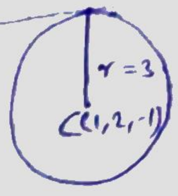
ANALYTIC GEOMETRY

: CSE - 2015 :

① (e) For what the value of a, the plane $ax-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and hence find the point of contact.

→ If the given plane touches the given sphere, then the radius of the sphere is equal to the perpendicular distance of the plane from centre of the sphere.



Centre of the given sphere is $C(1,2,-1)$

Radius of the given sphere is $r = \sqrt{1+4+1+3} = 3$

Distance of the plane from $C \equiv p = \dfrac{a\cdot 1 - 2\cdot 2 + 1\cdot(-1) + 12}{\sqrt{a^2+4+1}}$

$p = r$ [since the plane touches the sphere] $\Rightarrow p^2 = r^2$

$\Rightarrow \dfrac{(a+7)^2}{a^2+5} = 3^2 \Rightarrow a^2+49+14a = 9a^2+45$

$\Rightarrow 8a^2 - 14a - 4 = 0 \Rightarrow a = 2, -\frac{1}{4}$.

The required positive value of a for which the plane touches the sphere is 2.

$\therefore \boxed{a=2}$

Then, the plane becomes $2x-2y+z+12=0$ —— ①

If (α,β,γ) is the point of contact, then, tangent plane to the sphere at (α,β,γ) is $\alpha x+\beta y+\gamma z-(x+\alpha)-2(y+\beta)+(z+\gamma)-3=0$

$(\alpha-1)x+(\beta-2)y+(\gamma+1)z+(-\alpha-2\beta+\gamma-3)=0$ —— ②

If the plane ① touches the sphere at (α,β,γ), the two planes ① & ② are the same. Therefore,

$$\frac{\alpha-1}{2} = \frac{\beta-2}{-2} = \frac{\gamma+1}{1} = \frac{-\alpha-2\beta+\gamma-3}{12}$$

$\dfrac{\alpha-1}{2} = \dfrac{\beta-2}{-2} \Rightarrow -2\alpha+2 = 2\beta-4$
$\Rightarrow 2\alpha+2\beta-6=0 \Rightarrow \alpha+\beta-3=0$ —— ③

$\dfrac{\alpha-1}{2} = \dfrac{\gamma+1}{1} \Rightarrow \alpha-2\gamma-3=0$ —— ④

$\dfrac{\alpha-1}{2} = \dfrac{-\alpha-2\beta+\gamma-3}{12} \Rightarrow 7\alpha+2\beta-\gamma-3=0$ —— ⑤

Solving ③, ④ & ⑤, we get $\alpha = -1,\ \beta = 4,\ \gamma = -2$

$\therefore$ The point of contact is $(-1, 4, -2)$

②(d) If $6x = 3y = 2z$ represents one of the three mutually perpendicular generators of the cone $5yz - 8xz - 3xy = 0$, obtain the equation of the other two generators.

$\rightarrow$ L_1: $6x = 3y = 2z \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ —— ①

The other two generators lie on the plane $\perp$ar to line L_1.

Any plane through $(0,0,0)$ is $Ax + By + Cz = 0$.

It is $\perp$ar to line ① $\equiv$ Normal to this plane is parallel to ①.

$\therefore \frac{A}{1} = \frac{B}{2} = \frac{C}{3}$. $\therefore$ the plane $\perp$ar to line ① is given by.

$$x + 2y + 3z = 0 \text{ —— ②}$$

Let l, m, n be the dcs of the lines of intersection of cone given & the plane ②.

Then, the cone satisfies the dcs of this line.

$\therefore 5yz - 8zx - 3xy = 0 \Rightarrow 5mn - 8nl - 3lm = 0$ —— ③

Also, the lines with dcs l, m, n lies on plane ②. Hence,

$l + 2m + 3n = 0$ —— ④

$l = -(2m + 3n)$ —— ⑤

Putting in ③: $5mn + 8n(2m+3n) + 3m(2m+3n) = 0$

$\Rightarrow 5mn + 16mn + 24n^2 + 6m^2 + 9mn = 0$

$\Rightarrow m^2 + 5mn + 4n^2 = 0 \Rightarrow m^2 + 4mn + mn + 4n^2 = 0$

$\Rightarrow (m+n)(m+4n) = 0 \Rightarrow m + n = 0$ or $m + 4n = 0$

$\frac{m}{1} = \frac{n}{-1}$ and $\frac{m}{4} = \frac{n}{-1}$.

⑤ $\equiv l = -(2m + 3n)$.

(i) $m = -n$: $l = -(-2n + 3n)$

$l = -n$

$\Rightarrow \frac{l}{1} = \frac{n}{-1}$

$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$

$\cancel{\text{(ii) } m = -4n}$

$\cancel{l = -(2m + 3(-4m))}$

$\cancel{l = -(2m - 12m)}$

$\cancel{l = 10m}$

$\cancel{\Rightarrow \frac{m}{1} = \frac{l}{10}}$

(ii) $m = -4n$: $\quad l = -(2(-4n)+3n) = 5n$

$\Rightarrow \frac{l}{-5} = \frac{n}{-1} \quad \Rightarrow \frac{l}{-5} = \frac{m}{4} = \frac{n}{-1}$

$\therefore$ The two lines whose drs are $1, 1, -1$ and $-5, 4, -1$ and pass through the vertex $(0,0,0)$ of the cone are the reqd generators.

The eqn of reqd generators are:

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \quad \& \quad \frac{x}{-5} = \frac{y}{4} = \frac{z}{-1}$$

5 (c)(i) Obtain the equation of the plane passing through the points $(2,3,1)$ and $(4,-5,3)$ and parallel to x-axis.

$\longrightarrow$ Plane through $(2,3,1) \equiv A(x-2) + B(y-3) + C(z-1) = 0$ — ①

It passes through $(4,-5,3) \equiv A(4-2) + B(-5-3) + C(3-1) = 0$

$A - 4B + C = 0$ — ②

It is parallel to x-axis $\Rightarrow$ Normal to plane ① is $\perp$ar to x-axis whose drs are $1, 0, 0$. Hence

$A \cdot 1 + B \cdot 0 + C \cdot 0 = 0 \Rightarrow A = 0.$

② $\equiv \quad -4B + C = 0 \Rightarrow \frac{C}{4} = \frac{B}{1}$

$\therefore \frac{A}{0} = \frac{B}{1} = \frac{C}{4}$ — ③.

Eliminating A, B, C between ① and ③

$0(x-2) + 1(y-3) + 4(z-1) = 0 \Rightarrow y + 4z = 7$ which is the required plane.

3 (c)(ii) Verify that the lines $\frac{x-a+d}{d-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{d+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar. If yes, then find the equation of planes in which they lie.

$\longrightarrow$ The line L_1: $\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}$ passes through $(b-c, b, b+c)$ & has drs $\beta-\gamma, \beta, \beta+\gamma$.

The line L_2: $\frac{x-(a-d)}{d-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{d+\delta}$ passes through $(a-d, a, a+d)$ and has drs $d-\delta, d, d+\delta$.

Cond^n for coplanarity $\equiv \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ where ——— (1)

(x_1,y_1,z_1) & (x_2,y_2,z_2) are any points on L_1 & L_2 respectively

& l_1, m_1, n_1 & l_2, m_2, n_2 are their respective drs.

Now putting the points in LHS of (1) $\quad C_1 \to C_1 + C_3$

$$\begin{vmatrix} (b-c)-(a-d) & b-a & (b+c)-(a+d) \\ \beta-\gamma & \beta & \beta+\gamma \\ \alpha-\delta & \alpha & \alpha+\delta \end{vmatrix} = \begin{vmatrix} 2(b-a) & b-a & (b+c)-(a+d) \\ 2\beta & \beta & \beta+\gamma \\ 2\alpha & \alpha & \alpha+\delta \end{vmatrix}$$

$$= 2\begin{vmatrix} b-a & b-a & (b+c)-(a+d) \\ \beta & \beta & \beta+\gamma \\ \alpha & \alpha & \alpha+\delta \end{vmatrix} = 0$$ as two rows are same.

Hence the two lines are coplanar.

Eq^n of plane containing them is given by $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-(b-c) & y-(b) & z-(b+c) \\ \beta-\gamma & \beta & \beta+\gamma \\ \alpha-\delta & \alpha & \alpha+\delta \end{vmatrix} = 0$$

$$\Rightarrow C_1 \to C_1+C_3 \quad \begin{vmatrix} x+z-2b & y-b & z-(b+c) \\ 2\beta & \beta & \beta+\gamma \\ 2\alpha & \alpha & \alpha+\delta \end{vmatrix} = 0$$

$$C_1 \to C_1 - 2C_2 \quad \begin{vmatrix} x-2y+z & y-b & z-(b+c) \\ 0 & \beta & \beta+\gamma \\ 0 & \alpha & \delta+\alpha \end{vmatrix} = 0$$

$$\Rightarrow (x-2y+z)\,[\beta\delta + \alpha\beta - \alpha\beta - \alpha\gamma] = 0$$

$\Rightarrow \boxed{x-2y+z=0}$ is the equation of plane containing the two lines

4(c) Two perpendicular ~~lines~~ tangent planes to the paraboloid $x^2+y^2=2z$ intersect in a straight line. in the plane $x=0$. obtain the curve to which this straight line touches.

→ Let the line of intersection be given by $my+nz=\lambda,\ x=0$ — (L1)

Any plane through this line is $my+nz-\lambda+kx=0$ ← (1)

⇒ $kx+my+nz-\lambda=0$ — (1) Let $l=k,\ m=m,\ n=n$ & $p=\lambda$

Given paraboloid: $x^2+y^2=2z$ — (2) On comparing with $ax^2+by^2=2cz$,

we have $a=b=1,\ c=1$

If plane (1) is tangent plane to the paraboloid (2), then

$\frac{l^2}{a}+\frac{m^2}{b}+\frac{2pn}{c}=0 \Rightarrow \frac{k^2}{1}+\frac{m^2}{1}+\frac{2\lambda n}{1}=0 \Rightarrow k^2+m^2+2\lambda n=0$

It gives two values of k. Let these two values of k be k_1, k_2

Then, $k_1k_2=\frac{C}{A}=m^2+2\lambda n \Rightarrow k_1k_2=m^2+2\lambda n$ — (3)

The k_1, k_2 gives two perpendicular tangent planes when put in (1)

Then, drs of normals to these two tangent planes are ~~k_1, l, m~~ k_1, m, n and k_2, m, n.

Since, the two planes are ⊥ar to each other, we have

$k_1k_2+m.m+nn=0 \Rightarrow k_1k_2+m^2+n^2=0$

$\Rightarrow m^2+2\lambda n+m^2+n^2=0$ [from (3)]

$\Rightarrow 2m^2+2\lambda n+n^2=0$ — (4)

Now, to prove that line L1 touches the paraboloid (2), we have to find the envelope of (L) which satisfies condⁿ (4).

Eliminating λ between (L1) and (4), the line of intersection of two ⊥ar tangent planes is given by

$2m^2+n^2+2n(my+nz)=0,\ x=0$

$\Rightarrow 2\left(\frac{m}{n}\right)^2+2y\left(\frac{m}{n}\right)+(1+2z)=0,\ x=0$ [Dividing by n^2]

It is quadratic in m/n, so envelope is given by $B^2=4AC,\ x=0$

$4y^2=4.2.(1+2z),\ x=0$

$\Rightarrow \boxed{y^2=2(2z+1),\ x=0}$ which is the required curve.