x - \tfrac{4}{3}\cot x$$

$$= -\tfrac{2}{3}\operatorname{cosec}^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta - \tfrac{4}{3}\cot \tfrac{1}{2}\theta. \qquad [\because x = \theta/2]$$

Example 2 : *Evaluate* $\int (1 + x^2)^{3/2}\, dx$.

Solution : Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

Then $\int (1 + x^2)^{3/2}\, dx = \int \sec^2\theta \sec^3\theta\, d\theta = \int \sec^5\theta\, d\theta$.

Now we shall form a reduction formula for $\int \sec^n\theta\, d\theta$. Proceeding as in article 1.5 (a), we get

$$\int \sec^n\theta\, d\theta = \frac{\sec^{n-2}\theta\tan\theta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\theta\, d\theta.$$

$$\therefore \quad \int \sec^5\theta\, d\theta = \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{4}\int \sec^3\theta\, d\theta$$

$$= \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{4}\left[\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\int \sec\theta\, d\theta\right]$$

$$= \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{8}\sec\theta\tan\theta + \tfrac{3}{8}\log(\sec\theta + \tan\theta)$$

$$= \tfrac{1}{4}[(1 + x^2)^{3/2} . x] + \tfrac{3}{8}x(1 + x^2)^{1/2} + \tfrac{3}{8}\log\{x + \sqrt{(1 + x^2)}\}.$$

Comprehensive Exercise 1

Evaluate the following integrals :

1. (i) $\int \sin^6 x\, dx$. (ii) $\int_0^{\pi/2} \sin^6 x\, dx$. **(Kanpur 2005)**

(iii) $\int_0^{\pi/2} \cos^9 x\, dx$. (iv) $\int_0^{\pi/2} \cos^{10} x\, dx$.

2. (i) $\int_0^{\pi/4} \tan^5\theta\, d\theta$. (ii) $\int_0^{a} x^5 (2a^2 - x^2)^{-3}\, dx$.

(iii) $\int \sec^3 x\, dx$. (iv) $\int_0^{\pi/4} \sec^3 x\, dx$.

3. (i) $\int_0^{a} (a^2 + x^2)^{5/2}\, dx$. (ii) $\int_0^{\pi/4} \sin^2\theta\cos^4\theta\, d\theta$.

(iii) $\int \tan^6 x\, dx$.

4. Show that $\int_0^{a} \frac{x^4}{\sqrt{(a^2 - x^2)}}\, dx = \frac{3a^4\pi}{16}$.

5. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$, and deduce the value of I_5.

(Kanpur 2005, 12; Avadh 06, 11; Bundelkhand 06; Purvanchal 14)

6. If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, prove that $n(I_{n-1} + I_{n+1}) = 1$.

(Kanpur 2005, 12; Avadh 06)

Answers 1

1. (i) $-\frac{1}{6}\sin^5 x \cos x - \frac{5}{24}\sin^3 x \cos x - \frac{5}{16}\sin x \cos x + \frac{5}{16}x$. (ii) $\frac{5\pi}{32}$.

(iii) $\frac{128}{315}$. (iv) $\frac{63\pi}{512}$.

2. (i) $\frac{1}{2}[\log 2 - \frac{1}{2}]$. (ii) $\frac{1}{2}[\log 2 - \frac{1}{2}]$.

(iii) $\frac{1}{2}\sec x \tan x + \frac{1}{2}\log(\sec x + \tan x)$. (iv) $\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(\sqrt{2} + 1)$.

3. (i) $\frac{a^6}{48}[67\sqrt{2} + 15 \log \tan(\frac{3}{8}\pi)]$. (ii) $\frac{1}{48} + \frac{\pi}{64}$.

(iii) $\frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$.

5. $\frac{1}{2}(\log 2 - \frac{1}{2})$.

1.7 Gamma Function

The definite integral $\int_0^{\infty} e^{-x} x^{n-1} \, dx$ is called the *second Eulerian integral* and is denoted by the symbol $\Gamma(n)$ [read as Gamma n].

Properties of Gamma function. **(Commit to memory)**

$$\boldsymbol{\Gamma(n+1) = n\,\Gamma n \,;\, \Gamma 1 = 1 \,;\, \Gamma \tfrac{1}{2} = \sqrt{\pi}.}$$

$\boldsymbol{\Gamma(n) = (n-1)!}$ **provided $\boldsymbol{n}$ is a positive integer.**

Thus $\Gamma(10) = 9!$.

Also $$\Gamma\frac{9}{2} = \frac{7}{2}\Gamma\frac{7}{2} = \frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}$$

$$= \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}.$$

1.8 Value of $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ in terms of Γ Function, where m and n are positive integers

We have already derived in article 1.6 that

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x \, dx.$$

$$\therefore \quad \int_0^{\pi/2} \sin^m x \cos^n x\, dx$$

$$= \left[\frac{\sin^{m+1} x \cos^{n-1} x}{m+n}\right]_0^{\pi/2} + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

$$= 0 + \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx$$

i.e.,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(n-1)}{(m+n)}\int_0^{\pi/2} \sin^m x \cos^{n-2} x\, dx \qquad ...(1)$$

Now four cases arise according as m and n take different types of values, odd or even.

Case I : When m and n are both even :

Successively applying the formula (1) till the power of $\cos x$ becomes zero, we have

$$\int_0^{\pi/2} \sin^m x \,.\, \cos^n x\, dx$$

$$= \frac{(n-1)}{(m+n)} \cdot \frac{(n-3)}{(m+n-2)} \cdot \frac{(n-5)}{(m+n-4)} \cdots \frac{1}{(m+2)}\int_0^{\pi/2} \sin^m x\, dx.$$

Also $\displaystyle\int_0^{\pi/2} \sin^m x\, dx = \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdot \frac{m-5}{m-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}.$ [See article 1.3]

Therefore,

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{(n-1)(n-3)(n-5)\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots(m+2)} \times \frac{(m-1)(m-3)(m-5)\ldots 3\,.1}{m(m-2)(m-4)\ldots 4\,.2} \cdot \frac{\pi}{2}$$

$$= \frac{\left\{\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\left(\frac{n-5}{2}\right)\cdots\frac{1}{2}\right\}\left\{\left(\frac{m-1}{2}\right)\left(\frac{m-3}{2}\right)\cdots\frac{1}{2}\right\}}{\left\{\left(\frac{m+n}{2}\right)\left(\frac{m+n-2}{2}\right)\cdots\frac{4}{2}\cdot\frac{2}{2}\right\}} \cdot \frac{\pi}{2}$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Similarly, the cases for other values of m and n may be considered and it may be verified that the result is true in other cases too.

Thus for all positive integral values of m and n, we have

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right).\,\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}. \qquad \textbf{(Remember)}$$

Walli's Formula : [An easy way to evaluate $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$ where m and n are + ive integers]. We have $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{(m-1)(m-3)(m-5)\ldots(n-1)(n-3)(n-5)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots} \times k,$$

where k is $\frac{1}{2}\pi$ if m and n are both even, otherwise $k = 1$. The last factor in each of the three products is either 1 or 2. In case any of m or n is 1, we simply write 1 as the only factor to replace its product. This formula is equally applicable if any of m or n is zero provided we put 1 as the only factor in its product and we regard 0 as even.

Illustrative Examples

Example 1 : *Evaluate* $\int_0^{\pi/2} \sin^4 x \cos^2 x \, dx$.

(Rohilkhand 2014)

Solution : We know that

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}.$$

$$\therefore \text{ the given integral} = \frac{\Gamma\left(\frac{4+1}{2}\right)\cdot\Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{4+2+2}{2}\right)} = \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 4} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\,.3\,.2.\,1} = \frac{\pi}{32}.$$

$$\left[\because \Gamma(n+1) = n\,\Gamma n \text{ and } \Gamma\tfrac{1}{2} = \sqrt{\pi}\right]$$

Alternate Solution : Using Walli's formula, the given integral

$$= \frac{3\,.1\,.1}{6\,.4\,.2}\cdot\frac{\pi}{2} = \frac{\pi}{32}.$$

Example 2 : *Evaluate* $\int_0^{\pi/2} \sin^6 \theta \, d\theta$.

Solution : We have $\int_0^{\pi/2} \sin^6 \theta \, d\theta = \int_0^{\pi/2} \sin^6 \theta \cos^0 \theta \, d\theta$. **(Note)**

Now $m = 6$, $n = 0$; using the Gamma function, we have the given integral

$$= \frac{\Gamma\left(\frac{6+1}{2}\right).\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{6+0+2}{2}\right)} = \frac{\Gamma\frac{7}{2}\cdot\Gamma\frac{1}{2}}{2\,\Gamma 4} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,.\sqrt{\pi}}{2\,.\,(3\,.2\,.\,1)} = \frac{5\pi}{32}.$$

Otherwise, by Walli's formula, the given integral

$$= \frac{5\,.3\,.1}{6\,.4\,.2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}.$$

Example 3 : *Evaluate* $\int_0^{\pi/6} \sin^2 6\theta \cos^5 3\theta \, d\theta$.

Solution : To bring the given integral into the form of Gamma function, put $3\theta = x$, so that $3\, d\theta = dx$. Also for limits, $x = 0$ at $\theta = 0$ and $x = \pi/2$ at $\theta = \pi/6$.

$\therefore$ the given integral

$$= \frac{1}{3}\int_0^{\pi/2} \sin^2 2x \cos^5 x \, dx$$

$$= \frac{1}{3}\int_0^{\pi/2} (2 \sin x \cos x)^2 \cos^5 x \, dx = \frac{4}{3}\int_0^{\pi/2} \sin^2 x \cos^7 x \, dx$$

$$= \frac{4}{3} \frac{\Gamma\frac{3}{2} \,.\, \Gamma 4}{2\,\Gamma\frac{11}{2}} = \frac{4}{3} \cdot \frac{\Gamma\frac{3}{2} \,.\, 3\,.2\,.1}{2 \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma\frac{3}{2}} = \frac{64}{945}.$$

Example 4 : *Evaluate* $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta \, d\theta$.

Solution : The given integral $= \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta$. **(Note)**

Now put $\sqrt{2} \sin\theta = \sin x$, so that $\sqrt{2} \cos\theta \, d\theta = \cos x \, dx$.

Also when $\theta = 0$, $\sin x = \sqrt{2} \sin 0 = 0$ giving $x = 0$

and when $\theta = \frac{1}{4}\pi$, $\sin x = \sqrt{2} \sin(\pi/4) = 1$ giving $x = \frac{1}{2}\pi$.

Hence the given integral

$$= \int_0^{\pi/2} (1 - \sin^2 x)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos x \, dx = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^3 x \,.\, \cos x \, dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4 x \, dx = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sin^0 x \,.\cos^4 x \, dx$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\Gamma\left(\frac{0+1}{2}\right) \cdot \Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{0+4+2}{2}\right)} = \frac{1}{\sqrt{2}} \frac{\Gamma\frac{5}{2} \cdot \Gamma\frac{1}{2}}{2\,\Gamma 3}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}\sqrt{\pi}}{2\,.2\,.1} = \frac{3\pi}{16\sqrt{2}}.$$

Example 5 : *Evaluate* $\int_0^1 x^4 (1 - x^2)^{5/2} \, dx$.

Solution : Here we put $x = \sin\theta$, so that $dx = \cos\theta \, d\theta$.

And now the new limits are $\theta = 0$ to $\theta = \pi/2$.

Thus the given integral $\int_0^{\pi/2} \sin^4\theta \, (1 - \sin^2\theta)^{5/2} \cos\theta \, d\theta$

$$= \int_0^{\pi/2} \sin^4\theta \,.\, \cos^5\theta \, \cos\theta \, d\theta = \int_0^{\pi/2} \sin^4\theta \cos^6\theta \, d\theta$$

$$= \frac{3.1.5.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2}, \qquad \text{[by Walli's formula]}$$

$$= \frac{3\pi}{512}.$$

Example 6 : *Show that* $\int_0^a x^4 (a^2 - x^2)^{1/2}\, dx = \frac{\pi a^6}{32}.$

(Bundelkhand 2012)

Solution : Put $x = a \sin\theta$ so that $dx = a\cos\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{1}{2}\pi$.

$$\therefore \text{ the given integral} = \int_0^{\pi/2} a^4 \sin^4\theta . a\cos\theta . a\cos\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta = a^6 \frac{\Gamma\frac{5}{2} . \Gamma\frac{3}{2}}{2\Gamma 4}$$

$$= a^6 \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}\, \frac{1}{2}\sqrt{\pi}}{2.3.2.1} = \frac{\pi a^6}{32}.$$

Example 7 : *Evaluate* $\int_0^{\pi/4} \sin^4 x \cos^2 x\, dx.$

Solution : The given integral

$$I = \int_0^{\pi/4} (\sin^2 x \cos^2 x) \sin^2 x\, dx$$

$$= \int_0^{\pi/4} \frac{1}{4}(4\sin^2 x \cos^2 x) \cdot \frac{1}{2}(2\sin^2 x)\, dx$$

$$= \frac{1}{8}\int_0^{\pi/4} \sin^2 2x (1 - \cos 2x)\, dx.$$

Put $2x = t$, so that $2\, dx = dt$.

Also when $x = 0$, $t = 0$ and when $x = \pi/4$, $t = \pi/2$.

$$\therefore \qquad I = \frac{1}{8}\int_0^{\pi/2} \sin^2 t (1 - \cos t) \cdot \frac{1}{2}\, dt$$

$$= \frac{1}{16}\left[\int_0^{\pi/2} \sin^2 t\, dt - \int_0^{\pi/2} \sin^2 t \cos t\, dt\right]$$

$$= \frac{1}{16}\left[\frac{1}{2} \cdot \frac{\pi}{2} - \frac{1.1}{3.1}\right] = \frac{1}{16}\left[\frac{\pi}{4} - \frac{1}{3}\right].$$

Example 8 : *Evaluate* $\int_0^a x^2 \sqrt{(ax - x^2)}\, dx.$

Solution : We have $\int_0^a x^2 \sqrt{(ax - x^2)}\, dx = \int_0^a x^{5/2} \sqrt{(a - x)}\, dx.$

Now put $x = a \sin^2 \theta$, so that $dx = 2a \sin \theta \cos \theta \, d\theta$, and the new limits are $\theta = 0$ to $\theta = \pi/2$.

Thus the given integral

$$= \int_0^{\pi/2} (a \sin^2 \theta)^{5/2} (a \cos^2 \theta)^{1/2} . 2a \sin \theta \cos \theta \, d\theta$$

$$= 2a^4 \int_0^{\pi/2} \sin^6 \theta \cos^2 \theta \, d\theta$$

$$= 2a^4 \frac{\Gamma \frac{7}{2} . \Gamma \frac{3}{2}}{2 \, \Gamma \, 5} = 2a^4 \left[\frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \, . \, \frac{1}{2} \sqrt{\pi}}{2 .4 .3 .2 .1} \right] = \frac{5\pi a^4}{128}.$$

Example 9 : *Evaluate* $\displaystyle\int_0^{\infty} \frac{x^4 \, dx}{(a^2 + x^2)^4}$.

Solution : Put $x = a \tan \theta$, so that $dx = a \sec^2 \theta \, d\theta$ and the new limits are $\theta = 0$ to $\theta = \pi/2$.

$\therefore$ the given integral

$$= \int_0^{\pi/2} \frac{a^4 \tan^4 \theta .a \sec^2 \theta \, d\theta}{(a^2 + a^2 \tan^2 \theta)^4} = \frac{1}{a^3} \int_0^{\pi/2} \frac{\tan^4 \theta \, d\theta}{\sec^6 \theta}$$

$$= \frac{1}{a^3} \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta = \frac{1}{a^3} \frac{\Gamma \frac{5}{2} \cdot \Gamma \frac{3}{2}}{2 \, \Gamma \, 4} = \frac{1}{a^3} \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \, . \, \frac{1}{2} \sqrt{\pi}}{2 .3 .2 .1} = \frac{\pi}{32 \, a^3}.$$

Comprehensive Exercise 2

Evaluate the following integrals :

1. (i) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx.$ **(Bundelkhand 2009, 10)**

(ii) $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx.$ (iii) $\displaystyle\int_0^{\pi/2} \sin^5 x \cos^8 x \, dx.$

(iv) $\displaystyle\int_0^{\pi/2} \sin^{12} x \cos^{18} x \, dx.$

2. (i) $\displaystyle\int_0^{\pi/8} \cos^3 4x \, dx.$ (ii) $\displaystyle\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2} \, dx.$

(iii) $\displaystyle\int_0^{\pi/2} \cos^5 x \sin 3x \, dx.$ (iv) $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^4 x \cos 2x \, dx.$

(v) $\displaystyle\int_0^{\pi/6} \cos^4 3\phi \sin^3 6\phi \, d\phi.$

3. (i) $\displaystyle\int_0^{1} x^2 (1 - x^2)^{3/2} \, dx.$ (ii) $\displaystyle\int_0^{1} x^4 (1 - x^2)^{3/2} \, dx.$

(iii) $\int_0^1 x^6 (1 - x^2)^{1/2}\, dx.$ (iv) $\int_0^a x^2 (a^2 - x^2)^{3/2}\, dx.$

4. (i) $\int_0^1 x^m (1 - x)^n\, dx.$ (ii) $\int_0^1 x^{3/2} (1 - x)^{3/2}\, dx.$

(iii) $\int_0^1 x^{3/2} \sqrt{(1 - x)}\, dx.$

(iv) $\int_0^{2a} x^m \sqrt{(2ax - x^2)}\, dx$, m being a positive integer.

(Kanpur 2007, 12; Bundelkhand 07)

5. (i) $\int_0^{2a} x^5 \sqrt{(2ax - x^2)}\, dx.$ (ii) $\int_0^a x^3 (2ax - x^2)^{3/2}\, dx.$

(iii) $\int_0^a x^2 (2ax - x^2)^{5/2}\, dx.$ (iv) $\int_0^a \frac{x^4}{(x^2 + a^2)^4}\, dx.$

6. (i) $\int_0^a x^2 \sqrt{\left(\frac{a - x}{a + x}\right)}\, dx.$ (ii) $\int_0^a x \sqrt{\left(\frac{a^2 - x^2}{a^2 + x^2}\right)}\, dx.$

(iii) $\int_a^b (x - a)^m (b - x)^n\, dx.$

7. Prove that $\int_0^1 \frac{dx}{\sqrt{(1 - x^n)}} = \frac{\sqrt{\pi}\, \Gamma(1/n)}{n\, \Gamma\{\frac{1}{2} + (1/n)\}}$.

8. Show that $\int_0^\infty \frac{x^4\, dx}{(1 + x^2)^4} = \frac{\pi}{32}$.

9. If m, n are positive integers, then prove that

$$\int_0^1 x^{m-1} (1 - x)^{n-1}\, dx = \int_0^1 x^{n-1} (1 - x)^{m-1}\, dx = \frac{1 \,.\, 2 \,.\, 3 \,\ldots\ldots\, (m - 1)}{n\,(n + 1)\,\ldots\ldots\,(n + m - 1)}.$$

Answers 2

1. (i) $\frac{2}{15}$. (ii) $\frac{3\pi}{512}$. (iii) $\frac{8}{1287}$.

(iv) $\frac{\Gamma(\frac{13}{2})\,\Gamma(\frac{19}{2})}{2\,\Gamma(16)}$.

2. (i) $\frac{1}{6}$. (ii) $\frac{5\pi}{2048}$.

(iii) $\frac{1}{3}$. (iv) $\frac{2}{315}$. (v) $\frac{1}{15}$.

3. (i) $\frac{\pi}{32}$. (ii) $\frac{3\pi}{256}$. (iii) $\frac{5\pi}{256}$.

(iv) $\frac{\pi a^6}{32}$.

4. (i) $\frac{\Gamma(m + 1)\,\Gamma(n + 1)}{\Gamma(m + n + 2)}$. (ii) $\frac{3\pi}{128}$.

(iii) $\dfrac{\pi}{16}$. (iv) $a^{m+2}\dfrac{(2m+1)(2m-1)\ldots\ldots 3.1}{(m+2)(m+1)m(m-1)\ldots\ldots 2.1}$.

5. (i) $\dfrac{33}{16}\pi a^7$. (ii) $a^7\left(\dfrac{9\pi}{32}-\dfrac{23}{35}\right)$. (iii) $a^8\left[\dfrac{45\pi}{256}-\dfrac{2}{7}\right]$.

(iv) $\dfrac{1}{a^3}\cdot\dfrac{1}{16}\left[\dfrac{\pi}{4}-\dfrac{1}{3}\right]$. **6.** (i) $a^3\left(\frac{1}{4}\pi-\frac{2}{3}\right)$. (ii) $\frac{1}{4}a^2(\pi-2)$.

(iii) $(b-a)^{m+n+1}\left[\dfrac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+2+2)}\right]$.

1.9 Integration of $x^n \sin mx$ and $x^n \cos mx$

(a) $\int x^n \sin mx\,dx$. To form the reduction formula, integrating by parts regarding $\sin mx$ as the 2nd function, we have

$$\int x^n \sin mx\,dx = -\frac{x^n\cos mx}{m}+\int nx^{n-1}\frac{\cos mx\,dx}{m}$$

$$=-\frac{x^n\cos mx}{m}+\frac{n}{m}\left[\frac{x^{n-1}\sin mx}{m}-\frac{n-1}{m}\int x^{n-2}\sin mx\,dx\right],$$

[again integrating by parts regarding $\cos mx$ as the 2nd function]

$$=-\frac{x^n\cos mx}{m}+\frac{nx^{n-1}}{m^2}\sin mx-\frac{n(n-1)}{m^2}\int x^{n-2}\sin mx\,dx.$$

Above is the required reduction formula. Successively applying this formula we are left with $\int x\sin mx\,dx$ or $\int \sin mx\,dx$ according as n is odd or even.

(b) $\int x^n \cos mx\,dx$. Integrating by parts regarding $\cos mx$ as the 2nd function, we have

$$\int x^n\cos mx\,dx=\frac{x^n\sin mx}{m}-\int\frac{nx^{n-1}\sin mx\,dx}{m}$$

$$=\frac{x^n\sin mx}{m}-\frac{n}{m}\int x^{n-1}\sin mx\,dx$$

$$=\frac{x^n\sin mx}{m}+\frac{n}{m}\left[x^{n-1}\cdot\left(\frac{\cos mx}{m}\right)-\frac{n-1}{m}\int x^{n-2}\cos mx\,dx\right],$$

[again integrating by parts regarding $\sin mx$ as 2nd function]

$$=\frac{x^n\sin mx}{m}+\frac{nx^{n-1}\cos mx}{m^2}-\frac{n(n-1)}{m^2}\int x^{n-2}\cos mx\,dx,$$

which is the required reduction formula.

1.10 Reduction Formulae for $\int x\sin^n x\,dx$ and $\int x\cos^n x\,dx$.

(Bundelkhand 2009; Meerut 13)

Let $I_n=\int x\sin^n x\,dx=\int (x\sin^{n-1}x).\sin x\,dx$ **(Note)**

$$= (x \sin^{n-1} x) \,.\, (-\cos x) + \int \cos x \,[\sin^{n-1} x + x\,(n-1) \sin^{n-2} x \cos x]\, dx,$$

integrating by parts regarding $\sin x$ as 2nd function

$$= -x \sin^{n-1} x \cos x + \int \sin^{n-1} x \cos x \, dx + (n-1) \int x \sin^{n-2} x \cos^2 x \, dx$$

$$= -x \sin^{n-1} x \cos x + \frac{1}{n} \sin^n x + (n-1) \int x \sin^{n-2} x \,.\, (1 - \sin^2 x)\, dx$$

$$= -x \sin^{n-1} x \cos x + \frac{1}{n} \sin^n x + (n-1) \int x \sin^{n-2} x \, dx - (n-1) \int x \sin^n x \, dx$$

$$= -x \sin^{n-1} x \cos x + \frac{1}{n} \sin^n x + (n-1) I_{n-2} - (n-1) I_n .$$

Transposing the last term to the left, we have

$$I_n (1 + n - 1) = -x \sin^{n-1} x \cos x + \frac{1}{n} \sin^n x + (n-1) I_{n-2}$$

or $$n I_n = -x \sin^{n-1} x \cos x + \frac{1}{n} \sin^n x + (n-1) I_{n-2}$$

or $$\boldsymbol{I_n = -\frac{x}{n} \,.\, \sin^{n-1} x \,.\, \cos x + \frac{\sin^n x}{n^2} + \frac{(n-1)}{n} I_{n-2}},$$

which is the required reduction formula.

Similarly, reduction formula for $\int x \cos^n x \, dx$ is

$$\boldsymbol{I_n = \int x \cos^n x \, dx = \frac{x \cos^{n-1} x \sin x}{n} + \frac{\cos^n x}{n^2} + \frac{(n-1)}{n} I_{n-2}}.$$

1.11 Reduction Formulae for $\int e^{ax} \sin^n bx \, dx$ and $\int e^{ax} \cos^n bx \, dx$.

(a) Let $\boldsymbol{I_n = \int e^{ax} \sin^n bx \, dx}$

$$= \frac{e^{ax}}{a} \sin^n bx - \frac{nb}{a} \int e^{ax} \sin^{n-1} bx \cos bx \, dx, \qquad ...(1)$$

integrating by parts taking e^{ax} as the 2nd function.

Now $\boldsymbol{\int e^{ax} \sin^{n-1} bx \cos bx \, dx}$

$$= \frac{e^{ax}}{a} (\sin^{n-1} bx \cos bx)$$

$$- \int \frac{e^{ax}}{a} [(n-1)\, b \sin^{n-2} bx \cos^2 bx - b \sin^n bx]\, dx,$$

integrating by parts taking e^{ax} as the 2nd function

$$= \frac{e^{ax}}{a}(\sin^{n-1} bx \cos bx)$$
$$- \frac{b}{a}\int e^{ax}[(n-1)\sin^{n-2} bx(1-\sin^2 bx) - \sin^n bx]\,dx$$

$$= \frac{e^{ax}}{a}(\sin^{n-1} bx \cos bx) - \frac{b}{a}\int e^{ax}[(n-1)\sin^{n-2} bx - n\sin^n bx]\,dx$$

$$= \frac{e^{ax}}{a}(\sin^{n-1} bx \cos bx) - (n-1)\frac{b}{a}\int e^{ax}\sin^{n-2} bx\,dx + \frac{nb}{a}I_n.$$

Substituting this value in (1), we get

$$I_n = \frac{e^{ax}}{a}\sin^n bx - \frac{nb}{a^2}e^{ax}\sin^{n-1} bx\cos bx$$
$$+ n(n-1)\frac{b^2}{a^2}\int e^{ax}\sin^{n-2} bx\,dx - n^2\frac{b^2}{a^2}I_n.$$

Transposing the last term to L.H.S., we get

$$\left(1+\frac{n^2b^2}{a^2}\right)I_n = \frac{e^{ax}}{a^2}(a\sin bx - nb\cos bx)\sin^{n-1} bx + n(n-1)\frac{b^2}{a^2}I_{n-2}.$$

$$\therefore \qquad I_n = \frac{e^{ax}}{a^2+n^2b^2}(a\sin^n bx - nb\sin^{n-1} bx\cos bx) + \frac{n(n-1)}{a^2+n^2b^2}I_{n-2},$$

which is the required reduction formula.

Similarly, $\int \boldsymbol{e^{ax}\cos^n bx\,dx}$

$$= \frac{e^{ax}}{a^2+n^2b^2}(a\cos^n bx + nb\sin bx\cos^{n-1} bx) + \frac{n(n-1)}{a^2+n^2b^2}I_{n-2}.$$

Note : The above formulae should not be applied when n is small. In that case $\sin^n bx$ and $\cos^n bx$ are converted in terms of multiples of angles.

1.12 Reduction Formulae For $\int x^n e^{ax}\sin bx\,dx$ and $\int x^n e^{ax}\cos bx\,dx$.

We know that $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{r}\sin(bx-\phi)$,

where $r = \sqrt{(a^2+b^2)}$ and $\phi = \tan^{-1}(b/a)$.

Now $\frac{1}{r}\int e^{ax}\sin(bx-\phi)\,dx = \frac{1}{r}\left[\frac{1}{r}e^{ax}\sin\{(bx-\phi)-\phi\}\right]$

$$= \frac{1}{r^2}\int e^{ax}\sin(bx-2\phi).$$

Similarly $\frac{1}{r^2}\int e^{ax}\sin(bx-2\phi)\,dx = \frac{1}{r^3}e^{ax}\sin(bx-3\phi)$, and so on.

Now $\int x^n e^{ax}\sin bx\,dx$ can be easily evaluated by repeatedly integrating by parts taking function of the type $e^{ax}\sin bx$ as the 2nd function.

Similarly we can obtain a reduction formula for

$$\int x^n e^{ax}\cos bx\,dx.$$

1.13 Reduction Formulae for $\int \cos^m x \sin nx\,dx$. (Meerut 2013B)

Let $I_{m,n}=\int \cos^m x\sin nx\,dx$. Integrating by parts regarding $\sin nx$ as the 2nd function, we have

$$I_{m,n}=\frac{\cos^m x(-\cos nx)}{n}-\int\frac{m\cos^{m-1}x.(-\sin x)(-\cos nx)\,dx}{n}$$

$$=\frac{-\cos^m x\cos nx}{n}-\frac{m}{n}\int\cos^{m-1}x\cos nx.\sin x\,dx. \qquad \text{...(1)}$$

Now $\sin\{(n-1)x\}=\sin(nx-x)=\sin nx\cos x-\cos nx\sin x$.

$\therefore \qquad \cos nx\sin x=\sin nx\cos x-\sin(n-1)x$.

$$\therefore \qquad I_{m,n}=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int\cos^{m-1}x\{\sin nx\cos x-\sin(n-1)x\}\,dx$$

$$=-\frac{\cos^m x\cos nx}{n}-\frac{m}{n}\int\cos^m x\sin nx\,dx$$

$$+\frac{m}{n}\int\cos^{m-1}x\sin(n-1)x\,dx.$$

Transposing the middle term to the left and simplifying, we get

$$I_{m,n}=-\frac{\cos^m x\cos nx}{m+n}+\frac{m}{m+n}\int\cos^{m-1}x\sin(n-1)x\,dx,$$

which is the required reduction formula.

Deduction : If in the above integral we take the limits of integration as 0 to $\frac{1}{2}\pi$, we find that

$$\int_0^{\pi/2}\cos^m x\sin nx\,dx=\frac{1}{m+n}+\frac{m}{m+n}\int_0^{\pi/2}\cos^{m-1}x\sin(n-1)x\,dx.$$

1.14 Reduction Formulae for $\int \cos^m x\cos nx\,dx$

Let $\qquad I_{m,n}=\int\cos^m x.\cos nx\,dx$

$$=\cos^m x\cdot\left(\frac{\sin nx}{n}\right)-\int m\cos^{m-1}x.(-\sin x)\left(\frac{\sin nx}{n}\right)dx,$$

integrating by parts taking $\cos nx$ as the 2nd function

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \,.\, \sin nx \sin x \, dx.$$

But $\quad \cos(n-1)x = \cos nx \cos x + \sin nx \sin x.$

$\therefore \quad \sin nx \sin x = \cos(n-1)x - \cos nx \cos x.$

Hence $\quad I_{m,n} = \dfrac{\cos^m x \,.\sin nx}{n} + \dfrac{m}{n} \displaystyle\int \cos^{m-1} x \{\cos(n-1)x - \cos nx \cos x\}\, dx$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \cos(n-1)x\, dx - \frac{m}{n} \int \cos^m x \cos nx \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}.$$

Transposing the last term to the left, we have

$$\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

or $$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1},$$

which is the required reduction formula.

Deduction : $\displaystyle\int_0^{\pi/2} \cos^m x \cos nx \, dx$

$$= \left[\frac{\cos^m x \sin nx}{m+n}\right]_0^{\pi/2} + \frac{m}{(m+n)} \int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x\, dx$$

$$= 0 + \frac{m}{(m+n)} \int_0^{\pi/2} \cos^{m-1} x \cos(n-1)x\, dx$$

or $$I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx = \frac{m}{m+n} I_{m-1,n-1}.$$

Illustrative Examples

Example 1 : *If* $u_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx$ *and* $n > 1$,

show that $\quad u_n + n(n-1) u_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}.$

Hence evaluate $\displaystyle\int_0^{\pi/2} x^5 \sin x \, dx.$

(Kanpur 2005; Gorakhpur 05; Avadh 07; Meerut 12B)

Solution : We have $u_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx$

$$= \left[x^n \,.\, (-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} n \,.\, x^{n-1} \,.\, (-\cos x)\, dx,$$

[Integrating by parts taking $\sin x$ as the 2nd function]

$$= 0 + n \int_0^{\pi/2} x^{n-1} \cos x \, dx$$

$$= n \left[\left\{ x^{n-1} . \sin x \right\}_0^{\pi/2} - \int_0^{\pi/2} (n-1) . x^{n-2} . \sin x \, dx \right],$$

again integrating by parts

$$= n . (\tfrac{1}{2} \pi)^{n-1} - n (n-1) \int_0^{\pi/2} x^{n-2} \sin x \, dx.$$

Thus $\quad u_n = n (\tfrac{1}{2} \pi)^{n-1} - n (n-1) u_{n-2}.$ ...(1)

$\therefore \quad u_n + n (n-1) u_{n-2} = n (\tfrac{1}{2} \pi)^{n-1}.$ **Proved.**

Now to evaluate $\int_0^{\pi/2} x^5 \sin x \, dx$, put $n = 5$ in (1).

Then $\quad u_5 = 5 (\tfrac{1}{2} \pi)^{5-1} - 5 (5-1) u_3$

$$= 5 \cdot (\tfrac{1}{2} \pi)^4 - 20 [3 (\tfrac{1}{2} \pi)^{3-1} - 3 (3-1) u_1], \text{ putting } n = 3 \text{ in (1)}$$

$$= \frac{5}{16} \pi^4 - 15\pi^2 + 120 u_1.$$

Now $\quad u_1 = \int_0^{\pi/2} x \sin x \, dx$

$$= \left[x . (-\cos x) \right]_0^{\pi/2} + \int_0^{\pi/2} \cos x \, dx$$

$$= \left[0 + \sin x \right]_0^{\pi/2} = \left[\sin \frac{\pi}{2} - \sin 0 \right] = 1 .$$

Hence $\quad u_5 = \int_0^{\pi/2} x^5 \sin x \, dx = \dfrac{5\pi^4}{16} - 15\pi^2 + 120.$

Example 2(i) : *Integrating by parts twice or otherwise, obtain a reduction formula for*

$$I_m = \int_0^{\infty} e^{-x} \sin^m x \, dx, \text{ where } m \geq 2$$

in the form $(1 + m^2) I_m = m (m-1) I_{m-2}$ *and hence evaluate* I_4. **(Agra 2014)**

Solution : We have $I_m = \int_0^{\infty} e^{-x} \sin^m x \, dx$

$$= \left[\sin^m x . (-e^{-x}) \right]_0^{\infty} + \int_0^{\infty} m \sin^{m-1} x \cos x \, e^{-x} \, dx,$$

integrating by parts taking e^{-x} as the second function

$$= 0 + m \int_0^{\infty} (\sin^{m-1} x \cos x) . e^{-x} \, dx$$

$$= m\left[\sin^{m-1} x \cos x\,.(-e^{-x})\right]_0^{\infty}$$

$$-m\int_0^{\infty}[-\sin^m x+(m-1)\sin^{m-2} x\cos^2 x]\,.(-e^{-x})\,dx$$

$$= 0+m\int_0^{\infty} e^{-x}[-\sin^m x+(m-1)\sin^{m-2} x(1-\sin^2 x)]\,dx$$

$$= m\int_0^{\infty} e^{-x}[(m-1)\sin^{m-2} x-m\sin^m x]\,dx$$

$$= m(m-1)I_{m-2}-m^2 I_m$$

or $(1+m^2)I_m = m(m-1)I_{m-2}$ **Proved.**

or $$I_m = \frac{m(m-1)}{1+m^2} I_{m-2}. \qquad ...(1)$$

To evaluate I_4, putting $m=4$ in (1), we get

$$I_4 = \frac{4(4-1)}{1+16} I_2 = \frac{12}{17} I_2$$

$$= \frac{12}{17}\left[\frac{2(2-1)}{1+4} I_0\right] = \frac{24}{85} I_0, \qquad [\text{To get } I_2, \text{ we put } m=2 \text{ in (1)}]$$

$$= \frac{24}{85}\int_0^{\infty} e^{-x}\sin^0 x\,dx = \frac{24}{85}\int_0^{\infty} e^{-x}\,dx = -\frac{24}{85}\left[e^{-x}\right]_0^{\infty} = \frac{24}{85}.$$

Example 2(ii) : *If* $I_m = \int_0^{\infty} e^{-ax}\sin^m x\,dx$, $(a>0, m\geq 0)$; *prove that* $(m^2+a^2)I_m = m(m-1)I_{m-2}$ *and hence evaluate* $\int_0^{\infty} e^{-x}\sin^4 x\,dx.$

(Kanpur 2006)

Solution : Proceed as in Ex. 2(i).

Example 3 : *Evaluate* $\int_0^{\infty} x e^{-2x}\cos x\,dx.$

(Rohilkhand 2014)

Solution : The given integral

$$I = \int_0^{\infty} x\,.(e^{-2x}\cos x)\,dx.$$

Integrating by parts taking $e^{-2x}\cos x$ as the 2nd function, we have

$$I = \left[x\frac{e^{-2x}}{r}\cos(x-\phi)\right]_0^{\infty} - \int_0^{\infty} 1\cdot\frac{e^{-2x}}{r}\cos(x-\phi)\,dx,$$

where $\phi = \tan^{-1}(b/a) = \tan^{-1}(-1/2)$

and $r = \sqrt{(a^2+b^2)} = \sqrt{(4+1)} = \sqrt{5}$

$$= \left[\lim_{x \to \infty} \frac{1}{r} \frac{x}{e^{2x}} \cos(x - \phi) - 0 \right] - \frac{1}{r} \int_0^{\infty} e^{-2x} \cos(x - \phi)\, dx$$

$$= 0 - \frac{1}{\sqrt{5}} \int_0^{\infty} e^{-2x} \cos(x - \phi)\, dx = -\frac{1}{\sqrt{5}} \left[\frac{1}{\sqrt{5}} e^{-2x} \cos(x - 2\phi) \right]_0^{\infty}$$

$$= -(1/5)\,[0 - e^0 \cos(-2\phi)] = \tfrac{1}{5} \cos 2\phi,$$

where $\phi = \tan^{-1}(b/a) = \tan^{-1}(-\tfrac{1}{2})$

$$= \tfrac{1}{5} [\{1 - \tan^2 \phi\}/\{1 + \tan^2 \phi\}], \text{ where } \tan \phi = -\tfrac{1}{2}$$

$$= \tfrac{1}{5} [(1 - \tfrac{1}{4})/(1 + \tfrac{1}{4})] = \tfrac{1}{5} \cdot \tfrac{3}{5} = \tfrac{3}{25}.$$

Example 4 : *Prove that*

$$\int_0^{\pi/2} \cos^m x \sin mx\, dx = \frac{1}{2^{m+1}} \left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \ldots + \frac{2^m}{m} \right].$$

Solution : Proceeding as in article 1.13 and taking $n = m$, we first establish the reduction formula

$$I_{m,m} = \frac{1}{2m} + \tfrac{1}{2} I_{m-1,m-1}. \qquad \ldots(1)$$

$$\therefore \quad I_{m,m} = \frac{1}{2m} + \frac{1}{2} \left[\frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2} \right].$$

$$\left[\because \text{ from (1), } I_{m-1,m-1} = \frac{1}{2(m-1)} + \tfrac{1}{2} I_{m-2,m-2} \right]$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2} I_{m-2,m-2}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3} I_{m-3,m-3}, \text{ and so on.}$$

Finally,
$$I_{m,m} = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \ldots + \frac{1}{2^{m-2} . 3} + \frac{1}{2^{m-1} .2} + \frac{1}{2^{m-1}} I_{1,1}.$$

But
$$I_{1,1} = \int_0^{\pi/2} \cos x \sin x\, dx = \left[\tfrac{1}{2} \sin^2 x \right]_0^{\pi/2} = \tfrac{1}{2}.$$

$$\therefore \quad I_{m,m} = \frac{1}{2^{m+1}} \left[2 + \frac{2^2}{2} + \frac{2^3}{3} + \ldots + \frac{2^m}{m} \right].$$

Example 5 : *Prove that if n be a positive integer,*

$$\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}.$$

(Kanpur 2008)

Solution : Proceeding as in article 1.14 and taking $m = n$, we first establish the reduction formula

$$I_{m,n}=\frac{n}{n+n}I_{n-1,n-1}=\frac{1}{2}I_{n-1,n-1}. \qquad ...(1)$$

Putting $(n-1)$ for n in (1), we have $I_{n-1,n-1}=\frac{1}{2}I_{n-2,n-2}$.

$\therefore$ $$I_{n,n}=\frac{1}{2}\cdot\frac{1}{2}I_{n-2,n-2}=\frac{1}{2^2}I_{n-2,n-2}.$$

Thus by repeated application of (1), we get

$$I_{n,n}=\frac{1}{2^n}I_{n-n,n-n}=\frac{1}{2^n}I_{0,0}.$$

But $$I_{0,0}=\int_0^{\pi/2}\cos^0 x\cos 0x\,dx=\int_0^{\pi/2}1\,.\,dx=\Big[\,x\,\Big]_0^{\pi/2}=\frac{\pi}{2}.$$

$\therefore$ $$I_{n,n}=\frac{1}{2^n}\cdot\frac{1}{2}\pi=\frac{\pi}{2^{n+1}}.$$

Example 6 : *Find the reduction formula for the integral* $\int\frac{\sin nx}{\sin x}dx$ *and show that* $\int_0^{\pi}\frac{\sin nx}{\sin x}dx=\pi$ *or* 0, *according as n is odd or even.* **(Kanpur 2009; Rohilkhand 14)**

Solution : To find the required reduction formula, consider

$$\sin nx-\sin(n-2)x=2\cos(n-1)x\sin x$$

or $$\frac{\sin nx}{\sin x}-\frac{\sin(n-2)x}{\sin x}=2\cos(n-1)x,\quad\text{dividing both sides by }\sin x$$

or $$\frac{\sin nx}{\sin x}=2\cos(n-1)x+\frac{\sin(n-2)x}{\sin x}.$$

Integrating both the sides, we have

$$\int\frac{\sin nx}{\sin x}dx=\frac{2\sin(n-1)x}{(n-1)}+\int\frac{\sin(n-2)x}{\sin x}dx,$$

which is the required reduction formula.

Now let

$$I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}dx=\left\{\frac{2\sin(n-1)x}{n-1}\right\}_0^{\pi}+\int_0^{\pi}\frac{\sin(n-2)x}{\sin x}dx=0+I_{n-2}.$$

Hence, $I_n=I_{n-2}=I_{n-4}=I_{n-6}=\dots$

i.e., when n is even

$$I_n=I_2=\int_0^{\pi}\frac{\sin 2x}{\sin x}dx=2\int_0^{\pi}\cos x\,dx=2\Big[\sin x\Big]_0^{\pi}=0$$

and *when n is odd*

$$I_n=I_1=\int_0^{\pi}\frac{\sin x}{\sin x}dx=\int_0^{\pi}dx=\pi.$$

Comprehensive Exercise 3

Evaluate the following integrals :

1. (i) $\int_0^{\pi/2} x^3 \sin 3x \, dx.$ (ii) $\int_0^{\pi} x \sin^2 x \cos x \, dx.$

(iii) $\int_0^{1} x^6 \sin^{-1} x \, dx.$ **(Kanpur 2008)** (iv) $\int_0^{a} \sqrt{(a^2 - x^2)} \left\{\cos^{-1}\left(\frac{x}{a}\right)\right\}^2 dx.$

2. (i) $\int_0^{\pi} x \sin^3 x \, dx.$ (ii) $\int x \sin^4 x \, dx.$

(iii) $\int_0^{\pi/2} x \cos^3 x \, dx.$ (iv) $\int e^x (x \cos x + \sin x) \, dx.$

3. (i) $\int x^2 e^{x \cos \alpha} \sin (2x \sin \alpha) \, dx.$ (ii) $\int_0^{1} (\sin^{-1} x)^4 \, dx.$

(iii) $\int_1^{\infty} \frac{x^4 + 1}{x^2 (x^2 + 1)^2} \, dx.$ (iv) $\int_1^{\infty} \frac{x^2 + 3}{x^6 (x^2 + 1)} \, dx.$

4. If $u_n = \int_0^{\pi/2} x^n \sin mx \, dx$, prove that $u_n = \frac{n \pi^{n-1}}{m^2 \,.\, 2^{n-1}} - \frac{n (n-1)}{m^2} u_{n-2}$,

if m is of the form $4r + 1$. **(Kanpur 2011)**

5. If $I_n = \int_0^{\pi/2} x^n \sin (2p + 1) x \, dx$, prove that

$$I_n + \frac{n (n-1)}{(p+1)^2} I_{n-2} = (-1)^p \frac{n}{(2p+1)^2} \left(\frac{\pi}{2}\right)^{n-1},$$

where n and p are positive integers.

Hence deduce that $\int_0^{\pi/2} x^3 \sin 3x \, dx = \frac{2}{27} - \frac{\pi^2}{12}.$

6. If $u_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$ and $n > 1$, prove that

$$u_n = \frac{(n-1)}{n} u_{n-2} + \frac{1}{n^2}.$$

Hence deduce that $u_5 = \frac{149}{225}.$ **(Gorakhpur 2006; Rohilkhand 07)**

7. Prove that if n be a positive integer greater than unity, then

$$\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}.$$ **(Avadh 2004; Kanpur 10)**

8. If $I_{(m,n)} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$, prove that $I_{(m,n)} = \left\{\dfrac{m(m-1)}{m^2 - n^2}\right\} I_{(m-2,n)}$.

(Purvanchal 2014)

9. Prove that $\int_0^{\pi} \left(\dfrac{\sin n\theta}{\sin \theta}\right)^2 d\theta = n\pi$.

10. If $S_n = \int_0^{\pi/2} \dfrac{\sin(2n-1)x}{\sin x}\, dx$, $V_n = \int_0^{\pi/2} \left(\dfrac{\sin nx}{\sin x}\right)^2 dx$, ($n$ is an integer), show that $S_{n+1} - S_n = 0$, $V_{n+1} - V_n = S_{n+1}$.

Answers 3

1. (i) $-\dfrac{\pi^2}{12} + \dfrac{2}{27}$. (ii) $-\frac{4}{9}$. (iii) $\dfrac{\pi}{14} - \dfrac{16}{245}$.

(iv) $\dfrac{\pi a^2}{8}(1 + \frac{1}{6}\pi^2)$.

2. (i) $\frac{2}{3}\pi$.

(ii) $-\dfrac{x\sin^3 x\cos x}{4} + \dfrac{\sin^4 x}{16} + \dfrac{3}{16}x^2 - \dfrac{3}{16}x\sin 2x - \dfrac{3}{32}\cos 2x$.

(iii) $\frac{1}{3}[\pi - \frac{7}{3}]$.

(iv) $\frac{1}{2}e^x[x(\cos x + \sin x) - \cos x]$.

3. (i) $\frac{1}{2}x^2 e^{ax}\sin(bx - \phi) - \frac{1}{2}xe^{ax}\sin(bx - 2\phi) + \frac{1}{4}e^{ax}\sin(bx - 3\phi)$,

where $a = 2\cos\alpha$, $b = 2\sin\alpha$ and $\phi = \alpha$.

(ii) $\frac{1}{16}\pi^4 - 3\pi^2 + 24$. (iii) $\frac{3}{2} - \frac{1}{4}\pi$.

(iv) $\frac{1}{30}(58 - 15\pi)$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ..." so that the following statements are complete and correct.

1. $\int \sin^n x\, dx = \ldots\ldots + \dfrac{n-1}{n}\int \sin^{n-2} x\, dx.$

2. $\int \tan^n x\, dx = \ldots\ldots - \int \tan^{n-2} x\, dx.$

3. $\int \sec^n x\,dx = \ldots\ldots + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx.$

4. $\int_0^{\pi/2} \sin^6 \theta\,d\theta = \ldots\ldots.$

5. $\int_0^{\pi/2} \sin^4 x \cos^6 x\,dx = \ldots\ldots.$

6. If $u_n = \int_0^{\pi/2} \theta \sin^n \theta\,d\theta$ and $n > 1$, then $u_n = \dfrac{(n-1)}{n} u_{n-2} + \ldots\ldots.$

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, then

(a) $I_n + I_{n-2} = \dfrac{1}{n}$

(b) $I_n + I_{n-2} = \dfrac{1}{n-1}$

(c) $I_n + I_{n-2} = \dfrac{1}{n-2}$

(d) $I_n + I_{n-2} = \dfrac{n}{n-1}.$

8. If $I_n = \int \cot^n x\,dx$, then

(a) $I_n = \dfrac{\cot^{n-1} x}{n-1} + \int \cot^{n-2} x\,dx$

(b) $I_n = -\dfrac{\cot^{n-1} x}{n-1} + \int \cot^{n-2} x\,dx$

(c) $I_n = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$

(d) $I_n = \dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$

True or False:

Write 'T' for true and 'F' for false statement.

9. $\int \operatorname{cosec}^n x\,dx = -\dfrac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx.$

10. $\int_0^{\pi/2} \cos^6 x\,dx = \dfrac{5\pi}{16}.$

11. If $u_n = \int_0^{\pi/2} x^n \sin x\,dx$ and $n > 1$, then $u_n + n(n-1)\,u_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}.$

12. $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx = \dfrac{1}{15}$.

Answers

1. $-\dfrac{1}{n}\sin^{n-1} x \cos x.$

2. $\dfrac{\tan^{n-1} x}{n-1}$.

3. $\dfrac{\sec^{n-2} x \tan x}{n-1}$.

4. $\dfrac{5\pi}{32}$.

5. $\dfrac{3\pi}{512}$.

6. $\dfrac{1}{n^2}$.

7. (b).

8. (c).

9. T.

10. F.

11. T.

12. F.

Chapter 2

Reduction Formulae Continued (For Irrational Algebraic and Transcendental Functions)

2.1 Reduction Formulae for $\int x^m (a+bx^n)^p\, dx$

$\int x^m (a+bx^n)^p\, dx$ can be connected with any one of the following six integrals:

(i) $\int x^{m-n} (a+bx^n)^p\, dx$, **(ii)** $\int x^m (a+bx^n)^{p-1}\, dx$,

(iii) $\int x^{m+n} (a+bx^n)^p\, dx$, **(iv)** $\int x^m (a+bx^n)^{p+1}\, dx$,

(v) $\int x^{m-n} (a+bx^n)^{p+1}\, dx$, **(vi)** $\int x^{m+n} (a+bx^n)^{p-1}\, dx$.

Rule for Connection : In order to connect the given integral with any one of the six integrals we use the following rule :

Let $P = x^{\lambda+1} (a+bx^n)^{\mu+1}$, where λ and μ are the smaller of the indices of x and $(a+bx^n)$ in the two expressions whose integrals we want to connect.

Find (dP/dx) and rearrange this as a linear combination of the expressions whose integrals are to be connected.

Finally integrate both sides and transpose suitably to get the required reduction formula.

(i) **To connect $\int x^m (a+bx^n)^p\,dx$ with $\int x^{m-n}(a+bx^n)^p\,dx$.**

Here $\lambda = m-n$ and $\mu = p$ (choosing smaller indices for λ and μ).

Let us take $P = x^{\lambda+1}(a+bx^n)^{\mu+1} = x^{m-n+1}(a+bx^n)^{p+1}$.

$$\therefore \quad (dP/dx) = (m-n+1)\,x^{m-n}(a+bx^n)^{p+1} + x^{m-n+1}(p+1)(a+bx^n)^p\,bnx^{n-1}$$

$$= (m-n+1)\,x^{m-n}(a+bx^n)^p(a+bx^n) + (p+1)\,bn\,x^m(a+bx^n)^p$$

$$= a(m-n+1)\,x^{m-n}(a+bx^n)^p + b(m-n+1)\,x^m(a+bx^n)^p + (p+1)\,bn\,x^m(a+bx^n)^p$$

$$= a(m-n+1)\,x^{m-n}(a+bx^n)^p + bx^m(a+bx^n)^p(m-n+1+pn+n)$$

i.e., $$(dP/dx) = a(m-n+1)\,x^{m-n}(a+bx^n)^p + b(m+pn+1)\,x^m(a+bx^n)^p.$$

Thus (dP/dx) is expressed as a linear combination of the two expressions whose integrals are to be connected.

Integrating both the sides, we have

$$P = a(m-n+1)\int x^{m-n}(a+bx^n)^p\,dx + b(m+np+1)\int x^m(a+bx^n)^p\,dx.$$

Now putting the value of P and transposing suitably, we get

$$b(pn+m+1)\int x^m(a+bx^n)^p\,dx$$

$$= x^{m-n+1}(a+bx^n)^{p+1} - a(m-n+1)\int x^{m-n}(a+bx^n)^p\,dx$$

or $$\int \boldsymbol{x^m(a+bx^n)^p\,dx}$$

$$= \boldsymbol{\frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(pn+m+1)} - \frac{a(m-n+1)}{b(pn+m+1)}\int x^{m-n}(a+bx^n)^p\,dx.}$$

(ii) To connect $\int x^m(a+bx^n)^p\,dx$ *with* $\int x^m(a+bx^n)^{p-1}\,dx$.

Here $\lambda = m$ and $\mu = p-1$; (choosing smaller indices for λ and μ).

Now let $P = x^{\lambda+1}(a+bx^n)^{\mu+1} = x^{m+1}(a+bx^n)^p$.

$$\therefore \quad \frac{dP}{dx} = (m+1)\,x^m(a+bx^n)^p + x^{m+1}.\,p\,(a+bx^n)^{p-1}\,bn\,x^{n-1}$$

$$= (m+1)\,x^m(a+bx^n)^p + pn\,.\,x^m(a+bx^n)^{p-1}\,bx^n \qquad \textbf{(Note)}$$

$$= (m+1)\,x^m(a+bx^n)^p + pn\,.\,x^m(a+bx^n)^{p-1}(a+bx^n-a)$$

$$= (pn+m+1)\,x^m(a+bx^n)^p - a\,pn\,x^m(a+bx^n)^{p-1}.$$

Thus (dP/dx) has been expressed as a linear combination of the two expressions whose integrals are to be connected.

Integrating both the sides, we have

$$P = (pn + m + 1)\int x^m (a + bx^n)^p\,dx - apn\int x^m (a + bx^n)^{p-1}\,dx.$$

Now putting the value of P, dividing by $(pn + m + 1)$ and transposing suitably, we get

$$\boldsymbol{\int x^m (a + bx^n)^p\,dx = \frac{x^{m+1}(a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1}\int x^m (a + bx^n)^{p-1}\,dx.}$$

(iii) To connect $\int x^m (a + bx^n)^p\,dx$ with $\int x^{m+n} (a + bx^n)^p\,dx$.

Here $\lambda = m$ and $\mu = p$.

Now let $P = x^{\lambda+1}(a + bx^n)^{\mu+1} = x^{m+1}(a + bx^n)^{p+1}$.

$$\therefore \quad \frac{dP}{dx} = x^{m+1}(p + 1)(a + bx^n)^p\,nbx^{n-1} + (m + 1)x^m (a + bx^n)^{p+1}$$

$$= bn(p + 1)x^{m+n}(a + bx^n)^p + (m + 1)x^m (a + bx^n)^p (a + bx^n) \quad \textbf{(Note)}$$

$$= bn(p + 1)x^{m+n}(a + bx^n)^p + (m + 1)ax^m (a + bx^n)^p + b(m + 1)x^{m+n}(a + bx^n)^p$$

$$= b(np + n + m + 1)x^{m+n}(a + bx^n)^p + (m + 1)ax^m (a + bx^n)^p,$$

which is linear combination of the two expressions whose integrals are to be connected.

Integrating both sides, we have

$$P = b(np + n + m + 1)\int x^{m+n}(a + bx^n)^p\,dx + (m + 1)a\int x^m (a + bx^n)^p\,dx.$$

Now putting the value of P, dividing by $a(m + 1)$ and transposing suitably, we get

$$\boldsymbol{\int x^m (a + bx^n)^p\,dx = \frac{x^{m+1}(a + bx^n)^{p+1}}{a(m + 1)} - \frac{b(np + n + m + 1)}{a(m + 1)}\int x^{m+1}(a + bx^n)^p\,dx.}$$

(iv) To connect $\int x^m (a + bx^n)^p\,dx$ with $\int x^m (a + bx^n)^{p-1}\,dx$.

Here $\lambda = m$ and $\mu = p$, λ being the lesser index of x, and μ being the lesser index of $(a + bx^n)$ in both the integrals.

Now let $P = x^{\lambda+1}(a + bx^n)^{\mu+1} = x^{m+1}(a + bx^n)^{p+1}$.

$$\therefore \quad \frac{dP}{dx} = x^{m+1}(p + 1)(a + bx^n)^p.\,nbx^{n-1} + (m + 1)x^m (a + bx^n)^{p+1}$$

$$= n(p + 1)x^m (a + bx^n)^p\,bx^n + (m + 1)x^m (a + bx^n)^{p+1} \quad \textbf{(Note)}$$

$$= n(p+1)x^m(a+bx^n)^p(a+bx^n-a)+(m+1)x^m(a+bx^n)^{p+1}$$

$$= n(p+1)x^m(a+bx^n)^{p+1} - an(p+1)x^m(a+bx^n)^p + (m+1)x^m(a+bx^n)^{p+1}$$

$$= (np+n+m+1)x^m(a+bx^n)^{p+1} - an(p+1)x^m(a+bx^n)^p$$

i.e., (dP/dx) is a linear combination of the two expressions whose integrals are to be connected.

Integrating both the sides, we have

$$P = \{n(p+1)+m+1)\}\int x^m(a+bx^n)^{p+1}\,dx - an(p+1)\int x^m(a+bx^n)^{p+1}\,dx.$$

Now putting the value of P, dividing by $\{an(p+1)\}$ and transposing suitably,

we get $\displaystyle\int \boldsymbol{x^m(a+bx^n)^p\,dx}$

$$\boldsymbol{= -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{(np+n+m+1)}{an(n+1)}\int x^m(a+bx^n)^{p+1}\,dx.}$$

(v) To connect $\int x^m(a+bx^n)^p\,dx$ with $\int x^{m-n}(a+bx^n)^{p+1}\,dx$.

Here $\lambda = m-n$ and $\mu = p$, [as also in case (i)].

$\therefore$ $P = x^{\lambda+1}(a+bx^n)^{\mu+1} = x^{m-n+1}(a+bx^n)^{p+1}.$

$\therefore$ $$\frac{dP}{dx} = x^{m-n+1}(p+1)(a+bx^n)^p . nbx^{n-1} + (m-n+1)x^{m-n}(a+bx^n)^{p+1}$$

$$= bn(p+1)x^m(a+bx^n)^p + (m-n+1)x^{m-n}(a+bx^n)^{p+1}$$

= a linear combination of the two expressions whose integrals are to be connected.

$\therefore$ integrating both sides, we have

$$P = bn(p+1)\int x^m(a+bx^n)^p\,dx + (m-n+1)\int x^{m-n}(a+bx^n)^{p+1}\,dx$$

or $$bn(p+1)\int x^m(a+bx^n)^p\,dx = P - (m-n+1)\int x^{m-n}(a+bx^n)^{p+1}\,dx.$$

Now putting the value of P and dividing by $bn(p+1)$, we get

$$\boldsymbol{\int x^m(a+bx^n)^p\,dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{bn(p+1)} - \frac{(m-n+1)}{bn(p+1)}\int x^{m-n}(a+bx^n)^{p+1}\,dx.}$$

(vi) To connect $\int x^m(a+bx^n)^p\,dx$ with $\int x^{m+n}(a+bx^n)^{p-1}\,dx$.

Here $\lambda = m$ and $\mu = p-1$.

$\therefore \quad P = x^{\lambda+1}(a + bx^n)^{\mu+1} = x^{m+1}(a + bx^n)^p.$

And $\quad \dfrac{dP}{dx} = x^{m+1}.\, p\,(a + bx^n)^{p-1}\, bn\, x^{n-1} + (m+1)\, x^m\,(a + bx^n)^p$

$$= bp\, nx^{m+n}(a + bx^n)^{p-1} + (m+1)\, x^m\,(a + bx^n)^p.$$

Thus (dP/dx) is expressed as a linear combination of the two expressions whose integrals are to be connected.

Integrating both sides, we have

$$P = b\,pn \int x^{m+n}(a + bx^n)^{p-1}\,dx + (m+1)\int x^m(a + bx^n)^p\,dx$$

or
$$(m+1)\int x^m(a + bx^n)^p\,dx = P - bn\,p\int x^{m+n}(a + bx^n)^{p-1}\,dx.$$

Now putting the value of P and dividing by $(m+1)$, we get

$$\boldsymbol{\int x^m(a + bx^n)^p\,dx = \frac{x^{m+1}(a + bx^n)^p}{(m+1)} - \frac{bnp}{(m+1)}\int x^{m+n}(a + bx^n)^{p-1}\,dx.}$$

Illustrative Examples

Example 1 : *If I_n denotes $\int_0^1 x^p(1 - x^q)^n\,dx$, where p, q and n are positive, prove that $(nq + p + 1)\, I_n = nq\, I_{n-1}$.*

Hence evaluate I_n when n is a positive integer.

Solution : Here we have to connect

$$\int_0^1 x^p(1 - x^q)^n\,dx \quad \text{with} \quad \int_0^1 x^p(1 - x^q)^{n-1}\,dx.$$

$\therefore$ Here $\lambda =$ lesser index of $x = p$;

$\mu =$ lesser index of $(1 - x^q) = n - 1$.

$\therefore \quad P = x^{\lambda+1}(1 - x^q)^{\mu+1} = x^{p+1}(1 - x^q)^n.$

Hence $\quad \dfrac{dP}{dx} = (p+1)\, x^p(1 - x^q)^n + x^{p+1}.\, n(1 - x^q)^{n-1}.\,(-\, qx^{q-1})$

$$= (p+1)\, x^p(1 - x^q)^n + nqx^p(1 - x^q)^{n-1}.\,(-\, x^q)$$

$$= (p+1)\, x^p(1 - x^q)^n + nqx^p(1 - x^q)^{n-1}.\,\{(1 - x^q) - 1\} \qquad \textbf{(Note)}$$

$$= (p+1)\, x^p(1 - x^q)^n + nqx^p(1 - x^q)^n - nqx^p(1 - x^q)^{n-1}$$

$$= (p + 1 + nq)\, x^p(1 - x^q)^n - nqx^p(1 - x^q)^{n-1}.$$

Thus (dP/dx) is expressed as a linear combination of the two expressions whose integrals are to be connected. Therefore integrating both sides, we have

$$P = (p + 1 + nq)\int x^p(1 - x^q)^n\,dx - nq\int x^p(1 - x^q)^{n-1}\,dx.$$

$\therefore \quad (p + 1 + nq) \int_0^1 x^p (1 - x^q)^n \, dx$

$$= \left[x^{p+1} (1 - x^q)^n \right]_0^1 + nq \int_0^1 x^p (1 - x^q)^{n-1} \, dx,$$

putting the value of P, transposing and also putting the limits of integration

$$= 0 + nq \int_0^1 x^p (1 - x^q)^{n-1} \, dx.$$

Thus $\quad (qn + p + 1) I_n = nq I_{n-1}$ ...(1) **Proved.**

or $\quad I_n = \dfrac{nq}{qn + p + 1} \cdot I_{n-1}$

$$= \frac{nq}{qn + p + 1} \cdot \left[\frac{(n-1) q}{\{(n-1) q + p + 1\}} I_{n-2} \right],$$

putting $(n - 1)$ for n in (1) to get I_{n-1} in terms of I_{n-2}.

Proceeding similarly by successive reduction, we have finally

$$I_n = \frac{nq}{nq + p + 1} \cdot \frac{(n-1) q}{(n-1) q + p + 1} \cdots \frac{q}{q + p + 1} I_0.$$

But $\quad I_0 = \displaystyle\int_0^1 x^p (1 - x^q)^0 \, dx = \int_0^1 x^p \, dx = \left[\frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1}.$

$\therefore \quad I_n = \dfrac{nq}{nq + p + 1} \cdot \dfrac{(n-1) q}{(n-1) q + p + 1} \cdots \dfrac{q}{q + p + 1} \cdot \dfrac{1}{p+1}.$

Example 2 : *If I_n denotes $\int_0^a (a^2 - x^2)^n \, dx$, and $n > 0$, prove that*

$I_n = \dfrac{2na^2}{2n + 1} I_{n-1}.$ **(Avadh 2005; Kanpur 06)**

Hence evaluate $\int_0^a (a^2 - x^2)^3 \, dx.$

Solution : We have $\quad I_n = \int_0^a (a^2 - x^2)^n .1 \, dx$ **(Note)**

$$= \left[(a^2 - x^2)^n . x \right]_0^a - \int_0^a n (a^2 - x^2)^{n-1} (-2x) . x \, dx,$$

integrating by parts taking unity as the secnod function

$$= 0 + 2n \int_0^a (a^2 - x^2)^{n-1} x^2 \, dx, \qquad [\because n > 0]$$

$$= -2n \int_0^a (a^2 - x^2)^{n-1} . \{(a^2 - x^2) - a^2\} \, dx,$$

$[\because x^2 = -\{(a^2 - x^2) - a^2\}]$

$$= -2n\int_0^a (a^2-x^2)^n\,dx + 2na^2\int_0^a (a^2-x^2)^{n-1}\,dx$$

$$= -2n\,I_n + 2na^2\,I_{n-1}.$$

$$\therefore \quad (1+2n)\,I_n = 2na^2\,I_{n-1}$$

or $$I_n = \frac{2na^2}{2n+1}\,I_{n-1} \qquad \text{...(1)} \quad \textbf{Proved.}$$

$$\therefore \quad I_3 = \tfrac{6}{7}\,a^2\,I_2, \text{ putting } n = 3 \text{ in (1)}$$

$$= \tfrac{6}{7}\,a^2.\,[\tfrac{4}{5}\,a^2\,I_1], \text{ putting } n = 2 \text{ in (1) to get } I_2 \text{ in terms of } I_1$$

$$= \frac{24}{35}\,a^4 I_1.$$

Thus $$I_3 = \int_0^a (a^2-x^2)^3\,dx = \frac{24a^4}{35}\int_0^a (a^2-x^2)\,dx$$

$$= \frac{24a^4}{35}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{24a^4}{35}\left[a^3 - \frac{a^3}{3}\right] = \frac{16a^7}{35}.$$

2.2 Reduction Formulae for $\int \frac{dx}{(x^2+a^2)^n}$, where n is Positive

(Bundelkhand 2010; Meerut 12)

Let $I_n = \int \frac{1}{(x^2+a^2)^n}\,dx$. To form a reduction formula for I_n, we shall integrate by parts $\int \frac{1}{(x^2+a^2)^{n-1}}\,dx$, taking unity as the second function. Thus

$$\int \frac{1}{(x^2+a^2)^{n-1}}\cdot 1\,dx = \frac{x}{(x^2+a^2)^{n-1}} - \int x\cdot \frac{-(n-1)}{(x^2+a^2)^n}\cdot 2x\,dx$$

or $$I_{n-1} = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^n}\,dx \qquad \textbf{(Note)}$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{1}{(x^2+a^2)^{n-1}}\,dx$$

$$- 2(n-1)\,a^2\int \frac{1}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\,I_{n-1} - 2(n-1)\,a^2\,I_n.$$

$$\therefore \quad 2(n-1)\,a^2\,I_n = \frac{x}{(x^2+a^2)^{n-1}} + (2n-2-1)\,I_{n-1}$$

or $$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1},$$

which is the required reduction formula.

2.3 Reduction Formulae for $\int x^m \sqrt{(2ax - x^2)}\, dx$; m being a Positive Integer

(Bundelkhand 2010)

Let $$I_m = \int x^m \sqrt{(2ax - x^2)}\, dx = \int x^{m+1/2} \sqrt{(2a - x)}\, dx.$$

Integrating by parts taking $\sqrt{(2a-x)}$ as the 2nd function, we have

$$I_m = x^{m+1/2} \frac{(2a-x)^{3/2}}{(\frac{3}{2}).(-1)} - \int \left(m + \frac{1}{2}\right) x^{m-1/2} \frac{(2a-x)^{3/2}}{(\frac{3}{2}).(-1)} dx$$

$$= -\frac{2}{3} x^{m-1} x^{3/2} (2a-x)^{3/2} + \frac{2m+1}{3} \int x^{m-1/2} (2a-x) \sqrt{(2a-x)}\, dx$$

$$= -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{2m+1}{3} \int 2a x^{m-1/2} \sqrt{(2a-x)}\, dx$$

$$- \frac{2m+1}{3} \int x^{m-1/2}. x \sqrt{(2a-x)}\, dx$$

$$= -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{(2m+1)\, 2a}{3} \int x^{m-1} x^{1/2} \sqrt{(2a-x)^{1/2}}\, dx$$

$$- \frac{2m+1}{3} \int x^{m-1/2}. x^{1/2} x^{1/2} (2a-x)^{1/2}\, dx$$

$$= -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{2(2m+1)\, a}{3} \int x^{m-1} \sqrt{(2ax - x^2)}\, dx$$

$$- \frac{2m+1}{3} \int x^m \sqrt{(2ax - x^2)}\, dx$$

$$= -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{2(2m+1)}{3} 2I_{m-1} - \frac{2m+1}{3} I_m.$$

Transposing the last term to the left, we have

$$\left(1 + \frac{2m+1}{3}\right) I_m = -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{2(2m+1)\, a}{3} I_{m-1}$$

or $$\frac{2(m+2)}{3} I_m = -\frac{2}{3} x^{m-1} (2ax - x^2)^{3/2} + \frac{2(2m+1)\, a}{3} I_{m-1}$$

or $$I_m = -\frac{x^{m-1} (2ax - x^2)^{3/2}}{m+2} + \frac{(2m+1)\, a}{m+2} I_{m-1},$$

which is the required reduction formula.

2.4 Reduction Formulae for $\int e^{mx} x^n \, dx$ and $\int \frac{e^{mx}}{n^n} dx$, $(n > 0)$

(a) $\int e^{mx} x^n \, dx$, $(n > 0)$.

We have $\int e^{mx} x^n \, dx = x^n \frac{e^{mx}}{m} - \int n x^{n-1} \frac{e^{mx}}{m} dx$,

integrating by parts taking e^{mx} as the 2nd function

$$= \frac{x^n e^{mx}}{m} - \frac{n}{m} \int x^{n-1} e^{mx} \, dx,$$

which is the required reduction formula.

By repeated application of this formula the integral shall ultimately reduce to $\int x^0 e^{mx} \, dx$ and we have

$$\int x^0 e^{mx} \, dx = \int e^{mx} \, dx = e^{mx}/m.$$

(b) $\int \frac{e^{mx}}{n^n} dx$, $(n > 0)$.

We have $\int \frac{e^{mx}}{x^n} dx = \int e^{mx} . x^{-n} \, dx$

$$= e^{mx} \cdot \frac{x^{-n+1}}{-n+1} - \int \frac{x^{-n+1}}{-n+1} \cdot m e^{mx} \, dx,$$

integrating by parts regarding x^{-n} as the second function

$$= \frac{-e^{mx}}{(n-1) x^{n-1}} + \frac{m}{n-1} \int \frac{e^{mx}}{x^{n-1}} dx,$$

which is the required reduction formula.

2.5 Reduction Formulae for $\int a^x x^n \, dx$ and $\int (a^x / x^n) \, dx$

(a) $\int a^x x^n \, dx$.

Integrate by parts taking a^x as the second function. The required reduction formula is

$$\int a^x x^n \, dx = \frac{a^x x^n}{\log a} - \frac{n}{\log a} \int x^{n-1} a^x \, dx.$$

(b) $\int (a^x/x^n)\,dx.$

We have $\int (a^x/x^n)\,dx = \int a^x x^{-n}\,dx.$

Now integrate by parts taking x^{-n} as the 2nd function.

2.6 Reduction Formulae for $\int x^m (\log x)^n \,dx$.

Integrating by parts regarding x^m as the 2nd function, we get

$$\int x^m (\log x)^n \,dx = (\log x)^n . \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1}\,dx,$$

which is the required reduction formula.

Illustrative Examples

Example 1 : *Evaluate* $\int_0^\infty e^{-x} x^n \,dx$, *n being a positive integer.*

(Rohilkhand 2007; Kanpur 10, 12)

Solution : Integrating by parts regarding e^{-x} as the 2nd function, we get

$$\int_0^\infty e^{-x} x^n \,dx = \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n e^{-x} x^{n-1}\,dx.$$

Now $\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x}$, which is of the form $\frac{\infty}{\infty}$.

$\therefore$ differentiating the numerator and the denominator separately, we get

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \ldots = \lim_{x\to\infty} \frac{n(n-1)\ldots 1}{e^x} = 0.$$

Hence $\quad \left[-x^n e^{-x}\right]_0^\infty = -\lim_{x\to\infty} x^n e^{-x} - 0 = 0 - 0 = 0.$

Therefore $\int_0^\infty e^{-x} x^n \,dx = n \int_0^\infty e^{-x} x^{n-1}\,dx.$ ...(1)

Now applying the reduction formula (1) repeatedly, we get

$$\int_0^\infty e^{-x} x^n \,dx = n(n-1)(n-2)\ldots 2 . 1 \int_0^\infty e^{-x} x^0 \,dx$$

$$= n! \int_0^\infty e^{-x}\,dx = n! \left[-e^{-x}\right]_0^\infty$$

$$= n! \left[-\frac{1}{e^x}\right]_0^\infty = (n!) .1 = n!.$$

Example 2 : *Evaluate* $\int_0^1 x^m (\log x)^n\, dx$, *when* $m \geq 0$ *and* n *is an integer* ≥ 0.

Solution : Let $I_{m,n} = \int_0^1 x^m (\log x)^n\, dx$.

Integrating by parts taking x^m as the second function, we have

$$I_{m,n} = \left[\frac{x^{m+1} (\log x)^n}{m+1}\right]_0^1 - \frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx$$

$$= \frac{1}{m+1}\left[1^{m+1} (\log 1)^n - \lim_{x \to 0} x^{m+1} (\log x)^n\right]$$

$$- \frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx.$$

But $\qquad \log 1 = 0$.

Also $\qquad \lim_{x \to 0} x^{m+1} (\log x)^n = \lim_{x \to 0} \frac{(\log x)^n}{x^{-(m+1)}}, \qquad$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{n (\log x)^{n-1} . (1/x)}{-(m+1)\, x^{-(m+2)}}$$

$$= \lim_{x \to 0} \left(-\frac{n}{m+1}\right) \frac{(\log x)^{n-1}}{x^{-(m+1)}}.$$

Proceeding in this way, we ultimately have

$$\lim_{x \to 0} x^{m+1} (\log x)^n$$

$$= (\text{some number}) \times \lim_{x \to 0} \frac{1}{x^{-(m+1)}}$$

$$= \text{some number} \times \lim_{x \to 0} x^{m+1} = 0.$$

$$\therefore \qquad I_{m,n} = -\frac{n}{(m+1)} I_{m,n-1} \qquad \ldots(1)$$

$$= -\frac{n}{(m+1)} \cdot \left(-\frac{n-1}{m+1}\right) I_{m,n-2}, \text{ applying (1)}$$

$$= (-1)^2 \frac{n(n-1)}{(m+1)^2} I_{m,n-1} = \ldots = \ldots$$

Proceeding similarly by successive application of (1), we have ultimately

$$I_{m,n} = (-1)^n \frac{n(n-1)(n-2)\ldots 2.1}{(m+1)^n} I_{m,0}.$$

But $\qquad I_{m,0} = \int_0^1 x^m (\log x)^0\, dx = \int_0^1 x^m\, dx$

$$= \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}.$$

$$\therefore \qquad I_{m,n} = (-1)^n \cdot \frac{n!}{(m+1)^n} \cdot \frac{1}{(m+1)}$$

$$= (-1)^n \frac{n!}{(m+1)^{n+1}}.$$

Example 3 : *Find the reduction formula for* $\int \{x^m/(\log x)^n\}\, dx$.

Solution : We have

$$\int \frac{x^m}{(\log x)^n} dx = \int x^{m+1} \left[\frac{1}{(\log x)^n} \cdot \frac{1}{x}\right] dx$$

$$= \int x^{m+1} \cdot \left[(\log x)^{-n} \frac{1}{x}\right] dx. \qquad \textbf{(Note)}$$

Now integrating by parts regarding x^{m+1} as the first function, we have

$$\int \frac{x^m\, dx}{(\log x)^n} = x^{m+1} \frac{(\log x)^{-n+1}}{-n+1} - \int (m+1)\, x^m \frac{(\log x)^{-n+1}}{-n+1} dx$$

$$= -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m}{(\log x)^{n-1}} dx,$$

which is the required reduction formula.

Comprehensive Exercise 1

1. Prove the reduction formula

$$\int (a^2 + x^2)^{n/2} dx = \frac{x\,(a^2 + x^2)^{n/2}}{(n+1)} + \frac{na^2}{(n+1)} \int (a^2 + x^2)^{(n/2)-1} dx.$$

Hence evaluate $\int (x^2 + a^2)^{5/2} dx$. **(Bundelkhand 2005)**

2. Find a reduction formula for $\int x^m (1 + x^2)^{n/2} dx$,

where m and n are positive integers.

Hence evaluate $\int x^5 (1 + x^2)^{7/2} dx$.

3. If $I_{m,n} = \int \frac{x^m\, dx}{(1+x^2)^n}$, prove that

$$2(n-1)\, I_{m,n} = -\, x^{m-1} (x^2 + 1)^{1-n} + (m-1)\, I_{m-2,\, n-1}.$$

4. If $\phi(n) = \int_0^x \frac{x^n\, dx}{\sqrt{(x-1)}}$, prove that

$$(2n+1)\,\phi(n) = 2x^n \sqrt{(x-1)} + 2n\,\phi(n-1).$$

5. If I_n denotes $\int_0^\infty \frac{1}{(a^2+x^2)^n}\,dx$,

where n is a positive integer ≥ 2, prove that

$$I_n = \frac{2n-3}{2a^2(n-1)} I_{n-1}.$$

Hence or otherwise evaluate $\int_0^\infty \frac{1}{(a^2+x^2)^4}\,dx.$

6. If $I_m = \int_0^{2a} x^m \sqrt{(2ax-x^2)}\,dx$, prove that

$$2^m\, m\,!\,.\,(m+2)\,!\,I_m = a^{m+2}\,(2m+1)\,!\,\pi.$$

Hence or otherwise evaluate $\int_0^{2a} x^3 \sqrt{(2ax-x^2)}\,dx.$ **(Bundelkhand 2007; 10)**

7. If $I_n = \int x^n (a-x)^{1/2}\,dx$, prove that

$$(2n+3)\,I_n = 2an\,I_{n-1} - 2x^n (a-x)^{3/2}.$$ **(Bundelkhand 2011)**

Hence evaluate $\int_0^a x^2 \sqrt{(ax-x^2)}\,dx.$

8. If $u_n = \int x^n (a^2-x^2)^{1/2}\,dx$, prove that

$$u_n = -\frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}\,a^2\,u_{n-2}.$$

Hence evaluate $\int_0^a x^4 \sqrt{(a^2-x^2)}\,dx.$

9. Show that $\int_0^\infty e^{-ax} x^n\,dx = \frac{n\,!}{a^{n+1}}$,

where a is a positive quantity and n is a positive integer.

10 Evaluate $\int_0^1 (\log x)^4\, x^m\,dx.$

11. If m and n are positive integers, and

$$f(m, n) = \int_0^1 x^{n-1} (\log x)^m\,dx,$$

prove that

$$f(m, n) = -(m/n)\, f(m-1, n).$$

Deduce that $f(m, n) = (-1)^m.\, m\,!/n^{m+1}$.

12. Evaluate $\int_0^\infty \frac{x}{(1+e^x)}\,dx.$

Answers 1

1. $\dfrac{x(a^2+x^2)^{5/2}}{6}+\dfrac{5a^2}{24}x(a^2+x^2)^{3/2}+\dfrac{5a^4}{16}\left[x\sqrt{(a^2+x^2)}+a^2\sin^{-1}\dfrac{x}{a}\right]$.

2. $\dfrac{1}{9}(1+x^2)^{9/2}\left[x^4-\dfrac{4}{11}x^2(1+x^2)+\dfrac{8}{143}(1+x^2)^2\right]$.

5. $\dfrac{5\pi}{32a^7}$.

6. $\dfrac{7\pi a^5}{8}$.

7. $\dfrac{5\pi a^4}{128}$.

8. $\dfrac{\pi a^6}{32}$.

10. $\dfrac{24}{(m+1)^5}$.

12. $\dfrac{\pi^2}{12}$.

□

Chapter

3

Beta and Gamma Functions

Euler's Integrals : Beta and Gamma Functions :

3.1 Beta Function

(Meerut 2012; Kashi 13, 14)

Definition : *The definite integral*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad \text{for } m > 0, n > 0$$

is called the **Beta function** *and is denoted by* $\boldsymbol{B}(m,n)$ [*read as "Beta m, n"*].

Thus $$\mathbf{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

where m, n are any positive numbers, integral or fractional.

Beta function is also called the **Eulerian integral of the first kind.**

3.2 Elementary Properties of Beta Function

(Meerut 2012B; Lucknow 11)

(i) Symmetry of Beta function i.e., B (m, n) = B (n, m) :

We have $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, by the def. of the Beta function

$$= \int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{m-1}x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$= \mathbf{B}(n, m)$, by the def. of Beta function.

Hence $\mathbf{B}(m, n) = \mathbf{B}(n, m)$.

(ii) If *m* or *n* is a positive integer, *B* (*m*, *n*) can be evaluated in an explicit form:

Case I : When *n* is a positive integer : If $n = 1$, the result is obvious because

$$\mathbf{B}(m, 1) = \int_0^1 x^{m-1}(1-x)^{1-1}\,dx = \int_0^1 x^{m-1}\,dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}.$$

So let us take $n > 1$. We have

$$\mathbf{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^1 (1-x)^{n-1}x^{m-1}\,dx$$

$$= \left[(1-x)^{n-1}\cdot\frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1)\cdot\frac{x^m}{m}\,dx,$$

integrating by parts taking x^{m-1} as the second function

$$= 0 + \frac{n-1}{m}\cdot\int_0^1 x^m(1-x)^{n-2}\,dx \qquad [\because n > 1]$$

$$= \frac{n-1}{m}\cdot\int_0^1 x^{(m+1)-1}(1-x)^{(n-1)-1}\,dx$$

$$= \frac{n-1}{m}\mathbf{B}(m+1, n-1).$$

By the repeated application of this process, we get

$$\mathbf{B}(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot\frac{n-3}{m+2}\cdots\cdots\frac{1}{m+n-2}\mathbf{B}(m+n-1, 1)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2} (1-x)^0 \, dx$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2} \, dx$$

$$= \frac{(n-1)\,!}{m\,(m+1)\,(m+2) \cdots (m+n-2)} \cdot \left[\frac{x^{m+n-1}}{m+n-1} \right]_0^1 .$$

$$\therefore \qquad \mathbf{B}\,(m, n) = \frac{1}{m\,(m+1)\,(m+2) \cdots (m+n-2)\,(m+n-1)} .$$

Case II : When m is a positive integer : Since the Beta function is symmetrical in m and n *i.e.*, $\mathbf{B}\,(m, n) = \mathbf{B}\,(n, m)$, therefore by case I, we conclude that

$$\mathbf{B}\,(m, n) = \frac{(m-1)\,!}{n\,(n+1)\,(n+2) \cdots (n+m-2)\,(n+m-1)} .$$

(iii) If both m and n are positive integers, then

$$\mathbf{B}\,(\boldsymbol{m}, \boldsymbol{n}) = \frac{(\boldsymbol{m}-1)\,!\,(\boldsymbol{n}-1)\,!}{(\boldsymbol{m}+\boldsymbol{n}-1)\,!} .$$

From (ii), we have

$$\mathbf{B}\,(m, n) = \frac{(n-1)\,!}{m\,(m+1)\,(m+2) \cdots (m+n-2)\,(m+n-1)}$$

$$= \frac{(n-1)\,!\,(m-1)\,!}{(m+n-1)\,(m+n-2) \cdots (m+1)\,m\,(m-1)\,!} ,$$

writing the denominator in the reversed order and multiplying the Nr and Dr by $(m-1)\,!$

$$= \frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)\,!} .$$

Illustrative Examples

Example 1 : *Express the following integrals in terms of Beta function :*

(i) $\int_0^1 x^m (1-x^2)^n \, dx, m > -1, n > -1$; **(Lucknow 2010)**

(ii) $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} \, dx$ **(Meerut 2013B)**

(iii) $\int_0^1 x^{m-1} (1-x^2)^{n-1} \, dx$. **(Garhwal 2003)**

Solution : (i) We have

$$\int_0^1 x^m (1-x^2)^n \, dx = \int_0^1 x^{m-1} (1-x^2)^n \, . \, x \, dx \qquad \textbf{[Note]}$$

$$= \int_0^1 y^{(m-1)/2} (1-y)^n \cdot \frac{dy}{2}, \text{ putting } x^2 = y \text{ so that } 2x\,dx = dy$$

$$= \frac{1}{2}\int_0^1 y^{(m-1)/2} (1-y)^n \, dy$$

$$= \frac{1}{2}\int_0^1 y^{[(m+1)/2]-1} (1-y)^{(n+1)-1} \, dy$$

$$= \frac{1}{2} \mathbf{B}\left\{\frac{1}{2}(m+1), n+1\right\}.$$

(ii) We have $\displaystyle\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}}\,dx = \int_0^1 x^2 (1-x^5)^{-1/2}\,dx$

$$= \int_0^1 x^2 \cdot \frac{1}{x^4} (1-x^5)^{-1/2} . x^4 \, dx$$

$$= \int_0^1 x^{-2} (1-x^5)^{-1/2} x^4 \, dx$$

$$= \int_0^1 y^{-2/5} (1-y)^{-1/2} \cdot \frac{1}{5}\,dy, \quad \text{putting } x^5 = y \text{ so that } 5x^4\,dx = dy$$

$$= \frac{1}{5}\int_0^1 y^{-2/5} (1-y)^{-1/2}\,dy$$

$$= \frac{1}{5}\int_0^1 y^{(3/5)-1} (1-y)^{(1/2)-1}\,dy$$

$$= \frac{1}{5}\mathbf{B}\left(\frac{3}{5}, \frac{1}{2}\right).$$

(iii) Proceed as in part (i).

Example 2 : *Prove that*

$$\int_0^a (a-x)^{m-1} . x^{n-1}\,dx = a^{m+n-1}\,\mathbf{B}(m, n) = \frac{a^{m+n-1}\,\Gamma m\,\Gamma n}{\Gamma(m+n)}.$$

Solution : We have

$$\int_0^a (a-x)^{m-1} x^{n-1}\,dx$$

$$= \int_0^1 (a-ay)^{m-1} (ay)^{n-1}\, a\,dy, \quad \text{putting } x = ay$$

$$= \int_0^1 a^{(m-1)+(n-1)+1} (1-y)^{m-1} y^{n-1}\,dy$$

$$= a^{m+n-1} \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$$

$$= a^{m+n-1} \mathbf{B}(n, m) = a^{m+n-1} \mathbf{B}(m, n) \qquad [\because \mathbf{B}(m, n) = \mathbf{B}(n, m)]$$

$$= \frac{a^{m+n-1} \Gamma m \, \Gamma n}{\Gamma(m+n)}. \qquad \left[\because \mathbf{B}(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}\right]$$

Example 3 : *Show that if m, n are positive, then*

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx = (b-a)^{m+n-1} . \mathbf{B}(m, n)$$

$$= (b-a)^{m+n-1} . \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}.$$

(Agra 2003; Avadh 04)

Solution : The given integral is

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx.$$

Put $x = a + (b-a) y$ so that $dx = (b-a) \, dy$.

Also when $x = a, y = 0$ and when $x = b, y = 1$.

$$\therefore \int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx$$

$$= \int_0^1 [(b-a) y]^{m-1} [b-a-(b-a) y]^{n-1} . (b-a) \, dy$$

$$= \int_0^1 (b-a)^{m-1} . y^{m-1} . (b-a)^{n-1} . (1-y)^{n-1} . (b-a) \, dy$$

$$= (b-a)^{m+n-1} \int_0^1 y^{m-1} (1-y)^{n-1} \, dy$$

$$= (b-a)^{m+n-1} \mathbf{B}(m, n)$$

$$= (b-a)^{m+n-1} . \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}. \qquad \left[\because \mathbf{B}(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}\right]$$

Example 4 : *Show that* $\displaystyle\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} \, dx = \frac{1}{(a+b)^m . a^n} \mathbf{B}(m, n).$

Solution : The given integral

$$I = \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+bx)^{m+n}} \, dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} . \left(\frac{1-x}{a+bx}\right)^{n-1} . \frac{1}{(a+bx)^2} \, dx. \qquad \textbf{[Note]}$$

Put $\quad \dfrac{x}{a+bx}=\dfrac{y}{a+b}$ so that $\dfrac{(a+bx)\,.\,1-x\,.\,b}{(a+bx)^2}\,dx=\dfrac{dy}{a+b}$

i.e., $\quad \dfrac{1}{(a+bx)^2}\,dx=\dfrac{dy}{a\,(a+b)}\cdot$

Further $\quad \dfrac{1-x}{a+bx}=\dfrac{1}{a}\dfrac{a-ax}{a+bx}=\dfrac{1}{a}\left[\dfrac{a+bx-ax-bx}{a+bx}\right]=\dfrac{1}{a}\left[1-\dfrac{x\,(a+b)}{a+bx}\right]=\dfrac{1-y}{a}\cdot$

Also when $x=0, y=0$ and when $x=1, y=1$.

$$\therefore \quad I=\int_0^1\left(\frac{y}{a+b}\right)^{m-1}\left(\frac{1-y}{a}\right)^{n-1}\cdot\frac{dy}{a\,(a+b)}$$

$$=\frac{1}{(a+b)^m\,.\,a^n}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$=\frac{\mathbf{B}\,(m,n)}{(a+b)^m\,.\,a^n}\cdot$$

Example 5 : *Prove that* $\dfrac{\mathbf{B}\,(m+1,n)}{\mathbf{B}\,(m,n)}=\dfrac{m}{m+n}\cdot$

Solution : We have $\mathbf{B}\,(m+1,n)=\mathbf{B}\,(n,m+1)$

[By the symmetry of Beta function]

$$=\int_0^1 x^{n-1}(1-x)^{(m+1)-1}\,dx$$

$$=\int_0^1 (1-x)^m\,x^{n-1}\,dx \qquad \textbf{[Note]}$$

$$=\left[(1-x)^m\cdot\frac{x^n}{n}\right]_0^1-\int_0^1 m\,(1-x)^{m-1}(-1)\cdot\frac{x^n}{n}\,dx,$$

(integrating by parts)

$$=0+\frac{m}{n}\int_0^1 x^{n-1}.\,x\,(1-x)^{m-1}\,dx$$

$$=\frac{m}{n}\int_0^1 x^{n-1}\,[1-(1-x)]\,(1-x)^{m-1}\,dx$$

$$=\frac{m}{n}\left[\int_0^1 x^{n-1}(1-x)^{m-1}\,dx-\int_0^1 x^{n-1}(1-x)^m\,dx\right]$$

$$=\frac{m}{n}\left[\mathbf{B}\,(n,m)-\mathbf{B}\,(n,m+1)\right]$$

$$=\frac{m}{n}\,\mathbf{B}\,(m,n)-\frac{m}{n}\,\mathbf{B}\,(m+1,n)$$

or $\quad \left(1+\dfrac{m}{n}\right)\mathbf{B}\,(m+1,n)=\dfrac{m}{n}\,\mathbf{B}\,(m,n)$ $\quad$ [By transposition]

or $\quad (n+m)\,\mathbf{B}\,(m+1,n)=m\,\mathbf{B}\,(m,n)$

or $$\frac{\mathbf{B}(m+1, n)}{\mathbf{B}(m, n)}=\frac{m}{m+n}.$$

3.3 Another form of Beta Function

$$\mathbf{B}(\boldsymbol{m}, \boldsymbol{n})=\int_0^{\infty} \frac{\boldsymbol{x}^{\boldsymbol{m}-1}}{(\mathbf{1}+\boldsymbol{x})^{\boldsymbol{m}+\boldsymbol{n}}} \boldsymbol{d x}, \boldsymbol{m}>\mathbf{0}, \boldsymbol{n}>\mathbf{0}.$$

Proof : By the definition of Beta function, we have

$$\mathbf{B}(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1} d x.$$

Put $x=\dfrac{1}{1+y}$ so that $dx=-\dfrac{1}{(1+y)^2} dy$.

Also when $x \rightarrow 0, y \rightarrow \infty$ and when $x=1, y=0$.

$$\therefore \mathbf{B}(m, n)=\int_{\infty}^0 \frac{1}{(1+y)^{m-1}} \cdot\left[1-\frac{1}{1+y}\right]^{n-1} \cdot\left[-\frac{1}{(1+y)^2}\right] d y$$

$$=\int_0^{\infty} \frac{1}{(1+y)^{m-1+2}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} d y=\int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} d y$$

$$=\int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} d x. \qquad \text{[By a property of definite integrals] ...(1)}$$

Again since Beta function is symmetrical in m and n, we have

$$\mathbf{B}(m, n)=\mathbf{B}(n, m)=\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} d x, \text{ by (1)}.$$

Thus $$\mathbf{B}(m, n)=\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} d x=\int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} d x, \quad m>0, n>0.$$

Illustrative Examples

Example 1 : *Prove that* $\displaystyle\int_0^{\infty} \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}} d x=0, m>0, n>0$.

Solution : The given integral is

$$=\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} d x-\int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} d x$$

$$=\mathbf{B}(m, n)-\mathbf{B}(n, m)=\mathbf{B}(m, n)-\mathbf{B}(m, n)=0.$$

Example 2 : *Express* $\displaystyle\int_0^{\infty} \frac{x^{m-1}}{(a+b x)^{m+n}} d x$ *in terms of Beta function where* $m>0, n>0, a>0, b>0$.

Solution : In the given integral put $bx=ay$ *i.e.*, $x=(a/b) y$ so that $dx=(a/b)\, dy$. When $x=0, y=0$ and when $x \rightarrow \infty, y \rightarrow \infty$.

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \int_0^\infty \left(\frac{a}{b}y\right)^{m-1} \cdot \frac{1}{(a+ay)^{m+n}} \cdot \frac{a}{b}\,dy$$

$$= \int_0^\infty \frac{a^{m-1}y^{m-1}a}{b^{m-1}.\,a^{m+n}(1+y)^{m+n}b}\,dy = \frac{1}{a^n b^m}\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \frac{1}{a^n b^m}\mathbf{B}(m,n). \qquad \text{[By article 3.3]}$$

3.4 Gamma Function

(Gorakhpur 2006; Meerut 12; Kashi 13, 14; Kanpur 14)

The definite integral

$$\int_0^\infty e^{-x}x^{n-1}\,dx, \text{for } n > 0$$

is called the **Gamma Function** and is denoted by $\Gamma(n)$ [read as "Gamma n"]. Thus

$$\boldsymbol{\Gamma(n) = \int_0^\infty e^{-\infty}x^{n-1}\,dx, \text{ for } n > 0.}$$

Gamma function is also called **Eulerian integral of the second kind**.

3.5 Elementary Properties of Gamma Function

To prove that
(i) $\Gamma(n+1) = n\Gamma(n)$, *where* $n > 0$
and(ii) $\Gamma(n) = (n-1)!$, *where* n *is a positive integer.*

(Gorakhpur 2006; Kashi 14)

Proof : By the definition of gamma function, we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{(n+1)-1}\,dx = \int_0^\infty x^n e^{-x}\,dx$$

$$= \left[-e^{-x}x^n\right]_0^\infty + \int_0^\infty e^{-x}.\,nx^{n-1}\,dx, \qquad ...(1)$$

integrating by parts taking e^{-x} as the second function.

Now $\quad \lim\limits_{x\to\infty}\dfrac{x^n}{e^x} = \lim\limits_{x\to\infty}\dfrac{x^n}{1+x+(x^2/2!)+\ldots+(x^n/n!)+\ldots}$

$$= \lim_{x\to\infty}\frac{1}{\frac{1}{x^n}+\frac{1}{x^{n-1}}+\ldots+\frac{1}{n!}+\frac{x}{(n+1)!}+\ldots\ldots} = \frac{1}{\infty} = 0.$$

$\therefore$ from (1), we get $\Gamma(n+1) = 0 + n\displaystyle\int_0^\infty e^{-x}x^{n-1}\,dx$, $[\because n > 0]$

$= n\,\Gamma(n)$, which proves the result (i).

(ii) We have $\Gamma(n) = \Gamma[(n-1)+1] = (n-1)\Gamma(n-1)$. $[\because \Gamma(n+1) = n\Gamma(n)]$
Similarly $\Gamma(n-1) = (n-2)\Gamma(n-2)$, ... etc.

Hence if n is a + ive integer, then proceeding as above, we get

$$\Gamma(n) = (n-1)(n-2)\ldots 2.1\,\Gamma(1).$$

But $$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\,dx = \int_0^\infty e^{-x} x^0\,dx = \int_0^\infty e^{-x}.\,1\,dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[\lim_{x\to\infty}\frac{1}{e^x} - e^0\right] = -[0-1] = 1.$$

Hence $\Gamma(n) = (n-1)(n-2)\ldots 2.1.1 = (n-1)!$ if n is a + ive integer.

Remember : $\Gamma(n) = (n-1)\,\Gamma(n-1)$, where $n > 1$ and $\Gamma(1) = 1$.

Also it may be remarked that $\Gamma(0) = \infty$ and $\Gamma(-n) = \infty$ where n is a positive integer.

3.6 Some Transformations of Gamma Function

We have $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx. \qquad \ldots(1)$$

(i) Put $x = ay$ so that $dx = a\,dy$; when $x = 0, y = 0$ and when $x \to \infty, y \to \infty$.

$$\therefore \qquad \Gamma(n) = \int_0^\infty e^{-ay} a^n y^{n-1}\,dy.$$

Hence $$\int_0^\infty e^{-ay} y^{n-1}\,dy = \frac{\Gamma(n)}{a^n}.$$ **(Remember)**

(Kanpur 2006)

(ii) In (1) if we put $x = \log(1/y)$ or $y = e^{-x}$ so that $dy = -e^{-x}\,dx$,

then $$\Gamma(n) = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy.$$ **(Kanpur 2006)**

(iii) In (1) if we put $x^n = y$ so that $nx^{n-1}\,dx = dy$, we get

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-(y)^{1/n}}\,dy$$

or $$\int_0^\infty e^{-(y)^{1/n}}\,dy = n\,\Gamma(n) = \Gamma(n+1).$$

3.7 Relation Between Beta and Gamma Functions

To prove that $\mathbf{B}(\boldsymbol{m}, \boldsymbol{n}) = \dfrac{\Gamma(\boldsymbol{m})\,\Gamma(\boldsymbol{n})}{\Gamma(\boldsymbol{m}+\boldsymbol{n})}$, **where** $\boldsymbol{m} > \mathbf{0}, \boldsymbol{n} > \mathbf{0}$.

(Kanpur 2009, 12; Kashi 13, 14; Rohilkhand 13)

Proof : We have

$$\frac{\Gamma(m)}{z^m} = \int_0^\infty e^{-zx} x^{m-1}\,dx.$$ [See article 3.6, part (ii)]

$\therefore \qquad \Gamma(m) = z^m \int_0^\infty e^{-zx} . x^{m-1} \, dx = \int_0^\infty z^m e^{-zx} x^{m-1} \, dx.$

Multiplying both sides by $e^{-z} z^{n-1}$, we get

$$\Gamma(m) e^{-z} z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} \, dx. \qquad \ldots(1)$$

Now integrating both sides of (1) with respect to z from 0 to ∞, we get

$$\Gamma(m) \int_0^\infty e^{-z} z^{n-1} \, dz = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1} \, dx \right] dz$$

or

$$\Gamma(m) \Gamma(n) = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} \, dz \right] x^{m-1} \, dx$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} \, dx \qquad \text{[By article 3.6, part (ii)]}$$

$$= \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$= \Gamma(m+n) . B(m, n), \text{ by § 3 .}$$

$$\therefore \qquad \mathbf{B}(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}.$$

Thus remember that $\displaystyle\int_0^1 x^{m-1} (1-x)^{n-1} \, dx = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$. **(Kanpur 2009)**

Corollary : $\boldsymbol{\Gamma(n) \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}}$, **where $\boldsymbol{0 < n < 1}$.** **(Avadh 2008)**

Proof : We know that $\mathbf{B}(m, n) = \displaystyle\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx$, [See § 3]

and $\qquad \mathbf{B}(m, n) = \dfrac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$, where $m > 0$ and $n > 0$.

$$\therefore \qquad \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \, dx.$$

Putting $m + n = 1$ or $m = 1 - n$ in the above relation, we get

$$\frac{\Gamma(1-n) \Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x} \, dx, \text{ where } 0 < n < 1.$$

[Note that $m > 0 \Rightarrow 1 - n > 0 \Rightarrow n < 1$.]

But $\Gamma(1) = 1$. Also

$$\int_0^\infty \frac{x^{n-1}}{1+x} \, dx = \frac{\pi}{\sin n\pi}. \qquad \textbf{[Remember it]}$$

$$\therefore \qquad \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

3.8 The Value of $\Gamma\left(\frac{1}{2}\right)$

To prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. **(Agra 2000; Meerut 12)**

Proof : We know that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ...(1)

If we take $m = \frac{1}{2}$, $n = \frac{1}{2}$, then from (1), we have

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2. \qquad [\because \Gamma(1) = 1]$$

Thus
$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = B\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$
$$= \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx,$$

by the definition of Beta function

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$$

Now put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$$\therefore \qquad \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta$$
$$= 2\int_0^{\pi/2} d\theta = 2\Big[\,\theta\,\Big]_0^{\pi/2}$$
$$= 2\left[\tfrac{1}{2}\pi - 0\right] = \pi.$$

Taking square root of both the sides, we get

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}.$$ **(Remember)**

Important Deduction : *To prove that* $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

Proof : Let $\displaystyle I = \int_0^\infty e^{-x^2}\,dx$.

Put $x^2 = z$ so that $2x\,dx = dz$

or
$$dx = \frac{1}{2x}\,dz = \frac{1}{2\sqrt{z}}\,dz = \frac{1}{2}z^{-1/2}\,dz.$$

Also when $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \qquad I = \int_0^\infty e^{-z}\,\frac{1}{2}z^{-1/2}\,dz = \frac{1}{2}\int_0^\infty e^{-z}z^{1/2-1}\,dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Hence $\displaystyle\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. **(Remember)**

3.9 Value of $\displaystyle\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$ in terms of Γ, where $m > -1, n > -1$

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Proof : Put $\sin^2 \theta = x$ so that $2 \sin \theta \cos \theta \, d\theta = dx$

or $2 \sin \theta . \sqrt{(1 - \sin^2 \theta)} \, d\theta = dx$ or $2x^{1/2} . \sqrt{(1-x)} \, d\theta = dx$.

$\therefore$ $\displaystyle d\theta = \frac{dx}{2x^{1/2}(1-x)^{1/2}}$.

Also when $\theta = 0, x = 0$ and when $\theta = \frac{1}{2}\pi, x = 1$.

$$\therefore \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \int_0^{\pi/2} (1 - \sin^2 \theta)^{m/2} . \sin^n \theta \, d\theta$$

$$= \int_0^1 (1-x)^{m/2} . x^{n/2} . \frac{dx}{2x^{1/2}(1-x)^{1/2}}$$

$$= \frac{1}{2}\int_0^1 x^{(n-1)/2}(1-x)^{(m-1)/2}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{\{(n+1)/2\}-1}(1-x)^{\{(m+1)/2\}-1}\,dx$$

$$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right), \text{ provided } m > -1 \text{ and } n > -1$$

$$= \frac{1}{2}\frac{\Gamma\frac{1}{2}(m+1)\,\Gamma\frac{1}{2}(n+1)}{\Gamma\frac{1}{2}(m+1+n+1)} \qquad \left[\because \mathbf{B}(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\Gamma\frac{1}{2}(m+1)\,\Gamma\frac{1}{2}(n+1)}{2\,\Gamma\frac{1}{2}(m+n+2)}.$$

3.10 Some Important Transformations of Beta Function

Beta function can be transformed into many other forms. A few of them are given below.

(i) We know that $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \mathbf{B}(m, n)$.

Now $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$

Making the substitution $y = 1/x$ in the last integral, we get

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}}.$$

$$\therefore \qquad \mathbf{B}(m, n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Hence $\displaystyle \int_0^1 \frac{\boldsymbol{x^{m-1} + x^{n-1}}}{\boldsymbol{(1+x)^{m+n}}}\,\boldsymbol{dx} = \mathbf{B}(\boldsymbol{m}, \boldsymbol{n}) = \frac{\Gamma(\boldsymbol{m})\,\Gamma(\boldsymbol{n})}{\Gamma(\boldsymbol{m}+\boldsymbol{n})}.$

(ii) We know that $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \mathbf{B}(m, n)$.

If we put $x = \frac{ay}{b}$, so that $dx = \frac{a}{b}\,dy$, we get

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy.$$

$$\therefore \qquad \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy = \frac{1}{a^m b^n}\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \frac{1}{a^m b^n}\,\mathbf{B}(m, n).$$

Hence $\displaystyle \int_0^\infty \frac{\boldsymbol{y^{m-1}}}{\boldsymbol{(ay+b)^{m+n}}}\,\boldsymbol{dy} = \frac{\Gamma(\boldsymbol{m})\,\Gamma(\boldsymbol{n})}{\boldsymbol{a^m b^n}\,\Gamma(\boldsymbol{m}+\boldsymbol{n})}.$

Again putting $y = \tan^2\theta$ *i.e.*, $dy = 2\tan\theta\sec^2\theta\,d\theta$ in the integral just obtained, we get

$$\int_0^{\pi/2} \frac{\mathbf{sin}^{2m-1}\theta\,\mathbf{cos}^{2n-1}\theta\,\boldsymbol{d}\theta}{(\boldsymbol{a}\,\mathbf{sin}^2\theta + \boldsymbol{b}\,\mathbf{cos}^2\theta)^{m+n}} = \frac{\Gamma(\boldsymbol{m})\,.\Gamma(\boldsymbol{n})}{\mathbf{2}\,\boldsymbol{a}^m\boldsymbol{b}^n\,\Gamma(\boldsymbol{m}+\boldsymbol{n})}.$$

(iii) We know that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \mathbf{B}(m, n)$.

Putting $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\,d\theta$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

$$\therefore \quad \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\mathbf{B}(m,n)}{2} = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}.$$

This result may also be written in the form

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},$$

by putting $2m-1=p$ and $2n-1=q$.

(iv) We know that $\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \mathbf{B}(m,n)$.

Putting $y = \dfrac{x-b}{a-b}$, so that $dy = \dfrac{dx}{a-b}$, we have

$$\int_0^1 y^{m-1}.(1-y)^{n-1}\,dy = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1}\left(\frac{a-x}{a-b}\right)^{n-1}\cdot\frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx.$$

$$\therefore \quad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

or
$$\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\,\mathbf{B}(m,n)$$

$$= (a-b)^{m+n-1}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

3.11 Duplication Formula

To prove that $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$, **where** $m>0$.

(Agra 2001, 03; Kanpur 09; Rohilkhand 13, 13B; Avadh 13; Purvanchal 14)

Proof : We know that

$$\mathbf{B}(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \text{ where } m>0, n>0.$$

If we take $n=m$, then

$$\mathbf{B}(m,m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)}. \qquad \text{...(1)}$$

Again by the definition of Beta function, we have

$$\mathbf{B}(m, m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx.$$

Let us put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

Then $$\mathbf{B}(m, m) = \int_0^{\pi/2} \sin^{2(m-1)}\theta\,.\cos^{2(m-1)}\theta\,.2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,.\cos^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi \cdot \frac{d\phi}{2},\ \text{putting } 2\theta = \phi \text{ so that } d\theta = \tfrac{1}{2}\,d\phi$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi = \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi \qquad \textbf{[Note]}$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1}\phi\,.\cos^0\phi\,d\phi \qquad \textbf{[Note]}$$

$$= \frac{1}{2^{2m-2}} \frac{\Gamma\frac{1}{2}(2m-1+1)\,\Gamma\frac{1}{2}(0+1)}{2\,\Gamma\frac{1}{2}(2m-1+0+2)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}. \qquad ...(2)$$

$[\because \Gamma(\frac{1}{2}) = \sqrt{\pi}]$

Now equating the two values of $B(m, m)$ obtained in (1) and (2), we get

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,.\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}$$

or $$\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m). \qquad \textbf{(Remember)}$$

3.12 Value of $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$, where n is a Positive Integer

Let $$A = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right). \qquad ...(1)$$

Writing the above expression in the reverse order, we have

$$A = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right). \qquad \ldots(2)$$

Multiplying (1) and (2), we get

$$A^2 = \Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\ldots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

$$= \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi} \quad \text{[See corollary of article 3.7]} \qquad \ldots(3)$$

To calculate this expression, we factorize $1 - x^{2n}$.

Now the roots of the equation $x^{2n} - 1 = 0$ are given by

$$x = (1)^{1/2n} = (\cos 2r\pi + i\sin 2r\pi)^{1/2n}$$

$$= \cos\frac{r\pi}{n} + i\sin\frac{r\pi}{n}, \quad \text{where } r = 0, 1, 2, \ldots, 2n-1.$$

Hence, we have $1 - x^{2n}$

$$= (1-x)(1+x)\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\left(x-\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\ldots$$

$$\ldots\left(x-\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)\left(x-\cos\frac{n-1}{n}\pi+i\sin\frac{n-1}{n}\pi\right)$$

$$= (1-x^2)\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots$$

$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

$$\therefore \qquad \frac{1-x^{2n}}{1-x^2} = \left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots$$

$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

Putting $x = 1$ and $x = -1$ respectively, we have in the limit,

$$n = \left(2-2\cos\frac{\pi}{n}\right)\left(2-2\cos\frac{2\pi}{n}\right)\ldots\left(2-2\cos\frac{n-1}{n}\pi\right)$$

and
$$n = \left(2+2\cos\frac{\pi}{n}\right)\left(2+2\cos\frac{2\pi}{n}\right)\ldots\left(2+2\cos\frac{n-1}{n}\pi\right).$$

Multiplying these, we get

$$n^2 = 2^{2n-2}\sin^2\frac{\pi}{n}\cdot\sin^2\frac{2\pi}{n}\ldots\sin^2\frac{n-1}{n}\pi$$

or
$$n = 2^{n-1}.\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi.$$

Hence, from (3), we get

$$A^2 = \frac{\pi^{n-1}}{n/2^{n-1}} = \frac{(2\pi)^{n-1}}{n} \quad \text{or} \quad A = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}.$$

Remark : The value of $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n} \pi$ can also be found by using the trigonometrical identity

$$\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin\left(\theta + \frac{\pi}{n}\right) \sin\left(\theta + \frac{2\pi}{n}\right) \sin\left(\theta + \frac{3\theta}{n}\right) \ldots$$

$$\ldots \sin\left(\theta + \frac{n-1}{n}\pi\right).$$

From the above identity, we have

$$\frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin \theta} \cdot n = 2^{n-1} \sin\left(\theta + \frac{\pi}{n}\right) \sin\left(\theta + \frac{2\pi}{n}\right) \ldots \sin\left(\theta + \frac{n-1}{n}\pi\right).$$

Taking limit as $\theta \to 0$, we get

$$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \ldots \sin \frac{n-1}{n} \pi .$$

3.13 Values of the Integrals $\int_0^\infty e^{-ax} \cos bx \, . \, x^{m-1} \, dx$ and $\int_0^\infty e^{-ax} \sin bx \, . \, x^{m-1} \, dx$

We have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} \, dx = \int_0^\infty e^{-(a-ib)x} x^{m-1} \, dx = \frac{\Gamma(m)}{(a-ib)^m}$$

[See article 3.6, part (i)]

$$= (a-ib)^{-m} \Gamma(m). \qquad \ldots(1)$$

Let us first separate $(a-ib)^{-m}$ into real and imaginary parts.

Put $a = k \cos \alpha$ and $b = k \sin \alpha$ so that

$$\alpha = \tan^{-1}(b/a) \text{ and } k = \sqrt{(a^2 + b^2)}.$$

Then $(a-ib)^{-m} = [k(\cos \alpha - i \sin \alpha)]^{-m} = k^{-m} (\cos \alpha - i \sin \alpha)^{-m}$

$= k^{-m} (\cos m\alpha + i \sin m\alpha)$, by De-Moivre's theorem.

Now from (1), we have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} \, dx = k^{-m} (\cos m\alpha + i \sin m\alpha) \Gamma(m)$$

or
$$\int_0^\infty e^{-ax} (\cos bx + i \sin bx) x^{m-1} \, dx = \frac{\Gamma(m)}{k^m} (\cos m\alpha + i \sin m\alpha),$$

[$\because e^{i\theta} = \cos \theta + i \sin \theta$, by Euler's theorem]

or
$$\int_0^\infty e^{-ax} \cos bx \, . \, x^{m-1} \, dx + i \int_0^\infty e^{-ax} \sin bx \, . \, x^{m-1} \, dx$$

$$= \frac{\Gamma(m)}{k^m} \cos m\alpha + i \frac{\Gamma(m)}{k^m} \sin m\alpha. \qquad \ldots(2)$$

Equating real and imaginary parts in (2), we get

$$\int_0^\infty e^{-ax}\cos bx \,.\, x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}\cos m\alpha,$$

and $$\int_0^\infty e^{-ax}\sin bx \,.\, x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}\sin m\alpha,$$

where $k = \sqrt{(a^2+b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Deductions : (i) If we put $a = 0$, then $\alpha = \pi/2$ and $k = b$.

Hence $$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)}{b^m}\cos\frac{m\pi}{2}$$

and $$\int_0^\infty x^{m-1}\sin bx\,dx = \frac{\Gamma(m)}{b^m}\sin\frac{m\pi}{2}.$$

(ii) If we put $m = 1$, then

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{\Gamma(1)}{k}\cos\alpha = \frac{k\cos\alpha}{k^2} = \frac{a}{a^2+b^2}$$

and $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{\Gamma(1)}{k}\sin\alpha = \frac{k\sin\alpha}{k^2} = \frac{b}{a^2+b^2}.$$

Illustrative Examples

Example 1 : *Evaluate the following integrals.*

(i) $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx,$ **(Garhwal 2000)** *(ii)* $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx,$

(iii) $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^6}.$

Solution : (i) We have

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8\,dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$= \mathbf{B}(9, 15) - \mathbf{B}(15, 9),$ [By article 3.3]

$= \mathbf{B}(9, 15) - \mathbf{B}(9, 15),$ by symmetry of Beta function

$= 0.$

(ii) We have $$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4\,dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9\,dx}{(1+x)^{15}}$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= \mathbf{B}(5, 10) + \mathbf{B}(10, 5) = \mathbf{B}(5, 10) + \mathbf{B}(5, 10)$$

$$= 2\,\mathbf{B}(5, 10) = 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}$$

$$= 2 \cdot \frac{4.3.2.1}{14.13.12.11.10} = \frac{1}{5005}.$$

(iii) Let $I = \int_0^\infty \frac{x\,dx}{1 + x^6}$.

Put $x^6 = y$ or $x = y^{1/6}$, so that $dx = \frac{1}{6} y^{-5/6}\,dy$.

$$\therefore \qquad I = \frac{1}{6}\int_0^\infty \frac{y^{1/6}.\,y^{-5/6}}{1 + y}\,dy = \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1 + y}\,dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{(1/3)-1}}{(1 + y)^{(1/3)+(2/3)}}\,dy = \frac{1}{6} B\left(\frac{1}{3}, \frac{2}{3}\right), \qquad \text{[By article 3.3]}$$

$$= \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma\left(\frac{1}{3} + \frac{2}{3}\right)} = \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\left(1 - \frac{1}{3}\right)}{\Gamma 1} = \frac{1}{6} \cdot \frac{\pi}{\sin\frac{1}{3}\pi}$$

$$\left[\because \; \Gamma n\,\Gamma(1 - n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{6} \cdot \frac{\pi}{(\sqrt{3}/2)} = \frac{1}{6} \cdot \frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Example 2 : *Show that* $\int_0^1 \frac{dx}{(1 - x^n)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n + 1/2)}$. **(Lucknow 2010)**

Solution : Let $x^n = \sin^2\theta$ *i.e.*, $x = \sin^{2/n}\theta$ so that

$$dx = \frac{2}{n}\sin^{(2/n)-1}\theta\cos\theta\,d\theta.$$

Then
$$\int_0^1 \frac{dx}{\sqrt{(1 - x^n)}} = \frac{2}{n}\int_0^{\pi/2} \frac{\sin^{(2/n)-1}\theta\cos\theta\,d\theta}{\cos\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta\cos^0\theta\,d\theta$$

$$= \frac{2}{n} \cdot \frac{\Gamma(1/n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(1/n + 1/2)}$$

$$= \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n + 1/2)}.$$

Example 3 : *Evaluate* $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1 + x)^{m+n}}\,dx$. **(Kumaun 2002)**

Solution : We have

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1 + x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}\,dx}{(1 + x)^{m+n}} + \int_0^1 \frac{x^{n-1}\,dx}{(1 + x)^{m+n}}. \qquad \text{...(1)}$$

Now in the second integral on the R.H.S. of (1), we put $x = 1/y$ so that $dx = -(1/y^2)\, dy$; also when $x \to 0, y \to \infty$ and when $x = 1, y = 1$.

$$\therefore \quad \int_0^1 \frac{x^{n-1}\, dx}{(1+x)^{m+n}} = \int_\infty^1 \frac{(1/y)^{n-1}}{(1+1/y)^{m+n}} \left(-\frac{1}{y^2}\, dy\right)$$

$$= -\int_\infty^1 \frac{y^{m+n}\, dy}{(1+y)^{m+n} \cdot y^{n-1} \cdot y^2} = \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\, dy \qquad \textbf{(Note)}$$

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx. \qquad \left[\because \int_a^b f(x)\, dx = \int_a^b f(y)\, dy\right]$$

Now from (1), we have

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx, \text{ by a property of definite integrals}$$

$$= \mathbf{B}(m, n) = \frac{\Gamma(m)\, \Gamma(n)}{\Gamma(m+n)}. \qquad \text{[Refer article 3.3 and 3.7]}$$

Example 4 : *Show that* $\displaystyle\int_0^{\pi/2} (\tan x)^n\, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$, *where* $-1 < n < 1$.

Solution : We have

$$\int_0^{\pi/2} \tan^n x\, dx = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x}\, dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\, dx$$

$$= \frac{\Gamma\frac{1}{2}(n+1) \cdot \Gamma\frac{1}{2}(-n+1)}{2\,\Gamma\frac{1}{2}(n-n+2)},$$

where $-n+1 > 0$ *i.e.*, $n < 1$ and $n+1 > 0$ *i.e.*, $n > -1$

$$= \frac{1}{2} \Gamma\frac{1}{2}(n+1)\, \Gamma\frac{1}{2}(1-n) = \frac{1}{2} \Gamma\frac{1}{2}(n+1)\, \Gamma\left[1 - \frac{1}{2}(n+1)\right]$$

$$= \frac{1}{2} \frac{\pi}{\sin\frac{1}{2}(n+1)\pi}, \qquad \left[\because \Gamma(n)\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}, \right.$$

by corollary to article 3.7]

$$= \frac{\pi}{2} \cdot \frac{1}{\sin\left(\frac{1}{2}\pi + \frac{1}{2}n\pi\right)}$$

$$= \frac{\pi}{2} \cdot \frac{1}{\cos\left(\frac{1}{2}n\pi\right)}$$

$$= \frac{\pi}{2} \sec \frac{n\pi}{2}, \text{ where } -1 < n < 1.$$

Example 5 : *Prove that*

(i) $\displaystyle\int_0^\infty x^{2n-1} e^{-ax^2} dx = \frac{\Gamma(n)}{2a^n}$; (ii) $\displaystyle\int_0^\infty x^m e^{-ax^n} dx = \frac{\Gamma[(m+1)/n]}{na^{(m+1)/n}}$;

(iii) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$.

Solution : (i) Let

$$I = \int_0^\infty x^{2n-1} e^{-ax^2} dx = \int_0^\infty x^{2n-2} e^{-ax^2} x\, dx.$$

Put $ax^2 = z$ so that $2ax\,dx = dz$. When $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \qquad I = \int_0^\infty \left(\frac{z}{a}\right)^{n-1} e^{-z} \frac{1}{2a} dz = \frac{1}{2a^n}\int_0^\infty e^{-z} z^{n-1} dz$$

$$= \frac{1}{2a^n}\Gamma(n), \text{ by definition of Gamma function.}$$

(ii) Let $\displaystyle I = \int_0^\infty x^m e^{-ax^n} dx = \int_0^\infty \frac{x^m}{x^{n-1}} e^{-ax^n} x^{n-1} dx$ **[Note]**

$$= \int_0^\infty x^{m-n+1} e^{-ax^n} x^{n-1} dx.$$

Put $ax^n = t$ so that $nax^{n-1} dx = dt$. Also when $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$\therefore \qquad I = \int_0^\infty \left(\frac{t}{a}\right)^{(m-n+1)/n} e^{-t} \cdot \frac{1}{na} dt, \qquad \left[\because ax^n = t \Rightarrow x = \left(\frac{t}{a}\right)^{1/n}\right]$$

$$= \frac{1}{na \,.\, a^{(m-n+1)/n}}\int_0^\infty t^{\{(m+1)/n\}-1} e^{-t} dt$$

$$= \frac{1}{na^{(m+1)/n}}\Gamma\{(m+1)/n\}, \text{ by the definition of Gamma function.}$$

(iii) Let $\displaystyle I = \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_0^1 \frac{dx}{\sqrt{\{\log(1/x)\}}} = \int_0^1 \left(\log\frac{1}{x}\right)^{-1/2} dx.$

Put $\log(1/x) = y$ *i.e.*, $1/x = e^y$ *i.e.*, $x = e^{-y}$ so that $dx = -e^{-y} dy$.
Also when $x \to \infty, y \to \infty$ and when $x = 1, y = 0$.

$$\therefore \qquad I = -\int_\infty^0 y^{-1/2} e^{-y} dy$$

$$= \int_0^\infty e^{-y} y^{1/2-1} dy = \Gamma(\tfrac{1}{2}), \qquad \text{by the def. of Gamma function}$$

$$= \sqrt{\pi}.$$

Example 6 : *Evaluate the integral*

$$\int_a^b (x-a)^p (b-x)^q \, dx, \text{ where } p \text{ and } q \text{ are positive integers.}$$

Solution : Let $I = \int_a^b (x-a)^p (b-x)^q \, dx.$

Put $x = a\cos^2\theta + b\sin^2\theta$ so that

$$dx = -2a\cos\theta\sin\theta\, d\theta + 2b\sin\theta\cos\theta\, d\theta$$

i.e., $dx = 2(b-a)\cos\theta\sin\theta\, d\theta.$

Also $x - a = a\cos^2\theta + b\sin^2\theta - a = b\sin^2\theta - a(1-\cos^2\theta)$

$$= b\sin^2\theta - a\sin^2\theta = (b-a)\sin^2\theta$$

and $b - x = b - a\cos^2\theta - b\sin^2\theta = b(1-\sin^2\theta) - a\cos^2\theta$

$$= (b-a)\cos^2\theta.$$

To find the limits for θ, when $x = a$, we have

$$a = a\cos^2\theta + b\sin^2\theta$$

i.e., $(b-a)\sin^2\theta = 0$ *i.e.,* $\sin^2\theta = 0$ as $a \neq b$ *i.e.,* $\theta = 0$

and when $x = b$, we have $b = a\cos^2\theta + b\sin^2\theta$

i.e., $(a-b)\cos^2\theta = 0$ *i.e.,* $\cos^2\theta = 0$ as $a \neq b$ *i.e.,* $\theta = \pi/2.$

Thus the new limits for θ are 0 to $\pi/2$. Hence the given integral

$$I = \int_0^{\pi/2} (b-a)^p \sin^{2p}\theta \,.(b-a)^q \cos^{2q}\theta \,.2(b-a)\cos\theta\sin\theta\, d\theta$$

$$= 2(b-a)^{p+q+1} \int_0^{\pi/2} \sin^{2p+1}\theta \cos^{2q+1}\theta\, d\theta$$

$$= 2(b-a)^{p+q+1} \frac{\Gamma\{\frac{1}{2}(2p+1+1)\}\,\Gamma\{\frac{1}{2}(2q+1+1)\}}{2\,\Gamma(2p+1+2q+1+2)},$$

provided $2p+1 > -1$ and $2q+1 > -1$ *i.e.,* $p > -1$ and $q > -1$ which is so because p and q are given to be + ive integers

$$= (b-a)^{p+q+1} \frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+1+1)}$$

$$= (b-a)^{p+q+1} \frac{p\,!\,q\,!}{(p+q+1)\,!},$$

because $\Gamma(n+1) = n\,!$ if n is a positive integer.

Example 7 : *Find the value of* $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right).$

Solution : We know that $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}},$

where n is a positive integer.

Putting $n = 9$ in the above relation, we get

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right) = \frac{(2\pi)^{(9-1)/2}}{9^{1/2}} = \frac{(2\pi)^4}{3} = \frac{16}{3}\pi^4.$$

Example 8 : *Show that*

(i) $2^n\,\Gamma(n+\frac{1}{2}) = 1.3.5.\ldots(2n-1)\sqrt{\pi}$, *where n is a + ive integer,* **(Lucknow 2009)**

(ii) $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi \sec \pi x$, *provided* $-1 < 2x < 1$.

Solution : (i) We have

$$\Gamma(n+\tfrac{1}{2}) = (n-\tfrac{1}{2})\,\Gamma(n-\tfrac{1}{2}) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}(2n-1)(2n-3)(2n-5)\ldots 3.1.\sqrt{\pi}.$$

$\therefore \qquad 2^n\,\Gamma(n+\frac{1}{2}) = 1.3.5.\ldots(2n-1)\sqrt{\pi}.$

(ii) We have $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = (\frac{1}{2}-x)\,\Gamma(\frac{1}{2}-x).(\frac{1}{2}+x)\,\Gamma(\frac{1}{2}+x)$

$$= \left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(\frac{1+2x}{2}\right)$$

$$= \left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(1-\frac{1-2x}{2}\right)$$

$$= \left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1-2x}{2}\pi\right)} = \left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\sin\left(\frac{1}{2}\pi - x\pi\right)}$$

$$= \left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\cos x\pi} = \left(\frac{1}{4}-x^2\right)\cdot\pi\sec\pi x.$$

Example 9 : *With certain restrictions on the values of* a, b, m *and* n, *prove that*

$$\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy = \frac{\Gamma(m)\,\Gamma(n)}{4a^m b^n}.$$

Solution : Let us denote the given integral by I. Then

$$I = \int_0^\infty e^{-ax^2}x^{2m-1}\,dx \times \int_0^\infty e^{-by^2}y^{2n-1}\,dy = I_1 \times I_2.$$

To evaluate I_1, put $ax^2 = t$ so that $2ax\,dx = dt$.

$$\therefore \qquad I_1 = \int_0^\infty e^{-t}(t/a)^{(2m-1)/2}\cdot\frac{dt}{2\sqrt{(at)}} = \frac{1}{2a^m}\int_0^\infty e^{-t}t^{m-1}\,dt = \frac{\Gamma(m)}{2a^m},$$

provided a and m are + ive.

Similarly, $I_2 = \dfrac{\Gamma(n)}{2b^n}$, provided b and n are + ive.

Hence $\quad I = \dfrac{\Gamma m \, \Gamma n}{4a^m \, b^n}$.

Example 10 : *Show that the sum of the series*

$$\frac{1}{n+1} + m \frac{1}{n+2} + \frac{m(m+1)}{2!} \cdot \frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!} \cdot \frac{1}{n+4} + \ldots$$

is $\dfrac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)}$, *where* $-1 < n < 1$.

Solution : We have $\dfrac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)} = \mathbf{B}(n+1, 1-m)$

$$= \int_0^1 x^n (1-x)^{-m} \, dx$$

$$= \int_0^1 x^n \left[1 + mx + \frac{m(m+1)}{2!} x^2 + \frac{m(m+1)(m+2)}{3!} x^3 + \ldots\right] dx$$

$$= \int_0^1 \left[x^n + mx^{n+1} + \frac{m(m+1)}{2!} x^{n+2} + \frac{m(m+1)(m+2)}{3!} x^{n+3} + \ldots\right] dx$$

$$= \left[\frac{x^{n+1}}{n+1} + m \frac{x^{n+2}}{n+2} + \frac{m(m+1)}{2!} \frac{x^{n+3}}{n+3} + \frac{m(m+1)(m+2)}{3!} \frac{x^{n+4}}{n+4} + \ldots\right]_0^1$$

$$= \frac{1}{n+1} + m . \frac{1}{n+2} + \frac{m(m+1)}{2!} \cdot \frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!} \frac{1}{n+4} + \ldots$$

Example 11 : *Prove that* $\displaystyle\int_0^\infty \frac{\sin bz}{z} \, dz = \frac{\pi}{2}$.

Solution : We have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz \, dx \, dz$$

$$= \int_0^\infty \left[\frac{e^{-xz}}{-z}\right]_0^\infty \sin bz \, dz, \text{ on first integrating w.r.t. } x$$

$$= \int_0^\infty \frac{\sin bz}{z} \, dz. \qquad \ldots(1)$$

Again on first integrating w.r.t. z, we have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz \, dx \, dz$$

$$= \int_0^\infty \left[\int_0^\infty e^{-xz} \sin bz \, dz\right] dx$$

$$= \int_0^\infty \frac{b}{b^2 + x^2} \, dx,$$ [See article 3.13, Deduction (ii)]

$$= \left[\tan^{-1}\frac{x}{b}\right]_0^\infty$$

$$= \frac{\pi}{2}. \qquad ...(2)$$

Hence equating the two values (1) and (2) of I, we have

$$\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}.$$

Example 12 : *Show that*

$$\int_0^\infty \cos(bz^{1/n})\, dz = \frac{1}{b^n}\Gamma(n+1) \,.\, \cos\frac{n\pi}{2}.$$

(Kanpur 2014; Agra 14)

Solution : Put $z^{1/n} = x$ *i.e.*, $z = x^n$, so that $dz = nx^{n-1}\, dx$.

$$\therefore \quad \int_0^\infty \cos(bz^{1/n})\, dz = \int_0^\infty \cos(bx)\,.\,nx^{n-1}\, dx$$

$$= n\int_0^\infty x^{n-1}\cos(bx)\, dx$$

$$= \text{real part of } n\int_0^\infty e^{-ibx}x^{n-1}\, dx$$

$$= \text{real part of } n\frac{\Gamma(n)}{(ib)^n}$$

$$= \text{real part of } \frac{n\,\Gamma(n)}{b^n}\cdot(\cos\tfrac{1}{2}\pi + i\sin\tfrac{1}{2}\pi)^{-n}$$

$$= \text{real part of } \frac{\Gamma(n+1)}{b^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{b^n}\cdot\Gamma(n+1)\,.\cos\left(\frac{n\pi}{2}\right).$$

Example 13 : *Show that* $\displaystyle\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 1.$

(Gorakhpur 2005; Kanpur 07; Lucknow 10; Rohilkhand 13)

Solution : We have

$$I = \int_0^\infty \frac{x^c}{c^x}\, dx = \int_0^\infty \frac{x^c}{e^{\log c^x}}\, dx = \int_0^\infty \frac{x^c}{e^{x\log c}}\, dx = \int_0^\infty e^{-x\log c}x^c\, dx.$$

Put $x\log c = y$ so that $(\log c)\, dx = dy$.

When $x = 0$, we have $y = 0$ and when $x \to \infty, y \to \infty$ because $c > 1 \Rightarrow \log c > 0$.

$$\therefore \quad I = \int_0^\infty e^{-y}\left(\frac{y}{\log c}\right)^c \frac{dy}{\log c} = \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-y}y^{(c+1)-1}\, dy$$

$$= \frac{1}{(\log c)^{c+1}}\Gamma(c+1),$$

provided $c + 1 > 0$ which is so because $c > 1$.

Comprehensive Exercise 1

1. Prove that

(i) $\displaystyle\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}$. **(Kanpur 2010)**

(ii) $\displaystyle\int_0^2 (8 - x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$. **(Kumaun 2008)**

2. Show that $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \frac{\Gamma(m)\,\Gamma(n)}{a^n (1+a)^m\, \Gamma(m+n)}$.

Hint. Put $\dfrac{x(1+a)}{a+x} = y$.

3. Show that $\displaystyle\int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), p > -1, q > -1.$

Deduce that $\displaystyle\int_0^2 x^4 (8 - x^3)^{-1/3}\, dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right).$

4. Prove that $B(m, n) = B(m+1, n) + B(m, n+1)$ for $m > 0, n > 0$.

(Kanpur 2005, 11; Gorakhpur 05; Bundelkhand 11; Avadh 06, 11, 14)

5. Prove that

(i) $\displaystyle\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}.$

(ii) $\displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n).$

6. Show that, if $m > -1$, then

$$\int_0^\infty x^m e^{-n^2 x^2}\, dx = \frac{1}{2n^{m+1}} \Gamma\left(\frac{m+1}{2}\right).$$

7. Prove that

(i) $\displaystyle\int_0^1 x^m (1 - x^n)^p\, dx = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right).$

(ii) $\displaystyle\int_0^1 (1 - x^n)^{1/n}\, dx = \frac{1}{n} \frac{[\Gamma(1/n)]^2}{2\,\Gamma(2/n)}$.

8. Show that $\Gamma(0\cdot 1)\,\Gamma(0\cdot 2)\,\Gamma(0\cdot 3) \ldots \Gamma(0\cdot 9) = \dfrac{(2\pi)^{9/2}}{\sqrt{(10)}}$.

9. Show that

(i) $\displaystyle\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$

(ii) $\displaystyle\int_0^1 \frac{x^2\, dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}.$

10. Show that $\displaystyle\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta = 4\int_0^{\infty} \frac{x^2\, dx}{1+x^4} = \pi\sqrt{2}.$

11. Show that the perimeter of a loop of the curve $r^n = a^n \cos n\theta$ is

$$\frac{a}{n} \cdot 2^{(1/n)-1} \cdot \frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

12. Prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

13. Show that $\displaystyle\int_0^{\pi/2} \sin^p\theta\, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$

14. Show that

(i) $\displaystyle\int_0^{\infty} xe^{-\alpha x} \cos\beta x\, dx = \frac{(\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}$

(ii) $\displaystyle\int_0^{\infty} xe^{-\alpha x} \sin\beta x\, dx = \frac{2\alpha\beta}{(\alpha^2+\beta^2)^2}.$

15. Prove that $\displaystyle\int_{-\infty}^{\infty} \cos\left(\frac{1}{2}\pi x^2\right) dx = 1.$

16. Show that $\displaystyle B(m, m) \,.\, B\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}.$ **(Rohilkhand 2005)**

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "…… " so that the following statements are complete and correct.

1. The definite integral $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$, for $m > 0, n > 0$ is called the …… .

2. The definite integral $\displaystyle\int_0^{\infty} e^{-x} x^{n-1}\, dx$, for $n > 0$ is called the …… .

(Garhwal 2001)

3. $\displaystyle B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\dots\dots}.$

4. $\dfrac{B(m+1,n)}{B(m,n)} = \dfrac{m}{\ldots\ldots}$.

5. For $m > 0, n > 0, \displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \ldots\ldots$.

6. For $m > 0, n > 0, \displaystyle\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\, dx = \ldots\ldots$.

7. For $n > 0, \Gamma(n+1) = \ldots\ldots \Gamma(n)$.

8. If n is a positive integer, then $\Gamma(n) = \ldots\ldots$.

9. If $0 < n < 1$, then $\Gamma(n)\,\Gamma(1-n) = \ldots\ldots$.

10. $\Gamma\left(\frac{1}{2}\right) = \ldots\ldots$.

11. If $m > -1, n > -1$, then

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \frac{\Gamma\left(\dfrac{m+1}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right)}{\ldots\ldots}$$

12. $\displaystyle\int_0^\infty e^{-x^2}\, dx = \ldots\ldots$.

13. For $m > 0, \Gamma(m)\,\Gamma\left(m + \dfrac{1}{2}\right) = \dfrac{\sqrt{\pi}}{\ldots\ldots}\,\Gamma(2m)$.

14. For $a > 0, n > 0, \displaystyle\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{\ldots\ldots}$.

15. The value of $\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{2}{3}\right)$ is .... **(Agra 2002)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

16. For $m > 0, n > 0$,

(a) $B(m,n) = \dfrac{\Gamma(m)}{\Gamma(n)}$

(b) $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

(c) $B(m,n) = \dfrac{\Gamma(n)}{\Gamma(m)}$

(d) $B(m,n) = \dfrac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$

17. The value of the integral $\displaystyle\int_0^\infty e^{-x} x^{-1/2}\, dx$ is

(a) $\dfrac{\sqrt{\pi}}{2}$

(b) $\dfrac{\pi}{2}$

(c) $\sqrt{\pi}$

(d) π **(Kumaun 2008)**

18. For $m > 0, n > 0$,

(a) $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$ (b) $B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

(c) $B(m, n) = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$ (d) $B(m, n) = \int_0^\infty \frac{x^{m}}{(1+x)^{m+n}}\, dx$

19. If $a > 0$ and $n > 0$, then the value of the integral $\int_0^\infty e^{-ax} x^{n-1}\, dx$ is

(a) $a^n \Gamma(n)$ (b) $a^{-n} \Gamma(n)$

(c) $\frac{\Gamma(n)}{2a^n}$ (d) $\frac{\Gamma(n)}{n^2}$

20. The value of $\int_0^1 x^4 (1-x)^3\, dx$ is

(a) $\frac{1}{280}$ (b) $\frac{1}{180}$

(c) $\frac{1}{380}$ (d) $\frac{1}{80}$ **(Garhwal 2001)**

21. The value of the integral $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$

(c) $\frac{\pi}{16}$ (d) $\frac{\pi}{32}$ **(Garhwal 2001)**

22. The value of $\int_0^\infty \frac{x^{m-1}\, dx}{(1+x)^{m+n}}$ is

(a) $\Gamma(m) + \Gamma(n)$ (b) $\frac{\Gamma(m)}{\Gamma(n)}$

(c) $\Gamma(m) . \Gamma(n)$ (d) $\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ **(Garhwal 2002)**

23. If $m, n > 0$, then the value of $\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx$ is equal to

(a) $\frac{\Gamma(m)}{n^m}$ (b) $\frac{\Gamma(n)}{n^m}$

(c) $\frac{\Gamma(n)}{m^n}$ (d) $\frac{\Gamma(m)}{m^m}$ **(Garhwal 2003)**

True or False:

Write 'T' for true and 'F' for false statement.

24. $\int_0^\infty e^{-x} x^{1/2} \, dx = \Gamma\left(\frac{1}{2}\right)$.

25. $\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \pi \sqrt{2}$.

26. $\int_0^\infty \frac{x \, dx}{1 + x^6} = \frac{\pi}{3 \sqrt{3}}$.

27. $\int_0^\infty \frac{x^8 (1 - x^6)}{(1 + x)^{24}} \, dx = 1$.

28. For $m > 0, n > 0, B(m, n) = \int_0^1 x^m (1 - x)^n \, dx$.

29. For $m > 0, n > 0, B(m, n) = \int_0^\infty \frac{x^{n-1}}{(1 + x)^{m+n}} \, dx$.

30. For $m > 0, n > 0, \int_0^\infty \frac{x^{m-1}}{(1 + x)^{m+n}} \, dx = \int_0^\infty \frac{x^{n-1}}{(1 + x)^{m+n}} \, dx$.

31. $\Gamma(6) = 120$.

32. For $m > 0, n > 0, B(m, n) = \frac{\Gamma(m) \Gamma(n)}{2 \Gamma(m + n)}$.

33. $\int_0^\infty e^{-x^2} \, dx = \frac{\sqrt{\pi}}{4}$.

34. $B(m + 1, n) + B(m, n + 1) = B(m + 1, n + 1)$. **(Agra 2003)**

Answers

1. Beta function. **2.** Gamma function. **3.** $\Gamma(m + n)$.

4. $m + n$. **5.** $B(m, n)$. **6.** 0. **7.** n. **8.** $(n - 1)!$.

9. $\frac{\pi}{\sin n\pi}$. **10.** $\sqrt{\pi}$. **11.** $2\Gamma\left(\frac{m + n + 2}{2}\right)$. **12.** $\frac{\sqrt{\pi}}{2}$.

13. 2^{2m-1}. **14.** a^n. **15.** $\frac{2\pi}{\sqrt{3}}$. **16.** (b). **17.** (c). **18.** (a).

19. (b). **20.** (a). **21.** (d). **22.** (d). **23.** (a). **24.** *F*.

25. *T*. **26.** *T*. **27.** *F*. **28.** *F*. **29.** *T*. **30.** *T*.

31. *T*. **32.** *F*. **33.** *F*. **34.** *F*.

Chapter 4

Multiple Integrals (Double and Triple Integrals, Change of Order of Integration)

4.1 Double Integrals

The concept of double integral is an extension of the concept of a definite integral to the case of two arguments (*i.e.* a two dimensional space). Let a function $f(x, y)$ of the independent variables x and y be continuous inside some domain (region) A and on its boundary. Divide the domain A into n subdomains $A_1, A_2, \ldots, A_n$ of areas $\delta A_1, \delta A_2, \ldots, \delta A_n$. Let (x_r, y_r) be any point inside the rth elementary area δA_r. Form the sum

$$S_n = f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_r, y_r)\,\delta A_r + \ldots + f(x_n, y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r. \qquad \ldots(1)$$

Now take the limit of the sum (1) as $n \to \infty$ in such a way that the largest of the areas δA_r approaches to zero. This limit, if it exists, is called the **double integral** of the function $f(x, y)$ over the domain A. It is denoted by $\iint_A f(x, y)\, dA$ and is read as "the double integral of $f(x, y)$ over A".

Suppose the domain (region) A is divided into rectangular partitions by a network of lines parallel to the coordinate axes. Let dx be the length of a sub-rectangle and dy be its width so that $dx\,dy$ is an element of area in Cartesian coordinates. The integral $\iint f(x, y)\, dA$ is written as $\iint_A f(x, y)\, dx\, dy$ and is called the *double integral* of $f(x, y)$ over the region A.

4.2 Properties of a Double Integral

I. If the region A is partitioned into two parts, say A_1 and A_2, then

$$\iint_A f(x, y)\, dx\, dy = \iint_{A_1} f(x, y)\, dx\, dy + \iint_{A_2} f(x, y)\, dx\, dy.$$

Similarly for a sub-division of A into three or more parts.

II. The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term. Thus

$$\iint_A [f_1(x, y) + f_2(x, y) + f_3(x, y) + \ldots]\, dx\, dy$$

$$= \iint_A f_1(x, y)\, dx\, dy + \iint_A f_2(x, y)\, dx\, dy + \iint_A f_3(x, y)\, dx\, dy + \ldots$$

III. A constant factor may be taken outside the integral sign. Thus

$$\iint_A m\, f(x, y)\, dx\, dy = m \iint_A f(x, y)\, dx\, dy,$$

where m is a constant.

4.3 Evaluation of Double Integrals

(a) If the region A be given by the inequalities $a \le x \le b$, $c \le y \le d$, then the double integral

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dx\, dy$$

$$= \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx, \qquad \ldots(1)$$

or

$$\iint_A f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dy\, dx$$

$$= \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy \qquad \ldots(2)$$

i.e., in this case the order of integration is immaterial, provided the limits of integration are changed accordingly.

Important Note : In formula (1) the definite integral $\int_c^d f(x, y)\, dy$ is calculated first. During this integration x is regarded as a constant. While in the formula (2) the definite integral $\int_a^b f(x, y)\, dx$ is calculated first and during this integration y is regarded as a constant.

(b) If the region A is bounded by the curves

$y = f_1(x)$, $y = f_2(x)$, $x = a$ and $x = b$, then

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dx\, dy$$

$$= \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right] dx,$$

where the integration with respect to y is performed first treating x as a constant.

Similarly, if the region A is bounded by the curves

$x = f_1(y)$, $x = f_2(y)$, $y = c$, $y = d$, we have

$$\iint_A f(x, y)\, dx\, dy = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y)\, dy\, dx$$

$$= \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x, y)\, dx \right] dy,$$

where the integration with respect to x is performed first treating y as a constant.

Remember : While evaluating double integrals, first integrate w.r.t. the variable having variable limits (treating the other variable as constant) and then integrate w.r.t. the variable with constant limits.

Remark : In the double integral $\int_a^b \int_c^d f(x, y)\, dx\, dy$, it is generally understood that the limits of integration c to d are those of y and the limits of integration a to b are those of x. However this is not a standard convention. Some authors regard these limits in the reverse order *i.e.* they regard the limits c to d as those of x and the limits a to b as those of y. So it is better to write this double integral as $\int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dx\, dy$ so that there in no confusion about the limits. However in the double integral $\int_a^b \int_{f_1(x)}^{f_2(x)} F(x, y)\, dx\, dy$, there is no confusion about the limits. Obviously the variable limits are those of y because they are in terms of x and so the constant limits must be those of x. Here the first integration must be performed with respect to y regarding x as constant.

Illustrative Examples

Example 1 : *Evaluate the following double integrals :*

(i) $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy$

(ii) $\int_1^2 \int_0^x \dfrac{dx\, dy}{x^2 + y^2}$. **(Kanpur 2006; Purvanchal 14)**

Solution : (i) We have $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy = \int_0^a \left[x^2 y + \dfrac{y^3}{3} \right]_{y=0}^{b} dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_0^a \left[bx^2 + \frac{b^3}{3} \right] dx = \left[b\,\frac{x^3}{3} + \frac{b^3}{3}\,x \right]_0^a = \frac{ba^3}{3} + \frac{b^3 a}{3} = \tfrac{1}{3}\,ab\,(a^2 + b^2).$$

(ii) We have $\displaystyle\int_1^2 \int_0^x \frac{dx\,dy}{x^2 + y^2} = \int_1^2 \left[\int_0^x \frac{dy}{x^2 + y^2} \right] dx$

$$= \int_1^2 \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_{y=0}^{x} dx \qquad \text{(integrating w.r.t. } y \text{ treating } x \text{ as constant)}$$

$$= \int_1^2 \left[\frac{1}{x} (\tan^{-1} 1 - \tan^{-1} 0) \right] dx = \frac{\pi}{4} \int_1^2 \frac{dx}{x} = \frac{\pi}{4} \Big[\log x \Big]_1^2$$

$$= \tfrac{1}{4}\pi\,[\log 2 - \log 1] = \tfrac{1}{4}\pi \log 2.$$

Example 2 : *Evaluate*

(i) $\displaystyle\int_0^3 \int_1^2 xy\,(1 + x + y)\,dx\,dy.$ *(ii)* $\displaystyle\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1 + x^2 + y^2}.$

(Gorakhpur 2005; Kanpur 12; Avadh 14)

Solution : (i) We have $\displaystyle\int_0^3 \int_1^2 xy\,(1 + x + y)\,dx\,dy$

$$= \int_0^3 \left[x \cdot \frac{y^2}{2} + x^2 \cdot \frac{y^2}{2} + x \cdot \frac{y^3}{3} \right]_{y=1}^{2} dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^3 \left[\frac{x}{2}(4 - 1) + \frac{x^2}{2}(4 - 1) + \frac{x}{3}(8 - 1) \right] dx$$

$$= \int_0^3 \left[\left(\frac{3}{2} + \frac{7}{3} \right) x + \frac{3}{2} x^2 \right] dx = \left[\frac{23}{6} \cdot \frac{x^2}{2} + \frac{3}{2} \cdot \frac{x^3}{3} \right]_0^3$$

$$= \frac{23}{6} \cdot \frac{9}{2} + \frac{27}{2} = \frac{123}{4} = 30\,\frac{3}{4}.$$

(ii) We have $\displaystyle\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1 + x^2 + y^2}$

$$= \int_0^1 \frac{1}{\sqrt{(1 + x^2)}} \left[\tan^{-1} \frac{y}{\sqrt{(1 + x^2)}} \right]_{y=0}^{\sqrt{(1+x^2)}} dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^1 \frac{1}{\sqrt{(1 + x^2)}} \left[\tan^{-1} 1 - \tan^{-1} 0 \right] dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{(1 + x^2)}}$$

$$= \frac{\pi}{4} \Big[\log \{ x + \sqrt{(1 + x^2)} \} \Big]_0^1 = \frac{\pi}{4} \log (1 + \sqrt{2}).$$

Example 3 : *Evaluate* $\displaystyle\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} \sqrt{(a^2 - x^2 - y^2)}\,dy\,dx.$

Solution : Here the variable limits are those of x and so the first integration must be performed w.r.t. x taking y as constant.

$$\therefore \int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\, dy\, dx = \int_0^a \left[\int_0^{\sqrt{(a^2-y^2)}} \sqrt{\{(a^2-y^2)-x^2\}}\, dx \right] dy$$

$$= \int_0^a \left[\frac{x\sqrt{(a^2-y^2-x^2)}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1} \frac{x}{\sqrt{(a^2-y^2)}} \right]_{x=0}^{\sqrt{(a^2-y^2)}} dy,$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^a \left[0 + \frac{a^2-y^2}{2} \cdot \frac{\pi}{2} \right] dy = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} \right] = \frac{1}{6} \pi a^3.$$

Example 4 : *Show that* $\displaystyle\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \int_0^{x/2} x\, dx\, dy.$

Solution : We have

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \left[y \int_0^{y/2} dx \right] dy = \int_1^2 y \Big[x \Big]_0^{y/2} dy,$$

(integrating w.r.t. x treating y as a constant)

$$= \int_1^2 y \left[\frac{y}{2} - 0 \right] dy = \frac{1}{2} \int_1^2 y^2\, dy = \frac{1}{2} \left[\frac{y^3}{3} \right]_1^2 = \frac{1}{6} [8-1] = \frac{7}{6} \qquad \text{...(1)}$$

Again $$\int_1^2 \int_0^{x/2} x\, dx\, dy = \int_1^2 x \left[\int_0^{x/2} dy \right] dx = \int_1^2 x \Big[y \Big]_0^{x/2} dx,$$

(integrating w.r.t. y treating x as a constant)

$$= \int_1^2 x \left[\frac{x}{2} - 0 \right] dx = \frac{1}{2} \int_1^2 x^2\, dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_1^2 = \frac{1}{6} (8-1) = \frac{7}{6} \qquad \text{...(2)}$$

From (1) and (2), we see that

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \int_0^{x/2} x\, dx\, dy.$$

Examples on the region of integration (*Double Integration*)

Example 5 : *Evaluate* $\iint x^2 y^2\, dx\, dy$ *over the region* $x^2 + y^2 \le 1$.

Solution : Let R denote the region $x^2 + y^2 \le 1$. Then R is the region in the xy-plane bounded by the circle $x^2 + y^2 = 1$. The limits of integration for this region can be expressed either as

$$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$$

or as $$-\sqrt{(1-y^2)} \le x \le \sqrt{(1-y^2)}, -1 \le y \le 1.$$

Because from the equation of the circle $x^2 + y^2 = 1$, we have $x^2 = 1 - y^2$ so that $x = \pm \sqrt{(1-y^2)}$. Thus for a fixed value of y, x varies from $-\sqrt{(1-y^2)}$ to $\sqrt{(1-y^2)}$ in the area bounded by the circle $x^2 + y^2 = 1$. Also y varies from -1 to 1 to cover the whole area of the circle $x^2 + y^2 = 1$. Therefore if the first integration is to be performed w.r.t. x regarding y as constant, then

$$\iint_R x^2 y^2 \, dx \, dy = \int_{y=-1}^{1} \int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2 y^2 \, dx \, dy$$

$$= \int_{y=-1}^{1} y^2 \left[\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2 \, dx \right] dy$$

$$= \int_{-1}^{1} y^2 \left[2 \int_{x=0}^{\sqrt{(1-y^2)}} x^2 \, dx \right] dy = \int_{-1}^{1} 2y^2 \left[\frac{x^3}{3} \right]_0^{\sqrt{(1-y^2)}} dy$$

$$= \int_{-1}^{1} \tfrac{2}{3} y^2 (1-y^2)^{3/2} \, dy = 2 \cdot \tfrac{2}{3} \int_0^1 y^2 (1-y^2)^{3/2} \, dy.$$

Put $y = \sin\theta$ so that $dy = \cos\theta \, d\theta$;

when $y = 0, \theta = 0$ and when $y = 1, \theta = \pi/2$.

$$\therefore \qquad \iint_R x^2 y^2 \, dx \, dy = \frac{4}{3} \int_0^{\pi/2} \sin^2\theta \, (1 - \sin^2\theta)^{3/2} . \cos\theta \, d\theta$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta = \frac{4}{3} \cdot \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{\pi}{24}.$$

Example 6 : *Find by double integration the area of the region bounded by the circle* $x^2 + y^2 = a^2$. **(Agra 2007; Kanpur 09)**

Solution : The area of a small element situated at any point (x, y) is $dx \, dy$. To find the area bounded by the circle $x^2 + y^2 = a^2$, the region of integration R can be expressed as $-a \le y \le a, -\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)}$,

where the first integration is to be performed w.r.t. x regarding y as constant.

$\therefore$ the required area

$$= \iint_R dx \, dy = \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} 1 \, . \, dx \, dy$$

$$= \int_{-a}^{a} \left[2 \int_0^{\sqrt{(a^2-y^2)}} 1 . \, dx \right] dy = 2 \int_{-a}^{a} \Big[x \Big]_0^{\sqrt{(a^2-y^2)}} dy$$

$$= 2 \int_{-a}^{a} \sqrt{(a^2 - y^2)} \, dy = 2.2 \int_0^a \sqrt{(a^2 - y^2)} \, dy \qquad \textbf{(Note)}$$

$$= 4 \left[\frac{y\sqrt{(a^2 - y^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_0^a = 4 \left[0 + \frac{a^2}{2} \sin^{-1} 1 \right]$$

$$= 4 \, . \tfrac{1}{2} a^2 \, . \tfrac{1}{2} \pi = \pi a^2.$$

Example 7 : *Evaluate* $\iint (x^2 + y^2) \, dx \, dy$ *over the region in the positive quadrant for which* $x + y \le 1$. **(Rohilkhand 2012; Avadh 14)**

Solution : The region of integration R is the area bounded by the coordinate axes and the straight line $x+y=1$. Therefore the region R is bounded by $y=0$, $y=1-x$ and $x=0$, $x=1$.

Therefore

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dx\,dy,$$

the first integration to be performed w.r.t. y regarding x as constant

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx = \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{3\times 4}\right]_0^1 = \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{12}\right] = \frac{1}{6}.$$

Example 8 : *Evaluate* $\iint xy(x+y)\,dx\,dy$ *over the area between* $y=x^2$ *and* $y=x$.

(Gorakhpur 2005, 06)

Solution : Draw the given curves $y=x^2$ and $y=x$ in the same figure. The two curves intersect at the points whose abscissae are given by $x^2=x$ or $x(x-1)=0$ *i.e.*, $x=0$ or 1. When $0<x<1$, we have $x>x^2$. So the area of integration can be considered as lying between the curves $y=x^2$, $y=x$, $x=0$ and $x=1$.

Therefore the required integral

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{x}(x^2y+xy^2)\,dy\right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx = \int_0^1 \left[\left(\frac{x^4}{2}+\frac{x^4}{3}\right) - \left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx = \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{9}{168} = \frac{3}{56}.$$

Example 9 : *Prove by the method of double integration that the area lying between the parabolas* $y^2=4ax$ *and* $x^2=4ay$ *is* $\frac{16}{3}a^2$.

Solution : Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissae are given by $(x^2/4a)^2=4ax$ *i.e.*, $x(x^3-64a^3)=0$ *i.e.*, $x=0$ and $x^3=64a^3$. Thus the two parabolas intersect at the points where $x=0$ and $x=4a$.

Now the area of a small element situated at any point $(x, y) = dx\,dy$.

$\therefore$ the required area

$$= \int_{x=0}^{4a}\int_{y=x^2/4a}^{\sqrt{(4ax)}} dx\,dy = \int_0^{4a}\left[y\right]_{x^2/4a}^{\sqrt{(4ax)}} dx$$

$$= \int_0^{4a} \left[2\sqrt{a} . x^{1/2} - \frac{1}{4a} . x^2 \right] dx = \left[2\sqrt{a} . x^{3/2} . \frac{2}{3} - \frac{1}{4a} . \frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4}{3} \sqrt{a} . (4a)^{3/2} - \frac{1}{12a} . 64a^3 = \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2 .$$

Comprehensive Exercise 1

Evaluate the following double integrals :

1. (i) $\displaystyle\int_0^2 \int_0^{\sqrt{(4+x^2)}} \frac{dx\,dy}{4 + x^2 + y^2}$. **(Rohilkhand 2005)**

(ii) $\displaystyle\int_1^a \int_1^b \frac{dx\,dy}{xy}$.

(iii) $\displaystyle\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x + y)\, dy\,dx$. **(Kanpur 2007, 11)**

(iv) $\displaystyle\int_0^1 \int_0^{x^2} e^{y/x}\, dx\,dy$.

(v) $\displaystyle\int_1^2 \int_0^{3y} y\, dy\,dx$.

(vi) $\displaystyle\int_0^2 \int_0^{\sqrt{(2x - x^2)}} x\, dx\,dy$. **(Kanpur 2008)**

2. (i) $\displaystyle\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{\{(1 - x^2)(1 - y^2)\}}}$. (ii) $\displaystyle\int_0^1 \int_0^{\sqrt{(1-y^2)}} 4y\, dy\,dx$.

(iii) $\displaystyle\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\,dy$.

(iv) $\displaystyle\int_2^3 \int_0^{y-1} \frac{dy\,dx}{y}$.

(v) $\displaystyle\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\, dy\,dx$.

(vi) $\displaystyle\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} (x + y)\, dx\,dy$.

3. Show that (i) $\displaystyle\int_1^2 \int_3^4 (xy + e^y)\, dx\,dy = \int_3^4 \int_1^2 (xy + e^y)\, dy\,dx$.

(ii) $\displaystyle\int_0^1 dx \int_0^1 \frac{x - y}{(x + y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x - y}{(x + y)^3}\, dx$.

Find the values of the two integrals.

4. (i) Evaluate the double integral $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} x^2 y \, dx \, dy$.

Mention the region of integration involved in this double integral.

(ii) Evaluate $\iint x^2 y^3 \, dx \, dy$ over the circle $x^2 + y^2 = a^2$.

(Rohilkhand 2013B)

5. Evaluate $\iint (x + y + a) \, dx \, dy$ over the circular area $x^2 + y^2 \le a^2$.

6. Evaluate $\iint x^2 y^2 \, dx \, dy$ over the region bounded by $x = 0, y = 0$ and $x^2 + y^2 = 1$.

(Avadh 2012)

7. Evaluate $\iint xy \, dx \, dy$ over the region in the positive quadrant for which $x + y \le 1$.

8. Evaluate $\iint e^{2x+3y} \, dx \, dy$ over the triangle bounded by $x = 0, y = 0$ and $x + y = 1$.

9. Evaluate $\iint \dfrac{xy}{\sqrt{(1-y^2)}} \, dx \, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

10. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$, by double integration.

11. Compute the value of $\iint_R y \, dx \, dy$, where R is the region in the first quadrant bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

12. Find the mass of a plate in the form of a quadrant of an ellipse $x^2/a^2 + y^2/b^2 = 1$ whose density per unit area is given by $\rho = kxy$.

Answers 1

1. (i) $\frac{1}{4}\pi \log(1 + \sqrt{2})$. (ii) $(\log b)(\log a)$. (iii) -2.

(iv) $\frac{1}{2}$. (v) 7. (vi) $\frac{1}{2}\pi$.

2. (i) $\frac{1}{4}\pi^2$. (ii) $\frac{4}{3}$. (iii) $3/35$.

(iv) $1 - \log(3/2)$. (v) $\pi a^4/8$. (vi) $2a^3/3$.

3. (ii) $\frac{1}{2}$ and $-\frac{1}{2}$.

4. (i) $a^5/15$. The area of the circle $x^2 + y^2 = a^2$ in the positive quadrant.

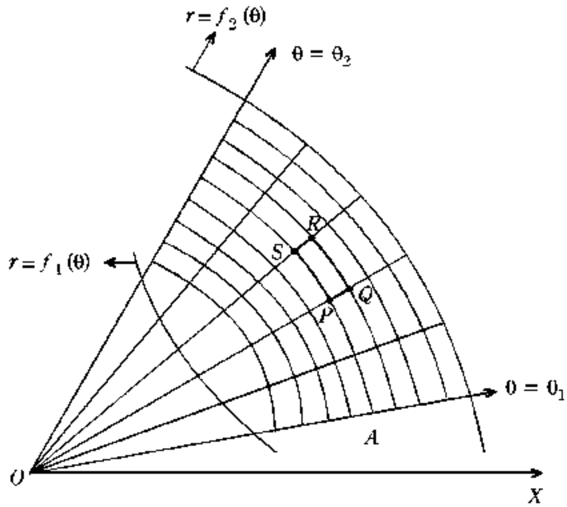
(ii) 0. **5.** πa^3. **6.** $\pi/96$.

7. $\dfrac{1}{24}$. **8.** $\frac{1}{6}(e-1)^2(2e+1)$. **9.** $\frac{1}{6}$.

10. πab. **11.** $ab^2/3$. **12.** $k a^2 b^2/8$.

4.4 To Express a Double Integral in Terms of Polar Coordinates

Let a function $f(r, \theta)$ of the polar coordinates (r, θ) be continuous inside some region A and on its boundary. Let the region A be bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.



Divide the area A into elements by a series of concentric circular arcs with centre at origin and successive radii differing by equal amounts and a series of straight lines drawn through the origin at equal intervals of angles. Let δr be the distance between two consecutive circles and $\delta\theta$ be the angle between two consecutive lines. There is thus a network of elementary areas (say n in number) of which a typical one is $PQRS$. If P is the point (r, θ), the area of the element $PQRS$ situated at the point P is $\frac{1}{2}(r + \delta r)^2 \delta\theta - \frac{1}{2} r^2 \delta\theta = r \delta\theta \, \delta r$, by neglecting the term $\frac{1}{2}(\delta r)^2 \delta\theta$ being an infinitesimal of higher order.

Now by the definition of the double integral of $f(r, \theta)$ over the region A, we have

$$\iint_A f(r, \theta) \, dA = \lim_{\substack{n \to \infty \\ \delta r \to 0, \, \delta\theta \to 0}} \sum_{k=1}^{n} f(r_k, \theta_k) \, r_k \, \delta\theta \, \delta r,$$

where $r_k \, \delta\theta \delta r$ is the area of the element situated at the point (r_k, θ_k).

Using the area of integration, this double integral is generally written as

$$\int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta) \, d\theta \, dr, \quad \text{or} \quad \int_{\theta_1}^{\theta_2} d\theta \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta) \, dr.$$

The first integration is performed with respect to r, keeping θ as a constant. After substituting the limits for r, the second integration with respect to θ is performed.

Remark : The area of the typical element $PQRS$ situated at the point $P(r, \theta)$ can also be found as below :

We have $OP = r$, $OQ = r + \delta r$ so that $PQ = \delta r$. Also PS is the arc of a circle of radius r subtending an angle $\delta\theta$ at the centre of the circle and so arc $PS = r\,\delta\theta$. Therefore the area of the element $PQRS$ is $\delta r . \, r \, \delta\theta$ *i.e.*, $r \, \delta\theta \, \delta r$.

Illustrative Examples

Example 1 : *Evaluate* $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr$.

Solution : We have

$$\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \cos\theta \, . \, a^3 (1+\cos\theta)^3 \, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} \cos\theta \, (1+3\cos\theta+3\cos^2\theta+\cos^3\theta) \, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} [\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta] \, d\theta$$

$$= 2 . \frac{a^3}{3}\int_0^{\pi/2} [3\cos^2\theta+\cos^4\theta] \, d\theta, \quad \left[\because \int_0^{\pi} \cos^n\theta \, d\theta = 0 \text{ or } 2\int_0^{\pi/2} \cos^n\theta \, d\theta \text{ accordingas } n \text{ is odd or even}\right]$$

$$= \frac{2a^3}{3}\left[3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3.1}{4.2} \cdot \frac{\pi}{2}\right] = \frac{2a^3}{3} \cdot \frac{3\pi}{4}\left[1+\frac{1}{4}\right]$$

$$= \frac{2a^3}{3} \cdot \frac{3\pi}{4} \cdot \frac{5}{4} = \frac{5\pi a^3}{8}.$$

Example 2 : *Evaluate* $\iint \frac{r \, d\theta \, dr}{\sqrt{(a^2+r^2)}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Solution : In the equation of the lemniscate $r^2 = a^2 \cos 2\theta$, putting $r = 0$, we get $\cos 2\theta = 0$ *i.e.*, $2\theta = \pm \pi/2$ *i.e.*, $\theta = \pm \pi/4$. Therefore for one loop of the given lemniscate θ varies from $-\pi/4$ to $\pi/4$ and r varies from 0 to $a\sqrt{(\cos 2\theta)}$.

Therefore the required integral

$$= \int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} \frac{r \, d\theta \, dr}{\sqrt{(a^2+r^2)}}$$

$$= \int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \frac{1}{2}(a^2+r^2)^{-1/2}(2r) \, d\theta \, dr$$

$$= \int_{-\pi/4}^{\pi/4} \left[(a^2+r^2)^{1/2}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [a(1+\cos 2\theta)^{1/2} - a] \, d\theta$$

$$= 2a\int_0^{\pi/4}[(2\cos^2\theta)^{1/2}-1]\,d\theta = 2a\int_0^{\pi/4}(\sqrt{2}\cos\theta - 1)\,d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\pi/4} = 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left[1-\frac{\pi}{4}\right] = \frac{a}{2}(4-\pi).$$

Example 3 : *Find by double integration the area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.*

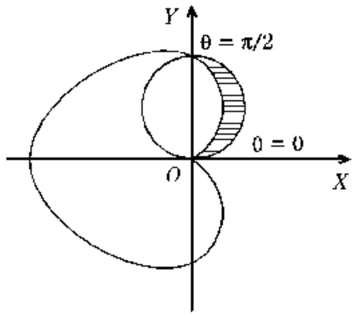
Solution : The given circle is $r = a\sin\theta$ and the cardioid is $r = a(1-\cos\theta)$. Note that the given circle passes through the pole and the diameter through the pole makes an angle $\pi/2$ with the initial line.

Eliminating r between the two equations, we have

$$a\sin\theta = a(1-\cos\theta)$$

or $$1 = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = \tan\frac{\theta}{2}$$

or $$\tfrac{1}{2}\theta = \tfrac{1}{4}\pi \text{ i.e., } \theta = \pi/2.$$



Thus the two curves meet at the point where $\theta = \pi/2$. Also for both the curves $r = 0$ when $\theta = 0$ and so the two curves also meet at the pole O where $\theta = 0$. To cover the required area the limits of integration for r are $a(1-\cos\theta)$ to $a\sin\theta$ and for θ are 0 to $\pi/2$. Therefore the required area

$$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,d\theta\,dr$$

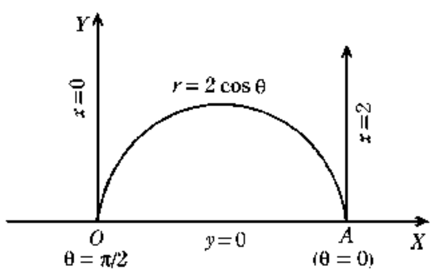
$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi/2}[a^2\sin^2\theta - a^2(1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}[\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\pi}{2} + 2.1 - \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{a^2}{2}\left[2-\frac{\pi}{2}\right] = \frac{a^2}{4}(4-\pi).$$

Example 4 : *Transform the integral* $\displaystyle\int_0^2\int_0^{\sqrt{(2x-x^2)}}\frac{x\,dx\,dy}{\sqrt{(x^2+y^2)}}$ *by changing to polar coordinates and hence evaluate it.* **(Kumaun 2008)**

Solution : From the limits of integration it is obvious that the region of integration is bounded by $y = 0$, $y = \sqrt{(2x-x^2)}$ and $x = 0$, $x = 2$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 - 2x = 0$ between the lines $x = 0$, $x = 2$ and lying above the axis of x *i.e.*, the line $y = 0$.



Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is

$$r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0, \text{ or } r = 2\cos\theta.$$

From the figure it is obvious that r varies from 0 to $2\cos\theta$ and θ varies from 0 to $\pi/2$. Note that at the point A of the circle, $\theta = 0$ and at the point O, $r = 0$ and so from $r = 2\cos\theta$, we get $\theta = \pi/2$ at O.

The polar equivalent of elementary area $dx\,dy$ is $r\,d\theta\,dr$.

$$\therefore \qquad \iint_A f(x, y)\,dx\,dy = \iint_A f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr,$$

where A is the region of integration.

Hence transforming to polar coordinates, the given double integral

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{r}\,r\,d\theta\,dr = \int_0^{\pi/2} \cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\cos\theta\,.4\cos^2\theta\,d\theta = 2\int_0^{\pi/2}\cos^3\theta\,d\theta = 2\,.\frac{2}{3} = \frac{4}{3}\cdot$$

Comprehensive Exercise 2

1. (i) Evaluate $\displaystyle\int_0^{\pi}\int_0^{a\sin\theta} r\,d\theta\,dr.$ **(Kashi 2013)**

(ii) Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\,d\theta\,dr.$

(iii) Evaluate $\displaystyle\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\,d\theta\,dr.$ **(Agra 2003)**

2. Evaluate $\iint r^2\,d\theta\,dr$ over the area of the circle $r = a\cos\theta$. **(Kanpur 2010)**

3. Integrate $r\sin\theta$ over the area of the cardioid $r = a(1+\cos\theta)$, lying above the initial line. **(Kanpur 2010)**

4. Find the mass of a loop of the lemniscate $r^2 = a^2\sin 2\theta$ if density $\rho = kr^2$.

5. Find by double integration the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

6. Find by double integration the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1+\cos\theta) = 1$.

Transform the following double integrals to polar coordinates and hence evaluate them :

7. (i) $\displaystyle\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}} (a^2 - x^2 - y^2)\,dx\,dy.$

(ii) $\int_0^1 \int_x^{\sqrt{(2x-x^2)}} (x^2 + y^2)\, dx\, dy.$

(iii) $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} y^2 \sqrt{(x^2+y^2)}\, dx\, dy.$

Answers 2

1. (i) $\frac{1}{4}\pi a^2.$ (ii) $\frac{a^2}{6}\cdot$ (iii) $\frac{16}{15}a^4.$

2. $\frac{4a^3}{9}\cdot$ **3.** $\frac{4a^3}{3}\cdot$ **4.** $\frac{\pi k a^4}{16}\cdot$

5. $\frac{1}{4}a^2(\pi+8).$ **6.** $\frac{9\pi+16}{12}\cdot$

7. (i) $\int_0^{\pi/2}\int_0^a (a^2-r^2)\, r\, d\theta\, dr; \frac{\pi a^4}{8}\cdot$ (ii) $\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^3\, d\theta\, dr; \left(\frac{3\pi}{8}\right) - 1.$

(iii) $\int_0^{\pi/2}\int_0^a r^4 \sin^2\theta\, d\theta\, dr; \frac{\pi a^5}{20}\cdot$

4.5 Triple Integrals

Let the function $f(x, y, z)$ of the point $P(x, y, z)$ be continuous for all points within a finite region V and on its boundary. Divide the region V into n parts; let $\delta V_1, \delta V_2, \ldots, \delta V_n$ be their volumes. Take a point in each part and form the sum

$$S_n = f(x_1, y_1, z_1)\,\delta V_1 + f(x_2, y_2, z_2)\,\delta V_2 + \ldots + f(x_n, y_n, z_n)\,\delta V_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r. \qquad \ldots(1)$$

Then the limit to which the sum (1) tends when n tends to infinity and the dimensions of each sub-division tend to zero, is called the **triple integral** of the function $f(x, y, z)$ over the region V. This is denoted by

$$\iiint_V f(x, y, z)\, dV \quad \text{or} \quad \iiint_V f(x, y, z)\, dx\, dy\, dz.$$

4.6 Evaluation of Triple Integrals

(a) If the region V be specified by the inequalities

$$a \le x \le b,\ c \le y \le d,\ e \le z \le f,$$

then the triple integral

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b \int_c^d \int_e^f f(x, y, z)\, dx\, dy\, dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\, dz.$$

Here the order of integration is immaterial and the integration with repsect to any of x, y and z can be performed first.

(b) If the limits of z are given as functions of x and y, the limits of y as functions of x while x takes the constant values say from $x = a$ to $x = b$, then

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz.$$

The integration with respect to z is performed first regarding x and y as constants, then the integration w.r.t. y is performed regarding x as a constant and in the last we perform the integration w.r.t. x.

Illustrative Examples

Example 1 : *Evaluate* $\int_{y=0}^{3} \int_{x=0}^{2} \int_{z=0}^{1} (x + y + z)\, dz\, dx\, dy.$

Solution : The given integral

$$= \int_{y=0}^{3} \int_{x=0}^{2} \left\{ \int_0^1 (x + y + z)\, dz \right\} dx\, dy$$

$$= \int_{y=0}^{3} \int_{x=0}^{2} \left[xz + yz + \frac{z^2}{2} \right]_0^1 dx\, dy = \int_0^3 \left\{ \int_0^2 (x + y + \tfrac{1}{2})\, dx \right\} dy$$

$$= \int_0^3 \left[\frac{x^2}{2} + xy + \frac{x}{2} \right]_0^2 dy = \int_0^3 (3 + 2y)\, dy = \left[3y + \frac{2y^2}{2} \right]_0^3 = 18.$$

Example 2 : *Evaluate the following integrals.*

(*i*) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dx\, dy\, dz;$

(*ii*) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz.$

Solution : (i) We have

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dx\, dy\, dz = \int_0^1 \int_0^{1-x} xy \left[\frac{z^2}{2} \right]_0^{1-x-y} dx\, dy,$$

integrating w.r.t. z regarding x and y as constants

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} xy\, \{(1 - x) - y\}^2\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} x\, [y(1 - x)^2 - 2(1 - x)\, y^2 + y^3]\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 x \left[\frac{(1-x)^2 y^2}{2} - \frac{2(1-x)\, y^3}{3} + \frac{y^4}{4} \right]_0^{1-x} dx,$$

integrating w.r.t. y regarding x as constant

$$= \frac{1}{24}\int_0^1 x\,[6\,(1-x)^4 - 8\,(1-x)^4 + 3\,(1-x)^4]\,dx$$

$$= \frac{1}{24}\int_0^1 x\,(1-x)^4\,dx = \frac{1}{24}\int_0^{\pi/2} \sin^2\theta\cos^8\theta.\,2\sin\theta\cos\theta\,d\theta,$$

putting $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$

$$= \frac{1}{12}\int_0^{\pi/2} \sin^3\theta\cos^9\theta\,d\theta = \frac{1}{12}\cdot\frac{2.8.6.4.2}{12.10.8.6.4.2} = \frac{1}{720}.$$

(ii) Here the integrand $x^2 + y^2 + z^2$ is a symmetrical expression in x, y and z and therefore the limits of integration can be assigned at pleasure. We have the given integral

$$= \int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=-a}^{a} (x^2 + y^2 + z^2)\,dx\,dy\,dz$$

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\int_{x=0}^{a} (x^2 + y^2 + z^2)\,dx\,dy\,dz,$$

because $x^2 + y^2 + z^2$ is an even function of x

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{x^3}{3} + (y^2 + z^2)\,x\right]_0^a dy\,dz,$$

integrating w.r.t. x regarding y and z as constants

$$= 2\int_{z=-c}^{c}\int_{y=-b}^{b}\left[\frac{a^3}{3} + ay^2 + az^2\right] dy\,dz$$

$$= 4\int_{z=-c}^{c}\int_0^b \left[\frac{a^3}{3} + az^2 + ay^2\right] dy\,dz,$$

because $\dfrac{a^3}{3} + az^2 + ay^2$ is an even function of y

$$= 4\int_{z=-c}^{c}\left[\frac{a^3}{3}y + az^2 y + \frac{ay^3}{3}\right] dz,$$ integrating w.r.t. y regarding z as constant

$$= 4\int_{z=-c}^{c}\left[\frac{a^3 b}{3} + abz^2 + \frac{ab^3}{3}\right] dz = 8\int_0^c \left[\frac{a^3 b}{3} + abz^2 + \frac{ab^3}{3}\right] dz$$

$$= 8\left[\frac{a^3 b}{3}z + ab\frac{z^3}{3} + \frac{ab^3}{3}z\right]_0^c$$

$$= \frac{8}{3}(a^3 bc + abc^3 + ab^3 c) = \frac{8}{3}abc\,(a^2 + b^2 + c^2).$$

Example 3: *Evaluate* $\displaystyle\int_0^4\int_0^{2\sqrt z}\int_0^{\sqrt{(4z-x^2)}} dz\,dx\,dy.$

Solution : The given triple integral is

$$= \int_0^4\int_0^{2\sqrt z}\left[\int_0^{\sqrt{(4z-x^2)}} dy\right] dz\,dx = \int_0^4\int_0^{2\sqrt z}\Big[\,y\,\Big]_0^{\sqrt{(4z-x^2)}} dz\,dx$$

$$= \int_0^4 \left[\int_0^{2\sqrt{z}} \sqrt{(4z - x^2)}\, dx\right] dz = \int_0^4 \left[\frac{x}{2}\sqrt{(4z - x^2)} + \frac{4z}{2}\sin^{-1}\frac{x}{2\sqrt{z}}\right]_0^{2\sqrt{z}} dz$$

$$= \int_0^4 \left[0 + \frac{4z}{2}\sin^{-1}\frac{2\sqrt{z}}{2\sqrt{z}}\right] dz = \int_0^4 2z \cdot \frac{\pi}{2}\, dz = \int_0^4 \pi z\, dz$$

$$= \pi\left[\frac{z^2}{2}\right]_0^4 = \frac{\pi}{2}[16] = 8\pi.$$

Example 4 : *Find the volume of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.* **(Rohilkhand 2013B)**

Solution : Here the region of integration V to cover the volume of the tetrahedron can be expressed as $0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y$.

Therefore the required volume of the tetrahedron

$$= \iiint_V dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz \qquad \textbf{(Note)}$$

$$= \int_0^1 \int_0^{1-x} \left[z\right]_0^{1-x-y} dx\,dy = \int_0^1 \int_0^{1-x} (1 - x - y)\, dx\,dy$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2\, dx = \frac{1}{2}\left[\frac{(1-x)^3}{3.(-1)}\right]_0^1$$

$$= -\frac{1}{6}[0 - 1] = \frac{1}{6}.$$

Example 5 : *Evaluate* $\iiint (x + y + z)\, dx\,dy\,dz$ *over the tetrahedron* $x = 0$, $y = 0$, $z = 0$ *and* $x + y + z = 1$.

Solution : The region of integration V for the given tetrahedron can be expressed as

$$0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y.$$

Hence the required triple integral $= \iiint_V (x + y + z)\, dx\,dy\,dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\, dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)z + \frac{z^2}{2}\right]_0^{1-x-y} dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \left[(x+y)(1-x-y) + \frac{(1-x-y)^2}{2}\right] dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left(x + y + \frac{1-x-y}{2}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)(1+x+y)\, dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} [1-(x+y)^2]\,dx\,dy = \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x} dx$$ **(Note)**

$$= \frac{1}{2}\int_0^1 \left(1 - x - \frac{1}{3} + \frac{x^3}{3}\right) dx = \frac{1}{2}\int_0^1 \left(\frac{2}{3} - x + \frac{x^3}{3}\right) dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{3\times 4}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

Example 6 : *Evaluate* $\iiint z^2\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Solution : Here the region of integration can be expressed as
$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}, -\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}$.
$\therefore$ the required triple integral

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} z^2\,dx\,dy\,dz$$

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left[\frac{z^3}{3}\right]_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} dx\,dy$$

$$= \frac{1}{3}\int_{-1}^{1} \left[\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} 2\,(1-x^2-y^2)^{3/2}\,dy\right] dx$$

$$= \frac{2}{3}\int_{-1}^{1} \left[\int_{-\pi/2}^{\pi/2} [(1-x^2)\cos^2\theta]^{3/2}.\sqrt{(1-x^2)}.\cos\theta\,d\theta\right] dx$$

[putting $y = \sqrt{(1-x^2)}\sin\theta$ so that $dy = \sqrt{(1-x^2)}\cos\theta\,d\theta$;
also when $y = 0, \theta = 0$ and when $y = \sqrt{(1-x^2)}, \theta = \pi/2$]

$$= \frac{2}{3}\int_{-1}^{1} \left[2.\int_0^{\pi/2} (1-x^2)^2\cos^4\theta\,d\theta\right] dx$$

$$= \frac{4}{3}\int_{-1}^{1} (1-x^2)^2.\frac{3.1}{4.2}\cdot\frac{\pi}{2}\,dx = \frac{\pi}{4}\int_{-1}^{1} (1-x^2)^2\,dx$$

$$= \frac{\pi}{4}.2\int_0^1 (1-2x^2+x^4)\,dx = \frac{\pi}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{\pi}{2}\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{\pi}{2}\cdot\frac{8}{15} = \frac{4\pi}{15}.$$

Comprehensive Exercise 3

Evaluate the following integrals :

1. (i) $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2 yz\,dx\,dy\,dz.$

(ii) $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz.$

(iii) $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x + y + z)\, dy\, dx\, dz.$

(iv) $\int_{0}^{\log 2} \int_{0}^{x} \int_{0}^{x + \log y} e^{x+y+z}\, dx\, dy\, dz.$

2. (i) $\int_{0}^{1} \int_{y^2}^{1} \int_{0}^{1-x} x\, dy\, dx\, dz.$

(ii) $\int_{0}^{1} \int_{0}^{1-x} \int_{0}^{1-x-y} \dfrac{dx\, dy\, dz}{(1 + x + y + z)^3}.$ **(Kanpur 2008; Avadh 13)**

(iii) $\int_{1}^{3} \int_{1/x}^{1} \int_{0}^{\sqrt{(xy)}} xyz\, dx\, dy\, dz.$

(iv) $\int_{0}^{\pi/2} d\theta \int_{0}^{a \sin \theta} dr \int_{0}^{(a^2 - r^2)/a} r\, dz.$

3. (i) $\int_{0}^{a} \int_{0}^{x} \int_{0}^{x+y} e^{x+y+z}\, dx\, dy\, dz.$

(ii) $\int_{0}^{a} \int_{0}^{a-x} \int_{0}^{a-x-y} x^2\, dx\, dy\, dz.$

4. Evaluate the triple integral of the function $f(x, y, z) = x^2$ over the region V enclosed by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$.

(Rohilkhand 2012; Avadh 12)

5. Find the volume of the tetrahedron bounded by the plane $x/a + y/b + z/c = 1$ and the coordinate planes.

6. (i) Evaluate $\iiint \dfrac{dx\, dy\, dz}{(x + y + z + 1)^3}$ over the region $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1.$

(Avadh 2013)

(ii) Evaluate $\iiint xyz\, dx\, dy\, dz$ over the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ **(Kanpur 2011)**

(iii) Evaluate $\iiint (z^5 + z)\, dx\, dy\, dz$ over the sphere $x^2 + y^2 + z^2 = 1.$

(iv) Evaluate $\iiint_R u^2 v^2 w\, du\, dv\, dw$, where R is the region $u^2 + v^2 \leq 1, 0 \leq w \leq 1.$

Answers 3

1. (i) 1. (ii) $(e - 1)^3$. (iii) 0.

(iv) $\dfrac{8}{3} \log 2 - \dfrac{19}{9}\cdot$ **2.** (i) $\dfrac{4}{35}\cdot$ (ii) $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)\cdot$

(iii) $\frac{1}{6}\left(\frac{26}{3}-\log 3\right)$. (iv) $\frac{5a^3\pi}{64}$. **3.** (i) $\frac{1}{8}(e^{4a}-6e^{2a}+8e^{a}-3)$.

(ii) $\frac{a^5}{60}$. **4.** $\frac{a^5}{60}$. **5.** $\frac{abc}{6}$.

6. (i) $\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$. (ii) 0. (iii) 0. (iv) $\frac{\pi}{48}$.

4.7 Change of Order of Integration

If in a double integral the limits of integration of both x and y are constant, we can generally integrate $\iint f(x,y)\,dx\,dy$ in either order. But if the limits of y are functions of x, we must first integrate w.r.t. y regarding x as constant and then integrate w.r.t. x. In this case the order of integration can be changed only if we find the new limits of x as functions of y and the new constant limits of y. This is usually best obtained from geometrical considerations as will be clear from the examples that follow.

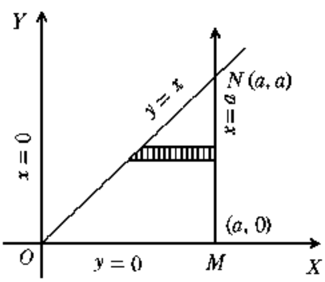
Illustrative Examples

Example 1 : *Change the order of integration in the double integral*

$$\int_0^a \int_0^x f(x,y)\,dx\,dy.$$

(Kashi 2013)

Solution : In the given integral the limits of integration are given by the straight lines $y=0$, $y=x$, $x=0$ and $x=a$. Draw these lines bounding the region of integration in the same figure. We observe that the region of integration is the area ONM.



In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as constant and then w.r.t. x.

To reverse the order of integration, we have to integrate first w.r.t. x regarding y as constant and then w.r.t. y. This is done by dividing the area ONM into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line ON (*i.e.*, $y=x$) and terminating on the line MN (*i.e.*, $x=a$). Thus for this region ONM, x varies from y to a and y varies from 0 to a.

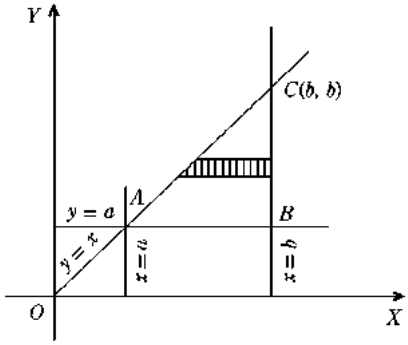
Hence by changing the order of integration, we have

$$\int_0^a \int_0^x f(x,y)\,dx\,dy = \int_0^a \int_y^a f(x,y)\,dy\,dx.$$

Example 2 : *Prove that* $\int_a^b dx \int_a^x f(x,y)\,dy = \int_a^b dy \int_y^b f(x,y)\,dx.$

Solution : Let $I=\int_a^b dx \int_a^x f(x,y)\,dy.$

We are required to change the order of integration in the integral I. In the integral I the limits of integration of y are given by the straight lines $y = a$ and $y = x$. Also the limits of integration of x are given by the straight lines $x = a$ and $x = b$. Draw the straight lines $y = a, y = x, x = a$ and $x = b$, bounding the region of integration, in the same figure. We observe that the region of integration is the area of the triangle ABC.



In the integral I we are required to integrate first w.r.t. y and then w.r.t. x. To reverse the order of integration we have to integrate first w.r.t. x and then w.r.t. y. This is done by dividing the area ABC into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line AC (*i.e.*, $y = x$) and terminating on the line BC (*i.e.*, $x = b$). Thus for the region ABC, x varies from y to b and y varies from a to b. Hence by changing the order of integration, we have

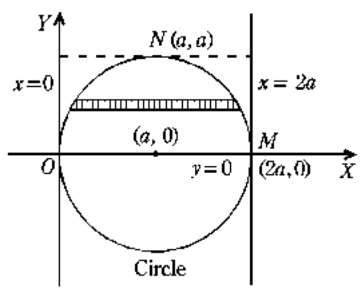
$$\int_a^b dx \int_a^x f(x, y)\, dy = \int_a^b dy \int_y^b f(x, y)\, dx.$$

Example 3 : *Change the order of integration in* $\displaystyle\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} f(x, y)\, dx\, dy.$

(Meerut 2013B)

Solution : In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = \sqrt{(2ax - x^2)}$ *i.e.*, $y^2 = 2ax - x^2$ *i.e.*, $(x - a)^2 + y^2 = a^2$ which is a circle with centre $(a, 0)$ and radius a. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = 2a$.

Draw the curves $(x - a)^2 + y^2 = a^2$, $y = 0$, $x = 0$ and $x = 2a$, bounding the region of integration, in the same figure. From figure we observe that the area of integration is $OMNO$.



In the given integral we are required to integrate first w.r.t. y regarding x as a constant and then w.r.t. x.

To reverse the order of integration, divide the area $OMNO$ into strips parallel to the x-axis. These strips will have their extremities on the portions ON and NM of the circle.

Solving the equation of circle $(x - a)^2 + y^2 = a^2$ for x, we get

$$(x - a)^2 = a^2 - y^2 \text{ i.e., } x - a = \pm \sqrt{(a^2 - y^2)} \qquad \text{i.e.,} \qquad x = a \pm \sqrt{(a^2 - y^2)}.$$

So for the region $OMNO$, x varies from $a - \sqrt{(a^2 - y^2)}$ to $a + \sqrt{(a^2 - y^2)}$ and y varies from 0 to a.

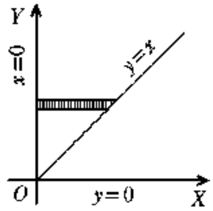
Therefore, changing the order of integration, the given double integral transforms to $\displaystyle\int_0^a \int_{a - \sqrt{(a^2 - y^2)}}^{a + \sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx.$

Example 4 : *Change the order of integration in the double integral*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$$

and hence find its value. **(Agra 2000, 02; Kumaun 01; Avadh 07; Kashi 14; Purvanchal 14)**

Solution : In the given integral the limits of integration are given by the lines $y = x$, $y = \infty$, $x = 0$ and $x = \infty$. Therefore the region of integration is bounded by $x = 0$, $y = x$ and, an infinite boundary. In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by first integrating w.r.t. y along a strip drawn parallel to the y-axis and then integrating w.r.t. x along all such strips so drawn as to cover the whole region of integration.



If we want to reverse the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by dividing this area into strips parallel to the x-axis. So we take strips parallel to the x-axis starting from the line $x = 0$ and terminating on the line $y = x$. Now the limits for x are 0 to y and the limits for y are 0 to ∞.

Hence by changing the order of integration, we have

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \frac{e^{-y}}{y} \Big[x \Big]_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot y\, dy = \int_0^\infty e^{-y}\, dy = \left[\frac{e^{-y}}{-1} \right]_0^\infty = 1.$$

Example 5 : *Change the order of integration in the integral*

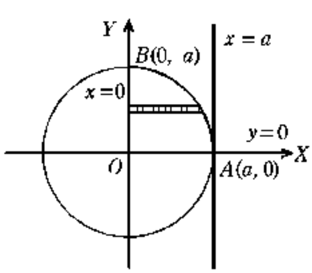
$$\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} f(x, y)\, dx\, dy.$$

Solution : In the given integral the limits of integration of y are given by the straight line $y = 0$ (*i.e.*, the x-axis) and the curve

$$y = \sqrt{(a^2 - x^2)} \text{ i.e., } y^2 = a^2 - x^2 \text{ i.e., } x^2 + y^2 = a^2$$

which is a circle with centre at the origin and radius a. Again the limits of integration of x are given by the lines $x = 0$ and $x = a$.

We draw the curves $y = 0, x^2 + y^2 = a^2, x = 0$ and $x = a$, giving the limits of integration, in the same very figure and we observe that the region of integration is the area OAB of the quadrant of the circle $x^2 + y^2 = a^2$.



To change the order of integration in the given integral, we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covering the area OAB by strips drawn parallel to the x-axis. These strips start from the line OB (*i.e.*, $x = 0$) and terminate on the arc AB of the circle $x^2 + y^2 = a^2$. So on these strips x varies from 0 to $\sqrt{(a^2 - y^2)}$. Also to cover the area OAB, y varies from 0 to a. Hence by changing the order of integration, we have the given integral

$$= \int_0^a \int_0^{\sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx.$$

Example 6 : *Change the order of integration in*

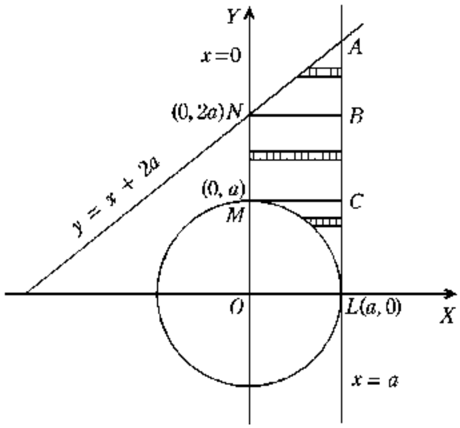
$$\int_0^a \int_{\sqrt{(a^2 - x^2)}}^{x + 2a} f(x, y)\, dx\, dy.$$

Solution : Here the area of integration is bounded by the curves

$$y = \sqrt{(a^2 - x^2)} \quad i.e., \quad x^2 + y^2 = a^2$$

which is a circle with centre $(0, 0)$ and radius a, $y = x + 2a$ which is a straight line passing through $(0, 2a)$, $x = 0$ *i.e.*, the y-axis and the line $x = a$ which is a line parallel to the y-axis at a distance a from the origin.

We draw the curves $x^2 + y^2 = a^2$, $y = x + 2a$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $MLANM$.



To reverse the order of integration, cover this area of integration $ML\,ANM$ by strips parallel to the x-axis. Draw the lines MC and NB parallel to the x-axis so that the region of integration $ML\,ANM$ is divided into three portions MLC, $NMCB$ and NAB.

For the region MLC, x varies from the arc ML of the circle $x^2 + y^2 = a^2$ to the line $x = a$ *i.e.*, x varies from $\sqrt{(a^2 - y^2)}$ to a and y varies from 0 to a.

For the region $NMCB$, x varies from 0 to a and y varies from a to $2a$.

For the region NBA, x varies from $y - 2a$ to a and y varies from $2a$ to $3a$.

Therefore, changing the order of integration, the given integral transforms to

$$\int_0^a \int_{\sqrt{(a^2 - y^2)}}^a f(x, y)\, dy\, dx + \int_0^{2a} \int_0^a f(x, y)\, dy\, dx + \int_{2a}^{3a} \int_{y - 2a}^a f(x, y)\, dy\, dx.$$

Comprehensive Exercise 4

Change the order of integration in the following integrals.

1. $\int_0^1 \int_x^{x(2 - x)} f(x, y)\, dx\, dy.$

2. $\int_0^3 \int_1^{\sqrt{(4 - y)}} (x + y)\, dy\, dx.$ **(Meerut 2013)**

3. $\displaystyle\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x,y)\,dx\,dy.$ **(Kanpur 2005; Avadh 11)**

4. $\displaystyle\int_0^{a}\int_{mx}^{lx} f(x,y)\,dx\,dy.$

5. $\displaystyle\int_0^{2a}\int_{x^2/4a}^{3a-x} f(x,y)\,dx\,dy.$

6. $\displaystyle\int_0^{a}\int_0^{b/(b+x)} f(x,y)\,dx\,dy.$

7. $\displaystyle\int_0^{a}\int_x^{a^2/x} f(x,y)\,dx\,dy.$ **(Kanpur 2010)**

8. $\displaystyle\int_c^{a}\int_{(b/a)\sqrt{(a^2-x^2)}}^{b} f(x,y)\,dx\,dy,$ where $c<a$.

9. $\displaystyle\int_0^{a/2}\int_{x^2/a}^{x-(x^2/a)} f(x,y)\,dx\,dy.$

10. $\displaystyle\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x,y)\,dx\,dy.$

11. $\displaystyle\int_0^{ab/\sqrt{(a^2+b^2)}}\int_0^{(a/b)\sqrt{(b^2-y^2)}} f(x,y)\,dy\,dx.$

12. $\displaystyle\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\,d\theta\,dr.$ **(Kanpur 2009)**

13. Change the order of integration in the double integral

$$\int_0^{a}\int_0^{x}\frac{\phi'(y)\,dx\,dy}{\sqrt{\{(a-x)(x-y)\}}}$$ and hence find its value.

Answers 4

1. $\displaystyle\int_0^{1}\int_{1-\sqrt{(1-y)}}^{y} f(x,y)\,dy\,dx.$

2. $\displaystyle\int_1^{2}\int_0^{4-x^2}(x+y)\,dx\,dy.$

3. $\displaystyle\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x,y)\,dy\,dx+\int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx.$

4. $\displaystyle\int_0^{am}\int_{y/l}^{y/m} f(x,y)\,dy\,dx+\int_{am}^{al}\int_{y/l}^{a} f(x,y)\,dy\,dx.$

5. $\int_0^a \int_0^{\sqrt{(4ay)}} f(x, y)\, dy\, dx + \int_a^{3a} \int_0^{3a-y} f(x, y)\, dy\, dx.$

6. $\int_0^{b/(a+b)} \int_0^a f(x, y)\, dy\, dx + \int_{b/(a+b)}^1 \int_0^{b(1-y)/y} f(x, y)\, dy\, dx.$

7. $\int_0^a \int_0^y f(x, y)\, dy\, dx + \int_a^\infty \int_0^{a^2/y} f(x, y)\, dy\, dx.$

8. $\int_0^{b\sqrt{\{1-(c^2/a^2)\}}} \int_{a\sqrt{\{1-(y^2/b^2)\}}}^a f(x, y)\, dy\, dx + \int_{b\sqrt{\{(1-(c^2/a^2)\}}}^b \int_c^a f(x, y)\, dy\, dx.$

9. $\int_0^{a/4} \int_{\frac{1}{2}[a - \sqrt{(a^2-4ay)}]}^{\sqrt{(ay)}} f(x, y)\, dy\, dx.$

10. $\int_0^a \int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx + \int_0^a \int_{a+\sqrt{(a^2-y^2)}}^{2a} f(x, y)\, dy\, dx + \int_a^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dy\, dx.$

11. $\int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x, y)\, dx\, dy + \int_{ab/\sqrt{(a^2+b^2)}}^a \int_0^{(b/a)\sqrt{(a^2-x^2)}} f(x, y)\, dx\, dy.$

12. $\int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r, \theta)\, dr\, d\theta.$

13. $\int_0^a \int_y^a \dfrac{\phi'(y)\, dy\, dx}{\sqrt{\{(a-x)(x-y)\}}} = \pi\, [\phi(a) - \phi(0)].$

To evaluate the integral put $x = a\sin^2\theta + y\cos^2\theta$.

4.8 Change of Variables in a Double Integral

Sometimes, the evaluation of a double integral becomes more convenient by a suitable change of variables from one system to another system.

Let the variables in the double integral $\iint_A f(x, y)\, dx\, dy$ be changed from x, y to u, v where $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then on substituting for x and y, the double integral is transformed to $\iint_{A'} F(u, v)\, J\, du\, dv$, where $J(u, v)$ is the Jacobian of x, y w.r.t. u, v *i.e.*,

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix},$$

and A' is the region in the uv-plane corresponding to the region A in the xy-plane. Thus remember that $dx\, dy = J\, du\, dv$.

Special case : Change to polar coordinates from the cartesian co-ordinates :

To change the variables from cartesian to polar coordinates we put $x = r\cos\theta$, $y = r\sin\theta$. In this case

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

and therefore $dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr$.

This change is specially useful when the region of integration is a circle or a part of a circle.

Illustrative Examples

Example 1 : *Transform* $\iint f(x, y)\,dx\,dy$ *by the substitution* $x + y = u$, $y = uv$.

Solution : We have $x + y = u$ and $y = uv$. ...(1)

From these, we have

$$x = u - y = u - uv \quad \text{and} \quad y = uv. \qquad ...(2)$$

$$\therefore \quad \frac{\partial x}{\partial u} = 1 - v,\ \frac{\partial x}{\partial v} = -u,\ \frac{\partial y}{\partial u} = v \quad \text{and} \quad \frac{\partial y}{\partial v} = u.$$

$$\therefore \quad J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u.$$

$$\therefore \quad dx\,dy = J\,du\,dv = u\,du\,dv.$$

Hence the given integral transforms to

$$\iint F(u, v)\,u\,du\,dv.$$

Example 2 : *Transform* $\iint f(x, y)\,dx\,dy$ *to polar coordinates.*

Solution : We have $x = r\cos\theta$, $y = r\sin\theta$.

$$\text{Now} \quad J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$\therefore \quad dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr.$$

Hence the given integral transforms to $\iint F(r, \theta)\,r\,d\theta\,dr$.

Example 3 : *Evaluate* $\iint \sqrt{(a^2 - x^2 - y^2)}\,dx\,dy$ *over the semi-circle* $x^2 + y^2 = ax$ *in the positive quadrant.*

Solution : Here the region of integration is a semi-circle. Therefore, for the sake of convenience, changing to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = ax$, we have

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = a\, r \cos \theta \quad \text{or} \quad r^2 (\sin^2 \theta + \cos^2 \theta) = a r \cos \theta$$

or $\qquad r = a \cos \theta.$

The equation $r = a \cos \theta$ represents a circle passing through the pole and diameter through the pole along the initial line.

For the given region r varies from 0 to $a \cos \theta$ and θ varies from 0 to $\pi/2$.

$$\therefore \qquad \iint \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy = \int_0^{\pi/2} \int_0^{a \cos \theta} \sqrt{(a^2 - r^2)}\, .\, r\, d\theta\, dr,$$

$$[\because x^2 + y^2 = r^2 \text{ and } dx\, dy = r\, d\theta\, dr]$$

$$= \int_0^{\pi/2} \left[\int_0^{a \cos \theta} - \frac{1}{2} (a^2 - r^2)^{1/2} . (-2r)\, dr \right] d\theta \qquad \textbf{(Note)}$$

$$= \int_0^{\pi/2} \left[- \frac{1}{2} \cdot \frac{2}{3} (a^2 - r^2)^{3/2} \right]_0^{a \cos \theta} d\theta$$

$$= - \frac{1}{3} \int_0^{\pi/2} (a^3 \sin^3 \theta - a^3)\, d\theta = - \frac{a^3}{3} \left[\frac{2}{3.1} - \frac{\pi}{2} \right] = \frac{1}{3} a^3 \left(\frac{1}{2} \pi - \frac{2}{3}\right).$$

Comprehensive Exercise 5

1. Transform $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$, by the substitution $x + y = u, y = uv$.

2. By using the transformation $x + y = u, y = uv$, show that

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\, dy = \frac{1}{2} (e - 1).$$

3. By using the transformation $x + y = u, y = uv$, prove that

$$\iint \{xy (1 - x - y)\}^{1/2}\, dx\, dy$$

taken over the area of the triangle bounded by the lines

$x = 0, y = 0, x + y = 1$ is $2\pi/105$.

4. Evaluate $\iint (x^2 + y^2)^{7/2}\, dx\, dy$ over the circle $x^2 + y^2 = 1$.

5. Evaluate $\iint xy (x^2 + y^2)^{3/2}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

6. Evaluate $\iint e^{-(x^2 + y^2)}\, dx\, dy$ over the circle $x^2 + y^2 = a^2$.

Answers 5

1. $\int_0^a \int_0^1 F(u, v)\, u\, du\, dv.$

4. $2\pi/9$.

5. $1/14$.

6. $\pi (1 - e^{-a^2})$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ..." so that the following statements are complete and correct.

1. The value of the double integral $\int_0^3 \int_1^2 dx\, dy$ is .... **(Agra 2002)**

2. The value of the double integral $\int_0^1 \int_0^1 xy\, dx\, dy$ is ....

3. The value of the double integral $\int_0^1 \int_0^x xy\, dx\, dy$ is ....

4. The value of the double integral $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$ is ....

5. The value of the triple integral $\int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz$ is ....

6. The value of the triple integral $\int_1^2 \int_1^2 \int_1^3 dx\, dy\, dz$ is ....

7. The value of the double integral $\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} dx\, dy$ is ....

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

8. The value of the double integral $\int_{\theta=0}^{2\pi} \int_{r=0}^{a} r\, d\theta\, dr$ is

(a) πa^2 (b) $\dfrac{\pi a^2}{2}$

(c) πa (d) $2\pi a^2$.

9. The value of the triple integral $\int_0^1 \int_0^1 \int_0^1 xyz\, dx\, dy\, dz$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$

(c) $\frac{1}{4}$ (d) 1.

10. The value of the double integral $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} dy\, dx$ is

(a) πa^2 (b) $2\pi a^2$

(c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a^2}{4}$.

11. The value of the triple integral $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \cos^2 y \cos^2 z\, dx\, dy\, dz$ is

(a) $\frac{\pi^2}{16}$ (b) $\frac{\pi}{64}$

(c) $\frac{\pi^3}{8}$ (d) $\frac{\pi^3}{64}$.

12. The value of the triple integral $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$ is

(a) e^3 (b) $\frac{e^3}{4}$

(c) $(e-1)^3$ (d) $(e+1)^3$.

13. The value of the triple integral $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos y \cos z\, dx\, dy\, dz$ is

(a) 1 (b) $\frac{\pi}{2}$

(c) π (d) $\frac{3\pi}{2}$. **(Rohilkhand 2005)**

14. The value of $\int_0^{\pi/2} \int_0^{\sin\theta} r\, d\theta\, dr$ is equal to

(a) $\int_0^{\pi/2} \sin\theta\, d\theta$ (b) $\int_0^{\sin\theta} \frac{\pi}{2} r\, dr$

(c) $\int_0^{\pi/2} \frac{\sin^2\theta}{2} d\theta$ (d) none of these.

(Garhwal 2003)

True or False:

Write 'T' for true and 'F' for false statement.

15. The value of the double integral $\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a} r\, d\theta\, dr$ is $\frac{\pi a^2}{2}$.

16. The value of the double integral $\int_{-a}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx\, dy$ is πa^2.

17. The value of the double integral $\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} x\, dx\, dy$ is 0.

Answers

1. 3.
2. $\frac{1}{4}$.
3. $\frac{1}{8}$.
4. $\dfrac{\pi a^2}{2}$.
5. 8.
6. 2.
7. $\dfrac{\pi a^2}{2}$.
8. (a).
9. (b).
10. (d).
11. (d).
12. (c).
13. (a).
14. (c).
15. T.
16. T.
17. T.

Chapter 5

Dirichlet's and Liouvill's Integrals

5.1 Dirichlet's Theorem for Three Variables

If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \frac{\Gamma(l)\, \Gamma(m)\, \Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \leq 1$. **(Agra 2001; Kanpur 14; Avadh 14; Kashi 14; Purvanchal 14)**

Proof : Let us first consider the double integral

$$I_2 = \iint x^{l-1} y^{m-1} \, dx \, dy,$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x+y \leq 1$.

Obviously the region of integration of I_2, in the 2-dimensional Euclidean space, is bounded by the straight lines $x=0, y=0$ and $x+y=1$. The limits of integration for this region can be expressed as $0 \leq x \leq 1, 0 \leq y \leq 1-x$.

$$\therefore \qquad I_2 = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1} y^{m-1} \, dx \, dy$$

$$= \int_0^1 x^{l-1} \left[\frac{y^m}{m} \right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{m} x^{l-1} (1-x)^m \, dx$$

$$= \frac{1}{m} \int_0^1 x^{l-1} (1-x)^{m+1-1} \, dx = \frac{1}{m} \mathbf{B}(l, m+1),$$

by the def. of Beta function

$$= \frac{1}{m} \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{1}{m} \frac{\Gamma(l)\,.\,m\,\Gamma(m)}{\Gamma(l+m+1)}, \qquad [\because \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}. \qquad \textbf{(Remember)} \qquad ...(1)$$

This is Dirichlet's theorem for two variables.

Now consider the double integral $U_2 = \iint x^{l-1} y^{m-1} \, dx\, dy$,

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \leq h$.

We have $x + y \leq h \Rightarrow \frac{x}{h} + \frac{y}{h} \leq 1$.

So putting $x/h = u$ and $y/h = v$ so that $dx = h\,du$ and $dy = h\,dv$, the integral U_2 becomes

$$U_2 = \iint (hu)^{l-1} (hv)^{m-1} h^2 \, du\, dv$$

$$= h^{l+m} \iint u^{l-1} v^{m-1} \, du\, dv, \text{ where } u + v \leq 1$$

$$= h^{l+m} \frac{\Gamma(L)\,\Gamma(m)}{\Gamma(l+m+1)}, \text{ by (1).} \qquad ...(2)$$

Now we consider the triple integral

$$I_3 = \iiint x^{l-1} y^{m-1} z^{n-1} \, dx\, dy\, dz,$$

subject to the condition $x + y + z \leq 1$ *i.e.*, $y + z \leq 1 - x$ and $0 \leq x \leq 1$.

We have

$$I_3 = \int_{x=0}^1 \left[\iint y^{m-1} z^{n-1} \, dy\, dz \right] x^{l-1} \, dx, \text{ where } y + z \leq 1 - x$$

$$= \int_0^1 (1-x)^{m+n} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} x^{l-1} \, dx, \quad \text{by using (2)}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n+1-1} \, dx$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \mathbf{B}(l, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}, \qquad \text{which proves the required result.}$$

Remark : The triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = h^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \leq h$.

Alternative proof of Dirichlet's theorem for three variables :

Let $$I_3 = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \leq 1$.

Obviously the region of integration, in the 3-dimensional Euclidean space, is the volume bounded by the coordinate planes $x=0, y=0, z=0$ and the plane $x+y+z=1$. After a little geometric consideration, we observe that the limits of integration for this region can be expressed as

$$0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y.$$

Hence the triple integral I_3 may be written as

$$I_3 = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{n} \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n\, dx\, dy$$

$$= \frac{1}{n} \int_0^1 x^{l-1} \left[\int_0^{1-x} y^{m-1} \{(1-x)-y\}^n\, dy\right] dx.$$

To integrate w.r.t. y, put $y=(1-x)\,t$ so that $dy=(1-x)\,dt$; also when $y=0, t=0$ and when $y=1-x, t=1$.

$\therefore$ the required integral

$$I_3 = \frac{1}{n} \int_0^1 x^{l-1} \left[\int_0^1 (1-x)^{m-1} t^{m-1} \{(1-x)^n (1-t)^n\} (1-x)\, dt\right] dx$$

$$= \frac{1}{n} \int_0^1 \int_0^1 x^{l-1} (1-x)^{m+n} t^{m-1} (1-t)^n\, dx\, dt$$

$$= \frac{1}{n} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx \times \int_0^1 t^{m-1} .(1-t)^n\, dt$$

$$= \frac{1}{n} B(l, m+n+1)\, B(m, n+1),$$

(by the definition of Beta function)

$$= \frac{1}{n} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} \cdot \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

$$[\because \Gamma(n+1) = n\,\Gamma(n)]$$

Note : Dirichlet's theorem holds good even if the condition is taken as $x + y + z < 1$ in place of $x + y + z \leq 1$.

Corollary : *Evaluate without using Dirichlet's Theorem* $\iiint x^p y^q z^r \, dx\, dy\, dz$, *where x, y, z are always positive and* $x + y + z \leq 1$.

5.2 Dirichlet's Theorem for *n* Variables

The theorem states that $\iint \dots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \dots x_n^{l_n - 1} \, dx_1\, dx_2 \dots dx_n$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2) \dots \Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \dots + l_n)},$$

where the integral is extended to all positive values of the variables $x_1, x_2, \dots, x_n$ *subject to the condition* $x_1 + x_2 + \dots + x_n \leq 1$. **(Meerut 2013B)**

Proof : We shall prove the theorem by mathematical induction. To start the induction we shall first show that the theorem is true for two variables *i.e.*, for $n = 2$.

So let us consider the integral $I_2 = \iint x_1^{l_1 - 1} x_2^{l_2 - 1} \, dx_1\, dx_2$

subject to the condition $x_1 + x_2 \leq 1$.

Now proceeding as in § 5.1, show that $I_2 = \dfrac{\Gamma(l_1)\,\Gamma(l_2)}{\Gamma(1 + l_1 + l_2)}$. ...(1)

The result (1) shows that the theorem is true for two variables *i.e.*, for $n = 2$.

Now assume as our induction hypothesis that the theorem is true for n variables *i.e.*, assume that

$$I_n = \iint \dots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \dots x_n^{l_n - 1} \, dx_1\, dx_2 \dots dx_n$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2) \dots \Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \dots + l_n)}, \quad \text{...(2)}$$

subject to the condition $x_1 + x_2 + \dots + x_n \leq 1$.

If the condition be $x_1 + x_2 + \dots + x_n \leq h$, then putting

$\dfrac{x_1}{h} = u_1, \dfrac{x_2}{h} = u_2, \dots, \dfrac{x_n}{h} = u_n$, so that

$dx_1 = h\, du_1,\ dx_2 = h\, du_2, \dots, dx_n = h\, du_n$, we have

$$\iint \dots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \dots x_n^{l_n - 1} \, dx_1\, dx_2 \dots dx_n$$

$$= h^{l_1 + l_2 + \dots + l_n} \iint \dots \int u_1^{\,l_1 - 1} u_2^{\,l_2 - 1} \dots u_n^{\,l_n - 1} \, du_1 \, du_2 \dots du_n$$

subject to the condition $u_1 + u_2 + \dots + u_n \leq 1$

$$= h^{l_1 + l_2 + \dots + l_n} \frac{\Gamma(l_1)\,\Gamma(l_2)\dots\Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \dots + l_n)}, \qquad \text{...(3)}$$

using the assumed result (2).

Now for $n + 1$ variables the condition is

$$x_1 + x_2 + \dots + x_n + x_{n+1} \leq 1$$

i.e., $\quad x_2 + x_3 + \dots + x_n + x_{n+1} \leq 1 - x_1$, and $0 \leq x_1 \leq 1$.

We then have

$$\iint \dots \int x_1^{\,l_1 - 1} x_2^{\,l_2 - 1} \dots x_n^{\,l_n - 1} x_{n+1}^{\,l_{n+1} - 1} \, dx_1 \, dx_2 \dots dx_n \, dx_{n+1},$$

where $x_1 + x_2 + \dots + x_{n+1} \leq 1$

$$= \int_{x_1 = 0}^{1} x_1^{\,l_1 - 1} \left[\iint \dots \int x_2^{\,l_2 - 1} \dots x_{n+1}^{\,l_{n+1} - 1} \, dx_2 \dots dx_{n+1} \right] dx_1$$

$$= \int_{x_1 = 0}^{1} x_1^{\,l_1 - 1} \cdot \frac{\Gamma(l_2)\,\Gamma(l_3)\dots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + l_3 + \dots + l_n + l_{n+1})} \cdot (1 - x_1)^{l_2 + l_3 + \dots + l_{n+1}} \, dx_1,$$

using (3)

$$= \frac{\Gamma(l_2)\,\Gamma(l_3)\dots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + \dots + l_n + l_{n+1})} \cdot \int_0^1 x_1^{\,l_1 - 1} (1 - x_1)^{(1 + l_2 + l_3 + \dots + l_{n+1}) - 1} \, dx_1$$

$$= \frac{\Gamma(l_2)\,\Gamma(l_3)\dots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + \dots + l_n + l_{n+1})} \cdot \frac{\Gamma(l_1)\,\Gamma(1 + l_2 + \dots + l_{n+1})}{\Gamma(1 + l_1 + l_2 + \dots + l_n + l_{n+1})}$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2)\dots\Gamma(l_{n+1})}{\Gamma(1 + l_1 + l_2 + \dots + l_{n+1})}. \qquad \text{...(4)}$$

The result (4) shows that the theorem holds for $(n + 1)$ variables if it holds for n variables. But we have seen that the theorem is true for two variables. Hence by mathematical induction the theorem is true for all values of n.

Illustrative Examples

Example 1 : Evaluate $\iint x^{2l - 1} y^{2m - 1} \, dx \, dy$ *for all positive values of x and y such that* $x^2 + y^2 \leq c^2$.

Solution : Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of x and y subject to the condition

$$\left(\frac{x}{c}\right)^2 + \left(\frac{y}{c}\right)^2 \le 1.$$

Put $\quad (x/c)^2 = u$ *i.e.*, $x = cu^{1/2}$, so that $dx = \frac{1}{2} cu^{-1/2}\, du$,

and $\quad (y/c)^2 = v$ *i.e.*, $y = cv^{1/2}$, so that $dy = \frac{1}{2} cv^{-1/2}\, dv$.

Then the required integral

$$I = \iint (cu^{1/2})^{2l-1} (cv^{1/2})^{2m-1} \cdot \frac{1}{2} cu^{-1/2} \cdot \frac{1}{2} cv^{-1/2}\, du\, dv$$

$$= \frac{1}{4} c^{2l+2m} \iint u^{l-1} v^{m-1}\, du\, dv, \text{ where } u, v \text{ take all + ive values}$$

subject to the condition $u + v \le 1$

$$= \frac{1}{4} c^{2l+2m} \cdot \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}, \text{ by Dirichlet's theorem.}$$

Example 2 : *Find the value of* $\iint \dots \int dx_1\, dx_2 \dots dx_n$ *extended to all positive values of the variables, subject to the condition* $x_1^2 + x_2^2 + \dots + x_n^2 < R^2$.

Solution : Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of $x_1, x_2, \dots, x_n$ subject to the condition

$$\frac{x_1^2}{R^2} + \frac{x_2^2}{R^2} + \dots + \frac{x_n^2}{R^2} < 1.$$

Put $\quad (x_1/R)^2 = u_1$ *i.e.*, $x_1 = Ru_1^{1/2}$, so that $dx_1 = \frac{1}{2} Ru_1^{-1/2}\, du_1$,

$(x_2/R)^2 = u_2$ *i.e.*, $x_2 = Ru_2^{1/2}$, so that $dx_2 = \frac{1}{2} Ru_2^{-1/2}\, du_2$, and so on.

Then the required integral

$$I = \iint \dots \int \left(\frac{1}{2}\right)^n R^n u_1^{-1/2} u_2^{-1/2} \dots u_n^{-1/2}\, du_1\, du_2 \dots du_n$$

$$= \left(\frac{R}{2}\right)^n \iint \dots \int u_1^{(1/2)-1} u_2^{(1/2)-1} \dots u_n^{(1/2)-1}\, du_1\, du_2 \dots du_n,$$

subject to the condition $u_1 + u_2 + \dots + u_n < 1$

$$= \left(\frac{R}{2}\right)^n \frac{\{\Gamma(\frac{1}{2})\}^n}{\Gamma(1 + n \cdot \frac{1}{2})}, \text{ by Dirichlet's theorem}$$

$$= \left(\frac{R}{2}\right)^n \cdot \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2}n)}. \qquad [\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}]$$

Example 3 : *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$

Solution : Since the equation $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ does not change by putting $-x$ for x, $-y$ for y and $-z$ for z, therefore the surface represented by this equation is symmetrical in all the eight octants.

So the volume of the solid surrounded by this surface = $8\times$ the volume of the portion of this solid lying in the positive octant.

Now the volume of a small element situated at any point $(x, y, z) = dx\,dy\,dz$.

$\therefore$ the volume of the solid in the positive octant

$$= \iiint dx\,dy\,dz,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} \leq 1$.

Now put $(x/a)^{2/3} = u, (y/b)^{2/3} = v, (z/c)^{2/3} = w$

i.e., $x = au^{3/2}, y = bv^{3/2}, z = cw^{3/2}$

so that $dx = \frac{3}{2} au^{1/2}\,du, dy = \frac{3}{2} bv^{1/2}\,dv, dz = \frac{3}{2} cw^{1/2}\,dw$.

$\therefore$ the volume in the positive octant

$$= \iiint \frac{27}{8} abc\, u^{(3/2)-1} v^{(3/2)-1} w^{(3/2)-1}\,du\,dv\,dw,$$

where $u + v + w \leq 1$

$$= \frac{27}{8} abc \frac{[\Gamma(3/2)]^3}{\Gamma(\frac{9}{2}+1)} = \frac{27}{8} abc \cdot \frac{(\frac{1}{2}\cdot\sqrt{\pi})^3}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{27}{8} abc \cdot \frac{\pi}{8} \cdot \frac{32}{27.35} = \frac{\pi\,abc}{8} \cdot \frac{4}{35}.$$

Hence the required volume

$$= 8 \cdot \frac{\pi\,abc}{8} \cdot \frac{4}{35} = \frac{4\pi\,abc}{35}.$$

5.3 Liouville's Extension of Dirichlet's Theorem

If the variables x, y, z are all positive such that

$$h_1 \leq x + y + z \leq h_2,$$

then the triple integral

$$\iiint \boldsymbol{f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz}$$

$$= \boldsymbol{\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\,du.}$$

(Garhwal 2002; Kashi 14)

Proof: Let $I = \iiint x^{l-1} y^{m-1} z^{n-1}\,dx\,dy\,dz$, integrated over some region.

Subject to the condition $x + y + z \leq u$, we have by Dirichlet's theorem

$$I = u^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}. \qquad ...(1)$$

If the condition be $x + y + z \leq u + \delta u$, then

$$I = (u + \delta u)^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}. \qquad ...(2)$$

Therefore the value of the integral I extended to all such positive values of the variables as make the sum of the variables lie between u and $u + \delta u$ is

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} [(u + \delta u)^{l+m+n} - u^{l+m+n}],$$

[subtracting (2) from (1)]

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n} \left[\left(1 + \frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n} \left[1 + (l+m+n)\frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} (l+m+n)\, u^{l+m+n-1}\, \delta u,$$

to the first order of approximation

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} u^{l+m+n-1}\, \delta u.$$

Now consider the integral $\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$,

subject to the condition $h_1 \le x + y + z \le h_2$.

If $x + y + z$ lies between u and $u + \delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

which arises from supposing the sum of the variables to lie between u and $u + \delta u$ is ultimately equal to $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} f(u)\,.\, u^{l+m+n-1}\, \delta u$.

Therefore the whole integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz, \quad \text{where } h_1 \le x + y + z \le h_2,$$

is equal to $$\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\,.\, u^{l+m+n-1}\, du.$$

Remark : The above theorem holds good even if we take the condition as $h_1 < x + y + z < h_2$ in place of $h_1 \le x + y + z \le h_2$.

Illustrative Examples

Example 1 : *Find the value of* $\iiint \log(x+y+z)\, dx\, dy\, dz$, *the integral extending over all positive values of* x, y, z *subject to the condition* $x + y + z < 1$.

(Kanpur 2014; Purvanchal 14)

Solution : Here the integral is to be extended for all positive values of x, y and z such that $0 < x + y + z < 1$.

$\therefore$ the required integral

$$= \iiint \log(x+y+z)\,dx\,dy\,dz, \text{ where } 0 < x+y+z < 1$$

$$= \iiint \log(x+y+z)\,x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz \qquad \textbf{(Note)}$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 (\log u)\,u^{1+1+1-1}\,du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{1}{\Gamma(3)} \int_0^1 u^2 \log u\,du, \qquad [\because \Gamma(1) = 1]$$

$$= \frac{1}{2.1}\left[\left((\log u)\cdot\frac{u^3}{3}\right)_0^1 - \int_0^1 \frac{1}{u}\cdot\frac{u^3}{3}\,du\right],$$

integrating by parts taking u^2 as the second function

$$= \frac{1}{2}\left[0 - \frac{1}{3}\lim_{u\to 0} u^3 \log u - \frac{1}{3}\int_0^1 u^2\,du\right]$$

$$= -\frac{1}{6}\left[\frac{u^3}{3}\right]_0^1, \qquad \left[\because \lim_{u\to 0} u^3 \log u = 0\right]$$

$$= -\frac{1}{18}.$$

Note : $\displaystyle \lim_{u\to 0} u^3 \log u = \lim_{u\to 0} \frac{\log u}{1/u^3} = \lim_{u\to 0} \frac{1/u}{-3/u^4} = \lim_{u\to 0} -\frac{1}{3}u^3 = 0.$

Example 2 : *Prove that*

$$\iiint \frac{dx\,dy\,dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \frac{\pi^2 a^2}{8},$$

the integral being extended for all positive values of the variables for which the expression is real. **(Garhwal 2002; Avadh 08)**

Solution : The given expression is real when $x^2 + y^2 + z^2 < a^2$.

Therefore the required integral is to be extended to all positive values of x, y and z such that

$$0 < x^2 + y^2 + z^2 < a^2 \quad \text{i.e.,} \quad 0 < x^2/a^2 + y^2/a^2 + z^2/a^2 < 1.$$

Put $(x^2/a^2) = u_1$, $(y^2/a^2) = u_2$ and $(z^2/a^2) = u_3$

i.e., $x = a u_1^{1/2}$, $y = a u_2^{1/2}$ and $z = a u_3^{1/2}$

so that $dx = \frac{1}{2} a u_1^{-1/2}\,du_1$, $dy = \frac{1}{2} a u_2^{-1/2}\,du_2$ and $dz = \frac{1}{2} a u_3^{-1/2}\,du_3$.

With these substitutions the given condition reduces to

$$0 < u_1 + u_2 + u_3 < 1$$

and the required integral becomes

$$= \iiint \frac{(\frac{1}{2})^3 \cdot a^3 u_1^{-1/2} u_2^{-1/2} u_3^{-1/2}\,du_1\,du_2\,du_3}{a\sqrt{\{1 - (u_1 + u_2 + u_3)\}}}$$

$$= \frac{a^2}{8} \iiint \frac{u_1^{1/2-1} u_2^{1/2-1} u_3^{1/2-1} \, du_1 \, du_2 \, du_3}{\sqrt{\{1-(u_1+u_2+u_3)\}}}$$

$$= \frac{a^2}{8} \cdot \frac{[\Gamma(1/2)]^3}{\Gamma(\frac{3}{2})} \cdot \int_0^1 u^{3/2-1} \cdot \frac{1}{\sqrt{(1-u)}} \, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2}{8} \cdot \frac{[\sqrt{\pi}]^3}{\frac{1}{2} \cdot \sqrt{\pi}} \cdot \int_0^{\pi/2} \frac{\sin\theta \, .2 \sin\theta \cos\theta \, d\theta}{\sqrt{(1-\sin^2\theta)}}, \text{ putting } u = \sin^2\theta \text{ etc.}$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi a^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^2}{8}.$$

Example 3 : *Prove that when x and y are positive and* $x + y < h$,

$$\iint f'(x+y)\, x^{l-1} y^{-l} \, dx\, dy = \frac{\pi}{\sin l\pi} [f(h) - f(0)].$$ **(Kumaun 2008)**

Solution : The given integral

$$I = \iint f'(x+y)\, x^{l-1} y^{(1-l)-1} \, dx\, dy, \text{ where } 0 < x + y < h$$

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(l+1-l)} \int_0^h f'(u)\, u^{l+(1-l)-1} \, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(1)} \int_0^h f'(u)\, du$$

$$= \frac{\pi}{\sin \pi l} \Big[f(u)\Big]_0^h = \frac{\pi}{\sin \pi l} [f(h) - f(0)].$$

Example 4 : *Evaluate* $\iiint x^\alpha y^\beta z^\gamma (1-x-y-z)^\lambda \, dx\, dy\, dz$ *over the interior of the tetrahedron formed by the coordinate planes and the plane* $x + y + z = 1$.

Solution : Here the region of integration is bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$. So the variables x, y, z take all positive values subject to the condition

$$0 < x + y + z < 1.$$

Hence the given integral

$$= \iiint x^{(\alpha+1)-1} y^{(\beta+1)-1} z^{(\gamma+1)-1} [1-(x+y+z)]^\lambda \, dx\, dy\, dz$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{\alpha+1+\beta+1+\gamma+1-1} (1-u)^\lambda \, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{(\alpha+\beta+\gamma+3)-1} (1-u)^{(\lambda+1)-1} \, du$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} B(\alpha+\beta+\gamma+3, \lambda+1)$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \frac{\Gamma(\alpha+\beta+\gamma+3)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}.$$

Example 5 : *Evaluate*

$$\iiint \sqrt{(a^2 b^2 c^2 - b^2 c^2 x^2 - c^2 a^2 y^2 - a^2 b^2 z^2)}\, dx\, dy\, dz$$

taken throughout the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Solution : The given ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is symmetrical in all the eight octants. Let us first evaluate the given integral over the region of the ellipsoid which lies in the positive octant *i.e.*, where x, y, z are all positive.

Put $x^2/a^2 = u$, $y^2/b^2 = v$, $z^2/c^2 = w$.

Then $x = au^{1/2}$, $dx = \frac{1}{2} a u^{-1/2}\, du$ etc.

Now the given integral extended over the positive octant of the given ellipsoid is

$$I = abc \iiint \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)}\, dx\, dy\, dz, \text{ where } 0 < x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1$$

$$= abc \iiint \sqrt{(1 - u - v - w)} \cdot \frac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

where $0 < u + v + w \leq 1$

$$= \frac{a^2 b^2 c^2}{8} \iiint u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} \sqrt{\{1 - (u + v + w)\}}\, du\, dv\, dw$$

$$= \frac{a^2 b^2 c^2}{8} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)} \int_0^1 \sqrt{(1 - t)}\, .\, t^{1/2 + 1/2 + 1/2 - 1}\, dt,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2 b^2 c^2}{8} \cdot \frac{(\sqrt{\pi})^3}{\frac{1}{2} . \sqrt{\pi}} \int_0^1 (1 - t)^{(3/2)-1}\, t^{(3/2)-1}\, dt$$

$$= \frac{a^2 b^2 c^2}{8} \cdot 2\pi \cdot \frac{\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(3)} = \frac{\pi^2 a^2 b^2 c^2}{32}.$$

Hence if the integration is extended throughout the ellipsoid, the given integral

$$= 8I = 8 \cdot \frac{\pi^2 a^2 b^2 c^2}{32} = \frac{\pi^2 a^2 b^2 c^2}{4}.$$

Example 6 : *Evaluate* $\displaystyle\iint \sqrt{\left\{\frac{1 - x^2/a^2 - y^2/b^2}{1 + x^2/a^2 + y^2/b^2}\right\}}\, dx\, dy$

where $x^2/a^2 + y^2/b^2 \leq 1$.

Solution : The ellipse $x^2/a^2 + y^2/b^2 = 1$ is symmetrical in all the four quadrants.

Let us first evaluate the given integral over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant *i.e.*, where x and y are both positive.

Put $x^2/a^2 = u$, $y^2/b^2 = v$.

Then $\quad x = au^{1/2}, dx = \frac{1}{2} au^{-1/2}\, du,$

$y = bv^{1/2}, dy = \frac{1}{2} bv^{-1/2}\, dv.$

$\therefore$ the given integral extended over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant is given by

$$I = \iint \sqrt{\left(\frac{1-u-v}{1+u+v}\right)} \cdot \frac{1}{2} abu^{-1/2} v^{-1/2}\, du\, dv, \quad \text{where} \quad 0 < u + v \leq 1$$

$$= \frac{ab}{4} \iint \sqrt{\left\{\frac{1-(u+v)}{1+(u+v)}\right\}}\, u^{(1/2)-1} v^{(1/2)-1}\, du\, dv$$

$$= \frac{ab}{4} \cdot \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} \int_0^1 \sqrt{\left(\frac{1-t}{1+t}\right)} \cdot t^{1/2+1/2-1}\, dt$$

$$= \frac{\pi ab}{4} \cdot \int_0^1 \frac{1-t}{\sqrt{(1-t^2)}}\, dt = \frac{\pi ab}{4} \int_0^{\pi/2} \frac{1-\sin\theta}{\cos\theta} \cos\theta\, d\theta,$$

putting $t = \sin\theta$ so that $dt = \cos\theta\, d\theta$

$$= \frac{\pi ab}{4} \int_0^{\pi/2} (1 - \sin\theta)\, d\theta = \frac{\pi ab}{4} \Big[\theta + \cos\theta\Big]_0^{\pi/2}$$

$$= \frac{\pi ab}{4}\left[\left(\frac{\pi}{2} + 0\right) - \left(0 + 1\right)\right] = \frac{\pi ab}{4}\left(\frac{\pi}{2} - 1\right).$$

Hence the given integral extended over the whole region of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 = 4.I = \pi ab\,(\tfrac{1}{2}\pi - 1).$$

Example 7: *Prove that* $I = \iiiint dx\, dy\, dz\, dw$, *for all positive values of the variables for which* $x^2 + y^2 + z^2 + w^2$ *is not less than* a^2 *and not greater than* b^2 *is* $\pi^2 (b^4 - a^4)/32$.

Solution : We have to evaluate I subject to the condition

$$a^2 < x^2 + y^2 + z^2 + w^2 < b^2.$$

Putting $x^2 = u_1$ *i.e.*, $x = u_1^{1/2}, dx = \frac{1}{2} u_1^{-1/2}\, du_1$ etc., we get

$$I = \iiiint \frac{1}{2} u_1^{-1/2} \cdot \frac{1}{2} u_2^{-1/2} \cdot \frac{1}{2} u_3^{-1/2} \cdot \frac{1}{2} u_4^{-1/2}\, du_1\, du_2\, du_3\, du_4$$

subject to the condition $a^2 < u_1 + u_2 + u_3 + u_4 < b^2$

or $$I = \frac{1}{16} \iiiint u_1^{(1/2)-1} u_2^{(1/2)-1} u_3^{(1/2)-1} u_4^{(1/2)-1}\, du_1\, du_2\, du_3\, du_4$$

$$= \frac{1}{16} \cdot \frac{[\Gamma(\frac{1}{2})]^4}{\Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2})} \int_{a^2}^{b^2} t^{1/2+1/2+1/2+1/2-1}\, dt,$$

by Liouville's theorem

$$= \frac{(\sqrt{\pi})^4}{16\,\Gamma(2)} \int_{a^2}^{b^2} t\, dt = \frac{\pi^2}{16} \cdot \left[\frac{t^2}{2}\right]_{a^2}^{b^2} = \frac{\pi^2}{32}(b^4 - a^4).$$

Comprehensive Exercise 1

1. (i) Show that the integral $\iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz$ integrated over the region in the first octant below the surface $(x/a)^p + (y/b)^q + (z/c)^r = 1$ is,

$$\frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p) \, \Gamma(m/q) \, \Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}.$$

(Avadh 2011)

(ii) Show that if l, m, n are all positive,

$$\iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2) \, \Gamma(m/2) \, \Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2)},$$

where the triple integral is taken throughout the part of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, which lies in the positive octant.

2. Prove that the area in the positive quadrant between the curve $x^n + y^n = a^n$ and the coordinate axes is

$$\frac{a^2 [\Gamma(1/n)]^2}{2n \, \Gamma(2/n)}.$$

(Kanpur 2009)

3. (i) Find the volume in the positive octant of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(ii) Find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(iii) Evaluate $\iiint dx \, dy \, dz$, where $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

4. Evaluate $\iiint xyz \, dx \, dy \, dz$ for all positive values of the variables throughout the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(Kanpur 2011)**

5. The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in the points A, B, C. Use Dirichlet's integral to evaluate the mass of the tetrahedron $OABC$, the density at any point (x, y, z) being $k \, xyz$. **(Garhwal 2003)**

6. Evaluate the integral $\iiint x^2 yz \, dx \, dy \, dz$ over the volume enclosed by the region $x, y, z, \ge 0$ and $x + y + z \le 1$. **(Agra 2003)**

7. (i) Evaluate the double integral

$$\iint_D x^{1/2} y^{1/2} (1 - x - y)^{2/3} \, dx \, dy$$

over the domain D bounded by the lines $x = 0, y = 0, x + y = 1$.

(ii) Evaluate $\iint_T x^{1/2} y^{1/2} (1 - x - y)^{3/2} \, dx \, dy$, where T is the region bounded by $x \ge 0, y \ge 0, x + y \le 1$.

(iii) Evaluate $\iiint e^{x+y+z} \, dx \, dy \, dz$ taken over the positive octant such that $x + y + z \le 1$. **(Kanpur 2005, 10)**

8. (i) Evaluate $\iiint x^{-1/2} y^{-1/2} z^{-1/2} (1 - x - y - z)^{1/2}\, dx\, dy\, dz$ extended to all positive values of the variables subject to the condition $x + y + z < 1$. **(Kanpur 2007)**

(ii) Prove that $\iiint \dfrac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \dfrac{\pi^2}{8}$, the integral extended to all positive values of the variables for which the expression is real.

(iii) If S is a unit sphere with its centre at the origin, then prove that

$$\iiint_S \frac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \pi^2.$$

9. Evaluate $\iiint_R (x + y + z + 1)^2\, dx\, dy\, dz$, where R is the region defined by

$$x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1.$$

10. Show that $\displaystyle\iint \left(\frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right)^{1/2} dx\, dy = \frac{\pi}{8}(\pi - 2)$

over the positive quadrant of the circle $x^2 + y^2 = 1$.

11. Evaluate $\displaystyle\iiint \sqrt{\left(\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}\right)}\, dx\, dy\, dz$ integral being taken over all positive values of x, y, z such that $x^2 + y^2 + z^2 \leq 1$.

12. (i) Evaluate $\iint_R \sqrt{(x^2 + y^2)}\, dx\, dy$

where R is the region in the xy-plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(ii) Evaluate $\iint_R \sqrt{(x^2 + y^2)}\, dx\, dy$ where R is the region $x^2 + y^2 \leq a^2$.

13. Evaluate the integral

$$\iiint_R \sqrt{(1 - x^2 - y^2 - z^2)}\, dx\, dy\, dz$$

where R is the region interior to the sphere $x^2 + y^2 + z^2 = 1$.

14. Find the mass of the region bounded by the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

if the density varies as the square of the distance from its centre.

15. Prove that $\displaystyle\iiint \frac{dx\, dy\, dz}{(x + y + z + 1)^3} = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$

throughout the volume bounded by the coordinate planes and the plane $x + y + z = 1$.

(Rohilkhand 2013)

Answers 1

3. (i) $\pi\, abc/6$. (ii) $\frac{4}{3}\pi\, abc$. (iii) $\frac{4}{3}\pi\, abc$.

4. $a^2b^2c^2/48$. **5.** $k\, a^2b^2c^2/720$. **6.** $1/2520$.

7. (i) $27\pi/1760$. (ii) $2\pi/315$. (iii) $(e-2)/2$.

8. (i) $\pi^2/4$. **9.** $31/60$.

11. $\frac{1}{8}\pi\left[B\left(\frac{3}{4},\frac{1}{2}\right)-B\left(\frac{5}{4},\frac{1}{2}\right)\right]$. **12.** (i) $38\pi/3$. (ii) $2\pi a^3/3$.

13. $\pi^2/4$. **14.** $8\pi\, abck\,(a^2+b^2+c^2)/30$, where k is constant.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …" so that the following statements are complete and correct.

1. If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\ldots\ldots},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le 1$.

2. If the variables x, y, z are all positive such that $h_1 \le x+y+z \le h_2$, then the triple integral

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\ldots\ldots}\int_{h_1}^{h_2} F(t)\, t^{l+m+n-1}\, dt.$$

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

3. If $x, y, z \ge 0$ and $h_1 \le x+y+z \le h_2$, the value of

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz \text{ is equivalent to}$$

(a) $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$

(b) $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$

(c) $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n}\, du$

(d) $\dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n}\, du$ **(Garhwal 2002)**

4. Dirichlet's theorem can be generalized for n variables, where

(a) $n \leq 4$ (b) $n \leq 100$

(c) n is any positive integer (d) none of these **(Kumaun 2008)**

True or False:

Write 'T' for true and 'F' for false statement.

5. $$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{3}\int_0^1 \frac{u^2}{(u+1)^3}\,du,$$

where the region of integration is the volume bounded by the coordinate planes and the plane $x+y+z=1$.

6. $$\iiint (x+y+z+1)^2\,dx\,dy\,dz = \frac{1}{2}\int_0^1 u^2\,(u+1)^2\,du,$$

where the region of integration is the volume bounded by the coordinate planes and the plane $x+y+z=1$.

7. If l and m are both positive, then the double integral

$$\iint x^{l-1}\,y^{m-1}\,dx\,dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)},$$

where the integral is extended to all positve values of the variables x and y subject to the condition $x+y \leq 1$.

Answers

1. $\Gamma(l+m+n+1)$. **2.** $\Gamma(l+m+n)$. **3.** (a).

4. (c). **5.** *F*. **6.** *T*.

7. *F*.

Chapter

6

Areas of Curves

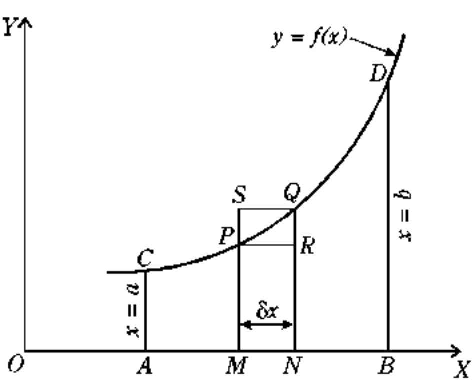
6.1 Quadrature

The process of finding the area of any bounded portion of a curve is called **quadrature.**

6.2 Areas of Curves given by Cartesian Equations

If $f(x)$ is a continuous and single valued function of x, then the area bounded by the curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ is

$$\int_a^b y\,dx, \quad or \quad \int_a^b f(x)\,dx.$$



Proof. Let CD be the arc of the curve $y = f(x)$ and AC and BD be the two ordinates $x = a$ and $x = b$.

Consider $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, the two neighbouring points on the curve. Draw PM and QN perpendiculars to the axis of x, then

$PM = y$, $QN = y + \delta y$ and $MN = \delta x$.

Draw PR and QS perpendiculars to NQ and MP produced respectively. The area $AMPC$ depends upon the position of P on the curve. Let A denote the area $AMPC$ and $A + \delta A$ be the area $ANQC$. Then the area

$$MNQP = \text{area } ANQC - \text{area } AMPC$$
$$= A + \delta A - A = \delta A.$$

But clearly this area δA (*i.e.*, the area $MNQP$) lies in magnitude between the areas of the rectangles $MNRP$ and $MNQS$.

Thus, we have

Area of the rectangle $MNQS > \delta A >$ area of the rectangle $MNRP$

i.e., $(y + \delta y)\,\delta x > \delta A > y\,\delta x$ or $y + \delta y > \dfrac{\delta A}{\delta x} > y.$

Now as $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$. Therefore we have

$$\frac{dA}{dx} = y = f(x), \quad \text{or} \quad dA = y\,dx.$$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$\int_{x=a}^{x=b} dA = \int_a^b y\,dx \quad \text{or} \quad \left[A\right]_{x=a}^{x=b} = \int_a^b y\,dx$$

or $$(\text{Area } A \text{ when } x = b) - (\text{Area } A \text{ when } x = a) = \int_a^b y\,dx$$

or $$\text{Area } ABDC - 0 = \int_a^b y\,dx$$

or $$\textbf{Area } \boldsymbol{ABDC} = \int_a^b \boldsymbol{y\,dx} = \int_a^b \boldsymbol{f(x)\,dx}.$$

Similarly, it can be shown that the area bounded by the curve $x = f(y)$, the axis of y and the abscissae $y = a$ and $y = b$ is

$$\int_a^b \boldsymbol{x\,dy}, \quad \textit{or} \quad \int_a^b \boldsymbol{f(y)\,dy}.$$

Note 1 : In choosing the limits of integration, the lower limit of integration should be taken as the smaller value of the independent variable while the greater value gives us the upper limit of integration.

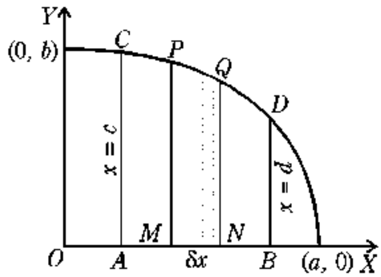
Note 2 : If the curve is symmetrical about x-axis or y-axis or both, then we shall find the area of one symmetrical part and multiply it by the number of symmetrical parts to get the whole area.

Illustrative Examples

Example 1 : *Find the area bounded by the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *the ordinates* $x = c$, $x = d$ *and the x-axis.* **(Meerut 2000)**

Solution : Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$



giving $\qquad y = \dfrac{b}{a}\sqrt{(a^2 - x^2)}$. ...(1)

$\therefore$ the required area = the area $ABDC$

$$= \int_c^d y\,dx = \int_c^d \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx,$$

from (1)

$$= \frac{b}{a}\left[\frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_c^d$$

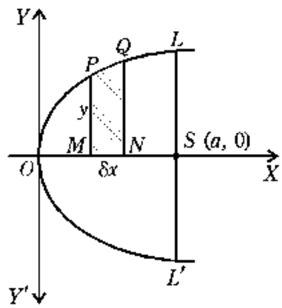
$$= \frac{b}{2a}\left[d\sqrt{(a^2 - d^2)} - c\sqrt{(a^2 - c^2)} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a}\right)\right].$$

Example 2 : *Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*

(Agra 2005; Avadh 05; Bundelkhand 08)

Solution : Latus rectum is a line through the focus $S(a, 0)$ and perpendicular to x-axis *i.e.*, its equation is $x = a$. Also the curve is symmetrical about x-axis.

$\therefore$ the required area LOL'



$$= 2 \times \text{ area } OSL = 2 \cdot \int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{(4ax)}\,dx,$$

$$[\because\ y^2 = 4ax, \text{ i.e., } y = \sqrt{(4ax)}]$$

$$= 2\sqrt{(4a)}\left[\frac{2}{3}x^{3/2}\right]_0^a = \frac{8}{3}\sqrt{(a)}\,.\,a^{3/2} = \frac{8}{3}a^2.$$

Example 3 : *Find the area of a loop of the curve*

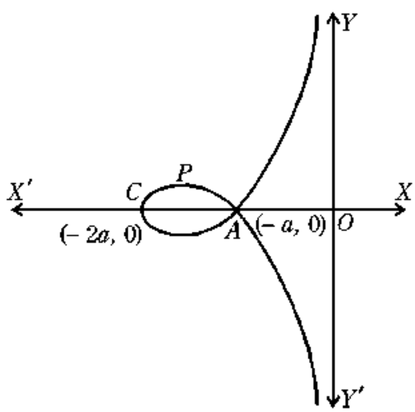
$$xy^2 + (x + a)^2(x + 2a) = 0.$$

Solution : The curve is symmetrical about x-axis.

Putting $y = 0$, we get

$$x = -a \quad \text{and} \quad x = -2a.$$

The loop is formed between $x = -2a$ and $x = -a$.



To find the area of the loop, we first shift the origin to the point $(-a, 0)$. The equation of the curve then becomes

$$(x - a)y^2 + \{(x - a) + a\}^2 (x - a + 2a) = 0$$

or $\qquad y^2(x - a) + x^2(x + a) = 0$

or $\qquad y^2 = \dfrac{x^2(a + x)}{a - x}$. ...(1)

Note that the shifting of the origin only changes the equation of the curve and has no effect on its shape. Now the origin being at the point A, the new limits for the loop are $x = -a$ to $x = 0$.

$\therefore$ required area of the loop $= 2 \times$ area CPA

$$= 2 \int_{-a}^{0} y\, dx, \text{ [the value of } y \text{ to be put from (1)]}$$

$$= 2 \int_{-a}^{0} \left\{ - x \sqrt{\left(\frac{a + x}{a - x} \right)} \right\} dx, \qquad \text{[Note that in the equation (1), for}$$

the portion CPA, $y = -x \sqrt{\{(a + x)/(a - x)\}}$]

$$= 2 \int_{-a}^{0} \frac{-x (a + x)}{\sqrt{(a^2 - x^2)}} dx, \text{ multiplying the numerator and the}$$

denominator by $\sqrt{(a + x)}$

$$= 2 \int_{\pi/2}^{0} \frac{-(-a \sin \theta)(a - a \sin \theta)}{a \cos \theta} \cdot (-a \cos \theta)\, d\theta,$$

putting $x = -a \sin \theta$ and $dx = -a \cos \theta\, d\theta$

$$= -2a^2 \int_{\pi/2}^{0} (\sin \theta - \sin^2 \theta)\, d\theta = 2a^2 \int_{0}^{\pi/2} (\sin \theta - \sin^2 \theta)\, d\theta$$

$$= 2a^2 [1 - \tfrac{1}{2} \cdot \tfrac{1}{2} \pi], \text{ by Walli's formula}$$

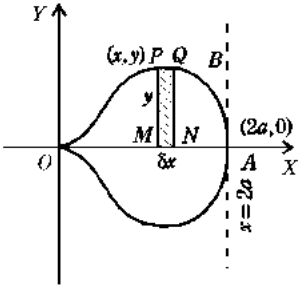
$$= 2a^2 (1 - \tfrac{1}{4} \pi).$$

Example 4 : *Find the whole area of the curve* $a^2 y^2 = x^3 (2a - x)$.

(Meerut 2006B; Bundelkhand 12; Avadh 13; Rohilkhand 14)

Solution : The given curve is $a^2 y^2 = x^3 (2a - x)$. ...(1)

It is symmetrical about x-axis and it cuts the x-axis at the points $(0, 0)$ and $(2a, 0)$. The curve does not exist for $x > 2a$ and $x < 0$. Thus the curve consists of a loop lying between $x = 0$ and $x = 2a$.



$\therefore$ the required area $= 2 \times$ area $OBA = 2 \int_{0}^{2a} y\, dx$

$$= 2 \int_{0}^{2a} \frac{x^{3/2} \sqrt{(2a - x)}}{a} dx, \text{ from (1).}$$

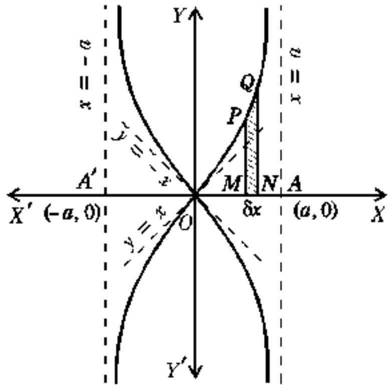
Now put $x = 2a \sin^2 \theta$ so that $dx = 4a \sin \theta \cos \theta\, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = 2a$, $\theta = \frac{1}{2} \pi$.

$$\therefore \text{ the required area} = \frac{2}{a} \int_{0}^{\pi/2} (2a)^{3/2} \sin^3 \theta . \sqrt{(2a)} . \cos \theta . 4a \sin \theta \cos \theta\, d\theta$$

$$= 32a^2 \int_{0}^{\pi/2} \sin^4 \theta \cos^2 \theta\, d\theta = 32a^2 . \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula.}$$

$$= \pi a^2.$$



Example 5 : *Find the whole area between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes.*

Solution : The given curve is symmetrical about both the axes and passes through the origin. The tangents at $(0, 0)$ are given by $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$ are the tangents at the origin.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$.

The asymptotes parallel to x-axis are given by $y^2 + a^2 = 0$ which gives two imaginary asymptotes.

$\therefore$ the required area

$$= 4 \times \text{ area lying in the first quadrant}$$

$$= 4 \int_0^a y\, dx = 4 \int_0^a \sqrt{\left(\frac{a^2 x^2}{a^2 - x^2}\right)} dx,$$

[$\because$ from the equation of the given curve, $y^2 = a^2 x^2 / (a^2 - x^2)$]

$$= 4 \int_0^a \frac{ax\, dx}{\sqrt{(a^2 - x^2)}} = -2a \int_0^a \frac{-2x\, dx}{\sqrt{(a^2 - x^2)}} = -2a \left[\frac{(a^2 - x^2)^{1/2}}{1/2}\right]_0^a$$

$$= -4a[0 - a] = 4a^2.$$

Comprehensive Exercise 1

1. Find the area bounded by the axis of x, and the following curves and the given ordinates :

(i) $y = \log x ; x = a, x = b \ (b > a > 1)$.

(ii) $xy = c^2 ; x = a, x = b, (a > b > 0)$. **(Kashi 2012)**

2. (i) Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.

(ii) Show that the area cut off a parabola by any double ordinate is two third of the corresponding rectangle contained by that double ordinate and its distance from the vertex.

3. (i) Find the area of the quadrant of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

(Bundelkhand 2010; Kanpur 11)

(ii) Find the whole area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

(Avadh 2010; Rohilkhand 10B)

4. (i) Trace the curve $ay^2 = x^2 (a - x)$ and show that the area of its loop is $8a^2/15$.

(Avadh 2008)

(ii) Find the area of the loop of the curve $3ay^2 = x(x - a)^2$.

(iii) Find the area of the loop of the curve $y^2 = x(x - 1)^2$.

5. Find the area

(i) of the loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$ or $y^2(a + x) = x^2(a - x)$.

(ii) of the portion bounded by the curve and its asymptotes. **(Meerut 2004)**

6. (i) Trace the curve $y^2(2a - x) = x^3$ and find the entire area between the curve and its asymptotes. **(Avadh 2011)**

(ii) Find the area between the curve $y^2(4 - x) = x^2$ and its asymptote.

(Avadh 2012; Kanpur 14; Bundelkhand 14)

(iii) Find the whole area of the curve

$$a^2 x^2 = y^3(2a - y).$$

7. (i) Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.

(Rohilkhand 2009B)

(ii) Find the area enclosed by the curve $xy^2 = a^2(a - x)$ and y-axis.

(iii) Trace the curve $a^2y^2 = a^2x^2 - x^4$ and find the whole area within it.

(Rohilkhand 2012; Avadh 12, Bundelkhand 14)

8. (i) Prove that the area of a loop of the curve $a^4y^2 = x^4(a^2 - x^2)$ is $\pi a^2/8$.

(ii) Show that the whole area of the curve $a^4y^2 = x^5(2a - x)$ is to that of the circle whose radius is a, as 5 to 4. **(Kanpur 2010)**

9. (i) Find the area between the curve $y^2(a - x) = x^3$ (cissoid) and its asymptotes. Also find the ratio in which the ordinate $x = a/2$ divides the area.

(ii) Find the area of the loop of the curve $y^2(a - x) = x^2(a + x)$.

(Purvanchal 2011)

10. Trace the curve $y^2(a + x) = (a - x)^3$. Find the area between the curve and its asymptote. **(Purvanchal 2007)**

Answers 1

1. (i) $b \log(b/e) - a \log(a/e)$ (ii) $c^2 \log(a/b)$.

2. (i) $(45)/4$. **3.** (i) $\pi ab/4$. (ii) πab.

4. (ii) $8a^2/(15\sqrt{3})$. (iii) $8/(15)$.

5. (i) $\frac{1}{2}a^2(4 - \pi)$. (ii) $\frac{1}{2}a^2(4 + \pi)$.

6. (i) $3\pi a^2$. (ii) $(64)/3$. (iii) πa^2.

7. (i) $4\pi a^2$. (ii) πa^2 (iii) $4a^2/3$.

9. (i) $3\pi a^2/4$; $(3\pi - 8) : (3\pi + 8)$. (ii) $\frac{1}{2}a^2(4 - \pi)$.

10. $3\pi a^2$.

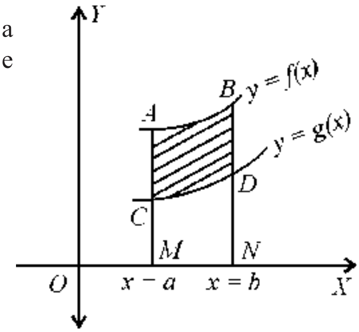
6.3 Area Between Two Curves

It is a clear from the adjacent figure that the area lying between the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a, x = b$ is

$$= \text{area } ABNM - \text{area } CDNM$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$= \int_a^b \{f(x)\,dx - g(x)\}\,dx.$$



Illustrative Examples

Example 1 : *Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4by$.*

(Bundelkhand 2011; Rohilkhand 10)

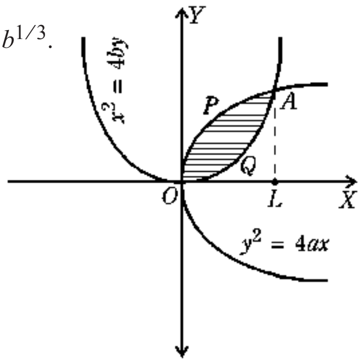
Solution : Solving the equations of the two given curves, we have

$$y^4 = 16a^2(4by) = 64a^2 by.$$

$\therefore \quad y(y^3 - 64a^2 b) = 0$, giving $y = 0, 4a^{2/3} b^{1/3}$.

When $y = 0$, $x = 0$ and when $y = 4a^{2/3} b^{1/3}$, $x = 4a^{1/3} b^{2/3}$.

Hence, the points of intersection of the given curves are $O(0, 0)$ and $A(4a^{1/3} b^{2/3}, 4b^{1/3} a^{2/3})$.



$\therefore$ the required area (*i.e.*, the shaded area)

= area $OPAL$ − area $OQAL$

$$= \int_0^{4a^{1/3} b^{2/3}} y\,dx, \text{ from the curve } y^2 = 4ax$$

$$- \int_0^{4a^{1/3} b^{2/3}} y\,dx, \text{ from the curve } x^2 = 4by$$

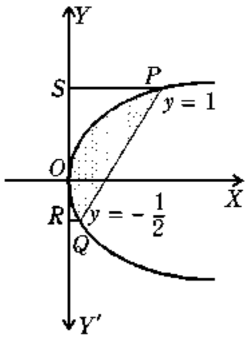
(Note that for the required area x varies from 0 to $4a^{1/3} b^{2/3}$)

$$= \int_0^{4a^{1/3} b^{2/3}} \sqrt{(4ax)}\,dx - \int_0^{4a^{1/3} b^{2/3}} \left(\frac{x^2}{4b}\right) dx$$

$$= 2\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^{4a^{1/3} b^{2/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{1/3} b^{2/3}}$$

$$= \frac{4\sqrt{a}}{3}[8\sqrt{(a)}\,.\,b] - \frac{1}{12b}(64ab^2) = \frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab.$$

Example 2 : *Find the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$.*



Solution : The given curves are $y^2 = 2x$, ...(1)

and $y = 4x - 1$. ...(2)

The two curves have been shown in the figure.

Solving (1) and (2) for y we have

$$y^2 = 2 \cdot \tfrac{1}{4}(y + 1) \quad \text{or} \quad 2y^2 - y - 1 = 0$$

or $$(y - 1)(2y + 1) = 0. \quad \therefore \quad y = -\tfrac{1}{2}, 1.$$

Thus the curves (1) and (2) intersect at the points where

$$y = -\tfrac{1}{2} \quad \text{and} \quad y = 1.$$

Now the required area of the segment POQ (*i.e.*, the dotted area)

= the area bounded by the st. line $y = 4x - 1$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

− the area bounded by the parabola $y^2 = 2x$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

$$= \int_{-1/2}^{1} x\,dy, \text{ from (2)} - \int_{-1/2}^{1} x\,dy, \text{ from (1)}$$

$$= \int_{-1/2}^{1} \frac{1}{4}(y + 1)\,dy - \int_{-1/2}^{1} \frac{1}{2}y^2\,dy$$

$$= \frac{1}{4}\left[\frac{1}{2}y^2 + y\right]_{-1/2}^{1} - \frac{1}{6}\left[y^3\right]_{-1/2}^{1} = \frac{1}{4}\left[\frac{3}{2} - \left(\frac{1}{8} - \frac{1}{2}\right)\right] - \frac{1}{6}\left(1 + \frac{1}{8}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + \frac{3}{8}\right) - \frac{1}{6} \cdot \frac{9}{8} = \frac{1}{4} \cdot \frac{15}{8} - \frac{1}{6} \cdot \frac{9}{8} = \frac{15}{32} - \frac{3}{16} = \frac{9}{32}.$$

Example 3 : *If $P(x, y)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and S be the sectorial area bounded by the curve, the x-axis and the line joining the origin to P, show that $x = a \cos(2S/ab)$, $y = b \sin(2S/ab)$.*

Solution : The given ellipse is shown in the figure. We have

S = the sectorial area OAP

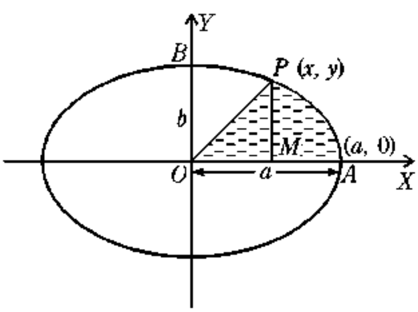
(*i.e.*, the dotted area)

= the area of the $\Delta\, OMP$

+ the area PMA

$$= \frac{1}{2} OM \cdot MP + \int_x^a y\,dx,$$

for the ellipse



$$= \frac{1}{2}xy + \int_x^a \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx$$

[$\because$ from the equation of the ellipse, $y = (b/a)\sqrt{(a^2 - x^2)}$]

$$= \frac{1}{2} x . \frac{b}{a} \sqrt{(a^2 - x^2)} + \frac{b}{a} \left[\frac{x}{2} \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} \right]_x^a$$

$$= \frac{bx}{2a} \sqrt{(a^2 - x^2)} + \frac{b}{a} \left[0 + \frac{1}{2} a^2 . \frac{\pi}{2} - \frac{x}{2} \sqrt{(a^2 - x^2)} - \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} \right]$$

$$= \frac{bx}{2a} \sqrt{(a^2 - x^2)} + \frac{b}{a} \cdot \frac{1}{2} a^2 \left(\frac{\pi}{2} - \sin^{-1} \frac{x}{a} \right) - \frac{bx}{2a} \sqrt{(a^2 - x^2)}$$

$$= \frac{ab}{2} \left(\frac{\pi}{2} - \sin^{-1} \frac{x}{a} \right) = \frac{ab}{2} \cos^{-1} \frac{x}{a} .$$

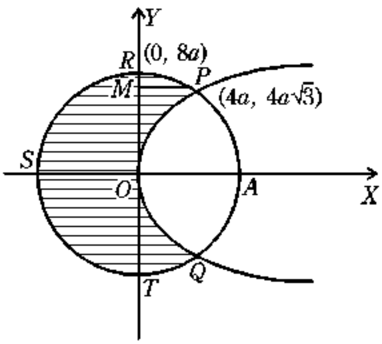
Thus $\quad S = \dfrac{ab}{2} \cos^{-1} \dfrac{x}{a}.$

$\therefore \quad \cos^{-1} \dfrac{x}{a} = \dfrac{2S}{ab}$ or $\dfrac{x}{a} = \cos \dfrac{2S}{ab}$ or $x = a \cos \dfrac{2S}{ab}.$

Also $\quad y = \dfrac{b}{a} \sqrt{(a^2 - x^2)} = \dfrac{b}{a} \sqrt{\{a^2 - a^2 \cos^2 (2S/ab)\}} = b \sin \dfrac{2S}{ab}.$

Example 4 : *Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\dfrac{16}{3} a^2 [8\pi - \sqrt{3}]$.*

Solution : $x^2 + y^2 = 64a^2$ is a circle with centre $(0, 0)$ and radius $8a$ and $y^2 = 12ax$ is a parabola whose vertex is at $(0, 0)$ and latus rectum $12a$. Both the curves are symmetrical about x-axis. Solving the two equations, the co-ordinates of the common point P are $(4a, 4a\sqrt{3})$. Draw PM perpendicular from P to the y-axis.



Now the area of the larger portion of the circle (*i.e.*, the shaded area)

= the area $PRSTQOP$

= the area of the semi-circle RST + 2 area OPR

$= \frac{1}{2} . \pi (8a)^2 + 2$ [area OPM + area MPR]

$$= \frac{1}{2} \pi (8a)^2 + 2 \int_0^{4a\sqrt{3}} x\, dy, \text{ for } y^2 = 12ax \; + 2 \int_{4a\sqrt{3}}^{8a} x\, dy, \quad \text{for} \quad x^2 + y^2 = 64a^2$$

$$= 32\pi a^2 + 2 \int_0^{4a\sqrt{3}} \frac{y^2}{12a} dy + 2 \int_{4a\sqrt{3}}^{8a} \sqrt{(64a^2 - y^2)}\, dy$$

$$= 32\pi a^2 + \frac{1}{6a} \left[\frac{y^3}{3} \right]_0^{4a\sqrt{3}} + 2 \left[\frac{1}{2} y \sqrt{(64a^2 - y^2)} + \frac{64a^2}{2} \sin^{-1} \frac{y}{8a} \right]_{4a\sqrt{3}}^{8a}$$

$$= 32\pi a^2 + \frac{1}{6a} \left[\frac{64 \times 3\sqrt{3} a^3}{3} \right] + 2 [\{0 - 8a^2\sqrt{3}\} + 32a^2 \{\sin^{-1} 1 - \sin^{-1} (\sqrt{3}/2)\}]$$

$$= 32\pi a^2 + \frac{32\sqrt{3}a^2}{3} - 16a^2\sqrt{3} + \frac{32}{3} a^2 \pi$$

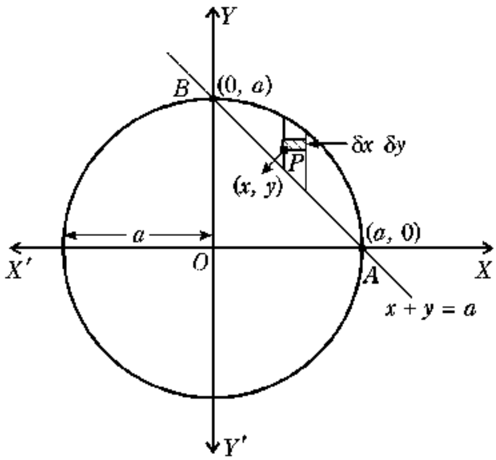
$$= \frac{128}{3} a^2 \pi - \frac{16}{3} a^2 \sqrt{3} = \frac{16}{3} a^2 (8\pi - \sqrt{3}).$$

Example 5 : *Find by double integration the area of the region enclosed by the curves* $x^2 + y^2 = a^2,\ x + y = a$ (*in the first quadrant*).

Solution : The given equations of the circle

$$x^2 + y^2 = a^2$$

[centre $(0, 0)$ and radius a] and of the straight line $x + y = a$ (with equal intercepts a on both the axes) can be easily traced as shown in the figure.



The required area is the area bounded by the arc AB and the line AB. To find it with the help of double integration take any point $P(x, y)$ in this portion and consider an elementary area $\delta x\, \delta y$ at P. The required area can now be covered by first moving y from the straight line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving x from 0 to a.

$\therefore$ the required area $= \displaystyle\int_{x=0}^{a} \int_{y=(a-x)}^{\sqrt{(a^2 - x^2)}} dx\, dy,$ the first integration to be performed w.r.t. y whose limits are variable

$$= \int_0^a \Big[y \Big]_{(a-x)}^{\sqrt{(a^2 - x^2)}} dx = \int_0^a [\sqrt{(a^2 - x^2)} - (a - x)]\, dx$$

$$= \left[\left\{ \frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}(x/a) \right\} - ax + \frac{1}{2} x^2 \right]_0^a$$

$$= \frac{1}{2} a^2 . \left(\frac{1}{2}\pi\right) - a^2 + \frac{1}{2} a^2 = \frac{1}{2} a^2 \left(\frac{1}{2}\pi - 1\right) = \frac{1}{4} a^2 (\pi - 2).$$

Note : The required area can also the covered by first moving x from the st. line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving y from 0 to a.

Comprehensive Exercise 2

1. Find the common area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

(Meerut 2004B, 08; Agra 14)

2. (i) Find the area included between $y^2 = 4ax$ and $y = mx$.

(ii) Find the area of the segment cut off from the parabola $y^2 = 4x$ by the line $y = 8x - 1$.

3. (i) Find the area common to the two curves $y^2 = ax, x^2 + y^2 = 4ax$.

(Meerut 2005B, 06, 09B)

(ii) Find the area lying above x-axis and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$. **(Bundelkhand 2007)**

4. (i) Show that the area included between the parabolas

$$y^2 = 4a(x + a), y^2 = 4b(b - x) \quad \text{is} \quad \frac{8}{3}(a + b)\sqrt{(ab)}.$$

(Rohilkhand 2013)

(ii) Show that the area common to the ellipses $a^2x^2 + b^2y^2 = 1, b^2x^2 + a^2y^2 = 1$, where $0 < a < b$, is $4(ab)^{-1}\tan^{-1}(a/b)$.

5. If A is the vertex, O the centre and P any point (x, y) on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that

$$x = a\cosh(2S/ab), y = b\sinh(2S/ab),$$

where S is the sectorial area OPA.

6. Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{1}{2}ab(\theta - e\sin\theta)$, where θ is the eccentric angle of the point to which the radius vector is drawn.

7. Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

(Purvanchal 2009)

8. Find the area included between the parabola $x^2 = 4ay$ and the curve

$$y = 8a^3/(x^2 + 4a^2).$$ **(Rohilkhand 2008B)**

9. Find by double integration the area bounded by the curves $y(x^2 + 2) = 3x$ and $4y = x^2$.

10. Find by double integration the area lying between the parabola $y = 4x - x^2$ and the straight line $y = x$.

Answers 2

1. $16a^2/3$. **2.** (i) $8a^2/3m^3$. (ii) $9/(64)$.

3. (i) $a^2\left(3\sqrt{3} + \frac{4}{3}\pi\right)$. (ii) $a^2\left[\frac{1}{4}\pi - \frac{2}{3}\right]$.

7. $4\pi + 9\sin^{-1}\left\{\frac{1}{3}\sqrt{(7/3)}\right\} - 8\sin^{-1}\left\{\frac{1}{2}\cdot\sqrt{(7/3)}\right\}$. **8.** $\left[2\pi - \frac{4}{3}\right]a^2$.

9. $(3/2)\log 3 - (2/3)$.

6.4 Areas of Curves given by Parametric Equations

To find the area of a curve given by parametric equations is explained by the following examples.

Illustrative Examples

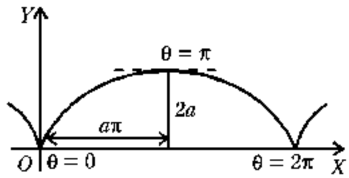
Example 1 : *Find the area included between the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *and its base.* **(Meerut 2007B; Purvanchal 07; Kashi 13)**

Solution : The parametric equations of the given cycloid are

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta).$$

We have $dx/d\theta = a(1 - \cos\theta)$,

$dy/d\theta = a\sin\theta$.



$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\frac{1}{2}\theta.$$

In this curve $y = 0$ when $a(1 - \cos\theta) = 0$ *i.e.*, $\cos\theta = 1$ *i.e.*, $\theta = 0$.

When $\theta = 0$, $x = a(0 - \sin 0) = 0$, $y = 0$ and $dy/dx = \cot 0 = \infty$. Thus the curve passes through the point (0, 0) and the axis of y is tangent at this point.

In this curve y is **maximum** when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. When $\theta = \pi$, $x = a(\pi - \sin\pi) = a\pi$, $y = 2a$, $dy/dx = \cot\frac{1}{2}\pi = 0$. Thus at the point $\theta = \pi$, whose cartesian co-ordinates are $(a\pi, 2a)$, the tangent to the curve is parallel to x-axis. This curve does not exist in the region $y > 2a$.

In this curve y cannot be – ive because $\cos\theta$ cannot be greater than 1. Thus one complete arch of the given cycloid is as shown in the figure.

Now this cycloid is symmetrical with respect to the line $x = a\pi$ (axis of the cycloid) and its base is the x-axis. Therefore the required area

$$= 2\int_{x=0}^{a\pi} y\,dx = 2\int_{\theta=0}^{\pi} y\frac{dx}{d\theta}\cdot d\theta$$

$$= 2\int_0^{\pi} a(1 - \cos\theta) . a(1 - \cos\theta)\,d\theta = 2a^2\int_0^{\pi}(1 - \cos\theta)^2\,d\theta$$

$$= 2a^2\int_0^{\pi}(2\sin^2\tfrac{1}{2}\theta)^2\,d\theta = 8a^2\int_0^{\pi}\sin^4\tfrac{1}{2}\theta\,d\theta$$

$$= 8a^2\int_0^{\pi/2}\sin^4\phi\,2d\phi, \text{ putting } \tfrac{1}{2}\theta = \phi \text{ so that } \tfrac{1}{2}d\theta = d\phi$$

$$= 16a^2\int_0^{\pi/2}\sin^4\phi\,d\phi = 16a^2\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}, \text{ by Walli's formula}$$

$$= 3\pi a^2.$$

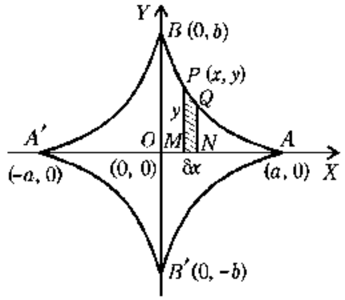
Example 2 : *Find the whole area of the curve (hypocycloid) given by the equations*

$$x = a\cos^3 t, y = b\sin^3 t.$$

(Gorakhpur 2005; Rohilkhand 09; Kashi 11)

Solution : Eliminating t from the given equations the cartesian equation of the curve is obtained as $(x/a)^{2/3} + (y/b)^{2/3} = 1$ *i.e.*, $\{(x/a)^2\}^{1/3} + \{(y/b)^2\}^{1/3} = 1$.

Since the powers of x and y are all even, the curve is symmetrical about both the axes. It does not pass through the origin. It cuts the axis of x at the points $(\pm a, 0)$ and the axis of y at the points $(0, \pm b)$. The tangent at the point $(a, 0)$ is x-axis. At the point B, $x = 0$ and $t = \frac{1}{2}\pi$. At the point A, $x = a$ and $t = 0$.



$\therefore$ the required area $= 4 \times$ area OAB

$$= 4 \int_{x=0}^{a} y\,dx = 4 \int_{t=\pi/2}^{0} y \cdot \frac{dx}{dt} \cdot dt$$

$$= 4 \int_{\pi/2}^{0} b \sin^3 t \,.(-3a \cos^2 t \sin t)\,dt, \qquad \text{(putting for } y \text{ and } dx/dt)$$

$$= 12ab \int_0^{\pi/2} \sin^4 t \cos^2 t\,dt \qquad \textbf{(Note)}$$

$$= 12ab \cdot \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{3}{8}\pi\,ab.$$

Comprehensive Exercise 3

1. Find the area included between the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ and its base. **(Agra 2005)**
2. Find the area of a loop of the curve $x = a \sin 2t$, $y = a \sin t$ or
$$a^2 x^2 = 4y^2 (a^2 - y^2).$$
3. Show that the area bounded by the cissoid $x = a \sin^2 t$, $y = (a \sin^3 t)/\cos t$ and its asymptote is $3\pi a^2/4$.
(Purvanchal 2006; Avadh 09; Rohilkhand 11; Purvanchal 14)
4. Find the area of the loop of the curve
$$x = a(1 - t^2),\ y = at(1 - t^2), \text{ where } -1 \le t \le 1.$$

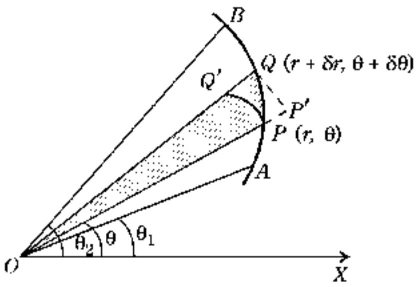
Answers 3

1. $3\pi a^2$.
2. $4a^2/3$
3. $3\pi a^2/4$.
4. $8a^2/(15)$.

6.5 Areas of Curves given by Polar Equations

If $r = f(\theta)$ be the equation of a curve in polar coordinates where $f(\theta)$ is a single valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to $\frac{1}{2}\int_{\theta=\theta_1}^{\theta_2} r^2\,d\theta$.

Proof : Let OAB be the area of the curve $r = f(\theta)$ between the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$.



Let $P(r, \theta)$ be any point on the curve between A and B. Take a point $Q(r + \delta r, \theta + \delta\theta)$ on the curve very near to P and draw the radius vector OQ. Let the sectorial areas AOP and AOQ be denoted by A and $A + \delta A$ respectively.

Then the curvilinear area

$$OPQO = A + \delta A - A = \delta A.$$

Also we have $OP = r$; $OQ = r + \delta r$

and $\angle POQ = \delta\theta$.

The area of the circular sector POQ'

$$= \frac{1}{2}(\text{radius} \times \text{arc}) = \frac{1}{2} r \cdot r\delta\theta = \frac{1}{2} r^2 \,\delta\theta,$$

and the area of the circular sector $P'OQ$

$$= \frac{1}{2}(r + \delta r) \cdot (r + \delta r)\,\delta\theta = \frac{1}{2}(r + \delta r)^2 \,\delta\theta.$$

Now, area $POQ' <$ area $OPQ <$ area $P'OQ$,

i.e., $\frac{1}{2} r^2 \delta\theta < \delta A < \frac{1}{2}(r + \delta r)^2 \delta\theta,$

i.e., $\frac{1}{2} r^2 < \delta A / \delta\theta < \frac{1}{2}(r + \delta r)^2.$

Proceeding to limits as $\delta\theta \to 0$, we get

$$\frac{dA}{d\theta} = \frac{1}{2} r^2 \quad \text{or} \quad dA = \frac{1}{2} r^2 \, d\theta. \qquad \therefore \left[A \right]_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta.$$

Now the L.H.S. = the value of A for θ equal to θ_2 − the value of A for θ equal to θ_1

= (the area AOB) − 0 = area AOB.

Hence the required area $AOB = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.$

Note : In some cases it is more convenient to find the required area by using double integration. In that case the area is given by

$$\int_{\theta = \theta_1}^{\theta_2} \int_{r = 0}^{f(\theta)} r \, d\theta \, dr, \; (\theta_1 < \theta_2).$$

Remember : The number of loops in $r = a \cos n\theta$ or $r = a \sin n\theta$ is n or $2n$ according as n is odd or even.

Illustrative Examples

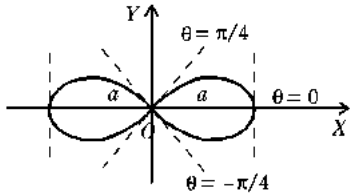
Example 1 : *Find the area of the curve* $r^2 = a^2 \cos 2\theta$.

(Agra 2006, 07; Rohilkhand 07; Meerut 10B)

Solution : The given curve is symmetrical about the initial line $\theta = 0$ and about the pole. Putting $r = 0$ in the given equation of the curve, we get

$$\cos 2\theta = 0 \quad \text{or} \quad 2\theta = \pm \tfrac{1}{2}\pi \quad \text{or} \quad \theta = \pm \tfrac{1}{4}\pi.$$

Thus two consecutive values of θ for which r is zero are $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$. Therefore for one loop of the curve θ varies from $-\pi/4$ to $\pi/4$.



When $\frac{1}{2}\pi < 2\theta < \frac{3}{2}\pi$ *i.e.*, $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, r^2 is negative *i.e.*, r is imaginary. Therefore this curve does not exist in the region

$$\tfrac{1}{4}\pi < \theta < \tfrac{3}{4}\pi.$$

Hence this curve has only two loops as shown in the figure.

$\therefore$ whole area of the curve $= 2 \times$ area of one loop

$$= 2\int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2\, d\theta = \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta, \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta, \qquad \text{[by a property of definite integrals]}$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{2a^2}{2} = a^2.$$

Example 2 : *Find the area of the cardioid $r = a(1 + \cos\theta)$.*

(Meerut 2003, 04B, 10B; Kashi 12)

Solution : The given curve is symmetrical about the initial line since its equation remains unaltered when θ is changed into $-\theta$.

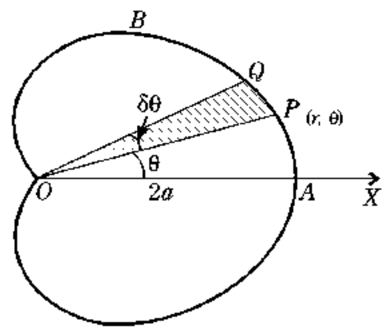
We have $r = 0$, when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. Therefore the line $\theta = \pi$ is tangent at the pole to the curve. Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = 2a$.

When θ increases from 0 to π, r decreases from $2a$ to 0. Thus the curve is as shown in the figure.

Now the required area

$$= 2 \times \text{area of the upper half of the curve}$$

$$= 2\int_0^{\pi} \frac{1}{2} r^2\, d\theta$$



$$= 2\int_0^{\pi} \frac{1}{2} a^2 (1 + \cos\theta)^2\, d\theta, \qquad [\because r = a(1 + \cos\theta)]$$

$$= a^2 \int_0^{\pi} (2\cos^2 \tfrac{1}{2}\theta)^2\, d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^4 \frac{1}{2}\theta\, d\theta.$$

Now put $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2} d\theta = d\phi$.

Also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = \pi/2$.

$\therefore$ the required area $= 8a^2 \int_0^{\pi/2} \cos^4 \phi\, d\phi = 8a^2 \cdot \dfrac{3.1}{4.2} \cdot \dfrac{\pi}{2}$, by Walli's formula

$$= 3\pi a^2/2.$$

Example 3 : *Find the area of a loop of the curve* $r = a \cos 3\theta + b \sin 3\theta$.

(Meerut 2000)

Solution : In the given equation of the curve put $a = k \cos \alpha, b = k \sin \alpha$ so that

$$k = \sqrt{(a^2 + b^2)} \text{ and } \alpha = \tan^{-1}(b/a).$$

Thus the given equation reduces to $r = k \cos 3\theta \cos \alpha + k \sin 3\theta \sin \alpha$

or $$r = k \cos(3\theta - \alpha) = k \cos 3(\theta - \tfrac{1}{3}\alpha).$$ **(Note)**

Now rotating the initial line through an angle $\alpha/3$, the given equation of the curve becomes

$$r = k \cos 3(\theta + \tfrac{1}{3}\alpha - \tfrac{1}{3}\alpha) = k \cos 3\theta.$$ **(Note)**

It should be noted that the rotation of the initial line changes only the equation of the curve and has no effect on its shape. Therefore the area of a loop of the given curve is the same as the area of a loop of the curve $r = k \cos 3\theta$.

The curve $r = k \cos 3\theta$ is symmetrical about the initial line.

Putting $r = 0$ in it we have, $\cos 3\theta = 0$ *i.e.*, $3\theta = \pm \pi/2$ *i.e.*, $\theta = \pm \pi/6$.

$\therefore$ one loop of this curve lies between $\theta = -\pi/6$ and $\theta = +\pi/6$ and it is symmetrical about the initial line.

$$\therefore \quad \text{the required area} = 2 \cdot \int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta, \qquad \text{[By symmetry]}$$

$$= \int_0^{\pi/6} k^2 \cos^2 3\theta \, d\theta.$$

Now put $3\theta = t$, so that $3\, d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta = \pi/6$, $t = \pi/2$.

$$\therefore \quad \text{the required area} = \frac{k^2}{3} \int_0^{\pi/2} \cos^2 t \, dt = \frac{k^2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = \frac{k^2}{12} \pi$$

$$= (a^2 + b^2)\pi/12. \qquad [\because k^2 = a^2 + b^2]$$

Comprehensive Exercise 4

1. Find the area between the following curves and the given radii vectors :
   (i) The spiral $r\theta^{1/2} = a; \theta = \alpha, \theta = \beta$.
   (ii) The parabola $l/r = 1 + \cos\theta; \theta = 0, \theta = \alpha$.
2. Find the area of the loop of the curve $r = a\theta \cos\theta$ between $\theta = 0$ and $\theta = \pi/2$.
   (Kanpur 2009)
3. (i) Find the area of one loop of $r = a \cos 4\theta$. **(Rohilkhand 2007)**
   (ii) Find the area of a loop of the curve $r = a \sin 3\theta$.
4. (i) Find the whole area of the curve $r = a \sin 2\theta$. **(Bundelkhand 2009)**
   (ii) Find the whole area of the curve $r = a \cos 2\theta$.
5. (i) Find the whole area of the curve $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.
   (ii) Find the area of the cardioid $r = a(1 - \cos\theta)$.

6. (i) Show that the area of the limacon $r = a + b\cos\theta$, $(b < a)$ is equal to

$$\pi\left(a^2 + \tfrac{1}{2}b^2\right).$$

(ii) Prove that the sum of the areas of the two loops of the limacon $r = a + b\cos\theta$, $(b > a)$ is equal to $\pi(2a^2 + b^2)/2$.

7. Calculate the ratio of the area of the larger to the area of the smaller loop of the curve $r = \frac{1}{2} + \cos 2\theta$.

8. Show that the area of a loop of $r = a\cos n\theta$ is $\pi a^2/4n$, n being integral. Also prove that the whole area is $\pi a^2/4$ or $\pi a^2/2$ according as n is odd or even.

9. Trace the curve $r = \sqrt{3}\cos 3\theta + \sin 3\theta$, and find the area of a loop.

Answers 4

1. (i) $\frac{1}{2}a^2\log(\beta/\alpha)$, (ii) $\frac{1}{4}l^2\left[\tan\frac{1}{2}\alpha + \frac{1}{3}\tan^3\frac{1}{2}\alpha\right]$. **2.** $\dfrac{\pi a^2}{96}(\pi^2 - 6)$.

3. (i) $\pi a^2/(16)$. (ii) $\pi a^2/(12)$. **4.** (i) $\pi a^2/2$. (ii) $\pi a^2/2$.

5. (i) $\frac{1}{2}\pi(a^2 + b^2)$. (ii) $3\pi a^2/2$.

7. $\dfrac{4\pi + 3\sqrt{3}}{2\pi - 3\sqrt{3}}$. **9.** $\pi/3$.

6.6 Area Bounded by Two Curves (Polar Form)

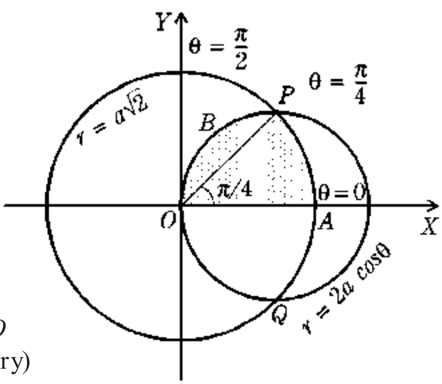
To find the area bounded by two curves given in polar form is explained by the following examples.

Illustrative Examples

Example 1 : *Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a\cos\theta$.*

(Meerut 2005, 11; Kumaun 08; Rohilkhand 11B; Kanpur 14; Purvanchal 14)

Solution : The given equations of circles are $r = a\sqrt{2}$ and $r = 2a\cos\theta$. The first equation represents a circle with centre at pole and radius $a\sqrt{2}$. The second equation repre- sents a circle passing through the pole and the diameter through the pole as the initial line. Both these circles are symmetrical about the initial line. Eliminating r between the two equations, we have at the points of intersection



$$a\sqrt{2} = 2a\cos\theta, \text{ i.e., } \cos\theta = 1/\sqrt{2},$$

i.e., $\theta = \pm\pi/4$.

Thus at P, $\theta = \pi/4$. For the circle $r = 2a\cos\theta$, at O, $r = 0$ and so $\cos\theta = 0$ *i.e.,* $\theta = \frac{1}{2}\pi$.

Now the required area $=$ Area $OQAPBO$

$= 2$ (area $OAPBO$), (by symmetry)

$= 2\,[\text{Area } OAP + \text{ Area } OPBO]$

$$= 2\left[\frac{1}{2}\int_0^{\pi/4} r^2\,d\theta, \text{ for the circle } r = a\sqrt{2}\right.$$

$$\left. + \frac{1}{2}\int_{\pi/4}^{\pi/2} r^2\,d\theta, \text{ for the circle } r = 2a\cos\theta\right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2\,d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2\,d\theta$$

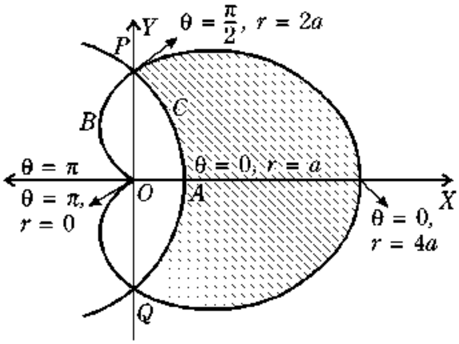
$$= 2a^2\Big[\theta\Big]_0^{\pi/4} + 2a^2\int_{\pi/4}^{\pi/2} (1 + \cos 2\theta)\,d\theta$$

$$= 2a^2\left(\frac{\pi}{4}\right) + 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \tfrac{1}{2}\pi a^2 + \tfrac{1}{2}\pi a^2 - a^2 = \pi a^2 - a^2 = a^2(\pi - 1).$$

Example 2 : *Find the ratio of the two parts into which the parabola $2a = r(1 + \cos\theta)$ divides the area of the cardioid $r = 2a(1 + \cos\theta)$.*

Solution : Eliminating r between the given equations of the curves, we get



$$2a(1 + \cos\theta) = 2a/(1 + \cos\theta)$$

or $\quad (1 + \cos\theta)^2 = 1$

or $\quad \cos\theta(\cos\theta + 2) = 0$

or $\quad \cos\theta = 0, \quad [\because \cos\theta \neq -2]$

or $\quad \theta = \pm\pi/2.$

Thus at the point of intersection P of the two curves, $\theta = \pi/2$.

Now area of the whole cardioid

$$= 2 \times \frac{1}{2}\int_0^{\pi} r^2\,d\theta, \quad \text{(by symmetry)}$$

$$= \int_0^{\pi} 4a^2(1 + \cos\theta)^2\,d\theta = 4a^2\int_0^{\pi} (2\cos^2\tfrac{1}{2}\theta)^2\,d\theta$$

$$= 16a^2\int_0^{\pi} \cos^4\tfrac{1}{2}\theta\,d\theta = 16a^2\int_0^{\pi/2} \cos^4\phi\,.2\,d\phi,$$

(putting $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2}d\theta = d\phi$;

also when $\theta = 0, \phi = 0$ and when $\theta = \pi, \phi = \frac{1}{2}\pi$)

$$= 32a^2\,\frac{3.1}{4.2}\cdot\frac{\pi}{2} = 6\pi a^2. \qquad \text{...(1)}$$

Area $OACPO = \dfrac{1}{2}\displaystyle\int_0^{\pi/2} r^2\,d\theta$, for the parabola $r = \dfrac{2a}{1 + \cos\theta}$

$$= \frac{1}{2}\cdot 4a^2\int_0^{\pi/2} \frac{d\theta}{(1 + \cos\theta)^2} = \frac{a^2}{2}\int_0^{\pi/2} \sec^4\tfrac{1}{2}\theta\,d\theta$$

$$= \frac{1}{2} a^2 \int_0^{\pi/2} (1 + \tan^2 \tfrac{1}{2} \theta) \sec^2 \tfrac{1}{2} \theta \, d\theta$$

$$= a^2 \left[\tan \tfrac{1}{2} \theta + \tfrac{1}{3} \tan^3 \tfrac{1}{2} \theta \right]_0^{\pi/2} = a^2 (1 + \tfrac{1}{3}) = \frac{4a^2}{3}. \quad \text{...(2)}$$

Also area $OPBO = \dfrac{1}{2} \displaystyle\int_{\pi/2}^{\pi} r^2 \, d\theta$, for the cardioid $r = 2a (1 + \cos \theta)$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} 4a^2 (1 + \cos \theta)^2 \, d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} [1 + 2 \cos \theta + \cos^2 \theta] \, d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} (1 + 2 \cos \theta + \tfrac{1}{2} + \tfrac{1}{2} \cos 2\theta) \, d\theta$$

$$= 2a^2 \left[\tfrac{3}{2} \theta + 2 \sin \theta + \tfrac{1}{4} \sin 2\theta \right]_{\pi/2}^{\pi} = \frac{3}{2} \pi a^2 - 4a^2. \quad \text{...(3)}$$

Adding (2) and (3) and multiplying by 2, we get the whole area included between the two curves *i.e.*, the area of the smaller portion of the cardioid

$$= 2 \times \left[\frac{4}{3} a^2 + \left(\frac{3}{2} \pi a^2 - 4a^2 \right) \right] = a^2 \left[3\pi - \frac{16}{3} \right]$$

$$= \tfrac{1}{3} a^2 [9\pi - 16]. \quad \text{...(4)}$$

Also the shaded area (*i.e.*, the area of the larger portion of the cardioid)

$$= (\text{Area of the whole cardioid}) - (\text{unshaded area}) \ i.e., = (1) - (4)$$

$$= 6 \pi a^2 - \tfrac{1}{3} a^2 (9\pi - 16) = \tfrac{1}{3} a^2 (9\pi + 16). \quad \text{...(5)}$$

$\therefore$ Ratio of the two parts $= \dfrac{\text{Larger area}}{\text{Smaller area}} = \dfrac{\frac{1}{3} a^2 (9\pi + 16)}{\frac{1}{3} a^2 (9\pi - 16)} = \dfrac{9\pi + 16}{9\pi - 16}.$

Example 3 : *Find the area lying between the cardioid $r = a (1 - \cos \theta)$ and its double tangent.*

Solution : Let PQ be the double tangent of the cardioid. Clearly it is perpendicular to OX *i.e.*, it must be inclined at an angle of 90° to the initial line

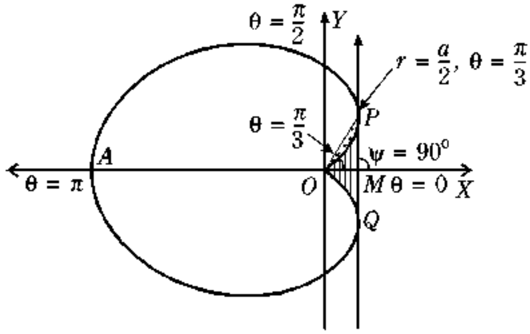
i.e., $\psi = 90^\circ$ at P.

Also we know that at any point of a curve,

$$\psi = \theta + \phi. \quad \text{...(1)}$$

Now $\tan \phi = r (d\theta / dr) = r / (dr / d\theta) = a (1 - \cos \theta) / (a \sin \theta)$,

$[\because \ r = a (1 - \cos \theta)]$



$$= \frac{2 \sin^2 \frac{1}{2} \theta}{2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta} = \tan \frac{1}{2} \theta.$$

$\therefore \quad \phi = \frac{1}{2} \theta.$

Putting the value of ϕ in (1), we get $\psi = \theta + \frac{1}{2} \theta = \frac{3}{2} \theta$.

Since at P, $\psi = \frac{1}{2} \pi$, therefore at P, $\frac{1}{2} \pi = \frac{3}{2} \theta$ or $\theta = \pi/3$.

$\therefore$ the vectorial angle of the point of contact P of the double tangent is $\pi/3$ *i.e.*, 60^o. Substituting this value of θ in the equation of the curve, we get the radius vector

$$OP = a(1 - \cos 60^o) = a/2.$$

Thus in the triangle OPM,

$$OP = \frac{1}{2} a, \angle POM = 60^o, \angle PMO = 90^o.$$

$\therefore \quad OM = \frac{1}{2} a \cos 60^o = \frac{1}{2} a \cdot \frac{1}{2}$ and $PM = \frac{1}{2} a \sin 60^o = \frac{1}{2} a (\sqrt{3}/2)$.

$\therefore \quad$ area of the triangle $OPM = \frac{1}{2} OM \,.\, PM = \frac{1}{2} (\frac{1}{4} a) (\sqrt{3} a/4) = (1/32) a^2 \sqrt{3}$.

Also the sectorial area OPO of the cardioid $r = a(1 - \cos \theta)$ *i.e.*, the dotted area

$$= \frac{1}{2} \int_0^{\pi/3} r^2 \, d\theta = \frac{1}{2} \int_0^{\pi/3} a^2 (1 - \cos \theta)^2 \, d\theta$$

$$= \frac{1}{2} a^2 \int_0^{\pi/3} (1 - 2 \cos \theta + \tfrac{1}{2} + \tfrac{1}{2} \cos 2\theta) \, d\theta$$

$$= \frac{1}{2} a^2 \left[\frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\pi/3}$$

$$= \frac{1}{2} a^2 (\tfrac{1}{2} \pi - \sqrt{3} + \tfrac{1}{8} \sqrt{3}) = \frac{1}{16} a^2 (4\pi - 7\sqrt{3}).$$

Hence the required area (*i.e.*, the area shaded by vertical lines)

$$= 2 \,[\text{area of } \Delta \, OPM - \text{area of sector } OPO]$$

$$= 2 \left[\frac{1}{32} a^2 \sqrt{3} - \frac{1}{16} a^2 (4\pi - 7\sqrt{3}) \right] = \frac{1}{16} a^2 (15\sqrt{3} - 8\pi).$$

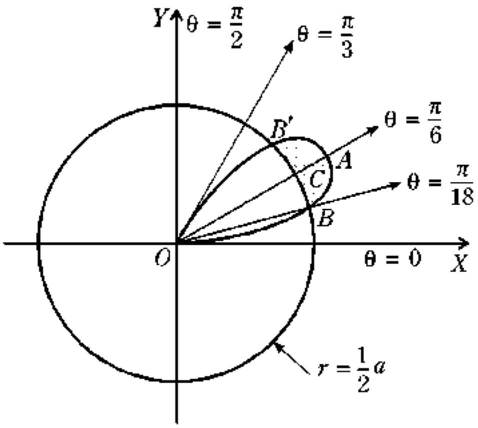
Example 4 : *Find the area of a loop of the curve* $r = a \sin 3\theta$ *outside the circle* $r = a/2$ *and hence find the whole area of the curve outside the circle* $r = a/2$.

Solution : Eliminating r between the two given equations, we get *i.e.,* $(a/2) = a \sin 3\theta \sin 3\theta = \frac{1}{2}$

i.e., $3\theta = \pi/6$ or $5\pi/6$

i.e., $\theta = \pi/18$ or $5\pi/18$

i.e., $\theta = 10^{\circ}$ or 50°.



Thus the loop of the curve $r = a \sin 3\theta$ lying between $\theta = 0$ and $\theta = \pi/3$ intersects the circle $r = a/2$ at the points B and B' where $\theta = 10^{\circ}$ at B and $\theta = 50^{\circ}$ at B'. This loop is symmetrical about OA and $\theta = \pi/6$ at A.

Now the required area of a loop of the curve $r = a \sin 3\theta$ lying outside the circle $r = a/2$

= the area $BAB'CB$

(*i.e.*, the shaded area)

$= 2 \times$ area $BACB$, (by symmetry)

$= 2 \times$ [(area of the curve $r = a \sin 3\theta$ between the radii vectors OB and OA *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$) $-$ (area of the circle $r = a/2$ between the radii vectors OB and OC *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$)]

$$= 2\left[\frac{1}{2}\int_{\pi/18}^{\pi/6} r^2\, d\theta, \text{ for the curve } r = a \sin 3\theta \right.$$

$$\left. - \frac{1}{2}\int_{\pi/18}^{\pi/6} r^2\, d\theta, \text{ for the circle } r = \frac{a}{2}\right]$$

$$= \int_{\pi/18}^{\pi/6} a^2 \sin^2 3\theta\, d\theta - \int_{\pi/18}^{\pi/6} \frac{a^2}{4}\, d\theta = \frac{a^2}{2}\int_{\pi/18}^{\pi/6} (1 - \cos 6\theta)\, d\theta - \frac{a^2}{4}\Big[\,\theta\,\Big]_{\pi/18}^{\pi/6}$$

$$= \frac{a^2}{2}\left[\theta - \frac{\sin 6\theta}{6}\right]_{\pi/18}^{\pi/6} - \frac{a^2}{4}\left[\frac{\pi}{6} - \frac{\pi}{18}\right] = \frac{a^2}{2}\left[\left\{\frac{\pi}{6} - \frac{\sin \pi}{6}\right\} - \left\{\frac{\pi}{18} - \frac{1}{6}\sin\frac{\pi}{3}\right\}\right] - \frac{a^2}{4}\cdot\frac{\pi}{9}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{6} - \left\{\frac{\pi}{18} - \frac{1}{6}\cdot\frac{\sqrt{3}}{2}\right\}\right] - \frac{a^2\pi}{36} = \frac{a^2}{2}\left[\frac{\pi}{9} + \frac{\sqrt{3}}{12}\right] - \frac{a^2\pi}{36} = \frac{a^2}{72}[2\pi + 3\sqrt{3}].$$

Again the curve $r = a \sin 3\theta$ has 3 equal loops ($\because n = 3$ which is odd).

$\therefore$ whole area of the curve $r = a \sin 3\theta$ outside the circle $r = a/2$

$= 3 \times$ area $BAB'CB$ *i.e.*, 3 times the shaded area

$$= 3 \times \frac{1}{72} a^2 [2\pi + 3\sqrt{3}] = \frac{1}{24} a^2 [2\pi + 3\sqrt{3}].$$

Comprehensive Exercise 5

1. Find the area outside the circle $r = 2a \cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.

2. Find the total area inside $r = \sin\theta$ and outside $r = 1 - \cos\theta$.

3. Find by double integration the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.
4. Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos \theta)$. **(Meerut 2007)**
5. Find the area of the portion included between the cardioids
$$r = a(1 + \cos \theta) \quad \text{and} \quad r = a(1 - \cos \theta).$$
6. Show that the area contained between the circle $r = a$ and the curve $r = a \cos 5\theta$ is equal to three-fourth of the area of the circle.
7. Find the area between the curve $r = a(\sec \theta + \cos \theta)$ and its asymptote. **(Purvanchal 2010)**
8. O is the pole of the lemniscate $r^2 = a^2 \cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.

Answers 5

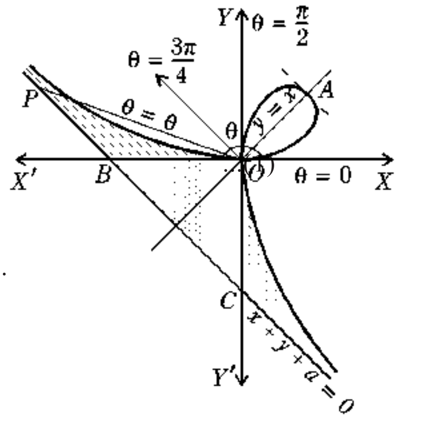
1. $\pi a^2/2$. 2. $1 - (\pi/4)$. 3. $a^2 \{1 - (\pi/4)\}$.
4. $a^2 \left(\frac{5}{4}\pi - 2\right)$. 5. $2a^2 \left(\frac{3}{4}\pi - 2\right)$. 7. $5\pi a^2/4$. 8. $\frac{1}{8} a^2 (3\sqrt{3} - 4)$.

6.7 Evaluation of Area by Changing the Cartesian Equation to Polar Form

Sometime it is convenient to evaluate the area of a given curve by changing its cartesian equation to polar form by using the substitution
$$x = r\cos\theta \text{ and } y = r\sin\theta.$$

Illustrative Examples

Example 1 : *Find the area of a loop of the folium $x^3 + y^3 = 3axy$.* **(Meerut 2001)**



Solution : Changing the equation of the curve $x^3 + y^3 = 3axy$ into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$, we have $(r\cos\theta)^3 + (r\sin\theta)^3 = 3a(r\cos\theta).(r\sin\theta)$

or $\quad r = 3a\cos\theta\sin\theta/(\cos^3\theta + \sin^3\theta)$.

From (1), $r = 0$ when $\theta = 0$ and when $\theta = \pi/2$.

$\therefore$ the loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop
$$= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}\right)^2 d\theta, \text{ putting for } r \text{ from (1)}$$

$$= \frac{9a^2}{2}\int_0^{\pi/2}\frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}\,d\theta = \frac{9a^2}{2}\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta,$$

dividing the numerator and the denominator by $\cos^6\theta$.

Now put $1+\tan^3\theta = t$ so that

$$3\tan^2\theta\sec^2\theta\,d\theta = dt.$$

Also when $\theta = 0, t = 1$ and when

$$\theta \to \pi/2, t \to \infty.$$

$\therefore$ area of the loop $= \dfrac{9a^2}{2}\displaystyle\int_1^{\infty}\frac{1}{t^2}\cdot\frac{dt}{3} = \frac{3a^2}{2}\left[-\frac{1}{t}\right]_1^{\infty} = \frac{3a^2}{2}.$

Comprehensive Exercise 6

1. Find the area of a loop of the curve $x^4 + y^4 = 4a^2xy$.
2. Find the area of a loop of the curve $(x^2 + y^2)^2 = 4axy^2$.
3. Prove that the area of a loop of the curve $x^6 + y^6 = a^2y^2x^2$ is $\pi a^2/12$.
4. Find the area of a loop of the curve $x^4 + 3x^2y^2 + 2y^4 = a^2xy$.
5. Prove that the area of a loop of the curve $x^5 + y^5 = 5ax^2y^2$ is five times the area of one loop of the curve $r^2 = a^2\cos 2\theta$. **(Purvanchal 2014)**

Answers 6

1. $a^2(\pi/2)$. **2.** $\pi a^2/4$. **4.** $\frac{1}{4}a^2\log 2$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "... ...", so that the following statements are complete and correct.

1. The process of finding the area of any bounded portion of a curve is called ....
2. If $f(x)$ is a continuous and single valued function of x then the area bounded by the curve $y = f(x)$ the axis of x and the ordinates $x = a$ and $x = b$ is ....
3. The area between the curve $r = ae^{m\theta}$ and the given radii vectors $\theta = \alpha, \theta = \beta$ is ....
4. The curve $r = a\sin 3\theta$ has ...... loops.
5. The area bounded by the axis of x, and the curve $y = c\cosh\left(\dfrac{x}{c}\right)$ and the ordinates $x = 0, x = a$ is ....

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

6. The area bounded by the axis of x, and the curve $y = \sin^2 x$ and the given ordinates $x = 0,\ x = \dfrac{\pi}{2}$ is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi^2}{4}$ (c) $\dfrac{\pi}{2}$ (d) π

7. The loop of the curve $3ay^2 = x(x - a)^2$ will lie between

(a) $x = 0, x = a$ (b) $x = -a, x = a$ (c) $x = 0, x = -a$ (d) $y = 0, y = a$

8. The area of one loop of the curve $r^2 = a^2 \cos 2\theta$ is

(a) a^2 (b) $\dfrac{a^2}{2}$ (c) $\dfrac{3a^2}{2}$ (d) $\dfrac{a^2}{4}$

9. The whole area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

(a) $\dfrac{\pi}{2} ab$ (b) πab (c) $\pi^2 ab$ (d) $\dfrac{2\pi}{3} ab$

(Agra 2006; Bundelkhand 06, 08)

10. The area of the curve $r = a$ is

(a) πa^2 (b) $2\pi a$ (c) $2\pi a^2$ (d) $4\pi a^2$ **(Rohilkhand 2006)**

True or False:

Write 'T' for true and 'F' for false statement.

11. If $r = f(\theta)$ be the equation of a curve in polar co-ordinates where $f(\theta)$ is a single-valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to $\dfrac{1}{2}\displaystyle\int_{\theta=\theta_1}^{\theta=\theta_2} r^2\, d\theta$.

12. The number of loops in $r = a \cos n\theta$ is n or $2n$ according as n is even or odd.

13. The area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\dfrac{3}{8}\pi a^2$.

Answers

1. quadrature **2.** $\displaystyle\int_a^b f(x)\, dx$ **3.** $\dfrac{a^2}{4m}(e^{2m\beta} - e^{2m\alpha})$

4. three **5.** $c^2 \sinh \dfrac{a}{c}$. **6.** (a). **7.** (a).

8. (b). **9.** (b). **10.** (a). **11.** *T*. **12.** *F*.

13. *T*.

Chapter

7

Rectification (Lengths of Arcs and Intrinsic Equations of Plane Curves

7.1 Rectification

The process of finding the length of an arc of a curve between two given points is called ***rectification.***

7.2 Lengths of Curves

(Meerut 2009B)

If s denotes the arc length of a curve measured from a *fixed point* to any point on it, then as proved in Unit I on Differential Calculus, we have

$$\frac{ds}{dx} = \pm \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

where + ive or –ive sign is to be taken before the radical sign according as x increases or decreases as s increases. Hence if s increases as x increases, we have

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \text{ or } ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx.$$

Integrating, we have $s = \int_a^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx,$

where a is the abscissa of the fixed point from which s is measured.

Hence the arc length of the curve $\boldsymbol{y = f(x)}$ included between two points for which $x = a$ and $x = b$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx, \quad (b > a).$$

Sometimes it is more convenient to take y as the independent variable. Then the length of the arc of the curve $\boldsymbol{x = f(y)}$ between $y = a$ and $y = b$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy, \quad (b > a).$$

Remark : Suppose we have to find the length of the arc of a curve (whose cartesian equation is given) lying between the points (x_1, y_1) and (x_2, y_2). We can use either of the two formulae

$$s = \int_{x_1}^{x_2} \sqrt{\{1 + (dy/dx)^2\}}\, dx \quad \text{and} \quad s = \int_{y_1}^{y_2} \sqrt{\{1 + (dx/dy)^2\}}\, dy.$$

If we feel any difficulty in integration while using one of these two formulae, we must try the other formula also.

Illustrative Examples

Example 1 : *Show that the length of the curve $y = \log \sec x$ between the points where $x = 0$ and $x = \frac{1}{3}\pi$ is $\log(2 + \sqrt{3})$.* **(Kanpur 2005; Rohilkhand 14)**

Solution : The given curve is $y = \log \sec x$. ...(1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x = \tan x.$$

Now $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x. \qquad ...(2)$$

If the arc length s of the given curve is measured from $x = 0$ in the direction of x increasing, we have $\frac{ds}{dx} = \sec x$ or $ds = \sec x\, dx$.

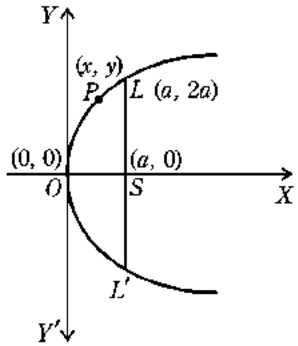
Therefore if s_1 denotes the arc length from $x = 0$ to $x = \frac{1}{3}\pi$, then

$$\int_0^{s_1} ds = \int_0^{\pi/3} \sec x\, dx = \Big[\log(\sec x + \tan x)\Big]_0^{\pi/3}$$

or $$s_1 = [\log(\sec \tfrac{1}{3}\pi + \tan \tfrac{1}{3}\pi) - \log 1] = \log(2 + \sqrt{3}).$$

Example 2 : *Find the length of the arc of the parabola $y^2 = 4ax$ extending from the vertex to an extremity of the latus rectum.* **(Meerut 2009)**

Solution : The given equation of parabola is $y^2 = 4ax$. ...(1)



The point $O(0,0)$ is the vertex of the parabola and the point $L(a, 2a)$ is an extremity of the latus rectum LSL'. We have to find the length of arc OL. Defferentiating (1) w.r.t. x, we get $2y(dy/dx) = 4a$.

$\therefore$ $\quad dy/dx = 2a/y \quad$ or $\quad dx/dy = y/2a$.

Now $$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2}$$

$$= \frac{1}{4a^2}(4a^2 + y^2). \qquad ...(2)$$

If 's' denotes the arc length of the parabola measured from the vertex O to any point $P(x, y)$ towards the point L, then s increases as y increases. Therefore ds/dy will be positive. So extracting the square root of (2) and keeping the positive sign, we have

$$\frac{ds}{dy} = \frac{1}{2a}\sqrt{(4a^2 + y^2)}, \quad \text{or} \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\, dy.$$

Let s_1 denote the arc length OL. Then

$$\int_0^{s_1} ds = \int_0^{2a} \frac{1}{2a}\sqrt{(4a^2 + y^2)}\, dy$$

or $$s_1 = \frac{1}{2a}\left[\frac{y}{2}\sqrt{(4a^2 + y^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^{2a}$$

$$= \frac{1}{2a}[a\sqrt{(4a^2 + 4a^2)} + 2a^2\log\{2a + \sqrt{(8a^2)}\} - 0 - 2a^2\log(2a)]$$

$$= \frac{1}{2a}[2\sqrt{2}a^2 + 2a^2\log\{(2a + 2\sqrt{2}a)/2a\}]$$

$$= \frac{2a^2}{2a}[\sqrt{2} + \log(1 + \sqrt{2})] = a[\sqrt{2} + \log(1 + \sqrt{2})].$$

Example 3 : *Find the perimeter of the loop of the curve* $3ay^2 = x^2(a - x)$.

(Meerut 2000, 04, 06B, 07B, 11B; Purvanchal 10; Kashi 14)

Solution : The given curve is $3ay^2 = x^2(a - x)$. ...(1)

Here the curve is symmetrical about the x-axis. Putting $y = 0$, we get $x = 0$, $x = a$. So the loop lies between $x = 0$ and $x = a$. Differentiating (1) w.r.t. x, we get

$$6ay\frac{dy}{dx} = 2ax - 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{x(2a - 3x)}{6ay}.$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2(2a - 3x)^2}{36a^2y^2} = 1 + \frac{x^2(2a - 3x)^2}{12a\,x^2(a - x)}$$

[Substituting for $3ay^2$ from (1)]

$$= 1 + \frac{(2a - 3x)^2}{12a(a - x)} = \frac{12a^2 - 12ax + (2a - 3x)^2}{12a(a - x)} = \frac{(4a - 3x)^2}{12a(a - x)}.$$

$\therefore$ the required length of the loop

= twice the length of the half loop lying above the x-axis,

[By symmetry]

$$= 2\int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = 2\int_0^a \sqrt{\left\{\frac{(4a - 3x)^2}{12a(a - x)}\right\}}\, dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a \frac{(4a - 3x)}{\sqrt{(a - x)}} dx = \frac{1}{\sqrt{(3a)}} \int_0^a \frac{3(a - x) + a}{\sqrt{(a - x)}} dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a \left[\frac{3(a - x)}{\sqrt{(a - x)}} + \frac{a}{\sqrt{(a - x)}} \right] dx$$

$$= \frac{1}{\sqrt{(3a)}} \int_0^a [3\sqrt{(a - x)} + a(a - x)^{-1/2}] dx$$

$$= \frac{1}{\sqrt{(3a)}} \left[-3 \cdot \frac{2}{3} (a - x)^{3/2} - a \cdot 2(a - x)^{1/2} \right]_0^a$$

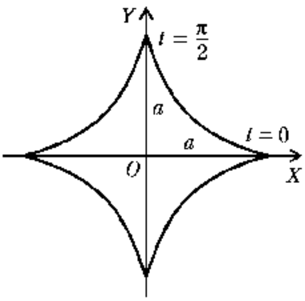
$$= \frac{1}{\sqrt{(3a)}} \left[2a^{3/2} + 2a^{3/2} \right] = \frac{4a}{\sqrt{3}}.$$

Example 4 : *Find the length of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$.

(Meerut 2002, 03, 13B; Agra 05; Purvanchal 08; Kashi 12)

Solution : The given astroid is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The curve is symmetrical in all the four quadrants. For the arc of the curve in the first quadrant x varies from 0 to a. Differentiating (1), w.r.t. x, we get



$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0 \text{ so that}$$

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}.$$

$\therefore$ the required whole length of the curve

= $4 \times$ length of the curve lying in the Ist quadrant

$$= 4 \int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx = 4 \int_0^a \sqrt{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)} dx$$

$$= 4 \int_0^a \frac{\sqrt{(x^{2/3} + y^{2/3})}}{x^{1/3}} dx = 4 \int_0^a \frac{\sqrt{(a^{2/3})}}{x^{1/3}} dx$$

$$= 4a^{1/3} \int_0^a x^{-1/3} dx = 4a^{1/3} \left[\frac{3}{2} x^{2/3} \right]_0^a = 6a.$$

Comprehensive Exercise 1

1. (i) Find the arc length of the curve $y = \frac{1}{2} x^2 - \frac{1}{4} \log x$ from $x = 1$ to $x = 2$.

(Meerut 2012B)

(ii) Find the length of the curve $y = \log \dfrac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

(Meerut 2004B; Agra 06; Avadh 08; Kanpur 11; Rohilkhand 13; Kashi 13)

2. (i) Show that in the catenary $y = c \cosh(x/c)$, the length of arc from the vertex to any point is given by $s = c \sinh(x/c)$.

(ii) If s be the length of the arc of the catenary $y = c \cosh(x/c)$ from the vertex $(0, c)$ to the point (x, y), show that $s^2 = y^2 - c^2$.

3. (i) Find the length of an arc of the parabola $y^2 = 4ax$ measured from the vertex.

(ii) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.

4. (i) Find the length of the arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of the latus rectum. **(Kanpur 2008; Purvanchal 09)**

(ii) Find the length of the arc of the parabola $x^2 = 8y$ from the vertex to an extremity of the latus rectum.

5. (i) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $y = 3x$.

(ii) Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$. **(Gorakhpur 2006; Purvanchal 06)**

6. (i) Find the perimeter of the curve $x^2 + y^2 = a^2$. **(Avadh 2010; Rohilkhand 13B)**

(ii) Find the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a). **(Bundelkhand 2010)**

7. (i) Show that the length of the arc of the curve $x^2 = a^2(1 - e^{y/a})$ measured from the origin to the point (x, y) is $a \log\{(a + x)/(a - x)\} - x$. **(Rohilkhand 2010B)**

(ii) Prove that the length of the loop of the curve $3ay^2 = x(x - a)^2$ is $4a/\sqrt{3}$. **(Meerut 2005B, 08, 09B)**

8. (i) Find the perimeter of the loop of the curve $9ay^2 = (x - 2a)(x - 5a)^2$.

(ii) Show that the whole length of the curve $x^2(a^2 - x^2) = 8a^2 y^2$ is $\pi a \sqrt{2}$. **(Bundelkhand 2006; Purvanchal 11)**

Answers 1

1. (i) $\frac{3}{2} + \frac{1}{4}\log 2$. (ii) $\log\left(e + \frac{1}{e}\right)$

3. (i) $\frac{1}{4a}\left[y\sqrt{(y^2 + 4a^2)} + 4a^2 \log\left\{\frac{y + \sqrt{(y^2 + 4a^2)}}{2a}\right\}\right]$.

(ii) $2a[\sqrt{2} + \log(1 + \sqrt{2})]$.

4. (i) $a[\sqrt{2} + \log(1 + \sqrt{2})]$.

(ii) $2[\sqrt{2} + \log(1 + \sqrt{2})]$.

5. (i) $a\left[\frac{2\sqrt{(13)}}{9} + \log\left\{\frac{2 + \sqrt{(13)}}{3}\right\}\right]$.

6. (i) $2a\pi$. (ii) $\frac{1}{27}a[13\sqrt{(13)} - 8]$.

8. (i) $4a\sqrt{3}$.

7.3 Equations of the Curve in Parametric form (Meerut 2009B)

If the equations of the curve be given in the parametric form $x = f(t), y = \phi(t)$, then s is obviously a function of t. In this case if we measure the arc length s in the direction of t increasing, we have

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$

On integrating between proper limits, the required length

$$s = \int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$ **(Meerut 2003)**

Illustrative Examples

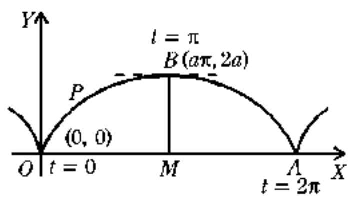
Example 1 : *Show that 8a is the length of an arch of the cycloid whose equations are* $x = a(t - \sin t), y = a(1 - \cos t)$. **(Meerut 2006; Rohilkhand 08; Kashi 11; Avadh 12; Purvanchal 14)**

Solution : The given equations of the cycloid are

$$x = a(t - \sin t), y = a(1 - \cos t).$$

We have $dx/dt = a(1 - \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 - \cos t)} = \frac{2 \sin \frac{1}{2} t \cos \frac{1}{2} t}{2 \sin^2 \frac{1}{2} t} = \cot \frac{1}{2} t.$$



Now $y = 0$ when $\cos t = 1$ *i.e.,* $t = 0$. At $t = 0, x = 0, y = 0$ and $dy/dx = \infty$. Thus the curve passes through the point (0, 0) and the tangent there is perpendicular to the x-axis.

Again y is maximum when $\cos t = -1$ *i.e.,* $t = \pi$. When $t = \pi, x = a\pi, y = 2a, dy/dx = 0$. Thus at the point $(a\pi, 2a)$ the tangent to the curve is parallel to the x-axis.

Also in this curve y cannot be negative. Thus an arch OBA of the given cycloid is as shown in the figure. It is symmetrical about the line BM which is the axis of the cycloid.

We have $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= \{a(1 - \cos t)\}^2 + (a \sin t)^2$$

$$= a^2 \{(2 \sin^2 \tfrac{1}{2} t)^2 + (2 \sin \tfrac{1}{2} t \cos \tfrac{1}{2} t)^2\}$$

$$= 4a^2 \sin^2 \tfrac{1}{2} t (\sin^2 \tfrac{1}{2} t + \cos^2 \tfrac{1}{2} t) = 4a^2 \sin^2 \tfrac{1}{2} t. \quad \text{...(1)}$$

If s denotes the arc length of the cycloid measured from the cusp O to any point P towards the vertex B, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (1), we have

$$ds/dt = 2a \sin \tfrac{1}{2} t, \quad \text{or} \quad ds = 2a \sin \tfrac{1}{2} t\, dt.$$

At the cusp $O, t = 0$, and at the vertex $B, t = \pi$.

Now the length of the arch $OBA = 2 \times$ length of the arc OB

$$= 2\int_0^{\pi} 2a \sin \tfrac{1}{2} t \, dt = 4a\left[-2\cos \tfrac{1}{2} t\right]_0^{\pi} = -8a\left[\cos \tfrac{1}{2} t\right]_0^{\pi}$$

$$= -8a\,[0-1] = 8a.$$

Example 2 : *Find the length of the loop of the curve* $x = t^2, y = t - \frac{1}{3} t^3$.

(Kanpur 2010)

Solution : Eliminating the parameter t from $x = t^2$ and $y = t - \frac{1}{3} t^3$, we get $y^2 = x(1 - \frac{1}{3} x)^2$ as the cartesian equation of the curve and hence we observe that the curve is symmetrical about the x-axis. The loop of the curve extends from the point $(0, 0)$ to the point $(3, 0)$. Putting $y = 0$ in $y = t - \frac{1}{3} t^3$, we get $t = 0$ and $t = \sqrt{3}$. Therefore the arc of the upper half of the loop extends from $t = 0$ to $t = \sqrt{3}$.

Now the required length of the loop

$$= 2 \times \text{ length of the half of the loop which lies above } x\text{-axis}$$

$$= 2\int_0^{\sqrt{3}} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt = 2\int_0^{\sqrt{3}} \sqrt{\{(2t)^2 + (1 - \tfrac{1}{3} \cdot 3t^2)^2\}}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(1 + 2t^2 + t^4)}\, dt = 2\int_0^{\sqrt{3}} (1 + t^2)\, dt$$

$$= 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2\,[\sqrt{3} + \sqrt{3}] = 4\sqrt{3}.$$

Example 3 : *Show that the length of an arc of the curve*

$x \sin t + y \cos t = f'(t)$, $x \cos t - y \sin t = f''(t)$ *is given by*

$s = f(t) + f''(t)$, *where c is the constant of integration.*

Solution : The given equations of the curve are $x \sin t + y \cos t = f'(t)$ ...(1)

and $x \cos t - y \sin t = f''(t)$. ...(2)

Multiplying (1) by $\sin t$ and (2) by $\cos t$ and adding, we get

$$x(\sin^2 t + \cos^2 t) = \sin t . f'(t) + \cos t . f''(t)$$

or $$x = \sin t f'(t) + \cos t f''(t). \qquad ...(3)$$

Again, multiplying (1) by $\cos t$ and (2) by $\sin t$ and subtracting, we get

$$y = \cos t f'(t) - \sin t f''(t). \qquad ...(4)$$

Now differentiating (3) and (4) w.r.t. t, we get

$$dx/dt = \cos t f'(t) + \sin t f''(t) + \cos t f'''(t) - \sin t f''(t)$$

$$= [f'(t) + f'''(t)] \cos t$$

and $$dy/dt = -[f'(t) + f'''(t)] \sin t.$$

Now if s be the arc length in the direction of t increasing, then

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{[\cos^2 t\,\{f'(t) + f'''(t)\}^2 + \sin^2 t\,\{f'(t) + f'''(t)\}^2]}$$

$$= [f'(t) + f'''(t)]\sqrt{(\cos^2 t + \sin^2 t)} = f'(t) + f'''(t).$$

Integrating both sides, we have $s = \int [f'(t) + f'''(t)]\, dt + c$

$$= f(t) + f''(t) + c, \text{ where } c \text{ is the constant of integration.}$$

Comprehensive Exercise 2

1. (i) Find the whole length of the curve (astroid) $x = a\cos^3 t,\ y = a\sin^3 t$.

(Rohilkhand 2011)

(ii) Find the whole length of the curve (Hypocycloid) $x = a\cos^3 t,\ y = b\sin^3 t$.

2. Rectify the curve or find the length of an arch of the curve

$$x = a(t + \sin t), y = a(1 - \cos t).$$ **(Rohilkhand 2009B)**

3. Prove that the length of an arc of the cycloid $x = a(t + \sin t), y = a(1 - \cos t)$ from the vertex to the point (x, y) is $\sqrt{(8ay)}$. **(Bundelkhand 2007; Meerut 12)**

4. Find the length of the arc of the curve

$$x = e^t \sin t, y = e^t \cos t, \text{ from } t = 0 \text{ to } t = \tfrac{1}{2}\pi.$$

(Kumaun 2008; Kanpur 09)

5. Show that in the epi-cycloid for which

$$x = (a + b)\cos\theta - b\cos\{(a + b)/b\}\,\theta,$$
$$y = (a + b)\sin\theta - b\sin\{(a + b)/b\}\,\theta,$$

the length of the arc measured from the point $\theta = \pi b/a$ is

$$\{4b(a + b)/a\}\cos\{(a/2b)\,\theta\}.$$

6. In the ellipse $x = a\cos\phi, y = b\sin\phi$, show that $ds = a\sqrt{(1 - e^2\cos^2\phi)}\,d\phi$, and hence show that the whole length of the ellipse is

$$2\pi a\left[1 - \left(\frac{1}{2}\right)^2 \cdot \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \cdot \frac{e^6}{5} - \dots\right],$$

where e is the eccentricity of the ellipse. **(Meerut 2005)**

Answers 2

1. (i) $6a$. (ii) $4(b^2 + ab + a^2)/(b + a)$.

2. $8a$. **4.** $\sqrt{2}\,[e^{\pi/2} - 1]$.

7.4 Equation of the Curve in Polar Form

For the curve $r = f(\theta)$, if we measure the arc length s in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta.$$

On integrating between proper limits, the required length

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta.$$ **(Meerut 2003)**

If the equation of the curve be $\theta = f(r)$, then the required length is given by

$$s = \int_{r_1}^{r_2} \sqrt{\left\{1 + \left(r \frac{d\theta}{dr}\right)^2\right\}}\, dr.$$

Illustrative Examples

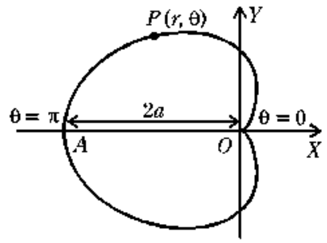
Example 1 : *Find the perimeter of the cardioid $r = a(1 - \cos\theta)$.*

(Meerut 2007; Bundelkhand 11)

Solution : The given curve is $r = a(1 - \cos\theta)$. ...(1)

It is symmetrical about the initial line.

We have $r = 0$ when $\cos\theta = 1$ *i.e.*, $\theta = 0$. Also r is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = 2a$. As θ increases from 0 to π, r increases from 0 to $2a$. So the curve is as shown in the figure.



By symmetry, the perimeter of the cardioid $= 2 \times$ the arc length of the upper half of the cardioid.

Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta = a \sin\theta.$$

We have

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta$$

$$= a^2 (2 \sin^2 \tfrac{1}{2}\theta)^2 + a^2 (2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta)^2$$

$$= 4a^2 \sin^2 \tfrac{1}{2}\theta (\sin^2 \tfrac{1}{2}\theta + \cos^2 \tfrac{1}{2}\theta) = 4a^2 \sin^2 \tfrac{1}{2}\theta. \qquad ...(2)$$

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.*, the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, then s increases as θ increases. Therefore $ds/d\theta$ will be positive.

Hence from (2), we have

$$ds/d\theta = 2a \sin \tfrac{1}{2}\theta, \quad \text{or} \quad ds = 2a \sin \tfrac{1}{2}\theta\, d\theta. \qquad ...(3)$$

At the cusp O, $\theta = 0$ and at the vertex A, $\theta = \pi$.

$\therefore$ the length of the arc $OPA = \int_0^{\pi} 2a \sin \tfrac{1}{2}\theta\, d\theta$

$$= 4a\left[-\cos\frac{\theta}{2}\right]_0^{\pi} = -4a\left[\cos\frac{\theta}{2}\right]_0^{\pi} = -4a(0 - 1) = 4a.$$

$\therefore$ the perimeter of the cardioid $= 2 \times 4a = 8a$.

Example 2 : *Find the length of the arc of the equiangular spiral $r = ae^{\theta \cot\alpha}$ between the points for which radii vectors are r_1 and r_2.*

(Kanpur 2007)

Solution : The given curve is $r = ae^{\theta \cot\alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = ae^{\theta \cot\alpha} \cdot \cot\alpha = r \cot\alpha, \text{ from (1)}.$$

$\therefore \qquad d\theta/dr = 1/(r \cot \alpha)$ *i.e.*, $(r\, d\theta/dr) = \tan \alpha$. ...(2)

If s denotes the arc length of the given curve measured in the direction of r increasing, we have

$$\frac{ds}{dr} = \sqrt{\left\{1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right\}} \qquad \textbf{(Note)}$$

$$= \sqrt{(1 + \tan^2 \alpha)} = \sqrt{(\sec^2 \alpha)} = \sec \alpha, \text{ from (2)}$$

or $\qquad ds = \sec \alpha\, dr.$ **(Meerut 2001B)**

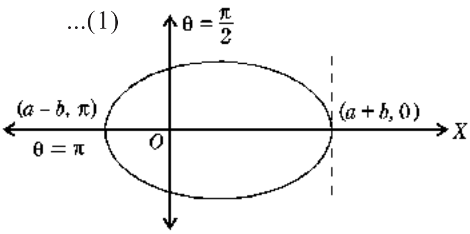
Let s_1 denote the required arc length *i.e.*, from $r = r_1$ to $r = r_2$.

Then $\displaystyle\int_0^{s_1} ds = \int_{r_1}^{r_2} \sec \alpha\, dr = (\sec \alpha) \Big[r \Big]_{r_1}^{r_2}$ or $s_1 = (\sec \alpha)(r_2 - r_1)$.

Example 3 : *Prove that the perimeter of the limacon* $r = a + b \cos \theta$, *if* b/a *be small, is approximately* $2 \pi a \left(1 + \frac{1}{4} b^2/a^2\right)$.

Solution : The given curve is

$r = a + b \cos \theta, (a > b)$. ...(1)



Note that b/a is given to be small so we must have $b < a$. The curve (1) is symmetrical about the initial line and for the portion of the curve lying above the initial line θ varies from $\theta = 0$ to $\theta = \pi$.

By symmetry, the perimeter of the limacon

$= 2 \times$ the arc length of the upper half of the limacon.

Now differentiating (1) w.r.t. θ, we have

$$dr/d\theta = - b \sin \theta.$$

We have

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = (a + b \cos \theta)^2 + (- b \sin \theta)^2$$

$$= a^2 + b^2 \cos^2 \theta + 2ab \cos \theta + b^2 \sin^2 \theta = a^2 + b^2 + 2ab \cos \theta.$$

If we measure the arc length s in the direction of θ increasing,

we have $\qquad ds/d\theta = \sqrt{(a^2 + b^2 + 2ab \cos \theta)}$

or $\qquad ds = \sqrt{(a^2 + b^2 + 2ab \cos \theta)}\, d\theta.$

The arc length of the upper half of the limacon

$$= \int_0^{\pi} \sqrt{(a^2 + b^2 + 2ab \cos \theta)}\, d\theta = a \int_0^{\pi} \left(1 + \frac{2b}{a} \cos \theta + \frac{b^2}{a^2}\right)^{1/2} d\theta$$

$$= a \int_0^{\pi} \left[1 + \frac{b}{a} \cos \theta + \frac{1}{2} \cdot \frac{b^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2\,!} \left(4 \frac{b^2}{a^2} \cos^2 \theta\right)\right] d\theta$$

[Expanding by binomial theorem and neglecting powers of b/a higher than two because b/a is small]

$$= a \int_0^{\pi} \left[1 + \frac{b}{a} \cos \theta + \frac{1}{2} \frac{b^2}{a^2} (1 - \cos^2 \theta)\right] d\theta$$

$$= a\int_0^{\pi}\left[1+\frac{b}{a}\cos\theta+\frac{1}{2}\frac{b^2}{a^2}\sin^2\theta\right]d\theta$$

$$= a\left[\left\{\theta+\frac{b}{a}\sin\theta\right\}_0^{\pi}+\frac{1}{2}\frac{b^2}{a^2}2\int_0^{\pi/2}\sin^2\theta\,d\theta\right]$$

$$= a\left[\pi+\frac{1}{2}\frac{b^2}{a^2}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= a\pi\left[1+\frac{b^2}{4a^2}\right].$$

$\therefore$ the perimeter of the limacon

$$= 2\times a\pi\,[1+(b^2/4a^2)] = 2a\pi\,[1+(b^2/4a^2)].$$

Example 4 : *If s be the length of the curve $r = a \tanh\frac{1}{2}\theta$ between the origin and $\theta = 2\pi$, and Δ be the area under the curve between the same two points, prove that $\Delta = a(s - a\pi)$.*

Solution : The given curve is $r = a\tanh\frac{1}{2}\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get $dr/d\theta = a\cdot\frac{1}{2}\operatorname{sech}^2\frac{1}{2}\theta$.

We have $\left(\dfrac{ds}{d\theta}\right)^2 = r^2+\left(\dfrac{dr}{d\theta}\right)^2 = a^2\tanh^2\frac{1}{2}\theta+\dfrac{a^2}{4}\operatorname{sech}^4\frac{1}{2}\theta$

$$= \tfrac{1}{4}a^2\,[4\tanh^2\tfrac{1}{2}\theta+\operatorname{sech}^4\tfrac{1}{2}\theta] = \tfrac{1}{4}a^2\,[4(1-\operatorname{sech}^2\tfrac{1}{2}\theta)+\operatorname{sech}^4\tfrac{1}{2}\theta]$$

$$= \tfrac{1}{4}a^2\,[2-\operatorname{sech}^2\tfrac{1}{2}\theta]^2. \qquad ...(2)$$

If we measure the arc length s in the direction of θ increasing, we have

$$ds/d\theta = \tfrac{1}{2}a\,(2-\operatorname{sech}^2\tfrac{1}{2}\theta)$$

[Retaining + ive sign while taking the square root of (2)]

or $$ds = \tfrac{1}{2}a\,(2-\operatorname{sech}^2\tfrac{1}{2}\theta)\,d\theta.$$

Now at the origin $r = 0$ and putting $r = 0$ in (1), we get $\theta = 0$.

$\therefore$ the arc length of the given curve between the origin ($\theta = 0$) and $\theta = 2\pi$ is given by

$$s = \frac{1}{2}a\int_0^{2\pi}(2-\operatorname{sech}^2\tfrac{1}{2}\theta)\,d\theta = \tfrac{1}{2}a\int_0^{2\pi}2d\theta-\tfrac{1}{2}a\int_0^{2\pi}\operatorname{sech}^2\tfrac{1}{2}\theta\,d\theta$$

$$= \tfrac{1}{2}a\,.\,2\Big[\theta\Big]_0^{2\pi}-\tfrac{1}{2}a\Big[2\tanh\tfrac{1}{2}\theta\Big]_0^{2\pi} = 2a\pi-a\tanh\pi. \qquad ...(3)$$

Also the area between the radii vectors $\theta = 0, \theta = 2\pi$ and the curve is

$$\Delta = \frac{1}{2}\int_0^{2\pi}r^2\,d\theta = \frac{1}{2}a^2\int_0^{2\pi}\tanh^2\frac{1}{2}\theta\,d\theta$$

$$= \tfrac{1}{2}a^2\int_0^{2\pi}(1-\operatorname{sech}^2\tfrac{1}{2}\theta)\,d\theta = \tfrac{1}{2}a^2\Big[\theta-2\tanh\tfrac{1}{2}\theta\Big]_0^{2\pi}$$

$$= \tfrac{1}{2} a^2 [2\pi - 2 \tanh \pi] = a^2 [\pi - \tanh \pi]$$

$$= a [a\pi - a \tanh \pi] = a [(2a\pi - a \tanh \pi) - a\pi] = a (s - a\pi), \text{ from (3).}$$

7.5 Equation of the Curve in Pedal Form

sLet $p = f(r)$ be the equation of the curve and r_1 and r_2 be the values of r at two given points of the curve. Then by differential calculus we know that

$$\frac{ds}{dr} = \frac{r}{\sqrt{(r^2 - p^2)}} \quad \text{or} \quad ds = \frac{r}{\sqrt{(r^2 - p^2)}} dr,$$

where s increases as r increases.

On integrating between proper limits, the required length

$$\boldsymbol{s = \int_{r_1}^{r_2} \frac{r}{\sqrt{(r^2 - p^2)}} dr.}$$

The value of p should be put in terms of r from the equation of the curve.

Important Remark : If the curve is symmetrical about one or more lines, then find out the length of one symmetrical part and then multiply it by the number of symmetrical parts.

Illustrative Examples

Example 1 : *Prove the formula* $s = \int \frac{r\, dr}{\sqrt{(r^2 - p^2)}}$.

Show that the arc of the curve $p^2 (a^4 + r^4) = a^4 r^2$ *between the limits* $r = b$, $r = c$ *is equal in length to the arc of the hyperbola* $xy = a^2$ *between the limits* $x = b, x = c$.

Solution : From differential calculus, we know that

$$\tan \phi = r \frac{d\theta}{dr} \text{ and } \frac{ds}{dr} = \sqrt{\left[1 + \left(r \frac{d\theta}{dr}\right)^2\right]}.$$

$$\therefore \quad \frac{ds}{dr} = \sqrt{(1 + \tan^2 \phi)} = \sqrt{(\sec^2 \phi)} = \sec \phi$$

$$= \frac{1}{\cos \phi} = \frac{1}{\sqrt{(1 - \sin^2 \phi)}} = \frac{1}{\sqrt{\{1 - (p^2/r^2)\}}} \qquad [\because p = r \sin \phi]$$

$$= \frac{r}{\sqrt{(r^2 - p^2)}}.$$

Thus $\quad ds = \dfrac{r}{\sqrt{(r^2 - p^2)}} dr.$

Integrating between the given limits, we get $s = \int \dfrac{r}{\sqrt{(r^2 - p^2)}} dr.$ ...(1)

Now the given curve is $p^2 (a^4 + r^4) = a^4 r^2$ or $p^2 = a^4 r^2 / (a^4 + r^4)$.

We have $r^2 - p^2 = r^2 - \dfrac{a^4 r^2}{(a^4 + r^4)} = \dfrac{r^6}{(a^4 + r^4)}$. ...(2)

Therefore from (1), the arc of the given curve between the limits $r = b, r = c$ is

$$= \int_b^c \frac{r\,dr}{\sqrt{(r^2 - p^2)}} = \int_b^c \frac{r\,dr}{\sqrt{\{r^6/(a^4 + r^4)\}}} \qquad \text{[From (2)]}$$

$$= \int_b^c \frac{r\sqrt{(a^4 + r^4)}}{r^3}\,dr = \int_b^c \frac{\sqrt{(a^4 + r^4)}}{r^2}\,dr. \qquad ...(3)$$

Also, for the hyperbola $xy = a^2$ *i.e.*, $y = a^2/x$, $dy/dx = -a^2/x^2$.

$\therefore$ the arc length of the hyperbola $xy = a^2$ between the limits $x = b, x = c$

$$= \int_b^c \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx = \int_b^c \sqrt{\left\{1 + \frac{a^4}{x^4}\right\}}\,dx$$

$$= \int_b^c \frac{\sqrt{(x^4 + a^4)}}{x^2}\,dx = \int_b^c \frac{\sqrt{(r^4 + a^4)}}{r^2}\,dr$$

[Changing the variable from x to r by a property of definite integrals]

$$= \int_b^c \frac{\sqrt{(a^4 + r^4)}}{r^2}\,dr. \qquad ...(4)$$

From (3) and (4) we observe that the two lengths are equal.

Comprehensive Exercise 3

1. Find the entire length of the cardioid $r = a(1 + \cos\theta)$.

(Purvanchal 2007; Rohilkhand 09, 11B)

2. Find the perimeter of the curve $r = a(1 + \cos\theta)$ and show that arc of the upper half is bisected by $\theta = \pi/3$. **(Gorakhpur 2005; Purvanchal 07)**

3. Prove that the line $4r\cos\theta = 3a$ divides the cardioid $r = a(1 + \cos\theta)$ into two parts such that lengths of the arc on either side of the line are equal.

4. Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi/3$.

5. Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$.

6. Find the length of the arc of the equiangular spiral $r = a\,e^{\theta\cot\alpha}$, taking $s = 0$ when $\theta = 0$.

7. Find the length of any arc of the cissoid $r = a(\sin^2\theta/\cos\theta)$.

8. Show that the whole length of the limacon $r = a + b\cos\theta$, $(a > b)$ is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.

Answers 3

1. $8a$.

5. $4a\sqrt{3}$.

6. $a\sec\alpha\,[e^{\theta\cot\alpha} - 1]$.

7. $f(\theta_2) - f(\theta_1)$, where $f(\theta) = a\sqrt{(\sec^2\theta + 3)} - a\sqrt{3}\log\{\cos\theta + \sqrt{(\cos^2\theta + \frac{1}{3})}\}$.

7.6 Intrinsic Equations

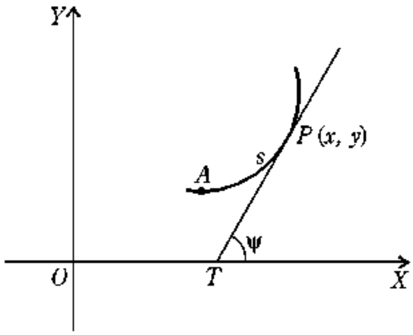
Definition : By the *intrinsic equation* of a curve we mean a relation between s and ψ, where s is the length of the arc AP of the curve measured from a fixed point A on it to a variable point P, and ψ is the angle which the tangent to the curve at P makes with a fixed straight line usually taken as the positive direction of the axis of x.

The co-ordinates s and ψ are known as **Intrinsic Co-ordinates**.

(a) To find the intrinsic equation from the cartesian equation :

Let the equation of the given curve be $y = f(x)$. Take A as the fixed point on the curve from which s is measured and take the axis of x as the fixed straight line with reference to which ψ is measured. Let $P(x, y)$ be any point on the curve and PT be the tangent at the point P to the curve.

Let arc $AP = s$ and $\angle PTX = \psi$.



Now, we have $\tan \psi = dy/dx = f'(x)$. ...(1)

Let a be the abscissa of the point A from which s is measured. Then

$$s = \int_a^x \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\, dx$$

$$= \int_a^x \sqrt{[1 + \{f'(x)\}^2]}\, dx. \qquad \text{...(2)}$$

Eliminating x between (1) and (2), we obtain the required intrinsic equation.

Note : To find the intrinsic equation from the parametric equations we use $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$ and then proceed as in case (a).

(b) Intrinsic equation from Polar equation :

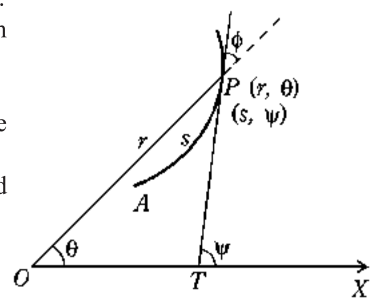
Let the equation of the given curve be $r = f(\theta)$.

Take A as the fixed point on the curve from which s is measured.

Let P be any point (r, θ) on the curve.

Let arc $AP = s$ and $\angle PTX = \psi$, where OX is the initial line.

If ϕ is the angle between the radius vector and the tangent at P, then



$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{f(\theta)}{f'(\theta)}, \qquad \text{...(1)}$$

and $\psi = \theta + \phi$. ...(2)

Let α be the vectorial angle of the point A. Then we have

$$s = \int_\alpha^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

$$= \int_\alpha^\theta \sqrt{[\{f(\theta)\}^2 + \{f'(\theta)\}^2]}\, d\theta \qquad \text{...(3)}$$

Eliminating θ and ϕ between (1), (2) and (3), we get a relation between s and ψ, which is the intrinsic equation of the curve.

(c) Intrinsic equation from Pedal Equation :

Let the pedal equation of the curve be $p = f(r)$. ...(1)

Then $$s = \int_a^r \frac{r\,dr}{\sqrt{(r^2 - p^2)}}, \qquad ...(2)$$

the arc length s being measured from the point $r = a$.

Also the radius of curvature $\rho = \dfrac{ds}{d\psi} = r\dfrac{dr}{dp}$. ...(3)

Eliminating p and r between (1), (2) and (3), we obtain the required intrinsic equation.

Illustrative Examples

Example 1 : *Show that the intrinsic equation of the parabola $y^2 = 4ax$ is*

$$s = a \cot\psi \operatorname{cosec}\psi + a \log(\cot\psi + \operatorname{cosec}\psi),$$

ψ being the angle between the x-axis and the tangent at the point whose arcual distance from the vertex is s.

Solution : The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \quad \tan\psi = dy/dx = 4a/2y = 2a/y$. ...(2)

If s denotes the arc length of the parabola measured from the vertex (0, 0) in the direction of y increasing, then

$$\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \sqrt{\left\{1 + \frac{y^2}{4a^2}\right\}} \qquad \left[\because \frac{dx}{dy} = \frac{y}{2a}\right]$$

$$= \sqrt{\left\{\frac{4a^2 + y^2}{4a^2}\right\}} = \frac{1}{2a}\sqrt{(4a^2 + y^2)}.$$

$$\therefore \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy.$$

Integrating, $$\int_0^s ds = \frac{1}{2a}\int_0^y \sqrt{(4a^2 + y^2)}\,dy$$

or $$s = \frac{1}{2a}\left[\frac{1}{2}y\sqrt{(4a^2 + y^2)} + \frac{1}{2}\cdot 4a^2 \log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^y$$

$$= (1/2a)\,[\tfrac{1}{2}y\sqrt{(4a^2 + y^2)} + \tfrac{1}{2}\cdot 4a^2 \log\{y + \sqrt{(4a^2 + y^2)}\} - \tfrac{1}{2}\cdot 4a^2 \log 2a]$$

$$= \frac{1}{4a}\left[y\sqrt{(4a^2 + y^2)} + 4a^2 \log\frac{y + \sqrt{(4a^2 + y^2)}}{2a}\right]. \qquad ...(3)$$

Now to obtain the intrinsic equation of the given parabola we eliminate y between (2) and (3). From (2), we have $y = 2a \cot\psi$. Putting this value of y in (3), we get

$$s = \frac{1}{4a}\left[2a \cot\psi \sqrt{(4a^2 + 4a^2 \cot^2\psi)} \right.$$
$$\left. + 4a^2 \log \frac{2a \cot\psi + \sqrt{(4a^2 + 4a^2 \cot^2\psi)}}{2a}\right]$$
$$= \frac{1}{4a}[(2a \cot\psi) . 2a\sqrt{(1 + \cot^2\psi)} + 4a^2 \log\{\cot\psi + \sqrt{(1 + \cot^2\psi)}\}]$$
$$= a \cot\psi \operatorname{cosec}\psi + a \log(\cot\psi + \operatorname{cosec}\psi),$$

which is the required intrinsic equation.

Example 2 : *Show that the intrinsic equation of the cycloid*

$$x = a(t + \sin t), y = a(1 - \cos t)$$

is $s = 4a \sin\psi$.

Hence or otherwise find the length of the complete cycloid.

(Meerut 2001, 06B, 07, 10; Kanpur 04; Avadh 04, 09, 10; Rohilkhand 07B)

Solution : The given equations of the cycloid are

$$x = a(t + \sin t), y = a(1 - \cos t). \qquad ...(1)$$

We have $dx/dt = a(1 + \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)} = \frac{2 \sin\frac{1}{2}t \cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t} = \tan\frac{1}{2}t.$$

Hence $\tan\psi = dy/dx = \tan\frac{1}{2}t$ or $\psi = \frac{1}{2}t$. ...(2)

If s denotes the arc length of the cycloid measured from the vertex (*i.e.*, the point $t = 0$) to any point P (*i.e.*, the point 't') in the direction of t increasing, then

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt$$
$$= \int_0^t \sqrt{\{a^2(1 + \cos t)^2 + a^2 \sin^2 t\}}\, dt$$
$$= \int_0^t \sqrt{\{2a^2(1 + \cos t)\}}\, dt$$
$$= 2a \int_0^t \cos\frac{1}{2}t\, dt = 2a\left[2 \sin\frac{1}{2}t\right]_0^t = 4a \sin\frac{1}{2}t \qquad ...(3)$$

Eliminating t from (2) and (3), we get $s = 4a \sin\psi$, ...(4)

which is the required intrinsic equation of the cycloid.

Second Part : In the intrinsic equation (4) of the cycloid the arc length s has been measured from the vertex *i.e.*, the point $\psi = 0$. At a cusp, we have $t = \pi$ and $\psi = \pi/2$. If s_1 denotes the length of the arc extending from the vertex to a cusp, then from (4), we have $s_1 = 4a \sin\frac{1}{2}\pi = 4a$.

$\therefore$ the whole length of an arch of the cycloid $= 2 \times 4a = 8a$.

Example 3 : *Find the intrinsic equation of the cardioid* $r = a(1 + \cos\theta)$, *and hence, or otherwise, prove that* $s^2 + 9\rho^2 = 16a^2$, *where* ρ *is the radius of curvature at any point, and s is the length of the arc intercepted between the vertex and the point.*

(Meerut 2005B)

Solution : The given curve is $r = a(1 + \cos\theta)$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = -a\sin\theta$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

$$= -\cot\tfrac{1}{2}\theta = \tan(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta).$$

Therefore $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$, so that

$$\psi = \theta + \phi = \theta + \tfrac{1}{2}\pi + \tfrac{1}{2}\theta = \tfrac{1}{2}\pi + \tfrac{3}{2}\theta$$

or $$\tfrac{1}{2}\theta = \tfrac{1}{3}(\psi - \tfrac{1}{2}\pi). \qquad \text{...(2)}$$

If s denotes the arc length of the cardioid measured from the vertex (*i.e.*, $\theta = 0$) to any point P (*i.e.*, $\theta = \theta$) in the direction of θ increasing, then

$$s = \int_0^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$$

$$= 2a\int_0^\theta \sqrt{\{(1+\cos\theta)^2 + \sin^2\theta\}}\, d\theta$$

$$= 2a\int_0^\theta \sqrt{\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}}\, d\theta$$

$$= 2a\int_0^\theta \sqrt{\{2(1+\cos\theta)\}}\, d\theta$$

$$= 2a\int_0^\theta \cos\tfrac{1}{2}\theta\, d\theta$$

$$= 2a\left[2\sin\tfrac{1}{2}\theta\right]_0^\theta = 4a\sin\tfrac{1}{2}\theta. \qquad \text{...(3)}$$

Eliminating θ between (2) and (3), we get $s = 4a\sin\{\frac{1}{3}(\psi - \frac{1}{2}\pi)\}$, ...(4)

which is the required intrinsic equation.

Also $\quad \rho = \dfrac{ds}{d\psi} = \dfrac{4a}{3}\cos\frac{1}{3}(\psi - \frac{1}{2}\pi)$, from (4)

or $$3\rho = 4a\cos\tfrac{1}{3}(\psi - \tfrac{1}{2}\pi). \qquad \text{...(5)}$$

Squaring and adding (4) and (5), we get

$$s^2 + 9\rho^2 = (4a)^2\{\sin^2\tfrac{1}{3}(\psi - \tfrac{1}{2}\pi) + \cos^2\tfrac{1}{3}(\psi - \tfrac{1}{2}\pi)\}$$

$$= 16a^2 \,.\, 1 = 16a^2.$$

Example 4 : *Find the intrinsic equation of the equiangular spiral* $p = r\sin\alpha$.

(Meerut 2000, 01, 04, 06, 09, 10B)

Solution: The given pedal equation of the curve is $p = r\sin\alpha$. s ...(1)

Differentiating (1) w.r.t. r, we have $dp/dr = \sin\alpha$.

$$\therefore \qquad \rho = \frac{ds}{d\psi} = r\frac{dr}{dp} = \frac{r}{dp/dr} = \frac{r}{\sin\alpha} = r\,\text{cosec}\,\alpha. \qquad \text{...(2)}$$

If we measure the arc length s from the point $r = 0$ in the direction of r increasing, we have

$$s = \int_0^r \frac{r\,dr}{\sqrt{(r^2 - p^2)}} = \int_0^r \frac{r\,dr}{\sqrt{(r^2 - r^2\sin^2\alpha)}} = \int_0^r \sec\alpha\,dr$$

$$= \sec\alpha\int_0^r dr = \sec\alpha\Big[\,r\,\Big]_0^r = r\sec\alpha. \qquad \text{...(3)}$$

Eliminating r between (2) and (3), we have

$$\frac{(ds/d\psi)}{s} = \frac{\text{cosec}\,\alpha}{\sec\alpha} = \cot\alpha \qquad \text{[Dividing (2) by (3)]}$$

or $\qquad ds/s = \cot\alpha\,d\psi.$

Integrating,

$$\log s = \psi\cot\alpha + \log a, \text{ where } a \text{ is constant of integration}$$

or $\qquad \log(s/a) = \psi\cot\alpha$

or $\qquad s = a\,e^{\psi\cot\alpha},$

which is the required intrinsic equation of the curve.

Example 5 : *Find the intrinsic equation of the curve for which the length of the arc measured from the origin varies as the square root of the ordinate. Find also parametric equations of the curve in terms of any parameter.*

Solution : Let s denote the arc length of the curve measured from the origin to any point $P(x, y)$ such that s increases as y increases. As given $s \propto \sqrt{y}$ so that $s = \lambda\sqrt{y}$, where λ is some constant.

Choosing this constant $\lambda = \sqrt{(8a)}$, we have **(Note)**,

$$s = \sqrt{(8ay)}$$

or $\qquad s^2 = 8ay. \qquad \text{...(1)}$

Now differentiating (1) w.r.t. y, we have

$$2s\,(ds/dy) = 8a$$

or $\qquad ds/dy = 4a/s. \qquad \text{...(2)}$

Now we know that $dy/ds = \sin\psi$.

$$\therefore \qquad \sin\psi = dy/ds = s/4a \qquad \text{[From (2)]}$$

or $\qquad s = 4a\sin\psi$, which is the required intrinsic equation.

Again from (1), we have $y = \dfrac{s^2}{8a} = \dfrac{16a^2\sin^2\psi}{8a}$ $\qquad [\because s = 4a\sin\psi]$

$$= a(1 - \cos 2\psi). \qquad \text{...(3)}$$

Also $\qquad \dfrac{ds}{dx} = \dfrac{ds}{d\psi}\cdot\dfrac{d\psi}{dx} = 4a\cos\psi\dfrac{d\psi}{dx}$ $\qquad \left[\because \dfrac{ds}{d\psi} = 4a\cos\psi\right]$

or $\qquad \dfrac{1}{\cos\psi} = 4a\cos\psi\dfrac{d\psi}{dx}$ $\qquad \left[\because \dfrac{dx}{ds} = \cos\psi\right]$

or $\qquad dx = 4a\cos^2\psi\,d\psi = 2a(1 + \cos 2\psi)\,d\psi. \qquad \text{...(4)}$

If $x = 0$ when $\psi = 0$, then integrating (4), we get

$$\int_0^x dx = 2a \int_0^\psi (1 + \cos 2\psi)\, d\psi$$

or $$x = 2a \left[\psi + \tfrac{1}{2} \sin 2\psi \right]_0^\psi$$

or $$x = a\,[2\psi + \sin 2\psi]. \qquad ...(5)$$

So from (3) and (5), the required parametric equations of the curve are

$$x = a\,(2\psi + \sin 2\psi)$$

and $$y = a\,(1 - \cos 2\psi),$$

which are the parametric equations of a cycloid.

Comprehensive Exercise 4

1. Prove that the intrinsic equation of the parabola $x^2 = 4ay$ is
$$s = a \tan\psi \sec\psi + a \log(\tan\psi + \sec\psi).$$

2. Find the intrinsic equation of the parabola $y^2 = 4ax$. Hence deduce the length of the arc measured from the vertex to an extremity of the latus rectum.

3. Show that the intrinsic equation of the semi-cubical parabola
$3ay^2 = 2x^3$ is $9s = 4a\,(\sec^3\psi - 1)$. **(Meerut 2005, 09B; Rohilkhand, 08B)**

4. Find the intrinsic equation of the catenary $y = c \cosh(x/c)$.
(Rohilkhand 2007; Kumaun 08; Kanpur 14)

Hence show that $c\rho = c^2 + s^2$, where ρ is the radius of curvature.

5. Prove that the intrinsic equation of the curve
$x = a\,(1 + \sin t),\ y = a\,(1 + \cos t)$ is $s + a\psi = 0$.

6. Find the intrinsic equation of the cardioid $r = a\,(1 - \cos\theta)$.
(Meerut 2007B; Avadh 05, 12; Rohilkhand 12)

7. Find the intrinsic equation of $r = a\,e^{\theta \cot \alpha}$, where s is measured from the point $(a, 0)$.

8. Find the intrinsic equation of the spiral $r = a\theta$, the arc being measured from the pole.

9. Find the intrinsic equation of the curve $p^2 = r^2 - a^2$.

10. In the four-cusped astroid $x^{2/3} + y^{2/3} = a^{2/3}$, show that

(i) $s = \frac{3}{4} a \cos 2\psi$, s being measured from the vertex;

(ii) $s = \frac{3}{2} a \sin^2\psi$, s being mesured from the cusp on x-axis; **(Purvanchal 2014)**

(iii) whole length of the curve is $6a$.

11. Find the cartesian equation of the curve whose intrinsic equation is $s = c \tan\psi$ when it is given that at $\psi = 0, x = 0$ and $y = c$.

Answers 4

2. $s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi), a \{\sqrt{2} + \log (1 + \sqrt{2})\}$.

4. $s = c \tan \psi$.

6. $s = 8a \sin^2 \frac{1}{6} \psi$.

7. $s = a \sec \alpha \, [e^{(\psi - \alpha) \cot \alpha} - 1]$.

8. $s = \frac{1}{2} a \, [\theta \sqrt{(1 + \theta^2)} + \log \{\theta + \sqrt{(1 + \theta^2)}\}]$, where $\psi = \theta + \tan^{-1} \theta$.

9. $s = \frac{1}{2} a \psi^2$.

11. $y = c \cosh (x/c)$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …", so that the following statements are complete and correct.

1. The process of finding the length of an arc of a curve between two given points is called ……. **(Kumaun 2008)**

2. The arc length of the curve $y = f(x)$ included between two points for which $x = a$ and $x = b$ $(b > a)$ is …… .

3. The arc length of the curve $y = \frac{1}{2} x^2 - \frac{1}{4} \log x$ from $x = 1$ to $x = 2$ is …… .

4. If $r = a \, e^{\theta \cot \alpha}$, then $ds =$ …… . **(Meerut 2001, 03)**

5. $\dfrac{ds}{dr} = \sqrt{\ldots\ldots}$ **(Meerut 2001)**

6. The length of an arch of the cycloid whose equations are $x = a \, (t - \sin t), y = a \, (1 - \cos t)$ is …… **(Agra 2005)**

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

7. If the equations of the curve be given in the parametric form $x = f(t), y = \phi \, (t)$, and the arc length s is measured in the direction of t increasing, then on integrating between the proper limits, the required length s is given as

(a) $\displaystyle\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \, dt$

(b) $\displaystyle\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)\right\}} \, dt$

(c) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^3 + \left(\frac{dy}{dt}\right)^3\right\}}\, dt$

(d) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^{1/2} + \left(\frac{dy}{dt}\right)^{1/2}\right\}}\, dt.$

(Meerut 2003)

8. For the curve $r = a\,(1 + \cos\theta)$, $\frac{ds}{d\theta}$ is

(a) $2 \cos\frac{1}{2}\theta$ (b) $2a \cos\frac{1}{2}\theta$

(c) $a \cos\frac{1}{2}\theta$ (d) $\frac{3}{2} a \cos\frac{1}{2}\theta$

9. If $x = a\cos^3 t$, $y = a\sin^3 t$, then $\left(\frac{ds}{dt}\right)^2$ is

(a) $(a \sin t \cos t)^2$ (b) $(\sin t \cos t)^2$

(c) $(3\,a \sin t \cos t)^2$ (d) $3a \sin t \cos t$

10. The entire length of the cardioid $r = a\,(1 + \cos\theta)$ is

(a) $8\,a$ (b) $4\,a$

(c) $6\,a$ (d) $2\,a$ **(Rohilkhand 2008)**

True or False:

Write 'T' for true and 'F' for false statement.

11. The length of the arc of the curve $x = f(y)$ between $y = a$ and $y = b$, $(b > a)$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy.$$

12. The relation between s and ψ for any curve is called its polar equation.

13. If the equation of the curve be $\theta = f(r)$, then the arc length from $r = r_1$ to $r = r_2$ is given by

$$\int_{r_1}^{r_2} \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}\, dr.$$

14. If the equation of the curve be $r = f(\theta)$, then the arc length from $\theta = \theta_1$ to $\theta = \theta_2$ is given by

$$\int_{\theta_1}^{\theta_2} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, dr.$$

(Meerut 2003)

Answers

1. Rectification.

2. $\int_a^b \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx.$

3. $\frac{3}{2}+\frac{1}{4}\log 2.$

4. $r \operatorname{cosec} \alpha \, d\theta.$

5. $\left\{1+\left(r\frac{d\theta}{dr}\right)^2\right\}.$

6. $8a.$

7. (a).

8. (b).

9. (c).

10. (a).

11. *T*.

12. *F*.

13. *T*.

14. *F*.

Chapter 8

Volumes and Surfaces of Solids of Revolution

8.1 Revolution : Definitions

Solid of Revolution : If a plane area is revolved about a fixed line lying in its own plane, then the body so generated by the revolution of the plane area is called a solid of revolution.

Surface of Revolution : If a plane curve is revolved about a fixed line lying in its own plane, then the surface generated by the perimeter of the curve is called a surface of revolution.

Axis of Revolution : The fixed straight line, say AB, about which the area revolves is called the axis of revolution or axis of rotation.

8.2 Volumes of Solids of Revolution

(a) The axis of rotation being *x*-axis :

If a plane area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis revolves about the x-axis then the volume of the solid thus generated is

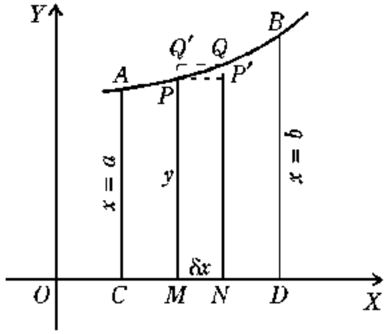
$$\int_a^b \pi y^2 \, dx = \int_a^b \pi \, [f(x)]^2 \, dx,$$

where $y = f(x)$ is a finite, continuous and single valued function of x in the interval $a \le x \le b$.

Or

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ about the x-axis is $\int_a^b \pi y^2 \, dx$.

Proof : Let AB be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut the x-axis and $f(x)$ is a continuous function of x in the interval (a, b).



Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$. Draw the ordinates PM and QN. Also draw PP' and QQ' perpendiculars to these ordinates.

Let V denote the volume of the solid generated by the revolution of the area $ACMP$ about the x-axis and let the volume of revolution obtained by revolving the area $ACNQ$ about x-axis be $V + \delta V$, so that volume of the solid generated by the revolution of the strip $PMNQ$ about the x-axis is δV.

Now $PM = y$, $QN = y + \delta y$ and $MN = (x + \delta x) - x = \delta x$. Then the volume of the solid generated by revolving the area $PMNP' = \pi y^2 \delta x$ and the volume of the solid generated by revolving the area $Q'MNQ = \pi (y + \delta y)^2 \delta x$.

Also the volume of the solid generated by the revolution of the area $PMNQP$ (*i.e.*, the volume δV) lies between the volumes of the right circular cylinders generated by the revolution of the areas $PMNP'P$ and $MNQQ'$ *i.e.*, δV lies between $\pi y^2 \delta x$ and $\pi (y + \delta y)^2 \delta x$

or $\qquad (\delta V/\delta x)$ lies between πy^2 and $\pi (y + \delta y)^2$

i.e., $\qquad \pi y^2 < (\delta V/\delta x) < \pi (y + \delta y)^2.$

In the limiting position as $Q \to P$, $\delta x \to 0$ (and therefore $\delta y \to 0$), we have

$$dV/dx = \pi y^2 \quad \text{or} \quad dV = \pi y^2 \, dx.$$

Hence $$\int_a^b \pi y^2 \, dx = \int_a^b dV = \Big[V \Big]_{x=a}^{x=b}$$

= (value of V for $x = b$) − (value of V for $x = a$)

= volume generated by the area $ACDB - 0$

= volume of the solid generated by the revolution of the given area $ACDB$ about the axis of x.

$\therefore$ the required volume $= \pi \int_a^b y^2 \, dx.$ **(Meerut 2003)**

(b) The axis of rotation being y-axis :

Similarly, it can be shown that the *volume of the solid generated by the revolution about y-axis of the area between the curve $x = f(y)$, the y-axis and the two abscissae $y = a$ and $y = b$ is given by* $\int_a^b \pi x^2 \, dy$.

Important Remarks :

(i) If the given curve is symmetrical about x-axis and we have to find the volume generated by the revolution of the area about x-axis, then in such case we shall revolve only one of the two symmetrical areas and **shall not double it** as in the case of area or length. Obviously each of the two symmetrical parts will generate the same volume.

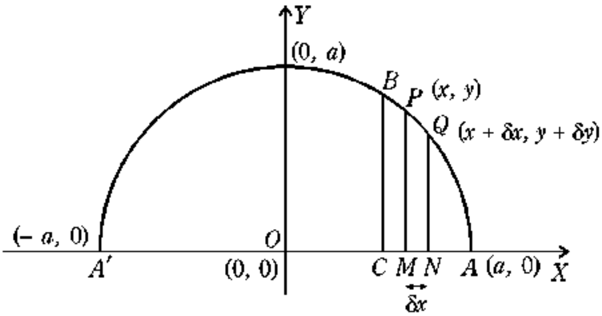
(ii) If the curve is symmetrical about x-axis and it is required to find the volume generated by the revolution of the area about y-axis, then the volume generated **will be twice** the volume generated by half of the symmetrical portion of the curve.

Illustrative Examples

Example 1 : *Show that the volume of a sphere of radius a is* $\frac{4}{3}\pi a^3$.

(Bundelkhand 2010; Avadh 10)

Solution : The sphere is generated by the revolution of a semi-circular area about its bounding diameter. The equation of the generating circle of radius a and centre as origin is $x^2 + y^2 = a^2$.



Let AA' be the bounding diameter about which the semi-circle revolves.

Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.

We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the diameter AA' is

$$= \pi . PM^2 . MN = \pi y^2 \delta x = \pi (a^2 - x^2) \delta x.$$

Also the semi-circle is symmetrical about the y-axis and for the portion of the curve lying in the first quadrant x varies from 0 to a.

$\therefore$ the required volume of the sphere

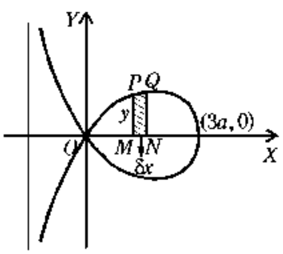
$$= 2\int_0^a \pi (a^2 - x^2)\, dx = 2\pi \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = 2\pi \left[a^3 - \frac{1}{3} a^3\right] = \frac{4}{3}\pi a^3.$$

Example 2 : *The curve* $y^2 (a + x) = x^2 (3a - x)$ *revolves about the axis of* x. *Find the volume generated by the loop.* **(Meerut 2004; Bundelkhand 05)**

Solution : The given curve is $y^2 (a + x) = x^2 (3a - x)$. ...(1)

It is symmetrical about x-axis. Putting $y = 0$ in (1), we get $x = 0$ and $x = 3a$ *i.e.*, a loop is formed between $(0, 0)$ and $(3a, 0)$.

The volume generated by the revolution of the whole loop about x-axis is the same as the volume generated by the revolution of the upper half of the loop about x-axis.



Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of x is

$$= \pi PM^2 . MN = \pi y^2 \delta x.$$

$\therefore$ the required volume generated by the loop

$$= \int_0^{3a} \pi y^2 \, dx = \pi \int_0^{3a} \frac{x^2 (3a - x)}{a + x} \, dx, \qquad \text{from (1)}$$

$$= \pi \int_0^{3a} \left\{ - x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a} \right\} dx, \text{ dividing the Nr. by the Dr.}$$

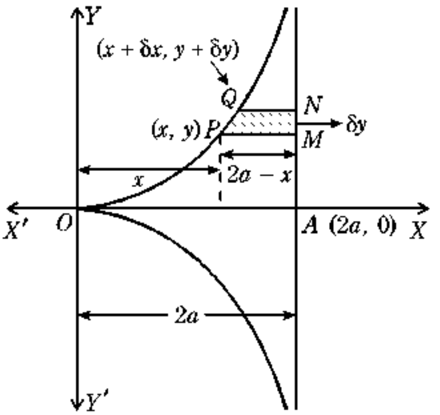
$$= \pi \left[- \frac{x^3}{3} + \frac{4ax^2}{2} - 4a^2 x + 4a^3 \log (x + a) \right]_0^{3a}$$

$$= \pi \left[- 9a^3 + 18a^3 - 12a^3 + 4a^3 (\log 4a - \log a) \right]$$

$$= \pi \left[- 3a^3 + 4a^3 \log 4 \right] = \pi a^3 \left[8 \log 2 - 3 \right].$$

Example 3 : *Find the volume of the solid generated by the revolution of the cissoid $y^2 (2a - x) = x^3$ about its asymptote.* **(Meerut 2007; Kanpur 14)**

Solution : The given curve is $y^2 (2a - x) = x^3$. Its shape is as shown in the figure. Equating to zero the coefficient of highest power of y, the asymptote parallel to the axis of y is $x = 2a$. Take an elementary strip $PMNQ$ perpendicular to the asymptote $x = 2a$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.



We have $PM = 2a - x$ and $MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = 2a$ is

$$= \pi \, . \, PM^2 . \, MN = \pi (2a - x)^2 \, \delta y.$$

The given curve is symmetrical about x-axis and for the portion of the curve above x-axis y varies from 0 to ∞.

$$\therefore \text{ the required volume} = 2 \int_{y=0}^{\infty} \pi (2a - x)^2 \, dy. \qquad \text{...(1)}$$

From the given equation of the curve $y^2 (2a - x) = x^3$ we observe that the value of x cannot be easily found in terms of y. Hence for the sake of integration we change the independent variable from y to x. **(Note)**

The curve is $y^2 = \dfrac{x^3}{2a - x}$;

$$\therefore \qquad 2y \frac{dy}{dx} = \frac{(2a - x) \, . \, 3x^2 - x^3 (-1)}{(2a - x)^2} = \frac{2 (3a - x) x^2}{(2a - x)^2}$$

or
$$dy = \frac{(3a - x) x^2}{(2a - x)^2} \cdot \frac{\sqrt{(2a - x)}}{x \sqrt{x}} \, dx = \frac{(3a - x) \sqrt{x} \sqrt{(2a - x)}}{(2a - x)^2} \, dx.$$

Also when $y = 0, x = 0$ and when $y \to \infty, x \to 2a$.

Hence from (1), the required volume

$$= 2\pi \int_{x=0}^{2a} (2a - x)^2 \left[\frac{(3a - x) \sqrt{x} \sqrt{(2a - x)}}{(2a - x)^2} \right] dx$$

$$= 2\pi \int_0^{2a} (3a - x)\sqrt{x}\sqrt{(2a - x)}\, dx.$$

Now put $x = 2a \sin^2 \theta$ so that $dx = 4a \sin\theta \cos\theta\, d\theta$. When $x = 0, \theta = 0$ and when $x = 2a, \theta = \pi/2$. Therefore the required volume

$$= 2\pi \int_0^{\pi/2} (3a - 2a \sin^2 \theta)\sqrt{(2a)} \sin\theta \sqrt{[2a(1 - \sin^2\theta)]} \times 4a \sin\theta \cos\theta\, d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2} (3 \sin^2\theta \cos^2\theta - 2 \sin^4\theta \cos^2\theta)\, d\theta$$

$$= 16\pi a^3 \left[\frac{3\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)} - \frac{2\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(4)}\right]$$

$$= 16\pi a^3 \left[\frac{3 \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2.2.1} - \frac{2 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2.3.2.1}\right]$$

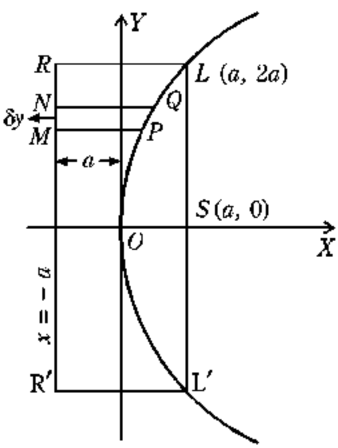
$$= 16\pi a^3 \left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2 a^3.$$

Note : If the given curve is $y^2(a - x) = x^3$, then the required volume can be obtained by putting a for $2a$ in the above Exercise. The volume so obtained is $\frac{1}{4}\pi^2 a^3$.

Important Remark : *When we are to revolve an area about a line which is neither the x-axis nor the y-axis we must take an elementary strip which is perpendicular to the line of revolution as explained in the above example.*

Example 4 : *The area between a parabola and its latus rectum revolves about the directrix. Find the ratio of the volume of the ring thus obtained to the volume of the sphere whose diameter is the latus rectum.*

Solution : Let the parabola be $y^2 = 4ax$. Then the directrix is the line $x = -a$. Let LL' be the latus rectum. The area $LOL'SL$ is revolved about the directrix. The volume of the ring thus obtained = the volume V_1 of the cylinder formed by the revolution of the rectangle $LL'R'R$ about the directrix – the volume V_2 of the reel formed by the revolution of the arc LOL' about the directrix.



Now the volume V_1 of the cylinder $= \pi r^2 h = \pi (LR)^2 . LL' = \pi (2a)^2 . 4a = 16\pi a^3$.

To find the volume V_2 of the reel consider an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc OL and PM, QN are perpendiculars from P and Q on the directrix.

We have $PM = a + x$ and $MN = \delta y$.

$\therefore$ the volume V_2 of the reel

$$= 2\int_0^{2a} \pi (a + x)^2\, dy \qquad \text{[By symmetry about } x\text{-axis]}$$

$$= 2\int_0^{2a} \pi\,(a^2 + 2ax + x^2)\,dy = 2\pi \int_0^{2a} \left(a^2 + 2a \cdot \frac{y^2}{4a} + \frac{y^4}{16a^2}\right) dy$$

$[\because x = y^2/4a]$

$$= 2\pi \left[a^2 y + \frac{1}{2}\frac{y^3}{3} + \frac{1}{16a^2} \cdot \frac{y^5}{5}\right]_0^{2a}$$

$$= 2\pi \left[2a^3 + \frac{4}{3}a^3 + \frac{2}{5}a^3\right] = 2\pi a^3 \cdot \frac{56}{15} = \frac{112\pi a^3}{15}.$$

$\therefore$ Volume of the ring = volume of the cylinder − volume of the reel

$$= V_1 - V_2 = 16\pi a^3 - \frac{112}{15}\pi a^3 = \frac{128}{15}\pi a^3.$$

Volume of the sphere whose diameter is the latus rectum $4a$ *i.e.*, the radius is $2a$

$$= \tfrac{4}{3}\pi r^3 = \tfrac{4}{3}\pi\,(2a)^3 = \tfrac{32}{3}\pi a^3.$$

$\therefore$ the required ratio $= \dfrac{128\pi a^3/15}{32\pi a^3/3} = \dfrac{4}{5}\cdot$

Comprehensive Exercise 1

1. (i) Find the volume of a hemisphere.

(ii) Find the volume of a spherical cap of height h cut off from a sphere of radius a. **(Kanpur 2010)**

2. (i) A segment is cut off from a sphere of radius a by a plane at a distance $\frac{1}{2}a$ from the centre. Show that the volume of the segment is 5/32 of the volume of the sphere.

(ii) The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

3. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle about the major axis. **(Rohilkhand 2010)**

4. (i) Find the volume of the solid generated by the revolution of an arc of the catenary $y = c \cosh (x/c)$ about the x-axis. **(Meerut 2009B; Purvanchal 11)**

(ii) Find the volume of the solid generated by the revolution of the curve $y = a^3/(a^2 + x^2)$ about its asymptote. **(Meerut 2009)**

5. If the hyperbola $x^2/a^2 - y^2/b^2 = 1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and two planes perpendicular to the axis of x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.

6. (i) Find the volume formed by the revolution of the loop of the curve $y^2 (a + x) = x^2 (a - x)$ about the axis of x. **(Kanpur 2008)**

(ii) Find the volume of the solid generated by the revolution of the loop of the curve $y^2 = x^2 (a - x)$ about the axis of x. **(Kanpur 2011)**

7. Show that the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2 (2 - x)$ about x-axis is $\frac{4}{3}\pi$. **(Meerut 2005)**

8. The area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant revolves about x-axis. Find the volume of the solid generated. **(Agra 2014)**

9. Find the volume of the solid obtained by revolving the loop of the curve

$$a^2 y^2 = x^2 (2a - x)(x - a) \text{ about } x\text{-axis.}$$

10. A basin is formed by the revolution of the curve $x^3 = 64y$, $(y > 0)$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water it will hold ?

11. Show that the volume of the solid generated by the revolution of the curve $(a - x) y^2 = a^2 x$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.
(Meerut 2004B, 06B; Kumaun 08; Rohilkhand 12)

12. The figure bounded by a quadrant of a circle of radius a and tangents at its extremities revolves about one of the tangents. Prove that the volume of the solid generated is $(\frac{5}{3} - \frac{1}{2}\pi)\pi a^3$.

13. The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated. **(Rohilkhand 2008; Bundelkhand 09)**

Answers 1

1. (i) $\frac{2}{3}\pi a^3$. (ii) $\pi h^2 \left[a - \frac{1}{3} h\right]$.

2. (ii) $\frac{4}{5}\pi a^3$.

4. (i) $\dfrac{\pi c^2}{2}\left[x + \dfrac{c}{2}\sinh\dfrac{2x}{c}\right]$. (ii) $\dfrac{\pi^2 a^3}{2}$.

6. (i) $2a^3\pi\left[\log 2 - \frac{2}{3}\right]$. (ii) $\dfrac{1}{12}\pi a^4$.

8. $\dfrac{16}{105}\pi a^3$.

9. $\dfrac{23}{60}\pi a^3$.

10. $\dfrac{1536}{5}\pi$ cubic inches.

13. $\dfrac{2}{75}\sqrt{5}\,\pi a^3$.

8.3 Volume of a Solid of Revolution when the Equations of the Generating Curve are given in Parametric Form

(i) If the curve is given by the parametric equations, say $x = \phi(t)$, $y = \psi(t)$, then the volume of the solid generated by the revolution about ***x*-axis** of the area bounded by the curve, the axis of x and the ordinates at the points where $t = a$ and $t = b$ is

$$= \int_a^b \pi y^2 \frac{dx}{dt}\, dt = \pi \int_a^b \{\psi(t)\}^2 \phi'(t)\, dt.$$

(ii) The volume of the solid generated by the revolution about y-axis of the area between the curve $x = \phi(t)$, $y = \psi(t)$, the y-axis and the abscissae at the points where $t = a$, $t = b$ is

$$= \int_a^b \pi x^2 \frac{dy}{dt}\, dt = \pi \int_a^b \{\phi(t)\}^2 . \psi'(t)\, dt.$$

Illustrative Examples

Example 1 : *Find the volume of the solid formed by revolving the cycloid*

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

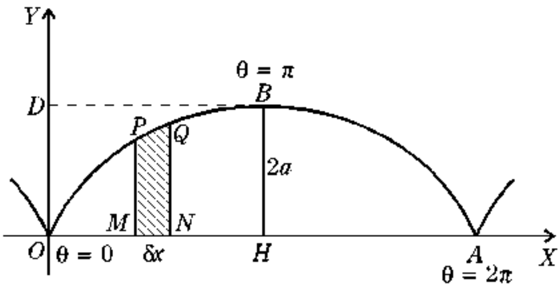
(*i*) *about its base* (*ii*) *about the y-axis.*

Solution : The given equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta). \quad \text{...(1)}$$

(i) The arc OBA is revolved about the base *i.e.*, the x-axis. For the arc OBA, θ varies from 0 to 2π and at B, $\theta = \pi$.

Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.



We have $PM = y$ and $MN = \delta x$.

Now the volume of the elementary disc formed by revolving the strip $PMNQ$ about the base (*i.e.*, the x-axis) is $\pi\, PM^2 . MN = \pi y^2 \delta x$.

Now the cycloid is symmetrical about the line BH.

$\therefore$ the required volume

$$= 2\int \pi y^2\, dx, \text{ the limits of integration being extended from } O \text{ to } B$$

$$= 2\pi \int_{\theta=0}^{\pi} y^2 \frac{dx}{d\theta}\, d\theta = 2\pi \int_0^{\pi} a^2 (1 - \cos\theta)^2 a(1 - \cos\theta)\, d\theta \quad [\text{From (1)}]$$

$$= 2\pi \int_0^{\pi} a^3 (1 - \cos\theta)^3\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^3 d\theta = 16\pi a^3 \int_0^{\pi} \sin^6\frac{\theta}{2}\, d\theta$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6\phi\, d\phi, \text{ putting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2\, d\phi$$

$$= 32\pi a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = 5\pi^2 a^3.$$

(ii) When the curve revolves about y-axis, the required volume of the solid generated

= the volume generated by the revolution of the area $OABDO$ about y-axis

− the volume generated by the revolution of the area $OBDO$ about the y-axis. ...(2)

Also at $A, \theta = 2\pi$; at $B, \theta = \pi$ and at $O, \theta = 0$.

Now the area $OABD$ is bounded by the arc AB of the cycloid and the axis of y. Therefore volume of the solid generated by the revolution of the area $OABDO$ about y-axis

$$= \int_{\theta=2\pi}^{\pi} \pi x^2\, dy = \int_{\theta=2\pi}^{\pi} \pi x^2 \frac{dy}{d\theta}\, d\theta$$

$$= \pi \int_{2\pi}^{\pi} a^2 (\theta - \sin\theta)^2 a \sin\theta\, d\theta \qquad \text{[From (1)]}$$

$$= \pi \int_{2\pi}^{\pi} a^2 (\theta^2 - 2\theta\sin\theta + \sin^2\theta)\, a\sin\theta\, d\theta$$

$$= \pi a^3 \int_{2\pi}^{\pi} (\theta^2\sin\theta - 2\theta\sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3 \int_{2\pi}^{\pi} [\theta^2\sin\theta - \theta(1-\cos 2\theta) + \tfrac{1}{4}(3\sin\theta - \sin 3\theta)]\, d\theta \qquad \textbf{(Note)}$$

$$= \pi a^3 \Big[\theta^2 . (-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \tfrac{1}{2}\theta^2 + \theta(\tfrac{1}{2}\sin 2\theta) - 1(-\tfrac{1}{4}\cos 2\theta) - \tfrac{3}{4}\cos\theta + \tfrac{1}{12}\cos 3\theta\Big]_{2\pi}^{\pi},$$

the values of the integrals $\int \theta^2 \sin\theta\, d\theta$ and $\int \theta\cos 2\theta\, d\theta$e have been written after applying integration by parts

$$= \pi a^3 [(\pi^2 - 2 - \tfrac{1}{2}\pi^2 + \tfrac{1}{4} + \tfrac{3}{4} - \tfrac{1}{12}) - (-4\pi^2 + 2 - 2\pi^2 + \tfrac{1}{4} - \tfrac{3}{4} + \tfrac{1}{12})]$$

$$= \pi a^3 [\tfrac{13}{2}\pi^2 - \tfrac{8}{3}]. \qquad ...(3)$$

Again volume of the solid generated by the revolution of the area $OBDO$ about y-axis

$$= \int_{\theta=0}^{\pi} \pi x^2\, dy = \int_{\theta=0}^{\pi} \pi x^2 \frac{dy}{d\theta}\, d\theta = \pi \int_0^{\pi} a^2(\theta - \sin\theta)^2 . a\sin\theta\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 - 2\theta\sin\theta + \sin^2\theta)\sin\theta\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2\sin\theta - 2\theta\sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3 \int_0^{\pi} [\theta^2\sin\theta - \theta(1-\cos 2\theta) + \tfrac{1}{4}(3\sin\theta - \sin 3\theta)]\, d\theta$$

$$= \pi a^3 \Big[\theta^2(-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \tfrac{1}{2}\theta^2 + \theta(\tfrac{1}{2}\sin 2\theta) - 1(-\tfrac{1}{4}\cos 2\theta) - \tfrac{3}{4}\cos\theta + \tfrac{1}{12}\cos 3\theta\Big]_0^{\pi}$$

$$= \pi a^3 \left[\left(\pi^2 - 2 - \tfrac{1}{2}\pi^2 + \tfrac{1}{4} + \tfrac{3}{4} - \tfrac{1}{12}\right) - \left(2 + \tfrac{1}{4} - \tfrac{3}{4} + \tfrac{1}{12}\right)\right]$$

$$= \pi a^3 \left(\tfrac{1}{2}\pi^2 - \tfrac{8}{3}\right). \qquad \text{...(4)}$$

$\therefore$ from (2), the required volume $=$ (3) $-$ (4)

$$= \pi a^3 \left[\tfrac{13}{2}\pi^2 - \tfrac{8}{3}\right] - \pi a^3 \left[\tfrac{1}{2}\pi^2 - \tfrac{8}{3}\right] = \pi a^3 [6\pi^2] = 6\pi^3 a^3.$$

Example 2 : *Find the volume of the solid generated by the revolution of the tractrix* $x = a\cos t + \frac{1}{2} a \log \tan^2 (t/2)$, $y = a \sin t$ *about its asymptote.*

(Meerut 2000, 05B; Rohilkhand 06; Avadh 09, 11; Kashi 12; Purvanchal 14)

Solution : The given curve is

$$x = a\cos t + \tfrac{1}{2} a \log \tan^2 (t/2), \; y = a \sin t. \qquad \text{...(1)}$$

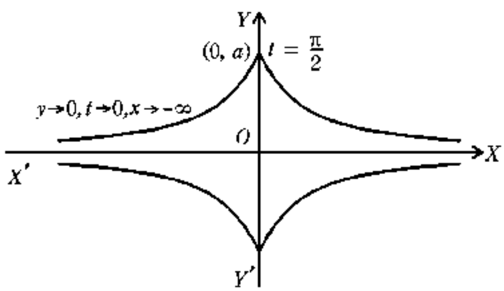
$$\therefore \quad \frac{dx}{dt} = -a \sin t + \frac{1}{2} a \cdot \frac{1}{\tan^2 (t/2)} \cdot 2 \tan (t/2) \sec^2 (t/2) \cdot \frac{1}{2}$$

$$= -a \sin t + \frac{a}{2 \sin (t/2) \cos (t/2)} = -a \sin t + \frac{a}{\sin t}$$

$$= a \frac{(1 - \sin^2 t)}{\sin t} = a \frac{\cos^2 t}{\sin t} \qquad \text{...(2)}$$

Now the given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, x-axis.

For the portion of the curve lying in the second quadrant y varies from a to 0, t varies from $\pi/2$ to 0 and x varies from 0 to $-\infty$.



$\therefore$ the required volume

$$= 2 \int_{-\infty}^{0} \pi y^2 \, dx$$

$$= 2 \int_{0}^{\pi/2} \pi y^2 \frac{dx}{dt} \cdot dt$$

$$= 2\pi \int_{0}^{\pi/2} a^2 \sin^2 t \,.\, \frac{a \cos^2 t}{\sin t} \, dt \qquad \text{[From (1) and (2)]}$$

$$= 2\pi a^3 \int_{0}^{\pi/2} \cos^2 t \sin t \, dt = 2\pi a^3 \frac{1}{3.1} = \frac{2}{3} \pi a^3.$$

Comprehensive Exercise 2

1. Find the volume of the solid generated by the revolution of the cycloid

$$x = a(\theta + \sin\theta), \; y = a(1 - \cos\theta), \; -\pi \le \theta \le \pi,$$

(i) about the x-axis, (ii) about the base.

2. Show that the volume of the solid generated by the revolution of the cycloid

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta), 0 \leq \theta \leq \pi,$$

about the y-axis is $\pi a^3 \left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$.

3. Prove that the volume of the reel formed by the revolution of the cycloid

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$$

about the tangent at the vertex is $\pi^2 a^3$.

4. Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2, y = t - \frac{1}{3}t^3$ is $\frac{3}{4}\pi$.

5. Find the volume of the spindle shaped solid generated by revolving the astroid

$x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.

6. Find the volume of the solid generated by the revolution of the cissoid

$x = 2a\sin^2 t, y = 2a\sin^3 t/\cos t$ about its asymptote.

(Kanpur 2006; Bundelkhand 14)

Answers 2

1. (i) $\pi^2 a^3$, (ii) $5\pi^2 a^3$. **5.** $\frac{32}{105}\pi a^3$. **6.** $2\pi^2 a^3$.

8.4 Volume of Solid of Revolution when the Equation of the Generating Curve is given in Polar Co-ordinates

If the equation of the generating curve is given in polar co-ordinates, say $r = f(\theta)$, and the curve revolves about the axis of x, the volume generated

$$s = \pi \int_{x=a}^{b} y^2\, dx = \pi \int_{\theta=\alpha}^{\beta} y^2 \frac{dx}{d\theta}\, d\theta,$$

where α and β are the values of θ at the points where $x = a$ and $x = b$ respectively.

Now $x = r\cos\theta$ and $y = r\sin\theta$. Therefore the volume

$$= \pi \int_{\theta=\alpha}^{\beta} r^2 \sin^2\theta \frac{d}{d\theta}(r\cos\theta)\, d\theta,$$

in which the value of r in terms of θ must be substituted from the equation of the given curve.

A similar procedure can be adopted in case the curve revolves about the axis of y.

Alternative method in the case of polar curves :

The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ *and radii vectors* $\theta = \theta_1, \theta = \theta_2$

(i) about the initial line $\theta = 0$ (i.e., the x-axis) is $\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta\, d\theta,$

(ii) about the line $\theta = \pi/2$ (i.e., the y-axis) is $\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos\theta\, d\theta,$

(iii) about any line $(\theta = \gamma)$ **is** $\displaystyle\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin(\theta - \gamma)\, d\theta,$

where in each of the above three formulae the value of r in terms of θ must be substituted from the equation of the given curve.

Note : The above results are important and should be committed to memory.

8.5 Volume of the Solid Generated by the Revolution when The Axis of Rotation being any Line

If, however, the axis of rotation is neither x-axis nor y-axis, but is any other line CD, then *the volume of the solid generated by the revolution about CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD on the aixs is*

$$\int_{OC}^{OD} \pi\,(PM)^2\, d\,(OM),$$

where PM is the perpendicular drawn from any point P on the curve to the axis of rotation and O is some fixed point on the axis of rotation.

Illustrative Examples

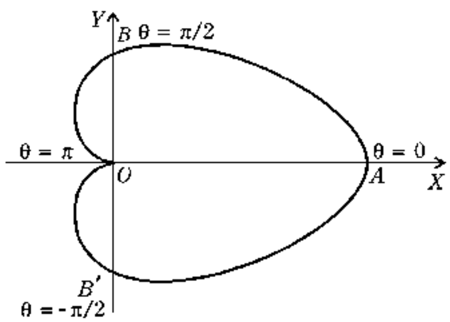
Example 1 : *The cardioid* $r = a(1 + \cos\theta)$ *revolves about the initial line. Find the volume of the solid thus generated.*

(Meerut 2001, 03, 07B; Agra 06, 07, 08; Rohilkhand 13, 13B)

Solution : The given curve is $r = a(1 + \cos\theta)$. ...(1)

It is symmetrical about the initial line. We have $r = 0$ when $\cos\theta = -1$ *i.e.*, $\theta = \pi$.

Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = 2a$. As θ increases from 0 to π, r decreases from $2a$ to 0. Hence the shape of the curve is as shown in the figure. For the upper half of the curve, θ varies from 0 to π.



$\therefore$ the required volume

$$= \frac{2}{3}\int_0^{\pi} \pi r^3 \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1 + \cos\theta)^3 \sin\theta\, d\theta \quad \text{[From (1)]}$$

$$= -\frac{2}{3}\pi a^3 \int_0^{\pi} (1 + \cos\theta)^3 (-\sin\theta)\, d\theta \qquad \textbf{(Note)}$$

$$= -\frac{2}{3}\pi a^3 \left[\frac{(1 + \cos\theta)^4}{4}\right]_0^{\pi}, \text{ using power formula}$$

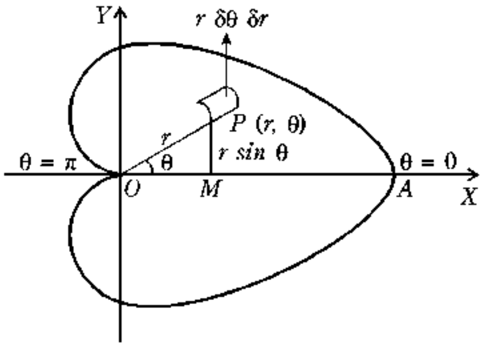
$$i.e., \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= -\frac{1}{6}\pi a^3 (0 - 2^4) = \frac{8}{3}\pi a^3.$$

Aliter : (By double integration)

Take a small element $r\delta\theta\,\delta r$ at any point $P(r,\theta)$ lying within the area of the upper half of the cardioid. Draw PM perpendicular to OX. Then $PM = r\sin\theta$. The volume of the elementary ring formed by revolving the element $r\,\delta\theta\,\delta r$ about OX

$$= 2\pi\,(r\sin\theta)\,r\,\delta\theta\,\delta r$$
$$= 2\pi r^2 \sin\theta\,\delta\theta\,\delta r.$$



$\therefore$ the required volume formed by revolving the whole cardioid about the initial line

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\,d\theta\,dr$$

$$= \int_0^{\pi} 2\pi\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta\,d\theta = \frac{2\pi}{3}\int_0^{\pi} a^3(1+\cos\theta)^3\sin\theta\,d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^{\pi}(1+\cos\theta)^3(-\sin\theta)\,d\theta = -\frac{2\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= -\frac{2\pi a^3}{3}\cdot\frac{1}{4}[0-2^4] = \frac{2}{3}\cdot\pi a^3\cdot\frac{1}{4}\cdot 16 = \frac{8}{3}\pi a^3.$$

Example 2 : *Find the volume of the solid formed by revolving one loop of the curve* $r^2 = a^2\cos 2\theta$ *about the initial line.* **(Rohilkhand 2007B)**

Solution : For the upper half of the loop θ varies from 0 to $\pi/4$. Here the curve is revolving about the initial line (*i.e.*, x-axis).

$$\therefore \text{ the required volume} = \frac{2}{3}\pi\int_0^{\pi/4} r^3\sin\theta\,d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi/4}\{a\sqrt{(\cos 2\theta)}\}^3\sin\theta\,d\theta \qquad [\because r^2 = a^2\cos 2\theta]$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi/4}(2\cos^2\theta-1)^{3/2}\sin\theta\,d\theta. \qquad \textbf{(Note)}$$

Put $\sqrt{2}\cos\theta = \sec\phi$ so that $-\sqrt{2}\sin\theta\,d\theta = \sec\phi\tan\phi\,d\phi$.

When $\theta = 0, \phi = \pi/4$ and when $\theta = \pi/4, \phi = 0$.

$\therefore$ the required volume

$$= \frac{2\pi a^3}{3}\int_{\pi/4}^{0}(\sec^2\phi-1)^{3/2}\frac{(-\sec\phi\tan\phi)}{\sqrt{2}}\,d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}\tan^4\phi\sec\phi\,d\phi = \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}(\sec^2\phi-1)^2\sec\phi\,d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4}(\sec^5\phi-2\sec^3\phi+\sec\phi)\,d\phi. \qquad ...(1)$$

Also we know the reduction formula

$$\int \sec^n \phi \, d\phi = \frac{\sec^{n-2} \phi \tan \phi}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} \phi \, d\phi.$$

[Establish it here]

$$\therefore \qquad \int_0^{\pi/4} \sec^5 \phi \, d\phi = \left[\frac{\sec^3 \phi \tan \phi}{4} \right]_0^{\pi/4} + \frac{3}{4} \int_0^{\pi/4} \sec^3 \phi \, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4} \left\{ \left[\frac{\sec \phi \tan \phi}{2} \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec \phi \, d\phi \right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4} \left\{ \frac{\sqrt{2}}{2} + \frac{1}{2} \left[\log (\sec \phi + \tan \phi) \right]_0^{\pi/4} \right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8} \log (\sqrt{2} + 1) = \frac{7\sqrt{2}}{8} + \frac{3}{8} \log (\sqrt{2} + 1),$$

$$\int_0^{\pi/4} \sec^3 \phi \, d\phi = \left[\frac{\sec \phi \tan \phi}{2} \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \sec \phi \, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2} \log (\sqrt{2} + 1)$$

and $$\int_0^{\pi/4} \sec \phi \, d\phi = \log (\sqrt{2} + 1).$$

Hence the required volume from (1) is

$$= \frac{\sqrt{2}\pi a^3}{3} \left[\frac{7\sqrt{2}}{8} + \frac{3}{8} \log (\sqrt{2} + 1) - 2 \left\{ \frac{\sqrt{2}}{2} + \frac{1}{2} \log (\sqrt{2} + 1) \right\} + \log (\sqrt{2} + 1) \right]$$

$$= \frac{\sqrt{2}\pi a^3}{3} \left[\frac{3}{8} \log (\sqrt{2} + 1) - \frac{\sqrt{2}}{8} \right]$$

$$= \frac{\pi a^3 \sqrt{2}}{24} [3 \log (\sqrt{2} + 1) - \sqrt{2}].$$

Aliter : The equation of the given curve is

$$r^2 = a^2 \cos 2\theta \quad \text{or} \quad r^4 = a^2 r^2 (\cos^2 \theta - \sin^2 \theta).$$

Changing to cartesians, the equation becomes

$$(x^2 + y^2)^2 = a^2 (x^2 - y^2) \quad \text{or} \quad y^4 + y^2 (2x^2 + a^2) + x^4 - a^2 x^2 = 0.$$

Solving for y^2, we have

$$y^2 = [-(2x^2 + a^2) \pm \sqrt{\{(2x^2 + a^2)^2 - 4(x^4 - a^2 x^2)\}}] / 2.$$

Neglecting the negative sign because y^2 cannot be –ive, we have

$$y^2 = \frac{-(2x^2 + a^2) + \sqrt{(8a^2 x^2 + a^4)}}{2} = \frac{-(2x^2 + a^2) + 2\sqrt{2}a\sqrt{(x^2 + \frac{1}{8}a^2)}}{2}.$$

Now for one loop of the given curve x varies from 0 to a.

$$\therefore \text{the required volume} = \pi \int_0^a y^2 \, dx$$

$$= \frac{\pi}{2} \int_0^a [-2x^2 - a^2 + 2\sqrt{2}a\sqrt{(x^2 + \tfrac{1}{8}a^2)}] \, dx$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}x^3 - a^2x + 2\sqrt{2}a \cdot \frac{x}{2}\sqrt{(x^2 + \tfrac{1}{8}a^2)}\right.$$
$$\left. + 2\sqrt{2}a \cdot \frac{1}{16}a^2 \log\{x + \sqrt{(x^2 + \tfrac{1}{8}a^2)}\}\right]_0^a$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}a^3 - a^3 + 2\sqrt{2}a \cdot \frac{a}{2} \cdot \frac{3a}{2\sqrt{2}}\right.$$
$$\left. + \frac{1}{8}\sqrt{2}a^3\left\{\log\left(a + \frac{3a}{2\sqrt{2}}\right) - \log\frac{a}{2\sqrt{2}}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{5}{3}a^3 + \frac{3}{2}a^3 + \frac{1}{8}\sqrt{2}a^3 \log\left\{\frac{a(2\sqrt{2} + 3)}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{a}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3 \log(2\sqrt{2} + 3)\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3 \log(\sqrt{2} + 1)^2\right]$$

$$= \frac{\pi a^3}{2}\left[2 \cdot \frac{1}{8}\sqrt{2}\log(\sqrt{2} + 1) - \frac{1}{6}\right] = \frac{\pi a^3}{2}\left[\frac{1}{4}\sqrt{2}\log(\sqrt{2} + 1) - \frac{1}{6}\right]$$

$$= \frac{\pi a^3}{24}\left[3\sqrt{2}\log(\sqrt{2} + 1) - 2\right] = \frac{\pi a^3\sqrt{2}}{24}\left[3\log(\sqrt{2} + 1) - \sqrt{2}\right].$$

Example 3 : *Show that if the area lying within the cardioid* $r = 2a(1 + \cos\theta)$ *and without the parabola* $r(1 + \cos\theta) = 2a$ *revolves about the initial line, the volume generated is* $18\pi a^3$.

Solution : The equation of the cardioid is

$$r = 2a(1 + \cos\theta), \qquad \text{...(1)}$$

and that of the parabola is $r = 2a/(1 + \cos\theta)$. ...(2)

Equating the values of r from (1) and (2), we get

$$2a(1 + \cos\theta) = 2a/(1 + \cos\theta)$$

or $\quad (1 + \cos\theta)^2 = 1$

or $\quad \cos\theta(\cos\theta + 2) = 0.$

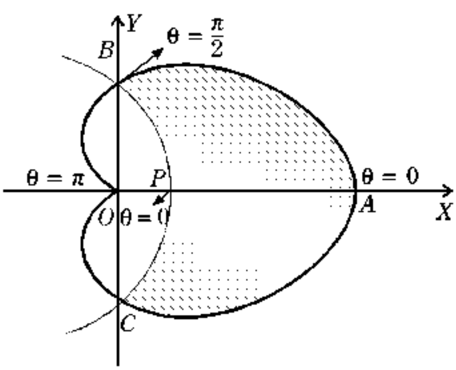
Now $\cos\theta \neq -2$.

Therefore $\cos\theta = 0$ *i.e.*, $\theta = \pi/2, -\pi/2$.

Thus the curves (1) and (2) intersect where $\theta = \pi/2$ and $\theta = -\pi/2$.

Also both the curves are symmetrical about the initial line (*i.e.*, x-axis). The required volume is generated by revolving the upper half of the shaded area about the initial line.

$\therefore$ the required volume = Volume generated by the revolution of the area $OABO$ of the cardioid − volume generated by the revolution of the area $OPBO$ of the parabola



$$= \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta - \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta$$

(for cardioid) (for parabola)

$$= \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right]\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\int_0^{\pi/2}[(1+\cos\theta)^3-(1+\cos\theta)^{-3}](-\sin\theta)\, d\theta \qquad \textbf{(Note)}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4} - \frac{(1+\cos\theta)^{-2}}{-2}\right]_0^{\pi/2}, \text{ using power formula}$$

$$= \frac{-16\pi a^3}{3}\left[\frac{1}{4}(1-16)+\frac{1}{2}\left(1-\frac{1}{4}\right)\right]$$

$$= \frac{-16}{3}\pi a^3\left[-\frac{15}{4}+\frac{3}{8}\right]$$

$$= \left(-\frac{16}{3}\pi a^3\right)\left(\frac{-27}{8}\right)$$

$$= 18\pi a^3.$$

Comprehensive Exercise 3

1. Find the volume of the solid generated by the revolution of $r = 2a\cos\theta$ about the initial line.
2. Find the volume of the solid generated by the revolution of the cardioid $r = a(1-\cos\theta)$ about the initial line. **(Rohilkhand 2010)**
3. The arc of the cardioid $r = a(1+\cos\theta)$, specified by $-\pi/2 \le \theta \le \pi/2$, is rotated about the line $\theta = 0$, prove that the volume generated is $\frac{5}{2}\pi a^3$.
4. Show that the volume of the solid formed by the revolution of the curve $r = a + b\cos\theta$ $(a > b)$ about the initial line is $\frac{4}{3}\pi a(a^2+b^2)$. **(Meerut 2008)**
5. Find the volume of the solid generated by revolving one loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{1}{2}\pi$. **(Meerut 2006)**

Answers 3

1. $\frac{4}{3}\pi a^3$. **2.** $\frac{8}{3}\pi a^3$. **4.** $\frac{4}{3}\pi a(a^2+b^2)$. **5.** $(\pi^2 a^3)/4\sqrt{2}$.

8.6 Surfaces of Solids of Revolution (Agra 2014)

(a) Revolution about the axis of x: *To prove that the curved surface of the solid generated by the revolution, about x-axis, of the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis is*

$$\int_{x=a}^{x=b} 2\pi y\, ds,$$

where s is the length of the arc measured from $x = a$ to any point (x, y).

Or

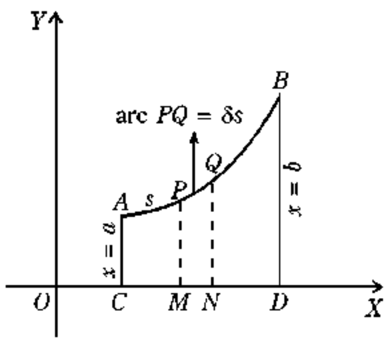
Show that the area of the surface of the solid obtained by revolving about x-axis the arc of the curve intercepted between the points whose abscissae are a and b is

$$\int_a^b 2\pi y \frac{ds}{dx}\, dx.$$

Proof : Let AB be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut x-axis and $f(x)$ is a continuous function of x in the interval (a, b).

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$.

Let the length of the arc AP be s and arc $AQ = s + \delta s$ so that arc $PQ = \delta s$.



Draw the ordinates PM and QN. Let S denote the curved surface of the solid generated by the revolution of the area $CMPA$ about the x-axis. Then the curved surface of the solid generated by the revolution of the area $MNQP = \delta S$.

We shall take it as an axiom that the curved surface of the solid generated by the revolution of the area $MNQP$ about the x-axis lies between the curved surfaces of the right circular cylinders whose radii are PM and NQ and which are of the same thickness (height) δs. There is no loss in assuming so because ultimately Q is to tend to P.

Thus δS lies between $2\pi y \delta s$ and $2\pi(y + \delta y)\delta s$

i.e., $\quad 2\pi y\,\delta s < \delta S < 2\pi(y + \delta y)\,\delta s \quad$ or $\quad 2\pi y < (\delta S/\delta s) < 2\pi(y + \delta y)$.

Now as Q approaches P *i.e.*, $\delta s \to 0$, δy will also tend to zero. Hence by taking limits as $\delta s \to 0$, we have

$$\frac{dS}{ds} = 2\pi y \quad \text{or} \quad dS = 2\pi y\, ds.$$

$$\therefore \quad \int_{x=a}^{x=b} 2\pi y\, ds = \int_{x=a}^{x=b} dS = \Big[\, S \,\Big]_{x=a}^{x=b}$$

$= $ (the value of S when $x = b$) $-$ (the value of S when $x = a$)

$= $ surface of the solid generated by the revolution of the area $ACDB - 0$.

$\therefore$ the required curved surface $= \displaystyle\int_{x=a}^{x=b} 2\pi y\, ds$

$$= \int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} dx, \text{ where } \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

(b) Axis of revolution as y-axis : *Similarly the curved surface of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and the y-axis is*

$$2\pi \int_{y=a}^{y=b} x\, ds \text{ or } S = 2\pi \int_{y=a}^{b} x \frac{ds}{dy} dy, \text{ where } \frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

Important Remark : If an arc length revolves about x-axis, the basic formula for the surface of revolution in all cases is $\int 2\pi y\, ds$, between the suitable limits. If we want to integrate w.r.t. x, we shall change ds as $(ds/dx)\, dx$ and adjust the limits accordingly.

A similar transformation can be made if we want to integrate w.r.t. y or with respect to θ or w.r.t. some parameter, say t.

Illustrative Examples

Example 1 : *Find the curved surface of a hemisphere of radius a.*

(Agra 2005; Kanpur 14)

Solution : A hemisphere is generated by the revolution of a quadrant of a circle about one of its bounding radii.

Let the equation of the circle be $x^2 + y^2 = a^2$. ...(1)

Let the hemisphere be formed by revolving about x-axis the arc of the circle (1) lying in the first quadrant.

Differentiating (1), w.r.t. x, we get

$$2x + 2y\,(dy/dx) = 0 \quad \text{or} \quad dy/dx = -\,x/y.$$

Therefore $\dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \dfrac{x^2}{y^2}\right\}}$

$$= \sqrt{\left\{\frac{y^2 + x^2}{y^2}\right\}} = \sqrt{\left(\frac{a^2}{y^2}\right)} \qquad \text{[From (1)]}$$

$$= a/y.$$

For the arc of the circle (1) lying in the first quadrant x varies from 0 to a.

$\therefore$ the required surface

$$= 2\pi \int_{x=0}^{a} y\, ds = 2\pi \int_0^a y \frac{ds}{dx} \cdot dx$$

$$= 2\pi \int_0^a y \cdot \frac{a}{y} dx = 2\pi \int_0^a a\, dx = 2\pi a \Big[\, x \,\Big]_0^a$$

$$= 2\pi a \,.\, a = 2\pi a^2.$$

Example 2 : *Find the surface generated by the revolution of an arc of the catenary $y = c \cosh (x/c)$ about the axis of x.* **(Meerut 2000, 04B, 07, 07B, 10; Rohilkhand 14)**

Solution : The given curve is, $y = c \cosh (x/c)$. ...(1)

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx} = c \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c} \cdot$$

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \sinh^2 \frac{x}{c}\right\}} = \cosh \frac{x}{c} \qquad ...(2)$$

If the arc be measured from the vertex ($x = 0$) to any point (x, y), then the required surface formed by the revolution of this arc about x-axis

$$= \int_{x=0}^{x} 2\pi y \frac{ds}{dx} dx = 2\pi \int_0^x c \cosh \frac{x}{c} \cdot \cosh \frac{x}{c} dx, \text{ from (1) and (2)}$$

$$= \pi c \int_0^x 2 \cosh^2 \frac{x}{c} dx = \pi c \int_0^x \left[1 + \cosh \frac{2x}{c}\right] dx \qquad \textbf{(Note)}$$

$$= \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^x = \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]$$

$$= \pi c \left[x + c \sinh \frac{x}{c} \cosh \frac{x}{c}\right].$$

Example 3 : *Prove that the surface of the prolate spheroid formed by the revolution of the ellipse of eccentricity e about its major axis is equal to* $2 \times$ *area of the ellipse* $\times$ $[\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e]$.

Solution : [Note : Prolate spheroid is generated by the revolution of an ellipse about its major axis]

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, ...(1)

the x-axis being the major axis so that $a > b$.

The parametric equations of (1) are $x = a \cos t, y = b \sin t$.

$\therefore \qquad dx/dt = -a \sin t$ and $dy/dt = b \cos t$.

We have $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)}$$

$$= \sqrt{\{a^2 \sin^2 t + a^2 (1 - e^2) \cos^2 t\}},$$

[$\because$ for the ellipse $b^2 = a^2 (1 - e^2)$]

$$= a\sqrt{(1 - e^2 \cos^2 t)}. \qquad ...(2)$$

Now the ellipse (1) is symmetrical about y-axis and for the arc of the ellipse lying in the first quadrant t varies from 0 to $\pi/2$. At the point $(a, 0)$ we have $t = 0$ and at the point $(0, b)$ we have $t = \pi/2$.

Hence the required surface S formed by the revolution of the ellipse (1) about the x-axis

$$= 2 \int 2\pi y \, ds, \text{ between the suitable limits}$$

$$= 4\pi \int_0^{\pi/2} y \frac{ds}{dt} dt = 4\pi \int_0^{\pi/2} b \sin t \,.\, a\sqrt{(1 - e^2 \cos^2 t)} \, dt,$$

[$\because$ $y = b \sin t$ and $ds/dt = a \sqrt{(1 - e^2 \cos^2 t)}$, from (2)]

$$= 4\pi\, ab \int_0^{\pi/2} \sin t\, \sqrt{(1 - e^2 \cos^2 t)}\, dt.$$

Put $e \cos t = z$ so that $-e \sin t\, dt = dz$. When $t = 0, z = e$ and when $t = \frac{1}{2}\pi$, $z = 0$.

$$\therefore \qquad S = -4\pi\, ab \int_e^0 \frac{1}{e} \sqrt{(1 - z^2)}\, dz = \frac{4\pi ab}{e} \int_0^e \sqrt{(1 - z^2)}\, dz$$

$$= \frac{4\pi ab}{e} \left[\frac{z}{2} \sqrt{(1 - z^2)} + \frac{1}{2} \sin^{-1} z \right]_0^e = \frac{4\pi ab}{e} \left[\frac{e}{2} \sqrt{(1 - e^2)} + \frac{1}{2} \sin^{-1} e \right]$$

$$= 2\pi ab \left[\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e \right]$$

$$= 2 \times \text{area of the ellipse} \times \left[\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e \right].$$

Remark : The solid of revolution formed by revolving an ellipse about its minor axis is called an **oblate spheroid**.

Example 4 : *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.*

(Bundelkhand 2011)

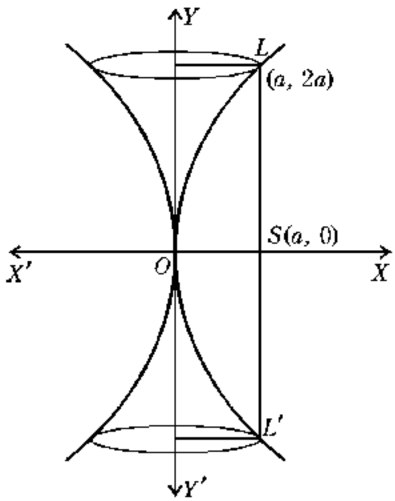
Solution : The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $dy/dx = 2a/y$.

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{4a^2}{y^2}\right\}}$$

$$= \sqrt{\left\{1 + \frac{4a^2}{4ax}\right\}} = \sqrt{\left(\frac{x + a}{x}\right)}.$$

The required curved surface is generated by the revolution of the arc LOL' (LSL' is the latus rectum), about the tangent at the vertex *i.e.*, y-axis. The curve is symmetrical about x-axis and for the arc OL, x varies from 0 to a.



$\therefore$ the required surface

$$= 2 \int_{x=0}^{a} 2\pi x \frac{ds}{dx}\, dx$$

$$= 4\pi \int_0^a x \sqrt{\left(\frac{x + a}{x}\right)}\, dx$$

$$= 4\pi \int_0^a \sqrt{(x^2 + ax)}\, dx$$

$$= 4\pi \int_0^a \sqrt{\left\{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right\}}\, dx \quad \textbf{(Note)}$$

$$= 4\pi \left[\frac{1}{2}\left(x + \frac{a}{2}\right) \sqrt{(x^2 + ax)} - \frac{1}{2} \cdot \frac{a^2}{4} \log\left\{\left(x + \frac{a}{2}\right) + \sqrt{(x^2 + ax)}\right\} \right]_0^a$$

$$\left[\because \int \sqrt{(x^2 - a^2)}\, dx = \tfrac{1}{2} x \sqrt{(x^2 - a^2)} - \tfrac{1}{2} a^2 \log \{x + \sqrt{(x^2 - a^2)}\} \right]$$

$$= 4\pi \left[\tfrac{1}{2} \cdot \tfrac{3}{2} a\, a\sqrt{2} - \tfrac{1}{8} a^2 \log \{\tfrac{3}{2} a + a\sqrt{2}\} + \tfrac{1}{8} a^2 \log (\tfrac{1}{2} a)\right]$$

$$= 4\pi \left[\tfrac{3}{4} a^2 \sqrt{2} - \tfrac{1}{8} a^2 \log \{(\tfrac{3}{2} a + a\sqrt{2})/(\tfrac{1}{2} a)\}\right] = \pi a^2 \left[3\sqrt{2} - \tfrac{1}{2} \log (3 + 2\sqrt{2})\right]$$

$$= \pi a^2 \left[3\sqrt{2} - \tfrac{1}{2} \log (\sqrt{2} + 1)^2\right] \qquad \textbf{(Note)}$$

$$= \pi a^2 \left[3\sqrt{2} - \log (\sqrt{2} + 1)\right].$$

Comprehensive Exercise 4

1. Find the surface of a sphere of radius a. **(Kanpur 2006)**
2. Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$ where a is the radius of the sphere and h the distance between the planes. **(Kanpur 2009)**
3. Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum. **(Rohilkhand 2011)**
4. Find the surface generated by the revolution of an arc of the catenary $y = c \cosh (x/c)$ about the axis. of x, between the planes $x = a$ and $x = b$.
5. For a catenary $y = a \cosh (x/a)$, prove that $aS = 2V = \pi a\,(ax + sy)$, where s is the length of the arc from the vertex, S and V are respectively the area of the curved surface and volume of the solid generated by the revolution of the arc about x-axis.
6. Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis. **(Meerut 2005, 06)**
7. Find the surface of the solid formed by the revolution, about the axis of y, of the part of the curve $ay^2 = x^3$ from $x = 0$ to $x = 4a$ which is above the x-axis.

Answers 4

1. $4\pi a^2$.
3. $\frac{8}{3} \pi a^2 [2\sqrt{2} - 1]$.
4. $\pi c \left[(b - a) + \frac{c}{2} \sinh \frac{2b}{c} - \frac{c}{2} \sinh \frac{2a}{c}\right]$.
6. $8\pi \left[1 + \frac{4\pi}{3\sqrt{3}}\right]$.
7. $\frac{128}{1215} \pi a^2 [125\sqrt{(10)} + 1]$.

8.7 Surface Formula for Parametric Equations

Suppose the equation of the curve is given in parametric form $x = f(t), y = \phi(t)$, t being the variable parameter. Then the curved surface of the solid formed by the revolution about the x-axis

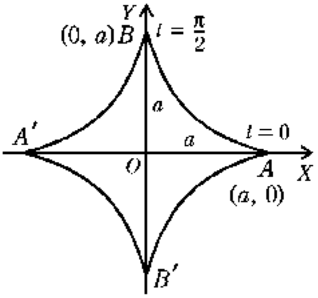
$$= \int 2\pi y \frac{ds}{dt}\, dt, \text{ between the suitable limits}$$

where $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}.$$

Illustrative Examples

Example 1 : *Find the surface of the solid generated by revolution of the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ *or* $x = a\cos^3 t, y = a\sin^3 t$ *about the x-axis.* **(Meerut 2006, 09; Kanpur 05; Rohilkhand 07, 09, 11B; Avadh 11; Kashi 12)**

Solution : The parametric equations of the the curve are $x = a\cos^3 t, y = a\sin^3 t$.



$$\therefore \quad \frac{dx}{dt} = -3a\cos^2 t\sin t$$

and $$\frac{dy}{dt} = 3a\sin^2 t\cos t.$$

Hence $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

$$= \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$$

$$= \sqrt{[9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)]} = 3a\sin t\cos t.$$

Also the given curve (astroid) is symmetrical about both the axes and for the curve in the first quadrant, t varies from 0 to $\pi/2$.

$$\therefore \text{ the required surface} = 2\int_{t=0}^{\pi/2} 2\pi y\frac{ds}{dt}\,dt$$

$$= 4\pi\int_0^{\pi/2} a\sin^3 t \,.\, 3a\sin t\cos t\,dt = 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt$$

$$= 12\pi a^2\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = 12\pi a^2\left[\tfrac{1}{5} - 0\right] = \frac{12\pi a^2}{5}.$$

Example 2 : *Prove that the surface of the solid generated by the revolution of the tractrix*

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t, y = a\sin t$$

about its asymptote is equal to the surface of a sphere of radius a.

(Gorakhpur 2006; Meerut 09)

Solution : The given tractrix is

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t, y = a\sin t.$$

$$\therefore \quad \frac{dx}{dt} = -a\sin t + a\frac{\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}\cdot\frac{1}{2} = a\left(-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

and $$\frac{dy}{dt} = a\cos t.$$

Hence $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \sqrt{\left\{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t\right\}} = \frac{a\cos t}{\sin t}.$$

The given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, x-axis. For the arc of the curve lying in the second quadrant t varies from 0 to $\frac{1}{2}\pi$.

$\therefore$ the required surface $= 2 . \int_0^{\pi/2} 2\pi y \frac{ds}{dt}\, dt$ **(Note)**

$$= 4\pi \int_0^{\pi/2} a \sin t . \frac{a \cos t}{\sin t}\, dt = 4\pi a^2 \int_0^{\pi/2} \cos t\, dt$$

$$= 4\pi a^2 \Big[\sin t\Big]_0^{\pi/2} = 4\pi a^2$$

$=$ the surface of a sphere of radius a.

Comprehensive Exercise 5

1. Find the surface area of the solid generated by revolving the cycloid
$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta) \text{ about the } x\text{-axis.}$$
2. Find the area of the surface generated by revolving an arch of the cycloid
$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta) \text{ about the tangent at the vertex.}$$
3. The portion between two consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the area of the surface so formed is to the area of the cycloid as 64 : 9.
4. Prove that the surface area of the solid generated by the revolution, about the x-axis of the loop of the curve
$$x = t^2, y = t - \tfrac{1}{3}t^3 \text{ is } 3\pi.$$
5. Prove that the surface of the oblate spheroid formed by the revolution of the ellipse of the semi-major axis a and eccentricity e is
$$2\pi a^2 \left[1 + \frac{1 - e^2}{2e} \log\left(\frac{1 + e}{1 - e}\right)\right].$$

Answers 5

1. $\frac{64}{3}\pi a^2$.
2. $\frac{32}{3}\pi a^2$.

8.8 Surface Formula for Polar Equations

Suppose the equation of the curve is given in the polar form $r = f(\theta)$. Then the curved surface generated by the revolution about the initial line, of the arc intercepted between the radii vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} \mathbf{2\pi\,(r \sin\theta)} \frac{ds}{d\theta}\, d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}. \quad [\because y = r\sin\theta]$$

Note. In some cases we may use the formula

$$S = \int 2\pi y \frac{ds}{dr}\, dr, \text{ where } \frac{ds}{dr} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}.$$

8.9 Curved Surface Generated by Revolution about any Axis

If the given arc AB is revolved about a line CD other than the coordinate axes, then the curved surface thus generated is

$$= 2\pi \int (PM)\, ds, \qquad \text{(between the proper limits of integration)}$$

where PM is the perpendicular drawn from any point P on the arc AB to the axis of revolution CD and ds is the length of an element of the arc AB at the point P.

Illustrative Examples

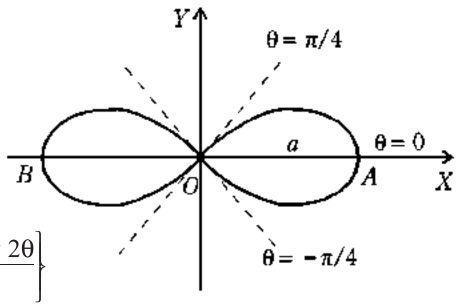
Example 1 : *Find the surface of the solid generated by the revolution of the lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.* **(Meerut 2004, 10B, 11; Rohilkhand 08B; Agra 14; Purvanchal 14)**

Solution : The given curve is $r^2 = a^2 \cos 2\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$2r \frac{dr}{d\theta} = -\ 2a^2 \sin 2\theta$$

or $$\frac{dr}{d\theta} = \frac{-\ a^2 \sin 2\theta}{r}.$$



$\therefore$ $$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\left\{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}\right\}}$$

$$= \frac{1}{r} \sqrt{\{r^2 \cdot a^2 \cos 2\theta + a^4 \sin^2 2\theta\}}$$

$$= \frac{1}{r} \sqrt{\{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta\}}, \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= a^2/r. \qquad ...(2)$$

The given curve is symmetrical about the initial line and about the pole.

Putting $r = 0$ in (1), we get $\cos 2\theta = 0$ giving $2\theta = \pm \frac{1}{2}\pi$ *i.e.*, $\theta = \pm \frac{1}{4}\pi$.

Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$.

There are two loops in the curve and for the upper half of one of these two loops θ varies from 0 to $\frac{1}{4}\pi$.

$\therefore$ the required surface

$= 2 \times$ the surface generated by the revolution of one loop

$$= 2 \cdot \int_0^{\pi/4} 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta$$

$$= 4\pi \int_0^{\pi/4} r \sin\theta \cdot \frac{a^2}{r}\, d\theta \qquad \text{[From (2)]}$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta = 4\pi a^2 \Big[-\cos\theta\Big]_0^{\pi/4}$$

$$= 4\pi a^2 [-(1/\sqrt{2}) + 1] = 4\pi a^2 [1 - (1/\sqrt{2})].$$

Example 2 : *A circular arc revolves about its chord. Find the area of the surface generated, when* 2α *is the angle subtended by the arc at the centre.*

Solution : Let the parametric equations of the circle be

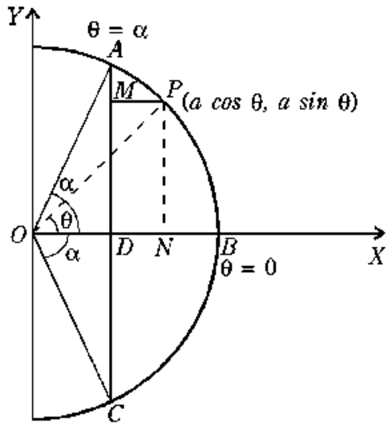
$$x = a\cos\theta, y = a\sin\theta, \qquad ...(1)$$

θ being the parameter.

Take any point $P(a\cos\theta, a\sin\theta)$ on the circular arc ABC which is symmetrical about the x-axis and which subtends an angle 2α at the centre O so that $\angle AOB = \alpha$.

We have $OD = OA\cos\alpha = a\cos\alpha$. Draw PM perpendicular from P to AC, the axis of rotation. Then

$$PM = ON - OD = a\cos\theta - a\cos\alpha. \qquad ...(2)$$



For the upper half of the arc to be rotated *i.e.*, for the arc BA, θ varies from 0 to α.

Also $$\frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{a^2\sin^2\theta + a^2\cos^2\theta\}} = a.$$

$\therefore$ the required surface

$= 2\times$ surface generated by the revolution of the arc BA about the chord AC

$$= 2\times\int_0^{\alpha} 2\pi(PM)\frac{ds}{d\theta}\,d\theta$$

$$= 4\pi\int_0^{\alpha}(a\cos\theta - a\cos\alpha)\,.\,a\,.\,d\theta \qquad \text{[From (2)]}$$

$$= 4\pi a^2\Big[\sin\theta - \theta\cos\alpha\Big]_0^{\alpha} = 4\pi a^2[\sin\alpha - \alpha\cos\alpha].$$

Comprehensive Exercise 6

1. Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.
2. Find the surface of the solid formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line. **(Purvanchal 2006, 10; Kashi 11)**
3. The arc of the cardioid $r = a(1 + \cos\theta)$ included between $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ is rotated about the line $\theta = \frac{1}{2}\pi$. Find the area of the surface generated. **(Purvanchal 2010)**
4. A quadrant of a circle of radius a revolves about its chord. Show that the surface of the spindle generated is
$$2\pi a^2\sqrt{2}\,(1 - \tfrac{1}{4}\pi).$$
5. The lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$. **(Meerut 1993, 2005B)**

Answers 6

1. $4\pi a^2$. **2.** $\frac{32}{5}\pi a^2$. **3.** $\frac{48}{5}\sqrt{2}\,\pi a^2$.

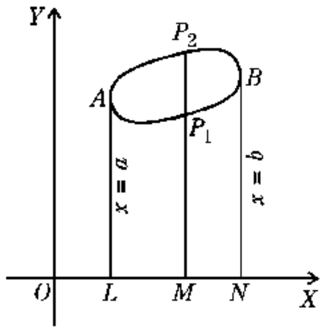
8.10 Theorems of Pappus and Guldin (Agra 2014)

State and prove the theorems of Pappus and Guldin.

Theorem 1 : Volume of a Solid of Revolution :

If a closed plane curve revolves about a straight line in its plane which does not intersect it, the volume of the ring thus obtained is equal to the area of the region enclosed by the curve multiplied by the length of the path described by the centroid of the region.

Proof : Let AP_1BP_2A be the closed plane curve and let it rotate about the axis of x.



Let AL $(x = a)$ and BN $(x = b)$ be the tangents to the curve parallel to the y-axis $(a < b)$. Also let any ordinate meet the curve at P_1, P_2 and let $MP_1 = y_1$, $MP_2 = y_2$ so that y_1, y_2 are functions of x.

Now volume of the ring generated by the revolution of the closed curve AP_1BP_2A about the axis of x

= volume generated by the area $ALNBP_2A$

− volume generated by the area $ALNBP_1A$

$$= \pi \int_a^b y_2^2\,dx - \pi \int_a^b y_1^2\,dx = \pi \int_a^b (y_2^2 - y_1^2)\,dx. \qquad ...(1)$$

Also if $\bar{y}$ be the ordinate of the centroid of the area of the closed curve, then

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1 + y_2)(y_2 - y_1)\,dx}{A} = \frac{\frac{1}{2}\int_a^b (y_2^2 - y_1^2)\,dx}{A}, \qquad ...(2)$$

where A is the area of the closed curve. [See the chapter on centre of gravity]

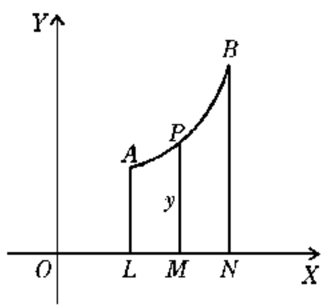
Hence from (1) and (2), the required volume $= 2\pi A\,\bar{y} = A \times 2\pi\bar{y}$

= area of the closed curve × circumference of the circle of radius $\bar{y}$

= (area of the curve) × (length of the arc described by the centroid of the region bounded by the closed curve).

Theorem 2 : Surface of a solid of revolution :

If an arc of a plane curve revolves about a straight line in its plane, which does not intersect it, the surface of the solid thus obtained is equal to the arc multiplied by the length of the path described by the centroid of the arc.



Proof : Let l be the length of the arc AB and let it revolve about OX.

Let the abscissae of the extremities A and B of the arc be a and b.

Then the surface generated by the revolution of the arc AB about x-axis is

$$= \int_{x=a}^{x=b} 2\pi\, y\, ds \qquad ...(1)$$

Also we know that (see the chapter on centre of gravity) the ordinate $\bar{y}$, of the centroid of the arc from $x = a$ to $x = b$, of length l, is given by

$$\bar{y} = \frac{\int_{x=a}^{b} y\, ds}{l} \qquad ...(2)$$

From (1) and (2), we get the required surface $= 2\pi \bar{y} l = l \times 2\pi \bar{y}$

= (length of the arc) $\times$ (length of the path described by the centroid of the arc).

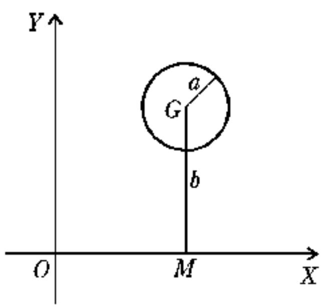
Note 1 : The closed curve or arc in the above theorems must not cross the axis of revolution but may be terminated by it.

Note 2 : When the volume or surface generated is known, the theorems may be applied to find the position of the centroid of the generating area or arc.

Illustrative Examples

Example 1 : *Find the volume and surface-area of the anchor-ring generated by the revolution of a circle of radius a about an axis in its own plane distant b from its centre* $(b > a)$.

Solution : Here the given curve (circle) does not intersect the axis of rotation, so Pappus theorem can be applied.



In this case A

= area of the region of the closed curve

= area of the circle of radius $a = \pi a^2$

and $l =$ length of the arc of the curve

= circumference of the circle $= 2\pi a$.

As the centroid of the area of a circle and also of its circumference lies at the centre, so $\bar{y} = b$ in both the cases and hence the length of the path described by the C.G. $= 2\pi b$.

Now by Pappus theorem, the required volume of the anchor-ring

= (area of the circle)

$\times$ (circumference of the circle generated by the centroid)

$= \pi a^2 . 2\pi b = 2\pi^2 a^2 b.$

And the surface area of the anchor-ring

= (arc length of the circle)

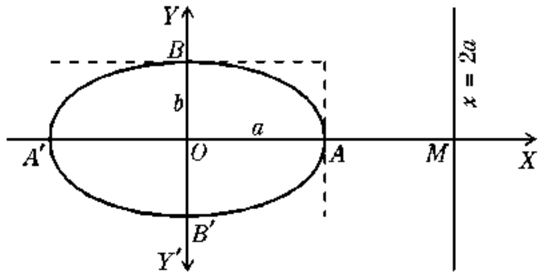
$\times$ (circumference of the circle generated by the centroid)

$= 2\pi a . 2\pi b = 4\pi^2 ab.$

Example 2 : *Show that the volume generated by the revolution of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *about the line* $x = 2a$ *is* $4\pi^2 a^2 b$.

Solution : Area of the given ellipse is πab. ...(1)

The C.G. of the ellipse will describe a circle of radius $2a$ when revolved about the line $x = 2a$. Hence the length of the arc described by the C.G. $= 2\pi(2a) = 4\pi a$.



$\therefore$ by Pappus theorem the required volume

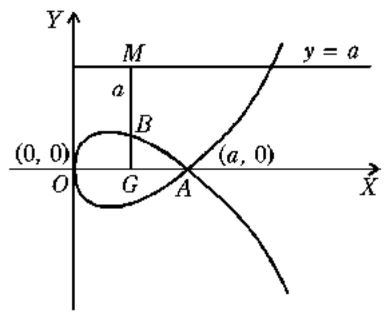
$$= \text{(area of the ellipse)} \times \text{(length of the arc described by its C.G.)}$$

$$= \pi ab \,.\, 4\pi a = 4\pi^2 a^2 b.$$

Example 3 : *The loop of the curve $2ay^2 = x(x - a)^2$ revolves about the straight line $y = a$. Find the volume of the solid generated.*

Solution : The given curve is $2ay^2 = x(x - a)^2$. ...(1)

The curve (1) is symmetrical about the x-axis and the loop lies between $x = 0$ and $x = a$.



Differentiating (1) w.r.t. x, we get

$$4ay\,(dy/dx) = 2x(x - a) + (x - a)^2$$
$$= 3x^2 - 4ax + a^2.$$

Now $(dy/dx) = 0$ when $3x^2 - 4ax + a^2 = 0$

or when $x = a/3$ which gives from (1),

$y = (a\sqrt{2})/(3\sqrt{3})$ *i.e.*, $< a$ showing that the loop does not intersect the straight line $y = a$.

By symmetry the C.G. of the loop lies on x-axis *i.e.*, the distance of the C.G. from the axis of revolution $(y = a)$ is a. When the loop is rotated about $y = a$, its C.G. will describe a circle of radius a whose perimeter is $2\pi a$.

Also the area A of the loop

$$= 2\int_0^a y\,dx = 2\int_0^a \frac{(x - a)\sqrt{x}}{\sqrt{(2a)}}\,dx, \qquad \left[\because \text{ from (1), } y = \frac{(x - a)\sqrt{x}}{\sqrt{(2a)}}\right]$$

$$= \sqrt{\left(\frac{2}{a}\right)}\int_0^a (x^{3/2} - ax^{1/2})\,dx$$

$$= \sqrt{\left(\frac{2}{a}\right)}\left[\frac{x^{5/2}}{5/2} - \frac{ax^{3/2}}{3/2}\right]_0^a = \frac{4}{15}\sqrt{2}\,a^2.$$

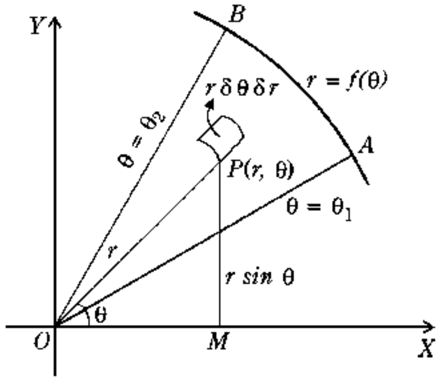
$\therefore$ by Pappus theorem, the required volume

$$= 2\pi a \times A = 2\pi a \times \frac{4}{15}\sqrt{2}a^2 = \frac{8}{15}\sqrt{2}\pi a^3.$$

Example 4 : *Prove that the volume of the solid formed by the rotation about the line $\theta = 0$ of the area bounded by the curve $r = f(\theta)$ and the lines*

$$\theta = \theta_1, \theta = \theta_2 \quad \textit{is} \quad \frac{2}{3}\pi\int_{\theta_1}^{\theta_2} r^3 \sin\theta\, d\theta.$$

Solution : Let OAB be the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$. We have to find the volume formed by the revolution of area OAB about the initial line OX.



Take any point (r, θ) inside the area OAB and take a small element of the area $r\,\delta\theta\,\delta r$ at the point P. Drop PM perpendicular from P to the axis of rotation OX. We have

$$PM = OP \sin\theta = r \sin\theta.$$

Now the volume of the ring formed by revolving the element of area $r\,\delta\theta\,\delta r$ about OX

$$= 2\pi r \sin\theta \,.\, r\delta\theta\,\delta r = 2\pi\, r^2 \sin\theta\,\delta\theta\,\delta r.$$

Therefore the whole volume formed by revolving the area OAB about OX

$$= \int_{\theta = \theta_1}^{\theta_2} \int_{r = 0}^{f(\theta)} 2\pi\, r^2 \sin\theta\, d\theta\, dr$$

$$= \int_{\theta = \theta_1}^{\theta_2} 2\pi \sin\theta \left[\frac{r^3}{3}\right]_0^{f(\theta)} d\theta$$

$$= \frac{2}{3}\pi \int_{\theta = \theta_1}^{\theta_2} [f(\theta)]^3 \sin\theta\, d\theta = \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta\, d\theta,$$

where r is to be replaced from the equation of the curve $r = f(\theta)$.

Note : Proceeding as above we can also show that the volume of the solid formed by the rotation of the above mentioned area about the line $\theta = \dfrac{\pi}{2}$ is equal to

$$\frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \cos\theta\, d\theta.$$

Comprehensive Exercise 7

Use Pappus theorem to find :

1. The position of the centroid of a semi-circular area.
2. The volume generated by the revolution of an ellipse having semi-axes a and b about a tangent at the vertex.
3. Find by using Pappus theorem the volume of the ring generated by the revolution of an ellipse of eccentricity $1/\sqrt{2}$ about a straight line parallel to the minor axis and situated at a distance from the centre equal to three times the major axis.
4. Find the volume of the ring generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the line $r\cos\theta + a = 0$, given that the centroid of the cardioid is at a distance $5a/6$ from the origin.

5. A semi-circular bend of lead has a mean radius of 8 inches; the initial diameter of the pipe is 4 inches and the thickness of the lead is $\frac{1}{2}$ inch. Applying the theorem of Pappus and Guldin find the volume of the lead and its weight, given that 1 cubic inch of lead weighs $0 \cdot 4$ lb.
[**Hint.** Internal diameter of pipe = 4 inches.
Thickness of metal = $\frac{1}{2}$ inch
$\therefore$ external diameter of the pipe = $4 + 1 = 5$ inches.
$\therefore$ area of lead = $\frac{1}{4}\pi(5^2 - 4^2) = \frac{9}{4}\pi$.
The centroid of this area is at a distance of 8 inches from the axis of rotation. Therefore the length of path traced out by its centroid in describing a semi-circle $= 8\pi$ inches.
$\therefore$ volume of the lead $= 8\pi \times \frac{9}{4}\pi = 18\pi^2$ cu. inch.
$\therefore$ weight of the pipe = volume $\times$ density $= 18\pi^2 \times 0 \cdot 4$ lb. $= 71 \cdot 1$ lb]

6. State the theorems of Pappus and Guldin. **(Meerut 2008)**

Answers 7

1. $4a/3\pi$.
2. $2\pi^2 a^2 b$ or $2\pi^2 ab^2$.
3. $6\sqrt{2}\pi^2 a^3$, where a is the semi-major axis.
4. $\frac{11}{2}\pi^2 a^2$.

Objective Type Questions

Fill in the Blanks:

Fill in the blanks "… …", so that the following statements are complete and correct.

1. The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$ about the x-axis is …… . **(Meerut 2003)**

2. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about the initial line is …… . **(Meerut 2001)**

3. If the equation of the curve in the polar form is $r = f(\theta)$, then the curved surface generated by the revolution about the initial line of the arc intercepted between the radii vectors $\theta = \alpha$ and $\theta = \beta$ is $\int_{\theta=\alpha}^{\theta=\beta} 2\pi(r\sin\theta)\frac{ds}{d\theta}d\theta$, where $\frac{ds}{d\theta} =$ …… .

4. If the equations of the curve in parametric form are $x = f(t), y = \phi(t)$, t being the variable parameter, then the curved surface of the solid formed by the revolution about the x-axis is $\int 2\pi y \frac{ds}{dt}\, dt$, between the suitable limits, where

$\frac{ds}{dt} = \ldots\ldots$.

Multiple Choice Questions:

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

5. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about any line ($\theta = \gamma$) is

(a) $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \cos(\theta - \gamma)\, d\theta$

(b) $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin(\theta - \gamma)\, d\theta$

(c) $\int_{\theta_1}^{\theta_2} \pi r^2 \sin(\theta - \gamma)\, d\theta$

(d) $\int_{\theta_1}^{\theta_2} \pi r^2 \cos(\theta - \gamma)\, d\theta$

6. The volume of the paraboloid generated by the revolution about the x-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$ is

(a) $2\pi a h^2$ (b) $2\pi a h$

(c) $\frac{2}{3}\pi a h^2$ (d) $\frac{2}{3}\pi a h$ **(Rohilkhand 2005)**

7. The curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the lines $y = a, y = b$ and y-axis is

(a) $\int_a^b \pi x\, ds$ (b) $\int_a^b 2\pi x\, ds$

(c) $\int_a^b \frac{2}{3}\pi x\, ds$ (d) $\int_a^b \pi^2 x\, ds$

True or False:

Write 'T' for true and 'F' for false statement.

8. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about the initial line $\theta = 0$ is

$$\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin\theta\, d\theta.$$

9. If an arc length revolves about x-axis, the basic formula for the surface of revolution in all cases is $\int 2\pi\, y\, ds$, between the suitable limits.

Answers

1. $\int_a^b \pi y^2\, dx$.

2. $\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$.

3. $\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$.

4. $\sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$.

5. (b).

6. (a).

7. (b).

8. T.

9. T.