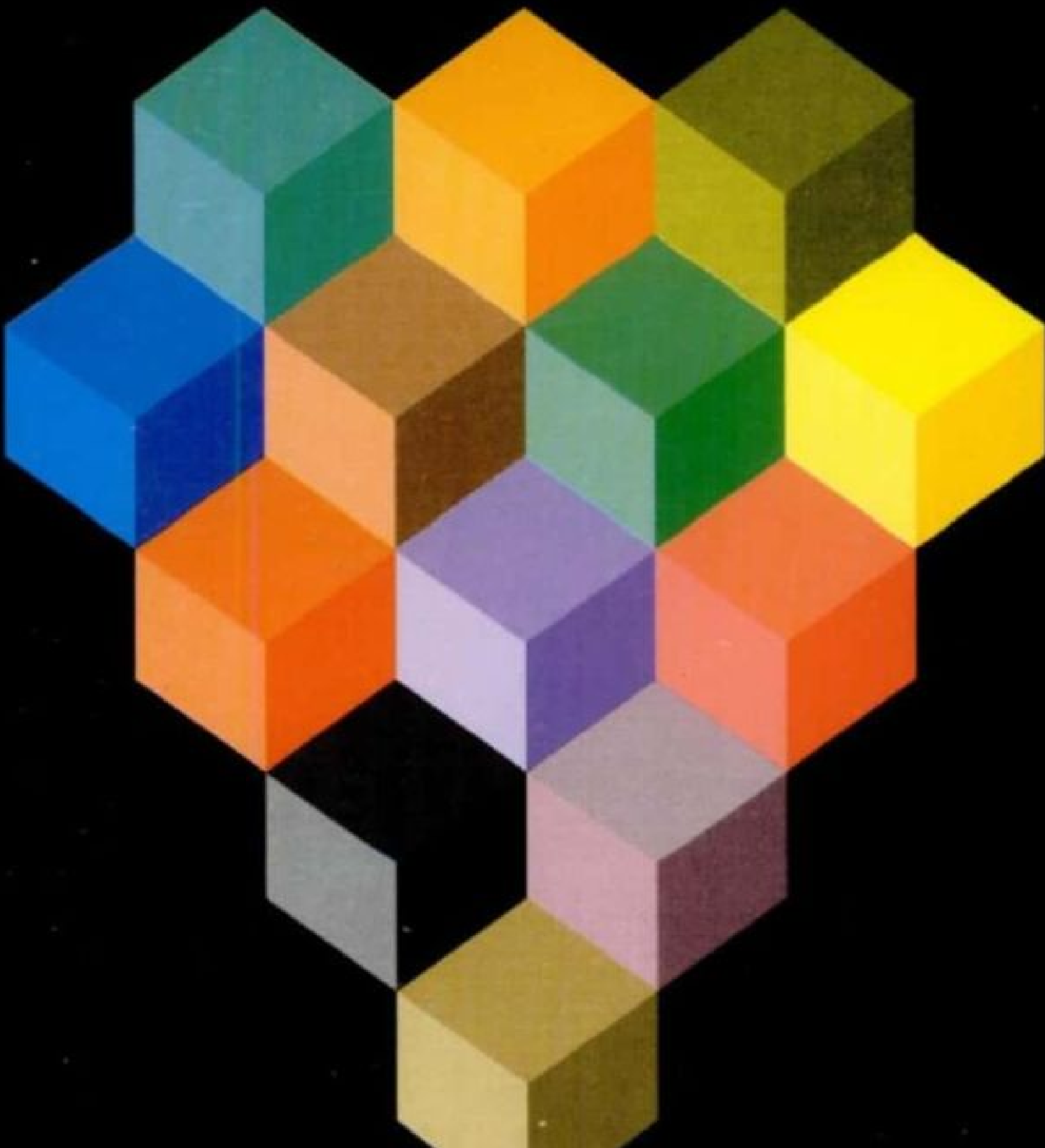
Krishna's Series

DIFFERENTIAL CALCULUS



KRISHNA Prakashan Media (P) Ltd. Meerut

Copyrighted material

Scanned with CamScanner

Krishna's Series

DIFFERENTIAL CALCULUS

(Fully Solved)

(For Degree and Honours Students of All Indian Universities, various Competitive Examinations like P.C.S., I.A.S. and Engineering Students)

By

A.R. Vasishtha
Retired Head, Deptt. of Mathematics
Meerut College, Meerut

Dr. S.K. Sharma
Retired Head, Deptt. of Mathematics
N.A.S. College, Meerut

&

A.K. Vasishtha
M.Sc., Ph.D.,
C.C.S. University, Meerut

SPECIMEN
With Best Compliments From
AUTHORS & PUBLISHERS

KRISHNA Prakashan Media (P) Ltd.
KRISHNA HOUSE, 11, Shivaji Road, Meerut - 250 001 (U.P.), INDIA

Copyrighted material
Scanned with CamScanner

Krishna's Series

DIFFERENTIAL CALCULUS

[FULLY SOLVED]

First Edition : 1991

Twenty First Edition : 2009

This book or any part thereof may not be reproduced in any form without the writter permission of the publishers and authors.

Book Code No. : 446-21

Price : Rs. 250.00 Only

Published by : Satyendra Rastogi "*Mitra*"
for **KRISHNA** Prakashan Media (P) Ltd.
11, Shivaji Road, Meerut - 250 001 (U.P.) India.
Phones : 0121-2642946, 2644766 Fax : 0121-2645855
Website : www.krishnaprakashan.com
e-mail : info@krishnaprakashan.com

Printed at : Raj Printers, Meerut

Copyrighted material

Scanned with CamScanner

CONTENTS

Copyrighted material

Scanned with CamScanner

UNIT - II

Copyrighted material

Scanned with CamScanner

Note : The chapter 1 of Unit-I, chapter 1 of Unit-II and two chapters of Unit-III of this book are not in the syllabus of C.C.S. University, Meerut. The chapters 2 to 9 cover the syllabus of Unit I and the chapters 2 to 7 of Unit-II cover the syllabus of Unit II.

Copyrighted material

Scanned with CamScanner

1

Differentiation

§ 1. Derivative (or differential coefficient) of a function.

Let $y = f(x)$ be a function of x defined in the interval (a, b). For a small increment δx in x, let the corresponding increment in the value of y be δy. The quotient $\delta y/\delta x$ is called the *average rate of change* of y with respect to x in the interval $(x, x + \delta x)$. If the increment ratio $\delta y/\delta x$ tends to a finite limit as $\delta x \to 0$, then this limit is called the **differential coefficient** of y with respect to x and is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx} f(x) \text{ or } f'(x) \text{ or } Df(x) \text{ or } Dy \text{ etc.}$$

Thus $\frac{dy}{dx}$ = the diff. coeff. of y w.r.t. x = $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

The diff. coeff. dy/dx is also called the *rate of change* of y with respect to x.

Now if $y = f(x)$ and $y + \delta y = f(x + \delta x)$, we have

$$\delta y = f(x + \delta x) - f(x).$$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x},$$ provided the limit exists. The process of finding the diff. coeff. of a given function $f(x)$ is called **differentiation.**

The differential coefficient of $f(x)$ for $x = a$ is defined as $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, *provided the limit exists.* It is denoted by

$$\left(\frac{dy}{dx}\right)_{x = a} \text{ or } (y')_a \text{ or } f'(a).$$

§ 2. Some standard results.

To find the differential coefficients of some standard functions from the first principles.

1. Differential coefficient of a constant. Let $f(x) = c$, where c is a constant. Then $f(x + \delta x) = c$, because $f(x)$ takes the same value c for every value of x.

Now by the definition of a diff. coeff., we have

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{c - c}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{0}{\delta x} = \lim_{\delta x \to 0} 0 = 0.$$

Thus, the diff. coeff. of a constant is zero.

Copyrighted material

Scanned with CamScanner

2. Diff. coeff. of the product of a constant and a function.

We have, $\frac{d}{dx}\{cf(x)\} = \lim_{\delta x \to 0} \frac{cf(x+\delta x) - cf(x)}{\delta x}$

$$= c \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = c\frac{d}{dx}f(x).$$

3. Derivative of a sum or a difference of two functions.

Let $f(x) = f_1(x) \pm f_2(x)$.

Then $f(x+\delta x) = f_1(x+\delta x) \pm f_2(x+\delta x)$.

$$\therefore \quad \frac{d}{dx}f(x) = \lim_{\delta x \to 0} \frac{\{f_1(x+\delta x) \pm f_2(x+\delta x)\} - \{f_1(x) \pm f_2(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{f_1(x+\delta x) - f_1(x)}{\delta x} \pm \lim_{\delta x \to 0} \frac{f_2(x+\delta x) - f_2(x)}{\delta x}$$

$$= \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x).$$

4. Differential coefficient of the product of two functions.

(Delhi 1982)

Let $y = f_1(x) f_2(x)$. Then $y + \delta y = f_1(x+\delta x) \cdot f_2(x+\delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f_1(x+\delta x) f_2(x+\delta x) - f_1(x) \cdot f_2(x)}{\delta x},$$

[by the def. of a diff. coeff.]

$$= \lim_{\delta x \to 0} \left[f_1(x+\delta x) \cdot \frac{f_2(x+\delta x) - f_2(x)}{\delta x} + f_2(x) \frac{f_1(x+\delta x) - f_1(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} f_1(x+\delta x) \frac{f_2(x+\delta x) - f_2(x)}{\delta x}$$

$$+ \lim_{\delta x \to 0} f_2(x) \cdot \frac{f_1(x+\delta x) - f_1(x)}{\delta x}$$

$$= f_1(x) \cdot \frac{d}{dx}f_2(x) + f_2(x) \cdot \frac{d}{dx}f_1(x).$$

$\therefore$ **the diff. coeff. of the product of two functions**

= (first function) × (diff. coeff. of the second function)

+ (second function) × (diff. coeff. of the first function).

5. Differential coefficient of the quotient of two functions.

Let $y = f_1(x)/f_2(x)$. Then $y + \delta y = f_1(x+\delta x)/f_2(x+\delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\left[\frac{f_1(x+\delta x)}{f_2(x+\delta x)} - \frac{f_1(x)}{f_2(x)}\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f_1(x+\delta x) \cdot f_2(x) - f_1(x) \cdot f_2(x+\delta x)}{\delta x \cdot f_2(x+\delta x) \cdot f_2(x)}\right]$$

Copyrighted material

Scanned with CamScanner

$$= \lim_{\delta x \to 0} \left[\frac{f_1(x+\delta x) f_2(x) - f_1(x) f_2(x) - f_1(x) f_2(x+\delta x) + f_1(x) \cdot f_2(x)}{\delta x \cdot f_2(x+\delta x) \cdot f_2(x)} \right]$$

$$= \lim_{\delta x \to 0} \left[\frac{f_2(x) \cdot \left\{ \dfrac{f_1(x+\delta x) - f_1(x)}{\delta x} \right\} - f_1(x) \cdot \left\{ \dfrac{f_2(x+\delta x) - f_2(x)}{\delta x} \right\}}{f_2(x+\delta x) f_2(x)} \right]$$

$$= \frac{f_2(x) \cdot \dfrac{d}{dx} f_1(x) - f_1(x) \cdot \dfrac{d}{dx} f_2(x)}{[f_2(x)]^2}.$$

$\therefore$ **the diff. coeff. of the quotient of two functions**

$$= \frac{\text{(diff. coeff. of Nr.) (Dr.)} - \text{(Nr.) (diff. coeff. of Dr.)}}{\text{square of the denominator}}.$$

6. $\mathbf{(d/dx)\, x^n = nx^{n-1}}$.

Let $y = x^n$. Then $y + \delta y = (x + \delta x)^n$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x} = \lim_{\delta x \to 0} x^n \left[\frac{\left(1 + \dfrac{\delta x}{x}\right)^n - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} x^n \left[\frac{1 + n\dfrac{\delta x}{x} + \dfrac{n(n-1)}{2!} \dfrac{(\delta x)^2}{x^2} + \ldots - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!} \frac{\delta x}{x^2} + \text{terms containing higher powers of } \delta x \right]$$

$= x^n(n/x) = nx^{n-1}$.

7. $\mathbf{(d/dx)\, e^x = e^x}$.

Let $y = e^x$. Then $y + \delta y = e^{x + \delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x}$$

$$= \lim_{\delta x \to 0} e^x \left[\frac{e^{\delta x} - 1}{\delta x} \right] = \lim_{\delta x \to 0} e^x \left[\frac{1 + \delta x + (\delta x)^2/2! + \ldots - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} e^x \left[1 + \frac{\delta x}{2!} + \text{terms containing higher powers of } \delta x \right] = e^x.$$

8. $\mathbf{(d\ dx)\, a^x = a^x \log_e a}$.

Let $y = a^x$. Then $y + \delta y = a^{x + \delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x}$$

$$= \lim_{\delta x \to 0} a^x \left[\frac{a^{\delta x} - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} a^x \left[\frac{1 + \delta x \log a + \dfrac{(\delta x)^2 (\log a)^2}{2!} + \ldots - 1}{\delta x} \right]$$

Copyrighted material

Scanned with CamScanner

$$= \lim_{\delta x \to 0} a^x [\log a + (\delta x/2!) (\log a)^2 + \text{terms containing higher powers of } \delta x]$$

$$= a^x \log a.$$

9. $(d/dx) \log_e x = 1/x$.

Let $y = \log x$. Then $y + \delta y = \log (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\log(x + \delta x) - \log x}{\delta x} = \lim_{\delta x \to 0} \frac{\log \{(x + \delta x)/x\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log \left(1 + \dfrac{\delta x}{x}\right)}{\delta x} = \lim_{\delta x \to 0} \frac{\dfrac{\delta x}{x} - \dfrac{(\delta x)^2}{2x^2} + \ldots}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{1}{x} - \frac{\delta x}{2x^2} + \ldots\right] = \frac{1}{x}.$$

10. $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \log_e a}$.

We have $\log_a x = \log_e x \,.\, \log_a e = \log_a e \,.\, \log_e x$, where $\log_a e$ is simply a constant.

$$\therefore \quad \frac{d}{dx} (\log_a x) = (\log_a e) \frac{d}{dx} (\log_e x) = (\log_a e) \frac{1}{x} = \frac{1}{x} \cdot \log_a e$$

$$= \frac{1}{x \log_e a}, \text{ since } \log_a e \,.\, \log_e a = 1.$$

11. $(d/dx) \sin x = \cos x$.

Let $y = \sin x$. Then $y + \delta y = \sin (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin (x + \delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2 \cos \{x + (\delta x/2)\} \sin (\delta x/2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos \{x + (\delta x/2)\} \sin (\delta x/2)}{(\delta x/2)}$$

$$= \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin (\delta x/2)}{(\delta x/2)}$$

$$= (\cos x) \cdot 1 = \cos x.$$

12. $(d/dx) \cos x = -\sin x$.

Let $y = \cos x$. Then $y + \delta y = \cos (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\cos (x + \delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2 \sin \{x + (\delta x/2)\} \sin (\delta x/2)}{\delta x}$$

$$= \lim_{\delta x \to 0} -\sin \left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin (\delta x/2)}{(\delta x/2)} = (-\sin x) \cdot 1 = -\sin x.$$

Copyrighted material

Scanned with CamScanner

13. $(d/dx)\tan x = \sec^2 x$. **(Delhi 1982)**

Let $y = \tan x$. Then $y + \delta y = \tan(x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\tan(x+\delta x) - \tan x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left[\dfrac{\sin(x+\delta x)}{\cos(x+\delta x)} - \dfrac{\sin x}{\cos x}\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x+\delta x)\cos x - \sin x \cos(x+\delta x)}{\delta x \cos(x+\delta x)\cdot\cos x}$$

$$= \lim_{\delta x \to 0} \frac{\sin\{(x+\delta x) - x\}}{\delta x \cdot \cos(x+\delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x+\delta x)\cos x}\right] = \frac{1}{\cos^2 x} = \sec^2 x.$$

14. $(d/dx)\cot x = -\operatorname{cosec}^2 x$.

For proof proceed as in the case of $\tan x$.

15. $(d/dx)\sec x = \sec x \tan x$.

Let $y = \sec x$. Then $y + \delta y = \sec(x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{1}{\cos(x+\delta x)} - \dfrac{1}{\cos x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos x - \cos(x+\delta x)}{\delta x \cdot \cos(x+\delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \frac{2\sin(x+\frac{1}{2}\delta x)\sin(\delta x/2)}{\delta x \cdot \cos(x+\delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\sin(x+\frac{1}{2}\delta x)}{\cos(x+\delta x)\cos x} \cdot \frac{\sin(\delta x/2)}{(\delta x/2)}\right]$$

$$= \frac{\sin x}{\cos x \cos x} = \sec x \tan x.$$

16. $(d/dx)\operatorname{cosec} x = -\operatorname{cosec} x \cot x$.

For proof proceed as in the case of $\sec x$.

17. $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{(1-x^2)}}$.

Let $y = \sin^{-1} x$. Then $x = \sin y$ and so $x + \delta x = \sin(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sin(y + \delta y) - \sin y$.

$$\therefore \quad 1 = \frac{\sin(y+\delta y) - \sin y}{\delta x}, \qquad \text{[on dividing both sides by } \delta x]$$

$$\text{or} \quad 1 = \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Copyrighted material

Scanned with CamScanner

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim_{\delta y \to 0} \frac{\delta y}{\delta x},$$

$[\because \delta y \to 0 \text{ when } \delta x \to 0]$

$$= \left[\lim_{\delta y \to 0} \frac{2\cos(y + \frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y}\right] \cdot \frac{dy}{dx}$$

$$= \left[\lim_{\delta y \to 0} \cos(y + \tfrac{1}{2}\delta y) \cdot \frac{\sin(\frac{1}{2}\delta y)}{\delta y/2}\right] \frac{dy}{dx}$$

$$= (\cos y) \,.\, (dy/dx).$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}.$$

18. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{(1-x^2)}}$.

For proof proceed exactly as in the case of $\sin^{-1} x$.

19. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.

Let $y = \tan^{-1} x$. Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\qquad \delta x = \tan(y+\delta y) - \tan y.$

$$\therefore \quad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \left[\lim_{\delta y \to 0} \left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}\right\} \Big/ \delta y\right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y - y)}{\delta y . \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$= \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

20. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$.

For proof proceed exactly as in the case of $\tan^{-1} x$.

Copyrighted material

Scanned with CamScanner

5:51:12 91%

books.google.co.in/books?i

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim_{\delta y \to 0} \frac{\delta y}{\delta x},$$

$[\because \delta y \to 0 \text{ when } \delta x \to 0]$

$$= \left[\lim_{\delta y \to 0} \frac{2\cos(y + \frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y}\right] \cdot \frac{dy}{dx}$$

$$= \left[\lim_{\delta y \to 0} \cos(y + \tfrac{1}{2}\delta y) \cdot \frac{\sin(\frac{1}{2}\delta y)}{\delta y/2}\right] \frac{dy}{dx}$$

$$= (\cos y) \,.\, (dy/dx).$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}.$$

18. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{(1-x^2)}}.$

For proof proceed exactly as in the case of $\sin^{-1} x$.

19. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}.$

Let $y = \tan^{-1} x$. Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\quad \delta x = \tan(y+\delta y) - \tan y.$

$$\therefore \quad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \left[\lim_{\delta y \to 0} \left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}\right\} \Big/ \delta y\right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y - y)}{\delta y \,.\, \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$= \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

20. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}.$

For proof proceed exactly as in the case of $\tan^{-1} x$.

Copyrighted material

21. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2-1)}}.$

Let $y = \sec^{-1} x$. Then $x = \sec y$ and so $x + \delta x = \sec(y+\delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sec(y+\delta y) - \sec y$.

Scanned with CamScanner

5:51:18 91%
books.google.co.in/books?i

$\delta y \qquad \delta x$

Copyrighted material

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sin (y + \delta y) - \sin y}{\delta y} \cdot \lim_{\delta y \to 0} \frac{\delta y}{\delta x},$$

$$[\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \left[\lim_{\delta y \to 0} \frac{2 \cos (y + \frac{1}{2}\delta y) \sin (\frac{1}{2}\delta y)}{\delta y} \right] \cdot \frac{dy}{dx}$$

$$= \left[\lim_{\delta y \to 0} \cos (y + \tfrac{1}{2}\delta y) \cdot \frac{\sin (\frac{1}{2}\delta y)}{\delta y/2} \right] \frac{dy}{dx}$$

$$= (\cos y) \,.\, (dy/dx).$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1 - \sin^2 y)}} = \frac{1}{\sqrt{(1 - x^2)}}.$$

18. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\cos^{-1} x) = -\dfrac{1}{\sqrt{(1 - x^2)}}.$

For proof proceed exactly as in the case of $\sin^{-1} x$.

19. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\tan^{-1} x) = \dfrac{1}{1 + x^2}.$

Let $y = \tan^{-1} x$. Then $x = \tan y$ and so $x + \delta x = \tan (y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\qquad \delta x = \tan (y + \delta y) - \tan y.$

$$\therefore \quad 1 = \frac{\tan (y + \delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\tan (y + \delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \left[\lim_{\delta y \to 0} \left\{ \frac{\sin (y + \delta y)}{\cos (y + \delta y)} - \frac{\sin y}{\cos y} \right\} \Big/ \delta y \right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin (y + \delta y) \cos y - \cos (y + \delta y) \sin y}{\delta y \cos (y + \delta y) \cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin (y + \delta y - y)}{\delta y . \cos (y + \delta y) \cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos (y + \delta y) \cos y} \right]$$

$$= \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

20. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\cot^{-1} x) = -\dfrac{1}{1 + x^2}.$

For proof proceed exactly as in the case of $\tan^{-1} x$.

Copyrighted material

Scanned with CamScanner

5:51:29 91%

books.google.co.in/books?i

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim_{\delta y \to 0} \frac{\delta y}{\delta x}, \quad [\because \delta y \to 0 \text{ when } \delta x \to 0]$$

$$= \left[\lim_{\delta y \to 0} \frac{2\cos(y+\frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y} \right] \cdot \frac{dy}{dx}$$

$$= \left[\lim_{\delta y \to 0} \cos(y+\tfrac{1}{2}\delta y) \cdot \frac{\sin(\frac{1}{2}\delta y)}{\delta y/2} \right] \frac{dy}{dx}$$

$$= (\cos y) . (dy/dx).$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}}.$$

18. $\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{(1-x^2)}}$.

For proof proceed exactly as in the case of $\sin^{-1} x$.

19. $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$.

Let $y = \tan^{-1} x$. Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$.
As $\delta x \to 0$, δy also $\to 0$.

Now $\quad \delta x = \tan(y+\delta y) - \tan y$.

$$\therefore \quad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \left[\lim_{\delta y \to 0} \left\{ \frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y} \right\} \Big/ \delta y \right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y - y)}{\delta y . \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y} \right]$$

$$= \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

20. $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$.

For proof proceed exactly as in the case of $\tan^{-1} x$.

Copyrighted material

21. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2-1)}}$.

Let $y = \sec^{-1} x$. Then $x = \sec y$ and so $x + \delta x = \sec(y + \delta y)$.
As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sec(y + \delta y) - \sec y$.

Scanned with CamScanner

5:51:38 91%
books.google.co.in/books?i

21. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2-1)}}$.

Let $y = \sec^{-1} x$. Then $x = \sec y$ and so $x + \delta x = \sec(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sec(y + \delta y) - \sec y$.

$$\therefore \quad 1 = \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{1}{\cos(y+\delta y)} - \frac{1}{\cos y}\right] \Big/ \delta y$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\cos y - \cos(y+\delta y)}{\delta y \cdot \cos y \cos(y+\delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{2\sin(y+\frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y \cdot \cos y \cos(y+\delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin(y+\frac{1}{2}\delta y)}{\cos y \cos(y+\delta y)} \cdot \frac{\sin(\frac{1}{2}\delta y)}{\frac{1}{2}\delta y}\right]$$

$$= \frac{dy}{dx} \cdot \frac{\sin y}{\cos y \cos y} = \frac{dy}{dx} \cdot \sec y \tan y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2-1)}}.$$

22. $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{(x^2-1)}}$.

For proof proceed as in the case of $\sec^{-1} x$.

§ 3. Differential coefficient of a function of a function.

Consider the function $\log \sin x$. Here $\log(\sin x)$ is a function of $\sin x$ whereas $\sin x$ is itself a function of x. Thus we have the case of a function of a function.

Let $y = f\{\phi(x)\}$.

Put $t = \phi(x)$. Then $t + \delta t = \phi(x + \delta x)$. As $\delta x \to 0$, δt also $\to 0$.

We have

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta t} \cdot \frac{\delta t}{\delta x} = \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta t}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta t}{\delta x}\right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\delta y}{\delta t}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x}\right) \cdot \text{since } \delta t \to 0 \text{ when } \delta x \to 0$$

$$= \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Thus, *if y is a function of t and t is a function of x, then y is a function of x and we have*

Copyrighted material

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Similarly, *if y is a function of u, u is a function of v and v is a function of x, then y is also a function of x, and we have*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

This is known as the **chain rule of differentiation**. We can extend

Scanned with CamScanner

5:51:48
books.google.co.in/books?i

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}.$$

Similarly, *if y is a function of u, u is a function of v and v is a function of x, then y is also a function of x, and we have*

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}.$$

This is known as the **chain rule of differentiation**. We can extend it still further.

Cor. To prove that $\dfrac{dy}{dx}\times\dfrac{dx}{dy}=1$.

Let $y=f(x)$. Then $y+\delta y=f(x+\delta x)$. As $\delta x\to 0$, δy also $\to 0$. We have

$$\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta y}=1.$$

Taking limits of both sides when $\delta x\to 0$, we get

$$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta y}\right)=1 \text{ or } \left(\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\right)\left(\lim_{\delta x\to 0}\frac{\delta x}{\delta y}\right)=1$$

or $\left(\lim_{\delta x\to 0}\dfrac{\delta y}{\delta x}\right)\cdot\left(\lim_{\delta y\to 0}\dfrac{\delta x}{\delta y}\right)=1$, since $\delta y\to 0$ as $\delta x\to 0$

or $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$. Thus $\dfrac{dy}{dx}=\dfrac{1}{dx/dy}$.

§ 4. Hyperbolic functions.

These are defined as follows :

$$\sinh x=\tfrac{1}{2}(e^x-e^{-x});\ \cosh x=\tfrac{1}{2}(e^x+e^{-x});$$

$$\tanh x=\frac{\sinh x}{\cosh x}=\frac{e^x-e^{-x}}{e^x+e^{-x}};\ \coth x=\frac{\cosh x}{\sinh x}=\frac{e^x+e^{-x}}{e^x-e^{-x}};$$

$$\operatorname{sech} x=\frac{1}{\cosh x};\ \text{and } \operatorname{cosech} x=\frac{1}{\sinh x}.$$

Remember the following relations between hyperbolic functions.

(i) $\cosh^2 x-\sinh^2 x=1$; (ii) $\sinh 2x=2\sinh x\cosh x$;
(iii) $1-\tanh^2 x=\operatorname{sech}^2 x$; (iv) $\coth^2 x-1=\operatorname{cosech}^2 x$;
(v) $\cosh 2x=\cosh^2 x+\sinh^2 x$;
(vi) $\tanh 2x=\dfrac{2\tanh x}{1+\tanh^2 x}$.

To get any of these relations first write the corresponding relation between circular functions. Then convert each circular function into the corresponding hyperbolic function and also change the sign of the term which contains the product of two sines.

Copyrighted material

Derivatives of hyperbolic functions.

We have, $\dfrac{d}{dx}\sinh x=\dfrac{d}{dx}\{\tfrac{1}{2}(e^x-e^{-x})\}=\tfrac{1}{2}(e^x+e^{-x})=\cosh x$.

Similarly we can find the derivatives of other hyperbolic functions. Remember the following results.

$$\frac{d}{dx}\sinh x=\cosh x,\ \frac{d}{dx}\cosh x=\sinh x,\ \frac{d}{dx}\tanh x=\operatorname{sech}^2 x,$$

Scanned with CamScanner

5:51:58 91%
books.google.co.in/books?i 2

We have, $\frac{d}{dx}\sinh x = \frac{d}{dx}\{\frac{1}{2}(e^x - e^{-x})\} = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

Similarly we can find the derivatives of other hyperbolic functions. Remember the following results.

$$\frac{d}{dx}\sinh x = \cosh x, \frac{d}{dx}\cosh x = \sinh x, \frac{d}{dx}\tanh x = \text{sech}^2 x,$$

$$\frac{d}{dx}\coth x = -\text{cosech}^2 x, \frac{d}{dx}\text{sech } x = -\text{sech } x \tanh x,$$

$$\frac{d}{dx}\text{cosech } x = -\text{cosech } x \coth x.$$

We note that these results are similar to the corresponding results on circular funcions. The only change is that here the derivatives of $\sinh x$, $\cosh x$ and $\tanh x$ have + ive sign placed before them and the derivatives of the remaining three functions have − ive sign placed before them.

§ 5. Inverse hyperbolic functions.

Let $y = \sinh^{-1} x$. Then $\sinh y = x$. Therefore $\cosh y = \sqrt{(\sinh^2 y + 1)} = \sqrt{(x^2 + 1)}$. Adding these, we get $\sinh y + \cosh y = x + \sqrt{(x^2 + 1)}$

or $\frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = x + \sqrt{(x^2+1)}$ or $e^y = x + \sqrt{(x^2+1)}$.

$\therefore \quad y = \log[x + \sqrt{(x^2+1)}]$.

Thus $\quad \sinh^{-1} x = \log[x + \sqrt{(x^2+1)}]$.

Similarly, $\quad \cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]$;

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x};\ \coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1};$$

$$\text{sech}^{-1} x = \log\frac{1+\sqrt{(1-x^2)}}{x};\ \text{cosech}^{-1} x = \log\frac{1+\sqrt{(1+x^2)}}{x}.$$

Derivatives of Inverse Hyperbolic functions.

Let $\sinh^{-1} x = y$. Then $x = \sinh y$.

$$\therefore \quad \frac{d}{dx}(x) = \frac{d}{dx}(\sinh y) \text{ or } 1 = \left[\frac{d}{dy}(\sinh y)\right]\frac{dy}{dx} = \cosh y\frac{dy}{dx}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh^2 y + 1)}} = \frac{1}{\sqrt{(x^2+1)}}.$$

$$\therefore \quad \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{(x^2+1)}}.$$

Similarly, $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{(x^2-1)}}$;

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2};\ \frac{d}{dx}\coth^{-1} x = -\frac{1}{x^2-1}.$$

Copyrighted material

$$\frac{d}{dx}\text{sech}^{-1} x = \frac{1}{x\sqrt{(1-x^2)}};\ \frac{d}{dx}\text{cosech}^{-1} x = -\frac{1}{x\sqrt{(1+x^2)}}.$$

§ 6. Some more methods of differentiation.

(a) Logarithmic differentiation. When a function consists of the product or the quotient of a number of functions, we take the logarithm and differentiate. This process, called *logarithmic differentiation*, is also useful when a funcion of x is raised to a power which is itself a function of x.

(b) Implict functions. If the relation between y and x is represented by an equation from which y cannot be easily expressed in terms of x, then y is called an *implicit function* of x. But if x is given

Scanned with CamScanner

5:52:07 91%
books.google.co.in/books?i
Copyrighted material

Derivatives of hyperbolic functions.

We have, $\frac{d}{dx}\sinh x = \frac{d}{dx}\{\frac{1}{2}(e^x - e^{-x})\} = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

Similarly we can find the derivatives of other hyperbolic functions. Remember the following results.

$$\frac{d}{dx}\sinh x = \cosh x,\ \frac{d}{dx}\cosh x = \sinh x,\ \frac{d}{dx}\tanh x = \text{sech}^2 x,$$

$$\frac{d}{dx}\coth x = -\text{cosech}^2 x,\ \frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x,$$

$$\frac{d}{dx}\text{cosech}\, x = -\text{cosech}\, x \coth x.$$

We note that these results are similar to the corresponding results on circular funcions. The only change is that here the derivatives of $\sinh x$, $\cosh x$ and $\tanh x$ have + ive sign placed before them and the derivatives of the remaining three functions have − ive sign placed before them.

§ 5. Inverse hyperbolic functions.

Let $y = \sinh^{-1} x$. Then $\sinh y = x$. Therefore $\cosh y = \sqrt{(\sinh^2 y + 1)} = \sqrt{(x^2 + 1)}$. Adding these, we get $\sinh y + \cosh y = x + \sqrt{(x^2 + 1)}$

or $\frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = x + \sqrt{(x^2+1)}$ or $e^y = x + \sqrt{(x^2 + 1)}$.

$\therefore\ y = \log[x + \sqrt{(x^2 + 1)}]$.

Thus $\sinh^{-1} x = \log[x + \sqrt{(x^2 + 1)}]$.

Similarly, $\cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]$;

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x};\ \coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1};$$

$$\text{sech}^{-1} x = \log\frac{1+\sqrt{(1-x^2)}}{x};\ \text{cosech}^{-1} x = \log\frac{1+\sqrt{(1+x^2)}}{x}.$$

Derivatives of Inverse Hyperbolic functions.

Let $\sinh^{-1} x = y$. Then $x = \sinh y$.

$$\therefore\ \frac{d}{dx}(x) = \frac{d}{dx}(\sinh y) \text{ or } 1 = \left[\frac{d}{dy}(\sinh y)\right]\frac{dy}{dx} = \cosh y\,\frac{dy}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh^2 y + 1)}} = \frac{1}{\sqrt{(x^2+1)}}.$$

$$\therefore\ \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{(x^2+1)}}.$$

Similarly, $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{(x^2 - 1}}$;

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2};\ \frac{d}{dx}\coth^{-1} x = -\frac{1}{x^2-1}.$$

Copyrighted material

$$\frac{d}{dx}\text{sech}^{-1} x = \frac{1}{x\sqrt{(1-x^2)}};\ \frac{d}{dx}\text{cosech}^{-1} x = -\frac{1}{x\sqrt{(1+x^2)}}.$$

Scanned with CamScanner

5:52:17 91%
books.google.co.in/books?

$$\frac{d}{dx}\operatorname{sech}^{-1}x = \frac{1}{x\sqrt{(1-x^2)}};\ \frac{d}{dx}\operatorname{cosech}^{-1}x = -\frac{1}{x\sqrt{(1+x^2)}}.$$

§ 6. Some more methods of differentiation.

(a) Logarithmic differentiation. When a function consists of the product or the quotient of a number of functions, we take the logarithm and differentiate. This process, called *logarithmic differentiation*, is also useful when a funcion of x is raised to a power which is itself a function of x.

(b) Implict functions. If the relation between y and x is represented by an equation from which y cannot be easily expressed in terms of x, then y is called an *implicit function* of x. But if x is given directly in terms of x, then y is called an *explicit function* of x.

To find dy/dx in the case of implicit functions, differentiate each term of the given equation w.r.t. 'x' and solve the resulting equation for dy/dx.

Note. Here the value of dy/dx shall usually contain both x and y.

(c) Parametric equations. Sometimes x and y are both expressed in terms of a third variable, say, t. This variable is called a parameter and the equations thus given are known as parametric equations. In the case of parametric equations $x = f_1(t)$ and $y = f_2(t)$, we find dy/dx in the following manner :

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy/dt}{dx/dt}.$$

(d) Trigonometrical transformations. Sometimes it is easy to differentiate after making some trigonometrical transformations.

Following formulae of trigonometry are of frequent use :

(i) $\cos x = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2)$;

(ii) $\sin x = 2\tan\frac{1}{2}x/(1+\tan^2\frac{1}{2}x)$;

(iii) $\cos x = (1-\tan^2\frac{1}{2}x)/(1+\tan^2\frac{1}{2}x)$;

(iv) $\tan x = 2\tan\frac{1}{2}x/(1-\tan^2\frac{1}{2}x)$;

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$;

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$;

(vii) $2\tan^{-1}x = \tan^{-1}2x/(1-x^2)$;

(viii) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$;

(ix) $\sin 3x = 3\sin x - 4\sin^3 x$;

(x) $\cos 3x = 4\cos^3 x - 3\cos x$.

(e) Differentiation of a function w.r.t. a function.

Suppose we have to differentiate $f(x)$ w.r.t. $\phi(x)$ · Put $\phi(x) = t$.

Copyrighted material

Then

$$\frac{d f(x)}{d\phi(x)} = \frac{d f(x)}{dt} = \frac{d f(x)}{dx}\cdot\frac{dx}{dt} = \frac{d f(x)}{dx}\Big/\frac{dt}{dx}.$$

Hence $\dfrac{d f(x)}{d\phi(x)} = \dfrac{d f(x)}{dx}\Big/\dfrac{d\phi(x)}{dx}.$

§ 7. List of Standard results to be Committed to Memory.

Scanned with CamScanner

5:52:23 91%
books.google.co.in/books?i

Then

$$\frac{d f(x)}{d \phi(x)} = \frac{d f(x)}{dt} = \frac{d f(x)}{dx} \cdot \frac{dx}{dt} = \frac{d f(x)}{dx} \Big/ \frac{dt}{dx}.$$

Hence $\dfrac{d f(x)}{d \phi(x)} = \dfrac{d f(x)}{dx} \Big/ \dfrac{d \phi(x)}{dx}$.

§ 7. List of Standard results to be Committed to Memory.

$\dfrac{d}{dx} x^n = nx^{n-1}$ | $\dfrac{d}{dx} \tan x = \sec^2 x$

$\dfrac{d}{dx} e^x = e^x$ | $\dfrac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

$\dfrac{d}{dx} \log_e x = \dfrac{1}{x}$ | $\dfrac{d}{dx} \sec x = \sec x \tan x$

$\dfrac{d}{dx} \log_a x = \dfrac{1}{x} \log_a e$ | $\dfrac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$\dfrac{d}{dx} a^x = a^x \log_e a$ | $\dfrac{d}{dx} \sinh x = \cosh x$

$\dfrac{d}{dx} \sin x = \cos x$ | $\dfrac{d}{dx} \cosh x = \sinh x$

$\dfrac{d}{dx} \cos x = -\sin x$ | $\dfrac{d}{dx} \tanh x = \operatorname{sech}^2 x$

$\dfrac{d}{dx} \coth x = -\operatorname{cosech}^2 x$ | $\dfrac{d}{dx} \cot^{-1} x = -\dfrac{1}{1+x^2}$

$\dfrac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$ | $\dfrac{d}{dx} \sec^{-1} x = \dfrac{1}{x\sqrt{(x^2-1)}}$

$\dfrac{d}{dx} \operatorname{cosech} x = -\operatorname{cosech} x \coth x$ | $\dfrac{d}{dx} \operatorname{cosec}^{-1} x = -\dfrac{1}{x\sqrt{(x^2-1)}}$

$\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\sqrt{(1-x^2)}}$ | $\dfrac{d}{dx} \sinh^{-1} x = \dfrac{1}{\sqrt{(x^2+1)}}$

$\dfrac{d}{dx} \cos^{-1} x = -\dfrac{1}{\sqrt{(1-x^2)}}$ | $\dfrac{d}{dx} \cosh^{-1} x = \dfrac{1}{\sqrt{(x^2-1)}}$

$\dfrac{d}{dx} \tan^{-1} x = \dfrac{1}{1+x^2}$ | $\dfrac{d}{dx} \tanh^{-1} x = \dfrac{1}{1-x^2}$

Solved Examples

Ex. 1. *Find the differential coefficients of the following w.r.t. x.*

(i) $\sqrt{(2x - x^{-2})}$ **(ii)** $\dfrac{\sqrt{(5-2x)}}{2x+1}$ **(iii)** $\log \dfrac{2}{x}$

(iv) $\sqrt{\{(1+\sin x)/(1-\sin x)\}}$ **(v)** $\dfrac{ax^2+b}{ax^2-b} + \dfrac{ax^2-b}{ax^2+b}$

(vi) $\log(\log x)$.

Copyrighted material

Sol. **(i)** $\dfrac{d}{dx}(2x - x^{-2})^{1/2} = \frac{1}{2}(2x - x^{-2})^{-1/2}(2 + 2x^{-3})$

$= (2x - x^{-2})^{-1/2}(1 + x^{-3}) = \dfrac{1 + x^{-3}}{\sqrt{(2x - x^{-2})}}$.

(ii) $\dfrac{d}{dx}\left\{\dfrac{\sqrt{(5-2x)}}{2x+1}\right\} = \dfrac{d}{dx}\{(5-2x)^{1/2} \cdot (2x+1)^{-1}\}$

Scanned with CamScanner

5:52:40 91%
books.google.co.in/books?i

Copyrighted material

Sol. **(i)** $\frac{d}{dx}(2x - x^{-2})^{1/2} = \frac{1}{2}(2x - x^{-2})^{-1/2}(2 + 2x^{-3})$

$= (2x - x^{-2})^{-1/2}(1 + x^{-3}) = \frac{1 + x^{-3}}{\sqrt{(2x - x^{-2})}}$.

(ii) $\frac{d}{dx}\left\{\frac{\sqrt{(5-2x)}}{2x+1}\right\} = \frac{d}{dx}\{(5-2x)^{1/2} \cdot (2x+1)^{-1}\}$

$= \frac{1}{2}(5-2x)^{-1/2}(-2)(2x+1)^{-1} - (2x+1)^{-2} \cdot 2 \cdot (5-2x)^{1/2}$

$= \frac{-1}{(2x+1)\sqrt{(5-2x)}} - \frac{2\sqrt{(5-2x)}}{(2x+1)^2}$

$= \frac{-(2x+1) - 2(5-2x)}{(2x+1)^2\sqrt{(5-2x)}} = \frac{2x-11}{(2x+1)^2\sqrt{(5-2x)}}$.

(iii) $\frac{d}{dx}\log\frac{2}{x} = \frac{d}{dx}(\log 2 - \log x) = -\frac{1}{x}$.

(iv) We have $\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)} = \sqrt{\left\{\frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)}\right\}}$

$= \sqrt{\left\{\frac{(1+\sin x)^2}{1-\sin^2 x}\right\}} = \frac{1+\sin x}{\cos x} = \sec x + \tan x.$

$\therefore \quad \frac{d}{dx}\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)} = \frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x.$

(v) $\frac{d}{dx}\left(\frac{ax^2+b}{ax^2-b} + \frac{ax^2-b}{ax^2+b}\right) = \frac{d}{dx}\left(\frac{2a^2x^4+2b^2}{a^2x^4-b^2}\right)$

$= \frac{8a^2x^3(a^2x^4-b^2) - 4a^2x^3(2a^2x^4+2b^2)}{(a^2x^4-b^2)^2}$

$= -\frac{16a^2b^2x^3}{(a^2x^4-b^2)^2}$.

(vi) $\frac{d}{dx}\{\log(\log x)\} = \frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log x}$.

Ex. 2. *Find the differential coefficients of the following w.r.t. x.*

(i) $7\sin x + 2\log x - e^x + (x^2 - 7x + 4)$

(ii) $(\cos x) \cdot (\log x)$ **(iii)** $e^{\sin^{-1} x}$

(iv) $e^{ax}\cos(bx + c)$ **(v)** $\log \sec x$

(vi) $a^{2x}\sinh 2x$ **(vii)** $\log\frac{1+\sqrt{x}}{1-\sqrt{x}}$

(viii) $\frac{e^x}{1+x}$ **(ix)** $\frac{\tan x}{x+e^x}$

(x) $\log[\sqrt{(1+\log x)} - \sin x]$ **(xi)** $\tan(\log\tan^{-1}\sqrt{x})$.

Sol. **(i)** Let $y = 7\sin x + 2\log x + e^x + (x^2 - 7x + 4)$.

Then $\frac{dy}{dx} = 7.\frac{d}{dx}(\sin x) + 2\frac{d}{dx}(\log x) - \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2 - 7x + 4)$

$= 7\cos x + 2/x - e^x + 2x - 7.$

Copyrighted material

Scanned with CamScanner

5:52:45 books.google.co.in/books? 91%

Copyrighted material

(ii) Let $y = (\cos x) \cdot (\log x)$.

Then $\dfrac{dy}{dx} = (\cos x) \dfrac{d}{dx} (\log x) + (\log x) \dfrac{d}{dx} (\cos x)$

$= (\cos x)(1/x) + (\log x)(-\sin x)$

$= (1/x) \cos x - (\sin x) \log x.$

(iii) Let $y = e^{\sin^{-1} x}$. Then $\dfrac{dy}{dx} = e^{\sin^{-1} x} \dfrac{d}{dx} \sin^{-1} x$

$= e^{\sin^{-1} x} \cdot \dfrac{1}{\sqrt{(1-x^2)}}$.

(iv) Let $y = e^{ax} \cos (bx + c)$.

Then $\dfrac{dy}{dx} = e^{ax} \cdot \dfrac{d}{dx} \cos (bx + c) + \cos (bx + c) \cdot \dfrac{d}{dx} e^{ax}$

$= e^{ax} [\{-\sin (bx + c)\} b] + \cos (bx + c) . e^{ax} . a$

$= e^{ax} [a \cos (bx + c) - b \sin (bx + c)].$

(v) Let $y = \log \sec x$. Then $\dfrac{dy}{dx} = \dfrac{1}{\sec x} \cdot \dfrac{d}{dx} (\sec x)$

$= \dfrac{1}{\sec x} \sec x \tan x = \tan x.$

(vi) Let $y = a^{2x} \sinh 2x$. Then $\dfrac{dy}{dx} = a^{2x} . \dfrac{d}{dx} \sinh 2x + \sinh 2x \cdot \dfrac{d}{dx} a^{2x}$

$= 2a^{2x} \cosh 2x + (\sinh 2x) \cdot (a^{2x} \log a) \cdot 2$

$= 2a^{2x} [\cosh 2x + (\log a) \sinh 2x].$

(vii) Let $y = \log \dfrac{1 + \sqrt{x}}{1 - \sqrt{x}} = \log (1 + \sqrt{x}) - \log (1 - \sqrt{x})$.

Then $\dfrac{dy}{dx} = \dfrac{1}{1 + \sqrt{x}} \cdot \dfrac{d}{dx} (1 + \sqrt{x}) - \dfrac{1}{1 - \sqrt{x}} \cdot \dfrac{d}{dx} (1 - \sqrt{x})$

$= \dfrac{1}{2\sqrt{x}} \left[\dfrac{1}{1 + \sqrt{x}} + \dfrac{1}{1 - \sqrt{x}}\right]$

$= \dfrac{1}{2\sqrt{x}} \cdot \dfrac{2}{1 - x} = \dfrac{1}{(1 - x)\sqrt{x}}$.

(viii) Let $y = e^x/(1 + x)$.

Then $\dfrac{dy}{dx} = \dfrac{(1 + x) \cdot \dfrac{d}{dx} e^x - e^x \cdot \dfrac{d}{dx} (1 + x)}{(1 + x)^2}$

$= \dfrac{(1 + x) e^x - e^x \cdot 1}{(1 + x)^2} = \dfrac{x e^x}{(1 + x)^2}$.

(ix) Let $y = \tan x/(x + e^x)$.

Then $\dfrac{dy}{dx} = \dfrac{(x + e^x) \sec^2 x - (\tan x)(1 + e^x)}{(x + e^x)^2}$

$= \dfrac{x \sec^2 x + e^x (\sec^2 x - \tan x) - \tan x}{(x + e^x)^2}$

Copyrighted material

(x) Let $y = \log [\sqrt{(1 + \log x)} - \sin x]$.

Scanned with CamScanner

5:52:52 91%
books.google.co.in/books?i
Copyrighted material

(x) Let $y = \log[\sqrt{(1+\log x)} - \sin x]$.

Then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1+\log x)} - \sin x} \cdot \left[\dfrac{1}{2x\sqrt{(1+\log x)}} - \cos x\right]$.

(xi) Let $y = \tan(\log \tan^{-1}\sqrt{x})$.

Then $\dfrac{dy}{dx} = \sec^2(\log\tan^{-1}\sqrt{x})\left[\dfrac{1}{\tan^{-1}(\sqrt{x})} \cdot \dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}}\right]$

$$= \frac{\sec^2(\log\tan^{-1}\sqrt{x})}{(2\sqrt{x})(1+x)\tan^{-1}(\sqrt{x})}.$$

Problems involving logarithmic differentiation, implicit functions and parametric equations.

Ex. 3. *Find (dy/dx), when*

(i) $y = x^{(x^x)}$ **(Delhi 1983,79)**

(ii) $y = (\tan x)^{\log x} + (\cot x)^{\sin x}$,

(iii) $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$, (iv) $y = x^x + x^{\sin x}$.

Sol. (i) Taking logarithm of both sides, we get

$\log y = (x^x)\log x$.

Now differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x^x . \frac{1}{x} + \log x . \frac{d}{dx}(x^x). \quad ...(1)$$

Let $z = x^x$. Then $\log z = x \log x$.

Differentiating w.r.t. x, we have

$$\frac{1}{z}\frac{dz}{dx} = x\frac{1}{x} + \log x. \quad \therefore \quad \frac{dz}{dx} = x^x(1+\log x).$$

Now (1) gives, $\dfrac{dy}{dx} = x^{(x^x)} \cdot [x^{x-1} + x^x(\log x)\cdot(1+\log x)]$.

(ii) We have $y = (\tan x)^{\log x} + (\cot x)^{\sin x}$.

Let $(\tan x)^{\log x} = u$ and $(\cot x)^{\sin x} = v$.

Then $y = u + v$. $\therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Now $\log u = (\log x)(\log\tan x)$.

$$\therefore \frac{1}{u}\cdot\frac{du}{dx} = (\log x)\cdot\frac{\sec^2 x}{\tan x} + \frac{1}{x}\log\tan x$$

or $\dfrac{du}{dx} = (\tan x)^{\log x}.[\cot x \sec^2 x \log x + (1/x)\log\tan x]$.

Again $\log v = (\sin x)(\log\cot x)$.

$$\therefore \frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\cot x}(-\operatorname{cosec}^2 x) + \cos x \log\cot x$$

or $\dfrac{dv}{dx} = (\cot x)^{\sin x}[\cos x\log\cot x - \sec x]$.

Hence $\dfrac{dy}{dx} = (\tan x)^{\log x}[\cot x\sec^2 x\log x + (1/x)\log\tan x]$
$+ (\cot x)^{\sin x}[\cos x\log\cot x - \sec x]$.

Copyrighted material

(iii) Here $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$.

Let $(\cot x)^{\cot x} = u$ and $(\cosh x)^{\cosh x} = v$.

Then $y = u + v$. Therefore $dy/dx = du/dx + dv/dx$.

Scanned with CamScanner

5:52:57 91%
books.google.co.in/books?i

Copyrighted material

(iii) Here $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$.

Let $(\cot x)^{\cot x} = u$ and $(\cosh x)^{\cosh x} = v$.

Then $y = u + v$. Therefore $dy/dx = du/dx + dv/dx$.

Now $\log u = \cot x . \log \cot x$.

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \cot x . \frac{1}{\cot x}(-\operatorname{cosec}^2 x) - \operatorname{cosec}^2 x . \log \cot x$$

or $$\frac{du}{dx} = -\operatorname{cosec}^2 x\,(1 + \log \cot x).$$

Again $\log v = \cosh x \log \cosh x$.

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \cosh x \cdot \frac{1}{\cosh x} \cdot \sinh x + \sinh x \log \cosh x$$

or $$\frac{dv}{dx} = v \sinh x\,(1 + \log \cosh x).$$

Hence

$$\frac{dy}{dx} = -u \operatorname{cosec}^2 x\,(1 + \log \cot x) + v \sinh x\,(1 + \log \cosh x)$$

$$= -(\cot x)^{\cot x} \operatorname{cosec}^2 x\,(1 + \log \cot x) + (\cosh x)^{\cosh x} \sinh x\,(1 + \log \cosh x).$$

(iv) We have $y = x^x + (x)^{\sin x}$.

Let $x^x = u$ and $(x)^{\sin x} = v$.

Then $y = u + v$ and $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Now $\log u = x \log x$.

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \log x \quad \text{or} \quad \frac{du}{dx} = x^x(1 + \log x).$$

Again $\log v = (\sin x) \log x$.

$$\therefore \quad \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) . \frac{1}{x} + (\cos x) \log x$$

or $$\frac{dv}{dx} = x^{\sin x} \cdot \left[\frac{1}{x}\sin x + (\cos x)\log x\right].$$

Hence $$\frac{dy}{dx} = x^x . (1 + \log x) + (x)^{\sin x}\left[\frac{1}{x}\sin x + (\cos x)\log x\right].$$

Ex. 4. *Find* $\dfrac{dy}{dx}$ *if*

(*i*) $y = (x)^{\tan x} + (\sin x)^{\cos x}$. **(Delhi 1980)**

(*ii*) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$.

Sol. (i) Let $u = (x)^{\tan x}$ and $v = (\sin x)^{\cos x}$.

$\therefore \quad y = u + v \Rightarrow (dy/dx) = (du/dx) + (dv/dx)$. ...(1)

Now $u = (x)^{\tan x} \Rightarrow \log u = \tan x \log x$.

$\therefore$ differentiating w.r.t. x, we have

$$(1/u)(du/dx) = (\sec^2 x) \cdot \log x + (\tan x)/x$$

Copyrighted material

or $$\frac{du}{dx} = (x)^{\tan x}\left[\sec^2 x \log x + \frac{\tan x}{x}\right].$$

Scanned with CamScanner

5:53:04 91%
books.google.co.in/books?i
Copyrighted material

or $\dfrac{du}{dx} = (x)^{\tan x}\left[\sec^2 x \log x + \dfrac{\tan x}{x}\right]$.

Similarly $\dfrac{dv}{dx} = (\sin x)^{\cos x}\,[(-\sin x)\cdot \log \sin x + (\cos x)\cdot \cot x]$.

$\therefore$ from (1), we get

$$(dy/dx) = (du/dx) + (dv/dx)$$
$$= (x)^{\tan x}\,[\sec^2 x \log x + (\tan x)/x]$$
$$+ (\sin x)^{\cos x}\,[\cos x \cot x - \sin x \cdot \log \sin x].$$

(ii) Proceed exactly as in part (i).

Ex. 5. *Find (dy/dx) if $\sin y = \log_{\sin x} \cos x$.* **(Delhi 1980)**

Sol. We have

$$\sin y = \log_{\sin x} \cos x = \frac{\log_e \cos x}{\log_e \sin x}. \qquad \textbf{[Note]}$$

Differentiating both sides w.r.t. x, we have

$$(\cos y)\cdot \frac{dy}{dx} = \frac{(-\tan x)\cdot \log \sin x - (\cot x)\cdot \log \cos x}{(\log_e \sin x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(\tan x) \log \sin x + (\cot x) \log \cos x}{(\cos y)\,(\log \sin x)^2}.$$

Ex. 6. *Find (dy/dx), when*

(i) $x = a\,(t - \sin t),\ y = a\,(1 - \cos t)$.

(ii) $x = a\,(\cos t + \log \tan t/2),\ y = a \sin t$.

(iii) $y = \tan^{-1} \dfrac{2t}{1 - t^2},\ x = \sin^{-1} \dfrac{2t}{1 + t^2}$.

(iv) $x = a\sqrt{\left(\dfrac{t^2 - 1}{t^2 + 1}\right)},\ y = at\sqrt{\left(\dfrac{t^2 - 1}{t^2 + 1}\right)}$.

Sol. (i) We have $x = a\,(t - \sin t), y = a\,(1 - \cos t)$.

$\therefore \quad \dfrac{dx}{dt} = a\,(1 - \cos t)$ and $\dfrac{dy}{dt} = a \sin t$.

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a\,(1 - \cos t)} = \dfrac{2 \sin t/2 \cos t/2}{2 \sin^2 t/2} = \cot t/2.$

(ii) Here $x = a\,(\cos t + \log \tan t/2), y = a \sin t$.

$$\therefore \quad \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan t/2}(\sec^2 t/2)\cdot \tfrac{1}{2}\right]$$

$$= a\left(-\sin t + \frac{1}{2 \sin t/2 \cos t/2}\right)$$

$$= a\left[\frac{1}{\sin t} - \sin t\right] = a\left[\frac{1 - \sin^2 t}{\sin t}\right] = \frac{a \cos^2 t}{\sin t}.$$

Again $(dy/dt) = a \cos t$.

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = a \cos t \cdot \dfrac{\sin t}{a \cos^2 t} = \tan t.$

Copyrighted material

(iii) Let $t = \tan \theta$. Then

$$y = \tan^{-1} \frac{2 \tan \theta}{1 - \tan^2 \theta} = \tan^{-1} \tan 2\theta = 2\theta = 2 \tan^{-1} t.$$

$(dy/dt) = 2/(1 + t^2)$

Scanned with CamScanner

5:53:10 91%

books.google.co.in/books?i

Copyrighted material

(iii) Let $t = \tan\theta$. Then

$$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}\tan 2\theta = 2\theta = 2\tan^{-1}t.$$

$\therefore$ $(dy/dt) = 2/(1+t^2)$.

Again $$x = \sin^{-1}\frac{2t}{1+t^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}t.$$

$\therefore$ $(dx/dt) = 2/(1+t^2)$.

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2/(1+t^2)}{2/(1+t^2)} = 1.$$

(iv) Here $x = a\sqrt{\left\{\dfrac{t^2-1}{t^2+1}\right\}}$ and $y = at\sqrt{\left\{\dfrac{t^2-1}{t^2+1}\right\}}$.

We have, $\log y = \log a + \log t + \frac{1}{2}\log(t^2-1) - \frac{1}{2}\log(t^2+1)$.

$$\therefore \quad \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{t}{t^2-1} - \frac{t}{t^2+1} = \frac{t^4+2t^2-1}{t(t^4-1)}$$

or $$\frac{dy}{dt} = a\sqrt{\left(\frac{t^2-1}{t^2+1}\right)}\cdot\frac{t^4+2t^2-1}{(t^4-1)}.$$

Again $\log x = \log a + \frac{1}{2}\log(t^2-1) - \frac{1}{2}\log(t^2+1)$.

$$\therefore \quad \frac{1}{x}\frac{dx}{dt} = \frac{t}{t^2-1} - \frac{t}{t^2+1} = \frac{2t}{t^4-1}$$

or $$\frac{dx}{dt} = a\sqrt{\left(\frac{t^2-1}{t^2+1}\right)}\cdot\frac{2t}{t^4-1}.$$

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^4+2t^2-1}{2t}.$$

****Ex. 7 (a).** *If* $\sin y = x\sin(a+y)$, *prove that*

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$ **(Delhi 1982,75,72)**

(b) *If* $x^y = e^{x-y}$, *prove that*

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$ **(Delhi 1981, 76; Meerut 81S)**

(c) *If* $x = e^{\tan^{-1}\{(y-x^2)/x^2\}}$, *find* (dy/dx), *expressing it as a function of x only.*

(d) *If* $x/(x-y) = \log\{a/(x-y)\}$, *prove that* $(dy/dx) = 2 - (x/y)$. **(Delhi 1983, 74)**

Sol. (a) We have $x = \sin y/\sin(a+y)$.

Differentiating both sides w.r.t. x, we get

$$1 = \frac{\cos y\sin(a+y) - \sin y\cos(a+y)}{\sin^2(a+y)}\cdot\frac{dy}{dx}$$

Copyrighted material

or $$1 = \frac{\sin\{(a+y)-y\}}{\sin^2(a+y)}\frac{dy}{dx}.$$ Hence $$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Scanned with CamScanner

5:53:15 91%
books.google.co.in/books?i
Copyrighted material

or $1 = \dfrac{\sin\{(a+y)-y\}}{\sin^2(a+y)}\dfrac{dy}{dx}$. Hence $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

(b) We have, $x^y = e^{x-y}$.

$\therefore$ $y \log x = (x-y)\log e = x - y$ or $y + y\log x = x$

or $y = x/(1+\log x)$. Hence $\dfrac{dy}{dx} = \dfrac{1\,(1+\log x) - x\cdot(1/x)}{(1+\log x)^2}$

$= \log x/(1+\log x)^2$.

(c) We have $x = e^{\tan^{-1}\{(y-x^2)/x^2\}}$.

Therefore $\log x = \tan^{-1}\dfrac{y-x^2}{x^2}\log e$

or $\dfrac{y-x^2}{x^2} = \tan\log x$ or $y = x^2 + x^2\tan\log x$.

$\therefore$ $\dfrac{dy}{dx} = 2x + x^2(\sec^2\log x)\cdot\dfrac{1}{x} + 2x\cdot\tan\log x$

$= x\,[2 + \sec^2\log x + 2\tan\log x]$.

(d) We have $x/(x-y) = \log\{a/(x-y)\}$.

Differentiating w.r.t. x, we have

$$\frac{d}{dx}\left(\frac{x}{x-y}\right) = \frac{d}{dx}\left(\log\frac{a}{x-y}\right) = \frac{d}{dx}[\log a - \log(x-y)]$$

or $\dfrac{1\cdot(x-y) - \{1-(dy/dx)\}\cdot x}{(x-y)^2} = \left[0 - \dfrac{1}{(x-y)}\left(1 - \dfrac{dy}{dx}\right)\right]$

or $(x-y) - x + x\,(dy/dx) = -(x-y) + (x-y)\,(dy/dx)$

or $y\,(dy/dx) = 2y - x$ i.e., $dy/dx = 2 - (x/y)$. **Proved.**

Ex. 8. *Find (dy/dx) if $y = \tan^{-1}\{(ax-b)/(bx+a)\}$.*

(Delhi 1980)

Sol. We have $y = \tan^{-1}\left(\dfrac{ax-b}{bx+a}\right)$.

$\therefore$ $\dfrac{dy}{dx} = \dfrac{1}{1+(ax-b)^2/(bx+a)^2}\cdot\dfrac{d}{dx}\left(\dfrac{ax-b}{bx+a}\right)$

or $\dfrac{dy}{dx} = \dfrac{(bx+a)^2}{(bx+a)^2+(ax-b)^2}\cdot\dfrac{a\,(bx+a) - b\,(ax-b)}{(bx+a)^2}$

$= \dfrac{a^2+b^2}{(a^2+b^2)x^2+(a^2+b^2)} = \dfrac{1}{1+x^2}$.

□

Copyrighted material
Scanned with CamScanner

5:54:05 91%
books.google.co.in/books?i

3
Expansions of Functions

§ 1. Taylor's series.

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[a, a+h]$. Then for every positive integral value of n, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad \ldots(1)$$

where $R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h)$, $(0 < \theta < 1)$.

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots} \quad \ldots(2)$$

The series given in (2) is known as **Taylor's infinite series** for the expansion of $f(a+h)$ as a **power series** in h.

§ 2. Maclaurin's series.

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n, \quad \ldots(1)$$

where $R_n = \frac{x^n}{n!}f^{(n)}(\theta x)$, $(0 < \theta < 1)$.

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$\mathbf{f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots} \quad \ldots(2)$$

The series given in (2) is known as **Maclaurin's infinite series** for the expansion of $f(x)$ as a **power series** in x. Maclaurin's series is a particular case of Taylor's series. If in Taylor's series we put $a = 0$ and $h = x$ we get Maclaurin's series.

Maclaurin's expansion of $f(x)$ fails if any of the functions $f(x)$, $f'(x)$, $f''(x)$, ... becomes infinite or discontinuous at any point of the interval $[0, x]$ or if R_n does not tend to zero as $n \to \infty$.

Copyrighted material

§ 3. Formal expansions of functions.

We have seen that for the validity of the expansion of a function $f(x)$ as an infinite Maclaurin's series, it is necessary that $R_n \to 0$ as $n \to \infty$. But to examine the behaviour of R_n as $n \to \infty$ is a difficult job because in many cases it is not possible to find a general expression for the nth derivatives of the functions to be expanded. So in this

Scanned with CamScanner

5:54:12 91%
books.google.co.in/books?i

Copyrighted material

§ 3. Formal expansions of functions.

We have seen that for the validity of the expansion of a function $f(x)$ as an infinite Maclaurin's series, it is necessary that $R_n \to 0$ as $n \to \infty$. But to examine the behaviour of R_n as $n \to \infty$ is a difficult job because in many cases it is not possible to find a general expression for the nth derivatives of the functions to be expanded. So in this chapter we shall simply obtain *formal expansion* of a function $f(x)$ without showing that $R_n \to 0$ as $n \to \infty$.

Such an expansion will not give us any idea of the range of values of x for which the expansion is valid. To obtain such an expansion of $f(x)$ we have only to calculate the values of its derivatives for $x = 0$ and substitute them in the infinite Maclaurin's series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \ldots\ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots.$$

For the convenience of the students we shall now give *formal proofs* of Maclaurin's and Taylor's theorems without bothering about the nature of R_n as $n \to \infty$.

Maclaurin's theorem.

(Gorakhpur 87; Meerut 84; Magadh 84; Bihar 82; Kashmir 83)

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$. Assuming that $f(x)$ can be expanded as an infinite power series in x, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

Proof. Suppose $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$ ...(1)

Let the expansion (1) be differentiable term by term any number of times. Then by successive differentiation, we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \ldots,$$
$$f''(x) = 2.1 A_2 + 3.2 A_3 x + 4.3 A_4 x^2 + \ldots,$$
$$f'''(x) = 3.2.1 A_3 + 4.3.2 A_4 x + \ldots, \text{ and so on.}$$

Putting $x = 0$ in each of these relations, we get

$$f(0) = A_0, f'(0) = A_1, f''(0) = 2!A_2, f'''(0) = 3!A_3, \ldots$$

Substituting theses values of $A_0, A_1, A_2, \ldots$ in (1), we get

$$\mathbf{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots}$$

This is **Maclaurin's theorem.** If we denote $f(x)$ by y, then Maclaurin's theorem can be written in either of the following ways :

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$

Copyrighted material

or $$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots$$

Taylor's theorem. **(Gorakhpur 1988; Bihar 82; Vikram 88; Meerut 85; Magadh 87; Jiwaji 90)**

Scanned with CamScanner

5:54:18 91%
books.google.co.in/books?i
Copyrighted material

or $$y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \dots + \frac{x^n}{n!} y_n(0) + \dots$$

Taylor's theorem. (Gorakhpur 1988; Bihar 82; Vikram 88; Meerut 85; Magadh 87; Jiwaji 90)

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[a, a+h]$. Assuming that $f(a+h)$ can be expanded as an infinite power series in h, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + \dots$$

Proof. Suppose $f(a+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \dots$ ...(1)

Let the expansion (1) be differentiable term by term any number of times w.r.t. h. Then by successive differentiation w.r.t. h, we have

$$f'(a+h) = A_1 + 2A_2 h + 3A_3 h^2 + \dots,$$
$$f''(a+h) = 2.1A_2 + 3.2A_3 h + \dots,$$
$$f'''(a+h) = 3.2.1A_3 + \dots, \text{ and so on.}$$

Putting $h = 0$ in each of the above relations, we get

$f(a) = A_0, f'(a) = A_1, f''(a) = 2!A_2, f'''(a) = 3!A_3$, and so on.

$$\therefore \quad A_0 = f(a), A_1 = f'(a), A_2 = \frac{1}{2!} f''(a), A_3 = \frac{1}{3!} f'''(a),$$
and so on.

Substituting these values of $A_0, A_1, A_2, A_3, \dots$ in (1), we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^{(n)}(a) + \dots}$$

This is **Taylor's theorem.** Another useful form is obtained on replacing h by $(x-a)$. Thus

$$\mathbf{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots,}$$

which is an expansion of $f(x)$ as a power series in $(x-a)$.

Note. If we expand $f(x+h)$, by Taylor's theorem, as a power series in h, then the result is as follows :

$$\mathbf{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x) + \dots}$$

§ 4. Some Important Expansions.

1. **Expansion of e^x.** (*Exponential series*).

Let $f(x) = e^x$. Then $f(0) = e^0 = 1$;

$f^{(n)}(x) = e^x$ so that $f^{(n)}(0) = e^0 = 1$, where $n = 1, 2, 3, \dots$

Substituting these values in Maclaurin's series

Copyrighted material

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

we get $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

Scanned with CamScanner

5:54:24 91%
books.google.co.in/books?i
2
Copyrighted material

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

we get $\quad \mathbf{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots}$

2. Expansion of sin x. (*Sine series*)

Let $f(x) = \sin x$. Then $f(0) = 0$,

$f'(x) = \cos x \quad$ so that $f'(0) = 1$,

$f''(x) = -\sin x \quad$ so that $f''(0) = 0$,

$f'''(x) = -\cos x \quad$ so that $f'''(0) = -1$, and so on.

In general, $f^n(x) = \sin(x + \frac{1}{2}n\pi)$ so that $f^n(x) = \sin(\frac{1}{2}n\pi)$.

When $n = 2m$, $f^n(0) = \sin m\pi = 0$ and when $n = 2m + 1$,

$f^n(0) = \sin\{\frac{1}{2}(2m + 1)\pi\} = \sin(m\pi + \frac{1}{2}\pi)$

$$= (-1)^m \sin(\tfrac{1}{2}\pi) = (-1)^m.$$

Substituting these values in Maclaurin's series, we get

$$\sin x = 0 + x.1 + 0 + \frac{x^3}{3!}(-1) + 0 + \ldots$$
$$+ 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots$$

or $\quad \mathbf{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots}$ **(Meerut 1985)**

Similarly, we may obtain the *Cosine series* :

$$\mathbf{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + \ldots}$$

3. Expansion of log (1 + x). **(Meerut 1976, 89; Luck. 83)**

Let $f(x) = \log(1 + x)$. Then $f(0) = \log 1 = 0$;

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

so that $\quad f^n(0) = (-1)^{n-1}(n-1)!$, where $n = 1, 2, 3, \ldots$

$\therefore \quad f'(0) = (-1)^{1-1}(1-1)! = 1$,

$f''(0) = (-1)^{2-1}(2-1)! = -1!$,

$f'''(0) = (-1)^{3-1}(3-1)! = 2!$,

$f^{iv}(0) = (-1)^{4-1}(4-1)! = 3!$, and so on,

Substituting the values of $f(0), f'(0), f''(0)$, etc. in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots,$$

we get $\quad \log(1 + x) = 0 + x.1 - \frac{x^2}{2!}\cdot 1! + \frac{x^3}{3!}\cdot 2! - \frac{x^4}{4!}\cdot 3! + \ldots$

$$+ \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \ldots$$

Copyrighted material

or $\quad \mathbf{\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots}$

Scanned with CamScanner

5:54:29 91%
books.google.co.in/books?i 2
Copyrighted material

or $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

Note. The function $\log x$ does not possess Maclaurin's series expansion because it is not defined at $x = 0$.

4. Expansion of $(1+x)^n$. (*Binomial series*).

Let $f(x) = (1+x)^n$. Then $f(0) = 1$;

$$f^m(x) = n(n-1)(n-2)\ldots(n-m+1)(1+x)^{n-m}$$

so that $f^m(0) = n(n-1)\ldots(n-m+1)$, where $m = 1, 2, 3, \ldots$

$\therefore$ $f'(0) = n, f''(0) = n(n-1), f'''(0) = n(n-1)(n-2)$ and so on.

Substituting the values of $f(0), f'(0), f''(0)$ etc. in Maclaurin's series for $f(x)$, we get

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-m+1)}{m!}x^m + \ldots$$

Solved Examples

Ex. 1. *Expand the following by Maclaurin's theorem :*

(i) a^x

(ii) $\tan x$ **(Meerut 1981; Vikram 87; Agra 81; Kashmir 84)**

(iii) $\log \sec x$ **(Rohilkhand 1991; Meerut 82, 86, 88, 95, 98)**

(iv) $\log \cos x$ (v) $\sec x$

(vi) $\log(\sec x + \tan x)$.

Sol. (i) Let $f(x) = a^x$. Then $f(0) = a^0 = 1$,

$f'(x) = a^x \log a$ so that $f'(0) = \log a$,

$f''(x) = a^x (\log a)^2$ so that $f''(0) = (\log a)^2$,

$f'''(x) = a^x (\log a)^3$ so that $f'''(0) = (\log a)^3$, and so on.

In general, $f^n(x) = a^x (\log a)^n$ so that $f^n(0) = (\log a)^n$.

Now by Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \quad a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots + \frac{x^n}{n!}(\log a)^n + \ldots$$

(ii) Let $y = \tan x$. Then $(y)_0 = \tan 0 = 0$,

$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ so that

$(y_1)_0 = 1 + (y)_0^2 = 1 + 0 = 1$,

Copyrighted material

$y_2 = 2yy_1$ so that $(y_2)_0 = 2(y)_0(y_1)_0 = 2 \times 0 \times 1 = 0$,

$y_3 = 2y_1y_1 + 2yy_2 = 2y_1^2 + 2yy_2$ so that $(y_3)_0 = 2 \times 1^2 + 0 = 2$,

Scanned with CamScanner

5:54:35 91%
books.google.co.in/books?i

$y_2 = 2yy_1$ so that $(y_2)_0 = 2\,(y)_0\,(y_1)_0 = 2 \times 0 \times 1 = 0,$

$y_3 = 2y_1y_1 + 2yy_2 = 2y_1^2 + 2yy_2$ so that $(y_3)_0 = 2 \times 1^2 + 0 = 2,$

$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$ so that

$$(y_4)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$

$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6y_2^2 + 8y_1y_3 + 2yy_4$ so that

$$(y_5)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \frac{x^4}{4\,!}(y_4)_0 + \frac{x^5}{5\,!}(y_5)_0 + \ldots$$

$$\therefore \quad \tan x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot 2 + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot 16 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

(iii) Let $y = \log \sec x$. Then $(y)_0 = \log \sec 0 = \log 1 = 0,$

$y_1 = \dfrac{1}{\sec x} \cdot \sec x \tan x = \tan x$ so that $(y_1)_0 = 0,$

$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2$ so that $(y_2)_0 = 1 + (y_1)_0^2 = 1,$

$y_3 = 2y_1y_2$ so that $(y_3)_0 = 2\,(y_1)_0\,(y_2)_0 = 0,$

$y_4 = 2y_2^2 + 2y_1y_3$ so that $(y_4)_0 = 2 \times 1^2 + 0 = 2,$

$y_5 = 4y_2y_3 + 2y_2y_3 + 2y_1y_4 = 6y_2y_3 + 2y_1y_4$ so that

$$(y_5)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$

$y_6 = 6y_3^2 + 6y_2y_4 + 2y_2y_4 + 2y_1y_5 = 6y_3^2 + 8y_2y_4 + 2y_1y_5$ so that

$$(y_6)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \frac{x^4}{4\,!}(y_4)_0 + \ldots$$

$$\therefore \quad \log \sec x = 0 + x\,.\,0 + \frac{x^2}{2\,!}\cdot 1 + \frac{x^3}{3\,!}\cdot 0 + \frac{x^4}{4\,!}\cdot 2 + \frac{x^5}{5!}\cdot 0 + \frac{x^5}{6\,!}\cdot 16 + \ldots$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$$

(iv) Let $y = \log \cos x$. Then $(y)_0 = 0,$

$y_1 = \dfrac{-\sin x}{\cos x} = -\tan x \quad$ so that $(y_1)_0 = 0,$

$y_2 = -\sec^2 x = -(1 + \tan^2 x) = -(1 + y_1^2) = -1 - y_1^2$

so that $(y_2)_0 = -1 - 0 = -1,$

$y_3 = -2y_1y_2$ so that $(y_3)_0 = 0,$

Copyrighted material

$y_4 = -2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = -2\,(-1)^2 - 0 = -2,$

$(y_5)_0 = 0,\ (y_6)_0 = -16$, and so on.

[For calculation of $(y_5)_0$ and $(y_6)_0$ proceed as in part (iii)]

Now substituting these values in Maclaurin's theorem, we get

Scanned with CamScanner

5:54:40 91%
books.google.co.in/books?i
Copyrighted material

$y_4 = -2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = -2(-1)^2 - 0 = -2$,
$(y_5)_0 = 0$, $(y_6)_0 = -16$, and so on.

[For calculation of $(y_5)_0$ and $(y_6)_0$ proceed as in part (iii)]

Now substituting these values in Maclaurin's theorem, we get

$$\log\cos x = 0 + x.0 + \frac{x^2}{2!}\cdot(-1) + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot(-2) + \frac{x^5}{5!}\cdot 0 + (x^6/6!).(-16) + \ldots$$
$$= -(x^2/2) - (x^4/12) - (x^6/45) + \ldots$$

(v) Let $y = \sec x$. Then $(y)_0 = \sec 0 = 1$,

$y_1 = \sec x \tan x$ so that $(y_1)_0 = 1 \times 0 = 0$,

$y_2 = \sec x \sec^2 x + \sec x \tan x \tan x = \sec^3 x + \sec x \tan^2 x$
$= \sec^3 x + \sec x(\sec^2 x - 1) = 2\sec^3 x - \sec x = 2y^3 - y$

so that $(y_2)_0 = 2 \times 1^3 - 1 = 1$,

$y_3 = 6y^2y_1 - y_1$ so that $(y_3)_0 = 0 - 0 = 0$,

$y_4 = 6y^2y_2 + 12yy_1^2 - y_2$ so that $(y_4)_0 = 6 \times 1^2 \times 1 + 0 - 1 = 5$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots$$

$$\therefore \quad \sec x = 1 + x.0 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot 5 + \ldots$$
$$= 1 + (x^2/2!) + (5x^4/4!) + \ldots$$

(vi) Let $y = \log(\sec x + \tan x)$. Then $(y)_0 = \log(1+0) = 0$,

$$y_1 = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \text{ so that } (y_1)_0 = \sec 0 = 1,$$

$y_2 = \sec x \tan x$ so that $(y_2)_0 = 1 \times 0 = 0$,

$y_3 = \sec x \sec^2 x + \sec x \tan^2 x = \sec^3 x + \sec x(\sec^2 x - 1)$

$= 2\sec^3 x - \sec x = 2y_1^3 - y_1$ so that $(y_3)_0 = 2 \times 1^3 - 1 = 1$,

$y_4 = 6y_1^2y_2 - y_2$ so that $(y_4)_0 = 0$,

$y_5 = 6y_1^2y_3 + 12y_1y_2^2 - y_3$ so that $(y_5)_0 = 6 \times 1^2 \times 1 + 0 - 1 = 5$,

and so on.

Substituting these values in Maclaurin's theorem, we get

$$\log(\sec x + \tan x) = 0 + x.1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}\cdot 1 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 5 + \ldots$$
$$= x + (x^3/6) + (x^5/24) + \ldots$$

**Ex. 2. *Use Maclaurin's formula to show that*

Copyrighted material

$$e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$$

(Garhwal 1983; Kanpur 80; Gorakhpur 83)

Scanned with CamScanner

5:54:47 91%
books.google.co.in/books?i
Copyrighted material

$$e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \dots$$

(Garhwal 1983; Kanpur 80; Gorakhpur 83)

Sol. Let $y = e^x \sec x$. Then $(y)_0 = 1$,

$y_1 = e^x \sec x + e^x \sec x \tan x = y + y \tan x$ so that $(y_1)_0 = 1$,

$y_2 = y_1 + y_1 \tan x + y \sec^2 x$ so that $(y_2)_0 = 1 + 0 + 1 = 2$,

$y_3 = y_2 + y_2 \tan x + 2y_1 \sec^2 x + 2y \sec^2 x \tan x$ so that

$(y_3)_0 = 2 + 2 = 4$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \dots$$

Ex. 3. *Expand the following functions by Maclaurin's theorem :*

***(i)** $e^{\sin x}$ **(Meerut 1990S; Gorakhpur 86; G.N.U. 81; K.U. 89)**

(ii) $e^{x \cos x}$

(iii) $e^x \log (1 + x)$

(iv) $\log (1 + \sin x)$ **(Gorakhpur 1989; Kanpur 86, Meerut 92)**

Sol. (i) Let $y = e^{\sin x}$. Then $(y)_0 = e^{\sin 0} = e^0 = 1$,

$y_1 = e^{\sin x} \cos x = y \cos x$ so that $(y_1)_0 = (y)_0 \cos 0 = 1 \times 1 = 1$,

$y_2 = y_1 \cos x - y \sin x$ so that $(y_2)_0 = 1 \times 1 - 1 \times 0 = 1$,

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$= y_2 \cos x - 2y_1 \sin x - y \cos x$ so that $(y_3)_0 = 1 - 0 - 1 = 0$,

$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$ so that $(y_4)_0 = -3$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \frac{x^4}{4!} (y_4)_0 + \dots$$

$$\therefore \quad e^{\sin x} = 1 + x . 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-3) + \dots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$$

(ii) Let $y = e^{x \cos x}$. Then $(y)_0 = e^0 = 1$,

$$y_1 = e^{x \cos x} (\cos x - x \sin x) = y (\cos x - x \sin x)$$

so that $(y_1)_0 = 1 . (1 - 0) = 1$,

$y_2 = y_1 (\cos x - x \sin x) + y (-2 \sin x - x \cos x)$ giving

$(y_2)_0 = 1 . (1 - 0) + 1 . 0 = 1$,

$y_3 = y_2 (\cos x - x \sin x) + y_1 (-2 \sin x - x \cos x)$

$+ y_1 (-2 \sin x - x \cos x) + y (-3 \cos x + x \sin x)$

Copyrighted material

$= y_2 (\cos x - x \sin x) + 2y_1 (-2 \sin x - x \cos x)$

Scanned with CamScanner

5:54:53 91%
books.google.co.in/books?i

$$= y_2(\cos x - x\sin x) + 2y_1(-2\sin x - x\cos x) + y(-3\cos x + x\sin x)$$

so that $(y_3)_0 = 1.(1-0) + 0 + 1.(-3) = -2$,

$$y_4 = y_3(\cos x - x\sin x) + 3y_2(-2\sin x - x\cos x) + 3y_1(-3\cos x + x\sin x) + y(4\sin x + x\cos x)$$

giving $(y_4)_0 = -2.(1-0) + 0 + 3.1.(-3+0) + 0 = -11$,

$$y_5 = y_4(\cos x - x\sin x) + 4y_3(-2\sin x - x\cos x) + 6y_2(-3\cos x + x\sin x) + 4y_1(4\sin x + x\cos x) + y(5\cos x - x\sin x)$$

so that $(y_5)_0 = -11 + 6.1.(-3) + 5 = -24$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^{x\cos x} = 1 + x.1 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot(-2) + \frac{x^4}{4!}\cdot(-11) + \frac{x^5}{5!}\cdot(-24) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} + \ldots$$

(iii) Let $y = e^x \log(1+x)$. Then $(y)_0 = e^0 \log 1 = 1 \times 0 = 0$,

$$y_1 = e^x \log(1+x) + e^x(1+x)^{-1} = y + e^x(1+x)^{-1}$$

so that $(y_1)_0 = 0 + 1 = 1$,

$y_2 = y_1 + e^x(1+x)^{-1} - e^x(1+x)^{-2}$ so that $(y_2)_0 = 1 + 1 - 1 = 1$,

$$y_3 = y_2 + e^x(1+x)^{-1} - e^x(1+x)^{-2} - e^x(1+x)^{-2} + 2e^x(1+x)^{-3}$$

$$= y_2 + e^x(1+x)^{-1} - 2e^x(1+x)^{-2} + 2e^x(1+x)^{-3}$$

so that $(y_3)_0 = 1 + 1 - 2 + 2 = 2$,

$y_4 = y_3 + e^x(1+x)^{-1} - 3e^x(1+x)^{-2} + 6e^x(1+x)^{-3} - 6e^x(1+x)^{-4}$ so that $(y_4)_0 = 2 + 1 - 3 + 6 - 6 = 0$,

$y_5 = y_4 + e^x(1+x)^{-1} - 4e^x(1+x)^{-2} + 12e^x(1+x)^{-3} - 24e^x(1+x)^{-4} + 24e^x(1+x)^{-5}$ so that

$(y_5)_0 = 0 + 1 - 4 + 12 - 24 + 24 = 9$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^x \log(1+x) = 0 + x.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}\cdot 2 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 9 + \ldots$$

$$= x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{9x^5}{5!} + \ldots$$

(iv) Let $y = \log(1 + \sin x)$. Then $(y)_0 = 0$,

$y_1 = \dfrac{\cos x}{1 + \sin x}$ so that $(y_1)_0 = 1$,

Copyrighted material

$$y_2 = \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$= -\dfrac{1}{1+\sin x}$ so that $(y_2)_0 = -1$,

Scanned with CamScanner

5:55:00 91%

books.google.co.in/books?i 2

$$y_2 = \frac{-\sin x\,(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= -\frac{1}{1+\sin x} \text{ so that } (y_2)_0 = -1,$$

$$y_3 = \frac{\cos x}{(1+\sin x)^2} = \frac{\cos x}{1+\sin x}\cdot\frac{1}{1+\sin x} = -y_1y_2$$

so that $(y_3)_0 = -1.(-1) = 1,$

$y_4 = -y_1y_3 - y_2^2$ so that $(y_4)_0 = -1.1-(-1)^2 = -1-1 = -2,$

$y_5 = -y_1y_4 - y_2y_3 - 2y_2y_3 = -y_1y_4 - 3y_2y_3$ so that

$(y_5)_0 = -1.(-2) - 3.(-1).1 = 2+3 = 5,$ and so on.

Substituting these values in Maclaurin's theorem, we get

$$\log(1+\sin x) = 0 + x.1 + (x^2/2!).(-1) + (x^3/3!).1 + (x^4/4!).(-2) + (x^5/5!).5 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + \ldots$$

Ex. 4. *Expand by Maclaurin's theorem $e^x/(1+e^x)$ as far as the term x^3.*

Sol. Let $y = \dfrac{e^x}{1+e^x} = \dfrac{1+e^x-1}{1+e^x} = 1 - \dfrac{1}{1+e^x}.$

Then $(y)_0 = \dfrac{e^0}{1+e^0} = \frac{1}{2},$

$$y_1 = 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x}\,\frac{1}{1+e^x} = y\,(1-y) = y - y^2 \text{ so that}$$

$(y_1)_0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$

$y_2 = y_1 - 2yy_1$ so that $(y_2)_0 = \frac{1}{4} - 2.\frac{1}{2}.\frac{1}{4} = 0,$

$y_3 = y_2 - 2y_1^2 - 2yy_2$ so that $(y_3)_0 = 0 - 2.(1/4)^2 - 0 = -1/8,$

and so on.

Substituting these values in Maclaurin's theorem, we get

$$\frac{e^x}{1+e^x} = \frac{1}{2} + x.\frac{1}{4} + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}\cdot(-1/8) + \ldots$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{48}x^3 + \ldots$$

Ex. 5. *Apply Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e(1+e^x)$ to the term containing x^4.*

(Gorakhpur 1981; Kanpur 88; Ranchi 84)

Sol. Let $y = \log_e(1+e^x)$. Then $(y)_0 = \log_e(1+e^0) = \log_e 2,$

$$y_1 = \frac{e^x}{1+e^x} = \frac{(1+e^x)-1}{1+e^x} = 1 - \frac{1}{1+e^x} \text{ so that } (y_1)_0 = 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

Copyrighted material

$$y_2 = 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)}\cdot\frac{1}{1+e^x} = y_1(1-y_1) = y_1 - y_1^2$$

so that $(y_2)_0 = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4},$

$y_3 = y_2 - 2y_1y_2$ so that $(y_3)_0 = \frac{1}{4} - 2.\frac{1}{2}.\frac{1}{4} = 0,$

$y_4 = y_3 - 2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = 0 - 2.(\frac{1}{4})^2 - 0 = -1/8,$

Scanned with CamScanner

5:55:04 90%
books.google.co.in/books?i

$$y_1 = \frac{e^x}{1+e^x} = \frac{(1+e^x)-1}{1+e^x} = 1 - \frac{1}{1+e^x} \text{ so that } (y_1)_0 = 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

Copyrighted material

$$y_2 = 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)} \cdot \frac{1}{1+e^x} = y_1(1-y_1) = y_1 - y_1^2$$

so that $(y_2)_0 = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$,

$y_3 = y_2 - 2y_1y_2$ so that $(y_3)_0 = \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = 0$,

$y_4 = y_3 - 2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = 0 - 2 \cdot (\frac{1}{4})^2 - 0 = -1/8$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots$$

$\therefore \quad \log(1+e^x)$

$$= \log 2 + x \cdot \tfrac{1}{2} + \frac{x^2}{2!} \cdot \tfrac{1}{4} + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-1/8) + \ldots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$$

Ex. 6. *Find the first four terms in the expansion of* $\log(1 + \tan x)$ *in powers of x.* **(Magadh 1986)**

Sol. Let $y = \log(1+\tan x)$. Then $(y)_0 = \log(1 + \tan 0) = 0$.

Now $e^y = 1 + \tan x$. Differentiating both sides w.r.t. x, we get

$$e^y y_1 = \sec^2 x. \qquad \ldots(1)$$

Putting $x = 0$ on both sides of (1), we get

$$e^0 (y_1)_0 = 1 \text{ or } (y_1)_0 = 1.$$

Differentiating (1), we get

$$e^y y_1^2 + e^y y_2 = 2\sec^2 x \tan x$$

or $$e^y (y_1^2 + y_2) = 2 \sec^2 x \tan x. \qquad \ldots(2)$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 0 \quad \text{or} \quad (y_2)_0 = -1.$$

Differentiating (2), we get

$$e^y y_1 (y_1^2 + y_2) + e^y (2y_1y_2 + y_3) = 2\sec^4 x + 4\sec^2 x \tan^2 x$$

or $$e^y (y_1^3 + 3y_1y_2 + y_3) = 2\sec^4 x + 4 \sec^2 x \tan^2 x. \qquad \ldots(3)$$

Putting $x = 0$ in (3), we get

$$1 + 3 \cdot 1 \cdot (-1) + (y_3)_0 = 2 \text{ or } (y_3)_0 = 4.$$

Differentiating (3), we get

$$e^y y_1 (y_1^3 + 3y_1y_2 + y_3) + e^y (3y_1^2 y_2 + 3y_2^2 + 3y_1y_3 + y_4)$$
$$= 8\sec^4 x \tan x + 8 \sec^2 x \tan^3 x + 8 \sec^4 x \tan x$$

or $$e^y (y_1^4 + 6y_1^2 y_2 + 4y_1y_3 + 3y_2^2 + y_4)$$
$$= 16 \sec^4 x \tan x + 8 \sec^2 x \tan^3 x. \qquad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$1 + 6 \cdot 1 \cdot (-1) + 4 \cdot 1 \cdot 4 + 3 \cdot (-1)^2 + (y_4)_0 = 0$$

Copyrighted material

Scanned with CamScanner

5:55:11 90%
books.google.co.in/books?i

$1 + 6 . 1 . (-1) + 4 . 1 . 4 + 3 . (-1)^2 + (y_4)_0 = 0$

Copyrighted material

or $\quad (y_4)_0 + 14 = 0$ or $(y_4)_0 = -14$.

Now substituting these values in Maclaurin's theorem, we get

$$\log(1 + \tan x) = 0 + x . 1 + \frac{x^2}{2!} \cdot (-1) + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!} \cdot (-14) + \ldots$$
$$= x - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{7}{12}x^4 + \ldots$$

Ex. 7. *Apply Maclaurin's theorem to obtain terms upto x^4 in the expansion of $\log(1 + \sin^2 x)$.* **(Rohilkhand 1979)**

Sol. Let $y = \log(1 + \sin^2 x)$. Then $(y)_0 = 0$.

Now $\quad e^y = 1 + \sin^2 x$.

Differentiating, we get

$$e^y y_1 = 2 \sin x \cos x = \sin 2x. \qquad \ldots(1)$$

Putting $x = 0$ in (1), we get

$$e^0 (y_1)_0 = 0 \text{ or } (y_1)_0 = 0.$$

Differentiating (1), we get

$$e^y (y_1^2 + y_2) = 2 \cos 2x. \qquad \ldots(2)$$

Putting $x = 0$ in (2), we get

$$(y_2)_0 = 2.$$

Differentiating (2), we get

$$e^y [(y_1^2 + y_2) y_1 + 2y_1 y_2 + y_3] = -4 \sin 2x$$

or
$$e^y (y_1^3 + 3y_1 y_2 + y_3) = -4 \sin 2x. \qquad \ldots(3)$$

Putting $x = 0$ in (3), we get

$$(y_3)_0 = 0.$$

Differentiating (3), we get

$$e^y y_1 (y_1^3 + 3y_1 y_2 + y_3) + e^y (3y_1^2 y_2 + 3y_2^2 + 3y_1 y_3 + y_4) = -8 \cos 2x$$

or
$$e^y (y_1^4 + 6y_1^2 y_2 + 4y_1 y_3 + 3y_2^2 + y_4) = -8 \cos 2x. \qquad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$3 . 2^2 + (y_4)_0 = -8 \text{ or } (y_4)_0 = -20.$$

Now substituting these values in Maclaurin's theorem, we get

$$\log(1 + \sin^2 x) = 0 + x . 0 + \frac{x^2}{2!} \cdot 2 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!}(-20) + \ldots$$
$$= x^2 - \tfrac{5}{6}x^4 + \ldots$$

****Ex. 8.** *Show that*

(i) $\quad e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2 x^4}{4!} - \dfrac{2^2 x^5}{5!} + \dfrac{2^3 x^7}{7!} + \ldots$

$$+ 2^{n/2} \cos \frac{n\pi}{4} \cdot \frac{x^n}{n!} + \ldots$$

(Meerut 1974, 96; Kanpur 85; Rohilkhand 88, 89; Jhansi 88)

Copyrighted material

(ii) $\quad e^x \sin x = x + x^2 - \dfrac{2}{3!}x^3 - \dfrac{2^2}{5!}x^5 - \ldots + \sin(\tfrac{1}{4}n\pi) \dfrac{2^{n/2}}{n!} x^n + \ldots$

Scanned with CamScanner

5:55:17 90%
books.google.co.in/books?i

(ii) $e^x \sin x = x + x^2 - \dfrac{2}{3\,!}x^3 - \dfrac{2^2}{5\,!}x^5 - \ldots + \sin\left(\tfrac{1}{4}n\pi\right)\dfrac{2^{n/2}}{n\,!}x^n + \ldots$

(Meerut 1983, 95, 96 BP; Avadh 87; Lucknow 82; Gorakhpur 81; Agra 80, 84, 89)

Sol. (i) Let $y = e^x \cos x$. Then $(y)_0 = e^0 \cos 0 = 1$,

$y_1 = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$

so that $(y_1)_0 = 1\,(1 - 0) = 1$,

$y_2 = e^x (\cos x - \sin x) + e^x (-\sin x - \cos x) = -2e^x \sin x$

so that $(y_2)_0 = 0$,

$y_3 = -2e^x \sin x - 2e^x \cos x = -2e^x (\sin x + \cos x)$

so that $(y_3)_0 = -2$,

$y_4 = -2e^x (\sin x + \cos x) - 2e^x (\cos x - \sin x)$

$= -4e^x \cos x = -2^2 y$ so that $(y_4)_0 = -2^2$,

$y_5 = -2^2 y_1$ so that $(y_5)_0 = -2^2$,

$y_6 = -2^2 y_2$ so that $(y_6)_0 = 0$,

$y_7 = -2^2 y_3$ so that $(y_7)_0 = 2^3$, and so on.

In general

$y_n = (1 + 1)^{n/2} \cos (x + n \tan^{-1} 1) = 2^{n/2} \cos (x + n\pi/4)$

so that $(y_n)_0 = 2^{n/2} \cos\left(\tfrac{1}{4}n\pi\right)$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \ldots + \frac{x^n}{n\,!}(y_n)_0 + \ldots$$

$$= 1 + x \,.\, 1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot(-2) + \frac{x^4}{4\,!}\cdot(-2^2) + \frac{x^5}{5\,!}\cdot(-2^2)$$

$$+ \frac{x^6}{6\,!}\cdot 0 + \frac{x^7}{7\,!}\cdot 2^3 + \ldots + \frac{x^n}{n\,!}2^{n/2}\cos\left(\tfrac{1}{4}n\pi\right) + \ldots$$

$$= 1 + x - \frac{2x^3}{3\,!} - \frac{2^2x^4}{4\,!} - \frac{2^2x^5}{5\,!} + \frac{2^3x^7}{7\,!} + \ldots + 2^{n/2}\cos\left(\tfrac{1}{4}n\pi\right)\frac{x^n}{n\,!} + \ldots$$

(ii) Proceed as in part (i).

****Ex. 9.** *Apply Maclaurin's theorem to prove that*

(i) $e^{ax} \sin bx = bx + abx^2 + \dfrac{3a^2b - b^3}{3\,!}x^3 + \ldots$

$$+ \frac{(a^2 + b^2)^{n/2}}{n\,!}x^n \sin\{n \tan^{-1}(b/a)\} + \ldots$$

(Meerut 1984)

(ii) $e^{ax} \cos bx = 1 + ax + \dfrac{a^2 - b^2}{2}x^2 + \dfrac{a\,(a^2 - 3b^2)}{3\,!}x^3 + \ldots$

$$+ \frac{(a^2 + b^2)^{n/2}}{n\,!}x^n \cos\{n \tan^{-1}(b/a)\} + \ldots$$

Copyrighted material

(Meerut 1998, 84; 83S, 77; Gorakhpur 89; Allahabad 84; Kurukshetra 82; Magadh 83; Delhi 88; G.N.U. 85)

Hence deduce that

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2\,!}\cos 2\alpha + \frac{x^3}{3\,!}\cos 3\alpha + \ldots$$

(Kanpur 1982)

Scanned with CamScanner

5:55:23 90%
books.google.co.in/books?i

(Meerut 1998, 84; 83S, 77; Gorakhpur 89; Allahabad 84; Kurukshetra 82; Magadh 83; Delhi 88; G.N.U. 85)

Hence deduce that

$$e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \frac{x^3}{3!}\cos 3\alpha + \ldots$$

(Kanpur 1982)

Sol. (i) Let $y = e^{ax}\sin bx$. Then $(y)_0 = e^0 \sin 0 = 0$,

$$y_1 = ae^{ax}\sin bx + be^{ax}\cos bx = ay + be^{ax}\cos bx$$

so that $(y_1)_0 = b$,

$$y_2 = ay_1 + abe^{ax}\cos bx - b^2e^{ax}\sin bx$$
$$= ay_1 - b^2y + abe^{ax}\cos bx \text{ so that } (y_2)_0 = ab - 0 + ab = 2ab,$$
$$y_3 = ay_2 - b^2y_1 + a^2be^{ax}\cos bx - ab^2e^{ax}\sin bx$$
$$= ay_2 - b^2y_1 - ab^2y + a^2be^{ax}\cos bx$$

so that $(y_3)_0 = 2a^2b - b^3 + a^2b = 3a^2b - b^3$, and so on.

In general,

$$y_n = (a^2 + b^2)^{n/2}\sin\{bx + n\tan^{-1}(b/a)\}$$

so that $(y_n)_0 = (a^2 + b^2)^{n/2}\sin\{n\tan^{-1}(b/a)\}$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!}\cdot(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$
$$= 0 + \frac{x}{1!}\cdot b + \frac{x^2}{2!}\cdot(2ab) + \frac{x^3}{3!}(3a^2b - b^3) + \ldots$$
$$+ \frac{x^n}{n!}(a^2 + b^2)^{n/2}\sin\{n\tan^{-1}(b/a)\} + \ldots$$
$$= bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + \ldots$$
$$+ \frac{(a^2 + b^2)^{n/2}}{n!}x^n\sin\{n\tan^{-1}(b/a)\} + \ldots$$

(ii) Let $y = e^{ax}\cos bx$. Then $(y)_0 = e^0\cos 0 = 1$,

$$y_1 = ae^{ax}\cos bx - be^{ax}\sin bx = ay - be^{ax}\sin bx \text{ so that } (y_1)_0 = a,$$
$$y_2 = ay_1 - abe^{ax}\sin bx - b^2e^{ax}\cos bx = ay_1 - b^2y - abe^{ax}\sin bx$$

so that $(y_2)_0 = a^2 - b^2$,

$$y_3 = ay_2 - b^2y_1 - a^2be^{ax}\sin bx - ab^2e^{ax}\cos bx$$
$$= ay_2 - b^2y_1 - ab^2y - a^2be^{ax}\sin bx$$

so that $(y_3)_0 = a(a^2 - b^2) - b^2a - ab^2 = a(a^2 - 3b^2)$, and so on.

In general, $y_n = (a^2 + b^2)^{n/2}\cos\{bx + n\tan^{-1}(b/a)\}$ so that

$$(y_n)_0 = (a^2 + b^2)^{n/2}\cos\{n\tan^{-1}(b/a)\}.$$

Substituting these values in Maclaurin's theorem, we get

Copyrighted material

$$e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2!}x^2 + \frac{a(a^2 - 3b^2)}{3!}x^3 + \ldots$$
$$+ \frac{(a^2 + b^2)^{n/2}}{n!}x^n\cos\{n\tan^{-1}(b/a)\} + \ldots$$

Deduction. Putting $a = \cos\alpha$ and $b = \sin\alpha$, we get

$(y_n)_0 = (\cos^2\alpha + \sin^2\alpha)^{n/2}\cos(n\tan^{-1}\tan\alpha) = \cos n\alpha$ so that

$(y_1)_0 = \cos\alpha$, $(y_2)_0 = \cos 2\alpha$, $(y_3)_0 = \cos 3\alpha$, etc.

Scanned with CamScanner

5:55:30 90%
books.google.co.in/books?i
Copyrighted material

$$e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2\,!}x^2 + \frac{a\,(a^2 - 3b^2)}{3\,!}x^3 + \ldots$$
$$+ \frac{(a^2 + b^2)^{n/2}}{n\,!}x^n \cos\{n \tan^{-1}(b/a)\} + \ldots$$

Deduction. Putting $a = \cos\alpha$ and $b = \sin\alpha$, we get
$(y_n)_0 = (\cos^2\alpha + \sin^2\alpha)^{n/2}\cos(n\tan^{-1}\tan\alpha) = \cos n\alpha$ so that
$(y_1)_0 = \cos\alpha$, $(y_2)_0 = \cos 2\alpha$, $(y_3)_0 = \cos 3\alpha$, etc.

$$\therefore \quad e^{x\cos\alpha}\cos(x\sin\alpha)$$
$$= 1 + x\cos\alpha + (x^2/2\,!)\cos 2\alpha + (x^3/3\,!)\cos 3\alpha + \ldots$$

Ex. 10. *Prove that*

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots$$

(Rohilkhand 1980, 87S)

Sol. Let $y = \sin^{-1}x$. Then $y_1 = 1/\sqrt{(1 - x^2)}$

or $\quad (1 - x^2)\,y_1^2 - 1 = 0.$

Differentiating again, we get

$$(1 - x^2)\,2y_1y_2 - 2xy_1^2 = 0$$

or $\quad (1 - x^2)\,y_2 - xy_1 = 0$, since $2y_1 \neq 0$.

Now differentiating n times by Leibnitz's theorem, we get

$$(1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - n^2y_n = 0.$$

Putting $x = 0$ in the above relations, we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

and $\quad (y_{n+2})_0 = n^2\,(y_n)_0.$ ...(1)

Putting $n = 1, 2, 3, \ldots$ in (1), we get
$(y_3)_0 = 1^2\,(y_1)_0 = 1^2$, $(y_4)_0 = 2^2\,(y_2)_0 = 0$, $(y_5)_0 = 3^2\,(y_3)_0 = 3^2 . 1^2$,
$(y_6)_0 = 4^2\,(y_4)_0 = 0$, $(y_7)_0 = 5^2\,(y_5)_0 = 5^2 . 3^2 . 1^2$, and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots$$

$$\therefore \quad \sin^{-1}x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot 1^2 + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot 3^2 . 1^2$$
$$+ \frac{x^6}{6\,!}\cdot 0 + \frac{x^7}{7\,!}\cdot 5^2 . 2^2 . 1^2 + \ldots$$
$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \ldots$$

Ex. 11. *Expand* $\tan^{-1}x$ *by Maclaurin's theorem. Write also the general term.* **(Delhi 1981; Bundelkhand 82; Kashmir 87)**

Sol. Let $y = \tan^{-1}x$. Proceeding as in Ex. 36 on page 50, we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0,$$

Copyrighted material

and $\quad (y_{n+2})_0 = -\{(n + 1)\,n\}\,(y_n)_0.$ ...(1)

Scanned with CamScanner

5:55:36 90%
books.google.co.in/books?i
Copyrighted material

and $$(y_{n+2})_0 = -\{(n+1)\,n\}\,(y_n)_0. \qquad \text{...(1)}$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$(y_3)_0 = -(2\,.\,1)\,(y_1)_0 = -2\,!,\ (y_4)_0 = -(3\,.\,2)\,(y_2)_0 = 0,$$

$$(y_5)_0 = -(4\,.\,3)\,(y_3)_0 = -(4\,.\,3)\,.\,(-2\,!) = 4\,!, \text{ etc.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots$$

$$\therefore \quad \tan^{-1}x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot(-2\,!) + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot(4\,!) + \ldots$$

$$= x - (x^3/3) + (x^5/5) - \ldots$$

The general term in this expansion is $(x^n/n\,!)\,(y_n)_0$.

So we need the value of $(y_n)_0$. Putting $(n-2)$ in place of n in (1), we get

$$(y_n)_0 = -\{(n-1)\,(n-2)\}(y_{n-2})_0$$

$$= [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)]\,(y_{n-4})_0.$$

Now there arise two cases :

Case I. When n is even, we have

$$(y_n)_0 = [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)]\ldots[-(3)\,(2)]\,(y_2)_0$$

$$= 0, \text{ since } (y_2)_0 = 0.$$

Case II. When n is odd, we have

$$(y_n)_0 = [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)\}]\ldots[-(4)\,(3)]\,[-(2)\,(1)]\,(y_1)_0$$

$$= (-1)^{(n-1)/2}\,(n-1)\,!, \text{ since } (y_1)_0 = 1.$$

Thus in the expansion of $\tan^{-1}x$, the coefficient of x^n is 0 if n is even and is $\dfrac{(-1)^{(n-1)/2}\,(n-1)\,!}{n\,!}$ *i.e.*, $\dfrac{(-1)^{(n-1)/2}}{n}$ if n is odd.

$$\therefore \quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{[(2n-1)-1]/2}}{2n-1}x^{2n-1} + \ldots$$

$$= x - \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots$$

Ex. 12. *Expand* $\log\{1 - \log(1-x)\}$ *in powers of* x *by Maclaurin's theorem as far as the term* x^3.

By substituting $x/(1+x)$ *for* x *deduce the expansion of* $\log\{1 + \log(1+x)\}$ *as far as the term in* x^3.

Sol. Let $y = \log\{1 - \log(1-x)\}$. Then $(y)_0 = 0$.

Now $e^y = 1 - \log(1-x)$. Differentiating, we get

$$e^y\,y_1 = (1-x)^{-1}. \qquad \text{...(1)}$$

Copyrighted material

Putting $x = 0$ in (1), we get $(y_1)_0 = 1$.

Differentiating (1), we get

$$e^y\,y_1^2 + e^y\,y_2 = (1-x)^{-2}$$

Scanned with CamScanner

5:55:43 90%
books.google.co.in/books?i
Copyrighted material

Putting $x = 0$ in (1), we get $(y_1)_0 = 1$.

Differentiating (1), we get

$$e^y y_1^2 + e^y y_2 = (1-x)^{-2}$$

or $$e^y (y_1^2 + y_2) = (1-x)^{-2} \qquad \text{...(2)}$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 1 \quad \text{or} \quad (y_2)_0 = 0.$$

Differentiating (2), we get

$$e^y (y_1^3 + 3y_1 y_2 + y_3) = 2(1-x)^{-3} \qquad \text{...(3)}$$

Putting $x = 0$ in (3), we get

$$1 + (y_3)_0 = 2 \quad \text{or} \quad (y_3)_0 = 1.$$

Substituting these values in Maclaurin's theorem, we get

$$\log\{1 - \log(1-x)\} = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 1 + \ldots$$

$$= x + (x^3/6) + \ldots \qquad \text{...(A)}$$

Now substituting $x/(1+x)$ for x on both sides of (A), we get

$$\log\left\{1 - \log\left(1 - \frac{x}{1+x}\right)\right\} = \frac{x}{1+x} + \frac{1}{6}\left(\frac{x}{1+x}\right)^3 + \ldots$$

or $$\log\{1 + \log(1+x)\} = x(1+x)^{-1} + (1/6)x^3(1+x)^{-3} + \ldots$$

$$= x\left\{1 + (-1)x + \frac{(-1)(-2)}{1.2}x^2 + \ldots\right\} + \frac{1}{6}x^3\{1 + (-3)x + \ldots\},$$

on expanding by binomial theorem

$$= (x - x^2 + x^3 + \ldots) + (\tfrac{1}{6}x^3 + \ldots)$$

$$= x - x^2 + \tfrac{7}{6}x^3 + \ldots$$

Ex. 13. *Expand $\{x + \sqrt{(1+x^2)}\}^m$ in ascending powers of x and find the general term also.* **(Agra 1978; Kanpur 78)**

Sol. Let $y = \{x + \sqrt{(1+x^2)}\}^m$. Proceeding as in Ex. 39 (a) on page 52, we get

$$(y)_0 = 1, (y_1)_0 = m, (y_2)_0 = m^2,$$

and $$(y_{n+2})_0 = (m^2 - n^2)(y_n)_0. \qquad \text{...(1)}$$

Now putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$(y_3)_0 = (m^2 - 1^2)(y_1)_0 = (m^2 - 1^2)m,$$

$$(y_4)_0 = (m^2 - 2^2)(y_2)_0 = (m^2 - 2^2)m^2,$$

$$(y_5)_0 = (m^2 - 3^2)(y_3)_0 = (m^2 - 3^2)(m^2 - 1^2)m,$$

$$(y_6)_0 = (m^2 - 4^2)(y_4)_0 = (m^2 - 4^2)(m^2 - 2^2)m^2, \text{ etc.}$$

In general,

if n is odd, $(y_n)_0 = \{m^2 - (n-2)^2\} \ldots (m^2 - 3^2)(m^2 - 1^2)m$

and if n is even, $(y_n)_0 = \{m^2 - (n-2)^2\} \ldots (m^2 - 4^2)(m^2 - 2^2)m^2$ ...(2)

Copyrighted material

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots$$

$$\therefore \quad \{x + \sqrt{(1+x^2)}\}^m = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-1^2)}{3!}x^3$$

Scanned with CamScanner

5:55:48 90%
books.google.co.in/books?i
Copyrighted material

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \ldots + \frac{x^n}{n\,!}(y_n)_0 + \ldots$$

$$\therefore \quad \{x + \sqrt{(1+x^2)}\}^m = 1 + mx + \frac{m^2}{2\,!}x^2 + \frac{m\,(m^2-1^2)}{3\,!}x^3 + \frac{m^2\,(m^2-2^2)}{4\,!}x^4 + \frac{m\,(m^2-1^2)\,(m^2-3^2)}{5\,!}x^5 + \ldots$$

The general term $= \dfrac{x^n}{n\,!}(y_n)_0$, where $(y_n)_0$ is given by (2).

Ex. 14. *Expand log $\{x + \sqrt{(1+x^2)}\}$ in ascending powers of x and find the general term.* **(Meerut 1991; K.U. 85; Bihar 84)**

Sol. Let $y = \log\{x + \sqrt{(1+x^2)}\}$. Then

$$y_1 = \frac{1}{x + \sqrt{(1+x^2)}} \cdot \left\{1 + \frac{2x}{2\sqrt{(x^2+1)}}\right\} = \frac{1}{\sqrt{(x^2+1)}}.$$

Therefore $y_1^2\,(x^2+1) - 1 = 0$.

Differentiating again, we get

$$(x^2+1)\,2y_1y_2 + 2xy_1^2 = 0$$

or $\quad (x^2+1)\,y_2 + xy_1 = 0$, since $2y_1 \neq 0$.

Now differentiating n times by Leibnitz's theorem, we get

$$(x^2+1)\,y_{n+2} + ny_{n+1}\,.\,2x + \frac{n\,(n-1)}{2\,!}y_n\,.\,2 + xy_{n+1} + ny_n = 0$$

or $\quad (x^2+1)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0.$

Putting $x = 0$ in the above relations, we get

$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$, and

$$(y_{n+2})_0 = -\,n^2\,(y_n)_0. \qquad \ldots(1)$$

Now putting $n = 1, 3, 5, \ldots$ in (1), we get

$(y_3)_0 = -\,1^2\,(y_1)_0 = -\,1^2$, $(y_5)_0 = (-\,3^2)\,(y_3)_0 = (-\,3^2)\,(-\,1^2) = 3^2\,.\,1^2$,

$(y_7)_0 = (-\,5^2)\,(y_5)_0 = (-\,5^2)\,(-\,3^2)\,(-\,1^2) = -\,5^2\,.\,3^2\,.\,1^2$, etc.

In general, if n is odd, we have

$$(y_n)_0 = \{-\,(n-2)^2\}\,\{-\,(n-4)^2\}\ldots(-\,5^2)\,(-\,3^2)\,(-\,1^2)$$
$$= (-1)^{(n-1)/2}\,(n-2)^2\,(n-4)^2\ldots5^2\,.\,3^2\,.\,1^2. \qquad \ldots(2)$$

Again, putting $n = 2, 4, 6\ldots$ in (1), we get

$(y_4)_0 = -\,2^2\,(y_2)_0 = 0,\ (y_6)_0 = -\,4^2\,(y_4)_0 = 0$, etc.

Thus, if n is even, we have $(y_n)_0 = 0$.

Now by Maclaurin's theorem, we have

$$\log[x + \sqrt{(1+x^2)}] = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots$$

$$= 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}(-\,1^2) + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}(3^2\,.\,1^2) + \ldots$$

Copyrighted material

$$= x - \frac{x^3}{3\,!}1^2 + \frac{x^5}{5\,!}(3^2\,.\,1^2) - \frac{x^7}{7\,!}(5^2\,.\,3^2\,.\,1^2) + \ldots$$

The general term $= (x^n/n\,!)\,(y_n)_0$, where $(y_n)_0$ is given by (2) when

Scanned with CamScanner

5:55:57 90%
books.google.co.in/books?i

$$= x - \frac{x^3}{3!} 1^2 + \frac{x^5}{5!} (3^2 . 1^2) - \frac{x^7}{7!} (5^2 . 3^2 . 1^2) + \ldots$$

The general term $= (x^n/n!)\,(y_n)_0$, where $(y_n)_0$ is given by (2) when n is odd and $(y_n)_0 = 0$, when n is even.

Putting $2n-1$ in place of n in (2), we find that

$$(y_{2n-1})_0 = (-1)^{n-1} (2n-3)^2 \ldots 5^2 . 3^2 . 1^2.$$

Thus $\log [x + \sqrt{(1+x^2)}]$

$$= x - 1^2 . \frac{x^3}{3!} + 1^2 . 3^2 . \frac{x^5}{5!} - 1^2 . 3^2 . 5^2 . \frac{x^7}{7!} + \ldots$$

$$+ (-1)^{(n-1)} 1^2 . 3^2 . 5^2 \ldots (2n-3)^2 . \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

**Ex. 15. *Expand $e^{a \sin^{-1} x}$ by Maclaurin's theorem and find the general term. Hence show that*

$$e^\theta = 1 + \sin\theta + \frac{1}{2!} \sin^2\theta + \frac{2}{3!} \sin^3\theta + \ldots$$

(Meerut 1982, 84, 94; Allahabad 87; Rohilkhand 78; Agra 77; Magadh 77; Ranchi 75; Indore 73; Lucknow 83, 79; Kanpur 87, 89)

Sol. Let $y = e^{a \sin^{-1} x}$. Proceeding as in Ex. 37 on page 51, we get

$$y(0) = 1, y_1(0) = a, y_2(0) = a^2,$$

and $$y_{n+2}(0) = (n^2 + a^2)\, y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$y_3(0) = (1^2 + a^2)\, y_1(0) = (1^2 + a^2)\, a, y_4(0) = (2^2 + a^2)\, y_2(0)$$
$$= (2^2 + a^2)\, a^2, y_5(0) = (3^2 + a^2)\, y_3(0) = (3^2 + a^2)(1^2 + a^2)\, a,$$
$$y_6(0) = (4^2 + a^2)\, y_4(0) = (4^2 + a^2)(2^2 + a^2)\, a^2, \text{ etc.}$$

In general,

$$y_n(0) = \begin{cases} a\,(1^2 + a^2)(3^2 + a^2) \ldots [(n-2)^2 + a^2] & \text{if } n \text{ is odd} \\ a^2 (2^2 + a^2)(4^2 + a^2) \ldots [(n-2)^2 + a^2] & \text{if } n \text{ is even.} \end{cases}$$

Substituting these values in Maclaurin's expansion

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots,$$

we get

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{2!} x^2 + \frac{a\,(1^2 + a^2)}{3!} x^3 + \frac{a^2 (2^2 + a^2)}{4!} x^4 + \ldots \qquad \ldots(2)$$

The general term is $(x^n/n!)\, y_n(0)$, where $y_n(0)$ is as given above.

Now putting $x = \sin\theta$ and $a = 1$ in (2), we get

$$e^\theta = 1 + \sin\theta + \frac{1}{2!} \sin^2\theta + \frac{2}{3!} \sin^3\theta + \ldots$$

Ex. 16. *Use Maclaurin's theorem to show that*

Copyrighted material

$$e^{m \cos^{-1} x} = e^{m\pi/2} \left[1 - mx + \frac{m^2}{2!} x^2 - \frac{m\,(1^2 + m^2)}{3!} x^3 + \frac{m^2 (2^2 + m^2)}{4!} x^4 - \ldots \right]$$

Scanned with CamScanner

5:56:03 90%
books.google.co.in/books?i
Copyrighted material

$$e^{m\cos^{-1}x} = e^{m\pi/2}\left[1 - mx + \frac{m^2}{2!}x^2 - \frac{m(1^2+m^2)}{3!}x^3 + \frac{m^2(2^2+m^2)}{4!}x^4 - \ldots\right]$$

(Agra 1983; Delhi 82)

Sol. Proceed as in Ex. 15.

Ex. 17. *Expand* sin $(m \sin^{-1} x)$ *by Maclaurin's theorem as far as* x^5. *Hence expand* $\sin m\theta$ *in powers of* $\sin \theta$.

(Meerut 1978, 93; G.N.U. 72; Kanpur 88; Rohilkhand 83, 85, 90; Lucknow 75)

Sol. Let $y = \sin(m \sin^{-1} x)$. Proceeding as in Ex. 34 on page 49, we get

$$y(0) = 0, y_1(0) = m, y_2(0) = 0,$$

and $$y_{n+2}(0) = (n^2 - m^2)\, y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$y_3(0) = (1^2 - m^2)\, y_1(0) = (1^2 - m^2)\, m,$$
$$y_4(0) = (2^2 - m^2)\, y_2(0) = 0,$$
$$y_5(0) = (3^2 - m^2)\, y_3(0) = (3^2 - m^2)(1^2 - m^2)\, m, \text{ etc.}$$

Substituting these values in Maclaurin's theorem, we get

$$\sin(m \sin^{-1} x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \ldots$$
$$= mx + \frac{m(1^2 - m^2)}{3!}x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}x^5 + \ldots$$

Putting $x = \sin\theta$ on both sides, we get

$$\sin m\theta = m \sin\theta + \frac{m(1^2 - m^2)}{3!}\sin^3\theta + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}\sin^5\theta + \ldots$$

Ex. 18. *Show that*

$$(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \frac{2.2^2.4^2}{6!}x^6 + \ldots$$

(Meerut 1983, 84, 86)

Deduce that

$$\theta^2 = 2 \cdot \frac{\sin^2\theta}{2!} + 2^2 \cdot \frac{2\sin^4\theta}{4!} + 2^2.4^2\frac{2\sin^6\theta}{6!} + \ldots$$

(Rajasthan 1987; Meerut 86 P)

Sol. Let $y = (\sin^{-1} x)^2$. Proceeding as in Ex. 32 on page 47, we get

$$y(0) = 0, y_1(0) = 0, y_2(0) = 2,$$

and $$y_{n+2}(0) = n^2 y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

Copyrighted material

$$y_3(0) = 1^2 y_1(0) = 0, y_4(0) = 2^2 y_2(0) = 2^2.2,$$

Scanned with CamScanner

5:56:08 90%
books.google.co.in/books?i

$y_3(0) = 1^2 y_1(0) = 0, y_4(0) = 2^2 y_2(0) = 2^2 . 2,$
$y_5(0) = 3^2 y_3(0) = 0, y_6(0) = 4^2 y_4(0) = 4^2 . 2^2 . 2$, etc.

Hence by Maclaurin's theorem, we get

$$(\sin^{-1} x)^2 = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \dots$$
$$= \frac{2}{2!} x^2 + \frac{2.2^2}{4!} x^4 + \frac{2.2^2 . 4^2}{6!} x^6 + \dots$$

Now putting $\sin^{-1} x = \theta$ or $x = \sin\theta$ on both sides, we get the required expansion for θ^2.

Ex. 19. (a) *By Maclaurin's theorem or otherwise find the expansion of $y = \sin(e^x - 1)$ upto and including the term in x^4.* **(Gorakhpur 1982, 88)**

(b) *Also show that $x = y - \frac{1}{2}y^2 + \dots$* **(Allahabad 1973)**

Sol. (a) Let $y = \sin(e^x - 1)$. Then $(y)_0 = \sin 0 = 0$,

$y_1 = [\cos(e^x - 1)] . e^x$ so that $(y_1)_0 = (\cos 0) . e^0 = 1,$

$y_2 = [\cos(e^x - 1)] . e^x - [\sin(e^x - 1)] . e^{2x} = y_1 - ye^{2x}$

so that $(y_2)_0 = (y_1)_0 - (y)_0 e^0 = 1 - 0 = 1,$

$y_3 = y_2 - y_1 e^{2x} - 2y e^{2x}$ so that $(y_3)_0 = 1 - 1 - 0 = 0,$

$y_4 = y_3 - y_2 e^{2x} - 4y_1 e^{2x} - 4y e^{2x}$ so that $(y_4)_0 = -5$, etc.

Hence by Maclaurin's theorem, we get

$$\sin(e^x - 1) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$$
$$= 0 + x . 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-5) + \dots$$
$$= x + \tfrac{1}{2}x^2 - \tfrac{5}{24}x^4 + \dots$$

(b) We have

$y = \sin(e^x - 1) \Rightarrow e^x - 1 = \sin^{-1} y \Rightarrow e^x = 1 + \sin^{-1} y.$ ...(1)

Differentiating (1) w.r.t. 'y', we get

$e^x . x_1 = 1/\sqrt{(1 - y^2)}$, where $x_1 = dx/dy$. ...(2)

From (2), we get $(1 - y^2) x_1^2 = e^{-2x}$.

Differentiating it w.r.t. 'y', we get

$(1 - y^2) 2x_1 x_2 - 2yx_1^2 = e^{-2x}(-2x_1)$

or $(1 - y^2) x_2 - yx_1 = -e^{-2x}$, since $2x_1 \neq 0$. ...(3)

From (1), we have $x = \log(1 + \sin^{-1} y)$. ...(4)

Now putting $y = 0$ in (4), (2) and (3), we get

$(x)_0 = \log(1 + 0) = 0, e^0 . (x_1)_0 = 1/\sqrt{(1 - 0)}$ giving $(x_1)_0 = 1$,

$(x_2)_0 = -e^0 = -1.$ [Note that $(x)_{y=0} = 0$]

Hence by Maclaurin's theorem, we get

Copyrighted material

$$x = (x)_0 + y(x_1)_0 + (y^2/2!)(x_2)_0 + \dots$$
$$= 0 + y . 1 + (y^2/2!)(-1) + \dots = y - \tfrac{1}{2}y^2 + \dots$$

•**Ex. 20. (a)** *If $y = (\sin^{-1} x)/\sqrt{(1 - x^2)}$, where $-1 < x < 1$ and $-\pi/2 < \sin^{-1} x < \pi/2$,*

prove that $(1 - x^2) y_{n+1} - (2n + 1) xy_n - n^2 y_{n-1} = 0.$

Also if $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$

Scanned with CamScanner

5:56:13 90%
books.google.co.in/books?i

Copyrighted material

$$x = (x)_0 + y\,(x_1)_0 + (y^2/2\,!)\,(x_2)_0 + \ldots$$
$$= 0 + y\,.\,1 + (y^2/2\,!)\,(-1) + \ldots = y - \tfrac{1}{2}y^2 + \ldots$$

***Ex. 20. (a)** *If $y = (\sin^{-1}x)/\sqrt{(1-x^2)}$, where $-1 < x < 1$ and $-\pi/2 < \sin^{-1}x < \pi/2$,*

prove that $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$

Also if $y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$

prove that $(n+1)\,a_{n+1} = n\,a_{n-1}$ and hence obtain the general term of the expansion. **(Lucknow 1976)**

(b) *Expand $(\sin^{-1}x)/\sqrt{(1-x^2)}$ in powers of x upto three terms.*

Sol. (a) Here $y = (\sin^{-1}x)/\sqrt{(1-x^2)}$. ...(1)

$\therefore$ $y^2(1-x^2) = (\sin^{-1}x)^2$.

Differentiating w.r.t. x, we get

$$2yy_1(1-x^2) - 2xy^2 = 2\,(\sin^{-1}x)/\sqrt{(1-x^2)} = 2y.$$

Since $2y \neq 0$, therefore $y_1(1-x^2) - xy = 1$

i.e. $y_1(1-x^2) - xy - 1 = 0.$ ...(2)

Differentiating (2) n times by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2) + ny_n(-2x) + \frac{n(n-1)}{2\,!}y_{n-1}\cdot(-2) - xy_n - ny_{n-1} = 0$$

or $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$ ...(3)

Now putting $x = 0$ in (1), (2) and (3), we get

$(y)_0 = 0,\ (y_1)_0 = 1$ and $(y_{n+1})_0 = n^2\,(y_{n-1})_0$. ...(4)

By Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + (x^2/2\,!)\,(y_2)_0 + \ldots + (x^n/n\,!)\,(y_n)_0 + \ldots$$

Also we are given that

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

Comparing the coefficients of x^n in the two expansions for y, we get

$$a_n = (y_n)_0/n\,!.$$

$$\therefore \quad \frac{a_{n+1}}{a_{n-1}} = \frac{(y_{n+1})_0}{(n+1)\,!} \div \frac{(y_{n-1})_0}{(n-1)\,!} = \frac{(y_{n+1})_0}{(y_{n-1})_0}\cdot\frac{(n-1)\,!}{(n+1)\,!}$$

$$= n^2\cdot\frac{1}{n(n+1)}, \qquad \text{from (4)}$$

$$= \frac{n}{n+1}.$$

$\therefore$ $(n+1)\,a_{n+1} = n\,a_{n-1}$ **Proved.**

or $a_{n+1} = \dfrac{n}{n+1}a_{n-1}.$ ...(5)

Copyrighted material

Now $a_0 = (y)_0 = 0,\ a_1 = (y_1)_0 = 1$. Putting $n = 1, 3, 5, \ldots$ in (5), we get $a_2 = \frac{1}{2}a_0 = 0,\ a_4 = \frac{3}{4}a_2 = 0,\ a_6 = \frac{5}{6}a_4 = 0$, etc.

Scanned with CamScanner

5:56:20 90%
books.google.co.in/books?i

Now $a_0 = (y)_0 = 0$, $a_1 = (y_1)_0 = 1$. Putting $n = 1, 3, 5, \ldots$ in (5), we get $a_2 = \frac{1}{2} a_0 = 0$, $a_4 = \frac{3}{4} a_2 = 0$, $a_6 = \frac{5}{6} a_4 = 0$, etc.

Thus $a_n = 0$ if n is even *i.e.*, $a_{2m} = 0$.

Again putting $n = 2, 4, 6, \ldots$ in (5), we get

$$a_3 = \tfrac{2}{3} a_1 = \tfrac{2}{3}, a_5 = \tfrac{4}{5} \cdot a_3 = \tfrac{4}{5} \cdot \tfrac{2}{3}, a_7 = \tfrac{6}{7} a_5 = \tfrac{6}{7} \cdot \tfrac{4}{5} \cdot \tfrac{2}{3}, \text{ etc.}$$

In general, if n is odd, we have

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

Thus $$a_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

(b) As found in part (a), we have $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 2/3$, $a_4 = 0$, $a_5 = \frac{2}{3} \cdot \frac{4}{5}$, etc.

$$\therefore \quad \frac{\sin^{-1} x}{\sqrt{(1-x^2)}} = x + \tfrac{2}{3} x^3 + \frac{2.4}{3.5} x^5 + \ldots$$

Ex. 20. (c) *If* $y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots$, *prove that* $(n+1)(n+2)\, a_{n+2} = n^2 a_n$. **(Meerut 1990 P)**

Sol. Let $y = \sin^{-1} x$. ...(1)

Then $$y_1 = \frac{1}{\sqrt{(1-x^2)}}. \quad \ldots(2)$$

$$\therefore \quad (1-x^2) y_1^2 - 1 = 0.$$

Differentiating again, we get

$$(1-x^2)\, 2y_1 y_2 - 2xy_1^2 = 0 \text{ or } 2y_1 [(1-x^2) y_2 - xy_1] = 0$$

or $$(1-x^2) y_2 - xy_1 = 0, \quad \ldots(3)$$

since $2y_1 \neq 0$.

Now differentiating (3) n times by Leibnitz's theorem, we get

$$(1-x^2) y_{n+2} + n \cdot y_{n+1} \cdot (-2x) + \frac{n(n-1)}{1.2} y_n \cdot (-2) - y_{n+1} \cdot x - ny_n \cdot 1 = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0. \quad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$(y_{n+2})_0 = n^2 (y_n)_0. \quad \ldots(5)$$

By Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!} (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \ldots + \frac{x^n}{n!} (y_n)_0 + \ldots$$

Also we are given that

$$y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Equating the coefficients of x^n in the two expansions for y, we get

Copyrighted material

$$a_n = \frac{(y_n)_0}{n!}.$$

$$\therefore \quad \frac{a_{n+2}}{} = \frac{(y_{n+2})_0}{} \cdot \frac{n!}{} = \frac{(y_{n+2})_0}{} \cdot \frac{1}{}$$

Scanned with CamScanner

5:56:29 90%
books.google.co.in/books?i 2

$$a_n = \frac{(y_n)_0}{n\,!}.$$

$$\therefore \quad \frac{a_{n+2}}{a_n} = \frac{(y_{n+2})_0}{(n+2)\,!} \cdot \frac{n\,!}{(y_n)_0} = \frac{(y_{n+2})_0}{(y_n)_0} \cdot \frac{1}{(n+2)(n+1)}$$

$$= \frac{n^2}{(n+2)(n+1)}, \text{ substituting for } \frac{(y_{n+2})_0}{(y_n)_0} \text{ from (5).}$$

Hence $(n+1)(n+2)\,a_{n+2} = n^2 a_n$.

Ex. 20 (d). *If $y = \sin \log (x^2 + 2x + 1)$, prove that*

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as for as x^6.

(Allahabad 1989; Agra 85)

Sol. Here $y = \sin \log (x^2 + 2x + 1) = \sin \log (x+1)^2$ ...(1)

$$\therefore \quad y_1 = [\cos \log (x+1)^2] \cdot \frac{1}{(x+1)^2} \cdot 2\,(x+1)$$

$$= [\cos \log (x+1)^2] \cdot \frac{2}{x+1}. \qquad \text{...(2)}$$

Squaring both sides of (2), we get

$$(x+1)^2 y_1^2 = 4 \cos^2 \log (x+1)^2 = 4\,[1 - \sin^2 \log (x+1)^2]$$
$$= 4\,(1 - y^2)$$

or $(x+1)^2 y_1^2 + 4y^2 - 4 = 0.$ ...(3)

Differentiating (3), we get

$$(x+1)^2\, 2y_1 y_2 + 2\,(x+1)\,y_1^2 + 8yy_1 = 0$$

or $2y_1\,[(x+1)^2 y_2 + (x+1)\,y_1 + 4y] = 0$

or $(x+1)^2 y_2 + (x+1)\,y_1 + 4y = 0,$ ...(4)

since $2y_1 \neq 0$.

Differentiating (4) n times by Leibnitz's theorem, we get

$$(x+1)^2 y_{n+2} + {}^nC_1 \cdot y_{n+1} \cdot 2\,(x+1) + {}^nC_2 \cdot y_n \cdot 2$$
$$+ (x+1)\,y_{n+1} + {}^nC_1 \cdot y_n \cdot 1 + 4y_n = 0$$

or $(x+1)^2 y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0.$ ...(5)

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 2,\ (y_2)_0 + (y_1)_0 + 4(y)_0 = 0 \text{ or } (y_2)_0 = -2.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 + (2n+1)\,(y_{n+1})_0 + (n^2+4)\,(y_n)_0 = 0$$

or $(y_{n+2})_0 = -[(2n+1)\,(y_{n+1})_0 + (n^2+4)\,(y_n)_0].$ ...(6)

Now putting $n = 1, 2, 3, 4$ in (6), we get

$$(y_3)_0 = -[3\,(y_2)_0 + 5\,(y_1)_0] = -[3 \cdot (-2) + 5 \cdot 2] = -4,$$
$$(y_4)_0 = -[5\,(y_3)_0 + 8\,(y_2)_0] = -[5 \cdot (-4) + 8 \cdot (-2)] = 36,$$

Copyrighted material

Some pages are omitted from this book preview.

Scanned with CamScanner