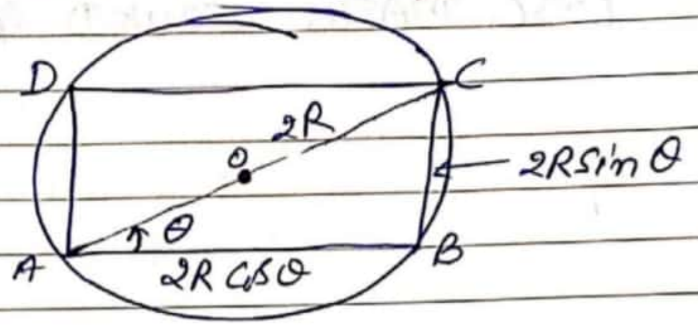
IFS 2018 PAPER-I

(Sec-A)

DATE

1 (a) Show that the maximum rectangle inscribed in a circle is a square. (8)



Let ABCD be the rectangle inscribed in a circle of radius R.

Let $\angle BAC = \theta$

$AB = 2R\cos\theta$, $BC = 2R\sin\theta$

Area, $A = (2R\cos\theta)(2R\sin\theta)$

$= 2R^2 \cdot \sin 2\theta$

for max area, $\frac{dA}{d\theta} = 0 \Rightarrow 4R^2\cos 2\theta = 0$

$\cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ in $[0, 2\pi]$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$ in $[0, \pi]$.

But no rectangle is possible for $\theta = 3\pi/4$

So, we discard it

$\frac{d^2A}{d\theta^2} = -8R^2\sin 2\theta < 0$ at $\theta = \frac{\pi}{4}$

Hence, A is maximum when $\theta = \pi/4$

Then $AB = 2R\cos\frac{\pi}{4} = 2R \cdot \frac{1}{\sqrt{2}} = \sqrt{2}R = BC$

Hence ABCD becomes square.

DATE

1.(c) If $f : [a,b] \to R$ be continuous in $[a,b]$ and derivable in (a,b), where $0<a<b$, show that for $c \in (a,b)$
$$f(b)-f(a) = c.f'(c)\log(b/a).$$ (8)

Cauchy's Mean Value Theorem

Two functions f and g are

i) cont on $[a,b]$ ii) derivable in (a,b)

iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$, then there exist atleast one point $c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

Here, take $g(x) = \log x$ in $[a,b]$
$0<a<b$

Applying Cauchy's MVT

$\exists$ some $c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{\log b - \log a} = \frac{f'(c)}{(1/c)}.$$

$$\Rightarrow \quad f(b)-f(a) = c.f'(c)\log b/a.$$

Hence proved.

Scanned by CamScanner

Scanned with CamScanner

DATE

2(c) If ϕ and ψ be two functions derivable in $[a,b]$ and $\phi(x)\psi'(x) - \psi(x)\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x)=0$ in $[a,b]$, there lies exactly one root of $\psi(x)=0$. (10).

Let α and β be two consecutive roots of $\phi(x)=0$ in $[a,b]$ and $\alpha<\beta$.

To prove that only one root of $\psi(x)=0$ lies between α and β.

If possible, let $\psi(x)=0$ has no root in (α,β).

Consider the function $F(x) = \dfrac{\phi(x)}{\psi(x)}$

$$F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0 \quad \& \quad F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0$$

$(\because \phi(\alpha)=0=\phi(\beta)$ and $\psi(\alpha)\neq 0, \psi(\beta)\neq 0)$

$\psi(x)\neq 0$ in $[\alpha,\beta]$

$\therefore F(x)$ is continuous in $[\alpha,\beta]$

$$F'(x) = \frac{\phi'(x)\psi(x) - \psi'(x)\phi(x)}{[\psi(x)]^2} \text{ exist in } [\alpha,\beta]$$

$\therefore F(x)$ satisfies all conditions of Rolle's Theorem in $[\alpha,\beta]$

$\therefore F'(\gamma)=0$ where $\alpha<\gamma<\beta$

But by given condition $\phi'(x)\psi(x)-\psi'(x)\phi(x)>0$

$\therefore F'(x)\neq 0$ in (α,β) and we get contradiction.

Hence $\psi(x)$ has atleast one root in (α,β)

classmate

PAGE

Scanned by CamScanner

DATE

By similar argument it can be shown that between two roots of $\psi(x)=0$ there is a root of $\phi(x)=0$.

Now, we prove that there is exactly one root of $\psi(x)=0$ between α,β.

If possible let γ and δ be two roots of $\psi(x)=0$ in (α,β) ie $\alpha<\gamma<\delta<\beta$.

Between γ and δ there would exist a root of $\phi(x)=0$ contradicting that α and β are consecutive roots of $\phi(x)=0$

Hence, there is only one root of $\psi(x)=0$ between α and β.

DATE

3(b). If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Chain Rule

$$\frac{\partial f(u,v)}{\partial (x)} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f(u,v)}{\partial x^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,

$$\frac{\partial^2 f(u,v)}{\partial y^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2\right] + \left[\frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$\cdot\ u = e^x \cdot \cos y$

$$\frac{\partial u}{\partial x} = e^x\cdot\cos y = u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} = u$$

$$\frac{\partial u}{\partial y} = -e^x\cdot\sin y = -v$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y} = -e^x\cos y = -u$$

$v = e^x \cdot \sin y$

$$\frac{\partial v}{\partial x} = e^x\cdot\sin y = v$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x} = v$$

$$\frac{\partial v}{\partial y} = e^x\cdot\cos y = u$$

$$\frac{\partial^2 v}{\partial y^2} = -e^x\cdot\sin y = -v$$

Using these values

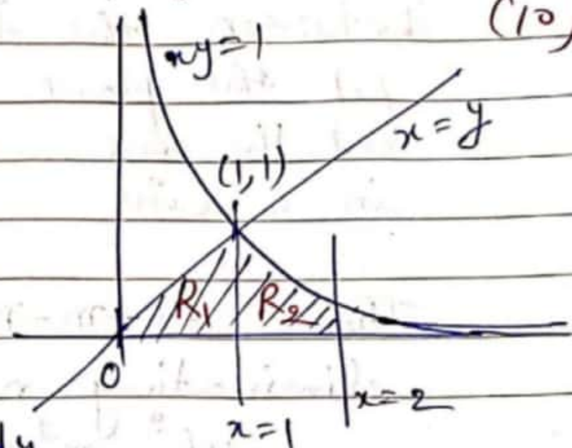
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 f}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= \cancel{\frac{\partial f}{\partial u}(u-u)} + \frac{\partial^2 f}{\partial u^2}(u^2+v^2) + \cancel{\frac{\partial f}{\partial v}(v-v)} + \frac{\partial^2 f}{\partial v^2}(v^2+u^2)$$

$$= (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

3(d) Evaluate $\iint_R (x^2+xy)\,dx\,dy$ over the region R bounded by $xy=1$, $y=0$, $y=x$ and $x=2$. (10)

We split the ~~area~~ region of integration in two parts.



$$I = \iint_R (x^2+xy)\,dx\,dy$$

$$= \iint_{R_1} (x^2+xy)\,dx\,dy + \iint_{R_2} (x^2+xy)\,dx\,dy$$

$$= \int_0^1\int_0^x (x^2+xy)\,dy\,dx + \int_1^2\int_0^{1/x} (x^2+xy)\,dy\,dx$$

$$= \int_0^1 \left. x^2\cdot y + x\frac{y^2}{2}\right|_0^x dx + \int_1^2 \left. x^2 y + x\cdot\frac{y^2}{2}\right|_0^{1/x} dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{2}\right) dx + \int_1^2 \left(x^2\cdot\frac{1}{x} + \frac{x}{2}\cdot\frac{1}{x^2}\right) dx$$

$$= \int_0^1 \frac{3}{2}x^3\,dx + \int_1^2 \left(x + \frac{1}{2x}\right) dx$$

$$= \frac{3}{2}\times\frac{1}{4}\left[x^4\right]_0^1 + \left[\frac{x^2}{2} + \frac{1}{2}\log x\right]_1^2$$

$$= \frac{3}{8}\cdot 1 + \frac{4-1}{2} + \frac{1}{2}(\log 2 - \log 1)$$

$$= \frac{3+12}{8} + \frac{1}{2}\log 2 = \frac{15}{8} + \frac{1}{2}\log 2.$$

DATE

4(b) Show that the functions $u = x+y+z$, $v = xy+yz+zx$ and $w = x^3+y^3+z^3-3xyz$ are dependent and find the relation between them.

Given functions are dependent if their Jacobian vanishes.

$$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ y+z & x-y & x-z \\ 3x^2-3yz & 3y^2-3xz-3x^2+3yz & 3z^2-3xy-3x^2+3yz \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ y+z & x-y & x-z \\ x^2-yz & y^2-x^2+z(y-x) & z^2-x^2+y(z-x) \end{vmatrix}$$

$$= 3(y-x)(z-x)\left[-(z+x+y)-(-1)(y+x+z)\right]$$

$$= 0 \quad \therefore u, v, w \text{ are dependent.}$$

Now

$$w = x^3+y^3+z^3-3xy+z$$

$$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= (x+y+z)\left[(x+y+z)^2-3(xy+yz+zx)\right]$$

$$= u\left[u^2-3v\right]$$

$\boxed{w = u^3-3uv}$ $\Rightarrow$ required relation.