LINEAR ALGEBRA

: CSE - 2012 :

1(c) Prove or disprove the following statement :
If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis of $\mathbb{R}^5$ and V is a two dimensional subspace of $\mathbb{R}^5$, then V has a basis made of just two members of B.

→ Let the given basis $B = \{b_1, b_2, b_3, b_4, b_5\}$ be the standard basis of $\mathbb{R}^5$. Then $b_1 = (1,0,0,0,0)$, $b_2 = (0,1,0,0,0)$, $b_3 = (0,0,1,0,0)$, $b_4 = (0,0,0,1,0)$ and $b_5 = (0,0,0,0,1)$.

Consider a basis $W = \{(1,1,1,1,0), (1,1,1,1,1)\}$ of V.

Then let $a, b \in R$, then $a(1,1,1,1,0) + b(1,1,$

Let $\alpha, \beta \in V$. Then $a\alpha + b\beta \in V$. [since $a, b \in \mathbb{R}$ and $\alpha, \beta \in$ ″]

because: $\alpha = (a+b, a+b, a+b, a+b, b)$, where $a, b, p, q \in \mathbb{R}$
$\beta = (p+q, p+q, p+q, p+q, q)$, then

$$a_1\alpha + b_1\beta = (a_1a + a_1b + b_1p + b_1q,\ a_1a + a_1b + b_1p + b_1q,\ a_1a + b_1b + pb_1 + b_1q,\ a_1a + a_1b + b_1p + b_1q,\ a_1b + b_1q)$$

$$= (r+s, r+s, r+s, r+s, r) \quad \text{where } r = a_1a + b_1p, \ r_2 = a_1b + b_1q$$

$\therefore a_1\alpha + b_1\beta \in V \Rightarrow V$ is a subspace of $\mathbb{R}^5$.

Basis of V is $W = \{(1,1,1,1,0), (1,1,1,1,1)\} \Rightarrow V = \{a+b, a+b, a+b, a+b, b\}$

Now Taking the basis of $\mathbb{R}^5$ two at a time, we have

b_1 & b_2: $(a, b, 0,0,0)$ is the span of b_1, b_2
b_1 & b_3: $(a, 0, b, 0, 0)$ is the span of b_1, b_3
b_1 & b_4: $(a, 0, 0, b, 0)$ is the span of b_1 & b_4
b_1 & b_5: $(a, 0,0,0,b)$ is the span of b_1 & b_5
b_2 & b_3: $(0, a, b, 0, 0)$ is the span of b_2 & b_3
b_2 & b_4: $(0, a, 0, b, 0)$ is the span of b_2 & b_4
b_2 & b_5: $(0, a, 0, 0, b)$ is the span of b_2 & b_5
b_3 & b_4: $(0,0,a,b,0)$ is the span of b_3 & b_4
b_3 & b_5: $(0,0,a,0,b)$ is the span of b_3 & b_5

b_4 & b_5: $(0,0,0,a,b)$ is the span of b_4 & b_5

$\therefore$ No two members of B span V. Hence, the basis of V is not made up of just 2 members of B

1(d) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(\alpha, \beta, \gamma) = (\alpha + 2\beta - 3\gamma,\ 2\alpha + 5\beta - 4\gamma,\ \alpha + 4\beta + \gamma)$. Find a basis and dimension of the image of T and the kernel of T.

→ (i) Image of T: Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$ $\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$ — 1

The echelon form of A has 2 non-zero rows therefore rank of $A = 2 < 3$.

Hence, the given image of T has dimension = 2.

The basis is $S = \{(1,2,-3), (0,1,2)\}$.

(ii) Null space of T: $N_A(T) = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 \mid T[(\alpha, \beta, \gamma)] = (0,0,0)\}$.

Let ~~α_r~~ $= (\alpha, \beta, \gamma) \in N_A(T)$, then, $T(\alpha, \beta, \gamma) = 0$.

i.e. $(\alpha + 2\beta - 3\gamma,\ 2\alpha + 5\beta - 4\gamma,\ \alpha + 4\beta + \gamma) = (0,0,0)$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing to echelon form, we get

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad [\text{from } 1]$$

Therefore, $\beta + 2\gamma = 0 \Rightarrow \beta = -2\gamma$

$\alpha + 2\beta - 3\gamma = 0 \Rightarrow \alpha = -2\beta + 3\gamma$
$= 4\gamma + 3\gamma = 7\gamma$

$$\therefore \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 7\gamma \\ -2\gamma \\ \gamma \end{bmatrix} = \gamma \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Basis of $N_A(T) = \{(7, -2, 1)\}$

& dim. $N_A(T) = \underline{\underline{1}}$

2(a)(i) Let V be the vector space of all 2x2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

$\longrightarrow$ $V = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / \ a,b,c,d \in \mathbb{R} \}$.

$W = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / \ ad-bc=0;\ a,b,c,d \in \mathbb{R} \}$.

Now: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, $|A| = |B| = 0$.

$\therefore A, B \in W$. Now, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A+B| \neq 0$. i.e. $A+B \notin W$.

$\therefore A \in W,\ B \in W$, but $A+B \notin W \Rightarrow$ W is not a subspace of V. as internal composition is violated.

2(a)(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation:

$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$
$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$
$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$\longrightarrow$ Let $A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then, the system of equations reduce to

$$\begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_3 + 5x_4 + x_5 = 0 \Rightarrow x_5 = -2x_3 - 5x_4$.

$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$

$\Rightarrow x_1 = -2x_2 - 3x_3 - 2x_4 - 4x_5 = -2x_2 - 3x_3 - 2x_4 - 4(-2x_3 - 5x_4)$

$= -2x_2 + 5x_3 + 18x_4$.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 + 5x_3 + 18x_4 \\ x_2 \\ x_3 \\ x_4 \\ -2x_3 - 5x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 18 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix}.$$

Then, the required basis is $S = \{(-2,1,0,0,0), (5,0,1,0,-2), (18,0,0,1,-5)\}$.

The dimension$(W) = 3$

2(b)(i) Consider the linear mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (3x+4y, 2x-5y)$. Find the matrix A relative to basis $\{(1,0),(0,1)\}$ and the matrix B relative to basis $\{(1,2),(2,3)\}$

$\longrightarrow$ $f(1,0) = (3,2) = 3(1,0) + 2(0,1)$

$f(0,1) = (4,-5) = 4(1,0) + (-5)(0,1)$

Matrix of T wrt basis $\{(1,0),(0,1)\}$ is $\begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$

Now: Let $(x,y) = a(1,2) + b(2,3)$ where $a,b \in \mathbb{R}$

$\Rightarrow (x,y) = (a+2b, 2a+3b)$

$\Rightarrow x = a+2b, \quad y = 2a+3b$

$\Rightarrow 2x - y = 2a+4b-2a-3b = b$

$\Rightarrow b = 2x-y$, $\quad x = a+2b \Rightarrow a = x-2b$

$a = x-4x+2y$

$a = -3x+2y$

$\therefore (x,y) = (-3x+2y)(1,2) + (2x-y)(2,3)$,

$T(1,2) = (3+8, 2-10) = (11,-8) = (-49)(1,2) + 30(2,3)$

$T(2,3) = (6+12, 4-15) = (18,-11) = (-76)(1,2) + 47(2,3)$

$\therefore$ Matrix of T wrt basis $\{(1,2),(2,3)\}$ is given by

$$\begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

2 (b) (ii) If λ is a characteristic root of a non-singulae matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of adj A.

→ Let λ be a characteristic root of a non-singular matrix A and X be the corresponding characteristic vector.

Since A is non-singular A^{-1} exists. and $A^{-1} = \frac{\operatorname{adj} A}{|A|}$ — ①

Now $AX = \lambda X$

Premultiplying both sides with A^{-1}, we have

$$A^{-1}AX = \lambda A^{-1}X \Rightarrow IX = \lambda A^{-1}X$$

$$\Rightarrow \lambda^{-1}X = A^{-1}X \Rightarrow A^{-1}X = \lambda^{-1}X$$

$$\Rightarrow \frac{1}{|A|}(\operatorname{adj} A)X = \lambda^{-1}X \quad [\text{from ①}]$$

$$\Rightarrow (\operatorname{adj} A)X = \lambda^{-1}|A|X = \frac{|A|}{\lambda}X.$$

$$\Rightarrow (\operatorname{adj} A)X = \frac{|A|}{\lambda}X.$$

$\therefore \frac{|A|}{\lambda}$ is the eigen root of adj A if λ is the eigen value of A.

2(c) Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P^T H P$ is a diagonal matrix.

$\longrightarrow$ $$[H|I] = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 1 & i & i & 1 & 0 \\ 0 & -i & -3 & -(2-i) & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + iR_1$, $R_3 \to R_3 - (2-i)R_1$

$C_2 \to C_2 - iC_1$
$C_3 \to C_3 - (2+i)C_1$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & 1-2i \\ 0 & -i & -3 & -(2-i) & 2i+1 & 6 \end{array}\right]$$

$R_3 \to R_3 + iR_2$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & 1-2i \\ 0 & 0 & -4 & -3+i & 4i+1 & 8+i \end{array}\right]$$

$R_2 \to 4R_2 + iR_3$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 4 & 0 & i-1 & 4+i & 3 \\ 0 & 0 & -4 & -3+i & 4i+1 & 8+i \end{array}\right]$$

$$\therefore P = \begin{bmatrix} 1 & -i & -(2+i) \\ i-1 & 4+i & 3 \\ -3+i & 4i+1 & 8+i \end{bmatrix} \quad \& \quad P^T H P = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$