IMS (INSTITUTE OF MATHEMATICAL SCIENCES)

IAS MATHEMATICS (OPT.)-2017

PAPER - II : SOLUTIONS

1(a) 2017 $\rightarrow$ Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$, $n = 1, 2, 3, \ldots$ Show that the sequence $x_1, x_2, x_3, \ldots$ is convergent.

Sol Given that $x_1 = 2$, $x_{n+1} = \sqrt{x_n + 20}$

$n = 1, 2, 3, \ldots$ —— (1)

if $n=1$: $x_2 = \sqrt{x_1 + 20} = \sqrt{2+20}$

$= \sqrt{22} > 2 = x_1$

$\therefore x_2 > x_1$

let us assume that $x_{k+1} > x_k$ —— (2)

we have

$x_{k+1} + 20 > x_k + 20$

$\Rightarrow \sqrt{x_{k+1} + 20} > \sqrt{x_k + 20}$

$\Rightarrow x_{k+2} > x_{k+1}$

$\therefore$ By Mathematical Induction,

$x_{n+1} > x_n \ \forall n \in N$

i.e. $x_n < x_{n+1} \ \forall n \in N$.

$\therefore (x_n)$ is an increasing sequence.

we have $x_1 = 2 < 5$ —— (3)

$x_2 = \sqrt{x_1 + 20} = \sqrt{2+20} = \sqrt{22} < 5$

let us assume that

$x_k < 5$ —— (4)

we have

$x_k + 20 < 20 + 5$

$\Rightarrow \sqrt{x_k + 20} < \sqrt{25} = 5$

$\Rightarrow x_{k+1} < 5$

$\therefore$ By Mathematical Induction,

$x_n < 5 \ \forall n \in N$.

$\therefore$ The given sequence (x_n) is monotonically increasing and bounded above

$\therefore$ (x_n) is convergent and converges to its least upper bound.

1(b) 2017 Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n.

Sol let G be a group of order n.

let $G_1 = A(G)$ be the group of all permutations of the set G and is of order $n!$ and degree n.

For any $a \in G$, define a map

$f_a : G \to G$ s.t $f_a(x) = ax$ — (1)

$\forall x, y \in G$ s.t $x = y \Rightarrow ax = ay$

$\Rightarrow f_a(x) = f_a(y)$

$\therefore f_a : G \to G$ is well-defined.

Again $f_a(x) = f_a(y)$ $\forall x, y \in G$

$\Rightarrow ax = ay$

$\Rightarrow x = y$ (by using cancellation laws in G)

$\therefore f_a : G \to G$ is 1-1.

Also, for any $y \in G$, $\exists\, a^{-1}y \in G$

s.t $f_a(a^{-1}y) = a(a^{-1}y) = (aa^{-1})y$
$= y$

$\therefore f_a : G \to G$ is an onto.

$\therefore f_a \in A(G)$.

let $K = \{ f_a \,/\, f_a : G \to G : f_a(x) = ax \} \subseteq A(G)$.

To p.T $K \leq A(G)$

clearly $K \neq \phi$. as $f_e \in K$.

Let $f_a, f_b \in K$

we have $(f_b f_{b^{-1}})(x) = f_b(f_{b^{-1}}(x))$

$= f_b(b^{-1}x) = b(b^{-1}x)$

$= (bb^{-1})x = ex = x$

$\forall x \in G.$

$= f_e(x)$

$\therefore f_b f_{b^{-1}} = f_e$. (identity element in G)

$\Rightarrow f_{b^{-1}} = (f_b)^{-1}$

Also $(f_a f_b)(x) = f_a(f_b(x)) = f_a(bx)$ (by ①)

$= a(bx)$

$= (ab)(x)$

$= f_{ab}(x)$

$\therefore f_a f_b = f_{ab}$.

$\therefore f_a (f_b)^{-1} = f_a f_{b^{-1}} = f_{ab^{-1}} \in K$.

$\therefore \underline{\underline{K \leq A(G)}}$

let us define a mapping $\psi : G \to K$

such that $\psi(a) = f_a \quad \forall a \in G$ ②

TO S.T ψ is well-defined, 1-1

$\forall a, b \in G$; $a = b$

$\Leftrightarrow ax = bx$

$\Leftrightarrow f_a(x) = f_b(x) \quad \forall x \in G$

$\Leftrightarrow f_a = f_b$

$\Leftrightarrow \psi(a) = \psi(b)$

ψ is obviously onto.

TO S.T ψ is homomorphism: we have

$\psi(ab) = f_{ab} = f_a f_b = \psi(a)\psi(b)$.

$\therefore \psi$ is 1-1, onto and homomorphism. $\therefore G$ is isomorphic to K.

1C 2013 2017) find the supremum and infimum of $\frac{x}{\sin x}$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Sol let $f(x) = \frac{x}{\sin x} \;\forall x \in \left(0, \frac{\pi}{2}\right]$.

then $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$ —— (1)

let $g(x) = \sin x - x\cos x$

then $g'(x) = \cos x - (\cos x - x\sin x)$

$= x\sin x$

$> 0 \;\forall x \in \left(0, \pi/2\right]$

$\therefore$ $g(x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right]$.

$\because x \in \left(0, \frac{\pi}{2}\right] \Rightarrow 0 < x \leq \frac{\pi}{2}$

$\Rightarrow g(0) < g(x) \leq g(\pi/2)$

$\Rightarrow 0 < g(x) \leq 1$

$\therefore f'(x) > 0 \;\forall x \in \left(0, \frac{\pi}{2}\right]$.

f is an increasing on $\left(0, \frac{\pi}{2}\right]$.

$\therefore$ infimum $f = \lim_{x \to 0} \frac{x}{\sin x} = 1$

supremum $f = f(\pi/2) = \frac{\pi/2}{\sin \pi/2} = \frac{\pi}{2}$

Q.1(d) Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f(1/z)$?

Sol:- As the function $f(z)$ has no singularity in the finite part of the plane; it can be expanded in Taylor's series in any circle $|z| = K$; where K is arbitrarily large.

$$\therefore f(z) = \sum_{r=0}^{\infty} A_r z^r$$

Also, if $f(z)$ has no singularity at $z = \infty$; $f(1/z)$ has none at $z = 0$. Moreover, since $f(z)$ has no singularity in finite part of plane; we have

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \qquad \therefore f(1/z) = \sum_{n=0}^{\infty} A_n z^{-n}$$

Since; $f(z)$ has a pole of order 'n' at infinity. $f(1/z)$ has a pole of order 'n' at zero (0).

$$\therefore f\left(\frac{1}{z}\right) = \sum_{s=1}^{n} \frac{B_s}{z^s} + \phi(z).$$

where; $\phi(z)$ is a regular function of z.

Hence; $$\phi(z) + \sum_{s=1}^{n} \frac{B_s}{z^s} = \sum_{r=0}^{\infty} A_r z^{-r}$$

$$\therefore \phi(z) = A_0 = \text{a constant}$$

and $$A_{n+1} = A_{n+2} = \cdots\cdots = 0$$

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} A_n z^{-n} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots\cdots$$

$f\left(\frac{1}{z}\right)$ is a removable singularity of degree n.

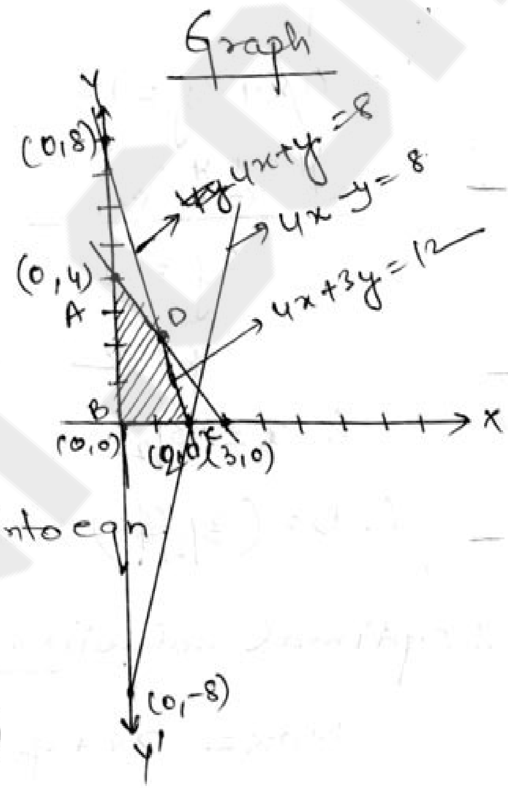
1(e) 2017 → Using Graphical method, find the maximum value of $2x+y$

subject to
$$4x+3y \le 12$$
$$4x+y \le 8$$
$$4x-y \le 8$$
$$x, y \ge 0$$

Sol^n:

1(e) Use graphical max $= 2x+y$

subject to
$$4x+3y \le 12$$
$$4x+y \le 8$$
$$4x-y \le 8$$
$$x, y \ge 0$$



By converting all constraints into eqn

$$4x+3y=12 \quad \text{—(i)}$$
$$4x+y=8 \quad \text{—(ii)}$$
$$4x-y=8 \quad \text{—(iii)}$$

From eqn —(i)

$4x+3y=12$

Putting $x=0$; $y=4$

Putting $y=0$; $x=3$

x	0	3
y	4	0

From eqn —(ii)

$4x+y=8$

Putting $x=0$; $y=8$

Putting $y=0$; $x=2$

x	0	2
y	8	0

From eqn (ii)

$-4x + y = 8$

Putting $x = 0;\ y = 8$

Putting $y = 0;\ x = 2$

x	0	2
y	8	0

From eqn (iii)

$4x - y = 8$

Putting $x = 0;\ y = -8$

Putting $y = 0;\ x = 2$

x	0	2
y	-8	0

~~Region~~ our shaded region ABCD which satisfies the given condition.

From eqn (i) and (ii) we find D.

$$\therefore\ 4x + 3y = 12$$
$$4x + y = 8$$
$$(-)\quad (-)\quad (-)$$
$$2y = 4$$
$$y = 2$$
$$\therefore\ x = 3/2$$
$$\therefore\ D = (3/2, 2)$$

$\therefore$ Optimal solution

$$\text{Max} = 2x + y.$$

at point $a(0,4) = 2\cdot(0) + 4 = 4$

$b(0,0) = 2(0) + 0 = 0$

$c(2,0) = 2(2) + 0 = 4$

$d = (3/2, 2) = 2\cdot\left(\frac{3}{2}\right) + 2 = 5.$

$\therefore$ Max. value of z is at point D $(3/2, 2)$ and its value is $\underline{5}$.

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Q.2(a) Let $f(t) = \int_0^t [x]\,dx$

where $[x]$ denotes the largest integer less than or equal to 'x'.

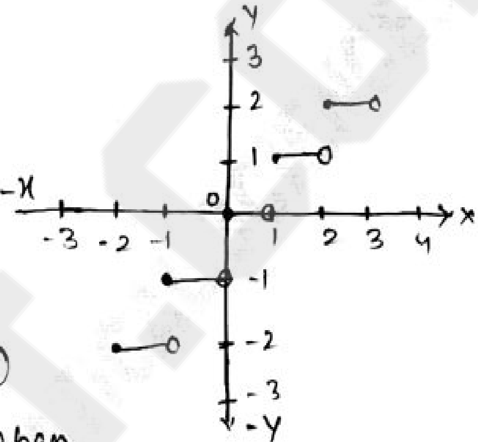
AS -2017

(i) Determine all the real numbers 't' at which 'f' is differentiable.

(ii) Determine all the real numbers 't' at which 'f' is continuous but not differentiable.

Sol:-

$$[x] = \begin{cases} \vdots \\ 1 \;;\; 1 \le x < 2 \\ 0 \;:\; 0 \le x < 1 \\ -1 \;;\; -1 \le x < 0 \\ -2 \;;\; -2 \le x < -1 \\ \vdots \end{cases} \quad \text{--- (1)}$$



Note that, we have an open circle on the right of each step, where the function "jumps up" to the next step.

Thus, the function $[x]$ is discontinuous at every integer point, and continuous at non-integer points.

Now, $f(t) = \int_0^t [x]\,dx$ is defined by integrating the greatest integer function $[x]$ from 0 to t.

For example:

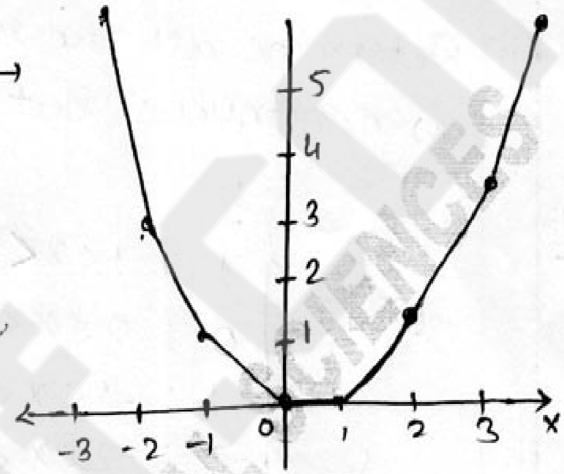
$$f(3.5) = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^{3.5} [x]\,dx$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^{3.5} 3\,dx$$

$$= 0 + 1 + 2 + 3(0.5) = 4.5.$$

$$f(x) = \begin{cases} \vdots \\ x-1 & ; \text{ if } 1 \le x < 2 \\ 0 & ; \text{ if } 0 \le x < 1 \\ -x & ; \text{ if } -1 \le x < 0 \\ -2x-1 & ; \text{ if } -2 \le x < -1 \\ \vdots \end{cases} \quad \text{——(2)}$$

graphing this function, →



we know that.
there are corners in this graph at integer values of x;

Hence, the given function,

$$f(t) = \int_0^t [x]\,dx$$

is continuous $\forall\ x \in \mathbb{R}$, but is not differentiable at the points where the integrand is discontinuous;
ie at all integer points.

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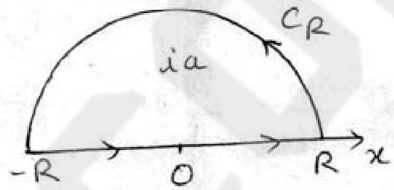
Q 2(b) IAS-2017
~~3(c)~~ Prove that $\int_0^\infty \frac{x\sin mx}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-ma}$ by contour integration.

Sol^n → Consider the integral $\int_C f(z)\,dz$,

where $f(z)\,dz = \frac{z e^{imz}}{z^2+a^2}$

taken round the closed the contour C consisting of real axis from $-R$ to R, and the upper half of the large circle $|z| = R$.



Poles of $f(z)$ are given by $z^2+a^2=0$ i.e., $z = \pm ia$

Hence by making $R \to \infty$, relation (1) reduces to

$$\int_{-\infty}^{\infty} \frac{x e^{imx}\,dx}{x^2+a^2} = 2\pi i \times \frac{e^{-am}}{2}$$

Equating imaginary parts we get -

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-am}.$$

2(c) 2017. Let F be a field and $F[x]$ denote the ring of polynomials over F in a single variable x. For $f(x), g(x) \in F[x]$ with $g(x) \neq 0$, show that there exist $q(x), r(x) \in F[x]$ such that degree $r(x) <$ degree$(g(x))$ and $f(x) = q(x) \cdot g(x) + r(x)$.

Sol^n: Let us consider the set

$$S = \{ f(x) - h(x)g(x) / h(x) \in F[x] \}$$

For $0(x) \in F[x]$,

$$f(x) = f(x) - 0(x)\, g(x) \in S$$

$$\therefore S \neq \phi$$

Let $0 \in S$. Then by definition of S,

$\exists\, q(x) \in F[x]$ so that $0 = f(x) - q(x)\, g(x)$

$$\text{i.e. } f(x) = q(x)\, g(x) + 0(x)$$

$$\text{i.e. } f(x) = q(x)\, g(x) + r(x) \text{ where } r(x) = 0$$

$\therefore$ the theorem is proved.

Let $0(x) \notin S$. Then every polynomial in S is a non-zero polynomials and hence non-negative degree.

Let $r(x) \in S$ be a polynomial of least degree.

By definition of S, there exist $q(x) \in F[x]$ so that

$$r(x) = f(x) - q(x)\, g(x)$$

$$\text{i.e. } f(x) = g(x)\, q(x) + r(x) \quad \text{——— (1)}$$

Let $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,\ a_n \neq 0$

So that $\deg g(x) = n$

Now we have to prove that $\deg r(x) < n$.

i.e. $\deg r(x) < \deg g(x)$.

If possible, suppose that $m = \deg r(x) \geq n$.

Let $r(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_m x^m, \quad c_m \neq 0$.

Now we have

$$c_m a_n^{-1} x^{m-n} g(x) = c_m a_n^{-1} x^{m-n} [a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n]$$

$$= c_m a_n^{-1} a_0 x^{m-n} + c_m a_n^{-1} a_1 x^{m-n+1} + \cdots + c_m a_n^{-1} a_{n-1} x^{m-1} + c_m x^m.$$

$$\therefore r(x) - c_m a_n^{-1} x^{m-n} g(x)$$

$$= (c_{m-1} x^{m-1} + \cdots + c_0) - (c_m a_n^{-1} a_{n-1} x^{m-1} + \cdots + c_m a_n^{-1} a_0 x^{m-n})$$

$$\therefore r(x) = c_m a_n^{-1} x^{m-n} g(x) + \alpha(x) \quad \text{——— } (2)$$

where $\alpha(x) = (c_{m-1} - c_m a_n^{-1} a_{n-1}) x^{m-1} + \cdots + c_0$

$\Rightarrow \deg \alpha(x) \leq m-1$

i.e. $\deg \alpha(x) \leq \deg r(x) - 1$

i.e. $\deg \alpha(x) < \deg r(x)$

$\therefore$ from (1) & (2)

$$\alpha(x) = f(x) - g(x)\{q(x) + c_m a_n^{-1} x^{m-n}\}$$

$$= f(x) - g(x)\beta(x)$$

where $\beta(x) = q(x) + c_m a_n^{-1} x^{m-n} \in F[x]$

$\therefore \alpha(x) \in S$

Now we have $\alpha(x), r(x) \in S$ and $\deg \alpha(x) < \deg r(x)$

This is a contradiction. Since $r(x)$ is the polynomial of least degree in S.

$\therefore$ our supposition is wrong.

Hence $\deg r(x) < n$

i.e. $\deg r(x) < \deg g(x)$

Uniqueness of $q(x)$ and $r(x)$:

If possible, suppose that

$$f(x) = q'(x)g(x) + r'(x)$$

where $r'(x) = 0$ (or) $\deg r'(x) < \deg g(x)$.

Then $q(x)g(x) + r(x) = q'(x)g(x) + r'(x)$

i.e. $(q(x) - q'(x))g(x) = r'(x) - r(x)$

If $q(x) - q'(x) \neq 0$ then

$\deg(q(x) - q'(x))g(x) = \deg(q(x) - q'(x)) + \deg g(x)$

i.e. $\deg(r'(x) - r(x)) \geq \deg g(x)$

This is a contradiction because

$\deg r(x) < \deg g(x)$ and

$\deg r'(x) < \deg g(x)$

$\therefore q(x) - q'(x) = 0$ and $r'(x) - r(x) = 0$

$\Rightarrow q'(x) = q(x)$ & $r'(x) = r(x)$

Hence $q(x), r(x) \in F[x]$ are unique.

Note: (1) The polynomials $q(x)$ and $r(x)$ of the above theorem are called the quotient and the remainder.

(2). In the above theorem if $r(x) = 0$ then we say that $q(x)$ divides $f(x)$ (or) $g(x)$ is a factor of $f(x)$

3(a) IAS -2017 → Show that the groups $Z_5 \times Z_7$ and Z_{35} are isomorphic.

Sol let $(Z_{35} = \{0, 1, 2, 3 \ldots 34\}, +)$ be a cyclic group of order 35.

let $Z_5 \times Z_7 = \{(a, b) / a \in Z_5, b \in Z_7\}$ be a cyclic ($\because \gcd(5,7)=1$)

where $(Z_5 = \{0, 1, 2, 3, 4\}, +)$

and $(Z_7 = \{0, 1, 2, 3, 4, 5, 6\}, +)$

are two cyclic groups such that $O(Z_5) = 5$ and $O(Z_7) = 7$

Define a mapping $f: Z_{35} \to Z_5 \times Z_7$

such that $f(a) = (a \oplus_5 0, a \oplus_7 0)$ ——(1)

Now we shall show that f is well-defined:

let $a, b \in Z_5$ s.t $a = b$

$\Rightarrow a +_5 0 = b +_5 0$

$\therefore$ f is well-defined.

To S.T 1-1: let $(a +_5 0, a \oplus_7 0) = (b \oplus_5 0, b \oplus_7 0$

$\Rightarrow a = b$

$\therefore f$ is 1-1

To show onto:

for every $(a \oplus_5 0, a \oplus_7 0) \in Z_5 \times Z_7$

$\exists\, a \in Z_{35}$ such that

$f(a) = (a \oplus_5 0, a \oplus_7 0)$ by definition.

$\therefore f$ is onto.

To show homomorphism:

Let $a, b \in Z_{35}$, then

$$f(a \oplus_{35} b) = \left((a \oplus_{35} b) \oplus_5 0, (a \oplus_{35} b) \oplus_7 0\right)$$

$$= (a \oplus_{35} b, a \oplus_{35} b)$$

$$= (a \oplus_5 0, a \oplus_7 0) \oplus_{35} (b \oplus_5 0, b \oplus_7 0)$$

$$= f(a) \oplus_{35} f(b)$$

$\therefore f$ is a homomorphism

$\therefore Z_5 \times Z_7 \cong Z_{35}$.

8(b) 2017 → Let $f = u+iv$ be an analytic function on the unit disc. $D = \{z \in \mathbb{C} / |z|<1\}$ Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ at all points of D.

Sol'n: Given that the function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain.

$$\therefore f'(z) = \frac{\partial f}{\partial x} \quad \text{--- ①} \quad (\text{or}) \quad f'(z) = -i\frac{\partial f}{\partial y} \quad \text{--- ②}$$

Since the analytic function has derivatives of all orders.

$$① \equiv f''(z) = \frac{\partial}{\partial x}(f'(z))$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} \quad \text{--- ③}$$

$$② \equiv f''(z) = -i\frac{\partial}{\partial y}(f'(z))$$

$$= -i\frac{\partial}{\partial y}\left(-i\frac{\partial f}{\partial y}\right) = (-1)\frac{\partial^2 f}{\partial y^2} \quad \text{--- ④}$$

from ③ & ④ we have

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 f = 0 \quad \text{--- ⑤}$$

which is valid for any analytic function $f(z)$

i.e. If $f(z) = u(x,y) + iv(x,y)$ is an analytic in a domain D then from (5)

$$\nabla^2 f = 0$$

$$\Rightarrow \nabla^2(u+iv) = 0$$

$$\Rightarrow \nabla^2 u + i\nabla^2 v = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

3(c) 2017 Solve the following linear programming problem by Simplex method:

maximize $z = 3x_1 + 5x_2 + 4x_3$

Subject to

$2x_1 + 3x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1, x_2, x_3 \ge 0$.

Sol^n: Simplex method:-

(2)

Max $z = 3x_1 + 5x_2 + 4x_3$

Subject to:

$2x_1 + 3x_2 \le 8$

$2x_2 + 5x_3 \le 10$

$3x_1 + 2x_2 + 4x_3 \le 15$

$x_1, x_2, x_3 \ge 0$

Soln:- Standard form of given problem.

$Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$

S.C

$$\left.\begin{aligned} 2x_1 + 3x_2 + 0x_3 + S_1 + 0S_2 + 0S_3 &= 8 \\ 0x_1 + 2x_2 + 5x_3 + 0S_1 + S_2 + 0S_3 &= 10 \\ 3x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 &= 15 \end{aligned}\right] -(i)$$

Where $x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$

or, $S_1, S_2, S_3 \rightarrow$ Slack variables.

For initial basic feasible soln

Free variable $= 6-3 = 3$

From -(i) we get, put $x_1, x_2, x_3 = 0$

$S_1 = 8,\ S_2 = 10,\ S_3 = 15$

IBFS $(x_1, x_2, x_3, S_1, S_2, S_3) = (0, 0, 0, 8, 10, 15)$

Now:

	C_j	3	5	4	0	0	0		
CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3	b	θ
0	S_1	2	3	0	1	0	0	8	8/3 ←
0	S_2	0	2	5	0	1	0	10	5
0	S_3	3	2	4	0	0	1	15	$\frac{15}{2}$
$Z_j =$	$\Sigma C_B a_{ij}$	0	0	0	0	0	0		
$C_j =$	$C_j - Z_j$	3	5 ↑	4	0	0	0		

∴ 3 is key element

∴ incoming variable $= x_2$ and outgoing variable $= S_1$

	C_j	3	5	4	0	0	0		
CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3	B	θ
5	x_2	2/3	1	0	1/3	0	0	8/3	—
0	S_2	-2/3	0	5/2	-1/3	1/2	0	7/3	14/15 ←
0	S_3	5/6	0	2	-1/3	0	1/2	29/6	29/12
$Z_j =$	$\Sigma C_B a_{ij}$	10/3	5	0	5/3	0	0		
$C_j =$	$c_j - z_j$	-1/3	0	4 ↑	-5/3	0	0		

∴ 5/2 = key element

∴ incoming variable $= x_3$ ∴ outgoing variable $= S_2$

	C_j	3	5	4	0	0	0		
CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3		
5	x_2	2/3	1	0	1/3	0	0	8/3	4
4	x_3	-4/15	0	1	-2/15	1/5	0	14/15	—
0	S_3	$\boxed{\frac{41}{60}}$	0	0	-1/5	-1/5	1/2	89/60	$\frac{89}{41}$ ←

$z_j =$	$\Sigma C_B a_j$	$34/15$	5	4	$37/15$	$4/5$	0		
$C_j =$	$c_j - z_j$	$11/15$ ↑	0	0	$-\frac{37}{15}$	$-4/5$	0		

$\therefore$ $41/60$ = key element $\therefore$ $\text{E.V} = x_1$ and $\text{O.V} = S_3$

CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3	B	θ
5	x_2	0	$3/2$	0	$94/41$	$\frac{12}{41}$	$-30/41$	$75/41$	
4	x_3	0	0	$15/4$	$-65/82$	$75/164$	$30/41$	$465/82$	
3	x_1	1	0	0	$-12/11$	$\frac{-12}{41}$	$\frac{30}{41}$	$89/41$	
$z_j =$	$\Sigma C_B a_{ij}$	3	$15/2$	15	$304/41$	$99/41$	$60/41$	$\frac{1572}{41} \Rightarrow Z_{max}$	
$C_j =$	$c_j - z_j$	0	$-5/2$	-11	$-304/41$	$-99/41$	$-60/41$		

From above table all C_j's ≤ 0 $\therefore$ optimal BFS = $x_1 = 89/41$, $x_2 = 75/41$, $x_2 = 465/82$ $\therefore$ Max $z = \frac{1572}{41}$

4.(a) For a function $f : \mathbb{C} \to \mathbb{C}$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all k = 1, 2, 3,...., show that f is a polynomial.

***Sol** ^{n}:*

YOUR SELF

4(b) IAS-2017 → Find the initial basic feasible solution of the following transportation problem using Vogel's approximation method and find the cost.

Destinations:

	D_1	D_2	D_3	D_4	D_5	
O_1	4	7	0	3	6	14
Origins O_2	1	2	−3	3	8	9
O_3	3	−1	4	0	5	17
	8	3	8	13	8	40

Demand; Supply

Sol: Clearly the TPP is a balanced problem and is of 3×5 order matrix.

$\therefore$ There are $m+n-1 = 3+5-1 = 7$ basic (cells) feasible solutions.

Let us find initial basic feasible solutions by using Vogel's approximation method:

4 (7)	7 (0)	0 (0)	3 (0)	6 (7)	~~14~~ ~~7~~ 1	(3)(1)(1)(2)(2)(6)
1 (1)	2 (0)	−3 (8)	3 (0)	8 (0)	~~9~~ 1	(4)(1)(2)(7)
3 (0)	−1 (3)	4 (0)	0 (13)	5 (1)	~~17~~ ~~4~~ 1	(1)(1)(3)(1)(1)(5)
~~8~~ 7	3	8	13	8		
(2)	(3)	(3)	(3)	(1)		
(2)	(3)		(3)	(1)		
(2)			(3)	(1) (1)(1)(1)		

$\therefore$ IBFS is $\begin{pmatrix} 7, 0, 0, 0, 7; \\ 1, 0, 8, 0, 0; \\ 0, 3, 0, 13, 1 \end{pmatrix}$.

$\therefore$ Minimum cost

$= 7(4) + 1(1) + 3(-1) + 8(-3) + 13(0) + 7(6) + 1(5)$

$= 49.$

Let us find the cost by using modified method:-

Let $\Delta_{ij} = u_i + v_j - c_{ij}$ ——— (1)

u_i's & v_j's are helping variables corr. to rows and columns respectively, and c_{ij}'s costs.

$\Delta_{ij} = 0$ corr. to basic cells.

$\therefore \Delta_{11} = 0 \Rightarrow u_1 + v_1 = 4$
$\Delta_{15} = 0 \Rightarrow u_1 + v_5 = 6$
$\Delta_{21} = 0 \quad u_2 + v_1 = 1$
$\Delta_{23} = 0 \quad u_2 + v_3 = -3$
$\Delta_{32} = 0 \quad u_3 + v_2 = -1$
$\Delta_{34} = 0 \quad u_3 + v_4 = 0$
$\Delta_{35} = 0 \quad u_3 + v_5 = 5$

$\Rightarrow$ let $u_3 = 0$
then $v_4 = 0$
$v_2 = -1$
$v_5 = 5$
$u_1 = 1$
$v_1 = 3$
$u_2 = -2$
$v_3 = -1$

Let us find Δ_{ij} corr. to non-basic cells:

$\Delta_{12} = -7 \qquad \Delta_{24} = -5$
$\Delta_{13} = 0 \qquad \Delta_{25} = -5$
$\Delta_{14} = -2 \qquad \Delta_{31} = 0$
$\Delta_{22} = -5 \qquad \Delta_{33} = -5$

$\therefore \Delta_{ij} \leq 0$
$\therefore$ optimality has been obtained
$\therefore$ OBFS is same as IBFS.

$\therefore$ Min. cost 49.

4(c) Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100.

IAS -2017

Sol.: We wish to show that, we can re-arrange the terms of $\sum x_n$ to form a series whose sum converges to 100.

Add together, in order, just enough of the first positive terms of $\sum x_n$ so that there sum exceeds 100.

Say we need, n_1 terms to do so, then the partial sum $S_{n_1} > 100$.

We can always do so no matter how large the sum (100) is, since the series of positive terms diverges to infinity (∞).

To this sum $S_{n_1}+$, in order, just enough of the first negative terms of $\sum x_n$ to make the resulting sum less than 100. Say we need n_2 negative terms, then the partial sum

$$S_{n_1+n_2} < 100$$

We can always do so no matter how far the +ve terms took us to right of 100, since the series of negative terms diverges to $-\infty$ (infinity).

Now, Add just enough of the next positive terms, say n_3 +ve terms to get the sum to exceed

100 again. we know have the partial sum

$$S_{n_1+n_2+n_3} > 100$$

Now, again add just enough of the next negative terms say n_4 negative terms, so that the sum is less than 100 again, we now have the partial sum –

$$S_{n_1+n_2+n_3+n_4} < 100$$

we continue to repeat this process, adding each time just enough new positive terms to make the sum exceed 100 and then just enough new negative terms to make the sum less than 100.

Now, we notice, that all these partial sums differs from 100 by, atmost, 1+ve or 1 -ve term. These partial sums must be closing in on 100 – since the original series Σx_n converges, its terms x_n go to zero (0) as $n \to \infty$. [i.e $x_n \to 0 \ \forall\ n \to \infty$]

we can get the partial sums as close to 100 as we wish (within any ϵ), if we go out far enough in the series (to where all the remaining terms have their absolute values less than ϵ). So, the partial sums converges to 100, which proves that the series of re-arranged terms converges to 100.

$\boxed{\text{Note:}}$ The given question is a restatement of Riemann's Re-arrangement Theorem.

Q5.(a) Solve $(D^2-2DD'+D'^2)z = e^{x+2y}+x^3+\sin 2x$,

where $D=\frac{\partial}{\partial x}$, $D'=\frac{\partial}{\partial y}$, $D^2=\frac{\partial^2}{\partial x^2}$, $D'^2=\frac{\partial^2}{\partial y^2}$. (10)

Sol: Given Eqn can be written as

$$(D-D')^2 z = e^{x+2y}+x^3+\sin 2x \quad \text{—(1)}$$

Its auxiliary eqn $\Rightarrow (m-1)^2=0 \Rightarrow m=1,1$

$\therefore$ C.F. $= \phi_1(y+x)+x\,\phi_2(y+x)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

→ PI corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}e^{x+2y} = e^{x+2y}$$

→ PI corresponding to x^3,

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3$$

$$= \frac{1}{D^2}\left(1+2\frac{D'}{D}+\cdots\right)x^3 = \frac{1}{D^2}x^3 = \frac{1}{D}\frac{x^4}{4} = \frac{x^5}{20}$$

→ PI corresponding to $\sin 2x$

$$= \frac{1}{(D-D')^2}\sin 2x = \frac{1}{(D-D')^2}\sin(2x+0y)$$

$$= \frac{1}{(2-0)^2}\iint \sin v\,dv\,dv, \quad \text{where } v=2x+0y$$

$$= -\frac{1}{4}\int \cos v\,dv = -\frac{1}{4}\sin v = -\frac{1}{4}\sin 2x$$

$\therefore$ Gen Sol

$$z = \phi_1(y+x)+x\,\phi_2(y+x)+e^{x+2y}+\frac{x^5}{20}-\frac{1}{4}\sin 2x.$$

5(a)) Explain the main steps of the Gauss-Jordan method and apply this method to find the inverse of the matrix:

IFoS-2017

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

Sol: Let us explain the main steps of the Gauss-Jordan's method:

— This is done by augmenting the matrix A by the identity matrix I of the order same as that of A.

— using elementary row operations on the augmented matrix $[A|I]$

— we reduce the matrix A to the form I and in the process the matrix I is transformed to A^{-1}.

i.e $[A \mid I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I \mid A^{-1}]$

Let us find an inverse of

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

we have

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 6 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 4 & -3 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_2$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & -3 & -3 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & -3/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_3 \to \frac{1}{2}R_3 \end{array}$$

$$= [I \mid A^{-1}]$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

5(c). Write the Boolean expression.

$z(y+z)(x+y+z)$

in its simplest form using Boolean postulate rules. Mention the rules used during simplification. Verify your result by constructing the truth table for the given expression and for its simplest form?

Sol. Let $A = z(y+z)(x+y+z)$

$= (zy+z^2)(x+y+z)$

$= zyx + zy^2 + z^2y + z^2x + z^2y + z^3$

$= zyx + zy + zy + zx + zy + z^3$ $\quad [\because A^2 = A;\ A^3 = A]$

$= zy(x+1) + zy + zx + zy + z$

$= zy + zy + zx + z(y+1)$

$= zy + zx + z$ $\quad [\because 1+A = 1]$

$= zy + z(x+1)$

$= zy + z$

$= z(y+1)$ $\quad [\because 1+y=1]$

$A = z$.

x	y	z	$z(y+z)$	$(x+y+z)$	$z(y+z)(x+y+z)$	z
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	0	1	0	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	1	1	1	1
1	1	0	0	1	0	0
1	1	1	1	1	1	1

$\therefore z(y+z)(x+y+z) = z$ Hence proved.

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5(d) Let Γ be a closed curve in XY-plane and let S denote the region bounded by the curve Γ. Let

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y) \quad \forall\, (x,y) \in S.$$

If 'f' be prescribed at each point (x,y) of S and 'w' is prescribed on the boundary Γ of S, then prove that any solution $w = w(x,y)$, satisfying these conditions, is unique?

Sol:- Let $w = w_1(x,y)$; $w = w_2(x,y)$ be two solutions satisfying

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y) \quad \forall\, (x,y) \in S \quad \text{——} (1)$$

in S together with the prescribed boundary conditions on Γ.

Let $W(x,y) = w_1(x,y) - w_2(x,y)$ —— (2)

Then $\dfrac{\partial^2 W}{\partial x^2} + \dfrac{\partial^2 W}{\partial y^2} = f - f = 0$ —— (3)

Also, on Γ ; $W = 0$; since ; $w_1 = w_2$ on Γ

Now;

$$I = \iint_S \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dx\,dy$$

$$I = \iint_S \left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + w\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right\} dx\,dy$$

using (3)

$$I = \iint_S \left\{\frac{\partial}{\partial x}\left(w\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left[w\frac{\partial w}{\partial y}\right]\right\} dx\,dy$$

$$= \oint_{\Gamma} \left(w \frac{\partial w}{\partial x} dy - w \frac{\partial w}{\partial y} dx \right)$$

$$= 0 \qquad (\text{Since; } w = 0 \text{ on } \Gamma)$$

Now; $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \geq 0$.

But $I = 0$; $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 0$ in S

which will be true only if

$$\frac{\partial w}{\partial x} = 0 = \frac{\partial w}{\partial y} \text{ at each point of } S$$

Now; shows that $w =$ constant in S.

Since, $w = 0$ on Γ, we infer from continuity of w; that $w = 0$ throughout S.

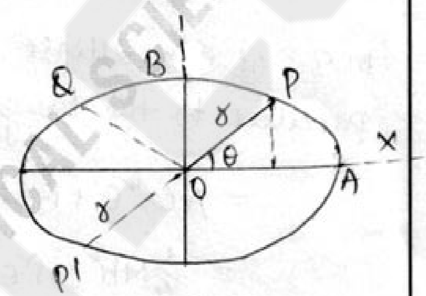
This gives $w_1 = w_2$; which establishes the result.

5(e) → Show that the moment of Interia of an elliptic area of mass M and Semi-axis a and b about a semi diameter of length r is $\frac{1}{4}M\frac{a^2b^2}{r^2}$. Further prove that the moment of interia about a tangent is $\frac{5M}{4}p^2$, where p is the ⊥lar distance from the centre of the ellipse to the tangent.

Sol^n: (1) Let PP' be Semi-diameter of length r of an elliptic area of mass M and Semi-axes a and b.

Equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \text{—— ①}$$



If PP' make an angle θ with OX then coordinates of P are $(r\cos\theta, r\sin\theta)$.

Since P lies on equation ①

$$\therefore (r^2/a^2)\cos^2\theta+(r^2/b^2)\sin^2\theta=1$$

$$\Rightarrow b^2\cos^2\theta+a^2\sin^2\theta=\frac{a^2b^2}{r^2} \quad \text{—— ②}.$$

Now M.I of the ellipse about $OX = A = \frac{1}{4}Mb^2$

and M.I of the ellipse about $OY = B = \frac{1}{4}Ma^2$

Also P.I of the ellipse about OX and $OY = F = 0$ (By symmetry)

$\therefore$ M.I of the ellipse about the diameter PP'

$$= A\cos^2\theta+B\sin^2\theta-F\sin2\theta$$

$$=\frac{1}{4}Mb^2\cos^2\theta+\frac{1}{4}Ma^2\sin^2\theta-0$$

$$=\frac{1}{4}M(b^2\cos^2\theta+a^2\sin^2\theta)$$

$$=\frac{M}{4}\cdot\frac{a^2b^2}{r^2}$$

(2) Equation of the tangent to the ellipse is

$$y=mx+\sqrt{(a^2m^2+b^2)}$$

where $m = \tan\theta$, if tangent is inclined at an angle θ to the axis of x.

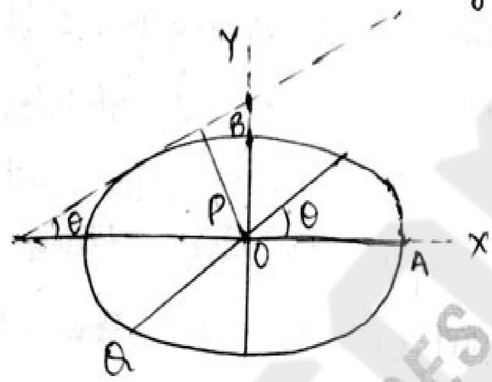
If p is the $\perp$lar from the centre $(0,0)$ on the tangent

(1) then

$$p = \frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(1+m^2)}}$$

$$= \frac{\sqrt{(a^2\tan^2\theta+b^2)}}{\sqrt{(1+\tan^2\theta)}}$$

$$= \sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$$



M.I of the ellipse about the diameter PQ which is parallel to the tangent :

$$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$$

$$= \frac{1}{4}Mb^2\cos^2\theta + \frac{1}{4}Ma^2\sin^2\theta - 0$$

$$= \frac{1}{4}M\left(b^2\cos^2\theta + a^2\sin^2\theta\right)$$

$$= \frac{1}{4}Mp^2, \text{ from } (2)$$

$\therefore$ M.I of the ellipse about the tangent

= Its. M.I about the parallel line through C.G 'O'

+ M.I of mass M at O about the tangent

$$= \frac{1}{4}Mp^2 + Mp^2 = \frac{5}{4}Mp^2$$

6.(a) Find a complete integral of the PDE

$$2(pq + yp + qx) + x^2 + y^2 = 0$$

(15)

Sol. Charpit's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

Here, $f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0$ — ①

$$\therefore \frac{dp}{2q+2x} = \frac{dq}{2p+2y} = \frac{dz}{-p(2q+2y) - q(2p+2x)} = \frac{dx}{-(2q+2y)} = \frac{dy}{-(2p+2x)}$$

$$= \frac{dp + dq + dx + dy}{(2q+2x) + (2p+2y) - (2q+2y) - (2p+2x)}$$

$$= \frac{d(p+q+x+y)}{0}$$

So that $(p+x) + (q+y) = a$. — ②

Re-writting ①,

$$2(p+x)(q+y) + (x-y)^2 = 0$$

or $(p+x)(q+y) = \frac{-(x+y)^2}{2}$ — ③

Now, $(p+x) - (q+y) = \sqrt{[(p+x)+(q+y)]^2 - 4(p+x)(q+y)}$

$$= \sqrt{a^2 + 2(x-y)^2} \quad \text{by ② \& ③}$$

— ④

② + ④ ⟹ $2(p+x) = a + \sqrt{a^2 + 2(x-y)^2}$

② − ④ ⟹ $2(q+y) = a - \sqrt{a^2 + 2(x-y)^2}$

from these we get p & q as

$p = -x + \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$, $q = -y + \frac{a}{2} - \frac{1}{2}\sqrt{a^2+2(x-y)^2}$

$\therefore$ $dz = p\,dx + q\,dy$ becomes

$= -(x\,dx + y\,dy) + \frac{a}{2} \times (dx + dy) + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2} \times (dx - dy)$

$dz = -\frac{1}{2} d(x^2 + y^2) + \frac{a}{2} d(x+y) + \sqrt{2} \times \frac{1}{2}\sqrt{\frac{a^2}{2} + (x-y)^2}\; d(x-y)$

Put $x - y = t$ s.t. $d(x-y) = dt$ — (5)

$\therefore$ $dz = -\frac{1}{2} d(x^2+y^2) + \frac{a}{2} d(x+y) + \frac{1}{\sqrt{2}}\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2}\, dt$

$$\therefore z = -\frac{x^2+y^2}{2} + a \cdot \frac{x+y}{2} + \frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2} + \frac{(a/\sqrt{2})^2}{2}\log\left[t + \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2}\right]\right] + b$$

Putting back the value of t, the required complete integral is

$$z = -\frac{(x^2+y^2)}{2} + \frac{a(x+y)}{2} + \frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2} + (x-y)^2} + \frac{a^2}{2}\log\left\{x - y + \sqrt{\frac{a^2}{2} + (x-y)^2}\right\}\right] + b.$$

6(b) IFoS-2017 → For given equidistant u_{-1}, u_0, u_1 and u_2, a value is interpolated by Lagrange's formula. Show that it may be written in the form

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$ where

$$x+y=1.$$

Sol

RHS: $yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$

$$= (1-x)u_0 + xu_2 + \frac{(1-x)\left[(1-x)^2-1\right]}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0.$$

$$= u_0 - xu_0 + xu_2 + \frac{(1-x)\left[x^2-2x\right]}{3!}\Delta(\Delta u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(\Delta u_0)$$

$$= u_0 - x u_0 + x u_2 + \frac{x(1-x)(x-2)}{3!}\Delta(u_0 - u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(u_1 - u_0)$$

$$= u_0 - x u_0 + x u_2 + \frac{x(1-x)(x-2)}{3!}\left[(u_1 - u_0) - (u_0 - u_{-1})\right] + \frac{x(x^2-1)}{3!}\left[(u_2 - u_1) - (u_1 - u_0)\right].$$

$$= u_0 - x u_0 + x u_2 + \frac{x(1-x)(x-2)}{3!}(u_1 - 2u_0 + u_{-1}) + \frac{x(x^2-1)}{3!}\left[u_2 - 2u_1 + u_0\right].$$

$$= u_0\left[(1-x) - \frac{1}{3}x(1-x)(x-2) + \frac{x(x^2-1)}{3!}\right] + u_2\left[x + \frac{x(x^2-1)}{3!}\right] + u_1\left[\frac{x(1-x)(x-2)}{3!} - \frac{1}{3}x(x^2-1)\right]$$

$$+ \frac{x(1-x)(x-2)}{3!}u_{-1}.$$

$$= \frac{x(1-x)(x-2)}{3!}u_{-1} + \left[\frac{x^3-2x^2-x+2}{2}\right]u_0 + \frac{x^2(1-x)}{2}u_1 + \frac{x(x^2+5)}{6}u_2$$

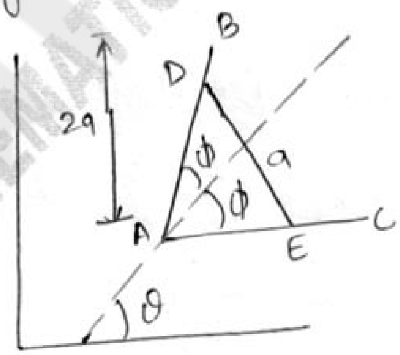
$$= u_x$$

Q.6(c) Two uniform rods AB, AC, each of mass 'm' and length '2a', are smoothly hinged together at 'A' and move on a horizontal plane. At time 't', the mass centre of rods is at the points (ξ, η) referred to fixed perpendicular axes Ox, Oy in the plane, and the rods make angles $\theta \pm \phi$ with Ox. Prove that the kinetic energy of the system is

$$m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right].$$

Also derive Lagrange's equations of motion for the system if an external force with components [X, Y] along the axes acts at 'A'.

Sol: Let, two uniform rods AB, AC each of mass 'm' and length = 2a'. smoothly hinged together at 'A'.



For the two rods:

$$T = \frac{1}{2}m\left[\dot{x}_D^2 + \dot{y}_D^2\right] + \frac{1}{2}\left[\frac{m\,4a^2}{12}\right](\dot{\theta} + \dot{\phi})^2 + \frac{1}{2}m\left[\dot{x}_E^2 + \dot{y}_E^2\right] + \frac{1}{2}\left[\frac{m\,4a^2}{12}\right](\dot{\theta} - \dot{\phi})^2 \quad (1)$$

$G = \{\xi, \eta\}$ = centre of mass

From the diagram

$x_D = \xi - a\sin\phi\sin\theta \qquad y_D = \eta + a\sin\phi\cos\theta$

$x_E = \xi + a\sin\phi\sin\theta \qquad y_E = \eta - a\sin\phi\cos\theta$

$$\therefore \dot{x}_D = \dot{\xi} - a\cos\phi\sin\theta\,\dot{\phi} - a\sin\phi\cos\theta\cdot\dot{\theta}$$

$$\dot{x}_E = \dot{\xi} + a\cos\phi\sin\theta\cdot\dot{\phi} + a\sin\phi\cos\theta\cdot\dot{\theta}$$

$$\dot{y}_D = \dot{\eta} + a\cos\phi\cos\theta\,\dot{\phi} - a\sin\phi\sin\theta\,\dot{\theta}$$

$$\dot{y}_E = \dot{\eta} - a\cos\phi\cos\theta\,\dot{\phi} + a\sin\phi\sin\theta\,\dot{\theta}$$

So;

$$\dot{x}_D^2 + \dot{x}_E^2 = 2\dot{\xi}^2 + 2(a\cos\phi\sin\theta\cdot\dot{\phi} + a\sin\phi\cos\theta\,\dot{\theta})^2$$

$$\dot{x}_D^2 + \dot{x}_E^2 = 2\dot{\xi}^2 + 2a^2\cos^2\phi\sin^2\theta\,\dot{\phi}^2 + 2a^2\sin^2\phi\cos^2\phi\,\dot{\theta}^2$$
$$+ 4a^2\sin\phi\cos\phi\sin\theta\cdot\cos\theta\,\dot{\phi}\dot{\theta}. \quad \text{—(1)}$$

Similarly.;

$$\dot{y}_D^2 + \dot{y}_E^2 = 2\dot{\eta}^2 + 2a^2\cos^2\phi\cos^2\theta\,\dot{\phi}^2 + 2a^2\sin^2\phi\sin^2\theta\cdot\dot{\theta}^2$$
$$- 4a^2\sin\phi\cos\phi\sin\theta\cdot\cos\theta\,\dot{\phi}\dot{\theta}. \quad \text{—(2)}$$

$\therefore$ from (1)+(2)

$$\dot{x}_D^2 + \dot{x}_E^2 + \dot{y}_D^2 + \dot{y}_E^2 = 2\left[\dot{\xi}^2 + \dot{\eta}^2 + a^2\cos^2\phi\,\dot{\phi}^2 + a^2\sin^2\phi\,\dot{\theta}^2\right] \quad \text{(3)}$$

Rearranging (1)

$$T = \frac{1}{2}m\left[\dot{x}_D^2 + \dot{y}_D^2 + \dot{x}_E^2 + \dot{y}_E^2\right] + \frac{1}{3}\frac{ma^2}{2}\left[2\dot{\phi}^2 + 2\dot{\theta}^2\right]$$

using (2) in above equation

$$T = m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\frac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right]$$

proved.

Now, constant force [X, Y] acts at A

$$\therefore V = -\int_A F\cdot dr.$$

co-ordinates of A = $[\xi - a\cos\phi\cos\theta,\ \eta - a\cos\phi\sin\theta]$

$$\therefore V = X(a\cos\phi\cos\theta - \xi) + Y(a\cos\phi\sin\theta - \eta] + \text{const}$$

$$L = T - V$$

$$L = m\left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\tfrac{1}{3} + \sin^2\phi\right)a^2\dot{\theta}^2 + \left(\tfrac{1}{3} + \cos^2\phi\right)a^2\dot{\phi}^2\right]$$
$$+ X[\xi - a\cos\phi\cos\theta] + Y[\eta - a\cos\phi\sin\theta] + \text{const.}$$

Equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

For ξ $\quad \frac{d}{dt}\left[2m\dot{\xi}\right] - X = 0$

$$\Rightarrow \ddot{\xi} - \frac{X}{2m} = 0 \quad \text{—— (I)}$$

For η $\rightarrow$ $\quad \ddot{\eta} - \frac{Y}{2m} = 0$ —— II.

For ϕ $\rightarrow \frac{d}{dt}\left[\left(\frac{1}{3} + \cos^2\phi\right)2a^2\dot{\phi}\right] - \left[2\sin\phi\cos\phi\, a^2\dot{\theta}^2 - 2\sin\phi\cos\phi\, a^2\dot{\phi}^2 + Xa\sin\phi\cos\theta + Ya\sin\phi\,\cancel{\sin}\theta\right] = 0$

$$\Rightarrow 2a^2\left[\tfrac{1}{3} + \cos^2\phi\right]\ddot{\phi} - a^2\left(\sin 2\phi\,\dot{\theta}^2 - \sin 2\phi\,\dot{\phi}^2\right) - a\sin\phi\,(X\cos\theta + Y\sin\theta) = 0$$

$$\Rightarrow 2a^2\left(\tfrac{1}{3} + \cos^2\phi\right)\ddot{\phi} - a^2\sin 2\phi\,(\dot{\theta}^2 - \dot{\phi}^2) - a\sin\phi\,(X\cos\theta + Y\sin\theta) = 0 \quad \text{—— (III)}$$

For θ:-

$$\frac{d}{dt}\left[\left(\tfrac{1}{3} + \sin^2\phi\right]2a^2\dot{\theta}\right] - \left(Xa\cos\phi\sin\theta - Ya\cos\phi\cos\theta\right) = 0$$

$$= 2a^2\left(\tfrac{1}{3} + \sin^2\phi\right)\ddot{\theta} - a\cos\theta\,(X\sin\theta - Y\cos\theta) = 0 \quad \text{—— (IV)}$$

7.(a) Reduce the Eqn

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canonical form and hence solve it. —(1) (15)

Sol: Given Eqn can be re-written as-

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x}p - \frac{x^2}{y}q = 0 \quad \text{— (2)}$$

Comparing (2) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

Here, $R = y^2$, $S = -2xy$, $T = x^2$

$S^2 - 4RT = 0 \Rightarrow$ PDE (1) is parabolic.

The λ-quadratic equation, $R\lambda^2 + S\lambda + T = 0$

reduces to, $y^2\lambda^2 - 2xy\lambda + x^2 = 0$

or $(y\lambda - x)^2 = 0 \Rightarrow \lambda = \frac{x}{y}, \frac{x}{y}$

The corresponding characteristic Eqn is-

$$\frac{dy}{dx} + \frac{x}{y} = 0 \quad \text{or} \quad xdx + ydy = 0$$

So that, $\frac{x^2}{2} + \frac{y^2}{2} = c_1$, c_1 being arbitrary constant.

Let us choose,

$$u = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{and} \quad v = \frac{x^2}{2} - \frac{y^2}{2} \quad \text{—(3)}$$

in such a way that u and v are independent functions of x, y as varified by Jacobian—

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} x & y \\ x & -y \end{vmatrix}$$

$$= -2xy \neq 0.$$

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

$$= x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \quad \text{using (3)} \quad \text{—(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \quad \text{—(5)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$$

$$= 1\cdot\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + x\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{by (3)}$$

—(6)

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{—(7)}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) \quad \text{—(8)}$$

using above values in (2) and simplifying

$$4x^2y^2\left(\frac{\partial^2 z}{\partial v^2}\right) = 0 \Rightarrow \frac{\partial^2 z}{\partial v^2} = 0. \quad \text{—(9)}$$

which is required canonical form.

Integrating (9) partially w.r.t v, $\frac{\partial z}{\partial v} = \phi(u)$

Again " " " , $z = v\cdot\phi(u) + \psi(u)$

$$\therefore z = \frac{(x^2-y^2)}{2}\,\phi\left(\frac{x^2+y^2}{2}\right) + \psi\left(\frac{x^2+y^2}{2}\right).$$

7(b) → Derive the formula

$$\int_a^b y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\right]$$

Is there any restriction on n? State that condition. What is the error bound in the case of Simpson's 3/8 rule?

Sol'n: We know that the Newton's – Cotes formula (A general quadrature for equidistant ordinates)

$$I = \int_a^b f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{2}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0 + \cdots\right]$$

Putting $n=3$ in the quadrature formula and taking the curve through (x_i, y_i); $i=0,1,2,3$ as a polynomial of third order so that differences above the third order vanish.

we get $\displaystyle\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx$

$$= 3h\left[y_0 + \tfrac{3}{2}\Delta y_0 + \tfrac{3}{4}\Delta^2 y_0 + \tfrac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}\left(y_0+3y_1+3y_2+3y_3\right).$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3+3y_4+3y_5+y_6\right)$$

and so on.

Adding all these integrals from x_0 to x_0+nh.

where 'n' is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx$$

$$= \frac{3h}{8}\Big[(y_0+3y_1+3y_2+y_3)+(y_3+3y_4+3y_5+y_6) + \cdots + (y_{n-3}+3y_{n-2}+3y_{n-1}+y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+\cdots+y_{n-3})\Big]$$

which is known as Simpson's 3/8 rule.

Note: while applying Simpson's 3/8 rule, the number of sub-intervals should be taken as multiple of 3.

→ The error in Simpson's 3/8 rule is given by

$$\boxed{E = -\frac{3h^5}{80} y^{iv}(\bar{x})},$$ where $y^{iv}(\bar{x})$ is the largest value of the fourth order derivatives.

Note: It may be noted that the errors in Simpson's 1/3 and 3/8 rules are of the same order. However, if we consider the magnitudes of the error terms, Simpson's 1/3 rule is superior to to Simpson's 3/8 rule. Hence Simpson's 3/8 rule, is not so accurate as Simpson's rule, the dominant term in the error of this formula being $-\frac{3}{80}h^5 y^{iv}(\bar{x})$

7(C). A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d. If V and v be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vortex of the cone, then Prove that

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2-V^2)/2K}$$

where K is the pressure divided by the density and is constant.

Sol^n: Let u be the velocity at a distance x from the end A, the equation of motion is

$$u\frac{du}{dx} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

(Since the motion is steady)

$$\Rightarrow \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \quad \text{as } p = k\rho$$

Integrating $\frac{1}{2}u^2 = -k\log\rho + C$

$$\Rightarrow \log\rho - \log A_1 = -\frac{u^2}{2k}$$

$$\Rightarrow \rho = A_1 e^{-u^2/2k} \quad \text{——— (1)}$$

Boundary conditions are

(i) $\rho = \rho_1$ when $u = v$, (ii) $\rho = \rho_2$ when $u = V$

Subjecting (i) & (ii) we obtain $\rho_1 = A_1 e^{-v^2/2k}$ and $\rho_2 = A_1 e^{-V^2/2k}$

This $\Rightarrow \frac{\rho_1}{\rho_2} = e^{(V^2-v^2)/2k}$ ——— (2)

By the equation of continuity

flux at A = flux at B

$$\pi\left(\frac{d}{2}\right)^2 v.\rho_1 = \pi\left(\frac{D}{2}\right)^2 .V.\rho_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{V}{v}\cdot\frac{D^2}{d^2}$$

Now (2) becomes $\frac{V}{v}\cdot\frac{D^2}{d^2} = e^{(V^2-v^2)/2k}$

$$\Rightarrow \frac{v}{V} = \frac{D^2}{d^2}e^{(v^2-V^2)/2k}.$$

8.(a) Given the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \; ; \; t>0 \quad \text{where } c^2 = \frac{T}{m}, \quad -(1)$$

T is the constant tension in the string and m is the mass per unit length of the string.

i) Find the appropriate solution of the above wave equation.

ii) Find also the solution under the conditions

$y(0,t)=0$, $y(l,t)=0$ for all t, and

$\left.\frac{dy}{dt}\right|_{t=0} = 0$, $y(x,0) = a\sin\frac{\pi x}{l}$, $0<x<l$, $a>0$.

(20)

Sol: i) Assume that the solution of (1) is of the form

$y(x,t) = X(t)\,T(t)$.

Then, $\frac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$

Substituting in (1), $XT'' = c^2 X'' T$

ie. $$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T''}{T} \quad -(2)$$

Clearly the LHS of (2) is a function of x only and the ~~right~~ RHS is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say).

Then (2) leads to the ODEs —

$$\frac{d^2X}{dx^2} - kX = 0 \quad -(3) \qquad \frac{d^2T}{dt^2} - kc^2T = 0 \quad -(4)$$

Solving (3) and (4), we get

i) when $k>0$ and equal to p^2, then

$$X = c_1 e^{px} + c_2 e^{-px} \; ; \; T = c_3 e^{cpt} + c_4 e^{-cpt}$$

ii) when $k<0$ and equal to $-p^2$, then

$$X = c_5 \cos px + c_6 \sin px$$

$$T = c_7 \cos cpt + c_8 \sin cpt$$

iii) when $k=0$, $X = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus various possible solutions of wave-eqn (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad -(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad -(6)$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad -(7)$$

Of these three solutions, we choose that solution which is consistent with the physical nature of the problem. As the problem deals with vibrations, y must be a periodic function of x and t.

Hence, required solution is —

$$y = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt). \quad -(8)$$

ii) Given boundary conditions

$y(0,t) = 0$ — (ii) $y(l,t) = 0$ — (iii)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — (iv) ; $y(x,0) = a \sin \frac{\pi x}{l}$ — (v)

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

(ii) gives, $y(0,t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

for this to be true for all time, $c_1 = 0$

$\therefore y(x,t) = \cancel{c_2 \sin cpt + c}$
$c_2 \sin px(c_3 \cos cpt + c_4 \sin cpt)$ — (vi)

$$\frac{\partial y}{\partial t} = c_2 \sin px\left[c_3(-cp \sin cpt) + c_4(cp \cos cpt)\right]$$

By (iv) $\Rightarrow$ $\left.\frac{\partial y}{\partial t}\right|_{t=0} = c_2 \sin px \cdot (c_4 cp) = 0$

$\Rightarrow c_2 c_4 cp = 0$

If $c_2 = 0$, (vi) will give trivial sol, $y(x,t) = 0$.

$\therefore$ the only possibility is $c_4 = 0$

(vi) becomes, $y(x,t) = c_2 c_3 \sin px \cos cpt$ — (vii)

$\therefore$ By (iii), $y(l,t) = c_2 c_3 \sin pl \cos cpt = 0 \;\; \forall t$

Since c_2 and $c_3 \neq 0$, $\therefore \sin pl = 0$

$\Rightarrow pl = n\pi \quad \therefore p = \frac{n\pi}{l}, \; n \in \mathbb{Z}$.

Hence, (vii) reduces to,

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

By condition (v), $y(x,0) = c_2 c_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$

which will get satisfied if we take

$c_2 c_3 = a$ and $n = 1$.

Hence the required solution is —

$$y(x,t) = a \sin \frac{\pi x}{l} \cdot \cos \frac{\pi ct}{l}.$$

Q8(b) Write an Algorithm in the form of a flow chart for Newton-Raphson Method. Describe the Cases of failure of this Method.

Solution: To find a Root of Eq^n $f(x)=0$

The Iterative Scheme for Newton-Raphson Method is given by.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This Method can be fail in following Cases :-

(i) If $f'(x)=0$

(ii) If $f(x)$ is not Continuously differentiable.

(iii) If starting point x_0 is outside the Range of Gaurante-ed Convergence.

$f(x)=0$ is the Equation whose Root is to be found

$df(x)$ is derivative of $f(x)$ w.r.t. x

x_0 is the value of n^{th} Iteration

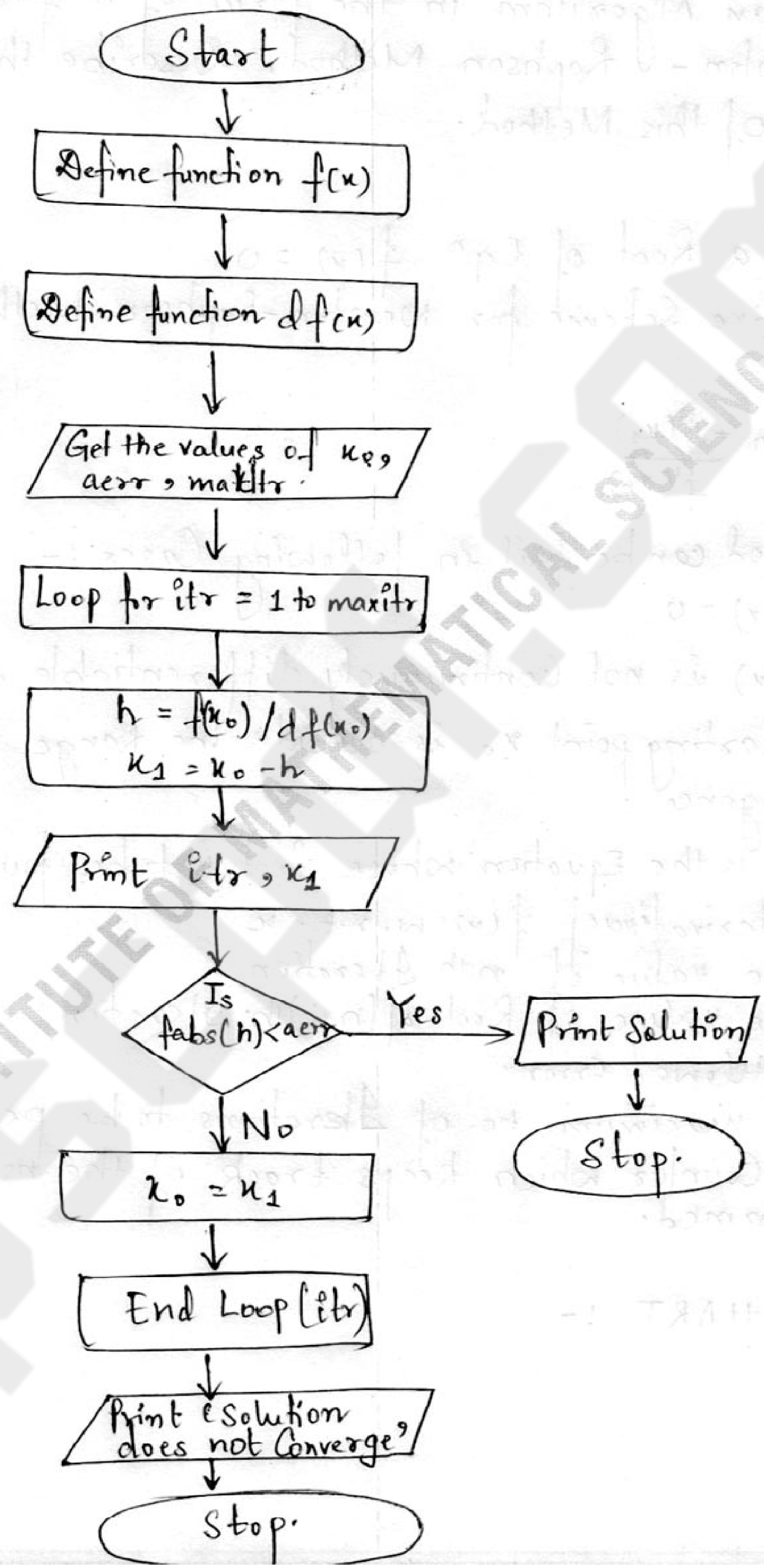
x_1 is the value of Root of $(n+1)$th Iteration

aerr is Allowed Error.

maxitr is maximum no. of Iterations to be performed.

itr is a Counter which keeps track of the no. of Iterations performed.

FLOW CHART :-


Start
Define function f(x)
Define function df(x)
Get the values of x0, aerr, maxitr
Loop for itr = 1 to maxitr
h = f(x0) / df(x0)
x1 = x0 - h
Print itr, x1
Is fabs(h)<aerr
Yes
Print Solution
Stop.
No
x0 = x1
End Loop (itr)
Print 'Solution does not Converge'
Stop.

8(C) 2017 → If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right), r^2 = x^2+y^2+z^2$$

then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$. Further, determine the streamlines.

Sol^n: Here $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$, $w = \frac{3z^2-r^2}{r^5} = \frac{3z^2}{r^5} - \frac{1}{r^3}$ —①

where $r = x^2+y^2+z^2$ ——②.

from ②, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$ —③

from ①, ② and ③, we have

$$\frac{\partial u}{\partial x} = 3z\left[\frac{1}{r^5} + (-5x)r^{-6}\frac{\partial r}{\partial x}\right] = \frac{3z}{r^5} - \frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y} = 3z\left[\frac{1}{r^5} + (-5y)r^{-6}\frac{\partial r}{\partial y}\right] = \frac{3z}{r^5} - \frac{15y^2z}{r^7}$$

$$\frac{\partial w}{\partial z} = \frac{6z}{r^5} - 15z^2r^{-6}\frac{\partial r}{\partial z} + 3r^{-4}\frac{\partial r}{\partial z} = \frac{6z}{r^5} - \frac{15z^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r}$$

$$= \frac{9z}{r^5} - \frac{15z^3}{r^7}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2)$$

$$= \frac{15z}{r^5} - \frac{15z}{r^7}\times r^2 = 0$$

Since the equation of continuity is satisfied by the given values of u, v and w, the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(u\,dx + v\,dy + w\,dz),$$ by definition of ϕ.

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right]$$

$$= \frac{r^2 dz - 3z(x\,dx + y\,dy + z\,dz)}{r^5} = \frac{r^3 dz - 3r^2 z\,dr}{(r^3)^2} = d\left(\frac{z}{r^3}\right)$$
using ②

Integrating, $\boxed{\phi = \frac{z}{r^3}}$

[Omitting constant of integration, for it has no significance in ϕ] In spherical polar coordinates (r, θ, ϕ), we know that $z = r\cos\theta$. Hence required potential is given by

$$\phi = \frac{(r\cos\theta)}{r^3} = \frac{\cos\theta}{r^2}$$

We now obtain the streamlines. The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \text{ i.e., } \frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2-r^2)/r^5}$$

$$\Rightarrow \frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - r^2} \quad \text{—— ③}$$

Taking the first two members of ③ and simplifying we get

$$dx/x = dy/y \Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating $\log x - \log y = \log c_1$ i.e: $x/y = c_1$, c_1 being a constant. —— ④

$$\text{Now, each member in ③} = \frac{x\,dx + y\,dy + z\,dz}{3x^2 z + 3y^2 z + 3z^3 - r^2 z}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{3z(x^2+y^2+z^2) - r^2 z}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{3z(x^2+y^2+z^2) - z(x^2+y^2+z^2)} = \frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)}, \text{ by ②} \quad \text{—— ⑤}$$

Taking the first member in ③ and ⑤, we get

$$\frac{2}{3}\frac{dx}{x} = \frac{1}{2}\,\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Integrating $\frac{2}{3}\log x = \frac{1}{2}\log(x^2+y^2+z^2) + \log c_2$

$x^{2/3} = c_2 (x^2+y^2+z^2)^{1/2}$, c_2 being an arbitrary constant.

The required streamlines are the curves of intersection of ④ and ⑥