Hat Trey

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Date : 3/9/19

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

196/250

PAPER - II : FULL SYLLABUS

TEST CODE: TEST-12: IAS(M)/11-AUG.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 32 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name KATTA RAVI TEJA

Roll No. 0830234

Test Centre ORN

Medium ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			08	35
	(c)			08	
	(d)			08	
	(e)			03	
2	(a)			15	
	(b)			14	43
	(c)			14	
	(d)				
3	(a)				
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	(c)				
	(d)				
4	(a)				
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	(d)				
5	(a)			04	
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6	(a)				
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7	(a)			08	
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	(c)			13	
	(d)				
Total Marks					

196/280

P.T.O.

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SECTION – A

1. (a) Find whether the following statements are true or false. Justify your answer.

(i) $\mathbb{Z} \times \mathbb{Z}$ is a cyclic group.

(ii) $\{a + b\sqrt{2} \in \mathbb{R} \mid a, b$ are rational numbers$\}$ is a cyclic group under usual addition of real numbers.

(iii) $G = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$ is a group under the operation (a, b) (c, d) (ac, bd) but not a cyclic group.

(iv) The number of elements of the subgroup $<a^{10}>$ in the cyclic group $<a>$ of order 30 is 10.

(v) The symmetric group S_n contains a cyclic group of order n. **[10]**

(i) $\mathbb{Z}\times\mathbb{Z} = \{(a,b) \mid a,b \in \mathbb{Z}\}$ → Group with addition.

↳ additive identity is $(0,0)$

If $\mathbb{Z}\times\mathbb{Z}$ is cyclic group then let (x,y) be its generator $\Rightarrow \forall (a,b) \in \mathbb{Z}\times\mathbb{Z}$ $\exists n \in \mathbb{N}$, ∋

$(a,b) = n(x,y) \Rightarrow a = nx$ & $b = ny$.

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$\Rightarrow a, b$ has to be multiples of x, y respectively which is not the case always

$\Rightarrow \boxed{\text{False}}$; $\mathbb{Z}\times\mathbb{Z}$ is not a cyclic group

(ii) Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$ be a cyclic group

$\Rightarrow (a + b\sqrt{2}) = n(x + \sqrt{2}y)$. $\Rightarrow a = nx$; $b = ny$.

$\Rightarrow a, b$ has to be multiples of x, y which is not always the case $\Rightarrow \boxed{\text{False}}$.

$\boxed{(a,b)\odot(c,d) = (ac, bd)}$

(iii)

$\odot$	(1,1)	(1,-1)	(-1,1)	(-1,-1)
(1,1)	$\boxed{(1,1)}$	(1,-1)	(-1,1)	(-1,-1)
(1,-1)	(1,-1)	$\boxed{(1,1)}$	(-1,-1)	(-1,1)
(-1,1)	(-1,1)	(-1,-1)	$\boxed{(1,1)}$	(1,-1)
(-1,-1)	(-1,-1)	(-1,1)	(1,-1)	$\boxed{(1,1)}$

→ identity row.

→ every rows & column has identity element.

→ all elements belong to G

$\left[(a,b)(c,d)\right][e,f] = \left[(ac,bd).(e,f)\right] = [a(ce), b(df)]$

$= [a,b][ce,df]$

$\Rightarrow$ associative $= [a,b]\{[c,d][e,f]\}$

$\Rightarrow \boxed{G \text{ is a group}}$ but not cyclic since every element has order $2 \Rightarrow \boxed{\text{True}}$

(iii) $\boxed{\text{False}}$ Given $O(a) = 30 \Rightarrow a^{30} = e$

Consider $\langle a^{10} \rangle = \{e, a^{10}, a^{20}, a^{30}\} = \{e, a^{10}, a^{20}\}$

$\Rightarrow \boxed{O(\langle a^{10} \rangle) = 3}$ not 10

(v) S_n consists of element $\alpha = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$

$\ni \boxed{\alpha^n = e} \Rightarrow \boxed{H = \langle \alpha \rangle}$ is a cyclic group of order n

$\Rightarrow \boxed{\text{True}}$

1. (b) Let R be a commutative ring. If a, b, ∈ R are nilpotent then show that so are a ± b and ar for any r ∈ R. Give an example of a non commutative ring in which a, b are nilpotent but a + b is not. **[10]**

Let $a^m = 0$ and $b^n = 0$ for some $m, n \in N$.

Consider $(a+b)^{m+n} = a^{m+n} + (m+n)a^{m+n-1}b^1 + \cdots$

$\cdots + {}^{m+n}C_m a^m \cdot b^n + \cdots$

$(\because R \text{ is a commutative ring})$ $\cdots + (m+n)ab^{m+n-1} + b^{m+n}$

$= 0 + 0 + 0 \cdots + 0 = 0$

$\therefore a+b$ is a nilpotent element.

Similarly $a-b$ is also nilpotent element

Consider $(ar)^m = (ar)(ar) \cdots (ar)$ m times

$= (a \cdots a)(r \cdots r)$ m times each $[\because R \text{ is commutative}]$

$= a^m r^m = 0$

$\therefore$ ar is also nilpotent element.

(ii) Take the ring of (2×2) Matrices with matrix multiplication and addition.
It is a non-commutative ring.

Consider $a=\begin{bmatrix}0&1\\0&0\end{bmatrix}$; $a^2=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$b=\begin{bmatrix}0&0\\1&0\end{bmatrix} \Rightarrow b^2=\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow$ a, b are nilpotent elements.

$a+b=\begin{bmatrix}0&1\\1&0\end{bmatrix} \Rightarrow (a+b)^2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\neq 0$

$\therefore$ a+b is not nilpotent because $(a+b)^n = I$ if n is even
$= a+b$ if n is odd.

1. (c) Test for convergence the series :

$1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+.....,x>0.$ **[10]**

here $U_n = \frac{x^{2n}}{(2n)!}$ $\forall n \geq 0$

$\frac{U_n}{U_{n+1}} = \frac{x^{2n}(n+1)!}{n!\, x^{2n+2}} = \frac{n+1}{x^2}$ $\boxed{\text{Let } x \neq 0}$

$\frac{U_n}{U_{n+1}} = \frac{x^{2n}(2n+2)}{(2n)(x^{2n+2})} = \frac{n+1}{nx^2}$

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$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{1}{x^2}\right) = 1/x^2$

$\Rightarrow \sum U_n$ is convergent $|x|<1$ $(x\neq 0)$
is divergent $|x|>1$

when $\boxed{|x|=1}$

$\sum U_n = \sum \frac{1}{2n}$.

Consider $\sum V_n = \sum 1/n$.

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n}{2n} = 1/2$ (limit is finite) and $\sum V_n$ is divergent (p-test) $\Rightarrow \sum U_n$ is divergent

if $\boxed{|x|=0} \Rightarrow \boxed{\sum U_n = 1}$ $\therefore$ convergent

$\therefore$ $\boxed{\begin{array}{l}\sum U_n \text{ is convergent if } |x|<1 \\ \quad\text{divergent if } |x|\geq 1\end{array}}$

1. (d) If f(z) = u + iv is analytic function and u – v = e^x. (cos y – sin y), find f(z) in terms of z. **[10]**

Given $u-v = e^x(\cos y - \sin y)$

$f(z) = u + iv$

$i f(z) = iu - v$

$\Rightarrow f(z)(1+i) = (u-v) + i(u+v)$

let $\boxed{F(z) = U + iV}$ such that — ①

$F(z) = (i+1)f(z)$
$U = u - v$
$V = u + v$
$\Rightarrow F(z)$ is also analytic function

Given $U(x,y) = e^x(\cos y - \sin y)$

$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y)$

$$\frac{\partial u}{\partial y} = e^{x}(-\sin y - \cos y)$$

By Milne's method

$$f'(z) = \left[\frac{\partial u}{\partial x}(z,0)\right] - i\left[\frac{\partial u}{\partial y}(z,0)\right]$$

$$= (e^{z}) - i(-e^{z}) \quad [\text{From } ①]$$

$$\Rightarrow f'(z)[1+i] = e^{z}[1+i]$$

$$\Rightarrow f'(z) = e^{z}$$

integrating on both sides with z, we get

$$\boxed{f(z) = e^{z} + C}$$

where 'c' is arbitrary complex number such that $c = (1+i)c_1$, c_1 is a real number.

1. (e) A firm has two bottling plants, one located at Coimbatore and, other at Chennai. Each plant produces three drinks, Coca-cola, Fanta, and Thumps-up named A, B, and C, respectively. The number of bottles produced per day are, as follows :

	Plant at	
	Coimbatore (E)	Chennai(F)
Coca-cola (A)	15,000	15,000
Fanta (B)	30,000	10,000
Thumps – up(C)	20,000	50,000

A market survey indicates that, during the month of april, there will be a demand of 200,000 bottles of Coca-cola, 400,000 bottles of Fanta, and 440,000 bottles of Thumps-up. The operating cost per day for plants at Coimbatore and Chennai is 600 and 400 monetary, units respectively. For how many days each plant be run in April so as to, minimize the production cost, while still meeting the market demand ? **[10]**

Let the number of days that plants at Coimbatore and Chennai are run be e, f respectively.

The problem can be framed as

$$\text{Min } W = 600e + 400f$$

$$15000e + 15000f \geq 200{,}000$$

$$30000e + 10000f \geq 400{,}000$$

$$20000e + 50000f \geq 440000$$

$$e, f \geq 0$$

Dual of the problem can be written as

$$3x + 3y + 2z \leq 600$$

$$3x + y + 5z \leq 400$$

$$\boxed{x, y, z \geq 0}$$

$$\text{Max } Z = 40x + 40y + 44z$$

$$\Rightarrow 3x + 3y + 2z + s_1 = 600$$

$$3x + y + 5z + s_2 = 400$$

2. (a) (i) Show that any infinite cyclic group is isomorphic to < **Z**, + > the group of integers.

(ii) Any finite cyclic group of order n is isomorphic to $\mathbf{Z}_n$ the group of integers addition modulo n.

(iii) Prove that if an ideal U of a ring R contains a unit of R, then U = R.

[6+6+5=17]

(i) Let G be an infinite cyclic group

$G = \langle a \rangle$.

Consider $\phi : G \longrightarrow (\mathbb{Z}, +)$

$\phi(a^n) = n \quad ; n \in \mathbb{Z}$

$n_1 = n_2$

$\Rightarrow a^{n_1} = a^{n_2}$

$\Rightarrow \phi(a^{n_1}) = \phi(a^{n_2})$ → well defined.

One-one

$\Rightarrow \phi(a^{n_1}) = \phi(a^{n_2})$

$\Rightarrow n_1 = n_2 \Rightarrow$ one-one

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onto

$\forall n \in \mathbb{Z} \ \exists \ a^{n_1} \in G$

homomorphism $\phi(a^{n_1} \cdot a^{n_2}) = \phi(a^{n_1+n_2}) = n_1 + n_2$

$= \phi(a^{n_1}) + \phi(a^{n_2})$

$\therefore G \cong (\mathbb{Z}, +)$ isomorphic

(ii) G is finite cyclic group of order n! $G = \langle a \rangle$

Consider $\phi : G \longrightarrow \mathbb{Z}_n$ Since $a^n \in G$

$\boxed{\phi(a^{x}) = x \bmod n}$ $\Rightarrow 0 \le x < n$

$x_1 = x_2$

$\Rightarrow x_1 \bmod n = x_2 \bmod n$

$\Rightarrow \phi(a^{x_1}) = \phi(a^{x_2})$ well defined

one-one $\phi(a^{x_1}) = \phi(a^{x_2})$

$\Rightarrow x_1 \bmod n = x_2 \bmod n$

since $0 \le x_1, x_2 < n$ $\Rightarrow \boxed{x_1 = x_2}$

onto $\forall x_1 \in \mathbb{Z}_n$; $\exists a^{x_1} \in G$.

homomorphism $\phi(a^{x_1} \cdot a^{x_2}) = \phi(a^{x_1+x_2}) = (x_1+x_2) \bmod n$

$= x_1 \bmod n + x_2 \bmod n$

$\Rightarrow$ From (1), (2), (3) $= \phi(x_1) + \phi(x_2)$

$\boxed{G \approx \mathbb{Z}_n}$

(iii) let U be an ideal of R.

$\Rightarrow \boxed{U \subseteq R}$ -(1)

~~let $r \in R$ and $1 \in U$~~ u be a unit $u \in U$

$\Rightarrow uv = 1$ in R.

$\Rightarrow u \in U$ & $v \in R$ $\Rightarrow uv \in U$ [U is an ideal].

$\Rightarrow 1 \in U$

$\Rightarrow r \in R$; $1 \in U \Rightarrow r \cdot 1 \in U$ [U is an ideal)

$\Rightarrow r \in U \Rightarrow \boxed{R \subseteq U}$ -(2)

from (1) & (2) $\boxed{U = R}$.

2. (b) (i) Determine the values of a, b, c for which the function

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \dfrac{\left(x+bx^2\right)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$$

is continuous at x = 0.

(ii) Show that the function f defined by $f(x)=\dfrac{1}{x^2}$, $x\neq 0$, is uniformly continuous on $[a,\infty)$ where $a>0$, but not uniformly continuous on $(0,\infty)$. **[16]**

(i) For continuity

$$\boxed{\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\sin(a+1)x+\sin x}{x} = a+2.$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} = \lim_{x\to 0}\frac{(1+bx)^{1/2}-1}{bx} = \lim_{x\to 0}\frac{(1+\frac{1}{2}bx-\frac{1}{4}b^2x^2\ldots)-1}{bx}$$
$$= 1/2$$

$f(0) = c$

$$\Rightarrow \boxed{a+2 = c = 1/2}$$

$$\therefore \boxed{a=-3/2;\ b\neq 0\in\mathbb{R};\ c=1/2}$$ are the values of a, b, c.

(ii) Consider $[a,\infty]$ $(a>0)$

$$|f(x)-f(y)| = \left|\frac{1}{x^2}-\frac{1}{y^2}\right| = \left|\frac{(y-x)(y+x)}{x^2y^2}\right|$$

$$f(x)=\frac{1}{x^2};\quad f'(x)=\frac{-2}{x^3} \Rightarrow |f'(x)|\le 2/a^3\ [\forall x\ge a]$$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)| \leq 2/a^3$$

$$\Rightarrow |f(x)-f(y)| < 2/a^3 \, |x-y| < \xi$$

$\forall \xi>0$ $\Rightarrow |f(x)-f(y)| < \xi$; there exists $\delta = \frac{a^3\xi}{2}$

such that $|x-y| < \delta \Rightarrow$ $\boxed{\text{uniform Convergence}}$

Consider $(0,\infty)$

Let us suppose $f(x)$ is uniform continuous

$\Rightarrow |f(x)-f(y)| < \varepsilon$ and $|x-y| < \delta$

~~let~~ there exists $n \in N$ such that $\frac{1}{\sqrt{n}} < \delta$

let $x = \frac{1}{\sqrt{n}}$; $y = \frac{1}{\sqrt{n+1}}$ both $x, y < \delta$ $\Rightarrow |x-y| < \delta$

$\Rightarrow |f(x)-f(y)| = |(n)-(n+1)| = 1 \nless \xi.$

$\therefore f(x)$ is not continuous so it can be uniformaly continuous.

2. (c) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, a > b > 0.$ **[17]**

$$I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$$

let $z = e^{i\theta}$; $|z| = 1$ $\Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$dz = ie^{i\theta}\,d\theta = iz\,d\theta$

$|z|=1$

$$\Rightarrow I = \int_C \frac{1}{\left[a+\frac{b}{2}(z+1/z)\right]^2}\frac{dz}{iz}$$

$$= \int_C \frac{4z}{(b(z^2+1)+2za)^2}\frac{dz}{i\cancel{z}}$$

$$= \frac{4}{b^2 i}\int_C \frac{z\,dz}{\left(z^2+\frac{2za}{b}+1\right)^2} = \frac{4}{bi}\int \frac{z\,dz}{[(z-\alpha)(z-\beta)]^2}$$

here $|\alpha\beta| = 1$ here $|\alpha| < 1 \Rightarrow |\beta| > 1$.

Residue at $z=\alpha \Rightarrow \lim_{z\to\alpha}\left[\frac{d}{dz}(z-\alpha)^2 f(z) = (z-\alpha)^2\frac{z}{(z-\alpha)^2(z-\beta)^2}\right]$

$$= \frac{1}{(z-\beta)^2} - \frac{2z}{(z-\beta)^3}$$

$$= \frac{1}{(\alpha-\beta)^2} - \frac{2\alpha}{(\alpha-\beta)^3}$$

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$$I = \frac{4}{bi}\left[2\pi i\left(\frac{\alpha}{\alpha-\beta}\right)\right] = \frac{8\pi}{b}\left(\frac{\alpha}{\alpha-\beta}\right)$$

$$I = \frac{4}{b^2 i}(2\pi i)(\text{Residue})$$

$$= \frac{8\pi}{b^2}\left[\frac{1}{(\alpha-\beta)^2}\left(1-\frac{2\alpha}{\alpha-\beta}\right)\right]$$

Now $(z-\alpha)(z-\beta) = z^2+\frac{2za}{b}+1$

$$\alpha,\beta = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2} = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2}-1} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\alpha = \frac{-a-\sqrt{a^2-b^2}}{b}; \quad \beta = \frac{-a+\sqrt{a^2-b^2}}{b}$$

$$\Rightarrow I = \frac{\cancel{8}4\pi}{b}\left(\frac{(+a+)\sqrt{a^2-b^2})\cancel{/b}}{(-\cancel{2}\sqrt{a^2-b^2})\cancel{/b}}\right)$$

$$\boxed{I = \frac{4\pi}{b}\left(1+\frac{a}{\sqrt{a^2-b^2}}\right)}$$

$$\boxed{I = \frac{2\pi a}{(a^2-b^2)^{3/2}}}$$

$$I = \frac{\cancel{8}2\pi}{\cancel{b^2}}\left(\frac{1}{\cancel{4}\left(\frac{a^2-b^2}{\cancel{b^2}}\right)}\right)\left(1+\frac{\cancel{2}(a+\sqrt{a^2-b^2})\cancel{/b}}{(-2\sqrt{a^2-b^2}\cancel{/b})}\right) = \frac{2\pi}{(a^2-b^2)}\left(\frac{a}{\sqrt{a^2-b^2}}\right)$$

3. (a) (i) Show that the set of all polynomials with even coefficients is a prime ideal in **Z**[x].

(ii) Let **Z**[i] denote the set of all complex numbers of the form a + bi where a and b are integers. Determine all the prime elements in **Z**[i]. **[17]**

4. (d) A department head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the tasks be allocated, one to a man, so as to minimize the total men hours ? **[12]**

		Subordinates			
		I	II	III	IV
Task	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

SECTION – B

5. (a) Find the complete integral of $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. **[10]**

$f(x, y, z, p, q) = (x+y)(p+q)^2 + (x-y)(p-q)^2 - 1$

Using Charpit's method

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

here $f_x = (p+q)^2 + (p-q)^2 = 2(p^2+q^2)$

$f_y = (p+q)^2 - (p-q)^2 = 4pq$

$f_p = 2(x+y)(p+q) + 2(x-y)(p-q) = 4(xp+yq)$

$f_q = 2(x+y)(p+q) - 2(x-y)(p-q) = 4(xq+yp)$

$f_z = 0$

OK

$$\Rightarrow \frac{dx}{-4(xp+yq)} = \frac{dy}{-4(xq+yp)} = \frac{dp}{2(p^2+q^2)} = \frac{dq}{4pq}.$$

$$\Rightarrow \frac{dp}{p^2+q^2} = \frac{dq}{2pq} \Rightarrow 2pq\,dp - p^2\,dq = q^2\,dq$$

$$\Rightarrow \frac{2p\,dp}{q} - \frac{p^2}{q^2}dq = dq$$

$$\Rightarrow p^2/q = q + C$$

$$\Rightarrow \boxed{p^2 = q^2 + Cq}.$$

$$\frac{dx}{xp+yq} = \frac{dy}{xq+yp}$$

$$\frac{x\,dx - y\,dy}{x^2p + \cancel{xyq} - \cancel{xyq} - y^2p} = \frac{2(x\,dx - y\,dy)}{(x^2-y^2)p} = \frac{2\,dq}{2\,4pq} \Rightarrow \ln(x^2-y^2) = \frac{1}{2}\ln q + c$$

$$q = c\,(x^2-y^2)^2$$

$$\frac{y\,dx - x\,dy}{\cancel{xyp} + y^2q - x^2q - \cancel{xyp}} = \frac{y\,dx - x\,dy}{(y^2-x^2)q} =$$

5. (b) Solve $(D^2 + DD' - 6D'^2)\,z = x^2 \sin(x + y)$. **[10]**

Complementary solution

auxillary equation $m^2 + m - 6 = 0$

$$(m+3)(m-2) = 0$$

$$m = -3, 2.$$

$$\Rightarrow \boxed{z_c = \phi_1(y-3x) + \phi_2(y+2x)}$$

Particular integral

$$\Rightarrow z_p = \frac{1}{(D+3D')(D-2D')}\left[x^2\sin(x+y)\right]$$

$$= \frac{1}{(D+3D')}\int x^2 \sin(x + c - 2x)\cdot dx$$

08

$$= \frac{1}{(D+3D')} \int x^2 \sin(c-x)\,dx$$

$$= \frac{1}{D+3D'} \left[x^2 \cos(c-x) - \int 2x \cos(c-x) \right] = \frac{1}{D+3D'} \left[x^2 \cos(c-x) + 2x \sin(c-x) - \int 2 \sin(c-x) \right]$$

$$= \frac{1}{D+3D'} \left[(x^2-2)\cos(x+y) + 2x \sin(x+y) \right]$$

$$= \int \left[(x^2-2) \cos(4x+C) + 2x \sin(4x+C) \right] dx$$

$$= (x^2-2) \frac{\sin(4x+C)}{4} + \int \left[2x \frac{\sin(4x+C)}{4} + 2x \sin(4x+C) \right] dx$$

$$y_p = \frac{(x^2-2)}{4} \frac{\sin(4x+C)}{4} + \int \frac{3x}{2} \sin(4x+C)\,dx \rightarrow \frac{3x}{2} \frac{\cos(4x+C)}{-4} + \int \frac{3}{8} \cos(4x+C)$$

$$\frac{8x^2-13}{32}$$

$$\therefore y = y_c + y_p = \phi_1(4-3x) + \phi_2(4+x) - \frac{3x}{8}\cos(4x+C) + \left(\quad \right) \sin(4x+C)$$

5. (c) The Quadratic equation $x^4 - 4x^2 + 4 = 0$ has a double root at $x = \sqrt{2}$. Starting with $x_0 = 1.5$, compute three successive iterations to the roots by Newton-Raphson method. Does the result converge Quadratically or linearly ? **[10]**

here $f(x) = x^4 - 4x^2 + 4$

$f'(x) = 4x^3 - 8x$

Newton-Ralphson method

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n)} = x - \frac{(x^4 - 4x^2 + 4)}{4x^3 - 8x}$$

$$\boxed{x_{n+1} = \frac{3x_n^4 - 4x_n^2 - 4}{4x_n^3 - 8x_n}}$$

04

Never write cheating things

let $x_0 = 1.5$

$\Rightarrow x_1 = 1.45833$

$x_2 = 1.436605$

$\boxed{x_3 = 1.42549}$ 3rd iteration.

Error $= |\sqrt{2} - x_3| = +0.011$

Convergence

$|x_n - x_{n-1}| \; C|x_{n+1} - x_n| \le C|x_n - x_{n-1}|^p$

$\Rightarrow$ quadratically.

$p = \dfrac{\ln\left(|x_{n+1} - x_n|\right) + \ln C}{\ln\left(|x_n - x_{n-1}|\right)}$

5. (d) Convert the following :

(i) $(41.6875)_{10}$ to binary number

(ii) $(101101)_2$ to decimal number

(iii) $(AF63)_{16}$ to decimal number

(iv) $(101111011111)_2$ to hexadecimal number **[10]**

(i) $(41.6875)_{10}$

$$\begin{array}{r|l}
2 & 41 \\
2 & 20 - 1 \\
2 & 10 - 0 \\
2 & 5 - 0 \\
2 & 2 - 1 \\
2 & 1 - 0 \\
 & 0 - 1
\end{array}$$

$(101001)_2$

x	$2 \times x$	fraction	integer
0.6875	1.375	0.375	1
0.375	0.75	0.75	0
0.75	1.5	0.5	1
0.5	1	0	1

$(0.6875)_{10} = (0.1011)_2$

$\therefore \boxed{(41.6875)_{10} = (101001.1011)_2}$

(ii) $(101101)_2 = 1\times2^0 + 0\times2^1 + 1\times2^2 + 1\times2^3 + 0\times2^4 + 2^5\times1$

$= 1 + 0 + 4 + 8 + 0 + 32$

$= (45)_{10}$

(iii) $(AF63)_{16} = 3\times16^0 + 6\times16^1 + 15\times16^2 + 10\times16^3$

$= 3 + 96 + 3840 + 40960$

$= (44899)_{10}$

(iv) $(1011|1101|1111)_2 = (BDF)_{16}$

B D F

5. (e) The x component of velocity is $u = x^3 + z^4 + 6$, and the y component is $v = y^3 + z^4$. Find the simplest z component of velocity that satisfies continuity. **[10]**

For equation of continuity to satisfy

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

here $\frac{\partial u}{\partial x} = 3x^2$; $\frac{\partial v}{\partial y} = 3y^2$

$\Rightarrow 3x^2 + 3y^2 + \frac{\partial w}{\partial z} = 0$

$\Rightarrow \frac{\partial w}{\partial z} = -3(x^2+y^2)$

integrating on both sides

$$\boxed{w = -3z(x^2+y^2) + f(x,y)}$$

There can be many possibilities for $f(x,y)$.

Let us assume $f(x,y) = 0$ for simplicity.

$\Rightarrow \boxed{w = -3z(x^2+y^2)}$ is the z component of velocity.

6. (a) Find the integral surface of $x^2p + y^2q + z^2 = 0$, $p = \partial z / \partial x$, $q = \partial z / \partial y$ which passes through the hyperbola $xy = x + y$, $z = 1$. **[08]**

6. (b) Reduce the equation
$x^2(y-1)r - x(y^2-1)\, s + y\,(y-1)t + xy\, p - q = 0.$
to canonical form and hence solve it. **[13]**

7. (a) Find Lagrange's interpolation polynomial fitting the points
$y(1) = -3, y(3) = 0, y(4) = 30, y(6) = 132.$
Hence find y(5). **[10]**

Given :

x	1	3	4	6
y	-3	0	30	132

here $x=3$ is a root of $y(x)$

$\Rightarrow y(x) = (x-3)\,h(x) \Rightarrow h(x) = \frac{y(x)}{(x-3)}$

$\Rightarrow$

x	1	4	6
h	3/2	30	44

08

$$h(x) = \frac{(x-4)(x-6)}{(1-4)(1-6)} h(1) + \frac{(x-1)(x-6)}{(4-1)(4-6)} h(4) + \frac{(x-1)(x-4)}{(6-1)(6-4)} h(6)$$

$$h(x) = \frac{(x^2-10x+24)}{(-3)(-5)}\left(\frac{3}{2}\right) + \frac{(x^2-7x+6)}{(3)(-2)}\left(\frac{5}{30}\right) + \frac{(x^2-5x+4)}{5\times 2}\left(\frac{22}{44}\right)$$

$$= \cancel{x^2\left(\frac{1}{10} - 5 + \frac{22}{5}\right) + x\left(-1 + 35 - 22\right) + \left[\frac{24}{10} - 30 + \frac{88}{5}\right]}$$

$$= x^2\left(\frac{1}{10} - 5 + \frac{22}{5}\right) + x\left(-1 + 35 - 22\right) + \left[\frac{12}{5} - 30 + \frac{88}{5}\right]$$

$$h(x) = \frac{-x^2}{20} + 12x - 10$$

$$y(x) = (x-3)h(x) = (x-3)\left(\frac{-x^2}{20} + 12x - 10\right) = \boxed{\frac{-x^3}{20} + 13.5x^2 - 46x + 30}$$

$$\therefore \boxed{y(5) = 72}$$

7. (b) A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h (ft.)	10	11	12	13	14
A (sq. ft.)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h/A}$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices. **[12]**

Given $\frac{dh}{dt} = -48\sqrt{\frac{h}{A}}$

$$\Rightarrow \frac{dh}{\sqrt{h}} = \frac{-48}{\sqrt{A}}\,dt$$

10′

$$\int_0^t dt = \int_{14}^{10} -\frac{1}{48}\sqrt{\frac{A}{h}}\,dh$$

$$\Rightarrow 48 \int_0^t dt = \int_{10}^{14} \sqrt{\frac{A}{h}}\, dh.$$

From the table

h	10	11	12	13	14
$\sqrt{\frac{A}{h}}$	9.7468	9.8627	10	10.1905	10.4540

Using Simpson's $\frac{1}{3}$rd rule.

$$I = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4)\right]$$

$$= \frac{1}{3}\left[(9.7468 + 10.4540) + 4(9.8627 + 10.1905) + 2(10)\right]$$

$$\boxed{I = 40.137}.$$

$$\Rightarrow 48t = 40.137$$

$$\therefore \boxed{t = 0.8362 \text{ units}}$$

7. (c) Use the Classical fourth-order Runge-Kutta method with h=0.2 to calculate a solution at x=0.4 for the initial value problem $\frac{du}{dx} = 4 - x^2 + u, u(0) = 0$ on the interval [0, 0.4]. **[13]**

Given $f(x,u) = \frac{du}{dx} = 4 - x^2 + u$.

here $x_0 = 0$; $u_0 = 0$; $h = 0.2$

(i) $\boxed{x = 0.2}$

$K_1 = h f(x_0, u_0) = 0.8$

$K_2 = h f\left(x_0 + \frac{h}{2}, u_0 + \frac{K_1}{2}\right) = 0.878$

$K_3 = h f\left(x_0 + \frac{h}{2}, u_0 + \frac{K_2}{2}\right) = 0.8858$

$K_4 = h f(x_0 + h, u_0 + K_3) = 0.96916$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.8828$

$\therefore \boxed{y(0.2) = u(0) + K = 0.8828}$

(ii) $\boxed{x = 0.4}$

$K_1 = h f(x_1, y_1) = 0.96856$

$K_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 1.055416$

$K_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 1.06410$

$k_4 = h\, f(x_1+h,\ y_1+k_3) = 1.15738.$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 1.06083.$

$\therefore\ y_2 = y_1 + k = 1.94362$

$\boxed{y(0.4) = 1.94362}.$

7. (d) (i) Draw the circuit diagram for $\bar{F} = A\bar{B}C + \bar{C}B$ using NAND to NAND logic long.

(ii) In a Boolean Algebra B, for any a and b prove that ab′ + a′ b = 0 if and only if a = b.

(iii) Design a logic circuit having three inputs A, B, C such that output is 1 when A=0 or whenever B=C=1. Also obtain logic circuit using only NAND gates.

[15]

(i) $\bar{F} = A\bar{B}C + \bar{C}B \Rightarrow \cancel{F = \overline{A\bar{B}C+\bar{C}B}}$

$= \overline{\overline{A\bar{B}C} \cdot \overline{\bar{C}B}}$

$\boxed{\bar{F} = \overline{\overline{A\cdot\bar{B}\cdot C} \cdot \overline{\bar{C}\cdot B}}}$

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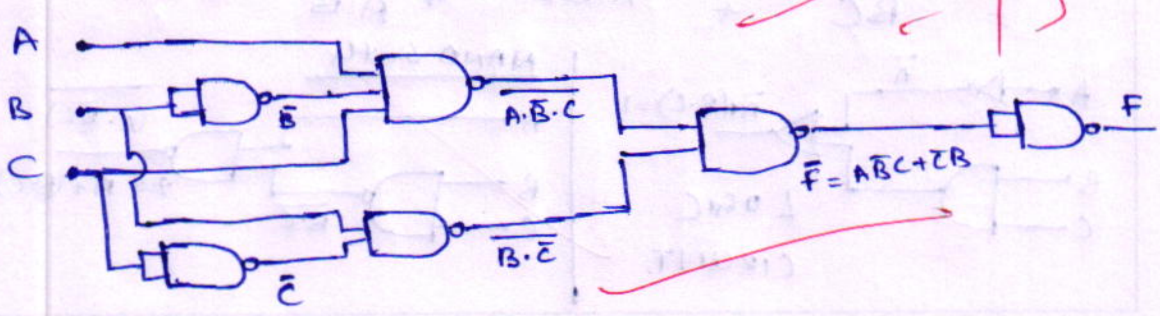


(ii) Consider truth table

a	b	a'	b'	ab'	a'.b	ab'+a'.b
1	1	0	0	0	0	0
1	0	0	1	1	0	1
0	1	1	0	0	1	1
0	0	1	1	0	0	0

From $a'b + a.b' = 0$ only when $a=1;\ b=1$ (or) $a=0;\ b=0$

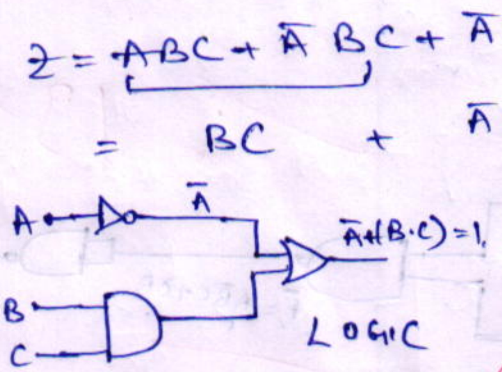
$$\therefore \boxed{a'b + ab' = 0 \text{ if and only if } a=b}$$

(iii)

A	B	C	Y	Z
1	1	1	1	ABC
1	1	0	0	$AB\bar{C}$
1	0	1	0	$A\bar{B}C$
1	0	0	0	$A\bar{B}\bar{C}$
0	1	1	1	$\bar{A}BC$
0	1	0	1	$\bar{A}B\bar{C}$
0	0	1	1	$\bar{A}\bar{B}C$
0	0	0	1	$\bar{A}\bar{B}\bar{C}$

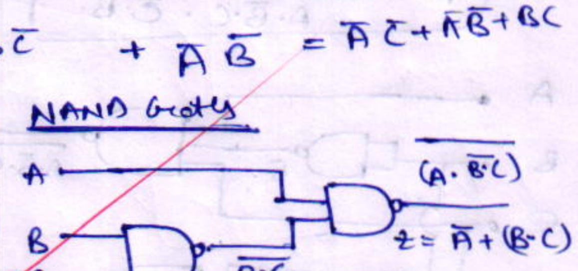
$$Z = \underbrace{ABC + \bar{A}BC}_{} + \bar{A}B\bar{C} + \underbrace{\bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}}_{}$$

$$= BC + \bar{A}B\bar{C} + \bar{A}\bar{B} = \bar{A}\bar{C} + \bar{A}\bar{B} + BC$$



LOGIC CIRCUIT.

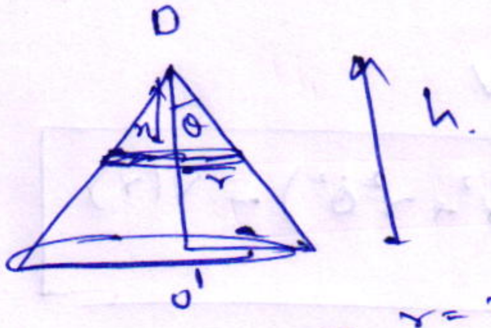
NAND Gates



8. (a) (i) Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis.

(ii) Write Hamilton's equations for a particle of mass m moving in a plane under a force which is some function of distance form the origin. **[08+12=20]**

(a)



$\tan\theta = \frac{r}{x} \Rightarrow r = x\tan\theta$

M.I of elementary disc about OO' $= \frac{1}{2}\left[\rho(\pi r^2)(dx)\right]\left[r^2\right]$

$$\Rightarrow M.I = \int_0^h \frac{\pi\rho}{2}(x\tan\theta)^4 . dx$$

$$= \frac{\pi\rho}{2}\left[\frac{x^5}{5}\right]_0^h \tan^4\theta$$

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$$= \frac{\pi\rho h^5 \tan^4\theta}{10} = \frac{\pi\rho h a^4}{10}$$

but $M = \left(\frac{1}{3}\rho\pi a^2 h\right)$

$$\therefore \boxed{M.I = \frac{3Ma^2}{10}}$$

(b) $T = \frac{1}{2}(m)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\boxed{T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)}$$ movement in plane

since force dependent on radius, potential also depends on r $\boxed{V(r) = -\int F\,dr}$

$$\Rightarrow L = T - V$$

$$\Rightarrow \boxed{L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} \; ; \; p_\theta = \frac{\partial L}{\partial \dot{\theta}}$$

$$\Rightarrow \boxed{p_r = m\dot{r}} \; ; \; \boxed{p_\theta = mr^2\dot{\theta}}$$

$$H = T + V = \frac{1}{2} m\left[\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2 r^4}\right] + V(r)$$

$$\boxed{H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + V(r)}$$

<u>Hamilton's equations of motion</u> are as follows

$$\dot{r} = \frac{\partial H}{\partial p_r} \Rightarrow \dot{r} = \frac{p_r}{2m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} \Rightarrow \dot{p}_r = -\left[-\frac{p_\theta^2}{2mr^3} + V'(r)\right] \Rightarrow 2m\ddot{r} = \frac{p_\theta^2}{2mr^3} - V'(r)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \dot{p}_\theta = -\left[0\right] = 0 \Rightarrow \boxed{r^2\dot{\theta} = K = \text{constant}}$$

8. (b) A particle of mass m moves in a conservative forces field. Find (i) the Lagrangian function and (ii) the equation of motion in cylindrical coordinates (ρ, ϕ, z). **[15]**

(i) $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$

$$\boxed{T = \frac{1}{2} m [\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)}$$

$V = V(\rho, \phi, z)$

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$L = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z)}$$

(ii) $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0}$ equation of motion

$\Rightarrow$ $\cancel{m\ddot{\rho} + \frac{\partial V}{\partial \rho} = 0}$

$m\ddot{\rho} - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$

$\Rightarrow \boxed{m\ddot{\rho} + \frac{\partial V}{\partial \rho} - m\rho\dot{\phi}^2 = 0}$ — (1)

$\Rightarrow \boxed{m\ddot{\phi} + \frac{\partial V}{\partial \phi} = 0}$ — (2)

(1), (2), (3) are the required equations of motion.

$\Rightarrow \boxed{m\ddot{z} + \frac{\partial V}{\partial z} = 0}$ — (3)

8. (c) Prove that liquid motion is possible when velocity at (x, y, z) is given by

$$u=\frac{3x^2-r^2}{r^5}, v=\frac{3xy}{r^5}, w=\frac{3xz}{r^5}, \text{ where } r^2 = x^2 + y^2 + z^2$$

are the stream lines are the intersection of the surfaces, $(x^2 + y^2 + z^2)^3 = c\,(y^2 + z^2)^2$, by the planes passing through Ox. Is this motion irrational. **[15]**

$\frac{\partial u}{\partial x} = $ $u = \frac{3x^2}{r^5} - \frac{1}{r^3}$

$\frac{\partial u}{\partial x} = \frac{6x}{r^5} - \frac{15x^2}{r^6}\left(\frac{x}{r}\right) + \frac{3}{r^4}\left(\frac{x}{r}\right) = \frac{9x}{r^5} - \frac{15x^3}{r^7}$

$\frac{\partial v}{\partial y} = \frac{3x}{r^5} - \frac{15xy}{r^6}\left(y/r\right) = \frac{3x}{r^5} - \frac{15xy^2}{r^7}$

$\frac{\partial w}{\partial z} = \frac{3x}{r^5} - \frac{15xz}{r^6}\left(z/r\right) = \frac{3x}{r^5} - \frac{15xz^2}{r^7}$

3

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \frac{15x}{r^5} - \frac{15x}{r^7}(x^2+y^2+z^2)$$

$$= \cancel{\frac{15x}{r^5}} - \cancel{\frac{15x}{r^7}}(r^2) = 0$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0}$$ $\Rightarrow$ fluid motion is possible.

Streamlines

$$\frac{dx}{3x^2-r^2} = \frac{dy}{3xy} = \frac{dz}{3xz}.$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z} \Rightarrow \ln y = \ln c_1 + \ln z$$

$$\boxed{y = c_1 z} \quad \text{ox plane} \; (1)$$

$$= \frac{x\,dx + y\,dy + z\,dz}{3x(x^2+y^2+z^2) - xr^2} = \frac{2(y\,dy + z\,dz)}{3x(y^2+z^2)}$$

$$\frac{d(r^2)}{x(3r^2-r^2)} = \frac{d(y^2+z^2)}{3\cancel{x}(y^2+z^2)} \Rightarrow \frac{d(r^2)}{2r^2} = \frac{d(y^2+z^2)}{3(y^2+z^2)}$$

integrating on both sides

$$\Rightarrow \frac{1}{2}\ln r^2 = \frac{1}{3}\ln(y^2+z^2)$$

$$\Rightarrow \boxed{(x^2+y^2+z^2)^3 = (c_2(y^2+z^2))^2} \; (2)$$

intersection of (1) & (2) gives stream lines

irrotational

$$\therefore \boxed{\nabla\times F = 0}$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ \frac{3x^2-r^2}{r^5} & \frac{3xy}{r^5} & \frac{3xz}{r^5} \end{vmatrix} = i\left(\cancel{-\frac{15xzy}{r^6}} + \cancel{\frac{15xzy}{r^6}}\right) - j\left(\cancel{\frac{3z}{r^5}} - \cancel{\frac{15x^2z}{r^7}} + \cancel{\frac{15x^2z}{r^7}} - \cancel{\frac{3z}{r^5}}\right) + k\left(\cancel{\frac{3y}{r^5}} - \cancel{\frac{15x^2y}{r^7}} + \cancel{\frac{15x^2y}{r^7}} - \cancel{\frac{3y}{r^5}}\right)$$

$$= 0$$

ROUGH SPACE

$x+y=u$

$x-y=v$

$(x+y)(p^2+2pq+q^2) = xp^2 + \cancel{2xpq} + xq^2 + \cancel{yp^2} + 2ypq + \cancel{yq^2}$

$(x-y)(p^2-2pq+q^2) = xp^2 - \cancel{2xpq} + xq^2 - \cancel{yp^2} + 2ypq - \cancel{yq^2}$

$= 2xp^2 + 2xq^2 + 4ypq.$

$= 2x(p^2+q^2)$

$2xp^2$