IFoS 2019

Date
DELTA Pg No

1. Using Cauchy's Integral formula -

$$\oint_C \frac{dz}{(z^2+4)^2} \quad \text{where } C : |z-i| = 2$$

let $\boxed{z - i = u}$ then

$$z^2+4 = (z+2i)(z-2i) = (u+3i)(u-i)$$

So $$\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2} = \frac{1}{(u+3i)^2(u-i)^2}$$

let $$f(u) = \frac{1}{(u+3i)^2} \quad \text{--- (1)}$$

then $$\oint_C \frac{dz}{(z^2+4)^2} = \oint_C \frac{f(u)\,du}{(u-i)^2}$$

where $C : |u| = 2$

Note that $f(u)$ is analytic within and on closed contour $|u| = 2$ and $u_0 = i$ is the only singular point within $C : |u| = 2$

so by using Cauchy's integral formula: $\boxed{f^n(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$

by putting $n = 1$, $z_0 = u_0 = i$, we get

ROUGH

$$f'(u_0) = \frac{1!}{2\pi i} \oint_C \frac{f(u)\, du}{(u-u_0)^2}$$

i.e. $\oint_C \frac{f(u)}{(u-i)^2} du = 2\pi i\, f'(i)$

$f(u) = \frac{1}{(u+3i)^2} \Rightarrow f'(u) = \frac{-2}{(u+3i)^3}$

$f'(i) = \frac{-2}{64 \times -i} = \frac{2}{64 i}$

$\Rightarrow \oint_C \frac{f(u)}{(u-i)^2} du = 2\pi \cancel{i} \times \frac{2}{64 \cancel{i}} = \frac{\pi}{16}$

i.e. $\oint \frac{f(u)}{(u-i)^2} du = \oint_C \frac{1}{(u+3i)^2 (u-i)^2} du$

using $z - i = u$ we get

$$\boxed{\oint_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}}$$

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(c) $f(z)$ is analytic in domain D.
$|f(z)| =$ constant, non-zero in D.
T.S.T. $f(z)$ is constant in D.

let $f(z) = u + iv$ — (0)
then $|f(z)| \neq 0$, constant gives
$\sqrt{u^2+v^2} \neq 0$; $\sqrt{u^2+v^2} = k$ — (1)

Since $f(z)$ is analytic so
$f'(z) = u_x + iv_x$ — (2)
Using (1),
$u^2 + v^2 = k^2 \Rightarrow$
$uu_x + vv_x = 0$ — (3)
&
$uu_y + vv_y = 0$
But since $f(z)$ is analytic so it
satisfies CR equations i.e.
$u_x = v_y$, $u_y = -v_x$ so,
$uu_x + vv_x = 0$ — (4) and $-uv_x + vu_x = 0$ (using (3)).

i.e. $uu_x + vv_x = vu_x - uv_x = 0$
comparing coefficients we get
either $u = v$ and $v = -u \Rightarrow u = -u \Rightarrow$ $u = 0$, hence $v = 0$
(i)

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(OR) $u_x = 0$, $v_x = 0$

i.e. ~~$u_y = v_y = 0$~~

$\Rightarrow u = f(y) + c$, $v = g(y) + c'$

But $u_x = v_y$ and $v_x = -u_y$ (by CR equations)

i.e. $v_y = g'(y) = u_x = 0 \Rightarrow \boxed{v = c'}$

and

$u_y = f'(y) = -v_x = -0 = 0$

$\Rightarrow \boxed{u = c}$

Hence $f(z) = u + iv = c + ic'$

$\Rightarrow f(z) = $ constant

3. Classify the singular point $z=0$ of the fun $f(z) = \frac{e^z}{z+\sin z}$ and obtain the principal part of Laurent series expansion.

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z \times e^z}{z+\sin z} = \lim_{z\to 0} \frac{e^z}{\frac{z+\sin z}{z}}$$

$$= \lim_{z\to 0} \frac{e^z}{1+\frac{\sin z}{z}} = \frac{\lim_{z\to 0} e^z}{\lim_{z\to 0} 1 + \lim_{z\to 0} \frac{\sin z}{z}}$$

$$= \frac{1}{1+1} = \frac{1}{2} \neq 0, \text{ finite}$$

so, $z=0$ is a pole of order 1.

Now let us expand $f(z) = \frac{e^z}{z+\sin z}$

$$f(z) = \frac{e^z}{z+\sin z} \Rightarrow \text{using } e^z = 1+z+\frac{z^2}{2!}+\cdots$$

& $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

$$f(z) = \frac{e^z}{2z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} = \frac{e^z}{2z\left[1 - \left(\frac{z^2}{2\times 3!} - \frac{z^4}{2\times 5!} + \cdots\right)\right]}$$

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$$= \frac{e^z}{2z}\left(1-\left(\frac{z^2}{2\times 3!}-\frac{z^4}{2\times 5!}+\cdots\right)\right)$$

$$= \frac{e^z}{2z}\left(1+\frac{z^2}{2\times 3!}+O(z^3)\right)$$

$$= \frac{1}{2}\left[\frac{e^z}{z}\right]\left[1+\frac{z^2}{12}+O(z^3)\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+O(z^3)\right]\left[1+\frac{z^2}{12}+O(z^3)\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}+1+\left(\frac{z}{2!}+\frac{z}{12}\right)+O(z^2)\right]$$

$$= \frac{1}{2z}+\frac{1}{2}+\frac{7z}{24}+O(z^2)$$

so principal part is $\boxed{\frac{1}{2z}}$

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