IFOS 2014

Q.1 For three vectors show that:

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.

Sol^n... Taking LHS:

$$= \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}$$

$$\ldots\left[\because (A\times(B\times C)) = (A.C)\vec{B} - (A.B)\vec{C}\right]$$

$$= (a.c)b - (a.b)c + (a.b)c - (b.c)a + (a.c)b - (b.c)a$$

$$\ldots\left[\because \vec{A}.\vec{B} = \vec{B}.\vec{A}\right]$$

$$= 0$$

Q.2 For the vector $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$, examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A} = \nabla\phi$.

Sol^n... $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ (Given)

$\bar{A}$ is irrotational if $\nabla\times\bar{A} = 0$.

$$\nabla\times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{\Sigma x^2} & \frac{y}{\Sigma x^2} & \frac{z}{\Sigma x^2} \end{vmatrix} = \left(\frac{-2z2y}{(\Sigma x^2)^2} + \frac{2yz}{(\Sigma x^2)^2}\right)\hat{i} + \left(\frac{-2zx}{(\Sigma x^2)^2} + \frac{2zx}{(\Sigma x^2)^2}\right)\hat{j}$$

$$+ \left(\frac{-2xy}{(\Sigma x^2)^2} + \frac{2xy}{(\Sigma x^2)^2}\right)\hat{k} = \vec{0}$$

$\therefore$ $\bar{A}$ is irrotational.

Now, $\bar{A} = \nabla\phi$

$$\Rightarrow \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Comparing: $\frac{\partial\phi}{\partial x} = \frac{x}{(\Sigma x^2)}$ — ① ; $\frac{\partial\phi}{\partial y} = \frac{y}{(\Sigma x^2)}$ — ②

$\frac{\partial\phi}{\partial z} = \frac{z}{(\Sigma x^2)^2}$ — ③

~~Basi~~ Integrating eqⁿ ①: $\boxed{\phi = \frac{1}{2}\log(x^2+y^2+z^2) + f(y) + f(z)}$ — ④

Partially diffⁿ wrt 'y' & 'z' respectively, we get:

$\frac{\partial\phi}{\partial y} = \frac{y}{x^2+y^2+z^2} + f'(y)$

Comparing with eqⁿ ②:

$f'(y) = 0$

$f(y) = C_1$

$\frac{\partial\phi}{\partial z} = \frac{z}{x^2+y^2+z^2} + f'(z)$

Comparing with eqⁿ ③:

$f'(z) = 0$

$f(z) = C_2$

$\therefore$ From eqⁿ ④: $\boxed{\phi = \frac{1}{2}\log(x^2+y^2+z^2) + A}$ ... [where $A = C_1 + C_2$]

Q.3 Evaluate $\iint_S (\nabla\times\bar{A})\cdot\vec{n}\,ds$ for $\bar{A} = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ and S is the surface of hemisphere $x^2+y^2+z^2 = 16$ above xy plane.

Solⁿ... The C is the boundary of S i.e. circle $x^2+y^2 = 16$, $z=0$

$\therefore$ By Stokes Theorem:

$$\iint_S (\nabla \times \vec{A})\cdot \vec{n}\, ds = \oint_C \vec{A}\cdot d\vec{r}$$

$$= \oint_C \left[(x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy\right]$$

... [$\because$ on C, z=0 & dz=0]

Converting to parametric form:

$x = 4\cos t$, $y = 4\sin t$, $z=0$, $0 \le t \le 2\pi$

$$= \oint_0^{2\pi} \left[(16\cos^2 t + 4\sin t - 4)(-4\sin t) + 48\cos t \sin t(4\cos t)\right] dt$$

$$= \int_0^{2\pi} (-64\cos^2 t \sin t - 16\sin^2 t + 16\sin t + 192\cos^2 t \sin t)\, dt$$

$$= 128\int_0^{2\pi} \cos^2 t \sin t\, dt - 16\int_0^{2\pi} \sin^2 t\, dt + 16\int_0^{2\pi} \sin t\, dt$$

$$= 128\times 0 - 16\int_0^{2\pi}\left(\frac{1-\cos 2t}{2}\right) dt + 16\left[-\cos t\right]_0^{2\pi}$$

$$= -16\left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} + 0 = -16\pi$$

$$\therefore \boxed{\iint_S (\nabla \times \vec{A})\cdot \vec{n}\, ds = -16\pi}$$ Ans.

Scanned with CamSc

Q.4 Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2+y^2=4$, $z=0$ & $z=3$

Soln let S denote the closed surface bounded by cylinder $x^2+y^2=4$ and planes $z=0$ & $z=3$.

Acc. to Divergence Theorem:

$$\iint_S (\vec{F}\cdot\vec{n})\,ds = \iiint_V \operatorname{div}\vec{F}\, dV$$

Volume Integral: $\iiint_V \operatorname{div}\bar{A}\, dV = \iiint_V (\nabla\cdot\bar{A})\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4-4y+2z)\, dV$$

$$= 2\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=3} (2-2y+z)\, dx\, dy\, dz.$$

$$= 2\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \left[2z-2yz+\frac{z^2}{2}\right]_0^3 dx\, dy$$

$$= 2\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \left[6-6y+\frac{9}{2}\right] dx\, dy$$

$$= 2\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \left[\frac{21}{2}-6y\right] dx\, dy$$

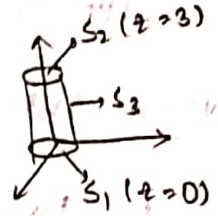
$$= 2\int_{x=-2}^{x=2} \left[\frac{21}{2}y-3y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 2\int_{x=-2}^{x=2} \left[\frac{21}{2}\cdot 2\sqrt{4-x^2}\right] dx = 42\int_{x=-2}^{x=2} \sqrt{4-x^2}\, dx$$

$$= 42 \cdot 2\int_0^2 \sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[\cancel{2}\cdot\frac{\pi}{\cancel{2}}\right] = 84\pi$$



Surface Integral: $\iint_S (\vec{A}\cdot\vec{n})\,ds$

Here the surface S has 3 faces:

(1) $S_1 \rightarrow$ the base i.e. $z=0$ plane

(2) $S_2 \Rightarrow$ the top i.e. $z=3$ plane

(3) $S_3 \Rightarrow$ the convex portion of cylinder.

For S_1: Normal is towards $-\hat{k}$ directⁿ & $z=0$

$$\therefore \iint_{S_1} (\vec{A}\cdot\vec{n})\,ds = \iint_{S_1} (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot(-\hat{k})\,ds$$

$$= \iint_{S_1} -z^2\,ds = 0.$$

For S_2: Normal is towards $\hat{k}$ direction & $z=3$.

$$\therefore \iint_{S_2} (4x\hat{i} - 2y^2\hat{j} + 9\hat{k})\cdot(\hat{k})\,ds = \iint_{S_2} 9\,ds$$

$$= 9\cdot(\pi \times 2^2)$$

$$= 36\pi \quad \ldots\left[\because \text{area of } S_2 = \pi r^2 = 4\pi\right]$$

For S_3: Vector normal to S_3 i.e. $x^2+y^2=4$ is $\nabla(x^2+y^2-4)$

$$= 2x\hat{i} + 2y\hat{j}$$

$$\therefore \hat{n} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \frac{2x\hat{i}+2y\hat{j}}{4} = \frac{x\hat{i}+y\hat{j}}{2} \quad \ldots\left[\because x^2+y^2=4 \text{ on } S_3\right]$$

$$\therefore \text{on } S_3,\ (\vec{A}\cdot\vec{n}) = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$

$$= 2x^2 - y^3$$

Also ds = elementary area on the surface S_3.

$= 2\,d\theta\,dz$ ... [polar co-ordinates $ds = r\,d\theta\,dz$ & $r=2$]

$\therefore \iint_{S_3} (\vec{A}\cdot\vec{n})\,ds = \iint_{S_3} (2x^2 - y^3)\,2\,d\theta\,dz$ .... [$x = 2\cos\theta$, $y = 2\sin\theta$]

$$= \int_{z=0}^{z=3}\int_{\theta=0}^{2\pi} 2(8\cos^2\theta - 8\sin^3\theta)\,d\theta\,dz.$$

$$= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)\,[z]_0^3\,d\theta$$

$$= 16.3\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta$$

$$= 48\left[\int_{\theta=0}^{2\pi} \cos^2\theta\,d\theta - \int_{\theta=0}^{2\pi} \sin^3\theta\,d\theta\right]$$

$$= 48\left[\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)_0^{2\pi} - 0\right]$$ ... [$\sin^3\theta$ is a odd function]

$$= 48\cdot\frac{2\pi}{2} = 48\pi.$$

$$\therefore \iint_S \vec{A}\cdot\vec{n}\,ds = \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3}\right)(\vec{A}\cdot\vec{n})\,ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi.$$

$$\therefore \iint_S (\vec{A}\cdot\vec{n})\,ds = \iiint_V (\nabla\cdot\vec{A})\,dV$$

Hence the divergence theorem is proved.