Fluid Mechanics (Theory).

① Representation of frame

- Eulerian ≡ Looks at a point fixed in space

$u = F(x, y, z, t)$ $\quad v = F_2(x, y, z, t)$ $\quad w = F_3(x, y, z, t)$.

[Easier to integrate]

- Lagrangian ≡ Looks at each particle in space

Let (x_0, y_0, z_0) be coordinates of a chosen particle at t_0.

At a later time t,

$x = f_1(x_0, y_0, z_0, t)$ $\quad y = f_2(x_0, y_0, z_0, t)$ $\quad z = f_3(x_0, y_0, z_0, t)$.

– $\vec{u}$ ≡ velocity = (u, v, w)

(2) Acceleration

$$u + du = U(x,y,z,t) + \frac{\partial u}{dx}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \frac{\partial u}{\partial t}\Delta t$$

$$\lim_{\Delta t \to 0}\left(\frac{du}{\Delta t} = \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial u}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial u}{\partial t}\right)$$

$$\underbrace{\frac{du}{dt}}_{\text{material derivative}} = \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial u}{\partial t}}_{\text{temporal acceleration}}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

* Cylindrical

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}$$

$$\boxed{\begin{aligned} a_r &= \frac{DV_r}{Dt} - \frac{V_\theta^2}{r}. \\ a_\theta &= \frac{DV_\theta}{Dt} + \frac{V_r V_\theta}{r} \\ a_z &= \frac{DV_z}{Dt} \end{aligned}}$$

* Spherical

$$a_r = \frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{V_\theta^2 + V_\phi^2}{r}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi} + \frac{V_r V_\theta}{r} - \frac{V_\phi^2 \cot\theta}{r}$$

$$a_\phi = \frac{\partial V_\phi}{\partial t} + V_r \frac{\partial V_\phi}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\phi}{\partial \theta} + \frac{V_\phi}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_r V_\phi}{r} + \frac{V_\phi V_\theta \cot\theta}{r}$$

$$\nabla \longrightarrow \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}.$$

$$\longrightarrow \frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Lines tangent at which, give velocity at a point at any time instant t.

$ds \rightarrow \hat{t} \equiv (u\hat{i} + v\hat{j} + w\hat{k})$

$|ds \times \hat{t}| = 0.$

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}}$$

$ds = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

- Cannot intersect $\equiv$ as can't have 2 velocities at one point at same time

$\Rightarrow$ $\boxed{\text{Surface orthogonal to streamlines : } u\,dx + v\,dy + w\,dz = 0}$

- Path lines

Path of any particle (x_0, y_0, z_0 at t_0) over entire time

$$\frac{dx}{dt} = u(x_0, y, z, t)$$

$$\frac{dy}{dt} = v(x, y, z, t)$$

$$\frac{dz}{dt} = w(x, y, z, t)$$

$$\boxed{\begin{array}{l} \text{I Cs} \Rightarrow \\ x(t_0) = x_0 \\ y(t_0) = y_0 \\ z(t_0) = z_0 \end{array}}$$

$\boxed{\text{Steady flow} \equiv \text{Pathline} = \text{Streamlines}}$

- Velocity potential

ϕ is velocity potential if $\vec{q} = -\text{grad}\,\phi$ (velocity vector)

↳ Existence if $\nabla \times q = 0$ (irrotational or conservative)

- $\phi(x, y, z, t) = C_i$ are equipotential surfaces for diff c_i's.

- For an irrotational flow of incompressible fluids

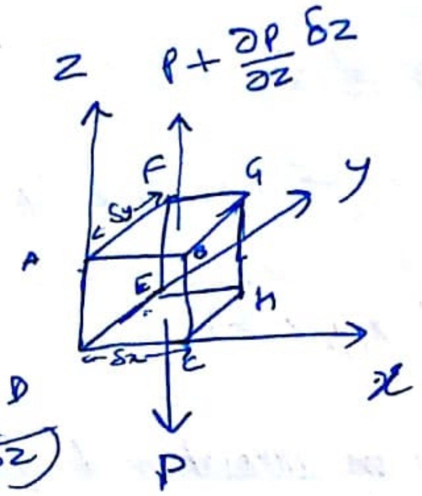
↳ $\nabla^2 \phi = 0$ (Combining $\nabla \times q = 0$ & Continuity $\nabla \cdot q = 0$)

(4) Euler Equations

Z-direction ≡

$$m a_z = B_Z - \frac{\partial p}{\partial z}(\delta x\, \delta x\, \delta z)$$

$Z\rho\,\delta x\,\delta y\,\delta z$



$$\rho\,\cancel{\delta x\,\delta y\,\delta z}\,a_z = Z\rho\,\cancel{\delta x\,\delta y}\,\delta z - \frac{\partial p}{\partial z}\cancel{(\delta x\,\delta y\,\delta z)}$$

$$\boxed{\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}}$$

(5) Continuity Equation.

Along x ≡ Inflow - Outflow = $(\rho\,\delta y\,\delta z)u$ ~~$-(\rho\,\delta y\,\delta z)\left[u + \frac{\partial u}{\partial x}\delta x\right]$~~

$$= (\rho\,\delta y\,\delta z)u - \left[(\rho\,\delta y\,\delta z)u + \delta x.\frac{\partial}{\partial x}(\rho\,\delta y\,\delta z\,u) \cdots\right]$$

$$= -\delta x\,\delta y\,\delta z\,\frac{\partial(\rho u)}{\partial x}$$

Along y = $-\delta x\,\delta y\,\delta z\,\frac{\partial(\rho v)}{\partial y}$ $\quad Z \equiv -\delta x\,\delta y\,\delta z\,\frac{\partial(\rho w)}{\partial z}$

$$x+y+z = \frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z)$$

Equating ⇒ $\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$ — General Continuity Equation

$$= \frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

For homogeneous, incompressible fluids $\boxed{\frac{D\rho}{Dt} = 0}$

↓

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

Problems

$\boxed{Q}$ $u = x+y+2t \qquad v = 2y+t$. Determine Lagrange coordinates x_0, y_0 at time t.

$$\frac{dx}{dt} = x+y+2t \qquad \frac{dy}{dt} - 2y = t \Rightarrow e^{-\int 2\,dt} = e^{-2t}$$

$$y.e^{-2t} = \int te^{-2t} + c = \frac{1}{-2}te^{-2t} - \frac{1}{4}e^{-2t} + c$$

$$\frac{dx}{dt} - x = ce^{2t} + \frac{1}{4}(6t-1) \qquad y = ce^{2t} - \frac{(2t+1)}{4}$$

$$x.e^{-t} = c_2 + ce^{t} + \int \frac{(6t-1)}{4} e^{-t}\,dt =$$

$$x = c_2 e^{t} + ce^{2t} - \frac{(6t+5)}{4}$$

Using x_0, y_0 at $t=0$. $\Rightarrow$ $y_0 = c - 1/4 \qquad x_0 = c_1 + c - 5/4$.

$$c = y_0 + 1/4 \qquad c_1 = x_0 - y_0 + 1$$

$$x = (x_0 - y_0 + 1)e^{t} + (y_0 + 1/4)e^{2t} - (6t+5)/4$$

$$y = (y_0 + 1/4)e^{2t} - (2t+1)/4.$$

$\boxed{Q}$ velocity $= (r^2 z\cos\theta, rz\sin\theta, z^2 t)$. Find acceleration.

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} = a_r$$

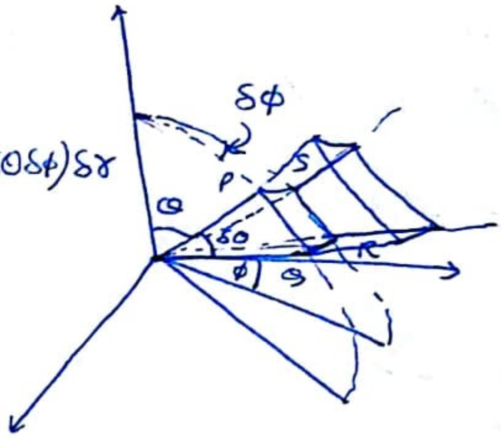
$$a_r = 2r^3z^2\cos^2\theta + z\sin\theta\,(-r^2 z\sin\theta) + z^2t\,(r^2\cos\theta) - rz^2\sin^2\theta$$

$$a_\theta = r^2 z\cos\theta . z\sin\theta + z\sin\theta(rz\cos\theta) + z^2t\,(r\sin\theta) + r^2z^2\sin\theta\cos\theta.$$

$$a_z = z^2 + z^2t\,(2zt)$$

$-v_\theta^2/r$	a_r
$+\frac{v_r v_\theta}{r}$	a_θ

8) Continuity for Spherical Coordinates.



$PQRS \Rightarrow (\rho r\delta\theta \, r\sin\theta\delta\phi \, u_r)$.

$P'Q'R'S' \Rightarrow (\rho r\delta\theta \, r\sin\theta\delta\phi \, u_r) - \frac{\partial}{\partial r}(r^2\rho u_r \sin\theta\delta\theta\delta\phi)\delta r$

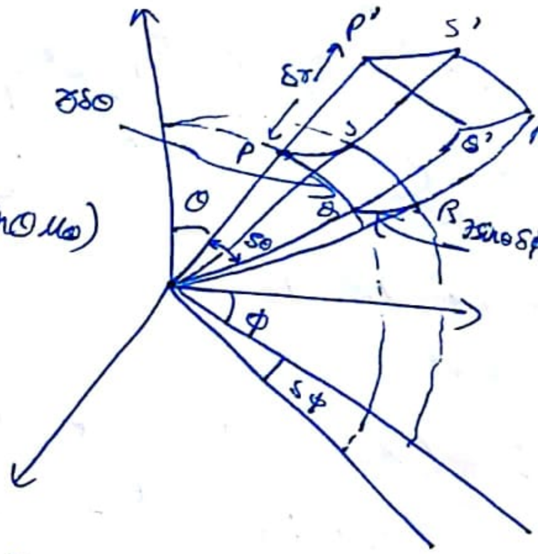
Net $\Rightarrow -\frac{\partial}{\partial r}(r^2\sin\theta \rho u_r)\delta\theta\delta\phi\delta r$

Illy other faces $\Rightarrow$

Net $= \frac{\partial}{\partial t}(\rho \, r\delta\theta \, \delta r \, r\sin\theta\delta\phi)$

Equating we get,

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta u_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0.$$



9) Symmetrical forms of Continuity Equations.

· Cylindrical (Take $h=1$)

$$\frac{\partial}{\partial t}(\rho 2\pi r.\delta r) = \rho q_r(2\pi r) - \left[\rho q_r 2\pi r + \delta r\frac{\partial}{\partial r}(\rho q_r 2\pi r)\right]$$

$$2\pi r\left(\frac{\partial\rho}{\partial t}\right)\cancel{\delta r} = -\cancel{\delta r}\frac{\partial}{\partial r}(\rho q_r 2\pi r)$$

If ρ is constant $\Rightarrow \frac{\partial}{\partial r}(r\rho q_r) = 0 \rightarrow r\cancel{\rho}q_r = \cancel{\rho}g(t)$

$\Rightarrow rq_r = g(t)$

If steady flow $\Rightarrow \boxed{rq_r = C}$

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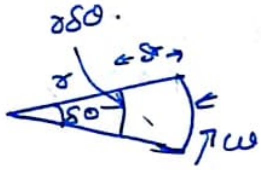
Q Fluid moves such that each particle describes circle in a plane moving with ω. Find Continuity equation.

$$\frac{\partial}{\partial t}\left(\rho \cancel{r}\,\delta\theta\,\delta r\right) = \left(-\cancel{r}\,\delta\theta \frac{\partial}{\cancel{r}\,\partial\theta}\left(\rho \cancel{r}\,\omega\,\delta r\right)\right)$$

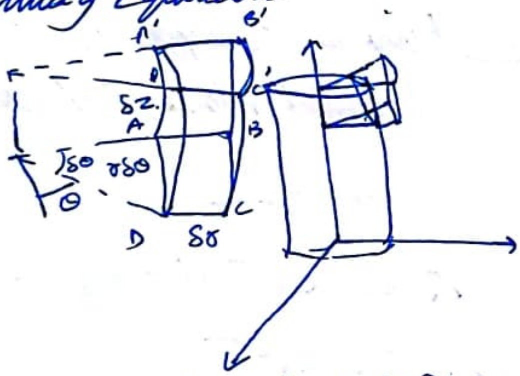


$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho\omega)}{\partial\theta}}$$ Ans

Q Mass of fluid so that lines of motion lie on surface of co-axial cylinders. Find Continuity Equation

u, v $\perp r$ and parallel to z

· No flow across $A'D'DA$



$$ABCD \equiv -\delta z \frac{\partial}{\partial z}\left(\rho v\, r\,\delta\theta\,\delta r\right)$$

$$D'C'CD \equiv -r\,\delta\theta \frac{\partial}{r\,\partial\theta}\left(\rho u\,\delta r\,\delta z\right).$$

$$\frac{\partial}{\partial t}\left(\rho r \cancel{\delta\theta\,\delta r\,\delta z}\right) + \cancel{\delta z}\frac{\partial}{\partial z}\left(\rho v r \cancel{\delta\theta\,\delta r}\right) + \cancel{r\,\delta\theta}\frac{\partial}{\cancel{r}\,\partial\theta}\left(\rho u \cancel{\delta r\,\delta z}\right) = 0.$$

$$\boxed{r\frac{\partial\rho}{\partial t} + r\frac{\partial(\rho v)}{\partial z} + \frac{\partial}{\partial\theta}(\rho u) = 0}$$ Ans

LEARN IT.

Q Particle moves on surface of sphere. Given ρ, θ (lat.), ϕ (longitude) of any element, ω and ω' angular velocities of element in latitude and longitude.

$RS = \delta r$

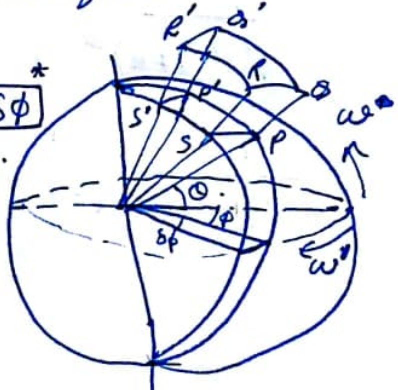
$\boxed{PS = r\cos\theta\,\delta\phi}$

$PP' = r\,\delta\theta.$

$$\rightarrow \text{Along } PP' \equiv -\frac{r\,\delta\theta\,\partial}{r\,\delta\theta}\left(r\omega . \delta r . r\cos\theta\,\delta\phi\right)$$

$$\text{Along } PS \equiv -\frac{r\cos\theta\,\delta\phi\,\partial}{r\cos\theta\,\delta\phi}\left(\rho . r\cos\theta\,\omega' . \delta r . r\,\delta\theta\right)$$

Equate as usual $\equiv \frac{\partial}{\partial t}(\,.\,.\,)$



Along $PP' \rightarrow r\omega$

$PS \rightarrow r\cos\theta\,\omega'$

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Q) Show continuity equation is satisfied.

$$u = \frac{k(x^2-y^2)}{(x^2+y^2)^2} \qquad v = \frac{2kxy}{(x^2+y^2)^2}$$

TS: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$$\frac{2kx}{(x^2+y^2)^2} - 2.\frac{k(x^2-y^2).2x}{(x^2+y^2)^3} + \frac{2kx}{(x^2+y^2)^2} - \frac{8kxy^2}{(x^2+y^2)^3}$$

$= 0$ Hence Possible

Q) Flow by a pipe $y = a + \left(\frac{kx^2}{a}\right)$ from $-a$ to a.
If initial velocity $= V$ at $x = -a$, find time taken.

$A_1 V_1 = A_2 V_2$

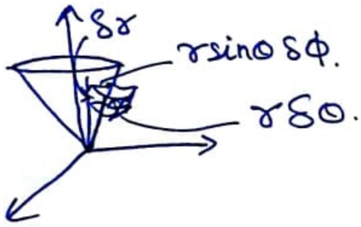
$$\frac{\cancel{\pi}(a+ka)^2 V}{\cancel{\pi}\left(a+\frac{kx^2}{a}\right)^2} = V_x = \frac{dx}{dt}$$

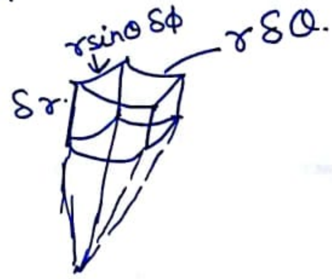
$$\int \frac{dx\left(a^2+\frac{k^2x^4}{a^2}+2kx^2\right)}{V(a+ka)^2} = \int dt$$

$$t = \frac{1}{Va^2(1+k)^2}\left[a^2x + \frac{k^2x^5}{5a^2} + \frac{2kx^3}{3}\right]_{-a}^{a}$$

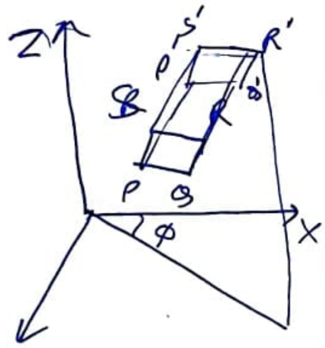
$$= \frac{2}{V(1+k)^2a^2}\left[a^3 + \frac{k^2}{5}a^3 + \frac{2}{3}ka^3\right] = \frac{2a}{V(1+k)^2}\left[1+\frac{2k}{3}+\frac{k^2}{5}\right]$$

Q Flow on co-axial cones.





Q Each particle of liquid ~~passes through~~ moves in a plane through z-axis.



PQRS lies in given plane.

$PP' \equiv$ no fluid along this $= \underline{r\sin\theta\,\delta\phi}$

$\underline{PQ = \delta r}$, $\underline{PS = r\delta\theta}$.

u v are velocity along them.

→ Now trivial construction of Continuity Equation

(5) Boundary Conditions.

- At any point on a solid surface inside a liquid. if normal to surface be $\vec{n}$ and $\vec{v}$ be velocity of fluid. Then, $\hat{v} \cdot \hat{n} = 0$. (at all surface points)

Ily if body is moving with $\vec{U}$ then $\boxed{\bar{v} \cdot \bar{n} = \bar{U} \cdot \bar{n}}$ or $\bar{v} - \bar{U} \cdot \bar{n} = 0$

Condition for a surface to be a boundary surface. (F)

$$\boxed{\frac{\partial F}{\partial t} + q \cdot \nabla F = 0}$$

$q = \vec{v}$

$$\frac{DF}{Dt} = 0.$$

* Normal velocity of boundary $= \bar{u} \cdot \hat{n} = \bar{u} \cdot \frac{\nabla F}{|\nabla F|}$.

IMPORTANT

$$\boxed{\hat{n} = \frac{\nabla F}{|\nabla F|}}$$

⑤ Rotational flow.

- Voriticity $\equiv$ $\boxed{\Omega = \text{curl } q}$

  L Vortex line $\equiv$ Line s.t tangent to it at each point is in the direction of Ω vector.

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

- Angular velocity $\equiv$ $\boxed{\omega = \frac{1}{2}\Omega = \frac{1}{2}\text{curl } q}$

- $\frac{\partial \phi}{\partial x} = -u \quad , \quad \frac{\partial \phi}{\partial y} = -v.$

Streamlines show each particle at an instant

Pathlines show a particle at all times

Streaklines look at a particular point in space

Q] Show that the surface $\frac{x^2}{a^2k^2t^4} + kt^2\left[\frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 1$ is a possible boundary surface.

$$-\frac{4x^2}{a^2k^2t^5} + 2kt\left[\frac{y^2}{b^2} + \frac{z^2}{c^2}\right] + u\left[\frac{2x}{a^2k^2t^4}\right] + v\left[\frac{2kt^2y}{b^2}\right] + w\left[\frac{2kt^2z}{c^2}\right] = 0$$

Putting $u = +\frac{2x}{t}$. $v = -\frac{y}{t}$. $w = -\frac{z}{t}$.

the equation is identically satisfied.

Now, check for (u, v, w) if continuity equation holds

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \text{ Hence possible}$$

Q] Show $\frac{x^2\tan^2 t}{a^2} + \frac{y^2\cot^2 t}{b^2} = 1$ is possible boundary and find normal velocity.

$$\frac{2x^2\tan t\sec^2 t}{a^2} - \frac{2y^2\cot t\operatorname{cosec}^2 t}{b^2} + u\left(\frac{2x\tan^2 t}{a^2}\right) + v\left(\frac{2y\cot^2 t}{b^2}\right) = 0$$

$$\underbrace{u = -\frac{x\sec^2 t}{\tan t} \qquad v = \frac{y\operatorname{cosec}^2 t}{\cot t}.}_{\Downarrow}$$

$$-\frac{\sec^2 t}{\tan t} + \frac{\operatorname{cosec}^2 t}{\cot t} \Rightarrow -\frac{\cos t}{\sin t\cos^2 t} + \frac{\sin t}{\cos t\sin^2 t} = 0.$$

Hence possible.

$$\hat{n} = \bar{q}\cdot\frac{\left(\frac{2x\tan^2 t}{a^2}\hat{i} + \frac{2y\cot^2 t}{b^2}\hat{j}\right)}{\sqrt{\left(\frac{2x\tan^2 t}{a^2}\right)^2 + \left(\frac{2y\cot^2 t}{b^2}\right)^2}}$$

Q. Find stream and paths of particles, the condition when

$$u = \frac{x}{1+t} \quad v = \frac{y}{2+t} \quad w = \frac{z}{3+t}$$ & stream line = pathlines

- $$\frac{dx(1+t)}{x} = \frac{dy(2+t)}{y} \Rightarrow \frac{dx}{x} - \frac{2dy}{y} = t\left(\frac{dy}{y} - \frac{dx}{x}\right)$$

$$\Rightarrow \log x - 2\log y = t(\log y - \log x) + \log c$$

$$\log\left(\frac{x}{y^2}\right) = \log\left(c\left(\frac{y}{x}\right)^t\right) \Rightarrow \boxed{\frac{x}{cy^2} = \left(\frac{y}{x}\right)^t} \quad -(1)$$

- $$\frac{dy(2+t)}{y} = \frac{dz(3+t)}{z} \Rightarrow \frac{2dy}{y} - \frac{3dz}{z} = t\left(\frac{dz}{z} - \frac{dy}{y}\right)$$

$$\cancel{\Rightarrow \log(x^2/y^3) = t\log(y/x) + \log c.}$$

$$\cancel{\Rightarrow \frac{xy^2}{cy^3 z} = \left(\frac{y}{x}\right)^t} \Rightarrow \boxed{\frac{y^2}{z^3} = c_2\left(\frac{z}{y}\right)^t} \quad -(2)$$

Intersection of (1) and (2) shows streamlines.

→ Path lines

$$\frac{dx}{dt} = \frac{x}{1+t} \Rightarrow x = c_3(1+t)$$
$$y = c_4(2+t)$$
$$z = c_5(3+t)$$

Streamlines and pathlines are same in case of steady motion.

- Pathlines (x_0, y_0, z_0) at t_0 $$\Rightarrow x = \frac{x_0}{1+t_0}(1+t),\ y = \frac{y_0}{(2+t_0)}(2+t),\ z = \frac{z_0}{(3+t_0)}(3+t).$$ (A)

- Streak lines → Let particle x_0, y_0, z_0 pass through x_1, y_1, z_1 at $t = s$ where $t_0 \le s \le t$

Use pathlines.

$$\rightarrow \frac{x_1(1+t_0)}{(1+s)}, \frac{y_1(2+t_0)}{2+s}, \frac{z_1(3+t_0)}{3+s} \equiv x_0, y_0, z_0$$

Substitute in (A) $$\Rightarrow (x, y, z) \equiv \left[\frac{x_1(1+t)}{1+s}, \frac{x_2(1+t)}{1+s}, \frac{x_3(1+t)}{1+s}\right]$$

⑨ Ex.11 (2.54)

In steady motion of homogenous liquid in surfaces $f_1 = a_1$, $f_2 = a_2$, define streamlines. Prove that most general values of velocity components u, v, w are

$$F(f_1, f_2)\frac{\partial(f_1 f_2)}{\partial(y,z)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z,x)}, \quad F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x,y)}.$$

$\rightarrow$ $f_1 = a_1$, $df_1 = 0 \Rightarrow \frac{\partial f_1}{\partial x_1}dx + \frac{\partial f_x}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0$

$f_2 = a_2$, $df_2 = 0 \Rightarrow \frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_3}{\partial z}dz = 0$

$$\frac{dx}{f_{1y}f_{2z} - f_{1z}f_{2y}} = \frac{dy}{f_{1z}f_{2x} - f_{1x}f_{2z}} = \frac{dz}{f_{1x}f_{2y} - f_{1y}f_{2x}}$$

or $\frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_3}$ $\quad J_1 \equiv \frac{\partial(f_1, f_2)}{\partial(y,z)}$; $J_2 = \frac{\partial(f_1, f_2)}{\partial(z,x)}$

$J_3 \equiv \frac{\partial(f_1, f_2)}{\partial(x,y)}$.

- Streamlines $\Rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow u = FJ_1,\ v = FJ_2,\ w = FJ_3.$

where F is arbitrary.

- Continuity $\equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow$ $F(J_{1x} + J_{2y} + J_{3z}) +$ $J_1F_x + J_2F_y + J_3F_z = 0$

By Jacobians $\Rightarrow$ $\boxed{J_{1x} + J_{2y} + J_{3z} = 0}$

$J_1F_x + J_2F_y + J_3F_z = 0 \Rightarrow \frac{\partial(F, f_1, f_2)}{\partial(x,y,z)} = 0$

F is dependent f_1, f_2.

$\Rightarrow$ F is a function of f_1, f_2 only. Hence, Proved

(Q) For $q = \frac{k^2(xj - yi)}{x^2+y^2}$. Show
- possible motion
- find streamlines
- Velocity potential exists. Find it.

$\rightarrow \frac{du}{dx} + \frac{\partial v}{\partial y} = 0.$

$\frac{k^2 y \cdot 2x}{(x^2+y^2)^2} + \frac{-2y \cdot x}{(x^2+y^2)} = 0$ ✓.

$\rightarrow \frac{dx}{(-y)} = \frac{dy}{x} \Rightarrow x^2+y^2 = C_2 \qquad z = C_1$

$\rightarrow \nabla \times q \equiv \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{-k^2 y}{y^2+x^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} \rightarrow i(0) + j(0) + \hat{k}\left(\frac{k^2}{x^2+y^2} - \frac{2k^2x^2}{(x^2+y^2)^2} - \frac{k^3}{x^2+y^2} + \cdots\right)$

$= 0$ Hence potential kind.

$\rightarrow \frac{\partial \phi}{\partial x} = \frac{yk^2}{x^2+y^2} \qquad \frac{\partial \phi}{\partial y} = \frac{-xk^2}{x^2+y^2}. \qquad \frac{\partial \phi}{\partial z} = 0$

~~$k^2 \tan^{-1}$~~ $\phi(x,y) = k^2 \tan^{-1}(x/y) + f(y).$

$\frac{\partial \phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2}.$ $\quad f'(y) = 0.$

So, $\boxed{\phi(x,y) = k^2 \tan^{-1}(x/y) + C.}$

equipotential $\Rightarrow k^2 \tan^{-1}(x/y) = k^2 \tan^{-1} C.$

(Q) $\phi = \frac{z}{r^3}\tan^{-1}\frac{y}{x}$, then find streamlines. ✱ UPSC Do AGAIN

$$u = -\frac{\partial\phi}{\partial x} = +\frac{3z \cdot x}{r^4 \cdot r}\tan^{-1}\frac{y}{x} + \frac{z}{r^3}\frac{x^2}{x^2+y^2}\cdot\frac{y}{x^2}$$

Remember

$$v = -\frac{\partial\phi}{\partial y} = \frac{3zy}{r^5}\tan^{-1}\left(\frac{y}{x}\right) - \frac{z}{r^3}\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}$$

$$w = -\frac{\partial\phi}{\partial z} = -\frac{\tan^{-1}\frac{y}{x}}{r^3} + \frac{3z\tan^{-1}\left(\frac{y}{x}\right)}{r^4}\frac{z}{r}$$

$$\frac{dx}{r^{-5}\left(3xz\tan^{-1}\frac{y}{x} + yz\right)} = \frac{dy}{r^{-5}\left(3yz\tan^{-1}\frac{y}{x} - xz\right)} = \frac{dz}{r^{-3}\tan^{-1}\frac{y}{x}\left(\frac{3z^2}{r^2}-1\right)}$$

$$\cancel{= \frac{x\,dx + y\,dy + z\,dz}{r^{-3}\tan^{-1}\left(\frac{y}{x}\right)(3z)\cdot r^{2} - 2r^{-3}\tan^{-1}\frac{y}{x}}} = \cancel{\frac{x\,dx + y\,dy}{2r^{-5}\left(3xyz\tan^{-1}\frac{y}{z}\right)}}$$

$$\cancel{= \frac{x\,dy + y\,dy + z\,dz}{2z}} = \cancel{\frac{x\,dx + y\,dy}{2r^{-2}}}$$

$$\cancel{\frac{x\,dx + y\,dy + z\,dz}{3z \cdot 2r^{-3}\tan^{-1}(y/x)}} = \cancel{\frac{x\,dx + y\,dy}{r^{-5}}}$$

$$\frac{x\,dx + y\,dy}{3(x^2+y^2)} = \frac{x\,dx + dy \cdot y + z \cdot dz}{2r^2}$$

$$\Rightarrow \boxed{\log(x^2+y^2+z^2) = \frac{2}{3}\log(x^2+y^2) + \log c}$$

$$r^{-5}\left(3x^2z\tan^{-1}\frac{y}{x} + \cancel{xyz} + 3y^2z\tan^{-1}\frac{1}{z}\right) - \cancel{xyz} + 3z^3 - \tan^{-1}\frac{y}{x}r^2$$

$$r^{-5}\tan^{-1}\frac{y}{x}\left(3r^2 \cdot\right)$$

9) $q \equiv \left[\frac{2M}{r^3}\cos\theta, \frac{M}{r^3}\sin\theta, 0\right]$.

Show velocity is of potential. Find ϕ and streamlines

TS: $\nabla \times q = 0 \rightarrow \frac{1}{r^2 \sin^2\theta} \begin{vmatrix} e_r & r e_\theta & r\sin\theta\, e_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{2M}{r^3}\cos\theta & \frac{M}{r^3}\sin\theta & 0 \end{vmatrix}$

$$\equiv e_r(0) + r e_\theta (0) + r\sin\theta\, e_\phi \left(-\frac{3M}{r^4}\sin\theta - \frac{2M}{r^3}\sin\theta\right)$$

$= 0$??

- $dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial \phi} d\theta.$

$(q_r, q_\theta, q_\phi) \equiv \left(-\frac{\partial F}{\partial r}, -\frac{1}{r}\frac{\partial F}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\right).$

- Streamlines.

$$\frac{dr}{q_r} = \frac{r\,d\theta}{q_\theta} = \frac{r\sin\theta\, d\phi}{q_\phi}.$$

SOLVE

$\boxed{Q}$ Show all conditions satisfied by

(*) - $\phi = \alpha x^2 + \beta y^2 + \gamma z^2$, bounding surface $F \equiv ax^4 + by^4 + cz^4 - \chi(t) = 0$ — (2)

[$\alpha, \beta, \gamma, a, b, c$ are f^{n}s of time]

TS:

(i) ϕ satisfies Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 2\alpha + 2\beta + 2\gamma = 0$$

(ii) F is bounding surface

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0.$$

$$x^4(\dot{a} - 8a\alpha) + y^4(\dot{b} - 8b\beta) + z^4(\dot{c} - 8c\gamma) - \dot{\chi} = 0. \quad \text{— (1)}$$

As all points of surface satisfy (1) and (2). So,

$$\frac{\dot{a} - 8a\alpha}{a} = \frac{\dot{b} - 8b\beta}{b} = \frac{\dot{c} - 8c\gamma}{c} = \frac{\dot{\chi}}{\chi}$$

$\log a = 8\int \alpha\, dt + \log \chi$

$\log b = 8\int \beta\, dt + \log \chi$

$\log c = 8\int \gamma\, dt + \log \chi$

Q. Find velocity potential and streamlines [2001]

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right).$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= -3\frac{xz}{r^5}dx - \frac{3yz}{r^5}dy - \frac{(3z^2-r^2)}{r^5}dz.$$

$$= \frac{(-3z(x\,dx + y\,dy + z\,dz) + r^2 dz)r}{r^5 \cdot r}$$

$$= \frac{-3zr\cdot(r\,dr) + r^3 dz}{(r^3)^2} = d\left(\frac{z}{r^3}\right)$$

$$\phi = \frac{z}{r^3} \Rightarrow z = r\cos\theta \Rightarrow \boxed{\phi = \frac{\cos\theta}{r^2}}$$ Ans

Streamlines $\rightarrow$ $$\frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2-r^2)/r^5}$$

$$\frac{x\,dy + y\,dy + z\,dz}{3\cancel{z}(x^2+y^2+z^2) - r^2\cancel{z}} = \frac{dx}{3x\cancel{z}} \Rightarrow \frac{2.(x\,dx + y\,dy + z\,dz)}{2r^2} = \frac{2}{3}\frac{dx}{x}.$$

$$\boxed{\frac{dr}{2r^2} = \frac{2}{3}\frac{dx}{x}}$$ $\rightarrow$ Answer

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(Q) Show $\phi = x f(r)$ is a possible potential. given that liquid speed $q \to 0$ as $r \to \infty$, deduce that surfaces of constant speed are $(r^2 + 3x^2) r^{-8} = \text{constant}$

- For ϕ to be possible
  - ϕ must satisfy Laplace.

$\rightarrow \dfrac{\partial \phi}{\partial x} = f(r) + x f'(r) \dfrac{x}{r}$

$\dfrac{\partial \phi}{\partial y} = x f'(r) \dfrac{y}{r}$ $\qquad \dfrac{\partial \phi}{\partial z} = \dfrac{x f'(r) z}{r}$

$(u, v, w) \equiv \left(- f(r) - x^2 \dfrac{f'(r)}{r}, - \dfrac{xy f'(r)}{r}, - \dfrac{xz f'(r)}{r} \right)$

- $\dfrac{\partial^2 \phi}{\partial x^2} \Rightarrow f'(r) \dfrac{x}{r} + 2x \dfrac{f'(r)}{r} - \dfrac{f'(r) x^3}{r^3} + \dfrac{x^3 f''(r)}{r^2}$
- $\dfrac{\partial^2 \phi}{\partial y^2} \Rightarrow \dfrac{x f'(r)}{r} + \dfrac{x f''(r) y^2}{r^2} - \dfrac{x y^2 f'(r)}{r^3}$
- $\dfrac{\partial^2 \phi}{\partial z^2} \Rightarrow \dfrac{x f'(r)}{r} + \dfrac{x f''(r) z^2}{r^2} - \dfrac{x z^2 f'(r)}{r^3}$

Adding $\Rightarrow$ $\cancel{\dfrac{5x f'(r)}{r}}$ $f'(r) \left[\dfrac{5x}{r} - \dfrac{x}{r} \right] + f''(r) \left[x \right]$

$\Rightarrow \quad \dfrac{4 f'(r)}{r} + f''(r) = 0 \Rightarrow \log(f'(r)) + 4 \log r = \log c_1$

$\Rightarrow f_1'(r) = c_1 r^{-4} \Rightarrow f(r) = -\dfrac{c_1}{3} \times r^{-3} + c_2$

As $q \to 0$ as $r \to \infty \Rightarrow \boxed{c_2 = 0}$

Surfaces of constant speed.

i.e. $q^2 =$ constant

Using $f(r) = -\frac{c}{3r^3}$

$$\bar{q} = \frac{c}{3r^3}\left(i - \frac{3x}{r^2}\bar{r}\right)$$

$$q \cdot q = \frac{c}{9r^8}\left(r^2 + 3x^2\right)$$

* Equations of motion of Inviscid Fluids

Pre-req $\equiv$ Continuity $=$ $r^2 v_r = F(t)$ (Sphere)

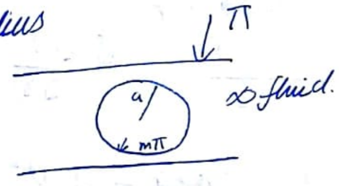
$r v_r = F(t)$ (Cylinder)

$\equiv$ Euler $\boxed{\frac{Dq}{Dt} = \bar{F} - \frac{\nabla p}{\rho}}$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} = F_{Ext} - \frac{1}{\rho}\frac{dp}{dr}.$$

Problems

Type 1 (i) Assuming Boyle's Law, show radius of sphere oscillates between a and na and find n.



Step 1 $\Rightarrow$ At any time t, at r', v', p'

cavity be r, v.

$$r'^2 v' = r^2 v = F(t). \Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}.$$

Step 2 $\Rightarrow$ Euler.

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p_1}{\partial r_1} \Rightarrow -\frac{F(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{P_1}{\rho}$$

$\rightarrow$ At $r = \infty$. $v' = 0$, $P_1 = \pi \Rightarrow C = \frac{\pi}{\rho}$

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\pi - P_1}{\rho}$$

Step 3 $\Rightarrow$ Boyles' $P_1 V_1 = P_2 V \Rightarrow$. $m\pi \cancel{\frac{4}{3}\pi} a^3 = P_0 \cancel{\frac{4}{3}\pi} r^3$. ($P_0$ at r) boundary

$P_0 = \frac{m\pi a^3}{r^3}$

Step 4 $\Rightarrow$ $F'(t) = \underline{\underline{2r v^2 + r^2 v \frac{dv}{dr}}}$.

Step 5 $\Rightarrow$ at $r' = r$, $v' = v$, $P_1 = P_0$

$$-\frac{3v^2}{2} - rv\frac{dv}{dr} + \cancel{\frac{1}{2}v^2} = \frac{\pi - P_0}{\rho} = \frac{\pi}{\rho}\left(1 - \frac{ma^3}{r^3}\right)$$

① When t not involved $\equiv F'(t) = d(r^2v). = 2rv + r^2\frac{dv}{dr}\frac{dr}{dt}$

$= 2rv^2 + r^2 v\frac{dv}{dr}.$

radius oscillates $a, na \Rightarrow v = 0$ when $r = a, na$.

$\Rightarrow \boxed{1 + 3m\log n - n^3 = 0}$

$\Uparrow$

$r = a, v = 0 \Rightarrow C' = \frac{2\pi a^3}{3\rho} - \frac{2a^3 m\pi}{\rho}\log a$

$\Uparrow$

r^2 both sides $\Rightarrow - 3r^2v^2 - 2r^3 v\,dv = \frac{\pi}{\rho}\left(1 - \frac{ma^3}{r^3}\right)dr\, r^2$

$-\int d(r^3v^2) = \frac{2\pi}{\rho}\left(\frac{r^3}{3} - 2ma^3\log r\right)$

$\Rightarrow \boxed{r^3v^2 = -\frac{2\pi}{\rho}\left(\frac{r^3}{3} - 2a^3m\log r\right) + C'}$

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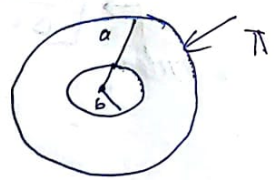
② Liquid between 2 parallel planes. Free Surface is circular cylinder of radius a, a cylinder of b is suddenly annihilated. Prove if Π is pressure at outer surface initial pressure at r from centre is $\frac{\Pi \log r - \log b}{\log a - \log b}$



$\rightarrow \quad r v = F(t).$

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{r}.$$

At $r \Rightarrow v, p$

$$\frac{F'(t)}{r} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

$$F'(t)\log r + \frac{1}{2}v^2 = C - \frac{p}{\rho}.$$

At $t=0 \Rightarrow$

$r=b, v=0, p=0 \Rightarrow F'(0)\log b = C$

$r=a, v=0, p=\pi \Rightarrow F'(0)\log a = C - \frac{\pi}{\rho}.$

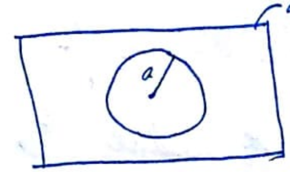
$$F'(0) = \frac{\pi}{\rho(\log b - \log a)} \qquad C = \frac{\pi \log b}{\rho(\log b - \log a)}$$

Initially at r, $v=0$

$$F'(0)\log r = \frac{\pi \log b}{\rho \log b - \log a} - \frac{p}{\rho}$$

$$\frac{\pi}{\cancel{\rho}}\left(\frac{\log r - \log b}{\log b - \log a}\right) = -\frac{p}{\cancel{\rho}} \Rightarrow \boxed{p = \pi\frac{\log r - \log b}{\log a - \log b}} \text{ Ans}$$

(3) Constant pressure at infinity.
Pressure at r' from centre ~~p_0~~ when radius cavity is r.



$$r'^2 v' = r^2 v = F(t).$$

Let p_∞ be Π.

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}.$$

$$\rightarrow \quad -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{p_1}{\rho} \qquad -C = \Pi/\rho.$$

$$(*)\ \boxed{-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p_1}{\rho}} \rightarrow \text{for } r'=r,\ v'=v\ \boxed{-\frac{F'(t)}{r} + \frac{1}{2}v^2 = \frac{\Pi}{\rho}} \ -(1)$$

Now find $F'(t)$ and v'^2

$$F'(t) = d(r^2 v) = 2rv^2 + r^2 v \frac{dv}{dr}.$$

At $r'=r$, $v'=v$.

$$\frac{1}{\cancel{r}}\left(\cancel{2dr}\,2rv^2 - 2r^2 v\,dv.\right) + \frac{1}{\cancel{r}}rv^2dr = \left(\frac{\Pi - p}{\rho}\right) r 2.dr.$$

$\times r$ both sides

$$\Rightarrow -3rv^2dr - 2r^2 v\,dv = 2\left(\frac{\Pi - p}{\rho}\right) r\,dr$$

$$-\left(d(r^3v^2)\right) = 2\left(\frac{\Pi - p}{\rho}\right)\frac{r^3}{3} + C.$$

$$-r^3v^2 = 2\left(\frac{\Pi - \cancel{p}^{\,0}}{\rho}\right)\frac{r^3}{3} + C.$$

Also at $r=r$, $p=0$
$r=a$, $v=0$

$$2\left(\frac{\Pi a^3}{}\right) + C = 0. \Rightarrow \boxed{r^3v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3)} \ -(2).$$

TOOLS AVAILABLE

(1) $F'(t) = \frac{d}{dt}(R^2 V) = \frac{d}{dt}\left(R^2 \frac{dR}{dt}\right)$

(2) Calculating $F'(t) \rightarrow F(t) = r^2 V$

$$F'(t) = 2r\frac{dr}{dt}V + r^2\frac{dv}{dt} = 2rV^2 + r^2\frac{dv}{dr}\frac{dr}{dt}$$

$$= 2rV^2 + r^2 V\frac{dv}{dr}$$ → always remove $\frac{d}{dt}$ term.

---

(1) $KE = \int^{\infty} \frac{1}{2}\rho\, 4\pi x^2 dx\, v'^2$

(Over entire mass)

$\cancel{x}'^2 v' = r^2 v$

$\Rightarrow \boxed{v' = \frac{r^2 v}{x}}$

(2) $PE = \int \pi\, 4\pi x^2 dx, \;/\; \int V\, dm$

(Can take inner or outer radii - Just one, look how it changes)

Type 2 Using energy method.

- $KE = PE$

$KE \downarrow$ $\int \frac{1}{2} 4\pi r^2 dr \rho v^2$; $PE \searrow \int p \cdot dv$

(1) Infinite fluid. Force $\equiv \frac{\mu}{r^{3/2}}$ to origin. Initial cavity (c).
Find time of filling up.

$$\Rightarrow r'^2 v' = \underbrace{r^2 v}_{\text{boundary condition}} = F(t)$$

At $t \Rightarrow$

$$KE = \int_r^\infty \frac{1}{2} 4\pi r'^2 dr' \rho v'^2 = 2\pi\rho \int_r^\infty r'^2 \cdot \frac{r^4 v^2}{r'^4} dr'$$

$$= 2\pi\rho r^4 v^2 \left[-\frac{1}{r_1}\right]_r^\infty = \boxed{2\rho\pi r^3 v^2}$$

$$\left(\int V\, dm\right) \qquad \boxed{-\frac{\partial V}{\partial r} = F}$$

$$PE = \int_r^c \frac{2\mu}{r'^{1/2}} \cdot 4\pi r'^2 dr' \rho.$$

$$\boxed{\begin{aligned} F &= -\frac{\mu}{r^{3/2}} \\ +\frac{\partial V}{\partial r} &= +\frac{\mu}{r^{3/2}} \\ \Rightarrow V &= \frac{2\mu}{r^{1/2}} \end{aligned}}$$

$$= \frac{16\pi\rho\mu(c^{5/2} - r^{5/2})}{5}$$

$$\frac{8\cancel{16}\cancel{\pi}\cancel{\rho}\mu}{5}(c^{5/2} - r^{5/2}) = \cancel{2}\cancel{\pi}\cancel{\rho}\, r^3 v^2.$$

$$v^2 = \frac{8\mu(c^{5/2} - r^{5/2})}{r^3} \Rightarrow \frac{dr}{dt} = -\left(\frac{8\mu(c^{5/2} - r^{5/2})}{5 \cdot r^3}\right)^{1/2}$$

(As r decreases with time)

$$\frac{-r^{3/2}\, dr}{\sqrt{\frac{8\mu}{5}}\sqrt{c^{5/2} - r^{5/2}}} = dt.$$

$$r^{5/2} = c^{5/2} \sin^2\theta.$$

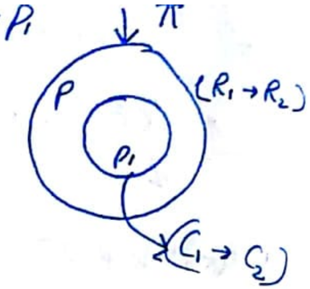
$$\frac{5}{2} r^{3/2} = c^{5/2}\, 2\sin\theta\cos\theta\, d\theta$$

$$\frac{c^{5/4}(-2/5) \cdot 2\cancel{c^{5/2}} \sin\theta \cancel{\cos\theta}\, d\theta \sqrt{5}}{\sqrt{8\mu} \cdot \cancel{c^{5/4}\cos\theta}} = dt.$$

$$\boxed{T = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}} \text{ Ans}$$

② Outside pressure Π. Initial pressure inside $= P_1$

Find velocity of inner surface (v) when R_2, C_2 is reached.



$\Rightarrow$ gas obeys Boyle's law.

$$\frac{4}{3}\pi(R_1^3 - C_1^3) = \frac{4}{3}\pi(R^3 - c^3)$$

$\cancel{r^2 v = F(t)}$ $\quad x^2 v = r^2 v_1 \leftarrow$ velocity at inner boundary.

$\cancel{-\frac{F'(t)}{r}} + \frac{1}{2}v^2 = \cancel{C - \frac{p}{\rho}} \Rightarrow$ Let (c, R) be intermediate config

$$\boxed{P_1 \cdot \frac{4}{3}\cancel{\pi} C_1^3 = P_2 \cdot \frac{4}{3}\cancel{\pi} C_2^3}$$

$$KE = \frac{1}{2}\int_r^R 4\pi x^2 \,\delta x\, \rho v^2 = 2\pi\rho \int_r^R x^2 \frac{r^4 v_1^2}{x^4}\, dx$$

$$= 2\pi\rho r^4 v_1^2 \left[\frac{1}{r} - \frac{1}{R}\right]$$

$$PE = \int_{R_1}^{R} \Pi \, 4\pi x^2 (\delta x) = 4\Pi\pi\left(\frac{R_1^3}{3} - \frac{R^3}{3}\right) + 4\pi P_1 C_1^3 \log\frac{c}{C_1}$$

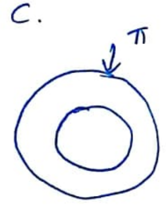
$$+ \int_{C_1}^{c} \frac{P_1 C_1^3}{x^3} \cdot 4\pi x^2\, dx =$$

$$\frac{4}{3}\pi\Pi(R_1^3 - R^3) + 4\pi P_1 C_1^3 \log\frac{c}{C_1} = 2\pi\rho c^4 v_1^2 \left[\frac{1}{c} - \frac{1}{R}\right]$$

At $c = C_2$, $R = R_2 \Rightarrow$

③ Fluid (ρ, volume = $\frac{4\pi c^3}{3}$) in spherical shell.
Π on external shell, no P on internal shell.
Initial $\equiv$ internal radius = $2c$.
Find velocity of inner surface when radius is c.

Done



At any time t, r and R be radii
v and V be velocity

In between $\quad r'^2 v' = r v = F(t)$.

$r'^2 v' = r^2 v = F(t)$.

$R^3 - r^3 = c^3 \Rightarrow R^3 = r^3 + c^3$

$$KE = \int_r^R \frac{1}{2}\rho(4\pi r_1^2)\,dr_1\, v'^2 \Rightarrow 2\pi\rho \int_r^R \frac{r_1^2 . r^4 v^2}{r_1^4}\,dr_1$$

$$= 2\pi\rho r^4 v^2 \left[\frac{1}{r} - \frac{1}{R}\right]$$

$$PE = \int_{2c}^{r} -\Pi .\, 4\pi r_1^2\,dr_1$$

$$= +\frac{4\pi\Pi}{3}\left(-r^3 + 8c^3\right).$$

$$KE = PE \Rightarrow \frac{4\pi\cancel{\Pi}}{3}\left(8c^3 - r^3\right) = 2\cancel{\pi}\rho r^4 v^2 \left[\frac{1}{r} - \frac{1}{(r^3+c^3)^{1/3}}\right]$$

Set $r = c$ to get $v \Rightarrow$

$$\frac{4\pi}{3} . 7\cancel{c^3} = 2\rho c^4 v^2 \left(\frac{1}{c} - \frac{1}{c \times 2^{1/3}}\right)$$

$$v = \left(\frac{14\pi}{3\rho}\,\frac{2^{1/3}}{2^{1/3} - 1}\right)^{1/2}$$

Type 3

(1) A steady inviscid incompressible flow with velocity $(fx, -fy, 0)$. Find pressure field $p(x,y,z)$ if $p(0,0,0) = p_0$ and $F = \cancel{-g\hat{k}} - gz\hat{k}$

$\Rightarrow$ Must satisfy equations of motion.

$$fx \cdot f = \cancel{X}^{0} - \frac{\partial p}{\rho \partial x} \qquad -f^2 y = \cancel{Y}^{0} - \frac{\partial p}{\rho \partial y}.$$

$$0 = -gz - \frac{1}{\rho}\frac{\partial p}{\partial z}.$$

$$\frac{\partial p}{\partial x} = -f^2 \rho x \;; \frac{\partial p}{\partial y} = f^2 \rho y \;; \frac{\partial p}{\partial z} = -\rho g z.$$

$$dp = -f^2 \rho x \, dx + f^2 \rho y \, dy - \rho g z \, dz.$$

$$\boxed{p = \left(-f^2 \rho x^2 + f^2 \rho y^2 - \rho g z^2\right)\frac{1}{2} + C ?\, P_0} \text{ Ans}$$

Q A fluid of density ρ bound by concentric spheres. Outer surface at uniform pressure Π and contracts R_1 to R_2. Hollow is filled with gas obeying Boyle's law with initial pressure P_1 and contracts C_1 to C_2. Find velocity of inner surface when (R_2, C_2) is reached.

$\rightarrow r_1^2 v_1 = \cancel{r_2}^2 v = F(t)$

$\rightarrow P_1 \frac{4}{3}\pi C_1^3 = P_2 \frac{4}{3}\pi C_2^3 \Rightarrow \left(P_2 = \frac{C_1^3}{C_2^3} P_1\right)$

$$KE = \int_{C_2}^{R_2} \frac{1}{2}\left(4\pi r_1^2 dr_1 \rho\right) \frac{C_2^4 v^2}{r_1^4}$$

$$= 2\pi\rho C_2^4 v^2 \left[\frac{1}{C_2} - \frac{1}{R_2}\right]$$

(No - sign)

$\Downarrow$

$$\text{Work done} = \int_{R_1}^{R_2} \Pi \, 4\pi R_2^2 (-dR_2) + \int_{C_1}^{C_2} \frac{P_1 C_1^3}{C^3} 4\pi C^2 (dc)$$

$$= \Pi\pi \cdot \frac{4}{3}(R_1^3 - R_2^3) + 4\pi P_1 C_1^3 \log\frac{C_2}{C_1}$$

By COM $\Rightarrow \boxed{R_1^3 - R_2^3 = C_1^3 - C_2^3}$

$$\underbrace{\frac{4}{3}\pi\Pi(C_1^3 - C_2^3) + 4\pi P_1 C_1^3 \log\frac{C_2}{C_1} = 2\pi\rho C_2^3 v^2\left(1 - \frac{C_2}{R_2}\right)}_{\text{Ans}}$$

9 Liquid of density between cylinders (a, b). Outer pressure on a is Π. Find velocity v of inner surface when inner radius is r.

At any time $t \equiv$ Inner (r, v) $p=0$
Outer (R, V) $p = \Pi$.

USELESS

For any point in fluid $\Rightarrow$ $r'v' = F(t)$

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'}$$

$$\frac{F'(t)}{r'} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

(ENERGY METHOD)

Integrating $\rightarrow$

$$F'(t)\log r' + \frac{1}{2}v'^2 = C - \frac{p}{\rho}$$

$$1 \quad F'(t)\log r + \frac{v^2}{2} = C.$$

$$2 \quad F'(t)\log R + \frac{V^2}{2} = C - \frac{\Pi}{\rho}.$$

$$\boxed{F'(t)\left(\log \frac{r}{R}\right) + \frac{v^2 - V^2}{2} = \frac{\Pi}{\rho}} \quad (A).$$

$rv = RV = F(t)$ (1). $\Rightarrow r\frac{dr}{dt} = R\frac{dR}{dt} = F(t) \Rightarrow \boxed{r\,dr = R\,dR = F(t)\,dt}$

$R^2 - r^2 = a^2 - b^2.$

$$F'(t) = \frac{d}{dt}(rv) = \frac{d}{dr}(rv)\cdot\frac{dr}{dt} = \frac{d}{dr}(rv)\cdot v. \quad — (2)$$

But, values in (A) of $F'(t)$, V from (1), (2)

$$v\frac{d}{dr}(rv)$$

*** Energy method**

When (r, R) $\quad r'v' = rv = F(t)$

$$\int_r^R \frac{1}{2}(4\pi r'^2 dr')\rho v'^2 = 2\pi\rho\int_r^R \cancel{r'^2}\frac{r^2v^2}{\cancel{r'^2}}\,dr'$$

$$KE = 2\pi\rho r^2 v^2 (R-r)$$

Potential Energy

$$\cancel{\int_a^R -(4\pi r'^2 dr')\pi} \quad \int_a^R (-2\pi r' h)\,dr'\,\pi$$

$$= \boxed{\frac{2\pi h \pi}{2}(a^2 - R^2)}$$

$$KE = \int_r^R \frac{1}{\cancel{2}}(\cancel{2}\pi \cancel{r'} h\,dr')\rho\,\frac{r^2v^2}{r'^2}$$

$$= \pi r^2 v^2 \rho h \log(R/r)$$

$$\boxed{a^2 - b^2 = R^2 - r^2}$$

$$\cancel{2}\cancel{\pi}h\pi(a^2 - R^2) = \cancel{\pi}r^2v^2\rho\log(R-r)$$

$$v^2 = \frac{\pi(a^2 - R^2)}{r^2\rho\log\left(\sqrt{r^2+a^2-b^2}/r\right)}$$

$$\boxed{= \frac{2\pi(b^2 - \cancel{r^2})}{r^2\rho\log\left(\frac{r^2+a^2-b^2}{r}\right)}}$$

Matches

* Sources and Sinks $\left(q_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; q_\theta = \frac{\partial \psi}{\partial r} \right)$

- $w = \phi + i\psi$ — ψ: Stream function; w: Complex potential; ϕ: Velocity potential.

- $(u, v) = \left(\frac{-\partial \phi}{\partial x}, \frac{-\partial \phi}{\partial y} \right) = \left(\frac{-\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right)$ (ϕ, ψ satisfy C-R also Laplace)

- $q = \left| \frac{dw}{dz} \right| = (u^2 + v^2)^{1/2}$ [Stagnation points $\Rightarrow q = 0$]

- To Show, velocity potential exist → Show irrotational
  - Evaluate Ω_z

→ Complex Potential $\equiv$ $\boxed{\text{Strength} = m, \text{ Rate of output} = 2\pi m}$

- $-m \log (z - z_0) = w = \phi + i\psi$
  - $\phi = -m \log r$
  - $\psi = -m\theta$

$$\boxed{\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2 + \beta^2) + i \tan^{-1}\frac{\beta}{\alpha}}$$

→ Images

- Plane boundaries at O $\Rightarrow$ $\boxed{n = \frac{\pi/2}{\alpha/2}}$ — $r, \alpha \rightarrow$ (1) $r^n, n\alpha$. (2) $\boxed{z \rightarrow z^n}$
  - (β_1, β_1) — $(n\beta_1, \beta_2 n)$ — Reflect in transform boundar
  - $w = -m\log(z^n - r^n e^{in\alpha})$

- Circle → m outside $\Rightarrow$ $\boxed{m\,(a^2/f), -m \text{ at } (O)}$ (Same if m inside)

- Milne Thomson $\equiv$ w before cylinder (no boundary flow)
  - After $|z| = a$ inserted → $\boxed{w_{new} = f(z) + \bar{f}(a^2/z)}$ for $|z| > a$

- Curves of $\phi = k_1$, $\psi = k_2$ intersect orthogonally

Then, $\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0$

- $\boxed{\frac{dw}{dz} = -u + iv}$

- Irrotational $\equiv \nabla \times q = 0$. (Not $\nabla^2 \psi$ or $\nabla^2 \phi$)

  If irrotational, then $\underline{\nabla^2 \phi / \nabla^2 \psi = 0}$.

- Slope of $\phi \equiv m_1 = \frac{-\partial \phi / \partial x}{\partial \phi / \partial y}$ (Same for ψ).

- Ovals of cassini $\Rightarrow rr' =$ constant.

  $\left| \frac{1}{z+a} - \frac{1}{z-a} \right| = \frac{2a}{|z+a||z-a|} = \frac{2a}{rr'}$

- $X - iY = \frac{1}{2} pi \int_C \underbrace{\left(\frac{dw}{dz}\right)^2 dz}_{\text{Residues}}$.

  $M = Re \left\{ -\frac{1}{2} ip \int_C z \left(\frac{dw}{dz}\right)^2 dz \right\}$

* For Reflection, Only that boundary on which source/sink does not lie.

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① Show $\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$ is possible.

Find streamlines.

- $\phi = \frac{1}{2}\left[\log(x+a)^2+y^2 - \log(x-a)^2+y^2\right]$

$$u = -\frac{\partial\phi}{\partial x} = \frac{-1}{2}\left[\frac{2(x+a)}{(x+a)^2+y^2} - \frac{2(x-a)}{(x-a)^2+y^2}\right]$$

$$v = -\frac{\partial\phi}{\partial y} = -\frac{1}{2}\left[\frac{2y}{(x+a)^2+y^2} - \frac{2y}{(x-a)^2+y^2}\right]$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left[\frac{2((x+a)^2+y^2) - \cancel{2(x+a)}\cdot 2(x+a)}{((x+a)^2+y^2)^2} - \frac{2((x-a)^2+y^2) - 2(x-a)}{((x-a)^2+y^2)^2}\right]$$

$2(x+a)^2$

$$\frac{\partial v}{\partial y} = -\frac{1}{2}\left[\frac{2((x+a)^2+y^2) - 2y\cdot 2y}{((x+a)^2+y^2)^2} - \frac{2((x-a)^2+y^2) - 2y\cdot 2y}{(x-a)^2+y^2}\right]$$

So, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

② Sources (m) at $(-a,0), (a,0)$, sink $(-2m)$ at $(0,0)$. Find streamlines and fluid speed at any point.

$$w = -m\log(z+a) - m\log(z-a) + 2m\log z.$$

$$= -m\log(x^2-y^2-a^2+i2xy) + m\log(x^2-y^2+i2xy)$$

$$\psi = -m\left(\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right) - \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)\right)$$

$$= -m\tan^{-1}\left[\frac{\frac{2xy}{x^2-y^2-a^2} - \frac{2xy}{x^2-y^2}}{\frac{(x^2-y^2-a^2)(x^2-y^2)+4x^2y^2}{(x^2-y^2-a^2)(x^2-y^2)}}\right]$$

$$= -m\tan^{-1}\left[\frac{\cancel{2x^3y} - \cancel{2xy^3} - \cancel{2x^3y} + \cancel{2xy^3} + 2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right]$$

$$= m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right] = \text{constant} = m\tan^{-1}\left(\frac{2}{\lambda}\right)$$

$$\text{Fluid speed} = \left| -\frac{m}{z+a} - \frac{m}{z-a} + \frac{2m}{z} \right|$$

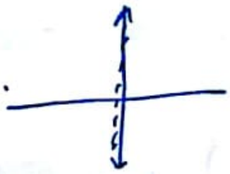
$$= \left| \frac{\cancel{-mz^2} + \cancel{amz} - \cancel{mz^2} - \cancel{amz} + \cancel{2mz^2} - 2ma^2}{z(z+a)(z-a)} \right|$$

$$= \left| \frac{2ma^2}{r r_1 r_2} \right| \quad \underline{\underline{\text{Ans}}}$$

③ Boundaries $\{-\frac{\pi}{4}, \frac{\pi}{4}\}$. Source $m(a,0)$, Sink $-m(b,0)$.

Find stream function.

$\rightarrow \quad \dfrac{\pi/2}{\pi/4} = 2 \Downarrow$



$m(a^2,0) \quad m(-a^2,0)$

$-m(b^2,0) \quad -m(-b^2,0)$.

$\text{Bdry} \equiv \{-\frac{\pi}{2}, \frac{\pi}{2}\}$

$$w = -m\log(z^2-a^2) - m\log(z^2+a^2) + m\log(z^2-b^2) + m\log(z^2+b^2).$$

$$= -m\log(r^4e^{i4\theta}-a^4) + m\log(r^4e^{i4\theta}-b^4)$$

$$\psi = -m\tan^{-1}\left(\frac{r^4\sin 4\theta}{r^4\cos 4\theta - a^4}\right) + m\tan^{-1}\left(\frac{r^4\sin 4\theta}{r^4\cos 4\theta - b^4}\right)$$

$$\psi = -m\tan^{-1}\left(\frac{r^8\sin 4\theta\cos 4\theta - b^4r^4\sin 4\theta - r^8 + a^4r^4\sin 4\theta}{r^8\cos^2 4\theta - a^4r^4\cos 4\theta - b^4r^4\cos 4\theta + a^4b^2}\right)$$

$$\boxed{\psi = -m\tan^{-1}\left(\frac{r^4\sin 4\theta\,(a^4-b^4)}{r^8 - r^4a^4\cos 4\theta - b^4\cos 4\theta + a^4b^4}\right)}$$ Ans

---

If Boundary at $\theta = 0, \frac{\pi}{3}$. $\Rightarrow \dfrac{\frac{\pi}{2}}{\frac{\pi}{6}} = 3$

$\Rightarrow \boxed{\theta = 0, \pi} \leftarrow$ Transformed

(i) Fluid fills space on the side of x axis which is a rigid boundary and Source m $(0,a)$, sink $(-m)$ $(0,b)$ and if pressure on negative side be same as P at ∞. Show resultant pressure on boundary is $\frac{\pi \rho m^2 (a-b)^2}{2ab(a+b)}$

→ To find pressure at boundary, find q at boundary

· $-m\,(0,bi)$
· $m\,(0,ai)$

· $m\,(0,-ai)$
· $-m\,(0,-bi)$

$$\frac{p}{\rho} + \frac{q^2}{2} = \text{constant} = \frac{p_0}{\rho} \quad \text{at } \infty.$$

$$w = -m\log(z^2+a^2) + m\log(z^2+b^2)$$

$$\cancel{\psi = -m\, Im}$$

$$\frac{p}{\rho} + \frac{q^2}{2} = \frac{p_0}{\rho}$$

$$\text{Find } \left|\frac{dw}{dz}\right|_{y=0} = \left| -\frac{m.2z}{(z^2+a^2)} + \frac{m.2z}{(z^2+b^2)} \right|$$

$$= \left| 2xm \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2} \right) \right|$$

$$q = \frac{2mx(a^2-b^2)}{(x^2+b^2)(x^2+a^2)} \quad \left(\frac{x}{x^2+b^2} - \frac{x}{x^2+a^2} \right)$$

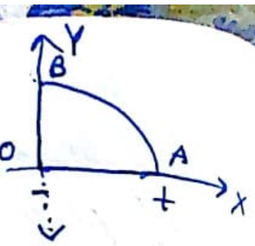
$$\frac{q^2}{2} = \frac{p_0 - p}{\rho} \Rightarrow p_0 - p \Rightarrow \boxed{\frac{1}{2}\rho q^2}$$

$$\text{Resultant pressure} \Rightarrow \frac{1}{2}\rho \int_0^\infty \frac{(2m)^2(a^2-b^2)^2 x^2}{((x^2+b^2)(x^2+a^2))^2}\,dx.$$

$$= 2\rho m^2 \int_0^\infty \frac{x^2(a^2-b^2)^2\,dx}{(x^2+b^2)^2(x^2+a^2)^2} = \frac{\pi \rho m^2 (a-b)^2}{2ab(a+b)} \quad \text{Ans}$$

$$\text{Assum } \frac{A}{x^2+a^2} + \frac{B}{(x^2+a^2)^2} + \frac{C}{(x^2+b^2)} + \cdots$$

⑤ Show that the streamline leaving either end at $\angle\alpha$ with radius is $\left(r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta)\right)$



Equivalent Image system

- m at $z=a$, $z=-a$
- $-m$ at $z=0$.

$$w = -m\log\left(\frac{z^2-a^2}{z}\right) = -m\log\left(z-a^2z^{-1}\right)$$

$$= -m\log\left(r\cos\theta + ir\sin\theta - a^2r^{-1}\cos\theta + ia^2r^{-1}\sin\theta\right)$$

$$= -m\log\left(\cos\theta\left(r-\frac{a^2}{r}\right) + i\left(r\sin\theta + \frac{a^2}{r}\sin\theta\right)\right)$$

$$\psi = -m\tan^{-1}\left[\frac{\sin\theta\,(r+a^2/r)}{\cos\theta\,(r-a^2/r)}\right]$$

$$\psi = -m\tan^{-1}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right\}$$

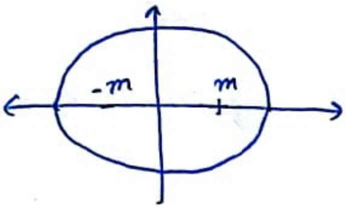
$$\boxed{\text{Streamline at angle } \alpha \equiv -m(\pi-\alpha)}$$

IMP Takeaway

$$\tan\,\cancel{\tan^{-1}}\left\{\frac{r^2+a^2}{r^2-a^2}\tan\theta\right) = \tan(\pi-\alpha) = -\tan\alpha$$

$$\boxed{(r^2+a^2)\tan\theta + (r^2-a^2)\tan\alpha = 0}$$ Ans

⑥ Source and sink at $(\pm a/2, 0)$ within $|z| = a$.
Show streamlines are $\left(r^2 - \frac{a^2}{4}\right)\left(r^2 - 4a^2\right) - 4a^2y^2 = ky\left(r^2 - a^2\right)$



$\Rightarrow$ As circular boundary, so we can use Milne Thomson theorem

– W/o boundary

$$w = f(z) = -m\log\left(z - \tfrac{a}{2}\right) + m\log\left(z + \tfrac{a}{2}\right)$$

$$= -m\log\left(\frac{z - a/2}{z + a/2}\right)$$

$-m\log(z - a/2) + m\log(z + a/2)$
$-m\log(2a^2 - az) + m\cancel{\log(2z)}$
$+ m\log(2a^2 + az) - m\cancel{\log(2z)}$
⊛ Continue from here.

After boundary $\Rightarrow$

$$w = f(z) + \bar{f}(a^2/z) = -m\log\left(\frac{z - a/2}{z + a/2}\right) - m\log\left(\frac{\frac{a^2}{z} - \frac{a}{2}}{\frac{a^2}{z} + \frac{a}{2}}\right)$$

$$= -m\log\left(z - \tfrac{a}{2}\right) - m\log\left(\frac{a^2}{z} - \frac{a}{2}\right) + m\log\left(a^2 + \frac{a^3}{2z} + \frac{az}{2} + \frac{a^2}{4}\right)$$

$$= -m\log\left(a^2 - \frac{a^3}{2z} - \frac{az}{2} + \frac{a^2}{4}\right) + m\log\left(\frac{5a^2}{4} + \frac{a^3(x - iy)}{2(x^2 + y^2)} + \frac{a}{2}(x + iy)\right)$$

$$= -m\tan^{-1}\left(\frac{-\frac{ay}{2} + \frac{a^3y}{2(x^2+y^2)}}{\frac{5a^2}{4} - \frac{ax}{2} - \frac{a^3x}{2(x^2+y^2)}}\right) + m\tan^{-1}\left(\frac{\frac{ay}{2} - \frac{a^3y}{2(x^2+y^2)}}{\frac{5a^2}{4} + \frac{ax}{2} + \frac{a^3x}{2(x^2+y^2)}}\right)$$

$$= -m\tan^{-1}\left(\frac{-ayx^2 - ay^3 + a^3y}{5a^2x^2 + 5a^2y^2 - 2ax^3 - 2axy^2 - 2a^3x}\right) + m\tan^{-1}\left(\frac{ayx^2 + ay^3 - a^3y}{5a^2x^2 + 5a^2y^2 + 2ax^3 + 2axy^2 + 2a^3x}\right)$$

$$= -m\tan^{-1}\left(\frac{ay(a^2 - r^2)}{5a^2r^2 - 2axr^2 - 2a^3x}\right) + m\tan^{-1}\Big($$

7 Infinite vortex rows

W = Due to an infinite row

vortices $-k$ at $z = na$

$$W = \frac{ik}{2\pi} \sum_{n=-\infty}^{\infty} \log (z - na)$$

$$= \frac{ik}{2\pi} \log \left[z(z^2 - a^2)(z^2 - 2^2a^2) \cdots\cdots \right]$$

$$= \frac{ik}{2\pi} \log \left[\frac{\pi z}{a} \left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right) \cdots \right]$$

$$+ \frac{ik}{2\pi} \log \left[(-1)^n \frac{a}{\pi} a^2 . 2^2a^2 \ldots n^2a^2 \right]$$

$$= \frac{ik}{2\pi} \log \left[\frac{\pi}{a} z \left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right) \cdots \right] + C$$

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(8) 2 infinite rows $[x = ra, y = b, k]$, $[x = ra, y = -b, -k]$.
Prove the street moves as a whole along its length.

$$w = \frac{ik}{2\pi} \sum_{-\infty}^{\infty} \log(z - na - ib) - \frac{ik}{2\pi} \sum \log(z - na + ib)$$

$$\cancel{\frac{ik}{2\pi} \log(z + i(y-b)) + \log}$$

For a pair, remember

$$\boxed{w = \frac{ik}{2\pi} \log\left(\sin \frac{\pi(z - ib)}{a}\right) - \frac{ik}{2\pi} \log\left(\sin \frac{\pi(z+ib)}{a}\right)}$$

Vortex Motion

$i \quad j \quad k$
$x \quad y \quad z$
$u \quad v \quad w$

① Vorticity.

$$\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

② $w = \frac{ik}{2\pi} \log(z - z_0) = \phi + i\psi$

$\phi = \frac{-k\theta_0}{2\pi}$, $\psi = \frac{k}{2\pi} \log r_0$

. $k \equiv$ strength of vortex

$\boxed{z_0 = r_0 e^{i\theta_0}}$

$u = -\frac{k}{2\pi} \frac{y - y_0}{r_0^2}$ $\quad$ $v = \frac{k}{2\pi} \frac{x - x_0}{r_0^2}$

$\boxed{q = -\frac{k}{2\pi r_0}} = \sqrt{u^2 + v^2}$ IMP

Mind

$\boxed{-u + iv = +\frac{dw}{dz}}$

③ Combination of vortices

- Vortex pair [Equal and opposite] $\Rightarrow \boxed{v = \frac{k}{2\pi d}}$ $k \quad v \quad -k$
- Two of same direction (sign) $\Rightarrow \boxed{\omega_0 = \frac{k_1 + k_2}{2\pi d^2}}$

* Finding $w \equiv$ superimpose velocity with (-) sign
  ↳ Add $\boxed{\frac{kz}{2\pi d}}$ in w $\Rightarrow$ $\left(\int \frac{k}{2\pi d} dz = \frac{kz}{2\pi d} \right)$.

Laplace $\equiv \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$

(4) Images

→ Plane (opposite) $\cdot k$ | $\cdot -k$.

→ Circular Cylinder (at A) -

(1)

(2)

- To Show stationary (At points of vortex)
  - Find w' by considering others (excluding that vortex)
  - Show $\left(\frac{dw'}{dz}\right) = 0$ at $z = a$ (vortex point)

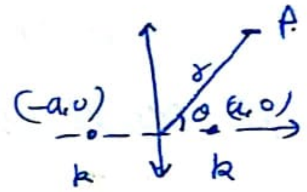
*Show fluid motion is stationary ≡ Show q at all vortices = 0

- Find stagnation points. $\frac{dw}{dz} = 0$

  (w ≡ include all vortices)

① 2 vortices of same strength, find rel. stream lines

$$w = \frac{ik}{2\pi}\log\left(re^{i\theta} - a\right) + \frac{ik}{2\pi}\log\left(re^{i\theta} + a\right).$$



r, θ, k, a given

$$= \frac{ik}{2\pi}\log\left(r\cos\theta - a + i\sin\theta . r\right)$$
$$+ \frac{ik}{2\pi}\log\left(r\cos\theta + a + ir\sin\theta\right)$$

$$\psi = \frac{k}{2\pi}\log\left[\left(r^2 + a^2 - 2ar\cos\theta\right)\left(r^2 + a^2 + 2ar\cos\theta\right)\right]^{1/2}.$$

Also, system rotates with $w = \frac{2k}{2\pi(2a)^2} = \boxed{\frac{k}{4\pi a^2}}$

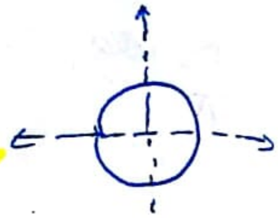
So, impose $\Rightarrow -wr \equiv \frac{-kr}{4\pi a^2}$ on system

$$\frac{\partial \psi_1}{\partial r} = -\frac{kr}{4\pi a^2} \Rightarrow \boxed{\psi_1 = \frac{-kr^2}{8\pi a^2}}$$ ← IMP

$$\psi_{Fin} = \psi + \psi_1$$

② n vortices (k) are symmetrical in a cylinder $|z|=a$, find motion of vortices and fluid.
(velocity of fluid, time to move a vortex around $|z|=a$) ③

$\Rightarrow$ n vortices are at.



$$\left(ae^{i\frac{2\pi}{n}}, ae^{i\frac{4\pi}{n}} \dots a^{i2\pi}\right) \equiv a^{\frac{i2k\pi}{n}}$$

$$w = \frac{ik}{2\pi}\log(z-z_0) + \frac{ik}{2\pi}\log(z-z_1) + \dots \frac{ik}{2\pi}\log(z-z_{n-1})$$

$$= \frac{ik}{2\pi}\log(z^n - a^n).$$

• Fluid velocity $= |U-iV| = \left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{nz^{n-1}}{z^n-a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n-a^n}\right|$

Velocity of vortex at $z = ae^{i2\pi} = a$

$$w' = w - \frac{ik}{2\pi}\log(z-a)$$

$$= \frac{ik}{2\pi}\log\left(\frac{z^n-a^n}{z-a}\right) = \frac{ik}{2\pi}\log\left(z^{n-1}+az^{n-2}+\dots a^{n-1}\right)$$

$$\left(\frac{dw}{dz}\right)_{z=a} = \frac{ik}{2\pi}\left(\frac{(n-1)z^{n-2}+a(n-2)z^{n-3}+\dots a^{n-2}}{na^{n-1}}\right)_a$$

$$= \frac{ik}{2\pi}\left(\frac{1+2+\dots n-1}{na}\right) = \frac{ik}{2\pi}\frac{\cancel{n}(n-1)}{2\cancel{n}a} = \frac{ik(n-1)}{4\pi a}$$

$$q = \left|\frac{dw}{dz}_{z=a}\right| = \frac{k(n-1)}{4\pi a}.$$

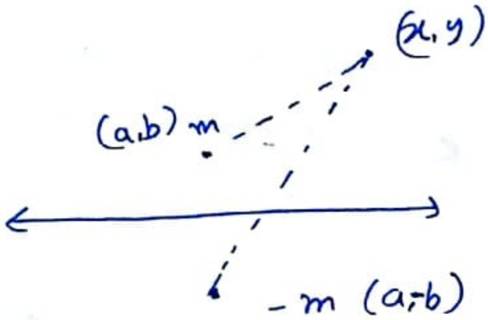
$$T = \frac{2\pi a \cdot 4\pi a}{k(n-1)} = \boxed{\frac{8\pi^2 a^2}{(n-1)k}}$$ Ans

③ Rigid wall at $y=0$ and a vortex (m).

Find surfaces of equal pressure (need x, y expressions)

Coordinates of vortex (a,b)

→ Image system is as shown.



$$w = \frac{im}{2\pi}\left(- \quad \cdots \quad \right)$$

⇒ For surfaces of equal pressure ⇒ q must be equal.

$$\boxed{\frac{p}{\rho} + \frac{q^2}{2} = \text{Constant}}$$

$$q = \sum \frac{-k}{2\pi r_0} = \frac{-k}{2\pi}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2}} + \frac{1}{\sqrt{(x-a)^2+(y+b)^2}}\right]$$

$$q^2 = \frac{k^2}{4\pi^2}\left[\frac{1}{(x-a)^2+(y-b)^2} + \frac{1}{(x-a)^2+(y+b)^2} + \frac{2}{\sqrt{(x-a)^2+(y-b)^2}\sqrt{(x-a)^2+(y+b)^2}}\right]$$

$$q^2 = \text{Constant} \Rightarrow \frac{(x-a)^2+(y+b)^2+(x-a)^2+(y-b)^2+2\sqrt{r_1 r_2}}{\left((x-a)^2+(y-b)^2\right)\left((x-a)^2+(y+b)^2\right)} = k$$

$$\frac{r_1^2+r_2^2+2r_1r_2}{r_1^2 r_2^2} = k \Rightarrow$$

Instead

$$u = -\frac{k}{2\pi}\frac{(y-y_0)}{r_1^2} \; ; \; v = \frac{k}{2\pi}\frac{(x-x_0)}{r_1^2}$$

$$q^2 = u^2 + v^2$$

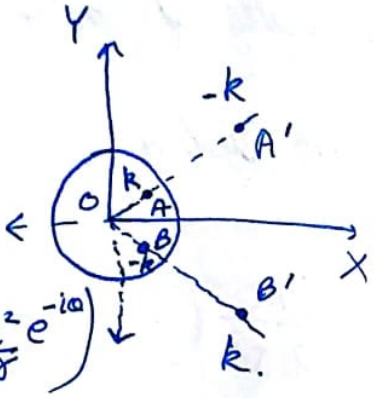
$$\frac{p}{\rho} + \frac{q^2}{2} = \text{Constant}$$

11.27

④ A pair of equal and opposite rectilinear vortices in $|z| = a$. Find path of each vortex.

$A = r, \theta \qquad A' = \frac{a^2}{r}, \theta$

$B = r, -\theta \qquad B' = \frac{a^2}{r}, -\theta$



$$w = \frac{ik}{2\pi}\log\left(z - r^{i\theta}\right) + \frac{ik}{2\pi}\log\left(z - \frac{a^2}{r}e^{-i\theta}\right) - \frac{ik}{2\pi}\log\left(z - re^{-i\theta}\right) - \frac{ik}{2\pi}\log\left(z - \frac{a^2}{r}e^{i\theta}\right)$$

Velocity of vortex at P is due to w of other 3 ⇒

$$w_1 = \frac{ik}{2\pi}\log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right) - \frac{ik}{2\pi}\log\left(re^{i\theta} - re^{-i\theta}\right) - \frac{ik}{2\pi}\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right)$$

$\psi = \ ⇙$

$$\left(r\cos\theta - \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta + \frac{a^2}{r}\sin\theta\right)$$

$\downarrow$

$$\frac{1}{2} \times \frac{k}{2\pi}\log\left[\left(r - \frac{a^2}{r}\right)^2\cos^2\theta + \left(r - \frac{a^2}{r}\right)^2\sin^2\theta\right]^{1/2} - \frac{k}{2\pi}\log(2r\sin\theta) - \frac{k}{2\pi}\log\left(r - \frac{a^2}{r}\right)$$

$$\psi = -\frac{k}{4\pi}\log\boxed{\left\{\frac{\left(r - \frac{a^2}{r}\right)^2 \times (2r\sin\theta)^2}{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta}\right\}} = b^2$$

Path ⇒ $b^2\left(r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right) = (r^2 - a^2)^2(4\sin^2\theta)$

$b^2\left(r^4 + a^4 - 2a^2r^2\cos 2\theta\right) = (r^2 - a^2)^2(4r^2\sin^2\theta)$

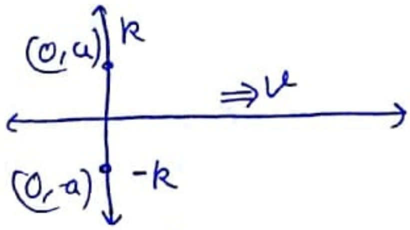
$b^2\left((r^2 - a^2)^2 + 2a^2r^2(1 - \cos 2\theta)\right) = (r^2 - a^2)^2 r^2\sin^2\theta$

⇒ $4a^2b^2r^2\sin^2\theta = (r^2 - a^2)^2(r^2\sin^2\theta - b^2)$ ⇐ Req'd form

(5) Two parallel equal and opposite vortices at distance $2a$, prove streamlines relative to vortices $\boxed{\log \frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a} = c}$

As a vortex pair (equal & opposite)

L move with constant $\boxed{v = \frac{k}{4\pi a}}$



$q = \frac{dw}{dz} \Rightarrow w = \frac{kz}{4\pi a}$ (to be ~~reversed~~ added to given w) (for relative streamlines)

$$w = \frac{ik}{2\pi} \log(z-ia) - \frac{ik}{2\pi} \log(z+ia) + \frac{kz}{4\pi a}$$

To find $\psi \Rightarrow$ get imaginary part of w

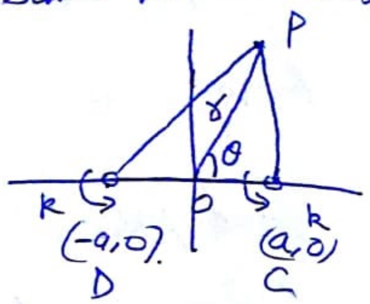
$$\psi = \frac{k}{2\pi} \log\sqrt{x^2+(y-a)^2} - \frac{k}{2\pi} \log\sqrt{x^2+(y+a)^2} + \frac{ky}{4\pi a}$$

$$\psi = \frac{k}{4\pi}\left[\log \frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a}\right] = C$$

$\log z = \log r + i\theta.$

Scanned by CamScanner

(5) Find relative streamlines for two same vortices with same spin and strength
(given θ is measured from origin, $2a$)



$$\omega = \frac{k_1+k_2}{2\pi(A_1A_2)^2} = \boxed{\frac{k}{4\pi a^2}}$$

$$CP = (r^2+a^2-2ar\cos\theta)^{1/2} \qquad DP = (r^2+a^2+2ar\cos\theta)^{1/2}$$

$$\psi = \frac{k}{2\pi}\log r_1 + \frac{k}{2\pi}\log r_2 = \frac{k}{2\pi}\log(r_1 r_2)$$

$$= \frac{k}{4\pi}\log((r^2+a^2)^2 - 4a^2r^2\cos^2\theta).$$

To find ψ relative we need to look at motion of vortices.

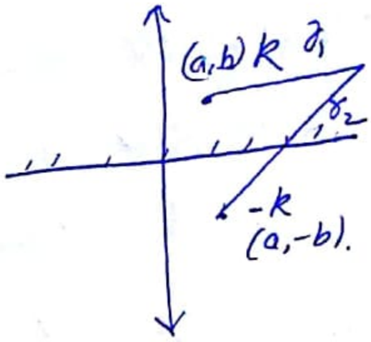
L Impose velocity $(-\omega r)$ on the system

$$\frac{\partial \psi_1}{\partial r} = -\omega r = -\frac{kr}{4\pi a^2} \Rightarrow \boxed{\psi_1 = -\frac{kr^2}{8\pi a^2}}$$

So, streamlines are (relative)

$$\psi+\psi_1 \equiv \frac{k}{4\pi}\left(\log(r^4+a^4-2a^2r^2\cos 2\theta) - \frac{r^2}{2a^2}\right) = \text{Constant}$$

⑥ To find surfaces of equal pressure.



$$\frac{p}{\rho}+\frac{q^2}{2} = \text{constant}$$

$\Rightarrow$ q^2 = constant gives required surfaces.

$$q^2 = u^2+v^2$$

$$u = \left(-\frac{k}{2\pi}\frac{(y-y_0)}{r_0^2}\right) = -\frac{k}{2\pi}\frac{(y-b)}{r_1^2}+\frac{k}{2\pi}\frac{(y+b)}{r_2^2}$$

$$v = \left(\frac{k}{2\pi}\frac{x-a}{r_1^2}\right)-\left(\frac{k}{2\pi}\frac{x+a}{r_2^2}\right)$$

- However, vortex pair is moving with velocity $\frac{k}{2\pi\cdot 2b} = \frac{k}{4\pi b}$ $\Rightarrow$ we need to superimpose this velocity.

$$u = -\frac{k}{2\pi}\frac{y-b}{r_1^2}+\frac{k}{2\pi}\frac{y+b}{r_2^2}-\frac{k}{4\pi b}$$

$$q^2 = \frac{k^2}{4\pi^2}\left[\frac{(x-a)^2+(y-b)^2}{r_1^4}+\frac{(x-a)^2+(y+b)^2}{r_2^4}-\frac{2(x-a)^2}{r_1^2r_2^2}+\frac{1}{4b^2} - 2\frac{(y^2-b^2)}{r_1^2r_2^2}-\cancel{2}\frac{(y+b)}{r_2^2}\cdot\frac{1}{\cancel{2}b}+\cancel{2}\cdot\frac{1}{\cancel{2}b}\frac{(y-b)}{r_1^2}\right]$$

$$= \frac{k^2}{\cancel{4\pi^2}}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\left((x-a)^2+(y^2-b^2)\right)}{r_1^2r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{(y+b)}{r_2^2}\right)+\frac{1}{4b^2}\right]$$

$= \underline{\text{constant}}$

$$r_1^2+r_2^2-2\left((x-a)^2+y^2-b^2)\right)+\frac{1}{b}\left((y r_2^2-b r_2^2)-y r_1^2-b r_1^2\right) = k r_1^2 r_2^2$$

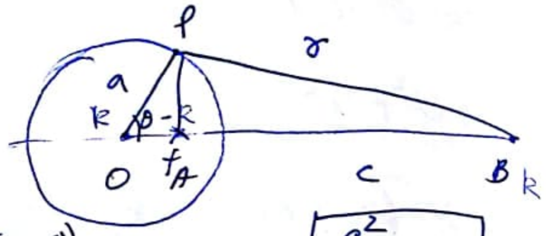
$$(y-b)^2+(y+b)^2-2(y^2-b^2)+\frac{1}{b}y(r_2^2-r_1^2)-(r_2^2+r_1^2) = k r_1^2 r_2^2$$

$4b^2$ ~~$+4y^2$~~.

$$r_2^2-r_1^2 = 4by.$$

(7) A vortex outside a cylinder (a) at distance c (k). Show speed of fluid at cylinder surface is $\frac{k}{2\pi a}\left(1-\frac{c^2-a^2}{r^2}\right)$, r is the distance of point from vortex.

· Equivalent system is as shown.



$$W = \frac{ik}{2\pi}\left(\log z + \log(z-c) - \log(z-f)\right)$$

$$\boxed{\frac{a^2}{f} = c}$$

$$\frac{dw}{dz} = \frac{ik}{2\pi}\left(\frac{1}{z} + \frac{1}{z-c} - \frac{1}{z-f}\right) = \frac{ik}{2\pi}\left(\frac{z^2-2fz+fc}{z(z-c)(z-f)}\right)$$

$$q = \left|\frac{dw}{dz}\right| = \frac{k}{2\pi}\frac{|z^2-2fz+fc|}{OP.AP.BP}$$

$$= \frac{k}{2\pi}\frac{2a(f-a\cos\theta)}{OP.AP.BP.}$$

$\Delta OPA \sim \Delta OAP.$

$$\frac{AP}{OP} = \frac{BP}{OB} \Rightarrow \boxed{AP = \frac{a r}{k}}$$

$$q = \frac{k}{2\pi}\frac{2a(f-a\cos\theta)}{a.\frac{ar}{c}.r.}$$

$$\left(\cos\theta = \frac{a^2+c^2-r^2}{2ac}\right)$$

$$= \frac{k}{\cancel{2}\pi}\frac{\cancel{2}c}{ar^2}\left(f - \cancel{a}.\frac{a^2+c^2-r^2}{2\cancel{a}c}\right)$$

$$= \frac{k\cancel{c}}{\pi a r^2}\left(\frac{2a^2-a^2-c^2+r^2}{2\cancel{c}}\right)$$

$$= \frac{k}{2\pi a}\left(\frac{a^2-c^2}{r^2}+1\right) \text{ Ans}$$

$$Z = ae^{i\theta}.$$

$$|z^2-2fz+fc| = |a^2e^{2i\theta} - 2afe^{i\theta} + a^2|$$

$$= |a^2 + a^2\cos 2\theta - 2af\cos\theta + i(a^2\sin 2\theta - 2af\sin\theta)|$$

$$= (\cancel{a^4} + a^4\cancel{\cos^2 2\theta} + 4a^2f^2\cancel{\cos^2\theta} + 2a^4\cos 2\theta - 4a^3f\cancel{\cos\theta\cos 2\theta} - 4a^3f\cos\theta + a^4\cancel{\sin^2 2\theta} + 4a^2f^2\cancel{\sin^2\theta} - 4a^3f\sin\theta\sin 2\theta)$$

$$= (a^4 + a^4 + 4a^2f^2 + 8a^3f(-\cos\theta) + 2a^4\cos 2\theta.)^{1/2}$$

$$= (\cancel{2a^4} + 4a^2f^2 - 8a^3f\cos\theta + 4a^4\cos^2\theta - \cancel{2a^4})^{1/2}$$

$$= 2a\left(f^2 - 2af\cos\theta + a^2\cos^2\theta\right)^{1/2}$$

$$= \boxed{2a(f - a\cos\theta)}$$

(8) Prove necessary and sufficient condition that vortex lines may be at right angles to streamlines are,

$$(u, v, w) = \mu\left(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z\right)$$

Vortex lines $\rightarrow$ $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \Rightarrow$ $\Omega_x = \partial w/\partial y - \frac{\partial v}{\partial z}$

$\Omega_y = \partial u/\partial z - \partial w/\partial x$

$\Omega_z = \partial v/\partial x - \partial u/\partial y$

Streamlines $\rightarrow$ $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$.

At right angles if

$$u\left(w_y - v_z\right) + v\left(u_z - w_x\right) + w\left(v_x - u_y\right) = 0$$

$\Downarrow$

This is the necessary and sufficient condition so that $u\,dx + v\,dy + w\,dz$ is a perfect differential.

$$u\,dx + v\,dy + w\,dz = \mu\,d\theta = \mu\left(\frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz\right)$$

$$\Rightarrow u = \mu\frac{\partial\theta}{\partial x},\ v = \mu\frac{\partial\theta}{\partial y},\ w = \mu\frac{\partial\theta}{\partial z}.$$

- If vorticity is constant $\Rightarrow$ Show $\nabla^2 u = \nabla^2 v = \nabla^2 w = 0$

(Use continuity equation)

Get 2 $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}$ from constant vorticity

Then $\frac{\partial^2 u}{\partial x^2}$ from continuity equation

(8) → $udx + vdy + wdz = d\Theta + \lambda d\chi$

Θ, λ, χ are functions of x, y, z, t

Prove vortex lines are lines of intersection of surfaces λ = constant and χ = constant.

- $udx + vdy + wdz = \frac{\partial \Theta}{\partial t} dt + \frac{\partial \Theta}{\partial x} dx + \frac{\partial \Theta}{\partial y} dy + \frac{\partial \Theta}{\partial z} dz + \lambda \left[\frac{\partial \chi}{\partial x} dx + \frac{\partial \chi}{\partial y} dy + \frac{\partial \chi}{\partial z} dz + \frac{\partial \chi}{\partial t} dt \right]$

- $u = \frac{\partial \Theta}{\partial x} + \lambda \frac{\partial \chi}{\partial x}$

  $v = \vdots \quad \vdots$

  $w = \vdots$

  $0 = \frac{\partial \Theta}{\partial t} + \lambda \frac{\partial \chi}{\partial t}$

- Spin components

$$2\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \Theta}{\partial z} + \lambda \frac{\partial \chi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \Theta}{\partial y} + \lambda \frac{\partial \chi}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 \Theta}{\partial y \partial z}} + \frac{\partial \lambda}{\partial y} \cdot \frac{\partial \chi}{\partial z} + \cancel{\lambda \frac{\partial^2 \chi}{\partial y \partial z}} - \cancel{\frac{\partial^2 \Theta}{\partial z \partial y}} - \cancel{\lambda \frac{\partial^2 \chi}{\partial y \partial z}} - \frac{\partial \lambda}{\partial z} \frac{\partial \chi}{\partial y}$$

$$= \begin{vmatrix} \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial y & \partial\chi/\partial z \end{vmatrix}$$

|||ly $2\Omega_y = \begin{vmatrix} \partial\lambda/\partial z & \partial\lambda/\partial x \\ \partial\chi/\partial z & \partial\chi/\partial x \end{vmatrix}$, $2\Omega_z$.

$$\left.\begin{array}{l} 2\left(\Omega_x \frac{\partial \lambda}{\partial x} + \Omega_y \frac{\partial \lambda}{\partial y} + \Omega_z \frac{\partial \lambda}{\partial z}\right) = 0 \\ \text{lly } \Omega_x \, \partial\chi/\partial x + \Omega_y \, \partial\chi/\partial y + \Omega_z \, \partial\chi/\partial z = 0 \end{array}\right\} \text{Implies the requirement}$$