⇒ Real Analysis :

1) Ordered Field

It satisfies 4 properties :

O1 : for $a, b \in \mathbb{R}$ exactly one of the following 3 holds (i) $a > b$ (ii) $a = b$ (iii) $a < b$

This is LAW OF TRICHOTOMY

O2 : for $a, b \in \mathbb{R}$ $a > b$, $b > c \Rightarrow a > c$

LAW OF TRANSITIVITY

O3 : $a > b \Rightarrow a + c > b + c$ Monotone property for $+^n$

O4 : $a > b$ & $c > 0 \Rightarrow ac > bc$ Monotone property for $\times^n$

eg $(\mathbb{R}, +, \cdot)$ & $(\mathbb{Q}, +, \cdot)$ are ordered fields

2) Each interval is infinite set but each infinite set is not an interval.

3) Extended Real Number System $R^* = [-\infty, \infty]$ [includes $\pm\infty$]

$\infty + x = x + \infty = -x + \infty = \infty - x = \infty$ $\frac{x}{\infty} = 0$ ..

$\infty - \infty$, $0 \times \infty$, $\frac{\infty}{\infty}$ etc are meaningless

4) Greatest Lower Bound or Infimum :

It is the maximum of the set of lower bounds of a bdd below set.

If 't' is infimum, for each $\epsilon > 0$, $\exists x \in S$, st $t < x < t + \epsilon$

Least Upper Bound or Supremum :

Min. of set of upper bounds of bdd above set

If t is supremum, $\forall \epsilon > 0, \exists x \in S$ st $t - \epsilon < x < t$

eg $S = \{2^n \mid n \in N\} - \{2^1, 2^2, 2^3, ---\}$

inf $= 2 \in S$; sup $= \infty$ (DNE)

5) Bounded Set : Both lower & upper bounds exist

* Every finite set is bdd & has inf & sup
* null set ϕ is bdd but has no inf & sup as every real no is upper | lower bound of set ϕ.

6) Sup | Inf need not belong to the set.

If sup(S) $\in S$ then it is the greatest member of the set

7) Completeness Property of $\mathbb{R}$:

Every non-empty subset of $\mathbb{R}$ which is bounded above has the supremum in $\mathbb{R}$ - Similar for infimum if it is bdd below.

Also $\mathbb{R}$ is a complete Ordered Field as it satisfies all the 3 axioms - Field, Order, Completeness

* $\mathbb{Q}$ is not ~~an~~ complete as $\left(1+\frac{1}{n}\right)^n \longrightarrow e$

or $S: \{x : x \in \mathbb{Q}^+ \ \& \ 0 < x^2 < 2\}$

8) Archimedian Property

If $a, b \in \mathbb{R}$ & $a > 0$ then $\exists$ $n (>0)$ st $n(a) > b$

Prove by contradiction, by using property of bdd above set & showing it can't have a lub.

9) Remember :

$$|x+y| \le |x|+|y|$$
$$|x-y| \ge \big||x|-|y|\big|$$
$$|xy| = |x||y|$$

10) Nbd of a point :

δ-nbd of a is $N(\delta, a) = N_\delta(a) = (a-\delta, a+\delta)$

$N_\delta(a) - \{a\}$ = deleted δ-nbd of a. $= N_\delta d(a)$

$S \subseteq \mathbb{R}$ is said to be nbd of $a \in \mathbb{R}$, if $\exists$ a $\delta > 0$ (however small) st $(a-\delta, a+\delta) \subset S$

* if $a \in \mathbb{Q} \subseteq \mathbb{R}$, then $\mathbb{Q}$ is not nbd of a as $(a-\delta, a+\delta) \not\subset \mathbb{Q}$
  it has Irrational nos.

* (a,b) is nbd of each of its points
* $[a,b]$ is nbd of each except the endpts a & b.
* Non-empty finite set cannot be a nbd of any of its pts
  set ϕ is nbd of each of its pts.
  N, Q, W, I — none of them are nbd of any of their pts

11) Interior Pt of a set :

Let $p \in S$, p is interior if S is a nbd of p.

* Every non-empty finite set has no interior point.

Interior of a set :

$Int(S) = S^\circ$ = set of all interior points of the set S.

$S = (a,b) \Rightarrow S^\circ = S$

$N^\circ = Z^\circ = \phi$

12) Open set : S is open $\iff (S^\circ = S)$

Union (of any no.) & Intersection of finite no of opensets is open.

Intersection of infinite no of opensets might not be open

eg $\bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{1}{n}\right) = \{0\} \neq$ open.

13) Limit Point of $S \subseteq \mathbb{R}$: (Accumulation | cluster | Condensation pt)

p is lmt pt is every nbd of p contains a pt of S other than p.

$$(p-\epsilon, p+\epsilon) \cap S - \{p\} \neq \phi.$$

* Lmt pt may not belong to set
* Lmt pts may be 0, unique, finite or infinite
* Finite Set has no limit pt.
* $\forall p \in \mathbb{R}$, p is limit pt of $S = \mathbb{Q}$
* p is lmt pt if $(p-\epsilon, p+\epsilon)$ has ∞ pts of S.

eg $S : \{-1, 1, -1, 1, -1, 1, \ldots\}$ Both -1 & 1 are limit pts.

* Every interior pt is a lp.

14) If $\sup(S) \notin S$, then $\sup(S) =$ limit pt of S.

Similar for $\inf(S)$

Proof: $\sup(S) = M$ $\exists x$ st $(M - \epsilon < x < M)$ for $\epsilon > 0$

$\therefore \exists x$ st $(M - \epsilon < x < M + \epsilon)$

$\therefore (M-\epsilon, M+\epsilon) \cap [S - \{M\} = S] \neq \phi$.

$M \notin S$.

15) Isolated Point :

If $p \in S$ is not lmt pt of $S \Rightarrow p$ is isolated pt

If $\forall e \in S$, e is isolated $\Rightarrow S$ is discrete set

16) Derived Set :

$D(S) = \{x \in \mathbb{R} \mid x$ is lmt pt of $S\}$
$=$ Set of all lmt pts of S.

Set of 1st species if only finite no of derived sets eg $\mathbb{N}$
Set of 2nd species if infinite no of derived sets eg. $\mathbb{R}$

$D(S) = \phi$ if S is finite set

Set of order $(n) \Rightarrow$ ~~(n)th D(S)~~ is $D^n(S)$ is finite
& $D^{n+1}(S)$ is empty

17) Adherent Point :
p is adherent to S if $\forall$ nbd N of $p \Rightarrow N \cap S \neq \phi$
[for lmt pt $N \cap (S - \{p\}) \neq \phi$]

p is adherent $\Leftrightarrow$ $p \in S$ or $p \in D(S)$

18) Closure of a set : Set of all adherent pts
$cl\, S = \bar{S} = S \cup D(S)$

Dense set : $S \subseteq \mathbb{R}$ is dense if $\bar{S} = \mathbb{R}$

Dense in itself : $S \subseteq D(S)$, ie, every pt is lmt pt.

Perfect set : $S = D(S)$, ie, $S = \bar{S}$

$\mathbb{Q} \subseteq D(\mathbb{Q}) = \mathbb{R}$ $\therefore$ Dense in itself but $\mathbb{Q} \neq D(\mathbb{Q})$ so not perfect

$\phi = D(\phi)$ $\therefore$ Dense in itself & perfect

$D(\mathbb{N}) = \phi$ $\therefore$ $\mathbb{N}$ is discrete & of 1st species
$\mathbb{R}$ is of 2nd species

eg $S = \{\frac{1}{n} \mid n \in N\} \subseteq R$. 0 is the only limit pt of S.

Cases : (i) $p < 0$ $\quad (-\infty, 0) \cap S = \phi$

(ii) $p > 1$ $\quad (1, \infty) \cap S = \phi$

(iii) $p = 1$ $\quad (\frac{1}{2}, 1) \cap S \neq \phi$

(iv) $0 < p < 1 \Rightarrow n \leq \frac{1}{p} < n+1$ for some N

$$\frac{1}{n-1} > 1/n \geq p > \frac{1}{n+1}$$

$\therefore \left(\frac{1}{n+1}, \frac{1}{n-1}\right) \cap S = \{1/n\}$, only finite.

19) Bolzano–Weierstrass Theorem : (sufficient condn for existence of lt pt)

Every infinite bounded set of real nos has a limit pt.

Converse need not be true eg $[0, \infty)$

20) Some results : $A, B \subseteq R$

a) $A \subset B \Rightarrow D(A) \subseteq D(B)$

b) $D(A \cup B) \Rightarrow D(A) \cup D(B)$

c) $D(A \cap B) \subseteq D(A) \cap D(B)$

d) $D(D(A)) \subseteq D(A)$

* Derived set of any infinite bdd set attains its bounds.

$\overline{\lim} S \rightarrow$ upper limit of $D(S)$

$\underline{\lim} S \rightarrow$ lower limit of $D(S)$

* Sup & Inf of a bdd set $S \in \bar{S}$ always

($\because$ $\sup(S) \in S$ or $\sup(S) \in D(S)$) (Result 14)

21) Closed Set :

$S \subseteq \mathbb{R}$ is closed if $S^c = \mathbb{R} - S$ is open set

or S is closed if $D(S) \subseteq S$

S is closed $\Leftrightarrow \bar{S} = S$

All lmt pts $\in S$ but not all pts of $S \in D(S)$.

eg S is non-empty finite set
$D(S) = \phi \subseteq S \Rightarrow S$ is closed
$\mathbb{N}$ is closed.
$\mathbb{Q}$ is not closed as $D(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{Q}$

22) Arbitrary no of $\cap \Rightarrow$ closed
finite no of $\cup \Rightarrow$ closed
infinite no of $\cup \Rightarrow$ may or may not be closed.
eg $S_n = \left[\frac{1}{n}, 1\right]$ $\bigcup_{n=1}^{\infty} S_n = (0,1]$

23) A is closed & B is open
$A - B$ is closed $\because A - B = A \cap B^c$ = closed $\cap$ closed
$B - A$ is open $\because B - A = B \cap A^c$ = open $\cap$ open

24) Compact Set :
non-empty $S \subseteq \mathbb{R}$ is compact if S is closed & bounded

Union of finite $\Rightarrow$ compact
intersection of arbitrary st $\cap \neq \phi \Rightarrow$ compact [$\because S = \phi$ is not compact as its not bdd]

25) Cover of a set :
S is a set & $\{G_\alpha\}$ is a family of sets.
$\{G_\alpha\}$ is cover if $\forall s \in S$, $s \in$ some set in $\{G_\alpha\}$
$S \subseteq \bigcup \{G_\alpha\}$ eg $G = \{(-n, n) \mid n \in \mathbb{N}\}$ is open cover of $\mathbb{R}$.

If all sets of $\{G_\alpha\}$ are open $\Rightarrow$ open cover

Finite Subcover

G is open cover of S. E is finite subcover if

i) E is contained in G

ii) E is finite collection

iii) E covers S.

Summary :

i)	Completeness Axiom	Every non-empty bdd above subset has its supremum in $\mathbb{R}$ (not true for $\mathbb{Q}$)
ii)	Archimedian Property	$a, b \in \mathbb{R}$, $a > 0$ then $\exists\ n > 0$ st $na > b$
iii)	Interior Point	$p \in S$ if $N_\delta(p) \subseteq S$ for some δ
iv)	Open Set	$S^\circ = S$ [Any no of $\cup$ or finite no of $\cap$ of open sets is open]
v)	Limit Point	Intersection of every nbd of a pt with the set has infinite entries. If $\sup(S) \notin S$, then $\sup(S) \in D(S)$
vi)	Isolated Point	$p \in S$ st p is not limit point
vii)	Discrete Set	Each point is isolated.
viii)	$D(S)$	set of all limit points
ix)	Adherent Point	$p \in S$ or $p \in D(S)$
x)	Closure of Set	$\bar{S} = S \cup D(S)$
xi)	Dense in itself	$S \subseteq D(S)$
	Perfect Set	$S = D(S)$
	Dense	$\bar{S} = R$ eg $\mathbb{Q}$
xii)	Closed set	$D(S) \subseteq S$ or S^c is open [Infinite $\cap$ or finite $\cup$]
xiii)	Compact Set	Closed + Bounded

⇒ SEQUENCES

1) It is a function $s: \mathbb{N} \longrightarrow \mathbb{R}$

Range of sequence = set of all distinct terms of the sequence.

$X = (x_n)$ & $Y = (y_n)$

Sum = $X+Y = (x_n + y_n)$
Diff. = $X-Y = (x_n - y_n)$
Multiplication = $XY = (x_n y_n)$
...

2) Bounds of a sequence:

Sequence $\{x_n\}$ is bdd if $\exists\ k, K \in \mathbb{R}$ st

$$k < x_n < K \quad \forall\ n \in \mathbb{N}$$

$\{x_n\}$ is bdd $\iff \exists\ M(>0)$ st $|x_n| \leq M \quad \forall n \in \mathbb{N}$

3) Limit of a sequence:

L is limit of (x_n) if $\forall\ \epsilon > 0$ (however small), $\exists\ K \in \mathbb{N}$ [where $k(\epsilon)$]

st $\quad |x_n - L| < \epsilon \quad \forall\ n \geq k(\epsilon)$

4) If $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, (x_n) is said to be cgt to x

If $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$, it is dgt.

Diverges to $-\infty$ if for any $k > 0$ (however large), $\exists\ m \in \mathbb{N}$

st $\quad x_n > K \quad \forall\ n \geq m$

5) Oscillatory Sequence: Neither cgs nor dgs to $+\infty$ or $-\infty$.

$(x_n) = (-1)^n$ = bdd osc. sequence $\quad (x_n) = ((-1)^n . n)$ = unbdd os.

6) Null Sequence: $\lim_{n\to\infty} x_n = 0$ then x_n is null sequence

Scanned with CamScanner

7) Thm: Every cgt sequence has a unique limit

8) Thm: Every cgt sequence is bdd.

Proof: given $\epsilon > 0$, $\exists$ K st $|x_n - x| < \epsilon \quad \forall\ n \geq k$

$\therefore\ |x_n| < |x| + \epsilon \quad \forall\ n \geq k$

Let $M = \sup\{|x_1|, |x_2|, \ldots, |x_{n-1}|, |x| + \epsilon\}$

$\therefore\ |x_n| \leq M \quad \forall\ n \in N$

converse is not true: eg $\{(-1)^n\}$

9) Thm: (x_n) $x_n \in \mathbb{R}$ & (a_n) $a_n \in \mathbb{R}^+$ are sequences & $\lim\limits_{n \to \infty} a_n = 0$

if for some $c > 0$ & some $m \in \mathbb{N}$, $|x_n - x| \leq c a_n \ \forall\ n \geq m$

$\Rightarrow \ \text{Lt}\ x_n = x$

10) Bernoulli's Inequality:

If $x > -1$, then $(1+x)^n \geq 1 + nx \quad \forall\ n \in \mathbb{N}$

11) Let $(x_n) \to x$ & $(y_n) \to y$

$(x_n \pm y_n) \to x \pm y \qquad (x_n y_n) \to (x \cdot y) \qquad (c x_n) \to cx$

Also if $y_n \neq 0$ & $y \neq 0$ $\left(\frac{x_n}{y_n}\right) \to \frac{x}{y}$

These rules apply to add^n, product of finite cgt sequences also

Also $\text{Lt}\ a_n^k = (\text{Lt}\,(a_n))^k$ where $k \in \mathbb{N}$ & (a_n) is cgt

12) Thm: if $x_n \geq 0 \ \forall\ n \in \mathbb{N}$ & $(x_n) \to x \Rightarrow x \geq 0$

if $x_n \leq y_n \ \forall\ n \in \mathbb{N}$ & $(x_n) \to x$, $(y_n) \to y \Rightarrow x < y$

if $a \leq x_n \leq b \ \forall\ n \in \mathbb{N}$ & $(x_n) \to x \Rightarrow a \leq x \leq b$

All the 3 above have cond^n that (x_n) & (y_n) are cgt.

13) Squeeze Theorem:

Let $X=(x_n)$, $Y=(y_n)$ & $Z=(z_n)$ are sequences st

$x_n \le y_n \le z_n \ \forall\ n \in \mathbb{N}$ & $lt(x_n) = lt(z_n)$, then

(y_n) is cgt & $lt(x_n) = lt(y_n) = lt(z_n)$

14) if $(x_n) \to x \Rightarrow (|x_n|) \to |x|$

15) Thm: (x_n) is a seq of +ve real nos. st $L = lt\left(\frac{x_{n+1}}{x_n}\right)$ exists.

If $L < 1$, then (x_n) cgs & $lt(x_n) = 0$

16) Cauchy's First Theorem on limits:

$$\lim_{n\to\infty} a_n = l \Rightarrow \lim_{n\to\infty} \frac{a_1+a_2+\dots+a_n}{n} = l$$

Proof. Let $b_n = a_n - l$ $\quad \lim_{n\to\infty} b_n = 0$

Also $x_n = \dfrac{a_1+a_2+\dots+a_n}{n} = \dfrac{b_1+b_2+\dots+b_n+nl}{n}$

We have to show that $\dfrac{b_1+b_2+\dots+b_n}{n} \to 0$

Hint: use given $\epsilon > 0$, $\exists\ k$ st $|b_n| < \frac{\epsilon}{2}\ \forall\ n \ge k$

Also $\{b_n\}$ is cgt $\Rightarrow \{b_n\}$ is bdd, ie, $\exists\ M$ st $|b_n| \le M\ \forall n$

Now $\left|\dfrac{\Sigma b_i}{n} - 0\right| \le \dfrac{1}{n}\left[|b_1|+|b_2|+\dots+|b_{k-1}| + |b_k|+\dots+|b_n|\right]$

$\le \dfrac{1}{n}\left[kM + \dfrac{(n-k)\epsilon}{2}\right] = \dfrac{kM}{n} + \dfrac{(n-k)\epsilon}{2}$

$< \dfrac{kM}{n} + \epsilon/2 \quad \left[\because \dfrac{n-k}{n} < 1\right]$

We need $\dfrac{kM}{n} < \dfrac{\epsilon}{2} \Rightarrow n > \dfrac{2mM}{\epsilon}$ if $p \in \mathbb{N}$ & $p > \dfrac{2mM}{\epsilon}$

then choose $q = \max\{p, m\}$

$\left|\dfrac{\Sigma b_i}{n} - 0\right| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon\ \forall\ n \ge q \quad \therefore lt \dfrac{\Sigma b_i}{n} \to 0$

17) Converse of Cauchy's 1st theorem need not be true.

eg $\{a_n\} = (-1)^n \Rightarrow x_n = 0$ n is even
$= -1/n$ n is odd

$\{x_n\} \to 0$ but $\{a_n\}$ is oscillatory.

18) Thm: $\{a_n\}$ is a +ve sequence & $\lim_{n\to\infty} a_n = l$

then $\lim_{n\to\infty} (a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{1/n} = l$

Proof- Let $b_n = \log a_n$ & apply Cauchy's 1st theorem

19) Cauchy's second theorem on limits:

(a_n) is a sequence st $a_n > 0\ \forall n$ & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$

then $\lim_{n\to\infty} (a_n)^{1/n} = l$

Define: $b_n = \frac{a_n}{a_{n-1}}$ & use above theorem

* Converse need not be true. eg. $a_n = 2^{-n+(-1)^n}$

Here $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} 2^{-1+\frac{(-1)^n}{n}} = 2^{-1} = 1/2$

& $\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n+(-1)^n}} = 2^{-1+(-1)^{n+1}-(-1)^n}$

$= 2^{-3}$ n is even
2^{-1} n is odd

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ DNE

eg. Try using this theorem if sequence has $(\ )^{1/n}$.

- Find $\lim (n!)^{1/n}$ $a_n = n!$. $\frac{a_{n+1}}{a_n} = (n+1) \to \infty$

$\therefore \lim (n!)^{1/n} \to \infty$ [or use $a_n = n$, $\lim (a_1 \cdot a_2 \cdots a_n)^{1/n} = \lim a_n = \infty$

Remember : $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ $\left[\text{Don't put } \lim\left(1-\frac{1}{n+1}\right)^n = \right]$

eg $x_n = \frac{n!}{n^n}$ $\frac{x_n}{x_{n+1}} = \frac{n!\,(n+1)^n}{n^n.(n+1)!} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$

$\therefore \lim \frac{x_n}{x_{n+1}} = e \Rightarrow \lim \frac{x_{n+1}}{x_n} = L = \frac{1}{e} < 1 \Rightarrow \lim x_n \longrightarrow 0$

⇒ MONOTONIC SEQUENCES :

1) Monotonically ↑ing $x_n \le x_{n+1}$
Monotonically ↓ing $x_n \ge x_{n+1}$
Strictly monotonically ↑ing $x_n < x_{n+1}$
Strictly monotonically ↓ing $x_n > x_{n+1}$
$\forall\ n \in \mathbb{N}$

2) THM : Every monotonically ↑ sequence which is bdd above converges to its lub / supremum.
Unbdd ↑ sequence diverges to $+\infty$
Similar for ↓ sequences, ie, bdd ⇒ cgs to glb
unbdd ⇒ dgs to $-\infty$

Proof : Easy using definition of lub. , ie, if $x = \sup(x_n)$ & $\epsilon > 0$
then $\exists$ m st $x-\epsilon < x_m \le x < x+\epsilon$

$$\Rightarrow x-\epsilon < x_m \le x_{m+1} \le x_{m+2} \cdots \le x < x+\epsilon$$
$$\Rightarrow x-\epsilon < x_n < x+\epsilon \quad \forall\ n \ge m$$
$$\Rightarrow |x_n - x| < \epsilon \quad \forall\ n \ge m.$$

3) Monotone Convergence Theorem :
A monotone sequence of real no is cgt ⇔ it is bounded

4) Limit Point of a sequence:

l is limit pt if every nbd of l contains ∞ terms of (x_n).

eg $(x_n) = ((-1)^n)$ has 2 limit pts $(-1, 1)$.

Limit of a sequence is a limit pt but converse need not be true.

Because limit pt doesn't exclude possibility of ∞ terms outside the nbd whereas limit has only finite terms outside its nbd.

5) Bolzano - Weierstrass Theorem for sequences:

Every bounded sequence has at least one limit point.

6) Cauchy's General Principle of convergence:

(x_n) cgs iff for each $\epsilon > 0$, $\exists$ $m \in I^+$ st
$|x_{n+p} - x_n| < \epsilon$ $\forall$ $n \geq m$ & $p \geq 1$

NC: $\lim x_n = l$

$|x_n - l| < \epsilon/2$ $\forall$ $n \geq m$

$|x_{n+p} - l| < \epsilon/2$ $\forall$ $n \geq m$ & $p \geq 1$

$|x_{n+p} - x_n| < |x_{n+p} - l| + |x_n - l|$
$< \epsilon/2 + \epsilon/2$
$< \epsilon$

SC:
$|x_{n+p} - x_n| < \epsilon$ $\forall$ $p \geq 1, n \geq m$
$\therefore$ $|x_{m+p} - x_m| < \epsilon$
$\Rightarrow x_m - \epsilon < x_{m+p} < x_m + \epsilon$ $\forall$ $p \geq 1$
$h = \min \{x_1, x_2 \ldots, x_{m-1}, x_m + \epsilon\}$
$k = \max \{ \quad " \quad \}$
$h \leq x_n \leq k$ $\forall n$
$\therefore$ (x_n) is bdd

By B-W $\Rightarrow$ (x_n) has a limit point say l.

Now pt $\lim x_n = l$

$|x_{m+p} - x_m| < \epsilon/3$ [for $n = m$] since L is limit pt $\exists$ $m_1 > m$ st $|x_{m_1} - l| < \epsilon/3$

Also $|x_{m_1} - x_m| < \epsilon/3$ [$\because m_1 > m$] using $\Rightarrow$ $|x_{m+p} - l| \leq |x_{m+p} - x_m| + |x_m - x_{m_1}| + |x_{m_1} - l|$
$= \epsilon$ $\therefore$ $|x_n - l| < \epsilon$ $\forall$ $n \geq m$

7) Cauchy Sequence

(x_n) is cauchy or fundamental sequence if

$\forall \epsilon > 0, \exists m \in I^+$ st $|x_{n+p} - x_n| < \epsilon \;\; \forall n \geq m, p \geq 1$

or $|S_p - S_q| < \epsilon \;\; \forall p, q \geq m$.

* Every cauchy sequence is bdd.
* If $X = \{x_n\}$ is cgt $\Rightarrow X$ is cauchy sequence

⇒ Example on Bernoullis :

1) $0 < b < 1$ then $lt(b^n) = 0$

Put $b = \frac{1}{1+a}$ where $a = \frac{1}{b} - 1 > 0$

$\therefore (1+a)^n \geq 1 + na \;\; \forall n$

$\frac{1}{(1+a)^n} \leq \frac{1}{(1+na)} \;\; \forall n \in \mathbb{N}$

$\therefore b^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$

$|b^n - 0| = < \frac{1}{na} = \left(\frac{1}{n}\right)\left(\frac{1}{a}\right)$ $\quad lt \frac{1}{n} = 0$ & $\frac{1}{a} > 0$

$\therefore lt \frac{1}{na} = 0$.

2) $c > 0$, then $\lim_{n \to \infty} (c^{1/n}) = 1$

(a) $c = 1 \Rightarrow (c^{1/n}) = \{1, 1, 1 \ldots\}$

$\lim_{n \to \infty} c^{1/n} = 1$

(b) $c > 1 \Rightarrow c^{1/n} = 1 + d_n \quad d_n > 0$

$c = (1+d_n)^n \geq 1 + nd_n$

$\frac{(c-1)}{n} \geq d_n$

$|c^{1/n} - 1| = |d_n| < \frac{(c-1)}{n}$

$(c-1) > 0$ & $\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \to \infty} |c^{1/n} - 1| = 0$

$\therefore$

(c) $c < 1 \Rightarrow c^{1/n} = \frac{1}{1+h_n} \quad h_n > 0$

$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} \leq \frac{1}{nh_n} \Rightarrow h_n \leq \frac{1}{nc}$

$|c^{1/n} - 1| = \left|\frac{1}{1+h_n} - 1\right| = \left|\frac{h_n}{1+h_n}\right| < h_n \leq \frac{1}{nc}$

$\frac{1}{n} \to 0$ as $\frac{1}{c} > 0$

$\therefore c^{1/n} \to 1$.

Examples on convergence :

1) Show that $x_n = \left(1+\frac{1}{n}\right)^n$ is cgt

$x_n = \left(1+\frac{1}{n}\right)^n \quad \forall\ n \in N \Rightarrow x_n = 1 + {}^nC_1 \cdot \frac{1}{n} + {}^nC_2 \frac{1}{n^2} + \cdots + {}^nC_n \frac{1}{n^n}$

$$x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$x_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

$n+1 > n \quad \forall\ n \in N$

$\frac{1}{n+1} < \frac{1}{n} \Rightarrow \frac{-1}{n+1} > \frac{-1}{n} \Rightarrow 1-\frac{1}{n+1} > 1-\frac{1}{n} \quad \forall\ n \in N$

$\therefore \quad x_{n+1} \geq x_n \Rightarrow (x_n)$ is $\uparrow$ sequence

Now $x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \cdots$

$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}} \qquad \left[\because 2^n \leq (n+1)!\right]$

$= 1 + \frac{\left(1-\frac{1}{2^n}\right)}{1-1/2} = 1 + 2\left(1-\frac{1}{2^n}\right)$

$= 3 - \frac{1}{2^{n-1}}$

$< 3 \quad \forall\ n$

$\therefore\ x_n < 3 \Rightarrow (x_n)$ is $\uparrow$ ing & bdd above.

2) $\{x_n\}$ is defd by $x_1 = \sqrt{7} \quad x_{n+1} = \sqrt{7+x_n}$

St $\{x_n\}$ cgs to +ve root of $x^2 - x - 7 = 0$

$\Rightarrow x_1 = \sqrt{7} \quad x_2 = \sqrt{7+\sqrt{7}} > \sqrt{7} = x_1$

Suppose $x_n > x_{n-1}$

$7 + x_n > 7 + x_{n-1}$

$\sqrt{7+x_n} > \sqrt{7+x_{n-1}}$

$\Rightarrow x_{n+1} > x_n$ [Hence $\uparrow$ by induction]

$\Rightarrow \quad x_1 = \sqrt{7} < 7$

$7 + x_1 < 14$

$\sqrt{7+x_1} < \sqrt{14} < \sqrt{49} = 7$

Let $x_n < 7$

$x_n + 7 < 14$

$\sqrt{7+x_n} < \sqrt{14} < \sqrt{49} = 7$

$\therefore$ Bdd above by Induction

$x_{n+1} = \sqrt{7+x_n}$

$\Rightarrow$ let $x_{n+1} = l = \sqrt{7+l}$

$l^2 = 7 + l$

$\Rightarrow l^2 - l - 7 = 0$

$l \to$ +ve root of $x^2 - x - 7 = 0$ as $x_n \geq \sqrt{7}$

3) Let $x_1 = 8$ & $x_{n+1} = \frac{x_n}{2} + 2$.

ST (x_n) is bdd and monotone. Find its limit.

$x_1 = 8 \quad x_2 = \frac{8}{2} + 2 = 6 \quad x_1 > x_2$

Let $x_n > x_{n+1} \Rightarrow \frac{x_n}{2} + 2 > \frac{x_{n+1}}{2} + 2 \Rightarrow x_{n+1} > x_{n+2}$

$\therefore (x_n)$ is MD.

$x_n > x_{n+1} \Rightarrow x_n > \frac{x_n}{2} + 2 \Rightarrow x_n > 4$

$\therefore (x_n)$ is bdd below as well as above

$\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{x_n}{2} + 2 \Rightarrow l = \frac{l}{2} + 2 \Rightarrow l = 4.$

4) $y_1 = \sqrt{p}$, $p > 0$, $y_{n+1} = \sqrt{p + y_n}$ $n \in \mathbb{N}$

ST (y_n) cgs & find limit

Easy to st (y_n) is $\uparrow$ing. $y_n < y_{n+1} \Rightarrow y_n < \sqrt{p + y_n}$

$\Rightarrow y_n^2 - y_n - p < 0 \Rightarrow \left[y_n - \frac{(1+\sqrt{1+4p})}{2}\right]\left[y_n - \frac{(1-\sqrt{1+4p})}{2}\right] < 0$

$\Downarrow$ ≥ 0 always

$\Rightarrow y_n < \frac{1+\sqrt{1+4p}}{2}$

$\Rightarrow y_n$ is bdd. Easy to find limit now

5) $y_n = \sqrt{n+1} - \sqrt{n}$ st (y_n) & $(\sqrt{n}\, y_n)$ cgs. Find their limits

$y_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$< \frac{1}{2\sqrt{n}}$

$\Rightarrow 0 < y_n < \frac{1}{2\sqrt{n}}$

$\lim 0 < \lim y_n < \lim \frac{1}{2\sqrt{n}}$

$0 < \lim y_n < 0 \Rightarrow \lim y_n = 0$

Also $\rightarrow$ $y_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$\frac{1}{2\sqrt{n+1}} < y_n < \frac{1}{2\sqrt{n}}$

$\frac{\sqrt{n}}{2\sqrt{n+1}} < \sqrt{n}\, y_n < \frac{1}{2}$

$x_n < \sqrt{n}\, y_n < z_n$

$\lim x_n = \lim z_n = 1/2$

$\therefore \lim \sqrt{n}\, y_n = 1/2$

Ques. $\{x_n\} = \sqrt{x_{n-1}y_{n-1}}$ $\{y_n\} \Rightarrow \frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$ $x_1 = \frac{1}{2}, y_1 = 1$

Prove: $x_{n-1} < x_n < y_n < y_{n-1}$ & deduce they reach same limit and $\frac{1}{2} < l < 1$

$x_1 < y_1$; $x_2 = \sqrt{x_1 y_1} < \sqrt{y_1 y_1} < y_1$ $\Rightarrow \frac{1}{y_2} = \frac{1}{2}\left(\frac{1}{x_2} + \frac{1}{y_1}\right) < \frac{1}{2}\left(\frac{2}{x_2}\right)$

$\Rightarrow x_2 < y_2$

$\therefore$ Assume $x_n < y_n \Rightarrow x_{n+1} = \sqrt{x_n y_n} < y_n$

$\frac{1}{y_{n+1}} = \frac{1}{2}\left(\frac{1}{x_{n+1}} + \frac{1}{y_n}\right) < \frac{1}{2} \cdot \frac{2}{x_{n+1}} \Rightarrow x_{n+1} < y_{n+1}$

$\Rightarrow$ $\boxed{x_n < y_n}$ $\Rightarrow \frac{2}{y_n} = \frac{1}{x_n} + \frac{1}{y_{n-1}} \Rightarrow \frac{2}{y_n} - \frac{1}{y_{n-1}} > \frac{1}{y_n}$

$\Rightarrow \frac{1}{y_n} > \frac{1}{y_{n-1}} \Rightarrow \boxed{y_{n-1} > y_n}$

$\therefore x_n = \sqrt{x_{n-1} y_{n-1}}$
$> \sqrt{x_{n-1} \cdot x_{n-1}} = x_{n-1}$

$\therefore$ $\boxed{x_{n-1} < x_n < y_n < y_{n-1}}$

$\therefore$ $l_1^2 = l_1 l_2$ $\frac{1}{l_2} = \frac{1}{2}\left(\frac{l_1 + l_2}{l_1 l_2}\right) \Rightarrow 2l_1 = l_1 + l_2$
$\boxed{l_1 = l_2}$

⇒ Infinite Series

1) $\{x_n\}$ is a sequence of real nos.
Sum of its terms is called Infinite series.

$$\sum_{n=1}^{\infty} x_n$$

If all terms are +ve, Σx_n is series of +ve terms
If terms are +ve & -ve alternatively, Alternating series.

2) n^{th} Partial Sum : $S_n = x_1 + x_2 + \cdots + x_n$
$\{S_n\}$ is the sequence of partial sums.

3) Σx_n cgs if $\{S_n\}$ cgs
Σx_n dgs if $\{S_n\} \to +\infty$ or $-\infty$
Oscillatory Series → neither cgs nor dgs
Oscillates finitely if $\{S_n\}$ is bdd & neither cgs nor dgs
Constant series is dgt unless 0. [whereas constt sequence is cgt]

4) Geometric Series : $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$

a) cgs if $|r| < 1$
b) dgs if $r \geq 1$
c) oscillates finitely if $r = -1$
d) oscillates infinitely if $r < -1$

5) Nature of series doesn't change if a finite no of terms are added or removed or each term is multiplied or divided by a fixed non-zero no.

6) Convergence Tests :

a) P-Test :

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p} + \dots$$

cgs : $p > 1$
dgs : $p \le 1$

b) n^{th} Term test :

If $\sum x_n$ cgs $\Rightarrow \lim_{n\to\infty} (x_n) = 0$

used for negation, ie,

$\lim_{n\to\infty} (x_n) \neq 0 \Rightarrow \sum x_n$ is not cgt.

* A +ve term series either cgs or (dgs to ∞)

* If $x_n > 0 \ \forall n$ & $\lim_{n\to\infty} x_n \neq 0$ then $\sum x_n$ dgs to $+\infty$

c) Comparison Test :

$\{x_n\}$ & $\{y_n\}$ are non-negative sequences & for some $k \in \mathbb{N}$

$0 \le x_n \le y_n$ for $n \ge k$

$\sum y_n$ cgs $\Rightarrow \sum x_n$ cgs
$\sum x_n$ dgs $\Rightarrow \sum y_n$ dgs

d) Limit Comparison Test :

$X = (x_n)$ & $Y = (y_n)$ are strictly +ve sequences

& $r = \lim \left(\frac{x_n}{y_n}\right)$

i) $r \neq 0$ (finite), then
$\sum x_n$ is cgt (dgt) $\Leftrightarrow \sum y_n$ is cgt (dgt)

ii) $r = 0$, $\sum y_n$ cgt $\Rightarrow \sum x_n$ cgt

iii) $r = \infty$, $\sum y_n$ dgt $\Rightarrow \sum x_n$ dgt

TIP: Use rationalisation only in case of square roots. o/w go for binomial expansion

eg $x_n = (n+1)^{1/3} - n^{1/3}$

$= n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3} - 1\right]$

$= n^{1/3}\left[1 + \frac{1}{3n} - \frac{1}{9n^2} + \dots - 1\right]$

$= \frac{1}{n^{2/3}}\left[\frac{1}{3} - \frac{1}{9n} + \dots\right]$

Take $y_n = 1/n^{2/3}$

$r = \lim \frac{x_n}{y_n} = 1/3$ = finite

$\sum y_n$ dgs as $\frac{2}{3} < 1 \Rightarrow \sum x_n$ dgs

* eg. $\sum x_n = \sum 1/n^{(1+1/n)}$

$x_n = \frac{1}{n \cdot n^{1/n}}$ $\quad y_n = 1/n$

$\frac{x_n}{y_n} = \frac{1}{n^{1/n}}$ $\quad \lim \frac{x_n}{y_n} = 1$ $\left[\because \lim (n)^{1/n} = 1 \text{ using cauchy's 2nd theorem}\right]$

$\therefore \sum y_n$ dgs $\Rightarrow \sum x_n$ dgs

e) D' Alembert's Ratio Test:

If Σu_n is a series of +ve terms st $\lim \frac{u_n}{u_{n+1}} = l$, then

i) $l > 1 \Rightarrow \Sigma u_n$ cgs
ii) $l < 1 \Rightarrow \Sigma u_n$ dgs
iii) $l = 1 \Rightarrow$ Test fails

Used when fractionals & combinations of powers is involved

eg. $\Sigma \frac{x^n}{3^n \cdot n^2}$, $x > 0$

$\lim \frac{u_n}{u_{n+1}} = \frac{3}{x}$ $\therefore$ $x > 3 \Rightarrow$ dgs
$x < 3 \Rightarrow$ cgs

$x = 3 \Rightarrow \Sigma \frac{1}{n^2} \Rightarrow$ cgs.
(Need to look separately)

f) Cauchy's Root Test:

If Σu_n is +ve term series

i) If $\lim (u_n)^{1/n} = l$, then
$l < 1 \Rightarrow$ cgs
$l > 1 \Rightarrow$ dgs
$l = 1 \Rightarrow$ fails
(remember with help of (e) + Cauchy's 2nd thm)

ii) If $\lim (u_n)^{1/n} = \infty$, Σu_n is dgt

* Root test is used when powers are involved
* It is more general than Ratio test because Cauchy's 2nd thm says if $\frac{u_{n+1}}{u_n}$ exists, $(u_n)^{1/n}$ also exists but converse might not be true.

g) Raabe's Test:

Σu_n is series of +ve terms st $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then

i) $l > 1 \Rightarrow$ cgs
ii) $l < 1 \Rightarrow$ dgs
iii) $l = 1 \Rightarrow$ fails

stronger than Ratio Test.

h) Logarithmic Test:

$$l = \lim_{n \to \infty} n \log \frac{u_n}{u_{n+1}}$$

i) $l > 1 \Rightarrow$ cgs
ii) $l < 1 \Rightarrow$ dgs
iii) $l = 1 \Rightarrow$ fails

Used when e is involved obv.

i) Gauss Test:

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\alpha_n}{n^{1+\delta}}$$

where $\delta > 0$ & (α_n) is a bdd seq.

$\lambda > 1 \Rightarrow$ cgs
$\lambda \leq 1 \Rightarrow$ dgs
↳ Here we can say for $\lambda = 1$ also!!

Use binomial expansions if needed.

Apply after failure of Raabe's / Ratio test.

Directly go for Gauss if Ratio test fails j> De Morgan's & Bertrand's Test

$$l = \lim_{n\to\infty} \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 \right\} \log n \right]$$

$l > 1 \Rightarrow$ cgs , $l < 1 \Rightarrow$ dgs

If e is there :

$$l = \lim_{n\to\infty} \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right]$$

$l > 1 \Rightarrow$ cgs $\quad l < 1 \Rightarrow$ dgs

k> Cauchy's Condensation Test

Let $\Sigma a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers.

$\Sigma a(n)$ cgs (or dgs) iff

$\Sigma 2^n a(2^n)$ cgs (or dgs)

Example : $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$

$$\Sigma 2^n a(2^n) = \frac{2^n}{2^n (n \log 2)(\log n \log 2)}$$

$$= \frac{1}{n(\log 2)(\log n \log 2)}$$

$n \log 2 < n$

$\dfrac{1}{\log(n \log 2)} > \dfrac{1}{(\log n)}$

Take $y_n = \dfrac{1}{n \log 2 (\log n)}$

$x_n \geq y_n \quad \Sigma y_n = \dfrac{1}{\log 2} \Sigma \dfrac{1}{n \log n}$ ↳ dgs

$\therefore \; x_n$ dgs.

l> Comparison Test :

$\Sigma u_n, \Sigma v_n$ are +ve-series & $h, k \in \mathbb{R}$ st $h v_n < u_n < k v_n \; \forall n$

Then both cgs or dgs together.

m> Leibnitz's test on Alternating Series :

$\Sigma (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

$u_n > 0 \; \forall n$

cgs if (i) $u_n \geq u_{n+1} \; \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

,ie, u_n should be a decreasing +ve sequence tending to 0.

n> Abel's Test :

If $\sum_{n=1}^{\infty} a_n$ is cgt & sequence $\{b_n\}$ is monotonic & bdd, then $\Sigma a_n \cdot b_n$ is cgt

o> Dirichilet's Test :

If $\sum_{n=1}^{\infty} a_n$ is a series whose nth partial sum $\{S_n\}$ is bdd & $\{b_n\}$ is a monotonic sequence converging to 0, then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Leibnitz is special case of Dirichilet's where $a_n = (-1)^n$ & $b_n = u_n$

SUMMARY

No	Test	Condition / Terms
1	P- Test	$\sum \frac{1}{n^p}$ $p>1 \longrightarrow$ cgs $p \le 1 \longrightarrow$ dgs
2	n^{th} Term Test	$\sum x_n$ cgs $\Rightarrow$ lt $x_n = 0$ or lt $x_n \neq 0 \Rightarrow \sum x_n$ doesn't converge
3	Comparison Test	$0 \le x_n \le y_n$ $\sum y_n$ cgs $\Rightarrow \sum x_n$ cgs $\sum x_n$ dgs $\Rightarrow \sum y_n$ dgs
4	Limit Comparison Test	$l =$ lt $\left(\frac{x_n}{y_n}\right)$ $l \neq 0$ (finite) cgs, dgs together $l = 0$ $\sum y_n$ cgt $\Rightarrow \sum x_n$ cgt $l = \infty$ $\sum y_n$ dgt $\Rightarrow \sum x_n$ dgt
5	D'Alembert's Ratio Test	$l =$ lt $\frac{u_n}{u_{n+1}}$ $L>1$ cgs $L<1$ dgs $L=1$ fails
6	Cauchy Root test	$l =$ lt $(u_n)^{1/n}$ Only in this case $L>1 \Rightarrow$ dgs $L<1 \Rightarrow$ cgs
7	Raabe's Test	$l =$ lt $n\left(\frac{u_n}{u_{n+1}} - 1\right)$ Same as ratio
8	Logarithmic Test	$l =$ lt $n \log\left(\frac{u_n}{u_{n+1}}\right)$ Same as ratio
9	Gauss Test	$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$ Tie breaker $\lambda = 1 \Rightarrow$ dgs
10	De Morgan & Bertrand's Test	$l =$ lt $\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right]$
11	Alternative	$l = \lim_{n\to\infty} \left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right]$
12	Cauchy Condensation Test	(a_n) is $\downarrow$ +ve seq ; $\sum a(n)$ & $\sum 2^n (a(2^n))$ cgs (or dgs) together
13	Comparison Test	$\sum u_n$ & $\sum v_n$ are +ve st $h v_n < u_n < k v_n \; \forall n$ cgs or dgs together
14	Leibnitz Test on alternating series	u_n is $\downarrow$ +ve sequence & lt $u_n = 0$ $\Rightarrow \sum (-1)^n u_n$ cgs.
15	Abel's Test	$\sum a_n$ is cgt & $\{b_n\}$ is monotonic -bdd $\Rightarrow \sum a_n b_n$ is cgt
16	Dirichlet's Test.	Partial sum $\{S_n\}$ of $\sum a_n$ is bdd & $\{b_n\}$ is monotonic $\longrightarrow 0$, $\Rightarrow \sum a_n b_n$ is cgt

→ Some examples:

1) Checking cgs of $\sum \sin\frac{1}{n}$ or $\sum \frac{1}{\sqrt{n}} \tan^{-1}\frac{1}{n}$

a) Use $y_n = \frac{1}{n}$ $\quad lt \frac{x_n}{y_n} = 1$ & y_n dgs ⇒ x_n dgs.

b) Expand & let $y_n = 1/n^{3/2}$

2) Check cgs of $\sum \frac{1}{(\log n)^{\log n}}$

we can find 'n' large enough so that $\log(\log n) > 2$ $\quad [\because \log(\log n) \to \infty]$

$$\log n\,[\log(\log n)] > 2\log n$$

$$\Rightarrow \log(\log n)^{\log n} > \log n^2$$

$$\Rightarrow (\log n)^{\log n} > n^2$$

$$\Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$$

↳ cgs

↳ cgs. ⇐

(b) $\sum r^{\log n}$ $\qquad (\log n)(\log r) = (\log r)(\log n)$ $\quad [\because x^n$ is comm.

$$\log r^{\log n} = \log n^{\log r}$$

$$r^{\log n} = n^{\log r}$$

$$\sum r^{\log n} = \sum \frac{1}{n^{-\log r}}$$

It cgs iff $-\log r > 1 \Rightarrow \log r < -1 \Rightarrow r < e^{-1} = r < \frac{1}{e}$

3) St $\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3} - n}{\log n}$ cgs. $\quad a_n = (n^3+1)^{1/3} - n \quad a_n = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{3}\frac{1}{5n^3}\ldots\right.$

$b_n = \frac{1}{\log n}$ → monotonic $\quad \sum a_n$ cgs

⇓

$\sum a_n b_n$ cgs by Abel's

4) $\sum \frac{(-1)^{n-1}}{n^p}$ $(p>0)$ $\quad \{a_n\} = (-1)^{n-1} \quad \{b_n\} = \frac{1}{n^p}$

$S_n = 0$ even n $\qquad b_n$ is monotonic → 0

$\quad = 1$ odd n $\qquad$ By ~~Abel~~ Dirichilet it cgs.

⇒ bdd

7) Absolute & Conditional Convergence :

$\sum u_n$ is absolutely cgt if $\sum |u_n|$ is cgt

If $\sum u_n$ cgs but not absolutely, then its conditional convergence

Absolute convergence $\Rightarrow$ convergence

Converse is not true eg $\sum \frac{(-1)^{n-1}}{n}$

8) Rearrangement of Terms :

$\sum b_n$ & $\sum a_n$ are said to be rearrangement if $\exists$ a 1-1 correspondence b/w the terms. (i.e, every a_n is some b_n & vice versa)

Rearrangement may change the nature of series completely.

9) Dirichlet Theorems :

(I) if $\sum_{n=1}^{\infty} a_n$ is a +ve-term series cgs to 's' then any derangement $\sum b_n$ of $\sum a_n$ also cgs to 's'

(II) if $\sum a_n$ is dgt, then so is $\sum b_n$

(III) If $\sum a_n$ is absolutely cgt, then any derangement $\sum b_n$ is also abs. cgt to same sum.

eg $1+1/3^2-1/2^2+1/5^2 \cdots \equiv 1-1/2^2+1/3^2 \cdots$

Abs cgt $\Leftarrow$ Abs cgt

10) Riemann's Theorem :

A conditionally cgt series can be made by derangement

(i) to converge to any real no.

(ii) to diverge to any $+\infty$ or $-\infty$

(iii) to oscillate finitely or infinitely.

17) Pringsheims Method :

Generally we have the series $\sum \frac{(-1)^{n-1}}{n}$ which is conditionally cgt whose sum is $\log 2$.

Now new arrangement has 'α' +ve & 'β' -ve terms alternatively. The new sum is $\log 2 + \frac{1}{2}\log\left(\frac{\alpha}{\beta}\right)$

eg Find sum of $1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots$

$\alpha = 1 \quad \beta = 2 \qquad S' = \log 2 + \frac{1}{2}\log(1/2) = \frac{1}{2}\log 2$

⇒ Product of Series :

1) Cauchy Product of 2 infinite series

$$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$$

$$= a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots$$

$$= c_1 + c_2 + c_3 + \cdots$$

$\therefore$ $c_n = \sum_{r=1}^{n} a_r b_{n-r+1}$ for each $n \in N$

if sum is $\sum_{n=0}^{\infty}$ then $c_n = \sum_{r=0}^{\infty} a_r b_{n-r}$

If 2 series cgs, the product may not cgs.

2) Conditions for convergence of $\sum c_n$

a) If $\sum a_n$ & $\sum b_n$ are non-negative & cgs to A & B then $\sum c_n$ cgs to AB.

b) If $\sum a_n$ & $\sum b_n$ are abs. cgt to A & B, then $\sum c_n$ is abs. cgt to AB

c) Merten's theorem : Both $\sum a_n$ & $\sum b_n$ cgs & one of them is abs. cgt, then $\sum c_n$ cgs to AB.

3) Abel's Test : Let $\sum_{n=1}^{\infty} a_n$ be cgt to A & $\sum_{n=1}^{\infty} b_n$ to B.

If $\sum c_n$ cgs, then $\sum c_n = AB$.

4) Cauchy product of 2 conditionally cgt series need not be cgt eg. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = \frac{(-1)^{n-1}}{n}$

Example ① ST $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ cauchy product with itself is not abs. cgt.

$a_n = b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$ $\quad c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$

$= (-1)^{n-1}\left[\frac{1}{1\cdot\sqrt{n}} + \frac{1}{\sqrt{2}\cdot\sqrt{n-1}} + \dots + \frac{1}{\sqrt{n}\cdot\sqrt{1}}\right]$

Now $r < n \Rightarrow \frac{1}{\sqrt{r}} > \frac{1}{\sqrt{n}}$

$\therefore |c_n| \geq \left|(-1)^{n-1}\cdot\left[\frac{1}{\sqrt{n}\cdot\sqrt{n}} + \frac{1}{\sqrt{n}\cdot\sqrt{n}} + \dots\right]\right|$

$\geq \left|\frac{n}{n}\right| \geq 1$

$\lim_{n\to\infty} c_n \neq 0$. Σc_n doesn't cgt absolutely

(ii) ST. Cauchy Product of 2 dgt series

$\sum_{n=1}^{\infty} a_n = 2 + 2^1 + 2^2 + \dots$ & $\sum_{n=1}^{\infty} b_n = -1 + 1 + 1 + \dots$ is cgt

★ Don't forget to show that Σa_n & Σb_n are dgt.

$a_n = \begin{cases} 2 & n=1 \\ 2^{n-1} & n\neq 1 \end{cases}$ $\quad b_n = \begin{cases} -1 & n=1 \\ 1 & n\neq 1 \end{cases}$

$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_{n-1} b_2 + a_n b_1 = 2\cdot 1 + 2^{2-1} + 2^{3-1} + \dots + 2^{n-1-1} - 2^{n-1}$

$= 2 + 2 + 2^2 + 2^3 + \dots + 2^{n-2} - 2^{n-1}$

$= 1 + (2^0 + 2^1 + \dots + 2^{n-2}) - 2^{n-1}$

$= 1 + \frac{1\cdot(2^{n-1}-1)}{2-1} - 2^{n-1} = 1 + 2^{n-1} - 1 - 2^{n-1} = 0$

$c_n = 0$ $\quad\therefore \Sigma c_n$ is cgt

(iii) ST $\left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \dots + \frac{1}{n\cdot 1}\right]$

Here we have to show that Σc_n cgs.

See page 21 in notes

⇒ Infinite Products :

1) (a_n) is a seq. $\prod_{n=1}^{\infty} a_n = a_1 \cdot a_2 \cdot a_3 \cdots a_n \cdots$ is inf. product

Partial Product : $P_n = \prod_{r=1}^{n} a_r$

(P_n) ⇒ sequence of partial products.

$\prod a_n$ cgs if (P_n) cgs.

2) Convergence of infinite products :

$P_n = \prod_{r=1}^{n} a_r$ is the n^{th} partial product of $\prod_{n=1}^{\infty} a_n$

a) If infinitely many factors 'a_n' are zero → dgs to 0

b) Finitely many a_n are 0 → if product after removal of 0 ans cgs then $\prod a_n$ cgs.

c) If (P_n) oscillates → $\prod a_n$ oscillates

d) If finite a_n are < 0, $\exists$ m st $a_n > 0 \ \forall n > m$ then $\prod a_n$ cgs if $\prod_{n=m+1}^{\infty} a_n$ cgs.

e) If no a_n is 0, then $\prod_{n=1}^{\infty} a_n$ has cgs if (P_n) cgs.

$\lim_{n\to\infty} P_n = P$, then $\prod a_n = P$ — eg $\prod \left(1 - \frac{1}{(n+1)^2}\right)$

$\lim_{n\to\infty} P_n = \infty$ — $\prod a_n$ dgs to ∞ — eg. $\prod (1 + 1/n)$

$\lim_{n\to\infty} P_n = 0$ — $\prod a_n$ dgs to 0. — eg $\prod_{n=2}^{\infty} (1 - 1/n)$

3) Convenient to write as $\prod_{n=1}^{\infty} (1 + a_n)$ & assume $a_n > -1$ st $\log(1+a_n)$ is defd

4) ST $\prod(1 - 1/n^2)$ is cgt ⇒ $\log P = \sum \log(1 - 1/n^2)$ — $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$

$a_n = \log\left(1 - \frac{1}{n^2}\right) = -\left[\frac{1}{n^2} + \frac{1}{2n^4} + \cdots\right] = \frac{-1}{n^2}[\quad] \Rightarrow \sum a_n$ is cgt

∴ $\log P = $ a finite no as $n \to \infty$ ⇒ $P \to$ a finite no.

5) Necessary condition for convergence:

if $\prod(1+a_n)$ is cgt $\Rightarrow \lim_{n\to\infty} a_n = 0$

6) General rules of convergence:

a) If $a_n \geq 0$ then $\sum a_n$ & $\prod(1+a_n)$ cgs | dgs together

b) $-1 < a_n \leq 0$ $\sum a_n$ & $\prod(1+a_n)$ cgs or dgs together

c) $0 \leq b_n < 1$ if $\sum b_n$ cgs $\Rightarrow \prod(1-b_n)$ cgs to non-zero finite lmt

if $\sum b_n$ dgs $\Rightarrow \prod(1-b_n)$ dgs to zero

d) $\sum a_n^2$ cgt $\Rightarrow$ $\prod(1+a_n)$ & $\sum a_n$ cgs or dgs together

e) $\sum a_n^2$ cgt $\Rightarrow$ $\sum a_n$ & $\sum \log(1+a_n)$ cgs or dgs together

$\sum a_n^2$ dgt & $\sum a_n$ cgs or oscillates finitely $\Rightarrow \prod(1+a_n)$ dgs to 0.

7) Absolute Convergence:

$\prod(1+a_n)$ is abs cgt if $\prod(1+|a_n|)$ is cgt

$\therefore$ $\prod(1+a_n)$ is abs cgt iff $\sum a_n$ is abs cgt

iff $\sum \log(1+a_n)$ is abs cgt

Factors of abs cgt product can be rearranged in any order w/o affecting the convergence

Examples:

1) $\prod_{n\geq1}^{\infty}\left(1+\frac{1}{n^\alpha}\right)_{\alpha>1} \Rightarrow$ cgt [6(a)]

2) $\prod_{n\geq1}^{\infty}\left(\frac{3n}{3n+1}\right) = \prod_{n\geq1}^{\infty}\left(1-\frac{1}{3n+1}\right)$ - dgs to 0 [6(c)(ii)]

3) $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right)$ $\quad a_n = \frac{(-1)^n}{n}$ $\quad \sum a_n^2 = \sum \frac{1}{n^2} \Rightarrow$ cgt

$\sum a_n \Rightarrow$ cgt by Leibnitz

$\therefore \quad \prod(1+a_n)$ is cgt $\quad$ [c(d)]

4) $\prod\left(1+\frac{(-1)^n}{n^\alpha}\right)$ $\quad \sum a_n$ cgt by leibnitz

$\sum a_n^2 = \sum \frac{1}{n^{2\alpha}}$ is cgt for $2\alpha > 1 \Rightarrow \alpha > 1/2$

5) $\prod\left(1+\left(\frac{nx}{n+1}\right)^n\right)$ $\quad a_n = \left(\frac{nx}{n+1}\right)^n \geq 0$ $\quad (a_n)^{1/n} = \frac{nx}{n+1} = \frac{x}{1+1/n}$

$\lim (a_n)^{1/n} = x$ $\quad x>1 \quad \sum a_n$ dgt $\Rightarrow \prod(1+a_n)$ dgt

$x<1 \quad \sum a_n$ cgt $\Rightarrow \prod(1+a_n)$ cgt

$x=1 \Rightarrow a_n = \left(\frac{n}{n+1}\right)^n$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} \neq 0$

$\lim a_n \neq 0$ & $a_n \geq 0 \Rightarrow \sum a_n$ is dgt (nth term test) $\Rightarrow \prod(1+a_n)$ is dgt

6) PT $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is abs cgt for any real x.

$1+a_n = \left(1+\frac{x}{n}\right)e^{-x/n} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{x^2}{2!\,n^2}-\frac{x^3}{3!\,n^3}-\ldots\right)$

$= \left[1-\frac{x^2}{2n^2}+\frac{x^3}{3n^3}-\ldots\right]$

$a_n = \frac{x^2}{n^2}\left[\frac{-1}{2}+\frac{x}{3n}-\ldots\right]$ $\quad |a_n| = \frac{x^2}{n^2}\left|\frac{-1}{2}+\frac{x}{3n}-\ldots\right|$

Let $b_n = 1/n^2$ $\quad \lim \frac{|a_n|}{b_n} = \frac{x^2}{2} =$ finite

$\Rightarrow \sum |a_n|$ cgs $\Rightarrow \prod(1+a_n)$ is abs cgt

7) Discuss cgs of $\prod \left(1+\frac{x^n}{x^{2n}+1}\right)$

$$a_n = \frac{x^n}{x^{2n}+1} \Rightarrow \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{x^n}{x^{2n}+1} \cdot \frac{x^{2n+2}+1}{x^{n+1}}\right| = \left|\frac{x^{2n+2}+1}{x^{2n+1}+x}\right|$$

(i) $|x|<1 \Rightarrow \left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{|x|}\left|\frac{x^{2n+2}+1}{x^{2n}+1}\right|$ lt $\left|\frac{a_n}{a_{n+1}}\right| = 1/|x| > 1$

$\Rightarrow \sum |a_n|$ is cgt $\Rightarrow \prod(1+a_n)$ is abs cgt

(ii) $|x|>1 \Rightarrow$ lt $\left|\frac{a_n}{a_{n+1}}\right| =$ lt $\left|x \cdot \frac{1+1/x^{2n+2}}{1+1/x^{2n}}\right| = |x| > 1$

$\Rightarrow \sum|a_n|$ is cgt $\Rightarrow \prod(1+a_n)$ is abs cgt

(iii) $x=1 \Rightarrow a_n = \frac{1}{2} \Rightarrow$ lt $a_n \neq 0 \Rightarrow \sum a_n$ is dgt

$\therefore \prod(1+a_n)$ is dgt

b) ST: $\prod (1+x^{2^n})$ cgs to $\frac{1}{1-x}$ if $|x|<1$

$$P_n = (1+x)(1+x^2)(1+x^4)(1+x^8) \cdots (1+x^{2^{n-1}})$$

$$P_n = \frac{1}{1-x}\left[(1-x^2)(1+x^2)(1+x^4) \cdots\right]$$

$$= \frac{1}{(1-x)}\left[(1-x^4)(1+x^4)(1+x^8) \cdots\right]$$

$$\vdots$$

$$P_n = \frac{1}{(1-x)}\left[1-x^{2^n}\right]$$

$$\lim_{n\to\infty} P_n = \frac{1}{1-x} \quad \left[\because |x|<1 \Rightarrow |x|^{2^n} \to 0 \text{ as } n \to \infty\right]$$

⇒ Limits & Continuity :

1) f is bounded on D if $\exists\ M \in \mathbb{R}^+$ st
$|f(x)| \le M \quad \forall\ x \in D$.

2) Limit of a function :
$A \subseteq \mathbb{R}$ & c is a limit pt of A.
Function $f : A \longrightarrow \mathbb{R}$
'L' is said to be the limit of f at c if
$\forall\ \epsilon > 0,\ \exists$ a $\delta > 0$ $(\delta(\epsilon))$ st if $x \in A$ & $0 < |x-c| < \delta$
$\Rightarrow |f(x) - L| < \epsilon$

To prove $\lim_{x\to c} f(x) \ne L$, we have to show that
for an $\epsilon > 0$ & any $\delta > 0$, $\exists\ x \in A$ st $0 < |x-c| < \delta$
$\Rightarrow |f(x) - L| > \epsilon$

Sequential Criterion : $\forall\ (x_n) \longrightarrow c \Rightarrow (f(x_n))$ cgs to L
$x_n \in A$ & $x_n \ne c$.

f doesn't have a limit at c iff $\exists$ a sequence (x_n)
st $(x_n) \longrightarrow c$ but $(f(x_n))$ doesn't cgs or
cgs not to L.

eg. $\lim_{x\to 0} \frac{1}{x}$ take $x_n = \frac{1}{n}$.

eg $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ take $x_n = \frac{1}{n\pi}$ $(x_n) \to 0$ $(f(x_n)) \to 0$
& $y_n = \frac{1}{2n\pi + \pi/2}$ $(y_n) \to 0$ $(f(y_n)) \to 1$

3) $\lim_{x\to c} f = L$ $\lim_{x\to c} g = M$
$\lim_{x\to c} (f \pm g) = L \pm M$
$\lim_{x\to c} (fg) = LM$ $\lim_{x\to c} (f/g) = L/M$ provided $M \ne 0$

4) Squeeze Theorem : $f(x) \le g(x) \le h(x) \quad \forall\, x \in A$

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \Rightarrow \lim_{x\to c} g(x) = L$$

use : $-x \le \sin x \le x$

$1 - \frac{1}{2}x^2 \le \cos x \le 1$

eg $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \Rightarrow$ $-1 \le \sin\frac{1}{x} \le 1$

$-x \le x\sin\frac{1}{x} \le x$

$\Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$

5) One-sided limit :

given $\epsilon > 0$, $\exists$ a $\delta > 0$ st $0 < x - c < \delta \Rightarrow |f(x) - L| < \epsilon$

then L is right hand limit at $x = c$.

Sly for LHL

6) Limits at ∞ & $L = \infty$:-

a) $\lim_{x\to\infty} f(x) = L \Rightarrow \forall\, \epsilon > 0$, $\exists$ a +ve $k(\epsilon)$ st $x \ge k \Rightarrow |f(x) - L| < \epsilon$

$\lim_{x\to-\infty} f(x) = L \Rightarrow \forall\, \epsilon > 0$, $\exists$ a +ve $k(\epsilon)$ st $x \le -k \Rightarrow |f(x) - L| < \epsilon$

b) $\lim_{x\to c} f(x) = +\infty \Rightarrow \forall\, k > 0$ (however large), $\exists$ a $\delta > 0$ st $0 < |x - c| < \delta \Rightarrow f(x) > k$

$\lim_{x\to c} f(x) = -\infty \Rightarrow \forall\, k > 0$ (however large), $\exists$ a $\delta > 0$ st $0 < |x - c| < \delta \Rightarrow f(x) < -k$

c) $\lim_{x\to\infty} f(x) = \infty \Rightarrow \forall\, k$, $\exists\, k'$, st $x \ge k' \Rightarrow f(x) > k$

$\lim_{x\to\infty} f(x) = -\infty \Rightarrow \forall k$, $\exists\, k'$ st $x \ge k' \Rightarrow f(x) < -k$

d) $\lim_{x\to-\infty} f(x) = \infty \Rightarrow \forall\, k$, $\exists\, k'$ st $x \le -k' \Rightarrow f(x) > k$

$\lim_{x\to-\infty} f(x) = -\infty \Rightarrow \forall k$, $\exists\, k'$ st $x \le -k' \Rightarrow f(x) < k$

7) Discontinuity: $\lim_{x \to c} f(x) \neq f(c)$ or

$\exists$ a seq (x_n) st $(x_n) \to c$ but $(f(x_n)) \not\to f(c)$

Removable: $\left[\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)\right] \neq f(c)$ ⇒ can be redefined & removed.

1st kind | Jump | Ordinary: LHL, RHL exist but not equal & f(c) equal to either

2nd Kind: Both LHL, RHL DNE

Mixed: One of LHL or RHL exists but not the other. It can be discontinuous or continuous from other side.

Infinite: one or both of LHL & RHL are $+\infty$ or $-\infty$.

8) DIRICHILET FUNCTION: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$

Imp Rejection example

It is discontinuous at every point of $\mathbb{R}$.

Easy to prove using sequence criterion.

$x = c \in \mathbb{Q}$, then let (x_n) be seq $\{x_n\} \in R - Q$ st $(x_n) \to c$
Here $(f(x_n)) \to 0$ but $f(c) = 1$

Sly for $x = c \in R - Q$ it can be proved.
Take $(x_n) = \left[\left(1 + \frac{1}{n}\right)^n\right]$

eg. $g: \mathbb{R} \to \mathbb{R}$ $g(x) = \begin{cases} 2x & x \in \mathbb{Q} \\ x+3 & x \notin \mathbb{Q} \end{cases}$ Find pts at which g is continuous.

We go by sequence method & form 2 sequences (a_n) & (b_n) for any given no 'x' at which 'g' is continuous.

Let $x \in \mathbb{R}$, for each $n \in \mathbb{N}$, $\exists$ a $a_n \in \mathbb{Q}$ & $b_n \in R - Q$
st $x - \frac{1}{n} < a_n < x + \frac{1}{n}$ & $x - \frac{1}{n} < b_n < x + \frac{1}{n}$

$\Rightarrow |a_n - x| < \frac{1}{n}$ & $|b_n - x| < \frac{1}{n} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$

Now we should have $\lim_{n\to\infty} (g(a_n)) = \lim_{n\to\infty} (g(b_n)) \Rightarrow \lim_{n\to\infty} (2a_n) = \lim_{n\to\infty} (b_n + 3)$

$\Rightarrow 2x = x + 3 \Rightarrow x = 3$ Now we can easily prove that 'g' is cont. at $x = 3$, i.e. LHL = RHL = f(3)

9) Imp: Uniform continuity: Let $k>0$ and let $f: \mathbb{R} \to \mathbb{R}$ st $|f(x) - f(y)| \le k|x-y| \ \forall \ x,y \in \mathbb{R}$, then '$f$' is continuous at every point $c \in \mathbb{R}$

10) Additive function: $f: \mathbb{R} \to \mathbb{R}$ & $f(x+y) = f(x) + f(y) \ \forall_{x,y \in \mathbb{R}}$

11) eg. $f, g: \mathbb{R} \to \mathbb{R}$ cont at 'c' & $h(x) = \operatorname{Max}\{f(x), g(x)\} \ \forall x \in \mathbb{R}$

ST $h(x) = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)| \ \forall x \in \mathbb{R}$

Use this to st 'h' is cont. at c.

$\Rightarrow$ 1st part is easy by simple expansion of the ~~formula~~ equation for different branches.

For continuity, f, g cont. $\Rightarrow (f+g), \frac{1}{2}(f+g), (f-g), |f-g|, \frac{1}{2}|f-g|$ all are continuous

$\Rightarrow h = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ is continuous

12) f is cont. and $f(x+y) = f(x) + f(y)$.

ST: $f(x) = a(x)$ where a is a constant $\forall x \in \mathbb{R}$

$\Rightarrow f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0 \quad \Rightarrow f(x + -x) = f(x) + f(-x) = 0$
$\Rightarrow f(x) = -f(x)$

Cases:

(a) $x \in \mathbb{Z}^+$ $\quad f(x) = f(1+1+\dots+1) = xf(1) = ax \quad a = f(1)$

(b) $x \in \mathbb{Z}^-$ $\quad f(x) = f(-y) = -f(y) = -y\, f(1) = xf(1) = a$
$y = -x \in \mathbb{Z}^+$

(c) $x = p/q$ $\quad f(p) = f(p/q \cdot q) = f(p/q + p/q + \dots) = qf(p/q) = ap$ [$\because p \in \mathbb{Z}$ & $f(p) =$
$\Rightarrow f(p/q) = a(p/q) \Rightarrow f(x) = ax$

(d) $x \in \mathbb{R}$ Let (x_n) be a seq $\to x$ where $x_n \in \mathbb{Q}$

$(f(x_n)) \to f(x) \quad f(x_n) = ax_n \quad$ [$\because f$ is cont]

$\lim f(x_n) = \lim(ax_n) = ax$

$\therefore f(x) = ax$

13) Thm: f is cont. at c iff $\forall\, \epsilon > 0$, $\exists$ a $\delta > 0$ st
$|f(x_1) - f(x_2)| < \epsilon \quad x_1, x_2 \in (c-\delta, c+\delta)$

14) f is cont. at $c \Rightarrow f$ is bdd in some nbd of 'c'.

15) Thm: If f is cont in $[a,b] \Rightarrow f$ is bdd in $[a,b]$

Proof: uses the result that given $\epsilon > 0$ (however small) $[a,b]$ can be partitioned into finite no of intervals st $|f(x_1) - f(x_2)| < \epsilon$ for x_1, x_2 in same interval [for f cont.]

$\therefore [a,b] = [a, a_1] \cup [a_1, a_2] \cdots [a_{n-1}, a_n = b]$

In $[a, a_1]$: $|f(x) - f(a)| < \epsilon \quad \therefore |f(x)| = |f(x) - f(a) + f(a)|$
$\leq |f(x) - f(a)| + |f(a)|$
$\leq \epsilon + f(a)$

Keep on doing like this $\Rightarrow |f(a_1)| \leq \epsilon + f(a)$

$|f(ax)| < n\epsilon + |f(a)| \quad \forall\, x \in [a_{n-1}, a_n]$

$\therefore$ since 'n' is finite $|f(x)|$ is bdd.

NOTE: It is not necessary in (a,b). eg $f(x) = 1/x$ in $(0,1)$

16) Thm: If 'f' is continuous in $[a,b] \Rightarrow$ it attains its bounds m & M

Proof: Assume it doesn't attain its bounds $M - f(x) \neq 0 \quad \forall\, x \in [a,b]$
then $f(x) \neq M \quad \forall\, x \in [a,b] \Rightarrow$

$\Rightarrow g(x) = \dfrac{1}{M - f(x)} \neq 0$ in $[a,b]$ & hence cont.

$\therefore \dfrac{1}{M - f(x)} \leq k \Rightarrow f(x) \leq M - \dfrac{1}{k} \Rightarrow \#$ as M is $\sup(f)$

17) Sign Preservation Theorem:

a) f is cont in $[a,b]$ & $a < c < b$ st $f(c) \neq 0$
then $f(x)$ has same sign as $f(c)$ $\forall\, x \in (c-\delta, c+\delta)$

b) f is cont in $[a,b]$ & $f(a) \cdot f(b) < 0 \Rightarrow \exists\, c \in (a,b)$ st $f(c) = 0$

c) Intermediate Value Theorem: f is cont. in $[a,b]$ & $f(a) \neq f(b)$
then every value b/w a & b is attained atleast once

18> Uniform Continuity :

Here given $\epsilon > 0$, $\exists$ a $\delta > 0$ that works for all 'x'

, ie, δ depends only on ϵ. not x.

$\Rightarrow$ f is uniformly continuous in I, if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depends on ϵ only) st $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where $x_1, x_2 \in I$

* UC is a global property.

* f is not UC if $\exists$ some $\epsilon > 0$ for which no $\delta > 0$ works, ie, for any $\delta > 0$ $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \epsilon$

eg ST: $f(x) = \frac{x}{x+1}$ is UC on $[0,2]$

Let $\epsilon > 0$ & $x_1, x_2 \in [0,2]$

$$|f(x_1) - f(x_2)| = \left|\frac{x_1}{x_1+1} - \frac{x_2}{x_2+1}\right| = \frac{|x_1 - x_2|}{|x_1+1||x_2+1|}$$

$0 \leq x_1 \leq 2$

$1 \leq x_1 + 1 \leq 3$

$1 \geq \frac{1}{x_1+1} \geq \frac{1}{3}$ sly $\frac{1}{x_2+1} \leq 1$

$\Rightarrow |f(x_1) - f(x_2)| \leq 1 \cdot 1 \cdot |x_1 - x_2| < \epsilon$ whenever $|x_1 - x_2| < \epsilon$

$\therefore$ Choose $\delta = \epsilon$

19> If f is continuous on $[a,b]$ then it is uniformly continuous on $[a,b]$

Take $a_n = \sqrt{n+2\epsilon} - \sqrt{n}$, $n \to \infty$, $\delta > 0$
& $a_n \to 0$ as
$\therefore |a_n - 0| < \delta$ for $n > m$
$= \sqrt{n+2\epsilon} - \sqrt{n} < \delta$
Take $x_1 = n+2\epsilon$, $x_2 = n$
δ doesn't satisfy UC cond'

Ques ST. $f(x) = x^2$ is not UC in $[a, \infty)$

enough to show 2 sequences $\{u_n\}$ & $\{v_n\}$ st $\lim_{n\to\infty} |u_n - v_n| = 0$ but $\lim_{n\to\infty} |f(u_n) - f(v_n)| \neq 0$

Let $u_n = \sqrt{n}$ $v_n = \sqrt{n+1}$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $|u_n - v_n| = |\sqrt{n} - \sqrt{n+1}| = \frac{1}{\sqrt{n}+\sqrt{n+1}} \to 0$ as $n \to \infty$

$|f(u_n) - f(v_n)| = |n - n+1| = 1 \neq 0$ as $n \to \infty$

→ Differentiability :

1) $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

LHD $= \lim_{h \to 0^+} \frac{f(c-h) - f(c)}{-h}$ RHD $= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

$= \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ $= \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$.

$f'(c)$ exists if LHD = RHD

2) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$

3) $f(-x) = f(x) \Rightarrow$ even fn.

$f'(-x) = -f'(x) \Rightarrow$ odd fn.

4) Interior Extremum Theorem :

Let c be an interior pt at which f(c) is extremum.

If f'(c) exists then $f'(c) = 0$

* it is possible than f'(c) doesn't exist at ext: eg $|x|$

* converse is not true · eg x^3

$f'(c) = 0$, then c is stationary point.

5) Rolle's Theorem : 3 conditions : (a) f is cont on [a,b]
(b) f' exists on (a,b)
(c) $f(a) = f(b) = 0$

then $\exists\ c \in (a,b)$ st $f'(c) = 0$

Proof : use fact that f attains its bounds in [a,b] as it is cont. & that extremum has $f'(c) = 0$ [using IET]

Rolle's Theorem fails if (a) f is not cont in [a,b]
(b) f' is not derivable in (a,b)
(c) $f(a) \neq f(b)$

eg. (1) Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$

ST: $a_0x^n + a_1x^{n-1} + \cdots + a_n$ vanishes atleast once in $(0,1)$.

Take: $f(x) = \frac{a_0x^{n+1}}{n+1} + \frac{a_1x^n}{n} + \frac{a_2x^{n-1}}{n-1} + \cdots + a_nx$

& $I = [0,1]$ & apply Rolls

(2) ST b/w any 2 roots of $e^x\cos x = 1$, $\exists$ atleast one root of $e^x \sin x - 1 = 0$

Take $f(x) = e^{-x} - \cos x$ where $e^a\cos a = 1$ & $e^b \cos b = 1$

$f(a) = e^{-a} - \cos a = 0$

$f(b) = e^{-b} - \cos b = 0$

$f'(c) = e^{-x} + \sin x = 0 \Rightarrow e^c \sin c = 1$.

6) Lagrange Mean Value Theorem:

2 conditions (a) f is cont. on $[a,b]$
(b) f is derivable on (a,b)

then $\exists\ c \in (a,b)$ st $f'(c) = \frac{f(b)-f(a)}{b-a}$

Proof: Take $\phi(x) = f(x) - f(a) - k(x-a)$ $\forall\ x \in [a,b]$
where $k = \frac{f(b)-f(a)}{b-a}$ & apply Rolles

Corr (I): $f'(x) = 0$ $\forall\ x \in (a,b) \Rightarrow f$ is constant

Corr (II): f is increasing $\Leftrightarrow f'(x) \geq 0$ $\forall\ x \in I$
f is decreasing $\Leftrightarrow f'(x) \leq 0$ $\forall\ x \in I$

eg ST: $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$; $x > 0$

Just use bruteforce by forming $F(x) = \log(1+x) - x + \frac{x^2}{2}$
& $G(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$

show $F'(x) \geq 0$ & $G'(x) \leq 0$

eg. Using LMVT in certain examples like

$$\frac{x}{1+x} < \log(1+x) < x$$

Here we show $f'(c) = \dfrac{f(x)-f(0)}{x-0}$ for $c \in (0,x)$

Then do manipulations to $0<c<x$ to reach

term 1 $< f'(c) <$ term 2

& then put $f'(c) = \dfrac{f(x)-f(0)}{x-0}$.

eg $f(t)=\log(1+t)$ $\quad t\in(0,x)$ $\quad f'(t)=\dfrac{1}{1+t}$

& $\exists\ c \in (0,x)$ st $f'(c) = \dfrac{\log(1+x)-0}{x}$.

$$0<c<x$$
$$1<1+c<1+x$$
$$1 > \frac{1}{1+c} > \frac{1}{1+x}$$
$$1 > f'(c) > \frac{1}{1+x} \Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$$
$$\Rightarrow \frac{x}{1+x} < \log(1+x) < x.$$

TRICK

eg. f is st f' is cont in $[a,b]$ & derivable in (a,b).

ST: $f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$ for $c\in(a,b)$

Manipulation

Try to get a $\phi(x)$ st $\phi(a)=\phi(b)$ & then apply rolle's theorem

Try to get $f''(c) = (k)$ from the condition

then form $\phi(x) \rightarrow$ replace 'a' by x & express $f(b)$ as ---

eg Here we form $\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 k$

$\phi(b)=f(b)$ & $\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 k$
$= f(b)$

& Apply rolle's, ie, $\exists\ c$ st $\phi'(c)=0$

this gives the reqd result

7) Cauchy's Mean Value Theorem :

f, g are cont on $[a,b]$ & derivable of (a,b) & $g'(x) \neq 0$ $\forall x \in (a,b)$

then $\exists\, c \in (a,b)$ st $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Proof: Let $\phi(x) = f(x) - f(a) - k\,[g(x) - g(a)]$ where $k = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$\phi(a) = \phi(b) = 0$

$\therefore \exists\, c$ st $\phi'(c) = 0$

$\phi'(c) = f'(c) - kg'(c) \Rightarrow \dfrac{f'(c)}{g'(c)} = k$.

eg ST $\dfrac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \dfrac{f''(c)}{g''(c)}$ —— 1

form $\phi(x) = [f(b) - f(x) - (b-x)f'(x)] - k[g(b) - g(x) - (b-x)g'(x)]$

$\phi(b) = \phi(a) = 0$ where k = LHS of 1

so $\phi'(c) = 0$.

eg. $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$

Cross multiply we get $f(c)g'(c) + f'(c)g(c) = f(a)g'(c) + f'(c)g(b)$

form $\phi(x) = f(a)\,g(x) + f(x)\,g(b) - f(x)g(x)$

$\phi(a) = \phi(b) = f(a)\,g(b)$

Apply Rolle's now.

8) Generalised Mean Value Theorem :

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Example: PT: $\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$ if $x>y>0$

take $f(x) = \left(1+\frac{1}{x}\right)^x$

$$f'(x) = \left(1+\frac{1}{x}\right)^x \left[\underbrace{\log\left(1+\frac{1}{x}\right) - \frac{1/x}{1+1/x}}_{\text{replace } 1/x \text{ by } x}\right]$$

* Now consider $\phi(x) = \log(1+x) - \frac{x}{1+x}$

$\phi'(x) = \frac{x}{1+x^2} > 0$

$\therefore \phi(x) > \phi(0)$ for $x>0$

$\Rightarrow \phi(x) > 0 \quad \forall x>0$

$\Rightarrow \phi(1/x) > 0 \quad \forall x>0$

$\Rightarrow f'(x) > 0 \Rightarrow x>y \Rightarrow f(x) > f(y)$ HP.

eg PT: $\frac{\tan x}{x} > \frac{x}{\sin x}$ $\quad 0<x<\pi/2$

$\frac{\tan x \sin x - x^2}{x \sin x} > 0$

consider $f(x) = \tan x \sin x - x^2$

$f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$

$= (1+\sec^2 x)\sin x - 2x$

$f''(x) = (2\sec x \sec x \tan x)\sin x + (1+\sec^2 x)\cos x - 2$

$= 2\sin^2 x \sec^3 x + \cos x + \sec x - 2$

$= 2\sin^2 x \sec^3 x + \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$

$\Rightarrow f''(x) > 0$

$f'(x) > f'(0) = 0$

$\therefore f(x) > f(0)$

HP.

⇒ Taylor's Theorem :

1) Let f be defined on $[a,b]$ st

(i) $f^{(n-1)}$ is cont. on $[a,b]$ (ii) f^{n} exists on (a,b)

then $\exists\ c \in (a,b)$ st

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+ \frac{(b-a)^p(b-c)^{n-p}}{p(n-1)!}f^{n}(c)$$

where p is a given +ve integer.

2) Remainder : $R_n(x) = T_{n+1} = \dfrac{(b-a)^p(b-c)^{n-p}}{p(n-1)!}f^{n}(c)$ Roche's form

$p=1 \Rightarrow R_n(x) = \dfrac{(b-a)(b-c)^{n-1}}{(n-1)!}f^{n}(c)$ Cauchy's form

$p=n \Rightarrow R_n(x) = \dfrac{(b-a)^n}{n!}f^{n}(c)$ Lagrange's form ✓

3) Taylor Series : $f(a+h) = f(a) + hf'(a) + \dfrac{h^2}{2!}f''(a) + \dots + \dfrac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \dots$

it cgs to $f(a+h)$ if $\lim_{n\to\infty} R_n = 0$ where $R_n = \dfrac{h^n}{n!}f^{n}(a+\theta h)$

($a=0$ $h=x$

Maclaurin's series : $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dots + \dfrac{x^n}{n!}f^{n}(0)$

↳ It is used to find expansion of functions.

eg Using taylor series ST: $1+x+x^2/2 < e^x < 1+x+x^2e^x/2$ $x>0$

$e^x = 1 + x + \dfrac{x^2e^{\theta x}}{2}$ $\theta \in (0,1)$.

4) To find taylor & maclaurin series, remember to show $R_n \to 0$ as $n \to \infty$

where $R_n = \frac{h^n}{n!} f^n(a + \theta h)$

eg. expand e^x.

$R_n = \frac{x^n}{n!} e^{\theta x}$ $\lim_{n\to\infty} R_n = e^{\theta x} \lim_{n\to\infty} \frac{x^n}{n!}$

$a_n = \frac{x^n}{n!}$ $\frac{a_n}{a_{n+1}} = \frac{x^n}{n!} \frac{n+1!}{x^{n+1}} = \frac{n+1}{x} = \infty > 1$

$\Rightarrow a_n \to 0$

$\therefore$ conditions of maclaurin series are satisfied

& $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

MAXIMA- MINIMA

5) DARBOUX THEOREM:

If f is differentiable on $I = [a,b]$ and if k is a no b/w $f'(a)$ & $f'(b)$ $\exists\ c \in (a,b)$ st $f'(c) = k$

Proof: $g(x) = f(x) - kx$

g is continuous on $[a,b] \Rightarrow$ it attains its bounds

Now $g'(a) = f'(a) - k < 0$ as $k > f'(a)$

& $g'(b) > 0$

$\therefore g$ ~~doesn't have~~ $\therefore \exists\ c \in (a,b)$ st $g'(c) = 0$

$\therefore f'(c) = k$

[$\because$ sign preservn thm]

6) Test to find Maxima - Minima :

Let $f' = f'' = f''' = \dots = f^{(n-1)}(x_0) = 0$ but $f^{n}(x_0) \neq 0$

then (I) n is even $\rightarrow$ $f^{n}(x) > 0 \rightarrow$ minima
$f^{n}(x) < 0 \rightarrow$ maxima

(II) n is odd $\longrightarrow$ neither maxima nor minima

Working Rule w/o going till n^{th} derivative :

(i) Find $f'(x)$ & let its roots be x_1, x_2, x_3

(ii) Test value of $f'(x)$ in succession near these points

(iii) If for $x_i \Rightarrow$ $x_i - \delta$, x_i, $x_i + \delta$

sign of $f'(x)$: $x_i - \delta$	$x_i + \delta$	
+	−	$\Rightarrow$ maxima at x_i
−	+	$\Rightarrow$ minima at x_i
+	+	$\Rightarrow$ neither maxima nor minima
−	−	

eg. useful in cases like $f(x) = (x-3)^5 (x+1)^4$

$$f'(x) = 5(x-3)^4 (x+1)^4 + (x-3)^5 \cdot 4(x+1)^3$$
$$= (x-3)^4 (x+1)^3 [5x + 5 + 4x - 12]$$
$$= (x-3)^4 (x+1)^3 (9x - 7)$$

$f'(x)$: + | -1 | − | $7/9$ | + | 3 | +

$-1 \Downarrow$ max, $7/9 \Downarrow$ min, $3 \Downarrow$ neither

7) L' Hospital's Rule for Indeterminate forms :

If $\frac{f(x)}{g(x)}$ has indeterminate form at $x \to a$ like $\frac{\infty}{\infty}, 0 \times \infty, \frac{0}{0}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Sometimes $\frac{\infty}{\infty}$ has to be converted to $\frac{0}{0}$, else it will not stop

Other form like $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0 can be reduced to either of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by manipulations & proceed as usual.

eg $\lim_{x \to 0} \left(\cot^2 x - \frac{1}{x^2}\right) = \lim_{x \to 0} \left(\frac{x^2\cos^2 x - \sin^2 x}{x^2 \sin^2 x}\right)$

$$= \lim_{x \to 0} \left[\frac{x^2\cos^2 x - \sin^2 x}{x^4}\right] \cdot \left[\frac{x^2}{\sin^2 x}\right]$$

Not doing this will lead to lot of chain L'Hospital

$$= \lim_{x \to 0} \left[\frac{x^2\left(1 - \frac{x^2}{2!} + \cdots\right)^2 - \left(x - \frac{x^3}{3!} + \cdots\right)^2}{x^4}\right] \cdot \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2$$

$$= \lim_{x \to 0} \left[\frac{x^2\left(1 - \frac{x^2}{2!} + 1 - \frac{x^2}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} - 1 + \frac{x^2}{3!} + \cdots\right)}{x^4}\right] \cdot 1$$

$$= \lim_{x \to 0} \left[\frac{x^2\left(2 - \frac{2x^2}{3}\right)\left(-\frac{x^2}{3} + \cdots\right)}{x^4}\right]$$

$$= \lim_{x \to 0} \left(\frac{-1}{3}\right)\left(2 - \frac{2x^2}{3} + \cdots\right) = \frac{-2}{3}.$$

eg $\lim_{x \to \infty} \left(1 + 1/x\right)^x = l$

$$\log l = \lim_{x \to \infty} x \log\left(1 + 1/x\right)$$

$$= \lim_{x \to \infty} \log\left(1 + 1/x\right) \Big/ 1/x$$

$$= \frac{\frac{1}{1 + 1/x} \cdot \frac{-1}{x^2}}{-1/x^2}$$

$$= \lim_{x \to \infty} \frac{x}{x+1}$$

$$= 1$$

$\therefore l = e.$

⇛ REIMANN INTEGRAL :

1) Reimann definition is only for bdd functions.

2) Norm of partition : $\|p\| = \max\{\delta_r \mid r=1,2,\dots n\}$
$= \max\{x_r - x_{r-1} \mid r=1,2,\dots n\}$
$= \max$ of length of all sub-intervals.

$(a=x_0)(x_1)(x_2)\dots(x_{n-1})(x_n)=b$

3) Upper Darboux Sum : $U(p,f)$: $\sum_{r=1}^{n} M_r \delta_r$ where M_r is sup of f in $I_r = [x_{r-1}, x_r]$

Lower Darboux Sum : $L(p,f)$: $\sum_{r=1}^{n} m_r \delta_r$ m_r is the inf of f.

Oscillatory sum : $\omega(p,f)$: $\sum_{r=1}^{n} O_r \delta_r$ $O_r = M_r - m_r$

4) $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$
$\Rightarrow$ $L(P,f)$ & $U(P,f)$ are bdd in $[a,b]$.

5) $\{L(P,f)\}$ is the set of all Lower Darboux Sum.

Lower Reimann Integral $= \sup\{L(P,f)\} = \int_{\underline{a}}^{b} f(x)\,dx$

Upper Reimann Integral $= \inf\{U(P,f)\} = \int_{a}^{\bar{b}} f(x)\,dx$

6) A bdd function is said to be R-integrable if
$$\int_{\underline{a}}^{b} f(x)\,dx = \int_{a}^{\bar{b}} f(x)\,dx \Rightarrow \int_{a}^{b} f(x)\,dx \Rightarrow \text{Reimann Integral of } f \text{ on } [a,b]$$

$R[a,b]$ = family of all bdd functions which are R-integrable on $[a,b]$

7) Every bdd function need not be riemann integrable
eg. Dirichlet's function.

8) To find R-Integral in $[a,b]$, take the partition

$$\left[a,\ a+\frac{(b-a)}{n},\ \ldots,\ a+\frac{r(b-a)}{n}\ a+\frac{(r+1)}{n},\ b\right] \quad r = 0,1,\ldots,n-1$$

or $[a,\ a+h,\ a+2h,\ \ldots a+(r-1)h,\ \ldots a+nh=b]$ $\quad h = \frac{b-a}{n}$

9) Remember: $\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left[\alpha + \frac{(n-1)}{2}\beta\right]\sin\frac{n\beta}{2}}{\sin\beta/2}$$

$$\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+(n-1)\beta) = \frac{\cos\left[\alpha+\frac{(n-1)\beta}{2}\right]\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

10) Lower R-Integral can't exceed upper R-Integral

11) If $f \in R[a,b] \Rightarrow$ (i) $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$ if $b \ge a$

(ii) $m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a)$ if $b \le a$

12) Thm: A bdd function f is integrable on $[a,b]$ iff for each $\epsilon > 0$, $\exists$ P of $[a,b]$ st $U(P,f) - L(P,f) < \epsilon$

13) Impt Conditions:

(a) f is cont. on $[a,b]$ ~~ocs~~

(b) f is monotonous & bdd on $[a,b]$

(c) f is bdd with finite discontinuties on $[a,b]$

(d) f is bdd & set of pts of discontinuity has finite no of limit points

} f is integrable on $[a,b]$

14) Reimann Sum: $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\, \delta_r \quad \xi_r \in [x_{r-1}, x_r]$

There can be infinitely many R-sums as ξ_r is arbitrary

15) Another definition of R-integral:

$f \in R[a,b]$ if $\lim_{\|P\|\to 0} S(P,f)$ exists & $\lim_{\|P\|\to 0} S(P,f) = \int_a^b f(x)dx$

Corr. 1 If f is integrable on $[a,b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} h f(a+rh) \qquad h = \frac{b-a}{n}$$

Corr 2: If f is integrable on $[0,1]$ then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

16) To evaluate limit of a sum, write it as $\lim_{n\to\infty} \sum f\left(\frac{r}{n}\right)\cdot\frac{1}{n}$

Replace $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx, $\lim_{n\to\infty} \sum$ by $\int_0^1$

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left(\frac{1}{n}\right) = \int_0^1 f(x)\, dx.$$

17) If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r) \qquad h = \left(\frac{b}{a}\right)^{1/n}$$

eg $f(x) = \begin{cases} \frac{1}{2^n} & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \quad n = 0,1,2,\ldots \\ 0 & x = 0 \end{cases}$

ST: $\int_0^1 f(x)dx = \frac{2}{3}$ despite ∞ pts of discont in $[0,1]$

$f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$ $n=0,1,2 \ldots$

$$f(x) = \begin{cases} \frac{1}{2^0} = 1 & \frac{1}{2} < x \le 1 \\ \frac{1}{2^1} & \frac{1}{4} < x \le \frac{1}{2} \\ \vdots & \\ \frac{1}{2^{n-1}} & \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ \frac{1}{2^n} & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n} \\ 0 & x = 0 \end{cases}$$

Pts of dc $= S = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots\right\}$ $D(S) = \{0\}$

which is finite

$\therefore f$ is bdd with finite no of lt pts of its set of dc

$\Rightarrow f$ is integrable

$$\therefore \int_{1/2^n}^{1} f(x)\,dx = \int_{1/2}^{1} 1\,dx + \int_{1/2^2}^{1/2} \tfrac{1}{2}\,dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} \tfrac{1}{2^{n-1}}\,dx$$

$$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^2}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-2}} - \frac{1}{2^{2n-1}}$$

$$= \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \cdots + \frac{1}{2^{2(n-1)}}\right]$$

$$= \frac{1}{2} \times \frac{\left[1 - \left(\frac{1}{4}\right)^n\right]}{1 - 1/4} = \frac{4}{6}\left[1 - \frac{1}{4^n}\right]$$

$$= \frac{2}{3}\left[1 - \frac{1}{4^n}\right]$$

$$\therefore \int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_{1/2^n}^{1} f(x)\,dx = \frac{2}{3}$$

18) Remember: $\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}+\dots\right) \longrightarrow \frac{\pi^2}{6}$; & $\frac{\Sigma(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$

19) $|f| \in R[a,b] \not\Rightarrow f \in R[a,b]$ eg. Dirichlet
But $f \in R[a,b] \Rightarrow |f| \in R[a,b]$

20) $f, g \in R[a,b] \Rightarrow f \pm g,\ fg,\ f^2,\ \alpha f + \beta g \in R[a,b]$

21) $f, g, h \in R[a,b]$ & $f \geq g \geq h \ \forall\ x \in [a,b]$ then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \geq \int_a^b h(x)\,dx$$

eg. $0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow e^0 < e^{x^2} < e^1$
$\Rightarrow 1 < e^{x^2} < e$

$$\therefore \int_0^1 1\,dx < \int_0^1 e^{x^2}dx < \int_0^1 e^1\,dx$$

$$\Rightarrow 1 < \int_0^1 e^{x^2}dx < e$$

22) If $f \in R[a,b]$ then $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx$

23) $f \in R[a,b] \Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

& $\int_a^b f(x)\,dx = \mu(b-a)$ where $\mu \in [m,M]$
$= f(c)(b-a)$ if f is cont. in $[a,b]$

$\Rightarrow \underline{\int_a^b} f(x)\,dx = \text{lub}\{L(P,f)\} \Rightarrow L(P,f) \leq \underline{\int_a^b} f(x)\,dx$

$\overline{\int_a^b} f(x)\,dx = \text{glb}\{U(P,f)\} \Rightarrow U(P,f) \geq \overline{\int_a^b} f(x)\,dx$

Also $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

$\therefore m(b-a) \leq L(P,f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(P,f) \leq M(b-a)$

$\Rightarrow$ If $\underline{\int} = \overline{\int}$, then $m(b-a) \leq L(P,f) \leq \int f(x)\,dx \leq U(P,f) \leq M(b-a)$

$\Rightarrow m \leq \frac{1}{b-a}\int f(x)\,dx \leq M \Rightarrow \int_a^b f(x)\,dx = (b-a)\mu \quad \mu \in [m,M]$ If f is cont. in $[a,b]$ $\exists\, c$ s.t. $f(c) = \mu$

24) If $f \in R[a,b]$ & $|f(x)| \leq K$ $\Rightarrow$ $\left|\int_a^b f(x)dx\right| \leq K(b-a)$

25) Functions defined by Integrals :

Let $f \in R[a,b]$, then $\forall\ t \in [a,b]$, $[a,t] \subset [a,b]$

& $f \in R[a,t]$

then $\phi(t) = \int_a^t f(x)dx$ $\quad t \in [a,b]$ is called integral function or indefinite integral of integrable function f.

(a) Integral function of an integrable is continuous.

Proof:- using cauchy's definition $|\phi(x_2) - \phi(x_1)| \leq K(x_2 - x_1)$

where $|f| \leq K$
as f is bdd
$\because f \in R[a,b]$

* FIRST FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS :

(b) If $f \in R[a,b]$ & f is continuous on $c \in [a,b]$

then $\phi(t)$ is derivable at c & $\phi'(c) = f(c)$.

Corr:- If f is continuous on $[a,b]$, then $\phi(t) = \int_a^t f(x)dx$ $\forall\ t \in [a,b]$

is derivable at every $x \in [a,b]$ & $\phi'(x) = f(x)$

(c) PRIMITIVE OR ANTIDERIVATIVE OF $f(x)$:

If $f \in R[a,b]$ and if $\exists\ \phi : [a,b] \longrightarrow \mathbb{R}$ st $\phi'(x) = f(x)$ $\forall x \in [a,b]$

then ϕ is the primitive or antiderivative of f.

Primitive is not unique !

If f is cont. on $[a,b]$ then $\phi(t) = \int_0^t f(x)dx$ is primitive of f.

* Continuity is not necessary for existence of primitive

eg. $\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$g(x) = \phi'(x)$ is not cont at $x=0$ but it has primitive $\phi(x)$.

(d) II^{nd} FUNDAMENTAL THEOREM ON INTEGRAL CALCULUS:

If $f \in R[a,b]$ & ϕ is its primitive then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Proof: Use LMVT in $[x_{r-1}, x_r]$ $\phi'(\xi_r) = \dfrac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}}$
[$\because \phi$ is derivable on $[a,b]$].

$$\therefore \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} \phi'(\xi_r)\delta_r = \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$\therefore \sum_{r=1}^{n} f(\xi_r)\delta_r = [\phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) - - + \phi(x_n) - \phi(x_{n-1})]$$

$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r)\delta_r = \phi(b) - \phi(a)$$

Corr. If ϕ' is continuous on $[a,b] \Rightarrow \int_a^b \phi'(x)\,dx = \phi(b) - \phi(a)$

(e) Mean value theorem of Integral Calculus:

If $g, f \in R[a,b]$ & $g(x) \geq 0$ (or) $g(x) \leq 0$ $\forall x \in [a,b]$

then $\exists\ \mu \in [m, M]$ st $\int_a^b f(x)g(x)\,dx = \mu \int_a^b g(x)\,dx$

Corr: If f is cont. on $[a,b]$, $\exists\ c \in [a,b]$ st $\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$

(e) eg. ST: $\frac{1}{\pi} < \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx < \frac{2}{\pi}$

$f(x) = \frac{1}{1+x^2}$ $g(x) = \sin \pi x$ $g(x) \geq 0 \ \forall x \in [0,1]$

& $f(x)$ is cont. & $g \in R[a,b]$

$\therefore \exists\, c$ st $\int_0^1 \frac{\sin \pi x}{1+x^2} = f(c) \int_0^1 \sin \pi x\, dx = \frac{2}{\pi(1+c^2)}$

$0 < c < 1 \Rightarrow 1 < 1+c^2 < 2$

$\Rightarrow 1 > \frac{1}{1+c^2} > \frac{1}{2} \Rightarrow \frac{2}{\pi} > \frac{2}{\pi(1+c^2)} > \frac{1}{\pi}$

$\Rightarrow \frac{1}{\pi} < I < \frac{2}{\pi}$

(f) Bonnet's Mean Value Theorem :

Let $g \in R[a,b]$ & let f be $\downarrow$ & ≥ 0,
then for some $\xi \in [a,b]$ $\int_a^b f(x) g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$

(OR)

Let $g \in R[a,b]$ & f be $\uparrow$ & ≥ 0
then $\exists\, \eta \in [a,b]$ st $\int_a^b f(x) g(x)\, dx = f(b) \int_{\eta}^b g(x)\, dx$

* Remember $\uparrow \Rightarrow \eta\, a \rightarrow$ & $\downarrow \Rightarrow \leftarrow \xi$ & limit has the in b/w elt.

eg ST: $\left| \int_a^b \frac{\sin x}{x} \right| \leq \frac{2}{a}$. Here $f(x) = \frac{1}{x}$ & $g(x) = \sin x$

(g) 2nd Mean Value Theorem

Let $g \in R[a,b]$ & f be bdd & monotonic on $[a,b]$, then
$\int_a^b fg = f(a) \int_a^{\xi} g + f(b) \int_{\xi}^b g$ Proof: Let f be $\downarrow$ & BMVT on $(f - f(b))$ & g

26) Integral as the limit of a sum :

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left(\frac{1}{n}\right).$$

eg Find $\lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\}^{1/n}$ let $A = \lim_{n\to\infty} \left\{ \frac{n!}{n^n} \right\}^{1/n}$

$$\log A = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(\frac{r}{n}\right) = \int_0^1 \log x\,dx$$

$$= \left[x\log x - x \right]_0^1 = -1 \quad \therefore \quad A = e^{-1} = 1/e.$$

☆☆ If f is cont. on $[a,b]$ & $f(x) \geq 0$ & $\int_a^b f(x)\,dx = 0$

then $f(x) = 0 \quad \forall \ x \in [a,b]$

We have $\phi(t) = \int_a^t f(x)\,dx \quad \forall \ t \in [a,b]$ as primitive of f.

So that $\phi'(c) = f(c) \quad \forall \ c \in [a,b]$

Now $\phi'(x) = f(x) \geq 0 \Rightarrow \phi(x)$ is increasing function

$\phi(a) = \int_a^a f(x)\,dx = 0$ & $\phi(b) = \int_a^b f(x)\,dx = 0$

Let $c \in (a,b)$ $\quad a \leq c \leq b$

$\Rightarrow \phi(a) \leq \phi(c) \leq \phi(b)$

$\Rightarrow 0 \leq \phi(c) \leq 0$

$\Rightarrow \phi(c) = 0$

$\Rightarrow \phi(x) \equiv 0 \quad \forall \ x \in [a,b]$, ie, ϕ is constant

$\therefore \phi'(x) = 0 = f(x)$ HP.

eg. By an example PT : $\int_a^b f'(x)\,dx = f(b) - f(a)$ is not always true

Take $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$f'(x)$ is not bdd & hence $\notin R[a,b]$ =

eg. ST: $\int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \cdots \quad |x|<1$

$$\frac{1}{1+t^4} = 1 - t^4 + t^8 - t^{12} + \cdots + (-1)^{n-1} t^{4n-4} + \frac{(-1)^n t^{4n}}{1+t^4}$$

$$\int_0^x \frac{1}{1+t^4} = x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots + (-1)^n \int_0^x \frac{t^{4n}}{1+t^4} dt$$

$$0 \le \left| \int_0^x \frac{t^{4n}}{1+t^4} dt \right| < \left| \int_0^x t^{4n} dt \right| < \left| \frac{x^{4n+1}}{4n+1} \right| < \frac{1}{4n+1}$$

$$\therefore \lim_{n\to\infty} \int_0^x \frac{t^{4n}}{1+t^4} = 0.$$

eg. If f is cont on $[0,1]$ st $\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx = \frac{\pi}{2} f(0)$

$$I = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2} dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2} dx$$

$$I_1 = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2} dx = f(\alpha_n) \int_0^{1/\sqrt{n}} \frac{n\,dx}{1+n^2x^2} \quad [\because \text{using } 1^{st} \text{ MVT}]$$

$$= f(\alpha_n) \tan^{-1}\sqrt{n} \longrightarrow f(0) \cdot \frac{\pi}{2} \quad \text{as } n \to \infty$$

$$|I_2| = \left| \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2} dx \right| = \left| f(\beta_n) (\tan^{-1} n - \tan^{-1}\sqrt{n}) \right|$$

$$\le M (\tan^{-1} n - \tan^{-1}\sqrt{n}) \to 0 \quad \text{as } n \to \infty$$

$M \hookrightarrow \sup |f(x)|$

$$\therefore I = I_1 = \frac{\pi}{2} f(0)$$

⇒ UNIFORM CONVERGENCE - Sequences & series of functions :

1) $\{f(n)\} = \{f_1, f_2, \ldots, f_n \ldots\}$ $f_n : I \to \mathbb{R}, n \in \mathbb{N}$

2) $\{f_n\}$ cgs pointwise if $\forall x$ the seq $\{f_1(x), f_2(x), \ldots, f_n(x) \ldots\}$ cgs. & the limiting value of each x gives $\lim_{n\to\infty} f_n(x) = f(x)$

ie, $\forall x \in I$, & $\epsilon > 0$, $\exists$ $m \in I^+$ st
$|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq m$ Here m depends on x & ϵ.

3) Pointwise cgs is not strong enough to transfer any of the properties like continuity, differentiability (or) integrability to the limit function.

eg: (a) $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$ $f(x) = \begin{cases} 0 & |x|<1 \\ 1/2 & |x|=1 \\ 1 & |x|>1 \end{cases}$ Continuity is not transferred

(b) $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ $f(x) = 0$ $\Rightarrow \underset{n\to\infty}{\text{lt}} f_n'(0) \neq f'(0)$

$\Downarrow$ $\Downarrow$

$f_n'(x) = \sqrt{n} \cos nx$ $f'(x) = 0$ $\forall x \in \mathbb{R}$ Limit of derivatives $\neq$ derivative of limit

(c) $f_n(x) = nx(1-x^2)^n$ $f(x) = 0$ [use L'Hospital] $\Rightarrow \underset{n\to\infty}{\text{lt}} \int_0^1 f_n\,dx \neq \int_0^1 f\,dx$

$\int_0^1 f_n(x)\,dx = \dfrac{n}{2(n+1)}$ $\int_0^1 f(x)\,dx = 0$ Limit of integrals $\neq$ Integral of limit.

* These properties are transmitted if $\{f_n\}$ is uniformly convergent.

4) Uniform Convergence : (UC) :

$\{f_n\}$ is uniformly convergent to a function 'f' on I if to each $\epsilon > 0$, $\exists$ $m \in I^+$ (depending on <u>ONLY</u> ϵ) st $|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq m$ & $\forall$ $x \in I$.

* uniform convergence $\Rightarrow$ pointwise convergence

But converse is not true eg $f_n(x) = x^n$ $x \in [0,1]$

5) Uniformly Bounded sequence of functions :

$\exists$ a +ve $k \in \mathbb{R}$ st $|f_n(x)| < K$ $\forall$ $n \in \mathbb{N}$ & $\forall$ $x \in I$

6) Cauchy's criterion for UC :

$\{f_n\}$ is UC $\Leftrightarrow$ for each $\epsilon > 0$ & $\forall$ $x \in I$, $\exists$ $m(x)$ st for every integer $p \geq 1$, $|f_{n+p}(x) - f_n(x)| < \epsilon$ $\forall$ $n \geq m$

7) M_n Test for UC :

Let $M_n = \sup \{ |f_n(x) - f(x)| \mid x \in I \}$

Then $\{f_n\}$ is UC iff $\lim_{n \to \infty} M_n = 0$

So if $M_n \not\to 0 \Rightarrow$ no UC.

egs: $f_n(x) = \dfrac{x}{1+nx^2}$ $x \in \mathbb{R}$ is UC on $[a,b]$

$f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$ is non-UC on $[0,1]$

8) Series of Real valued functions : $\Rightarrow \sum_{n=1}^{\infty} f_n$

Pointwise convergence, if the partial sum $S_n = \sum_{r=1}^{n} f_r$ cgs pointwise.

$\sum f_n$ is UC if the sequence $\{S_n\}$ is UC.

, ie, $\forall \epsilon > 0$ & $\forall x \in I$, $\exists$ +ve $m(\epsilon)$ st

$$|S_n(x) - f(x)| < \epsilon \quad \forall\ n \geq m$$

9) Cauchy's criterion for UC of series of functions :

$\forall\ \epsilon > 0$, $\exists\ m \in \mathbb{N}$ st $|S_{n+p}(x) - S_n(x)| < \epsilon \ \forall\ n \geq m,\ p \geq 0$

(not that much useful)

10) WEISTRASS' M-test :

$\sum f_n$ is uniformly (and absolutely) cgt on interval I,

if $\exists$ a cgt series $\sum_{n=1}^{\infty} M_n$ of non-negative terms, ie, $M_n \geq 0$

st $|f_n(x)| \leq M_n \quad \forall\ n \in N$ & $x \in I$.

We can't negate it, If $\sum M_n$ dgs !!!

eg. $0 < r < 1$ st $\sum r^n \cos nx$ is UC.

$|f_n(x)| = |r^n \cos nx| \leq r^n \quad [\because |\cos nx| \leq 1]$

$M_n = r^n$

$\sum r^n$ cgs for $0 < r < 1 \Rightarrow \sum f_n$ is UC.

eg. Check $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2} \Rightarrow f_n(x) = \frac{x}{(n+x^2)^2}$

Better find $\sup\{f_n(x)\}$ using $f_n'(x) = 0$ & $f_n''(x) < 0$

We get $M_n = \frac{1}{n^{3/2}}$ $\quad \sum M_n$ cgs $\Rightarrow \sum f_n$ cgs.

11) f_n is cont. & $\{f_n\}$ is UC $\Rightarrow$ f is cont on $[a,b]$

f_n is cont & $\sum f_n$ is UC $\Rightarrow$ f is cont on $[a,b]$

NOTE : Being UC is a sufficient condition, not necessary

NOTE : This gives an important negation test.

if f is not cont. & f_n is cont., then $\{f_n\}$ can't be UC

eg. $f_n(x) = x^n \quad x \in [0,1]$ Now $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

f is discont but f_n is cont

$\therefore$ f_n is not UC

eg. $f_n(x) = \dfrac{nx}{1+n^2x^2} \quad x \in [0,1]$ Here $f(x) = 0 \quad \forall x \in [0,1]$

Both f & f_n are cont, so we need to check separately. Use M_n test.

12) UC & $\int$
$\{f_n\}$ is UC to f on $[a,b]$ & each f_n is integrable
$\Rightarrow$ f is integrable & $\left\{ \int_a^b f_n(x)\,dx \right\}$ is UC to $\int_a^b f(x)\,dx$

$\sum f_n$ is UC to f & each f_n is integrable.
$\Rightarrow$ f is integrable & $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$ is UC to $\int_a^b f(x)\,dx$

ie $\sum \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

$\Rightarrow$ series is term by term integrable

$\therefore$ If ~~$\sum$~~ $\lim \int_a^b f_n(x)\,dx \neq \int_a^b f(x)\,dx$ then $\{f_n(x)\}$ is not UC to $f(x)$

eg. Show that $1-x+x^2-x^3+\cdots\ ;\ 0\le x\le 1$ is term by term integrable on $[0,1]$ but is not UC.

$$\Sigma x_n = \sum_{n=1}^{\infty}(-1)^n x^n$$

$$S_n = 1-x+x^2+\cdots+(-1)^n x^n$$

$$= \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \quad x=1 \\ \dfrac{1}{1+x} & x\in[0,1) \end{cases}$$

$\Rightarrow S(x)$ is dis cont.

$\Rightarrow S_n$ is not UC to $S(x)$

Now $\displaystyle\int_0^1 \frac{1}{1+x}\,dx = \log 2 = \int_0^1 S\,dx.$

$$\sum_{n=1}^{\infty}\int_0^1 f_n\,dx = \int_0^1 1.dx + \int_0^1 -x\,dx + \int_0^1 +x^2 dx \cdots$$

$$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \log 2.$$

13) UC & $\frac{d}{dx}$.

(a) If $\{f_n\}$ is st
(i) each f_n is diff. on $[a,b]$
(ii) each f_n' is cont on $[a,b]$
(iii) $\{f_n\}$ cgs uniformly to f on $[a,b]$
(iv) $\{f_n'\}$ cgs uniformly to g on $[a,b]$

then f is differentiable & $f'(x)=g(x)\ \forall\ x\in[a,b]$

(b) If Σf_n is st
(i) each f_n is diff on $[a,b]$
(ii) each f_n' is cont. on $[a,b]$
(iii) Σf_n cgs ~~uniformly~~ to f on $[a,b]$
(iv) $\Sigma f_n'$ cgs uniformly to g on $[a,b]$ ← Only check UC of $\Sigma f_n'$!!!

then f is differentiable & $f'(x)=g(x)\ \forall\ x\in[a,b]$

eg. ST: $f_n(x) = \dfrac{nx}{1+n^2x^2}$ can't be diff. term by term at $x=0$

$x \in [0,1]$

$f_n(x) = \dfrac{nx}{1+n^2x^2}$ $\quad f(x) = 0 \;\forall\; x \in [0,1]$

$\therefore f'(0) = 0$

$f_n'(0) = \lim_{h \to 0} \dfrac{f_n(h) - f_n(0)}{h} = \dfrac{nh}{1+n^2h^2 \cdot h} = \dfrac{n}{1+h^2n^2} = n$

$\therefore \lim f_n'(0) = \lim n = \infty$

NOTE: Term by Term $\int$ & $\frac{d}{dx}$ is not restricted to series only. It can be done for sequences also.

We check for $\int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n(x)$

& $\dfrac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\dfrac{d}{dx} f_n(x)\right)$

eg. ST for $\left\{ f_n = \dfrac{x}{1+nx^2} \right\}$ $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true if $x \neq 0$ & false if $x=0$. why?

Ans: $f(x) = 0$ & $f_n(x)$ is UC to $f(x)$

$f'(x) = 0 \Rightarrow f'(0) = 0$ & $f'(x) = 0$ for $x \neq 0$

$f_n'(x) = \dfrac{1-nx^2}{(1+nx^2)^2}$ for $x \neq 0$ $\lim_{n\to\infty} f_n'(x) = \dfrac{1-nx^2}{(1+nx^2)^2} = \dfrac{-x^2}{2(1+nx^2)x^2} = 0$

$\therefore$ for $x \neq 0$ $\lim_{n\to\infty} f_n'(x) = f'(x) = 0$

But $f_n'(0) = \lim_{h\to 0} \dfrac{f_n(h) - f_n(0)}{h} = \dfrac{1}{1+nh^2} = 1 \neq f'(0)$.

This is because $\{f_n'\}$ is not UC on any interval having 0 and the 3rd condn in the thm fails

Examples :

① ST $x=0$ is a pt of non-UC of series

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

$$\Rightarrow S_n(x) = \sum_{r=1}^{n} \frac{x^2}{(1+x^2)^r} = \frac{x^2\left[1-\frac{1}{(1+x^2)^n}\right]}{1-\frac{1}{1+x^2}} = (1+x^2) - \frac{1}{(1+x^2)^{n-1}}$$

$$\lim_{n\to\infty} S_n(x) = S(x) = \begin{cases}(1+x^2) & x\neq 0\\ 0 & x=0\end{cases}$$

For $x\neq 0$ & given ϵ $|S_n(x)-S(x)| = \frac{1}{(1+x^2)^{n-1}} < \epsilon$

if $(1+x^2)^{n-1} > \frac{1}{\epsilon} \Rightarrow (n-1)\log(1+x^2) > \log\left(\frac{1}{\epsilon}\right)$

$\Rightarrow$ if $n > 1 + \frac{\log(1/\epsilon)}{\log(1+x^2)}$

As $x\to 0$, $n\to\infty$, so its pt of non-UC.

if $x\in[a,\infty)$, $a>0$, then max value of $1+\frac{\log(1/\epsilon)}{\log(1+x^2)}$

is $1+\frac{\log(1/\epsilon)}{\log(1+a^2)}$

So if we take $m \geq 1+\frac{\log(1/\epsilon)}{\log(1+a^2)}$, then

$|S_n(x)-S(x)| < \epsilon \ \forall\ n\geq m$ Hence series is UC in $[a,\infty)$ $a>0$

② Test UC for $\sum_{n=1}^{\infty}\left[\frac{2n^2x^2}{e^{n^2x^2}} - \frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}\right]$

$S_n(x) = \frac{2n^2x^2}{e^{n^2x^2}}$ & $S(x)=0$

$\left|S_n(x)-S(x)\right| = \frac{2n^2x^2}{e^{n^2x^2}} < \epsilon$ now take $x=\frac{1}{n}\in[0,1]$

$\Rightarrow \frac{2}{e} < \epsilon$ $\therefore$ If we take $\epsilon < \frac{2}{e}$, it fails

Hence non UC.

Here ϵ can be any value.

TRICK : Don't forget to try putting some value of x in the interval for which all ϵ are not possible.

eg in previous one.

or $S_n(x) = \frac{n^2 x}{1+n^4x^2}$ ST it is not UC in $[0,1]$

Here $|S_n(x) - S(x)| = \left|\frac{n^2x}{1+n^4x^2}\right| < \epsilon$ Take $x = \frac{1}{n^2} \in [0,1]$

$\Rightarrow \frac{1}{2} < \epsilon$, $\therefore$ $\epsilon < \frac{1}{2}$ it fails.

Remember : M_n test can be used to negate as it is 'iff' but not Weistrass'

eg. Test $\sum_{n=0}^{\infty} xe^{-nx}$ in $x \in [0,1]$

$S_n(x) = x + xe^{-x} + xe^{-2x} + \cdots + xe^{-(n-1)x}$

$= \frac{x(1-e^{-nx})}{1-e^{-x}} = \frac{xe^x}{e^x-1}(1-e^{-nx})$

$S(x) = \frac{xe^x}{e^x-1}$ $x \neq 0$

0 $x=0$

$|S_n(x) - S(x)| = \frac{xe^x}{(e^x-1)e^{nx}} < \epsilon \Rightarrow \frac{e^{nx}(e^x-1)}{xe^x} > \frac{1}{\epsilon}$

$\Rightarrow$ If $\log(e^x-1) + nx - \log x - x > \log(1/\epsilon)$

$\log\left(x + \frac{x^2}{2!} + \cdots\right) + nx - \log x - x > \log(1/\epsilon)$

$nx > \log(1/\epsilon) + x - \log\left(1 + \frac{x}{2!} + \cdots\right)$

$\therefore n > \frac{\log(1/\epsilon) + x - \log\left(1+\frac{x}{2!}+\cdots\right)}{x} \to \infty$ $n \to \infty$ as $x \to 0$

$\therefore$ not UC

eg Test UC of $1+x+\frac{x^2}{2!}+ \dots$ in $[-1,1]$.

$f_n(x) = \frac{x^n}{n!}$ $\quad |f_n(x)| = \left|\frac{x^n}{n!}\right| \le \frac{1}{n!}$

REMEMBER : $n! \ge 2^n$ for $n > 3$

$\therefore |f_n(x)| \le \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$\Sigma M_n = \Sigma \left(\frac{1}{2}\right)^n$ is cgt.

14) Abel's Test :

(i) Σf_n is UC on $[a,b]$

(ii) $(g_n(x))$ is monotonically decreasing $\forall x \in [a,b]$

(iii) $|g_n(x)| < K$ $\quad K \in \mathbb{R}^+ \quad \forall x \in [a,b]$ & $n \in \mathbb{N}$

Then $\Sigma f_n(x) g_n(x)$ is UC on $[a,b]$

15) Dirichlet's Test :

(i) $|S_n(x)| < K (>0)$ $\forall x \in [a,b]$, $n \in \mathbb{N}$

(ii) $(g_n(x))$ is +ve $\downarrow$ seq UCing to 0 in $[a,b]$

then $\Sigma f_n(x) g_n(x)$ is UC on $[a,b]$

eg. $\Sigma \frac{(-1)^{n-1}}{n} x^n$ is UC on $[0,1]$

$\Rightarrow \Sigma f_n = \Sigma \frac{(-1)^{n-1}}{n}$ is cgt by Leibnitz & UC as it is independent of 'x'

Also $|g_n(x)| = |x^n| \le 1 \quad \forall x \in [0,1]$

$\Rightarrow$ By Abel's its UC

Remember : If seq is independent of 'x' & cgt $\Rightarrow$ it is UC

As UC condition says ϵ & m (st $n \ge m$) is independent of x & depends only on ϵ.

TIP (a) for $\frac{(-1)^{n-1}}{n}$ types go for Abel's or if you can see a cgt series

Otherwise go for dirichlets with $f_n(x)$ as $(-1)^{n-1}$

eg $\sum \frac{(-1)^{n-1}}{n+x^2}$ here $f = (-1)^{n-1}$ & $g_n(x) = \frac{1}{n+x^2}$

Now $S_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd} \end{cases} \Rightarrow$ bdd

& $g_n(x) = \frac{1}{n+x^2}$ cgs to 0 uniformly.

So use dirichlet & hence its UC

(b) cos, sin terms $\Rightarrow$ bdd generally so choose dirichlet

eg $f_n(x) = \sin nx$

Σ $S_n(x) = \sin x + \sin 2x + \dots + \sin nx$ $\quad x \in (0, 2\pi)$

$$= \frac{\sin\left[x + \frac{(n-1)x}{2}\right] \cdot \sin\left(\frac{nx}{2}\right)}{\sin \frac{x}{2}}$$

$$|S_n(x)| \le \frac{1}{|\sin \frac{x}{2}|} = \left|\operatorname{cosec} \frac{x}{2}\right|$$

Now in $(0, 2\pi)$ even $\operatorname{cosec} \frac{x}{2}$ is bdd

Imp ST: $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}\right) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$ $\forall x \in R$

$f_n(x) = \frac{1}{n^3+n^4x^2}$ $\quad |f_n(x)| \le \frac{1}{n^3}$ $\quad M_n = \frac{1}{n^3}$ ΣM_n is cgt $\Rightarrow \Sigma f_n$ cgs UC.

$f_n'(x) = \frac{-2x(n^4)}{(n^3+n^4x^2)^2} = \frac{-2x}{n^2(1+nx^2)^2}$ $\quad |f_n'(x)| \le \frac{2x}{n^2(1+nx^2)^2}$ [can't do $\frac{c}{n^2}$ with x]

Iti maxima occurs at $x = 1/\sqrt{3n}$
$|f_n'(x)| \le 1/n^{5/2} \Rightarrow$ UC $\quad \therefore \frac{d}{dx}(\Sigma f) = \Sigma \frac{d}{dx}(f)$. [$\because \Sigma f_n$ & $\Sigma f_n'$ are UC]

eg. ST: $S_n(x) = \dfrac{nx}{1+n^2x^2}$ can't be differentiated term by term at $x=0$.

$S(x) = \lim\limits_{n\to\infty} S_n(x) = 0$

$\therefore S'(x) = 0 \quad \forall\ x \in \mathbb{R}$

$S_n'(0) = \lim\limits_{h\to 0} \dfrac{S_n(0+h) - S_n(0)}{h} = \dfrac{\frac{nh}{1+n^2h^2} - 0}{h} = \dfrac{n}{1+n^2h^2} = n$

Improper Integrals :

1) Reimann needs that range of integration is finite & f remains bdd., but if either or both become infinite it is improper integral.

2) 1^{st} Kind : either a or b (or both) are infinite but f is bdd. eg $\int_{-\infty}^{0} e^{2x} dx$

2^{nd} Kind : Range $[a,b]$ is finite but f has one or more pts of ∞ discontinuity, ie, f is not bounded eg $\int_{0}^{1} \frac{1}{x(1-x)} dx$

3^{rd} Kind : Both the range of integration & f is unbounded eg $\int_{0}^{\infty} e^{x} dx$

3) $\int_{a}^{\infty} f(x)\, dx = \lim_{t \to \infty} \int_{a}^{t} f(x)\, dx$ Improper integral cgs if limit on RHS exists.

$\int_{-\infty}^{b} f(x)\, dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)\, dx$

$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)\, dx + \lim_{t_2 \to \infty} \int_{c}^{t_2} f(x)\, dx \qquad c \in \mathbb{R}$

Here t_1 & t_2 are different.

If $f(x)$ becomes infinite at $x=b$ $\int_{a}^{b} f(x)\, dx = \lim_{\epsilon \to 0} \int_{a}^{b-\epsilon} f(x)\, dx$

at $x=a$ $\int_{a}^{b} f(x)\, dx = \lim_{\epsilon \to 0} \int_{a+\epsilon}^{b} f(x)\, dx$

at both $\Rightarrow \int_{a}^{b} f(x)\, dx = \lim_{\epsilon_1 \to 0} \int_{a+\epsilon_1}^{c} f(x)\, dx + \lim_{\epsilon_2 \to 0} \int_{c}^{b-\epsilon_2} f(x)\, dx$

$= \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{a+\epsilon_1}^{b-\epsilon_2} f(x)\, dx$

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eg. $\int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \int_{\sqrt{2}}^{t} \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \int_{\pi/4}^{\sec^{-1}(t)} \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta}$

$= \lim_{t\to\infty} \left[\sec^{-1}t - \frac{\pi}{4}\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

eg $\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\, dx$ Put $x = \tan\theta$ & solve.

Remember : $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

→ $\int\cos$ is +ve sign & cos before sin in $a(\,) + b(\,)$

4) Comparison Tests

Test - I : $f, g \geq 0$ & $f(x) \leq g(x)$ $\forall x \in (a, b]$

and 'a' is the only pt of ∞ discontinuity

(i) $\int_a^b g(x)\,dx$ cgs $\Rightarrow \int_a^b f(x)\,dx$ cgs

(ii) $\int_a^b f(x)\,dx$ dgs $\Rightarrow \int_a^b g(x)\,dx$ cgs.

Test - II : $\lim_{x\to a^+} \frac{f(x)}{g(x)} = l \neq 0$, finite, then f & g cgs(only) together

$l = 0 \Rightarrow \int g(x)$ cgs $\Rightarrow \int f(x)$ cgs.

$l = +\infty \Rightarrow \int g(x)$ dgs $\Rightarrow \int f(x)$ dgs

Same as sequence tests !!

5) * $\int_a^b \frac{1}{(x-a)^n}dx$ cgs iff $n<1$ [similarly $\int_a^b \frac{1}{(b-x)^n}$]

* a is the pt of ∞ dc & $\lim_{x\to a^+} (x-a)^n f(x)$ exists & is non-zero finite

then $\int_a^b f(x)dx$ cgs iff $n<1$.

*6) f is +ve on $(a,b]$ & a is the only pt of ∞ dc.

then $\int_a^b f(x)dx$ cgs if $\exists$ +ve no $n<1$ & a fixed +ve M

st $f(x) \le \frac{M}{(x-a)^n}$ $\forall x\in(a,b]$

7) $\int_a^b f(x)dx$ dgs if $\exists$ a no $n\ge 1$ & a fixed +ve G

st $f(x) \ge \frac{G}{(x-a)^n}$ $\forall x\in(a,b]$

⇓

Application of Test - II & pt. 5.

Examples:

ⓐ $\int_0^{\pi/2} \frac{\sin x}{x^p}dx$ $p\le 1$ it is cgt [∵ All proper integrals are cgt]

↳ note equality also

check for $p>1$: $f(x) = \frac{1}{x^{p-1}}\cdot\frac{\sin x}{x}$ $g(x) = \frac{1}{x^{p-1}}$

$\lim_{x\to 0} \frac{f}{g} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \ne 0$ ∴ cgs & dgs together

$g(x)$ cgs for $p-1<1 \Rightarrow p<2$

∴ $p<2 \Rightarrow$ cgt $p\ge 2 \Rightarrow$ dgt

(b) If both (a & b are ∞ dc pts, then check for both separately.

I cgs if both I_1 & I_2 cgs

I dgs if either I_1 or I_2 dgs

eg $\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$ cgs as it cgs at both $x=0$ & $x=1$

(c) $\int_0^2 \frac{\log x}{\sqrt{2-x}} dx$ $f(x) = \frac{\log x}{\sqrt{2-x}}$ 0, 2 are pts of ∞ dc.

$$\int_0^2 f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{I_1} + \underbrace{\int_1^2 f(x)\,dx}_{I_2}$$

Check I_1 : at $x=0$ $\log(x) \le 0$ for $\in (0,1]$

$\therefore$ consider $-f(x)$.

Take $g(x) = \frac{1}{x^n}$ → Remember technique

Now $\lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{\sqrt{2-x}}$ $\left[\lim_{x\to 0^+} x^n \log x = 0 \text{ if } n>0\right]$ ⇓ Very impt

$= 0$ if $n>0$

Take n b/w 0 & 1 ⇒ $\int g$ is cgt $\therefore \int f$ is cgt by comp test

I_2 is clearly cgt at $x=2$

$\therefore I$ is cgt in $[0,2]$

(d) $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ $g(x) = \frac{1}{(x-1)^n}$ $\lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)^n \sqrt{x}}{\log x}$

$l = \lim_{x\to 1^+} \left[n(x-1)^{n-1}\sqrt{x} + \frac{(x-1)^n}{2\sqrt{x}} x\right] x = 1$ for $n=1$ ⇒ dgs.

↳ Don't put $(x-1)=0$ directly, there's powers can be 0 also for some n!!

(e) Check only ∞ dc $\therefore \int_0^1 \frac{\log x}{1-x^2}\,dx$

$$\lim_{x\to 1} \frac{-\log x}{1-x^2} = \lim_{x\to 1} \frac{-1}{x \times -2x} = +1/2$$

$\therefore$ only $x=0$ is ∞ dc at which It is cgt

$\left[\because g(x) = \frac{1}{x^n}, 0 < n < 1\right]$

(f) Check $\int_0^1 \frac{(x^p + x^{-p})\log(1+x)}{x}\,dx$

(i) $p = 0$ $f(x)$ is not ∞ at $x=0$ & hence proper

(ii) $p > 0$ $f(x) = \left(x^p + \frac{1}{x^p}\right)\frac{\log(1+x)}{x}$ $g(x) = \frac{1}{x^p}$

$$\lim_{x\to 0} \frac{f}{g} = \lim_{x\to 0} (x^{2p}+1)\frac{\log(1+x)}{x} = 1 \cdot \lim \frac{\frac{1}{1+x}}{1} = 1.$$

$\therefore 0 < p < 1 \Rightarrow$ cgt

$p \geq 1 \Rightarrow$ dgt

(iii) $p < 0$ $p = -q$. $f(x) = \left(\frac{1}{x^q} + x^q\right)\frac{\log(1+x)}{x}$ $g(x) = \frac{1}{x^q}$

again $\lim_{x\to 0}\frac{f}{g} = 1$.

$\therefore 0 < q < 1 \Rightarrow 0 < -p < 1 \Rightarrow -1 < p < 0$ cgt

$q > 1 \Rightarrow -p > 1 \Rightarrow p < -1$ dgt.

(g) $\int_0^1 x^{n-1}\log x\,dx$ $n-1 \geq 0$ cases are trivial

for $n-1 < 0$ $f(x) = -x^{n-1}\log x$ $g(x) = 1/x^p$

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+} -x^{p+n-1}\log x = \begin{cases} 0 & p+n-1 > 0 \\ \infty & p+n-1 \leq 0 \end{cases}$$

$\therefore 0 < p < 1$ & $p > 1-n \Rightarrow$ cgt $\Rightarrow 1-n < 1 \Rightarrow n > 0$

$p = 1 \Rightarrow g(x)$ is dgt & $L = \infty \Rightarrow f(x)$ is dgt $\Rightarrow 1 \leq 1-n$
& $p \leq 1-n$ $\Rightarrow n \leq 0$

$\therefore$ $n > 0$ cgt
$n \leq 0$ dgt.

Don't forget $L = \infty$
& g dgt $\Rightarrow$ f dgt

(h) $\int_0^1 (\log 1/x)^n dx$ $\quad f(x) = (\log 1/x)^n$

$I = \underbrace{\int_0^a f(x)\,dx}_{I_1} + \underbrace{\int_a^1 f(x)\,dx}_{I_2}$

Check: $I_1 \Rightarrow n \le 0$ proper

$n > 0 \Rightarrow 0$ is dc pt

$g(x) = 1/x^p$.

$\lim_{x\to 0} \frac{f}{g} = \lim x^p \left(\log\frac{1}{x}\right)^n = 0$

choose $0 < p < 1$

$\Rightarrow f(x)$ is cgt

Check $I_2 \Rightarrow n \ge 0$ proper

$n < 0 \Rightarrow 1$ is dc pt

let $g(x) = \frac{1}{(1-x)^{-n}}$

instead of some p/q, go for n if you can get a finite f/g limit

$\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \left[\lim_{x\to 1^-} \frac{\log 1/x}{1-x}\right]^n$

$= \left[\lim_{x\to 1^-} \frac{-1/x^2 \cdot x}{-1}\right]^n$

$= 1$

$g(x)$ is cgt if $0 < -n < 1$

$\Rightarrow -1 < n < 0$

$\therefore \int_0^1 f(x)\,dx$ is cgt for $-1 < n < 0$

(i) ST: $\int_0^1 \frac{\operatorname{cosec} x}{x^n} dx$ is dgt if $n \ge 1$

$f(x) = \frac{\operatorname{cosec} x}{x^n}$ $\quad \left|\frac{\operatorname{cosec} x}{x^n}\right| \ge \frac{1}{|x^n|}$ $\quad \forall\, x \in (0,1]$

$f(x) \ge \frac{1}{x^n} \quad \forall\, x \in (0,1]$

$g(x)$ is dgt if $n \ge 1$ Hence $f(x)$ is dgt.

8) Absolute convergence: if $\int_a^b |f(x)|\,dx$ is cgt

AC $\Rightarrow$ cgt.

It is useful if $f(x)$ changes sign in the interval.

Here make use of (6), (7) eg $\int_0 \frac{\sin 1/x}{\sqrt{x}} dx$ $\quad |f(x)| \le \frac{1}{\sqrt{x}}$

$\Downarrow$

$\int |f(x)|$ is cgt.

9) Convergence at ∞, f is +ve :

similar to earlier Test - I & Test - II.

10) $\int_a^\infty \frac{dx}{x^n}$ $(a>0)$ is convergent iff $n>1$ (opposite of earlier one)

dgt iff $n \le 1$

do not forget to change limits for this reqt before testing

eg. $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} = \int_0^a \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} + \int_a^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}$

proper

$a>0$ now apply test

eg $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ Now $e^{x^2} > x^2$

$e^{-x^2} < \frac{1}{x^2}$

cgt. $\Leftarrow$ cgt

eg $\int_1^\infty x^n e^{-x}dx$ cgs as e^{-x} goes to 0 rapidly.

choose $g(x) = \frac{1}{x^2}$ $\lim \frac{f}{g} = 0$ $\int g$ cgt $\Rightarrow \int f$ cgt.

11) Abel's Test : If $\int_a^\infty f(x)\,dx$ cgs & $g(x)$ is bdd & monotonic

then $\int_a^\infty f(x)\,g(x)\,dx$ cgs at ∞

12) Dirichilet's Test : If $\int_a^t f(x)\,dx$ is bdd $\forall$ $t \ge a$

& $g(x)$ is bdd & monotonic for $x \ge a$ & $\longrightarrow 0$ as $x \longrightarrow \infty$

then $\int_a^\infty f(x)\,g(x)\,dx$ cgs at ∞

eg. $\displaystyle\int_1^\infty \frac{\sin x^m}{x^n}\,dx$

$m=0 \quad I = \sin 1 \displaystyle\int_1^\infty \frac{1}{x^n}\,dx$ cgs for $n>1$

$m \neq 0$ Put $x^m = t \quad x = t^{1/m} \quad dx = \frac{1}{m} t^{1/m - 1}\,dt$

$$I = \int_1^\infty \frac{\sin t}{t^{n/m}} \cdot \frac{1}{m} \cdot t^{1/m-1}\,dt = \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{\frac{n-1}{m}+1}}$$

Let $f(t) = \sin t$ & $g(t) = \dfrac{1}{t^{\frac{n-1}{m}+1}}$

and apply Dirichilet.

* Try to get $\frac{1}{x}$ term for $g(x)$

eg $\displaystyle\int_1^\infty \sin x^2\,dx$ Here take $f(x) = 2x \sin x^2$ & $g(x) = \frac{1}{2x}$