Partial Differential Equations.

① Formation (Eliminate arbitrary constants)

If constants > variables then higher order derivatives.

Q. $z = a(x+y) + b$

$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = a$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}}$$

- $z = axe^y + \frac{a^2e^{2y}}{2} + b$

$\frac{\partial z}{\partial x} = ae^y \qquad \frac{\partial z}{\partial y} = axe^y + a^2e^{2y}$

$= x \cdot \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$

- $(x-h)^2 + (y-k)^2 + z^2 = \lambda^2.$

$2(x-h) + 2z\frac{\partial z}{\partial x} = 0 \qquad 2(y-k) + 2z\frac{\partial z}{\partial y} = 0$

$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 + z^2 = \lambda^2$

- DE of all spheres of radius λ with centre in xy-plane.

DE of all right circular cones whose axis coincide with z-axis

→ Let semi-vertical angle be α. Vertex at $(0,0,c)$.

$$\boxed{x^2 + y^2 = (z-c)^2\tan^2\alpha}$$

– c, α are arbitrary

$\cancel{2}x = \cancel{2}(z-c)\tan^2\alpha\frac{dz}{dx} \quad ; \quad \cancel{2}y = \cancel{2}(z-c)\tan^2\alpha\frac{dz}{dy}$

$\therefore \frac{dz}{dy} \cdot \frac{1}{y} = \frac{dz}{dx} \cdot \frac{1}{x}$. Ans

15 Show that DE of all cones having vertex at origin is $px+qy=z$. Verify $yz+zx+xy=0$ satisfies above equation.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

$$\hookrightarrow \cancel{2}ax+\cancel{2}fy\frac{dz}{dx}+\cancel{2}gz+\cancel{2}gx\frac{dz}{dx}+\cancel{2}hy+cz\frac{dz}{dx}=0$$

$$ax+hy+gz+\frac{\partial z}{\partial x}(gx+fy+cz)=0 \quad -(1)$$

$$\hookrightarrow 2by+2cz\frac{dz}{dy}+2fz+2fy\frac{dz}{\partial y}+2gx\frac{dz}{dy}+2hx=0.$$

$$(hx+by+fz)+\frac{\partial z}{\partial y}(gx+fy+cz)=0. \quad -(2).$$

$$(1)x+(2)y=0$$

$$(ax^2+hxy+gzx+hxy+by^2+fzy)+(gx+fy+cz)(px+qy)=0$$

$$-z\left(-cz^2-\cancel{hxy}-g\cancel{z}x-f\cancel{z}y\right)+\cancel{(gx+fy+cz)}(px+qy)=0.$$

$$\Rightarrow \boxed{px+qy=z}$$

$$y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial x}+z+y=0. \qquad z+y\frac{\partial z}{\partial y}+x\frac{\partial z}{\partial y}+x=0.$$

$$p(x+y)+z+y=0 \qquad q(x+y)+x+z=0.$$

$$p=-\frac{(z+y)}{x+y} \qquad q=-\frac{(x+z)}{(x+y)}.$$

$$\boxed{px+qy} = \frac{\cancel{-zx+xy-xy-zy}}{\cancel{x+y}} \quad \frac{-zx-yx-xy-zy}{x+y}.$$

or

$$\boxed{px+qy-z} = \frac{-zx-yx-xy-zy-zx-zy}{x+y}=0$$

$$\boxed{\text{So, } px+qy=z} \quad \text{Ans}$$

ation.

,y) =0

0.

.0.

(8) $\log(az-1) = x + ay + b.$

$$a\frac{\partial z}{\partial x} = (az-1) \qquad a\frac{\partial z}{\partial y} = a(az-1).$$

$$az - 1 = Z_y = q \qquad \cancel{ap = q \Rightarrow a = \frac{q}{p}} \quad a = \left(\frac{1+q}{2}\right)$$

$$\cancel{\frac{1+q}{2} = z} \qquad \cancel{\log q = x - 1} \quad \left(\frac{1+q}{2}\right)P = q \qquad \frac{1}{(az-1)}\,aP = 1 \qquad \frac{aq}{az-1} = 1$$

$$\boxed{\frac{1+q}{2} \cdot \frac{\partial z}{\partial x} = q}. \text{ Ans}$$

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad (1) \qquad \frac{2y}{b^2} + \frac{2z}{c^2}\frac{dz}{dy} = 0 \quad (2) \qquad \frac{2z}{c^2}$$

$$\frac{\cancel{2}\,dz}{\cancel{c^2}\,dy}\frac{dz}{dx} + \frac{\cancel{2}z}{\cancel{c^2}}\frac{\partial^2 z}{dx\,\partial y} = 0 \quad \text{— One solution.}$$

$$(1)x + (2)y \Rightarrow \frac{2x^2}{a^2} + \frac{2y^2}{b^2} + \frac{2xzp}{c^2} + \frac{2yzq}{c^2} = 0$$

$$\cancel{2}c^2 - \frac{\cancel{2}z^2}{\cancel{c^2}} + \frac{\cancel{2}xzp}{\cancel{c^2}} + \frac{\cancel{2}yzq}{\cancel{c^2}} = 0$$

$$-2zp + xp^2 + zp + zq + yq^2 = 0$$
$$+ xzp' + yzq'$$

(1.1) Elimination of arbitrary function

Eliminate ϕ to form PDE

when z also

Else direct

(*). $\phi(x+y+z,\ x^2+y^2-z^2)=0$.

Solve Let, $x+y+z=u$, $x^2+y^2-z^2=v$ } $\phi(u,v)=0$

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial\phi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial\phi}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}=0$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$\frac{\partial\phi}{\partial u}(1+p)+\frac{\partial\phi}{\partial v}(2x-2pz)=0 \qquad -(1)$$

$$\frac{\partial}{\partial y} \Rightarrow \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}=0$$

$$\frac{\partial\phi}{\partial u}+q\frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial v}(2y-2qz)=0 \qquad -(2)$$

From (1) and (2)

$$\frac{\partial\phi/\partial u}{\partial\phi/\partial v}=\underbrace{\frac{2pz-2x}{1+p}=\frac{2qz-2y}{1+q}}$$

$$(2pz-2x)(1+q)=(2qz-2y)(1+p) \quad \text{Ans}$$

· Eliminate f, F

$$f(x-at) + F(x+at) = y$$

$$\frac{\partial y}{dx} = f'(x-at) + F'(x+at) \qquad \frac{\partial y}{dt} = (-a) f'(x-at) + a F'(x+at)$$

$$\frac{\partial^2 y}{dx^2} = f''(x-at) + F''(x+at) \qquad \frac{\partial^2 y}{\partial t^2} = a^2 \left(f''(x-at) + F''(x+at) \right)$$

$$\boxed{\frac{d^2 y}{dt^2} = a^2 \frac{d^2 y}{dx^2}} \quad \text{Ans}$$

· $z = e^{ax+by} f(ax-by)$

$$\frac{\partial z}{\partial x} = a e^{ax+by} f(ax-by) + a e^{ax+by} f'(ax-by).$$

$$\frac{\partial z}{\partial y} = b e^{ax+by} f(ax-by) - b e^{ax+by} f'(ax-by).$$

$$bp + aq = f(ax-by)\, e^{ax+by} (2ab)$$

$$\boxed{bp + aq = 2ab z} \quad \text{Ans}$$

· $z = f(x^2-y) + g(x^2+y).$

$$p = 2x f'(x^2-y) + 2x g'(x^2+y) \qquad q = -f'(x^2-y) + g'(x^2+y).$$

$$p' = 2f'(x^2-y) + 2x f''(x^2-y).2x + 2g'(x^2+y) + 4x^2 g''(x^2+y).$$

$$= 2[f'(x^2-y) + g'(x^2+y)] + 4x^2 [f''(x^2-y) + g''(x^2+y)].$$

$$q' = f''(x^2-y) + g''(x^2+y).$$

$$\boxed{p' = \frac{2.p}{2x} + 4x^2 q'} \quad \text{Ans}$$

Find the D.E. of all surfaces of revolution having z-axis as the axis of rotation.

Such surfaces are : $z = \phi\left[(x^2+y^2)^{1/2}\right]$.

$$\frac{\partial z}{\partial x} = p = \phi'\left(\sqrt{x^2+y^2}\right) \cdot \frac{2x}{2\sqrt{x^2+y^2}}$$

$$q = \phi'\left(\sqrt{x^2+y^2}\right) \cdot \frac{2y}{2\sqrt{x^2+y^2}}$$

$$\frac{\cancel{2}\cancel{\phi}\, q}{\cancel{2}\, y} = \frac{\cancel{2}\cancel{\phi}\, p}{\cancel{2}\, x} \Rightarrow \boxed{yp = xq} \quad \text{Ans}$$

② Linear PDE of order one. Lagrange's method,

Types $\equiv$ $\boxed{Pp + Qq = R \longleftrightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.}$

① Easily Separable

Q. $(x^2+y^2 \cdot 2)p - xy\,q = xz.$

$$\frac{dx}{x^2+2y^2} = \frac{dy}{(-xy)} = \frac{dz}{xz.}$$

$\boxed{yz = c_1}$

$$\frac{dx}{dy} = -\frac{x}{y} - \frac{2y}{x}. \Rightarrow x\frac{dx}{dy} + \frac{x^2}{y} = -2y$$

$$x^2 = t \quad 2x\frac{dx}{dy} = \frac{dt}{dy}. \Rightarrow \frac{1}{\cancel{2}}\frac{dt}{dy} + \frac{2t}{y} = -2y \times 2.$$

$$e^{\int \frac{2}{y} dy} = y^2 \quad \Rightarrow \quad y^2 \times t = \int y^2 \times (-4y)\,dy \Rightarrow y^2x^2 = -y^4 +$$

$$\boxed{y^2x^2 + y^4 = c_2}$$

Solution $\equiv$ $\boxed{\phi\,(yz,\ y^2x^2 + y^4) = 0}$

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(2) Only one solution obtained, 2nd by putting

TIP: Getting 1st solution shall give an obvious hint

$\boxed{Q}$ $$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z^2-y^2-2x^3)}$$

(dz - 2y dy also)

$$\frac{y}{z} = a \text{ or } \boxed{\frac{z}{y} = a}$$

$$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)}$$

$$(ay-y^2-2x^3)dx + x(2y-a)dy = 0$$

$\llcorner\ a-2y \qquad 2y-a.$ $\qquad \frac{(M_y - N_x)}{N} = -\frac{2}{x}.$

$IF = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$, Mult by $\frac{1}{x^2}$ $\Rightarrow$ we get exact equation.

$$\frac{1}{x^2}(ay-y^2-2x^3)dx + \frac{1}{x}(2y-a)dy = 0$$

$$\int \frac{(ay-y^2)}{x^2} - 2x\,dx + 0 = C.$$

$$\boxed{(ay-y^2)\times\left(-\frac{1}{x}\right) - x^2 = b}$$

$$\boxed{\phi\left(\frac{z}{y}, \frac{(ay-y^2)}{(-x)} - x^2\right) = 0}$$ Ans

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5) $p + 3q = z + \cot(y - 3x)$

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot(y-3x)}$$

$3x - y = c_1$

$$\frac{dy}{3} = \frac{dz}{z + \cot c_1} \Rightarrow y = 3\log(z + \cot c_1) + c_2$$

$$\phi\left(3x - y,\ y - 3\log\left(z + \cot(3x-y)\right)\right) = 0 \ \underline{\text{Ans}}$$

(3) Find P_1, Q_1, R_1 such that $P_1P + Q_1Q + R_1R = 0$

Then $\boxed{P_1dx + Q_1dy + R_1dz = 0}$

Q $(mz - ny)p + (nx - lz)q = ly - mx.$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

$l, m, n \Rightarrow lmz - lny + mnx - mlz + lny - nmx = 0.$

$$= \frac{ldx + mdy + ndz}{0} \Rightarrow \boxed{lx + my + nz = C_1}$$

$x, y, z \Rightarrow mzx - nxy + nxy - lzy + lyz - mzx = 0$

$$\boxed{x^2 + y^2 + z^2 = C_2}$$

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$$

ME

Q $(y - 2x)p + (x + yz)q = x^2 + y^2.$

$$\frac{dx}{y - 2x} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}$$

$x, -y, z \Rightarrow x^2 - y^2 + z^2 = C_1$

$y, x, -1 \Rightarrow xy - z = C_2.$

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⑧ E

$$\frac{dx}{x(y^2-z)} = \frac{dy}{-y(x^2-z)} = \frac{dz}{z(x^2-y^2)}.$$

$x, y, -1 \Rightarrow x^2+y^2-z = c_1$

$1/x, 1/y, 1/z \Rightarrow xyz = c_2.$

$\Big\}\ \phi(x^2+y^2-z, xyz) = 0$

⑨ ME

$$\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y}$$

$-2x, 1, 1 \rightarrow -x^2+y+z = c_1$

$y, x, -2z \rightarrow xy - z^2 = c_2.$

$\Big\}\ \phi(c_1, c_2) = 0$

E-9 Solve

⑩ M

$$\frac{dx}{y^3x-2x^4} = \frac{dy}{2y^4-x^3y} = \frac{dz}{9z(x^3-y^3)} \Leftarrow (*)$$

$\frac{3}{x}, \frac{3}{y}, \frac{1}{z} \Rightarrow \boxed{xyz^{1/3} = c_1}$

$(2y^4-x^3y)dx = (y^3x-2x^4)dy.$

Divide both sides by (x^3y^3)

$$dx\left(\frac{2y}{x^3}-\frac{1}{y^2}\right) = dy\left(\frac{1}{x^2}-\frac{2x}{y^3}\right)$$

$$\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right)$$

$$d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0.$$

$$\Rightarrow \boxed{\frac{y}{x^2}+\frac{x}{y^2} = c_2.}$$

Does not work

$2y^4-x^3y\,dx = dy(y^3x-2x^4)$

$$\frac{dy}{dx} = \frac{2y^4-x^3y}{y^3x-2x^4}.$$

$y = mx$

$$\frac{dy}{dx} = m + x\frac{dm}{dx}$$

$$m + x\frac{dm}{dx} = \frac{2m^4-m}{m^3-2}.$$

$$\frac{m^3-2}{2m^4-m^3-m+2}\,dm = \frac{dx}{x}.$$

$\frac{1}{8}\cdot(8m^3-3m^2-1 \quad +3m^2-15.$

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{c_1} \Rightarrow y-x=\frac{xy}{c_1} \quad — (1).$$

$$\boxed{\frac{1}{x}, -\frac{1}{y}, \frac{1}{nc_1}} \quad \Rightarrow x-y=-\frac{xy}{c_1} + k.(nxy).$$

$$\boxed{xy+\frac{z}{nc_1}=c_2} \quad — (2).$$

$\boxed{Q}$
$$\frac{dx}{3x+y-z}=\frac{dy}{x+y-z}=\frac{dz}{2(z-y)}$$

$$1, -3, +1 \Rightarrow \boxed{x-3y+z=c_1.}$$

Put z in (1),(2).

$$\cancel{x+y-(c_1-x+3y)\,dx + (-3x-y+c_1-x+3y)\,dy = 0.}$$

$$\frac{dy}{dx}=\frac{4y+c_1}{2x+4y+c_1.} \qquad u=4y+c_1$$

$$\frac{du}{dx}=\frac{4dy}{dx}.$$

$$\frac{1}{4}\frac{du}{dx}=\frac{u}{2x+u} \Rightarrow \frac{dx}{du}=\left(\frac{2x}{u}+1\right)\frac{1}{4}.$$

$$IF = e^{-\int\frac{2}{4u}du} = 1/\sqrt{u}$$

$$\frac{x}{\sqrt{u}} = \int\frac{1}{4\sqrt{u}}\,du + c$$

Q $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$.

$1,1,1 \Rightarrow x+y+z = c_1$
$\frac{1}{x}, \frac{1}{y}, \frac{1}{z}. \Rightarrow xyz = c_2$.

$\boxed{\phi(x+y+z, xyz) = 0}$ Ans

Q $\dfrac{dx}{y^2-z^2} = \dfrac{dy}{-xy} = \dfrac{dz}{-xz}$.

$x, y, z \Rightarrow x^2+y^2+z^2 = c_2$ $\quad \frac{y}{z} = c_1 \quad \Rightarrow \boxed{\phi\left(x^2+y^2+z^2; \frac{y}{z}\right) = 0}$

Generally think basic $\rightarrow$ constants, x-y-z, $\frac{1}{x}$-$\frac{1}{y}$-$\frac{1}{z}$
$\hookrightarrow$ If not $(y-x)$

(4) Find P_1, Q_1, R_1 such that

$$\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1 P + Q_1 Q + R_1 R}$$ and $d(P_1P + Q_1Q + R_1R) = \underline{N^r}$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

(8) E

$$\underbrace{\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}} = \underbrace{\frac{dy-dz}{-(y-z)}}.$$

$$\boxed{(x-y)^2(x+y+z) = C_2.} \qquad \boxed{\frac{x-y}{y-z} = C_1.}$$

(8) ME

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \underbrace{\frac{dz}{2xz}.}_{\boxed{y = C_1 z}}$$

$$\frac{2(x\,dx + y\,dy + z\,dz)}{x^3 + y^2x + z^2\cancel{x}} = \frac{dy}{\cancel{x}xy}$$

$$\log(x^2+y^2+z^2) = \cancel{\tfrac{1}{2}}\log y + C. \Rightarrow \frac{x^2+y^2+z^2}{y} = C_2.$$

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9 $(x^2-yz)p+(y^2-zx)q=z^2-xy$. (Can also use $d(z-x) \to$ and finish)

$$\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dx-dy}{(x+y+z)(x-y)}$$
$$\frac{dy-dz}{y^2-z^2+x(y-z)}=\frac{dy-dz}{(x+y+z)(y-z)}$$
$$\boxed{\frac{x-y}{y-z}=C_1}$$

$$\frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}=\frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$
$$\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$
$$\frac{x\,dx+y\,dy+z\,dz}{x+y+z}=\frac{dx+dy+dz}{1}$$

$$\Rightarrow \frac{(x+y+z)^2}{2}=\frac{x^2+y^2+z^2}{2}+\frac{C_2}{2}$$
$$\boxed{(x+y+z)^2-(x^2+y^2+z^2)=C_2}$$

9 M $$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)}$$

$$\frac{dx-dy}{z(x-y)}=\frac{dz}{z(x-y)}\Rightarrow \boxed{x-y=z+C_1}$$

~~$\frac{dx-dy-dz}{0} \Rightarrow x-y-z=C$~~ $(x,-y,0)$

$$\frac{x\,dx-y\,dy}{x^3-xy^2-\cancel{xyz}-yx^2+y^3+\cancel{xyz}}\Rightarrow 2\frac{x\,dx-y\,dy}{(x-y)(x^2-y^2)}=\frac{2\,dz}{z(x-y)}$$

$$\boxed{\frac{x^2-y^2}{z^2}=C_2}$$

TIP: Remember 2 in x^2+y^2/x^2-y^2.

5

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}.$$

$$\frac{dx+dy}{\sin(x+y)+\cos(x+y)} \qquad \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z} \Rightarrow \frac{dt}{\cos t+\sin t} = \frac{dz}{z}.$$

$$\Rightarrow \frac{dt/\sqrt{2}}{\sin(\pi/4+t)}. = \frac{dz}{z} \Rightarrow \sqrt{2}\log z = \int \operatorname{cosec}\left(t+\frac{\pi}{4}\right)dt$$

$$= \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$$

$$\boxed{z^{\sqrt{2}} = c_1 \tan\left(\frac{x+y}{2}+\frac{\pi}{8}\right)}$$

$$dx-dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy)$$

$$x-y = \log(\sin t+\cos t) - \log c_2 \Rightarrow \boxed{e^{y-x}\left[\sin(x+y)+\cos(x+y)\right] = c_2}$$

Q E

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)}$$

$$*\ \frac{d(x+y)}{(x+y)^3} = \frac{d(x-y)}{(x-y)^3} \Rightarrow \boxed{\frac{1}{(x-y)^2} - \frac{1}{(x+y)^2} = C_1}$$

$$*\ \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{2}{z}dz}{0} \Rightarrow \boxed{\frac{xy}{z^2} = C_2}$$

Q ME

$$\frac{dx}{2x^2+y^2+z^2-2yz-xz-xy} = \frac{dy}{x^2+2y^2+z^2-yz-2zx-xy} = \frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}$$

$$\frac{d(x-y)}{(x^2-y^2)+z(x-y)} = \frac{d(y-z)}{(y^2-z^2)+\cancel{z}\,x(y-z)} \Rightarrow \boxed{\frac{x-y}{y-z} = C_1}$$

$$\cancel{dx+dy+dz}$$

$$\frac{d(z-x)}{(z-x)(x+y+z)} = \frac{d(y-z)}{(y-z)+z(y-z)} \Rightarrow \boxed{\frac{z-x}{y-z} = C_2}$$

$\ast$ 2001

(7) $$\frac{dx}{my(x+y)-nz^2} = \frac{dy}{nz^2 - lx(x+y)} = \frac{dz}{(lx-my)z}$$

$$\frac{dx+dy}{(x+y)} = \frac{dz}{(-z)} \Rightarrow \boxed{z(x+y) = c_1}$$

VVI

$$\frac{lx\,dx + my\,dy + nz\,dz}{-lnz^2x + mnz^2y + lnxz^2 - mnz^2y = 0} \Rightarrow \boxed{lx^2 + my^2 + nz^2 = c_2}$$

2006 $\ast$

(8) $$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

$$\underbrace{\frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}}_{\boxed{y = c_1 z}}$$

$\boxed{0, -2y, 1 \Rightarrow}$ (Try to manufacture D^r of dx)

$$\frac{-2y\,dy + dz}{(z-y^2-2x^3)\cancel{(z-2y^2)}} = \frac{dx}{x\cancel{(z-2y^2)}}$$

$t = z - y^2$

$$\frac{dt}{t-2x^3} = \frac{dx}{x}$$

$$\frac{dt}{dx} - \frac{t}{x} = -2x^2. \qquad IF = \frac{1}{x}$$

$$\frac{t}{x} = -x^2 + c_2. \Rightarrow z - y^2 = -x^3 + c_2 x$$

$$\boxed{\frac{(z-y^2+x^3)}{x} = c_2}$$

Q] Tangent plane to a surface at (x, y, z) is

$$\boxed{\underbrace{p(X-x) + q(Y-y)}_{\text{Find } p,q \text{ for given surface.}} = Z-z}$$

$$\Rightarrow \boxed{p(x-\alpha)+q(y-\beta) = (z-\gamma) \quad \text{through } (\alpha, \beta, \gamma)}$$

Easier to remember / Clearer notation

* Formulate PDE for surfaces whose tangent planes form a tetrahedron of constant volume with coordinate planes. [2005]

Surface $\equiv z = f(x, y)$. Tangent at (x, y, z)

Let plane be $\Rightarrow$ $p(X-x) + q(Y-y) = Z-z$.

$$pX + qY - Z = px + qy - z.$$

$$\Rightarrow \frac{X}{\frac{px+qy-z}{p}} + \frac{Y}{\frac{px+qy-z}{q}} + \frac{Z}{-(px+qy-z)} = 1.$$

As tetrahedron of constant volume $V = \frac{1}{6}\underbrace{abc}$ intercepts on axes by tangent

$$-\frac{(px+qy-z)^3}{pq} = \text{Constant (take } a).$$

$$\boxed{(px+qy-z)^3 + apq = 0}$$ Ans

(9) 1992 Find surface $4yzp + q + 2y = 0$ through $y^2+z^2=1,\ x+z=2$

Ey//

$$\underbrace{\frac{dx}{4yz} = \frac{dy}{1}} = \frac{dz}{-2y} \Rightarrow -2y\,dy = dz$$

$$-\frac{y^2}{2} = z + c_1 \Rightarrow \boxed{y^2+z=c_1}$$

$$-2x = 2z^2 + c_2 \longrightarrow \boxed{x+z^2=c_2}$$

$\cdot\ y^2+z^2=1 \qquad x+z=2 \Rightarrow z=t,\ x=2-t.$

$\hookrightarrow y^2 = 1-t^2 \Rightarrow y = \sqrt{1-t^2}$

$$c_1 + c_2 = 3 \Rightarrow \boxed{y^2+z^2+x+z=3}$$

(8) 2008 $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ via $(x=1, y=0)$

Ey//

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)}$$

$2x^2y - x \quad yz - 2x^2y \quad x - yz \quad \equiv x, y, \frac{1}{2} \Rightarrow x^2+y^2+z=c_1$

$z, y, x \Rightarrow \cancel{2}xyz - z + z - \cancel{2x^2} + \cancel{2x^2} - 2xyz \equiv xz + y = c_2.$

$z+1 = c_1 \qquad z = c_2 \Rightarrow c_1 - 1 = c_2.$

$\boxed{x^2+y^2+z-1 = xz+y}$ Ans

(9) 2009 $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$. $xy = x+y$. $z=1$.

$\frac{1}{x} = \frac{1}{y} + c_1$ $\quad \frac{1}{y} + \frac{1}{z} = c_2 \Rightarrow y = c_2 - 1$

$xy = x+y \Rightarrow \frac{1}{y} + \frac{1}{x} = 1$

$1 - \frac{2}{y} = c_1 \Rightarrow 1 - \frac{2}{c_2 - 1} = c_1.$

$c_2 - 1 - 2 = c_1 c_2 - c_1$

$\frac{y+z-3yz}{yz} = \left(\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{x}-\frac{1}{y}\right) - \left(\frac{1}{x}-\frac{1}{y}\right)$

$\frac{y+z-3yz}{yz} = \left(\frac{y-x}{xy}\right)\left[\frac{z+y-yz}{yz}\right]$

$xy^2 + \cancel{xyz} - 3xy^2z = yz + y^2 - y^2z - xz - xy + \cancel{xyz}$

OR.

$\frac{1}{x} + \frac{1}{z} = c_1$ $\quad \frac{1}{y} + \frac{1}{z} = c_2.$

$\frac{x+y}{xy} + \frac{2}{z} = c_1 + c_2 \Rightarrow \boxed{c_1 + c_2 = 3}$

$\boxed{\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3}$ Ans

(5) Orthogonal Surfaces

Q. Find surface which intersects $z(x+y) = c(3z+1)$ orthogonally and through $x^2+y^2=1$, $z=1$.

$\rightarrow$ Surface $\equiv \frac{z(x+y)}{3z+1} = c$. $\left(f(x,y,z) = c \right)$

$$\frac{x+y}{3z+1} - 3\frac{z(x+y)}{(3z+1)^2} = \frac{(x+y)(3z+1-3z)}{(3z+1)^2}$$

$$\frac{\partial f}{\partial x} = \frac{z}{3z+1} \qquad \frac{\partial f}{\partial y} = \frac{z}{3z+1} \qquad \frac{\partial f}{\partial z} = \frac{x+y}{(3z+1)^2}$$

IMP

So, orthogonal surface is solution of

$$p\frac{z}{3z+1} + q\left(\frac{z}{3z+1}\right) = \frac{x+y}{(3z+1)^2}.$$

$$\frac{dx(3z+1)}{z} = \frac{dy(3z+1)}{z} = \frac{dz(3z+1)^2}{(x+y)}$$

$$\cancel{\frac{(dx+dy)(3z+1)}{2z} = \frac{dz(3z+1)^2}{(x+y)}}$$

$$\cancel{\frac{(x+y)^2}{2} = 2z^3+z^2+C_2.}$$

$$\frac{dx}{(3z+1)z} = \frac{dy}{z(3z+1)} = \frac{dz}{(x+y)}$$

$$\boxed{x-y=C_1}$$

$x,\ y,\ -z(3z+1)$ (*)

$$\cancel{\frac{dx+dy}{2(3z+1)z} = \frac{dz}{(x+y)}} \Rightarrow \frac{(x+y)^2}{2} = 2z^3+z^2+C_2.$$

$$\Rightarrow \boxed{(x+y)^2 = 4z^3+2z^2+C_2}$$

$$\boxed{2xy = 6+C_2-1}$$

$$x^2+y^2-2xy = C_1^2. \Rightarrow \boxed{2xy = 1-C_1^2}$$

$$5+C_2 = 1-C_1^2.$$

$$5+(x+y)^2-4z^3-2z^2 = 1-(x-y)^2.$$

??

$$\frac{x\,dx+y\,dy}{\frac{z}{3z+1}(x+y)} = \frac{dz}{\frac{(x+y)}{(3z+1)^2}}$$

$$\frac{(x^2+y^2)}{2} = z^3+\frac{z^2}{2}+C_2$$

$$\Rightarrow \frac{x^2+y^2}{2} - z^3 - \frac{z^2}{2} = \phi(x-y). \Rightarrow \phi(x-y) = -2.$$ (*)

- $Pp + Qq = R$ are orthogonal to $Pdx + Qdy \pm Rdz = 0$

* Find family of surfaces orthogonal to

$$(y+z)p + (z+x)q = x+y.$$

$$\boxed{P\,dx + Q\,dy + R\,dz = 0}$$

$\Rightarrow$ Such surface is $\Rightarrow$ $(y+z)dx + (z+x)dy + (x+y)dz = 0$

$$\rightarrow \boxed{xy + yz + zx = C_1} \quad \text{Ans}$$

* Find family orthogonal to $\phi\left[z(x^2+y^2)^2, x^2-y^2\right] = 0$

$u = z(x^2+y^2)^2; \; v = (x^2-y^2)$

(1) First find PDE

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\left(2z(x+y) + p(x+y)^2\right) + \frac{\partial\phi}{\partial v}(2x) = 0$$

$$\frac{\partial}{\partial y} \Rightarrow \frac{\partial\phi}{\partial u}\left(2z(x+y) + q(x+y)^2\right) + \frac{\partial\phi}{\partial v}(-2y) = 0$$

Equate $\dfrac{\partial\phi/\partial u}{\partial\phi/\partial v} \Rightarrow \dfrac{2z(x+y)+p(x+y)^2}{-\cancel{2}x} = \dfrac{2z(x+y)+q(x+y)^2}{y}$

$$\Rightarrow 2zy + py(x+y) = -2zx - qx(x+y)$$

$$\Rightarrow \boxed{py + qx + 2z = 0.}$$

Orthogonal Surface $\Rightarrow$ $y\,dx + x\,dy - 2z\,dz = 0$ *

$$\boxed{xy - z^2 = C_1} \quad \text{Ans}$$

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(6) Linear PDE with n variables (Form $n-1$ equations)

(Q) $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z) = xyz.$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}$$

$$\boxed{x/y = C_1} \qquad \boxed{y/z = C_2}$$

$$\frac{yz\,dx + xz\,dy + xy\,dz}{3xyz} = \frac{du}{\cancel{xyz}}$$

$$\boxed{xyz = 3u + C_3}$$

(Q) $(y+z+w)\dfrac{\partial w}{\partial x} + (z+x+y)\dfrac{\partial w}{\partial y} + (x+y+w)\dfrac{\partial w}{\partial z} = x+y+z.$

$$\frac{dx}{y+z+w} = \frac{dy}{z+x+w} = \frac{dz}{x+y+w} = \frac{dw}{x+y+z.}$$

$$\frac{d(x-y)}{-(x-y)} = \frac{dy-dz}{-(y-z)} = \frac{dz-dw}{-(z-w).} = \frac{dx+dy+dz+dw}{3(x+y+z+w)}$$

$$\frac{x-y}{y-z} = C_1 \;;\; \frac{y-z}{z-w} = C_2 \;;\; (z-w)(x+y+z+w)^{1/3} = C_3$$

$$\phi\left(\frac{x-y}{y-z}, \frac{y-z}{z-w}, (z-w)(x+y+z+w)^{1/3}\right) = 0$$

$\boxed{8}$ $(x_3-x_1)p_1 + x_2p_2 - x_3p_3 + x_2^2 - (x_1x_2 + x_2x_3) = 0$

$$\frac{dx_1}{x_3-x_2} = \underbrace{\frac{dx_2}{x_2} = \frac{dx_3}{-x_3}}_{\boxed{x_2x_3 = C_1}} = \frac{dz}{x_2^2-(x_1x_2+x_2x_3)}$$

$$\frac{dx_1}{dx_3} = \frac{x_1}{x_3} - 1 \Rightarrow e^{\int -\frac{1}{x_3}} = \frac{1}{x_3}$$

$$\frac{x_1}{x_3} = \int -\frac{1}{x_3}\,dx_3 = -\ln(x_3) + C_2$$

$$\boxed{\frac{x_1}{x_3} + \ln(x_3) = C_2}$$

$$dx_1 + dx_2 + dx_3 = 0 \Rightarrow \boxed{x_1 + x_2 + x_3 = C_2}$$

$$\frac{x_2\,dx_1 + x_1\,dx_2}{-x_2^2 + x_2x_3 + x_1x_2} = \frac{dz}{x_2^2 - (x_1x_2 + x_2x_3)}$$

* $Pp + Qq = R$ has a solution $z = \phi(x, y)$.
$\hookrightarrow (1)$

Then $z = \phi(x, y)$ is a surface whose normal at any point (x, y, z) is $(p, q, -1)$.

- $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ represents curves such that tangent at any point has d.r.s (P, Q, R)

(1) is $P.p + Q.q. + R.(-1) = 0$.

This shows $z = \phi(x, y)$ is equivalent to $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

* Find surface orthogonal to $Pp + Qq = R$.

$P dx + Q dy + R dz = 0$

(5) Transform $yz_x - xz_y = 0$ in polar and show that solution represents surfaces of revolution.

$\Rightarrow x = r\cos\theta \quad y = r\sin\theta \qquad r^2 = x^2 + y^2 \quad \theta = \tan^{-1}(y/x)$

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{x}{r}\frac{\partial z}{\partial r} + \frac{\partial z}{\partial \theta}\cdot\frac{(-\sin\theta)}{r} = \boxed{\cos\theta\, z_r - \frac{\sin\theta}{r} z_\theta}$$

$$z_y = \frac{\partial z}{\partial r}\frac{y}{r} + \frac{\partial z}{\partial \theta}\cdot\frac{\cos\theta}{r} = \boxed{\sin\theta\, z_r + \frac{\cos\theta}{r} z_\theta}$$

$$r\sin\theta\left(\cos\theta\, z_r - \frac{\sin\theta}{r} z_\theta\right) - r\cos\theta\left(\sin\theta\, z_r + \frac{\cos\theta}{r} z_\theta\right) = 0.$$

$$z_r(r\sin\theta\cos\theta - r\sin\theta\cos\theta) + z_\theta(-1) = 0.$$

$$\frac{\partial z}{\partial \theta} = 0. \Rightarrow \boxed{z = f(r)}$$

$z =$ surface of revolution

TOPIC 3: Charpit's Method
(Order 1, any degree)

$\Rightarrow f(x, y, z, p, q) = 0$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}.$$

STEPS:

(1) Find f_x, f_y, f_z, f_p, f_p

(2) Select a pair containing atleast p or q in integral.

(3) Use it in ~~①~~ f to get p and q and
$dz = p\,dx + q\,dy$ which on integration gives complete integral

[If $p = a$ and ~~or~~ $q = b$ obtained, then simply putting in f gives solution ∟ complete integral]

• Finding singular integral
 ↳ get complete solution with arbit constants
 ↳ Eliminate arbit constants
 ∟ Solution + $\frac{\partial S}{\partial a} = 0$ + $\frac{\delta S}{\partial b} = 0$ . . . so on.

get Parti Singular solution

Q. Find complete integral $z = px + qy + p^2 + q^2$.

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}$$

$\cancel{f_x =}$ $f \equiv z - px - qy - p^2 - q^2 = 0$.

$f_x = -p \quad f_y = -q \quad f_z = 1.$

$$\frac{dp}{0} = \frac{dq}{0} \Rightarrow \begin{matrix} p = a \\ q = b. \end{matrix}$$

Solution $\equiv$ $\boxed{z = ax + by + a^2 + b^2}$

Q. $z^2(p^2 z^2 + q^2) = 1.$

$f = p^2 z^4 + q^2 z^2 - 1 = 0.$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}.$$

$f_x = 0, \; f_y = 0 \quad f_z = 4p^2 z^3 + 2q^2 z$

$f_p = 2z^4 p \qquad f_q = 2z^2 q$

$$\frac{dp}{p(4p^2z^3 + 2q^2z)} = \frac{dq}{q(4p^2z^3 + 2q^2z)} = \frac{dz}{-2z^4p^2 - 2z^2q^2} = \frac{dx}{-2z^4p} = \frac{dy}{-2z^2q}$$

$\boxed{\frac{p}{q} = c_1}$ Put in $f \Rightarrow$ $q^2(c_1^2 z^2 + 1) = \frac{1}{z^2}$

$$\Rightarrow q = \frac{1}{z\sqrt{c_1^2 z^2 + 1}} ; \; p = \frac{c_1}{z\sqrt{c_1^2 z^2 + 1}}$$

$$dz = \frac{c_1\,dx + dy}{z\sqrt{c_1^2 z^2 + 1}} \Rightarrow z\sqrt{c_1^2z^2+1}\,dz = c_1\,dx + dy$$

$$\boxed{c_1 x + y + c_2} = \int t \cdot t\,\frac{dt}{c_1^2}$$

$c_1^2 z^2 + 1 = t^2$

$\cancel{2}c_1^2 z\,dz = \cancel{2}t\,dt$

$$= \frac{t^3}{3c_1^2} = \boxed{\frac{(c_1^2 z_1^2 + 1)^{3/2}}{3c_1^2}}$$

(8) $f(x, y, z, p, q) = 0$ $\quad [16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0]$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0}$$

$$\frac{dp}{p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)}$$

$p = aq$. — Put in f

$16a^2q^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$.

$$q^2 = \frac{4(1 - z^2)}{z^2(16a^2 + 9)} \Rightarrow q = \frac{2\sqrt{1-z^2}}{z\sqrt{16a^2+9}}$$

$$p = \frac{2a\sqrt{1-z^2}}{z\sqrt{16a^2+9}}$$

$$dz = p\,dx + q\,dy \Rightarrow \frac{z\sqrt{16a^2+9}\,dz}{2\sqrt{1-z^2}} = a\,dx + dy$$

$= \frac{dy}{-2zq}$

$1 - z^2 = t^2$

$-\cancel{2}z\,dz = \cancel{2}t\,dt$.

$$\int \frac{\sqrt{16a^2+9}}{2}\,\frac{-t\,dt}{t} = ax + y + c_1$$

$$\Rightarrow -\frac{\sqrt{16a^2+9}}{2}\sqrt{1-z^2} = ax + y + c_1 \quad \underline{\text{Ans}}$$

9 * Find complete and singular integrals of

$$2xz - px^2 - 2qxy + pq = 0$$

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{2z - \cancel{2px} - 2qy + \cancel{2px}} = \frac{dq}{\cancel{-2qx + 2qx}} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)}$$

$$= \frac{dx}{\cancel{+x^2+q}} = \frac{dy}{\cancel{-2xy+p}}$$

$$\qquad x^2 - q \qquad\qquad p - 2xy.$$

$$\boxed{q = a}$$

$$p = \frac{2xz - 2axy}{x^2 - a}$$

$$\frac{dz}{(z-ay)} = \frac{2x(z-ay)\,dx}{(x^2-a)} + \frac{a\,dy}{(z-ay)}$$

$$\log(z-ay) = \log(x^2-a) + c$$

$$\boxed{\frac{z-ay}{x^2-a} = c_1}$$

$$z - ay = c_1 x^2 - ac_1.$$

$$\cancel{x^2 = 0.} \qquad \cancel{-y = 0} \qquad \boxed{\cancel{z = c_1 \ -a}}$$

$$-y = -c_1 \Rightarrow \boxed{c_1 = y}$$

$$x^2 - a = 0 \Rightarrow \boxed{a = x^2}$$

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1999 [LEAVING]

(8) $p^2+q^2-2px-2qy+2xy=0 \Rightarrow (p-x)^2+(q-y)^2-(x-y)^2=0.$

$f_x=-2p+2y \quad f_y=-2q+2x \quad f_z=0. \quad f_p=2p+(-2x)$

$f_q=2q-2y.$

$$\frac{dp}{-2p+2y}=\frac{dq}{-2q+2x}=\frac{dz}{-2p^2+2px+2q^2+2qy}=\frac{dx}{-2p+2x}=\frac{dy}{2y-2q}.$$

$$\frac{(dp-dy)}{\cancel{2q-2p}}=\frac{-(dq-dx)}{\cancel{2p-2q}} \Rightarrow -dx-dy+dp+dq=0.$$

$$(p-x)+(q-y)=a \Rightarrow q-y=a-(p-x)$$

Putting in $f \Rightarrow$.

$$2(p-x)^2-2a(p-x)+\{a^2-(x-y)^2\}=0.$$

$$p-x=\frac{2a\pm\sqrt{4a^2-8a^2+8(x-y)^2}}{4}.$$

$$p=x+\frac{1}{2}\left[a\pm\sqrt{\{2(x-y)^2-a^2\}}\right]$$

$$q=y+a-p+x.=y+\frac{1}{2}\left[a\mp\sqrt{2(x-y)^2-a^2}\right]$$

$$dz=p\,dx+q\,dy.$$

2001, 06

9

$$p^2x + q^2y = z.$$

$$f_x = p^2,\ f_y = q^2,\ f_z = -1,\ f_p = 2px,\ f_q = 2qy$$

$$\frac{dp}{p^2 - p} = \frac{dq}{q^2 - q} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dx}{-2px} = \frac{dy}{-2qy}.$$

$$\frac{2px\,dp + p^2\,dx}{\cancel{2p^3x} - 2p^2x - \cancel{2p^3x}} = \frac{2qx\,dq + q^2\,dy}{-2q^2y}$$

$$\frac{d(p^2x)}{\cancel{-2}p^2x} = \frac{d(q^2y)}{\cancel{-2}q^2y} \Rightarrow p^2x = a\,q^2y.$$

$$q = \sqrt{\frac{z}{y(1+a)}} \qquad p^2x = \frac{a\cancel{y}z}{x\cancel{y}(1+a)} \Rightarrow p = \sqrt{\frac{az}{x(1+a)}}$$

$$\frac{dz}{\sqrt{z}} = \sqrt{\frac{a}{x(1+a)}}\,dx + \sqrt{\frac{1}{y(1+a)}}\,dy.$$

$$\boxed{\sqrt{1+a}\,\sqrt{z} = \sqrt{a}\,\sqrt{x} + \sqrt{y} + C.}$$ Ans

(3) $2z+p^2+qy+2y^2=0$

$f_x=0,\ f_y=q+4y,\ f_z=2,\ f_p=2p,\ f_q=y.$

$$\frac{dp}{0+p.2}=\frac{dq}{q+4y+2q}=\frac{dz}{-2p^2-yq}=\frac{dx}{-2p}=\frac{dy}{-y}.$$

$p=-x+a. \quad \Rightarrow \quad -\frac{(2z+(a-x)^2+2y^2)}{y}=q.$

$$dz=(a-x)\,dx-\frac{2z}{y}dy-\frac{(a-x)^2}{y}dy-2y\,dy.$$

*(Multiply both sides by $2y^2$).

$$2y^2dz+4yz\,dy=2y^2(a-x)\,dx-2y(a-x)^2dy-4y^3dy$$

$$d(2y^2z)=d\left(y^2(a-x)^2\right)-4y^3dy$$

$$\boxed{2y^2z=y^2(a-x)^2-y^4+b}$$ Ans

$\boxed{9}$ $Z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y).$

$$f \equiv \frac{1}{2}(p^2+q^2) + pq - qx - yp + xy - z = 0$$

$$\frac{dp}{-q+y+p(-1)} = \frac{dq}{-p+x-q} = \frac{dz}{-p(2p+q-y)-q(q+p-x)} =$$

$$\frac{dx}{-(p+q-y)} = \frac{dy}{-(q+p-x)}.$$

$$\cancel{dp+dq} \quad \cancel{\frac{dp-dq}{y-x}} = \cancel{\frac{dx-dy}{y-x}} \Rightarrow p - \cancel{q = x - y} \pm a.$$

$$\Rightarrow \cancel{(p-x) = (q-y) + a}.$$

$$dp = dx \Rightarrow \boxed{p = x+a} \qquad \boxed{q = y+b}$$

$$\boxed{Z = \frac{1}{2}\left((x+a)^2 + (y+b)^2\right) + ab}$$ Ans

* Standard forms

① Only p and q / reduce to p and q. $\boxed{Z = ax + by + c.}$

- 89, 94 $\Rightarrow$ $pq = x^m y^n z^{2l}$. $\Rightarrow$ $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = x^m y^n z^{2l}$.

$$\frac{z^{-l}\partial z}{x^m \partial x}\;\frac{z^{-l}\partial z}{y^n \cdot \partial y} = 1.$$

Put $x^m dx = dX$
$y^n dy = dY$
$z^{-l} dz = dZ$
$\Downarrow$
$X = x^{m+1}/m+1$
$Y = y^{n+1}/n+1$
$Z = z^{-l+1}/1-l$

$$F \Rightarrow \underbrace{\frac{\partial Z}{\partial X}\frac{\partial Z}{\partial Y}}_{F(p,q)} = 1.$$

Solution of this type is $\boxed{Z = aX + bY + c.}$
where $F(a,b) = 1 \Rightarrow ab = 1. \Rightarrow b = 1/a$.

$Z = aX + \frac{1}{a}Y + C.$ $\Leftarrow$ Substitute values

- $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1.$

$x+y = X^2$, $x-y = Y^2$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{Z_X}{2X} + \frac{Z_Y}{2Y}$

$q = \frac{Z_X}{2X} - \frac{Z_Y}{2Y}$

$p+q = \frac{Z_X}{X}$ $\qquad p-q = \frac{Z_Y}{Y}$

$(1) \equiv \cancel{X^2}\cdot\frac{Z_X^2}{\cancel{X^2}} + \cancel{Y^2}\cdot\frac{(Z_Y)^2}{\cancel{Y^2}} = 1 \Rightarrow P^2 + Q^2 = 1$

Solution $\equiv$ $Z = aX + bY + C$ s.t $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1-a^2}$

$\boxed{Z = aX + \sqrt{1-a^2}\,Y + C}$

(2) Clairaut's form

$$\boxed{z = px + qy + f(p,q)}$$

$\Downarrow$

Sol^n $\equiv$ $z = ax + by + f(a,b)$

1989 $\quad (px+qy-z)^2 = 1+p^2+q^2$

$\Rightarrow \quad z = px + qy \pm \sqrt{1+p^2+q^2}$

In standard form, complete integral

$$z = Ax + By \pm \sqrt{1+A^2+B^2}$$

(3) Only p, q and z $\quad f(p,q,z)$

- Let $\underline{u = x + ay}$
- $\underline{p \to \frac{dz}{du}}$, $\underline{q \to a\frac{dz}{du}}$ and solve DE of 1st order
- Put $u = x+ay$ in solution of DE

- $z^2(p^2+q^2+1) = k^2$. $\quad$ 1996

$\Rightarrow z^2\left(\left(\frac{\partial z}{\partial u}\right)^2 + a^2\left(\frac{\partial z}{\partial u}\right)^2 + 1\right) = k^2 \Rightarrow \frac{\partial z}{\partial u} = \pm\sqrt{\frac{k^2-z^2}{z^2(1+a^2)}}$

$\Rightarrow \quad \underline{\pm\sqrt{(1+a^2)}\,(k^2-z^2)^{1/2} = u+b = x+ay+b}$

(4) If $F(x,y,z,p,q) = f_1(x,p) - f_2(y,q) = 0$

Express $\equiv$ $f_1(x,p) = a$, $f_2(y,q) = a$

Put $p = f(a,x)$, $q = f(a,y)$

Then put $\boxed{dz = p\,dx + q\,dy}$

Incorporate z terms in dz.

- $py + qx + pq = 0$. [1990]

$$\frac{p}{x+p} = -\frac{q}{y} = a \Rightarrow p = a(x) + ap \Rightarrow p = \frac{ax}{1-a}$$

$$\Rightarrow q = -ay$$

$$\int dz = \int \frac{ax}{1-a}\,dx + \int -ay\,dy$$

$$\Rightarrow z = \frac{a(x^2)}{2(1-a)} - \frac{ay^2}{2} + b$$

- $z^2(p^2+q^2) = x^2+y^2$. [1989]

$Z = z^2/2$.

$dZ = z\,dz$.

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$$

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2+y^2$$

$$P^2 - x^2 = y^2 - Q^2 = a^2$$

$$\Rightarrow P = \sqrt{a^2+x^2}, \quad Q = \sqrt{y^2-a^2}$$

$$\int dZ = \int \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$$

$$\frac{z^2}{2} = Z = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\{x+\sqrt{a^2+x^2}\} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\log\{y+\sqrt{y^2-a^2}\} + b$$

Ans

$$\bullet\ 2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2+y^2)$$

$$2q^2(p^2 - 4x^4) = 8x^2y^2q^2 - 3x^2y^2$$

$$2q^2(p^2 - 4x^4) = x^2y^2(8q^2-3)$$

$$\frac{(p^2-4x^4)}{x^2} = \frac{y^2(8q^2-3)}{2q^2} = 4a^2$$

$$\Rightarrow p^2 = 4a^2x^2 + 4x^4 \quad ; \quad 8q^2a^2 - 8q^2y^2 = -3y^2$$

$$\Rightarrow q^2 = \frac{3y^2}{8(y^2-a^2)}$$

$$dz = 2x(a^2+x^2)^{1/2}dx + \sqrt{\frac{3}{2}}\ \frac{y}{2\sqrt{y^2-a^2}}dy$$

$$\boxed{z = \frac{2}{3}(x^2+a^2)^{3/2} + \sqrt{\frac{3}{2}} \cdot \frac{1}{2} \cdot \sqrt{y^2-a^2} + b}$$

TOPIC 4 : Cauchy Method of Characteristics

Q Find characteristics of $pq = z$ and determine integral surface passing through $x=0,\ y^2=z$.

STEP 1 : Write curve in Parametric form

$\boxed{x = x_0(\lambda) = 0}$ $\boxed{y = y_0(\lambda) = \lambda}$ $\boxed{z = z_0(\lambda) = \lambda^2}$

STEP 2 : Let initial values p_0, q_0, x_0, y_0, z_0

Satisfy $\Rightarrow p_0 q_0 = z_0$ or $\boxed{p_0 q_0 = \lambda^2}$

STEP 3 : $\boxed{z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)}$

$2\lambda = 0 + q_0 \Rightarrow \boxed{q_0 = 2\lambda}$

$\boxed{p_0 = \frac{\lambda}{2}}$

Now, all initials obtained.

STEP 4 : $f(x,y,z,p,q) = pq - z = 0$

Write characteristic equations [Charpits with (-) sign]

① $\rightarrow \frac{dx}{dt} = \frac{\partial f}{\partial p} = q$; ② $\frac{dy}{dt} = \frac{\partial f}{\partial q} = p$.

③ $\rightarrow \frac{dz}{dt} = pf_p + qf_q = 2pq$.

④ $\rightarrow \frac{dp}{dt} = -f_x - pf_z = p \Rightarrow p = Ce^t \rightarrow \boxed{p = \frac{\lambda}{2}e^t}$

⑤ $\rightarrow \frac{dq}{dt} = q \Rightarrow \boxed{q = 2\lambda e^t}$

① = ⑤ $\Rightarrow x = q + C_1 \Rightarrow C_1 = x_0 - q_0 = -2\lambda$

$\hookrightarrow \boxed{x = q - 2\lambda}$

② = ④ $\Rightarrow y = p + C_2 \Rightarrow \boxed{y = p + \lambda/2}$

① $\Rightarrow \frac{dx}{dt} = 2\lambda e^t \Rightarrow \boxed{x = 2\lambda e^t - 2\lambda}$

② $\Rightarrow \frac{dy}{dt} = \frac{\lambda}{2}e^t \Rightarrow \boxed{y = \frac{\lambda}{2} \times (e^t + 1)}$

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$$\frac{dz}{dt} = 2\lambda^2 e^{2t} \Rightarrow \boxed{z = \lambda^2 e^{2t}}$$

STEP 5:

Characteristics given by.

$x(t),\ y(t),\ z(t)$

$x(t) = 2\lambda e^{t} - 2\lambda$, $y(t) = \frac{\lambda}{2}(e^{t}+1)$, $z(t) = \lambda^2 e^{2t}$

STEP 6:

To find integral surface eliminate t, λ from x, y, z.

$$2y = \sqrt{z} + \lambda \Rightarrow \boxed{\lambda = 2y - \sqrt{z}}$$

$$x = 2\sqrt{z} - 4y + 2\sqrt{z}$$

$$\boxed{(x+4y)^2 = 16z}$$

Ans

TIP FOR STEP 4:

1. First directly solvable
2. Then equate RHS
3. x and y as f^n of t by substituting values from 1

(8) Find solution of $Z = \frac{(p^2+q^2)}{2} + pq - py - qx + xy$ passing through x-axis.

- $Z - \frac{(p^2+q^2)}{2} + py + qx - xy - pq = 0.$

- Initial values $\Rightarrow (x_0, y_0, z_0, p_0, q_0)$

Curve $\equiv$ $\boxed{x_0(\lambda) = \lambda,\ y_0(\lambda) = 0,\ z_0(\lambda) = 0}$

$$-\left(\frac{p_0^2+q_0^2}{2}\right) + q_0\lambda - p_0 q_0 = 0 \rightarrow -\frac{q_0^2}{\cancel{2}} + 2q_0\lambda = 0$$

$$\boxed{q_0 = 2\lambda}$$

$$z_0'(\lambda) = p_0 \cdot 1 + 0 \Rightarrow \boxed{p_0 = 0}$$

- Characteristic equations

$$\frac{dx}{dt} = -(y - q - p) \qquad \frac{dy}{dt} = -(x - p - q)$$

$$\frac{dz}{dt} = -py + pq + p^2 - qx + pq + q^2 \Rightarrow (p+q)^2 - (py+qx)$$

$$\frac{dp}{dt} = -(-q + y - p) \qquad \frac{dq}{dt} = -(+x - p - q)$$

$$\boxed{x = p + \lambda} \qquad \boxed{y = q - 2\lambda}$$

$$\frac{dp}{dt} = +2\lambda + p \Rightarrow \log(2\lambda + p) = +t + C.$$

$$2\lambda + p = ce^{+t} \Rightarrow \boxed{p = 2\lambda(e^{+t} - 1)}$$

(1)

$$\boxed{q = \lambda(1 + e^t)}$$

$$\boxed{x = 2\lambda e^t - \lambda} \quad (+) \quad \boxed{y = \lambda(e^t - 1)}$$

$$\boxed{\begin{aligned} \lambda &= x - 2y \\ e^t &= \frac{x-y}{x-2y} \end{aligned}}$$

$$\frac{dz}{dt} = 5\lambda^2 e^{2t} - 3\lambda e^t \Rightarrow \boxed{z = \frac{5}{2}\lambda^2(e^{2t} - 1) - 3\lambda^2(e^t - 1)}$$

$\boxed{Q}$ 2001

$$z + px + qy - 1 - pqx^2y^2 = 0. \qquad z = 0,\ x = y.$$

① Initial = $x_0 = \lambda,\ y_0 = \lambda,\ z_0 = 0.$

$z_0' = x_0' p_0 + y_0' q_0 \Rightarrow p_0 + q_0 = 0$

$p_0\lambda - p_0\lambda - 1 + p_0^2\lambda^4 = 0 \Rightarrow \boxed{p_0^2 = \frac{1}{\lambda^4}}$ $\begin{cases} p_0 = 1/\lambda^2 \\ q_0 = -1/\lambda^2 \end{cases}$

② Equations.

$$\frac{dx}{dt} = f_p = x - qx^2y^2. \qquad \frac{dy}{dt} = f_q = y - px^2y^2$$

$$\frac{dz}{dt} = pf_p + qf_q = px + qy - 2pqx^2y^2$$

$$\frac{dp}{dt} = -f_x - pf_z = -p + 2pqxy^2 - p = -2p + 2pqxy^2$$

$$\frac{dq}{dt} = -f_y - qf_z = -2q + 2pqxy^2.$$

③.

$$2p\frac{dx}{dt} + x\frac{dp}{dt} = 0 \Rightarrow 2px\frac{dx}{dt} + x^2\frac{dp}{dt} = 0 \Rightarrow px^2 = C_1$$

Illy $qy^2 = C_2$

$$\frac{dx}{dt} = x - x^2 c_2 \Rightarrow \frac{dx}{x - c_2x^2} = dt \Rightarrow \frac{dx}{x(1-c_2x)} = dt.$$

$$\frac{1}{x} + \frac{c_2}{1-c_2x}\, dx = dt. \qquad p = c_2x.$$

$$\ln x - \left(\frac{dp}{p-1}\right) = dt$$

$$\ln x - \ln(c_2x - 1) = dt.$$

$$\ln\left(\frac{x}{c_2x-1}\right) = t + c_3.$$

$$\ln\left(\frac{c_2x-1}{x}\right) = -t - c_3$$

$$\ln\left(c_2 - \frac{1}{x}\right) = -t - c_3$$

$$c_2 - \frac{1}{x} = e^{-t}C_3.$$

$$\boxed{x = \frac{1}{c_2 + c_3e^{t}}}$$

Illy $\frac{dy}{dt} = y - c_1y^2$

$$\Rightarrow \boxed{y = \frac{1}{c_1 + c_4e^{-t}}}$$

$$\Rightarrow \boxed{\begin{aligned} q &= C_2\left(c_1 + c_4e^{-t}\right)^2 \\ p &= c_1\left(c_2 + c_3e^{-t}\right)^2. \end{aligned}}$$

Illy get z also

TOPIC 5 : Linear PDE with Constant Coefficients.

$D \equiv \partial/\partial x \qquad D' = \partial/\partial y$

• $F(D, D')z = f(x,y) \equiv Sol^n \equiv \boxed{CF + PI}$

$\rightarrow \left(A_0 D^n + A_1 D^{n-1} D' + \dots A_n D'^n\right) z = f(x,y).$

Replace D by m, D' by 1.

Auxiliary $\Rightarrow A_0 m^n + \dots \; A_n = 0.$
$\Downarrow$.

(1) m_1, m_2, m_3 distinct

$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \dots$

(2) $m = m'$ repeated

$C.F = \phi_1(y + m'x) + x\phi_2(y + m'x) + x^2\phi_3(y + m'x)$.

(3) Factor $D \rightarrow \phi(y)$

repeated $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \dots$

(4) Factor $D' \rightarrow \phi(x)$.

repeated $\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \dots$
$y^{m-1}\phi_{m-1}(x)$

OR if directly.

$\prod_i (b_i D - a_i D') \rightarrow$ Same
$\Downarrow$

$\phi(by + ax)$ Sign reversed

$(D - mD') \rightarrow \phi(y + mx)$

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* p $\quad q \quad r \quad s \quad t$

$\frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} \quad \frac{\partial^2 z}{\partial x^2} \quad \frac{\partial^2 z}{\partial x \partial y} \quad \frac{\partial^2 z}{\partial y^2}.$

$\boxed{Q}$ $(D^3 D'^2 + D^2 D'^3) z = 0.$

$\cancel{(m^3+m^2) = 0} \rightarrow m = 0, 0, -1 \Rightarrow \phi_1(y) + x\phi_2(y) + \phi_3(y-x)$

Instead, in such cases $\Rightarrow$

$D^2 D'^2 (D + D') z = 0.$

$z = \phi_1(y) + x\,\phi_2(y) + \phi_3(x) + y\,\phi_4(x) + \phi_5(y-x)$ Ans

[If common can be taken out, then take them out first]

Finding Particular Integrals $F(D,D')y = f(x,y)$

① Type I $\boxed{f(x,y) = \phi(ax+by)}$

- $F(a,b) \neq 0$ and $F(D,D')$ homogenous of order n.

$$PI = \frac{1}{F(D,D')}\phi(ax+by) = \frac{1}{F(a,b)} \underbrace{\iiint\!\!\int}_{n \text{ times}} f(v)\,dv$$

where $v = ax+by$

OR

- $F(a,b) = 0 \rightarrow \frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{b^n n!}\phi(ax+by)$

Q $r+s-2t = (2x+y)^{1/2}$.

C.F $\Rightarrow (D^2 + DD' - 2D'^2)y = 0$.

$m^2+m-2=0 \rightarrow -2, 1$ $\boxed{\phi_1(y-2x) + \phi_2(y+x)}$

P.I $\Rightarrow$. $\frac{1}{(D+2D')(D-D')}(2x+y)^{1/2}$.

$$\frac{1}{4.1}\iint U^{1/2}\,dU = \frac{1.\cancel{2}.\cancel{2}}{\cancel{4}.3.5}U^{5/2}$$

$\boxed{8}$ $r + 5s + 6t = \frac{1}{(y-2x)}$.

$(D^2 + 5DD' + 6D'^2)y = 0$

$\hookrightarrow CF \Rightarrow m^2 + 5m + 6 = 0 \Rightarrow m = -3, -2.$

$\hookrightarrow \phi_1(y-3x) + \phi_2(y-2x)$

$PI \Rightarrow \frac{1}{(D+3D')(D+2D')} (y-2x)^{-1}$

$\frac{1}{(D+2D')} \int U^{-1} \cancel{=}$ $\qquad U = y-2x.$

$\frac{\log u}{D+2D'} = \frac{1}{D+2D'} \log(y-2x) = \frac{x}{1} \log(y-2x)$

$(b=1)$.

$z = \phi_1(y-2x) + \phi_2(y-3x) + x\log(y-2x)$ Ans

Q) $(D_x^3 - 7D_x D_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$

CF ⇒ $m = -1, -2, 3 \Rightarrow \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

PI ⇒ $\dfrac{1}{(D+D')(D+2D')(D-3D')}\left[\sin(x+2y) + e^{3x+y}\right]$.

$$\frac{1}{3.5.(-5)}\iiint \sin u \, du + \frac{1}{(D-3D')\,4.5}\iint e^{v} dv$$

$$-\frac{(+1)}{75}\cos(x+2y) + \frac{1}{20} \times \underbrace{\frac{1}{(D-3D')} e^{3x+y}}_{x e^{3x+y}} \quad \left(\frac{x^n}{n!\,b^n}\right)$$

$$\boxed{-\frac{\cos(x+2y)}{75} + \frac{x e^{3x+y}}{20}} = PI$$

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(2) Type 2.

$$f(x,y) = x^m y^n$$

- Expand $\frac{1}{F(D,D')}$ at N^r position.

$\frac{D'}{D}$ when deg y is less

$\frac{D}{D'}$ when deg x is less

Q) $(D^2 + 3DD' + 2D'^2)z = x + y.$

C.F $\Rightarrow \phi_1(y-2x) + \phi_2(y-x)$

$$PI \Rightarrow \frac{1}{D^2\left(1 + \frac{3D'}{D} + 2\left(\frac{D'}{D}\right)^2\right)}(x+y)$$

$$\frac{1}{D^2}\left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right) + \left(\quad\right)\right](x+y)$$

$$\frac{1}{D^2}\left(x+y - 3\int 1\,dx.\right) \Rightarrow \frac{1}{D^2}(y-2x)$$

$$\Rightarrow \frac{1}{D}yx - x^2 = \boxed{\frac{yx^2}{2} - \frac{x^3}{3}}$$

$$\boxed{z = \phi_1(y-2x) + \phi_2(y-x) + \frac{x^2y}{2} - \frac{x^3}{3}}$$ Ans

Q $(D^2D' - 2DD'^2 + D'^3)y = 1/x^2.$

$D'(D^2 - 2DD' + D'^2)y = 0$

C.F $\Rightarrow$ $\phi_1(y+x) + x\phi_2(y+x) + \phi_3(x)$

PI $\Rightarrow$ $\frac{1}{(D-D')^2} \cdot \frac{1}{D'} \times \frac{1}{x^2} = \frac{1}{(D-D')^2} \frac{y}{x^2}$

$\frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}\left(\frac{y}{x^2}\right) = \frac{\left(1-\frac{D'}{D}\right)^{-2}}{D^2}\left(\frac{y}{x^2}\right)$

$= \frac{1}{D^2}\left[1 + \frac{2D'}{D} + \cdots\right]\left(\frac{y}{x^2}\right).$

$= \frac{1}{D^2}\left[\frac{y}{x^2} + \frac{2}{D}\left(\frac{1}{x^2}\right)\right] = \frac{1}{D^2}\left(\frac{y}{x^2} - \frac{2}{x}\right).$

$= \frac{1}{D}\left(-\frac{y}{x} - 2\log x\right) = -y\log x - 2\int \log x\,dx.$

$= \boxed{-y\log x - 2x\log x + 2x}$. Ans

$\boxed{(1+x)^{-n} = 1 - nx + {}^{n}C_2 x^2 \cancel{- {}^{n}C_3 x^3}}$

$(1+x)^m = 1 + mx + \cdots$

$\forall m$ +ve or -ve

③ Type 3.

$$\boxed{y + mx = c}$$

General Method

$$PI = \frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')} f(x,y)$$

$$\frac{1}{D-mD'} f(x,y) = \int f(x, c-mx) \qquad \boxed{c = y+mx}$$

$\boxed{Q}$ $(D-D')(D+2D')z = (y+1)e^x$

$$C.F = \boxed{\phi_1(y+x) + \phi_2(y-2x)}$$

Same

(Not)

$$PI \Rightarrow \frac{1}{(D-D')(D+2D')} (y+1)e^x. \qquad \boxed{y = c + 2x}$$

$$\frac{1}{D-D'}\left\{\frac{1}{D+2D'}(y+1)e^x\right\}.$$

$$\left\{\int f(x, c+2x)\right\} \Rightarrow \int (c+2x+1)e^x$$

$$\left\{(c+1)e^x + 2xe^x - 2e^x\right\}$$

$$\frac{1}{D-D'}\left\{e^x(c+1+2x-2)\right\}.$$

$$\frac{1}{D-D'}\{e^x(y-1)\} \qquad \boxed{y = c' - x}$$

$$\int e^x(c'-x-1)\,dx = (c'-x-1)e^x + e^x = \boxed{ye^x}$$

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2009 MUG IT UP!

(8) * $(D^2 - DD' - 2D'^2)Z = (2x^2 + xy - y^2)\sin xy - \cos xy$

C.F $\equiv$ $m = 2, -1 \Rightarrow \phi_1(y+2x) + \phi_2(y-x)$

PI $\Rightarrow$ $\dfrac{1}{(D+D')}\left\{\dfrac{1}{D - 2D'}\left[(2x^2+xy-y^2)\sin xy - \cos xy\right]\right\}$

$\cancel{\dfrac{1}{D+D'}\left\{\int\left[\{2x^2 + x(c-2x) - (c-2x)^2\}\sin\{x\cdot(c-2x)\} - \cos x(c-2x)\right]\right.}$

$\cancel{\dfrac{1}{D+D'}\left\{\int\left[\{cx - (c-2x)^2\}\sin(x\cdot(c-2x)) - \cos(x(c-2x))\right]\right.}$

$\left(\dfrac{1}{D-2D'}\right)\left\{\dfrac{1}{D+D'}\left[\underbrace{(2x^2+xy-y^2)}_{(2x-y)(x+y)}\sin xy - \cos xy\right]\right\}$

$\dfrac{1}{D-2D'}\left\{\int(2x-(c+x))(x+c+x)\sin x(c+x) - \cos x(c+x)\right\}$ $\boxed{y = c+x}$

$\dfrac{1}{D-2D'}\left\{\int(x-c)(2x+c)\sin(cx+x^2) - \cos(cx+x^2)\right\}$

$\dfrac{1}{D-2D'}\left\{-(x-c)\cos(cx+x^2) + \int\cancel{\cos(cx+x^2)} - \int\cancel{\cos(cx+x^2)}\right\}$

$\dfrac{1}{D-2D'}(y-2x)\cos xy = \int(c'-4x)\cos(c'x - 2x^2)\,dx.$

$= \sin(c'x - 2x^2) = \boxed{\sin xy}$

$\boxed{8}$ $(r + s - 6t) = y\cos x.$

$(D^2 + DD' - 6D'^2)y = 0$

C.F $\Rightarrow$ $\phi_1(y-3x) + \phi_2(y+2x).$

PI $\Rightarrow$ $\frac{1}{(D+3D')(D-2D')} y\cos x$

$\frac{1}{(D+3D')} \int \underbrace{(c-2x)\cos x} \rightarrow c\sin x - 2x\sin x + 2\int \sin x.$

$\frac{1}{D+3D'}\left\{(c-2x)\sin x - 2\cos x\right\} = y\sin x - 2\cos x.$

$\int (c'+3x)\sin x - 2\cos x . dx$

$-c'\cos x - 3x\cos x . + \cancel{\int 3\cos x\, dx} . + 2\sin x$

$\boxed{\sin x - \cos x . y.}$ Ans

$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

4.30 Q. 10 left $\rightarrow$ Wave equation

ALITER Approach

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x}\cos y.$$

Finding PI $\Rightarrow$ $\frac{1}{D^2+D'^2} e^{-1.x+0.y} \cos y.$

$$= e^{-x} \frac{1}{(D-1)^2+D'^2} \cos y = e^{-x} \frac{1}{D^2+D'^2-2D+1} \cos(y)$$

$$= e^{-x} \frac{1}{D^2-(1)^2-2D+1} \cos y = e^{-x} \frac{(-1)}{2D} \cos y$$

$$= \boxed{-\frac{1}{2} x e^{-x} \cos y.}$$

(4) Solutions under geometrical conditions

(8) Find surface through $z = x = 0$, $z - 1 = x - y = 0$

$r - 4s + 4t = 0$.

$D^2 - 4DD' + 4D'^2 = 0 \Rightarrow (D - 2D')^2 = 0$. $m = 2, 2$.

$z = \phi_1(y + 2x) + x\phi_2(y + 2x)$

$\phi_1(y) + 0 = 0$, $y = x \Rightarrow \phi_1(y + 2x) = 0$.

$\cancel{\phi_1(3y) + y\phi_2(3y) = z}$

$z = x\phi_2(y + 2x). \Rightarrow z = 1, x = y$.

$1 = x\phi_2(3x) \Rightarrow \phi_2(3x) = \frac{3}{3x} \Rightarrow \phi_2(y + 2x) = \frac{3}{y + 2x}$

So, $\boxed{z = \frac{3x}{y + 2x}}$ is the surface

[8] Find surface $r-2s+t=6$ touching $z=xy$ along its section by $y=x$.

$$(D^2-2DD'+D'^2)z=6 \Rightarrow (D-D')^2 z=6$$

$$CF = \phi_1(y+x) + x\,\phi_2(y+x)$$

$$PI = \frac{1}{(D-D')^2}6 \Rightarrow \frac{1}{D^2}\left[1-\frac{D'}{D}\right]^{-2}6 \Rightarrow \frac{1}{D^2}\left[1+\frac{2D'}{D}+\cdots\right]6$$

$$= 3x^2$$

$$z_1 \equiv \text{Solution} \Rightarrow \phi_1(y+x) + \phi_2(y+x).x + 3x^2.$$

- As it touches $z=xy$

So, value of p, q for z and z_1 must be equal when $y=x$.

① - $p \Rightarrow \phi_1'(y+x) + \phi_2(y+x) + x\phi_2'(y+x) + 6x = y.$

② - $q \Rightarrow \phi_1'(y+x) + x\phi_2'(y+x) = x.$

① - ② $\Rightarrow \phi_2(y+x) + 6x = y-x.$

$y=x \Rightarrow \phi_2(2y) + 6y = 0.$

$\phi_2(2y) = -6(y) = -3(2y)$

$\phi_2(y+x) = -3(y+x)$

$\phi_2'(y+x) = -3$

$\phi_1'(y+x) = 4x \Rightarrow \phi_1'(2x) = 2.(2x) \Rightarrow \phi_1'(x) = 2x$

$\phi_1(x) = x^2 + C \Rightarrow \phi_1(x+y) = (x+y)^2 + C.$

$z_1 = (x+y)^2 + C + x(-3y-3x) + 3x^2.$

To remove $C \Rightarrow z_1 = z$ and $y=x$

TOPIC 6 : Reducible PDEs to constant coefficients

- Form $\equiv \left(a_0 x^n D^n + a_1 x^{n-1} y D^{n-1} D' + \ldots + a_n y^n D'^n\right) Z = f(x, y)$.

STEPS $\Rightarrow$ $x = e^u$ $\quad$ $y = e^v$

- $x^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$ $\quad$ $y^3 D'^3 = D_2(D_2 - 1)(D_2 - 2)$
- Replace and solve, substitute $u = \log x$, $v = \log y$

Q $x^2(\partial^2 z/\partial x^2) - 4xy(\partial^2 z/\partial x \partial y) + 4y^2(\partial^2 z/\partial y^2) + 6y(\partial z/\partial y) = x^3 y^4$

$x = e^u$, $y = e^v$ $\quad$ $\frac{\partial}{\partial u} = D_1$ $\quad$ $\frac{\partial}{\partial v} = D_2$

$$\left[D_1(D_1 - 1) - 4D_1 D_2 + 4D_2(D_2 - 1) + 6D_2\right] z = e^{3u} e^{4u}$$

$$\left[D_1^2 - 4D_1 D_2 + 4D_2^2 - D_1 - 4D_2 + 6D_2\right] z = e^{3u} e^{4v}.$$

$$\left[(D_1 - 2D_2)^2 - (D_1 - 2D_2)\right] z = e^{3u} e^{4v}.$$

$$\left[(D_1 - 2D_2)(D_1 - 2D_2 - 1)\right] z = e^{3u + 4v}.$$

$$C.F = \phi(v + 2u) + e^{u} \phi_2(v + 2u).$$

$$PI = \frac{1}{(3-8)(3-8-1)} e^{3u+4v} = \frac{1}{30} e^{3u} e^{4v} = \boxed{\frac{1}{30} x^3 y^4}$$ Ans

(8) $(x^2 D^2 - y^2 D'^2) z = xy$

$x = e^u$, $y = e^v$ $\quad \frac{\partial}{\partial u} = D_1 \quad \frac{\partial}{\partial v} = D_2.$

$$\left(D_1(D_1-1) - D_2(D_2-1)\right) z = e^{u+v}$$

$$\left(D_1^2 - D_2^2 - (D_1 - D_2)\right) z = (D_1 - D_2)(D_1 + D_2 - 1) z = e^{u+v}.$$

$$C.F \Rightarrow \phi_1(v+u) + e^{u} \phi_2(v-u)$$

$$= \boxed{\phi_1(\log(xy)) + x\,\phi_2(\log(y/x))}$$

$$PI \Rightarrow 1 \cdot \frac{1}{D_1 - D_2} e^{u+v} = \frac{u}{1} e^{u+v} = \boxed{xy \log x}$$

(9) $x^2 r - y^2 t + px - qy = \log x.$

$x = e^u \quad y = e^v$

$$D_1(D_1-1) - D_2(D_2-1) + D_1 - D_2 = u$$

$$(D_1^2 - D_2^2) z = u \Rightarrow C.F \Rightarrow \phi_1(v+u) + \phi_2(v-u) = \boxed{\phi_1(\log xy) + \phi_2(\log \tfrac{y}{x})}$$

$$PI \Rightarrow \frac{1}{(D_1+D_2)(D_1-D_2)} u \Rightarrow \frac{(-1)}{D_2^2} \left(1 + \frac{D_1}{D_2}\right)^{-1} \left(1 - \frac{D_1}{D_2}\right)^{-1} u$$

$$-\frac{1}{D_2^2} \left[1 - \frac{D_1}{D_2} + \frac{D_1^2}{D_2^2} + \cdots\right] \left[1 + \frac{D_1}{D_2} + \frac{D_1^2}{D_2^2} + \cdots\right] u$$

$$-\frac{1}{D_2^2} \left[1 + \frac{D_1}{D_2} - \frac{D_1}{D_2} + \cdots\right] u \Rightarrow -u \frac{v^2}{2} = \boxed{-\log x \cdot \frac{(\log y)^2}{2}}$$

* $D^2 - 3DD' + 2D'^2 - D + 3D' - 2 \equiv (D - D' - 2)(D - 2D' + 1)$

* Let $x = e^u$, $y = e^v$ $\rightarrow$ $u = \log x$, $v = \log y$

$D \equiv \partial/\partial x$ $D' \equiv \partial/\partial y$; $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$

Then . . . . .

Chapter 5 Non-homogenous Linear PDE with constant coefficients.

(1) Finding C.F

- If reducible

  Let factors be $(bD - aD' - c)$ — If $b=0 \Rightarrow \boxed{aD'+c}$

  $\to e^{\frac{xc}{b}}\,\phi(by+ax)$ if $b \neq 0$.

  $e^{-\frac{yc}{a}}\,\phi(by+ax)$ if $a \neq 0$.

  – Repeated roots $\equiv e^{xc/b}\left[\phi_1(by+ax) + x\phi_2(by+ax) + \dots\right]$

- If irreducible

  Let $z = Ae^{hx+ky}$ be trial solution and put it.

  Find relation h, k

  Solution (general) is $z = \sum A e^{\bigcirc}$

Tip: Factoring → Put $D = 0, 1, -1, D', -D'$
Same for D'

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② Finding P.I $\quad F(D, D')z = f(x,y)$.

- $f(x,y) = e^{ax+by}$

$$PI = \frac{1}{F(a,b)} e^{ax+by}$$

or

$$e^{ax+by} \frac{1}{F(D+a, D'+b)} \cdot 1 \quad \text{if } F(a,b)=0$$

- $f(x,y) = \sin(ax+by)$.
  $\cos(ax+by)$

$$\frac{1}{F(D,D')} \sin ax+by$$

Put $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$

- $f(x,y) = x^m y^m$

$$\underbrace{F(D,D')^{-1}}_{\Downarrow \text{ Expand}} x^m y^m$$

Questions

① $(3D-5)(7D'+2)DD'(2D+3D'+5)z=0$

$\boxed{bD-aD'-c}$

$$e^{\frac{5x}{3}}(\phi_1(3y)) + \underbrace{e^{-\frac{2y}{7}}\phi_2(-7\cancel{y})}_{\times} + \phi_3(y) + \phi_4(x) + e^{-\frac{5x}{2}}\phi_5(2y-3x)$$

$$\boxed{e^{-\frac{2y}{7}}\phi_2(7x)}$$

As $b=0$
Take $\boxed{aD'+c}$ for
computing $\phi(by+ax)$

② $(2D^4-3D^2D'+D'^2)z=0 \leftarrow$ ✱

$(2x-3x+1)$

Looks irreducible but $\Rightarrow$

$2D^4-2D^2D' \cancel{+} D^2D'+D'^2$

$2D^2(D^2-D') - D'(D^2-D') = (2D^2-D')(D^2-D')$

Both irreducible $\Rightarrow$

- $2D^2-D'$

Assume $Ae^{hx+ky} \equiv 2Ah^2e^{hx+ky} - Ake^{hx+ky} = 0$

$2h^2=k.$

$$Z = \sum Ae^{hx+2h^2y}$$

- D^2-D'

Assume $A'e^{h'x+k'y} \equiv A'h'^2 - A'k' = 0 \Rightarrow h'^2=k'$

$$Z = \sum A'e^{h'x+h'^2y}$$

$$Z = \sum Ae^{hx+2h^2y} + \sum A'e^{h'x+h'^2y}$$

③ $(D^2+DD'+D'-1)z = \sin(x+2y)$

$$\begin{array}{r} D+D'-1 \\ D+1\overline{)\,D^2+DD'+D'-1\,(} \\ D^2\quad +D \\ \hline DD'-D+D'-1 \\ DD'\quad +D' \\ \hline \end{array}$$

$D=-1$

$(D+1)(D+D'-1)z = \sin(x+2y)$

$C.F \equiv e^{-x}\phi_1(y) + e^{x}\phi_2(y-x)$

$PI \Rightarrow \dfrac{1}{D^2+DD'+D'-1}\sin(x+2y) = \dfrac{1}{-1-2+D'-1}\sin(x+2y)$

$= \dfrac{D'+4}{(D'-4)(D'+4)}\sin(x+2y) = \dfrac{D'+4}{-4-16}\sin(x+2y) = -\dfrac{1}{20}(2\cos + 4\sin)$

④ $(D-D'-1)(D-D'-2)z = \sin(2x+3y)$

$C.F \Rightarrow e^{x}\phi(y+x) + e^{2x}\phi(y+x)$

$PI \Rightarrow \dfrac{1}{D^2+D'^2-2DD'-3D+3D'+2}\sin(2x+3y)$

$\dfrac{1}{-4-9+12-3D+3D'+2}\sin(2x+3y)$

$\dfrac{1}{-3D+3D'+1}\sin(2x+3y)$ [Easier : Multiply by D in N^r, D^r]

$\dfrac{1-(3D'-3D)}{1-(3D'-3D)^2}\sin(2x+3y) = \dfrac{3D-3D'+1}{1-[9.-9+9.-4+18.6]}\sin(2x+3y)$

$\dfrac{1}{10}\left[3.2\cos(2x+3y) - 3.3\cos(2x+3y) + \sin(2x+3y)\right] = \dfrac{1}{10}\left[\sin(2x+3y) - 3\cos(2x+3y)\right]$

Ans

⑤ $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

$D = D'$ $\quad D'^2 - D'^2 - 2D'^2$ ✗

$\boxed{D = -D'}$ $\quad \cancel{D'^2 + D'^2 + 2D'}$ ✓

$$D + D' \overline{\left) D^2 - DD' - 2D'^2 + 2D + 2D' \right.}$$

Quotient: $D - 2D' + 2$

$D^2 + DD'$

$-2DD' - 2D'^2$

$(D + D')(D - 2D' + 2)z = \sin(2x+y)$.

C.F ⇒ $\phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$.

PI ⇒ $\dfrac{1}{-4 + 2 + 2 + 2D + 2D'}\sin(2x+y)$.

$\dfrac{1}{2}\ \dfrac{D - D'}{D^2 - D'^2}\sin(2x+y) = \dfrac{1}{2}\ \dfrac{2\cos(2x+y) - \cos(2x+y)}{-2^2 + 1}$

$= \boxed{-\dfrac{1}{6}\cos(2x+y)}$

sin)

⑥ $s + p - q = z + xy$

$(DD' + D - D' - 1)z = xy$.

$CF = (D - 1)(D' + 1) \Rightarrow e^{x}\phi_1(y) + e^{-y}\phi_2(x)$.

$aD' + c \quad a = 1, c = 1$.

in]

PI ⇒ $-\dfrac{1}{1 - (D - D' + DD')}xy = -\left[1 + D - D' + DD'\right]xy$

$= -(xy + y - x + 1)$. ✳

+3y)

(2x+3y))

$(1 - x)^{-1} = 1 + x$

Instead use separate expansions

(7) $r - s + 2q = z + x^2y^2$ IMPORTANT

$$(D^2 - DD' + 2D' - 1)z = x^2y^2.$$

Cannot be resolved, so Ae^{hx+ky}.

$$A(h^2 - hk + 2k - 1) = 0 \Rightarrow \boxed{k = \frac{h^2-1}{h-2}}$$

$$C.F = \sum A e^{hx + \frac{h^2-1}{h-2}y}.$$

$$PI \Rightarrow \frac{(-1)}{1+(-2D'+DD'-D^2)} x^2y^2.$$

$$(-1)\left[1 + 2D' - DD' + D^2 + (-2D'+DD'-D^2)^2 + \dots\right] x^2y^2$$

$$4D'^2 + D^2D'^2 - 4D'^2D + 4D'D^2.$$

$$(-1)(x^2y^2 + 4x^2y - 4xy + 2y^2 + 4 \cdot 2x^2 \cdot 4$$

$$+ 8x^2 + 4 - 4(4x) + 16y$$

$$(-1)\left[1 - (-2D'+DD'-D^2) + (-2D'+DD'-D^2)^2 - (-2D'+DD'-D^2)^3 + \dots\right] x^2y^2$$

$$\to -12D^2D'^2$$

$$(-1)\left[x^2y^2 + 4x^2y - 4xy + 2y^2 + 8x^2y + 4xy + 16x + 16y + 12\cdot 4 + 4\right]$$ Ans

(8) $(D-3D'-2)^2 z = 2e^{2x}\sin(y+3x)$

C.F $\Rightarrow$ $e^{2x}\left[\phi_1(y+3x) + x\,\phi_2(y+3x)\right]$

PI $\Rightarrow$ $\dfrac{1}{(D-3D'-2)^2}\, 2e^{2x}\sin(y+3x)$

$$2e^{2x}\frac{1}{(D+\cancel{2}-3D'-\cancel{2})^2}\sin(y+3x) = \frac{2e^{2x}}{D^2+9D'^2-6DD'}\sin(y+3x)$$

$$= \cancel{\frac{2}{-9-9+18}}\ 0$$

$$\frac{2e^{2x}}{(D-3D')^2}\sin(y+3x)$$

$$= \boxed{2e^{2x}\frac{x^2}{2!}\sin(y+3x)}\ \text{Ans}$$

$\boxed{\text{IMPORTANT}}$

$\dfrac{x^n}{b^n\, n!}$ Not only sin also tan or. $f(ax+by)$ in general

(9) $r - 3s + 2t - p + 2q = (2+4x)e^{-y}$

C.F $\Rightarrow$ $(D^2 - 3DD' + 2D'^2 - D + 2D')z = (2+4x)e^{-y}$

$$D-2D'\,\overline{)\,D^2-3DD'+2D'^2-D+2D'\,(}\ D-D'-1$$
$$D^2-$$
$$-DD'+2D'^2$$

$D = 2D' \Rightarrow 0$

$(D-2D')(D-D'-1)z = (2+4x)e^{-y}$

C.F $\Rightarrow$ $\phi_1(y+2x) + e^{x}\phi_2(y+x)$

PI $\Rightarrow$ $e^{-y}\dfrac{1}{(D-2(D'-1))(D-(D'\cancel{-1})\cancel{-1})}(2+4x)$

$$e^{-y}\frac{1}{(D-2D'+2)(D-D')}(2+4x) = \cancel{e^{-y}\frac{1}{D^2-DD'-2DD'+2D'^2+2D-2D'}(2+4x)}$$

$$e^{-y}\frac{1}{2D}\left(1-\frac{D'}{D}\right)^{-1}\left(1+\frac{D-2D'}{2}\right)^{-1}(2+4x) = \frac{e^{-y}}{2D}\left(1+\frac{D'}{D}+\cdots\right)\left(1-\left(\frac{D-2D'}{2}\right)+\cdots\right)2+4x$$

$$\frac{e^{-y}}{\cancel{2}D}\left(1-\frac{D}{2}\right)\overset{1}{\cancel{2}}+\overset{2}{\cancel{4}}x = \boxed{x^2e^{-y}}\ \text{A}$$

(10) $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

$D = 1$ satisfies

$$D-1 \,)\, D^2 - DD' + D' - 1 \,(\, D - D' + 1.$$
$$\underline{D^2 \quad - D}$$
$$D + D' - DD' - 1$$
$$D' - DD'$$

$$(D-1)(D-D'+1)z = \cos(x+2y) + e^y.$$

C.F ⇒ $\boxed{e^{x}\phi_1(y) + e^{-x}\phi_2(y+x)}$

P.I ⇒ $\cos x+2y$ ⇒ $\dfrac{1}{\cancel{-1} + \cancel{2} + D' - 1}\cos(x+2y) = \boxed{\dfrac{\sin(x+2y)}{2}}$

e^{0x+1y} ⇒ $\cancel{\dfrac{1}{(D-1)(D-(D'+1)+1)}e^y}$ $\cancel{e^y \dfrac{1}{(D-1)(D-D')} \times 1}$.

$$\frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{(D-D'+1)}\left\{\frac{1}{0-1}e^y\right\}$$

$$= (-1) \times \frac{1}{D - D' \cancel{-1} \cancel{+1}}e^y$$

$$= -\frac{e^y}{D} \times \left(1 - \frac{D'}{D}\right)^{-1} 1.$$

$$= -\frac{e^y}{D} \times (1) = \boxed{-xe^y}$$

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(11) $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$

$(D - D' + 2)(D + D' - 1)z = e^{x-y} - x^2y$

$(D-D')(D+D')$
$+2(D+D') + D' - 2$
$-D$

$(D+D')(D-D'+2)$
$-(D-D'+2)$

C.F $\Rightarrow$ $e^{-2x}\phi_1(y+x) + e^{x}\phi_2(y-x)$.

PI $\Rightarrow$

$e^{x-y} \Rightarrow \quad \dfrac{1}{4\cdot(-1)}e^{x-y} = -\dfrac{1}{4}e^{x-y}$

$x^2y \Rightarrow \dfrac{-1}{2}\left(1+\dfrac{D-D'}{2}\right)^{-1}\left(1-(D+D')\right)^{-1} x^2y.$

$\Rightarrow \dfrac{-1}{2}\left[1 - \left(\dfrac{D-D'}{2}\right) + \left(\dfrac{D-D'}{2}\right)^2 - \left(\dfrac{D-D'}{2}\right)^3 + \cdots\right]\left[1 + (D+D') + (D+D')^2 + (D+D')^3\right]$

$\dfrac{-1}{2}\left[1 - \dfrac{D}{2} + \dfrac{D'}{2} + \dfrac{D^2}{4} + \dfrac{D'^2}{4} - \dfrac{DD'}{2} + \dfrac{3}{8}D^2D' + \cdots\right]\left[1 + D + D' + D^2 + D'^2 + 2DD' + 3D^2D' + \cdots\right]x^2y$

$\dfrac{-1}{2}\left[1 + D\left(-\dfrac{1}{2}+1\right) + D'\left(\dfrac{1}{2}+1\right) + DD'\left(-\dfrac{1}{2}+\dfrac{1}{2}-\dfrac{1}{2}+2\right) + D^2\left(-\dfrac{1}{2}+\dfrac{1}{4}+1\right) + D'^2\left(\dfrac{1}{2}+\dfrac{1}{4}+1\right)\right.$
$\left. D^2D'\left(\dfrac{3}{8} + 2 + \dfrac{1}{2} + \dfrac{1}{4} \cancel{-\dfrac{1}{2}} \cancel{+1}\right)\right] x^2y$

$\dfrac{-1}{2}\left(1 + \dfrac{D}{2} + \dfrac{3D'}{2} + DD' \times \dfrac{3}{2} + \dfrac{3}{4}D^2 + \dfrac{21}{8}D^2D'\right)x^2y.$

$\dfrac{-1}{2}\left(x^2y + xy + \dfrac{3}{2}x^2 + 3x + \dfrac{3}{2}y + \dfrac{21}{4}\right).$

$PI = -\dfrac{1}{4}e^{x-y} + \dfrac{1}{2}\left(x^2y + xy + \dfrac{3}{2}x^2 + 3x + \dfrac{3}{2}y + \dfrac{21}{4}\right)$ Ans

TOPIC 8 Reduction to Normal forms

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

① Classification.

- 2 variable

Check $S^2 - 4RT$ $\begin{cases} > 0 & \text{Hyperbolic} \\ = 0 & \text{Parabolic} \\ < 0 & \text{Elliptic} \end{cases}$

- 3 variable

– Check $A = \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix}$ → u are Coefficients

– Find eigenvalues of A.

- If all non-zero ⊕ ! Same sign → Hyperbolic
- One is zero → Parabolic
- All non-zero ⊕ Same sign → Ellipse

② Find characteristics.

- Consider λ - quadratic $R\lambda^2 + S\lambda + T = 0$
- Solutions of $\frac{dy}{dx} + \lambda_i(x, y) = 0$ are characteristics

classify $xy\,r - (x^2-y^2)\,s - xy\,t + py - qx = 2(x^2-y^2)$

$$S^2-4RT = (x^2-y^2)^2 + 4x^2y^2 = (x^2+y^2)^2 > 0. \quad \forall x, y$$

Hence hyperbolic at all points

- Classify $u_{xx} + u_{yy} + u_{zz} = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1,1,1$$

$\boxed{Q}$ Find characteristics of $y^2 r - x^2 t = 0$.

$(R\lambda^2 + S\lambda + T = 0)$

λ-equation $\equiv \quad y^2\lambda^2 + 0.\lambda - x^2 = 0 \Rightarrow \lambda = \pm \frac{x}{y}$

$$\frac{dy}{dx} + \frac{x}{y} = 0 \Rightarrow \boxed{y^2+x^2=C_1} \text{ — circles}$$

$$\frac{dy}{dx} - \frac{x}{y} = 0 \Rightarrow \boxed{y^2-x^2=C_2} \text{ — parabolas.}$$

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(3) Reduction of Hyperbolic equation

- given $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ — (1)

$\rightarrow$ Hyperbolic $\equiv S^2 - 4RT > 0$.

Step 1 $\equiv$ Solve $R\lambda^2 + S\lambda + T = 0 \Rightarrow \lambda_1, \lambda_2$

Step 2 $\equiv$ Solve $\frac{dy}{dx} + \lambda_1 = 0$, $\frac{dy}{dx} + \lambda_2 = 0$

Step 3 $\equiv$ Solution $u(x, y) = c_1$, $v(x, y) = c_2$ obtained

Step 4 $\equiv$ Find p, q, r, s, t in terms of u, v

$\boxed{\frac{\partial z}{\partial x} = \alpha\frac{\partial z}{\partial u} + \beta\frac{\partial z}{\partial v} \text{ Similarly.}}$

Step 5 $\equiv$ Substitute values of p, r, q, s, t in (1)

$\boxed{Q}$ Reduce $t - s + p - q(1 + 1/x) + \frac{z}{x} = 0$ to canonical form.

$-\lambda + 1 = 0 \Rightarrow \boxed{\lambda = 1.}$

1st $\Rightarrow$ $S^2 - 4RT = 1 - 4.0 = 1 > 0$ hyperbolic (Proceed)

$\frac{dy}{dx} + 1 = 0 \Rightarrow \boxed{\underset{u}{\underset{\parallel}{y+x}} = c_1}$

Select v such that $J(u,v) \neq 0$

$\boxed{v = x}$ $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$

$u = y + x$; $v = x$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ $\qquad q = \frac{\partial z}{\partial u} + 0$

$$s = \frac{\partial}{\partial y}(z_u + z_v) = \frac{\partial z_u}{\partial u}\frac{\partial u}{\partial y} + \cancel{\frac{\partial z_u}{\partial v}\frac{\partial v}{\partial y}} + \frac{\partial z_v}{\partial u}\frac{\partial u}{\partial y} + \cancel{\frac{\partial z_v}{\partial v}\frac{\partial v}{\partial y}}$$

$$s = z_{uu} + z_{vu}$$

$$t = \frac{\partial}{\partial y}(z_u) = z_{uu} + 0$$

$$\cancel{z_{uu}} - \cancel{z_{uu}} - z_{vu} + \cancel{z_u} + z_v - \cancel{z_u} - \frac{z_u}{x} + \frac{z}{x}$$

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Q $\dfrac{\partial^2 z}{\partial x^2} = x^2 \dfrac{\partial^2 z}{\partial y^2}$

$\lambda^2 - x^2 = 0 \Rightarrow \lambda = \pm x$

$\dfrac{dy}{dx} \pm x = 0 \Rightarrow \boxed{y \pm \dfrac{x^2}{2} = c} = u, v$

$p = \dfrac{\partial z}{\partial x} = z_u u_x + z_v v_x = x z_u - x z_v$

$q = \dfrac{\partial z}{\partial y} = z_u u_y + z_v v_y = z_u + z_v.$

$r = \dfrac{\partial}{\partial x}(x(z_u - z_v)) = z_u - z_v + x\left[z_{uu} u_x + z_{uv} v_x - z_{vu} v_x - z_{vv} u_x\right]$

$= z_u - z_v + x\left[x z_{uu} - x z_{uv} + x z_{vv} - x z_{vu}\right]$

$= z_u - z_v + x^2\left[z_{uu} + z_{vv} - 2z_{uv}\right]$

$t = \dfrac{\partial}{\partial y}(z_u + z_v) = z_{uu} + 2z_{uv} + z_{vv}$

Putting in ① $\Rightarrow$ $z_u - z_v + x^2\left[\cancel{z_{uu}} + \cancel{z_{vv}} - 2z_{uv}\right] = x^2\left[\cancel{z_{uu}} + \cancel{z_{vv}} + 2z_{uv}\right]$

$\boxed{z_u - z_v = 4x^2 z_{uv}.}$ Ans

(4) Reduction of parabolic equation $(S^2 - 4RT = 0)$

- Find one root of $R\lambda^2 + S\lambda + T = 0$

- 2nd equation $\Rightarrow \frac{dy}{dx} + \lambda_1 = 0$ gives $u = c_1$

$\boxed{v \text{ take such that } J(u,v) \neq 0}$

TIP: Candidates for $v \equiv x \pm y,\ x,\ y$
$x^n \pm y^n$

$\boxed{8}$ $r+2s+t=0.$

$\lambda^2+2\lambda+1=0 \Rightarrow \lambda=-1,-1.$

$dy-dx=0 \Rightarrow \underbrace{y-x}_{u}=c_1$

$\boxed{\text{Take } v=x}$

$J \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$

$p=\frac{\partial z}{\partial x} = -z_u + z_v \qquad q = \frac{\partial z}{\partial y} = z_u$

$r = z_{uu} + (z_{uv}) - z_{uv} + z_{vv}$

$s = -z_{uu} + z_{vu}$

$t = z_{uu}$

$\cancel{z_{uu}} - 2\cancel{z_{uv}} + z_{vv} + 2\cancel{z_{vu}} - 2\cancel{z_{uu}} + \cancel{z_{uu}} = 0$

$\frac{\partial^2 z}{\partial v^2}=0 \Rightarrow \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)=0 \Rightarrow \frac{\partial z}{\partial v} = \phi(u).$

$z = v\phi(u) + \psi(u)$

$z = x\phi(y-x) + \psi(y-x)$ Ans

$\boxed{5}$ $y^2 r - 2xy s + x^2 t = \frac{y^2}{x} p + \frac{x^2}{y} q.$

$y^2\lambda^2 - 2xy\lambda + x^2 = 0 \qquad \frac{2xy \pm \sqrt{0}}{2y^2} = \frac{x}{y}.$

$\frac{dy}{dx} + \frac{x}{y} = 0 \Rightarrow \underbrace{\frac{y^2}{2} + \frac{x^2}{2}}_{u} = c_1. \qquad J \begin{bmatrix} x & y \\ x & -y \end{bmatrix} \neq 0$

Take $\boxed{v = \frac{x^2}{2} - \frac{y^2}{2}}$

$p = \frac{\partial z}{\partial x} = x z_u + x z_v \qquad q = \frac{\partial z}{\partial y} = y z_u - y z_v$

$r = \frac{\partial}{\partial x}\left[x z_u + x z_v\right] = z_u + z_v + x\left[z_{uu} x + 2 z_{uv} x + z_{vv} x\right]$

$t = \frac{\partial}{\partial y}\left(y z_u - z_v\right) = z_u - z_v + y\left[y z_{uu} - 2y z_{uv} + y z_{vv}\right]$

$s = \frac{\partial}{\partial y}\left[x z_u + x z_v\right] = x\left[y z_{uu} - \cancel{y z_{uv}} + \cancel{y z_{uv}} - y z_{vv}\right]$

Substituting $\Rightarrow$

$\cancel{y^2 z_u} + \cancel{y^2 z_v} + \cancel{x^2 y^2} z_{uu} + x^2 y^2 \cancel{2 z_{uv}} + x^2 y^2 z_{vv}$

$-2\cancel{x^2 y^2} z_{uu} + 2x^2 y^2 z_{vv} + \cancel{x^2 z_u} - \cancel{x^2 z_v} \cancel{+ x^2 y^2 z_{uu}} - \cancel{x^2 y^2 2 z_{uv}}$

$+ x^2 y^2 z_{vv} = \cancel{y^2 z_u} + \cancel{y^2 z_v} + \cancel{x^2 z_u} - \cancel{x^2 z_v}.$

$z_{vv} = 0. \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0$

(5) Reducing Elliptic equation.

- $S^2 - 4RT < 0$

$$R\lambda^2 + S\lambda + T = 0$$

$\Rightarrow \lambda_1, \lambda_2$ be complex roots.

$$\frac{dy}{dx} + \lambda_1 = 0 \Rightarrow u = f_1(x,y) + i f_2(x,y)$$

$$\frac{dy}{dx} + \lambda_2 = 0 \Rightarrow v = f_1(x,y) - i f_2(x,y)$$

$\Rightarrow$ Consider $\alpha = f_1(x,y)$, $\beta = f_2(x,y)$ as previous u, v. and reduce.

Q $r + x^2 t = 0.$ (Only reduce)

$\lambda^2 + x^2 = 0 \Rightarrow \lambda = \pm ix.$

$\frac{dy}{dx} \pm ix = 0 \Rightarrow y \pm \frac{ix^2}{2} = 0.$

$\alpha = y \; ; \; \beta = \frac{x^2}{2}.$

$p = \frac{\partial z}{\partial x} = \cancel{z_\alpha} + x z_\beta \qquad q = \frac{\partial z}{\partial y} = z_\alpha$

$r = z_\beta + x^2 z_{\beta\beta} \qquad t = z_{\alpha\alpha}.$

$z_\beta + x^2 z_{\beta\beta} + x^2 z_{\alpha\alpha} \Rightarrow z_\beta + 2\beta(z_{\beta\beta} + z_{\alpha\alpha}) = 0.$

Q $xr + t = x^2.$ (Only reduce)

$x\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm \frac{i}{\sqrt{x}}.$

$\frac{dy}{dx} \pm \frac{i}{\sqrt{x}} = 0 \Rightarrow y \pm 2\sqrt{x}\, i = 0$

$\alpha = y \qquad \beta = 2\sqrt{x}.$

$p = \frac{\partial z}{\partial x} = \frac{z_\beta \cancel{\sqrt{x}}}{\sqrt{x}}. \qquad q = \frac{\partial z}{\partial y} = z_\alpha$

$r = -\frac{1}{2}\frac{z_\beta}{x^{3/2}} + \frac{1}{\sqrt{x}}\left[\frac{z_{\beta\beta}}{\sqrt{x}}\right] \qquad t = z_{\alpha\alpha}.$

$-\frac{1}{2}\frac{z_\beta}{\sqrt{x}} + \frac{z_{\beta\beta}}{\cancel{x}} + z_{\alpha\alpha} = x^2.$

$$\boxed{z_{\alpha\alpha} + z_{\beta\beta} = \frac{1}{\beta} z_\beta + \frac{\beta^4}{16}}$$ Ans

* Separation of variables

- $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ $\qquad u(x,0) = 6e^{-3x}$

Assume solution as $X(x)\,T(t)$.

$$X'T = 2XT' + XT \Rightarrow \frac{X' - X}{2X} = \frac{T'}{T} = k.$$

$$X' - X = 2kX \qquad\qquad \frac{T'}{T} = k \Rightarrow \boxed{T = c\,e^{kt}}$$

$$X' - (2k+1)X = 0 \Rightarrow X = C_2\,e^{(2k+1)x}$$

$$u(x,t) = C_1 C_2\, e^{(2k+1)x}\, e^{kt}$$

$$u(x,0) = C_1 C_2\, e^{(2k+1)x} = 6e^{-3x}$$

$$\Rightarrow 2k+1 = -3 \Rightarrow k = -2. \qquad C_1 C_2 = 6$$

So, $\boxed{u(x,t) = 6e^{-3x}e^{-2t}}$ Ans

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Wave equation

(1) A string is stretched to two points l apart. Motion is started by displacing string at $y = a\sin\frac{\pi x}{l}$ and released at $t=0$. Find displacement at x at time t.

This motion is given by

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}}$$

Conditions:
$y(0,t) = 0$, $y(l,t) = 0$.
$y(x,0) = a\sin\frac{\pi x}{l}$

$\boxed{\text{Also, } \frac{dy}{dt}(x,0) = 0}$ ← No initial velocity condition

Let Y be $X(x)T(t)$. Then we have

$$X(x)T''(t) = c^2 X''(x) T(t) \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k.$$

$$X'' - kX = 0 \qquad T'' - kc^2 T = 0.$$

As $D^2 - m^2 = 0$ can have real, imaginary, zero roots

$c_1 e^{k_1 x} + c_2 e^{k_2 x}$ $\quad$ $a_1 \cos + \sin x$

fits wave profile

Hence k must $-$ve or $k = -m^2$

$$X(x) = (c_1 \cos\sqrt{k}\,x + c_2 \sin\sqrt{k}\,x) \quad T(t) = (c_3 \cos c\sqrt{k}\,t + c_4 \sin c\sqrt{k}\,t)$$

$$y(x,t) = (c_1 \cos\sqrt{k}\,x + c_2 \sin\sqrt{k}\,x)(c_3 \cos c\sqrt{k}\,t + c_4 \sin c\sqrt{k}\,t)$$

$y(0,t) = 0 \Rightarrow c_1 = 0$, $\frac{\partial y}{\partial t}(x,0) = 0 \Rightarrow c_4 = 0$

$y(l,t) = 0 \Rightarrow c_2(\sin\sqrt{k}\,l)(c_3 \cos(c\sqrt{k}\,t)) = 0 \Rightarrow \sqrt{k}\,l = n\pi.$
$\Rightarrow \sqrt{k} = \frac{n\pi}{l}$

$$\boxed{y(x,t) = c_2 c_3 \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi c\, t}{l}\right)}$$

Using last condition
$c_2 c_3 \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l}$
$\hookrightarrow \boxed{c_2 c_3 = a;\ n = 1}$

② Same as ①. $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$.

Obtain $y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$.

$C_2 C_3 = b_n$ $\boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$

$$y(x,t) = \sum b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}.$$

$$y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_1^\infty b_n \sin \frac{n\pi x}{l}$$

$$y_0 \left(\frac{3\sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right) = b_1 \sin \frac{\pi x}{l} + b_3 \sin \frac{3\pi x}{l}.$$

$\boxed{b_1 = \frac{3y_0}{4} ;\ b_3 = -\frac{y_0}{4}}$ (IMPORTANT) $\boxed{b_2 = b_4 = b_5 = 0}$

Even if some constants here.
$\mu x(l-x) = \frac{\pi c}{l} \sum n b_n \sin\left(\frac{\pi x n}{l}\right)$
$\Downarrow$
$\frac{\pi c n}{l} b_n = \frac{2}{l} \int_0^l \mu x(l-x) \sin\left(\frac{n\pi x}{l}\right)$

Here not RHS

③ $y(x,0) = \mu x(l-x)$

$$\mu x(l-x) = \sum_1^\infty b_n^{(*)} \sin \frac{\pi x}{l}.$$

$$\Rightarrow b_n = \frac{2}{l} \int_0^l \mu x(l-x) \sin \frac{n\pi x}{l} dx.$$

$$= \frac{2\mu}{l} \left[\left(\frac{-l\,l x}{n\pi} \cos \frac{n\pi x}{l} \right)_0^l + \int_0^l \frac{l\,l}{n\pi} \cos \frac{n\pi x}{l} dx. - \int_0^l x^2 \sin \frac{n\pi x}{l} dx \right]$$

$$\left[(-1)^n \frac{l^3}{n\pi} + \frac{l^2\, l}{n\pi(n\pi)} \cancel{\sin \frac{n\pi x}{l}}\Big|_0^{l} \overset{0}{} - \frac{l x^2}{n\pi} \cos \frac{n\pi x}{l} \Big]_0^l - \int \frac{2xl}{n\pi} \cos \frac{n\pi x}{l} dx \right]$$

cancel

$$\frac{2l^3}{n\pi}(-1)^n - \left[\frac{2x\,l\,l}{n\pi(n\pi)} \cancel{\sin \frac{n\pi x}{l}} \right]_0^l - \int_0^l \frac{2l\,l}{(n\pi)^2} \sin \frac{n\pi x}{l} dx$$

$$\frac{2l^3}{n\pi}(-1)^n + \left(\frac{2l^2 l}{(n\pi)^3} \left(-\cos \frac{n\pi x}{l} \right) \right) \Bigg]_0^l \Rightarrow \frac{2l^3}{(n\pi)^3} \left(1 - (-1)^n\right).$$

$$\Rightarrow \frac{2\mu}{l}\left(\frac{2l^3}{(n\pi)^3}\left(1-(-1)^n\right)\right)$$

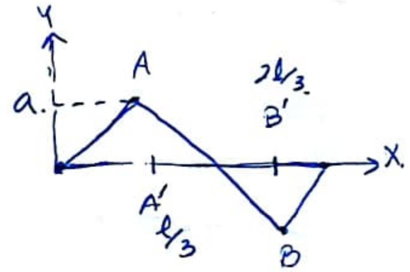
$$b_n \Rightarrow \frac{4\mu l^2}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$\boxed{y(x,t) = \frac{4\mu l^2}{\pi^3}\sum \frac{1-(-1)^n}{n^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.}$$

Can take $n = 2m-1$ as for even n 0

(4) Points of trisection of a string are pulled aside through same distance on opposite sides and string is released from rest. Find displacement at time t.

– Let stretched by a ($= AA'$)



$y(0,t)=0 \quad y(l,t)=0.$

$$\left(\frac{dy}{dt}\right)_{t=0} = 0$$

$$y(x,0) = \begin{cases} \frac{3a}{l}x & 0 \le x \le l/3 \\ 3a\left(1-\frac{2x}{l}\right) & l/3 \le x \le 2l/3 \\ 3a\left(\frac{x}{l}-1\right) & \frac{2l}{3} \le x \le l. \end{cases}$$

$$\rightarrow y(x,t) = \sum b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$

$$y(x,0) = \sum b_n \sin\frac{n\pi x}{l}.$$

$$b_n = \frac{2}{l}\left[\int_0^{l/3}\frac{3ax}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} 3a\left(1-\frac{2x}{l}\right)\sin\frac{n\pi x}{l} + \int_{2l/3}^{l} 3a\left(\frac{x}{l}-1\right)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{18a}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

(5) D'Alembert's Solution.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$U = x + ct \quad V = x - ct.$ Let $y = y(x+ct, x-ct)$
$= y(U, V)$

Transform in terms of u, v.

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial U}\frac{\partial U}{\partial t} + \frac{\partial y}{\partial V}\frac{\partial V}{\partial t} = c\,y_u - c\,y_v \,; \quad \frac{\partial y}{\partial x} = y_u + y_v$$

$$\frac{\partial^2 y}{\partial t^2} = c\left[y_{uu}\cdot c + y_{uv}(-c) - y_{vu}\cdot c + c\,y_{vv}\right] = c^2\left[y_{uu} - 2y_{uv} + y_{vv}\right]$$

$$\frac{\partial^2 y}{\partial x^2} = y_{uu} + 2y_{uv} + y_{vv}$$

Substituting

$$\cancel{y_{uu}} - 2y_{uv} + \cancel{y_{vv}} = \cancel{y_{uu}} + 2y_{uv} + \cancel{y_{vv}} \Rightarrow y_{uv} = 0$$

$$\frac{\partial^2 y}{\partial u \partial v} = 0 \Rightarrow \frac{\partial y}{\partial u} = f(u) \Rightarrow \boxed{y = \phi(u) + \psi(v)}$$

$$\boxed{y(x,t) = \phi(x+ct) + \psi(x-ct)}$$ ← General D'Alemberts Solution.

Put conditions for solution

Heat Equation

- $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{c^2 T} = (-\lambda^2)$ -ve needed.

We need a solution decreasing with time. So T must be e^{-kt}

$$y(x,t) = (C_1 \cos \lambda x + C_2 \sin \lambda x)\, e^{-c^2 \lambda^2 t}.$$

---

① Initial Conditions are non-zero

- Split solution $y = y_s + y_t$

Steady state (y_s) / transient (y_t).

$y_s \rightarrow \dfrac{\partial y}{\partial t} = 0 \Rightarrow \dfrac{\partial^2 y_s}{\partial x^2} = 0$ and find $y_s = Ax + b$ using given conditions

Now y_t conditions:
- Boundary $= 0$
- Initial $\equiv y(x,0) - y_s$

② Insulation $\Rightarrow \dfrac{\partial u}{\partial x} = 0$ (Not $\dfrac{\partial u}{\partial t} = 0$)

① Solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Subject to

(1) U is finite for $t \to \infty$

(2) $\frac{\partial U}{\partial x} = 0$ for $x=0$, $x=l$

(3) $u = lx - x^2$ for $t=0$ between $x=0$ and $x=l$

$$\frac{T'(t)}{T(t)\alpha^2} = \frac{X''}{X} = \lambda .$$

$\lambda = 0 . \Rightarrow \quad (C_1 x + C_2) C_3 .$

$\lambda < 0 \Rightarrow . \quad (C_4 \cos\sqrt{\lambda}x + C_5 \sin\sqrt{\lambda}x) C_6 e^{-k^2\alpha^2 t} .$
$=(-k^2)$

$\lambda > 0 \Rightarrow \quad T(t) = e^{k^2\alpha^2 t} \to \infty$ as $t \to \infty$ (Rejected)

Using (2) $\Rightarrow$ $C_1 = 0$, $C_5 = 0$. , $\sqrt{\lambda} = \frac{n\pi}{l}$.

→ * $C_2 C_3$ and $C_4 \cos \frac{n\pi x}{l} C_6 \, e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}}$

are two contenders

So, general solution = Sum of above

$$u \equiv a_0 + \sum a_n \cos(n\pi x/l) \, e^{-n^2\pi^2\alpha^2 t/l^2}$$

$$lx - x^2 = a_0 + \sum a_n \cos(n\pi x/l) .$$

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\, dx = \boxed{\frac{l^2}{6}}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2) \cos\frac{n\pi x}{l}\, dx = \boxed{-\frac{4l^2}{n^2\pi^2}}$$ when n is even

(2) Insulated rod l, with ends at A (0°C), B(100°C) at steady state. If B is suddenly reduced to 0° find temp at distance x from A at time t.

Let Eq^n be $\rightarrow$ $\boxed{\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}}$

• Steady State

Heat flow is independent of time

So, heat equation $\Rightarrow$ $\dfrac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = ax + b$

$u = 0$ at $x = 0$, $u = 100$ for $x = l \Rightarrow b = 0,\ a = \frac{100}{l}$.

$\boxed{u_{(x,0)} = \dfrac{100x}{l}}$ $\leftarrow$ Initial condition.

$u(0,t) = 0 \qquad u(l,t) = 0$ $\leftarrow$ Boundary Condition

• $u(x,t) = \left(\cancel{C_1 \cos}^{\,0} \sqrt{k}\,x + C_2 \sin \sqrt{k}\,x\right) e^{-kc^2 t}$.

$u(0,t) = 0 \Rightarrow C_1 = 0, \quad u(l,t) = 0 \Rightarrow \sqrt{k} = \frac{n\pi}{l}$

$u(x,t) = \sum a_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2\pi^2 c^2 t}{l^2}}$

$\quad u(x,0) = \sum a_n \sin \frac{n\pi x}{l} = \frac{100x}{l}$.

$a_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l}\, dx = \left(\frac{-200}{n\pi} \cos n\pi\right)$

$a_n = \boxed{\dfrac{200}{n\pi} (-1)^{n+1}}$

---

* If change is A to 20°C, B to 80°C

$\rightarrow u(x,t) = \underbrace{u_s(x)}_{\text{steady state}} + u_t(x,t)$

$u_s(0) = 20$
$u_s(l) = 80$
$u_s(x) = 20 + \frac{60x}{l}$

$u_t(x,0) = u(x,0) - u_s(x) \qquad u_t(0) = u_t(l) = 0$
$= \frac{100x}{l} - \left(\frac{60x}{l} + 20\right)$

③ Bar of length 1, one end at 10°C and other end is insulated. $y(x,0) = 1-x \quad 0<x<1.$

$y(0,t) = 10$

Use,

$$\frac{1}{C^2}\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad \frac{\partial y(1,t)}{\partial x} = 0.$$

Let $y(x,t) = X(x)\,T(t).$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{c^2 T(t)} = -\lambda^2/\lambda^2/0.$$

① $(c_1 \cos \lambda x + c_2 \sin \lambda x)\, e^{-\lambda^2 c^2 t}$

② $(c_1 e^{\lambda x} + c_2 e^{-\lambda x})\, e^{\lambda^2 c^2 t}.$

③ $(ax+b)\,c$

Solution $\Rightarrow y = y_1 + y_2$ — transient . $\boxed{y = kx + c}$

Steady state .

$\Downarrow$

0 at $x = l$.

At steady state $\Rightarrow \frac{\partial^2 y}{\partial x^2} = 0 \Rightarrow \frac{\partial y}{\partial x} = k$

$\Rightarrow y(x,t) = c_1 \Rightarrow c_1 = 10$

$\boxed{y_1 = 10}$

To find $y_2 \equiv \boxed{y_2(0,t) = 0 \;;\; \frac{\partial y(1,t)}{\partial x} = 0.}$

As converges for $t \to \infty \Rightarrow$ ② ✗

③ $\Rightarrow ac = 0 \;,\; ac = -1 \,,\; bc = 2 \Rightarrow$ Not possible

So ① ✓.

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$y_2(0,t)=0 \Rightarrow C_1 = 0$

$\frac{\partial y_2}{\partial x}(1,t)=0 \Rightarrow \left(-\cancel{C_1\lambda \sin\lambda x} + \lambda C_2 \cos\lambda x\right) e^{0} = 0$

$$\lambda \cancel{x} = \frac{(2n+1)\pi}{2} \Rightarrow \lambda = \frac{(2n+1)\pi}{2}$$

$$y_2 = \sum a_n \sin\left(\frac{2n+1}{2}\right)\pi x \, e^{-\left(\frac{2n+1}{2}\right)^2 \pi^2 c^2 t}$$

$$y_2(x,0) = 1-x = \sum a_n \sin\left(\frac{2n+1}{2}\right)\pi x.$$

$$a_n = \frac{2}{1}\int_0^1 (1-x)\sin\frac{(2n+1)}{2}\pi x \, dx$$ Ans

Laplace Equation

- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$$\frac{X''}{X} = -\frac{Y''}{Y} = k$$

① $p^2 \rightarrow (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py)$

② $-p^2 \rightarrow (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$

③ $0 \rightarrow (C_1 x + C_2)(C_3 y + C_4)$.

* Select as per boundary conditions

Q ① → An infinite plate with an end at u_0 (breadth $= \pi$) other edges at 0°. Find temp at steady state

- $u(0, y) = 0$
- $u(\pi, y) = 0$.
- $u(x, \infty) = 0$
- $u(x, 0) = u_0 \quad 0 \le x < \pi$

→ $u(x,\infty) = 0$ → Not by ③. , $u(0,y) = 0$ → Not by ①

So, Only ② is possible.

$$(C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$u(0, y) = 0 \Rightarrow \boxed{C_1 = 0}$. $\quad u(\pi, y) = 0 \Rightarrow \boxed{p = \frac{n\pi}{\pi} = n}$

$u(x, \infty) = 0 \Rightarrow \boxed{C_3 = 0}$

$$u(x,y) = C_2 C_4 \sin nx \, e^{-ny} = \sum b_n \sin nx \, e^{-ny}$$

$$u_0 = u(x, 0) = \sum b_n \sin nx \Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx = \frac{2u_0}{n\pi}[1 - (-1)^n]$$

Rejection ① ⇒ $C_1 + C_2 = 0$. $\quad (x, \infty) \Rightarrow$

$C_3 (C_1 e^{p\pi} - C_1 e^{-p\pi}) = 0 \Rightarrow C_3 = 0$

Trivial X.

② $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $u(0,y) = u(l,y) = u(x,0) = 0$
$u(x,a) = \sin \frac{n\pi x}{l}$.

Candidates ≡

① $\cancel{x}.(c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

② $(c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$

③ $(c_9 x + c_{10})(c_{11} y + c_{12})$

① ⇒ $c_1 + c_2 = 0$ $\cancel{(c_1 e^{pl} + c_2 e^{-pl}) \cancel{c_3} = 0}$
$(c_1 e^{pl} - c_1 e^{-pl})(c_3 \cos py + c_4 \sin py) = 0$
$\forall y = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) = 0 \Rightarrow$ Trivial ⓧ

③ ⇒ $c_{10} = 0$, $l c_9 (c_{11} y + c_{12}) = 0 \Rightarrow c_9 = 0 \Rightarrow X = 0 \Rightarrow$ Trivial ⓧ

So ② holds ⇒

$u(0,y) = 0 \Rightarrow c_5 = 0$, $u(l,y) = 0 \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$.

$u(x,0) = 0 \Rightarrow \left(c_6 \sin \frac{n\pi}{l} x\right)(c_7 + c_8) = 0 \Rightarrow c_7 = -c_8$
($c_6 = 0$ not else trivial)

$$u(x,y) = \sum b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$\Rightarrow \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l} \left(e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}\right)$$

$$\boxed{b_n = \frac{1}{e^{n\pi a/l} - e^{-n\pi a/l}}}$$

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(3) Laplace in Polar Coordinates

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Assume $R(r)\,\Theta(\theta)$

$$r^2 R''\Theta + rR'\Theta + R\Theta'' = 0$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = k.$$

$$r^2R'' + rR' - kR = 0 \quad ; \quad \Theta'' + k\Theta = 0.$$

→ $k = p^2\ (>0) \Rightarrow (c_1 r^{p} + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$

→ $k = -p^2\ (<0) \Rightarrow [c_5 \cos(p\log r) + c_6 \sin(p\log r)](c_7 e^{p\theta} + c_8 e^{-p\theta})$

→ $k = 0 \Rightarrow (c_9 \log r + c_{10})(c_{11}\theta + c_{12})$

(Select as per boundary conditions)

($r \to \log r$) in previous forms

$r \to \log r$

$c_1 \cos\lambda r + c_2 \sin\lambda r$ $c_1 \cos(\lambda \log r) + c_2 \sin(\lambda \log r)$

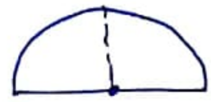
$c_1 e^{\lambda r} + c_2 e^{-\lambda r}$ $c_1 e^{\lambda \log r} + c_2 e^{-\lambda \log r} = c_1 r^{\lambda} + c_2 r^{-\lambda}$

$c_1 r + c_2$ $c_1(\log r) + c_2$

④ Diameter of semi-circular plate at 0°C and temp along semi-circular boundary is:

$$u(a,\theta) = \begin{cases} 50\,\theta & \theta \in [0, \pi/2] \\ 50(\pi-\theta) & \theta \in [\pi/2, \pi] \end{cases}$$

Find steady state temperature.



0°C

→ Boundary Conditions

$$u(r,0)=0 \qquad u(r,\pi)=0.$$

$$u(a,\theta) = \begin{cases} 50\,\theta \\ 50(\pi-\theta). \end{cases}$$

→ Candidates

① $(C_1 r^p + C_2 r^{-p})(C_3 \cos p\theta + C_4 \sin p\theta)$

② $(C_5 \cos(p \log r) + C_6 \sin(p \log r))(C_7 e^{p\theta} + C_8 e^{-p\theta})$

③ $(C_9 \log r + C_{10})(C_{11}\theta + C_{12})$

For $r \to 0$, must be ~~bounded~~ → ③ X, ② X

$u=0$ when $r=0$

~~Only ②.~~

~~$u(r,0)=0 \Rightarrow C_3=0.$~~

~~$u(r,\pi)=0 \Rightarrow p\pi=n\pi \Rightarrow p=n.$~~

$\boxed{u=0 \text{ when } r=0}$ ← Extra

Extra ⇒ $\boxed{u=0,\ r=0}$

Only ① ⇒ $C_3=0$, $p=n$, $C_2=0$.

$$u(r,\theta) = \sum C_n r^n \sin n\theta$$

$$u(a,\theta) = \sum C_n a^n \sin n\theta = \begin{cases} 50\,\theta & \theta \le \frac{\pi}{2} \\ 50(\pi-\theta) & \theta \le \pi \end{cases}$$

Let $B_n = \boxed{C_n a^n}$

$$B_n = \frac{2}{\pi}\left(\int_0^{\pi/2} 50\,\theta \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} 50(\pi-\theta) \sin n\theta \, d\theta\right)$$

$$C_n = \frac{200}{\pi n^2} \sin \frac{n\pi}{2} \times \frac{1}{a^n}.$$

$$\boxed{u(r,\theta) = \sum \frac{200}{\pi n^2} \sin \frac{n\pi}{2} \cdot \frac{r^n}{a^n} \sin n\theta}$$ Ans

1999 Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-x}\cos y$ which $\to 0$ as $x \to \infty$ and $\cos y$ at $x=0$

① First get Particular Solution ⇒

$$\frac{1}{D^2+D'^2} e^{-x}\cos y = e^{-x}\frac{1}{(D-1)^2+D'^2}\cos y$$

$$= e^{-x}\frac{1}{\cancel{D^2}+\cancel{D'^2}-2D+\cancel{1}}\cos y. = \boxed{\frac{e^{-x}}{(-2)}x\cos y.}$$

⇓

→ 0 as $x \to \infty$ ⇓

② General Solution ≡ CF + PI

To Solve $\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0$ using given conditions

$z(x,y) \to 0$ as $x \to \infty$

$z(0,y) = \cos y$

Let $z = X(x)\,Y(y)$.

$$-\frac{Y''(y)}{Y(y)} = +\frac{X''(x)}{X(x)} = n^2 \text{ as } z(0,y) = \cos y.$$

$$X(x) = Ae^{nx} + Be^{-nx} \qquad Y(y) = C\cos ny + D\sin ny.$$

- As $x \to \infty$ $z(x,y) \to 0 \Rightarrow A = 0$
- $z(0,y) = \cos y. \Rightarrow \cos y = E\cos ny + F\sin ny$

$n=1,\ F=0,\ E=1.$ (Equality holds)

So, $CF = e^{-x}\cos y.$

$$\boxed{Z = e^{-x}\cos y - \frac{x}{2}e^{-x}\cos y.}$$