UPSC Civil Services Main 1979 - Mathematics Linear Algebra

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Question 1(a) *State and prove Cayley-Hamilton Theorem.*

Solution. See 1987, question 5(a). ∎

Question 1(b) *Reduce the quadratic expression* $x^2 + 2y^2 + 2z^2 + 2xy + 2xz$ *to the canonical form.*

Solution. Completing the squares: given form $= (x + y + z)^2 + (y - z)^2$. Put $X = x+y+z, Y = y-z, Z = z$ to get the canonical form $= X^2 + Y^2$. The expression is positive semi-definite.

Alternate solution: See 1981 question 1(b) for an alternate method of canonicalization. ∎

Question 2(a) *Find the elements* p, q, r *such that the product* $\mathbf{BA}$ *of the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 4 & 1 & 2 \\ -10 & 2 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & r & 1 \end{pmatrix}$$

is of the form

$$\mathbf{BA} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

Hence solve the set of equations $\mathbf{Ax} = \mathbf{y}$, *where* $\mathbf{x}$ *is the column vector* (x_1, x_2, x_3), *and* $\mathbf{y}$ *is the column vector* $(0, 8, -4)$.

Solution.

$$\mathbf{BA} = \begin{pmatrix} 1 & 2 & 1 \\ p+4 & 2p+1 & p+2 \\ q+4r-10 & 2q+r+2 & q+2r+4 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix}$$

Thus $p+4=0, q+4r-10=0, 2q+r+2=0 \Rightarrow p=-4, r=\frac{22}{7}, q=-\frac{18}{7}$.

Now solving $\mathbf{Ax}=\mathbf{y}$ is the same as solving $\mathbf{BAx}=\mathbf{By}$ because $|\mathbf{B}| \neq 0$.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -7 & -2 \\ 0 & 0 & \frac{54}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -\frac{18}{7} & \frac{22}{7} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ \frac{148}{7} \end{pmatrix}$$

Thus $\frac{54}{7}x_3 = \frac{148}{7} \Rightarrow x_3 = \frac{74}{27}$. $-7x_2 - 2x_3 = 8 \Rightarrow -7x_2 = 2x_3 + 8 = \frac{364}{27}$, so $x_2 = -\frac{52}{27}$. $x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 = \frac{104}{27} - \frac{74}{27} = \frac{10}{9}$.

Thus $x_1 = \frac{10}{9}, x_2 = -\frac{52}{27}, x_3 = \frac{74}{27}$ is the required solution. ∎

Question 3(a) *If $\mathcal{S}$ and $\mathcal{T}$ are subspaces of a finite dimensional vector space, then show that*

$$\dim(\mathcal{S}+\mathcal{T}) = \dim\mathcal{S} + \dim\mathcal{T} - \dim(\mathcal{S}\cap\mathcal{T})$$

Solution. See 1988, question 1(b). ∎

Question 3(b) *Determine the value of a for which the following system of equations:*

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + 2x_2 + x_3 &= -2 \\ x_1 + x_2 + (a-5)x_3 &= a \end{aligned}$$

has (1) a unique solution (2) no solution.

Solution. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & a-5 \end{vmatrix} = 2a - 10 - 3 - a + 5 + 3 - 1 = a - 6$

1. If $a - 6 \neq 0$ i.e. $a \neq 6$, the system has a unique solution.
2. If $a = 6$, the system is inconsistent as the third equation becomes $x_1 + x_2 + x_3 = 6$, which is inconsistent with the first. So there is no solution.

∎

Paper II

Question 4(a) *Prove that any two finite dimensional vector spaces of the same dimension are isomorphic.*

Solution. See 1987 question 4(b). ∎

Question 4(b) *Define the dual space of a finite dimensional vector space $\mathcal{V}$ and show that it has the same dimension as $\mathcal{V}$.*

Solution. Let $\mathcal{V}^* = \{f : \mathcal{V} \longrightarrow \mathbb{R}, f \text{ a linear transformation}\}$. Then $\mathcal{V}^*$ is a vector space for the usual pointwise addition and scalar multiplication of functions: for all $\mathbf{v} \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$, $(f+g)(\mathbf{v}) = f(\mathbf{v}) + g(\mathbf{v}), (\alpha f)(\mathbf{v}) = \alpha f(\mathbf{v})$.

Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be a basis for $\mathcal{V}$. Define n linear functionals $v_1^*, \ldots, v_n^*$ by $v_i^*(\mathbf{v_j}) = \delta_{ij}$, and $v_i^*(\sum_{j=1}^n \alpha_j \mathbf{v_j}) = \sum_{j=1}^n \alpha_j v_i^*(\mathbf{v_j}) = \alpha_i$.

Then $v_1^*, \ldots, v_n^*$ are linearly independent — $\sum_{i=1}^n \alpha_i v_i^* = 0 \Rightarrow (\sum_{i=1}^n \alpha_i v_i^*)(\mathbf{v_j}) = \alpha_j = 0,\ 1 \leq j \leq n$.

$v_1^*, \ldots, v_n^*$ generate $\mathcal{V}^*$ — if $f \in \mathcal{V}^*$, then $f = \sum_{i=1}^n f(\mathbf{v_i}) v_i^*$. Clearly $(\sum_{i=1}^n f(\mathbf{v_i}) v_i^*)(\mathbf{v_j}) = \sum_{i=1}^n f(\mathbf{v_i}) v_i^*(\mathbf{v_j}) = f(\mathbf{v_j})$, so the two sides agree on $\mathbf{v_1}, \ldots, \mathbf{v_n}$, and hence by linearity on all of $\mathcal{V}$.

Thus $v_1^*, \ldots, v_n^*$ is a basis of $\mathcal{V}^*$, so $\dim \mathcal{V}^* = \dim \mathcal{V}$. $\mathcal{V}^*$ is called the dual of $\mathcal{V}$. ∎

Question 4(c) *Show that every finite dimensional inner product space $\mathcal{V}$ over the field of complex numbers has an orthonormal basis.*

Solution. Let $\mathbf{w_1}, \ldots, \mathbf{w_n}$ be a basis of $\mathcal{V}$. We will convert it into an orthonormal basis of $\mathcal{V}$ by the Gram-Schmidt orthonormalization process.

Starting with $i = 1$, define

$$\mathbf{v_i} = \mathbf{w_i} - \sum_{j=1}^{i-1} \frac{\langle \mathbf{w_i}, \mathbf{v_j} \rangle}{||\mathbf{v_j}||^2} \mathbf{v_j}$$

Each $\mathbf{v_i}$ is non-zero, as otherwise $\mathbf{w_i}$ can be written as a linear combination of $\mathbf{w_j}, j < i$, but $\mathbf{w_1}, \ldots, \mathbf{w_n}$ are linearly independent.

Now we can prove by induction on i that $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = 0$ for all $j < i$ — this is enough because $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \overline{\langle \mathbf{v_j}, \mathbf{v_i} \rangle}$. Suppose it is true for all $k < i$. Then $\langle \mathbf{v_i}, \mathbf{v_j} \rangle = \langle \mathbf{w_i}, \mathbf{v_j} \rangle - \sum_{m=1}^{i-1} \frac{\langle \mathbf{w_i}, \mathbf{v_m} \rangle}{||\mathbf{v_m}||^2} \langle \mathbf{v_m}, \mathbf{v_j} \rangle = \langle \mathbf{w_i}, \mathbf{v_j} \rangle - \frac{\langle \mathbf{w_i}, \mathbf{v_j} \rangle}{||\mathbf{v_j}||^2} \langle \mathbf{v_j}, \mathbf{v_j} \rangle = 0$. Thus $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are mutually orthogonal. They are linearly independent, as $\sum_{i=1}^n a_i \mathbf{v_i} = \mathbf{0} \Rightarrow \langle \sum_{i=1}^n a_i \mathbf{v_i}, \mathbf{v_j} \rangle = a_j ||\mathbf{v_j}||^2 = 0 \Rightarrow a_j = 0$ for all $i \leq j \leq n$. Replacing $\mathbf{v_i}$ by $\frac{\mathbf{v_i}}{||\mathbf{v_i}||}$ gives us an orthonormal basis of $\mathcal{V}$. ∎

Question 5(a) *Define the rank and nullity of a linear transformation. If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional vector spaces over a field, and T is a linear transformation of $\mathcal{V}$ into $\mathcal{W}$, prove that*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$$

Solution. See 1998 question 3(a). ∎

Question 5(b) *Define a positive definite form. State and prove a necessary and sufficient condition for a quadratic form to be positive definite.*

Solution. See 1992 question 2(c). ∎

Question 5(c) *Show that the mapping* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *defined by* $T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z)$ *is a linear transformation. Find its nullity.*

Solution.

$$\begin{aligned}
T(a\mathbf{x} + b\mathbf{y}) &= T(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \\
&= (ax_1 + by_1 - ax_2 - by_2 + 2ax_3 + 2by_3, 2ax_1 + 2by_1 + ax_2 + by_2, \\
&\quad\ -ax_1 - by_1 - 2ax_2 - 2by_2 + 2ax_3 + 2by_3) \\
&= aT(x_1, x_2, x_3) + bT(y_1, y_2, y_3)
\end{aligned}$$

Thus T is a linear transformation.

If $(x, y, z) \in$ the null space of T, then $x - y + 2z = 0, 2x + y = 0, -x - 2y + 2z = 0 \Rightarrow y = -2x, z = -\frac{3x}{2}$. Thus the null space is $\{(x, -2x, -\frac{3x}{2}) \mid x \in \mathbb{R}\} = \{(2, -4, -3)x \mid x \in \mathbb{R}\}$.

Thus nullity $T = 1$. ∎

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Question 1(a) *Define the rank of a matrix. Prove that a system of equations* $\mathbf{Ax} = \mathbf{b}$ *is consistent if and only if* $\text{rank}\,(\mathbf{A}, \mathbf{b}) = \text{rank}\,\mathbf{A}$*, where* $(\mathbf{A}, \mathbf{b})$ *is the augmented matrix of the system.*

Solution. See 1987 question 3(a). ∎

Question 1(b) *Verify the Cayley Hamilton Theorem for the matrix* $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$*, and hence find* $\mathbf{A}^{-1}$*.*

Solution. The Cayley Hamilton theorem is — Every matrix $\mathbf{A}$ satisfies its characteristic equation $|x\mathbf{I} - \mathbf{A}| = 0$. In the current problem, $|x\mathbf{I} - \mathbf{A}| = \begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = x^2 - 4x + 4 - 1 = x^2 - 4x + 3$. Thus we need to show that $\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \mathbf{0}$. Now $\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, so $\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, verifying the Cayley Hamilton Theorem.

$\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A} - 4\mathbf{I}) = -3\mathbf{I} \Rightarrow \mathbf{A}^{-1} = -\frac{1}{3}(\mathbf{A} - 4\mathbf{I}) = -\frac{1}{3}(\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$. ∎

Question 2(a) *Prove that if* $\mathbf{P}$ *is any non-singular matrix of order* n*, then the matrices* $\mathbf{P}^{-1}\mathbf{AP}$ *and* $\mathbf{A}$ *have the same characteristic polynomial.*

Solution. The characteristic polynomial of $\mathbf{P}^{-1}\mathbf{AP}$ is $|x\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| = |x\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||x\mathbf{I} - \mathbf{A}||\mathbf{P}| = |x\mathbf{I} - \mathbf{A}|$ which is the characteristic polynomial of $\mathbf{A}$. ∎

Question 2(b) *Find the eigenvalues and eigenvectors of the matrix* $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

Solution. The characteristic equation of $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ is $\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow -(9 - \lambda^2) - 16 = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \lambda = 5, -5$.

If (x_1, x_2) is an eigenvector for $\lambda = 5$, then $\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2$. Thus $(2x, x), x \in \mathbb{R}, x \neq 0$ gives all eigenvectors for $\lambda = 5$, in particular, we can take $(2, 1)$ as an eigenvector for $\lambda = 5$.

If (x_1, x_2) is an eigenvector for $\lambda = -5$, then $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow 4x_1 + 2x_2 = 0 \Rightarrow x_2 = -2x_1$. Thus $(x, -2x), x \in \mathbb{R}, x \neq 0$ gives all eigenvectors for $\lambda = -5$, in particular, we can take $(1, -2)$ as an eigenvector for $\lambda = -5$. ∎

Question 3(a) *Find a basis for the vector space* $\mathcal{V} = \{p(x) \mid p(x) = a_0 + a_1x + a_2x^2\}$ *and its dimension.*

Solution. Let $f_1 = 1, f_2 = x, f_3 = x^2$, then f_1, f_2, f_3 are linearly independent, because $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \Rightarrow \alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0$(zero polynomial) $\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$.

f_1, f_2, f_3 generate $\mathcal{V}$ because $p(x) = a_0 + a_1x + a_2x^2 = a_0 f_1 + a_1 f_2 + a_2 f_3$ for any $p(x) \in \mathcal{V}$. Thus $\{f_1, f_2, f_3\}$ is a basis for $\mathcal{V}$ and its dimension is 3. ∎

Question 3(b) *Find the values of the parameter* λ *for which the system of equations*

$$\begin{aligned} x + y + 4z &= 1 \\ x + 2y - 2z &= 1 \\ \lambda x + y + z &= 1 \end{aligned}$$

will have (i) unique solution (ii) no solution.

Solution. The system will have the unique solution given by $\begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ if $\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{vmatrix} = 1(2+2) + 4(1-2\lambda) - 1(1+2\lambda) \neq 0$. Thus $4 + 4 - 8\lambda - 1 - 2\lambda \neq 0 \Rightarrow \lambda \neq \frac{7}{10}$.

When $\lambda = \frac{7}{10}$, the system is

$$\begin{aligned} x + y + 4z &= 1 \\ x + 2y - 2z &= 1 \\ 7x + 10y + 10z &= 10 \end{aligned}$$

This system has no solution as it is inconsistent: $4(x+y+4z)+3(x+2y-2z) = 7x+10y+10z = 7$, but the third equation says that $7x + 10y + 10z = 10$. Thus there is a unique solution if $\lambda \neq \frac{7}{10}$, and no solution if $\lambda = \frac{7}{10}$. ∎

Question 3(c) *If $\mathcal{V}$ is a finite dimensional vector space and $\mathcal{M}$ is a subspace of $\mathcal{V}$, then show that each vector $\mathbf{x} \in \mathcal{V}$ can be uniquely expressed as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in \mathcal{M}$ and $\mathbf{z} \in \mathcal{M}^{\perp}$, the orthogonal complement of $\mathcal{M}$.*

Solution. Let $\mathbf{v_1}, \ldots, \mathbf{v_m}$ be any orthonormal basis of $\mathcal{M}$, where $m = \dim \mathcal{M}$. Given $\mathbf{x} \in \mathcal{V}$, let $\mathbf{y} = \sum_{i=1}^{m} \langle \mathbf{x}, \mathbf{v_i} \rangle \mathbf{v_i}$, and $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Clearly $\mathbf{y} \in \mathcal{M}$, and $\mathbf{x} = \mathbf{y} + \mathbf{z}$. Now $\langle \mathbf{z}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \langle \mathbf{y}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \sum_{j=1}^{m} \langle \mathbf{x}, \mathbf{v_j} \rangle \langle \mathbf{v_j}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \langle \mathbf{x}, \mathbf{v_i} \rangle = 0$. So $\langle \mathbf{z}, \mathbf{v_i} \rangle = 0, i = 1, \ldots, m \Rightarrow \langle \mathbf{z}, \mathbf{m} \rangle = 0$ for every $\mathbf{m} \in \mathcal{M}$, so $\mathbf{z} \in \mathcal{M}^{\perp}$.

Now if $\mathbf{x} = \mathbf{y}' + \mathbf{z}'$, then $\mathbf{y} - \mathbf{y}' = \mathbf{z}' - \mathbf{z}$. But $\mathbf{y} - \mathbf{y}' \in \mathcal{M}, \mathbf{z}' - \mathbf{z} \in \mathcal{M}^{\perp}$, so $\langle \mathbf{y} - \mathbf{y}', \mathbf{z}' - \mathbf{z} \rangle = 0 \Rightarrow \langle \mathbf{y} - \mathbf{y}', \mathbf{y} - \mathbf{y}' \rangle = 0 \Rightarrow ||\mathbf{y} - \mathbf{y}'|| = 0 \Rightarrow \mathbf{y} - \mathbf{y}' = \mathbf{0} \Rightarrow \mathbf{z}' - \mathbf{z} = \mathbf{0}$. Thus $\mathbf{y} = \mathbf{y}', \mathbf{z} = \mathbf{z}'$ and the representation is unique. ∎

Question 3(d) *Find one characteristic value and corresponding characteristic vector for the operators T on $\mathbb{R}^3$ defined as*

1. *T is a reflection on the plane $x = z$.*
2. *T is a projection on the plane $z = 0$.*
3. *$T(x, y, z) = (3x + y + z, 2y + z, z)$.*

Solution.

1. $T(x, y, z) = (z, y, x)$ because the midpoint of (x, y, z) and (z, y, x) lies on the plane $x = z$. $T(1, 0, 0) = (0, 0, 1), T(0, 1, 0) = (0, 1, 0), T(0, 0, 1) = (1, 0, 0)$. Thus it is clear that 1 is an eigenvalue, and $(0, 1, 0)$ is a corresponding eigenvector.
2. $T(1, 0, 0) = (1, 0, 0), T(0, 1, 0) = (0, 1, 0), T(0, 0, 1) = (0, 0, 0)$. Clearly 1 is an eigenvalue with $(1, 0, 0)$ or $(0, 1, 0)$ as eigenvectors.
3. $T(1, 0, 0) = (3, 0, 0), T(0, 1, 0) = (1, 2, 0), T(0, 0, 1) = (1, 1, 1)$. Clearly $(1, 0, 0)$ is an eigenvector, corresponding to the eigenvalue 3.

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Question 1(a) *State and prove the Cayley Hamilton theorem and verify it for the matrix* $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$. *Use the result to determine* $\mathbf{A}^{-1}$.

Solution. See 1987 question 5(a) for the Cayley Hamilton theorem.

The characteristic equation of $\mathbf{A}$ is $\begin{vmatrix} x-2 & -3 \\ -3 & x-5 \end{vmatrix} = 0$, or $(x-2)(x-5) - 9 = 0 \Rightarrow x^2 - 7x + 1 = 0$. The Cayley Hamilton theorem implies that $\mathbf{A}^2 - 7\mathbf{A} + \mathbf{I} = \mathbf{0}$.

$\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 21 \\ 21 & 34 \end{pmatrix}$.

Now $\mathbf{A}^2 - 7\mathbf{A} + \mathbf{I} = \begin{pmatrix} 13 & 21 \\ 21 & 34 \end{pmatrix} - 7\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

So the theorem is verified. $\mathbf{A}^2 - 7\mathbf{A} + \mathbf{I} = \mathbf{0} \Rightarrow (\mathbf{A} - 7\mathbf{I})\mathbf{A} = -\mathbf{I} \Rightarrow \mathbf{A}^{-1} = 7\mathbf{I} - \mathbf{A}$. Thus $\mathbf{A}^{-1} = 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$. ∎

Question 1(b) *Let Q be the quadratic form*

$$Q = 5x_1^2 + 5x_2^2 + 2x_3^2 + 8x_1x_2 + 4x_1x_3 + 4x_2x_3$$

By using an orthogonal change of variables reduce Q to a form without the cross terms i.e. with terms of the form $a_{ij}x_ix_j, i \neq j$.

Solution. The matrix of the qiven quadratic form Q is $\mathbf{A} = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}
&\begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = 0 \\
\Rightarrow\quad & (5-\lambda)(5-\lambda)(2-\lambda) - 4(5-\lambda) - 4(8-4\lambda) + 16 + 16 - 4(5-\lambda) = 0 \\
\Rightarrow\quad & (\lambda^2 - 10\lambda + 25)(2-\lambda) - 20 + 4\lambda - 32 + 16\lambda + 12 + 4\lambda = 0 \\
\Rightarrow\quad & -\lambda^3 + 12\lambda^2 + \lambda(-25 + 4 + 16 + 4 - 20) + 50 - 20 - 32 + 12 = 0 \\
\Rightarrow\quad & \lambda^3 - 12\lambda^2 + 21\lambda - 10 = 0
\end{aligned}$$

Thus the eigenvalues are $\lambda = 1, 1, 10$. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 10$, then

$$\begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0} \Rightarrow \begin{array}{rcll} -5x_1 + 4x_2 + 2x_3 & = & 0 & (i) \\ 4x_1 - 5x_2 + 2x_3 & = & 0 & (ii) \\ 2x_1 + 2x_2 - 8x_3 & = & 0 & (iii) \end{array}$$

Subtracting (ii) from (i), we get $-9x_1 + 9x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow x_1 = 2x_3$. Thus taking $x_3 = 1$, we get $(2, 2, 1)$ as an eigenvector for $\lambda = 10$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$, then

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0} \Rightarrow \begin{array}{rcl} 4x_1 + 4x_2 + 2x_3 & = & 0 \\ 2x_1 + 2x_2 + x_3 & = & 0 \end{array}$$

Take $x_3 = 0, x_1 = 1 \Rightarrow x_2 = -1$ to get $(1, -1, 0)$ as an eigenvector for $\lambda = 1$. Take $x_1 = x_2 = 1 \Rightarrow x_3 = -4$ to get $(1, 1, -4)$ as another eigenvector for $\lambda = 1$, orthogonal to the first.

Thus

$$\mathbf{O} = \begin{pmatrix} \frac{2}{\sqrt{9}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{2}{\sqrt{9}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{9}} & 0 & -\frac{4}{\sqrt{18}} \end{pmatrix}$$

is an orthogonal matrix such that $\mathbf{O}'\mathbf{A}\mathbf{O} = \mathbf{O}^{-1}\mathbf{A}\mathbf{O} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. If (X_1, X_2, X_3) are new variables, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{O} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ takes $Q(x_1, x_2, x_3)$ to $10X_1^2 + X_2^2 + X_3^2$. ∎

Note: If the orthogonal transformation was not required for the diagonalization, we could do it easily by completing squares:

$$\begin{aligned}
& 5x_1^2+5x_2^2+2x_3^2+8x_1x_2+4x_1x_3+4x_2x_3 \\
= \; & 5[x_1^2+\frac{8}{5}x_1x_2+\frac{4}{5}x_1x_3]+5x_2^2+2x_3^2+4x_2x_3 \\
= \; & 5[x_1+\frac{4}{5}x_2+\frac{2}{5}x_3]^2+(5-\frac{16}{25})x_2^2+(2-\frac{4}{5})x_3^2+(4-\frac{16}{5})x_2x_3 \\
= \; & 5[x_1+\frac{4}{5}x_2+\frac{2}{5}x_3]^2+\frac{9}{5}(x_2^2+\frac{4}{9}x_2x_3)+\frac{6}{5}x_3^2 \\
= \; & 5[x_1+\frac{4}{5}x_2+\frac{2}{5}x_3]^2+\frac{9}{5}[x_2+\frac{2}{9}x_3]^2+\frac{10}{9}x_3^2 \\
= \; & 5X^2+\frac{9}{5}Y^2+\frac{10}{9}Z^2
\end{aligned}$$

where $X = x_1+\frac{4}{5}x_2+\frac{2}{5}x_3, Y = x_2+\frac{2}{9}x_3, Z = x_3$, or $x_3 = Z, x_2 = Y-\frac{2}{9}Z, x_1 = X-\frac{4}{5}Y-\frac{2}{9}Z$.

Question 2(a) *Define a vector space. Show that the set $\mathcal{V}$ of all real-valued functions on $[0,1]$ is a vector space over the set of real numbers with respect to the addition and scalar multiplication of functions.*

Solution. See 1984 question 4(a). ∎

Question 2(b) *If zero is a root of the characteristic equation of a matrix $\mathbf{A}$, show that the corresponding linear transformation cannot be one to one.*

Solution. If zero is a root of $|\mathbf{A}-\lambda\mathbf{I}| = 0$, the characteristic equation of $\mathbf{A}$, then 0 is an eigenvalue of $\mathbf{A}$, so there is a non-zero eigenvector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$, thus $\mathbf{A}$ is not 1-1. ∎

Question 2(c) *Show that a linear transformation $\mathbf{T}$ from a Euclidean space $\mathcal{V}$ to $\mathcal{V}$ is orthogonal if and only if the matrix corresponding to it with respect to any orthonormal basis is orthogonal.*

Solution. $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$ is said to be orthogonal if $\langle \mathbf{T}(\mathbf{u}), \mathbf{T}(\mathbf{v})\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$ for any $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

Lemma 1. $\mathbf{T}$ is orthogonal iff $\mathbf{T}$ takes an orthonormal basis to an orthonormal basis.

Proof: Let $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ be an orthonormal basis. Then

1. $\langle \mathbf{T}(\mathbf{v_i}), \mathbf{T}(\mathbf{v_j})\rangle = \langle \mathbf{v_i}, \mathbf{v_j}\rangle = 0$ if $i \neq j$
2. $\langle \mathbf{T}(\mathbf{v_i}), \mathbf{T}(\mathbf{v_i})\rangle = \langle \mathbf{v_i}, \mathbf{v_i}\rangle = 1$
3. If $\sum_{i=1}^n \alpha_i \mathbf{T}(\mathbf{v_i}) = 0$, then $\langle \sum_{i=1}^n \alpha_i \mathbf{T}(\mathbf{v_i}), \mathbf{v_j}\rangle = \alpha_j = 0$ for all j, so $\mathbf{T}(\mathbf{v_i})$ are linearly independent.

Thus $\mathbf{T}(\mathbf{v_1}), \ldots, \mathbf{T}(\mathbf{v_n})$ form an orthonormal basis.

Conversely, let $\mathbf{T}(\mathbf{v_1}), \ldots, \mathbf{T}(\mathbf{v_n})$ be an orthonormal basis of $\mathcal{V}$. Let $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v_i}, \mathbf{w} = \sum_{i=1}^n \beta_i \mathbf{v_i}$, then $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^n \alpha_i \beta_i$ and $\langle \mathbf{T}(\mathbf{v}), \mathbf{T}(\mathbf{w}) \rangle = \langle \sum_{i=1}^n \alpha_i \mathbf{T}(\mathbf{v_i}), \sum_{i=1}^n \beta_i \mathbf{T}(\mathbf{v_i}) \rangle = \sum_{i=1}^n \alpha_i \beta_i$. Thus $\langle \mathbf{T}(\mathbf{v}), \mathbf{T}(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$, so $\mathbf{T}$ is orthogonal.

Lemma 2. Let $\mathbf{T}^*$ be defined by $\langle \mathbf{T}(\mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{T}^*(\mathbf{w}) \rangle$. Then $\mathbf{T}^*$ is a linear transformation, and $\mathbf{T}$ is orthogonal iff $\mathbf{T}^*\mathbf{T} = \mathbf{T}\mathbf{T}^* = \mathbf{I}$.

Proof: The fact that $\mathbf{T}^*$ is a linear transformation can be easily checked. If $\mathbf{T}$ is orthogonal, then $\langle \mathbf{v}, \mathbf{T}^*\mathbf{T}(\mathbf{w}) \rangle = \langle \mathbf{T}(\mathbf{v}), \mathbf{T}(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$, so $\mathbf{T}^*\mathbf{T} = \mathbf{I}$. From this and the fact that $\mathbf{T}$ is 1-1, it follows that $\mathbf{T}\mathbf{T}^* = \mathbf{I}$.

Lemma 3. If the matrix of $\mathbf{T}$ w.r.t. the orthonormal basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is $\mathbf{A} = (a_{ij})$, then the matrix of $\mathbf{T}^*$ is the transpose, i.e. (a_{ji}).

Proof: $\mathbf{T}(\mathbf{v_i}) = \sum_{j=1}^n a_{ij} \mathbf{v_j}$. Let $\mathbf{T}^*(\mathbf{v_i}) = \sum_{j=1}^n b_{ij} \mathbf{v_j}$. Now $b_{ij} = \langle \mathbf{T}^*(\mathbf{v_i}), \mathbf{v_j} \rangle = \langle \mathbf{v_i}, \mathbf{T}(\mathbf{v_j}) \rangle = \langle \mathbf{v_i}, \sum_{k=1}^n a_{jk} \mathbf{v_k} \rangle = a_{ji}$. Since $\mathbf{T}\mathbf{T}^* = \mathbf{I}$, $\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}' = \mathbf{I}$, so $\mathbf{A}$ is orthogonal.

The converse is also obvious now. ∎

Question 3(a) *Investigate for what values of λ and μ does the following system of equations*

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

have (1) a unique solution (2) no solution (3) an infinite number of solutions?

Solution.

1. A unique solution exists when $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} \neq 0$, whatever μ may be. Thus $2\lambda - 6 - (\lambda - 3) \neq 0 \Rightarrow \lambda \neq 3$. Thus for all $\lambda \neq 3$ and for all μ we have a unique solution given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 10 \\ \mu \end{pmatrix}$

2. A unique solution does not exist if $\lambda = 3$. If $\mu \neq 10$, then the second and third equations are inconsistent. Thus if $\lambda = 3, \mu \neq 10$, the system has no solution.

3. If $\lambda = 3, \mu = 10$, then the system is $x+y+z = 6, x+2y+3z = 10$. The coefficient matrix is of rank 2, so the space of solutions is one dimensional. $y + 2z = 4 \Rightarrow y = 4 - 2z$, and thus $x = 2 + z$. The space of solutions is $(2 + z, 4 - 2z, z)$ for $z \in \mathbb{R}$.

∎

Question 3(b) *Let $(x_i, y_i), i = 1, \ldots, n$ be n points in the plane, no two of them having the same abscissa. Find a polynomial $f(x)$ of degree $n-1$ which takes the value $f(x_i) = y_i, 1 \leq i \leq n$.*

Solution. Let $f(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$. We want to determine $a_0, \ldots, a_{n-1}$ such that

$$\mathbf{A}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & & & \\ 1 & x_n & \ldots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

This is possible as $|\mathbf{A}| \neq 0$, as $x_1, \ldots, x_n$ are distinct. $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.

Note: We can also use Lagrange's interpolation formula from numerical analysis, giving

$$f(x) = \sum_{i-1}^{n} y_i \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

The two methods give the same polynomial, which is unique. ∎

Paper II

Question 4(a) *Find a set of three orthonormal eigenvectors for the matrix* $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix}$

Solution. The characteristic equation of $\mathbf{A}$ is

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & \sqrt{3} \\ 0 & \sqrt{3} & 6-\lambda \end{vmatrix} = 0$$

Thus $(3-\lambda)(4-\lambda)(6-\lambda) - 3(3-\lambda) = 0 \Rightarrow \lambda = 3, \lambda^2 - 10\lambda + 21 = 0$. Thus the eigenvalues of $\mathbf{A}$ are $3, 3, 7$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 7$. Then

$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & -3 & \sqrt{3} \\ 0 & \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-4x_1 = 0, -3x_2 + \sqrt{3}x_3 = 0, \sqrt{3}x_2 - x_3 = 0$. Thus $x_1 = 0, x_3 = \sqrt{3}x_2$ with $x_2 \neq 0$ gives any eigenvector for $\lambda = 7$. Take $x_2 = 1$ to get $(0, 1, \sqrt{3})$, and normalize it to get $(0, \frac{1}{2}, \frac{\sqrt{3}}{2})$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 3$. Then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_2 + \sqrt{3}x_3 = 0$. Thus $(x_1, -\sqrt{3}x_3, x_3)$ with $x_1, x_3 \in \mathbb{R}$ gives any eigenvector for $\lambda = 3$. We can take $x_1 = 1, x_3 = 0$, and $x_1 = 0, x_3 = 1$ to get $(1, 0, 0), (0, -\sqrt{3}, 1)$ as eigenvectors for $\lambda = 3$ — these are orthogonal and therefore span the the eigenspace of $\lambda = 3$. Orthonormal vectors are $(1, 0, 0), (0, -\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Thus the required orthonormal vectors are $(0, \frac{1}{2}, \frac{\sqrt{3}}{2}), (1, 0, 0), (0, -\frac{\sqrt{3}}{2}, \frac{1}{2})$.

In fact

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & \sqrt{3} \\ 0 & \sqrt{3} & 6 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

∎

Question 4(b) *Show that if $A = \mathbf{X'AX}$ and $B = \mathbf{X'BX}$ are two quadratic forms one of which is positive definite and $\mathbf{A}, \mathbf{B}$ are symmetric matrices, then they can be expressed as linear combinations of squares by an appropriate linear transformation.*

Solution. Let $\mathbf{B}$ be positive definite. Then there exists an orthogonal real non-singular matrix $\mathbf{H}$ such that $\mathbf{H'BH} = \mathbf{I_n}$, the unit matrix of order n. $\mathbf{A}$ is real-symmetric $\Rightarrow \mathbf{H'AH}$ is real symmetric. There exists $\mathbf{K}$ a real orthogonal matrix such that $\mathbf{K'H'AHK}$ is a diagonal matrix i.e. $\mathbf{K'H'AHK} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $\mathbf{H'AH}$.

Now $\mathbf{K'H'BHK} = \mathbf{K'I_nK} = \mathbf{I_n}$. Then $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{HK} \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ diagonalizes $\mathbf{A}, \mathbf{B}$ simultaneously.

$$\begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda_1 X_1^2 + \ldots + \lambda_n X_n^2 \qquad \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \mathbf{B} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = X_1^2 + \ldots + X_n^2$$

Note that $\lambda_1, \dots, \lambda_n$ are the roots of $|\mathbf{A} - \lambda\mathbf{B}| = 0$ because $|\mathbf{A} - \lambda\mathbf{B}| = |\mathbf{H'}||\mathbf{A} - \lambda\mathbf{B}||\mathbf{H}| = |\mathbf{H'AH} - \lambda\mathbf{H'BH}| = |\mathbf{H'AH} - \lambda\mathbf{I_n}| = \prod_{i=1}^{n}(\lambda - \lambda_i)$.

∎

UPSC Civil Services Main 1982 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{V}$ be a vector space. If $\dim \mathcal{V} = n$ with $n > 0$, prove that*

1. *any set of n linearly independent vectors is a basis of $\mathcal{V}$.*
2. *$\mathcal{V}$ cannot be generated by fewer than n vectors.*

Solution. From 1983 question 1(a) we get that any two bases of $\mathcal{V}$ have n elements.

1. Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be n linearly independent vectors in $\mathcal{V}$. Then $\mathbf{v_1}, \ldots, \mathbf{v_n}$ generate $\mathcal{V}$ — if $\mathbf{v} \in \mathcal{V}$ is such that $\mathbf{v}$ is not a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_n}$, then $\mathbf{v}, \mathbf{v_1}, \ldots, \mathbf{v_n}$ are linearly independent, so $\dim \mathcal{V} > n$ which is not true. Thus $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is a basis of $\mathcal{V}$ — here we have used the technique used to complete any linearly independent set to a basis.
2. $\mathcal{V}$ cannot be generated by fewer than n vectors, because then it will have a basis consisting of less than n elements, which contradicts the fact that $\dim \mathcal{V} = n$.

$\blacksquare$

Question 1(b) *Define a linear transformation. Prove that both the range and the kernel of a linear transformation are vector spaces.*

Solution. Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces. A mapping $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is said to be a linear transformation if

1. $\mathbf{T}(\mathbf{v_1} + \mathbf{v_2}) = \mathbf{T}(\mathbf{v_1}) + \mathbf{T}(\mathbf{v_2})$.
2. $\mathbf{T}(\alpha\mathbf{v}) = \alpha\mathbf{T}(\mathbf{v})$ for any $\alpha \in \mathbb{R}, \mathbf{v} \in \mathcal{V}$.

Range of $\mathbf{T} = \mathbf{T}(\mathcal{V})$, kernel of $\mathbf{T} = \{\mathbf{v} \mid \mathbf{T}(\mathbf{v}) = 0\}$. If $\mathbf{w_1}, \mathbf{w_2} \in \mathbf{T}(\mathcal{V})$, then $\mathbf{w_1} = \mathbf{T}(\mathbf{v_1}), \mathbf{w_2} = \mathbf{T}(\mathbf{v_2})$ for some $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{V}$, $\alpha\mathbf{w_1} + \beta\mathbf{w_2} = \alpha\mathbf{T}(\mathbf{v_1}) + \beta\mathbf{T}(\mathbf{v_2}) = \mathbf{T}(\alpha\mathbf{v_1} + \beta\mathbf{v_2})$. But $\alpha\mathbf{v_1} + \beta\mathbf{v_2} \in \mathcal{V} \therefore \alpha\mathbf{w_1} + \beta\mathbf{w_2} \in \mathbf{T}(\mathcal{V})$, thus $\mathbf{T}(\mathcal{V})$ is a subspace of $\mathcal{W}$. Note that $\mathbf{T}(\mathcal{V}) \neq \emptyset \because \mathbf{0} \in \mathbf{T}(\mathcal{V})$ so $\mathbf{T}(\mathcal{V})$ is a vector space.

If $\mathbf{v_1}, \mathbf{v_2} \in \operatorname{kernel}\mathbf{T}$ then $\mathbf{T}(\mathbf{v_1}) = \mathbf{0}, \mathbf{T}(\mathbf{v_2}) = \mathbf{0}$. Now $\mathbf{T}(\alpha\mathbf{v_1} + \beta\mathbf{v_2}) = \alpha\mathbf{T}(\mathbf{v_1}) + \beta\mathbf{T}(\mathbf{v_2}) = \mathbf{0} \Rightarrow \alpha\mathbf{v_1} + \beta\mathbf{v_2} \in \operatorname{kernel}\mathbf{T}$. Thus $\operatorname{kernel}\mathbf{T}$ is a subspace. $\operatorname{kernel}\mathbf{T} \neq \emptyset, bf0 \in \operatorname{kernel}\mathbf{T}$ so $\operatorname{kernel}\mathbf{T}$ is a vector space. ∎

Question 2(a) *Reduce the matrix*

$$\begin{pmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix}$$

to row echelon form.

Solution. Let the given matrix be called $\mathbf{A}$.

Operation $R_1 - R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix}$

Operation $R_2 - R_1, R_3 - R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix}$

Operation $-\frac{1}{5}R_2, -\frac{1}{2}R_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$

Operation $R_3 - R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Operation $R_1 - 4R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\operatorname{rank}\mathbf{A} = 2$ and the row echelon form is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ∎

Question 2(b) *If $\mathcal{V}$ is a vector space of dimension n and $\mathbf{T}$ is a linear transformation on $\mathcal{V}$ of rank r, prove that $\mathbf{T}$ has nullity $n - r$.*

Solution. See 1998 question 3(a). ∎

Question 2(c) *Show that the system of equations*

$$\begin{aligned} 3x + y - 5z &= -1 \\ x - 2y + z &= -5 \\ x + 5y - 7z &= 2 \end{aligned}$$

is inconsistent.

Solution. From the first two equations, $(3x+y-5z)-2(x-2y+z) = -1-2(-5) = 9 \Rightarrow x+5y-7z = 9$. But this is inconsistent with the third equation, hence the overall system in inconsistent. ∎

Question 3(a) *Prove that the trace of a matrix is equal to the sum of its characteristic roots.*

Solution. The characteristic polynomial of $\mathbf{A}$ is $|\lambda\mathbf{I}-\mathbf{A}| = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$. Thus the sum of the roots of $|\lambda\mathbf{I}-\mathbf{A}| = -p_1 = a_{11} + a_{22} + \ldots + a_{nn} = \operatorname{tr}\mathbf{A}$. Thus the trace of $\mathbf{A}$ = sum of the eigenvalues of $\mathbf{A}$. ∎

Question 3(b) *If* $\mathbf{A}, \mathbf{B}$ *are two non-singular matrices of the same order, prove that* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same eigenvalues.*

Solution. See 1995 question 2(b). ∎

Question 3(c) *Find the eigenvalues and eigenvectors of the matrix* $\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Solution. The characteristic equation of $\mathbf{A}$ is $(\cos\theta-\lambda)(-\cos\theta-\lambda) - \sin^2\theta = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$.

If (x_1, x_2) is an eigenvector for $\lambda = 1$, then

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Thus $x_1(\cos\theta-1)+x_2\sin\theta = 0, x_1\sin\theta+x_2(-\cos\theta-1) = 0$. We can take $x_1 = 1+\cos\theta, x_2 = \sin\theta$.

Similarly if (x_1, x_2) is an eigenvector for $\lambda = -1$, then

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Thus $x_1(\cos\theta+1)+x_2\sin\theta = 0, x_1\sin\theta+x_2(-\cos\theta+1) = 0$. We can take $x_1 = 1-\cos\theta, x_2 = -\sin\theta$ as an eigenvector. ∎

Paper II

Question 4(a) *If $\mathcal{V}$ is finite dimensional and if $\mathcal{W}$ is a subspace of $\mathcal{V}$, then show that $\mathcal{W}$ is finite dimensional and* $\dim \mathcal{W} \leq \dim \mathcal{V}$.

Solution. If $\mathcal{W} = \{\mathbf{0}\}$ then $\dim \mathcal{W} = 0 \leq \dim \mathcal{V}$. If $\mathcal{W} \neq \{\mathbf{0}\}$, let $\mathbf{v_1} \in \mathcal{W}, \mathbf{v_1} \neq \mathbf{0}$. Let $\mathcal{W}_1$ be the space spanned by $\mathbf{v_1}$ then $\mathcal{W}_1$ is of dimension 1. If $\mathcal{W}_1 = \mathcal{W}$, then $\dim \mathcal{W} = 1 \leq \dim \mathcal{V}$.

If $\mathcal{W}_1 \neq \mathcal{W}$, then there exists a $\mathbf{v_2} \in \mathcal{W}, \mathbf{v_2} \notin \mathcal{W}_1$. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent — if $a\mathbf{v_1} + b\mathbf{v_2} = 0$, then if $b \neq 0$ then $\mathbf{v_2} = -\frac{a}{b}\mathbf{v_1} \Rightarrow \mathbf{v_2} \in \mathcal{W}_1$, which is not true, hence $b = 0 \Rightarrow a = 0$. Now let $\mathcal{W}_2$ be the space spanned by $\mathbf{v_1}, \mathbf{v_2}$ then $\mathcal{W}_2$ is of dimension 2. If $\mathcal{W}_2 = \mathcal{W}$, then $\dim \mathcal{W} = 2 \leq \dim \mathcal{V}$.

We continue the same reasoning as above, but this process must stop after at most r steps where $r \leq n$, otherwise we would have found $n + 1$ linearly independent vectors in $\mathcal{V}$, which is not possible. After r steps, we would have $\mathbf{v_1}, \ldots, \mathbf{v_r}$ which are linearly independent and span $\mathcal{W}$. Thus $\dim \mathcal{W} \leq \dim \mathcal{V}$, and $\mathcal{W}$ is finite dimensional. ∎

Question 5(a) *State and prove the Cayley-Hamilton Theorem when the eigenvalues are all different.*

Solution. See 1987 question 5(a). ∎

Question 5(b) *When are two real symmetric matrices said to be congruent? Is congruence an equivalence relation? Also show how you can find the signature of* $\mathbf{A}$.

Solution. Two matrices $\mathbf{A}, \mathbf{B}$ are said to be congruent to each other if there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}$.

Congruence is an equivalence relation:

- Reflexive: $\mathbf{A} \equiv \mathbf{A} \because \mathbf{A} = \mathbf{I}'\mathbf{A}\mathbf{I}$, $\mathbf{I}$ is the unit matrix.
- Symmetric: $\mathbf{A} \equiv \mathbf{B} \Rightarrow \mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B} \Rightarrow \mathbf{A} = (\mathbf{P}^{-1})'\mathbf{B}\mathbf{P}^{-1} \Rightarrow \mathbf{B} \equiv \mathbf{A}$.
- Transitive: $\mathbf{A} \equiv \mathbf{B}, \mathbf{B} \equiv \mathbf{C} \Rightarrow \mathbf{A} \equiv \mathbf{C}$ — $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}, \mathbf{Q}'\mathbf{B}\mathbf{Q} = \mathbf{C} \Rightarrow \mathbf{Q}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{Q} = \mathbf{C} \Rightarrow \mathbf{A} \equiv \mathbf{C}$ because $\mathbf{P}\mathbf{Q}$ is nonsingular as both $\mathbf{P}, \mathbf{Q}$ are nonsingular.

Given a symmetric matrix $\mathbf{A}$, we first prove that there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \text{diagonal}[\alpha_1, \alpha_2, \ldots, \alpha_r, 0, \ldots, 0]$ where r is the rank of $\mathbf{A}$.

We will prove this by induction on the order n of the matrix $\mathbf{A}$. If $n = 1$, there is nothing to prove. Assume that the result is true for all matrices of order $< n$.

Step 1. We first ensure that we have $a_{11} \neq 0$. If it is 0, but some other $a_{kk} \neq 0$, we interchange the k-th row with the first row and the k-th column with the first column, to get $\mathbf{B} = \mathbf{P}'\mathbf{A}\mathbf{P}$, where $b_{11} = a_{kk} \neq 0$. Note that $\mathbf{P}$ is the elementary matrix $\mathbf{E}_{1k}$ (see 1983 question 2(a)), and is hence nonsingular and symmetric, so $\mathbf{B}$ is symmetric.

If all a_{ii} are 0, but some $a_{ij} \neq 0$. We add the j-th row to the i-th row and the j-the column to the i-th column by multiplying $\mathbf{A}$ by $\mathbf{E}_{ij}(1)$ and its transpose, to get $\mathbf{B} = \mathbf{E}_{ij}(1)\mathbf{A}\mathbf{E}_{ij}(1)'$

— now $b_{ii} = a_{ij} + a_{ji} \neq 0$. B is still symmetric, and we can shift b_{ii} to the leading place as above.

(Note that if all $a_{ij} = 0$, we stop.)

Thus we start with $a_{11} \neq 0$. We subtract $\frac{a_{1k}}{a_{11}}$ times the first row from the k-th row and $\frac{a_{1k}}{a_{11}}$ times the first column from the k-th column, by performing $\mathbf{B} = \mathbf{E}_{k1}(-\frac{a_{1k}}{a_{11}})\mathbf{A}\mathbf{E}_{k1}(-\frac{a_{1k}}{a_{11}})'$ Repeating this for all k, $2 \leq k \leq n$, we get $\mathbf{P_1'AP_1} = \begin{pmatrix} a_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix}$, where $\mathbf{A}_1$ is $n-1 \times n-1$ and $\mathbf{P_1}$ is nonsingular. Now by induction, $\exists \mathbf{P_2}, n-1 \times n-1$ such that $\mathbf{P_2'AP_2} =$ diagonal$[\beta_2, \ldots, \beta_r, 0, \ldots, 0]$, rank $\mathbf{A_1}$ = rank $\mathbf{A} - 1$. Now set $\mathbf{P} = \mathbf{P}_1 \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix}$ to get the result.

Now that we have $\mathbf{P'AP} =$ diagonal$[\alpha_1, \alpha_2, \ldots, \alpha_r, 0, \ldots, 0]$, let us assume that $\alpha_1, \ldots, \alpha_s$ are positive, the rest are negative. Then let $\alpha_i = \beta_i^2, 1 \leq i \leq s, -\alpha_j = \beta_j^2, s+1 \leq j \leq r$. Set $\mathbf{Q} =$ diagonal$[\beta_1^{-1}, \ldots, \beta_r^{-1}, 1, \ldots, 1]$. Then $\mathbf{x'Q'P'APQx} = x_1^2 + \ldots + x_s^2 - x_{s+1}^2 - x_r^2$. Thus we can find the signature of $\mathbf{A}$ by looking at the number of positive and negative squares of the RHS. ■

Question 5(c) *Derive a set of necessary and sufficient conditions that the real quadratic form $\sum_{j=1}^3 \sum_{i=1}^3 a_{ij}x_ix_j$ be positive definite.*

Is $4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy$ positive definite?

Solution. For the first part, see 1992 question 2(c).

$$\begin{aligned}
Q(x,y,z) &= 4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + 9y^2 + 2z^2 + 8yz + -\frac{9}{2}yz - \frac{9}{4}y^2 - \frac{9}{4}z^2 \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}y^2 - \frac{1}{4}z^2 - \frac{7}{2}yz \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}(y^2 - \frac{1}{27}z^2 - \frac{14}{27}yz) \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}(y - \frac{7}{27}z)^2 - \frac{1}{4}z^2 - \frac{49}{108}z^2
\end{aligned}$$

So set $X = 2x + \frac{3}{2}y + \frac{3}{2}z, Y = y - \frac{7}{27}z, Z = z$, then $Q(x,y,z)$ is transformed to $X^2 + \frac{27}{4}Y^2 - \frac{76}{108}Z^2$. Hence $Q(x,y,z)$ is is not positive definite. ■

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Question 1(a) *Let $\mathcal{V}$ be a finitely generated vector space. Show that $\mathcal{V}$ has a finite basis and any two bases of $\mathcal{V}$ have the same number of vectors.*

Solution. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a generating set for $\mathcal{V}$, we assume that $\mathbf{v}_i \neq \mathbf{0}, 1 \leq i \leq m$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent, then it is a basis of $\mathcal{V}$. Otherwise, there exists a $\mathbf{v}_k$ that depends linearly on $\{\mathbf{v}_i \mid 1 \leq i \leq m, i \neq k\}$. This latter set is also a generating set, and we rename it $\{\mathbf{u}_1, \ldots, \mathbf{u}_{m-1}\}$. We now apply the same reasoning to it — either it is linearly independent and hence a basis, or we can drop an element from it and it still remains a generating set. In a finite number of steps, we reach $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ such that $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is linearly independent and a generating set, thus $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is a basis of $\mathcal{V}$.

Note: An alternative approach leading to the same result is to pick the maximal linearly independent subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. There are only 2^m such subsets, so we can do so in a finite number of steps (in the above procedure we dropped the dependent elements one at a time to reach the maximal linearly independent subset). Now to be a basis, the maximal linearly independent subset $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_r\} \subseteq \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ needs to generate $\mathcal{V}$. But this is immediate, as for each $\mathbf{v}_i$, either $\mathbf{v}_i \in S$ or $S \cup \{\mathbf{v_i}\}$ is linearly dependent — in that case $\sum_{j=1}^r a_j \mathbf{x}_j + b\mathbf{v_i} = 0$, but not all a_j, b are 0. Now if $b = 0$ then $\sum_{j=1}^r a_j \mathbf{x}_j = 0 \Rightarrow a_j = 0$ for $1 \leq j \leq r$, as S is linearly independent, and this contradicts the statement that not all a_j, b are 0. So $b \neq 0$, hence $\mathbf{v}_i$ is a linear combination of S, hence S generates $\mathcal{V}$ and is a basis.

Any two bases have the same number of elements: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be two bases of $\mathcal{V}$. Assume wlog that $m \leq n$. Now since $\mathbf{w}_1 \in \mathcal{V}$, $\mathbf{w}_1$ is generated by the basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, thus $\mathbf{w}_1 = \sum_{j=1}^m a_j \mathbf{v}_j$. There must be at least one non-zero a_k, as $\mathbf{w}_1 \neq \mathbf{0}$. Now the set $\{\mathbf{v}_i \mid 1 \leq i \leq m, i \neq k\} \cup \{\mathbf{w}_1\}$ generates the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ (since $\mathbf{v}_k = \frac{1}{a_k}\mathbf{w}_1 - \sum_{j=1, j\neq k}^m \frac{a_j}{a_k}\mathbf{v}_j$) and hence generates $\mathcal{V}$.

Now we have $\mathbf{w}_2 = \sum_{i=1, i\neq k}^m a_i \mathbf{v}_i + b\mathbf{w_1}$. At least one of the $a_i \neq 0$, otherwise we have a linear equation between $\mathbf{w}_1$ and $\mathbf{w}_2$, but these are linearly independent. We replace $\mathbf{v}_i$ by $\mathbf{w}_2$, and the result is also a generating set as above. Continuing, after m steps, we get a subset $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ which is a generating set. Now if $n > m$, we would have $\mathbf{w}_n = \sum_{i=0}^{m} a_i \mathbf{w}_i$, but this is not possible as the $\mathbf{w}_i$ were a basis, and thus linearly independent. Hence $n = m$, and the two bases have equal number of elements. ∎

Question 1(b) *Let $\mathcal{V}$ be the vector space of polynomials of degree ≤ 3. Determine whether the following vectors of $\mathcal{V}$ are linearly dependent or independent: $u = t^3 - 3t^2 + 5t + 1, v = t^3 - t^2 + 8t + 2, w = 2t^3 - 4t^2 + 9t + 5$.*

Solution. Let $au + bv + cw = 0$. Then

$$\begin{aligned} a + b + 2c &= 0 & (1) \\ -3a - b - 4c &= 0 & (2) \\ 5a + 8b + 9c &= 0 & (3) \\ a + 2b + 5c &= 0 & (4) \end{aligned}$$

From (4) - (1) we get $b + 3c = 0$. Substituting $b = -3c$ in (2), $c = -3a \Rightarrow b = 9a$. Now from (1), $a + 9a - 6a = 0 \Rightarrow a = 0 \Rightarrow b = c = 0$. Thus $au + bv + cw = 0 \Rightarrow a = b = c = 0$, so the vectors are linearly independent. ∎

Question 1(c) *For any linear transformation $T : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ prove that*

$$\operatorname{rank} T \leq \min(\dim \mathcal{V}_1, \dim \mathcal{V}_2)$$

Solution. By definition, $\operatorname{rank} T = \dim T(\mathcal{V}_1)$. Clearly $T(\mathcal{V}_1)$ is a subspace of $\mathcal{V}_2$ and therefore $\dim T(\mathcal{V}_1) \leq \dim \mathcal{V}_2$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $\mathcal{V}_1$, then $T(\mathcal{V}_1)$ is generated by $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$ — If $\mathbf{w} \in T(\mathcal{V}_1)$, then there exists $\mathbf{v} \in \mathcal{V}_1$ such that $T(\mathbf{v}) = \mathbf{w}$. But $\mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i, a_i \in \mathbb{R}$, therefore $\mathbf{w} = T(\mathbf{v}) = \sum_{i=1}^{n} a_i T(\mathbf{v}_i) \Rightarrow \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$ is a generating system for $T(\mathcal{V}_1) \Rightarrow \dim T(\mathcal{V}_1) \leq n$. Thus $\operatorname{rank} T = \dim T(\mathcal{V}_1) \leq \min(\dim \mathcal{V}_1, \dim \mathcal{V}_2)$. ∎

Question 2(a) *Show that every non-singular matrix can be expressed as a product of elementary matrices.*

Solution. We first list all the elementary matrices:

1. $\mathbf{E}_{ij}$ = the matrix obtained by interchanging the i-th and j-th rows (or the i-th and j-th columns of the unit matrix. For example, if $n = 4$, then

$$E_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. $\mathbf{E}_i(\alpha)$ is the matrix obtained by multiplying the i-th row of the unit matrix by $\alpha =$ the matrix obtained by multiplying the i-th column of the unit matrix by α.

3. $\mathbf{E}_{ij}(\beta) =$ the matrix obtained by adding β times the j-th row to the i-th row of the unit matrix.

4. $(\mathbf{E}_{ij}(\beta))' =$ transpose of $\mathbf{E}_{ij}(\beta) =$ the matrix obtained by adding β times the j-th column to the i-th column of the unit matrix.

All elementary matrices are non-singular. In fact $|\mathbf{E}_{ij}| = -1$, $|\mathbf{E}_i(\alpha)| = \alpha$, $|\mathbf{E}_{ij}(\beta)| = |(\mathbf{E}_{ij}(\beta))'| = 1$.

We now prove the result.

(1) Let $\mathbf{C} = \mathbf{AB}$. Then any elementary row transformation on $\mathbf{AB}$ is equivalent to subjecting $\mathbf{A}$ to the same row transformation. Let $\mathbf{A} = \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix}_{m\times n}$ and $\mathbf{B} = \begin{pmatrix} C_1 & \dots & C_p \end{pmatrix}_{n\times p}$.

Then $\mathbf{AB} = \begin{pmatrix} R_1C_1 & \dots & R_1C_p \\ \vdots & & \vdots \\ R_mC_1 & \dots & R_mC_p \end{pmatrix}_{m\times p}$. Thus if any elementary row transformation i.e. (i) Intercanging two rows (ii) Multiplying a row by a scalar (iii) Adding a scalar multiple of a row to another row, is carried out on $\mathbf{A}$, the same will be carried out on $\mathbf{AB}$ and vice versa. Similarly any column transformation on $\mathbf{B}$ is equivalent to the same column transformation on $\mathbf{AB}$.

(2) Multiplying by an elementary matrix $\mathbf{E}_{ij}, \mathbf{E}_i(\alpha), \mathbf{E}_{ij}(\beta)$ on the left is the same as performing the corresponding elementary row operation on the matrix. Multiplying the matrix by an elementary matrix to the right is equal to subjecting the matrix to the corresponding column transformation. We write $\mathbf{A} = \mathbf{IA}$. Now interchanging the i-th and j-th row of $\mathbf{A}$ is equivalent to doing the same on $\mathbf{I}$ in $\mathbf{IA}$ (result (1) above), which is the same as $\mathbf{E}_{ij}\mathbf{A}$. Similar results hold for the other two row transformations. Writing $\mathbf{A}$ as $\mathbf{AI}$ gives the corresponding result for column transformations.

(3) We now prove that if $\mathbf{A}$ is a matrix of rank $r > 0$, then there exist $\mathbf{P}, \mathbf{Q}$ products of elementary matrices such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ where $\mathbf{I}_r$ is the unit matrix of order r. Since $\mathbf{A} \neq \mathbf{0}$, $\mathbf{A}$ has at least one non-zero element, say a_{ij}. By interchanging the i-th row with the first row and the j-th column with the first column, we get a new matrix $\mathbf{B}_{ij} = (b_{ij})$ such that $b_{11} \neq 0$. This simply means that there exist elementary matrices $\mathbf{P_1}, \mathbf{Q_1}$ such that $\mathbf{P_1AQ_1} = \mathbf{B}$. We multiply $\mathbf{P_1AQ_1}$ by $\mathbf{P_2} = \mathbf{E}_1(b_{11}^{-1})$ to obtain $\mathbf{P_2P_1AQ_1} = \mathbf{C} = \begin{pmatrix} 1 & * & \dots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{pmatrix}$. Subtracting suitable multiples of the first row from the remaining rows of $\mathbf{C}$ and suitable multiples of the first column from the remaining columns, we get the new

matrix $\mathbf{D}$ of the form $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}^* & \\ 0 & & & \end{pmatrix}$. Thus we have proved that there exist $\mathbf{P}^*, \mathbf{Q}^*$ products of elementary matrices such that $\mathbf{P}^*\mathbf{A}\mathbf{Q}^* = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A}^* & \\ 0 & & & \end{pmatrix}$. We carry on the same process on $\mathbf{A}^*$ without affecting the first row and column, and in r steps we get $\mathbf{P}^{**}\mathbf{A}\mathbf{Q}^{**} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}$, where $\mathbf{P}^{**}, \mathbf{Q}^{**}$ are products of elementary matrices. Note that $\mathbf{E} = \mathbf{0}$ because rank $\mathbf{A} = r$.

Now if $\mathbf{A}$ is nonsingular, then $\mathbf{P}^{**}\mathbf{A}\mathbf{Q}^{**} = \mathbf{I}$. Inverting the elementary matrices (the inverse of an elementary matrix is elementary), we get that $\mathbf{A}$ is a product of elementary matrices. ∎

Question 2(b) *Reduce the matrix $\mathbf{A}$ to its normal form, and hence or otherwise determine its rank.*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution. Interchange of R_1 and $R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$

$R_3 - 3R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$

$R_3 + 5R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$

$-\frac{1}{2}R_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$

$R_3 + R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Interchanging C_2, C_4, $\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$C_3 - 2C_2, C_4 - C_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. $C_4 - C_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Thus we have $\mathbf{P}(3 \times 3)$ and $\mathbf{Q}(4 \times 4)$ both products of elementary matrices such that $\mathbf{PAQ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which is the normal form of $\mathbf{A}$. Clearly the rank of $\mathbf{A}$ is 3. ∎

Question 2(c) *Show that the equations*

$$\begin{aligned} x + y + z &= 3 \\ 3x - 5y + 2z &= 8 \\ 5x - 3y + 4z &= 14 \end{aligned}$$

are consistent and solve them.

Solution. The coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & -5 & 2 \\ 5 & -3 & 4 \end{pmatrix}$.

$\det \mathbf{A} = 1(-20+6) - 1(12-10) + 1(-9+25) = 0$, thus $\operatorname{rank} \mathbf{A} < 3$. Actually $\operatorname{rank} \mathbf{A} = 2$, since $\begin{vmatrix} 1 & 1 \\ 3 & -5 \end{vmatrix} \neq 0$.

The augmented matrix $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 3 & -5 & 2 & 8 \\ 5 & -3 & 4 & 14 \end{pmatrix}$.

$R_2 - 3R_1, R_3 - 5R_1 \Rightarrow \mathbf{B} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & -8 & -1 & -1 \end{pmatrix}$

$R_3 - R_2 \Rightarrow \mathbf{B} \sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -8 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\operatorname{rank} \mathbf{B} = 2$, because $\begin{vmatrix} 1 & 1 \\ 0 & -8 \end{vmatrix} \neq 0$.

Since $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{B} = 2$, the system is consistent, and the space of solutions has dimension 1.

Now $x + y = 3 - z, 3x - 5y = 8 - 2z$, subtracting the second from 3 times the first we get $8y = 1 - z \Rightarrow y = \frac{1-z}{8}$. $x = 3 - z - \frac{1-z}{8} = \frac{23-7z}{8}$. Thus the solutions are given by $(\frac{23-7z}{8}, \frac{1-z}{8}, z)$, $z \in \mathbb{R}$. ∎

Question 3(a) *Prove that a square matrix satisfies its characteristic equation. Use this result to find the inverse of*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. The first part is the Cayley-Hamilton theorem, see 1987 question 3(a).

The characteristic equation of $\mathbf{A}$ is

$$\begin{aligned} |\mathbf{A}-\lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 3 \\ 3 & 1 & 1-\lambda \end{vmatrix} &= 0 \\ \Rightarrow -\lambda(\lambda^2-3\lambda+2-3)-(1-\lambda-9)+2(1-6+3\lambda) &= 0 \\ \Rightarrow \lambda^3-3\lambda^2-8\lambda+2 &= 0 \end{aligned}$$

Thus $\mathbf{A}^3-3\mathbf{A}^2-8\mathbf{A}+2\mathbf{I}=\mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A}^2-3\mathbf{A}-8\mathbf{I})=-2\mathbf{I}$, or $\mathbf{A}^{-1}=-\frac{1}{2}(\mathbf{A}^2-3\mathbf{A}-8\mathbf{I})$.

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix} \\ \therefore \mathbf{A}^{-1} &= \frac{1}{2}\left[-\begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix}+\begin{pmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{pmatrix}+\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{pmatrix} \end{aligned}$$

∎

Note: In this case, we were required to use this method to find the inverse. An alternate method of finding the inverse by performing elementary row and column operations is shown in 1985 question 1(c).

Question 3(b) *Find the eigenvalues and eigenvectors of*

$$\mathbf{A} = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Solution.

$$\begin{aligned} |\mathbf{A}-x\mathbf{I}| &= \begin{vmatrix} 8-x & -6 & 2 \\ -6 & 7-x & -4 \\ 2 & -4 & 3-x \end{vmatrix} = 0 \\ &\Rightarrow (8-x)(x^2-10x+21-16)+6(6x-18+8)+2(24-14+2x)=0 \\ &\Rightarrow -x^3+18x^2-85x+40+36x-60+20+4x=0 \\ &\Rightarrow x^3-18x^2+45x=0 \end{aligned}$$

Thus the eigenvalues are $0, 3, 15$.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 0, then $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$.

Thus $8x_1-6x_2+2x_3=0, -6x_1+7x_2-4x_3=0, 2x_1-4x_2+3x_3=0 \Rightarrow x_1=\frac{1}{2}x_3, x_2=x_3$. Thus $(1,2,2)$ is an eigenvector for 0, in general $(x/2, x, x), x \neq 0$ is an eigenvector for 0.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 3, then $\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$. Thus $5x_1 - 6x_2 + 2x_3 = 0, -6x_1 + 4x_2 - 4x_3 = 0, 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2, x_3 = -2x_2$. Thus $(2, 1, -2)$ is an eigenvector for 3, in general $(2x, x, -2x), x \neq 0$ is an eigenvector for 3.

If (x_1, x_2, x_3) is an eigenvector for the eigenvalue 15, then $\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$. Thus $-7x_1 - 6x_2 + 2x_3 = 0, -6x_1 - 8x_2 - 4x_3 = 0, 2x_1 - 4x_2 - 12x_3 = 0 \Rightarrow x_1 = 2x_3, x_2 = -2x_3$. Thus $(2, -2, 1)$ is an eigenvector for 15, in general $(2x, -2x, x), x \neq 0$ is an eigenvector for 15. ∎

Question 3(c) *Show that the eigenvalues of an upper or lower triangular matrix are just the diagonal elements of the matrix.*

Solution. Let $\mathbf{A} = (a_{ij})$, such that $a_{ij} = 0$ for $i < j$, i.e. $\mathbf{A}$ is upper triangular. Now

$$|x\mathbf{I} - \mathbf{A}| = (x - a_{11})(x - a_{22}) \ldots (x - a_{nn})$$

showing that $|x\mathbf{I} - \mathbf{A}| = 0 \Rightarrow x = a_{11}, a_{22}, \ldots, a_{nn}$. Thus the eigenvalues of $\mathbf{A}$ are $a_{11}, a_{22}, \ldots, a_{nn}$.

Similarly for a lower triangular matrix. ∎

Paper II

Question 4(a) *Prove that a necessary and sufficient condition that a linear transformation* $\mathbf{A}$ *on a unitary space is Hermitian is that* $\langle \mathbf{Ax}, \mathbf{x} \rangle$ *is real for all* $\mathbf{x}$.

Solution. A unitary space is an old name for an inner product space. Let $\mathcal{V}$ be an inner product space over $\mathbb{C}$, and $\langle \mathbf{Av}, \mathbf{v} \rangle$ be real for all $\mathbf{v} \in \mathcal{V}$. Then since

$$\langle \mathbf{A}(\mathbf{v} + \mathbf{w}), \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{Av}, \mathbf{v} \rangle + \langle \mathbf{Aw}, \mathbf{w} \rangle + \langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle$$

$\langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle$ is real (because $\langle \mathbf{A}(\mathbf{v} + \mathbf{w}), \mathbf{v} + \mathbf{w} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle - \langle \mathbf{Aw}, \mathbf{w} \rangle$ is real). Hence

$$\langle \mathbf{Av}, \mathbf{w} \rangle + \langle \mathbf{Aw}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{Av} \rangle + \langle \mathbf{v}, \mathbf{Aw} \rangle \qquad (1)$$

because z real $\Rightarrow z = \overline{z}$.

Also,

$$\langle \mathbf{A}(\mathbf{v} + i\mathbf{w}), \mathbf{v} + i\mathbf{w} \rangle = \langle \mathbf{Av}, \mathbf{v} \rangle + \langle \mathbf{A}(i\mathbf{w}), i\mathbf{w} \rangle - i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle$$

thus $-i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle$ is real. Thus

$$-i\langle \mathbf{Av}, \mathbf{w} \rangle + i\langle \mathbf{Aw}, \mathbf{v} \rangle = -i\langle \mathbf{w}, \mathbf{Av} \rangle + i\langle \mathbf{v}, \mathbf{Aw} \rangle \qquad (2)$$

Multiplying (1) by i and adding to (2), we get

$$2i\langle \mathbf{Aw}, \mathbf{v} \rangle = 2i\langle \mathbf{v}, \mathbf{Aw} \rangle$$

Thus $\mathbf{A} = \mathbf{A}^*$, so $\mathbf{A}$ is Hermitian.

Conversely, if $\langle \mathbf{Aw}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{Aw} \rangle$, then $\langle \mathbf{Av}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{Av} \rangle = \overline{\langle \mathbf{Av}, \mathbf{v} \rangle} \Rightarrow \langle \mathbf{Av}, \mathbf{v} \rangle$ is real. ∎

Question 4(b) *If* $\mathbf{A}$ *is a linear transformation on an* n*-dimensional vector space, then prove that*

1. $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{A}'$.
2. $\operatorname{nullity} \mathbf{A} = n - \operatorname{rank} \mathbf{A}$.

Solution. We know that $\operatorname{rank} \mathbf{A} = r$ if $\mathbf{A}$ has a minor of order r different from 0, and all minors of order $> r$ are 0. Thus $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{A}'$.

For the second part, see 1998 question 3(a). ∎

Question 4(c) *Show that a real symmetric matrix* $\mathbf{A}$ *is positive definite if and only if there exists a real non-singular matrix* $\mathbf{P}$ *such that* $\mathbf{A} = \mathbf{P}\mathbf{P}'$.

Solution. $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{P}\mathbf{P}'\mathbf{x} =$ sum of squares > 0 (because $\mathbf{P}'\mathbf{x} \neq \mathbf{0}$ as $\mathbf{P}$ is non-singular).

Conversely: Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ be a basis of $\mathbb{R}^n$. We will use this to construct a new basis $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ which satisfies $\mathbf{e_i}\mathbf{A}\mathbf{e_j} = \delta_{ij}$, as follows:

$$
\begin{aligned}
\mathbf{e_1} &= \frac{\mathbf{x_1}}{\sqrt{\mathbf{x_1}\mathbf{A}\mathbf{x_1}}} \\
\mathbf{y_2} &= \mathbf{x_2} - (\mathbf{x_2}\mathbf{A}\mathbf{e_1})\mathbf{e_1} \\
\mathbf{e_2} &= \frac{\mathbf{y_2}}{\sqrt{\mathbf{y_2}\mathbf{A}\mathbf{y_2}}} \\
&\ldots \\
\mathbf{y_i} &= \mathbf{x_i} - \sum_{j=1}^{i-1} (\mathbf{x_i}\mathbf{A}\mathbf{e_j})\mathbf{e_j} \\
\mathbf{e_i} &= \frac{\mathbf{y_i}}{\sqrt{\mathbf{y_i}\mathbf{A}\mathbf{y_i}}} \\
&\ldots
\end{aligned}
$$

$\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ are linearly independent — if $\sum_{i=1}^n a_i \mathbf{e_i} = \mathbf{0}$, then take the largest i such that $a_i \neq 0$, this allows us to express $\mathbf{x_i}$ in terms of the other basis vectors, which is not possible. Inductively we can also verify that $\mathbf{e_i}\mathbf{A}\mathbf{e_i} = 1$, and $\mathbf{e_i}\mathbf{A}\mathbf{e_j} = 0$ if $i \neq j$. This is the Gram-Schmidt orthonormalization process - we exploit the property that any positive definite matrix $\mathbf{A}$ gives rise to an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}'\mathbf{A}\mathbf{y}$.

Now consider the matrix $\mathbf{Q} = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} & \ldots & \mathbf{e_n} \end{pmatrix}$. Now if $\mathbf{B} = \mathbf{Q}'\mathbf{A}\mathbf{Q}$ then $b_{ij} = \mathbf{e_i}\mathbf{A}\mathbf{e_j}$, thus $\mathbf{B} = \mathbf{I}_n$. Since $\mathbf{Q}$ consists of linearly independent columns, it is invertible, and thus $\mathbf{A} = \mathbf{Q}'^{-1}\mathbf{Q}^{-1}$. Setting $\mathbf{P} = \mathbf{Q}'^{-1}$, we have $\mathbf{A} = \mathbf{P}\mathbf{P}'$. ∎

Question 5(a) *If* $\mathbf{S}$ *is a skew symmetric matrix of order* n *and if* $\mathbf{I} + \mathbf{S}$ *is non-singular, then prove that* $\mathbf{A} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$ *is an orthogonal matrix of order* n.

Solution. See 1999, question 2(b). ∎

Question 5(b) *Under what circumstances will the real $n \times n$ matrix*

$$\mathbf{A} = \begin{pmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ & & & \dots & \\ a & a & a & \dots & x \end{pmatrix}$$

be (1) positive semidefinite (2) positive definite.

Solution. The eigenvalues of the given matrix can be computed as follows:

$$\begin{vmatrix} x-\lambda & a & a & \dots & a \\ a & x-\lambda & a & \dots & a \\ a & a & x-\lambda & \dots & a \\ & & & \dots & \\ a & a & a & \dots & x-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-\lambda & a-x+\lambda & a-x+\lambda & \dots & a-x+\lambda \\ a & x-\lambda-a & 0 & \dots & 0 \\ a & 0 & x-\lambda-a & \dots & 0 \\ & & & \dots & \\ a & 0 & 0 & \dots & x-\lambda-a \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-\lambda+(n-1)a & 0 & 0 & \dots & 0 \\ a & x-\lambda-a & 0 & \dots & 0 \\ a & 0 & x-\lambda-a & \dots & 0 \\ & & & \dots & \\ a & 0 & 0 & \dots & x-\lambda-a \end{vmatrix} = 0$$

$$\Rightarrow (x-\lambda+(n-1)a)(x-\lambda-a)^{n-1} = 0$$

Thus the eigenvalues are $x-a$ (repeated $n-1$ times) and $x+(n-1)a$. For positive definite, $\lambda > 0 \Rightarrow x > a, x > (n-1)(-a)$. If $a > 0$, this reduces to $x > a$, if $a \leq 0$, this reduces to $x > (n-1)(-a)$.

For positive semi-definite, $\lambda \geq 0$. By the same reasoning, if $a > 0$, then $x \geq a$, otherwise $x \geq (n-1)(-a)$. ∎

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Question 1(a) *If $\mathcal{W}_1, \mathcal{W}_2$ are finite dimensional subspaces of a vector space $\mathcal{V}$, then show that $\mathcal{W}_1 + \mathcal{W}_2$ is finite dimensional and* $\dim \mathcal{W}_1 + \dim \mathcal{W}_2 = \dim(\mathcal{W}_1 + \mathcal{W}_2) + \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$.

Solution. See 1988, question 1(b). ∎

Question 1(b) *If $\mathbf{A}$ and $\mathbf{B}$ are n-rowed square non-zero matrices such that $\mathbf{AB} = \mathbf{0}$, then show that both $\mathbf{A}$ and $\mathbf{B}$ are singular. If both $\mathbf{A}$ and $\mathbf{B}$ are singular, and $\mathbf{AB} = \mathbf{0}$, does it imply that $\mathbf{BA} = \mathbf{0}$? Justify your answer.*

Solution. If $\mathbf{A}$ were non-singular, then $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{0} \Rightarrow \mathbf{B} = \mathbf{0}$. Thus $\mathbf{A}$ is singular, and similarly $\mathbf{B}$ is singular.

$\mathbf{AB} = \mathbf{0}$ does not imply that $\mathbf{BA} = \mathbf{0}$. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \mathbf{0}$. ∎

Question 2(a) *Show that row-equivalent matrices have the same rank.*

Solution. See 1986 question 3(b). ∎

Question 2(b) *A linear transformation T on a vector space $\mathcal{V}$ with finite basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ is non-singular if and only if the vectors $\alpha_1 T, \alpha_2 T, \ldots, \alpha_n T$ are linearly independent in $\mathcal{V}$. When this is the case, show that T has an inverse T^{-1} with $TT^{-1} = T^{-1}T = I$.*

Solution. If $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent, then T is one-one: Let $\mathbf{v} \in \mathcal{V}$. Then $\mathbf{v} = \sum_{i=1}^n a_i\alpha_i, T(\mathbf{v}) = \sum_{i=1}^n a_i T(\alpha_i)$. If $T(\mathbf{v}) = 0$, then $\sum_{i=1}^n a_i T(\alpha_i) = 0 \Rightarrow a_i = 0, 1 \leq i \leq n$, because $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent. Thus $T(\mathbf{v}) = 0 \Rightarrow \mathbf{v} = \mathbf{0}$, so T is one-one.

T is onto: $\dim T(\mathbf{v}) = \dim \mathcal{V} = \mathbf{n}$.

Thus T is invertible, in fact $T^{-1}(T(\alpha_i)) = \alpha_i$.

T^{-1} is a linear transformation: Let $T^{-1}(\mathbf{v}) = \mathbf{u}, \mathbf{T^{-1}}(\mathbf{w}) = \mathbf{x}$. Then $T(\mathbf{u}) = \mathbf{v}, T(\mathbf{x}) = \mathbf{w}$. Let $T^{-1}(a\mathbf{v} + b\mathbf{w}) = \mathbf{z}$, then $T(\mathbf{z}) = a\mathbf{v} + b\mathbf{w} = aT(\mathbf{u}) + bT(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{x}) \Rightarrow \mathbf{z} = a\mathbf{u} + b\mathbf{x}$. Thus $T^{-1}(a\mathbf{v} + b\mathbf{w}) = aT^{-1}(\mathbf{v}) + bT^{-1}(\mathbf{w})$, so T^{-1} is linear. It is obvious that $TT^{-1} = T^{-1}T = I$, as this is true for the basis elements by definition, and extends to all vectors by linearity.

Conversely if T is non-singular, then $a_1T(\alpha_1) + a_2T(\alpha_2) + \ldots + a_nT(\alpha_n) = 0 \Rightarrow T(\sum_{i=1}^n a_i\alpha_i) = 0 \Rightarrow \sum_{i=1}^n a_i\alpha_i = 0 \Rightarrow a_i = 0, 1 \leq i \leq n$ because $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent. Thus $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent. ∎

Question 2(c) *Solve the following system of equations:*

$$\begin{aligned}
3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\
2x_1 + 5x_2 + 5x_3 - 18x_4 &= 9 \\
4x_1 - x_2 - x_3 + 8x_4 &= 7
\end{aligned}$$

Solution. Let the coefficient matrix be

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 & -5 \\ 2 & 5 & 5 & -18 \\ 4 & -1 & -1 & 8 \end{pmatrix}$$

Doubling R_1, and subtracting $R_2 + R_3$, we get

$$\mathbf{A} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 5 & 5 & -18 \\ 4 & -1 & -1 & 8 \end{pmatrix}$$

Thus the rank of $\mathbf{A}$ is 2.

The augmented matrix

$$\mathbf{B} = \begin{pmatrix} 3 & 2 & 2 & -5 & 8 \\ 2 & 5 & 5 & -18 & 9 \\ 4 & -1 & -1 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 5 & -18 & 9 \\ 4 & -1 & -1 & 8 & 7 \end{pmatrix}$$

Thus the rank of $\mathbf{B}$ is also 2, so the system is consistent.

Since the rank of $\mathbf{A}$ is 2, the space of solutions has rank $= 4 - 2 = 2$. Adding twice the third equation to the first we get $11x_1 + 11x_4 = 22 \Rightarrow x_1 = 2 - x_4$. Substituting this in the third equation, we get $x_2 = 1 - x_3 + 4x_4$. Thus the required solution system is $(2 - x_4, 1 - x_3 + 4x_4, x_3, x_4)$, where x_3, x_4 take any value in $\mathbb{R}$. ∎

Question 3(a) *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the field F, and let T be a linear transformation from $\mathcal{V}$ to $\mathcal{W}$. If $\mathcal{V}$ is finite dimensional show that* $\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$.

Solution. See question 1(a) from 1992. ∎

Question 3(b) *Let $\mathbf{A}$ be a square matrix and $\mathbf{T}$ be non-singular. Let $\widetilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Show that*

1. *$\mathbf{A}$ and $\widetilde{\mathbf{A}}$ have the same eigenvalues.*
2. $\operatorname{tr} \mathbf{A} = \operatorname{tr} \widetilde{\mathbf{A}}$.
3. *If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue, then $\mathbf{T}^{-1}\mathbf{x}$ is an eigenvector of $\widetilde{\mathbf{A}}$ corresponding to the same eigenvalue.*

Solution.

1. The eigenvalues of $\mathbf{A}$ are roots of $|x\mathbf{I} - \mathbf{A}| = 0$. The eigenvalue of $\widetilde{\mathbf{A}}$ are roots of $0 = |x\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T}| = |\mathbf{T}^{-1}x\mathbf{I}\mathbf{T} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T}| = |\mathbf{T}^{-1}||x\mathbf{I} - \mathbf{A}||\mathbf{T}| = |x\mathbf{I} - \mathbf{A}|$, so the eigenvalues are the same.
2. $\operatorname{tr} \mathbf{AB} = \operatorname{tr} \mathbf{BA}$, so $\operatorname{tr} \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \operatorname{tr} \mathbf{A}\mathbf{T}\mathbf{T}^{-1} = \operatorname{tr} \mathbf{A}$.
3. If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}(\mathbf{T}^{-1}\mathbf{x}) = \mathbf{T}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{T}^{-1}\mathbf{x}$.

∎

Question 3(c) *A 3×3 matrix has the eigenvalues 6, 2, -1. The corresponding eigenvectors are $(2, 3, -2), (9, 5, 4), (4, 4, -1)$. Find the matrix.*

Solution. Let $\mathbf{P} = \begin{pmatrix} 2 & 9 & 4 \\ 3 & 5 & 4 \\ -2 & 4 & -1 \end{pmatrix}$, and let $\mathbf{A}$ be the required matrix. Then $\mathbf{AP} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, therefore $\mathbf{A} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{P}^{-1}$. A simple calculation gives $\mathbf{P}^{-1} = \begin{pmatrix} -21 & 25 & 16 \\ -5 & 6 & 4 \\ 22 & -26 & -17 \end{pmatrix}$, note that $|\mathbf{P}| = 1$. Now $\begin{pmatrix} 2 & 9 & 4 \\ 3 & 5 & 4 \\ -2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 18 & -4 \\ 18 & 10 & -4 \\ -12 & 8 & 1 \end{pmatrix}$.
Thus $\mathbf{A} = \begin{pmatrix} 12 & 18 & -4 \\ 18 & 10 & -4 \\ -12 & 8 & 1 \end{pmatrix}\begin{pmatrix} -21 & 25 & 16 \\ -5 & 6 & 4 \\ 22 & -26 & -17 \end{pmatrix} = \begin{pmatrix} -430 & 512 & 352 \\ 516 & 620 & 396 \\ 234 & 226 & -173 \end{pmatrix}$

A longer way would be to set $\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. Then $\mathbf{AP} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. This yields three systems of linear equations which have to be solved. ∎

Question 4(a) *Let $\mathcal{V}$ be the set of all functions from a non-empty set into a field K. For any function $f, g \in \mathcal{V}$ and any scalar $k \in K$, let $f+g$ and kf be functions in $\mathcal{V}$ defined by $(f+g)(x) = f(x) + g(x), (kf)(x) = kf(x)$ for every $x \in X$. Prove that $\mathcal{V}$ is a vector space over K.*

Solution. $\mathcal{V} = \{f \mid f : X \longrightarrow K\}$.

1. $f, g \in \mathcal{V} \Rightarrow (f+g)(x) = f(x) + g(x) \Rightarrow f + g : X \longrightarrow K \Rightarrow f + g \in \mathcal{V}$.
2. The zero function namely $0(x) = 0 \forall x \in X$ is the additive identity of $\mathcal{V}$ i.e. $f + 0 = 0 + f = f \forall f \in \mathcal{V}$.
3. $f \in \mathcal{V} \Rightarrow -f \in \mathcal{V}$ where $(-f)(x) = -f(x)$ anf $f + (-f) = 0 = (-f) + f$.
4. $(f + g) + h = f + (g + h)$ for every $f, g, h \in \mathcal{V}$.
5. If $f \in \mathcal{V}, k \in K$, then $(kf)(x) = kf(x) \forall x \in X$, so $kf \in \mathcal{V}$ and $k(f + g) = kf + kg$.
6. If $k, k' \in K, f \in \mathcal{V}$, then $k(k'f) = (kk')f$.
7. If $1 \in K$ is the multiplicative identity, then $1f = f$ for every f.
8. $k, k' \in K, f \in \mathcal{V} \Rightarrow (k + k')f(x) = kf(x) + k'f(x) \Rightarrow (k + k')f = kf + k'f$.

Thus $\mathcal{V}$ is a vector space over K. ∎

Question 4(b) *Find the eigenvalues and basis for each eigenspace of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

.

Solution. The characteristic equation of $\mathbf{A}$ is

$$\begin{array}{rcrcl} & & |\mathbf{A} - \lambda\mathbf{I}| & = & 0 \\ \Rightarrow & & \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} & = & 0 \\ \Rightarrow & & (1-\lambda)(\lambda+5)(\lambda-4) + 18(1-\lambda) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) & = & 0 \\ \Rightarrow & & (1-\lambda)(\lambda^2 + \lambda - 20) + 18 - 18\lambda - 9\lambda - 18 + 36 + 18\lambda & = & 0 \\ \Rightarrow & & \lambda^2 + \lambda - 20 - \lambda^3 - \lambda^2 + 20\lambda - 9\lambda + 36 & = & 0 \\ \Rightarrow & & \lambda^3 - 12\lambda - 16 & = & 0 \end{array}$$

Thus $\lambda = -2, 4, -2$. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 4$.

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Thus $-3x_1 - 3x_2 + 3x_3 = 0, 3x_1 - 9x_2 + 3x_3 = 0, 6x_1 - 6x_2 = 0 \Rightarrow x_1 = x_2, x_3 = 2x_1$. We can take $(1, 1, 2)$ as an eigenvector corresponding to $\lambda = 4$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = -2$.

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Thus $3x_1 - 3x_2 + 3x_3 = 0 \Rightarrow x_2 = x_1 + x_3$. $(1, 1, 0), (0, 1, 1)$ can be taken as eigenvectors for $\lambda = -2$.

Clearly $(1, 1, 2)$ is a basis for the eigenspace for $\lambda = 4$. $(1, 1, 0), (0, 1, 1)$ is a basis for the eigenspace for $\lambda = -2$. ∎

Question 4(c) *Let a vector space $\mathcal{V}$ have finite dimension and let $\mathcal{W}$ be a subspace of $\mathcal{V}$ and $\mathcal{W}^0$ the annihilator of $\mathcal{W}$. Prove that* $\dim \mathcal{W} + \dim \mathcal{W}^0 = \dim \mathcal{V}$.

Solution. Let $\dim \mathcal{V} = n$, $\dim \mathcal{W} = m$, $\mathcal{W} \subseteq \mathcal{V}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$ so chosen that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis of $\mathcal{W}$. Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of $\mathcal{V}^*$ i.e. $v_i^*(\mathbf{v}_j) = \delta_{ij}$. We shall show that $\mathcal{W}^0$ has $\{v_{m+1}^*, \ldots, v_n^*\}$ as a basis.

By definition of the dual basis $v_i^*(\mathbf{v}_j) = 0$ when $1 \leq i \leq m$ and $m + 1 \leq j \leq n$. Since $v_j^*, m + 1 \leq j \leq n$ annihilate the basis of $\mathcal{W}$, it follows that $v_j(\mathbf{w}) = \mathbf{0}$ for all $\mathbf{w} \in \mathcal{W}$. Thus $\{v_{m+1}^*, \ldots, v_n^*\} \subseteq \mathcal{W}^0$, and are linearly independent, being a subset of a linearly independent set.

Let $f \in \mathcal{W}^0$, then $f = \sum_{i=1}^n a_i v_i^*$. We shall show that $a_i = 0$ for $1 \leq i \leq m$, thus f is a linear combination of $\{v_{m+1}^*, \ldots, v_n^*\}$. By definition of $\mathcal{W}^0$, $f(\mathbf{v}_1) = 0, \ldots, f(\mathbf{v}_m) = 0$, therefore $(\sum_{i=1}^n a_i v_i^*)(\mathbf{v}_j) = \sum_{i=1}^n a_i \delta_{ij} = a_j = 0$ when $1 \leq j \leq m$. Thus $\{v_{m+1}^*, \ldots, v_n^*\}$ is a basis of $\mathcal{W}^0$, hence $\dim \mathcal{W}^0 = n - m$, hence $\dim \mathcal{W} + \dim \mathcal{W}^0 = n = \dim \mathcal{V}$. ∎

Question 5(a) *Prove that every matrix satisfies its characteristic equation.*

Solution. See 1987 question 3(a). ∎

Question 5(b) *Find a necessary and sufficient condition that the real quadratic form* $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ *be a positive definite form.*

Solution. See 1991 question 1(c) and 1992 question 1(c) ∎

Question 5(c) *Prove that the rank of the product of two matrices cannot exceed the rank of either of them.*

Solution. See 1987 question 1(b). ∎

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Question 1(a) *If $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite dimensional subspaces of a vector space $\mathcal{V}$, then show that $\mathcal{W}_1 + \mathcal{W}_2$ is finite dimensional and*

$$\dim \mathcal{W}_1 + \dim \mathcal{W}_2 = \dim(\mathcal{W}_1 + \mathcal{W}_2) + \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$$

Solution. See 1988 question 1(b). ∎

Question 1(b) *Let* $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Prove that the set $\{M_1, M_2, M_3, M_4\}$ forms the basis of the vector space of 2×2 matrices.*

Solution. See 2006 question 1(a). ∎

Question 1(c) *Find the inverse of the matrix* $\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Using operation $R_2 - R_1, R_3 - R_1$, we get

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Now with operation $R_1 - 3(R_2 + R_3)$ we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Thus the inverse of $\mathbf{A}$ is $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. ∎

Question 2(a) *If $T : \mathcal{V} \longrightarrow \mathcal{W}$ is a linear transformation from an n-dimensional vector space $\mathcal{V}$ to a vector space $\mathcal{W}$, then prove that rank(T) + nullity(T) = n.*

Solution. See 1992 question 1(a). ∎

Question 2(b) *Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where $\mathbf{v}_1 = (1, 1, 1), \mathbf{v}_2 = (1, 1, 0), \mathbf{v}_3 = (1, 0, 0)$. Express $(2, -3, 5)$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined as $T(\mathbf{v}_1) = (1, 0), T(\mathbf{v}_2) = (2, -1), T(\mathbf{v}_3) = (4, 3)$. Find $T(2, -3, 5)$.*

Solution. Let $(2, -3, 5) = a(1, 1, 1) + b(1, 1, 0) + c(1, 0, 0)$. Then $a + b + c = 2, a + b = -3, a = 5 \Rightarrow a = 5, b = -8, c = 5$. Thus $(2, -3, 5) = 5\mathbf{v}_1 - 8\mathbf{v}_2 + 5\mathbf{v}_3$.

$T(2, -3, 5) = 5T(\mathbf{v}_1) - 8T(\mathbf{v}_2) + 5T(\mathbf{v}_3) = 5(1, 0) - 8(2, -1) + 5(4, 3) = (9, 23)$. ∎

Question 2(c) *Reduce the following matrix into echelon form:* $\begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix}$.

Solution. $\mathbf{A} = \begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{pmatrix}$ by exchanging R_1 and R_3.

Now $R_2 + 4R_1, R_3 - 6R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{pmatrix}$.

$R_3 + R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix}$.

Multiply R_2 by $\frac{1}{9}$ to get $\mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{pmatrix}$.

Now $R_1 - 2R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{pmatrix}$ which is the required form. ∎

Question 3(a) *Show that if λ is an eigenvalue of matrix $\mathbf{A}$, then λ^n is an eigenvalue of $\mathbf{A}^n$, where n is a positive integer.*

Solution. If $\mathbf{x}$ is an eigenvector for λ, then $\mathbf{A}^n\mathbf{x} = \mathbf{A}^{n-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{A}^{n-1}\mathbf{x}$. Repeating this process, we get the result. ∎

Question 3(b) *Determine if the vectors $(1,-2,1),(2,1,-1),(7,-4,1)$ are linearly independent in $\mathbb{R}^3$.*

Solution. If possible, let $a(1,-2,1)+b(2,1,-1)+c(7,-4,1) = \mathbf{0}$. Then $a+2b+7c = 0, -2a+b-4c = 0, a-b+c = 0$. Adding the last two we get $a = -3c$, and from the third we then get $b = -2c$. These values satisfy the first equation also, hence letting $c = -1$ we get $3(1,-2,1)+2(2,1,-1)-(7,-4,1) = \mathbf{0}$. Thus the vectors are linearly dependent. ∎

Question 3(c) *Solve*

$$2x_1 + 3x_2 + x_3 = 9 \quad (1)$$

$$x_1 + 2x_2 + 3x_3 = 6 \quad (2)$$

$$3x_1 + x_2 + 2x_3 = 8 \quad (3)$$

Solution. $2(2)-(1) \Rightarrow x_2 + 5x_3 = 3 \Rightarrow x_2 = 3 - 5x_3$. Substituting x_2 in (2), $x_1 = 7x_3$. Now substituting x_1, x_2 in (3), we get $21x_3 + 3 - 5x_3 + 2x_3 = 8 \Rightarrow x_3 = \frac{5}{18}, x_2 = \frac{29}{18}, x_1 = \frac{35}{18}$, which is the required solution.

(Using Cramer's rule would have been lengthy.) ∎

Paper II

Question 4(a) *Let $\mathcal{V}$ be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let $\mathcal{V}_e$ be the subset of all even functions $f, f(-x) = f(x)$, and $\mathcal{V}_o$ be the subset of all odd functions $f, f(-x) = -f(x)$. Prove that*

1. *$\mathcal{V}_e$ and $\mathcal{V}_o$ are subspaces of $\mathcal{V}$*
2. *$\mathcal{V}_e + \mathcal{V}_o = \mathcal{V}$*
3. *$\mathcal{V}_e \cap \mathcal{V}_o = \{0\}$*

Solution.

1. Let $f, g \in \mathcal{V}_e$, then $\alpha f + \beta g \in \mathcal{V}_e$ for all $\alpha, \beta \in \mathbb{R}$, because $(\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x) = \alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x)$, thus $\mathcal{V}_e$ is a subspace of $\mathcal{V}$. Similarly, if $f, g \in \mathcal{V}_o$, then $\alpha f + \beta g \in \mathcal{V}_o$ for all $\alpha, \beta \in \mathbb{R}$, because $(\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x) = -\alpha f(x) - \beta g(x) = -(\alpha f + \beta g)(x)$, thus $\mathcal{V}_e$ is a subspace of $\mathcal{V}$.
2. Let $f(x) \in \mathcal{V}$. Define $F(x) = \frac{f(x)+f(-x)}{2}, G(x) = \frac{f(x)-f(-x)}{2}$. Then $F(-x) = F(x) \Rightarrow F \in \mathcal{V}_e$, $G(x) = -G(x) \Rightarrow G \in \mathcal{V}_o$ and $f(x) = F(x) + G(x)$. Thus $\mathcal{V}_e + \mathcal{V}_o = \mathcal{V}$.
3. If $f \in \mathcal{V}_e \cap \mathcal{V}_o$, then $f(-x) = f(x) \because f \in \mathcal{V}_e, f(-x) = -f(x) \because f \in \mathcal{V}_o$. Thus $2f(-x) = 0$ for all $x \in \mathbb{R}$, so $f = 0 \Rightarrow \mathcal{V}_e \cap \mathcal{V}_o = \{0\}$.

∎

Question 4(b) *Find the dimension and basis of the solution space S of the system*

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 0 \\ 3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 &= 0 \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

by performing $R_3 - R_1 - 2R_2$.

Thus rank $\mathbf{A} < 3$. Actually rank $\mathbf{A} = 2$, because if $\mathbf{A} = (C_1, C_2, C_3, C_4, C_5)$, where C_i are columns, then C_1 and C_3 are linearly independent.

Adding the first two equations, we get $4x_5 = -2x_1 - 4x_2 - 5x_3$. Subtracting 3 times the second from the first, we get $4x_4 = -2x_1 - 4x_2 - 7x_3$. From these we can see that $\mathbf{X_1} = (2, 0, 0, -1, -1), \mathbf{X_3} = (0, 1, 0, -1, -1), \mathbf{X_3} = (0, 0, 4, -5, -7)$ are three independent solutions. Since rank $\mathbf{A} = 2$, the dimension of the solution space S is 3, and $\{\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}\}$ is its basis. ∎

Question 4(c) *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of a finite dimensional vector space $\mathcal{V}$. Prove that $(\mathcal{W}_1 + \mathcal{W}_2)^0 = \mathcal{W}_1^0 \cap \mathcal{W}_2^0$.*

Solution. Let $\mathcal{V}^*$ be the dual of $\mathcal{V}$ i.e. $\mathcal{V}^* = \{f \mid f : \mathcal{V} \longrightarrow \mathbb{R},\ f \text{ linear}\}$. Then $\mathcal{W}^0 = \{f \mid f \in \mathcal{V}^*, \forall \mathbf{w} \in \mathcal{W}. f(\mathbf{w}) = \mathbf{0}\}$. $\mathcal{W}^0$ is a vector subspace of $\mathcal{V}^*$ of dimension $\dim \mathcal{V} - \dim \mathcal{W}$.

If $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then $\mathcal{W}_2^0 \subseteq \mathcal{W}_1^0$, because if $f \in \mathcal{W}_2^0, f(\mathbf{w}) = \mathbf{0} \forall \mathbf{w} \in \mathcal{W}_\mathbf{2}$, and therefore $f(\mathbf{w}) = \mathbf{0} \forall \mathbf{w} \in \mathcal{W}_\mathbf{1}$, so $f \in \mathcal{W}_1^0$.

Now $\mathcal{W}_1 \subseteq \mathcal{W}_1 + \mathcal{W}_2$ and $\mathcal{W}_2 \subseteq \mathcal{W}_1 + \mathcal{W}_2$, so $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_1^0$ and $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_2^0$, thus $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_1^0 \cap \mathcal{W}_2^0$.

Conversely, if $f \in \mathcal{W}_1^0 \cap \mathcal{W}_2^0$, then $f(\mathbf{w}_1) = 0, f(\mathbf{w}_2) = 0$ for all $\mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2$. Now any $\mathbf{w} \in \mathcal{W}_1 + \mathcal{W}_2$ is of the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, so $f(\mathbf{w}) = f(\mathbf{w}_1) + f(\mathbf{w}_2) = 0$, because f is linear. Thus $f \in (\mathcal{W}_1 + \mathcal{W}_2)^0$.

Thus $(\mathcal{W}_1 + \mathcal{W}_2)^0 = \mathcal{W}_1^0 \cap \mathcal{W}_2^0$. ∎

Question 5(a) *Let* $\mathbf{H} = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix}$. *Find* $\mathbf{P}$ *so that* $\mathbf{P}'\mathbf{H}\overline{\mathbf{P}}$ *is diagonal. Find the signature of* $\mathbf{H}$.

Solution.

$$\begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $(1-i)R_1$ from R_2, and adding $2iR_1$ to R_3, we get

$$\begin{pmatrix} 1 & 1+i & 2i \\ 0 & 2 & -5i \\ 0 & 5i & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ 2i & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $(1+i)C_1$ from C_2, and adding $-2iC_1$ to C_3, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -5i \\ 0 & 5i & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ 2i & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & -1-i & -2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $\frac{5}{2}iR_2$ from R_3, and adding $\frac{5}{2}iC_2$ to C_3 we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{19}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ \frac{5}{2}+\frac{9}{2}i & -\frac{5}{2}i & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & -1-i & \frac{5}{2}-\frac{9}{2}i \\ 0 & 1 & \frac{5}{2}i \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\mathbf{P} = \begin{pmatrix} 1 & -1+i & \frac{5}{2}+\frac{9}{2}i \\ 0 & 1 & -\frac{5}{2}i \\ 0 & 0 & 1 \end{pmatrix}$

Index = Number of positive entries = 2. Signature = Number of positive entries - Number of negative entries = 1. ∎

Question 5(b) *Prove that every matrix is a root of its characteristic polynomial.*

Solution. This is the Cayley Hamilton theorem, proved in Question 5(a), 1987. ∎

Question 5(c) *If* $\mathbf{B} = \mathbf{AP}$, *where* $\mathbf{P}$ *is nonsingular and* $\mathbf{A}$ *orthogonal, show that* $\mathbf{PB}^{-1}$ *is orthogonal.*

Solution. $\mathbf{B}^{-1} = \mathbf{P}^{-1}\mathbf{A}^{-1}$, so $\mathbf{PB}^{-1} = \mathbf{PP}^{-1}\mathbf{A}^{-1} = \mathbf{A}^{-1}$. Now $(\mathbf{A}^{-1})'\mathbf{A}^{-1} = (\mathbf{AA}')^{-1} = \mathbf{I}$. Similarly $\mathbf{A}^{-1}(\mathbf{A}^{-1})' = (\mathbf{A}'\mathbf{A})^{-1} = \mathbf{I}$, so $\mathbf{PB}^{-1}$ is orthogonal. ∎

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Question 1(a) *If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three $n \times n$ matrices, show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$. Show by an example that matrix multiplication is non-commutative.*

Solution. Let $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}), \mathbf{C} = (c_{ij})$. Let $\mathbf{BC} = (\beta_{ij}), \mathbf{AB} = (\alpha_{ij})$. Then the ij-th element of the RHS $= \sum_{k=1}^{n} \alpha_{ik} c_{kj}$. But $\alpha_{ik} = \sum_{l=1}^{n} a_{il} b_{lk}$, so the ij-th element of the RHS $= \sum_{k=1}^{n} \sum_{l=1}^{n} a_{il} b_{lk} c_{kj}$.

Similarly, the ij-th element of the LHS $= \sum_{l=1}^{n} a_{il} \beta_{lj} = \sum_{l=1}^{n} a_{il} \sum_{k=1}^{n} b_{lk} c_{kj}$. Thus $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. But $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $\mathbf{AB} \neq \mathbf{BA}$. ∎

Question 1(b) *Examine the correctness or otherwise of the following statements:*

1. *The division law is not valid in matrix algebra.*
2. *If $\mathbf{A}, \mathbf{B}$ are square matrices each of order n, and $\mathbf{I}$ is the corresponding unit matrix, then the equation*
$$\mathbf{AB} - \mathbf{BA} = \mathbf{I}$$
can never hold.

Solution.

1. True. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.

2. True. We have proved in 1987 question 5(c) that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues. Trace of $\mathbf{AB}-\mathbf{BA}$ = trace of $\mathbf{AB}$ - trace of $\mathbf{BA}$ = sum of the eigenvalues of $\mathbf{AB}$ - sum of the eigenvalues of $\mathbf{BA}$ = 0. But trace of I_n = n, thus $\mathbf{AB}-\mathbf{BA}=\mathbf{I}$ can never hold.

∎

Question 1(c) *Find a* 3×3 *matrix* $\mathbf{X}$ *such that*

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution. $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \mathbf{3-2(-1)-2(2)} = \mathbf{1}$, so $\mathbf{A}$ is non-singular. Hence $\mathbf{X}=\mathbf{A}^{-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. A simple calculation gives $\mathbf{A}^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Thus

$$\mathbf{X} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 8 & 9 \\ 1 & 3 & 3 \\ 2 & 7 & 7 \end{pmatrix}$$

. ∎

Question 2(a) *If* $\mathcal{M},\mathcal{N}$ *are two subspaces of a vector space* $\mathcal{S}$, *then show that their dimensions satisfy*

$$\dim\mathcal{M}+\dim\mathcal{N} = \dim(\mathcal{M}\cap\mathcal{N})+\dim(\mathcal{M}+\mathcal{N})$$

Solution. See 1998 question 1(b). ∎

Question 2(b) *Find a maximal linearly independent subsystem of the system of vectors* $\mathbf{v}_1=(2,-2,-4), \mathbf{v}_2=(1,9,3), \mathbf{v}_3=(-2,-4,1), \mathbf{v}_4=(3,7,-1)$.

Solution. $\mathbf{v}_1,\mathbf{v}_2$ are linearly independent because $a\mathbf{v}_1+b\mathbf{v}_2=(2a+b,-2a+9b,-4a+3b)=\mathbf{0}\Rightarrow a=b=0$.

$\mathbf{v}_3$ is dependent on $\mathbf{v}_1,\mathbf{v}_2$. If $\mathbf{v}_3=a\mathbf{v}_1+b\mathbf{v}_2$, then $(2a+b,-2a+9b,-4a+3b)=(-2,-4,1)\Rightarrow a=-\frac{7}{10}, b=-\frac{6}{10}$.

Similarly $\mathbf{v}_4$ is dependent on $\mathbf{v}_1,\mathbf{v}_2$. If $\mathbf{v}_4=a\mathbf{v}_1+b\mathbf{v}_2$, then $(2a+b,-2a+9b,-4a+3b)=(3,7,-1)\Rightarrow a=b=1$.

Thus the maximally linearly independent set is $\{\mathbf{v}_1,\mathbf{v}_2\}$. ∎

Question 2(c) *Show that the system of equations*

$$\begin{aligned} 4x + y - 2z + w &= 3 \\ x - 2y - z + 2w &= 2 \\ 2x + 5y - w &= -1 \\ 3x + 3y - z - 3w &= 1 \end{aligned}$$

although consistent is not uniquely solvable. Determine a general solution using x as a parameter.

Solution. The coefficient matrix $\mathbf{A} = \begin{pmatrix} 4 & 1 & -2 & 1 \\ 1 & -2 & -1 & 2 \\ 2 & 5 & 0 & -1 \\ 3 & 3 & -1 & -3 \end{pmatrix}$.

The augmented matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 & -2 & 1 & 3 \\ 1 & -2 & -1 & 2 & 2 \\ 2 & 5 & 0 & -1 & -1 \\ 3 & 3 & -1 & -3 & 1 \end{pmatrix}$.

Add R_2 to R_4, and subtract R_1, to get

$$\mathbf{A} \sim \begin{pmatrix} 4 & 1 & -2 & 1 \\ 1 & -2 & -1 & 2 \\ 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{B} \sim \begin{pmatrix} 4 & 1 & -2 & 1 & 3 \\ 1 & -2 & -1 & 2 & 2 \\ 2 & 5 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\begin{vmatrix} 1 & -2 & 1 \\ -2 & -1 & 2 \\ 5 & 0 & -1 \end{vmatrix} = 1 - 16 + 5 \neq 0$, it follows that rank $\mathbf{A}$ = rank $\mathbf{B} = 3$, so the system is consistent. Since rank $\mathbf{A} = 3$, the space of solutions is of dimension 1.

Subtracting the second equation from the fourth, we get $2x + 5y - 5w = -1$. But $2x + 5y - w = -1$, so $w - 5w = 0 \Rightarrow w = 0$.

Now $y - 2z = 3 - 4x, -2y - z = 2 - x \Rightarrow -5z = 8 - 9x \Rightarrow z = \frac{9x-8}{5}$. Now $y = 3 - 4x + 2\frac{9x-8}{5} = \frac{-2x-1}{5}$. Thus the space of solutions is $(x, \frac{-2x-1}{5}, \frac{9x-8}{5}, 0)$. The system does not have a unique solution. ∎

Question 3(a) *Show that every square matrix satisfies its characteristic equation. Using this result or otherwiseshow that if*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then $\mathbf{A}^4 - 2\mathbf{A}^3 - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = \mathbf{A}$, *where* $\mathbf{I}$ *is the* 3×3 *identity matrix.*

Solution. The first part is the Cayley Hamilton theorem. See 1987 Question 5(a).

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A} - x\mathbf{I}| = 0$, thus

$$\begin{vmatrix} 1-x & 0 & 2 \\ 0 & -1-x & 1 \\ 0 & 1 & -x \end{vmatrix} = (1-x)(x^2+x-1) = -x^3+2x-1 = 0$$

By the Cayley Hamilton Theorem, $\mathbf{A}^3 - 2\mathbf{A} + \mathbf{I} = \mathbf{0}$.

Thus $\mathbf{A}^4 = 2\mathbf{A}^2 - \mathbf{A}$, and $2\mathbf{A}^3 = 4\mathbf{A} - 2\mathbf{I}$. Hence $\mathbf{A}^4 - 2\mathbf{A}^3 - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = 2\mathbf{A}^2 - \mathbf{A} - 4\mathbf{A} + 2\mathbf{I} - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = \mathbf{A}$ as required. ∎

Question 3(b) 1. *Show that a square matrix is singular if and only if at least one of its eigenvalues is 0.*

2. *The rank of an $n \times n$ matrix $\mathbf{A}$ remains unchanged if it is premultiplied or postmultiplied by a nonsingular matrix, and that* $\mathrm{rank}(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}) = \mathrm{rank}(\mathbf{A})$.

Solution.

1. The characteristic polynomial of $\mathbf{A}$ is $|\mathbf{A} - x\mathbf{I}|$. Putting $x = 0$, we see that the constant term in the characteristic polynomial is $|\mathbf{A}|$. Thus if $\mathbf{A}$ has 0 as an eigenvalue iff 0 is a root of the characteristic polynomial iff $|\mathbf{A}| = 0$.

2. Let $\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_m \end{pmatrix}$, where each $\mathbf{R}_i$ is $1 \times n$, i.e. $\mathbf{A}$ is $m \times n$. Now $\mathrm{rank}(\mathbf{A})$ is the dimension of the row space of $\mathbf{A}$, i.e. the space generated by $\mathbf{R}_1, \ldots, \mathbf{R}_m$. Let $\mathbf{P} = (p_{ij})$ be an $m \times m$ nonsingular matrix. Then $\mathbf{B} = \mathbf{P}\mathbf{A} = \begin{pmatrix} p_{11}\mathbf{R}_1 + p_{12}\mathbf{R}_2 + \ldots + p_{1m}\mathbf{R}_m \\ p_{21}\mathbf{R}_1 + p_{22}\mathbf{R}_2 + \ldots + p_{2m}\mathbf{R}_m \\ \vdots \\ p_{m1}\mathbf{R}_1 + p_{m2}\mathbf{R}_2 + \ldots + p_{mm}\mathbf{R}_m \end{pmatrix}$.

   Thus the rows of $\mathbf{P}\mathbf{A} \subset$ the row space of $\mathbf{A}$, being linear combinations of rows of $\mathbf{A}$. Writing $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}$, we get that the row space of $\mathbf{A} \subset$ the row space of $\mathbf{B}$, so $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{B})$.

   Let $\mathbf{Q}$ be non-singular $n \times n$, and $\mathbf{C} = \mathbf{A}\mathbf{Q}$. It can be proved as above that the column space of $\mathbf{A}$ = the column space of $\mathbf{C}$, thus $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{C})$.

   Now by using the above results, $\mathrm{rank}(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}) = \mathrm{rank}(\mathbf{X}\mathbf{A}) = \mathrm{rank}(\mathbf{A})$.

∎

Paper II

Question 4(a) *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of a vector space $\mathcal{V}$, then show that* $\dim(\mathcal{V}_1 + \mathcal{V}_2) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) - \dim(\mathcal{V}_1 \cap \mathcal{V}_2)$.

Solution. See 1998, question 1(b). ∎

Question 4(b) *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the same field F and $\dim \mathcal{V} = n$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$. Show that a map $f : \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \longrightarrow \mathcal{W}$, can be uniquely extended to a linear transformation $T : \mathcal{V} \longrightarrow \mathcal{W}$ whose restriction to the given basis is f i.e. $T(\mathbf{e}_i) = f(\mathbf{e}_i)$.*

Solution. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, define $T(\mathbf{v}) = \sum_{i=1}^n a_i f(\mathbf{e}_i)$. Clearly $T(\mathbf{e}_i) = f(\mathbf{e}_i)$. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, $\mathbf{w} = \sum_{i=1}^n b_i \mathbf{e}_i$, then

$$\begin{aligned} T(\alpha\mathbf{v} + \beta\mathbf{w}) &= T(\sum_{i=1}^n (\alpha a_i + \beta b_i)\mathbf{e}_i) \\ &= \sum_{i=1}^n (\alpha a_i + \beta b_i) f(\mathbf{e}_i) \\ &= \alpha \sum_{i=1}^n a_i f(\mathbf{e}_i) + \beta \sum_{i=1}^n b_i f(\mathbf{e}_i) \\ &= \alpha T(\mathbf{v}) + \beta T(\mathbf{w}) \end{aligned}$$

Thus T is a linear transformation.

If U is any other linear transformation satisfying $U(\mathbf{e}_i) = f(\mathbf{e}_i)$, then for any $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, by linearity, $T(\mathbf{v}) = T(\sum_{i=1}^n a_i \mathbf{e}_i) = \sum_{i=1}^n a_i T(\mathbf{e}_i) = \sum_{i=1}^n a_i f(\mathbf{e}_i) = \sum_{i=1}^n a_i U(\mathbf{e}_i) = U(\mathbf{v})$. Since this is true for every $\mathbf{v}$, we have $T = U$. ∎

Question 5(a)

1. *If A and B are two linear transformations and if A^{-1} and B^{-1} exist, show that $(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$.*

2. *Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.*

3. *Prove that the eigenvalues of a Hermitian matrix are real.*

Solution.

1. Clearly $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$, $(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}B = I$. Thus AB is invertible and its inverse is $B^{-1}A^{-1}$.

2. If $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ then $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. Thus $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial and therefore the same eigenvalues.

3. See 1993 question 2(c).

∎

Question 5(b) *Reduce $2x^2 + 4xy + 5y^2 + 4x + 13y - \frac{1}{4} = 0$ to canonical form.*

Solution.

$$\begin{aligned}
LHS &= 2(x+y+1)^2 - 2y^2 - 2 + 5y^2 + 9y - \frac{1}{4} \\
&= 2(x+y+1)^2 + 3(y^2+3y) - \frac{9}{4} \\
&= 2(x+y+1)^2 + 3(y+\frac{3}{2})^2 - \frac{27}{4} - \frac{9}{4} \\
&= 2X^2 + 3Y^2 - 9 \qquad\qquad \text{where } X = x+y+1, Y = y+\frac{3}{2}
\end{aligned}$$

$2X^2 + 3Y^2 - 9 = 0 \Rightarrow \frac{X^2}{9/2} + \frac{Y^2}{3} = 1$. Thus the given equation is an ellipse. ∎

Question 5(c) *Find the reciprocal of the matrix $\mathbf{T} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Then show that the transform of the matrix $\mathbf{A} = \frac{1}{2}\begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix}$ by $\mathbf{T}$ i.e. $\mathbf{TAT}^{-1}$ is a diagonal matrix. Determine the eigenvalues of the matrix $\mathbf{A}$.*

Solution. $|\mathbf{T}| = -1(-1) + 1(1) = 2$. So

$$\mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Here A_{ij} denotes the cofactor of a_{ij}. Now

$$\begin{aligned}
\mathbf{TAT}^{-1} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 4a & 0 & 0 \\ 0 & 4b & 0 \\ 0 & 0 & 4c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}
\end{aligned}$$

Thus $\mathbf{TAT}^{-1}$ is diagonal. Now the eigenvalues of $\mathbf{A}$ and $\mathbf{TAT}^{-1}$ are the same, so the eigenvalues of $\mathbf{A}$ are a, b, c. ∎

UPSC Civil Services Main 1987 - Mathematics Linear Algebra

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Question 1(a) *1. Find all the matrices which commute with the matrix* $\begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}$.

2. Prove that the product of two $n \times n$ symmetric matrices is a symmetric matrix if and only if the matrices commute.

Solution.

1.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \qquad 7a + 5b &= 7a - 3c & (i) \\ -3a - 2b &= 7b - 3d & (ii) \\ 7c + 5d &= 5a - 2c & (iii) \\ -3c - 2d &= 5b - 2d & (iv) \end{aligned}$$

(i) and (iv) $\Rightarrow 5b = -3c$. From (ii) we get $d = a + 3b$, and from (iii) we get the same thing: $5a - 9c = 5a + 15b = 5d$, or $d = a + 3b$. Thus the required matrices are $\begin{pmatrix} a & b \\ -\frac{5}{3}b & a + 3b \end{pmatrix}$, a, b arbitrary.

2. Given $\mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$. Suppose $\mathbf{AB} = \mathbf{BA}$, then $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' = \mathbf{BA} = \mathbf{AB} \Rightarrow \mathbf{AB}$ is symmetric. Let $\mathbf{AB}$ be symmetric. Then $\mathbf{AB} = (\mathbf{AB})' = \mathbf{B}'\mathbf{A}' = \mathbf{BA}$, so $\mathbf{A}$ and $\mathbf{B}$ commute. Thus $\mathbf{AB}$ is symmetric $\Leftrightarrow \mathbf{AB} = \mathbf{BA}$ when $\mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$.

∎

Question 1(b) *Show that the rank of the product of two square matrices* $\mathbf{A}, \mathbf{B}$ *each of order* n *satisfies the inequality*

$$r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$$

where $r_{\mathbf{C}}$ *stands for the rank of* $\mathbf{C}$*, a square matrix.*

Solution. There exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{PA} = \begin{pmatrix} \mathbf{G} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{G}$ is a $r_{\mathbf{A}} \times n$ matrix of rank $r_{\mathbf{A}}$. Now $\mathbf{PAB} = \begin{pmatrix} \mathbf{G} \\ \mathbf{0} \end{pmatrix} \mathbf{B}$ has at most $r_{\mathbf{A}}$ non-zero rows obtained on multiplying $r_{\mathbf{A}}$ non-zero rows of $\mathbf{G}$ with $\mathbf{B}$. Thus $r_{\mathbf{PAB}}$, which is the same as rank $r_{\mathbf{AB}}$ as $\mathbf{P}$ is non-singular, $\leq r_{\mathbf{A}}$.

Similarly there exists a non-singular matrix $\mathbf{Q}$ such that $\mathbf{BQ} = \begin{pmatrix} \mathbf{H} & \mathbf{0} \end{pmatrix}$, where $\mathbf{H}$ is a $n \times r_{\mathbf{B}}$ matrix of rank $r_{\mathbf{B}}$. Now $\mathbf{ABQ} = \mathbf{A} \begin{pmatrix} \mathbf{H} & \mathbf{0} \end{pmatrix}$ has at most $r_{\mathbf{B}}$ non-zero, columns, so $r_{\mathbf{ABQ}} \leq r_{\mathbf{B}}$. Now $r_{\mathbf{ABQ}} = r_{\mathbf{AB}}$ as $|\mathbf{Q}| \neq 0$, so $r_{\mathbf{AB}} \leq r_{\mathbf{B}}$, hence $r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$.

Let $S(\mathbf{A})$ denote the space generated by the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ where $\mathbf{r}_i$ is the ith row of $\mathbf{A}$, then $\dim(S(\mathbf{A})) = r_{\mathbf{A}}$, similarly $\dim(S(\mathbf{B})) = r_{\mathbf{B}}$. Let S denote the space generated by the rows of $\mathbf{A}$ and $\mathbf{B}$. Clearly $\dim(S) \leq \dim(S(\mathbf{A})) + \dim(S(\mathbf{B})) = r_{\mathbf{A}} + r_{\mathbf{B}}$. Clearly $S(\mathbf{A}+\mathbf{B}) \subseteq S$. Therefore $r_{\mathbf{A}+\mathbf{B}} \leq \dim(S) \leq r_{\mathbf{A}} + r_{\mathbf{B}}$.

Now there exist non-singular matrices $\mathbf{P}, \mathbf{Q}$ such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ or $\mathbf{A} = \mathbf{P}^{-1} \begin{pmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{Q}^{-1}$. Let $\mathbf{C} = \mathbf{P}^{-1} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r_{\mathbf{A}}} \end{pmatrix} \mathbf{Q}^{-1}$. Then $\mathbf{A}+\mathbf{C} = \mathbf{P}^{-1} \begin{pmatrix} \mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r_{\mathbf{A}}} \end{pmatrix} \mathbf{Q}^{-1} = \mathbf{P}^{-1}\mathbf{Q}^{-1}$, so $\mathbf{A}+\mathbf{C}$ is nonsingular.

Now rank $\mathbf{B} = \operatorname{rank}((\mathbf{A}+\mathbf{C})\mathbf{B}) \leq \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{CB})$. But $\operatorname{rank}(\mathbf{CB}) \leq \operatorname{rank}(\mathbf{C}) = n - r_{\mathbf{A}}$. Thus $r_{\mathbf{B}} \leq r_{\mathbf{AB}} + n - r_{\mathbf{A}} \Rightarrow r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}}$. Hence $r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$. ∎

Question 1(c) *If* $1 \leq a \leq 5$*, find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & a \\ 2 & 2a-2 & -a-2 & 3a-1 \\ 3 & a+2 & -3 & 2a+1 \end{pmatrix}$$

Solution. $|\mathbf{A}| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & a-1 \\ 2 & 2a-4 & -a-4 & 3a-3 \\ 3 & a-1 & -6 & 2a-2 \end{vmatrix}$ by carrying out the operations $\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1$. Thus $|\mathbf{A}| = (a-1) \begin{vmatrix} 2 & -3 & 1 \\ 2a-4 & -a-4 & 3 \\ a-1 & -6 & 2 \end{vmatrix} = (a-1) \begin{vmatrix} 0 & 0 & 1 \\ 2a-10 & -a+5 & 3 \\ a-5 & 0 & 2 \end{vmatrix} = (a-1)(a-5)^2$.

Thus $|\mathbf{A}| \neq 0$ when $a \neq 1, a \neq 5$. So for $1 < a < 5, \operatorname{rank} \mathbf{A} = 4$.
If $a = 5$,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 5 \\ 2 & 8 & -7 & 14 \\ 3 & 7 & -3 & 11 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 4 \\ 2 & 6 & -9 & 12 \\ 3 & 4 & -6 & 8 \end{pmatrix} \qquad (\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 4 \\ 0 & 6 & -9 & 12 \\ 0 & 4 & -6 & 8 \end{pmatrix} \qquad (\mathbf{R}_2 - \mathbf{R}_1, \mathbf{R}_3 - 2\mathbf{R}_1, \mathbf{R}_4 - 3\mathbf{R}_1) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\mathbf{R}_3 - 3\mathbf{R}_2, \mathbf{R}_4 - 2\mathbf{R}_2)
\end{aligned}$$

which has rank 2, as $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \neq 0$, showing that rank of $\mathbf{A}$ is 2 when $a = 5$.
If $a = 1$,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 2 & -2 & -5 & 0 \\ 3 & 0 & -6 & 0 \end{pmatrix} \qquad (\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1)
\end{aligned}$$

which has rank 3 since $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 2 & -2 & -5 \end{vmatrix} \neq 0$, showing that rank of $\mathbf{A}$ is 3 when $a = 1$. ∎

Question 2(a) *If the eigenvalues of a matrix* $\mathbf{A}$ *are* $\lambda_j, j = 1, 2, \ldots n$ *and if* $f(x)$ *is a polynomial in* x, *show that the eigenvalues of the polynomial* $f(\mathbf{A})$ *are* $f(\lambda_j), j = 1, 2, \ldots n$.

Solution. Let $\mathbf{x}_r$ be an eigenvector of λ_r. Then $\mathbf{A}^k\mathbf{x}_r = \mathbf{A}^{k-1}(\mathbf{A}\mathbf{x}_r) = \lambda_r\mathbf{A}^{k-1}\mathbf{x}_r = \ldots = \lambda_r^k\mathbf{x}_r$. Thus the eigenvalues of $\mathbf{A}^k$ are $\lambda_j^k, j = 1, 2, \ldots, n$.

Let $f(x) = a_0 + a_1x + \ldots + a_mx^m$. Then $(a_0\mathbf{I} + a_1\mathbf{A} + \ldots + a_m\mathbf{A}^m)\mathbf{x}_r = (a_0 + a_1\lambda_r + \ldots + a_m\lambda_r^m)\mathbf{x}_r = f(\lambda_r)\mathbf{x}_r$. Thus the eigenvalues of $f(\mathbf{A})$ are $f(\lambda_j), j = 1, 2, \ldots n$. ∎

Question 2(b) *If $\mathbf{A}$ is skew-symmetric, then show that $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$, where $\mathbf{I}$ is the corresponding identity matrix, is orthogonal.*

Hence construct an orthogonal matrix if $\mathbf{A} = \begin{pmatrix} 0 & \frac{a}{b} \\ -\frac{a}{b} & 0 \end{pmatrix}$.

Solution. For the orthogonality of $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$, see question 2(a) of 1999.

$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & -\frac{a}{b} \\ \frac{a}{b} & 1 \end{pmatrix}$, and $\mathbf{I} + \mathbf{A} = \begin{pmatrix} 1 & \frac{a}{b} \\ -\frac{a}{b} & 1 \end{pmatrix} \Rightarrow (\mathbf{I} + \mathbf{A})^{-1} = \frac{b}{a^2+b^2}\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$.

Thus $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} b & -a \\ a & b \end{pmatrix}\begin{pmatrix} b & -a \\ a & b \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2 - a^2 & -2ab \\ 2ab & b^2 - a^2 \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{pmatrix}$, which is the required orthogonal matrix. ∎

Question 2(c)

1. *If $\mathbf{A}$ and $\mathbf{B}$ are arbitrary square matrices of which $\mathbf{A}$ is non-singular, show that $\mathbf{AB}$ and $\mathbf{BA}$ have the same characteristic polynomial.*

2. *Show that a real matrix $\mathbf{A}$ is orthogonal if and only if $|\mathbf{Ax}| = |\mathbf{x}|$ for all $\mathbf{x}$.*

Solution.

1. $\mathbf{BA} = \mathbf{A^{-1}ABA}$. Thus the characteristic polynomial of $\mathbf{BA}$ is $|\mathbf{x}\mathbf{I} - \mathbf{BA}| = |\mathbf{x}\mathbf{A^{-1}A} - \mathbf{A^{-1}ABA}| = |\mathbf{A}^{-1}||\mathbf{x}\mathbf{I} - \mathbf{AB}||\mathbf{A}| = |\mathbf{x}\mathbf{I} - \mathbf{AB}|$ which is the characteristic polynomial of $\mathbf{AB}$.

2. If $\mathbf{A}$ is orthogonal, i.e. $\mathbf{A}'\mathbf{A} = \mathbf{I}$, then $|\mathbf{Ax}| = \sqrt{\mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}} = \sqrt{\mathbf{x}'\mathbf{x}} = |\mathbf{x}|$.

   Conversely $|\mathbf{Ax}| = |\mathbf{x}| \Rightarrow \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{x} \Rightarrow \mathbf{x}'(\mathbf{A}'\mathbf{A} - \mathbf{I})\mathbf{x} = 0$ for all $\mathbf{x}$. Thus $\mathbf{A}'\mathbf{A} - \mathbf{I} = \mathbf{0}$, so $\mathbf{A}$ is orthogonal.

Note that is $\mathbf{A} = (a_{ij})$ is symmetric, and $\sum_{i,j=1}^n a_{ij}x_ix_j = 0$ for all $\mathbf{x}$, then choose $\mathbf{x} = \mathbf{e_i}$ to get $\mathbf{e_i}'\mathbf{A}\mathbf{e_i} = a_{ii} = 0$, and choose $\mathbf{x} = \mathbf{e_i}+\mathbf{e_j}$ to get $0 = \mathbf{x}'\mathbf{A}\mathbf{x} = a_{ii}+2a_{ij}+a_{jj} = 2a_{ij} \Rightarrow a_{ij} = 0$. (Here $\mathbf{e_i}$ is the i-th unit vector.) Thus $\mathbf{A} = \mathbf{0}$. ∎

Question 3(a) *Show that a necessary and sufficient condition for a system of linear equations to be consistent is that the rank of the coefficient matrix is equal to the rank of the augmented matrix. Hence show that the system*

$$\begin{aligned} x + 2y + 5z + 9 &= 0 \\ x - y + 3z - 2 &= 0 \\ 3x - 6y - z - 25 &= 0 \end{aligned}$$

is consistent and has a unique solution. Determine this solution.

Solution. Let the system be $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is $m \times n$, $\mathbf{x}$ is $n \times 1$ and $\mathbf{b}$ is $m \times 1$. Let $\operatorname{rank} \mathbf{A} = r$. $\mathbf{A} = [\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}]$ where each $\mathbf{c_j}$ is an $m \times 1$ column. We can assume without loss of generality that $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ are linearly independent, $r = \operatorname{rank} \mathbf{A}$. The system is now

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \ldots + x_n\mathbf{c_n} = \mathbf{b}$$

, where $\mathbf{x}' = (x_1, \ldots, x_n)$. Suppose $\operatorname{rank}([\mathbf{A}\ \mathbf{b}]) = r$. This means that out of $n+1$ columns, exactly r are independent. But by assumption, $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ are linearly independent, therefore these vectors form a basis for the column space of $[\mathbf{A}\ \mathbf{b}]$. Consequently there exist $\alpha_1, \ldots, \alpha_r$ such that $\alpha_1\mathbf{c_1} + \alpha_2\mathbf{c_2} + \ldots + \alpha_r\mathbf{c_r} = \mathbf{b}$. This gives us the required solution $\{\alpha_1, \ldots, \alpha_r, 0, \ldots, 0\}$ to the linear system.

Conversely, let the system be consistent. Let $\mathbf{A} = [\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}]$ as before, with $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ linearly independent, $r = \operatorname{rank} \mathbf{A}$. Since the column space of $\mathbf{A}$, i.e. the space generated by $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ has dimension r, each c_j for $r+1 \leq j \leq n$ is linearly dependent on $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$. Since there exist $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1\mathbf{c_1} + \alpha_2\mathbf{c_2} + \ldots + \alpha_n\mathbf{c_n} = \mathbf{b}$, $\mathbf{b}$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$. But each c_j for $r+1 \leq j \leq n$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$, therefore $\mathbf{b}$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$. Thus the space generated by $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$ also has dimension r, so $\operatorname{rank}([\mathbf{A}\ \mathbf{b}]) = r = \operatorname{rank} \mathbf{A}$.

The coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 3 \\ 3 & -6 & -1 \end{pmatrix}$. $|\mathbf{A}| = 24 \neq 0$, so $\operatorname{rank} \mathbf{A} = 3$. The augmented matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 & 5 & -9 \\ 1 & -1 & 3 & 2 \\ 3 & -6 & -1 & 25 \end{pmatrix}$ has rank ≤ 3, but since $\begin{vmatrix} 1 & 2 & 5 \\ 1 & -1 & 3 \\ 3 & -6 & -1 \end{vmatrix} \neq 0$, it has rank 3. Thus the given system is consistent.

Subtracting the second equation from the first we get $3y + 2z + 11 = 0$. Subtracting 3 times the second equation from the third, we get $3y + 10z + 19 = 0$. Clearly $z = -1, y = -3 \Rightarrow x = 2$. Thus $(2, -3, -1)$ is the unique solution. In fact the only solution of the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} -9 \\ 2 \\ 25 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

∎

Question 3(b) *In an n-dimensional vector space the system of vectors $\mathbf{x}_j, j = 1, \ldots, r$ are linearly independent and can be expressed linearly in terms of the vectors $\mathbf{y}_k, k = 1, \ldots, s$. Show that $r \leq s$.*

Find a maximal linearly independent subsystem of the linear forms

$$\begin{aligned} f_1 &= x + 2y + z + 3t \\ f_2 &= 4x - y - 5z - 6t \\ f_3 &= x - 3y - 4z - 7t \\ f_4 &= 2x + y - z \end{aligned}$$

Solution. Let $\mathcal{W}$ be the subspace spanned by $\mathbf{y}_k, k = 1, \ldots, s$. Then $\dim \mathcal{W} \leq s$. Since $\mathbf{x}_j \in \mathcal{W}, j = 1, \ldots, r$ because $\mathbf{x}_j$ is a linear combination of $\mathbf{y}_k, k = 1, \ldots, s$, and $\mathbf{x}_j, j = 1, \ldots, r$ are linearly independent, $\dim \mathcal{W} \geq r \Rightarrow r \leq s$.

Clearly f_1 and f_4 are linearly independent. f_2 is linearly expressible in terms of f_1 and f_4 because $f_2 = af_1 + bf_4 \Rightarrow a + 2b = 4, 2a + b = -1, a - b = 5, 3a = -6 \Rightarrow a = -2, b = 3$ satisfy all four, hence $f_2 = -2f_1 + 3f_4$. Similarly $f_3 = -\frac{7}{3}f_1 + \frac{5}{3}f_4$. Thus $\{f_1, f_4\}$ is a maximally independent subsystem. ∎

Paper II

Question 4(a) *Let $T : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation. If $\mathcal{V}$ is finite dimensional, show that*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$$

Solution. See question 1(a) of 1992. ∎

Question 4(b) *Prove that two finite dimensional vector spaces $\mathcal{V}, \mathcal{W}$ over the same field $\mathcal{F}$ are isomorphic if they are of the same dimension n.*

Solution. Let $\dim \mathcal{V} = \dim \mathcal{W} = n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $\mathcal{V}$, and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be a basis of $\mathcal{W}$. Define $T : \mathcal{V} \longrightarrow \mathcal{W}$ by $T(\mathbf{v}_i) = \mathbf{w}_i$ and if $\mathbf{v} \in \mathcal{V}, \mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i, a_i \in \mathbb{R}$ then $T(\mathbf{v}) = \sum_{i=1}^n a_i T(\mathbf{v}_i)$. Then

1. T is a linear transformation. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i, \mathbf{u} = \sum_{i=1}^n b_i \mathbf{v}_i$ then

$$\begin{aligned} T(\alpha\mathbf{v} + \beta\mathbf{u}) &= T(\sum_{i=1}^n (\alpha a_i + \beta b_i) T(\mathbf{v}_i) \\ &= \sum_{i=1}^n (\alpha a_i + \beta b_i) T(\mathbf{v}_i) \\ &= \alpha \sum_{i=1}^n a_i T(\mathbf{v}_i) + \beta \sum_{i=1}^n b_i T(\mathbf{v}_i) \\ &= \alpha T(\mathbf{v}) + \beta T(\mathbf{u}) \end{aligned}$$

2. T is 1-1. Let $T(\mathbf{v}) = \mathbf{0}$, where $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$. Then $\mathbf{0} = T(\mathbf{v}) = \sum_{i=1}^n a_i T(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{w}_i \Rightarrow a_i = 0, i = 1, \ldots, n$, because $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent. Thus $T(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$.

3. T is onto. If $\mathbf{w} \in \mathcal{W}$ and $\mathbf{w} = \sum_{i=0}^n b_i \mathbf{w}_i$, then $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \sum_{i=1}^n b_i \mathbf{v}_i$.

∎

Note: The converse of 4(b) is also true i.e. if $T : \mathcal{V} \longrightarrow \mathcal{W}$ is an isomorphism i.e. $\mathcal{V}, \mathcal{W}$ are isomorphic, then $\dim \mathcal{V} = \dim \mathcal{W}$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $\mathcal{V}$. Then $\{\mathbf{w}_1 = T(\mathbf{v}_1), \ldots, \mathbf{w}_n = T(\mathbf{v}_n)\}$ is a basis of $\mathcal{W}$.

$\mathbf{w}_1, \ldots, \mathbf{w}_n$ **are linearly independent**. If $\sum_{i=0}^n b_i \mathbf{w}_i = \mathbf{0}$, then $\sum_{i=0}^n b_i T\mathbf{v}_i = \mathbf{0} \Rightarrow T(\sum_{i=0}^n b_i \mathbf{v}_i) = \mathbf{0} \Rightarrow \sum_{i=0}^n b_i \mathbf{v}_i = \mathbf{0} \Rightarrow b_i = 0$ for $1 \leq i \leq n$, because $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

$\mathbf{w}_1, \ldots, \mathbf{w}_n$ **generate W**. If $\mathbf{w} \in \mathcal{W}$, then there exists a $\mathbf{v} \in \mathcal{V}$ such that $T(\mathbf{v}) = \mathbf{w}$, because T is onto. Let $\mathbf{v} = \sum_{i=0}^n b_i \mathbf{v}_i$, then $\mathbf{w} = T(\mathbf{v}) = T(\sum_{i=0}^n b_i \mathbf{v}_i) = \sum_{i=0}^n b_i T(\mathbf{v}_i) = \sum_{i=0}^n b_i \mathbf{w}_i$.

Question 5(a) *Prove that every square matrix is the root of its characteristic polynomial.*

Solution. This is the Cayley Hamilton Theorem. Let $\mathbf{A}$ be a matrix of order n. Let

$$|\mathbf{A} - x\mathbf{I}| = a_o + a_1 x + \ldots + a_n x^n$$

Then we wish to show that

$$a_o \mathbf{I} + a_1 \mathbf{A} + \ldots + a_n \mathbf{A}^n = \mathbf{0}$$

Suppose the adjoint of $\mathbf{A} - x\mathbf{I}$ is $\mathbf{B}_0 + \mathbf{B}_1 x + \ldots + \mathbf{B}_{n-1} x^{n-1}$, where $\mathbf{B}_i$ are matrices of order n. Then by definition of the adjoint,

$$(\mathbf{A} - x\mathbf{I})(\mathbf{B}_0 + \mathbf{B}_1 x + \ldots + \mathbf{B}_{n-1} x^{n-1}) = |\mathbf{A} - x\mathbf{I}|\mathbf{I}$$

Substituting for $|\mathbf{A} - x\mathbf{I}|$ the expression $a_o + a_1 x + \ldots + a_n x^n$ and equating coefficients of like powers, we get

$$\begin{aligned}
\mathbf{A}\mathbf{B}_0 &= a_o \mathbf{I} \\
\mathbf{A}\mathbf{B}_1 - \mathbf{B}_0 &= a_1 \mathbf{I} \\
\mathbf{A}\mathbf{B}_2 - \mathbf{B}_1 &= a_2 \mathbf{I} \\
&\ldots \\
\mathbf{A}\mathbf{B}_{n-1} - \mathbf{B}_{n-2} &= a_{n-1} \mathbf{I} \\
-\mathbf{B}_{n-1} &= a_n \mathbf{I}
\end{aligned}$$

Multiplying these equations successively by $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^n$ on the left and adding, we get $\mathbf{0} = a_o \mathbf{I} + a_1 \mathbf{A} + \ldots + a_n \mathbf{A}^n$, which was to be proved. ∎

Question 5(b) *Determine the eigenvalues and the corresponding eigenvectors of*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Solution.

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & -2 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -2 & \lambda - 2 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \quad (\lambda-2)^2(\lambda-3) - 2(\lambda-2) + 2(-\lambda+2) - 2 - 2 - (\lambda-3) &= 0 \\ \Rightarrow \quad (\lambda^2 - 4\lambda + 4)(\lambda - 3) - 5\lambda + 7 &= 0 \\ \Rightarrow \quad \lambda^3 - 7\lambda^2 + 11\lambda - 5 = (\lambda - 1)(\lambda^2 - 6\lambda + 5) &= 0 \\ \Rightarrow \quad \lambda = 1, 5, 1 & \end{aligned}$$

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 5$. Then

$$\begin{pmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $3x_1 - 2x_2 - x_3 = 0, -x_1 + 2x_2 - x_3 = 0, -x_1 - 2x_2 + 3x_3 = 0 \Rightarrow x_1 = x_2 = x_3$. Thus $(1, 1, 1)$ is an eigenvector for $\lambda = 5$. In fact (x, x, x) with $x \neq 0$ are eigenvectors for $\lambda = 5$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$. Then

$$\begin{pmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + 2x_2 + x_3 = 0$. We can take $\mathbf{x_1} = (1, 0, -1)$ and $\mathbf{x_2} = (0, 1, -2)$ as eigenvectors for $\lambda = 1$. These are linearly independent, and all eigenvectors for $\lambda = 1$ are linear combinations of $\mathbf{x_1}, \mathbf{x_2}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

∎

Question 5(c) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* n *square matrices over* F. *Show that* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same eigenvalues.*

Solution. If $\mathbf{A}$ is non-singular, then

$$\mathbf{BA} = \mathbf{A}^{-1}\mathbf{ABA} \Rightarrow |x\mathbf{I} - \mathbf{BA}| = |x\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{ABA}| = |\mathbf{A}^{-1}||x\mathbf{I} - \mathbf{AB}||\mathbf{A}| = |x\mathbf{I} - \mathbf{AB}|$$

Thus the characteristic polynomials of $\mathbf{AB}$ and $\mathbf{BA}$ are the same, so they have the same eigenvalues.

If $\mathbf{A}$ is singular, then let $\text{rank}(\mathbf{A}) = \mathbf{r}$. Then there exist $\mathbf{P}, \mathbf{Q}$ non-singular such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Now $\mathbf{PABP}^{-1} = \mathbf{PAQQ}^{-1}\mathbf{BP}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{Q}^{-1}\mathbf{BP}^{-1}$. Let $\mathbf{Q}^{-1}\mathbf{BP}^{-1} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix}$, where $\mathbf{B}_1$ is $r \times r$, $\mathbf{B}_2$ is $r \times n-r$, $\mathbf{B}_3$ is $n-r \times r$, $\mathbf{B}_4$ is $n-r \times n-r$. Then $\mathbf{PABP}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, so the characteristic roots of $\mathbf{AB}$ are the same as those of $\mathbf{B}_1$, along with 0 repeated $n-r$ times.

Now $\mathbf{Q}^{-1}\mathbf{BAQ} = \mathbf{Q}^{-1}\mathbf{BP}^{-1}\mathbf{PAQ} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_3 & \mathbf{0} \end{pmatrix}$ so the characteristic roots of $\mathbf{BA}$ are the same as those of $\mathbf{B}_1$, along with 0 repeated $n-r$ times. Thus $\mathbf{BA}$ and $\mathbf{AB}$ have the same characteristic roots. ∎

UPSC Civil Services Main 1988 - Mathematics Linear Algebra

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Question 1(a) *Show that a linear transformation of a vector space $\mathcal{V}_m$ of dimension m into a vector space $\mathcal{V}_n$ of dimension n over the same field can be represented as a matrix. If $\mathbf{T}$ is a linear transformation of $\mathcal{V}_2$ into $\mathcal{V}_4$ such that $\mathbf{T}(3,1) = (4,1,2,1)$ and $\mathbf{T}(-1,2) = (3,0,-2,1)$, then find the matrix of $\mathbf{T}$.*

Solution. Let $\mathbf{v_i}, i = 1, \ldots, m$ be a basis of $\mathcal{V}_m$ and $\mathbf{w_j}, j = 1, \ldots, n$ be a basis of $\mathcal{V}_n$. If

$$\mathbf{T}(\mathbf{v_i}) = \sum_{j=1}^{n} a_{ji}\mathbf{w_j}, \quad i = 1, \ldots, m$$

then $\mathbf{T}$ corresponds to the $n \times m$ matrix $\mathbf{A}$ whose (i,j)'th entry is a_{ij}. In fact $(\mathbf{v_1}, \ldots, \mathbf{v_m}) = (\mathbf{w_1}, \ldots, \mathbf{w_n})\mathbf{A}$.

It can be easily seen that

$$\begin{aligned}
\mathbf{e_1} &= (1,0) = \frac{2}{7}(3,1) - \frac{1}{7}(-1,2) \\
\mathbf{e_2} &= (0,1) = \frac{1}{7}(3,1) + \frac{3}{7}(-1,2)
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbf{T}(\mathbf{e_1}) &= \frac{2}{7}(4,1,2,1) - \frac{1}{7}(3,0,-2,1) \\
&= \frac{1}{7}(5,2,6,1) \\
&= \frac{1}{7}(5\mathbf{e_1^*} + 2\mathbf{e_2^*} + 6\mathbf{e_3^*} + \mathbf{e_4^*}) \\
\mathbf{T}(\mathbf{e_2}) &= \frac{1}{7}(4,1,2,1) + \frac{3}{7}(3,0,-2,1) \\
&= \frac{1}{7}(13,1,-4,4) \\
&= \frac{1}{7}(13\mathbf{e_1^*} + \mathbf{e_2^*} - 4\mathbf{e_3^*} + 4\mathbf{e_4^*})
\end{aligned}$$

Thus $\mathbf{T}$ corresponds to the matrix $\frac{1}{7}\begin{pmatrix} 5 & 13 \\ 2 & 1 \\ 6 & -4 \\ 7 & 4 \end{pmatrix}$ w.r.t. the standard basis. ∎

Question 1(b) *If $\mathcal{M}, \mathcal{N}$ are finite dimensional subspaces of $\mathcal{V}$, then show that* $\dim(\mathcal{M} + \mathcal{N}) = \dim \mathcal{M} + \dim \mathcal{N} - \dim(\mathcal{M} \cap \mathcal{N})$.

Solution. Let $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ be a basis of $\mathcal{M} \cap \mathcal{N}$ where $\dim(\mathcal{M} \cap \mathcal{N}) = r$. Complete $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to a basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}\}$ of $\mathcal{M}$, where $\dim \mathcal{M} = m + r$. Complete $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ to a basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{w_1}, \ldots, \mathbf{w_n}\}$ of $\mathcal{N}$, where $\dim \mathcal{N} = n + r$. We shall show that $\mathscr{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}, \mathbf{w_1}, \ldots, \mathbf{w_n}\}$ is a basis of $\mathcal{M} + \mathcal{N}$, proving the result.

If $\mathbf{u} \in \mathcal{M} + \mathcal{N}$, then $\mathbf{u} = \mathbf{v} + \mathbf{w}$ for some $\mathbf{v} \in \mathcal{M}, \mathbf{w} \in \mathcal{N}$. Since $\mathscr{B}$ is a superset of the bases of $\mathcal{M}, \mathcal{N}$, $\mathbf{v}, \mathbf{w}$ can be written as linear combination of elements of $\mathscr{B} \Rightarrow \mathbf{u}$ can be written as a linear combination of elements of $\mathscr{B}$. Thus $\mathscr{B}$ generates $\mathcal{M} + \mathcal{N}$.

We now show that the set $\mathscr{B}$ is linearly independent. If possible let

$$\sum_{i=1}^{n} \alpha_i \mathbf{v_i} + \sum_{i=1}^{m} \beta_i \mathbf{w_i} + \sum_{i=1}^{r} \gamma_i \mathbf{u_i} = \mathbf{0}$$

Since $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} = -\sum_{i=1}^{m} \beta_i \mathbf{w_i} - \sum_{i=1}^{r} \gamma_i \mathbf{u_i}$ it follows that $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{N}$. Therefore $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{M} \cap \mathcal{N} \Rightarrow \sum_{i=1}^{n} \alpha_i \mathbf{v_i} = \sum_{i=1}^{r} \eta_i \mathbf{u_i}$ for $\eta_i \in \mathbb{R}$. This means that $\sum_{i=1}^{n} \alpha_i \mathbf{v_i} - \sum_{i=1}^{r} \eta_i \mathbf{u_i} = 0$. But $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}, \mathbf{v_1}, \ldots, \mathbf{v_m}\}$ are linearly independent, so $\alpha_i = 0, 1 \leq i \leq n$. Similarly we can show that $\beta_i = 0, 1 \leq i \leq m$. Then the linear independence of $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ shows that $\gamma_i = 0, 1 \leq i \leq r$. Thus the vectors in $\mathscr{B}$ are linearly independent and form a basis of $\mathcal{M} + \mathcal{N}$, showing that the dimension of $\mathcal{M} + \mathcal{N}$ is $m + n + r = (m + r) + (n + r) - r$, which completes the proof. ∎

Question 1(c) *Determine a basis of the subspace spanned by the vectors* $\mathbf{v_1} = (1, 2, 3), \mathbf{v_2} = (2, 1, -1), \mathbf{v_3} = (1, -1, -4), \mathbf{v_4} = (4, 2, -2)$.

Solution. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent because if $\alpha\mathbf{v_1} + \beta\mathbf{v_2} = \mathbf{0}$ then $\alpha + 2\beta = 0, 2\alpha + \beta = 0, 3\alpha - \beta = 0 \Rightarrow \alpha = \beta = 0$. If $\mathbf{v_3} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then the three linear equations $\alpha + 2\beta = 1, 2\alpha + \beta = -1, 3\alpha - \beta = -4$ should be consistent — clearly $\alpha = -1, \beta = 1$ satisfy all three, showing $\mathbf{v_3} = \mathbf{v_2} - \mathbf{v_1}$. Again suppose $\mathbf{v_4} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then the three linear equations $\alpha + 2\beta = 4, 2\alpha + \beta = 2, 3\alpha - \beta = -2$ should be consistent — clearly $\alpha = 0, \beta = 2$ satisfy all three, showing $\mathbf{v_4} = 2\mathbf{v_2}$.

Hence $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis for the vector space generated by $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$. ∎

Question 2(a) *Show that it is impossible for* $\mathbf{S} = \begin{pmatrix} a_1 & b \\ b & a_2 \end{pmatrix}, b \neq 0$ *to have identical eigenvalues.*

Solution. We know given $\mathbf{S}$ symmetric $\exists\mathbf{O}$ orthogonal so that $\mathbf{O'SO} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where λ_1, λ_2 are eigenvalues of $\mathbf{S}$. If $\lambda_1 = \lambda_2$, then we have $\mathbf{S} = \mathbf{O'}^{-1}(\lambda\mathbf{I})\mathbf{O}^{-1} = \lambda(\mathbf{OO'})^{-1} = \lambda\mathbf{I} \Rightarrow \mathbf{S} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Thus if $b \neq 0$, $\mathbf{S}$ cannot have identical eigenvalues. ∎

Question 2(b) *Prove that the eigenvalues of a Hermitian matrix are all real and the eigenvalues of a skew-Hermitian matrix are either zero or pure imaginary.*

Solution. See question 2(a), year 1998. ∎

Question 2(c) *If* $\mathbf{x'Ax} > 0$ *for all* $\mathbf{x} \neq \mathbf{0}$, $\mathbf{A}$ *symmetric, then for all* $\mathbf{y} \neq \mathbf{0}$ $\mathbf{y'A}^{-1}\mathbf{y} > 0$. *If* λ *is the largest eigenvalue of* $\mathbf{A}$, *then*

$$\lambda = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x'Ax}}{\mathbf{x'x}}$$

.

Solution. Clearly $\mathbf{A} = \mathbf{A'A}^{-1}\mathbf{A} \therefore \mathbf{x'Ax} = \mathbf{x'A'A}^{-1}\mathbf{Ax} = \mathbf{y'A}^{-1}\mathbf{y}$ where $\mathbf{y} = \mathbf{Ax}$ for any $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$. Since $|\mathbf{A}| \neq 0$, any vector $\mathbf{y}$ can be written as $\mathbf{Ax}$, by taking $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. Thus $\mathbf{x'Ax} > 0 \Rightarrow \mathbf{y'A}^{-1}\mathbf{y} > 0$ for all $\mathbf{y} \neq \mathbf{0}$.

Let $M = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} \neq 0}} \frac{\mathbf{x'Ax}}{\mathbf{x'x}}$. Let $\mathbf{O}$ be an orthogonal matrix such that $\mathbf{O'AO} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$.

Let $\mathbf{0} \neq \mathbf{x} = \mathbf{Oy}$, then $\mathbf{x'x} = \mathbf{y'O'Oy} = \mathbf{y'y}$. Now $\mathbf{x'Ax} = \mathbf{y'O'AOy} = \sum_i \lambda_i y_i^2 \leq \lambda\mathbf{y'y}$ where λ is the largest eigenvalue of $\mathbf{A}$. Thus $\lambda \geq \frac{\mathbf{x'Ax}}{\mathbf{y'y}} = \frac{\mathbf{x'Ax}}{\mathbf{x'x}}$, so $\lambda \geq M$. On the other hand, if $\mathbf{x} \neq \mathbf{0}$ is an eigenvector corresponding to λ, then $\mathbf{x'Ax} = \lambda\mathbf{x'x} \Rightarrow \lambda = \frac{\mathbf{x'Ax}}{\mathbf{x'x}} \leq M$. Thus $\lambda = M$ as required. ∎

Question 3(a) *By converting* $\mathbf{A}$ *to an echelon matrix, determine its rank, where*

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 2 & 8 & 9 \\ 0 & 0 & 4 & 6 & 5 & 3 \\ 0 & 2 & 3 & 1 & 4 & 7 \\ 0 & 3 & 0 & 9 & 3 & 7 \\ 0 & 0 & 5 & 7 & 3 & 1 \end{pmatrix}$$

Solution. Consider

$$\mathbf{A}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 1 & 4 & 3 & 0 & 5 \\ 2 & 6 & 1 & 9 & 7 \\ 8 & 5 & 4 & 3 & 3 \\ 9 & 3 & 7 & 7 & 1 \end{pmatrix}$$

Interchange the first row with the third, then third with fourth, fourth with fifth and fifth with sixth to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 2 & 6 & 1 & 9 & 7 \\ 8 & 5 & 4 & 3 & 3 \\ 9 & 3 & 7 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now perform $\mathbf{R_3} - 2\mathbf{R_1}, \mathbf{R_4} - 8\mathbf{R_1}, \mathbf{R_5} - 9\mathbf{R_1}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & -2 & -5 & 9 & -3 \\ 0 & -27 & -20 & 3 & -37 \\ 0 & -33 & -20 & 7 & -44 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Interchange the second and the third row, and perform $-\frac{1}{2}\mathbf{R_2}, \frac{1}{2}\mathbf{R_3}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & -27 & -20 & 3 & -37 \\ 0 & -33 & -20 & 7 & -44 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Perform $\mathbf{R_4} + 27\mathbf{R_2}, \mathbf{R_5} + 33\mathbf{R_2}$ to get

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{95}{2} & -\frac{237}{2} & \frac{7}{2} \\ 0 & 0 & \frac{125}{2} & -\frac{283}{2} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Operation $\mathbf{R_4} - \frac{95}{2}\mathbf{R_3}, \mathbf{R_5} - \frac{125}{2}\mathbf{R_3}$ yields

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & -\frac{759}{4} & \frac{7}{2} \\ 0 & 0 & 0 & -\frac{941}{4} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now multiply $\mathbf{R_4}$ with $-\frac{4}{759}$

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & -\frac{941}{4} & \frac{11}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Performing $\mathbf{R_5} + \frac{941}{4}\mathbf{R_4}$ results in

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & 0 & \frac{11}{2} - \frac{941\times 7}{1882} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which can be converted to

$$\mathbf{A}' \sim \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \\ 0 & 1 & \frac{5}{2} & -\frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{14}{759} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is an echelon matrix. Its rank is clearly 5, so the rank of $\mathbf{A} = 5$. ∎

Question 3(b) *Given* $\mathbf{AB} = \mathbf{AC}$ *does it follow that* $\mathbf{B} = \mathbf{C}$*? Can you provide a counterexample?*

Solution. It does not follow that $\mathbf{B} = \mathbf{C}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \mathbf{AB} = \mathbf{0}$$

$\mathbf{C} = \mathbf{0} \Rightarrow \mathbf{AC} = \mathbf{0}$, but $\mathbf{B} \neq \mathbf{C}$. ∎

Question 3(c) *Find a nonsingular matrix which diagonalizes* $\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}$ *simultaneusly. Find the diagonal form of* $\mathbf{A}$.

Solution.

$$|\mathbf{A} - \lambda\mathbf{B}| = 0 \Rightarrow \begin{vmatrix} -2\lambda & -1-\lambda & 2\lambda \\ -1-\lambda & -1-2\lambda & 1+2\lambda \\ 2\lambda & 1+2\lambda & -3\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -2\lambda & -1-\lambda & 0 \\ -1+\lambda & 0 & 0 \\ 2\lambda & 1+2\lambda & -\lambda \end{vmatrix} = 0$$

Thus $\lambda = 0, 1, -1$. This shows that the matrices are diagonalizable simultaneously.

We now determine $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_i} = \mathbf{0}, i = 1, 2, 3$. For $\lambda = 0$, let $\mathbf{x_1}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_1} = \mathbf{0}$. Thus

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-x_2 = 0, -x_1 - x_2 + x_3 = 0, x_2 = 0$. Thus $\mathbf{x_1}' = (1, 0, 1)$.

For $\lambda = 1$, let $\mathbf{x_2}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_2} = \mathbf{0}$. Thus

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -3 & 3 \\ 2 & 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 - 2x_2 + 2x_3 = 0, -2x_1 - 3x_2 + 3x_3 = 0, 2x_1 + 3x_2 - 3x_3 = 0 \Rightarrow x_2 - x_3 = 0 \Rightarrow x_1 = 0$. Thus we may take $\mathbf{x_2}' = (0, 1, 1)$.

For $\lambda = -1$, let $\mathbf{x_3}' = (x_1, x_2, x_3)$ be such that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_3} = \mathbf{0}$. Thus

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $2x_1 - 2x_3 = 0, x_2 - x_3 = 0, -2x_1 - x_2 + 3x_3 = 0 \Rightarrow x_1 = x_2 = x_3$. Thus we may take $\mathbf{x_3}' = (1, 1, 1)$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ so that

$$\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

∎

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Question 1(a) *Find a basis for the null space of the matrix* $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

Solution. $\mathbf{A}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $\mathbf{A}(\mathbf{e_1}) = 3\mathbf{e_1^*},\ \mathbf{A}(\mathbf{e_2}) = \mathbf{e_1^*} + \mathbf{e_2^*},\ \mathbf{A}(\mathbf{e_3}) = -\mathbf{e_1^*} + 2\mathbf{e_2^*}$, where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is the standard basis of $\mathbb{R}^3$ and $\mathbf{e_1^*}, \mathbf{e_2^*}$ is the standard basis of $\mathbb{R}^2$. Thus $\mathbf{A}(a, b, c) = \mathbf{e_1^*}(3a + b - c) + \mathbf{e_2^*}(b + 2c)$. Consequently, $(a, b, c) \in$ null space of $\mathbf{A}$ if and only if $3a + b - c = 0, b + 2c = 0 \Rightarrow b = -2c, a = c$. Thus null space of $\mathbf{A}$ is $\{(c, -2c, c) \mid c \in \mathbb{R}\}$. Note that $\operatorname{rank} \mathbf{A} = 2$, so the null space has dimension 1. A basis for the null space is $(1, -2, 1)$, any other multiple of this can also be regarded as a basis. ∎

Question 1(b) *If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$ then prove that* $\dim \mathcal{V}/\mathcal{W} = \dim \mathcal{V} - \dim \mathcal{W}$.

Solution. Let $\mathbf{v_1}, \ldots, \mathbf{v_r}$ be a basis of $\mathcal{W}$, $\dim \mathcal{W} = r$. Let $\mathbf{v_{r+1}}, \ldots, \mathbf{v_n}$ be $n-r$ vectors in $\mathcal{V}$ so chosen that $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is a basis of $\mathcal{V}$, $\dim \mathcal{V} = n$. We will show that $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ is a basis of $\mathcal{V}/\mathcal{W} \Rightarrow \dim \mathcal{V}/\mathcal{W} = n - r$.

First we show linear independence:

$$
\begin{aligned}
& \sum_{i=r+1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W}) = 0 \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} + \mathcal{W} = \mathbf{0} + \mathcal{W} \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{W} \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} = \sum_{i=1}^{r} -\alpha_i \mathbf{v_i} \text{ (say)} \\
\Rightarrow\ & \sum_{i=1}^{n} \alpha_i \mathbf{v_i} = \mathbf{0} \\
\Rightarrow\ & \alpha_i = 0, 1 \leq i \leq n \text{ } (\mathbf{v_i} \text{ are linearly independent.})
\end{aligned}
$$

Thus $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ are linearly independent.

If $\mathbf{v} + \mathcal{W}$ is any element of $\mathcal{V}/\mathcal{W}$, then $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}$ as $\mathbf{v} \in \mathcal{V}$. Therefore $\mathbf{v} + \mathcal{W} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i} + \mathcal{W} = \sum_{i=1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W}) = \sum_{i=r+1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W})$ because $\mathbf{v_1} + \mathcal{W} = \ldots = \mathbf{v_r} + \mathcal{W} = \mathcal{W}$. Thus $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ generate $\mathcal{V}/\mathcal{W}$. Hence $\dim \mathcal{V}/\mathcal{W} = n - r = \dim \mathcal{V} - \dim \mathcal{W}$ ∎

Question 1(c) *Show that all vectors (x_1, x_2, x_3, x_4) in the vector space $\mathcal{V}_4(\mathbb{R})$ which obey $x_4 - x_3 = x_2 - x_1$ form a subspace $\mathcal{V}$. Show further that $\mathcal{V}$ is spanned by $\xi_1 = (1, 0, 0, -1), \xi_2 = (0, 1, 0, 1), \xi_3 = (0, 0, 1, 1)$.*

Solution. If $\mathbf{y} = (y_1, y_2, y_3, y_4), \mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathcal{V}$ then $\alpha\mathbf{y} + \beta\mathbf{z} = (a_1, a_2, a_3, a_4) \in \mathcal{V}$ because

$$
\begin{aligned}
a_4 - a_3 &= (\alpha y_4 + \beta z_4) - (\alpha y_3 + \beta z_3) \\
&= \alpha(y_4 - y_3) + \beta(z_4 - z_3) \\
&= \alpha(y_2 - y_1) + \beta(z_2 - z_1) \quad \because y_4 - y_3 = y_2 - y_1, z_4 - z_3 = z_2 - z_1 \\
&= a_2 - a_1
\end{aligned}
$$

Thus $\mathcal{V}$ is a subspace of $\mathcal{V}_4(\mathbb{R})$. Note that $\mathcal{V} \neq \emptyset$.

Clearly ξ_1, ξ_2, ξ_3 are linearly independent $\Rightarrow \dim \mathcal{V} \geq 3$. But $\mathcal{V} \neq \mathcal{V}_4(\mathbb{R})$ because $(1, 0, 0, 0) \notin \mathcal{V} \therefore \dim \mathcal{V} < 4 \Rightarrow \dim \mathcal{V} = 3$.

Hence ξ_1, ξ_2, ξ_3 is a basis of $\mathcal{V}$ and therefore span $\mathcal{V}$. ∎

Question 2(a) *Let $\mathbf{P}$ be a real skew-symmetric matrix and $\mathbf{I}$ the corresponding unit matrix. Show that $\mathbf{I} - \mathbf{P}$ is non-singular. Also show that $\mathbf{Q} = (\mathbf{I} + \mathbf{P})(\mathbf{I} - \mathbf{P})^{-1}$ is orthogonal.*

Solution. We have proved (question 2(a), year 1998) that the eigenvalues of a skew-Hermitian and therefore of a skew-symmetric matrix are zero or pure imaginary. This means $|\mathbf{I} - \mathbf{P}| \neq 0$ because 1 cannot be an eigenvalue of $\mathbf{P}$.

$\mathbf{Q}'\mathbf{Q} = [(\mathbf{I} - \mathbf{P})^{-1}]'(\mathbf{I} + \mathbf{P})'(\mathbf{I} + \mathbf{P})(\mathbf{I} - \mathbf{P})^{-1} = (\mathbf{I} + \mathbf{P})^{-1}(\mathbf{I} - \mathbf{P})(\mathbf{I} + \mathbf{P})(\mathbf{I} - \mathbf{P})^{-1}$. But $(\mathbf{I} - \mathbf{P})(\mathbf{I} + \mathbf{P}) = \mathbf{I} - \mathbf{P}^2 = (\mathbf{I} + \mathbf{P})(\mathbf{I} - \mathbf{P})$, therefore $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$. Similarly $\mathbf{Q}\mathbf{Q}' = \mathbf{I} \Rightarrow \mathbf{Q}$ is orthogonal. ∎

Related Results:

1. If $\mathbf{S}$ is skew-Hermitian, then $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}$ is unitary. Conversely, if $\mathbf{A}$ is unitary, then $\mathbf{A}$ can be written as $\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}$ for some skew-Hermitian matrix $\mathbf{S}$ provided -1 is not an eigenvalue of $\mathbf{A}$.

   Proof:

   $$\begin{aligned} \overline{\mathbf{A}}' &= (\overline{(\mathbf{I} - \mathbf{S})^{-1}})'\overline{(\mathbf{I} + \mathbf{S})}' = (\mathbf{I} - \overline{\mathbf{S}}')^{-1}(\mathbf{I} + \overline{\mathbf{S}}') \\ &= (\mathbf{I} + \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S}) \\ \therefore \mathbf{A}\overline{\mathbf{A}}' &= (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S}) \\ &= (\mathbf{I} + \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S}) = \mathbf{I} \\ &\qquad \because (\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})^{-1} = (\mathbf{I} - \mathbf{S}^2)^{-1} = (\mathbf{I} + \mathbf{S})^{-1}(\mathbf{I} - \mathbf{S})^{-1} \end{aligned}$$

   Similarly $\overline{\mathbf{A}}'\mathbf{A} = \mathbf{I}$, so $\mathbf{A}$ is unitary.

   Now $\mathbf{A}(\mathbf{I} - \mathbf{S}) = \mathbf{I} + \mathbf{S} \Rightarrow \mathbf{A} - \mathbf{I} = (\mathbf{A} + \mathbf{I})\mathbf{S} \Rightarrow \mathbf{S} = (\mathbf{A} + \mathbf{I})^{-1}(\mathbf{A} - \mathbf{I})$. It can be checked as above that $\mathbf{S}$ is skew-Hermitian. Note that $|\mathbf{A} + \mathbf{I}| \neq 0$.

2. If $\mathbf{H}$ is Hermitian, then $\mathbf{A} = (\mathbf{H} + i\mathbf{I})^{-1}(\mathbf{H} - i\mathbf{I})$ is unitary and every unitary matrix can be thus represented provided it does not have -1 as its eigenvalue.

3. If $\mathbf{S}$ is real, $\mathbf{S}' = -\mathbf{S}$ and $\mathbf{S}^2 = -\mathbf{I}$, then $\mathbf{S}$ is orthogonal and of even order, and there exist non-null vectors $\mathbf{x}, \mathbf{y}$ such that $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{y} = 1$, $\mathbf{x}'\mathbf{y} = 0$, $\mathbf{S}\mathbf{x} + \mathbf{y} = \mathbf{0}, \mathbf{S}\mathbf{y} = \mathbf{x}$.

   Proof: $\mathbf{S}'\mathbf{S} = -\mathbf{S}\mathbf{S} = \mathbf{I}$, so $\mathbf{S}$ is orthogonal, $|\mathbf{S}| \neq 0 \Rightarrow \mathbf{S}$ is of even order.

   Choose $\mathbf{y}$ such that $\mathbf{y}'\mathbf{y} = 1$. Then $\mathbf{y}'\mathbf{S}\mathbf{y} = (\mathbf{y}'\mathbf{S}\mathbf{y})' = \mathbf{y}'\mathbf{S}'\mathbf{y} = -\mathbf{y}'\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}'\mathbf{S}\mathbf{y} = 0$. Set $\mathbf{x} = \mathbf{S}\mathbf{y}$, then $\mathbf{y}'\mathbf{x} = 0$, $\mathbf{S}\mathbf{x} + \mathbf{y} = \mathbf{0}$. In addition, $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{S}'\mathbf{S}\mathbf{y} = \mathbf{y}'\mathbf{y} = 1$.

Question 2(b) *Show that an $n \times n$ matrix $\mathbf{A}$ is similar to a diagonal matrix if and only if the set of eigenvectors of $\mathbf{A}$ includes a set of n linearly independent vectors.*

Solution. See question 2(c) of 1998. ∎

Question 2(c) *Let r_1, r_2 be distinct eigenvalues of a matrix $\mathbf{A}$ and let $\xi_\mathbf{i}$ be an eigenvector corresponding to $r_i, i = 1, 2$. If $\mathbf{A}$ is Hermitian, show that $\overline{\xi_\mathbf{1}}'\xi_\mathbf{2} = 0$.*

Solution. See question 2(c) of 1993. ∎

Question 3(a) *Find the roots of the equation* $|x\mathbf{A}-\mathbf{B}|=0$ *where* $\mathbf{A}=\left(\begin{smallmatrix}1&1\\1&4\end{smallmatrix}\right)$*, and* $\mathbf{B}=\left(\begin{smallmatrix}0&3\\3&0\end{smallmatrix}\right)$*. Use the result to show that the real quadratic forms* $F=x_1^2+2x_1x_2+4x_2^2$*,* $G=6x_1x_2$ *can be simultaneously reduced by a non-singular linear substitution to* $y_1^2+y_2^2, y_1^2-3y_2^2$*.*

Solution. $|x\mathbf{A}-\mathbf{B}|=\begin{vmatrix}x & x-3\\ x-3 & 4x\end{vmatrix}=4x^2-(x-3)^2\Rightarrow \pm 2x=x-3\Rightarrow x=-3,1.$

Let $\mathbf{x_1}=(x_1,x_2)$ be a row vector such that $(\mathbf{A}-\mathbf{B})\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)=\mathbf{0}$.

$$\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\mathbf{0}\Rightarrow x_1-2x_2=0$$

We take $x_1=2, x_2=1$, so $\mathbf{x_1}=(2,1)$.

Let $\mathbf{x_2}=(x_1,x_2)$ be a row vector such that $(-3\mathbf{A}-\mathbf{B})\left(\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\right)=\mathbf{0}$.

$$\begin{pmatrix}-3 & -6\\ -6 & -12\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\mathbf{0}\Rightarrow x_1+2x_2=0$$

We take $x_1=-2, x_2=1$, so $\mathbf{x_2}=(-2,1)$.

$\mathbf{x_1Ax_1'}=(2,1)\left(\begin{smallmatrix}1&1\\1&4\end{smallmatrix}\right)\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)=(2,1)\left(\begin{smallmatrix}3\\6\end{smallmatrix}\right)=12.$

$\mathbf{x_2Ax_2'}=(-2,1)\left(\begin{smallmatrix}1&1\\1&4\end{smallmatrix}\right)\left(\begin{smallmatrix}-2\\1\end{smallmatrix}\right)=(-2,1)\left(\begin{smallmatrix}-1\\2\end{smallmatrix}\right)=4.$

Note that $\mathbf{x_1Ax_2'}=0$.

$\mathbf{x_1Bx_1'}=(2,1)\left(\begin{smallmatrix}0&3\\3&0\end{smallmatrix}\right)\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)=(2,1)\left(\begin{smallmatrix}3\\6\end{smallmatrix}\right)=12.$

$\mathbf{x_2Bx_2'}=(-2,1)\left(\begin{smallmatrix}0&3\\3&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-2\\1\end{smallmatrix}\right)=(-2,1)\left(\begin{smallmatrix}3\\-6\end{smallmatrix}\right)=-12.$

Note that $\mathbf{x_1Bx_2'}=0$.

Thus if $\mathbf{P}=[\mathbf{x_1}',\mathbf{x_2}']$, then $\mathbf{P'AP}=\left(\begin{smallmatrix}12&0\\0&4\end{smallmatrix}\right)$, and $\mathbf{P'BP}=\left(\begin{smallmatrix}12&0\\0&-12\end{smallmatrix}\right)$. Let $\mathbf{Q}=\left(\begin{smallmatrix}\frac{1}{\sqrt{12}}&0\\0&\frac{1}{2}\end{smallmatrix}\right)$, then $\mathbf{Q'P'APQ}=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\mathbf{Q'P'BPQ}=\left(\begin{smallmatrix}1&0\\0&-3\end{smallmatrix}\right)$ as desired. Thus the required non-singular linear transformation is $\mathbf{PQ}$. ∎

Question 3(b) *Show that* $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}1 & -\tan\frac{\theta}{2}\\ \tan\frac{\theta}{2} & 1\end{pmatrix}\begin{pmatrix}1 & \tan\frac{\theta}{2}\\ -\tan\frac{\theta}{2} & 1\end{pmatrix}^{-1}$.

Solution.

$$\begin{aligned} R.H.S &= \begin{pmatrix}1 & -\tan\frac{\theta}{2}\\ \tan\frac{\theta}{2} & 1\end{pmatrix}\begin{pmatrix}\cos^2\frac{\theta}{2} & -\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2}\end{pmatrix}\\ &= \begin{pmatrix}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2} & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}\end{pmatrix}=L.H.S\end{aligned}$$

∎

Question 3(c) *Verify the Cayley-Hamilton theorem for* $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

Solution. The characteristic equation for $\mathbf{A}$ is $\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -3\lambda + \lambda^2 + 2 = 0$

Thus according to the Cayley-Hamilton theorem $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = \mathbf{0}$.

$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$

$\begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix} - 3 \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus the Cayley Hamilton theorem is verified for $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. ∎

UPSC Civil Services Main 1990 - Mathematics
Linear Algebra

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1 Linear Algebra

Question 1(a) *State any definition of the determinant of an $n \times n$ matrix and show that the determinant function is multiplicative i.e.* $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ *for any two $n \times n$ matrices* $\mathbf{A}, \mathbf{B}$. *You may assume the matrices to be real.*

Solution. Let π be a permutation of $1, \ldots, n$. Define $\text{sign}(\pi)$ as follows: count the number of pairs of numbers that need to be interchanged to get to π from the identity permutation. If this is even, the sign is 1, and if it is odd, the sign is -1. Now if Π is the set of all permutations of $1, \ldots, n$, define

$$\det \mathbf{A} = \sum_{\pi \in \Pi} \text{sign}(\pi) \prod_i a_{i\pi(i)}$$

where a_{ij} are the elements of $\mathbf{A}$.

Note that the $\det \mathbf{A}$ is n-linear i.e. if we perform any row or column operation on $\mathbf{A}$ the determinant is unchanged. Also, if any two rows are swapped, the sign of the determinant changes. These are simple consequences of the above definition.

Consider the $2n \times 2n$ matrix

$$\mathbf{P} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & b_{11} & \ldots & b_{1n} \\ 0 & -1 & \ldots & 0 & b_{21} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & b_{n1} & \ldots & b_{nn} \end{pmatrix}$$

Then $\det \mathbf{P} = \det \mathbf{A} \det \mathbf{B}$, because if for any permutation π, $\pi(i) > n$ for $i \leq n$, then the corresponding element of the sum is 0 as $a_{i\pi(i)} = 0$. Thus $\pi(i) \leq n$ if $i \leq n$, and consequently $\pi(j) > n$ if $j > n$. So each permutation consists of a permutation of $1, \ldots, n$ and a permutation of $n+1, \ldots, 2n$, consequently we can factor the sum, to get $\det \mathbf{P} = \det \mathbf{A} \det \mathbf{B}$.

Now we perform a series of column operations to $\mathbf{P}$ — add $b_{11}\mathbf{C}_1 + \ldots + b_{n1}\mathbf{C}_n$ to $\mathbf{C}_{n+1}$, to get

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & c_{11} & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & c_{21} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & c_{n1} & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & 0 & b_{12} & \ldots & b_{1n} \\ 0 & -1 & \ldots & 0 & 0 & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -1 & 0 & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

where $\mathbf{C} = \mathbf{AB} = (c_{ij})$. Similarly add $b_{12}\mathbf{C}_1 + \ldots + b_{n2}\mathbf{C}_n$ to $\mathbf{C}_{n+2}$, ..., $b_{1n}\mathbf{C}_1 + \ldots + b_{nn}\mathbf{C}_n$ to $\mathbf{C}_{2n}$ to get

$$\begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & -1 \end{pmatrix} & \mathbf{0} \end{pmatrix}$$

We can now verify that $\det \mathbf{P} = \det \mathbf{C}$. Any permutation π that leads to a non-zero term in the determinant sum must have $\pi(j) = j - n$ for $j > n$, thus $p_{i\pi(i)} = -1, i > n$. Also $\pi(j) > n$ for $j \leq n$, so any such π can be written as a permutation of $1, \ldots, n$ followed by a series of swaps of the i-th number with the $(n+i)$-th number, which is $n+i$. Also $\text{sign}(\pi)$ is the same as the sign of the corresponding permutation π' of $1, \ldots, n$ — we first do π' by exchanges and then additionally swap the i-th element with the $(i+n)$-th element, for each $i \leq n$. Now if n is even, this involves an even number of additional swaps, and multiply by $(-1)^n$ corresponding to $p_{i\pi(i)}$ for $i > n$, otherwise we get an odd number of additional swaps, flipping the sign, but we still multiply by $(-1)^n = -1$.

Thus $\det \mathbf{P} = \det \mathbf{C} = \det \mathbf{A} \det \mathbf{B}$. ∎

Question 1(b) *Prove Laplace's formula for simulataneous expansion of the determinant by the first row and column; that given an $(n+1)\times(n+1)$ matrix in the block form $\mathbf{M} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \mathbf{D} \end{smallmatrix}\right)$, where α is a scalar, β is a $1 \times n$ matrix (a row vector), γ is a $n \times 1$ matrix (a column vector), and $\mathbf{D}$ is an $n \times n$ matrix, then* $\det \mathbf{M} = \alpha \det \mathbf{D} - \beta\mathbf{D}'\gamma'$, *where $\mathbf{D}'$ is the matrix of cofactors of $\mathbf{D}$ and $\beta\mathbf{D}'\gamma'$ stands for the matrix product of size* 1×1.

Solution. Let $\mathbf{M} = (a_{ij}), 1 \leq i, j \leq n+1$. Thus $\alpha = a_{11}$, $\beta = (a_{12} \ \ldots \ a_{1,n+1})$,

$\gamma = \begin{pmatrix} a_{21} \\ \vdots \\ a_{n+1,1} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} a_{22} & \dots & a_{2,n+1} \\ \vdots & & \vdots \\ a_{n+1,2} & \dots & a_{n+1,n+1} \end{pmatrix}$.

$\det \mathbf{M} = a_{11}|\mathbf{A}_{11}| - a_{12}|\mathbf{A}_{12}| + \ldots + (-1)^n a_{1,n+1}|\mathbf{A}_{1,n+1}|$ where $\mathbf{A}_{ij}$ is the minor corresponding to a_{ij} (formed by deleting the i-th row and j-th column of $\mathbf{A}$). Clearly $\mathbf{D} = \mathbf{A}_{11}$, so $\det \mathbf{M} = \alpha \det \mathbf{D} - \sum_{j=2}^{n+1}(-1)^j a_{1j} \det \mathbf{A}_{1j}$. Now

$$|\mathbf{A}_{1j}| = \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,j-1} & a_{2,j+1} & \dots & a_{2,n+1} \\ a_{31} & a_{32} & \dots & a_{3,j-1} & a_{3,j+1} & \dots & a_{3,n+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,j-1} & a_{n+1,j+1} & \dots & a_{n+1,n+1} \end{vmatrix}$$

Let $\mathbf{B}_{ij}$ be the minor of a_{ij} in $\mathbf{D}$. Expanding $|\mathbf{A}_{1j}|$ in terms of the first column, we get

$$|\mathbf{A}_{1j}| = a_{21}|\mathbf{B}_{2j}| - a_{31}|\mathbf{B}_{3j}| + \ldots + (-1)^{n+1} a_{n+1,1}|\mathbf{B}_{n+1,1}|$$

$$\begin{aligned} \det \mathbf{M} &= \alpha \det \mathbf{D} - \sum_{j=2}^{n+1}\sum_{i=2}^{n+1} a_{1j}a_{i1}|\mathbf{B}_{ij}|(-1)^i(-1)^j \\ &= \alpha \det \mathbf{D} - (a_{12}\ a_{13}\ \dots\ a_{1,n+1})(c_{ij}) \begin{pmatrix} a_{21} \\ \vdots \\ a_{n+1,1} \end{pmatrix} \\ &= \alpha \det \mathbf{D} - \beta \mathbf{D}'\gamma \end{aligned}$$

where $c_{ij} = (-1)^{i+j}|\mathbf{B}_{ij}|$, thus $\mathbf{D}' = (c_{ij})$ is the matrix of cofactors of $\mathbf{D}$. ∎

Question 1(c) *For* $\mathbf{M}$ *as in 1(b), if* $\mathbf{D}$ *is invertible, show that* $\det \mathbf{M} = \det \mathbf{D}(\alpha - \beta \mathbf{D}^{-1}\gamma)$*.*

Solution. If $\mathbf{D}$ is invertible, then $\mathbf{D}\mathbf{D}' = \mathbf{D}'\mathbf{D} = (\det \mathbf{D})\mathbf{I} \Rightarrow \mathbf{D}' = \mathbf{D}^{-1}\det \mathbf{D}$. So $\det \mathbf{M} = \alpha \det \mathbf{D} - \beta \mathbf{D}'\gamma = \alpha \det \mathbf{D} - \beta \mathbf{D}^{-1} \det \mathbf{D}\gamma = \det \mathbf{D}(\alpha - \beta \mathbf{D}^{-1}\gamma)$. ∎

Question 2(a) *Write the definition of the characteristic polynomial, eigenvalues and eigenvectors of a square matrix. Also say briefly something about the importance and/or applications of these notions.*

Solution. Let $\mathbf{A}$ be an $n \times n$ real or complex matrix. The polynomial $|\mathbf{x}\mathbf{I}_n - \mathbf{A}|$ is called the characteristic polynomial of $\mathbf{A}$. The roots of this polynomial are called the eigenvalues of $\mathbf{A}$. If λ is an eigenvalue of $\mathbf{A}$, then all the non-zero vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ are called eigenvectors of $\mathbf{A}$ corresponding to λ.

Many problems in mathematics and other sciences require finding eigenvalues and eigenvectors of an operator.

- Eigenvalues can be used to find a very simple matrix for an operator — either diagonal or a block diagonal form. This can be used to compute powers of matrices quickly.
- If one wishes to solve a linear differential system like $\mathbf{x}' = \mathbf{A}\mathbf{x}$, or study the local properties of a nonlinear system, finding the diagonal form of the matrix can give us a decoupled form of the system, allowing us to find the solution or understand its qualitative behavior, like its stability and oscillatory behavior.
- The calculation of Google's Pagerank is essentially the computation of the principal eigenvector (corresponding to the eigenvalue with the largest absolute value) of a very large matrix (the adjacency matrix of the web graph) — this is used to find the relative importance of documents on the World Wide Web. Similar calculations are used to compute the stationary distribution of a Markov system.
- In mechanics, the eigenvectors of the inertia tensor are used to define the principal axes of a rigid body, which are important in analyzing the rotation of the rigid body.
- Eigenvalues can be used to compute low rank approximations to matrices, which help in reducing the dimensionality of various problems. This is used in statistics and operations research to explain a large number of observables in terms of a few hidden variables + noise.
- Eigenvalues can help us determine the form of a quadric or higher dimensional surface — see the relevant section in year 1999.
- In quantum mechanics, states are represented by unit vectors, while observable quantities (like position and energy) are represented by Hermitian matrices. The basic problem in any quantum system is the determination of the eigenvalues and eigenvectors of the energy matrix. The eigenvalues are the observed values of the observable quantity, and discreteness of the eigenvalues leads to the quantization of the observed values.

∎

Question 2(b) *Show that a Hermitian matrix possesses a set of eigenvectors which form an orthonormal basis. State briefly how or why a general $n \times n$ complex matrix may fail to possess n linearly independent eigenvectors.*

Solution. Let $\mathbf{H}$ be Hermitian, and $\lambda_1, \ldots, \lambda_n$ its eigenvalues, not necessarily distinct. Let $\mathbf{x_1}$ with norm 1 be an eigenvector corresponding to λ_1. Then there exists (from a result analogous to the result used in question 3(a), year 1995) a unitary matrix $\mathbf{U}$ such that $\mathbf{x_1}$ is its first column. Therefore

$$\mathbf{U_1}^{-1}\mathbf{H}\mathbf{U_1} = \overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1} = \begin{pmatrix} \lambda_1 & \mathbf{L} \\ \mathbf{0} & \mathbf{H_1} \end{pmatrix}$$

where $\mathbf{H_1}$ is $(n-1)\times(n-1)$ and $\mathbf{L}$ is $(n-1)\times 1$. Since $\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1}$ is Hermitian, it follows that $\mathbf{L}=\mathbf{0}$. Consequently

$$\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1} = \begin{pmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H_1} \end{pmatrix}$$

Now $\mathbf{H_1}$ is Hermitian with eigenvalues $\lambda_2, \ldots, \lambda_n$. Repeating the above argument, we find $\mathbf{U_2^*}$ an $(n-1)\times(n-1)$ unitary matrix such that

$$\overline{\mathbf{U_2^*}}'\mathbf{H_1}\mathbf{U_2^*} = \begin{pmatrix} \lambda_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{H_2} \end{pmatrix}$$

If $\mathbf{U_2} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{U_2^*} \end{pmatrix}$ then $\mathbf{U_2}$ is unitary, and

$$\overline{\mathbf{U_2}}'\overline{\mathbf{U_1}}'\mathbf{H}\mathbf{U_1}\mathbf{U_2} = \begin{pmatrix} \lambda_1 & 0 & \mathbf{0} \\ 0 & \lambda_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H_2} \end{pmatrix}$$

Repeating this process or by induction, we can get $\mathbf{U}$ unitary such that

$$\overline{\mathbf{U}}'\mathbf{H}\mathbf{U} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

If $\mathbf{U} = [\mathbf{C_1}, \mathbf{C_2}, \ldots, \mathbf{C_n}]$, then $\mathbf{C_1}, \mathbf{C_2}, \ldots, \mathbf{C_n}$ are eigenvectors of $\mathbf{H}$ and form an orthonormal system.

A complex matrix $\mathbf{A}$ would fail to have n eigenvectors which are linearly independent if $\mathbf{A}$ is not diagonalizable i.e. we cannot find $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix. For example if $\mathbf{A} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$, $c \neq 0$, and $\mathbf{x_1}, \mathbf{x_2}$ are two independent eigenvectors of $\mathbf{A}$, then $\mathbf{P} = [\mathbf{x_1}\ \mathbf{x_2}]$ would lead to $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \Rightarrow \mathbf{A} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ which is false. ∎

Question 2(c) *Define the minimal polynomial and show that a complex matrix is diagonalizable (i.e. conjugate to a diagonal matrix) if and only if the minimal polynomial has no repeated root.*

Solution. Given a complex $n \times n$ complex matrix $\mathbf{A}$, if $f(x)$ is a nonzero polynomial with complex coefficients of least degree such that $f(\mathbf{A}) = \mathbf{0}$, then $f(x)$ is called the *minimal polynomial* of $\mathbf{A}$. The Cayley-Hamilton therem tells us that any $n \times n$ complex matrix $\mathbf{A}$ satisfies the degree n polynomial equation $|\mathbf{A} - x\mathbf{I}| = 0$, so the minimal polynomial exists and is of degree $\leq n$.

A complex $n \times n$ matrix can be thought of as a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$, $\dim \mathcal{V} = n$. Let the minimal polynomial of $\mathbf{T}$ be $p(x)$, having distinct roots $c_1, \ldots, c_k$, so $p(x) = \prod_{j=1}^{k}(x - c_j)$. We shall show that $\mathbf{T}$ is diagonalizable.

If $k = 1$, then the minimal polynomial is $x - c$, thus $\mathbf{T} - c\mathbf{I} = \mathbf{0}$, so $\mathbf{T} = c\mathbf{I}$ is diagonalizable. So assume $k > 1$.

Consider the polynomials $p_j = \prod_{\substack{i=1 \\ i \neq j}}^{k} \frac{x-c_i}{c_j-c_i}$. Clearly $p_j(c_i) = 0$ for $i \neq j$, and $p_i(c_i) = 1$. This implies that the polynomials $p_1, \ldots, p_k$ are linearly independent, and each one is of degree $k-1 < k$. Thus these form a basis of the space of polynomials of degree $\leq k-1$. Thus given any polynomial g of degree $\leq k-1$, $g = \sum_{i=1}^{k} \alpha_i p_i$, where $\alpha_i = g(c_i)$. In particular, $1 = \sum_{i=1}^{k} p_i, x = \sum_{i=1}^{k} c_i p_i$. Thus

$$\mathbf{I} = \sum_{i=1}^{k} p_i(\mathbf{T}), \quad \mathbf{T} = \sum_{i=1}^{k} c_i p_i(\mathbf{T})$$

Moreover $p_i(\mathbf{T})p_j(\mathbf{T}) = \mathbf{0}, i \neq j$ because $p_i(x)p_j(x)$ is divisible by the minimal polynomial of $\mathbf{T}$. Also $p_j(\mathbf{T}) \neq \mathbf{0}, 1 \leq j \leq k$, because the degree of p_j is less than k, the degree of the minimal polynomial of $\mathbf{T}$.

Set $\mathcal{V}_i = p_i(\mathbf{T})\mathcal{V}$, then $\mathcal{V} = \mathbf{I}(\mathcal{V}) = \sum_{i=1}^{k} p_i(\mathbf{T})\mathcal{V} = \mathcal{V}_1 + \ldots + \mathcal{V}_k$. We shall now show that $\mathcal{V}_i = \mathcal{V}_{c_i}$, the eigenspace of $\mathbf{T}$ with respect to c_i.

$\mathbf{v} \in \mathcal{V}_i \Rightarrow \mathbf{v} = p_i(\mathbf{T})\mathbf{w}$ for some $\mathbf{w} \in \mathcal{V}$. Since $(x - c_i)p_i$ is divisible by p, $(\mathbf{T} - c_i\mathbf{I})\mathbf{v} = \mathbf{0}$, so $\mathbf{T}\mathbf{v} = c_i\mathbf{v}$ so $\mathbf{v} \in \mathcal{V}_{c_i}$. Conversely, if $\mathbf{v} \in \mathcal{V}_{c_i}$, then $\mathbf{T}\mathbf{v} = c_i\mathbf{v}$, or $(\mathbf{T} - c_i\mathbf{I})\mathbf{v} = \mathbf{0} \Rightarrow p_j(\mathbf{T})\mathbf{v} = \mathbf{0}$ for $j \neq i$. Since $\mathbf{v} = p_i(\mathbf{T})\mathbf{v} + \ldots + p_k(\mathbf{T})\mathbf{v}$, we get $\mathbf{v} = p_i(\mathbf{T})\mathbf{v} \Rightarrow \mathbf{v} \in \mathcal{V}_i$.

Thus $\mathcal{V} = \sum_{i=1}^{k} \mathcal{V}_{c_i}$ so $\mathcal{V}$ has a basis consisting of eigenvectors, so $\mathbf{T}$ is diagonalizable.

Conversely let $\mathbf{T}$ be diagonalizable, then we shall show that the minimal polynomial of $\mathbf{T}$ has distinct roots. Let

$$\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

and out of $\lambda_1, \ldots, \lambda_n$, let $\lambda_1, \ldots, \lambda_k$ be distinct. Let $g(x) = (x - \lambda_1) \ldots (x - \lambda_k)$. Then $\mathbf{v} \in \mathcal{V} \Rightarrow \mathbf{v} = \mathbf{v_1} + \ldots + \mathbf{v_k}$ where $\mathbf{v_i} \in \mathcal{V}_{\lambda_i}$, the eigenspace of λ_i. Thus $g(\mathbf{T})(\mathbf{v}) = \mathbf{0}$, so $g(\mathbf{T}) = \mathbf{0}$. Thus $g(x)$ is divisible by the minimal polynomial of $\mathbf{T}$. Since $g(x)$ has all distinct roots, it immediately follows that the minimal polynomial also has all distinct roots. ∎

Question 3(a) *Show that a* 2×2 *matrix* $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *is expressible in the form* $\mathbf{LDU}$*, where* $\mathbf{L}$ *has the form* $\left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)$*,* $\mathbf{D}$ *is diagonal and* $\mathbf{U}$ *has the form* $\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ *If and only if either* $a \neq 0$ *or* $a = b = c = 0$*. Also show that when* $a \neq 0$ *the factorization* $\mathbf{M} = \mathbf{LDU}$ *is unique.*

Solution. Given $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, suppose $\mathbf{M} = \mathbf{LDU} = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & 0 \\ a_1\alpha & a_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & a_1\beta \\ a_1\alpha & a_1\alpha\beta + a_2 \end{smallmatrix}\right)$. Thus $\mathbf{M} = \mathbf{LDU} \Rightarrow a_1 = a,\ a_1\beta = b,\ a_1\alpha = c,\ a_1\alpha\beta + a_2 = d$. Thus if $a = 0$, then $b = c = 0$ and $d = a_2$, In this case, $\mathbf{M} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ whatever α, β may be, i.e. $\mathbf{M}$ can be represented as $\mathbf{LDU}$ in infinitely many ways.

If $a \neq 0$, then $a_1 = a,\ \beta = \frac{b}{a},\ \alpha = \frac{c}{a},\ a_2 = d - \frac{bc}{a}$ are uniquely determined. Thus $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{smallmatrix}\right)$ and has a unique representation.

Conversely, if $\mathbf{M} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)$, i.e. $a = b = c = 0$, then $\mathbf{M} = \left(\begin{smallmatrix} 1 & 0 \\ \alpha & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ for any $\alpha, \beta \in \mathbb{R}$. If $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a \neq 0$, then $\mathbf{M} = \mathbf{LDU}$ with $\mathbf{L} = \left(\begin{smallmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{smallmatrix}\right)$, $\mathbf{D} = \left(\begin{smallmatrix} a & 0 \\ 0 & d-\frac{bc}{a} \end{smallmatrix}\right)$, $\mathbf{U} = \left(\begin{smallmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{smallmatrix}\right)$ as shown above. ∎

Question 3(b) *Suppose a real matrix has eigenvalue λ, possibly complex. Show that there exists a real eigenvector for λ if and only if λ is real.*

Solution. If λ is real, then the $n \times n$ matrix $\mathbf{A} - \lambda\mathbf{I}$ defines a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Since $|\mathbf{A} - \lambda\mathbf{I}| = 0$, the rows are linearly dependent, so there exists $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Thus there exists a real eigenvector for λ.

Conversely, suppose $\mathbf{A}\mathbf{x} = \lambda\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}$. then $\lambda\mathbf{x} = \mathbf{A}\mathbf{x} = \overline{\mathbf{A}\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\mathbf{x} \Rightarrow (\lambda - \overline{\lambda})\mathbf{x} = \mathbf{0} \Rightarrow \lambda - \overline{\lambda} = 0 \because \mathbf{x} \neq \mathbf{0} \Rightarrow \lambda = \overline{\lambda}$ i.e. λ is real. ∎

Question 3(c) *If a 2×2 matrix $\mathbf{A}$ has order n, i.e. $\mathbf{A}^n = \mathbf{I}_2$, then show that $\mathbf{A}$ is conjugate to the matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ where $\theta = \frac{2\pi m}{n}$ for some integer m.*

Solution. Note: $\mathbf{A}$ has to be real, otherwise the result is false: if α_1, α_2 are two distinct n-th roots of unity such that $\alpha_1 \neq \overline{\alpha_2}$, then $\mathbf{A} = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$ has order n, but $\mathbf{A}$ is not conjugate to $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ whose eigenvalues are complex conjugates of each other.

$\mathbf{A}^n = \mathbf{I} \Rightarrow$ eigenvalues of $\mathbf{A}$ are n-th roots of unity. If $\mathbf{A}$ has repeated eigenvalues, then these can be 1 or -1, because eigenvalues of real matrices are complex conjugates of each other, so the repeated eigenvalues must be real, and they also must be roots of 1.

Case 1: $\mathbf{A}$ has eigenvalues $1, 1$. There exists $\mathbf{P}$ non-singular such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. Now $(\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^n = \left(\begin{smallmatrix} 1 & nc \\ 0 & 1 \end{smallmatrix}\right) = \mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \mathbf{P}^{-1}\mathbf{I_2}\mathbf{P} = \mathbf{I_2}$, so $nc = 0 \Rightarrow c = 0$. Thus $\mathbf{A}$ is conjugate to $\mathbf{I_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta = \frac{2\pi n}{n}$.

Case 2: $\mathbf{A}$ has eigenvalues $-1, -1$. There exists $\mathbf{P}$ non-singular such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left(\begin{smallmatrix} -1 & c \\ 0 & -1 \end{smallmatrix}\right)$. Now $\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \left(\begin{smallmatrix} -1 & nc \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & nc \\ 0 & 1 \end{smallmatrix}\right)$, according as n is odd or even. But $\mathbf{A}^n = \mathbf{I_2}$, therefore n is even and $c = 0$. Thus $\mathbf{A}$ is conjugate to $-\mathbf{I_2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta = \frac{2\pi m}{n}, m = \frac{n}{2}$.

Case 3: $\mathbf{A}$ has distinct eigenvalues λ_1, λ_2. Then $\lambda_1 = \overline{\lambda_2}$. If $\lambda_1 = \cos\theta + i\sin\theta$, with $\theta = \frac{2\pi m}{n}$, set $\mathbf{B} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The eigenvalues of $\mathbf{B}$ are roots of $\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0 \Rightarrow \lambda = \cos\theta \pm i\sin\theta$. Since $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues λ_1, λ_2 distinct, both are conjugate to $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and are therefore conjugate to each other. ∎

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Linear Algebra

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Question 1(a) *Let $\mathcal{V}(\mathbb{R})$ be the real vector space of all 2×3 matrices with real entries. Find a basis of $\mathcal{V}(\mathbb{R})$. What is the dimension of $\mathcal{V}(\mathbb{R})$.*

Solution. Let $\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{A_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{A_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathbf{B_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{B_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Clearly $\mathbf{A_i}, \mathbf{B_i},\ i = 1, 2, 3 \in \mathcal{V}(\mathbb{R})$. These generate $\mathcal{V}(\mathbb{R})$ because

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = a_1\mathbf{A_1} + a_2\mathbf{A_2} + a_3\mathbf{A_3} + b_1\mathbf{B_1} + b_2\mathbf{B_2} + b_3\mathbf{B_3}$$

for any arbitrary element $\mathbf{A} \in \mathcal{V}(\mathbb{R})$.

They are linearly independent because if the RHS in the above equation was equal to $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $a_i = 0, b_i = 0$ for $i = 1, 2, 3$. Thus $\mathbf{A_i}, \mathbf{B_i}, i = 1, 2, 3$ is a basis for $\mathcal{V}(\mathbb{R})$ and the dimension of $\mathcal{V}(\mathbb{R})$ is 6. ∎

Question 1(b) *Let $\mathbb{C}$ be the field of complex numbers and let $\mathbf{T}$ be the function from $\mathbb{C}^3$ to $\mathbb{C}^3$ defined by*

$$\mathbf{T}(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$$

1. *Verify that $\mathbf{T}$ is a linear transformation.*
2. *If $(a, b, c) \in \mathbb{C}^3$, what are the conditions on a, b, c so that (a, b, c) is in the range of $\mathbf{T}$? What is the rank of $\mathbf{T}$?*

3. *What are the conditions on a, b, c so that (a, b, c) is in the null space of $\mathbf{T}$? What is the nullity of $\mathbf{T}$?*

Solution. $\mathbf{T}(\mathbf{e_1}) = (1, 2, -1), \mathbf{T}(\mathbf{e_2}) = (-1, 1, -2), \mathbf{T}(\mathbf{e_3}) = (2, 0, 2)$. Clearly $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_3})$ are linearly independent. If

$$(-1, 1, 2) = \alpha(1, 2, -1) + \beta(2, 0, 2)$$

then $\alpha+2\beta = -1, 2\alpha = 1, -\alpha+2\beta = -2$, so $\alpha = \frac{1}{2}, \beta = -\frac{3}{4}$, so $\mathbf{T}(\mathbf{e_2})$ is a linear combination of $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_3})$. Thus rank of $\mathbf{T}$ is 2, nullity of $\mathbf{T}$ is 1.

If (a, b, c) is in the range of $\mathbf{T}$, then $(a, b, c) = \alpha(1, 2, -1) + \beta(2, 0, 2)$. Thus $\alpha + 2\beta = a, 2\alpha = b, -\alpha + 2\beta = c$. From the first two equations, $\alpha = \frac{b}{2}, \beta = \frac{a-\frac{b}{2}}{2}$. The equations would be consistent if $-\frac{b}{2} + a - \frac{b}{2} = c$, or $a = b + c$. So the condition for (a, b, c) to belong to the range of $\mathbf{T}$ is $a = b + c$.

If $(a, b, c) \in$ null space of $\mathbf{T}$, then $a - b + 2c = 0, 2a + b = 0, -a - 2b + 2c = 0$. Thus $3a+2c = 0$, so $a = -\frac{2c}{3}, b = \frac{4c}{3}$. Thus the conditions for (a, b, c) to belong to the null space of $\mathbf{T}$ are $3a+2c = 0, 3b = 4c$. Thus the null space consists of the vectors $\{(-\frac{2c}{3}, \frac{4c}{3}, c) \mid c \in \mathbb{R}\}$, showing that the nullity of $\mathbf{T}$ is 1. ∎

Question 1(c) *If $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$, express $\mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2$ as a linear polynomial in $\mathbf{A}$.*

Solution. Characteristic polynomial of $\mathbf{A}$ is $\left|\begin{smallmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{smallmatrix}\right| = (\lambda - 3)(\lambda - 1) - 2 = \lambda^2 - 4\lambda + 1$. By the Cayley Hamilton theorem, $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = \mathbf{0}$. Dividing the given polynomial by $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}$, we have

$$\begin{aligned}
& \mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 \\
= \; & \mathbf{A}^4(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 7\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2)(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 16\mathbf{A}^3 + 7\mathbf{A}^2 \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2 + 16\mathbf{A})(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 71\mathbf{A}^2 - 16\mathbf{A} \\
= \; & (\mathbf{A}^4 + 7\mathbf{A}^2 + 16\mathbf{A} + 71\mathbf{I})(\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}) + 268\mathbf{A} - 71\mathbf{I}
\end{aligned}$$

Since $\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} = \mathbf{0}$, $\mathbf{A}^6 - 4\mathbf{A}^5 + 8\mathbf{A}^4 - 12\mathbf{A}^3 + 14\mathbf{A}^2 = 268\mathbf{A} - 71\mathbf{I}$. ∎

Question 2(a) *Let $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear transformation defined by $\mathbf{T}(x_1, x_2) = (-x_2, x_1)$.*

1. *What is the matrix of $\mathbf{T}$ in the standard basis of $\mathbb{R}^2$?*

2. *What is the matrix of $\mathbf{T}$ in the ordered basis $\mathcal{B} = \{\alpha_\mathbf{1}, \alpha_\mathbf{2}\}$ where $\alpha_\mathbf{1} = (1, 2)$, $\alpha_\mathbf{2} = (1, -1)$?*

Solution. $\mathbf{T}(\mathbf{e_1}) = (0,1) = \mathbf{e_2}$, $\mathbf{T}(\mathbf{e_2}) = (-1,0) = -\mathbf{e_1}$. Thus $(\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2})) = (\mathbf{e_1}\ \mathbf{e_2})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So the matrix of $\mathbf{T}$ in the standard basis is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\mathbf{T}(\alpha_\mathbf{1}) = (-2,1), \mathbf{T}(\alpha_\mathbf{2}) = (1,1)$. If $(a,b) = x\alpha_\mathbf{1} + y\alpha_\mathbf{2}$, then $x+y=a, 2x-y=b$, so $x = \frac{a+b}{3}, y = \frac{2a-b}{3}$. This shows that

$$\begin{aligned}\mathbf{T}(\alpha_\mathbf{1}) &= (-2,1) = -\tfrac{1}{3}\alpha_\mathbf{1} - \tfrac{5}{3}\alpha_\mathbf{2}\\ \mathbf{T}(\alpha_\mathbf{2}) &= (1,1) = \tfrac{2}{3}\alpha_\mathbf{1} + \tfrac{1}{3}\alpha_\mathbf{2}\end{aligned}$$

Thus $(\mathbf{T}(\alpha_\mathbf{1})\ \mathbf{T}(\alpha_\mathbf{2})) = (\alpha_\mathbf{1}\ \alpha_\mathbf{2})\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$. Consequently the matrix of $\mathbf{T}$ in the ordered basis $\mathcal{B}$ is $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{5}{3} & \frac{1}{3} \end{pmatrix}$. ∎

Question 2(b) *Determine a non-singular matrix* $\mathbf{P}$ *such that* $\mathbf{P'AP}$ *is a diagonal matrix, where* $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$. *Is the matrix congruent to a diagonal matrix? Justify your answer.*

Solution. The quadratic form associated with $\mathbf{A}$ is $Q(x,y,z) = 2xy + 4xz + 6yz$. Let $x = X, y = X+Y, z = Z$ (thus $X = x, Y = y-x, Z = z$). Then

$$\begin{aligned} Q(X,Y,Z) &= 2X^2 + 2XY + 4XZ + 6XZ + 6YZ \\ &= 2X^2 + 2XY + 10XZ + 6YZ \\ &= 2(X + \frac{Y}{2} + \frac{5}{2}Z)^2 - \frac{Y^2}{2} - \frac{25}{2}Z^2 + YZ \\ &= 2(X + \frac{Y}{2} + \frac{5}{2}Z)^2 - \frac{1}{2}(Y-Z)^2 - 12Z^2 \end{aligned}$$

Put

$$\begin{aligned} \xi &= X + \frac{Y}{2} + \frac{5}{2}Z = \frac{x}{2} + \frac{y}{2} + \frac{5z}{2} \\ \eta &= Y - Z = -x + y - z \\ \zeta &= Z = z \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$Q(x,y,z)$ transforms to $2\xi^2 - \frac{1}{2}\eta^2 - 12\zeta^2$. Thus

$$\mathbf{P'AP} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -12 \end{pmatrix}$$

with $\mathbf{P} = \begin{pmatrix} 1 & -\frac{1}{2} & -3 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \end{pmatrix}$ Clearly $\mathbf{A}$ is congruent to a diagonal matrix as shown above. ∎

Question 2(c) *Reduce the matrix*

$$\begin{pmatrix} 1 & 3 & 4 & -5 \\ -2 & -5 & -10 & 16 \\ 5 & 9 & 33 & -68 \\ 4 & 7 & 30 & -78 \end{pmatrix}$$

to echelon form by elementary row transformations.

Solution. Let the given matrix be $\mathbf{A}$. Operations $\mathbf{R}_2 + 2\mathbf{R}_1, \mathbf{R}_3 - 5\mathbf{R}_1, \mathbf{R}_4 - 4\mathbf{R}_1 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & -6 & 13 & -43 \\ 0 & -5 & 14 & -58 \end{pmatrix}$$

Operations $\mathbf{R}_3 + 6\mathbf{R}_2, \mathbf{R}_4 + 5\mathbf{R}_2 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 4 & -28 \end{pmatrix}$$

Operations $\mathbf{R}_4 - 4\mathbf{R}_3 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 3 & 4 & -5 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Operation $\mathbf{R}_1 - 3\mathbf{R}_2 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 0 & 10 & -23 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Operations $\mathbf{R}_1 - 10\mathbf{R}_3, \mathbf{R}_2 + 2\mathbf{R}_3 \Rightarrow$

$$\mathbf{A} \approx \begin{pmatrix} 1 & 0 & 0 & 47 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the required row echelon form. The rank of $\mathbf{A}$ is 3. ∎

Question 3(a) $\mathbf{U}$ *is an n-rowed unitary matrix such that* $|\mathbf{I}-\mathbf{U}| \neq 0$, *show that the matrix* $\mathbf{H}$ *defined by* $i\mathbf{H} = (\mathbf{I}+\mathbf{U})(\mathbf{I}-\mathbf{U})^{-1}$ *is Hermitian. If* $e^{i\alpha_1}, \ldots, e^{i\alpha_n}$ *are the eigenvalues of* $\mathbf{U}$ *then* $\cot\frac{\alpha_1}{2}, \ldots, \cot\frac{\alpha_n}{2}$ *are eigenvalues of* $\mathbf{H}$.

Solution.

$$\begin{aligned} (i\mathbf{H})(\mathbf{I}-\mathbf{U}) &= (\mathbf{I}+\mathbf{U}) \\ \Rightarrow (\mathbf{I}-\overline{\mathbf{U}}')\overline{(i\mathbf{H})}' &= (\mathbf{I}+\overline{\mathbf{U}}') \end{aligned}$$

Substituting $\mathbf{I} = \overline{\mathbf{U}}'\mathbf{U}$, we have from the second equation that $\overline{\mathbf{U}}'(\mathbf{U}-\mathbf{I})\overline{(i\mathbf{H})}' = \overline{\mathbf{U}}'(\mathbf{U}+\mathbf{I})$. So $\overline{(i\mathbf{H})}' = -i\overline{\mathbf{H}}' = -(\mathbf{I}+\mathbf{U})(\mathbf{I}-\mathbf{U})^{-1} = -i\mathbf{H}$, so $\overline{\mathbf{H}}' = \mathbf{H}$, thus $\mathbf{H}$ is Hermitian.

If an eigenvalue of a nonsingular matrix $\mathbf{A}$ is λ, then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ $\because$ $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \Rightarrow \lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}$, note that $\lambda \neq 0$ $\because$ $|\mathbf{A}| \neq 0$. Thus the eigenvalues of $\mathbf{H}$ are

$$\begin{aligned} & \frac{1}{i}\frac{1+e^{i\alpha_j}}{1-e^{i\alpha_j}}, 1 \leq j \leq n \\ = & -i\frac{e^{i\alpha_j/2}+e^{-i\alpha_j/2}}{e^{-i\alpha_j/2}-e^{i\alpha_j/2}}, 1 \leq j \leq n \\ = & \frac{\frac{e^{i\alpha_j/2}+e^{-i\alpha_j/2}}{2}}{\frac{e^{-i\alpha_j/2}-e^{i\alpha_j/2}}{2i}}, 1 \leq j \leq n \\ = & \frac{\cot\alpha_j}{2}, 1 \leq j \leq n \end{aligned}$$

■

Question 3(b) *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with distinct eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Show that if* $\mathbf{A}$ *is non-singular then there exist* 2^n *matrices* $\mathbf{X}$ *such that* $\mathbf{X}^2 = \mathbf{A}$. *What happens in case* $\mathbf{A}$ *is a singular matrix?*

Solution. There exists $\mathbf{P}$ non-singular such that $\mathbf{P^{-1}AP} = diagonal[\lambda_1, \ldots, \lambda_n]$.

Let $\mathbf{Y}_1 = diagonal[\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}]$, and let $\mathbf{X} = \mathbf{PYP}^{-1}$. Then $\mathbf{X}^2 = \mathbf{PYP}^{-1}\mathbf{PYP}^{-1} = \mathbf{PY^2P}^{-1} = \mathbf{A}$. Thus any of the 2^n matrices formed by choosing a sign for each of the diagonal entries from $\mathbf{X} = \mathbf{P}\ diagonal[\pm\sqrt{\lambda_1}, \ldots, \pm\sqrt{\lambda_n}]\ \mathbf{P}^{-1}$ has the same property (note that they are all distinct).

If one of the eigenvalues is zero, the number of matrices $\mathbf{X}$ would become 2^{n-1}, since we would have one less choice. ■

Question 3(c) *Show that a real quadratic* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *is positive definite if and only if there exists a non-singular matrix* $\mathbf{B}$ *such that* $\mathbf{A} = \mathbf{B}'\mathbf{B}$.

Solution. If $\mathbf{A} = \mathbf{B}'\mathbf{B}$, then $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}'\mathbf{B}\mathbf{x} = \mathbf{X}'\mathbf{X}$, where $\mathbf{X} = \mathbf{B}\mathbf{x}$. Now if $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{B}\mathbf{x} \neq \mathbf{0}$, as $\mathbf{B}$ is nonsingular, and 0 is not its eigenvalue. Thus $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{X}'\mathbf{X} > 0$, so $\mathbf{x}'\mathbf{A}\mathbf{x}$ is positive definite.

Conversely, see the result used in the solution of question 2(c), year 1992. ■

UPSC Civil Services Main 1992 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{U}$ and $\mathcal{V}$ be vector spaces over a field K and let $\mathcal{V}$ be of finite dimension. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{U}$ be a linear transformation, prove that* $\dim \mathcal{V} = \dim \mathbf{T}(\mathcal{V}) + \dim \text{nullity}\, \mathbf{T}$.

Solution. See question 3(a), year 1998. ∎

Question 1(b) *Let $\mathcal{S} = \{(x, y, z) \mid x + y + z = 0, x, y, z \in \mathbb{R}\}$. Prove that $\mathcal{S}$ is a subspace of $\mathbb{R}^3$. Find a basis of $\mathcal{S}$.*

Solution. $\mathcal{S} \neq \emptyset$ because $(0, 0, 0) \in \mathcal{S}$. If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathcal{S}$ then $\alpha_1(x_1, y_1, z_1) + \alpha_2(x_2, y_2, z_2) \in \mathcal{S}$ because $(\alpha_1 x_1 + \alpha_2 x_2) + (\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1(x_1 + y_1 + z_1) + \alpha_2(x_2 + y_2 + z_2) = 0$. Thus $\mathcal{S}$ is a subspace of $\mathbb{R}^3$.

Clearly $(1, 0, -1), (1, -1, 0) \in \mathcal{S}$ and are linearly independent. Thus $\dim \mathcal{S} \geq 2$. However $(1, 1, 1) \notin \mathcal{S}$, so $\mathcal{S} \neq \mathbb{R}^3$. Thus $\dim \mathcal{S} = 2$ and $\{(1, 0, -1), (1, -1, 0)\}$ is a basis for $\mathcal{S}$. ∎

Question 1(c) *Which of the following are linear transformations?*

1. $\mathbf{T} : \mathbb{R} \longrightarrow \mathbb{R}^2$ *defined by* $\mathbf{T}(x) = (2x, -x)$.
2. $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *defined by* $\mathbf{T}(x, y) = (xy, y, x)$.
3. $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *defined by* $\mathbf{T}(x, y) = (x + y, y, x)$.
4. $\mathbf{T} : \mathbb{R} \longrightarrow \mathbb{R}^2$ *defined by* $\mathbf{T}(x) = (1, -1)$.

Solution.

1.

$$\begin{aligned}
\mathbf{T}(\alpha x+\beta y) &= (2\alpha x+2\beta y, -\alpha x-\beta y)\\
&= (2\alpha x, -\alpha x)+(2\beta y, -\beta y)\\
&= \alpha\mathbf{T}(x)+\beta\mathbf{T}(y)
\end{aligned}$$

Thus $\mathbf{T}$ is a linear transformation.

2. $\mathbf{T}(2(1,1)) = \mathbf{T}(2,2) = (4,2,2) \neq 2\mathbf{T}(1,1) = 2(1,1,1)$ Thus $\mathbf{T}$ is not a linear transformation.

3.

$$\begin{aligned}
\mathbf{T}(\alpha(x_1,y_1)+\beta(x_2+y_2)) &= \mathbf{T}(\alpha x_1+\beta x_2, \alpha y_1+\beta y_2)\\
&= (\alpha x_1+\beta x_2+\alpha y_1+\beta y_2, \alpha y_1+\beta y_2, \alpha x_1+\beta x_2)\\
&= \alpha(x_1+y_1, y_1, x_1)+\beta(x_2+y_2, y_2, x_2)\\
&= \alpha\mathbf{T}(x_1,y_1)+\beta\mathbf{T}(x_2,y_2)
\end{aligned}$$

Thus $\mathbf{T}$ is a linear transformation.

4. $\mathbf{T}(2(0,0)) = \mathbf{T}(0,0) = (1,-1) \neq 2\mathbf{T}(0,0)$ Thus $\mathbf{T}$ is not a linear transformation.

$\blacksquare$

Question 2(a) *Let* $\mathbf{T} : \mathcal{M}_{2,1} \longrightarrow \mathcal{M}_{2,3}$ *be a linear transformation defined by (with the usual notation)*

$$\mathbf{T}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2&1&3\\4&1&5\end{pmatrix}, \mathbf{T}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}6&1&0\\0&0&2\end{pmatrix}$$

Find $\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix}$.

Solution.

$$\begin{aligned}
\begin{pmatrix}x\\y\end{pmatrix} &= x\begin{pmatrix}1\\0\end{pmatrix} - y\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix}\\
\mathbf{T}\begin{pmatrix}x\\y\end{pmatrix} &= (x-y)\begin{pmatrix}2&1&3\\4&1&5\end{pmatrix} + y\begin{pmatrix}6&1&0\\0&0&2\end{pmatrix} = \begin{pmatrix}2x+4y & x & 3x-3y\\4x-4y & x-y & 5x-3y\end{pmatrix}
\end{aligned}$$

$\blacksquare$

Question 2(b) *For what values of η do the following equations*

$$\begin{aligned} x+y+z &= 1 \\ x+2y+4z &= \eta \\ x+4y+10z &= \eta^2 \end{aligned}$$

have a solution? Solve them in each case.

Solution. Since the determinant of the coefficient matrix $\left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{smallmatrix}\right)$ is 0, the system has to be consistent to be solvable.

Clearly $x+4y+10z = 3(x+2y+4z) - 2(x+y+z)$. Thus for the system to be consistent we must have $\eta^2 = 3\eta - 2$, or $\eta = 1, 2$.

If $\eta = 1$, then $x+y+z = 1, x+2y+4z = 1$ so $y+3z = 0$, or $y = -3z, x = 1+2z$. Thus the space of solutions is $\{(1+2z, -3z, z) \mid z \in \mathbb{R}\}$. Note that the rank of the coefficient matrix is 2, and consequently the space of solutions is one dimensional.

If $\eta = 2$, then $x+y+z = 1, x+2y+4z = 2$, so $y+3z = 1$ or $y = 1-3z$, hence $x = 2z$. Consequently, the space of solutions is $\{(2z, 1-3z, z) \mid z \in \mathbb{R}\}$. ∎

Question 2(c) *Prove that a necessary and sufficient condition of a real quadratic form* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *to be positive definite is that the leading principal minors of* $\mathbf{A}$ *are all positive.*

Solution. Let all the principal minors be positive. We have to prove that the quadratic form is positive definite. We prove the result by induction.

If $n = 1$, then $a_{11}x^2 > 0 \Leftrightarrow a_{11} > 0$. Suppose as induction hypothesis the result is true for $n = m$. Let $\mathbf{S} = \left(\begin{smallmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}_1' & k \end{smallmatrix}\right)$ be a matrix of a quadratic form in $m+1$ variables, where $\mathbf{B}$ is $m \times m$, $\mathbf{B}_1$ is $m \times 1$ and k is a single element. Since all principle minors of $\mathbf{B}$ are leading principal minors of $\mathbf{S}$, and are hence positive, the induction hypothesis gives that $\mathbf{B}$ is positive definite. This means that there exists a non-singular $m \times m$ matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{B}\mathbf{P} = \mathbf{I}_m$ (We shall prove this presently). Let $\mathbf{C}$ be an m-rowed column to be determined soon. Then

$$\begin{pmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{C}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}'_1 & k \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{B}\mathbf{P} & \mathbf{P}'\mathbf{B}\mathbf{C} + \mathbf{P}'\mathbf{B}_1 \\ \mathbf{C}'\mathbf{B}'\mathbf{P} + \mathbf{B}_1'\mathbf{P} & \mathbf{C}'\mathbf{B}\mathbf{C} + \mathbf{C}'\mathbf{B}_1 + \mathbf{B}'_1\mathbf{C} + k \end{pmatrix}$$

Let $\mathbf{C}$ be so chosen that $\mathbf{B}\mathbf{C} + \mathbf{B}_1 = \mathbf{0}$, or $\mathbf{C} = -\mathbf{B}^{-1}\mathbf{B}_1$. Then

$$\begin{pmatrix} \mathbf{P}' & \mathbf{0} \\ \mathbf{C}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{B} & \mathbf{B}_1 \\ \mathbf{B}'_1 & k \end{pmatrix} \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{B}\mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}'_1\mathbf{C} + k \end{pmatrix}$$

Taking determinants, we get $|\mathbf{P}'||\mathbf{S}||\mathbf{P}| = \mathbf{B}'_1\mathbf{C} + k$, because $\mathbf{P}'\mathbf{B}\mathbf{P} = \mathbf{I}_m$, and $\mathbf{B}'_1\mathbf{C} + k$ is a single element. Since $|\mathbf{S}| > 0$, it follows that $\mathbf{B}'_1\mathbf{C} + k > 0$, so let $\mathbf{B}'_1\mathbf{C} + k = \alpha^2$. Then $\mathbf{Q}'\mathbf{S}\mathbf{Q} = \mathbf{I}_{m+1}$ with $\mathbf{Q} = \begin{pmatrix} \mathbf{P} & \mathbf{C} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{0} & \alpha^{-1} \end{pmatrix}$. Thus the quadratic forms of $\mathbf{S}$ and $\mathbf{I}_{m+1}$ take the same values. Hence $\mathbf{S}$ is positive definite, so the condition is sufficient.

The condition is necessary - Since $\mathbf{x}'\mathbf{A}\mathbf{x}$ is positive definite, there is a non-singular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{I} \Rightarrow |\mathbf{A}||\mathbf{P}|^2 = 1 \Rightarrow |\mathbf{A}| > 0$.

Let $1 \leq r < n$. Let $x_{r+1} = \ldots = x_n = 0$, then we obtain a quadratic form in r variables which is positive definite. Clearly the determinant of this quadratic form is the $r \times r$ principal minor of $\mathbf{A}$ which shows the result.

Proof of the result used: Let $\mathbf{A}$ be positive definite, then there exists a non-singular $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{I}$.

We will prove this by induction. If $n = 1$, then the form corresponding to $\mathbf{A}$ is $a_{11}x^2$ and $a_{11} > 0$, so that $\mathbf{P} = (\sqrt{a_{11}})$.

Take

$$\mathbf{P_1} = \begin{pmatrix} 1 & -a_{11}^{-1}a_{12} & 0 & \ldots & 0 \\ 0 & & & & \\ \vdots & & & (n-1)\times(n-1) & \\ 0 & & & & \end{pmatrix}$$

then

$$\mathbf{P_1'}\mathbf{A}\mathbf{P_1} = \begin{pmatrix} a_{11} & 0 & a_{13} & \ldots & a_{1n} \\ 0 & & & & \\ a_{13} & & & & \\ \vdots & & & (n-1)\times(n-1) & \\ a_{1n} & & & & \end{pmatrix}$$

Repeating this process, we get a non-singular $\mathbf{Q}$ such that

$$\mathbf{Q}'\mathbf{A}\mathbf{Q} = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ \vdots & & (n-1)\times(n-1) & \\ 0 & & & \end{pmatrix}$$

Given the $(n-1) \times (n-1)$ matrix on the lower right, we get by induction $\mathbf{P}^*$ s.t. $\mathbf{P}^{*\prime}((n-1) \times (n-1)$ matrix$)\mathbf{P}^*$ is diagonal. Thus $\exists \mathbf{P}, |\mathbf{P}| \neq 0, \mathbf{P}'\mathbf{A}\mathbf{P} = [\alpha_1, \ldots, \alpha_n]$ say. Take $\mathbf{R} =$ diagonal$[\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n}]$, then $\mathbf{R}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{R} = \mathbf{I_n}$.

∎

Question 3(a) *State the Cayley-Hamilton theorem and use it to find the inverse of* $\left(\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right)$.

Solution. Let $\mathbf{A}$ be an $n \times n$ matrix. If $|\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$ is the characteristic equation of $\mathbf{A}$, then the Cayley-Hamilton theorem says that $\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \ldots + a_n\mathbf{I} = \mathbf{0}$ i.e. a matrix satisfies its characteristic equation.

The characteristic equation of $\mathbf{A} = \left(\begin{smallmatrix} 2 & 1 \\ 4 & 3 \end{smallmatrix}\right)$ is

$$\begin{vmatrix} 2-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 2 = 0$$

By the Cayley-Hamilton theorem, $\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I} = \mathbf{0}$, so $\mathbf{A}(\mathbf{A} - 5\mathbf{I}) = -2\mathbf{I}$, thus $\mathbf{A}^{-1} = -\frac{1}{2}(\mathbf{A} - 5\mathbf{I})$. Thus

$$\mathbf{A}^{-1} = -\frac{1}{2}\left[\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}$$

∎

Question 3(b) *Transform the following into diagonal form*

$$x^2 + 2xy, 8x^2 - 4xy + 5y^2$$

and give the transformation employed.

Solution. Let $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right), \mathbf{B} = \left(\begin{smallmatrix} 8 & -2 \\ -2 & 5 \end{smallmatrix}\right)$

$$\text{Let } 0 = |\mathbf{A} - \lambda\mathbf{B}| = \begin{vmatrix} 1 - 8\lambda & 1 + 2\lambda \\ 1 + 2\lambda & -5\lambda \end{vmatrix} = -5\lambda + 40\lambda^2 - 4\lambda^2 - 4\lambda - 1$$

Thus $36\lambda^2 - 9\lambda - 1 = 0$, so $\lambda = \frac{9 \pm \sqrt{81+144}}{72} = \frac{1}{3}, -\frac{1}{12}$.

Let (x_1, x_2) be the vector such that $(\mathbf{A} - \lambda\mathbf{B})\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \mathbf{0}$ with $\lambda = \frac{1}{3}$. Thus $-\frac{5}{3}x_1 + \frac{5}{3}x_2 = 0 \Rightarrow x_1 = x_2$. We take $\mathbf{x_1} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ so that $(\mathbf{A} - \lambda\mathbf{B})\mathbf{x_1} = \mathbf{0}$ with $\lambda = \frac{1}{3}$. Similarly, if (x_1, x_2) is the vector such that $(\mathbf{A} - \lambda\mathbf{B})\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \mathbf{0}$ with $\lambda = -\frac{1}{12}$, then $\frac{5}{3}x_1 + \frac{5}{6}x_2 = 0$, so $2x_1 + x_2 = 0$. We take $\mathbf{x_2} = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$.

Now

$$\begin{aligned}
\mathbf{x_1'Ax_1} &= \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = 3 \\
\mathbf{x_2'Ax_2} &= \left(\begin{smallmatrix} 1 & -2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -2 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right) = -3 \\
\mathbf{x_1'Ax_2} &= \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right) = 0
\end{aligned}$$

If $\mathbf{P} = (\mathbf{x_1}\ \mathbf{x_2})$, then $\mathbf{P'AP} = \left(\begin{smallmatrix} 3 & 0 \\ 0 & -3 \end{smallmatrix}\right)$, thus $x^2 + 2xy \approx 3X^2 - 3Y^2$ by $\mathbf{P} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -2 \end{smallmatrix}\right)$.

Similarly

$$\begin{aligned}
\mathbf{x_1'Bx_1} &= \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 8 & -2 \\ -2 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 6 \\ 3 \end{smallmatrix}\right) = 9 \\
\mathbf{x_2'Bx_2} &= \left(\begin{smallmatrix} 1 & -2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 8 & -2 \\ -2 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 12 \\ -12 \end{smallmatrix}\right) = 36 \\
\mathbf{x_1'Bx_2} &= \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 8 & -2 \\ -2 & 5 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 12 \\ -12 \end{smallmatrix}\right) = 0
\end{aligned}$$

Thus $\mathbf{P'BP} = \left(\begin{smallmatrix} 9 & 0 \\ 0 & 36 \end{smallmatrix}\right)$, so $8x^2 - 4xy + 5y^2$ is transformed to $9X^2 + 36Y^2$ by $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \mathbf{P}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ ∎

Question 3(c) *Prove that the characteristic roots of a Hermitian matrix are all real, and the characteristic roots of a skew Hermitian matrix are all zero or pure imaginary.*

Solution. For Hermitian matrices, see question 2(c), year 1995.

If $\mathbf{H}$ is skew-Hermitian, then $i\mathbf{H}$ is Hermitian, because $\overline{(i\mathbf{H})}' = \bar{i}\overline{\mathbf{H}}' = -i\overline{\mathbf{H}}' = i\mathbf{H}$ as $\mathbf{H} = -\overline{\mathbf{H}}'$. Thus the eigenvalues of $i\mathbf{H}$ are real. Therefore the eigenvalues of $\mathbf{H}$ are $-ix$ where $x \in \mathbb{R}$. So they must be 0 (if $x = 0$) or pure imaginary. ∎

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June 14, 2007

Question 1(a) *Show that the set $S = \{(1,0,0),(1,1,0),(1,1,1),(0,1,0)\}$ spans the vector space $\mathbb{R}^3$ but is not a basis set.*

Solution. The vectors $(1,0,0),(0,1,0),(1,1,1)$ are linearly independent, because $\alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)=\mathbf{0} \Rightarrow \alpha+\gamma=0, \beta+\gamma=0, \gamma=0 \Rightarrow \alpha=\beta=\gamma=0$.

Thus $(1,0,0),(1,1,0),(1,1,1)$ is a basis of $\mathbb{R}^3$, as $\dim_{\mathbb{R}} \mathbb{R}^3 = 3$.

Any set containng a basis spans the space, so S spans $\mathbb{R}^3$, but it is not a basis because the four vectors are not linearly independent, in fact $(1,1,0)=(1,0,0)+(0,1,0)$. ∎

Question 1(b) *Define rank and nullity of a linear transformation. If $\mathcal{V}$ is a finite dimensional vector space and $\mathbf{T}$ is a linear operator on $\mathcal{V}$ such that $\operatorname{rank}\mathbf{T}^2 = \operatorname{rank}\mathbf{T}$, then prove that the null space of $\mathbf{T}$ is equal to the null space of $\mathbf{T}^2$, and the intersection of the range space and null space of $\mathbf{T}$ is the zero subspace of $\mathcal{V}$.*

Solution. The dimension of the image space $\mathbf{T}(\mathcal{V})$ is called rank of $\mathbf{T}$. The dimension of the vector space kernel of $\mathbf{T} = \{\mathbf{v} \mid \mathbf{T}(\mathbf{v}) = \mathbf{0}\}$ is called the nullity of $\mathbf{T}$.

Now $\mathbf{v} \in$ null space of $\mathbf{T} \Rightarrow \mathbf{T}(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{T}^2(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{v} \in$ null space of $\mathbf{T}^2$. Thus null space of $\mathbf{T} \subseteq$ null space of $\mathbf{T}^2$. But we are given that $\operatorname{rank}\mathbf{T} = \operatorname{rank}\mathbf{T}^2$, so therefore nullity of $\mathbf{T}$ = nullity of $\mathbf{T}^2$, because of the nullity theorem — $\operatorname{rank}\mathbf{T} + \operatorname{nullity}\mathbf{T} = \dim\mathcal{V}$. Thus null space of $\mathbf{T}$ = null space of $\mathbf{T}^2$.

Finally if $\mathbf{v} \in$ range of $\mathbf{T}$, and $\mathbf{v} \in$ null space of $\mathbf{T}$, then $\mathbf{v} = \mathbf{T}(\mathbf{w})$ for some $\mathbf{w} \in \mathcal{V}$. Now

$$\begin{aligned} \mathbf{T}^2(\mathbf{w}) = \mathbf{T}(\mathbf{v}) &\Rightarrow \mathbf{w} \in \text{ null space of } \mathbf{T}^2 \\ &\Rightarrow \mathbf{w} \in \text{ null space of } \mathbf{T} \\ &\Rightarrow \mathbf{0} = \mathbf{T}(\mathbf{w}) = \mathbf{v} \end{aligned}$$

Thus range of $\mathbf{T} \cap$ null space of $\mathbf{T} = \{\mathbf{0}\}$. ∎

Question 1(c) *If the matrix of a linear operator* $\mathbf{T}$ *on* $\mathbb{R}^2$ *relative to the standard basis* $\{(1,0),(0,1)\}$ *is* $\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$, *find the matrix of* $\mathbf{T}$ *relative to the basis* $\mathbf{B} = \{(1,1),(-1,1)\}$.

Solution. Let $\mathbf{v_1} = (1,1), \mathbf{v_2} = (-1,1)$. Then $\mathbf{T}(\mathbf{v_1}) = (11)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) = (2,2) = 2\mathbf{v_1}$. $\mathbf{T}(\mathbf{v_2}) = (-11)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right) = (0,0) = \mathbf{0}$. So $(\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}) = (\mathbf{v_1}\ \mathbf{v_2})\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)$, so the matrix of $\mathbf{T}$ relative to the basis $\mathbf{B}$ is $\left(\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right)$. ∎

Question 2(a) *Prove that the inverse of* $\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix}$ *is* $\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix}$ *where* $\mathbf{A}, \mathbf{C}$ *are nonsingular matrices. Hence find the inverse of* $\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}$.

Solution.

$$\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{B}\mathbf{A}^{-1} - \mathbf{B}\mathbf{A}^{-1} & \mathbf{I}\end{pmatrix} = \text{Identity matrix.}$$

$$\begin{pmatrix}\mathbf{A}^{-1} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} & \mathbf{C}^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{B} & \mathbf{C}\end{pmatrix} = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{C}^{-1}\mathbf{B} + \mathbf{C}^{-1}\mathbf{B} & \mathbf{I}\end{pmatrix} = \text{Identity matrix, which shows}$$

the result.

Let $\mathbf{A} = \mathbf{C} = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ and $\mathbf{B} = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$. Then $\mathbf{A}^{-1} = \mathbf{C}^{-1} = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ and $\mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1} = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ Thus

$$\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}^{-1} = \begin{pmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{pmatrix}$$

∎

Question 2(b) *If* $\mathbf{A}$ *is an orthogonal matrix with the property that* -1 *is not an eigenvalue, then show that* $\mathbf{A} = (\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}$ *for some skew symmetric matrix* $\mathbf{S}$.

Solution. We want $\mathbf{S}$ skew symmetric such that $\mathbf{A}(\mathbf{I} + \mathbf{S}) = \mathbf{I} - \mathbf{S}$ i.e. $\mathbf{A} + \mathbf{A}\mathbf{S} = \mathbf{I} - \mathbf{S}$ or $\mathbf{A}\mathbf{S} + \mathbf{S} = \mathbf{I} - \mathbf{A}$ or $(\mathbf{I} + \mathbf{A})\mathbf{S} = \mathbf{I} - \mathbf{A}$. Let $\mathbf{S} = (\mathbf{I} + \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})$, note that $\mathbf{I} + \mathbf{A}$ is invertible because if $|\mathbf{I} + \mathbf{A}| = 0$, then -1 will be an eigenvalue of $\mathbf{A}$.

Note that the two factors of $\mathbf{S}$ commute, because $(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A}) = \mathbf{I}-\mathbf{A}^2 = (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})$, so $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1} = (\mathbf{I} + \mathbf{A})^{-1}(\mathbf{I} - \mathbf{A})$.

Now

$$\begin{aligned}
\mathbf{S}' &= (\mathbf{I}-\mathbf{A})'((\mathbf{I}+\mathbf{A})^{-1})' \\
&= (\mathbf{I}-\mathbf{A}')(\mathbf{I}+\mathbf{A}')^{-1} \\
&= (\mathbf{A}\mathbf{A}'-\mathbf{A}')(\mathbf{A}'\mathbf{A}+\mathbf{A}')^{-1} \\
&= (\mathbf{A}-\mathbf{I})\mathbf{A}'\mathbf{A}'^{-1}(\mathbf{A}+\mathbf{I})^{-1} \\
&= -(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} \\
&= -(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A}) \\
&= -\mathbf{S}
\end{aligned}$$

Thus $\mathbf{S}$ is skew symmetric, so $\mathbf{A} = (\mathbf{I}-\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}$ where $\mathbf{S} = (\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})$ ■

Question 2(c) *Show that any two eigenvectors corresponding to distinct eigenvalues of (i) Hermitian matrices (ii) unitary matrices are orthogonal.*

Solution. We first prove that the eigenvalues of a Hermitian matrix, and therefore of a symmetric matrix, are real.

Let $\mathbf{H}$ be Hermitian, and λ be one of its eigenvalues. Let $\mathbf{x} \neq \mathbf{0}$ be an eigenvector corresponding to λ. Thus $\mathbf{H}\mathbf{x} = \lambda\mathbf{x}$, so $\overline{\mathbf{x}}'\mathbf{H}\mathbf{x} = \overline{\mathbf{x}}'\lambda\mathbf{x}$. But $\overline{(\overline{\mathbf{x}}'\mathbf{H}\mathbf{x})}' = (\mathbf{x}'\overline{\mathbf{H}}\overline{\mathbf{x}})' = \overline{\mathbf{x}}'\overline{\mathbf{H}}'\mathbf{x} = \overline{\mathbf{x}}'\mathbf{H}\mathbf{x}$, because $\overline{\mathbf{H}}' = \mathbf{H}$. Note that $(\overline{\mathbf{x}}'\mathbf{H}\mathbf{x})' = \overline{\mathbf{x}}'\mathbf{H}\mathbf{x}$, since it is a single element, therefore $\overline{\mathbf{x}}'\mathbf{H}\mathbf{x}$ is real. Similarly $\overline{\mathbf{x}}'\mathbf{x} \neq 0$ is real, so $\lambda = \frac{\overline{\mathbf{x}}'\mathbf{H}\mathbf{x}}{\overline{\mathbf{x}}'\mathbf{x}}$ is real.

Let $\mathbf{H}$ be Hermitian, $\mathbf{H}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, $\mathbf{H}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ with $\lambda_1 \neq \lambda_2$. Clearly $\overline{\mathbf{x}}_2'\mathbf{H}\mathbf{x}_1 = \lambda_1\overline{\mathbf{x}}_2'\mathbf{x}_1$, $\overline{\mathbf{x}}_1'\mathbf{H}\mathbf{x}_2 = \lambda_2\overline{\mathbf{x}}_1'\mathbf{x}_2$. But $(\overline{\overline{\mathbf{x}}_2'\mathbf{H}\mathbf{x}_1})' = \overline{\mathbf{x}}_1'\overline{\mathbf{H}}'\mathbf{x}_2 = \lambda_1\overline{\mathbf{x}}_1'\mathbf{x}_2$. So $\lambda_2\overline{\mathbf{x}}_1'\mathbf{x}_2 = \overline{\mathbf{x}}_1'\mathbf{H}\mathbf{x}_2 = \overline{\mathbf{x}}_1'\overline{\mathbf{H}}'\mathbf{x}_2 = \lambda_1\overline{\mathbf{x}}_1'\mathbf{x}_2$ because $\overline{\mathbf{H}}' = \mathbf{H}$. Since $\lambda_1 \neq \lambda_2$, $\overline{\mathbf{x}}_1'\mathbf{x}_2 = 0$, so $\mathbf{x}_1, \mathbf{x}_2$ are orthogonal.

Let $\mathbf{U}$ be unitary, $\mathbf{U}\mathbf{x}_1 = \lambda_1\mathbf{x}_1, \mathbf{U}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$, where λ_1, λ_2 are distinct eigenvalues of $\mathbf{U}$ with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$. Thus $\overline{\mathbf{x}}_2'\overline{\mathbf{U}}'\mathbf{U}\mathbf{x}_1 = \overline{\lambda}_2\overline{\mathbf{x}}_2'\lambda_1\mathbf{x}_1$. Since $\overline{\mathbf{U}}'\mathbf{U} = \mathbf{I}$, $\overline{\lambda}_2\overline{\mathbf{x}}_2'\lambda_1\mathbf{x}_1 = \overline{\mathbf{x}}_2'\mathbf{x}_1$, so $(1-\overline{\lambda}_2\lambda_1)(\overline{\mathbf{x}}_2'\mathbf{x}_1) = 0$. But $1-\overline{\lambda}_2\lambda_1 = \overline{\lambda}_2\lambda_2 - \overline{\lambda}_2\lambda_1 = \overline{\lambda}_2(\lambda_2-\lambda_1) \neq 0$[1]. Thus $\overline{\mathbf{x}}_2'\mathbf{x}_1 = 0$, so $\mathbf{x}_1, \mathbf{x}_2$ are orthogonal. ■

Question 3(a) *A matrix* $\mathbf{B}$ *of order* n *is of the form* $\lambda\mathbf{A}$, *where* λ *is a scalar and* $\mathbf{A}$ *has 1 everywhere except the diagonal, which has* μ. *Find* λ, μ *so that* $\mathbf{B}$ *may be orthogonal.*

Solution. $\mathbf{A} = \begin{pmatrix} \mu & 1 & \dots & 1 \\ 1 & \mu & \dots & 1 \\ \dots & & \dots & \\ 1 & 1 & \dots & \mu \end{pmatrix}$. $\mathbf{B} = \lambda\mathbf{A}$. Thus

$$\mathbf{B}'\mathbf{B} = \begin{pmatrix} \lambda\mu & \lambda & \dots & \lambda \\ \lambda & \lambda\mu & \dots & \lambda \\ \dots & & \dots & \\ \lambda & \lambda & \dots & \lambda\mu \end{pmatrix} \begin{pmatrix} \lambda\mu & \lambda & \dots & \lambda \\ \lambda & \lambda\mu & \dots & \lambda \\ \dots & & \dots & \\ \lambda & \lambda & \dots & \lambda\mu \end{pmatrix} = \mathbf{B}\mathbf{B}' = \mathbf{B}^2$$

[1]We used here the fact that all eigenvalues of a unitary matrix have modulus 1. If $\mathbf{U}\mathbf{x} = \lambda\mathbf{x}$, then $\overline{\mathbf{x}}'\overline{\mathbf{U}}' = \overline{\lambda}\overline{\mathbf{x}}'$. Thus $\overline{\mathbf{x}}'\overline{\mathbf{U}}'\mathbf{U}\mathbf{x} = \lambda\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}$, so $\overline{\mathbf{x}}'\mathbf{x} = \lambda\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}$. Now $\overline{\mathbf{x}}'\mathbf{x} \neq 0$, so $\lambda\overline{\lambda} = 1$.

Clearly each diagonal element of $\mathbf{BB}'$ is $\lambda^2\mu^2 + (n-1)\lambda^2$, and each nondiagonal element is $2\lambda^2\mu + (n-2)\lambda^2$. Thus $\mathbf{B}$ will be orthogonal if $2\lambda^2\mu + (n-2)\lambda^2 = 0, \lambda^2\mu^2 + (n-1)\lambda^2 = 1$. Since $\lambda \neq 0$, $\mu = \frac{2-n}{2} = 1 - \frac{n}{2}$, and $\lambda^2 = \frac{1}{(1-\frac{n}{2})^2+n-1} = \frac{1}{1-n+\frac{n^2}{4}+n-1} = \frac{4}{n^2}$, thus $\lambda = \pm\frac{2}{n}$. ∎

Question 3(b) *Find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix}$$

by reducing it to its normal form.

Solution.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{C_2} + \mathbf{C_1}, \mathbf{C_3} - 3\mathbf{C_1}, \mathbf{C_4} - 6\mathbf{C_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{R_2} - \mathbf{R_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 5 & 8 & -12 & -19 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operation $\mathbf{R_3} - 2\mathbf{R_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 5 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -3 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Interchanging $\mathbf{C_3}$ and $\mathbf{C_4}$ we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -10 & -6 \\ 5 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\mathbf{R_3} - 5\mathbf{R_1} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -10 & -6 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operation $\frac{1}{4}\mathbf{R_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -6 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Operation $\mathbf{C_3} + \frac{5}{2}\mathbf{C_2}, \mathbf{C_4} + \frac{3}{2}\mathbf{C_2} \Rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 1 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Thus the normal form of $\mathbf{A}$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ so $\operatorname{rank} A = 3$. $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ -3 & -2 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 1 & 1 & -\frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathbf{PAQ}$ is the normal form. ∎

Question 3(c) *Determine the following form as definite, semidefinite or indefinite*

$$2x_1^2 + 2x_2^2 + 3x_3^2 - 4x_2x_3 - 4x_3x_1 + 2x_1x_2$$

Solution. Completing the squares of the given form (say $Q(x_1, x_2, x_3)$):

$$\begin{aligned} Q(x_1, x_2, x_3) &= 2(x_1 + \frac{1}{2}x_2 - x_3)^2 + \frac{3}{2}x_2^2 + x_3^2 - 2x_2x_3 \\ &= 2(x_1 + \frac{1}{2}x_2 - x_3)^2 + (x_3 - x_2)^2 + \frac{1}{2}x_2^2 \end{aligned}$$

Thus Q can be written as the sum of 3 squares with positive coefficients, so it is positive definite. ∎

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Question 1(a) *Show that* $f_1(t) = 1, f_2(t) = t-2, f_3(t) = (t-2)^2$ *forms a basis of* $\mathcal{P}_3 = \{$*Space of polynomials of degree* $\leq 2\}$. *Express* $3t^2 - 5t + 4$ *as a linear combination of* f_1, f_2, f_3.

Solution. If $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \equiv 0$, then α_3 being the coefficient of t^2 is equal to 0. Then coefficient of t is α_2 so it must be 0, hence $\alpha_1 = 0$. Thus f_1, f_2, f_3 are linearly independent. Since $\{1, t, t^2\}$ is a basis for $\mathcal{P}_3$, its dimension is 3, hence f_1, f_2, f_3 is a basis of $\mathcal{P}_3$.

Now by Taylor's expansion $p(t) = 3t^2 - 5t + 4 = p(2) + p'(2)(t-2) + \frac{p"(2)}{2!}(t-2)^2 = 6 + 7(t-2) + 3(t-2)^2 = 6f_1 + 7f_2 + 3f_3$. ∎

Question 1(b) *Let* $\mathbf{T} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ *be defined by*

$$\mathbf{T}(a, b, c, d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d), a, b, c, d \in \mathbb{R}$$

Verify that $\text{rank}(\mathbf{T}) + \text{nullity}(\mathbf{T}) = \dim(\mathcal{V}_4(\mathbb{R})$.

Solution. Let

$$\begin{aligned}
\mathbf{T}(1,0,0,0) &= (1,1,1) = \mathbf{v_1} \\
\mathbf{T}(0,1,0,0) &= (-1,0,1) = \mathbf{v_2} \\
\mathbf{T}(0,0,1,0) &= (1,2,3) = \mathbf{v_3} \\
\mathbf{T}(0,0,0,1) &= (1,-1,-3) = \mathbf{v_4}
\end{aligned}$$

$\mathbf{T}(\mathbb{R}^4)$ is generated by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ and therefore a maximal independent subset of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ will form a basis of $\mathbf{T}(\mathbb{R}^4)$. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent because if $\alpha\mathbf{v_1} + \beta\mathbf{v_2} = \mathbf{0}$, then $\alpha - \beta = 0, \alpha = 0$ so $\alpha = \beta = 0$.

$\mathbf{v_3}$ is dependent on $\mathbf{v_1}, \mathbf{v_2}$, because if $\mathbf{v_3} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then $\alpha - \beta = 1, \alpha = 2, \alpha + \beta = 3 \Rightarrow \alpha = 2, \beta = 1 \therefore \mathbf{v_3} = 2\mathbf{v_1} + \mathbf{v_2}$.

$\mathbf{v_4}$ is dependent on $\mathbf{v_1}, \mathbf{v_2}$, because if $\mathbf{v_4} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then $\alpha - \beta = 1, \alpha = -1, \alpha + \beta = -3 \Rightarrow \alpha = -1, \beta = -2 \therefore \mathbf{v_4} = -\mathbf{v_1} - 2\mathbf{v_2}$.

Thus $\mathbf{v_1}, \mathbf{v_2}$ is a basis of $\mathbf{T}(\mathbb{R}^4)$, so $\operatorname{rank}\mathbf{T} = 2$.

Now $(a, b, c, d) \in \ker\mathbf{T} \Leftrightarrow a - b + c + d = 0, a + 2c - d = 0, a + b + 3c - 3d = 0$ Choosing particular values of a, b, c, d, we see that $(1, 2, 0, 1), (-1, 1, 1, 1) \in \ker\mathbf{T}$ and are linearly independent, so $\dim\ker\mathbf{T} \geq 2$. But $(1, 2, 0, 1), (-1, 1, 1, 1)$ generate $\ker\mathbf{T}$, because if $(a, b, c, d) \in \ker\mathbf{T}$, and $(a, b, c, d) = \alpha(1, 2, 0, 1) + \beta(-1, 1, 1, 1)$, then $\alpha - \beta = a, 2\alpha + \beta = b, \beta = c, \alpha + \beta = d$, so $\alpha = a + c, \beta = c$ and these satisfy the remaining equations $2\alpha + \beta = b, \alpha + \beta = d$, because $(a, b, c, d) \in \ker\mathbf{T}$ and therefore $a - b + c + d = 0, a + 2c - d = 0$. Thus $(a, b, c, d) = (a + c)(1, 2, 0, 1) + c(-1, 1, 1, 1)$, so $\dim\ker\mathbf{T} = \operatorname{nullity}\mathbf{T} = 2$

Hence $\operatorname{rank}\mathbf{T} + \operatorname{nullity}\mathbf{T} = 4 = \dim(\mathbb{R}^4)$, as required. ∎

Question 1(c) *If* $\mathbf{T}$ *is an operator on* $\mathbb{R}^3$ *whose basis is* $\mathbf{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *such that*

$$[\mathbf{T} : \mathbf{B}] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

find a matrix of $\mathbf{T}$ *w.r.t. a basis* $\mathbf{B_1} = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$.

Solution. The basis $\mathbf{B}$ is the standard basis, hence the representation of $\mathbf{B_1}$ in this basis is as given. (Note that if $\mathbf{B}$ were some other basis, we would write $\mathbf{B_1}$ in that basis, and then continue as below.) Let $\mathbf{v_1} = (0, 1, -1), \mathbf{v_2} = (1, -1, 1), \mathbf{v_3} = (-1, 1, 0)$. Then

$$\begin{aligned} \mathbf{T}(\mathbf{v_1}) &= (0, 1, -1) = \mathbf{v_1} \\ \mathbf{T}(\mathbf{v_2}) &= (0, 0, 0) = \mathbf{0} \\ \mathbf{T}(\mathbf{v_3}) &= (1, -1, 0) = -\mathbf{v_3} \end{aligned}$$

Thus

$$[\mathbf{T} : \mathbf{B_1}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note: The main idea behind the above solution is to express $\mathbf{T}(\mathbf{v_i}) = \sum_{i=0}^{n} \alpha_{ii}\mathbf{v_i}$. Now we solve for α_{ii} to get the matrix for $\mathbf{T}$ in the new basis.

An alternative is to compute $\mathbf{P}^{-1}[\mathbf{T} : \mathbf{B}]\mathbf{P}$, where $\mathbf{P}$ is given by $[\mathbf{v_1}, \dots, \mathbf{v_n}] = [\mathbf{e_1}, \dots, \mathbf{e_n}]\mathbf{P}$ or $\mathbf{P} = [\mathbf{v_1}', \dots, \mathbf{v_n}']$. Show that this is true. ∎

Question 2(a) *If* $\mathbf{A} = \langle a_{ij} \rangle$ *is an* $n \times n$ *matrix such that* $a_{ii} = n, a_{ij} = r$ *if* $i \neq j$, *show that*

$$[\mathbf{A} - (n - r)\mathbf{I}][\mathbf{A} - (n - r + nr)\mathbf{I}] = \mathbf{0}$$

Hence find the inverse of the $n \times n$ *matrix* $\mathbf{B} = \langle b_{ij} \rangle$ *where* $b_{ii} = 1, b_{ij} = \rho, i \neq j$ *and* $\rho \neq 1, \rho \neq \frac{1}{1-n}$.

Solution. Let $\mathbf{C} = \mathbf{A} - (n-r)\mathbf{I}$, then every entry of $\mathbf{C}$ is r. Let $\mathbf{D} = \mathbf{A} - (n-r+nr)\mathbf{I} = \mathbf{C} - nr\mathbf{I}$. Thus $\mathbf{CD} = \mathbf{C}^2 - nr\mathbf{C}$. Each entry of $\mathbf{C}^2$ is nr^2, which is the same as each entry of $nr\mathbf{C}$, so $\mathbf{CD} = \mathbf{0}$ as required.

The given equation implies

$$\mathbf{A}[\mathbf{A} - (2n - 2r + nr)\mathbf{I}] = -(n-r)(n-r-nr)\mathbf{I}$$

Let $\mathbf{A} = n\mathbf{B}$, where $r = \rho n$. Thus $\mathbf{A}$ satisfies the conditions for the equation to hold, so substituting $\mathbf{A}$ and r in the above equation

$$\begin{aligned} n\mathbf{B}[n\mathbf{B} - (2n - 2n\rho + n^2\rho)\mathbf{I}] &= -(n - n\rho)(n - n\rho - n^2\rho)\mathbf{I} \\ \mathbf{B}[\mathbf{B} - (2 - 2\rho + n\rho)\mathbf{I}] &= -(1-\rho)(1-\rho-n\rho)\mathbf{I} \\ \mathbf{B}^{-1} &= -\frac{1}{(1-\rho)(1-\rho-n\rho)}[\mathbf{B} - (2 - 2\rho + n\rho)\mathbf{I}] \end{aligned}$$

Thus the diagonal elements of $\mathbf{B}^{-1}$ are all $\frac{1-2\rho+n\rho}{(1-\rho)(1-\rho-n\rho)}$, while the off-diagonal elements are all $\frac{2-3\rho+n\rho}{(1-\rho)(1-\rho-n\rho)}$. ∎

Question 2(b) *Prove that the eigenvectors corresponding to distinct eigenvalues of a square matrix are linearly independent.*

Solution. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix $\mathbf{A}$. Let $a_1\mathbf{x_1} + \ldots + a_r\mathbf{x_r} = \mathbf{0}, a_i \in \mathbb{R}$. We shall show that $a_i = 0, 1 \leq i \leq r$.

Let $\mathbf{L_1} = \prod_{i=2}^{r}(\mathbf{A} - \lambda_i\mathbf{I})$. Note that the factors of $\mathbf{L_1}$ commute. Thus $\mathbf{L_1}\mathbf{x_2} = \prod_{i=3}^{r}(\mathbf{A} - \lambda_i\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x_2} = \mathbf{0}$ because $\mathbf{A}\mathbf{x_2} = \lambda_2\mathbf{x_2}$. Similarly $\mathbf{L_1}\mathbf{x_3} = \ldots = \mathbf{L_1}\mathbf{x_r} = \mathbf{0}$. Moreover $\mathbf{L_1}\mathbf{x_1} = (\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_r)\mathbf{x_1}$.

Consequently

$$\begin{aligned} \mathbf{0} &= \mathbf{L_1}(a_1\mathbf{x_1} + \ldots + a_r\mathbf{x_r}) \\ &= a_1\mathbf{L_1}\mathbf{x_1} \\ &= a_1(\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_r)\mathbf{x_1} \end{aligned}$$

$\lambda_1 - \lambda_i \neq 0, 2 \leq i \leq r$, and $\mathbf{x_1} \neq 0$ so $a_1 = 0$.

Similarly taking $\mathbf{L_i} = \prod_{\substack{j=1 \\ i \neq j}}^{r}(\mathbf{A} - \lambda_j\mathbf{I})$, we show that $a_i = 0$ for $1 \leq i \leq r$. Thus $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ are linearly independent. ∎

Question 2(c) *Determine the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

Solution. The characteristic polynomial of $\mathbf{A}$ is $|\lambda\mathbf{I} - \mathbf{A}| = (\lambda - 3)(\lambda - 2)(\lambda - 5)$.[1] Thus the eigenvalues of $\mathbf{A}$ are 3, 2, 5.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 3$ then

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_2 + 4x_3 = 0, -x_2 + 6x_3 = 0, 2x_3 = 0$, take $x_1 = 1, x_2 = 0$ to get $(1, 0, 0)$ as an eigenvector for $\lambda = 3$. All the eigenvectors are $(x_1, 0, 0), x_1 \neq 0$.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 2$ then

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + x_2 + 4x_3 = 0, 6x_3 = 0, 3x_3 = 0$, take $x_1 = 1$ to get $(1, -1, 0)$ as an eigenvector for $\lambda = 2$. All the eigenvectors are $(x_1, -x_1, 0), x_1 \neq 0$.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 5$ then

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \begin{pmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 + x_2 + 4x_3 = 0, -3x_2 + 6x_3 = 0$, take $x_3 = 1$ to get $(3, 2, 1)$ as an eigenvector for $\lambda = 5$. All eigenvectors are $(3x_3, 2x_3, x_3), x_3 \neq 0$. ∎

Question 3(a) *Show that the matrix congruent to a skew symmetric matrix is skew symmetric. Use the result to prove that the determinant of a skew symmetric matrix of even order is the square of a rational function of its elements.*

Solution. Let $\mathbf{B} = \mathbf{P}'\mathbf{A}\mathbf{P}$, be congruent to $\mathbf{A}$, and $\mathbf{A}' = -\mathbf{A}$. Then $\mathbf{B}' = \mathbf{P}'\mathbf{A}'\mathbf{P} = -\mathbf{P}'\mathbf{A}\mathbf{P} = -\mathbf{B}$, so $\mathbf{B}$ is skew symmetric.

We prove the second result by induction on m, where $n = 2m$ is the order of the skew symmetric matrix under consideration. If $m = 1$ then $\mathbf{A} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, |\mathbf{A}| = a^2$, so the result is true for $m = 1$.

Assume by induction that the result is true for all skew symmetric matrices of even order $< 2m$. If $\mathbf{A} \equiv \mathbf{0}$, there is nothing to prove. Otherwise there exists at least one non-zero element a_{ij}. Changing the first row with row j, we move a_{ij} in the first row. Changing

[1] Note that the determinant of an upper diagonal or a lower diagonal matrix is just the product of the elements on the main diagonal.

column 1 and column j, we get $-a_{ij}$ in the symmetric position. Now by multiplying the new matrix by suitable elementary matrices on the left and right, we get

$$\mathbf{P'AP} = \begin{pmatrix} 0 & a_{ij} & * & * & \dots & * \\ -a_{ij} & 0 & * & * & \dots & * \\ * & * & & & & \\ \dots & & & & \mathbf{A_{2m-2}} & \\ * & * & & & & \end{pmatrix}$$

Now we can find $\mathbf{P}^*$ a product of elementary matrices such that

$$\mathbf{P^{*'}P'APP^*} = \begin{pmatrix} 0 & a_{ij} & 0 & 0 & \dots & 0 \\ -a_{ij} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & & & & \\ \dots & & & & \mathbf{A_{2m-2}} & \\ 0 & 0 & & & & \end{pmatrix}$$

Thus $\det \mathbf{A}$ = determinant of a skew symmetric matrix of order 2 $\times$ determinant of a skew symmetric matrix of order $2(m-1)$. The induction hypothesis now gives the result. ∎

Question 3(b) *Find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{pmatrix}$$

where $aa' + bb' + cc' = 0, a, b, c$ *positive integers.*

Solution. The rank of a skew symmetric matrix is always even[2]. Since $\left|\begin{smallmatrix} 0 & c \\ -c & 0 \end{smallmatrix}\right| = c^2 > 0, \operatorname{rank} \mathbf{A} \geq 2$.

If $c' \neq 0$, then

$$-c'|\mathbf{A}| = \begin{vmatrix} 0 & c & -b & a' \\ cc' & 0 & ac' & b'c' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{vmatrix}$$

Adding $b'\mathbf{R_3} - a\mathbf{R_4}$ to $\mathbf{R_2}$, all the entries of the second row become 0, so $-c'|\mathbf{A}| = 0 \Rightarrow |\mathbf{A}| = 0 \Rightarrow \operatorname{rank} \mathbf{A} < 4 \Rightarrow \operatorname{rank} \mathbf{A} = 2$.

[2]We can prove this by induction on the order of the skew symmetric matrix $\mathbf{S}$. It is true if $\mathbf{S}$ is a 1×1 matrix, since it must be the 0-matrix, thus has rank 0. Now given an $(n+1) \times (n+1)$ matrix $\mathbf{S}$, we can write it as $\left(\begin{smallmatrix} \mathbf{C} & \mathbf{b} \\ -\mathbf{b'} & 0 \end{smallmatrix}\right)$, where $\mathbf{C}$ is skew symmetric, and hence has even rank by the induction hypothesis. If $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{C}$, then by a series of elementary operations on $\mathbf{S}$, we can transform it into $\mathbf{P'SP} = \left(\begin{smallmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0'} & 0 \end{smallmatrix}\right)$, so $\operatorname{rank} \mathbf{S} = \operatorname{rank} \mathbf{P'SP} = \operatorname{rank} \mathbf{C}$. If $\mathbf{b}$ is linearly independent of the columns of $\mathbf{C}$, then $\mathbf{b}$' is linearly independent of the rows of $\mathbf{C}$, so $\operatorname{rank} \mathbf{S} = \operatorname{rank}\left(\begin{smallmatrix} \mathbf{C} & \mathbf{b} \\ -\mathbf{b'} & 0 \end{smallmatrix}\right) = \operatorname{rank}[\mathbf{C}\ \mathbf{b}] + 1(\because [-\mathbf{b'}0]$ is independent of the rows of $[\mathbf{C}\ \mathbf{b}]) = \operatorname{rank} \mathbf{C} + 2$, which is also even.

If $c' = 0$,

$$a|\mathbf{A}| = \begin{vmatrix} 0 & c & -b & a'a \\ c & 0 & a & b'a \\ b & -a & 0 & 0 \\ -a'a & -b'a & 0 & 0 \end{vmatrix}$$

Adding $-b'\mathbf{R_3}$ to $\mathbf{R_4}$, we see that the fourth row has all 0's, hence rank $\mathbf{A} = 2$ as before.

Alternate solution:

$$-a|\mathbf{A}| = \begin{pmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ aa' & ab' & ac' & 0 \end{pmatrix}$$

Add $-c'\mathbf{R_2}$ and $b'\mathbf{R_3}$ to $\mathbf{R_4}$, then all entries of the last row become 0. So rank $\mathbf{A} < 4$, and by the reasoning above, rank $\mathbf{A} \geq 2$, rank $\mathbf{A} \neq 3$ so rank $\mathbf{A} = 2$. ∎

Question 3(c) *Reduce the following symmetric matrix to a diagonal form and interpret the results in terms of quadratic forms.*

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

Solution.

$$\begin{aligned} (x\ y\ z)\mathbf{A}\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) &= 3x^2 + 2y^2 + z^2 + 4xy - 2xz + 6yz \\ &= 3(x + \frac{2}{3}y - \frac{1}{3}z)^2 + \frac{2}{3}y^2 + \frac{2}{3}z^2 + \frac{22}{3}yz \\ &= 3(x + \frac{2}{3}y - \frac{1}{3}z)^2 + \frac{2}{3}(y + \frac{11}{2}z)^2 - \frac{117}{6}z^2 \\ &= 3X^2 + \frac{2}{3}Y^2 - \frac{117}{6}Z^2 \end{aligned}$$

where $X = x - \frac{2}{3}y - \frac{1}{3}z, Y = y + \frac{11}{2}z, Z = z$. This implies $z = Z, y = Y - \frac{11}{2}Z, x = X + \frac{2}{3}(Y - \frac{11}{2}Z) - \frac{1}{3}Z = X + \frac{2}{3}Y - 4Z$.

Then if $\mathbf{P} = \begin{pmatrix} 1 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{pmatrix}$, we have $\mathbf{P'AP} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{117}{6} \end{pmatrix}$.

The quadratic form associated with $\mathbf{A}$ is indefinite as it takes both positive and negative values. Note that $\mathbf{x'Ax}$ and $\mathbf{x'P'APx'}$ take the same values. ∎

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1 Linear Algebra

Question 1(a) *Let* $\mathbf{T}(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$ *be a linear transformation on* $\mathbb{R}^3$. *What is the matrix of* $\mathbf{T}$ *w.r.t. the standard basis? What is a basis of the range space of* $\mathbf{T}$*? What is a basis of the null space of* $\mathbf{T}$*?*

Solution.

$$\begin{aligned}
\mathbf{T}(\mathbf{e_1}) &= \mathbf{T}(1,0,0) = (3,-2,-1) = 3\mathbf{e_1} - 2\mathbf{e_2} - \mathbf{e_3} \\
\mathbf{T}(\mathbf{e_2}) &= \mathbf{T}(0,1,0) = (0,1,2) = \mathbf{e_2} + 2\mathbf{e_3} \\
\mathbf{T}(\mathbf{e_3}) &= \mathbf{T}(0,0,1) = (1,0,4) = \mathbf{e_1} + 4\mathbf{e_3} \\
\mathbf{T} \Longleftrightarrow \mathbf{A} &= \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}
\end{aligned}$$

Clearly $\mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. If $(3,-2,-1) = \alpha(0,1,2) + \beta(1,0,4)$, then $\beta = 3, \alpha = -2$, but $2\alpha + 4\beta \neq -1$, so $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. Thus $(3,-2,-1), (0,1,2), (1,0,4)$ is a basis of the range space of $\mathbf{T}$.

Note that $\mathbf{T}(x_1, x_2, x_3) = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0$, so the null space of $\mathbf{T}$ is $\{\mathbf{0}\}$, and the empty set is a basis. Note that the matrix of $\mathbf{T}$ is nonsingular, so $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. ∎

Question 1(b) *Let* $\mathbf{A}$ *be a square matrix of order* n. *Prove that* $\mathbf{Ax} = \mathbf{b}$ *has a solution* $\Leftrightarrow \mathbf{b} \in \mathbb{R}^n$ *is orthogonal to all solutions* $\mathbf{y}$ *of the system* $\mathbf{A'y} = \mathbf{0}$.

Solution. If $\mathbf{x}$ is a solution of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{y}$ is a solution of $\mathbf{A}'\mathbf{y} = \mathbf{0}$, then $\mathbf{b}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{y} = 0$, thus $\mathbf{b}$ is orthogonal to $\mathbf{y}$.

Conversely, suppose $\mathbf{b}'\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbb{R}^n$ which is a solution of $\mathbf{A}'\mathbf{y} = \mathbf{0}$. Let $\mathcal{W} = \mathbf{A}(\mathbb{R}^n) =$ the range space of $\mathbf{A}$, and $\mathcal{W}^\perp$ its orthogonal complement. If $\mathbf{A}'\mathbf{y} = \mathbf{0}$ then $\mathbf{x}'\mathbf{A}'\mathbf{y} = 0 \Rightarrow (\mathbf{Ax})'\mathbf{y} = 0$ for every $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{y} \in \mathcal{W}^\perp$. Conversely $\mathbf{y} \in \mathcal{W}^\perp \Rightarrow \forall \mathbf{x} \in \mathbb{R}^n.(\mathbf{Ax})'\mathbf{y} = 0 \Rightarrow \mathbf{x}'\mathbf{A}'\mathbf{y} = 0 \Rightarrow \mathbf{A}'\mathbf{y} = \mathbf{0}$. Thus $\mathcal{W}^\perp = \{\mathbf{y} \mid \mathbf{A}'\mathbf{y} = \mathbf{0}\}$. Now $\mathbf{b}'\mathbf{y} = 0$ for all $\mathbf{y} \in \mathcal{W}^\perp$, so $\mathbf{b} \in \mathcal{W} \Rightarrow \mathbf{b} = \mathbf{Ax}$ for some $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{Ax} = \mathbf{b}$ is solvable. ∎

Question 1(c) *Define a similar matrix and prove that two similar matrices have the same characteristic equation. Write down a matrix having* $1, 2, 3$ *as eigenvalues. Is such a matrix unique?*

Solution. Two matrices $\mathbf{A}, \mathbf{B}$ are said to be similar if there exists a matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. If $\mathbf{A}, \mathbf{B}$ are similar, say $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$, then characteristic polynomial of $\mathbf{B}$ is $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}\lambda\mathbf{IP} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. (Note that $|\mathbf{X}||\mathbf{Y}| = |\mathbf{XY}|$.) Thus the characteristic polynomial of $\mathbf{B}$ is the same as that of $\mathbf{A}$.

Clearly the matrix $\mathbf{A} = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$ has eigenvalues 1,2,3. Such a matrix is not unique, for example $\mathbf{B} = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right)$ has the same eigenvalues, but $\mathbf{B} \neq \mathbf{A}$. ∎

Question 2(a) *Show that*

$$\mathbf{A} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

is diagonalizable and hence determine $\mathbf{A}^5$.

Solution.

$$\begin{array}{rrcl} & |\mathbf{A} - \lambda\mathbf{I}| & = & 0 \\ \Rightarrow & \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} & = & 0 \\ \Rightarrow & (5-\lambda)[(4-\lambda)(-4-\lambda)+12] + 6[4+\lambda-6] - 6[6-3(4-\lambda)] & = & 0 \\ \Rightarrow & (5-\lambda)[\lambda^2-4] + 6[\lambda-2-3\lambda+6] & = & 0 \\ \Rightarrow & -\lambda^3+5\lambda^2+4\lambda-20-12\lambda+24 & = & 0 \\ \Rightarrow & \lambda^3-5\lambda^2+8\lambda-4 & = & 0 \end{array}$$

Thus $\lambda = 1, 2, 2$.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 1$, then

$$\begin{array}{rrcl} & \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \mathbf{0} \\ \Rightarrow & 4x_1 - 6x_2 - 6x_3 & = & 0 \\ & -x_1 + 3x_2 + 2x_3 & = & 0 \\ & 3x_1 - 6x_2 - 5x_3 & = & 0 \end{array}$$

Thus $x_1 = x_3, x_3 = -3x_2$, so $(-3, 1, -3)$ is an eigenvector for $\lambda = 1$.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 2$, then

$$\begin{aligned}
\begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \mathbf{0} \\
\Rightarrow \quad 3x_1 - 6x_2 - 6x_3 &= 0 \\
-x_1 + 2x_2 + 2x_3 &= 0 \\
3x_1 - 6x_2 - 6x_3 &= 0
\end{aligned}$$

Thus $x_1 - 2x_2 - 2x_3 = 0$, so taking $x_1 = 0, x_2 = 1$, $(0, 1, -1)$ is an eigenvector for $\lambda = 2$. Taking $x_1 = 4, x_2 = 1$, $(4, 1, 1)$ is another eigenvector for $\lambda = 2$, and these two are linearly independent.

Let $\mathbf{P} = \begin{pmatrix} -3 & 0 & 4 \\ 1 & 1 & 1 \\ -3 & -1 & 1 \end{pmatrix}$. A simple calculation shows that $\mathbf{P}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix}$.

Clearly $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Now $\mathbf{P}^{-1}\mathbf{A}^5\mathbf{P} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix}$.

$$\begin{aligned}
\mathbf{A}^5 &= \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix} \mathbf{P}^{-1} \\
&= \frac{1}{2} \begin{pmatrix} -3 & 0 & 4 \\ 1 & 1 & 1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} -3 & 0 & 128 \\ 1 & 32 & 32 \\ -3 & -32 & 32 \end{pmatrix} \begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 125 & -186 & -186 \\ -31 & 94 & 62 \\ 93 & -186 & -154 \end{pmatrix}
\end{aligned}$$

Note: Another way of computing $\mathbf{A}^5$ is given below. This uses the characteristic polynomial of $\mathbf{A} : \mathbf{A}^3 = 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}$ and not the diagonal form, so it will *not* be permissible here.

$$\begin{aligned}
\mathbf{A}^5 &= \mathbf{A}^2(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) \\
&= 5\mathbf{A}(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) - 8(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) + 4\mathbf{A}^2 \\
&= 25(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) - 76\mathbf{A}^2 + 84\mathbf{A} - 32\mathbf{I} \\
&= 49\mathbf{A}^2 - 116\mathbf{A} + 68\mathbf{I}
\end{aligned}$$

Now calculate $\mathbf{A}^2$ and substitute. ∎

Question 2(b) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices of order* n. *If* $\mathbf{I} - \mathbf{AB}$ *is invertible, then* $\mathbf{I} - \mathbf{BA}$ *is also invertible and*

$$(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$$

Show that $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same characteristic values.*

Solution.

$$\begin{array}{rll}
& (\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\mathbf{I} - \mathbf{BA}) & \\
& = \mathbf{I} - \mathbf{BA} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{ABA} & \\
& = [\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}] - \mathbf{B}[\mathbf{I} + (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB}]\mathbf{A} & (1) \\
\text{Now} & (\mathbf{I} - \mathbf{AB})^{-1}(\mathbf{I} - \mathbf{AB}) = (\mathbf{I} - \mathbf{AB})^{-1} - (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I} & \\
\therefore & (\mathbf{I} - \mathbf{AB})^{-1} = \mathbf{I} + (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} & \\
\text{Substituting in (1)} & (\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\mathbf{I} - \mathbf{BA}) & \\
& = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} = \mathbf{I} &
\end{array}$$

Thus $\mathbf{I} - \mathbf{BA}$ is invertible and $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$ as desired.

We shall show that $\lambda\mathbf{I} - \mathbf{AB}$ is invertible if and only if $\lambda\mathbf{I} - \mathbf{BA}$ is invertible. This means that if λ is an eigenvalue of $\mathbf{AB}$, then $|\lambda\mathbf{I} - \mathbf{AB}| = 0 \Rightarrow |\lambda\mathbf{I} - \mathbf{BA}| = 0$ so λ is an eigenvalue of $\mathbf{BA}$.

If $\lambda\mathbf{I} - \mathbf{AB}$ is invertible, then

$$\begin{array}{rll}
& (\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\lambda\mathbf{I} - \mathbf{BA}) & \\
& = \lambda\mathbf{I} - \mathbf{BA} + \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{ABA} & \\
& = \lambda[\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}] - \mathbf{B}[\mathbf{I} + (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB}]\mathbf{A} & (2) \\
\text{Now} & (\lambda\mathbf{I} - \mathbf{AB})^{-1}(\lambda\mathbf{I} - \mathbf{AB}) = \lambda(\lambda\mathbf{I} - \mathbf{AB})^{-1} - (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I} & \\
\therefore & \lambda(\lambda\mathbf{I} - \mathbf{AB})^{-1} = \mathbf{I} + (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} & \\
\text{Substituting in (2)} & (\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\lambda\mathbf{I} - \mathbf{BA}) & \\
& = \lambda\mathbf{I} + \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} = \lambda\mathbf{I} &
\end{array}$$

Thus $\lambda\mathbf{I} - \mathbf{BA}$ is invertible if $\lambda\mathbf{I} - \mathbf{AB}$ is invertible. The converse is obvious as the situation is symmetric, thus $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

We give another simple proof of the fact that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

1. Let 0 be an eigenvalue of $\mathbf{AB}$. This means that $\mathbf{AB}$ is singular, i.e. $0 = |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{BA}|$, so $\mathbf{BA}$ is singular, hence 0 is an eigenvalue of $\mathbf{BA}$.

2. Let $\lambda \neq 0$ be an eigenvalue of $\mathbf{AB}$ and let $\mathbf{x} \neq \mathbf{0}$ be an eigenvector corresponding to λ, i.e. $\mathbf{ABx} = \lambda\mathbf{x}$. Let $\mathbf{y} = \mathbf{Bx}$. Then $\mathbf{y} \neq \mathbf{0}$, because $\mathbf{Ay} = \mathbf{ABx} = \lambda\mathbf{x} \neq \mathbf{0}$ as $\lambda \neq 0$. Now $\mathbf{BAy} = \mathbf{BABx} = \mathbf{B}(\mathbf{ABx}) = \lambda\mathbf{Bx} = \lambda\mathbf{y}$. Thus λ is an eigenvalue of $\mathbf{BA}$.

∎

Question 2(c) *Let $a, b \in \mathbb{C}, |b| = 1$ and let $\mathbf{H}$ be a Hermitian matrix. Show that the eigenvalues of $a\mathbf{I} + b\mathbf{H}$ lie on a straight line in the complex plane.*

Solution. Let t be as eigenvalue of $\mathbf{H}$, which has to be real because $\mathbf{H}$ is Hermitian. Clearly $a + tb$ is an eigenvalue of $a\mathbf{I} + b\mathbf{H}$. Conversely, if λ is an eigenvalue of $a\mathbf{I} + b\mathbf{H}$, then $\frac{\lambda - a}{b}$ (note $b \neq 0$ as $|b| = 1$) is an eigenvalue of $\mathbf{H}$.

Clearly $a + tb$ lies on the straight line joining points a and $a + b$:

$$z = (1 - x)a + x(b - a), \;\; x \in \mathbb{R}$$

For the sake of completeness, we prove that the eigenvalues of a Hermitian matrix $\mathbf{H}$ are real. Let $\mathbf{z} \neq \mathbf{0}$ be an eigenvector corresponding to the eigenvalue t.

$$\begin{aligned}
\mathbf{Hz} &= t\mathbf{z} \\
\Rightarrow \overline{\mathbf{z}}'\mathbf{Hz} &= t\overline{\mathbf{z}}'\mathbf{z} \\
\Rightarrow \overline{\overline{\mathbf{z}}'\mathbf{Hz}}' &= \bar{t}\overline{\mathbf{z}}'\mathbf{z} \\
\text{But } \overline{\overline{\mathbf{z}}'\mathbf{Hz}} &= \overline{\mathbf{z}}'\overline{\mathbf{H}}'\mathbf{z} = \overline{\mathbf{z}}'\mathbf{Hz} = t\overline{\mathbf{z}}'\mathbf{z} \\
\Rightarrow t\overline{\mathbf{z}}'\mathbf{z} &= \bar{t}\overline{\mathbf{z}}'\mathbf{z} \\
\Rightarrow t &= \bar{t} \quad \because \overline{\mathbf{z}}'\mathbf{z} \neq 0
\end{aligned}$$

∎

Question 3(a) *Let $\mathbf{A}$ be a symmetric matrix. Show that $\mathbf{A}$ is positive definite if and only if its eigenvalues are all positive.*

Solution. $\mathbf{A}$ is real symmetric so all eigenvalues of $\mathbf{A}$ are real. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $\mathbf{A}$, not necessarily distinct. Let $\mathbf{x_1}$ be an eigenvector corresponding to λ_1. Since λ_1 and $\mathbf{A}$ are real, $\mathbf{x_1}$ is also real. Replacing $\mathbf{x_1}$ if necessary by $\mu\mathbf{x_1}$, μ suitable, we can assume that $||\mathbf{x_1}|| = \sqrt{\mathbf{x_1'x_1}} = 1$.

Let $\mathbf{P_1}$ be an orthogonal matrix with $\mathbf{x_1}$ as its first column. Such a $\mathbf{P_1}$ exists, as will be shown at the end of this result. Clearly the first column of the matrix $\mathbf{P_1}^{-1}\mathbf{AP_1}$ is equal to $\mathbf{P_1}^{-1}\mathbf{Ax} = \lambda_1\mathbf{P_1}^{-1}\mathbf{x} = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{pmatrix}$, because $\mathbf{P_1}^{-1}\mathbf{x}$ is the first column of $\mathbf{P_1}^{-1}\mathbf{P} = \mathbf{I}$. Thus $\mathbf{P_1}^{-1}\mathbf{AP_1} = \begin{pmatrix} \lambda_1 & \mathbf{L} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \mathbf{P_1'AP_1}$ where $\mathbf{B}$ is $(n-1) \times (n-1)$ symmetric. Since $\mathbf{P_1'AP_1}$ is symmetric, it follows that $\mathbf{P_1}^{-1}\mathbf{AP_1} = \mathbf{P_1'AP_1} = \begin{pmatrix} \lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}$. Induction now gives that there exists an $(n-1) \times (n-1)$ orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q'BQ} = \begin{pmatrix} \lambda_2 & 0 & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$

where $\lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of $\mathbf{B}$. Let $\mathbf{P_2} = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{smallmatrix}\right)$, then $\mathbf{P_2}$ is orthogonal and $\mathbf{P_2'P_1'AP_1P_2} = \text{diagonal}[\lambda_1, \ldots, \lambda_n]$. Set $\mathbf{P} = \mathbf{P_1P_2 \ldots P_n}$, and $(y_1, \ldots, y_n)\mathbf{P'} = \mathbf{x}$ then $\mathbf{x'Ax} = \mathbf{y'P'APy} = \sum_{i=0}^n \lambda_i^2 y_i^2$.

Since $\mathbf{P}$ is non-singular, quadratic forms $\mathbf{x'Ax}$ and $\sum_{i=0}^n \lambda_i^2 y_i^2$ assume the same values. Hence $\mathbf{A}$ is positive definite if and only if $\sum_{i=0}^n \lambda_i^2 y_i^2$ is positive definite if and only if $\lambda_i > 0$ for all i.

Result used: If $\mathbf{x_1}$ is a real vector such that $||\mathbf{x_1}|| = \sqrt{\mathbf{x_1'x_1}} = 1$ then there exists an orthogonal matrix with $\mathbf{x_1}$ as its first column.

Proof: We have to find real column vectors $\mathbf{x_2}, \ldots, \mathbf{x_n}$ such that $||\mathbf{x_i}|| = 1, 2 \leq i \leq n$ and $\mathbf{x_2}, \ldots, \mathbf{x_n}$ is an orthonormal system i.e. $\mathbf{x_i'x_j} = 0, i \neq j$. Consider the single equation $\mathbf{x_1'x} = 0$, where $\mathbf{x}$ is a column vector to be determined. This equation has a non-zero solution, in fact the space of solutions is of dimension $n-1$, the rank of the coefficient matrix being 1. If $\mathbf{y_2}$ is a solution, we take $\mathbf{x_2} = \frac{\mathbf{y_2}}{||\mathbf{y_2}||}$ so that $\mathbf{x_1'x_2} = 0$.

We now consider the two equations $\mathbf{x_1'x} = 0$, $\mathbf{x_2'x} = 0$. Again the number of unknowns is more than the number of equations, so there is a solution, say $\mathbf{y_3}$, and take $\mathbf{x_3} = \frac{\mathbf{y_3}}{||\mathbf{y_3}||}$ to get $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ mutually orthogonal.

Proceeding in this manner, if we consider $n-1$ equations $\mathbf{x_1'x} = 0, \ldots, \mathbf{x_{n-1}'x} = 0$, these will have a nonzero solution $\mathbf{y_n}$, so we set $\mathbf{x_n} = \frac{\mathbf{y_n}}{||\mathbf{y_n}||}$. Clearly $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ is an orthonormal system, and therefore $\mathbf{P} = [\mathbf{x_1}, \ldots, \mathbf{x_n}]$ is an orthogonal matrix having $\mathbf{x_1}$ as a first column. ∎

Question 3(b) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be square matrices of order* n*, show that* $\mathbf{AB} - \mathbf{BA}$ *can never be equal to the identity matrix.*

Solution. Let $\mathbf{A} = \langle a_{ij} \rangle$ and $\mathbf{B} = \langle b_{ij} \rangle$. Then

$$\begin{aligned} \operatorname{tr} \mathbf{AB} &= \text{Sum of diagonal elements of } \mathbf{AB} \\ &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} = \sum_{k=1}^n \sum_{i=1}^n b_{ki} a_{ik} = \operatorname{tr} \mathbf{BA} \end{aligned}$$

Thus $\operatorname{tr}(\mathbf{AB} - \mathbf{BA}) = \operatorname{tr} \mathbf{AB} - \operatorname{tr} \mathbf{BA} = 0$. But the trace of the identity matrix is n, thus $\mathbf{AB} - \mathbf{BA}$ can never be equal to the identity matrix. ∎

Question 3(c) *Let* $\mathbf{A} = \langle a_{ij} \rangle, 1 \leq i, j \leq n$. *If* $\sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}| < |a_{ii}|$, *then the eigenvalues of* $\mathbf{A}$ *lie in the disc*

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}|$$

Solution. See the solution to question 2(c), year 1997. We showed that if $|\lambda - a_{ii}| > \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|$ then $|\lambda\mathbf{I} - \mathbf{A}| \neq 0$, so λ is not an eigenvalue of $\mathbf{A}$. Thus if λ is an eigenvalue, then $|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|$, so λ lies in the disc described in the question. ∎

UPSC Civil Services Main 1996 - Mathematics Linear Algebra

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Question 1(a) *In $\mathbb{R}^4$ let $\mathcal{W}_1$ be the space generated by $\{(1,1,0,-1),(2,4,6,0)\}$ and let $\mathcal{W}_2$ be space generated by $\{(-1,-2,-2,2),(4,6,4,-6),(1,3,4,-3)\}$. Find a basis for the space $\mathcal{W}_1+\mathcal{W}_2$.*

Solution. Let $\mathbf{v_1}=(1,1,0,-1)$, $\mathbf{v_2}=(2,4,6,0)$, $\mathbf{v_3}=(-1,-2,-2,2)$, $\mathbf{v_4}=(4,6,4,-6)$, $\mathbf{v_5}=(1,3,4,-3)$. Since $\mathbf{w}\in\mathcal{W}_1+\mathcal{W}_2$ can be written as $\mathbf{w}=\mathbf{w_1}+\mathbf{w_2}$, and $\mathbf{w_1}=\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v_2}$ and $\mathbf{w_2}=\alpha_3\mathbf{v_3}+\alpha_4\mathbf{v_4}+\alpha_5\mathbf{v_5}$, it follows that $\mathbf{w}$ is a linear combination of $\mathbf{v_i}\Rightarrow\mathcal{W}_1+\mathcal{W}_2$ is generated by $\{\mathbf{v_i},1\le i\le 5\}$. Thus a maximal independent subset of $\{\mathbf{v_i},1\le i\le 5\}$ will be a basis of $\mathcal{W}_1+\mathcal{W}_2$.

Clearly $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent. If possible, let $\mathbf{v_3}=\lambda_1\mathbf{v_1}+\lambda_2\mathbf{v_2}$, then the four equations

$$\begin{aligned}\lambda_1+2\lambda_2 &= -1\\ \lambda_1+4\lambda_2 &= -2\\ 0\lambda_1+6\lambda_2 &= -2\\ -\lambda_1+0\lambda_2 &= 2\end{aligned}$$

should be consistent and provide us λ_1,λ_2. Clearly the third and fourth equations give us $\lambda_1=-2,\lambda_2=-\frac{1}{3}$ which do not satisfy the first two equations. Thus $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ are linearly independent.

If possible let $\mathbf{v_4}=\lambda_1\mathbf{v_1}+\lambda_2\mathbf{v_2}+\lambda_3\mathbf{v_3}$. Then

$$\lambda_1+2\lambda_2-\lambda_3=4 \tag{1a}$$
$$\lambda_1+4\lambda_2-2\lambda_3=6 \tag{1b}$$
$$0\lambda_1+6\lambda_2-2\lambda_3=4 \tag{1c}$$
$$-\lambda_1+0\lambda_2+2\lambda_3=-6 \tag{1d}$$

Adding (1b) and (1d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. Solving (1a) and (1b) we get $\lambda_3 = -2, \lambda_1 = 2$. These values satisfy all the four equations, so $\mathbf{v_4} = 2\mathbf{v_1} - 2\mathbf{v_3}$.

If possible let $\mathbf{v_5} = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \lambda_3\mathbf{v_3}$. Then

$$\lambda_1 + 2\lambda_2 - \lambda_3 = 1 \tag{2a}$$
$$\lambda_1 + 4\lambda_2 - 2\lambda_3 = 3 \tag{2b}$$
$$0\lambda_1 + 6\lambda_2 - 2\lambda_3 = 4 \tag{2c}$$
$$-\lambda_1 + 0\lambda_2 + 2\lambda_3 = -3 \tag{2d}$$

Adding (2b) and (2d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. (2c) then gives us $\lambda_3 = -2$, and (2a) now gives $\lambda_1 = -1$, which satisfies all equations. Thus $\mathbf{v_5} = -\mathbf{v_1} - 2\mathbf{v_3}$. Hence $\{(1,1,0,-1),(2,4,6,0),(-1,-2,-2,2)\}$ is a basis of $\mathcal{W}_1 + \mathcal{W}_2$. ∎

Question 1(b) *Let $\mathcal{V}$ be a finite dimensional vector space and $\mathbf{v} \in \mathcal{V}, \mathbf{v} \neq \mathbf{0}$. Show that there exists a linear functional f on $\mathcal{V}$ such that $f(\mathbf{v}) \neq 0$.*

Solution. Complete $\mathbf{v}$ to a basis of $\mathcal{V}$, say $\{\mathbf{v_1} = \mathbf{v}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$, where $\dim \mathcal{V} = n$. Define $f(\mathbf{v_j}) = \delta_{1j}$ and $f(\sum_{j=1}^{n} a_j\mathbf{v_j}) = \sum_{j=1}^{n} a_j f(\mathbf{v_j})$.

Clearly f is a linear functional over $\mathcal{V}$, and $f(\mathbf{v}) = f(\mathbf{v_1}) = 1$. Note that $f(\mathbf{v_j}) = 0, j > 1$ and if any $\mathbf{w} \in \mathcal{V}, \mathbf{w} = \sum_i a_i\mathbf{v_i}, f(\mathbf{w}) = a_1$. ∎

Question 1(c) *Let $\mathcal{V} = \mathbb{R}^3, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ be a basis of $\mathcal{V}$. Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be such that $\mathbf{T}(\mathbf{v_i}) = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}, 1 \leq i \leq 3$. By writing the matrix of $\mathbf{T}$ w.r.t. another basis show that the matrices*

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are similar.

Solution. Clearly $\mathbf{A}$ is the matrix of $\mathbf{T}$ w.r.t. the basis $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. Note that

$$[\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}), \mathbf{T}(\mathbf{v_3})] = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})\mathbf{A}$$

Let

$$\begin{aligned} \mathbf{w_1} &= \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3} \\ \mathbf{w_2} &= \mathbf{v_1} - \mathbf{v_2} \\ \mathbf{w_3} &= \mathbf{v_2} - \mathbf{v_3} \\ \Rightarrow \mathbf{T}(\mathbf{w_1}) &= 3\mathbf{w_1}, \mathbf{T}(\mathbf{w_2}) = \mathbf{T}(\mathbf{w_3}) = 0 \end{aligned}$$

We now show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is a basis for $\mathcal{V}$, i.e. these are linearly independent.

Let $\alpha\mathbf{w_1} + \beta\mathbf{w_2} + \gamma\mathbf{w_3} = \mathbf{0}$, then $(\alpha+\beta)\mathbf{v_1} + (\alpha-\beta+\gamma)\mathbf{v_2} + (\alpha-\gamma)\mathbf{v_3} = \mathbf{0}$. But $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent, therefore $\alpha+\beta = 0, \alpha-\beta+\gamma = 0, \alpha-\gamma = 0 \Rightarrow \alpha = \beta = \gamma = 0 \Rightarrow \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are linearly independent.

The matrix of $\mathbf{T}$ w.r.t. the basis $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is clearly $\mathbf{B}$. Note that the choice of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is suggested by the shape of $\mathbf{B}$.

If $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})\mathbf{P}, |\mathbf{P}| \neq 0$ then $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, so $\mathbf{A}$ and $\mathbf{B}$ are similar. ∎

Question 2(a) *Let $\mathcal{V} = \mathbb{R}^3$ and $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{V}$ be a linear map defined by*

$$\mathbf{T}(x, y, z) = (x + z, -2x + y, -x + 2y + z)$$

What is the matrix of $\mathbf{T}$ w.r.t. the basis $(1, 0, 1), (-1, 1, 1), (0, 1, 1)$? Using this matrix write down the matrix of $\mathbf{T}$ with respect to the basis $(0, 1, 2), (-1, 1, 1), (0, 1, 1)$.

Solution. Let $\mathbf{v_1} = (1, 0, 1), \mathbf{v_2} = (-1, 1, 1), \mathbf{v_3} = (0, 1, 1)$. $\mathbf{T}(x, y, z) = (x+z, -2x+y, -x+2y+z) = \alpha\mathbf{v_1}+\beta\mathbf{v_2}+\gamma\mathbf{v_3}$, say. This means $\alpha-\beta = x+z, \beta+\gamma = -2x+y, \alpha+\beta+\gamma = -x+2y+z$. This implies $\alpha = x+y+z, \beta = y, \gamma = -2x$. Thus $\mathbf{T}(x, y, z) = (x+y+z)\mathbf{v_1} + y\mathbf{v_2} - 2x\mathbf{v_3}$. Hence

$$[\mathbf{T}(\mathbf{v_1})\ \mathbf{T}(\mathbf{v_2})\ \mathbf{T}(\mathbf{v_3})] = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}$$

Let $\mathbf{w_1} = (0, 1, 2), \mathbf{w_2} = (-1, 1, 1), \mathbf{w_3} = (0, 1, 1)$. Then

$$[\mathbf{w_1}\ \mathbf{w_2}\ \mathbf{w_3}] = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}] \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} [\mathbf{T}(\mathbf{w_1})\ \mathbf{T}(\mathbf{w_2})\ \mathbf{T}(\mathbf{w_3})] &= [\mathbf{T}(\mathbf{v_1})\ \mathbf{T}(\mathbf{v_2})\ \mathbf{T}(\mathbf{v_3})]\mathbf{P} \\ &= [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}]\mathbf{A}\mathbf{P} \\ &= [\mathbf{w_1}\ \mathbf{w_2}\ \mathbf{w_3}]\mathbf{P}^{-1}\mathbf{A}\mathbf{P} \end{aligned}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the matrix of $\mathbf{T}$ w.r.t. basis $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

∎

Question 2(b) *Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional vector spaces such that $\dim\mathcal{V} \geq \dim\mathcal{W}$. Show that there is always a linear map of $\mathcal{V}$ onto $\mathcal{W}$.*

Solution. Let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}$ be a basis of $\mathcal{W}$, and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ be a basis of $\mathcal{V}, n \geq m$. Define

$$\begin{aligned} \mathbf{T}(\mathbf{v_i}) &= \mathbf{w_i}, \quad i = 1, 2, \ldots, m \\ \mathbf{T}(\mathbf{v_i}) &= \mathbf{0}, \quad i = m+1, \ldots, n \end{aligned}$$

and for any $\mathbf{v} \in \mathcal{V}, \mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}, \mathbf{T}(\mathbf{v}) = \sum_{i=1}^{m} \alpha_i \mathbf{T}(\mathbf{v_i})$.

Clearly $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is linear. $\mathbf{T}$ is onto, since if $\mathbf{w} \in \mathcal{W}, \mathbf{w} = \sum_{i=1}^{m} a_i \mathbf{w_i}$, then $\mathbf{T}(\sum_{i=1}^{m} a_i \mathbf{v_i}) = \sum_{i=1}^{m} a_i \mathbf{T}(\mathbf{v_i}) = \mathbf{w}$, proving the result. ∎

Question 2(c) *Solve by Cramer's rule*

$$\begin{aligned} x + y - 2z &= 1 \\ 2x - 7z &= 3 \\ x + y - z &= 5 \end{aligned}$$

Solution.

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -7 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -2 \\ -5 & -7 & -7 \\ 0 & 0 & -1 \end{vmatrix} = -2$$

$$x = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 5 & 1 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 4 & 0 & 1 \end{vmatrix}}{D} = \frac{-31}{-2} = \frac{31}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 1 & 5 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 1 & 4 & 1 \end{vmatrix}}{D} = \frac{13}{-2} = -\frac{13}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 4 \end{vmatrix}}{D} = \frac{-8}{-2} = 4$$

∎

Question 3(a) *Find the inverse of the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

by computing its characteristic polynomial.

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}|\mathbf{A}-\lambda\mathbf{I}| &= \begin{vmatrix}-\lambda & 1 & 0 & 0\\ 0 & -\lambda & 1 & 0\\ 0 & 0 & -\lambda & 1\\ 1 & 0 & 0 & -\lambda\end{vmatrix}\\ &= -\lambda[-\lambda^3]-1[1]=\lambda^4-1=0\end{aligned}$$

Thus by the Cayley-Hamilton theorem, $\mathbf{A}^4=\mathbf{I}$, so $\mathbf{A}^{-1}=\mathbf{A}^3$.

$$\mathbf{A}^2=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

$$\mathbf{A}^3=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}=\mathbf{A}^{-1}$$

∎

Question 3(b) *If $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ matrices such that $\mathbf{AB}=\mathbf{BA}$, show that $\mathbf{AB}$ and $\mathbf{BA}$ have a common characteristic vector.*

Solution. Let λ be any eigenvalue of $\mathbf{A}$ and let $\mathcal{V}_\lambda$ be the eigenspace of $\mathbf{A}$ corresponding to λ. We show that $\mathbf{B}(\mathcal{V}_\lambda)\subseteq\mathcal{V}_\lambda$. Let $\mathbf{v}\in\mathcal{V}_\lambda$, then $\mathbf{A}(\mathbf{Bv})=\mathbf{B}(\mathbf{Av})=\mathbf{B}(\lambda\mathbf{v})=\lambda\mathbf{Bv}\Rightarrow\mathbf{Bv}\in\mathcal{V}_\lambda$.

Consider $\mathbf{B}^*:\mathcal{V}_\lambda\longrightarrow\mathcal{V}_\lambda$ such that $\mathbf{B}^*(\mathbf{v})=\mathbf{B}(\mathbf{v})$ — note that $\mathbf{B}^*$ is a restriction of $\mathbf{B}$ to $\mathcal{V}_\lambda$ and we have already shown that $\mathbf{B}(\mathcal{V}_\lambda)\subseteq\mathcal{V}_\lambda$.

Let μ be an eigenvalue of $\mathbf{B}^*$, then μ is also an eigenvalue of $\mathbf{B}$ (because a basis of $\mathcal{V}_\lambda$ can be extended to a basis of $\mathcal{V}$, and in this basis $\mathbf{B}=\left(\begin{smallmatrix}\mathbf{B}^* & \mathbf{C}\\ \mathbf{0} & \mathbf{D}\end{smallmatrix}\right)$ for some matrices $\mathbf{C},\mathbf{D}$). Let $\mathbf{v}\in\mathcal{V}_\lambda$ be an eigenvector of $\mathbf{B}^*$ corresponding to μ, by definition $\mathbf{v}\neq\mathbf{0}$. Then $\mathbf{Bv}=\mathbf{B}^*\mathbf{v}=\mu\mathbf{v}$. Thus $\mathbf{A}$ and $\mathbf{B}$ have a common eigenvector $\mathbf{v}$, note that $\mathbf{Av}=\lambda\mathbf{v}$ as $\mathbf{v}\in\mathcal{V}_\lambda$. ∎

Question 3(c) *Reduce to canonical form the orthogonal matrix*

$$\mathbf{O}=\begin{pmatrix}\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3}\end{pmatrix}$$

Solution. Before solving this particular problem, we present a general discussion about orthogonal matrices. An orthogonal matrix satisfies $\mathbf{O}'\mathbf{O}=\mathbf{I}$, so its determinant is 1 or -1, here we focus on the case where $|\mathbf{O}|=1$. If λ is an eigenvalue of $\mathbf{O}$ and $\mathbf{x}$ a corresponding eigenvector, then $|\lambda|^2\mathbf{x}'\mathbf{x}=(\mathbf{Ox})'\mathbf{Ox}=\mathbf{x}'\mathbf{O}'\mathbf{Ox}=\mathbf{x}'\mathbf{x}$, so $|\lambda|=1$. Since the characteristic polynomial has real coefficients, the eigenvalues must be real or in complex conjugate pairs. Thus for a matrix of order 3, at least one eigenvalue is real, and must be 1 or -1. Since $|\mathbf{O}| = 1$, one real value must be 1, and the three possibilities are $\{1,1,1\}, \{1,-1,-1\}$ and $\{1, e^{i\theta}, e^{-i\theta}\}$.

Here we consider the third case, as the given matrix has 1 and $\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ as eigenvalues, proved later.

Let $\mathbf{Z} = \mathbf{X_1} + i\mathbf{X_2}$ be an eigenvector corresponding to the eigenvalue $e^{i\theta}$. Let $\mathbf{X_3}$ be the eigenvector corresponding to the eigenvalue 1. Since $\mathbf{Z}$ and $\mathbf{X_3}$ correspond to different eigenvalues, these are orthogonal, i.e. $\mathbf{Z}'\mathbf{X_3} = (\mathbf{X_1'} + i\mathbf{X_2'})\mathbf{X_3} = 0 \Rightarrow \mathbf{X_1'X_3} = 0, \mathbf{X_2'X_3} = 0$. Note that $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}$ are real vectors. Since $\mathbf{OZ} = e^{i\theta}\mathbf{Z} = (\cos\theta + i\sin\theta)(\mathbf{X_1} + i\mathbf{X_2})$. Equating real and imaginary parts we get

$$\begin{aligned}
\mathbf{OX_1} &= \mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta \\
\mathbf{OX_2} &= \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta \\
\therefore \mathbf{X_1'O'OX_1} &= (\mathbf{X_1'}\cos\theta - \mathbf{X_2'}\sin\theta)(\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta) \\
\Rightarrow \quad \mathbf{X_1'X_1} &= \mathbf{X_1'X_1}\cos^2\theta - \mathbf{X_2'X_1}\cos\theta\sin\theta - \mathbf{X_1'X_2}\sin\theta\cos\theta + \mathbf{X_2'X_2}\sin^2\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\sin^2\theta - \mathbf{X_2'X_2}\sin^2\theta + 2\mathbf{X_1'X_2}\cos\theta\sin\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\sin\theta - \mathbf{X_2'X_2}\sin\theta + 2\mathbf{X_1'X_2}\cos\theta \quad (1)
\end{aligned}$$

(Note that $\sin\theta \neq 0$ since we are considering the case where $e^{i\theta}$ is complex.) Similarly

$$\begin{aligned}
\mathbf{X_2'O'OX_1} &= (\mathbf{X_1'}\sin\theta + \mathbf{X_2'}\cos\theta)(\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta) \\
\Rightarrow \quad \mathbf{X_2'X_1} &= \mathbf{X_1'X_1}\sin\theta\cos\theta - \mathbf{X_1'X_2}\sin^2\theta - \mathbf{X_2'X_2}\sin\theta\cos\theta + \mathbf{X_2'X_1}\cos^2\theta \\
\Rightarrow \quad 0 &= \mathbf{X_1'X_1}\cos\theta - \mathbf{X_1'X_2}\cos\theta - 2\mathbf{X_1'X_2}\sin\theta \quad (2)
\end{aligned}$$

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding, we get $\mathbf{X_1'X_1} - \mathbf{X_2'X_2} = 0$ or $\mathbf{X_1'X_1} = \mathbf{X_2'X_2}$, so from (2), $\mathbf{X_1X_2} = 0$, i.e. $\mathbf{X_1}, \mathbf{X_2}$ are orthogonal.

Thus $\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}$ are mutually orthogonal. We can assume that $\mathbf{X_1'X_1} = \mathbf{X_2'X_2} = 1$, replacing $\mathbf{Z}$ by $\lambda\mathbf{Z}, \lambda \in \mathbb{R}$ if necessary. Similarly we can take $\mathbf{X_3'X_3} = 1$. Let $\mathbf{P} = [\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}]$ so that $\mathbf{P'P} = \mathbf{I}$. Now

$$\begin{aligned}
\mathbf{O}[\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}] &= [\mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta, \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta, \mathbf{X_3}] \\
&= [\mathbf{X_1}\ \mathbf{X_2}\ \mathbf{X_3}] \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow \mathbf{P}^{-1}\mathbf{OP} = \mathbf{P'OP} &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

which is the canonical form of $\mathbf{O}$ when the eigenvalues are $1, e^{i\theta}, e^{-i\theta}$.

Solution of given problem.

$$\begin{aligned}
\mathbf{O} &= \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\
|\mathbf{O} - \lambda\mathbf{I}| &= \begin{vmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} - \lambda & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} - \lambda \end{vmatrix} = \frac{1}{27}\begin{vmatrix} 2-3\lambda & -2 & 1 \\ 2 & 1-3\lambda & -2 \\ 1 & 2 & 2-3\lambda \end{vmatrix} \\
&= \frac{1}{27}[(2-3\lambda)^2(1-3\lambda) + 4(2-3\lambda) + 1(3+3\lambda) + 2(6-6\lambda)] \\
&= -\frac{1}{27}[27\lambda^3 - 45\lambda^2 + 45\lambda - 27] \\
&= -\frac{1}{3}[(\lambda-1)(3\lambda^2 - 2\lambda - 3)]
\end{aligned}$$

Thus $\lambda = 1,\ \frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ are eigenvalues of $\mathbf{O}$.

Thus the canonical form of $\mathbf{O}$ is derived from above, where $\cos\theta = \frac{1}{3}, \sin\theta = \frac{2\sqrt{2}}{3}$:

$$\begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $\mathbf{P}$ can be determined as follows (this is not needed for this problem, but is given for completeness):

1. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$, then

$$\begin{aligned}
-\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 0 \\
\frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}x_3 &= 0 \\
\frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 &= 0
\end{aligned}$$

   Thus $x_2 = 0, x_1 - x_3 = 0$, so we can take $(1, 0, 1)$ as an eigenvector.

2. The vectors $\mathbf{X_1}, \mathbf{X_2}$ in the above discussion are determined by the requirements

$$\begin{aligned}
\mathbf{OX_1} &= \mathbf{X_1}\cos\theta - \mathbf{X_2}\sin\theta \\
\mathbf{OX_2} &= \mathbf{X_1}\sin\theta + \mathbf{X_2}\cos\theta
\end{aligned}$$

where $\cos\theta = \frac{1}{3}, \sin\theta = \frac{2\sqrt{2}}{3}$. This gives us the following equations

$$\begin{aligned}
2x_{11} - 2x_{12} + x_{13} &= x_{11} - x_{21}2\sqrt{2} && (3)\\
2x_{11} + x_{12} - 2x_{13} &= x_{12} - x_{22}2\sqrt{2} && (4)\\
x_{11} + 2x_{12} + 2x_{13} &= x_{13} - x_{23}2\sqrt{2} && (5)\\
2x_{21} - 2x_{22} + x_{23} &= x_{11}2\sqrt{2} + x_{21} && (6)\\
2x_{21} + x_{22} - 2x_{23} &= x_{12}2\sqrt{2} + x_{22} && (7)\\
x_{21} + 2x_{22} + 2x_{23} &= x_{13}2\sqrt{2} + x_{23} && (8)
\end{aligned}$$

Adding the last 3 equations, we get $\sqrt{2}x_{21} = x_{11} + x_{12} + x_{13}$. Subtracting equation (6) from (8), $\sqrt{2}x_{22} = x_{13} - x_{11}$, and from (7) $\sqrt{2}x_{23} = x_{11} - x_{12} + x_{13}$. Substituting these in the first 3 equations and simplifying, we get $x_{11} = -x_{13}$. Setting $x_{11} = 0, x_{12} = 1$, we get $(0,1,0), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ as a possible solution for $\mathbf{X_1}, \mathbf{X_2}$.

Putting these together we get

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

We can now verify that $\mathbf{OP} = \mathbf{P}\begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ∎

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1 Linear Algebra

Question 1(a) *Let $\mathcal{V}$ be the vector space of polynomials over $\mathbb{R}$. Find a basis and the dimension of $\mathcal{W} \subseteq \mathcal{V}$ spanned by*

$$
\begin{aligned}
v_1 &= t^3 - 2t^2 + 4t + 1 \\
v_2 &= 2t^3 - 3t^2 + 9t - 1 \\
v_3 &= t^3 + 6t - 5 \\
v_4 &= 2t^3 - 5t^2 + 7t + 5
\end{aligned}
$$

Solution. v_1 and v_2 are linearly independent, because if $\alpha v_1 + \beta v_2 = 0$, then $\alpha + 2\beta = 0,\ -2\alpha - 3\beta = 0,\ 4\alpha + 9\beta = 0,\ \alpha - \beta = 0 \Rightarrow \alpha = \beta = 0$.

v_3 depends linearly on v_1, v_2 — if $\alpha v_1 + \beta v_2 = v_3$, then $\alpha + 2\beta = 1,\ -2\alpha - 3\beta = 0,\ 4\alpha + 9\beta = 6,\ \alpha - \beta = -5 \Rightarrow \alpha = -3, \beta = 2$ which satisfy all the equations. Thus $v_3 = -3v_1 + 2v_2$.

v_4 depends linearly on v_1, v_2 — if $\alpha v_1 + \beta v_2 = v_4$, then $\alpha + 2\beta = 2,\ -2\alpha - 3\beta = -5,\ 4\alpha + 9\beta = 7,\ \alpha - \beta = 5 \Rightarrow \alpha = 4, \beta = -1$ which satisfy all the equations. Thus $v_4 = 4v_1 - v_2$.

Thus $\dim_{\mathbb{R}} \mathcal{W} = 2$ and v_1, v_2 is a basis of $\mathcal{W}$. ∎

Question 1(b) *Verify that $\mathbf{T}(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. Find its range, rank, null space and nullity.*

Solution. Let $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$. Then

$$\begin{aligned}
\mathbf{T}(\alpha\mathbf{x} + \beta\mathbf{y}) &= \mathbf{T}(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\
&= (\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \alpha x_2 + \beta y_2) \\
&= (\alpha(x_1 + x_2), \alpha(x_1 - x_2), \alpha x_2) + (\beta(y_1 + y_2), \beta(y_1 - y_2), \beta y_2) \\
&= \alpha\mathbf{T}(x_1, x_2) + \beta\mathbf{T}(y_1, y_2)
\end{aligned}$$

Thus $\mathbf{T}$ is linear.

$$\begin{aligned}
\mathbf{T}(\mathbf{e_1}) = \mathbf{T}(1, 0) = (1, 1, 0) \\
\mathbf{T}(\mathbf{e_2}) = \mathbf{T}(0, 1) = (1, -1, 1)
\end{aligned}$$

Clearly $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2})$ are linearly independent. Since $T(\mathbb{R}^2)$ is generated by $\mathbf{T}(\mathbf{e_1})$ and $\mathbf{T}(\mathbf{e_2})$, the rank of $\mathbf{T}$ is 2.

The range of $\mathbf{T}$ $= \{\alpha\mathbf{T}(\mathbf{e_1}) + \beta\mathbf{T}(\mathbf{e_2}), \alpha, \beta \in \mathbb{R}\}$
$= \{\alpha(1, 1, 0) + \beta(1, -1, 1)\}$
$= \{(\alpha + \beta, \alpha - \beta, \beta) \mid \alpha, \beta \in \mathbb{R}\}$

To find the null space of $\mathbf{T}$, if $\mathbf{T}(x_1, x_2) = (0, 0, 0)$, then $x_1 + x_2 = 0,\ x_1 - x_2 = 0,\ x_2 = 0$, so $x_1 = x_2 = 0$. Thus the null space of $\mathbf{T}$ is $\{\mathbf{0}\}$, and nullity $\mathbf{T} = 0$. ∎

Question 1(c) *Let $\mathcal{V}$ be the space of 2×2 matrices over $\mathbb{R}$. Determine whether the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ are dependent where*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}$$

Solution. If $\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C} = 0$, then

$$\begin{aligned}
\alpha + 3\beta + \gamma &= 0 && (1) \\
2\alpha - \beta - 5\gamma &= 0 && (2) \\
3\alpha + 2\beta - 4\gamma &= 0 && (3) \\
\alpha + 2\beta &= 0 && (4)
\end{aligned}$$

From (4), we get $\alpha = -2\beta$. This, together with (3) gives $\gamma = -\beta$. These satisfy (1) and (2) also, so taking $\beta = 1, \alpha = -2, \gamma = -1$ gives us $-2\mathbf{A} + \mathbf{B} - \mathbf{C} = 0$. Thus $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are dependent. ∎

Question 2(a) *Let $\mathbf{A}$ be an $n \times n$ matrix such that each diagonal entry is μ and each off-diagonal entry is 1. If $\mathbf{B} = \lambda\mathbf{A}$ is orthogonal, determine λ, μ.*

Solution. Clearly $\mathbf{A}$ is symmetric. Let $\mathbf{A} = (a_{ij})$. $\mathbf{B}'\mathbf{B} = \mathbf{B}\mathbf{B}' = \lambda^2\mathbf{A}^2 = \mathbf{I} \Longrightarrow \sum_{k=1}^{n} \lambda^2 a_{ik} a_{kj} = \delta_{ij}$

Taking $i = j = 1$, we get $\lambda^2(\mu^2 + n - 1) = 1$ Taking $i = 1, j = 2$, we get $\lambda^2(2\mu + n - 2) = 0$. Thus $\mu = -(n-2)/2$ and $\lambda^2[(n-2)^2/4 + n - 1] = 1$. Simplifying, $\lambda^2[n^2 - 4n + 4 + 4n - 4]/4 = 1$, which means $\lambda^2 = \frac{4}{n^2}$, or $\lambda = \pm\frac{2}{n}$. ∎

Question 2(b) *Show that*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

is diagonalizable over $\mathbb{R}$. *Find* $\mathbf{P}$ *such that* $\mathbf{P}^{-1}AP$ *is diagonal and hence find* $\mathbf{A}^{25}$.

Solution. Characteristic equation of $\mathbf{A}$ is

$$\begin{aligned} \begin{vmatrix} 2-x & -1 & 0 \\ -1 & 2-x & 0 \\ 2 & 2 & 3-x \end{vmatrix} &= 0 \\ \Rightarrow \quad (2-x)(2-x)(3-x) + 1(-(3-x)) &= 0 \\ (3-x)(4-4x+x^2-1) &= 0 \end{aligned}$$

Thus the eigenvalues are 3, 3, 1.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$.

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 - x_2 = 0, -x_1 + x_2 = 0, 2x_1 + 2x_2 + 2x_3 = 0$. Take $x_1 = 1$, then $x_2 = 1, x_3 = -2$, so $(1, 1, -2)$ is an eigenvector with eigenvalue 1.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 3$.

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + x_2 = 0$. Take $x_1 = 1, x_3 = 0$, then $x_2 = -1$, so $(1, -1, 0)$ is an eigenvector with eigenvalue 3. Take $x_1 = 0, x_3 = 1$, then $x_2 = 0$, so $(0, 0, 1)$ is also an eigenvector for eigenvalue 3.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ then $\mathbf{AP} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ or $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Now $\mathbf{P}^{-1}\mathbf{A}^{25}\mathbf{P} = (\mathbf{P}^{-1}\mathbf{AP})^{25} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix}$. Thus $\mathbf{A}^{25} = \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix} \mathbf{P}^{-1}$

$|\mathbf{P}| = -2$, so $\mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

$$\begin{aligned}
\mathbf{A}^{25} &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3^{25} & 0 \\ 0 & 0 & 3^{25} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3^{25} & 0 \\ 1 & -3^{25} & 0 \\ -2 & 0 & 3^{25} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+3^{25}}{2} & \frac{1-3^{25}}{2} & 0 \\ \frac{1-3^{25}}{2} & \frac{1+3^{25}}{2} & 0 \\ -1+3^{25} & -1+3^{25} & 3^{25} \end{pmatrix}
\end{aligned}$$

∎

Question 2(c) *Let* $\mathbf{A} = [a_{ij}]$ *be a square matrix of order* n *such that* $|a_{ij}| \leq M$*. Let* λ *be an eigenvalue of* $\mathbf{A}$*, show that* $|\lambda| \leq nM$*.*

Solution. We first prove the following:

Lemma: If $\mathbf{A} = [a_{ij}]$ and $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \leq a_{ii}$ then $|\mathbf{A}| \neq 0$.

If $|\mathbf{A}| = 0$ then there exist $x_1, \ldots, x_n \in \mathbb{C}$ not all zero such that

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
&\ldots \\
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &= 0 \\
&\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= 0
\end{aligned}$$

Let $|x_i| = \max(|x_1|, |x_2|, \ldots, |x_n|)$, so $|\frac{x_j}{x_i}| \leq 1$ for all j.

$$\begin{aligned}
0 &= \left| a_{ii} - (-a_{i1}\frac{x_1}{x_i} - a_{i2}\frac{x_2}{x_i} - \ldots - a_{in}\frac{x_n}{x_i}) \right| \\
&\geq |a_{ii}| - \left| a_{i1}\frac{x_1}{x_i} + a_{i2}\frac{x_2}{x_i} + \ldots + a_{in}\frac{x_n}{x_i} \right| \\
&\geq |a_{ii}| - |a_{i1}| - |a_{i2}| - \ldots - |a_{in}|
\end{aligned}$$

which contradicts the premise $\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \leq a_{ii}$ Thus $|\mathbf{A}| \neq 0$.

Now the lemma tells us that if $|\lambda - a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$ then $|\lambda \mathbf{I} - \mathbf{A}| \neq 0$, so λ is not an eigenvalue of $\mathbf{A}$. Thus $|\lambda| \leq |\lambda - a_{ii}| + |a_{ii}| \leq \sum_{j=1}^{n} |a_{ij}| \leq nM$ as desired.

∎

Question 3(a) *Define a positive definite matrix and show that a positive definite matrix is always non-singular. Show that the converse is not always true.*

Solution. Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix. $\mathbf{A}$ is said to be positive definite if the associated quadtratic form

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} > 0$$

for all $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$ in $\mathbb{R}^n$.

If $|\mathbf{A}| = 0$ then rank $\mathbf{A} < n$, which means that columns of $\mathbf{A}$ i.e. $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are linearly dependent i.e. there exist real numbers $x_1, x_2, \dots, x_n$ not all zero such that

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \dots + x_n\mathbf{c_n} = \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \mathbf{0} \Longrightarrow \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = 0$$

where $(x_1, x_2, \dots, x_n) \neq (0, 0, \dots, 0)$, which means that $\mathbf{A}$ is not positive definite. Thus $\mathbf{A}$ is positive definite $\Longrightarrow |\mathbf{A}| \neq 0$.

The converse is not true. Take

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then $|\mathbf{A}| = -1$, but

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1$$

so $\mathbf{A}$ is not positive definite.

∎

Question 3(b) *Find the eigenvalues and their corresponding eigenvectors for the matrix*

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Solution. The characteristic equation for $\mathbf{A}$ is

$$\begin{aligned} 0 &= |\mathbf{A} - x\mathbf{I}| \\ &= \begin{pmatrix} 6-x & -2 & 2 \\ -2 & 3-x & -1 \\ 2 & -1 & 3-x \end{pmatrix} \\ &= (6-x)((3-x)^2 - 1) + 2(-6+2x+2) + 2(2-6+2x) \\ &= (6-x)(9-6x+x^2) - 6 + x - 8 + 4x - 8 + 4x \\ 0 &= x^3 - 12x^2 + 36x - 32 \\ &= (x-2)(x^2 - 10x + 16) \end{aligned}$$

Thus the eigenvalues are 2, 2, 8.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 2$.

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $4x_1 - 2x_2 + 2x_3 = 0, -2x_1 + x_2 - x_3 = 0, 2x_1 - x_2 + x_3 = 0$. Take $x_1 = 1, x_2 = 0$, then $x_3 = -2$, so $(1, 0, -2)$ is an eigenvector with eigenvalue 2. Take $x_1 = 0, x_2 = 1$, then $x_3 = 1$, so $(0, 1, 1)$ is an eigenvector with eigenvalue 2.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 8$.

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 - 2x_2 + 2x_3 = 0, -2x_1 - 5x_2 - x_3 = 0, 2x_1 - x_2 - 5x_3 = 0$. From the last two, we get $x_2 + x_3 = 0$, and from the first we get $x_1 = 2x_3$. Take $x_3 = 1$, then $x_2 = -1, x_1 = 2$, so $(2, -1, 1)$ is an eigenvector with eigenvalue 8. ∎

Question 3(c) *Find* $\mathbf{P}$ *invertible such that* $\mathbf{P}$ *reduces* $Q(x, y, z) = 2xy + 2yz + 2zx$ *to its canonical form.*

Solution. The matrix of $Q(x, y, z)$ is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

which has all diagonal entries 0, so we cannot complete squares right away.

$$\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\mathbf{A}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Add the second row to the first and the second column to the first.

$$\begin{pmatrix}2&1&2\\1&0&1\\2&1&0\end{pmatrix} = \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\mathbf{A}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$$

Subtract $\frac{1}{2}R_1$ from R_2 and $\frac{1}{2}C_1$ from C_2.

$$\begin{pmatrix}2&0&2\\0&-\frac{1}{2}&0\\2&0&0\end{pmatrix} = \begin{pmatrix}1&1&0\\-\frac{1}{2}&\frac{1}{2}&0\\0&0&1\end{pmatrix}\mathbf{A}\begin{pmatrix}1&-\frac{1}{2}&0\\1&\frac{1}{2}&0\\0&0&1\end{pmatrix}$$

Subtract R_1 from R_3 and C_1 from C_3.

$$\begin{pmatrix}2&0&0\\0&-\frac{1}{2}&0\\0&0&-2\end{pmatrix} = \begin{pmatrix}1&1&0\\-\frac{1}{2}&\frac{1}{2}&0\\-1&-1&1\end{pmatrix}\mathbf{A}\begin{pmatrix}1&-\frac{1}{2}&-1\\1&\frac{1}{2}&-1\\0&0&1\end{pmatrix}$$

Thus $\mathbf{P} = \begin{pmatrix}1&-\frac{1}{2}&-1\\1&\frac{1}{2}&-1\\0&0&1\end{pmatrix}$ and $\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix}2&0&0\\0&-\frac{1}{2}&0\\0&0&-2\end{pmatrix}$

So $Q(x,y,z) \longrightarrow 2X^2 - \frac{1}{2}Y^2 - 2Z^2$.

Alternative Solution. Let $x = X, y = X+Y, z = Z$

$$\begin{aligned} Q(x,y,z) &= 2X^2 + 2XY + 2ZX + 2ZY + 2ZX \\ &= 2[X^2 + XY + 2ZX + ZY] \\ &= 2[(X + \frac{Y}{2} + Z)^2 - \frac{Y^2}{4} - Z^2] \end{aligned}$$

Put $\xi = X + Y/2 + Z, \eta = Y, \zeta = Z$, so $X = \xi - \eta/2 - \zeta, Y = \eta, Z = \zeta$.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}X\\Y\\Z\end{pmatrix} = \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-\frac{1}{2}&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix} = \begin{pmatrix}1&-\frac{1}{2}&-1\\1&\frac{1}{2}&-1\\0&0&1\end{pmatrix}\begin{pmatrix}\xi\\\eta\\\zeta\end{pmatrix}$$

Thus $Q(x,y,z) \longrightarrow 2\xi^2 - \eta^2/2 - 2\zeta^2$, and $\mathbf{P} = \begin{pmatrix}1&-\frac{1}{2}&-1\\1&\frac{1}{2}&-1\\0&0&1\end{pmatrix}$ as before. Note that we put $x = X, y = X+Y, z = Z$ to create one square term to complete the squares. ∎

UPSC Civil Services Main 1998 - Mathematics Linear Algebra

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Question 1(a) *Given two linearly independent vectors* $(1, 0, 1, 0)$ *and* $(0, -1, 1, 0)$ *of* $\mathbb{R}^4$*, find a basis of* $\mathbb{R}^4$ *which includes them.*

Solution. Let $\mathbf{v_1} = (1, 0, 1, 0), \mathbf{v_2} = (0, -1, 1, 0)$. Clearly these are linearly independent. Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ be the standard basis. Then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ generate $\mathbb{R}^4$. We have to find four vectors out of these which are linearly independent and include $\mathbf{v_1}, \mathbf{v_2}$.

If $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{e_1} = 0$, then $\alpha + \gamma = 0, -\alpha = 0, \alpha + \beta = 0 \Rightarrow \alpha = \beta = \gamma = 0$. Therefore $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}$ are linearly independent.

We now show that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_4}$ are linearly independent. Let $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{e_1} + \delta\mathbf{e_4} = 0$ then $\delta = 0$, and therefore $\alpha = \beta = \gamma = 0$ because $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}$ are linearly independent.

Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_4}$ is a basis of $\mathbb{R}^4$.

Note that $\mathbf{e_2} = \mathbf{v_1} - \mathbf{v_2} - \mathbf{e_1}, \mathbf{e_3} = \mathbf{v_1} - \mathbf{e_1}$. ∎

Question 1(b) *If* $\mathcal{V}$ *is a finite dimensional vector space over* $\mathbb{R}$ *and if* f *and* g *are two linear transformations from* $\mathcal{V}$ *to* $\mathbb{R}$ *such that* $f(\mathbf{v}) = 0$ *implies* $g(\mathbf{v}) = 0$*, then prove that* $g = \lambda f$ *for some* $\lambda \in \mathbb{R}$.

Solution. If $g = 0$, take $\lambda = 0$, so $g(\mathbf{v}) = 0 = 0f(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$.

If $g \neq 0$, then $f \neq 0$. Thus $\exists \mathbf{v} \in \mathcal{V}$ such that $f(\mathbf{v}) \neq 0 \Rightarrow \exists \mathbf{w} \in \mathcal{V}$ such that $f(\mathbf{w}) = 1$ (Note that $f(\frac{\mathbf{v}}{f(\mathbf{v})}) = 1$).

Thus $\mathcal{V}/\ker f \backsimeq \mathbb{R}$, or $\dim(\ker f) = n - 1$. Similarly $\ker g$ has dimension $n - 1$. In fact, $\ker f = \ker g \because \ker f \subseteq \ker g$ and $\dim(\ker f) = \dim(\ker g)$. Let $\{\mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of $\ker f$ and extend it to $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ a basis of $\mathcal{V}$. Then $g = \lambda f$ with $\lambda = g(\mathbf{v_1})/f(\mathbf{v_1}) \because$ if $\mathbf{v} = \alpha_1\mathbf{v_1} + \ldots + \alpha_n\mathbf{v_n}$, then $g(\mathbf{v}) = \alpha_1 g(\mathbf{v_1}) = \alpha_1\lambda f(\mathbf{v_1}) = \lambda f(\mathbf{v})$. ∎

Question 1(c) *Let* $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *be defined by* $\mathbf{T}(x_1, x_2, x_3) = (x_2, x_3, -cx_1 - bx_2 - ax_3)$ *where* a, b, c *are fixed real numbers. Show that* $\mathbf{T}$ *is a linear transformation of* $\mathbb{R}^3$ *and that* $\mathbf{A}^3 + a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$ *where* $\mathbf{A}$ *is the matrix of* $\mathbf{T}$ *w.r.t. the standard basis of* $\mathbb{R}^3$.

Solution. Let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$. Then

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_2 + \beta y_2, \alpha x_3 + \beta x_3, -c(\alpha x_1 + \beta y_1) - b(\alpha x_2 + \beta y_2) - a(\alpha x_3 + \beta y_3)) \\ &= \alpha(x_2, x_3, -cx_1 - bx_2 - ax_3) + \beta(y_2, y_3, -cy_1 - by_2 - ay_3) \\ &= \alpha\mathbf{T}(\mathbf{x}) + \beta\mathbf{T}(\mathbf{y})\end{aligned}$$

Thus $\mathbf{T}$ is linear.

Clearly

$$\begin{aligned}\mathbf{T}(1, 0, 0) &= (0, 0, -c) \\ \mathbf{T}(0, 1, 0) &= (1, 0, -b) \\ \mathbf{T}(0, 0, 1) &= (0, 1, -a)\end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}$$

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$$\begin{aligned}\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -c & -b & -a - \lambda \end{vmatrix} &= 0 \\ -\lambda^2(a + \lambda) - b\lambda - c &= 0 \\ \lambda^3 + a\lambda^2 + b\lambda + c &= 0\end{aligned}$$

Now by the Cayley-Hamilton theorem $\mathbf{A}^3 + a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$. ∎

Question 2(a) *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are two matrices of order* 2×2 *such that* $\mathbf{A}$ *is skew-Hermitian and* $\mathbf{AB} = \mathbf{B}$ *then show that* $\mathbf{B} = \mathbf{0}$.

Solution. We first of all prove that eigenvalues of skew-Hermitian matrices are 0 or pure imaginary. Let $\mathbf{A}$ be skew-Hermitian, i.e. $\overline{\mathbf{A}}' = -\mathbf{A}$ and let λ be its characteristic root. If $\mathbf{x}$ is an eigenvector of λ, then

$$\begin{aligned}\mathbf{Ax} &= \lambda\mathbf{x} \\ \Rightarrow \quad \overline{\mathbf{x}}'\lambda\mathbf{x} &= \overline{\mathbf{x}}'\mathbf{Ax} \\ &= -\overline{\mathbf{x}}'\overline{\mathbf{A}}'\mathbf{x} \\ &= -\overline{\mathbf{Ax}}'\mathbf{x} \\ &= -\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}\end{aligned}$$

Thus $\lambda = -\overline{\lambda}$ $\because \overline{\mathbf{x}}'\mathbf{x} \neq 0$, showing that the real part of λ is 0.

Now if $\mathbf{B} \neq \mathbf{0}$ and $\mathbf{c_1}, \mathbf{c_2}$ are the columns of $\mathbf{B}$, then $\mathbf{c_1} \neq \mathbf{0}$ or $\mathbf{c_2} \neq \mathbf{0}$. $\mathbf{AB} = \mathbf{B}$ means that $\mathbf{Ac_1} = \mathbf{c_1}$ and $\mathbf{Ac_2} = \mathbf{c_2}$. Since either $\mathbf{c_1} \neq \mathbf{0}$ or $\mathbf{c_2} \neq \mathbf{0}$, 1 must be an eigenvalue of $\mathbf{A}$, which is not possible. Hence $\mathbf{c_1} = \mathbf{0}$ and $\mathbf{c_2} = \mathbf{0}$, which means $\mathbf{B} = \mathbf{0}$. ∎

Question 2(b) *If* $\mathbf{T}$ *is a complex matrix of order* 2×2 *such that* $\operatorname{tr}\mathbf{T} = \operatorname{tr}\mathbf{T}^2 = 0$*, then show that* $\mathbf{T}^2 = \mathbf{0}$.

Solution. Let λ_1, λ_2 be the eigenvalues of $\mathbf{T}$, then λ_1^2, λ_2^2 are the eigenvalues of $\mathbf{T}^2$. Given that

$$\begin{aligned} \operatorname{tr}\mathbf{T} &= \lambda_1 + \lambda_2 = 0 \\ \operatorname{tr}\mathbf{T}^2 &= \lambda_1^2 + \lambda_2^2 = 0 \end{aligned}$$

$0 = \lambda_1^2 + \lambda_2^2 = \lambda_1^2 + (-\lambda_1)^2 \Rightarrow \lambda_1 = 0$ and from $\lambda_1 + \lambda_2 = 0$ we get $\lambda_1 = \lambda_2 = 0$. The characteristic equation of $\mathbf{T}$ is $(x - \lambda_1)(x - \lambda_2) = 0$, or $x^2 = 0$. By Cayley-Hamilton theorem, we immediately get $\mathbf{T}^2 = \mathbf{0}$. ∎

Question 2(c) *Prove that a necessary and sufficient condition for an* $n \times n$ *real matrix* $\mathbf{A}$ *to be similar to a diagonal matrix is that the set of characteristic vectors of* $\mathbf{A}$ *includes a set of* n *linearly independent vectors.*

Solution.

Necessity: By hypothesis there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Let $\mathbf{P} = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]$, where each $\mathbf{c_i}$ is an n-row column vector.

$$\mathbf{A}[\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}] = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]\mathbf{D} = [\lambda_1\mathbf{c_1}, \lambda_2\mathbf{c_2}, \dots, \lambda_n\mathbf{c_n}]$$

so $\mathbf{A}\mathbf{c_i} = \lambda_i\mathbf{c_i}$ for $i = 1, \dots, n$. Thus $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are characteristic vectors of $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$. Since $\mathbf{P}$ is nonsingular, $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are linearly independent. Thus the set of characteristic vectors of $\mathbf{A}$ includes a set of n linearly independent vectors.

Sufficiency: Let $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ be n linearly independent eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$. Thus $\mathbf{A}\mathbf{c_i} = \lambda_i\mathbf{c_i}$ for $i = 1, \dots, n$. Let $\mathbf{P} = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]$, then $\mathbf{P}$ is nonsingular (otherwise 0 is an eigenvalue of $\mathbf{P}$, so $\exists \mathbf{x} = (x_1, \dots, x_n) \neq \mathbf{0}$ such that $\mathbf{P}\mathbf{x} = \mathbf{0} \Rightarrow x_1\mathbf{c_1} + \dots + x_n\mathbf{c_n} = \mathbf{0} \Rightarrow \mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are not linearly independent.). Clearly

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

∎

Question 3(a) *Let* $\mathbf{A}$ *be a* $m \times n$ *matrix. Show that the sum of the rank and nullity of* $\mathbf{A}$ *is* n.

Solution. The matrix $\mathbf{A}$ can be regarded as a linear transformation $\mathbf{A} : \mathcal{F}^n \longrightarrow \mathcal{F}^m$ where $\mathcal{F}$ is the field to which the entries of $\mathbf{A}$ belong, and the bases for $\mathcal{F}^n, \mathcal{F}^m$ are standard bases.

Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation, where $\dim(\mathcal{V}) = n, \dim(\mathcal{W}) = m$. We shall show that $\dim(\mathbf{T}(\mathcal{V})) + \dim(\text{kernel}\,\mathbf{T}) = n$.

Take $\mathbf{v_{n-r+1}}, \dots, \mathbf{v_n}$ to be any basis of kernel $\mathbf{T}$, where $\dim(\text{kernel}\,\mathbf{T}) = r$. Complete it to a basis $\mathbf{v_1}, \dots, \mathbf{v_{n-r+1}} \dots, \mathbf{v_n}$ of $\mathcal{V}$. We shall show that $\mathbf{T}(\mathbf{v_1}), \dots, \mathbf{T}(\mathbf{v_{n-r}})$ are linearly independent and generate $\mathbf{T}(\mathcal{V})$, thus $\dim(\mathbf{T}(\mathcal{V})) = n - r$.

If $\mathbf{w} \in \mathbf{T}(\mathcal{V})$, then $\exists \mathbf{v} \in \mathcal{V}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{w}$. If $\mathbf{v} = \alpha_1 \mathbf{v_1} + \ldots + \alpha_n \mathbf{v_n}, \alpha_i \in \mathcal{F}$, then $\mathbf{w} = \mathbf{T}(\mathbf{v}) = \alpha_1 \mathbf{T}(\mathbf{v_1}) + \ldots + \alpha_{n-r}\mathbf{T}(\mathbf{v_{n-r}})$ because $\mathbf{T}(\mathbf{v_i}) = \mathbf{0}$ for $i > n - r$. Thus $\mathbf{T}(\mathcal{V})$ is generated by $\mathbf{T}(\mathbf{v_1}), \dots, \mathbf{T}(\mathbf{v_{n-r}})$.

If $\alpha_1 \mathbf{T}(\mathbf{v_1}) + \ldots + \alpha_{n-r}\mathbf{T}(\mathbf{v_{n-r}}) = \mathbf{0}$, then $\mathbf{T}(\alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r}\mathbf{v_{n-r}}) = \mathbf{0}$. This implies $\alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r}\mathbf{v_{n-r}} \in \text{kernel}\,\mathbf{T} \Rightarrow \alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r}\mathbf{v_{n-r}} = \alpha_{n-r+1}\mathbf{v_{n-r+1}} + \ldots + \alpha_n \mathbf{v_n}$. But $\mathbf{v_1}, \dots, \mathbf{v_n}$ are linearly independent, so $\alpha_i = 0$ for $i = 1, \dots, n$. Hence $\mathbf{T}(\mathbf{v_1}), \dots, \mathbf{T}(\mathbf{v_{n-r}})$ are linearly independent, so they form a basis for $\mathbf{T}(\mathcal{V})$. Thus $\dim(\mathbf{T}(\mathcal{V})) + \dim(\text{kernel}\,\mathbf{T}) = n$. ∎

Question 3(b) *Find all real* 2×2 *matrices* $\mathbf{A}$ *with real eigenvalues which satisfy* $\mathbf{A}\mathbf{A}' = \mathbf{I}$.

Solution. Since $\mathbf{A}\mathbf{A}' = \mathbf{I}$, $|\mathbf{A}| = \pm 1$. If $|\mathbf{A}| = 1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $a^2 + b^2 = 1, c^2 + d^2 = 1, ac + bd = 0, ad - bc = 1$. Let $a = \cos\theta, b = \sin\theta$. Then

$$\begin{array}{c} c\cos\theta + d\sin\theta = 0 \\ -c\sin\theta + d\cos\theta = 1 \end{array} \Rightarrow \begin{array}{c} c\cos\theta\sin\theta + d\sin^2\theta = 0 \\ -c\sin\theta\cos\theta + d\cos^2\theta = \cos\theta \end{array} \Rightarrow d = \cos\theta, c = -\sin\theta$$

Thus $\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, θ is real.

Now the eigenvalues of $\mathbf{A}$ are given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

So $(\cos\theta - \lambda)^2 + \sin^2\theta = 0$, or $\lambda^2 - 2\lambda\cos\theta + 1 = 0$. Thus

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$$

Since the eigenvalues of $\mathbf{A}$ are real, $\sin\theta = 0$, so $\cos\theta = \pm 1$. Thus

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

If $|\mathbf{A}| = -1$, $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $|\mathbf{JA}| = 1$. Also $\mathbf{JA}(\mathbf{JA})' = \mathbf{JAA}'\mathbf{J}' = \mathbf{JJ}' = \mathbf{I}$. Thus

$$\mathbf{JA} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{A} = \mathbf{J}^{-1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

Now the eigenvalues of $\mathbf{A}$ are given by

$$0 = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda + \sin\theta & -\cos\theta \\ -\cos\theta & \lambda - \sin\theta \end{vmatrix} = \lambda^2 - \sin^2\theta - \cos^2\theta = \lambda^2 - 1$$

Hence $\lambda = \pm 1$, so the eigenvalues are always real. Thus the possible values of $\mathbf{A}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \text{ for all real } \theta$$

∎

Question 3(c) *Reduce to diagonal matrix by rational congruent transformation the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

Solution. The corresponding quadratic form is

$$\begin{aligned} & x^2 + z^2 + 4xy - 2xz + 6yz \\ = \;& (x + 2y - z)^2 - 4y^2 + 10yz \\ = \;& (x + 2y - z)^2 - 4(y - \frac{5}{4}z)^2 + \frac{25}{4}z^2 \\ = \;& X^2 - 4Y^2 + \frac{25}{4}Z^2 \end{aligned}$$

where $X = x + 2y - z, Y = y - 5z/4, Z = z$. From this we get $z = Z, y = Y + 5Z/4, x = X - 2Y - \frac{3}{2}Z$. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{25}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & \frac{5}{4} & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -\frac{3}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

∎

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Question 1(a) *Let $\mathcal{V}$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $f, g, h \in \mathcal{V}$ are linearly independent where $f(t) = e^{2t}$, $g(t) = t^2$ and $h(t) = t$.*

Solution. Let $a, b, c \in \mathbb{R}$ and let $ae^{2t} + bt^2 + ct = 0$ for all t. Setting $t = 0$ shows that $a = 0$. From $t = 1$ we get $b + c = 0$, and $t = -1$ gives $b - c = 0$, hence $b = c = 0$. Thus f, g, h are linearly independent. ∎

Question 1(b) *If the matrix of the linear transformation $\mathbf{T}$ on $\mathcal{V}_2(\mathbb{R})$ with respect to the basis $B = \{(1, 0), (0, 1)\}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then what is the matrix of $\mathbf{T}$ with respect to the ordered basis $B_1 = \{(1, 1), (1, -1)\}$.*

Solution. $\mathbf{T}(\mathbf{e_1}) = \mathbf{e_1} + \mathbf{e_2}, \mathbf{T}(\mathbf{e_2}) = \mathbf{e_1} + \mathbf{e_2}$. Let $\mathbf{v_1} = (1, 1) = \mathbf{e_1} + \mathbf{e_2}, \mathbf{v_2} = (1, -1) = \mathbf{e_1} - \mathbf{e_2}$. Then $\mathbf{T}(\mathbf{v_1}) = \mathbf{T}((1, 1)) = (2, 2) = 2\mathbf{e_1} + 2\mathbf{e_2} = 2\mathbf{v_1}$. $\mathbf{T}(\mathbf{v_2}) = \mathbf{T}((1, -1)) = (0, 0) = \mathbf{0}$. Thus the matrix of $\mathbf{T}$ with respect to the ordered basis $B_1 = \{(1, 1), (1, -1)\}$ is $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. ∎

Question 1(c) *Diagonalize the matrix* $\mathbf{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

Solution. The characteristic equation is

$$\begin{aligned}
0 &= \begin{vmatrix} 4-x & 2 & 2 \\ 2 & 4-x & 2 \\ 2 & 2 & 4-x \end{vmatrix} \\
&= (4-x)((4-x)^2-4)+2(4-8+2x)-2(8-2x-4) \\
&= (4-x)(12-8x+x^2)-8+4x-8+4x \\
&= 48-32x+4x^2-12x+8x^2-x^3-16+8x \\
&= -(x^3-12x^2+36x-32) \\
&= -(x-2)(x^2-10x+16) \quad \because\ 2 \text{ is a root} \\
&= -(x-2)(x-2)(x-8)
\end{aligned}$$

The characteristic roots are 2, 2, 8.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 8$, then

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 = x_2 = x_3$, so $(1, 1, 1)$ is an eigenvector for $\lambda = 8$.

Similarly for $\lambda = 2$,

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1+x_2+x_3 = 0$. Take $x_1 = 1, x_2 = 0$, so $(1, 0, -1)$ is an eigenvector. Take $x_1 = 0, x_2 = 1$, so $(0, 1, -1)$ is an eigenvector for $\lambda = 2$. These eigenvectors are linearly independent. Thus if $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$, then $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

To check, verify that $\mathbf{AP} = \mathbf{P}\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ ∎

Question 2(a) *Test for congruency the matrices* $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$*. Prove that* $\mathbf{A}^{2n} = \mathbf{B}^{2m} = \mathbf{I}$ *where* m, n *are positive integers.*

Solution. $\mathbf{A}$ and $\mathbf{B}$ are not congruent, because $\mathbf{A}$ is symmetric and $\mathbf{B}$ is not. If $\mathbf{A} \equiv \mathbf{B}$ then $\exists\mathbf{P}$ non-singular such that $\mathbf{P}'\mathbf{AP} = \mathbf{B}$ which implies that $\mathbf{B}$ should be symmetric.

$$\begin{aligned}
\mathbf{A}^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbf{B}^2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Hence $\mathbf{A}^{2n} = (\mathbf{A}^2)^n = \mathbf{I}$, and $\mathbf{B}^{2m} = (\mathbf{B}^2)^m = \mathbf{I}$. ∎

Question 2(b) *If $\mathbf{A}$ is a skew symmetric matrix of order n then prove that $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}$ is orthogonal.*

Solution.

$$\begin{aligned}
\mathbf{O} &= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} \\
\mathbf{O}\mathbf{O}' &= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}((\mathbf{I}+\mathbf{A})^{-1})'(\mathbf{I}-\mathbf{A})' \\
&= (\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \quad \textit{as } \mathbf{A}' = -\mathbf{A} \\
&= (\mathbf{I}-\mathbf{A})[(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})[\mathbf{I}-\mathbf{A}^2)]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})[(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})]^{-1}(\mathbf{I}+\mathbf{A}) \\
&= (\mathbf{I}-\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \\
&= \mathbf{I}
\end{aligned}$$

Similarly it can be shown that $\mathbf{O}'\mathbf{O} = \mathbf{I}$. Hence $\mathbf{O}$ is orthogonal. ∎

Question 2(c) *Test for positive definiteness the quadratic form $2x^2 + y^2 + 2z^2 + 2xy - 2zx$.*

Solution. The given form is
$\frac{1}{2}(4x^2 + 3y^2 + 4z^2 + 4xy - 4zx)$
$= \frac{1}{2}((2x + y - z)^2 + y^2 + 3z^2 + 2yz)$
$= \frac{1}{2}((2x + y - z)^2 + (y + z)^2 + 2z^2)$

Now $2x + y - z = 0, y + z = 0, z = 0$ implies $x = y = z = 0$. Hence the form is positive definite. ∎

Question 2(d) *Reduce the equation*

$$x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - y - 2z + 6 = 0$$

into canonical form and determine the nature of the quadric.

Solution. Consider $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$. Its matrix is

$$\mathbf{S} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Its characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - 3\lambda^2 = 0$. Thus $\lambda = 0, 0, 3$.

We next determine the characteristic vectors. For $\lambda = 0$, we get

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus $x_1 - x_2 + x_3 = 0$. Take $(1, 1, 0)$ and $(-1, 1, 2)$ as orthogonal characteristic vectors corresponding to $\lambda = 0$.

For $\lambda = 3$, we get

$$\begin{pmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This yields $x_1 = -x_2 = x_3$. Take $(1, -1, 1)$ as the characteristic vector for $\lambda = 3$.

Thus if

$$\mathbf{O} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$$

Then $\mathbf{O'SO} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Let

$$\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or

$$\begin{aligned} x &= \frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} - \frac{Z}{\sqrt{6}} \\ y &= -\frac{X}{\sqrt{3}} + \frac{Y}{\sqrt{2}} + \frac{Z}{\sqrt{6}} \\ z &= \frac{X}{\sqrt{3}} + \frac{2Z}{\sqrt{6}} \end{aligned}$$

Thus the given equation can be transformed to

$$\begin{gathered} 3X^2 + \tfrac{X}{\sqrt{3}} + \tfrac{Y}{\sqrt{2}} - \tfrac{Z}{\sqrt{6}} + \tfrac{X}{\sqrt{3}} - \tfrac{Y}{\sqrt{2}} - \tfrac{Z}{\sqrt{6}} - \tfrac{2X}{\sqrt{3}} - \tfrac{4Z}{\sqrt{6}} + 6 = 0 \\ \Rightarrow 3X^2 - Z\sqrt{6} + 6 = 0 \\ \Rightarrow \sqrt{\tfrac{3}{2}}X^2 = Z - \sqrt{6} \end{gathered}$$

Shifting the origin to $(0, 0, \sqrt{6})$, we get $X^2 = \sqrt{\frac{2}{3}}Z$, showing that the equation is a parabolic cylinder. ∎

1 Reduction of Quadrics

For the sake of completeness, we give the complete theoretical discussion for the above question.

Let

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

It can be expressed in matrix form as

$$\begin{pmatrix} x & y & z & 1 \end{pmatrix} \begin{pmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = 0$$

Let the 4×4 matrix be $\mathbf{Q}$.

Step I. Consider

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \mathbf{S} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

1. Find the characteristic roots of $\mathbf{S}$ — $\lambda_1, \lambda_2, \lambda_3$.

2. Find characteristic vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_2}$ corresponding to $\lambda_1, \lambda_2, \lambda_3$ which are orthogonal. These on normalization give us

$$\mathbf{O} = \begin{pmatrix} \frac{\mathbf{v_1}}{||\mathbf{v_1}||} & \frac{\mathbf{v_2}}{||\mathbf{v_2}||} & \frac{\mathbf{v_3}}{||\mathbf{v_3}||} \end{pmatrix}$$

Thus we get $\mathbf{O'SO}$ =diagonal$(\lambda_1, \lambda_2, \lambda_3)$. Let

$$\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives us three equations expressing x, y, z in term of X, Y, Z. Substituting in F, we get

$$F(X,Y,Z) = \lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + 2UX + 2VY + 2WZ + d = 0$$

(Note that d is unaffected.)

Note 1 *Since* $\mathbf{O}$ *is orthogonal, the transformation* $\mathbf{O} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ *is just a rotation of the axes, and therefore the nature of the quadric is unaffected.*

Step II. We now consider 3 possibilities (ρ is rank of the matrix):

1. $\rho(\mathbf{S}) = 3 \Rightarrow \lambda_1\lambda_2\lambda_3 \neq 0$. Shift the origin to $(-\frac{U}{\lambda_1}, -\frac{V}{\lambda_2}, -\frac{W}{\lambda_3})$, i.e. $x = X + \frac{U}{\lambda_1}, y = Y + \frac{V}{\lambda_2}, z = Z + \frac{W}{\lambda_3}$. (Actually we are just completing the squares.) F gets transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d_2 = 0$$
2. $\rho(\mathbf{S}) = 2$. One characteristic root, say $\lambda_3 = 0$. Shift the origin to $(-\frac{U}{\lambda_1}, -\frac{V}{\lambda_2}, 0)$, and F gets transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$
3. $\rho(\mathbf{S}) = 1$. Two characteristic roots, say $\lambda_2 = \lambda_3 = 0$. Shift the origin to $(-\frac{U}{\lambda_1}, 0, 0)$, and F gets transformed to
$$\lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

Note that $\rho(\mathbf{S}) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$. Then $F(x,y,z)$ is no longer a quadric, it is a plane.

Step III. Observe that $\rho(\mathbf{S}) \leq \rho(\mathbf{Q}) \leq 4, \rho(\mathbf{S}) \leq 3$.

1. Let $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 3$. As shown above, $F(x,y,z) = 0$ is transformed to
$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d_2 = 0$$
$|\mathbf{Q}| = \lambda_1\lambda_2\lambda_3 d_2 \Rightarrow d_2 = \frac{|\mathbf{Q}|}{|\mathbf{S}|}$. Thus the quadric is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = -\frac{|\mathbf{Q}|}{|\mathbf{S}|}$, which is a central quadric i.e. a quadric surface with a center, e.g., a sphere, ellipsoid, or hyperboloid, depending upon the signs and magnitudes of the eigenvalues. If the right hand side has positive sign (maybe by multiplying the equation with -1), then look at the signs of the coefficients of the l.h.s. If all are positive, it is an ellipsoid, further if all are equal, it is a sphere. If 1 or 2 are negative, it is a hyperboloid. If all 3 are negative, the surface is the empty set.
2. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 3$. $|\mathbf{Q}| = \lambda_1\lambda_2\lambda_3 d_2 = 0 \Rightarrow d_2 = 0$, because $\lambda_1\lambda_2\lambda_3 \neq 0$. Thus the quadric becomes $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, which is a cone.
3. $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 2$
$$F(x,y,z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$
$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & w_2 \\ 0 & 0 & w_2 & d_2 \end{pmatrix}$$
$|\mathbf{Q}| = -\lambda_1\lambda_2 w_2^2$. Since $\rho(\mathbf{Q}) = 4$, $w_2 \neq 0$. Shifting the origin to $(0, 0, -\frac{d_2}{2w_2})$ we get
$$F(x,y,z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z = 0$$
where $w_2^2 = -|\mathbf{Q}|/\lambda_1\lambda_2$. The surface is a paraboloid.

4. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 2$

$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z + d_2 = 0$$

$\rho(\mathbf{Q}) = 3 \Rightarrow |\mathbf{Q}| = -\lambda_1\lambda_2 w_2^2 \Rightarrow w_2 = 0$. Since

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}$$

and $\rho(\mathbf{Q}) = 3$, $d_2 \neq 0$. Thus

$$F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 + d_2 = 0$$

The quadric is a hyperbolic or elliptic cylinder.

5. $\rho(\mathbf{Q}) = 2, \rho(\mathbf{S}) = 2$

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix}$$

$\rho(\mathbf{Q}) = 2 \Rightarrow d_2 = 0$, and $F(x, y, z) = \lambda_1 x^2 + \lambda_2 y^2 = 0$. The quadric is a pair of distinct planes or a point, if $\lambda_1 = \lambda_2 \neq 0$.

6. $\rho(\mathbf{Q}) = 4, \rho(\mathbf{S}) = 1$

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

$$\mathbf{Q} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & w_2 \\ 0 & v_2 & w_2 & d_2 \end{pmatrix}$$

which shows that $\rho(\mathbf{Q}) = 4$ is not possible.

7. $\rho(\mathbf{Q}) = 3, \rho(\mathbf{S}) = 1$

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z + d_2 = 0$$

$\rho(\mathbf{Q}) = 3 \Rightarrow$ both v_2 and w_2 cannot be 0. Suppose $v_2 \neq 0$. Shift the origin to $(0, -\frac{d_2}{2v_2}, 0)$.

$$F(x, y, z) = \lambda_1 x^2 + 2v_2 y + 2w_2 z = 0$$

Rotate the axes by

$$\begin{aligned} x &= X \\ y &= \frac{v_2}{\sqrt{v_2^2+w_2^2}}Y - \frac{w_2}{\sqrt{v_2^2+w_2^2}}Z \\ z &= \frac{w_2}{\sqrt{v_2^2+w_2^2}}Y + \frac{v_2}{\sqrt{v_2^2+w_2^2}}Z \end{aligned}$$

$$F(x,y,z) = \lambda_1 X^2 + 2v_3 Y = 0 \qquad v_3 = \sqrt{v_2^2+w_2^2}$$

Thus the quadric is a parabolic cylinder.

8. $\rho(\mathbf{Q}) = 2, \rho(\mathbf{S}) = 1$ $\rho(\mathbf{Q}) = 2 \Rightarrow v_2 = w_2 = 0, d_2 \neq 0.$

$$F(x,y,z) = \lambda_1 X^2 + d_2 = 0$$

The quadric is two parallel planes.

9. $\rho(\mathbf{Q}) = 1, \rho(\mathbf{S}) = 1$ The quadric immediately reduces to $F(x,y,z) = \lambda_1 X^2 = 0$, so it represents two coincident planes $x = 0$.

$\rho(\mathbf{Q})$	$\rho(\mathbf{S})$	Surface	Canonical form
4	3	central quadric	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = -\frac{\lvert\mathbf{Q}\rvert}{\lvert\mathbf{S}\rvert}$
3	3	cone	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$
4	2	paraboloid	$\lambda_1 x^2 + \lambda_2 y^2 + 2w_2 z = 0, w_2^2 = -\frac{\lvert\mathbf{Q}\rvert}{\lambda_1\lambda_2}$
3	2	elliptic or hyperbolic cylinder	$\lambda_1 x^2 + \lambda_2 y^2 + d_2 = 0$
2	2	pair of distinct planes if $\lambda_1\lambda_2 < 0$ point if $\lambda_1\lambda_2 > 0$	$\lambda_1 x^2 + \lambda_2 y^2 = 0$
4	1	Not possible	
3	1	parabolic cylinder	$\lambda_1 X^2 + 2v_3 Y = 0, v_3 = \sqrt{v_2^2+w_2^2}$
2	1	pair of parallel planes	$\lambda_1 X^2 + d_2 = 0$
1	1	Two coincident planes	$\lambda_1 X^2 = 0$

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Question 1(a) *Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and let*

$$\mathcal{T} = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathcal{V}\}$$

Define $(\mathbf{x}, \mathbf{y}) + (\mathbf{x_1}, \mathbf{y_1}) = (\mathbf{x} + \mathbf{x_1}, \mathbf{y} + \mathbf{y_1})$ *in* $\mathcal{T}$ *and* $(\alpha + i\beta)(\mathbf{x}, \mathbf{y}) = (\alpha\mathbf{x} - \beta\mathbf{y}, \beta\mathbf{x} + \alpha\mathbf{y})$ *for every* $\alpha, \beta \in \mathbb{R}$. *Show that* $\mathcal{T}$ *is a vector space over* $\mathbb{C}$.

Solution.

1. $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{T} \Rightarrow \mathbf{v_1} + \mathbf{v_2} \in \mathcal{T}$
2. $(\mathbf{0}, \mathbf{0})$ is the additive identity where $\mathbf{0}$ is the zero vector in $\mathcal{V}$.
3. If $(\mathbf{x}, \mathbf{y}) \in \mathcal{T}$, then $(-\mathbf{x}, -\mathbf{y}) \in \mathcal{T}$, and $(\mathbf{x}, \mathbf{y}) + (-\mathbf{x}, -\mathbf{y}) = (\mathbf{0}, \mathbf{0})$
4. Clearly $\mathbf{v_1} + \mathbf{v_2} = \mathbf{v_2} + \mathbf{v_1}$ and $(\mathbf{v_1} + \mathbf{v_2}) + \mathbf{v_3} = \mathbf{v_1} + (\mathbf{v_2} + \mathbf{v_3})$ as addition is commutative and associative in $\mathcal{V}$.
5. $z \in \mathbb{C}, \mathbf{v} \in \mathcal{T} \Rightarrow z\mathbf{v} \in \mathcal{T}$
6. $1\mathbf{v} = (1 + i0)(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$
7.

$$\begin{aligned}
& (\alpha + i\beta)((\mathbf{x_1}, \mathbf{y_1}) + (\mathbf{x_2}, \mathbf{y_2})) \\
= \ & (\alpha(\mathbf{x_1} + \mathbf{x_2}) - \beta(\mathbf{y_1} + \mathbf{y_2}), \beta(\mathbf{x_1} + \mathbf{x_2}) + \alpha(\mathbf{y_1} + \mathbf{y_2})) \\
= \ & (\alpha\mathbf{x_1} - \beta\mathbf{y_1}, \beta\mathbf{x_1} + \alpha\mathbf{y_1}) + (\alpha\mathbf{x_2} - \beta\mathbf{y_2}, \beta\mathbf{x_2} + \alpha\mathbf{y_2}) \\
= \ & (\alpha + i\beta)(\mathbf{x_1}, \mathbf{y_1}) + (\alpha + i\beta)(\mathbf{x_2}, \mathbf{y_2})
\end{aligned}$$

8.

$$
\begin{aligned}
& ((\alpha+i\beta)(\gamma+i\delta))(\mathbf{x},\mathbf{y}) \\
= \; & (\alpha\gamma-\beta\delta+i(\alpha\delta+\beta\gamma))(\mathbf{x},\mathbf{y}) \\
= \; & ((\alpha\gamma-\beta\delta)\mathbf{x}-(\alpha\delta+\beta\gamma)\mathbf{y},(\alpha\gamma-\beta\delta)\mathbf{y}+(\alpha\delta+\beta\gamma)\mathbf{x}) \\
= \; & (\alpha(\gamma\mathbf{x}-\delta\mathbf{y})-\beta(\delta\mathbf{x}+\gamma\mathbf{y}),\beta(\gamma\mathbf{x}-\delta\mathbf{y})+\alpha(\delta\mathbf{x}+\gamma\mathbf{y})) \\
= \; & (\alpha+i\beta)((\gamma\mathbf{x}-\delta\mathbf{y}),(\delta\mathbf{x}+\gamma\mathbf{y})) \\
= \; & (\alpha+i\beta)((\gamma+i\delta)(\mathbf{x},\mathbf{y}))
\end{aligned}
$$

Thus $\mathcal{T}$ is a vector space over $\mathbb{C}$. ∎

Question 1(b) *Show that if λ is a characteristic root of a non-singular matrix $\mathbf{A}$, then λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$.*

Solution.

$$
\begin{aligned}
\mathbf{A}\mathbf{v} &= \lambda\mathbf{v} \qquad \mathbf{v}\neq\mathbf{0} \\
\Rightarrow \quad \mathbf{A}^{-1}\mathbf{A}\mathbf{v} &= \lambda\mathbf{A}^{-1}\mathbf{v} \\
\Rightarrow \quad \mathbf{A}^{-1}\mathbf{v} &= \lambda^{-1}\mathbf{v}
\end{aligned}
$$

Thus λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$. ∎

Question 2(a) *Prove that a real symmetric matrix $\mathbf{A}$ is positive definite if and only if $\mathbf{A}=\mathbf{B}\mathbf{B}'$ for some non-singular $\mathbf{B}$. Show also that $\mathbf{A}=\begin{pmatrix}1&2&3\\2&5&7\\3&7&11\end{pmatrix}$ is positive definite and find $\mathbf{B}$ such that $\mathbf{A}=\mathbf{B}\mathbf{B}'$. (Here $\mathbf{B}'$ is the transpose of $\mathbf{B}$.)*

Solution. If $\mathbf{A}=\mathbf{B}\mathbf{B}'$ for some non-singular $\mathbf{B}$, then $\mathbf{x}'\mathbf{A}\mathbf{x}=\mathbf{x}'\mathbf{B}\mathbf{B}'\mathbf{x}$ where $\mathbf{x}\neq\mathbf{0}$ is a column vector. Since $|\mathbf{B}|\neq 0$, $\mathbf{B}'\mathbf{x}\neq\mathbf{0}\Longrightarrow\mathbf{x}'\mathbf{B}.(\mathbf{B}'\mathbf{x})$ is the sum on n squares, at least one of which is non-zero. Thus $\mathbf{x}'\mathbf{A}\mathbf{x}>0$ whenever $\mathbf{x}\neq\mathbf{0}$, showing that $\mathbf{A}$ is positive definite.

Conversely, if $\mathbf{A}$ is positive definite, then $\exists\mathbf{P}$ non-singular such that $\mathbf{P}'\mathbf{A}\mathbf{P}=\mathbf{I_n}$. Thus $\mathbf{A}=\mathbf{P}'^{-1}\mathbf{P}^{-1}$. Letting $\mathbf{B}=\mathbf{P}'^{-1}$ we get $\mathbf{A}=\mathbf{B}\mathbf{B}'$ as required.

The existence of $\mathbf{P}$ can be found by induction on n. Let

$$
\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & \ldots & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}
$$

Define

$$
\mathbf{Q}=\begin{pmatrix} 1 & -\frac{a_{12}}{a_{11}} & \ldots & -\frac{a_{1n}}{a_{11}} \\ 0 & 1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & 1 \end{pmatrix}
$$

Then $\mathbf{Q}$ is non-singular, and $\mathbf{Q'AQ} = \begin{pmatrix} a_{11} & 0 \\ 0 & \mathbf{S} \end{pmatrix}$, where $\mathbf{S}$ is $(n-1)\times(n-1)$ positive definite. Let $\mathbf{Q}^*$ be a $(n-1)\times(n-1)$ non-singular matrix such that $\mathbf{Q}^{*\prime}\mathbf{S}\mathbf{Q}^*$ is diagonal, by induction. Then let $\mathbf{Q_1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{Q}^* \end{pmatrix}$, and let $\mathbf{P} = \mathbf{Q_1 Q}$. Then $\mathbf{P'AP}$ is diagonal $(b_{11}, b_{22}, \ldots, b_{nn})$. Let $\mathbf{B} =$ diagonal $(\frac{1}{\sqrt{b_{11}}}, \ldots, \frac{1}{\sqrt{b_{nn}}})$. Then $\mathbf{B'P'APB} = \mathbf{I_n}$.

The quadratic form $Q(x,y,z)$ associated with the given matrix $\mathbf{A}$ is given by

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + 5y^2 + 11z^2 + 4xy + 6xz + 14yz$$

Completing the squares we get $Q(x,y,z) = (x+2y+3z)^2 + (y+z)^2 + z^2$, so $\mathbf{A}$ is positive definite, as $z = 0, y+z = 0, x+2y+3z = 0 \Longrightarrow x = y = z = 0$.

If $\mathbf{B}$ is a 3×3 matrix such that

$$\mathbf{B}' \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ y+z \\ z \end{pmatrix}$$

then $\mathbf{x'BB'x} = Q = \mathbf{x'Ax}$, so $\mathbf{A} = \mathbf{BB'}$ as $\mathbf{A}$ and $\mathbf{BB'}$ are both symmetric. Clearly

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$$

and it can easily be verified that $\mathbf{A} = \mathbf{BB'}$. ∎

Question 2(b) *Prove that a system* $\mathbf{Ax} = \mathbf{B}$ *of non-homogeneous equations in* n *unknowns has a unique solution provided the coefficient matrix is non-singular.*

Solution. If $\mathbf{A}$ is non-singular, then the system is consistent because the rank of the coefficient matrix $\mathbf{A} = n =$ rank of the $n \times n+1$ augmented matrix $(\mathbf{A}, \mathbf{B})$. If $\mathbf{x_1}, \mathbf{x_2}$ are two solutions, then

$$\begin{aligned} & \mathbf{Ax_1} = \mathbf{B} = \mathbf{Ax_2} \\ \Longrightarrow\ & \mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0} \\ \Longrightarrow\ & \mathbf{A}^{-1}\mathbf{A}(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0} \\ \Longrightarrow\ & \mathbf{x_1} = \mathbf{x_2} \end{aligned}$$

Thus the unique solution is given by the column vector $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$.

∎

Question 2(c) *Prove that two similar matrices have the same characteristic roots. Is the converse true? Justify your claim.*

Solution. Let $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ then characteristic polynomial of $\mathbf{B}$ is $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}\lambda\mathbf{I}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. (Note that $|\mathbf{X}||\mathbf{Y}| = |\mathbf{X}\mathbf{Y}|$.) Thus the characteristic polynomial of $\mathbf{B}$ is the same as that of $\mathbf{A}$, so both $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic roots.

The converse is not true. Let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial $(\lambda - 1)^2$ and thus the same characteristic roots. But $\mathbf{B}$ can never be similar to $\mathbf{A}$ because $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{B}$ whatever $\mathbf{P}$ may be. ∎

UPSC Civil Services Main 2001 - Mathematics Linear Algebra

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Question 1(a) *Show that the vectors* $(1,0,-1),(0,-3,2)$ *and* $(1,2,1)$ *form a basis of the vector space* $\mathbb{R}^3(\mathbb{R})$.

Solution. Since $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, it is enough to prove that these are linearly independent. If possible, let

$$a(1,0,-1) + b(0,-3,2) + c(1,2,1) = 0$$

This implies

$$a + c = 0, -3b + 2c = 0, -a + 2b + c = 0$$

Solving for c, $c + \frac{4}{3}c + c = 0$, so $c = 0$, hence $a = b = 0$. (Note that if these linearly independent vectors were not a basis, they could be completed into one, but in $\mathbb{R}^3$ any four vectors are linearly dependent, so this is a maximal linearly independent set, hence it is a basis.

Alternate Solution. Since $\dim(\mathbb{R}^3) = 3$, to show that $(1,0,-1),(0,-3,2)$ and $(1,2,1)$ form a basis it is enough to show that these vectors generate $\mathbb{R}^3$. In fact, given (x_1,x_2,x_3), we can always find a,b,c s.t. $(x_1,x_2,x_3) = a(1,0,-1) + b(0,-3,2) + c(1,2,1)$ as follows: $a + c = x_1, -3b + 2c = x_2, -a + 2b + c = x_3$. Thus $(c - x_1) + 2(2c - x_2)/3 + c = x_3$, so $c + \frac{4}{3}c + c = x_1 + \frac{2}{3}x_2 + x_3$. Thus $c = \frac{3x_1+2x_2+3x_3}{10}$, $a = x_1 - c = \frac{7x_1-2x_2-3x_3}{10}$, and $b = \frac{2c-x_2}{3} = \frac{x_1-x_2+x_3}{5}$. ∎

Question 1(b) *If* λ *is a characteristic root of a non-singular matrix* $\mathbf{A}$*, then prove that* $\frac{|\mathbf{A}|}{\lambda}$ *is a characteristic root of* **Adj A**.

Solution. If μ is a characteristic root of $\mathbf{A}$, then $a\mu$ is a characteristic root of $a\mathbf{A}$ for a constant a, because if $\mathbf{A}\mathbf{v} = \mu\mathbf{v}$, $\mathbf{v} \neq 0$ a vector, then $a\mathbf{A}\mathbf{v} = a\mu\mathbf{v}$. Hence the result.

If λ is the characteristic root of $\mathbf{A}$, $|\mathbf{A}| \neq 0$, then $\lambda \neq 0$, and λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$, because $\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Longrightarrow \mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$.

Since $\mathbf{Adj\ A} = \mathbf{A}^{-1}|\mathbf{A}|$, it follows that $\frac{|\mathbf{A}|}{\lambda}$ is a characteristic root of **Adj A**. ∎

Question 2(a) *If* $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *show that for all integers* $n \geq 3, \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$. *Hence determine* $\mathbf{A}^{50}$.

Solution. Characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} = 0$$

or $(\lambda-1)(\lambda^2-1) = \lambda^3-\lambda^2-\lambda+1 = 0$. From the Cayley-Hamilton theorem, $\mathbf{A}^3-\mathbf{A}^2-\mathbf{A}+\mathbf{I} = 0 \Rightarrow \mathbf{A}^3 = \mathbf{A} + \mathbf{A}^2 - \mathbf{I}$. Thus the result is true for $n = 3$. Suppose the theorem is true for $n = m$ i.e. $\mathbf{A}^m = \mathbf{A}^{m-2} + \mathbf{A}^2 - \mathbf{I}$. We shall prove it for $m + 1$.

$$\begin{aligned} \mathbf{A}^{m+1} &= \mathbf{A}^m\mathbf{A} \\ &= (\mathbf{A}^{m-2} + \mathbf{A}^2 - \mathbf{I})\mathbf{A} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^3 - \mathbf{A} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^2 + \mathbf{A} - \mathbf{A} - \mathbf{I} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^2 - \mathbf{I} \end{aligned}$$

The result follows by induction.

Let $n = 2m$. Using successively $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$, we get $\mathbf{A}^{2m} = m\mathbf{A}^2 - (m-1)\mathbf{I}$. Now

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

so

$$\begin{aligned} \mathbf{A}^{50} &= 25\mathbf{A}^2 - 24\mathbf{I} \\ &= \begin{pmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix} \end{aligned}$$

∎

Question 2(b) *When is a square matrix* $\mathbf{A}$ *said to be congruent to a square matrix* $\mathbf{B}$*? Prove that every matrix congruent to a skew-symmetric matrix is skew-symmetric.*

Solution. $\mathbf{A} \equiv \mathbf{B}$ if $\exists \mathbf{P}$ nonsingular, s.t. $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}$. If $\mathbf{S}' = -\mathbf{S}$ then $(\mathbf{P}'\mathbf{S}\mathbf{P})' = \mathbf{P}'\mathbf{S}'\mathbf{P} = -(\mathbf{P}'\mathbf{S}\mathbf{P})$, so $\mathbf{P}'\mathbf{S}\mathbf{P}$ is also skew-symmetric. ∎

Question 2(c) *Determine the orthogonal matrix* $\mathbf{P}$ *such that* $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ *is diagonal where* $\mathbf{A} = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$.

Solution. The characteristic equation is

$$\begin{aligned}
\begin{vmatrix} \lambda - 7 & -4 & 4 \\ -4 & \lambda + 8 & 1 \\ 4 & 1 & \lambda + 8 \end{vmatrix} &= 0 \\
(\lambda - 7)((\lambda + 8)^2 - 1) + 4(-4 - 4\lambda - 32) + 4(-4 - 4\lambda - 32) &= 0 \\
\lambda^3 + 9\lambda^2 - 81\lambda - 729 &= 0 \\
(\lambda + 9)(\lambda^2 - 81) &= 0
\end{aligned}$$

Thus $\lambda = 9, -9, -9$.

1. $\lambda = 9$. If (x_1, x_2, x_3) is the eigenvector corresponding to $\lambda = 9$, we get

$$\begin{aligned}
2x_1 - 4x_2 + 4x_3 &= 0 \\
-4x_1 + 17x_2 + x_3 &= 0 \\
4x_1 + x_2 + 17x_3 &= 0
\end{aligned}$$

   From the second and third we get $18x_2 + 18x_3 = 0$. Take $x_2 = 1$. Then $x_3 = -1, x_1 = 4$, so $(4, 1, -1)$ is an eigenvector for $\lambda = 9$.

2. $\lambda = -9$. If (x_1, x_2, x_3) is the eigenvector corresponding to $\lambda = -9$, we get

$$\begin{aligned}
-16x_1 - 4x_2 + 4x_3 &= 0 \\
-4x_1 - x_2 + x_3 &= 0 \\
4x_1 + x_2 - x_3 &= 0
\end{aligned}$$

   There is only one equation $4x_1 + x_2 - x_3 = 0$. Take $x_1 = 0, x_2 = 1$, then $x_3 = 1$, so $(0, 1, 1)$ is an eigenvector. Take $x_1 = -1, x_2 = 2$, then $x_3 = -2$, so $(-1, 2, -2)$ is another eigenvector. These two are orthogonal to each other and are eigenvectors for $\lambda = -9$. Note that to make the second vector orthogonal to the first, we needed to ensure $x_2 = -x_3$, then the equation suggested values for x_1, x_2.

Let

$$\mathbf{P} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} & -\frac{1}{\sqrt{18}} \end{pmatrix}$$

Clearly $\mathbf{P}'\mathbf{P} = \mathbf{I}$, since the columns of $\mathbf{P}$ are mutually orthogonal unit vectors. Moreover from $\mathbf{A}\mathbf{x} = \mathbf{x}\lambda$ for the eigenvalues and eigenvectors, it follows that $\mathbf{AP} = \mathbf{P}\begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

Thus $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, which is diagonal as required. ∎

Question 2(d) *Show that the real quadratic form*

$$\Phi = n(x_1^2 + x_2^2 + \ldots + x_n^2) - (x_1 + x_2 + \ldots + x_n)^2$$

in n variables is positive semi-definite.

Solution. Consider the expression

$$\begin{aligned} E &= (X - x_1)^2 + \ldots + (X - x_n)^2 \\ &= nX^2 - 2X(x_1 + \ldots + x_n) + (x_1^2 + x_2^2 + \ldots + x_n^2) \end{aligned}$$

Clearly E being the sum of squares is non-negative, i.e. $E \geq 0$. Let

$$A = \frac{(x_1 + x_2 + \ldots + x_n)}{n} \qquad B = \frac{(x_1^2 + x_2^2 + \ldots + x_n^2)}{n}$$

Then $E = n(X^2 - 2AX + B) = n((X - A)^2 + B - A^2)$. When $X = A$, $E = n(B - A^2) = \Phi$, and since $E \geq 0$, $\Phi \geq 0$.

If $x_1 = x_2 = \ldots = x_n = 1$, then $\Phi = 0$ showing that Φ is actually positive semi-definite.

Alternate solution. By Cauchy's inequality

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Setting $b_1 = b_2 = \ldots = b_n = 1$, we get

$$n\left(\sum_{i=1}^{n} a_i^2\right) - \left(\sum_{i=1}^{n} a_i\right)^2 \geq 0$$

showing that Φ is positive semi-definite. ∎

UPSC Civil Services Main 2002 - Mathematics
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Question 1(a) *Show that the mapping* $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *where* $\mathbf{T}(a, b, c) = (a - b, b - c, a + c)$ *is linear and non-singular.*

Solution.

$$\begin{aligned} \mathbf{T}(\lambda a, \lambda b, \lambda c) &= (\lambda(a - b), \lambda(b - c), \lambda(a + c)) \\ &= \lambda(a - b, b - c, a + c) \\ &= \lambda\mathbf{T}(a, b, c) \end{aligned}$$

$$\begin{aligned} \mathbf{T}(a, b, c) + \mathbf{T}(x, y, z) &= (a - b, b - c, a + c) + (x - y, y - z, x + z) \\ &= (a - b + x - y, b - c + y - z, a + c + x + z) \\ &= \mathbf{T}(a + x, b + y, c + z) \end{aligned}$$

Thus $\mathbf{T}$ is linear.

Now we show that

$$\begin{aligned} \mathbf{T}(1, 0, 0) &= (1, 0, 1) \\ \mathbf{T}(0, 1, 0) &= (-1, 1, 0) \\ \mathbf{T}(0, 0, 1) &= (0, -1, 1) \end{aligned}$$

are linearly independent.

$$\begin{aligned} & a_1(1, 0, 1) + a_2(-1, 1, 0) + a_3(0, -1, 1) = 0 \\ \Rightarrow\ & a_1 - a_2 = 0, a_2 - a_3 = 0, a_1 + a_3 = 0 \\ \Rightarrow\ & a_1 = a_2 = a_3 = 0 \end{aligned}$$

Thus $(1,0,1),(-1,1,0),(0,-1,1)$ are linearly independent.

Since $(1,0,0),(0,1,0),(0,0,1)$ generate $\mathbb{R}^3$, $(1,0,1),(-1,1,0),(0,-1,1)$ generate $\mathbf{T}(\mathbb{R}^3)$, hence $\dim(\mathbf{T}(\mathbb{R}^3)) = 3$. Thus $\mathbf{T}$ is non-singular.

Alternatively,

$$\mathbf{T}(a,b,c) = (0,0,0) \iff a-b=0, b-c=0, a+c=0 \implies a=b=c=0$$

Thus $\mathbf{T}$ is 1-1, therefore it is onto, which shows it is nonsingular. ∎

Question 1(b) *Prove that a square matrix* $\mathbf{A}$ *is non-singular if and only if the constant term in its characteristic polynomial is different from* 0.

Solution. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

Then characteristic polynomial of $\mathbf{A} = \mathbf{Det}(x\mathbf{I}-\mathbf{A})$. $\mathbf{I} = n \times n$ unit matrix. Clearly

$$\mathbf{Det}(x\mathbf{I}-\mathbf{A}) = x^n - \sum_{i=1}^{n} a_{ii}x^{n-1} + \ldots + (-1)^n\mathbf{Det}\mathbf{A}$$

Thus $\mathbf{A}$ is nonsingular iff the constant term in the characteristic polynomial of $\mathbf{A} \neq 0$. ∎

Question 2(a) *Let* $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ *be a linear mapping given by* $T(a,b,c,d,e) = (b-d, d+e, b, 2d+e, b+e)$. *Obtain bases for its null space and range space.*

Solution. Clearly

$$\begin{aligned}
T(1,0,0,0,0) &= (0,0,0,0,0) \\
T(0,1,0,0,0) &= (1,0,1,0,1) \\
T(0,0,1,0,0) &= (0,0,0,0,0) \\
T(0,0,0,1,0) &= (-1,1,0,2,0) \\
T(0,0,0,0,1) &= (0,1,0,1,1)
\end{aligned}$$

are generators of the range space of T. In fact, if $\mathbf{v_1} = (1,0,1,0,1), \mathbf{v_2} = (-1,1,0,2,0), \mathbf{v_3} = (0,1,0,1,1)$ then $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3}$ generate $T(\mathbb{R}^5)$. We now show that $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3}$ are linearly independent. Let $\alpha_1\mathbf{v_1}+\alpha_2\mathbf{v_2}+\alpha_3\mathbf{v_3} = 0$. Then $\alpha_1-\alpha_2 = 0, \alpha_2+\alpha_3 = 0, \alpha_1 = 0 \Rightarrow \alpha_2 = \alpha_3 = 0$. Thus $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3}$ are linearly independent over $\mathbb{R} \Rightarrow T(\mathbb{R}^5)$ is of dimension 3 with basis $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3}$.

Thus the null space is of dimension 2, because dim(null space) + dim(range space) = dim(given vector space = $\mathbb{R}^5$) = 5. Since $\mathbf{e_1} = (1,0,0,0,0)$ and $\mathbf{e_3} = (0,0,1,0,0)$ belong to the null space of T, and both are linearly independent over $\mathbb{R}$, $\mathbf{e_1},\mathbf{e_3}$ is a basis of the null space of T. ∎

Question 2(b) *Let* $\mathbf{A}$ *be a* 3×3 *real symmetric matrix with eigenvalues* $0, 0, 5$. *If the corresponding eigenvectors are* $(2, 0, 1), (2, 1, 1), (1, 0, -2)$ *then find the matrix* $\mathbf{A}$.

Solution. Let $\mathbf{P} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$, then $\mathbf{P^{-1}AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, so $\mathbf{A} = \mathbf{P} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \mathbf{P^{-1}}$.

A simple calculation shows that $\mathbf{P^{-1}} = \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix}$, therefore

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -1 & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & 0 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Thus $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$ is the required symmetric matrix with $0, 0, 5$ as eigenvalues. ∎

Question 2(c) *Solve the following system of linear equations:*

$$\begin{aligned} x_1 - 2x_2 - 3x_3 + 4x_4 &= -1 \\ -x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 &= 0 \\ 2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 &= 17 \end{aligned}$$

Solution. There are three equations in 5 unknowns, therefore the rank of the coefficient matrix ≤ 3. Since $\begin{vmatrix} 1 & -2 & -3 \\ -1 & 3 & 5 \\ -2 & 1 & -2 \end{vmatrix} = 1(-6-5) + 2(2-10) + (-3)(-1-6) = -6$, the rank of the coefficient matrix is 3. Using Cramer's rule we solve the system

$$\begin{aligned} x_1 - 2x_2 - 3x_3 &= -1 - 4x_4 && (1) \\ -x_1 + 3x_2 + 5x_3 &= 5x_4 + 2x_5 && (2) \\ 2x_1 + x_2 - 2x_3 &= 17 - 3x_4 + 4x_5 && (3) \end{aligned}$$

$$
\begin{aligned}
x_1 &= -\frac{1}{6}\begin{vmatrix} -1-4x_4 & -2 & -3 \\ 5x_4+2x_5 & 3 & 5 \\ 17-3x_4+4x_5 & 1 & -2 \end{vmatrix} \\
&= -\frac{1}{6}[(-1-4x_4)(-11) - 3(5x_4+2x_5-51+9x_4-12x_5) + 2(-10x_4-4x_5-85+15x_4-20x_5)] \\
&= -\frac{1}{6}[-6+44x_4-42x_4+10x_4+30x_5-48x_5] \\
&= 1-2x_4+3x_5
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= -\frac{1}{6}\begin{vmatrix} 1 & -1-4x_4 & -3 \\ -1 & 5x_4+2x_5 & 5 \\ 2 & 17-3x_4+4x_5 & -2 \end{vmatrix} \\
&= -\frac{1}{6}[-10x_4-4x_5-85+15x_4-20x_5-8-32x_4+51-9x_4+12x_5+30x_4+12x_5] \\
&= -\frac{1}{6}[-42-6x_4] \\
&= 7+x_4
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= -\frac{1}{6}\begin{vmatrix} 1 & -2 & -1-4x_4 \\ -1 & 3 & 5x_4+2x_5 \\ 2 & 1 & 17-3x_4+4x_5 \end{vmatrix} \\
&= -\frac{1}{6}[51-9x_4+12x_5-5x_4-2x_5-34+6x_4-8x_5-20x_4-8x_5+7+28x_4] \\
&= -\frac{1}{6}[24-6x_5] \\
&= -4+x_5
\end{aligned}
$$

The solution space is $(1-2x_4+3x_5, 7+x_4, -4+x_5, x_4, x_5)$, where $x_4, x_5 \in \mathbb{R}$ (arbitrarily). Note that the vector space of solutions is of dimension 2.

Alternate Method.

$$
\begin{aligned}
x_2+2x_3 &= -1+x_4+2x_5 \text{ adding (1) and (2)} && (4) \\
7x_2+8x_3 &= 17+7x_4+8x_5 \text{ adding } 2\times(2) \text{ and (3)} && (5) \\
6x_3 &= -24+6x_5 \text{ using } 7\times(4) \text{ - (5)} && (6) \\
x_2 &= 7+x_4 \text{ from (4) and (6)} && (7) \\
x_1 &= 1-2x_4+3x_5 \text{ using (1), (6), (7)} && (8)
\end{aligned}
$$

The solution space is as shown above. ∎

Question 2(d) *Use Cayley-Hamilton theorem to find the inverse of the following matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

Solution. Characteristic polynomial is given by $|x\mathbf{I} - \mathbf{A}| = 0$, where $\mathbf{I}$ is the 3×3 unit matrix.

$$\begin{aligned} \begin{vmatrix} x & -1 & -2 \\ -1 & x-2 & -3 \\ -3 & -1 & x-1 \end{vmatrix} &= 0 \\ x[x^2 - 3x + 2 - 3] + 1[-x + 1 - 9] - 2[1 + 3x - 6] &= 0 \\ x^3 - 3x^2 - 8x + 2 &= 0 \end{aligned}$$

By Cayley-Hamilton theorem, $\mathbf{A}^3 - 3\mathbf{A}^2 - 8\mathbf{A} + 2\mathbf{I} = 0$, or $\mathbf{A}(\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) = -2\mathbf{I}$. Thus

$$\begin{aligned} \mathbf{A}^{-1} &= -\frac{1}{2}(\mathbf{A}^2 - 3\mathbf{A} - 8\mathbf{I}) \\ &= -\frac{1}{2}\left[\begin{pmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 0 & 3 & 6 \\ 3 & 6 & 9 \\ 9 & 3 & 3 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}\right] \\ &= -\frac{1}{2}\begin{pmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

Check $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. ∎

UPSC Civil Services Main 2003 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{S}$ be any non-empty subset of a vector space $\mathcal{V}$ over the field F. Show that the set $\{a_1\mathbf{x_1} + \ldots + a_n\mathbf{x_n} \mid a_1, \ldots, a_n \in F,\ \mathbf{x_1}, \ldots \mathbf{x_n} \in \mathcal{S},\ n \in \mathbb{N}\}$ is the subspace generated by $\mathcal{S}$.*

Solution. Let $\mathcal{W}$ be the subset mentioned above. Let $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{W}$ and $a, b \in F$. Then $\mathbf{w_1} = a_1\mathbf{x_1} + \ldots a_r\mathbf{x_r}$, where $a_1, \ldots, a_r \in F, \mathbf{x_1}, \ldots \mathbf{x_r} \in \mathcal{S}$ and $\mathbf{w_2} = b_1\mathbf{y_1} + \ldots b_s\mathbf{y_s}$ where $b_1, \ldots b_s \in F, \mathbf{y_1}, \ldots, \mathbf{y_s} \in \mathcal{S}$. Now $\alpha\mathbf{w_1} + \beta\mathbf{w_2} = c_1\mathbf{z_1} + \ldots + c_{r+s}\mathbf{z_{r+s}}$, where $c_i = \alpha a_i, 1 \leq i \leq r,\ c_{j+r} = \beta b_j, 1 \leq j \leq s$, and $\mathbf{z_i} = \mathbf{x_i}, 1 \leq i \leq r,\ \mathbf{z_{j+r}} = \mathbf{y_j}, 1 \leq j \leq s$. Clearly $c_j \in F, \mathbf{z_j} \in \mathcal{S}$ for $1 \leq j \leq r+s$, showing that for any $\mathbf{w_1}, \mathbf{w_2} \in \mathcal{W}, \alpha, \beta \in F, \alpha\mathbf{w_1} + \beta\mathbf{w_2} \in \mathcal{W}$, moreover $\mathcal{W} \neq \emptyset$ as $\mathcal{S} \subseteq \mathcal{W}$ and $\mathcal{S} \neq \emptyset$. Thus $\mathcal{W}$ is a subspace of $\mathcal{V}$. ∎

Question 1(b) *If* $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, *then find the matrix represented by* $2\mathbf{A}^{10} - 10\mathbf{A}^9 + 14\mathbf{A}^8 - 6\mathbf{A}^7 - 3\mathbf{A}^6 + 15\mathbf{A}^5 - 21\mathbf{A}^4 + 9\mathbf{A}^3 + \mathbf{A} - \mathbf{I}$.

Solution. The characteristic equation of $\mathbf{A}$ is

$$|\mathbf{A} - x\mathbf{I}| = \begin{vmatrix} 2-x & 1 & 1 \\ 0 & 1-x & 0 \\ 1 & 1 & 2-x \end{vmatrix} = (2-x)^2(1-x) - (1-x) = 0$$

or $(1-x)(4-4x+x^2) - 1 + x = 3 - 7x + 5x^2 - x^3 = 0$, or $x^3 - 5x^2 + 7x - 3 = 0$. By the

Cayley-Hamilton theorem, we get $\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I} = \mathbf{0}$. Now

$$\begin{aligned}
& 2\mathbf{A}^{10} - 10\mathbf{A}^9 + 14\mathbf{A}^8 - 6\mathbf{A}^7 - 3\mathbf{A}^6 + 15\mathbf{A}^5 - 21\mathbf{A}^4 + 9\mathbf{A}^3 + \mathbf{A} - \mathbf{I} \\
= \; & 2\mathbf{A}^7[\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I}] - 3\mathbf{A}^3[\mathbf{A}^3 - 5\mathbf{A}^2 + 7\mathbf{A} - 3\mathbf{I}] + \mathbf{A} - \mathbf{I} \\
= \; & 2\mathbf{A}^7\mathbf{0} - 3\mathbf{A}^3\mathbf{0} + \mathbf{A} - \mathbf{I} \\
= \; & \mathbf{A} - \mathbf{I} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\end{aligned}$$

which is the required value. ■

Question 2(a) *Prove that the eigenvectors corresponding to distinct eigenvalues of a square matrix are linearly independent.*

Solution. Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of the square matrix $\mathbf{A}$.

We will show that if any subset of the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ is linearly dependent, then we can find a smaller set that is also linearly dependent — but this leads to a contradiction as the eigenvectors are all non-zero.

Suppose, without loss of generality, that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ are linearly dependent. Thus there exist $\alpha_1, \ldots \alpha_r \in \mathbb{R}$, not all zero, such that

$$\alpha_1\mathbf{x_1} + \ldots + \alpha_r\mathbf{x_r} = \mathbf{0} \tag{1}$$

Thus $\mathbf{A}(\alpha_1\mathbf{x_1} + \ldots + \alpha_r\mathbf{x_r}) = \mathbf{0} \Rightarrow \alpha_1\lambda_1\mathbf{x_1} + \ldots + \alpha_r\lambda_r\mathbf{x_r} = \mathbf{0}$. Multiplying (1) by λ_1 and subtracting, we have $\alpha_2(\lambda_2 - \lambda_1)\mathbf{x_2} + \ldots + \alpha_r(\lambda_r - \lambda_1)\mathbf{x_r} = \mathbf{0}$. Now $\alpha_i \neq 0 \Rightarrow \alpha_i(\lambda_i - \lambda_1) \neq 0$, so not all $\alpha_i(\lambda_i - \lambda_1)$ can be zero, so we have a smaller set $\mathbf{x_2}, \ldots, \mathbf{x_r}$ which is also linearly dependent. This leads us to the contradiction mentioned above, hence the vectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}$ must be linearly independent. ■

Question 2(b) *If* $\mathbf{H}$ *is a Hermitian matrix, then show that* $(\mathbf{H} + i\mathbf{I})^{-1}(\mathbf{H} - i\mathbf{I})$ *is a unitary matrix. Also show that every unitary matrix* $\mathbf{A}$ *can be written in this form provided* 1 *is not an eigenvalue of* $\mathbf{A}$.

Solution. See related results of 1989, question 2(b). ■

Question 2(c) *If* $\mathbf{A} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ *then find a diagonal matrix* $\mathbf{D}$ *and a matrix* $\mathbf{B}$ *such that* $\mathbf{A} = \mathbf{BDB'}$ *where* $\mathbf{B'}$ *denotes the transpose of* $\mathbf{B}$.

Solution. Let $\mathbb{Q}(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be the quadratic form associated with $\mathbf{A}$. Then

$$\begin{aligned} \mathbb{Q}(x_1, x_2, x_3) &= 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3 \\ &= 6[x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3]^2 + \frac{7}{3}x_2^2 + \frac{7}{3}x_3^2 - \frac{2}{3}x_2x_3 \\ &= 6[x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3]^2 + \frac{7}{3}[x_2 - \frac{1}{7}x_3]^2 + \frac{16}{7}x_3^2 \end{aligned}$$

Let $X_1 = x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3, X_2 = x_2 - \frac{1}{7}x_3, X_3 = x_3$ and $\mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{pmatrix}$. Then

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \mathbf{BDB}' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix} \mathbf{D} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

where $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{B}' \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Thus $\mathbf{A} = \mathbf{BDB}'$ where $\mathbf{D} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ \frac{1}{3} & -\frac{1}{7} & 1 \end{pmatrix}$ ∎

Question 2(d) *Reduce the quadratic form given below to canonical form and find its rank and signature:*

$$x^2 + 4y^2 + 9z^2 + u^2 - 12yx + 6zx - 4zy - 2xu - 6zu$$

Solution. Let

$$\begin{aligned} \mathbb{Q}(x, y, z, u) &= x^2 + 4y^2 + 9z^2 + u^2 - 12yx + 6zx - 4zy - 2xu - 6zu \\ &= (x - 6y + 3z - u)^2 - 32y^2 + 32yz - 12yu \\ &= (x - 6y + 3z - u)^2 - 32(y^2 - yz + \frac{3}{8}yu) \\ &= (x - 6y + 3z - u)^2 - 32(y - \frac{1}{2}z + \frac{3}{4}u)^2 + 8z^2 + 18u^2 - 24uz \\ &= (x - 6y + 3z - u)^2 - 32(y - \frac{1}{2}z + \frac{3}{4}u)^2 + 8(z - \frac{3}{2}u)^2 \end{aligned}$$

Put

$$\begin{aligned} X &= x - 6y + 3z - u \\ Y &= y - \frac{1}{2}z + \frac{3}{4}u \\ Z &= z - \frac{3}{2}u \\ U &= u \end{aligned}$$

so that $\mathbb{Q}(x, y, z, u)$ is transformed to $X^2 - 32Y^2 + 8Z^2$. We now put $X^* = X, Y^* = \sqrt{32}Y, Z^* = \sqrt{8}Z, U^* = U$ to get ${X^*}^2 - {Y^*}^2 + Z^{*2}$ as the canonical form of $\mathbb{Q}(x, y, z, u)$.

Rank of $\mathbb{Q}(x, y, z, u) = 3 =$ rank of the associated matrix. Signature of $\mathbb{Q}(x, y, z, u) =$ number of positive squares - number of negative squares $= 2 - 1 = 1$. ∎

UPSC Civil Services Main 2004 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{S}$ be the space generated by the vectors $\{(0, 2, 6), (3, 1, 6), (4, -2, -2)\}$. What is the dimension of $\mathcal{S}$? Find a basis for $\mathcal{S}$.*

Solution. $(0, 2, 6), (3, 1, 6)$ are linearly independent, because $\alpha(0, 2, 6) + \beta(3, 1, 6) = \mathbf{0} \Rightarrow 3\beta = 0, 2\alpha + \beta = 0 \Rightarrow \alpha = \beta = 0$. Thus $\dim \mathcal{S} \geq 2$.

If possible let $(4, -2, -2) = \alpha(0, 2, 6) + \beta(3, 1, 6)$, then $4 = 3\beta, -2 = 2\alpha + \beta, -2 = 6\alpha + 6\beta$ should be consistent. Clearly $\beta = \frac{4}{3}, \alpha = \frac{1}{2}(-2 - \frac{4}{3}) = -\frac{5}{3}$ from the first two equations, and these values satisfy the third. Thus $(4, -2, -2)$ is a linear combination of $(0, 2, 6)$ and $(3, 1, 6)$.

Hence $\dim \mathcal{S} = 2$ and $\{(0, 2, 6), (3, 1, 6)\}$ is a basis of $\mathcal{S}$, being a maximal linearly independent subset of a generating system. ∎

Question 1(b) *Show that $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ where $f(x, y, z) = 3x + y - z$ is a linear transformation. What is the dimension of the kernel? Find a basis for the kernel.*

Solution.

$$\begin{aligned}
f(\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2)) &= f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) \\
&= 3(\alpha x_1 + \beta x_2) + \alpha y_1 + \beta y_2 - (\alpha z_1 + \beta z_2) \\
&= \alpha(3x_1 + y_1 - z_1) + \beta(3x_2 + y_2 - z_2) \\
&= \alpha f(x_1, y_1, z_1) + \beta f(x_2, y_2, z_2)
\end{aligned}$$

Thus f is a linear transformation.

Easy solution for this particular example. Clearly $(1, 0, 0)$ does not belong to the kernel, therefore the dimension of the kernel is ≤ 2. A simple look at f shows that $(0, 1, 1)$ and $(1, -1, 2)$ belong to the kernel and are linearly independent, thus the dimension of the kernel is 2 and $\{(0, 1, 1), (1, -1, 2)\}$ is a basis for the kernel.

General solution. Clearly $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is onto, thus the dimension of the range of f is 1. From question 3(a) of 1998, dimension of nullity of f + dimension of range of f = dimension of domain of f, so the dimension of the nullity of $f = 2$. Given this, we can pick a basis for the kernel by looking at the given transformation. ∎

Question 2(a) *Show that* $\mathbf{T}$ *the linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^4$ *represented by the matrix*

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

is one to one. Find a basis for its image.

Solution. $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the standard basis for $\mathbb{R}^3$. Then

$$\begin{aligned} \mathbf{T}(\mathbf{e_1}) &= (1, 0, 2, -1) = \mathbf{v_1} \\ \mathbf{T}(\mathbf{e_2}) &= (3, 1, 1, 1) = \mathbf{v_2} \\ \mathbf{T}(\mathbf{e_3}) &= (0, -2, 1, 2) = \mathbf{v_3} \end{aligned}$$

By linearity, if $\mathbf{T}(a, b, c) = a\mathbf{v_1} + b\mathbf{v_2} + c\mathbf{v_3} = \mathbf{0}$, then $a + 3b = 0, b - 2c = 0, 2a + b + c = 0, -a + b + 2c = 0 \Rightarrow a = b = c = 0$. Thus $\mathbf{T}$ is one-one. Also $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ forms a basis for the image, since $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ generates $\mathbb{R}^3$, and $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set. ∎

Question 2(b) *Verify whether the following system of equations is consistent:*

$$\begin{aligned} x + 3z &= 5 \\ -2x + 5y - z &= 0 \\ -x + 4y + z &= 4 \end{aligned}$$

Solution. The first equation gives $x = 5 - 3z$, the second now gives $5y = z + 10 - 6z = 10 - 5z \Rightarrow y = 2 - z$. Putting these values in the third equation we get $4 = -5 + 3z + 8 - 4z + z = 3$, hence the given system is inconsistent.

Alternative. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 5 & -1 \\ -1 & 4 & 1 \end{pmatrix}$ be the coefficient matrix and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 3 & 5 \\ -2 & 5 & -1 & 0 \\ -1 & 4 & 1 & 4 \end{pmatrix}$ be the augmented matrix, then it can be shown that rank $\mathbf{A} = 2$ and rank $\mathbf{B} = 3$, which implies that the system is inconsistent. For consistency the ranks should be equal. This procedure will be longer in this particular case. ∎

Question 2(c) *Find the characteristic polynomial of the matrix* $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$*. Hence find* $\mathbf{A}^{-1}$ *and* $\mathbf{A}^6$.

Solution. The characteristic polynomial of $\mathbf{A}$ is given by $|x\mathbf{I} - \mathbf{A}| = \begin{vmatrix} x-1 & -1 \\ 1 & x-3 \end{vmatrix} = (x-1)(x-3)+1 = x^2 - 4x + 4$.

The Cayley-Hamilton theorem states that $\mathbf{A}$ satisfies its characteristic equation i.e. $\mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I} = \mathbf{0} \Rightarrow (\mathbf{A} - 4\mathbf{I})\mathbf{A} = \mathbf{A}(\mathbf{A} - 4\mathbf{I}) = -4\mathbf{I}$. Thus $\mathbf{A}^{-1} = -\frac{\mathbf{A}-4\mathbf{I}}{4} = -\frac{1}{4}\begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

From $\mathbf{A}^2 - 4\mathbf{A} + 4\mathbf{I} = \mathbf{0}$ we get

$$\begin{aligned}
\mathbf{A}^2 &= 4\mathbf{A} - 4\mathbf{I} \\
\mathbf{A}^3 &= 4\mathbf{A}^2 - 4\mathbf{A} = 4(4\mathbf{A} - 4\mathbf{I}) - 4\mathbf{A} = 12\mathbf{A} - 16\mathbf{I} \\
\mathbf{A}^6 &= (12\mathbf{A} - 16\mathbf{I})^2 = 144\mathbf{A}^2 - 384\mathbf{A} + 256\mathbf{I} = 144(4\mathbf{A} - 4\mathbf{I}) - 384\mathbf{A} + 256\mathbf{I} \\
&= 192\mathbf{A} - 320\mathbf{I} = \begin{pmatrix} -128 & 192 \\ -192 & 256 \end{pmatrix}
\end{aligned}$$

∎

Question 2(d) *Define a positive definite quadratic form. Reduce the quadratic form* $x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3$ *to canonical form. Is this quadratic form positive definite?*

Solution. If $\mathbb{Q}(x_1, \ldots, x_n) = \sum_{\substack{i=1 \\ j=1}}^{n} a_{ij}x_ix_j, a_{ij} = a_{ji}$ is a quadratic form in n variables with $a_{ij} \in \mathbb{R}$, then it is said to be positive definite if $Q(\alpha_1, \ldots, \alpha_n) > 0$ whenever $\alpha_i \in \mathbb{R}, i = 1, \ldots, n$ and $\sum_i \alpha_i^2 > 0$.

Let the qiven be $\mathbb{Q}(x_1, x_2, x_3)$. Then

$$\begin{aligned}
\mathbb{Q}(x_1, x_2, x_3) &= x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 \\
&= (x_1 + x_2)^2 + x_3^2 + 2x_2x_3 - x_2^2 \\
&= (x_1 + x_2)^2 + (x_2 + x_3)^2 - 2x_2^2
\end{aligned}$$

Let $X_1 = x_1 + x_2, X_2 = x_2, X_3 = x_2 + x_3$ i.e.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

then $\mathbb{Q}(x_1, x_2, x_3)$ is transformed to $X_1^2 - 2X_2^2 + X_3^2$. Since $\mathbb{Q}(x_1, x_2, x_3)$ and the transformed quadratic form assume the same values, $\mathbb{Q}(x_1, x_2, x_3)$ is an indefinite form. The canonical form of $\mathbb{Q}(x_1, x_2, x_3)$ is $Z_1^2 - Z_2^2 + Z_3^2$ where $Z_1 = X_1, Z_2 = \sqrt{2}X_2, Z_3 = X_3$. ∎

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Question 1(a) *Find the values of k for which the vectors $(1,1,1,1),(1,3,-2,k),(2,2k-2,-k-2,3k-1)$ and $(3,k-2,-3,2k+1)$ are linearly independent in $\mathbb{R}^4$.*

Solution. The given vectors are linearly independent if the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{pmatrix}$$

is non-singular.

Now

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & k-1 \\ 2 & 2k-4 & -k-4 & 3k-3 \\ 3 & k-1 & -6 & 2k-2 \end{vmatrix} = \begin{vmatrix} 2 & -3 & k-1 \\ 2k-4 & -k-4 & 3k-3 \\ k-1 & -6 & 2k-2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & k-1 \\ 2k-4 & -k-4 & 3k-3 \\ k-5 & 0 & 0 \end{vmatrix} = (k-5)[-9k+9+(k-1)(k+4)] \neq 0$$

Clearly $(k-5)[-9k+9+(k-1)(k+4)] = 0 \Leftrightarrow k = 1,5$. Thus the vectors are linearly independent when $k \neq 1,5$. ∎

Question 1(b) *Let $\mathcal{V}$ be the vector space of polynomials in x of degree $\leq n$ over $\mathbb{R}$. Prove that the set $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{V}$. Extend this so that it becomes a basis for the set of all polynomials in x.*

Solution. $\{1, x, x^2, \ldots, x^n\}$ are linearly independent over $\mathbb{R}$ — If $a_0 + a_1x + \ldots + a_nx^n = 0$ where $a_i \in \mathbb{R}, 0 \leq i \leq n$, then we must have $a_i = 0$ for every i because the non-zero polynomial $a_0 + a_1x + \ldots + a_nx^n$ can have at most n roots in $\mathbb{R}$ whereas $a_0 + a_1x + \ldots + a_nx^n = 0$ for every $x \in \mathbb{R}$.

Every polynomial in x of degree $\leq n$ is clearly a linear combination of $1, x, x^2, \ldots, x^n$ with coefficients from $\mathbb{R}$. Thus $\{1, x, x^2, \ldots, x^n\}$ is a basis for $\mathcal{V}$.

We shall show that $\mathscr{S} = \{1, x, x^2, \ldots, x^n, x^{n+1}, \ldots\}$ is a basis for the space of all polynomials.

(i) Linear Independence: Let $\{x^{i_1}, \ldots, x^{i_r}\}$ be a finite subset of $\mathscr{S}$. Let $n = \max\{i_1, \ldots, i_r\}$, then $\{x^{i_1}, \ldots, x^{i_r}\}$ being a subset of the linearly independent set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent, which shows the linear independence of $\mathscr{S}$.

(ii) Let f be any polynomial. If degree of f is m, then f is a linear combination of $\{1, x, x^2, \ldots, x^m\}$, which is a subset of $\mathscr{S}$. Thus $\mathscr{S}$ is a basis of $\mathcal{W}$, the space of all polynomials over $\mathbb{R}$. ∎

Question 2(a) *Let $\mathbf{T}$ be a linear transformation on $\mathbb{R}^3$ whose matrix relative to the standard basis of $\mathbb{R}^3$ is $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{pmatrix}$. Find the matrix of $\mathbf{T}$ relative to the basis $\mathscr{B} = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$.*

Solution. Let the vectors of the given basis be $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$. $(\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2}), \mathbf{T}(\mathbf{v_3})) = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 5 & 3 & 4 \\ 10 & 6 & 7 \end{pmatrix}$.

If $(a, b, c) = \alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3}$, then $\alpha + \beta = a, \alpha + \beta + \gamma = b, \alpha + \gamma = c$ therefore $\alpha = a - b + c, \beta = b - c, \gamma = b - a$. Consequently

$$\begin{aligned} \mathbf{T}(\mathbf{v_1}) &= 7\mathbf{v_1} - 5\mathbf{v_2} + 3\mathbf{v_3} \\ \mathbf{T}(\mathbf{v_1}) &= 6\mathbf{v_1} - 3\mathbf{v_2} + 0\mathbf{v_3} \\ \mathbf{T}(\mathbf{v_1}) &= 3\mathbf{v_1} - 3\mathbf{v_2} + 4\mathbf{v_3} \end{aligned}$$

This shows that the matrix of $\mathbf{T}$ with respect to given basis $\mathscr{B}$ is $\begin{pmatrix} 7 & 6 & 3 \\ -5 & -3 & -3 \\ 3 & 0 & 4 \end{pmatrix}$ ∎

Question 2(b) *If $\mathbf{S}$ is a skew-Hermitian matrix, then show that $\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}$ is a unitary matrix. Show that a unitary matrix $\mathbf{A}$ can be expressed in the above form provided -1 is not an eigenvalue of $\mathbf{A}$.*

Solution. See related results of question 2(a) year 1989. ∎

Question 2(c) *Reduce the quadratic form*

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3 + 4x_1x_3$$

to a sum of squares. Also find the corresponding linear transformation, index and signature.

Solution.

$$\begin{aligned}
\mathcal{Q}(x_1, x_2, x_3) &= 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 4x_2x_3 + 4x_1x_3 \\
&= 6[x_1^2 - \frac{2}{3}x_1x_2 + \frac{2}{3}x_1x_3 + \frac{1}{9}x_2^2 + \frac{1}{9}x_3^2 - \frac{2}{9}x_2x_3] \\
&\quad + \frac{7}{3}x_2^2 + \frac{7}{3}x_3^2 - \frac{8}{3}x_2x_3 \\
&= 6[x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3]^2 + \frac{7}{3}[x_2 - \frac{4}{7}x_3]^2 + \frac{33}{21}x_3^2
\end{aligned}$$

Put $X_1 = x_1 - \frac{1}{3}x_2 + \frac{1}{3}x_3, X_2 = x_2 - \frac{4}{7}x_3, X_3 = x_3$, so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{1}$$

and $\mathcal{Q}(x_1, x_2, x_3)$ is transformed to $6X_1^2 + \frac{7}{3}X_2^2 + \frac{33}{21}X_3^2$. Let

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}}Z_1 \\ \sqrt{\frac{3}{7}}Z_2 \\ \sqrt{\frac{7}{11}}Z_3 \end{pmatrix}$$

then $\mathcal{Q}(x_1, x_2, x_3)$ is transformed to $Z_1^2 + Z_2^2 + Z_3^2$, which is its canonical form. Thus $\mathcal{Q}(x_1, x_2, x_3)$ is positive definite. The Index of $\mathcal{Q}(x_1, x_2, x_3) =$ Number of positive squares in its canonical form $= 3$. The signature of $\mathcal{Q}(x_1, x_2, x_3) =$ Number of positive squares - the number of negative squares in its canonical form $= 3$.

The required linear transformation which transforms $\mathcal{Q}(x_1, x_2, x_3)$ to sums of squares is given by (1), and the linear transformation which transforms it to its canonical form is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{7} \\ 0 & 1 & \frac{4}{7} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{7}} & 0 \\ 0 & 0 & \sqrt{\frac{7}{11}} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

∎

UPSC Civil Services Main 2006 - Mathematics Linear Algebra

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Question 1(a) *Let $\mathcal{V}$ be a vector space of all 2×2 matrices over the field* F. *Prove that $\mathcal{V}$ has dimension 4 by exhibiting a basis for $\mathcal{V}$.*

Solution. Let $\mathbf{M_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{M_2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{M_3} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{M_4} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We will show that $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a basis of $\mathcal{V}$ over F.

$\underline{\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\} \text{ generate } \mathcal{V}}$. Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{V}$. Then $\mathbf{A} = a\mathbf{M_1} + b\mathbf{M_2} + c\mathbf{M_3} + d\mathbf{M_4}$, where $a, b, c, d \in F$. Thus $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a set of generators for $\mathcal{V}$ over F.

$\underline{\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\} \text{ are linearly independent over } F}$. If $a\mathbf{M_1} + b\mathbf{M_2} + c\mathbf{M_3} + d\mathbf{M_4} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{0}$ for $a, b, c, d \in F$, then clearly $a = b = c = d = 0$, showing that $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ are linearly independent over F.

Hence $\{\mathbf{M_1}, \mathbf{M_2}, \mathbf{M_3}, \mathbf{M_4}\}$ is a basis of $\mathcal{V}$ over F and $\dim \mathcal{V} = 4$. ■

Question 1(b) *State the Cayley-Hamilton theorem and using it find the inverse of $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.*

Solution. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{I_n}$ be the $n \times n$ identity matrix. Then the n-degree polynomial $|\mathbf{xI_n} - \mathbf{A}|$ is called the charateristic polynomial of $\mathbf{A}$. The Cayley-Hamilton theorem states that every matrix is a root of its characteristic polynomial:

$$\begin{array}{ll} \text{if} & |\mathbf{xI_n} - \mathbf{A}| = x^n + a_1x^{n-1} + \ldots + a_n \\ \text{then} & \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \ldots + a_n\mathbf{I_n} = \mathbf{0} \end{array}$$

$|\mathbf{xI_n} - \mathbf{A}| = 0$ is called the characteristic equation of $\mathbf{A}$.

Let $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$. The characteristic equation of $\mathbf{A}$ is $0 = \begin{vmatrix} x-1 & -3 \\ -2 & x-4 \end{vmatrix} = (x-1)(x-4) - 6 = x^2 - 5x - 2$.

By the Cayley-Hamilton Theorem, $\mathbf{A}^2 - 5\mathbf{A} - 2\mathbf{I_2} = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A} - 5\mathbf{I_2}) = (\mathbf{A} - 5\mathbf{I_2})\mathbf{A} = 2\mathbf{I_2}$. Thus $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A} - 5\mathbf{I_2})$, so $\mathbf{A}^{-1} = \frac{1}{2}\left[\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right] = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$ ■

Question 2(a) *If* $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *is defined by* $\mathbf{T}(x, y) = (2x - 3y, x + y)$*, compute the matrix of* $\mathbf{T}$ *with respect to the basis* $\mathscr{B} = \{(1, 2), (2, 3)\}$.

Solution. It is obvious that $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation. Clearly

$$\begin{aligned}\mathbf{T}(\mathbf{v_1}) &= \mathbf{T}(1, 2) = (-4, 3)\\ \mathbf{T}(\mathbf{v_2}) &= \mathbf{T}(2, 3) = (-5, 5)\end{aligned}$$

Let $(a, b) = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, where $a, b, \alpha, \beta \in \mathbb{R}$, then $\alpha + 2\beta = a, 2\alpha + 3\beta = b \Rightarrow \alpha = 2b - 3a, \beta = 2a - b$. Thus $\mathbf{T}(\mathbf{v_1}) = 18\mathbf{v_1} - 11\mathbf{v_2}, \mathbf{T}(\mathbf{v_2}) = 25\mathbf{v_1} - 15\mathbf{v_1}$, so $(\mathbf{v_1}, \mathbf{v_2})\mathbf{T} = (\mathbf{T}(\mathbf{v_1}), \mathbf{T}(\mathbf{v_2})) = (\mathbf{v_1}, \mathbf{v_2})\begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$. Thus the matrix of $\mathbf{T}$ with respect to the basis $\mathscr{B}$ is $\begin{pmatrix} 18 & 25 \\ -11 & -15 \end{pmatrix}$ ∎

Question 2(b) *Using elementary row operations, find the rank of* $\mathbf{A} = \begin{pmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$

Solution. Operations $\mathbf{R_1} - 2\mathbf{R_3}, \mathbf{R_2} - \mathbf{R_4}$ give

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operation $\mathbf{R_3} - \mathbf{R_1}$ gives

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -9 & -5 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Operations $\mathbf{R_3} + 4\mathbf{R_2}, \mathbf{R_4} - \mathbf{R_2} \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

$\mathbf{R_4} + \frac{2}{9}\mathbf{R_3} \Rightarrow$

$$\mathbf{A} \sim \begin{pmatrix} 1 & 2 & 6 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -9 & -5 \\ 0 & 0 & 0 & -\frac{1}{9} \end{pmatrix}$$

Clearly $|\mathbf{A}| = 1 \Rightarrow \operatorname{rank} \mathbf{A} = 4$. ∎

Question 2(c) *Investigate for what values of λ and μ the equations*

$$\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\lambda z &= \mu \end{aligned}$$

have (1) no solution (2) a unique solution (3) infinitely many solutions.

Solution. (2) The equations will have a unique solution for all values of μ if the coefficient matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{pmatrix}$ is non-singular. i.e. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & \lambda - 1 \end{vmatrix} = \lambda - 1 - 2 \neq 0$ i.e. $\lambda \neq 3$. Thus for $\lambda \neq 3$ and for all μ we have a unique solution which can be obtained by Cramer's rule or otherwise.

(1) If $\lambda = 3, \mu \neq 10$ then the system is inconsistent and we have no solution.

(3) If $\lambda = 3, \mu = 10$, the system will have infinitely many solutions obtained by solving $x+y = 6-z, x+2y = 10-3z \Rightarrow x = 2+z, y = 4-2z$, z is any real number. ∎

Question 2(d) *Find the quadratic form $q(x,y)$ corresponding to the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix}$$

Is this quadratic form positive definite? Justify your answer.

Solution. The quadratic form is

$$\begin{aligned} q(x,y) &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 5x^2 - 6xy + 8y^2 \\ &= 5[x^2 - \frac{6}{5}xy + \frac{8}{5}y^2] \\ &= 5[(x - \frac{3}{5}y)^2 + \frac{31}{25}y^2] \end{aligned}$$

Clearly $q(x,y) > 0$ for all $(x,y) \neq (0,0), (x,y) \in \mathbb{R}^2$. Thus $q(x,y)$ is positive definite. In fact, $q(x,y) = 0 \Rightarrow x - \frac{3}{5}y = 0, y = 0 \Rightarrow x = y = 0$. ∎