2010

1. If v_1, v_2, v_3 are the velocities at three points A, B, C of the path of a projectile, where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$ and if t_1, t_2 are the times of describing the arcs AB, BC respectively. Prove $v_3 t_1 = v_1 t_2$ and $\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}$

A. Given v_1, v_2, v_3 are velocities with inclination to horizon as $\alpha, \alpha-\beta, \alpha-2\beta$

Horizontal Motion

$$\therefore v_1 \cos\alpha = v_2 \cos(\alpha-\beta) = v_3 \cos(\alpha-2\beta) \quad \text{--- (1)}$$

Vertical motion

$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - g t_1 \quad \text{--- (2)}$$

Also $$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - g t_2 \quad \text{--- (3)}$$

Now $$t_1 = \left(v_1 \sin\alpha - v_2 \sin(\alpha-\beta)\right)/g$$

and $$t_2 = \left\{v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)\right\}/g$$

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - v_2 \sin(\alpha-\beta)}{v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

$$= \frac{v_1\left[\sin\alpha - \cos\alpha \tan(\alpha-\beta)\right]}{v_3\left[\cos(\alpha-2\beta)\tan(\alpha-\beta) - \sin(\alpha-2\beta)\right]} \quad \text{(Using (1))}$$

$$= \frac{v_1\left[\cos(\alpha-\beta)\sin\alpha - \cos\alpha \sin(\alpha-\beta)\right]}{v_3\left[\cos(\alpha-2\beta)\sin(\alpha-\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)\right]}$$

$$\therefore \frac{t_1}{t_2} = \frac{v_1 \sin(\alpha - (\alpha-\beta))}{v_3 \sin[(\alpha-\beta) - (\alpha - 2\beta)]}$$

$$\frac{t_1}{t_2} = \frac{v_1 \sin\beta}{v_3 \sin\beta}$$

$$\Rightarrow \boxed{v_3 t_1 = v_1 t_2}$$

Now from (1)

$$v_1 = v_2 \frac{\cos(\alpha-\beta)}{\cos\alpha} \Rightarrow \frac{1}{v_1} = \frac{1}{v_2}\frac{\cos\alpha}{\cos(\alpha-\beta)}$$

Similarly $\frac{1}{v_3} = \frac{1}{v_2}\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$

So $$\frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\left[\frac{\cos(\alpha-2\beta) + \cos\alpha}{\cos(\alpha-\beta)}\right]$$

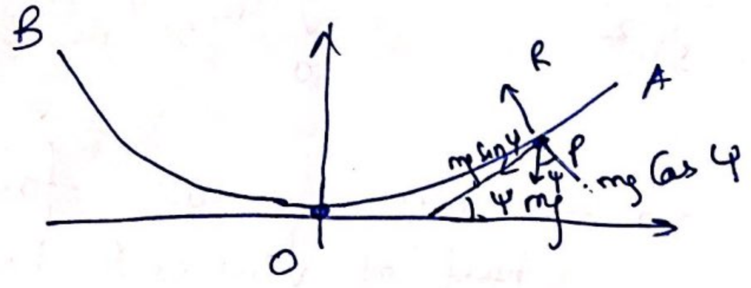
$$= \frac{1}{v_2}\left[\frac{2\cancel{\cos(\alpha-\beta)}\cos\beta}{\cancel{\cos(\alpha-\beta)}}\right]$$

$$\boxed{\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}}$$

2. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling

down the second half.

A:



Now $R - mg\cos\psi = \dfrac{mv^2}{\rho}$ $\qquad \rho = \dfrac{ds}{d\psi}$

$$-mg\sin\psi = m\frac{d^2s}{dt^2}$$

Also $s = 4a\sin\psi$, $\dfrac{ds}{d\psi} = 4a\cos\psi$,

$\therefore \dfrac{d^2s}{dt^2} = \dfrac{-g}{4a}s$ $\qquad s^2 = 8ay$

$$\left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

At $s = 4a$, $ds/dt = 0$

$$0 = -g4a + A \Rightarrow A = 4ag$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\left(16a^2 - s^2\right)$$

$$\frac{ds}{dt} = -\left(\frac{g}{4a}\right)^{1/2}\sqrt{(16a^2 - s^2)}$$ (-ve sign because it is coming down)

$$\frac{ds}{\sqrt{16a^2 - s^2}} = -\left(\frac{g}{4a}\right)^{1/2} dt$$

Now at $s = 4a$, $y = 2a$

at $y = a$, $s = 2\sqrt{2}a$

$$\int_{4a}^{2\sqrt{2}a} \frac{ds}{\sqrt{16a^2 - s^2}} = -\int_0^{t_1} \left(\frac{g}{4a}\right)^{1/2} dt$$

$t_1 \rightarrow$ Time period for first half of vertical height

$$\left[\sin^{-1}\left(\frac{s}{4a}\right)\right]_{4a}^{2\sqrt{2}a} = -t_1 \left(\frac{g}{4a}\right)^{1/2}$$

$$-\pi/4 = -t_1 \sqrt{g/4a}$$

$$\therefore \boxed{t_1 = \pi/4 \sqrt{4a/g}}$$

Now for $t_2 \rightarrow$ time for second half of vertical height

$$\int_{2\sqrt{2}a}^{0} \frac{ds}{\sqrt{16a^2 - s^2}} = -\int_0^{t_2} +\left(\frac{g}{4a}\right)^{1/2} dt$$

$$\left[\sin^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0} = -t_2 \sqrt{\frac{g}{4a}}$$

$$\therefore \boxed{t_2 = \sqrt{\frac{4a}{g}} \frac{\pi}{4}}$$

$$\therefore \boxed{t_1 = t_2}$$

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Q.3. A particle moves with central acceleration $\mu(r^5 - 9r)$ being projected from an apse at a distance $\sqrt{3}$ with velocity $3\sqrt{2\mu}$. Show that the path is the curve $x^4 + y^4 = 9$.

A. Given $P = \mu(r^5 - 9r)$

Now $$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

$$u + \frac{d^2u}{d\theta^2} = \frac{\mu(r^5 - 9r)}{h^2u^2} \qquad r = 1/u$$

$$u + \frac{d^2u}{d\theta^2} = \frac{\mu(1 - 9u^4)}{h^2u^7} = \frac{\mu}{h^2}\left[\frac{1}{u^7} - \frac{9}{u^3}\right]$$

Multiply by $2\,du/d\theta$ and integrate both sides

$$u^2 + \left(du/d\theta\right)^2 = \frac{2\mu}{h^2}\left[-\frac{1}{6}\frac{1}{u^6} + \frac{1}{2}\frac{9}{u^2}\right] + A'$$

$$\therefore u^2 + \left(du/d\theta\right)^2 = \frac{\mu}{3h^2}\left[\frac{-1 + 27u^4}{u^6}\right] + A'$$

Now $$v^2 = h^2\left[u^2 + \left(du/d\theta\right)^2\right]$$

$$\therefore h^2\left[u^2 + \left(du/d\theta\right)^2\right] = v^2 = \frac{\mu}{3}\left[-\frac{1}{u^6} + \frac{27}{u^2}\right] + A$$

Now at an apse $r = \sqrt{3}$, $v = 3\sqrt{2\mu}$, $\left(du/d\theta\right)^2$

$$\therefore \frac{h^2}{3} = 18\mu = \frac{\mu}{3}\left[-27 + 27 \times 3\right] + A$$

$$\Rightarrow \boxed{h^2 = 54\mu} \text{ and } \boxed{A = 0}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \left(\frac{27u^4 - 1}{162u^6}\right)$$

$$162u^6\left(\frac{du}{d\theta}\right)^2 = 27u^4 - 162u^8 - 1$$

$$\cancel{162}u^6\left(\frac{du}{d\theta}\right)^2 = \cancel{162}\left[\frac{u^4}{6} - u^8 - \frac{1}{162}\right]$$

$$u^6\left(\frac{du}{d\theta}\right) = \left[\left(\frac{1}{36}\right)^2 - \left(u^4 - \frac{1}{12}\right)^2\right]$$

$$\int \frac{u^3\,du}{\sqrt{\left(\frac{1}{36}\right)^2 - \left(u^4 - \frac{1}{12}\right)^2}} = \int d\theta$$

$$\frac{dt/4}{\sqrt{(1/36)^2 - t^2}} = \theta + C$$

$$\frac{1}{4}\sin^{-1}\left(36u^4 - 3\right) = \theta + C$$

at $u = \frac{1}{\sqrt{3}}$, $\theta = 0$ $\therefore$ $C = \pi/2$

$$36u^4 - 3 = \cos 4\theta$$

$$36u^4 - 3 = 2\cos^2 2\theta - 1$$

$$36u^4 - 3 = 2\left(2\cos^2\theta - 1\right)^2 - 1$$

$$36u^4 - 3 = 2\left(4\cos^4\theta + 1 - 4\cos^2\theta\right) - 1$$

$$36u^4 - 3 = 8\cos^4\theta - 8\cos^2\theta + 1$$

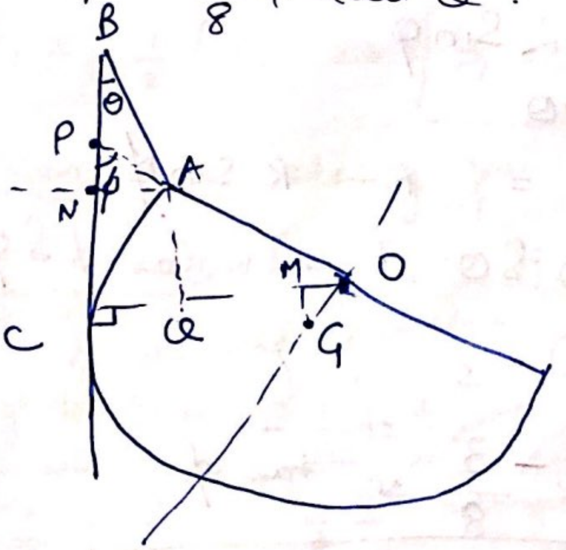
$$9u^4 = 2\cos^4\theta - 2\cos^2\theta + 1$$

$$9u^4 = \cos^4\theta + \left(1 - \cos^2\theta\right)^2$$

$$9u^4 = \sin^4\theta + \cos^4\theta$$
$$9 = r^4\sin^4\theta + r^4\cos^4\theta$$
$$\therefore \boxed{x^4 + y^4 = 9}$$ as $x = r\cos\theta$, $y = r\sin\theta$

4' Solid hemisphere is supported by a string fixed to point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ & ϕ are the inclination of the string and the plane base of the hemisphere to the vertical, prove by using the principal of virtual work that $\tan\phi = \frac{3}{8} + \tan\theta$.

A



G → Centre of Gravity
AB → String
OC is horizontal because BC is tangent to hemisphere.
Now B is the fixed point.

Let $AB = l$, $OA = R$ $\therefore OG = \frac{3}{8}R$

$\angle GOM = \phi$ $\therefore GM = OG\sin\phi = \frac{3}{8}R\sin\phi$

Now $AQ = R\cos\phi$

In $\triangle ABN$ $\quad AN = AB\sin\theta = l\sin\theta$

$BN = AB\cos\theta = l\cos\theta$

Now <u>Vertical distance of G from B</u>

is $BN + AQ + MG$

$\therefore x = l\cos\theta + R\cos\phi + \frac{3}{8}R\sin\phi$

Now By principle of virtual work

$W\delta x = 0$

$W\left[-l\sin\theta\,\delta\theta - R\sin\phi\,\delta\phi + \frac{3}{8}R\cos\phi\,\delta\phi\right] = 0$

$\therefore l\sin\theta\,\delta\theta = R\delta\phi\left(\frac{3}{8}\cos\phi - \sin\phi\right)$ — (1)

Now $CQ = AN$

$CQ = OC - OQ$

$CQ = R - R\sin\phi$

$AN = l\sin\theta$

$\therefore l\sin\theta = R - R\sin\phi$

$l\cos\theta\,\delta\theta = -R\cos\phi\,\delta\phi$ — (2)

Divide (1) by (2)

$\tan\theta = -\frac{3}{8} + \tan\phi$

$\therefore \boxed{\tan\phi = \frac{3}{8} + \tan\theta}$