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Date 9/7/20

IFoS 2015 Calculus

1) Let $f(x)$ be a real valued function defined on $(-5, 5)$ such that

$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt$$

Let $f^{-1}(x)$ be the inverse of $f(x)$. Find $(f^{-1})'(2)$. (8)

We know that

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(t)}, \quad \text{where } f(t) = x.$$

Given $$e^{-x} \cdot f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt \quad \text{—} ①$$

Differentiating both sides w.r.t. x

$$-e^{-x} f(x) + e^{-x} f'(x) = \frac{d}{dt}\left(\int_0^x \sqrt{t^4+1}\, dt \right)$$

$$= \sqrt{x^4+1}$$

At, $x = 0$

$$-f(0) + f'(0) = \sqrt{0+1} = 1$$

Also, from ①

$$f(0) = 2 + 0 \Rightarrow \boxed{f(0) = 2}$$

$$\therefore f'(0) = 1 + 2 = 3$$

$$\therefore \left.\frac{d}{dx} f^{-1}(x)\right|_{x=2} = \frac{1}{f'(0)} \quad \because f(0) = 2$$

$$= \frac{1}{3}.$$

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1.(d) For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t}\,dt$.

Evaluate $f(e) + f\left(\frac{1}{e}\right)$. (8)

Sol: $I_1 = f(e) = \int_1^e \frac{\log t}{1+t}\,dt$

$$I_2 = f\left(\frac{1}{e}\right) = \int_1^{1/e} \frac{\log t}{1+t}\,dt$$

$$= \int_1^e \frac{\log(1/y)}{1+\frac{1}{y}} \cdot \left(\frac{-dy}{y^2}\right) \quad \left[\text{Putting } t = \frac{1}{y}\right.$$

$$= \int_1^e \frac{\log y}{1+y} \cdot \frac{dy}{y} = \int_1^e \frac{\log t}{(1+t)} \frac{dt}{t}$$

$$\therefore I_1 + I_2 = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log t}{1+t} + \frac{\log t}{1+t} \cdot \frac{dt}{t}$$

$$= \int_1^e \frac{\log t}{1+t}\left(1+\frac{1}{t}\right)dt$$

$$= \int_1^e \frac{\log t}{t}\,dt = \left.\frac{(\log t)^2}{2}\right|_1^e = \frac{1}{2}.$$

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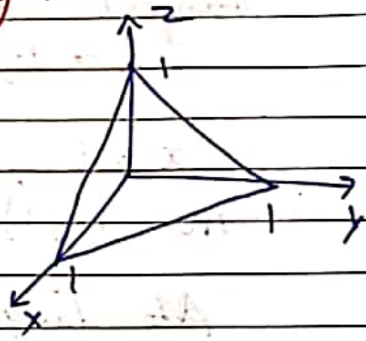
2(i) Consider the three-dimensional region R bounded by $x+y+z=1$, $y=0$, $z=0$, $x=0$. Evaluate

$$\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz.$$

Let R be the region bounded by the given tetrahedron.



$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} z(x)^2+(y^2)z + \frac{z^3}{3}\Big|_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (\cancel{x^2}+y^2)(1-x-y) + \frac{1}{3}(1-x-y)^3\,dy\,dx$$

$x^2(1-x-y)+$

$$= \int_0^1 x^2(1-x)y - x^2\frac{y^2}{2} + (1-x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(1-x-y)^4}{12}\Big|_0^{1-x} dx$$

$$= \int_0^1 x^2(1-x)^2 - \frac{1}{2}x^2(1-x)^2 + \frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4 + \frac{1}{12}(1-x)^4\,dx$$

$$= \int_0^1 \frac{1}{2}(x^2+x^4-2x^3) + \frac{2}{12}(1-x)^4\,dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right) - \frac{1}{30}(1-x)^5\Big|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right) - \frac{1}{30}(0-1) = \frac{1}{20}.$$

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2(d) Find the area enclosed by the curve in which the plane $z=2$ cuts the ellipsoid,
$$\frac{x^2}{25}+y^2+\frac{z^2}{5}=1. \quad (10).$$

The intersection of plane $z=2$ with the ellipsoid is given by

$$\frac{x^2}{25}+y^2+\frac{(2)^2}{5}=1 \Rightarrow \frac{x^2}{25}+y^2=\frac{1}{5}$$

ie $\frac{x^2}{5}+\frac{y^2}{1/5}=1$ (say S_1) in space $z=2$.

The area enclosed by this curve is an ellipse.

We take projection on xy- plane

$$A=\iint_D \sqrt{z_x^2+z_y^2+1}\, dA \qquad \left(\begin{array}{l} z=2 \\ z_x=0 \\ z_y=0 \end{array}\right)$$

$$=\int_{-\sqrt5}^{\sqrt5}\int_{-\frac{1}{5}\sqrt{5-x^2}}^{\frac{1}{5}\sqrt{5-x^2}} 1\, dy\, dx \qquad \left(\begin{array}{l} D \text{ is region of} \\ \text{projection on } xy\text{-plane} \end{array}\right)$$

$$=\int_{-\sqrt5}^{\sqrt5}\frac{2}{5}\sqrt{5-x^2}\, dx \qquad \begin{array}{l} \text{Put } x=\sqrt5\sin\theta \\ dx=\sqrt5\cos\theta d\theta \end{array}$$

$$=\frac{2}{5}\times 2\int_0^{\pi/2}\sqrt{5-5\sin^2\theta}\,\sqrt5\cos\theta d\theta = 4\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$=4\times\frac{1}{2}\times\frac{\pi}{2}=\pi.$$

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3(b) If $\sqrt{x+y}+\sqrt{y-x}=c$, find $\frac{d^2y}{dx^2}$. (10)

$$\left(\sqrt{y+x}+\sqrt{y-x}\right)^2=c^2$$

$$(y+\cancel{x})+(y-\cancel{x})+2\sqrt{y^2-x^2}=c^2$$

$$2y-c^2=2\sqrt{y^2-x^2}$$

$$\cancel{4y^2}-4c^2y+c^4=4(\cancel{y^2}-x^2)$$

$$-4c^2y=-4x^2-c^4$$

$$y=\frac{1}{c^2}x^2+\frac{c^2}{4}$$

Differentiating w.r.t x

$$\frac{dy}{dx}=\frac{2}{c^2}x$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2}=\frac{2}{c^2}}$$

4.d) $L = \lim_{x \to 0} \left(\frac{2+\cos x}{x^3 \sin x} - \frac{3}{x^4} \right)$

$= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^4 \sin x}$

$= \lim_{x \to 0} \frac{x(2+\cos x) - 3\sin x}{x^5} \cdot \frac{x}{\sin x}$

$= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^5} \cdot 1$ $\left(\frac{0}{0} \text{ form} \quad \lim_{x \to 0} \frac{x}{\sin x} = 1 \right)$

$= \lim_{x \to 0} \frac{2 + \cos x - x\sin x - 3\cos x}{5x^4}$ $\left(\frac{0}{0} \text{ form} \right)$ L-Hospital

$= \lim_{x \to 0} \frac{2\sin x - \sin x - x\cos x}{20x^3}$ $\left(\frac{0}{0} \text{ form} \right)$

$= \lim_{x \to 0} \frac{\cancel{\cos x} - \cancel{\cos x} + x\sin x}{60x^2}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{60} = \frac{1}{60}$