3

Motion in Two Dimensions

(Under finite Forces)

3·01. Dynamical Equations of Motion. *To determine dynamical equations of motion in two dimensions when the forces acting on the body are finite.*

The motion of a rigid body consists of two independent motions *viz.*,

(i) the motion of centre of gravity, and

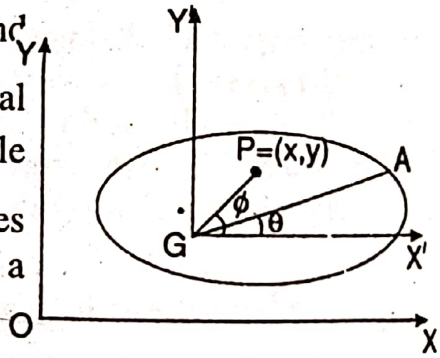
(ii) the motion about the centre of gravity.

Motion of Centre of Gravity.

Cartesian Method

Motion of C.G. states that the motion of centre of gravity is such that the total mass M of the rigid body is allowed to act at the C.G. and all the external forces are transferred parallel to themselves to act at the C.G. of the body.

Consider a particle m of the rigid body at point P whose coordinates referred to two axes fixed in space of two dimensions, OX, OY are (x, y). Now the effective forces acting on the particle are $m\ddot{x}$ and $m\ddot{y}$, let X, Y be the components of the external forces acting at P. By D' Alembert's principle $(X - m\ddot{x}), (Y - m\ddot{y})$ together with similar forces acting on all other particles of the body form a system in statical equilibrium, thus we have



$$\Sigma (X - m\ddot{x}) = 0, \Sigma (mY - m\ddot{y}) = 0$$

and $$\Sigma [x(Y - m\ddot{y}) - y((X - m\ddot{x})] = 0$$

and $$\left.\begin{aligned} &\Rightarrow \Sigma m\ddot{x} = \Sigma X, \Sigma m\ddot{y} = \Sigma Y \\ &\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma (xY - yX) \end{aligned}\right\} \quad ...(1)$$

Let (x_G, y_G) be the co-ordinates of the centre of gravity refered to axes OX and OY and (x', y') be the co-ordinates of the point P referred to parallel axes GX' and GY' through G.

$\therefore\ x = x_G + x', y = y_G + y'$

then $Mx_G = \Sigma mx, My_G = \Sigma my$ (where $\Sigma m = M$)

$\Rightarrow \ddot{x}\, M_G = \Sigma m\ddot{x}$ and $M\, \ddot{y}_G = \Sigma m\ddot{y}$.

Thus the first two equations of (1) reduces to

$$M\ddot{x}_G = \Sigma X \text{ and } M\ddot{y}_G = \Sigma Y \qquad ...(2)$$

Motion Relative to Centre of gravity

Third equation of (1) gives

$$\Sigma m\,[(x_G + x')(\ddot{y}_G + \ddot{y}') - (y_G + y')(\ddot{x}_G + \ddot{x}')]$$
$$= \Sigma\,[(x_G + x')\,Y - (y_G + y')\,X]$$

or $$(x_G \ddot{y}_G - y_G \ddot{x}_G)\,\Sigma m + x_G\, \Sigma m\ddot{y}' + \ddot{y}_G\, \Sigma m x'$$
$$- y_G\, \Sigma m\ddot{x}' - \ddot{x}_G\, \Sigma m y' + \Sigma m\,(x'\ddot{y}' - y'\ddot{x}')$$
$$= x_G\, \Sigma Y - y_G\, \Sigma X + \Sigma\,(x' Y - y' X) \qquad ...(3)$$

where $\Sigma m = M$.

By (2), first term on L.H.S. of (3) cancels the first two terms on the R.H.S. of (3).

Again $\dfrac{\Sigma m x'}{\Sigma m}$ and $\dfrac{\Sigma m y'}{\Sigma m}$ give the coordinates of G with respect to axes GX' and GY'

i.e. $\Sigma m x' = 0,\ \Sigma m y' = 0 \quad \Rightarrow \Sigma m\ddot{x}' = 0,\ \Sigma m\ddot{y}' = 0$

Thus (3) reduces to

$$\Sigma m\,(x'\ddot{y}' - y'\ddot{x}') = \Sigma\,(x'Y - y'X) \qquad ...(4)$$

$$\frac{d}{dt}\,\Sigma m\,(x'\dot{y}' - y'\dot{x}') = \Sigma\,(x'Y - y'X) \qquad ...(5)$$

Let GA be a line fixed on the body which makes an angle θ with GX

Let $GP = r$ and $\angle PGX' = \phi$.

$$\phi = \theta + \angle AGP.$$

Since the body turns about G, $\angle AGP$ remains constant.

$\Rightarrow \dot{\phi} = \dot{\theta}$ and $\ddot{\phi} = \ddot{\theta}$.

Again the velocity of m at point P is $r\dot{\phi}$ perpendicular the GP, its moment about G is $r\dot{\phi}\,.\, r = r^2\dot{\phi}$

$\therefore\ \Sigma m\,(x'\dot{y}' - y'\dot{x}') = \Sigma m r^2 \dot{\phi}$

or $\ \Sigma m\,(x'\dot{y}' - y'\dot{x}') = \Sigma m r^2 \dot{\theta} = \dot{\theta}\ \Sigma m r^2 = Mk^2\dot{\theta}$

where Mk^2 is the moment of inertia of the body about G.

Hence equation (5) may be put as

$$\frac{d}{dt}(Mk^2\dot{\theta}) = \Sigma(x'Y - y'X) \quad \text{or} \quad Mk^2\ddot{\theta} = L \qquad ...(6)$$

where L is the moment of the external forces about G.

Thus the equations of motion of the body are $M\ddot{x}_g = \Sigma X$, $M\ddot{y}_g = \Sigma Y$ and are known as equations of motion of the centre of gravity

and $\qquad \Sigma m(x\ddot{y}' - y\ddot{x}') = \Sigma(x'Y - y'X)$

known as equation of motion about the centre of gravity, this can also be put as $\quad Mk^2\ddot{\theta} = L$

where L is the moment of external forces about G.

This states that the sum of the moments, of the effective forces about the centre of gravity G, is equal to the sum of the moments of the external forces about G.

Vector Method .

Let $\mathbf{r}_G$ be the position vector of the **C.G.** and **F** the external forces acting at any particle m of the body, then we have $M\dfrac{d^2\mathbf{r}_G}{dt^2} = \Sigma\mathbf{F}$.

But $\qquad \mathbf{r}_G = x_G\mathbf{i} + y_G\mathbf{j}$ and $\mathbf{F} = X\,\mathbf{i} + Y\,\mathbf{j}$

where (x_g, y_g) are the co-ordinates of **C.G.** and X, Y the components of the forces **F** parallel of the axes.

$\therefore \quad$ (1) gives ; $M\left[\dfrac{d^2x_G}{dt^2}\mathbf{i} + \dfrac{d^2y_G}{dt^2}\mathbf{j}\right] = \Sigma(X\,\mathbf{i} + Y\,\mathbf{j})$.

Equating coefficients of **i** and **j** on both sides, we get

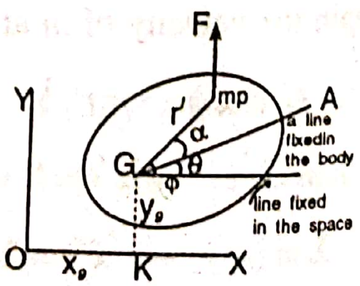
$$M\frac{d^2x_G}{dt^2} = \Sigma X \qquad ...(2) \qquad \text{and} \qquad M\frac{d^2y_G}{dt^2} = \Sigma Y \qquad ...(3)$$

These are the equations of the centre of gravity.

Let $\mathbf{r}'$ be the position vector, of the particle m at P, relative to G, and **F** the external forces acting on it, then we have

$$\Sigma\mathbf{r}' \times \frac{d^2\mathbf{r}'}{dt^2} = \Sigma\mathbf{r}' \times \mathbf{F} \Rightarrow \frac{d}{dt}\Sigma m\mathbf{r}' \times \frac{d\mathbf{r}'}{dt} = \Sigma\mathbf{r}' \times \mathbf{F} \qquad ...(4)$$

Now let θ be the angle that a line GA fixed in the body makes with a line GB fixed in the space, and let ϕ be the angle which the line joining P to G makes with the line GB (fixed in the space), then as obvious from the adjoining figure, we have $\quad \phi = \theta + \angle AGP = \theta + \alpha$.

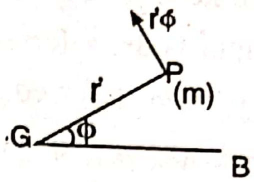


$\therefore \quad \dfrac{d\phi}{dt} = \dfrac{d\theta}{dt}, \qquad [\because \angle AGP = \alpha \text{ is constant }]$

Let $Gm = r'$

$\therefore$ the velocity of m relative to G

$= r'\dfrac{d\phi}{dt}$ in a direction perpendicular to r' in the plane AGP.



If $\hat{e}_1$, $\hat{e}_2$ be the unit vectors along and perpendicular to $\mathbf{r}'$ in the AGP plane, then we have

$\mathbf{r}' = r'\hat{\mathbf{e}}_1$ and $\dfrac{d\mathbf{r}'}{dt} = \mathbf{r}'\dfrac{d\phi}{dt}\hat{\mathbf{e}}_2$

$$\Rightarrow \Sigma m\mathbf{r}' \times \frac{d\mathbf{r}}{dt} = \Sigma m\,(r'\hat{\mathbf{e}}_1) \times \mathbf{r}'\frac{d\phi}{dt}\hat{\mathbf{e}}_2$$

$$= \Sigma m r'^2 \frac{d\theta}{dt}\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 \quad \left[\because \frac{d\phi}{dt} = \frac{d\theta}{dt} + \frac{d\alpha}{dt} = \frac{d\theta}{dt}\right]$$

$= \dfrac{d\theta}{dt}\Sigma m r'^2\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector normal to the plane AGP.

$= \dfrac{d\theta}{dt}(\Sigma m r'^2)\,\hat{\mathbf{n}}$

$= \dfrac{d\theta}{dt}(M k^2)\,\mathbf{n}$ where k is the radius of gyration of the body about G

$$= \left(M k^2 \frac{d\theta}{dt}\right)\hat{\mathbf{n}}.$$

Also we have, moment of the forced F about $G = \Sigma\mathbf{r}' \times \mathbf{F}$

$= \Sigma p' F\,\hat{\mathbf{n}}$ where p is the lwength of the perpendicular from G upon the direction of the force $\mathbf{F}$

$\therefore$ Equation (4) reduces to $\dfrac{d}{dt}\left(M k^2 \dfrac{d\theta}{dt}\right)\hat{\mathbf{n}} = (\Sigma p' F)\,\hat{\mathbf{n}}$ ...(5)

Equating coefficients of $\hat{\mathbf{n}}$ on both sides, we get

$\dfrac{d}{dt}\left(M k^2 \dfrac{d\theta}{dt}\right) = \Sigma p' F$ ...(6) $\Rightarrow M k^2 \dfrac{d^2\theta}{dt^2} = \Sigma p' F$ ...(7)

Let (x', y') be the co-ordinates of P relative to G and X, Y the components of $\mathbf{F}$ in the directions of the axes, scalar moment, of the force $\mathbf{F}$ about G is $p'F$ which is equivalent to $(x'Y - y'X)$.

$\therefore$ (6th) equation may be written as $\dfrac{d}{dt}\left(M k^2 \dfrac{d\theta}{dt}\right) = \Sigma(x'Y - y'X)$

$\Rightarrow M k^2 \dfrac{d^2\theta}{dt^2} = \Sigma(x'Y - y'X)$. ...(8)

Equations (2), (3) and (7) are the dynamical equations of motion of rigid body moving in two dimensions, under finite forces.

3·02. Kinetic Energy . *When a body is moving in two dimensions, then to express the kinetic energy in terms of the motion of the centre of inertia and the motion relative to the centre of inertia.*

(Meerut 84; Agra 81, 89; Raj 85; Patna 83)

At any time t, let $\mathbf{r}_G$ be the position vector of the centre of gravity of G of the rigid body, referred to an origin O ; and let $\mathbf{r}$ be the position vector of a particle m, referred to an origin O, then we have $\mathbf{r} = \mathbf{r}_G + \mathbf{r}'$

where $\mathbf{r}'$ is the $p.v$ of the particle of mass m w.r.t. C. G.
Now let T be the kinetic energy of the body, then we get

$$T = \tfrac{1}{2} \Sigma m \dot{\mathbf{r}}^2 \quad ...(1) \qquad = \tfrac{1}{2} \Sigma m (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}')^2$$

$$= \tfrac{1}{2} \Sigma m \dot{\mathbf{r}}_G^{\,2} + \tfrac{1}{2} \Sigma m \dot{\mathbf{r}}'^2 + \Sigma m \dot{\mathbf{r}}_G \cdot \dot{\mathbf{r}}'$$

$$= \tfrac{1}{2} \dot{\mathbf{r}}_G^{\,2} \Sigma m + \tfrac{1}{2} \Sigma m \dot{\mathbf{r}}'^2 + \dot{\mathbf{r}}_G \cdot \Sigma m \dot{\mathbf{r}}'$$

But $\dfrac{\Sigma m \mathbf{r}'}{\Sigma m} = 0,$

[$\because$ $\mathbf{r}'$ is the position vector of the centroid relative to the centroid itself.]

$\therefore$ $\Sigma m \mathbf{r}' = 0$, and so $\Sigma m \dot{\mathbf{r}}' = 0$,

$$\therefore \quad T = \tfrac{1}{2} M \dot{\mathbf{r}}_g^{\,2} + \tfrac{1}{2} \Sigma m \dot{\mathbf{r}}'^2 \quad [\because \Sigma m = M] \qquad ...(2)$$

Another form . Let $\mathbf{v}_G$ be the velocity of centre of gravity and let $\hat{\mathbf{e}}_2$ be the unit vector perpendicular to the direction of $\mathbf{r}'$ then we readily obtain

$$\mathbf{v}_G = \frac{d\mathbf{r}_g}{dt} = \dot{\mathbf{r}}_G$$

and $$\dot{\mathbf{r}}'^2 = \left(\mathbf{r}' \frac{d\phi}{dt} \hat{\mathbf{e}}_2 \right)^2 = \mathbf{r}'^2 \left(\frac{d\theta}{dt} \right)^2 \left[\therefore \frac{d\phi}{dt} = \frac{d\theta}{dt} \text{ and } \hat{\mathbf{e}}_2^{\,2} = \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = 1 \right]$$

$$\therefore \ (2) \Rightarrow T = \tfrac{1}{2} M \mathbf{v}_G^{\,2} + \tfrac{1}{2} \Sigma m r'^2 \left(\frac{d\theta}{dt} \right)^2$$

$$= \tfrac{1}{2} M v_G^{\,2} + \tfrac{1}{2} \left(\frac{d\theta}{dt} \right)^2 \Sigma m r'^2 \qquad [\because \mathbf{v}_G^2 = v_G^2]$$

$$= \tfrac{1}{2} M v_G^{\,2} + \tfrac{1}{2} M k^2 \left(\frac{d\theta}{dt} \right)^2 \qquad ...(3)$$

where k is the radius of gyration of the body about the centre of intertia.
Hence equation (3) expresses that ;
The total kinetic energy of a rigid body moving in two dimensions is equal to the kinetic energy of a particle of mass M placed at the centre of inertia

and moving with it together with the kinetic energy of the body relative to the centre of inertia.

Equation (3) can also be put as

K.E. of the body =(K.E. due to translation)+ (K.E. due to rotation) ...(4)

3·03. Moment of the Momentum . *To find the moment of momentum of the body about the fixed origin O, when the body is moving in two dimensions.*

At any time t, let $\mathbf{r}_G$ be the position vector of the centre of gravity G of the body referred the origin O, and let $\mathbf{r}$ be the position vector of a particle of mass m, referred to the origin O,

then we have $\mathbf{r} = \mathbf{r}_G + \mathbf{r}'$; where $\mathbf{r}'$ is the position vector of the particle of mass m w.r.t. G.

Now let $\mathbf{H}$ be the moment of momentum (or angular momentum) of the body about O, then we have $= \Sigma \mathbf{r} \times m\dot{\mathbf{r}}$

$$= \Sigma m \times \dot{\mathbf{r}} = \Sigma m\,(\mathbf{r}_G + \mathbf{r}') \times (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}')$$

$$= \Sigma m\mathbf{r}_G \times \dot{\mathbf{r}}_G + \Sigma m\mathbf{r}_G \times \dot{\mathbf{r}}' + \Sigma m\mathbf{r}' \times \dot{\mathbf{r}}_G + \Sigma m\mathbf{r}' \times \dot{\mathbf{r}}'$$

$$= \mathbf{r}_G \times \dot{\mathbf{r}}_G\, \Sigma m + \mathbf{r}_G \times \Sigma m\dot{\mathbf{r}}' + (\Sigma m\mathbf{r}') \times \dot{\mathbf{r}}_G' + \Sigma m\mathbf{r}' \times \dot{\mathbf{r}}' \qquad \text{...(1)}$$

But $\dfrac{\Sigma m\mathbf{r}'}{\Sigma m} = \mathbf{0}$, being position vector of C.G. relative to C.G.

$\therefore\ \Sigma m\mathbf{r}' = \mathbf{0}$ and so $\Sigma m\dot{\mathbf{r}}' = \mathbf{0}$

$$\therefore\ (1) \Rightarrow \mathbf{H} = \mathbf{r}_G \times \dot{\mathbf{r}}_G\ \Sigma m + \Sigma m\mathbf{r}' \times \dot{\mathbf{r}}'$$

$$= \mathbf{r}_G \times M\,\dot{\mathbf{r}}_G + \Sigma \mathbf{r}' \times m\dot{\mathbf{r}}' \qquad [\because\ \Sigma m = M]$$

$$= \mathbf{r}_G \times M\,\mathbf{v}_G + \Sigma \mathbf{r}' \times m\dot{\mathbf{r}}' \qquad \text{...(2)}$$

Another form. Let $\hat{\mathbf{n}}$ be the unit vector parallel to $\mathbf{H}$, then we get

$$\mathbf{r}_G \times M\,\mathbf{v}_G = M\,\mathbf{r}_G \times \mathbf{v}_G$$

$$= (M\,v_G\,p)\,\hat{\mathbf{n}}$$

[using the definition of moment ; p is the length of the perpendicular from the origin O on the direction of the velocity $\mathbf{v}_g$ of centre of gravity].

But we have $\Sigma \mathbf{r}' \times m\dot{\mathbf{r}}' = \left(Mk^2 \dfrac{d\theta}{dt}\right)\hat{\mathbf{n}}$ [3.01]

and $\mathbf{H} = H\,\hat{\mathbf{n}}$

$$\therefore\ (2) \Rightarrow H\hat{\mathbf{n}} = M\,v_g\,p\,\hat{\mathbf{n}} + Mk^2 \frac{d\theta}{dt}\,\hat{\mathbf{n}}$$

Equating coefficients of $\hat{\mathbf{n}}$ on both sides, we get

$$H = M v_g p + Mk^2 \frac{d\theta}{dt} \quad ...(3)$$

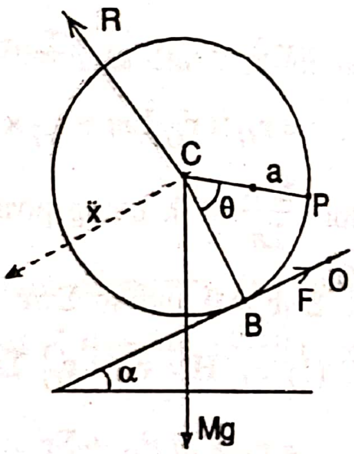
This equation expresses that the moment of momentum (or angular momentum) of a rigid body about a fixed point O is equal to the angular momentum about O of a single particle of mass M (equal to mass of the body concentrated at its C.G. and moving with the centroid's velocity), together with the angular momentum of the body in motion relative to the C.G.

Equation (3) can also be written as.
Angular momentum of the rigid body
= Angular momentum of centre of inertia
+ angular momentum relative to the centre of inertia.

3·04. *A uniform sphere rolls down an inclined plane, rough enough to prevent any sliding ; to discuss the motion.*

(Meerut 83, Agra 88, Ranchi 83)

Initially, the sphere was at rest with its points P in contact with O. During the motion, after any time t, let the centre "C" of the sphere describes a distance x on the inclined plane and θ is the angle through which the sphere turns. Thus CP a line fixed in the body, makes an angle θ with the normal to the plane, a line fixed in the space.



Let F be the frictional force and R the normal reaction at the point of contact B, then equations of motion of C.G. of the body are

$$M \frac{d^2x}{dt^2} = Mg \sin\alpha - F. \quad ...(1)$$

Since there is no motion perpendicular to the plane, we have

$$M\ddot{y} = 0 = Mg\cos\alpha - R \quad \text{or} \quad Mg\cos\alpha = R. \quad ...(2)$$

Also equation of motion about the centre of gravity is

$$Mk^2 \frac{d^2\theta}{dt^2} = F.a. \quad ...(3)$$

Since there is no sliding, so we have $OB = \text{arc } PB$

$$\Rightarrow x = a\theta, \dot{x} = a\dot{\theta} \text{ and } \ddot{x} = a\ddot{\theta}, \quad ...(4)$$

$\therefore$ (3) gives $M.\dfrac{k^2}{a^2}\dfrac{d^2x}{dt^2} = F.a$ $\quad [\because \ddot{x} = a\ddot{\theta}]$

Substituting the value of F from here in (1), we readily get

$$\frac{d^2x}{dt^2}\left(1+\frac{k^2}{a^2}\right)=g\sin\alpha \quad \text{or} \quad \frac{d^2x}{dt^2}=\frac{a^2g\sin\alpha}{a^2+k^2} \qquad ...(5)$$

i.e. the sphere rolls down with a constant acceleration $\frac{a^2g\sin\alpha}{a^2+k^2}$

$(5) \Rightarrow \frac{dx}{dt}=\frac{a^2g\sin\alpha}{a^2+k^2}t+C$; and C,

the constant of intergration vanishes as t and $\dot{x}$ vanish together.

Intergrating again , $x=\frac{1}{2}\frac{a^2g\sin\alpha}{a^2+k^2}t^2$;

because constant of intergration again vanishes as x and t vanish simultaneously.

Now we shall discuss various cases :

(i) If the body be a solid sphere, $k^2=\frac{2}{5}a^2$ and then equation (5) implies.

$$\ddot{x}=\frac{5}{7}g\sin\alpha.$$

(ii) If the body be hollow sphere, $k^2=\frac{2}{3}a^2$ $\quad\therefore \ddot{x}=\frac{3}{5}g\sin\alpha.$

(iii) If the body be circular disc, $k^2=\frac{1}{2}a^2$ $\quad\therefore \ddot{x}=\frac{2}{3}g\sin\alpha.$

(iv) If the body be circular ring , $k^2=a^2$ $\quad\therefore \ddot{x}=\frac{1}{2}g\sin\alpha.$

Pure rolling : Eliminating $\frac{d^2x}{dt^2}$ from (5), and (1), we get

$$F=Mg\sin\alpha-\frac{5}{7}Mg\sin\alpha=\frac{2}{7}Mg\sin\alpha \quad \left(\because k^2=\frac{2a^2}{5}\right)$$

Also from (2) $R=Mg\cos\alpha$.

In order that there may be no sliding $\frac{F}{R}$ must be less than μ *i.e.* for pure rolling $F<\mu R$ *i.e.* $\mu>\frac{F}{R}=\frac{2}{7}\tan\alpha$.

ILLUSTRATIVE EXAMPLES

Ex. 1. *A uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is* α , *show that the least coefficient of friction between it and the plane, so that it may roll and not slide, is* $\frac{1}{3}\tan\alpha$. *If the cylinder be hollow, and of small thickness, the least value is* $\frac{1}{2}\tan\alpha$.

Sol. At any time t, let the axis of the cylinder describe a distance x and θ be the angle turned Arguing as in 3.04, we have

$x = a\theta.$ [$\because$ there is no sliding]

Also the equations of a C.G. are given by

$$M\frac{d^2x}{dt^2} = Mg\sin\alpha - F \quad ...(1) \text{ and } \quad 0 = Mg\cos\alpha - R \qquad ...(2)$$

Again taking moments abouts the axis through G, the centre of gravity of the body, we have

$$Mk^2\frac{d^2\theta}{dt^2} = F\times a \Rightarrow M\frac{k^2}{a}\cdot\frac{d^2x}{dt^2} = F\times a \qquad ...(3)$$

whence elimination of $M\frac{d^2x}{dt^2}$ in between (1) and (3), we get

$$\frac{k^2}{a}(Mg\sin\alpha - F) = F\times a \Rightarrow F = \frac{k^2}{a^2+k^2}Mg\sin\alpha \qquad ...(4)$$

But $\quad R = Mg\cos\alpha \qquad ...(5)$

$\therefore$ For pure rolling, $\mu > \frac{F}{R} = \frac{k^2}{a^2+k^2}\tan\alpha$.

But when cylinder is solid, we have $k^2 = \frac{1}{2}a^2$, $\Rightarrow \mu > \frac{\frac{1}{2}a^2}{a^2+\frac{1}{2}a^2}\tan\alpha$

$$= \frac{1}{3}\tan\alpha$$

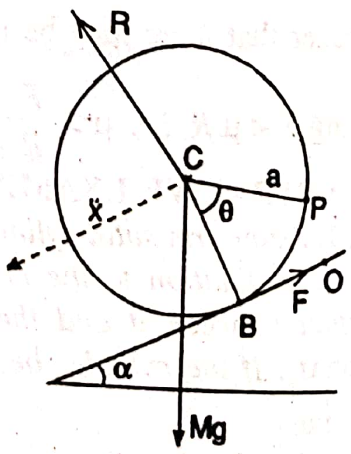
In case of hollow cylinder, we have

$$k^2 = a^2, \Rightarrow \mu > \frac{a^2}{a^2+a^2}\tan\alpha = \frac{1}{2}\tan\alpha.$$

Ex. 2. *A cylinder rolls down a smooth plane whose inclination to the horizontial is* α, *unwrapping, as it goes, a fine string fixed to the highest point of the plane ; find its acceleration and the tension of the string.*

Sol. When the cylinder has rolled down a distance x along the plane, let T be the tension in the string and in this time (say t), let θ be the angle turned by the cylinder, then as the string is tight, the motion is of pure rolling

i.e. $\quad$ arc $BP = OB \Rightarrow x = a\theta \qquad ...(1)$



$\therefore \dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

equations of motion of the centre of gravity of the cylinder are

$$M\frac{d^2x}{dt^2} = Mg\sin\alpha - T \qquad ...(2)$$

and $M\dfrac{d^2y}{dt^2}=0=Mg\cos\alpha-R.$ ...(3)

Now taking moments about the centre , we have

$$Mk^2\ddot{\theta}=T\times \text{ i.e. } M.\tfrac{1}{2}a^2\ddot{\theta}=T\times a$$

or $\frac{1}{2}M\ddot{x}=T.$ [$\because$ $\ddot{x}=a\ddot{\theta}$] ...(4)

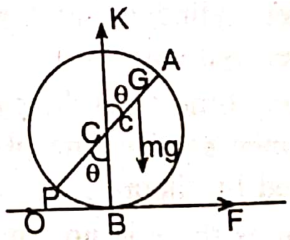
$\therefore$ (5) and (2), gives $\frac{3}{2}M\ddot{x}=Mg\sin\alpha$ i.e. $\ddot{x}=\frac{2}{3}g\sin\alpha$

$\Rightarrow T=\frac{1}{2}m\ddot{x}=\frac{1}{2}M(\frac{2}{3}g\sin\alpha)=\frac{1}{3}Mg\sin\alpha$

Ex. 3. *A circular cylinder, whose centre of inertia is at a distance c from axis, rolls on a horizontal plane. If it be just started from a position of unstable equilibrium. Show that the normal reaction of the plane when the centre of mass is in its lowest position is* $\left[1+\dfrac{4c^2}{(a-c)^2+k^2}\right]$ *times its weight, where k is the radius of gyration about an axis through the centre of mass.*

Sol. Initially the point of contact P of the cylinder was at O when its centre of gravity was vertically above the centre of the figure .

At any time t let the radius through G turn through an angle θ.



Referred to O as origin and horizontal and vertical line as axes, the co-ordinates (x, y) of G fiven by $x=a\theta+c\sin\theta$, $y=a+c\cos\theta$,

[$\because$ $CG=c$.]

Equations of motion of C.G. are

$$m\frac{d^2x}{dt^2}=m\frac{d^2}{dt^2}(a\theta+c\sin\theta)=F \quad ...(1)$$

and $$m\frac{d^2y}{dt^2}=m\frac{d^2}{dt^2}(a+c\cos\theta)=R-mg\ . \quad ...(2)$$

Also energy equation gives

$$\tfrac{1}{2}m\,[(\dot{x}^2+\dot{y}^2)+k^2\dot{\theta}^2]=\text{work done by the forces .}$$

i.e. $$\tfrac{1}{2}m\,[(a\dot{\theta}+c\cos\theta\dot{\theta})^2+(-c\sin\theta\dot{\theta})^2]$$

$$+\tfrac{1}{2}mk^2\dot{\theta}^2=mg(c-c\cos\theta)$$

Let ω be the angular velocity when G is in its lowest position

i.e. $\dot{\theta}=\omega$ when $\theta=\pi$; thus we have

$$\tfrac{1}{2}\, m\ [(a-c)^2 + k^2]\ \omega^2 = 2mgc \Rightarrow \omega^2 = \frac{4gc}{k^2 + (a-c)^2}.$$

Now (2) gives $R = mg - mc\,(\sin\,\theta\ddot{\theta} + \cos\,\theta\dot{\theta}^2)$

$$= mg - mc\ \cos\,\pi\,.\,\omega^2$$

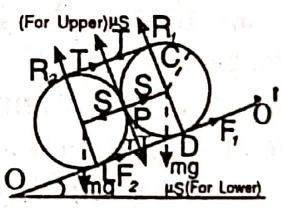
(since in the lowest position $\theta = \pi;\ \dot{\theta} = \omega$)

$$= mg + mc\,\frac{4cg}{k^2 + (a-c)^2} = mg\left[1 + \frac{4c^2}{k^2 + (a-c)^2}\right].$$

Ex. 4. *Two equal cylinders, of mass m, are bound together by an elastic string, whose tension is T, and roll with their axes horizontal down a rough plane of inclination α. Show that their acceleration is*

$$\tfrac{2}{3}\, g\ \sin\,\alpha\left[1 - \frac{2\mu T}{mg\ \sin\,\alpha}\right],$$ *where μ is the coefficient of friction between the cylinders .*

Sol. Let R_1 , F_1 be the normal reaction and friction on the upper cylinder and R_2 , F_2 be the normal reaction and friction on the lower cylinder due to the plane. Let S be the normal reaction between the two cylinders at P. The force μS acts away from the plane for upper cylinder and towards the plane for the lower cylinder .



At any time t let the cylinders move through a distance z along the plane, and θ be the angle turned by them .

Then as there is no slipping, we have

$$z = a\theta \Rightarrow \ddot{z} = a = a\,\ddot{\theta}. \qquad ...(1)$$

Equations of motion of the upper cylinder are given by

$$m\ddot{z} = mg\ \sin\,\alpha + 2T - F_1 - S \qquad ...(2)$$

$$0 = R_1 - mg\ \cos\,\alpha + \mu S \qquad ...(3)$$

and $\quad mk^2\,\ddot{\theta} = F_1 \times a - \mu S \times a. \qquad ...(4)$

where as the equations of motion for the lower cylinder are given by

$$m\ddot{z} = mg\ \sin\,\alpha - 2T - F_2 + S \qquad ...(5)$$

$$0 = R_2 - mg\ \cos\,\alpha - \mu S \qquad ...(6)$$

and $\quad mk^2\,\ddot{\theta} = F_2 \times a - \mu S \times a \qquad ...(7)$

Comparing (4) and (7), we have $F_1 = F_2$.

Subtracting (2) from (5), we have $S = 2T$. ...(8)

Also from (4), $F_1 = \frac{mk^2}{a}\ \ddot{\theta} + \mu S$, where $k^2 = a^2$

$$= \tfrac{1}{2}\ m\ddot{z} + 2\mu T \qquad \text{[From (1) and (8)]}$$

$\therefore$ (2) $\Rightarrow m\ddot{z} = mg \sin\alpha + 2T - (\tfrac{1}{2}\ m\ddot{z} + 2\mu T) - 2T \quad [\because S = 2T]$

or $$\ddot{z} = \frac{2}{3}\ g \sin\alpha \left[1 - \frac{2\mu T}{mg \sin\alpha}\right].$$

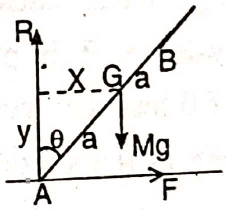
3·05. Slipping of rods.

A unoform rod is held in a vertical position with one end resting upon a perfectly rough table and when released rotates about the end in contact with the table . To discuss the motion.

(Meerut 1984 ; Agra 86, 88)

Let AB be the rod having length $2a$ and mass M.

Let the rod which is rotating about A makes an angle θ with the vertical at any time t .



Taking A point as the origin and horizontal and vertical lines as axes , the coordinate (x, y) of centre of mass G are given by

$$x = a \sin\theta,\ y = a\cos\theta$$

$$\therefore\ \dot{x} = a\cos\theta\dot{\theta},\ \dot{y} = -a\sin\theta\dot{\theta}$$

and $\ddot{x} = -a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta},\ \ddot{y} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}$.

Let F be the frictional force and R the normal reaction at A . Now the equation of motion of C.G. are

$$M\frac{d^2x}{dt^2} = M\,[a\cos\theta\ddot{\theta} - a\sin\theta\dot{\theta}^2] = F \qquad ...(1)$$

$$M\frac{d^2y}{dt^2} = M\,[-a\sin\theta\ddot{\theta} - a\cos\theta\dot{\theta}^2] = R - Mg \qquad ...(2)$$

Again energy of the rod $= \tfrac{1}{2}\ M\,[(\dot{x}^2 + \dot{y}^2 + \tfrac{1}{3}\ a^2\ \dot{\theta}^2)]$

$$[\because\ v^2 = \dot{x}^2 + \dot{y}^2,\ k^2 = \tfrac{1}{3}\ a^2]$$

$$= \frac{1}{2}\ m\left[(a\dot{\theta})^2 + \frac{1}{3}\ a^2\ \dot{\theta}^2\right] = \frac{2}{3}\ Ma^2\,\dot{\theta}^2$$

and work done by the forces $= Mg\,(a - a\cos\theta)$

Hence from energy equation , we have

$$\tfrac{2}{3}\, Ma^2\dot{\theta}^2 = Mg\ (a - a\cos\theta) \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}\ (1-\cos\theta)^* \qquad ...(3)$$

Differentiating (3) with respect to t, we have $\ddot{\theta} = \dfrac{3g}{4a} \sin\theta$ ...(4)

Putting the values of $\dot{\theta}$ and $\ddot{\theta}$ from (3) and (4) in (1) and (2), we get

$F = \dfrac{3}{4} Mg \sin\theta\,(3\cos\theta - 2)$ and $R = \dfrac{1}{4} Mg\,(1-3\cos\theta)^2$

We observe that R does not change its sign and vanishes when $\cos\theta = \frac{1}{3}$. Hence the end **A does not leave the plane.**

From the value of F, we see that F changes its sign as θ passes through the angle $\cos^{-1}\left(\frac{2}{3}\right)$; thus its direction is then reversed .

At $\cos\theta = \frac{1}{3}$, $R = 0$, hence the ratio $\dfrac{F}{R}$ **becomes infinite where**

$\cos\theta = \frac{1}{3}$, hence unless the plane be infinitely rough there will be sliding at this value of θ. In practice the end A of the rod begins to slip for some value of θ less then $\cos^{-1}\left(\frac{1}{3}\right)$. The end A will slip backwards or forward according as the slipping takes place before of after the inclination of the rod is $\cos^{-1}\left(\frac{2}{3}\right)$.

ILLUSTRATIVE EXAMPLES

Ex. 1. *A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ. If it be then released show that it will commence to slide if*

$$\mu < \left(\frac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}\right) \qquad \textbf{(Agra 91)}$$

Sol. Let AB be the rod having length $2a$ and mass m . Let F be the force

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*Equation (3) can also be obtained by taking moments about G, then

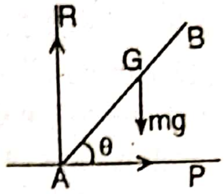
$$M\frac{a^2}{3}\ddot{\theta} = Ra\sin\theta - Fa\cos\theta = Mg\sin\theta - Ma^2\ddot{\theta}. \quad \text{[From (1) and (2)]}$$

or $\quad \ddot{\theta} = \dfrac{3g}{4a}\sin\theta$

Multiplying by $2\dot{\theta}$ and integrating, we get $\dot{\theta}^2 = -\dfrac{3g}{2a}\cos\theta + C$

When $\theta = 0, \dot{\theta} = 0 \Rightarrow C = \dfrac{3g}{2a}$. $\therefore\ \dot{\theta}^2 = \dfrac{3g}{2a}(1-\cos\theta)$

of friction sufficient to prevent sliding and R the normal reaction. With reference to point A as the origin, the coordinated of point G *i.e.* C.G. are ($a\cos\theta$, $a\sin\theta$). the coordinates of point G before the motion begins are ($a\cos\alpha$, $a\sin\alpha$).



Thus the vertical distance moved by the C.G. is ($a\sin\alpha - a\sin\theta$).

Equations of motion of C.G. are

$$m\frac{d^2x}{dt^2} = m\,[-a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}\,] = F \qquad ...(1)$$

and. $$m\frac{d^2y}{dt^2} = m\,[-a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}\,] = R - mg \qquad ...(2)$$

The equation of energy gives

$$\tfrac{1}{2}m\,[(\dot{x}^2+\dot{y}^2)+\tfrac{1}{3}a^2\dot{\theta}^2] = mg\,(a\sin\alpha - a\sin\theta)$$

$$\Rightarrow \tfrac{1}{2}m\,(a^2\dot{\theta}^2+\tfrac{1}{3}a^2\dot{\theta}^2) = amg\,(\sin\alpha - \sin\theta)$$

$$\Rightarrow \tfrac{2}{3}a^2\dot{\theta}^2 = ga\,(\sin\alpha - \sin\theta) \quad \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta) \qquad ...(3)$$

Differentiating (3) w.r.t., to t, we get $\ddot{\theta} = \dfrac{-3g}{4a}\cos\theta$ ...(4)

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (3) and (4) in (1) and (2), we get

$$F = m\left[-a\cos\theta\,.\frac{3g}{2a}(\sin\alpha-\sin\theta) - a\sin\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{3}{4}mg\cos\theta\,(3\sin\theta - 2\sin\alpha) = \frac{3}{4}mg\cos\alpha\sin\alpha,$$

when $\theta = \alpha$

and $$R = mg + m\left[-a\sin\theta\,.\frac{3g}{2a}(\sin\alpha-\sin\theta) + a\cos\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{1}{4}mg\,[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$$

$$= \tfrac{1}{4}mg\,(4 - 3\cos^2\alpha), \qquad \text{when } \theta = \alpha$$

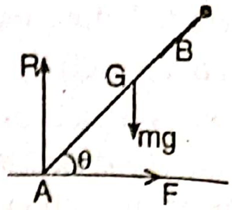
$$= \tfrac{1}{4}mg\,[1 + 3(1-\cos^2\alpha)] = \tfrac{1}{4}mg\,(1+3\sin^2\alpha)$$

The end A will commence to slide if $\mu < \dfrac{F}{R}$ *i.e.* $\mu < \dfrac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}$.

Ex. 2. *The lower end of a uniform rod, inclined intially at an angle α to the horizon is placed on a smooth horizontal table. A horizontal force is applied to its lower end of such a magnitude that the rod rotates in vertical* *plane with constant angular velocity ω. Show that when the rod is inclined at an angle θ to the horizon the magnitude of the force is*

$mg \cot\theta - ma\omega^2 \cos\theta$ where m is the mass of the rod .

Sol. Let the horizontal force applied at the lower end A of the rod be F. Let at any time t, θ be the angle that the rod makes with the horizontal. Since the rod rotates with a uniform angular velocity $\omega \therefore \dot{\theta} = \omega$ (const).



$\Rightarrow \ddot{\theta} = 0$ ...(2)

The equation of motion of G along the vertical

$$R - mg = m\frac{d^2}{dt^2}(a \sin\theta) = ma(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta})$$

$= - ma \sin\theta \,.\, \omega^2$ from (1) and (2) ...(3)

Since the end A is not fixed, the equation of horizontal motion of C.G. is not written .

Again taking moments about G, we have

$mk^2\ddot{\theta} = Fa \sin\theta - Ra \cos\theta \Rightarrow F = R \cot\theta$ $\{\because \ddot{\theta} = 0$ from (2)$\}$

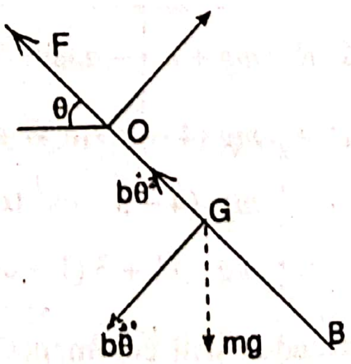
$\Rightarrow F = (mg - ma \sin\theta \,.\, \omega^2) \cot\theta$ from (3)

$\Rightarrow F = mg \cot\theta - ma\omega^2 \cos\theta.$

Ex. 3. *A rough uniform rod , of length 2a , is placed on a rough table at right angles to its edge ; if its centre of gravity be initially at distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\frac{\mu a^2}{a^2 + 9b^2}$ where μ is the coefficient of fricton .*

(Kanpur 89)

Sol. Initially the rod was at right angles to the edge of the rough table, now it has turned through an angle θ.



Let there be no sliding when the rod has turned through this angle. Let A and R be the normal reaction and the force of friction on the rod. Acceleration of G along and perpendicular to GO are respectively $b\dot{\theta}^2$ and $b\ddot{\theta}$. Equations of motion of centre of gravity G are

$mb\ddot{\theta} = mg \cos\theta - R$ ...(1)

and $\quad mb\,\dot{\theta}^2 = F - mg \sin\theta \qquad ....(2)$

Taking moments about O, the point of contact of the rod and table, we have $mk^2\ddot{\theta} = mg\, b\cos\theta, \Rightarrow m\left(b^2 + \frac{a^2}{3}\right)\ddot{\theta} = mg\, b\cos\theta$

$$\Rightarrow \ddot{\theta} = \frac{3gb}{a^2 + 3b^2}\cos\theta \qquad ...(3)$$

Multiplying (3) by $2\dot{\theta}$ and integrating , we get $\dot{\theta}^2 = \frac{6gb}{a^2+3b^2}\sin\theta$

The constant of integration vanishes as initially when $\theta = 0, \dot{\theta} = 0$. Putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ in (1) and (2) from (3) and (4), we have

$$R = -gb\,.\frac{3bg}{a^2+3b^2}\cos\theta + mg\cos\theta = \frac{mga^2}{a^2+3b^2}\cos\theta$$

and $F = mg\sin\theta + mb\,\frac{6gb}{a^2+3b^2}\sin\theta = mg\,\frac{a^2+9b^2}{a^2+3b^2}\sin\theta.$

The sliding commences when

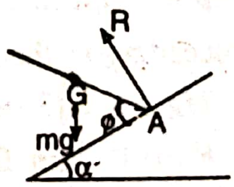
$F = \mu R$ *i.e.* when $\quad mg\,\frac{a^2+9b^2}{a^2+3b^2}\sin\theta = \mu\,\frac{mg\,a^2}{a^2+3b^2}\cos\theta$

or when $\quad \tan\theta = \frac{\mu a^2}{a^2+9b^2}.$

Ex. 4. *A uniform rod of mass m, is placed at right angle to a smooth plane of inclination α with one end in contact with it. The rod is then released. Show that when the inclination to the plane is φ, the reaction of the plane will be*

(Meerut 1980)

$$mg\,\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha$$



Sol. As there is no force acting along the plane, so initially there is no motion along the plane. The C.G. *i.e.* point G moves perpendicular to the plane.

Let ϕ be the angle which the rod makes with the plane after time t. Taking A as the origin, the plane as x-axis and a line perpendicular to to the plane as y-axis, the co-ordinates of G are

$$x = a\cos\phi, \quad y = a\sin\phi.$$

Equations of motion of point G are

$$m\ddot{y} = m(a \cos\phi\, \ddot{\phi} - a \sin\phi\, \dot{\phi}^2) = R - mg \cos\alpha \qquad ...(1)$$

and $$m\frac{a^2}{3}\ddot{\phi} = -R\,.\,a\cos\phi \qquad ...(2)$$

Also from energy equation, we have

$$\tfrac{1}{2}ma^2\cos^2\phi\,\dot{\phi}^2 + \tfrac{1}{2}\frac{ma^2}{3}\dot{\phi}^2 = \text{work done by gravity}$$

$$= mg\,a\cos\alpha\,(1-\sin\phi)$$

or $$\dot{\phi}^2 = \frac{6g(1-\sin\phi)}{a(1+3\cos^2\phi)}\cos\alpha \qquad ...(3)$$

Differentiating (3) w.r.t. t, we get

$$\dot{\phi}\,\ddot{\phi} = \frac{3g\cos\alpha}{a}\left[\frac{-\cos\phi}{(1+3\cos^2\phi)} + \frac{6\cos\phi\sin\phi(1-\sin\phi)}{(1+3\cos^2\phi)^2}\right]\dot{\phi}$$

$$= -\frac{3g}{a}\cos\alpha\left[\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\right]\cos\phi\,.\,\dot{\phi}$$

or $$\ddot{\phi} = -\frac{3g}{a}\cos\phi\cos\alpha\left[\frac{1+3(1-\sin\phi)^2}{(1+3\cos^2\phi)^2}\right]$$

Putting the value of $\ddot{\phi}$ in (2), we get

$$R = mg\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha.$$

Ex. 5. *A uniform rod is held nearly vertically with one end resting on an imperfectly rough plane. It is released from rest and falls forward. The inclination to the vertical at any instant is θ. Prove that*

(i) *If the coefficient of friction is less than a certain finite amount, the lower end of the rod will slip backward before* $\sin^2(\theta/2) = \left(\frac{1}{6}\right)$.

(ii) *However great the coefficient of friction may be, the lower end will begin to slip forward at a value of* $\sin^2(\theta/2)$ *between* $\frac{1}{6}$ *and* $\frac{1}{3}$.

Sol. (i) Proceeding in the same way as in **3.05,** we get

$$F = \tfrac{3}{4}Mg\sin\theta(3\cos\theta - 2) \quad \text{and} \quad R = \tfrac{1}{4}mg(1-3\cos\theta)^2$$

Obviously $F = 0$ if $\sin\theta = 0$ or $3\cos\theta - 2 = 0$

i.e. if $\theta = 0$ or $\cos\theta = \frac{2}{3}$

i.e. if $\theta = 0$ or $1 - 2\sin^2(\theta/2) = \frac{2}{3}$ or $\sin^2(\theta/2) = \frac{1}{6}$.

The value of F is positive when θ takes all intermediate values between $\theta = 0$ and $\theta = \cos^{-1}\frac{2}{3}$ and is continuous function of θ, hence between these two values of θ where F vanishes, F has a maximum value for some

θ. Let F_1 be the maximum value. We observe that for $0 \le \theta \le \cos^{-1} \frac{2}{3}$ the value of $R \le Mg$.

Thus there is a finite value of μ for which $F_1 > \mu R$ and therefore for this value of μ, sliding will take place before $\cos^{-1} \frac{2}{3}$ *i.e.* before $\sin^2 \frac{\theta}{2} = \frac{1}{6}$. Since F is positive (in the forward direction) hence the slipping will start in the backward direction.

(ii) We observe from the value of F that if $\cos\theta > 3/2$, F changes its sign, *i.e.* the direction of the friction is reversed if

$$F' = -F = \frac{3}{4} mg\ (2 - 3\cos\theta)$$

Now the slipping may start when $F' > \mu R$

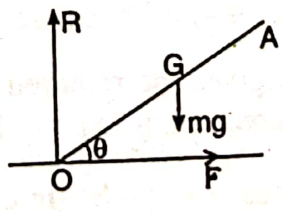
i.e. when $3\sin\theta\,(2 - 3\cos\theta) > \mu\,(1 - 3\cos\theta)^2$ ...(1)

As θ increases from $\cos^{-1} \frac{2}{3}$ to $\cos^{-1} \frac{1}{3}$, the term on the left hand side increases while the right hand side term decreases from 1 to 0. Therefore for some value of θ between $\cos^{-1} \frac{2}{3}$ and $\cos^{-1} \frac{1}{3}$ *i.e.* for $\sin^2(\theta/2)$ between $\frac{1}{6}$ and $\frac{1}{2}$ the condition (1) is satisfied and the slipping will then start in the forward direction.

Ex. 6. *A uniform rod is placed with one end in contact with a horizontal table, and is then at an inclination α to the horizon and is allowed to fall. when it becomes horizontal, show that its angular velocity is* $\left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$ *whether the plane is perfectly smooth or perfectly rough. Show also that the end of the rod will not leave the plane in either case.*

Sol. Let at any instant t the rod makes an angle θ with the horizontal. Let R and F be the normal reaction and friction at the instant with O as origin, the co-ordinates of C.G. are $x = a\cos\theta$, $y = a\sin\theta$.

Case I. *When plane is perfectly rough and O is fixed.*



Then energy equation gives

$\frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} mk^2\dot{\theta}^2 =$ work done by gravity

$$\Rightarrow \frac{1}{2} m\,(a^2\dot{\theta}^2 + \frac{1}{3} a^2\dot{\theta}^2) = mga\,(\sin\alpha - \sin\theta)$$

$$\dot{\phi}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta). \quad ...(1)$$

When the rod becomes horizontal *i.e.* when $\theta = 0$, the angular velocity $\dot{\theta} = \omega$ (say) is given by $\omega^2 + \frac{3g}{2a}\sin\alpha$ or $\omega = \left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$

Differentiating (1) w.r.t. 't' we get $\ddot{\theta} = \dfrac{-3g}{4a}\cos\theta$. ...(2)

The equation of motion of C.G. is

$$R - mg = m\frac{d^2}{dt^2}(a\sin\theta) = ma(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta})$$

$$\Rightarrow R = mg + ma\left[-\sin\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) + \cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

[substituting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (1) and (2)]

$$= \tfrac{1}{4} mg\,(4 - 6\sin\alpha\sin\theta + 6\sin^2\theta - 3\cos^2\theta)$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 - 9\sin^2\alpha\sin^2\theta + 9\sin^2\theta]$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\,(1 - \sin^2\alpha)]$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\cos^2\alpha]$$

This shows that R is always positive, therefore the end O of the rod never leaves the plane.

Case II. *When the plane is perfectly smooth.*

In this case there is no horizontal forces, hence C.G. descends in a vertical line *i.e.* the only velocity of G being along the vertical direction

$y = a\sin\theta,\ \dot{y} = a\cos\theta\dot{\theta}$.

The energy equation gives $\tfrac{1}{2}m\dot{y}^2 + \tfrac{1}{2}mk^2\dot{\theta}^2$ = work done by gravity

i.e. $\tfrac{1}{2}m(a^2\cos^2\theta\dot{\theta}^2 + \tfrac{1}{3}a^2\dot{\theta}^2) = mg(a\sin\alpha - a\sin\theta)$

or $\dot{\theta}^2(\cos^2\theta + \tfrac{1}{3}) = \left(\dfrac{2g}{a}\right)(\sin\alpha - \sin\theta)$ ...(1)

when the rod becomes horizontal *i.e.* when $\theta = 0$, the angular velocity $\dot{\theta} = \omega$ (say) is given by

$$\omega^2(1 + \tfrac{1}{3}) = \frac{2g}{a}\sin\alpha \Rightarrow \omega^2 = \frac{3g}{2a}\sin\alpha \Rightarrow \omega = \left(\frac{3g}{2a}\sin\alpha\right)^{1/2}.$$

This gives the required result in the case of plane being smooth.

Differentiating (1), we have

$$\ddot{\theta}(\cos^2\theta + \tfrac{1}{3}) - \dot{\theta}^2\sin\theta\cos\theta = -\left(\frac{g}{a}\right)\cos\theta.$$

$$\Rightarrow \ddot{\theta}(\cos^2\theta + \tfrac{1}{3}) - \sin\theta\cos\theta\left[\frac{(2g/a)(\sin\alpha - \sin\theta)}{\cos^2\theta + \frac{1}{3}}\right] = -\left(\frac{g}{a}\right)\cos\theta$$

$$\Rightarrow \ddot{\theta}(\cos^2\theta + \tfrac{1}{3})^2 = -\left(\frac{g}{a}\right)\cos\theta\,[\sin^2\theta - 2\sin\alpha\sin\theta + \tfrac{4}{3}]$$

$$= -(g/a)\cos\theta\,[(\sin\theta - \sin\alpha)^2 + \tfrac{1}{3} + \cos^2\alpha] \quad ...(3)$$

Again taking moments about G, we have

$$m\frac{a^2}{3}\ddot{\theta} = -Ra\cos\theta \text{ or } R = -\tfrac{1}{3}a\sec\theta \,.\, m\ddot{\theta}$$

$$\Rightarrow R = \frac{mg}{3}\left[\frac{(\sin\theta - \sin\alpha)^2 + \frac{1}{3} + \cos^2\alpha}{(\cos^2\theta + \frac{1}{3})^2}\right]$$

from (2) by putting the value of $\ddot{\theta}$

$$\Rightarrow R = mg\left[\frac{1 + 3\cos^2\alpha + 3(\sin\theta - \sin\alpha)^2}{(1 + 3\cos\theta)^2}\right]$$

we observe that R is positive for every value of α and θ. Hence the end never leaves the plane.

3·06 . *A uniform straight rod slides down in a vertical plane its end being in contact with two smooth planes, one horizontal and the other vertical. If it started from rest at an angle α with the horizontal ; to discuss the motion.*

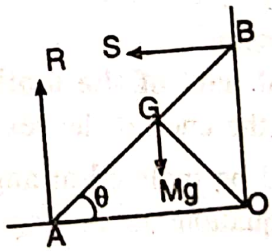
(Meerut 1987, 84, 83)

Let at any instant t, the rod makes an angle θ with the horizontal. Let R and S be the reactions at the ends A and B of the rod AB whose length is $2a$ and mass M .

With reference to point O as origin, the co-ordinates of G *i.e.* centre of gravity are $x = a\cos\theta, y = a\sin\theta$

$$\therefore \ddot{x} = -a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta},$$

$$\ddot{y} = -a\sin\theta\,\dot{\theta}^2 + a\cos\theta\ddot{\theta}$$



The equations of motion of C.G. are $M\ddot{x} = S$

$$\Rightarrow M(-a\cos\theta\dot{\theta}^2 - a\sin\theta\,\ddot{\theta}) = S \quad ...(1)$$

and $M\ddot{y} = R - Mg$

$$\Rightarrow M(-a\sin\theta\dot{\theta}^2 + a\cos\theta\,\ddot{\theta}) = R - Mg$$

Energy equation gives

$$\tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}Mk^2\dot{\theta}^2 = \text{work done by the gravity .}$$

$$\Rightarrow \tfrac{1}{2}M(a^2\dot{\theta}^2 + \tfrac{1}{3}a^2\dot{\theta}^2) = Mga(\sin\alpha - \sin\theta)$$

$$\Rightarrow \dot{\theta}^2 = \left(\frac{3g}{2a}\right)(\sin\alpha - \sin\theta). \quad ...(3)$$

Differentiating (3) w.r.t. t, we get $\ddot{\theta} = -\left(\dfrac{3g}{4a}\right)\cos\theta \quad ...(4)$

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ in (1) and (2), we have

$$S = M\left[-a \cos\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{3}{4} Mg \cos\theta\,(3\sin\theta - 2\sin\alpha) \qquad \text{...(5)}$$

$$R = Mg + M\left[-a\sin\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{1}{4} Mg\,[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$$

$$= \tfrac{1}{4} Mg\,[1 - 6\sin\theta\sin\alpha + 9\sin^2\theta]$$

$$= \tfrac{1}{4} Mg\,[1 - \sin^2\alpha + \sin^2\alpha - 6\sin\theta\sin\alpha + 9\sin^2\theta]$$

$$= \tfrac{1}{4} Mg\,[(3\sin\theta - \sin\alpha)^2 + \cos^2\alpha] \qquad \text{...(6)}$$

From (5), we observe that $S = 0$ when $\sin\theta = \frac{2}{3}\sin\alpha$ and S will be negative when this value of θ is reached. Hence the end B leaves the wall when $\sin\theta = \frac{2}{3}\sin\alpha$.

Again from (6), we observe that R is always positive *i.e.* the end A never leaves the plane

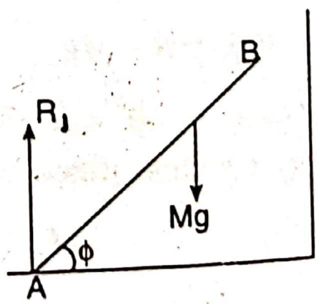
Further when the end B leaves the plane $\sin\theta = \frac{2}{3}\sin\alpha$ and $S = 0$ thus equations of motion (1), (2), (3) and (4) cease to hold good for further motion .

Putting $\sin\theta = \frac{2}{3}\sin\alpha$ in (3), the angular velocity of the rod now becomes $\left(\frac{g}{2a}\sin\alpha\right)^{1/2}$, this will be the initial angular velocity for the next part of the motion .

Second part of the motion .

When the end B leaves the wall, let R_1 be the normal reaction at A. Let the rod be inclined at angle ϕ to the horizontal.

The equations of motion are



$$M\ddot{x} = 0 \qquad \text{...(1)}$$

$$M\ddot{y} = R_1 - Mg \qquad \text{...(2)}$$

and $$M\frac{a^2}{3}\ddot{\phi} = -R_1\, a\cos\phi \qquad \text{...(3)}$$

As $y = a\sin\phi$, $\therefore \ddot{y} = -a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi}$

Hence from (2) and (3), we get

$$\left(\tfrac{1}{3} + \cos^2\phi\right)\left(\frac{d^2\phi}{dt^2}\right) - \sin\phi\cos\phi\left(\frac{d\phi}{dt}\right)^2 = -\frac{g}{a}\cos\phi \qquad \text{...(4)}$$

Integrating it, we get $(\frac{1}{3}+\cos^2\phi)\left(\frac{d\phi}{dt}\right)^2=-\frac{2g}{a}\sin\phi+C$ ...(5)

when $\sin\phi=\frac{2}{3}\sin\alpha$, $\frac{d\phi}{dt}=\sqrt{\left(\frac{g}{2a}\sin\alpha\right)}$

$\therefore\ \frac{g\sin\alpha}{2a}[\frac{1}{3}+1-\frac{4}{9}\sin^2\alpha]=-\frac{2g}{a}\cdot\frac{2}{3}\sin\alpha+C$

or $$C=\frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right)$$

Hence from (5), we have

$$(\tfrac{1}{3}+\cos^2\phi)\left(\frac{d\phi}{dt}\right)^2=\frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right)-\frac{2g}{a}\sin\phi. \quad ...(6)$$

When $\phi=0$ *i.e.* when rod reaches the horizontal plane, let its angular velocity be Ω, then

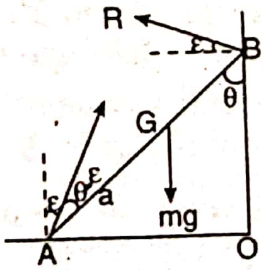
$$\Omega^2(\tfrac{1}{3}+1)=\frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right)\Rightarrow\ \Omega^2=\frac{3g}{2a}\left(1-\frac{\sin^2\alpha}{9}\right)\sin\alpha. \quad ...(7)$$

Ex. 7. *A heavy rod, of length 2a is placed in a vertical plane with its ends in contact with a rough vecrtical wall and an equally rough horizontal plane. the coefficient of friction being* tan ε. *Show that it will begin to slip down if its initial inclination to the vartical is greater that* 2ε. *Prove also that the inclination* θ *of the rod to the vertical at any time is given by* $\ddot{\theta}\ (k^2+a^2\cos 2\varepsilon)-a^2\dot{\theta}^2\sin 2\varepsilon=ag\sin(\theta-2\varepsilon)$

Sol. Let AB be the rod of length $2a$ and mass m. When AB makes an angle θ with the vertical and let R and S be the resultant reactions at B and A respectively .

Writting equations of motion of centre of mass G , we have

$$m\frac{d^2}{dt^2}(a\sin\theta)=-S\sin\varepsilon+R\cos\varepsilon \quad ...(1)$$



and $$m\frac{d^2}{dt^2}(a\cos\theta)=R\sin\varepsilon+S\cos\varepsilon-mg \quad ...(2)$$

Taking moments about G, we have

$$mk^2\ddot{\theta}=Sa\sin(\theta-\varepsilon)-Ra\cos(\theta-\varepsilon) \quad ...(3)$$

From (2), we have $ma(\cos\theta\ddot{\theta}-\sin\theta\dot{\theta}^2)=R\cos\varepsilon-S\sin\varepsilon$ ...(4)

From (2), we have

$$ma\,(\sin\theta\ddot{\theta}+\cos\theta\dot{\theta}^2)=mg-R\sin\varepsilon-S\cos\varepsilon \qquad ...(5)$$

On solving equations (4) and (5), we have

$$R=mg\sin\varepsilon+ma\cos(\theta+\varepsilon)\ddot{\theta}-ma\sin(\theta+\varepsilon)\dot{\theta}^2 \qquad ...(6)$$

$$S=mg\cos\varepsilon-ma\sin(\theta+\varepsilon)\ddot{\theta}-ma\cos(\theta+\varepsilon)\dot{\theta}^2 \qquad ...(7)$$

Putting the values of R and S in (3), we have

$$mk^2\ddot{\theta}=a\sin(\theta-\varepsilon)\,[mg\cos\varepsilon-ma\sin(\theta-\varepsilon)\ddot{\theta}-ma\cos(\theta+\varepsilon)\dot{\theta}^2]$$

$$-a\cos(\theta-\varepsilon)\,[mg\sin\varepsilon+ma\cos(\theta+\varepsilon)\ddot{\theta}-ma\sin(\theta+\varepsilon)\dot{\theta}^2]$$

$$=m\,ga\sin(\theta-2\varepsilon)-ma^2\ddot{\theta}\cos 2\varepsilon+ma^2\dot{\theta}^2\sin 2\varepsilon]$$

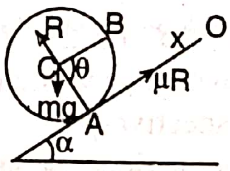
or $\ddot{\theta}\,(k^2+a^2\cos 2\varepsilon)-a^2\dot{\theta}^2\sin 2\varepsilon=ag\sin(\theta-2\varepsilon)$, which gives θ.

If $\theta > 2\varepsilon$, it is obvious that $\ddot{\theta}$ is positive and hence the rod starts slipping if $\theta > 2\varepsilon$.

3·07. When rolling and sliding are combined .

An imperfectly rough sphere moves from rest down a plane inclined at an angle α to the horizon, to determine the motion .

Let C be the centre of sphere whose radius is a . Let in time t the sphere have turned through an angle θ *i.e.* let CB be a radius (a line fixed in the body) which was initially normal to the plane, makes an angle θ with the normal CA during this period .



Let us suppose that the friction is not sufficient to produce pure rolling therefore the sphere slides as well as turns . So the maximum friction μR acts up the plane, μ being the coefficient of friction . Let x be the distance described by the centre of gravity C parallel to the inclined plane in time t , and θ the angle through which the sphere turns .

As there is no motion perpendicular to the plane, so the C.G. of the sphere always moves parallel to the plane. The equations of motion are

$$m\ddot{x}=mg\sin\alpha-\mu R \qquad ...(1) \qquad 0=R-mg\cos\alpha \qquad ...(2)$$

and $$m\,\tfrac{2}{5}a^2\ddot{\theta}=\mu Ra \qquad ...(3)$$

From (1) and (2), we have $\ddot{x}=g\,(\sin\alpha-\mu\cos\alpha)$ ...(4)

Integrating (4) w.r.t. 't' we get $\dot{x}=g\,(\sin\alpha-\mu\cos\alpha)\,t$ ...(5)

Integrating (5) again, we get $x=g\,(\sin\alpha-\mu\cos\alpha)\dfrac{t^2}{2}$ ...(6)

Constants of integration vanish as $\dot{x} = 0$, $x = 0$ when $t = 0$

From (2) and (3), we get $a\ddot{\theta} = \frac{5}{2}\mu g \cos\alpha$.

Integrating it, we get $a\dot{\theta} = \frac{5}{2}\mu g\, t \cos\alpha$.

Integrating it again , we get $\theta = \frac{5\mu g}{4} t^2 \cos\alpha$

The constants of integration vanish as $\dot{\theta} = 0$, $\theta = 0$ when $t = 0$.
The velocity of the point of contact A down the plane
= velocity of C , the centre of sphere, + velocity of A relative to C,

$$= \dot{x} - a\dot{\theta}$$
$$= g(\sin\alpha - \mu\cos\alpha)\, t - \tfrac{5}{2}\mu g t \cos\alpha$$
$$= \tfrac{1}{2} g(2\sin\alpha - 7\mu\cos\alpha)\,. \qquad \text{...(8)}$$

Equation (8) gives rise to the following three cases :
First case . If $2\sin\alpha > 7\mu\cos\alpha$ *i.e.* if $\mu < \frac{2}{7}\tan\alpha$.

In this case, velocity of the point of contact is positive for all values of t *i.e.* it does not vanish, hence the point of contact always slides down and the maximum friction μR acts. The sphere never rolls. The equations of motion established above govern the entire motion.
Second case . If $2\sin\alpha = 7\mu\cos\alpha$ *i.e.* if $\mu = \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is zero for all values of t and therefore motion of the sphere is that of pure rolling throughout and the maximum friction μR is always exerted .
Third case . $2\sin\alpha < 7\mu\cos\alpha$ *i.e.* if $\mu > \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is negative *i.e.* if the maximum friction μR were allowed to act, the point of contact will slide up the plane which is impossible because that amount of friction will only act which is just sufficient to keep the point of contact at rest. Hence in this case the motion is of pure rolling from the very start and remains the same throughout and the maximum friction μR is not exerted. Therefore in this case the equations of motion established above do not hold good.
Let F be the frictional force now in play, then equations of motion are

$$m\ddot{x} = mg\sin\alpha - F \quad \text{...(9)} \qquad 0 = R - mg\cos\alpha \quad \text{...(10)}$$

and $m\,\frac{2}{5}a^2\ddot{\theta} = Fa$

Beacause the point of contacty is at rest, we have

$$\dot{x} - a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta} \qquad \text{...(12)}$$

From (9), (11) and (12), we have $\ddot{x} = a\dot{\theta} = \frac{5}{7} g \sin\alpha$

Integrating above, we get $\dot{x} = a\dot{\theta} = \frac{5}{7} g\, t \sin\alpha$

Again integrating above, we get $x = a\theta = \dfrac{5}{14} gt^2 \sin\alpha,$ ...(14)

the constants of integration vanish as $\dot{x} = 0, x = 0,$ when $t = 0$

ILLUSTRATIVE EXAMPLES

Ex. 1. *A hoop is projected with velocity V down on inclined plane of inclination* α *, the coefficient of friction being* μ *(>* $\tan\alpha$*). It has initialy such a backward spin* Ω *that after a time* t_1 *it starts moving uphill and continues to do so for a time* t_2 *after which it once more descends. The motion being in a vertical at right angles to the given inclined plane, show that* $(t_1 + t_2)\, g \sin\alpha = a\Omega - V.$

Sol. Let C be the centre of the hoop and CB its radius (a line fixed in the body) makes an angle θ with CA which is normal to the plane (CA is a line fixed in space), after time t . Initially CB was normal to the plane. Initially the velocity of the point of contact A down the plane

= velocity of centre C + velocity of A relative to $C = V + a\Omega$, which is a positive quantity

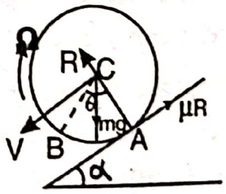
Hence the point of contact slides down and friction μR acts up the plane.

The equations of motion are

$m\ddot{x} = mg \sin\alpha - \mu R$ ...(1)

$0 = R - mg \cos\alpha$ ...(2) and

$ma^2\ddot{\theta} = -\mu Ra$ ...(3)



From (1) and (2), we have

$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$, integrating it, we get

$\dot{x} = g(\sin\alpha - \mu\cos\alpha)\, t +$ constant

when $\dot{x} = V, t = 0.$ $\therefore$ constant $= V$

Therefore $\dot{x} = g(\sin\alpha - \mu\cos\alpha)\, t + V$ ...(4)

From (1) and (3), we get

$a\ddot{\theta} = -\mu g \cos\alpha$, integrating it, we get

$a\dot{\theta} = -\mu gt \cos\alpha +$ constant ; When $t = 0, \dot{\theta} = \Omega$ constant $= a\Omega$

Therefore $a\dot{\theta} = -\mu gt \cos\alpha + a\Omega$ ...(5)

The hoop will cease to move downwards, when $\dot{x} = 0$ *i.e.* from (4),

$$t_1 = \frac{V}{g(\mu \cos \alpha - \sin \alpha)} \quad ...(6)$$

Obviously the velocity of the point of contact is $\dot{x} + a\dot{\theta}$, even when $\dot{x} = 0$, for the hoop to move uphill $a\dot{\theta}$ should be positive. It follows that throughout the downward motion $\dot{x} + a\dot{\theta}$ is always positive. Therefore when moving downward pure rolling does not take place. Thus the equations established above are true throughout the downward motion.

Putting the value of t_1 from (6) in (5), we get

$$a\dot{\theta} = a\Omega - \frac{\mu V \cos \alpha}{(\mu \cos \alpha - \sin \alpha)}$$ since $a\dot{\theta}$ is positive, the hoop begins to move uphill.

When the hoop starts moving uphill. The initial velocity of the centre is zero and $a\dot{\theta}$ is positive with the sense of the direction as θ.

Initial velocity of the point of contact $= 0 - a\dot{\theta}$ which is negative.

Thus initially the velocity of the point of contact is in the downward direction hence the friction μR acts upwards. Equations of motion are

$$m\ddot{y} = -mg \sin \alpha + \mu R \quad ...(1) \qquad 0 = R - mg \cos \alpha \quad ...(2)$$

$$ma^2 \ddot{\phi} = -\mu Ra \quad ...(3)$$

on eliminating R, we get $\ddot{y} = (\mu \cos \alpha - \sin \alpha) g$ and $a\ddot{\phi} = -\mu g \cos a$

Intetgrating these two equations, with the initial conditions, we get

$$\dot{y} = g(\mu \cos \alpha - \sin \alpha) t \quad ...(4)$$

and $$a\dot{\phi} = -\mu gt \cos \alpha + a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \quad ...(5)$$

$$\left[\because \text{ when } t = 0, \dot{y} = 0, a\dot{\phi} = a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}\right]$$

Rolling commences when the velocity of the point of contact is zero *i.e.*

$$\dot{y} - a\dot{\phi} = 0 \Rightarrow \dot{y} = a\dot{\phi}$$

$$\Rightarrow g(\mu \cos \alpha - \sin \alpha)\, t' = -\mu g t' \cos \alpha + a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}$$

$$\Rightarrow gt' (2\mu \cos \alpha - \sin \alpha) = a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}$$

This gives value of t'

$\therefore$ At this time $\dot{y} = gt' (\mu \cos \alpha - \sin \alpha)$ from (4),

When Rolling commences. Equations of motion are

$$m\ddot{z} = F - mg \sin \alpha \quad ...(1) \qquad ma^2 \ddot{\psi} = -Fa \quad ...(2)$$

and $\quad \dot{z} - a\dot{\psi} = 0 \quad ...(3)$

Solving these equations, we get $\quad F = \frac{1}{2} mg \sin \alpha$

Since $\mu > \tan \alpha \Rightarrow \mu R > \tan \alpha \; mg \cos \alpha \Rightarrow \mu R > mg \sin \alpha$.
We observe that $F < \mu R$, so the condition of pure rolling is satisfied, and hence the equations of motion holds good for the motion .

From (1), we have , $m\ddot{z} = F - mg \sin \alpha = \frac{1}{2} mg \sin \alpha - mg \sin \alpha$.

i.e. $\ddot{z} = -\frac{1}{2} g \sin \alpha$; integrating it, we get $\dot{z} = -\frac{1}{2} gt \sin \alpha + K$

when $t = 0, \dot{z} = \dot{y} = gt' (\mu \cos \alpha - \sin \alpha)$, $\therefore K = gt' (\mu \cos \alpha - \sin \alpha)$

Therefore $\quad \dot{z} = -\frac{1}{2} gt \sin \alpha + gt' (\mu \cos \alpha - \sin \alpha)$

The hoop ceases to move up the hill if $\dot{z} = 0$. Let this happen after time t''.

$$\therefore 0 = -\tfrac{1}{2} gt'' \sin \alpha + gt' (\mu \cos \alpha - \sin \alpha)$$

or $\quad t'' = 2 \dfrac{(\mu \cos \alpha - \sin \alpha) t'}{\sin \alpha}$

$$\therefore t_2 = t' + t'' = t' + 2 \frac{(\mu \cos \alpha - \sin \alpha)}{\sin \alpha} t' = \left(\frac{2\mu \cos \alpha - \sin \alpha}{\sin \alpha} \right) t'$$

$$= \left(\frac{2\mu \cos \alpha - \sin \alpha}{\sin \alpha} \right) \cdot \frac{1}{g (2\mu \cos \alpha - \sin \alpha)} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

$$= \frac{1}{g \sin \alpha} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

Hence the total time is $t_1 + t_2$

$$= \frac{V}{g (\mu \cos \alpha - \sin \alpha)} + \frac{1}{g \sin \alpha} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

$$= \frac{1}{g \sin \alpha} \left[a\Omega - \frac{\mu V \cos \alpha - V \sin \alpha}{\mu \cos \alpha - \sin \alpha} \right] = \frac{1}{g \sin \alpha} (a\Omega - V)$$

or $\qquad (t_1 + t_2) \; g \sin \alpha = a\Omega - V$

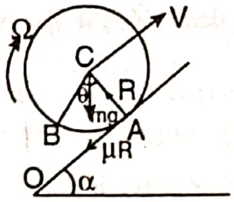
Ex. 2. *A sphere, of radius a is projected up an inclined plane with a velocity V and angular velocity* Ω *in the sense which would cause it to roll up ,* $V > a\Omega$*, and the coefficient of friction* $\frac{2}{7} \tan \alpha$*; show that the sphere will cease to ascend at the end of a time* $\dfrac{5V + 2a\Omega}{5 g \sin \alpha}$*, where* α *is the inclination of the plane.* **(Meerut 81)**

Sol. Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA normal to the plane (CA is a line fixed in space. Initial CB was normal to the plane. Initial velocity of the point of contact A up the plane

= Velocity of the centre C + velocity of A relative to C

$= V - a\Omega > 0$ as $V > a\Omega$.

Hence the friction μR acts down the plane, implying, that the sphere slides as well as turns .



Equations of motion are $m\ddot{x} = mg \sin\alpha - \mu R$ ...(1)

$0 = R - mg\cos\alpha$ ...(2) and $m \cdot \frac{2a^2}{5}\ddot{\theta} = \mu R a$ ...(3)

Eliminating R from (1) and (2), we have

$\ddot{x} = -g(\sin\alpha + \mu\cos\alpha)$, integrating it, we get

$\dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + K$.

Now when $t = 0, \dot{x} = V$, $\therefore K = V$

Therefore $\dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + V$ ...(4)

Similarly, we have $a\ddot{\theta} = \frac{5}{2}\mu g t \cos\alpha$, integrating it with intial conditions

i.e. when $t = 0, \dot{\theta} = \Omega$, we get $a\dot{\theta} = \frac{5}{2}\mu g t\cos\alpha + a\Omega$ ...(5)

The velocity of the point contact $= \dot{x} - a\dot{\theta}$. Rolling commences, say after time t_1 when $\dot{x} - a\dot{\theta} = 0$.

or $-g(\sin\alpha + \mu\cos\alpha)t_1 + V - \frac{5\mu}{2}gt_1\cos\alpha - a\Omega = 0$

or $t_1 = \dfrac{2V - 2a\Omega}{g(7\mu\cos\alpha + 2\sin\alpha)}$

Putting this value of $t = t_1$ in (4), we get

$$\dot{x} = V - g(\sin\alpha + \mu\cos\alpha)\left[\frac{2V - 2a\Omega}{g(7\mu\cos\alpha + 2\sin\alpha}\right]$$

$$= \frac{5\mu V\cos\alpha + 2a\Omega(\sin\alpha + \mu\cos\alpha)}{7\mu\cos\alpha + 2\sin\alpha} = V_1 \text{ (say).}$$

When rolling begins *i.e.* when the point of contact has been brought to rest, let F be the friction which is sufficient for pure rolling. Because the point of contact is at rest, so friction will try to keep it at rest if possible , hence the friction F acts upwards.

Equations of motion are $m\ddot{y} = -mg\sin\alpha + F$ ...(1)

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and $\quad m \cdot \frac{2a^2}{5} \ddot{\phi} = -Fa$ ...(2)

Since, throughout the motion the point of contact is at rest so

$\dot{y} - a\dot{\phi} = 0$ or $\dot{y} = a\dot{\phi} \Rightarrow \ddot{y} = a\ddot{\phi}$

Solving equations (1) and (2), we get $F = \frac{2}{7} \cdot mg \sin\alpha$

Again $\mu R = \mu \cdot mg \cos\alpha > \frac{2}{7} \tan\alpha \cdot mg \cos\alpha$ *i.e.* $> \frac{2}{7} mg \sin\alpha$.

Therefore the condition $F < \mu R$ is satisfied.

Putting the value of F in (1), we get $\ddot{y} = -\frac{5}{7} g \sin\alpha$.

Integrating it with intial conditions *i.e.* when $t = 0$, $\dot{y} = V_1$, we get

$\dot{y} = -\frac{5}{7} gt \sin\alpha + V_1$.

The sphere will cease to ascend when $\dot{y} = 0$, let this happen after time t_2.

$$\therefore\ 0 = -\frac{5}{7} gt_2 \sin\alpha + V_1 \quad \text{or} \quad t_2 = \frac{7V_1}{5g \sin\alpha}$$

$\therefore$ The total time of ascent $= t_1 + t_2$

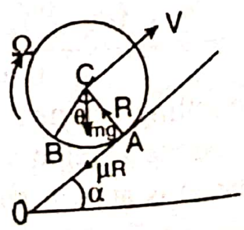
$$= \frac{2V - 2a\Omega}{g(7\mu \cos\alpha + 2\sin\alpha)} + \frac{7}{5g \sin\alpha} \times \left\{ \frac{5\mu V \cos\alpha + 2a\Omega(\sin\alpha + \mu \cos\alpha)}{7\mu \cos\alpha + 2\sin\alpha} \right\}$$

$$= \frac{10(V - a\Omega)\sin\alpha + 35\mu V \cos\alpha + 14a\Omega(\sin\alpha + \mu\cos\alpha)}{5g \sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V(7\mu\cos\alpha + 2\sin\alpha) + 2a\Omega(7\mu\cos\alpha + 2\sin\alpha)}{5g\sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V + 2a\Omega}{5g\sin\alpha}.$$

Ex. 3. *If a sphere be projected up an inclined plane, for which* $\mu = \frac{1}{7}\tan\alpha$, *with velocity V and an initial angular velocity* Ω *(in the direction in which it would roll up), and if* $V > a\Omega$, *show that the friction acts downwards at first and upwards afterwards, and prove that the whole time during which the sphere rises is* $\frac{17V + 4a\Omega}{18g\sin\alpha}$.



Sol. Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA, the normal to the plane (CA is a line fixed in the space). Initially CB was normal to the plane.

Initial velocity of the point of contact A up the plane

$= $ Velocity of the centre $C +$ velocity of A relative to C
$= V - a\Omega > 0$, since $V > a\Omega$.

Hence the velocity of the point of contact A is up the plane, thus the friction μR acts down the plane.

The sphere therefore slides as well as turns.

Equations of motion are

$$m\ddot{x} = -mg \sin \alpha - \mu R \quad ...(1) \quad 0 = R - mg \cos \alpha \quad ...(2)$$

and $$m\frac{2a^2}{5}\ddot{\theta} = \mu Ra \quad ...(3)$$

Eilminating R from (1) and (2), we get

$$m\ddot{x} = -mg \sin \alpha - \mu (mg \cos \alpha) = -mg \sin \alpha - \tfrac{1}{7} \tan \alpha \,.\, mg \cos \alpha$$
$$= -\tfrac{8}{7} mg \sin \alpha \quad (\mu = \tfrac{1}{7} \tan \alpha)$$

or $\ddot{x} = -\frac{8}{7} g \sin \alpha.$

Similarly, we have $m\frac{2a}{5}\ddot{\theta} = \mu R = \frac{1}{7} \tan \alpha \; mg \cos \alpha = \frac{1}{7} mg \sin \alpha$

or $$a\ddot{\theta} = \frac{5}{14} g \sin \alpha \quad ...(5)$$

Integrating (4) and (5) with intial conditions *i.e.* when $t = 0, \dot{x} = V$ and

$\dot{\theta} = \Omega$, we get $$\dot{x} = -\tfrac{8}{7} gt \sin \alpha + V \quad ...(6)$$

and $$a\dot{\theta} = \frac{5}{14} gt \sin \alpha + a\Omega \quad ...(7)$$

Let the velocity of the point of contact *i.e.* $\dot{x} - a\dot{\theta}$ be zero after time t_1 (then the point of contact is brought to rest)

i.e. $\dot{x} - a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta}$

$\Rightarrow -\frac{8}{7} gt_1 \sin \alpha + V = \frac{5}{14} gt \sin \alpha + a\Omega$ (Putting the values of $\dot{x}$ and $\dot{\theta}$)

$\Rightarrow t_1 = \frac{2(V - a\Omega)}{3g \sin \alpha}$. Putting this value of t_1 in (6), we get

$\dot{x} = V - \frac{16}{21}(V - a\Omega) = \frac{5V + 16a\Omega}{21} = V_1$ (say).

When the point of contact has been brought to rest, the pure rolling will commence if there is enough friction to keep the point of contact at rest. Let F be the force of friction sufficient for pure rolling. Equations of motion are $m\ddot{y} = -mg \sin \alpha + F$, $m\frac{2a^2}{5}\ddot{\phi} = -Fa$ Also $\dot{y} - a\dot{\phi} = 0$

Solving these equations, we get $F = \frac{2}{7} mg \sin \alpha$

while $\quad \mu R = \frac{1}{7} \tan \alpha \, mg \cos \alpha = \frac{1}{7} mg \sin \alpha$.

Hence we observe that $F > \mu R$.

From this we conclude that the friction required for pure rolling is more than the maximum friction that can be exerted by the plane, so the pure rolling is impossible .

Inspite of exerting the maximum friction μR upwards, the friction cannot keep the point of contact at rest . Hence the sphere slides as well as turns.

The equations of motion are $m\ddot{y} = -mg \sin \alpha + \mu R$ ...(i)

$0 = R - mg \cos \alpha$ ...(ii) and $m \frac{2a^2}{5} \ddot{\phi} = -\mu Ra$ ...(iii)

i.e. $m\ddot{y} = -mg \sin \alpha + \frac{1}{7} \tan \alpha \, . \, mg \cos \alpha = -\frac{6}{7} mg \sin \alpha$

$\dot{y} = -\frac{6}{7} gt \sin \alpha + V_1$

The sphere will cease to ascend when $\dot{y} = 0$, let this happen after time t_2 . $\therefore 0 = -\frac{6}{7} gt_2 \sin \alpha + V_1$ or $t_2 = (7V_1/6g \sin \alpha)$.

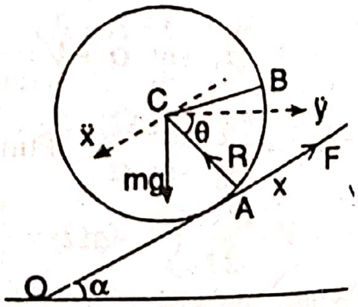
Hence the whole time of ascent

$$= t_1 + t_2 = \frac{2(V - a\Omega)}{3g \sin \alpha} + \frac{7}{6g \sin \alpha} \cdot \left(\frac{5V + 16a\Omega}{21} \right)$$

$$= \frac{12(V - a\Omega) + 5V + 16a\Omega}{18g \sin \alpha} = \frac{17V + 4a\Omega}{18g \sin \alpha} .$$

Ex. 4. *An inclined plane of mass M is capable of moving freely on a smooth horizontal plane . A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity . If y be the horizontal distance advanced by the inclined plane and x the part of the plane rolled over by the sphere, prove that $(M + m) y = mx \cos \alpha$ and $\frac{7}{5} x - y \cos a = \frac{1}{2} gt^2 \sin \alpha$, where α is the inclination of the plane to the horizon .*

Sol. There are two accelerations of the centre C , one $\ddot{x}$ down the plane and other $\ddot{y}$ in a horizontal direction .



The actual acceleration of C parallel to the plane $= \ddot{x} - \ddot{y} \cos \alpha$.

Equations of motion of the sphere are

$$m(\ddot{x} - \ddot{y} \cos \alpha) = mg \sin \alpha - F \quad ...(1)$$

$$m\ddot{y} \sin \alpha = mg \cos \alpha - R \quad ...(2)$$

and $\quad m \frac{2a^2}{5} \ddot{\theta} = Fa$ ...(3)

Since it is a case of pure rolling $x = a\theta \Rightarrow \dot{x} = a\dot{\theta}$ ...(4)
Equation of motion of the plane is given by

$M\ddot{y} = R \sin \alpha - F \cos \alpha$ ...(5)
From (1) and (3) on adding, we have

$\frac{7}{5}\ddot{x} - \ddot{y} \cos \alpha = g \sin \alpha$ {from (4) $\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}$ }

Integrating above, we get $\frac{7}{5}\dot{x} - \dot{y} \cos \alpha = gt \sin \alpha$

Integrating again $\frac{7}{5} x - y \cos \alpha = \frac{1}{2} gt^2 \sin \alpha$.

The constants of integrating vanish since initiailly all $\dot{x}$, $\dot{y}$, x and y are zero. Equation (5) is $M\ddot{y} = R \sin \alpha - F \cos \alpha$

$= (mg \cos \alpha - m\ddot{y} \sin \alpha) \sin \alpha + (m\ddot{x} - m\ddot{y} \cos \alpha - mg \sin \alpha) \cos \alpha$
[Putting the values of F and R from (1) and (2)]

or $M\ddot{y} = -m\ddot{y} (\cos^2 \alpha + \sin^2 \alpha) + m\ddot{x} \cos \alpha$

$= -m\ddot{y} + m\ddot{x} \cos \alpha$

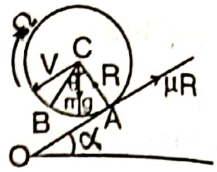
$\Rightarrow (M + m)\ddot{y} = m\ddot{x} \cos \alpha$

Integrating, we get $(M + m)\dot{y} = m\dot{x} \cos \alpha$.
Again integrating, we get $(M + m) y = mx \cos \alpha$.

The contants of integrating vanish as initiailly $\dot{x}$, $\dot{y}$, x and y are all zero.

Ex. 5. *A uniform sphere, of radius a, is rotating about a horizontal diameter with angular velocity Ω and is gently placed on a rough plane which is inclined at an angle α to the horizontal, the sense of rotation being such as to tend to cause the sphere to move up the plane along the line of greatest slope. Show that, if the coefficient of friction be* tan α, *the centre of the sphere will remain at rest for a time* $\frac{2a\Omega}{5g \sin \alpha}$ *and will then move downwards with acceleration* $\frac{5}{7} \sin \alpha$. *If the body be a thin circular hoop instead of sphere, show that the time is* $\frac{a\Omega}{g \sin \alpha}$ *and the acceleration* $\frac{1}{2} g \sin \alpha$.

Sol. The sphere before being placed gently on the inclined plane was rotating with an angular velocity Ω about the horizontal diameter. Hence initially the velocity of the centre is zero .
The sense of rotation at the time of placing the sphere on inclined plane is such that it tends to cause the sphere to move up the plane, that means



sense of Ω is as shown in the figure. The initial velocity of the point of contact A down the plane

= Velocity of the centre C + velocity of A relative to C

$= 0 + a\Omega$, which is a positive quantity .

Hence the initial velocity of the point of contact is down the plane, so the friction μR acts up the plane.

Equations of motion are $m\ddot{x} = mg \sin\alpha - \mu R$ ...(1)

$0 = R - mg \cos\alpha$ ...(2) and $mk^2\ddot{\theta} = -\mu Ra$ ...(3)

where $\mu = \tan\alpha$

Eilminating R from (1) and (2), we get

$m\ddot{x} = mg \sin\alpha - \tan\alpha . mg \cos\alpha = 0 \Rightarrow \ddot{x} = 0 \Rightarrow \dot{x} = 0$ ...(4)

From (2) and (3), we get (Initially when $t = 0, \dot{x} = 0$)

$mk^2\ddot{\theta} = -\tan\alpha\,(mg \cos\alpha)\, a = -mga \sin\alpha$ or $k^2\ddot{\theta} = -ga \sin\alpha$.

Integrating it, we get $k^2\dot{\theta} = -gat \sin\alpha + k^2\Omega$

From equation (4) and (5), we observe that the centre of the sphere does not move at all , but the sphere goes on revolving .

The sphere will cease to rotate when $\dot{\theta} = 0$

$\therefore$ From (5), we get $0 = -gat \sin\alpha + k^2\Omega$ or $t = \dfrac{k^2\Omega}{ga \sin\alpha}$

For sphere $k^2 = \frac{2}{5}a^2$, and for the hoop $k^2 = a^2$, hence the sphere will remain at rest for a time $\dfrac{2}{5}\dfrac{a\Omega}{g \sin\alpha}$ and for the hoop this time will be $\dfrac{a\Omega}{g \sin\alpha}$.

Now when $\dot{x}$ and $a\dot{\theta}$ become zero, the velocity of the point of contact $(\dot{x} + a\dot{\theta})$ becomes zero , therefore pure rolling may commence provided the friction is sufficient for pure rolling. Let F be the value of friction sufficient for pure rolling .

The equations of motion are $m\ddot{y} = mg \sin\alpha - F$ ...(i)

$mk^2\ddot{\phi} = Fa$ ...(ii) a nd $\dot{y} - a\dot{\phi} = 0$ ...(iii)

as $\dot{y} - a\dot{\phi} = 0 \Rightarrow \dot{y} = a\dot{\phi} \Rightarrow \ddot{y} = a\ddot{\phi}$)

Solving (i) and (ii) with the help of (iii) we get $F = \dfrac{mg \sin \alpha}{1 + (a^2/k^2)}$ which is obviously less then $mg \sin \alpha$.

When $F < \mu R$, the rolling continues and the equations (i),(ii) and (iii) hold good .

From (ii) we get $k^2\phi = \dfrac{ag \sin \alpha}{1 + (a^2/k^2)}$ or $\ddot{y} = \dfrac{ga^2 \sin \alpha}{(a^2 + k^2)}$ $(\because \ a\ddot{\phi} = \ddot{y})$

Putting $k^2 = \frac{2}{5} a^2, \ddot{y}$ *i.e.* acceleration in case of sphere is $\frac{5}{7} g \sin \alpha$.

Putting $d^2 = a^2, \ddot{y}$ *i.e.* acceleration in case of hoop is $\frac{1}{2} g \sin \alpha$.

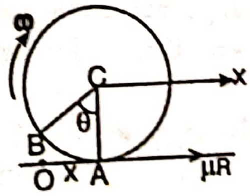
Ex. 6. *A homogeneous sphere of radius a , rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time* $\dfrac{2\,\omega a}{7\mu g}$ *and that then the sphere will roll with angular velocity* (2 ω/7) . **(Agra 1987, 86 ; Garhwal 90)**

Sol. Since the sphere is gently placed on the table, the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω .
Initial velocity of the point contact=initial velocity of the centre C+ Initial velocity of the point of contact with respect to C.

$= \ 0 + a\omega$ in direction $\leftarrow$

Hence the point of contact will slip in the direction ($\leftarrow$), therefore full friction μR acts in the direction ($\rightarrow$).

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns, then at any time t, equations of motion are,



$$m\ddot{x} = \mu R \quad ...(1) \quad (\text{Here } R = mg)$$

$$\text{and } \ m\frac{2a^2}{5}\ddot{\theta} = -\mu Ra \quad ...(2)$$

Therefore from (1) $\ddot{x} = \mu g$ and from (2)

$\frac{2}{5} a\ddot{\theta} = -\mu g$

Integrating these equations, we get

$$\dot{x} = \mu g t \quad ...(3) \quad \text{and} \quad a\dot{\theta} = -\tfrac{5}{2}\mu g t + a\omega \quad ...(4)$$

Since initially when $t = 0, \dot{x} = 0, \dot{\theta} = \omega$.

Velocity of the point contact $= \dot{x} - a\dot{\theta}$

Hence the point of contact will come to rest when $\dot{x} - a\dot{\theta} = 0$

i.e. when $\mu g t - (-\frac{5}{2}\mu g t + a\omega) = 0$ or when $t = \dfrac{2a\omega}{7\mu g}$

Therefore after time $\dfrac{2a\omega}{7\mu g}$ the slipping will stop and pure rolling will commence . Putting this value of t in (4), we get $\dot{\theta} = \dfrac{2\omega}{7}$

when rolling commences , the equations of motions are

$$m\ddot{x} = F \quad \text{...(i)}, \qquad m\frac{2a^2}{5}\ddot{\theta} = -Fa \quad \text{...(ii)}$$

and $\dot{x} - a\dot{\theta} = 0$ ...(iii)

From (i) and (ii) with the help of (iii) , we get

$ma\ddot{\theta} = F$ and $\frac{2}{5}ma\,\ddot{\theta} = -F$ $\quad (\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta})$

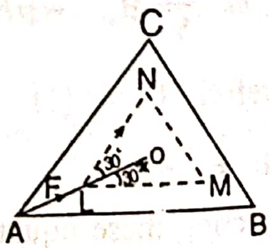
Adding these two equations , we get

$$\tfrac{7}{5}ma\ddot{\theta} = 0 \text{ or } \ddot{\theta} = 0 \Rightarrow \dot{\theta} = \text{const.} = \frac{2\omega}{7}$$

Ex. 7. *Three uniform spheres, edch of radius a and of mass m attract one another according to the law of the inverse square of the distance. Initially they are placed on a perfectly rough horizontal plane with their centres forming a triangle whose sides are each of length 4a. Show that the velocity of their centres when they collide is* $\left(\gamma\dfrac{5m}{14a}\right)^{1/2}$ *where γ is the constant of graviation .* **(Agra 1986)**

Sol. Let A , B and C be the points of contact of the spheres with the horizontal plane, when they are initially at rest. ABC is an equilateral triangle of side $4a$. Let O be the centre of the triangle ABC .

Due to the symmetry of the attraction , the spheres will move in the way that their points of contact with the horizontal plane always form equilateral triangle



Let L, M, N be the new positions of the points of contact with the horizontal plane after time t.

Let $OL = x$,

By geometry, we observe that

$$OL = x\frac{LM}{\sqrt{3}} \left(\because \tfrac{1}{2}\frac{LM}{x} = \cos 30^\circ \right)$$

Therefore initially $x = \left(\dfrac{4a}{\sqrt{3}}\right)$ because initially the side of the triangle is $4a$.

Now when the spheres collide , $x = \left(\frac{2a}{\sqrt{3}},\right)$ because in this case the side of the triangle will become $2a$ (As radius of each sphere is a, so the distance between their centres will be $2a$)

Let L be the point of contact of the first sphere with horizontal plane at time t.

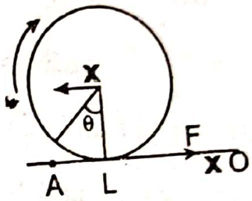
Force of attraction on this sphere due to other two spheres is

$$= \left(\frac{\gamma m^2}{LM^2} \cos 30^\circ + \frac{\gamma m^2}{LN^2} \cos 30^\circ\right) \text{ in the direction } LO$$

$$= \frac{\gamma m^2}{3x^2} \cdot \frac{\sqrt{3}}{2} + \frac{\gamma m^2}{3x^2} \cdot \frac{\sqrt{3}}{2} \qquad (LM = LN = x\sqrt{3})$$

$$= \frac{\gamma m^2}{\sqrt{3}\, x^2} \text{ in the direction } LO \text{ . i.e. towards}$$

x decreasing .



As the plane is perfectly rough , there is pure rolling thus the force of friction at the point of contact is F and acts opposite to the tendency of the motion of the point of contact , *i.e.* F acts towards x decreasing .

The equations of motion of the first sphere are

$$m\ddot{x} = -\left(\frac{\gamma m^2}{x^2\sqrt{3}}\right) - F \qquad \text{...(1)}$$

$$m\left(\frac{2a^2}{5}\right)\ddot{\theta} = -Fa \qquad \text{...(2)}$$

Since there is no slipping, the velocity of the point of contact $\dot{x} + a\dot{\theta}$ is zero. *i.e.* $\dot{x} = -a\dot{\theta} \Rightarrow \ddot{x} = -a\ddot{\theta}$ ...(3)

From (1), (2) and (3) on eliminating F and $a\ddot{\theta}$, we have $\ddot{x} = -\frac{5\gamma m}{7x^2\sqrt{3}}$

Integrating, we get $(\dot{x})^2 = \frac{10\gamma m}{7\sqrt{3}x} + K$.

Now, when $x = \frac{4a}{\sqrt{3}}, \dot{x} = 0, \quad \therefore K = -\frac{10\gamma m}{7\sqrt{3}} \cdot \frac{\sqrt{3}}{4a}$

$$\therefore (\dot{x})^2 = \frac{10\gamma m}{\sqrt{3}}\left(\frac{1}{x} - \frac{\sqrt{3}}{4a}\right) \qquad \text{...(4)}$$

When the spheres collide *i.e.* when $x = \frac{2a}{\sqrt{3}}$; from (4), the velocity at that time is $(\dot{x})^2 = \frac{10\gamma m}{7\sqrt{3}}\left(\frac{\sqrt{3}}{2a} - \frac{\sqrt{3}}{4a}\right)$ or $\dot{x} = \left(\gamma \frac{5m}{14a}\right)^{1/2}$