Q2. Find the value of q for which the following set of linear algebraic equations can have non-trivial solution

$$2x + 3y = 0$$
$$6x + qy = 0$$

Soln

NON-TRIVIAL SOLUTION

The system of equation in which the determinant of the coefficient is zero is called NON-TRIVIAL SOLUTION.

TRIVIAL SOLUTION

The system of equation in which the determinant of the coefficient is not zero but the solution are $x = y = z = 0$ is called TRIVIAL SOLUTION.

$$2x + 3y = 0$$
$$6x + qy = 0$$

$$AX = B$$

$$\begin{bmatrix} 2 & 3 \\ 6 & q \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 6 & q \end{vmatrix} = 2q - 18 = 0$$

$$\Rightarrow 2q = 18$$

$$\boxed{q = 9}$$ Ans

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Q3. A set of simultaneous linear algebraic equations is represented in a matrix form as shown below:

$$\begin{bmatrix} 0 & 0 & 0 & 4 & 13 \\ 2 & 5 & 5 & 2 & 10 \\ 0 & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 2 & 3 & 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 46 \\ 161 \\ 61 \\ 30 \\ 81 \end{bmatrix}$$

The value (rounded off to the nearest integer) of x_3 is ___

Soln

METHOD 1

$$4x_4 + 13x_5 = 46 \quad \text{--- (1)}$$

$$2x_1 + 5x_2 + 5x_3 + 2x_4 + 10x_5 = 161 \quad \text{--- (2)}$$

$$2x_3 + 5x_4 + 3x_5 = 61 \quad \text{--- (3)}$$

$$4x_4 + 5x_5 = 30 \quad \text{--- (4)}$$

$$2x_1 + 3x_2 + 2x_3 + x_4 + 5x_5 = 81 \quad \text{--- (5)}$$

Solving (1) and (4),

$$\begin{aligned} \cancel{4x_4} + 13x_5 &= 46 \\ \cancel{4x_4} + 5x_5 &= 30 \\ (-) \quad (-) \quad & (-) \\ \hline 8x_5 &= 16 \end{aligned}$$

$$x_5 = 2$$

$$x_4 = \frac{46 - (13 \times 2)}{4} = \frac{46 - 26}{4} = \cancel{\frac{20}{4}} \; 5$$

$$2x_3 + 5(5) + 3(2) = 61$$

$$2x_3 + 25 + 6 = 61$$

$$2x_3 + 31 = 61$$

$$2x_3 = 30$$

$$x_3 = 15$$

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$$2x_1 + 5x_2 + 5(15) + 2(5) + 10(2) = 161$$

$$2x_1 + 5x_2 + 75 + 10 + 20 = 161$$

$$2x_1 + 5x_2 = 56 \quad \text{—} (6)$$

Putting Values of x_3, x_4 & x_5 in eq^n 5

$$2x_1 + 3x_2 + 2(15) + 5 + 5(2) = 81$$

$$2x_1 + 3x_2 + 30 + 5 + 10 = 81$$

$$2x_1 + 3x_2 = 36 \quad \text{—} (7)$$

Solving (6) and (7),

$$\begin{array}{l} \cancel{2x_1} + 5x_2 = 56 \\ \cancel{2x_1} + 3x_2 = 36 \\ \quad (-) \quad (-) \\ \hline \cancel{(-)} \\ 2x_2 = 20 \\ x_2 = 10 \end{array}$$

$$2x_1 + 3(10) = 36$$

$$2x_1 + 30 = 36$$

$$2x_1 = 6$$

$$\boxed{x_1 = 3}$$

Solution:

$x_1 = 3$

$x_2 = 10$

$x_3 = 15$

$x_4 = 5$

$x_5 = 2$

METHOD 2

You can also solve it using ROW OPERATIONS

Q4. For what values of λ and μ the system of equations has no solution

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

Soln

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 1 & 4 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ \mu \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 6 \\ 1 & 4 & \lambda \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{bmatrix}$$

~~For no solution~~

System of Eqns

- Consistent
  - Unique soln
    - $|A| \neq 0$
    - $\rho(A) = \rho(C) = n$
  - Infinite soln
    - $|A| = 0$
    - $\rho(A) = \rho(C) \neq n$
- Inconsistent
  - No soln
    - $|A| = 0$
    - $\rho(A) \neq \rho(C)$

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If $\lambda = 6$ $|A| = 0$ $\Rightarrow$ Either No soln or ∞ soln
$\rho(A) = 2$

Now if $C = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & 6 & \mu \end{bmatrix}$ If $\mu = 20$
$\rho(C) = 2$

But for No solution $\rho(A) \neq \rho(C)$
$\Rightarrow$ $\rho(C)$ should not be 2.
$\Rightarrow$ $\mu \neq 20$

So for $[\lambda = 6 \ \& \ \mu \neq 20]$ the system has <u>NO SOLUTION</u>

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Q6. $[P]_{4\times2}\ [Q]_{2\times4}\ [R]_{4\times1}$

A: Min. no. of multiplications to obtain PQR

Case 1 PQR can be obtained by $[PQ]\times[R]$

$[\cancel{PQ}]$ No. of multiplications required to obtain PQ $= 4\times2\times4$
$= 32$

No. of multiplications required to obtain $[PQ]_{4\times4}\times[R]_{4\times1}$
$[PQR]$ $= 4\times4\times1$
$= 16$

Total multiplications required $= 48$

Case 2 PQR can be also obtained by $[P]\times[QR]$

No. of multiplications required to obtain $[QR] = 2\times4\times1 = 8$

No. of multiplications required to obtain $[PQR] = [P]_{4\times2}\ [QR]_{2\times1}$
$= 4\times2\times1$
$= 8$

Total multiplications required $= 8+8 = 16$

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Min no. of multiplications required $= \min\{48, 16\}$

$= 16$ Ans

MULT

B. Min no. of Additions required

Case 1 PQR can be obtained by $[PQ] \times [R]$

No. of additions req to obtain $[PQ] = [P]_{4\times2}\,[Q]_{2\times4}$

$= 4\times(2-1)\times4$

$= 4\times1\times4$

$= 16$

No. of addition req to obtain $[PQR] = [PQ]_{4\times4} \times [R]_{4\times1}$

$= 4\times(4-1)\times1$

$= 4\times3\times1$

$= 12$

Total additions req $= 16+12$

$= 28$

Case 2 PQR can be obtained by $[P] \times [QR]$

No. of additions req to obtain $[QR] = [Q]_{2\times4} \times [R]_{4\times1}$

$= 2\times(4-1)\times1$

$= 2\times3\times1 = 6$

No. of additions req to obtain $[PQR] = [P]_{4\times2}\,[Q]_{2\times1}$

$= 4\times(2-1)\times1$

$= 4\times1\times1 = 4$

Total additions req $= 10$

Min no. of additions req $= \min\{28, 10\} = 10$ Ans

ADDITION

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Q7. Find the rank of the matrix by reducing it to canonical form

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

Canonical form $\longrightarrow$ Row Reduced Echelon form

Soln

$$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix} \xrightarrow{R_2 \to 7R_2} \begin{bmatrix} 6 & 0 & 4 & 4 \\ -14 & 98 & 56 & 126 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 6 & 0 & 4 & 4 \\ -14 & 98 & 56 & 126 \\ 0 & 84 & 56 & 116 \end{bmatrix} \xrightarrow{R_1 \to \frac{14}{6} R_1} \begin{bmatrix} 14 & 0 & \frac{28}{3} & \frac{28}{3} \\ -14 & 98 & 56 & 126 \\ 0 & 84 & 56 & 116 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 14 & 0 & 28/3 & 28/3 \\ 0 & 98 & 196/3 & 406/3 \\ 0 & 84 & 56 & 116 \end{bmatrix} \xrightarrow{R_2 \to \frac{84}{98} R_2} \begin{bmatrix} 14 & 0 & \frac{28}{3} & \frac{28}{3} \\ 0 & 84 & 56 & 116 \\ 0 & 84 & 56 & 116 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 14 & 0 & \frac{28}{3} & \frac{28}{3} \\ 0 & 84 & 56 & 116 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{14}} \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 84 & 56 & 116 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2/84} \begin{bmatrix} 1 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 2/3 & \frac{29}{21} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

CANONICAL FORM

Rank = No. of Non-zero rows in Canonical form

= 2 Ans

Given :

The system of equations are,

$$x+2y-3z=a$$

$$2x+3y+3z=b$$

$$5x+9y-6z=c$$

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It is in the form of a non-homogenous equation and is given by,

$$AX = B$$

where, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 3 \\ 5 & 9 & -6 \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

The augmented matrix is given by,

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 2 & 3 & 3 & : & b \\ 5 & 9 & -6 & : & c \end{bmatrix}$$

By elementary transformation,

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-2a \\ 0 & -1 & 9 & : & c-5a \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B] = \begin{bmatrix} 1 & 2 & -3 & : & a \\ 0 & -1 & 9 & : & b-5a \\ 0 & 0 & 0 & : & c-b-3a \end{bmatrix}$$

$$\rho(A:B) = 2$$

For system to be consistent,

$$\rho(A:B) = \rho(A)$$

The above condition is only possible if

$$(c-b-3a) = 0$$

$$3a+b-c = 0$$

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Given :

$$Q = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ \frac{-6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & \frac{-3}{7} \end{bmatrix}$$ is an orthogonal matrix.

Method 1

If $Q_{n\times n}$ is an orthogonal matrix then, $QQ^T = I$

The inverse of the matrix, $Q^{-1} = Q^T$

$$Q^{-1} = Q^T = \begin{bmatrix} \frac{3}{7} & \frac{-6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{-3}{7} \end{bmatrix}$$

Hence, the correct option is (C).

Method 2

Inverse of a matrix is given by, $Q^{-1} = \frac{[\text{Adj}\,Q]}{|Q|}$

$$\text{Adj}\,Q = \begin{bmatrix} \frac{-21}{49} & \frac{42}{49} & \frac{-14}{49} \\ \frac{-14}{49} & \frac{-21}{49} & \frac{-42}{49} \\ \frac{-42}{49} & \frac{-14}{49} & \frac{21}{49} \end{bmatrix}$$

$$|Q| = \frac{3}{7}\left(\frac{-9}{49} - \frac{12}{49}\right) - \frac{2}{7}\left(\frac{18}{49} - \frac{4}{49}\right) + \frac{6}{7}\left(\frac{-36}{49} - \frac{6}{49}\right)$$

$$|Q| = \frac{3}{7} \times \left(\frac{-21}{49}\right) - \frac{2}{7} \times \left(\frac{14}{49}\right) + \frac{6}{7} \times \left(\frac{-42}{49}\right)$$

$$|Q| = \frac{-63 - 28 - 252}{49 \times 7} = \frac{-343}{343} = -1$$

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$$Q^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{-6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & \frac{-3}{7} \end{bmatrix}$$

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1.8 (D)

Given : $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

By elementary transformation,

$R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above form is echelon form and there exist four non-zero rows.

So, $\rho(A) = 4$

Hence, the correct option is (D).

1.9 (B)

Given : $AX = B$

This represents a non-homogenous equation.

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Augmented matrix is given by,

$$[A:B]=\begin{bmatrix} 1 & 0 & 1 & 0 & : & 0 \\ 0 & 1 & 0 & 1 & : & 0 \\ 1 & 1 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 1 & : & 1 \end{bmatrix}$$

By elementary transformations,

$R_3 \to R_3 - R_1$

$$[A:B]=\begin{bmatrix} 1 & 0 & 1 & 0 & : & 0 \\ 0 & 1 & 0 & 1 & : & 0 \\ 0 & 1 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$[A:B]=\begin{bmatrix} 1 & 0 & 1 & 0 & : & 0 \\ 0 & 1 & 0 & 1 & : & 0 \\ 0 & 0 & -1 & -1 & : & 0 \\ 0 & 0 & 0 & 1 & : & 1 \end{bmatrix}$$

There exists four non-zero rows.

So, $\rho(A)=4$

$\rho(A:B)=4$

and number of variables $n=4$

$$\rho(A)=\rho(A:B)=n=4$$

This shows, the given non-homogenous equatio

has a unique solution.

Hence, the correct option is (B).

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Given : A matrix is defined as, $A = [a_{ij}]$

where, $1 \le i, j \le n$ and $n \ge 3$ and $a_{ij} = i.j$

For $n = 3 \Rightarrow j \le 3$,

Let order of matrix be 2×3.

Then, $[A]_{2\times3} = \begin{bmatrix} 1\times1 & 1\times2 & 1\times3 \\ 2\times1 & 2\times2 & 2\times3 \end{bmatrix}$

$$[A]_{2\times3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

In the above matrix, there exist a linear relationship as given below,

$$R_2 = 2R_1$$

Thus, only row R_1 is an independent row.

So, $\rho(A) = 1$

For $n = 4 \Rightarrow j \le 4$,

Now, let order of matrix be 3×4.

Then, $[A]_{3\times4} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}_{3\times4}$

In the above matrix, there exist a linear relationship as given below,

$$R_2 = 2R_1 \text{ and } R_3 = 3R_1$$

Thus, only row R_1 is an independent row.

So, $\rho(A) = 1$

Therefore, for any order of given matrix, rank of A is always equal to 1.

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Given : P is a 3×3 matrix and $P \neq 0$

So, its rank can be 0, 1, 2, 3.

Since, $P \neq 0$ which means P is not a null matrix.

Therefore, rank zero is not possible.

$$Px = 0 \text{ and } Py = 0, \; P \neq 0$$

Therefore, $x = 0$, $y = 0$ i.e. they form null space which means P has two nullity.

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Rank = Dimension – Nullity

$\rho = 3 - 2 = 1$

Hence, the correct option is (B).

Key Point

(i) Rank of the matrix is also known as **dimensions of the Range space** of matrix.

(ii) Number of vectors required for the formation of null space is called **nullity**.

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Q13. State and prove Lagrange's identity

Soln

$$\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2$$

OR

$$\|u \times v\|^2 + (u \cdot v)^2 = |u|^2 |v|^2$$

LHS

$$\|u \times v\| = \|u\| \|v\| \sin\theta$$

m-1

$$\|u \times v\|^2 = \|u\|^2 \|v\|^2 \sin^2\theta$$

$$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$$

$$= \|u\|^2 \|v\|^2 - \|u\|^2 \|v\|^2 \cos^2\theta$$

$$= \|u\|^2 \|v\|^2 - (u \cdot v)^2$$

$\because [u \cdot v = \|u\| \|v\| \cos\theta]$

$$u \cdot u = \|u\| \|u\| \cos 0^{1}$$

$$= \|u\| \|u\|$$

$$(u \cdot u) = \|u\|^2$$

Similarly

$$(v \cdot v) = \|v\|^2$$

$$\boxed{\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2}$$

OR

$$\boxed{\|u \times v\|^2 = (u \cdot u)(v \cdot v) - (u \cdot v)^2}$$

Method 2

Take any vectors u & v in terms of scalar & prove

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Q15. Normalize each vector

a) $u = (5, -7)$ b) $v = (1, 2, -2, 4)$ c) $w = \left(\frac{1}{2}, \frac{-1}{3}, \frac{3}{4}\right)$

Soln a) $\hat{u} = \frac{u}{\|u\|} = \frac{(5,-7)}{\sqrt{5^2+(-7)^2}} = \cancel{\frac{(5,-7)}{\sqrt{25+49}} = \frac{(5,-7)}{\sqrt{76}}}$

$\cancel{\left(\frac{5}{\sqrt{60}}, \frac{-7}{\sqrt{60}}\right)}$ $\frac{5}{\sqrt{76}}$

$$\hat{u} = \frac{u}{\|u\|} = \frac{(5,-7)}{\sqrt{5^2+(-7)^2}} = \frac{(5,-7)}{\sqrt{25+49}} = \frac{(5,-7)}{\sqrt{74}}$$

$$\left(\frac{5}{\sqrt{74}}, \frac{-7}{\sqrt{74}}\right)$$ Ans

b) $$\hat{v} = \frac{v}{\|v\|} = \frac{(1,2,-2,4)}{\sqrt{1^2+2^2+(-2)^2+(4)^2}}$$

$$= \frac{(1, 2, -2, 4)}{\sqrt{1+4+4+16}}$$

$$= \frac{(1, 2, -2, 4)}{\sqrt{25}} = \left(\frac{1}{5}, \frac{2}{5}, \frac{-2}{5}, \frac{4}{5}\right)$$ Ans

c) $$\hat{w} = \frac{\left(\frac{1}{2}, \frac{-1}{3}, \frac{3}{4}\right)}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{-1}{3}\right)^2+\left(\frac{3}{4}\right)^2}} = \frac{\left(\frac{1}{2}, \frac{-1}{3}, \frac{3}{4}\right)}{\sqrt{\frac{133}{144}}}$$

$$= \frac{12\left(\frac{1}{2}, \frac{-1}{3}, \frac{3}{4}\right)}{\sqrt{133}} = \left(\frac{6}{\sqrt{133}}, \frac{-4}{\sqrt{133}}, \frac{9}{\sqrt{133}}\right)$$ Ans

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Q16. Let $u = (1, 2, -2)$, $v = (3, -12, 4)$ $k = -3$

a) Find $\|u\|$, $\|v\|$, $\|u+v\|$, $\|ku\|$

b) Verify that $\|ku\| = |k|\,\|u\|$ and

$\|u+v\| \le \|u\| + \|v\|$

Soln

a) $\|u\| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$ Ans

$\|v\| = \sqrt{3^2 + (-12)^2 + 4^2} = \sqrt{9+144+16} = \sqrt{169} = 13$ Ans

$\|ku\| = |k|\,\|u\| = |-3|\,\|u\| = 3 \times 3 = 9$ Ans

$\cancel{\|u+v\| = \sqrt{(u+v)^2}}$

$u+v = (1+3), (2-12), (-2+4)$

$= 4, -10, 2$

$\|u+v\| = \sqrt{4^2 + (-10)^2 + (2)^2} = \sqrt{16+100+4}$

$= \sqrt{120}$ Ans

b) $\|ku\| = \sqrt{(-3\times1)^2 + (-3\times2)^2 + (-3\times-2)^2}$

$= \sqrt{(-3)^2 + (-6)^2 + (6)^2}$

$= \sqrt{9+36+36} = \sqrt{81} = 9$

$|k|\,\|u\| = |-3| \times 3 = 3 \times 3 = 9$ LHS = RHS

$\|u+v\| = \sqrt{120}$ $\|u\| = 3$ $\|v\| = 13$

LHS RHS $\|u\| + \|v\| = 16$ LHS $\le$ RHS

QED

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Q17. Write $V=(2,5)$ as a linear combination of u_1 and u_2

a) $u_1 = (1,2)$ $u_2 = (3,5)$

b) $u_1 = (3,-4)$ $u_2 = (2,-3)$

Soln

a) $(2,5) = a(1,2) + b(3,5)$

$(2,5) = (a, 2a) + (3b, 5b)$

$(2,5) = (a+3b, 2a+5b)$

Now, $a+3b = 2$ — (1)

$2a+5b = 5$ — (2)

Solving (1) and (2)

$\cancel{2a} + 6b = \cancel{4}$

$\cancel{2a} + 5b = 5$

$\cancel{(-)} \quad (-) \quad (-)$

$b = -1$ $\quad a = \cancel{2-3(-3)}$

$= \cancel{2+6} = 8$

$\boxed{\cancel{a=8}}$ $\boxed{\cancel{b=-3}}$

$a = 2-3(-1)$

$= 2+3 = 5$

$\boxed{\cancel{V = 8u_1 - 3u_2}}$

$a=5$ $b=-1$

$\boxed{V = 5u_1 - u_2}$ Ans

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b) $(2,5) = a(3,-4) + b(2,-3)$

$(2,5) = (3a,-4a) + (2b-3b)$

$(2,5) = (3a+2b, -4a-3b)$

Now, $3a+2b = 2$ — (1)

$-4a-3b = 5$ — (2)

$4(3a+2b) = 4(2)$

$3(-4a-3b) = 3(5)$

$12a + 8b = 8$

$-12a - 9b = 15$

$-b = 23$

$b = -23$

$a = \frac{2-2b}{3}$

$= \frac{2-2(-23)}{3}$

$= \frac{2+46}{3} = \frac{48}{3}$ (16)

$\boxed{V = 16u_1 - 23u_2}$ Ans

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Q18. Find an equation of the Hyperplane H in R^4 that

a) Contains $P(1, 2, -3, 2)$ and is normal to $u = [2, 3, -5, 6]$

b) contains $P(3, -1, 2, 5)$ and is parallel to $2x_1 - 3x_2 + 5x_3 - 7x_4 = 4$

Soln a) The coefficients of the unknowns of an eqn of H are the components of normal vector u.

Thus, eqn of H is of the form

$$2x_1 + 3x_2 - 5x_3 + 6x_4 = k$$

Substitute P into this eqn

$$2(1) + 3(2) - 5(-3) + 6(2) = k$$

$$2 + 6 + 15 + 12 = k$$

$$k = 35$$

Hence the eqn of Hyperplane is

$$\boxed{2x_1 + 3x_2 - 5x_3 + 6x_4 = 35}$$

Ans

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b) Let the plane parallel to H be H'

$H \parallel H'$ iff their normal directions are parallel or antiparallel (opp. direction)

Hence equ^n of H' of the form

$$2x_1 - 3x_2 + 5x_3 - 7x_4 = k$$

Substitute P into this equ^n

$$2(3) - 3(-1) + 5(2) - 7(5) = k$$

$$6 + 3 + 10 - 35 = k$$

$$k = -16$$

Hence equ^n of Hyperplane H' is

$$\boxed{2x_1 - 3x_2 + 5x_3 - 7x_4 = -16}$$ Ans

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Q19. Consider the following curve C in $\mathbb{R}^3$ where

$0 \le t \le 5$

$$F(t) = t^3\hat{i} - t^2\hat{j} + (2t-3)\hat{k}$$

a) find the point P on C corresponding to $t = 2$

b) find the initial point Q and the terminal point Q'

c) find the unit tangent vector T to the curve C when $t = 2$

Soln a) Put $t = 2$

$$P = F(2) = (2)^3\hat{i} - (2)^2\hat{j} + [2(2)-3]\hat{k}$$
$$= 8\hat{i} - 4\hat{j} + \hat{k}$$

b) t ranges from 0 to 5.

$$Q = f(0) = 0\hat{i} - 0\hat{j} - 3\hat{k} = (0, 0, -3)$$
$$Q' = f(5) = 5^3\hat{i} - 5^2\hat{j} + 7\hat{k} = (125, -25, 7)$$

c) $$V(t) = \frac{d}{dt}F(t) = [3t^2, -2t, 2]$$

When $t = 2$, $V(2) = [12, -4, 2]$

$$\|V\| = \sqrt{12^2 + (-4)^2 + (2)^2}$$
$$= \sqrt{144 + 16 + 4}$$
$$= \sqrt{164}$$

$$T = \left[\frac{12}{\sqrt{164}}, \frac{-4}{\sqrt{164}}, \frac{2}{\sqrt{164}}\right]$$

or $$\left[\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}\right]$$

or $$\frac{6}{\sqrt{41}}\hat{i} - \frac{2}{\sqrt{41}}\hat{j} + \frac{1}{\sqrt{41}}\hat{k}$$

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Q1. If matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$ find the transpose of these matrices i.e $(AB)^T$

Soln $(AB)^T = B^T A^T$

$B = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$ $B^T = \begin{bmatrix} 4 & 5 \\ 6 & 9 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

M-1

$$B^T A^T = \begin{bmatrix} 4 & 5 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (4\times2)+(5\times4) & (4\times1)+(5\times3) \\ (6\times2)+(9\times4) & (6\times1)+(9\times3) \end{bmatrix} = \begin{bmatrix} 8+20 & 4+15 \\ 12+36 & 6+27 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 19 \\ 48 & 33 \end{bmatrix}$$

Ans

M-2

$$(AB)^T \Rightarrow AB = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} (2\times4)+(4\times5) & (2\times6)+(4\times9) \\ (1\times4)+(3\times5) & (1\times6)+(3\times9) \end{bmatrix}$$

$$= \begin{bmatrix} 8+20 & 12+36 \\ 4+15 & 6+27 \end{bmatrix} = \begin{bmatrix} 28 & 48 \\ 19 & 33 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 28 & 48 \\ 19 & 33 \end{bmatrix}^T = \begin{bmatrix} 28 & 19 \\ 48 & 33 \end{bmatrix}$$ Ans

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Q5. For what values of α and β the following has ∞ solutions

$$x+y+z=5$$
$$x+3y+3z=9$$
$$x+2y+\alpha z=\beta$$

$$A=\begin{bmatrix}1&1&1\\1&3&3\\1&2&\alpha\end{bmatrix} \qquad C=\begin{bmatrix}1&1&1&5\\1&3&3&9\\1&2&\alpha&\beta\end{bmatrix}$$

For ∞ solutions $|A|=0$.

$$\Rightarrow 1\begin{vmatrix}3&3\\2&\alpha\end{vmatrix}-1\begin{vmatrix}1&3\\1&\alpha\end{vmatrix}+1\begin{vmatrix}1&3\\1&2\end{vmatrix}=0$$

$$(3\alpha-6)-(\alpha-3)+(2-3)=0$$
$$3\alpha-6-\alpha+3+(-1)=0$$
$$2\alpha-3-1=0$$
$$2\alpha-4=0$$
$$\alpha=4/2=2$$

If $\alpha=2$ $\quad A=\begin{bmatrix}1&1&1\\1&3&3\\1&2&2\end{bmatrix}$

Converting A into echelon form

$$\begin{bmatrix}1&1&1\\1&3&3\\1&2&2\end{bmatrix}\xrightarrow{R_3\to R_3-R_1}\begin{bmatrix}1&1&1\\1&3&3\\0&1&1\end{bmatrix}\xrightarrow{R_2\to R_2-R_1}\begin{bmatrix}1&1&1\\0&2&2\\0&1&1\end{bmatrix}$$

$$\xrightarrow{R_3\to 2R_3}\begin{bmatrix}1&1&1\\0&2&2\\0&2&2\end{bmatrix}\xrightarrow{R_3\to R_3-R_2}\begin{bmatrix}1&1&1\\0&2&2\\0&0&0\end{bmatrix}$$

$\rho(A)=2$

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$$C = \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & 2 & \beta \end{bmatrix}$$

Converting C into Echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & 2 & \beta \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 0 & 1 & 1 & \beta-5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & \beta-5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 2R_3} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 2(\beta-5) \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 2(\beta-5)-4 \end{bmatrix}$$

To make $\rho(C) = 2$

$2(\beta-5) - 4 = 0$

Also for infinite solutions, $\rho(A) = \rho(C)$

Here $\rho(A) = \rho(C) = 2$

$\Rightarrow 2(\beta-5) - 4 = 0$

$\Rightarrow \cancel{2}(\beta-5) = \cancel{4}\ 2$

$\beta = 7$

Hence for INFINITE SOLUTIONS,

$$\boxed{\alpha = 2 \text{ and } \beta = 7}$$ Ans

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Q20. Find the dimension and basis

a) $x - y + 2z = 0$

$2x + y + z = 0$

$5x + y + 4z = 0$

Soln

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 5 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \to 5/2\, R_2} \begin{bmatrix} 1 & -1 & 2 \\ 5 & \frac{5}{2} & \frac{5}{2} \\ 5 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -1 & 2 \\ 5 & 5/2 & 5/2 \\ 0 & \frac{-3}{2} & \frac{3}{2} \end{bmatrix}$$

$$\xrightarrow[R_3 \to 2R_3]{R_2 \to 2R_2} \begin{bmatrix} 1 & -1 & 2 \\ 10 & 5 & 5 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 10R_1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 15 & -15 \\ 0 & -3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to 5R_3} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 15 & -15 \\ 0 & -15 & 15 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 15 & -15 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 15 & -15 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - y + 2z = 0$$
$$15y - 15z = 0$$

Leading unknown (2) ; Free variable (1)
x (z)

~~dim = 1~~

$\boxed{\dim W = 1}$

Let $z = k$

$y = k$ $\quad x = y - 2z$

$x = k - 2k$

$x = -k$

Putting free variable $z = 1$

$\Rightarrow k = 1$

$\Rightarrow x = -1$ & $y = 1$

Soln $[-k \; k \; k]$

Basis $\boxed{u_1 = (-1, 1, 1)}$ Ans

$\dim W = 1$

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b) $x + 2y - 3z = 0$

$2x + 5y + 2z = 0$

$3x - y - 4z = 0$

Soln

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 3 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 2 \\ 0 & -7 & 5 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 8 \\ 0 & -7 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 7R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 8 \\ 0 & 0 & 61 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 8 \\ 0 & 0 & 61 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y - 3z = 0$

$y + 8z = 0$

$61z = 0$

No. of free variables $= 0$

Hence $\dim W = 0$

This system does not have a Basis

c) $x + 2y + 3z + t = 0$

$2x + 4y + 7z + 4t = 0$

$3x + 6y + 10z + 5t = 0$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 4 \\ 3 & 6 & 10 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 4 \\ 3 & 6 & 10 & 5 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 7 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 2y + 3z + t = 0$

$z + 2t = 0$

Free variables $= \{y, t\}$ $\quad \dim W = 2$

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Now,

Let $t = k_1$ and $y = k_2$

$z = -2k_1$

$x = -2y - 3z - t$

$= -2k_2 - 3(-2k_1) - k_1$

$= -2k_2 + 6k_1 - k_1$

$= -2k_2 + 5k_1$

~~Soln $[(2k_2+5k_1), (k_2), (-2k_1)]$~~

~~Now, Put $k_1 = 1$ and $k_2 = 0$~~

~~$u_1 = [5, 0, -2]$~~

~~Put $k_2 = 1$ and $k_1 = 0$~~

~~$u_2 = [2, 1, 0]$~~

~~Basis vectors $\{(5, 0, -2)$ and $(2, 1, 0)\}$ Ans~~

Soln $[(2k_2+5k_1), k_2, (-2k_1), k_1]$

Put $k_1 = 1$ & $k_2 = 0$

$u_1 = [5, 0, -2, 1]$

Put $k_2 = 1$ & $k_1 = 0$

$u_2 = [-2, 1, 0, 0]$

$\dim W = 2$

Basis $\{u_1 = (5, 0, -2, 1),\ u_2 = (-2, 1, 0, 0)\}$

Ans

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Q14. $u = (1, -2, -4)$

$v = (3, 5, 1)$

$w = (2, 1, -3)$

a) $3u - 2v$

$3(1, -2, -4) - 2(3, 5, 1)$

$(3, -6, -12) - (6, 10, 2)$

$(3-6, -6-10, -12-2)$

$(-3, -16, -14)$

b) $5u + 3v - 4w$

$5(1, -2, -4) + 3(3, 5, 1) - 4(2, 1, -3)$

$(5, -10, -20) + (9, 15, 3) - (8, 4, -12)$

$(5+9-8, -10+15-4, -20+3+12)$

$(6, 1, -5)$

c) $u \cdot v = (1 \times 3) + (-2 \times 5) + (-4 \times 1)$

$= 3 - 10 - 4 = -11$

$u \cdot w = (1 \times 2) + (-2 \times 1) + (-4 \times -3)$

$= \cancel{2} - \cancel{2} + 12$

$= 12$

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$$V \cdot W = (3 \times 2) + (5 \times 1) + (1 \times -3)$$
$$= 6 + 5 - 3$$
$$= 8$$

d) $\|u\| = \sqrt{1^2 + (-2)^2 + (-4)^2}$
$$= \sqrt{1 + 4 + 16} = \sqrt{21}$$

$$\|v\| = \sqrt{3^2 + 5^2 + 1^2}$$
$$= \sqrt{9 + 25 + 1} = \sqrt{35}$$

e) $\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|} = \dfrac{-11}{\sqrt{21}\sqrt{35}}$

f) $d(u, v) = \|u - v\|$

$$u - v = [(1-3), (-2-5), (-4-1)]$$
$$= (-2, -7, -5)$$

$$\|u - v\| = \sqrt{(-2)^2 + (-7)^2 + (-5)^2} = \sqrt{4 + 49 + 25}$$
$$= \sqrt{78}$$

g) $\text{proj}(u, v) = \dfrac{u \cdot v}{\|v\|^2} v$

$$= \frac{-11}{35}[3, 5, 1] = \left[\frac{-33}{35}, \frac{-55}{35}, \frac{-11}{35}\right]$$

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