Mains Test Series - 2018

Test - 04 (Answer Key)

Paper - II (PDE, NA&CP, MECHANICS & FLUID DYNAMICS)

1(a) Solve the following differential equation:

$$(D^3-4D^2D'+5DD'^2-2D'^3)z = e^{y+2x}+(y+x)^{1/2}.$$

Sol'n: Here auxiliary equation is $m^3-4m^2+5m-2=0$

$$\Rightarrow (m-1)^2(m-2)=0 \quad \text{so that } m=1,1,2$$

$$\therefore C.F = \phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+2x)$$

Now, P.I corresponding to e^{y+2x}

$$= \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}e^{y+2x}$$

$$= \frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2}e^{y+2x}\right\}$$

$$= \frac{1}{(D-2D')}\frac{1}{(2-1)^2}\iint e^v dv, \text{ where } v=y+x, \text{ by formula (i)}$$

$$= \frac{1}{D-2D'}\int e^v dv dv = \frac{1}{D-2D'}e^v$$

$$= \frac{1}{(1.D-2D')^1}e^{y+2x} = \frac{x}{1.1!}e^{y+x} = xe^{y+x} \quad \text{—— (2)}$$

[using formula (ii) with $a=2, b=1, m=1$]

finally, P.I corresponding to $(y+x)^{1/2}$

$$= \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3}(y+x)^{1/2} = \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$$

$$= \frac{1}{(D-D')^2}\frac{1}{1\cdot 2\cdot 1}\int v^{1/2}\,dv, \text{ where } v = y+x,$$ using formula (i)

$$= -\frac{1}{D-D'}\cdot\frac{2}{3}v^{3/2} = -\frac{2}{3}\frac{1}{(D-D')^2}(y+x)^{3/2}$$

$$= -\frac{2}{3}\cdot\frac{x^2}{1^2\cdot 2!}(y+x)^{3/2}$$

$$= -\left(\frac{x^2}{3}\right)(y+x)^{3/2} \quad \text{[using formula (ii) with } a=b=1, m=2] \quad \text{— (3)}$$

from (1), (2) and (3), the required general solution is

$$z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) + xe^{y+x} - \left(\frac{x^2}{3}\right)(y+x)^{3/2}$$

1(b) Find a complete integral of $px+qy = z(1+pq)^{1/2}$.

Sol^n: Given $f(x,y,z,p,q) = px+qy - z(1+pq)^{1/2} = 0$ — (1)

Here usual Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p-p(1+pq)^{1/2}} = \frac{dq}{q-q(1+pq)^{1/2}} = \cdots \text{ so that } \frac{dp}{p} = \frac{dq}{q} \quad \text{— (2)}$$

$$\Rightarrow \log p = \log a + \log q \Rightarrow p = aq \quad \text{— (3)}$$

Using (2), (1) $\Rightarrow q(ax+y) = z(1+aq^2)^{1/2} \Rightarrow q^2[(ax+y)^2 - az^2]$

$$\therefore q = \frac{z}{[(ax+y)^2-az^2]^{1/2}} \text{ and } p = aq = \frac{az}{[(ax+y)^2-az^2]^{1/2}}$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{z(adx+dy)}{\sqrt{\{(ax+y)^2-az^2\}}} \Rightarrow \frac{dz}{z} = \frac{adx+dy}{\sqrt{[(ax+y)^2-az^2]}}$$

Let $ax+y=\sqrt{a}\,u$ so that $a\,dx+dy=\sqrt{a}\,du$.

$\therefore \frac{dz}{z}=\frac{\sqrt{a}\,du}{\sqrt{(au^2-az^2)}} \Rightarrow \frac{du}{dz}=\frac{\sqrt{(u^2-z^2)}}{z}=\sqrt{\left\{\left(\frac{u}{z}\right)^2-1\right\}}$,

which is linear homogeneous equation. To solve it, we put

$$\frac{u}{z}=v \Rightarrow u=vz \quad \text{so that} \quad \frac{du}{dz}=v+z\frac{dv}{dz}$$

$$\therefore v+z\frac{dv}{dz}=\sqrt{(v^2-1)} \Rightarrow \frac{dz}{z}=\frac{dv}{\sqrt{(v^2-1)}-v}$$

$$\Rightarrow \left(\frac{1}{z}\right)dz=-\left[\sqrt{(v^2-1)}+v\right]dv, \text{ on rationalization.}$$

Integrating, $\log z=-\left[\frac{v}{2}\sqrt{(v^2-1)}-\frac{1}{2}\log\{v+\sqrt{(v^2-1)}\}\right]-\frac{v^2}{2}+b$,

where, $v=u/z=(ax+y)/z\sqrt{a}$

1(c) Find the positive root of the equation $10\int_0^x e^{-x^2}dt-1=0$ correct upto 6 decimal places by using Newton Raphson method. Carry out computations only for three iterations.

Sol'n: Given that $10\int_0^x e^{-x^2}dt-1=0$

$$\Rightarrow 10e^{-x^2}\int_0^x dt-1=0$$

$$\Rightarrow 10e^{-x^2}(x-0)-1=0$$

$$\Rightarrow 10e^{-x^2}x-1=0$$

Let $f(x)=10xe^{-x^2}-1=0$

$f(0)=-1,\ f(1)=\frac{10}{e}-1=\frac{10-e}{e}=\frac{10-e}{e}>0$

$\&\ f(2)=\frac{10(2)}{e^4}-1=\frac{20-e^4}{e^4}=\frac{20-54}{e^4}<0.$

$\therefore$ The equation $f(x)=0$ has two +ve roots in the intervals $(0,1)$ and $(1,2)$.

Also $f'(x) = 10e^{-x^2}(1-2x^2)$

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

Let the initial approximation be $x_0 = 0.1$ in the interval $(0,1)$.

The Newton iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \quad \text{—— (1)}$$

putting $n=0$ in (1), the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.1 - \frac{f(0.1)}{f'(0.1)} = 0.100336$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.100336 - \frac{f(0.100336)}{f'(0.100336)}$$

$$= 0.100336$$

$\therefore$ with $x_0 = 0.1$, we get the first root as 0.100336 correct to six decimal places.

Similarly, if we take initial approximation $x_0 = 0.1$ or 1.6 we get the second root as 1.679631 which is correct to six decimal places.

1(d) Construct a switching table for each of the switching function represented by the following Boolean expression.

(i) $x'yz + yz'$, (ii) $x'(yz+y')$ (iii) $xz+yz'+x'y$

Soln.

(i)

x	y	z	$x'yz$	yz'	$x'yz+yz'$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	1
0	1	1	1	0	1
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	0	1	1
1	1	1	0	0	0

(ii)

x	y	z	x'	y'	yz	$y'+yz$	$x'(yz+y')$
0	0	0	1	1	0	1	1
0	0	1	1	1	0	1	1
0	1	0	1	0	0	0	0
0	1	1	1	0	1	1	1
1	0	0	0	1	0	1	0
1	0	1	0	1	0	1	0
1	1	0	0	0	0	0	0
1	1	1	0	0	1	1	0

(iii)

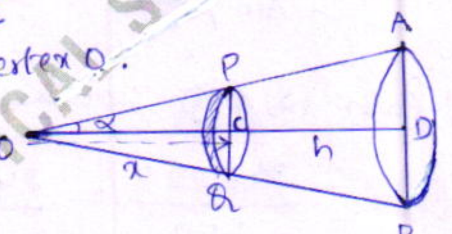
x	y	z	x'	z'	xz	yz'	$x'y$	$xz+yz'+x'y$
0	0	0	1	1	0	0	0	0
0	0	1	1	0	0	0	0	0
0	1	0	1	1	0	1	1	1
0	1	1	1	0	0	0	1	1
1	0	0	0	1	0	0	0	0
1	0	1	0	0	1	0	0	1
1	1	0	0	1	0	1	0	1
1	1	1	0	0	1	0	0	1

1(e) Find the M.I of a right solid cone of mass M, height h and radius of whose basis is a, about its axis.

Sol'n: Let O be the vertex of the right solid cone of mass M, height h and radius of whose basis is a. If α is the semi-vertical angle and ρ the density of the cone, then

$$M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha \quad \text{—} \ (1).$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.



$\therefore$ mass of the disc,

$$\delta m = \rho\pi x^2 \tan^2\alpha\, \delta x$$

M.I of this elementary disc about axis OD.

$$= \frac{1}{2}\delta m\, CP^2 = \frac{1}{2}\left(\rho\pi x^2 \tan^2\alpha\,\delta x\right)x^2\tan^2\alpha = \frac{1}{2}\rho\pi x^4 \tan^4\alpha\,\delta x.$$

$\therefore$ M.I of the cone about axis OD.

$$= \int_0^h \frac{1}{2}\rho\pi x^4 \tan^4\alpha\, dx = \rho\frac{\pi}{10}h^5\tan^4\alpha = \frac{3}{10}Mh^2\tan^2\alpha \text{ from (1)}$$

$$\left(\because \tan\alpha = a/h\right)$$

$$= \frac{3}{10}Ma^2$$

2(a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$. What is the order of this partial differential equation?

Sol'n: Let $u = x+y+z$, $v = x^2+y^2-z^2$

$$\therefore \phi(u,v) = 0 \quad \text{—} \ (1)$$

Differentiating (1) partially w.r.t x

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$\Rightarrow \frac{\partial\phi}{\partial u}(1+0+p)+\frac{\partial\phi}{\partial v}(2x+0+(-2z)p)=0$

$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v}=\frac{-(2x-2zp)}{1+p}$ —— ②

Differentiating ① partially w.r.t y

$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right]=0$

$\Rightarrow \frac{\partial\phi}{\partial u}(0+1+q)+\frac{\partial\phi}{\partial v}(0+2y+(-2z)q)=0$

$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v}=\frac{-(2y-2zq)}{1+q}$ —— ③

from ② & ③ we have

$$\frac{2x-2zp}{1+p}=\frac{2y-2zq}{1+q}$$

$\Rightarrow x-zp+xq-\cancel{zpq}=y-zq+yp-\cancel{zpq}$

$\Rightarrow x-y=p(z+y)+q(-z-x)$

$\Rightarrow p(x+y)-q(x+z)=x-y$

which is the required partial differential equation is of order 1.

2(b) Find the surface whose tangent planes cutoff an intercept of constant length k from the axis of z.

Sol^n: we know that the equation of the tangent plane at point (x,y,z) to a surface is given by

$$p(X-x)+q(Y-y)=Z-z \quad \text{—— ①}$$

where X, Y, Z denote current coordinates of any point on the surface ①, since ① cuts an intercept k on the z-axis, it follows that ① must pass through the point $(0,0,k)$. Hence putting $x=0$, $y=0$ and $z=k$ in ①, we obtain

$$px+qy = z-k \quad \text{—— (2)}$$

which is well known Lagrange's linear equation.
For (2), the Lagrange's auxiliary equations are given by

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-k} \quad \text{—— (3)}$$

Taking the first two fractions of (3),

$$\left(\tfrac{1}{x}\right)dx - \left(\tfrac{1}{y}\right)dy = 0.$$

Integrating, $\log x - \log y = \log c_1 \Rightarrow x/y = c_1$ —— (4).
Again, taking the first and 3rd fraction of (3), we have

$$[1/(z-k)]\,dz - \left(\tfrac{1}{x}\right)dx = 0$$

Integrating, $\log(z-k) - \log x = \log c_2 \Rightarrow (z-k)/x = c_2$ —— (5)
from (4) and (5), the required surface is given by

$\phi\,[y/x, (z-k)/x] = 0$, ϕ being an arbitrary function.

2(c) A reservior discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h(ft):	10	11	12	13	14
A (sq-ft)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.
Estimate the time taken for the water level to fall from 14 to 10ft. above the sluices.

Sol'n: The ~~rate of~~ fall of the surface is given by

$$\frac{dh}{dt} = -48\frac{\sqrt{h}}{A}$$

$$\Rightarrow dt = -\frac{1}{48}\frac{A}{\sqrt{h}}\,dh$$

$$\Rightarrow t = -\frac{1}{48}\int_{14}^{10} \frac{A}{\sqrt{h}}\,dh$$

$$= \frac{1}{48}\int_{10}^{14} \frac{A}{\sqrt{h}}\,dh$$

h	10	11	12	13	14
$\sqrt{h}$	3.1623	3.3166	3.464	3.6056	3.7412
A	950	1070	1200	1350	1530
$\frac{A}{48\sqrt{h}}$	6.2587 (y_0)	6.7212 (y_1)	7.2169 (y_2)	7.8005 (y_3)	8.519 (y_4)

Using Simpson's $\frac{1}{3}$rd rule

$$t = \frac{h}{3}\left[(y_0+y_4) + 4(y_1+y_3) + 2y_2\right]$$

$$= \frac{1}{3}\left[(6.2587 + 8.519) + 4(6.7212 + 7.8005) + 2(7.2169)\right]$$

$$= 29.1$$

$$\simeq 29 \text{ minutes.}$$

2(d) → Use Hamilton's equations to find the equations of motion of a projectile in space.

Sol'n: Let (x, y, z) be the coordinates of a projectile in space at time t, if K and V are the kinetic and potential energies, then

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \text{ and } V = mgz$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$

Here x, y, z are the generalised coordinates

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Since L doesnot contain t explicitly, therefore

$$H = T + V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + mgz$$

$$\Rightarrow H = \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + mgz \quad \text{(using relations ①)}$$

Hence the six Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \quad \text{---} (H_1) \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{---} (H_2)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \quad \text{---} (H_3) \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad \text{---} (H_4)$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \quad \text{---} (H_5) \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \text{---} (H_6)$$

from (H_1) & (H_2), we have $\ddot{x} = \frac{1}{m}\dot{p}_x = 0$ — ②

from (H_3) & (H_4), we have $\ddot{y} = \frac{1}{m}\dot{p}_y = 0$ — ③

from (H_5) & (H_6) we have $\ddot{z} = \frac{1}{m}\dot{p}_z$ i.e. $\ddot{z} = -g$ — ④

Equations ②, ③ and ④ are the equations of motion of a projectile in space.

3(a) → Reduce the equation $x^2 r - 2xys + y^2 t - xp + 3yq = \frac{8y}{x}$ to canonical form.

Sol'n: Given $x^2 r - 2xys + y^2 t - xp + 3yq - 8y/x = 0$ — (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$. showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $x^2\lambda^2 - 2xy\lambda + y^2 = 0 \Rightarrow (x\lambda - y)^2 = 0$ so that $\lambda = y/x, y/x$

The corresponding characteristic equation is

$dy/dx + y/x = 0 \Rightarrow \frac{1}{y}dy + \frac{1}{x}dx = 0$ so that $xy = C_1$

choose $u = xy$ and $v = x$ — (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0$$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, by (2) — (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}$, by (2) — (4)

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left[y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right] + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}.$$

$$= y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{— (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v} \quad \text{———} (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right), \text{ by } (4)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2}, \text{ by } (2) \text{ — } (7)$$

using (2), (3), (4), (5), (6) and (7) in (1), we have

$$x^2\left[y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right] - 2xy\left[\frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v}\right]$$

$$+ y^2x^2\frac{\partial^2 z}{\partial u^2} - x\left[y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] + 3yx\frac{\partial z}{\partial u} - \frac{8y}{x} = 0$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial v^2} - x\frac{\partial z}{\partial v} = \frac{8y}{x} \Rightarrow v^2\frac{\partial^2 z}{\partial v^2} - v\frac{\partial z}{\partial v} = \frac{8u}{v^2} \text{ by } (2)$$

$$\Rightarrow (v^2 D'^2 - vD')z = 8u/v^2, \text{ where } D \equiv \frac{\partial}{\partial u}, D' = \partial/\partial v \quad \text{———} (8)$$

To solve (8), let $u = e^X$ and $v = e^Y$ so that $X = \log u$, $Y = \log v$ — (9)

Then (8) becomes $\{D'(D'-1) - D'\}z = 8e^{X-2Y}$

$$\Rightarrow D'(D'-2)z = 8e^{X-2Y}$$

$$C.F = \phi(X) + e^{2Y}\psi(X) = \phi(\log u) + v^2\psi(\log u), \text{ by } (9)$$

$$= F(u) + v^2G(u) = F(xy) + x^2G(xy), \text{ by } (2)$$

$$P.I = \frac{1}{D'(D'-1)} 8e^{X-2Y}$$

$$= 8e^{x-2y}\frac{1}{(D'-2)(D'-2-2)}\cdot 1$$

$$= 8(e^x/e^{2Y})(1/8)(1-D'/2)^{-1}(1-D'/4)^{-1}\cdot 1$$

$$= u/v^2, \text{ by } (9)$$

$$= (xy)/x^2 = y/x \quad \text{using } (2)$$

Hence the required general solution of (1), is given by

$$z = F(xy) + x^2 G(xy) + y/x, \quad F, G \text{ being arbitrary functions.}$$

3(b) Using Newton's forward interpolation formula, show that

$$\sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Sol^n: If $S_n = \sum n^3$, then $S_{n+1} = \sum(n+1)^3$

$$\therefore \Delta S_n = S_{n+1} - S_n = \sum(n+1)^3 - \sum n^3 = (n+1)^3$$

Then $\Delta^2 S_n = \Delta S_{n+1} - \Delta S_n$

$$= (n+2)^3 - (n+1)^3 = 3n^2 + 9n + 7$$

$$\Delta^3 S_n = \Delta^2 S_{n+1} - \Delta^2 S_n$$

$$= [3(n+1)^2 + 9(n+1) + 7] - (3n^2 + 9n + 7)$$

$$= 6n + 12$$

$$\Delta^4 S_n = \Delta^3 S_{n+1} - \Delta^3 S_n = [6(n+1)+12] - [6n+12] = 6$$

and $\Delta^5 S_n = \Delta^5 \Delta_n = \text{---} = 0.$

Since the first term of the given series is 1, therefore

taking $n=1$, $S_1=1$, $\Delta S_1=8$, $\Delta^2 S_1=19$, $\Delta^3 S_1=18$, $\Delta^4 S_1=6$.
Substituting these in the Newton's forward interpolation formula i.e.

$$S_n = S_1 + (n-1)\Delta S_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 S_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 S_1 + \frac{(n-1)(n-2)(n-3)(n-4)}{4!}\Delta^4 S_1,$$

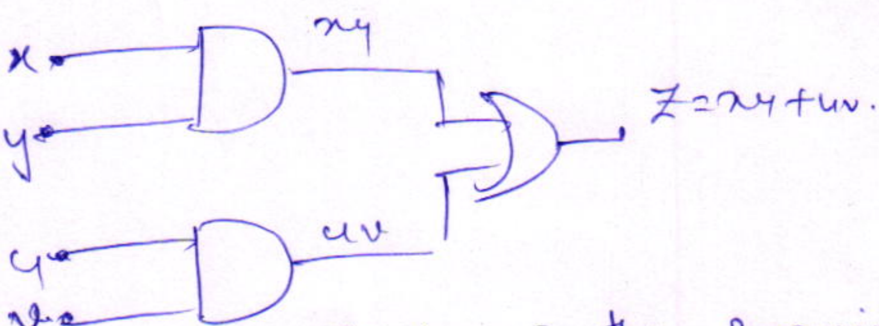
$$S_n = 1 + 8(n-1) + \frac{19}{2}(n-1)(n-2) + 3(n-1)(n-2)(n-3) + \tfrac{1}{4}(n-1)(n-2)(n-3)(n-4)$$

$$= \frac{1}{4}(n^4+2n^3+n^2)$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2$$

3(C) Use only AND and OR logic gates to construct a logic circuit for the boolean expression $z = xy + uv$.

Sol^n $Z = xy + uv$.



which is the required logic circuit

4(a) A string is stretched and fastened to two points l apart, motion is is started by displacing the string into the form $y = m(lx - x^2)$ from which it is released at time $t=0$. find the displacement of any point on the string at a distance x from one end at time t.

→ Sol^n: Given that

$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right) \quad \text{—— (1)}$$

Boundary conditions

$$y(0,t) = 0, \quad y(l,t) = 0, \quad \forall t > 0 \quad \text{—— (2)}$$

Initial conditions:

$$y(x,0) = m(lx - x^2) \quad \text{—— (3)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ; \; 0 < x < l. \quad \text{—— (4)}$$

Suppose that (1) has the solution of the form

$$y(x,t) = X(x)\, T(t) \quad \text{—— (5)}$$

Substituting this value of y in (1), we have

$$XT'' = a^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = \mu \text{ (say)}$$

$$\Rightarrow \frac{X''}{X} = \mu \quad \text{and} \quad \frac{T''}{T} = \mu a^2$$

$$\Rightarrow X'' - \mu X = 0 \quad \text{—— (6)} \qquad \Rightarrow T'' - \mu a^2 T = 0 \quad \text{—— (7)}$$

Using (2), (5) gives

$$X(0)\, T(t) = 0 \quad \text{and} \quad X(l)\, T(t) = 0$$

Since $T(t) \equiv 0$ leads to $y \equiv 0$ $\forall t$

so suppose that $T(t) \neq 0$.

Then equation (8) gives

$$\boxed{X(0) = 0} \quad \text{and} \quad \boxed{X(l) = 0} \quad \text{—— (9)}$$

which are boundary conditions.

we now solve (6) under Boundary conditions (9).

Three cases arise.

Case(1): Let $\mu=0$. Then the solution of (6) is

$$X(x) = Ax+B$$

Using boundary conditions (9), (10) gives

$$X(0) = A(0)+B \Rightarrow \boxed{0=B}$$

and $X(l) = Al+B \Rightarrow 0=Al+0$

$$\Rightarrow \boxed{A=0} \quad (\because l\neq 0)$$

$$\therefore \boxed{X(x) = 0}$$

This leads to $y\equiv 0$, which doesnot satisfy equations (3) and (4).

So we reject $\mu=0$.

Case(2): Let $\mu=\lambda^2$, $\lambda\neq 0$. Then solution of (6) is

$$X(x) = Ae^{\lambda x}+Be^{-\lambda x}$$

Using (9), (11) gives

$$X(0)=0 = A+B \Rightarrow A+B=0 \quad \text{—— (i)}$$

and $X(l) = 0 = Ae^{\lambda l}+Be^{-\lambda l}$

Solving the above equation we get

$$Ae^{\lambda l} - Ae^{-\lambda l} = 0$$

$$\Rightarrow A(e^{\lambda l}-e^{-\lambda l})=0$$

$$\Rightarrow \boxed{A=0} \quad (\because e^{\lambda l}-e^{-\lambda l}\neq 0)$$

$\therefore$ from (i); $\boxed{B=0}$

$$\Rightarrow \boxed{X(x) = 0}$$

This leads to $y\equiv 0$ which doesnot satisfy (3) & (4)

so we reject $\mu=\lambda^2$.

Case ③ Let $\mu = -\lambda^2$; $(\lambda \neq 0)$

Then solution of ⑥

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad \text{— (13)}$$

Using ⑨, ⑬ gives

$$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$$

and $X(l) = 0 = 0 + B\sin\lambda l$

$$\Rightarrow B\sin\lambda l = 0$$

$$\Rightarrow \sin\lambda l = 0$$

Here we taken $B \neq 0$.

Since otherwise $X \equiv 0$ so that

which doesnot satisfy ③ & ④.

Now $\sin\lambda l = 0$

$$\Rightarrow \lambda l = n\pi, \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow \lambda = \frac{n\pi}{l}, \quad n = 1, 2, 3, \ldots$$

$\therefore$ from ⑬, we have

$$X(x) = B\sin\frac{n\pi x}{l}; \quad n = 1, 2, 3, \ldots,$$

Hence non-zero solutions $X_n(x)$ of ⑥ are given by

$$\boxed{X_n(x) = B_n \sin\frac{n\pi x}{l}} \quad \text{— (14)}$$

from ⑦ $\quad T'' - \mu a^2 T = 0$

$$\Rightarrow T'' + \lambda^2 a^2 T = 0 \quad (\because \mu = -\lambda^2)$$

$$\Rightarrow T'' + \frac{n^2\pi^2}{l^2} a^2 T = 0 \quad \left(\because \lambda = \frac{n\pi}{l}\right)$$

whose general solution is

$$T_n(t) = C_n \cos\left(\frac{n\pi a t}{l}\right) + D_n \sin\left(\frac{n\pi a t}{l}\right)$$

$\therefore\ y_n(x,t) = X_n(x)\, T_n(t)$

$$= B_n \sin\frac{n\pi x}{l}\left[C_n \cos\left(\frac{n\pi a t}{l}\right) + D_n \sin\left(\frac{n\pi a t}{l}\right)\right]$$

$$= \left[E_n \cos\left(\frac{n\pi a t}{l}\right) + F_n \sin\left(\frac{n\pi a t}{l}\right)\right] \sin\frac{n\pi x}{l} \quad \text{--- (15)}$$

are solutions of (1) satisfying (2).

Here $E_n = B_n C_n$ and $F_n = B_n D_n$.

In order to obtain a solution, also satisfying (3) and (4), we consider more general solution $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$.

i.e. $$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\left(\frac{n\pi a t}{l}\right) + F_n \sin\left(\frac{n\pi a t}{l}\right)\right\}\sin\frac{n\pi x}{l} \quad \text{--- (16)}$$

Differentiating (16) partially w.r.t. 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{E_n\left(-\sin\left(\frac{n\pi a t}{l}\right)\right)\cdot\frac{n\pi a}{l} + F_n \cos\left(\frac{n\pi a t}{l}\right)\cdot\frac{n\pi a}{l}\right\}\sin\frac{n\pi x}{l} \quad \text{--- (17)}$$

putting $t=0$ in equations (16) and (17) and using the equations (3) & (4), we get

(16) $\equiv$ $$m(lx - x^2) = \sum_{n=1}^{\infty}\left\{E_n \cos\left(\frac{n\pi a(0)}{l}\right) + F_n \sin\frac{n\pi a(0)}{l}\right\}\sin\frac{n\pi x}{l}.$$

$$\Rightarrow m(lx - x^2) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}.$$

and

(17) $\equiv$ $$0 = \sum_{n=1}^{\infty}\left\{-E_n(0) + F_n\cdot\frac{n\pi a}{l}\right\}\sin\frac{n\pi x}{l}$$

$$\Rightarrow \sum_{n=1}^{\infty} F_n \frac{n\pi a}{l}\sin\frac{n\pi x}{l} = 0.$$

where $F_n = \frac{2}{n\pi a}\int_0^l (0)\sin\frac{n\pi x}{l} = 0$

and $E_n = \frac{2}{l}\int_0^l m(lx-x^2)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2m}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2m}{l}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \int (l-2x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2m}{l}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} + \int 2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2m}{l}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{2m}{l}\left[(0-0) - \frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$= \frac{2m}{l}\left[-\frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$= -\frac{4l^2m}{n^3\pi^3}(\cos n\pi - 1)$$

$$E_n = \begin{cases} 0 & \text{if } n = 2p;\ p = 1,2,3,\ldots \\ \frac{8l^2m}{n^3\pi^3} & \text{if } n = 2p-1;\ p = 1,2,3,\ldots \end{cases}$$

$\therefore$ From (16)

$$y(x,t) = \sum_{p=1}^{\infty}\frac{8l^2m}{(2p-1)^3\pi^3}\cos\frac{(2p-1)\pi at}{l}\sin\frac{(2p-1)\pi x}{l}$$

$$= \frac{8ml^2}{\pi^3}\sum_{p=1}^{\infty}\frac{1}{(2p-1)^3}\cos\frac{(2p-1)\pi at}{l}\sin\frac{(2p-1)\pi x}{l}$$

4(b) Solve $20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$ by Gauss - Seidal method.

Soln: we write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad (1)$$

By Gauss-Seidel method, system (1) can be written as

$$x^{k+1} = \frac{1}{20}(17 - y^k + 2z^k)$$

$$y^{k+1} = \frac{1}{20}(-18 - 3x^{k+1} + z^k)$$

$$z^{k+1} = \frac{1}{20}(25 - 2x^{k+1} + 3y^{k+1})$$

Now taking $x^{(0)} = 0$ (ie, $x = y = z = 0$) we obtain the following iterations.

First iteration

put $k = 0$
$$x^{(1)} = \frac{1}{20}(17 - y^{(0)} + 2z^{(0)}) = 0.8500$$

$$y^{(1)} = \frac{1}{20}(-18 - 3x^{(1)} + z^{(0)}) = -1.0275$$

$$z^{(1)} = \frac{1}{20}(25 - 2x^{(1)} + 3y^{(1)}) = 1.0109$$

Second iteration:

put $k = 1$
$$x^{(2)} = \frac{1}{20}(17 - y^{(1)} + 2z^{(1)}) = 1.0025$$

$$y^{(2)} = \frac{1}{20}(-18 - 3x^{(2)} + z^{(1)}) = -0.9998$$

$$z^{(2)} = \frac{1}{20}(25 - 2x^{(2)} + 3y^{(2)}) = 0.9998.$$

Third iteration

put $k = 2$
$$x^{(3)} = \frac{1}{20}(17 - y^{(2)} + 2z^{(2)}) = 1.0000$$

$$y^{(3)} = \frac{1}{20}(-18 - 3x^{(3)} + z^{(2)}) = -1.0000$$

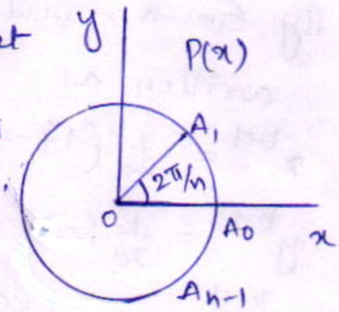
$$z^{(3)} = \frac{1}{20}(25 - 2x^{(3)} + 3y^{(3)}) = 1.0000$$

$\therefore$ The solution is given by $x = 1$, $y = -1$ & $z = 1$.

4(c) If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid.

Sol'n: from the fig., the n vortices are at $A_0, A_1, A_2 \cdots A_{n-1}$ such that

$\angle A_0OA_1 = \angle A_1OA_2 = \cdots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$



The coordinates of the points A_r are given by

$z = z_r = ae^{(2\pi/n)ir}$ where $r = 0, 1, 2 \cdots n-1$.

These are n roots of the equation $z^n - a^n = 0$

$[\text{For } z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi ri}]$

Hence $z^n - a^n = (z - z_0)(z - z_1) \cdots (z - z_{n-1})$

The complex potential due to n vortices at P is given by

$$w = \frac{ik}{2\pi}\left[\log(z - z_0) + \log(z - z_1) + \cdots + \log(z - z_{n-1})\right]$$

$$= \frac{ik}{2\pi}\log(z - z_0)(z - z_1)\cdots(z - z_{n-1}) = \frac{ik}{2\pi}\log(z^n - a^n) \quad \text{—— (1)}$$

For the point A_0, $z = a$ so that $r = a, \theta = 0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{ik}{2\pi}\log(z - a) = \frac{ik}{2\pi}\left[\log(z^n - a^n) - \log(z - a)\right]$$

$$\phi' + i\psi' = \frac{ik}{2\pi}\left[\log(r^n e^{in\theta} - a^n) - \log(re^{i\theta} - a)\right]$$

$$\psi' = \frac{k}{4\pi}\left[\log(r^{2n} + a^{2n} - 2r^n a^n \cos n\theta) - \log(r^2 + a^2 - 2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{k}{4\pi}\left[\frac{2nr^{2n-1} - 2nr^{n-1}a^n\cos n\theta}{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta} - \frac{2r - 2a\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi}\left[\frac{2nr^n a^n \sin n\theta}{r^{2n} - 2r^n a^n \cos n\theta + a^{2n}} - \frac{2ra\cos\theta}{r^2 + a^2 - 2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a} = \frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right) - \left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right] = \frac{k}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta} - \frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\lim_{x\to 0}\frac{F(x)}{G(x)} = \lim_{x\to 0}\frac{F'(x)}{G'(x)} = \lim_{x\to 0}\frac{F''(x)}{G''(x)}$ $\left[\text{form } \frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta} - \frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta} - \frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta\to 0$$

$$= \frac{k}{4\pi}[0+0] = 0$$

finally $\frac{\partial\psi'}{\partial\theta} = \frac{k}{4\pi a}(n-1)$, $\frac{\partial\psi'}{\partial\theta} = 0$ as $r\to a$, $\theta\to 0$..

Consequently, the velocity q_0 of the Vortex A_0 is given by

$$q_0 = \left[\left(\frac{\partial\psi'}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2} = \frac{k(n-1)}{4\pi a}$$

This proves that the whole of velocity is along the tangent and there is no velocity along the normal to the circle. Hence the vortices will move round the cylinder with uniform velocity $\frac{k(n-1)}{4\pi a}$.

The time of one complete revolution

$$= \frac{\text{distance}}{\text{velocity}} = \frac{2\pi a}{k(n-1)/4\pi a} = \frac{8\pi^2 a^2}{(n-1)k}$$

5(a) Solve $(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x) - \frac{1}{2}$.

Sol^n: Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$ — (1)

Also, Let $D = \partial/\partial x$, $D' = \partial/\partial y$, $D_1 = \partial/\partial u$ and $D_1' = \partial/\partial v$

Then the given equation reduces to

$$[D_1(D_1-1) - D_1D_1' - 2D_1'(D_1'-1) + D_1 - 2D_1']z = \log y - \log x - \tfrac{1}{2}$$

$$\Rightarrow (D_1^2 - D_1D_1' - 2D_1'^2)z = v - u - \tfrac{1}{2}$$

$$\Rightarrow (D_1 - 2D_1')(D_1 + D_1') = v - u - \tfrac{1}{2}$$

$$\therefore C.F = \phi_1(v+2u) + \phi_2(v-u)$$

$$= \phi_1(\log y + 2\log x) + \phi_2(\log y - \log x)$$

$$= \phi_1 \log(yx^2) + \phi_2 \log(y/x)$$

$$= f_1(yx^2) + f_2(y/x)$$

where f_1 and f_2 are arbitrary functions

$$P.I = \frac{1}{D_1^2 - D_1D_1' - 2D_1'^2}\left(v - u - \tfrac{1}{2}\right) = \frac{1}{D_1^2(1 - D_1'/D_1 - 2D_1'^2/D_1^2)}\left(v - u - \tfrac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{1 - \left(\frac{D_1'}{D_1} + \frac{2D_1'^2}{D_1^2}\right)\right\}^{-1}\left(v - u - \tfrac{1}{2}\right) = \frac{1}{D_1^2}\left(1 + \frac{D_1'}{D_1} + \cdots\right)\left(v - u - \tfrac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{v - u - \tfrac{1}{2} + \frac{1}{D_1} D_1'\left(v - u - \tfrac{1}{2}\right)\right\} = \frac{1}{D_1^2}\left(v - u - \tfrac{1}{2} + \frac{1}{D_1}\cdot 1\right)$$

$$= \frac{1}{D_1^2}\left(v - u - \tfrac{1}{2} + u\right) = \frac{1}{D_1^2}\left(v - \tfrac{1}{2}\right) = \left(v - \tfrac{1}{2}\right)\frac{u^2}{2}$$

$$= \tfrac{1}{2}u^2v - \tfrac{1}{4}u^2$$

$$= \tfrac{1}{2}(\log x)^2 \log y - (1/4)(\log x)^2, \text{ using (1)}$$

$$\therefore z = f_1(yx^2) + f_2(y/x) + \tfrac{1}{2}(\log x)^2 \log y - (1/4)(\log x)^2.$$

5(b) Use Euler's method with step size $h=0.15$ to compute the approximate value of $y(0.6)$, correct upto five decimal places from the initial value problem.

$y' = x(y+x)-2$, $y(0)=2$.

Sol^n: Given that

$f(x,y) = y' = x(y+x)-2$; $y(0)=2$
$h=0.15$

we shall compute the value of y at $x=0.15,\ 0.3,\ 0.45$ and 0.6

By Euler's method

$$y_n = y_{n-1} + h\,f(x_{n-1}, y_{n-1}).$$

$y_1 = y_0 + h\,f(x_0, y_0)$ where $y_0 = 2$, $x_0 = 0$

$\therefore \underset{=y(0.15)}{y_1} = 2 + (0.15) f(0,2) = 1.7$

$\underset{=y(0.3)}{y_2} = y_1 + h\,f(x_1, y_1) = 1.7 + (0.15) f(0.15, 1.7)$
$= 1.441625$

$\underset{=y(0.45)}{y_3} = y_2 + h\,f(x_2, y_2) = 1.441625 + (0.15) f(0.3, 1.441625)$
$= 1.219998.$

$\underset{=y(0.6)}{y_4} = y_3 + h\,f(x_3, y_3)$
$= 1.219998 + (0.15) f(0.45, 1.21998)$

$= 1.03272.$

$\therefore y(0.6) = 1.03272.$

5(C) (i) For any Boolean variables x and y, show that $x+xy=x$.
(ii) write the dual of each Boolean expression
(a) $a(a'+b)=ab$, (b) $(a+1)(a+0)=a$, (c) $(a+b)(b+c)=ac+b$.

Solⁿ: (i) $L.H.S = x+xy = x\cdot 1+xy$

$= x(1+y)$

$= x\cdot 1 \quad (\because 1+y=1)$

$= x$

$\therefore x+xy=x$

(ii) (a) The dual of
$a(a'+b)=ab$ is

$a(a'+b)=a+a'b$

(b) The dual of
$(a+1)(a+0)=a$
is $a0+a1=a$

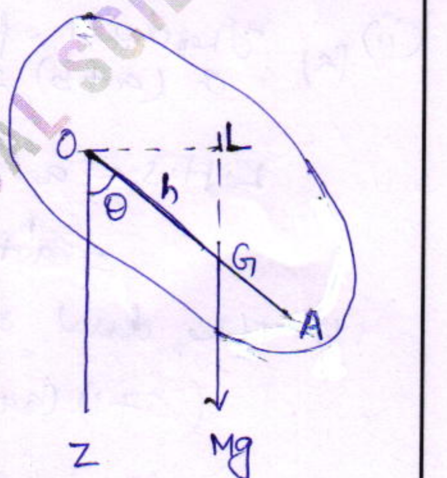
(c) The dual of
$(a+b)(b+c)=ac+b$ is
$ab+bc=(a+c)b$.

5(d). write the Hamiltonian function and equation of motion of the compound pendulum.

sol^n: At time t, let θ be the angle between the vertical plane through the fixed axis (plane fixed in space) and the plane through the C.G. 'G' and the fixed axis (plane fixed in the body). Let $OG = h$.

If T and V are the kinetic and potential energies of the pendulum then

$T = \frac{1}{2}Mk^2\dot{\theta}^2$ and $V = -Mgh\cos\theta$



$\therefore L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta$

Here θ is the only generalised coordinate.

$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}Mk^2\dot{\theta}$ ——— (1)

since L does not t explicity.

$\therefore H = T + V = \frac{1}{2}Mk^2\dot{\theta}^2 - Mgh\cos\theta$

$= \frac{1}{2Mk^2}p_\theta^2 - Mgh\cos\theta$, from (1)

Hence the two Hamilton's equations are

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mgh\sin\theta$ — (H_1) and

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{Mk^2}p_\theta$ —— (H_2)

Differentiating (H_2) and substituting from (H_1), we get

$\ddot{\theta} = \frac{1}{Mk^2}\dot{p}_\theta = \frac{1}{Mk^2}(-Mgh\sin\theta)$

$\Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta$

which is the equation of motion of a compound pendulum.

5(e) Determine the streamlines and the path of the particles $u = x/(1+t)$, $v = y/(1+t)$, $w = z/(1+t)$.

Sol^n: The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

$$\Rightarrow \underset{-(1)}{\frac{dx}{x}} = \underset{-(2)}{\frac{dy}{y}} = \underset{-(3)}{\frac{dz}{z}}$$

By integrating (1) & (2), we have

$\log x = \log y + \log A$, A is integration constant.

$$\Rightarrow x = Ay \quad \text{———} (4)$$

By integrating (2) & (3), we have

$\log x = \log z + \log B$, B is an integration constant.

$$\Rightarrow x = Bz \quad \text{———} (5)$$

Hence the streamlines are given by the intersection of (1) and (2). The differential equation of path lines given by

$$q = \frac{dr}{dt}$$

This $\Rightarrow \dfrac{dx}{dt} = \dfrac{x}{1+t}$, $\dfrac{dy}{dt} = \dfrac{y}{1+t}$, $\dfrac{dz}{dt} = \dfrac{z}{1+t}$

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = \frac{dt}{1+t}, \quad \frac{dz}{z} = \frac{dt}{1+t}$$

Integrating we get

$$\log x = \log(1+t) + \log a$$
$$\log y = \log(1+t) + \log b$$
$$\log z = \log(1+t) + \log c$$

$\Rightarrow x = a(1+t)$, $y = b(1+t)$, $z = c(1+t)$. These give required pathlines.

6(a) Find the integral surface of the partial differential equation $(x-y)p+(y-x-z)q=z$ through the circle $z=1$, $x^2+y^2=1$.

Sol'n: Given $(x-y)p+(y-x-z)q=z$. —— ①

Lagrange's auxiliary equations of ① are.

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z} \quad \text{—— ②}$$

Choosing 1, 1, 1 as multipliers, each fraction of ②

$$=\frac{dx+dy+dz}{0}$$

$\therefore dx+dy+dz=0$ so that $x+y+z=c_1$ —— ③

Taking the last two fractions of ② and using ③ we get

$$\frac{dy}{y-(c_1-y)}=\frac{dz}{z} \Rightarrow \frac{2dy}{2y-c_1}-\frac{2dz}{z}=0$$

Integrating it, $\log(2y-c_1)-2\log z=\log c_2 \Rightarrow (2y-c_1)/z^2=c_2$ —— ④

The given curve is given by $z=1$, $x^2+y^2=1$ —— ⑤

Putting $z=1$ in ③ & ④, we get

$$x+y=c_1-1 \text{ and } y-x=c_2+1 \quad \text{—— ⑥}$$

but $2(x^2+y^2)=(x+y)^2+(y-x)^2$. —— ⑦

using ⑤ and ⑥, ⑦ becomes

$$2=(c_1-1)^2+(c_2+1)^2 \Rightarrow c_1^2+c_2^2-2c_1+2c_2=0 \quad \text{—— ⑧}$$

Putting the values of c_1 and c_2 from ③ & ④ in ⑧, required integral surface is

$$(x+y+z)^2+(y-x-z)^2/z^4-2(x+y+z)+2(y-x-z)/z^2=0$$

$$\Rightarrow z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0$$

6(b)) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$.

Sol^n: Given that $z = p^2 - q^2$

$\Rightarrow f(x, y, z, p, q) = p^2 - q^2 - z = 0$ —(1)

Now we are to find the integral surface of (1) which is passing through the parabola

$y = 0, \quad 4z + x^2 = 0$

whose parametric equations are

$y = 0, \quad x = \lambda, \quad z = -\frac{\lambda^2}{4}$.

i.e. $x = f_1(\lambda), \quad y = f_2(\lambda), \quad z = f_3(\lambda)$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = f_1(\lambda) = \lambda, \quad y_0 = f_2(\lambda) = 0, \quad z_0 = f_3(\lambda) = -\frac{\lambda^2}{4}$.

Now we find the initial values p_0 and q_0 by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ and

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

i.e. $f(\lambda, 0, -\frac{\lambda^2}{4}, p_0, q_0) = 0$

$\Rightarrow -\frac{\lambda}{2} = p_0(1) + q_0(0)$ and $p_0^2 - q_0^2 + \frac{\lambda^2}{4} = 0$

$\Rightarrow \boxed{-\frac{\lambda}{2} = p_0}$ $\Rightarrow q_0^2 = \frac{\lambda^2}{4} + p_0^2$

$\Rightarrow q_0^2 = \frac{\lambda^2}{4} + \frac{\lambda^2}{4}$

$\Rightarrow \boxed{q_0 = \frac{\lambda}{\sqrt{2}}}$

$\therefore x_0 = \lambda, \quad y_0 = 0, \quad z_0 = -\frac{\lambda^2}{4}, \quad p_0 = -\frac{\lambda}{2}, \quad q_0 = \frac{\lambda}{\sqrt{2}}$ & $t = t_0 = 0$

(2)

Now the characteristic equations of ①
are

$x'(t) = \frac{\partial f}{\partial p} = 2p$ —— ③

$y'(t) = \frac{\partial f}{\partial q} = -2q$ —— ④

$z'(t) = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = p(2p) + q(-2q) = 2(p^2 - q^2) = 2z$ —— ⑤ $(\because z = p^2 - q^2)$

$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -0 - p(-1) = p$ —— ⑥

$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -0 - q(-1) = q$. —— ⑦

from ③ & ⑥:

$\frac{x'(t)}{2} = p'(t) \Rightarrow \frac{dx}{2} = dp$

$\Rightarrow dx = 2dp$

$\Rightarrow x = 2p + c_1$

Using the initial values

$x_0 = 2p_0 + c_1$

$\Rightarrow \lambda = 2\left(-\frac{\lambda}{2}\right) + c_1 \Rightarrow c_1 = 2\lambda$

$\therefore \boxed{x = 2p + 2\lambda.}$ —— ⑧

from ④ & ⑦:

$-\frac{y'(t)}{2} = q'(t) \Rightarrow -\frac{dy}{2} = dq \Rightarrow dy = -2q$

$\Rightarrow y = -2q + c_2$

using the initial values

$y_0 = -2q_0 + c_2$

$\Rightarrow 0 = -2\left(\frac{\lambda}{\sqrt{2}}\right) + c_2$

$\Rightarrow c_2 = \sqrt{2}\lambda$.

$\therefore \boxed{y = -2q + \lambda\sqrt{2}}$ —— ⑨

from ⑥: $\frac{dp}{dt} = p \Rightarrow \frac{dp}{p} = dt \Rightarrow \log p = t + \log c_3$

$\Rightarrow p = c_3 e^t$.

from ⑦: $\frac{dq}{dt} = q \Rightarrow \frac{dq}{q} = dt \Rightarrow \log q = t + \log c_4$

$\Rightarrow q = c_4 e^t$

Using the initial values

$p_0 = c_3 e^{t_0}$ and $q_0 = c_4 e^{t_0}$

$\Rightarrow -\frac{\lambda}{2} = c_3$ and $\frac{\lambda}{\sqrt{2}} = c_4$

$\therefore \boxed{p = -\frac{\lambda}{2} e^t}$ and $\boxed{q = \frac{\lambda}{\sqrt{2}} e^t}$

$\therefore$ from (8) & (9),

$x = 2p + 2\lambda = 2\left(-\frac{\lambda}{2}\right) + 2\lambda = -\lambda e^t + 2\lambda$

i.e. $\boxed{x = -\lambda e^t + 2\lambda}$

and $y = -2\left(\frac{\lambda}{\sqrt{2}}\right)e^t + \sqrt{2}\lambda = -\sqrt{2}\lambda e^t + \sqrt{2}\lambda$

$\Rightarrow \boxed{y = \lambda\sqrt{2}(1 - e^t)}$

from (3), $z'(t) = 2z$

$\Rightarrow \frac{dz}{z} = 2dt$

$\Rightarrow \log z = 2t + \log c_5$

$\Rightarrow$ using initial values, we get

$\log z_0 = 2t_0 + \log c_5 \Rightarrow c_5 = z_0$

$\therefore \log z = 2t + \log z_0$

$\Rightarrow z = z_0 e^{2t}$

$\Rightarrow \boxed{z = -\frac{\lambda^2}{4} e^{2t}}$

$\therefore$ The required characteristics of (1) are given by

$\boxed{x = \lambda(2 - e^t)}$ —(i), $\boxed{y = \lambda\sqrt{2}(1 - e^t)}$ —(ii); $\boxed{z = -\frac{\lambda^2}{4} e^{2t}}$ —(iii).

Now eliminating e^t and λ from (i), (ii) & (iii)

from (i) ; $\lambda = \frac{x}{2 - e^t}$

from (ii) $y = \frac{x}{(2-e^t)}\sqrt{2}(1-e^t) = \frac{\sqrt{2}x(1-e^t)}{2-e^t}$

$\Rightarrow y(2-e^t) = \sqrt{2}x(1-e^t)$

$\Rightarrow 2y - ye^t = \sqrt{2}x - \sqrt{2}xe^t$

$\Rightarrow (\sqrt{2}x - y)e^t = \sqrt{2}x - 2y$

$\Rightarrow e^t = \frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}$

from (1) $\lambda = \frac{x}{2 - \left(\frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}\right)} = \frac{(\sqrt{2}x - y)x}{2\sqrt{2}x - 2y - \sqrt{2}x + 2y}$

$\Rightarrow \lambda = \frac{(\sqrt{2}x - y)x}{\sqrt{2}x}$

$\Rightarrow \boxed{\lambda = \frac{\sqrt{2}x - y}{\sqrt{2}}}$

from (iii):

$z = \frac{-\lambda^2 e^{2t}}{4}$

$= -\frac{1}{4}\left(\frac{\sqrt{2}x - y}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}\right)^2$

$= \frac{-(\sqrt{2}x - 2y)^2}{8}$

$= \frac{-[2x^2 + 4y^2 - 4\sqrt{2}xy]}{8}$

$= \frac{-[x^2 - 2\sqrt{2}xy + 2y^2]}{4}$

$\Rightarrow \boxed{4z + (x - \sqrt{2}y)^2 = 0}$ which is the required integral surface of (1)

6(c) → find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0, y=b$. The edges $x=0, x=a, y=0$ are kept at temperature zero while the edge $y=b$ is kept at $100°C$.

Sol: The steady state temperature $u(x,y)$ is the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{—— (1)}$$

subject to boundary conditions

$$u(0,y)=u(a,y)=0 \quad \text{—— (2)}$$

$$u(x,0)=0 \quad \text{—— (3)}$$

$$u(x,b)=100 \quad \text{—— (4)}$$

suppose (1) has the solution of the form

$$u(x,y)=X(x)\,Y(y) \quad \text{—— (5)}$$

$\therefore$ from (1), $X''Y+XY''=0$

$$\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=\mu \text{ (say)}$$

$$\Rightarrow X''-\mu X=0 \quad \text{—— (6)} \quad \text{and} \quad Y''+\mu Y=0 \quad \text{—— (7)}$$

using B.C. (2), (5) gives $X(0)Y(y)=0$ & $X(a)Y(y)=0$

$\Rightarrow X(0)=0$ & $X(a)=0$ —— (8)

then $Y(y)\neq 0$ since otherwise $u\equiv 0$ which doesnot satisfy (4)

we now solve (6) under B.C (8):

Three cases arise.

case(i): Let $\mu=0$. Then (6) $\equiv X''=0 \Rightarrow X(x)=Ax+B$

Using B.C. (8) $A = B = 0$
$\Rightarrow X(x) = 0$ which leads to $u = 0$.
so reject $\mu = 0$

case(2): Let $\mu = \lambda^2$ where $\lambda \neq 0$
$\therefore$ from (6): $X'' - \lambda X = 0$
$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$
Using B.C. (8): we get $A = B = 0$
$\Rightarrow X(x) = 0$
This leads to $u = 0$
so reject $\mu = \lambda^2$

case(3): Let $\mu = -\lambda^2$, where $\lambda \neq 0$.
Then solution of (6) is given by
$X(x) = A\cos\lambda x + B\sin\lambda x$
Using B.C (8): $A = 0$ and $A\cos\lambda a + B\sin\lambda a = 0$
$\Rightarrow B\sin\lambda a = 0$
$\Rightarrow \sin\lambda a = 0$. ($\because B \neq 0$ otherwise we will get $X(x) = 0$ and hence $u = 0$)
$\Rightarrow \lambda a = n\pi$
$\Rightarrow \lambda = \frac{n\pi}{a}$, $n = 1, 2, \ldots$ — (9)

Hence non-zero solution $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$ — (10)

and $\mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$.

$\therefore$ from (7): $Y'' - \frac{n^2\pi^2}{a^2} Y = 0$
whose general solution is
$Y_n(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}$ — (11)

Using (3), (5) gives $0 = X(x) Y(y)$ so that $Y(0) = 0$ ($\because X(x) \neq 0$, for otherwise we will get $u \equiv 0$)

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ —

putting $y=0$ in (11), and using $Y_n(0)=0$,

we have $0 = C_n + D_n \Rightarrow D_n = -C_n$.

Then (16) reduces to

$$Y_n(y) = C_n\left(e^{n\pi y/a} - e^{-\frac{n\pi y}{a}}\right) = 2\sinh\left(\frac{n\pi y}{a}\right)$$

$$\therefore u_n(x,y) = X_n(x)\, Y_n(y)$$

$$u_n(x,y) = E_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi y}{a}$$

are solutions of (1) satisfying (2) and (3).

Here $E_n = 2B_nC_n$.

In order to satisfy (4), we now consider more general solution given by

$$u(x,y) = \sum u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \quad \text{——— (12)}$$

putting $y=b$ in (12) and using (4),

we get

$$f(x) = 100 = \sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

where $E_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int f(x) \sin\frac{n\pi x}{a}\,dx$

$$\Rightarrow E_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int (100)\sin\frac{n\pi x}{a}\,dx$$

$$= \frac{200}{a\sinh\left(\frac{n\pi b}{a}\right)}\left[\frac{-\cos\frac{n\pi x}{a}}{n\pi/a}\right]_0^a$$

$$\Rightarrow E_n = \frac{200}{n\pi}\left[1-(-1)^n\right]\operatorname{cosech}\left(\frac{n\pi b}{a}\right) = \begin{cases} 0, & \text{if } n=2m,\ m=1,2,3,\ldots \\ \frac{(400)\operatorname{cosec}\{(2m-1)\pi b/a\}}{(2m-1)\pi}, & \text{if } n=2m-1,\ m=1,2,\ldots \end{cases}$$

$\therefore$ (12) reduces to

$$u(x,y) = \sum_{m=1}^{\infty} E_{2m-1} \sin\frac{(2m-1)\pi x}{a} \sinh\frac{(2m-1)\pi y}{a}.$$

(or) $$u(x,y) = \frac{400}{\pi}\sum_{m=1}^{\infty} \sin\frac{(2m-1)\pi x}{a} \sinh\frac{(2m-1)\pi y}{a} \operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

7(c) Using Gauss-Jordan method, find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Sol^n: Writing the given matrix side by side with unit matrix of order 3, we have

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1/2 \end{array}\right] \quad \begin{array}{l} R_2 \to \frac{R_2}{2} \\ R_3 \to \frac{R_3}{2} \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -2 & 1/2 & 1/2 & 1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - \frac{3}{2}R_3 \\ R_3 \to R_3/2 \end{array}$$

Hence the inverse of the given matrix is

$$\begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

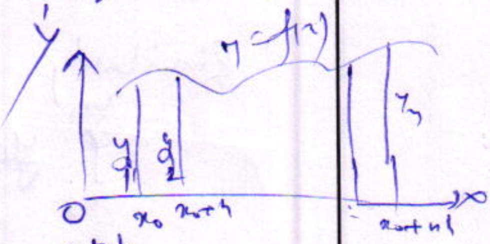
7(b) Derive the formula

$$\int_a^b y\,dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\Big]$$

Is there any restriction on n? State that condition. What is the error bound in the case of Simpson's $\frac{1}{3}$ rule?

Sol^n: Let $I=\int_a^b y\,dx$, where $y=f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x_0, x_1, x_2, \ldots x_n$.

Let us divide the interval (a,b) into n-subintervals of width h so that $x_0=a$, $x_1=x_0+h$, $x_2=x_0+2h$, $\ldots\ldots x_n=x_0+nh=b$.



Then

$$I=\int_{x_0}^{x_0+nh} y\,dx = h\int_0^n f(x_0+ph)\,dp$$

by putting $x=x_0+ph$, $dx=h\,dp$

Approximating y by Newton's forward difference formula, we obtain

$$I=h\int_0^{n}\left[y_0+p\Delta y_0+\frac{p(p-1)}{2}\Delta^2 y_0+\frac{p(p-1)(p-2)}{6}\Delta^3 y_0+\cdots\right]dp$$

which gives on simplification

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad \text{--- (1)}$$

This is known as Newton-Cotes quadrature formula.

from this general formula, we deduce the different integration formulae by putting $n = 1, 2, 3 \ldots$ etc.

Simpson's $\frac{1}{3}$-rd rule: putting $n=2$ in (1), and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola i.e, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_2 = x_0+2h} y\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\ldots \ldots$$

and finally

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Summing up we obtain,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots y_{n-2}) + y_n\right]$$

which is known as Simpson's $\frac{1}{3}$-rule.

This rule requires the division of the whole range into an even no. of subintervals of width h.

The error in the Simpsons Rule is given by

$$\int_a^b y\,dx = \frac{h}{3}\Big[y_0 + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2}) + y_n\Big]$$

$$= -\frac{(b-a)}{180} h^4 y^{iv}(\bar{x})$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivates.

7(C) Apply Runge-Kutta method of order 4 to find approximate value of y for $x=0.2$, in steps of 0.1, if $\frac{dy}{dx} = x+y^2$, given that $y=1$ where $x=0$.

Sol'n: Given that $f(x,y) = x+y^2$

Here we take $h=0.1$ and carry out the Calculations in two steps.

Step 1: $x_0 = 0,\ y_0 = 1,\ h = 0.1$

$$K_1 = hf(x_0, y_0) = (0.1)f(0.1) = 0.1$$

$$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2) = (0.1)f(0.05, 1.1) = 0.1152$$

$$K_3 = hf(x_0 + h/2,\ y_0 + \frac{K_2}{2}) = (0.1)f(0.05, 0.1152) = 0.1168$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3) = (0.1)f(0.1,\ 1.1168) = 0.1347$$

$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}(0.1 + 0.2304 + 0.2336 + 0.1347) = 0.1165$$

$$\therefore y(0.1) = y_0 + K = 1.1165$$

Step 2: $x_1 = x_0 + h = 0.1,\quad y_1 = 1.1165,\quad h = 0.1$

$$K_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.1165) = 0.1347$$

$$K_2 = hf(x_1 + \frac{1}{2}h,\ y_0 + \frac{1}{2}K_1) = (0.1)f(0.15, 1.1838) = 0.1551$$

$$K_3 = hf(x_1 + \frac{1}{2}h,\ y_0 + K_2/2) = (0.1)f(0.15, 1.194) = 0.1576$$

$$K_4 = hf(x_1 + h,\ y_1 + K_3) = (0.1)f(0.2, 1.1576) = 0.1823$$

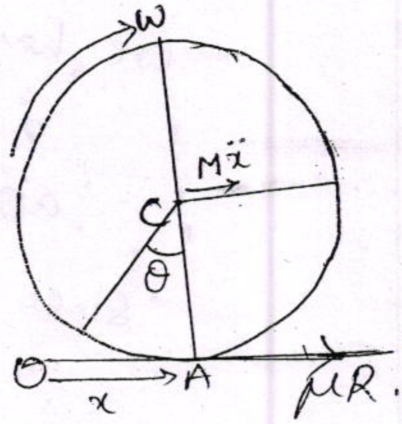
$$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1571$$

Hence $y(0.2) = y_1 + K = 1.2736$

8(a): A homogeneous sphere of radius a, rotating with angular velocity ω about horizontal diameter, is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\frac{2\omega a}{7\mu g}$ and that then the sphere will roll with angular velocity $\frac{2\omega}{7}$.

Sol^n:

As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.



Initial velocity of the point of contact

= Initial velocity of the centre C + initial velocity of the point of contact w.r.t the centre C.

$= 0 + a\omega$ in the direction from right to left,

i.e, the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t

the equations of motion are

$$M\ddot{x} = \mu R, \quad \text{where } R = Mg \quad \text{——— (1)}$$

$$\text{and } Mk^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu Ra \quad \text{——— (2)}$$

From (1), we have $\ddot{x} = \mu g$ ——— (3)

and from (2), we have

$$a\ddot{\theta} = -\frac{5}{2}\mu g \quad \text{——— (4)}$$

Integrating (3) & (4),

we have

$$\dot{x} = \mu g t + C_1 \text{ and}$$

$$a\dot{\theta} = -\frac{5}{2}\mu g t + C_2$$

Since initially when $t=0$, $\dot{x}=0$, $\dot{\theta}=\omega$

$\therefore C_1 = 0$ and $C_2 = a\omega$.

$$\therefore \dot{x} = \mu g t \quad \text{——— (5)}$$

$$\text{and } a\dot{\theta} = -\frac{5}{2}\mu g t + a\omega \quad \text{——— (6)}$$

velocity of the point of contact $= \dot{x} - a\dot{\theta}$

$\therefore$ The point of contact will come to rest

when $\dot{x} - a\dot{\theta} = 0$.

i.e. when $\mu g t - \left(-\frac{5}{2}\mu g t + a\omega\right) = 0$

i.e. when $t = \dfrac{2a\omega}{7\mu g}$

Therefore after time $\dfrac{2a\omega}{7\mu g}$ the slipping will stop and pure rolling will commence.

putting this value of t in (6), we get

$$\dot{\theta} = \frac{2\omega}{7}$$

when rolling commences, let F be the frictional force. Therefore the equations of motion are

$$M\ddot{x} = F, \quad \text{—— (7)}$$

$$M\cdot\frac{2}{5}a^2\ddot{\theta} = -Fa \quad \text{—— (8)}$$

and $\dot{x} - a\dot{\theta} = 0$ —— (9)

From (9) $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

now from (7) & (8), we get

$$M\ddot{x} = F = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow M\ddot{x} = -\frac{2}{5}Ma\ddot{\theta}$$

$$\Rightarrow \ddot{x} = -\frac{2}{5}a\ddot{\theta}$$

$$\Rightarrow a\ddot{\theta} = -\frac{2}{5}a\ddot{\theta} \quad (\because \ddot{x} = a\ddot{\theta})$$

$$\Rightarrow \frac{7}{5}a\ddot{\theta} = 0$$

$$\Rightarrow \ddot{\theta} = 0. \quad (\because \frac{7}{5}a \neq 0)$$

Integrating

$$\dot{\theta} = \text{constant}$$

$$\dot{\theta} = \frac{2\omega}{7}.$$

8.(c) → A Source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of source is

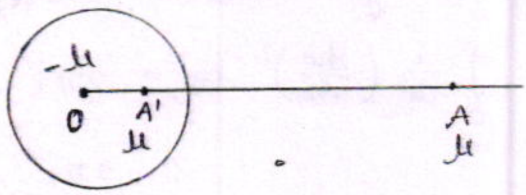
$$2\pi\rho\mu^2 a^2 / r(r^2-a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure?

Sol^n: ⇒ Let X & Y be the components of the required force. then we have to prove that

$$\sqrt{X^2+Y^2} = \frac{2\pi\rho\mu^2 a^2}{r(r^2-a^2)}$$

$\Rightarrow r > a$



By Blasius theorem,

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

where C represents the boundary of the disc.

Since $2\pi\rho\mu$ represents the mass of the fluid emitted at A hence strength of the sources is μ.

The image of source $+\mu$ at A ($OA = r$) is a source $+\mu$ at the inverse point A' such that $OA \cdot OA' = a^2$ and sink $-\mu$ at O.

Then $OA' = \frac{a^2}{r} = r'$, (say).

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to the object system and its image system with no rigid boundary. Hence.

$w = -\mu \log(z-r) - \mu \log(z-r') + \mu \log(z-0)$

$\frac{dw}{dz} = -\mu\left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z}\right]$

$\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{(z-r)^2} + \frac{1}{(z-r')^2} + \frac{1}{z^2} + \frac{2}{(z-r)(z-r')} - \frac{2}{z(z-r')} - \frac{2}{z(z-r)}$

The function $\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2$ has poles $z=0$ and $z=r'$ within C. Residue at $z=0$ is the sum of coefficients of $\frac{1}{z}$ which is equal to

$$\left[-\frac{2}{z-r'} - \frac{2}{z-r}\right]_{z=0} = 2\left(\frac{1}{r'} + \frac{1}{r}\right)$$

Residue at $z=r'$ = sum of coefficients of $\frac{1}{z-r'}$

$$= \frac{2}{r'-r} - \frac{2}{r'} + \frac{2}{r'} + \frac{2}{r} = \frac{2r}{(r'-r)r} = \frac{2a^2}{(a^2-r^2)r}$$

By Cauchy's residue theorem,

$$\int_C \frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 dz = 2\pi i \,.\, (\text{sum of residues within C})$$

$$= 2\pi i \frac{2a^2}{(a^2-r^2)r}$$

we have seen that

$$X - iY = \frac{i\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{i\rho}{2}\cdot\frac{2\pi i\, 2a^2}{(a^2-r^2)}\frac{\mu^2}{r} = \frac{2\pi a^2\mu^2\rho}{r(r^2-a^2)}$$

$$\Rightarrow X = \frac{2a^2\pi\mu^2\rho}{r(r^2-a^2)}, \quad Y=0$$

$$\Rightarrow \sqrt{X^2+Y^2} = \frac{2\pi a^2\mu^2\rho}{r(r^2-a^2)}$$

This also declares that the force is purely along $\overrightarrow{OA}$, the disc will be urged to move along OA. Also the cylinder is attacked towards the source, and sketch of the streamlines reveals that the pressure is greater on the opposite side of the disc than that of the source.