Chapter 3

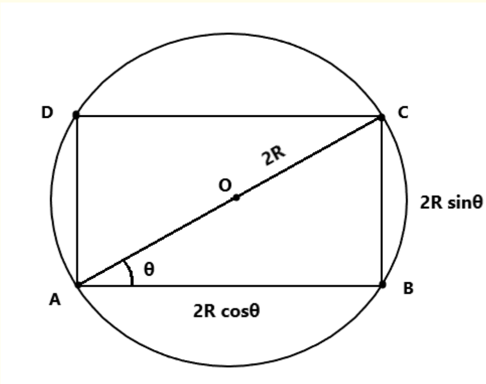
2018

3.1 Section-A

Question-1(a) Show that the maximum rectangle inscribed in a circle is a square.

[8 Marks]

Solution: Let $ABCD$ be the rectangle inscribed in a circle of radius R.



Let

$$\begin{aligned}\angle BAC &= \theta \\ AB &= 2R\cos\theta \\ BC &= 2R\sin\theta \\ \text{Area } A &= (2R\cos\theta)(2R\sin\theta) \\ &= 2R^2\cdot\sin 2\theta\end{aligned}$$

$$\text{For max. area, } \frac{dA}{d\theta} = 0 \Rightarrow \quad 4R^2\cos 2\theta = 0$$

$$\cos 2\theta = 0 \quad \Rightarrow \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{in } [0, 2\pi]$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4} \text{ in } [0, \pi]$$

But no rectangle is possible for $\theta = 3\frac{\pi}{4}$ so, we discard it

$$\frac{d^2A}{d\theta^2} = -8R^2\sin 2\theta < 0 \text{ at } \theta = \frac{\pi}{4}$$

Hence, A is maximum when $\theta = \dfrac{\pi}{4}$ Then

$$AB = 2R\cos\frac{\pi}{4} = 2R\cdot\frac{1}{\sqrt{2}} = \sqrt{2}R = BC$$

Hence, $ABCD$ becomes square.

Question-1(b) Given that Adj A $= \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **and det** A $= 2$. **Find the matrix** A.

[8 Marks]

Solution: We know,

$$A^{-1} = \frac{adj\,A}{|A|}$$

$$\Rightarrow \quad A = |A| \quad (adj\,A)^{-1}$$

$$|\operatorname{adj} A| = |A|^2 = 4(\because |adj(A)| = |A|^{n-1})$$

we find the adjoint of the given matrix:

$$adj(adj\,A) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}^{\top} = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

$$\therefore A = \mid A \mid (adj\,A)^{-1}$$

$$= 2 \times \frac{1}{4} \cdot \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Question-1(c) If f : [a, b] $\to$ R **be continuous in** [a, b] **and derivable in** (a, b), **where** $0 < a < b$, **show that for** $c \in (a, b)$

$$f(b) - f(a) = cf'(c)\log(b/a)$$

[8 Marks]

Solution: Cauchy's Mean Value Theorem Two functions f and g are i) continuous on $[a, b]$ ii) derivable in (a, b) iii) $g'(x) \neq 0 \quad \forall x \in (a, b)$, then there exist atleast one point

$C \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Here, take

$$g(x) = \log x \text{ in}[a, b] \quad 0 < a < b$$

Applying Cauchy's MVT $\exists$ some $c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{\log b - \log a} = \frac{f'(c)}{(1/c)}$$

$$\Rightarrow \quad f(b) - f(a) = c \cdot f'(c) \log \frac{b}{a}$$

Hence,proved.

Question-1(d) Find the equations of the tangent planes to the ellipsoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

which pass through the line

$$x - y - z = 0 = x - y + 2z - 9$$

[8 Marks]

Solution: Any plane through the line

$$x - y - z = 0 = x - y + 2z - 9$$

is

$$(x - y - z) + \lambda(x - y + 2z - 9) = 0$$

$$x(1 + \lambda) - (1 + \lambda)y - (1 - 2\lambda) = 9\lambda - (1)$$

If it touches the conicoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

i.e.

$$\frac{2}{27}x^2 + \frac{2}{9}y^2 + \frac{1}{9}z^2 = 1$$

then

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$\big(ax^2 + by^2 + cz^2 - 1, \quad (x + my + nz = p)\big)$$

$$\frac{27}{2}(1 + \lambda)^2 + \frac{9}{2}(1 + \lambda)^2 + 9(1 - 2\lambda)^2 = (9\lambda)^2$$

$$3\left(\lambda^2 + 2\lambda + 1\right) + \left(\lambda^2 + 2\lambda + 1\right) + 2(4)^2 - 4\lambda + 1\big) = 2 \times 9\lambda^2$$

$$12\lambda^2 + 6 = 18\lambda^2$$

$$6\lambda^2 = 6 \quad \Rightarrow \quad \lambda = 1, -1$$

Hence, from (1) required tangent planes are

$$2x - 2y + z = 9$$

;

$$z = 3$$

Question-1(e) Prove that the eigenvalues of a Hermitian matrix are all real.

[8 Marks]

Solution: Let A be a Hermitian matrix

$$\therefore \quad A^{\theta} = A - (1).Here, A^{\theta} = \text{conjugate transpose}$$

Let λ be an eigenvalue of A and x be corresponding eigenvector of λ.

$$\therefore \quad Ax = \lambda x$$

$$\Rightarrow (Ax)^{\theta} = (\lambda x)^{\theta}$$
$$\Rightarrow \quad x^{\theta} \cdot A^{\theta} = \bar{\lambda} \cdot x^{\theta}$$

Using eq. (1),

$$\Rightarrow \quad x^{\theta} A = \bar{\lambda} x^{\theta}$$

Post mutiplying x both sides,

$$\left(x^{\theta} A\right) x = \left(\bar{\lambda} x^{\theta}\right) x$$
$$x^{\theta}(Ax) = \bar{\lambda}\left(x^{\theta} x\right)$$
$$x^{\theta}(\lambda x) = \bar{\lambda}\left(x^{\theta} x\right)$$
$$\lambda\left(x^{\theta} x\right) = \bar{\lambda}\left(x^{\theta} x\right)$$

[λ is a scalar]

$$(\lambda - \bar{\lambda})\left(x^{\theta} x\right) = 0$$

$$\therefore \lambda - \bar{\lambda} = 0 \quad [\because x \neq 0 \therefore x^{\theta} x \neq 0]$$
$$\therefore \lambda = \bar{\lambda}$$
$$\Rightarrow \lambda \text{ is real.}$$

Question-2(a) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 4$, $z = 2$.

[10 Marks]

Solution: Let $P(x_1, y_1, z_1)$ be any point on the cylinder then the eqn. of the generator through P are

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3}$$

This generator meets the plane $z = 2$ in the point

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{2 - z_1}{3}$$

i.e.

$$\left[\frac{3x_1 - z_1 + 2}{3}, \frac{3y_1 + 2z_1 - 4}{3}, 2\right]$$

$\therefore$ The generator intersect the given curve if

$$\frac{1}{9}(3x_1 - z_1 + 2)^2 + \frac{1}{9}(3y_1 + 2z_1 - 4)^2 = 4$$

$\therefore$ The locus of P $P(x_1, y_1, z_1)$ or the required eqn of cylinder is

$$(3x - z + 2)^2 + (3y + 2z - 4)^2 = 36$$

$$\left(9x^2 + z^2 + 4 + 6xz - 4z + 12x\right) + \left(9y^2 + 4z^2 + 16 + 12yz - 16z - 24y\right) = 36$$

$$9x^2 + 9y^2 + 5z^2 - 6xz + 12yz - 20z - 24y - 12x$$
$$= 16$$

Question-2(b) **Show that the matrices A** $= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ **are congruent.**

[10 Marks]

Solution: **Sylvestor's Law of Inertia**

Two symmetric $n \times n$ matrices are congruent if and only if their diagonal representations have same rank, index and signature.

Rank = no of non-zero eigen-values
index = no of of positive eigen-values
signature = no of of positive eigen-values- = no of of negative eigen-values

Also, two symmetric matrices (as well as skew-symmetric) are congruent if they have the same rank.

$$|A| = 1(6 - 1) - 1(3 + 1) - (1 + 2)$$
$$= 5 - 4 - 3 = -2 \neq 0$$
$$\therefore P(A) = 3$$

$$(B) = 1(0 - 4) + 3(0 - 6) = -4 - 18 = -22 \neq 0$$

$$\therefore \quad P(B) = 3$$

Hence A and B are congruent.

Question-2(c) If ϕ and ψ be two functions derivable in $[\mathrm{a}, \mathrm{b}]$ and $\phi(\mathrm{x})\psi'(\mathrm{x}) - \psi(\mathrm{x})\phi'(\mathrm{x}) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a, b]$, there lies exactly one root of $\psi(x) = 0$.

[10 Marks]

Solution: Let α and β be two consecutive roots of $\phi(x) = 0$ in $[a, b]$ and $\alpha < \beta$.
We are required to prove that only one root of $\psi(x) = 0$ lies between α and β.
If possible, let $\psi(x) = 0$ has no root in (α, β).
Consider the function $F(x) = \dfrac{\phi(x)}{\psi(x)}$

$$F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0 \quad \& \quad F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0$$

$$(\because \phi(\alpha) = 0 = \phi(\beta)$$

and

$$\psi(\alpha) \neq 0, \psi(\beta) \neq 0$$

$$\psi(x) \neq 0, in[\alpha, \beta]$$

$\therefore$ $F(x)$ is continuous in $[\alpha, \beta]$

$$F'(x) = \frac{\phi'(x)\psi(x) - \psi'(x)\phi(x)}{[\psi(x)]^2}$$

exist in$(\alpha, \beta]$.
$\therefore$ $F(x)$ satisfies all condition of Rolle's Theorem in $[\alpha, \beta]$ $\therefore F'(r) = 0$ where $\alpha < r < \beta$
but by given condition

$$\phi'(x)\psi xj - \psi'(x)\phi(x) > 0$$

$\therefore F'(x) \neq 0$ in (α, β) and we get contradiction.
Hence, $\psi(x)$ has atleast one root in (α, β).
By similar argument, it can be shown that between two roots of $\psi(x) = 0$, there is a root of $\phi(x) = 0$.
Now, we prove that there is exactly one root of $\psi(x) = 0$ between α, β.
If possible, let r and δ two roots of $\psi(x) = 0$ in (α, β), i.e.,

$$\alpha < r < \delta < \beta$$

.
Between r and δ, there would exist a root of $\phi(x) = 0$. This contradicts that roots of α and β are consecutive roots of $\phi(x) = 0$.
Hence, there is only one root of $\varphi(x) = 0$ between α and β.

Question-2(d) Show that the vectors $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$

[10 Marks]

Solution: Let

$$x_1(1, 0, -1) + x_2(1, 2, 1) + x_3(0, -3, 2) = (0, 0, 0)$$

$$(x, +x_2, 2x_2 - 3x_3, -x_1 + x_2 + 2x_3) = (0, 0, 0)$$

where;

$$x_1, x_2, x_3 \in R$$

solving these, we get $x_1 = x_2 = x_3 = 0$

$$\Rightarrow \alpha_1, \alpha_2, \alpha_3 \text{ are L.I.}$$

Again, let $\beta = (x, y, z) \in R^3$ and

$$\beta = a(1, 0, -1) + b(1, 2, 1) + c(0, -3, 2)$$

$$(x, y, z) = (a + b, 2b - 3c, -a + b + 2c)$$

$$a + b = x$$

$$2b - 3c = y$$

$$-a + b + 2c = z$$

$$\Rightarrow a = \frac{1}{10}(7x - 2y - 3z)$$

$$\Rightarrow b = \frac{1}{10}(3x + 2y + 3z)$$

$$\Rightarrow c = \frac{1}{5}(x - y + z)$$

(elimentary row operations)

$$\therefore (x, y, z) = \frac{1}{10}(7x - 2y - 3z)(1, 0, -1) + \frac{1}{10}(3x + 2y + 3z)(1, 2, 1) + \frac{1}{5}(x - y + z)(0, -3, 2)$$

$\forall \beta \in R^3$ Using this, we write

$$(1, 0, 0) = \frac{7}{10}(1, 0, -1) + \frac{3}{10}(1, 2, 1) + \frac{1}{5}(0, -3, 2)$$

similarly (0,1,0) and (0,0,1)

Question-3(a) Find the equation of the tangent plane that can be drawn to the sphere

$$x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$$

through the straight line

$$3x - 4y - 8 = 0 = y - 3z + 2$$

[10 Marks]

Solution: The eqn of any plane through given line

$$3x - 4y - 8 + \lambda(y - 3z + 2) = 0$$

$$3x - (y - \lambda)y - 3\lambda z = 8 - 2\lambda - (1)$$

If this plane touches the sphere then. length of prependicular from centre of sphere to plane= Radius of sphere Centre $(1, -3, -1)$ radius $= \sqrt{(1 + 9 + 1 - 8)} = \sqrt{3}$

$$\frac{3(1) - (4 - \lambda)(-3) - 3\lambda(-1) - 8 + 2\lambda}{\sqrt{9 + (4 - \lambda)^2 + 9\lambda^2}} = \pm\sqrt{3}$$

$$-5 + 12 - 3\lambda + 3\lambda + 2\lambda = \pm\sqrt{3} \cdot \sqrt{9 + 16 + \lambda^2 - 8\lambda + 9\lambda^2}$$

$$(2\lambda + 7)^2 = 3\left(10\lambda^2 - 8\lambda + 25\right)$$

$$4\lambda^2 + 28\lambda + 49 = 30\lambda^2 - 24\lambda + 75$$

$$26\lambda^2 - 52\lambda + 26 = 0$$

$$\lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1$$

Hence, Required Eqn of plane from (1)

$$3x - 3y - 3z = 6$$

i.e.,

$$x - y - z = 2$$

Question-3(b) If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(u^2 + v^2\right)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

[10 Marks]

Solution: Chain Rule

$$\frac{\partial f(u, v)}{\partial(x)} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f(u,v)}{\partial x^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial x}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly,

$$\frac{\partial^2 f(u,v)}{\partial y^2} = \left[\frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 f}{\partial u^2}\cdot\left(\frac{\partial u}{\partial y}\right)^2\right] + \left[\frac{\partial f}{\partial v}\cdot\frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 f}{\partial v^2}\cdot\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$u = e^x \cdot cosy$$

$$\frac{\partial u}{\partial x} = e^x \cdot \cos y = u$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} = u$$

$$\frac{\partial u}{\partial y} = -e^x \cdot \sin y = -v$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial u}{\partial y} = -e^x \cos y = -u$$

$$v = e^x \cdot \sin y$$

$$\frac{\partial v}{\partial x} = e^x \cdot \sin y = v$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x} = v$$

$$\frac{\partial v}{\partial y} = e^x \cdot \cos y = u$$

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial u}{\partial y} = -e^x \sin y = -v$$

Using these values

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2 f}{\partial u^2}\cdot\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$+\frac{\partial f}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial^2 f}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$= \frac{\partial f}{\partial u}(u-u) + \frac{\partial^2 f}{\partial u^2}\left(u^2+v^2\right) + \frac{\partial f}{\partial v}(v-v) + \frac{\partial^2 f}{\partial v^2}\left(u^2+v^2\right)$$

$$= \left(u^2+v^2\right)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Question-3(c) Let $\mathrm{T} : \mathrm{V}_2(\mathrm{R}) \to \mathrm{V}_2(\mathrm{R})$ **be a linear transformation defined by** $\mathrm{T}(\mathrm{a},\mathrm{b}) = (\mathrm{a}, \mathrm{a}+\mathrm{b})$. **Find the matrix of** T, **taking** $\{\mathrm{e}_1, \mathrm{e}_2\}$ **as a basis for the domain and {(1,1),(1,-1)} as a basis for the range.**

[10 Marks]

Solution:

$$\text{Let } (x, y) = x_1(1,1) + x_2(1,-1)$$
$$(x, y) = (x_1 + x_2, x_1 - x_2)$$
$$x_1 + x_2 = x$$
$$x_1 - x_2 = y$$
$$\Rightarrow x = \frac{x+y}{2}$$
$$\Rightarrow x_2 = \frac{x-y}{2}$$
$$\therefore (x, y) = \frac{(x+y)}{2}(1,1) + \left(\frac{x-y}{2}\right)(1,-1)$$
$$T(e_1) = T(1,0) = (1, 1+0) = (1,1)$$
$$= \frac{1+1}{2}(1,1) + \frac{1-1}{2}(1,-1),$$
$$= 1 \cdot (1,1) + 0 \cdot (1,-1)$$
$$T(e_2) = T(0,1) = (0, 0+1) = (0,1)$$
$$= \frac{0+1}{2}(1,1) + \frac{(0-1)}{2}(1,-1)$$
$$= \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$$

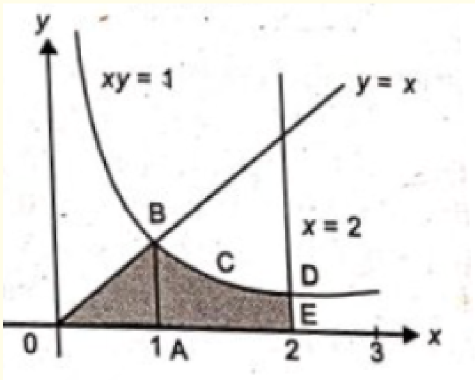
Matrix of $L.T.$ is represented by writing coordinatis of $T(e_1)$ and $T(e_2)$ as columns of matrix.

$$[T] = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$$

Question-3(d) Evaluate $\iint_{\mathrm{R}} (\mathrm{x}^2 + \mathrm{xy})\, \mathrm{dx}$ **dy over the region** R **bounded by** $\mathrm{xy} = 1$, $\mathrm{y} = 0$ $y = x$ **and** $x = 2$

[10 Marks]

Solution: The shaded region is shown in the region of integration.



Hence,

$$\iint\limits_R \left(x^2+xy\right)\cdot dx\cdot dy = \iint\limits_{OAB}\left(x^2+xy\right)\cdot dx\cdot dy + \iint\limits_{ABCDEA}\left(x^2+xy\right)\cdot dxdy$$
$$= \int_0^1 dx\int_0^x \left(x^2+xy\right)\cdot dy + \int_1^2 dx\cdot\int_0^{1/x}\left(x^2+xy\right)\cdot dy$$
$$= \int_0^1 \left[x^2\cdot[y]_0^x + x\left[\frac{y^2}{2}\right]_0^x\right]\cdot dx + \int_1^2\left[x^2\cdot[y]_0^{1/x} + x\cdot\left[\frac{y^2}{2}\right]_0^{1/x}\right]\cdot dx$$
$$= \int_0^1\left[x^3+\frac{x^3}{2}\right]dx + \int_1^2\left[x+\frac{1}{2x}\right]dx$$
$$= \frac{3}{8}\left[x^4\right]_0^1 + \left[\frac{x^2}{2}+\frac{1}{2}\ln(x)\right]_1^2$$
$$= \frac{3}{8}+\left(2-\frac{1}{2}\right)+\frac{1}{2}\ln 2$$
$$= \frac{15}{8}+0.34657 = 2.22157$$

Question-4(a) Find the equations of the straight lines in which the plane $2x+y-z=0$ cuts the cone $4\mathrm{x}^2-\mathrm{y}^2+3\mathrm{z}^2=0$. Find the angle between the two straight lines.

[10 Marks]

Solution: Equation of plane : $2x+y-z=0$
Equation of cone : $4x^2-y^2+3z^2=0$
Let l, m, n be the direction cosines of any one line of section

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Since, it lies in the plane and on the cone,
We have, $2l+m-n=0-(1)$ and $4l^2-m^2+3n^2=0-(2)$ From equation (1), $n=2l+m$
Putting in equation (2), we have

$$4l^2-m^2+3(2l+m)^2=0$$
$$4l^2-m^2+12l^2+3m^2+12lm=0$$
$$8l^2+m^2+6lm=0$$
$$(4l+m)(2l+m)=0$$
$$m=-4l \text{ or } -2l$$

On solving, we get

$$l=1, m=-2, n=0$$
$$\text{and, } l=-1, m=4, n=2$$

Hence, equation of lines:

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

Or

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}$$

and angle between two lines:

$$\cos\theta = \frac{l_1 \cdot l_2 + m_1 \cdot m_2 + n_1 \cdot n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$
$$\therefore \cos\theta = \frac{1(-1) + 4(-2) + 0}{\sqrt{(1)^2 + (-2)^2 + 0} \cdot \sqrt{(-1)^2 + (4)^2 + (2)^2}}$$
$$\cos\theta = \frac{-1-8}{\sqrt{5} \cdot \sqrt{21}}$$
$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{105}}\right) = 151.74°$$

Question-4(b) Show that the functions $u = x + y + z, v = xy + yz + zx$ and $\mathrm{w = x^3 + y^3 + z^3 - 3xyz}$ are dependent and find the relation between them.

[10 Marks]

Solution: From question, we have and

$$u = x + y + z$$
$$v = xy + yz + zx$$
$$w = x^3 + y^3 + z^3 - 3xyz$$

Now,

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial u}{\partial z} = 1$$
$$\frac{\partial v}{\partial x} = (y+z)$$
$$\frac{\partial v}{\partial y} = (x+z)$$
$$\frac{\partial v}{\partial z} = (y+x)$$
$$\frac{\partial w}{\partial x} = 3x^2 - 3yz$$
$$\frac{\partial w}{\partial y} = 3y^2 - 3xz$$
$$\frac{\partial w}{\partial z} = 3z^2 - 3xy$$
$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ (y+z) & (z+x) & (x+y) \\ 3\left(x^2-yz\right) & 3\left(y^2-zx\right) & 3\left(z^2-xy\right) \end{vmatrix}$$

$$\begin{aligned} &= (z+x)\left(3\left(z^2-xy\right)\right) - 3\left(y^2-zx\right)(x+y) \\ &-3\left(z^2-xy\right)(y+z) + 3\left(x^2-yz\right)(x+y) \\ &+3\left(y^2-zx\right)(y+z) - 3\left(x^2-yz\right)(z+x) \\ &= 3\left\{\left(z^2-xy\right)(x-y) + \left(y^2-zx\right)(z-x) \right. \\ &\left. + \left(x^2-yz\right)(y-z)\right\} \\ &= 3\left\{z^2x - x^2y - z^2y + x^2y + y^2z - z^2x - xy^2 \right. \\ &\left. +zx^2 + x^2y - y^2z - x^2z + yz^2\right\} \\ &= 0 \end{aligned}$$

Hence, u, v and w are functionally dependent.

We know that:

$$\begin{aligned} x^3+y^3+z^3-3xyz &= (x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right) \\ &= (x+y+z)\left[(x+y+z)^2 - 3(xy+yz+zx)\right] \\ \therefore w &= u(u^2-3v) \\ &= u^3-3uv \end{aligned}$$

Hence, relation between u, v and w is given by:

$$w = u^3 - 3u.v$$

Question-4(c) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$.

[10 Marks]

Solution: Let the equation of the hyperbolic paraboloid be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z \quad \ldots \quad (1)$$

The equations of the generator which belong to λ -system are

$$\frac{x}{a} - \frac{y}{b} = \lambda z \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}$$

$$or, \frac{x}{a} - \frac{y}{b} - \lambda z = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} + 0, z - \frac{2}{\lambda} = 0$$

Let (l_1, m_1, n_1) be the direction ratios of the generator, then we have

$$\frac{l_1}{a} - \frac{m_1}{b} - \lambda n_1 = 0 \quad \text{and} \quad \frac{l_1}{a} + \frac{m_1}{b} + 0.n_1 = 0$$

Solving for l_1, m_1, n_1, we have

$$\frac{l_1}{\frac{\lambda}{b}} = \frac{m_1}{\frac{-\lambda}{a}} = \frac{n_1}{\frac{2}{ab}}$$

or

$$\frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2} \quad \ldots \quad (2)$$

Similarly, if l_2, m_2, n_2, be the direction-ratios of any generator

$$\frac{x}{a} - \frac{y}{b} = \frac{2}{\mu} \text{ and } \frac{x}{a} + \frac{y}{b} = \mu z$$

of μ -system, then proceeding as above, we have

$$\frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2} \quad \ldots \quad (3)$$

Since the two generators (2) and (3) are perpendicular

$$\therefore \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

i.e.,

$$a^2\lambda\mu - b^2\lambda\mu + 4 = 0 \Rightarrow \left(a^2 - b^2\right)\lambda\mu + 4 = 0$$

$$\Rightarrow \quad \left(a^2 - b^2\right)\left(\frac{2}{z}\right) + 4 = 0$$

$\left[\because \text{ Point of intersection of two generators are } x = a\frac{\lambda+\mu}{\lambda\mu}, y = b\frac{\mu-\lambda}{\lambda\mu}, z = \frac{2}{\lambda\mu}\right]$

$$\Rightarrow \quad a^2 - b^2 + 2z = 0$$

Hence, the required locus is the curve of intersection of the hyperbolic paraboloid and the plane $a^2 - b^2 + 2z = 0$.

Question-4(d) If $(\mathrm{n}+1)$ vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set.

[10 Marks]

Solution: Consider

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n + a_{n+1}\alpha = 0$$

Where $a_1, a_2, \ldots, a_{n+1} \in R$.
Claim: $a_{n+1} \neq 0$ Let, if possible, $a_{n+1} = 0$

$$\therefore \quad a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$$

But, it is given that $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n, \alpha$ are linearly independent.

$$a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_n = 0 \ .$$

which implies that $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ are linearly independent, which is a contradiction. (Given $\alpha_1, \alpha_2, \ldots \alpha_n, \alpha$ are linearly dependent.)
Hence, $a_{n+1} \neq 0$.

$$\therefore \quad a_{n+1}\alpha = -\left(a_1\alpha_1 + a_2\alpha_2 + \cdots a_n\alpha_n\right)$$
$$\Rightarrow \alpha = -\frac{a_1}{a_{n+1}}\alpha_1 - \frac{a_2}{a_{n+1}}\alpha_2 + \ldots - \frac{a_n}{a_{n+1}}\alpha_n$$

3.2 Section-B

Question-5(a) Find the complementary function and particular integral for the equation

$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

and hence the general solution of the equation.

[8 Marks]

Solution: Given ODE is

$$\left(D^2 - 1\right) y = xe^x + \cos^2 x$$

Auxiliary Equation :

$$m^2 - 1 = 0 \Rightarrow m = 1, -1$$
$$\therefore \quad C \cdot I = c_1 e^x + c_2 e^{-x}$$

$$\begin{aligned}
P.I. &= \frac{1}{D^2-1}\left(xe^x + \cos^2 x\right) \\
&= \frac{1}{D^2-1} xe^x + \frac{1}{D^2-1}\cos^2 x \\
&= e^x \frac{1}{(D+1)^2 - 1} x + \frac{1}{D^2-1}\left(\frac{1+\cos 2x}{2}\right) \\
&= e^x \frac{1}{D^2+2D} x + \frac{1}{(D^2-1)} \cdot \frac{1}{2} + \frac{1}{D^2-1}\frac{\cos 2x}{2} \\
&= e^x \cdot \frac{1}{2D\left(1+\frac{D}{2}\right)} x + \frac{1}{D^2-1}\frac{1}{2}e^{0x} + \frac{\cos 2x}{2(-4-1)} \\
&= e^x \cdot \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x + \frac{1}{2(0-1)} + \frac{\cos 2x}{-10} \\
&= e^x \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \ldots\right) x - \frac{1}{2} - \frac{\cos 2x}{10} \\
&= \frac{e^x}{2}\left(\frac{1}{D} - \frac{1}{2} + \frac{D}{4}\right) x - \frac{1}{2} - \frac{1}{10}\cos 2x \\
&= \frac{e^x}{2}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) - \frac{1}{2} - \frac{1}{10}\cos 2x
\end{aligned}$$

Hence, General Solution is given by

$$y = C.I. + P.I.$$
$$y = C_1e^x + C_2e^{-x} + \frac{e^x}{8}(2x^2 - 2x + 1) - \frac{1}{10}\cos 2x - \frac{1}{2}$$

Question-5(b) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \log x (x > 0)$ by the method of variation of parameters.

[8 Marks]

Solution: Given

$$(D^2 - 2D + 1)y = xe^x \log x$$

Auxiliary Equation:

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0 \Rightarrow m = 1, 1$$
$$y_c = (C_1 + C_2x)e^x$$

Let $u = e^x, \quad v = xe^x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & 1 \cdot e^x + xe^x \end{vmatrix}$$
$$= e^{2x}[1 + x - x] = e^{2x} \neq 0$$

Hence Solutions are Independent.

$$P.I. = Au + Bv$$
$$A = -\int \frac{vR}{W}dx$$
$$A = -\int \frac{xe^x \cdot xe^x \log x}{e^{2x}}dx$$
$$= -\int x^2 \log x dx = -\int (\log x)x^2 dx$$
$$= -\left[(\log x)\frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}dx\right] \text{ (by parts)}$$
$$= \frac{-x^3}{3}\log x + \frac{x^3}{9} = \frac{-1}{9}x^3(3\log x - 1)$$

$$\begin{aligned} B &= \int \frac{uR}{W} dx \\ &= \int \frac{e^x \cdot xe^x \log x}{e^{2x}} dx \\ &= \int x \log x dx = \int (\log x) x dx \\ &= (\log x)\frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} dx \qquad \text{(by parts)} \\ &= \frac{x^2}{2}\log x - \frac{x^2}{4} = \frac{1}{4}x^2(2\log x - 1) \\ \therefore \quad y_p &= \frac{-e^x.x^3}{9}(3\log x - 1) + \frac{x^3.e^x}{4}(2\log x - 1) \\ &= x^3.e^x\left[\frac{1}{6}\log x - \frac{5}{36}\right] \end{aligned}$$

General Solution:

$$y = y_c + y_p$$
$$y = (C_1 + C_2 x)e^{2x} + \frac{x^3.e^x}{36}(6\log x - 5)$$

Question-5(c) If the velocities in a simple harmonic motion at distances a, b and c from a fixed point on the straight line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by

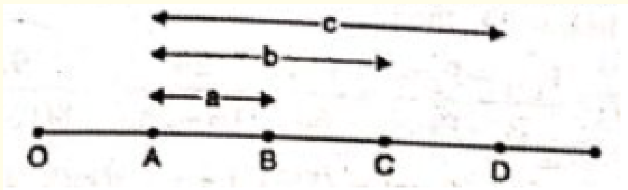
$$\frac{4\pi^2}{\mathrm{T}^2}(\mathrm{b}-\mathrm{c})(\mathrm{c}-\mathrm{a})(\mathrm{a}-\mathrm{b}) = \begin{vmatrix} \mathrm{u}^2 & \mathrm{v}^2 & \mathrm{w}^2 \\ \mathrm{a} & \mathrm{b} & \mathrm{c} \\ 1 & 1 & 1 \end{vmatrix}$$

[8 Marks]

Solution: Let O be the centre of the force and A be the fixed point such that

$$AB = a, AC = b, AD = c$$

and let $OA = s$ and amplitude be A.



$$\begin{aligned} \therefore \quad OB &= s + a \\ OC &= s + b \\ \text{and} \quad OD &= s + c \end{aligned}$$

Velocities at B, C and D are u, v, w respectively

$$\begin{aligned} \therefore u^2 &= \mu\left[A^2 - (s+a)^2\right] - (1) \\ v^2 &= \mu\left[A^2 - (s+b)^2\right] - (2) \\ w^2 &= \mu\left[A^2 - (s+c)^2\right] - (3) \end{aligned}$$

or

$$\frac{u^2}{\mu} = \left(A^2 - s^2\right) - 2as - a^2$$
$$\frac{v^2}{\mu} = \left(A^2 - s^2\right) - 2bs - b^2$$
$$\frac{w^2}{\mu} = \left(A^2 - s^2\right) - 2cs - c^2$$

or

$$\left(\frac{u^2}{\mu} + a^2\right) + 2as + s^2 - A^2 = 0 - (4)$$
$$\left(\frac{v^2}{\mu} + b^2\right) + 2bs + s^2 - A^2 = 0 - (5)$$
$$\left(\frac{w^2}{\mu} + b^2\right) + 2cs + s^2 - A^2 = 0 - (6)$$

From (4),(5) and (6) eliminating s and $s^2 - A^2$ using determinants, we get

$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

Property of determinant

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$
$$\Rightarrow -\mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = -\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Solving the determinant, we get

$$\mu(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} - (7)$$

But,

$$T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \frac{4\pi^2}{T^2}$$

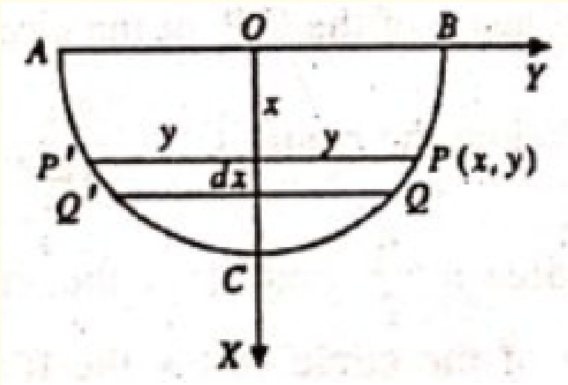
Putting μ in (7), we get

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Question-5(d) From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Find the depth of the centre of pressure of the remainder part.

[8 Marks]

Solution: Let OC be the axis of x



By symmetry, it is evident that the C.P. lies on OX. Consider an elementary strip $PQQ'P'$ of width dx at a depth x below O. Then dS = area of the strip $= 2ydx$, p = intensity of pressure at any point of the strip $= \rho gx$ If $\bar{x}$ be the depth of the C.P. of the semicircular lamina below O, we have

$$\bar{x} = \frac{\int xpdS}{\int pdS} = \frac{\int_0^a x\rho gx \cdot 2ydx}{\int_0^a \rho gx \cdot 2ydx} = \frac{\int_0^a x^2ydx}{\int_0^a xydx}$$

The parametric equations of the circle are

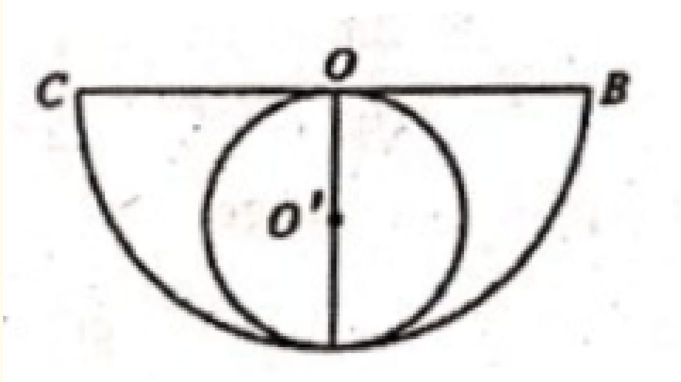
$$x = a\cos t$$
$$y = a\sin t$$
$$\therefore \quad dx = -a\sin tdt$$

$$\therefore \bar{x} = \frac{\int_{\pi/2}^a a^2\cos^2 t \cdot a\sin t(-a\sin tdt)}{\int_{\pi/2}^a a\cos t \cdot a\sin t(-a\sin tdt)}$$

$$= \frac{a\int_0^{\pi/2}\cos^2 t\sin^2 tdt}{\int_0^{\pi/2}\cos t\sin^2 tdt} = \frac{a\left(\frac{1.1}{4.2}\frac{\pi}{2}\right)}{\frac{1}{3.1}} = \frac{3}{16}\pi a$$

Again, x_1 = depth of the C.P. of the semi-circle below

$$O = \frac{3\pi}{16}a$$



$$P_1 = \text{ Pressure on the semi-circle}$$
$$= w \cdot \frac{1}{2}\pi a^2 \cdot \frac{4a}{3\pi} = \frac{2}{3}a^3 w$$

Again depth of the C.P. of the circle of radius $\frac{1}{2}a$ below the centre O' is $\frac{A^2}{4H}$, where A is its radius $= \frac{a}{2}$ and H is the depth of the centre of the circle below the free surface

$$= OO' = \frac{a}{2}$$

$\therefore \quad \frac{A^2}{4H} = \frac{(a/2)^2}{4(a/2)} = \frac{a}{8} \therefore x_2 =$ depth of the C.P. of circle below

$$O = \frac{a}{2} + \frac{a}{8} = \frac{5a}{8}$$

$P_2 =$ pressure on the circle

$$= w \cdot \pi \left(\frac{a}{2}\right)^2 \cdot \frac{a}{2} = \frac{1}{8}w\pi a^3$$

If $\bar{x}$ be the depth of the C.P. of the remainder below O, then

$$\bar{x} = \frac{P_1 x_1 - P_2 x_2}{P_1 - P_2} = \frac{3a\pi}{64} \cdot \frac{24}{(16 - 3\pi)} = \frac{9\pi a}{8(16 - 3\pi)}$$

Question-5(e) **f** $\overrightarrow{\mathrm{r}} = \mathrm{x}\hat{\mathrm{i}} + \mathrm{y}\hat{\mathrm{j}} + \mathrm{z}\hat{\mathrm{k}}$ **and** $\mathrm{f(r)}$ **is differentiable, show that**

$$\mathrm{div}[\mathrm{f(r)}\overrightarrow{\mathrm{r}}] = \mathrm{rf}'(\mathrm{r}) + 3\mathrm{f(r)}$$

Hence or otherwise show that $\mathrm{div}\left(\frac{\vec{r}}{r^3}\right) = 0$

[8 Marks]

Solution: We know that

$$\mathrm{div}(\phi A) = (\mathrm{grad}\,\phi) \cdot A + \phi\,\mathrm{div}(A)$$

$$\begin{aligned}
\therefore \nabla \cdot (f(r)\vec{r}) &= [\nabla f(r)] \cdot \vec{r} + f(r)\nabla \cdot \vec{r} \\
&= [f'(r)\nabla r] \cdot \vec{r} + f(r)[1+1+1] \\
&= \left[f'(r)\frac{\vec{r}}{r}\right] \cdot \vec{r} + 3f(r) \\
&= rf'(r) + 3f(r) \quad [\because \vec{r} \cdot \vec{r} = r^2]
\end{aligned}$$

Now, taking $f(r) = \dfrac{1}{r^3}$,

$$\begin{aligned}
\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) &= r\left(\frac{-3}{r^4}\right) + 3 \cdot \frac{1}{r^3} \\
&= \frac{-3}{r^3} + \frac{3}{r^3} = 0.
\end{aligned}$$

Question-6(a) Solve the differential equation $(y^2 + 2x^2y)\, dx + (2x^3 - xy)\, dy = 0.$

[10 Marks]

Solution: We have $(y^2 + 2x^2y)\, dx + (2x^3 - xy)\, dy = 0 - (1)$ Here,

$$M = y^2 + 2x^2y$$

$$\text{and} \qquad N = 2x^3 - xy$$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2y + 2x^2$$

and

$$\frac{\partial N}{\partial x} = 6x^2 - y$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is not exact. (1)

$$\Rightarrow \left(y^2dx - xydy\right) + \left(2x^2ydx + 2x^3dy\right) = 0$$

$$\Rightarrow y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$$

$$\Rightarrow x^0y^1(1.ydx - 1xdy) + x^2y^0(2ydx + 2xdy) = 0$$

Comparing it with

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$$

we have $a = 0, b = 1, m = 1, n = -1, c = 2\ d = 0, p = 2, q = 2$

$$\text{Also} \quad \frac{m}{n} = \frac{1}{-1} = -1$$

$$\text{and} \quad \frac{p}{q} = \frac{2}{2} = 1$$

$$\therefore \quad \frac{m}{n} \neq \frac{p}{q}$$

$$\text{Let} \qquad \text{I.F.} = x^\alpha y^\beta$$
$$\therefore \quad \frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n}$$
$$\text{and} \quad \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$$

$\Rightarrow$
$$\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$$

and
$$\frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$$

$\Rightarrow$
$$\alpha+1 = -\beta-2$$

and
$$\alpha+3 = \beta+1$$

$\Rightarrow$
$$\alpha+\beta = -3$$

and
$$\alpha-\beta = -2$$

Solving, we get,
$$\alpha = -\frac{5}{2}, \beta = -\frac{1}{2}$$
$$\therefore I.F = x^\alpha y^\beta = x^{-5/2} y^{-1/2}$$

Multiplying (1) by $x^{-5/2}y^{-1/2}$, we get
$$x^{-5/2}y^{-1/2}\left(y^2+2x^2y\right)dx + x^{-5/2}y^{-1/2}\left(2x^3-xy\right)dy = 0$$

$\Rightarrow$
$$\left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2}-x^{-3/2}y^{1/2}\right)dy = 0$$

This equation is exact. $\therefore$ The general solution
$$\int^x \left(x^{-5/2}y^{3/2}+2x^{-1/2}y^{1/2}\right)dx = c$$

$\Rightarrow$
$$y^{3/2}\cdot\frac{x^{-3/2}}{-3/2} + 2y^{1/2}\cdot\frac{x^{1/2}}{1/2} = c$$
$$\Rightarrow \quad -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$$

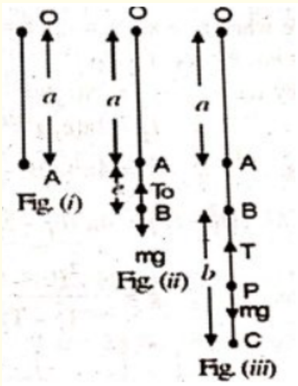
$\Rightarrow$
$$-x^{-3/2}y^{3/2} + 6x^{1/2}y^{1/2} = c'$$

$\left(\text{ Putting } c' = \frac{3}{2}c\right)$

Question-6(b) Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_1 and C_2 are the statical extensions due to these weights and g is the acceleration due to gravity. Show that $g = \dfrac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}$.

[15 Marks]

Solution: Let one end of an elastic string of natural length a and modulus of elasticity λ be attached to the fixed point O and with the other end A, a mass m be attached .(Refer figure (i))



Due to weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB = e$ then tension T_0 in the string will balance the weight of the particle.(Refer figure (ii)) Thus, at B, we get or

$$mg = T_0$$
$$mg = \lambda(e/a) - (1)$$

Let the particle be now pulled down a further distance $BC(= b$, say) and released. Let P be the position of the particle at any subsequent time t. Let $BP = x$ and let T be tension in the string. Then equation of motion of the particle is or

$$\begin{aligned} m\left(d^2x/dt^2\right) &= mg - T = mg - \lambda(e+x)/a \\ &= mg - \lambda(e/a) - \lambda(x/a) \\ \text{or} \quad m\left(d^2x/dt^2\right) &= -\lambda(x/a), \text{ using } (1) \\ \text{or} \quad d^2x/dt^2 &= -(\lambda/am)x - (2) \end{aligned}$$

which is of standard form $d^2x/dt^2 = -\mu x$ of S.H.M., where $\mu = \lambda/am$. Here centre of oscillation is B, from which x is measured and amplitude $= BC = b$. The periodic time T of S.H.M. represented by (2) is given by

$$\begin{aligned} T &= 2\pi/\mu^{1/2} = 2\pi/(\lambda/am)^{1/2} \\ &= 2\pi(am/\lambda)^{1/2} \\ &= 2\pi(e/g)^{1/2}, \text{ by } (1) \quad -(3) \end{aligned}$$

Equation (3) by taking $c(= AB)$ as statical extension corresponding to mass m. Then, time period T is given by

$$T = 2\pi(e/g)^{1/2} - (i)$$

Here when $m = m_1, e = c_1, T = t_1$ and when

$$m = m_2, e = c_2, T = t_2$$

So by (i),

$$t_1 = 2\pi \left(c_1/g\right)^2$$

and

$$t_2 = 2\pi \left(c_2/g\right)^{1/2}$$

Thus,

$$t_1^2 - t_2^2 = 4\pi^2 \left(c_1/g - c_2/g\right)$$

or

$$g\left(t_1^2 - t_2^2\right) = 4\pi^2 \left(c_1 - c_2\right)$$

Hence,

$$g = \frac{4\pi^2 \left(C_1 - C_2\right)}{T_1^2 - T_2^2}$$

Question-6(c) Show that $\overrightarrow{\mathrm{F}} = (2\mathrm{xy} + \mathrm{z}^3)\,\hat{\mathrm{i}} + \mathrm{x}^2\hat{\mathrm{j}} + 3\mathrm{xz}^2\hat{\mathrm{k}}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from $(1, -2, 1)$ to $(3, 1, 4)$.

[15 Marks]

Solution: Here,

$$\vec{\nabla} \times \overrightarrow{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}x^2\right) + \hat{j}\left[\frac{\partial}{\partial t}\left(2xy + z^3\right) - \frac{\partial}{\partial x}3xz^2\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}\left(2xy + z^3\right)\right]$$

$$= 0 + \hat{j}\left(3z^2 - 3z^2\right) + \hat{k}(2x - 2x) = 0$$

For a conservative force field, $\vec{\nabla} \times \overrightarrow{F} = 0$. Work done

$$W = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \left(F_x \cdot dx + F_y \cdot dy + F_z dz\right)$$

$$= \int_A^B \left(2xy + z^3\right) dx + x^2 dy + 3xz^2 dz$$

$$\int_{(1,-2,1)}^{(3,1,4)} \left(2xydx + x^2dy\right) + \left(z^3dx + 3xz^2dz\right)$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\left(x^2y\right) + d\left(xz^3\right) = \int_{(1,-2,1)}^{(3,1,4)} d\left(x^2y + xz^3\right)$$

$$= \left[x^2y + xz^3\right]_{1,-2,1}^{3,1,4}$$

$$= 9 + 3\cdot(4)^3 - 1(-2) - 1(1)^3$$

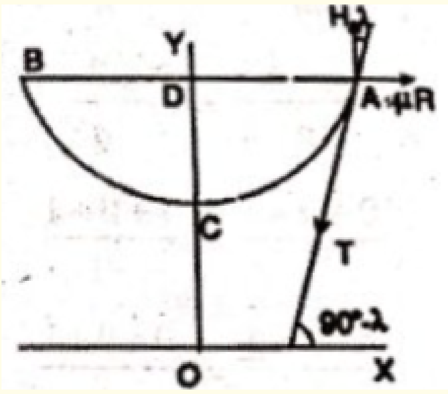
$$= 201 + 2 - 1 = 202$$

Question-7(a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \frac{1 + (1+\mu^2)^{\frac{1}{2}}}{\mu}$$

[10 Marks]

Solution: Let AB be the maximum span. Hence the end links A and B are in limiting equilibrium each under three forces namely the normal reaction $R \perp$ to AB (upwards) the force of friction μR along the fixed horizontal *rod* outwards and the tension T along the tangent at $A(or B)$



If S is the resultant of R and μR at $A($ say $)$ inclined at λ (the angle of friction) to R, then the tension at A must balance S and therefore it is inclined at $(90° - \lambda)$ to the horizon. That is ψ at

$$A = (90° - \lambda)$$

, and

$$\tan\lambda = \frac{\mu R}{R} = \mu$$

$\therefore$ Maximum span $AB = 2x$

$$= 2c \log(\sec\psi + \tan\psi)$$

$$= 2c \log\{\sec(90-\lambda) + \tan(90-\lambda)\}$$

$$= 2c \log\{\operatorname{cosec}\lambda + \cot\lambda\}$$

$$= 2c \log\left\{\sqrt{\left(1+\frac{1}{\mu^2}\right)} + \frac{1}{\mu}\right\}$$

$$= 2c \log\left\{\frac{\sqrt{(\mu^2+1)}+1}{\mu}\right\}$$

And the length of chain $ACB = 2s = 2c$ tan $\psi = 2c$ tan $(90-\lambda)$

$$= 2c\cot\lambda = \frac{2c}{\mu} - (2)$$

$\Rightarrow$

$$\frac{\text{Maximum span } AB}{\text{Length of the chain } ACB}$$
$$= \mu \log \left\{ \frac{1+\sqrt{(1+\mu^2)}}{\mu} \right\}$$

[from (1) and (2) by division]

Question-7(b) Solve:

$$\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$$

[10 Marks]

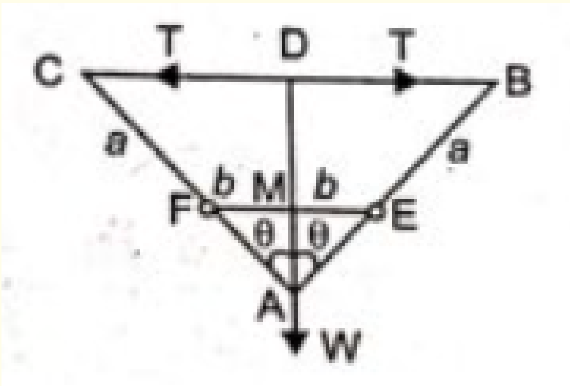
Solution:

$$\frac{dy}{dx} = \frac{2(2x+3y)+5}{3y+2x+4}$$
$$\text{let} \quad 2x+3y = t$$
$$2+3\frac{dy}{dx} = \frac{dt}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx}-2\right)$$
$$\frac{1}{3}\frac{dy}{dx} - \frac{2}{3} = \frac{2t+5}{t+4}$$
$$\frac{1}{3}\frac{dt}{dx} = \frac{2t+5}{t+4} + \frac{2}{3}$$
$$\Rightarrow = \frac{6t+15+2t+8}{3(t+4)}$$
$$\frac{1}{3}\frac{dt}{dx} = \frac{8t+23}{3(t+4)}$$
$$\int \left(\frac{t+4}{8t+23}\right) dt = \int dx$$
$$\frac{1}{8}\int \left(\frac{8t+32}{8t+32}\right) dt = x+c$$
$$= \frac{1}{8}\int \left(\frac{8t+23+9}{8t+23}\right) dt$$
$$= x+c$$
$$= \frac{1}{8}\int dt + \int \frac{9}{8t+23} dt$$
$$= x+c$$
$$= \frac{1}{8}|t| + \frac{9}{8}\int \frac{du}{u}$$
$$= \frac{1}{8}\left[2x+3y+\frac{9}{8}\ln(8t+23)\right] = x+c$$

Question-7(c) A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance $2\mathrm{b}$ apart. A weight W is suspended from A. Find the thrust in the rod BC.

[10 Marks]

Solution: ABC is framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2b$. Each of the rods AB and AC is of length a.



Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let, $\angle BAD = \theta = \angle CAD$. Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes. The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

$$\text{We have, } BC = 2BD = 2AB\sin\theta$$
$$= 2a\sin\theta$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME\cot\theta = b\cot\theta$$

The equation of virtual work is

$$T\delta(2a\sin\theta) + W\delta(b\cot\theta) = 0$$

or

$$2aT\cos\theta\delta\theta - bW\operatorname{cosec}^2\theta\delta\theta = 0$$

or

$$\left(2aT\cos\theta - bW\operatorname{cosec}^2\theta\right)\delta\theta = 0$$

or

$$2aT\cos\theta - bW\operatorname{cosec}^2\theta = 0$$
$$[\because \delta\theta \neq 0]$$

or

$$2aT\cos\theta = bW\operatorname{cosec}^2\theta$$

or

$$T = \frac{Wb}{2a}\operatorname{cosec}^2\theta\sec\theta$$

But in the position of equilibrium,

$$BC = \frac{3}{2}a \text{ and so } BD = \frac{3}{4}a$$

$$\text{Therefore, } \sin\theta = \frac{BD}{AB} = \frac{\frac{3}{4}a}{4} = \frac{3}{4}$$

$$\text{and } \cos\theta = \sqrt{(1-\sin^2\theta)}$$

$$= \sqrt{\left(1-\frac{9}{16}\right)} = \frac{1}{4}\sqrt{7}$$

$$\therefore \quad = \frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}} = \frac{32}{9\sqrt{7}}\frac{b}{a}W$$

Question-7(d) Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of a circle).

[10 Marks]

Solution: Consider $\vec{r} + \frac{1}{k}\hat{N}$, where $\vec{r}(s)$ is a unit-speed curve with s as arc length parameter.

$$\frac{d}{ds}\left(\vec{r} + \frac{1}{k}\hat{N}\right) = \frac{d\vec{r}}{ds} + \frac{1}{k}\frac{d\hat{N}}{ds}$$

$$= \hat{T} + \frac{1}{k}(\tau\hat{B} - k\hat{T})$$

$$= \frac{\tau}{k}\hat{B} \qquad \left[\begin{array}{l}\text{Serret-Frenet} \\ \frac{d\hat{N}}{ds} = \tau\hat{B} - k\hat{T}\end{array}\right.$$

$$= 0 \qquad (\because \text{ Torsion } = 0)$$

It implies that vector $\left(\vec{r} + \frac{1}{k}\hat{N}\right)$ is a constant rector, say, $\vec{a}$.

$\therefore \vec{r} + \frac{1}{k}\hat{N} = \vec{a}$
$\Rightarrow |\vec{r} - \vec{a}| = \left|\frac{-1}{k}\hat{N}\right| = \frac{1}{k}$
Since curvature is constant, let $\frac{1}{\kappa} = c \Rightarrow |\vec{r} - \vec{a}| = c$

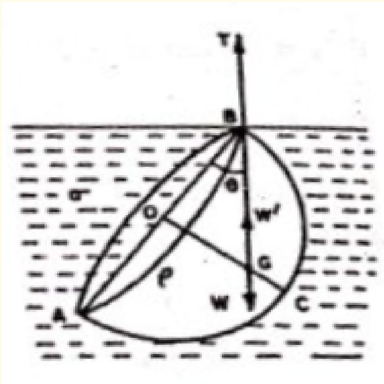
It is the equation of a sphere. Since, torsion is 0, hence curvature α lies in a plane, i.e., α is a part of circle.

Question-8(a) A solid hemisphere floating in a liquid is completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and tension of the string.

[15 Marks]

Solution: ACB is the hemisphere of radius a and density ρ. Density of liquid is σ. Since the hemisphere is completely immersed in the liquid, the weight of the body and force of buoyancy act at the same point G. Here all the forces W, W' and T act along the same vertical line BG.

$$OG = \frac{3a}{8}; OB = a$$



(i) In

$$\Delta BOG, \quad \tan\theta = \frac{OG}{OB} = \frac{\frac{3a}{8}}{a} = \frac{3}{8}$$

$$\therefore \quad \theta = \tan^{-1}\frac{3}{8}$$

(ii) Further $T = W - W'$

$$= \frac{2}{3}\pi a^3\rho g - \frac{2}{3}\pi a^3\sigma g = \frac{2}{3}\pi a^3(\rho - \sigma)_g$$

Question-8(b) A snowball of radius r(t) **melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process.**

[15 Marks]

Solution: Let $\frac{dr}{dt} = k$ (uniform) , $density = \rho$, fixed.

$$\Rightarrow M = \left(\frac{4}{3}\pi r^3\right)\rho$$

$$\Rightarrow \frac{dM}{dt} = (4\pi\rho)r^2\frac{dr}{dt}$$

$$= 4\pi\rho\left(\frac{3M}{4\pi p}\right)^{2/3} k$$

$$= k_1 M^{2/3},$$

$$\text{where } k_1 = \frac{4\pi\rho \cdot 3^{2/3} \cdot k}{(4\pi\rho)^{1/3}}$$

$$\Rightarrow M^{-2/3}dM = k_1 dt$$

Integrating, we get:

$$3M^{1/3} = k_1 t + k_2$$

Let M_0 be initial mass of snow ball. $\therefore M(0) = M_0$ and $M(1) = \frac{M_0}{2}$

$\Rightarrow k_2 = 3M_0^{1/3}$ and $k_1 = 3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}$

We want to calculate time t when $M = 0$
ie.

$$0 = \left[3\left(\frac{M_0}{2}\right)^{1/3} - 3M_0^{1/3}\right) t + 3M_0^{1/3}$$

$$\Rightarrow t = \frac{-3}{3/2^{1/3} - 3} = \frac{-1}{2^{-1/3} - 1} = 4.85$$

Therefore, the it will take 4.85 hours for the entire mass of the snowball to melt.

Question-8(c) For a curve lying on a sphere of radius a and such that the torsion is never 0**, show that**

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = a^2$$

[10 Marks]

Solution: Let vector point $r(s)$ lies on a sphere with centre r_0 and radius a.

$$\therefore |(r - r_0)| = a$$

$$|r - r_0|^2 = a^2 \quad \Rightarrow \quad (r - r_0)\cdot(r - r_0) = a^2 \quad \ldots(*)$$

Differentiating w.r.t s

$$\frac{dr}{ds}(x - r_0) + (r - r_0)\cdot\frac{dr}{ds} = 0$$

$$2\frac{dr}{ds}\cdot(r - r_0) = 0$$

$$\Rightarrow \quad (r - r_0)\cdot t = 0 \quad \ldots(1)$$

Again differentiating w.r.t s

$$\begin{aligned}\frac{dr}{ds}\cdot t+(r-r_0)\cdot\frac{dt}{ds}&=0\\ t\cdot t+(r-r_0)\cdot(kn)&=0\\ 1+(r-r_0)\cdot(kn)&=0 \qquad \text{(serret-frenet)}\\ (r-r_0)\cdot n&=\frac{-1}{k} \qquad \ldots(2)\end{aligned}$$

Again differentiating w.r.t s

$$\begin{aligned}\frac{dr}{ds}\cdot n+(r-r_0)\cdot\frac{dn}{ds}&=\frac{1}{k^2}\cdot k'\\ t\cdot n+(r-r_0)\cdot(\tau b-kt)&=\frac{k'}{k^2} \qquad \text{(serret-frenet)}\\ 0+(r-r_0)\cdot(\tau b)-(r-r_0)\cdot(kt)&=\frac{k'}{k^2}\\ (r-r_0)\cdot b&=\frac{k'}{k^2\tau} \quad [\text{using}(1)] \quad \ldots(3)\end{aligned}$$

From $(1),(2),(3)$ we see that The components of $(r-r_0)$ with respect to t,n,b are $0,-\frac{1}{k}$ and $\frac{k'}{k^2\tau}$ Hence,

$$r-r_0=\frac{-1}{k}n+\left(\frac{k'}{k^2\tau}\right)b$$

From $(*)$ we get,

$$\begin{aligned}a^2=(r-r_0)\cdot(r-r_0)&=\left(\frac{-1}{k}n+\frac{k'}{k^2\tau}b\right)\cdot\left(\frac{-1}{k}n+\frac{k'}{k^2\tau}b\right)\\ &=\frac{1}{k^2}n\cdot n-\frac{k'}{k^3\tau}b\cdot n-\frac{k'}{k^3\tau}n\cdot b+\left(\frac{k'}{k^2\tau}\right)^2 b\cdot b\\ &=\frac{1}{k^2}+\left(\frac{k'}{k^2T}\right)^2 \quad [\because n\cdot n=1=b\cdot b\Rightarrow n\cdot b=0]\end{aligned}$$