IFoS 2010

2010 (1) $\cos(x+y) \to$ IF of

$$\boxed{y\,dx + [y+\tan(x+y)]\,dy = 0}$$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = \sec^2(x+y)$

$y\cos(x+y)\,dx + [\cos(x+y)] \times [y+\tan(x+y)]\,dy = 0$

$\frac{\partial M}{\partial y} = -y\sin(x+y) + \cos(x+y)$

$\frac{\partial N}{\partial x} = -y\sin(x+y) + \cos(x+y)$

$\therefore$ exact since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

so, $\int_{y\ \text{const}} y\cos(x+y)\,dx + \int_{\text{terms not containing } x} \cos(x+y)(y+\tan(x+y)) = C$

ie. $\boxed{-y\sin(x+y) = C}$

$\Rightarrow \boxed{y = c_0 \operatorname{cosec}(x+y)}$

(2) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

AE : $(D^2 - 2D + 1)y = 0$

$(D-1)^2 y = 0 \Rightarrow D = 1, 1.$

so C.F is $\boxed{y_c = (C_1 + C_2 x)e^x}$

PI $y_p = \frac{1}{(D-1)^2} xe^x \sin x$

= imaginary part of $\frac{1}{(D-1)^2} xe^{x+ix}$

$$= e^{x+ix} \times \frac{1}{(D+1+i-1)^2} x$$

$$\left[\text{using } \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x)\right]$$

$$= e^{x+ix} \times \frac{1}{i^2 (1+\frac{D}{i})^2} x$$

$$= -e^{x+ix} \times \left(1+\frac{D}{i}\right)^{-2} (x)$$

$$= -e^{x+ix} \times \left(1 - \frac{2D}{i}\right)(x) \quad \left[\text{using } (1+x)^{-2} = 1-2x+3x^2 - - \right]$$

$$= -e^{x+ix}\left(x - \frac{2}{i}\right)$$

so, $y_p =$ imaginary part of $-e^{x}(\cos x + i\sin x)(x+2i)$

$$\boxed{y_p = -2e^{x}\cos x - xe^{x}\sin x}$$

so g.s. is $y = y_c + y_p = (c_1 + c_2 x)e^{x} - 2e^{x}\cos x - xe^{x}\sin x$.

(3) $\frac{dy}{dx} = \sin^2(x-y+6)$ —(1)

Put $x - y + 6 = z$ Diff. wrt x,

$$1 - \frac{dy}{dx} = \frac{dz}{dx}$$

(1) becomes, $1 - \frac{dz}{dx} = \sin^2 z$

i.e. $\frac{dz}{dx} = \cos^2 z$

$$\sec^2 z\, dz = dx$$

Integrating both sides we get

$$\boxed{\tan z = x + c}$$

i.e. $\boxed{\tan(x-y+6) = x+c}$.

4. $$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2+1)\,y = 0.$$

Comparing with standard 2^{nd} order equation: $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ — (1)

$$P(x) = 2x \;;\; Q(x) = x^2+1$$

Then by replacing dependent variable y and eliminating first order term method;

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int 2x dx} = e^{-\frac{x^2}{2}}$$

let $y = uv$ be the solution Then (1) transforms to

$$\frac{d^2v}{dx^2} + Iv = R' \quad \text{where}$$

$$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = x^2+1 - \frac{1}{4}(4x^2) - \frac{1}{2}x2 = 0.$$

$$R' = \frac{R}{u} = 0.$$

i.e. $\dfrac{d^2v}{dx^2} = 0$

$\Rightarrow \dfrac{dv}{dx} = c$

$\Rightarrow \boxed{v = cx + d}$

$\dfrac{d^2v}{dx^2} + Iv = 0$

$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$x^2+1-\frac{1}{4}(4x^2)$

$-\frac{1}{2}x^2$

$1-1=0$

so gs of ① is $\boxed{y = uv = e^{-\frac{x^2}{2}}(cx+d)}$

5. $\left(\dfrac{d}{dx}-1\right)^2\left(\dfrac{d^2}{dx^2}+1\right)^2 y = x + e^x$

$(D-1)^2(D^2+1)^2 y = x + e^x$

AE is: $m = 1, 1,\ m = i, i, -i, -i$

CE is $y_c = (c_1 + c_2 x)e^x + (c_3 x + c_4)\cos x$
$+ (c_5 x + c_6)\sin x$

for PI,

$y_p = \dfrac{1}{(D-1)^2(D^2+1)^2}(x + e^x)$

$= \dfrac{1}{(D^2+1)^2} \times (1-D)^{-2}[x] + \dfrac{1}{(D-1)^2(D^2+1)^2} e^x$

$= \dfrac{1}{(D^2+1)^2}(1+2D)(x) + \dfrac{1}{(D-1)^2} \times \dfrac{e^x}{1^2+1}$

$\Big[$ using $(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$ and
$\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$ if $f(a) \neq 0 \Big]$

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$$= \frac{1}{(D^2+1)^2}(x+2) + \frac{1}{2} \times \frac{x^2}{2!} e^x \quad \left[\text{using } \frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \right]$$

$$= (1 - 2D^2 + 3D^4 \ldots)(x+2) + \frac{x^2}{4} e^x$$

$$= x+2 + \frac{x^2}{4} e^x$$

so $\boxed{\text{GS is } y = y_c + y_p = (c_1 + c_2 x)e^x + (c_3 x + c_4)\cos x + (c_5 x + c_6)\sin x + (x+2) + \frac{x^2}{4} e^x}$

6. Using M.V.P → Method of variation of parameter

$$(x^2-1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = (x^2-1)^2$$

Dividing by x^2-1,

$$\frac{d^2y}{dx^2} - \frac{2x}{x^2-1}\frac{dy}{dx} + \frac{2}{x^2-1}y = (x^2-1) \quad \text{--- (1)}$$

Comparing with standard 2nd order equation,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

$$P(x) = \frac{-2x}{x^2-1}, \quad Q(x) = \frac{2}{x^2-1}, \quad R(x) = x^2-1$$

since $P(x) + xQ(x) = 0$ so,

$y = x'$ is a solution

[$\because$ $y = x^m$ is a sol. if $m(m-1) + mPx + Qx^2$

so let $u = x$, then let

$y = uv$ be the gs of (1),

Then replacing dependent variable and

$$y = xv \rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

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$$\frac{d^2y}{dx^2} = x\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right) + \frac{du}{dx}$$

$$\frac{d^2y}{dx^2} = x\frac{d^2u}{dx^2} + 2\frac{du}{dx}$$

$2 = \frac{4x^2}{x^2-1} + \frac{2}{x^2-1}$

$\frac{2x^2-2-2x^2}{x^2-1} = \frac{-2}{x^2-1}$

$a^2+aP+Q=0$

e^{ax}

$\Rightarrow$ (1) becomes,

$$x\frac{d^2u}{dx^2} + \frac{2du}{dx} - \frac{2x}{x^2-1}\left(u + x\frac{du}{dx}\right) + \frac{2}{x^2-1}(xu) = x^2-1$$

$$\Rightarrow x\frac{d^2u}{dx^2} + 2\frac{du}{dx} - \frac{2ux}{x^2-1} - \frac{2x^2}{x^2-1}\frac{du}{dx} + \frac{2xu}{x^2-1} = x^2-1$$

$$\frac{d^2u}{dx^2} - \frac{2}{x(x^2-1)}\frac{du}{dx} = \frac{x^2-1}{x}$$

Put $\frac{du}{dx} = V$ then $\frac{d^2u}{dx^2} = \frac{dV}{dx}$

$$\frac{dV}{dx} - \frac{2}{x(x^2-1)}V = \frac{x^2-1}{x}$$

which is first order linear diff equation

$$IF = e^{\int P dx} = e^{\int -\frac{2}{x(x-1)(x+1)}dx}$$

$$= e^{\int\left(\frac{2}{x} - \frac{1}{x-1} - \frac{1}{x+1}\right)dx}$$

$$= e^{2\log x - \log(x-1) - \log(x+1)}$$

$$= e^{\log\frac{x^2}{(x+1)(x-1)}} = \frac{x^2}{x^2-1}$$

sol. is $\boxed{V \times \frac{x^2}{x^2-1} = \frac{x^2}{2} + C}$

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Replacing v by $\frac{du}{dx}$ we get

$$\frac{du}{dx} = \frac{x^2}{2} \times \frac{x^2-1}{x^2} + c \times \frac{x^2-1}{x^2}$$

$$du = \left(\frac{x^2-1}{2} + c - \frac{c}{x^2}\right) dx$$

Integrating both sides we get

$$\boxed{u = \frac{x^3}{6} - \frac{x}{2} + cx + \frac{c}{x} + c'}$$