D' Alembert's principle

(And Equations of Motion of a Rigid Body)

§ 1. Motion of a Particle.

The motion of a particle is dertermined by Newton's second law of motion, which states that "*the rate of change of momentum in any direction is proportional to the applied force in that direction*". From this law, we deduce the formula $P = mf$, where f is the acceleration of the particle of mass m in the direction of the applied force P.

If (x, y, z) be the coordinates of a moving particle of mass m, at any time t and X, Y, Z be the components of the external forces acting on it parallel to the axes, then by Newton's second law of motion the equations of motion of the particle are

$$m\ddot{x} = X, m\ddot{y} = Y, m\ddot{z} = Z.$$

§ 2. Motion of a Rigid Body.

A *rigid body* is an assemblage of particles rigidly connected together such that the distance between any two constituent particles does not change on account of the effect of forces. **[Meerut 95 BP]**

For a rigid body we assume that

(i) the mutual actions between its two particles act along the straight line joining them,

(ii) the action and reaction between the two particles are equal and opposite.

In considering the motion of a rigid body, we write the equations of motion of the particles of the body according to the equations in § 1. But here the external forces acting on a particle of the body include, together with the applied forces, the unknown inner forces acting due to the action of the rest of the body on it.

D' Alembert proposed a method which enables us to obtain all the necessary equations without writing down the equations of motion of all particles and without considering the unknown inner forces. This important principle is based on the following rule which is a natural consequence of Newton's third law of motion.

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

§ 3. Definitions.

Impressed forces. *The external forces acting on a body are called 'impressed forces'.* For example, the weight of the body is the impressed force on the body. In case a body is tied to a string then the tension in the string is also an impressed force on the body. **[Meerut 95 BP]**

Effective forces. *The effective force on a particle is defined as the product of its mass m and its acceleration f.* If a particle of mass m is situated at the point (x, y, z) at time t, then the effective forces on this particle at this time t are $m\ddot{x}, m\ddot{y}, m\ddot{z}$ parallel to the co-ordinate axes.

[Meerut 95BP]

§ 4. D' Alembert's Principle.

The reversed effective forces acting on each particle of the body and the impressed (external) forces on the system are in equilibrium.

[Meerut 80, 81, 82, 83, 84, 85, 86, 87, 88, 90, 91P, 92, 92S, 93, 94; Raj. 80, 82]

Let (x, y, z) be the coordinates of a particle of mass m, of a rigid body which is in motion, at any time t. If f is the resultant of component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$ of m then the effective force on the particle is mf. Let F denote the resultant of the impressed forces and R the resultant of the internal forces (mutual actions) on the particle. Then by Newton's second law of motion, mf is the resultant of F and R. Thus $-mf$ (reversed effective force), F and R are in equilibrium. This holds good for every particle of the body.

Thus the system of forces $\Sigma(-mf)$, ΣF and ΣR are in equlibrium, where the summation sign Σ has been simply used in the sense that we have extended our consideration to all the particles of the body.

But the internal actions and reactions of different particles of a body are in equilibruim amongst themselves *i.e.*, the system of forces ΣR are in equilibrium. Therefore the system of forces $\Sigma(-mf)$ and ΣF are in equilibrium.

Hence the reversed effective forces acting on each particle of the body and the impressed (external) forces on the system are in equilibrium.

Vector Method.

Consider a rigid body in motion. At time t, let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ and $\mathbf{R}$ the external and internal forces respectively acting on it.

$\therefore$ By Newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + \mathbf{R} \quad \text{or} \quad \mathbf{F} + \mathbf{R} - m\frac{d^2\mathbf{r}}{dt^2} = 0$$

i.e. the forces $\mathbf{F}, \mathbf{R}, -m\dfrac{d^2\mathbf{r}}{dt^2}$ acting on a particle of mass m are in equilibrium.

Now applying the same argument to every particle of the rigid body, the forces $\Sigma\,\mathbf{F}, \Sigma\,\mathbf{R}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles of the body.

But the internal forces acting on the body form pairs of equal and opposite forces. $\therefore\ \Sigma\,\mathbf{R} = \mathbf{0}$.

Thus the forces $\Sigma\,\mathbf{F}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium

i.e. $$\Sigma\,\mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{0}.$$

Hence the reversed effective forces acting on each particle of the body and the impressed (external) forces on the system are in equilibrium.

Note. The above D' Alembert's principle reduces the problem of dynamics to the problem of statics. Thus we mark all the external forces of the system and mark the effective forces in opposite directions and then solve this problem as a problem of statics. If the forces are coplanar, then we equate to zero the resolved parts of all these forces in two mutually perpendicular directions and take moments about a suitable point.

§ 5. General Equations of motion of a body.

To deduce the general equations of motion of a rigid body from D' Alembert's principle. **[Meerut 80, 81, 82, 83, 89, 92, 94, 96; Raj. 82]**

Let X, Y, Z be the components, parallel to the axes, of the external forces acting on a particle of mass m whose coordinates are (x, y, z) at time t, referred to any set of rectangular axes. Then the reversed effective forces parallel to the axes on the particle m are $-m\ddot{x}, -m\ddot{y}, -m\ddot{z}$. Thus the components of the resultant of the external forces and the reversed effective forces acting on the particle m parallel to the axes are $X - m\ddot{x}$, $Y - m\ddot{y}$ and $Z - m\ddot{z}$ respectively.

By D' Alembert's principle the forces whose components are $X - m\ddot{x}, Y - m\ddot{y}, Z - m\ddot{z}$ acting at the particle m at (x, y, z) together with similar forces acting at each other particle of the body, form a system in equilibrium.

Hence as in statics the six conditions of equilibrium* are

$$\Sigma\,(X - m\ddot{x}) = 0, \Sigma\,(Y - m\ddot{y}) = 0, \Sigma\,(Z - m\ddot{z}) = 0,$$
$$\Sigma\,\{y\,(Z - m\ddot{z}) - z\,(Y - m\ddot{y})\} = 0, \Sigma\,\{z\,(X - m\ddot{x}) - x\,(Z - m\ddot{z})\} = 0$$

* The necessary and sufficient conditions for the equilibrium of a rigid body under the action of forces acting at different points of the body are that

$$\Sigma X_r = 0, \Sigma Y_r = 0, \Sigma Z_r = 0, L = \Sigma(y_r Z_r - z_r Y_r) = 0,$$
$$M = \Sigma(z_r X_r - x_r Z_r) = 0 \text{ and } N = \Sigma(x_r Y_r - y_r X_r) = 0.$$

and $\Sigma\{x(Y - m\ddot{y}) - y(X - m\ddot{x})\} = 0$

where the summation is extended to all the particles of the body.

These six equations can be written as

$$\Sigma m\ddot{x} = \Sigma X \quad \text{...(1)} \qquad \Sigma m\ddot{y} = \Sigma Y \quad \text{...(2)}$$

$$\Sigma m\ddot{z} = \Sigma Z \quad \text{...(3)} \qquad \Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY) \quad \text{...(4)}$$

$$\Sigma m(z\ddot{x} - x\ddot{z}) = \Sigma(zX - xZ) \quad \text{...(5)}$$

and $$\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX). \quad \text{...(6)}$$

These equations (1) to (6) are the general equations of motion of a body.

Equations (1), (2), (3) state that the *sums of the components, parallel to the coordinates axes of the effecive forces are respectively equal to the sums of the components parallel to the same axes of the external (impressed) forces.*

Equations (4), (5), (6) state the *sums of the moments about the axes of coordinates of the effective forces are respectively equal to the sums of the moments about the same axes of the external (impressed) forces.*

The equations (1), (2) and (3) can be written as $\frac{d}{dt}(\Sigma m\dot{x}) = \Sigma X$, $\frac{d}{dt}(\Sigma m\dot{y}) = \Sigma Y$ and $\frac{d}{dt}(\Sigma m\dot{z}) = \Sigma Z$.

These show that the *rate of change of linear momentum of the system in any direction is equal to the total external force in that direction.*

The equations (4), (5) and (6) can be written as

$$\frac{d}{dt}\{\Sigma m(y\dot{z} - z\dot{y})\} = \Sigma(yZ - zY), \frac{d}{dt}\{\Sigma m(z\dot{x} - x\dot{z})\} = \Sigma(zX - xZ)$$

and $\frac{d}{dt}\{\Sigma m(x\dot{y} - y\dot{x})\} = \Sigma(xY - yX)$.

These show that *the rate of change of angular momentum (moment of momentum) about any given axis is equal to the total moment of all the external forces about that axis.*

Vector Method. Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ the external forces acting on it.

Then by D' Alembert's principle, we have

$$\Sigma \mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = \mathbf{0}$$

or $$\Sigma m\frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{F}. \quad \text{...(1)}$$

Taking cross product by $\mathbf{r}$, we have

$$\Sigma m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{r} \times \mathbf{F}. \quad \text{...(2)}$$

Equations (1) and (2) are in general vector equations of motion of a rigid body.

Deduction of general equations of motion in scalar form.

To deduce the general equations of motion of a rigid body, we put
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \text{and} \quad \mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$$
in (1) and (2) where (x, y, z) are the cartesian coordinates of the particle m and X, Y, Z are the components of the force $\mathbf{F}$ parallel to the coordinate axes.

So substituting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ in (1) and (2), we get

$$\Sigma m\,(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k}) = \Sigma\,(X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}) \qquad \text{...(3)}$$

and
$$\Sigma\,\{m\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \times (\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j} + \ddot{z}\,\mathbf{k})\}$$
$$= \Sigma\,\{(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \times (X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k})\}$$

or
$$\Sigma m\,\{(y\ddot{z} - z\ddot{y})\,\mathbf{i} + (z\ddot{x} - x\ddot{z})\,\mathbf{j} + (x\ddot{y} - y\ddot{x})\,\mathbf{k}\}$$
$$= \Sigma\,\{(yZ - zY)\,\mathbf{i} + (zX - xZ)\,\mathbf{j} + (xY - yX)\,\mathbf{k}\} \qquad \text{...(4)}$$

Equating the coefficients of **i**, **j**, **k** on the two sides of the equations (3) and (4), we get the six equations of motion of the rigid body in cartesion form.

§ 6. Linear Momentum.

The linear momentum of a body in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity of a body of mass M, then we have

$$\bar{x} = \frac{\Sigma\, mx}{\Sigma\, m} = \frac{\Sigma\, mx}{M}. \qquad [\because\ \Sigma\, m = M]$$

$\therefore\ \Sigma\, mx = M\bar{x}$. Similarly, $\Sigma\, my = M\bar{y}$ and $\Sigma\, mz = M\bar{z}$.

Differentiating these relations w.r.t. 't', we get

$$\Sigma\, m\dot{x} = M\dot{\bar{x}},\ \Sigma\, m\dot{y} = M\dot{\bar{y}},\ \text{and}\ \Sigma\, m\,\dot{z} = M\dot{\bar{z}}.$$

Hence the result.

§ 7. Motion of the Centre of Inertia.

To show that the centre of inertia of a body moves as if all the mass of the body were collected at it and if all the external forces acting on the body were acting on it in directions parallel to those in which they act.

[Meerut 81; Raj. 80]

If $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of inertia of a body of mass M, then as in § 6, we have

$$\Sigma\, mx = M\bar{x},\ \Sigma\, my = M\bar{y},\ \Sigma\, mz = M\bar{z}.$$

Differentiating twice w.r.t. 't', we get

$$\Sigma m \ddot{x} = M\ddot{\bar{x}}, \Sigma m \ddot{y} = M\ddot{\bar{y}} \text{ and } \Sigma m \ddot{z} = M\ddot{\bar{z}} \quad ...(1)$$

But from the general equations of motion of a body, we get (see § 5)

$$\Sigma m \ddot{x} = \Sigma X, \Sigma m \ddot{y} = \Sigma Y \text{ and } \Sigma m \ddot{z} = \Sigma Z. \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$M\ddot{\bar{x}} = \Sigma X, M\ddot{\bar{y}} = \Sigma Y \text{ and } M\ddot{\bar{z}} = \Sigma Z.$$

These are the equations of motion of a particle of mass M placed at the centre of inertia of the body, and acted on by forces $\Sigma X, \Sigma Y, \Sigma Z$ parallel to the original directions of the forces acting on the different points of the body.

This proves the theorem.

Vector method. Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle m of the body and $\mathbf{F}$ the external force acting on it. Then the equation of motion of the body is

$$\Sigma m \frac{d^2\mathbf{r}}{dt^2} = \Sigma \mathbf{F}. \quad ...(1)$$

If $\bar{\mathbf{r}}$ is the position vector of the centre of inertia of the body, then we have

$$\bar{\mathbf{r}} = \frac{\Sigma m\mathbf{r}}{\Sigma m} = \frac{\Sigma m\mathbf{r}}{M} \quad \text{or} \quad \Sigma m\mathbf{r} = M\bar{\mathbf{r}}.$$

$$\therefore \Sigma m \frac{d^2\mathbf{r}}{dt^2} = M\frac{d^2\bar{\mathbf{r}}}{dt^2}. \quad ...(2)$$

From (1) and (2), we have

$$M\frac{d^2\bar{\mathbf{r}}}{dt^2} = \Sigma \mathbf{F}. \quad ...(3)$$

This is the vector form of the equation of motion of a particle of mass M placed at the centre of inertia of the body and acted upon by the external forces $\Sigma \mathbf{F}$.

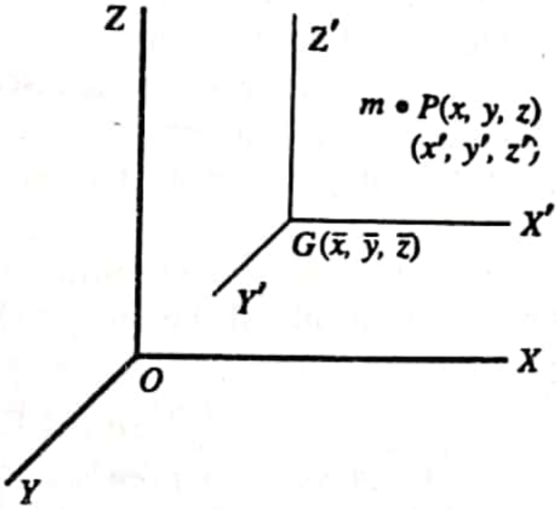
Deduction of the equations of motion of the centre of inertia in scalar form. Substituting $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ in (3) and equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from the two sides we can get the equations of motion of the centre of inertia in scalar form.

Note. The proposition discussed in § 7 is called *the principle of conservation of motion of translation*. From this it follows that the motion of C. G. is independent of rotation.

§ 8. Motion Relative to the Centre of Inertia.

To show that the motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body. **[Meerut 1991S; Kanpur 82; Raj. 79, 80, 81, 83]**

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity (centre of inertia) G of the body referred to the rectangular axes OX, OY, OZ through a fixed point O. Let GX', GY', GZ' be the axes through G parallel to the axes OX, OY, OZ respectively.



If (x, y, z) and (x', y', z') are the coordinates of a particle of mass m at P referred to the coordinate axes OX, OY, OZ and parallel axes GX', GY', GZ' respectively, then

$$x = \bar{x} + x', y = \bar{y} + y', z = \bar{z} + z'.$$

$$\therefore \quad \ddot{x} = \ddot{\bar{x}} + \ddot{x}', \ddot{y} = \ddot{\bar{y}} + \ddot{y}', \ddot{z} = \ddot{\bar{z}} + \ddot{z}'.$$

Now consider the equation $\Sigma m (y\ddot{z} - z\ddot{y}) = \Sigma (yZ - zY)$, which becomes

$$\Sigma m \{(\bar{y} + y')(\ddot{\bar{z}} + \ddot{z}') - (\bar{z} + z')(\ddot{\bar{y}} + \ddot{y}')\} = \Sigma \{(\bar{y} + y') Z - (\bar{z} + z') Y\}$$

or

$$\Sigma m (y'\ddot{z}' - z'\ddot{y}') + \bar{y}\ddot{\bar{z}} \Sigma m + \bar{y} \Sigma m\ddot{z}' + \ddot{\bar{z}} \Sigma my' - \bar{z}\ddot{\bar{y}} \Sigma m - \bar{z} \Sigma m\ddot{y}' - \ddot{\bar{y}} \Sigma mz'$$
$$= \Sigma (y'Z - z'Y) + \bar{y} \Sigma Z - \bar{z} \Sigma Y. \quad ...(1)$$

Now referred to GX', GY', GZ' as axes the coordinates of G are $(0, 0, 0)$.

$$\therefore \frac{\Sigma mx'}{\Sigma m} = 0 \text{ or } \Sigma mx' = 0.$$

Similarly, $\Sigma my' = 0, \Sigma mz' = 0$.

$$\therefore \Sigma m\ddot{x}' = 0, \Sigma m\ddot{y}' = 0, \Sigma m\ddot{z}' = 0.$$

Also from § 7, we have $M\ddot{\bar{x}} = \Sigma X, M\ddot{\bar{y}} = \Sigma Y, M\ddot{\bar{z}} = \Sigma Z$.

Thus from eqn. (1), we get

$$\Sigma m (y'\ddot{z}' - z'\ddot{y}') + \bar{y}\ddot{\bar{z}} M - \bar{z}\ddot{\bar{y}} M = \Sigma (y'Z - z'Y) + \bar{y} \Sigma Z - \bar{z} \Sigma Y$$

$$\Sigma\, m\,(y'\ddot{z}' - z'\ddot{y}') + \bar{y}\,\Sigma\, Z - \bar{z}\,\Sigma\, Y = \Sigma\,(y'Z - z'Y) + \bar{y}\,\Sigma\, Z - \bar{z}\,\Sigma\, Y$$

or $$\Sigma\, m\,(y'\ddot{z}' - z'\ddot{y}') = \Sigma\,(y'Z - z'Y).$$

or Similarly, we get the other two equations as

$$\Sigma\, m\,(z'\ddot{x}' - x'\ddot{z}') = \Sigma\,(z'X - x'Z)$$

and $$\Sigma\, m\,(x'\ddot{y}' - y'\ddot{x}') = \Sigma\,(x'Y - y'X).$$

But these equations are the same as would have been obtained if we had regarded the centre of gravity as fixed point.

Hence the proposition.

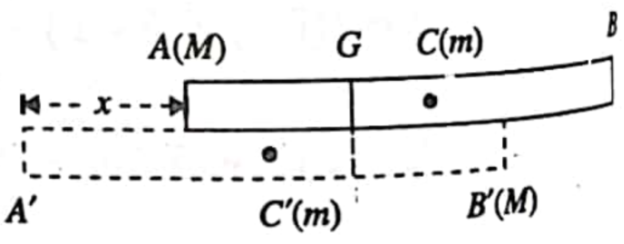
Note 1. The proposition discussed in § 8 is called the *principle of conservation of motion of rotation*. From this it follows that the motion round the centre of inertia is independent of its motion of translation.

Note 2. The two propositions discussed in § 7 and 8 together prove the principle of the independence of the motion of translation and rotation.

Solved Examples

Ex. 1. *A rough uniform board, of mass m and length 2a, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time.* **[Meerut 78, 80, 82, 91P, 91S, 93S; Kanpur 81, 82]**

Sol. Here the external forces are, (i) the weights of the board and the man acting vertically downwards and (ii) the reaction of the horizontal plane acting vertically upwards. Thus there are no external forces in the horizontal direction, therefore by D' Alembert's principle, the C.G of the system will remain at rest. As a matter of fact as the man moves forward, the board slips backwards, keeping the position of C. G. of the system unchanged.



Let AB be the position of the board when the man of mass M is at A.

$\therefore$ Distance of C. G. of the system from A (towards B)

$$= \frac{M.0 + m.AC}{M+m} = \frac{M.0 + m.a}{M+m} = \frac{ma}{M+m} = x_1 \text{ (say)}.$$

$$(\because AC = BC = a)$$

Let $A'B'$ be the position of the board when the man reaches the other end of the board. If the board slips through a distance $AA' = x$ (backwards) during the time the man walks from A to B, then in this position the distance of C. G. of the system from A (towards B)

$$= \frac{M.AB' + m.AC'}{M+m} = \frac{M.(2a-x)+m(a-x)}{M+m} = x_2 \text{ (say).}$$

Since the position of the C. G., 'G' of the system remains unchanged,

$$\therefore x_1 = x_2$$

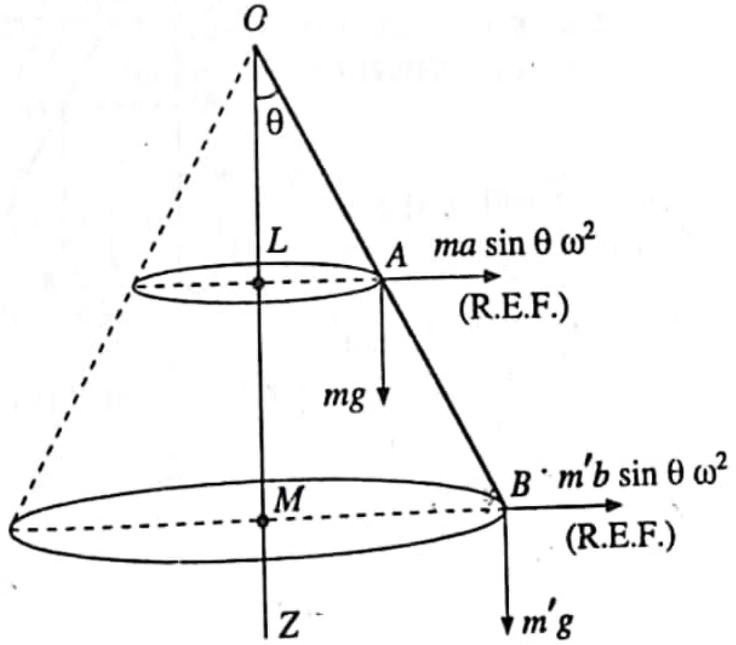
or
$$\frac{ma}{M+m} = \frac{m(2a-x)+m(a-x)}{M+m}$$

or $\quad ma = 2aM + ma - (M+m)x$ or $x = 2aM/(m+M)$,

which is the required distance.

Ex. 2. *Find the motion of the rod OAB, with two masses m and m' attached to it at A and B respectively, when it moves round the vertical as a conical pendulum with uniform angular velocity, the angle θ which the rod makes with the vertical being constant.*

Sol. Let the light rod OAB with two masses m and m' attached at A and B respectively such that $OA = a$ and $OB = b$, revolve round the vertical OZ as a conical pendulum with uniform angular velocity ω, making a constant angle θ with the vertical. In this position the masses m and m' move in circles on horizontal planes with radii $AL = a \sin\theta$ and $BM = b \sin\theta$ with centres at L and M respectively.



The motion about the vertical being with uniform angular velocity, the effective forces are entirely inwards. The rod being taken as light, there is no effective force on any element of the rod due to its weight. The effective forces on the particles of masses m and m' at A and B are $ma \sin\theta\,\omega^2$ and $m'b \sin\theta\omega^2$ along AL and BM respectively.

By D' Alembert's principle the external forces *i.e.*, the weights mg, $m'g$ and the reaction at O, and the reversed effective forces $ma \sin\theta\omega^2$ along LA and $m'b \sin\theta\omega^2$ along MB are in equilibrium.

Now to avoid reaction at O, taking moments about the point O, we get

$$ma \sin\theta\omega^2 . OL + m'b \sin\theta\omega^2 . OM - mg . LA - m'g . MB = 0$$

or $$(ma \sin\theta . a \cos\theta + m'b \sin\theta . b \cos\theta)\,\omega^2 = g\,(ma \sin\theta + m'b \sin\theta)$$

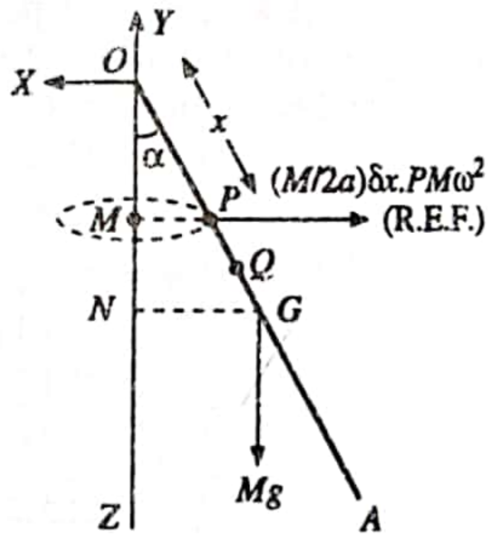
or $$\omega^2 = \frac{(ma + m'b)\,g}{(ma^2 + m'b^2)\cos\theta}, \text{ since } \sin\theta \neq 0.$$

This determines the motion of the rod.

Ex. 3. *A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}(3g/4a\omega^2)$.*

[Meerut 80, 81, 84P, 92, 95BP; Rohilkhand 83]

Sol. Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, inclined at a constant angle α to OZ. Let $PQ = \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$.



The element PQ will make a circle in the horizontal plane with radius PM $(= x \sin\alpha)$ and the centre at M. Since the rod revolves with uniform angular velocity, the only effective force on this element is

$$\frac{M}{2a}\delta x . PM\,\omega^2 \text{ along } PM.$$

Thus the reversed effective force on the element PQ is $\frac{M}{2a}\delta x . x \sin\alpha . \omega^2$ along MP.

Now by D' Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight Mg and reaction at O, are in equilibrium.

To avoid reaction at O, taking moments about O, we get

$$\Sigma\left[\left(\frac{M}{2a}\delta x . \omega^2 . x \sin\alpha\right) . OM\right] - Mg . NG = 0$$

or $$\int_0^{2a} \frac{M}{2a}\omega^2 x^2 \sin\alpha \cos\alpha\, dx - Mg . a \sin\alpha = 0$$

or $\qquad \frac{M}{2a}\omega^2 . \{\frac{1}{3}(2a)^3\} . \sin\alpha\cos\alpha - Mg\,a\sin\alpha = 0 \qquad (\because OM = x\cos\alpha)$

or $$Mg\,a\sin\alpha\left(\frac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0.$$

$\therefore$ either $\sin\alpha = 0$ *i.e.* $\alpha = 0$

or $\qquad \frac{4a}{3g}\omega^2\cos\alpha - 1 = 0$ *i.e.* $\cos\alpha = \frac{3g}{4a\omega^2}$.

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

Note. If $\omega^2 < \frac{3g}{4a}$, then $\cos\alpha > 1$, therefore in this case $\cos\alpha = \frac{3g}{4a\omega^2}$ gives an impossible value of α *i.e.* when $\omega^2 < \frac{3g}{4a}$, then $\alpha = 0$ is the only possible value of α.

Ex. 4. *A rod, of length 2a, revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that $\omega^2 = 3g/(4a\cos\alpha)$.*

Prove also that the direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$.

(Raj. 82)

Sol. Refer figure of last Ex. 3. Proceeding as in last Ex. 3, we get

$$\cos\alpha = \frac{3g}{4a\omega^2} \text{ i.e., } \omega^2 = \frac{3g}{4a\cos\alpha}. \qquad ...(1)$$

Second part

Let the horizontal and vertical components of the reaction at the hinge O be X and Y respectively (as shown in the figure). Then resolving the forces horizontally and vertically, we get

$$X = \Sigma\frac{M}{2a}\delta x . PM . \omega^2 = \int_0^{2a}\frac{M}{2a}\omega^2 x\sin\alpha\,dx \quad [\because PM = x\sin\alpha]$$

$$= \frac{M}{2a}\omega^2\left\{\tfrac{1}{2}(2a)^2\right\}\sin\alpha = Ma\,\omega^2\sin\alpha$$

and $\qquad Y = Mg.$

If the reaction at O makes an angle θ with the vertical, then

$$\tan\theta = \frac{X}{Y} = \frac{Ma\omega^2\sin\alpha}{Mg} = \frac{a}{g}\cdot\left(\frac{3g}{4a\cos\alpha}\right)\sin\alpha$$

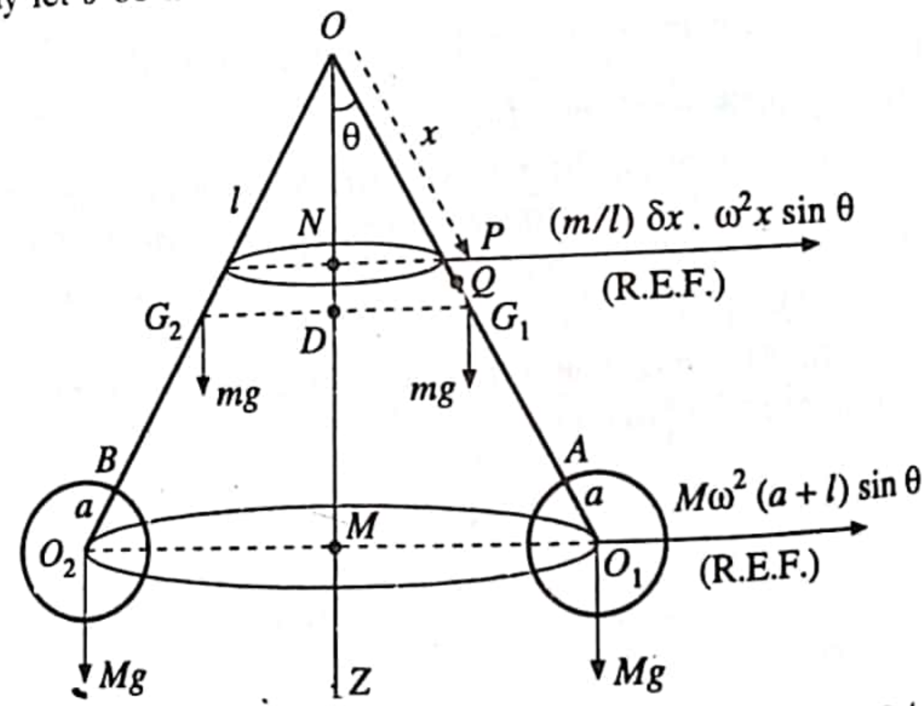
[substituting for ω^2 from (1)]

or $\theta = \tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$.

Ex. 5. *Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two uniform thin rods, each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Governor of a steam Engine, about a vertical line through O with the angular velocity ω. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ, given by the equation*

$$\cos\theta = \frac{g}{\omega^2} \cdot \frac{M(l+a) + \frac{1}{2}ml}{M(l+a)^2 + \frac{1}{3}ml^2}.$$

Sol. Let OA, OB be two rods, each of length l and mass m attached freely to a point O. Let two spheres each of mass M and radius a be attached to the other ends of the two rods. When the motion is steady let θ be the inclination of each of the rods to the vertical.



Consider the motion of one of the rods, say the rod OA, and the sphere with centre at O_1 attached to its end A. If $PQ = \delta x$ is an element of the rod at P, then its mass is $(m/l)\,\delta x$.

$\therefore$ the reversed effective force at the element δx at P

$$= \frac{m}{l}\delta x \,.\, \omega^2 \,.\, PN = \frac{m}{l}\delta x\omega^2 \,.\, x\sin\theta \text{ along } NP.$$

Also the reversed effective force on the sphere

$$= M\omega^2 \,.\, O_1M = M\omega^2 (a+l)\sin\theta \text{ along } MO_1$$

The external forces on the rod OA and sphere with centre at O_1 are the weights mg and Mg and the reaction at O.

Now to avoid reaction at O, taking moments about O, we get

$$\Sigma \frac{m}{l} \delta x \omega^2 x \sin\theta \,.\, ON + M\omega^2 (a+l) \sin\theta \,.\, OM$$

$$- mg \,.\, DG_1 - Mg \,.\, MO_1 = 0$$

or $$\int_0^l \frac{m}{l} \omega^2 x^2 \sin\theta \,.\, \cos\theta \, dx + M\omega^2 (a+l)^2 \sin\theta \cos\theta$$

$$- mg \frac{l}{2} \sin\theta - Mg (a+l) \sin\theta = 0$$

or $$[\omega^2 \,.\, \{\tfrac{1}{3} ml^2 + M(a+l)^2\} \cos\theta - g \{\tfrac{1}{2} ml + M(a+l)\}] \sin\theta = 0.$$

$\therefore$ Either $\sin\theta = 0$, *i.e.* $\theta = 0$ which is inadmissible

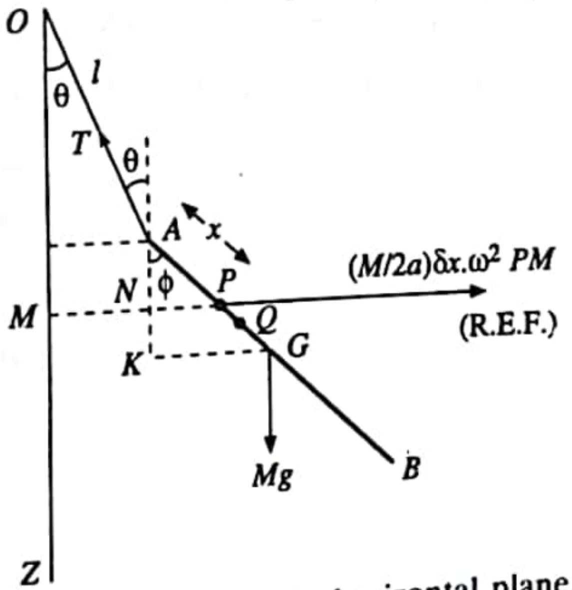
or $$\omega^2 \{\tfrac{1}{3} ml^2 + M(a+l)^2\} \cos\theta - g \{\tfrac{1}{2} ml + M(a+l)\} = 0$$

giving $$\cos\theta = \frac{g}{\omega^2} \cdot \frac{M(a+l) + \frac{1}{2} ml}{M(a+l)^2 + \frac{1}{3} ml^2}.$$

Ex. 6. *A rod, of length 2a, is suspended by a string of length l, attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be θ and ϕ respectively, show that*

$$\frac{3l}{a} = \frac{(4 \tan\theta - 3 \tan\phi) \sin\phi}{(\tan\phi - \tan\theta) \sin\theta}.$$

[Meerut 80, 81, 82, 83, 83P, 86, 88, 90, 91, 92S, 94, 96; Kanpur 81, 83; Raj. 79, 83]



Sol. Let the rod AB of length $2a$ and mass M be suspended by a string OA of length l. Let θ and ϕ be the inclinations of the string and the rod to the vertical respectively.

Consider an element $PQ\ (= \delta x)$ of the rod at a distance x from A, then mass of this element is $(M/2a)\, \delta x$.

As the rod revolves with uniform angular velocity ω, about the vertical OZ, the element δx will describe a circle of radius PM in the horizontal plane.

The reversed effective force on element δx is

$$\frac{M}{2a} \delta x . \omega^2 . PM = \frac{M}{2a} \delta x . \omega^2 . \ (l \sin\theta + x \sin\phi), \text{ along } MP.$$

The external forces acting on the rod are (i) tension T at A along AO, and (ii) its weight Mg acting vertically downwards at its middle point G.

Resolving horizontally and vertically the forces acting on the rod, we get

$$T \sin\theta = \Sigma \frac{M}{2a} \delta x \omega^2 (l \sin\theta + x \sin\phi)$$

or $$T \sin\theta = \frac{M}{2a} \omega^2 \int_0^{2a} (l \sin\theta + x \sin\phi)\, dx$$

or $$T \sin\theta = \frac{M}{2a} \omega^2 \left[lx \sin\theta + \tfrac{1}{2} x^2 \sin\phi \right]_0^{2a}$$

or $$T \sin\theta = M\omega^2 (l \sin\theta + a \sin\phi), \quad ...(1)$$

and $$T \cos\theta = Mg. \quad ...(2)$$

Now taking moments about A of all the forces acting on the rod AB, we get

$$-Mg.\,KG + \Sigma \frac{M}{2a} \delta x \omega^2 (l \sin\theta + x \sin\phi)\,.\,AN = 0$$

or $$Mg\, a \sin\phi = \frac{M\omega^2}{2a} \int_0^{2a} (l \sin\theta + x \sin\phi)\, x \cos\phi\, dx$$

$$= \frac{M}{2a} \omega^2 \left[\tfrac{1}{2} l x^2 \sin\theta + \tfrac{1}{3} x^3 \sin\phi \right]_0^{2a} \cos\phi$$

$$= aM\omega^2 (l \sin\theta + \tfrac{4}{3} a \sin\phi) \cos\phi$$

or $$g \tan\phi = \tfrac{1}{3} \omega^2 (3l \sin\theta + 4a \sin\phi). \quad ...(3)$$

Dividing (1) by (2), we get

$$\tan\theta = \frac{\omega^2}{g} (l \sin\theta + a \sin\phi)$$

or $$\omega^2 = g \tan\theta / (l \sin\theta + a \sin\phi).$$

Putting this value of ω^2 in (3), we get

$$g \tan\phi = \frac{1}{3} \cdot \frac{g \tan\theta\, (3l \sin\theta + 4a \sin\phi)}{(l \sin\theta + a \sin\phi)}$$

or $$3 \tan\phi\, (l \sin\theta + a \sin\phi) = \tan\theta\, (3l \sin\theta + 4a \sin\phi)$$

or $$3l \sin\theta\, (\tan\phi - \tan\theta) = a \sin\phi\, (4 \tan\theta - 3 \tan\phi)$$

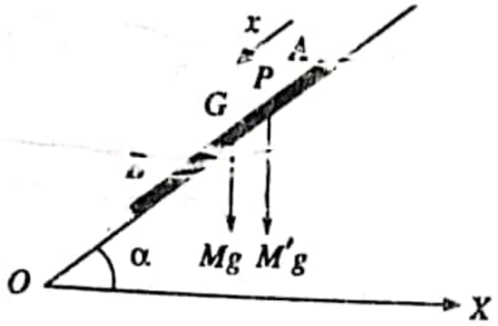
or $$\frac{3l}{a} = \frac{(4 \tan\theta - 3 \tan\phi) \sin\phi}{(\tan\phi - \tan\theta) \sin\theta}.$$

Ex. 7. *A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M', starting from the upper end, walks down the plank so that it does not move; show that he gets to the other end in time*

$$\sqrt{\left\{ \frac{2M'a}{(M + M') g \sin\alpha} \right\}},$$ *where a is the length of the plank.*

[Meerut 75, 80, 84, 85, 87, 89, 97; Kanpur 82; Raj. 77]

Sol. Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an anlge α to the horizon. A man of mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance $AP = x$ in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C.G. of the plank and the man from A in this position, then



$$\bar{x} = \frac{M.AG + M'.AP}{M + M'} = \frac{M.(a/2) + M'x}{M + M'}.$$

Differentiating twice w.r.t. 't', we get

$$\ddot{\bar{x}} = \frac{M'}{M + M'}\ddot{x}. \quad \text{...(1)}$$

Now the total weight $(M + M')g$ will act vertically downwards at the C.G, of the system.

$\therefore$ The equation of motion of the C.G. of the system is given by

$$(M + M')\ddot{\bar{x}} = (M + M')g \sin \alpha. \quad \text{...(2)}$$

$\therefore$ From (1) and (2), we get

$$M'\ddot{x} = (M + M')g \sin \alpha.$$

Integrating, we get $M'\dot{x} = (M + M')g \sin \alpha . t + c_1$.

But initially when $t = 0, \dot{x} = 0$. $\therefore c_1 = 0$.

$$\therefore M'\dot{x} = (M + M')g \sin \alpha . t.$$

Integrating again, we get $M'x = (M + M')g \sin \alpha . \frac{1}{2}t^2 + c_2$.

Initially when $t = 0, x = 0$. $\therefore c_2 = 0$.

$$\therefore M'x = (M + M')g \sin \alpha . \frac{1}{2}t^2,$$

or
$$t = \sqrt{\left\{\frac{2M'x}{(M + M')g \sin \alpha}\right\}}.$$

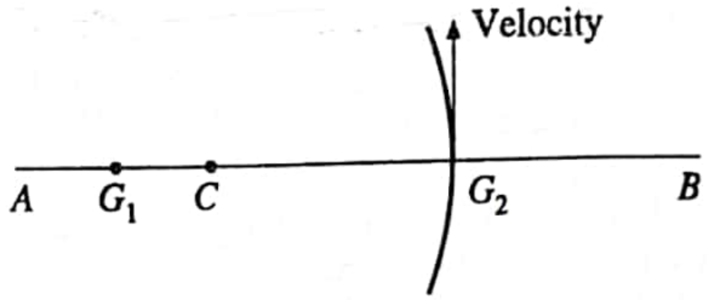
Putting $x = AB = a$, the time to reach the other end B of the plank is given by

$$t = \sqrt{\left\{\frac{2M'a}{(M + M')g \sin \alpha}\right\}}.$$

Ex. 8. *A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts, what is the subsequent motion of the two parts ?*

(Meerut 77, 79, 95)

Sol. Let the rod AB revolving on a smooth horizontal plane about one end A, which is fixed, break into two parts AC and CB. Obviously the part AC of the rod will continue to revolve about the fixed end A with the same angular velocity as before.



The part BC of the rod also has the same angular velocity about A and its centre of gravity G_2 has a linear velocity along the tangent at G_2 to the circle with centre at A and AG_2 as radius. Hence the part BC of the rod will fly off along the tangent line (the direction of linear velocity) at G_2 to the circle with centre at A and AG_2 as radius. But since the motion of a body about its centre of inertia is the same as if the centre of interia was fixed and the same forces acted on the body (see § 8), hence the part BC will also continue rotating about G_2 with the same angular velocity.

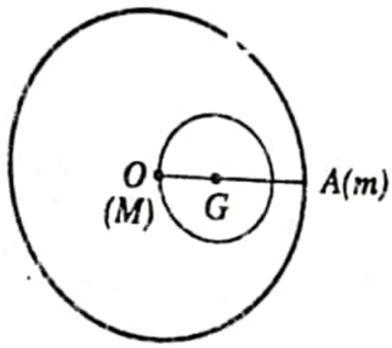
Hence the part BC will move along the tangent at G_2 to the circle with A as centre and AG_2 as radius with the velocity acquired by its centre of inetia G_2 before breaking and this part will also go on rotating about its centre of inertia G_2 with the same angular velocity as that of the part AC.

Ex. 9. *A circular board is placed on a smooth horizonal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board ?*

Sol. Let O be the centre and M the mass of the board which is placed on a smooth horizontal plane. Here we consider the motion of a system of rigid bodies consisting board and boy. Initially, if the boy of mass, say m, be at A on the edge of the board, then the C.G. 'G' of the system will be on OA, and

$$OG = \frac{M \cdot 0 + m \cdot OA}{M + m} = \frac{ma}{M + m}.$$

The external forces acting on the system of rigid bodies consisting of board and boy are (i) weight of the board acting vertically downwards, (ii) weight of the boy also acting vertically downwards and (iii) reaction of the smooth plane acting vertically upwards. Since all these forces are acting vertically, hence there is no external force on the system in the horizontal direction during the motion. Hence by D' Alembert's principle the C.G. 'G' of the system will remain at rest when the boy runs round the edge of the board. Hence as the boy runs round the edge of the board at a uniform rate, the centre O of the board will describe a circle of radius $OG = ma/(M+m)$ round the centre at G.



§ 9. Impulse of a Force.

The **impulse** *of a force acting on a particle in any interval of time is defined to be the change in momentum produced.*

Thus due to a force F, if the velocity of a particle of mass m changes from v_1 to v_2 in time t, then the impulse I is given by

$$I = mv_2 - mv_1 = m(v_2 - v_1)$$

$$= m \int_{t_1}^{t_2} dv = \int_{t_1}^{t_2} m \frac{dv}{dt} dt$$

$$= \int_{t_1}^{t_2} F\,dt, \quad \text{since } F = m\frac{dv}{dt}.$$

Thus the impulse of the force F is the time integral of the force.

Now let the force F increase indefinitely and the interval $(t_2 - t_1)$ decrease to a very small quantity such that the time integral $\int_{t_1}^{t_2} F\,dt$ remains finite. Such a force is called **impulsive force.**

Note. The measurement of an impulsive force is impracticable but it can be measured by the change in momentum produced.

§ 10. An Important Rule.

The effect of an impulse on a body remains the same even if all the finite forces acting simultaneously on it are neglected.

Let I be the impulse due to an impulsive force F which acts for a time τ. If f is the finite force acting simultaneously on the body, then

$$m(v_2 - v_1) = \int_0^{\tau} F\,dt + \int_0^{\tau} f\,dt = I + f\tau.$$

Since $f\tau \to 0$ as $\tau \to 0$, $\quad \therefore I = m(v_2 - v_1)$.

This shows that the finite force f acting on the body may be neglected in forming the equations.

§ 11. General Equations of Motion under Impulsive Forces.

To determine the general equations of motion of a system acted on by a number of impulses at a time. [Meerut 95]

Let u, v, w and u', v', w' be the velocities parallel to the axes respectively before and after the action of impulsive forces on the particle of mass m. If X', Y', Z' are the resolved parts of the total impulse on m parallel to the axes, then

$$\Sigma m\,(u' - u) = \Sigma \int_0^{\tau} X\,dt = \Sigma X'$$

or $$\Sigma m\,u' - \Sigma m\,u = \Sigma X'. \quad \text{...(1)}$$

Similarly $$\Sigma\, mv' - \Sigma\, mv = \Sigma\, Y' \quad \text{...(2)}$$

and $$\Sigma\, mw' - \Sigma\, mw = \Sigma\, Z' \quad \text{...(3)}$$

i.e., the change in momentum parallel to any of the axes is equal to the total impulse of the external forces parallel to the corresponding axis.

Hence the change in momentum parallel to any of the axes of the whole mass M, supposed collected at the centre of inertia and moving with it, is equal to the impulse of the external force parallel to the correspondng axis.

Again we have the equation

$$\Sigma\, m\,(y\ddot{z} - z\ddot{y}) = \Sigma\,(yZ - zY)$$

or $$\frac{d}{dt}\Sigma\, m(y\dot{z} - z\dot{y}) = \Sigma\,(yZ - zY).$$

Integrating this, we have

$$\left[\Sigma\, m\,(y\dot{z} - z\dot{y})\right]_0^{\tau} = \Sigma\left[y\int_0^{\tau} Z\,dt - z\int_0^{\tau} Y\,dt\right].$$

Since the time interval τ is so small that the body has not moved during this interval, we may take x, y, z as constants. Thus the above equation becomes

$$\Sigma\, m\,\{y\,(w' - w) - z\,(v' - v)\} = \Sigma\,(yZ' - zY')$$

or $$\Sigma\, m\,(yw' - zv') - \Sigma\, m\,(yw - zv) = \Sigma\,(yZ' - zY'). \quad \text{...(4)}$$

Similarly,

$$\Sigma\, m\,(xv' - yu') - \Sigma\, m\,(xv - yu) = \Sigma\,(xY' - yX') \quad \text{...(5)}$$

and $$\Sigma\, m\,(zu' - xw') - \Sigma\, m\,(zu - xw) = \Sigma\,(zX' - xZ'). \quad \text{...(6)}$$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Vector method. Let **I** and **I′** be the resultant external and internal impulses acting on the particle of mass m at P. Also let the velocity of m change from $\mathbf{v}_1$ to $\mathbf{v}_2$.

$\because$ Impulses = change in momentum,

$$\therefore \mathbf{I} + \mathbf{I}' = m(\mathbf{v}_2 - \mathbf{v}_1)$$

or $$\Sigma \mathbf{I} + \Sigma \mathbf{I}' = \Sigma m\mathbf{v}_2 - \Sigma m\mathbf{v}_1. \quad ...(1)$$

But $\Sigma \mathbf{I}' = \mathbf{0}$, by Newton's third law.

$$\therefore \text{ we get, } \Sigma \mathbf{I} = \Sigma m\mathbf{v}_2 - \Sigma m\mathbf{v}_1$$

i.e. the total external impulse applied to the system of particles is equal to the change of linear momentum produced.

Now let $\vec{OP} = \mathbf{r}$, then from (1), we get

$$\Sigma \mathbf{r} \times (\mathbf{I} + \mathbf{I}') = \Sigma \mathbf{r} \times m(\mathbf{v}_2 - \mathbf{v}_1)$$

or $$\Sigma \mathbf{r} \times \mathbf{I} = \Sigma \mathbf{r} \times m\mathbf{v}_2 - \Sigma \mathbf{r} \times m\mathbf{v}_1 \qquad (\text{Since } \Sigma \mathbf{r} \times \mathbf{I}' = \mathbf{0})$$

i.e. the total vector sum of the moments of the external impulses about any point O is equal to the increase in the angular momentum produced about the same point.

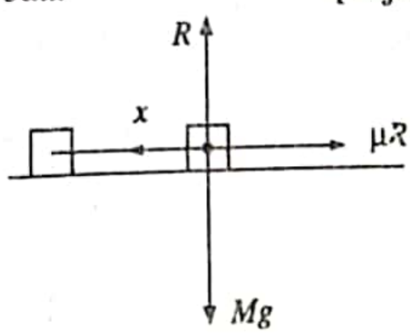
Solved Examples

Ex. 1. *A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity of the ball and cannon at the moment when it leaves the cannon is u. Show that the cannon will recoil a distance*

$$\left(\frac{mu}{M+m}\right)^2 \cdot \frac{1}{2\mu g},$$

along the plane, m being the mass of the ball. **[Raj. 80]**

Sol. Let I be the impulse between the cannon and the ball. If v is the velocity of the ball and V the velocity of cannon in opposite directions, then the relative velocity of the ball and cannon at the moment the ball leaves the cannon is



$$v + V = u \text{ (given).} \quad ...(1)$$

Also since, impulse = change in momentum

$$\therefore I = m(v - 0) \qquad \text{(for the ball)}$$

and $$I = M(V - 0) \qquad \text{(for the cannon)}$$

$$\therefore mv = MV \text{ or } v = \frac{MV}{m}. \quad ...(2)$$

Substituting the value of v from (2) in (1), we get

$$\frac{MV}{m} + V = u \text{ or } V(M+m) = mu$$

or $$V = mu/(M+m) \quad ...(3)$$

If the cannon moves through a distance x in the direction opposite to the direction of motion of the ball in time t, then on the rough plane, for the cannon the equation of motion is

$$M\ddot{x} = -\mu R = -\mu Mg$$

or $$\ddot{x} = -\mu g.$$

Multiplying both sides by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = -2\mu gx + C.$$

But initially when $x = 0, \dot{x} = V$ (*i.e.* starting velocity of the cannon).

$$\therefore \quad C = V^2.$$

$$\therefore \quad \dot{x}^2 = V^2 - 2\mu gx.$$

When the cannon comes to rest, $\dot{x} = 0$.

$$\therefore \quad 0 = V^2 - 2\mu gx$$

or $$x = \frac{V^2}{2\mu g} = \left(\frac{mu}{M+m}\right)^2 \cdot \frac{1}{2\mu g},$$

[substituting for V from (3)]

which is the required distance through which the cannon will recoil.

Ex. 2. *Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons, of mass M, throws a ball of mass m towards the other which reaches him in time t. Prove that the first person will begin to slide along the plane with velocity $ma/(Mt)$.*

Sol. Let the first person throw a ball of mass m with velocity u and begin to slide along the plane with velocity v. If I is the impulse between the ball and the first person, then since impulse = change in momentum, we have

$$I = M(v-0) \quad \text{for the first person}$$

and $$I = m(u-0) \quad \text{for the ball}$$

$$\therefore \quad mu = Mv \quad ...(1)$$

As the ball reaches the second person at a distance a in time t,

$$\therefore \quad a = ut$$

Substituting $u = a/t$ from (2) in 91), we get

$$\frac{ma}{t} = Mv \quad \therefore \quad v = \frac{ma}{Mt}. \quad \text{Ans.}$$

□□□