CSE 2015

Q.1 Find the angle betn the surfaces $x^2+y^2+z^2-9=0$ and $z=x^2+y^2-3$ at $(2,-1,2)$

Soln let $f_1 = x^2+y^2+z^2-9$ and $f_2 = x^2+y^2-z-3$

$\therefore$ Grad $f_1 = \nabla\phi_1 = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$ ... $\left[\because \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right]$

Grad $f_2 = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$.

let $n_1$ and $n_2$ be grad $f_1$ and grad $f_2$ at point $(2,-1,2)$ respectively.

$\therefore$ $n_1 = 4\hat{i} - 2\hat{j} + 4\hat{k}$ & $n_2 = 4\hat{i} - 2\hat{j} - 2\hat{k}$

Here $n_1$ and $n_2$ are the normals to the surfaces and $\theta$ is the angle between them.

$\therefore$ $\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k})}{(\sqrt{4^2+2^2+4^2})(\sqrt{4^2+2^2+1^2}}$ .... $(A\cdot B = |A||B|\cos\theta)$

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$$\cos\theta = \frac{16+4-4}{\sqrt{36}\cdot\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$\therefore$ Angle betn the surfaces $= \theta = \cos^{-1}\left(\frac{16}{6\sqrt{21}}\right)$.

— x —

Q.2 Find the value of '$\lambda$' and '$\mu$' so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at $(1,-1,2)$.

Soln let $f_1 = \lambda x^2 - \mu yz - (\lambda+2)x$ and $f_2 = 4x^2y + z^3 - 4$.

$\therefore$ Grad $f_1 = \nabla f_1 = [2\lambda x - (\lambda+2)]\hat{i} - \mu z\hat{j} - \mu y\hat{k}$

Grad $f_2 = \nabla f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

let $n_1$ and $n_2$ be value of $\nabla f_1$ and $\nabla f_2$ at $(1,-1,2)$.

$\therefore$ $n_1 = (\lambda-2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}$

$n_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}$

$\because$ $n_1$ and $n_2$ intersect orthogonally.

$\therefore$ $\cos 90° = 0 = \frac{n_1 \cdot n_2}{|n_1||n_2|}$

$\Rightarrow n_1 \cdot n_2 = 0$

$\Rightarrow -8(\lambda-2) - 8\mu + 12\mu = 0$

$\Rightarrow -8\lambda + 4\mu = -16$

$\Rightarrow \mu - 2\lambda = -4 \Rightarrow \boxed{2\lambda - \mu = 4}$ — (1)

Also point $(1,-1,2)$ lies on surface $\Rightarrow \lambda(1)^2 - \mu(-1)(2) = (\lambda+2)(1)$

$\Rightarrow \lambda + 2\mu = \lambda + 2$

$\Rightarrow \boxed{\mu = 1}$

$\therefore$ from (1): $2\lambda - \mu = 4$

$\Rightarrow 2\lambda - 1 = 4$

$\Rightarrow \boxed{\lambda = 5/2}$ Ans.

Q. A vector field is given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. Verify that the field $\vec{F}$ is irrotational or not. Find the scalar potential.

Soln: A vector field $\vec{F}$ is irrotational or not when curl $\vec{F} = 0$ or not.

$$\therefore \text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + (2xy - 2xy)\hat{k}$$

$$= \vec{0}$$

$\because \nabla \times \vec{F} = 0 \quad \therefore \vec{F}$ is irrotational.

Now $\vec{F}$ can be written as grad of a scalar field (let $\phi$)

$\Rightarrow \vec{F} = \nabla\phi$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\Rightarrow (x^2+xy^2)\hat{i} + (y^2+x^2y)\hat{j} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} \quad \text{--- (1)}$$

Comparing coefficients: $\frac{\partial\phi}{\partial x} = x^2+xy^2$

Integrating $\Rightarrow$ $\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + f(y)$ —(2) ...( $f(y)$ is function of $y$ )

Diffn wrt $y$: $\frac{\partial\phi}{\partial y} = \frac{\cancel{2}x^2y}{\cancel{2}} + f'(y) = x^2y + f'(y)$

From (1): $\frac{\partial\phi}{\partial y} = y^2+x^2y = x^2y + f'(y)$

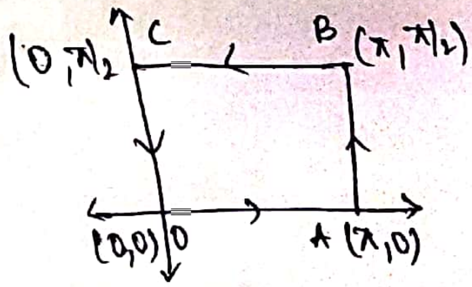
$\Rightarrow f'(y) = y^2$

$\Rightarrow f(y) = \frac{y^3}{3} + c$

$\therefore$ From (2): $\boxed{\phi = \frac{x^3}{3} + \frac{x^2y^2}{2} + \frac{y^3}{3} + c}$

Q.4 Evaluate $\int_C e^{-x}(\sin y\, dx + \cos y\, dy)$ where $C$ is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

Soln... The path $C$ can be broken into paths OA, AB, BC and CO.




$\therefore \int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$.

Along OA: $y=0$; $dy=0$ and $x$ varies from 0 to $\pi$

$\therefore \int_{OA} e^{-x}(\sin y\, dx + \cos y\, dy) = \int_0^{\pi} e^{-x}(\sin 0\, dx + (\cos 0)\cdot 0) = 0.$

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Along AB: $dx=0$, $x=\pi$ & $y$ varies from 0 to $\pi/2$.

$$\int_{AB} = \int_0^{\pi/2} e^{-\pi}(\cos y\, dy) = e^{-\pi}\left[\sin y\right]_0^{\pi/2} = e^{-\pi}.$$

Along BC: $dy=0$, $y=\pi/2$ and $x$ varies from $\pi$ to 0.

$$\int_{BC} = \int_{\pi}^{0} e^{-x}(\sin \pi/2)\, dx = \int_{\pi}^{0} e^{-x}\, dx = \left[-e^{-x}\right]_{\pi}^{0}$$

$$= e^{-\pi} - 1$$

Along CO: $y=0$ $x=0$, $dx=0$ and $y$ varies from $\pi/2$ to 0.

$$\int_{CO} = \int_{\pi/2}^{0} \cos y\, dy = \left[\sin y\right]_{\pi/2}^{0} = -1.$$

$\therefore$ $$\int_C e^{-x}(\sin y\, dx + \cos y\, dy) = e^{-\pi} + e^{-\pi} - 1 - 1.$$

$$= 2(e^{-\pi} - 1)$$

Ans.