ANALYTIC GEOMETRY

: IFoS - 2011 :

(1)(c) A variable plane is at a constant distance 'p' from the origin and meets the axes at A, B & C. P.T. the locus of centroid of tetrahedron OABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$

→ Any plane that has intercepts a, b, c is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. — (1)

Then centroid of OABC is $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$;

Distance of plane (1) from origin is $p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$

$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

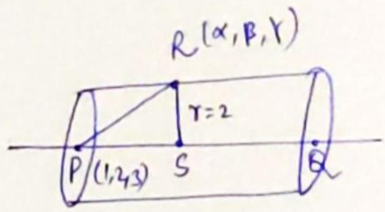
$\Rightarrow \left(\frac{4}{a}\right)^2+\left(\frac{4}{b}\right)^2+\left(\frac{4}{c}\right)^2=\frac{16}{p^2}$. Since centroid is $\left(\frac{a}{4},\frac{b}{4},\frac{c}{4}\right)$,

$\Rightarrow \boxed{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z}=\frac{16}{p^2}}$ which is the required locus.

(4) (a) Find the equations of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.

→ Eqn of axis PQ is :

$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$ — (1)



It passes through (1,2,3). Let P(1,2,3).

Let $R(\alpha,\beta,\gamma)$ be any point on the surface of the cylinder.

Then projection of line PR on line PQ is given as:

$(\alpha-1)\frac{2}{3}+(\beta-2)\frac{1}{3}+(\gamma-3)\cdot\frac{2}{3}=PS$

Now: $RP^2=RS^2+PS^2$

$\Rightarrow (\alpha-1)^2+(\beta-2)^2+(\gamma-3)^2=4+\frac{1}{9}[2(\alpha-1)+(\beta-2)+2(\gamma-3)]^2$

$\Rightarrow 5\alpha^2+8\beta^2+5\gamma^2-4\alpha\beta-8\alpha\gamma+2\alpha-16\beta+4\gamma-10=0$.

∴ Reqd eqn. $5x^2+8y^2+5z^2-4xy-8zx+2x-16y+4z-10=0$

4 (b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$.

→ Any plane parallel to the given plane is $lx+my+nz=p$ — ①

Tangent plane to the ellipsoid at any point (α,β,γ) is given by

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1 \quad — ②$$

If the plane ① is a tangent plane at (α,β,γ) to the ellipsoid, then, planes ① & ② are the same.

$$\therefore \frac{\alpha}{a^2 l}=\frac{\beta}{b^2 m}=\frac{\gamma}{c^2 n}=\frac{1}{p} \Rightarrow \alpha=\frac{a^2 l}{p},\ \beta=\frac{b^2 m}{p},\ \gamma=\frac{c^2 n}{p}.$$

Since (α,β,γ) lie on the ellipsoid: $\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1$

$$\therefore \frac{a^2 l^2}{p^2}+\frac{b^2 m^2}{p^2}+\frac{c^2 n^2}{p^2}=1 \Rightarrow p^2=a^2 l^2+b^2 m^2+c^2 n^2.$$

$\therefore$ The required eqn of tangent planes to the ellipsoid parallel to the given plane is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\pm\sqrt{a^2 l^2+b^2 m^2+c^2 n^2}$$

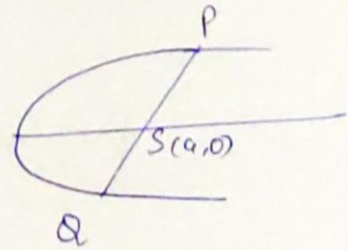
4 (c) Prove that the semi-latus rectum of a conic is a harmonic mean between the segments of any focal chords.

→ Let us consider the parabola $y^2=4ax$ — ① as the given conic.

Its latus rectum is $4a$.

$\therefore$ Semi-latus rectum length $=2a$ — ②

Let $S\equiv(a,0)$ be the focus of the parabola & PSQ be any focal chord.



Let $P\equiv(at^2, 2at)$. then,

$$Q=\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$

$$SP^2 = a^2(1-t^2)^2+4a^2t^2$$
$$= a^2(1+t^2)^2 \Rightarrow SP=a(1+t^2)$$

$$SQ = a\left(1+\frac{1}{t^2}\right)$$

Harmonic mean of SP & SQ is $\dfrac{2\,SP.SQ}{SP+SQ}$

$$\Rightarrow \frac{2a^2(1+t^2)(1+\frac{1}{t^2})}{a\left[1+t^2+1+\frac{1}{t^2}\right]} \Rightarrow \frac{2a(1+t^2+1+\frac{1}{t^2})}{(1+t^2+1+\frac{1}{t^2})} = 2a.$$

$\therefore$ Harmonic mean of focal chord segments = length of semi latus rectum

Hence proved

4(d) Tangent planes at two points P & Q of a paraboloid meet in the line RS. S.T. the plane through RS and middle point of PQ is parallel to the axis of the paraboloid.

$\rightarrow$ Let the given paraboloid be $ax^2+by^2=2cz$ — ①

Its axis is the z-axis.

Let $P(\alpha_1,\beta_1,\gamma_1)$ & $Q(\alpha_2,\beta_2,\gamma_2)$ be any two points on paraboloid

Then $a\alpha_1^2+b\beta_1^2=2c\gamma_1$ — ② & $a\alpha_2^2+b\beta_2^2=2c\gamma_2$ — ③

Tangent planes to ① at P & Q are

$a\alpha_1x+b\beta_1y-cz=\gamma_1c$ — ④ & $a\alpha_2x+b\beta_2y-cz=\gamma_2c$ — ⑤

These tangent planes meet in a line RS.

Eqn of plane through RS.

$$a\alpha_1x+b\beta_1y-cz-\gamma_1c+\lambda(a\alpha_2x+b\beta_2y-cz-\gamma_2c)=0 \quad \text{— ⑥}$$

It passes through the mid point of PQ i.e. $\left(\frac{\alpha_1+\alpha_2}{2},\frac{\beta_1+\beta_2}{2},\frac{\gamma_1+\gamma_2}{2}\right)$

$\therefore$

$$\left[a\alpha_1\cdot\tfrac{1}{2}(\alpha_1+\alpha_2)+b\beta_1\tfrac{1}{2}(\beta_1+\beta_2)-c\gamma_1-c\tfrac{1}{2}(\gamma_1+\gamma_2)\right]+$$
$$\lambda\left[a\alpha_2\cdot\tfrac{1}{2}(\alpha_1+\alpha_2)+b\beta_2\tfrac{1}{2}(\beta_1+\beta_2)-c\gamma_2-c\tfrac{1}{2}(\gamma_1+\gamma_2)\right]=0$$

$$\Rightarrow (a\alpha_1^2+b\beta_1^2-2c\gamma_1)+a\alpha_1\alpha_2+b\beta_1\beta_2-c(\gamma_1+\gamma_2)+$$
$$\lambda\left[(a\alpha_2^2+b\beta_2^2-2c\gamma_2)+a\alpha_1\alpha_2+b\beta_1\beta_2-c(\gamma_1+\gamma_2)\right]=0$$

$\Rightarrow \quad (1+\lambda)\left[a\alpha_1\alpha_2 + b\beta_1\beta_2 - c(\gamma_1+\gamma_2)\right] = 0$

$\Rightarrow \quad \lambda = -1$

Putting in ⑥ : $\left[a\alpha_1 x + b\beta_1 y - c(\gamma_1 + z)\right] - \left[a\alpha_2 x + b\beta_2 y - c(\gamma_2 + z)\right] = 0$

$\Rightarrow \; a(\alpha_1 - \alpha_2)x + b(\beta_1 - \beta_2)y - c(\gamma_1 - \gamma_2) = 0.$

DRs of normal to this plane are $(\alpha_1-\alpha_2)a,\ (\beta_1-\beta_2)b,\ 0$.

The drs of z-axis is $(0,0,1)$.

$\therefore \; (\alpha_1-\alpha_2)a \cdot 0 + (\beta_1-\beta_2)b \cdot 0 + 0 \cdot 1 = 0.$

$\therefore$ The plane through RS & middle point of PQ is parallel to the z-axis ie. the axis of paraboloid.