IFoS 2018

5 (a) Solve the DE
$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

$$(D^2-1)y = xe^x + \cos^2 x$$

Auxiliary Eqn: $m^2 - 1 = 0 \Rightarrow m = 1, -1$

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{1}{D^2-1}(xe^x + \cos^2 x)$$

$$= \frac{1}{D^2-1} x \cdot e^x + \frac{1}{D^2-1} \cdot \cos^2 x$$

$$= e^x \frac{1}{(D+1)^2-1} x + \frac{1}{D^2-1}\left(\frac{1+\cos 2x}{2}\right)$$

$$= e^x \frac{1}{D^2+2D} x + \frac{1}{D^2-1} \cdot \frac{1}{2} + \frac{1}{2} \frac{1}{D^2-1} \cos 2x$$

$$= e^x \cdot \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} \cdot x - \frac{1}{2}(1-D^2)^{-1} \cdot 1 + \frac{1}{2} \frac{\cos 2x}{((-4)-1)}$$

$$= e^x \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \dots\right) x - \frac{1}{2} - \frac{1}{10}\cos 2x$$

$$= e^x \left(\frac{1}{2D} - \frac{1}{4} + \frac{D}{8}\right) x - \frac{1}{2} - \frac{\cos 2x}{10}$$

$$= e^x \left(\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8}\right) - \frac{1}{2} - \frac{\cos 2x}{10}$$

Gen Sol: $y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{4}\left(x^2 - x + \frac{1}{2}\right) - \frac{\cos 2x}{10} - \frac{1}{2}$.

5(b) Solve, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \log x \quad (x>0)$

by the method of variation of parameters.

$$(D^2 - 2D + 1)y = xe^x \log x$$

Auxiliary Eqn : $m^2 - 2m + 1 = 0$

$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$y_c = (c_1 + c_2 x)e^x$$

Let $u = e^x$, $v = xe^x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & 1.e^x + xe^x \end{vmatrix}$$

$$= e^{2x}[1 + x - x] = e^{2x} \neq 0$$

Hence solutions are independent.

$$P.I. = Au + Bv$$

$$A = -\int \frac{vR}{W}\,dx$$

$$A = -\int \frac{x e^x \cdot x e^x \log x}{e^{2x}}\, dx$$

$$= -\int x^2 \log x \, dx = -\int (\log x)\, x^2 \, dx$$

$$= -\left[(\log x)\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\, dx\right] \quad \text{(by parts)}$$

$$= \frac{-x^3}{3}\log x + \frac{x^3}{9} = \frac{-1}{9}x^3(3\log x - 1)$$

$$B = \int \frac{uR}{W}\, dx$$

$$= \int \frac{e^x \cdot x e^x \log x}{e^{2x}}\, dx$$

$$= \int x \log x \, dx = \int (\log x)\, x \, dx$$

$$= (\log x)\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\, dx \quad \text{(by parts)}$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} = \frac{1}{4}x^2(2\log x - 1)$$

$$\therefore \; y_p = \frac{-e^x \cdot x^3}{9}(3\log x - 1) + \frac{x^3 \cdot e^x}{4}(2\log x - 1)$$

$$= x^3 \cdot e^x\left[\frac{1}{6}\log x - \frac{5}{36}\right]$$

Gen Sol: $y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{2x} + \frac{x^3 e^x}{36}(6\log x - 5).$$

6(a) Solve the DE.

~~6(a)~~ $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$

$$M\,dx+N\,dy=0$$

$M = y^2+2x^2y$ | $N = 2x^3-xy$

$\frac{\partial M}{\partial y} = 2y+2x^2$ | $\frac{\partial N}{\partial x} = 6x^2-y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given D.E. is not exact.

Let $x^m y^n$ be integrating factor.
Multiplying with it, we get

$$\left(x^m y^{n+2}+2x^{m+2}\cdot y^{n+1}\right)dx+\left(2x^{m+3}\cdot y^n-x^{m+1}\cdot y^{n+1}\right)dy=0$$

If this is exact, then $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ —— (A)

$$(n+2)x^m y^{n+1}+2(n+1)x^{m+2}\cdot y^n$$
$$=2(m+3)x^{m+2}\cdot y^n-(m+1)x^m\cdot y^{n+1}$$

$\therefore\ n+2=-(m+1) \Rightarrow m+n=-3$

$2(n+1)=2(m+3) \qquad m-n=-2$

$\therefore\ m=-5/2,\ n=-1/2$

$\therefore\ x^{-5/2}\cdot y^{-1/2}$ is integrating factor.

Eqn (*) becomes

$$\left(x^{-5/2} \cdot y^{3/2} + 2x^{-1/2} \cdot y^{1/2}\right)dx + \left(2x^{1/2} \cdot y^{1/2} - x^{-3/2} \cdot y^{1/2}\right)dy = 0$$

Gen solution:

$$\int M\,dx \quad + \quad \int N\,dy$$

Taking y as const $\qquad$ excluding terms of x

$$\int \left(x^{-5/2} \cdot y^{3/2} + 2x^{-1/2} \cdot y^{1/2}\right)dx + \int 0\,dy = 0.$$

y = constant

$$\left(\frac{-2}{3}\right) x^{-3/2} \cdot y^{3/2} + 2 \times \frac{2}{1} x^{1/2} \cdot y^{1/2} = c$$

$$\boxed{-\frac{2}{3} x^{-3/2} \cdot y^{3/2} + 4x^{1/2} \cdot y^{1/2} = c}$$

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7(b) solve

$$\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$$

Let $2x+3y = v$

$$\Rightarrow 2+3\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{3}\left[\frac{dv}{dx}-2\right] = \frac{2v+5}{v+4}$$

$$\frac{dv}{dx} = 2+\frac{6v+15}{v+4} = \frac{8v+23}{v+4}$$

$$\frac{v+4}{8v+23}\,dv = dx$$

$$\frac{1}{8}\left[\frac{8v+32}{8v+23}\right]dv = dx$$

$$\left(1+\frac{9}{8v+23}\right)dv = 8\,dx$$

$$\Rightarrow v+\frac{9}{8}\log(8v+23) = 8x+C$$

$$\Rightarrow (2x+3y)+\frac{9}{8}\log[8(2x+3y)+23] = 8x+C$$

$$\Rightarrow (3y-6x)+\frac{9}{8}\log(16x+24y+23) = C$$

IFoS 2018 ODE

8(b). A snowball of radius $r(t)$ melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process.

Let $\frac{dr}{dt} = k$ (uniform), density $= \rho$, fixed

$$M = \left(\frac{4}{3}\pi r^3\right)\rho \Rightarrow \frac{dM}{dt} = (4\pi\rho)r^2 \frac{dr}{dt}$$

$$\frac{dM}{dt} = 4\pi\rho\left(\frac{3M}{4\pi\rho}\right)^{2/3} \cdot k \Rightarrow$$

$$\Rightarrow \frac{dM}{dt} = k_1 M^{2/3}, \quad \text{where } k_1 = \frac{4\pi\rho \cdot 3^{2/3} \cdot k}{(4\pi\rho)^{2/3}}$$

$$M^{-2/3}\, dM = k_1\, dt$$

Integrating, $\quad 3M^{1/3} = k_1 t + k_2$

Let M_0 be initial mass of snowball

$$\therefore M(0) = M_0 \quad \text{and} \quad M(1) = \frac{M_0}{2}$$

$$\Rightarrow k_2 = 3M_0^{1/3} \quad \& \quad k_1 = 3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}$$

We want to calculate time t, when $M = 0$

i.e. $$0 = \left[3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}\right]t + 3M_0^{1/3}$$

$$t = \frac{-3}{3/2^{1/3} - 3} = \frac{-1}{2^{-1/3} - 1} = 4.85 \text{ hours}$$