Mains Test Series - 2020

Test - 10, Paper - II, Batch - I

Answer key (full syllabus)

1(a) Give an example of a ring in which some prime ideal not a maximal ideal.

Sol^n: Let $R = Z + Z$ be the direct sum of the ring of integers. Let $I = 0 + Z$ be an ideal of R. Now R/I is a prime ideal since R/I is isomorphic to Z which is an integral domain.

But I is not a maximal ideal since R/I is isomorphic to Z. Since Z is not a field we obtain I is not a maximal ideal.

An integral domain R is said to be a Euclidean ring if for every $a \neq 0$ in R there is defined a non-negative integer $d(a)$ such that

(1) For all $a, b \in R$ both non-zero, $d(a) \leq d(ab)$.

(2) For any $a, b \in R$ both non-zero, $\exists\, t, r \in R$ such that $a = tb + r$ where either $r = 0$ (or) $d(r) < d(b)$.

1(b). If R is a ring with unit element 1 and ϕ is a homomorphism of R into an integral domain R' such that kernel of ϕ is different from R, prove that $\phi(1)$ is the unit element of R'.

Sol'n: observe first that if $\phi(1)=0$,

then for any element $a \in R$

we have $\phi(a) = \phi(a.1)$

$= \phi(a)\phi(1)$

$= 0.$

But by assumption we have

$\ker\phi \neq R$

So we obtain $\phi(1) \neq 0$.

Then for any $y \in R'$

we have $\phi(1)y = \phi(1.1)y$

$= \phi(1)\phi(1)y.$

Then we have $\phi(1)y = \phi(1)(1)y$

and so $\phi(1)(y - \phi(1)y) = 0.$

Since R' is an integral domain and $\phi(1) \neq 0$

we obtain $\phi(1)y = y \;\; \forall\, y \in R'.$

Hence $\phi(1)$ is an identity element of R'.

1(C) Examine the convergence of $\int_0^1 \left(\log \frac{1}{x}\right)^n dx$.

Sol'n: $$\int_0^1 \left(\log \frac{1}{x}\right)^n dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx + \int_a^1 \left(\log \frac{1}{x}\right)^n dx \quad \text{—(1)}$$

where $0<a<1$.

0 and 1 are the points of infinite discontinuity of the integrals on the right.

Let $f(x) = \left(\log \frac{1}{x}\right)^n$

Convergence of $\int_0^a \left(\log \frac{1}{x}\right)^n dx$ at 0.

$$\lim_{x\to 0+} \left(\log \frac{1}{x}\right)^n = 1 \text{ if } n=0$$
$$= 0 \text{ if } n<0$$

$\therefore$ The integral is proper if $n\le 0$.

0 is the only point of infinite discontinuity if $n>0$.

For $n>0$, take $g(x)=\frac{1}{x^p}$, $0<p<1$.

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} x^p \left(\log \frac{1}{x}\right)^n = 0.$$

Also $\int_0^a g(x)\, dx$ converges, since $0<p<1$.

$\therefore$ $\int_0^a f(x)\, dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx$ converges.

Combining all cases, $\int_0^a \left(\log \frac{1}{x}\right)^n dx$ converges for all n.

Convergence of $\int_a^1 \left(\log \frac{1}{x}\right)^n dx$ at 1.

The integral is proper if $n\ge 0$ and 1 is the only point of infinite discontinuity if $n<0$.

For $n<0$, take $g(x) = \frac{1}{(1-x)^{-n}}$

$$\lim_{x\to 1^-}\frac{f(x)}{g(x)} = \lim_{x\to 1^-}\frac{\left(\log \frac{1}{x}\right)^n}{1-x} = 1$$

which is non-zero and finite.

But $\int_a^1 g(x)\,dx = \int_0^1 \frac{dx}{(1-x)^{-n}}$ is convergent if $-n<1$

i.e., if $n>-1$.

$\therefore$ By Comparison test, $\int_a^1 f(x)dx = \int_a^1 \left(\log \frac{1}{x}\right)^n dx$ is convergent if $-1<n<0$.

Hence from ①, $\int_0^1 \left(\log \frac{1}{x}\right)^n dx$ is convergent if $-1<n<0$.

1(e) construct the dual of the primal problem:

Maximize $Z = 2x_1 + x_2 + x_3$:

Subject to the Constraints. $x_1 + x_2 + x_3 \geq 6$,

$3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 + 3x_2 - 6x_3 = 1$ and $x_1, x_2, x_3 \geq 0$

Sol^n: Max $Z = 2x_1 + x_2 + x_3$ ——— ①

Changing the Constraint 2 from $=$ to $\leq$:-

$$3x_1 - 2x_2 + 3x_3 \leq 3 \Rightarrow 3x_1 - 2x_2 + 3x_3 \geq 3,$$

$$-3x_1 + 2x_2 - 3x_3 \leq -3,$$

Changing the Constraint 3 from $=$ to $\leq$:-

$$-4x_1 + 3x_2 - 6x_3 \geq 1 \Rightarrow -4x_1 + 3x_2 - 6x_3 \leq 1,$$

$$4x_1 - 3x_2 + 6x_3 \leq -1$$

$\therefore$ The Constraints are

$$-x_1 - x_2 - x_3 \leq -6$$

$$3x_1 - 2x_2 + 3x_3 \leq 3$$

$$-3x_1 + 2x_2 - 3x_3 \leq -3$$

$$-4x_1 + 3x_2 - 6x_3 \leq 1$$

$$4x_1 - 3x_2 + 6x_3 \leq -1$$

Converting to duality :-

Min $Z_1 = -6y_1 + 3y_2 - 3y_3 + y_4 - y_5$

S.C $-y_1 + 3y_2 - 3y_3 - 4y_4 + 4y_5 \geq 2$

$-y_1 - 2y_2 + 2y_3 + 3y_4 - 3y_5 \geq 1$

$-y_1 + 3y_2 - 3y_3 - 6y_4 + 6y_5 \geq 1$

$y_1, y_2, y_3, y_4, y_5 \geq 0$

It can be written as :-

$$\text{Min } Z_1 = -3y_1 + 3(y_2 - y_3) + 1(y_4 - y_5)$$

$$\text{S.C } -y_1 + 3(y_2 - y_3) - 4(y_4 - y_5) \geq 2$$

$$-y_1 - 2(y_2 + y_3) + 3(y_4 - y_5) \geq 1$$

$$-y_1 + 3(y_2 - y_3) - 6(y_4 - y_5) \geq 1$$

Let $y_2 - y_3 = u_1$, $y_4 - y_5 = u_2$

$$\therefore \text{Min } Z = -6y_1 + 3u_1 + u_2$$

$$\text{S.C } -y_1 + 3u_1 - 4u_2 \geq 2$$

$$-y_1 - 2u_1 + 3u_2 \geq 1$$

$$-y_1 + 3u_1 - 6u_2 \geq 1$$

where $y_1 \geq 0$, u_1, u_2 are unrestricted.

2(a)(i) Give an example of a non-abelian group H of order 3^5 such that each element of G is of order 3. Also, give an example of a non-abelian group H of order 54 such that H has an element of order 12.

(ii) Let ϕ be a group homomorphism from Z_{30} to Z_6 such that $\text{Ker}(\phi) = \{0, 6, 12, 18, 24\}$. Prove that ϕ is onto. Also, find all possiblities for $\phi(1)$.

Sol'n: (i) Let $H = Z_3 \oplus Z_3 \oplus G$, where G is the group.

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in Z_3 \right\}$$ is non-abelian group of order 27, under matrix multiplication such that each non identity element of G has order 3.

Since G is non-abelian, we conclude that H is non-abelian.

It is clear that each element of H is of order 3.

For the second part, let $H = Z_4 \oplus G$, where G is the group $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} ; a, b, c \in Z_3 \right\}$. Then H is a non-abelian group and $\text{ord}(H) = 54$.

Let $a = (1, B)$, where B is a non identity element of G. Then by theorem (Let $H_1, \ldots, H_n$ be finite groups, and let $d = (h_1, h_2 \ldots h_n) \in D = H_1 \oplus H_2 \cdots \oplus H_n$. Then $\text{Ord}(d) = \text{Ord}((h_1, h_2 \ldots h_n)) = \text{lcm}(\text{ord}(h_1), \text{ord}(h_2) \ldots \text{ord}(h_n))$.

2(a)(ii) Sol^n: Since $Z_{30}/\text{Ker}(\phi) \cong \phi(Z_{30}) \subset Z_6$

by theorem (Let ϕ be a group homomorphism from a group G to group H. Then Ker(ϕ) is a normal subgroup of G and $G/\text{Ker}(\phi) \cong \phi(G)$ (the image of G under ϕ).) and

$\text{Ord}(\text{Ker}(\phi)) = 5$,

we conclude that $\text{ord}(Z_{30}/\text{Ker}(\phi)) = \text{Ord}(\phi(Z_{30})) = 30/5 = 6$.

Hence $\phi(Z_{30}) = Z_6$.

Thus, ϕ is onto. Now since $Z_{30} = \langle 1 \rangle$ is cyclic and a group homomorphism from Z_{30} to Z_6 is determined by $\phi(1)$ and ϕ is onto, we conclude that $\text{ord}\,\phi(1) = 6$.

Hence there are $\phi(6) = 2$ elements in Z_6 of order 6, namely, 1 and 5. Thus, all possibilities for $\phi(1)$ are: (1) $\phi(1) = 1$. (2) $\phi(1) = 5$.

2(b) (i) If continuous function $f(x):[0,1]\to[0,1]$, then there exists a point $c\in[0,1]$, such that $f(c)=c$.

(ii) Prove that $\int_0^{\pi/2}\left(\frac{\pi}{2}-x\right)\tan x\,dx=\frac{1}{2}\pi\log 2$.

Sol'n: (i) Let $f(x)-x\neq 0\ \forall x\in[0,1]$,

then $f(x)-x$ must be +ve for some α in $[0,1]$

and it must be −ve for some β in $[0,1]$,

otherwise $f(x)$ will not be in $[0,1]$.

Thus, continuous $f(x)-x$ assumes positive to negative values from α to β in $[0,1]$ and

$\therefore$ by the intermediate value theorem, there exists a point c lying between α and β, and so in $[0,1]$ such that

$f(c)-c=0$, i.e. $f(c)=c$ for some $c\in[0,1]$.

(ii) Sol'n: Let

$$I_1=\int_0^{\pi/2}\left(\frac{\pi}{2}-x\right)\tan x\,dx \quad \text{—— (1)}$$

$$I_1=\int_0^{\pi/2}\left(\frac{\pi}{2}-\frac{\pi}{2}+x\right)\tan\left(\frac{\pi}{2}-x\right)dx$$

$$=\int_0^{\pi/2}x\cot x\,dx \qquad \left[\int uv=u\int v-\int u'\int v\right]$$

$$I_1=\left[x\int\cot x\,dx-\int\log(\sin x)\,dx\right]_0^{\pi/2}$$

Now
$\lim_{x\to0}x\log\sin x=\lim_{x\to0}\frac{\log\sin x}{1/x}$
$=\lim_{x\to0}\frac{\frac{1}{\sin x}\cos x}{-1/x^2}$ ($\frac{\infty}{\infty}$ form)
$=\lim_{x\to0}-\frac{x^2\cos x}{\sin x}$ ($\frac{0}{0}$ form)
$=\lim_{x\to0}\frac{-2x\cos x+x^2\sin x}{\cos x}$
$=\frac{0}{1}=0$

$$\Rightarrow I_1=\left[x\log(\sin x)\right]_0^{\pi/2}-\int_0^{\pi/2}\log(\sin x)\,dx$$

$$=\left[\frac{\pi}{2}\log 1-\lim_{x\to0}x\log\sin x\right]-\int_0^{\pi/2}\log\sin x\,dx$$

$$=0-0-\int_0^{\pi/2}\log\sin x\,dx$$

$$\Rightarrow I_1=-I \quad \text{where } I=\int_0^{\pi/2}\log\sin x\,dx$$

then $I=\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-x\right)dx$ $\quad\boxed{\because\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx}$

$$= \int_0^{\pi/2} \log \cos x \, dx \longrightarrow (3)$$

Adding (2) & (3) we get

$$2I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log\left\{\frac{\sin 2x}{2}\right\} = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx.$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2\, (x)_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log \sin 2x - \frac{\pi}{2} \log 2$$

Now put $2x = t$, so that $2\,dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$$\therefore 2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x \, dx - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2}\log 2$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2}\log 2$$

$$2I = I - \frac{\pi}{2}\log 2.$$

$$\Rightarrow I = -\frac{\pi}{2}\log 2$$

$$\therefore \text{from (1)}\quad I_1 = -I = -\left(-\frac{\pi}{2}\log 2\right) = \frac{\pi}{2}\log 2.$$

$$\therefore \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right)\tan x \, dx = \frac{\pi}{2}\log 2$$

2(c) → Evaluate the following integrals, justifying your procedures. For (c) and (d) you should also state why the integrals is well defined (i.e. independent of the path taken).

(i) $\int_C \frac{2\,dz}{z^2-1}$, where C is the circle with radius $\frac{1}{2}$, centre 1, positively oriented.

(ii) $\int_C \left(e^z + \frac{1}{z}\right)dz$, where C is the lower half of the circle with radius 1, centre 0, -vely oriented.

(iii) $\int_C z e^{z^2} dz$ (iv) $\int_C \cosh z\,dz$.

Sol'n: (i) Notice that

$$\frac{2}{z^2-1} = \frac{1}{z-1} - \frac{1}{z+1}$$

Then $\int_C \frac{2}{z^2-1} dz = \int_C \frac{1}{z-1} dz - \int_C \frac{1}{z+1} dz$

on the other hand $\int_C \frac{1}{z-1} dz = 2\pi i$

by Cauchy integral formula, $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz$

on the other hand, since $\frac{1}{(z+1)}$ is analytic on and inside C, then $\int_C \frac{1}{z+1} dz = 0$.

by Cauchy's theorem.

$$\therefore \int_C \frac{2}{z^2-1} dz = 2\pi i$$

(ii) Notice that the integrand $f(z) = e^z - \frac{1}{z}$ is analytic on C.

The function $F(z) = e^z - \log z$

serves as an antiderivative of $f(z)$. Hence $\log z$ is a branch of the logarithm chosen with the branch cut on the +ve imaginary axis. i.e,

$$\operatorname{Log} z = \ln r + i\theta, \quad \left(r>0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}\right).$$

Thus

$$\int_C \left(e^z - \frac{1}{z}\right) dz = \Big[\left(e^z - \log z\right)\Big]_1^{-1} = \frac{1}{e} - \pi i - e + 2\pi i$$

$$= \frac{1}{e} - e + \pi i.$$

(iii) Since the integrand $f(z) = ze^{z^2}$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is

$$F(z) = \frac{e^{z^2}}{2}.$$

Thus $$\int_C ze^{z^2} dz = \frac{e^{z^2}}{2}\Big]_0^i = \frac{1}{2e} - \frac{1}{2}.$$

(iv) Since the integrand $f(z) = \cosh z$ is analytic, the integral is path independent. An antiderivative of $f(z)$ is $F(z) = \sinh z$.

Thus $$\int_C \cosh z \, dz = \Big[\sinh z\Big]_{\pi i}^{2\pi i} = \sinh(2\pi i) - \sinh(\pi i)$$

$$= 0$$

3(a) Let p be an odd prime number. Show that $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in Z_p \right\}$ is a non-abelian group of order p^3, under matrix multiplication, such that each non identity element of G has order p.

Sol'n: We will show that G is a group with p^3 elements.

Now let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Then the entry in the first row & third column of AB is 2. But the entry in the first row and third column of BA is 1.

Hence $AB \neq BA$.

Thus G is non-abelian.

Let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, $a, b, c \in Z_p$. Then

$$A^p = \begin{bmatrix} 1 & pa & p(p-1)/2\,ac + pb \\ 0 & 1 & pc \\ 0 & 0 & 1 \end{bmatrix}.$$

But $pa = p(p-1)/2\,ac + pb = pc = 0$ in Z_p.

Hence $A^p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3(b)(i). Show that the infinite product $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^{\alpha}}\right)$ is convergent if $\alpha > \frac{1}{2}$.

Sol'n: The given product is

$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^{\alpha}}\right) = \prod_{n=2}^{\infty}(1+a_n) \quad \text{where } a_n = \frac{(-1)^n}{n^{\alpha}}.$$

Now

$$\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^{\alpha}} \text{ converges if } \alpha > 0 \text{ (by Leibnitz's test)}$$

Also

$$\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty}\frac{1}{n^{2\alpha}} \text{ converges if } 2\alpha > 1$$

i.e, if $\alpha > \frac{1}{2}$.

Hence the given product converges if $\alpha > \frac{1}{2}$.

3(b)(ii) Show that the series for which the sum of first n terms $f_n(x) = \frac{nx}{1+n^2x^2}$; $0 \le x \le 1$.

Cannot be differentiated term by term at $x=0$. What happens at $x \neq 0$?

Sol^n: Given $f_n(x) = \frac{nx}{1+n^2x^2}$; $0 \le x \le 1$

For $f_n(x)$ to be term by term differentiable $\lim_{n\to\infty} f_n'(x)$ must be equal to $f'(x)$, where $f(x)$ is a limit function.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2}$$

$$= \lim_{n\to\infty} \frac{x}{2nx^2} \quad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$f(x) = 0 \Rightarrow f'(x) = 0 \quad \forall x \in \mathbb{R}.$$

Differentiating f_n with respect to x:

$$f_n'(x) = \frac{(1+n^2x^2)n - nx(2n^2x)}{(1+n^2x^2)^2}$$

$$= \frac{n+n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2}$$

$$\boxed{f_n'(x) = \frac{n-n^3x^2}{(1+n^2x^2)^2}}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{n-n^3x^2}{(1+n^2x^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0 ; & x \neq 0 \\ \infty ; & x = 0 \end{cases}$$

Thus, $f_n(x)$ is not differentiable term by term at $x=0$ and differentiable term by term at $\forall x \neq 0$.

3(C) → Use Simplex method to solve the following LPP.

Maximise $Z = 6x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$.

Obtain an alternative optimal basic feasible solution if it exists.

Sol'n: The objective function of the given LPP is of maximization type.

Now we write the given LPP is standard form

$$\text{Max } Z = 6x_1 + 4x_2 + 0S_1 + 0S_2 + 0S_3 - M.A$$

Subject to

$$2x_1 + 3x_2 + S_1 + 0S_2 + 0A = 30$$

$$3x_1 + 2x_2 + 0S_1 + S_2 + 0A = 24$$

$$x_1 + x_2 + 0S_1 + 0S_2 - S_3 + A = 3$$

$$x_1, x_2, S_1, S_2, S_3, A \geq 0$$

where S_1, S_2 are slack variables.

S_3 is the surplus variable and A is the artificial variable.

Now the IBFS is $x_1 = x_2 = S_3 = 0$ (non-basic)

$S_1 = 30, S_2 = 24;\ A = 3$ (basic)

for which $Z = -3M$

Now we put the above information in the Simple tableau.

	C_j	6	4	0	0	0	$-M$		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	A	b	θ
0	S_1	2	3	1	0	0	0	30	30/2
0	S_2	3	2	0	1	0	0	24	24/3
$-M$	A	1	1	0	0	-1	1	3	$3/1 \rightarrow$
$z_j = \Sigma C_B a_{ij}$		$-M$	$-M$	0	0	M	$-M$	$-3M$	
$C_j = c_j - z_j$		$6+M \uparrow$	$M+4$	0	0	$-M$	0		

From the above table

x_1 is the entering variable, A is the outgoing variable and omit its column in the next simplex and (1) is the key element and all other elements in its column equal to zero.

Then the revised simplex table is

	C_j	6	4	0	0	0		
C_B	basis	x_1	x_2	S_1	S_2	S_3	b	θ
0	S_1	0	1	1	0	2	24	12
0	S_2	0	-1	0	1	3	15	$5 \rightarrow$
6	x_1	1	1	0	0	-1	3	$-ve$
$z_j = \Sigma C_B a_{ij}$		6	6	0	0	-6		
$C_j = c_j - z_j$		0	-2	0	0	$+6 \uparrow$		

From the above table S_3 is entering variable and S_2 is the outgoing variable and 3 is the key element and make it unity and all other elements in its column equal to zero.

Then the revised simplex table is

	c_j	6	4	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b
0	s_1	0	$5/3$	1	$-2/3$	0	14
0	s_3	0	$-1/3$	0	$1/3$	1	5
6	x_1	1	$2/3$	0	$1/3$	0	8
$z_j = C_B a_{ij}$		6	4	0	2	0	
$C_j = c_j - z_j$		0	0	0	-2	0	

Since all $C_j \leq 0$, an optimum solution has been reached. The optimum basic feasible solution is

$$\boxed{x_1 = 8,\ x_2 = 0,\ z_{max} = 48}$$

From the optimum tableau we observe that the net evaluation corresponding to non-basic variable x_2 is zero. This is an indication for the existence of an alternate basic feasible solution.
Thus we can bring x_2 into basis in place of s_1 or s_3.
The resulting new basic solutions will also be an optimum solution.
Introducing x_2 and dropping s_1, the alternative optimum table is:

	c_j	6	4	0	0	0	
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b
4	x_2	0	1	$3/5$	$-2/5$	0	$42/5$
0	s_3	0	0	$1/5$	$1/5$	1	$39/5$
6	x_1	1	0	$-2/5$	$3/5$	0	$12/5$
$z_j = \Sigma C_B a_{ij}$		6	4	0	2	0	48
$C_j = c_j - z_j$		0	0	0	-2	0	

$\therefore$ An alternative optimum basic feasible solution is

$$\boxed{x_1 = 12/5,\ x_2 = 42/5,\ z_{max} = 48}$$

4(a). Find all $c \in Z_3$ such that $Z_3[x]/\langle x^3+cx^2+1\rangle$ is a field.

Sol'n: $Z_3[x]/\langle x^3+cx^2+1\rangle$ is a field

$\Leftrightarrow$ the ideal generated by x^3+cx^2+1 is maximal ideal $\Leftrightarrow$ x^3+cx^2+1 is irreducible in $Z_3[x]$.

Let $f(x) = x^3+cx^2+1$

Since $f(x)$ has degree 3, if $f(x)$ is reducible then one of the factor should have degree one i.e. $f(x)$ should have a root in Z_3.

$f(0) = 1 \neq 0 \pmod 3$

$f(1) = 2+c$

$f(2) = 9+c \equiv c \pmod 3$

So, if $c \neq 0$ and $c \not\equiv 1$ in Z_3, then $f(x)$ is irreducible. So for $c \equiv 2$, $f(x)$ is irreducible.

4(b) Let $f(x) = \begin{cases} \frac{x^2}{2}+4 & \text{if } x \geq 0 \\ -\frac{x^2}{2}+2 & \text{if } x<0 \end{cases}$

Is f Riemann integrable in the interval $[-1,2]$? why? Does there exist a function g such that $g'(x)=f(x)$? Justify your answer.

Sol^n: Given $f(x) = \begin{cases} \frac{x^2}{2}+4 & \text{if } x \geq 0 \\ -\frac{x^2}{2}+2 & \text{if } x<0 \end{cases}$

To check the continuity of $f(x)$ at $x=0$

$$\lim_{x\to 0^-} f(x+h) = \lim_{h\to 0^-} \left(\frac{0+h}{2}\right)^2+4 = 4$$

$$\lim_{x\to 0^-} f(x+h) = \lim_{h\to 0^-} -\left(\frac{0+h}{2}\right)^2+2 = 2$$

and $f(0)=4$

$\therefore f(x+h) = f(0) \neq f(x-h)$

Hence $f(x)$ is discontinuous at $x=0$ which is finite and countable. Therefore $f(x)$ is Riemann integrable in $[-1,2]$.

Suppose, $g(x)$ exist such that $g'(x)=f(x)$

$$g(x) = \begin{cases} \int_0^x f(x)\,dx \; ; \text{ if } x \geq 0 \\ \int_{-1}^x f(x)\,dx \; ; \text{ if } x<0 \end{cases}$$

$$g(x) = \begin{cases} \int_0^x \left(\frac{x^2}{2}+4\right)dx \; ; \; 0 \leq x < 2 \\ \int_{-1}^x \left(-\frac{x^2}{2}+2\right)dx \; ; \; -1 \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ \left(-\frac{x^3}{6}+2x\right)_{-1}^{x} & ; \ -1 \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ -\frac{x^3}{6}+2x+\frac{11}{6} & ; \ -1 \le x < 0 \end{cases}$$

But, $\lim_{x \to 0} g(x)$ does not exist thus not continuous at $x=0$.

$\therefore$ there does not exist a function $g(x)$ such that

$$g'(x) = f(x).$$

Hence the result.

4(c) (i) using residue theorem, evaluate $\int_C \frac{e^z dz}{z(z-1)^2}$ where C is circle $|z|=2$.

Sol'n: Let $I = \int_C \frac{e^z dz}{z(z-1)^2}$, where C is circle $|z|=2$.

Here centre is $z=0$ and radius $=2$.

$\therefore z=0,1$ are poles lying within C.

$z=0$ is a simple pole.

$$Res(z=0) = \lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0} z\left[\frac{e^z}{z(z-1)^2}\right]$$

$$= \lim_{z\to 0}\frac{e^z}{(z-1)^2} = \frac{e^0}{(0-1)^2} = 1$$

$z=1$ is a pole of order 2. Take $f(z) = \frac{\phi(z)}{(z-1)^2}$

where $\phi(z) = \frac{e^z}{z}$.

$$Res(z=1) = \frac{\phi'(1)}{1!}, \quad \phi'(z) = \frac{e^z.z - 1.e^z}{z^2}$$

$$\phi'(1) = \frac{e^1.1-e^1}{1^2} = 0$$

$\therefore Res(z=1) = 0$.

By Cauchy's residue theorem,

$$I = 2\pi i \text{ (sum of residues within C)}$$

$$= 2\pi i\left[Res(z=0) + Res(z=1)\right]$$

$$= 2\pi i[1+0]$$

$$= 2\pi i$$

4(c)(iii) Let $f(z) = u+iv$ be an analytic function. Find $f(z)$ as a function of z), when $2u+3v = 13(x^2-y^2)+2x+3y$.

Sol^n: Let $f(z) = u+iv$

we have $2u+3v = 13(x^2-y^2)+2x+3y$

$$2\frac{\partial u}{\partial x}+3\frac{\partial v}{\partial x} = 13(2x)+2 = 26x+2 \quad \text{--- (1)}$$

and $$2\frac{\partial u}{\partial y}+3\frac{\partial v}{\partial y} = 13(-2y)+3 = -26y+3$$

$$\Rightarrow 2\left(-\frac{\partial v}{\partial x}\right)+3\left(\frac{\partial u}{\partial x}\right) = -26y+3 \quad \text{--- (2)}$$

(∵ by Cauchy Riemann eq^n $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

solving (1) & (2), we get

$$\frac{\partial u}{\partial x} = 4x-6y+1 = \phi_1(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x} = 6x+4y = \phi_2(x,y)$$

Now $$f'(z) = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$= \phi_1(z,0)+i\phi_2(z,0) \quad \text{by Milne's method.}$$

$$\therefore f(z) = \int\left[\phi_1(z,0)+i\phi_2(z,0)\right]dz+C$$

$$= \int\{(4z+1)+i(6z)\}dz+C$$

$$= 2z^2+z+i3z^2+C$$

$$= (2+3i)z^2+z+C.$$

4(d) (i) Consider the problem of assigning four operations to four machines. The assignment costs in rupees are given below. operator O_1 cannot be assigned to machine M_3. Also O_3 cannot be assigned to M_4. Find the optimal assignment.

operator	M_1	M_2	M_3	M_4
O_1	5	5	–	2
O_2	7	4	2	3
O_3	9	3	5	–
O_4	7	2	6	7

(ii) Suppose that in problem (i) a fifth machine M_5 is made available, its respective assignment costs to the four operators are 2, 1, 2 & 8. The new machine M_5 replaces an existing one if the replacement can be justified economically. Reformulate the problem as an assignment model & find the optimal solution.

Soln: The problem indicates that two assignments are impossible. Therefore, we put a very large cost 10 in the cells (1, 3) and (2, 4), then proceed as usual. and then performing first row reduction and column reduction, we get Table I.

5	5	10	2
7	4	2	3
9	3	5	10
7	2	6	7

3	3	8	0
5	2	0	1
6	0	2	7
5	0	4	5

Table.II

0	3	8	0
2	2	0	1
3	0	2	7
2	0	4	5

Draw the minimum no. of horizontal and vertical lines to cover all zeros of table 1, which shows that the no. of lines $N = 3 < 4 = m$, order of matrix.

The smallest element among the uncovered elements by the lines is 1. we subtract 1 each uncovered element and add 1 to the element at the point of intersection of lines, given by table II

0	4	9	0
1	2	0	0
2	0	2	6
1	0	4	4

$\longrightarrow$

Table III

~~0~~	5	9	0
	3	0	~~0~~
1	0	1	5
0	~~0~~	3	3

Again $N = 3 < = 4 = m$
Now smallest element among uncovered elements = 3.
perform the same operation which will give modified cost matrix in table III.

Now $N = 4 = m$ = order of matrix.
an optimum assignment has been reached.
The assignment is given by
$O_1 \to M_4$, $O_2 \to M_3$, $O_3 \to M_2$, $O_4 \to M_1$
Minimum cost $= 2+2+3+7 = 14$.

(iii)

5	5	10	2	2
7	4	2	3	1
9	3	5	10	2
7	2	6	7	8
0	0	0	0	0

table (i)

In the 2nd part, it is an unbalanced Assignment problem. so O_5 is taken with zero costs. which is given by in Table (i)

performing first row reduction and then column reduction, we get

3	3	8	0	0
6	2	1	2	0
7	1	3	8	0
5	0	4	5	6
0	0	0	0	0

Table (i)

As $N = 4 < 5 = m$, order of matrix.

subtract smallest uncovered element 1 from each uncovered element and add 1 to the element at the point of intersection of lines.

2	3	7	0	0
5	3	0	2	0
6	1	2	8	0
4	0	3	5	6
0	1	0	1	1

→ Therefore the new assignment is given by

$O_1 \to M_4$, $O_2 \to M_3$

$O_3 \to M_5$, $O_4 \to M_2$

$O_5 \to M_1$ (idle)

∴ Minimum cost $= 2+2+2+2 = 8$

Clearly it is justified economically if M_5 replaces M_1 (which is existing in part A)

5(a). Find the surface which intersects the surfaces of the system $z(x+y)=c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2=1$, $z=1$.

Sol^n: The given system of surfaces is given by

$$f(x,y,z) \equiv \frac{z(x+y)}{3z+1} = c \quad \text{——} (1).$$

$$\therefore \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \quad \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \quad \frac{\partial f}{\partial z} = (x+y)\frac{1.(3z+1)-z.3}{(3z+1)^2}$$

$$= \frac{x+y}{(3z+1)^2}$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

$$\Rightarrow z(3z+1)p + z(3z+1)q = x+y \quad \text{——} (2)$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \quad \text{——} (3)$$

Taking the first two fractions of (3), we get $dx-dy=0$ so that $x-y=C_1$ —— (4)

Choosing $x, y, -z(3z+1)$ as multipliers, each fraction of (3)

$$= \frac{x\,dx + y\,dy - z(3z+1)dz}{0}$$

$$\therefore x\,dx + y\,dy - 3z^2dz - z\,dz = 0$$

Integrating $\frac{1}{2}x^2 + \frac{1}{2}y^2 - 3\left(\frac{z^3}{3}\right) - \frac{1}{2}z^2 = \frac{1}{2}C_2$

$$\Rightarrow x^2+y^2-2z^3-z^2 = C_2 \quad \text{——} (5)$$

Hence any surface which is orthogonal to (1) has equation of the form $x^2+y^2-2z^3-z^2 = \phi(x-y)$, ϕ being an arbitrary function —— (6)

In order to get the desired surface passing through the circle $x^2+y^2=1$, $z=1$, we must choose $\phi(x-y) = -2$. Thus, the required particular surface is $x^2+y^2-2z^3-z^2=-2$.

5(b) Solve $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x+y)$.

Sol^n The given equation can be written as -

$$(D+D')(D-2D'+2)z = e^{2x+3y} + xy + \sin(2x+y) \quad \text{--- (1)}$$

$\therefore$ CF $= \phi_1(y-x) + e^{-2x}\phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y} = \frac{1}{(2+3)(2-6+2)} e^{2x+3y}$$

$$= -\frac{1}{10} e^{2x+3y}$$

P.I. corresponding to xy

$$= \frac{1}{(D+D')(D-2D'+2)} xy$$

$$= \frac{1}{D(1+D'/D)\cdot 2\{1+(D/2-D')\}} xy$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left\{1+\left(\frac{D}{2}-D'\right)\right\}^{-1} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\cdots\right)\left\{1-\left(\frac{D}{2}-D'\right)+\left(\frac{D}{2}-D'\right)^2+\cdots\right\} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\cdots\right)\left(1-\frac{D}{2}+D'-DD'+\cdots\right) xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\cdots\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2y}{2}-\frac{xy}{2}+\frac{x^2}{2}-x-\frac{x^3}{6}+\frac{x^2}{4}\right]$$

$$= \frac{x^2y}{4} + \frac{3x^2}{8} - \frac{xy}{4} - \frac{x}{2} - \frac{x^3}{12}$$

P.I. corresponding to $\sin(2x+y)$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{-2^2 - (2\cdot 1) - 2(-1^2) + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \sin(2x+y)$$

$$= \frac{D-D'}{2(D^2 - D'^2)} \sin(2x+y)$$

$$= \frac{1}{2} \frac{1}{-2^2 - (-1^2)} (D-D') \sin(2x+y)$$

$$= -\left(\frac{1}{6}\right)(D-D') \sin(2x+y)$$

$$= -\left(\frac{1}{6}\right)\left[D \sin(2x+y) - D' \sin(2x+y)\right]$$

$$= -\left(\frac{1}{6}\right)\left[2\cos(2x+y) - \cos(2x+y)\right].$$

$$= -\left(\frac{1}{6}\right)\cos(2x+y).$$

The required solution is $z = CF + PI$

$$z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - \frac{1}{10}e^{2x+3y}$$

$$+ \frac{1}{4}x^2y + \frac{3}{8}x^2 - \frac{1}{4}xy - \frac{1}{2}x - \frac{1}{12}x^3 - \frac{1}{6}\cos(2x+y)$$

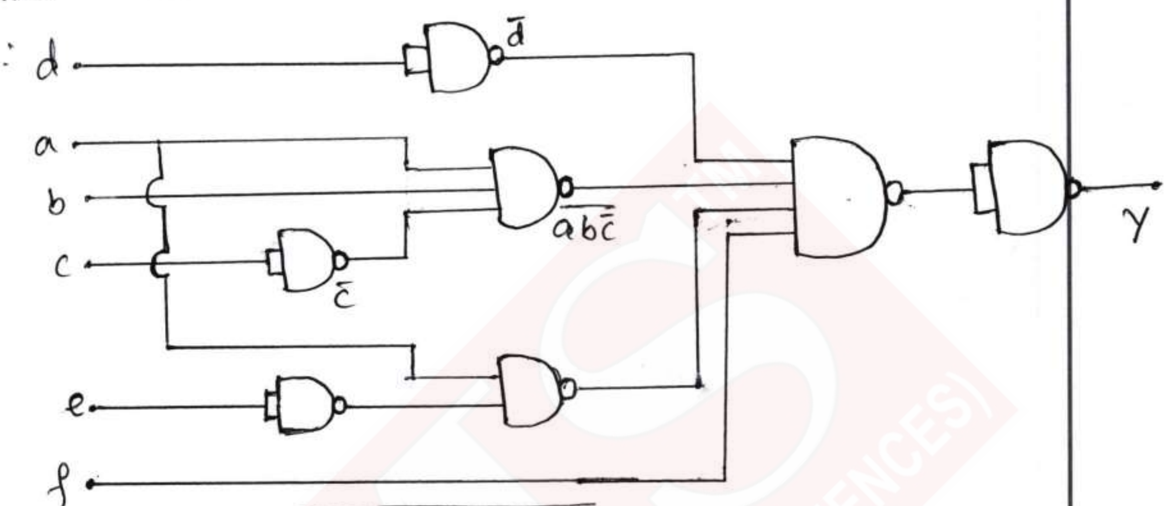
5(d) (i) Realize the following expression by using NAND gates only:

$$g = (\bar{a}+\bar{b}+c)\,d\,(\bar{a}+e)f$$

where $\bar{x}$ denotes the complement of x.

(ii) Find the decimal equivalent of $(357.32)_8$.

Sol'n:



$$Y = \overline{\overline{(\overline{ab\bar{c}})\,\bar{d}\,(\overline{a\bar{e}})f}}$$

$$= \overline{\overline{(\bar{a}+\bar{b}+c)\,\bar{d}\,(\bar{a}+e)f}}$$

$$= \overline{\overline{(\bar{a}+\bar{b}+c)}+\bar{\bar{d}}+\overline{(\bar{a}+e)}+\bar{f}}$$

$$= (\bar{a}+\bar{b}+c)\,\bar{d}\,(\bar{a}+e)f$$

(ii) $(357.32)_8 = 3\times 8^2 + 5\times 8^1 + 7\times 8^0 + 3\times 8^{-1} + 2\times 8^{-2}$

$$= 3\times 64 + 40 + 7 + \tfrac{3}{8} + \tfrac{2}{8^2}$$

$$= 192 + 40 + 7 + 0.375 + 0.03125$$

$$= (239.40625)_{10}$$

5(e) Find the stream lines and paths of the particles for the two dimensional velocity field:

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Sol: we have

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Step-I. To determine stream lines.

Streamline are the solution of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Putting the value This $\frac{(1+t)}{x} dx = \frac{dy}{y} = \frac{dz}{0}$

$$\Rightarrow \left(\frac{1+t}{x}\right) dx = \frac{dy}{y}, \quad \frac{dy}{y} = \frac{dz}{0}$$

$$\Rightarrow (1+t)\log x = \log y + \log a, \quad dz = 0$$

$$\Rightarrow x^{1+t} = ay, \quad z = b.$$

These two equations represent stream lines.

Step-II. To determine path lines.

Path lines are the solutions of

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0.$$

This

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = dt, \quad dz = 0.$$

Integrating,

$$\log x = \log(1+t) + \log a,$$

$$\log y = t - \log b, \quad z = c.$$

or $\quad x = a(1+t), \; (y/b) = e^t, \; z = c$

or $\quad y = b e^{[(x/a)-1]}, \; z = c.$

These two equations represent path lines.

6(a) Reduce the equation

$$y^2\left(\frac{\partial^2 z}{\partial x^2}\right) - 2xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) + x^2\left(\frac{\partial^2 z}{\partial y^2}\right)$$

$$= \left(\frac{y^2}{x}\right)\left(\frac{\partial z}{\partial x}\right) + \left(\frac{x^2}{y}\right)\left(\frac{\partial z}{\partial y}\right)$$ to canonical form and hence solve it.

Sol'n: Re-writing, $y^2 r - 2xys + x^2 t - (y^2/x)p - (x^2/y)q = 0$ — ①

Comparing ① with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

Here $R = y^2$, $S = -2xy$, $T = x^2$ so that $S^2 - 4RT = 0$,

showing that ① is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$y^2\lambda^2 - 2xy\lambda + x^2 = 0 \Rightarrow (y\lambda - x)^2 = 0$

$\Rightarrow \lambda = x/y, x/y$

The corresponding characteristic equation is $\frac{dy}{dx} + \frac{x}{y} = 0$

$\Rightarrow x\,dx + y\,dy = 0$

$\Rightarrow \frac{x^2}{2} + y^2/2 = C_1$

Choose $u = x^2/2 + y^2/2$ and $v = \frac{x^2}{2} - \frac{y^2}{2}$ — ②

where we have chosen $v = \frac{x^2}{2} - \frac{y^2}{2}$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -2xy \neq 0.$$

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$, using ② — ③

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$, using ② — ④

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by ③}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right), \text{ using } (2) \quad \text{——} (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ by } (4)$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{——} (6).$$

and $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left\{y\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)\right\}$ by (4)

$$= y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) \quad \text{——} (7)$$

using (3), (4), (5), (6) and (7) in (1) we finally get

$$4x^2y^2\left(\partial^2 z/\partial v^2\right) = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial v^2} = 0 \quad \text{——} (8)$$

which is the required canonical form.

Integrating (8) partially w.r.t 'v' $\dfrac{\partial z}{\partial v} = \phi(u)$ —— (9)

Integrating (9) partially w.r.t 'v', $z = v\phi(u) + \psi(v)$

$$z = \left[(x^2-y^2)/2\right]\phi\left\{(x^2+y^2)/2\right\} + \psi\left\{(x^2+y^2)/2\right\}$$

$$\Rightarrow z = (x^2-y^2)F(x^2+y^2) + G(x^2+y^2),$$

F, G being arbitrary constants.

6(b)(i) Given the following data, evaluate $f(3)$ using Lagrange's interpolating polynomial.

x	1	2	5
$f(x)$	1	4	10

(ii) Solve the following system of equations.

$$2x_1 - x_2 = 7,\ -x_1 + 2x_2 - x_3 = 1,\ -x_2 + 2x_3 = 1$$

using Gauss-Seidel method of iteration and perform the first-five iterations. The exact solution is $(6\ 5\ 3)^T$.

Sol'n: (i) Using Lagrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$\therefore f(3) = \frac{(3-2)(3-5)}{(1-2)(1-5)}(1) + \frac{(3-1)(3-5)}{(2-1)(2-5)}(4) + \frac{(3-1)(3-2)}{(5-1)(5-2)}(10)$$

$$= 6.4999$$

(ii) The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}(7+x_2) \\ x_2 &= \tfrac{1}{2}(1+x_1+x_3) \\ x_3 &= \tfrac{1}{2}(1+x_2) \end{aligned}\right\} \text{---} (1)$$

By Gauss-Seidal method, system (1) can be written as

$$x_1^{k+1} = \tfrac{1}{2}\left(7 + x_2^{(k)}\right)$$

$$x_2^{k+1} = \tfrac{1}{2}\left(1 + x_1^{(k+1)} + x_3^{k}\right)$$

$$x_3^{K+1} = \frac{1}{2}\left(1 + x_2^{K+1}\right) \text{ where } K = 0,1,2,3\ldots$$

Now taking $x^{(0)} = 0$, we obtain the following iterations.

K=0:

$$x_1^{(1)} = \frac{1}{2}(7+0) = 7/2 = 3.5$$

$$x_2^{(1)} = \frac{1}{2}(1+3.5+0) = \frac{4.5}{2} = 2.25$$

$$x_3^{(1)} = \frac{1}{2}(1+2.25) = \frac{1}{2}(3.25) = 1.625$$

K=1:

$$x_1^{(2)} = \frac{1}{2}\left(7 + x_2^{(1)}\right) = \frac{1}{2}(7+2.25) = \frac{9.25}{2} = 4.625$$

$$x_2^{(2)} = \frac{1}{2}\left(1 + x_1^{(2)} + x_3^{(1)}\right) = \frac{1}{2}(1+4.625+1.625) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1 + x_2^{(2)}\right) = \frac{1}{2}(1+3.625) = 2.3125$$

K=2:

$$x_1^{(3)} = \frac{1}{2}\left(7 + x_2^{(2)}\right) = \frac{1}{2}(7+3.625) = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}\left(1 + x_1^{(3)} + x_3^{(2)}\right) = \frac{1}{2}(1+5.3125+2.3125) = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}\left(1 + x_2^{(3)}\right) = \frac{1}{2}(1+4.3125) = 2.6563.$$

K=3:

$$x_1^{(4)} = \frac{1}{2}\left(7 + x_2^{(3)}\right) = \frac{1}{2}(7+4.3125) = 5.6563$$

$$x_2^{(4)} = \frac{1}{2}\left(1 + x_1^{(4)} + x_3^{(3)}\right) = \frac{1}{2}(1+5.6563+2.6563) = 4.6563$$

$$x_3^{(4)} = \frac{1}{2}\left(1 + x_2^{(4)}\right) = \frac{1}{2}(1+4.6563) = 2.8282$$

K=4:

$$x_1^{(5)} = \frac{1}{2}\left(7 + x_2^{(4)}\right) = \frac{1}{2}(7+4.6563) = 5.8282$$

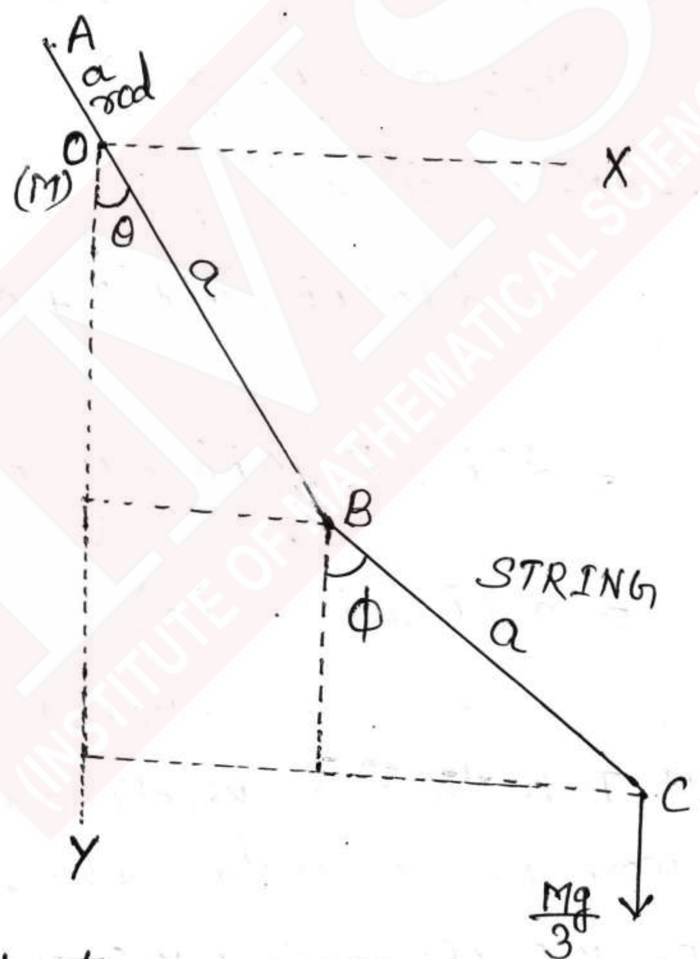
$$x_2^{(5)} = \frac{1}{2}\left(1 + x_1^{(5)} + x_3^{(4)}\right) = \frac{1}{2}(1+5.8282+2.8282) = 4.8282$$

$$x_3^{(5)} = \frac{1}{2}\left(1 + x_2^{(5)}\right) = \frac{1}{2}(1+4.8282) = 2.9141.$$

which is good approximation to the exact solution $(6\ 5\ 3)^T$.

6(c) A uniform straight rod of length $2a$ is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of length a to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{(a/g)}$.

Sol:



Let M be the mass of the rod AB of length $2a$, BC the string and $M/3$ the mass at C.

At time t, let the rod and the

String make angle θ and ϕ to the vertical respectively.

Referred to the middle point O of the rod AB as origin. horizontal and vertical lines OX and OY through O as axes, the co-ordinates of C are given by

$$x_C = a(\sin\theta + \sin\phi),$$

$$y_C = a(\cos\theta + \cos\phi).$$

$$\therefore \; v_C^2 = \dot{x}_C^2 + \dot{y}_C^2$$

$$= a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$$

$$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)]$$

$$= a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

($\because \theta, \phi$ are small)

If T be the total kinetic energy and W the work function of the system, then

$T =$ K.E. of the rod + K.E. of the particle at C

$$= \left[\frac{1}{2}M\cdot\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}Mv_0^2\right] + \frac{1}{2}\left(\frac{1}{3}M\right)v_C^2$$

$$= \frac{1}{6}Ma^2\dot{\theta}^2 + \frac{1}{6}Ma^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

$$= \frac{1}{6}Ma^2(2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) \qquad (\because v_0 = 0)$$

and $W = mg \cdot 0 = \frac{1}{3}Mg \cdot y_c + C$

$$= \frac{1}{3}Mga(\cos\theta + \cos\phi) + C$$

Lagranges θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\text{down30 dedel}\,\dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e. $$\frac{d}{dt}\left[\frac{1}{6}Ma^2(4\dot{\theta} + 2\dot{\phi})\right] - 0 = \frac{1}{3}Mga(-\sin\theta)$$

$$= -\frac{1}{3}Mga\theta, \qquad (\because \theta \text{ is small})$$

or $2\ddot{\theta} + \ddot{\phi} = -c\theta$, ——— ①

(where $c = g/a$)

And Lagranges ϕ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

i.e. $\frac{d}{dt}\left[\frac{1}{6}Ma^2(2\dot{\phi}+2\dot{\theta})\right]-0=\frac{1}{3}Mga(-\cos\phi)$

$$=-\frac{1}{3}Mga\phi$$

$\because \phi$ is small

or $\ddot{\theta}+\ddot{\phi}=-c\phi,$ ——— (2)

(where $c=g/a$)

Equations (1) and (2) can be written as

$(2D^2+c)\theta+D^2\phi=0$ and

$D^2\theta+(D^2+c)\phi=0$

eliminating ϕ between these two equations, we get

$$\left[(D^2+c)(2D^2+c)-D^4\right]\theta=0$$

or $(D^4+3cD^2+c^2)\theta=0$ ——— (3)

Let the solution of (3) be given by

$\theta=A\cos(pt+B)$

$\therefore D^2\theta=-p^2\theta$ and $D^4\theta=p^4\theta.$

Substituting in (2), we get

$$(p^4 - 3cp^2 + c^2)\theta = 0$$

or $p^4 - 3cp^2 + c^2 = 0$ $\quad (\because \theta \neq 0)$

$$\therefore p^2 = \frac{3c \pm \sqrt{(9c^2 - 4c^2)}}{2}$$

$$= \left(\frac{3 \pm \sqrt{5}}{2}\right) c$$

$$= \left(\frac{3 \pm \sqrt{5}}{2}\right) \frac{g}{a}$$

$\therefore$ one value of p^2 is $p_1^2 = \left(\frac{3 - \sqrt{5}}{2}\right) \frac{g}{a}$

$\therefore$ one period of principal oscillation

$$= \frac{2\pi}{p_1} = 2\pi \sqrt{\left[\frac{2}{3 - \sqrt{5}} \cdot \frac{a}{g}\right]}$$

$$= 2\pi \sqrt{\left[\frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \cdot \frac{a}{g}\right]}$$

$$= 2\pi \sqrt{\left[\frac{6 + 2\sqrt{5}}{4} \cdot \frac{a}{g}\right]}$$

$$= 2\pi \sqrt{\left[\left(\frac{\sqrt{5}+1}{2}\right)^2 \frac{a}{g}\right]}$$

$$= (\sqrt{5}+1)\pi \sqrt{(a/g)}.$$

7(a) Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the parabola $x=0$, $y^2 = z$.

Sol^n Given equation is $pq = z \rightarrow (1)$

We are to find its integral surface which passes through the given parabola.

$$x=0,\ y^2 = z \longrightarrow (2)$$

Rewriting (2) in parametric form, we have

$x=0$, $y=\lambda$, $z=\lambda^2$ where λ being the parameter $\rightarrow (3)$

Let initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = 0$, $y_0 = y_0(\lambda) = \lambda$, $z_0 = z_0(\lambda) = \lambda^2$ $\rightarrow (4A)$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfies (1), so

$$p_0 q_0 = z_0 \quad (\text{or}) \quad p_0 q_0 = \lambda^2 \text{ by } (4A)$$

Also we have $z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$

So that $2\lambda = p_0 \times 0 + q_0 \times 1$ (or) $q_0 = 2\lambda$ by (4A) $\rightarrow (6)$

Solving (5) and (6) $\quad p_0 = \lambda/2$, $q_0 = 2\lambda \rightarrow (4B)$

Collecting relations (4A) and (4B) together, initial values x_0, y_0, z_0, p_0, q_0 are given by.

$x_0 = 0$, $y_0 = \lambda$, $z_0 = \lambda^2$, $p_0 = \lambda/2$, $q_0 = 2\lambda$

where $t = t_0 = 0$ $\rightarrow$ (7')

Rewriting (1') let $f(x, y, z, p, q) = pq - z = 0$ $\rightarrow$ (8')

The usual characteristic equations of (8') are given by

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = q \rightarrow (9)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = p \rightarrow (10)$$

$$\frac{\partial z}{\partial t} = p\left(\partial f/\partial p\right) + q\left(\partial f/\partial q\right) = 2pq \rightarrow (11')$$

$$\partial p/\partial t = -(\partial f/\partial y) - p(\partial f/\partial z) = p \rightarrow (12)$$

$$\frac{\partial q}{\partial t} = -(\partial f/\partial y) - q(\partial f/\partial z) = q \rightarrow (13')$$

from (9) and (13') $\left(\frac{dx}{dt}\right) - \left(\frac{\partial q}{dt}\right) = 0$ so that

$x - q = c_1$ $\rightarrow$ (14') where c_1 is an arbitrary constant. using initial values (7'), (14') gives

$x_0 - q_0 = c_1$ (or) $0 - 2\lambda = c_1$ (or) $c_1 = -2\lambda$, then

(14) becomes $x - q = -2\lambda$ (or) $x = q - 2\lambda$. $\rightarrow$ (15)

from (10) and (12) $\left(\frac{dy}{dt}\right) - \left(dp/dt\right) = 0$ so that

$$y - p = c_2 \rightarrow (16)$$

where c_2 is an arbitrary constant. using initial values (7), (16) gives $y_0 - p_0 = c_2$ (or)

$\lambda - \lambda/2 = c_2$ (or) $c_2 = \lambda/2$. Then (16) becomes

$$y - p = \lambda/2 \text{ (or) } y = p + (\lambda/2) \rightarrow (17)$$

from (12) $(\frac{1}{p}) dp = dt$ so that $\log p - \log c_3 = t$

$$\text{(or) } p = c_3 e^t \rightarrow (18)$$

using initial values (7), (18) gives $p_0 = c_3 e^0$ (or)

$$\lambda/2 = c_3$$

Hence (18) reduces to $p = (\lambda/2) e^t \rightarrow (19)$

from (13), $\frac{1}{q} dq = dt$ so that $\log q - \log c_4 = t$

$$\text{(or) } q = c_4 e^t \rightarrow (20)$$

using initial values (7), (20) gives $q_0 = c_4 e^0$

$$\text{(or) } 2\lambda = c_4$$

Hence (20) reduces to $q = 2\lambda e^t \rightarrow (21)$

using (21), (15) becomes $x = 2\lambda e^t - 2\lambda$ (or)

$$x = 2\lambda(e^t - 1) \rightarrow (22)$$

using (19), (17) becomes $y = (\lambda/2) e^t + \lambda/2$

$$\text{(or) } y = (\lambda/2)(e^t + 1) \rightarrow (23)$$

Substituting value of p and q from (19) and (21) in (11), we get

$$\frac{\partial z}{\partial t} = 2\{(\lambda/2)e^t\}\{2\lambda e^t\}$$

(or)

$$dz = 2\lambda^2 e^{2t} dt.$$

integrating $z = \lambda^2 e^{2t} + c_5$, c_5 being arbitrary constant $\longrightarrow$ (24)

using initial value of (7), (24) gives

$z_0 = \lambda^2 e^0 + c_5$ (or) $\lambda^2 = \lambda^2 + c_5$ (or)

$c_5 = 0$. Then (24) reduces to $z = \lambda^2 e^{2t}$ (or)

$$z = \lambda^2 (e^t)^2 \longrightarrow (25)$$

The required characteristics of (1) are given by (22), (23) and (25)

To find the required integral surface of (1) we now proceed to eliminate two parameters t and λ from three equations (22), (23) and (25).

solving (22) and (23) for e^t and λ in (25)

we have.

$$z = \{(4y-x)^2/16\} \times \{(x+4y)^2/(4y-x)^2\}$$

(or) $16z = (4y+x)^2$ which is the required integral surface of (1) passing through (2).

7(b) Given $\frac{dy}{dx} = y - x$ where $y(0)=2$, using the Runge-Kutta fourth order method, find $y(0.1)$ and $y(0.2)$ compare the approximate solution with its exact solution ($e^{0.1} = 1.10517$, $e^{0.2} = 1.2214$).

Sol'n: Given $x=0$, $y(0)=2$

$$\frac{dy}{dx} = f(x,y) = y - x \quad , h = 0.1$$

To find $y(0.1) = ?$, $y(0.2) = ?$

For $y(0.1)$

$$y(0) = 2 \quad , x_0 = 0, \quad h = 0.1$$

$$y(0.1) = y(0) + K$$

$$K = \frac{1}{6}(K_1 + K_4 + 2(K_2 + K_3))$$

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$\therefore K_1 = 0.1 f(0, 2) = 0.1 \times 2 = 0.2$$

$$K_2 = 0.1 f(0.05, 2.1) = 0.1 \times (2.05) = 0.205$$

$$K_3 = 0.1 f(0.05, 2.1025) = 0.1 \times (2.0525) = 0.20525$$

$$K_4 = 0.1 f(0.1, 2.20525) = 0.1(2.10525) = 0.210525$$

$$K = \frac{1}{6}\left[0.2 + 0.210525 + 2(0.205 + 0.20525)\right]$$

$$K = \frac{1}{6}\left[0.410525 + 0.8205\right] = 0.20517$$

$$y(0.1) = y(0) + K = 2 + 0.20517 = 2.20517$$

For $y(0.2)$: $y(0.1) = 2.20517$, $h = 0.1$, $x = 0.1$

$K_1 = 0.1 f(0.1, 2.20517) = 0.1 \times 2.10517$
$= 0.210517$

$K_2 = 0.1 f(0.15, 2.3104) = 0.1 \times 2.10517$
$= 0.21604$

$K_3 = 0.1 f(0.15, 2.3132) = 0.1 \times 2.163$
$= 0.2163$

$K_4 = 0.1 f(0.2, 2.42147) = 0.1 \times 2.22147 = 0.22215$

$K = \frac{1}{6}\left[0.210517 + 0.22215 + 2(0.21604 + 0.2163)\right]$

$K = \frac{1}{6}(1.2974) = 0.2162$

$y(0.2) = y(0.1) + k = 2.20517 + 0.2162 = 2.42137$

$\therefore \boxed{y(0.1) = 2.20517 \; ; \; y(0.2) = 2.42137}$

Error $= |y(0.1) - e^{0.1}| = |2.20517 - 1.10517|$
$\simeq 1.1$

Error $= |y(0.2) - e^{0.2}| = |2.42137 - 1.2214|$
$\simeq 1.2$

Note: If $y(0) = 1$; then it will be negligible error from exact solution.

7(c). If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$$

Prove that the liquid motion is possible and the velocity potential is $\cos\theta/r^2$. Also determine the stream lines.

Sol: Given $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$, $w = \frac{3z^2-r^2}{r^5}$.

Since $r^2 = x^2+y^2+z^2$

hence $\frac{\partial r}{\partial x} = \frac{x}{r}$ e.t.c.

Step-I. To prove that the liquid motion is possible. For this we have to prove that the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied.

$$\frac{\partial u}{\partial x} = \frac{3z}{r^{10}}(r^5 - 5r^3x^2),$$

$$\frac{\partial v}{\partial y} = \frac{3z}{r^{10}}(r^5 - 5r^3y^2),$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}\left[(6z-2z)r^5 - 5r^3(3z^2-r^2)z\right]$$

This $\Rightarrow$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

$$= \frac{3z}{r^{10}}\left[2r^5 - 5r^3(r^2-z^2)\right] + \frac{1}{r^{10}}\left(9zr^5 - 15r^3z^3\right] = 0$$

Hence the result.

Step-II. The show that $\phi = \cos\frac{\theta}{r^2}$.

$$\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= -u\,dx - v\,dy - w\,dz$$

$$= -\frac{1}{r^5}\left[3xz\,dx + 3yz\,dy + (3z^2-r^2)dz\right]$$

$$= -\frac{1}{r^5}\left[3z(x\,dx + y\,dy + z\,dz) - r^2 dz\right]$$

$$= -\frac{1}{r^5}\left[3z\,d\left(\frac{r^2}{2}\right) - r^2 dz\right]$$

$$= -\frac{3z}{r^4}dr + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right).$$

Integrating, $\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$,

neglecting constant of integration.

Aliter. $\frac{\partial \phi}{\partial x} = -u = \frac{-3xz}{r^5}$

Integrating w.r.t. x,

$$\phi = -\frac{3z}{2}\int (2x)(x^2+y^2+z^2)^{-5/2}\,dx$$

$$= \left(\frac{-3z}{2}\right)\left(\frac{-2}{3}\right)(x^2+y^2+z^2)^{-3/2}$$

or $\phi = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

$$= \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2},$$

on neglecting constant of integration.

Step-III. stream lines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Putting the values of respective terms,

$$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2-r^2} = \frac{x\,dx+y\,dy+z\,dz}{3z(x^2+y^2+z^2)-r^2z}$$
$$\quad (1) \qquad (2) \qquad (3) \qquad (4)$$

Taking the ratios (1) and (2),

$$\frac{dx}{x} = \frac{dy}{y}.$$

Integration yields the result

$$\log x = \log y + \log a$$

or $\quad x = ay \quad$ ——— (5)

By (1) and (4),

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2r^2}$$

or $$\frac{4\,dx}{x} = 3\left(\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}\right).$$

Integrating,

$$4\log x = 3\log(x^2+y^2+z^2) + \log b$$

or $\quad x^4 = b(x^2+y^2+z^2)^3. \quad$ ——— (6)

The (5) and (6) equations represent stream lines.

8(a) Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length, whose one end is kept at $10°C$ and other end is insulated. Further, it is given that $y(x,0) = 1-x$, $0<x<1$.

Solution:-

Suppose the bar be placed along the x-axis with its one end (which is at $10°C$) at origin and the other end at $x=1$ (which is insulated, so the flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity).

Then we are to solve

$$\frac{\partial y}{\partial t} = k\left(\partial^2 y/\partial x^2\right). \quad \text{——— (1)}$$

with B.C. $y_x(1,t)=0$; $y(0,t)=10$ ——(2)

with I.C. $y(x,0) = 1-x$; $0<x<1$ ——(3)

Let ; $y(x,t) = u(x,t)+10$ ——(4)

i.e $u(x,t) = y(x,t) - 10$ ——(5)

using (4) or (5) in, (1),(2) and (3) reduce to

$$\frac{\partial u}{\partial t} = k\left(\partial^2 u/\partial x^2\right) \quad \text{—— (6)}$$

$$u_x(1,t)=0 \;,\; u(0,t)=0 \quad \text{—— (7)}$$

$$u(x,0) = y(x,0)-10 = -(x+9) \quad \text{—— (8)}$$

Suppose that (6) has solutions of the form

$$u(x,t) = X(x)T(t) \quad \text{——— (9)}$$

substituting this value of 'u' in (6), we get

$$XT' = kX''T \quad \text{or} \quad \boxed{\frac{X''}{X} = \frac{T'}{kT}} \quad \text{——— (10)}$$

Since, x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ.

$$\therefore \quad X'' - \mu X = 0 \quad \text{—— (11)}$$

$$T' = \mu k T \quad \text{—— (12)}$$

Using (7), (9) gives

$$X'(1)T(t) = 0 \quad \text{and} \quad X(0)T(t) = 0 \quad \text{—— (13)}$$

Since, $T(t) = 0$, leads to $u = 0$,

so we suppose that $T(t) \neq 0$

$$\therefore \text{ from (13)} \quad X'(1) = 0 \quad \text{and} \quad X(0) = 0 \quad \text{—— (14)}$$

We now solve (11) under B.C. (14).

Three cases arises.

Case (i) Let $\mu = 0$. Then solution of (11) is

$$X(x) = Ax + B \quad \text{—— (15)}$$

from (15) $\quad X'(x) = A \quad$ —— (15')

Using B.C. (14), (15) and (15)' gives $0 = A$ & $0 = B$

So from (15), $X(x) \equiv 0$,

which lead to $u = 0$. So reject $\mu = 0$

Case ii) Let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{—— (16)}$$

So that $\quad X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} \quad$ —— (16)'

Using B.C. (14), (16) & (16)', give

$$0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda} \quad \text{and} \quad 0 = A + B$$

These give $A = B = 0$ so that $X(x) = 0$ and hence $u(x) \equiv 0$ and so we reject $\mu = \lambda^2$.

Case III) Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{—— (17)}$$

so that $\quad X'(x) = -A\lambda\sin x + B\lambda\cos x$ —— (17)'

using B.C. (14), (17) & (17)' give

$$0 = -A\lambda\sin x + B\lambda\cos x \quad \text{and} \quad 0 = A$$

These gives ; $A = 0$ and $\cos\lambda = 0$ —— (18)

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u = 0$

Now, $\cos\lambda = 0 \Rightarrow \lambda = \frac{1}{2}(2n-1)\pi$; $n = 1, 2, 3 \ldots$

$$\therefore \mu = -\lambda^2 = -\frac{1}{4}(2n-1)^2\pi^2 \quad \text{—— (19)}$$

Hence, non-zero solution $X_n(x)$ of (17) are given by

$$X_n(x) = B_n \sin\left\{\frac{1}{2}(2n-1)\pi x\right\}$$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4}T \quad \text{or} \quad \frac{dT}{T} = -C_n^2\,dt \quad \text{—— (20)}$$

where ; $C_n^2 = +\frac{1}{4}(2n-1)^2\pi^2 k$ —— (21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ —— (22)

So; $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7). Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{—— (23)}$$

Putting $t=0$ in (23) and using (8), we have

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \quad \text{—— (24)}$$

Multiply both sides of (24) by $\sin\{\frac{1}{2}(2m-1)\pi x\}$ and then integrating w.r.t 'x' from 0 to 1, we get.

$$\Rightarrow -\int_0^1 (x+9)\sin\{\tfrac{1}{2}(2m-1)\pi x\}\,dx.$$

$$= \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\cdot\sin\frac{(2m-1)\pi x}{2}\,dx. \quad \text{—— (25)}$$

But $\int_0^1 \sin\frac{(2n-1)\pi x}{2}\cdot\sin\frac{(2m-1)\pi x}{2}\,dx = \left.\begin{matrix}0, & \text{if } m\neq n\\ 1; & \text{if } m=n\end{matrix}\right] \text{—— (26)}$

using (26), (25) gives.

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$$

$$\therefore E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)}{2}\pi x\,dx$$

$$\therefore E_n = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2\pi^2}{4}}\right\}\right]_0^1$$

[on using chain rule of integration by parts].

$$\therefore E_n = \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \left\{\begin{matrix}\because \cos\frac{(2n-1)\pi}{2}=0\\ \text{and } \sin\frac{(2n-1)\pi}{2}=(-1)^{n-1}\end{matrix}\right\} \text{—— (27)}$$

using (23) and (24), the required solution is given by

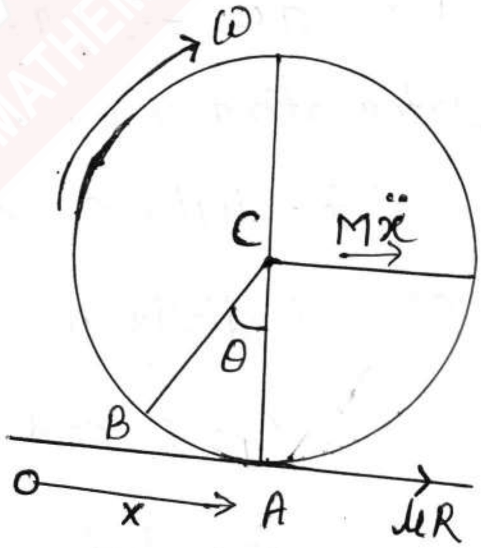
$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}\cdot e^{-c_n^2 t}$$

where c_n and E_n are given by (21) and (27) respectively.

8(c) A homogeneous sphere of radius a, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. show that there will be slipping at the point of contact for a time $(2a\omega/7\mu g)$, and that then the sphere will roll with angular velocity $(2\omega/7)$.

Sol: As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.

Initial velocity of the point of contact = Initial velocity of the centre C + Initial velocity of the point of contact with respect to the centre C = $0 + a\omega$ in the direction from right to left,



i.e., the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t the equations of motion are

$$M\ddot{x} = \mu R, \text{ (where } R = Mg) \quad \text{———} (1)$$

and $$MK^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu R a. \quad \text{———} (2)$$

From (1), we have $\ddot{x} = \mu g$ and from (2), we have $a\ddot{\theta} = -\frac{5}{2}\mu g$.

Integrating these equations, we have

$$\dot{x} = \mu g t + c_1 \text{ and } a\dot{\theta} = -\frac{2}{5}\mu g t + c_2.$$

Since initially when $t=0$, $\dot{x}=0$, $\dot{\theta}=\omega$.

$$\therefore c_1 = 0 \text{ and } c_2 = a\omega.$$

$$\therefore \dot{x} = \mu g t, \quad \text{———} (3)$$

and $$a\dot{\theta} = -\frac{5}{2}\mu g t + a\omega \quad \text{———} (4)$$

velocity of the point of contact $= \dot{x} - a\dot{\theta}$

$\therefore$ The point of contact will come to rest when $\dot{x}-a\dot{\theta}=0$, i.e. when

$$\mu g t-\left(-\tfrac{5}{2}\mu g t+a\omega\right)=0 \quad \text{or when}$$

$$t=(2a\omega/7\mu g).$$

Therefore after time $(2a\omega/7\mu g)$ the slipping will stop and pure rolling will commence. Putting this value of t in (4), we get

$$\dot{\theta}=(2\omega/7).$$

When rolling commences, let F be the frictional force. Therefore the equations of motion are

$$M\ddot{x}=F \quad \text{——— (5)}$$

$$M\cdot\tfrac{2}{5}a^2\ddot{\theta}=-Fa, \quad \text{——— (6)}$$

$$\text{and} \quad \dot{x}-a\dot{\theta}=0 \quad \text{——— (7)}$$

From (7) $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$.

Now from (5) and (6), we get

$$M\ddot{x}=F=-\tfrac{2}{5}Ma\ddot{\theta}$$

$$\text{or} \quad a\ddot{\theta}=-\tfrac{2}{5}a\ddot{\theta} \qquad (\because \ddot{x}=a\ddot{\theta})$$

$\Rightarrow \tfrac{7}{5}a\ddot{\theta}=0 \Rightarrow \ddot{\theta}=0$ Integrating $\dot{\theta}=\text{Constant}=\tfrac{2}{7}\omega$.