ANALYTIC GEOMETRY

: CSE-2016 :

① (d) Find the equation of the sphere which passes through the circle $x^2+y^2=4;\ z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

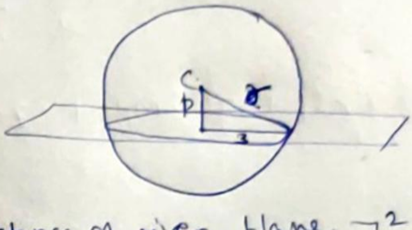
→ Any sphere through the given circle $x^2+y^2=4,\ z=0$ is

$x^2+y^2+z^2-0^2-4+\lambda z=0 \Rightarrow x^2+y^2+z^2+\lambda z-4=0$ —— ①

Its centre is $C(0,0,-\lambda/2)$ and radius is $r=\sqrt{\frac{\lambda^2}{4}+4}=\frac{\sqrt{\lambda^2+16}}{2}$.

$\perp$ar Distance of the centre from the plane $x+2y+2z=0$ is given by

$$p=\frac{0.1+0.2+\left(\frac{-\lambda}{2}\right).2}{\sqrt{1^2+2^2+2^2}}=\frac{-\lambda}{3}$$



Radius of the circle in which the plane cuts the sphere is given as

$R^2=[\text{Radius of sphere ①}]^2-[\text{Distance of given plane from centre of sphere ①}]^2$

$$9=\frac{\lambda^2+16}{4}-\frac{\lambda^2}{9}$$

$$\Rightarrow 9=\frac{9\lambda^2+144-4\lambda^2}{36}$$

$\Rightarrow \cancel{5}\lambda^2=\cancel{180}\ 36 \Rightarrow \lambda=\pm 6$

∴ The required spheres are $x^2+y^2+z^2+6z-4=0$ and $x^2+y^2+z^2-6z-4=0$

① (e) Find the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{4}=z-3$ and $y-mx=z=0$. For what value of 'm' will the two lines intersect?

→ Given lines are L_1: $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{1}$ —— ①

L_2: $y-mx=0,\ z=0 \Rightarrow \frac{x}{1}=\frac{y}{m}=\frac{z}{0}$ —— ②

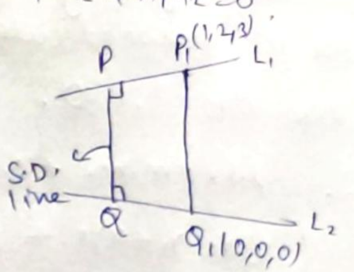
DRs of line ① are $2,4,1$

DRs of line ② are $1,m,0$.

Let the drs of the S.D. line between them be l_1,m_1,n_1.

Then, the SD line is $\perp$ar to both L_1 and L_2

WKT condition for perpendicularity is $l_1l_2+m_1m_2+n_1n_2=0$



Hence: $2l_1+4m_1+n_1=0$.

$l_1+mm_1+0n_1=0$

$\Rightarrow \quad \frac{l_1}{-m} = \frac{m_1}{1} = \frac{n_1}{2m-4} = \frac{1}{\sqrt{m^2+1+(2m-4)^2}}$

$\therefore \quad l_1 = \frac{-m}{\sqrt{m^2+1+4m^2+16-16m}} = \frac{-m}{\sqrt{5m^2-16m+17}}$

$m_1 = \frac{1}{\sqrt{5m^2-16m+17}}, \quad n_1 = \frac{2m-4}{\sqrt{5m^2-16m+17}}$ —— (3)

The shortest distance between the two ~~poti~~ lines is the projection of a line joining any two points on L_1 and L_2 respectively.

Let us take the point on L_1 as $P_1(1,2,3)$ & on L_2 as $Q_1(0,0,0)$

Then S.D. = Projection of P_1Q_1 on line with dcs given by (3)

$= l_1(x_2-x_1)+m_1(y_2-y_1)+n_1(z_2-z_1)$

$SD = \frac{1}{\sqrt{5m^2-16m+17}}\left[-m(1-0)+1(2-0)+(2m-4)(3-0)\right]$

$$\boxed{SD = \frac{1}{\sqrt{5m^2-16m+17}}[5m-10]}$$

For the two lines to intersect, S.D. between them is zero

$\therefore \frac{(5m-10)}{\sqrt{5m^2-16m+17}} = 0 \quad \Rightarrow 5m-10=0 \quad \Rightarrow m=2$

3(a) Find the surface generated by a line which intersects the lines $y=a=z$, $x+3z=a=y+z$ and is parallel to the plane $x+y=0$?

→ The given lines are $L_1: \frac{x}{1} = \frac{y-a}{0} = \frac{z-a}{0} = r_1$ (say) —— (1)

$L_2: \frac{x-a}{3} = \frac{y-a}{1} = \frac{z}{-1} = r_2$ (say) —— (2)

Any point on L_1: $P(r_1, a, a)$

Any point on L_2: $Q(3r_2+a, r_2+a, -r_2)$

DRs of ~~#~~ join of PQ = $3r_2-r_1+a,\ r_2,\ -r_2-a$.

The line joining PQ is parallel to the plane $x+y=0 \Rightarrow$ The join of PQ is perpendicular to the normal to the plane $x+y=0$ which has drs $1,1,0$.

$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$\Rightarrow \quad (3r_2 - r_1 + a)1 + r_2 \cdot 1 + (-r_2 - a) \cdot 0 = 0$

$\Rightarrow \quad 4r_2 - r_1 + a = 0 \quad \Rightarrow r_1 = 4r_2 + a.$

DRs of PQ: $\quad 3r_2 - r_1 + a,\ r_2,\ -r_2 - a \equiv 3r_2 - (4r_2 + a) + a,\ r_2,\ -(r_2+a)$
$\qquad \equiv -r_2,\ r_2,\ -(r_2 + a)$

$\therefore$ Eqn of PQ which passes through $P(r_1, a, a) = P(4r_2 + a, a, a)$ & whose drs are $-r_2, r_2, -(r_2+a)$ is

$$\frac{x-(4r_2+a)}{-r_2} = \frac{y-a}{r_2} = \frac{z-a}{-r_2-a} \quad \text{--- (3)}$$

Now

$$\frac{y-a}{r_2} = \frac{z-a}{-r_2-a} \Rightarrow (-r_2 - a)(y-a) = (z-a)r_2$$

$\Rightarrow \quad -yr_2 - ay + ar_2 + a^2 = zr_2 - ar_2$

$\Rightarrow (z + y - 2a)\, r_2 = a^2 - ay$

$$\Rightarrow \quad r_2 = \frac{a(a-y)}{y+z-2a}$$

Now: from (3):

$$\frac{x - 4r_2 + a}{-r_2} = \frac{y-a}{r_2}$$

$\Rightarrow \quad (x - 4r_2 + a)\cancel{r_2} = -\cancel{r_2}(y-a)$

$\Rightarrow \quad x + \cancel{a} - 4r_2 = \cancel{a} - y$

$$\Rightarrow \quad 4r_2 = x + y \Rightarrow \frac{4a(a-y)}{y+z-2a} = x+y$$

$\Rightarrow \quad 4a^2 - 4ay = xy + y^2 + xz + yz - 2ax - 2ay$

$\Rightarrow \quad y^2 + xy + xz + yz - 2ax + 2ay - 4a^2 = 0$ which is the surface generated by the line PQ.

4(b) Show that the cone $3yz - 2zx - 2xy = 0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ is a generator belonging to one such set, find the other two.

→ The eqn of given cone is $3yz - 2zx - 2xy = 0$ — ①

Clearly, the coeff. of x^2, y^2, z^2 are zero each. Hence, the sum of coeff. of x^2, y^2 and $z^2 = 0$.

Therefore, we can say that the cone given by ① has an infinite set of three mutually perpendicular generators.

If the line $L_1 : \frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ — ② is one of such a set, then the other two lie on the plane perpendicular to this line.

Any plane through line L_1 is $Ax + By + Cz = 0$. — ③

This plane is ⊥ar to line ② ⇒ The normal to this plane is parallel to line ②. Hence the dcs of normal to the plane ③ i.e. A, B, C and the drs of line ② 1, 1, 2 are proportional ⇒ $\frac{A}{1} = \frac{B}{1} = \frac{C}{2} = k$ (say)

Putting in ②
$$kx + ky + 2kz = 0$$
$$\Rightarrow x + y + 2z = 0 \quad — ④$$

The required generators lie on the intersection of the given cone ① and the plane ②.

If l, m, n be the dcs of the required line, then, this dcs satisfy the cone ①

$$\therefore 3yz - 2zx - 2xy = 0$$
$$\Rightarrow 3mn - 2nl - 2lm = 0 \quad — ⑤$$

Also, the line with dcs l, m, n lie on plane ④. So, it is ⊥ar to the normal to plane ④

Hence, $l + m + 2n = 0$.

$$\Rightarrow n = \frac{-(l+m)}{2} \quad — ⑥$$

Putting the value of n in ⑤: $3mn - 2nl - 2ml = 0$

$$\Rightarrow -3m\frac{(l+m)}{2} + (l+m)l - 2lm = 0 \Rightarrow -\frac{3lm}{2} - \frac{3m^2}{2} + l^2 + lm - 2lm = 0$$

$$\Rightarrow -3lm - 3m^2 + 2l^2 - 2lm = 0 \Rightarrow 2l^2 - 5lm - 3m^2 = 0$$

Dividing by m^2: $2\left(\frac{l}{m}\right)^2 - 5\left(\frac{l}{m}\right) - 3 = 0$

$$2\left(\frac{l}{m}\right)^2 - 6\left(\frac{l}{m}\right) + \left(\frac{l}{m}\right) - 3 = 0 \Rightarrow \left(2\frac{l}{m}+1\right)\left(\frac{l}{m}-3\right) = 0$$

$2\frac{l}{m} + 1 = 0$ & $\frac{l}{m} - 3 = 0$

$\frac{l}{-1} = \frac{m}{2}$ $\quad\quad$ $\frac{l}{3} = \frac{m}{1}$

$n = -\frac{(l+m)}{2} \Rightarrow$ (i) $m = -2l$:

$n = -\frac{(l-2l)}{2}$

$\frac{n}{-1} = \frac{l}{-2} \Rightarrow \cancel{\frac{l}{-1} = \frac{n}{-1/2}}$

$\cancel{\Rightarrow \frac{l}{-2} = \frac{n}{-4}}$

(ii) $l = 3m$:

$n = -\frac{(3m+m)}{2} = -2m$

$\Rightarrow \frac{m}{1} = \frac{n}{-2}$

$\therefore \frac{l}{-2} = \frac{m}{4} = \frac{n}{-1}$ & $\frac{l}{3} = \frac{m}{1} = \frac{n}{-2}$

$\therefore$ The required generators have d.r.s $-1, 2, -2$ & $3, 1, -2$.

They pass through the vertex (origin). Hence, equation of reqd generators are

$$\frac{x}{-2} = \frac{y}{4} = \frac{z}{-1} \text{ and } \frac{x}{3} = \frac{y}{1} = \frac{z}{-2}$$

4(d) Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2+by^2+cz^2=1$.

$\rightarrow$ The given conicoid is $ax^2+by^2+cz^2=1$ — ①

Let any tangent plane to this conicoid be $lx+my+nz=p$ — ②

Then, the condition of tangency is $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$

$\Rightarrow$ ② $\equiv lx+my+nz = \pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$

Let the three mutually $\perp$ar tangent planes be

$$l_1x+m_1y+n_1z = \pm\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}, \quad l_2x+m_2y+n_2z = \pm\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}}$$

and $$l_3x+m_3y+n_3z = \pm\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$$

Squaring and adding each equation of these tangents on the corr. sides, we have

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$

$$=\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}+\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}+\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}$$

On solving: we get

$$x^2\Sigma l_1^2+y^2\Sigma m_1^2+z^2\Sigma n_1^2+2xy\Sigma l_1m_1+2yz\Sigma m_1n_1+2xz\Sigma n_1l_1$$

$$=\frac{\Sigma l_1^2}{a}+\frac{\Sigma m_1^2}{b}+\frac{\Sigma n_1^2}{c} \quad \text{——— (3)}$$

where $\Sigma l_1^2 = l_1^2+l_2^2+l_3^2$, $\Sigma m_1^2 = m_1^2+m_2^2+m_3^2$, $\Sigma n_1^2 = n_1^2+n_2^2+n_3^2$

$\Sigma l_1m_1 = l_1m_1+l_2m_2+l_3m_3$, $\Sigma n_1l_1 = n_1l_1+n_2l_2+n_3l_3$ and

$\Sigma m_1n_1 = m_1n_1+m_2n_2+m_3n_3$.

Since the three lines are mutually $\perp$ar, we have

$\Sigma l_1^2 = \Sigma m_1^2 = \Sigma n_1^2 = 1$ and $\Sigma l_1m_1 = \Sigma m_1n_1 = \Sigma n_1l_1 = 0$

$\therefore$ (3) $\equiv$ $\boxed{x^2+y^2+z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ which is the required locus.