Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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Linear Algebra

Vector Spaces

(1) V is vector space over field F. If following is satisfied
(a) Vector addition & scalar multiplication should be defined.
(b) $(V,+)$ should be abelian group.
(c) 4 properties should be satisfied $\forall\ \alpha, \beta \in V$ & $a, b \in F$;

$a(\alpha+\beta) = a\alpha + a\beta$
$\alpha(a+b) = a\alpha + b\alpha$ } natural dist. properties
$(ab)\alpha = a(b\alpha)$ } Associativity
$1 \cdot \alpha = \alpha$ } multiplication by unity.

Note: If $V \subseteq F$ then V can't be vector space
& $F \subseteq V$ then V will be vector space.

(2) For vector space, we get following obvious results from manipulation of distributive law

$a\vec{0} = \vec{0}$; $0 \cdot \vec{\alpha} = \vec{0}$, $a(\alpha-\beta) = a\alpha - a\beta$

$a\alpha = \vec{0} \Rightarrow a = 0$ or $\alpha = \vec{0}$

(3) Subspace of vector space
Let V be vector space & $W \subseteq V$.
W is subspace of V if $\forall\ \alpha, \beta \in W$ & $a, b \in F$;
$a\alpha + b\beta \in W$

(4) Note that (x_1, x_2, x_3) s.t. $2x_1 + 3x_2 + x_3 = 0$ form ~~subspace~~ vector space. But (x_1, x_2, x_3) s.t. $2x_1 + 3x_2 + x_3 = 2$ doesn't.

(5) Let $S \subseteq V$. Smallest subspace of V containing S is called subspace generated or spanned by S. Shown by $\{S\}$.

(6) Linear sum of 2 subspaces
$W_1 + W_2 = \{\alpha + \beta \mid \alpha \in W_1\ \&\ \beta \in W_2\}$
$W_1 + W_2$ is also a subspace of V & $W_1 + W_2 = \{W_1 \cup W_2\}$
(span of $W_1 \cup W_2$)

Date
Page

(7) $\{\{w\}\} = \{w\}$
bcoz $\{\{w\}\}$ is smallest subspace containing $\{w\}$ but $\{w\}$ itself is a subspace. hence.

(8) Linear dependence of vectors
v_1, v_2, v_3 are L.D if $\exists\ \alpha_1, \alpha_2, \alpha_3$ not all zero s.t.
$\sum \alpha_i v_i = 0$

(9) Echelon form of Matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 2 \\ 0 & 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of echelon = no. of non-zero rows.

Row Reduced echelon = The first non-zero element in row is 1 & it is the only non-zero element from its column

$$\begin{bmatrix} 1 & 5 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(10) Solving $Ax = B$
Reduce augmented matrix $[A|B]$ to echelon form.
Then :-
case (i) Rank A = Rank $[A|B]$ = no. of variables
Unique solution exists.
case (ii) Rank A = Rank $[A|B]$ < no. of variables.
infinite sol^n
case (iii) Rank A $\neq$ Rank $[A|B]$
no solution.

Similarly for $Ax = 0$ (~~$A \neq$~~)
case (i) Rank A = number of variables
then trivial sol^n $x = 0$
case (ii) Rank A < no. of variables
non-trivial sol^n

Basis & Dimensions

① Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq V$. S is called basis of V if
(i) α_1 to α_n are L.I. (ii) $\{S\} = V$

② How to show given subset is basis of V
e.g. show $S = \{1, x, \ldots, x^n\}$ is basis of polynomials upto degree n.
→ (i) Showing L.I. → let $\alpha_1(1) + \alpha_2(x) + \ldots + \alpha_{n+1}x^n = 0$
$\qquad = 0(1) + 0(x) + 0(x^2) \ldots + 0(x^n)$
$\therefore \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$ $\quad\therefore$ L.I.

(ii) Showing $L(S) = V$
We know that $L(S) \subseteq V$
Now, let $v \in V$ & $v = b_0 + b_1 x + \ldots + b_n x^n$
$= b_0(1) + b_1(x) + \ldots + b_n(x^n)$
$\therefore v \in L(S)$ $\quad\therefore V \subseteq L(S)$ $\quad\therefore L(S) = V$, $\therefore$ basis.

Seems simple but remember the proper method.

③ Dimension = no. of basis vectors

④ e.g. Show that $\mathbb{R}$ is vector space of infinite dim. over $\mathbb{Q}$.
→ Consider $\{1, \pi, \pi^2, \ldots \pi^n\}$ → This becomes LI $\forall n$. How?
let $\alpha_0 + \alpha_1\pi + \alpha_2\pi^2 + \ldots + \alpha_n\pi^n = 0$ $(\alpha_0, \alpha_1, \ldots \alpha_n \in \mathbb{Q})$
then π is sol^n of polynomial $\alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n = 0$
But π is transcendental. $\therefore$ contradiction. $\therefore$ infinite.

⑤ Show that sol^n of $\frac{d^2y}{dx^2} + 4y = 0$ is a 2 dimensional vector space
→ $(D^2+4)y = 0 \Rightarrow y = c_1\cos 2x + c_2\sin 2x$ $\therefore$ c_1 & c_2 become parameters
every sol^n can be shown as l.c. of $\cos 2x$ & $\sin 2x$.
Now, Wronskian becomes $\begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2$ $\therefore$ nonzero $\therefore$ 2 are LI
$\therefore$ basis $\therefore$ 2 dimensi

Classmate
Date
Page

(6) Let W_1, W_2 be subspaces of V.
Then $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$\longrightarrow$ $W_1 \cap W_2$ is also a subspace.
Let $S = \{\gamma_1, \gamma_2 \dots \gamma_k\}$ be basis of $W_1 \cap W_2$.
Then S can be extended to form basis of W_1 & basis of W_2.
Let basis $W_1 = \{\gamma_1, \gamma_2, \dots, \gamma_k, b_1, b_2 \dots b_m\}$
basis $W_2 = \{\gamma_1, \gamma_2, \dots, \gamma_k, c_1, c_2 \dots c_n\}$
& then show basis $W_1 + W_2 = \{\gamma_1, \gamma_2, \dots, \gamma_k, b_1 \dots b_m, c_1 \dots c_n\}$ (10)

(7) How to check if 2 matrices are row equivalent?
Reduce both of them to row reduced echelon form.
Their nonzero rows should be same.

Similarly, easiest way to find dimension of given subspace is going for row reduced form of basis vectors matrix.

(8) Quotient space
Let W be subspace of vector space V.
Let $\alpha \in V$. Then $W+\alpha$ is coset of W.
If $W+\alpha = W+\beta$, then $\alpha - \beta \in W$.

Now, $\frac{V}{W}$ is set of all cosets $\{W+\alpha \,/\, \alpha \in V\}$.

This forms vectorspace over F for following 2 operations.
$(W+\alpha) + (W+\beta) = W + (\alpha+\beta)$
$a(W+\alpha) = W + a\alpha$

(9) $\mathrm{Dim}\left(\frac{V}{W}\right) = \mathrm{Dim}(V) - \mathrm{Dim}(W)$
$\longrightarrow$ Let basis of W be $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$
extend this to form basis of V as $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots \beta_m\}$
Then show basis of (V/W) is $\{W+\beta_1, W+\beta_2, \dots W+\beta_m\}$

So, for problems with dimension; generally try to start with basis of smaller subspace & extend it for basis of bigger space. This gives elegant answer.

(10) e.g. Give basis of P_4/P_2 & verify $\dim(P_4/P_2) = \dim(P_4) - \dim(P_2)$

$\rightarrow$ $\frac{P_4}{P_2} = \{ P_2 + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_i \in \mathbb{R} \}$

$= \{ (P_2 + a_0 + a_1x + a_2x^2) + (a_3x^3 + a_4x^4) \mid a_i \in \mathbb{R} \}$

$= \{ P_2 + a_3x^3 + a_4x^4 \}$

$\therefore$ basis of P_4/P_2 are $\{ P_2 + x^3, P_2 + x^4 \}$ $\therefore \dim = 2$.

& $\dim(P_4) - \dim(P_2) = 5 - 3 = 2$

Linear Transformation

(1) Let $U(F)$ & $V(F)$ be 2 vector spaces.
The $T: U \to V$ is a linear transformation if
$T(\alpha a + \beta b) = aT(\alpha) + bT(\beta)$; $\alpha, \beta \in U(F)$
& $a, b \in F$.

(2) Properties of linear transformation
(a) $T(\bar{0}) = \hat{0}$ where $\bar{0} \in U$ & $\hat{0} \in V$.
(b) $T(-\alpha) = -T(\alpha)$
(c) $T(\alpha - \beta) = T(\alpha) - T(\beta)$

(3) Determination of linear transformation
Let $S = \{\alpha_1, \dots \alpha_n\}$ be basis of U.
Let $\{\delta_1, \delta_2 \dots \delta_n\}$ be a set of vectors in V.
Then $\exists$ a unique transformation s.t. $T(\alpha_i) = \delta_i$

Ans $\to$ Obv. define
$T(\alpha) = \beta_1 \delta_1 + \beta_2 \delta_2 + \dots + \beta_n \delta_n$
where $\alpha = \beta_1 \alpha_1 + \beta_2 \alpha + \dots + \beta_n \alpha_n$

(4) ex. Describe lin. transformation
$T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(2,3) = (4,5)$ & $T(1,0) = (0,0)$
$\to$ Show that $(2,3)$ & $(1,0)$ form basis of $\mathbb{R}^2$ & then define T.

(5) Sum of L.T. is also a L.T. $(T_1 + T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$
Similarly scalar multi. of L.T. is L.T. $(aT)(\alpha) = aT(\alpha)$

Multiplication of L.T.
Let $H: U \to V$ & $T: V \to W$
then $(TH)(\alpha) = T[H(\alpha)]$ where $\alpha \in U$

This provides algebra of L.T. $\Rightarrow T(H + H') = TH + TH'$
$a(TH) = (aT)H = T(aH)$

(6) Linear operator: It is linear transformation from U to U.
The algebra of lin. operators is also very intuitive.
$AI = IA = A$ (I is identity op) $AO = O$, $A(B+C) = AB + AC$

(7) L.T. form a vector space
→ let $L(U,V)$ be set of all L.T. from $U \to V$.
$L(U,V)$ is vector space for following vector add^n & scalar mul^n.
(i) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$
(ii) $(aT)(\alpha) = aT(\alpha)$ $\forall \alpha \in U$ & $T, H \in L(U,V)$ & $a \in F$.

(8) Prove $\dim L(U,V) = mn$ where $\dim(U) = m$ & $\dim(V) = n$.
→ Let basis of U be $B_1 = \{\alpha_1, \dots \alpha_m\}$
V be $B_2 = \{\beta_1, \dots \beta_n\}$
Consider lin. trans. T_{ij} s.t. $T(\alpha_i) = \beta_j$
& $T(\alpha_k) = 0$ $\forall k \neq i$.

We have such mn transformations T_{ij}.
Easy to show they form basis for $L(U,V)$

e.g. Find dimension of $L(\mathbb{C}^3, \mathbb{R}^2)$
→ here $\mathbb{C}^3$ is vector space over $\mathbb{C}$ & $\mathbb{R}^2$ over $\mathbb{R}$. $\therefore$ $L(\mathbb{C}^3, \mathbb{R}^2)$
dimension doesn't exist.

Find dim. of $(V, \mathbb{R}^2)$ where $V = \mathbb{C}^3$ over $\mathbb{R}$.
$= 6 \times 2 = 12$

classmate

Date

Page

(9) Range of L.T. is subspace of V.

Nullspace or Kernel of L.T. $= \{\alpha / T(\alpha) = 0\}$

Nullspace is also a subspace of V.

(10) Rank of L.T. = dim. of Range

Nullity of L.T. = dim. of kernel

Rank + Nullity = dim U

Proof :- Obviously you start by assuming basis of null space are $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$

Expand them to form basis of V $\{\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m\}$

Then show basis of Range are $\{T(\beta_1), \dots, T(\beta_m)\}$

(11) Singular transformation = nullspace contains atleast 1 non-zero vector

Non-singular = doesn't contain.

(12) For a non-singular L.T. $\Leftrightarrow$ images of ind. vectors are also independent.

L.T. is non-singular $\Leftrightarrow$ it is one one.

T is non-sing on finite dimension vector space $\Leftrightarrow$ invertible.

(13) If dim U = dim V. Following statements are equivalent.

1. T is non singular.
2. T is invertible
3. Range T = V
4. If $\{\alpha_1, \dots, \alpha_n\}$ is basis of V then $\{T(\alpha_1) \dots T(\alpha_n)\}$ = basis V

(14) So in questions where dim. of U & V are same, showing T is invertible means showing it is non-singular.

(15)

(16)

(15) Matrix of L.T. $\rightarrow$ Columns are images of basis.

Remember that transformation matrix is always w.r.t. ~~to~~ particular basis of U & V.

Very straight forward, no need to get confused.

(16) e.g. $T(x,y,z) = (3x+2y-4z,\ x-5y+3z)$

$B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,3), (2,5)\}$

$\rightarrow (a,b) = (-5a+2b)(1,3) + (3a-b)(2,5)$

now, $T(1,1,1) = (1,-1) = -7(1,3) + 4(2,5)$

$T(1,1,0) = (5,-4) = -33(1,3) + 19(2,5)$

$T(1,0,0) = (3,1) = -13(1,3) + 8(2,5)$

$\therefore$ Matrix $\begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

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Linear Equations

(1) For solution A & [A|B] should have same rank.

(2) To solve $Ax=B$; reduce [A|B] to echelon form.

case (i) Rank A = Rank [A|B] = no. of variable
unique solutions.

case (ii) Rank A = Rank [A|B] < no. of variables
infinite solutions

case (iii) Rank $A \neq$ Rank [A|B]
no solution.

(3) Cramer's rule.

Let given system be

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & \dots & & a_{nn} \end{vmatrix}$$

& Δ_i obtained by replacing i^{th} column by column of $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Then, $x_1 = \frac{\Delta_1}{\Delta}$, $x_2 = \frac{\Delta_2}{\Delta}$, ..., $x_n = \frac{\Delta_n}{\Delta}$

(Remember no -ve sign needed, Very elegant).

(4) Solving $Ax=0$

Where we have equations in n variables

Reduce given eq. to echelon form.

case (i) If rank $(A) = r = n$ then trivial solution $x=0$.

case (ii) rank $(A) = r < n$ then $\exists$ $n-r$ linearly independent solⁿ.

Total solⁿ ...

Eigen values & eigen vectors

① $|A-\lambda I| = 0$ characteristic eq. & roots are eigen values.
(= latent roots, characteristic roots, proper roots)

Set of eigen values = Spectrum of A.

② eigenvectors (char. vectors) $Ax_0 = \lambda x_0$ (x has to be non-zero)
of course kx_0 is also eigenvectors.
& 1 eigenvector can't correspond to more than 1 eigenvalue

③ Thm:- Eigenvectors corresponding to different eigenvalues are linearly independent.

→ Obv. we assume linear dependence, take out ind. vectors & then go for contradiction using $Ax = \lambda x$ relation.

Let $x_1, x_2 \ldots, x_m$ eigenvectors be linearly dependent.
let $x_1, x_2 \ldots, x_r$ be L.I. & $x_1, x_2 \ldots, x_r$ & x_{r+1} L.D. $(r < m)$
$\therefore \exists\, k_1, k_2 \ldots, k_{r+1}$ s.t.

$k_1x_1 + k_2x_2 + \cdots + k_{r+1}x_{r+1} = 0$ — ①

$\therefore A(k_1x_1 + k_2x_2 + \cdots + k_{r+1}x_{r+1}) = A(0) = 0$

$\therefore k_1(\lambda_1 x_1) + k_2(\lambda_2 x_2) + \cdots + k_{r+1}(\lambda_{r+1}x_{r+1}) = 0$ — ②

$\therefore$ ② $- \lambda_{r+1} \times$ ① gives

$k_1(\lambda_1 - \lambda_{r+1})x_1 + \cdots + k_r(\lambda_r - \lambda_{r+1})x_r = 0$

$\because x_1$ to x_r are L.I. & all λ's distinct $\therefore k_1 = k_2 \ldots = k_r = 0$

from ① $\Rightarrow k_{r+1} = 0$ #.

④ Eigenvalues of triangular matrix or Diagonal matrix are diagonal values.

For a non-singular matrix, no eigenvalue is 0.

For singular, at least one eigenvalue is 0.

If λ is eigenvalue of A, $\lambda + k$ is eigenvalue of $A + kI$.
λ^2 is eigenvalue of A^2
λ^{-1} " of A^{-1}

(5) A & $C^{-1}AC$ have same eigenvalues.

$\rightarrow$ $C^{-1}AC - \lambda I = C^{-1}AC - C^{-1}\lambda I C = C^{-1}(A-\lambda I)C$

$\therefore |C^{-1}AC - \lambda I| = |C^{-1}||A-\lambda I||C| = |A-\lambda I|$

(6) Eigenvalues of hermitian matrix are real.

$\rightarrow$ let A be hermitian $\therefore A^\theta = A$

let $Ax = \lambda x \Rightarrow x^\theta A x = x^\theta \lambda x \Rightarrow (x^\theta A x)^\theta = (x^\theta \lambda x)^\theta$

$x^\theta A^\theta x = \bar{\lambda} x^\theta x$

$\therefore x^\theta A x = \bar{\lambda} x^\theta x$

$\therefore \lambda x^\theta x = \bar{\lambda} x^\theta x \Rightarrow \lambda = \bar{\lambda}$ $\therefore$ real.

Corollary $\rightarrow$ eigenvalues of real symmetric matrix are real.

(7) Eigenvalues of skew hermitian matrix are purely imaginary or 0.

$\rightarrow$ let A be skew hermitian $\therefore A^\theta = -A$.

$\therefore (iA)^\theta = \bar{i}(-A) = -i \cdot -A = iA$

$\therefore iA$ is hermitian & its eigenvalue is real. $\therefore$ so on.

Same goes for skew-symmetric matrix.

(8) Eigenvalues of unitary matrix are of unit modulus.

$\rightarrow$ unitary means $A^\theta A = I$.

let $Ax = \lambda x \Rightarrow (Ax)^\theta = (\lambda x)^\theta \Rightarrow x^\theta A^\theta = \bar{\lambda} x^\theta$

$\therefore (x^\theta A^\theta)(Ax) = \bar{\lambda} x^\theta \lambda x$

$\therefore x^\theta x = |\lambda|^2 x^\theta x$ $\therefore |\lambda|^2 = 1$.

Corollary - Eigenvalues of orthogonal matrix are ± 1.

Matrix		Eigenvalue
Real Symmetric		real
Hermitian ($A^\theta = A$)		real
Skew Hermitian	$A^\theta = -A$	Pure imaginary or 0.
Skew Symmetric	$A^T = -A$	Pure imaginary or 0.
unitary	$A^\theta A = I$	$\lvert\lambda\rvert = 1$
orthogonal		$\lvert\lambda\rvert = 1$

Whenever skew is involved, eigenvalue doesn't become real; pure imag. or 0

(9)

(10)

⑨ Caley Hamilton theorem

Every matrix satisfies its characteristic equation.

→ Trick is we express adj $(A-\lambda I)$ as polynomial in λ of degree $n-1$ & then multiply $(A-\lambda I)\,\mathrm{adj}\,(A-\lambda I)$.

Obv., adj $(A-\lambda I)$ at most will have degree $n-1$.

$\therefore$ let adj $(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots\dots + B_{n-2}\lambda + B_{n-1}$

Where B_i are square matrices of order $n \times n$.

now,

$$(A-\lambda I).\,\mathrm{adj}\,(A-\lambda I) = |A-\lambda I|\, I$$

$$\therefore (A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \dots\dots + B_{n-2}\lambda + B_{n-1}) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots\dots + a_n)I$$

$$\therefore \quad -IB_0 = (-1)^n I$$

$$AB_0 - B_1 = (-1)^n a_1 I$$

$$\vdots$$

$$AB_{n-2} - B_{n-1} = (-1)^n a_{n-1} I$$

$$-AB_{n-1} = (-1)^n a_n I$$

Premultiplying each eq. by A^n, A^{n-1} successively, we get on adding.

$$O = (-1)^n (A^n + a_1 A^{n-1} + \dots + a_n I)$$

Thus A satisfies.

⑩ e.g. find inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ using caley-hamilton.

→ Char. eq. $A^2 - 5A - 2I = 0$ multiplying by A^{-1}

we get $A^{-1} = \frac{1}{2}[A - 5I]$

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(11) Now in all questions involving A^n, don't go for caley-hamilt

some questions need different tricks.

(a) e.g. $A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$ s.t. $A^n = A^{n-2} + A^2 - I \quad \forall\, n \geq 3.$

Here you use mathematical induction for $n \geq 3$.

(b) e.g. $P = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$ find P^{50}.

→ finding P^2, P^3, P^4 makes you realise $P^n = \begin{vmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{vmatrix}$

& so on.

(c) Find P which diagonalises $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$.

Hence calculate A^4

→ Here find P using eigenvectors & then

$P^{-1} A P = D$ (D has eigenvalues as diagonal entries)

$\therefore P^{-1} A^4 P = D^4$

$\therefore A^4 = P D^4 P^{-1}$

(12) Minimal polynomial of matrix A

Lowest degree polynomial $f(x)$ s.t. $f(A) = 0$

Minimal polynomial is always a factor of characteristic polynomial; but need not always be equal to it.

for I_n; characteristic polynomial is $(x-1)^n = 0$

whereas minimal polynomial is $x - 1 = 0$

Matrices basics, properties

(1) Idempotent Matrix $A^2 = A$
Involuntary Matrix $A^2 = I$
Nilpotent Matrix $\exists\, n$ s.t. $A^n = 0$ smallest such n = index of nilpotent A.

(2) Trace (AB) = Trace (BA) $\rightarrow$ this implies $AB - BA = I$ is never possible. remember this.

(3) Symmetric matrix $A^T = A$
Skew-symmetric matrix $A^T = -A$

Every matrix is sum of a sym. & skew sym. (Obs $A + A^T$ & $A - A^T$)

(4) Transpose conjugate or tranjugate of $A = A^\theta = (\bar{A})^T$

Hermitian matrix $A^\theta = A$
skew-hermitian matrix $A^\theta = -A$ (if A is herm; iA is skew herm)
Similarly, A is unique sum of herm & skew herm $(A + A^\theta, A - A^\theta)$
$\Rightarrow A = P + iQ$, P & Q hermitian $\left(P = \frac{A + A^\theta}{2}, Q = \frac{A - A^\theta}{2i}\right)$

(5) Determinants
Cofactor is obtained by $(-1)^{i+j}$ multiplied by minor.

Value of det. of odd order skew symmetric mat. is 0. $\because |A| = (-1)^n |A|$

(6) Adjoint Matrix
First write cofactors & then take transpose.
Don't forget to take transpose.

$$A \cdot Adj(A) = |A|\, I \qquad \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ \vdots & & \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \end{vmatrix} = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & & \ddots \end{vmatrix}$$

($\because$ sum of multiplication of a row to cofactors of other row = 0)

$$\therefore A^{-1} = \frac{1}{|A|} adj(A)$$

(7) If A, B are non-singular; $Adj(AB) = adj(A) \cdot adj(B)$
$\rightarrow$ involves simple manipulations using $A \cdot adj A = |A| I_n$

(8) Orthogonal $A^T A = I$
Unitary $A^\theta A = I$

(9) Matrices can be partitioned & then their multiplication can be shown in terms of submatrices multi. etc.
Very intuitive stuff.
It gives $A = \begin{bmatrix} P & O \\ O & Q \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}$

(10) Elementary Matrices
Obtained by performing single elementary op. on I.
E_{ij}, $E_i(k)$, $E_{ij}(k)$, $E_{ij}^T(k)$

(11) Reduction of matrix to normal form.
Thm. If A is matrix of rank r; $\exists$ non singular matrices P & Q s.t. $PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$

$\longrightarrow$ Since Rank is r; A can be transformed to $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ by elementary transformations.
Multiplication of all elt. row matrices gives P
& — " — elt. column matrices gives Q.
Remember that pre-multiplication is ~~by~~ for row transformation.
Post — " — for column transformation.

(12) Every non-singular matrix is a product of elementary matrices.
Corollary: Multiplication by non-sing. matrix doesn't change rank
(show non-sing. matrix ~~to~~ as mul. by elt. matrices)

(13)

(13) (a) e.g. Reduce matrix A $\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to connonical form & find rank.

→ Connonical implies. using both row & column transformation to reduce given A to $\begin{vmatrix} I_r & 0 \\ 0 & 0 \end{vmatrix}$. This obv. gives rank.

(b) e.g. find P&Q s.t. $PAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ & $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 3R_1$ & $C_2 \longrightarrow C_2 - C_1$ & $C_3 \to C_3 - C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So row changes will be reflected in P & column changes will be reflected in Q.

Note P&Q are not unique.

(c) e.g. express A as multiplication of elt. matrices.

→ Easy convert to I_n & then take inverses of elt. matrices appropriately.

(d) If A is rank r; ∃ non-singular P s.t. $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$. G = rank $r \times n$

⟶ We know ∃ P&Q s.t. $PAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ ∴ $PA = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] Q^{-1}$

Now Q are column transformations, ∴ they can't change last $n-r$ rows. ∴ $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$

Similarly ∃ Q s.t. $AQ = [H | 0]$ H is $m \times r$

(e) Above result implies $\text{rank}(AB) \le \text{Rank } A$
& $\text{rank}(AB) \le \text{Rank } B$

(14) Finding inverse using elementary row operations.

$\rightarrow \quad A = I_3 A$

now by performing row op., make ~~I~~ LHS I & I_3 on RHS turns into A^{-1}.

Example from diagonalizable matrices part

(1) Find orthogonally similar matrix that diagonalizes

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$, Also find the diagonal matrix

$\rightarrow$ Now eigenvalues turn out to be 0, 0, 14.

It is better to first find eigenvector corresponding to eigenvalue 14.

It turns out to be $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$.

Then get any eigenvector corresponding to 0 and 3^{rd} eigenvector (corr. to 0 again) would be cross product of above 2 since we want orthogonal.

If we find eigenvectors of 0 first, ensuring both are $\perp$ to each other not so easy. $\therefore$ This way works.

Similarity of Matrices

① 2 matrices are similar $\Rightarrow \exists\, C$ s.t. $A = C^{-1}BC$
Obv. Similar matrices have same char. eq. & hence eigenvalues.
But if 2 matrices have same eigenvalues, doesn't imply they are similar.

② Let λ_0 be common eigenvalue of ~~A~~ A & $P^{-1}AP$.
Then if X_0 is corresponding eigenvector for A, the eigenvector of $P^{-1}AP$ is $P^{-1}X_0$

③ If A is similar to B & B is diagonal; the eigenvalues of A are diagonal entries of B. Obv.
Such A is diagonalisable

④ Diagonalizable matrix
A is diagonalizable if $\exists\, P$ s.t. $P^{-1}AP = D$ (D is diagonal)

⑤ Thm: A is diagonalizable $\Leftrightarrow$ A has n ind. eigenvectors.
$\Rightarrow$ Let A be diagonalizable $\therefore \exists\, P$ s.t. $AP = PD$
Let $P = [x_1, x_2, \ldots, x_n]$ & $D = \text{dia}[\lambda_1, \lambda_2, \ldots, \lambda_n]$
$\therefore AP = PD$
$\Rightarrow A[x_1, x_2 \ldots x_n] = [x_1, x_2 \ldots x_n]\,\text{dia}[\lambda_1, \lambda_2 \ldots \lambda_n]$
$\therefore [Ax_1, Ax_2 \ldots Ax_n] = [\lambda_1 x_1, \lambda_2 x_2 \ldots \lambda_n x_n]$
$\therefore Ax_i = \lambda_i x_i$ $\therefore$ they are eigenvectors & $\because P$ is nonsing $\Rightarrow$ ind.
Inverse easy.

Note that, order of (eigen) column vectors of P & order of eigenvalues in diagonal should be same. Also we have $P^{-1}AP = D$ (P^{-1} is first & not P)

Every square matrix is not ~~diag~~ diagonalizable.
If all eigenvalues are all distinct $\Rightarrow$ diagonalizable.
If some eigenvalues are repeated $\Rightarrow$ may or may not be diagonalizable
If A & B have same eigenvalues & they are distinct $\Rightarrow$ A & B are similar.

$$\begin{bmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{bmatrix} \; \& \; \begin{bmatrix} 0 & -6 & -16 \\ 0 & 17 & 45 \\ 0 & -6 & -16 \end{bmatrix}$$

(7) e.g. P.T. these 2 matrices are similar
$\longrightarrow$ find eigenvalues, they turn out to be same & distinct

(8) One point regarding eigenvalues.
Every $(n \times n)$ matrix will have n degree characteristic equation. Hence, n eigenvalues will always be there; either real or imaginary.
Now, we know that distinct eigenvalues $\Rightarrow$ distinct eigenvectors
So if all eigenvalues are distinct; we will get diagonalizable.
if some are repeated, we can't be sure.

(9) Algebraic multiplicity $r \Rightarrow$ eigenvalue λ_0 is repeated r times.
Geometric multiplicity $s \Rightarrow$ eigenvector corresponding to λ_0
$\exists\ s$ lin. indep. eigenvectors.
Obv. $s \leq r$.

Now, if for each eigenvalue, alg. mul = geo-mul $\Rightarrow$ diagonalizable.

(10) example. S.T. $A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is diagonalizable & find transforming matrix, diagonal matrix.
$\longrightarrow$ simple; ~~fg~~ find eigenvalues & then eigenvectors.
Transf. matrix is given by eigenvectors as columns
& Diag. matrix is given by eigenvalues as diag. entries.

(11) In questions asking to show A is not diagonalizable; just show algebraic & geometric multiplicity is not same for some eigenvalue.

(12)

(13)

(14)

(15)

(16)

Orthogonal Vectors

(12) Norm of a complex vector

$\|X\| = \sqrt{X^{\theta} X} = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$

$\|X\| = 1 \Rightarrow$ unit vector or normal vector.

(13) Orthogonal set $\rightarrow$ Any 2 vectors orthogonal

Orthonormal set $\rightarrow$ Orthogonal + all vectors unit length.

(14) Orthogonally similar matrices

$B = P^{-1} A P$ where P is orthogonal.

Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

Obv: that is bcoz eigenvectors of real symmetric matrix are mutually orthogonal.

(15) Thm: 2 eigenvectors belonging to 2 distinct eigenvalues of real symmetric matrix are orthogonal.

$\rightarrow$ Let $A x_1 = \lambda_1 x_1$ & $A x_2 = \lambda_2 x_2$

consider $\lambda_1 x_2^T x_1 = x_2^T (\lambda_1 x_1) = x_2^T (A x_1)$

$= (x_2^T A) x_1 = (x_2^T A^T) x_1$

$= (A x_2)^T x_1 = (\lambda_2 x_2)^T x_1 = \lambda_2 x_2^T x_1$

$\therefore (\lambda_1 - \lambda_2) x_2^T x_1 = 0 \Rightarrow x_2^T x_1 = 0 \quad (\because \lambda_1 \neq \lambda_2)$

(16) Examples of diagonalising real sym. matrix using ortho. matrix

$\rightarrow$ Find eigenvectors $\rightarrow$ normalise them $\rightarrow$ job done.

Again remember that if matrix of eigenvectors is P, then

$P^{-1} A P = D$. (first comes P^{-1})

(17) Unitarily Similar Matrices. $\therefore P^{\theta} P = 1$

$B = P^{-1}AP$ where P is unitary

Every hermitian matrix is unitarily similar to a diagonal matrix.

Bcoz a hermitian matrix has n mutually orthogonal eigenvectors in complex vectorspace.

(18) Solving ex. asking to find unitary matrix that diagonalises given hermitian

→ similar to what we did earlier exactly;

find eigenvectors → normalise → get transforming matrix.

Just remember, complex norm defn → intuitive

$$\left\| \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} \right\| = \sqrt{|1-2i|^2 + 1^2} = \sqrt{5+1} = \sqrt{6}$$

Simple theory A orthogonal $= (I-S)^{-1}(I+S)$ (S skew symmetric

(1) S skew symmetric. P.T. $(I-S)$ non-singular

→ Skew sym. ⇒ eigen value 0 or imaginary $\therefore \lambda \neq 1 \therefore$ proved

(2) Let $A = (I+S)(I-S)^{-1}$ then A is orthogonal

→ $A^T = (I-S^{-1})^T (I+S)^T = (I+S)^{-1}(I-S)$

$\therefore A^TA = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$ $\left(\because (I-S)(I+S) = (I+S)(I-S)\right)$

$= I$ $\therefore$ ortho.

P.T. $A = (I-S)^{-1}(I+S)$

→ ~~$A = (I+S)(I-S)^{-1}$~~

$\therefore$ ~~$(I-S)^{-1}$~~ $A(I-S) = (I-S)^{-1}$ ~~$(I+S)$~~ $\therefore$ ~~$A(I-S) = (I+S)$~~

$\therefore$ ~~$(I-S)^{-1}(I-S)^{-1}(I+S)(I-S)$~~

$(I+S)(I-S) = (I-S)(I+S)$

$(I-S)^{-1}(I+S)(I-S) = (I+S)$

So $(I-S)^{-1}(I+S) = (I+S)(I-S)^{-1}$

simil...

(4) If λ is eigenvalue of S then $\frac{1+\lambda}{1-\lambda}$ is e.v. of A.
for same eigenvector X (for both)

→ ~~S.T.~~ $SX = \lambda X$ $\therefore$ $SX + X = (\lambda+1)X$

$\therefore (S+I)X = (\lambda+1)X$ — ①

also $(S-I)X = (\lambda-1)X$

$\therefore$ $X = (S-I)^{-1}(\lambda-1)X$ — ② ($\because$ $(S-I)$ non singular)

$\therefore$ ① & ② give

$(S+I)(S-I)^{-1}X = \frac{(\lambda+1)}{(\lambda-1)}X$ & so on.

(5) If A is ortho with property that -1 is not eigenvalue, P.T. A is expressible as $(I+S)(I-S)^{-1}$ for some skew symmetric matrix S.

→ In all such questions, assume it is true & arrive at S in terms of A with manipulations.

let $A = (I+S)(I-S)^{-1}$

$A(I-S) = (I+S)$ $\therefore$ $AI - AS = I + S$

$\therefore$ $AI - I = S(A+I)$ $\therefore S = (A-I)(A+I)^{-1}$

now we can find $(I+S)$ & $(I-S)^{-1}$ and show it actually becomes A.