Chapter 6

2015

6.1 Section-A

Question-1(a) Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$

[8 Marks]

Solution:

Let upper triangular matrix,

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\implies \mathrm{A}^2 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}$$

$$\implies A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} x^3 & x^2y+xyz+yz^2 \\ 0 & z^3 \end{bmatrix}$$

It is given that

$$A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$

$$\implies \begin{bmatrix} x^3 & x^2y+xyz+yz^2 \\ 0 & z^3 \end{bmatrix} = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$

$$\therefore x^3 = 8 \Rightarrow x = 2,$$

$$z^3 = 27 \Rightarrow z = 3,$$

$$x^2y + xyz + yz^2 = -57 \Rightarrow 4y + 6y + 9y = -57 \Rightarrow y = 3$$

$$\therefore A = \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}$$

Question-1(b) Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x,y,z) = (2y+z, x-4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1,1,1),(1,1,0),(1,0,0)\}$$

[8 Marks]

Solution:

$$\text{Given, } G(x,y,z) = (2y+z, x-4y, 3x)$$
$$\text{Basis, } S = \{(1,1,1),(1,1,0),(1,0,0)\}$$

$$\begin{aligned}\text{Let } (x,y,z) =& a(1,1,1)+b(1,1,0)+c(1,0,0)\\ =&(a+b+c, a+b, a)\end{aligned}$$

$$\therefore \quad x = a+b+c, \quad y = a+b, \quad z = a$$
$$\text{i.e. } a = z, \quad b = y-z, \quad c = x-y$$
$$\therefore \quad (x,y,z) = z(1,1,1)+(y-z)(1,1,0)+(x-y)(1,0,0)$$

$$\begin{aligned}G(1,1,1) &= (3,-3,3)\\ &= 3(1,1,1)+(-6)(1,1,0)+6(1,0,0)\\ G(1,1,0) &= (2,-3,3)\\ &= 3(-1,1,1)+(-6)(1,1,0)+(-1)(1,0,0)\\ G(1,0,0) &= (0,1,3)\\ &= 3(1,1,1)+(-2)(1,1,0)+(-3)(1,0,0)\end{aligned}$$

$$\therefore [M]_S = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & -1 \\ 3 & -2 & -3 \end{bmatrix}^\top = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & -1 & -3 \end{bmatrix}$$

Question-1(c) Let $f(x)$ be a real-valued function defined on the interval (-5,5) such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}dt$ for all $x \in (-5,5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

[8 Marks]

Solution:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(t)}, \text{ where } f(t) = x$$

.

Here,

$$e^{-x}f(x) = 2 + \int_0^x \sqrt{x^4+1}dt \qquad \ldots(1)$$

Differentiating both sides w.r.t x,

$$-e^{-x}f(x) + e^{-x}f'(x) = 0 + \sqrt{x^4+1}$$

Put, $x = 0$

$$-f(0) + f'(0) = 1$$

Also, ptting x=0 in (1),($f(0) = 2 + 0 \Rightarrow f(0) = 2$

$$\therefore f'(0) = 3$$

$$\therefore \frac{d}{dx}f^{-1}(x)\Big|_{x=2} = \frac{1}{f'(0)} = \frac{1}{3}$$

Question-1(d) **For** $x > 0$, **let** $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. **Evaluate** $f(e) + f\left(\frac{1}{e}\right)$

[8 Marks]

Solution:

$$I_1 = f(e) = \int_1^e \frac{\log t}{1+t}dt$$

$$I_2 = f\left(\frac{1}{e}\right) = \int_1^{1/e} \frac{\log t}{1+t}dt$$

$$= \int_1^e \frac{\log(1/y)}{1+\frac{1}{y}} \cdot \left(-\frac{dy}{y^2}\right) \quad \left[\begin{array}{c}\text{Puting} \\ t = \frac{1}{y}\end{array}\right]$$

$$\int_1^e \frac{\log y}{1+y} \cdot \frac{dy}{y} = \int_1^e \frac{\log t}{(1+t)} \cdot \frac{dt}{t}$$

$$\therefore I_1 + I_2 = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log t}{1+t} + \frac{\log t}{1+t}\frac{dt}{t}$$

$$= \int_1 \frac{\log t}{1+t}\left(1 + \frac{1}{t}\right)dt$$

$$= \int \frac{\log t}{t}dt = \frac{(\log t)^2}{2}\Big|_1^e = \frac{1}{2}$$

Question-1(e) **The tangent at** $(a\cos\theta, b\sin\theta)$ **on the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.**

[8 Marks]

Solution: Equation of the tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

is

$$\Rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1...(i)$$

The joint equation of the lines. joining the points of intersection of (i) and the auxiliary circle $x^2 + y^2 = a^2$ to the origin, which is the center of the circle, is

$$x^2 + y^2 = a^2\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2$$

Since, these lines are at right angles co-efficient of x^2 + co-efficient of $y^2 = 0$

$$\Rightarrow 1 - a^2\left(\frac{\cos^2\theta}{a^2}\right) + 1 - a^2\left(\frac{\sin^2\theta}{b^2}\right) = 0$$

$$\Rightarrow \sin^2\theta\left(1 - \frac{a^2}{b^2}\right) + 1 = 0$$

$$\Rightarrow \sin^2\theta\left(b^2 - a^2\right) + b^2 = 0$$

$$\Rightarrow \sin^2\theta\left[a^2\left(1-e^2\right) - a^2\right] + a^2\left(1-e^2\right) = 0$$

$$\Rightarrow \left(1 + \sin^2\theta\right)a^2e^2 = a^2$$

$$\Rightarrow e = \frac{1}{\sqrt{\left(1+\sin^2\theta\right)}}$$

Question-2(a) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$.

[10 Marks]

Solution:

$$Dim(U+W) = \mathrm{Dim}(U) + \mathrm{Dim}(W) - \mathrm{Dim}(U\cap W)$$
$$= 4 + 4 - \mathrm{Dim}(U\cap W)$$
$$\therefore \mathrm{Dim}(U\cap W) = 8 - \mathrm{Dim}(U+W)$$

$U + W$ is a subspace of V $\therefore \dim(U+W) \le \dim(V)$

$$\therefore \quad \dim(U+W) \le 6$$

$\Rightarrow$ Dim $(\cup \cap w) \ge 8 - 6$ ie, $\dim(U\cap W) \ge 2$ Also, $U \cap W$ is a subspace of U

$$\therefore \dim(U\cup W)' \le \dim(U)$$

i.e

$$\dim(U\cap W) \le 4$$

Hence, Possible values of $\dim(U\cup W)$ are 2,3 or 4 . Result: Intersection of two subspaces is a subspace.

Question-2(b) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution:

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$$

[10 Marks]

Solution:

$$Ax = B$$

$$[A : B] \quad \sim \quad \begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1$$

$$R_3 + R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b - 2a \\ 0 & -4 & 10 & c - a \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b - 2a \\ 0 & 0 & 0 & -5a + 2b + c \end{bmatrix}$$

Now this system has solution of

$$\text{Rank}(A; B) = \text{Rank}(A) = 2$$

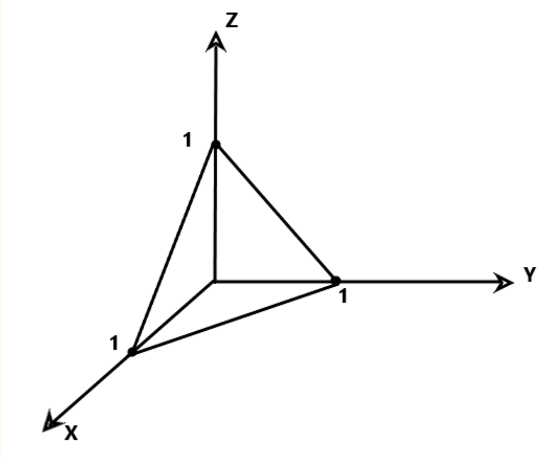
which is possible only if

$$-5a + 2b + c = 0$$

Question-2(c) Consider the three-dimensional region R bounded by $x+y+z = 1, y = 0, z = 0$. Evaluate $\iiint_R (x^2 + y^2 + z^2)\, dxdydz$.

[10 Marks]

Solution: Let R be the region bounded by the given tetrahedron.



$$
\begin{aligned}
I &= \int_0^1 \int_0^1 \int_0^{1-x} \left(x^2 + y^2 + z^2\right) dzdydx \\
&= \int_0^1 \int_0^{1-x} 2\left(x^2 + y^2\right) z + \left.\frac{z^3}{3}\right|_0^{1-x-y} dydx \\
&= \int_0^1 x^2(1-x)y - x^2 \cdot \frac{y^2}{2} + (1-x)\frac{y^3}{3} - \frac{y^4}{4} - \left.\frac{(1-x-y)^4}{12}\right|_0^{1-x} dx \\
&= \int_0^1 x^2(1-x)^2 - \frac{1}{2}x^2(1-x)^2 + \frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4 + \frac{1}{12}(1-x)^4 dx \\
&= \int \frac{1}{2}\left(x^2 + x^4 - 2x^3\right) + \frac{2}{12}(1-x)^4 dx \\
&= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^y}{5} - \frac{2x^4}{4}\right) - \left.\frac{1}{30}(1-x)^5\right|_0^1 \\
&= \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right) - \frac{1}{30}(0-1) \\
&= \frac{1}{20}
\end{aligned}
$$

Question-2(d) Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid

$$\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$$

[10 Marks]

Solution: The intersection of plane $z = 2$ with the ellipsoid is given by

$$\frac{x^2}{25} + y^2 + \frac{(2)^2}{5} = 1 \quad \Rightarrow \frac{x^2}{25} + y^2 = \frac{1}{5}$$

ie

$$\frac{x^2}{5} + \frac{y^2}{1/5} = 1$$

(say S_1) in space. $Z = 2$. The area enclosed by this true is an ellipse.
We take projection on $xy - plane$.

$$A = \iint_D \sqrt{z_x^2 + z_y^2 + 1} dA$$
$$= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\frac{1}{5}\sqrt{5-x^2}}^{\frac{1}{5}\sqrt{5-x^2}} 1.dydx$$
$$= \int_{-\sqrt{5}}^{\sqrt{5}} \frac{2}{5}\sqrt{5-x^2}dx \quad \Big| \quad \begin{array}{l} \text{Put } x = \sqrt{5}\sin\theta \\ dx = \sqrt{5}\,\text{cssos}\,\theta \end{array}$$
$$= \frac{2}{5} \times 2 \int_0^{\pi/2} \sqrt{5 - 5\sin^2\theta}\sqrt{5}\cos\theta d\theta$$
$$= 4\int_0^{\pi/2} \cos^2\theta d\theta$$
$$= 4 \times \frac{1}{2} \times \frac{\pi}{2}$$
$$= \pi$$

Question-3(a) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$

[10 Marks]

Solution: Minimal polynomial of a matrix is a monic polynomial of least degree such that

$$p(A) = 0$$

First, let us find the characteristic polynomial

$$\begin{vmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)[(3+\lambda)(\lambda-3)+8) + 2(18 - 6\lambda - 12) + 2(-12 + 9 + 3\lambda) = 0$$
$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda + 10 = 0$$
$$\Rightarrow (\lambda+1)\left(\lambda^2 - 5\lambda + 10\right) = 0$$

Hence, we have 3 possibilities for minimal polynomial,

$$(\lambda+1), -\left(\lambda^2 - 5\lambda + 10\right)$$

or

$$\lambda^3 - 4\lambda^2 + 5\lambda + 10$$

. Let us check one by one. clearly

$$A + I \neq 0$$

$$A^2 - 5A + 10I = \begin{bmatrix} 0 & 4 & -4 \\ -12 & 14 & -8 \\ -6 & 4 & 2 \end{bmatrix} \neq 0$$

By Cayley-Hamilton theorem,

$$A^3 - 4A^2 + 5A + 10I = 0$$

Here, minimal polynomial is

$$x^3 - 4x^2 + 5x + 10$$

Question-3(b) If $\sqrt{x+y} + \sqrt{y-x} = c$, find $\frac{d^2y}{dx^2}$.

[10 Marks]

Solution:

$$\begin{aligned} (\sqrt{y+x} + \sqrt{y-x})^2 &= c^2 \\ (y+x) + (y-x) + 2\sqrt{y^2-x^2} &= c^2 \\ 2y - c^2 &= -2\sqrt{y^2-x^2} \\ 4y^2 - 4c^2y + c^4 &= 4\left(y^2 - x^2\right) \\ -4c^2y &= -4x^2 - c^4 \\ y &= \frac{1}{c^2}x^2 + \frac{c^2}{4} \end{aligned}$$

Differentiating wrt x,

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x}{c^2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{2}{c^2} \end{aligned}$$

Question-3(c) A rectangular box, open at the top, is said to have a volume of 32 cubic meters. Find the dimensions of the box so that the total surface is a minimum.

[10 Marks]

Solution:

$$\begin{aligned} &v = xyz = 32 \quad \text{(given)} \\ &S = xy + 2yz + 2zx, \quad \text{where } x, y \text{ z are dimension} \end{aligned}$$

$$S = xy + 2y \cdot \frac{32}{xy} + 2x - \frac{32}{xy}$$
$$= xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$
$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}; \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$
$$r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad S = \frac{\partial^2 S}{\partial x \partial y} = 1 \quad , \quad t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

for stationary points,

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial S}{\partial y} = 0$$
$$y - \frac{64}{x^2} = 0$$

&

$$x - \frac{64}{y^2} = 0$$
$$y = \frac{64}{x^2} \Rightarrow x \cdot \left(\frac{64}{x^2}\right)^2 = 64 \quad \Rightarrow \quad x = 4$$
$$\therefore \quad (4, 4)$$

is stationary point

$$\Rightarrow \quad y = 4$$

Also,

$$rt - s^2 = 4 - 1 = 3 > 0$$

&,

$$r > 0$$

$\therefore (4, 4)$ is a point of minima.

$$\therefore \quad x = 4, \quad y = 4, \quad z = \frac{32}{4 \times 4} = 2$$

Question-3(d) Find the equation of the plane containing the straight line $y + z = 1, x = 0$ and parallel to the straight line $x - z = 1,\ y = 0.$

[10 Marks]

Solution: Eqn of plane through the line

$$y + z - 1 = 0, x = 0$$

is

$$\lambda x + y + z - 1 = 0$$

other line ie.

$$x - z = 1, \quad y = 0$$

ie.
$$\frac{x}{1} = \frac{z+1}{1} = \frac{y}{0}$$
Plane is parallel to this line
$$\therefore \quad \lambda \cdot 1 + 1.1 + 1 \cdot 0 = 0$$
$$\lambda = -1$$
Hence eqn of plane:
$$-x + y + z - 1 = 0$$
.

Question-4(a) Find a 3×3 orthogonal matrix whose first two rows are
$$\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$$
and
$$\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$
.

[10 Marks]

Solution: Let given two rows vectors are
$$\vec{u} = \left[\begin{array}{ccc} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{array}\right]$$
$$\vec{v} = \left[\begin{array}{ccc} 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{array}\right]$$
Let $\vec{w}$ be the third row vector which makes the given matrix orthogonal.
Then $\vec{w}$ is obtained by the cross product of $\vec{u}$ and $\vec{v}$.
$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} i & j & k \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} \\
&= i\left(\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right) + j\left(0 + \frac{1}{3\sqrt{2}}\right) + k\left(\frac{1}{3\sqrt{2}} - 0\right) \\
&= \frac{-4i}{3\sqrt{2}} + \frac{j}{3\sqrt{2}} + \frac{k}{3\sqrt{2}} \\
&= \frac{1}{3\sqrt{2}}(-4i + j + k)
\end{aligned}$$
Hence, unit vector, $w = \left[\begin{array}{ccc} \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{array}\right]$
Hence, the orthogonal matrix will be
$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$$

Question-4(b) Find the locus of the variable straight line that always intersects $x = 1,\ y = 0;\ y = 1,\ z = 0;\ z = 1,\ x = 0.$

[10 Marks]

Solution: The plane passes through given lines are respectively, given by

$$x - 1 + py = 0$$
$$y - 1 + qz = 0$$
$$z - 1 + rx = 0$$

These planes will intersect in a lines if the determinant formed the coefficients of x, y and z is 0, i.e.,

$$\begin{vmatrix} 1 & p & 0 \\ 0 & 1 & q \\ r & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & p & 0 \\ 0 & 1 & q \\ r & 0 & 1 \end{vmatrix} = 0$$

$$1 + pqr = 0$$

$$1 + \left(\frac{1-x}{y}\right)\left(\frac{1-y}{z}\right)\left(\frac{1-z}{x}\right) = 0$$

$$xyz + (1-x)(1-y)(1-z) = 0$$

Question-4(c) Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$.

[10 Marks]

Solution: Any normal to the parabola $y^2 = 4ax$ is $\ldots (i)$

$$y = mx - 2am - am^3 \quad \ldots (ii)$$

Let, $(x_1, \dot{y}_1)$. be the pole of (2) with respect to (i), then (ii) is the polar of (x_1, y_1) w.r.t. (i) i.e.

$$yy_1 = 2a\,(x + x_1)\,...(iii)$$

comparing (ii) and (iii), we get

$$\frac{2a}{m} = \frac{y_1}{1}$$

$$= \frac{(2ax_1)}{(-2am - am^3)}$$

Hence, we get

$$x_1 = -2a - am^2 \ldots (iv)$$

And,

$$y_1 = 2a/m...(v)$$

Eliminating m between $(iv)\&(v)$ we get

$$y_1^2\,(x_1 + 2a) + 4a^3 = 0$$

$\therefore$ The required locus of (x_1, y_1) is

$$(x + 2a)y^2 - 4a^3 = 0$$

Question-4(d) Evaluate

$$\lim_{x \to 0} \left(\frac{2 + \cos x}{x^3 \sin x} - \frac{3}{x^4} \right)$$

[10 Marks]

Solution:

$$\begin{aligned}
L &= \lim_{x \to 0} \left(\frac{2 + \cos x}{x^3 \sin x} - \frac{3}{x^4} \right) \\
&= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^4 \sin x} \\
&= \lim_{x \to 0} \frac{x(2 + \cos x) - 3\sin x}{x^5} \cdot \frac{x}{\sin x} \\
&= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^5} \cdot 1 \\
&= \lim_{x \to 0} \frac{2 + \cos x - x\sin x - 3\,\mathrm{css}\,x}{5x^4} \\
&= \lim_{x \to 0} \frac{2\sin x - \sin x - x\cos x}{20x^3} \\
&= \lim_{x \to 0} \frac{\cos x - \cos x + x\sin x}{60x^2} \\
&= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{60} = \left(\frac{1}{60} \right).
\end{aligned}$$

6.2 Section-B

Question-5(a) Reduce the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0, \quad p = \dfrac{dy}{dx}$ **to Clairaut's form. Hence, find the singular solution of the equation.**

[8 Marks]

Solution:

$$x^2p^2 + yp(2x + y) + y^2 = 0$$

Put $y = u$ and $xy = v$
$\therefore dy = du$ and $xdy + ydx = dv$

$$P = \frac{dv}{du} = \frac{xdy+ydx}{dy} = x + y\frac{dx}{dy}$$
$$\Rightarrow P = x + y\left(\frac{1}{p}\right)$$
$$\Rightarrow \quad p = \frac{y}{P-x}$$

The given DE transforms to

$$x^2\frac{y^2}{(P-x)^2} + y\frac{y}{(P-x)}(2x + y) + y^2 = 0$$
$$x^2 + (2x + y)(P - x) + (P - x)^2 = 0$$
$$x^2 + (2xP + yP - 2x^2 - xy) + (P^2 + x^2 - 2xP) = 0$$
$$P^2 + Py - xy = 0$$
$$P^2 + Pu - V = 0$$
$$V = Pu + P^2$$

Which is in Claiuraut's form

$$y = px + f(p)$$

General solution is given by

$$v = cu + c^2$$
$$\Rightarrow xy = cy + c^2$$
$$\Rightarrow c^2 + yc - xy = 0$$

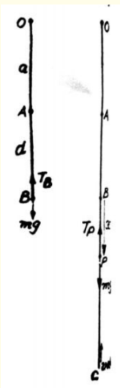
To obtain the singular solution, we equate the discriminant of above equation to 0

$$\Rightarrow y^2 - 4(-xy) = 0$$
$$\Rightarrow y^2 + 4xy = 0$$
$$\Rightarrow y + 4x = 0$$

Question-5(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point.

[8 Marks]

Solution: Let, OA $= a$ be the natural length of the string whose one end O is fixed. Let B be the position of equilibrium of a particle of mass m attached to the other end A of the string and let AB $= e$. Then at B, the weight of the particle = the tension T_o in the string.



$\therefore$

$$mg = \lambda(e/a) = (mg)(e/a) \text{ as}$$
$$\lambda = mg \qquad \ldots(i) \text{ (given)}$$

Thus, here

$$e = a \qquad \ldots(ii)$$

Now, the particle is pulled down to a point C such that OC $= 4a$ (given) and then let go. The particle will start to move towards B from rest from C

Let P be its position after time t, where BP $= x$ At P, the forces acting on the particle are its weight mg acting vertically downwards and tension T $= \lambda(x+e)/a$ acting vertically upwards.

Then, the equation of motion of the particle at P is Or,

$$\frac{md_x^2}{dt^2} = mg - \mathrm{T} = mg - \lambda(e+x)/a$$
$$= mg - \lambda(e/a) - \lambda(x/a)$$
$$\frac{md_x^2}{dt^2} = -\lambda(x/a)$$

i.e.

$$\frac{d_x^2}{dt^2} = -(\lambda/am)x,$$

using Or,

$$\frac{d_x^2}{dt^2} = -(mg/am)x$$
$$= -(g/a)x$$

as

$$\lambda = mg \qquad \ldots(iii)$$

Or,

$$v\left(\frac{dv}{dx}\right) = -(g/a)x$$

Or,

$$2vdv = -(2g/a)xdx$$

Integrating,

$$v^2 = -(g/a)x^2 + K \qquad \ldots(iv)$$

Where, K being a constant. At the point C, when

$$x = \mathrm{BC} = \mathrm{OC} - \mathrm{OB}$$
$$= 4a - (a+e)$$
$$= 4a - (a+a)$$

i.e.,

$$x = 2a, \quad v = 0$$

Hence, (iv) reduces to

$$0 = -(g/a)(2a)^2 + K$$

So that,

$$K = 4ag$$

. From

$$(iv), \quad v^2 = 4ag - (g/a)x^2$$

Or,

$$(dx/dt) = (g/a)\left(4a^2 - x^2\right)$$
$$\cdots (v)$$

When the particle reaches A, let the velocity of the particle be V. Then, putting $x = -a$ and $v = V$ in (v), we get Or,

$$\mathrm{V}^2 = (g/a)\left(4a^2 - a^2\right)$$
$$\mathrm{V} = (3ag)^{1/2} ...(vi)$$

From

$$(v), \quad \frac{dx}{dt} = \left(\frac{g}{a}\right)^{1/2}\left(4a^2 - x^2\right)$$

Or,

$$dt = \left(\frac{g}{a}\right)^{1/2} \frac{dx}{\left(4a^2 - x^2\right)^{1/2}} \ldots (vii)$$

Where we have taken negative sign on R.H.S. due to the fact that in moving from C towards B, x decreases as t increases. Let, t_1 be the time taken from C to A. Then integrating (vii) between $t = 0$ to $t = t_1$ and corresponding limits $x = -2a$ to $x = -a$ we get,

$$\int_0^1 dt = \left(\frac{a}{g}\right)^{1/2} \int_{2a}^{a} \frac{dx}{\left(4a^2 - x^2\right)^{1/2}}$$
$$= \left(\frac{a}{g}\right)^{1/2} \left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$
$$\text{Or,} \quad t_1 = (a/g)^{1/2}\left\{\cos^{-1}(-1/2) - \cos^{-1}\right\}$$
$$= (a/g)^{1/2}\left\{\pi - \cos^{-1}(1/2)\right\}$$
$$\text{Thus,} \quad t_1 = (a/g)^{1/2}(\pi - \pi/3)$$
$$= (a/g)^{1/2}(2\pi/3)$$

Thus, particle has velocity V in upward direction and it moves above A. But thee string becomes slack in upward motion from A so the S.H.M. ceases at A and the particle moves vertically upwards freely under gravity till its velocity V is destroyed. Let t_2 be the time taken by the particle from A till its velocity V becomes zero. Then, using formula

$$v = u - gt, \text{ we get}$$
$$0 = \mathrm{V} - gt_2$$

Or,

$$t_2 = \mathrm{V/g}$$
$$= (3ag)^{1/2}/g$$
$$= (3a/g)^{1/2}$$

Conditions being the same, the particle will take time t_2 in falling freely back to A. Again from A to C the time taken by the particle will be t_1 (which was taken by it to move from C to A). So, the required time

$$= 2(t_1 + t_2) = 2\{a/g)^{1/2}(2\pi/3) + (3a/g)^{1/2}\}$$
$$= (a/g)^{1/2}(4\pi/3 + 2\sqrt{3})$$

Question-5(c) Find the curvature and torsion of the curve $x = a\cos t, y = a\sin t, z = bt$.

[8 Marks]

Solution: The position vector $(\vec{r})$ of the curve at any point of the time (t) can be given as:

$$\vec{r}(t) = a\cos t\hat{\imath} + a\sin t\hat{\jmath} + bt\hat{k}$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -a\cos t\hat{i} - a\sin t\hat{j}$$

$$\Rightarrow \frac{d^3\vec{r}}{dt^3} = a\sin t\hat{i} - a\cos t\hat{j}$$

(1) **Curvature**:

$$\kappa = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{\partial^2\vec{r}}{\partial t^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} \qquad \ldots(1)$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & -a\cos t & 0 \\ -acost & -asint & 0 \end{vmatrix} = ab\sin t\hat{i} + ab\cos t\hat{j} + a^2\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + a^4}$$
$$= \sqrt{a^2b^2\left(\sin^2 t + \cos^2 t\right) + a^4}$$
$$= \sqrt{a^2\left(a^2 + b^2\right)}$$
$$= a\left(a^2 + b^2\right)^{1/2}$$

$$\Rightarrow \left|\frac{dr}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$
$$= \left(a^2 + b^2\right)^{1/2}$$

$$\therefore \quad \text{From (1),} \quad \kappa = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} = \frac{a\left(a^2 + b^2\right)^{1/2}}{\left(a^2 + b^2\right)^{3/2}}$$

$$\Rightarrow \kappa = \frac{a}{a^2 + b^2}$$

(2) **Torsion**:

$$\tau = \frac{\left[\frac{dr}{dt}\,\frac{d^2r}{\partial t^2}\,\frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

$$\left[\frac{dr}{dt}\,\frac{d^2r}{dt^2}\,\frac{d^3r}{dt^3}\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \text{-asint} & \text{acost} & b \\ \text{-acost} & \text{-asint} & 0 \\ \text{asint} & \text{-acost} & 0 \end{vmatrix}$$

$$= b\left(a^2\cos^2 t + a^2\sin^2 t\right)$$

$$= a^2 b$$

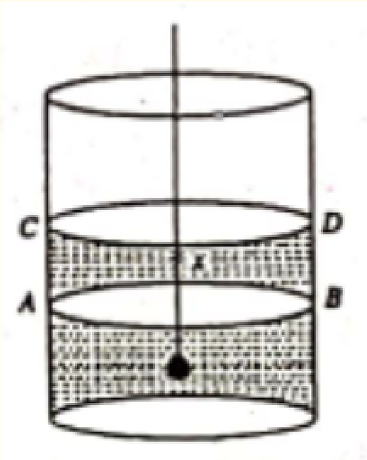
$$\Rightarrow \tau = \frac{a^2 b}{(a^2(a^2+b^2))}$$

$$\Rightarrow \tau = \frac{b}{a^2+b^2}$$

Question-5(d) A cylindrical vessel on a horizontal circular base of radius a is filled with a liquid of density w with a height h. If a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface.

[8 Marks]

Solution: Let the level of the liquid in the vessel be AB before the immersion of the sphere. After the sphere is immersed, let the level of the liquid be CD. If x be the increased height when the level is raised the $AC = BD = x$.



Since the volume of the liquid displaced by the sphere must be equal to the volume of the sphere, so we have $\Rightarrow$

$$\pi a^2 x = \frac{4}{3}\pi c^3$$

$$x = \frac{4}{3}\left(\frac{c^3}{a^2}\right)$$

Now, the whole pressure on the curved surface before immersion

$$\begin{aligned} &= \mathrm{P}_1 = 2\pi ah \cdot \tfrac{1}{2}h \cdot wg \\ &= \pi ah^2 wg \end{aligned}$$

Whole pressure on the curved surface after

$$\begin{aligned} \text{immersion } &= \mathrm{P}_2 \\ &= 2\pi a(h+x) \cdot \frac{1}{2}(h+x)wg \\ &= \pi a(h+x)^2 wg \end{aligned}$$

$\therefore$ Increase of whole pressure on the curved surface $= \mathrm{P}_2 - \mathrm{P}_1$

$$\begin{aligned} &= \pi awg\left[(h+x)^2 - h^2\right] \\ &= \pi awg\left(x^2 + 2hx\right) \\ &= \pi awgx(x+2h) \end{aligned}$$

$$= \pi awg\frac{4}{3} \cdot \left(\frac{c^3}{a^2}\right)\left(\frac{4}{3} \cdot \frac{c^3}{a^2} + 2h\right)$$

$$= \frac{8\pi}{3a} wgc^3 \left(h + \frac{2c^3}{3a^2}\right) \cdots$$

Question-5(e) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

[8 Marks]

Solution: The given equation is

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

Put $x = e^z$ so that $z = \log x$ and let $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation reduces to

$$[\mathrm{D}(\mathrm{D}-1) + 3\mathrm{D} + 1]y = \frac{1}{(1-e^x)^2}$$

$$\left(\mathrm{D}^2 + 2\mathrm{D} + 1\right) y = \frac{1}{(1-e^x)^2}$$

Auxiliary equation is $D^2 + 2D + 1 = 0$.
$\Rightarrow (\mathrm{D}+1)^2 = 0 \quad \Rightarrow \quad \mathrm{D} = -1, -1 \therefore \quad$ C.F. $= (c_1 + c_2 z)\, e^{-z} = (c_1 + c_2 \log x)\, \frac{1}{x}$
Now,

$$P.I. = \frac{1}{\mathrm{D}^2 + 2\mathrm{D} + 1} \cdot \frac{1}{(1-e^z)^2} = \frac{1}{(\mathrm{D}+1)^2} \cdot \frac{1}{(1-e^z)^2}$$

$$= \frac{1}{(D+1)(D+1)} \cdot \frac{1}{(1-e^z)^2} == \frac{1}{(D+1)}\left[\frac{1}{(D+1)}(1-e^z)^2\right]$$

$$= \frac{1}{\mathrm{D}+1} e^{-z} \int \frac{1}{(1-e^z)^2} \cdot e^z dz \quad \left[\because \frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx\right]$$

$$= \frac{1}{\mathrm{D}+1} \cdot e^{-z} \int (1-t)^{-2} dt, \text{ where } e^x = t$$

$$= \frac{1}{\mathrm{D}+1} . e^{-z} \left[- \int (1-t)^{-2} (-dt) \right]$$

$$= \frac{1}{D+1} . e^{-z} \frac{-(1-t)^{-1}}{-1} \left[\because \int [f(x)]^n \cdot f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \frac{1}{D+1} \cdot e^{-z} \cdot \frac{1}{1-t} = \frac{1}{D+1} \cdot e^{-z} \frac{1}{1-e^z} \left[\because t = e^t \right]$$

$$= \frac{1}{\mathrm{D}+1} \cdot \frac{e^{-z}}{1-e^z}$$

$$= e^{-z} \int \frac{e^{-z}}{1-e^z} \cdot e^z dz$$

$$= e^{-z} \int \frac{dz}{1-e^z}$$

$$= e^{-z} \int \frac{e^{-z}}{e^{-z}-1} dz \text{ (Mult. num and den by } e^{-z})$$

$$= -e^{-z} \int \frac{-e^{-z}}{e^{-z}-1} dz$$

$$= -e^{-z} \log \left(e^{-z} - 1 \right)$$

$$= -\frac{1}{x} \log \left(\frac{1}{x} - 1 \right)$$

$$= -\frac{1}{x} \log \left(\frac{1-x}{x} \right) = \frac{1}{x} \log \left(\frac{1-x}{x} \right)^{-1}$$

$$= \frac{1}{x} \log \frac{x}{1-x}$$

Hence the complete solution is

$$y = (c_1 + c_2 \log x) \frac{1}{x} + \frac{1}{x} \log \frac{x}{1-x}$$

Question-6(a) Solve

$$x \frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3 y = 8x^3 \sin x^2$$

by changing the independent variable.

[10 Marks]

Solution:

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - \left(4x^2\right)y = 8x^2 \sin x^2$$

Comparing it with

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + \varphi y = R$$

$$p = -\frac{1}{x}, \quad Q = -4x^2, \quad R = -8x^2 \sin x^2$$

Let

$$\left(\frac{dz}{dx}\right)^2 = \pm a^2 Q = -4x^2$$

(for $a = 1$)

$$\frac{dz}{dx} = 2x \quad \therefore \quad z = x^2$$

(Note that $\frac{dz}{dx} = e^{-\int P dx}$ is not working here)

$$\text{Now,} \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{2 + \left(\frac{-1}{x}\right)2x}{4x^2}$$

$$Q_1 = \frac{Q}{(dz)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{R}{(d2/dx)^2} = \frac{8x^2 \sin x^2}{4x^2} = 2\sin x^2$$

Transformed Eqn is:

$$\frac{d^2y}{dz^2} + p_1\frac{dy}{dz} + \varphi_1 y = R_1$$

$$\frac{d^2y}{dz^2} - 1(y) = 2\sin z$$

$$\text{ie.} \quad \left(D'^2 - 1\right)y = 2\sin 2$$

Auxiliary Eqn: $\quad D'^2 - 1 = 0$

$$D' = 1, -1$$

$$c \cdot F \cdot = c_1 e^2 + c_2 e^{-2}$$

$$= c_1 e^{x^2} + c_2 e^{-x^2}$$

$$p \cdot I_1 = \frac{1}{D'^2 - 1} 2\sin z$$

$$= \frac{2}{(-1^2) - 1}\sin z = -\sin z$$

$$= -\sin x^2$$

Hence, complete solution is:

$$y = CF + PI$$

$$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$$

Question-6(b) The forces P, Q and R act along three straight lines $y = b$, $z = -c$, $z = c$, $x = -a$ and $x = a$, $y = -b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action.

[10 Marks]

Solution: The forces X, Y, Z act along the lines

$$y = b, z = -c; \quad z = c, x = -a; \quad x = a, \quad y = -b$$

The equations of these lines are

$$\frac{x-0}{1} = \frac{y-b}{0} = \frac{z+c}{0}, \frac{x+a}{0} = \frac{y-0}{1} = \frac{z-c}{0}, \frac{x-a}{0} = \frac{y+b}{0} = \frac{z-0}{1}$$

The forces acting on the body are as follows : (i) A force X acting at the point $(0, b, -c)$ along the line whose d.c's a $< 1, 0, 0 >$ (ii) A force Y acting at the point $(-a, 0, c)$ along the line whose de 'c $< 0, 1, 0 > < 0.0, 1 > \therefore$ The components of the forces parallel to the axes are

$$X_1 = X.1 = X, \quad X_2 = Y.0 = 0, \quad X_3 = Z.0 = 0$$
$$Y_1 = X.0 = 0, \quad Y_2 = Y.1 = Y, \quad Y_3 = Z.0 = 0$$
$$Z_1 = X.0 = 0, \quad Z_2 = Y.0 = 0, \quad Z_3 = Z.1 = Z$$

If the system reduces to a single force $R = (X.Y.Z\ )$ arting at Ω. couple $G = (L, M, N)$, then

$$X = \Sigma X_1 = X_1 + X_2 + X_3 = X + 0 + 0 = X$$
$$Y = \Sigma Y_1 = 0 + Y + 0 = Y$$

and

$$Z = \Sigma Z_1 = 0 + 0 + Z = Z$$

To find L, M, N (i) Consider

$$\hat{i}L_1 + j\hat{M}_1 + \hat{k}N_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ X_1 & Y_1 & Z_1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & -c \\ X & 0 & 0 \end{vmatrix}$$
$$= \hat{i}(0) - \hat{j}(cX) + \hat{k}(-bX)$$
$$\therefore \quad L_1 = 0, M_1 = -cX, N_1 = -bX$$
$$\therefore \hat{i}L_2 + jM_2 + \hat{k}N_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{h} \\ -a & 0 & c \\ 0 & Y & 0 \end{vmatrix}$$
$$= \hat{i}(-cY) - \hat{j}(0) + \hat{k}(-aY)$$
$$\therefore \quad L_2 = -cY, \quad M_2 = 0; \quad N_2 = -aY$$
$$\text{(iii) } \hat{i}L_3 + \hat{j}M_3 + \hat{k}N_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_3 & y_3 & z_3 \\ X_3 & Y_3 & Z_3 \end{vmatrix}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & -b & 0 \\ 0 & 0 & Z \end{vmatrix}$$
$$= \hat{i}(-bZ) - \hat{j}(aZ) + \hat{k}(0)$$

$$\therefore \quad L_3 = -bZ, M_3 = -aZ, N_3 = 0$$

Here

$$L = \Sigma L_1 = -(bZ + cY), M = \sum M_1 = -(cX + aZ)$$

and

$$N = \Sigma N_1 = -(bX + aV$$

The system is equivalent to a single form if

$$LX + MY + NZ = 0$$

Substituting the values of L, M in

$$- \mid (bZ + cY)X + (cX + aZ)Y + (bX + aY)Z1 - n$$

or

$$2|aYZ + bZX + cXY| = 0$$

or

$$\frac{a}{X} + \frac{b}{Y} + \frac{c}{Z} = 0$$

which is the required condition. The equations of the line of action of the single force is of the central axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = 0$$

$\therefore$ The equations of the line of action of the single resultant force are any two of the following three:

$$L - yZ + zY = 0, M - zX + xZ = 0, N - xY + yX = 0$$

or

$$\begin{aligned} -(bZ + cY) - yZ + zY = 0 \\ -(cX + aZ) - zX + xZ = 0 \\ -(bX + aY) - xY + yX = 0 \end{aligned}$$

Dividing these equations by YZ, ZX and XY respectively, we get

$$\begin{aligned} -\left(\tfrac{b}{Y} + \tfrac{c}{Z}\right) - \tfrac{y}{Y} + \tfrac{z}{Z} = 0, -\left(\tfrac{c}{Z} + \tfrac{a}{X}\right) - \tfrac{z}{Z} + \tfrac{x}{X} = 0 \\ -\left(\tfrac{b}{Y} + \tfrac{a}{X}\right) - \tfrac{x}{X} + \tfrac{y}{Y} = 0 \end{aligned}$$

Using (2), we have

$$\frac{a}{X} - \frac{y}{Y} + \frac{z}{Z} = 0; \frac{b}{Y} - \frac{z}{Z} + \frac{x}{X} = 0, \frac{c}{Z} - \frac{x}{X} + \frac{y}{Y} = 0$$

or

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

Hence, the equations to its line of action are any two of the three

$$\frac{y}{Y} - \frac{z}{Z} - \frac{a}{X} = 0, \quad \frac{z}{Z} - \frac{x}{X} - \frac{b}{Y} = 0, \quad \frac{x}{X} - \frac{y}{Y} - \frac{c}{Z} = 0$$

Question-6(c) **Solve**

$$\left(D^4+D^2+1\right)y=e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right),$$

where $D\equiv\dfrac{d}{dx}$.

[10 Marks]

Solution: Solve the differential equation

$$\left(D^4+D^2+1\right)y=e^{-x/2}\cos\left(x\frac{\sqrt{3}}{2}\right)$$

Sol. Here the auxiliary equation is $m^4+m^2+1=0$ or

$$\left(m^2+1\right)^2-m^2=0$$

$$\left(m^2+m+1\right)\left(m^2-m+1\right)=0$$

$$\begin{aligned}
m&=\frac{-1\pm\sqrt{(1-4)}}{2},\frac{1\pm\sqrt{(1-4)}}{2}\\
&=-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i,\frac{1}{2}\pm\frac{1}{2}\sqrt{3}i\\
\therefore\quad C.F.&=c_1e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_2\right)+c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x+c_4\right)
\end{aligned}$$

$$\begin{aligned}
\therefore\quad \text{Also P.I.}&=\frac{1}{D^4+D^2+1}e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=e^{-x/2}\frac{1}{\left(D-\frac{1}{2}\right)^4+\left(D-\frac{1}{2}\right)^2+1}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos\left(\frac{1}{2}\sqrt{3}x\right)
\end{aligned}$$

We observe that $D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}$ becomes zero on putting $D^2=-3/4$.
So $D^2+\frac{3}{4}$ must be one of its factors.
By actual division, we get the other factor.

$$\begin{aligned}
\text{So the P.I.}&=e^{-x/2}\frac{1}{\left(D^2+\frac{3}{4}\right)\left(D^2-2D+\frac{7}{4}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=e^{-x/2}\frac{1}{\left(-\frac{3}{4}-2D+\frac{7}{4}\right)\left(D^2+\frac{3}{4}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=e^{-x/2}\frac{1+2D}{\left(1-4D^2\right)\left(D^2+\frac{3}{4}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=e^{-x/2}\frac{1+2D}{\left[1-4(-3/4)\right]\left(D^2+\frac{3}{4}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\\
&=\frac{1}{4}e^{-x/2}\frac{1}{D^2+\frac{3}{4}}(1+2D)\cos\left(\frac{1}{2}\sqrt{3}x\right)
\end{aligned}$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+\frac{3}{4}}\left[\cos\left(\frac{1}{2}\sqrt{3}x\right) - \sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+\frac{3}{4}}\cos\left(\frac{1}{2}\sqrt{3}x\right) - \frac{1}{4}\sqrt{3}e^{-x/2}\frac{1}{D^2+4}\sin\left(\frac{1}{2}\sqrt{3}x\right)$$

$$= \frac{1}{4}e^{-x/2}\cdot\frac{x}{2\cdot\left(\frac{1}{2}\sqrt{3}\right)}\sin\left(\frac{1}{2}\sqrt{3}x\right) - \frac{1}{4}\sqrt{3}e^{-x/2}\left[-\frac{x}{2.\left(\frac{1}{2}\sqrt{3}\right)}\cos\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$= \frac{x}{4\sqrt{3}}e^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right) + \frac{x}{4}e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$= \frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right) + \frac{x}{4}e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

Hence the general solution is $y = (\text{ C.F. }) + (\text{ P.I. })$

$$y = c_1e^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x + c_2\right) + c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x + c_4\right)$$

$$+\frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right) + \frac{1}{4}xe^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$y = e^{-x/2}\left[c_1\cos\left(\frac{1}{2}\sqrt{3}x\right) + c_2\sin\left(\frac{1}{2}\sqrt{3}x\right)\right] + c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x + c_4\right)$$

$$+\frac{1}{12}\sqrt{3}xe^{-x/2}\sin\left(\frac{1}{2}\sqrt{3}x\right) + \frac{1}{4}xe^{-x/2}\cos\left(\frac{1}{2}\sqrt{3}x\right)$$

$$y = e^{-x/2}\left[\left(\frac{1}{4}x + c_1\right)\cos\left(\frac{1}{2}\sqrt{3}x\right) + \left(\frac{1}{12}\sqrt{3}x + c_2\right)\sin\left(\frac{1}{2}\sqrt{3}x\right)\right]$$

$$+c_3e^{x/2}\cos\left(\frac{1}{2}\sqrt{3}x + c_4\right)$$

Question-6(d) Examine if the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F} = \operatorname{grad}\phi$.

[10 Marks]

Solution: $\vec{F}$ is irrotational if Curl $| F = \nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= i\left(3x^2z^2 - 3x^2z^2\right) + j\left(6xyz^2 - 6xyz^2\right)$$
$$+ k\left(2xz^3 - 2xz^3\right) = 0$$

$\therefore \vec{F}$ is irrotational.

$$\vec{F} = grad\phi = \nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\left.\begin{array}{l}\therefore \frac{\partial \phi}{\partial x} = 2xyz^3 \\ \frac{\partial \phi}{\partial y} = x^2z^3 \\ \frac{\partial \phi}{\partial z} = 3x^2yz^2\end{array}\right\} \Rightarrow \phi = x^2yz^3 + c$$

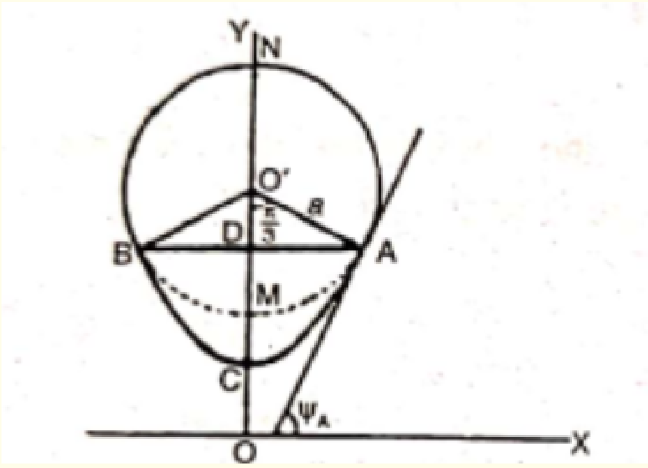
$\therefore$ Scalar potential,

$$\phi = x^2yz^3 + C$$

Question-7(a) Determine the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley.

[15 Marks]

Solution: Let, $ANBMA$ be the circular pulley of radius a and $ANBCA$ the endless chain hanging over it.



Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain.

$$= \frac{2}{3}(\text{circumference of the pulley})$$

$$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a$$

Let, the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e., the vertex, CO'N the axis and OX the directrix of this catenary.

Let, OC $= c =$ the parameter of the catenary. The tangent at A will be perpendicular to the radius O$'$A $\therefore$ If the tangent at A is inclined at an angle ψ_{A} to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B)$$

$$= \frac{1}{2}\left(\frac{1}{3} \cdot 2\pi\right) = \frac{\pi}{3}$$

From the triangle $AO'D$, we have

$$\text{DA} = \text{O}'\text{A} \sin\frac{1}{3}\pi = a\sqrt{3/2}$$

$\therefore$ From $x = c\log(\tan\psi + \sec\psi)$; for the point A, we have

$$x = \mathrm{DA} = c\log\cdot(\tan\psi_{\mathrm{A}} + \sec\psi_{\mathrm{A}})$$

$$\text{Or, } \frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$

$$= c\log(\sqrt{3}+2)$$

$$\therefore \quad c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $s = c$ tan ψ applied for the point A, we have

$$\text{arc}\,\mathrm{CA} = c\tan\psi_{\mathrm{A}} = c\tan\frac{1}{3}\pi = c\sqrt{3}$$

$$= \frac{3a}{2\log(2+\sqrt{3})}$$

Hence, the total length of the chain = arc ABC+ length of the chain in contact with the pulley

$$= 2\cdot(\text{arc}\,\mathrm{CA}) + \frac{4}{3}\pi a$$

$$= 2\frac{3a}{2\log(2+\sqrt{3})} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}$$

Question-7(b) Using divergence theorem, evaluate

$$\iint_S \left(x^3dydz + x^2ydzdx + x^2zdydx\right)$$

where S is the surface of the sphere $x^2+y^2+z^2=1.$

[15 Marks]

Solution: Divergence Theorem states that

$$\iint_S (\vec{F}\cdot\hat{n})dS = \iiint_V (\nabla\cdot\vec{F})dV$$

Here,

$$I = \iint_S \left(x^3dydz + x^2ydzdx + x^2zdydx\right)$$

$$= \iint_S \left(x^3i + x^2yj + x^2zk\right)\cdot\hat{n}dS$$

$$\therefore \vec{F} = x^3i + x^2yj + x^2z\hat{k}$$

$$\Rightarrow \nabla \cdot \vec{F} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$
$$= 3x^2 + x^2 + x^2 = 5x^2$$

Hence,

$$I = \iiint\limits_V 5x^2 dV$$

where V is volume of sphere $x^2 + y^2 + z^2 = 1$.
Converting to spherical corrdinates

$$x = r\sin\theta\cos\phi$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

Limits:
r varies from 0 to 1
θ varies from 0 to π
ϕ varies from 0 to 2π

$$\therefore \iiint\limits_V 5x^2 dV = 5\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\sin\theta\cos\phi)^2 r^2 \sin\theta dr d\theta d\phi$$
$$= 5\int_0^1\int_0^{\pi}\int_0^{2\pi} r^4 \sin^3\theta\cos^2\phi dr d\theta d\phi$$
$$= 5\int_0^1\int_0^{\pi}\int_0^{2\pi} r^4 \sin^3\theta \left[\frac{1+\cos 2\phi}{2}\right] dr d\theta d\phi,$$
$$= \frac{5}{2}\int_0^1\int_0^{\pi} r^4 \sin^3\theta \left[\phi + \frac{\sin 2\phi}{2}\right]_0^{2\pi} dr d\theta$$
$$= \frac{5}{2} \times \left[\frac{r^5}{5}\right]_0^1 [2\pi + 0]\int_0^{\pi}\sin^3\theta d\theta$$
$$= \frac{1}{2} \times 2\pi \times 2 \times \int_0^{\pi/2}\sin^3\theta d\theta$$
$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx,\ \text{if } f(2a-x) = f(x)\right]$$
$$= 2\pi \times \frac{2}{3} = \frac{4\pi}{3}(\text{ Using Walli's formula for definite integral})$$

Walli's Formula:

$$I = \int_0^{\pi/2}\sin^n x dx = \int_0^{\pi/2}\cos^n x dx$$

If n is even,

$$I = \frac{(n-1)(n-3)(n-5)\ldots 3\cdot 1}{n(n-2)(n-4)\cdots 4\cdot 2} \times \frac{\pi}{2}$$

If n is odd,

$$I = \frac{(n-1)(n-3)(n-5)\cdots\cdot 4\cdot 2}{n(n-2)(n-4)\cdots\cdot 1}$$

Question-7(c) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t.

[10 Marks]

Solution: Let, the particle start from rest from a fixed point O. Let, P be the position of the particle at any time t such that OP $= x$.
Let, v be its velocity at P. Here, the force of resistance is μv (given) which acts in vertical upward direction.
The weight mg of the particle acts in vertically downward direction.

Then, the equation of motion of the particle at any time t is

$$m\ddot{x} = mg - \mu v$$
$$\frac{dv}{dt} = g - \left(\frac{\mu}{m}\right) v$$

Or,

$$dt = \left\{\frac{\frac{1}{(g-\mu v)}}{m}\right\} dv$$

On Integration, we have,

$$t = -\left(\frac{m}{\mu}\right)\log\left(g - \frac{\mu v}{m}\right) + \text{A}$$

where A is a constant.
Initially, at Point O, when $t = 0, v = 0$. Hence,

$$A = \left(\frac{m}{\mu}\right)\log g$$
$$\therefore \quad t = -\left(\frac{m}{\mu}\right)\log\left(g - \frac{\mu v}{m}\right) + \left(\frac{m}{\mu}\right)\log g$$
$$= -\left(\frac{m}{\mu}\right)\log\left(1 - \frac{\mu v}{gm}\right)$$

Or,

$$\log\left(1 - \frac{\mu v}{gm}\right) = -\frac{\mu t}{m}$$
$$\text{Or,} \quad 1^2 - \left(\frac{\mu}{gm}\right) v = e^{-\mu t/m}$$

Or,

$$v = \frac{dx}{dt} = \left(\frac{gm}{\mu}\right)\left\{1 - e^{-\frac{\mu t}{m}}\right\}$$

Or,

$$dx = \left(\frac{gm}{\mu}\right)\left(1 - e^{-\frac{\mu t}{m}}\right) dt$$

On integration, we have

$$x = \left(\frac{gm}{\mu}\right)\left\{t + \left(\frac{m}{\mu}\right) e^{-\mu t/m}\right\} + \text{B}$$

where B is a constant.
Initially, at Point O, when $t=0, x=0$

$$\Rightarrow \mathrm{B}=-\frac{gm^2}{\mu^2}$$

Then,

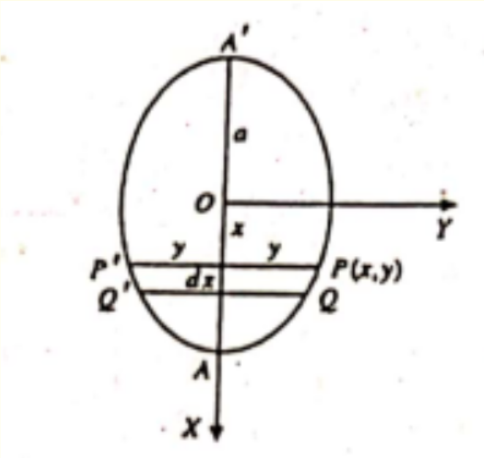
$$x=\left(\frac{gm}{\mu}\right)\left\{t+\left(\frac{m}{\mu}\right)e^{-\mu t/m}\right\}-\frac{gm^2}{\mu^2}$$
$$\text{or}\quad x=\left(\frac{gm^2}{\mu^2}\right)\left\{e^{-(\mu/m)t}-1+\frac{\mu t}{m}\right\}$$

Question-8(a) An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse.

[15 Marks]

Solution: Take the major axis and minor axis respectively as the axes of x and y Then the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



By symmetry it is clear that the C.P. $(\bar{x},\bar{y})$ will lie on the line AOA$'$ i.e., x -axis. $\therefore$

$$\bar{y}=0$$

Take an elementary strip PQQ$'$P$'$ at a depth x below O, the centre of the ellipse, and of width dx. Then $d\,\mathrm{S}=$ area of the elementary strip $=2ydx$ $p=$ intensity of pressure at any point of the strip $=\rho g(a+x)$, where ρ is the density of the liquid. $\therefore\ \bar{x}=$ depth of the C.P. of the ellipse below Point,

$$\mathrm{O}=\frac{\int xpd\,\mathrm{S}}{\int pd\,\mathrm{S}},$$

between suitable limits

$$\frac{\int_{-a}^{a}x\rho g(a+x)2ydx}{\int_{-a}^{a}\rho g(a+x)2ydx}=\frac{\int_{-a}^{a}xy(a+x)dx}{\int_{-a}^{a}y(a+x)dx}$$

The parametric equations of the ellipse (1) are

$$x = a\cos t, y = b\sin t$$
$$\therefore \quad dx = -a\sin t dt$$

Also when, $x = a, \cos t = 1 \Rightarrow t = 0$ and when $x = -a, \cos t = -1 \Rightarrow t = \pi$

$$\begin{aligned}
\therefore \bar{x} &= \frac{\int_\pi^0 a\cos t \cdot b\sin t(a + a\cos t)(-a\sin t dt)}{\int_\pi^0 b\sin t(a + a\cos t)(-a\sin t dt)} \\
&= \frac{a\int_0^\pi \cdot \left(\cos t\sin^2 t + \cos^2 t\sin^2 t\right) dt}{\int_0^\pi \left(\sin^2 t + \cos t\sin^2 t\right) dt} \\
&= \frac{a\left[\int_0^\pi \cos t\sin^2 t dt + \int_0^\pi \cos^2 t\sin^2 t dt\right]}{\int_0^\pi \sin^2 t dt + \int_0^\pi \cos t\sin^2 t dt} \\
&= \frac{a\left[0 + 2\int_0^{\pi/2} \cos^2 t\sin^2 t dt\right]}{2\int_0^{\pi/2} \sin^2 t dt + 0}
\end{aligned}$$

$$= \frac{a\left(\frac{1\cdot 1}{4\cdot 2} \cdot \frac{\pi}{2}\right)}{\frac{1}{2} \cdot \frac{\pi}{2}} = \frac{a}{4}$$

Now, the C.P. of the ellipse will coincide with the focus, if

$$\bar{x} = ae$$

i.e., if

$$\frac{a}{4} = ae$$
$$e = \frac{1}{4}$$

Question-8(b) **If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.**

[10 Marks]

Solution: The boundary C of the surface S is the circle $x^2 + y^2 = a^2$, $z = 0$. Suppose, $x = a\cos t$, $y = a\sin t$, $z = 0$ are parametric eqns of C, where $0 \leq t \leq 2\pi$ By Stokes Theorem,

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C [y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_C ydx + (x - 2xz)dy + xydz$$

$$= \int_C ydx + xdy \quad (\because \text{ On } C_1, z = 0, dz = 0)$$

Converting to parametric form:

$$\iint_s (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_C y dx + x dy$$

$$= \int_0^{2\pi} [a \sin t(-a \sin t) + a\cos t(a \cos t)]dt$$

$$= \int_0^{2\pi} a^2 \left(\cos^2 t - \sin^2 t\right) dt$$

$$= a^2 \int_0^{2\pi} \cos dt = a^2 \left[\frac{\sin 2t}{2}\right]_0^{2\pi} = 0$$

Question-8(c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path.

[15 Marks]

Solution: Here, the central acceleration varies inversely as the cube of the distance *i.e.* $\mathrm{P} = \frac{\mu}{r^3} = \mu u^3$, where μ is a constant. If V is the velocity of a particle along a circle of radius a, then Or,

$$\frac{\mathrm{V}^2}{a} = [\mathrm{P}]_{r=a} = \frac{\mu}{a^3}$$

$$\mathrm{V} = \sqrt{\left(\frac{\mu}{a^2}\right)}$$

$\therefore$ The velocity of projection $v_1 = \sqrt{2}\ \mathrm{V}$

$$= \sqrt{\left(\frac{2\mu}{a^2}\right)}$$

The differential equation of the path is

$$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mathrm{P}}{u^2} = \frac{\mu u^3}{u^2} = \mu u$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$= \mu u^2 + \mathrm{A}...(i)$$

Where, A is a constant. But initially when $r = a$ i.e.,

$$u = \frac{1}{a}, \frac{du}{d\theta} = 0 \text{ (at an apse}$$

and

$$v = v_1 = \sqrt{\left(\frac{2\mu}{a^2}\right)}$$

$\therefore$ From equation, we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + \text{A}$$

$$\therefore h^2 = 2\mu \text{ and A} = \frac{\mu}{a^2}$$

Substituting the values of h^2 and A in (1) we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2$$

Or,

$$= \frac{1 - a^2u^2}{a^2}$$

Or,

$$\sqrt{2}a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)}$$

Or,

$$\frac{d\theta}{\sqrt{2}} = \frac{adu}{\sqrt{(1 - a^2u^2)}}$$

On integration, we have,

$$\left(\frac{\theta}{\sqrt{2}}\right) + \text{B} = \sin^{-1}(au),$$

where B is a constant.
But initially, when

$$u = \frac{1}{a}, \theta = 0$$

$$B = \sin^{-1} 1 = \frac{1}{2}\pi$$

$$\therefore \quad \left(\frac{\theta}{\sqrt{2}}\right) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$$

For,

$$au = \frac{a}{r}$$

$$\frac{a}{r} = \sin\left\{\frac{1}{2}\pi + \left(\frac{\theta}{\sqrt{2}}\right)\right\}$$

or,

$$a = r\cos\left(\frac{\theta}{\sqrt{2}}\right),$$

which is the required equation of the path.