# REAL ANALYSIS

## CSE PYQs

# 2020

## 1. 1c

Prove that the sequence $(a_n)$ satisfying the condition $|a_{n+1} - a_n| \leq \alpha |a_n - a_{n-1}|$, $0<\alpha<1$ for all natural numbers $n \geq 2$, is a Cauchy sequence. 10

## 2. 2b

Prove that the function $f(x) = \sin x^2$ is *not* uniformly continuous on the interval $[0, \infty[$. 15

## 3. 3c

If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, $x \neq y$

then show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = (1 - 4 \sin^2 u) \sin 2u$ 20

## 4. 4b

Show that $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log_e (1 + \sqrt{2})$ 15

# 2019

## 1. 1b

Show that the function

$$f(x, y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & (x, y)\neq(1,\ -1),\ (1,\ 1)\\ 0, & (x, y)=(1,\ 1),\ (1,\ -1)\end{cases}$$

is continuous and differentiable at $(1, -1)$. 10

## 2. 1c

Evaluate

$$\int_0^{\infty}\frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx,\quad a>0,\ a\neq 1.$$ 10

## 3. 2c 2019

Using differentials, find an approximate value of $f(4\cdot1,\ 4\cdot9)$ where

$$f(x, y)=\left(x^3+x^2y\right)^{\frac{1}{2}}.$$ 15

## 4. 3a 2019

Discuss the uniform convergence of

$$f_n(x)=\frac{nx}{1+n^2x^2},\ \forall x\in\mathbb{R}\,(-\infty,\ \infty)$$

$$n=1, 2, 3, \ldots.$$ 15

# 5. 4a

Find the maximum value of $f(x, y, z) = x^2y^2z^2$ subject to the subsidiary condition $x^2 + y^2 + z^2 = c^2$, $(x, y, z > 0)$. 15

# 6. 4c

**Discuss the convergence of** $\int_1^2 \frac{\sqrt{x}}{l_n x} dx$. **15**

# 2018

## 7. 1b 2018

Prove the inequality : $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$. 10

## 8. 1d 2018

Find the range of $p(>0)$ for which the series :

$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \ldots$, $a > 0$, is

(i) absolutely convergent and (ii) conditionally convergent. 10

## 9. 2c 2018

Show that if a function $f$ defined on an open interval $(a, b)$ of $\mathbb{R}$ is convex, then $f$ is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous. 15

## 10. 4a 2018

Suppose $\mathbb{R}$ be the set of all real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a function such that the following equations hold for all $x, y \in \mathbb{R}$ :

(i) $f(x + y) = f(x) + f(y)$

(ii) $f(xy) = f(x) f(y)$

Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$, or, $f(x) = x$. 20

# 2017

## 11. 1a 2017

Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$, $n = 1, 2, 3, \ldots$ . Show that the sequence $x_1, x_2, x_3, \ldots$ is convergent. *10*

## 12. 1c

Find the supremum and the infimum of $\dfrac{x}{\sin x}$ on the interval $\left(0, \dfrac{\pi}{2}\right]$. *10*

## 13. 2a 2017

Let

$$f(t) = \int_0^t [x]\, dx,$$

where $[x]$ denotes the largest integer less than or equal to $x$.

(i) Determine all the real numbers $t$ at which $f$ is differentiable.

(ii) Determine all the real numbers $t$ at which $f$ is continuous but not differentiable. *15*

## 14. 4c

Let $\sum\limits_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum\limits_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum\limits_{n=1}^{\infty} x_n$ that converges to 100. *20*

# 2016

## 15. 1b 2016

For the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = x^2 \sin\frac{1}{x},\ 0 < x < \infty,$$

show that there is a differentiable function $g : \mathbb{R} \to \mathbb{R}$ that extends f. *10*

## 16. 1c

Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following :

$$x_1 = \frac{1}{2},\ y_1 = 1 \text{ and } x_n = \sqrt{x_{n-1}\, y_{n-1}},\quad n = 2, 3, 4, \ldots$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right),\quad n = 2, 3, 4, \ldots$$

Prove that

$$x_{n-1} < x_n < y_n < y_{n-1},\quad n = 2, 3, 4, \ldots$$

and deduce that both the sequences converge to the same limit $l$, where $\frac{1}{2} < l < 1$. *10*

## 17. 2a 2016

Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

is conditionally convergent. (If you use any theorem(s) to show it, then you must give a proof of that theorem(s).) *15*

# 18. 3b

Find the relative maximum and minimum values of the function

$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. *15*

# 19. 4b 2016

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbb{R}$. *15*

# 2015

## 20. 1c 2015

Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$. 10

## 21. 2b 2015

Is the function

$$f(x)=\begin{cases}\frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n}\\ 0, & x=0\end{cases}$$

Riemann integrable? If yes, obtain the value of $\int_0^1 f(x)\,dx$. 15

## 22. 3b 2015

Test the series of functions $\sum_{n=1}^{\infty}\frac{nx}{(1+n^2x^2)}$ for uniform convergence. 15

## 23. 4b

Find the *absolute* maximum and minimum values of the function $f(x, y)=x^2+3y^2-y$ over the region $x^2+2y^2\le 1$. 15

# 2014

## 24. 1b

Test the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$. 10

## 25. 2b 2014

Integrate $\int_0^1 f(x)\,dx$, where

$$f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & , \quad x \in \,]0, 1] \\ 0 & , \quad x = 0 \end{cases}$$ 15

## 26. 3b

Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x, y) = \begin{cases} \frac{xy(3x^2 - 2y^2)}{x^2 + y^2} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = (0, 0) \end{cases}$$

Also, discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0, 0)$. 15

## 27. 4b

Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multipliers. 15

# 2013

## 28. 1c 2013

Let $f(x) = \begin{cases} \dfrac{x^2}{2} + 4 & \text{if } x \geq 0 \\ \dfrac{-x^2}{2} + 2 & \text{if } x < 0 \end{cases}$

Is f Riemann integrable in the interval $[-1, 2]$ ? Why ? Does there exist a function g such that $g'(x) = f(x)$ ? Justify your answer. *10*

## 29. 2c 2013

Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n + x^2}$, is uniformly convergent but not absolutely for all real values of x. *13*

## 30. 2d 2013

Show that every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals. *14*

# 31. 3c

Let $f(x, y) = y^2 + 4xy + 3x^2 + x^3 + 1$. At what points will $f(x, y)$ have a maximum or minimum ? *10*

# 32. 3d 2013

Let $[x]$ denote the integer part of the real number $x$, i.e., if $n \le x < n + 1$ where $n$ is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in $[-1, 2]$ ? If not, explain why. If it is integrable, compute $\int_{-1}^{2} ([x]^2 + 3)\, dx$. *10*

# 2012

## 33. 1b 2012

(b) Let

$$f_n(x) = \begin{cases} 0, & \text{if } x < \dfrac{1}{n+1}, \\ \sin\dfrac{\pi}{x}, & \text{if } \dfrac{1}{n+1} \leqslant x \leqslant \dfrac{1}{n}, \\ 0, & \text{if } x > \dfrac{1}{n}. \end{cases}$$

Show that $f_n(x)$ converges to a continuous function but not uniformly. 12

## 34. 1e 2012

Show that the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{\pi}{\pi+1}\right)^n n^6$ is convergent. 12

# 35. 2b

Let $f(x, y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 1, & \text{if } (x, y) = (0, 0). \end{cases}$

Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at (0, 0) though $f(x, y)$ is not continuous at (0, 0). 15

# 36. 2d

(d) Find the minimum distance of the line given by the planes $3x + 4y + 5z = 7$ and $x - z = 9$ from the origin, by the method of Lagrange's multipliers. 15

# 37. 3b 2012

(b) Let $f(x)$ be differentiable on [0, 1] such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x)\,dx = 1$. Prove that

$$\int_0^1 x f(x) f'(x)\,dx = -\tfrac{1}{2}.$$ 15

# 38. 4b 2012

(b) Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify. 20

# 2011

## 39. 1b 2011

(b) Let $S = (0, 1]$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \le 1$ (in $\mathbb{R}$). Is f uniformly continuous on S ? Justify your answer. 12

## 40. 2b 2011

(b) Let $f_n(x) = nx(1 - x)^n$, $x \in [0, 1]$

Examine the uniform convergence of $\{f_n(x)\}$ on $[0, 1]$. 15

## 41. 2d

(d) Find the shortest distance from the origin (0, 0) to the hyperbola

$$x^2 + 8xy + 7y^2 = 225$$ 15

# 43. 3b 2011

(b) Show that the series for which the sum of first n terms

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \le x \le 1$$

cannot be differentiated term-by-term at x = 0. What happens at $x \neq 0$ ? 15

# 44. 4b 2011

(b) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2}$, then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}, \text{ for all } x. \quad 20$$

# 2010

## 45. 1c 2010

(c) Discuss the convergence of the sequence $\{x_n\}$

where $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$. 12

## 46. 1d 2010

(d) Define $\{x_n\}$ by $x_1 = 5$ and

$x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. 12

Show that the sequence converges to $\dfrac{(1+\sqrt{17})}{2}$.

## 47. 2c 2010

(c) Define the function

$$f(x) = x^2 \sin\frac{1}{x}, \quad \text{if } x \neq 0$$
$$= 0, \quad \text{if } x = 0$$

Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Justify your answer. 15

# 48. 2d 2010

(d) Consider the series $\sum_{n=0}^{\infty} \frac{x^2}{\left(1+x^2\right)^n}$.

Find the values of $x$ for which it is convergent and also the sum function.

Is the convergence uniform ? Justify your answer. 15

# 49. 3c 2010

(c) Let $f_n(x) = x^n$ on $-1 < x \leq 1$ for $n = 1, 2, \ldots$. Find the limit function. Is the convergence uniform ? Justify your answer. 15

# 50. 3d

(d) Find the maxima, minima and saddle points of the surface $Z = \left(x^2 - y^2\right) e^{(-x^2-y^2)/2}$. 15

# 2009

## 51. 1c

(c) State Rolle's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root of $e^x \sin x = 1$. 2+10=12

## 52. 1d

$$\text{Let } f(x) = \begin{cases} \dfrac{|x|}{2} + 1 & \text{if } x < 1 \\ \dfrac{x}{2} + 1 & \text{if } 1 \le x < 2 \\ -\dfrac{|x|}{2} + 1 & \text{if } 2 \le x \end{cases}$$

What are the points of discontinuity of f, if any ? What are the points where f is not differentiable, if any ? Justify yours answers. 12

## 53. 2c 2009

Show that the series :

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot4}{3\cdot6}\right)^2 + \dots\dots\dots + \left(\frac{1\cdot4\cdot7\dots\dots(3n-2)}{3\cdot6\cdot9\dots\dots3n}\right)^2 + \dots\dots\dots$$

converges. 15

## 54. 2d 2009

Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then $f([a, b]) = [c, d]$ for some real numbers $c$ and $d$, $c \le d$. 15

## 55. 3c 2009

Show that :

$$\underset{x \to 1}{\mathrm{Lt}} \sum_{n=1}^{\infty} \frac{n^2 x^2}{n^4 + x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1} .$$

Justify all steps of your answer by quoting the theorems you are using. 15

## 56. 3d 2009

Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point. 15