Ex. 12. *Solve graphically the following linear programming problem.*

$$\text{Min. } Z = 3x_1 + 5x_2$$

Rectangular Snip

$$\text{s.t. } -3x_1 + 4x_2 \le 12$$

$$2x_1 - x_2 \ge -2$$

$$2x_1 + 3x_2 \ge 12$$

$$x_1 \le 4,\ x_2 \ge 2,$$

$$x_1, x_2 \ge 0.$$

[Meerut 88]

Sol. Step one. First consider the constraints as equalities.

$$-3x_1 + 4x_2 = 12$$

$$2x_1 - x_2 = -2$$

$$2x_1 + 3x_2 = 12$$

$$x_1 = 4,\ x_2 = 2.$$

Step two. Here draw lines in two dimensional plane.

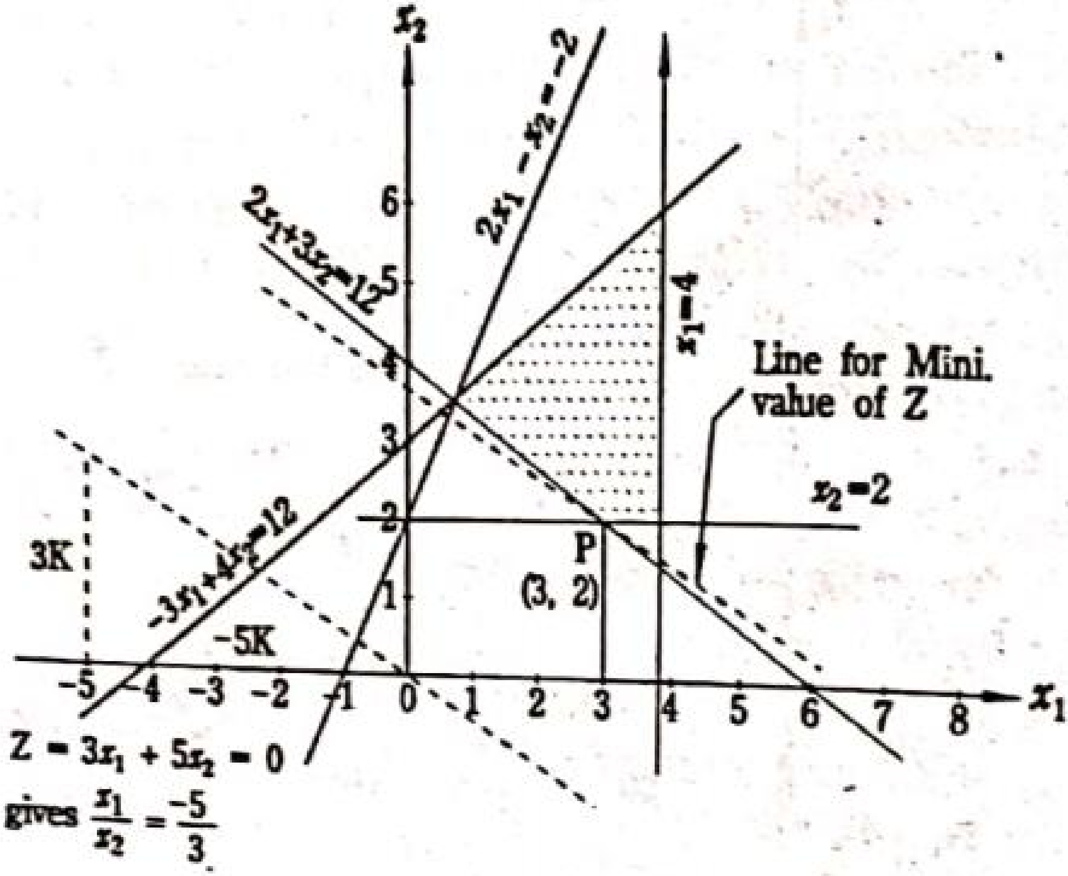

Fig. 1·5

Step three. The shaded region in fig. 1·5 is the permissible region for values of x_1 and x_2.

Step four. $Z = 3x_1 + 5x_2 = 0$ gives $\frac{x_1}{x_2} = \frac{-5}{3}$.

Draw this line through O (dotted line) and continue drawing lines parallel to it till we reach the point P of the permissible region which is nearest to the region.

$\therefore$ Z is min. at P (3, 2) which is the point of intersection of lines $x_2 = 2$ and $2x_1 + 3x_2 = 12$.

$\therefore$ Z is Min. when $x_1 = 3, x_2 = 2$

and mini. $Z = 3 \times 3 + 5 \times 2 = 19$

Note. In this problem we ...

Ex. 5. *Using simplex Algorithm solve the problem*

$$\text{Max. } Z = 2x_1 + 5x_2 + 7x_3$$

subject to
$$3x_1 + 2x_2 + 4x_3 \leq 100$$
$$x_1 + 4x_2 + 2x_3 \leq 100$$
$$x_1 + x_2 + 3x_3 \leq 100$$
$$x_1, x_2, x_3 \geq 0$$

[Agra 86]

Sol. The equations obtained by introducing slack variables x_4, x_5, x_6 are as follows :

$$x_1 + 4x_2 + 2x_3 \qquad + x_5 \qquad = 100$$
$$x_1 \quad + x_2 + 3x_3 \qquad + x_6 = 100$$

Taking $x_1 = 0, x_2 = 0, x_3 = 0$ we get $x_4 = 100, x_5 = 100, x_6 = 100$ which is starting B.F.S.

All computation work is done in the following table 3·11.

Table 3·11

B	C_B	c_j / X_B	2 / Y_1	5 / Y_2	7 / Y_3	0 / Y_4	0 / Y_5	0 / Y_6	Mini. Ratio X_B / Y_3
Y_4	0	100	3	2	$\boxed{4}$	1	0	0	25(Mini-)
Y_5	0	100	1	4	2	0	1	0	$\longrightarrow$ 50
Y_6	0	100	1	1	3	0	0	1	100/3
$Z' = C_B X_B = 0$		Δ_j	2	5	7 ↑	0 ↓	0	0	X_3 / Y_2
Y_3	0	25	3/4	1/2	1	1/4	0	0	50
Y_5	0	50	– 1/2	$\boxed{3}$	0	– 1/2	1	0	50/3 (Min.) $\longrightarrow$
Y_6	0	25	– 5/4	– 1/2	0	– 3/4	0	1	Neg.
$Z = C_B X_B = 175$		Δ_j	– 13/4	3/2 ↑	0	– 7/4	0 ↓	0	
Y_3	7	50/3	5/6	0	1	1/3	–1/6	0	
Y_2	5	50/3	–1/6	1	0	–1/6	1/3	0	
Y_6	0	100/3	–4/3	0	0	– 5/6	1/6	1	
$Z = C_B X_B = 200$		Δ_j	– 3	0	0	– 3/2	– 1/2	0	

Since all Δ_j's are zero or negative, so the solution is optimal.

$\therefore$ Optimal solution is $x_1 = 0, x_2 = 50/3, x_3 = 50/3$ and Max. $Z = 200$.

Problem having no Feasible Solution

Ex. 9. *Solve the L.P.P.*

$$\text{Max. } Z = -x_1 - x_2$$

$$\text{s.t. } \quad 3x_1 + 2x_2 \geq 30$$

$$-2x_1 + 3x_2 \le -30$$
$$x_1 + x_2 \le 5.$$
$$x_1, x_2 \ge 0.$$

Sol. Multiplying second constraint by -1 (to make the right hand side positive), and adding slack and surplus variables, the given problem reduces to the form.

Max. $Z = -x_1 - x_2 + 0.x_3 + 0.x_4 + 0.x_5$

s.t.
$$3x_1 + 2x_2 - x_3 = 30$$
$$2x_1 - 3x_2 - x_4 = 30$$
$$x_1 + x_2 + x_5 = 5$$

In order to get a unit matrix I_3 we have to add two artificial variables x_6 and x_7 in the first two constraints. Thus assigning a large negative price vector $-M$ to the artificial variables the given problem reduces to the following form

Max. $Z = -x_1 - x_2 + 0.x_3 + 0.x_4 + 0.x_5 - Mx_6 - Mx_7$
$$3x_1 + 2x_2 - x_3 + 0.x_4 + 0.x_5 + x_6 + 0.x_7 = 30$$
$$2x_1 - 3x_2 + 0.x_3 - x_4 + 0.x_5 + 0.x_6 + x_7 = 30$$
$$x_1 + x_2 + 0.x_3 + 0.x_4 + x_5 + 0.x_6 + 0.x_7 = 5$$

Taking $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$ we get $x_5 = 5, x_6 = 30, x_7 = 30$, which is the starting B.F.S.

All computation work is done in the following table 3·18.

Table 3·18

B	C_B	c_j / X_B	-1 / Y_1	-1 / Y_2	0 / Y_3	0 / Y_4	0 / Y_5	$-M$ / A_1	$-M$ / A_2	Mini Ratio X_B/Y_1
A_1	$-M$	30	3	2	-1	0	0	1	0	10.
A_2	$-M$	30	2	-3	0	-1	0	0	1	15
A_6	0	5	$\boxed{1}$	1	0	0	1	0	0	5 (Mini)
$Z = C_B X_B = -60M$		Δ_j	$5M-1$ ↑	$-M$	$-M$	$-M$	0 ↓	0	0	
A_1	$-M$	15	0	-1	-1	0	-3	1	0	
A_2	$-M$	20	0	-5	0	-1	-2	0	1	
Y_1	-1	5	1	1	0	0	1	0	0	
$Z = 35M - 5$		Δ_j	0	$-6M$	$-M$	$-M$	$-5M$	0	0	

Here no $\Delta_j > 0$. Hence the optimality condition is satisfied and therefore this solution is optimal. $\therefore$ Optimal solution is

$x_1 = 5, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 15, x_7 = 20.$

Here the artificial vectors A_1, A_2 appear in the basis at positive level, which immediately indicates that the given problem has no feasible solution.

Ex. 13. Max. $Z = 2x_1 + 3x_2$

s.t.
$$-x_1 + 2x_2 \le 4$$
$$x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9, \quad x_1, x_2 \text{ unrestricted.}$$

Sol. Taking $x_1 = x_1' - x_1''$, $x_2 = x_2' - x_2''$ s.t.

$x_1', x_1'', x_2', x_2'' \geq 0$ and introducing the slack variables x_3, x_4 and x_5 the equations can be written as

$$\text{Max.} \quad Z = 2x_1' - 2x_1'' + 3x_2' - 3x_2''$$

$$\begin{aligned} -x_1' + x_1'' + 2x_2' - 2x_2'' + x_3 \qquad\qquad &= 4 \\ x_1' - x_1'' + x_2' - x_2'' \qquad + x_4 \qquad &= 6 \\ x_1' - x_1' + 3x_2' - 3x_2'' \qquad\qquad + x_5 &= 9 \end{aligned}$$

Taking $x_1' = 0 = x_1'' = x_2' = x_2''$, we get $x_3 = 4, x_4 = 6, x_5 = 9$ which is the starting B.F.S.

All computation work is done is the following table.

Table 3·22

B	C_B	C_j / X_B	2 / Y_1'	−2 / Y_1''	3 / Y_2'	−3 / Y_2''	0 / Y_3	0 / Y_4	0 / Y_5	Mini Ratio X_B/Y_3'
Y_3	0	4	−1	1	$\boxed{2}$	−2	1	0	0	2 (Mini) →
Y_4	0	6	1	−1	1	−1	0	1	0	6
Y_5	0	9	1	−1	3	−3	0	0	1	3
$Z = C_B X_B = 0$		Δ_j	2	−2	3 ↑	−3	0 ↓	0	0	X_B/Y_1'
Y_2'	3	2	−1/2	1/2	1	−1	1/2	0	0	Neg.
Y_2	3	4	3/2	−3/2	0	0	−1/2	1	0	8/3
Y_3	3	3	$\boxed{5/2}$	−5/2	0	0	−3/2	0	1	6/5 (Mini) →
$Z = C_B X_B = 6$		Δ_j	7/2 ↑	−7/2	0	0	−3/2	0	0 ↓	X_B/Y_3
Y_2'	3	13/5	0	0	1	−1	1/5	0	1/5	13
Y_4	0	11\5	0	0	0	0	$\boxed{2/5}$	1	−3/5	11/2(Mini) →
Y_1'	2	6/5	1	−1	0	0	$\boxed{-3/5}$	0	2/5	Neg.
$Z = C_B X_B = 51/5$		Δ_j	0	0	0	0	3/5 ↑	0 ↓	−7/5	
Y_2'	3	3/2	0	0	1	−1	0	−1/2	1/2	
Y_3	0	11/2	0	0	0	0	1	5/2	−3/2	
Y_1'	2	9/2	1	−1	0	0	0	3/2	−1/2	
$Z = C_B X_B = 27/2$		Δ_j	0	0	0	0	0	−3/2	−1/2	

Since no $\Delta_j > 0$, the optimal solution to the given L.P.P. is

$x_1' = 9/2, x_1'' = 0, x_2' = 3/2, x_2'' = 0$ and Max. $Z = 27/2.$

i.e. $x_1 = x_1' - x_1'' = 9/2, x_2 = x_2' - x_2'' = 3/2$

Max. $Z = 27/2.$

Ex. 4. *Find the dual of the following L.P.P.*

$$\text{Min} \quad Z = x_1 + x_2 + x_3$$

$$\text{s.t.} \quad x_1 - 3x_2 + 4x_3 = 5$$

$$x_1 - 2x_2 \le 3$$

$$2x_2 - x_3 \ge 4$$

$x_1, x_2 \ge 0, x_3$ *is unrestricted in sign.* **[Meerut 87, 90 (TDC), 94]**

Sol. First we shall write the given problem in the standard primal form as follows.

(i) It is minimization problem, so all the constraints must contain the sign $\ge$.

(ii) The variable x_3 is unrestricted in sign.

$\therefore$ We write $x_3 = x_3' - x_3''$ where $x_3', x_3'' \ge 0$

(iii) The first constraint (equation) is equivalent to

$$x_1 - 3x_2 + 4(x_3' - x_3'') \ge 5$$

and

$$x_1 - 3x_2 + 4(x_3' - x_3'') \le 5$$

The second can be written as

$$-x_1 + 3x_2 - 4x_3' + 4x_3'' \ge -5.$$

(iv) Multiplying second constraint by -1, we have

$$-x_1 + 2x_2 + 0.(x_3' - x_3'') \ge -3$$

Thus the given problem in the standard primal form is

$$\text{Min.} \quad Z = x_1 + x_2 + x_3' - x_3'' = (1, 1, 1, -1)\,[x_1, x_2, x_3', x_3'']$$

$$= \mathbf{c}\mathbf{x}$$

$$\text{s.t.} \quad x_1 - 3x_2 + 4x_3' - 4x_3'' \ge 5$$

$$-x_1 + 3x_2 - 4x_3' + 4x_3'' \ge -5$$

$$-x_1 + 2x_2 + 0.x_3' - 0.x_3'' \ge -3$$

$$0.x_1 + 2x_2 - x_3' + x_3'' \ge 4$$

or

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ -1 & 3 & -4 & 4 \\ -1 & 2 & 0 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_3'' \end{bmatrix} \ge \begin{bmatrix} 5 \\ -5 \\ -3 \\ 4 \end{bmatrix}$$

or

$$A\mathbf{x} \ge \mathbf{b}$$

and

$$x_1, x_2, x_3', x_3'' \ge 0.$$

$\therefore$ The dual of the given problem is

Max. $Z_D = \mathbf{b'w} = (5, -5, -3, 4)\,[w_1', w_1'', w_2, w_3]$

$$= 5w_1' - 5w_1'' - 3w_2 + 4w_3$$

s.t. $A'\mathbf{w} \le \mathbf{c}'$

or
$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 3 & 2 & 2 \\ 4 & -4 & 0 & -1 \\ -4 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1' \\ w_1'' \\ w_2 \\ w_3 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

or
$$w_1' - w_1'' - w_2 + 0.w_3 \le 1$$
$$-3w_1' + 3w_1'' + 2w_2 + 2w_3 \le 1$$
$$4w_1' - 4w_1'' + 0.w_2 - w_3 \le 1$$
$$-4w_1' + 4w_1'' + 0.w_2 + w_3 \le -1$$

and
$$w_1', w_1'', w_2, w_3 \ge 0.$$

Writing $w_1' - w_1'' = w_1$, the dual problem is

Max. $Z_D = 5w_1 - 3w_2 + 4w_3$

s.t.
$$w_1 - w_2 \le 1$$
$$-3w_1 + 2w_2 + 2w_3 \le 1$$
$$4w_1 - w_3 = 1$$
$$w_2, w_3 \ge 0, w_1 \text{ unrestricted in sign.}$$

Ex. 7. *Formulate the following L. P. P. into dual problem and hence solve it :*

$$\text{Min} \quad Z_P = 3x_1 - 2x_2 + 4x_3$$

s.t.
$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$

and $x_1, x_2, x_3 \geq 0.$

Sol. In the standard primal form the given L.P.P. can be written as follows :

$$\text{Min. } Z_P = 3x_1 - 2x_2 + 4x_3$$

s.t.
$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$-7x_1 + 2x_2 + x_3 \geq -10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

and $$x_1, x_2, x_3 \geq 0.$$

The dual of the above L.P.P. can be given by

$$\text{Max. } Z_D = 7w_1 + 4w_2 - 10w_3 + 3w_4 + 2w_5$$

s.t. $$3w_1 + 6w_2 - 7w_3 + w_4 + 4w_5 \leq 3$$

$$5w_1 + w_2 + 2w_3 - 2w_4 + 7w_5 \leq -2$$

$$4w_1 + 3w_2 + w_3 + 5w_4 - 2w_5 \leq 4$$

and $$w_1, w_2, \ldots, w_5 \geq 0.$$

For the solution of this dual problem by simplex method first we multiply the second constraints by -1 so that its requirement become positive and in doing so the inequality is also reversed.

i.e., the second constraint reduce to

$$-5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 \geq 2.$$

Using slack variables w_6, w_8 in the first and third constraints and the surplus variable w_7 and artificial variable w_9 in the second constraints, and assigning a large negative price M to the arificial variable, the dual problem reduce to the following form.

$$\text{Max. } Z_D = 7w_1 + 4w_2 - 10w_3 + 3w_4 + 2w_5 + 0.w_6 + 0.w_7 + 0.w_8 - Mw_9$$

s.t. $$3w_1 + 6w_2 - 7w_3 + w_4 + 4w_5 + w_6 = 3$$

$$-5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 - w_7 + w_9 = 2$$

$$4w_1 + 3w_2 + w_3 + 5w_4 - 2w_5 + w_8 = 4$$

and $$w_1, w_2, \ldots, w_9 \geq 0.$$

Taking $w_1 = 0 = w_2 = w_3 = w_4 = w_5 = w_7$, we have $w_6 = 3, w_8 = 4, w_9 = 2$, which is the starting B.F.S. All computational work is shown in the table 6·4 given on page 201.

Since all $\Delta_j < 0$, so this solution is optimal $x_1 = 0 = x_2 = x_3$, $x_4 = 4/5, x_5 = 0, x_6 = 11/5, x_7 = 0, x_8 = 0, x_9 = 2/5$. But the artificial vector, appear in the basis at the positive level which indicates that the dual problem has no feasible solution.

Hence the given problem (primal) also has no solution.

(See § 3·10 Case III).

Table 6·4

B	C_B	c_j	7	4	-10	3	2	0	0	0	$-M$	Mini-Ratio W_B/W_4
		W_B	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	A_1	
W_6	0	3	3	6	-7	1	4	1	0	0	0	3/1
A_1	M	2	-5	-1	-2	2	-7	0	-1	0	1	2/2
W_8	6	4	4	3	1	$\boxed{5}$	-2	0	0	1	0	4/5 (Min) →
$Z_D = -2M$		Δ_j	$7-5M$	$4-M$	$-10m-2m$	$3+2M$ ↑	$2-7M$	0	$-M$	0 ↓	0	
W_6	0	11/5	11/5	27/5	$-36/5$	0	22/5	1	0	$-1/5$	0	
A_1	$-M$	2/5	$-33/5$	$-11/5$	$-12/5$	0	$-31/5$	0	-1	$-2/5$	1	
W_4	3	4/5	4/5	3/5	1/5	1	$-2/5$	0	0	1/5	0	
$Z_D = \frac{12}{5} - \frac{2}{5}M$		Δj	$\frac{23-33M}{5}$	$\frac{11-11M}{5}$	$\frac{-53-12M}{5}$	0	$\frac{16-31M}{5}$	0	$-M$	$\frac{-3-2M}{5}$	0	

Ex. 5. *An airline that operates seven days a week has timetable shown below. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover.*

Delhi-Jaipur Flight No.	Depart	Arrive	Jaipur-Delhi Flight No.	Depart	Arrive
1	7.00 A.M.	8.00 A.M.	101	8.00 A.M.	9.15 A.M.
2	8.00 A.M.	9.00 A.M.	102	8.30 A.M.	9.45 A.M.
3	1.30 P.M.	2.30 P.M.	103	12.00 Noon	1.15 P.M.
4	6.30 P.M.	7.30 P.M.	104	2.30 P.M.	6.45 P.M.

For each pair also mention the town where the crew should be based.

Sol.

Step 1. First we construct the table for layover times between flights when crew is based at Delhi. (so that they start from and come back to Delhi with minimum stay at jaipur)

Since the crew must have a minimum layover of 5 hours between flights, the layover time between flights 1 and 101 will be 24 hours. Also the layover times between flights 1 and 102, flights. 1 and 103, flights 1 and 104 are 24·5 hours, 28 hours, 9·5 hours. Similarly the layover times between other pair of flights may be calculated which are shown in table 10·21.

Table 10·21

Layover times when crew based at Delhi

Flights → ↓	101	102	103	104
1	24	24.5	28	9.5
2	23	23.5	27	8.5
3	17.5	18	21.5	27
4	12.5	13	16.5	22

Since the plane arrives at Delhi at 9·15 A.M. by flight No. 101 and will depart to Jaipur at 7·00 A.M. by flight No. 1 after 21·75 hours. Therefore, layover times between pair of flights No. 101 and 1 is 21·75 hours. Similarly, the layover times between other pairs of fights may be calculated.

The layover times between the pair of flights when the crew is based at Jaipur (so that they start from and come back to Jaipur with minimum stay at Delhi) are shown in table 10·22 or 10·23.

Table 10·22

Layover times when crew based at Jaipur

Flights → ↓	1	2	3	4
101	21.75	22.75	28.25	9.25
102	21.25	22.75	27.75	8.75
103	17.75	18.75	24.25	5.25
104	12.25	13.25	18.75	23.75

Table 10·23

Flights → ↓	101	102	103	104
1	21.75	21.25	17.75	12.25
2	22.75	22.25	18.75	13.25
3	28.25	27.75	24.25	18.75
4	9.25	8.75	5.25	23.75

Step 2. To avoid the fractions we consider either the layovers times in terms of quarter hour as one unit of time or the layovers times for four weeks. Thus multiplying the matrices (table 10·21 and 10·23) by 4, the modified matrices are as follows. (table 10·24 and 10·25).

Table 10·24 (Crew based at Delhi)

Flights → ↓	101	102	103	104
1	96	98	112	38
2	92	94	108	34
3	70	72	86	108
4	50	52	66	88

Table 10·25 (Crew based at Jaipur)

Flights → ↓	101	102	103	104
1	87	85	71	49
2	91	89	75	53
3	113	111	97	75
4	37	35	21	95

Step 3. Now we combine the tables 10·24 and 10·25, choosing the base which gives a lesser time for each pairing. The layover times

marked with '*' denote the crue based at Jaipur. Otherwise, the crew is based at Delhi. Thus we get the following table :

Table 10·26 (Minimum layover times table)

87*	85*	71*	38
91*	86*	75*	34
70	72	86	75*
37*	35*	21*	88

Step 4. Subtracting the smallest element of each row from every element of the corresponding row and then subtracting the smallest element of each column from every element of the corresponding column, we get the following matrix (table 10·27)

Table 10·27

				L3	
	49*	45*	33*	$\boxed{0}$	✓③
	57*	53*	41*	$\cancel{0}$	✓①
	$\boxed{0}$	$\cancel{0}$	16	5*	L1
	16*	12*	$\boxed{0}$*	0	L2
				✓②	

Step 5. Giving the zero assignment we find the there is no assignment in row 2 and column 2, so we draw minimum number of lines to cover all the zeros as shown in table 10·27.

Step 6. Now subtracting the smallest uncovered element 33 from all uncovered elements and adding it to the elements which lies at the intersection of lines and leaving other elements as usual the matrix obtained is as follows : (table 10·28).

Table 10·28

			L2	L3	
	16*	12*	0*	0	✓①
	24*	20*	8*	$\boxed{0}$	✓⑤
	$\boxed{0}$	$\cancel{0}$	16	38*	L1
	16*	12*	$\boxed{0}$*	100	④
			✓②	✓③	

Step 7. Giving the zero assignments we can find that there is no assignments in row 1 and column 2, so we again draw minimum number of lines to cover all the zeros as shown in table 10·28.

Step 8. Proceeding again as in Step 6, the final matrix obtained is as follows (table 10·29).

Table 10·29

	101	102	103	104
1	4•	0•	0•	0
2	12•	8•	8•	0
3	0	0	28	50•
4	4•	0•	0•	100

Step 9. Giving the zero assignments, we get the following tables :

Table 10·30

	101	102	103	104
1	4•	$\boxed{0}$•	$\cancel{0}$•	$\cancel{0}$
2	12•	8•	8•	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	28	50•
4	4•	$\cancel{0}$•	$\boxed{0}$•	100

Table 10·31

	101	102	103	104
1	4•	$\cancel{0}$•	$\boxed{0}$•	$\cancel{0}$
2	12•	8•	8•	$\boxed{0}$
3	$\boxed{0}$	$\cancel{0}$	28	50•
4	4•	$\boxed{0}$•	$\cancel{0}$•	100

Two optimal assignments (tables 10·30 and 10·31) :

(i) 1 → 102 (Crew at Jaipur), 2 → 104 (Crew at Delhi)
3 → 101 (Crew at Delhi), 4 → 103 (Crew at Jaipur)

(ii) 1 → 103 (Crew at Jaipur) 2 → 104 (Crew at Delhi)
3 → 101 (Crew at Delhi), 4 → 102 (Crew at Jaipur),

In both the cases minimum layover times is 210 hours for four weeks *i.e.*, 52 hours 30 minutes per week.

3 respectively.

Ex. 2. *Solve the following transportation problem*

		To			
		1	2	3	Supply
	1	2	7	4	5
From	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
Demand		7	9	18	34

Sol.

Step 1. The initial B.F.S. of the above problem (by Vogel's method) is given by table 11·12.

(For initial B.F.S. see § 11·6, method 3 on page 365).

Total transportation cost

$= 5 \times 2 + 2 \times 1 + 7 \times 4 + 2 \times 6 + 8 \times 1 + 10 \times 2$

$=$ Rs. 80

Table 11·12

				a_i
	(2) 5	(7)	(4)	5
	(3)	(3)	(1) 8	8
	(5)	(4) 7	(7)	7
	(1) 2	(6) 2	(2) 10	14
b_j	7	9	18	

Step 2. Now we determine a set of u_i and v_j s.t. for each occupied cell (r, s), $c_{rs} = u_r + v_s$.

For this we choose $u_4 = 0$ (since row 4 contains maximum number of allocations).

Since $c_{41} = 1 = u_4 + v_1$, $c_{42} = 6 = u_4 + v_2$, $c_{43} = 2 = u_4 + v_3$

$\therefore$ $v_1 = 1 - u_4 = 1$, $v_2 = 6 - u_4 = 6$, $v_3 = 2 - u_4 = 2$.

Also $c_{11} = 2 = u_1 + v_1$, $c_{23} = 1 = u_2 + v_3$, $c_{32} = 4 = u_3 + v_2$.

$\therefore$ $u_1 = 2 - v_1 = 1$, $u_2 = 1 - v_3 = -1$, $u_3 = 4 - v_2 = -2$.

Step 3. Then we find the cell evaluations $u_i + v_j$ for each unoccupied cell (i, j) and enter at the upper right corner of the corresponding unoccupied cell.

Step 4. Then we find the cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ (*i.e.*, the difference of the upper right corner entry from the upper left corner entry) for each unoccupied cell (i, j) and enter at the lower right corner of the corresponding unoccupied cell.

Thus we get the following table.

Table 11·13

				u_i
	(2) 5	(7) (7) (0)	(4) (3) (1)	1 (u_1)
	(3) (0) (3)	(3) (5) (−2*)	(1) 8	−1 (u_2)
	(5) (−1) (6)	(4) 7	(7) (0) (7)	−2 (u_3)
	(1) 2	(6) 2	(2) 10	0 (u_4)
v_j	1 (v_1)	6 (v_2)	2 (v_3)	

Step 5. Since cell evaluation $d_{22} = -2 < 0$, so the solution under test is not optimal.

Step 6. Since minimum d_{ij} is $d_{22} = -2$ (negative), so we give maximum allocation to this cell from an occupied cell and make the necessary changes in other allocation as shown in table 11·14.

Table 11·14

5		
	+2	−2 8
	7	
2	−2 2	+2 10

Step 7. The new B.F.S. (allocations in independent positions) thus obtained is shown in table 11·15 For this B.F.S. Total transportation cost

$$= 5 \times 2 + 2 \times 1 + 2 \times 3 + 7 \times 4 + 6 \times 1 + 12 \times 2$$
$$= \text{Rs. } 76,$$

which is less than that for the initial B.F.S.

Table 11·15

				a_i
	(2) 5	(7)	(4)	5
	(3)	(3) 2	(1) 6	8
	(5)	(4) 7	(7)	7
	(1) 2	(6)	(2) 12	14
b_j	7	9	18	

Step 8. Proceeding as in step 2, 3 and 4 we get the following table.

				u_i
	(2) 5	(7) (5) (2)	(4) (3) (1)	1 (u_1)
	(3) (0) (3)	(3) 2	(1) 6	−1 (u_2)
	(5) (1) (4)	(4) 7	(7) (2) (5)	0 (u_3)
	(1) 2	(6) (4) (2)	(2) 12	0 (u_4)
b_j	1	4	2	
	(v_1)	(v_2)	(v_3)	

Since all $d_{ij} > 0$, hence the B.F.S. shown by table 11·16 is an optimal solution which is also unique.

Thus the solution of the given transportation problem is

From source 1 transport 5 units to destination 1,

From source 2 transport 2 and 6 units to destination 2 and 3 respectively,

From source 3 transport 7 units to destination 2,

and From source 4 transport 2 and 12 units to destinations 1 and 3 respectively.

And the total transportation cost (optimal) = Rs.76.

Ex. 3. *Solve the following transportation problem.*

	S_1	S_2	S_3	S_4	a_i
0_1	1	2	1	4	30
0_2	3	3	2	1	50
0_3	4	2	5	9	20
b_j	20	40	30	10	100

[Meerut 88]

Sol. By 'Lowest Cost Entry' method, we get the following B.F.S. of the problem.