IFoS-2016 → Paper II

5) (d) Apply Lagrange's interpolation formula to find $f(5)$ and $f(6)$. given that $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(7)=128$.

⇒ The Computational table,

				D_x	y_x	y_x/D_x
$x-1$	-1	-2	-6	$-12(x-1)$	2	$-\frac{1}{6(x-1)}$
1	$x-2$	-1	-5	$5(x-2)$	4	$\frac{4}{5(x-2)}$
2	1	$x-3$	-4	$-8(x-3)$	8	$-\frac{1}{(x-3)}$
6	5	4	$x-7$	$120(x-7)$	128	$\frac{16}{15(x-7)}$

$$\therefore \omega(x) = (x-1)(x-2)(x-3)(x-7)$$

$$\therefore f(x) = (x-1)(x-2)(x-3)(x-7)\left[-\frac{1}{6(x-1)} + \frac{4}{5(x-2)} - \frac{1}{(x-3)} + \frac{16}{15(x-7)}\right]$$

$$= \frac{1}{30}\left[-5(x-2)(x-3)(x-7) + 24(x-1)(x-3)(x-7) - 30(x-1)(x-2)(x-7) + 32(x-1)(x-2)(x-3)\right]$$

$$= \frac{1}{30}\left[-5(x^3-12x^2+41x-42) + 24(x^3-11x^2+31x-21) - 30(x^3-10x^2+23x-14) + 32(x^3-6x^2+11x-6)\right]$$

$$= \frac{1}{30}\left[(-5+24-30+32)x^3 + (60-264+300-192)x^2 + (-205+744-690+352)x + 210-504+420-192\right]$$

$$= \frac{1}{30}\left[21x^3 - 96x^2 + 201x - 66\right]$$

$$= \frac{1}{10}\left[7x^3 - 32x^2 + 67x - 22\right]$$

$\therefore f(x) = \frac{1}{10}[7x^3 - 32x^2 + 67x - 22]$

So $f(5) = \frac{1}{10}[(7\times125) - (32\times25) + (67\times5) - 22]$

$= 38.8$

and $f(6) = \frac{1}{10}[(7\times6^3) - (32\times36) + (67\times6) - 22]$

$= 74$

7) (a) Evaluate $\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}$ by simpson's $\frac{1}{3}$rd rule, by taking 12 equal sub-intervals.

$\Rightarrow$ Here $f(x) = \frac{1}{\sqrt{1-x^2}}$, $a=0$, $b=0.6$, $n=12$

$\Rightarrow h = \frac{0.6-0}{12} = 0.05$

x_i $i=0$ to 12	y_i $i=0$ to 12	y_i $i=0,12$	y_i $i=1,3,5,7,9,11$	y_i $i=2,4,6,8,10$
0.00	1.0000	1.0000	—	—
0.05	1.0013	—	1.0013	—
0.10	1.0050	—	—	1.0050
0.15	1.0114	—	1.0114	—
0.20	1.0206	—	—	1.0206
0.25	1.0328	—	1.0328	—
0.30	1.0483	—	—	1.0483
0.35	1.0675	—	1.0675	—
0.40	1.0911	—	—	1.0911
0.45	1.1198	—	1.1198	—
0.50	1.1547	—	—	1.1547
0.55	1.1974	—	1.1974	—
0.60	1.2500	1.2500	—	—

$\sum y_i = 2.2500 (=Y_1)$ $\quad \sum y_i = 6.4302 (=Y_2)$ $\quad \sum y_i = 5.3197 (Y_3)$

$\therefore$ by Simpson $1/3$rd rule,

$$\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}} dx = \frac{h}{3}\left[y_0 + y_{12} + 4(y_1+y_3+y_5+y_7+y_9+y_{11}) + 2(y_2+y_4+y_6+y_8+y_{10})\right]$$

$$= \frac{h}{3}[Y_1 + 4Y_2 + 2Y_3]$$

$$= 0.6435$$

7) (b) Find the cube root of 10 upto 5 significant figures by Newton-Raphson method.

$\Rightarrow$ Let $x = \sqrt[3]{10} \Rightarrow x^3 = 10$

Using Newton-Raphson Method to $f(x) \equiv x^3 - 10 = 0$, we have the iteration formula as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 10}{3x_n^2} = \frac{2x_n^3 + 10}{3x_n^2}$$

Taking $x_0 = 2$, we have successive approximations,

n	x_n	x_{n+1}
0	2	2.16667
1	2.16667	2.15450
2	2.15450	2.15443
3	2.15443	2.15443

$\therefore \sqrt[3]{10} = 2.1544$, correct upto 5-significant figure.

7) (c) Use classical Fourth-order Runge-Kutta method with $h = 0.2$ to calculate a solution at $x = 0.4$ for the initial value problem $\frac{dy}{dx} = x + y^2$ with initial condition $y = 1$ when $x = 0$.

$\Rightarrow$ For $y(0.2) \Rightarrow x_0 = 0$, $y_0 = 1$, $f(x, y) = x + y^2$ and $h = 0.2$

$\therefore K_1 = hf(x_0, y_0) = 0.2 f(0, 1) = 0.2$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.2 f(0.1, 1.1) = 0.262$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.2 f(0.1, 1.131) = 0.2758$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.2 f(0.2, 1.2758) = 0.3655$

$\therefore y_1 = y(0.2) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6} \times 1.6411$

$= 1.2735$

for $y_2(0.4) \Rightarrow x_1 = 0.2,\ y_1 = 1.2735,\ h = 0.2$

$$K_1 = hf(x_1, y_1) = 0.2f(0.2, 1.2735) = 0.3644$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2f(0.3, 1.4557) = 0.4838$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2f(0.3, 1.5154) = 0.5193$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2f(0.4, 1.7928) = 0.7228$$

$$\therefore y_2 = y(0.4) = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 1.2735 + \frac{1}{6} \times 3.0934$$

$$= 1.7891$$

$\therefore$ The solution at the point $x = 0.4$ is 1.7891