MAINS TEST SERIES - 2021

TEST -3, Paper-I, Answer Key

ODE, DYNAMICS & STATICS AND VECTOR ANALYSIS

1(a) solve $\frac{dy}{dx} = (x+y+4)/(x-y-6)$

sol^n: Given $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$ —— (1)

Let $x = X+h$, $y = Y+k$

so that $\frac{dy}{dx} = \frac{dY}{dX}$ —— (2)

Using (2), (1) reduces to

$$\frac{dy}{dx} = \frac{(X+Y)+(h+k+4)}{(X-Y)+(h-k-6)} \quad \text{—— (3)}$$

we choose h and k, such that $h+k+4=0$ & $h-k-6=0$ —— (4)

solving (4) $h=1, k=-5$

so by (2) $X = x-1$; $Y = y+5$ —— (5)

using (4), (3) reduces to

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} = \frac{1+\frac{Y}{X}}{1-\frac{Y}{X}} \quad \text{—— (6)}$$

putting $Y = XV$ and $\frac{dY}{dX} = v + X\left(\frac{dv}{dX}\right)$

(6) becomes

$$v + X\left(\frac{dv}{dX}\right) = \frac{1+v}{1-v}$$

$$\Rightarrow \frac{dX}{X} = \frac{1-v}{1+v^2}\,dv = \frac{dv}{1+v^2} - \frac{v}{1+v^2}\,dv$$

By Integrating $\Rightarrow \log x = \tan^{-1} v - \frac{1}{2}\log(1+v^2) + \frac{1}{2}\log c$

$\Rightarrow 2\log x + \log\left(1+\frac{y^2}{x^2}\right) - \log c = 2\tan^{-1}\frac{y}{x}$

as $v = \frac{y}{x}$

$\Rightarrow \log x^2 + \log\left(\frac{x^2+y^2}{c}\right) = 2\tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow x^2 + y^2 = ce^{2\tan^{-1}\left(\frac{y}{x}\right)}$

$\Rightarrow \boxed{(x-1)^2 + (y+5)^2 = ce^{-2\tan^{-1}\left(\frac{y+5}{x-1}\right)}}$

c being an arbitrary constant.

(1) (b) Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$

Sol^n Complementry Part :- $(D^2+5D+6)y=0$

Putting $D=m \Rightarrow m^2+5m+6=0$

$$m^2+2m+3m+6=0$$

$$(m+2)(m+3)=0$$

$$\Rightarrow m=-2,-3$$

$$CF = c_1 e^{-2x} + c_2 e^{-3x}$$

Particular Integral (PI)

$$PI = \frac{1}{(D^2+5D+6)} e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \frac{1}{(D+2)(D+3)} e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \frac{1}{(D+3)} e^{-2x}\int e^{2x}e^{-2x}\sec^2 x(1+2\tan x)\,dx$$

$$= \frac{1}{(D+3)} e^{-2x}\int \sec^2 x(1+2\tan x)\,dx$$

$\tan x = t \rightarrow dt = \sec^2 x\,dx$

$$= \frac{1}{D+3} e^{-2x}\int (1+2t)\,dt = \frac{1}{(D+3)} e^{-2x}(t+t^2)$$

$$= \frac{1}{(D+3)} e^{-2x}(\tan x+\tan^2 x)$$

$$= e^{-3x}\int e^{3x}e^{-2x}(\tan x+\sec^2 x-1)\,dx$$

$$= e^{-3x}\left[\int e^x\tan x\,dx + \int e^x\sec^2 x\,dx - \int e^x\,dx\right]$$

$$= e^{-3x}\left[\tan x \int e^{x}\,dx - \int \cancel{\sec^2 x\, e^{x}\,dx} + \int e^{x} \cancel{\sec^2 x\,dx} - e^{x}\right]$$

$$= e^{-3x}\left[\tan x - 1\right]e^{x} = e^{-2x}(\tan x - 1)$$

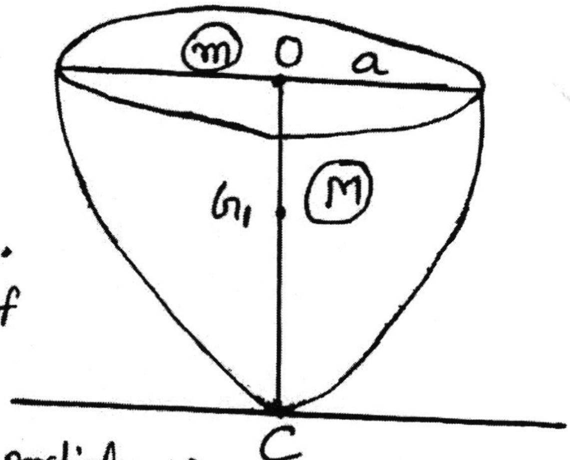
Therefore -

$$y(x) = CF + PI$$

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1)$$

1(C). A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.

Sol: C is the point of contact of the hemisphere and the plane and O is the centre of the base of the hemisphere. Let M be the mass of the hemisphere and



a be the radius. A particle of mass m is placed at O. The mass M of the hemisphere acts at G_1 where $OG_1 = 3a/8$.

If h be the height of the centre of gravity of the combined body consisting of the hemisphere and the mass m above the point of contact C, then

$$h = \frac{M \cdot \frac{5}{8}a + m \cdot a}{M + m}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = a$;

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = \infty$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

i.e., $\frac{1}{h} > \frac{1}{a} + \frac{1}{\infty}$

i.e., $\frac{1}{h} > \frac{1}{a}$

i.e., $h < a$

i.e., $\frac{\frac{5}{8}aM + am}{M+m} < a$

i.e., $\frac{5}{8}aM + am < aM + am$

i.e., $\frac{5}{8}aM < aM$

i.e., $\frac{5}{8}a < a$, which is so whatever may be the value of m. Hence for any value of m, the equilibrium is stable.

(1)(d) Find the values of constant λ and μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$, $4x^2y + z^3 = 4$ intersect orthogonally at the point $(1,-1,2)$.

Soln Let $\hat{n}_1$ and $\hat{n}_2$ are unit normal vector for surface $S_1=0$ and $S_2=0$ respectively.

where $S_1 = \lambda x^2 - \mu yz - (\lambda+2)x = 0$

$S_2 = 4x^2y + z^3 - 4 = 0$

$\vec{n}_1 = \nabla S_1 = [2\lambda x - (\lambda+2)]\hat{i} - \mu z\hat{j} - \mu y\hat{k}$

$\vec{n}_1$ at $(1,-1,2) = [2\lambda - \lambda - 2]\hat{i} - 2\mu\hat{j} + \mu\hat{k}$

$= (\lambda-2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}$

$\vec{n}_2 = \nabla S_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$\vec{n}_2$ at $(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$

For orthogonally at $(1,-1,2) \to$

$\vec{n}_1 \cdot \vec{n}_2 = 0$

$(\lambda-2)(-8) + (-2\mu)4 + \mu(12) = 0$

$\Rightarrow 2(\lambda-2) + \mu - 3\mu = 0 \Rightarrow \boxed{\mu = 2\lambda - 4}$

Point $(1,-1,2)$ should also lie on $S_1=0$ & $S_2=0 \Rightarrow \lambda + 2\mu - (\lambda+2) = 0 \Rightarrow \boxed{\mu = 1}$

Therefore $\lambda = \frac{\mu+4}{2} = \frac{1+4}{2} \Rightarrow \boxed{\lambda = 5/2}$

Required values of $\boxed{\lambda = 5/2, \mu = 1}$

1(e). Find $\int_C F \cdot dr$ on every path between $(0,0,1)$ and $(1, \pi/4, 2)$ where $F = (2xyz^2, (x^2z^2 + z\cos yz), (2x^2yz + y\cos yz))$

Sol^n: we have $F = 2xyz^2\,\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}$

$$\therefore \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)\hat{i} - (4xyz - 4xyz)\hat{j} + (2xz^2 - 2xz^2)\hat{k} = 0$$

$\therefore$ the given line integral is independent of path in space.

Let $F = \nabla\phi$,

Then $\frac{\partial \phi}{\partial x} = 2xyz^2$ hence $\phi = x^2yz^2 + f_1(y,z)$ — ①

$\frac{\partial \phi}{\partial y} = x^2z^2 + z\cos yz$ hence $\phi = x^2z^2y + \sin yz + f_2(x,z)$ — ②

$\frac{\partial \phi}{\partial z} = 2x^2yz + y\cos yz$ hence $\phi = x^2yz^2 + \sin yz + f_3(x,y)$ — ③

①, ②, ③ each represents ϕ. These agree if we choose

$f_1(y,z) = \sin yz$, $f_2(x,z) = 0$, $f_3(x,y) = 0$

$\therefore \phi = x^2yz^2 + \sin yz$ to which may be added any constant.

The given line integral is

$$\int_C d(x^2yz^2 + \sin yz) = \left[x^2yz^2 + \sin yz\right]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$= \pi + \sin \tfrac{\pi}{2}$$

$$= \pi + 1$$

(2)(a)(i) Solve $\sin x\left(\frac{dy}{dx}\right)+3y=\cos x$

Sol^n $\frac{dy}{dx}+\frac{3}{\sin x}y=\frac{\cos x}{\sin x}$ $\left[\frac{dy}{dx}+P(x)y=Q(x)\right]$

$IF = e^{\int P(x)dx}$ where $P(x)=\frac{3}{\sin x}$

$IF = e^{\int \frac{3}{\sin x}dx} = e^{3\ln\tan x/2}$

$= \tan^3 x/2$

General solution –

$y(IF) = \int Q.IF\,dx$ where $Q(x)=\frac{\cos x}{\sin x}$

$y.\tan^3 x/2 = \int \frac{\cos x}{\sin x}.\tan^3 x/2\,dx$

$= \int \frac{(2\cos^2 x/2-1)}{2\sin x/2\cos x/2}\tan^3 x/2\,dx$

$= \int\left(\tan^2 x/2 - \frac{1}{2}\tan^2 x/2\sec^2 x/2\right)dx$

$= \int(\sec^2 x/2-1)\,dx - \int \frac{1}{2}\tan^2 x/2\sec^2 x/2\,dx$

$= \frac{\tan x/2}{(1/2)} - x - \int t^2 dt$ [where $t=\tan x/2$]

$= 2\tan x/2 - x - \frac{1}{3}\tan^3 x/2 + c$

$\Rightarrow \boxed{y\tan^3 x/2 = 2\tan x/2 - x - \frac{1}{3}\tan^3 x/2 + c}$

2(a)(ii) Find the orthogonal trajectories of family of parabolas $y^2 = 4a(x+a)$, where a is parameter.

Sol^n: Given $y^2 = 4a(x+a)$, with a as parameter. —— ①

Differentiating ① $2y\left(\frac{dy}{dx}\right) = 4a$

$$\Rightarrow a = \frac{y}{2}\frac{dy}{dx} \quad \text{—— ②}$$

Eliminating a from ① & ②, we have

$$y^2 = 2y\left(\frac{dy}{dx}\right)\left\{x + \left(\frac{y}{2}\right)\left(\frac{dy}{dx}\right)\right\}.$$

$$y = 2x\left(\frac{dy}{dx}\right) + y\left(\frac{dy}{dx}\right)^2 \quad \text{—— ③}$$

which is differential equation of ①, Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ③

the differential equation of the required orthogonal trajectories is

$$y = -2x\frac{dx}{dy} + y\left(-\frac{dx}{dy}\right)^2$$

$$\Rightarrow y = -\frac{2x}{(dy/dx)} + \frac{y}{(dy/dx)^2}$$

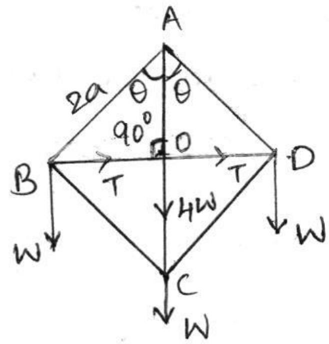
$$\Rightarrow y\left(\frac{dy}{dx}\right)^2 = -2x\left(\frac{dy}{dx}\right) + y$$

$$\Rightarrow y = 2x\left(\frac{dy}{dx}\right) + y\left(\frac{dy}{dx}\right)^2 \quad \text{—— ④}$$

which is the same as the differential equation ③ of the given system ①. Hence, the system of parabolas ① it self orthogonal, i.e. each member of the given system of parabolas intersects its own members orthogonally.

2(b) A square framework, formed of uniform heavy rods of equal weight W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.

Sol'n: ABCD is a square framework formed of four rods each of weight W & say of length $2a$. It is suspended from the point A and a weight W is suspended from each of the three lower corners B, C & D. A light rod along the horizontal diagonal BD prevents the system from collapsing. Let T be the thrust in the rod BD. The total weight 4W of the rods AB, BC, CD and DA can be taken as acting at O.



To find T we shall have to give the system a displacement in which BD must change. So replace the rod BD by two equal and opposite forces T and assume that $\angle BAC = \theta = \angle CAD$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta + \delta\theta$. The point A remains fixed and the points B, O, D and C change. The lengths of the rods AB, BC, CD and DA donot change while the length BD changes.

we have $BD = 2BO = 2AB\sin\theta = 4a\sin\theta$

the depth of each of the points B, C and D below the fixed point A

$$= AO = 2a\cos\theta$$

and the depth of C below $A = 2AO - 4a\cos\theta$.

By the principle of virtual work, we have

$$T\delta(4a\sin\theta) + 4W\delta(2a\cos\theta) + 2W\delta(2a\cos\theta) + W\delta(4a\cos\theta) = 0$$

$$\Rightarrow 4aT\cos\theta\,\delta\theta - 8aW\sin\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta = 0$$

$$\Rightarrow 4a[T\cos\theta - 4W\sin\theta]\delta\theta = 0$$

$$\Rightarrow T\cos\theta - 4W\sin\theta = 0 \qquad [\because \delta\theta \neq 0]$$

$$\Rightarrow T = 4W\tan\theta$$

But in the position of equilibrium $\theta = 45^\circ$

$\therefore T = 4W\tan 45^\circ = 4W =$ the total weight of the four rods.

(2)(c)(i) → Determine the constants a & b such that the curl of vector $\vec{A} = (2xy+3yz)\hat{i} + (x^2 - axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero

Sol^n: Given that curl $\vec{A} = 0$

$$\text{Curl}\,\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-3yz & x^2-axz-4z^2 & -3xy-byz \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}\left(-3x - bz + ax + 8z\right) - \hat{j}\left(-3y + 3y\right) + \hat{k}\left(\cancel{2x} - az - \cancel{2x} + 3z\right) = 0$$

$$\Rightarrow \hat{i}\left[x(a-3) + z(8-b)\right] - \hat{j}(0) + \hat{k}\left(-a+3\right)z = 0$$

i.e., $a-3=0$ and $8-b=0$ // $-a+3=0$

$$\Rightarrow a = 3,\ b = 8.$$

2(c)(ii) Show that

$\vec{V}(x,y,z) = 2xyz\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k}$

is irrotational and find a scalar function $u(x,y,z)$ such that

$\vec{v} = \text{grad}(u)$.

Sol^n $\text{curl}\,\vec{v} = \nabla \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z+2y & x^2y \end{vmatrix}$$

$$= \hat{i}(x^2 - x^2) - \hat{j}(2xy - 2xy) + \hat{k}(2xz - 2xz) = 0$$

Hence $\vec{V}$ is irrotational

So there exist $u(x,y,z)$ such that

$$\vec{V} = \text{grad}(u) = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$$

comparing $\frac{\partial u}{\partial x} = 2xyz$ —①

$\frac{\partial u}{\partial y} = (x^2z + 2y)$ —②, $\frac{\partial u}{\partial z} = x^2y$ —③

Integrating ① ②, ③ ⇒

①⇒ $u(x,y,z) = x^2yz + f_1(y,z)$

②⇒ $u(x,y,z) = x^2yz + y^2 + f_2(x,z)$

③⇒ $u(x,y,z) = x^2yz + f_3(x,y)$

Therefore $u(x,y,z) = x^2yz + y^2$

3.(a) Reduce the equation $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$, $(x>0)$ into the normal form and hence solve it.

Solution:

The given equation is $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$.

Rewriting it, we get,

$$y'' - \frac{2}{x}(1+x)y' + \frac{2(1+x)}{x^2}y = x \quad \text{—— (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2(1+x)}{x^2}, \quad R = x. \quad \text{—— (2)}$$

We choose $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -\frac{2}{x}(1+x)dx} = e^{\int\left(\frac{1}{x}+1\right)dx}$

$$= e^{\ln x + x} = x \cdot e^x. \quad \text{—— (3)}$$

Let the required general solution be $y = uv$ —— (4)

Then v is given by the normal form, —— (5)

$$\frac{d^2v}{dx^2} + Iv = S,$$

where,

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{2(1+x)}{x^2} - \frac{1}{4}\times\frac{4}{x^2}(1+x)^2 - \frac{1}{2}\times\frac{2}{x^2}$$

$$= \cancel{\frac{2}{x^2}} + \cancel{\frac{2}{x}} - 1 - \cancel{\frac{1}{x^2}} - \cancel{\frac{2}{x}} - \cancel{\frac{1}{x^2}}$$

$$= -1$$

$$S = \frac{R}{u} = \frac{x}{x \cdot e^{x}} = e^{-x}$$

Then (5) becomes, $\frac{d^2V}{dx^2} - V = e^{-x}$

$$\Rightarrow (D^2-1)v = e^{-x} \quad \text{where } D \equiv d/dx \quad \text{——— (6)}$$

The A.E. of (6) is given by $D^2-1=0$
$\Rightarrow D = \pm 1$

$$\therefore \text{C.F.} = c_1 e^{x} + c_2 e^{-x} \quad \text{——— (7).}$$

$$\text{P.I.} = \frac{1}{(D+1)(D-1)} \cdot e^{-x} = \frac{x}{1L} \times -\frac{1}{2} e^{-x} = \frac{-x \cdot e^{-x}}{2}$$

$$\therefore v = c_1 e^{x} + c_2 e^{-x} - \frac{x \cdot e^{-x}}{2} \quad \text{——— (8)}$$

$\therefore$ from (4), (3) and (8), we have

$$\boxed{y = c_1 x e^{2x} + c_2 x - \frac{x^2}{2}}$$ which is the required

solution.

Hence, the result

3(b) A particle moves in a plane under a central force which varies inversely as the square of the distance from the fixed point, find the orbit.

sol^n: We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \quad \text{——— (1)}$$

where P is the central acceleration assumed to be attractive.

Hence $P = \mu/r^2$, putting $P = \mu/r^2$ in (1)

we get $\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$

$$\Rightarrow \frac{h^2}{p^3}dp = \frac{\mu}{r^2}dr$$

$$\Rightarrow -2\frac{h^2}{p^3}dp = -\frac{2\mu}{r^2}dr$$

Integrating bothsides, we get

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad \text{——— (2)}$$

Let $v = v_0$ when $r = r_0$.

Then $v_0^2 = \frac{2\mu}{r_0} + C$ (or) $C = v_0^2 - \frac{2\mu}{r_0}$

Putting this value of C in (2), the pedal equation of the central orbit is

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0} \quad \text{——— (3)}$$

Case 1: Let $v_0^2 = \frac{2\mu}{r_0}$. Then the equation (3) becomes

$\frac{h^2}{p^2} = \frac{2\mu}{r}$ which is of the form $p^2 = ar$.

This is the pedal equation of a parabola referred to focus as pole. Hence in this case the orbit is a parabola with centre of force at the focus.

<u>Case 2</u>: Let $v_0^2 < \frac{2\mu}{r_0}$. In this case the equation ③ reduces to the form

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

This is the pedal equation of an ellipse referred to a focus as pole. Hence in this case the orbit is an ellipse with centre of force at its focus.

<u>Case 3</u>: Let $v_0^2 > \frac{2\mu}{r_0}$. In this case the equation ③ reduces to the form

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1$$

This is the pedal equation of a hyperbola referred to a focus as pole.

It represents that branch of the hyperbola which is nearer to the focus taken as pole.

Hence we conclude that under inverse square law the central orbit is always a conic with centre of force at the focus.

4(a)(ii) Find $L\{F(t)\}$ if $F(t) = \begin{cases} 1 & 0 < t < 2 \\ t & , t > 2 \end{cases}$

Soln

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$L\{F(t)\} = \int_0^2 e^{-st} dt + \int_2^\infty t e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[t\frac{e^{-st}}{-s} - \int \frac{e^{-st}}{(-s)}\right]_2^\infty$$

$$= -\frac{1}{s}\left[e^{-2s} - 1\right] + \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_2^\infty$$

$$= \frac{1-e^{-2s}}{s} + \left[-0 - 0 - \left(-\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s}\right)\right]$$

$$= \frac{1-e^{-2s}}{s} + \frac{2}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$$

$$= \frac{1}{s} + \frac{1}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}$$

$$\Rightarrow \boxed{L\{F(t)\} = \frac{1}{s} + \frac{1}{s}e^{-2s} + \frac{1}{s^2}e^{-2s}}$$

which is required solution.

4(a)(ii) Evaluate $L\{F(t)\}$ if

$$F(t) = \begin{cases} (t-1)^2 & t > 1 \\ 0 & 0 < t < 1 \end{cases}$$

Soln we know $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

Using given $F(t)$ —

$$L\{F(t)\} = \int_0^1 0\, e^{-st}\,dt + \int_1^\infty (t-1)^2 e^{-st}\,dt$$

$$= \left[(t-1)^2 \frac{e^{-st}}{(-s)} + \frac{1}{s}\int 2(t-1)\, e^{-st}\right]_1^\infty$$

$$= \left[(t-1)^2 \frac{e^{-st}}{(-s)} + \frac{2}{s}\left\{(t-1)\frac{e^{-st}}{(-s)} - \int \frac{e^{-st}}{-s}\right\}\right]_1^\infty$$

$$= \left[-\frac{(t-1)^2}{s} e^{-st} - \frac{2}{s^2}(t-1)e^{-st} - \frac{2}{s^3} e^{-st}\right]_1^\infty$$

$$= \left[-0 - 0 - 0 + 0 + 0 + \frac{2}{s^3} e^{-s}\right]$$

$$= \cancel{2/s^3 e^{-st}}\ \frac{2}{s^3} e^{-s}$$

Therefore $\boxed{L\{F(t)\} = \frac{2}{s^3} e^{-s}}$

4(a)(ii) Evaluate $L\{(t^2-3t+2)\sin 3t\}$

Sol^n We know $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$L\{(t^2-3t+2)\sin 3t\}$

Here $F(t) = \sin 3t$

$L\{F(t)\} = \frac{3}{s^2+9} = f(s)$

$L\{t^2 F(t)\} = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$

$= \frac{d}{ds}\left[\frac{(s^2+9)\cdot(0) - 3(2s)}{(s^2+9)^2}\right] = \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$

$= (-6)\frac{\left[(s^2+9)^2\cdot 1 - s\{2(s^2+9)\cdot 2s\}\right]}{(s^2+9)^4}$

$= \frac{(-6)(s^2+9)\left[s^2+9-4s^2\right]}{(s^2+9)^4} = \frac{6(3s^2-9)}{(s^2+9)^3}$

$L\{tF(t)\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$

Therefore –

$L\{(t^2-3t+2)\sin 3t\}$

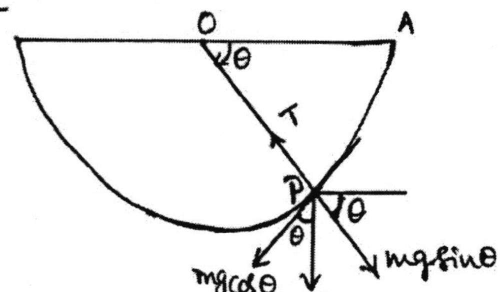
$= L(t^2\sin 3t) - 3L(t\sin 3t) + 2L(\sin 3t)$

$= \frac{6(3s^2-9)}{(s^2+9)^3} - \frac{18s}{(s^2+9)^2} + \frac{6}{(s^2+9)}$

4.(b) A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$.

Sol^n: Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Initially let the string be horizontal in the position OA such that $OA = l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$.

The forces acting on the particle at P are : (i) its weight mg acting vertically downwards and (ii) the tension T in the string along PO.



$\therefore$ the equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2} = mg\cos\theta \quad \text{——— (1)}$$

and $$m\frac{v^2}{l} = T - mg\sin\theta \quad \text{——— (2)}.$$

Also $$s = l\theta \quad \text{——— (3)}$$

from (1) and (3), we have $l\dfrac{d^2\theta}{dt^2} = g\cos\theta$

multiplying both sides by $2l(d\theta/dt)$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\sin\theta + A$$

But initially at the point A, $\theta=0$, $v=0$ $\therefore A=0$

$$\therefore v^2 = 2lg\sin\theta \quad \text{———} \quad (4)$$

The resultant acceleration of the particle at P

$$= \sqrt{(\text{Tangential accel.})^2 + (\text{Normal accel.})^2}$$

$$= \sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{l}\right)^2\right]} \quad \left[\because \text{Normal accel} = \frac{v^2}{\rho} = \frac{v^2}{l}\right]$$

$$= \sqrt{\left[(g\cos\theta)^2 + \left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$= g\sqrt{[1-\sin^2\theta + 4\sin^2\theta]}$$

$$= g\sqrt{(1+3\sin^2\theta)}$$

4(c) The acceleration of a particle at time t is given by -

$\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$

If the velocity and displacement $\vec{r}$ be zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any point t.

Sol^n Given $\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$ — ①

We know velocity $(\vec{v}) = \int \vec{a}\,dt$

$\Rightarrow \vec{v} = \frac{18\sin 3t}{3}\,\hat{i} + \frac{8}{2}\cos 2t\,\hat{j} + 3t^2\hat{k} + c_1$ — ②

and displacement $\vec{r} = \int \vec{v}\,dt$

$\vec{r} = -\frac{6\cos 3t}{3}\,\hat{i} + 4\frac{\sin 2t}{2}\,\hat{j} + t^3\hat{k} + c_1 t + c_2$ — ③

Given $\vec{v} = 0$ and $\vec{r} = 0$ at $t=0$

So using equation ② and ③ :-

$\Rightarrow 0 = 6\sin 0\,\hat{i} + 4\cos 0\,\hat{j} + 3\times 0\,\hat{k} + c_1$

$\Rightarrow c_1 = -4\hat{j}$

and Eq^n ③ :- $0 = -2\cos 0\,\hat{i} + 2\sin 0\,\hat{j} + 0\hat{k} + c_1\times 0 + c_2$

$\Rightarrow c_2 = 2\hat{i}$

Putting value of c_1 & c_2 :-

$\vec{v} = 6\sin 3t\,\hat{i} + (4\cos 2t - 4)\,\hat{j} + 3t^2\hat{k}$

$\vec{r} = (-2\cos 3t + 2)\,\hat{i} + 2\sin 2t\,\hat{j} + t^3\hat{k} - 4t\,\hat{j}$

4(d). Use the divergence Theorem to evaluate $\iint_S \vec{F}\cdot d\vec{s}$ when $\vec{F} = yx^2\vec{i} + (xy^2 - 3z^4)\vec{j} + (x^3+y^2)\vec{k}$ and S is the surface of the sphere of radius 4 with $z \le 0$ and $y \le 0$.

Sol'n: Let us call the region we want to do our triple integral over E, the region enclosed by all these surfaces. E is a portion of a sphere and so we are going to want to use spherical coordinates for the integration.

i.e. $x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi.$

The spherical limits are

$\pi \le \theta \le 2\pi,\ \pi/2 \le \varphi \le \pi,\ 0 \le \rho \le 4.$

Now,

$$\operatorname{div} F = \nabla\cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(yx^2, xy^2 - 3z^4, x^3+y^2\right)$$

$$= 4xy.$$

Thus, by Gauss Divergence theorem

$$\iint_S F\cdot dS = \iiint_E \operatorname{div} F\, dV$$

$$= \int_{\pi/2}^{\pi}\int_0^4\int_{\pi}^{2\pi} 4(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)\cdot\rho^2\sin\varphi\, d\theta\, d\varphi\, d\rho$$

$$= \int_{\pi/2}^{\pi}\int_0^4\int_{\pi}^{2\pi} 4\rho^4\sin^3\varphi\cos\theta\sin\theta\, d\theta\, d\varphi\, d\rho$$

let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$= \int_{\pi/2}^{\pi} \int_{0}^{4} 4\rho^4 \sin^3\varphi - \frac{1}{2}\cos^2\theta \Big]_{\pi}^{2\pi} d\varphi\, d\rho$$

$$= \int_{\pi/2}^{\pi} \int_{0}^{4} 4\rho^4 \sin^3\varphi \cdot 0\, d\varphi\, d\rho$$

$= 0.$

Section B.

(5)(a) Solve $xp^2 - 2yp + x + 2y = 0$

Sol^n Given $xp^2 - 2yp + x + 2y = 0$

$\Rightarrow (p^2+1)x + 2(1-p)y = 0$

$\Rightarrow y = \frac{-(p^2+1)}{2(1-p)} x$

$\Rightarrow y = \frac{(p^2+1)}{2(p-1)} x$

Differentiating w.r.t x –

$$\frac{dy}{dx} = \frac{p^2+1}{2(p-1)} + \frac{x}{2}\left[\frac{(p-1)2p - (p^2+1)}{(p-1)^2}\right]\frac{dp}{dx}$$

$$p - \frac{(p^2+1)}{2(p-1)} = \frac{x}{2}\frac{(p^2-2p-1)}{(p-1)^2}\frac{dp}{dx} \quad \left[\because \frac{dy}{dx} = p\right]$$

$$\Rightarrow \frac{(p^2-2p-1)}{2(p-1)} = \frac{x}{2}\frac{(p^2-2p-1)}{(p-1)^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{(p-1)} = \frac{dx}{x}$$

Integrating $\rightarrow \log(p-1) = \log x + \log c_1$ where c_1 is constant

$\Rightarrow p = 1 + c_1 x$

$\Rightarrow \frac{dy}{dx} = 1 + c_1 x \Rightarrow \boxed{y(x) = \frac{c_1 x^2}{2} + x + c_2}$

which is required sol^n, c_1, c_2 are integral constant.

(5)(b)(i) Evaluate $L^{-1}\left\{\frac{se^{-2\pi s/3}}{s^2+9}\right\}$

Sol^n Let $f(s) = \frac{s}{s^2+9}$

$\therefore F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{s}{s^2+9}\right\}$

$\Rightarrow F(t) = \cos 3t$

By second shifting theorem, we have –

$$L^{-1}\{e^{-s\pi} f(s)\} = \begin{cases} F(t-\pi), & t>\pi \\ 0, & t<\pi \end{cases}$$

or

$$L^{-1}\left\{\frac{e^{-2\pi/3 s}\, s}{s^2+9}\right\} = \begin{cases} F(t-2\pi/3) & t>2\pi/3 \\ 0, & t<2\pi/3 \end{cases}$$

$$= \begin{cases} \cos 3(t-2\pi/3) & t>2\pi/3 \\ 0 & t<2\pi/3 \end{cases}$$

$$= \begin{cases} \cos 3t & t>2\pi/3 \\ 0 & t<2\pi/3 \end{cases} = \cos 3t\, H(t-2\pi/3)$$

where $H(t-2\pi/3)$ is the Heaviside unit step function.

(5)(b)(ii) Find $L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\}$

Sol^n Let $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = f(t)$

$\log\left(1+\frac{1}{s^2}\right) = L\{f(t)\} = F(s)$

Now $\frac{d}{ds}F(s) = \frac{d}{ds}\left[\log(1+s^2) - \log s^2\right]$

$= \frac{2s}{1+s^2} - \frac{2}{s} = -2\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

$L^{-1}\left[\frac{d}{ds}F(s)\right] = -2\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{s}{s^2+1}\right)\right]$

$= -2(1-\cos t)$

i.e. $\Rightarrow$ $-t\,f(t) = -2(1-\cos t)$

$\Rightarrow f(t) = \frac{2(1-\cos t)}{t}$

$\Rightarrow L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{2(1-\cos t)}{t}$

$\therefore L^{-1}\left\{\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)\right\} = 2\int_0^t \frac{1-\cos t}{t}\,dt$

which is required solution

5(c) One end of a heavy uniform rod AB can slide along a fixed rough horizontal rod AC to which it is attached by a ring. B and C are joined by a string. When the rod is just on the point of slipping, the string is ⊥lar to the rod which makes an angle of 'α' with the vertical, Prove that the co-efficient of friction is given by

$$\mu = \frac{\tan\alpha}{2+\tan^2\alpha}.$$

Sol^n: Let the lines of action of the weight w and tension T in the string BC meet in O. Then the resultant reaction R_1 at A also passes through O.

By "m-n theorem" in ΔAOB,

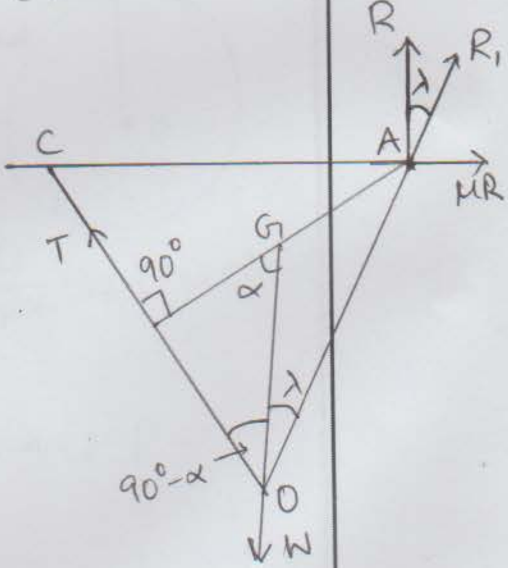
we have $(a+a)\cot\alpha = a\cot\lambda - a\cot(90^\circ-\alpha)$

$$\Rightarrow 2\cot\alpha = \cot\lambda - \tan\alpha$$

$$\Rightarrow \cot\lambda = 2\cot\alpha + \tan\alpha$$

$$= \frac{2}{\tan\alpha} + \tan\alpha$$

$$= \frac{2+\tan^2\alpha}{\tan\alpha}$$



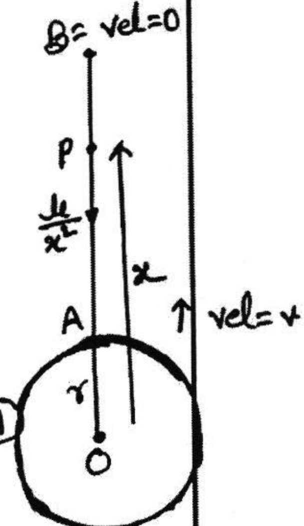
$$\therefore \mu = \tan\lambda = \frac{\tan\alpha}{2+\tan^2\alpha}$$

Q. 5.d) → A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]$$

where a is the radius of the earth.

Sol: Let O be the centre of the earth and A the point of projection on the surface.



If P is the position of the particle at any time t, such that $OP = x$, then the acceleration at $P = \mu/x^2$ directed towards O.

∴ the equation of motion of the particle P is $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$ —— (1)

But at the point A, on the surface of the earth, $x = a$ and $\frac{d^2x}{dt^2} = -g$

$\therefore -g = -\mu/a^2$ or $\mu = a^2 g$.

$$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}$$

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} + c$, where c is constant.

But when $x \to \infty$, $dx/dt \to 0$. $\therefore c = 0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} \text{ or } \frac{dx}{dt} = \frac{a\sqrt{(2g)}}{\sqrt{g}} \quad \text{—— (2)}$$

[Here +ive sign is taken because the particle is moving in the direction of x increasing.]

Seperating the variables, we have

$$dt = \frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx$$

Integrating between the limits $x=a$ to $x=a+h$, the required time t to reach a height h is given by

$$t = \frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx$$

$$= \frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$= \frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2} - a^{3/2}\right]$$

$$= \frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right]$$

5(e). Find the direction in which the directional derivative of $\phi(x,y) = \frac{x^2+y^2}{xy}$ at $(1,1)$ is zero and hence findout component of velocity of the vector $\bar{r} = (t^3+1)\hat{i} + t^2\hat{j}$ in the same direction at $t=1$.

Sol^n: Directional derivative

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x^2+y^2}{xy}\right)$$

$$= \hat{i}\left[\frac{x.y.2x - (x^2+y^2)y}{x^2y^2}\right] + \hat{j}\left[\frac{xy.2y - x(y^2+x^2)}{x^2y^2}\right]$$

$$= \hat{i}\left[\frac{x^2y - y^3}{x^2y^2}\right] + \hat{j}\left[\frac{xy^2 - x^3}{x^2y^2}\right]$$

Directional Derivative at $(1,1) = \hat{i}0 + \hat{j}0 = 0$

Since $(\nabla\phi)_{(1,1)} = 0$, the directional derivative of ϕ at $(1,1)$ is zero in any direction.

Again $\bar{r} = (t^3+1)\hat{i} + t^2\hat{j}$

velocity, $\bar{v} = \frac{d\bar{r}}{dt} = 3t^2\hat{i} + 2t\hat{j}$

velocity at $t=1$ is $= 3\hat{i} + 2\hat{j}$

The component of velocity in the same direction of velocity

$$= (3\hat{i} + 2\hat{j})\left(\frac{3\hat{i}+2\hat{j}}{\sqrt{9+4}}\right)$$

$$= \frac{9+4}{\sqrt{13}} = \sqrt{13}$$

(6)(a) Solve $(xy\sin(xy)+\cos xy)y\,dx+(xy\sin xy-\cos xy)x\,dy=0$

Soln Given equation is the form of -

$M\,dx+N\,dy=0$ where $M=y f_1(x,y)$

& $N=x f_2(x,y)$

So Integrating factor (IF) $=\dfrac{1}{Mx-Ny}$

$Mx-Ny = xy(xy\sin xy+\cos xy)$

$- xy(xy\sin xy-\cos xy)$

$=2xy\cos xy \neq 0$

$IF=\dfrac{1}{2xy\cos xy}$

Multiplying with IF the given Equation -

$$\left(\frac{x y^2\sin xy + y\cos xy}{2xy\cos xy}\right)dx+\left(\frac{x^2y\sin xy - x\cos xy}{2xy\cos xy}\right)dy=0$$

$$\Rightarrow \left(\frac{y}{2}\tan xy+\frac{1}{2x}\right)dx+\left(\frac{x\tan xy}{2}-\frac{1}{2y}\right)dy=0$$

which is exact differential Equation

So solution

$$\int_{y\text{-const}}\left(\frac{y}{2}\tan xy+\frac{1}{2x}\right)dx+\int -\frac{1}{2y}\,dy=C$$

$$\Rightarrow \frac{y}{2}\,\frac{\ln \sec xy}{y} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log a$$

where a is arbitary constant

$$\Rightarrow (\cancel{\sec xy})^{\cancel{1/2}\,\cancel{x}}\ \frac{x\sec xy}{y} = a$$

which is required solution

6(b) Use the method of variation of parameters to solve

$$y'' + y = \frac{1}{1+\sin x}$$

Sol^n Comparing given equation with

$$y'' + Py' + Qy = R$$

$$P = 0, \quad Q = 1, \quad R = \frac{1}{1+\sin x}$$

For complementary part –

$$y'' + y = 0 \Rightarrow (D^2+1)y = 0$$

Solving $D^2+1 = 0$, putting $D = m$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

Therefore –

$y_c = c_1 \cos x + c_2 \sin x$, where c_1, c_2 are arbitrary constants.

By variation of parameter method

Particular Integral (PI) of equation

$$y_p = A(x)\,u(x) + B(x)\,v(x)$$

where $u(x) = \cos x$, $v(x) = \sin x$

$$W = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1 \neq 0$$

$$A(x) = \int -\frac{v(x)R}{W}\,dx$$

$$= -\int \frac{\sin x}{1+\sin x}\,dx = -\int \frac{\sin x + 1 - 1}{1+\sin x}\,dx$$

$$= -\int\left[1 - \frac{1\,(1-\sin x)}{(1+\sin x)(1-\sin x)}\right]dx$$

$$= -\int\left(1 - \frac{1-\sin x}{1-\sin^2 x}\right)dx$$

$$= -\int\left(1 - \frac{1-\sin x}{\cos^2 x}\right)dx = -\int(1 - \sec^2 x + \tan x\sec x)\,dx$$

$$= -(x - \tan x + \sec x\tan x)$$

$$= \tan x - \sec x\tan x - x$$

$$B(x) = \int \frac{u(x)R}{W}\,dx = \int \frac{\cos x}{1+\sin x}\,dx$$

take $\sin x = t \Rightarrow dt = \cos x\,dx$

$$= \int \frac{dt}{1+t} = \log(1+\sin x)$$

General solution $y(x) = y_c + y_p$

$$y(x) = c_1\cos x + c_2\sin x$$
$$+ (\tan x - \sec x\tan x - x)\cos x$$
$$+ \sin x\log(1+\sin x)$$

which is required solution

(6)(d) By using Laplace transform method solve-

$[tD^2+(t-1)D-1]y=0$ if $y(0)=5$, $y(\infty)=0$

Sol^n Given $[tD^2+tD-D-1]y=0$

Taking Laplace both side -

$L[(tD^2+tD-D-1)y]=0$

We know $L\{tf(t)\}=-\frac{d}{ds}\{L\,f(t)\}$

$\Rightarrow -\frac{d}{ds}[s^2\bar{y}-sy(0)-y'(0)]-\frac{d}{ds}[s\bar{y}-y(0)]$

$-(s\bar{y}-y(0))-\bar{y}=0$

where $\bar{y}=L(y(t))$

$\Rightarrow -\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-y(0)\right]-\left[s\frac{d\bar{y}}{ds}+\bar{y}\right]$

$-s\bar{y}+y(0)-\bar{y}=0$

Given $y(0)=5$

$\Rightarrow -(s^2+s)\frac{d\bar{y}}{ds}-(2s+1+s+1)\bar{y}+10=0$

$\Rightarrow -(s^2+s)\frac{d\bar{y}}{ds}-(3s+2)\bar{y}+10=0$

$\Rightarrow \frac{d\bar{y}}{ds}+\frac{(3s+2)}{s^2+s}\bar{y}=\frac{10}{s^2+s}$ —①

Integrating factor (IF)

$$IF = e^{\int \frac{3s+2}{s^2+s} ds} = e^{\int \left(\frac{2}{s} + \frac{1}{s+1}\right) ds}$$

$$\Rightarrow IF = e^{2\log s + \log(s+1)} = (s+1)s^2$$

Solution of equation ① →

$$\bar{y}\, s^2(s+1) = \int s^2(s+1) \times \frac{10}{s(s+1)} ds$$

$$\Rightarrow \bar{y}\, s^2(s+1) = \int 10s\, ds + c$$

$$\Rightarrow \bar{y}\, s^2(s+1) = 5s^2 + c$$

where c is integrating constant

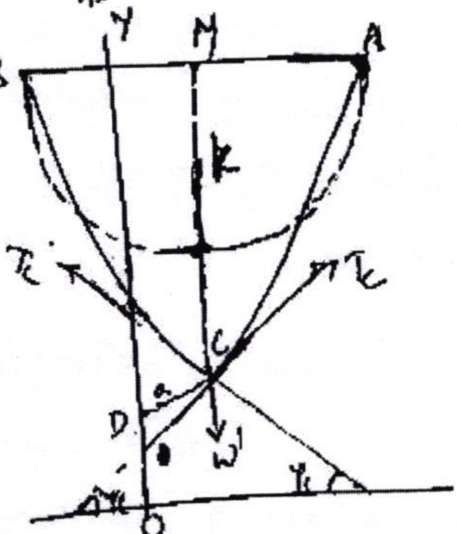
$$\Rightarrow \bar{y} = \frac{c}{(s+1)s^2} + \frac{5}{(s+1)}$$

$$\Rightarrow L(y(t)) = c\left[\frac{1}{(s+1)} - \frac{1}{s} + \frac{1}{s^2}\right] + \frac{5}{(s+1)}$$

taking inverse Laplace →

$$y(t) = c\left[e^{-t} - 1 + t\right] + 5e^{-t}$$

7(a) A uniform chain, of length $2l$ and weight $2w$, is suspended from two point in the horizontal line. A load w is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h. Show that the terminal tension is $\frac{1}{2}w \cdot \frac{h^2+2l^2}{hl}$.



Sol: Let a weight w be suspended at the middle point C of the spring suspended from two point A and B in the same horizontal line.

Let $MC = h$ (depth of the middle point C of the chain below the horizontal line AB).

If w' is the weight per unit length of the chain, then $w' = \frac{2w}{2l} = \frac{w}{l}$.

Let T_c, T_c be the tensions at the point C in the strings CA and CB acting along the tangents at C.

Let the tangents at the point C be inclined at an angle ψ_c to the horizontal.

Resolving vertically the forces acting at C, we have $2T_c \sin\psi_c = w$ —— (1)

Let D be the vertex of the Catenary of

which AC is an arc. If arc DC = a, then from

$T\sin\psi = w's$, we have

$$T_C \sin\psi_C = w'a \quad \text{——— (2)}$$

∴ from (1) and (2), we have

$$2w'a = w \Rightarrow a = \frac{w}{2w'} = \frac{l}{2} \qquad \left(\because w' = \frac{w}{l}\right)$$

Let y_A and y_C be the ordinates of the points A and C respectively.

Then at C, $y = y_C$, $s = \text{arc } DC = a = \frac{l}{2}$

and at A, $y = y_A$, $s = \text{arc } DA = \text{arc } DC + \text{arc } CA$
$= \frac{l}{2} + l = \frac{3l}{2}$.

∴ from $y^2 = s^2 + c^2$, we have

$$y_C^2 = \frac{l^2}{4} + c^2 \text{ and } y_A^2 = \frac{9}{4}l^2 + c^2$$

Subtracting, we have

$$y_A^2 - y_C^2 = 2l^2 \quad \text{——— (3)}$$

But $y_C = y_A - CM = y_A - h$

∴ from (3), we have

$$y_A^2 - (y_A - h)^2 = 2l$$

$$\Rightarrow 2h\, y_A = 2l^2 + h^2$$

$$\Rightarrow y_A = \frac{h^2 + 2l^2}{2h}$$

The terminal tension T_A at A

$$= w' y_A = \frac{w}{l} \cdot \frac{h^2 + 2l^2}{2h}$$

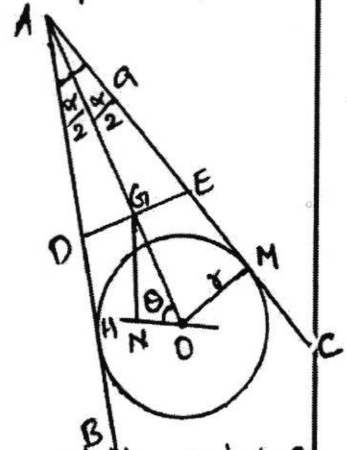
$$= \frac{1}{2} w \left(\frac{h^2 + 2l^2}{hl}\right)$$

:0

7(b). Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable (or) unstable equilibrium according as the length of the rod is

$2a >$ or $< 4r \operatorname{cosec}\alpha$.

Sol^n: Let AB and AC be two rods jointed at A and placed in a vertical plane on a smooth sphere of centre O and radius r. We have $\angle BAC = \alpha$.



Since the rods are tangential to the sphere, therefore $\angle BAO = \angle CAO = \frac{1}{2}\alpha$.

Suppose $AB = AC = 2a$.

If D and E are the middle points of the rods AB & AC, then the combined C.G of the rods is at the middle point G of ED which must be on AO. Suppose the rod AC touches the sphere at M. We have,

$$OM = r,\ AE = a,\ \angle AMO = 90^\circ,\ \angle AGE = 90^\circ.$$

Suppose AO makes an angle θ with the horizontal line OH through the fixed point O. Let z be the height of the C.G of the system above the horizontal through O. Then

$$z = GN = OG \sin\theta = (AO - AG)\sin\theta$$

$$= \left(r \operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\sin\theta$$

$$\therefore \frac{dz}{d\theta} = \left(r\operatorname{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha\right)\cos\theta$$

For the equilibrium of the rods, we must have

$$dz/d\theta = 0.$$

i.e, $(r\,\text{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha)\cos\theta = 0$ i.e. $\cos\theta = 0 \Rightarrow \theta = \pi/2$.

Thus in the position of equilibrium of the rods, the line AO must be vertical.

Also $d^2z/d\theta^2 = -(r\,\text{cosec}\tfrac{1}{2}\alpha - a\cos\tfrac{1}{2}\alpha)\sin\theta$

$= -r\,\text{cosec}\tfrac{1}{2}\alpha + a\cos\tfrac{1}{2}\alpha$, for $\theta = \pi/2$

The equilibrium will be stable (or) unstable according as the height z of the C.G of the system is minimum (or) maximum in the position of equilibrium.

i.e. according as $d^2z/d\theta^2$ is +ve (or) -ve at $\theta = \pi/2$

$\Rightarrow$ according as $a\cos\tfrac{1}{2}\alpha >$ (or) $< r\,\text{cosec}\tfrac{1}{2}\alpha$

$\Rightarrow$ according as $2a >$ (or) $< \dfrac{2r}{\cos\frac{1}{2}\alpha\,\sin\frac{1}{2}\alpha}$

$\Rightarrow$ according as $2a >$ (or) $< \dfrac{4r}{\sin\alpha}$

$\Rightarrow$ according as $2a >$ or $< 4r\,\text{cosec}\,\alpha$.

7(c) A particle moves with a central acceleration $\mu\left(r+a^4/r^3\right)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. prove that it describes the curve $r^2\left(2+\cos\sqrt{3}\theta\right)=3a^2$.

Sol^n: Here, the central acceleration

$$P=\mu\left(r+\frac{a^4}{r^3}\right)=\mu\left\{\frac{1}{u}+a^4u^3\right\}$$

where $u=1/r$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left[\frac{1}{u}+a^4u^3\right]$$

(or) $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\left[\frac{1}{u^3}+a^4u\right].$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating w.r.t to θ we have

$$h^2\left[2\cdot\frac{u^2}{2}+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left(\frac{-1}{2u^2}+\frac{a^4u^2}{2}\right)+A$$

(or) $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{u^2}+a^4u^2\right)+A \quad \text{eq}^n (1)$$

where A is constant.

Now initially the particle has been projected from an apse (say the point A) at a distance 'a' with velocity $2\sqrt{\mu a}$. Therefore, when $r=a$ i.e

$u = 1/a$, $\frac{du}{d\theta} = 0$ (at an apse) and $v = 2\sqrt{\mu a}$.

$\therefore$ from (1), we have

$$\underset{(i)}{4\mu a^2} = \underset{(ii)}{h^2\left[\frac{1}{a^2}\right]} = \underset{(iii)}{\mu^2\left(-a^2 + a^4 \cdot \frac{1}{a^2}\right)} + A$$

from (i) and (ii), we have $h^2 = 4\mu a^4$ and.

from (i) and (iii), we have

$$4\mu a^2 = 0 + A \quad \text{i.e,} \quad A = 4\mu a^2.$$

Substituting the values of h^2 and A in (1), we have

$$4\mu a^4\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{-1}{u^2} + a^4u^2\right) + 4\mu a^2$$

(or) $4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$

(or) $4a^4u^2\left(\frac{du}{d\theta}\right)^2 = \left(-1 - 3a^4u^4 + 4a^2u^2\right) \rightarrow (2)$

(or) $2a^2u\frac{du}{d\theta} = \sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}$ [taking square root]

(or) $$d\theta = \frac{2a^2u\,du}{\sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\sqrt{\left[\frac{-1}{3} - \left(a^4u^4 - \frac{4}{3}a^2u^2\right)\right]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\cdot\sqrt{\left[-\tfrac{1}{3}-(a^2u^2-\tfrac{2}{3})^2+\tfrac{4}{9}\right]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\,\sqrt{\left[(\tfrac{1}{3})^2-(a^2u^2-\tfrac{2}{3})^2\right]}}$$

(or) $\sqrt{3}\,d\theta = \dfrac{2a^2u\,du}{\sqrt{(\tfrac{1}{3})^2-(a^2u^2-\tfrac{2}{3})^2}}$

Substituting $a^2u^2-\frac{2}{3}=z$, so that $2a^2u\,du = dz$,

we have $\sqrt{3}\,d\theta = \dfrac{dz}{\sqrt{\left[(\tfrac{1}{3})^2-z^2\right]}}$

Integrating, $\sqrt{3}\theta + B = \sin^{-1}(3z)$ where B is a constant

(or) $\sqrt{3}\theta + B = \sin^{-1}(3a^2u^2-2)$. $\longrightarrow$ (3)

Now take the apse-line OA as the initial line.

Then initially

$r=a,\ u=1/a$ and $\theta=0$

$\therefore$ from (3), $0+B=\sin^{-1}(1)$ (or) $B=\frac{1}{2}\pi$.

putting $B=\frac{1}{2}\pi$ in (3), we have

$$\sqrt{3}\,\theta+\tfrac{1}{2}\pi = \sin^{-1}(3a^2u^2-2)$$

(or) $3a^2u^2-2 = \sin\left(\frac{1}{2}\pi+\sqrt{3}\theta\right) = \cos(\sqrt{3}\theta)$

(or) $\dfrac{3a^2}{r^2}-2 = \cos(\sqrt{3}\theta)$ (or)

$$3a^2-2r^2 = r^2\cos(\sqrt{3}\theta)$$

$$\therefore\ 3a^2 = r^2\left[2+\cos(\sqrt{3}\theta)\right],$$

which is the equation of the required curve.

8(a) If the directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at the point $(1,1,1)$ has maximum magnitude 15 in the direction parallel to the line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$, find the values of a, b, c.

Sol^n. Given $\phi = ax^2y + by^2z + cz^2x$

$$\text{grad}\,\phi = (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)\hat{k}$$

$$\text{grad}\,\phi \text{ at } (1,1,1) = (2a+c)\hat{i} + (a+2b)\hat{j} + (b+2c)\hat{k}$$

Given maximum directional derivative $= 15$

we know maximum directional derivative is $|\text{grad}\,\phi|$ in the direction of $\widehat{\text{grad}\,\phi}$.

$$|\text{grad}\,\phi| = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2} = 15$$

$$\widehat{\text{grad}\,\phi} = \frac{(2a+c)}{15}\hat{i} + \frac{(a+2b)}{15}\hat{j} + \frac{(b+2c)}{15}\hat{k} \quad \text{—(1)}$$

Unit vector of direction of line (Given maximum directional derivative)

$$= \frac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k} \quad \text{--- (2)}$$

Equating (1) and (2), we get

$$\frac{2a+c}{15} = \frac{2}{3}, \quad \frac{a+2b}{15} = -\frac{2}{3}, \quad \frac{b+2c}{15} = \frac{1}{3}$$

$$\Rightarrow 2a+c = 10, \quad a+2b = -10, \quad b+2c = 5$$

Solving three equation, we get

$$a = \frac{20}{9}, \quad b = -\frac{55}{9}, \quad c = \frac{50}{9}$$

8. (b) Given the space curve $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$, find (i) the curvature κ, (ii) the torsion τ.

Sol'n: (I) Given that $\vec{r} = ti + t^2 j + \frac{2}{3}t^3 k$

$$\Rightarrow \frac{d\vec{r}}{dt} = i + 2tj + 2t^2 k$$

$$\frac{d^2\vec{r}}{dt^2} = 0i + 2j + 4tk$$

and $$\frac{d^3\vec{r}}{dt^3} = 0i + 0j + 4k$$

Now $$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}$$

$$= i(8t^2 - 4t^2) + j(0 - 4t) + k(2-0)$$

$$= 4t^2 i - 4tj + 2k$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{16t^4 + 16t^2 + 4}$$

$$= 2\sqrt{4t^4 + 4t^2 + 1}$$

$$= 2\sqrt{(2t^2+1)^2}$$

$$= 2(2t^2+1)$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}$$

$$= (4t^2 i - 4tj + 2k) \cdot 4k = 8$$

$$\therefore \text{Curvature}(\kappa) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$$

$$= \frac{2(2t^2+1)}{\left[\sqrt{1+4t+4t^4}\right]^3}$$

$$= \frac{2(2t^2+1)}{(2t^2+1)^3} = \frac{2}{(2t^2+1)^2}$$

$$\text{Torsion } (\tau) = \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt}\ \frac{d^3\vec{r}}{dt^3}\right] \Big/ \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2$$

$$= \frac{8}{4(2t^2+1)^2} = \frac{2}{(2t^2+1)^2}$$

$$\therefore K = \tau = \frac{2}{(2t^2+1)^2}$$

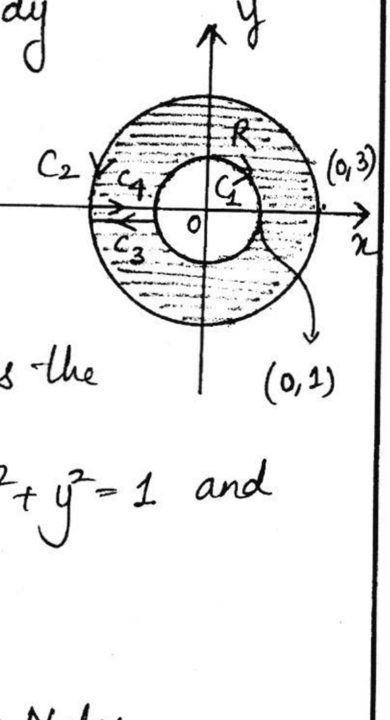
$\therefore$ For the space curve $x=t$, $y=t^2$, $z=\frac{2t^3}{3}$ the Curvature (K) and Torsion(τ) are same at every point.

8.(c). → Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

Solution :

By Green's theorem, we have,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



The boundary of the curve C is given by

$C : C_1 + C_2 + C_3 + C_4$ and R is the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=3$.

$$\therefore \int_C M\,dx + N\,dy = \int_{C_1+C_2+C_3+C_4} M\,dx + N\,dy$$

We also observe that in this question, along C_3 and C_4 i.e. $\int_{C_3+C_4} M\,dx + N\,dy = 0$.

$\because$ C_3 & C_4 are in opposite direction.

Along C_3 & C_4 : $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{C_3} 2x\,dx + \int_{C_4} 2x\,dx = \int_1^3 2x\,dx + \int_3^1 2x\,dx$$

$$= \int_1^3 2x\,dx - \int_1^3 2x\,dx = 0$$

Now,

$$L.H.S = \int_C M\,dx + N\,dy = \int_{C_1 + C_2} M\,dx + N\,dy$$

$$= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy \quad \text{——(1)}$$

Let

$$\int_{C_2} M\,dx + N\,dy = \int_{C_2} (2x - y^3)\,dx - xy\,dy$$

Putting $x = 3\cos\theta$, $y = 3\sin\theta$
$dx = -3\sin\theta$, $dy = 3\cos\theta$.

$$\int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} (6\cos\theta - 27\sin^3\theta)(-3\sin\theta)\,d\theta$$

$$- 27\cos^2\theta \sin\theta\,d\theta$$

$$= \int_0^{2\pi} (-18\cos\theta\sin\theta + 81\sin^4\theta -$$

$27 \sin\theta \cdot \cos^2\theta)\, d\theta$

$$= 0 + 81\cdot(4)\cdot \int_0^{\pi/2} \sin^4\theta\, d\theta + 27 \left.\frac{\cos^3\theta}{3}\right]_0^{2\pi}$$

$$= 81 \cdot \cancel{4} \cdot \frac{3}{\cancel{4}} \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 0$$

$$= \frac{243\pi}{4}$$

$$\int_{C_1} M\,dx + N\,dy = \int_{2\pi}^{0} (2\cos\theta - \sin^3\theta)(-\sin\theta\, d\theta) - \cos^2\theta \sin\theta\, d\theta$$ by putting $x = \cos\theta$, $y = \sin\theta$.
$\theta : 2\pi \to 0$.

$$= 0 + \int_{2\pi}^{0} \sin^4\theta\, d\theta - 0$$

$$= -4\int_0^{\pi/2} \sin^4\theta\, d\theta = -\cancel{4}\cdot\frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= -\frac{3\pi}{4}$$

$\therefore$ from (1),

$$\int_C M\,dx + N\,dy = -\frac{3\pi}{4} + \frac{243\pi}{4} = \frac{240\pi}{4}$$

$$\therefore \boxed{\int_C M\,dx + N\,dy = 60\pi} \quad \text{—— (A)}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R (3y^2 - y)\, dx\, dy$$

Using the polar co-ordinates,

$x = r\cos\theta,\ y = r\sin\theta.$

$dx\, dy = r\, dr\, d\theta.$

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} (3r^2\sin^2\theta - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{3r^4}{4}\sin^2\theta - \frac{r^3}{3}\sin\theta\right]_1^3 d\theta$$

$$= \int_0^{2\pi} \left(60\sin^2\theta - \frac{26}{3}\sin\theta\right) d\theta$$

$$= 60\left[\frac{\theta - \sin 2\theta}{2}\right]_0^{2\pi}$$

$$= 60 \cdot \frac{2\pi}{2} = 60\pi$$

$$\therefore \boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = 6\pi} \quad \text{———} \; (B)$$

$\therefore$ from (A) & (B), Green's theorem is verified.

Hence, the result.

8(d) If C is the curve given by

$r(t) = (1+2\sin t)\hat{i} + (1+5\sin^2 t)\hat{j} + (1+4\sin^3 t)\hat{k}$

$0 \le t \le \pi/2$ and F is the radial vector field $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

compute the work done by F on a particle moving along C.

Sol^n Work done (W) $= \int_C F \cdot dr = \int_C \left(F \cdot \frac{dr}{dt} dt\right)$

$F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$

radial vector so

$x = 1+2\sin t$, $y = 1+5\sin^2 t$, $z = 1+4\sin^3 t$

$$\frac{d\vec{r}}{dt} = 2\cos t\,\hat{i} + 10\sin t\cos t\,\hat{j} + 12\sin^2 t\cos t\,\hat{k}$$

$$W = \int_{t=0}^{\pi/2} \left[(1+2\sin t)\hat{i} + (1+5\sin^2 t)\hat{j} + (1+4\sin^3 t)\hat{k}\right] \cdot \left(2\cos t\,\hat{i} + 10\sin t\cos t\,\hat{j} + 12\sin^2 t\cos t\right) dt$$

$$= \int_{t=0}^{\pi/2} (2\cos t + 4\sin t\cos t + 10\sin t\cos t + 50\sin^3 t\cos t + 12\sin^2 t\cos t + 48\sin^5 t\cos t)\, dt$$

$$= \int_0^{\pi/2} (2\cos t + 2\sin 2t + 5\sin 2t + 50\sin^3 t\cos t + 12\sin^2 t\cos t + 48\sin^5 t\cos t)\,dt$$

$$= \left[2\sin t - \frac{7}{2}\cos 2t + \frac{25}{2}\sin^4 t + 4\sin^3 t + 8\sin^6 t\right]_0^{\pi/2}$$

$$= 2(1-0) - \frac{7}{2}(-1-1) + \frac{25}{2}(1-0) + 4(1-0) + 8(1-0)$$

$$= 2 + 7 + \frac{25}{2} + 4 + 8 = \frac{67}{2}$$