ANALYTIC GEOMETRY

: CSE - 2018 :

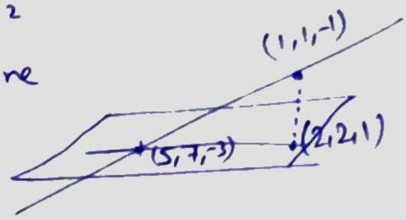
① 1(e) Find the projection of the straight line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1}$ on the plane $x+y+2z=6$

→ Any point on the given line is $(2r+1, 3r+1, -r-1)$.

If this point is the point of intersection of the given line and the given plane, then, this point lies on the given plane.

$\therefore x+y+2z=6 \Rightarrow 2r+1+3r+1-2r-2=6$

$\Rightarrow 3r = 6 \Rightarrow r = 2$



$\therefore$ Point of intersection of the given line and the given plane is

$(2r+1, 3r+1, -r-1) = (5, 7, -3)$

The point $(1,1,-1)$ lies on the given line. The drs of normal to the given plane are $1,1,2$.

Then, a line through $(1,1,-1)$ $\perp$ar to the given plane is

$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z+1}{2} = r_1$ (say)

Any point on this line is $(r_1+1, r_1+1, 2r_1-1)$.

If this point represents foot of the $\perp$ar from $(1,1,-1)$ to the plane, then this point lies on the given plane

$\therefore x+y+2z=6 \Rightarrow r_1+1+r_1+1+4r_1-2=6$

$\Rightarrow r_1 = 1$

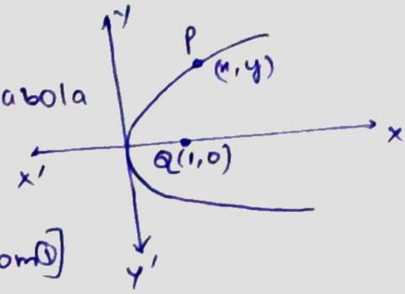
$\therefore$ Foot of $\perp$ar $\equiv (r_1+1, r_1+1, 2r_1-1)$

$= (2, 2, 1)$

$\therefore$ Required projection is a line through the point of intersection of the given line and the given plane and the foot of $\perp$ar from $(1,1,-1)$ to the plane.

$\therefore \frac{x-5}{5-2} = \frac{y-7}{7-2} = \frac{z+3}{-3-1} \Rightarrow \frac{x-5}{3} = \frac{y-7}{5} = \frac{z+3}{-4}$.

(2)(b) Find the shortest distance from the point (1,0) to the parabola $y^2 = 4x$.



$\rightarrow$ The given parabola is $y^2 = 4x$ —— (1)

Let $P(x,y)$ be a point on the parabola which is nearest to the point (0,1).

$$PQ = \sqrt{(x-1)^2 + y^2} = \sqrt{\left(\frac{y^2}{4}-1\right)^2 + y^2} \quad [\text{from (1)}]$$

$$= \sqrt{\frac{y^4}{16} - \frac{y^2}{2} + 1 + y^2} = \sqrt{\frac{y^4}{16} + \frac{y^2}{2} + 1}$$

Let $r = PQ^2 = \frac{y^4}{16} + \frac{y^2}{2} + 1$

We have to minimize r. $\frac{dr}{dy} = \frac{y^3}{4} + y = 0$

$\Rightarrow y(y^2+4) = 0 \Rightarrow y = 0, \pm 2i$.

Ignoring imaginary values of y, we have $y=0$.

$\frac{d^2r}{dy^2} = \frac{3y^2}{4} + 1 \Rightarrow$ at $y=0$: $\frac{d^2r}{dy^2} = 1 > 0$.

$\therefore$ Minima at $y = 0$.

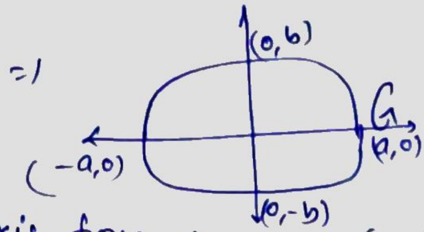
$y^2 = 4x \Rightarrow 0 = 4x \Rightarrow x = 0$.

$\therefore$ (0,0) is the closest point to (1,0).

$\therefore$ Shortest distance $= \sqrt{(1-0)^2 + (0-0)^2} = \underline{\underline{1.}}$

(2)(c): The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ revolves about the x-axis. Find the volume of the solid of revolution.

$\rightarrow$ The given ellipse eqn is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

The area revolves around x-axis from (-a,0) to (a,0). Then,

Volume of solid of revolution $\Rightarrow V = \pi\int_{-a}^{a} y^2\,dx = 2\pi b^2\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx$

$\Rightarrow V = 2\pi b^2\left[x - \frac{x^3}{3a^2}\right]_0^a \Rightarrow V = 2\pi b^2\left[a - \frac{a}{3}\right] = \underline{\underline{\frac{4\pi ab^2}{3}}}$

2(d) Find the shortest distance between the lines $a_1x+b_1y+c_1z+d_1=0$, $a_2x+b_2y+c_2z+d_2=0$ and the z-axis.

→ ~~Z-axis~~ Any plane through the given line is

$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0.$

$\Rightarrow (a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0$ —— ①

If this plane is parallel to z-axis whose DRs are 0,0,1, then normal to this plane is ⊥ar to z-axis. Therefore :

$l_1l_2+m_1m_2+n_1n_2=0$ i.e. $(a_1+\lambda a_2)0+(b_1+\lambda b_2)0+(c_1+\lambda c_2)1=0$

$\Rightarrow \lambda=-\dfrac{c_1}{c_2}$

$\therefore$ ① $\equiv \left(a_1-\dfrac{c_1}{c_2}a_2\right)x+\left(b_1-\dfrac{c_1}{c_2}b_2\right)y+\left(c_1-\dfrac{c_1}{c_2}c_2\right)z+\left(d_1-\dfrac{c_1}{c_2}d_2\right)=0$

$\Rightarrow (a_1c_2-a_2c_1)x+(b_1c_2-b_2c_1)y+(d_1c_2-d_2c_1)=0$ ②

is the plane parallel to z-axis.

Any point on z-axis is the origin O(0,0,0).

Distance between the two given lines is equal to the ⊥ar distance between O and the plane ② since z-axis is parallel to this plane

$\therefore$ ⊥ar distance from O(0,0,0) to the plane ② is given as $p=\dfrac{|0.(a_1c_2-a_2c_1)+0(b_1c_2-b_2c_1)+(d_1c_2-d_2c_1)|}{\sqrt{(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)^2}}$

$\therefore$ Required shortest distance between the two lines is given by $p=\dfrac{|d_1c_2-d_2c_1|}{\sqrt{(a_1c_2-a_2c_1)^2+(b_1c_2-b_2c_1)^2}}$

3(c) Find the equations of the generating lines of paraboloid $(x+y+z)(2x+y-z) = 6z$ which pass through the point $(1,1,1)$

→ Eqⁿ of generators of the λ and μ systems are given by

$$x+y+z = 6\lambda, \quad 2x+y-z = \frac{z}{\lambda}. \quad \text{---} (1)$$

and $$x+y+z = \frac{z}{\mu}, \quad 2x+y-z = 6\mu. \quad \text{---} (2)$$

If these lines pass through $(1,1,1)$, then

$1+1+1 = 6\lambda$ & $2+1-1 = 6\mu$ $\Rightarrow \lambda = \frac{1}{2}, \mu = \frac{1}{3}$.

∴ from (1) & (2): $x+y+z = 3, \quad 2x+y-z = 2z$

& $x+y+z = 3z, \quad 2x+y-z = 2$

$\Rightarrow x+y+z = 3, \quad 2x+y-3z = 0.$

and $x+y-2z = 0, \quad 2x+y-z = 2$ are the required generators.

3(d) Find the equations to the sphere in xyz-plane passing through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$

→ Let the eqⁿ of sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ ---(1)

It passes through:

(i) $(0,0,0) \equiv d = 0$. ∴ (1) ≡ $x^2+y^2+z^2+2ux+2vy+2wz = 0$ ---(2)

(ii) $(0,1,-1) \equiv 1+1+2v-2w = 0 \Rightarrow v-w+1 = 0$ ---(3)

(iii) $(-1,2,0) \equiv 1+4-2u+4v = 0 \Rightarrow -2u+4v+5 = 0.$ ---(4)

(iv) $(1,2,3) \equiv 1+4+9+2u+4v+6w = 0 \Rightarrow u+2v+3w+7 = 0.$ ---(5)

(3), (4), (5) gives $u = -\frac{15}{14}, \quad v = -\frac{25}{14}, \quad w = -\frac{11}{14}$.

Putting in (2):

$$x^2+y^2+z^2 - 2\left(\frac{15}{14}\right)x - 2\left(\frac{25}{14}\right)y - 2\left(\frac{11}{14}\right)z = 0.$$

$\Rightarrow 7(x^2+y^2+z^2) - 15x - 25y - 11z = 0$ which is the equation of the reqd sphere.

4(c) Find the equation of the cone with (0,0,1) as vertex and $2x^2 - y^2 = 4,\ z = 0$ as the guiding curve.

→ Any line through the vertex (0,0,1) is $\frac{x}{l} = \frac{y}{m} = \frac{z-1}{n}$ —— ①

It cuts $z=0$, then $\frac{x}{l} = \frac{y}{m} = -\frac{1}{n} \Rightarrow x = -\frac{l}{n},\ y = -\frac{m}{n}$.

Putting in $2x^2 - y^2 = 4$, we get

$\frac{2l^2}{n^2} - \frac{m^2}{n^2} = 4 \Rightarrow 2l^2 - m^2 = 4n^2. \Rightarrow 2l^2 - m^2 - 4n^2 = 0$ —— ②

Eliminating l, m, n between ① & ②,

the reqd eqn of the cone is $2x^2 - y^2 - 4(z-1)^2 = 0$

4(d) Find the equation of the plane parallel to $3x - y + 3z = 8$ and passing through the point (1,1,1)

→ Eqn of any plane parallel to the given plane is

$3x - y + 3z + d = 0$ —— ①

If it passes through (1,1,1), then $3 - 1 + 3 + d = 0$

$\Rightarrow d = -5.$

Putting in ①:

The reqd equation of the plane is $3x - y + 3z - 5 = 0$