# LINEAR ALGEBRA

## : CSE-2015 :

**1(a).** The vectors $V_1 = (1,1,2,4)$, $V_2 = (2,-1,-5,2)$, $V_3 = (1,-1,-4,0)$ and $V_4 = (2,1,1,6)$ are linearly independent. Is it true? Justify your answer

$\longrightarrow$ Let $A = [V_1^T \; V_2^T \; V_3^T \; V_4^T] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix}$.

If the rank of matrix $A_{4\times4}$ is equal to 4, then the given vectors $V_1, V_2, V_3$ & $V_4$ are L.I. otherwise L.D..

Let us convert A into echelon form.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 2 & -5 & -4 & 1 \\ 4 & 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -9 & -6 & -3 \\ 0 & -6 & -4 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$\therefore$ The echelon form of A has two non-zero rows. Therefore rank of $A \leq 2 < 4$. Hence, the statement is false. The vectors $V_1, V_2, V_3$ & $V_4$ are L.D.

**1(b)** Reduce the following matrix to row echelon form and hence find its rank. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$.

$\longrightarrow$
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ R_4 \to R_4 - 5R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5/3 & 2 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + \tfrac{2}{3}R_2 \quad \sim \begin{bmatrix} 1 & 0 & 5/3 & 2 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{---} (1) \quad R_3 \to R_3 \div (-3)$$

The matrix (1) is the row echelon form of A. It has two non-zero rows. Therefore,

Rank of $A = \rho(A) = \underline{\underline{2}}$

**2(a):** If the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, find $A^{30}$.

$\rightarrow$ Characteristic equation of A is given by $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[\lambda^2 - 1] = 0$$

$$\Rightarrow \lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$\Rightarrow \lambda^3 = \lambda^2 + \lambda - 1 \quad \text{—— (1)}$$

By Cayley-Hamilton's theorem, (1) is satisfied by matrix A.

$\therefore A^3 = A^2 + A - I$ —— (2)

Premultiplying 'A' on both sides, we have.

$A \cdot A^3 = A \cdot A^2 + A \cdot A - A \cdot I = A^3 + A^2 - A = A^2 + A - I + A^2 - A$ [from (2)]

$\Rightarrow A^4 = 2A^2 - I$

**Sly:** $A^6 = 2A^4 - A^2 \Rightarrow 2(2A^2 - I) - A^2 = 3A^2 - 2I$

$A^8 = 3A^4 - 2A^2 \Rightarrow 3(2A^2 - I) - 2A^2 = 4A^2 - 3I$

$\vdots$

$\therefore A^{30} = 15A^2 - 14I$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$A^{30} = 15\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$$

**2(c)** Find the eigen values and vectors of the matrix

$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

$\rightarrow$ Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda) - 1] - 1[(1-\lambda) - 3] + 3[1 - 3(5-\lambda)] = 0$$

$$\Rightarrow (1-\lambda)[4 - 6\lambda + \lambda^2] + [2 + \lambda] + 3[-14 + 3\lambda] = 0$$

$$\Rightarrow 4 - 4\lambda - 6\lambda + 6\lambda^2 + \lambda^2 - \lambda^3 + 2 + \lambda + 9\lambda - 42 = 0$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0 \Rightarrow \lambda = -2, 3, 6$$

Hence, eigen values of $A$ are $\underline{-2, 3, 6}$.

Eigen vectors corresponding to eigen values:

(i) $\underline{\lambda = -2}$: $(A - (-2)I)X = 0$

$\Rightarrow (A + 2I)X = 0$

$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\begin{bmatrix} 1 & 7 & 1 \\ 3 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & 7 & 1 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -20y = 0 \Rightarrow y = 0$

$x + 7y + z = 0$

$x + z = 0 \quad [\because y = 0]$

$x = -z$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

(ii) $\underline{\lambda = 3}$ $(A - 3I)X = 0$

$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore 5y + 5z = 0 \Rightarrow y = -z$

$x + 2y + z = 0$

$\Rightarrow x + 2(-z) + z = 0$

$\Rightarrow x = z$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(iii) $\underline{\lambda = 6}$ $(A - 6I)X = 0$

$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + 5R_1$, $R_3 \to R_3 - 3R_1$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$

$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore -4y + 8z = 0$

$y = 2z$

$x - y + z = 0$

$\to x - 2z + z = 0$

$\to x = z$

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Required Eigen Vectors corresponding to eigen values $\lambda = -2, 3$ and $6$ are $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ respectively

$\underline{3(a)}$ Let $V = \mathbb{R}^3$ and $T \in A(V)$, for all $a_i \in A(V)$, be defined by $T(a_1, a_2, a_3) = (2a_1 + 5a_2 + a_3, -3a_1 + a_2 - a_3, -a_1 + 2a_2 + 3a_3)$. What is the matrix $T$ relative to basis $V_1 = (1, 0, 1)$, $V_2 = (-1, 2, 1)$ and $V_3 = (3, -1, 1)$ ?

$\longrightarrow$ Let $(a, b, c) = x(1, 0, 1) + y(-1, 2, 1) + z(3, -1, 1)$

$\Rightarrow (a, b, c) = (x - y + 3z,\ 2y - z,\ x + y + z)$

$\Rightarrow$ Comparing on both sides, we have,

$a = x - y + 3z$, $\quad b = 2y - z$, $\quad c = x + y + z$

$\quad$ ⟶ ①, $\quad$ ⟶ ②, $\quad$ ⟶ ③

② + ③ $\Rightarrow$ $x + 3y = b + c$.

① + 3×② $\Rightarrow$ $\underline{x + 5y = a + 3b}$

$-2y = b + c - a - 3b$

$\Rightarrow y = \dfrac{a + 2b - c}{2}$ ⟶ ④

② $\Rightarrow$ $b = 2y - z$

$\Rightarrow b = (a + 2b - c) - z$ [from ④]

$\Rightarrow z = a + b - c$

③ $\Rightarrow$ $c = x + y + z$

$c = x + \dfrac{(a + 2b - c)}{2} + a + b - c$

$c = x + \frac{3a}{2} + 2b - \frac{3}{2}c$

$\Rightarrow x = -\frac{3}{2}a - 2b + \frac{5}{2}c$

$\Rightarrow x = \frac{1}{2}(-3a - 4b + 5c)$

$\therefore (a, b, c) = \frac{1}{2}(-3a - 4b + 5c)(1, 0, 1) + \frac{1}{2}(a + 2b - c)(-1, 2, 1) + (a + b - c)(3, -1, 1)$

$\therefore$ Now,

$T(1, 0, 1) = (2\cdot1 + 5\cdot0 + 1\cdot1, -3\cdot1 + 1\cdot0 - 1\cdot1, -1\cdot1 + 2\cdot0 + 3\cdot1)$

$= (3, -4, 2) = \left(\frac{17}{2}\right)(1, 0, 1) + \left(-\frac{7}{2}\right)(-1, 2, 1) + (-3)(3, -1, 1)$

$T(-1, 2, 1) = (-2 + 10 + 1, 3 + 2 - 1, 1 + 4 + 3)$

$= (9, 4, 8) = \left(-\frac{3}{2}\right)(1, 0, 1) + \left(\frac{9}{2}\right)(-1, 2, 1) + (5)(3, -1, 1)$

$T(3, -1, 1) = (6 - 5 + 1, -9 - 1 - 1, -3 - 2 + 3) = (2, -11, -2)$

$= (14)(1, 0, 1) + (-9)(-1, 2, 1) + (-7)(3, -1, 1)$

$\therefore$ Matrix of $T$ wrt given basis is $M = \begin{bmatrix} \frac{17}{2} & -\frac{3}{2} & 14 \\ -\frac{7}{2} & \frac{9}{2} & -9 \\ -3 & 5 & -7 \end{bmatrix}$

**4(b)** Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. Hence, find its basis.

$\longrightarrow$ Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

The number of non-zero rows in the echelon form of A gives the dimension of the set and the rows itself gives the basis.

Reducing A into echelon form.

$R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim$ $R_3 \to R_3 - 2R_2$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim$ $R_4 \to R_4 - R_3$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ — ①

The matrix ① gives the echelon form of A. There are three non-zero rows in the echelon form of A.

$\therefore$ Dimension of the set $= 3$.

Basis of the set $= \{(1,0,0,0),\ (0,1,0,0),(0,0,0,1)\}$.