10) S.T $f(x) = \dfrac{x(e^{1/x}-1)}{e^{1/x}+1}, x \neq 0$

$0, x = 0$

is continuous but not differentiable at $x=0$

Sol for f to be continuous

$\lim_{x\to 0^-} f = \lim_{x\to 0^+} = f(0)$

$\lim_{x\to 0^-} \dfrac{-x(e^{-1/x}-1)}{e^{-1/x}+1} = x\left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right)$

$\Rightarrow \dfrac{-x(1-e^{1/x})}{1+e^{1/x}} = \dfrac{x(e^{1/x}-1)}{e^{1/x}+1}$

$\Rightarrow \dfrac{x(e^{1/x}-1)}{e^{1/x}+1} = \dfrac{x(e^{1/x}-1)}{e^{1/x}+1}$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\therefore f$ is continuous at $x=0$

but at $x=0$ $f(x)$ is not defined

$\therefore f$ is not differentiable at $x=0$.

11) Evaluate $\iint y \dfrac{\sin x}{x} dx\,dy$

$y \le x \le \pi/2, \quad 0 \le y \le \pi/2$

Sol $\Rightarrow \iint_R y\left(\dfrac{\sin x}{x}dx\right)dy$

$= \iint_0^{\pi/2} dy \int_0^{\pi/2} \dfrac{\sin x}{x}\, dx$

$= \int_0^{\pi/2} y\,dy\,[\sin x]_y^{\pi/2}$ [∵ bracket function]

$= \int_0^{\pi/2} y \cdot dy\,(1-\sin y)$

$= \left[\dfrac{y^2}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} y \sin y\, dy$

$= \dfrac{\pi^2}{8} + [y\cos y - \sin y]_0^{\pi/2}$

$= \pi^2/8 - 1$

2a) If $xyz = a^3$ then S.T min value of $x^2+y^2+z^2 = 3a^2$

Sol Let $F = x^2+y^2+z^2 + \lambda(xyz - a^3)$

$\dfrac{\partial F}{\partial x} = 2x + \lambda yz = 0 \Rightarrow -2x^2 = \lambda xyz$

$\parallel^{ly}$ $\dfrac{\partial F}{\partial y} = 0 \Rightarrow \lambda xyz = -2y^2$

$\dfrac{\partial F}{\partial z} = 0 \Rightarrow \lambda xyz = -2z^2$

$\therefore x = y = z$ as $xyz = a^3$

possibilities are (a, a, a)

$(a, -a, -a)$, $(-a, a, -a)$, $(-a, -a, a)$

as am $\geq$ gm

$$\frac{x^2+y^2+z^2}{3} \geq \sqrt[3]{x^2y^2z^2}$$

$$\geq 3\sqrt[3]{a^6}$$

$$\geq a^2$$

$\therefore$ minimum value $= 3a^2$

3c) Evaluate $I = \int_0^\infty 2^{-ax^2} dx$ using γ function.

Sol

$$I = \int_0^\infty 2^{-ax^2} dx$$

$$= \int_0^\infty e^{-ax^2(\log 2)} dx$$

$$a \log 2 \, x^2 = y$$

$$a \log 2 \cdot 2x = dy$$

$$I = \int_0^\infty \frac{e^{-y}\, dy}{2a(\log 2)x}$$

$$= \int_0^\infty \frac{e^{-y}\, dy}{2a \log 2 \sqrt{y}} \times \sqrt{a\log 2}$$

$$= \frac{1}{2\sqrt{a\log 2}} \int_0^\infty e^{-y} y^{-1/2} dy$$

$$= \frac{1}{2\sqrt{a\log 2}} \Gamma(-1/2+1) = \frac{\sqrt{\pi}}{2\sqrt{a\log 2}}$$

4a] Let f be real valued function $f(x) = \begin{cases} 1/a^{r-1} \\ \frac{1}{a^r} < x \leq 1/a^{r-1} \\ 0, & x = 0 \end{cases}$

when 'a' is integer greater than '2', st $\int_0^1 f(x)\, dx$ exists and equal to $\frac{a}{a+1}$

Sol

$$f(x) = \begin{cases} \frac{1}{a^{1-1}} = 1, & 1/a < x \leq 1 \\ \frac{1}{a}, & \frac{1}{a^2} < x < \frac{1}{a} \\ \vdots \\ 0, & x = 0 \end{cases}$$

$\therefore f(x) \in [0, 1] \; \forall x \in [0, 1]$ as $a > 2$

$\Rightarrow f$ is bounded on $[0, 1]$

also it is continuous on $[0, 1]$

except at $0, 1/a, 1/a^2, 1/a^3 \ldots$

the set of points of discontinuity has only one limit point 0, hence 'f' is integrable on $[0,1]$

Consider $\int_{1/a^r}^{1} f(x) = \int_{1/a}^{1} f\,dx$

$+ \int_{1/a^2}^{1/a} f\,dx \; \dots + \int_{1/a^2}^{1/a^{r-1}} f\,dx$

$= \int_{1/a}^{1} 1\,dx + \int_{1/a^2}^{1/a} 1/a\,dx + \dots \int_{1/a^r}^{1/a^{r-1}} \frac{1}{a^{r-1}}\,dx$

$= (1 - 1/a) + (1/a - 1/a^2)\cdot 1/a$

$+ (1/a^2 - 1/a^3)\, 1/a^2$

$+ \dots$

$= (1 - 1/a)\left[1 + 1/a^2 + 1/a^4 + 1/a^{2(r-1)}\right]$

$= \frac{a-1}{a}\left[\frac{1 \cdot 1 - (1/a^2)^r}{1 - 1/a^2}\right]$

$S_n = \frac{a(1-r^n)}{1-r}$

$= \frac{a}{a+1} \quad \frac{\cancel{a-1}}{\cancel{a}} \cdot \frac{a^{\cancel{2}}}{(a+1)(\cancel{a-1})}$

(2)

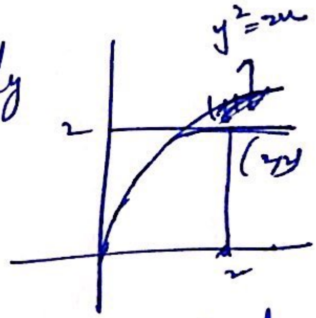
$[\because r \to \infty]$

$\int_0^1 f(x) = a/a+1$ ''

(ii) Evaluate $\iint_R \frac{y}{\sqrt{x^2+y^2+1}}$

$0 \le x \le y^2/2$; $0 \le y \le 2$

Sol: $I = \iint_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$



$= \int_{x=0}^{2}\int_{y=\sqrt{2x}}^{2} \frac{2y}{2\sqrt{1+x^2+y^2}}\,dy\,dx$

$= \int_{x=0}^{x=2} \left[(1+x^2+y^2)^{1/2}\right]_{y=\sqrt{x^2}}^{y=2} dx$

$= \int_{x=0}^{x=2} \left(\sqrt{1+4+x^2} - \sqrt{1+x^2+2x}\right) dx$

$= \int_{x=0}^{x=2} \sqrt{5+x^2} - (1+x)\,dx$

$= \left[\frac{1}{2}\, x\sqrt{5+x^2} + \frac{5}{2}\log\left|x+\sqrt{5+x^2}\right|\right]_0^2$

$- \left[x + x^2/2\right]_0^2$

$= 5/4 \log 5 - 1.$