LINEAR ALGEBRA

: CSE-2019 :

① (c) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2,1) = (5,7)$ and $T(1,2) = (3,3)$. If A is the matrix corresponding to transformation T with respect to the standard basis e_1, e_2, find the rank of matrix A.

→ Given that $T(1,2) = (3,3)$ & $T(2,1) = (5,7)$.

$e_1 = (1,0)$, $e_2 = (0,1)$. Then,

$T(1,2) = T(e_1) + 2T(e_2) = (3,3)$ ⟶ ① and $T(2,1) = 2T(e_1) + T(e_2) = (5,7)$ ⟶ ②

① × 2 − ② :

$3T(e_2) = (1,-1) \Rightarrow T(e_2) = \frac{1}{3}e_1 - \frac{1}{3}e_2$.

$T(e_1) + 2T(e_2) = (3,3) \Rightarrow T(e_1) = (3,3) - \left(\frac{2}{3}, -\frac{2}{3}\right) = \left(\frac{7}{3}, \frac{11}{3}\right)$

$\therefore T(e_1) = \frac{7}{3}e_1 + \frac{11}{3}e_2$

$$\therefore A = \begin{bmatrix} 7/3 & 1/3 \\ 11/3 & -1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 & 1 \\ 11 & -1 \end{bmatrix}$$

$|A| = \frac{1}{9}[-7-11] \neq 0$. $\Rightarrow$ Rank$(A) = \underline{\underline{2}}$.

① (d) If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$. Show that $AB = 6I_3$.

Use this result to solve the following system of equations $\begin{matrix} 2x+y+z=5 \\ x-y=6 \\ 2x+y-z=1 \end{matrix}$

→ $AB = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6\underline{\underline{I_3}}$.

Given system of equations is $BX = R$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $R = \begin{bmatrix} 5 \\ 6 \\ 1 \end{bmatrix}$. Also $|B| \neq 0$.

$\therefore$ B is non-singular $\Rightarrow$ B^{-1} exists and $B^{-1} = \frac{1}{6}A$.

Solution to the given system of equations is $X = B^{-1}\begin{bmatrix}5\\0\\1\end{bmatrix} = \frac{1}{6}A\begin{bmatrix}5\\0\\1\end{bmatrix}$

$$\Rightarrow X = \frac{1}{6}\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}\begin{bmatrix}5\\0\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5+0+1\\5+0+1\\15+0-3\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$$

$\therefore$ Reqd solution: $x=1,\ y=1,\ z=2$.

(2)(b) Let A and B be two orthogonal matrices of the same order and $\det A + \det B = 0$, show that $A+B$ is a singular matrix.

$\rightarrow$ A and B are orthogonal matrices $\Rightarrow$ $A^TA = I$ and $B^TB = I$

Now $|A| + |B| = 0$ [Given] $\Rightarrow |A| = -|B|$ —(1)

Let $|A| = x$, then $|B| = -x$ and $x \neq 0$ since A and B are orthogonal $\Rightarrow$ A and B are non-singular.

$$(A+B)^T = A^T + B^T = B^T + A^T$$
$$= B^T[I + (B^T)^{-1}A^T] = B^T[I + BA^T]$$
$$= B^T[AA^T + BA^T] = B^T[A+B]A^T$$

$\left[\begin{matrix}B^TB = I\\ \therefore (B^T)^{-1} = B\end{matrix}\right]$

~~$I(A+B)^TI$~~ Taking determinant both sides

$$|(A+B)^T| = |B^T(A+B)A^T| = |B^T||A+B||A^T|$$

$$\rightarrow |A+B| = |B||A+B||A| \quad [\text{Since } |A^T| = |A|]$$
$$= (-x)|A+B|x$$
$$= -x^2|A+B|$$

$$\rightarrow (1+x^2)|A+B| = 0$$

Since $x \neq 0 \Rightarrow 1+x^2 \neq 0$.

Therefore $|A+B| = 0$

$\therefore$ A+B is a singular matrix.

3 (c) Let $A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$. (i) find the rank of matrix A

(ii) find the dimension of the subspace $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}$

→ (i) Reducing A to echelon form:

$$A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \sim \begin{bmatrix} 1 & 1 & -8 & 1 \\ 5 & 7 & 2 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix} \sim$$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$

$$\begin{bmatrix} 1 & 1 & -8 & 1 \\ 0 & 2 & 42 & -4 \\ 0 & 1 & 21 & -2 \\ 0 & 1 & 21 & -2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{R_2}{2}$, $R_4 \to R_4 - \frac{R_2}{2}$

$$\sim \begin{bmatrix} 1 & 0 & -8 & 1 \\ 0 & 2 & 42 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{echelon form of A} \quad \text{—— (1)}$$

A in echelon form has 2 non-zero rows. Therefore

Rank of A = $\rho(A) = 2$

(ii) $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \right\}$

The given condition is $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

which is a homogeneous system of linear equations.

from, (1) we can write the reduced form of A in place of A in the given equation.

$$\therefore \begin{bmatrix} 1 & 0 & -8 & 1 \\ 0 & 2 & 42 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now: $2x_2 + 42x_3 - 4x_4 = 0$ and $x_1 - 8x_3 + x_4 = 0$

$\Rightarrow x_2 = -21x_3 + 2x_4$ and $x_1 = 8x_3 - x_4$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8x_3 - x_4 \\ -21x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 8 \\ -21 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis of subspace V is $\{[8, -21, 1, 0], [-1, 2, 0, 1]\}$

$\therefore \dim V = 2.$

4(a). State the Cayley-Hamilton theorem. Use this theorem to find A^{100} where $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

→ Cayley-Hamilton Theorem states that every square matrix satisfies its characteristic equation.

Characteristic equation of A is given by $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2-1)=0$$

$$\Rightarrow \lambda^2-\lambda^3-1+\lambda=0$$

$$\Rightarrow \lambda^3 = \lambda^2+\lambda-1 \quad \text{—①}$$

By Cayley-Hamilton's Theorem, A satisfies the equation ①

$$\therefore A^3 = A^2+A-I \quad \text{—②}$$

Premultiplying A on both sides,

$$A\cdot A^3 = A\cdot A^2 + A\cdot A - A\cdot I = A^3+A^2-A$$

$$A^4 = A^2+A-I+A^2-A \quad \text{[from ②]}$$

$$A^4 = 2A^2-I$$

Premultiplying both sides with A^2

$$\Rightarrow A^2\cdot A^4 = 2\cdot A^2\cdot A^2 - A^2 I = 2A^4-A^2$$

$$= 2(2A^2-I)-A^2$$

$$\Rightarrow A^6 = 3A^2-2I \qquad = 3A^2-2I$$

Premultiplying both side with A^2,

$$A^2\cdot A^6 = 3\cdot A^2\cdot A^2 - 2A^2 = 3A^4-2A^2 = 3(2A^2-I)-2A^2$$

$$A^8 = 4A^2-3I.$$

$\therefore$ for each even power of a greater than/= 2, we have

$$A^{2n} = nA^2-(n-1)I \quad \text{where } n\geq 1$$

$$\therefore A^{100} = 50A^2-49I$$

$$= 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^2 - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 50\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \cancel{25} & 1 & \cancel{0} \\ \cancel{25} & 0 & \cancel{1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$$

4