

IAS-MATHEMATICS (Opt.) - 2018

PAPER - I : SOLUTIONS

Q1) Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C = AB$ is a singular matrix.

Sol) Given that A be a 3×2 matrix
 B be a 2×3 matrix

$\therefore \rho(A) \leq 2$, & $\rho(B) \leq 2$
 and $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$.

Let $\rho(A) = r_1$; $\rho(B) = r_2$ and $\rho(AB) = r$

We know that \exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$,

where G is of order $r_1 \times 2$ and 0 is a zero matrix of order $(3-r_1) \times 2$.

Now by post multiplying both sides by B , we have $PAB = \begin{bmatrix} G \\ 0 \end{bmatrix}B$

$\therefore \rho(PAB) = \rho(AB) = r$.

\therefore rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix}B = r$
 Since the matrix G has only r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ 0 \end{bmatrix}B$ can not have more than r_1 non-zero rows.

\therefore Rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix}B \leq r_1$
 $\therefore r \leq r_1$

$$\begin{aligned}
 & \text{i.e } e(A\beta) \leq e(A) \quad (\text{i.e } A \text{ is the pre-factor}) \\
 \text{again } e(A\beta) &= [e(A\beta)^T] \\
 &= e(\beta^T A^T) \\
 &\leq e(\beta^T) \quad (\text{by using } ①) \\
 &= e(\beta) \quad \text{i.e } e(A\beta) \leq e(A) \\
 &= r \\
 \therefore e(A\beta) &\leq e(A). \quad \underline{\text{②}}
 \end{aligned}$$

From ① & ② we have
 $e(A\beta) \leq e(A)$ and $e(A\beta) \leq e(\beta)$
since A is of 3×2 order matrix
 β is of 2×3 order vector
 $A\beta$ is of 3×3 order matrix
and $e(A\beta) \leq 2$.
 $\therefore A\beta$ is singular matrix.

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1(b)

Express basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ as linear combination of $\alpha_1 = (2, -1)$ and $\alpha_2 = (1, 3)$.

Soln

Basis vectors

$$e_1 = (1, 0) \quad \text{and} \quad e_2 = (0, 1)$$

given vectors.

$$\alpha_1 = (2, -1) \quad \text{and} \quad \alpha_2 = (1, 3)$$

e_1 and e_2 can be represented as linear combination of given vectors.

$$e_1 = a\alpha_1 + b\alpha_2$$

$$e_2 = c\alpha_1 + d\alpha_2$$

$$e_1 = (1, 0) = (a)(2, -1) + (b)(1, 3)$$

$$2a + b = 1$$

$$-a + 3b = 0$$

by solving above equn we get -

$$a = \frac{3}{7}, \quad b = \frac{1}{7} \quad \text{--- (i)}$$

Similarly c and d

$$e_2 = (0, 1) = (c)(2, -1) + (d)(1, 3)$$

$$2c + d = 0$$

$$-c + 3d = 1$$

$$c = -\frac{1}{7}, \quad d = \frac{2}{7} \quad \text{--- (ii)}$$

so from (i) and (ii) we get

$$e_1 = \frac{3\alpha_1 + \alpha_2}{7} \quad \text{and} \quad e_2 = \frac{-\alpha_1 + 2\alpha_2}{7}$$

1(c)

Determine if $\lim_{z \rightarrow 1} (1-z) \tan \frac{\pi z}{2}$ exists or not.

If the limit exists, then find its value.

Sol^n

Given function f to be continuous at $z=1$, we must have

$$\lim_{z \rightarrow 1} f(z) = f(1)$$

$$\therefore f(1) = \lim_{z \rightarrow 1} (1-z) \tan\left(\frac{\pi z}{2}\right) + \dots \quad \textcircled{1}$$

Let $z = 1+h$, i.e. $(z-1)=h$, such that as $z \rightarrow 1, h \rightarrow 0$

\therefore by $\textcircled{1}$ -

$$f(1) = \lim_{h \rightarrow 0} \left\{ -h \cdot \tan\left(\frac{\pi}{2}(1+h)\right) \right\}$$

$$= \lim_{h \rightarrow 0} \left\{ -h \cdot \tan\left(\frac{\pi}{2} + \frac{\pi}{2}h\right) \right\}$$

$$= \lim_{h \rightarrow 0} \left\{ (-h) \left(-\cot\left(\frac{\pi}{2}h\right) \right) \right\}$$

$$= \lim_{h \rightarrow 0} \left\{ (-h) \left(-\cot\left(\frac{\pi}{2}h\right) \right) \right\}$$

$$\stackrel{z \rightarrow 1}{=} \lim_{h \rightarrow 0} h \cot\left(\frac{\pi}{2}h\right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{h}{\sin\left(\frac{\pi}{2}h\right)} \right) \cdot \cos\left(\frac{\pi}{2}h\right)$$

$$= \left\{ 1 \cdot \left(\frac{2}{\pi} \right) \right\} \cos 0 \quad \dots \quad \left[\because \lim_{y \rightarrow 0} \frac{y}{\sin y} = 1 \right]$$

$$\Rightarrow f(1) = \frac{2}{\pi}$$

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Q.1(d) find the limit $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2 - r^2}$

Sol: We have, the limit of given function be l .

$$l = \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2 - r^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} (n) \sum_{r=0}^{n-1} \sqrt{1 - \frac{r^2}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1 - \frac{r^2}{n^2}}$$

$$l = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \cdot \frac{1}{n} \cdot \sqrt{1 - \frac{r^2}{n^2}} \quad \text{--- (1)}$$

$\sum_{r=0}^{n-1} \cdot \frac{1}{n} \sqrt{1 - \frac{r^2}{n^2}}$ is in form of Reiman series.

$$\text{S.t. } R(n) = \sum_{r=0}^{n-1} \sqrt{1 - \frac{r^2}{n^2}} \times \frac{1}{n}$$

$$\left[\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=1}^n \Delta x f(x_r) \right]$$

$$\Delta x = \frac{b-a}{n}$$

$$x_r = a + (\Delta x) r$$

Using above

$$\frac{1}{n} = \frac{b-a}{n}$$

$$\therefore b-a = 1.$$

Also

$$x_0 = 0 = a + \left(\frac{1}{n}\right) 0$$

$$\therefore a = 0$$

$$x_0 = \frac{9}{n}$$

So

$$l = \int_0^1 \sqrt{1-x^2} dx$$

$$l = \frac{1}{2} \left[x\sqrt{1-x^2} + \sin^{-1}x \right]_0^1$$

$$l = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$\therefore l = \frac{\pi}{4}$$

Hence;

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2 - r^2} = \frac{\pi}{4}$$

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Q.1@) Find the projection of the straight line

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1} \text{ on the plane } x+y+2z=6.$$

Sol: Given line is

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1} = \alpha \text{ (say)}$$

any point on it is

$$x = 2\alpha + 1 ; y = 3\alpha + 1 ; z = -\alpha - 1.$$

∴ co-ordinate of the point $P'(2\alpha + 1, 3\alpha + 1, -\alpha - 1)$

If it passes through the given plane, then it must intersect, then the point of intersection can be find as,

$$\text{Given plane } \Rightarrow x+y+2z=6$$

$$2\alpha + 1 + 3\alpha + 1 + 2(-\alpha - 1) = 6$$

$$5\alpha + 2 - 2\alpha - 2 = 6$$

$$3\alpha = 6$$

$$\boxed{\alpha = 2}$$

∴ The co-ordinate of P is

$$[2 \times 2 + 1, 3 \times 2 + 1, -(2 + 1)]$$

$$P = [5, 7, -3].$$

Now, equation of Normal of plane through.

$A(3, 1, -1)$ is

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z+1}{2} = s \text{ --- say.}$$

∴ Co-ordinates of $P'(s+1, s+1, 2s-1)$

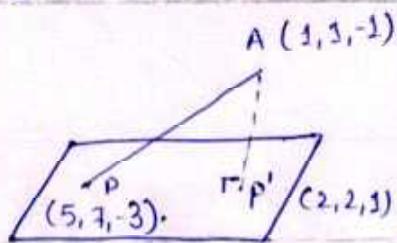
as P' is on Plane.

$$s+1 + s+1 + 2(2s-1) = 6$$

$$2s+2 + 4s - 2 = 6$$

$$6s = 6$$

$$\boxed{s = 1}$$



∴ Co-ordinates of P' (2, 2, 1)

and Direction ratios of $PP' = (3, 5, -4)$

∴ The Required Projection:-

$$\boxed{\frac{x-5}{3} = \frac{y-7}{5} = \frac{z+3}{-4}}$$

Q2) show that if A and B are similar $n \times n$ matrices then they have the same eigen values.

Sol) Given that A and B are similar $n \times n$ matrices

\therefore there exists invertible matrix

P such that $B = P^{-1}AP$.

$$\begin{aligned} \text{We have } B - \lambda I &= P^{-1}AP - \lambda I \\ &= P^{-1}A I P - \lambda P^{-1}P \\ &= P^{-1}A P - P^{-1}\lambda P \\ &= P^{-1}A P - P^{-1}\lambda (I P) \\ &= P^{-1}A P - P^{-1}(\lambda I)P \\ &= P^{-1}(A - \lambda I)P. \end{aligned}$$

$$\begin{aligned} \therefore |B - \lambda I| &= |P^{-1}(A - \lambda I)P| \\ &= |P^{-1}| |A - \lambda I| |P| \\ &= |A - \lambda I| |P^{-1}P| \\ &= |A - \lambda I| \end{aligned}$$

$$\therefore |B - \lambda I| = |A - \lambda I|.$$

\therefore A and B have the same characteristic polynomial and hence same characteristic (eigen) roots (values).

2(b)

find the shortest distance from the point $(1,0)$ to the parabola $y^2 = 4x$.

Solⁿ

The equation of the given curve is

$$y^2 = 4x$$

Let $P(x,y)$ be a point on the curve, which is nearest to point $(1,0)$.

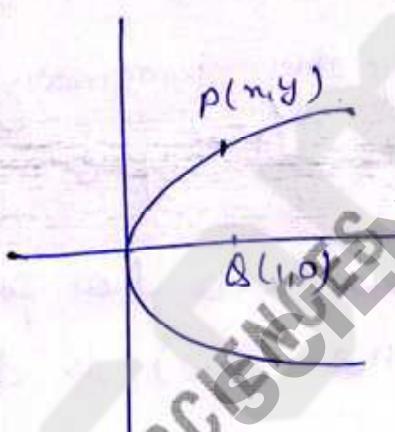
Now, distance between the points P and Q is given by:-

$$\begin{aligned} PQ &= \sqrt{(x-1)^2 + y^2} \\ &= \sqrt{\left(\frac{y^2}{4} - 1\right)^2 + (y^2)} \\ &= \sqrt{\left[\frac{y^4}{16} - \frac{y^2}{2} + 1\right] + (y^2)} \\ &= \sqrt{\frac{y^4}{16} + \frac{y^2}{2} + 1} \end{aligned}$$

$$\text{Let } z = PQ^2 = \frac{y^4}{16} + \frac{y^2}{2} + 1$$

Clearly z is maximum or minimum according as PQ is maximum. Also Q will be nearest to the point P if PQ is minimum.

$$\frac{dy}{dz} = \frac{y^3}{4} + y = 0$$



$$\Rightarrow y(y^2 + 4) = 0$$

$$\Rightarrow y=0, \quad y=\pm 2i$$

rejecting imaginary values
we get $y=0$.

Thus $x=0$ (on putting $y=0$ in $y^2=4x$)
Hence, $(0,0)$ is closest to the point $(1,0)$.

2(c)

The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the x-axis. find the volume of the solid of revolution.

Solⁿ

If y is given as a function of x, volume of the solid obtained by rotating the portion of the curve between $x=a$ and $x=b$ about the x-axis is given by

$$V = \int_a^b \pi y^2 dx.$$

By rotating the ellipse around the x-axis, we generate a solid of revolution called an ellipsoid whose volume can be calculated using the disk method.

We revolve around x-axis a thin vertical strip of height $y = f(x)$ and thickness dx .

This generates a disk of radius y and thickness dx whose volume is dV .

$$dV = (\text{area of the disk}) dx$$

$$dV = \pi r^2 dx$$

$$dV = \pi y^2 dx$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right)$$

$$\text{Substituting } y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right) \text{ in } dV = \pi y^2 dx$$

$$dV = \pi b^2 \left(1 - \frac{x^2}{a^2}\right)$$

We get the volume of the ellipsoid by filling it with a very large number of very thin disks, that is by integrating dV from $x=-a$ to $x=a$.

$$\text{Volume of the ellipsoid} = V = \int_{-a}^a dV = \int_{-a}^a \pi y^2 dx$$

$$V = \int_{-a}^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

On integrating we get -

$$V = \frac{4}{3} \pi a b^2$$

2(d)

→ find the shortest distance between the lines.
 $a_1x + b_1y + c_1z + d_1 = 0$
 $a_2x + b_2y + c_2z + d_2 = 0$

and the z-axis.

Soln

The plane through the given line is

$$(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$$

$$\text{or } (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0 \quad \textcircled{1}$$

If this plane is parallel to z-axis, whose d.c's are 0, 0, 1, then the normal to the plane

(i) is perpendicular to z-axis and we get

$$(a_1 + \lambda a_2) \cdot 0 + (b_1 + \lambda b_2) \cdot 0 + (c_1 + \lambda c_2) \cdot 1 = 0$$

$$\text{or, } \lambda = -c_1/c_2$$

∴ from ① the equation of the plane through the given line and parallel to z-axis is

$$(a_1x + b_1y + c_1z + d_1) - (c_1/c_2)(a_2x + b_2y + c_2z + d_2) = 0$$

$$\text{or, } (c_2a_1 - a_1c_2)x + (c_2b_1 - b_2c_1)y + (cd_2 - c_2d_1) = 0 \quad \textcircled{2}$$

Also any point on the z-axis can be taken as origin i.e. (0, 0, 0).

Required S.D. = length of perpendicular from (0, 0, 0)
 to the plane ②.

$$= \frac{(d_1c_2 - c_1d_2)}{\sqrt{(c_2a_1 - a_1c_2)^2 + (c_2b_1 - b_2c_1)^2}}$$

Q(6)

for the system of linear equations

$$x+3y-2z = -1$$

$$5y + 3z = -8$$

$$x-2y-5z = 7$$

determine which of the following statements are true and which are false.

- (i) The system has no solution
- (ii) The system has a unique solution
- (iii) The system has infinitely many solutions.

SOL

Given that $x+3y-2z = -1$

$$0x+5y+3z = -8$$

$$\rightarrow -2y-5z = 7$$

we write single matrix equation

$$AX = B \quad \text{①}$$

where $A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & 3 \\ 1 & -2 & -5 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

we have

$$\left[A | B \right] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 1 & -2 & -5 & 7 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & -5 & -3 & 8 \end{array} \right] \quad R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 \rightarrow R_3 + 2R_2$$

Clearly it is in echelon form
 $\therefore e(A) = e(A|z) = 2$ (number of unknown variables
a, y, z)

\therefore The given system of equations
are consistent and have
infinitely many solution

i. (i) and (ii) are false.

(iii) is true

Q(5) Let $f(x,y) = xy^x$ if $y > 0$
 $= -xy^x$, if $y \leq 0$

Determine which of $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ exists and which does not exist.

We have

$$\begin{aligned}\frac{\partial f}{\partial x}(0,1) &= \lim_{\delta x \rightarrow 0} \frac{f(0+\delta x, 1) - f(0,1)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{f(\delta x, 1) - f(0,1)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\delta x(1)^x - 0}{\delta x} = 1.\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial f}{\partial y}(0,1) &= \lim_{\delta y \rightarrow 0} \frac{f(0, 1+\delta y) - f(0,1)}{\delta y} \\ &= \lim_{\delta y \rightarrow 0} \frac{\delta y(1)^{\delta y} - 0}{\delta y} \\ &= 0\end{aligned}$$

$\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ both exist

3(c)
P-I

find the equations to the generating lines of the paraboloid $(x+y+2)(2x+y-z) = 6z$ which pass through the point (1,1,1).

Soln

The equation of the two generators of $\lambda-\mu$ system can be written as

$$x+y+2 = 6\lambda, 2x+y-z = z/\lambda \quad \text{--- (1)}$$

$$\text{and } x+y+2 = z/\mu, 2x+y-z = 6\mu \quad \text{--- (2)}$$

If these pass through the point (1,1,1) then.

$$3 = 6\lambda \text{ and } 2 = 6\mu \Rightarrow \lambda = 1/2, \mu = 1/3$$

\therefore from (1) + (2) the generators are given by :

$$x+y+2 = 3, 2x+y-z = 2z$$

$$\text{and } x+y+2 = 3z, 2x+y-z = 2$$

$$\text{i.e. } x+y+2 = 3, 2x+y-3z = 0 \quad \text{--- (3)}$$

$$\text{and } x+y+2 = 0, 2x+y-z = 2 \quad \text{--- (4)}$$

We can find that direction ratios of the generators given by (3) and (4) are 4,5,-1 and 1,-3,-1 respectively and as they pass through the given point (1,1,1) so their equations are.

$$\frac{x-1}{4} = \frac{y-1}{5} = \frac{z-1}{-1} \text{ and } \frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}$$

Q10. Find the equation of the sphere in xyz -plane passing through the points $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$.

Sol. Let the required sphere be
 $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ (1)
 Since it is passing through the point $(0, 0, 0)$,

$\therefore d = 0$
 $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ (2)
 Since it is passing through the points $(0, 1, -1)$, $(-1, 2, 0)$ and $(1, 2, 3)$.

$$\text{at } (0, 1, -1) \quad 1 + 1 + 2v - 2w = 0,$$

$$\text{at } (-1, 2, 0) \quad 1 + 4 - 2u + 4v = 0$$

$$\text{at } (1, 2, 3) \quad 1 + 4 + 9 + 2u + 4v + 6w = 0$$

Solving above three equations,

$$2 + 2v - 2w = 0$$

$$v - w = -1 \text{ or } w - v = 1 \quad \text{--- (i)}$$

$$4v - 2u = -5 \quad \text{--- (ii)}$$

$$u + 2v + 3w = -7 \quad \text{--- (iii)}$$

Putting w from (i) in (3)

$$u + 2v + 3(v+1) = -7$$

$$u + 5v = -10 \quad \text{--- (iv)}$$

using equation ⑪ & ⑫

$$4V - 2U = -5$$

$$4 + 5V = -10$$

$$U = -10 - 5V$$

$$4V - 2(-10 - 5V) = -5$$

$$4V + 20 + 10V = -5$$

$$14V = -25$$

$$\boxed{V = \frac{-25}{14}}$$

$$U + 5 \times \frac{-25}{14} = -10$$

$$U = -10 + \frac{125}{14}$$

$$U = \frac{-140 + 125}{14}$$

$$\boxed{U = \frac{-15}{14}}$$

Hence; $U = \frac{-15}{14}, V = \frac{-25}{14}$

Put these values in eqn ⑬

$$-\frac{15}{14} + 2 \times \frac{-25}{14} + 3W = -7$$

$$3W = -7 + \frac{15}{14} + \frac{50}{14}$$

$$3W = \frac{-98 + 65}{14} \Rightarrow W = \frac{-33}{3 \times 14} = \frac{-11}{14}$$

$$\therefore U = \frac{-15}{14}, V = \frac{-25}{14}, W = \frac{-11}{14}$$

Put these values in eqn ②

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$$

$$x^2 + y^2 + z^2 - 2 \times \frac{15}{14}x + 2 \times \frac{-25}{14}y + 2 \times \frac{-11}{14}z = 0$$

$$14x^2 + 14y^2 + 14z^2 - 30x - 50y - 22z = 0$$

is the required solution

or,

$$7x^2 + 7y^2 + 7z^2 - 15x - 25y - 11z = 0$$

or

$$7(x^2 + y^2 + z^2) - [15x + 25y + 11z] = 0$$

is the required equation of

Sphere.

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Q.4(a) find the minimum and maximum values of $x^4 - 5x^2 + 4$ on the interval $[2, 3]$.

Given ; $f(x) = x^4 - 5x^2 + 4$

$$f'(x) = 4x^3 - 10x^2$$

let $f'(x) = 0$

$$4x^3 - 10x^2 = 0$$

$$2x(2x^2 - 5) = 0$$

$$x = 0, \pm \sqrt{5}/2$$

These are the critical points of the $f(x)$, which $\notin [2, 3]$. Thus, $f(x)$ is monotonic on $[2, 3]$

Now, to check, whether it is monotonic increasing or decreasing -

$$f'(2) = 2 \times 2 (2 \times 4 - 5) = 4[8 - 5] = 4 \times 3 = 12 > 0$$

$$f'(3) = 2 \times 3 (2 \times 9 - 5) = 6[18 - 5] = 6 \times 13 = 78 > 0$$

Since, $f'(2) > 0$ thus $f(x)$ is monotonic increasing

since; $f'(2), f'(3) > 0$ and $f'(3) > f'(2)$.

Hence, $f(x)$ is monotonically increasing on $[2, 3]$.

$$\text{Thus; } f_{\min} = f(x) \Big|_{x=2} = (2)^4 - 5(2)^2 + 4 \\ = 16 - 20 + 4 \\ = 0$$

$$f_{\max} = f(x) \Big|_{x=3} = (3)^4 - 5(3)^2 + 4 \\ = 81 - 5 \times 9 + 4 \\ = 81 + 4 - 45 \\ = 85 - 45 = 40$$

$$\therefore f_{\min} = 0 \quad \& \quad f_{\max} = 40$$



Q.4 (b) Evaluate the integral $\int_0^a \int_{x/a}^x \frac{x dy dx}{x^2 + y^2}$?

Sol: Given integral is

$$\int_0^a \int_{x/a}^x \frac{x dy dx}{x^2 + y^2}$$

as the integral limits are variable with respect to x , we integrate first with respect to y , (treating x as constant) from inside out.

Thus;

$$I = \int_0^a \left[\int_{x/a}^x \frac{x dy}{x^2 + y^2} \right] dx$$

$$I = \int_0^a \left[x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_{x/a}^x \right] dx$$

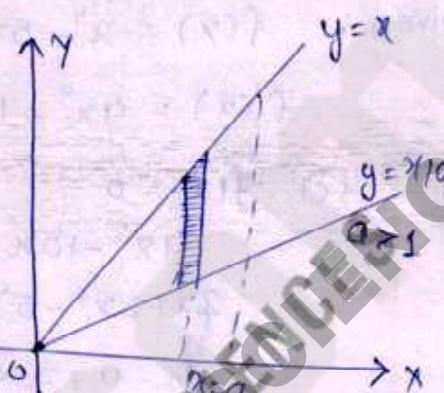
$$I = \int_0^a \left[\tan^{-1} \frac{x}{a} - \tan^{-1} \frac{x/a}{x} \right] dx$$

$$I = \int_0^a \left[\tan^{-1} 1 - \tan^{-1} \frac{1}{a} \right] dx$$

$$I = \int_0^a \left[\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{a} \right) \right] dx = \left[\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right] \int_0^a dx$$

$$I = \left[\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right] [x]_0^a = \left[\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right] [a - 0]$$

$$I = a \left[\frac{\pi}{4} - \tan^{-1} \frac{1}{a} \right]$$



4(c) → find the equation of the cone with $(0,0,1)$ as the vertex and $2x^2 - y^2 = 4$, as the guiding curve. $z=0$

Sol) The given base conic is $2x^2 - y^2 = 4, z=0$

Now the equations of any line through $(0,0,1)$ is

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-1}{n} \quad (1)$$

Since it meets the plane $z=0$

$$\text{where } \frac{x-0}{l} = \frac{y-0}{m} = \frac{0-1}{n}$$

$$\Rightarrow \frac{x}{l} = \frac{-1}{n}, \frac{y}{m} = \frac{-1}{n}$$

$$\Rightarrow x = -\frac{l}{n}, y = -\frac{m}{n}$$

$$\therefore (x, y, 0) = \left(-\frac{l}{n}, -\frac{m}{n}, 0\right)$$

Since it lies on the conic

$$2\left(\frac{x}{l}\right)^2 - \left(\frac{y}{m}\right)^2 = 4.$$

$$\Rightarrow 2\frac{l^2}{n^2} - \frac{m^2}{n^2} = 4. \quad (2)$$

Now eliminating l, m, n from (2) & (1) we have

$$2\left(\frac{x^2}{(z-1)^2}\right) - \frac{y^2}{(z-1)^2} = 4.$$

$\Rightarrow 2x^2 - y^2 = 4(z-1)^2$ which is the required equation of the cone.

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4(d)
P-I

find the equation of the plane parallel to $3x-y+3z=8$ and passing through the point $(1,1,1)$.

Solⁿ

We know that any plane, parallel to $3x-y+3z-8$ is $3x-y+3z+k=0$

Since this plane passes to point $(1,1,1)$ then,

$$3 \cdot 1 - 1 + 3 \cdot 1 + k = 0$$

$$\boxed{k = -5}$$

$\boxed{3x-y+3z-5=0}$, is the required plane.

SOLUTION : UPSC-IAS-MAINS-2018-MATHEMATICS-OPTIONAL-PAPER-1



Section - B

5 (a). Solve . $y'' - y = x^2 \cdot e^{2x}$.

Sol:- Given Differential Equation is

$$y'' - y = x^2 \cdot e^{2x}$$

It can be re-written as .

$$(D^2 - 1)y = x^2 \cdot e^{2x}$$

where; $D = \frac{dx}{dy}$

The auxillary equation

$$m^2 - 1 = 0 ; m = \pm 1$$

$$y_c = c_1 \cdot e^x + c_2 e^{-x}$$

Now for y_p -

$$y_p = \frac{1}{(D^2 - 1)} e^{2x} \cdot x^2$$

$$y_p = e^{2x} \cdot \frac{1}{((D+2)^2 - 1)} x^2 = e^{2x} \cdot \frac{1}{(D^2 + 4D + 3)} \cdot x^2$$

$$y_p = e^{2x} \cdot \frac{1}{3} \left[\frac{1}{1 + \left(\frac{D^2 + 4D}{3}\right)} \right] x^2$$

$$y_p = \frac{e^{2x}}{3} \left[1 + \frac{D^2 + 4D}{3} \right]^{-1} x^2$$

$$y_p = \frac{e^{2x}}{3} \left[1 - \frac{D^2 + 4D}{3} + \frac{(D^2 + 4D)^2}{9} \dots \right] x^2$$

$$y_p = \frac{e^{2x}}{3} \left[x^2 - \left(\frac{2+8x}{3} \right) + \frac{32}{9} \right] = \frac{e^{2x}}{3} \left[x^2 - \frac{8x}{3} + \frac{32}{9} \right]$$

General Solution -

$$\therefore y = y_c + y_p = c_1 e^x + c_2 e^{-x} + \frac{e^{2x}}{3} \left[x^2 - \frac{8x}{3} + \frac{32}{9} \right]$$

5(b)

IAS
2018

P-I

find the angle between the tangent at a general point of the curve whose equations are $x = 3t$, $y = 3t^2$, $z = 3t^3$ and the line $y = z - x = 0$

Soln

$$\vec{g}_1 = 3t\hat{i} + 3t^2\hat{j} + 3t^3\hat{k}$$

$$\frac{d\vec{g}_1}{dt} = 3\hat{i} + 6t\hat{j} + 9t^2\hat{k}$$

$$y = z - x = 0$$

$$y = 0, \quad x = z$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1} \Rightarrow [1, 0, 1]$$

$$\vec{g}_2 = \hat{i} + \hat{k}, \quad |\vec{g}_2| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

$$\vec{g}_2 = \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$$

$$\begin{aligned} \therefore \cos \theta &= \frac{\vec{g}_1 \cdot \vec{g}_2}{|\vec{g}_1| |\vec{g}_2|} = \frac{3 \cdot 1 + 6t \cdot 0 + 9t^2 \cdot 1}{\sqrt{9+36t^2+81t^4} \cdot \sqrt{2}} \\ &= \frac{3+9t^2}{3\sqrt{2} \cdot \sqrt{1+4t^2+9t^4}} \\ &= \frac{1+3t^2}{\sqrt{2} \cdot \sqrt{1+4t^2+9t^4}} \end{aligned}$$

$$\Rightarrow \boxed{\theta = \cos^{-1} \left[\frac{1+3t^2}{\sqrt{2} \cdot \sqrt{1+4t^2+9t^4}} \right]}$$

5(c) Solve: $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$.

So: Given D.E.

$$y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}.$$

It can be rewritten as:

$$(D^3 - 6D^2 + 12D - 8)y = 12e^{2x} + 27e^{-x}.$$

The Auxillary eq.

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$(m-2)(m^2 - 4m + 4) = 0$$

$$(m-2)(m-2)(m-2) = 0$$

$$m = 2, 2, 2.$$

$$y_c = C.F = (C_1 + C_2x + C_3x^2)e^{2x}.$$

$$y_p = \frac{1}{(D-2)^3} [12e^{2x} + 27e^{-x}]$$

$$y_p = \frac{12}{(D-2)^3} \cdot e^{2x} + \frac{27}{(D-2)^3} \cdot e^{-x}$$

$$y_p = 12 \cdot \frac{x^3}{3!} \cdot e^{2x} + 27x \cdot e^{-x} \cdot \frac{1}{(-1)^3}$$

$$y_p = 2x^3 \cdot e^{2x} + \frac{27}{-27} \cdot e^{-x}$$

$$y_p = 2x^3 \cdot e^{2x} - e^{-x}.$$

: The Solution for given D.E

$$y = y_c + y_p$$

$$y = (C_1 + C_2x + C_3x^2)e^{2x} + 2x^3e^{2x} - e^{-x}$$

Ques 5)(d)(i) Find the Laplace Transform of $f(t) = \frac{1}{\sqrt{t}}$.

Sol: Given: $f(t) = \frac{1}{\sqrt{t}}$

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_{t=0}^{\infty} e^{-st} \cdot \frac{1}{\sqrt{t}} dt$$

$$\text{Put } u = st \Rightarrow t = u/s$$

$$du = sdt \Rightarrow dt = \frac{du}{s}$$

$$\sqrt{t} = \sqrt{\frac{u}{s}} \Rightarrow \frac{1}{\sqrt{t}} = \frac{\sqrt{s}}{\sqrt{u}}$$

$$= \int_{u=0}^{\infty} e^{-u} \cdot \frac{\sqrt{s}}{\sqrt{u}} \cdot \frac{1}{s} du$$

$$= \frac{\sqrt{s}}{s} \int_{u=0}^{\infty} e^{-u} \cdot u^{-1/2} du$$

$$= \frac{1}{\sqrt{s}} \int_{u=0}^{\infty} e^{-u} \cdot u^{-1/2} du.$$

$$\text{put } w = \sqrt{u} \Rightarrow u = w^2$$

$$dw = \frac{1}{2\sqrt{u}} du.$$

$$du = 2\sqrt{u} dw$$

$$= \frac{1}{\sqrt{s}} \int_{w=0}^{\infty} e^{-w^2} \cdot \frac{1}{\sqrt{w}} \cdot 2\sqrt{w} dw$$

$$= \frac{2}{\sqrt{s}} \int_{w=0}^{\infty} e^{-w^2} dw = \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2}$$

$$\boxed{\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}}$$

$$\left[\because \int_{x=0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \right]$$

5(d)(ii) find the Inverse Laplace transform of

$$\frac{5s^2 + 3s + 6}{(s-1)(s-2)(s+3)}.$$

So!: Given $F(s) = \frac{5s^2 + 3s + 6}{(s-1)(s-2)(s+3)}$

$$\mathcal{L}^{-1} \left\{ \frac{5s^2 + 3s + 6}{(s-1)(s-2)(s+3)} \right\} = \mathcal{L}^{-1} \left\{ \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+3} \right\}$$

$$5s^2 + 3s + 6 = A(s-2)(s+3) + B(s-1)(s+3) + C(s-1)(s-2)$$

By Equating

$$A + B + C = 5$$

$$A + 2B - 3C = 3$$

$$-6A - 3B + 2C = -16$$

By solving above three equation.

we get $A = 2, B = 2, C = 1$

$$\therefore \mathcal{L}^{-1} \left[\frac{2}{s-1} + \frac{2}{s-2} + \frac{1}{s+3} \right]$$

$$f(t) = 2e^t + 2e^{2t} + e^{-3t}.$$

$$f(t) = 2[e^t + e^{2t}] + e^{-3t}$$

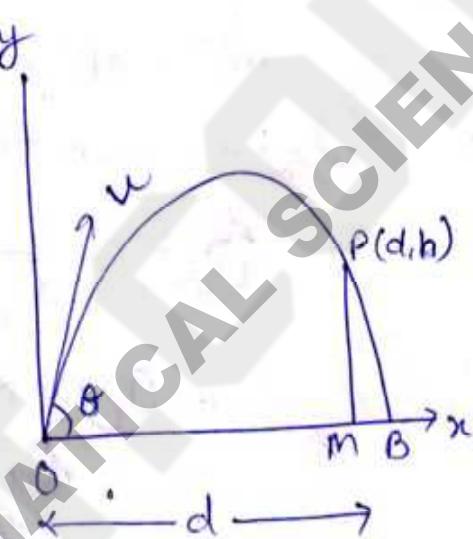
Q.5 (e) A particle projected from a given point on ground just clears a wall of height 'h' at a distance 'd' from the point of projection. If the particle moves in a vertical plane and if the horizontal range is 'R'. find the elevation of projection?

Sol: Let a particle projected from O with a velocity v at an angle θ to the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes.

Here

$$x = v \cos \theta t \quad \text{--- (1)}$$

$$y = v \sin \theta t - \frac{1}{2} g t^2 \quad \text{--- (2)}$$



The equation of trajectory is -

$$y = x \tan \theta - \frac{1}{2} \frac{g x^2}{v^2 \cos^2 \theta} \quad \text{--- (3)}$$

$$h = v \tan \theta \left[1 - \frac{x}{R} \right]$$

Now $y = h$
 $x = d$

[The particle just clears the wall PM of height h at a distance d from O and strikes ground at the point B at a distance R from O. Thus both the points (d, h) and $(R, 0)$ lie on the curve (3).]

$$\text{Therefore } h = d \tan \theta - \frac{1}{2} g \frac{d^2}{u^2 \cos^2 \theta} \quad \textcircled{4}$$

$$\text{and } 0 = R \tan \theta - \frac{1}{2} g \frac{R^2}{u^2 \cos^2 \theta} \quad \textcircled{5}$$

To eliminate u^2 , we multiply $\textcircled{4}$ by R^2 and $\textcircled{5}$ by d^2 and subtract. Thus we get

$$hR^2 = dR^2 \tan \theta - d^2 R \tan \theta$$

$$\text{or, } hR^2 = dR \tan \theta (R-d)$$

$$\tan \theta = \frac{h}{d} \left[1 - \frac{d}{R} \right]$$

$$\tan \theta = \frac{hR}{d[R-d]}$$

$$\theta = \tan^{-1} \left[\frac{hR}{d[R-d]} \right]$$

=====

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Q-6- (a) Solve: $\left(\frac{dy}{dx}\right)^2 y + 2 \frac{dy}{dx} x - y = 0$

\rightarrow i.e. $y p^2 + 2px - y = 0$ — (1) $\therefore p = \frac{dy}{dx}$

$$\therefore 2px = y - y p^2$$

$$\therefore x = \frac{1}{2} \left(\frac{y}{p} \right) - \frac{1}{2} (yp) — (2)$$

clearly it is solvable for x

diff. eqn (2) w.r.t. y

$$\therefore \frac{dx}{dy} = \frac{1}{2} \left[p - y \frac{dp}{dy} \right] - \frac{1}{2} \left[p + y \frac{dp}{dy} \right]$$

$$\therefore \frac{2}{p} = \frac{1}{p} - \left(\frac{y}{p^2} \right) \frac{dp}{dy} - p - y \frac{dp}{dy}$$

$$\text{i.e. } \left(\frac{1}{p} + p \right) = \frac{-y}{p^2} \frac{dp}{dy} - y \frac{dp}{dy}$$

$$\therefore p \left(\frac{1}{p^2} + 1 \right) = -y \frac{dp}{dy} \left(\frac{1}{p^2} + 1 \right)$$

$$\therefore p = -y \frac{dp}{dy}$$

$$\therefore \frac{1}{p} dp = -\frac{1}{y} dy$$

integrating both sides.

$$\log p + \log y = \log c$$

$$py = c$$

$$p = c/y$$

put value of p in eqn (2)

$$\therefore x = \frac{1}{2} \left(y \times \frac{y}{c} \right) - \frac{1}{2} \left(y \times \frac{c}{y} \right)$$

$$x = \frac{y^2}{2c} - \frac{c}{2}$$

6(b)

A particle moving with simple harmonic motion in a straight line has velocities v_1 and v_2 at distances x_1 and x_2 respectively from the centre of its path. find the period of its motion.

SolM

Let the equation of the S.H.M. with centre as origin be $\frac{d^2x}{dt^2} = -\mu x$. Then the time period $T = 2\pi/\sqrt{\mu}$.

If a be the amplitude of the motion, we have

$$v^2 = \mu(a^2 - x^2) \quad \text{--- (1)}$$

where v is the velocity at a distance x from the centre.

But when $x = x_1$, $v = v_1$.

and when $x = x_2$, $v = v_2$

Therefore from (1), we have.

$$v_1^2 = \mu(a^2 - x_1^2)$$

$$\text{and } v_2^2 = \mu(a^2 - x_2^2)$$

$$\text{These give } v_2^2 - v_1^2 = \mu \{ (a^2 - x_2^2) - (a^2 - x_1^2) \} \\ = \mu(x_1^2 - x_2^2)$$

$$\text{i.e. } \mu = (v_2^2 - v_1^2) / (x_1^2 - x_2^2)$$

Hence the time period $T = 2\pi/\sqrt{\mu}$

$$= 2\pi \sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2} \right)}$$

6(c) Solve $y'' + 16y = 32 \sec 2x$.

Sol: Given that $y'' + 16y = 32 \sec 2x \quad \text{--- (1)}$

The homogeneous equation is

$$y'' + 16y = 0.$$

Auxiliary Equation is

$$m^2 + 16 = 0$$

$$m = \pm 4i$$

$$\therefore y_c(x) = C_1 \cos 4x + C_2 \sin 4x$$

$$P.I. = \frac{1}{D^2 + 16} \cdot 32 \sec(2x)$$

$$= 32 \cdot \frac{1}{(D+4i)(D-4i)} \sec 2x,$$

$$= 32 \cdot \frac{1}{8i} \left[\frac{1}{D+4i} + \frac{1}{D-4i} \right] \sec 2x$$

$$P.I. = -4i \left[\frac{1}{D-4i} \mp \frac{1}{D+4i} \right] \sec 2x.$$

$$P.I. = -4i \left[\frac{1}{D-4i} \sec 2x - \frac{1}{D+4i} \sec 2x \right]$$

NOW:

$$\frac{1}{D-4i} \sec 2x = e^{4ix} \int e^{-4ix} \cdot \sec 2x dx,$$

$$= e^{4ix} \int \left[\frac{\cos 4x - i \sin 4x}{\cos 2x} \right] dx.$$

$$= e^{4ix} \int \left[\frac{\cos^2 2x - \sin^2 2x - 2i \sin 2x \cos 2x}{\cos 2x} \right] dx$$

$$I_1 = e^{i4x} \int \frac{(\cos 2x - i\sin 2x)^2}{\cos 2x} dx.$$

$$I_1 = e^{i4x} \int \left(\frac{2\cos^2 2x - 1 - 2i\sin 2x \cos 2x}{\cos 2x} \right) dx.$$

$$I_1 = e^{i4x} \int (2\cos 2x - 2i\sin 2x - \sec 2x) dx$$

$$I_1 = e^{i4x} [\sin 2x + i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right) + c]$$

Now

$$I_2 = \frac{1}{(D+4i)} \cdot \sec 2x = e^{-4ix} \int e^{4ix} \cdot \sec 2x dx.$$

$$I_2 = e^{-4ix} \int \frac{\cos 4x + i\sin 4x}{\cos 2x} dx.$$

$$I_2 = e^{-4ix} \int \left[\frac{2\cos^2 2x - 1 + 2i\sin 2x \cos 2x}{\cos 2x} \right] dx.$$

$$I_2 = e^{-4ix} \int [2\cos 2x - \sec 2x + 2i\sin 2x] dx$$

$$I_2 = e^{-4ix} [\sin 2x - i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right) + c]$$

$$P.I. = -4i [I_1 + I_2]$$

$$P.I. = -4i \left[e^{i4x} [\sin 2x + i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right)] \right. \\ \left. - e^{-4ix} [\sin 2x - i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right)] \right]$$

$$\therefore Y = Y_c + Y_p$$

$$Y = C_1 \cos 4x + C_2 \sin 4x - 4i \left[e^{i4x} [\sin 2x + i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right)] \right. \\ \left. - e^{-i4x} [\sin 2x - i\cos 2x - \log \left(\frac{\sec 2x + \tan 2x}{2} \right)] \right]$$

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Q.6 (i) If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$; then evaluate

$$\iint_S [x+z] dy dz + [y+z] dz dx + (x+y) dx dy]$$

using Gauss Divergence Theorem?

Sol: Let S is the surface of sphere

$$x^2 + y^2 + z^2 = a^2$$

then $\iint_S [x+z] dy dz + [y+z] dz dx + (x+y) dx dy]$

$\downarrow \quad \downarrow \quad \downarrow$
 i j k

$$F = [x+z] dy dz + [y+z] dz dx + (x+y) dx dy$$

$$\text{or } F = [x+z] \hat{i} + [y+z] \hat{j} + (x+y) \hat{k}$$

$$\iiint \nabla \cdot \vec{F} dV = \iint_S F \cdot dS \quad [\text{Gauss Divergence Theorem}]$$

$$\therefore \nabla \cdot \vec{F} = \nabla \cdot [(x+z) \hat{i} + (y+z) \hat{j} + (x+y) \hat{k}]$$

$$\nabla \cdot \vec{F} = 1 + 1 + 0 = 2.$$

$$\iiint \nabla \cdot \vec{F} dV = \iiint 2 dV$$

$$= 2 \iiint dV = 2 \times \frac{4}{3} \pi a^3$$

$[\because \text{volume of sphere} = \frac{4}{3} \pi a^3]$

$$\therefore \iiint \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot dS = \frac{8}{3} \pi a^3$$

7(a) Solve $(1+x^2)y'' + (1+x)y' + y = 4 \cos(\log(1+x))$

Sol:

$$(1+x^2)y'' + (1+x)y' + y = 4 \cos(\log(1+x))$$

$$\text{put } \log(1+x) = z.$$

Hence, the given DE can be rewritten as

$$(D(D-1)y + Dy + y) = 4 \cos z$$

$$(D(D-1) + D + 1)y = 4 \cos z$$

$$(D^2 + D - D + 1)y = 4 \cos z$$

$$(D^2 + 1)y = 4 \cos z$$

The Auxillary Eqn.

$$1 - m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore y_c : C.F = C_1 \cos z + C_2 \sin z$$

$$y_c = C_1 \cos(\log(1+x)) + C_2 \sin(\log(1+x)).$$

$$y_p = \frac{L \cdot 4 \cos z}{D^2 + 1} = 4 \times \frac{z}{2} \cdot \sin z$$

$$y_p = 2 \frac{\log(1+x)}{z} \cdot \sin(\log(1+x))$$

$$\therefore \text{General Solution} \Rightarrow y = y_c + y_p.$$

$$y = C_1 \cos \log(1+x) + C_2 \sin \log(1+x) + 2 \frac{\log(1+x)}{z} \sin(\log(1+x))$$

$$y = C_1 \cos \log(1+x) + C_2 \sin \log(1+x) + \log(1+x)^2 \cdot \sin(\log(1+x))$$

is the required solution

7(b) Find the Curvature and torsion of the curve.

$$\vec{r} = a(u - \sin u)\hat{i} + a(1 - \cos u)\hat{j} + bu\hat{k}$$

Sol:- Given; $\vec{r} = a(u - \sin u)\hat{i} + a(1 - \cos u)\hat{j} + bu\hat{k}$

$$\frac{d\vec{r}}{du} = (a - a\cos u)\hat{i} + a\sin u\hat{j} + bu\hat{k} \quad \text{--- (1)}$$

$$\begin{aligned}\therefore \left| \frac{d\vec{r}}{du} \right| &= \sqrt{a^2 - 2a^2 \cos^2 u + a^2 \cos^2 u + a^2 \sin^2 u + b^2} \\ &= \sqrt{2a^2(1 - \cos u) + b^2} \\ &= \sqrt{\frac{2a^2}{b^2} [1 - \cos u + 1]}\end{aligned}$$

$$\left| \frac{d\vec{r}}{du} \right| = \frac{a}{b} \sqrt{2(2 - \cos u)} \quad \text{--- (2)}$$

$$\text{Also: } \frac{d^2\vec{r}}{du^2} = a\sin u\hat{i} + a\cos u\hat{j} \quad \text{--- (3)}$$

$$\& \frac{d^3\vec{r}}{du^3} = a\cos u\hat{i} - a\sin u\hat{j} \quad \text{--- (4)}$$

$$\text{Now: } \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} = \begin{vmatrix} i & j & k \\ a - a\cos u & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix}$$

$$\begin{aligned}&= i[(a\sin u \cdot 0) - ab\cos u] - j[-ab\sin u] \\ &\quad + k(a^2\cos u - a^2\cos^2 u - a^2\sin^2 u)\end{aligned}$$

$$\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} = -ab\cos u\hat{i} + ab\sin u\hat{j} + a^2(\cos u - 1)\hat{k}$$

$$\begin{aligned}\left| \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} \right| &= \sqrt{a^2 b^2 \cos^2 u + a^2 b^2 \sin^2 u + a^4 (\cos u - 1)^2} \\ &= \sqrt{(ab)^2 (\cos^2 u + \sin^2 u) + a^4 (\cos^2 u + 1 - 2\cos u)} \\ &= \sqrt{ab^2 + a^4 (\cos u - 1)^2}\end{aligned}$$

$$\therefore \left| \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} \right| = \sqrt{a^2 b^2 + a^4 (\cos u - 1)^2}$$

$$= a \sqrt{b^2 + a^2 (\cos u - 1)^2}$$

$$\kappa = \frac{\left| \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} \right|}{\left| \frac{d\vec{r}}{du} \right|^3} = \frac{\sqrt{a^2 b^2 + a^4 (\cos u - 1)^2}}{(2a^2(1 - \cos u) + b^2)^{3/2}}$$

$$T = \frac{\left[\frac{d\vec{u}}{du}, \frac{d\vec{r}}{du^2}, \frac{d^3\vec{r}}{du^3} \right]}{\left| \frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} \right|} = \frac{[-abc\cos u \hat{i} + ab\sin u \hat{j} + a^2(\cos u - 1)\hat{k} - (a\cos u \hat{i} - a\sin u \hat{j})]}{[a^2 b^2 + a^4 (\cos u - 1)^2]}$$

$$T = \frac{-a^2 b \cos^2 u - a^2 b \sin^2 u}{(a^2 b^2 + a^4 (\cos u - 1)^2)}$$

$$T = \frac{-a^2 b (\cos^2 u + \sin^2 u)}{a^2 b^2 + a^4 (\cos u - 1)^2}$$

$$T = \frac{-a^2 b}{a^2 b^2 + a^4 (\cos u - 1)^2}$$

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7(c) Solve the initial value problem

$$y'' - 5y' + 4y = e^{2t}$$

$$y(0) = \frac{19}{12}; y'(0) = \frac{8}{3}.$$

Sol:- Given; $y'' - 5y' + 4y = e^{2t}$ & $y(0) = \frac{19}{12}; y'(0) = \frac{8}{3}$

$$(D^2 - 5D + 4)y = e^{2t}$$

$$\lambda \cdot E = m^2 - 5m + 4 = 0$$

$$(m-4)(m-1) = 0$$

$$m = 4, 1$$

$$Q.F = C_1 e^t + C_2 e^{4t}.$$

$$P.I. = \frac{1}{(D-1)(D-4)} e^{2t}$$

$$= \frac{1}{(D^2 - 5D + 4)} \cdot e^{2t}$$

$$= \frac{e^{2t}}{(2)^2 - 5 \cdot 2 + 4} = \frac{e^{2t}}{4 - 10 + 4}$$

$$P.I. = Y_p = -\frac{1}{2} e^{2t}$$

$$y = y_c + Y_p = C_1 e^t + C_2 e^{4t} - \frac{1}{2} e^{2t} \quad \text{--- (A)}$$

$$\text{Now}; \quad y(0) = \frac{19}{12}.$$

Put $t=0$ in (A) :

$$y(0) = C_1 + C_2 - \frac{1}{2} \Rightarrow C_1 + C_2 = \frac{19}{12} + \frac{1}{2} = \frac{25}{12} \quad \text{--- (1)}$$

$$y' = C_1 e^t + 4C_2 e^{4t} - e^{2t} \quad \text{--- (B)}$$

$$y'(0) = \frac{8}{3}$$

Put $y(t=0)$ in eqn (B)

$$y(0) = c_1 + 4c_2 - 1.$$

$$c_1 + 4c_2 = \frac{8}{3} + 1$$

$$c_1 + 4c_2 = \frac{11}{3} \quad \text{--- (2)}$$

from (1) & (2)

$$\frac{11}{3} = \frac{25}{12} - c_2 + 4c_2$$

$$3c_2 = \frac{11}{3} - \frac{25}{12} = \frac{44 - 25}{12} = \frac{19}{12}$$

$$c_2 = \frac{19}{36}$$

$$c_1 + c_2 = \frac{25}{12}$$

$$c_1 + \frac{19}{36} = \frac{25}{12}$$

$$c_1 = \frac{75 - 19}{36} = \frac{56}{36} = \frac{14}{9}$$

$$\therefore c_1 = \frac{14}{9} \quad \& \quad c_2 = \frac{19}{36}$$

$$\boxed{\therefore y = \frac{14}{9} e^t + \frac{19}{36} e^{4t} - \frac{1}{2} e^{2t}}$$

is the required solution.

7(d) → find α and β such that $x^\alpha y^\beta$ is an integrating factor of

$$(4y^2 + 3xy)dx - (3xy + 2x^2)dy = 0 \text{ and}$$

solve the equation.

Sol Given that $(4y^2 + 3xy)dx - (3xy + 2x^2)dy = 0$ ①

let $x^\alpha y^\beta$ be an integrating factor of ①

$$\text{Then } x^\alpha y^\beta (4y^2 + 3xy)dx - x^\alpha y^\beta (3xy + 2x^2)dy = 0$$

$$\Rightarrow (4x^\alpha y^{\beta+2} + 3x^\alpha y^{\beta+1})dx + (3x^\alpha y^{\beta+1} + 2x^{\alpha+2}y^\beta)dy = 0$$

clearly it is in the form of

$\frac{M}{y} dx + \frac{N}{x} dy = 0$

and is exact.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad ③$$

$$\text{Here } \frac{\partial M}{\partial y} = 4(\beta+2)x^\alpha y^{\beta+1} + 3(\beta+1)x^{\alpha+1}y^\beta$$

$$\frac{\partial N}{\partial x} = -3(\alpha+1)x^\alpha y^{\beta+1} - 2(\alpha+2)x^{\alpha+2}y^\beta.$$

$$③ \Rightarrow 4(\beta+2)x^\alpha y^{\beta+1} + 3(\beta+1)x^{\alpha+1}y^\beta =$$

$$-3(\alpha+1)x^\alpha y^{\beta+1} - 2(\alpha+2)x^{\alpha+2}y^\beta$$

$$\Rightarrow 4(\beta+2) = -3(\alpha+1) ; 3(\beta+1) = -2(\alpha+2)$$

$$\Rightarrow -3\beta - 4\alpha - 11 = 0 ; -3\beta - 7 = 0$$

$$\Rightarrow 3\beta + 4\alpha + 11 = 0 \quad (i) \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \alpha = -5 & \beta = 1$$

$$2\alpha + 3\beta + 7 = 0 \quad (ii) \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \alpha = -5 & \beta = 1$$

$$\therefore \textcircled{A} = \int y^5 (4y^2 + 3xy) dx - \int y^5 y (3xy + 2x^2) dy = 0$$

$$\Rightarrow y \frac{(4y^2 + 3xy)}{x^5} dx - y \frac{(3xy + 2x^2)}{x^5} dy = 0$$

$$\Rightarrow \left(\frac{4y^3}{x^5} + \frac{3y^2}{x^4} \right) dx - \left(\frac{3y^2}{x^4} + \frac{2y}{x^3} \right) dy = 0$$

Since it is an exact
 \because The general solution is
 given by

$$\int \left(\frac{4y^3}{x^5} + \frac{3y^2}{x^4} \right) dx + \int 0 dy = C$$

$y = \text{const}$

$$\Rightarrow \frac{+4y^3}{-4x^4} - \frac{3y^2}{3x^3} = C$$

$$\Rightarrow \boxed{\frac{y^3}{x^4} + \frac{y^2}{x^3} = C}$$

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8(a). Let $\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$. Show that $\text{curl}(\text{curl } \vec{v}) = \text{curl}(\text{curl } \vec{v}) = \text{grad}(\text{div } \vec{v}) - \nabla^2 \vec{v}$.

Sol

$$\text{let } \vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$\text{Then } \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\nabla \times \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \hat{k}$$

$$\therefore \nabla \times (\nabla \times \vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \nabla \cdot \left[\left\{ \frac{\partial}{\partial y} \left(\frac{\partial v_2}{\partial z} - \frac{\partial v_1}{\partial x} \right) - \frac{\partial}{\partial z} \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \right\} \hat{i} \right]$$

$$= \nabla \cdot \left[\left\{ \left(\frac{\partial^2 v_2}{\partial y \partial z} + \frac{\partial^2 v_3}{\partial z \partial x} \right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) \right\} \hat{i} \right]$$

$$= \nabla \cdot \left[\left\{ \frac{\partial}{\partial z} \left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) \right\} \hat{i} \right]$$

$$= \nabla \cdot \left[\left\{ \frac{\partial}{\partial x} \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) - \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) \right\} \hat{i} \right]$$

$$= \nabla \cdot \left[\left\{ \frac{\partial}{\partial x} (\nabla \cdot \vec{v}) - (\nabla^2 \vec{v})_1 \right\} \hat{i} \right]$$

$$= \nabla \cdot \left[\left\{ \frac{\partial}{\partial x} (\nabla \cdot \vec{v}) \right\} \hat{i} \right] - \nabla^2 \vec{v}_1 \hat{i}$$

$$= \nabla \cdot (\nabla \cdot \vec{v}) - \nabla^2 \vec{v}_1.$$

$$\therefore \boxed{\nabla \times (\nabla \times \vec{v}) = \text{grad}(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}} \text{ proved}$$

8(b) Evaluate the line integral $\int_C -y^3 dx + x^3 dy + z^3 dz$ using Stoke's theorem. Here 'C' is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. The orientation on 'C' corresponds to counterclockwise motion in the x-y plane?

Sol.

$$\text{Let us evaluate } \int_C (-y^3 dx + x^3 dy + z^3 dz)$$

by using Stokes theorem.

Let us suppose that 'S' is the part of the plane cut by the cylinder.

The curve 'C' is oriented counter-clockwise when viewed from the end of the normal vector \hat{n} .

$$\therefore \hat{n} = \frac{i + j + k}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$$

Let us apply Stokes theorem

$$\iint_S (-y^3 dx + x^3 dy + z^3 dz) = \iint_S (\nabla \times F) \cdot \hat{n} dS$$

$$\text{Let } F = P i + Q j + R k \quad (1)$$

$$\text{where } P = -y^3, Q = x^3, R = z^3$$

$$\therefore \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & z^3 \end{vmatrix}$$

$$= i [0] - j [0] + k [3x^2 + 3y^2]$$

$$= 3(x^2 + y^2)k$$

we have

$$(\nabla \times F) \cdot \hat{n} = \frac{1}{\sqrt{3}} (x^{\nu} + y^{\nu})$$

$$= \sqrt{3} (x^{\nu} + y^{\nu})$$

$$\begin{aligned} \therefore \textcircled{1} &= \oint_C (y^3 dx + x^3 dy + z^3 dz) = \iint_S \sqrt{3} (x^{\nu} + y^{\nu}) ds \\ &= \sqrt{3} \iint_S (x^{\nu} + y^{\nu}) ds \\ &= \sqrt{3} \iint_S ds \quad (\because x^{\nu} + y^{\nu} = 1) \end{aligned}$$

The projection of the surface's onto the xy -plane is circle $x^{\nu} + y^{\nu} = 1$ of radius 1.

\therefore Representing the equation of the plane in the form

$$z = 1 - x - y \text{ and using formula}$$

$$\iint_S ds = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

$$= \iint_R \sqrt{1 + (-1)^2 + (-1)^2} dxdy$$

$$= \sqrt{3} \iint_R dxdy$$

$$= \sqrt{3} \pi (1)^2$$

$$\therefore \textcircled{2} = \oint_C (-y^3 dx + x^3 dy + z^3 dz) = \frac{\sqrt{3}}{7} \pi (\sqrt{3} \pi)$$

8(c) Let $\vec{F} = xy^2\hat{i} + (y+x)\hat{j}$. Integrate $(\nabla \times \vec{F}) \cdot \hat{k}$ over the region in the first quadrant bounded by the curves $y=x^2$, and $y=x$ using Green's Theorem?

Sol: The given equation.

$$\vec{F} = xy^2\hat{i} + (y+x)\hat{j} = P\hat{i} + Q\hat{j}$$

Now, $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x+y & 0 \end{vmatrix}$

$$\vec{\nabla} \times \vec{F} = -\frac{\partial}{\partial z}(x+y)\hat{i} - j \frac{\partial}{\partial z}(xy^2) + \hat{k}\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}xy^2\right]$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

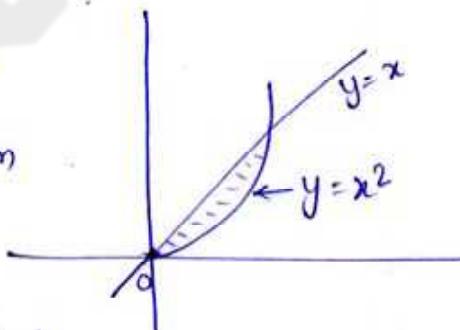
$$\vec{\nabla} \times \vec{F} = 2\hat{k}[1 - 2xy]$$

$$\therefore (\vec{\nabla} \times \vec{F}) \cdot \hat{k} = [1 - 2xy]$$

NOW; Using Green's Theorem

$$\iint_R (1 - 2xy) dx dy$$

where R be the shaded region.



$$\begin{aligned} I &= \iint_R (1 - 2xy) dx dy = \int_0^1 \left[y - xy^2 \right]_{x^2}^x dx \\ &= \int_0^1 [x - x^2 - x(x^2 - x^4)] dx \\ &= \int_0^1 [x - x^2 - x^3 + x^5] dx = \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^6}{6} \right]_0^1 \\ &= \left[\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \right] = \frac{7}{60} \end{aligned}$$

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8(d) find $f(y)$ such that $(2xe^y + 3y^2)dy + (3x^2 + f(y))dx = 0$ is exact and hence solve.

Sol: Given; $(2xe^y + 3y^2)dy + (3x^2 + f(y))dx = 0$

Given $Mdx + Ndy = 0$

and also this is exact

$$\text{i.e } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \quad \dots \quad (1)$$

$$\frac{\partial N}{\partial x} = 2e^y \quad ; \quad \frac{\partial M}{\partial y} = f'(y) \quad \dots \quad (2)$$

Hence; from (1) & (2)

$$f'(y) = 2e^y.$$

$$f(y) = 2e^y$$

Thus, the given eqn transforms to

$$[3x^2 + 2e^y] dx + [2xe^y + 3y^2] dy = 0$$

Then, the given solution is

$$\int M dx + \int N dy = \int \text{terms in } N \text{ not containing } x$$

taking y=constant

$$\Rightarrow x^3 + 2xe^y + y^3 = C$$

$$\underline{x^3 + 2xe^y + y^3} = \underline{C}$$