P-2 IFS 2019

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7(a) Given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$

Find $y(0.1)$ and $y(0.2)$ by fourth order Runge-Kutta method. (15)

$x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2 + y^2$, $h = 0.1$

$$y(x_1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_0, y_0) = 0.1(0^2 + 1^2) = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$$

$$= 0.1 f(0.05, 1.05) = 0.1105$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 f(0.05, 1.05525) = 0.111605$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1 f(0.1, 1.111605) = 0.124567$$

$$\therefore y(0.1) = 1 + \frac{1}{6}(0.1 + 2 \times 0.1105 + 2 \times 0.111605 + 0.124567)$$

$$= 1.111463$$

Now, for $y(0.2)$,

$x_1 = 0.1$, $y_1 = 1.111463$, $h = 0.1$

$$y(x_2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

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$$k_1 = h f(x_1, y_1)$$

$$= 0.1 f(0.1, 1.111463) = 0.124535$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left(0.1 + \frac{0.1}{2}, 1.111463 + \frac{0.124535}{2}\right)$$

$$= 0.1 f(0.15, 1.173731) = 0.140014$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 f(0.15, 1.18147)$$

$$= 0.141837$$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

$$= 0.1 f(0.2, 1.2533)$$

$$= 0.161076$$

$$y(0.2) = 1.111463 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.253015$$

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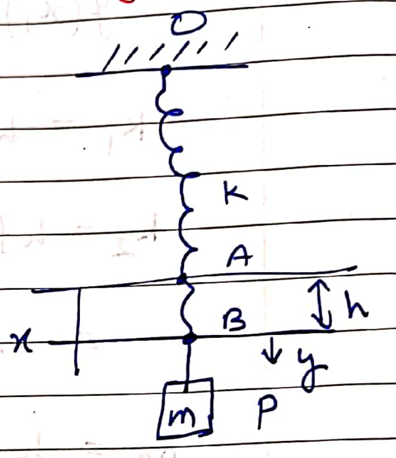
7(b) Consider a mass-spring system consisting of a mass m and a linear spring of stiffness k hanging from a fixed point. Find the equation of motion using the Hamiltonian Method, assuming that the displacement x is measured from the unstretched position of the string. (10)



Let A be untretched position.

$$mg = kh$$

$$h = \frac{mg}{k}$$

h, is stretch in the spring due to mass m.

• B is equilibrium position.

Now at Position P,

$$T = \frac{1}{2} m\dot{y}^2$$

$$V = -mgy + \frac{kx^2}{2}, \quad x = y + h$$

$$\therefore \quad L = T - V$$

$$= \frac{1}{2} m\dot{y}^2 + mgy - \frac{k}{2}(y+h)^2$$

$$P = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$H = P\dot{y} - L$$

$$= m\dot{y}^2 - \frac{m\dot{y}^2}{2} - mgy + \frac{k}{2}(y+h)^2$$

$$H = \frac{P^2}{2m} - mgy + \frac{k}{2}(y+h)^2, \quad \left(\dot{y} = \frac{P}{M}\right)$$

Now,

$$\dot{p} = \frac{-\partial H}{\partial y} = mg - k(y+h)$$

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\therefore \ddot{y} = \frac{\dot{p}}{m} = \frac{mg - k(y+h)}{m}$$

$$= g - \frac{k}{m}y - k\frac{h}{m}$$

$$\ddot{y} = \frac{-k}{m}y \qquad \left(\because h = \frac{mg}{k}\right)$$

Put $y = x - h$

$$\therefore \ddot{x} = \frac{-k}{m}(x-h)$$

$$\ddot{x} = \frac{-k}{m}\left(x - \frac{mg}{k}\right)$$

Which is the required equation of motion.

7c) Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy = z + c$.

(15m)

Given Surface is,

$$f(x, y, z) = 2y - x^2 \qquad \text{—(1)}$$

Hyperboloids of the one-parameter system is

$$xy = z + C \qquad \text{—(2)}$$

Then the system of D.Es. of the given curves of intersection of (1) and (2) is

$$-2x\,dx + 2\,dy = 0 \; ;$$

$$y\,dx + x\,dy - dz = 0$$

Solving these equation for dx, dy, dz

$$\frac{dx}{-2-0} = \frac{dy}{0-2x} = \frac{dz}{-2x^2-2y}$$

$$\frac{dx}{1} = \frac{dy}{x} = \frac{dz}{x^2+y}$$

Hence the system of D.E. of the required orthogonal trajectories of the given curves is

$$-x\,dx + dy + 0\,dz = 0 \; ;$$

$$dx + x\,dy + (x^2+y)\,dz = 0$$

$$\Rightarrow \quad \frac{dx}{(x^2+y)} = \frac{dy}{x(x^2+y)} = \frac{dz}{-x^2-1}$$

From the first two.

$$\frac{dx}{(x^2+y)} = \frac{dy}{x(x^2+y)}$$

$$\Rightarrow \quad x\,dx = dy$$

$$\Rightarrow \quad x^2/2 = y + c_1$$

Taking $c_1 = 0$, $\boxed{x^2 = 2y}$ — ★

from first and third, and using ★

$$\frac{dx}{x^2 + x^2/2} = \frac{dz}{-(x^2+1)}$$

$$2\frac{(x^2+1)}{x^2}\,dx = -3\,dz$$

$$2\left(1+\frac{1}{x^2}\right)dx = -3\,dz$$

$$\Rightarrow \quad 2\left(x - \frac{1}{x}\right) = -3z + c'$$

c' being an arbitrary constant.

Hence required family of orthogonal trajectories is given by

$$x^2 = 2y \quad \text{and} \quad 3z + 2\left(x - \frac{1}{x}\right) = c'.$$