1

Functions of a Real Variable, Limits, Continuity and Differentiability

§ 1. **Definitions.**

Differential Calculus is the study of the rates of change of continuously varying co-related quantities and with the help of these rates, it studies the problems and properties connected with such quantities.

A constant is a quantity which, during any set of mathematical operations, retains the same value.

A variable is a quantity, or a symbol representing a number, which is capable of assuming different values.

A continuous variable is a variable that can take all the numerical values between two given numbers.

An independent variable is one which may take up any arbitrary value that may be assigned to it.

A dependent variable is a symbol which can assume its value as a result of some other variable taking some assigned value.

Functions. If one quantity depends upon another, so that it assumes a definite value when a system of definite values is given to other *i.e.*, if a symbol y has one definite value for every value of the variable x, **then y is called a function of x** and is written as $y = f(x)$, or $y = F(x)$, or $y = \phi(x)$ etc.

The function itself is a *dependent variable*, and the variables to which values are given are *independent variables*. Thus $u = f(x, y, z)$ represents that u is a function of several variables x, y, z.

Domain of a variable x consists of the totality of values that it can take. If a variable x takes all real values between a and b, its domain is represented by the interval (a, b). If a and b belong to the domain (a, b), it is said to be a **closed domain** or a **closed interval** and if a and b do not belong to the domain (a, b), then it is an **open domain** or an **open interval.**

Explicit and Implicit functions. A function is said to be *explicit* when it is expressed directly in terms of the independent variable or variables. Thus a function is explicit if it is written as $y = f(x)$. But a function y which instead of being written in the form $y = f(x)$ is written in the form $f(x, y) = 0$ is called an *implicit* function of x.

Scanned by TapScanner

Even and odd functions. A function $f(x)$ is said to be an *even function of* x if $f(-x) = f(x)$ and an *odd function of* x if

$$f(-x) = -f(x).$$

Algebraic functions. Functions which consist of a finite number of terms involving powers and root of the independent variable, are called algebraic functions *e.g.*, $(x^2 + \sqrt{x})$, $2x^2 + 9x + 7$, $x^{7/2}$, etc.

Transcendental functions. All non-algebraic functions are called transcendental functions. For example, $\sin x$, $\sqrt{\tan x}$, $x^2 + \cos x$, $\sin^{-1} x$, e^x, 3^x, $\log x$, $\log(1 + x)$ etc. are transcendental functions.

§ 2. Limit. Definition. **(Delhi 1983, 81; Kashmir 83; Allahabad 72; Meerut 80)**

A function $f(x)$ is said to tend to the limit A as x tends to a, if, corresponding to any arbitrary chosen positive number ε, however small (but not zero), there exists a positive number δ, such that

$$|f(x) - A| < \varepsilon,$$

for all values of x for which $0 < |x - a| < \delta$.

We write it as $\lim_{x \to a} f(x) = A$.

Limit on the right : A function $f(x)$ is said to tend to the limit A as $x \to a$ **from the right**, if, corresponding to any arbitrary chosen +ive number ε, however small (but not zero), there exists a +ive number δ, such that

$$|f(x) - A| < \varepsilon,$$

for all values of x for which $0 < x - a < \delta$.

The limit of $f(x)$ as $x \to a$ from the right is called the *right hand limit* of $f(x)$ and is denoted by $\lim_{x \to a+0} f(x)$ or by $f(a + 0)$.

We calculate $f(a + 0)$ by evaluating $\lim_{h \to 0} f(a + h)$, where h is +ive and sufficiently small.

Limit on the left. A function $f(x)$ is said to tend to the limit A as $x \to a$ from the left, if corresponding to any arbitrary chosen +ive number ε, however small (but not zero), there exists a +ive number δ, such that

$$|f(x) - A| < \varepsilon,$$

for all values of x for which $0 < a - x < \delta$.

The limit of $f(x)$ as $x \to a$ from the left is called the *left hand limit* of $f(x)$ and is denoted by $\lim_{x \to a-0} f(x)$ or by $f(a - 0)$. We calculate $f(a - 0)$ by evaluating $\lim_{h \to 0} f(a - h)$, where h is +ive and sufficiently small.

Scanned by TapScanner

Note 1. $\lim_{x \to a} f(x)$ exists only if, limit on the left = limit on the right.

Note 2. If $f_1(x) \to A$ and $f_2(x) \to B$, as $x \to a$, where A and B are both finite, we have

(i) $\lim_{x \to a} \{f_1(x) \pm f_2(x)\} = A \pm B$,

(ii) $\lim_{x \to a} \{f_1(x) \cdot f_2(x)\} = A \cdot B$,

(iii) $\lim_{x \to a} \left\{\dfrac{f_1(x)}{f_2(x)}\right\} = \dfrac{A}{B}$, provided $B \neq 0$.

Note 3. The limit of $f(x)$ as $x \to a$ is not necessarily the same as the value of the function at $x = a$. In fact the limit of $f(x)$ as $x \to a$ may exist even if the function $f(x)$ is not defined at $x = a$.

Examples on Limits

Ex. 1. *The limit of a sum is equal to the sum of the limits.*

Sol. Let $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$.

Let ε be any arbitrary chosen positive number.

We can choose δ_1, δ_2 such that

$$|f_1(x) - A| < \tfrac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta_1,$$

and $$|f_2(x) - B| < \tfrac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta_2.$$

Let δ be any positive number smaller than both δ_1 and δ_2. Then

$|f_1(x) - A| < \frac{1}{2}\varepsilon$ and $|f_2(x) - B| < \frac{1}{2}\varepsilon$, for $0 < |x - a| < \delta$.

Now from algebra, we have

$$|\{f_1(x) + f_2(x)\} - (A + B)| = |\{f_1(x) - A\} + \{f_2(x) - B\}|$$
$$\leq |f_1(x) - A| + |f_2(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ i.e., } < \varepsilon.$$

$$\therefore \lim_{x \to a} \{f_1(x) + f_2(x)\} = A + B.$$

Note. Similarly we can prove that

$$\lim_{x \to a} [f_1(x) - f_2(x)] = A - B.$$

Ex. 2. *The limit of a product is equal to the product of the limits.* **(Meerut 1975)**

Sol. Let $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$. We shall prove the result when $A \neq 0$ and $B \neq 0$. A similar proof can be given when $A = 0$ or $B = 0$ or $A = 0$ and $B = 0$.

Scanned by TapScanner

We can write $f_1(x) f_2(x) - AB$

$$= [f_1(x) - A][f_2(x) - B] + A[f_2(x) - B] + B[f_1(x) - A].$$

$\therefore$ by algebra, we have $|f_1(x) f_2(x) - AB|$

$$\leq |f_1(x) - A| \cdot |f_2(x) - B| + |A| \cdot |f_2(x) - B| + |B| \cdot |f_1(x) - A|$$

Now $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$. Therefore, for a given ε, we can take δ such that

$$|f_1(x) - A| < \frac{\varepsilon}{3|B|} \text{ and } |f_2(x) - B| < \frac{\varepsilon}{3|A|},$$

when $0 < |x - a| < \delta$. Then

$$|f_1(x) f_2(x) - AB| < \frac{\varepsilon}{3} \cdot \frac{\varepsilon}{3|A||B|} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Since only small values of ε need by considered, we can take ε such that $\frac{\varepsilon}{3|A||B|} < 1$. Then $|f_1(x) f_2(x) - AB| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ *i.e.*, $< \varepsilon$, when $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} f_1(x) . f_2(x) = AB$.

Ex. 3. *Show that* $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ *does not exist.*

Sol. Let $f(x) = |x - 2| / (x - 2)$.

Here the right hand limit *i.e.*,

$$f(2 + 0) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} \frac{|2 + h - 2|}{(2 + h - 2)} = 1,$$

and left hand limit *i.e.*,

$$f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} \frac{|2 - h - 2|}{(2 - h - 2)}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Since $f(2 + 0) \neq f(2 - 0)$, therefore $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ does not exist.

Ex. 4. *Evaluate* $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$. **(Delhi 1981)**

Sol. We have

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$$

Scanned by TapScanner

$$= \lim_{x \to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x}$. **(Delhi 1980)**

Sol. We have

$$\lim_{x \to 0} \frac{(1+x)^{1/3} - (1-x)^{1/3}}{x} = \lim_{h \to 0} \frac{(1+h)^{1/3} - (1-h)^{1/3}}{h}$$

$$= \lim_{h \to 0} \frac{1 + \frac{1}{3}h + \ldots - (1 - \frac{1}{3}h + \ldots)}{h} = \lim_{h \to 0} \frac{(2/3)\,h + \ldots}{h} = \frac{2}{3}.$$

Ex. 6. *Evaluate the following limits if they exist :*

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. **(Delhi 1982)**

Sol. Let $f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2}$.

We have, Right Hand Limit *i.e.*, $f(2+0) = \lim_{h \to 0} f(2+h)$

$$= \lim_{h \to 0} \frac{(2+h-2)(2+h+2)}{2+h-2} = \lim_{h \to 0} \frac{h(h+4)}{h}$$

$$= \lim_{h \to 0} (h+4) = 4.$$

Also, Left Hand Limit *i.e.*, $f(2-0) = \lim_{h \to 0} f(2-h)$

$$= \lim_{h \to 0} \frac{(2-h-2)(2-h+2)}{2-h-2} = \lim_{h \to 0} \frac{-h(-h+4)}{-h}$$

$$= \lim_{h \to 0} (-h+4) = 4.$$

Since $f(2+0) = f(2-0)$, therefore $\lim_{x \to 2} f(x)$ exists and is equal to 4.

(b) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$. Proceed as in part (a).

Here the limit exists and is equal to 1.

(c) $\lim_{x \to 2} \frac{x^2 + 3x + 2}{x - 2}$.

Sol. Let $f(x) = \frac{x^2 + 3x + 2}{x - 2}$.

We have, right hand limit *i.e.*, $f(2+0) = \lim_{h \to 0} f(2+h)$

$$= \lim_{h \to 0} \frac{(2+h)^2 + 3(2+h) + 2}{2+h-2} = \lim_{h \to 0} \frac{12 + 7h + h^2}{h}$$

Scanned by TapScanner

$$= \lim_{h \to 0} \left(\frac{12}{h} + 7 + h\right) = \infty.$$

Also, left hand limit *i.e.*, $f(2-0) = \lim_{h \to 0} f(2-h)$

$$= \lim_{h \to 0} \frac{(2-h)^2 + 3(2-h) + 2}{2-h-2} = \lim_{h \to 0} \frac{12 - 7h + h^2}{-h}$$

$$= \lim_{h \to 0} \left(-\frac{12}{h} + 7 - h\right) = -\infty.$$

Since $f(2+0) \neq f(2-0)$, therefore $\lim_{x \to 2} f(x)$ does not exist.

(d) $\lim_{x \to 0} \frac{1+3x^2}{x}$. Let $f(x) = \frac{1+3x^2}{x}$.

We have R.H.L. *i.e.*, $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{1+3h^2}{h} = \lim_{h \to 0} \left(\frac{1}{h} + 3h\right) = \infty.$$

Again L.H.L. *i.e.*, $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{1+3h^2}{-h} = \lim_{h \to 0} \left(-\frac{1}{h} - 3h\right) = -\infty.$$

Since $f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} f(x)$ does not exist.

(e) *Prove that* $\lim_{x \to 0} (1+x)^{1/x} = e$.

Sol. Let $f(x) = (1+x)^{1/x}$. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1+h)^{1/h}$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{h} \cdot h + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)}{1.2} h^2 + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)\left(\frac{1}{h} - 2\right)}{1.2.3} h^3 + \ldots\right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{1!} + \frac{1.(1-h)}{2!} + \frac{1.(1-h)(1-2h)}{3!} + \ldots\right]$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (1-h)^{-1/h} = e.$$

Since both $f(0+0)$ and $f(0-0)$ exist and are equal to e, therefore $\lim_{x \to 0} (1+x)^{1/x} = e$.

Scanned by TapScanner

(f) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

(Lucknow 1982, 79)

Sol. Let $f(x) = \dfrac{\sin x}{x}$.

We have $f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h) = \lim\limits_{h \to 0} \dfrac{\sin h}{h}$

$$= \lim_{h \to 0} \frac{h - \dfrac{h^3}{3!} + \dfrac{h^5}{5!} - \ldots\ldots}{h} = \lim_{h \to 0} \left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \ldots\ldots\right) = 1.$$

Similarly $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Since $f(0+0) = f(0-0) = 1$, therefore $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$.

(g) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

Sol. Put $x = 1/y$. Then $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = \lim\limits_{y \to 0} y \sin(1/y)$.

Now let $f(y) = y \sin(1/y)$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin(1/h) = 0,$$

since $\sin(1/h)$ lies between -1 and 1.

Also $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (-h) \sin(-1/h) = \lim_{h \to 0} h \sin(1/h) = 0.$$

Since both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero, therefore $\lim\limits_{y \to 0} y \sin(1/y) = 0$. Hence $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = 0$.

(h) $\lim\limits_{x \to 0} \sin \dfrac{1}{x}$.

(Delhi 1975)

Sol. Let $f(x) = \sin(1/x)$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \sin \frac{1}{h}.$$

As $h \to 0$, the value of $\sin(1/h)$ oscillates between $+1$ and -1, passing through zero and intermediate values an infinite number of times. Thus there is no definite number A to which $\sin(1/h)$ tends as h tends to zero. Therefore the right hand limit $f(0+0)$ does not exist.

Scanned by TapScanner

Similarly the left hand limit $f(0-0)$ also does not exist. Hence $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

(i) $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$. Proceed as in part (h). This limit also does not exist.

(j) $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Sol. Let

$$f(x) = \frac{e^x - 1}{x} = \frac{1 + x + \frac{x^2}{2!} + \ldots - 1}{x} = \frac{x\left(1 + \frac{x}{2!} + \ldots\right)}{x}.$$

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{h\,[1 + (h/2!) + \ldots]}{h} = \lim_{h \to 0} \left(1 + \frac{h}{2!} + \ldots\right) = 1.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{-h\,[1 - (h/2!) + \ldots]}{-h} = 1.$$

Thus $f(0+0) = f(0-0) = 1$. Therefore $\lim_{x \to 0} (e^x - 1)/x = 1$.

(k) $\lim_{x \to 0} \frac{a^x - 1}{x}$. Let $f(x) = \frac{a^x - 1}{x}$

$$= \frac{1 + x \log a + (x^2/2!)(\log a)^2 + \ldots - 1}{x}$$

$$= \frac{x\left[\log a + \frac{1}{2}x(\log a)^2 + \ldots\right]}{x}.$$

Now

$$f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h\left[\log a + \frac{h}{2}(\log a)^2 + \ldots\right]}{h} = \log a.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \log a$.

Thus $f(0+0) = f(0-0) = \log a$. Therefore

$$\lim_{x \to 0} \frac{(a^x - 1)}{x} = \log a.$$

(l) $\lim_{x \to 0} \frac{1}{x} . e^{1/x}$.

Sol. Let $f(x) = \frac{1}{x} e^{1/x}$.

Scanned by TapScanner

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{h} e^{1/h} = \infty.$$

[Since both $(1/h)$ and $e^{1/h}$ tend to ∞ as $h \to 0$].

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -\frac{1}{h} e^{-1/h}$$

$$= \lim_{h \to 0} \frac{-1}{h\, e^{1/h}} = \lim_{h \to 0} \frac{-1}{h\left(1 + \frac{1}{h} + \frac{1}{2!}\frac{1}{h^2} + \dots\right)}$$

$$= \lim_{h \to 0} \frac{-1}{h + 1 + (1/2h) + \dots} = 0.$$

Since both the limits $f(0+0)$ and $f(0-0)$ are unequal, therefore $\lim_{x \to 0} \frac{1}{x} e^{1/x}$ does not exist.

(m) $\lim_{x \to 0} e^{-1/x}$. Let $f(x) = e^{-1/x}$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} e^{-1/h}$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}} = 0.$$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} e^{1/h} = \infty.$

Since $f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} e^{-1/x}$ does not exist.

(n) $\lim_{x \to 0} \frac{1}{1 - e^{1/x}}$. Let $f(x) = \frac{1}{1 - e^{1/x}}$.

We have $f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{1 - e^{1/h}} = 0.$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$

$$= \lim_{h \to 0} \frac{1}{1 - (1/e^{1/h})} = 1.$$

$\because \quad f(0+0 \neq f(0-0), \quad \therefore \quad \lim_{x \to 0} \frac{1}{1 - e^{1/x}}$ does not exist.

(o) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$. Let $f(x) = \frac{(1+x)^n - 1}{x}$.

We have $f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{(1+h)^n - 1}{h}$

Scanned by TapScanner

$$= \lim_{h \to 0} \frac{1 + nh + \frac{n(n-1)}{2!} h^2 + \ldots - 1}{h}$$

$$= \lim_{h \to 0} \frac{h\left[n + \frac{n(n-1)}{2!} h + \ldots\right]}{h}$$

$$= \lim_{h \to 0} \left[n + \frac{n(n-1)}{2!} h + \ldots\right] = n.$$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{(1-h)^n - 1}{-h}$

$$= \lim_{h \to 0} \frac{1 + n(-h) + \frac{n(n-1)}{2!}(-h)^2 + \ldots - 1}{-h} = n.$$

Thus $f(0+0) = f(0-0) = n$. Therefore $\lim_{x \to 0} f(x) = n$.

(p) $\lim_{x \to a} \frac{x^m - a^m}{x - a}$. Let $f(x) = \frac{x^m - a^m}{x - a}$.

We have $f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \frac{(a+h)^m - a^m}{a + h - a}$

$$= \lim_{h \to 0} \frac{a^m\left[\left(1 + \frac{h}{a}\right)^m - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{a^m}{h}\left[1 + m \cdot \frac{h}{a} + \frac{m(m-1)}{2!} \frac{h^2}{a^2} + \ldots - 1\right]$$

$$= \lim_{h \to 0} a^m \left[\frac{h}{a} + \frac{m(m-1)}{2} \frac{h}{a^2} + \ldots\right] = a^m \cdot \frac{m}{a} = ma^{m-1}.$$

Again $f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{(a-h)^m - a^m}{a - h - a} = ma^{m-1}$.

Thus $f(a+0) = f(a-0) = ma^{m-1}$.

Therefore $\lim_{x \to a} f(x) = ma^{m-1}$.

Ex. 7. *Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case :*

(*i*) $\lim_{x \to 2} \frac{2x^2 - 8}{x - 2}$;

(*ii*)* $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$;

(Delhi 1983, 79)

(*iii*) $\lim_{x \to 0} f(x)$, *where* $f(x)$ *is defined as*

$f(x) = x$, *when* $x > 0$

Scanned by TapScanner

$f(x) = 0, \quad$ when $x = 0$
$f(x) = -x, \quad$ when $x < 0$.

Sol. (i) Let $f(x) = \dfrac{2x^2 - 8}{x - 2}$.

Here the right hand limit, *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{2(2+h)^2 - 8}{2 + h - 2}$$

$$= \lim_{h \to 0} \frac{2(4 + 4h + h^2) - 8}{h} = \lim_{h \to 0} \frac{8h + 2h^2}{h} = \lim_{h \to 0} \frac{h(8 + 2h)}{h}$$

$$= \lim_{h \to 0} (8 + 2h) = 8.$$

Again the left hand limit, *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{2(2-h)^2 - 8}{2 - h - 2}$$

$$= \lim_{h \to 0} \frac{2(4 - 4h + h^2) - 8}{-h} = \lim_{h \to 0} \frac{-8h + 2h^2}{-h}$$

$$= \lim_{h \to 0} \frac{-h(8 - 2h)}{-h} = \lim_{h \to 0} (8 - 2h) = 8.$$

Since $f(2+0) = f(2-0) = 8$, therefore $\lim\limits_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$ exists and is equal to 8.

(ii) Let $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h \to 0} \frac{e^{1/h}[1 - (1/e^{1/h})]}{e^{1/h}[1 + (1/e^{1/h})]} = \lim_{h \to 0} \frac{1 - (1/e^{1/h})}{1 + (1/e^{1/h})} = 1.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h \to 0} \frac{(1/e^{1/h}) - 1}{(1/e^{1/h}) + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Since $f(0+0) \neq f(0-0)$, therefore $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

(iii) Here the right hand limit *i.e.*, $f(0+0) = \lim\limits_{h \to 0} f(0+h)$, where h is +ive and sufficiently small

Scanned by TapScanner

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h,$ [$\because h > 0$ and $f(x) = x$ when $x > 0$]

$= 0.$

Again the left hand limit *i.e.*, $f(0-0) = \lim_{h \to 0} f(0-h)$,

where h is +ive and sufficiently small

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h),$

[$\because -h < 0$ and $f(x) = -x$ when $x < 0$]

$= \lim_{h \to 0} h = 0.$

Since both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero, therefore $\lim_{x \to 0} f(x)$ exists and is equal to zero.

Ex. 8. *Define the limit of a function. Discuss the existence of the limit of the function* $f(x) = 1$, *if* $x < 1$; $f(x) = 2 - x$, *if* $1 < x < 2$; $f(x) = 2$, *if* $x \geq 2$ *at* $x = 1$ *and* $x = 2$. **(Delhi 1982)**

Sol. For definition of limit see § 2 on page 2.

At x = 1, we have, right hand limit

i.e., $f(1+0) = \lim_{h \to 0} f(1+h),$

where h is +ive and sufficiently small

$= \lim_{h \to 0} [2 - (1+h)] = \lim_{h \to 0} (1-h) = 1;$

and left hand limit *i.e.*, $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1) = 1.$

Since R.H.L. = L.H.L. = 1, therefore $\lim_{x \to 1} f(x)$ exists and is equal to 1.

At x = 2, we have, right hand limit

i.e., $f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2) = 2;$

and left hand limit

i.e., $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2 - (2-h)]$

$= \lim_{h \to 0} h = 0.$

$\because$ left hand limit $\neq$ right hand limit,

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist.

Scanned by TapScanner

§ 3. Continuity. Definition. **(Meerut 1983, 82, 81; Delhi 83, 81, 80)**

A function $f(x)$ defined for $x = a$ is said to be continuous at $x = a$ if

(i) $f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number.

(ii) the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.

Otherwise the function is **discontinuous** at $x = a$.

A function is said to be continuous in an interval (a, b) if it is continuous at every point of that interval.

Arithmetical Definition of Continuity. *A function $f(x)$ is said to be continuous at $x = a$, if for any arbitrarily chosen positive number ε, however small (but not zero), we can find a corresponding number δ such that*

$$|f(x) - f(a)| < \varepsilon$$

for all values of x for which $|x - a| < \delta$.

This definition, as given by Cauchy, is known as Cauchy's definition of continuity.

Note. On comparing the definitions of limit and continuity, we find that a function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a + 0) = f(a - 0) = f(a)$; otherwise it is discontinuous at $x = a$.

§ 4. Differentiability. Definition. **(Lucknow 1970; Mysore 71; Delhi 82, 80)**

A function $f(x)$ is said to be differentiable at $x = a$ if both

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, h > 0 \text{ and } \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}, h > 0,$$

exist and have a common value (finite or infinite). This common value is called the **derivative** of $f(x)$ at the point $x = a$ and is denoted by $f'(a)$. Another name for derivative is *differential coefficient.*

Progressive and regressive derivatives.

Progressive derivative or *the right hand differential coefficient* of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, h > 0.$$

It is denoted by $Rf'(a)$.

Regressive derivative or *the left hand differential coefficient* of $f(x)$ at $x = a$ is given by

Scanned by TapScanner

$$\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}, h > 0.$$

It is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$ if both $Rf'(a)$ and $Lf'(a)$ exist and are equal; otherwise it is said to be non-differentiable.

A necessary condition for the existence of a finite derivative.

Theorem. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.* **(Delhi 1982, 77, Meerut 86 S)**

Proof. Let $f(x)$ have a finite derivative $f'(a)$ at $x = a$. Then

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a). \quad ...(1)$$

Now $f(a+h)-f(a) = \dfrac{f(a+h)-f(a)}{h} \times h, h \neq 0.$

Taking limit of both sides when $h \to 0$, we get

$$\lim_{h \to 0} [f(a+h)-f(a)] = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \times \lim_{h \to 0} h$$

$$= f'(a) \times 0, \quad \text{[from (1)]}$$

$$= 0, \text{ since } f'(a) \text{ is finite.}$$

Now $\lim_{h \to 0} [f(a+h)-f(a)] = 0$

implies $\lim_{h \to 0} f(a+h) = f(a)$. ...(2)

From the condition (2) we see that $f(x)$ is necessarily continuous at $x = a$. Thus continuity is a necessary condition for the existence of a finite derivative.

In order to show that continuity is not a sufficient condition for differentiability, we should give an example of a function which is continuous at a point but is not differentiable at that point. The students can select one such example out of the many that are given ahead. The function given in Ex. 1 (a) is a very good illustration of a continuous function which is not differentiable.

Examples on Continuity and Differentiability

****Ex. 1 (a).** *Show that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$, where $|x|$ means the numerical value of x.* **(Kashmir 1983; Meerut 1981, 75; Delhi 81, 74, 73)**

Sol. To test the continuity of f (x) at x = 0.

We have $f(0) = |0| = 0$;

$$f(0+h) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h|$$

$$= \lim_{h \to 0} h = 0;$$

Scanned by TapScanner

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} |-h|$

$$= \lim_{h \to 0} h = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

To test the differentiability of f (x) at x = 0.

We have, $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$, where h is +ive and sufficiently small

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h| - |0|}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1;$$

and $Lf'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$, h being +ive and sufficiently small

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - |0|}{-h} = \lim_{h \to 0} \frac{h - 0}{-h}$$

$$= \lim_{h \to 0} (-1)$$

$$= -1.$$

Since $Rf'(0) \neq Lf'(0)$, therefore the function $f(x)$ is not differentiable at $x = 0$.

Ex. 1 (b). *Show that a function which is derivable at $x = a$ is necessarily continuous at $x = a$ but the converse is not true.* **(Delhi 1977)**

Sol. For the Ist part of the question see theorem on § 4 and for the second part see Ex. 1 (a).

Ex. 2. *Show that $f(x) = (x^2 - 1)/(x - 1)$ is continuous for all values of x except $x = 1$.* **(Meerut 1977)**

How may this function be defined to make it continuous at $x = 1$?

Sol. Let us first test the continuity of $f(x)$ at $x = a$ where $a \neq 1$. We have

$$f(a) = \frac{a^2 - 1}{a - 1} = \text{a definite real number because } a - 1 \neq 0;$$

$$f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \frac{(a+h)^2 - 1}{a + h - 1} = \frac{a^2 - 1}{a - 1};$$

and $f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \dfrac{(a-h)^2 - 1}{a - h - 1} = \dfrac{a^2 - 1}{a - 1}$.

Scanned by TapScanner

Thus $f(a+0) = f(a-0) = f(a)$. Therefore $f(x)$ is continuous at $x = a$ where $a \neq 1$.

From the formula defining $f(x)$, we have $f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ which is meaningless. Thus $f(x)$ has no definite value at $x = 1$. Therefore $f(x)$ is not continuous at $x = 1$. To make $f(x)$ continuous at $x = 1$, we shall find $\lim\limits_{x \to 1} f(x)$. We have

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \frac{(1+h)^2 - 1}{1+h-1} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0} (2+h) = 2;$$

and $$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{(1-h)^2 - 1}{1-h-1}$$

$$= \lim_{h \to 0} \frac{-2h + h^2}{-h}$$

$$= \lim_{h \to 0} (2-h) = 2.$$

Thus $\lim\limits_{x \to 1} f(x) = 2$. So if we take $f(x) = \dfrac{(x^2-1)}{(x-1)}$ when $x \neq 1$ and $f(x) = 2$ when $x = 1$, it will become continuous at $x = 1$.

Note. In a similar manner we can show that $f(x) = \dfrac{(x^2 - a^2)}{(x-a)}$ is continuous for all values of x except for $x = a$. **(Meerut 1969)**

Ex. 3. *Test the following functions for continuity :*

(a) $f(x) = x^3$ *at* $x = 2$.

Sol. We have $f(2) = 2^3 = 8$;

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^3 = 2^3 = 8;$$

and $$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} (2-h)^3 = 2^3 = 8.$$

Since $f(2) = f(2+0) = f(2-0)$, therefore $f(x)$ is continuous at $x = 2$.

(b) $f(x) = x \sin(1/x)$, $x \neq 0$ *and* $f(0) = 0$ *at* $x = 0$.

(Delhi 1983, 80; Meerut 74; Allahabad 74)

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Again $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h)$

Scanned by TapScanner

$$= \lim_{h \to 0} (-h) \sin\left(-\frac{1}{h}\right) = 0.$$

Also $f(0) = 0$. Thus $f(0+0) = f(0-0) = f(0)$. Therefore $f(x)$ is continuous at $x = 0$.

Note. If we are given that $f(0) = 2$, the above function becomes discontinuous at $x = 0$.

(c) $f(x) = \sin(1/x), x \neq 0$ *and* $f(0) = 0$ *at* $x = 0$. **(Meerut 1973)**

Sol. As shown in Ex. 3 part (h) on page 7, neither the left hand limit nor the right hand limit of the function $\sin(1/x)$ exists at $x = 0$. Hence the function $\sin(1/x)$ is discontinuous at $x = 0$.

(d) $f(x) = 2^{1/x}$, *when* $x \neq 0$; $f(0) = 0$ *at* $x = 0$.

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 2^{1/h} = \infty.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} 2^{-1/h} = 2^{-\infty} = 0.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is doscontinuous at $x = 0$.

Note. A function $f(x)$ is said to have an **infinite discontinuity** at a point $x = a$, if one or both of the limits $f(a+0)$ and $f(a-0)$ are infinite.

(e) $f(x) = \dfrac{1}{1 - e^{-1/x}}, x \neq 0$; $f(0) = 0$ *at* $x = 0$.

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$$

$$= \frac{1}{1 - e^{-\infty}} = \frac{1}{1-0} = 1.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{1 - e^{1/h}}$$

$$= \frac{1}{1 - e^{\infty}} = \frac{1}{1 - \infty} = 0.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is discontinuous at $x = 0$.

(f) $f(x) = x \cos(1/x)$ *for* $x \neq 0$; $f(0) = 0$ *at* $x = 0$.

Scanned by TapScanner

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \cos \frac{1}{h} = 0.$$

Again $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -h \cos\left(\frac{-1}{h}\right) = 0.$$

Also $f(0) = 0$. Thus $f(0+0) = f(0-0) = f(0)$. Therefore $f(x)$ is continuous at $x = 0$.

Ex. 4. *Define continuity of a function at a point and in an interval. Discuss the continuity of*
$f(x) = 2, x \le 0; f(x) = 3x + 2, 0 < x \le 1; f(x) = x/(x-1), 1 < x$
at $x = 0$ *and at* $x = 1$.

(Delhi 1983, 81)

Sol. For definition of continuity see § 3 on page 13.

(i) To test f (x) for continuity at x = 0. We have

$f(0) = 2$; Also right hand limit $= f(0+0) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} (3h + 2) = 2$$

and left hand limit $= f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (2) = 2.$$

Since L.H.L. = R.H.L. $= f(0) = 2$, the function is continuous at $x = 0$.

(ii) To test f (x) for continuity at x = 1. We have
$f(1) = 3.1 + 2 = 5$.

Also right hand limit $= f(1+0) = \lim_{h \to 0} f(1+h)$

$$= \lim_{h \to 0} \frac{1+h}{(1+h)-1} = \lim_{h \to 0} \frac{1+h}{h} = \infty,$$

and left hand limit $= f(1-0) = \lim_{h \to 0} f(1-h)$

$$= \lim_{h \to 0} [3(1-h) + 2] = \lim_{h \to 0} [5 - 3h] = 5.$$

Since right hand limit $\neq$ left hand limit, the function is not continuous at $x = 1$.

Ex. 5. *Discuss the continuity of the function* $f(x)$ *at* $x = \frac{1}{2}$ *where*
$f(x) = x, 0 \le x < \frac{1}{2}; f(x) = 1, x = \frac{1}{2}; f(x) = 1 - x, \frac{1}{2} < x \le 1$.

(Delhi 1979)

Sol. We have $f(x) = 1$, when $x = \frac{1}{2}$.

Scanned by TapScanner

Now R.H.L. $= \lim_{h \to 0} f(\frac{1}{2} + h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [1 - (\frac{1}{2} + h)]$, $[\because f(x) = 1 - x$ when $\frac{1}{2} < x \leq 1]$

$= \lim_{h \to 0} (\frac{1}{2} - h) = \frac{1}{2}$,

and L.H.L. $= \lim_{h \to 0} f(\frac{1}{2} - h)$

$= \lim_{h \to 0} (\frac{1}{2} - h) = \frac{1}{2}$, $[\because f(x) = x$ when $0 \leq x < \frac{1}{2}]$

Since R.H.L. = L.H.L. $\neq$ the value of $f(x)$ at $x = \frac{1}{2}$, therefore the function is discontinuous at $x = \frac{1}{2}$.

****Ex. 6.** *A function $\phi(x)$ is defined as follows :*

$\phi(x) = 0$ *for* $x = 0$

$\phi(x) = \frac{1}{2} - x$ *for* $0 < x < \frac{1}{2}$

$\phi(x) = \frac{1}{2}$ *for* $x = \frac{1}{2}$

$\phi(x) = \frac{3}{2} - x$ *for* $\frac{1}{2} < x < 1$

$\phi(x) = 1$ *for* $x = 1$.

Find the points of discontinuity. Also draw the graph of the function.

(Meerut 1990; Lucknow 79; Delhi 77)

Sol. We shall test the function for continuity at $x = 0, \frac{1}{2}$ and 1.

(i) For $x = 0$, we have $\phi(0) = 0$, $\phi(0 + 0) = \lim_{h \to 0} \phi(0 + h)$

$= \lim_{h \to 0} \phi(h) = \lim_{h \to 0} (\frac{1}{2} - h) = \frac{1}{2}$. Since $\phi(0) \neq \phi(0 + 0)$, the function $\phi(x)$ is discontinuous at $x = 0$.

(ii) For $x = \frac{1}{2}$, we have $\phi(\frac{1}{2}) = \frac{1}{2}$, $\phi(\frac{1}{2} - 0) = \lim_{h \to 0} \phi(\frac{1}{2} - h)$

$= \lim_{h \to 0} [\frac{1}{2} - (\frac{1}{2} - h)]$, (Note that $0 < \frac{1}{2} - h < \frac{1}{2}$)

$= \lim_{h \to 0} h = 0$.

Since $\phi(\frac{1}{2} - 0) \neq \phi(\frac{1}{2})$, the function $\phi(x)$ is discontinuous at $x = \frac{1}{2}$.

(iii) For $x = 1$, we have $\phi(1) = 1$, $\phi(1 - 0) = \lim_{h \to 0} \phi(1 - h)$

$= \lim_{h \to 0} [3/2 - (1 - h)]$, [Note that $\frac{1}{2} < 1 - h < 1$]

$= \lim_{h \to 0} (\frac{1}{2} + h) = \frac{1}{2}$.

Scanned by TapScanner

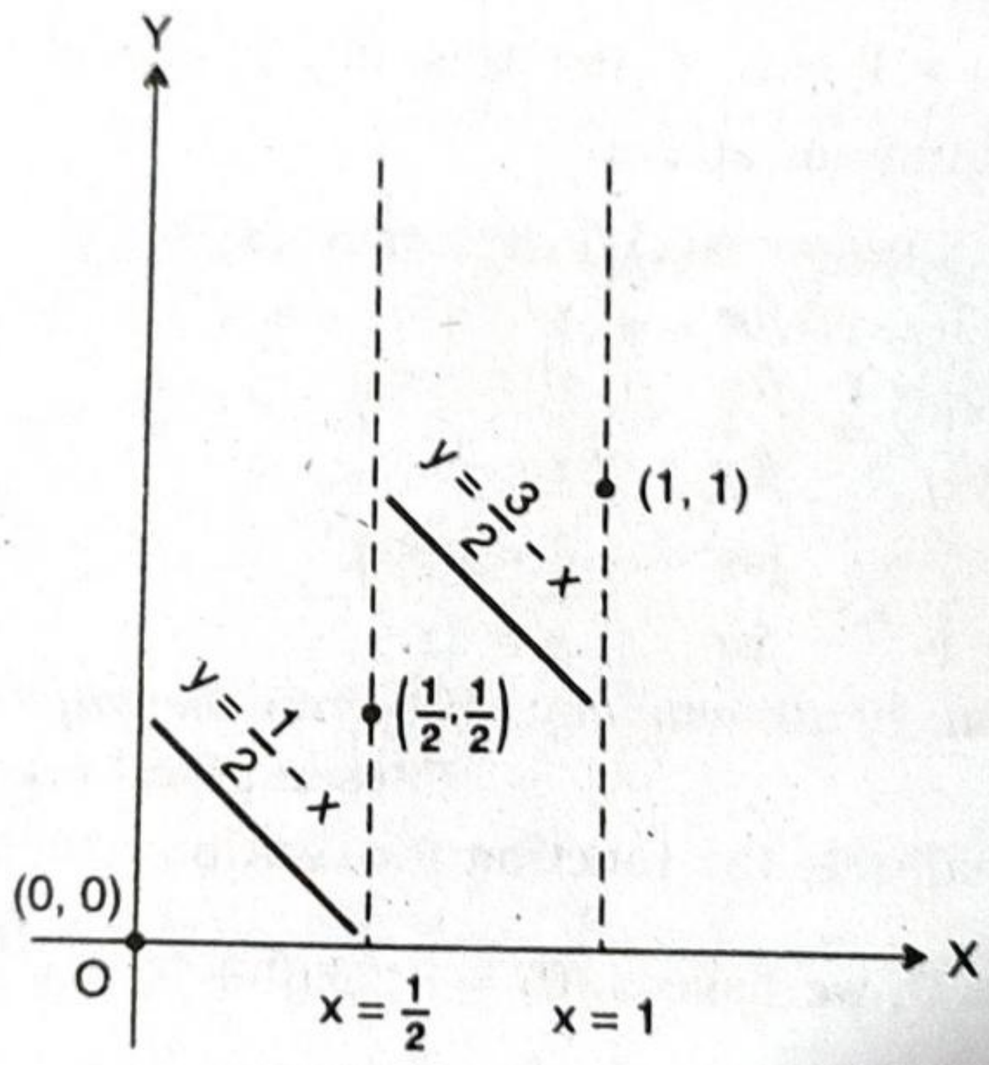
Since $\phi(1) \neq \phi(1-0)$, the function $\phi(x)$ is discontinuous at $x = 1$.

Hence the function $\phi(x)$ has three points of discontinuity at $x = 0, \frac{1}{2}$ and 1.

The graph of the function consists of the point (0, 0); the segment of the line $y = \frac{1}{2} - x, \ 0 < x < \frac{1}{2}$;

the point $(\frac{1}{2}, \frac{1}{2})$; the segment of the line

$$y = (3/2) - x, \ \tfrac{1}{2} < x < 1; \text{ and the point } (1, 1).$$



Thus the graph is as shown in the figure. From the graph we observe that the function is discontinuous at $x = 0, \frac{1}{2}$ and 1.

Ex. 7. *Define continuity of a function $f(x)$ at $x = a$. Discuss the continuity of the function $f(x)$ defined as :*

$$f(0) = 0; f(x) = 1 - x, \ 0 < x < 1; f(1) = 1;$$
$$f(x) = 2 - x, \ 1 < x < 2; f(2) = 0$$

at the points $x = 0, 1, 2$. **(Delhi 1980)**

Sol. For definition of continuity refer § 3 page 13.

(i) To test f (x) for continuity at x = 0. We have $f(0) = 0$.

Now $f(0+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1-h)$

$[\because f(x) = 1 - x \text{ for } 0 < x < 1]$

$= 1.$

As $f(0+0) \neq f(0)$, therefore $f(x)$ is discontinuous at $x = 0$.

(ii) To test f (x) for continuity at x = 1. We have $f(1) = 1$.

Scanned by TapScanner

Now $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [1-(1-h)]$

$= 0,$ $[\because f(x) = 1 - x \text{ for } 0 < x < 1]$

and $f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [2-(1+h)],$

$= 1.$ $[\because f(x) = 2 - x, \text{ for } 1 < x < 2]$

$\because$ $f(1-0) \neq f(1+0)$, therefore $f(x)$ is discontinuous at $x = 1$.

(iii) **To test f (x) for continuity at x = 2.** We have $f(2) = 0$.

Now $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2-(2-h)],$

$= 0.$ $[\because f(x) = 2 - x \text{ for } 1 < x < 2]$

Since $f(2) = f(2-0)$, $f(x)$ is continuous at $x = 2$.

Note that here $f(x)$ is not defined for $x < 2$. So the question of finding $f(2+0)$ does not arise. Here $f(x)$ will be continuous at $x = 2$ if $f(2) = f(2-0)$.

Ex. 8. *Discuss the points of discontinuity of the function given by:*

$$f(x) = \begin{cases} -x, & \text{for } x \le 0 \\ x, & \text{for } 0 < x \le 1 \\ 2-x, & \text{for } 1 < x \le 2 \\ 1, & \text{for } x > 2. \end{cases}$$

(Meerut 1973)

Sol. Here we shall test $f(x)$ for continuity at the points $x = 0, 1$ and 2. Obviously at all other points the function is continuous.

(i) **Continuity at x = 0.** We have $f(0) = 0$, because $f(x) = -x$ for $x = 0$.

Now left hand limit of $f(x)$ as $x \to 0$ *i.e.*,

$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} [-(-h)],$

$[\because f(x) = -x \text{ or } x < 0]$

$= \lim_{h \to 0} (h) = 0;$

and right hand limit of $f(x)$ as $x \to 0$ *i.e.*,

$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h,$

$[\because f(x) = x \text{ for } 0 < x \le 1]$

$= 0.$

Scanned by TapScanner

Thus $f(0-0) = f(0+0) = f(0)$ and so $f(x)$ is continuous at $x = 0$.

(ii) Continuity at x = 1. We have $f(1) = 1$, because $f(x) = x$ for $0 < x \leq 1$.

Now left hand limit of $f(x)$ as $x \to 1$ *i.e.*,

$$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h),$$

$$[\because f(x) = x \text{ for } 0 < x \leq 1]$$

$$= 1;$$

and $$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [2-(1+h)],$$

$$[\because f(x) = 2 - x \text{ for } 0 < x \leq 2]$$

$$= \lim_{h \to 0} (1-h) = 1.$$

Since $f(1-0) = f(1+0) = f(1)$, therefore $f(x)$ is continuous at $x = 1$.

(iii) Continuity at x = 2. We have $f(2) = 2 - 2 = 0$, because $f(x) = 2 - x$, for $1 < x \leq 2$.

Now left hand limit of $f(x)$ as $x \to 2$ *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2-(2-h)],$$

$$[\because f(x) = 2 - x, \text{ for } 1 < x \leq 2]$$

$$= \lim_{h \to 0} (h) = 0;$$

and $$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (1),$$

$$[\because f(x) = 1 \text{ for } x > 2]$$

$$= 1.$$

Since $f(2+0) \neq f(2-0)$, the function $f(x)$ is discontinuous at $x = 2$.

Ex. 9. *Determine the continuity of the function $f(x)$ defined as follows :*

$$f(0) = 0;\ f(x) = \tfrac{1}{2} - x,\ 0 < x < \tfrac{1}{2};\ f(\tfrac{1}{2}) = 0;$$

$$f(x) = 4x^2 - 1,\ \tfrac{1}{2} < x < 3/4;\ f(x) = 1 - x^2,\ 3/4 \leq x \leq 1.$$

(Meerut 1982)

Sol. Proceed exactly as in Ex. 8. The function is discontinuous at $x = 0, 3/4$.

Ex. 10 (a). *A function $f(x)$ is defined as follows :*

Scanned by TapScanner

$$f(x) = \begin{cases} (x^2/a) - a, & \text{when } x < a \\ 0, & \text{when } x = a \\ a - (a^2/x), & \text{when } x > a. \end{cases}$$

Prove that the function $f(x)$ is continuous at $x = a$. **(Delhi 1983, 78)**

Sol. We have $f(a) = 0$.

Also the right hand limit of $f(x)$ at $x = a$ *i.e.*,

$$f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left[a - \frac{a^2}{(a+h)}\right],$$

$$[\because f(x) = a - (a^2/x) \text{ for } x > a]$$

$$= a - (a^2/a) = a - a = 0;$$

and the left hand limit of $f(x)$ at $x = a$ *i.e.*,

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \left[\frac{(a-h)^2}{a} - a\right],$$

$$[\because f(x) = (x^2/a) - a \text{ for } x < a]$$

$$= (a^2/a) - a = a - a = 0.$$

$\because$ $f(a+0) = f(a-0) = f(a)$, therefore $f(x)$ is continuous at $x = a$.

Ex. 10 (b). *Discuss the continuity at $x = a$ of the function $f(x)$ given by :—*

$$\begin{aligned} f(x) &= (x^2/a) - a, && \textit{for} \quad 0 < x < a \\ &= 0, && \textit{for} \quad x = a \\ &= a - (a^3/x^2), && \textit{for} \quad x > a. \end{aligned}$$

Sol. Proceed exactly as in Ex. 10 (a).

Here $f(a+0) = f(a-0) = f(a) = 0$, therefore the given function is continuous at $x = a$.

Ex. 11. *Examine for continuity at $x = 0$ of the following functions*

(i) $f(x) = \cos(1/x)$, *when* $x \neq 0$ *and* $f(0) = 0$. **(Meerut 1981)**

(ii) $f(x) = \sin x \cos(1/x)$, *when* $x \neq 0$ *and* $f(0) = 0$.

Sol. (i) We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \cos(1/h).$$

As $h \to 0$ the value of $\cos(1/h)$ oscillates between $+1$ and -1. Obviously there is no definite number A to which $\cos(1/h)$ tends as $h \to 0$. Thus $f(0+0)$ does not exist. Therefore $f(x)$ is discontinuous at $x = 0$.

(ii) We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \sin h \cos(1/h) = 0 \times \text{a number which oscillates between } -1 \text{ and } +1 = 0.$$

Scanned by TapScanner

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \sin(-h) \cos(-1/h)$$

$$= -\lim_{h \to 0} \sin h \cos(1/h) = 0.$$

Also $f(0) = 0$. Thus $f(0+0) = f(0-0) = f(0)$. Hence the function $f(x)$ is continuous at $x = 0$.

Ex. 12. *Examine the function*

$f(x) = (x-a) \sin\{1/(x-a)\}$, *when* $x \neq a$ *and* $f(a) = 0$ *for continuity at* $x = a$.

Sol. We have $f(a+0) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} (a+h-a) \sin\{1/(a+h-a)\} = \lim_{h \to 0} h \sin(1/h) = 0.$$

Again

$$f(a-0) = \lim_{h \to 0} f(a-h)$$

$$= \lim_{h \to 0} (a-h-a) \sin\{1/(a-h-a)\}$$

$$= \lim_{h \to 0} -h \sin(-1/h) = 0.$$

Also $f(a) = 0$. Thus $f(a+0) = f(a-0) = f(a)$. Therefore $f(x)$ is continuous at $x = a$.

Ex. 13. *Investigate the points of continuity and discontinuity of the following function :*

$$f(x) = \frac{1}{x-a} \operatorname{cosec}\left(\frac{1}{x-a}\right), x \neq a;$$

$$f(x) = 0, \qquad x = a.$$

(Lucknow 1980)

Sol. We have $f(a+0) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \frac{1}{a+h-a} \operatorname{cosec}\left(\frac{1}{a+h-a}\right) = \lim_{h \to 0} \frac{1}{h \sin(1/h)}$$

$= +\infty$, since $h \sin(1/h) \to 0$ as $h \to 0$.

Again

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{1}{a-h-a} \operatorname{cosec}\left(\frac{1}{a-h-a}\right)$$

$$= \lim_{h \to 0} -\frac{1}{h} \cdot \frac{1}{\sin\{-(1/h)\}} = \lim_{h \to 0} \frac{1}{h \sin(1/h)}$$

$= +\infty$, since $h \sin(1/h) \to 0$ as $h \to 0$.

Also as given $f(a) = 0$.

Scanned by TapScanner

Since $f(a+0) = f(a-0) \neq f(a)$, the function $f(x)$ is discontinuous at $x = a$, having an infinite discontinuity of second kind.

Ex. 14. *Discuss the continuity at $x = 0, 1, 2$ of function $f(x)$ defined as follows :*

$f(x) = -x^2$ *for* $x \leq 0$, $f(x) = 5x - 4$ *for* $0 < x \leq 1$, $f(x) = 4x^2 - 3x$ *for* $1 < x < 2$ *and* $f(x) = 3x + 4$ *for* $x \geq 2$.

(Delhi 1982; Meerut 81)

Sol. (i) Conitinuity at x = 0. We have $f(0) = -0^2 = 0$;

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h)^2$$

$$= \lim_{h \to 0} -h^2 = 0;$$

and $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (5h - 4), \quad [\because 0 < h < 1]$$

$$= -4.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is discontinuous at $x = 0$.

(ii) Continuity at x = 1. We have $f(1) = 5 \times 1 - 4 = 1$;

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [4(1+h)^2 - 3(1+h)], \quad [\because 1 < 1 + h < 2]$$

$$= \lim_{h \to 0} [4h^2 + 5h + 1] = 1;$$

and $$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [5(1-h) - 4], \quad [\because 0 \leq 1 - h < 1]$$

$$= \lim_{h \to 0} (1 - 5h) = 1.$$

Thus $f(1+0) = f(1-0) = f(1)$. Therefore $f(x)$ is continuous at $x = 1$.

(iii) Continuity at x = 2. We have $f(2) = 3 \times 2 + 4 = 10$;

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [4(2-h)^2 - 3(2-h)]; \quad [\because 1 < 2 - h < 2]$$

$$= \lim_{h \to 0} [4h^2 - 13h + 10] = 10;$$

and $$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} [3(2+h) + 4]$$

$$= \lim_{h \to 0} (3h + 10) = 10.$$

Scanned by TapScanner

Thus $f(2) = f(2-0) = f(2+0)$. Therefore $f(x)$ is continuous at $x = 2$.

Ex. 15. *Consider the function $f(x) = x - [x]$, where x is a positive variable, and $[x]$ denotes the integral part of x and show that it is discontinuous for all integral values of x, and continuous for all others. Draw the graph.* **(Meerut 1983, 90)**

Sol. From the definition of the function $f(x)$, we have

$$f(x) = x - (n-1) \qquad \text{for } n-1 < x < n,$$
$$f(x) = 0 \qquad \text{for } x = n,$$
$$f(x) = x - n \qquad \text{for } n < x < n+1$$

and so on, where n is an integer.

We shall test the function $f(x)$ for continuity at $x = n$. We have $f(n) = 0$;

$$f(n+0) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} [(n+h) - n] = \lim_{h \to 0} h = 0;$$

$$[\because n < n + h < n + 1]$$

and $$f(n-0) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} [(n-h) - (n-1)],$$

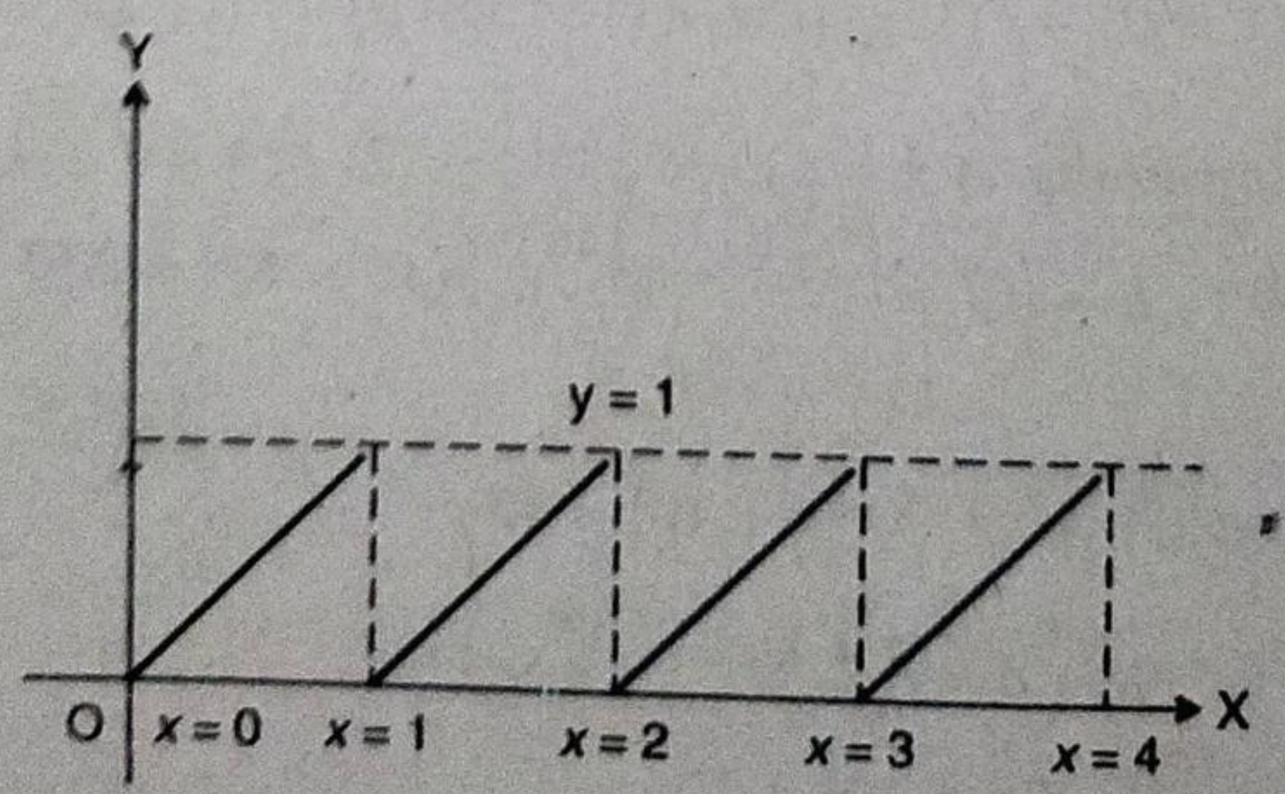
$$[\because n - 1 < n - h < n]$$

$$= \lim_{h \to 0} (1 - h) = 1.$$

Since $f(n+0) \neq f(n-0)$, the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. It is obviously continuous for all other values of x.

Since x is a positive variable, putting $n = 1, 2, 3, 4, 5, \ldots$, we observe that the graph of $f(x)$ consists of the following :

$y = x$ when $0 < x < 1$, $\quad y = 0$ when $x = 1$
$y = x - 1$ when $1 < x < 2$, $\quad y = 0$ when $x = 2$
$y = x - 2$ when $2 < x < 3$, $\quad y = 0$ when $x = 3$
$y = x - 3$ when $3 < x < 4$, $\quad y = 0$ when $x = 4$



Scanned by TapScanner

and so on.

The graph of the function thus obtained is shown by thick lines from $x = 0$ to $x = 4$.

Ex. 16. *Let $y = [x]$ or $E(x)$, where $E(x)$ denotes the integral part of x. Prove that this function is dis-continuous where x has an integral value. Also draw the graph.* **(Allahabad 1973)**

Sol. From the definition of $E(x)$, we have

$$E(x) = n - 1 \quad \text{for} \quad n - 1 \le x < n,$$
$$E(x) = n \quad \text{for} \quad n \le x < n + 1,$$
$$E(x) = n + 1 \quad \text{for} \quad n + 1 \le x < n + 2,$$

and so on where n is an integer.

We consider $x = n$. Then $E(n) = n$, $E(n - 0) = n - 1$ and $E(n + 0) = n$.

Since $E(n + 0) \ne E(n - 0)$, the function $E(x)$ is discontinuous at $x = n$ *i.e.*, when x has an integral value. Evidently it is continuous for all other values of x.

To draw the graph, we put

$n = \ldots\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots$, so that

$y = -4$, when $-4 \le x < -3$

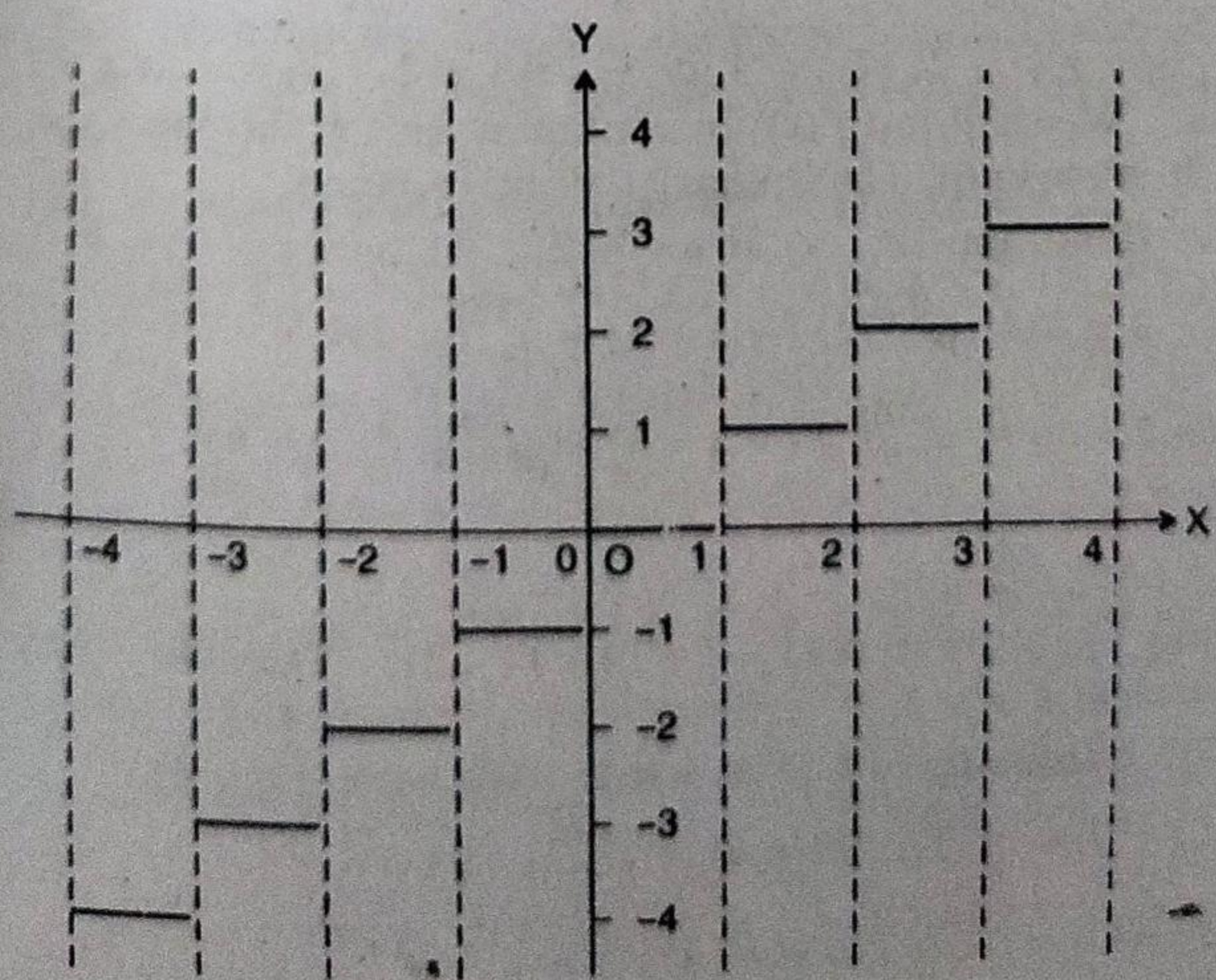
$y = -3$, when $-3 \le x < -2$

$y = -2$, when $-2 \le x < -1$

$y = -1$, when $-1 \le x < 0$

$y = 0$, when $0 \le x < 1$

$y = 1$, when $1 \le x < 2$



Scanned by TapScanner

$y = 2$, when $2 \le x < 3$

$y = 3$, when $3 \le x < 4$

$y = 4$, when $4 \le x < 5$ and so on.

The graph is shown by thick lines.

Ex. 17. *A function $f(x)$ is defined as follows :*

$f(x) = 1 + x$ *if* $x \le 2$ *and* $f(x) = 5 - x$ *if* $x \ge 2$.

Is the function continuous at $x = 2$? **(Lucknow 1982; Meerut 86, 88)**

Sol. We have $f(2) = 1 + 2$ or $5 - 2 = 3$;

$f(2+0) = \lim_{h \to 0} f(2+h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [5 - (2+h)]$, $[\because 2 + h > 2$ and $f(x) = 5 - x$ if $x > 2]$

$= \lim_{h \to 0} (3 - h) = 3$;

and $f(2-0) = \lim_{h \to 0} f(2-h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [1 + (2-h)]$, $[\because 2 - h < 2$ and $f(x) = 1 + x$ if $x < 2]$

$= \lim_{h \to 0} (3 - h) = 3$.

Since $f(2+0) = f(2-0) = f(2)$, the function $f(x)$ is continuous at $x = 2$.

Ex. 18. *Discuss the continuity of the function $f(x)$ defined as follows :*

$f(x) = x^2$ *for* $x < -2$, $f(x) = 4$ *for* $-2 \le x \le 2$, $f(x) = x^2$ *for* $x > 2$.

Sol. We shall test $f(x)$ for continuity only at the points $x = -2$ and 2. It is obviously continuous at all other points.

(i) **Continuity of f (x) at x = − 2.** We have $f(-2) = 4$;

$f(-2+0) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} 4 = 4$;

$f(-2-0) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} (-2-h)^2$, $[\because -2 - h < -2]$

$= 4$.

Thus $f(-2+0) = f(-2-0) = f(-2)$. Therefore $f(x)$ is continuous at $x = -2$.

(ii) **Continuity of f (x) at x = 2.** We have $f(2) = 4$;

$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4$.

and $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4$.

Scanned by TapScanner

Thus $f(x)$ is also continuous at $x = 2$.

Ex. 19. *Let*

$$f(x) = \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} + \dots \infty.$$

Is $f(x)$ continuous at the origin ? Give reasons for your answer.

Sol. Here the function $f(x)$ forms an infinite G.P. whose first terms a is $\frac{x^2}{1+x^2}$ and common ratio r is $\frac{1}{1+x^2}$ which is less than one when $x \neq 0$.

$$\therefore f(x) = \frac{a}{1-r} = \frac{x^2/(1+x^2)}{1-1/(1+x^2)} = \frac{x^2/(1+x^2)}{x^2/(1+x^2)} = 1,$$

provided $x \neq 0$.

Thus $f(x) = 1$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1.$$

Thus $f(0+0) \neq f(0)$ and so $f(x)$ is not continuous at $x = 0$.

Ex. 20. *Discuss the continuity and differentiability of the function $f(x)$ defined as follows :*

$$f(x) = 1 \text{ for } -\infty < x < 0, f(x) = 1 + \sin x \text{ for } 0 \le x < \pi/2,$$

$$f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2 \text{ for } \frac{\pi}{2} \le x < \infty.$$

Sol. We shall test $f(x)$ for continuity and differentiability at $x = 0$ and $\pi/2$. It is obviously continuous as well as differentiable at all other points.

(i) Continuity and differentiability of f (x) at x = 0.

We have $f(0) = 1 + \sin 0 = 1$; $f(0+0) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} (1 + \sin h) = 1$; and

$f(0-0) = \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} 1 = 1$. Thus $f(0) = f(0+0) = f(0-0)$ and therefore $f(x)$ is continuous at $x = 0$.

Now $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{(1 + \sin h) - (1 + \sin 0)}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1;$

Scanned by TapScanner

and $Lf'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \dfrac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{1 - (1 + \sin 0)}{-h} = \lim_{h \to 0} \frac{0}{-h} = \lim_{h \to 0} 0 = 0.$$

Since $Rf'(0) \neq Lf'(0)$, $f(x)$ is not differentiable at $x = 0$.

(ii) Continuity and differentiability of f (x) at x = $\pi/2$.

We have $f\left(\dfrac{\pi}{2}\right) = 2 + \left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\right)^2 = 2$;

$$f\left(\frac{\pi}{2} + 0\right) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0} \left[2 + \left\{\left(\frac{\pi}{2} + h\right) - \frac{\pi}{2}\right\}^2\right]$$

$$= \lim_{h \to 0} (2 + h^2) = 2;$$

and $f\left(\dfrac{\pi}{2} - 0\right) = \lim_{h \to 0} f\left(\dfrac{\pi}{2} - h\right) = \lim_{h \to 0} \left[1 + \sin\left(\dfrac{\pi}{2} - h\right)\right]$

$$= \lim_{h \to 0} (1 + \cos h) = 1 + 1 = 2.$$

Thus $f\left(\dfrac{\pi}{2} + 0\right) = f\left(\dfrac{\pi}{2} - 0\right) = f\left(\dfrac{\pi}{2}\right)$

and therefore $f(x)$ is continuous at $x = \pi/2$.

Now $Rf'\left(\dfrac{\pi}{2}\right) = \lim_{h \to 0} \dfrac{f(\pi/2 + h) - f(\pi/2)}{h}$

$$= \lim_{h \to 0} \frac{[2 + \{\pi/2 + h - \pi/2\}] - [2 + (\pi/2 - \pi/2)^2]}{h}$$

$$= \lim_{h \to 0} \frac{2 + h^2 - 2}{h} = \lim_{h \to 0} h = 0;$$

and $Lf'\left(\dfrac{\pi}{2}\right) = \lim_{h \to 0} \dfrac{f(\pi/2 - h) - f(\pi/2)}{-n}$

$$= \lim_{h \to 0} \frac{1 + \sin(\pi/2 - h) - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-1 + \cos h}{-h} = \lim_{h \to 0} \frac{1 - \cos h}{h} = \lim_{h \to 0} \frac{2 \sin^2 (h/2)}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin h/2}{h/2} \sin h/2\right] = 1 \times 0 = 0.$$

Thus $Rf'(0) = Lf'(0)$ and so $f(x)$ is differentiable at $x = \pi/2$.

Ex. 21. *Examine the following curve for continuity and differentiability :*

$y = x^2$ *for* $x \leq 0$, $y = 1$ *for* $0 < x < 1$, $y = 1/x$ *for* $x > 1$.

(Meerut 1986 P)

Sol. Let $y = f(x)$. We need to test $f(x)$ for continuity and differentiability at the points $x = 0$ and 1. It is obviously continuous and differentiable at all other points.

Scanned by TapScanner

At x = 0. We have $f(0) = 0^2 = 0$; $f(0+0) = \lim_{h \to 0} f(0+h)$ $= \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1$. Since $f(0+0) \neq f(0)$, the function $f(x)$ is not continuous at $x = 0$. Consequently it is also not differentiable at $x = 0$.

At x = 1. The function $f(x)$ is not defined at $x = 1$. Therefore it is discontinuous as well as non-differentiable at $x = 1$.

Ex. 22. *Show that $f(x) = |x-1|$, $0 \le x \le 2$ is not derivable at $x = 1$. Is it continuous in $[0, 2]$?* **(Delhi 1979, 77)**

Sol. (i) To test for continuity at x = 1. We have $f(1) = 0$.

Also

$$\text{R.H.L.} = f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} |1+h-1| = 0;$$

$$\text{and L.H.L.} = f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} |1-h-1| = 0.$$

Since $f(1+0) = f(1-0) = f(1)$, therefore $f(x)$ is continuous at $x = 1$.

In fact $f(x)$ is continuous at every point in the interval $[0, 2]$.

(ii) To test f (x) for differentiability at x = 1.

We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{|1+h-1| - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1;$$

$$\text{and } Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{|1-h-1| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1.$$

Since $Rf'(1) \neq Lf'(1)$, the function $f(x)$ is not differentiable at $x = 1$.

Ex. 23. *Show that the function $f(x)$, where*

$$f(x) = 2 + x \text{ if } x \ge 0; \; f(x) = 2 - x \text{ if } x < 0$$

is not derivable at the point $x = 0$. **(Delhi 1981, 78)**

Sol. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{2+h-2}{h} = 1,$$

$$\text{and } Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

Scanned by TapScanner

$$= \lim_{h \to 0} \frac{\{2-(-h)\}-2}{-h} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

Ex. 24. *Examine the following curve for continuity and differentiability at the points $x = \pm 1$:*

$$y = x - 1 \text{ for } x \geq 1, \; y = \tfrac{3}{2} + x \text{ for } x \leq -1,$$

$$y = 1 + x + x^2 + x^3 + \ldots \infty \text{ for } -1 < x < 1.$$

Sol. Obviously $y = \dfrac{1}{1-x}$ for $-1 < x < 1$. Let $y = f(x)$.

(i) **At x = 1.** We have $f(1) = 1 - 1 = 0$;

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h-1) = 0;$$

and $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \dfrac{1}{1-(1-h)} = \lim_{h \to 0} \dfrac{1}{h} = \infty.$

Since $f(1+0) \neq f(1-0)$, the function $f(x)$ is not continuous at $x = 1$. Consequently $f(x)$ is also not differentiable at $x = 1$.

(ii) **At x = − 1.** We have $f(-1) = \frac{3}{2} - 1 = \frac{1}{2}$;

$$f(-1+0) = \lim_{h \to 0} f(-1+h) = \lim_{h \to 0} \frac{1}{1-(-1+h)}$$

$$= \lim_{h \to 0} \frac{1}{2-h} = \tfrac{1}{2};$$

and $f(-1-0) = \lim_{h \to 0} f(-1-h) = \lim_{h \to 0} [\frac{3}{2} + (-1-h)]$

$$= \lim_{h \to 0} (\tfrac{1}{2} - h) = \tfrac{1}{2}.$$

Since $f(-1+0) = f(-1-0) = f(-1)$, the function $f(x)$ is continuous at $x = -1$.

Now $$Rf'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{1-(-1+h)} - (\frac{3}{2} - 1)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{2-h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{2-2+h}{2h(2-h)} = \lim_{h \to 0} \frac{1}{2(2-h)} = \tfrac{1}{4}.$$

Again $$Lf'(-1) = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{[\frac{3}{2} + (-1-h)] - (\frac{3}{2} - 1)}{-h}$$

Scanned by TapScanner

$$= \lim_{h \to 0} \frac{\frac{1}{2} - h - \frac{1}{2}}{-h} = \lim_{h \to 0} 1 = 1.$$

Since $Rf'(-1) \neq Lf'(-1)$, the function $f(x)$ is not differentiable at $x = -1$.

Ex. 25. *Examine the function defined below for continuity at* $x = 0$:

$$f(x) = \frac{\sin^2 ax}{x^2} \text{ for } x \neq 0, f(x) = 1 \text{ for } x = 0.$$

Sol. We have $f(0) = 1$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah}\right)^2 \cdot a^2 = 1 \cdot a^2 = a^2;$$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin^2(-ah)}{(-h)^2}$

$$= \lim_{h \to 0} \frac{\sin^2 ah}{h^2} = a^2.$$

Now $f(x)$ is continuous at $x = 0$ if and only if

$$f(0) = f(0+0) = f(0-0).$$

Therefore $f(x)$ is discontinuous at $x = 0$ unless $a = \pm 1$.

Ex. 26. *If* $f(x) = -x$ *for* $x \leq 0$ *and* $f(x) = +x$ *for* $x \geq 0$, *prove that* $f(x)$ *is continuous but not differentiable at* $x = 0$.

Sol. (i) To test f(x) for continuity at x = 0. We have

$$f(0) = 0; f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h = 0$$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h)$

$$= \lim_{h \to 0} h = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function $f(x)$ is continuous at $x = 0$.

(ii) To test f(x) for differentiability at x = 0.

We have $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h - 0}{h} = \lim_{h \to 0} 1 = 1.$$

Again $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{-(-h) - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Scanned by TapScanner

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

Ex. 27. *Give an example to show that a continuous function need not be differentiable.* **(Delhi 1980; Meerut 72)**

Sol. The students can give the example given in Ex. 1 (a) page 14. Here we shall give one more such example. Consider the function $f(x)$ defined as follows :

$$f(x) = x \sin(1/x) \text{ for } x \neq 0, f(0) = 0.$$

As already proved in part (b) of Ex. 3 page 16, this function $f(x)$ is continuous at $x = 0$. Now we shall show that the function $f(x)$ is not differentiable at $x = 0$. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}, \text{ which does not exist.}$$

Similarly $Lf'(0)$ does not exist. Hence $f(x)$ is not differentiable at $x = 0$.

****Ex. 28.** *If $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is continuous and differentiable at $x = 0$.* **(Delhi 1982, 80; Meerut 77)**

Sol. (i) **To test f (x) for continuity at x = 0.**

We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h)^2 \sin(-1/h)$$

$$= -\lim_{h \to 0} h^2 \sin \frac{1}{h} = 0.$$

Now $f(0+0) = f(0-0) = f(0)$ implies that $f(x)$ is continuous at $x = 0$.

(ii) **To test f (x) for differentiability at x = 0.** We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Again $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

Scanned by TapScanner

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h) - 0}{-h} = \lim_{h \to 0} \frac{-h^2 \sin(1/h)}{-h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Now $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$. The derivative of $f(x)$ at $x = 0$ has the value zero.

Ex. 29. *If* $f(x) = e^{-1/x^2} \sin \frac{1}{x}$ *for* $x \neq 0$ *and* $f(0) = 0$, *test the differentiability of* $f(x)$ *at* $x = 0$.

Sol. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h) - 0}{h} = \lim_{h \to 0} \frac{\sin(1/h)}{he^{1/h^2}}$$

$$= \lim_{h \to 0} \frac{\sin(1/h)}{h\left[1 + \frac{1}{h^2} + \frac{1}{h^4\, 2!} + \ldots\ldots\right]} = \lim_{h \to 0} \frac{\sin(1/h)}{h + \frac{1}{h} + \frac{1}{2h^3} + \ldots\ldots}$$

$$= \frac{\text{a finite quantity}}{0 + \infty} = 0, \qquad [\because \sin(1/h) \text{ oscillates between } -1 \text{ and } 1 \text{ and hence finite}]$$

Also $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{e^{-1/h^2} \sin(-1/h) - 0}{-h} = \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h)}{h} = 0$$

as before.

Now $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$ and its derivative is zero there.

Ex. 30. *If* $f(x) = \frac{xe^{1/x}}{1 + e^{1/x}}$ *for* $x \neq 0$ *and* $f(0) = 0$, *show that* $f(x)$ *is continuous at* $x = 0$, *but* $f'(0)$ *does not exist.*

(Delhi 1982; Meerut 85 S, 87)

Sol. Here $f(0) = 0$. Also right hand limit *i.e.*,

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{he^{1/h}}{1 + e^{1/h}} = \lim_{h \to 0} \frac{h}{e^{-1/h} + 1} \text{ (Note)} = \frac{0}{0+1} = 0,$$

and the left hand limit *i.e.*,

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

Scanned by TapScanner

$$= \lim_{h \to 0} \left[\frac{-he^{-1/h}}{1+e^{-1/h}}\right] = \frac{0}{1+0} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

We now proceed to find the derivative of $f(x)$ at $x = 0$.

We have,

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{he^{1/h}}{1+e^{1/h}}\right) - 0}{h} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{h \to 0} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{0+1} = 0$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{-he^{-1/h}}{1+e^{-1/h}} - 0}{-h} = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} = \frac{0}{1+0} = 0.$$

Since $Rf'(0) \neq Lf'(0)$, the derivative of $f(x)$ at $x = 0$ does not exist.

Ex. 31. *Let* $f(x) = x\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, $x \neq 0$; $f(0) = 0$.

Show that $f(x)$ is continuous but not derivable at $x = 0$.

Sol. We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} h\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h \to 0} h\frac{1 - e^{-2/h}}{1 + e^{-2/h}},$$

dividing Nr. and Dr. by $e^{1/h}$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -h \cdot \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{h \to 0} -h\frac{e^{-2/h} - 1}{e^{-2/h} + 1} = 0 \times \frac{0-1}{0+1} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

Scanned by TapScanner

Now to test f (x) for differentiability at x = 0, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \left[h \frac{e^{1/h} - e^{-1/h}}{e^{1/h} - e^{-1/h}} - 0 \right] \Big/ h = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1-0}{1+0} = 1,$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left[(-h) \cdot \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0 \right] \Big/ (-h)$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0-1}{0+1} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function is not differentiable at $x = 0$.

□

Scanned by TapScanner