Q. $F \propto \frac{1}{r^3}$

Let, $F = \frac{-K}{r^3}$ (attractive force)

Potential Energy: $-\int_{\infty}^{r} F \cdot dr = +\int_{\infty}^{r} \frac{K\,dr}{r^3} = \frac{-K}{2r^2}$

We will use polar equation; as particle moves under a Central force, motion is confined to plane.

Lagrangian $(L) = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{K}{2r^2}$

$\Rightarrow$ r, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

$\Rightarrow \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{K}{r^3} = 0$

$\Rightarrow \boxed{m\ddot{r} - mr\dot{\theta}^2 + \frac{K}{r^3} = 0}$ ...... (1)

θ, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow \boxed{mr^2\dot{\theta} = b}$ (const.) .. (2)

Let, $u = \frac{1}{r} \Rightarrow \frac{m\dot{\theta}}{u^2} = b \Rightarrow \boxed{\dot{\theta} = \frac{bu^2}{m}}$

Now; $r = \frac{1}{u} \Rightarrow \dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$

$\Rightarrow \dot{r} = \frac{-1}{u^2}\dot{\theta}\frac{du}{d\theta} = \frac{-b}{m}\frac{du}{d\theta}$. $\Rightarrow \ddot{r} = \frac{-b}{m}\frac{d}{dt}\left(\frac{du}{d\theta}\right)$

$\Rightarrow \ddot{r} = \frac{-b}{m}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$

$\Rightarrow \ddot{r} = \frac{-b}{m}\frac{d^2u}{d\theta^2}\cdot\dot{\theta}$

Scanned with CamScanner

$$\Rightarrow \ddot{r} = -\frac{b}{m} \cdot \frac{bu^2}{m} \frac{d^2u}{d\theta^2}$$

$$\Rightarrow \ddot{r} = \frac{-b^2u^2}{m^2} \frac{d^2u}{d\theta^2}$$

Substituting in (1), we get;

$$\cancel{m} \cdot \frac{+b^2u^2}{m^{\cancel{2}}} \frac{d^2u}{d\theta^2} + \cancel{m} \cdot \frac{\cancel{k}}{\cancel{k}} \frac{b^2u^{3}}{m^{\cancel{2}}} - ku^3 = 0$$

$$\Rightarrow \frac{b^2u^2}{m} \frac{d^2u}{d\theta^2} + \frac{b^2u^3}{m} - ku^3 = 0$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u - \frac{mk}{b^2} u = 0$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + \left(1 - \frac{mk}{b^2}\right) u = 0$$

Second degree equation; (Simple Harmonic Motion)

Hence; if $b^2 > mk$

$$\Rightarrow u = u_0 \cos\left[\sqrt{1 - \frac{mk}{b^2}} (\theta - \theta_0)\right]$$

$$\Rightarrow \boxed{r \cos\left(\sqrt{1 - \frac{mk}{b^2}} (\theta - \theta_0)\right) = r_0}$$

(r_0, θ_0) is any point on orbit.

$b^2 < mk$

$$\boxed{r \cosh\left(\sqrt{\frac{mk}{b^2} - 1} \, (\theta - \theta_0)\right) = r_0}$$

8(b) Evaluate the integral $\int_0^2 \frac{x}{1+x^3}\,dx$ using the trapezoidal rule with $h = 1/4$, correct to three decimal places.
(h is the length of subinterval). (10)

Sol: Let $y = \frac{x}{1+x^3}$

Then,

x	:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
y	:	0	$\frac{16}{25}$	$\frac{4}{9}$	$\frac{48}{91}$	$\frac{1}{2}$	$\frac{80}{189}$	$\frac{12}{35}$	$\frac{112}{407}$	$\frac{2}{9}$

Using trapezoidal rule,

$$\int_a^b f(x)\,dx = \frac{(b-a)}{(n)}\left[\frac{f(a)+f(b)}{2} + \sum_{h=1}^{n-1} f\left[a+h\cdot\left(\frac{b-a}{n}\right)\right]\right]$$

$$I = \int_0^2 \frac{x}{1+x^3}\,dx$$

$$= \frac{2-0}{8}\left[\frac{1}{2}\left(0+\frac{2}{9}\right)+\left(\frac{16}{25}+\frac{4}{9}+\frac{48}{91} + \frac{1}{2}+\frac{80}{189}+\frac{12}{35}+\frac{112}{407}\right)\right]$$

$$= 0.8161$$

$\simeq 0.816$, correct upto three decimal places.

Date
Page

8(c) Solve the following system of linear equations using Gaussian elimination method:

$$5x_1 + 2x_2 + x_3 = -2$$
$$6x_1 + 3x_2 + 2x_3 = 1$$
$$x_1 - x_2 + 2x_3 = 0$$

(10)

Sol:
$$\begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 6 & 3 & 2 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$R_2 \to R_2 - 6R_1, \quad R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 9 & -10 \\ 0 & 7 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$R_3 \to R_3 - \frac{7}{9} R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 9 & -10 \\ 0 & 0 & -11/9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -25/9 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 + 2x_3 = 0$

$9x_2 - 10x_3 = 1$

$-\frac{11}{9} x_3 = -\frac{25}{9} \Rightarrow \boxed{x_3 = \frac{25}{11}}$

$9x_2 - 10\left(\frac{25}{11}\right) = 1 \Rightarrow \boxed{x_2 = \frac{29}{11}}$

$x_1 = \frac{29}{11} - 2 \times \frac{25}{11} \Rightarrow \boxed{x_1 = \frac{-21}{11}}$

Scanned with CamScanner