Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

Please note that the notes on this particular topic are incomplete. Aspirants are advised to ensure that they cover the remaining part from other sources.

Numerical Analysis

(1) 2 Types of methods for solving equations

(a) Direct (b) Iterative

(a) Direct Methods

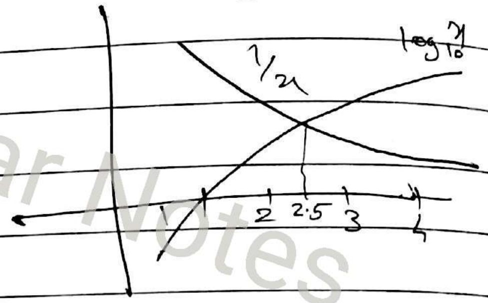
Graphical method.

Draw graph of $f(x)$ & see where it cuts x-axis.

If drawing $f(x)$ graph is difficult, split it as $f_1(x) = f_2(x)$ & see where 2 graphs intersect

i.e. find sol^n of $x \log_{10} x = 1$

i.e. $\log_{10} x = \frac{1}{x}$



(b) Iterative Method.

(b.1) Bisection Method

Find a & b s.t. $f(a)f(b) < 0$ & approximate root $\frac{a+b}{2}$

Then check $f\left(\frac{a+b}{2}\right)$ & based on its sign choose next interval.

e.g. find sol^n of $x^3 - 9x + 1$ lying bet^n 2 & 3

n	a_n (+ve)	b_n (-ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	-5.8
1	2.5	3	2.75	-2.9
2	2.75	3	2.88	-1.03
3	2.88	3	2.94	-0.05

& so on.

b2) Regula-Falsi Method (aka method of false position)

① Let x_0, x_1 be numbers s.t. $f(x_0)f(x_1) < 0$

then $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

② If $f(x_0)f(x_2) < 0$; choose (x_0, x_2) as next interval & apply formula for x_3. O/W go for (x_2, x_1) interval.

③ Error after i^{th} iteration is $|x_{i+1} - x_i|$. Stop when error is within prescribed bounds.

In all these methods, first try to find sol^n by putting $f(x)$ in calculator first. Gives you idea if your calculations are in right directions.

b3) Secant Method (Modified Regula falsi method)

This method doesn't care about signs.

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$ is formula.

Presentation in answer sheet

x_i	$f(x_i)$	$x_{i+1} = \frac{x_i' (f(x_i)) - x_i f(x_i')}{f(x_i) - f(x_i')}$

again check $|x_{i+1} - x_i|$ if error is prescribed.

Tip :- If $f(x) = x^3 + x^2 - 3x - 3$ then put eq. $ANS^3 + ANS^2 - 3ANS - 3$ in calculator & also put eq. for x_{i+1} in terms of ANS where ANS will be $f(x_i)$

THIS SAVES TIME!

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6.4) Newton Raphson Method

By basic taylor series $f(x_0+h) = f(x_0) + h f'(x_0)$

& ~~be~~ if x_0+h is root then $f(x_0+h) = 0$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ is the newton-raphson formula

e.g. finding value of $\frac{1}{\sqrt{N}}$ using Newton-raphson

let $x = \frac{1}{\sqrt{N}}$ $\therefore$ $\cancel{\frac{1}{x^2}}$ $x^2 = \frac{1}{N}$ let $f(x) = x^2 - \frac{1}{N}$

$\therefore f'(x) = 2x$

now, $x_{n+1} = x_n - \frac{x_n^2 - \frac{1}{N}}{2x_n} = \frac{x_n^2 + \frac{1}{N}}{2x_n}$

So just put $\frac{ANS^2 + 1/N}{2\,ANS}$ in calc. & find $x_1, x_2, x_3 \ldots$

for reasonable starting approximation x_0, find actual root in calc. first & take closest integer.

Newton-Raphson fails when we get $f'(x_i) = 0$

2) Order of Error

Let actual root be α & $\epsilon_{n+1} = x_{n+1} - \alpha$

& $\epsilon_n = x_n - \alpha$

then if $|\epsilon_{n+1}| \leq \lambda |\epsilon_n|^p$ then we say error is of order p. Higher the p; faster the iterations will converge to actual value.

λ is called asymptotic error constant.

If $P=1$ then obv. we want $\lambda < 1$.

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(3) Convergence of bisection method.
If we have been asked for iterations with given error ϵ;
we want N s.t. $\epsilon > \frac{(b-a)}{2^N}$. Quite intuitive.

Secant Method has order of error 1.618 i.e. $\frac{1+\sqrt{5}}{2}$

Newton raphson method has order of error 2
i.e. $|\epsilon_{n+1}| \leq P \cdot \epsilon_n^2$
But if α is a multiple root i.e. $f'(\alpha)=0$, then this is linear relationship.

e.g. $f(x) = (x-2)^4$ starting with 2.1 find 5 iterations & get order of convergence.
→ We get $\epsilon_n = x_n - 2$ (luckily we know actual root here)
We get $\epsilon_1 = \frac{3}{4}\epsilon_0$, $\epsilon_2 = \frac{3}{4}\epsilon_1$ ∴ linear convergence.

So don't worry after seeing convergence related questions, you can solve them.

classmate

Date
Page

Solving simultaneous linear equations

① Direction method for solving lin. eq.

ⓐ Forward Substitution Method.

We have lower triangular matrix here.

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ | & \text{———} & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \end{bmatrix}$$

$\therefore x_1 = b_1/a_{11}$

$x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}$

& so on. Trivial

ⓑ Backward Substitution method

We have Upper triangular matrix here. Rest same as above.

ⓒ GAUSSIAN ELIMINATION

Reduce given eq. to upper-triangular form & solve.

i.e.

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

This reduces after 2 row operations to

$$\begin{bmatrix} 1 & 3/2 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/2 \\ 7/2 \\ -15 \end{bmatrix}$$ & solve.

a_{11}, a_{22}, a_{33} diagonal elements are pivot elements.

You don't have to always convert pivot element to 1 if it is cumbersome.

Ⓓ GAUSS ELIMINATION WITH PARTIAL PIVOTING

If in row operation some pivot element becomes zero, we have a problem.

So in partial pivot we first inspect 1^{st} column & bring biggest element to the top. After doing necessary row operations of first column, we inspect last $(n-1)$

elements of column 2 for biggest guy & bring him to 2^{nd} row & so on.

e.g. Solve following with partial pivoting gauss elimination

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{array}\right]$$ & solve.

Pivoting helps in getting better answer when original pivot elements are very small. $\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -0.436 & 3.018 \end{array}\right]$ & so on.

Complete pivoting means finding largest guy in whole matrix & then moving both rows & columns. But this involves shifting variables hence is rarely used.

E) GAUSS- JORDAN ELIMINATION METHOD

(with partial pivoting)

Here we transform matrix A to diagonal matrix D i.e. identity.
so On completion of Gauss-Jordan

$$[A|B] \longrightarrow [I|D]$$. So we get sol^n D.

e.g. Solve: $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 3 & 4 \end{array}\right]$$ (We also do partial pivoting)

$$\rightarrow \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 0 & 1/4 & 5/4 & -1/2 \\ 0 & 11/4 & 15/4 & -1/2 \end{array}\right] \xrightarrow{\text{with pivoting}} \left[\begin{array}{ccc|c} 4 & 0 & -56/11 & 72/11 \\ 0 & 11/4 & 15/4 & -1/2 \\ 0 & 0 & 10/11 & -5/11 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

(F) Using Gauss Jordan for finding matrix inverse, then with partial

→ We augment Identity matrix, then with partial pivoting turn first matrix into identity matrix & new matrix adjacent to it is the inverse. Partial pivoting allowed

$$[A \mid I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I \mid A^{-1}]$$

e.g. find inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \\ 1 & 11/4 & 15/4 & 0 & -3/4 & 1 \end{array}\right] \xrightarrow{\text{after some steps}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] = [I \mid A^{-1}]$$

(G) GAUSSIAN ELIMINATION METHOD for finding matrix inverse.

Here also we augment identity matrix $[A \mid I]$.
but first we convert A to an upper triangular matrix & then convert it to I.

In Gauss-Jordan we eliminate middle step of upper triangular.
Remember this difference in exam.

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(H) Iterative Method

(2) Gauss - Seidel iteration Method

We use an improved component as soon as it becomes available.

following example clarifies the method.

e.g. Perform 4 iteration to solve using Gauss-Siedel with initial approximation $x^{(0)} = 0$

$$\begin{bmatrix} -8 & 1 & 1 \\ 1 & -5 & 1 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 7 \end{bmatrix}$$

$-8x_1 + x_2 + x_3 = 1$

$x_1 - 5x_2 + x_3 = 16$

$x_1 + x_2 - 4x_3 = 7$

$\therefore$ We approximate $x_1 = -\frac{1}{8}(1 - x_2 - x_3)$, $x_2 = -\frac{1}{5}(16 - x_1 - x_3)$

& $x_3 = -\frac{1}{4}(7 - x_1 - x_2)$

Since we want to use improved elt. as soon as it is available. So, we use

$$x_1^{(k+1)} = -\frac{1}{8}\left(1 - x_2^{(k)} - x_3^{(k)}\right)$$

$$x_2^{(k+1)} = -\frac{1}{5}\left(16 - x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = -\frac{1}{4}\left(7 - x_1^{(k+1)} - x_2^{(k+1)}\right)$$

after $k = 3$ we get $x_1 = -0.9966$, $x_2 = -3.9973$, $x_3 = -2.9985$ which is very close to actual answer $(-1, -4, -3)$.

My Own Discovery :- Sometimes above process might not converge. For Gauss-Seidel convergence we need diagonal heavy matrix. So try to arrange given equation in diagonal heavy manner for greater chance of convergence.

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$$4x - y + 8z = 26$$
$$5x + 2y - z = 6$$
$$x - 10y + 2z = -13$$

Putting $x = \frac{1}{4}(26 + y - 8z)$ $y = \frac{1}{2}(6 - 5x + z)$ $z = \frac{1}{2}(13 - x + 10y)$ doesn't converge & approximations keep getting absurdly large.

Instead put order with diagonal heavy elements

$$x = \frac{1}{5}(6 - 2y + z)$$

$$y = -\frac{1}{10}(-13 - x - 2z)$$

$$z = \frac{1}{8}(26 - 4x + y)$$

this converges rapidly in 2 iterations to actual answer i.e. (1, 2, 3).

INTERPOLATION

① Forward difference

$\Delta y_0 = y_1 - y_0$ $\Delta y_1 = y_2 - y_1$

upper triangle

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

example of forward difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.003			
		0.064		
0.3	0.067		0.017	
		0.081		0.002
0.5	0.148		0.019	
		0.100		
0.7	0.248			

② Backward Difference

$\nabla y_1 = y_1 - y_0$ $\therefore$ $\nabla y_1 = \Delta y_0$ & $\nabla^2 y_1 = \nabla y_1 - \nabla y_0$

ulta triangle

Backward Difference table is same as fwd diff. table

③ Shift operator E

$E f(x) = f(x+h)$ $E^n f(x) = f(x+nh)$ $E^{-1} f(x) = f(x-h)$

Some questions based on relations betn E, Δ, ∇ can be handled easily with formula $\Delta = E - 1$ & $\nabla = 1 - E^{-1}$

④ Polynomial function values with x spaced at equal interval

→ If $f(x)$ is polynomial of degree n then $\Delta^n y_i$ are constant for all x_i if they are equally spaced.

The converse is true i.e. if $\Delta^n y_i$ are same & x_i are equally spaced, it is polynomial function (!! may or may not ??)

e.g. Following is table of degree 5 polynomial & f(3) is wrong.
Find correct value

x	0	1	2	3	4	5	6
y	1	2	33	254	1054	3126	7777

Put f(3) as $254 + \epsilon$ & then draw diff. table & put $\Delta^5 y$ as constant.

(5) e.g. In questions of finding ~~m~~ missing terms do as follows (a) If we know values for m equally spaced x_i then we can fit polynomial of degree $m-1$ & hence $\Delta^m y$ will be zero.

x	0	1	2	3	4	5
y	0	–	8	15	–	35

e.g. find missing term

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0				
		x			
1	x		$8-2x$		
		$8-x$		$3x-9$	
2	8		$x-1$		$y-4x-12$
		7		$y-x-21$	
3	15		$y-22$		$93-4y+x$
		$y-15$		$72-3y$	
4	y		$50-2y$		
		$35-y$			
5	35				

$\therefore$ $y-4x-12=0$ & $93-4y+x=0$ gives $x=3$ & $y=24$

Easier than remembering any of Newton's formula!!

(6) Newton's Forward Interpolation Formula

Let $y_0, y_1, \dots y_n$ be values of $x_0, x_1, \dots x_n$ which are equally spaced by distance h.

Say we want to find value of f at x.

Let $u = \frac{x-x_0}{h}$ then

$$f(x) = y_0 + u\frac{\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)\dots(u-n+1)}{n!}\Delta^n y_0$$

Similar to taylor formula mon.

terms are like $u_{C_r} = \frac{u(u-1)\dots(u-r+1)}{r!} = \frac{u(u-1)\dots(u-r+1)}{r!}$

$\therefore$ r^{th} terms is like $u_{C_r}\,\Delta^r y_0$.

This formula is useful when x is close to x_0.
Remember if we value of function is available for $x = 1, 2, 3, 4, 5, 6$ & we want $f(3.1)$ then choose y_0 as $f(3)$ & use $\Delta y_3, \Delta^2 y_3$ etc. Just forget about y_1 & y_2 man.

(7) In questions where marks of students are given in ranges, first find data in ranges of cumulative.

e.g.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Find no. of candidates whose score lies in 45-50.
∴ We first present table with upper limit of marks.

Upper limit	40	50	60	70	80
Cumulative no.	31	73	124	159	190

& find value for 45.

(8) Newton's Backward interpolation formula

~~Again~~ put $u = \dfrac{x - x_n}{h}$. This formula works for x close to x_n.

Then $$f(x) = y_n + \frac{u \nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \dots + \frac{u(u+1)\dots(u+n-1)}{n!}\nabla^n y_n$$

So here essentially we use last slanted row of difference table.

— — → This is used for forward formula $\Delta^r y_0$.
— — — — —
— — — —
— — → This used for backward formula $\nabla^r y_n$

(9) Error in interpolation formula

It is simply equal to last term of the formula

i.e. error = $\frac{\Delta^n y_0}{n!} u(u-1)(u-2)\dots(u-n+1)$

& for backward formula

error = $\frac{\nabla^n y_n}{n!} u(u+1)(u+2)\dots(u+n-1)$

(10) Lagrange Interpolation Formula

Newton's formula needs equal interval x. Lagrange is cool & can work with unequal interval.

Let $x_0, x_1 \dots x_n$ be given & $y_0, y_1, \dots, y_n$ be given then

$$f(x) = \sum_{i=0}^{n} \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} f(x_i)$$

intuitive man formula.

(11) Lagrange inverse formula.

$$f^{-1}(y) = \sum_{i=0}^{n} \frac{(y-y_0)(y-y_1)\dots(y-y_{i-1})(y-y_{i+1})\dots(y-y_n)}{(y_i-y_0)(y_i-y_1)\dots(y_i-y_{i-1})(y_i-y_{i+1})\dots(y_i-y_n)} x_i$$

again intuitive man.

(12) Lagrange interpolation can be used to find partial fraction.

→ e.g. $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ find value of 1, 2, 3 for $3x^2+x+1$

x	1	2	3
$3x^2+x+1$	5	15	31

then $f(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} 5 + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$

$$\therefore \frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}$$

(13) e.g. By means of lagrange's formula prove that

$$y_1 = y_3 - 0.3\,[y_5 - y_{-3}] + 0.2\,[y_{-3} - y_{-5}]$$

this simply means we have $f(3), f(-3), f(5)$ & $f(-5)$ & want to find $f(1)$. So put in lagrange expression $x=1$

$$y_n = \frac{[x-(-3)]\,(x-3)(x-5)}{(-5+3)\;(-5-3)(-5-5)}\,y_{-5} + (\quad)\,y_{-3} + (\quad)\,y_3 + (\quad)\,y_5$$

& so on.

(14) Divided Difference formula - Newton

first divided difference $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$

2^{nd} divided difference $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$

3^{rd} divided difference $[x_0, x_1, x_2, x_3] = \dfrac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$

Divided Difference table

		1^{st} DD	2^{nd} DD	3^{rd} DD
x_0	y_0			
		$[x_0, x_1]$		
x_1	y_1		$[x_0, x_1, x_2]$	
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$
x_2	y_2		$[x_1, x_2, x_3]$	
		$[x_2, x_3]$		
x_3	y_3			

Again n^{th} DD are constant for n degree polynomial
& DD are independent of order i.e. $[x_0, x_1, x_2] = [x_1, x_0, x_2]$

$$f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + \ldots + (x-x_0)(x-x_1)\ldots(x-x_n)\,[x_0, x_1, x_2, \ldots, x_n]$$

The good thing is unlike lagrange, you can afterwards add some y_{n+1} value & don't have to make major changes to formula.

Date
Page

(15) Truncation Error.

When actual function is $P(x)$ & estimation is $f(x)$ then truncation error is $P(x) - f(x) = \epsilon$

& if h is step size, we got

$$\frac{h^2}{8} \max_{a \le x \le b} |f''(x)| \geq \epsilon \quad \text{bound on truncation error.}$$

(16) linear convergence of iteration

In these questions, we check error ϵ_k & see $\frac{\epsilon_k}{\epsilon_{k+1}}$. this ratio should tend to a constant.

Then we see linear convergence.

Could be asked in newton raphson questions.

Numerical Integration

These formulae are also called quadrature formula. (we are trying to find simple best fit curve for integration)

original our curve

(1) Trapezoidal Rule

It comes after putting $n=1$ in general quadrature formula.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\dots+y_{n-1})\right].$$

Error in trapezoidal rule is of order h^2

$$E = -\frac{(b-a)}{12} h^2 y''(\bar{x}) \quad \left(\text{where } y''(x) \text{ is largest of } y''_0, y''_1, \dots, y''_n\right)$$

(2) Simpson's $\frac{1}{3}^{rd}$ Rule (used when $n=2k$ $k \in \mathbb{N}$)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\dots+y_{n-2})+4(y_1+y_3+y_5\dots)\right]$$

i.e. $\frac{h}{3}\big[(y_0+y_n) + 2(\text{sum of even ordinates}) + 4(\text{sum of odd ordinates})\big]$

Error in Simpson's $\frac{1}{3}^{rd}$ rule is of order of h^4.

$$E = -\frac{(b-a)}{180} h^4 y^{iv}(\bar{x}) \rightarrow \text{highest } 4^{th} \text{ derivative.}$$

(3) Simpson's $\frac{3}{8}^{th}$ rule → (used when $n=3k$ $k \in \mathbb{N}$)

$$\int_{x_0}^{x_0+nh} f(x) = \frac{3h}{8}\big[(y_0+y_n)+2(\text{multiples of } 3) + 3(\text{remaining terms})\big]$$

Similar to $\frac{1}{3}$ rule i.e. $\frac{h}{3}\big[(y_0+y_n) + 2(\text{multiples of } 2) + 4(\text{remaining terms})\big]$

for $\frac{3}{8}^{th}$ rule, intervals should be multiple of 3.

(quite natural that coefficient of remaining terms in $\frac{3}{8}$ rule is smaller than coefficient of remaining terms in $\frac{h}{3}$ rule $\because \frac{3h}{8} > \frac{h}{3}$)

This is also h^4 order error but simpson $\frac{1}{3}$rd rule is superior when we consider magnitude of error.

e.g. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ by taking 7 ordinates.

→ 7 Ordinates means 7 points ⇒ 6 sections.

Gaussian quadrature formula.

→ Trapezoidal rule is $\int_a^b f(x) \approx C_1 f(a) + C_2 f(b)$ $\left(C_1 = \frac{b-a}{2} = C_2\right)$
(for 2 points)

In gaussian quadrature (2 point) we find x_1, x_2, C_1, C_2 s.t.

$$\int_a^b f(x)\,dx = C_1 f(x_1) + C_2 f(x_2) \qquad x_1, x_2 \in [a,b]$$

for 3 point we approximate $\int_a^b f(x) = C_1 f(x_1) + C_2 f(x_2) + C_3 f(x_3)$ & so on.

For 2 point we assume function is approximates by degree 3 polynomial. For 3 pt. we assume degree 5 & so on.

$$\therefore \int_a^b f(x) = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

now we assume $\int_a^b f(x)\,dx = C_1 f(x_1) + C_2 f(x_2)$

$$= C_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + C_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

Comparing coefficients & solving eq. gives

$$\int_a^b f(x)\,dx = C_1 f(x_1) + C_2 f(x_2)$$

$C_1 = C_2 = \frac{b-a}{2}$ $\quad x_1 = \left(\frac{b+a}{2}\right) - \frac{1}{\sqrt{3}}\left(\frac{b-a}{2}\right)$ $\quad x_2 = \left(\frac{b+a}{2}\right) + \frac{1}{\sqrt{3}}\left(\frac{b-a}{2}\right)$

(just like trapez)

So we go to mean and get points on $\frac{1}{\sqrt{3}}$ ratio on either side.

C_1, C_2 intuitive.

So on for n points.

Solving ODE with Numerical Methods

① Using Taylor's Series

e.g. If $y(x)$ satisfies $y' = x - y^2$ & $y(0) = 1$ find $y(0.1)$ correct to four decimal places

$\rightarrow$ $y(x) = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv} + \dots$

$y_0 = 1$

$y_0' = x - y^2 = -1$

$y''(x) = 1 - 2yy' = 3$

$y'''(x) = -2yy'' - 2y'^2 = -8$

$y^{iv}(x) = -2yy''' - 6y'y'' \Rightarrow 34$

$\therefore y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5$

$\therefore y(0.1) = 0.9138$

If we truncate terms after x^4; & if we want range of x for which above series will give y correct upto 4 decimal places

$\therefore \frac{31}{20}x^5 \le 0.00005 \quad \therefore x \le 0.128$

Tip. If initial value data is available for $x = 1$ & we want to find $f(1.2)$ with $h = 0.1$ then first find $f(1.1)$, $f'(1.1)$, $f''(1.1)$ etc. & with these numbers find $f(1.2)$

② Taylor Series Method for simultaneous ODE.

$\rightarrow$ not much difference. Here we use 2 Taylor Series

e.g. Solve following eqns. for $h = 0.3$

$y' = 1 + xz \quad z' = -xy$, initial value $x = 0, y = 0, z = 1$

$\rightarrow$ $y' = 1 + xz = 1$, $y'' = xz' + z = x(-xy) + z = 1$

$y''' = x(z')^2 + 2z' = x_0(-xy)^2 + 2(xy) = 0$

$\therefore y = 0 + (0.3)(1) + \frac{(0.3)^2}{2}(1) + \frac{(0.3)^3}{6}(0) = 0.345$

& $z' = -xy = 0$

$z'' = -xy' + y = 0$

$z''' = -xy'' + 2y' = 2$

$\therefore z_1 = 1 + (0.3)(0) + \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6}(2) + \dots = 1.009$

(3) Taylor Series for 2^{nd} order ODE.
Same as in first case, find y'', y''', y^{iv} put in formula
nothing new.

(4) EULER'S METHOD
We are given $\frac{dy}{dx} = f(x, y)$ & y_0

Then $y_1 = y(x_1) = y_0 + h f(x_0, y_0)$

$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$

$\vdots$

$y_n = y(x_n) = y_{n-1} + h f(x_{n-1}, y_{n-1})$

Problem is large error accumulates.

(5) $\therefore$ MODIFIED EULER'S METHOD
We use trapezoid rule here.

iteration formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

e.g. Solve by Euler's Modified Method
for $x = 0.02$ by taking $h = 0.01$

$y' = x^2 + y$, $y(0) = 1$

$\therefore f(x, y) = x^2 + y$, $x_1 = 0.01$ & $x_2 = 0.02$

$y_1^{(0)} = y_0 + h f(x_0, y_0) = 1 + (0.01)(x_0^2 + y_0) = 1.01$

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Euler's Modified formula gives

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(0)})\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.01\right] = 1.01005$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1,y_1^{(1)})\right] \rightarrow$$ (writing formula each time helps in minimizing errors)

$$= 1.01005$$

now,

$$y_2^{(0)} = y_1 + h f(x_1,y_1) = 1.02015.$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f(x_2,y_2^{(0)})\right]$$

$$= 1.01005 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.02015)\right]$$

$$= 1.020204$$

Remember, we need $\frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(n-1)})\right]$ & not $\frac{h}{2}(y_0 + y_1^{(n-1)})$. You have to calculate $f(x_1, y_1^{(n-1)})$ each time.

(6) Runge-Kutta Methods

These methods agree with Taylor's series solution upto the term in h^r where 'r' differs from method to method & is called order of that method.

(a) Euler's method is Runga-Kutta method of first order.

(b) Modified Euler's Method = Runge-Kutta of 2^{nd} order

(c) Runge Kutta of third order $\rightarrow$ $y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)$

where $K_1 = h f(x_0,y_0)$ $\quad K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = h f(x_0 + h, y_0 + K')$ where $K' = h f(x_0 + h, y_0 + K_1)$

(d) Fourth Order Runge-Kutta Method

This is commonly known as Runge-Kutta Method.

$K_1 = hf(x_0, y_0)$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ then $y_1 = y_0 + K$

If question is for higher marks & h is not prescribed, take appropriate h.

(e) Runge Kutta Method for a system of equation

$\frac{dy}{dx} = P = f(x, y, P)$ $\frac{dP}{dx} = f_2(x, y, P)$

With initial conditions $y(x_0) = y_0$ & $y'(x_0) = y'_0 = P_0$

Considering steps for x, y, P to be h, k, l respectively.

$K_1 = hf(x_0, y_0, P_0)$ $l_1 = hf_2(x_0, y_0, P_0)$

$K_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, P_0 + \frac{l_1}{2}\right)$ $l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2} + P_0 + \frac{l_1}{2}\right)$

$K_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, P_0 + \frac{l_2}{2}\right)$ $l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, P_0 + \frac{l_2}{2}\right)$

$K_4 = hf_1(x_0 + h, y_0 + K_3, P_0 + l_3)$ $l_4 = hf_2(x_0 + h, y_0 + K_3, P_0 + l_3)$

$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$ $P_1 = P_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Remember that arguments for f_1 & f_2 are same for k_i, l_i also all terms have multiplication by h only.

Also, when y' is not explicitely mentioned as a function, then $f_1\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, P_0+\frac{l_2}{2}\right) = P_0+\frac{l_2}{2}$ obv.

But sometimes the give $y' = 1+xP$. In that case calculation of f_1 needed in each step & you can't just put $P_0+\frac{l_2}{2}$.

Normal forms in boolean algebra

1) Disjunctive normal form = minterm cannonical form = sum of products

Remember minterm = product = abc etc.
maxterm = sum = a+b+c

2) Conjunctive normal form = maxterm cannonical form = product of sums.

Any logic circuit can be represented in these 2, equivalent.

For sum of products we consider output 1 of truth table.
For product of sums, we consider output 0 of truth table.

Date
Page

Basics of Representation of Numbers

(1) Significant figures are counted from left to right Starting from left most non zero digit

Number	Significant figures	No. of significant figures
37.89	3, 7, 8, 9	4
0.00082	8, 2	2
0.000620	6, 2, 0	3
3.506×10^4	3, 5, 0, 6	4

(2) Rounding off number

If we want to round off 0.125 we do it as 0.12 because 2 is even

f rounding off of 0.135 would be 0.14 because 3 is odd. (preference to even digit at end)

So, if preceding digit is odd, add 1 to round off 5 digit, if it is even, leave unchanged.

(3) |Error| = |true value − approx. value|

|Relative error| = $\left|\dfrac{\text{error}}{\text{True value}}\right|$

Percentage error = 100× relative error

(4) 3 types of error

(a) Inherent error (b) Round off error (c) Truncation error

(a) Inherent error is already present in statement of problem before solution.

(b) Round off error − Bcoz of rounding off

(5) Floating point representation.

$x = \pm (0.d_1 d_2 \dots d_n) \times 10^m$ m is exponent.

if d_1 is non zero, it is normalized representation.

How many digits are wanted in mantissa is mentioned.

For converting to floating point, 2 methods are there.

(a) Chopped F.P. → Say we want $n = 2$

$fl(537) = 0.53 \times 10^3$

(b) Rounded F.P. → again $n = 2$

$fl(537) = 0.54 \times 10^3$

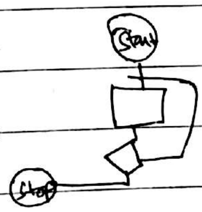
This n is called precision.

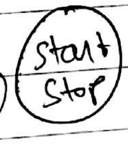
(6) Let x^* be actual no. & x its approximation then if $|x - x^*| \leq \frac{1}{2} 10^{-k}$ then we have x^* rounded to k decimal places.

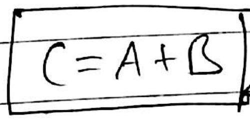
(7) Truncation Error

Sometimes functions are an infinite series but we calculate only for finite terms; the discarded terms give truncation error. e.g. Taylor series etc.

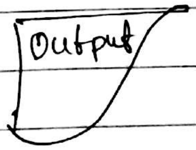
(8) Remember Algorithms means:- Step 1 : Read a, b, c
Step 2 : calculate f(a)
: etc.

Flow chart has  etc.

Flow chart symbols (a) 

(b) 

(calculations)

(c) 

(d) 

(e)