PDE 2014 CSE

Q.1 $(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$

C. Aux. eq^n

$(2m^2 - 5m + 2)z = 0$ (taking $D = m$, $D' = 1$)

$(2m-1)(m-2) = 0$

$\Rightarrow m = \frac{1}{2}, 2$

C.F. $z = \phi_1(y + \frac{1}{2}x) + \phi_2(y + 2x)$

P.I. $P.I = 24 \frac{1}{(2D^2 - 5DD' + 2D'^2)}(y-x)$

$\phi(ax+by)$

$a = -1,\ b = 1$

$F(D, D') = F(a, b) = 2(-1)^2 - 5(-1)(1) + 2(1)^2$

$= 2 + 5 + 2 = 9 \neq 0$

$= \frac{24}{9} \frac{(y-x)^3}{6}$

$= \frac{4}{9}(y-x)^3$

Gen. sol^n $= z = C.F + P.I$

$= \phi_1(y + \frac{x}{2}) + \phi_2(y + 2x) + \frac{4}{9}(y-x)^3$ Ans

Q.2

Q.2 Reduce to Canonical form

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

$$r - x^2 t = 0 \quad -(i)$$

$$\left[r = \frac{\partial^2 z}{\partial x^2} \quad s = \frac{\partial^2 z}{\partial x \partial y} \quad t = \frac{\partial^2 z}{\partial y^2} \quad p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y} \right]$$

Comparing eqn (i) with

$$Rr + Ss + Tt - f(x, y, z, p, q) = 0$$

$$\Rightarrow \quad R = 1, \; S = 0, \; T = -x^2$$

Now, quadratic eqn

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow \quad \lambda^2 - x^2 = 0$$

$$\Rightarrow \quad \lambda = \pm x \quad (\text{real \& distinct root of } \lambda)$$

$$\frac{dy}{dx} + x = 0 \quad -(ii) \qquad \Bigg| \qquad \frac{dy}{dx} - x = 0 \quad -(iii)$$

$$\Rightarrow \quad dy = -x\,dx \qquad \Bigg| \qquad dy = x\,dx$$

$$\Rightarrow \quad y + \frac{x^2}{2} = c_1 \qquad \Bigg| \qquad y - \frac{x^2}{2} = c_2$$

Now, to change x & y to u & v we consider

$$u(x, y) = y + \frac{x^2}{2} \quad \& \quad v(x, y) = y - \frac{x^2}{2}$$

Now $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= x\left[\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right]$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x\left[\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial x} - \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x} - \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x\left[x\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial v^2} - 2x\frac{\partial^2 z}{\partial u \partial v}\right]$$

$$r = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x^2\frac{\partial^2 z}{\partial u^2} + x^2\frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u \partial v}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial u}\frac{\partial v}{\partial z} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v}$$

putting value of $r, \& t$ in eq (i)

$$x^2\frac{\partial^2 z}{\partial u^2} + x^2\frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\frac{\partial^2 z}{\partial u^2} - x^2\frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u \partial v} = 0$$

$$\Rightarrow -4x^2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$$

$$\Rightarrow 4x^2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = 0$$

$$\Rightarrow \boxed{4(u-v)\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}} \rightarrow \text{Canonical form}$$

Ans

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Q.3. eqⁿ of vibrating string   B.C $\begin{cases} u(0,t)=0 \\ u(\pi,t)=0 \end{cases}$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---(i)}$$

Given $c^2=1$, length of string $= \pi$

$u(x,0) = f(x) = k(\sin x - \sin 2x)$ ---(ii)   Init $\frac{\partial u}{\partial t}(x,0) = 0$

Now, Assuming sol of $U(x,y) = X(x)\,T(t)$

here $X(x)$ is fun of x only & $T(t)$ is funⁿ of t only

Now Partially diff. $u(x,y)$ w.r.t x & t

$$\frac{\partial u}{\partial x} = X'(x)\,T(t) \qquad \frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$

putting in eqⁿ(i)

$$X(x)\,T''(t) = X''(x)\,T(t) \qquad (\because c^2=1)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \quad \text{---(iii)}$$

Since x & t are independent variable & left Hand side is a funⁿ of x & R.H.S is funⁿ of t. Therefore, both will be equal only if eq (iii) is equal to some constant μ

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \mu$$

$$\Rightarrow X''(x) - \mu X(x) = 0 \quad \text{---(IV)} \qquad T''(t) - \mu T(t) = 0 \quad \text{---(V)}$$

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Case(i) taking $\mu=0$

$X''(x)=0$

$\Rightarrow X(x)=Ax+B$

using B.C $\quad u(x,t)=X(x)T(t)$

$u(0,t)=0=X(0)T(t)$

$\Rightarrow \boxed{X(0)=0}$

Now $\quad u(\pi,t)=0=X(\pi)T(t)$

$X(x)=Ax+B \quad \Rightarrow \boxed{X(\pi)=0}$

$$\left.\begin{aligned} X(0)&=0=B \\ X(\pi)&=0=A \end{aligned}\right\} \Rightarrow A=B=0 \Rightarrow X(x)=0 \Rightarrow u(x,t)=0$$

does not satisfy (ii) initial condition

we reject $\mu=0$

Case ii) taking $\mu=\lambda^2$ $(\lambda\neq 0)$

$X''(x)-\lambda^2 X(x)=0$

$\Rightarrow X(x)=c_1e^{\lambda x}+c_2e^{-\lambda(x)}$

Now, $X(0)=0=c_1+c_2$

$X(\pi)=0=c_1e^{\lambda\pi}+c_2e^{-\lambda\pi}=0$

$\Rightarrow c_1=c_2=0 \Rightarrow X(x)=0 \Rightarrow u(x,t)=0$

$\Rightarrow$ rejecting $\mu=\lambda^2$

Case iii) $\mu=-\lambda^2$ $(\lambda\neq 0)$

$X''(x)+\lambda^2 X(x)=0$

$X(x)=c_1\cos\lambda x+c_2\sin\lambda x$

$X(0)=0 \quad c_1+c_2(0) \Rightarrow \boxed{c_1=0}$

$X(\pi)=0=c_2\sin\lambda\pi=0$

taking $c_2\neq 0$ & $\sin\lambda\pi=0 \Rightarrow \boxed{\lambda=n}$

$\Rightarrow \lambda\pi=n\pi$

$X(x) = \cancel{C_1 \cos nx + C_2 \sin nx}$ $\quad X(x) = C_2 \sin nx$

$T''(t) + \lambda^2 T(t) = 0$

$T(t) = C_3 \cos \lambda t + C_4 \sin \lambda t$

$T(t) = C_3 \cos nt + C_4 \sin nt$

$u(x,t) = \cancel{X(x)\, T(t)}$

$= \cancel{(C_1 \cos nx + C_2 \sin nx)(C_3 \cos nt + C_4 \sin nt)}$

$u_n(x,t) = [E_n \cos nt + F_n \sin nt] \sin nx$ $\quad$ [taking $C_2 C_3 = E_n$, $C_2 C_4 = F_n$]

more generalised soln

$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} [E_n \cos nt + F_n \sin nt] \sin nx$

partially diff. w.r.t. t

$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty} (-E_n \sin nt\,(n) + F_n \cos nt\,(n)) \sin nx$

$0 = \sum_{n=1}^{\infty} F_n \sin nx \Rightarrow \boxed{F_n = 0}$

$u(x,0) = k(\sin x - \sin 2x) = \sum_{n=1}^{\infty} E_n \sin nx$

fourier sine series

$\Rightarrow$

$\cancel{E_n = \frac{2}{\pi} \int_0^{\pi} k(\sin x - \sin 2x) \sin nx \, dx}$

$k \sin x - k \sin 2x = E_1 \sin x + E_2 \sin 2x + E_3 \sin 3x + \cdots$

$\boxed{E_1 = k \quad E_2 = -k}$ $\quad \boxed{E_n = 0 \ \forall n \geq 3}$

$\boxed{u(x,t) = k \cos t \sin x - k \cos 2t \sin 2x}$ Ans

Q.4

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0<x<1, \quad t>0 \quad \text{(i)}$$

$u(x,0) = 0 \quad 0 \le x \le 1$ — (ii)

$\frac{\partial u}{\partial t}(x,0) = x^2 \quad 0 \le x \le 1$ — (iii)

$u(0,t) = u(1,t) = 0 \quad \forall t$ (iv)

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ → one dimensional ~~wave~~ heat eqn

Taking $U(x,t) = X(x)T(t)$ ⇒ $\boxed{X(0)=0 \\ X(1)=0}$

partially diff. w.r.t. x & t

$\frac{\partial u}{\partial t} = X(x)T'(t)$ & $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$

⇒ $X(x)T'(t) = X''(x)T(t)$

⇒ $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \mu$ (say)

⇒ $X''(x) - \mu X(x) = 0$ | $T'(t) - \mu T(t) = 0$

Case i) $\mu = 0$

$X(x) = Ax + B$

$X(0) = 0 = B$

$X(1) = 0 = A$

⇒ $A = B = 0$ ⇒ $X(x) = 0$ ⇒ $u(x,t) = 0$ rejecting $\mu = 0$

(ii) $\mu = \lambda^2$

$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$X(0) = 0 = c_1 + c_2 = 0$

$X(1) = 0 = c_1 e^{\lambda} + c_2 e^{-\lambda} = 0$

→ $c_1 = c_2 = 0$ ⇒ $X(x) = 0$

⇒ $u(x,t) = 0$ rejecting $\mu = \lambda^2$

iii) $\mu = -\lambda^2$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$

$X(0) = C_1 + C_2(0) = 0$

$\Rightarrow C_1 = 0$

$X(1) = 0 \Rightarrow 0 = C_2 \sin\lambda$

taking $C_2 \neq 0$ $\quad \sin\lambda = 0$

$\Rightarrow \boxed{\lambda = n\pi}$

$\Rightarrow X(x) = C_2 \sin n\pi x$ $\quad\Big|\quad T'(t) + \lambda^2 T(t) = 0$

$T = C_3 e^{-\lambda^2 t}$

$\Rightarrow T = C_3 e^{-(n\pi)^2 t}$

$\Rightarrow \boxed{u_n(x,t) = E_n \sin n\pi x \; e^{-(n\pi)^2 t}}$ $\quad E_n = C_2 C_3$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x \; e^{-(n\pi)^2 t}$

partially diff. $u(x,t)$ w.r.t. t

$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} E_n \sin n\pi x \; e^{-(n\pi)^2 t} \; -(n\pi)^2$

$\frac{-x^3}{n^2\pi^2} = \sum_{n=1}^{\infty} E_n \sin n\pi x$

$\Rightarrow E_n = \frac{2}{1}\int_0^1 \frac{-x^3}{n^2\pi^2} \sin n\pi x \, dx$

$E_n = \frac{-2}{n^2\pi^2}\int_0^1 x^3 \sin n\pi x = \frac{2\cos n\pi}{(n\pi)^3} - \frac{12\cos n\pi}{(n\pi)^5}$

$= \cancel{\frac{6\cos n\pi}{(n\pi)^3}} \cancel{-\cos}$

$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2\cos n\pi}{(n\pi)^3} - 12\frac{\cos n\pi}{(n\pi)^5}\right) e^{-(n\pi)^2 t}$

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if $n = 2m$

$$u(x,t) = \begin{cases} \sum_{m=1}^{\infty} \left(\frac{2}{(2m\pi)^3} - \frac{12}{(2m\pi)^5} \right) e^{-(2m\pi)^2 t} & ; n = 2m \\ \sum_{m=1}^{\infty} \left(\frac{12}{((2m-1)\pi)^5} - \frac{2}{((2m-1)\pi)^3} \right) e^{-((2m-1)\pi)^2 t} & ; n = 2m-1 \end{cases}$$

Ans