D. Solve by the method of variation of parameters the following equation

$$(x^2-1)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=(x^2-1)^2 \quad 10$$

7. A. A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P.\frac{l}{h}+W.\frac{h^2+l^2}{2hl}\right] \quad 14$$

B. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}\right] \quad 13$$

C. A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle a to the horizontal plane. Prove that the acceleration of the wedge is,

$$\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha} \quad 13$$

8. A. (*i*) Show that

$\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3z^2x\vec{k}$

is a conservative field. Find its scalar potential and also the work done in moving a particle from (1, −2, 1) to (3, 1, 4). 5

(*ii*) Show that, $\nabla^2 f(r)=\left(\frac{2}{r}\right)f'(r)+f''(r)$,

where $r=\sqrt{x^2+y^2+z^2}$ 5

B. Use divergence theorem to evaluate,

$$\iint_S (x^3dy\,dz+x^2y\,dz\,dx+x^2z\,dy\,dx),$$

where S is the sphere, $x^2+y^2+z^2=1$. 10

C. If $\vec{A}=2y\vec{i}-z\vec{j}-x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2=8x$ in the first octant bounded by the planes $y=4$, $z=6$, evaluate the surface integral,

$$\iint_S \vec{A}.\hat{n}\,d\vec{S}. \quad 10$$

D. Use Green's theorem in a plane to evaluate the integral, $\int_C\left[(2x^2-y^2)dx+(x^2+y^2)dy\right]$, where C is the boundary of the surface in the xy-plane enclosed by, $y=0$ and the semi-circle, $y=\sqrt{1-x^2}$. 10

IFS 2010

PAPER-II

Instructions : *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

2. Answer any **four** parts from the following:

A. Let $G=\left\{\begin{bmatrix} a & a \\ a & a\end{bmatrix} \middle| a\in\mathbb{R}, a\neq 0\right\}$

Show that G is a group under matrix multiplication. 10

B. Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. 10

C. If $f : \mathbb{R} \to \mathbb{R}$ is such that

$f(x + y) = f(x) f(y)$

for all x, y in $\mathbb{R}$ and $f(x) \neq 0$ for any x in $\mathbb{R}$, show that $f'(x) = f(x)$ for all x in $\mathbb{R}$ given that $f'(0) = f(0)$ and the function is differentiable for all x in $\mathbb{R}$. 10

D. Determine the analytic function $f(z) = u + iv$ if $v = e^x (x \sin y + y \cos y)$. 10

E. A captain of a cricket team has to allot four middle-order batting positions to four batsmen. The average number of runs scored by each batsman at these positions are as follows. Assign each batsman his batting position for *maximum* performance:

Batting Position / Batsman	IV	V	VI	VII
A	40	25	20	35
B	36	30	24	40
C	38	30	18	40
D	40	23	15	33

2. A. A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum. 13

B. Prove or disprove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, .)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. 13

C. Using the method of contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2 (x^2+2x+2)}$$ 14

3. A. Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. 13

B. Evaluate

$$\iint_R (x - y + 1)\, dx\, dy$$

where R is the region inside the unit square in which $x + y \geq \frac{1}{2}$. 13

C. Solve the following linear programming problem by the simplex mothod: 14

Maximize $Z = 3x_1 + 4x_2 + x_3$

subject to

$$x_1 + 2x_2 + 7x_3 \leq 8$$
$$x_1 + x_2 - 2x_3 \leq 6$$
$$x_2, x_2, x_3 \geq 0$$

4. A. Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1) + n \geq 0$. Show that the function $d_n : R - \{0\} \to S$, where S is the set of all negative integers defined by $d_n(a) = d(a) + n$ for all $a \in R - \{0\}$ is a Euclidean valuation. 13

B. Obtain Laurent's series expansion of the function

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in the region $0 < |z + 1| < 2$. 13

C. *ABC* Electricals manufactures and sells two models of lamps, L_1 and L_2, the profit per unit being Rs 50 and Rs 30, respectively. The process involves two workers W_1 and W_2, who are available for 40 hours and 30 hours per week, respectively. W_1 assembles each unit of L_1 in 30 minutes and that of L_2 in 40 minutes. W_2 paints each unit of L_1 in 30 minutes and that of L_2 in 15 minutes. Assuming that all lamps made can be sold, determine the weekly production figures that maximize the profit. 14

Section-B

5. Answer any *four* parts from the following:

A. Find the general solution of

$$x(y^2 + z)\, p + y\,(x^2 + z)q = z(x^2 - y^2)$$ 10

B. Solve $x \log_{10} x = 1.2$ by regula falsi method. 10

C. Convert the following: 10
(i) $(736.4)_8$ to decimal number
(ii) $(41.6875)_{10}$ to binary number
(iii) $(101101)_2$ to decimal number
(iv) $(AF63)_{16}$ to decimal number
(v) $(101111011111)_2$ to hexadecimal number

D. Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same. 10

E. A two-dimensional flow field is given by $\psi = xy$. Show that–
(*i*) the flow is irrotational;
(*ii*) ψ and ϕ satisfy Laplace equation.
Symbols ψ and ϕ convey the usual meaning. 10

6. A. Using Lagrange interpolation, obtain an approximate value of sin(0.15) and a bound on the truncation error for the given data: 12
sin (0.1) = 0.09983, sin(0.2) = 0.19867

B. Draw a flow chart for finding the roots of the quadratic equation $ax^2 + bx + c = 0$. 12

C. Solve

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

given the conditions
(*i*) $u(0, t) = u(\pi, t) = 0,\ t > 0$
(*ii*) $u(x, 0) = \sin 2x,\ 0 < x < \pi$ 16

7. A. Find the general solution of $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + \sin(3x + 2y)$ 13

B. Show that $\phi = (x - t).(y - t)$ represents the velocity potential of an incompressible two-dimensional fluid. Further show that the streamlines at time t are the curves $(x - t)^2 - (y - t)^2 = \text{constant}$ 13

C. Find the interpolating polynomial for (0, 2), (1, 3), (2, 12) and (5, 147). 14

8. A. A mass m_1, hanging at the end of a string, draws a mass m_2 along the surface of a smooth table. If the mass on the table be doubled, the tension of the string is increased by one-half.
Show that $m_1 : m_2 : 2 : 1$. 13

B. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1$$

for $x = 0.1$ by Euler's method. 13

C. Show that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving under no external forces satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}.\nabla)\vec{q} + \nu\nabla^2\vec{\Omega}$$

where ν is the kinematic viscosity. 14

ANSWERS

PAPER-I

Section-A

1.(a): From question $P(t) = \{at^2 + bt + c\}$.
Let, $f(t)$ & $g(t) \in p(t)$
then, $f(t) = a_1t^2 + b_1t + c_1$.
$g(t) = a_2t^2 + b_2t + c_2$
then, $f(t) + g(t) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$
$\Rightarrow f(t) + g(t) \in p(t)$
$\because a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$
Also, $f(t) = g(t)$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$.
Also, $kf(t) = (ka_1)t^2 + (kb_1)t + kc_1 = i \in p(t)$.

PAPER-II

Section-A

1.(a) Given, $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a \in \mathbb{R}, a \neq 0 \right\}$.

Clearly (G,.) is a binary operator as the multiplication of two matrix of above group is also a matrix belong to same group.

(G,.) is also associative.

Now, to prove (G.) is group. It is suffice to show

(*i*) It has an identity.

(*ii*) It has an inverse.

Let, $\begin{bmatrix} x & x \\ x & x \end{bmatrix}$ be an identity of $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ then

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2ax & 2ax \\ 2ax & 2ax \end{bmatrix} = \begin{bmatrix} a & a \\ a & a \end{bmatrix} \Rightarrow 2ax = a$$

$$\Rightarrow \quad a(2x - 1) = 0 \; [\because a \neq 0)$$

$$\Rightarrow \quad x = \frac{1}{2}$$

i.e. $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ is an identity of G.

Now, let $\begin{bmatrix} y & y \\ y & y \end{bmatrix}$ be an inverse of G then,

$$\begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2ay & 2ay \\ 2ay & 2ay \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \quad y = \frac{1}{4a}$$

ie. $\begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix}$ is an inverse of $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

Hence, (G,.) is a group under matrix multiplication.

(b) As F is the field of order $32 = 2^5$.

Now, we also know that $\{0,1\}$ is the smallest field and every field of order 2^n is an extension of smallest possible Galois field $GF(2) = \{0,1\}$.

Now, from the fundamental theorem in the theory of finite field states that if $q = p^n$ is the number of the elements of a finite field $GF(q) = GF(p^n)$ where p = char $(GF(q))$ is the field characteristic which is prime and n is natural number. In this case, then every subfield of $GF(p^n)$ has order p^m, where m is natural divisor of n and vice-versa for every such divisor m exists unique subfield $GF(p^n)$.

Now, the given Galois field (GF) has 2^5 element and 5 is prime whose divisor are 1 and 5.

Hence, the only subfield of it have 2^1 and 2^5 elements which imply the subfields are $\{0,1\}$ and F itself.

(c) From question,

$$f(x + y) = f(x) f(y)$$

putting $\frac{x}{2}$ & $\frac{x}{2}$ in place of x & y we get,

$$f\left(\frac{x}{2} + \frac{x}{2}\right) = f\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right)$$

$$\Rightarrow \quad f(x) = \left\{f\left(\frac{x}{2}\right)\right\}^2$$

$$\Rightarrow \quad f(x) > 0 \quad \forall x \in \mathbb{R}$$

Also from question,

$$f(0) = f(0)$$

and f is differentiable for all x in $\mathbb{R}$.

Now,

$$f'(x) = \underset{h\to 0}{Lt} \frac{f(x+h)-f(x+0)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{f(x)f(h)-f(x)f(o)}{h}$$

$$= \underset{h\to 0}{Lt} f(x)\left[\frac{f(h)-f(0)}{h}\right]$$

$$= \underset{h\to 0}{Lt} f(x)f'(0)$$

$$= f(x) \underset{h\to 0}{lt} f'(0) = f(x)\, f(0)$$

$$= f(x+0) = f(x)$$

$$\Rightarrow f'(x) = f(x)\ \forall\ x \in \mathbb{R}$$

(d) The analytical function can be calculated with the help of Milne's method.

i.e. $f(z) = \int \phi_1(z,0)dz + i\int \phi_2(z,0)dz$

where, $\phi_1(x, y) = v_y$

& $\phi_2(x,y) = v_x$

Now, $v = e^x (x \sin y + y \cos y)$

$$v_x = e^x (x \sin y + y \cos y + \sin y)$$

$$= \phi_2 (x, y)$$

$$\Rightarrow \phi_2(z, 0) = e^z (z.0 + 0 + 0) = 0$$

$$v_y = e^x(x \cos y + \cos y - y \sin y)$$

$$= \phi_1 (x, y)$$

$$\Rightarrow \phi_1 (z,0) = e^z (z.1 + 1 - 0) = e^z (z + 1)$$

$$\Rightarrow f(z) = \int e^z(z+1)dz$$

$$= ze^z - e^z + e^z + c$$

$$\Rightarrow f(z) = ze^z + c$$

where c is an integrating constant.

(e) As the anignment is done for maximum performance which can achieve by converting into standard anignment problem which is of minimization type. It is done by converting into min(-z) hence the above table reduces to.

	IV	V	VI	VII
A	–40	–25	–20	–35
B	–36	–30	–24	–40
C	–38	–30	–18	–40
D	–40	–23	–15	–33

Substracting from the minimum value of each rows we get,

	IV	V	VI	VII
A	0	15	20	5
B	4	10	16	0
C	2	10	22	0
D	0	17	25	7

Substracting of minimum of column 2 and 3 to other element of it we get,

	IV	V	VI	VII
A	$\boxed{0}$	5	4	5
B	4	0	$\boxed{0}$	$\cancel{0}$
C	2	$\boxed{0}$	6	$\cancel{0}$
D	$\cancel{0}$	7	9	7

marks the rows having not any anignment and ticking column having (not marked) then tick row having anignment in marked column.

Draw lines through all unticked rows and ticked column. If number of these lines is equal to order of the matrix then it is optinal else not.

Now, substracting the minimum from remaining elements (not covered by the lines) we get,

$\cancel{0}$	1	$\boxed{0}$	1
4	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
2	$\cancel{0}$	6	$\boxed{0}$
$\boxed{0}$	3	5	3

Hence, the required batting position

A – VI, B – V, C – VII and D – IV

2.(a) Let dimension of the rectangular box is x, y and z where these represent length, breadth and height respectively.

then volume $V = xyz$

and the surface area of the rectangular box

$$= xy + 2z(x + y) = 12$$

defined a lagrangian function

$$F = xyz + \lambda[xy + 2z(x + y) - 12]$$

Then for extremum value $dF = 0$

$$\Rightarrow dF = \{yz + \lambda(y + 2)\}dx + \{xz + \lambda(x + 2z\}dy + \{xy + \lambda(2x + 2y)\}dz = 0$$

Now equating the coefficient we get,

$$yz + \lambda(y + 2z) = 0 \quad \text{... (1)}$$

$$xz + \lambda(x + 2z) = 0 \quad \text{... (2)}$$

$$xy + 2\lambda(x + y) = 0 \quad \text{... (3)}$$

$\Rightarrow$ (1) – (2) $\Rightarrow (y - x)z + \lambda(y - x) = 0$

$\Rightarrow \quad (y - x)(z + \lambda) = 0$

$\Rightarrow \quad y - x = 0$

Other factor cannot be zero

$\Rightarrow \quad y = x$

when, $\quad y = x$

(2) × 2 – (3)

$\Rightarrow x(y - 2z) + 2\lambda(x + y - x - 2z) = 0$

$\Rightarrow (2 + 2\lambda)(y - 2z) = 0$

$\Rightarrow \quad y = 2z$

$\Rightarrow$ The dimension of box is of the form

$$x = y = 2z$$

also, $xy + 2z(x + y) = 12$

$\Rightarrow \quad 12z^2 = 12 \Rightarrow z = 1$

Hence, dimension of box are 2, 2 and 1 respectively.

(b) Given $(\mathbb{R}, +)$ is a group where $\mathbb{R}$ denotes real number & $(\mathbb{R}^+, .)$ is a group where R^+ denotes positive real number to show that they are isomorphic. It is suffice to show that there exist as one-one onto mapping from R to $\mathbb{R}^+$.

define a function $f : R \to R^+$ such that

$$f(x) = e^x \quad \text{where, } x \in \mathbb{R}$$

then, $e^x > 0$ for all $x \in \mathbb{R}$

i.e. $\quad \forall x \in R \quad e^x \in \mathbb{R}^+$

Now, $\quad f(x + y) = e^{x+y}$

$$= e^x . e^y$$

i.e. It preserve composition.

i.e. f is a homomorphism.

Now f is one-one

as $\quad f(x) = f(y)$

$\Rightarrow \quad e^x = e^y \Rightarrow e^{x-y} = 0$

$\Rightarrow \quad x - y = 0 \Rightarrow x = y$

Also, f is onto

as any $y \in \mathbb{R}^+$

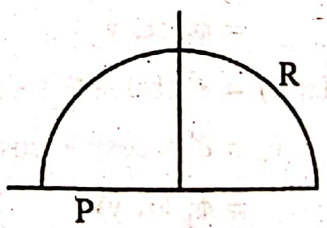
It can be written as e^x where $x = \log y$

(which is defined as $y > 0$)

i.e. f is one-one onto as it preserve composition

i.e. $\quad \mathbb{R} \equiv R^+$.

(c) Consider the contour having radius R in the upper half (Im > 0).



then let, $\quad f(z) = \dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$

then, $\quad \int_c f(z)dz = 2\pi i\, \mathbf{R}^+ f(z)$

c is the given contour.

Now, $\quad \int_c f(z)dz = \int_{-R}^{R} f(x)dx + \int_T f(z)dz$

Now, when $\mathbf{R} \to -\infty$ then

$$\int_c f(z)dz = \int_{-\infty}^{\infty} f(x)dx + \int_T f(z)dz$$

from Jordon's Lemma the second integral of Right hand side becomes zero.

Hence, the required integral reduces to

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\, f(z).$$

The poles of $f(z)$ are given by

$(z^2+1)^2 = 0 \Rightarrow z = \pm i$ (twice)

& $z^2 + 2z + 2 = 0 \quad \Rightarrow z = -1 \pm i$

Out of these only $+i$ (twice) & $z = -1 + i$ lies inside the given contour.

Now, Residue at $z = i$ can be calculated as

$$\lim_{z \to i} \phi'(i) \text{ where } \phi(z) = \frac{z^2}{(z+i)^2(z^2+2z+2)}$$

Now, $\phi(z) = \dfrac{z^2}{(z+i)^2(z^2+2z+2)}$.

Taking logarithm both sides we get,

$$\log f'(z) = 2\log z - 2\log(z+i) - \log(z^2+2z+2)$$

differentiating we get,

$$\frac{\phi'(z)}{\phi(z)} = \frac{2}{z} - \frac{2}{z+i} - \frac{2z+2}{z^2+2z+2}$$

$$\Rightarrow \quad \phi'(z) = \phi(z)\left[\frac{2}{z} - \frac{2}{z+i} - \frac{2z+2}{z^2+2z+2}\right]$$

$$\Rightarrow \quad \phi'(i) = \phi(i)\left[\frac{2}{i} - \frac{2}{2i} - \frac{2i+2}{1+2i}\right]$$

$$= \frac{-3}{25} + \frac{9i}{100}$$

Again, residue at $z = -1 + i$, can be given as.

$$\lim_{z \to -1+i} \frac{z^2}{(z^2+1)^2(z+1+i)} = \frac{3-4i}{25}$$

Hence, sum of residue $= -\dfrac{3}{25} + \dfrac{9i}{100} + \dfrac{3}{25} - \dfrac{4i}{25}$

$$= -\frac{7i}{100}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = 2\pi i\left(-\frac{7i}{100}\right)$$

$$= \frac{7\pi}{50}$$

3.(a) An element a is called idempotent if $a^2 = a$

Now, $z_n = \{0, 1, 3, \ldots\ldots n = p^r - 1\}$.

Clearly, $0^2 = 0$ & $1^2 = 1$

$\Rightarrow$ 0, 1 are idempotent of z_n.

Now, suppose $a \in z_n$ such that a is a non-zero non-unit idempotent.

i.e. $\quad a^2 = a$ in z_n.

This imply $\qquad a^2 \equiv a \pmod{n}$

$\Rightarrow \qquad a^2 - a \equiv 0 \pmod{n}$

$\Rightarrow \qquad a(a-1) \equiv 0 \pmod{n}$

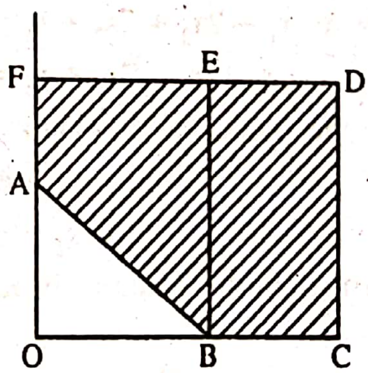
Hence, the solution of these congruence relations are

$$a = 0 \text{ or } 1 \pmod{n}$$

$$a = 0 \text{ or } 1 \text{ (contradiction)}$$

Hence, z_n has only two idempotent.

(b) $\displaystyle\iint_R (x - y + 1)dxdy$



The above integral can be broken into

$$= \iint_{ABEFA} (x-y+1)dxdy + \iint_{BCDEB} (x-y+1)dxdy$$

$$= \int_{x=0}^{1/2}\int_{1/2-x}^{1} (x-y+1)dydx + \int_{x=1/2}^{1}\int_{y=0}^{1} (x-y+1)dxdy$$

$$= \int_{x=0}^{1/2} \left. xy - \frac{y^2}{2} + y \right|_{1/2-x}^{1} dx + \int_{1/2}^{1} \left. xy - \frac{y^2}{2} + y \right|_0^1 dx$$

$$= \int_{x=0}^{1/2} \left\{ \left(x - \frac{1}{2} + 1 \right) - x\left(\frac{1}{2} - x \right)^2 + \frac{1}{2}\left(\frac{1}{2-x} \right)^2 - \left(\frac{1}{2} - x \right) \right\} dx + \int_{1/2}^{1} \left(x - \frac{1}{2} + 1 \right) dx$$

$$= \int_{0}^{1/2} \left\{ \left(x + \frac{1}{2} \right) + x^2 + \frac{x}{2} + \frac{1}{2}\left(x^2 - x + \frac{1}{4} \right) + x - \frac{1}{2} \right\} dx + \int_{1/2}^{1} \left(x + \frac{1}{2} \right) dx$$

$$= \int_{0}^{1/2} \left(\frac{3}{2}x^2 + 2x \right) dx + \left. \frac{x^2}{2} + \frac{x}{2} \right|_{1/2}^{1}$$

$$= \left. \frac{3}{2} \cdot \frac{x^3}{3} + x^2 \right|_{0}^{1/2} + \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{8} - \frac{1}{4} \right)$$

$$= \left(\frac{1}{2} \cdot \frac{1}{8} + \frac{1}{4} \right) + \left(1 - \frac{3}{8} \right)$$

$$= \frac{5}{16} + \frac{5}{8} = \frac{10+5}{16} = \frac{15}{16}$$

(c) Introducing the slack variable S_1 and S_2 then above L.P.P. reduces to

$$z = 3x_1 + 4x_2 + x_3$$

Subject to, $x_1 + 2x_2 + 7x_3 + s_1 = 8$ ... (1)

$x_1 + x_2 - 2x_3 + s_2 = 6$ ... (2)

choose the initial basic solution

$$x_1 = x_2 = x_3 = 0$$

then, $s_1 = 8$ & $s_2 = 6$

writing the above equations in tabular form, we get,

$c_i \rightarrow$		3	4	1	0	0			
		x_1	x_2	x_3	s_1	s_2	b	θ	
0	s_1	1	$\langle 2 \rangle$	7	1	0	8	4	$\leftarrow$ (lowest
0	s_2	1	1	−2	0	1	6	6	+ ive ratio)
$E_j \rightarrow$		0	0	0	0	0			
$C_j - E_j$		3	4	1	0	0			
		$\uparrow$							

Key column

		3	4	1	0	0		
		x_1	x_2	x_3	s_1	s_2	b	
4	x_2	½	1	7⁄2	½	0	4	
0	s_2	½	0	−11⁄2	−½	1	2	$\leftarrow$
E_j		2	4	14	2			
$C_j - E_j$		1	0	−13	−2			
		$\uparrow$						

		3	4	1			
		x_1	x_2	x_3	s_1	s_2	b
4	x_2	0	1	9	1	−1	2
3	x_1	1	0	−11	−1	2	4

$\therefore$ $x_1 = 4, \quad x_2 = 2, \quad x_3 = 0$

Hence,

$$z = 3x_1 + 4x_2 + x_3$$

$$= 3 \cdot 4 + 4 \cdot 2 + 0 = 20$$

4.(b) Let, $z + 1 = u$

then, $\dfrac{1}{(z+1)(z+3)}$ is reduced to

$$= \frac{1}{u(u+2)} \text{ is the region } 0 < |u| < 2$$

$$= \frac{1}{u.2\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \ldots\ldots\right)$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \frac{u^2}{32} - \frac{u^4}{64} \ldots\ldots$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} - \frac{(z+1)^4}{64} + \ldots$$

(c) Let weekly production of L_1 and L_2 are x and y units respectively then

Then the profit for manufacture

$= 50x + 30y = z$ (objective function)

Also Worker W_1 and W_2 remain available for 40 hours and 30 hours per week respectively.

then,

$$x.\frac{1}{2}+y.\frac{2}{3} \le 40 \Rightarrow 3x + 2y \le 240 \quad \dots (1)$$

also, $$x.\frac{1}{2}+y.\frac{1}{4} \le 30 \Rightarrow 2x + y \le 120 \dots (2)$$

Introducing slack variable S_1 and S_2 in the (1) and (2) and writing the standard form in tabular representation we get,

		50	30	0	0		
		x	y	s_1	s_2	b	θ
0	s_1	3	0	1	0	240	80
0	s_2	2	1	0	1	120	60
		50	30				

Key element

Key element

		50	30	0	0	
		x	y	s_1	s_2	b
0	s_1	0	$\frac{5}{2}$	1	$-\frac{3}{2}$	60
50	x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	60
		50	25	0	25	
		0	5			

	x	y	s_1	s_2	b
y	0	1	$\frac{2}{5}$	$-\frac{3}{5}$	24
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	60
	50	30			

		x	y	s_1	s_2	b
30	y	0	1	$\frac{2}{5}$	$-\frac{3}{5}$	24
50	x	1	0	$-\frac{1}{5}$	$\frac{4}{5}$	48
		50	30	2	22	
$C_j \rightarrow$		0	0	-2	-22	

i.e. No further optimalization as all the $C_j \le 0$

$$\Rightarrow \quad x = 48,\ y = 24$$

Hence, production of L_1 model = 48

production of L_2 model = 24

and the profit per week of the company

$$= 50 \times 48 + 30 \times 24 = 2400 + 720$$
$$= 3120$$

5.(a) The Lagrange equation is given by,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Choosing $x, -y, -1$ as multiplier we get,

$$\frac{xdx - ydy - dz}{x^2y^2 + x^2z - x^2y^2 - y^2z - zx^2 + zy^2}$$

$$\Rightarrow \quad xdx - ydy - dz = 0$$

$$\Rightarrow \quad x^2 - y^2 - 2z = c_1 \quad \dots (1)$$

Choosing $\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$ as multiplies we get,

$$\frac{\frac{1}{x}dx - \frac{1}{y}dy + \frac{1}{z}dz}{y^2 + z - x^2 - z + x^2 - y^2} = \frac{\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\Rightarrow \quad \log\frac{xz}{y} = \text{constant}$$

$$\Rightarrow \quad \frac{xz}{y} = C_2 \quad \dots (2)$$

Hence, the general solution of the above equation is given by

$$\phi\left(x^2 - y^2 - 2z, \frac{xz}{y}\right) = 0$$

where ϕ is an arbitrary function.

(b) Let, $\quad f(x) = x \log_{10} x - 1.2$

then, $\quad f(2) = 2 \times \log_2 - 1.2$

$$= 0.597 < 0$$

$$f(3) = 3 \log_3 - 1.2$$

$$= 0.231 > 0$$

i.e. there lies one root between 2 and 3.

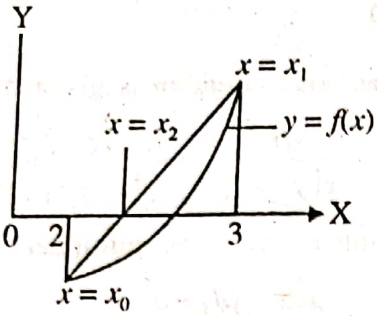
$$x = x_0 = 2, \quad x_1 = 3$$

At $\quad x = x_2\ y = 0$

i.e. it intersect x axis.

Scanned with CamScanner

then from figure



$$\frac{x_0 - x_2}{f(x_0)} = \frac{x_1 - x_0}{f(x_1) f(x_0)}$$

$$\Rightarrow \quad x_0 - x_2 = \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

$$\Rightarrow \quad x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} . f(x_0)$$

Now, $\quad x_0 = 2, \quad x_1 = 3$

$$\Rightarrow x_2 = x_0 - \frac{(x_1 - x_0)}{(x_1 \log x_1 - x_0 \log x_0)} (x_0 \log x_0 - 1.2)$$

$\Rightarrow \quad x_2 = 2.7210$

$f(x_2) < 0$

$\Rightarrow$ new $\quad x_0 = 2.7210, \quad x_1 = 3$

$\Rightarrow \quad x_2 = 2.7402$

Similarly $\quad x_0 = 2.7402, \quad x_1 = 3$

$\Rightarrow \quad x_2 = 2.740205$

$\Rightarrow$ Root of $\quad f(x) = 0, \quad x = 2.7402$

(c) i. $(736.4)_8 = 7 \times 8^2 + 3 \times 8^1 + 6 \times 8^0 + 4 \times 8^{-1}$

$= 448 + 24 + 6 + 0.5 = (478.5)_{10}$

(c) ii. It can be converted into binary by converting seperately the integral & fractional part into binary number

Now, binary conversion of 41

2	41	1
2	20	0
	10	0
	5	1
	2	0
	1	

$= (101001)_2$

& $\quad 0.6875 = 0.6875 \times 2 = 1.3750$

$0.375 \times 2 = 0.750 \Rightarrow (.6875)_2 = (1011)_2$

$.750 \times 2 = 1.500$

$.500 \times 2 = 1.000$

$\Rightarrow (41.6875)_{20} = (101001.1011)_2$

(c) iii.

$= 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 1 \times 2^0$

$= 32 + 8 + 4 + 1 = (45)_{10}$

(c) iv. $\quad (AF63)_{16} = 10 \times 16^3 + 15 \times 16^2$

$+ 6 \times 16 + 3 \times 16^0$

$= 40960 + 3840 + 96 + 3$

$= (44899)_{10}$

(c) v. $(101111011111)_2 = (BDF)_{16}$

Note: The above conservation has been done by making the binary number into group of four from right side by replacing it with the equivalent hexadecimal integer.

5.(e) The given stream function is

$\because \quad \psi = xy$

Then the difinition of stream function

$$u = -\frac{\partial \psi}{\partial y} = -x, \; \& \; v = +\frac{\partial \psi}{\partial x} = Y$$

$\Rightarrow \quad q = -x\hat{i} + y\hat{j}$

The flow is irrotational then

$$\bar{\nabla} \times \bar{q} = 0 \quad \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix} = 0$$

$\Rightarrow$ flow is irrotational.

$$\because \psi = xy \Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Also, $\dfrac{\partial \phi}{\partial x} = +\dfrac{\partial \psi}{\partial y} = +x \quad \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} = -y$

$\Rightarrow \quad \phi = x^2 - y^2 + c$

$\Rightarrow \quad \nabla^2 \phi = \nabla^2 (x^2 - y^2)$

$= 2 - 2 = 0$

i.e. both ψ and ϕ satisfy Laplace equation.

6.(a) Since two values of sine function is known. The value of sin(0.15) using Lagrange interpolation is given by,

$$\sin(0.15) = \frac{0.15-0.2}{0.1-0.2}\times 0.09983 + \frac{0.05-0.1}{0.2-0.1}\times 0.19867$$

$$= \frac{0.05}{0.1}\times 0.09983 + \frac{0.05}{0.1}\times 0.19867$$

$$= \frac{1}{2}(0.09983+0.19867)$$

$$= 0.14925$$

(c) The above equation represent the heat equation which can be solved by seperation of variable.

Let, $u(x, t) = X(x)\,T(t)$

where X, T are functions of x and t respectively.

Now, $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad 4X''(x)\,T(t) = X(x)\,T'(t)$$

$$\Rightarrow \quad \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x \text{ only}} = \underbrace{\frac{T''(t)}{4T(t)}}_{\text{function of } t \text{ only}} = \lambda \text{ (say)}$$

Hence, they must be equal to constant.

Also, for non-trivial solution, it can be shown that λ must be less than zero

say $\lambda = n^2$ then

$$\frac{X''(x)}{X(x)} = -n^2 \Rightarrow X''(x) = n^2\,X(x)$$

$$\Rightarrow \quad X(x) = A\cos nx + B\sin nx$$

Now, given condition

$$u(0, t) = u(\pi, t) = 0$$

Now, $$u(0, t) = X(0)\,T(t) = 0$$

$$= (A\cos 0 + B.0)\,T(t) = 0$$

$$= A = 0$$

Similarly, $u(\pi, t) = 0$

$$\Rightarrow \quad \sin n\pi = 0$$

$$\Rightarrow \quad n = m$$

where, $m = \pm 1, \pm 2, \pm 3$ .....

Now, $$\frac{T'(t)}{4T(t)} = -n^2$$

$$\Rightarrow \quad \frac{T'(t)}{T(t)} = -4m^2$$

$$\Rightarrow \quad T(t) = Ce^{-4m^2 t}.$$

Hence, the general solution can be written as

$$U(x, t) = \sum u(x,t)$$

$$= \sum (B\sin mx)\,Ce^{-4m^2 t}$$

$$U(x, t) = \sum_{m=1}^{\infty} F\sin mx\, e^{-4m^2 t}$$

where $F = BC$

Now, we calculate the value of F by Fourier sine series which is given by

$$F = \frac{2}{\pi}\int_0^{\pi} u(x,0)\sin mx\,dx$$

or, $$F = \frac{2}{\pi}\int_0^{\pi} \sin 2x.\sin mx\,dx$$

$$\Rightarrow \quad F = 0 \text{ or } m \neq 0$$

When $m = 2$ then,

$$F = \frac{2}{\pi}\int_0^{\pi}\sin^2 2x\,dx = \frac{2}{\pi}.\frac{\pi}{2} = 1$$

Hence, the general solution is given by,

$$U(x, t) = 1.\sin 2x\, e^{-4.4t}$$

$$U(x, t) = \sin 2x\, e^{-4t}$$

Which is the required general solution

7.(a) The complementary function is given by

$$Z = e^x\,\phi_1(y+x) + e^{2x}\,\phi_2(y+x)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Now, the particular integral is given by,

$$z = \frac{1}{(D-D'-1)(D-D'-2)}\left\{e^{2x-y} + \sin(3x+2y)\right\}$$

Now, we calculate the particular integral for both of them seperately.

Now, $\frac{1}{(D-D'-1)(D-D'-2)}e^{2x-y}$

$$=\frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}}e^{2x-y}=\frac{1}{2}e^{2x-y}$$

Now, $\frac{1}{(D-D'-1)(D-D'-2)}\sin(3x+2y)$

$$\frac{1}{(D-D')^2-3(D-D')+2}\sin(3x+2y)$$

$$=\frac{1}{D^2+D'^2-2DD'-3(D-D')+2}\sin(3x+2y)$$

$$=\frac{1}{-9.4+2.3.2-3(D-D')+2}\sin(3x+2y)$$

$$=\frac{1}{1-3(D-D')}\sin(3x+2y)$$

$$=\frac{1+3(D-D')}{1-9(D-D')^2}\sin(3x+2y)$$

$$=\frac{1+3(D-D')}{1-9(-9-4+12)}\sin(3x+2y)$$

$$=\frac{1}{10}[\sin(3x+2y)+9\cos(3x+2y)-4\cos(3x+2y)]$$

$$=\frac{1}{10}[\sin(3x+2y)+5\cos(3x+2y)]$$

$$=\frac{\sin(3x+2y)+5\cos(3x+2y)}{10}$$

Hence, the general solution is

$$z=e^x\,\phi_1(y+x)+e^{2x}\,\phi_2(y+x)$$

$$+\frac{1}{2}e^{2x-y}+\frac{\sin(3x+2y)}{10}+\frac{\cos(3x+2y)}{2}$$

(b) $$q=\nabla\phi=(y-t)\hat{i}+(x-t)\hat{j}$$

Now, if it represent incompressible two dimension flow then

$$\vec{\nabla}\times\vec{q}=0$$

Now, $\vec{\nabla}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (y-t) & (x-t) & 0\end{vmatrix}=0$

$\Rightarrow \phi$ represent flow of incompressible flow fluid.

Now, $$\frac{\partial\psi}{\partial y}=\frac{\partial\phi}{\partial x}=(y-t)$$

$$\Rightarrow \quad \psi=(y-t)^2+f(x) \quad \text{--(1)}$$

Also, $$\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

$$\Rightarrow \quad \psi=-(x-t)^2+f(y) \quad \text{--(2)}$$

from (1) and (2), we have,

$$\psi=(x-t)^2-(y-t)^2$$

(c) The interpolaring polynomial can be calculated using Lagrange interpolation formula.

$$y=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}\times 2$$

$$+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}\times 3$$

$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}\times 12$$

$$+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}\times 147$$

$$=\frac{x^3-8x^2+17x-10}{-10}\times 2+\frac{x^3-7x^2+10x}{4}\times 3$$

$$+\frac{x^3-6x^2+5x}{(-6)}\times 12+\frac{x^3-3x^2+2x}{60}\times 147$$

$$=x^3\left(-\frac{1}{5}+\frac{3}{4}-2+\frac{147}{60}\right)+x^2\left(\frac{8}{5}-\frac{21}{4}+12-\frac{147}{20}\right)$$

$$+\left(-\frac{34}{10}+\frac{30}{4}-10+\frac{294}{60}\right)x+2$$

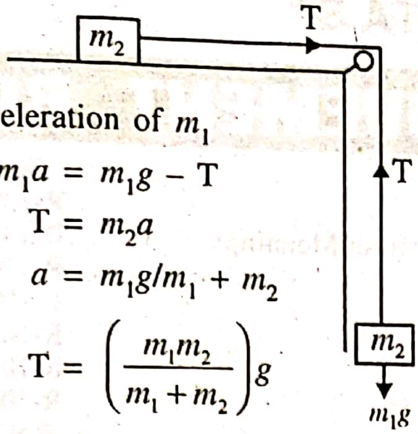
$$=x^3+x^2-x+2$$

$\Rightarrow$ Hence, the polynomial function is

$$y=x^3+x^2-x+2.$$

8.(b) In Case I,

from figure,



If a be the acceleration of m_1

then $m_1 a = m_1 g - T$

& $T = m_2 a$

$\Rightarrow$ $a = m_1 g / m_1 + m_2$

$\Rightarrow$ $T = \left(\dfrac{m_1 m_2}{m_1 + m_2}\right) g$

In Case II

When m_2 is doubled

then if a' be the acceleration of m_1 and T′ be tension in the string

$$m_1 g - T' = m_1 a'$$
$$T' = 2m_2 a'$$

$\Rightarrow$ $a'(m_1 + 2m_2) = m_1 g$

$\therefore$ $a' = \dfrac{m_1 g}{m_1 + 2m_2}$

$\therefore$ $T' = \dfrac{2m_1 m_2 g}{m_1 + 2m_2}$

from question, $T' = \dfrac{3}{2} T$

$\Rightarrow$ $\dfrac{2m_1 m_2 g}{m_1 + 2m_2} = \dfrac{3}{2} \dfrac{m_1 m_2 g}{m_1 + m_2}$

$3(m_1 + 2m_2) = 4(m_1 + m_2)$

$\Rightarrow$ $m_1 = 2m_2 \Rightarrow \dfrac{m_1}{m_2} = \dfrac{2}{1}$

i.e. $m_1 : m_2 = 2 : 1.$

(b) We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$

Then, $x = 0.0$, $y = 1.00$

then, $\dfrac{dy}{dx} = \dfrac{y-x}{y+x} = 1$

$\Rightarrow$ new y = old $y + 0.02\left(\dfrac{dy}{dx}\right) = 1.02$

$x = 0.02$, $y = 1.02$

$\dfrac{dy}{dx} = 0.9615$

$\Rightarrow$ new $y = 1.02 + 0.02 \times 0.9615 = 1.0392$

$x = 0.04$, $y = 1.0392$

$\dfrac{dy}{dx} = 0.926$

new $y = 1.0392 + 0.02(0.926) = 1.0577$

$x = 0.06$, $y = 1.0577$

$\dfrac{dy}{dx} = 0.893$

new $y = 1.0577 + 0.02 \times 0.893 = 1.0756$

$x = 0.08$ $y = 1.0756$

$\dfrac{dy}{dx} = 0.862$

new $y = 1.0756 + 0.02 \times 0.862 = 1.0928$

$y(0.10) = 1.0928.$

☆☆☆☆☆☆