ANALYTIC GEOMETRY

: IFoS - 2014 :

① (e) Prove that the locus of variable line which intersects the three lines $y = mx, z = c$, $y = -mx, z = -c$ and $y = z, mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$

→ Given lines are
$L_1 : y = mx, z = c$
$L_2 : y = -mx, z = -c$
$L_3 : y = z, mx = -c$

Any line intersecting L_1 & L_2 is $y - mx + \lambda_1(z - c) = 0 = y + mx + \lambda_2(z + c)$ — ①

If this line intersects L_3, then eliminating x, y & z from ① & L_3:

Putting $y = z$, & $mx = -c$ in ①

$z + c + \lambda_1(z - c) = 0 = z - c + \lambda_2(z + c)$

$z(1 + \lambda_1) + c(1 - \lambda_1) = 0$ & $z(1 + \lambda_2) - c(1 - \lambda_2) = 0$

$$\Rightarrow \frac{(\lambda_1 - 1)\cancel{c}}{1 + \lambda_1} = \frac{(1 - \lambda_2)\cancel{c}}{1 + \lambda_2}$$

$\Rightarrow (\lambda_1 - 1)(\lambda_2 + 1) = (1 + \lambda_1)(1 - \lambda_2)$

$\rightarrow \lambda_1\lambda_2 - \cancel{\lambda_2} + \cancel{\lambda_1} - 1 = 1 + \cancel{\lambda_1} - \cancel{\lambda_2} - \lambda_1\lambda_2$

$\rightarrow 2\lambda_1\lambda_2 - 2 = 0 \Rightarrow \lambda_1\lambda_2 = 1$

① ≡ $\lambda_1 = -(y - mx)/(z - c)$ & $\lambda_2 = -\frac{(y + mx)}{z + c}$

$$\therefore \frac{(y - mx)(y + mx)}{(z - c)(z + c)} = 1$$

$\Rightarrow \boxed{y^2 - m^2x^2 = z^2 - c^2}$ which is the required locus

② (c) Prove that every sphere passing through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ cut orthogonally, every sphere passing through the circle $x^2+z^2=r^2,\ y=0$

→ Ⓐ Any sphere through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ is

$$x^2+y^2+z^2-2ax+\lambda_1 z+r^2=0.$$

Its centre is $C_1\left(a, 0, -\frac{\lambda_1}{2}\right)$ — ① & Radius is

$$r_1=\sqrt{a^2+\frac{\lambda_1^2}{4}-r^2} \quad \text{— ②}$$

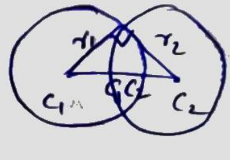
Sly, any sphere through the circle ~~$x^2+y^2=2ax$~~ $x^2+z^2=r^2,\ y=0$ is

$x^2+y^2+z^2+\lambda_2 y=r^2$ → Its centre is $C_2\left(0,-\frac{\lambda_2}{2},0\right)$ — ③

& radius is $r_2=\sqrt{\frac{\lambda_2^2}{4}+r^2}$ — ④

The condition for orthogonality is $(C_1C_2)^2=r_1^2+r_2^2$

$$r_1^2+r_2^2=a^2+\frac{\lambda_1^2}{4}+\frac{\lambda_2^2}{4}$$



$$C_1C_2=\sqrt{(a-0)^2+\left(\frac{\lambda_2}{2}-0\right)^2+\left(0-\frac{\lambda_1}{2}\right)^2}$$

$$=\sqrt{a^2+\frac{\lambda_2^2}{4}+\frac{\lambda_1^2}{4}}$$

$$C_1C_2^2=a^2+\frac{\lambda_1^2}{4}+\frac{\lambda_2^2}{4}=r_1^2+r_2^2$$

Hence, the spheres through the two systems always intersect orthogonally.

③ (b): A moving plane passes through a fixed point (2,2,2) and meets the coordinates axes at the point A, B, C all away from the origin. Find the locus of ~~sphere~~ centre of ~~passing the~~ the sphere passing through the points O, A, B, C.

→ Let the point A, B, C be $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$ respectively.
Then, the plane whose intercepts are a, b, c on the axes is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{— ①}$$

It passes through 2,2,2 $\Rightarrow \frac{2}{a}+\frac{2}{b}+\frac{2}{c}=1 \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$ — ②

Any sphere has equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{— ③}$$

It passes through:

(i) $O(0,0,0) \equiv d=0$ (ii) $A(a,0,0) \equiv u=-\frac{a}{2}$

(iii) $B(0,b,0) \equiv v=-\frac{b}{2}$ (iv) $C(0,0,c) \equiv w=-\frac{c}{2}$

Then ③: $x^2+y^2+z^2-(ax+by+cz)=0$

Its centre is $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$.

②: $\frac{2}{a}+\frac{2}{b}+\frac{2}{c}=1$.

Then locus of centre of the sphere which is $C\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ is

$$\boxed{x^{-1}+y^{-1}+z^{-1}=1}$$

3(d): Prove that the equation: $4x^2-y^2+2z^2-3yz+2xy+12x-11y+6z+4=0$ represents a cone with vertex at $(-1,-2,-3)$

→ Making the given equation homogeneous with the help of a new variable t, we have

$$F(x,y,z,t) = 4x^2-y^2+z^2-3yz+2xy+12xt-11yt+6zt+4t^2=0$$

Taking the partial diff. of F wrt x, y, z & t & equating to zero, we have

$$\frac{\partial f}{\partial x} = 8x+2y+12t=0 \equiv 4x+y+6t=0 \quad \text{— ①}$$

$$\frac{\partial f}{\partial y} = 2x-2y-3z-11t=0 \quad \text{— ②}$$

$$\frac{\partial f}{\partial z} = -3y+4z+6t=0 \quad \text{— ③}$$

$$\frac{\partial F}{\partial t} = 12x - 11y + 6z + 8t = 0 \quad \text{--- (4)}$$

Putting $t = 1$ in (1), (2), (3) & (4):

$4x + y + 6 = 0$, (5) $2x - 2y - 3z - 11 = 0$, (6) $-3y + 4z + 6 = 0$ and (7)

$12x - 11y + 6z + 8 = 0$. (8)

Solving (5), (6) & (8), we get $x = -1$, $y = -2$, $z = -3$

Putting in (7): $-3 \cdot -2 + 4 \cdot -3 + 6 = 0$

$\therefore$ It satisfies (7).

$\therefore$ The given equation represents a cone with vertex at $(-1, -2, -3)$.

(9) (b) Prove that the plane $ax + by + cz = 0$ cuts the cone in $\perp$ar ~~generator~~ lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ where cone is $yz + zx + xy = 0$

$\rightarrow$ Both the plane & the cone pass through the origin. Hence, their line of intersection passes through origin. Let them line of intersection have equation $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = 0$ --- (1)

Then, $al + bm + cn = 0$ and $mn + nl + lm = 0$

$$\Rightarrow n = -\frac{(al + bm)}{c} \Rightarrow -\frac{m(al + bm)}{c} - \frac{l(al + bm)}{c} + lm = 0$$

$$\Rightarrow -alm - bm^2 - al^2 - blm + clm = 0$$

$$\rightarrow a\left(\frac{l}{m}\right)^2 + (a + b - c)\frac{l}{m} + b = 0$$

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a} \Rightarrow \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \quad \text{[By symmetry]}$$

Then, if the two lines are $\perp$ar, then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \boxed{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0}$$ which is the reqd condition