Main Test Series – 2018
Test – 15 – Paper-I. Answer key

1(a) Find a basis and dimension of the subspace W of V spanned by the polynomials
$v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$, $v_3 = t^3 + 6t - 5$
$v_4 = 2t^3 - 5t^2 + 7t + 5$.

Sol^n: Since W is spanned by polynomials of degree 3.
∴ W is a subspace of the space $V_3(\mathbb{R})$ (the space of all real polynomials of degree ≤ 3)

W.k.T $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$
∴ The co-ordinate vectors of v_1, v_2, v_3, v_4 w.r.t the above basis are
$(1, 4, -2, 1)$, $(-1, 9, -3, 2)$, $(-5, 6, 0, 1)$ and $(5, 7, -5, 2)$
Now form the matrix A whose rows are these co-ordinate vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ -5 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 5R_1 \\ R_4 \to R_4 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 + R_2 \end{matrix}$$

which is in the echelon form

The non-zero rows of the echelon form of A form a basis of the subspace W.

i.e. the vectors $(1,4,-2,1)$, $(0,13,-5,3)$ form a basis for W.

$\therefore$ A basis for W consists of polynomials $t^3 - 2t^2 + 4t + 1$ and $3t^3 - 5t^2 + 13t$

$\therefore$ $\dim W = 2$.

1(b) → If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-\sqrt{3}i}{2} \end{bmatrix}$ then find trace of A^{102}.

Sol^n: let $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-\sqrt{3}i}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix}$ say

where $1, \omega, \omega^2$ are cube roots of unity.

If $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ then trace of $A^n = a^n + c^n + f^n$

$\therefore \operatorname{tr}(A^{102}) = a^{102} + c^{102} + f^{102}$

$\Rightarrow \operatorname{tr}(A^{102}) = 1^{(102)} + \omega^{(102)} + (\omega^2)^{(102)}$

$\Rightarrow \operatorname{tr}(A^{102}) = 1 + (\omega^3)^{34} + (\omega^3)^{68}$

$= 1 + (1)^{34} + (1)^{68}$

$= 1 + 1 + 1 = 3$

$\therefore \operatorname{tr}(A^{102}) = 3$

1(d) Let ϕ be a function of two variables defined as

$\phi(x,y) = \frac{x^3+y^3}{x-y}$, when $x \neq y$

$\phi(x,y) = 0$, when $x = y$.

Show that ϕ is discontinuous at the origin, but the first order partial derivatives exist at that point.

Soln: Suppose $(x,y) \to (0,0)$ along the curve $y = x - mx^3$.

Then $\lim_{x\to 0} \phi(x, x-mx^3) = \lim_{x\to 0} \frac{x^3 + (x-mx^3)^3}{mx^3}$

$= \lim_{x\to 0} \frac{x^3[1+(1-mx^2)^3]}{mx^3}$

$= \lim_{x\to 0} \frac{[1+(1-mx^2)^3]}{m}$

$= \frac{2}{m}$ which is different for different values of m

Thus $\lim_{(x,y)\to(0,0)} \phi(x,y)$ doesnot exist and so the given function is discontinuous at $(0,0)$.

Now, $\phi_x(0,0) = \lim_{h\to 0} \frac{[\phi(0+h,0) - \phi(0,0)]}{h} = \lim_{h\to 0} \frac{h^2-0}{h} = 0$

$\phi_y(0,0) = \lim_{k\to 0} \frac{[\phi(0,0+k), \phi(0,0)]}{k} = \lim_{k\to 0} \frac{-k^2-0}{k} = 0$

$\therefore$ first order partial derivatives exist at the origin. Hence the result.

1(e) Prove that the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie.

Sol: Given lines are coplanar, if

$$\begin{vmatrix} (a-d)-(b-c) & (a-b) & (a+d)-(b+c) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

Adding third column to first we get

$$\begin{vmatrix} 2(a-b) & a-b & (a+d)-(b+c) \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0$$

The first column being twice the second column, the determinant on the left vanishes, hence the given lines are coplanar.

Also the equation of the plane in which the two given lines lie is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} = 0 \quad \text{adding 3rd column to the first}$$

$$\Rightarrow \begin{vmatrix} (x+z-2a)-2(y-a) & y-a & z-a-d \\ 2\alpha-2(\alpha) & \alpha & \alpha+\delta \\ 2\beta-2(\beta) & \beta & \beta+\gamma \end{vmatrix} = 0, \quad \text{Subtracting twice second column from first}$$

$$\Rightarrow \begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix} = 0$$

$$\Rightarrow (x+z-2y)[\alpha(\beta+\gamma)-\beta(\alpha+\delta)] = 0$$

$$\Rightarrow x+z-2y = 0.$$

2(a) (i) Let W be the vectorspace of 3×3 antisymmetric matrices over K. Show that $\dim W = 3$ by exhibiting a basis of W.

(ii) If B is non-singular, Prove that the matrices A and $B^{-1}AB$ have the same determinant, A and B being both square matrices of order n.

Soln: (i) Let $W(K) = \left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \Big/ a, b, c \in K \right\}$

be the vectorspace of all 3×3 anti-symmetric matrices.

let $A = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \in W(K)$ then

$$A = a\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in L(S)$$

where $S = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq W(K)$

$\therefore A \in W(K) \Rightarrow A \in L(S)$

$\therefore L(S) = W(K)$

Clearly S is linearly independent subset of $W(K)$

$\therefore S$ is a basis of W and $\dim(W) = 3$.

(ii) we have $\det(B^{-1}AB) = (\det B^{-1})(\det A)(\det B)$

$= (\det B^{-1})(\det B)(\det A)$

$= (\det B^{-1}B)\det A$

$= (\det I)(\det A) = \det A$.

$\therefore A$ and $B^{-1}AB$ have the same determinant.

2(b) Find the dimension of the subspace
$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x+y+z+w=0,\ x+y+2z=0,\ x+3y=0\}$

Sol'n. $\because$ $x+y+z+w=0$
$x+y+2z=0$
$x+3y=0$

$\Rightarrow AX = 0$ —— (1)

where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$; $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_3$$

we write a single matrix equation

$$x+y+z+w=0$$
$$2y-z-w=0$$
$$z-w=0$$

$\Rightarrow \boxed{z=w}$: $2y-w-w=0$
$\Rightarrow 2y=-2w$
$\Rightarrow \boxed{y=-w}$

$\therefore x-w+w+w=0 \Rightarrow \boxed{x=-w}$

$\therefore W = \{(-w, -w, w, w) \in \mathbb{R}^4 \mid w \in \mathbb{R}\}$

Clearly it contains only one free variable w.

$\therefore \boxed{\dim(W)=1}$

2(c) Find the maximum and minimum values of $f(x,y) = x^2+3y^2+2y$ on the unit disc $x^2+y^2 \le 1$.

Sol^n: Extreme values of $f(x,y) = x^2+3y^2+2y$ are found at $f_x(x,y)=0$, $f_y(x,y)=0$

$$f_x = 2x = 0 \Rightarrow x = 0$$

$$f_y = 6y+2 = 0 \Rightarrow y = -\tfrac{1}{3}$$

$(0,-\tfrac{1}{3})$ lies within the unit disc

$f_{xx} = 2$, $f_{yy} = 6$ $f_{xy} = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(6) - 0$$
$$= 12 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 > 0$$

$\therefore$ minimum value is found at $(0,-\tfrac{1}{3})$

Minimum value $= -\tfrac{1}{3}$ at $(0,-\tfrac{1}{3})$

Maximum value found on the circumference of disc.

$\therefore$ put $x^2+y^2 = 1$

$$f(x,y) = 1-y^2+3y^2+2y$$

Let $g(y) = 1+2y+2y^2$

for $y \in (0,1)$, $g'(y) > 0 \ \forall \ y \in (0,1)$

$\therefore$ Maximum at $y=1$

$\therefore$ Maximum value at $(0,1)$

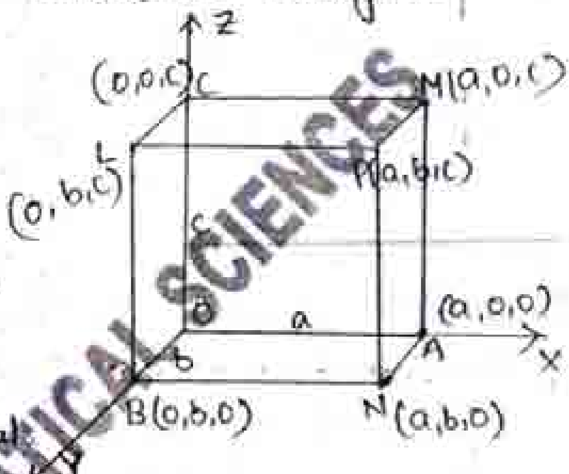
$\therefore$ $f(x,y) = 5$ at $(0,1)$

Maximum value $= 5$ at $(0,1)$

2(d)(i) If the edges of a rectangular parallepiped be a, b, c show that the angles between the four diagonals are given by $\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right]$

Sol: Take 'O', a corner of the rectangular parallelopiped as the origin and three edges OA, OB, OC through it (i.e 'O') as the axes.

Then the coordinates of the various corners are O(0,0,0), A(a,0,0), B(0,b,0), C(0,0,c), L(0,b,c), M(a,0,c), N(a,b,0), P(a,b,c). The four diagonals are AL, BM, CN and OP.



the d.c's of AL are proportional to $0-a, b-0, c-0 \Rightarrow -a, b, c$ using $(x_2-x_1, y_2-y_1, z_2-z_1)$

Similarly d.c's of BM are proportional to $a, -b, c$

d.c's of CN are proportional to $a, b, -c$

d.c's of OP are proportional to a, b, c

If α is the angle b/w the diagonals OP and AL then

$$\cos\alpha = \frac{a(-a)+b(b)+c(c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\therefore \alpha = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right) \quad \text{——— ①}$$

Similarly angle b/w OP & BM $= \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$

" " OP & CN $= \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$

" " AL & BM $= \cos^{-1}\left(\frac{-a^2-b^2+c^2}{a^2+b^2+c^2}\right)$

$$= \cos^{-1}\left|\frac{-(a^2+b^2-c^2)}{a^2+b^2+c^2}\right|$$

$= \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$ (when acute angle is taken)

This angle is the same as the angle b/w OP and CN. Similarly we can show that all other angle b/w other two diagonals are repeated and we get only three different angles as given by ①, ② and ③.

Hence the angles between four diagonals are given by $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$

2(d)(iii) Find the incentre of the tetrahedron formed by the planes $x=0, y=0, z=0$ and $x+y+z=a$.

sol'n: Evidently the planes $x=0, y=0$ and $z=0$ meet in $(0,0,0)$. Hence the incentre lies on the ⊥lar from $(0,0,0)$ to the plane $x+y+z=a$ and divides it in the ratio 3:1 [3 from the vertex $(0,0,0)$ & 1 from the plane $x+y+z=a$].

The equations of the ⊥lar from $(0,0,0)$ to the plane $x+y+z=a$ is $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=r$

Any point on this ⊥lar is (r,r,r). If it lies on the plane $x+y+z=a$, then we have $r+r+r=a \Rightarrow r=a/3$

∴ The ⊥lar from $(0,0,0)$ meets the plane $x+y+z=a$ in (r,r,r) i.e. $(a/3, a/3, a/3)$. Also the incentre divides the join of $(0,0,0)$ and $(a/3, a/3, a/3)$ in the ratio 3:1, the required incentre,

Therefore if (x_1, y_1, z_1) be the required incentre,

we have $x_1 = \frac{3.\frac{1}{3}a + 1.0}{3+1} = a/4$.

Similarly $y_1 = \frac{1}{4}a = z$

∴ The required incentre is $(a/4, a/4, a/4)$

3(a)(i) Let P_n denote the vectorspace of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$. Find the matrix of T w.r.to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T.

(ii) Let V be an n-dimensional vectorspace and $T: V \to V$ be an invertible linear operator. If $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis of V, show that $\beta' = \{Tx_1, Tx_2, \ldots, Tx_n\}$ is also a basis of V.

Sol'n: (i) Given $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$

basis for P_2 is $\{1, x, x^2\}$ and

basis for P_3 is $\{1, x, 1+x^2, 1+x^3\}$

Now

$$T(1) = \int_0^x 1\,dt = x = 0\cdot 1 + 1\cdot x + 0(1+x^2) + 0(1+x^3)$$

$$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = -\tfrac{1}{2}\cdot 1 + 0\cdot x + \tfrac{1}{2}(1+x^2) + 0(1+x^3)$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = -\tfrac{1}{3}\cdot 1 + 0\cdot x + 0\cdot(1+x^2) + \tfrac{1}{3}(1+x^3)$$

Matrix of T w.r.t bases β_1 and β_2 is

$$[T : \beta_1, \beta_2] = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Null space of T will be given by

$\int_0^2 p(t)dt = 0$ i.e. if $p(x) = a_0 + a_1x + a_2x^2$.

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a_0 = 0\ ;\ a_1 = 0;\ a_2 = 0$

$\therefore a_0 = a_1 = a_2 = 0$

$\therefore p(x) = 0$

$\therefore$ Null Space of T Contains only a single element $\{0\}$.

(ii) Since it is given that $T: V \to V$ is invertible. so, T must also be one-one and onto. Also T is linear.

so, $T(\alpha) = 0 \Leftrightarrow \alpha = 0$ —— ①

Again V is a vectorspace of dimension 'n', so any linearly independent set of dimension 'n' can form its basis. —— ②

Consider set $B' = \{Tx_1, Tx_2, --- Tx_n\}$

let $\alpha_1, \alpha_2, -- \alpha_n$ be n scalars such that

$\alpha_1Tx_1 + \alpha_2Tx_2 + --- \alpha_nTx_n = 0$ —— ③

by property of linear transformation ③ becomes

$T(\alpha_1x_1 + \alpha_2x_2 + --- \alpha_nx_n) = 0$

from ① $\alpha_1x_1 + \alpha_2x_2 + --- \alpha_nx_n = 0$.

$B(x_1, x_2 -- x_n)$ forms a basis of V. so must be LI

$\alpha_1 = \alpha_2 = --- = \alpha_n = 0$

$Tx_1, Tx_2 --- Tx_n$ are LI.

so, from ② set B' forms a basis for V.

3(b) (i) Find all the maxima and minima of the function given by $f(x,y) = x^3+y^3-63(x+y)+12xy$

(ii) If $v = At^{-1/2} e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$

Sol^n: we have.

$f_x(x,y) = 3x^2-63+12y$, $f_y(x,y) = 3y^2-63+12x$

$f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = 12$, $f_{yy}(x,y) = 6y$

The critical points of f are given by $f_x = 0$, $f_y = 0$

Thus $f_x(x,y) = 3x^2-63+12y = 0$

$f_x(x,y) - f_y(x,y) = 3(x^2-y^2) + 12(y-x) = 0$

$\Rightarrow x^2-21+4y = 0$, $(y-x)(4-y-x) = 0$

$\therefore \begin{cases} x^2-21+4x=0 \\ (x+7)(x-3)=0 \end{cases}$ (if $y=x$) and $\begin{cases} x^2-21+4y=0 \\ 4-y-x=0 \end{cases}$

Solving these, the four critical points are

$(-7,-7)$, $(3,3)$, $(5,-1)$, $(-1,5)$

At $(-7,-7)$

$A = f_{xx} = -42 < 0$ and $AC-B^2 = f_{xx}f_{yy} - (f_{xy})^2 = 1620 > 0$

So that the function is maximum at $(-7,-7)$

At $(3,3)$

$A = f_{xx} = 18 > 0$ and $AC-B^2 = f_{xx}f_{yy} - (f_{xy})^2 = 180 > 0$

So that the function is minimum at $(3,3)$

At each of the other points $(5,-1)$ and $(-1,5)$

$AC-B^2 = f_{xx}f_{yy} - (f_{xy})^2 = -324 < 0$.

So that the function is neither a maximum nor a minimum.

3(b)(ii) Sol^n: we have $v = At^{-1/2}e^{-x^2/4a^2t}$

$$\Rightarrow \frac{\partial v}{\partial x} = At^{-1/2}e^{-x^2/4a^2t}\left(\frac{-2x}{4a^2t}\right) = \frac{-x}{2a^2t}v$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{-1}{2a^2t}\left[v + x\frac{\partial v}{\partial x}\right]$$

$$= \frac{-1}{2a^2t}\left[v + x\left(\frac{-vx}{2a^2t}\right)\right]$$

$$= \frac{v}{4a^4t^2}\left(-2a^2t + x^2\right)$$

Again $\frac{\partial v}{\partial t} = At^{-1/2}e^{-x^2/4a^2t}\left(\frac{x^2}{4a^2t^2}\right) - A\frac{1}{2}t^{-3/2}e^{-x^2/4a^2t}$

$$= At^{-1/2}e^{-x^2/4a^2t}\left[\frac{x^2}{4a^2t^2} - \frac{1}{2t}\right]$$

$$= \frac{v}{4a^2t^2}\left(x^2 - 2a^2t\right)$$

clearly, $\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$

3(c) The generators through P of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meets the principal elliptic section at A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, show that P lies on the surface $z(\alpha x+\beta y)+\gamma(c^2+z^2)=0$

Sol^n:- Let the co-ordinates of P, A and B be (x_1, y_1, z_1), $(a\cos\theta, b\sin\theta, 0)$ and $(a\cos\phi, b\sin\phi, 0)$ respectively.

Also the co-ordinates of F, the mid point of AB are $\left[\frac{1}{2}(a\cos\theta+a\cos\phi), \frac{1}{2}(b\sin\theta+b\sin\phi), 0\right]$

$$= \left(a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ 0\right)$$

Direction ratios of the median PF through P are

$$x_1-a\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),\ y_1-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ z_1-0 \quad \text{--- (1)}$$

[The values of x_1, y_1, z_1 can be found as follows

The equation of the tangent to the given hyperboloid at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$ and

$\therefore$ meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0 \quad \text{——(i)}$$

which is the same as the line joining the points A and B

i.e, $\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2},\ z=0$ ——(ii)

Comparing (i) & (ii), we get

$$\frac{x_1/a^2}{\frac{1}{a}\cos\frac{\theta+\phi}{2}}=\frac{y_1/b^2}{\frac{1}{b}\sin\frac{\theta+\phi}{2}}=\frac{1}{\cos\frac{\theta-\phi}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos(\theta+\phi)/2}{\cos\frac{\theta-\phi}{2}},\quad \frac{y_1}{b}=\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \quad \text{——(iii)}$$

Again $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1$

$$\Rightarrow \frac{1}{\cos^2\frac{\theta-\phi}{2}}-\frac{z_1^2}{c^2}=1 \Rightarrow \frac{z_1^2}{c^2}=\sec^2\left(\frac{\theta-\phi}{2}\right)-1$$

$$=\tan^2\frac{\theta-\phi}{2}$$

$$\Rightarrow \frac{z_1}{c}=\pm\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$\therefore P(x_1,y_1,z_1)=\boxed{\left(\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ \pm\frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right)}$$

$\therefore$ from (1), we have.

$$a\,\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}=a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}=b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\quad \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}$$

$\Rightarrow \dfrac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right), \dfrac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right), \dfrac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$

$\Rightarrow a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2}, \; b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2}, \; c\tan\frac{\theta-\phi}{2}\operatorname{cosec}^2\frac{\theta-\phi}{2}.$

$\Rightarrow x_1, y_1, z_1\operatorname{cosec}^2\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\cot^2\frac{\theta-\phi}{2}\right)$ when $\frac{z_1}{c} = \tan\frac{\theta-\phi}{2}$

$\Rightarrow x_1, y_1, z_1\left(1+\frac{c^2}{z_1^2}\right)$

As PF is parallel to the plane

$\alpha x + \beta y + \gamma z = 0$

$\therefore \alpha x_1 + \beta y_1 + \gamma z_1\left(1+\frac{c^2}{z_1^2}\right) = 0$

$\Rightarrow (\alpha x_1 + \beta y_1)z_1 + \gamma(z_1^2 + c^2) = 0$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$

is $z(\alpha x + \beta y) + \gamma(z^2 + c^2) = 0$

— Hence proved

4(a) Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. Is A diagonalizable?

If yes find P such that $P^{-1}AP$ is diagonal.

Sol'n. The characteristic equation of A is

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)\left[\lambda^2 - 7\lambda + 12\right] - (6-2\lambda) - (\lambda-3) = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$\Rightarrow (\lambda-3)^2(\lambda-5) = 0$$

$$\Rightarrow \lambda = 3, 3, 5$$

$\therefore$ The eigen values of the matrix A are 3, 3, 5.

The eigen vectors x of A corresponding to the eigen value 3 are given by the equation.

$(A-3I)X=0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix of coefficients of these equations has rank 1. Therefore these equations have two linearly independent solutions. These equations reduce to the single equation

$x_1 + x_2 - x_3 = 0$

$\therefore X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are two linearly independent solutions of this equation.

Therefore X_1 and X_2 are two linearly independent eigen vectors of A corresponding to the eigen value 3. Thus the geometric multiplicity of the eigenvalue 3 is equal to its algebraic multiplicity.

Now the eigen vectors of A corresponding to the eigen value 5 are given by $(A-5I)X=0$

$$\Rightarrow \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix of coefficients of these equations has rank 2. Therefore these equations have $3-2=1$ linearly independent solution. These equations can be written as $-x_1+x_2-x_3=0$, $x_2-4x_3=0$.

From these, we get $x_1=1$, $x_2=2$, $x_3=1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 5.

The geometric multiplicity of the eigen value 5 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigen value of A is equal to its algebraic multiplicity therefore A is similar to a diagonal matrix.

Let $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

The columns of P are linearly independent eigen vectors of A corresponding to the eigen values 3, 3, 5 respectively. The matrix P will transform A to diagonal form D which is given by the relation

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D.$$

4(b). Find the values of a, b and c so that

$$\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$$

Sol^n: $\lim_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x}$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{ae^x - b\cos x + ce^{-x}}{x^2}\right)$$

$$= (1)\lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x^2} \quad \text{--- (1)} \quad \left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

Since the denominator of (1) $\to 0$ as $x\to 0$

but (1) $\to$ a finite limit 2.

$\therefore$ The numerator $ae^x - b\cos x + ce^{-x}$ must $\to 0$ as $x\to 0$

$\therefore a - b + c = 0$ --- (2)

Also, if the relation (2) holds, then

(1) $= \lim_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x^2}$ is of the form $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{2x} \quad \left(\tfrac{0}{0}\text{ form}\right)$$

For existence of the limit Nr $\to 0$ as $x\to 0$ $\therefore \boxed{\frac{a-c}{2} = 0}$ --- (3)

$$= \lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{2}$$

$$= \frac{a+b+c}{2} = 2 \text{ (given)}$$

$\Rightarrow a + b + c = 4$ --- (4)

$\therefore$ Solving (2), (3) and (4)

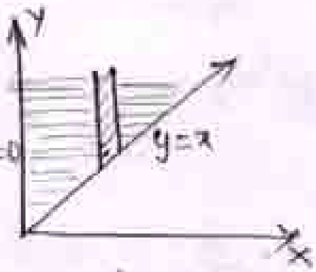
we get $\boxed{a = 1,\ b = 2,\ c = 1}$

4(c) Evaluate $\int_0^{\infty}\int_x^{\infty}\left(\frac{1}{y}\right)e^{-y/2}\,dy\,dx$ by changing the order of integration.

Soln: The limits of integration are given by the straight lines $y=x$

$y=\infty$, $x=0$ and $x=\infty$

i.e. the region of integration is bounded by $y=x$, $x=0$ and infinite boundary.



Hence taking the strips parallel y-axis the limits for x are from $x=0$ to $x=y$ and the limits for y are from $y=0$ to $y=\infty$

Hence changing the order of integration we have

$$\int_0^{\infty}\int_x^{\infty}\frac{1}{y}e^{-y/2}\,dy\,dx = \int_{y=0}^{\infty}\int_{x=0}^{x=y}\frac{1}{y}e^{-y/2}\,dx\,dy$$

$$= \int_{y=0}^{\infty}\frac{1}{y}e^{-y/2}\left[x\right]_{x=0}^{y}dy$$

$$= \int_{y=0}^{\infty}\frac{1}{y}e^{-y/2}\,(y)\,dy$$

$$= \int_{y=0}^{\infty}e^{-y/2}\,dy = \left[\frac{e^{-y/2}}{-1/2}\right]_0^{\infty}$$

$$= -2\left[e^{-y/2}\right]_0^{\infty} = -2[0-1]$$

$$= 2.$$

4(d) Reduce the equation

$$9x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+5=0$$

to the standard form.

What does it represent?

Sol^n: Comparing the given equation $F(x,y,z)=0$ with the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

we have $a=2$, $b=-7$, $c=2$, $f=-5$, $g=-4$, $h=-5$, $u=3$, $v=6$, $w=-3$, $d=5$

Now coordinates of the centre (x_1,y_1,z_1) of the given surface are given by

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0 \text{ and } \frac{\partial f}{\partial z}=0$$

$$4x_1-8z_1-10y_1+6=0 \Rightarrow 2x_1-5y_1-4z_1+3=0 \quad (1)$$

$$-14y_1-10z_1-10x_1+12=0 \Rightarrow 5x_1+7y_1+5z_1-6=0 \quad (2)$$

$$4z_1-10y_1-8x_1-6=0 \Rightarrow 4x_1+5y_1-2z_1+3=0 \quad (3)$$

solving (1), (2) and (3) we get

$$x_1=1/3,\ y_1=-1/3,\ z_1=4/3$$

$\therefore$ Centre of the given surface is $(1/3, -1/3, 4/3)$

Also $d' = ux_1+vy_1+wz_1+d$

$$=3(1/3)+6(-1/3)+(-3)(4/3)+5$$

$$=1-2-4+5=0 \quad (4)$$

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix}=0$$

$\Rightarrow (2-\lambda)[-(7+\lambda)(2-\lambda)-25]+5[-5(2-\lambda)-20]$
$-4[25-4(7+\lambda)]=0$

$\Rightarrow \lambda^3+3\lambda^2-90\lambda+216=0$

$\Rightarrow (\lambda-3)(\lambda^2+6\lambda-72)=0$

$\Rightarrow (\lambda-3)(\lambda-6)(\lambda+12)=0$

$\Rightarrow \lambda = 3, 6, -12.$

$\therefore$ Let $\lambda_1=3,\ \lambda_2=6,\ \lambda_3=-12$

$\therefore$ By rotation of axes the given equation transforms to

$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$

substituting the values of $\lambda_1, \lambda_2, \lambda_3, d'$

$\Rightarrow 3x^2+6y^2-12z^2+0=0$

$\Rightarrow x^2+2y^2-4z^2=0$

which is the required standard form and represents a cone.

Also the vertex of the cone is

$\left(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}\right)$.

5(a) Solve $(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$

Sol'n: Given that

$$(x^2+y^2)(1+p^2)-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$$

Put $x+y=u$, $x^2+y^2=v$.

$$\Rightarrow 1+\frac{dy}{dx}=\frac{du}{dx} \;;\; 2x+2y\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow 1+p=\frac{du}{dx} \;;\; 2x+2yp=\frac{dv}{dx}$$

$$\therefore \frac{dv}{du}=\frac{2(x+yp)}{1+p}$$

$$\Rightarrow P=\frac{2(x+yp)}{1+p} \;;\; \text{where } P=\frac{dv}{du},\ p=\frac{dy}{dx}$$

$$\Rightarrow P(1+p)=2(x+yp)$$

$$\Rightarrow P+Pp=2x+2yp \Rightarrow P-2x=p(2y-P)$$

$$\Rightarrow p=\frac{P-2x}{2y-P} \quad \text{———} (1)$$

using (1) the given equation becomes

$$(x^2+y^2)\left[1+\frac{P-2x}{2y-P}\right]^2-2(x+y)\left(1+\frac{P-2x}{2y-P}\right)\left(x+y\,\frac{P-2x}{2y-P}\right)+\left[x+y\left(\frac{P-2x}{2y-P}\right)\right]^2=0$$

$$\Rightarrow (x^2+y^2)\left[\frac{2y-2x}{2y-P}\right]^2-2(x+y)\left[\frac{2y-2x}{2y-P}\right]P\frac{(y-x)}{2y-P}+P^2\left(\frac{y-x}{2y-P}\right)^2=0$$

$$\Rightarrow (x^2+y^2)4(y-x)^2-4(x+y)(y-x)^2P+P^2(y-x)^2=0$$

$$\Rightarrow 4(x^2+y^2)-4P(x+y)+P^2=0$$

$$\Rightarrow 4v-4Pu+P^2=0$$

$$\Rightarrow v=Pu-\frac{P^2}{4}$$

which ~~is~~ of Clairaut's form and its solution is

$$v=uc-c^2/4$$

i.e., $x^2+y^2=(x+y)c-c^2/4$

5(b) Find the orthogonal trajectories of the family of circles $x^2+y^2+2fy+1=0$, f being a parameter.

Sol^n: Given $x^2+y^2+2fy+1=0$, where f is parameter. —— ①

Differentiating ① w.r.t 'x', $2x+2y\left(\frac{dy}{dx}\right)+2f\left(\frac{dy}{dx}\right)=0$ —— ②

from ① and ② $2fy=-(1+x^2+y^2)$ and $2f\,(dy/dx)=-\left[2x+2y\left(\frac{dy}{dx}\right)\right]$

on dividing, these give $\frac{2f\,(dy/dx)}{2fy}=\frac{2x+2y\,(dy/dx)}{1+x^2+y^2}$

$$\Rightarrow (1+x^2+y^2)(dy/dx)=2y\left[x+y\,(dy/dx)\right] \quad ③$$

which is the differential equation of ①, Replacing dy/dx by $-dx/dy$, the differential equation of the required orthogonal trajectories is

$$(1+x^2+y^2)\,(-dx/dy)=2y\left[x+y\,(-dx/dy)\right]$$

$$\Rightarrow (dx/dy)\,(y^2-x^2-1)=2xy$$

$$\Rightarrow 2xy\frac{dy}{dx}=y^2-x^2-1 \Rightarrow 2y\frac{dy}{dx}-\frac{1}{x}y^2=-\frac{x^2+1}{x} \quad ④$$

Putting $y^2=v$ so that $2y\,(dy/dx)=\frac{dv}{dx}$, ④ reduces to

$$\left(\frac{dv}{dx}\right)-\left(\frac{1}{x}\right)v=-(x^2+1)/x$$ which is linear equation —— ⑤

Integrating factor of ⑤ $=e^{\int(-1/x)\,dx}=e^{-\log x^{-1}}=x^{-1}=1/x$

and Solution $v/x=-\int\{(x^2+1)/x\}.\frac{1}{x}\,dx+c$

$$=-\int(1+x^{-2})\,dx+c$$

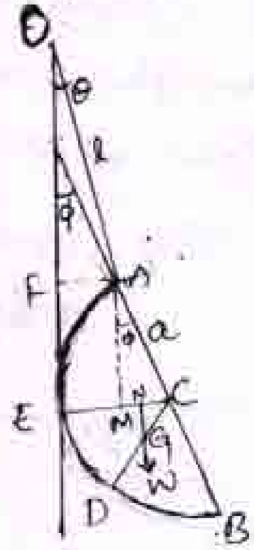
$$y^2/x=-x+(1/x)+c$$

$$\Rightarrow x^2+y^2-cx-1=0$$

$$\Rightarrow x^2+y^2+2gx-1=0$$ where $2g=-c$, g being parameter.

5(c) A Solid Sphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\phi = \frac{3}{8} + \tan\theta.$$



Soln: O is a fixed point in the wall to which one end of the string has been attached.

Let 'l' be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C.

The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that $CG = \frac{3}{8}a$.

The hemisphere touches the wall at E.

we have $\angle OEC = 90°$ so that EC is horizontal.

The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below $O = OF + AM + NG$
$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note that $\angle NCG = 90° - \angle ACM = 90° - (90° - \phi) = \phi$]

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed, the length of the string AO does not change so that the work done by its tension is zero and the point G is slightly displaced. The $\angle OEC$ remains 90°.

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above.

The equation of virtual work is

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0$$

$$\Rightarrow -l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$$

$$\Rightarrow l\sin\theta\,\delta\theta = a\left(\frac{3}{8}\cos\phi - \sin\phi\right)\delta\phi \quad (1)$$

from the figure $EC = a$.

Also $EC = EM + MC = FA + MC$
$= l\sin\theta + a\sin\phi$

$$a = l\sin\theta + a\sin\phi$$

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

$$\Rightarrow -l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi \quad (2)$$

Dividing (1) by (2), we get

$$-\tan\theta = \frac{3}{8} - \tan\phi$$

$$\tan\phi = \frac{3}{8} + \tan\theta$$

5(d) → Find the curvature and torsion of the circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$.

Sol'n: Given that

$$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + a\theta\cot\alpha\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\alpha\,\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j}$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & a\cot\alpha \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= \hat{i}\,(a^2\sin\theta\cot\alpha) + \hat{j}\,(-a^2\cos\theta\cot\alpha) + a^2\hat{k}$$

$$[\vec{r}'\ \vec{r}''\ \vec{r}'''] = \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d^3\vec{r}}{dt^3} = a^3\sin^2\theta\cot\alpha + a^3\cos^2\theta\cot\alpha$$

$$= a^3\cot\alpha$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{a^4\sin^2\theta\cot^2\alpha + a^4\cos^2\theta\cot^2\alpha + a^4}$$

$$= \sqrt{a^4(\cot^2\alpha + 1)}$$

$$= a^2\operatorname{cosec}\alpha$$

$$|\vec{r}'| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\alpha}$$

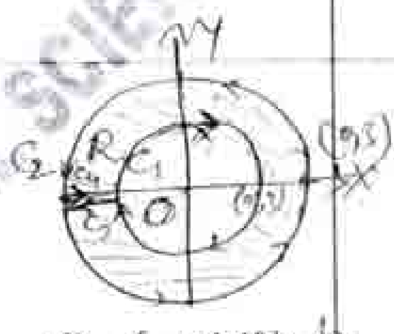
$$= \sqrt{a^2 + a^2\cot^2\alpha} = a\operatorname{cosec}\alpha$$

$$K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{a^2\operatorname{cosec}\alpha}{a^3\operatorname{cosec}^3\alpha} = \frac{1}{a}\sin^2\alpha$$

$$\tau = \frac{[\vec{r}'\ \vec{r}''\ \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2} = \frac{a^3\cot\alpha}{a^4\operatorname{cosec}^2\alpha} = \frac{1}{a}\frac{\cos\alpha}{\sin\alpha}\sin^2\alpha$$

$$= \frac{1}{a}\sin\alpha\cos\alpha$$

5(c) Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.



Sol: By Green's theorem, we have

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

The boundary of the curve C is given by $C: C_1+C_2+C_3+C_4$, and R is the region bounded by the circles $x^2+y^2=1$, and $x^2+y^2=9$

$\therefore \int M\,dx + N\,dy = \int_{C_1+C_2} M\,dx + N\,dy$.

Here note that along C_3 & C_4

i.e. $\int_{C_3+C_4} M\,dx + N\,dy = 0$

$\because$ C_3 & C_4 are in opposite directions

Along C_3: $y=0 \Rightarrow dy=0$

$\int_{C_3} 2x\,dx + \int_{C_4} 2x\,dx$

$= \int_3^1 2x\,dx + \int_1^3 2x\,dx$

$= \int_3^1 2x\,dx - \int_3^1 2x\,dx$

$= 0$

Now

LHS $= \int_C M\,dx + N\,dy = \int_{C_1+C_2} M\,dx + N\,dy$

$= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy \quad \text{(1)}$

Let $\int_{C_2} M\,dx + N\,dy = \int_{C_2} (2x-y^3)\,dx - xy\,dy$

putting $x = 3\cos\theta,\ y = 3\sin\theta$.
$dx = -3\sin\theta,\ dy = 3\cos\theta$

$$\int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} (6\cos\theta - 27\sin^3\theta)(-3\sin\theta)\,d\theta - 27\cos^2\theta\sin\theta\,d\theta$$

$$= \int_0^{2\pi} \left(-18\cos\theta\sin\theta + 81\sin^4\theta - 27\sin\theta\cos^2\theta\right) d\theta$$

$$= 0 + 81\cdot(4)\int_0^{\pi/2} \sin^4\theta\,d\theta + 27\,\frac{\cos^3\theta}{3}\Big]_0^{2\pi}$$

$$= 81\cdot 4\cdot \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 0$$

$$= \frac{243\pi}{4}.$$

$$\int_{C_1} M\,dx + N\,dy = \int_{2\pi}^{0} (2\cos\theta - \sin^3\theta)(-\sin\theta)\,d\theta - \cos^2\theta\sin\theta\,d\theta$$

by putting $x = \cos\theta$
$y = \sin\theta$
$\theta : 2\pi \to 0$

$$= 0 + \int_{2\pi}^{0} \sin^4\theta\,d\theta - 0$$

$$= -4\int_0^{\pi/2} \sin^4\theta\,d\theta = -4\cdot\frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2} = -3\pi/4$$

$\therefore$ from (1)

$$\int_C M\,dx + N\,dy = -\frac{3\pi}{4} + \frac{243\pi}{4} = \frac{240\pi}{4} = 60\pi$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3y^2 - y)\,dx\,dy$$

using the polar co-ordinates $x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{3} (3r^2\sin^2\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{3r^4}{4}\sin^2\theta - \frac{r^3}{3}\sin\theta\right)\Big|_1^3 d\theta = \int_0^{2\pi} \left(60\sin^2\theta - \frac{26}{3}\sin\theta\right) d\theta$$

$$= 60\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= 60\cdot\frac{2\pi}{2} = 60\pi$$

$\int_0^{2\pi}\sin\theta\,d\theta = 0$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$; $\ 2\int \left(\frac{1}{2} - \frac{\sin 2\theta}{2}\right)_0^{2\pi}$

$\therefore$ Green's theorem is verified

6(a) Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3mm, and one hour later has been reduced to 2mm, find an expression for the radius of the rain drop at any time.

Sol^n. Let r mm be the radius of the rain drop at time 't' hours from start. If V and S be volume and surface area of the rain drop, then we have

$$V = \tfrac{4}{3}\pi r^3 \text{ cubic mm} \quad \text{and} \quad S = 4\pi r^2 \text{ sq. mm} \quad \text{— (1)}$$

Given $dV/dt = -kS$ where $k(>0)$ is constant of proportionality.

using (1), this $\Rightarrow 4\pi r^2 (dr/dt) = -k(4\pi r^2)$

$$\Rightarrow dr = -k\,dt$$

Integrating $r = -kt + c$, where c is arbitrary constant. — (2)

Now, initially when $t=0$, $r = 3$mm. Then (2) $\Rightarrow c = 3$

Hence (2) reduces to $r = 3 - kt$ — (3)

Again, given that $r = 2$mm when $t = 1$ hour.

Hence (3) reduces to $2 = 3 - k$ so that $k = 1$.

with $k=1$, (3) reduces to $r = 3 - t$,

which is the required expression for radius r at any time t.

6(b) Solve $\left[(1+2x)^2 (d^2y/dx^2) - 6(1+2x)(dy/dx) + 16y = 8(1+2x)^2\right.$ given that $y(0) = 0$, $y'(0) = 2$.

Sol^n: Given $[(1+2x)^2 D^2 - 6(1+2x)D + 16]y = 8(1+2x)^2$ — (1).

Let $1+2x = e^z \Rightarrow \log(1+2x) = z$

Also let $D_1 \equiv d/dz$ — (2)

Then $(1+2x)D = 2D_1$, $(1+2x)^2 D^2 = 2^2 D_1(D_1-1)$ & so

① becomes $[2^2 D_1(D_1-1) - 6.2D_1 + 16]y = 8e^{2z}$

$$\Rightarrow (D_1-2)^2 y = 2e^{2z} \quad \text{--- ③}$$

Its auxiliary equation is $(D_1-2)^2 = 0$ so that $D_1 = 2, 2$.

$\therefore$ C.F $= (c_1 + c_2 z)e^{2z} = (c_1 + c_2 z)(e^z)^2 = [c_1 + c_2 \log(1+2x)](1+2x)^2$

where c_1 and c_2 are arbitrary constants.

$$P.I = \frac{1}{(D_1-2)^2} 2e^{2z} = 2\frac{z^2}{2!}e^{2z}, \text{ as } \frac{1}{(D_1-a)^n}e^{az} = \frac{z^n}{n!}e^{az}$$

$$= z^2(e^z)^2 = [\log(1+2x)]^2 (1+2x)^2, \text{ using ②}$$

$\therefore$ Solution is $y = [c_1 + c_2 \log(1+2x)](1+2x)^2 + [\log(1+2x)]^2(1+2x)^2$

$$\Rightarrow y = (1+2x)^2 [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2] \quad \text{--- ④}$$

$$\Rightarrow y(x) = (1+2x)^2 [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2] \quad \text{--- ⑤}$$

Differentiating both sides of ⑤ w.r.t 'x' we have.

$$y'(x) = (2\times 2)(1+2x)[c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]$$
$$+ (1+2x)^2 \left[\frac{2c_2}{1+2x} + \frac{2\log(1+2x)}{1+2x} \times 2\right] \quad \text{--- ⑥}$$

Putting $x=0$ in ⑤ & noting that $y(0)=0$ (given),

we get $0 = 1 \times [c_1 + (c_2 \times 0) + 0^2]$ so that $c_1 = 0$

Putting $x=0$ in ⑥ and noting that $y'(0) = 2$ (given), we get

$$2 = 4[c_1 + (c_2 \times 0) + 0^2] + 1 \times [2c_2 + (2\times 0 \times 2)]$$

so that $c_2 = 1$ as $c_1 = 0$

Putting the above values of c_1 & c_2 in ⑤, the required solution is

$$y(x) = (1+2x)^2 \log(1+2x)[1 + \log(1+2x)].$$

6(c). Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4 \sin x$.

Sol'n: Rewriting the given equation,

$$y_2 - (4/x)\times y_1 + (6/x^2)\times y = -x^2 \sin x \quad \text{——} \ (1)$$

comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = -x^2 \sin x$

Consider $y_2 - (4/x)\times y_1 + (6/x^2)\times y = 0$

$$\Rightarrow (x^2D^2 - 4xD + 6)y = 0, \quad D \equiv d/dx \quad \text{——} \ (2)$$

In order to apply the method of variation of parameters, we shall reduce (2) into linear differential equation with constant coefficients.

Let $x = e^z$ i.e. $\log x = z$ and let $D_1 \equiv d/dz$ —— (3)

then, $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$ and so (2) reduces to

$$\{D_1(D_1-1) - 4D_1 + 6\}y = 0 \Rightarrow (D_1^2 - 5D_1 + 6)y = 0$$

whose auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$

giving $D_1 = 2, 3$

$\therefore$ C.F of (1) $= c_1e^{2z} + c_2e^{3z} = c_1(e^z)^2 + c_2(e^z)^3 = c_1x^2 + c_2x^3$

Let $u = x^2$ and $v = x^3$. Also here $R = -x^2 \sin x$ — (5)

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0$

Hence P.I of (1) $= u f(x) + v g(x)$, where — (6)

$$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{x^3 \times (-x^2 \sin x)}{x^4}\,dx = \int x \sin x\,dx$$

$$= x(-\cos x) - \int \{1 \times (-\cos x)\}\,dx$$

$$= -x\cos x + \sin x \quad \text{——} \ (7)$$

and $g(x) = \int \frac{uR}{W}\,dx = \int \frac{x^2 \times (-x^2 \sin x)}{x^4}\,dx = -\int \sin x\,dx = \cos x$ — (8)

Using (5), (7) and (8), (6) reduces to

P.I of (1) $= x^2(-x\cos x + \sin x) + x^3\cos x = x^2\sin x$

Hence the required general solution is.

$Y = C.F + P.I$ i.e.

$y = c_1x^2 + c_2x^3 + x^2\sin x$, c_1 & c_2 being arbitrary constants.

6(d) Solve $(D^2+n^2)y = a\sin(nt+\alpha)$, if $y = Dy = 0$ when $t=0$

Sol^n: Given that $y'' + n^2y = a\sin(nt+\alpha)$

i.e. $y'' + n^2y = a(\sin nt\cos\alpha + \cos nt\sin\alpha)$ —— (1)

with initial conditions: $y(0)=0$, $y'(0)=0$ — (2)

Taking Laplace transform of bothsides of (1), we have

$L\{y''\} + n^2L\{y\} = a\cos\alpha\, L\{\sin nt\} + a\sin\alpha\, L\{\cos nt\}$

$\Rightarrow s^2L\{y\} - sy(0) - y'(0) + n^2L\{y\} = (an\cos\alpha)/(s^2+n^2) + (as\sin\alpha)/(s^2+n^2)$

$\Rightarrow (s^2+n^2)L\{y\} = (an\cos\alpha + as\sin\alpha)/(s^2+n^2)$, by (2)

$\Rightarrow L\{y\} = (an\cos\alpha)/(s^2+n^2)^2 + (as\sin\alpha)/(s^2+n^2)^2$ —— (3)

Taking inverse Laplace transform of bothsides of (3), we get

$y = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha\, L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}$ —(4)

Now, $L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+n^2}\right)\right\}$.

$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{s^2+n^2}\right\}$

Thus $L^{-1}\{s/(s^2+n^2)^2\} = (t/2n)\sin nt$ —— (5)

Let $f(s) = 1/(s^2+n^2)$ and $g(s) = 1/(s^2+n^2)$ —— (6)

$$\left.\begin{aligned} &\text{Then } F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n)\sin nt \\ &\text{and } G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n)\sin nt \end{aligned}\right\} \text{—(7)}$$

Now, by the convolution theorem, we have

$$L^{-1}\{f(s)\,g(s)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$\Rightarrow L^{-1}\left\{1/(s^2+n^2)^2\right\} = \int_0^t \frac{\sin nu}{n} \cdot \frac{\sin n(t-u)}{n}\,du. \text{ by } ⑥ \,\&\, ⑦$$

$$= \frac{1}{2n^2}\int_0^t \left[\cos n(t-2u) - \cos nt\right]du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} - t\cos nt + \frac{\sin nt}{2n}\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right] \quad \text{—} ⑧$$

Using ⑤ and ⑧, ④ reduces to

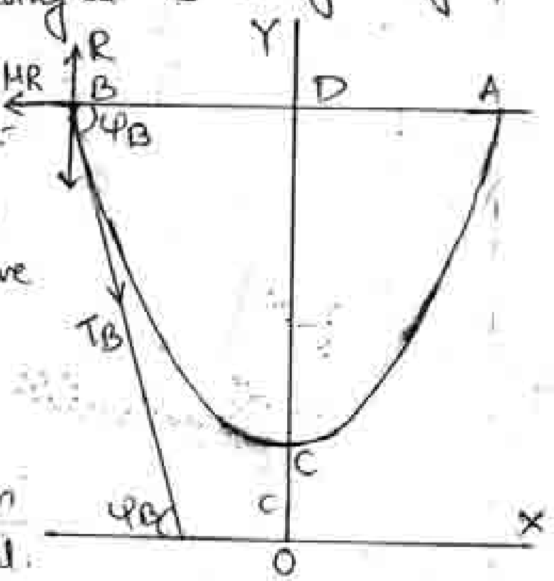
$$y = an\cos\alpha \cdot \frac{1}{2n^2}\left(\frac{\sin nt}{n} - t\cos nt\right) + a\sin\alpha \cdot \frac{1}{2n}\, t\sin nt$$

$$\Rightarrow y = \left(\frac{a}{2n^2}\right)\cos\alpha\,\sin nt - \left(\frac{at}{2n}\right)(\cos\alpha\cos nt - \sin\alpha\sin nt)$$

$$\Rightarrow y = \left(\frac{a}{2n^2}\right)\left[\cos\alpha\,\sin nt - nt\cos(t+nt)\right]$$

7(a) A heavy chain, of length $2l$, has one end tied at A and other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\left\{\lambda+\sqrt{1+\lambda^2}\right\}$, where $\frac{1}{\lambda}=\mu(2n+1)$ and μ is the coefficient of friction.

Sol'n: Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A. Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A. In this position of limiting equilibrium the forces acting on the ring are: (i) the weight $2nlw$ of the ring acting vertically downwards, (ii) the normal reaction R of the rod, the force of limiting friction μR of the rod acting in the direction AB and (iv) the tension T_B in the string at B acting along the tangent to the string at B.



For the equilibrium of the ring at B, resolving the forces acting on it horizontally & vertically, we have

$$\mu R = T_B\cos\psi_B \quad \text{—(1)}$$

and $$R = 2nlw + T_B\sin\psi_B \quad \text{—(2)}$$

where ψ_B is the angle of inclination of tangent at B to the horizontal.

Let C be the lowest point of catenary formed by the chain, OX be the directrix and $OC = c$ be the parameter. We have arc $CB = s_B = l$. By the formula $T\cos\psi = wc$, we have $T_B \cos\psi_B = wc$. Also by the formula $T \sin\psi = ws$. we have $T_B \sin\psi_B = ws_B = wl$.

putting these values in ① and ②, we have

$$\mu R = wc \text{ and } R = 2nlw + wl = (2n+1)wl$$

$$\therefore \mu(2n+1)wl = wc \Rightarrow \mu(2n+1)l = c$$

But it is given that $\mu(2n+1) = 1/\lambda$ — ③

Using the formula $s = c\tan\psi$ for the point B, we have

$$l = c\tan\psi_B$$

$$\therefore \tan\psi_B = l/c = \lambda \text{ — ④}$$

Now the required greatest possible distance of the ring from A $= AB = 2DB = 2x_B$

$$= 2c\log(\sec\psi_B + \tan\psi_B) \quad [\because x = c\log(\sec\psi + \tan\psi)]$$

$$= 2c\log[\tan\psi_B + \sqrt{(1-\tan^2\psi_B)}]$$

$$= \frac{2l}{\lambda}\log[\lambda + \sqrt{(1+\lambda^2)}]$$

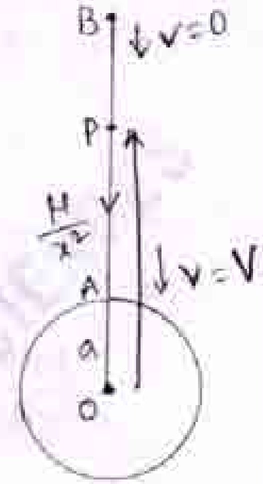
$$[\because \text{from ③}, c = l/\lambda \text{ \& from(4)}, \tan\psi_B = \lambda]$$

7(b). Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance $b\,(b>a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the time to travel from the surface to the centre of the earth is $\left[\sqrt{(a/g)}\sin^{-1}\sqrt{\frac{b}{(3b-2a)}}\right]$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.

Solⁿ: Let the particle fall from rest from the point B such that $OB = b$, where O is the centre of the earth.

Let P be the position of the particle at any time t measured from the instant it starts falling from B & let $OP = x$.

Acceleration at $P = \mu/x^2$ towards O. The equation of motion of P is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$,



which holds good for the motion from B to A. i.e. outside the surface of the earth.

But at the point A (on the earth surface)

$x = a$ and $d^2x/dt^2 = -g$

$\therefore -g = -\mu/a^2 \Rightarrow \mu = a^2 g$

$$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2} \quad \text{——— (1)}$$

Multiplying bothsides of (1) by $2(dx/dt)$ and then integrating w.r.t 't', we have $\left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} + A$, where A is a constant.

But at B, $x = OB = b$ and $dx/dt = 0$

$$\therefore 0 = \frac{2a^2g}{b} + A \Rightarrow A = -\frac{2a^2g}{b}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2a^2g\left(\frac{1}{x} - \frac{1}{b}\right) \quad \text{——— (2)}$$

If V is the velocity of the particle at the point A, then at A, $x = OA = a$ & $(dx/dt)^2 = V^2$

$$\therefore V^2 = 2a^2g\left(\frac{1}{a} - \frac{1}{b}\right) \quad \text{——— (3)}$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x<a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A

The acceleration at this point will be λx towards O, where λ is a constant.

The equation of motion (inside the earth) is $\frac{d^2x}{dt^2} = -\lambda x$, which holds good for the motion from A to O.

At A, $x = a$ and $d^2x/dt^2 = -g$. $\therefore \lambda = g/a$

$$\therefore \frac{d^2x}{dt^2} = -\frac{g}{a}x$$

Multiplying bothsides by $2(dx/dt)$ and then integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + B, \text{ where B is a constant. } \quad \text{—— (4)}$$

But at A, $x = OA = a$ and $\left(\frac{dx}{dt}\right)^2 = v^2 = 2a^2g\left(\frac{1}{a} - \frac{1}{b}\right)$, from (3)

$$\therefore 2a^2g\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{g}{a}a^2 + B \Rightarrow B = ag\left(\frac{3b-2a}{b}\right)$$

Substituting the value of B in (4), we have

$$(dx/dt)^2 = ag\left(\frac{3b-2a}{b}\right) - \frac{g}{a}x^2 \quad \text{—— (5)}$$

Putting $x = 0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{[ga(3b-2a)/b]}$

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{g}{a}\right)\left[a^2\frac{(3b-2a)}{b} - x^2\right]$$

$$= \frac{g}{a}(c^2 - x^2), \quad \text{where } c^2 = \frac{a^2}{b}(3b-2a)$$

$\therefore dx/dt = -\sqrt{(g/a)} \cdot \sqrt{(c^2-x^2)}$, the $-$ve sign being taken because the particle is moving in the direction of decreasing

$$dt = -\sqrt{a/g} \cdot \frac{dx}{\sqrt{c^2-x^2}} \quad \text{separating the variables}$$

Integrating from A to O, the required time t is given by

$$t = -\sqrt{\left(\frac{a}{g}\right)} \int_{x=a}^{0} \frac{dx}{\sqrt{(c^2-x^2)}}$$

$$= \sqrt{\left(\frac{a}{g}\right)} \int_{0}^{a} \frac{dx}{\sqrt{(c^2-x^2)}} = \sqrt{\left(\frac{a}{g}\right)} \left[\sin^{-1} x/c\right]_0^a$$

$$= \sqrt{\left(\frac{a}{g}\right)} \sin^{-1}\left(\frac{a}{c}\right) = \sqrt{a/g}\, \sin^{-1} \frac{a}{a\sqrt{\{(3b-2a)/b\}}}$$

Substituting for c.

$$= \sqrt{\left(\frac{a}{g}\right)} \sin^{-1} \sqrt{\left(\frac{b}{3b-2a}\right)}$$

7(c)ii, A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. show that greatest height attained by the particle is $\frac{1}{2k} \log \frac{g+ku^2}{g}$.

Sol^n: Let a particle of mass m be projected vertically upwards from a point O with velocity u. If v is the velocity of the particle at time t at a distance x from the starting point O, then the resistance on the particle is mkv^2 in the downward direction i.e. in the direction of x decreasing. The weight mg of the particle also acts vertically downwards. So the equation of motion of the particle during its upward motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$\Rightarrow v\frac{dv}{dx} = -(g+kv^2) \quad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

$\Rightarrow \frac{2kv\,dv}{g+kv^2} = -2k\,dx$, Separating the variables.

Integrating $\log(g+kv^2) = -2kx + A$, where A is constant.

But initially $x=0$, $v=u$; $\therefore A = \log(g+ku^2)$

$\therefore \log(g+kv^2) = -2kx + \log(g+ku^2)$

$\Rightarrow 2kx = \log(g+ku^2) - \log(g+kv^2)$

$\Rightarrow x = \frac{1}{2k}\log\frac{g+ku^2}{g+kv^2}$ —— ①.

which gives the velocity of the particle at a distance x.

If h is the greatest height attained by the particle then at $x=h$, $v=0$.

$\therefore$ from ①, we have

$$h = \frac{1}{2k}\log\left(\frac{g+ku^2}{g}\right).$$

7(c)(ii) A shot fired with velocity V at an elevation θ strikes a point P on the horizontal plane through the point of projection. If the point P is receding from the gun with velocity v, show that the elevation must be changed to ϕ, where $\sin 2\phi = \sin 2\theta + \frac{2v}{V}\sin\phi$.

Sol^n: Let O be the point of projection. When the point P is stationary, then the original range

$$OP = \frac{V^2 \sin 2\theta}{g}.$$

When the point P receds from O i.e, moves away from O in the direction of motion of the shot, then to hit at P the angle of projection is changed to ϕ.

$\therefore$ the new range $= \frac{V^2 \sin 2\phi}{g}$

Also in this case the time of flight $T = \frac{2V\sin\phi}{g}$.

During this time P moves away from its original position a distance $= v \cdot \frac{2V\sin\phi}{g}$.

In order to hit P, we should have

the new range = the original range + the distance moved by P in time T.

i.e., $$\frac{V^2\sin 2\phi}{g} = \frac{V^2\sin 2\theta}{g} + v\cdot\frac{2V\sin\phi}{g}$$

$$\Rightarrow \sin 2\phi = \sin 2\theta + \left(\frac{2v}{V}\right)\sin\phi.$$

Alternative solution: Let O be the point o projection. When the point P is stationary then the original range $OP = \frac{V^2 \sin 2\theta}{g}$.

When the point recedes from O with velocity v, then to hit at P the angle of projection is changed to ϕ. In this case the initial horizontal velocity of the shot relative to P is $V\cos\phi - v$ and the initial vertical velocity of the shot relative to P is $V \sin\phi$.

$\therefore$ In this case the range of the shot relative to P $= \left(\frac{2}{g}\right)(V\cos\phi - v) V\sin\phi$.

To hit P, we must have

$$\frac{2}{g}(V\cos\phi - v) V\sin\phi = \frac{V^2 \sin 2\theta}{g}$$

$$\Rightarrow \frac{V^2 \sin 2\phi}{g} - \frac{2}{g} v V \sin\phi = \frac{V^2}{g} \sin 2\theta$$

$$\Rightarrow \frac{V^2 \sin 2\phi}{g} = \frac{V^2}{g} \sin 2\theta + \frac{2v}{g} V \sin\phi$$

$$\Rightarrow \sin 2\phi = \sin 2\theta + \left(\frac{2v}{V}\right) \sin\phi$$

8(a)(ii) A vector function f is the product of a scalar function and the gradient of a scalar function. show that f. curl f = 0.

Sol'n: Let $f = \psi \, \mathrm{grad}\, \phi$

where ϕ and ψ are scalar functions.

we have $\mathrm{curl}\, f = \mathrm{curl}\,(\psi\, \mathrm{grad}\, \phi)$

we know that

$$\mathrm{curl}(\phi A) = (\mathrm{grad}\,\phi) \times A + \phi\, \mathrm{curl}\, A$$

$$\therefore \mathrm{curl}(\psi\, \mathrm{grad}\,\phi) = (\mathrm{grad}\,\psi) \times (\mathrm{grad}\,\phi) + \psi(\mathrm{curl}\, \mathrm{grad}\,\phi)$$

$$= (\mathrm{grad}\,\psi) \times (\mathrm{grad}\,\phi)$$

$(\because \mathrm{curl}\, \mathrm{grad}\,\phi = 0)$

$$\text{Now } f \cdot \mathrm{curl}\, f = (\psi\, \mathrm{grad}\,\phi) \cdot \{(\mathrm{grad}\,\psi) \times (\mathrm{grad}\,\phi)\}$$

$$= [\psi\, \mathrm{grad}\,\phi, \mathrm{grad}\,\psi, \mathrm{grad}\,\phi]$$

$$= \psi[\mathrm{grad}\,\phi, \mathrm{grad}\,\psi, \mathrm{grad}\,\phi]$$

$$= 0$$

[$\because$ the value of a scalar triple product is zero if two vectors are equal.]

8(a)iii) Suppose $F = i(e^x \cos y + yz) + j(xz - e^x \sin y) + k(xy + z)$. Is F conservative? If so, find f such that $F = \nabla f$.

Sol^n: Given $F = \hat{i}(e^x \cos y + yz) + \hat{j}(xz - e^x \sin y)$
$+ \hat{k}(xy + z)$.

F is conservative if $\nabla \times F = 0$

we have

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x\cos y + yz & xz - e^x \sin y & xy + z \end{vmatrix}$$

$$= \hat{i}(x - x) + j(y - y) + k(z - e^x \sin y + e^x \sin y - z)$$

$$= i(0) + j(0) + k(0)$$

$$= 0$$

Therefore F is a conservative field.

Let $F = \nabla f$.

$$\Rightarrow i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z} = i(e^x \cos y + yz) + j(xz - e^x \sin y) + k(xy + z)$$

Then

$$\frac{\partial f}{\partial x} = e^x \cos y + yz \Rightarrow f = e^x \cos y + xyz + f_1(y, z) \quad \text{---(1)}$$

$$\frac{\partial f}{\partial y} = xz - e^x \sin y \Rightarrow f = xyz + e^x \cos y + f_2(x, z) \quad \text{---(2)}$$

$$\& \ \frac{\partial f}{\partial z} = xy + z \Rightarrow xyz + \frac{z^2}{2} + f_3(x, y) \quad \text{---(3)}$$

Eqns (1), (2) & (3) each represents ϕ. These agree if we choose $f_1(y, z) = z^2/2$, $f_2(x, z) = \frac{z^2}{2}$, $f_3(x, y) = e^x \cos y$

$$\therefore f = e^x \cos y + xyz + \frac{z^2}{2} + c$$

where c is any constant

8(b) Prove that $\int_V (g \cdot \text{curl curl} f - f \cdot \text{curl curl} g)\,dv$

$$= \int_S \{(f \times \text{curl} g) - (g \times \text{curl} f)\} \cdot da$$

Sol^n: To prove $\iiint g \cdot (\nabla \times (\nabla \times f)) - f \cdot (\nabla \times (\nabla \times g))\,dv$

$$= \int (f \times \text{curl} g - g \times \text{curl} f)\,ds$$

By Gauss divergence theorem, we have

$$\iint \vec{F} \cdot d\vec{s} = \iiint \nabla \cdot \vec{F}\,dv$$

Here $F = (f \times \text{curl} g - g \times \text{curl} f)$

$$\therefore \nabla \cdot F = \nabla \cdot (f \times \text{curl} g - g \times \text{curl} f)$$

$$= \nabla \cdot (f \times \text{curl} g) - \nabla \cdot (g \times \text{curl} f)$$

$$= \text{curl} g \cdot (\nabla \times f) - f \cdot (\nabla \times \text{curl} g) - \text{curl} f \cdot \nabla \times g + g \cdot (\nabla \times \text{curl} f)$$

$$= (\nabla \times g) \cdot (\nabla \times f) - f \cdot (\nabla \times (\nabla \times g)) - (\nabla \times f) \cdot (\nabla \times g) + g \cdot (\nabla \times \nabla \times f)$$

$$= g \cdot [\nabla \times (\nabla \times f)] - f \cdot [\nabla \times (\nabla \times g)]$$

$$\therefore \iiint_V g \cdot (\nabla \times (\nabla \times f)) - f \cdot (\nabla \times (\nabla \times g))\,dv$$

$$= \iint_S (f \times \text{curl} g - g \times \text{curl} f)\,d\vec{a}$$

i.e. $\int_V (g \cdot \text{curl curl} f - f \cdot \text{curl curl} g)\,dv$

$$= \int_S \{(f \times \text{curl} g) - (g \times \text{curl} f)\}\,d\vec{a}$$

8(c) The acceleration a of a particle at any time $t \geq 0$ is given by $\vec{a} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\hat{k}$. If the velocity v and displacement r are zero $t=0$, and v and r at any time.

Sol^n: $\vec{a} = \frac{dv}{dt} = e^{-t}\hat{i} - 6(t+1)\hat{j} + 3\sin t\hat{k}$

Integrating we get

$$\vec{v} = \hat{i}\int e^{-t}dt - 6\hat{j}\int (t+1)dt + 3\hat{k}\int \sin t\, dt$$

$$= -e^{-t}\hat{i} - 6\hat{j}\left(\frac{t^2}{2}+t\right) + 3\hat{k}(-\cos t) + c$$

when $t=0$, $v=0$

$$\therefore 0 = -\hat{i} - 0\hat{j} + 3\hat{k}(-1) + c$$

$$\Rightarrow -3\hat{k} - \hat{i} + c = 0$$

$$\Rightarrow c = \hat{i} + 3\hat{k}$$

$$\therefore \vec{v} = (-e^{-t}+1)\hat{i} - 6\left(\frac{t^2}{2}+t\right)\hat{j} + (-3\cos t + 3)\hat{k}$$

$$\vec{v} = (1-e^{-t})\hat{i} - (3t^2+6t)\hat{j} - (3-3\cos t)\hat{k}$$

Integrating we get

$$\vec{r} = \hat{i}\int (1-e^{-t})dt - 3\hat{j}\int (3t^2+6t)dt - \hat{k}\int (3-3\cos t)dt$$

$$= (t+e^{-t})\hat{i} - 3\hat{j}(t^3+3t^2) - \hat{k}(3t - 3\sin t) + d$$

when $t=0$, $\vec{r}=0$

$$0 = \hat{i} - \hat{j}(0) - \hat{k}(0) + d \Rightarrow d = -\hat{i}$$

$$\therefore \vec{r} = (t-1+e^{-t})\hat{i} - (t^3+3t^2)\hat{j} + (3t-3\sin t)\hat{k}$$

8(d) By using Gauss divergence theorem evaluate $\iint_S (x^2+y^2)\,ds$, where S is the surface of the cone $z^2 = 3(x^2+y^2)$ bounded by $z=0$ and $z=3$.

Sol^n: Let S be the surface of the cone

$$z^2 = 3(x^2+y^2)$$

bounded by the planes $z=0$ and $z=3$. The plane $z=3$ cuts the surface $z^2=3(x^2+y^2)$ in the circle $x^2+y^2=3, z=3$. Let S_1 be the plane region bounded by this circle. Let S' be the closed surface consisting of the surface S & S_1.

Let us first put the integral $\iint_S (x^2+y^2)\,ds$ in the form $\iint_S F\cdot\hat{n}\,ds$

where $\hat{n}$ is a unit vector along the outward drawn normal to the surface S whose equation is $\phi(x,y,z) = 3(x^2+y^2)-z^2=0$

we have $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{6x\hat{i}+6y\hat{j}-2z\hat{k}}{\sqrt{36x^2+36y^2+4z^2}}$

$$= \frac{3x\hat{i}+3y\hat{j}-zk}{\sqrt{9(x^2+y^2)+z^2}}$$

$$= \frac{3x\hat{i}+3y\hat{j}-zk}{\sqrt{3z^2+z^2}} \quad \left(\because \text{on } S\ \ 3(x^2+y^2)=z^2\right)$$

$$= \frac{3x\hat{i}+3y\hat{j}-zk}{2z}$$

Now take $F = \frac{2z}{3}(x\hat{i}+y\hat{j})$. Then on S, $F\cdot\hat{n} = x^2+y^2$.

By Gauss divergence theorem, we have

$$\iint_{S'} F\cdot\hat{n}\,ds = \iiint_V \operatorname{div}F\,dv \quad (1)$$

where V is the volume enclosed by the closed surface S'.

we have $\operatorname{div}F = \operatorname{div}\left(\frac{2}{3}zx\hat{i}+\frac{2}{3}zy\hat{j}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2}{3}zx\right)+\frac{\partial}{\partial y}\left(\frac{2}{3}zy\right) = \frac{2}{3}z+\frac{2}{3}z = \frac{4z}{3}$$

$\therefore \iiint_V \operatorname{div} \vec{f}\, dV = \iiint_V \frac{4}{3} z\, dV$, where V is the volume bounded by $z=0, z=3$ & $x^2+y^2=z^2/3$

$$= \frac{4}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=-\sqrt{z^2/3-y^2}}^{\sqrt{z^2/3-y^2}} z\, dz\, dy\, dx.$$

$$= \frac{4}{3}\cdot 2\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=0}^{\sqrt{z^2/3-y^2}} z\, dz\, dy\, dx$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}} z\,[x]_{x=0}^{\sqrt{z^2/3-y^2}}\, dz\, dy$$

$$= 2\cdot\frac{8}{3}\int_{z=0}^{3}\int_{y=0}^{z/\sqrt{3}} z\left[\sqrt{\frac{z^2}{3}-y^2}\right] dz\, dy$$

$$= \frac{16}{3}\int_{z=0}^{3}\left[\frac{y}{2}\sqrt{\left(\frac{z^2}{3}-y^2\right)} + \frac{z^2}{6}\sin^{-1}\left(\frac{y}{z/\sqrt{3}}\right)\right]_{y=0}^{z/\sqrt{3}} z\, dz$$

$$= \frac{16}{3}\int_{z=0}^{3}\left[0+\frac{z^2}{6}\sin^{-1}(1)\right] z\, dz$$

$$= \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\int_0^3 z^3\, dz = \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\left[\frac{z^4}{4}\right]_0^3$$

$$= \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\cdot\frac{81}{4}$$

$$= 9\pi$$