1.c, 1.e, 2a, 3a

2014 March / Wednesday

NOTES APPOINTMENTS

12

IFS 2009 P-2

1.c) show that under the transformation
$$w = \frac{z-i}{z+i}$$
real axis in the z-plane is mapped into the circle $|w|=1$.
What portion of the z-plane corresponds to the interior of the circle. (10)

Here, $w = \frac{z-i}{z+i}$

Real axis of z-plane is, $y=0$
i.e. $z=x$

$\therefore$ $w = \frac{x-i}{x+i}$

$$|w| = \frac{|x-i|}{|x+i|} = \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} = 1$$

$\therefore$ y-axis is mapped into circle, $|w|=1$

Interior of circle, $|w|<1$

$\Rightarrow \left|\frac{z-i}{z+i}\right| < 1 \Rightarrow |x+iy-i| < |x+iy+i|$

i.e. $x^2+(y-1)^2 < (x)^2+(y+1)^2$

$-2y < 2y \Rightarrow \boxed{0 > y}$

Half-plane $y<0$ corresponds to interior of circle.

1.e) Evaluate $\int_C \frac{2z+1}{z^2+z} dz$ using Cauchy's Integral formula where C is $|z| = \frac{1}{2}$.

Cauchy's Integral formula: $f(z)$ is analytic over D bounded by closed curve C. $z_0 \in D$

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i \, f(z_0).$$

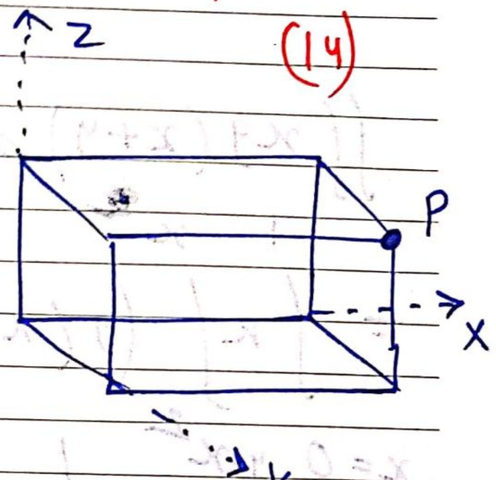
$$\int_C \frac{2z+1}{z^2+z} dz = \int_{|z|=\frac{1}{2}} \frac{f(z)}{z} dz$$

where $f(z) = \frac{2z+1}{z+1}$ which is analytic on domain, $|z| = \frac{1}{2}$

$$f(0) = \frac{0+1}{0+1} = 1$$

$$\therefore \int_C \frac{2z+1}{z^2+1} dz = (2\pi i) f(0) = 2\pi i$$

2(a) Find the dimensions of the largest rectangular parallelopiped that has three faces in the coordinates planes and one vertex in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (14)



Let point P (vertex) lies on given plane.

$\therefore$ sides of cuboid becomes x, y and z

Maximize, xyz

given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ —— (1)

Consider, $F(x,y,z) = xyz + \lambda\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)$

for critical points, $dF = 0$

i.e. $F_x = 0 = F_y = F_z$

$yz + \frac{\lambda}{a} = 0$, $xz + \frac{\lambda}{b} = 0$, $xy + \frac{\lambda}{c} = 0$

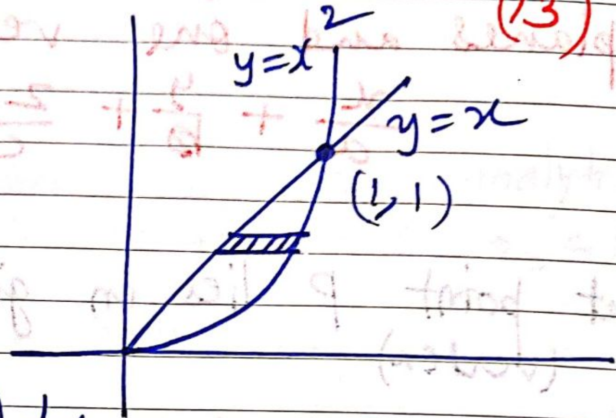
$-\lambda = ayz = bxz = cxy \Rightarrow \boxed{\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = \frac{1}{k}}$

using in (1) $\frac{ak}{a} + \frac{bk}{b} + \frac{ck}{c} = 1 \Rightarrow k = \frac{1}{3}$

$\therefore x = a/3$, $y = b/3$, $z = c/3$.

Evaluate

3a) $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$. (13)



$$\iint xy(x+y)\,dx\,dy$$

$$= \int_{x=0}^{1} x \int_{y=x^2}^{x} \left(y(x+y^2\,dy\right)dx$$

$$= \int_{x=0}^{1} x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_{y=x^2}^{x} dx$$

$$= \int_{0}^{1} x\left\{\frac{x}{2}(x^2-x^4)+\frac{1}{3}(x^3-x^6)\right\}dx$$

$$= \int_{0}^{1} x\left(\frac{5x^3}{6}-\frac{x^5}{2}-\frac{x^6}{3}\right)dx$$

$$= \int_{0}^{1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx = \left.\frac{\cancel{5}x^5}{\cancel{30}_{6}}-\frac{x^7}{14}-\frac{x^8}{24}\right|_{0}^{1}$$

$$= \frac{1}{6}-\frac{1}{14}-\frac{1}{24} = \frac{3}{56}$$

$\boxed{\text{Alternative}}$ Change of Order

$$= \int_{y=0}^{1} y \int_{x=y}^{\sqrt{y}} x\,(x+y)\,dx\,dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^3}{3} + y \cdot \frac{x^2}{2} \right]_{x=y}^{\sqrt{y}} dy$$

$$= \int_{0}^{1} y \left[\frac{1}{3}\left(y^{3/2} - y^3\right) + \frac{y}{2}\left(y - y^2\right) \right] dy$$

$$= \int_{0}^{1} y \left[\frac{1}{3} y^{3/2} - \frac{1}{3} y^3 + \frac{y^2}{2} - \frac{y^3}{2} \right] dy$$

$$= \int_{0}^{1} \left(\frac{1}{3} y^{5/2} - \frac{5}{6} y^4 + \frac{y^3}{2} \right) dy$$

$$= \frac{1}{3} \times \frac{2}{7} \cdot y^{7/2} - \frac{5}{6} \times \frac{y^5}{5} + \frac{y^4}{8} \Bigg|_0^1$$

$$= \frac{2}{21} - \frac{1}{6} + \frac{1}{8} = \frac{3}{56}$$