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Q1. i) $E_1 = 3(2u-4v)+5u+7v$
$= 6u-12v+5u+7v$
$= 11u-5v$

$[\because$ In a vector space, $u, v \in V$ and $a, b \in \mathbb{F}$
$a(u+v) = au+av$, $a(bu) = (ab)u$ $]$

ii) $E_2 = 3u-6(3u-5v)+7u$
$= 3u-18u+30v+7u$
$= -8u+30v$

iii) $E_3 = 2uv+3(2u+4v)$
$= 2uv+6u+12v$

iv) $E_4 = 5u-(3/v)+5u$
$= 10u-(3/v)$

Q2. a) Let $u = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and
$v = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$

$$K(u-v) = K[(x_1, x_2, \cdots, x_n) - (y_1, y_2, \cdots, y_n)]$$
$$= K(x_1, x_2, \cdots, x_n) - K(y_1, y_2, \cdots, y_n)$$
$$= Ku - Kv$$

b) $$u + u = (x_1, x_2, \cdots, x_n) + (x_1, x_2, \cdots, x_n)$$
$$= (x_1 + x_1, x_2 + x_2, \cdots, x_n + x_n)$$
$$= (2x_1, 2x_2, \cdots, 2x_n)$$
$$= 2(x_1, x_2, \cdots, x_n) = 2u.$$

Q3. $$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} a+b+2c \\ a+2b-c \\ a+3b+c \end{bmatrix}$$

$\Rightarrow$ $a+b+2c = 1$
$a+2b-c = -2$ $\quad \Rightarrow \quad a = -6,\ b = 3,\ c = 2$
$a+3b+c = 5$

$\therefore (1, -2, 5) = -6u_1 + 3u_2 + 2u_3$

(b) $(2,-5,3) = a(1,-3,2)+b(2,-4,-1)+c(1,-5,7)$

$= (a+2b+c, -3a-4b-5c, 2a-b+7c)$

$\Rightarrow \quad a+2b+c = 2 \qquad\qquad a+2b+c = 2$

$-3a-4b-5c = -5 \quad \Rightarrow \quad 2b-2c = 1$

$2a-b+7c = 3 \qquad\qquad -5b+5c = -1$

ie. $b-c = \frac{1}{2}$ and $b-c = \frac{1}{5}$

which is not possible.

Q4. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

a) $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$

$(\lambda-1)(\lambda-3)-8 = 0$

i.e. $\lambda^2-4\lambda+3-8 = 0$

i.e. $\lambda^2-4\lambda-5 = 0$

i.e. $(\lambda-5)(\lambda+1) = 0 \Rightarrow \lambda = 5, -1$

b) Latent vectors,

$\lambda = 5 \Rightarrow [A - 5I]x = 0$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{matrix} -4x + 2y = 0 \\ 4x - 2y = 0 \end{matrix}$$

i.e. $2x = y$

$(x, y) = (x, 2x) = x(1, 2)$

$\underline{\lambda = -1} \Rightarrow [A - (-1)I]x = 0$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \begin{matrix} 2x + 2y = 0 \\ 4x + 4y = 0 \end{matrix}$$

$\Rightarrow x = -y$

$(x, y) = (x, -x) = (1, -1)$

Hence Latent vectors are $(1, 2)$ and $(1, -1)$

c) Diagonalizing A i.e. $P^{-1}AP = D$

$$P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad P^{-1} = \frac{-1}{3}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\therefore \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

d) By Cayley Hamilton Theorem

$$A^2 - 4A - 5I = 0$$

$$A^2 = 4A + 5I$$

$$A^3 = 4A^2 + 5A$$

$$A^3 - 5A^2 + 3A + I = -A^2 + 8A + I$$

$$= -4A - 5I + 8A + I$$

$$= 4A - 4I = 4(A - I)$$

$$= 4\begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix} = 8\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

f) let $B = A^3 - 5A^2 + 3A + I$

$$= 8\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$\therefore B^2 = 8^2 \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = 8^2 \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

Q5. $v = xp_1 + yp_2 + zp_3$

$$t^2 + 4t - 3 = x(t^2 - 2t + 5) + y(2t^2 - 3t) + z(t+1)$$

$$t^2 + 4t - 3 = (x + 2y)t^2 + (-2x - 3y + z)t + (5x + z)$$

comparing coeff of t^2, t and constant term

$$x + 2y = 1$$
$$-2x - 3y + z = 4$$
$$5x + z = -3$$

$$\Rightarrow \quad x = \frac{-17}{11}, \quad y = \frac{14}{11}, \quad z = \frac{52}{11}$$

$$\therefore \boxed{v = \frac{-17}{11} p_1 + \frac{14}{11} p_2 + \frac{52}{11} p_3}$$

Q6. $M = xA + yB + zC$

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = x \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + z \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} x + y + z & x + 2y + z \\ x + 3y + 4z & x + 4y + 5z \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x + y + z &= 4 \\ x + 2y + z &= 7 \\ x + 3y + 4z &= 7 \end{aligned}\right\} \Rightarrow x = 2, \quad y = 3, \quad z = -1$$

It satisfies fourth eqn also.

$$x + 4y + 5z = 9$$

$$\therefore \boxed{M = 2A + 3B - C}$$

Q7. $V = P(t)$

a) W = polynomials with integral coefficients

Let $t+1 \in W$ and $\frac{1}{2} \in R$

$$\frac{1}{2}(t+1) = \frac{1}{2}t + \frac{1}{2} \notin W$$

$\Rightarrow$ W is not a subspace

b) W = polynomials with degree ≥ 6 & zero polynomial

Let $u = t^6 + t$

& $v = -t^6 + t \in W$

but $u + v = 2t \notin W \Rightarrow$ W is not a subspace

c) W = polynomials with even powers of t.

Let $u = a_0 + a_1 t^2 + a_2 t^4 + \dots + a_n t^{2n}$

$v = b_0 + b_1 t^2 + b_2 t^4 + \dots + b_m t^{2m}$

then $u + v = a_0 + b_0 + (a_1 + b_1) t^2 + (a_2 + b_2) t^4 + \dots$

$\therefore u + v \in W$ (only even powers)

also for any scalar $c \in \mathbb{R}$

$c \cdot u = ca_0 + ca_1 t^2 + ca_2 t^4 + \dots + ca_n t^{2n} \in W$ $\therefore$ W is a subspace

(7)

Q8. V = vector space of functions, $f: R \to R$

a) $W = \{ f(x) : f(1) = 0 \}$, Let $c \in \mathbb{R}$

Let $f(x), g(x) \in W \Rightarrow f(1) = 0, g(1) = 0$

$\therefore (f+g)(1) = f(1) + g(1) = 0 + 0 = 0$

$\Rightarrow f+g \in W$

$(c \cdot f)(1) = c \cdot f(1) = c \cdot 0 = 0 \Rightarrow c f \in W$

$\therefore$ W is a subspace of V.

b) $W = \{ f(x) : f(3) = f(1) \}$

Let $f, g \in W \Rightarrow f(3) = f(1)$ & $g(3) = g(1)$

$\therefore (f+g)(3) = f(3) + g(3) = f(1) + g(1) = (f+g)(1)$

$\Rightarrow f+g \in W$

$(c \cdot f)(3) = c \cdot f(3) = c \cdot f(1) = (c \cdot f)(1)$

$\Rightarrow cf \in W \Rightarrow$ W is subspace of V

c) $W = \{ f(t) : f(-x) = -f(x) \}$

Let $f, g \in W \Rightarrow f(-x) = -f(x)$ & $g(-x) = -g(x)$

$\therefore (f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x)$

$= -(f(x) + g(x)) = -(f+g)(x)$

$(c \cdot f)(-x) = c \cdot f(-x) = c(-f(x)) = -c \cdot f(x) = -(cf)(x)$

Q4. Two vectors are linearly independent if one is not a multiple of another.

a) $u = (1,2)$, $v = (3,-5)$

$u \neq kv$ for any $k \in \mathbb{R}$ $\therefore$ linearly independent

b) $u = (1,-3)$, $v = (-2,6)$

$(-2,6) = -2(1,-3)$ ie. $v = -2u$

$\therefore$ u & v are linearly dependent

c) $u = (1,2,-3)$, $v = (4,5,-6)$

$u \neq kv$ for any $k \in \mathbb{R}$ $\therefore$ linearly independent

d) $u = (2,4,-6)$, $v = (3,6,-12)$

$v \neq k.u$ for any $k \in \mathbb{R}$

$\therefore$ u & v are linearly independent

Q10. two vectors are linearly independent if one is not scalar multiple of another.

a) $u = 2t^2 + 4t - 3$, $v = 4t^2 + 8t - 6$

$v = 2u$ $\therefore$ <u>linearly dependent</u>

b) $u = 2t^2 - 3t + 4$, $v = 4t^2 - 3t + 2$

$v \neq ku$ for any $k \in \mathbb{R}$

$\therefore$ ~~not~~ <u>linearly independent</u>

c) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}$, $v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$

$v = -4u$ $\therefore$ <u>linearly dependent</u>

d) $u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

$v \neq ku$ for any $k \in R$

$\therefore$ <u>linearly independent</u>.

Q11. $u = (1, 1, 2)$, $v = (2, 3, 1)$
$w = (4, 5, 5)$ in $\mathbb{R}^3$.

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{vmatrix} = 1(15-5) - 1(10-4) + 2(10-12)$$
$$= 10 - 6 - 4 = 0$$

As determinant value is zero.
Hence given vectors are linearly dependent.

Q12. a) Here we have four vectors u_1, u_2, u_3 and u_4, each from $\mathbb{R}^3$ (dim = 3).
Number of vectors is more than dimension of vector space ($\mathbb{R}^3$). Hence given vectors are linearly dependent.

b) $u = (1, 2, 5)$, $v = (2, 5, 1)$, $w = (1, 5, 2)$

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 1 \\ 1 & 5 & 2 \end{vmatrix} = 1(10-5) - 2(4-1) + 5(10-5)$$
$$= 5 - 6 + 25 = 24 \neq 0.$$

$\therefore$ linearly independent.

12(c). $u = (1, 2, 3)$, $v = (0, 0, 0)$, $w = (1, 5, 6)$

~~(1, 2, 3)~~ Consider the linear combination

$$0 \cdot (1, 2, 3) + 10 \cdot (0, 0, 0) + 0 \cdot (1, 5, 6) = (0, 0, 0)$$

Here scalar components are not all zero.

Hence given vectors are linearly dependent.

Q13. $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$

Consider,

$$x \cdot \sin t + y \cdot \cos t + z \cdot t = 0.$$

This is possible only when $x = y = z = 0$

because $$x\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) + y\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) + z(t) = 0$$

Comparing coefficients, $y = 0$

$x + z = 0$

$x = 0$, $\Rightarrow$ $z = 0$.

Hence given functions are linearly independent.

Q14. u, v, w are linearly independent.

Consider,

$a(u+v) + b(u-v) + c(u-2v+w) = 0$

$(a+b+c)u + (a-b-2c)v + cw = 0$

as u, v, w are L.I.

$\therefore \quad a+b+c = 0$

$a-b-2c = 0 \quad \Rightarrow \quad a = 0$

$c = 0 \qquad b = 0$

$c = 0.$

Hence $u+v,\ u-v,\ u-2v+w$ are L.I.

Q.15 $u = (1+i, 2i)$, $v = (1, 1+i)$

Let $au + bv = 0$

$a(1+i, 2i) + b(1, 1+i) = (0, 0)$

$(a+ai+b,\ 2ai+b+bi) = (0, 0)$

$a(1+i)+b = 0$ & $(2a+b)i + b = 0$

ie. $a = 0$ & $b = 0$ & $a = 0,\ b = 0$

if a & b are from $\mathbb{R}$

$\therefore$ Linearly independent over real field.

But $(1+i, 2i) = (1+i)\left[1, \frac{2i}{1+i}\right]$

$$= (1+i)\left(1, \frac{2i}{1+i} \times \frac{1-i}{1-i}\right)$$

$$= (1+i)\left(1, \frac{\cancel{2}(i+1)}{\cancel{2}}\right)$$

$$= (1+i)(1, i+1)$$

$\therefore$ $u = (1+i)\, w$

Hence linearly dependent over Complex field

Q16. Vector space $= \mathbb{R}^3$

$\dim(\mathbb{R}^3) = 3$

a) We need minimum 3 vector to form basis. Hence given vectors do not form basis of $\mathbb{R}^3$

b) More than three vectors in $\mathbb{R}^3$ are linearly dependent. Hence given vectors do not form basis.

c) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 1(2+3) - 1(1-6) + 1(-1-4)$

$$= 5 + \cancel{5} - \cancel{5} = 5 \neq 0$$

$\therefore$ Given vectors are L.I. $\therefore$ form a Basis.

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d) $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{vmatrix} = 1(8-15) - 1(4-25) + 2(3-10)$

$= -7 + 21 - 14 = 0$

Given vectors are linearly dependent. Hence <u>do not form</u> a basis for $\mathbb{R}^3$.

Q17. a) $W = \{(a,b,c) : a+b+c = 0\}$

$= \{(a, b, -a-b)\}$

$= \{a(1,0,-1) + b(0,1,-1)\}$

$\therefore$ Basis$(W) = \{(1,0,-1), (0,1,-1)\}$

dim$(W) = 2$ as $(1,0,-1)$ & $(0,1,-1)$ are linearly independent.

b) $W = \{(a,b,c) : a = b = c\}$

$= \{(a,a,a)\} = \{a(1,1,1)\}$

Basis$(W) = \{(1,1,1)\}$

dim$(W) = \underline{1}$.

Q18. $W = \text{Span}\{(1,-2,5,-3), (2,3,1,-4), (3,8,-3,-5)\}$

a)
$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$\therefore \dim(W) = 2$

Basis $(W) = \{(1,-2,5,-3), (2,3,1,-4)\}$

(Taking vectors corresponding to non-zero rows in reduced matrix).

b) Including vectors $(0,0,1,0)$ and $(0,0,0,1)$ in ~~above~~ reduced matrix we get.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Extended Basis for $\mathbb{R}^4$ is

$\{(1,-2,5,-3), (2,3,1,-4), (0,0,1,0), (0,0,0,1)\}$

Q19.

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - 2R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix} \quad \begin{array}{l} R_4 - 2R_3 \\ R_5 - 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & -4 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ R_3 \leftrightarrow R_5 \end{array}$$

← Echelon form

Hence, for basis we take first, thired and fifth vectors (corresponding to non zero rows)

$$\text{Basis}(W) = \{(1,2,-1,3,4), (1,3,2,2,6), (2,7,3,3,9)\}$$

$u_1, u_3, u_5.$

Q20.

a) $$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 4 & -3 & -1 \\ 0 & 0 & -3 & -1 \end{bmatrix} \quad R_3 - R_2$$

$\therefore$ Rank $= 3$, Basis $= \{(1, 2, 0, -1), (0, 4, -3, -1), (0, 0, -3, -1)\}$

b) $$B = \begin{bmatrix} 1 & 3 & -1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -1 & -2 & -3 \\ 0 & -1 & 4 & 1 & -1 \\ 0 & -3 & -2 & -3 & -3 \\ 0 & -1 & 4 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & -2 & -3 \\ 0 & -1 & 4 & 1 & -1 \\ 0 & 0 & -14 & -6 & 0 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix}$$

~~Basis~~ Rank $= 4$

Basis $= \{(1, 3, -1, -2, -3), (0, -1, 4, 1, -1), (0, 0, -14, -6, 0), (0, 0, 0, -2, 2)\}$

Q21. $U = \text{span}\{(1,1,-1),(2,3,-1),(3,1,-5)\}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$U = \text{Span}\{(1,1,-1),(2,3,-1)\}$ $\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$W = \text{span}\{(1,-1,-3),(3,-2,-8),(2,1,-3)\}$

$$\begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$W = \text{Span}\{(1,-1,-3),(3,-2,-8)\}$ $\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Since reduced echelon form of U and W are same. Hence both have same dimension.

Also, we express the basis vectors of one subspace in terms of basis vectors of other.

$$(1,1,-1) = -5(1,-1,-3) + 2(3,-2,-8)$$
$$(2,3,-1) = -13(1,-1,-3) + 5(3,-2,-8)$$

& $$(1,-1,3) = 5(1,1,-1) + (-2)(2,3,-1)$$
$$(3,-2,-8) = 13(1,1,-1) + (-5)(2,3,-1)$$

$\therefore$ & $\boxed{U = W}$

Q 22. a) $x+2y+2z-s+3t=0$

$x+2y+3z+s+t=0$

$3x+6y+8z+s+5t=0.$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -5 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x+2y-5s-7t=0$, $z+2s-2t=0.$

$$\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2y+5s+7t \\ y \\ -2s+2t \\ s \\ t \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Dim $= 3$

Solution space $= \text{Span}\{(-2,1,0,0,0), (5,0,-2,1,0), (7,0,2,0,1)\}$

b) $x+2y+z-2t=0$

$2x+4y+4z-3t=0$

$3x+6y+7z-4t=0.$

$$\begin{bmatrix} 1 & 2 & 1 & -2 \\ 2 & 4 & 4 & -3 \\ 3 & 6 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -5/2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + 2y - \frac{5}{2}t = 0, \quad 2z + t = 0.$$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} -2y + \frac{5}{2}t \\ y \\ -t/2 \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t}{2} \begin{bmatrix} 5 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

dim = 2

Solution space = span $\{(-2, 1, 0, 0), (5, 0, -1, 1)\}$

c). $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

$$x + y + 2z = 0$$
$$2x + 3y + 3z = 0$$
$$x + 3y + 5z = 0.$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow x = 0, \quad y = 0, \quad z = 0$

$(x, y, z) = (0, 0, 0)$

dim of zero space is not defined.

Solution space = $(0, 0, 0)$.

~~[1 2 1 =~~

Q23. $F(x,y,z) = (x+y+z,\ 2x-3y+4z)$

$F : \mathbb{R}^3 \to \mathbb{R}^2$

Let $\underbrace{(x_1, y_1, z_1)}_{=u},\ \underbrace{(x_2, y_2, z_2)}_{=v} \in \mathbb{R}^3$ & $c \in \mathbb{R}$

$F(u+v) = F(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$= (x_1+x_2+y_1+y_2+z_1+z_2,\ 2x_1+2x_2-3y_1-3y_2+4z_1+4z_2)$

$= (x_1+y_1+z_1,\ 2x_1-3y_1+4z_1) + (x_2+y_2+z_2,\ 2x_2-3y_2+4z_2)$

$= F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$

$= F(u) + F(v)$

$F(c.u) = F(cx_1, cy_1, cz_1) = (cx_1+cy_1+cz_1,\ 2cx_1-3cy_1+4cz_1)$

$= c(x_1+y_1+z_1,\ 2x_1-3y_1+4z_1)$

$= cF(x_1, y_1, z_1) = c.F(u)$.

Q24. a) $F(x,y) = (xy, x)$

$$F(1,2) = (2,1)$$

$$F(10(1,2)) = F(10,20) = F(200,10)$$

$$10 \cdot F(1,2) = 10(2,1) = (20,10)$$

$$F(10(1,2)) \neq 10\,F(1,2) \Rightarrow \text{Not linear}$$

b) $F(x,y) = (x+3, 2y, x+y)$.

$$F(1,1) = (4,2,2) \quad , \quad F(2,2) = (5,4,4)$$

$$F((1,1)+(2,2)) = F(3,3) = (6,6,6)$$

$$F(1,1) + F(2,2) = (4,2,2) + (5,4,4)$$
$$= (9,6,6)$$

$$\therefore F(1,1) + F(2,2) \neq F[(1,1)+(2,2)]$$

$\therefore$ Not linear

c) $F(x,y,z) = (|x|, y+z)$

$$f(-1,0,0) = (1,0,0) \quad , \quad f(1,0,0) = (1,0,0)$$

$$f(-1,0,0) + f(1,0,0) = (2,0,0)$$

$$f[(-1,0,0) + (1,0,0)] = f(0,0,0) = (0,0,0)$$

Both values are different $\therefore$ Not linear

Q25. $F(A) = AM + MA$

Let $A, B \in V$, $c \in \mathbb{R}$

$$F(A+B) = (A+B)M + M(A+B)$$
$$= AM + BM + MA + MB$$
$$= AM + MA + BM + MB$$
$$= F(A) + F(B)$$

and $F(c \cdot A) = (c \cdot A)M + M(c \cdot A)$
$$= c[AM + MA]$$
$$= c \cdot F(A)$$

Hence F is linear.

Q26. $F(x, y, z) = (x - y + z + t,\ x + 2z - t,\ x + y + 3z - 3t)$

$$= x\begin{bmatrix}1\\1\\1\end{bmatrix} + y\begin{bmatrix}-1\\0\\1\end{bmatrix} + z\begin{bmatrix}1\\2\\3\end{bmatrix} + t\begin{bmatrix}1\\-1\\-3\end{bmatrix}$$

Taking first three vectors

$$\begin{vmatrix}1 & -1 & 1\\1 & 0 & 2\\1 & 1 & 3\end{vmatrix} = 1(-2) + 1(3-2) + 1(1) = 0$$

$$\begin{vmatrix}-1 & 1 & 1\\0 & 2 & -1\\1 & 3 & -3\end{vmatrix} = -1(-6+3) - 1(1) + 1(-2) = 0$$

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -3 \end{vmatrix} = 1(1) + 1(-3+1) + 1(1) = 0.$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & -3 \end{vmatrix} = 1(-6+3) - 1(-3+1) + 1(3-2) = -3+2+1 = 0.$$

$\therefore$ ~~dim~~ Rank$(F) = 2$

Image(F) = Span$\{(1,1,1), (-1,0,1)\}$

b) Kernel $F(x,y,z,t) = 0$.

$$x - y + z + t = 0$$
$$x + 2z - t = 0$$
$$x + y + 3z - 3t = 0.$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + 2z - t = 0$ & $y + z - 2t = 0$.

$$\text{Ker} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{bmatrix} -2z+t \\ -z+2t \\ z \\ t \end{bmatrix} = z\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Q27.

$$G(x,y,z,t) = (x+2y-z,\ y+z,\ x+y-2z)$$

a) $$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim(F) = 2$

$\text{Image}(G) = \text{Span}\{(1,0,1),\ (2,1,1)\}$

b) Kernel G. from reduced echelon form

$x - 3z = 0 \Rightarrow x = 3z$

$y + z = 0 \Rightarrow y = -z$

$$\text{Kernel}(G) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z \\ -z \\ z \end{bmatrix} = z\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$= \text{Span}\{(3,-1,1)\}.$

Q28. a) $F(x,y) = (x-y, x-2y)$
$= (0,0)$

$x-y=0$
$x-2y=0$ $\Rightarrow$ $x=0, y=0$

F is non-singular.

b) $G(x,y) = (2x-4y, 3x-6y)$
$=(0,0)$

$2x-4y=0$
$3x-6y=0$ $\Rightarrow$ $x=2y$

Take $x=2 \Rightarrow y=1$

$G(2,1) = (4-4, 6-6) = (0,0)$.

Q29. a) $F(x,y) = (3x+4y, 2x-5y)$
$E = \{(1,0), (0,1)\}$
$F(1,0) = (3,2) = 3(1,0)+2(0,1)$
$F(0,1) = (4,-5) = 4(1,0)-5(0,1)$

$$[M]_E = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$$

b) $S = \{(1,2), (2,3)\}$

$F(1,2) = (11, -8) = -49(1,2) + 30(2,3)$

$F(2,3) = (18, -11) = -76(1,2) + 47(2,3)$

$$[M]_S = \begin{bmatrix} -49 & 30 \\ -76 & 47 \end{bmatrix}^T = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

30. $G(x,y) = (2x - 7y, 4x + 3y)$

$S = \{(1,3), (2,5)\}$

$G(1,3) = (2 - 21, 4 + 9) = (-19, 13)$

$= 121(1,3) + (-70)(2,5)$

$G(2,5) = (4 - 35, 8 + 15) = (-31, 23)$

$= 201(1,3) + (-116)(2,5)$

$$[G]_S = \begin{bmatrix} 121 & -70 \\ 201 & -116 \end{bmatrix}^T = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}$$

b) $v = (4, -3)$

$= (-26)(1,3) + 15(2,5) \Rightarrow [v]_S = \begin{bmatrix} -26 \\ 15 \end{bmatrix}$

$$[G]_S [v]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix} \begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix}$$

$G(v) = G(4,-3) = (8 + 21, 16 - 9) = (29, 7)$

$= (-131)(1,3) + (80)(2,5) \Rightarrow [G(v)]_S = \begin{bmatrix} -131 \\ 80 \end{bmatrix}$