Chapter

4 Solution of Algebraic and Transcendental Equations

4.1 INTRODUCTION

Determination of roots of an equation of the form $f(x)=0$ has great importance in the fields of science and Engineering. In this chapter we consider some simple methods of obtaining approximate roots of algebraic and transcendental equations.

4.2 DEFINITIONS

1. Polynomial function :

A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

i.e. $f(x)=a_0x^n+a_1x^{n-1}+....+a_{n-1}x+a_n$, where $a_0 \neq 0$, the coefficients $a_0, a_1, ..., a_n$ are real constants and n is a non-negative integer.

2. Algebraic function :

A function which is a sum or difference or product of two polynomials is called an **algebraic function**; otherwise, the function is called a **transcendental** or **non-algebraic function.**

If $f(x)$ is an algebraic function, then the equation $f(x)=0$ is called an algebraic equation.

If $f(x)$ is a transcendental function, then the equation $f(x)=0$ is called a **transcendental equation.**

e.g: $f(x)=c_1e^x+c_2e^{-x}=0$; $f(x)=2\log x-\frac{\pi}{4}=0$; $f(x)=e^{5x}-\frac{x^3}{2}+3=0$

are examples of transcendental equations.

3. Root of an equation :

A number α (real or complex) is called a root (or solution) of an equation $f(x)=0$ if $f(\alpha)=0$. We also say that α is a zero of the function $f(x)$. Geometrically, the roots of an equation are the abscissae of the points where the graph of $y=f(x)$ cuts the x-axis.

The roots of the equation $f(x)=0$ can be obtained by the following two methods.

4.3 ITERATIVE METHODS

In the following section of this chapter, we deal with a number of iterative methods. The basic idea behind these methods is explained here.

Suppose, we have to find a root α of the equation $f(x)=0$. Let x_0 be an approximation to α. Using x_0, we generate a sequence of numbers $x_1, x_2, ...$. Under certain conditions this sequence converges to the root α. The method of generating better and better approximation from an initial guess is called an **Iteration method.**

Order of Convergence :

Let $\varepsilon_i = x_i - \alpha$ be the error in the i^{th} stage. If the sequence $\{x_i\}$ converges to α, then the sequence $\{\varepsilon_i\}$ converges to 0. Suppose error ε_i is related to $\varepsilon_{i+1} = x_{i+1} - \alpha$ by a formula $|\varepsilon_{i+1}| \le k|\varepsilon_i|^p$, where k are p are costants $k > 0, p \ge 1$, then we say that the convergence is of order p.

If $p = 1$, the convergence is said to be **linear.**

If $p = 2$, the convergence is said to be **quadratic.**

If $p = 3$, the convergence is said to be **cubic.**

We can clearly see that the convergence is faster if k is small and p is large.

4.4 DIRECT METHOD

We are familiar with the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$, using direct methods or analytical methods. Analytical methods for the solution of cubic and biquadratic equations are also available. However polynomial equations of degree greater than 4 are not solvable by analytical methods. Analytical methods are not useful in solving most of transcendental equations.

4.5 FALSE POSITION METHOD (REGULA - FALSI METHOD)

In the false position method we will find the root of the equation $f(x) = 0$. Consider two initial approximate values x_0 and x_1 near the required root so that $f(x)$ and $f(x_1)$ have different signs. This implies that a root lies between x_0 and x_1. The curve $f(x)$ crosses $x-$axis only once at the point x_2 lying between the points x_0 and x_1. Consider the point $\mathrm{A} = (x_0, f(x_0))$ and $\mathrm{B} = (x_1, f(x_1))$ on the graph and suppose they are connected by a straight line. Suppose this line cuts $x-$axis at x_2. We calculate the value of $f(x_2)$ at the point. If $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2 and value x_1 is replaced by x_2 (see Fig. (1)). Otherwise the root lies between x_2 and x_1 and the value of x_0 is replaced by x_2 (see Fig.(2)).

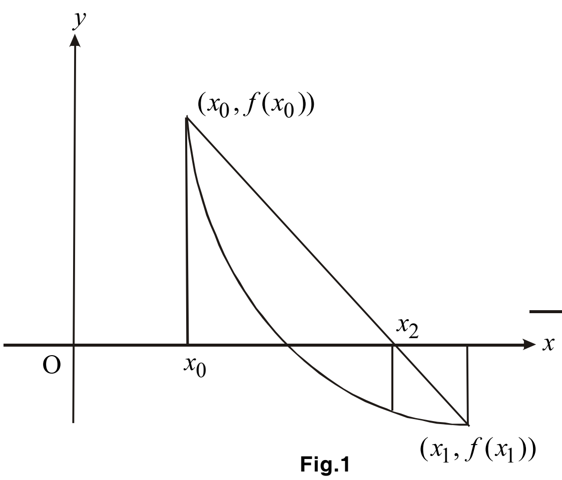


Fig.1

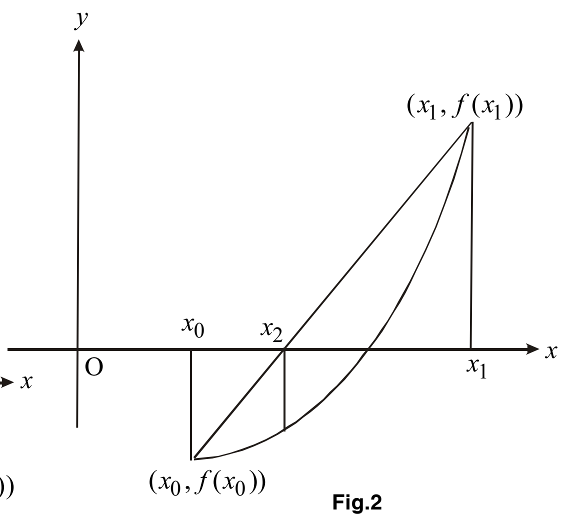


Fig.2

Another line is drawn by connecting the newly obtained pair of values. Again the point here the line cuts the $x-$axis is a closer approximation to the root. This process is repeated as many times as required to obtain the desired accuracy. It can be observed that the points $x_2, x_3, x_4, \ldots\ldots$ obtained converge to the expected root of the equation $y = f(x)$.

To obtain the equation to find the next approximation to the root.

Let $A = (x_0, f(x_0))$ and $B = (x_1, f(x_1))$ be the points on the curve $y = f(x)$. Then the equation to the chord AB is $\dfrac{y - f(x_0)}{x - x_0} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ .... (1)

At the point C where the line AB crosses the $x-$axis, we have $f(x) = 0$ *i.e.* $y = 0$.

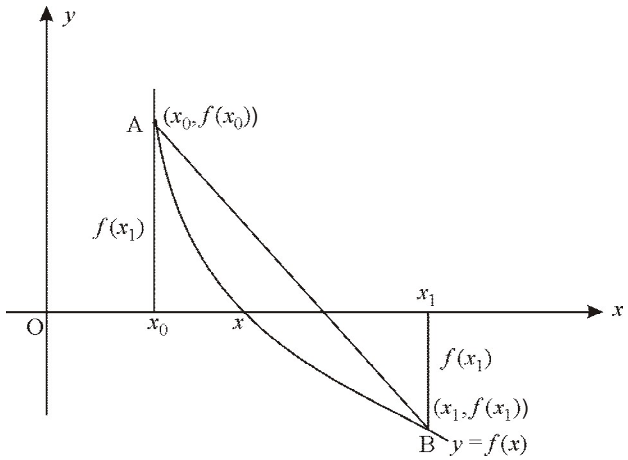


Fig.3

From (1), we get $x = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0)$ ... (2)

x given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0) = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \ldots (3)$$

Now we decide whether the root lies between x_0 and x_2 or x_2 and x_1.

We name that interval as (x_1, x_2). The line joining $(x_1, y_1), (x_2, y_2)$ meets $x-$axis at x_3 is given by $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearer to the exact root. We continue this procedure till the root is found to the desired accuracy.

The iteration process based on (3) is known as the **method of False position.**

The successive intervals where the root lies, in the above procedure are named as $(x_0, x_1), (x_1, x_2), (x_2, x_3)$, etc.., where $x_i < x_{i+1}$ and $f(x_i), f(x_{i+1})$ are of opposite signs.

Also $$\boxed{x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}}$$

SOLVED EXAMPLES

Example 1 : By using Regula-Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations.

[JNTU(A) June 2010 (Set No.1)]

Solution : Let us take $f(x) = x^4 - x - 10$, and $x_0 = 1.8$, $x_1 = 2$.

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$.

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, the equation $f(x) = 0$ has a root between x_0 and x_1.

The first order approximation of this root is

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.8)(4) - 2(-1.3)}{4 - (-1.3)} = \frac{7.2 + 2.6}{5.3} = \frac{9.8}{5.3} = 1.849$$

We find that $f(x_2) = -0.161$ so that $f(x_2)$ and $f(x_1)$ are of opposite signs. Hence, the root lies between x_2 and x_1 and the second order approximation of the root is

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)} = \frac{2(-0.161) - 1.849(4)}{-0.161 - 4} = \frac{7.7182}{4.161} = 1.8549$$

We find that $f(x_3) = f(1.8549) = -0.019$ so that $f(x_3)$ and $f(x_2)$ are of the same sign. Hence, the root doesnot lie between x_2 and x_3. But $f(x_3)$ and $f(x_1)$ are of opposite signs. So the root lies between x_3 and x_1 and the third-order approximation of the root is,

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{1.849(-0.019) - 1.8549(-0.161)}{-0.019 + 0.161} = \frac{0.2635}{0.142} = 1.8557$$

This gives the approximate value of x.

Example 2 : Find the root of the equation $x \log_{10}(x) = 1.2$ using False position method.

[JNTU Aug. 2005S, 2008S, (K)2009S, (A)June 2010, June 2011, May 2012 (Set No. 1)]

Solution : Let $f(x) = x \log_{10} x - 1.2$. Then

$$f(2) = 2 \times \log_{10}(2) - 1.2 = 2 \times 0.30103 - 1.2 = -0.59794$$

and $f(3) = 3 \times \log_{10}(3) - 1.2 = 3 \times 0.47712 - 1.2 = 0.23136$

Since $f(2)$ and $f(3)$ have opposite signs, the root lies between 2 and 3.

Consider $x_0 = 2$ and $x_1 = 3$

By False position method, $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{2 \times 0.23136 - 3 \times (-0.59794)}{0.23136 - (-0.59794)} = 2.7210$$

$f(x_2) = f(2.7210) = 2.721 \times \log_{10} 2.721 - 1.2 = -0.0171$

Now the root lies between 2.721 and 3.

$$x_3 = \frac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)} = \frac{2.721 \times 0.23136 - 3 \times (-0.0171)}{0.23136 - (-0.0171)} = 2.740$$

$f(x_3) = f(2.740) = 2.740 \times \log_{10}(2.740) - 1.2 = -0.00056$

Now, the root lies between 2.740 and 3.

$$\therefore\ x_4 = \frac{x_2 \cdot f(x_3) - x_3 \cdot f(x_2)}{f(x_3) - f(x_2)} = \frac{2.740 \times 0.23136 - 3 \times (-0.00056)}{0.23136 - (-0.00056)} = 2.7406$$

Hence the root is $x = 2.74$.

Example 3 : Find out the roots of the equation $x^3 - x - 4 = 0$ using False position method. **[JNTU (A)June 2010, June 2011 (Set No. 2), Dec 2011, Dec. 2013 (Set No. 1)]**

Solution : Let $f(x) = x^3 - x - 4 = 0$. Then $f(0) = -4$, $f(1) = -4$, $f(2) = 2$

Since $f(1)$ and $f(2)$ have opposite signs, the root lies between 1 and 2.

Consider $x_0 = 1$ and $x_1 = 2$.

By False position method, $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

i.e. $x_2 = \dfrac{(1 \times 2) - 2(-4)}{2 - (-4)} = \dfrac{2+8}{6} = \dfrac{10}{6} = 1.666 \Rightarrow f(1.666) = (1.666)^3 - 1.666 - 4 = -1.042$

Now, the root lies between 1.666 and 2.

$x_3 = \dfrac{1.666 \times 2 - 2 \times (-1.042)}{2 - (-1.042)} = 1.780$. Now $f(1.780) = (1.780)^3 - 1.780 - 4 = -0.1402$

Hence, the root lies between 1.780 and 2.

$x_4 = \dfrac{1.780 \times 2 - 2 \times (-0.1402)}{2 - (-0.1402)} = 1.794$. Now $f(1.794) = (1.794)^3 - 1.794 - 4 = -0.0201$

Hence, the root lies between 1.794 and 2.

$x_5 = \dfrac{1.794 \times 2 - 2 \times (-0.0201)}{2 - (-0.0201)} = 1.796$. Now $f(1.796) = (1.796)^3 - 1.796 - 4 = -0.0027$

Hence, the root lies between 1.796 and 2.

$$x_6 = \frac{1.796 \times 2 - 2 \times (-0.0027)}{2 - (-0.0027)} = 1.796 . \qquad \therefore \text{ The root is } 1.796.$$

Example 4 : Find the positive root of the equation $f(x) = x^3 - 2x - 5 = 0$

[JNTU (K)Nov. 2009S (Set No.1)]

Solution : Given equation is $f(x) = x^3 - 2x - 5 = 0$

We have $f(2) = -1, f(3) = 16$ Thus, a root lies between 2 and 3.

Take $x_0 = 2$, $x_1 = 3$

We have $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \dfrac{2.16 - 3.(-1)}{16+1} = \dfrac{32+3}{17} = \dfrac{35}{17} = 2.059$

Again $f(x_2) = -0.386$, and hence the root lies between 2.059 and 3.

Using $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \dfrac{3.(-0.386) - (2.059)(16)}{-0.386 - (16)} = 2.0812$

Repeating this process we obtain $x_4 = 2.0904$ and $x_5 = 2.0934$, etc.....

We observe that the correct value is 2.0945 and x_5 is corrected to two decimal places only. Thus it is clear that the process of convergence is very slow.

Example 5 : Find the root of the equation $2x - \log_{10} x = 7$, which lies between 3.5 and 4 by regula - falsi method. **[JNTU(A) June 2010 (Set No.4)]**

(or) Find a real root of the equation $2x - \log x = 7$, by successive approximate method.

[JNTU 2006 (Set No. 3)]

Solution : Let $f(x) = 2x - \log_{10} x - 7 = 0$. Take $x_0 = 3.5$, $x_1 = 4$

Then $x_2 = x_0 - \dfrac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0) = 3.5 - \dfrac{0.5}{0.3979 + 0.5441}(-0.5441) = 3.7888$

Now $f(x_2) = -0.0009$, $f(x_1) = 0.3979$

$\therefore$ The root lies between 3.7888 and 4.

$\therefore$ By taking $x_0 = 3.7888$ and $x_1 = 4$, we get $x_3 = 3.7888 - \dfrac{0.2112}{0.3988}(-0.0009) = 3.7893$.

Now $f(x_3) = 0.00004$.

Hence the required root corrected to three decimal places is 3.789.

Example 6 : Find a real root of $xe^x = 3$ using Regula - Falsi method.

[JNTU May 2006 (Set No.4)]

Solution : Let $f(x) = xe^x - 3$.

We have $f(1) = e - 3 = -0.2817 < 0$

$f(2) = 2.e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2.

Take $x_0 = 1$ and $x_1 = 2$.

The first approximation of the root by falsi method is

$$x_2 = x_0 - \left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right) \cdot f(x_0) = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817}$$

$= 1.0234$

Now $f(x_2) = f(1.0234) = (1.0234)\, e^{1.0234} - 3 = -0.1522 < 0$

$f(2) = 11.778 > 0$

$\therefore$ The root lies between 1.0234 and 2.

Taking $x_0 = 1.0234$ and $x_2 = 2$.

we get $$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(1.0234) f(2) - 2 f(1.0234)}{f(2) - f(1.0234)}$$

$$= \frac{(1.0234)(11.778) - 2(-0.1522)}{11.778 - (-0.1522)} = 1.036$$

Now $f(x_3) = (1.036)\, e^{1.036} - 3 = -0.0806 < 0$

$$\therefore x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 1.043 \text{ and } x_5 = 1.046$$

This gives approximate root.

Example 7 : Find a real root for $e^x \sin x = 1$, using Regula Falsi method.

[JNTU Sep. 2006, (H) June 2011 (Set No. 3)]

Solution : Given $e^x \sin x = 1$. Let $f(x) = e^x \sin x - 1 = 0$

We have $f(x_0) = f(0.5) = e^{0.5} \sin(0.5) - 1 = 0.790439 - 1 = -0.20956 < 0$

$f(x_1) = f(0.6) = e^{0.6} \sin(0.6) - 1 = 0.0288 > 0$

$\therefore$ The root lies between 0.5 and 0.6. By Regular Falsi method,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(0.5)(0.0288) - (0.6)(-0.20956)}{0.0288 - (-0.20956)} = \frac{0.140136}{0.23856} = 0.588$$

$$\therefore \quad f(x_2) = e^{0.588} \sin(0.588) - 1 = -0.00133 < 0$$

$\therefore$ Root lies between x_2 and x_1.

$$\text{Now } x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{0.588(0.0288) - 0.6(-0.00133)}{0.0288 + 0.00133} = 0.5885$$

$\therefore \quad f(x_3) = e^{0.5885}\sin(0.5885) - 1 = -0.0000818$

Since $f(x_3)$ is nearly equal to zero, the required root is 0.5885.

Example 8 : Find a real root of $xe^x = 2$ using Regula-falsi method.

[JNTU April 2007, (A) Nov. 2010 (Set No. 4)]

Solution : Let $f(x) = xe^x - 2 = 0$. Then

$f(0) = -2 < 0$; $f(1) = e - 2 = 2.7183 - 2 = 0.7183 > 0$

Take $x_0 = 0, x_1 = 1$.

$\therefore \quad x_2$ lies between 0 and 1.

By Regula - Falsi method,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0-(-2)}{0.7183-(-2)} = \frac{2}{2.7183} = 0.73575$$

$$f(x_2) = -0.46445 < 0$$

$\therefore \quad x_3$ lies between x_1 and x_3.

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{(0.73575)\,(0.7183) - (1)\,(0.46445)}{0.7183 + 0.46445}$$

$$= \frac{0.52848 + 0.46445}{1.18275} = \frac{0.992939}{1.18275} = 0.83951$$

$$\therefore \quad f(x_3) = -0.056339 < 0$$

Now x_4 lies between x_1 and x_3.

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)} = \frac{(0.83951)\,(0.7183) + 0.056339}{0.7183 + 0.056339}$$

$$= \frac{0.65935}{0.774639} = 0.851171$$

$$f(x_4) = -0.006227 < 0$$

Now x_5 lies between x_1 and x_4.

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)} = \frac{(0.851171)\,(0.7183) + 0.006227}{0.7183 + 0.006227}$$

$$= \frac{0.617623}{0.724527} = 0.85245$$

$$\therefore \quad f(x_5) = -0.0006756 < 0$$

Now x_6 lies between x_1 and x_5.

$$x_6 = \frac{x_5 f(x_1) - x_1 f(x_5)}{f(x_1) - f(x_5)} = \frac{(0.85245)\,(0.7183) + 0.0006756}{0.7183 + 0.0006756}$$

$$= \frac{0.612990}{0.71897} = 0.85260$$

$f(x_6) = -0.00002391 < 0$

$\therefore x_7$ lies between x_1 and x_6.

$$x_7 = \frac{x_6 f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)} = \frac{(0.85260)(0.7183) + 0.00002391}{0.7183 + 0.00002391}$$

$$= 0.85260$$

$\therefore$ The root of $xe^x - 2 = 0$ is 0.85260.

Example 9 : Find a real root of the equation, $\log x = \cos x$ using regula falsi method.

[JNTU (H) June 2011 (Set No. 4)]

Solution : Given equation is $\log x = \cos x$

Let $f(x) = \log x - \cos x$

$$f(1) = \log(1) - \cos(1)$$

$$= 0 - 0.5403 = -0.5403 < 0$$

$$f(2) = 0.6931 + 0.4161 = 1.1092 > 0$$

The root lies between 1 and 2.

Take $x_0 = 1$ and $x_1 = 2$.

The first approximation is

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_0)}$$

$$= \frac{(1)(1.1092) - (2)(-0.5403)}{1.1092 + 0.5403}$$

$$= \frac{1.1092 + 1.0806}{1.6495} = \frac{2.1898}{1.6495} = 1.3275$$

$f(x_2) = 0.2832 - 0.2409 = 0.0423 > 0$

$\therefore$ The root lies between x_0 and x_2

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = \frac{(1)(0.0423) - (1.3275)(-0.5403)}{(0.0423) + 0.5403}$$

$$= \frac{0.7595}{0.5826} = 1.3037$$

$f(x_3) = -0.1487 < 0$

The root lies between x_3 and x_2.

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)}$$

$$= \frac{(1.3037)\,(0.0423) - (1.3275)\,(-0.1487)}{0.0423 + 0.1487} = \frac{(0.0551) + (0.1973)}{0.191}$$

$$= \frac{0.2524}{0.191} = 1.3214$$

Thus we take the approximate value of the root is 1.3214.

Example 10 : Find the root of the equation $xe^x = \cos x$ using the Regula false method correct to four decimal places **[JNTU (A) May 2012 (Set No. 3)]**

Solution : Let $f(x) = \cos x - xe^x = 0$. We have, $f(0) = 1$ and $f(1) = -2.1779 < 0$

$\therefore$ A root lies between 0 and 1. Take $x_0 = 0$ and $x_1 = 1$.

By False method,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{0 - 1}{-2.1779 - 1} = 0.3146$$

$$f(x_2) = f(0.3146) = 0.5198 > 0$$

$$f(x_1) = -2.1779 < 0$$

$\therefore$ The root lies between 0.3146 and 1.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(1)(0.5198) - (0.3146)(-2.1779)}{(0.5198) + (2.1779)} = 0.4467$$

$$f(x_3) = f(0.4467) = 0.2035 > 0$$

$$f(x_1) = -2.1779 < 0$$

$\therefore$ The root lies between 0.4467 and 1.

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{1(0.2035) - (0.4467)(-2.1779)}{-0.2035 - 2.1779} = 0.4940$$

Continuing this process we get

$x_5 = 0.5099$; $x_6 = 0.5152$; $x_7 = 0.5169$

$x_8 = 0.5174$; $x_9 = 0.5176$; $x_{10} = 0.5177$

Thus we will take 0.5177 as correct root.

4.7 NEWTON - RAPHSON METHOD (NEWTON'S ITERATIVE METHOD)

The Newton - Raphson method is a powerful and elegent method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$, and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand

$$f(x_1) = f(x_0+h) = 0$$

$$f(x_0 + h) = f(x_0) + hf(x_0) + h^2 f''(x) + \ldots.$$

$$\Rightarrow f(x_0) + h\, f'(x_0) = 0 \;\; \Rightarrow h = -\frac{f(x_0)}{f'(x_0)} \qquad \text{(neglecting } h^2, h^3, \ldots)$$

Substituting this in x_1, we get, $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

x_1 is a better approximation than x_0.

Successive approximations are given by $x_2, x_3, \ldots, x_{n+1}$ where $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$.

This is called **Newton - Raphson formula.**

The iterative method starts with an initial approximation say x_0. Then a tangent is drawn from the corresponding point $f(x_0)$ on the curve $y = f(x)$. Let this tangent cuts the x-axis at a point say x_1 which will be a better approximation of the root. Now compute $f(x_1)$ and draw another tangent at the point $f(x_1)$ on the curve so that it cuts the x-axis at the point say x_2. The value of x_2 gives still better approximation and the process can be continued till the desired accuracy has been achieved.

Graphically this can be shown as in Fig. (4) .

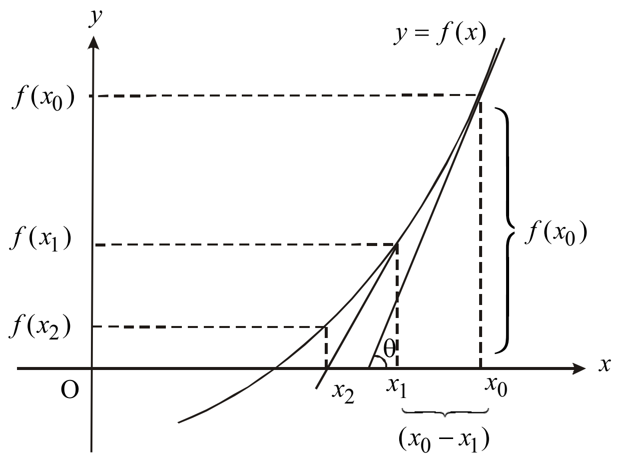


Fig. 4

1. CONVERGENCE OF NEWTON-RAPHSON METHOD

To examine the convergence of Newton-raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad ....(1)$$

We compare it with the general iteration formula

$$x_{i+1} = \phi(x_i)$$

$$\phi(x_i) = x_i - \frac{f(x_i)}{f'(x_i)} \qquad ...(2)$$

In general we write it as

$$\phi(x) = x - \frac{f(x)}{f'(x)} \qquad ...(3)$$

we have already noted that the iteration method converges if $|\phi'(x)| < |$

$\therefore$ Newton Raphson formula equation (1) converages, provided

$$|f(x)f''(x)| < |f'(x)|^2 \qquad ...(4)$$

In the considered interval, Newton - Raphson formula converges provided the initial approximation x_0 is chosen sufficiently close to the root and $f(x)$, $f'(x)$ and $f''(x)$ are continuous as bounded in any small interval containing the root.

2. QUADRATIC CONVERGENCE OF NEWTON-RAPHSON METHOD [JNTU (H) 2010 (Set No. 4)]

Suppose x_r is a root of $f(x) = 0$ and x_i is an estimate of x_r such that $|x_r - x_i| = h << 1$ then by Taylor Series expansion, we have

$$0 = f(x_r) = f(x_i + h) = f(x_i) + f'(x_i)(x_r - x_i) + \frac{f''(\xi)}{2}(x_r - x_i)^2 \text{ for some } \xi \in (x_r, x_i) \ldots (1)$$

By Newton-Raphson method, we know

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow \quad f(x_i) = f'(x_i)(x_i - x_{i+1}) \qquad \ldots (2)$$

Using (2) in (1), we get

$$0 = f'(x_i)(x_r - x_{i+1}) + \frac{f''(\xi)}{2}(x_r - x_i)^2$$

Suppose $e_i = (x_r - x_i)$, $l_{i+1} = x_r - x_{i+1}$, are the errors in the solution at i^{th} and $(i+1)^{th}$ iterations

$$\therefore \quad e_{i+1} = -\frac{f''(\xi)}{2f'(x_r)}e_i^2 \quad \Rightarrow \quad e_{i+1} \; \alpha \; e_i^2$$

$\therefore$ The Newton method is said to have quadratic convergence.

3. Newton-Raphson Extended Formula (or) Chebyshev's Formula of Third Order

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x)} - \dfrac{1}{2}\dfrac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0)$ for finding the root of the equation $f(x) = 0$.

Expanding $f(x)$ by using Taylor's series and neglecting the second order terms in the neighbourhood of x_0, we obtain

$f(x) = f(x_0) + (x - x_0) f'(x_0) \ldots = 0$

It gives $x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

This is the first approximation to the root.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \ldots(1)$$

Again expanding f(x) by Taylor's series and neglecting the third order terms,

we have, $f(x) = f(x_0) + (x - x_0) f'(x_0) + \dfrac{(x - x_0)^2}{2!} f''(x_0) + \ldots 0$

$$\therefore \; f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{(x_1 - x_0)^2}{2!} f''(x_0) = 0 \qquad \ldots(2)$$

Using equation 1, the equation 2 reduces to the form

$$f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^2} f''(x_0) = 0$$

$\therefore$ The Newton - Raphson extended formula or Chebysher's formula of third order is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{[f(x_0)]^2}{[f'(x_0)]^3} f''(x_0).$$

4. Merits and demerits of Newton - Raphson Method

Merits :

1. In this method convergence is quite fast provided the starting value is close to the desired root.
2. If the root is simple, the convergence is quadratic.
3. The accuracy of Newtons method for the function $f(x)$ which possess continues first and second derivatives can be estimated.

   If $\mathrm{M} = \max |f''(x)|$ and $m = \min |f''(x)|$ in an interval that contains the root α and the estimator x_1 and x_2, then $|x_2 - \alpha| \le (x - \alpha)^2 . \dfrac{\mathrm{M}}{m}$

   Thus the error decreases if $\left|(x_1 - \alpha)^2 . \dfrac{\mathrm{M}}{m}\right| < 1$.

4. Newton - Raphson iteration is a single point iteration.

5. This method can used for solving both algebraic and transcendental equations. It can also be used when the roots are complex.

Demerits :

1. In deriving the formula for this method, it is assumed that α is a not a repeated root of $f(x)=0$. In this case the convergence of the iteration is not guarented. Thus the Newton-Raphson method is not applicable to find the approximated values of a repeated root.

2. Most severe limitation in the use of this method is the requirement that $f'(x) \neq 0$ in the neighbourhood of the root α. Even a moderate value of $f'(x_0)$ may more than sampled by a large value of either $f(x_0)$ or $f'(x_0)$ to produce a value x that will result in a sequence that converges to a root that we are not interested.

SOLVED EXAMPLES

Example 1 : Apply Newton - Raphson method to find an approximate root, correct to three decimal places, of the equation $x^3-3x-5=0$, which lies near $x = 2$.

Solution : Here $f(x)=x^3-3x-5=0$ and $f'(x)=3(x^2-1)$.

$\therefore$ The Newton-Raphson iterative formula (6) yeilds in this case,

$$x_{i+1}=x_i-\frac{x_i^3-3x_i-5}{3(x_i^2-1)}=\frac{2x_i^3+5}{3(x_i^2-1)}, \quad i=0,1,2,...... \qquad ....(1)$$

To find the root near $x=2$, we take $x_0=2$. Then (1) gives

$$x_1=\frac{2x_0^3+5}{3(x_0^2-1)}=\frac{16+5}{3(4-1)}=\frac{21}{9}=2.3333, \qquad x_2=\frac{2x_1^3+5}{3(x_1^2-1)}=\frac{2\times(2.3333)^3+5}{3\left\{(2.3333)^2-1\right\}}=2.2806$$

$$x_3=\frac{2x_2^3+5}{3(x_2^2-1)}=\frac{2\times(2.2806)^3+5}{3\left\{(2.2806)^2-1\right\}}=2.2790, \qquad x_4=\frac{2\times(2.2790)^3+5}{3\left\{(2.2790)^2-1\right\}}=2.2790$$

Since x_3 and x_4 are identical upto 3 places of decimal, we take $x_4=2.279$ as the required root, correct to three places of the decimal.

Example 2 : Using the Newton-Raphson method, find the root of the equation $f(x)=e^x-3x$ that lies between 0 and 1. **[JNTU (A) June 2013 (Set No. 1)]**

Solution : Here $f(x)=e^x-3x$ and $f'(x)=e^x-3$.

$\therefore$ The Newton - Raphson iterative formula (6) yeilds

$$x_{i+1}=x_i-\frac{e^{x_i}-3x_i}{e^{x_i}-3}=\frac{(x_i-1)e^{x_i}}{(e^{x_i}-3)}, \quad i=0,1,2,.... \qquad .....(1)$$

Since the required root is supposed to lie between 0 and 1, we take x_0 to be the average of 0 and 1, *i.e.,* $x_0 = 0.5$. Then formula (1) yields.

$$x_1 = \frac{((0.5)-1)\,e^{0.5}}{e^{0.5}-3} = 0.61006, \qquad x_2 = \frac{(0.61006-1)\,e^{0.61006}}{e^{0.61006}-3} = 0.618996$$

$$x_3 = \frac{(0.618996-1)\,e^{0.618996}}{e^{0.618996}-3} = 0.619061, \quad x_4 = \frac{(0.619061-1)\,e^{0.619061}}{e^{0.619061}-3} = 0.619061$$

We observe that x_3 and x_4 are identical, we therefore, take $x \approx 0.619061$ as an approximate root of the given equation.

Example 3 : Using Newton-Raphson Method

(*a*) Find square root of a number (*b*) Find Reciprocal of a number

[JNTU Sep. 2008 (Set No.2)]

Solution : (*a*) Square root:

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found.

The solution to $f(x)$ is then $x = \sqrt{N}$. Here $f'(x) = 2x$. By Newton-Raphson technique,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - N}{2\,x_i} \Rightarrow x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$$

using the above iteration formula the square root of any number N can be found to any desired accuracy. For example (*i*) We will find the square root of N = 24.

Let the initial approximation be $x_0 = 4.8$.

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{2304+24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01+24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904+24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since $x_2 = x_3 = 4.898$, therefore, the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means, the square root of 24 is 4.898.

(ii) To find the square root of 10. **[JNTU Sep. 2008 (Set No. 2)]**

Let $x = \sqrt{10}$. Then $x^2 = 10$

Also let $f(x) = x^2 - 10 = 0$. Then $f'(x) = 2x$

Here, $a = 10$, $x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$

Now $f(3) = 9 - 10 = -1 < 0$ and $f(4) = 16 - 10 = 6 > 0$

$\therefore$ The root lies between 3 and 4.

Let x_0 be the approximate root of the given equation which is 3.8.

$$x_1 = \frac{1}{2}\left[3.8 + \frac{10}{3.8}\right] = 3.21579 \approx 3.216\,; \quad x_2 = \frac{1}{2}\left[3.216 + \frac{10}{3.216}\right] = 3.1627$$

$$x_3 = \frac{1}{2}\left[3.162 + \frac{10}{3.1627}\right] = 3.1627$$

$\therefore$ Since $x_2 = x_3 = 3.162$, therefore, the solution to $f(x) = x^2 - 10 = 0$ is 3.162. Thus we can take square root of 10 as 3.1627.

(*b*) Reciprocal:

Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found.

The solution to $f(x)$ is then $x = \frac{1}{N}$. Also, $f'(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton-Raphson technique,

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{\frac{-1}{x_i^2}} = x_i(2 - x_i N)\,.$$

For example, the calculation of reciprocal of 22 is as follows.

Assume the initial approximation be $x_0 = 0.045$.

$\therefore$ $x_1 = 0.045\,(2 - 0.045 \times 22) = 0.045\,(2 - 0.99) = 0.045\,(1.01) = 0.0454$

$x_2 = 0.0454\,(2 - 0.0454 \times 22) = 0.0454\,(2 - 0.9988) = 0.0454\,(1.0012) = 0.04545$

$x_3 = 0.04545\,(2 - 0.04545 \times 22) = 0.04545\,(2 - 0.9999) = 0.04545\,(1.0001) = 0.04545$

$x_4 = 0.04545\,(2 - 0.04545 \times 22) = 0.04545\,(2 - 0.9999) = 0.04545\,(1.00002) = 0.0454509$

$\therefore$ The reciprocal of 22 is 0.0454509.

Example 4 : Find the reciprocal of 18 using Newton - Raphson method

[JNTU 2004]

Solution : We have by Newton-Raphson method $x_{i+1} = x_i(2 - x_i N)$ **[Refer Ex.3(b)]**

Take the initial approximtion as $x_0 = 0.055 \cdot$ Then

$x_1 = 0.055\,(2 - 0.055 \times 18) = 0.055\,(1.01) = 0.0555$

$x_2 = 0.0555\,(2 - 0.0555 \times 18) = 0.0555\,(1.001) = 0.05555$

Since $x_1 = x_2$, therefore, the reciprocal of 18 is 0.05555.

Example 5 : Evaluate $\sqrt{28}$ to four decimal places by Newton's iterative method.

[JNTU (A) June 2013 (Set No. 2)]

Solution : Let $x = \sqrt{28}$ so that $x^2 - 28 = 0$ ..... (1)

Taking $f(x) = x^2 - 28$, Newton's iterative method gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - 28}{2x_i} = \frac{1}{2}\left(x_i + \frac{28}{x_i}\right) \quad ..... (2)$$

Now since $f(5) = -3$, $f(6) = 8$, a root of (1) lies between 5 and 6.

$\therefore$ Taking $x_0 = 5.5$, (2) gives $x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0}\right) = \frac{1}{2}\left(5.5 + \frac{28}{5.5}\right) = 5.29545$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = \frac{1}{2}\left(5.29545 + \frac{28}{5.29545}\right) = 5.2915$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2}\right) = \frac{1}{2}\left(5.2915 + \frac{28}{5.2915}\right) = 5.2915$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{28} = 5.2915$

Example 6 : Solve the equation $x^3 + 2x^2 + 0.4 = 0$ using Newton-Raphson method.

Solution : Here $f(x) = x^3 + 2x^2 + 0.4 = 0$, $f'(x) = 3x^2 + 4x$

By using the Newton-Raphson formula, the $(i+1)^{th}$ iteration is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad ..... (1) \qquad \text{where } i = 0, 1, 2, ...$$

Clearly, a root lies between –2 and –3, since $f(-2) = 0.4$, $f(-3) = -8.6$

We choose $x_0 = -2$ and obtain the successive iterative values as follows:

First approximation: Put $i = 0$ in the Newton-Raphson formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 - \frac{0.4}{4} = -2.1$$

Second approximation : Put $i = 1$, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2.1 - \frac{(-2.1)^3 + 2(-2.1)^2 + 0.4}{3(2.1)^2 - 4(2.1)}$$

$$= -2.1 + \frac{0.041}{4.83} = -2.0915$$

Third approximation : By putting $i = 2$ in equation (1), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -2.09145$$

Fourth approximation : By putting $i=3$ in (1), we get

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -2.09145$$

Since two iterative values (*i.e.,* third and fourth iterative values) coincide, we stop the process.

Hence the real root of the equation correct to 4 decimal places is – 2.09145.

Example 7 : Derive a formula to find the cube root of N using Newton Raphson method hence find the cube root of 15. **[JNTU (H) June 2011 (Set No. 1)]**

Solution : Let $f(x) = x^3 - \mathrm{N} = 0$, when N is the number whose cube root is to be found.

The solution to $f(x)$ is the $x = \mathrm{N}^{1/3}$

$f'(x) = 3x^2$

Using Newton - Raphson formula,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^3 - \mathrm{N}}{3x_i^2}$$

$$= \frac{3x_i^3 - x_i^3 + \mathrm{N}}{3x_i^2} = \frac{2x_i^3 + \mathrm{N}}{3x_i^2}$$

$$x_{i+1} = \frac{1}{3}\left(2x_i + \frac{\mathrm{N}}{x_i^2}\right) \qquad \ldots (1)$$

Using the above iteration formula, the cube root of any number can be found out.

To find the cube root of 15

Let $\mathrm{N} = 15$

Let the initial approximation be $x_0 = 2.4$

Substituting in (1),

$$x_1 = \frac{1}{3}\left(5 + \frac{15}{(2.5)^2}\right) = \frac{1}{3}\left(5 + \frac{15}{6.25}\right)$$

$$= \frac{1}{3}\left(5 + \frac{3}{1.25}\right) = \frac{1}{3}\left(\frac{6.25+3}{1.25}\right) = \frac{1}{3}\left(\frac{9.25}{1.21}\right) = 2.4666$$

Put $i=1$ in (1). Then

$$x_2 = \left(2x_1 + \frac{15}{x_1^2}\right)$$

$$= \frac{1}{3}\left[4.932 + \frac{15}{(2.466)^2}\right] = \frac{1}{3}\left[4.932 + \frac{15}{6.08}\right] = \frac{1}{3}[4.932 + 2.467] = 2.465$$

Put $i = 2$ in (1). Then

$$x_3 = \frac{1}{3}\left(2x_2 + \frac{15}{x_2^2}\right) = \frac{1}{3}\left[2 \times 2.405 + \frac{15}{(2.465)^2}\right] = \frac{1}{3}\left[4.93 + \frac{15}{6.076}\right]$$

$$= \frac{1}{3}[4.93 + 2.468] = \frac{1}{3}[7.3987] = 2.4662$$

Put $i = 3$ in (1). Then

$$x_4 = \frac{1}{3}\left[2x_3 + \frac{15}{(x_3)^2}\right]$$

$$= \frac{1}{3}\left[2 \times 2.4662 + \frac{15}{(2.4662)^2}\right] = \frac{1}{3}\left[4.9324 + \frac{15}{6.0821}\right] = 2.4661$$

The value is converging to 2.466.

We take $\sqrt[3]{15} = 2.466$.

Example 8 : Find by Newton's method, the real root of the equation $xe^x - 2 = 0$ correct to three decimal places.

Solution : Let $f(x) = xe^x - 2$ ..... (1).

Then $f(0) = -2$ and $f(1) = e - 2 = 0.7183$

So root of $f(x)$ lies between 0 and 1. It is near to 1. So we take $x_0 = 1$

And $f'(x) = xe^x + e^x$ and $f'(1) = e + e = 5.4366$

$\therefore$ By Newton's Rule

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1 - \dfrac{0.7183}{5.4366} = 0.8679$

$\therefore$ $f(x_1) = 0.0672,\ f'(x_1) = 4.4491$.

Thus second approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.8679 - \frac{0.0672}{4.4491} = 0.8528$$

$\therefore$ Required root is 0.853 correct to 3 decimal places.

Example 9 : Find a real root of the equation $x = e^{-x}$, using the Newton-Raphson method.

Solution : Let $f(x) = xe^x - 1 = 0$. Then $f'(x) = e^x + x\,e^x = (1+x)\,e^x$.

Let $x_0 = 1$, $x_1 = 1 - \dfrac{e-1}{2e} = \dfrac{1}{2}\left(1 + \dfrac{1}{e}\right) = 0.6839397$

$f(x_1) = 0.3553424,\ f'(x_1) = 3.337012,$

$$x_2 = 0.6839397 - \frac{0.3553424}{3.337012} = 0.5774545$$

Proceeding in the same way, $x_3 = 0.5672297,\ x_4 = 0.5671433$

Example 10 : Find the root of the equation $x\sin x + \cos x = 0$, using Newton-Raphson method.

Solution : Let $f(x) = x\sin x + \cos x = 0$,

$f'(x) = x\cos x$ we have $f(2) >$ and $f(3) < 0$

By using the formula $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$,

when $x_0 = 3,\ x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2.8092$

Continuing in this manner we get,

$x_2 = 2.7984,\ x_3 = 2.7984,\ x_4 = 2.7984$

$\therefore$ Root of the equation is 2.7984

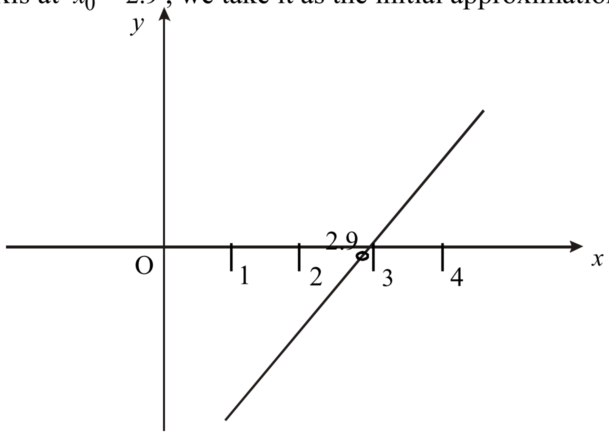
Example 11 : Using Newton-Raphson method find the root of the equation $x + \log_{10} x = 3.375$ corrected to four significant figures.

Solution : Let $y = x + \log_{10} x - 3.375$ ..... (1)

We obtain a rough estimate of the root by drawing the graph of (1) with the help of the following table.

x	1	2	3	4
y	–2.375	–1.074	0.102	1.227

Taking one unit along either axis = 0.1, the graph is as shown is figure below. Since the curve crosses x – axis at $x_0 = 2.9$, we take it as the initial approximation of the root.



Now we will apply Newton-Raphson method to

$f(x) = x + \log_{10} x - 3.375$. $\quad f'(x) = 1 + \frac{1}{x} . \log_{10} e$

$\therefore \quad f(2.9) = 2.9 + \log_{10} 2.9 - 3.375 \; = - 0.0126$

$\quad f'(2.9) = 1 + \frac{1}{2.9} \log_{10} e = 1.1497$

$\therefore$ The first approximation to the root is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.9109$

Thus $f(x_1) = -0.001$ and $f'(x_1) = 1.1492$

$\therefore$ The second approximation is given by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.9109 + \frac{0.0001}{1.1492} = 2.91099.$

Hence the required root corrected to four decimals is 2.911.

Example 12 : Find a real root for x tan$x$ + 1 = 0 using Newton Raphson method.

(or) Find the root of the equation $x \sin x + \cos x = 0$ using Newton Raphson method.

[JNTU Sep 2006, JNTU (A) June 2011 (Set No. 4)]

Solution : Given $f(x) = x \tan x + 1 = 0$

$\therefore \quad f'(x) = x \sec^2 x + \tan x$

Now $f(2) = 2 \tan 2 + 1 = - 3.370079 < 0$

and $f(3) = 3 \tan 3 + 1 = .572370 > 0$

$\therefore$ The root lies between 2 and 3. We take the average of 2 and 3.

Let $x_0 = \frac{2+3}{2} = 2.5$

Using Newton-Raphson method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{-.86755}{3.14808} = 2.77558$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.77558 - \frac{(-.06383)}{2.80004} = 2.798$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.798 - \frac{-0.0010803052}{2.7983} = 2.798$$

Since $x_2 = x_3$, therefore, the real root is 2.798.

Example 13 : Find a root of e^x sinx = 1 (near 1) using Newton Raphson's method.

[JNTU Sep. 2006, (H) June 2010 (Set No.3)]

Solution : Given $e^x \sin x = 1$

Let $f(x) = e^x \sin x - 1 \Rightarrow f'(x) = e^x (\sin x + \cos x)$

We have to find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ have opposite signs.

Then the root lies between x_1 and x_2.

$\therefore$ Root of the equation lies between x_1 and x_2.

$$f(0) = e^0 \sin 0 - 1 = -1 < 0$$

$$f(1) = e^1 \sin 1 - 1 = 1.287 > 0$$

$\therefore$ Root of the equation lies between 0 and 1.

By Newton-Raphson's method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Let $x_0 = \dfrac{1+0}{2} = 0.5$. Then $f(x_0) = e^{0.5} \sin(.5) - 1 = -.20956$ and

$$f'(x_0) = e^{0.5}[(\sin(.5) + \cos(.5)] = 2.237328$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{-.20956}{2.237328} = .593665.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = .593665 - \frac{e^{.593665}\sin(.593665) - 1}{e^{.593665}(\sin(.593665) + \cos(.593665))}$$

$$= 0.593665 - \frac{.01286}{2.51367} = .58854$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = .58854 - \frac{.000018127}{2.4983} = .58853$$

$\therefore \quad x_2 = x_3 = .58853.$

$\therefore \quad$ Root of the equation is .58853.

Example 14 : Find a real root of the equation $xe^x - \cos x = 0$ using Newton Raphson method. **[JNTU 2006S, JNTU(A) June 2009 (Set No.1) , Nov. 2010 (Set No. 4), May 2011]**

(or) Using Newton-Raphson's method, find a positive root of $\cos x - x\,e^x = 0$

[JNTU Sep. 2008S (Set No.1)]

Solution : Given $xe^x - \cos x = 0$. Let $f(x) = xe^x - \cos x = 0$

We have to find x_1 and x_2 such that $f(x_1)$ and $f(x_2)$ are of opposite signs.

$\therefore$ Root of the equation lies between x_1 and x_2.

$$f(x) = xe^x - \cos x$$

$$f'(x) = (x+1)e^x + \sin x$$

Now $f(0) = 0 - \cos 0 = -1 < 0$; $f'(0) = 1 + \sin 0 = 1$

$f(1) = e - \cos 1 = 2.177979 > 0$; $f'(1) = 6.27803$

Roots lies between 0 and 1.

Let $x_0 = \dfrac{x_1 + x_2}{2} = 0.5$

By Newton Raphson method, we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\therefore \; x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(-0.053221926)}{2.952507} = 0.51803$$

Now $f(x_1) = (0.51803)e^{0.51803} - \cos(0.51803) = 0.00083$

and $f'(x_1) = (1.51803)\, e^{0.51803} + \sin(0.51803) = 3.0435$

$$\therefore \; x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.51803 - \frac{0.00083}{3.0435} = 0.5178$$

Now $f(x_2) = (0.5178)\, e^{0.5178} - \cos(0.5178) = 0.00013$

and $f'(x_2) = 1.5178\, e^{0.5178} + \sin(0.5178) = 3.04234$

$$\therefore \; x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5178 - \frac{0.0003}{3.04234} = 0.5177$$

Now $f(x_3) = (0.5177)\, e^{0.5177} - \cos(0.5177) = -0.0001745$

and $f'(x_3) = (1.5177)\, e^{0.5177} + \sin(0.5177) = 3.04183$

$$\therefore \; x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.5177 + \frac{0.0001745}{3.04183} = 0.5177573$$

Since $x_3 = x_4 = 0.5177$,

$\therefore$ The desired root of the equation is 0.5177.

Example 15 : Find a real root of $x + \log_{10} x - 2 = 0$ using Newton Raphson method.

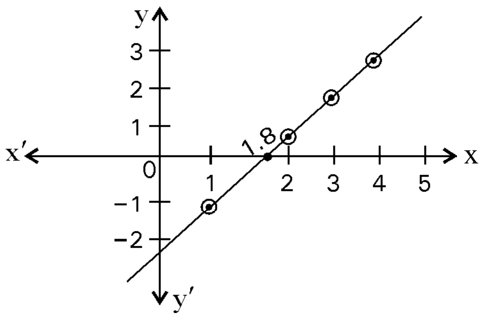
[JNTU April 2007 (Set No.3), (A) Nov. 2010 (Set No. 1)]

Solution : Let $y = x + \log_{10} x - 2$ ... (1)

We obtain a rough estimate of the root by drawing the graph of (1) with the help of the following table.

x	1	2	3	4
y	–1	0.3010	1.4771	2.6021

Since the curve crosses x-axis at $x_0 = 1.8$, we take it as the initial approximation of the root.



$f(x) = x + \log_{10} x - 2 \Rightarrow f'(x) = 1 + \dfrac{1}{x}\log_{10} e$

$\therefore f(1.8) = 1.8 + \log_{10} 1.8 - 2 = 1.8 + 0.2552725 - 2 = 0.0555272$

and $f'(1.8) = 1 + \dfrac{1}{1.8}\log_{10} e = 1 + \dfrac{0.4343}{1.8} = 1.2412778$

By Newton - Raphson method,

$$x_1 = x_0 - \frac{f(x_o)}{f'(x_0)} = 1.8 - \frac{f(1.8)}{f'(1.8)} = 1.8 - \frac{0.0555272}{1.2412778} = 1.7552661$$

Now $f(x_1) = -0.00013658$; $f'(x_1) = 1.247369$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.755470 + \frac{0.00013658}{1.247369} = 1.75558$$

Now $f(x_2) = -0.0000001238$, $f'(x_2) = 1.2473536$.

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.75558 + \frac{0.0000001238}{1.2473536} = 1.75558$$

$\therefore$ The real root of $x + \log_{10} x - 2 = 0$ is 1.75558.

Example 16 : Using Newton-Raphson method, find a positive root of $x^4 - x - 9 = 0$.

[JNTU (A) June 2009 (Set No.1)]

Solution : Let $f(x) = x^4 - x - 9$

Now $f(0) = -9 < 0$, $f(1) = -9 < 0$, $f(2) = 5 > 0$

$\therefore$ The root lies between 1 and 2.

Now $f(1.5) = -5.4375$, $f(1.75) = -1.3711$, $f(1.8) = 0.3024$, $f(1.9) = 2.1321$, $f(2) = 5$

The root lies between 1.75 and 1.8.

$f'(x) = 4x^3 - 1$

$\therefore f'(1.8) = 4(1.8)^3 - 1 = 22.328$

By Newton-Raphson method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

Since $f(x)$ and $f'(x)$ have same sign at 1.8, we choose 1.8 as starting point.

$i.e., x_0 = 1.8$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.8 - \frac{f(1.8)}{f'(1.8)} = 1.8 - \frac{0.3024}{22.328} = 1.8 - 0.0135 = 1.7865$$

Now $f(x_1) = f(1.7865) = (1.7865)^4 - 1.7865 - 9 = -0.6003 < 0$

and $f'(x_1) = 4\,(1.7865)^3 - 1 = 21.807$

$$\therefore \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.7865 + \frac{0.6003}{21.807} = 1.814$$

Now $f(x_2) = (1.814)^4 - 1.814 - 9 = 0.014$

and $f'(x_2) = 4\,(1.814)^3 - 1 = 22.8766$

$$\therefore \quad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.814 - \frac{0.014}{22.8766} = 1.8134$$

Now $f(x_3) = (1.8134)^4 - 1.8134 - 9 = 0.000303$

and $f'(x_3) = 4\,(1.8134)^3 - 1 = 22.8529$

$$\therefore \quad x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8134 - \frac{0.000303}{22.8529} = 1.8134$$

Since $x_3 = x_4 = 1.8134$, the desired root is 1.8134.

Example 17 : Find a real root of $x^3 - x - 2 = 0$. using Newton-Raphson method.

[JNTU (A)June 2009 (Set No.2)]

Solution : Let $f(x) = x^3 - x - 2$. Then $f'(x) = 3x^2 - 1$

Since $f(1) = 1 - 1 - 2 = -2$, $f(2) = 8 - 2 - 2 = 4$, one root lies between 1 and 2.

By Newton – Raphson method, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

We take $x_0 = 1$

$$i = 0, \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{2} = 2$$

$$i = 1, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{11} = 1.6364$$

$$i = 2,\ x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6364 - \frac{f(1.6364)}{f'(1.6364)}$$

$$= 1.6364 - \frac{0.7455}{7.0334} = 1.6364 - 0.106 = 1.5304$$

$$i = 3,\ x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.5304 - \frac{0.054}{6.02637} = 1.52144$$

$$i = 4,\ x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.52144 - \frac{0.0003584}{5.94434} = 1.5214$$

Since $x_4 = x_5$, the desired root is 1.5214.

Example 18 : By using Newton-Raphson method, find the root of $x^4 - x - 10 = 0$, correct to three places of decimal.

Solution : Let $f(x) = x^4 - x - 10$

We have $f(1) = -10 < 0$ and $f(2) = 4 > 0$

So there is a real root of $f(x) = 0$ lying between 1 and 2.

Now $f'(x) = 4x^3 - 1$

Here we take $x_0 = 2$ as first approximation

$x_0 = 2, f(x_0) = 4, f'(x_0) = 3$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{4}{31} = \frac{58}{31} = 1.871$$

Second Aproximation :

$f(x_1) = 0.3835, \quad f'(x_1) = 25.1988$

$$\therefore \quad x_2 = 1.871 - \frac{0.3835}{25.1988} = 1.85578$$

Third Approximation :

$f(x_2) = 0.004827, \quad f'(x_2) = 24.5646$

$$\therefore \quad x_3 = 1.85578 - \frac{0.004827}{24.5646} = 1.85558$$

Hence the root is 1.856 corrected to three places.

Example 19 : Find a real root of the equation $\cos x - x^2 - x = 0$ using Newton Raphson method. **[JNTU (H) Jan. 2012 (Set No. 1)]**

Solution : Given equation $f(x) = \cos x - x^2 - x = 0$

$f(0) = 1, f(1) = \cos(1) - 1 - 1 < 0$

The root lies between 0 and 1

We will use the formula, $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$ by Newton - Raphson method.

$f'(x) = -\sin x - 2x - 1$

Take $x_0 = 0$, $f'(x_0) = f'(0) = -1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(1)}{(-1)} = 1$$

$\therefore \; f(x_1) = f(1) = \cos(1) - 1 - 1 = 0.5403 - 2$

$= -1.4597$

and $f'(x_1) = -\sin(1) - 2 - 1 = -3 - 0.8414 = -3.8414$

$$\therefore \; x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{(-1.4597)}{-3.8414} = 1 - 0.3799 = 0.6201$$

$f(x_2) = \cos(0.6201) - (0.6201)^2 - (0.6201) = 0.8138 - (0.3845) - (0.6201)$

$= -(0.1908)$

$f'(x_2) = -\sin(0.6201) - 2(0.6201) - 1$

$= -(0.5811) - (1.2402) - 1 = -2.8213$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6201 - \frac{(-0.1908)}{(-2.8213)}$$

$= 0.6201 - 0.0676 = 0.5525$

$f(x_3) = 0.8512 - 0.3052 - (0.5525) = -0.0065$

$f'(x_3) = -(0.5248) - 1.105 - 1 = -2.6298$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.5525 - \frac{(-0.0065)}{-2.6298}$$

$= 0.5525 - 0.0024 = 0.5501$

$f(x_4) = 0.8524 - 0.8527 = -0.0003$

$\therefore$ (0.5501) is taken as the approximate value of the root.

Example 20 : Find a real root of the equation $3x - \cos x - 1 = 0$ using Newton Raphson method. **[JNTU (H) June 2012]**

Solution : Let $f(x) = 3x - \cos x - 1$

$f(0) = 0 - \cos 0 - 1 = -2 < 0$

$f(1) = 3 - \cos 1 - 1 = 1 - 0.5403 = 0.4597 > 0$

$\therefore$ The root lies between 0 and 1.

$f'(x) = 3 + \sin x$

By Newton-Rauphson method,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad \text{.... (1)}$$

$f'(1) = 3 + \sin(1) = 3 + 0.8414 = 3.8414$

Taking $i = 0$ and $x_0 = 1$ in (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{0.4597}{3.8414}$$

$$= 1 - 0.1196 = 0.8804$$

$\therefore f(x_1) = f(0.8804) = 2.6412 - 0.6368 - 1 = 1.0044$ and $f'(x_1) = 3.7709$

From (1), $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = (0.8804) - 0.2663 = 0.6141$

$\therefore f(x_2) = 1.8423 - 0.8172 - 1 = 0.0251$ and $f'(x_2) = 3.5762$

From (1), $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$$= 0.6141 - \frac{0.0251}{3.5762} = 0.6141 - 0.0070 = 0.6071$$

$f(x_3) = 1.8213 - 0.8213 - 1 = 0$

$\therefore$ The root of the equation is 0.6071

Example 21 : Find the real root of $x\log_{10} x = 1.2$ correct to five decimal places by using Newton's iterative method. **[JNTU (A) May 2012 (Set No. 4)]**

Solution : $f(x) = x\log_{10} x - 1.2$

$f(x) = -0.59794$

$f(3) = 0.23136$

Since $f(2)$ and $f(3)$ having opposite signs the root lies between 2 and 3.

$$f'(x) = x.\frac{1}{x}\log_{10} e + \log_{10} x \qquad (\because \log_{10} x = \log_e x.\log_{10} e)$$

$$= \frac{1}{\log_e 10} + \log_{10} x = \frac{1}{2.3025} + \log_{10} x = 0.4343 + \log_{10} x$$

By Newton's iteration method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Take, $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-0.5979}{0.7353} = 2 + 0.8131 = 2.8131$$

$$f(2.8131) = (2.8131)\log_{10}(2.8131) - 1.2 = 0.0636$$

$$f'(2.8131) = 0.8834$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8131 - \frac{0.0636}{0.8834} = 2.8131 - 0.0719 = 2.7412$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$f(x_2) = 1.2004 - 1.200 = 0.0004$$

$$f'(x_2) = 0.8722$$

$$\therefore\ x_3 = 2.7412 - \frac{0.0004}{0.8722} = 2.7414 - 0.0004 = 2.7408$$

The approximate value of the root is 2.7408

EXERCISE 4.1

1. Find a real root of the following equations using false position method correct to three decimal places:

(i) $x^3 - 4x - 9 = 0$ **(ii)** $x^6 - x^4 - x^3 - 1 = 0$ **(iii)** $x^3 - x^2 - 2 = 0$ over (1, 2)

2. Using regula-falsi method, find the real root correct to three decimal places:

(i) $2x - \log x = 6$ **(ii)** $xe^x - 2 = 0$ **(iii)** $x^2 - \log_x e = 12$ over (3,4)

3. Using Newton-Raphson method, find a root of the following equations correct to three decimal places:

(i) $e^x - x^3 + \cos 25x$ which is near 4.5 **(ii)** $3x = 1 + \cos x$

(iii) $x^3 - 8x - 4 = 0$ near 3 **(iv)** $2x - 3\sin x = 5$ near 3

4. Using Newton's method compute $\sqrt{41}$ correct to four decimal places.

ANSWERS

1. (*i*) 2.7065 (*ii*) 1.399 (*iii*) 1.69 **2.** (*i*) 3.257 (*ii*) 0.853 (*iii*) 3.646

3. (*i*) 4.5067 (*ii*) 0.6071 (*iii*) 3.051 (*iv*) 2.88324

4. 6.4032

Chapter

5 INTERPOLATION

5.1 INTRODUCTION

If we consider the statement $y = f(x),\ x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ..., x_n$ can be calculated. The problem now is if we are given the set of tabular values

x	x_0	x_1	x_2	...	x_n
y	y_0	y_1	y_2	...	y_n

satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, is it possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process of finding $\phi(x)$ is called **interpolation**. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation.

5.2 ERRORS IN POLYNOMIAL INTERPOLATION

Suppose the function $y(x)$ which is defined at the points $(x_i, y_i), i = 0, 1, 2, 3, ..., n$ is continuous and differentiable $(n+1)$ times. Let $\phi_n(x)$ be polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i, i = 0, 1, 2, 3, ..., n$ ...(1)

be the approximation of $y(x)$ using this $\phi_n(x_i)$ for other value of x, not defined by (1). The error is to be determined. Since $y(x) - \phi_n(x) = 0$ for $x = x_0, x_1, x_2, ..., x_n$ we put

$$y(x) - \phi_n(x) = \mathrm{L}\,\pi_{n+1}(x) \quad ...(2)$$

where $\pi_{n+1}(x) = (x - x_0)(x - x_1) .... (x - x_n)$ ...(3)

and L to be determined such that the equation (2) holds for any intermediate value of x such as $x = x', x_0 < x' < x_n$.

$$\text{Clearly } \mathrm{L} = \frac{y(x') - \phi_n(x')}{\pi_{n+1}(x')} \quad ...(4)$$

We construct a function $\mathrm{F}(x)$ such that $\mathrm{F}(x) = y(x) - \phi_n(x) - \pi_{n+1}(x)$ ...(5)

where L is given by (4).

We can easily see that $\mathrm{F}(x_0) = 0 = \mathrm{F}(x_1) = \mathrm{F}(x_n) = \mathrm{F}(x')$. Then $\mathrm{F}(x)$ vanishes $(n+2)$ times in the interval $[x_0, x_n]$. Then by repeated application of Rolle's theorem $\mathrm{F}'(x)$ must be equal to zero $(n+1)$ times, $\mathrm{F}''(x)$ must be zero n times ... in the interval $[x_0, x_n]$. Also $\mathrm{F}^{n+1}(x) = 0$ once in this interval. Suppose this point is $x = \xi, x_0 < \xi < x_n$.

Differentiate (5), $(n+1)$ times with respect to x and putting $x = \xi$, we get

$$y^{n+1}(\xi) - L\,\lfloor(n+1) = 0 \text{ which implies that } L = \frac{y^{n+1}(\xi)}{\lfloor n+1} \quad ...(6)$$

Comparing (4) and (6), we get, $y(x') - \phi_n(x') = \dfrac{y^{n+1}(\xi)}{\lfloor n+1}\,\pi_{n+1}(x')$

which can be written as $y(x) - \phi_n(x) = \dfrac{\pi_{n+1}(x)}{\lfloor n+1}\,y^{n+1}(\xi),\ x_0 < \xi < x_n$ ...(7)

This gives the required expression for error.

5.3 FINITE DIFFERENCES

1. Introduction :

In this chapter, we introduce what are called the forward, backward and central differences of a function $y = f(x)$. These differences are three standard examples of finite differences and play a fundamental role in the study of Differential calculus, which is an essential part of Numerical Applied Mathematics.

2. Forward Differences :

Consider a function $y = f(x)$ of an independent variable x. Let $y_0, y_1, y_2, ..., y_r$ be the values of y corresponding to the values $x_0, x_1, x_2, ..., x_r$ of x respectively. Then, the differences $y_1 - y_0, y_2 - y_1, ...$ are called the first forward differences of y, and we denote them by $\Delta y_0, \Delta y_1, ...$ That is $\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, ...$

In general, $\Delta y_r = y_{r+1} - y_r,\ r = 0, 1, 2, ...$ ...(1)

Here the symbol Δ is called the **Forward difference** operator.

The first forward differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, ...$

That is, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0,\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1, ...$

In general, $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r, r = 0,1,2,...$ ...(2)

Similarly, the n^th forward differences are defined by the formula

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r, r = 0,1,2,... \quad ...(3)$$

While using this formula for $n = 1$, use the notation $\Delta^0 y_r = y_r$.

If $f(x)$ is a constant function, i.e., if $f(x) = c$, a constant, then $y_0 = y_1 = y_2 = ... = c$ and we have $\Delta^n y_r = 0$ for $n = 1, 2, 3, ...$ and $r = 0, 1, 2, ...$ The symbol Δ^n is referred as the n^th forward difference operator.

Note: $\Delta f(x) = f(x+h) - f(x)$

3. Forward Difference Table :

The forward differences are usually arranged in tabular columns as shown in the following table called a Forward Difference Table.

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_0	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
--	--	Δy_3	---	---	---
x_n	y_n	---	---	---	---

Example : Finite Forward Difference Table for the function $y = x^3$

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

4. Backward Differences :

As mentioned earlier, let $y_0, y_1, y_2, ..., y_r, ...$ be the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2, ..., x_r, ...$ of x respectively. Then

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, ...$ are called the first backward differences.

In general, $\boldsymbol{\nabla y_r = y_r - y_{r-1}, r = 1, 2, 3, ...}$ ...(1)

The symbol ∇ is called the **Backward difference** operator. Like the operator Δ, this operator is also a Linear Operator.

Comparing expression (1) above with the expression (1) of previous section, we immediately note that $\nabla y_r = \Delta y_{r-1}, r = 0, 1, 2, ...$ ...(2)

The first backward differences of the first backward differences are called second backward differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3, ..., \nabla^2 y_r, ...$ *i.e.*,

$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \nabla^2 y_3 = \nabla y_3 - \nabla y_2, ...$

In general, $\boldsymbol{\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, \ r = 2, 3, ...}$ ...(3)

Similarly, the nth backward differences are defined by the formula

$$\boldsymbol{\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, \ r = n, \ n+1, ...} \quad ...(4)$$

While using this formula, for $n = 1$ we employ the notation $\nabla^0 y_r = y_r$.

If $y = f(x)$ is a constant function, then $y = c$, a constant, for all x, and we get $\nabla^n y_r = 0$ for all n.

The symbol ∇^n is referred to as the n[th] backward difference operator.

Note: $\nabla f(x) = f(x) - f(x-h)$

5. Backward Difference Table :

The backward differences can be exhibited as shown in the following table, called the Backward Difference Table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

6. Central Differences :

With $y_0, y_1, y_2, ..., y_r$ as the values of a function $y = f(x)$ corresponding to the values $x_1, x_2, ..., x_r, ...$ of x, we define the first Central differences $\delta y_{1/2}, \ \delta y_{3/2}, \ \delta y_{5/2}, ...$ as follows

$$\delta y_{1/2} = y_1 - y_0,\ \delta y_{3/2} = y_2 - y_1,\ \delta y_{5/2} = y_3 - y_2, \ldots,$$
$$\delta y_{r-1/2} = y_r - y_{r-1} \qquad \ldots(1)$$

The symbol δ is called the **Central difference** operator. This operator is a Linear operator.

Comparing expressions (1) above with expressions earlier used on Forward and Backward differences, we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1,\ \delta y_{3/2} = \Delta y_1 = \nabla y_2,\ \delta y_{5/2} = \Delta y_2 = \nabla y_3, \ldots$$

In general, $\delta y_{n+1/2} = \Delta y_n = \nabla y_{n+1},\ n = 0, 1, 2, \ldots$ ...(2)

The first central differences of the first central differences are called the second central differences and are denoted by $\delta^2 y_1,\ \delta^2 y_2,\ \delta^2 y_3,\ \ldots$. Thus,

$$\delta^2 y_1 = \delta_{3/2} - \delta_{1/2},\ \delta^2 y_2 = \delta_{5/2} - \delta_{3/2},\ \ldots$$
$$\delta^2 y_n = \delta y_{n+1/2} - \delta y_{n-1/2} \qquad \ldots(3)$$

Higher order Central differences are similarly defined. In general the n[th] central differences are given by :

(*i*) for odd n : $\delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1},\ r = 1, 2, \ldots$ ...(4)

(*ii*) for even n : $\delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2},\ r = 1, 2, \ldots$ ...(5)

while employing the formula (4) for $n = 1$, we use the notation $\delta^0 y_r = y_r$.

If y is a constant function, that is, if $y = c$, a constant, then $\delta^n y_r = 0$, for all $n \geq 1$.

The symbol δ^n is referred to as the n[th] central difference operator.

7. Central Difference Table :

The central differences can be displayed in a table as shown below. This is called a Central difference Table.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Ex. Given $f(-2) = 12, f(-1) = 16, f(0) = 15, f(1) = 18, f(2) = 20$ form the Central difference table and write down the values of $\delta y_{-3/2}, \delta^2 y_0$ and $\delta^3 y_{1/2}$ by taking $x_0 = 0$.

Solution : The Central difference table is

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
-2	12				
		4			
-1	16		-5		
		-1		9	
0	15		4		-14
		3		-5	
1	18		-1		
		2			
2	20				

Since $x_0 = 0$ and $h = 1$, we have $y_{-r} = f(x_0 - rh) = f(-r)$

From the above table, $\delta\, y_{-3/2} = \delta f(-3/2) = 4,\ \delta^2 y_0 = 4,\ \delta^3 y_{1/2} = -5$.

5.4 SYMBOLIC RELATIONS AND SEPARATION OF SYMBOLS

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by Symbolic methods.

Def. The averaging operator μ is defined by the equation

$$\mu y_r = \frac{1}{2}(y_{r+1/2} + y_{r-1/2}).$$

Def. The shift operator E is defined by the equation $\mathrm{E}\, y_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $\mathrm{E}^2 y_r = \mathrm{E}\,(\mathrm{E}\, y_r) = \mathrm{E}\,(y_{r+1}) = y_{r+2}$. Generalising $\mathrm{E}^n y_r = y_{r+n}$.

Def. Inverse operator E^{-1} is defined as $\mathrm{E}^{-1} y_r = y_{r-1}$

In general $\mathrm{E}^{-n} y_r = y_{r-n}$.

RELATIONSHIP BETWEEN Δ AND E.

We have $\Delta y_0 = y_1 - y_0 = \mathrm{E}\, y_0 - y_0 = (\mathrm{E} - 1) y_0$

$$\Rightarrow \Delta \equiv \mathrm{E} - 1 \text{ or } \mathrm{E} = 1 + \Delta \qquad ...(1)$$

SOME MORE RELATIONS

$$\Delta^3 y_0 = (\mathrm{E}-1)^3 y_0 = (\mathrm{E}^3 - 3\mathrm{E}^2 + 3\mathrm{E} - 1) y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = (E-1)^4 y_0 = (E^2 - 2E + 1)^2 y_0 = (E^4 + 4E^2 + 1 - 4E^3 - 4E + 2E^2) y_0$$
$$= (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

We can easily establish the following relations:

(*i*) $\nabla \equiv 1 - \mathrm{E}^{-1}$ (*ii*) $\delta \equiv \mathrm{E}^{1/2} - \mathrm{E}^{-1/2}$ (*iii*) $\mu \equiv \frac{1}{2}(\mathrm{E}^{1/2} + \mathrm{E}^{-1/2})$

(*iv*) $\Delta \equiv \nabla\, \mathrm{E} \equiv \mathrm{E}^{1/2}$ (*v*) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$ ...(2)

Proof: ***(iii)*** $\mu\, y_r = \frac{1}{2}(y_{r+1/2} + y_{r-1/2})$

$$= \frac{1}{2}[E^{1/2} y_r + E^{-1/2} y_r] = \frac{1}{2}[E^{1/2} + E^{-1/2}]\, y_r$$

$$\therefore \mu = \frac{1}{2}[E^{1/2} + E^{-1/2}].$$

(v) $\mu^2 \equiv \frac{1}{4}[E^{1/2} + E^{-1/2}]^2 \equiv \frac{1}{4}[E + E^{-1} + 2]$

$$\equiv \frac{1}{4}[(E^{1/2} - E^{-1/2})^2 + 4] \equiv \frac{1}{4}(\delta^2 + 4)$$

$$\therefore \mu^2 = \frac{1}{4}(\delta^2 + 4).$$

Def. The operator D is defined as $D\, y(x) = \frac{d}{dx}\left(y(x)\right)$.

Relation between the operators D and E

Using Taylor's series we have, $y\,(x+h) = y\,(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \ldots$

This can be written in symbolic form $E y_x = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots\right] y_x = e^{hD} \cdot y_x$

($\because$ The above series in brackets is the expansion of e^{hD})

$\therefore$ We obtain the relation $E \equiv e^{hD}$. ...(3)

Note : Using the relation (1), many identities can be obtained. This relation is used to separate the effect of E into powers of Δ. This method of separation is called the method of separation of symbols. Some examples are given.

5.5 DIFFERENCES OF A POLYNOMIAL

Result : If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is a constant.

Proof: Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_0 \neq 0$.

If h is the step-length, we know the formula for first forward difference

$$\Delta f(x) = f(x+h) - f(x).$$

$$= [a_0 (x+h)^n + a_1 (x+h)^{n-1} + \ldots + a_{n-1}(x+h) + a_n]$$

$$- [a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n]$$

$$= a_0 \left[\left\{x^n + n\,.\,x^{n-1}.\,h + \frac{n\,(n-1)}{2!} x^{n-2}.\,h^2 + \ldots\right\} - x^n\right]$$

$$+ a_1 \left[\left\{x^{n-1} + (n-1)\, x^{n-2}.\,h + \frac{(n-1)\,(n-2)}{2!} x^{n-3}.\,h^2 + \ldots\right\} - x^{n-1}\right] + \ldots + a_{n-1} h$$

$$= a_0 n h x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + \ldots + b_{n-3} x + b_{n-2}$$

where $b_2, b_3, \ldots, b_{n-2}$ are constants. Here this polynomial is of degree $(n-1)$.

Thus, the first difference of a polynomial of n[th] degree is a polynomial of degree $(n-1)$.

Now $\Delta^2 f(x) = \Delta\,[\Delta f(x)] = \Delta\,[a_0 nh\,.\,x^{n-1} + b_2 x^{n-2} + b_3 x^{n-3} + ... + b_{n-1}x + b_{n-2}]$

$= a_0 nh\,[(x+h)^{n-1} - x^{n-1}] + b_2\,[(x+h)^{n-2} - x^{n-2}] + ... + b_{n-1}\,[(x+h) - x]$

$= a_0 n\,(n-1)\,h^2 x^{n-2} + c_3 x^{n-3} + ... + c_{n-4}x + c_{n-3}$

where $c_3, ..., c_{n-3}$ are constants. This polynomial is of degree $(n-2)$.

Thus, the second difference of a polynomial of degree n is a polynomial of degree $(n-2)$. Continuing like this we get, $\Delta^n f(x) = a_0 n\,(n-1)\,(n-2)....\,2\,.\,1\,.\,h^n = a_0 h^n (n!)$

which is a constant.Hence the result.

Note 1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0; \Delta^{n+2} f(x) = 0, ...$

2. The converse of above result is also true. That is, if $\Delta^n f(x)$ is tabulated at equally spaced intervals and is a constant, then the function $f(n)$ is a polynomial of degree n.

Factorial notation :

The product of factors of which the first is *x* and the successive factors decrease by a constant difference is called a factorial polynomial function and is denoted $x^{(r)}$, *r* being a positive integer and is read as "*x* raised to the power *r* factorial". In general the interval of differences is h,

In particular we get $x^{(0)} = 1$

We define $x^{(r)} = x\,(x-h)\,(x-2h).....[x-(\overline{r-1})\,h]$

Also $\Delta x^{(r)} = (x+h)^{(r)} - x^{(r)}$

$= (x+h)\,x\,(x-h).....[x-(r-2)h] - x\,(x-h).....[x-(r-1)h]$

$= x\,(x-h)\,.....\,(x-(\overline{r-2})\,h)\,[(x+h) - x - (\overline{r-1})\,h]$

$= rhx^{(r-1)}$

Similarly, $\Delta^2 (x)^r = \Delta\,[\Delta\,x^{(r)}] = \Delta\,[hrx^{(r-1)}] = hr\Delta\,x^{(r-1)}$

$\Rightarrow \Delta^2 x^{(r)} = h^2 r\,(r-1)\,x^{(r-2)}$

and generally, $\Delta^m x^{(r)} = h^m r\,(r-1)......[r-(m-1)]\,x^{(r-m)}, m \le r$

$= 0, m > r$

$\Delta^m (x^{(m)}) = m!\,h^m$

Note : 1. If *x* is an integer greater than $n-1$, then $x^{(n)} = \dfrac{x!}{(x-n)!}$

2. For factorial notation, operator Δ is analogous to operator D.

3. $x^{(r-1)} = \dfrac{1}{r^h} x^{(r)}$

We will represent the given polynomial in Factorial notation.

SOLVED EXAMPLES

Example 1 : Represent the function $f(x)$ given by $f(x)=2x^4-12x^3+24x^2-30x+9$ and its successive differences in factorial notation.

Solution : Given $f(x)=2x^4-12x^3+24x^2-30x+9$

$$=2x^{(4)}+bx^{(3)}+cx^{(2)}+dx^{(1)}+9$$

$$=2x(x-1)(x-2)(x-3)+bx(x-1)(x-2)+cx(x-1)+dx+9$$

where a,b,c are constants to be determined.

Put $x=1$, we get $-7=d+9 \Rightarrow d=-16$

$x=2$ gives, $2(16)-12(8)+24(4)-30(2)+9$

$$=2c+2d+9$$

$$\Rightarrow 32-96+96-60=28-32$$

$$\Rightarrow c=4$$

$x=3$, gives $b=-2$

$$\therefore \; f(x)=2x^{(4)}-2x^{(3)}+4x^{(2)}-16x^{(1)}+9$$

$$\Delta f(x)=8x^{(3)}-6x^{(2)}+8x^{(1)}-16+0$$

$$\Delta^2 f(x)=24x^{(3)}-12x^{(1)}+8$$

$$\Delta^3 f(x)=48x^{(1)}-12$$

$$\Delta^4 f(x)=48$$

Example 2 : Find the function whose first difference is $9x^2+11x+5$

Solution : Let $f(x)$ be the required function, so that we have

$$\Delta f(x)=9x^2+11x+5$$

$$=9(x)(x-1)+bx+c$$

$$9x^2+11x+5=9x(x-1)+bx+c$$

Putting $x=0$, we get $5=c$

$x=1$, gives $9+11+5=b+c=b=20$

$$\therefore \; \Delta f(x)=9x^{(2)}+20x^{(1)}+5$$

Hence $f(x)=3x^{(3)}+10x^{(2)}+5x^{(1)}+k$, where k is a constant

$$\therefore f(x)=3x(x-1)(x-2)+10x(x-1)+5x+k$$

$$=3x^3+x^2+x+k$$

Example 3 : The following table gives a set of values of x and the corresponding values of $y = f(x)$.

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

Form the forward difference table and write down the values of $\Delta f(10), \Delta^2 f(10), \Delta^3 f(15)$ and $\Delta^4 f(15)$.

Solution : The forward difference table for the given values of x and y is as shown below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	19.97					
		1.54				
15	21.51		– 0.58			
		0.96		.67		
20	22.47		0.09		– 0.68	
		1.05		–0.01		0.72
25	23.52		0.08		+0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

We note that the values of x are equally spaced with step-length $h = 5$.

$\therefore \quad x_0 = 10, x_1 = 15, ..., x_5 = 35$ and

$y_0 = f(x_0) = 19.97$

$y_1 = f(x_1) = 21.51$

$y_5 = f(x_5) = 25.89$

From table, $\Delta f(10) = \Delta y_0 = 1.54$; $\quad \Delta^2 f(10) = \Delta^2 y_0 = -0.58$

$\Delta^3 f(15) = \Delta^3 y_1 = -0.01$; $\quad \Delta^4 f(15) = \Delta^4 y_1 = 0.04$

Example 4 : Construct a forward difference table from the following data

x	0	1	2	3	4
y_x	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1$, y_x and y_5.

Solution : The forward difference table for the given values of x and y is as shown below.

x	y_x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 0$	$y_0 = 1$				
		$\Delta y_0 = 0.5$			
$x_1 = 1$	$y_1 = 1.5$		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 0.7$		$\Delta^3 y_0 = 0$	
$x_2 = 2$	$y_2 = 2.2$		$\Delta^2 y_1 = 0.2$		$\Delta^4 y_0 = 0.4$
		$\Delta y_2 = 0.9$		$\Delta^3 y_1 = 0.4$	
$x_3 = 3$	$y_3 = 3.1$		$\Delta^2 y_2 = 0.6$		
		$\Delta y_3 = 1.5$			
$x_4 = 4$	$y_4 = 4.6$				

Now, $\Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1 = 4.6 - 3\,(3.1) + 3\,(2.2) - 1.5 = 0.4$

Again, we have

$$y_x = y_0 + {}^xC_1 \Delta y_0 + {}^xC_2 \Delta^2 y_0 + {}^xC_3 \Delta^3 y_0 + {}^xC_4 \Delta^4 y_0$$

$$= 1 + x\,(0.5) + \frac{1}{2!}\,(x\,(x-1))\,(0.2) + \frac{1}{3!}\,x\,(x-1)\,(x-2)\,(0)$$

$$+ \frac{1}{4!}\,x\,(x-1)\,(x-2)\,(x-3)\,(0.4)$$

$$= 1 + \frac{1}{2}x + \frac{1}{10}(x^2 - x) + \frac{1}{60}(x^4 - 6x^3 + 11x^2 - 6x)$$

$$\therefore\ y_x = \frac{1}{60}(x^4 - 6x^3 + 17x^2 + 18x + 60)$$

$$\Rightarrow y_5 = \frac{1}{60}\left((5)^4 - 6\,(5)^3 + 17\,(5)^2 + 18\,(5) + 60\right) = 7.5.$$

Example 5 : If $y = (3x+1)\,(3x+4)....(3x+22)$ prove that

$$\Delta^4 y = 136080\ (3x+13)\,(3x+16)\,(3x+19)\,(3x+22).$$

Solution : The given equation $y = (3x+1)\ (3x+4)....(3x+22)$ contains eight factors.

$$\therefore\ y = 3^8 (x + 1/3)\ (x + 4/3)....(x + 22/3) = 3^8 (x + 22/3)^8$$

$$\Delta y = 8.3^8 (x + 22/3)^7, \quad \Delta^2 y = 3^8.8\,.7\ (x + 22/3)^6$$

$$\Delta^3 y = 3^8.8\,.7\,.6\ (x + 22/3)^5 \text{ and } \Delta^4 y = 3^8.8\,.7\,.6\,.5\ (x + 22/3)^4$$

$$\therefore\ \Delta^4 y = 11022480\left(x + \frac{22}{3}\right)\left(x + \frac{22}{3} - 1\right)\left(x + \frac{22}{3} - 2\right)\left(x + \frac{22}{3} - 3\right)$$

$$= 136080\ (3x+22)\ (3x+19)\ (3x+16)\ (3x+13).$$

Example 6 : Evaluate (*i*) $\Delta \cos x$ (*ii*) $\Delta \log f(x)$ (*iii*) $\Delta^2 \sin(px+q)$ (*iv*) $\Delta \tan^{-1} x$ and (v) $\Delta^n e^{ax+b}$.

Solution : Let h be the interval of differencing

(*i*) $\Delta \cos x = \cos(x+h) - \cos x = -2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

(*ii*) $\Delta \log f(x) = \log f(x+h) - \log f(x) = \log\left(\frac{f(x+h)}{f(x)}\right)$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right]$$

(*iii*) $\Delta \sin(px+q) = \sin\,[p(x+h)+q] - \sin(px+q)]$

$$= 2\cos\left(px+q+\frac{ph}{2}\right)\sin\frac{ph}{2} = 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2}+px+q+\frac{ph}{2}\right)$$

$$\Delta^2 \sin\ (px+q) = 2\sin\frac{ph}{2}\Delta\left[\sin\left(px+q+\frac{1}{2}(\pi+ph)\right)\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left(px+q+2\,.\frac{1}{2}(\pi+ph)\right)$$

(*iv*) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+x\,(x+h)}\right] = \tan^{-1}\left[\frac{h}{1+x\,(x+h)}\right]$$

(*v*) $\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b} = e^{(ax+b)}(e^{ah}-1)$

$$\Delta^2 e^{ax+b} = \Delta\,[\Delta\,(e^{ax+b})] = \Delta\,[(e^{ah}-1)\,(e^{ax+b})] = (e^{ah}-1)^2\ \Delta\,(e^{ax+b})$$

$$= (e^{ah}-1)^2\ e^{ax+b} \qquad [\because\ \ e^{ah}-1 \text{ is a constant}]$$

Proceeding on, we get $\Delta^n(e^{ax+b}) = (e^{ah}-1)^n\ e^{ax+b}$.

Example 7 : If the interval of differencing is unity, prove that

$$\Delta\tan^{-1}\left(\frac{n-1}{n}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right)$$

[JNTU (K) June 2009, Nov. 2009S, (A) Dec. 2013 (Set No. 3)]

Solution : We know that $\Delta\tan^{-1}(x) = \tan^{-1}(x+h) - \tan^{-1}(x)$

Here the interval of difference is $h = 1$.

Thus we have, $\Delta\tan^{-1}\left(\frac{n-1}{n}\right) = \Delta\tan^{-1}\left(1-\frac{1}{n}\right)$

$$= \tan^{-1}\left(1-\frac{1}{n+1}\right) - \tan^{-1}\left(1-\frac{1}{n}\right)$$

$$= \tan^{-1}\left[\frac{\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n}\right)}{1+\left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n}\right)}\right] = \tan^{-1}\left[\frac{\left(\frac{1}{n}-\frac{1}{n+1}\right)}{1+\left(\frac{n}{n+1}\right)\frac{n-1}{n}}\right]$$

$$= \tan^{-1}\left(\frac{1}{2n^2}\right) \text{ on simplification}$$

Example 8 : Using the method of separation of symbols, show that

$$\Delta^n u_{x-n} = u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + ... + (-1)^n u_{x-n}.$$

[JNTU (A) Dec. 2013 (Set No. 3)]

Solution : To prove this result, we start with the right hand side. Thus,

$$u_x - nu_{x-1} + \frac{n(n-1)}{2}u_{x-2} + ... + (-1)^n u_{x-n}$$

$$= u_x - nE^{-1}u_x + \frac{n(n-1)}{2}E^{-2}u_x + ... + (-1)^n E^{-n}u_x$$

$$= \left[1 - nE^{-1} + \frac{n(n-1)}{2}E^{-2} + ... + (-1)^n E^{-n}\right]u_x = (1-E^{-1})^n u_x$$

$$= \left(1-\frac{1}{E}\right)^n u_n = \left(\frac{E-1}{E}\right)^n u_n = \frac{\Delta^n}{E^n}u_x = \Delta^n E^{-n}u_x$$

$$= \Delta^n u_{x-n} \text{ which is left-hand side.}$$

Hence the result.

Example 9 : Show that $e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + ...\right) = u_0 + u_1 x + u_2\frac{x^2}{2!} + ...$

Solution : $e^x\left(u_0 + x\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + ...\right)$

$$= e^x\left(1 + x\Delta + \frac{x^2}{2}\Delta^2 + ...\right)u_0$$

$$= e^x . e^{x\Delta}u_0 = e^{x(1+\Delta)}u_0 = e^{xE}u_0$$

$$= \left[1 + xE + \frac{x^2E^2}{2!} + ...\right]u_0$$

$$= u_0 + xu_1 + \frac{x^2}{2!}u_2 + ...$$

which is the required result.

Example 10 : Evaluate (*i*) $\Delta[f(x)\,g(x)]$ (*ii*) $\Delta\left[\frac{f(x)}{g(x)}\right]$.

Solution : Let h be the interval of differencing.

(*i*) $\Delta[f(x)\,g(x)] = f(x+h)\,g(x+h) - f(x)\,g(x)$

$$= f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$$

$$= f(x+h)\,[g(x+h) - g(x)] + g(x)\,[f(x+h) - f(x)]$$

$$= f(x+h)\,\Delta g(x) + g(x)\,\Delta f(x).$$

(*ii*) $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)} = \dfrac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)}$

$$= \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)}$$

$$= \frac{g(x)\,[f(x+h) - f(x)] - f(x)\,[g(x+h) - g(x)]}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\,\Delta f(x) - f(x)\,\Delta g(x)}{g(x+h)\,g(x)}.$$

Example 11 : (*i*) Show that $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0$ **[JNTU 2003]**

(*ii*) If $f(x) = e^{ax}$, show that $\Delta^n f(x) = (e^{ah} - 1)^n e^{ax}$

(*iii*) Show that $\Delta(f_i / g_i) = (g_1 \Delta f_i - f_i \Delta g_i) / g_i . g_{i+1}$

(*iv*) Show that $\Delta f_i^2 = (f_i + f_{i+1})\,\Delta f_i$. **[JNTU 2006 (Set No.4)]**

Solution : Let $y = f(x)$. The first finite forward difference is $\Delta y_k = y_{k+1} - y_k$.

Put $y_k = f(x_k) = f_k$, we get $\Delta f_k = f_{k+1} - f_k$.

The second difference is $\Delta^2 f_k = \Delta(\Delta f_k) = \Delta(f_{k+1} - f_k) = \Delta f_{k+1} - \Delta f_k$.

(*i*) $\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta^2 f_0 + \Delta^2 f_1 + \Delta^2 f_2 + \Delta^2 f_3 + ... + \Delta^2 f_{n-1}$

$$= \Delta f_1 - \Delta f_0 + \Delta f_2 - \Delta f_1 + \Delta f_3 - \Delta f_2 + \Delta f_4 - \Delta f_3 + ... + \Delta f_n - \Delta f_{n-1}$$

$$= \Delta f_n - \Delta f_0$$

(*ii*) Given $f(x) = e^{ax}$, we have $f(x+h) = e^{a\,(x+h)}$.

Here, h is the step size $x_{i+1} = x_i + h$

We have to show that $\Delta^n f(x) = (e^{ah} - 1)^n . e^{ax}$.

This can be proved by mathematical induction.

First we shall prove that this is true for $n = 1$.

$(e^{ah} - 1)^1 e^{ax} = e^{ah} . e^{ax} - e^{ax}$

$$= e^{ah+ax} - e^{ax} = e^{a(x+h)} - e^{ax} = f(x+h) - f(x) = \Delta f(x)$$

$\therefore\ \Delta f(x_i) = f(x_i + h) - f(x_i)$

Therefore, the result is true for $n = 1$.

Assume that the problem is true for $n - 1$.

Now consider, $\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$

$$= (e^{ah} - 1)^{n-1} e^{a(x+h)} - (e^{ah} - 1)^{n-1}. e^{ax}$$

$$= (e^{ah} - 1)^{n-1}. [e^{a(x+h)} - e^{ax}] = (e^{ah} - 1)^{n-1}. [e^{ax+ah} - e^{ax}]$$

$$= (e^{ah} - 1)^{n-1}. [e^{ax}(e^{ah} - 1)] = (e^{ah} - 1)^{n-1}. (e^{ah} - 1).e^{ax}$$

$$= (e^{ah} - 1)^{n-1+1}. e^{ax} = (e^{ah} - 1)^n. e^{ax}$$

$\therefore \Delta^n f(x) = (e^{ah} - 1)^n. e^{ax}$.

(*iii*) According to first forward difference, $\Delta\left(\dfrac{f_i}{g_i}\right) = \dfrac{f_{i+1}}{g_{i+1}} - \dfrac{f_i}{g_i}$

Now $\dfrac{g_i \Delta f_i - g_i \Delta g_i}{g_i \cdot g_{i+1}} = \dfrac{g_i(f_{i+1} - f_i) - f_i(g_{i+1} - g_i)}{g_i \cdot g_{i+1}}$

$$= \frac{g_i f_{i+1} - g_i f_i - f_i g_{i+1} + g_i f_i}{g_i \cdot g_{i+1}} = \frac{g_i f_{i+1} - f_i g_{i+1}}{g_i \cdot g_{i+1}}$$

$$= \frac{g_i f_{i+1}}{g_i \cdot g_{i+1}} - \frac{f_i g_{i+1}}{g_i \cdot g_{i+1}} = \frac{f_{i+1}}{g_{i+1}} - \frac{f_i}{g_i}$$

$$\therefore \quad \Delta\left(\frac{f_i}{g_i}\right) = \frac{g_i \Delta f_i - f_i \Delta g_i}{g_i \cdot g_{i+1}}$$

(*iv*) We know that $\Delta f_k = f_{k+1} - f_k$

$\therefore \ \Delta f_i^2 = f^2{}_{i+1} - f_i^2 = (f_{i+1} + f_i)(f_{i+1} - f_i) = (f_{i+1} + f_i)\Delta f_i$.

Example 12 : If $f(x) = u(x)\, v(x)$ show that $f[x_0, x_1] = u[x_0] \,.\, v[x_0, x_1] + u[x_0, x_1] v[x_1]$.

[JNTU 2006 (Set No 4)]

Solution : Given $f(x) = u(x)\, v(x)$

The first order divided difference between x_0 and x_1 is

$$y[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So, $f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$

$$u[x_0, x_1] = \frac{u[x_1] - u[x_0]}{x_1 - x_0}, \quad v[x_0, x_1] = \frac{v[x_1] - v[x_0]}{x_1 - x_0}$$

Thus, $u[x_0].v[x_0, x_1] + u[x_0, x_1].v[x_1] = u(x_0) \cdot \dfrac{v[x_1] - v[x_0]}{x_1 - x_0} + \dfrac{u[x_1] - u[x_0]}{x_1 - x_0} v[x_1]$

$$= \frac{1}{x_1 - x_0}\{u[x_0].v[x_1] - u[x_0].v[x_0] + u[x_1].v[x_1] - u[x_0].v[x_1]\}$$

$$= \frac{1}{x_1 - x_0}\{u[x_1].v[x_1] - u[x_0].v[x_0]\} = \frac{1}{x_1 - x_0}\left[f[x_1] - f[x_0]\right] = f[x_0, x_1].$$

Example 13 : Find the missing term in the following data.

x	0	1	2	3	4
y	1	3	9	–	81

Why this value is not equal to 3^3. Explain.

Solution : Consider $\Delta^4 y_0 = 0$ (we are given only 4 values)

$\Rightarrow$ $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values. We get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31.$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$.

Example 14 : If y_x is the value of y at x for which the fifth differences are constant and $y_1 + y_7 = -784$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.

Solution : Since fifth differences are constant, $\Delta^6 y_1 = 0$

$\Rightarrow (E - 1)^6 y_1 = 0$

$\Rightarrow (E^6 - 6_{c_1} E^5 + 6_{c_2} E^4 - 6_{c_3} E^3 + 6_{c_4} E^2 - 6_{c_5} E + 6_{c_6} 1)y_1 = 0$

$\Rightarrow y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$

$\Rightarrow (y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5) - 20y_4 = 0$

$\Rightarrow -784 - 6(686) + 15(1088) - 20y_4 = 0$

$\Rightarrow -784 - 4116 + 16320 - 20y_4 = 0 \Rightarrow 11420 - 20y_4 = 0$

or $20y_4 = 11420$ $\therefore y_4 = 571.$

Example 15 : If $f(x) = x^3 + 5x - 7$, form a table of forward differences taking $x = -1, 0, 1, 2, 3, 4, 5$. Show that the third differences are constant.

Solution : Here $f(-1) = -1 - 5 - 7 = -13.$

$f(0) = 0 - 7 = -7,$

$f(1) = 1 + 5 - 7 = -1,$

$f(2) = 8 + 10 - 7 = 11,$

$f(3) = 27 + 15 - 7 = 35,$

$f(4) = 64 + 20 - 7 = 77$

$f(5) = 125 + 25 - 7 = 143$

We form the difference table as follows:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–1	–13			
		6		
0	–7			
		6	0	
1	–1			
		12	6	6
2	11			
		24	12	6
3	35			
		42	18	6
4	77			
		66		
5	143		24	6

We note from the table that all the third forward differences are constant. This illustrates the result discussed in 1.5

Example 16 : Prove the results:

(*i*) $E\nabla = \Delta = \nabla E$

(*ii*) $\delta E^{\frac{1}{2}} = \Delta$

(*iii*) $h\Delta = \log(1+\Delta) = -\log(1-\Delta) = \sin^{-1}(\mu\delta)$

(*iv*) $1+\mu^2\delta^2 = \left(1+\frac{1}{2}\delta^2\right)^2$

(*v*) $E^{1/2} = \mu + \frac{1}{2}\delta$

(*vi*) $E^{-1/2} = \mu - \frac{1}{2}\delta$

(*vii*) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

(*viii*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$

(*ix*) $\nabla\Delta = \Delta - \nabla = \delta^2$

(*x*) $(1+\nabla)(1-\nabla) = 1$

(*xi*) $\mu\delta = \frac{1}{2}(\Delta+\nabla)$

Solution : (*i*) $(E\nabla)\mu_x = E(\nabla\mu_x) = E(\mu_x - \mu_{x-h})$

$$= E\mu_x - E\mu_{x-h} = \mu_{x+h} - \mu_x = \Delta\mu_x$$

$\therefore \quad E\nabla = \Delta$

Also $(\nabla E)\mu_x = \nabla(E\mu_x) = \nabla\mu_{x+h} = \mu_{x+h} - \mu_x = \Delta\mu_x$

$\therefore\ \nabla E = \Delta$

Hence $\quad E\nabla = \Delta = \nabla E$

(*ii*) $\delta\mu_{x+\frac{h}{2}} = (E^{1/2} - E^{-1/2})\mu_{x+\frac{h}{2}} = \mu_{x+h} - \mu_x = \Delta\mu_x$

$\therefore \delta E^{1/2} = \Delta$

(*iii*) We know $e^{hd} = E = 1+\Delta$

Taking logaritham

$$\therefore \quad hd \log e = \log(1+\Delta) \qquad \ldots(1)$$

Also $\nabla = 1 - E^{-1} \Rightarrow E^{-1} = 1 - \nabla$

i.e., $e^{-hd} = (1 - \nabla)$. Taking logarithms

$$-hd = \log(1-\nabla) \Rightarrow hd = -\log(1-\nabla)$$

$$\sinh(hd) = \frac{e^{hd} - e^{-hd}}{2} = \frac{E - E^{-\frac{1}{2}}}{2} = \left[\frac{E^{1/2} + E^{-1/2}}{2}\right]\left(E^{1/2} - E^{-1/2}\right) = \mu\delta$$

$$\therefore\ hd = \sinh^{-1}(\mu\delta)$$

(*iv*) $1 + \mu^2\delta^2 = 1 + \left(\frac{E^{1/2} + E^{-1/2}}{2}\right)(E^{1/2} - E^{-1/2})^2$

$$= 1 + \left(\frac{E - E^{-1}}{2}\right) = 4 + \frac{(E - E^{-1})^2}{4} = \frac{(E + E^{-1})^2}{2} \qquad \ldots(1)$$

Now $\left[1 + \frac{1}{2}\delta^2\right] = \left[1 + \frac{1}{2}(E^{1/2} - E^{-1/2})\right]^2 = \left[1 + \frac{1}{2}(E + E^{-1} - 2)\right]^2$

$$= \left[\frac{E + E^{-1}}{2}\right]^2 \qquad \ldots(2)$$

From (1) amd (2), $1 + \mu^2\delta^2 = \left(1 + \frac{1}{2}\delta^2\right)^2$

(*v*) $\mu + \frac{1}{2}\delta = \frac{E^{1/2} + E^{-1/2}}{2} + \frac{E^{1/2} - E^{-1/2}}{2} = E^{1/2}$

(*vi*) $\mu - \frac{\delta}{2} = \frac{E^{1/2} + E^{-1/2}}{2} - \frac{1}{2}(E^{1/2} - E^{-1/2}) = E^{-1/2}$

(*vii*) $\frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta = \frac{1}{2}\Delta(E^{-1} + 1) = \frac{1}{2}(E - 1)(E^{-1} + 1) = \frac{1}{2}(E - E^{-1}) = \mu\delta$

(*viii*) $\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} = \frac{1}{2}\delta\left[\delta + 2\sqrt{1 + \frac{\delta^2}{4}}\right]$

$$= \frac{1}{2}\delta\left[\delta + \sqrt{4 + \delta^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{4 + (E^{1/2} - E^{-1/2})^2}\right]$$

$$= \frac{1}{2}\delta\left[(E^{1/2} - E^{-1/2}) + \sqrt{(E^{1/2} + E^{-1/2})^2}\right]$$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})[E^{1/2} - E^{-1/2} + E^{1/2} + E^{-1/2}]$$

$$= \frac{1}{2} \times 2[E^{1/2} - E^{-1/2}]E^{1/2} = E - 1 = \Delta$$

(ix) $\Delta\nabla = (1-E^{-1})(E-1) = E + E^{-1} - 2 = (E^{1/2} - E^{-1/2})^2 = \delta^2$

Also $\Delta - \nabla = (E-1)-(1-E^{-1}) = E + E^{-1} - 2 = \delta^2 \quad \therefore\ \nabla\Delta = \Delta - \nabla = \delta^2$

(x) $(1+\Delta)(1-\nabla) = E[1-(1-E^{-1})] = EE^{-1} = 1 \qquad [\because \Delta = E-1, \nabla = 1-E^{-1}]$

(xi) $\frac{1}{2}(\Delta+\nabla) = \frac{1}{2}[E-1+1-E^{-1}] = \frac{1}{2}(E-E^{-1}) = \mu\delta$

Example 17 : If the interval of differencing is unity prove that

$\Delta\,[x\,(x+1)\,(x+2)\,(x+3)] = 4(x+1)\,(x+2)\,(x+3)$

[JNTU 2008 (Set No.4)]

Solution : Let $f(x) = x\,(x+1)\,(x+2)\,(x+3)$

$\Delta\,[x\,(x+1)\,(x+2)\,(x+3)] = f(x+h) - f(x)$. Then $h = 1$

$= (x+1)\,(x+2)\,(x+3)\,(x+4) - x\,(x+1)\,(x+2)\,(x+3)$

$= (x+1)\,(x+2)\,(x+3)\,[\,x+4-x]$

$= 4(x+1)\,(x+2)\,(x+3)$

Example 18 : Find the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$. **[JNTU 2008S, (H) Dec. 2011 (Set No. 1)]**

Solution : Let $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

First difference is given by $= \Delta\, f(x)$

$f(x+n) - f(x) = f(x+2) - f(x)$

$= (x+2)^4 - 12\,(x+2)^3 + 42\,(x+2)^2 - 30x\,(x+2) + 9 - 9x^4 + 12x^3 - 42x^2 + 30x - 9$

$= 8x^3 - 48x^2 + 56x + 28$

Second difference $= \Delta^2 f(x) = \Delta\,[\Delta f(x)]$

$= 8\,(x+2)^3 - 48\,(x+2)^2 + 56\,(x+2) + 28$

$= -8x^3 + 48x^2 - 56x - 28 = 48x^2 - 96x - 16$.

Example 19 : If the interval of differencing is unity, prove that $\Delta\, f(x) = \dfrac{-\Delta\, f(x)}{f(x) f(x+1)}$

[JNTU(H) June 2010 (Set No.1)]

Solution : We know that $\Delta\left(\dfrac{1}{f(x)}\right) = \dfrac{1}{f(x+h)} - \dfrac{1}{f(x)}$

$$= \frac{-[f(x+h) - f(x)]}{f(x)\, f(x+h)} = \frac{-\Delta\, f(x)}{f(x)\, f(x+h)}$$

Taking $h = 1$, we get

$$\Delta\left(\frac{1}{f(x)}\right) = \frac{-\Delta\, f(x)}{f(x)\, f(x+1)}$$

Hence the result.

Example 20 : Show that $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)] = 24\times 2^{10}\times 10!$ if $h=2$.

[JNTU(H) 2009 (Set No.)]

Solution : $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$

$$= \Delta^{10}[(-1)(-2)(-3)(-4)\,x^{10} + \text{terms containing powers of } x \text{ less than } 10]$$

$$= 24\Delta^{10}[x^{10}]$$

$$= 24\,\underline{|10}\,.\,2^{10} \qquad [\because \Delta^n f(x) = \underline{|n}\,h^n \text{ and } h=2]$$

5.6 INTERPOLATION

If we consider $y = f(x)$, $x_0 \le x \le x_n$ then we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function f(x) is single valued and continuous and known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, \ldots, x_n$ can be calculated. The problem now is if we are given the set of tabular values

$x:$	x_0	x_1	x_2	....	x_n
$y:$	y_0	y_1	y_2	....	y_n

satisfying the relation $y = f(x)$ and the explicit difinition of $f(x)$ is not known, is it possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process of finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation as $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation.

5.7 ERRORS IN POLYNOMIAL INTERPOLATION

Suppose the function $y(x)$ which is defined at the points (x_i, y_i), $i = 0, 1, 2, 3\ldots, n$ is continuous and differentiable $(n+1)$ times. Let $\phi_n(x)$ be the polynomial of degree not exceeding n such that $\phi_n(x_i) = y_i$, $i = 0, 1, 2, 3, \ldots, n$ ....(1) be the approximation of $y(x)$. Using this $\phi_n(x_i)$ for other value of x, not defined by (1), the error is to be determined.

Since $y(x) - \phi_n(x) = 0$ for $x = x_0, x_2, \ldots, x_n$

we put $y(x) - \phi_n(x) \mathrm{L} \prod\limits_{n+1}(x)$ ...(2)

where $\prod_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$ ...(3)

and L to be determined such that the equation (2) holds for any intermediate value of x such as $x = x'$, $x_0 < x' < x_n$.

Clearly $\mathrm{L} = \dfrac{y(x') - \phi_n(x')}{\prod\limits_{n+1}(x^1)}$ ...(4)

we construct a function F(x) such that

$F(x) = y(x) - \phi_n(x) - \prod\limits_{n+1}(x)$ ...(5)

where L is given by (4).

We can easily sec that $F(x_0) = 0 = F(x_1) = F(x_n) = F(x^1)$. Then F(x) vanishey $(n+2)$ times in the interval $[x_o, x_n]$. Then by replaced application of Rolle's theroam $F'(x)$ must be equal to zero $(n+1)$ times, $F''(x)$ must be zero in times in the interval $[x_0, x_n]$. Also $F^{n+1}(x) = 0$ once in this intervl. Suppose this point is $x = t$, $x_0 < t < x_n$.

Differentiating equation (3), $(n + 1)$ times w.r.t. x and putting $x = t$, we get

$y^{n+1}(t) - L\,(n + 1) = 0$ ....(6)

Comparing (4) and (6), we get

$$y(x') - \phi_n(x') = \frac{y^{n+1}(t)}{n+1} \prod_{n+1}(x')$$

which can be written as

$$y(x) - \phi_n = \frac{\prod_{n+1}(x)}{n+1} y^{n+1}(t), x_0 < t < x_n \qquad .....(7)$$

This gives the required expression for error.

5.8 NEWTON'S FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a polynomial of degree n and taken in the following form

$$y = f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) + \ldots + b_n(x - x_0)(x - x_1)\ldots.(x - x_{n-1}) \qquad \ldots(A)$$

This polynomial passes through all the points $[x_i, y_i]$ for $i = 0$ to n. Therefore, we can obtain the y_i's by substituting the corresponding x_i's as :

$$\begin{aligned} &\text{at } x = x_0, \; y_0 = b_0 \\ &\text{at } x = x_1, \; y_1 = b_0 + b_1(x_1 - x_0) \\ &\text{at } x = x_2, \; y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1) \qquad \ldots(1) \end{aligned}$$

Let 'h' be the length of interval such that x_i's represent

$x_0, x_0 + h, x_0 + 2h, x_0 + 3h, \; ....., x_0 + nh$.

This implies $x_1 - x_0 = h, x_2 - x_0 = 2h, x_3 - x_0 = 3h, ..., x_n - x_0 = nh$ ...(2)

From (1) and (2), we get

$$\begin{aligned} y_0 &= b_0 \\ y_1 &= b_0 + b_1 h \\ y_2 &= b_0 + b_1 2h + b_2(2h)h \\ y_3 &= b_0 + b_1 3h + b_2(3h)(2h)h + b_3(3h)(2h)h \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n &= b_0 + b_1(nh) + b_2(nh)(n-1)h + \ldots. + b_n(nh)[(n-1)h][(n-2)h] \qquad \ldots(B) \end{aligned}$$

Solving the above equations for $b_0, b_1, b_2, ..., b_n$, we get

$$b_0 = y_0$$

$$b_1 = \frac{y_1 - b_0}{h} = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

$$b_2 = \frac{y_2 - b_0 - b_1 2h}{2h^2} = \frac{y_2 - y_0 - \left(\dfrac{y_1 - y_0}{h}\right)2h}{2h^2}$$

$$= \frac{y_2 - y_0 - 2y_1 - 2y_0}{2h^2} = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$$\therefore\ b_2 = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly, we can see that

$$b_3 = \frac{\Delta^3 y_0}{3!h^3},\ b_4 = \frac{\Delta^4 y_0}{4!h^4}, \ldots,\ b_n = \frac{\Delta^n y_0}{n!h^n}$$

$$\therefore\ y = f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)$$

$$(x - x_1)(x - x_2) + \ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad \ldots(3)$$

If we use the relationship $x = x_0 + ph \Rightarrow x - x_0 = ph$, where $p = 0, 1, 2, \ldots, n$

then $x - x_1 = x - (x_0 + h) = (x - x_0) - h = ph - h = (p - 1)h$

$x - x_2 = x - (x_1 + h) = (x - x_1) - h = (p - 1)h - h = (p - 2)h$

....

....

$x - x_i = (p - i)h$

....

....

$x - x_{n-1} = [p - (n - 1)]h$

$\therefore$ Equation (3) becomes,

$$y = f(x) = f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0$$

$$+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots[p-(n-1)]}{n!}\Delta^n y_0 \quad \ldots(4)$$

This formula is known as ***Newton's forward interpolation formula*** **(or)** ***Newton Gregory forward interpolation formula.***

This is useful for interpolation near the beginning of a set of tabular values.

Newton's Backward Interpolation Formula

If we consider $y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1})$

$$+ a_3(x - x_n)(x - x_{n-1})(x - x_{n-2}) + \ldots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1) \quad \ldots(5)$$

and impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots, x_2, x_1, x_0$.

We obtain $y_n(x) = y_n + p\nabla y_n + \dfrac{p(p+1)}{2!}\nabla^2 y_n + \ldots + \dfrac{p(p+1)\ldots(p+\overline{n-1})}{n!}\nabla^n y_n + \ldots,$

where $p = \dfrac{x - x_n}{h}$. ... (6)

This uses tabular values to the left of y_n. Thus this formula is useful for interpolation near the end of the tabular values.

Formulae for Error in Polynomial Interpolation

If $y = f(x)$ is the exact curve and $y = \phi_n(x)$ is the interpolating polynomial curve, then the error in polynomial interpolation is given by

$$\text{Error} = f(x) - \phi_n(x) = \frac{(x - x_0)(x - x_1)....(x - x_n)}{(n+1)!} f^{n+1}(\xi) \quad ...(7)$$

for any x, where $x_0 < x < x_n$ and $x_0 < \xi < x_n$.

The error in Newton's forward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p-1)(p-2)....(p-n)}{(n+1)!} \Delta^{n+1} f(\xi) \text{ where } p = \frac{x - x_0}{h} \quad ...(8)$$

The error in Newton's backward interpolation formula is given by

$$f(x) - \phi_n(x) = \frac{p(p+1)(p+1)(p+n)}{(n+1)!} h^{n+1} y^{n+1} f(\xi)$$

where $p = \dfrac{x - x_n}{h}$ ...(9)

SOLVED EXAMPLES

Example 1 : The following data gives the melting points of an alloy of lead and zinc.

Percentage of lead in the alloy (p) :	50	60	70	80
Temperature (Q^0c) :	205	225	248	274

Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula.

Solution : The difference table is as under :

x	y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

We have $x_0 = 50$, $h = 10$

$x_0 + ph = 54,$

$50 + p(10) = 54$ or $p = 0.4$

By Newton's forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ...$$

$$\therefore \quad f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{0.4(0.4-1)(0.4-2)}{3!}(0)$$

$$= 205 + 8 - 0.36 = 212.64.$$

Melting point = 212.64

Example 2 : State appropriate interpolation formula which is to be used to calculate the value of exp(1.75) from the following data and hence evaluate it from the given data

x	1.7	1.8	1.9	2.0
$y = e^x$	5.474	6.050	6.686	7.389

[JNTU (A) June 2013 (Set No. 1)]

Solution : The difference table is as under :

x	y	Δ	Δ^2	Δ^3
1.7	5.474			
		0.576		
1.8	6.050		0.060	
		0.636		0.007
1.9	6.686		0.067	
		0.703		
2.0	7.389			

Let $f(x) = y = e^x$

$x_0 + ph = 1.75,\ x_0 = 1.7,\ h = 0.1$

$1.7 + p(0.1) = 1.75$ or $p = 0.5$

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\underline{|3}}\Delta^3 y_0 + \ldots$$

$$f(1.75) = 5.474 + 0.5 \times (0.576) + \frac{0.5\,(0.5-1)}{2}(0.060) + \frac{0.5\,(0.5-1)\,(0.5-2)}{6}(0.007)$$

$= 5.474 + 0.288 - 0.0075 + 0.0004375 = 5.7624375 - 0.0075 = 5.7549375$

$= 5.7549$ (Rounded up to four decimal places).

Example 3 : Applying Newton's forward interpolation formula, compute the value of $\sqrt{5.5}$, given that $\sqrt{5} = 2.236, \sqrt{6} = 2.449, \sqrt{7} = 2.646$ and $\sqrt{8} = 2.828$ correct upto three places of decimal.

Solution : Let $f(x) = \sqrt{x}$. The difference table is as under :

x	y	Δ	Δ^2	Δ^3
5	2.236			
		0.213		
6	2.449		-0.016	
		0.197		0.001
7	2.646		-0.015	
		0.182		
8	2.828			

We have

$x_0 + ph = 5.5,\ x_0 = 5,\ h = 1$

$\Rightarrow\ 5 + p(1) = 5.5 \quad \text{or} \quad p = 0.5$

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots$$

$$f(5.5) = 2.236 + 0.5 \times (0.213) + \frac{0.5(0.5-1)}{\lfloor 2}(-0.016)$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{\lfloor 3}(0.001)$$

i.e. $\sqrt{5.5} = 2.236 + 0.1065 + 0.00200 + 0.0000625$

$= 2.3445625 = 2.345$ (Rounded upto four decimal places).

Example 4 : If $\mu_0 = 1, \mu_1 = 0, \mu_2 = 5, \mu_3 = 22, \mu_4 = 57$ find $\mu_{0.5}$.

Solution : The difference table is as under :

x	μ_x	Δ	Δ^2	Δ^3	Δ^4
0	1				
		–1			
1	0		6		
		5		6	
2	5		12		0
		17		6	
3	22		18		
		35			
4	57				

We have $x_0 + ph = 0.5,\ x_0 = 0,\ h = 1$

$\Rightarrow\ 0 + p(1) = 0.5 \quad \text{or} \quad p = 0.5$

By Newton's Forward interpolation formula,

$$\mu_{0.5} = \mu_0 + 0.5\,\Delta\,\mu_0 + \frac{0.5(0.5-1)}{\lfloor 2}\Delta^2\mu_0 + \frac{0.5(0.5-1)(0.5-2)}{\lfloor 3}\Delta^3\mu_0 + \ldots$$

$$= 1 + (0.5)(-1) + \frac{0.5(-0.5)}{2}6 + \frac{0.5(-0.5)(-1.5)}{6}6$$

$$= 1 - 0.5 - 0.75 + 0.375 = 0.125.$$

Example 5 : Using Newton's forward interpolation formula, and the given table of values

x	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)$ when $x = 1.4$. **[JNTU (A) May 2011, June 2013 (Set No. 2)]**

Solution : The difference table is as under :

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.1	0.21				
		0.48			
1.3	0.69		0.08		
		0.56		0	
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

If we take $x_0 = 1.3$, then $y_0 = 0.69$, $\Delta y_0 = 0.56$, $\Delta^2 y_0 = 0.08$, $\Delta^3 y_0 = 0$, $h = 0.2, x = 1.3$

We have $x_0 + ph = 1.4$ or $1.3 + p\,(0.2) = 1.4 \Rightarrow p = \frac{1}{2}$

Using Newton's interpolation formula,

$$f(1.4) = 0.69 + \frac{1}{2} \times 0.56 + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{\lfloor 2} \times 0.08 = 0.69 + 0.28 - 0.01 = 0.96 \,.$$

Note : $x_0 = 1.3$ is taken so that $h < 1$.

Example 6 : Find the Newton's forward difference interpolating polynomial for the data :

x	0	1	2	3
$f(x)$	1	3	7	13

Solution : The difference table is as under :

x	$f(x)$	Δ	Δ^2	Δ^3
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

By Newton's Forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2\,!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\Delta^3 y_0 + \ldots.$$

Here $x_0 = 0$, $n = 1$ and $p = x$

Thus we have $f(x) = 1 + x\,(2) + \frac{x\,(x-1)}{\lfloor 2}(2) + \frac{x\,(x-1)\,(x-2)}{\lfloor 3}(0) + \ldots.$

$$= 1 + 2x + x^2 - x = x^2 + x + 1 \,.$$

Example 7 : The following table gives corresponding values of x and y. Construct the difference table and then express y as a function of x :

x	0	1	2	3	4
y	3	6	11	18	27

Solution : The difference table is as under :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

We have

$$x_0 + ph = x, \; x_0 = 0, \; h = 1$$

$$\Rightarrow \; 0 + p(1) = x \quad \text{or} \quad p = x$$

By Newton's forward interpolation formula,

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\Delta^3 y_0 + ...$$

$$i.e.\; f(x) = 3 + x\,(3) + \frac{x\,(x-1)}{\lfloor 2}(2) + \frac{x\,(x-1)\,(x-2)}{\lfloor 3}(0) + ...$$

$$i.e.\; f(x) = 3 + 3x + x^2 - x + 0$$

$$\text{or } f(x) = x^2 + 2x + 3 .$$

Example 8 : Consider the following data for $g(x) = (\sin x) / x^2$

x	0.1	0.2	0.3	0.4	0.5
$g(x)$	9.9833	4.9696	3.2836	2.4339	1.9177

[JNTU (A) 2003, Dec. 2013 (Set No. 1)]

Calculate $g(0.25)$ accurately using Newton's forward method of interpolation.

Solution : Newton's Forward interpolation formula is

$$f(x_0 + ph) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_0 + \frac{p\,(p-1)\,(p-2)}{\lfloor 3}\Delta^3 y_0 + ...$$

Let $x = x_0 + ph$, $x = 0.25$, $x_0 = 0.1$

Step interval $h = 0.2 - 0.1 = 0.1$

$$\therefore \; p = \frac{x - x_0}{h} = \frac{0.25 - 0.1}{0.1} = \frac{0.15}{0.1} = 1.5$$

The Newton's forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.1	9.9833				
		−5.0137			
0.2	4.9696		3.3277		
		−1.6860		−2.4914	
0.3	3.2836		0.8363		1.9886
		−0.8497		−0.5028	
0.4	2.4339		0.3335		
		−0.5162			
0.5	1.9177				

$$g(0.25) = 9.9833 + 1.5(-5.0137) + \frac{1.5 \times 0.5}{2} \times 3.3277 + \frac{1.5 \times 0.5 \times (-0.5)}{3 \times 2}$$

$$\times(-2.4919) + \frac{1.5 \times 0.5 \times (-0.5) \times (-1.5)}{4 \times 3 \times 2} \times 1.9886$$

$$= 9.9833 - 7.52 + 1.24789 + 0.1557 + 0.0466 = 3.9135$$

$\therefore\ g(0.25) = 3.9135$

Example 9 : For $x = 0, 1, 2, 3, 4$; $f(x) = 1, 14, 15, 5, 6$. Find $f(3)$ using Forward difference table. **[JNTU 2004, (A) June 2011 (Set No. 4)]**

Solution : Given

x	0	1	2	3	4
$f(x)$	1	14	15	5	6

Let $x = 3, h = 1, p = \dfrac{x - x_0}{h} = \dfrac{3-0}{1} = 3$. Then

$\Delta y_0 = 13,\quad \Delta^2 y_0 = -12,\quad \Delta^3 y_0 = 1$

$\Delta y_1 = 1,\quad \Delta^2 y_1 = -11,\quad \Delta^3 y_1 = 22,\quad \Delta^4 y_0 = 21$

$\Delta y_2 = -10,\quad \Delta^2 y_2 = 11,$

and $\Delta y_3 = 1$

$$\therefore\ f(x_0+ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + ... + \frac{p\,(p-1)(p-2)....(p-(n-1))}{n!}\Delta^n y_0$$

$$= 1 + 13\,(3) + \frac{3\,(2)}{2}(-12) + \frac{3\,(2)\,(1)}{3 \times 2 \times 1}(1) = 5.$$

Example 10 : Find the cubic polynomial which takes the following values :

$y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$. Hence, or otherwise, obtain $y(4)$.

Solution : We form the difference table as :

x	y	Δ	Δ^2	Δ^3
0	1			
		–1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here $h = 1$. Hence, take $x = x_0 + ph$ and $x_0 = 0$, we obtain $p = x$.
Substituting the value of p, we get

$$y(x) = 1 + x\,(-1) + \frac{x\,(x-1)}{2}(2) + \frac{x\,(x-1)\,(x-2)}{(6)}(6) = x^3 - 2x^2 + 1$$

which is the polynomial form which we obtained the above tabular values. To compute $y(4)$ we observe that $p = 4$. Hence formula gives $y(4) = 1 + 4\,(-1) + (12) + 24 = 33$ which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Note. This process of finding the value of y for some value of x outside the given range is called **extrapolation** and this example demonstrates the fact that if a tabulated function is polynomial, then interpolation and extrapolation would give exact values.

Example 11 : The population of a town in the decimal census was given below. Estimate the population for the 1895.

year x	1891	1901	1911	1921	1931
population y (thousands)	46	66	81	93	101

Solution : Putting $h = 10, x_0 = 1891, x = 1895$ in the formula $x = x_0 + ph$ we obtain $p = 2/5 = 0.4$

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		–5		
		15		2	
1911	81		–3		–3
		12		–1	
1921	93		–4		
		8			
1931	101				

$$\therefore\ y(1895) = 46 + (0.4)\,(20) + \frac{(0.4)\,(0.4-1)}{6}(-5) + \frac{(0.4-1)\,0.4\,(0.4-2)}{6}(2)$$

$$+ \frac{(0.4)\,(0.4-1)\,(0.4-2)\,(0.4-3)}{24}(-3)$$

$$= 54.45 \text{ thousands.}$$

Example 12 : In Ex. 11, estimate the population of the year 1925.

Solution : Here Interpolation is desired at the end of the table. Thus we use Newton's Backward difference interpolation formula. Take $x = x_n + ph$ with $x = 1925, x_n = 1931$ and $h = 10$. We obtain $p = -0.6$. Hence it gives

$$y(1925) = 101 - (0.6)\, 8 + \frac{(-0.6)\,((-0.6)+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)\,(-0.6+2)}{6}(-1)$$

$$+ \frac{(-0.6)\,(-0.6+1)\,(-0.6+2)\,(-0.6+3)}{24}(-3)$$

= 96.84 thousands.

Example 13 : In the table below the values of y are consecutive terms of a series of which the number 21.6 is the 6th term. Find the first and tenth terms of the series.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Solution : The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3.0		0
		9.1		0.6	
6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		0
		16.9		0.6	
8	51.2		4.8		
		21.7			
9	72.9				

From the difference table, it will be seen that third differences are constant and hence tabulated function represents a polynomial of third degree. We conclude that both interpolation and extra polation would yield exact results.

To obtain tenth term, we use formula with $x_0 = 3, x = 10, h = 1$ and $p = 7$ we get,

$$y\,(10) = 2.7 + (3.7)\;7 + \frac{(7)\,(6)}{1\,(2)}(2.4) + \frac{(7)\,(6)\,(5)}{(1)\,(2)\,(3)}(0.6)$$

$= 100$

To find the first term, we use formula with $x_n = 9, x = 1, h = 1$ and $p = -8$.

The student is advised to verify that the formula gives $y\,(1) = 0.1$.

Example 14 : Given sin 45⁰ = 0.7071, sin 50⁰ = 0.7660, sin 55⁰ = 0.8192 and sin 60⁰ = 0.8660, find sin 52⁰ using Newton's interpolation formula. Estimate the error.
[JNTU 2006 (Set No.2)]

Solution : Let $y = \sin x$ be the function. We construct the following difference table

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45	0.7071			
		0.0589		
50	0.7660		–0.0057	
		0.0532		–0.0007
55	0.8192		–0.0064	
		0.0468		
60	0.8660			

Here $x_0 = 45, y_0 = 0.7071, \Delta y_0 = 0.0589, \Delta^2 y_0 = -0.0057$ and $\Delta^3 y_0 = -0.0007$
Using Newton's Forward interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{1}{2!}p\,(p-1)\Delta^2 y_0 + \frac{1}{3!}p\,(p-1)\,(p-2)\Delta^3 y_0$$

where $p = \dfrac{x - x_0}{h}$. Let y_p be the value of y at $x = 52^0$.

$$\therefore \quad p = (52-45)/5 = 7/5 = 1.4$$

$$y_{52} = 0.7071 + (1.4)\,(0.0589) + \frac{1}{2}(1.4)\,(1.4-1)\,(-0.0057)$$

$$+\frac{1}{6}(1.4)\,(1.4-1)(1.4-2)(-0.0007)$$

$$= 0.7071 + 0.08246 - 0.001596 + 0.0000392 = 0.7880032$$

$$\therefore \quad \sin 52^0 = 0.7880032$$

$$\text{Error} = \frac{p\,(p-1)....(p-n)}{3!}\Delta^{n+1}y(c) = \frac{(1.4)\,(1.4-1)\,(1.4-2)}{3!}\Delta^3 y(c) \quad [\text{by taking } n = 2\,]$$

$$= \frac{(1.4)\,(1.4-1)\,(1.4-2)}{6}\Delta^3 y(c) = \frac{(1.4)\,(0.4)\,(-0.6)}{6}(-0.0007) = 0.0000392\,.$$

Example 15 : Find $f(2.5)$ using Newtons forward formula from the following table:

x	0	1	2	3	4	5	6
y	0	1	16	81	256	625	1296

[JNTU May 2006 (Set No.1)]

Solution : We have $x = 2.5$, $h = 1$, $p = \dfrac{x - x_0}{h} = \dfrac{2.5-0}{1} = 2.5$

$$\Delta y_0 = y_1 - y_0 = 1 - 0 = 1$$

$$\Delta y_1 = y_2 - y_1 = 16 - 1 = 15$$

$$\Delta y_2 = y_3 - y_2 = 81 - 16 = 65$$

$$\Delta y_3 = y_4 - y_3 = 256 - 81 = 175$$

$$\Delta y_4 = y_5 - y_4 = 1296 - 625 = 671$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 15 - 1 = 14$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = 65 - 15 = 50$$

$$\Delta^2 y_2 = \Delta y_3 - \Delta y_2 = 175 - 65 = 110$$

$$\Delta^2 y_3 = \Delta y_4 - \Delta y_3 = 674 - 175 = 499$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = 50 - 14 = 36$$

$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = 110 - 50 = 60$$

$$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2 = 499 - 110 = 389$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = 60 - 36 = 24$$

$$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1 = 389 - 60 = 329$$

$$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0 = 329 - 24 = 305$$

Using Newton Forward Difference Formula, we have

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3} \Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{\lfloor 4} \Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{\lfloor 5} \Delta^5 y_0$$

$$\therefore \quad f(2.5) = 0 + 2.5(1) + \frac{(2.5)(1.5)}{\lfloor 2}(14) + \frac{(2.5)(1.5)(.5)}{\lfloor 3}(36) + \frac{(2.5)(1.5)(.5)(-.5)}{\lfloor 4}(24)$$

$$+ \frac{(2.5)(1.5)(.5)(-.5)(-1.5)}{\lfloor 5}(305)$$

$$= 2.5 + 26.25 + 11.25 - 0.9375 + 3.5390 = 42.6015.$$

Example 16 : Find $y(1.6)$ using Newton's Forward difference formula from the table

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.5

[JNTU May 2006 (Set No.3)]

Solution : Let $x_0 = 1,\ h = 1.4 - 1 = .4,\ x_0 + ph = 1.6 \ \Rightarrow\ 1 + .4p = 1.6 \ \Rightarrow\ p = \frac{.6}{.4} = \frac{3}{2}$

We have $\Delta y_0 = y_1 - y_0 = 4.82 - 3.49 = 1.33$

$$\Delta y_1 = y_2 - y_1 = 5.96 - 4.82 = 1.14$$

$$\Delta y_2 = y_3 - y_2 = 6.5 - 5.96 = .54$$

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = 1.14 - 1.33 = -0.19$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = .54 - 1.14 = -.60$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = -0.60 + 0.19 = -0.41.$$

Using Newton's forward difference formula, we have

$$f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

i.e. $$f(1.6) = 3.49 + \frac{3}{2}(1.33) + \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\right)(-0.19)}{\lfloor 2} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(-0.41)}{\lfloor 3}$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625$$
$$= 5.4394.$$

Example 17 : Construct difference table for the following data.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate f(0.6). **[JNTU May 2007 (Set No. 2)]**

Solution :

x	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$.
0.1	0.003			
		0.064		
0.3	0.067		0.017	
		0.081		0.002
0.5	0.148		0.019	
		0.1		0.003
0.7	0.248		0.022	
		0.122		0.004
0.9	0.370		0.026	
		0.148		0.005
1.1	0.518		0.031	
		0.179		
1.3	0.697			

Here $x = 0.6$, $x_0 = 0.1$, $h = 0.2$, $y_0 = 0.003, \Delta y_0 = 0.064, \Delta^2 y_0 = 0.017, \Delta^3 y_0 = 0.002$

We have $x_0 + ph = x$

$$\Rightarrow \quad 0.1 + p(0.2) = 0.6 \Rightarrow p(0.2) = 0.5 \Rightarrow \quad p = \frac{0.5}{0.2} \quad \therefore p = 2.5$$

By Newton's forward difference formula,

$$y(x) = f(x_0 + ph) = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_0) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_0) + \ldots$$

$$i.e.,\ f(0.6) = 0.003 + (2.5)\,(0.064) + \frac{(2.5)(2.5-1)}{2}\,(0.017) + \frac{(25)(2.5-1)(2.5)(0.002)}{6}$$

$$= 0.003 + 0.16 + 0.031875 + 0.000625 \;=\; 0.1955$$

$$\therefore\ f(0.6) = 0.1955.$$

Example 18 : Find $y(54)$ given that $y(50) = 205, y(60) = 225, y(70) = 248$ and $y(80) = 274$. Using Newton's forward difference formula. **[JNTU (H) Jan. 2012 (Set No. 4)]**

Solution :

x	50	60	70	80
$y(x)$	205	225	248	274

Here, $h = 10, x_0 = 50, x_0 + ph = 55 \Rightarrow p = \dfrac{55-50}{10} = 0.5$

x	$y(x)$	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Using Newton's forward difference formula,

$$y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p_{-1})}{2!}\Delta^2 y_0 + \frac{p(p_{-1})(p_{-2})}{3}\Delta^3 y_0$$

$$y(55) = 205 + (0.5)(20) + \frac{(0.5)(-0.5)}{2}(3)$$

$$= 205 + 10 - 0.375 \;= 215 - 0.375 \;\;= 214.625$$

5.9 CENTRAL DIFFERENCE INTERPOLATION

As mentioned earlier, Newton's forward interpolation formula is useful to find the value of $y = f(x)$ at a point which is near the beginning value of x and the Newton's backward interpolation formula is useful to find the value of 'y' at a point which is near the terminal value of x. We now derive the interpolation formulas that can be employed to find the value of x which is around the middle to the specified values.

For this purpose, we take x_0 as one of the specified values of x that lies around the middle of the difference table and denote $x_0 - rh$ by x_{-r} and the corresponding value of y by y_{-r}. Then the middle part of the forward difference table will appear as shown below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
....	....					
x_{-4}	y_{-4}					
		Δy_{-4}				
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$			
		Δy_{-3}		$\Delta^3 y_{-4}$		
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$	
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$	
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$	
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		
x_3	y_3		$\Delta^2 y_2$			
		Δy_3				
x_4	y_4					
....	....					

From the table, we note the following :

$$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1},\ \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1},\ \Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1},$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \qquad \text{...(1) and so on.}$$

and $\Delta y_{-1} = \Delta y_{-2} + \Delta^2 y_{-2},\ \Delta^2 y_{-1} = \Delta^2 y_{-2} + \Delta^3 y_{-2},\ \Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2},$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2},\ \Delta^5 y_{-1} = \Delta^5 y_{-2} + \Delta^6 y_{-2} \text{ and so on.} \qquad \text{...(2)}$$

By using the expressions (1) and (2), we now obtain two versions of the following Newton's Forward interpolation formula :

$$y_p = \left[y_0 + p\,(\Delta y_0) + \frac{p\,(p-1)}{2!}(\Delta^2 y_0) + \frac{p\,(p-1)\,(p-2)}{3!}(\Delta^3 y_0) \right.$$

$$\left. + \frac{p\,(p-1)\,(p-2)\,(p-3)}{4!}(\Delta^4 y_0) + ... \right] \qquad \text{...(3)}$$

Here y_p is the value of y at $x = xp = x_0 + ph$.

1. Gauss's Forward Interpolation formula :

Substituting for $\Delta^2 y_0,\ \Delta^3 y_0, ...$ from (1) in the formula (3), we get,

$$y_p = \left[y_0 + p\,(\Delta\, y_0) + \frac{p(p-1)}{2!}\Big((\Delta^2 y_{-1}) + (\Delta^3 y_{-1})\Big) + \frac{p\,(p-1)\,(p-2)}{3!}(\Delta^3 y_{-1} \right.$$

$$\left. + \Delta^4 y_{-1}) + \frac{p\,(p-1)\,(p-2)\,(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots \right]$$

$$y_p = \left[y_0 + p\,\Delta\, y_0 + \frac{p\,(p-1)}{2!}(\Delta^2 y_{-1}) + \frac{(p+1)\,p(p-1)}{3!}(\Delta^3 y_{-1}) \right.$$

$$\left. + \frac{(p+1)\,(p-1)p(p-2)}{4!}(\Delta^4 y_{-1}) + \ldots \right]$$

Substituting for $\Delta^4 y_{-1}$ from (2), this becomes

$$y_p = \left[y_0 + p\,(\Delta\, y_0) + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^3 y_{-1}) \right.$$

$$\left. + \frac{(p+1)\,(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + \ldots \right] \quad \ldots(4)$$

This version of the Newton's Forward interpolation formula is known as the **Gauss's Forward interpolation formula.** We observe that the formula (4) contains y_0 and the even differences $\Delta^2 y_{-1},\ \Delta^4 y_{-2}, ...$ which lie on the line containing x_0 (called the central line) and the odd differences $\Delta\, y_0,\ \Delta^3 y_{-1}, ...$ which lie on the line just below this line, in the difference table.

Note. We observe from the difference table that

$\Delta y_0 = \delta y_{1/2},\ \ \Delta^2 y_{-1} = \delta^2 y_0,\ \Delta^3 y_{-1} = \delta^3 y_{1/2},\ \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula (4) can be rewritten in the notation of central differences as given below :

$$y_p = \left[y_0 + p\,\delta\, y_{1/2} + \frac{p\,(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)\,p\,(p-1)}{3!}\delta^3 y_{1/2} \right.$$

$$\left. + \frac{(p+1)\,p(p-1)\,(p-2)}{4!}\delta^4 y_0 + \ldots \right] \quad \ldots(5)$$

2. Gauss's Backward interpolation formula :

Next, let us substitute for $\Delta y_0,\ \Delta^2 y_0,\ \Delta^3 y_0, ...$ from (1) in the formula (3).
Thus we obtain.

$$y_p = \left[y_0 + p\,(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p\,(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p\,(p-1)\,(p-2)}{3!} \right.$$

$$\left. (\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p\,(p-1)\,(p-2)\,(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots \right]$$

$$= \left[y_0 + p\,(\Delta\, y_{-1}) + \frac{(p+1)p}{2!}(\Delta^2 y_{-1}) + \frac{(p+1)p\,(p-1)}{3!}(\Delta^3 y_{-1}) + \frac{(p+1)\,p\,(p-1)\,(p-2)}{4!}(\Delta^4 y_{-1}) + \ldots \right]$$

Substituting for $\Delta^3 y_{-1}$ and $\Delta^4 y_{-1}$ from (2), this becomes

$$y_p = \left[y_0 + p\,(\Delta\, y_{-1}) + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-2}) + \frac{(p+1)\,p(p-1)\,(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots \right]$$

$$= \left[y_0 + p\,(\Delta\, y_{-1}) + \frac{(p+1)p}{2!}(\Delta^2 y_{-1}) + \frac{(p+1)\,p(p-1)}{3!}(\Delta^3 y_{-2}) + \frac{(p+1)\,p(p-1)\,(p-2)}{4!}(\Delta^4 y_{-2}) + \ldots \right] \quad \ldots(6)$$

This version of the Newton's Forward interpolatin formula is known as the **Gauss's Backward interpolation formula.**

Observe that the formula (6) contains y_0 and the even differences $\Delta^2 y_{-1},\ \Delta^4 y_{-2},\ldots.$ which lie on the central line, and the odd differences $\Delta y_{-1},\ \Delta^3 y_{-2},\ldots.$ which lie on the line just above this line.

Note. In the notation of central differences, the formula (6) reads

$$y_p = \left[y_0 + p\,\delta\, y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)\,p\,(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)\,(p+1)\,p(p-1)}{4!}\delta^4 y_0 + \ldots \right] \quad \ldots(7)$$

SOLVED EXAMPLES

Example 1 : Find $f(2.5)$ using the following table

x	1	2	3	4
$f(x)$	1	8	27	64

Solution : Since the value required for interpolation is near the centre of the table, we can use Gauss forward formula by considering $x_0 = 2$. The central difference table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

Here $h = 2 - 1 = 1, x = 2.5, x_0 = 2$

$$p = \frac{x - x_0}{h} = \frac{2.5 - 2}{1} = 0.5$$

Using Gauss Forward interpolation formula,

$$\therefore\ f(2.5) = 8 + 0.5 \times 19 + \frac{(0.5-1)\ (0.5)}{2} \times 12 + \frac{(0.5-1)\ (0.5)\ (0.5+1)}{3 \times 2} \times 6$$

$$= 8 + 9.5 - 1.5 - 0.375 = 15.625.$$

Example 2 : From the following table values of x and $y = e^x$ interpolate values of y when $x = 1.91$.

x	1.7	1.8	1.9	2	2.1	2.2
e^x	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

Solution : The central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.7	5.4739					
		0.5757				
1.8	6.0496		0.0606			
		0.6363		0.0063		
1.9	6.6859		0.0669		0.0007	
		0.7032		0.0070		0.0001
2	7.3891		0.0739		0.0008	
		0.7771		0.0078		
2.1	8.1662		0.0817			
		0.8588				
2.2	9.0250					

Here $h = 1.8 - 1.7 = 0.1, x_0 = 1.9, x = 1.91;$

$$p = \frac{x - x_0}{h} = \frac{1.91 - 1.9}{0.1} = \frac{0.01}{0.1} = 0.1$$

According to Gauss Forward interpolation formula,

$$y_{1.91} = f(1.91) = 6.6859 + 0.1 \times 0.7032 + \frac{(0.1-1) \times 0.1}{2} \times 0.0669 +$$

$$\frac{(0.1-1)\ (0.1)\ (0.1+1)}{3 \times 2} \times 0.0070 + \frac{(0.1-2)\ (0.1-1)\ (0.1)\ (0.1+1)}{4 \times 3 \times 2} \times 0.0007$$

$$+ \frac{(0.1-2)\ (0.1-1)\ (0.1)\ (0.1+1)\ (0.1+2)}{5 \times 4 \times 3 \times 2} \times 0.0001 = 6.7531$$

Example 3 : From the following table find y when $x = 38$.

x	30	35	40	45	50
y	15.9	14.9	14.1	13.3	12.5

Solution : Since the value $x = 38$ is near the centre of the table we can use Gauss Backward interpolation formula starting from $x_0 = 40$. The central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	15.9				
		– 1.0			
35	14.9		0.2		
		– 0.8		– 0.2	
40	14.1		0.0		0.2
		– 0.8		0.0	
45	13.3		0.0		
		– 0.8			
50	12.5				

Here $h = 35 - 30 = 5, x_0 = 40, x = 38$

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{38 - 40}{5} = \frac{-2}{5} = -0.4$$

According to Gauss Backward formula,

$$y_{38} = f(38) = 14.1 + (-0.4)(-0.8) + \frac{(-0.4)(-0.4+1)}{2!} \times 0.0$$
$$+ \frac{(-0.4-1)(-0.4)(-0.4+1)}{3!} \times (0.0)$$
$$+ \frac{(-0.4-1)(-0.4)(-0.4+1)(-0.4+2)}{4!} \times 0.2$$

$= 14.4245.$

Example 4 : From the following table find y when $x = 1.35$

x	1	1.2	1.4	1.6	1.8	2
y	0.0	-0.112	-0.016	0.336	0.992	2

Solution : The central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0				
		– 0.112			
1.2	– 0.112		0.208		
		+0.096		0.048	
1.4	– 0.016		0.256		0
		0.352		0.048	
1.6	0.336		0.304		0
		0.656		0.048	
1.8	0.992		0.352		
		1.008			
2	2				

Here $h = 1.2 - 1 = 0.2, x_0 = 1.4, x = 1.35$

$$x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h} = \frac{1.35 - 1.4}{0.2} = \frac{-0.05}{0.2} = -0.25$$

According to Gauss Backward interpolation formula,

$$y_{1.35} = f(1.35) = -0.016 + (-0.25) \times 0.096 + \frac{(-0.25)(-0.25+1)}{2!} \times 0.256 + \frac{(-0.25-1)(-0.25)(-0.25+1)}{3!} \times 0.048$$

$$= -0.062125.$$

Example 5 : Use Gauss Forward interpolation formula to find $f(3.3)$ from the following table :

x	1	2	3	4	5
$y = f(x)$	15.30	15.10	15.00	14.50	14.00

Solution : The difference table for the given data is given below with $x_0 = 3$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 1$	$y_{-2} = 15.30$				
		-0.20			
$x_{-1} = 2$	$y_{-1} = 15.10$		0.10		
		-0.10		-0.50	
$x_0 = 3$	$y_0 = 15.00$		-0.40		0.90
		-0.50		0.40	
$x_1 = 4$	$y_1 = 14.50$		0.00		
		-0.50			
$x_2 = 5$	$y_2 = 14.00$				

From the table, we note that $y_0 = 15.00$; $\Delta y_0 = -0.50$, $\Delta^2 y_{-1} = -0.40$, $\Delta^3 y_{-1} = 0.40$ and $\Delta^4 y_{-2} = 0.90$.

Let $x_p = 3.3$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{3.3 - 3}{1} = 0.3$

The Gauss's Forward difference formula now becomes

$$f(3.3) = y_p = 15.00 + (0.3)(-0.50) + \frac{(0.3)(0.3-1)}{2!}(-0.40) + \frac{(0.3)(0.09-1)}{6}(0.40) + \frac{(0.3)(0.09-1)(0.3-2)}{24}(0.90)$$

$$= 14.9$$

Example 6 : Use Gauss's Forward interpolation formula to find $f(30)$ given that $f(21) = 18.4708, f(25) = 17.8144, f(29) = 17.1070, f(33) = 16.3432, f(37) = 15.5154$.

Solution : Let us take $x_0 = 29$ and prepare the following difference table :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-2} = 21$	$y_{-2} = 18.4708$				
		-0.6564			
$x_{-1} = 25$	$y_{-1} = 17.8144$		-0.0510		
		-0.7074		-0.0064	
$x_0 = 29$	$y_0 = 17.1070$		-0.0574		$.0018$
		-0.7648		-0.0046	
$x_1 = 33$	$y_1 = 16.3422$		-0.0620		
		-0.8268			
$x_2 = 37$	$y_2 = 15.5154$				

From the table, we find that $y_0 = 17.1070$; $\Delta y_0 = -0.7648$,

$\Delta^2 y_{-1} = -0.0574$, $\Delta^3 y_{-1} = -0.0046$, $\Delta^4 y_{-2} = .0018$

Let $x_p = 30$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{30-29}{4} = 0.25$

The Gauss's Forward difference formula now gives

$$f(30) = y_p = 17.1070 + (0.25)\,(-0.7648) + \frac{(0.25)\,(0.25-1)}{2}(-0.0574)$$

$$+\frac{(0.25)\,(0.0625-1)}{6}(-0.0046)$$

$$+\frac{(0.25)\,(0.0625-1)\,(0.25-2)}{24}(.0018) = 16.921.$$

Example 7 : Find the polynomial which fits the data in the following table using Gauss forward formula.

x	3	5	7	9	11
y	6	24	58	108	174

[JNTU (H) Jan. 2012 (Set No. 3)]

Solution : Take $x_0 + ph = x$. Here $x_0 = 3$ and $h = 2$ $\Rightarrow 3 + 2p = x \Rightarrow p = \dfrac{x-3}{2}$

Difference table is

x	y	Δy	$\Delta^2 y$	Δ^3	$\Delta^4 y$
3	6				
		18			
5	24		16		
		34		0	
7	58		16		0
		50		0	
9	108		16		
		66			
11	179				

Using the Gauss forward formula,

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0$$

$$= 6 + \left(\frac{x-3}{2}\right)(18) + \left(\frac{x-3}{2}\right)\left(\frac{x-5}{2}\right)(16)$$

$$= 6 + (9x - 27) + (x^2 - 8x + 15)\,(4)$$

$$= 4x^2 - 32x + 60 + 9x - 27 + 6$$

$$= 4x^2 - 23x + 39$$

Example 8 : Find by Gauss's Backward interpolating formula the value of y at $x = 1936$, using the following table :

x	1901	1911	1921	1931	1941	1951
y	12	15	20	27	39	52

Solution : Let us take $x_0 = 1931$ and construct the following difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-3} = 1901$	$y_{-3} = 12$					
		3				
$x_{-2} = 1911$	$y_{-2} = 15$		2			
		5		0		
$x_{-1} = 1921$	$y_{-1} = 20$		2		3	
		7		3		-10
$x_0 = 1931$	$y_0 = 27$		5		-7	
		12		-4		
$x_1 = 1941$	$y_1 = 39$		1			
		13				
$x_2 = 1951$	$y_2 = 52$					

From the table, we find that

$$y_0 = 27,\ \Delta y_{-1} = 7,\ \Delta^2 y_{-1} = 5,\ \Delta^3 y_{-2} = 3,\ \Delta^4 y_{-2} = -7,\ \Delta^5 y_{-3} = -10$$

Let $x_p = 1936$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{1936 - 1931}{10} = 0.5$

The Gauss's Backward difference formula now gives

$$y_p = 27 + (0.5)\,(7) + \frac{(0.5)\,(0.5+1)}{2}(5) + \frac{(0.5)\,(0.25-1)}{6}(3)$$

$$+ \frac{(0.5)\,(0.25-1)\,(0.5+2)}{24}(-7) + \frac{(0.5)\,(0.25-1)\,(0.25-4)}{120}(-10)$$

$$= 32.345.$$

This is the value of y for $x = 1936$.

Example 9 : Use Gauss's backward interpolation formula to find $f(32)$ given that $f(25) = 0.2707$, $f(30) = 0.3027$, $f(35) = 0.3386$, $f(40) = 0.3794$.

[JNTU (A) Nov. 2010 (Set No. 1), May 2012 (Set No. 2)]

Solution : Let us take $x_0 = 35$ and construct the following difference table :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-2} = 25$	$y_{-2} = 0.2707$			
		0.032		
$x_{-1} = 30$	$y_{-1} = 0.3027$		0.0039	
		0.0359		0.0010
$x_0 = 35$	$y_0 = 0.3386$		0.0049	
		0.0408		
$x_1 = 40$	$y_1 = 0.3794$			

From the table, we find that $y_0 = 0.3386$;

$$\Delta y_{-1} = 0.0359,\ \Delta^2 y_{-1} = 0.0049,\ \Delta^3 y_{-2} = 0.0010$$

Let $x_p = 32$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{32 - 35}{5} = -0.6$

The Gauss's backward difference formula now yields

$$f(32) = y_p = 0.3386 + (-0.6)(0.0359) + \frac{(-0.6)(-0.6+1)}{2}(0.0049)$$

$$+ \frac{(-0.6)(0.36-1)}{6}(0.0010) = 0.3165.$$

Example 10 : Given that $\sqrt{6500} = 80.6223$, $\sqrt{6510} = 80.6846$, $\sqrt{6520} = 80.7456$, $\sqrt{6530} = 80.8084$. Find $\sqrt{6526}$ by using Gauss's backward formula.

Solution : Here the given function is of the form $f(x) = \sqrt{x}$. Let us take $x_0 = 6520$ and construct the difference table below :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_{-2} = 6500$	$y_{-2} = 80.6223$			
		0.0623		
$x_{-1} = 6510$	$y_{-1} = 80.6846$		– 0.0004	
		0.0619		0.004
$x_0 = 6520$	$y_0 = 80.7465$		0	
		0.0619		
$x_1 = 6530$	$y_1 = 80.8084$			

From the table, we find

$$y_0 = 80.7465;\ \Delta y_{-1} = 0.0619,\ \Delta^2 y_{-1} = 0,\ \Delta^3 y_{-2} = 0.0004$$

Let $x_p = 6526$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{6526 - 6520}{10} = 0.6$

The Gauss's Backward interpolation formula gives

$$y_p = 80.7465 + (0.6)\,(0.0619) + \frac{(0.6)\,(0.6+1)}{2}(0) \;+ \frac{(0.6)\,(0.36-1)}{6}(0.0004)$$

$$= 80.7836.$$

Thus $\sqrt{6526} = 80.7836$.

Example 11 : Find $y(25)$, given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using Gauss forward difference formula. **[JNTU Sep. 2006, (H) June 2011 (Set No. 2,4)]**

Solution : Given

x	20	24	28	32
y	24	32	35	40

By Gauss Forward difference formula,

$$y\,(x) = y_0 + p\Delta_{y_0} + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)(p)(p-1)}{3!}\Delta^3 y_{-1} + \cdots$$

We take $x = 24$ as origin.

$$\therefore\; x_0 = 24,\, h = 4,\, x = 25,\; p = \frac{x - x_0}{3!} = \frac{25-24}{4} = .25$$

$\therefore$ Gauss forward difference table is as follows

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
20	–1	24			
			8 Δ_{y-1}		
24	0	32 y_0		–5 $\Delta^2 y-1$	
			3 Δy_0		7 $\Delta^3 y_{-1}$
28	1	35 y_1		2 $\Delta^2 y_0$	
			5 Δy_1		
32	2	40 y_2			

$\therefore$ By Gauss Forward interpolation formula, we have

$$y\,(25) = 32 + (.25)3 + \frac{(.25)(.25-1)}{2}(-5) + \frac{(.25+1)(.25)(.25-1)}{6}(7)$$

$$= 32 + .75 \;\; .46875 - .2734 = 32.945.$$

$\therefore \quad y\,(25) = 32.945.$

Example 12 : Using Gauss Backward difference formula, find $y(8)$ from the following table. **[JNTU Sep. 2006, May 2007 (Set No. 1)]**

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Solution : Given

x	0	5	10	15	20	25
y	7	11	14	18	24	32

The difference table is given below :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$x_{-2} = 0$	$y_{-2} = 7$					
		$\Delta y_{-2} = 4$	$\Delta^2 y_{-2} = -1$	$\Delta^3 y_{-2} = 2$	$\Delta^4 y_{-2} = -1$	
$x_{-1} = 5$	$y_{-1} = 11$					$\Delta^5 y_{-2} = 0$
		$\Delta y_{-1} = 3$	$\Delta^2 y_{-1} = 1$	$\Delta^3 y_{-1} = 1$	$\Delta^4 y_{-1} = -1$	
$x_0 = 10$	$y_0 = 14$					
		$\Delta y_0 = 4$	$\Delta^2 y_0 = 2$	$\Delta^3 y_0 = 0$		
$x_1 = 15$	$y_1 = 18$					
		$\Delta y_{-1}=6$	$\Delta^2 y_1 = 2$			
$x_2 = 20$	$y_2 = 24$					
		$\Delta_{y_{-2}=8}$				
$x_3 = 25$	$y_3 = 32$					

By Gauss Backward interpolation formula,

$$f(x) = y_p = y_0 + p\,\Delta\, y_{-1} + \frac{p(p+1)}{2!}\ \Delta^2_{y-1} + \frac{(p+1)\,p\,(p-1)}{3!}\ \Delta^3_{y-2}$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{4!}\ \Delta^4_{y-2} + \ldots$$

Here $\quad x_p = 8, y_0 = 14,\ x_0 = 10, h = 5$

and $\quad p = \dfrac{x_1 - x_0}{h} = \dfrac{8-10}{5} = \dfrac{-2}{5} = -0.4$

$$\therefore \quad f(8) = 14 - 0.4(3) + \frac{(-0.4)(-0.4+1)1}{2} + \frac{(-0.4+1)(-0.4)(-0.4-1)}{6}(2)$$

$$+ \frac{(-0.4-2)(-0.4+1)(-0.4)(-0.4-1)}{24}\ (-1)$$

$$= 14 - 1.2 + 0.112 + 0.0336 - 0.12$$

$\therefore\ y(8) = \ 12.7024.$

Example 13 : Find $f(22)$ from the Gauss forward formula. **[JNTU May 2007 (Set No. 4)]**

x	20	25	30	35	40	45
$f(x)$	354	332	291	260	231	204

Solution : The Difference table for the given data is given below with $x_0 = 25$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-1} 20	y_{-1} 354	Δy_{-1}				
		–22	$\Delta^2 y_{-1}$			
			–19	$\Delta^3 y{-1}$	$\Delta^4 y_{-1}$	
	y_0	Δy_0		29		$\Delta^5 y{-1}$
x_0 25	332	– 41	$\Delta^2 y_0$	$\Delta^3\ y_0$	–37	
			10			45
x_1 30	y_1 291	Δy_1		–8		
		– 31				
x_2 35	y_2 260	Δy_2	$\Delta^2 y_1$ 2		$\Delta^4 y_0 8$	
		– 29				
x_3 40	y_3 231			$\Delta^3 y_1 0$		
x_4 45	y_4 204	Δy_3	$\Delta^2 y_2$ 2			
		– 27				

From the table, we note that

$$y_0 = 332,\ \Delta y_0 = -41,\ \Delta^2 y_{-1} = -19,\ \Delta^3 y_{-1} = 8,\ \Delta^4 y_{-1} = -37,\ \Delta^5 y_{-1} = 45$$

Let $x_p = 22$. Then $p = \dfrac{x_p - x_0}{h} = \dfrac{22-25}{5} = \dfrac{-3}{5} = -0.6.$

Now the Gauss Forward formula gives,

$$f(22) = y_p = 332 + (-0.6)\,(-41) + \frac{(-0.6)\,(-0.6-1)}{2}\,(-19) + \frac{(-0.6)(-0.6-1)(-0.6+1)}{6}\,(-8)$$

$$+ \frac{(-0.6)(-0.6-1)(-0.6+1)(-0.6-2)}{24}\,(-37) +$$

$$+ \frac{(-0.6)(-0.6-1)(-0.6+1)(-0.6-2)(-0.6+2)}{120}\,(45)$$

$$= 332 + (0.6)\,(41) - \frac{((0.6)^2+0.6)}{2}(19) + \frac{(0.6)\,[(0.6)^2-(1)^2]}{6}(8)$$

$$- \frac{(0.6)\,[(0.6)^2-1^2]\,(0.6+2)}{24}(37)$$

$$- \frac{(0.6)\,[(0.6)^2-1]\,[(0.6)^2-2^2]}{120}(45).$$

$$= 332 + 24.6 - 9.12 - 0.512 + 1.5392 - 0.5241$$

Thus $f(22) = 347.9831$.

Example 14 : Find $f\,(2.36)$ from the following table :

x :	1.6	1.8	2.0	2.2	2.4	2.6
y :	4.95	6.05	7.39	9.03	11.02	13.46

[JNTU 2008 (Set No.4)]

Solution :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.6	4.95					
		1.1				
1.8	6.05		0.24			
		1.34		0.06		
2.0	7.39		0.30		–0.01	
		1.64		0.05		0.06
2.2	9.03		0.35		0.05	
		1.99		0.10		
2.4	11.02		0.45			
		2.44				
2.6	13.46					

Here $h = 1.8 - 1.6 = 0.2,\ x_0 = 2.4,\ \ x = 2.36$

$$x = x_0 + ph \Rightarrow 2.36 = 2.4 + (0.2)\,p$$

$$\Rightarrow \quad -0.04 = 0.2p \Rightarrow p = -\,0.2$$

Using the Gauss backward formula,

$$y_{2.36} = f(2.36)$$

$$= y_0 + p\,(\Delta y_0) + \frac{p(p+1)}{2!}(\Delta^2 y_0) + \frac{(p+1)(p)(p-1)}{3!}(\Delta^3 y_0)$$

$$= 11.02 + (-0.2)(1.99) + \frac{(-0.2)(0.8)}{2}(0.45) + \frac{(0.8)(-0.2)(-1.2)}{6}(0.10)$$

$= 11.02 - 0.398 - 0.036 + 0.0032$

$\therefore\ y_{2.36} = 10.5892.$

Example 15 : Given $f(2) = 10, f(1) = 8, f(0) = 5, f(-1) = 10$ estimate $f(1/2)$ by using Gauss's forward formula. **[JNTU (A) May 2012 (Set No. 4)]**

Solution : Tabulating the given values

x	-1	0	1	2
$f(x) = y$	10	5	8	10

We form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_{-1} = -1$	$y_{-1} = 10$				
		-5			
$x_0 = 0$	$y_0 = 5$		8		
		3		-9	
$x_1 = 1$	$y_1 = 8$		-1		
		2			
$x_2 = 2$	$y_2 = 10$				

Here, $x_0 = 0, y_0 = 5, \Delta y_0 = 3, \Delta^2 y_{-1} = 8$

$$x_p = \frac{1}{2} = 0.5,$$

$$p = \frac{x_p - x_0}{h} = \frac{0.5 - 0}{1} = 0.5$$

Using Gauss forward difference formula,

$$f(1/2) = f(x_p) = y_p = y_0 + p(\Delta y_0) + \frac{p(p_{-1})}{2}\Delta^2 y_{-1}$$

$$= 5 + (0.5)(3) + \frac{(0.5)\,(-0.5)}{2}.8$$

$$= 5 + 1.5 - 1 = 4.5$$

5.10 INTERPOLATION WITH UNEVENLY SPACED POINTS

In the previous sections we have derived interpolation formulae which are of great importance. But in those formulae the disadvantage is that the values of the independent variables are to be equally spaced. We desire to have interpolation formulae with unequally spaced values of the independent variables. We discuss Lagrange's Interpolation Formula which uses only function values.

1. Lagrange's Interpolation Formula :

Let $x_0, x_1, x_2, ..., x_n$ be the $(n+1)$ values of x which are not necessarily equally spaced. Let $y_0, y_1, y_2, ..., y_n$ be the corresponding values of $y = f(x)$. Let the polynomial of degree n for the function $y = f(x)$ passing through the $(n+1)$ points $(x_0, f(x_0))$ $(x_1, f(x_1))....(x_n, f(x_n))$ be in the following form

$$y = f(x) = a_0(x-x_1)(x-x_2)....(x-x_n) \ldots + a_1(x-x_0)(x-x_2)....(x-x_n)$$
$$+ a_2(x-x_0)(x-x_1)(x-x_3)....(x-x_n) + \ldots + a_n(x-x_0)(x-x_1)....(x-x_{n-1})$$

where $a_0, a_1, a_2, ..., a_n$ are constants. ...(1)

Since the polynomial passes through $(x_0, f(x_0)), (x_1, f(x_1)), ..., (x_n, f(x_n))$, the constants can be determined by substituting one of the values of $x_0, x_1, x_2, ..., x_n$ for x in the above equation.

Putting $x = x_0$ in (1) we get, $f(x_0) = a_0(x_0-x_1)(x_0-x_2)....(x_0-x_n)$

$$\Rightarrow a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)}$$

Putting $x = x_1$ in (1) we get, $f(x_1) = a_1(x_1-x_0)(x_1-x_2)....(x_1-x_n)$

$$\Rightarrow a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}$$

Similarly substituting $x = x_2$ in (1) we get, $a_2 = \dfrac{f(x_2)}{(x_2-x_0)(x_2-x_1)....(x_2-x_n)}$

Continuing in this manner and putting $x = x_n$ in (1), we get

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})}$$

Substituting the values of $a_0, a_1, a_2, ..., a_n$, we get

$$f(x) = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)} f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)....(x-x_n)}{(x_2-x_0)(x_2-x_1)....(x_2-x_n)} f(x_2) + ...$$
$$+ \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})} f(x_n)$$

This is known as Lagrange's Interpolation formula. This can be expressed as

$$f(x) = \sum_{k=0}^{n} f(x_k) \cdot \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x-x_j)}{(x_k-x_j)}.$$

Another form : $f(x) = \dfrac{(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_2)(x_1-x_3)...(x_1-x_n)} f(x_1) +$

$$\frac{(x-x_1)(x-x_3)(x-x_4)...(x-x_n)}{(x_2-x_1)(x_2-x_3)...(x_2-x_n)} f(x_2) + ... + \frac{(x-x_1)(x-x_2)...(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})} f(x_n)$$

SOLVED EXAMPLES

Example 1 : Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively. Use Lagrange interpolation. **[JNTU 2002, (A) May 2012 (Set No. 2)]**

Solution : We are given

$x_0 = 1, x_1 = 7, x_2 = 15, x = 10$ and

$y_0 = 168, y_1 = 192, y_2 = 336, y = ?$

The Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

On substitution, we have

$$y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)} \times 168 + \frac{(10-1)(10-15)}{(7-1)(7-15)} \times 192 + \frac{(10-1)(10-7)}{(15-1)(15-7)} \times 336$$

$$= \frac{-15}{84} \times 168 + \frac{-45}{-48} \times 192 + \frac{27}{112} \times 336$$

$$= -0.1786 \times 168 + 0.9375 \times 192 + 0.24 \times 336$$

$$= -30.005 + 180 + 81.01 = 231.005 \text{ approx.}$$

Example 2 : Using Lagrange formula, calculate $f(3)$ from the following table.

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

[JNTU May 03]

Solution : Given $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$

and $f(x_0) = 1, f(x_1) = 14, f(x_2) = 15, f(x_3) = 5, f(x_4) = 6, f(x_5) = 19$

From Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} f(x_3) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} f(x_4) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(x_5)$$

Here $x = 3$.

$$\therefore\ f(3) = \frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times 1 + \frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} \times 14 +$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times 15 + \frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} \times 5 +$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times 6 + \frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} \times 19$$

$$= \frac{12}{240} - \frac{18}{60} \times 14 + \frac{36}{48} \times 15 + \frac{36}{48} \times 5 - \frac{18}{60} \times 6 + \frac{12}{240} \times 19$$

$$= 0.05 - 4.2 + 11.25 + 3.75 - 1.8 + 0.95 = 10$$

$\therefore\ f(x_3) = 10$.

Example 3 : Using Lagrange's interpolation formula, find the value of $y(10)$ from the following table:

x	5	6	9	11
y	12	13	14	16

[JNTUAug. 2008S (Set No.2]

(or) Find $y(10)$, Given that $y(5) = 12$, $y(6) = 13$, $y(9) = 14$, $y(11) = 16$ using Lagrange's formula. **[JNTU(H) June 2010 (Set No.3)]**

Solution : Lagrange's interpolation formula is given by

$$f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2)$$

$$+ \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3) + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

Given $x_1 = 5, x_2 = 6, x_3 = 9, x_4 = 11$

Here $x = 10$, $f(x_1) = 12, f(x_2) = 13, f(x_3) = 14, f(x_4) = 16$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{4\times 1\times -1}{-1\times -4\times -6}\times 12+\frac{5\times 1\times -1}{1\times -3\times -5}\times 13+\frac{5\times 4\times -1}{4\times 3\times -2}\times 14+\frac{5\times 4\times 1}{6\times 5\times 2}\times 16$$

$$= 2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=14\frac{2}{3}=14.6666.$$

Example 4 : Given $u_0 = 580$, $u_1 = 556$, $u_2 = 520$ and $u_4 = 385$ find u_3.

Solution : Given data can be tabulated as follows.

x	0	1	2	4
u(x)	580	556	520	385

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_4 = 3$ and

$$f(x_0) = f(0) = u_0 = 580$$

$$f(x_1) = f(1) = u_1 = 556$$

$$f(x_2) = f(2) = u_2 = 520$$

$$f(x_4) = f(4) = u_4 = 385$$

By Lagrange's formula,

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_4)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_4)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_4)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)}f(x_4)$$

$$f(3)=\frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)}(580)+\frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)}(556)$$

$$+\frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)}(520)+\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)}(385)$$

$$=\frac{2\times 1\times -1}{-1\times -2\times -4}(580)+\frac{3\times 1\times -1}{1\times -1\times -3}(556)+\frac{3\times 2\times -1}{2\times 1\times -2}(520)+\frac{3\times 2\times 1}{4\times 3\times 2}(385)$$

$$= 145-556+780+96.25 = 465.25\,.$$

Example 5 : The values of a function $f(x)$ are given below for certain values of x

x	0	1	3	4
$f(x)$	5	6	50	105

Find the values of $f(2)$ using Lagrange's interpolation formula.

Solution : By Lagrange's interpolation formula,

$$f(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)+\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$
$$+\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)+\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$\therefore\ f(2) = \frac{(2-1)\,(2-3)\,(2-4)}{(0-1)\,(0-3)\,(0-4)}(5) + \frac{(2-0)\,(2-3)\,(2-4)}{(1-0)\,(1-3)\,(1-4)}(6) + \frac{(2-0)\,(2-1)\,(2-4)}{(3-0)\,(3-1)\,(3-4)}(50) + \frac{(2-0)\,(2-1)\,(2-3)}{(4-0)\,(4-1)\,(4-3)}(105)$$

$$= \frac{1\times -1\times -2}{-1\times -3\times -4}(5) + \frac{2\times -1\times -2}{1\times -2\times -3}(6) + \frac{2\times 1\times -2}{3\times 2\times -1}(50) + \frac{2\times 1\times -1}{4\times 3\times 1}(105)$$

$$= \frac{-5}{6} + 4 + \frac{100}{3} - \frac{35}{2} = \frac{-5+24+200-105}{6} = \frac{114}{6} = 19.$$

Example 6 : Given the values :

x	0	2	3	6
$f(x)$	–4	2	14	158

Using Lagrange's formula for interpolation find the value of $f(4)$.

Solution : Using Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_2)\,(x-x_3)\,(x-x_4)}{(x_1-x_2)\,(x_1-x_3)\,(x_1-x_4)}f(x_1) + \frac{(x-x_1)\,(x-x_3)\,(x-x_4)}{(x_2-x_1)\,(x_2-x_3)\,(x_2-x_4)}f(x_2) + \frac{(x-x_1)\,(x-x_2)\,(x-x_4)}{(x_3-x_1)\,(x_3-x_2)\,(x_3-x_4)}f(x_3) + \frac{(x-x_1)\,(x-x_2)\,(x-x_3)}{(x_4-x_1)\,(x_4-x_2)\,(x_4-x_3)}f(x_4)$$

Here $x = 4,\ x_1 = 0, x_2 = 2, x_3 = 3, x_4 = 6$

and $f(x_1) = -4, f(x_2) = 2, f(x_3) = 14, f(x_4) = 158$

$$\therefore\ f(4) = \frac{(4-2)\,(4-3)\,(4-6)}{(0-2)\,(0-3)\,(0-6)}(-4) + \frac{(4-0)\,(4-3)\,(4-6)}{(2-0)\,(2-3)\,(2-6)}(2) + \frac{(4-0)\,(4-2)\,(4-6)}{(3-0)\,(3-2)\,(3-6)}(14) + \frac{(4-0)\,(4-2)\,(4-3)}{(6-0)\,(6-2)\,(6-3)}(158)$$

$$= \frac{2\times 1\times(-2)}{-2\times -3\times -6}(-4) + \frac{4\times 1\times(-2)}{2\times -1\times -4}(2) + \frac{4\times 2\times -2}{3\times 1\times -3}(14) + \frac{4\times 2\times 1}{6\times 4\times 3}(158)$$

$$= \frac{-4}{9} - 2 + \frac{224}{9} + \frac{158}{9} = \frac{-4-18+224+158}{9} = 40.$$

Example 7 : State Lagrange's formula of interpolation, using unequal intervals. From an experiment, we get the following values of a function $f(x)$:

x	1	2	–4
$f(x)$	3	–5	4

Represent the function $f(x)$ approximately by a polynomial of degree 2.

Solution : Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_2)\,(x-x_3)}{(x_1-x_2)\,(x_1-x_3)}f(x_1) + \frac{(x-x_1)\,(x-x_3)}{(x_2-x_1)\,(x_2-x_3)}f(x_2) + \frac{(x-x_1)\,(x-x_2)}{(x_3-x_1)\,(x_3-x_2)}f(x_3)$$

Here $x_1 = 1, x_2 = 2, x_3 = -4;\quad f(x_1) = 3, f(x_2) = -5, f(x_3) = 4$

$$f(x) = 3 \times \frac{(x-2)\,(x+4)}{(1-2)\,(1+4)} + (-5)\frac{(x-1)\,(x+4)}{(2-1)\,(2+4)} + 4 \times \frac{(x-1)\,(x-2)}{(-4-1)\,(-4-2)}$$

$$= \frac{-3}{5}(x^2 + 2x - 8) - \frac{5}{6}(x^2 + 3x - 4) + \frac{4}{30}(x^2 - 3x + 2)$$

$$= \left(\frac{-3}{5} - \frac{5}{6} + \frac{4}{30}\right)x^2 + \left(\frac{-6}{5} - \frac{15}{6} - \frac{4}{10}\right)x + \left(\frac{24}{5} + \frac{10}{3} + \frac{4}{15}\right)$$

$$\therefore\ f(x) = \frac{-13}{10}x^2 - \frac{41}{10}x + \frac{42}{5} = \frac{-1}{10}(13x^2 + 41x - 84).$$

Example 8 : Find the interpolation polynomial for the following :

x	0	1	2	5
$f(x)$	2	3	12	147

Solution : By Lagrange's interpolation formula,

$$f(x) = \frac{(x-1)\,(x-2)\,(x-5)}{(0-1)\,(0-2)\,(0-5)}(2) + \frac{(x-0)\,(x-2)\,(x-5)}{(1-0)\,(1-2)\,(1-5)}(3)$$

$$+ \frac{(x-0)\,(x-1)\,(x-5)}{(2-0)\,(2-1)\,(2-5)}(12) + \frac{(x-0)\,(x-1)\,(x-2)}{(5-0)\,(5-1)\,(5-2)}(147)$$

$$= \frac{-1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2\,(x^3 - 6x^2 + 5x)$$

$$+ \frac{49}{20}(x^3 - 3x^2 + 2x)$$

$$= \frac{1}{20}(-4x^3 + 15x^3 - 40x^3 + 49x^3) + \frac{1}{20}(32x^2 - 105x^2 + 240x^2 - 147x^2)$$

$$+ \frac{1}{20}\ (-68x + 150x - 200x + 98x) + 2$$

$$= x^3 + x^2 - x + 2$$

Example 9 : Given $x = 1, 2, 3, 4$ and $f(x) = 1, 2, 9, 28$ respectively find $f(3.5)$ using Lagrange method of 2nd and 3rd order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

[JNTU (A) May 2013]

Solution : By Lagrange's interpolation formula,

$$f(x) = \sum_{k=0}^{n} f(x_k)\frac{(x-x_0)\ldots(x-x_{k-1})\,(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)\ldots(x_k-x_{k-1})\ldots(x_k-x_n)}$$

For four points (i.e., $n = 4$)

$$f(x) = \frac{(x-x_1)\,(x-x_2)\,(x-x_3)}{(x_0-x_1)\,(x_0-x_2)\,(x_0-x_3)}f(x_0) + \frac{(x-x_0)\,(x-x_2)\,(x-x_3)}{(x_1-x_0)\,(x_1-x_2)\,(x_1-x_3)}$$

$$f(x_1)+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$\therefore\ f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)$$

$$+\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)$$

$$=0.0625+(-0.625)+8.4375+8.75=16.625$$

$$\text{Now } f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$

$$+\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$=\frac{(x^2-5x+6)(x-4)}{-6}+(x^2-4x+3)(x-4)+\frac{(x^2-3x+2)(x-4)}{-2}(9)$$

$$+\frac{(x^2-3x+2)(x-3)}{6}(28)$$

$$=\frac{x^3-9x^2+26x-24}{-6}+x^3-8x^2+19x-12+\frac{x^3-7x^2+14x-8}{-2}(9)$$

$$+\frac{x^3-6x^2+11x-6}{6}(28)$$

$$=\Big[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x$$

$$+216+308x+28x^3-168x^2-168\Big]\Big/6=\frac{6x^3-18x^2+18x}{6}$$

i.e. $f(x)=x^3-3x^2+3x$

$\therefore\ f(3.5)=(3.5)^3-3(3.5)^2+3\,(3.5)=16.625$.

Example 10 : Find the unique polynomial P(x) of degree 2 or less such that P(1) = 1, P(3) = 27, P(4) = 64 using Lagrange interpolation formula.

[JNTU 2004, (A) Nov. 2010 (Set No. 2), May 2012 (Set No. 3)]

Solution : Given

x	1	3	4
P(x)	1	27	64

Here $x_0=1, x_1=3, x_2=4$; $f(x_0)=1, f(x_1)=27, f(x_3)=64$

By Lagrange's interpolation formula for three points,

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$=\frac{(x-3)(x-4)}{(1-3)(1-4)}\times 1+\frac{(x-1)(x-4)}{(3-1)(3-4)}\times 27+\frac{(x-1)(x-3)}{(4-1)(4-3)}\times 64$$

$$= \frac{1}{6}[48x^2 - 114x + 72] = 8x^2 - 19x + 12$$

$\therefore$ The polynomial $P(x) = 8x^2 - 19x + 12$.

Example 11 : The values of x and $\log_{10} x$ *are* (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871), find the $\log_{10} 301$.

Solution : By Lagrange's interpolation formula,

$$f(x) = \frac{(x - x_1)(x - x_2)...(x - x_n)}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)...(x - x_n)}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)...(x - x_n)}{(x_2 - x_0)(x_2 - x_1)...(x_2 - x_n)} f(x_2) + ...$$

$$+ \frac{(x - x_0)(x - x_1)...(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)...(x_n - x_{n-1})} f(x_n)$$

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} \times 2.4771 + \frac{1(-4)(-6)}{(4)(-1)(-3)} \times 2.4829$$

$$+ \frac{(1)(-3)(-6)}{(5)(1)(-2)} \times 2.4843 + \frac{(1)(-3)(-4)}{(7)(3)(2)} \times 2.4871$$

$$= 1.2739 + 4.9658 - 4.4717 + 0.7106 = 2.4786.$$

Example 12 : The function $y = \sin x$ is tabulated below

x	0	$\pi/4$	$\pi/2$
$y = \sin x$	0	0.70711	1.0

Using Lagrange's interpolation formula, find the value of $\sin(\pi/6)$.

Solution : We have

$$\sin\frac{\pi}{6} \approx \frac{(\pi/6 - 0)(\pi/6 - \pi/2)}{(\pi/4 - 0)(\pi/4 - \pi/2)}(0.70711) + \frac{(\pi/6 - 0)(\pi/6 - \pi/4)}{(\pi/2 - 0)(\pi/2 - \pi/4)}(1)$$

$$= \frac{8}{9}(0.70711) - \frac{1}{9} = \frac{4.65688}{9} = 0.51743.$$

Example 13 : Using Lagrange's interpolation formula, find the form of the function $y(x)$ from the following table :

x	0	1	3	4
y	–12	0	12	24

Solution : From the table, we observe $x = 1, y = 0$. Thus $x - 1$ is a factor.

Let $y(x) = (x - 1)R(x) \Rightarrow R(x) = \dfrac{y}{x - 1}$

Tabulating the values of x and $R(x)$, we get

x	0	3	4
$R(x)$	12	6	8

Using the Lagrange's interpolation formula,

$$R(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$= \frac{(x-3)(x-4)}{(-3)(-4)}(12) + \frac{(x-0)(x-4)}{(3-0)(3-4)}(6) + \frac{(x-0)(x-3)}{(4-0)(4-3)}(8)$$

$$= (x-3)(x-4) - 2x(x-4) + 2x(x-3) = x^2 - 5x + 12$$

Hence the required polynomial approximation to $y(x)$ is given by

$y(x) = (x-1)(x^2 - 5x + 12)$.

Example 14 : Find the interpolating polynomial $f(x)$ from the table.

x	0	1	4	5
$f(x)$	4	3	24	39

[JNTU 2008, (H) June 2009, (K) Nov.2009S (Set No.4)]

Solution : Given

$x_0 = 0,\ x_1 = 1,\ x_2 = 4,\ x_3 = 5$ and

$f(x_0) = 4,\ f(x_1) = 3,\ f(x_2) = 24,\ f(x_3) = 39$

Using Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$\therefore\ f(x) = \frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}(4) + \frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}(3)$$

$$+ \frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}(24) + \frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}(39)$$

$$= \frac{(x-1)[x^2-9x+20]}{-20}(4) + \frac{x[x^2-9x+20]}{12}(3)$$

$$+ \frac{x[x^2-6x+5]}{-12}(24) + \frac{x[x^2-5x+4]}{20}(39)$$

$$= \frac{x^3 - 9x^2 + 20x - x^2 + 9x - 20}{-5} + \frac{[x^3 - 9x^2 + 20x]}{4} - (2x^3 - 12x^2 + 10x)$$

$$+ \left(\frac{39x^3 - 195x^2 + 156x}{20} \right)$$

On simplification, $f(x) = 2x^2 - 3x + 4$

Example 15 : Using Lagrange's interpolation formula, find y (10) from the following table :

x	5	6	9	11
y	12	13	14	16

[JNTU 2008 (Set No.2)]

Solution : Lagrange's interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_2$$

$$\therefore f(10) = \frac{4(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{5(4)(-1)}{4(3)(-2)}(14) + \frac{5(4)(1)}{6(5)(2)}(16)$$

$$= 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3} = \frac{6 - 13 + 35 + 16}{3} = 14.666$$

or $y(10) = 14.67$

Example 16 : Find the parabola passing through points (0, 1) (1, 3) and (3,55) using lagrange's interpolation formula. **[JNTU 2008 (Set No.3)]**

Solution : Given points are (0, 1) (1, 3) (3, 55).

Lagrange's Interpolation formula is

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3) + \frac{(x-0)(x-1)}{(3-0)(3-1)}(55)$$

$$= \frac{x^2 - 4x + 3}{3} + \frac{x^2 - 3x}{-2}(3) + \frac{x^2 - x}{6}(55)$$

$$= \frac{2x^2 - 8x + 6 - 9x^2 + 27x + 55x^2 - 55x}{6}$$

$$= \frac{1}{6}\left[48x^2 - 36x + 6\right]$$

or $f(x) = 8x^2 - 6x + 1$

Example 17 : The following are the measurements T made on a curve recorded by the oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 1.6$. **[JNTU (H) June 2009 (Set No.1), May 2012]**

Solution : By Lagrange's interpolation formula,

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

We will use T and I in the above formula

$$\therefore\ f(1.6) = \frac{(1.6-2)(1.6-2.5)(1.6-3)}{(1.2-2)(1.2-2.5)(1.2-3)} f(1.2) + \frac{(1.6-1.2)(1.6-2.5)(1.6-3)}{(2-1.2)(2-2.5)(2-3)} f(2)$$

$$+ \frac{(1.6-1.2)(1.6-2)(1.6-3)}{(2.5-1.2)(2.5-2)(2.5-3)} f(2.5) + \frac{(1.6-1.2)(1.6-2)(1.6-2.5)}{(3-1.2)(3-2)(3-2.5)} f(3)$$

$$= \frac{(-0.4)(-0.9)(-1.4)}{(-0.8)(-1.3)(-1.8)}(1.36) + \frac{(0.4)(-0.9)(-1.4)}{(0.8)(-0.5)(-1)}(0.58) + \frac{(0.4)(-0.4)(-1.4)}{(1.3)(0.5)(-0.5)}(0.34)$$

$$+ \frac{(0.4)(-0.4)(-0.9)}{(1.8)(1)(0.5)}(0.20)$$

$$= \frac{-(0.6854)}{-(1.872)} + \frac{0.2923}{0.4} + \frac{0.0761}{-(0.325)} + \frac{0.0288}{0.9}$$

$$= 0.3661 + 0.7307 - 0.2341 + 0.032$$

$$= 0.8947$$

$\therefore\ I = 0.8947$

Example 18 : A curve passes through the points $(0,18),(1,10),(3,-18)$ and $(6,90)$. Find the slope of the curve at $x = 2$. **[JNTU(H) June 2009 (Set No.1)]**

Solution : We are given

x	0	1	3	6
y	18	10	–18	90

Since the arguments are not equally spaced, we will use Lagrange's formula.

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)} \cdot (18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)} \cdot (10)$$

$$+ \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)} \cdot (-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)} \cdot (90)$$

i.e., $f(x) = (x^2-4x+3)(x-6)(-1) + x(x^2-9x+18) + x(x^2-7x+6) + x(x^2-4x+3)$

$$= (-x^3+10x^2-27x+18) + (x^3-9x^2+18x) + (x^3-7x^2+6x) + (x^3-4x^2+3x)$$

$$= 2x^3 - 10x^2 + 18$$

$\therefore \; f'(x) = 6x^2 - 20x$

Thus the slope of the curve at $x = 2$ is given by $f'(2) = 6(4) - 20(2) = -16$

Example 19 : Using Lagrange's formula fit a polynomial to the data

X :	–1	0	2	3
Y :	–8	3	1	12

and hence find $y(1)$. **[JNTU (H) June 2009 (Set No.3)]**

Solution : Take $x_0 = -1, x_1 = 0, x_2 = 2, x_3 = 12$

$y(0) = -8, y(1) = 3, y(2) = 1, y(3) = 12$

Using Lagranges interpolation formula, we have

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y(x_3)$$

$$= \frac{(x-0)(x-2)(x-3)}{(-1-0)(-1-2)(-1-3)}(-8) + \frac{(x+1)(x-2)(x-3)}{(0+1)(0-2)(0-3)}(3)$$

$$+ \frac{(x+1)(x-0)(x-3)}{(2+1)(2-0)(2-3)}(1) + \frac{(x+1)(x-0)(x-2)}{(3+1)(3-0)(3-2)}(12)$$

$$= \frac{x(x^2-5x+6)}{-12}(-8) + \frac{(x+1)(x^2-5x+6)}{6}(3) + \frac{x(x^2-2x-3)}{-6}(1) + \frac{x(x^2-x-2)}{12}(12)$$

$$= \frac{2(x^3-5x^2+6x)}{3} + \frac{x^3-5x^2+6x+x^2-5x+6}{2} + \frac{x^3-2x^2-3x}{-6} + \frac{x^3-x^2-2x}{1}$$

$$= \frac{4x^3 - 20x^2 + 24x + 3x^3 - 12x^2 + 3x + 18 - x^3 + 2x^2 + 3x + 6x^3 - 6x^2 - 12x}{6}$$

$$= \frac{12x^3 - 36x^2 + 18x + 18}{6} = 2x^3 - 6x^2 + 3x + 3$$

$\therefore\ y(x) = 2x^3 - 6x^2 + 3x + 3$ is the required polynomial.

Put $x = 1$. We get $y(1) = 2$.

Example 20 : Given $u_1 = 22, u_2 = 30, u_4 = 82, u_7 = 106, u_8 = 206$, find u_6. Use Lagrange's interpolation formula. **[JNTU (K), (A)June 2009 (Set No.2)]**

Solution : Given data can be tabulated as follows:

x	1	2	4	7	8
u(x)	22	30	82	106	206

According to Lagrange's interpolation formula

$$f(x) = \frac{(x-x_2)(x-x_4)(x-x_7)(x-x_8)}{(x_1-x_2)(x_1-x_4)(x_1-x_7)(x_1-x_8)} f(x_1) +$$

$$\frac{(x-x_1)(x-x_4)(x-x_7)(x-x_8)}{(x_2-x_1)(x_2-x_4)(x_2-x_7)(x_2-x_8)} f(x_2) +$$

$$\frac{(x-x_1)(x-x_2)(x-x_7)(x-x_8)}{(x_4-x_1)(x_4-x_2)(x_4-x_7)(x_4-x_8)} f(x_4) +$$

$$\frac{(x-x_1)(x-x_2)(x-x_4)(x-x_8)}{(x_7-x_1)(x_7-x_2)(x_7-x_4)(x_7-x_8)} f(x_7) +$$

$$\frac{(x-x_1)(x-x_2)(x-x_7)(x-x_8)}{(x_8-x_1)(x_8-x_2)(x_8-x_4)(x_8-x_7)} f(x_8)$$

Putting $x = x_6$, we obtain

$$f(x_6) = u_6 = \frac{(x-2)(x-4)(x-7)(x-8)}{(1-2)(1-4)(1-7)(1-8)}(22)$$

$$+ \frac{(x-1)(x-4)(x-7)(x-8)}{(2-1)(2-4)(2-7)(2-8)}(30) + \frac{(x-1)(x-2)(x-7)(x-8)}{(4-1)(4-2)(4-7)(4-8)}(82)$$

$$+ \frac{(x-1)(x-2)(x-4)(x-8)}{(7-1)(7-2)(7-4)(7-8)}(106) + \frac{(x-1)(x-2)(x-4)(x-7)}{(8-1)(8-2)(8-4)(8-7)}(206)$$

$$f(6) = \frac{(6-2)(6-4)(6-7)(6-8)}{(3)(-6)(-7)}(22) + \frac{(6-1)(6-4)(6-7)(6-8)}{(1)(-2)(-5)(-7)}(30)$$

$$+\frac{(6-1)(6-2)(6-7)(6-8)}{(+3)(+2)(-3)(-4)}(82) \quad +\frac{(6-1)(6-2)(6-4)(6-8)}{(6)(5)(3)(-1)}(106)$$

$$+\frac{(6-1)(6-2)(6-4)(6-7)}{(7)(6)(4)(1)}(206)$$

$$=\frac{(4)(2)(2)}{21\times 6}\times(22)+\frac{10\times 2}{-60}\times(30)+\frac{20\times 2}{72}\times 82$$

$$+\frac{(5)(-16)}{-90}\times(106)+\frac{20\times(-2)}{7\times 24}\times(206)$$

$$=\frac{352}{126}-10+\frac{3280}{72}+\frac{848}{9}-\frac{8240}{168}$$

$$=2.7936-10+45.5+94.2-49.0476$$

$$=142.4936-59.0476=83.446.$$

Example 21 : Using Lagrange's formula, fit a polynomial to the data

X :	0	1	3	4
Y :	–12	0	6	12

Also find y at $x=2$. **[JNTU (K)June 2009(Set No.2)]**

Solution : Take $x_1=0$, $x_2=1$, $x_3=3$, $x_4=4$ and

$f(x_1)=-12$, $f(x_3)=6$, $f(x_2)=0$, $f(x_4)=12$

$$f(x)=\frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}f(x_1)+\frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}f(x_2)$$

$$+\frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}f(x_3)+\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}f(x_4)$$

$$=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12)+\frac{(x-0)(x-3)(x-4)}{(0+12)(0-6)(0-12)}(0)$$

$$+\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$$=(x-1)(x-3)(x-4)+\frac{x(x-1)(x-4)}{-1}+x(x-1)(x-3)$$

$$=(x-1)[(x-3)(x-4)-x(x-4)+x(x-3)]$$

$$=(x-1)[\cancel{x^2}-3x-4x+12-\cancel{x^2}+4x+x^2-3x]$$

$$f(x)=x^3-7x^2+18x-12$$

From this we get, $f(2)=8-28+36-12=4$.

Example 22 : Using Lagrange's's formula find y (6) given:

x	3	5	7	9	11
y	6	24	58	108	74

[JNTU (H) June 2010 (Set No. 1)]

Solution : $x_0 = 3, x_1 = 5, x_2 = 7, x_3 = 9, x_4 = 11$ and

$y_0 = 6, y_1 = 24, y_2 = 58, y_3 = 108, y_4 = 74$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} y_4$$

Here, $x = 6$

$$\therefore\ f(6) = \frac{(6-5)(6-7)(6-9)(6-11)}{(3-5)(3-7)(3-9)(3-11)}(6) + \frac{(6-3)(6-7)(6-9)(6-4)}{(5-3)(5-7)(5-9)(5-11)}(24)$$

$$+\frac{(6-3)(6-5)(6-9)(6-11)}{(7-3)(7-5)(7-9)(7-11)}(58) + \frac{(6-3)(6-5)(6-7)(6-11)}{(9-3)(9-5)(9-7)(9-11)}(108)$$

$$+\frac{(6-3)(6-5)(6-7)(6-9)}{(9-3)(9-5)(9-7)(9-11)}(74)$$

$$= \frac{(1)(-1)(-3)(-5)}{(-2)(-4)(-6)(-8)}(6) + \frac{(3)(-1)(-3)(-5)}{(2)(-2)(-4)(-6)}(24) + \frac{(3)(1)(-3)(-5)}{(4)(2)(-2)(-4)}.(58)$$

$$+\frac{(3)(1)(-1)(-5)}{(6)(4)(2)(-2)}(108) + \frac{(3)(1)(-1)(-3)}{(6)(4)(2)(-2)}(74)$$

$$= \frac{-15}{-64} + \frac{-45}{-4} + \frac{45}{64} \times (58) + \frac{15}{-96} \times (108) + \frac{9}{96}(74)$$

$$= .2343 + 11.25 + 40.7812 - 16.875 + 6.9375$$

$$= 43.328$$

Example 23 : Find $y(5)$ given that $y(0)=1$, $y(1)=3$, $y(3)=13$, and $y(8)=123$ using Lagrange's formula. **[JNTU (H) June 2010 (Set No. 4)]**

Solution : Given

x	0	1	3	8
y	1	3	13	128

Using Lagrange's formula,

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Take $x_0=0, x_1=1, x_2=3, x_3=8$ and

$y_0=1, y_1=3, y_2=13, y_3=128$

$$y(5)=\frac{(5-1)(5-3)(5-8)}{(0-1)(0-3)(0-8)}(1)+\frac{(5-0)(5-3)(5-8)}{(1-0)(1-3)(1-8)}(3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(4)(2)(-3)}{(-1)(-3)(-8)}(1)+\frac{(5)(-2)(-3)}{(1)(-2)(-7)}(3)+\frac{(5)(4)(-3)}{(3)(2)(-5)}(13)+\frac{(5)(4)(2)}{(8)(7)(5)}(128)$$

$$=1+\frac{45}{7}+26+\frac{128}{7}=1+6.4285+26+18.2857=51.7142$$

$\therefore \quad y(5)=51.7142$

Example 24 : Given that $y(3)=6, y(5)=24, y(7)=58, y(9)=108, y(11)=174$ find x when $y=100$, Using Lagranges formula. **[JNTU (H) Jan. 2012 (Set No. 2)]**

Solution : Here we will view x as a function of y.

y	6	24	58	108	174
x	3	5	7	9	11

By Lagrange's formula,

$$x=f(y)=\frac{(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_1-y_5)}f(y_1)$$

$$+\frac{(y-y_1)(y-y_3)(y-y_4)(y-y_5)}{(y_2-y_1)(y_2-y_3)(y_2-y_4)(y_2-y_5)}f(y_2)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_4)(y-y_5)}{(y_3-y_1)(y_3-y_2)(y_3-y_4)(y_3-y_5)}f(y_3)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_5)}{(y_4-y_1)(y_4-y_2)(y_4-y_3)(y_4-y_5)}f(y_4)$$

$$+\frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_5-y_1)(y_5-y_2)(y_5-y_3)(y_5-y_4)}$$

Taking $y=100$ and substituting the values, we get

$$x=\frac{(100-24)(100-58)(100-108)(100-174)}{(6-24)(6-58)(6-108)(6-174)}(3)$$

$$+\frac{(100-6)(100-58)(100-108)(100-174)}{(24-6)(24-58)(24-108)(24-174)}(5)$$

$$+\frac{(100-6)(100-24)(100-108)(100-174)}{(58-6)(58-24)(58-108)(58-174)}(7)$$

$$+\frac{(100-6)(100-24)(100-58)(100-174)}{(108-6)(108-24)(108-58)(108-174)}(9)$$

$$+\frac{(100-6)(100-24)(100-58)(100-108)}{(174-6)(174-24)(174-58)(174-108)}(11)$$

$$=\frac{(76)(42)(-8)(-74)}{(-18)(-52)(-102)(-168)}(3)+\frac{(94)(42)(-8)(-74)}{(18)(-34)(-84)(-150)}(5)+\frac{(94)(76)(-8)(-74)}{(52)(34)(-50)(-116)}(7)$$

$$+\frac{(94)(76)(42)(-74)}{(102)(84)(50)(-66)}(9)+\frac{(94)(76)(42)(-8)}{(168)(150)(116)(66)}(11)$$

$$=\frac{1889664}{16039296}\times 3-\frac{2337216}{7711200}\times 5+\frac{4229248}{10254400}\times 7+\frac{22203552}{28204400}\times 9+\frac{2400384}{192931200}\times 11$$

$$=0.3534-1.5154+2.8870+7.0675-0.1368$$
$$=10.3079-1.6522$$
$$=8.6557$$

Example 25 : Use Lagrange's interpolation formula to express the function

(*a*) $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ (*b*) $\dfrac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sums of partial fractions.

[JNTU (A) Jan. 2012 (Set No. 2)]

Sol. Given function is $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$

Denominator $=x^3-2x^2-x+2$

$=x^2(x-2)-1(x-2)$

$$= (x^2 - 1)(x - 2)$$

$$= (x+1)(x-1)(x-2)$$

Take $f(x) = x^2 + x - 3$

$f(-1) = 1 - 1 - 3 = -3$; $f(1) = 1 + 1 - 3 = -1$; $f(2) = 4 + 2 - 3 = 3$

We write the table as follows :

x	-1	1	2
$f(x)$	-3	-1	3

We will use the Lagrange's interpolation formula,

$$x^2 + x - 3 = L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_2)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$= \frac{(x-1)(x-2)}{(-1-1)(-1-2)}(-3) + \frac{(x+1)(x-2)}{(1+1)(1-1)}(-1) + \frac{(x+1)(x-1)}{(2+1)(2-1)}(3)$$

$$= \frac{(x-1)(x-2)}{-2} + \frac{(x+1)(x-2)}{2} + \frac{(x+1)(x-1)}{1}$$

$$\frac{x^2 + x - 3}{x^3 - 2x^2 - x + 2} = \frac{(x-1)(x-2)}{-2(x+1)(x-1)(x-2)} + \frac{(x+1)(x-2)}{2(x+1)(x-1)(x-2)} + \frac{(x+1)(x-1)}{(x+1)(x-1)(x-2)}$$

$$= \frac{-1}{2(x+1)} + \frac{1}{2(x-1)} + \frac{1}{(x-2)}$$

which is the required partil fractions form.

EXERCISE 5.1

1. (*i*) Using Newton's Forward formula, find the value of $f(1.6)$, if

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

(*ii*) Find $f(2.5)$ using the following table.

x	1	2	3	4
$f(x)$	1	8	27	64

[JNTU (A) June 2013 (Set No. 4)]

2. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$ find $f(1.28)$.

3. Construct Newton's Forward interpolation polynomial for the following data

x	4	6	8	10
y	1	3	8	16

Hence evaluate for $x = 5$.

4. Using Lagrange's interpolation formula find the value of y when $x = 10$, if the following values of x and y are given

x :	5	6	9	11
y :	12	13	14	16

5. Given $\log_{10} 654 = 2.8156, \log_{10} 658 = 2.8182, \log_{10} 659 = 2.8189,$ $\log_{10} 661 = 2.8202$ find by using Lagrange formula, the value of $\log_{10} 656$.

6. Using Lagrange's formula find the form of $f(x)$ given

x :	0	2	3	6
$f(x)$:	648	704	729	792

7. The population of certain town is shown in the following table

Year :	1921	1931	1941	1951	1961
Population in thouusands:	19.96	39.65	58.81	77.21	94.61

Estimate the population in the years 1936 and 1963. Also find the rate of growth of population in 1951 ?

8. Find the value of cos 1.747 using the values given in the table below :

x :	1.70	1.74	1.78	1.82	1.86
$\sin x$:	0.9916	0.9857	0.9781	0.9691	0.9584

9. Find $y(142)$ from the following data using Newton's Forward interpolation formula:

x :	140	150	160	170	180
$y(x)$	3.685	4.854	6.302	8.076	10.225

10. Using Lagrange's interpolation formula, find the interpolating polynomial that approximate the following function

x :	–4	–1	0	2	5
$f(x)$	1245	33	5	9	1335

11. Given $f(2) = 10, f(1) = 8, f(0) = 5, f(-1) = 10$ estimate $f(1/2)$ by using Gauss's forward formula.

12. Using Gauss's Forward interpolation formula estimate $f(32)$, given $f(25) = 0.2707, f(30) = 0.3027, f(35) = 0.3386, f(40) = 0.3794$.

13. Find the Lagrange interpolation polynomial for the function given that

x	0	–1	1
$f(x)$	1	2	3

14. Find the second difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing $h = 2$. **[JNTU 2008(Set No.2)]**

15. If the interval of differencing is unity, prove that $\Delta \frac{2^x}{x!} = \frac{2^x(1-x)}{(x+1)!}$.

[JNTU 2008 (Set No.4)]

16. Using Lagrange's formula, fit a polynomial to the data

x	0	1	3	4
y	–12	0	6	12

Also find y at $x = 2$. **[JNTU(K) Nov.2009S(SetNo.4)]**

ANSWERS

1. 554 **2.** 1.1312 **3.** 1.625 **4.** 19.4

5. 2.8168 **6.** $648 + 30x - x^2$ **7.** 49.3, 97.68, 1.8 thousand / yr.

8. – 0.175 **9.** 3.899 **10.** $3x^4 - 5x^3 + 6x^2 - 14x + 5$

11. 6 **12.** 0.317 **13.** $1 + \frac{1}{2}x + \frac{3}{2}x^2$

14. $48x^2 - 96x - 16$ **16.** $x^3 - 7x^2 + 18x - 12$

OBJECTIVE TYPE QUESTIONS

1. If $x^3 - x - 4 = 0$, by Bisection method first two approximations x_0 and x_1 are 1 and 2 then x_2 is

(A) 1.25 (B) 2.0 (C) 1.75 (D) 1.5

2. $\nabla y_5 =$

(A) $y_6 + 3y_5 + 3y_4 + y_3$ (B) $y_5 - 3y_4 - 3y_3 - y_2$

(C) $y_6 - 3y_5 + 3y_4 - y_3$ (D) $y_5 + 3y_4 + 3y_3 + y_2$

3. Gauss – Forward interpolation formula is used to interpolate the values of y for

(A) $0 < p < -\alpha$ (B) $-\alpha < p < 0$

(C) $-1 < p < 0$ (D) $0 < p < 1$

4. If first two approximations x_0 and x_1 are roots of $x^3 - x^3 + 1 = 0$ are 1 and 2 then x_2 by Regula Falsi method is

(A) 1.05 (B) 1.25 (C) 1.15 (D) 1.35

5. If first approximation root of $x^3 - 5x + 3 = 0$ is $x_0 = 0.64$ then x_1 by Newton-Raphson method is

(A) 0.825 (B) 0.6565 (C) 0.721 (D) 0.6724

5. $-0.06, 0.5$	**6.** 0.44	**7.** 3.32	**8.** -0.175	**9.** 1.17
10. 4.25 km /sec, 5.5 km/sec., 6.75 km/sec.			**11.** 5.33 cm/sec., -45.6 cm / sec^2	
12. 0.223	**13.** 135	**14.** 232.869	**15.** 2.25	
16. Min. at 0 is 2 and Max. at 1 is 0.25				

7.6 NUMERICAL INTEGRATION

We know that a definite integral of the form $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$, enclosed between the limits $x = a$ and $x = b$. This integration is possible only if $f(x)$ is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows :

Given a set of $(n+1)$ data points $(x_i, y_i), i = 0, 1, 2, ..., n$ of the function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to evaluate $\int_{x_0}^{x_n} f(x)\,dx$.

The problem of numerical integration, like that of numerical differentiation is solved by replacing $f(x)$ with an interpolating polynomial $P_n(x)$ and obtaining $\int_{x_0}^{x_n} P_n(x)\,dx$ which is approximately taken as the value of $\int_{x_0}^{x_n} f(x)\,dx$. Numerical Integration is also known as Numerical quadrature.

7.7 NEWTON-COTE'S QUADRATURE FORMULA (GENERAL QUADRATURE FORMULA)

This is the most popular and widely used numerical integration formula. It forms the basis for a number of numerical integration methods known as Newton-Cote's methods.

Derivation of Newton-Cotes formula.

Let the interval $[a, b]$ be divided into n equal subintervals such that

$a = x_0 < x_1 < x_2 < ..... < x_n = b$. Then $x_n = x_0 + nh$.

Newton's forward difference formula is

$$y(x) = y(x_0 + ph) = P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... \quad .... (1)$$

where $p = \frac{x - x_0}{h}$.

Now, instead of $f(x)$ we will replace it by this interpolating polynomial.

$\therefore \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_n} P_n(x)\,dx$, where $P_n(x)$ is an interpolating polynomial of degree n

$$= \int_{x_0}^{x_0+nh} P_n(x)\,dx = \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] dx$$

Since $x = x_0 + ph,\ dx = h.dp$ and hence the above integral becomes

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0 + \ldots\right] dp$$

$$= h\left[y_0(p) + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - 3.\frac{p^3}{3} + 2.\frac{p^2}{2}\right)\Delta^3 y_0 + \ldots\ldots \right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right) + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \ldots \right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 \right.$$

$$\left. + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \cdots \right] \qquad \ldots(2)$$

This is called **Newton-Cote's** quadrature formula. From this general formula, we can get different integration formulae by putting $n = 1, 2, 3, \ldots\ldots.$

7.8 TRAPEZOIDAL RULE [JNTU 2007S, 2008S, (H) Dec. 2011S (Set No. 1)]

Here the function $f(x)$ is approximated by a first - order polynomial $P_1(x)$ which passes through two points.

Putting $n = 1$ in the above general formula, all differences higher than the first will become zero (since other differences do not exist if $n = 1$) and we get

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right] = h\left[y_1 + \frac{1}{2}(y_2 - y_1)\right] = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_2}^{x_3} f(x)\,dx = \int_{x_0+2h}^{x_0+3h} f(x)\,dx = \frac{h}{2}(y_2 + y_3)$$

....

Finally, $$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Hence $\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + ... + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$

$$= \frac{h}{2}(y_0+y_1)+\frac{h}{2}(y_1+y_2)+....+\frac{h}{2}(y_{n-1}+y_n)$$

$$= \frac{h}{2}[\,(y_0+y_n)+2\,(y_1+y_2+y_3+.....+y_{n-1})\,] \qquad ..... (3)$$

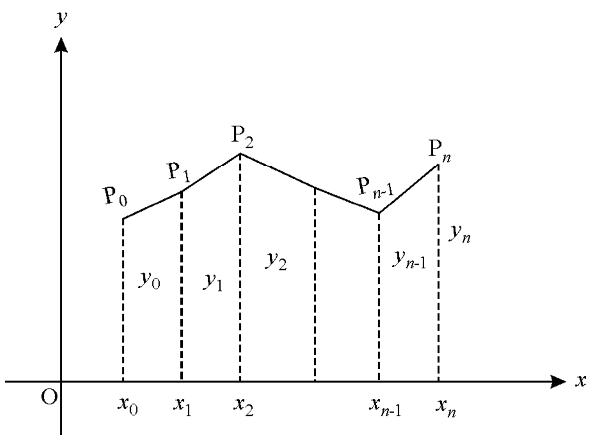
Thus

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[\text{(Sum of the first and last ordinates)} + 2\text{ (Sum of the remaining ordinates)}]$$

This is known as **Trapezoidal Rule.**

Geometrical Interpretation :

Consider the points $P_0(x_0,y_0), P_1(x_1,y_1), P_2(x_2,y_2),......., P_n(x_n,y_n)$. Suppose the curve $y = f(x)$ passing through the above points be approximated by the union of the line segments joining $(P_0,P_1),(P_1,P_2),(P_2,P_3),......,(P_{n-1},P_n)$.



Geometrically, the curve $y = f(x)$ is replaced by n straight line segments joining the points (x_0,y_0) and (x_1,y_1); (x_1,y_1) and (x_2,y_2);.....;(x_{n-1},y_{n-1}) and (x_n,y_n). The area bounded by the curve $y = f(x)$, x - axis and the ordinates $x = x_0$ and $x = x_n$ is then approximately equal to the sum of the areas of the n trapeziums as shown in the figure.

The total area is given by

$$\frac{h}{2}(y_0+y_1)+\frac{h}{2}(y_1+y_2)+\frac{h}{2}(y_2+y_3)+....+\frac{h}{2}(y_{n-1}+y_n)$$

$$= \frac{h}{2}[\,y_0+2\,(y_1+y_2+y_3+.....+y_{n-1})+y_n\,] = \int_{x_0}^{x_n} f(x)dx \text{ (approximately).}$$

Note. Though this method is very simple for calculation purposes of numerical integration, the error in this case is significant. The accuracy of the result can be improved by increasing the number of intervals or by decreasing the value of h.

7.9 SIMPSON'S 1/3 RULE [JNTU (H) Dec. 2011S (Set No. 2)]

This is another popular method. Here, the function $f(x)$ is approximated by a second order polynomial $P_2(x)$ which passes through three successive points.

Putting $n = 2$ in Newton-Cotes quadrature formula *i.e.* by replacing the curve $y = f(x)$ by $n/2$ parabolas, we have

$$\int_{x_0}^{x_2} f(x)\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + \frac{2\,(4-3)}{12}\Delta^2 y_0\right] = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - 2y_1 + y_0)\right] = 2h\left[\frac{1}{6}y_0 + \frac{2}{3}y_1 + \frac{1}{6}y_2\right]$$

$$= \frac{2h}{6}[y_0 + 4y_1 + y_2] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

....

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, we obtain

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \ldots\ldots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}[\,(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots. + (y_{n-2} + 4y_{n-1} + y_n)\,]$$

$$= \frac{h}{3}[(y_0 + y_n) + 4\,(y_1 + y_3 + y_5 + \ldots. + y_{n-1}) + 2\,(y_2 + y_4 + y_6 + \ldots. + y_{n-2})] \quad \ldots (4)$$

$$= \frac{h}{3}[\text{(Sum of the first and last ordinates)} + 4\,\text{(Sum of the odd ordinates)}$$
$$+ 2\,\text{(Sum of the remaining even ordinates)}\,]$$

with the convention that $y_0, y_2, y_4, \ldots., y_{2n}$ are even ordinates and $y_1, y_3, y_5, \ldots.., y_{2n-1}$ are odd ordinates.

This is known as **Simpson's 1/3 Rule** or simply **Simpson's Rule**. It should be noted that this rule requires the given interval must be divided into an even number of equal sub-intervals of width h.

7.10 SIMPSON'S 3/8 RULE

Simpson's 1/3 rule was derived using three points that fit a quadratic equation. We can extend this approach by incorporating four successive points so that the rule can be exact for a polynomial *f*(*x*) of degree 3. Putting *n* = 3 in Newton-Cote's quadrature formula, all differences higher than the third will become zero and we obtain

$$\int_{x_0}^{x_3} f(x)\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3\,(6-3)}{12}\Delta^2 y_0 + \frac{3\,(3-2)^2}{24}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly $\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$ and so on.

Adding all these integrals, from x_0 to x_n, where n is a multiple of 3, we get

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \ldots.. + \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \ldots. + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)]$$

$$= \frac{3h}{8}[(y_0 + y_n) + 3\,(y_1 + y_2 + y_4 + y_5 + \ldots. + y_{n-1}) + 2\,(y_3 + y_6 + y_9 + \ldots. + y_n)] \quad \ldots.\ (5)$$

Equation (5) is called **Simpson's 3/8 rule** which is applicable only when *n* is a multiple of 3. This rule is not so accurate as Simpson's 1/3 rule.

Note : While there is no restriction for the number of intervals in Trapezoidal rule, number of sub-intervals n in the case of Simpson's $\frac{1}{3}$ rule must be even, for Simpson's $\frac{3}{8}$ rule must be multiple of 3.

SOLVED EXAMPLES

Example 1 : Evaluate $\int_0^1 x^3\,dx$ with five sub-intervals by Trapezoidal rule.

Solution : Here $a = 0, b = 1, n = 5$ and $y = f(x) = x^3$; $\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

The values of x and y are tabulated below:

x	0	0.2	0.4	0.6	0.8	1
y	0 y_0	0.008 y_1	0.064 y_2	0.216 y_3	0.512 y_4	1 y_5

By Trapezoidal rule,

$$\int_0^1 x^3\,dx = \frac{h}{2}[(\text{sum of the first and last ordinates}) + 2\,(\text{sum of the remaining ordinates})]$$

$$= \frac{0.2}{2}[(0+1) + 2(0.008 + 0.064 + 0.216 + 0.512)] = 0.26$$

Example 2 : Evaluate $\int_0^{\pi} t \sin t \, dt$ using the Trapezoidal rule.

Solution : Divide the interval $(0, \pi)$ into six parts each of width $h = \pi/6$.

The values of $f(t) = t \sin t$ are given below.

t	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
$f(t) = y$	0 y_0	0.2618 y_1	0.9069 y_2	1.5708 y_3	1.8138 y_4	1.309 y_5	0 y_6

By Trapezoidal rule,

$$\int_0^{\pi} t \sin t \, dt = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{\pi}{12}[(0+0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.309)]$$

$$= \frac{\pi}{12}(11.7246) = 3.0695 \approx 3.07.$$

Example 3 : Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's rule. Hence obtain approximate value of $\log_e 2$. **[JNTU (A) Dec. 2013 (Set No. 1)]**

Solution : Divide the interval (1,2) into eight parts each of width $h = 0.125$.

The values of x and y are tabulated below:

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$(1/x) = y$	1 y_0	0.8888 y_1	0.8 y_2	0.7272 y_3	0.6666 y_4	0.6153 y_5	0.5714 y_6	0.5333 y_7	0.5 y_8

By Simpson's 1/3 rule,

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}[\text{(sum of the first and last ordinates)}$$

$$+ 4\text{ (sum of the odd ordinates)} + 2\text{ (sum of the remaining even ordinates)}]$$

$$= \frac{h}{3}[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0.125}{3}[(1 + 0.5) + 4(0.8888 + 0.7272 + 0.6153 + 0.5333) + 2(0.8 + 0.6666 + 0.5714)]$$

$$= \frac{0.125}{3}[1.5 + 11.0584 + 4.076] = \frac{0.125}{3}(16.6344) = 0.6931$$

By actual integration, $\int_1^2 \frac{dx}{x} = (\log x)_1^2 = \log 2 - \log 1 = \log 2$

Hence $\log 2 = 0.6931$, correct to four decimal places.

Example 4 : Evaluate $\int_0^2 e^{-x^2}\,dx$ using Simpson's rule taking $h = 0{\cdot}25$.

[JNTU 2006, 2007 (Set No.2)]

Solution : The values of $y = f(x) = e^{-x^2}$ are given below:

x	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
x^2	0.0625	0.25	0.5625	1.00	1.5625	2.25	3.0625	4.00
y	0.93941 y_0	0.7788 y_1	0.56978 y_2	0.36788 y_3	0.20961 y_4	0.1054 y_5	0.04677 y_6	0.0183 y_7

By Simpson's $\frac{1}{3}$rd rule, we have

$$\int_0^2 e^{-x^2}\,dx = \frac{h}{3}[(y_0 + y_7) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{0\cdot 25}{3}[(0\cdot 93941 + 0\cdot 0183) + 4(0\cdot 7788 + 0\cdot 36788 + 0\cdot 1054)$$

$$+2(0\cdot 56978 + 0\cdot 20961 + 0\cdot 04677)]$$

$$= \frac{0\cdot 25}{3}[(0\cdot 95771 + 5\cdot 00832 + 1\cdot 65232]$$

$$= \frac{0\cdot 25}{3}(7\cdot 61835) = 0.63486.$$

Example 5 : A rocket is launched from the ground. Its acceleration measured every 5 seconds is tabulated below. Find the velocity and the position of the rocket at t =40 seconds. Use trapezoidal rule as well as Simpson's rule.

t	0	5	10	15	20	25	30	35	40
$a(t)$	40.0	45.25	48.50	51.25	54.35	59.48	61.5	64.3	68.7

[JNTU 2006, (A) Dec. 2013 (Set No. 2)]

Solution : If s is the distance travelled in time t and v is the velocity at time t, then

$$\frac{dv}{dt} = a$$

Integrating, we get

$$\therefore \quad (v)_{t=0}^{40} = \int_0^{40} a\,dt$$

Here $h = 5,\ a_0 = 40.0,\ a_1 = 45.25,\ a_2 = 48.50,\ a_3 = 51.25,\ a_4 = 54.35,\ a_5 = 59.48,$

$a_6 = 61.5,\ a_7 = 64.3$ and $a_8 = 68.7$

By Trapezoidal rule, we have

$$\text{The required velocity} = \frac{h}{2}[(a_0 + a_8) + 2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7)]$$

$$= \frac{5}{2}[40.0 + 68.7) + 2(45.25 + 48.50 + 51.25 + 54.35 + 59.48 + 61.5 + 64.3)]$$

$$= \frac{5}{2}[108.7 + 2(384.63)] = \frac{5}{2}(877.96) = 2194.9$$

Position of the rocket at $t = 40$ seconds = (2194.9) (40) = 87796

By Simpson's rule, we have

$$\text{The required velocity} = \frac{h}{3}[(a_0 + a_8) + 2(a_2 + a_4 + a_6) + 4(a_1 + a_3 + a_5 + a_7)]$$

$$= \frac{5}{3}[(40.0 + 68.7) + 2(48.5 + 54.35 + 61.5) + 4(45.25 + 51.25 + 59.48 + 64.3)]$$

$$= \frac{5}{3}(108.7 + 328.7 + 881.123) = 2197.5$$

Position of the rocket at $t = 40$ seconds = (2197.5) (40) = 87900.

Example 6 : Evaluate the following integral using Simpson's $\frac{1}{3}$ rule for $n = 4$.

$$\int_1^2 \frac{e^x}{x}\,dx$$

Solution : Given $y = f(x) = \frac{e^x}{x}$, $a = 1$, $b = 2$ and $n = 4$

$$\therefore\ h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

$\therefore$ The values of x and y are tabulated below:

x	1	1.25	1.5	1.75	2
e^x	2.71828	3.4903	4.4817	5.7546	7.3890
$y = \frac{e^x}{x}$	2.71828 y_0	2.7922 y_1	2.9878 y_2	3.2883 y_3	3.69452 y_4

By Simpson's rule, we have

$$\int_1^2 \frac{e^x}{x}\,dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.25}{3}[(2.71828 + 3.69452) + 4(2.7922 + 3.2883) + 2(2.9878)]$$

$$= \frac{0.25}{3}[6.4128 + 24.322 + 5.9756] = \frac{0.25}{3}(36.7104) = 3.0592\,.$$

Example 7 : Evaluate $\int_0^1 \frac{1}{1+x}\,dx$

(*i*) by Trapezoidal rule and Simpson's $\frac{1}{3}$ rule.

[JNTU(H) June 2009, (K) May 2010, (H) Dec. 2011S, 2012]

(*ii*) using Simpson's $\frac{3}{8}$ rule. **[JNTU (H) Dec. 2011S (Set No. 3)]**

Solution : We divide the interval [0, 1] into six (multiple of 3) subintervals.

The values of x and y are tabulated below :

x	0	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	1
$y=\frac{1}{1+x}$	1 y_0	0.8571 y_1	0.75 y_2	0.6666 y_3	0.6 y_4	0.5454 y_5	0.5 y_6

(*i*) By Trapezoidal rule,

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{12}[(1+0.5)+2(0.8571+0.75+0.6666+0.6+0.5454)] = 0.69485$$

(*ii*) By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{18}[(1+0.5)+4(0.8571+0.6666+0.5454)+2(0.75+0.6)]$$

$= 0.6931$, correct to four decimal places

(ii) By Simpson's $\frac{3}{8}$ rule,

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)]$$

$$= \frac{3}{(6)(8)}[(1+0.5)+3(0.8571+0.75+0.6+0.5454)+2(0.6666)]$$

$$= \frac{1}{16}[1.5+8.2575+1.3332]$$

$= \frac{11.0907}{16} = 0.6932$, correct to 4 decimal places.

Example 8 : Given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x \, dx$ by Simpson's $\frac{3}{8}$ rule. **[JNTU 2006 (Set No.1)]**

Solution : Here $h = 0.2$, $y_0 = 1.3863$, $y_1 = 1.4351$, $y_2 = 1.4816$, $y_3 = 1.5261$, $y_4 = 1.5686$, $y_5 = 1.6094$ and $y_6 = 1.6487$

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_4^{5.2} \log x \, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3(0.2)}{8}[(1.3863 + 1.6487) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)]$$

$$= \frac{0.6}{8}[3.035 + 18.2841 + 3.0522]$$

$$= \frac{0.6}{8}(24.3713) = 1.827847.$$

Example 9 : Evaluate $\int_0^1 \sqrt{1 + x^4} \, dx$ using Simpson's $\frac{3}{8}$ rule.

Solution : We know that Simpson's $\frac{3}{8}$ rule is applicable only when n is a multiple of 3. Thus we should divide the interval (0, 1) into six equal parts each of width, $h = \frac{1}{6}$. The values of $y = f(x) = \sqrt{1 + x^4}$ are as follows.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1 y_0	1.0003857 y_1	1.006154 y_2	1.0307764 y_3	1.0943175 y_4	1.217478 y_5	1.4142136 y_6

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_0^1 \sqrt{1 + x^4} \, dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$

$$= \frac{3}{48}[(1 + 1.4142136) + 3\ (1.0003857 + 1.006154 + 1.0943175 + 1.217478) + 2(1.0307764)]$$

$$= \frac{1}{16}[2.4142136 + 12.955 + 2.0615528] = \frac{1}{16}(17.430772) = 1.08942.$$

Example 10 : Evaluate $\int_0^6 \frac{1}{1+x}dx$ by using (*i*) Simpson's $\frac{1}{3}$ rule (*ii*) Simpson's $\frac{3}{8}$ rule and compare the result with its actual value. **[JNTU (A) Dec. 2013 (Set No. 4)]**

Solution : All the formulae are applicable if n, the number of intervals is a multiple of six. So we divide the interval (0, 6) into equal parts each of width, $h = \frac{6-0}{6} = 1$.

The values of $y = f(x)$ are given below.

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1 y_0	0.5 y_1	0.3333 y_2	0.25 y_3	0.2 y_4	0.1666 y_5	0.1428 y_6

(*i*) By Simpson's $\frac{1}{3}$ rule,

$$\int_0^6 \frac{1}{1+x}dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{3}[(1+0.1428)+4(0.5+0.25+0.1666)+2(0.3333+0.2)]$$

$$= \frac{1}{3}(1.1428+3.6664+1.0666) = \frac{1}{3}(5.8758) = 1.9586$$

(*ii*) By Simpson's $\frac{3}{8}$ rule,

$$\int_0^6 \frac{1}{1+x}dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}[(1+0.1428)+3(0.5+0.3333+0.2+0.1666)+2(0.25)]$$

$$= \frac{3}{8}[1.1428+3.5997+0.5] = \frac{3}{8}(5.2425) = 1.9659$$

By actual integration,

$$\int_0^6 \frac{1}{1+x}dx = [\log(1+x)]_0^6 = \log 7 - \log 1 = \log 7$$

$$= 1.94591$$

Example 11 : Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{3}{8}$ rule taking $h = \frac{1}{6}$. Hence obtain an approximate value of π.

Solution : The values of x and y are tabulated below.

x	0	1/6	2/6	3/6	4/6	5/6	1
$\frac{1}{1+x^2}=y$	1 y_0	0.973 y_1	0.9 y_2	0.8 y_3	0.6923 y_4	0.5901 y_5	0.5 y_6

By Simpson's $\frac{3}{8}$ rule,

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{3h}{8}[(y_0+y_6)+3\,(y_1+y_2+y_4+y_5)+2\,(y_3)]$$

$$=\frac{3\,(1/6)}{8}[\,(1+0.5)+3\,(0.973+0.9+0.6923+0.5901)+2\,(0.8)]$$

$$=\frac{1}{16}[1.5+9.4662+1.6]=\frac{1}{16}(12.5662)=0.7854, \text{ correct to 4 decimal places}$$

By actual integration,

$$\int_0^1 \frac{dx}{1+x^2}=(\tan^{-1}x)_0^1=\tan^{-1}(1)-\tan^{-1}(0)=\frac{\pi}{4}$$

$$\therefore\ \frac{\pi}{4}=0.7854 \Rightarrow \pi=3.1416$$

Example 12 : Evaluate $\int_0^1 \sqrt{1+x^3}\,dx$ taking $h = 0.1$ using *i*) Simpson's $\frac{1}{3}$ *rd* rule. *ii*) Trapezoidal rule. **[JNTU 2006, (A) Dec. 2013, (Set No. 3)]**

Solution : Here $a = 0$, $b = 1$, $h = 0.1$. So $n=\frac{b-a}{h}=\frac{1-0}{0.1}=10$

The values of x and y are tabulated below.

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$y=\sqrt{1+x^3}$	1	1.0005	1.0034	1.0134	1.0315	1.0606	1.1027	1.1589	1.2296	1.3149	1.4142
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

i) By Simpson's $\frac{1}{3}$ rule,

$$\int_0^1 \sqrt{1+x^3}\,dx=\frac{h}{3}\,[(\text{Sum of the first and last ordinates}) + 4(\text{Sum of the odd ordinates})$$

$$+\,2(\text{ sum of the remaining even ordinates})]$$

$$=\frac{h}{3}[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)]$$

$$= \frac{0.1}{3}\left[(1+1.4142)+4(1.0005+1.0134+1.0606+1.1589+1.3149)+ 2(1.0034+1.0315+1.1027+1.2296)\right]$$

$$= \frac{0.1}{3}(2.4142+22.1932+8.7344)=1.1114\,.$$

ii) By Trapezoidal rule,

$$\int_0^1 \sqrt{1+x^3}\,dx = \frac{h}{2}\left[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$= \frac{0.1}{2}\left[(1+1.4142)+2(1.0005+1.0034+1.0134+1.0315+1.0606+ 1.1027+1.1589+1.2296+1.3149)\right]$$

$$= \frac{0.1}{2}(2.4142+19.831)=1.11226\,.$$

Example 13 : The table below shows the temperature $f(t)$ as a function of time

t	1	2	3	4	5	6	7
$f(t)$	81	75	80	83	78	70	60.

Use Simpson's 1/3 method to estimate $\int_1^7 f(t)dt$.

[JNTU 2006, 2007, (H) Dec. 2011S (Set No. 1)]

Solution : Here $h = 1$ and $y_0 = 81, y_1 = 75, y_2 = 80, y_3 = 83, y_4 = 78, y_5 = 70, y_6 = 60.$

By Simpson's $\frac{1}{3}$ rule,

$$\int_1^7 f(t)dt = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(81+60)+4(75+83+70)+2(80+78)\right]$$

$$= \frac{1}{3}\left[141+912+316\right] = \frac{1369}{3} = 456.3333$$

Example 14 : Evaluate $\int_{0.6}^{2.0} y\,dx$ using Trapezoidal rule.

[JNTU 2007,(H) Dec. 2011S (Set No. 2)]

x	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y	1.23	1.58	2.03	4.32	6.25	8.38	10.23	12.45

Solution : We have $h = 0.2$ and $y_0 = 1.23, y_1 = 1.58, y_2 = 2.03, \; y_3 = 4.32, y_4 = 6.25,$ $y_5 = 8.38, \; y_6 = 10.23,$ and $y_7 = 12.45$

By Trapezoidal rule,

$$\int_{0.6}^{2.0} y\,dx = \frac{h}{2}\left[(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)\right]$$

$$= \frac{0.2}{2}[(1.23 + 12.45) + 2(1.58 + 2.03 + 4.32 + 6.25 + 8.38 + 10.23 + 12.45)]$$

$$= (0.1)\,[13.68 + 90.48] = 10.416.$$

Example 15 : Using Simpson's 3/8th rule evaluate $\int_0^6 \frac{dx}{1+x^2}$ by dividing the range into 6 equal parts. **[JNTU 2008 (Set No.3)]**

Solution : Here $a = 0$, $b = 6$ and $n = 6$ $\therefore h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

The values of x and y are tabulated below:

x	0	1	2	3	4	5	6
$f(x) = \frac{1}{1+x^2}$	1	0.5	0.2	0.1	0.058824	0.03846	0.027027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\left(\frac{3}{8}\right)^{th}$ rule,

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3}{8}\left[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.03846) + 2(0.1)\right]$$

$$= \frac{3}{8}\left[1.027027 + 2.391852 + 0.2\right] = \frac{3}{8}(3.618879)$$

$$= 1.35708.$$

Example 16 : Calculate $\int_1^2 \frac{dx}{x}$ using Simpson's rule and Trapezoidal rule. Take $h = 0.25$ in the given range. **[JNTU 2008S(Set No.2)]**

Solution : Here $h = 0.25$ and $n = \frac{2-1}{0.25} = 4$.

So we cannot use Simpson's rule. Hence we will use Trapezoidal rule.

The values of $y = f(x) = 1/x$ are given below.

x	1	1.25	1.50	1.75	2.0
$y=f(x)$	1 y_0	0.8 y_1	0.6666 y_2	0.5714 y_3	0.5 y_4

By Trapezoidal rule, $\int_1^2 \frac{dx}{x} = \frac{h}{2}[(y_0+y_4)+2(y_1+y_2+y_3)]$

$$= \frac{0.25}{2}[(1+0.5)+2(0.8+0.6666+0.5714)] = 0.697$$

Example 17 : Evaluate $\int_0^\pi \sin x\, dx$ by dividing the range into 10 equal parts using

(*i*) Trapezoidal rule

(*ii*) Simpson's $\frac{1}{3}$ rule. **[JNTU(H) June 2009 (Set No.2), June 2013]**

Solution : Here $n=10$ and $h = \frac{\pi-0}{10} = \frac{\pi}{10}$

$\therefore$ The table of values is

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y=\sin x$	0 y_0	0.3090 y_1	0.5878 y_2	0.8090 y_3	0.9511 y_4	1.0 y_5	0.9511 y_6	0.8090 y_7	0.5878 y_8	0.3090 y_9	0 y_{10}

(*i*) By Trapezoidal rule,

$$\int_0^\pi \sin x\, dx = \frac{\pi}{20}[(0+0)+2\,(0.3090+0.5878+0.8090+0.9511+1.0$$

$$+\,0.9511+0.8090+0.5878+0.3090)]$$

$= 1.9843$ (approximately)

(*ii*) By Simpson's rule,

$$\int_0^\pi \sin x\, dx = \frac{\pi}{30}[(0+0)+4\,(0.3090+0.8090+1+0.8090+0.3090)$$

$$+\,2\,(0.5878+0.9511+0.9511+0.5878)]$$

$= 2.0009$

Example 18 : Evaluate $\int_0^4 e^x dx$ using Trapezoidal and Simpson's rule. Also compare your result with the exact value of the integral. **[JNTU (A) June 2009 (Set No.2)]**

Solution : Here $b-a=4-0=4$. Divide into four equal parts. $h=4/4=1$.

Hence, the table is

x	0	1	2	3	4
$y = e^x$	1	2.71828	7.3890	20.0855	54.5981
	y_0	y_1	y_2	y_3	y_4

There are 5 ordinates $(n = 4)$.

We can use both Trapezoidal and Simpson's rule.

(i) By Traezoidal rule,

$$\int_0^4 e^x dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{2}[(1 + 54.5981) + 2\,(2.71828 + 7.3890 + 20.0855)]$$

$$= \frac{1}{2}[55.5981 + 2\,(30.19278)] = 57.992$$

(ii) By Simpson's rule,

$$\int_0^4 e^x dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{3}[(1 + 54.5981) + 4\,(2.71828 + 20.0855) + 2\,(7.3890)]$$

$$= \frac{1}{3}[55.5981 + 91.21512 + 14.7780] = 53.864$$

(iii) By actual integration, $\int_0^4 e^x dx = (e^x)_0^4 = e^4 - 1 = 53.5981$. Here, the value by Simpson's rule is closer to the actual value than the value by Trapezoidal rule.

Note : The accuracy of the result can be improved by increasing the number of intervals and decreasing the value of h. Refer Solved Ex.19.

Example 19 : Compute $\int_0^4 e^x dx$ by Simpson's one-third rule with 10 subdivisions.

[JNTU (A) June 2009 (Set No.3)]

Solution : Here $b - a = 4 - 0 = 4$, $n = 10$ and $h = \frac{b-a}{n} = \frac{4}{10} = 0.4$

Hence the table is

x	0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
$y = e^x$	1	1.4918	2.2255	3.3201	4.9530	7.3890	11.0232	16.444	24.5325	36.5982	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$\int_0^4 e^x dx = \frac{h}{3}[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{0.4}{3}[(1 + 54.5981) + 4(1.4918 + 3.3201 + 7.3890 + 16.4446 + 36.5982)$$

$$+ 2(2.255 + 4.9530 + 11.0232 + 24.5325)]$$

$$= \frac{0.4}{3}[55.5981 + 4(65.2437) + 2(42.7342)]$$

$$= \frac{0.4}{3}(402.013) = 53.6055$$

Example 20 : When a train is moving at 30 m/sec, steam is shut off and brakes are applied. The speed of the train per second after t seconds is given by

Time (t)	0	5	10	15	20	25	30	35	40
Speed (v)	30	24	19.5	16	13.6	11.7	10	8.5	7.0

Using Simpson's rule, determine the distance moved by the train in 40 seconds.

[JNTU (A) 2009 (Set No.4)]

Solution : We know that $\frac{dS}{dt} = v$

$\therefore\ S = \int v\, dt$

To get S, we have to integrate v

$$\therefore\ S = \int_0^{40} v\, dt = \frac{5}{3}[(30+7) + 4(24+16+11.7+8.5) + 2(19.5+13.6+10)]$$

(using Simpson's 1/3 rule)

$$= \frac{5}{3}(37 + 240.8 + 86.2) = \frac{5(364)}{3} = 606.6667 \text{ meters.}$$

Example 21 : Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ taking $h = \pi/6$. **[JNTU (K)June 2009 (Set No.4)]**

Solution : Let $y = e^{\sin x}$.

Length of interval is $\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}$

$\therefore$ The values of y are calculated as points taking $h = \frac{\pi}{6}$.

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6} = \frac{\pi}{3}$	$\frac{3\pi}{6} = \frac{\pi}{2}$
$y = e^{\sin x}$	1 y_0	1.6487 y_1	2.3774 y_2	2.71828 y_3

Here $n = 3$. We will use Trapezoidal rule.

By Trapezoidal rule, $\int_0^{\pi/2} e^{\sin x} dx = \frac{h}{2}[(y_0 + y_3) + 2(y_1 + y_2)]$

$$= \frac{\pi}{12}[(1 + 2.71828) + 2(1.6487 + 2.3774)]$$

$$= \frac{\pi}{12}[(11.77048) = 3.0815$$

Example 22 : Evaluate $\int_0^{\pi/2} e^{\sin x} dx$ correct to four decimal places by Simpson's three-eighth rule. **[JNTU (A) May 2012 (Set No. 1)]**

Solution : Here $b - a = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

Simpson's 3/8 rule is applicable only when n is a multiple of 3.

So we divide $\left[0, \frac{\pi}{2}\right]$ into six equal parts.

$\therefore\ h = \frac{b-a}{n} = \frac{\pi/2}{6} = \frac{\pi}{12}$

The values of $y = e^{\sin x}$ are calculated as follows.

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12} = \frac{\pi}{6}$	$\frac{3\pi}{12} = \frac{\pi}{4}$	$\frac{4\pi}{12} = \frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12} = \frac{\pi}{2}$
$\sin x$	0	0.2588	0.5	0.7071	0.8660	0.9659	1
$y = e^{\sin x}$	1 y_0	1.2954 y_1	1.6487 y_2	2.0281 y_3	2.3774 y_4	2.6272 y_5	2.7183 y_6

By Simpson's three - eighth rule,

$$\int_0^{\pi/2} e^{\sin x} dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\pi}{96}[(1 + 2.7183) + 3\ (1.2954 + 1.6487 + 2.3774 + 2.6272) + 2(2.0281)]$$

$$= \frac{3\pi}{96}(3.7183 + 23.8461 + 4.0562) = \frac{3\pi}{96}(31.6206)$$

$$= 3.1043$$

REVIEW QUESTIONS

1. Derive the formula to evaluate $\int_a^b y\ dx$ using Trapezoidal rule.

[JNTU 2007S, 2008S, (H) Dec. 2011S (Set No. 1)]

2. Derive the formula to evaluate $\int_a^b y\,dx$ using Simpson's $\frac{1}{3}$ rule.
[JNTU (H) Dec. 2011S (Set No. 2)]

3. Derive the formula to evaluate $\int_a^b y\,dx$ using Simpson's $\frac{3}{8}$ rule.

EXERCISE 7.2

1. Use the Trapezoidal rule with n = 4 to estimate $\int_0^1 \frac{dx}{1+x^2}$, correct to four decimal places. **[JNTU 2007S, 2008S, (H) June 2011 (Set No. 1)]**

2. Evaluate $\int_0^\pi \frac{\sin x}{x}\,dx$ by using (*i*) Trapezoidal rule (*ii*) Simpson's 1/3 rule taking $n = 6$.
[JNTU (H) June 2011 (Set No. 1)]

3. (*a*) Evaluate $\int_0^1 e^{-x^2}\,dx$ taking h = 0.2 using (i) Simpson's $\frac{1}{3}rd$ rule (ii) Trapezoidal rule.
[JNTU 2007S, 2008S (Set No. 1)]

(*b*) Evaluate $\int_1^{1.4} e^{-x^2}\,dx$ by taking $h = 0.1$ using Simpson's rule.
[JNTU (K) 2011S (Set No. 2)]

4. Evaluate $\int_1^2 (x^3+1)\,dx$ using Simpson's 3/8 rule, dividing the range into three equal parts.

5. (*a*) Evaluate $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$ using (*i*) Simpson's 1/3 rule (*ii*) Simpson's 3/8 rule considering six sub - intervals.

(*b*) Evaluate $\int_0^{\pi/2} \sin x\,dx$ by Simpsoon's $\frac{1}{3}rd$ rule and compare with exact value.

[JNTU (A) June 2011 (Set No. 3)]

6. Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$ using Simpon's $\frac{3}{8}$ rule with $n = 6$.

7. (*a*) Evaluate $\int_0^6 \frac{1}{1+x}\,dx$ using (*i*) Trapezoidal rule (*ii*) Simpson's 3/8 rule and compare it with the actual value.

(*b*) Evaluate $\int_1^2 \frac{dx}{1+x}$ using Simpson's rule with $h = 0.1$ **[JNTU (K) 2011S (Set No. 3)]**

8. Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ by dividing the range into six equal parts.

9. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (*i*) Trapezoidal rule (*ii*) Simpson's $\frac{1}{3}$ rule (*iii*) Simpson's 3/8 rule and compare the result in each case with its actual value.

[JNTU 2008 (Set No. 3)]

10. Given that

Time, t	1	2	3	4	5	6	7
Temp. $f(t)$	81	75	80	83	78	70	60

Evaluate $\int_1^7 f(t)\, dt$ using Simpson's $\frac{1}{3}$ rule. **[JNTU 2006S (Set No.1)]**

11. Given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x\, dx$ by using (*i*) Trapezoidal rule (*ii*) Simpson's rule

(*iii*) Simpson's 3/8 rule **[JNTU 2006 (Set No.1)]**

12. The table below shows the velocities of a moped which starts from rest at fixed intervals of time. Find the distance travelled by the moped in 20 minutes.

Time, t(min)	2	4	6	8	10	12	14	16	18	20
Velocity, v (km/min.)	0	10	18	25	29	32	20	11	5	2

13. A curve is drawn to pass through the points given by the following table:

x	7.47	7.48	7.49	7.50	7.51	7.52
y	1.93	1.95	1.98	2.01	2.03	2.06

Find the area bounded by the curve, the x - axis and the lines $x = 7.47, x = 7.52$.

14. The table below shows the velocities of a car at various intervals of time. Find the distance covered by the car using Simpson's 1/3 rule.

Time (min.)	0	2	4	6	8	10	12
Velocity (km/hr)	0	22	30	27	18	7	0

15. The velocity $v\,(m/\text{sec})$ of a particle at distance $S\,(m)$ from a point on its path is given by the following table:

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by using Simpson's 1/3 rule. Compare your answer with Simpson's 3/8 rule.

Chapter

8 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

8.1 INTRODUCTION

Many problems in science and engineering can be formulated into ordinary differential equations. The analytical methods of solving differential equations are applicable only to a selected class of differential equations. Quite often equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods for solving such differential equations.

8.2 SOLUTION OF A DIFFERENTIAL EQUATION

The solution of an ordinary differential equation in which x is the independent variable and y is the dependent variable usually means finding an explicit expression for y in terms of a finite number of elementary functions of x; for example, polynomial, trigonometric or exponential functions. If such an explicit relation is found, then the solution is known as the closed form or finite form of solution. In the absence of such a solution, we have to resort to numerical methods of solution.

In this chapter we mainly concentrate on the numerical solution of ordinary differential equations and discuss the following methods :

1. Taylor's series method
2. Euler's method
3. Modified Euler method
4. Picard's method of successive approximation
5. Runge - Kutta method
6. Predictor Corrector methods : Adams Moulton method

To describe various numerical methods for the solution of ordinary differential equations, we consider the general first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{..... (1) with the initial condition } y(x_0) = y_0 .$$

The methods will yield the solution in one of the two forms :

(*i*) A series for y in terms of powers of x, from which the values of y can be obtained by direct substitution.

(*ii*) A set of tabulated values of y corresponding to different values of x.

The methods of Taylor and Picard belong to calss (*i*). In these methods, y in (1) is approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as **single - step methods**. The methods of Euler, Runge - Kutta, Adams - Bashforth, Milne, etc., belong to class (*ii*). These methods are called **step-by-step methods or marching** methods because the values of y are computed by short steps ahead for equal intervals h of the independent variable.

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne, Adams-Bashforth, Adams-Moulton, etc., may be applied for finding y over a wide range of x-values. Therefore, Milne and Adams methods requires 'starting values' which are usually obtained by Taylor's series or Runge-Kutta methods.

8.3 INITIAL AND BOUNDARY CONDITIONS

An ordinary differential equation of n th order is of the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0 \qquad \ldots.. \quad (2)$$

Its general solution will contain n arbitrary constants and it will be of the form

$$f(x, y, c_1, c_2, \ldots, c_n) = 0 \qquad \ldots.. \quad (3)$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \ldots, c_n$ can be determined. Problems in which $y, y', \ldots, y^{(n-1)}$ are all specified at the same value of x, say x_0, are called **initial-value** problems. If the conditions on y are prescribed at n distinct points, then the problems are called **boundary - value** problems. Problems in which function is prescribed at k different points and $(n-k)$ derivatives are prescribed at the same point are called mixed value problems.

In this chapter, we shall describe some numerical methods to solve initial value problems.

8.4 TAYLOR - SERIES METHOD

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots.. \quad (1)$$

given the initial condition $y(x_0) = y_0$

$y(x)$ can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots. + \frac{(x - x_0)^n}{n!} y^n(x_0) + \ldots. \qquad \ldots(2)$$

where $y^i(x_0)$ is the i th derivative of $y(x)$ at $x = x_0$.

The value of $y(x)$ can be obtained if we know the values of its derivatives.

Differentiating (1), we have

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[f(x, y)] = \frac{\partial}{\partial x}[f(x, y)] + \frac{\partial}{\partial y}[f(x, y)] \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f = f_x + f \cdot f_y \qquad \ldots \quad (3)$$

where f denotes the function $f(x, y)$ and f_x and f_y denote the partial derivatives of the function $f(x, y)$ with respect to x and y, respectively.

Similarly, we can obtain $y''' = f_{xx} + 2f.f_{xy} + f^2 f_{yy} + f_x f_y + f.f_y{}^2$ ..... (4)

and other higher derivatives of y.

If we let $x - x_0 = h$ $(i.e.\ x = x_1 = x_0 + h)$, we can write the Taylor's series as

$$y(x) = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + ... \qquad ..... (5)$$

$$i.e., \quad y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ...$$

From the above equation knowing the value of $y(x_0)$; the higher derivatives $y'(x_0), y''(x_0), ...$ may be computed and the value of y at the neighbouring point $x_0 + h$ may be found out.

On finding the value y_1 for $x = x_1$ using (2) or (5), y', y'', y''' etc. can be found at $x = x_1$ by means of (1), (3) and (4). Then y can be expanded about $x = x_1$.

$$\text{Thus } y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ...$$

Similarly expanding $y(x)$ in a Taylor series about the point x_1, we will get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + ...$$

Similarly expanding $y(x)$ at a general point x_n, we will get

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + ... \qquad ..... (6)$$

$$\text{where } y_n^r = \left(\frac{d^r y}{dx^r}\right)_{(x_n, y_n)}$$

Equation (6) can be used to get the value of y_{n+1}. For this, the exact value of y_n must be known from the previous step. Since (6) is an infinite series, we have to truncate at some term to have the numerical value calculated. Thus the value of y_n can be got approximately, without much error. Further equation (6) can be written as

$$y_{n+1} = y_n + h\, y_n' + \frac{h^2}{2} y_n'' + O(h^3) \qquad ..... (7)$$

$O(h^3)$ means "terms invloving third and higher powers of h^3" and read as "order of h^3". So if (7) is taken to determine y_{n+1} leaving the terms $O(h^3)$, the truncation error in the estimation of y_{n+1} is kh^3 where k is some constant. The Taylor series used is said to be of the second order.

In general, if we retain, for calculation purpose, the terms upto and including h^n and neglect terms h^{n+1} and higher powers of h in the R.H.S. of (7), the error will be proportional to the $(n+1)$ th power of the setp-size. The Taylor's algorithm is said to be of the n th order. The truncation error is $O(h^{n+1})$. By including more number of terms in the R.H.S. of (7), the error can be reduced further.

If h is small and the terms after n terms are neglected, the error is $\frac{h^n}{n!} f^n(\theta)$, where $x_0 < \theta < x_1$ if $x_1 - x_0 = h$.

8.5 MERITS AND DEMERITS OF THE TAYLOR SERIES

The Taylor series method is a single step method and works well so long as the successive derivatives of y can be calculated in an easy manner. But if $f(x, y)$ is some what complicated, then the evaluation of higher order derivatives may become tedious. This is the demerit of the Taylor's series method and therefore, has little application for computer programs. Also this method is particularly unsuitable if $f(x, y)$ is given in a tabular form.

However, this method will be very useful for finding initial starting values for powerful numerical methods such as Runge-Kutta, Milne's method and Adams-Bashforth which will be discussed subsequently.

SOLVED EXAMPLES

Example 1 : Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$.

Solution : Given equation is $y' = f(x, y)$ where $f(x, y) = x^2 + y^2$.

Differentiating repeatedly w.r.t.x, we get

$$y' = \frac{dy}{dx} = x^2 + y^2$$

$$\therefore \; y'' = \frac{d^2y}{dx^2} = 2x + 2y.y'; \quad y''' = \frac{d^3y}{dx^3} = 2 + 2(y')^2 + 2y.y''; \quad y^{iv} = \frac{d^4y}{dx^4} = 6y'.y'' + 2y.y'''$$

At $x = 0, y = 0$, so we have $y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 2,\ y^{(iv)}(0) = 0$

The Taylor series for $y(x)$ near $x = 0$ is given by

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{(iv)}(0) + \ldots = 0 + 0 + 0 + \frac{x^3}{3!}.2 + 0 + \ldots$$

$$= \frac{x^3}{3} + \text{ (higher order terms neglected)}$$

Hence $y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$

Note: Notice that Taylor's series method rests on the successive evaluation of $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ etc., using the given equation $\frac{dy}{dx} = f(x, y)$.

Example 2 : Solve $y' = x - y^2$, $y(0) = 1$ using Taylor's series method and compute $y(0.1), y(0.2)$. **[JNTU (A) Dec. 2013 (Set No. 1)]**

Solution : The derivatives of y are given by

$$y' = x - y^2 \;;\quad y'' = 1 - 2y\,y'\;;\quad y''' = -2\left[(y')^2 + y\,y''\right]$$

$$y^{iv} = -2\left[2y'\,y'' + y'\,y'' + y\,y'''\right] = -2\left[3\,y'\,y'' + y\,y'''\right]$$

Here $x_0 = 0, y_0 = 1$ and $h = 0.1$

Now

$$y_0' = -1,\; y_0'' = 1 - 2\,(1)\,(-1) = 3,\; y_0''' = -2\left[(-1)^2 + (1)\,(3)\right] = -8,$$

$$y_0^{iv} = -2\left[3(-1)(3) + (1)(-8)\right] = -2\left[-9 - 8\right] = 34$$

By Taylor's series, we have $y_1 = y_0 + \dfrac{h}{1!}y_0' + \dfrac{h^2}{2!}y_0'' + \dfrac{h^3}{3!}y_0''' + \dfrac{h^4}{4!}y_0^{iv} + ...$

$$\therefore\; y_1 = y(0.1) = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34) + ...$$

$$= 1 - 0.1 + 0.015 - 0.00133 + 0.00014 + ...$$

$$= 0.91381$$

Now, take $x_1 = 0.1, h = 0.1$ and $y_1 = 0.91381$

We calculate $y_1', y_1'', y_1''', y_1^{iv}, ...,$

$$y_1' = x_1 - y_1^2 = 0.1 - (0.91381)^2 = 0.1 - 0.8350487 = -\;0.735$$

$$y_1'' = 1 - 2y_1\;y_1' = 1 - 2(0.91381)\,(-0.735) = 1 + 1.3433 = 2.3433$$

$$y_1''' = -2\left[(y_1')^2 + y_1\;y_1''\right] = -2\left[(-\,0.735)^2 + (0.91381)\,(2.3433)\right]$$

$$= -2\left[0.540225 + 2.141331\right] = -5.363112$$

$$y_1^{iv} = -2\left[3y_1'\;y_1'' + y_1\;y_1'''\right] = -2\left[3\,(-0.735)(2.3433) + (0.91381)(-5.363112)\right]$$

$$= -2\left[-5.16697 - 4.90087\right] = 20.133567$$

We take $y_2 = y_1 + hy_1' + \dfrac{h^2}{2!}y_1'' + \dfrac{h^3}{3!}y_1''' + \dfrac{h^4}{4!}y^{iv} + ...$ using the Taylor's series method.

$$\therefore\; y_2 = y(0.2) = 0.91381 + (0.1)\,(-0.735) + \frac{(0.1)^2}{2}(2.3433)$$

$$+ \frac{(0.1)^3}{6}(-5.363112) + \frac{(0.1)^4}{24}(20.13567) + ...$$

$$= 0.91381 - 0.0735 + 0.0117 - 0.00089 + 0.00008 = 0.8512$$

Proceeding like this it is possible to get the values of y at various values of x.

Example 3 : Using Taylor series method, find an approximate value of y at $x = 0.2$ for the differential equation $y' - 2y = 3e^x$, $y(0) = 0$. **[JNTU (H) June 2010 (Set No.1)]**

Compare the numerical solution obtained with exact solution.

(OR) Using the Taylor's series method, solve $\frac{dy}{dx} = 2y + 3e^x, y(0) = 0$ at $x = 0.1, 0.2$

[JNTU (A) June 2011 (Set No. 4)]

Solution : Given equation can be written as $y' = 2y + 3e^x$

Differentiating repeatedly *w.r.t.* 'x', we get

$$y'' = 2y' + 3e^x; \quad y''' = 2y'' + 3e^x; \quad y^{iv} = 2y''' + 3e^x$$

Here $x_0 = 0, y_0 = 0, x_1 = 0.2, h = 0.2$

$$\therefore\ y_0' = 2y_0 + 3e^0 = 2 \times 0 + 3 \times 1 = 3; \quad y_0'' = 2y_0' + 3e^0 = 2 \times 3 + 3 \times 1 = 9$$

$$y_0''' = 2y_0'' + 3e^0 = 2 \times 9 + 3 \times 1 = 21; \quad y_0^{iv} = 2y_0''' + 3e^0 = 2 \times 21 + 3 \times 1 = 45$$

We have the Taylor algorithm $y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + ...$

$$\therefore\ y(0.2) = y_1 = 0 + \frac{0.2}{1!}(3) + \frac{(0.2)^2}{2}(9) + \frac{(0.2)^3}{6}(21) + \frac{(0.2)^4}{24}(45) + ...$$

$$= 0.6 + 0.18 + 0.028 + 0.003 = 0.811$$

We can get the analytical solution of the given differential equation as follows.

The equation is $\frac{dy}{dx} - 2y = 3e^x$

which is a linear equation in y.

Here $P = -2$, $Q = 3e^x$. I.F. $= e^{\int P dx} = e^{-2\int dx} = e^{-2x}$

$\therefore$ General solution is $y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$

i.e., $y\, e^{-2x} = \int 3e^x\, e^{-2x} dx + c = 3\int e^{-x} dx + c = -3e^{-x} + c$

$\therefore\ y = -3e^x + c\, e^{2x}$. When $x = 0, y = 0$. So $0 = -3 + c$ or $c = 3$

$\therefore$ The particular solution is $y = -3e^x + 3e^{2x}$

Putting $x = 0.2$ in the above particular solution,

$$y = -3e^{0.2} + 3e^{0.4} = -3(1.2214) + 3(1.4918) = -3.6642 + 4.4754 = 0.8112$$

Note : Using Taylor's series method, $y(0.2) = 0.811$

Using the exact solution, $y(0.2) = 0.8112$

$\therefore$ The difference between the values is 0.0002

Example 4 : Employ Taylor's method to obtain approximate value of $y(1.1)$ and $y(1.3)$, for the differential equation $y' = x \cdot y^{1/3},\ y(1) = 1$. Compare the numerical solution obtained with exact solution. **[JNTU (A) Dec. 2013 (Set No. 4)]**

Solution : The derivatives of y are given by

$$y' = x \cdot y^{1/3} \qquad \text{.... (1)}$$

$$y'' = x \cdot \frac{1}{3} \cdot y^{-2/3} y' + y^{1/3} = \frac{1}{3} x^2\ y^{-1/3} + y^{1/3} \qquad \text{.... (2)}$$

$$y''' = \frac{x^2}{3}\left(-\frac{1}{3}\right) y^{-4/3} y' + \frac{2x}{3} y^{-1/3} + \frac{1}{3} y^{-2/3} y' \qquad \text{.... (3)}$$

Step 1: We have the Taylor algorithm $y_1 = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ...$ .... (4)

Here $x_0 = 1, y_0 = 1, h = 0.1$.
Putting $x_0 = 1, y_0 = 1$ in (1), (2) and (3), we get

$$y_0' = 1\,(1)^{1/3} = 1,\quad y_0'' = \frac{1}{3}(1)^2 (1)^{-1/3} + (1)^{1/3} = \frac{4}{3} \text{ and } y_0''' = -\frac{1}{9} + \frac{2}{3} + \frac{1}{3} = \frac{8}{9}$$

Hence substituting the values of y_0, y_0', y_0'', y_0''' in (4), we get

$$y_1 = y(1.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{6}\left(\frac{8}{9}\right) + ...$$

$$= 1 + 0.1 + 0.0066 + 0.000148 = 1.1067481 \approx 1.1067.$$

Thus we have evaluated $y(1.1)$.

Step 2: Let us find $y(1.2)$. We start with (x_1, y_1) as the starting value $x_1 = x_0 + h = 1.1$

We have by the Taylor's algorithm , $y_2 = y_1 + h\, y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + ...$ .... (5)

Putting $x_1 = 1.1$ and $y_1 = 1.1067$ in (1), (2) and (3)

$$y_1' = x_1\ y_1^{1/3} = (1.1)\,(1.1067)^{1/3} = 1.13782$$

$$y_1'' = \frac{1}{3} x_1^2\ y_1^{-1/3} + y_1^{1/3} = \frac{1}{3}(1.1)^2 (1.1067)^{1/3} + (1.1067)^{1/3} = \frac{1}{3}(1.21)\,(0.96677) + 1.03437$$

$$= 0.38993 + 1.03437 = 1.4243$$

and $y_1''' = 0.9297$

Substituting the above in (5), we get

$$y_2 = y(1.2) = 1.1067 + (0.1)\,(1.13782) + \frac{(0.1)^2}{2}(1.4243) + \frac{(0.1)^3}{6}(0.9297)$$
$$+ \text{ (higher order terms neglected)}$$

$$= 1.1067 + 0.113782 + 0.00712 + 0.00015495 = 1.2277569 \approx 1.2278.$$

Thus we obtained $y(1.2)$.

Step 3 : Now we start with (x_2, y_2) as the starting value, where $x_2 = x_1 + h = 1.2$

We have by the Taylor's algorithm, $y_3 = y_2 + h\, y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + ...$ ..... (6)

Putting $x_2 = 1.2$ and $y_2 = 1.2278$ in (1) and (2),

$$y_2' = x_2\, y_2^{1/3} = (1.2)\,(1.2278)^{1/3} = 1.28496$$

$$y_2'' = \frac{1}{3}x_2^2\, y_2^{-1/3} + y_2^{1/3} = \frac{1}{3}(1.2)^2(1.2278)^{-1/3} + (1.2278)^{1/3}$$

$$= \frac{1}{3}(1.44)\,(0.93388) + 1.070802 = 0.44826 + 1.070802 = 1.51906$$

Substituting the above in (6), we obtain

$$y_3 = 1.2278 + (0.1)\,(1.28496) + \frac{(0.1)^2}{2}(1.51906) + \quad \text{(higher order terms neglected)}$$

$$= 1.2278 + 0.128496 + 0.0075953 = 1.3638913$$

$\therefore\ y_3 \approx 1.3639$

ANALYTICAL SOLUTION :

The equation is $\dfrac{dy}{dx} = x \cdot y^{1/3}$

Separating the variables, $\dfrac{dy}{y^{1/3}} = x\,dx$ or $y^{\frac{-1}{3}}\,dy = x\,dx$

Integrating, $\dfrac{3}{2}y^{2/3} = \dfrac{x^2}{2} + c$. When $x = 1, y = 1$ $\quad\therefore\ \dfrac{3}{2} = \dfrac{1}{2} + c \Rightarrow c = 1$

Hence the particular solution is $\dfrac{3}{2}y^{2/3} = \dfrac{x^2}{2} + 1$ or $y^{2/3} = \dfrac{1}{3}(x^2 + 2)$ ..... (7)

Putting $x = 1.1$ in (7), $y^{2/3} = \dfrac{1}{3}(1.21 + 2) = \dfrac{3.21}{3} = 1.07$

$\therefore\ y = (1.07)^{3/2} = 1.1068$ $\quad$ *i.e.,* $y\,(1.1) = y_1 = 1.1068$

Putting $x = 1.2$ in (7), $y^{2/3} = \dfrac{1}{3}(1.44 + 2) = \dfrac{3.44}{3} = 1.1467$

$\therefore\ y = (1.1467)^{3/2} = 1.2278$

Putting $x = 1.3$ in (7), $y^{2/3} = \dfrac{1}{3}(1.69 + 2) = \dfrac{3.69}{3} = 1.23$

$\therefore\ y = (1.23)^{3/2} = 1.364136 \approx 1.364$

Thus we can tabulate the values as follows :

x	Taylor's series method y	Exact solution y
1	1	1
1.1	1.1067	1.1068
1.2	1.2278	1.2278
1.3	1.3639	1.364

We notice that the values of y in the last two columns are sufficiently close to one another.

Example 5 : Solve $y' = x^2 - y,\ y(0) = 1$ using Taylor's series method and compute $y(0.1), y(0.2), y(0.3)$, and $y(0.4)$ (correct to 4 decimal places). **[JNTU (A) June 2010 (Set No.3)]**

Solution : Given equation is $y' = x^2 - y$ ... (1)

Differentiating (1) successively, we get

$$y'' = 2x - y' \quad \text{... (2)} \qquad y''' = 2 - y'' \quad \text{... (3) and} \quad y^{iv} = -y''' \quad \text{... (4)}$$

Step 1. The Taylor algorithm gives $y_1 = y_0 + h\,y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$ ...(5)

Here $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Putting $x_0 = 0,\ y_0 = 1$ in (1), (2), (3) and (4), we obtain

$$y_0' = x_0^2 - y_0 = -1; \qquad y_0'' = 2x_0 - y_0' = 0 - (-1) = 1$$

$$y_0''' = 2 - y_0'' = 2 - 1 = 1; \qquad y_0^{iv} = -y_0''' = -1$$

Hence substituting the above in (5), we get

$$y_1 = y(0.1) = 1 + (0.1)(-1) + \frac{0.01}{2}(1) + \frac{0.001}{6}(1) + \frac{0.0001}{24}(-1) + \ldots$$

$$= 1 - 0.1 + 0.005 + 0.01666 - 0.0000416 + \ldots = 0.905125 \approx 0.9051 \text{ (4 decimal places)}$$

Step 2. We start with (x_1, y_1) as the starting value where $x_1 = x_0 + h = 0 + 0.1 = 0.1$

From the Taylor's algorithm $y_2 = y_1 + h\,y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$ ..... (6)

Putting $x_1 = 0.1$ and $y_1 = 0.905125$ in (1), (2), (3) and (4),

$$y_1' = x_1^2 - y_1 = 0.01 - 0.905125 = -0.895125; \qquad y_1'' = 2x_1 - y_1' = 0.2 + 0.895125 = 1.095125$$

$$y_1''' = 2 - y_1'' = 2 - 1.095125 = 0.904875; \ y_1^{iv} = -y_1''' = -0.904875$$

Substituting the above in (6),

$$y_2 = y(0.2) = 0.905125 + (0.1)(-0.895125) + \frac{0.01}{2}(1.095125)$$

$$+ \frac{0.001}{6}(0.904875) + \frac{0.0001}{24}(-0.904875) + \ldots$$

$$= 0.905125 - 0.0895125 + 0.00547562 + 0.000150812 - 0.00000377$$

$$= 0.8212351 \approx 0.8212 \text{ (4 decimal places)}$$

Similarly $y(0.3) = 0.7492$ (4 decimals) and $y(0.4) = 0.6897$ (4 decimal places)

Note: Solve the equation $\frac{dy}{dx} = x - y^2$ with the conditions $y(0) = 1$ and $y'(0) = 1$. Find y(0.2) and y(0.4)using Taylor's series method. **[JNTU Aug. 2008S (Set No.1)]**

Take $x_0 = 0, y_0 = 1, h = 0.2$ and substitute these values in (1), (2), (3), (4) and then in (5) to find $y = y(0.2)$. Now take $x_1 = x_0 + h = 0 + 0.2 = 0.2$ and substitute these values in (6) to find $y_2 = y(0.4)$.

Example 6 : Tabulate $y(.1)$, $y(.2)$ and $y(.3)$ using Taylor's series method given that $y' = y^2 + x$ and $y(0) = 1$. **[JNTU 2006, 2006S (Set No.2, 3), (A) Nov. 2010, (Set No. 2)]**

Solution : Given $y' = y^2 + x$ ... (1)

and $y(0) = 1$ ... (2)

Differentiating (1) w.r.t. 'x', we get

$$y'' = 2y\,y' + 1 \qquad \text{... (3)}$$

$$y''' = 2y\,y'' + 2\,(y')^2 \qquad \text{... (4)}$$

$$y^{iv} = 2y\,y''' + 6y'\,y'' \qquad \text{... (5)}$$

and so on.

We have $x_0 = 0$ and $y_0 = 1$. Putting these in equations (1), (3), (4) and (5), we obtain

$$y'_0 = (1)^2 + 0 = 1$$

$$y''_0 = 2(1)\,(1) + 1 = 3$$

$$y'''_0 = 2(1)\,(3) + 2\,(1)^2 = 8$$

$$y^{iv}_0 = 2(1)\,(8) + 6(1)\,(3) = 34$$

Take $h = 0.1$

Step 1: We know by Taylor's series expansion,

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y_0^{iv} + \cdots \qquad \text{... (6)}$$

On Substituting the values of $y_0, y'_0, y''_0,$ etc. in (6), we get

$$y\,(0.1) = y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(34) + \cdots$$

$$= 1 + 0.1 + 0.015 + 0.001333 + 0.000416$$

$$= 1.116749$$

Step 2: Now we will find $y(0.2)$. We start with (x_1, y_1) as the starting value.

Here $x_1 = x_0 + h = 0 + 0.1$ and $y_1 = 1.116749$.

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$$y'_1 = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$$

$$y''_1 = 2y_1y'_1 + 1 = 2(1.116749)\,(1.3471283) + 1 = 4.0088$$

$$y_1''' = 2y_1 y_1'' + 2(y_1')^2 = 2(1.116749)(4.0088) + 2(1.347128)^2$$
$$= 8.95365 + 3.6295 = 12.5831$$
$$y_1^{iv} = 2y_1\, y_1''' + 6y_1'\, y_1''$$
$$= 2(1.116749)\,(12.5831) + 6(1.3471283)\,(4.0088)$$
$$= 28.104329 + 32.4022 = 60.50653$$

By Talyor's series expansion,

$$y_2 = y_1 + h\, y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv} + \ldots$$

$$= 1.116749 + (0.1)\,(1.3471283) + \left(\frac{0.01}{2}\right)(4.0088) + \left(\frac{0.001}{6}\right)(12.5831)$$

$$+ \left(\frac{0.0001}{24}\right)(60.50653)$$

$$= 1.116749 + 0.1347128 + 0.020044 + 0.002097 + 0.000252$$

i.e., $y\,(0.2) = 1.27385$

Step 3: Let us find y (0.2). We start with (x_2, y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

and $y_2 = 1.27385$

Substituting the values of x_2 and y_2 in equations (1), (3), (4) and (5), we get

$$y_2' = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$$

$$y_2'' = 2y_2\, y_2' + 1 = 2\,(1.27385)\,(1.82269) + 1 = 5.64366$$

$$y_2''' = 2y_2\, y_2'' + 2(y_2')^2 = 2(1.27385)\,(5.64366) + 2(1.82269)^2$$
$$= 14.37835 + 6.64439 = 21.02274$$

$$y_2^{iv} = 2y_2 y_2''' + 6y_2' y_2''$$
$$= 2(1.27385)\,(21.02274) + 6\,(1.82269)\,(5.64366)$$
$$= 53.559635 + 61.719856 = 115.27949$$

By Taylor's series expansion,

$$y_3 = y_2 + h\, y_2' + \frac{h^2}{2!} y_2'' + \frac{h^3}{3!} y_2''' + \frac{h^4}{4!} y_2^{iv} + \ldots$$

$$= 1.27385 + (0.1)\,(1.82269) + \left(\frac{0.01}{2}\right)(5.64366) + \left(\frac{0.001}{6}\right)(21.02274)$$

$$+ \left(\frac{0.0001}{24}\right)(115.27949)$$

$$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033$$
$$= 1.48831$$

Thus we can tabulate the values as follows :

x	y
0	1
0.1	1.116749
0.2	1.27385
0.3	1.48831

Note: Using Taylor's series method, solve $y' = xy + y^2$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3$

[JNTU Aug. 2008S, (K) June 2009 (Set No.2)]

Proceeding as in the above problem, the student can easily get the solution as $y(0.1) = 1.1167, y(0.2) = 1.2767$ and $y(0.3) = 1.5023$.

Example 7 : Solve $y' = x + y$, given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's series method. **[JNTU 2008R (Set No. 3)]**

Solution : Given $y' = x + y$ ... (1)

and $y(0) = 1$

Differentiating (1) w.r.t. 'x', we get

$$y'' = 1 + y' \quad \text{... (2)}$$
$$y''' = y'' \quad \text{... (3)}$$
$$y^{iv} = y''' \quad \text{... (4)}$$

and so on.

We have $x_0 = 1, y_0 = 0$ and $h = 0.1$.

Putting these values in equations (1), (2), (3) and (4), we obtain

$$y_0' = x_0 + y_0 = 1 + 0 = 1$$
$$y_0'' = 1 + y_0' = 1 + 1 = 2$$
$$y_0''' = y_0'' = 2$$
$$y_0^{iv} = 2, \text{ etc.,}$$

Step 1 : By Taylor's series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots$$

$$\therefore \; y_1 = y(1.1) = 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2) + \frac{(0.1)^5}{120}(2) + \cdots$$
$$= 0.1 + 0.01 + 0.00033 + 0.00000833 + 0.000000166 + \cdots$$
$$= 0.11033847.$$

Step 2 : Now we will find $y(0.2)$. We start with (x_1, y_1) as the starting value.

Here $x_1 = 1.1$ and $y_1 = 0.11033847$

Putting these values of x_1 and y_1 in (1), (2), (3) and (4), we get

$$y_1' = x_1 + y_1 = 1.1 + 0.11033847 = 1.21033847$$

$$y_1'' = 1 + y_1' = 2.21033847$$
$$y_1''' = y_1'' = y_1^{iv} = y_1^{v} = 2.21033847$$

By Taylor's series expansion,

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \cdots$$

$$\therefore\ y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1}(1.21033847) + \frac{(0.1)^2}{2}(2.21033847)$$
$$+\frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^4}{24}(2.21033847) + \cdots$$
$$= 0.11033847 + 0.121033847 + 2.21033847\,(0.005 + 0.0016666 + \cdots)$$
$$= 0.24280160$$

Analytical Solution :

The equation is $\dfrac{dy}{dx} - y = x$

I.F. $= e^{-x}$

The general solution is $y \cdot e^{-x} = \int xe^{-x}dx + c = -(x+1)e^{-x} + c$

or $y = -(x+1) + ce^{x}$

we have $y(1) = 0 \Rightarrow 0 = -2 + ce \ \therefore\ c = 2e^{-1}$

Hence the solution is $y = -x - 1 + 2e^{x-1}$

Thus $y(1.1) = -1.1 - 1 + 2e^{0.1} = 0.11034$

$y(1.2) = -1.2 - 1 + 2e^{0.2} = 0.2428$

We can tabulate the values as follows :

x	Taylor's series method (y)	Exact solution (y)
1.1	0.11033847	0.11034
1.2	0.2428016	0.2428

Example 8 : Use Taylor's series method to find the approximate value of y when $x = 0.1$ given $y(0) = 1$ and $y' = 3x + y^2$. **[JNTU(K) May 2010 (Set No.1)]**

Solution : Given $y' = 3x + y^2$ ... (1)

and $y(0) = 1$

Differentiating (1) successively w.r.t. 'x', we get

$$y'' = 3 + 2yy' \quad \text{... (2)}$$
$$y''' = 2[yy'' + (y')^2] \quad \text{... (3)}$$
$$y^{iv} = 2[yy''' + 3y' \cdot y''] \quad \text{... (4)}$$

Here $x_0 = 0, y_0 = 1$. We have to find y_1. Take $h = 0.1$

Putting these values in (1), (2), (3),(4) and (5), we obtain

$$y_0' = 3x_0 + y_0^2 = 1$$

$$y_0'' = 3 + 2y_0 y_0' = 3 + 2(1)(1) = 3 + 2 = 5$$

$$y_0''' = 2[y_0 y_0'' + (y_0')^2] = 2(5+1) = 12$$

$$y_0^{iv} = 2[y_0 y_0''' + 3y_0' \cdot y_0''] = 2[12+15] = 54$$

By Taylor's series method,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

$$= 1 + 0.1(1) + \frac{(0.1)^2}{2}(5) + \frac{(0.1)^3}{6}(12) + \frac{(0.1)^4}{24}(54) + \ldots$$

$$= 1 + 0.1 + 0.025 + 0.002 + 0.000225 + \ldots$$

$$= 1.127$$

Example 9 : Find by Taylor's series method the value of y at $x = 0.1$ to five places of decimal from

$$\frac{dy}{dx} = x^2 y - 1, y(0) = 1$$ **[JNTU(A) May2010 (Set No.1)]**

Solution : Given

$$y' = x^2 y - 1 \quad \ldots (1)$$

Differentiating (1) successively w.r.t.'x' we get

$$y'' = 2xy + x^2 y' \quad \ldots (2)$$

$$y''' = 2y + 4xy' + x^2 y'' \quad \ldots (3)$$

$$y^{iv} = 6y' + 6xy'' + x^2 y''' \quad \ldots (4)$$

and so on

We have $x_0 = 0, y_0 = 1$ and $h = 0.1$

Substituting these values in equations (1), (2), (3), and (4), we obtain

$$y_0' = x_0^2 y_0 - 1 = -1$$

$$y_0'' = 2x_0 y_0 + x_0^2 y_0' = 0$$

$$y_0''' = 2y_0 + 4x_0 y_0' + x_0^2 y_0'' = 2(1) = 2$$

$$y_0^{iv} = 6y_0' + 6x_0 y_0'' + x_0^2 y_0''' = 6(-1) = -6$$

By Taylor's sereies, we have

$$y_1 = y(0.1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$$

$$= 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(0) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(-6) + \ldots$$

$$= 1 - 0.1 + 0 + 0.00033 - 0.000025 + \ldots.$$

$$= 0.9003$$

Note: Similarly $y_2 = y(0.2) = 0.80256$

Example 10 : Solve $\frac{dy}{dx} = xy + 1$ and $y(0) = 1$ using Taylor's series method and compute $y(0.1)$.

[JNTU(H) June2010(Set No.3)]

Solution : Given $y' = xy + 1$ ... (1)

Differentiating (1) successively w.r.t. 'x', we get

$$y'' = xy' + y \quad \ldots (2)$$

$$y'' = xy'' + 2y' \quad \ldots (3)$$

$$y^{iv} = xy''' + 3y'' \quad \ldots (4)$$

and so on.

We have $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Substituting these values in equations (1), (2), (3) and (4), we obtain

$$y_0' = x_0 y_0 + 1 = 0 + 1 = 1$$

$$y_0'' = x_0 y_0' + y_0 = 0 + 1 = 1$$

$$y_0''' = x_0 y_0'' + 2y_0' = 0 + 2(1) = 2$$

$$y_0^{iv} = x_0 y_0''' + 3y_0'' = 0 + 3(1) = 3$$

By Taylor's series, we have

$$y_1 = y(0.1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0'' + \frac{h^4}{4!}y_0^{iv} + \ldots$$

$$= 1 + (0.1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(3) + \ldots$$

$$= 1 + 0.1 + 0.005 + 0.00033 + 0.0000125 + \ldots$$

$$= 1.1053425$$

$$= 1.1053 \text{ correct to four decimal places}$$

Example 11 : Solve the equation $\frac{dy}{dx} = x - y^2$ with the conditions $y(0) = 1$ and $y'(0) = 1$. Find $y(0.2)$ and $y(0.4)$ using Taylor's series method.

[JNTU 2008 (Set No.4)]

Solution : We have $y' = x - y^2$, $y(0) = 1$ and $y'(0) = 1$

Differentiating $y' = x - y^2$ repeatedly, we find

$$y'' = 1 - 2yy', \quad y''(0) = 1 - 2(1)(1) = 1 - 2 = -1$$

$$y''' = 2[yy'' + (y')^2], \quad y'''(0) = -2[1(-1) + 1] = 0$$

$$y^{iv} = -2[yy''' + y'y'' + 2y'y''],\ y^{iv}(0) = -2[0-1-2] = 6$$

By Taylor's series expansion,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots$$

$$= 1 + x + \frac{x^2}{2}(-1) + 0 + \frac{x^4}{24}(6) + \ldots = 1 + x - \frac{x^2}{2} + \frac{x^4}{4} + \ldots$$

$$\therefore\ y(0.2) = 1 + 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{4} + \ldots$$

$$= 1.2 - 0.02 + 0.0004 = 1.1804.$$

$$y(0.4) = 1 + 0.4 - \frac{(0.4)^2}{2} + \frac{(0.4)^4}{4} + \ldots$$

$$= 1.4 - 0.08 + 0.0064 = 1.3264.$$

8.6 Taylor Series Method for Simultaneous First order Differential Equations.

The equations of the type $\frac{dy}{dx} = f(x,y,z)$ and $\frac{dz}{dx} = g(x,y,z)$ with initial conditions $y(x_0) = y_0$, $z(x_0) = z_0$ (Here x is independent variable while y and z are dependent) can be solved by Taylor's series method as explained through the following example.

SOLVED EXAMPLES

Example 1 : Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$ given $\frac{dy}{dx} = x+z$, $\frac{dz}{dx} = x - y^2$ and $y(0) = 2$, $z(0) = 1$ by using Taylor's series method.

[JNTU 2008R, (K) June 2009, 2009S, (H) June 2010 (Set No. 2)]

Solution : Given

$y' = x + z$

Take $x_0 = 0$, $y_0 = 2$, $h = 0.1$

We have to find $y_1 = y(0.1)$ and $y_2 = y(0.2)$

Now $y' = x + z$

$$y'' = 1 + z'$$

$$y''' = z''$$

$$y^{iv} = z'''$$

and so on.

$z' = x - y^2$

Take $x_0 = 0$, $z_0 = 1$, $h = 0.1$

We have to find

$z_1 = z(0.1)$ and $z_2 = z(0.2)$

Now $z' = x - y^2$

$$z'' = 1 - 2y \cdot y'$$

$$z''' = -2[y \cdot y'' + (y')^2]$$

and so on.

By Taylor's series, for y_1 and z_1, we have

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \quad \text{... (1)}$$

$$\text{and } z_1 = z(0.1) = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \cdots \quad \text{... (2)}$$

We have

$$y_0 = 2$$
$$y_0' = x_0 + z_0 = 0 + 1 = 1$$
$$y_0'' = 1 + z_0' = 1 + x_0 - y_0^2 = 1 + 0 - 4 = 3$$
$$y''' = z_0'' = 1 - 2y_0 \cdot y_0' = 1 - 2(2)(1) = 1 - 4 = -3$$
$$y_0^{iv} = z_0''' = -2[y_0 \cdot y_0'' + (y_0')^2] = -2[2\cdot(-3) + 1] = 10$$

$$z_0 = 1$$
$$z_0' = x_0 - y_0^2 = 0 - 4 = -4$$
$$z_0'' = 1 - 2y_0 \cdot y_0' = 1 - 2(2)(1) = 1 - 4 = -3$$
$$z_0''' = -2[y_0 \cdot y_0'' + (y_0')^2] = 10$$

Substituting these in (1) and (2), we get

$$y_1 = y(0.1) = 2 + (0.1)(1) + \frac{0.01}{2}(-3) + \frac{0.001}{6}(-3) + \cdots$$
$$= 2 + 0.1 - 0.015 - 0.0005 + \cdots = 2.0845 \text{ (Correct to four decimal places)}$$

$$z_1 = z(0.1) = 1 + (0.1)(-4) + \frac{0.01}{2}(-3) + \frac{0.001}{6}(10) + \cdots$$
$$= 1 - 0.4 - 0.015 + 0.00166 + \cdots = 0.5867 \text{ (correct to four decimal places)}$$

By Taylor's series for y_2 and z_2, we have

$$y_2 = y(0.2) = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \cdots \quad \text{... (3)}$$

$$\text{and } z_2 = z(0.2) = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \cdots \quad \text{... (4)}$$

Now we have

$$x_1 = 0.1, \text{h} = 0.1$$
$$y_1 = 2.0845$$
$$y_1' = x_1 + z_1 = 0.1 + 0.5867 = 0.6867$$
$$y_1'' = 1 + z_1' = 1 + x_1 - y_1^2 = 1 + 0.1 - (2.0845)^2 = -3.2451$$
$$y_1''' = z_1'' = 1 - 2y_1 \cdot y_1' = 1 - 2\,(2.0845)\,(0.6867) = -1.8628$$

$$z_1 = 0.5867$$
$$z_1' = x_1 - y_1^2 = 0.1 - (2.0845)^2 = -4.2451$$
$$z_1''' = -2[y_1 \cdot y_1'' + (y_1')^2] = -2\,[(2.0845)\,(-3.2451) + (0.6867)^2] = -2\,[-6.7644 + 0.4716] = (-2)\,(-6.2928) = 12.5856$$

Substituting these values in (3) and (4), we get

$$y_2 = y(0.2) = 2.0845 + (0.1)(0.6867) + \frac{0.01}{2}(-3.2451) + \frac{0.001}{6}(-1.8628) + \cdots$$
$$= 2.0845 + 0.06867 - 0.0162 - 0.0003104 + \cdots$$
$$= 2.1367 \text{ (correct to four decimal places)}$$

$$z_2 = z(0.2) = 0.5867 + (0.1)(-4.2451) + \frac{0.01}{2}(-1.8628) + \frac{0.001}{6}(12.5856) + \cdots$$
$$= 0.5867 - 0.42451 - 0.009314 + 0.0020976 + \cdots$$
$$= 0.15497.$$

8.7 TAYLOR SERIES METHOD FOR SECOND ORDER DIFFERENTIAL EQUATION

Any differential equation of the second or higher order is best treated by transforming the given equation into a first order differential equation which can be solved as usual.

Consider, for example the second order differential equation:

$$y'' = f(x, y, y'), y(x_0) = y_0 \text{ and } y'(x_0) = y'_0$$

Substituting $\frac{dy}{dx} = z$ ... (1)

the above equation reduces to

$$z' = \frac{dz}{dx} = f(x, y, z) \quad \text{... (2)}$$

with initial conditions

$$y(x_0) = y_0 \quad \text{... (3)}$$

and $z(x_0) = z_0 = y'_0$ ... (4)

Now, we resort to solve (2) together with (3) and (4) using Taylor series method.

$$z_1 = z_0 + hz'_0 + \frac{h^2}{2!}z''_0 + \frac{h^3}{3!}z'''_0 + \ldots \quad \text{... (5)}$$

where $z_1 = z(x = x_1)$ and $x_1 - x_0 = h$

Now $y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \ldots$ becomes

$$y_1 = y_0 + hz_0 + \frac{h^2}{2!}z'_0 + \frac{h^3}{3!}z''_0 + \ldots, \text{ using (1)} \quad \text{... (6)}$$

Equation (2) gives z' and differentiating it, we get $z'', z''', \ldots$ Hence $z'_0, z''_0, z'''_0, \ldots$ can be obtained and using (6) and (5) we can get y_1 and z_1. Since we know y_1 and z_1 we can get $z'_1, z''_1, z'''_1, \ldots$ at (x_1, y_1).

Again using $z_2 = z_1 + \frac{h}{1!}z'_1 + \frac{h^2}{2!}z''_1 + \ldots$, we get z_2 and using $y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \ldots$, we get y_2 since we can calculate $y'_1, y''_1, \ldots$ from (1)

SOLVED EXAMPLES

Example 1 : Evaluate the values of $y(1.1)$ and $y(1.2)$ from $y'' + y^2 y' = x^3; y(1) = 1$, $y'(1) = 1$ by using Taylor series method. **[JNTU (A)June 2009 (Set No.4)]**

Solution : Given equation is $y'' + y^2 y' = x^3$ .... (1)

Put $y' = z$ so that (1) becomes $z' + y^2 z = x^3$

$\therefore\ z' = x^3 - y^2 z$ ... (2)

Given $y_0 = y(1) = 1$ and $z_0 = y_0^0 = 1$ .... (3)

Now we solve (2) given $z_0 = z(1) = 1$ and $x_0 = 1$.

Here $z_1 = z_0 + hz_0^0 + \dfrac{h^2}{2!} z_0'' + \ldots$ ... (4)

From (2), we have $z'' = 3x^2 - y^2 z' - 2zyy'$ and $y'' = z'$

$z''' = 6x - 2yz' - y^2 z'' - 2[yy' + yz'y' + yzy'']$ and $y''' = z''$

$\therefore z_0' = x_0^3 - y_0^2 z_0 = 1 - 1 = 0$

$z_0'' = 3x_0^2 - y_0^2 z_0' - 2z_0 y_0 y_0' = 3 - 0 - 2 = 1$

$z_0''' = 6x_0 - 2y_0 z_0' - y_0^2 z_0'' - 2[(y_0 y_0' + y_0 y_0' z_0' + y_0 z_0 y_0''] = 6 - 0 - 1 - 2[1 + 0 + 0] = 3$

Substituting in (4), we get

$$z_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(3) + \ldots = 1.1005$$

By Taylor series for y_1,

$$y_1 = y(0.1) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots$$

$$= 1 + (0.1) z_0 + \frac{0.01}{2!} z_0' + \frac{0.001}{3!} z_0'' + \ldots.$$

$$= 1 + 0.1 + 0 + \frac{0.001}{6} = 1.1002$$

Similarly $y_2 = y(x_2) = y_1 + \dfrac{h}{1!} y_1' + \dfrac{h^2}{2!} y_1'' + \ldots.$

$$= 1.1002 + \frac{0.1}{1} z_1 + \frac{0.01}{2} z_1' + \frac{0.001}{6} z_1'' + \ldots. \qquad ..... (5)$$

Now $z_1' = x_1^3 - y_1^2 z_1 = (0.1)^3 - (1.1002)^2 (1.1005) = -1.3311$

$z_1'' = 3x_1^2 - y_1^2 z_1' - 2z_1 y_1 y_1' = 3(0.01) - (1.1002)^2 (-1.3311)\ -2(1.1005)(1.1002)(1.1008)$

$= 0.03 + 1.6112 - 2.6656 = -1.0244$

Using in (5),

$$y_2 = 1.1002 + 0.1\,(1.1005) + \frac{0.01}{2}(-1.3311) + \frac{0.001}{6}(-1.0244) + ... = 1.2034$$

$\therefore\ y\,(0.1) = 1.1002$ and $y\,(0.2) = 1.2034$

EXERCISE 8.1

1. Given the differential equation $y' = x^2 + y^2,\ y\,(0) = 1.$ Obtain $y\,(0.25)$ and $y\,(0.5)$ by Taylor's series method.

2. Solve $\frac{dy}{dx} = xy + 1$ and $y\,(0) = 1$ using Taylor's series method and compute $y\,(0.1)$.

[JNTU (H) June 2010 (Set No.3)]

3. Evaluate $y\,(0.2)$ and $y\,(0.4)$ correct to four decimal places by Taylor's series method if $y(x)$ satisfies $y' = 1 - 2xy$ and $y\,(0) = 0.$ **[JNTU (H) Dec. 2011 (Set No. 3)]**

4. Employ Taylor's method to obtain approximate value of $y\,(1.1)$ and $y\,(1.2)$ for the differential equation $\frac{dy}{dx} = x + y,\ y\,(1) = 0$. Compare the final result with the value of the explicit solution. **[JNTU 2008 (Set No. 3)]**

5. Given the differential equation $\frac{dy}{dx} = x^2 y - 1,\ y\,(0) = 1$. Compute $y\,(0.1)$ by Taylor's series method. **[JNTU (A) June 2010 (Set No.1)]**

(OR) Find by Taylor's series method the value of y at $x = 0.1$ to five places of decimals from $\frac{dy}{dx} = x^2 y - 1,\ y\,(0) = 1$. **[JNTU (A) June 2011 (Set No. 1)]**

6. Solve $y' = xy^2 + y,\ y\,(0) = 1$ using Taylor's series method and compute $y\,(0.1)$ and $y\,(0.2)$.

7. Use Taylor's series method to solve the differential equation

$\frac{dy}{dx} = \frac{1}{x^2 + y},\ y\,(4) = 4$ and compute $y\,(4.2)$ and $y\,(4.4)$.

8. Using Taylor's series method, obtain the solution of $\frac{dy}{dx} = (x^3 + xy^2)\,e^{-x},\ y\,(0) = 1$ for $x = 0.1, 0.2, 0.3$ **[JNTU (A) June 2010, 2011 (Set No. 3)]**

9. Evaluate $y(0.4)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies $y' = xy + 1,\ y\,(0) = 1$ taking $h = 0.2$.

10. Find $y(\cdot 1)$, $y(\cdot 2)$ and $y(\cdot 3)$ using Taylor's series method given that $\frac{dy}{dx} = 1 - y, y(0) = 0.$

[JNTU 2007S, 2008S (Set No. 1)]

11. Find $y(0.1)$, $z(0.1)$ given $\frac{dy}{dx} = z - x,\ \frac{dz}{dx} = y + x$ and $y(0) = 1$, $z(0) = 1$ by using Taylor's series method.

12. Find $x(0.1)$, $y(0.1)$, $x(0.2)$, $y(0.2)$ given $\frac{dx}{dt} = ty + 1$, $\frac{dy}{dt} = -tx$ and $x(0) = 0$, $y(0) = 1$ by using Taylor's series method.

13. Estimate the value of $y(0.1)$ from $y'' = xy' + y, y(0) = 1, y'(0) = 0$ by using Taylor series method.

ANSWERS

1. 1.3333, 1.81667	**2.** 1.1053	**3.** 0.1948, 0.3599
4. 0.11033847, 0.2428016	**5.** 0.9003	**6.** 1.111, 1.248
7. 4.0098, 4.0185	**8.** 1.0047, 1.01812, 1.03995	
9. 2.588419	**10.** 0.095, 0.181, 0.2587	
11. 1.1003, 1.1100	**12.** 0.105, 0.9987, 0.21998, 0.9972	**13.** 1.005

8.8 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ ..... (1)

Given that $y = y_0$ for $x = x_0$ ..... (2)

It is required to obtain the solution of (1) subject to the condition (2).

The equation is $dy = f(x, y)\, dx$

Integrating (1) in the interval (x_0, x), we get

$$\int_{x=x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx$$

$$i.e., \quad (y)_{x=x_0}^{x} = \int_{x_0}^{x} f(x, y)dx \quad i.e., \quad y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)dx$$

$$\boxed{\text{or } \; y(x) = y_0 + \int_{x_0}^{x} f(x, y)dx} \qquad ..... (3)$$

We find that the R.H.S of (3) contains the unknown y under the integral sign. An equation of this kind is called an **integral equation** and it can be solved by a process of successive approximations.

Picard's method gives a sequence of functions $y^{(1)}(x), y^{(2)}(x), y^{(3)}(x), ...$ which form a sequence of approximations to y converging to $y(x)$.

To get the first approximation $y^{(1)}(x)$, put $y = y_0$ in the integrand of (3). We get

$$y^{(1)}(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx \qquad ..... (4)$$

Since $f(x, y_0)$ is a function of x, it is possible to evaluate the integral.

After getting the first approximation $y^{(1)}$ for y, we use this instead of y in $f(x, y)$ of (3) and then integrate to get the second approximation $y^{(2)}$ for y as

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx \qquad ..... (5)$$

Similarly, a third approximation $y^{(3)}$ for y is

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})dx \qquad \text{..... (6)}$$

Proceeding in this way, we get the n^{th} approximation $y^{(n)}$ for y as

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx, n = 1, 2, 3, \ldots \qquad \text{..... (7)}$$

$$\boxed{\text{or } y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx, \ n = 1, 2, 3, \ldots}$$

Equation (7) gives the general iterative formula for y. Iterations are repeated until the two successive approximations $y^{(i)}$ and $y^{(i-1)}$ are sufficiently close.

Equation (7) is known as Picard's iteration formula. It gives a sequence of approximations $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}, \ldots$ each giving a better result than the preceeding one. Since this method involves actual integration, sometimes it may not be possible to carry out the integration. This method is not convenient for computer based solutions.

This method is illustrated through the following examples.

SOLVED EXAMPLES

Example 1 : Find an approximate value of y for $x = 0.1$, $x = 0.2$, if $\frac{dy}{dx} = x + y$ and $y = 1$ at $x = 1$ using Picard's method. Check your answer with the exact particular solution.

Solution : Consider $\frac{dy}{dx} = f(x, y)$ where $y = y_0$ at $x = x_0$.

Here $f(x, y) = x + y$, $x_0 = 0$ and $y_0 = 1$.

By Picard's method, a sequence of successive approximations to y are given by

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

The integral equation representing the given problem is

$$y^{(n)} = 1 + \int_0^x (x + y^{(n-1)})\, dx \qquad \text{..... (1)}$$

Here $x = 0, y = 1$.

First approximation:

For $n = 1$, equation (1) becomes $y^{(1)} = 1 + \int_0^x (x + y_0)dx$

$$\therefore \ y^{(1)} = 1 + \int_0^x (x+1)\, dx = 1 + x + \frac{x^2}{2}$$

Second approximation:

For $n = 2$, equation (1) becomes

$$y^{(2)} = 1 + \int_0^x (x + y_1)\,dx$$

$$\therefore\ y^{(2)} = 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2} \right) \right] dx = 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2} \right) dx = 1 + x + x^2 + \frac{x^3}{6}$$

When $x = 0.1,\ y^{(2)} = 1 + 0.1 + 0.01 + \dfrac{0.001}{6} = 1.1101$

When $x = 0.2,\ y^{(2)} = 1 + 0.2 + 0.04 + \dfrac{0.008}{6} = 1.2413$

Third approximation:

Putting $n = 3$ in (1), we have

$$y^{(3)} = 1 + \int_0^x (x + y_2)\,dx$$

$$\therefore\ y^{(3)} = 1 + \int_0^x \left[x + \left(1 + x + x^2 + \frac{x^3}{6} \right) \right] dx = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} \qquad \ldots (2)$$

Thus y is found as a power series in x. It is clear that the resulting expressions are too big, as we proceed to higher approximations. Hence appropriate value of y is $y^{(3)}$. The method therefore has very limited applications.

For $x = 0.1,\ y = 1 + 0.1 + 0.01 + \dfrac{1}{3}(0.001) + \dfrac{1}{24}(0.001)$, using (2)

$$= 1 + 0.1 + 0.01 + 0.0003333 + 0.0000041$$

$$= 1.1103374 \approx 1.1103 \text{ (correct to 4 decimal places)}$$

For $x = 0.2,\ y = 1 + 0.2 + (0.2)^2 + \dfrac{(0.2)^3}{3} + \dfrac{(0.2)^4}{24}$, using (2)

$$= 1 + 0.2 + 0.04 + 0.0026666 + 0.00006666$$

$$= 1.242733 \approx 1.2427 \text{ (correct to 4 decimal places)}$$

We can get a better value by continuing the procedure and getting the subsequent approximations.

Note. To find y for $x = 0.2$ it will be better if we take $x = 0.1,\ y = 1.1103$ as the initial conditions and start again instead of simply putting $x = 0.2$ on R.H.S. of (2). In this case $y(0.2) = 1.2428$

ANALYTICAL SOLUTION :

The exact solution of $\frac{dy}{dx} = x + y,\ y(0) = 1$ can be found as follows.

The equation can be written as $\frac{dy}{dx} - y = x$

This is a linear equation in y.

Here $P = -1,\ Q = x \quad \therefore\ \text{I.F.} = e^{\int P dx} = e^{\int (-1)dx} = e^{-x}$

General solution is $y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$

i.e., $ye^{-x} = \int xe^{-x} dx + c = -(x+1)\, e^{-x} + c$ or $y = -(x+1) + ce^{x}$

When $x = 0,\ y = 1$ *i.e.,* $1 = -(0+1) + c$ or $c = 2$

Hence the particular solution of the equation is

$y = -(x+1) + 2e^{x} = 2e^{x} - x - 1$

For $x = 0.1,\ y = 2e^{0.1} - 0.1 - 1 = 2(1.1052) - 0.1 - 1 = 1.1104$

For $x = 0.2,\ y = 2e^{0.2} - 0.2 - 1 = 2\,(1.2214) - 0.2 - 1 = 1.2428$

These values of y agree well with the numerical solution got by Picard's method.

The above resutls are tabulated as follows :

x	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	Exact solution
0.1	1.105	1.1101	1.1103	1.1104
0.2	1.22	1.2413	1.2427	1.2428

Example 2 : Find the value of y for $x = 0.4$ by Picard's method, given that $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 0$. **[JNTU (A) June 2009 (Set No. 3), Dec. 2013 (Set No. 1, 3)]**

Solution : Here $f(x, y) = x^2 + y^2,\ x_0 = 0,\ y_0 = 0$

By Picard's method, $y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\, dx = 0 + \int_0^x (x^2 + y_0^2)\, dx = \int_0^x (x^2 + y_0^2)\, dx \quad .... (1)$

For the first approximation, replacing y_0 in the integrand by 0

$$\therefore\ y^{(1)} = \int_0^x (x^2 + 0)\, dx = \frac{x^3}{3}$$

For the second approximation, from (1)

$$y^{(2)} = \int_0^x \left[x^2 + \left(y^{(1)}\right)^2\right] dx = \int_0^x \left[x^2 + \left(\frac{x^3}{3}\right)^2\right] dx = \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx = \frac{x^3}{3} + \frac{x^7}{63}$$

Calculation of $y^{(3)}$ is tedious and hence approximate value is $y^{(2)}$.

For $x = 0.4$, $y = \dfrac{(0.4)^3}{3} + \dfrac{(0.4)^7}{63} = 0.021333 + 0.00026$

$= 0.0213663 \approx 0.0214$ (correct to 4 decimal places)

Example 3 : Solve $\dfrac{dy}{dx} = 2x - y,\ y(1) = 3$ by Picard's method.

Solution : Here $f(x, y) = 2x - y, x_0 = 1, y_0 = 3$

Using Picard's method, $y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$ *i.e.,* $y = 3 + \int_1^x (2x - y)\, dx$ ..... (1)

First approximation. Put $y = 3$ in $2x - y$, giving

$$y^{(1)} = 3 + \int_1^x (2x-3)\, dx = 3 + \left(2.\frac{x^2}{2} - 3x\right)_1^x = 3 + (x^2 - 3x)_1^x$$

$$= 3 + \left[(x^2 - 3x) - (1-3)\right] = 3 + (x^2 - 3x + 2) = x^2 - 3x + 5 \quad \text{..... (2)}$$

Second approximation. Put $y = x^2 - 3x + 5$ in $2x - y$, giving

$$y^{(2)} = 3 + \int_1^x \left[2x - (x^2 - 3x + 5)\right] dx = 3 + \int_1^x (-x^2 + 5x - 5)\, dx$$

$$= 3 + \left(\frac{-x^3}{3} + \frac{5x^2}{2} - 5x\right)_1^x = 3 + \left(\frac{-x^3}{3} + \frac{5x^2}{2} - 5x\right) - \left(-\frac{1}{3} + \frac{5}{2} - 5\right)$$

$$= \frac{35}{6} - 5x + \frac{5x^2}{2} - \frac{x^3}{3} \quad \text{..... (3)}$$

Third approximation. Put $y = \dfrac{35}{6} - 5x + \dfrac{5x^2}{2} - \dfrac{x^3}{3}$ in $2x - y$, giving

$$y^{(3)} = 3 + \int_1^x \left[2x - \left(\frac{35}{6} - 5x + \frac{5x^2}{2} - \frac{x^3}{3}\right)\right] dx = 3 + \int_1^x \left(\frac{-35}{6} + 7x - \frac{5x^2}{2} + \frac{x^3}{3}\right) dx$$

$$= 3 + \left(\frac{-35}{6}x + \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{12}\right)_1^x = 3 + \left(\frac{-35}{6}x + \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{12}\right) - \left(\frac{-35}{6} + \frac{7}{2} - \frac{5}{6} + \frac{1}{12}\right)$$

$$= \frac{71}{12} - \frac{35}{6}x + \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{12} \quad \text{.... (4)}$$

Calculation of $y^{(4)}$ is tedious and hence approximate value of y is $y^{(3)}$ which is given by (4).

Example 4 : Find the value of y at $x = 0.1$ by Picard's method, given that

$$\frac{dy}{dx} = \frac{y-x}{y+x}, \quad y(0) = 1$$ **[JNTU (A) June 2010, 2011 (Set No.1)]**

(or) Obtain $y(0.1)$ given $y' = \dfrac{y-x}{y+x}$, $y(0) = 1$ by Picard's method.

[JNTU Aug. 2008S, (K) June 2009 (Set No. 2)]

Solution : Here $f(x,y) = \dfrac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$.

By Picards method, $y = y_0 + \displaystyle\int_{x_0}^{x} f(x,y)\,dx = y_0 + \int_0^x \frac{y-x}{y+x}\,dx$ ..... (1)

For the first approximation, in the integrand on the R.H.S. of (1), y is replaced by its initial value 1.

$$\therefore \; y^{(1)} = 1 + \int_0^x \frac{1-x}{1+x}\,dx = 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right)dx$$

$$= 1 + \left[-x + 2\log(1+x)\right]_0^x = 1 + \left[-x + 2\log(1+x)\right] - \left(0 + 2\log(1+0)\right)$$

$$= 1 - x + 2\log(1+x) \qquad .... \; (2)$$

For the second approximation, from (1),

$$y^{(2)} = 1 + \int_0^x \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}\,dx = 1 + \int_0^x \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}\,dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1+2\log(1+x)}\right]dx = 1 + x - 2\int_0^x \frac{x}{1+2\log(1+x)}\,dx$$

which is very difficult to integrate.

Hence we use the first approximation (2) itself as the value of y.

$$\therefore \; y(x) = y^{(1)} = 1 - x + 2\log(1+x)$$

Putting $x = 0.1$, we obtain

$$y(0.1) = 1 - 0.1 + 2\log(1.1) = 1 - 0.1 + 0.1906203 = 1.0906204$$

≈ 1.0906 (correct to 4 decimals)

Example 5 : Given that $\dfrac{dy}{dx} = 1 + xy$ and $y(0) = 1$, compute $y(.1)$ and $y(.2)$ using Picards method. **[JNTU 2006 (Set No. 1)]**

Solution : Here $f(x, y) = 1 + xy$, $x_0 = 0$ and $y_0 = 1$

By Picard's method, a sequence of successive approximations to y are given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx$$

The integral equation representing the given problem is

$$y_n = 1 + \int_0^x (1 + x\, y_{n-1})\, dx$$

First approximation. we have

$$y_1 = 1 + \int_0^x (1 + x\, y_0)\, dx = 1 + \int_0^x (1 + x)\, dx$$

$$= 1 + \left(x + \frac{x^2}{2}\right)_0^x = 1 + x + \frac{x^2}{2}$$

Second approximation. We have

$$y_2 = 1 + \int_0^x (1 + x\, y_1)\, dx = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)_0^x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

Third approximation. We have

$$y_3 = 1 + \int_0^x (1 + x\, y_2)\, dx = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right] dx$$

$$= 1 + \int_0^x \left[1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right] dx$$

$$= 1 + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{12} + \frac{x^6}{48}\right]_0^x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{12} + \frac{x^6}{48} \qquad \ldots (1)$$

It is clear that the resulting expressions are too big, as we proceed to higher approximations. Hence we use the third approximation and taking $x = 0.1$ in (1), we obtain

$$y\,(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{12} + \frac{(0.1)^6}{48}$$

$$= 1 + 0.1 + 0.005 + 0.00033 + 0.0000125 + 0.00000025 + 0.00000002$$

$$= 1.10534$$

Putting $x = 0.2$ in (1), we obtain

$$y\,(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{8} + \frac{(0.2)^5}{12} + \frac{(0.2)^6}{48} = 1.222868$$

Example 6 : Solve $y' = y - x^2$, $y(0) = 1$, by Picard's method upto the fourth approximation. Hence find the value of $y(0.1)$, $y(0.2)$. **[JNTU 2008R, (A) Nov. 2010 (Set No. 1)]**

Solution : Here $f(x, y) = y - x^2$, $x_0 = 0$ and $y_0 = 1$.

By Picard's method, we have

$$y = y_0 + \int_{x_0}^{x} f(x, y)\,dx = 1 + \int_0^x (y - x^2)\,dx \qquad \text{... (1)}$$

First approximation : Put $y = 1$ in $y - x^2$, giving

$$y^{(1)} = 1 + \int_0^x (1 - x^2)\,dx = 1 + \left(x - \frac{x^3}{3}\right)_0^x = 1 + x - \frac{x^3}{3}$$

Second approximation : Put $y = 1 + x - \frac{x^3}{3}$ in $y - x^2$, giving

$$y^{(2)} = 1 + \int_0^x \left(1 + x - \frac{x^3}{3} - x^2\right)dx = 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}$$

Third approximation : Using this again in (1), we obtain

$$y^{(3)} = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} - x^2\right)dx$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}\right)dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}$$

Fourth approximation : Using this again in (1), we obtain

$$y^{(4)} = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60} - x^2\right)dx$$

$$= 1 + \int_0^x \left(1 + x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} - \frac{x^5}{60}\right)dx$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{60} - \frac{x^6}{360} \qquad \text{... (2)}$$

Calculation of $y(5)$ is tedious and hence approximate value of y is $y^{(4)}$ which is given by equation (2).

Putting $x = 0.1$ in (2), we obtain

$$y(0.1) = 1+0.1+\frac{(0.1)^2}{2}-\frac{(0.1)^3}{6}-\frac{(0.1)^4}{24}-\frac{(0.1)^5}{60}-\frac{(0.1)^6}{360}$$

$$= 1 + 0.1 + 0.005 - 0.0001666 - 0.00000416 - 0.000000166 - 0.00000000277$$

$$= 1.104829$$

Putting $x = 0.2$ in (2), we obtain

$$y(0.2) = 1+0.2+\frac{(0.2)^2}{2}-\frac{(0.2)^3}{6}-\frac{(0.2)^4}{24}-\frac{(0.2)^5}{60}-\frac{(0.2)^6}{360}$$

$$= 1 + 0.2 + 0.02 - 0.0013333 - 0.00006666 - 0.000005333 - 0.0000001777$$

$$= 1.21859$$

Note : In getting the value $y(0.2)$ we could have started with $x_0 = 0.1$ and $y_0 = 1.104829$ to get a closer value of $y(0.2)$.

We will adopt this procedure.

Now $$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

$$\therefore \quad y^{(1)} = 1.104829 + \int_{0.1}^{x} (y_0 - x^2)dx = 1.104829 + \left(y_0 x - \frac{x^3}{3} \right)_{0.1}^{x}$$

$$= 1.104829 + 1.104829x - \frac{x^3}{3} - (0.1)(1.104829) + \frac{(0.1)^3}{3}$$

$$= 0.994346 + 1.104829x - \frac{x^3}{3}$$

$$y^{(2)} = 1.104829 + \int_{0.1}^{x} \left(0.994346 + 1.104829x - \frac{x^3}{3} - x^2 \right) dx$$

$$= 1.104829 + \left(0.994346x + 1.104829\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} \right)_{0.1}^{x}$$

$$= 1.104829 + 0.994346(x - 0.1) + \frac{1.104829}{2}(x^2 - 0.01) - \frac{1}{12}[x^4 - (0.1)^4]$$

$$-\frac{1}{3}[x^3 - 0.001]$$

Hence $$y^{(2)}(0.2) = 1.104829 + 0.994346(0.2 - 0.1) + \frac{1.104829}{2}(0.04 - 0.01)$$

$$-\frac{1}{2}[(0.2)^4 - (0.1)^4] - \frac{1}{3}[(0.2)^3 - 0.001]$$

$$= 1.2177527$$

Example 7 : Obtain Picard's second approximate solution of the initial value problem

$\frac{dy}{dx} = \frac{x^2}{y^2+1}, y(0) = 0$. **[JNTU(A) June2010(Set No.3)]**

Solution : We have

$$f(x,y) = \frac{x^2}{y^2+1}, \quad x_0 = 0, y_0 = 0$$

By Picard's method, a sequence of successive approximations to y are given by

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

The integral equation representing the given problem is

$$y^{(n)} = \int_0^x \frac{x^2}{[y^{(n-1)}]^2+1} dx \qquad \dots (1)$$

First approximation:

For $n = 1$, equation (1) becomes

$$y^{(1)} = \int_0^x \frac{x^2}{y_0^2+1} dx = \int_0^x x^2 dx = \frac{x^3}{3}$$

Second approximation:

For $n = 2$, equation (2) becomes

$$y^{(2)} = \int_0^x \frac{x^2}{[y'']^2+1} dx = \int_0^x \frac{x^2}{\frac{x^6}{9}+1} dx = 9\int_0^x \frac{x^2}{x^6+9} dx$$

$$= 3\int_0^x \frac{3x^2}{(x^3)^2+3^2} dx = \tan^{-1}\left(\frac{x^3}{3}\right)$$

Example 8 : Solve $y' = x^2 + y^2, y(0) = 1$ using picard's method.

[JNTU (H) Jan. 2012 (Set No. 4)]

Solution : Here $f(x,y) = y' = x^2 + y^2$ and $x_0 = 0, y_0 = 1$.

By Picard's method, a sequence of successive approximations are given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \qquad \dots (1)$$

First Approximation :

For $n = 1$, equation (1) becomes

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 1 + \int_0^x f(x,1)\, dx$$

$$= 1 + \int_0^x (x^2 + 1)\, dx = 1 + \left(\frac{x^3}{3} + x\right)_0^x = 1 + x + \frac{x^3}{3}$$

Second Approximation :

For $n = 2$, equation (2) becomes

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx = 1 + \int_0^x f\left(x, 1 + x + \frac{x^3}{3}\right) dx$$

$$= 1 + \int_0^x \left[x^2 + \left(1 + x + \frac{x^3}{3}\right)^2\right] dx$$

$$= 1 + \int_0^x \left[x^2 + 1 + x^2 + \frac{x^6}{9} + 2x + \frac{2x^4}{3} + \frac{2x^3}{3}\right] dx$$

$$= 1 + \int_0^x \left[1 + 2x + 2x^2 + \frac{2x^3}{3} + \frac{2x^4}{3} + \frac{x^6}{9}\right]$$

$$= 1 + \left(x + 2.\frac{x^2}{2} + 2.\frac{x^3}{3} + \frac{2}{3}.\frac{x^4}{4} + \frac{2}{3}.\frac{x^5}{5} + \frac{1}{9}.\frac{x^7}{7}\right)_0^x$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

This is the approximate value of y (since higher approximations results in big expressions).

EXERCISE 8.2

1. Using Picard's method, obtain the solution of $\frac{dy}{dx} = x - y^2,\ y(0) = 1$ and compute $y(0.1)$ correct to four decimal places.
2. Solve $y' = x^2 + y^2,\ y(0) = 1$ using Picard's method. **[JNTU (H) Dec. 2011S (Set No. 4)]**
3. Solve $y' + y = e^x,\ y(0) = 0$ using Picard's method. **[JNTU (H) Dec. 2011S (Set No. 4)]**
4. Given $\frac{dy}{dx} = xe^y,\ y(0) = 0$ determine $y(0.1), y(0.2)$ and $y(1)$ using Picard's method. Compare the numerical solution obtained with exact solution.

5. Find the value of y for $x = 0.25, 0.5, 1$ by Picard's method, given that $\frac{dy}{dx} = \frac{x^2}{y^2+1}$, $y(0) = 0$.

6. Solve $\frac{dy}{dx} = 1 + 2xy$, $y(0) = 0$ by Picard's method.

7. Using Picard's method, obtain the solution of $y' = x + y^2$, $y(0) = 1$.

8. Find an approximate value of y for $x = 0.2$ if $\frac{dy}{dx} = x - y$, $y(0) = 1$ using Picard's method. Compare the numerical solution obtained with exact solution.

9. Find the successive approximate solution of the differential equation $y' = y, y(0) = 1$ by Picard's method and compare it with exact solution. **[JNTU (H) Dec. 2012]**

ANSWERS

1. $y = 1 - x + \frac{5}{2}x^2 - 2x^3 + x^4 - \frac{1}{4}x^5$; 0.9138

2. $y = 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7$

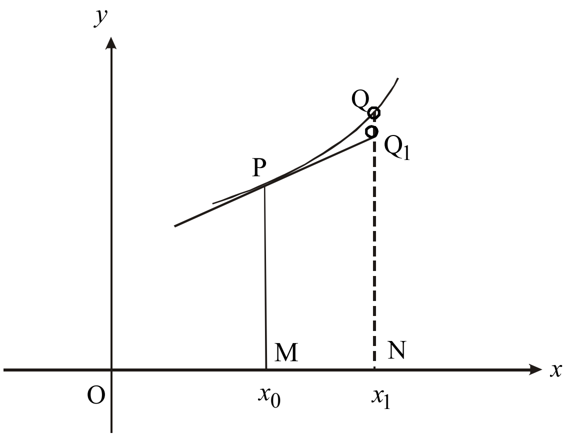
3. $y = e^x - \frac{x^2}{2} - \frac{x^4}{24} - 1$ 4. $y = e^{\frac{x^2}{2}-1}$; 0.005, 0.0202, 0.6487 5. 0.005, 0.042, 0.321

6. $y = x + \frac{2}{3}x^2 + \frac{4}{15}x^5$ 7. $y = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$

8. $y = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$; 0.83746

8.9 EULER'S METHOD

We have so far discussed the methods which yield the solution of a differential equation in the form of a function. We will now describe the methods which gives the solution in the form of a set of tabulated values.



Suppose we wish to solve the equation $\frac{dy}{dx} = f(x, y)$ subject to the condition that $y(x_0) = y_0$.

The solution of this differential equation subject to the given condition represents a curve $y = g(x)$ whose slope at any point (x, y) is $f(x, y)$. We note that the curve $y = g(x)$ passes through (x_0, y_0) and the slope of the curve at (x_0, y_0) is $f(x_0, y_0)$.

Suppose we want y at $x_1 = x_0 + h$ where h is 'small'. In the interval (x_0, x_1), Euler's method suggests that we replace the part of the curve PQ with the line segment PQ_1, (which is tangent at P to the curve) passing through $P(x_0, y_0)$ and having slope $f(x_0, y_0)$. The (approximate) value of $y(x_1)$ is taken to be Q_1N and not the exact QN (see figure).

Thus in the interval (x_0, x_1), we approximate the curve by the tangent at the point (x_0, y_0).

The equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0) = f(x_0, y_0)(x - x_0) \qquad \left(\because \frac{dy}{dx} = f(x, y)\right)$$

i.e., $y = y_0 + (x - x_0) f(x_0, y_0)$

This is the value of y on the tangent at $x = x_0$.

Then the value of y at $x = x_1$ is given by

$$y = y_0 + (x_1 - x_0) f(x_0, y_0) = y_0 + h f(x_0, y_0)$$

This gives the approximate value of y at $x = x_1$. We shall denote this by y_1.

After determining y_1 (approximately) at $x = x_1$, we will start with this (x_1, y_1), in place of (x_0, y_0) and find (x_2, y_2) where y_2 is the approximate value of y at $x = x_2$.

This is given by

$$y_2 = y_1 + h f(x_1, y_1)$$

Similarly, y at $x = x_3$ is given by

$$y_3 = y_2 + h f(x_2, y_2)$$

In general, we obtain a recursive relation as

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n), n = 0, 1, 2, \ldots}$$

This is known as Euler algorithm and can be used recursively to evaluate $y_1, y_2, \ldots$ (*i.e.,*) $y(x_1), y(x_2), \ldots$, starting from the initial condition $y(x_0) = y_0$. Note that this does not involve any derivatives. A new value of y is determined using the previous value of y as the initial condition. Note that the term $h f(x_n, y_n)$ represents the incremental value of y and $f(x_n, y_n)$ is the slope of y at (x_n, y_n).

To obtain reasonable accuracy with Euler's method, we have to take a smaller value of h. It may happen that the sequence of approximations may deviate considerably from the exact values of y. As such, the method is likely to give erroneous results as we move away from the initial point.

Hence we introduce a modification to this method and present this in the next section.

SOLVED EXAMPLES

Example 1 : Solve by Euler's method, $y' = x + y,\ y(0) = 1$ and find $y(0.3)$ taking step size $h = 0.1$. Compare the result obtained by this method with the result obtained by analytical method. **[JNTU (A) Dec. 2013 (Set No. 2)]**

Solution : Here $f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1$ and $h = 0.1$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ ..... (1)

Taking $n = 0,\ y_1 = y_0 + h\, f(x_0, y_0)$ *i.e.,* $y(0.1) = 1 + 0.1\, f(0,1) = 1 + 0.1(0+1) = 1.1$

Next, we have $x_1 = x_0 + h = 0 + 0.1 = 0.1$; Here $y_1 = 1.1$.

Hence $y_2 = y_1 + h\, f(x_1, y_1)$ [taking $n = 1$ in (1)]

$$= 1.1 + (0.1) f(0.1, 1.1) = 1.1 + (0.1)\,(0.1 + 1.1)$$

i.e., $y(0.2) = 1.1 + (0.1)\,(1.2) = 1.1 + 0.12 = 1.22$

Now $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$; $y_2 = 1.22$

$y_3 = y_2 + h\, f(x_2, y_2)$ [taking $n = 2$ in (1)]

$$= 1.22 + (0.1)\ f(0.2, 1.22) = 1.22 + (0.1)\,(0.2 + 1.22) = 1.22 + 0.142$$

i.e., $y(0.3) = 1.362$

To compare with exact solution :

Let us now find the exact solution of the given differential equation.

The equation is $\dfrac{dy}{dx} = x + y$ *i.e.,* $\dfrac{dy}{dx} - y = x$ ... (2)

which is linear in y. Comparing with $\dfrac{dy}{dx} + \text{P}y = \text{Q}$, $\text{P} = -1, \text{Q} = x$

The integrating factor (I.F) is $e^{\int \text{P}dx} = e^{-x}$

The general solution of (2) is $y\,(\text{I.F}) = \int \text{Q} \times (\text{I.F})\, dx + c$

i.e., $y\, e^{-x} = \int x e^{-x}\, dx + c = -(x+1)\, e^{-x} + c$

or $y = c\, e^{x} - (x+1)$

Given that when $x = 0,\ y = 1$.

So $1 = -(1+0) + ce^{0} = -1 + c \Rightarrow c = 2$

$\therefore$ Particular solution of (2) is $y = 2\, e^{x} - (x+1)$ ... (3)

Hence $y(0.1) = 2e^{0.1} - 0.1 - 1 = 2(1.10517) - 0.1 - 1 = 1.11034$, using (3)

$$y(0.2) = 2e^{0.2} - 0.2 - 1 = 2(1.2214) - 0.2 - 1 = 1.2428$$

$$y(0.3) = 2e^{0.3} - 0.3 - 1 = 2(1.34985) - 0.3 - 1 = 1.3997$$

We shall tabulate the results as follows:

x	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Exact y	1	1.11034	1.2428	1.3997

The values of y deviate from the exact value as x increases. (This indicates that the method is not that accurate. This necessitates a modification for the method.)

Note. If we compute $y(0.1)$ for the above problem by Taylor series of order 4,

$$y(0.1) = 1.110333$$

But by Euler method, $y(0.1) = 1.1$

Because of the restricted step size, Euler method is not commonly used for integration of differential equation. We could apply Taylor's algorithm of higher order to obtain better accuracy (higher the order-better the accuracy). However, the necessity of calculating higher derivatives makes Taylor's algorithm completely unsuitable for high speed computer for general integration purposes.

Example 2 : Using Euler's method, solve for y at $x = 2$ from $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, taking step size (*i*) $h = 0.5$ (*ii*) $h = 0.25$. **[JNTU (H) June 2010 (Set No.4)]**

Solution : Here $f(x, y) = 3x^2 + 1$, $x_0 = 1$, $y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n, y_n)$ .... (1)

(*i*) $h = 0.5$

Taking $n = 0$ in (1), we have

$y_1 = y_0 + h\, f(x_0, y_0)$ .... (2)

i.e., $y_1 = y(1.5) = 2 + 0.5\, f(1,2) = 2 + 0.5\left[3(1)^2 + 1\right] = 2 + 0.5\,(4) = 4$

Now $x_1 = x_0 + h = 1 + 0.5 = 1.5$

From (1), taking $n = 1$, we have

$$y_2 = y(2.0) = y_1 + h\, f(x_1, y_1) = 4 + 0.5\, f(1.5, 4) = 4 + 0.5\left[3\,(1.5)^2 + 1\right] = 7.875$$

(*ii*) $h = 0.25$

$$y_1 = y(1.25) = 2 + 0.25\, f(1,2) = 2 + 0.25\left[3\,(1)^2 + 1\right] = 3 \quad \text{[using (2)]}$$

$$y_2 = y(1.5) = 3 + 0.25\left[3\,(1.25)^2 + 1\right] = 4.42188$$

$$y_3 = y\,(1.75) = 4.42188 + 0.25\left[3\,(1.5)^2 + 1\right] = 6.35938$$

$$y_4 = y\,(2) = 6.35938 + 0.25\left[3\,(1.75)^2 + 1\right] = 8.90626$$

Notice the difference in values of $y\,(2)$ in both cases (*i.e.,* when $h = 0.5$ and when h = 0.25). The accuracy is improved significantly when h is reduced to 0.25. (Exact solution of the equation is $y = x^3 + x$ and with this $y\,(2) = y_2 = 10$.)

Example 3 : Given $y' = x^2 - y$, $y(0) = 1$, find correct to four decimal places the value of $y(0.1)$, by using Euler's method. **[JNTU 2008, (H) June 2009 (Set No.4)]**

Solution : We have $f(x, y) = x^2 - y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

By Euler's method,

$$y_{n+1} = y_n + h\,f\,(x_n, y_n)$$

$$\therefore\ y_1 = y_0 + h\,f\,(x_0, y_0) = 1 + (0,1)\,.f\,(0,1)$$

$$= 1 + (0.1)\,(0 - 1) = 1 - 0.1 = 0.9$$

i.e, $y(0.1) = 0.9$.

Example 4 : Use Eulers method to find $y\,(0.1), y\,(0.2)$ given $y' = (x^3 + xy^2)\,e^{-x}, y\,(0) = 1$.

[JNTU 2008S (Set No.2)]

Solution : Here $h = 0.1, f\,(x, y) = (x^3 + xy^2)\,e^{-x}, x_0 = 0, y_0 = 1, x_1 = 0.1, x_2 = 0.2$

By Euler's algorithm,

$$y_1 = y_0 + hf\,(x_0, y_0) = y_0 + h\,(x_0^3 + x_0 y_0^2)\,e^{-x_0} = 1 + (0.1)\,(0 + 0)\,e^{-0} = 1$$

$$y_2 = y_1 + hf\,(x_1, y_1) = y_1 + h(x_1^3 + x_1 y_1^2)\,e^{-x_1}$$

$$= 1 + (0.1)\,[(0.1)^3 + (0.1)\,(1)^2]\,e^{-0.1} = 1 + (0.1)\,(0.101)\,(0.9048) = 1.0091$$

Example 5 : Using Euler's method, solve numerically the equation, $y' = x + y,\ y\,(0) = 1$, for $x = 0.0\,(0.2)\,1.0$. Check your answer with the exact solution.

[JNTU (A)June 2009 (Set No.2)]

Solution : Here $h = 0.2, f\,(x, y) = x + y$ and $x_0 = 0, y_0 = 1$

Euler's algorithm is $y_{n+1} = y_n + hf\,(x_n, y_n)$ .... (1)

Taking $n = 0,\ y_1 = y_0 + hf\,(x_0, y_0) = y_0 + h\,(x_0 + y_0) = 1 + (0.2)\,(0 + 1) = 1.2$

Next we have $x_1 = x_0 + h = 0 + 0.2 = 0.2$ and $y_1 = 1.2$

Hence $y_2 = y_1 + hf\,(x_1, y_1)$ [Taking $n = 1$ in (1)]

$$= 1.2 + (0.2)\,(x_1 + y_1) = 1.2 + (0.2)\,(0.2 + 1.2) = 1.48$$

Now $x_2 = x_1 + h = 0.2 + 0.2 = 0.4, y_2 = 1.48$

$y_3 = y_2 + hf(x_2, y_2)$ [Taking $n = 2$ in (1)]

$= 1.48 + (0.2)(x_2 + y_2) = 1.48 + (0.2)(0.4 + 1.48) = 1.856$

$x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

Similarly $y_4 = y_3 + hf(x_3, y_3) = y_3 + h(x_3 + y_3)$

$= 1.856 + (0.2)(0.6 + 1.856) = 2.3472$

Now $x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

$y_5 = y_4 + hf(x_4, y_4) = y_4 + h(x_4 + y_4) = 2.3472 + (0.2)(0.8 + 2.3472) = 2.97664$

To compare with exact solution :

Let us now find the exact solution of the given differential equation.

Given equation can be written as $\frac{dy}{dx} - y = x$ which is linear in y.

I. F. $= e^{\int P\,dx} = e^{-x}$

Hence the general solution is $ye^{-x} = \int xe^{-x}dx + c = -(x+1)e^{-x} + c$

or $y = ce^{x} - (x+1)$

Given that when $x = 0, y = 1$

$\Rightarrow 1 = -(1+0) + ce^{0} = -1 + c \Rightarrow c = 2$

$\therefore$ The (particular) solution of the given equation is $y = 2e^{x} - (x+1)$

Hence $y(0.2) = 2e^{0.2} - (0.2+1) = 1.2428$

$y(0.4) = 2e^{0.4} - (0.4+1) = 1.5836$

$y(0.6) = 2e^{0.6} - (0.6+1) = 2.0442$

$y(0.8) = 2e^{0.8} - (0.8+1) = 2.6511$

$y(1.0) = 2e - (1+1) = 3.4366$

We shall tabulate the results as follows :

x	0	0.2	0.4	0.6	0.8	1.0
Euler y	1	1.2	1.48	1.856	2.3472	2.94664
Exact y	1	1.2428	1.5836	2.0442	2.6511	3.4366

We notice that the values of y deviates from the exact values as x increases.

Example 6 : Solve numerically using Eulers method $y' = y^2 + x$, $y(0) = 1$. Find $y(0.1)$ and $y(0.2)$. **[JNTU(K) May 2010 (Set No.1)]**

Solution : Given $y' = y^2 + x$, $y(0) = 1$

Here $f(x, y) = y^2 + x$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$ and $x_2 = 0.2$

We have to find y_1 and y_2. Take $h = 0.1$

By Euler algorithm,

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad \text{... (1)}$$

Taking $n = 0$ in (1), we have

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1 f(0,1) = 1 + 0.1(1) = 1.1$$

i.e., $y(0.1) = 1.1$

Now $x_1 = x_0 + h = 0 + 0.1 = 0.1$

From (1), taking $n = 1$, we have

$$y_2 = y_1 + hf(x_1, y_1) = 1.1 + 0.1 f(0.1, 1.1)$$

i.e., $y(0.2) = 1.1 + (0.1)[(1.1)^2 + 0.1] = 1.1 + 0.131 = 1.231$

Example 7 : Compute y at $x = 0.25$ by Euler's method given $y' = 2xy, y(0) = 1$.

[JNTU(K) May 2010 (Set No.2)]

Solution : Given $y' = 2xy$ and $y(0) = 1$

Here $f(x, y) = 2xy, \; x_0 = 0, \; y_0 = 1$

We have to find y_1 i.e., $y(0.25)$. Take $h = 0.25$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(2x_0 y_0)$$

i.e., $y(0.25) = 1 + (0.25)(0) = 1$

Exact Solution: Solving $\dfrac{dy}{dx} = 2xy$, we get

$$\log y = x^2 + c \quad i.e., \; y = e^{x^2} + c$$

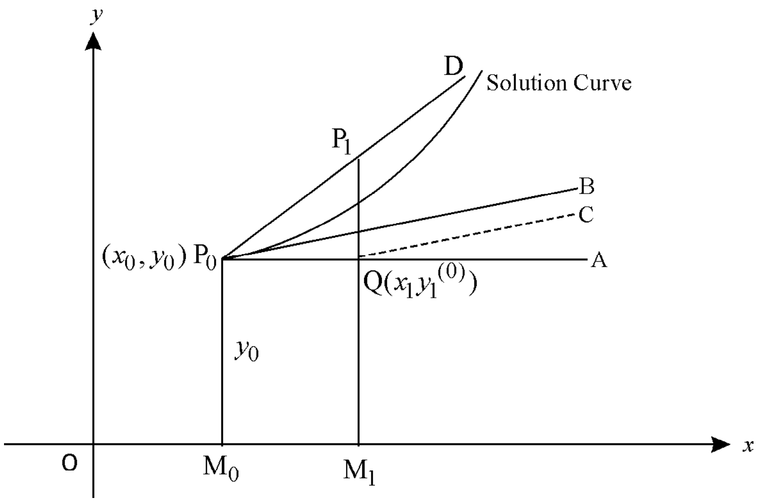
using $y(0) = 1$, $1 = e^0 + c \Rightarrow c = 0$

$\therefore$ The solution of $y' = 2xy, \; y(0) = 1$ is $y = e^{x^2}$

Hence $y(0.25) = e^{(0.25)^2} = 1.0645$

Note: We notice that the value of y deviates from the exact value. Hence we require to use Modified Euler method for the above problem.

8.10 MODIFIED EULER'S METHOD



Let P_0A be the tangent at (x_0, y_0) to the solution curve. In the interval (x_0, x_1), by Euler's method, we approximate the curve by the tangent P_0A.

$$\therefore \; y_1^{(0)} = y_0 + h\, f(x_0, y_0) \qquad \ldots.. \; (1)$$

The point $(x_1, y_1^{(0)})$ is on the line P_0A. Let it be Q_1. At Q_1 we compute the slope of the curve *i.e.,* the value of $\frac{dy}{dx}$ and draw the line P_0B with that slope.

$\therefore$ Slope of $Q_1B = f\,(x_1, y_1^{(0)})$.

Now take the average of the two slopes at $f(x_0, y_0)$ and $f\,(x_1, y_1^{(0)})$ and get the line Q_1C. Hence slope of $Q_1C = \dfrac{f(x_0, y_0) + f\,(x_1, y_1^{(0)})}{2}$

Now draw a line P_0D through $P_0(x_0, y_0)$ parallel to Q_1C and this line is taken as approximation to the curve in the interval (x_0, x_1).

The equation of the line P_0D is $y - y_0 = \dfrac{f(x_0, y_0) + f\,(x_1, y_1^{(0)})}{2}(x - x_0) \qquad \ldots.. \; (2)$

The point at which this line intersects the ordinate $x = x_1 = x_0 + h$ is taken to be the point (x_1, y_1).

Putting $x = x_1 = x_0 + h$ in (2), we obtain

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \qquad \ldots.. \; (3)$$

A further improvement to this is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \text{ etc.,}$$

In general, we have the formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)}) \right], \; n = 0, 1, 2, \ldots$$

where $y_1^{(n)}$ is the n th approximation to y_1.

The procedure will be terminated depending on the accuracy required. If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are almost equal, we stop there and take $y_1 \approx y_1^{(k)}$.

Now we start with this (x_1, y_1) and find (x_2, y_2).

$\therefore \; y_2^{(0)} = y_1 + h\, f(x_0 + h, y_1)$, from (1)

Better approximation $y_2^{(1)}$ is obtained from (3)

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0 + h, y_1) + f(x_2, y_2^{(0)}) \right]$$

We repeat this step until y_2 becomes stationary. Then we proceed to estimate y_3 as above so on.

Note. The difference between Euler's method and Modified Euler's method is that in the latter we take the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(0)})$ instead of the slope at (x_0, y_0) in the former method. Further we repeat this procedure until difference between $y_1^{(k+1)}$ and $y_1^{(k)}$ is negligible.

SUMMARY OF THE METHOD:

$\frac{dy}{dx} = f(x, y)$ given that $y = y_0$ at $x = x_0$.

To find $y(x_1) = y_1$ at $x = x_1 = x_0 + h$:

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)}) \right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

....

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(k)}) \right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value as y_1.

Now we have $\frac{dy}{dx} = f(x, y)$ with $y = y_1$ at $x = x_1$.

To get $y_2 = y(x_2) = y(x_1 + h)$ we use the above procedure again.

SOLVED EXAMPLES

Example 1 : Using modified Euler method find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x$, $y(0) = 0$.

(or) Solve numerically $y' = y + e^x$, $y(0) = 0$ for $x = 0.2, 0.4$ by modified Euler's method.

[JNTU(K) June 2009 (Set No. 3)]

Solution : Here $f(x, y) = y + e^x$, $x_0 = 0$, $y_0 = 0$ and $h = 0.2$

To find y_1 *i.e.* $y(0.02)$

Using Euler's formula $y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 0 + (0.2)\, f(0,0) = (0.2)\,(0 + e^0) = 0.2$

Now $x_1 = 0.2$ and $f(x_1, y_1^{(0)}) = f(0.2, 0.2) = 0.2 + e^{0.2} = 0.2 + 1.2214 = 1.4214$

We have $y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0, 1, 2, ...$ ... (1)

First Approximation to y_1 :

The value of $y_1^{(1)}$ can therefore be determined by using the formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \qquad \text{[Putting } n = 0 \text{ in (1)]}$$

$$= 0 + \frac{0.2}{2}\left[1 + 1.4214\right] = 0.24214$$

Second Approximation to y_1 :

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \qquad \text{[Putting } n = 1 \text{ in (1)]}$$

$$= 0 + \frac{0.2}{2}\left[1 + f(0.2, 0.24214)\right] = (0.1)\left[1 + (0.24214 + e^{0.2})\right] = 0.2463$$

Third Approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \qquad \text{[Putting } n = 2 \text{ in (1)]}$$

$$= 0 + \frac{0.2}{2}\left[1 + f(0.2, 0.2463)\right] = (0.1)\left[1 + (0.2463 + e^{0.2})\right]$$

$= 0.2468$, correct to 4 decimal places

Fourth Approximation to y_1 :

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \qquad \text{[Putting } n = 3 \text{ in (1)]}$$

$$= (0.1)\left[1 + f(0.2, 0.2468)\right] = (0.1)\left[1 + (0.2468 + 1.2214)\right] = 0.2468$$

Since the values of $y_1^{(3)}$ and $y_1^{(4)}$ are equal, we take

$y_1 = y(0.2) = 0.2468$ approximately.

To find y_2 *i.e.* $y(0.4)$

We take $x_1 = 0.2,\ y_1 = 0.2468$ and $x_2 = 0.4,\ h = 0.2$

$$\therefore\ f(x_1, y_1) = f(0.2, 0.2468) = 0.2468 + e^{0.2} = 0.2468 + 1.2214 = 1.4682$$

Euler's formula gives

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1)$$

$$= 0.2468 + (0.2)(1.4682) = 0.5404$$

First approximation to y_2 is given by

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + f(0.4, 0.5404)\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + (0.5404 + e^{0.4})\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + (0.5404 + 1.4918)\right] = 0.5968$$

A better approximation $y_2^{(2)}$ is obtained from

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + f(0.4, 0.5968)\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + (0.5968 + 1.4918)\right]$$

$= 0.6025$, correct to four decimal places.

Next approximation $y_2^{(3)}$ is given by

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + f(0.4, 0.6025)\right]$$

$$= 0.603$$

Next approximation $y_2^{(4)}$ is given by

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + f(0.4, 0.603)\right]$$

$$= 0.2468 + (0.1)\left[1.4682 + (0.603 + 1.4918)\right] = 0.6031$$

Next approximation $y_2^{(5)}$ is given by

$$y_2^{(5)} = 0.2468 + (0.1)\left[1.4682 + (0.6031 + 1.4918)\right]$$
$$= 0.6031, \text{ correct to four decimal places}$$

Since $y_2^{(4)} = y_2^{(5)} = 0.6031$, we have $y_2 = y(0.4) = 0.6031$

Hence we conclude that the value of y when $x = 0.2$ is 0.2468 and the value of y when $x = 0.4$ is 0.6031.

Example 2 : Solve the differential equation : $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$ by modified Euler's method and compute $y(0.02)$ and $y(0.04)$.

Solution : Here $f(x, y) = x^2 + y,\ x_0 = 0,\ y_0 = 1$ and $h = 0.02$

To find y_1 *i.e.* $y(0.02)$

$$f(x_0, y_0) = f(0,1) = 0 + 1 = 1$$

Using Euler's formula $y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 1 + (0.02)\,(1) = 1.02$

Now $x_1 = 0.02$ and $f(x_1, y_1^{(0)}) = f(0.02, 1.02) = (0.02)^2 + 1.02 = 1.0204$

First Approximation to y_1

The value of $y_1^{(1)}$ can be calculated by using the formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + (0.01)\left[1 + 1.0204\right] = 1.0202$$

Second Approximation to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1 + (0.01)\left[1 + f(0.02, 1.0202)\right]$$
$$= 1 + (0.01)\left[1 + (0.02)^2 + 1.0202\right] = 1.0202$$

Since $y_1^{(1)} = y_1^{(2)} = 1.0202$, therefore, we take $y_1 = y(0.02) = 1.0202$

To find y_2 *i.e.* $y(0.04)$

Now $x_1 = 0.02,\ y_1 = 1.0202,\ x_2 = 0.04$ and $h = 0.02$

$$\therefore\ f(x_1, y_1) = f(0.02, 1.0202) = (0.02)^2 + 1.0202 = 1.0206$$

Euler's formula gives $y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 1.0202 + 0.02(1.0206) = 1.0406$

First Approximation to y_2 :

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$
$$= 1.0202 + (0.01)\left[1.0206 + f(0.04, 1.0406)\right]$$
$$= 1.0202 + (0.01)\left[1.0206 + (0.04)^2 + 1.0406\right] = 1.0408$$

Second Approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.0202 + (0.01)[1.0206 + f(0.04, 1.0408)]$$

$$= 1.0202 + (0.01)\left[1.0206 + (0.04)^2 + 1.0408\right] = 1.0408$$

Since $y_2^{(0)} = y_2^{(2)} = 1.0408$, we take $y_2 = y(0.04) = 1.0408$

Hence we conclude that the value of y when $x = 0.02$ is 1.0202 and the value of y when $x = 0.04$ is 1.0408.

Example 3 : Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ compute $y(0.1)$ in steps of 0.02 using Euler's modified method.

Solution : Here $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$ and $h = 0.02$

To find y_1 *i.e.* y (0.02)

$$f(x_0, y_0) = f(0, 1) = \frac{1-0}{1+0} = 1$$

Using Euler's formula $y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 1 + (0.02)\,(1) = 1.02$

Now $x_1 = 0.02$ and $f(x_1, y_1^{(0)}) = f(0.02, 1.02) = \frac{1.02 - 0.02}{1.02 + 0.02} = 0.9615$

First Approximation to y_1 :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + (0.01)[1 + 0.9615] = 1.0196$$

Second Approximation to y_1 :

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + (0.01)[1 + f(0.02, 1.0196)] = 1 + (0.01)\left[1 + \frac{1.0196 - 0.02}{1.0196 + 0.02}\right]$$

$$= 1 + (0.01)\left[1 + \frac{0.9996}{1.0396}\right] = 1.0196$$

Since $y_1^{(1)} = y_1^{(2)}$, we take $y_1 = y(0.02) = 1.0196$.

To find y_2 i.e. $y(0.04)$

Now $x_1 = 0.02$, $y_1 = 1.0196$, $x_2 = 0.04$ and $h = 0.02$

$$\therefore\ f(x_1, y_1) = f(0.02, 1.0196) = \frac{1.0196 - 0.02}{1.0196 + 0.02} = \frac{0.9996}{1.0396} = 0.9615$$

Euler's formula gives

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 1.0196 + (0.02)\,(0.9615) = 1.0388$$

First Approximation to y_2 :

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.0196 + (0.01)\left[0.9615 + f(0.04, 1.0388)\right]$$

$$= 1.0196 + (0.01)\left[0.9615 + \frac{1.0388 - 0.04}{1.0388 + 0.04}\right]$$

$= 1.0196 + (0.01)[0.9615 + 0.9258] = 1.0385$, correct to four decimal places.

Second Approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 1.0196 + (0.01)\left[0.9615 + f(0.04, 1.0385)\right]$$

$$= 1.0196 + (0.01)\left[0.9615 + \frac{1.0385 - 0.04}{1.0385 + 0.04}\right]$$

$= 1.0196 + (0.01)\left[0.9615 + \dfrac{0.9985}{1.0785}\right] = 1.0385$, correct to four decimal places

Since $y_2^{(1)} = y_2^{(2)} = 1.0385$, we take $y_2 = y(0.04) = 1.0385$

To find y_3 i.e. $y(0.06)$

Now $x_2 = 0.04$, $y_2 = 1.0385$, $x_3 = 0.06$ and $h = 0.02$

$$\therefore\ f(x_2, y_2) = f(0.04, 1.0385) = \frac{1.0385 - 0.04}{1.0385 + 0.04} = \frac{0.9985}{1.0785} = 0.9258$$

Euler's formula gives

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2) = 1.0385 + (0.02)\,(0.9258) = 1.057$$

First Approximation to y_3 :

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(0)})\right] = 1.0385 + (0.01)\left[0.9258 + f(0.06, 1.057)\right]$$

$$= 1.0385 + (0.01)\left[0.9258 + \frac{1.057 - 0.06}{1.057 + 0.06}\right]$$

$$= 1.0385 + (0.01)\left[0.9258 + \frac{0.997}{1.117}\right] = 1.057$$, correct to four decimal places.

Since $y_3^{(0)} = y_3^{(1)} = 1.057$, we take $y_3 = y(0.06) = 1.057$

To find y_4 *i.e.* y (0.08)

Now $x_3 = 0.06,\ y_3 = 1.057,\ x_4 = 0.08$ and $h = 0.02$

$$\therefore\ f(x_3, y_3) = f(0.06, 1.057) = \frac{1.057 - 0.06}{1.057 + 0.06} = \frac{0.997}{1.117} = 0.8926$$

Euler's formula gives

$$y_4^{(0)} = y_3 + h\ f(x_3, y_3) = 1.057 + (0.02)\ (0.8926) = 1.0748$$

First Approximation to y_4 :

$$y_4^{(1)} = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^{(0)})\right]$$

$$= 1.057 + (0.01)\left[0.8926 + f(0.08, 1.0748)\right]$$

$$= 1.057 + (0.01)\left[0.8926 + \frac{1.0748 - 0.08}{1.0748 + 0.08}\right]$$

$$= 1.057 + (0.01)\left[0.8926 + \frac{0.9948}{1.1548}\right] = 1.0745$$

Second Approximation to y_4 :

$$y_4^{(2)} = y_3 + \frac{h}{2}\left[f(x_3, y_3) + f(x_4, y_4^{(1)})\right]$$

$$= 1.057 + (0.01)\left[0.8926 + f(0.08, 1.0745)\right]$$

$$= 1.057 + (0.01)\left[0.8926 + \frac{1.0745 - 0.08}{1.0745 + 0.08}\right]$$

$$= 1.057 + (0.01)\left[0.8926 + \frac{0.9945}{1.1545}\right] = 1.0745$$

Since $y_4^{(1)} = y_4^{(2)}$, therefore we take $y_4 = y\ (0.08) = 1.0745$

To find y_5 *i.e.* y (0.1)

Now $x_4 = 0.08,\ y_4 = 1.0745,\ x_5 = 0.1$ and $h = 0.02$

$$\therefore\ f(x_4, y_4) = f(0.08, 1.0745) = \frac{1.0745 - 0.08}{1.0745 + 0.08} = \frac{0.9945}{1.1545} = 0.8614$$

Euler's formula gives

$$y_5^{(0)} = y_4 + h\ f(x_4, y_4)$$

$$= 1.0745 + (0.02)\ f(0.1, 1.0745) = 1.0745 + (0.02)\ (0.8614) = 1.0917$$

First Approximation to y_5 :

$$y_5^{(1)} = y_4 + \frac{h}{2}\left[f(x_4, y_4) + f(x_5, y_5^{(0)})\right] = 1.0745 + (0.01)\left[0.8614 + f(0.1, 1.0917)\right]$$

$$= 1.0745 + (0.01)\left[0.8614 + \frac{1.0917 - 0.1}{1.0917 + 0.1}\right] = 1.0745 + (0.01)\left[0.8614 + \frac{0.9917}{1.1917}\right]$$

$$= 1.0914$$

Second Approximation to y_5 :

$$y_5^{(2)} = y_4 + \frac{h}{2}\left[f(x_4, y_4) + f(x_5, y_5^{(1)})\right] = 1.0745 + (0.01)\left[0.8614 + f(0.1, 1.0914)\right]$$

$$= 1.0745 + (0.01)\left[0.8614 + \frac{1.0914 - 0.1}{1.0914 + 0.1}\right] = 1.0745 + (0.01)\left[0.8614 + \frac{0.9914}{1.1914}\right]$$

$$= 1.0914$$

Since $y_5^{(1)} = y_5^{(2)} = 1.0914$, we take $y_5 = y(0.1) = 1.0914$

Hence $y(0.1) = 1.0914$ (approximately)

The results are tabulated as follows :

x	new y
0.0	0.9615
0.02	1.0196
0.02	1.0196
0.02	1.0388
0.04	1.0385
0.04	1.0385
0.04	1.057
0.06	1.057
0.06	1.0748
0.08	1.0745
0.08	1.0745
0.08	1.0917
0.1	1.0914
0.1	1.0914

Example 4 : Given $\frac{dy}{dx} = -xy^2$, $y(0) = 2$. Compute $y(0.2)$ in steps of 0.1, using modified Euler's method. **[JNTU (H) Dec. 2011S (Set No. 1)]**

Solution : Here $\frac{dy}{dx} = f(x, y) = -xy^2$, $x_0 = 0$, $y_0 = 2$ and $h = 0.1$

To find y_1 i.e. $y(0.1)$

$$f(x_0, y_0) = f(0, 2) = 0$$

Using Euler's formula

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0) = 2 + (0.1)\,(0) = 2$$

Now $x_1 = 0.1$ and $f(x_1, y_1^{(0)}) = f(0.1, 2) = -(0.1)\,(4) = -0.4$

First Approximation to y_1 :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 2 + (0.05)\left[0 + (-0.4)\right] = 2 - 0.02 = 1.98$$

Second Approximation to y_1 :

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 2 + (0.05)\left[0 + f(0.1, 1.98)\right]$$

$$= 2 + (0.05)\left[-(0.1)\,(1.98)^2\right] = 2 - 0.019602 = 1.9804$$

Third approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 2 + (0.05)\,[0 + f(0.1, 1.9804)]$$

$$= 2 + (0.05)\left[-(0.1)\,(1.9804)^2\right] = 2 - 0.196099 = 1.9804$$

Since $y_1^{(2)} = y_1^{(3)} = 1.9804$, therefore $y_1 = y\,(0.1) = 1.9804$

To find y_2 i.e. $y\,(0.2)$

Now $x_1 = 0.1,\ y_1 = 1.9804,\ x_2 = 0.2$ and $h = 0.1$

$\therefore\ f(x_1, y_1) = f(0.1, 1.9804) = -(0.1)\,(1.9804)^2 = -0.3922$

Euler's formula gives

$$y_2^{(0)} = y_1 + h\, f(x_1, y_1) = 1.9804 + (0.1)\,(-0.3922) = 1.94118$$

First Approximation to y_2 :

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.9804 + (0.05)\left[-0.3922 + f(0.2, 1.94118)\right]$$

$$= 1.9804 + (0.05)\left[-0.3922 + (-0.2)\,(1.94118)^2\right] = 1.9804 - 0.05729 = 1.9231$$

Second Approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.9804 + (0.05)\left[-0.3922 + f(0.2, 1.9231)\right]$$

$$= 1.9804 + (0.05)\left[-0.3922 + (-0.2)\,(1.9231)^2\right]$$

$$= 1.9804 - 0.056934 = 1.9238$$

Third approximation to y_2 :

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$
$$= 1.9804 + (0.05)[-0.3922 + f(0.2, 1.9238)]$$
$$= 1.9804 + (0.05)\left[-0.3922 + (-0.2)(1.9238)^2\right]$$
$$= 1.9804 - 0.05662 = 1.9238$$

Since $y_2^{(2)} = y_2^{(3)} = 1.9238$, therefore we take $y_2 = y(0.2) = 1.9238$

Hence we conclude that the value of y when $x = 0.2$ is 1.9238

The results are tabulated as shown below.

x	new y
0.0	2
0.1	1.98
0.1	1.9804
0.1	1.9804
0.1	1.94118
0.2	1.9231
0.2	1.9238
0.2	1.9238

Example 5 : Find the solution of $\frac{dy}{dx} = x - y$, $y(0) = 1$ at $x = 0.1, 0.2, 0.3, 0.4$ and 0.5 using modified Euler's method. **[JNTU 2006, 2007S (Set No. 3)]**

Solution : We have

$$\frac{dy}{dx} = f(x, y) = x - y \text{ and } x_0 = 0,\ y_0 = 1,\ h = 0.1$$

To find y_1 i.e. y (0.1)

$$f(x_0, y_0) = f(0, 1) = 0 - 1 = -1$$

Using Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.1)(-1) = 1 - 0.1 = 0.9$$

Now $x_1 = 0.1$ and $f(x_1, y_1^{(0)}) = f(0.1, 0.9) = 0.1 - 0.9 = -0.8$

First Approximation to y_1 :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1 + \frac{0.1}{2}[-1 - 0.8] = 1 - 0.09 = 0.91$$

Second Approximation to y_1 :

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + \frac{0.1}{2}[-1 + f(0.1, 0.91)]$$

$$= 1 + \frac{0.1}{2}[-1 + (0.1 - 0.91)] = 1 + \frac{0.1}{2}[-1.81] = 1 - 0.0905 = 0.9095$$

Third Approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}\ [-1 + f(0.1, 0.9095)] = 1 + \frac{0.1}{2}\ [-1 + (0.1 - 0.9095)]$$

$$= 1 + \frac{0.1}{2}\ (-1.8095) = 1 - 0.090475 = 0.909525$$

Since $y_1^{(2)} = y_1^{(3)} = 0.9095$, we have

$$y_1 = y\,(0.1) = 0.\,9095$$

To find y_2 i.e. $y(0.2)$

Now $x_1 = 0.1$, $y_1 = 0.9095$, $x_2 = 0.2$ and $h = 0.1$

$\therefore f(x_1, y_1) = f(0.1, 0.9095) = 0.1 - 0.9095 = -0.8095$

Euler's formula gives

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.9095 + (0.1)\ (-0.\,8095)$$

$$= 0.82855$$

First Approximation to y_2 :

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 0.9095 + \frac{0.1}{2}[-0.8095 + f(0.2, 0.82855] = 0.9095 + \frac{0.1}{2}\ (-0.8095 - 0.62855)$$

$$= 0.9095 - 0.0719 = 0.8376$$

Second Approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.9095 + \frac{0.1}{2}[-0.8095 + f(0.2, 0.8376)]$$

$$= 0.9095 + \frac{0.1}{2}\ [-0.8095 - 0.6376] = 0.9095 - 0.072355$$

$$= 0.837145$$

Third Approximation to y_2 :

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.9095 + \frac{0.1}{2}\ [-0.8095 + f(0.2 - 0.837145)] = 0.9095 + \frac{0.1}{2}\ (-1.\,446645)$$

$$= 0.9095 - 0.07233 = 0.83716$$

Since $y_2^{(2)} = y_2^{(3)} = 0.8371$, we have

$$y_2 = y\,(0.2) = 0.8371$$

To find y_3 i.e. $y(0.3)$

Now $x_2 = 0.2$, $y_2 = 0.8371$, $x_3 = 0.3$ and $h = 0.1$

$\therefore f(x_2, y_2) = f(0.2, 0.8371) = 0.2 - 0.8371 = -0.6371$

Euler's formula gives

$$y_3^{(0)} = y_2 + hf(x_2, y_2) = 0.8371 + 0.1\,(-0.6371) = 0.7734$$

First Approximation to y_3 :

$$y_3^{(1)} = y_2 + \frac{h}{2}\,[f(x_2, y_2) + f(x_3, y_3^{(0)})]$$
$$= 0.8371 + \frac{0.1}{2}[-0.6371 + f(0.3, 0.7734)] = 0.8371 + \frac{0.1}{2}(-0.6371 - 0.4734)$$
$$= 0.8371 - 0.0555 = 0.7816$$

Second Approximation to y_3 :

$$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(1)})]$$
$$= 0.8371 + \frac{0.1}{2}[-0.6371 + f(0.3, 0.7816)]$$
$$= 0.8371 + \frac{0.1}{2}(-1.1187) = 0.8371 - 0.056$$
$$= 0.7811$$

Third Approximation to y_3 :

$$y_3^{(3)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(2)})]$$
$$= 0.8371 + \frac{0.1}{2}[-0.6371 + f(0.3, 0.7811)]$$
$$= 0.8371 + \frac{0.1}{2}(-1.1182) = 0.8371 - 0.05591 = 0.7812$$

Fourth Approximation to y_3 :

$$y_3^{(4)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(3)})]$$

$$= 0.8371 + \frac{0.1}{2}[-0.3671 + f(0.3, 0.7812)] = 0.8371 - 0.0559 = 0.7812$$

Since $y_3^{(3)} = y_3^{(4)}$, we have

$$y_3 = y(0.3) = 0.7812$$

To find y_4 i.e. $y(0.4)$

Now $x_3 = 0.3$, $y_3 = 0.7812$, $x_4 = 0.4$ and $h = 0.1$

$\therefore f(x_3, y_3) = f(0.3, 0.7812) = 0.3 - 0.7812 = -0.4812$

Euler's formula gives

$$y_4^{(0)} = y_3 + hf(x_3, y_3) = 0.7812 + 0.1\,(-0.4812) = 0.7331$$

First Approximation to y_4 :

$$y_4^{(1)} = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_3, y_4^{(0)})]$$

$$= 0.7812 + \frac{0.1}{2}\ [-0.4812 + f(0.3, 0.7331)] = 0.7812 + \frac{0.1}{2}\ (-0.4812 - 0.4331)$$

$$= 0.7812 - 0.0457 = 0.7355$$

Second Approximation to y_4 :

$$y_4^{(2)} = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_3, y_4^{(1)})]$$

$$= 0.7812 + \frac{0.1}{2}[-0.4812 + f(0.3, 0.7355)] = 0.7812 + \frac{0.1}{2}\ [-0.4812 - 0.4355]$$

$$= 0.7812 - 0.0458 = 0.7354$$

Third Approximation to y_4 :

$$y_4^{(3)} = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_3, y_4^{(2)})]$$

$$= 0.7812 + \frac{0.1}{2}\ [-0.4812 + f(0.3, 0.7354)] = 0.7812 + \frac{0.1}{2}(-0.4812 - 0.4354)$$

$$= 0.7812 - 0.0458 = 0.7354$$

Since $y_4^{(2)} = y_4^{(3)} = 0.7354$, we have

$$y_4 = y(0.4) = 0.7354$$

To find y_5 i.e $y(0.5)$

Now $x_4 = 0.4$, $y_4 = 0.7354$, $x_5 = 0.5$ and $h = 0.1$

$\therefore f(x_4, y_4) = f(0.4, 0.7354) = 0.4 - 0.7354 = -0.3354$

Euler's formula gives

$$y_5^{(0)} = y_4 + hf(x_4, y_4) = 0.7354 + 0.1\ (-0.3354)$$

$$= 0.7354 - 0.03354 = 0.70186.$$

First Approximation to y_5 :

$$y_5^{(1)} = y_4 + \frac{h}{2}[f(x_4, y_4) + f(x_4, y_5^{(0)})]$$

$$= 0.7354 + \frac{0.1}{2}[-0.3354 + f(0.4, 0.70186)] = 0.7354 + \frac{0.1}{2}(-0.3354 - 0.30186)$$

$$= 0.7354 - 0.03186 = 0.7035.$$

Second Approximation to y_5 :

$$y_5^{(2)} = y_4 + \frac{h}{2}[f(x_4, y_4) + f(x_4, y_5^{(1)})]$$

$$= 0.7354 + \frac{0.1}{2}[-0.3354 + f(0.4, 0.7035)] = 0.7354 + \frac{0.1}{2}(-0.3354 - 0.30354)$$

$$= 0.7354 - 0.0319 = 0.7035$$

Since $y_5^{(1)} = y_5^{(2)} = 0.7035$, therefore, $y_5 = y(0.5) = 0.7035$

The above results are tabulated as shown below.

x	new y	x	new y
0.0	0.9	0.3	0.7331
0.1	0.91	0.4	0.7355
0.1	0.9095	0.4	0.7354
0.1	0.9095	0.4	0.7354
0.1	0.82855	0.4	0.70186
0.2	0.8376	0.5	0.7035
0.2	0.837145	0.5	0.7035
0.2	0.83716		
0.2	0.7734		
0.3	0.7816		
0.3	0.7811		
0.3	0.7812		
0.3	0.7812		

Example 6 : Find $y(0.1)$ and $y(0.2)$ using Euler's modified formula given that $\frac{dy}{dx} = x^2 - y,\ y(0) = 1.$ **[JNTU 2006, (H) June 2011 (Set No. 4)]**

Solution : Here $\frac{dy}{dx} = f(x, y) = x^2 - y,\ x_0 = 0,\ y_0 = 1$ and $h = 0.1$

To find y_1 i.e. $y(0.1)$

$f(x_0, y_0) = f(0,1) = 0 - 1 = -1$

Using Euler's formula

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0) = 1 + (0.1)(-1) = 0.9.$$

Now $x_1 = 0.1$ and $f(x_1, y_1^{(0)}) = f(0.1, 0.9) = (0.1)^2 - 0.9 = -0.89$

First Approximation to y_1 :

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.1}{2}[-1 + f(0.1, -0.89)] = 1 + \frac{0.1}{2}(-1 + 0.9) = 1 - 0.005 = 0.995$$

Second Approximation to y_1 :

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.1}{2}[-1 + f(0.1, 0.995)] = 1 + \frac{0.1}{2}(-1 - 0.985)$$

$$= 1 - 0.09925 = 0.90075$$

Third Approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.1}{2}[-1 + f(0.1, 0.90075)] = 1 + \frac{0.1}{2}(-1 - 0.89075) = 0.90546$$

Fourth Approximation to y_1 :

$$y_1^{(4)} = y_0 + \frac{h}{4}[f(x_0, y_0) + f(x_1, y_1^{(3)})] = 1 + \frac{0.1}{2}[-1 + f(0.1, 0.90546)]$$

$$= 1 + \frac{0.1}{2}(-1 - 0.89546) = 1 - 0.09477 = 0.90523$$

Fifth Approximation to y_1 :

$$y_1^{(5)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(4)})] = 1 + \frac{0.1}{2}[-1 + f(0.1, 0.90523)]$$

$$= 1 + \frac{0.1}{2}(-1 - 0.89523) = 1 - 0.09476 = 0.90523$$

Since $y_1^{(4)} = y_1^{(5)}$, we have

$y_1 = y(0.1) = 0.90523$

To find y_2 i.e. y (0.2)

Now $x_1 = 0.1$, $y_1 = 0.90523$, $x_2 = 0.2$ and $h = 0.1$

$\therefore f(x_1, y_1) = f(0.1, 0.90523) = 0.01 - 0.90523 = -0.8952$

Euler's formula gives

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.90523 + 0.1 f(0.1, 0.90523)$$

$$= 0.90523 + 0.1(-0.8952) = 0.8157$$

First Approximation to y_2 :

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})] = 0.90523 + \frac{0.1}{2}[-0.8952 + f(0.2, 0.8157)]$$

$$= 0.90523 + \frac{0.1}{2}[-0.8952 + (0.04 - 0.8157)] = 090523 - 0.0835 = 0.8217$$

Second Approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 0.90523 + \frac{0.1}{2}[-0.8952 + f(0.2, 0.8217)]$$

$$= 0.90523 + \frac{0.1}{2}[-0.8952 + (0.04 - 0.8217)]$$

$$= 0.90523 - 0.08385 = 0.8214.$$

Third Approximation to y_2 :

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.90523 + \frac{0.1}{2}[-0.8952 + f(0.2, 0.8214)]$$

$$= 0.90523 + \frac{0.1}{2}[-0.8952 + (0.04 - 0.8214)] = 0.90523 - 0.08383 = 0.8214$$

Since $y_2^{(2)} = y_2^{(3)} = 0.8214$, therefore $y_2 = y\,(0.2) = 0.8214$.
The above results are tabulated as follows :

x	new y
0.0	0.9
0.1	0.995
0.1	0.90075
0.1	0.90546
0.1	0.90523
0.1	0.90523
0.1	0.8157
0.2	0.8217
0.2	0.8214
0.3	0.8214

Example 7 : Given $y' = x + \sin y$, $y(0) = 1$ compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. **[JNTU 2007S, 2007, (H) Dec. 2011S (Set No. 2)]**

Solution : Here $f(x, y) = x + \sin y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.2$.

Using Euler's formula,

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + 0.2 f(0,1) = 1 + 0.2(0 + \sin 1) = 1.163$$

Now $x_1 = 0.2$ and $f(x_1, y_1^{(0)}) = f(0.2, 1.163) = 0.2 + \sin(1.163) = 1.12$

We have $y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0,1,2,.....$ ... (1)

To find y_1 i.e. $y(0.2)$

The value of $y_1^{(1)}$ can be determined by using the formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \text{ [Putting } n = 0 \text{ in (1)]}$$

$$= 1 + \frac{0.2}{2}\,[\sin 1 + 1.12] = 1.1961$$

Repeating the procedure again

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \quad \text{[Putting } n = 1 \text{ in (1)]}$$

$$= 1 + \frac{0.2}{2}[\sin 1 + 1.1961] = 1.2038$$

Next approximation to y_1 is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] \qquad \text{[Putting } n = 2 \text{ in (1)]}$$

$$= 1 + \frac{0.2}{2}\left[0.8414 + 1.2038\right] = 1.20452$$

Next approximation to y_1 is given by

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] \qquad \text{[Putting } n = 3 \text{ in (1)]}$$

$$= 1 + \frac{0.2}{2}\left[0.8414 + 1.20452\right] = 1.2046$$

Similarly $y_1^{(5)} = 1 + \frac{0.2}{2}\left[0.8414 + 1.2046\right] = 1.2046$

Since $y_1^{(4)} = y_1^{(5)} = 1.2046$, therefore, $y_1 = y(0.2) = 1.2046$.

To find y_2 i.e. $y(0.4)$

We take $x_1 = 0.2, y_1 = 1.2046$ and $x_2 = 0.4, h = 0.2$

$\therefore\ f(x_1, y_1) = f(0.2, 1.2046) = 0.2 + \sin(1.2046) = 1.1337$

Euler's formula gives

$$y_2^{(0)} = y_1 + hf(x_1, y_1) = 1.2046 + (0.2)(1.1337) = 1.4313$$

First approximation to y_2 is given by

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.2046 + (0.1)\,[1.1337 + 1.4313] = 1.4611$$

Next approximation to $y_2^{(2)}$ is given by

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.2046 + (0.1)\,[1.1337 + 1.4611] = 1.4641$$

Similarly $y_2^{(3)} = 1.2046 + (0.1)\,[1.1337 + 1.4641] = 1.4644$

and $y_2^{(4)} = 1.2046 + (0.1)\,[1.1337 + 1.4644] = 1.4644$

Since $y_2^{(3)} = y_2^{(4)} = 1.4644$, therefore, $y_2 = y(0.4) = 1.4644$.

The above results are tabulated as follows :

x	New y
0.0	1.163
0.2	1.1961
0.2	1.2038
0.2	1.20452
0.2	1.2046
0.2	1.2046
0.4	1.4313
0.4	1.4611
0.4	1.4641
0.4	1.4644
0.4	1.4644

Example 8 : Using modified Euler's method, find an approximate value of y when $x = 1.3$ given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, y(1) = 1$ **[JNTU (A) Dec. 2011]**

Solution : Here $\frac{dy}{dx} = f(x, y) = \frac{1}{x^2} - \frac{y}{x} = \frac{1 - xy}{x^2}, x_0 = 1, y_0 = 1$ and $h = 0.3$

To find y_1 *i.e.,* $y(1.3)$

$$f(x_0, y_0) = f(1,1) = \frac{1-1}{1} = 0$$

Using Euler's formula,

$$y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1 + 0.3(0) = 1$$

Now $x_1 = x_0 + h = 1.3$ and $f(x_1, y_1^{(0)}) = f(1.3, 1) = \frac{1 - 1.3(1)}{(1.3)^2} = -0.1775$

First Approximation to y_1

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, -0.1775)] = 1 + \frac{0.3}{2}\left[\frac{1 - (1.3)(-0.1775)}{(1.3)^2}\right]$$

$$= 1 + \frac{0.3}{2}\left(\frac{1 + 0.23075}{1.69}\right) = 1.1092$$

Second Approximation to y_1

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, 1.1092)] = 1 + \frac{0.3}{2}\left[\frac{1-(1.3)\,(1.1092)}{(1.3)^2}\right]$$

$$= 1 + \frac{0.3}{2}\left(\frac{-0.44196}{1.69}\right) = 0.961$$

Third Approximation to y_1

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, 0.961)] = 1 + \frac{0.3}{2}\left[\frac{1-(1.3)\,(0.961)}{(1.3)^2}\right]$$

$$= 1 + \frac{0.3(-0.2493)}{3.38} = 0.9778$$

Fourth Approximation to y_1

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, 0.778)] = 1 + \frac{0.3}{2}\left[\frac{1-(1.3)\,(0.778)}{(1.3)^2}\right]$$

$$= 0.999$$

Fifth Approximation to y_1

$$y_1^{(5)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(4)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, 0.999)] = 1 + \frac{0.3}{2}\left[\frac{1-(1.3)\,(0.999)}{(1.3)^2}\right]$$

$$= 0.9735$$

Sixth Approximation to y_1

$$y_1^{(6)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(5)})\right]$$

$$= 1 + \frac{0.3}{2}[0 + f(1.3, 0.9735)] = 1 + \frac{0.3}{2}\left[\frac{1-(1.3)\,(0.999)}{(1.3)^2}\right]$$

$$= 0.9735$$

Since $y_1^{(5)} = y_1^{(6)} = 0.9735$, therefore, $y_1 = y(0.3) = 0.9735$

Example 9 : Solve $\frac{dy}{dx} = 1 - y, y(0) = 0$ in the range $0 \le x \le 0.3$ by taking $h = 0.1$ by the modified Euler's method. **[JNTU (A) May 2012 (Set No. 4)]**

Solution : Here $\frac{dy}{dx} = 1 - y$. So $f(x, y) = y' = 1 - y$ and $x_0 = 0, y_0 = 0, h = 0.1$

To find y_1 *i.e.*, $y(0.1)$

$f(x_0, y_0) = f(0, 0) = 1 - 0 = 1$

Using Euler's formula,

$y_1^{(0)} = y_0 + hf(x_0, y_0) = 0 + (0.1)(1) = 0.1$

Now $x_1 = x_0 + h = 0.1$ and $f(x_1, y_1^{(0)}) = f(0.1, 0.1) = 1 - 0.1 = 0.9$

First Approximation to y_1

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 0 + \frac{0.1}{2}[1 + 0.9] = 0.095$$

Second Approximation to y_1

$f(x_1, y_1^{(1)}) = f(0.1, 0.095) = 1 - 0.095 = 0.905$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 0 + \frac{0.1}{2}[1 + 0.905] = 0.09525$$

Third Approximation to y_1

$f(x_1, y_1^{(2)}) = f(0.1, 0.09525) = 1 - 0.09525 = 0.90475$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 0 + \frac{0.1}{2}[1 + 0.90475] = 0.0952375$$

Since $y_1^{(2)} = y_1^{(3)}$, we take $y_1 = y(0.1) = 0.0952$

To find y_2 *i.e.*, $y(0.2)$

Now $x_1 = 0.1, y_1 = 0.0952, x_2 = 0.2$ and $h = 0.1$

$\therefore f(x_1, y_1) = f(0.1, 0.0952) = 1 - 0.0952 = 0.9048$

Using Euler's formula gives

$y_2^{(0)} = y_1 + hf(x_1, y_1) = 0.0952 + (0.1)(0.9048) = 0.18568$

First Approximation to y_2

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] = 0.0952 + \frac{0.1}{2}[0.9048 + f(0.2, 0.18568)]$$

$$= 0.0952 + \frac{0.1}{2}[0.9048 + 1 - 0.18568] = 0.18115$$

Second Approximation to y_2

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] = 0.0952 + \frac{0.1}{2}[0.9048 + f(0.2, 0.18115)]$$

$$= 0.0952 + \frac{0.1}{2}[0.9048 + 1 - 0.18115] = 0.18138$$

Third Approximation to y_2

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] = 0.0952 + \frac{0.1}{2}[0.9048 + f(0.2, 0.18138)]$$

$$= 0.0952 + \frac{0.1}{2}[0.9048 + 1 - 0.18138] = 0.18137$$

Since $y_2^{(2)} = y_2^{(3)}$, we take $y_2 = y(0.2) = 0.1814$

To find y_3 *i.e.,* $y(0.3)$

Now $x_2 = 0.2, y_2 = 0.1814, x_2 = 0.3$ and $h = 0.1$

$\therefore f(x_2, y_2) = f(0.2, 0.1814) = 1 - 0.1814 = 0.8186$

Using Euler's formula gives

$$y_3^{(0)} = y_2 + hf(x_2, y_2) = 0.1814 + (0.1)(0.8186) = 0.26326$$

First Approximation to y_3

$$y_3^{(1)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(0)})\right] = 0.1814 + \frac{0.1}{2}[0.8186 + f(0.3, 0.26326)]$$

$$= 0.1814 + \frac{0.1}{2}[0.8186 + 1 - 0.26326] = 0.259167$$

Second Approximation to y_3

$$y_3^{(2)} = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right]$$

$$= 0.1814 + \frac{0.1}{2}[0.8186 + f(0.3, 0.259167)]$$

$$= 0.1814 + \frac{0.1}{2}[0.8186 + 1 - 0.259167] = 0.2593716$$

Similarly $y_3^{(3)} = 0.2593614$

Since $y_3^{(1)} = y_3^{(2)}$, we take $y_3 = y(0.3) = 0.2593$

Exact Solution :

$$\frac{dy}{dx} = 1 - y \Rightarrow \frac{dy}{1-y} = dx \text{ (Variables Separable)}$$

Integrating, $\int \frac{dy}{1-y} = \int dx + c \Rightarrow \log(1-y) = -x + c$. $\therefore\ 1 - y = e^{-x}c$

At $x = 0, y = 0$, we get $1 - 0 = c \Rightarrow c = 1$

$\therefore\ 1 - y = e^{-x}$ or $y = 1 - e^{-x}$

Now $y(0.1) = 1 - e^{-0.1} = 0.09516258$

$y(0.2) = 1 - e^{-0.2} = 0.18126927$

$y(0.3) = 1 - e^{-0.3} = 0.259181779$

The results are tabulated as follows :

x	Modified Euler	Exact solution
0.1	0.0952	0.09516
0.2	0.1814	0.18127
0.3	0.2593	0.25918

EXERCISE 8.3

1. Given $\frac{dy}{dx} = xy,\ y(0) = 1$, find $y(0.1)$ using Euler's method. **[JNTU (H) June 2011 (Set No. 2)]**
2. Solve by Euler's method, $y' + y = 0$ given $y(0) = 1$ and find $y(0.04)$ taking step size $h = 0.01$.
3. Given that $\frac{dy}{dx} = 3x^2 + y,\ y(0) = 4$ compute $y(0.25)$ and $y(0.5)$ using Euler's method.
4. Solve by Euler's method $\frac{dy}{dx} = \frac{2y}{x}$ given $y(1) = 2$ and find $y(2)$. **[JNTU (H) June 2011 (Set No. 2)]**
5. Using Euler's method, find the value of y when $x = 0.6$ given that $y(0) = 0$ and $y' = 1 - 2xy$.
6. Using Euler's method, solve for y at $x = 0.1$ from $y' = x + y + xy,\ y(0) = 1$ taking step size $h = 0.025$.
7. Using Euler's method, find an approximate value of y corresponding to $x = 2.5$ given that $\frac{dy}{dx} = \frac{x+y}{y}$ and $y = 2$ when $x = 2$.
8. Solve the first order differential equation $\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1$ and estimate $y(0.1)$ using Euler's method (5 steps).

9. Using Modified Euler's method, find the value of y when $x = 0.1, 0.2$ and 0.3 given that $y' = 1 - y,\ y(0) = 0$.

10. Find $y(0.5)$, $y(1)$ and $y(1.5)$, given that $y' = 4 - 2x,\ y(0) = 2$ with $h = 0.5$ using Modified Euler's method. **[JNTU 2007S, (H) June 2011 (Set No. 3)]**

11. Using Modified Euler's formula, solve for $y(0.1)$ given that $y' = x^2 + y,\ y(0) = 1$.

12. Using Modified Euler's method, obtain $y(0.25)$ given $y' = 2xy,\ y(0) = 1$.

13. Given that $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$, determine $y(0.1)$ and $y(0.2)$ using Modified Euler's method.

14. Solve the following by Modified Euler's method :

$\frac{dy}{dx} = x^2 + y,\ y(0) = 1$ at $x = 0.02, 0.04$ and 0.06 with $h = 0.02$.

15. Find $y(1.2)$ and $y(1.4)$ by Modified Euler's method given $y' = \log(x + y),\ y(0) = 2$ taking $h = 0.2$.

16. If $\frac{dy}{dx} = x + \sqrt{y}$, use Modified Euler's method to approximate y when $x = 0.6$ in steps of 0.2 given that $y = 1$ at $x = 0$.

17. Using Modified Euler's method, find an approximate value of y when $x = 0.3$, given that $\frac{dy}{dx} = x + y,\ y(0) = 1$. **[JNTU (A) June 2010, 2011, May 2012 (Set No.2)]**

18. Solve the differential equation :

$\frac{dy}{dx} = 2 + \sqrt{xy},\ y(1) = 1$ by Modified Euler's method and obtain y at $x = 2$ in steps of 0.2.

(or) Given $\frac{dy}{dx} - \sqrt{xy} = 2,\ y(1) = 1$ find the value of $y(2)$ in steps of 0.2 using modified Euler's method . **[JNTU (A) June 2013 (Set No. 1)]**

19. Solve numerically $y' = y + e^x,\ y(0) = 0$ for $x = 0.2, 0.4$ by Euler's method. **[JNTU (K) June 2009 (Set No.3)]**

20. Using modified Euler's method, find an approximate value of y when $x = 1.3$, given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}.y(1) = 1$ **[JNTU (A) Dec. 2010]**

ANSWERS

(1) 1.0611 **(2)** 0.9606 **(4)** 7.2 **(5)** 0.4748

(6) 1.1448 **(7)** 3.028 **(8)** 1.0928 **(9)** 0.095, 0.18098, 0.2588

(10) 2.25, 2.45, 2.55 **(11)** 1.1055 **(12)** 1.0625

(13) 1.17266, 1.25066 **(14)** 1.0202, 1.0408, 1.0619

(15) 2.5351, 2.6531 **(16)** 1.8861 **(17)** 1.4004

(18) 5.051 **(19)** 0.24214, 0.59116

8.11 RUNGE - KUTTA METHODS

The previous methods used for numerical solution of initial value problems are restricted due to either slow convergence or due to labour involved in finding the higher order derivatives, especially in Taylor's series method. But, Runge-Kutta (R - K) method does not require the determination of higher order derivatives and give greater accuracy. These methods possess the advantage of requiring only the function values at some selected points on the subinterval. They agree with Taylor's series solution upto the terms of h^r where r differs from method to method and is known as the order of that Runge-Kutta method. Hence Runge-Kutta methods are known by their order.

Euler's method and Modified Euler's method are the Runge - Kutta methods of first and second order respectively.

These methods are called single-step methods, since they require only the value of y_i to determine y_{i+1}. Thus, R-K methods are self-starting.

Merits and Demerits of Runge-Kutta Method :

The principal advantage of R-K method is the self starting feature and consequently the ease of programming. One disadvantage of R-K method is the requirement that the function $f(x, y)$ must be evaluated for several slightly different values of x and y in every step of the function. This repeated determination of $f(x, y)$ may result in a less efficient method with respect to computing time than other methods of comparable accuracy in which previously determined values of the dependent variable are used in subsequent steps.

1. First - order Runge - Kutta method :

We know that, by Euler's method,

$$y_1 = y_0 + h\, f(x_0, y_0) = y_0 + h\, y_0' \qquad [\because y' = f(x, y)]$$

Expanding L. H. S. by Taylor's series, we get $y_1 = y(x_0 + h) = y_0 + h\, y_0' + \frac{h^2}{2!} y_0'' + ...$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, Euler's method is the R - K method of the first order.

2. Second - order Runge - Kutta method :

The modified Euler's mehod gives

$$y_1^{(1)} = y_0 + h\, f(x_0, y_0)$$

and $$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + h\, f_0)\right] \quad .... (1)$$

where $f_0 = f(x_0, y_0)$

If we now set

$k_1 = h\, f_0 ; \qquad k_2 = h\, f(x_0 + h, y_0 + k_1)$

then equation (1) becomes $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

which is the second order Runge-Kutta formula.

$\therefore$ The seconder order Runge-Kutta formula is

$$\boxed{\begin{aligned} &\mathbf{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)} \\ &\text{where } k_1 = h\, f(x_0, y_0) \\ &\text{and } k_2 = h\, f(x_0 + h, y_0 + k_1) \end{aligned}}$$

Since the derivations of third and fourth order R - K methods are tedious, we state them below for use.

3. Third-order Runge-Kutta method :

The third order R - K method is defined by the equation

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = h\, f(x_0, y_0)$;

$$k_2 = h\, f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1)$$

and $k_3 = h\, f(x_0 + h,\ y_0 + 2k_2 - k_1)$

4. Fourth - order Runge-Kutta method : [JNTU (A) June 2011 (Set No.4)]

This method is most commonly used in practice and is often referred to as 'Runge-Kutta method' only without any reference to the order.

Working Rule: To solve the differential equation $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$

by Runge - Kutta fourth order method:

Calculate successively

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1)$$

$$k_3 = h\, f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2)$$

and $k_4 = h\, f(x_0 + h,\ y_0 + k_3)\cdot$ Then

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Now starting from (x_1, y_1) and repeating the process, we get (x_2, y_2) etc.

8.12 ADVANTAGES OF RUNGE - KUTTA METHOD OVER TAYLOR'S SERIES

Though approximately the R-K method is the same as Taylor's polynomial, R-K formula does not require prior calculation of higher derivatives of $y(x)$ as the Taylor's series method does. Since the differential equations arising in application are often complicated, the calculation of derivatives may be very difficult. In R-K method, the computation of $f(x, y)$ at various positions, instead of derivatives are calculated and this function occurs in the given equation. To evaluate y_{n+1}, we need information only at the point (x_n, y_n). Informations at the points y_{n-1}, y_{n-2} etc., are not directly required. Thus R-K methods are one - step methods.

SOLVED EXAMPLES

Example 1 : Using Runge-Kutta method of second order, compute y (2.5) from $\frac{dy}{dx} = \frac{x+y}{x},\ y(2) = 2$, taking $h = 0.25$.

Solution : Here $f(x, y) = \frac{x+y}{x}$

First Step: $x_0 = 2,\ y_0 = 2$ and $h = 0.25$

$$\therefore\ k_1 = h\, f(x_0, y_0) = (0.25)\ f(2,2) = (0.25)\ (2) = 0.5$$

$$k_2 = h\, f(x_0 + h,\ y_0 + k_1) = (0.25)\left[f(2.25,\ 2.5)\right] = (0.25)\left(\frac{2.25 + 2.5}{2.25}\right) = 0.528$$

Hence $y_1 = y(2.25) = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(0.5 + 0.528) = 2.514$

Second Step: Now starting from (x_1, y_1), we get (x_2, y_2).

Again apply R -K method replacing (x_0, y_0) with (x_1, y_1).

Here $x_1 = x_0 + h = 2 + 0.25 = 2.25,\ y_1 = 2.514,\ h = 0.25$

$\therefore\ k_1 = h\ f(x_1, y_1) = (0.25)\ f(2.25, 2.514) = (0.25)\left(\frac{2.25 + 2.514}{2.25}\right) = 0.5293$

$k_2 = h\ f(x_1 + h, y_1 + k_1) = (0.25)\left[f(2.25 + 0.25, 2.514 + 0.5293)\right]$

$= (0.25)\left[f(2.5, 3.0433)\right] = (0.25)\left(\frac{2.5 + 3.0433}{2.5}\right) = 0.55433$

Hence $y_2 = y(2.5) = y_1 + \frac{1}{2}(k_1 + k_2) = 2.514 + \frac{1}{2}(0.5293 + 0.55433) = 3.0558.$

Example 2 : Obtain the values of y at x = 0.1, 0.2 using Runge-kutta method of (i) second order (ii) third order (iii) fourth order for the differential equation $y' + y = 0,\ y(0) = 1$.

Solution : Given equation can be written as $y' = -y,\ y(0) = 1$. Here $f(x, y) = -y$.

(i) Second order:

Step 1: $x_0 = 0,\ y_0 = 1,\ h = 0.1$

Now $k_1 = h\ f(x_0, y_0) = (0.1)\ f(0,1) = (0.1)\ (-1) = -\ 0.1$

and $k_2 = h\ f(x_0 + h, y_0 + k_1) = (0.1)\ f(0,1,\ 0.9) = (0.1)\ (-0.9) = -\ 0.09$

$\therefore\ y_1 = y(0.1) = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(-0.1 - 0.09) = 1 - 0.095 = 0.905$

Now starting from (x_1, y_1), we get (x_2, y_2). Again apply R-K method replacing (x_0, y_0) by (x_1, y_1).

Step 2: $x_1 = x_0 + h = 0.1,\ y_1 = 0.905,\ h = 0.1$

$\therefore\ k_1 = h\ f(x_1, y_1) = (0.1)\left[f(0.1, 0.905)\right] = (0.1)\ (-\ 0.905) = -\ 0.0905$

$k_2 = h\ f(x_1 + h,\ y_1 + k_1) = (0.1)\left[f(0.2,\ 0.905 - 0.0905)\right]$

$= (0.1)\left[f(0.2,\ 0.8145)\right] = (0.1)\ (-\ 0.8145) = -\ 0.08145$

Hence $y_2 = y(0.2) = y_1 + \frac{1}{2}(k_1 + k_2)$

$= 0.905 + \frac{1}{2}(-\ 0.0905 - 0.08145) = 0.905 - 0.085975 = 0.819025$

(ii) Third order :

Step 1: $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\therefore k_1 = h\ f(x_0, y_0) = (0.1)\ f(0,1) = -\ 0.1$

$k_2 = h\ f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1) = (0.1)\left[f(0.05, 0.95)\right] = (0.1)\ (-0.95) = -\ 0.095$

$0\ k_3 = h\ f(x_0 + h,\ y_0 + 2k_2 - k_1) = (0.1)\left[f(0.1, 0.9)\right] = (0.1)\ (-0.9) = -\ 0.09$

Hence $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3) = 1 + \frac{1}{6}(-\ 0.1 - 0.38 - 0.09) = 1 - 0.095 = 0.905$

Step 2: $x_1 = x_0 + h = 0.1,\ y_1 = 0.905,\ h = 0.1$

$\therefore\ k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 0.905) = (0.1)\,(-0.905) = -0.0905$

$$k_2 = h\,f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1) = (0.1)\left[f(0.1 + 0.05, 0.905 - 0.04525)\right]$$

$$= (0.1)\left[f(0.15, 0.85975)\right] = (0.1)\,(-0.85975) = -0.085975$$

$$k_3 = h\,f(x_1 + h,\ y_1 + 2k_2 - k_1) = (0.1)\left[f(0.2, 0.905 - 0.17195 + 0.0905)\right]$$

$$= (0.1)\left[f(0.2, 0.82355)\right] = (0.1)\,(-0.82355) = -0.082355$$

Hence $y_2 = y\,(0.2) = y_1 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

$$= 0.905 + \frac{1}{6}(-0.0905 - 0.3439 - 0.082355) = 0.905 - 0.0861258 = 0.818874$$

(*iii*) Fourth order:

Step 1: $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$\therefore\ k_1 = h\,f(x_0, y_0) = (0.1)\,f(0,1) = (0.1)\,(-1) = -0.1$

$$k_2 = h\,f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1) = (0.1)\left[f(0.05, 0.95)\right] = (0.1)\,(-0.95) = -0.095$$

$$k_3 = h\,f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2) = (0.1)\left[f(0.05, 0.9525)\right] = (0.1)\,(-0.9525) = -0.09525$$

$$k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\left[f(0.05, 0.90475)\right] = (0.1)\,(-0.90475) = -0.090475$$

Hence $y_1 = y\,(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1 + \frac{1}{6}(-0.1 - 0.19 - 0.1905 - 0.090475) = 1 - 0.0951625 = 0.9048375$$

Step 2: $x_1 = x_0 + h = 0.1,\ y_1 = 0.9048375,\ h = 0.1$

$\therefore\ k_1 = h\,f(x_1, y_1) = (0.1)\left[f(0.1, 0.9048375)\right] = (0.1)\,(-0.9048375) = -0.09048375$

$$k_2 = h\,f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1) = (0.1)\left[f(0.15, 0.8595956)\right]$$

$$= (0.1)\,(-0.8595956) = -0.08595956$$

$$k_3 = h\,f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_2) = (0.1)\left[f(0.15, 0.8618577)\right]$$

$$= (0.1)\,(-0.8618577) = -0.08618577$$

and $k_4 = h\,f(x_1 + h,\ y_1 + k_3) = (0.1)\left[f(0.2, 0.8186517)\right]$

$$= (0.1)\,(-0.8186517) = -0.8186517$$

Hence $y_2 = y\,(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.9048375 + \frac{1}{6}(-0.09048375 - 0.171919 - 0.1723714 - 0.08186517)$$

$$= 0.9048375 - 0.0861065 = 0.8187309$$

So the value of y when $x = 0.2$ is 0.8187 correct to four decimal places.

Note. Exact solution of the given differential equation is $y = e^{-x}$. Hence the exact solution of y when $x = 0.1$ is 0.9048 and when $x = 0.2$ is 0.8187. It can be seen that this fourth order method is an accurate method.

Tabular values are :

x	Second order	Third order	Fourth order	Exact value
0.1	0.905	0.905	0.9048375	0.9048374
0.2	0.819025	0.818874	0.8187309	0.8187307

Example 3 : Apply the fourth order Runge - Kutta method, to find an approximate value of y when $x = 1.2$, in steps of 0.1, given that : $y' = x^2 + y^2,\ y(1) = 1.5$

Solution : Here $f(x, y) = x^2 + y^2$ and we take $h = 0.1$ and carry out the calculations in two steps.

Step 1. $x_0 = 1,\ y_0 = 1.5,\ h = 0.1$

$$\therefore\ k_1 = h\,f(x_0, y_0) = (0.1)\,f(1, 1.5) = (0.1)\left[(1)^2 + (1.5)^2\right] = 0.325$$

$$k_2 = h\,f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1) = (0.1)\left[f(1 + 0.05, 1.5 + 0.1625)\right]$$

$$= (0.1)\left[f(1.05, 1.6625)\right] = (0.1)\left[(1.05)^2 + (1.6625)^2\right] = 0.3866$$

$$k_3 = h\,f(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2) = (0.1)\left[f(1.05, 1.6933)\right]$$

$$= (0.1)\left[(1.05)^2 + (1.6933)^2\right] = 0.39698$$

and $k_4 = h\,f(x_0 + h,\ y_0 + k_3) = (0.1)\left[f(1.05, 1.8969)\right]$

$$= (0.1)\left[(1.05)^2 + (1.8969)^2\right] = 0.4808$$

Hence $y_1 = y\,(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1.5 + \frac{1}{6}(0.325 + 0.7732 + 0.7939 + 0.4808) = 1.5 + 0.39548 = 1.89548 \approx 1.8955$$

Step 2. $x_1 = x_0 + h = 1 + 0.1 = 1.1,\ y_1 = 1.8955,\ h = 0.1$

$$\therefore\ k_1 = h\,f(x_1, y_1) = (0.1)\,f(1.10, 1.8955) = (0.1)\left[(1.10)^2 + (1.8955)^2\right] = 0.4803$$

$$k_2 = h\,f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1) = h\left[f(1.1 + 0.05, 1.8955 + 0.24015)\right]$$

$$= h\left[f(1.15, 2.13565)\right] = (0.1)\left[(1.15)^2 + (2.13565)^2\right] = 0.58835$$

$$k_3 = h\,f(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_2) = (0.1)\,f(1.15, 2.189675)$$

$$= (0.1)\left[(1.15)^2 + (2.189675)^2\right] = 0.6117$$

and $k_4 = h\,f(x_1 + h,\ y_1 + k_3) = (0.1)\,f(1.2, 2.5072) = (0.1)\left[(1.2)^2 + (2.5072)^2\right] = 0.7726$

Hence $y_2 = y\,(1.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1.8955 + \frac{1}{6}(0.4803 + 1.1767 + 1.2234 + 0.7726)$$

$$= 1.8955 + \frac{1}{6}(3.653) = 1.8955 + 0.6088 = 2.5043$$

Example 4 : Using Runge-Kutta method, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$

Solution : Here $y' = f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0, y_0 = 1$ and $h = 0.2$

$$\therefore\ k_1 = h\, f(x_0, y_0) = (0.2)\, f(0,1) = (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = (0.2)\left[f(0.1, 1.1)\right] = (0.2)\left(\frac{1.1-0.1}{1.1+0.1}\right) = 0.2\left(\frac{1}{1.2}\right) = 0.16666$$

$$k_3 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = (0.2)\left[f(0.1, 1.08333)\right] = (0.2)\left(\frac{1.08333-0.1}{1.08333+0.1}\right) = 0.16619$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = (0.1)\left[f(0.2, 1.16619)\right] = (0.1)\left(\frac{1.16619-0.2}{1.16619+0.2}\right) = 0.07072$$

Hence $y_1 = y\,(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1 + \frac{1}{6}(0.2 + 0.33332 + 0.33238 + 0.07072) = 1 + 0.15607 = 1.15607$$

Example 5 : Use Runge - Kutta method to evaluate $y(0.1)$ and $y(0.2)$ given that $y' = x + y$, $y(0) = 1$. **[JNTU 2007 (Set No.3), (A) May 2011]**

Solution : Here $f(x, y) = x + y$, $x_0 = 0, y_0 = 1$

Step 1. $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore\ k_1 = h\, f(x_0, y_0) = (0.1)\, f(0,1) = (0.1)\,(0+1) = 0.1$$

$$k_2 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = (0.1)\left[f(0.05, 1.05)\right] = (0.1)\,(0.05 + 1.05) = 0.11$$

$$k_3 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = (0.1)\left[f(0.05, 1.055)\right] = (0.1)\,(0.05 + 1.055) = 0.1105$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = (0.1)\left[f(0.1, 1.1105)\right] = (0.1)\,(0.1 + 1.1105) = 0.12105$$

Hence $y_1 = y\,(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) = 1 + 0.1103416 = 1.11034$$

To find $y_2 = y(0.2)$, we again start from $(x_1, y_1) = (0.1, 1.11034)$

Step. 2. $x_1 = x_0 + h = 0 + 0.1 = 0.1, y_1 = 1.11034$ and $h = 0.1$

$\therefore\ k_1 = h\ f(x_1, y_1) = (0.1)\left[f(0.1, 1.11034)\right] = (0.1)\ (0.1 + 1.11034) = 0.121034$

$k_2 = h\ f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) = (0.1)\left[f(0.15, 1.170857)\right]$

$= (0.1)\ (0.15 + 1.170857) = 0.1320857$

$k_3 = h\ f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = (0.1)\left[f(0.15, 1.1763829)\right]$

$= (0.1)\ (0.15 + 1.1763829) = 0.1326382$

$k_4 = h\ f(x_1 + h, y_1 + k_3) = (0.1)\left[f(0.2, 1.2429783)\right]$

$= (0.1)\ (0.2 + 1.2429783) = 0.1442978$

Hence $y_2 = y\,(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.11034 + \frac{1}{6}(0.121034 + 0.2641714 + 0.2652764 + 0.1442978)$

$= 1.11034 + 0.1324632 = 1.242803$

So the value of y when $x = 0.2$ is 1.2428 correct to four decimal places.

Example 6 : Find $y(.1)$ and $y(.2)$ using Runge-Kutta 4th order formula given that $y' = x^2 - y$ and $y(0) = 1$. **[JNTU 2006, (A) Nov. 2010 (Set No. 1, 4)]**

Solution : Here $y' = f(x, y) = x^2 - y,\ x_0 = 0,\ y_0 = 1$ and $h = 0.1$

Step1. To find *y*(0.1)

$\therefore k_1 = h\,f(x_0, y_0) = (0.1)\,f(0,1) = (0.1)(0-1) = -0.1$

$k_2 = h\,f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05,\ 0.95)$

$= (0.1)\ [(0.05)^2 - 0.95] = -\,0.09475$

$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05,\ 0.952625)$

$= (0.1)\ [(0.05)^2 - 0.952625] = -\,0.095$

$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.1)\ f(0.1,\ 0.905)$

$= (0.1)\ [(0.1)^2 - 0.905] = -\,0.0895$

Hence $y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}[-\,0.1 - 0.1895 - 0.19 - 0.0895] = 0.9052.$

Step 2. To find *y*(0.2)

Now we have $x_1 = x_0 + h = 0.1,\ y_1 = 0.9052$ and $h = 0.1$

$\therefore k_1 = h\,f(x_1, y_1) = (0.1)f\,(0.1,\ 0.9052) = (0.1)\ [0.01 - 0.9052] = -\,0.08952$

$$k_2 = hf\left(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15,\ 0.86044)$$

$$= (0.1)\,[(0.15)^2 - 0.86044] = -\,0.08379$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h,\ y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15,\ 0.8633)$$

$$= (0.1)\,[(0.15)^2 - 0.8633] = -\,0.0841$$

$$k_4 = hf\,(x_1 + h,\ y_1 + k_3) = (0.1)\,f(0.2,\ 0.8211)$$

$$= (0.1)\,[(0.2)^2 - 0.8211] = -\,0.07811$$

Hence $y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.9052 + \frac{1}{6}(-0.08952 - 0.16758 - 0.1682 - 0.07811) = 0.8213.$$

Example 7 : Solve the following using R-K fourth method $y' = y{-}x,\ y(0) = 2,\ h = 0.2$. Find $y(0.2)$. **[JNTU 2008, (H) June 2009 (Set No.4)]**

Solution : Here $x_0 = 0,\ y_0 = 2,\ h = 0.2,\ x_1 = x_0 + h = 0.2$ and $f(x, y) = y - x$

By R–K method of fourth order,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad ...(1)$$

Where $k_1 = h.f\,(x_0, y_0) = (0.2)\,f\,(0, 2)$

$$= (0.2)\,(2{-}0) = 0.4$$

$$k_2 = h.f\left(x_0 + \frac{h}{2},\ y_0 + \frac{1}{2}k_1\right) = (0.2).f\,(0.1,\ 2.2)$$

$$= (0.2)\,(2.2 - 0.1) = 0.42.$$

$$k_3 = h.f\left(x_0 + \frac{h}{2},\ y_0 + \frac{1}{2}k_2\right) = (0.2).f\,(0.1,\ 2.21)$$

$$= (0.2)(2.21 - 0.1) = 0.422$$

and $k_4 = h.f\,(x_0 + h,\ y_0 + k_3) = (0.2)\ .\ f(0.2,\ 2.422)$

$$= (0.2)\,(2.422 - 0.2) = 0.4444$$

Hence, using (1)

$$y\,(0.2) = y_1 = 2 + \frac{1}{6}[0.4 + 2(0.42 + 0.422) + 0.4444]$$

$$= 2 + 0.4214 = 2.4214.$$

Example 8 : Solve $\frac{dy}{dx} = xy$ using R-K Method for $x = 0.2$ given $y(0) = 1$, taking $h = 0.2$. **[JNTU 2008, 2008S(Set No.1)]**

Solution : Here $f(x, y) = xy, x_0 = 0, y_0 = 1$ and $h = 0.2$

$$k_1 = hf(x_0, y_0) = (0.2)\,f(0, 1) = 0$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ \ y_0 + \frac{1}{2}k_1\right) = (0.2\,)\,f(0.1, 1)$$

$$= (0.2)\,(1.1) = 0.0202$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ \ y_0 + \frac{1}{2}k_2\right) = (0.2)\,f(0.1, 1.01)$$

$$= (0.2)\,(\,0.1)\,(1.01) = 0.0202$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)\,f(0.2,\ 1.0202)$$

$$= (0.2)\,(0.2)\,(1.0202) = 0.040808$$

By R – K method,

$$y_1 = y(0.2) = y_0 + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$$

$$= 1 + \frac{1}{6}[0 + 2(0.02 + 0.0202) + 0.040808]$$

$$= 1.0202$$

Example 9 : Compute $y(0.1)$ and $y(0.2)$ by Runge - Kutta method of 4th order for the differential equation $y' = xy + y^2, y\,(0) = 1$. **[JNTU (A) June 2009, (H) Dec. 2012]**

Solution : Here $y' = f\,(x, y) = xy + y^2, x_0 = 0, y_0 = 1$ and $h = 0.1$

To find $y\,(0.1)$

By fourth order Runge - Kutta method,

$$K_1 = h\,.\,f\,(x_0, y_0) = (0.1)\,(x_0 y_0 + y_0^2) = (0.1)\,(0 + 1) = 0.1$$

$$K_2 = h\,.\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_1\right)$$

$$= h\ f\ (0.05, 1.05) = (0.1)[(0.05)\,(1.05) + (1.05)^2] = 0.1155$$

$$K_3 = h\,.\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}K_2\right)$$

$$= (0.1)\,f\,(0.05, 1.05775) = (0.1)\,[(0.05)\,(1.05775) + (1.05775)]^2 = 0.11217$$

$$K_4 = h\,.\,f\,(x_0 + h, y_0 + K_3) = (0.1)\,f\,(0.1, 1.11217)$$

$$= (0.1)\,[(0.1)(1.11217) + (1.11217)^2] = 0.1248$$

$$\therefore\ y_1 = y\,(0.1) = y_0 + \frac{1}{6}(\mathrm{K}_1 + 2\mathrm{K}_2 + 2\mathrm{K}_3 + \mathrm{K}_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.231 + 0.22434 + 0.1248) = 1.1133$$

To find $y\,(0.2)$

Now starting from (x_1, y_1) we get (x_2, y_2).

Again apply Runge - Kutta method replacing (x_0, y_0) by (x_1, y_1).

Now we have $x_1 = x_0 + h = 0.1, y_1 = 1.1133$ and $h = 0.1$.

$$\mathrm{K}_1 = h\,.\,f\,(x_1, y_1) = (0.1)\,.\,f\,(0.1, 1.1133)$$

$$= (0.1)\,[(0.1)\,(1.1133) + (1.1133)^2] = 0.1351$$

$$\mathrm{K}_2 = h\,.\,f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}\mathrm{K}_1\right)$$

$$= (0.1)\,f\,(0.15, 1.18085) = (0.1)\,[(0.15)\,(1.18085) + (1.18085)^2] = 0.1571$$

$$\mathrm{K}_3 = h\,.\,f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}\mathrm{K}_2\right)$$

$$= (0.1)f(0.15, 1.19185) = (0.1)\,[(0.15)\,(1.19185) + (1.19185)]^2 = 0.1599$$

$$\mathrm{K}_4 = h\,.\,f\,(x_1 + h, y_1 + \mathrm{K}_3) = (0.1)f(0.2, 1.2732)$$

$$= (0.1)\,[(0.2)\,(1.2732) + (1.2732)^2] = 0.1876$$

Hence $y_2 = y(0.2) = y_1 + \frac{1}{6}(\mathrm{K}_1 + 2\mathrm{K}_2 + 2\mathrm{K}_3 + \mathrm{K}_4)$

$$= 1.1133 + \frac{1}{6}(0.1351 + 0.3142 + 0.3198 + 0.1876) = 1.2728$$

Example 10 : Solve $y' = x - y$ given that $y(1) = 0.4$. Find $y(1.2)$ using R–K method.
[JNTU(K)May 2010 (Set No.2)]

Solution : Since h is not mentioned in the question, we take $h = 0.1$

Here $f(x, y) = x - y, x_0 = 1, y_0 = 0.4, x_1 = 1.1, x_2 = 1.2$

Step 1: By fourth order R – K method,

$$k_1 = hf(x_0, y_0) = (0.1)(x_0 - y_0) = (0.1)(1 - 0.4) = 0.06$$

$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$$

$$= (0.1)f(1 + 0.05, 0.4 + 0.03) = (0.1)f(1.05, 0.43)$$

$$= (0.1)(1.05 – 0.43) = 0.062$$

$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2)$$

$$= (0.1)f(1.05, 0.4 + 0.031) = (0.1)f(1.05, 0.431)$$

$$= (0.1)(1.05 - 0.431) = (0.1)(0.619) = 0.0619$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.1)f(1.1, 0.4 + 0.0619) = (0.1)f(1.1, 0.4619)$$

$$= (0.1)(1.1 – 0.4619) = (0.1)(0.6381) = 0.06381$$

$$\therefore \quad y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.4 + \frac{1}{6}[0.06 + 2(0.062 + 0.0619) + 0.06381]$$

$$= 0.4 + \frac{1}{6}(0.37161) = 0.4619$$

To find $y_2 = y(0.2)$, we again start from $(x_1, y_1) = (1.1, 0.4619)$

Step 2 : $x_1 = x_0 + h = 1 + 0.1 = 1.1, \quad y_1 = 0.4619$ and $h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = (0.1)f(1.1, 0.4619)$$

$$= (0.1)(1.1 – 0.4619) = 0.70191$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right)$$

$$= (0.1)f(1.1 + 00.5, 0.4619 + 0.350955)$$

$$= (0.1)f(1.15, 0.81285) = (0.1)(1.15 - 0.81285)$$

$$= 0.03371$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right)$$

$$= (0.1)f(1.15, 0.4619 + 0.01686)$$

$$= (0.1)f(1.15, 0.47876)$$

$$= (0.1)(1.15 - 0.47876) = 0.67124$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)f(1.1+0.1, 0.4619 + 0.67124)$$

$$= (0.1)f(1.2, 1.13314)$$

$$= (0.1)(1.2 - 1.13314) = 0.06686$$

Hence $$y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.4619 + \frac{1}{6}[0.70191 + 2(0.03371 + 0.67124) + 0.06686]$$

$$= 0.4619 + \frac{1}{6}(2.17867) = 0.825$$

Example 11 : Find $y(0.1)$ and $y(0.2)$ using Runge Kutta fourth order formula given that $\frac{dy}{dx} = x + x^2 y$ and $y(0) = 1$. **[JNTU (H) June 2012]**

Solution : Here $f(x, y) = y' = x + x^2 y = x(1 + xy), x_0 = 0, y_0 = 1$ and $h = 0.1$

To find y_1 *i.e.,* $y(0.1)$

Now $k_1 = hf(x_0, y_0) = (0.1) f(0,1) = 0.1\,(0) = 0$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) f(0.05,1)$$

$$= (0.1)\,[0.05\,(1 + 0.05)] = 0.00525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) f(0.05,1.002625)$$

$$= (0.1)\,[0.05\,(1 + 0.0501312)] = 0.00525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1,1.00525)$$

$$= (0.1)\,[0.1\,(1 + 0.100525)] = 0.011$$

By Runge - Kutta Fourth order formula,

$$y_1 = y_0 + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$$

$$= 1 + \frac{1}{6}[(0 + 0.011) + 2(0.00525 + 0.00525)] = 1.0053$$

To find y_2 *i.e.,* $y(0.2)$

Here $x_1 = 0.1, y_1 = 1.0053, x_2 = 0.2$ and $h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = (0.1) f(0.1,1.0053) = (0.1)[0.1\,(1 + 0.10053)] = 0.0110053$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1) f(0.15,1.0108) = (0.1)\,[0.15\,(1 + 0.15162)] = 0.0173$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1) f(0.15,1.01395) = (0.1)\,[0.15\,(1 + 0.1521)] = 0.01728$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2,1.02258)$$

$$= (0.1)\,[(0.2)\,(1 + 0.204516)] = 0.0241$$

By Runge - Kutta Fourth order formula,

$$y_2 = y(0.2) = y_1 + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$$

$$= 1.0053 + \frac{1}{6}[(0.0110053 + 0.0241) + 2(0.0173 + 0.01728)]$$

$$= 1.0053 + \frac{1}{6}(0.0351053 + 0.06916) = 1.02268$$

Example 12 : Using Runge-Kutta method of fourth order find $y(0.1), y(0.2)$ and $y(0.3)$, given that $\frac{dy}{dx} = 1 + xy, y(0) = 2$. **[JNTU (A) May 2012 (Set No. 2)]**

Solution : Here $\frac{dy}{dx} = 1 + xy$, so $f(x, y) = y' = 1 + xy, h = 0.1, x_0 = 0, y_0 = 2, x_1 = 0.1, x_2 = 0.2$ and $x_3 = 0.3$

To find y_1 *i.e.,* $y(0.1)$

$$k_1 = h \cdot f(x_0, y_0) = (0.1) f(0, 2) = (0.1)(1 + 0) = 0.1$$

$$k_2 = h \cdot f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 2.05)$$

$$= (0.1)[1 + (0.05)(2.05)] = 0.11025$$

$$k_3 = h \cdot f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 2.055125)$$

$$= (0.1)[1 + (0.05)(2.055125)] = 0.11028$$

$$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 2.11028)$$

$$= (0.1)[1 + (0.1)(2.11028)] = 0.1211$$

Hence $y_1 = y(0.1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, using R - K method of fourth order.

$$= 2 + \frac{1}{6}[(0.1 + 0.1211) + 2(0.11025 + 0.11028)]$$

$$= 2 + \frac{1}{6}(0.66216) = 2.11036$$

To find y_2 *i.e.,* $y(0.2)$

We have $x_1 = 0.1, y_1 = 2.11036$ and $h = 0.1$

$$k_1 = hf(x_1, y_1) = (0.1) f(0.1, 2.11036)$$

$$= (0.1)[1 + (0.1)(2.11036)] = 0.1211$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 2.17091)$$

$= (0.1)[1 + (0.15)(2.17091)] = 0.1325$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 2.17661)$$

$= (0.1)\,[1 + (0.15)(2.17661)] = 0.13265$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 2.24301)$

$= (0.1)\,[1 + (0.2)\,(2.24301)] = 0.14486$

Hence $y_2 = y(0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = y_1 + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$

$$= 2.11036 + \frac{1}{6}[(0.1211 + 0.14486) + 2(0.1325 + 0.13265)]$$

$$= 2.11036 + \frac{1}{6}(0.79626) = 2.24307$$

To find y_3 *i.e.,* $y(0.3)$

We have $x_2 = 0.2, y_2 = 2.24307$ and $h = 0.1$

$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 2.24307)$

$= (0.1)[1 + (0.2)(2.24307)] = 0.1449$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 2.31552)$$

$= (0.1)[1 + (0.25)(2.31552)] = 0.1579$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 2.32202)$$

$= (0.1)\,[1 + (0.25)(2.32202)] = 0.15805$

$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 2.40112)$

$= (0.1)\,[1 + (0.3)\,(2.40112)] = 0.1720$

Hence $y_3 = y(0.3) = y_2 + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$

$$= 2.24307 + \frac{1}{6}[(0.1449 + 0.1720) + 2(0.1579 + 0.15805)]$$

$$= 2.24307 + \frac{1}{6}(0.9488) = 2.4012$$

Thus $y(0.1) = 2.11036, y(0.2) = 2.24307$ and $y(0.3) = 2.4012$.

Example 13 : Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$. **[JNTU (A) May 2012 (Set No. 3)]**

Solution : Here $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, so $f(x, y) = y' = \frac{y^2 - x^2}{y^2 + x^2}$

and $x_0 = 0, y_0 = 1, h = 0.2, x_1 = 0.2, x_2 = 0.4$.

To find y_1 *i.e.,* $y(0.2)$

$$k_1 = hf(x_0, y_0) = (0.2) f(0,1) = (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.2) f(0+0.1, 1+0.1) = (0.2) f(0.1, 1.1)$$

$$= (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right] = (0.2)\left(\frac{1.2}{1.22}\right) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.2) f(0.1, 1+0.09836) = (0.2) f(0.1, 1.09836)$$

$$= (0.2)\left[\frac{(1.09836)^2 - (0.1)^2}{(1.09836)^2 + (0.1)^2}\right] = (0.2)\left(\frac{1.19639}{1.21639}\right) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2) f(0+0.2, 1+0.1967) = (0.2) f(0.2, 1.1967)$$

$$= (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right] = (0.2)\left(\frac{1.3921}{1.4721}\right) = 0.1891$$

Hence $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, using R - K method of fourth order.

$$= 1 + \frac{1}{6}[(0.2 + 0.1891) + 2(0.19672 + 0.1967)]$$

$$= 1.19599 \approx 1.196$$

To find y_2 *i.e.,* $y(0.4)$

We have $x_1 = 0.2, y_1 = 1.196$ and $h = 0.2$

$$k_1 = hf(x_1, y_1) = (0.2) f(0.2, 1.196) = (0.2)\left(\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 + (0.2)^2}\right)$$

$$= (0.2)\left(\frac{1.3904}{1.4704}\right) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.2)f(0.2 + 0.1, 1.196 + 0.09456)$$

$$= (0.2)f(0.3, 1.29056) = (0.2)\left[\frac{(1.29056)^2 - (0.3)^2}{(1.29056)^2 + (0.3)^2}\right]$$

$$= (0.2)\left(\frac{1.5755}{1.7555}\right) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.2)f(0.2 + 0.1, 1.196 + 0.08975) = (0.2)f(0.3, 1.28575)$$

$$= (0.2)\left[\frac{(1.28575)^2 - (0.3)^2}{(1.28575)^2 + (0.3)^2}\right] = (0.2)\left(\frac{1.56315}{1.74315}\right) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.2)f(0.2 + 0.2, 1.196 + 0.1793)$$

$$= (0.2)f(0.4, 1.3753) = (0.2)\left[\frac{(1.3753)^2 - (0.4)^2}{(1.3753)^2 + (0.4)^2}\right]$$

$$= (0.2)\left(\frac{1.73145}{2.05145}\right) = 0.1688$$

Hence $y_2 = y(0.4) = y_1 + \frac{1}{6}[(k_1 + k_4) + 2(k_2 + k_3)]$

$$= 1.196 + \frac{1}{6}[(0.1891 + 0.1688) + 2(0.1795 + 0.1793)]$$

$$= 1.196 + \frac{1}{6}(1.0755) = 1.37525$$

Thus $y(0.2) = 1.196$ and $y(0.4) = 1.37525$

8.13 Runge-Kutta Method for Simultaneous First Order Differential Equations.

The equations of the type $\frac{dy}{dx} = f_1(x, y, z)$ and $\frac{dz}{dx} = f_2(x, y, z)$ with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by R-K method as explained through the following example.

Formulae for the application of Runge-kutta method are as follows:

$$k_1 = h\,f_1(x_0, y_0, z_0)$$
$$l_1 = h\,f_2(x_0, y_0, z_0)$$
$$k_2 = h\,f_1(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1)$$
$$l_2 = h\,f_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$
$$k_3 = h\,f_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$
$$l_3 = h\,f_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$
$$k_4 = h\,f_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$l_4 = h\,f_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$\therefore \quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
and $$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Having got (x_1, y_1, z_1), we get (x_2, y_2, z_2) by repeating the above algorithm once again starting from (x_1, y_1, z_1).

SOLVED EXAMPLES

Example 1 : Find $y(0.1), z(0.1), y(0.2)$ and $z(0.2)$ from the system of equations, $y' = x + z$, $z' = x - y^2$ given $y(0) = 2, z(0) = 1$ using Runge - Kutta method of fourth order.

[JNTU(H) June 2009, (K) May 2010 (Set No.4)]

Solution : We have $y' = f_1(x, y, z) = x + z$ and $z' = f_2(x, y, z) = x - y^2$

and $x_0 = 0, y_0 = 2, z_0 = 1$. Also $h = 0.1$

Now $k_1 = h\,.\,f_1(x_0, y_0, z_0) = (0.1) f_1(0, 2, 1) = (0.1)\,(0 + 1) = 0.1$ $\quad [\because f_1 = x + z]$

$l_1 = h\,.\,f_2(x_0, y_0, z_0) = (0.1) f_2(0, 2, 1) = (0.1)\,(0 - 4) = -\,0.4$ $\quad [\because f_2 = x - y^2]$

$$k_2 = h\,.\,f_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= (0.1)\,.\,f_1(0.05, 2.05, 0.8) = (0.1)\,(0.05 + 0.8) = 0.085$$
$$l_2 = h\,.\,f_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= (0.1)\,.\,f_2(0.05, 2.05, 0.8) = (0.1)\,[0.05 - (2.05)^2] = -\,0.41525$$

$$k_3 = h \, . \, f_1\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$= (0.1) \, . \, f_1(0.05, 2.0425, 0.79238) = (0.1)\,(0.05+0.79238) = 0.084238$$

$$l_3 = h \, . \, f_2\left(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2, z_0+\frac{1}{2}l_2\right)$$

$$= (0.1)\; f_2\;(0.05, 2.0425, 0.79238)$$

$$= (0.1)\,[0.05-(2.0425)^2] = -\,0.4122$$

$$k_4 = h \, . \, f_1(x_0+h, y_0+k_3, z_0+l_3)$$

$$= (0.1) f_1\;(0.1,\; 2.084238,\; 0.5878) = (0.1)\,(0.1+0.5878) = 0.06878$$

$$l_4 = h \, . \, f_2(x_0+h, y_0+k_3, z_0+l_3)$$

$$= (0.1)\,[0.1-(2.084238)^2] = -0.4244$$

$\therefore$ By Runge - Kutta method, we have

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 2+\frac{1}{6}[0.1+2\,(0.085+0.084238)+0.06878] = 2.0845$$

$$z_1 = z_0 + \frac{1}{6}[l_1+2l_2+2l_3+l_4]$$

$$= 1+\frac{1}{6}[-0.4-2\,(0.41525+0.4122)+0.4122] = 0.5868$$

Hence $y\,(0.1) = 2.0845$ and $z\,(0.1) = 0.5868$.

Repeat the above procedure to compute $y(0.2)$ and $z(0.2)$ and this is left as an exercise to the reader.

8.14 RUNGE-KUTTA METHOD FOR SECOND ORDER DIFFERENTIAL EQUATION

Any differential equation of second or Higher order differential equations are best treated by transforming the given equation into a system of first order simultaneous differential equations which can be solved as usual.

Consider, for example the second order differential equation:

$$y'' = f(x, y, y'),\; y(x_0) = y_0,\;\; y'(x_0) = y'_0$$

Substituting $\dfrac{dy}{dx} = z$ ... (1)

we get $\dfrac{dz}{dx} = \dfrac{d^2y}{dx^2} = f(x, y, z)$, using (1) ... (2)

Given $y(x_0) = y_0$ and $y'(x_0) = z(x_0) = y'_0$

Equations (1) and (2) constitute the equivalent system of simultaneous equations where $f_1(x, y, z) = z,\;\; f_2(x, y, z) = f(x, y, z)$ given. Also $y(0)$ and $z(0)$ are given.

SOLVED EXAMPLES

Example 1 : Solve $y'' - x(y')^2 + y^2 = 0$ using R-K method for x = 0.2 given $y(0) = 1, y'(0) = 0$ taking $h = 0.2$. **[JNTU(A)May 2010S]**

Solution : Given equation is a second order differential equation.

Substituting $\frac{dy}{dx} = z = f_1(x, y, z)$ ... (1)

The given equation reduces to

$$\frac{dz}{dx} = xz^2 - y^2 = f_2(x, y, z) \quad \text{... (2)}$$

Given $x_0 = 0, y_0 = 1, z_0 = y'_0 = 0$. Also $h = 0.2$

By R – K algorithm,

$$k_1 = hf_1(x_0, y_0, z_0) = (0.2)f_1(0,1,0) = (0.2)(0) = 0$$

$$l_1 = hf_2(x_0, y_0, z_0) = (0.2)f_2(0,1,0) = (0.2)[0-(1)^2] = -0.2$$

$$k_2 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.2)\,f_1(0.1,\ 1, -0.1) = (0.2)(-0.1) = -\,0.02$$

$$l_2 = hf_2(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1)$$

$$= (0.2)\,f_2(0.1,\ 1, -0.1) = (0.2)[(0.1)(-0.1)^2-1]$$

$$= -0.1998$$

$$k_3 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= (0.2)\,f_1(0.1,\ 0.99, -\,0.0999)$$

$$= (0.2)\ (-\,0.0999) \qquad [\because\ f_1 = z]$$

$$= -\,0.01998$$

$$l_3 = hf_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$= (0.2)\,f_2(0.1, 0.99, -0.0999)$$

$$= (0.2)\ [(0.1)(-0.0999)^2-(0.99)^2] \qquad [\because\ f_2 = xz^2 - y^2]$$

$$= (0.2)\ (-0.9791) = -\,0.1958$$

$$k_4 = hf_1(x_0 + h_1, y_0 + k_3, z_0 + l_3)$$

$$= (0.2)\,f_1(0.2,\ 0.98, -\,0.1958)$$

$$= (0.2)\ (-0.1958) = -\,0.0392$$

$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2)f_2(0.2, 0.98, -0.1958)$

$= (0.2)\,[(0.2)(-0.1958)^2 - (0.98)^2]$

$= (0.2)\,(-0.9527324) = -0.1905$

$$\therefore \qquad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$i.e., \qquad y(0.2) = 1 + \frac{1}{6}[0 + 2(-0.02 - 0.01998) - 0.0392]$$

$$= 1 + \frac{1}{6}(-0.11916) = 0.98014$$

Example 2 : Use Runge-Kutta method to find $y(0.1)$ for the equation $y'' + xy' + y = 0$, $y(0) = 1, y'(0) = 0$.

Solution : Substituting $\frac{dy}{dx} = z = f_1(x, y, z)$ ... (1)

The given equation reduces to

$$\frac{dz}{dx} = -xz - y = f_2(x, y, z) \qquad \text{... (2)}$$

Given $x_0 = 0, y_0 = 1, z_0 = y_0' = 0$. Also $h = 0.1$

By Runge-Kutta algorithm,

$k_1 = hf_1(x_0, y_0, z_0) = (0.1)f_1(0,1,0) = (0.1)(0) = 0$ $\qquad [\because f_1 = z]$

$l_1 = hf_2(x_0, y_0, z_0) = (0.1)f_2(0,1,0) = (0.1)(-1) = -0.1$ $\qquad [\because f_2 = -xz - y]$

$k_2 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= (0.1)f_1(0.05, 1, -0.05) = (0.1)(-0.05) = -0.005$

$l_2 = hf_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$

$= (0.1)f_2(0.05, 1, -0.05)$

$= (0.1)[-(0.05)(-0.05) - 1] = (0.1)(-0.9975)$

$= -0.09975$

$k_3 = hf_1\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$= (0.1)f_1(0.05, 0.9975, -0.0499)$

$= (0.1)\,(-0.0499) = -0.00499$

$l_3 = hf_2\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$

$= (0.1)f_2(0.05, 0.9975, -0.0499)$

$= (0.1)\,[-(0.05)(-0.0499) - 0.9975]$

$= (0.1)\,(-0.995005) = -0.09950$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$
$$= (0.1) f_1(0.1, 0.99511, -0.0995)$$
$$= (0.1)\ (-\ 0.0995) = -\ 0.00995$$

$$l_4 = (0.1) f_2(0.1, 0.99511, -0.0995)$$
$$= (0.1)[-(0.1)(-0.0995) - 0.99511] = -0.0985$$

$$\therefore \quad y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

i.e., $y(0.1) = 1 + \frac{1}{6}[0 + 2(-0.005 - 0.00499) - 0.00995]$

$$= 1 + \frac{1}{6}(-0.02993) = 0.9950$$

REVIEW QUESTIONS

1. Write the merits and demerits of Runge-Kutta Method.
2. Write the Runge - Kutta fourth order formulae. **[JNTU (A) June 2011 (Set No. 4)]**

EXERCISE 8.4

1. Use Runge - Kutta method of second order to find y when $x = 0.3$ in steps of 0.1, given that : $\frac{dy}{dx} = \frac{1}{2}(1+x)\,y^2$, $y(0) = 1$.
2. Obtain the values of y at $x = 0.1, 0.2$ using Runge - Kutta method of (i) second order (ii) third order (iii) fourth order for the differential equation $y' = x - 2y$, $y(0) = 1$ taking $h = 0.1$.
3. Given that $y' = y - x$, $y(0) = 2$ find $y(0.2)$ using Runge-Kutta method. Take $h = 0.1$ **[JNTU 2008 (Set No. 1), JNTU (H) June 2009 (Set No. 4)]**
4. Using Runge- Kutta method of fourth order,

   (i) Compute $y(1.1)$ for the equation $y' = 3x + y^2$, $y(1) = 1.2$

   (ii) Find $y(0.2)$ given $\frac{dy}{dx} = x + y$, $y(0) = 1$ taking $h = 0.2$
5. Using Runge - Kutta method of order 4, compute $y(2.5)$ for the equation

   $$\frac{dy}{dx} = \frac{x+y}{x}, \quad y(2) = 2$$
6. Using Runge-Kutta method, find $y(0.4)$ for the differential equation $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$. Take $h = 0.2$.
7. Apply the fourth order R - K method, to find $y(0.2)$ and $y(0.4)$ given that :

   $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Take $h = 0.1$

8. Using Runge - Kutta method of fourth order, solve
$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$. Find $y(0.2)$ and $y(0.4)$ **[JNTU(K) June 2009, May 2012 (Set No. 3)]**

9. Using R - K method, find $y(0.3)$ given that : $\frac{dy}{dx} + y + xy^2 = 0$, $y(0) = 1$, taking $h = 0.1$

10. Estimate $y(0.2)$, given $y' = 3x + \frac{1}{2}y$, $y(0) = 1$ by using Runge - Kutta method, taking $h = 0.1$

11. Evaluate $y(0.8)$ using R-K method given $y' = (x+y)^{1/2}$, $y = 0.41$ at $x = 0.4$
[JNTU (K) Nov. 2009S (Set No.4)]

12. Using Runge - Kutta method of 4th order find the solution of $\frac{dy}{dx} = x^2 + 0.25y^2$, $y(0) = -1$ on $[0, 0.5]$ with $h = 0.1$. **[JNTU (A) June 2013 (Set No. 4)]**

13. Solve $y'' - xy' + y = 0$ using R-K method for $x = 0.2$ given $y(0) = 1, y'(0) = 0$ taking $h = 0.2$.

ANSWERS

1. 1.2073 **2.** (*i*) 0.825, 0.6905 (*ii*) 0.8234, 0.6878 (*iii*) 0.8234, 0.6879
3. 2.4214 **4.** (*i*) 1.7278 **5.** 3.058 **6.** 0.02136 **7.** 1.0207, 1.038
8. 1.19598, 1.3751 **9.** 0.7144 **10.** 1.16722 **11.** 0.8489 **13.** 0.97993

8.15 PREDICTOR - CORRECTOR METHODS

So far we have discussed many methods for obtaining numerical solution of the differential equation $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. ... (1)

We divide the range for x into a number of subintervals of equal width. If x_i and x_{i-1} are two consecutive step locations, then $x_i = x_{i-1} + h$. For each x_i approximate values of y are calculated using a suitable recursive formula. These values are $y_0, y_1, y_2, \ldots$

All the earlier methods require information only from the last computed point (x_i, y_i) to estimate the next point (x_{i+1}, y_{i+1}). Therefore, all these methods are called **single-step** methods. They do not make use of information available at the earlier steps, y_{i-1}, y_{i-2} etc., even when they are available. It is possible to improve the efficiency of estimation by using the information at several previous points. Methods that use information from more than one previous point to compute the next point are called **multistep** methods. Sometimes, a pair of multistep methods are used in conjunction with each other, one for predicting the value of y_{i+1} and the other for correcting the predicted value of y_{i+1}. Such methods are called **Predictor - Corrector** methods.

For example, in solving equation (1) we used Euler's formula

$$y_{i+1} = y_i + h\, f(x_i, y_i), \quad i = 0, 1, 2, \ldots \qquad \ldots (2)$$

We improved this value by Modified Euler's method

$$y_{i+1} = y_i + \frac{1}{2}h\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(1)})\right] \quad \text{... (3)}$$

where $y_{i+1}^{(1)}$ is same as y_{i+1} of equation (2)

Here we obtained initially a crude estimate of y_{i+1} and subsequently refined it by means of a more accurate formula. This method is a **predictor - corrector** method. As the name suggests, we first predict a value for y_{i+1} (here as $y_{i+1}^{(1)}$) by using a certain formula and then correct this value by using a different formula. Hence equation (2) is used as the **predictor** and the equation (3) is used as the **corrector**.

A predictor formula is used to predict the value of y_{n+1} at x_{n+1} and a corrector formula is used to correct the error and to improve the value of y_{n+1}.

Multistep methods are not self starting. They need more information than the initial value condition. In the predictor - corrector (multistep) methods, four prior values are needed for finding the value of y at x_n. If a method uses four previous points, say y_0, y_1, y_2 and y_3, then all these values must be obtained before the method is actually used. These values, known as starting values, can be obtained using any of the single - step methods discussed earlier.

We have two popular predictor - corrector methods, namely : Milne's method and Adams - Bashforth - Moulton method. In this chapter we will discuss these methods.

8.16 MILNE'S PREDICTOR - CORRECTOR FORMULAE

Suppose we want to solve the equation $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0$ ... (1)

numerically.

Starting from y_0, we have to estimate successively

$$y_1 = y(x_0 + h) = y(x_1),\quad y_2 = y(x_0 + 2h) = y(x_2),\quad y_3 = y(x_0 + 3h) = y(x_3)$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0),\quad f_1 = f(x_0 + h, y_1),\quad f_2 = f(x_0 + 2h, y_2),\quad f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\,.\,\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + ... \quad \text{... (2)}$$

where $n = \frac{x - x_0}{h}$ *i.e.* $x = x_0 + nh$ in the relation

$$y_4 = y_0 + \int_{x_0}^{x_4} f(x, y)\, dx$$

$$\therefore\ y_4 = y_0 + \int_{x_0}^{x_0+4h} \left[f_0 + n\,\Delta f_0 + \frac{n(n-1)}{2\,!}\,\Delta^2 f_0 + \ldots \right] dx$$

$$= y_0 + h\int_0^4 \left(f_0 + n\,\Delta f_0 + \frac{n(n-1)}{2}\,\Delta^2 f_0 + \ldots \right) dn \ \text{(putting } x = x_0 + nh, dx = h\,dn\text{)}$$

$$= y_0 + h\left[f_0 n + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 f_0 + \ldots \right]_0^4$$

$$= y_0 + h\left[4 f_0 + 8\,\Delta f_0 + \frac{1}{2}\left(\frac{64}{3} - 8\right)\Delta^2 f_0 + \frac{1}{6}(64 - 64 + 16)\,\Delta^3 f_0 + \ldots \right]$$

$$= y_0 + h\left[4 f_0 + 8\,\Delta f_0 + \frac{20}{3}\,\Delta^2 f_0 + \frac{8}{3}\,\Delta^3 f_0 + \frac{14}{45}\Delta^4 f_0 + \ldots \right] \qquad \ldots\ldots (3)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4 = y_0 + h\left[4 f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0) + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0) \right]$$

$$= y_0 + h\left[\left(4 - 8 + \frac{20}{3} - \frac{8}{3}\right) f_0 + \left(8 - \frac{40}{3} + 8\right) f_1 + \left(\frac{20}{3} - 8\right) f_2 + \frac{8}{3} f_3 \right]$$

$$= y_0 + h\left[\frac{8}{3} f_1 - \frac{4}{3} f_2 + \frac{8}{3} f_3 \right]$$

$$i.e.,\ \ y_4^p = y_0 + \frac{4h}{3}(2 f_1 - f_2 + 2 f_3) \qquad \ldots\ldots (4)$$

$$= y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

which is called a predictor (the superscript 'p' indicating that it is a predicted value).

The formula (3) can be used to predict the value of y_4 when those of y_0, y_1, y_2 and y_3 are known.

In general, $$\boxed{y_{n+1}^p = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n')} \qquad \ldots\ldots (5)$$

$$i.e.,\ \ y_{n+1}^p = y_{n-3} + \frac{4h}{3}(2 f_{n-2} - f_{n-1} + 2 f_n)$$

Equation (5) is called **Milne's predictor** formula. The superscript 'p' indicates that y_{n+1}^p is a predicted value.

CORRECTOR FORMULA

To obtain Milne's corrector formula, we substitute Newton's forward interpolation formula given by the equation (2) in the relation

$$y_2 = y_0 + \int_{x_0}^{x_2} f(x, y)\, dx \qquad \text{..... (6)}$$

and get $y_2 = y_0 + \int_{x_0}^{x_2+2h} \left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ... \right] dx$

$$= y_0 + h\int_0^2 \left[f_0 + n\Delta f_0 + \frac{n^2 - n}{2}\Delta^2 f_0 + ... \right] dn \text{ (putting } x = x_0 + nh, dx = h\, dn\text{)}$$

$$= y_0 + h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{2} - \frac{n^2}{2}\right)\Delta^2 f_0 + ... \right]_0^2$$

$$= y_0 + h\left[2f_0 + 2\Delta f_0 + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 f_0 - \frac{4}{15}\cdot\frac{1}{24}\Delta^4 f_0 + ... \right]$$

$$= y_0 + h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0 - \frac{1}{90}\Delta^4 f_0 + ... \right]$$

Neglecting fourth and higher order differences and expressing Δf_0 and $\Delta^2 f_0$ in terms of the function values, we get

$$y_2 = y_0 + h\left[2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0) \right] = y_0 + \frac{h}{3}[f_0 + 4f_1 + f_2] \qquad \text{..... (7)}$$

i.e., $y_2^c = y_0 + \frac{h}{3}[y_0' + 4y_1' + y_2']$

In general, $\boxed{\boldsymbol{y_{n+1}^c = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + f_{n+1}^p\right]}}$ ..... (8)

(or) $y_{n+1}^c = y_{n-1} + \frac{h}{3}[y_{n-1}' + 4y_n' + y_{n+1}']$

Equation (8) is called **Milne's corrector** formula; The superscript $\boldsymbol{c}$ indicates that y_{n+1}^c is a corrected value and the superscript '$\boldsymbol{p}$' on R. H. S. indicates that the predicted value of y_{n+1} should be used for computing the value of $f(x_{n+1}, y_{n+1})$.

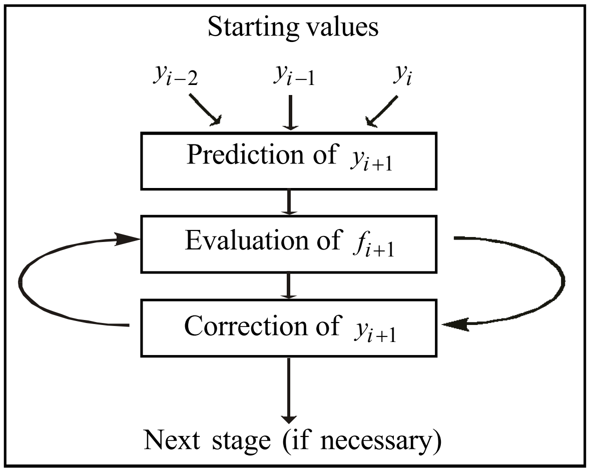
The value of y_4 obtained from equation (4) can therefore be corrected by using equation (7).

Hence we predict from

$$\boldsymbol{y_{n+1}^p = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)} \qquad \text{..... (9)}$$

and correct using

$$\boldsymbol{y_{n+1}^c = y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}^p)} \qquad \text{..... (10)}$$



Implementation of Predictor - Corrector method

Note 1: Knowing four consecutive values of y namely $y_{n-3}, y_{n-2}, y_{n-1}$ and y_n, we compute y_{n+1} using equation (9). Use this y_{n+1} on the R. H. S. of equation (10) to get y_{n+1} after correction. To refine the value further, we can use this latest y_{n+1} on the R. H. S. of (10) and get a better y_{n+1}.

Note 2 : To apply both Milne's and Adams Predictor - Corrector methods, we require four previous values of y. If in any problem, these values are not given, we can find them using Picard's method or Taylor's series method or Euler's method or Runge-Kutta method.

SOLVED EXAMPLES

Example 1 : Use Milne's predictor - corrector method to obtain the solution of the equation $y' = x - y^2$ at $x = 0.8$ given that $y(0) = 0, y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$.

Solution : Here $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, h = 0.2$ and

$y_0 = 0, y_1 = 0.02, y_2 = 0.0795, \quad y_3 = 0.1762$.

Also $f(x, y) = x - y^2 = y'$ ... (1)

By Milne's predictor formula

$$y_{n+1}^p = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$\therefore \; y_4^p = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3) \qquad \text{... (2)}$$

From (1) ,

$$y'_1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$$

$$y'_2 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$$

$$y'_3 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5689$$

Substituting these in equation (2), we predict the value of $y(0.8)$ as

$$y_4^p = 0 + \frac{4(0.2)}{3}(2\times 0.1996 - 0.3937 + 2\times 0.5689) = \frac{(0.8)}{3}(1.1433) = 0.30488$$

Now $y_4' = x_4 - y_4^2 = 0.8 - (0.30488)^2 = 0.7070$

Now we obtain the corrected value of $y(0.8)$ using Milne's corrector formula as

$$y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 0.0795 + \frac{0.2}{3}(0.3937 + 4\times 0.5689 + 0.7070) = 0.0795 + 0.2251 = 0.3046$$

$\therefore$ Corrected value of y at $x = 0.8$ is 0.3046.
Hence $y(0.8) = 0.3046$

Note. We can again use corrector formula to refine the estimate.
Now $y_4' = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

To refine y_4 further use $y_4^c = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$ with $y_4' = 0.7072$

$$\therefore\ y_4^c = 0.0795 + \frac{0.2}{3}(0.3937 + 4\times 0.5689 + 0.7072) = 0.0795 + 0.2251 = 0.3046$$

Example 2 : Use Milne's method to find $y(0.8)$ and $y(1.0)$ from $y' = 1 + y^2,\ y(0) = 0$. Find the initial values $y(0.2), y(0.4)$ and $y(0.6)$ from the Runge - Kutta method.

Solution :

To find initial values using R - K method

Here $f(x, y) = 1 + y^2$ and we take $h = 0.2$ and carry out the calculations in three steps.

Step 1. Here $x_0 = 0, y_0 = 0, h = 0.2$

$$\therefore\ k_1 = h\,f(x_0, y_0) = (0.2)\,f(0,0) = (0.2)\,(1+0) = 0.2$$

$$k_2 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = (0.2)\,[f(0.1, 0.1)] = (0.2)\,(1.01) = 0.202$$

$$k_3 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = (0.2)\,[f(0.1, 0.101)] = (0.2)\left[1 + (0.101)^2\right] = 0.20204$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = (0.2)\,[f(0.2, 0.20204)] = (0.2)\left[1 + (0.20204)^2\right] = 0.20816$$

Hence $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0 + \frac{1}{6}(0.2 + 0.404 + 0.40408 + 0.20816)$

$= 2027$, correct to four decimal places.

Step 2. $x_1 = 0.2, y_1 = 0.2027, h = 0.2$

$$\therefore\ k_1 = h\,f(x_1, y_1) = (0.2)\,[f(0.2, 0.2027)] = (0.2)\left[1 + (0.2027)^2\right] = 0.2082$$

$$k_2 = h\, f(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_1) = (0.2)\left[f(0.3, 0.3068)\right] = (0.2)\left[1 + (0.3068)^2\right] = 0.2188$$

$$k_3 = h\, f(x_1 + \frac{1}{2}h,\, y_1 + \frac{1}{2}k_2) = (0.2)\left[f(0.3, 0.3121)\right] = (0.2)\left[1 + (0.3121)^2\right] = 0.2195$$

$$k_4 = h\, f(x_1 + h,\, y_1 + k_3) = (0.2)\left[f(0.4, 0.4222)\right] = (0.2)\left[1 + (0.4222)^2\right] = 0.2356$$

Hence $y_2 = y\,(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 0.2027 + \frac{1}{6}(0.2082 + 0.4376 + 0.439 + 0.2356) = 0.2027 + 0.2201$$

$= 0.4228$, correct to four decimal places.

Step 3. $x_2 = 0.4,\ y_2 = 0.4228,\ h = 0.2$

Proceeding as above, we get $y_3 = y\,(0.6) = 0.6841$

To find y_4 using Milne's method.

Now, knowing y_0, y_1, y_2, y_3 we will find y_4

$y'_1 = 1 + y_1^2 = 1 + (0.2027)^2 = 1.0411$; $\quad y'_2 = 1 + y_2^2 = 1 + (0.4228)^2 = 1.1787$

$y'_3 = 1 + y_3^2 = 1 + (0.6841)^2 = 1.4681$

By Milne's predictor formula,

$$y_4^p = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + y'_3)$$

$$= 0 + \frac{4}{3}(0.2)\,[\,2\,(1.0411) - 1.1787 + 2\,(1.4681)] = 1.0239$$

Now $y'_4 = 1 + y_4^2 = 1 + (1.0239)^2 = 2.0484$

To correct this value of $y\,(0.8)$, we use the Milne's corrector formula,

$$y_4^c = y_2 + \frac{h}{3}(y'_2 + 4y'_3 + y'^{p}_4)$$

$$= 0.4228 + \frac{0.2}{3}[\,1.1787 + 4\,(1.4681) + 2.0484\,] = 0.4228 + 0.6066 = 1.0294$$

To find y (1.0)

Milne's predictor formula at $n = 4$ is

$$y_5^p = y_1 + \frac{4h}{3}(2y'_2 - y'_3 + 2y'_4)$$

Now $y'_4 = 1 + y_4^2 = 1 + (1.0294)^2 = 2.05966$

$$\therefore\ y_5 = 0.2027 + \frac{4}{3}(0.2)\,[\,2\,(1.1787) - 1.4681 + 2\,(2.05966)\,] = 0.2027 + 1.3356 = 1.5383$$

i.e. $y\,(1.0) = 1.5383$, correct to four decimal places

To correct this value of $y(1.0)$, we use the Milne's corrector formula at $n=4$.

That is $y_5^c = y_3 + \frac{h}{3}[y_3' + 4y_4' + y_5']$.

Now $y_5' = 1 + y_5^2 = 1 + (1.5383)^2 = 3.3664$

$\therefore\ y_5 = y(1.0) = 0.6841 + \frac{0.2}{3}[1.4681 + 4(2.05966) + 3.3664] = 0.6841 + 0.87154 = 1.5556$

Example 3 : Use Milne's method to find $y(0.3)$ from $y' = x^2 + y^2, y(0) = 1$. Find the initial values $y(-0.1), y(0.1)$ and $y(0.2)$ from the Taylor's series method.

Solution : Here $x_0 = 0, y_0 = 1$.

Given equation is $y' = f(x,y) = x^2 + y^2$

Differentiating successively w.r.t. x, we get

$y'' = 2x + 2yy'$; $y''' = 2 + 2[y\,y'' + (y')^2]$

At $x = 0, y = 1$. $\quad\therefore\ y'(0) = 1, y''(0) = 2\times 0 + 2\times 1\times 1 = 2$ and $y'''(0) = 2 + 2(1\times 2 + 1) = 8$

The Taylor series for $y(x)$ near $x = 0$ is given by

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + ...$$

Substituting the above values, we get

$$y(x) = 1 + x + x^2 + \frac{4x^3}{3} + ...$$

$$\therefore\ y(-0.1) = 1 - 0.1 + (-0.1)^2 + \frac{4(-0.1)^3}{3} + ... = 0.9087$$

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} + ... = 1.1113$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} + ... = 1.2506$$

Now $x_{-1} = -0.1, x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3, h = 0.1$

and $y_{-1} = 0.9087, y_0 = 1, y_1 = 1.1113, y_2 = 1.2506$

$\therefore\ y_0' = f(x_0, y_0) = 0 + 1 = 1 = f_0$; $\quad y_1' = f(x_1, y_1) = (0.1)^2 + (1.1113)^2 = 1.2449 = f_1$

$y_2' = f(x_2, y_2) = (0.2)^2 + (1.2506)^2 = 1.6040 = f_2$

Now, knowing y_{-1}, y_0, y_1 and y_2 we will find y_3.

By Milne's predictor formula,

$$y_3^p = y_{-1} + \frac{4h}{3}(2f_0 - f_1 + 2f_2) \quad .... (1)$$

$$= 0.9087 + \frac{0.4}{3}(2 - 1.2449 + 3.2080) = 1.4371$$

Now $y'_3 = f(x_3, y_3) = (0.3)^2 + (1.4371)^2 = 2.1552 = f_3$

Now we obtain the corrected value of $y(0.3)$.

Using Milne's corrector formula,

$$y_3^c = y_1 + \frac{h}{3}(f_1 + 4f_2 + f_3) \qquad \text{.... (2)}$$

$$= 1.1113 + \frac{0.1}{3}(1.2449 + 6.4160 + 2.1552) = 1.4385.$$

Hence $y(0.3) = 1.4385$

Note. We can use this $y(0.3)$ on the R. H. S. of (2) and get an improved value of y_4.

Example 4 : Find the solution of $\frac{dy}{dx} = x - y$ at $x = 0.4$ subject to the condition $y = 1$ at $x = 0$ and $h = 0.1$ using Milne's method. Use Euler's modified method to evaluate $y(0.1)$, $y(0.2)$ and $y(0.3)$. **[JNTU 2007 (Set No. 4)]**

Solution : Here $y' = f(x, y) = x - y$, $y_0 = 1$ and $h = 0.1$

To find initial values using Euler's modified method.

From solved Example **5** on page 827, we have

$y_1 = y(0.1) = 0.9095$, $y_2 = y(0.2) = 0.8371$ and $y_3 = y(0.3) = 0.7812$

Using the values of y_0, y_1, y_2 and y_3, we have to find y_4 by Milne's method.

$y'_1 = f(x_1, y_1) = x_1 - y_1 = 0.1 - 0.9095 = -0.8095$

$y'_2 = f(x_2, y_2) = x_2 - y_2 = 0.2 - 0.8371 = -0.6371$

$y'_3 = f(x_3, y_3) = x_3 - y_3 = 0.6 - 0.7812 = -0.1812$

By Milne's predictor formula,

$$y_4^p = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + y'_3)$$

$$= 1 + \frac{4(0.1)}{3}[-1.619 + 0.6371 - 0.1812]$$

$$= 1 - 0.15508 = 0.84492$$

Now $y'_4 = y'^p_4 = f(x_4, y_4) = x_4 - y_4 = 0.4 - 0.84492 = -0.44492$

To correct this value of y_4 i.e. $y(0.4)$, we use the Milne's corrector formula.

$$y_4^c = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'^p_4\right)$$

$$= 0.8371 + \frac{0.1}{3}(-0.6371 - 0.7248 - 0.44492)$$

$$= 0.8371 - 0.06023 = 0.7769$$

$\therefore \quad y(0.4) = y_4 = 0.7769$

EXERCISE 8.5

1. Given $y' = x^2(1+y)$ and $y(1)=1, y(1.1)=1.233, y(1.2)=1.548, y(1.3)=1.974$. Estimate $y(1.4)$ using Milne's predictor - corrector method.
2. Solve numerically, using Milne's method $y' = 1+xy^2, y(0)=1$. Take the starting values $y(0.1)=1.105, y(0.2)=1.223, y(0.3)=1.355$. Find the value of $y(0.4)$.
3. Given the differential equation $y' = \frac{2y}{x}$ with $y(1)=2$, compute $y(2)$ by Milne's method. Find the starting values using Runge - Kutta method taking $h=0.25$.
4. Use Milne's method to find $y(0.8)$ and $y(1.0)$ given : $y' = \frac{1}{x+y}, y(0)=2$ and $y(0.2)=2.0933, y(0.4)=2.1755, y(0.6)=2.2493$.
5. Given $y' = y - x^2, y(0)=1$ and the starting values $y(0.2)=1.12186, y(0.4)=1.4682$, $y(0.6)=1.7379$, evaluate $y(0.8)$ using Milne's predictor - corrector method.

   (or) Find $y(0.8)$ by Milne's method for $\frac{dy}{dx} = y - x^2, y(0)=1$ obtain the starting values by Taylor's series method. **[JNTU (A) June 2013 (Set No. 3)]**
6. Using Milne's predictor and corrector formulae, find $y(4.4)$ given :
   $5xy' + y^2 - 2 = 0, y(4)=1, y(4.1)=1.0049, y(4.2)=1.0097$ and $y(4.3)=1.0143$.
7. Use Milne's method to find $y(0.4)$ from $y' = xy + y^2, y(0)=1$. Find the initial values $y(0.1), y(0.2)$ and $y(0.3)$ from the Taylor's series method.
8. Calculate $y(0.6)$ by Milne's predictor-corrector method given $y' = x+y, y(0)=1$ with $h=0.2$. Obtain the required data by Taylor's series method.
9. Compute $y(0.6)$ given $y' = x+y, y(0)=1$ with $h=0.2$ using Milne's predictor - corrector method .
10. Use Milne's predictor - corrector method to obtain the solution of the equation $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ at $x = 1.4$ given that $y(1) = 1, y(1.1) = 0.996, y(1.2) = 0.986, y(1.3) = 0.972$
11. Determine $y(0.8)$ and $y(1.0)$ by Milne's predictor - corrector method when $\frac{dy}{dx} = x - y^2, y(0) = 0$. [**Hint :** Refer Solved Example 1] **[JNTU (A) June 2013 (Set No. 2)]**

ANSWERS

1. 2.575	**2.** 1.5	**3.** 8.00	**5.** 2.0111	**6.** 1.01874
7. 1.83698	**8.** 2.0442	**9.** 2.0439	**10.** 0.949	