IAS MATHEMATICS (OPT.)-2009

PAPER - II : SOLUTIONS

12M.

1.(a) If $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers. Show that $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and $\mathbb{R}_+$ under multiplication $(\mathbb{R}_+, \cdot)$ are isomorphic. Similarly if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers, are $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, \cdot)$ isomorphic? Justify.

Sol^n: Given that $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers.

Let $(\mathbb{R}, +)$ be the group of real numbers under addition and $(\mathbb{R}_+, \times)$ be the group of positive real numbers under multiplication.

Define $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$f(a) = e^a \quad \forall a \in \mathbb{R} \quad (1)$$

Clearly f is well-defined.

Now let $a, b \in \mathbb{R}$ such that

$$f(a) = e^a \ \& \ f(b) = e^b$$

Now we have

$$f(a+b) = e^{a+b} \quad (\text{by } (1))$$
$$= e^a . e^b$$
$$= f(a) . f(b)$$

Hence f is a homomorphism.

To show that $f: \mathbb{R} \to \mathbb{R}_+$ is 1-1 :-

Suppose that $f(a) = f(b)$

$\Rightarrow e^a = e^b$

$\Rightarrow a = b$.

Whence f is 1-1.

To show that f is onto :-

Let $b \in \mathbb{R}_+$ $\exists \log_e b \in \mathbb{R}$ such that

$f(\log_e b) = e^{\log_e b} = b$.

$\therefore f$ is onto.

consequently, f is an isomorphism of $(\mathbb{R}, +)$ onto $(\mathbb{R}_+, \times)$

i.e. $(\mathbb{R}_+, \cdot)$ is an isomorphic image of $(\mathbb{R}, +)$

Second part:

Given $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers

Let $(\mathbb{Q}, +)$ be the group of ~~all~~ rational numbers. and

$(\mathbb{Q}_+, \times)$ be the group of all positive rational numbers.

Here $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, \times)$ are not necessarily isomorphic.

Suppose $f: \mathbb{Q} \to \mathbb{Q}_+$ is an isomorphism of groups.

Now $2 \in Q_+$, $\exists\, x \in Q$ such that

$$2 = f(x)$$
$$= f\left(\frac{x}{2}+\frac{x}{2}\right)$$
$$= f(x/2)\cdot f(x/2)$$
$$= \left[f(x/2)\right]^2$$

But there is no rational number y such that $2 = y^2$.

So, there does not exist any isomorphism between $(Q,+)$ and $(Q_+,\times)$.

12M 2009 ⟶ 2

1.(b) Determine the number of homomorphisms from the additive group Z_{15} to the additive group Z_{10} (Z_n is the cyclic group of order n).

Sol Let $(Z_{15} = \{[0], [1], \dots [14]\}, +)$ and $(Z_{10} = \{[0], [1], \dots [9]\}, +)$ be the two given groups.

We have $Z_{15} = \langle [1] \rangle$ the cyclic group generated by $[1]$.

Let $f: Z_{15} \to Z_{10}$ be a homomorphism.

For any $[a] \in Z_{15}$, $f([a]) = a f([1])$ shows that f is completely known if $f[1]$ is known.

Now $o(f([1]))$ devides $o([1])$ and $|Z_{10}|$

i.e $o\, f([1])$ devides 15 and 10.

Hence $o(f[1]) = 1$ or 5.

Thus $f([1]) = [0]$ or $[2]$

If $f([1]) = [0]$ then f is the trivial homomorphism which maps every element to $[0]$.

On the otherhand, $f([1]) = [2]$

$\Rightarrow f([a]) = [2a] \quad \forall a \in Z_{15}$

Thus $f([a] + [b]) = f([a+b])$

$= [2(a+b)]$

$= [2a + 2b]$

$= [2a] + [2b]$

$= f([a]) + f([b])$,

proving that the function $f: Z_{15} \to Z_{10}$ defined by $f([a]) = [2a] \ \forall [a] \in Z_{15}$ is a homomorphism.

Hence there are two homomorphisms from Z_{15} into Z_{10}

2009 1.(c) → State Rolle's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root of $e^x \sin x = 1$

Soln :

Rolle's Theorem: Suppose that f is Continuous on $I=[a,b]$ that the derivative f' exists at every point of (a,b) and $f(a)=f(b)=0$. then there exists at least one point $c \in (a,b)$ such that $f'(c)=0$.

Sol^n: Let $x=a$ & $x=b$ be two distinct roots of the given equation $e^x \cos x = 1$

$\therefore e^a \cos a = 1$ & $e^b \cos b = 1$

$\Rightarrow \cos a = e^{-a}$ & $\cos b = e^{-b}$ ——— ①

Let $f(x) = -\cos x + e^{-x} \ \forall \ x \in [a,b]$

(i) Since $\cos x$ & e^{-x} are Continuous in $[a,b]$

$\therefore f(x)$ is Continuous in $[a,b]$

(ii) $f'(x) = \sin x - e^{-x}$

which exists for all $x \in (a,b)$

$\therefore f$ is derivable in (a,b)

(iii) $f(a) = -\cos a + e^{-a}$

$= 0$ (by ①)

& $f(b) = -\cos b + e^{-b}$

$= 0$ (by ①)

$\therefore f(a) = f(b) = 0$

$\therefore$ the Conditions of Rolle's theorem are satisfied.

$\therefore \exists$ at least one point $c \in (a,b)$ such that $f'(c)=0$

$\Rightarrow f'(c) = \sin c - e^{-c} = 0$

$\Rightarrow \sin c = e^{-c}$

$\Rightarrow e^{c} \sin c - 1 = 0$

$\Rightarrow x = c \in (a,b)$ is a root of the equation

$$e^{x} \sin x - 1 = 0$$

$\therefore e^{\sin x} - 1$ has at least one root between any two roots of the equation

$$e^{x} \cos x = 1$$

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1.(d) 12M Let $f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{|x|}{2}+1 & \text{if } 2\le x. \end{cases}$

what are the points of discontinuity of f, if any? what are the points where f is not differentiable, if any? Justify yours answers.

Sol^n: Given that

$$f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{|x|}{2}+1 & \text{if } 2\le x. \end{cases}$$

i.e., $$f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x. \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x. \end{cases}$$

f is linear function over the various subintervals.

$\Rightarrow$ f is continuous and differentiable over each subinterval.

The only doubtful points are the breaking points $x=0$ and $x=2$

At $x=0$, $f(x) = \frac{0}{2}+1 = 1$

i.e, $f(0) = 1$

Now

LHL: $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} -\frac{x}{2}+1$

$= 1$

RHL: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x}{2}+1$

$= 1$

$\Rightarrow \lim_{x\to 0-} f(x) = 1 = \lim_{x\to 0+} f(x)$

$\therefore$ f is continuous at $x=0$.

Also

LHD: $Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0-} \frac{-\frac{x}{2}+1-1}{x}$

$= \lim_{x\to 0-} \frac{-x}{2x}$

$= -\frac{1}{2}$.

RHD:

$Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0+} \frac{\frac{x}{2}+1-1}{x}$

$= \lim_{x\to 0+} \frac{x}{2}/x = \frac{1}{2}$.

$\Rightarrow Lf'(0) \neq Rf'(0)$

$\therefore$ f is not differentiable at $x=0$.

At $x=2$, $f(x) = -\frac{2}{2}+1 = 0$; i.e., $f(2)=0$

Now <u>LHL</u>: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x}{2}+1 = 1+1 = 2$

<u>RHL</u>: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} -\frac{x}{2}+1 = -1+1 = 0$

$\Rightarrow \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore$ f is not continuous at $x=2$

$\therefore$ f is not differentiable at $x=2$

Hence f is continuous for all values of x except at $x=2$.

and also f is differentiable for all values of x except at $x=0$ & $x=2$

1(e) Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$.

IAS-2009 P-II

Assume that the zeroes of the denominator are simple. Show that the sum of residues of $f(z)$ at its poles is equal to $\dfrac{a_{n-1}}{b_n}$.

Sol^n: Let $f(z) = \dfrac{a_0}{b_0 + b_1 z}$; where $b_1 \neq 0$.

$$= \frac{a_0}{b_1\left[\frac{b_0}{b_1} + z\right]}$$

$z = \dfrac{-b_0}{b_1}$ is a pole of order 1 i.e. simple pole.

The residue at $z = -b_0/b_1$ is

$$= \lim_{z \to -b_0/b_1} \left(\frac{b_0}{b_1} + z\right) \frac{a_0}{b_1\left(b_0/b_1 + z\right)}$$

$$= \frac{a_0}{b_1}$$

Now let us assume that $f(z) = \dfrac{a_0 + a_1 z}{b_0 + b_1 z + b_2 z^2}$

$$= \frac{a_0 + a_1 z}{b_2\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)}$$

Let $z = \alpha, \beta$ be the simple poles of

$$\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)$$

[Now since α, β are the roots of $z^2 + \dfrac{b_1}{b_2} z + \dfrac{b_0}{b_2}$

$$z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2} = (z - \alpha)(z - \beta)$$

$$= z - (\alpha + \beta) + \alpha\beta \quad]$$

Residue at $z = \alpha$ is $= \lim_{z \to \alpha} (z - \alpha) \dfrac{a_0 + a_1 z}{b_2 (z - \alpha)(z - \beta)}$

$$= \frac{a_0 + a_1\alpha}{b_2(\alpha - \beta)}$$

Residue at $z=\beta$ is $= \lim_{z\to\beta} (z-\beta) \dfrac{a_0 + a_1 z}{b_2(z-\alpha)(z-\beta)}$

$$= \frac{a_0 + a_1\beta}{b_2(\beta - \alpha)}$$

$$= -\frac{a_0 + a_1\beta}{b_2(\alpha - \beta)}$$

$\therefore$ Sum of the residues of $f(z) = \dfrac{a_0 + a_1 z}{b_2\left(z^2 + \frac{b_1}{b_2} z + \frac{a_0}{b_0}\right)}$

$$= \frac{1}{b_2}\left[\frac{a_0 + a_1\alpha}{\alpha - \beta} - \frac{a_0 + a_1\beta}{\alpha - \beta}\right]$$

$$= \frac{1}{b_2}\left[\frac{a_1(\alpha - \beta)}{\alpha - \beta}\right] = \frac{a_1}{b_2}$$

Let $f(z) = \dfrac{a_0 + a_1 z + a_2 z^2}{b_0 + b_1 z + b_2 z^2 + b_3 z^3}$

$$= \frac{a_0 + a_1 z + a_2 z^2}{b_3\left[(z-\alpha)(z-\beta)(z-\gamma)\right]}$$

where α, β, γ are the simple poles.

Residue at $z=\alpha$ is $\lim_{z\to\alpha} \cancel{(z-\alpha)} \dfrac{a_0 + a_1 z + a_2 z^2}{b_3 \cancel{(z-\alpha)}(z-\beta)(z-\gamma)}$

$$= \frac{a_0 + a_1\alpha + a_2\alpha^2}{b_3(\alpha - \beta)(\alpha - \gamma)}$$

Residue at $z=\beta$ is $\lim_{z\to\beta} \cancel{(z-\beta)} \dfrac{a_0 + a_1 z + a_2 z^2}{b_3(z-\alpha)\cancel{(z-\beta)}(z-\gamma)}$

$$= \frac{a_0 + a_1\beta + a_2\beta^2}{b_3(\beta-\alpha)(\beta-\gamma)}$$

Residue at $z=\gamma$ is $= \lim_{z\to\gamma} \cancel{(z-\gamma)} \dfrac{a_0+a_1 z+a_2 z^2}{b_3(z-\alpha)(z-\beta)\cancel{(z-\gamma)}}$

$$= \frac{a_0 + a_1\gamma + a_2\gamma^2}{b_3(\gamma-\alpha)(\gamma-\beta)}$$

$\therefore$ Sum of the residues of $f(z)$ at its poles α, β, γ

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} + \frac{a_0+a_1\beta+a_2\beta^2}{(\beta-\alpha)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} - \frac{a_0+a_1\beta+a_2\beta^2}{(\alpha-\beta)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\alpha-\gamma)(\beta-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{(a_0+a_1\alpha+a_2\alpha^2)(\beta-\gamma) - (a_0+a_1\beta+a_2\beta^2)(\alpha-\gamma) + (a_0+a_1\gamma+a_2\gamma^2)(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0(0)+a_1(0)+a_2(\alpha^2(\beta-\gamma)-\beta^2(\alpha-\gamma)+\gamma^2(\alpha-\beta))}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_2[\alpha^2\beta-\gamma\alpha^2-\beta^2\alpha+\gamma\beta^2+\gamma^2\alpha-\gamma^2\beta]}{\alpha^2\beta-\alpha\beta\gamma-\alpha^2\gamma+\alpha\gamma^2-\beta^2\alpha+\beta^2\gamma+\alpha\beta\gamma-\gamma^2\beta}\right]$$

$$= \frac{a_2}{b_3} \quad (b_3 \neq 0)$$

$\therefore$ we conclude that

$$f(z) = \frac{a_0+a_1 z+a_2 z^2+\cdots+a_{n-1}z^{n-1}}{b_0+b_1 z+b_2 z^2+\cdots+b_n z^n}$$

the sum of the residues of $f(z)$ at its poles is $= \dfrac{a_{n-1}}{b_n}$ where $b_n \neq 0$.

Que:- 2(c) IAS -2009

How many proper non-zero ideals does the ring Z_{12} have? Justify your answer.

How many ideals does the ring $Z_{12} \oplus Z_{12}$ have? Why?

Sol^n:-

consider, the ring $R = Z_{12}$. Ideals are the subgroup of the additive group of 'R' and subgroup of cyclic groups are cyclic, so the proper ideals are

$$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$$

$$\langle 3 \rangle = \{0, 3, 6, 9\}$$

$$\langle 4 \rangle = \{0, 4, 8\}$$

$$\langle 6 \rangle = \{0, 6\}.$$

Here, $\langle 2 \rangle$ & $\langle 3 \rangle$ are prime and maximal,

$\langle 4 \rangle$ & $\langle 6 \rangle$ neither prime nor maximal.

$\therefore$ Z_{12} has six ideals including $\{0\}$ and Z_{12} itself.

Thus, $Z_{12} \oplus Z_{12}$ has $6 \times 6 = 36$ ideals.

15 → 2009 2.(b) Show that the alternating group on four letters A_4 has no subgroup of order 6. (8)

Sol Given that A_4 is an alternating group on four letters w.r.t product of permutations.

$\therefore O(A_4) = \frac{4!}{2} = 12$.

Now I (identity permutation) $\in A_4$

all the 3-cycles are in A_4:
$(123)(132), (124)(142), (134)(143), (234)(243)$.

we have all the possible disjoint products of two transpositions.

They are $(12)(34), (13)(42), (14)(23)$

Now we show that A_4 has no subgroup of order 6 even though $6 \mid O(A_4)$.

because: if possible suppose that such a subgroup H exists.

Then $O(H) = 6$
$O(A_4) = 12$
$\therefore (A_4 : H) = 2$
$\Rightarrow H \trianglelefteq A_4$

$\therefore$ The quotient group $\frac{A_4}{H}$ is defined and is of order 2.

i.e $\frac{A_4}{H} = \{Hg \mid g \in A_4\}$

$= \{HI, Hg\}$; $g \neq I$.

$(Hg)^2 = H \quad \forall g \in A_4$;

$\Rightarrow Hg \cdot Hg = H \quad \forall g \in A_4$

$\Rightarrow Hg \cdot g = H \quad \forall g \in A_4$

$\Rightarrow Hg^2 = H \quad \forall g \in A_4$

$\Rightarrow g^2 \in H$

Now $(1\,2\,3) \in A_4$

$\Rightarrow (1\,2\,3)^2 = (1\,3\,2) \in H$

Similarly $(1\,3\,2)^2 = (1\,2\,3) \in H$

By the same reasoning

$(1\,4\,2), (1\,2\,4), (1\,4\,3)\,(1\,3\,4),$

$(2\,3\,4)(2\,4\,3)\,(2\,4\,3)$ are also distinct elements of H

of course $I \in H$

Thus H contains atleast 9 elements

$\therefore O(H) \geq 9$

This contradicts our assumption that $o(H) = 6$.

$\therefore A_4$ has no subgroup of order 6.

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15M 2009 → Show that $Z[x]$ is a unique factorization

3.(a) domain that is not a principal ideal domain (Z is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain? ($Z[x]$ is the ring of polynomials in the variable x with integer).

Soln: W.K.T the ring Z of integers U.F.D. Now we have to S.T $Z[x]$ is a U.F.D.

Let $f(x)$ be any non-zero, non-unit element of $Z[x]$.

W.K.T if Z be a UFD and $0 \neq f(x) \in Z[x]$

Then $f(x) = af_1(x)$, ——(1) where $a = c(f)$

and $f_1(x)$ is a primitive in $Z[x]$.

We know that, if Z is is U.F.D

and $f_1(x)$ is primitive polynomial in $Z[x]$.

then $f_1(x)$ can be factored in a unique way as the product of irreducible elements (polynomials) in $Z[x]$.

(11)

$\therefore$ We write

$$f_1(x) = q_1(x)\, q_2(x) \cdots q_n(x). \quad \text{——} (2)$$

where each $q_i(x)$ is irreducible in $R[x]$ and further this representation is unique upto associates.

Since $a \in Z$ and Z is a U.F.D.

we can write a in a unique way as a finite product of irreducible elements of R, say

$$a = d_1 d_2 \cdots d_m \text{ —— } (3).$$

Since Z is a U.F.D

any irreducible element of Z is an irreducible element of $Z[x]$. So that each d_i is an irreducible element of $Z[x]$.

$\therefore$ from (1), (2), (3), we see that

$$f(x) = d_1 d_2 \cdots d_m\, q_1(x)\, q_2(x) \cdots q_n(x).$$

which is a unique representation of $f(x)$ as a product of irreducible elements in $Z[x]$.

Hence $Z[x]$ is a U.F.D.

If possible ~~$Z[x]$~~ suppose that (12)

$Z[x]$ is a P.I.D., where Z is an

integral domain with unity

We know that an integral domain Z with unity is a field if and only if $Z[x]$ is a principal ideal domain.

$Z[x]$ is a principal ideal domain.

$\therefore$ Z is a field

but which is impossible.

Hence $Z[x]$ is not a P.I.D.

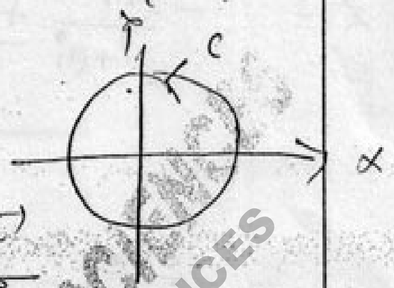
third part:

We know that, every principal ideal domain is a U.F.D

$\therefore$ It is not possible to give an example of principal ideal domain that is not a unique factorization domain.

4(a) IAS 2009 P-II → If α, β, γ are real nos s.t $\alpha^2 > \beta^2+\gamma^2$. Show that

$$\int_0^{2\pi} \frac{d\theta}{(\alpha+\beta\cos\theta+\gamma\sin\theta)} = \frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}.$$



Sol: let $z = e^{i\theta}$ then

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} \;; dz = ie^{i\theta}d\theta \Rightarrow \boxed{dz = iz\,d\theta}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \oint_C \frac{dz}{iz\left[\alpha+\beta\left(\frac{z+z^{-1}}{2}\right)+\gamma\left(\frac{z-z^{-1}}{2i}\right)\right]}$$

$$= \oint_C \frac{2\,dz}{z[2\alpha i+\beta i(z+z^{-1})+\gamma(z-z^{-1})]}$$

$$= \oint \frac{2dz}{2\alpha zi + z^2\beta i+\beta i+\gamma z^2-\gamma}$$

$$= \oint_C \frac{2\,dz}{(\gamma+\beta i)z^2+2\alpha z i+(-\gamma+\beta i)}$$

~~where~~ $$= \oint_C \frac{2\,dz}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]} \quad (1)$$

where C is the circle of unit radius with centre at the origin

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The poles of $\dfrac{2}{(\gamma+\beta i)\left[z^2+\left(\dfrac{2\alpha i}{\gamma+\beta i}\right)z+\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$

are the single poles.

$$z = \frac{-\dfrac{2\alpha i}{\gamma+\beta i} \pm \sqrt{\dfrac{-4\alpha^2}{(\gamma+\beta i)^2}-4(1)\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)}}{2(1)}$$

$$= \frac{-\dfrac{2\alpha i}{\gamma+\beta i} \pm \dfrac{2}{\gamma+\beta i}\sqrt{-\alpha^2-(-\gamma+\beta i)(\gamma+\beta i)}}{2}$$

$$= \frac{\dfrac{-2\alpha i}{\gamma+\beta i} \pm \dfrac{2i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)}}{2}$$

$$= \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)} \quad \text{——— (2)}$$

Given that $\alpha^2 > \beta^2+\gamma^2$

$\therefore z = \dfrac{-\alpha i + i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}{\gamma+\beta i} = z_1$ say, lies inside 'C' because $|z|<1$.

Now residue at z_1

$$= \lim_{z\to z_1}(z-z_1)\frac{2}{(\gamma+\beta i)\left[z^2+\left(\dfrac{2\alpha i}{\gamma+\beta i}\right)z+\left(\dfrac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$$

$$= \underset{z \to z_1}{Lt} \frac{2}{(\gamma + \beta i)\left[z + \dfrac{\alpha i + i\sqrt{\alpha^2 - (\beta^2 + \gamma^2)}}{\gamma + \beta i}\right]}$$

$$= \frac{\quad}{i\sqrt{\alpha^2 - (\beta^2 + \gamma^2)}}$$

$$\therefore (1) \equiv \oint_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = 2\pi i \left(\text{residue at } z_1\right)$$

$$= \frac{2\pi}{\sqrt{\alpha^2 - (\beta^2 + \gamma^2)}}.$$

4(b) IAS 2009 P-II

Maximize : $Z = 3x_1 + 5x_2 + 4x_3$

subject to :
$$2x_1 + 3x_2 \le 8$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$2x_2 + 5x_3 \le 10$$
$$x_1 \geq 0.$$

Sol^n →

Converting in standard form :-

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

S.C.
$$2x_1 + 3x_2 + s_1 = 8$$
$$3x_1 + 2x_2 + 4x_3 + s_2 = 15$$
$$2x_2 + 5x_3 + s_3 = 10$$
$$x_1, x_2, x_3, \ s_1, s_2, s_3 \geq 0$$

where s_1, s_2, s_3 are slack variables.

Now the IBFS is :-

$x_1 = x_2 = x_3 = 0$, $s_1 = 8$, $s_2 = 15$, $s_3 = 10$

Max $Z = 0$

The initial simplex table is :-

	c_j	3	5	4	0	0	0		
C_B	B	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
0	s_1	2	(3)	0	1	0	0	8	$8/3$ →
0	s_2	3	2	4	0	1	0	15	$15/2$
0	s_3	0	2	5	0	0	1	10	5
$z_j = \Sigma C_B a_{ij}$		0	0	0	0	0	0	0	
$c_j = c_i - z_j$		3	5 ↑	4	0	0	0		

Here s_1 is outgoing variable x_2 is incoming variable.

The key element is 3. making it unity and all other elements in that column to zero The new table is :—

		3	5	4	0	0	0		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
5	x_2	$2/3$	1	0	$1/3$	0	0	$8/3$	—
0	S_2	$5/3$	0	4	$-2/3$	1	0	$29/3$	$29/12$
0	S_3	$-4/3$	0	(5)	$-2/3$	0	1	$14/3$	$14/15$ →
	Z_j	$10/3$	5	0	$5/3$	0	0	$40/3$	
	C_j	$-1/3$	0	4 ↑	$-5/3$	0	0		

Here coming variable is x_3 and outgoing variable is S_3.

The key element here is (5).
Making it unity and all other elements in that column as 0.
The new simplex table is given by.

		3	5	4	0	0	0		
C_B	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
5	x_2	$2/3$	1	0	$1/3$	0	0	$8/3$	4
0	S_2	$41/15$	0	0	$-2/15$	1	$-4/5$	$\frac{89}{15}$	$\frac{89}{41}$ →
4	x_3	$-4/15$	0	0	$-2/15$	0	$1/5$	$\frac{14}{15}$	—
	Z_j	$34/15$	5	4	$17/15$	0	$4/5$		
	C_j	$11/15$ ↑	0	0	$17/15$	0	$-4/5$		

Here x_1 is incoming variable and S_2 is outgoing variable.

The key element is $41/15$, making it unity and all other elements in that column as 0. The new simplex table is :—

C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b
5	x_1	0	1	0	$15/41$	$-10/41$	[illegible]$/41$	$50/41$
3	x_2	1	0	0	$-2/41$	$15/41$	$-12/41$	$89/41$
4	x_3	0	0	1	[illegible]$/41$	$4/41$	$5/41$	$62/41$
	z_j	3	5	4	$15/41$	$11/41$	$24/41$	$\frac{765}{41}$
	c_j	0	0	0	$-15/41$	$-11/41$	$-24/41$	

Since all $c_j = 0$ hence optimality obtained

$x_1 = \frac{89}{41}$, $x_2 = \frac{50}{41}$, $x_3 = \frac{62}{41}$

$$\boxed{\text{Max } z = \frac{765}{41}}$$

5(a). Show that the differential equation of all cones which have their vertex at the origin is $px+qy=z$. Verify that this equation is satisfied by the surface $yz+zx+xy=0$.

SOLUTION

General solution of all cones which are having vertex at origin is given by

$$ax^2+by^2+cz^2+2gxz+2fyz+2hxy=0 \qquad \ldots(1)$$

differentiating w.r.t. 'x'

$$2ax+2cz\frac{\partial z}{\partial x}+2gz+2gx\frac{\partial z}{\partial x}+2fy\frac{\partial z}{\partial x}+2hy=0$$

$$2(ax+gz+hy)+2p(cz+gx+fy)=0 \qquad \ldots(2)$$

differentiating w.r.t. 'y'

$$2by+2cz\frac{\partial z}{\partial y}+2fz+2fy\frac{\partial z}{\partial y}+2hx=0$$

$$2(by+fz+hx)+2q(cz+gx+fy)=0 \qquad \ldots(3)$$

$$(2)\times x+(3)\times y \Rightarrow ax^2+gxz+hxy+p(cxz+gx^2+fxy)$$

$$by^2+fyz+hxy+q(cyz+gxy+fy^2)=0 \qquad \text{from}(1)$$

$$(-cz^2-gxz-fyz)+(px+qy)(cz+gx+fy)=0$$

$$-z(cz+gx+fy)+(px+qy)(cz+gx+fy)=0$$

$$\therefore \quad \boxed{px+qy=z}$$

$\therefore$ differential equation given by $px+qy=z$

Given surface $xy+yz+zx=0$.

Partially differenting w.r.t x

$$y+yp+z+xp=0$$

$$p=-\frac{(y+z)}{x+y}$$

partially differenting w.r.t. 'y'

$$x+yq+z+xq=0$$

$$q=\frac{-(x+z)}{(x+y)}$$

Putting is

$$px+qy=-\frac{x(y+z)}{x+y}-y\frac{(x+z)}{x+y}$$

$$=\frac{-xy-xz-xy-yz}{x+y}$$

$$(xy+yz+zx=0)$$

$$=\frac{-xy}{-xy/z}$$

$$\boxed{px+qy=z}$$

$\therefore$ Given equation satifies $px+qy=z$

5(b). (i) Form the partial differential equation by eliminating the arbitrary function f given by $f(x^2+y^2, z-xy)=0$

(ii) Find the integral surface of $x^2p+y^2q-z^2=0$ which passes through the curve $xy=x+y$, $z=1$.

SOLUTION

(i) Given $f(x^2+y^2, z-xy)=0$ ...(1)

Let $x^2+y^2=u$ and $z-xy=v$

Differentiating (1) w.r.t. x

$$\frac{\partial f}{\partial x}(u,v) = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) \quad ...(2)$$

Differentiating (1) w.r.t y

$$\frac{\partial f(u,v)}{dy} = \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) \quad ...(3)$$

$$(2) \equiv \frac{\partial f}{\partial u}(2x+D.p)+\frac{\partial f}{\partial v}(-y+p)=0 \quad ...(4)$$

$$(3) \equiv \frac{\partial f}{\partial u}(2y+D.p)+\frac{\partial f}{\partial v}(-x+q)=0 \quad ...(5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{2x}{2y} = \frac{p-y}{q-x}$$

$$\boxed{yp-xq=y^2-x^2}$$

(ii) Lagranges auxiliary equation given by

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2} \quad ...(1)$$

takign first two parts

$$\frac{-1}{x}=\frac{-1}{y}+c_1 \Rightarrow \frac{1}{x}-\frac{1}{y}+c_1=0 \quad ...(2)$$

taking first and 3rd part

$$\frac{-1}{x}=\frac{-1}{z}+c_2 \Rightarrow \frac{1}{x}-\frac{1}{z}+c_2=0 \quad ...(3)$$

Given $z=1$; $\quad xy = x+y$

$$\frac{1}{x}+\frac{1}{y} = 1$$

from (2) and (3)

$$1-\frac{2}{y}+c_1 = 0;\ 1-\frac{1}{y}+1+c_2 = 0.$$

$$\frac{1}{y}=\frac{1+c_1}{2} \Rightarrow 1-\frac{(1+c_1)}{2}+1+c_2=0$$

$$\therefore \frac{3}{2}+c_2-\frac{c_1}{2}=0$$

$$\therefore \frac{3}{2}+\frac{1}{z}-\frac{1}{x}+\frac{1}{2x}-\frac{1}{2y}=0$$

$$\frac{3}{2}+\frac{1}{z}-\frac{1}{2x}-\frac{1}{2y}=0$$

$$xz+yz=2xy+3xyz$$

Q. 5(c)(i) The equation $x^2+ax+b=0$ has two real roots α and β. Show that the iterative method given by $\rightarrow$ $x_{k+1} = -(ax_k+b)/x_k$; $k=0,1,2,\dots$ is convergent near $x=\alpha$; if $|\alpha|>|\beta|$.

Sol. The iteration is given by

$$x_{k+1} = \frac{-(ax_k+b)}{x_k} = g(x_k) \quad \text{(say)}$$

$k=0,1,2,\dots$

we know that if $g(x)$ and $g'(x)$ are continuous in an interval about a Root 'α' of the equation $x=g(x)$ and if $|g'(x)|<1$ for all 'x' in the interval, then the successive approximation $x_1, x_2 \dots$ given by

$$x_k = g(x_{k-1}) \quad — \; k=1,2,3\dots$$

convergent to root 'α' provided that the initial approximation x_0 is choosen in the interval.

These iterations converges to 'α'

if $|g'(x)|<1$

Near α ; i.e $|g'(x)| = \left|\frac{-b}{x^2}\right| < 1$

$g'(x)$ is convergent near 'α'. If the iterations converge to 'α', then we require.

$$|g'(\alpha)| = \left|\frac{-b}{\alpha^2}\right| < 1$$

$\therefore |b| < |\alpha|^2$ i.e $|\alpha|^2 > b$ —— ①

Given that α, β are the root of the equation

$x^2 + ax + b = 0$

$\alpha + \beta = -a$

$\alpha\beta = b \Rightarrow |\alpha||\beta| = |b|$ —— ②

Using ② in ①

$\Rightarrow |\alpha|^2 > |b| = |\alpha||\beta|$

$\Rightarrow |\alpha|^2 > |\alpha||\beta|$

$\Rightarrow |\alpha| > |\beta|$

Now, if $x_k = \frac{-b}{x+a}$.

The integration $x_{k+1} = \frac{-b}{x_k + a} = g(x_k)$ let.

cgs to α, if $|g'(x)| = \left|\frac{b}{(x+a)^2}\right|$

an interval containing 'α' in particular.
we require —

$|g'(\alpha)| = \left|\frac{b}{(\alpha+a)^2}\right| < 1$

$(\alpha+a)^2 > |b|$

but we have $\alpha + \beta = -a$; $\alpha\beta = b$

$\beta^2 > |b|$

$\beta^2 > |\alpha||\beta|$

$|\beta| > |\alpha|$

$x_{k+1} = \frac{-b}{x_k + a}$ is cgt near $x = \alpha$, if $|\beta| > |\alpha|$

Hence, the result

Also;

if $x = -\frac{(x^2+b)}{a}$

$x_k = -\frac{(x_k^2+b)}{a} = g(x_f)$ let

converges to α if $|g'(x)| = \left|\frac{2x}{a}\right| < 1$

in an interval containing α.

In Particular.

$|g'(\alpha)| = \left|\frac{2\alpha}{a}\right| < 1$

$|2\alpha| < |a|$ $\quad \begin{bmatrix} \alpha+\beta = -a \\ |\alpha+\beta| = |a| \end{bmatrix}$

$2|\alpha| < |\alpha+\beta|$

$\therefore x_{k+1} = \frac{-x_k^2 + b}{x_k}$ is convergent

near $x = a$; if $2|\alpha| < |\alpha+\beta|$.

6(a) Find the characteristic of $y^2r - x^2t = 0$; where r & t have thier usual meaning ?

SOLUTION

Given $\quad y^2r - x^2t = 0 \quad$ .....(1)

Comparing (1) with

$Rr + Ss + Tt + F(x, y, z, p, q) = 0$

here $\quad R = y^2, S = 0, T = -x^2$

so that λ–quadratic equation

$$R\lambda^2 + S\lambda + T = 0$$

reduces to $\quad y^2\lambda^2 + 0 - x^2 = 0$

$$\lambda^2 = \left(\frac{x}{y}\right)^2 \Rightarrow \lambda = \pm\frac{x}{y}$$

$$\lambda_1 = \frac{x}{y} \;\&\; \lambda_2 = -\frac{x}{y}$$

Charactristics equation becomes

$$\frac{dy}{dx} + \frac{x}{y} = 0 \quad ; \quad \frac{dy}{dx} - \frac{x}{y} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad ; \quad \frac{dy}{dx} = \frac{x}{y}$$

Integrating these two equation

$$\frac{y^2 + x^2}{2} = c_1 \qquad \frac{y^2 - x^2}{2} = c_2$$

In ordu to reduce (1) to its cononical from we choose

$$u = \frac{y^2 - x^2}{2} \quad \& \quad v = \frac{y^2 - x^2}{2} \quad .....(2)$$

Which are the required families of characteristic

Hence, these are the families of circles and Hyperbola.

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\right)\right] = x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$r = x^2\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u\,\partial v}$$

6(b). Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$. Where D and D′ represent $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$.

SOLUTION

Auxiliary equation given by

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$$m = -1, 2$$

$\therefore$ C.F. $= \phi_1(y-x) + \phi_2(y+2x)$

$$\text{P.I.} = \frac{(2x^2 + xy - y^2)\sin(xy) - \cos xy}{(D^2 - DD' - 2D'^2)}$$

$$= \frac{1}{(D+D')(D-2D')}\left[(2x^2 + xy - y^2)\sin xy - \cos xy\right] \qquad (\because y - x = a)$$

$$= \frac{1}{(D-2D')}\int\left[\left(2x^2 + x(a+x) - (a+x)^2\right)\sin(a+x)x - \cos x(a+x)\right]dx$$

$$= \frac{1}{(D-2D')}\int\left[\left[-a^2 - ax + 2x^2\right]\sin(ax + x^2) - \cos(x^2 + ax)\right]dx$$

$$= \frac{1}{D-2D'}\int\left[(2x+a)(x-a)\sin(ax+x^2) - \cos(x^2+ax)\right]dx$$

$$= \frac{1}{D-2D'}(x-a)\frac{\cos(ax+x^2)}{-1} + \int\cos(ax+x^2)dx - \int\cos(ax+x^2)dx$$

$$= \frac{(y-2x)}{D-2D'}\cos(xy) \qquad (\because y + 2x = b)$$

$$= \int(b-4x)\cos(xb - 2x^2)\,\text{ex}$$

$$= \sin(xb - 2x^2)$$

$$\text{P.I.} = \sin(xy)$$

$\therefore$ G.S. $= \phi_1(x-y) + \phi_2(y+2x) + \sin(xy)$

6(c). A tightly stretched string has its ends fixed at $x=0$ and $x=l$. At time $t=0$, the string is given a shape defined by $f(x)=\mu x(l-x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t > 0$.

SOLUTION

Weve equation given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(1)$$

string of length 'l'.

Boundary conditions given by

$$u(0,t) = u(l, t) = 0 \qquad \ldots(2)$$

Initial conditions given by

$$u(x,0) = \mu x(l-x) \qquad \ldots(3)$$

$$u_t(x,0) = 0 \qquad \ldots(4)$$

Let $u(x,t) = X(x)T(t)$ be the trail solution from(2) $X(0)T(t) = X(l)T(t) = 0$ for some $t > 0$, there exist $T(t)\neq 0$.

$\therefore \qquad X(0)=X(l) = 0 \qquad \ldots(5)$

from(1) $\qquad XT'' = c^2X''T$

$$\frac{T''}{c^2T} = \frac{X''}{X} = \mu \quad \text{(say)} \qquad \ldots(6)$$

from(6) $X''-\mu X = 0$; $X(0) = X(l) = 0$

Case-(i)

$\mu = 0$, $X = Ax + B$

from(5) A=0; B=0 we reject $\mu=0$

Case-(ii)

$\mu = \lambda^2$, $X''-\lambda^2X=0 \Rightarrow X = Ae^{\lambda x}+Be^{-\lambda x}$

from (5) A=0; B=0 we riject $\mu = \lambda^2$

Case-(iii)

$\mu = -\lambda^2$, $X''+\lambda^2X=0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x$

from (5) A=0; B=0 $0=A\cos\lambda l+B\sin(\lambda l)$

$\therefore$ $A = 0$, $\lambda l = n\pi$

$$\lambda = \left(\frac{n\pi}{l}\right)$$

from(6) $T''-\mu c^2T = 0$

putting $\mu = -\lambda^2 = -\left(\frac{n\pi}{l}\right)^2$

$$T_n'' + \left(\frac{n\pi c}{l}\right)^2 T_n = 0$$

$$\therefore \qquad T_n = C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)$$

$$\therefore \qquad u(x,t) = \sum_{n=1}^{\infty} X_n T_n = \sum_{n=1}^{\infty}\left(E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

from(4) $\quad u_t(x,0) = 0$

$$\left(\frac{n\pi c}{l}\right)F_n . \sin\left(\frac{n\pi x}{l}\right) = 0$$

$\therefore$ Fn = 0

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) = \mu x(l-x)$$

$$\mathrm{E_n} = \frac{2}{l}\int_0^l \mu x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\frac{\mu x(l-x)\cos\left(\frac{n\pi x}{l}\right)}{\frac{-n\pi}{l}}\right]_0^1 - \frac{2}{l}\int_0^l \frac{\mu(l-2x)\cos\left(\frac{n\pi x}{l}\right)}{-\left(\frac{n\pi}{l}\right)}$$

$$= \frac{2}{l}\left[\frac{\mu(l-2x)\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2} - \frac{(-2\mu)\left(-\cos\frac{n\pi x}{l}\right)}{(n\pi/l)^3}\right]_0^l$$

$$= \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n]$$

$$\therefore \quad \boxed{u(x,t) = \frac{4\mu l^2}{\pi^3}\sum\frac{(1-(-1)^n)}{n^3}\cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)}$$

Required Solution

7(a) IAS 2009 P-II

Develop an algorithm for Regula-falso method to find a root of $f(x)=0$ starting with two initial iterates x_0 and x_1 to the root such that sign $(f(x_0)) \neq$ sign $(f(x_1))$. Take n as the maximum number of iterations allowed and eps be the prescribed error.

Sol^n Regula falsi Method Algorthm:-

1. Start.
2. Read value of $x0$, $x1$ and e

   Here $x0$ and $x1$ are the two initial guesses

   e is the degree of accuracy or the absolute error i.e. the stopping criteria
3. Computer function values $f(x0)$ and $f(x1)$
4. Check wether the product of $f(x0)$ and $f(x1)$ is negative or not.

   If it is positive take another initial guesses.

   If it is negative then goto step 5.
5. Determine:-

   $x = [x0 * f(x1) - x1 * f(x0)] / (f(x1) - f(x0))$
6. Check wether the product of $(x1)$ and $f(x)$ is negative or not.

   If it is negative, then assign $x0 = x$;

   If it is positive, assign $x1 = x$;
7. Check wether the value of $f(x)$ is greater than 0.00001 or not.

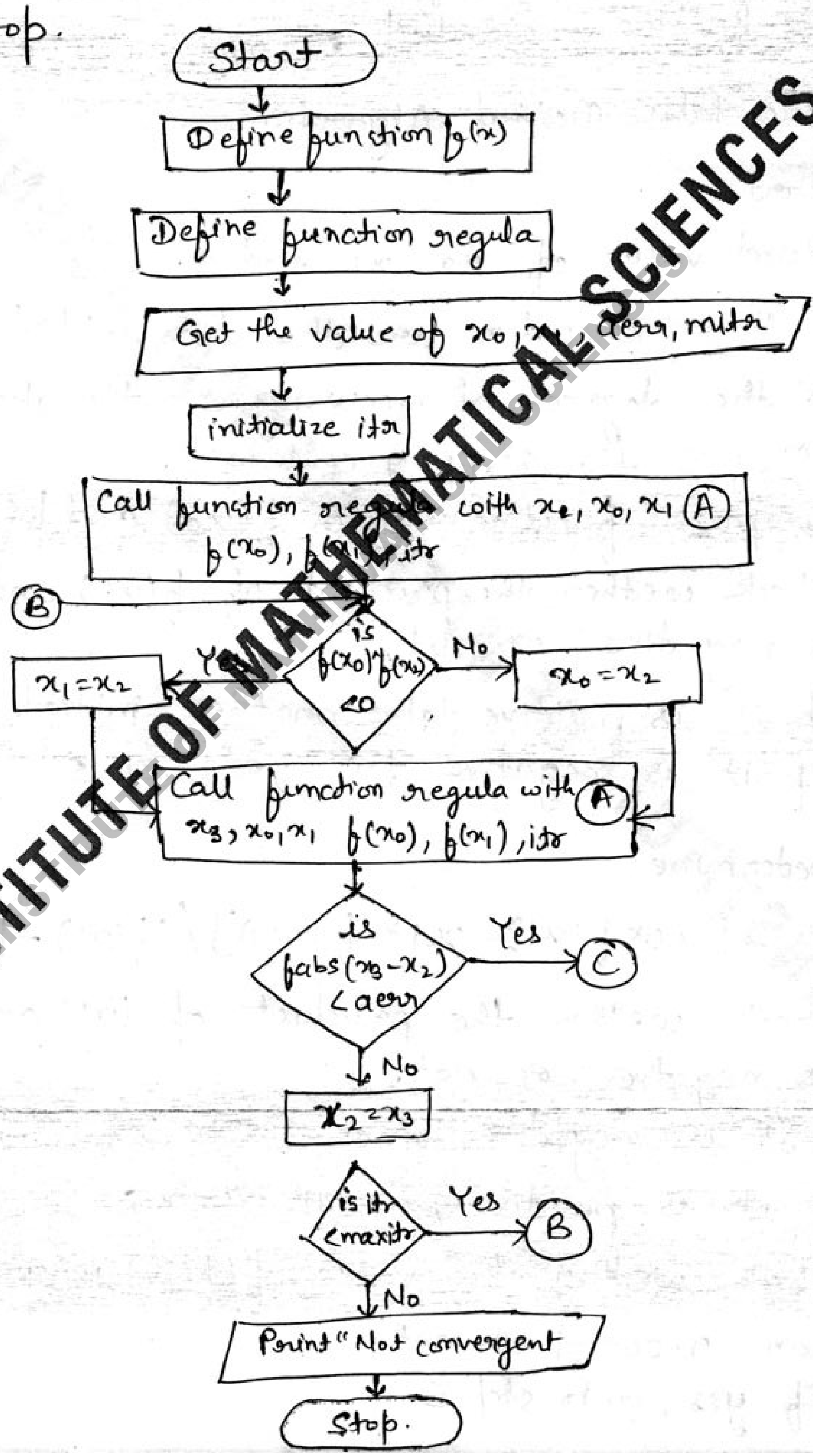
   If yes, goto step 5.

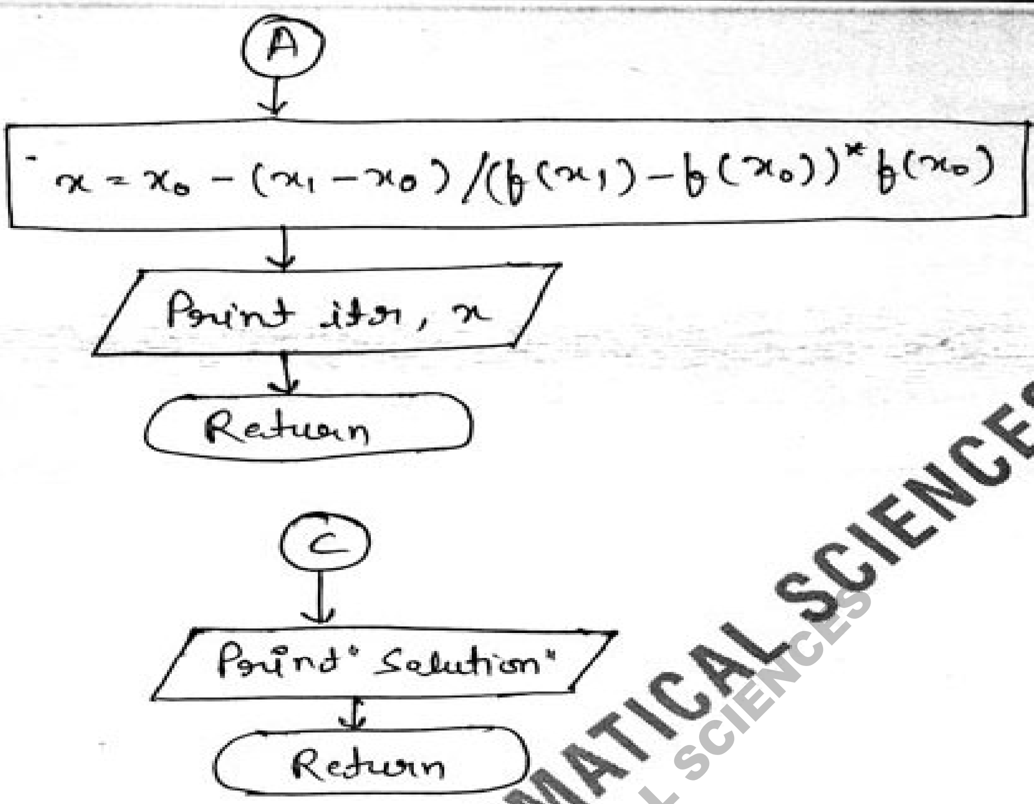
If no, goto step 8.

Here the value 0.0001 is the desired degree of accuracy, and hence the stopping criteria.

8. Display the root as x.

9. Stop.




A
x = x0 - (x1 - x0)/(f(x1) - f(x0))*f(x0)
Print itr, x
Return
C
Print "Solution"
Return

Q. Using Lagrange's interpolation formula, Calculate the value of f(3), from the following table of values of x and $f(x)$.

7(b)

x	0	1	2	4	5	6
$y=f(x)$	1	14	15	5	6	19

Sol:- The given values are.

$x_0 = 0$; $x_1 = 1$; $x_2 = 2$; $x_3 = 4$; $x_4 = 5$; $x_5 = 6$

$f(0) = y_0 = 1$; $y_1 = 14$; $y_2 = 15$; $y_3 = 5$; $y_4 = 6$; $y_5 = 19$

for $x = 3$ Lagrange Interpolation formula is given by.

$$f(3) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} \cdot f(0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} f(1)$$

$$\vdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} f(5)$$

$$f(3) = \frac{(2)(1)(-1)(-2)(-3)}{(-1)(-2)(-4)(-5)(-6)} \times 1 + \frac{(3)(1)(-1)(-2)(-3)}{(1)(-1)(-3)(-4)(-5)} \times 14$$

$$+ \frac{(3)(2)(-1)(-2)(-3)}{(2)(1)(-2)(-3)(-4)} \times 15 + \frac{(3)(2)(1)(-2)(-3)}{(4)(3)(2)(-1)(-2)} \times 5$$

$$+ \frac{(3)(2)(1)(-1)(-3)}{(5)(4)(3)(1)(-1)} \times 6 + \frac{(3)(2)(1)(-1)(-2)}{(6)(5)(4)(2)(1)} \times 19 ,$$

$$f(3) = \frac{1}{20} - \frac{21}{5} + \frac{45}{4} + \frac{15}{4} - \frac{9}{5} + \frac{19}{20}$$

$$f(3) = \left[\frac{1}{20} + \frac{19}{20}\right] + \left[\frac{45+15}{4}\right] - \left[\frac{+21+9}{5}\right]$$

$$f(3) = 1 + 15 - 6 = 16 - 6 = 10$$

$$\boxed{f(3) = 10}$$

Q. 7(c) Find the value of $y(1.2)$ using Runge-kutta fourth order method with step size $h=0.2$ from the initial value problem:

$$y' = xy$$
$$y(1) = 2.$$

Sol:- $f(x,y) = xy$

$x_0 = 1,\ y_0 = 2,\ h = 0.2$

$y_1 = y_0 + k = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = hf(x_0, y_0) = 0.2 f(1,2) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_2 = 0.2 f[1.1, 2.2) = 0.484.$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_3 = 0.2 f(1.1 + 2.242) = 0.49324$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$k_4 = 0.2 f(1.2, 2.49324) = 0.59838$

$k = 0.49214,$

$y_1 = y_0 + k = 2 + 0.49214 = 2.49214.$

Now, $x_1 = x_0 + h = 1 + 0.2 = 1.2$

$y_1 = 2.49214.$

Now; To calculate $y_2 = y(1.2) = y_1 + K$

$K_1 = hf(x_1, y_1) = 0.59811$

$K_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right) = 0.2f(1.3, 2.79120)$

$K_2 = 0.72571.$

$K_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{K_2}{2}\right) = 0.2f(1.3, 2.855)$

$K_3 = 0.74230$

$K_4 = hf(x_1+h, y_1+K_3) = 0.90564$

$K = \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$

$K = \frac{1}{6}\left[0.59811 + 2[0.72571 + 0.74280] + 0.90564\right]$

$K = 0.74026$

$y_2 = 2.49214 + 0.74026$

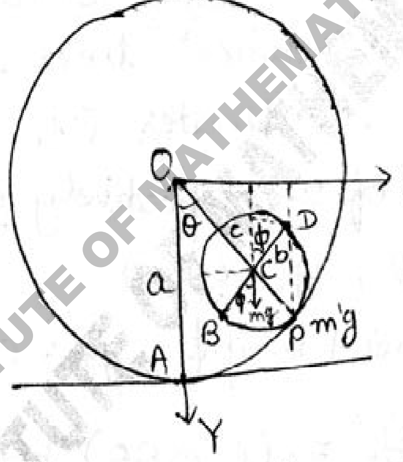
$\boxed{y_2 = 3.2340}$

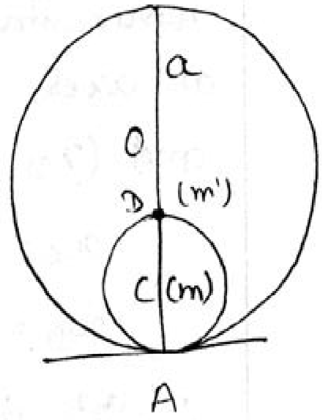
8(a) IAS-2009 P-II

A perfectly rough sphere of mass m and radius b rests at the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m' and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length

$$(a-b)\cdot\frac{4m'+7m/5}{m+m'(2-a/b)}$$

Soln





Let O be the centre of the fixed spherical cavity and C the centre of the sphere of mass m and radius b resting at the lowest point A of the cavity. A particle of mass m' is attached at the highest point D_0 of the sphere. In time t, let the line OC joining centres and the diameter B_0D_0

turn through angles θ and ϕ respectively from the verticals, i.e. at time t B & D correspond to the point B_0 and D_0 at time $t=0$.

Since there is no slipping b/w sphere and cavity, therfore if P is their point of contact at time t, then,

Arc AP = Arc PB i.e. $a\theta = b(\theta+\phi)$

or $b\phi = (a-b)\theta = c\theta$, where $a-b=c$ (say)

$$\therefore b\phi = c\theta \quad \text{--- (1)}$$

Referred to the centre O as origin, horizontal and vertical lines OX and OY as axes, the coordinates (x_C, y_C) of C and (x_D, y_D) of D respectively are

$$x_C = c\sin\theta, \quad y_C = c\cos\theta$$

$$x_D = c\sin\theta + b\sin\phi, \quad y_D = c\cos\theta - b\cos\phi$$

$$\therefore v_C^2 = \dot{x}_C^2 + \dot{y}_C^2 = (c\cos\theta\dot{\theta})^2 + (-c\sin\theta\dot{\theta})^2 = c^2\dot{\theta}^2$$

and
$$v_D^2 = \dot{x}_D^2 + \dot{y}_D^2 = (c\cos\theta\dot{\theta} + b\cos\phi\dot{\phi})^2 + (-c\sin\theta\dot{\theta} + b\sin\phi\dot{\phi})^2$$

$$= c^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2bc\dot{\theta}\dot{\phi}\cos(\theta+\phi) = c^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2bc\dot{\theta}\dot{\phi}$$

($\because \theta$ and ϕ are small)

If T be the kinetic energy and W the work function of the system, then we have.

$T =$ K.E. of the sphere + K.E. of the particle

$$= \left[\frac{1}{2} m \cdot \frac{2}{5} b^2 \dot{\phi}^2 + \frac{1}{2} m v_C^2\right] + \left[\frac{1}{2} m' v_D^2\right]$$

$$= \frac{1}{2} m \left(\frac{2}{5} b^2 \dot{\phi}^2 + c^2 \dot{\theta}^2\right) + \frac{1}{2} m' \left(c^2 \dot{\theta}^2 + b^2 \dot{\phi}^2 + 2bc \dot{\theta} \dot{\phi}\right)$$

$$= \frac{1}{2} m \left(\frac{2}{5} b^2 \dot{\phi}^2 + b^2 \dot{\phi}^2\right) + \frac{1}{2} m' \left(b^2 \dot{\phi}^2 + b^2 \dot{\phi}^2 + 2b \cdot b \dot{\phi} \cdot \dot{\phi}\right)$$

(using ①)

$$= \frac{1}{10} b^2 (7m + 20m') \dot{\phi}^2$$

and $W = -mg(OC' - y_C) + m'g(y_D - OD_0)$

$$= -mg(c - c\cos\theta) + m'g\{c\cos\theta - b\cos\phi - (a - 2b)\}$$

$$= (m + m') cg \cos\theta - m' bg \cos\phi + C$$

$$= (m + m') cg \cos(b\phi/c) - m' bg \cos\phi + C$$

$(\because c\theta = b\phi)$

$\therefore$ Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{5} b^2 (7m + 20m') \dot{\phi}\right] - 0 = -(m + m') cg \cdot \frac{b}{c} \sin\left(\frac{b}{c}\phi\right) + m' bg \sin\phi$

or $\frac{1}{5} b^2 (7m + 20m') \ddot{\phi} = -(m + m') bg \cdot \frac{b}{c} \phi + m' bg \phi$ $(\because \phi$ is small$)$

or $b\left(\frac{7}{5} m + 4m'\right) \ddot{\phi} = -\frac{g}{c}\left[(m + m') - \frac{c}{b} m'\right] \phi$

or $b\left(4m'+\frac{7}{5}m\right)\ddot{\phi} = \frac{-g}{a-b}\left[(m+m') - \frac{a-b}{b}m'\right]\phi$

$[\because c = a-b]$

$$\ddot{\phi} = \frac{-g}{a-b} \cdot \frac{m+(2-a/b)m'}{4m'+\frac{7}{5}m}\phi = -\mu\phi \text{ (say)}$$

which represent S.H.M.

$\therefore$ The length of simple equivalent pendulum is

$$\frac{g}{\mu} = (a-b)\,\frac{4m'+7m/5}{m+m'(2-a/b)}$$

8(b) IAS-2009 P-II

An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r=c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5}\mu\right)^{1/2} \cdot c^{5/4}$.

Soln Let v be the velocity, p the pressure at a distance x from the origin then the equations of motion and continuity are respectively.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and $x^2 v = f(t)$ so that $v = \frac{f(t)}{x^2}$, $\frac{\partial v}{\partial t} = \frac{f'(t)}{x^2}$

$$\therefore \frac{f'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\mu x^{-3/2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

integrating $\frac{f'(t)}{x} + \frac{1}{2}v^2 = \frac{2\mu}{\sqrt{x}} - \frac{p}{\rho} + C$ —— (1)

Boundary conditions are,

2. When $x=\infty$, $v=0$, $p=0$
3. When $x=r$ (radius of cavity), $p=0$, $v=\dot{r}$
4. When $r=c$, $v=0$ so that $f(t)=0$.
5. Let T be the required time of filling the cavity.

Subjecting (1) to the co-ordinates (2) & (3) –

$0+0=0-0+C$ and $\frac{-f'(t)}{r} + \frac{1}{2}(\dot{r}^2) = \frac{2\mu}{\sqrt{r}} - 0 + C$

or. $\frac{-f'(t)}{r} + \frac{1}{2}\dot{r}^2 = \frac{2\mu}{\sqrt{r}}$

Since $r^2(\dot{r}) = f(t)$, $r^2 dr = f(t)\, dt$.

Multiplying by $2f(t)\,dt$ or $2r^2 dr$,

$$\frac{-2f'(t)f(t)\,dt}{r} + \frac{f^2(t)}{r^4}\cdot r^2\,dr = \frac{4\mu}{\sqrt{r}}r^2 dr$$

or, $$d\left[\frac{-f^2(t)}{r}\right] = 4\mu r^{3/2} dr$$

Integrating $$\frac{-f^2(t)}{r} = 4\mu \frac{2}{5} r^{5/2} + A \quad \text{—} ⑥$$

Subjecting ⑥ to ④ –

$$0 = \frac{8\mu}{5} c^{5/2} + A$$

Now ⑥ $$\Rightarrow \frac{-(r^2\dot{r})^2}{r} = \frac{8\mu}{5}(r^{5/2} - c^{5/2})$$

$$\Rightarrow \frac{dr}{dt} = -\left[\frac{8\mu}{5r^3}(c^{5/2} - r^{5/2})\right]^{1/2}$$

[negative sign taken as velocity increases when r decreases]

$$-\int_c^0 \frac{r^{3/2}}{(c^{5/2} - r^{5/2})^{1/2}} dr = \int_0^T \left(\frac{8\mu}{5}\right)^{1/2} dt$$

or $$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2}\,dr}{(c^{5/2} - r^{5/2})^{1/2}} \quad \text{—} ⑦$$

put $r^{5/2} = c^{5/2}\sin^2\theta$, $\frac{5}{2} r^{3/2} dr = c^{5/2}\cdot 2\sin\theta\cdot\cos\theta\,d\theta$

$$T = \left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{4}{5} c^{5/2} \frac{\sin\theta\cos\theta\,d\theta}{c^{5/4}\cos\theta}$$

$$= \left(\frac{5}{8\mu}\right)^{1/2} \cdot \frac{4}{5} c^{5/4} (-\cos\theta)_0^{\pi/2}$$

or. $$T = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}$$