IFOS-2015

① $u(x,y) = \cos x \sinh y$

$\frac{\partial u}{\partial x} = -\sin x \sinh y \quad \frac{\partial u}{\partial y} = \cos x \cosh y.$

Since $f(z) = u(x,y) + i\,v(x,y)$ is Analytical by Cauchy Riemann Equation we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\Rightarrow \frac{\partial v}{\partial y} = -\sin x \sinh y \Rightarrow v = -\sin x \cosh y + f(x).$

Now $\frac{\partial v}{\partial x} = -\cos x \cosh y + f'(x) = -\frac{\partial u}{\partial y}$ [as per C.R equn]

$\Rightarrow f'(x) = 0 \quad \Rightarrow f(x) = c.$

$\therefore \boxed{v(x,y) = -\sin x \cosh y + c}.$

Using Milne Thomas Equation we can say that

$f(z) = \int (\psi_1(z,0) - i\,\psi_2(z,0))\,dz + c \quad \psi_1(x,y) = \frac{\partial u}{\partial x}$

$\psi_2(x,y) = \frac{\partial u}{\partial y}.$

$f(z) = \int (0 - i(\cos z))\,dz + c.$

$\boxed{f(z) = -i\sin z + c}$ is the analytic function.

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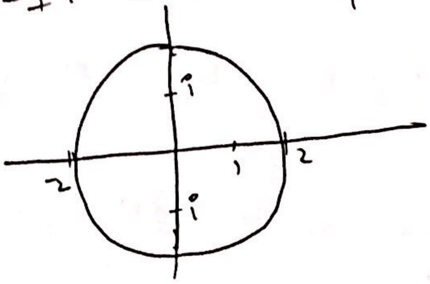
2. $\displaystyle\int_r \frac{z^2}{(z^2+1)(z-1)^2}\,dz$ where r is the circle $|z|=2$.

Singularities of the given function are..

$z^2+1=0 \qquad z=1$

$z=\pm i$ and $z=1$ of order two.



We can see that all the singularities are within the curve $|z|=2$.

Using Cauchy's Residue theorem. we can see that

$$\oint_c f(z)\,dz = 2\pi i \sum \text{Residues}.$$

Residue at $z=i$ $\Rightarrow \dfrac{(i)^2}{(i+i)(i-1)^2} = \dfrac{-1}{2i(-1+1-2i)} = \dfrac{-1}{4}$

Residue at $z=-i$ $\quad \dfrac{(-i)^2}{(-i-i)(-i-1)^2} = \dfrac{+1}{(+2i)(-\cancel{1}+\cancel{1}+2i)} = \dfrac{-1}{4}.$

Residue at $z=1$.

Order two.

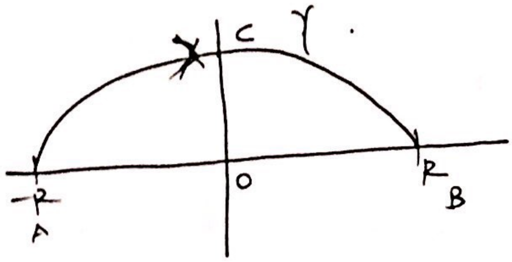
$$\Rightarrow \lim_{z\to 1} \frac{d}{dz}\frac{z^2}{(z^2+1)} = \frac{(z^2+1)(2z)-z^2(2z)}{(z^2+1)^2}$$

$$= \lim_{z\to 1}\frac{2z}{(z^2+1)^2} = \frac{2}{(2)^2} = \frac{1}{2}.$$

$$\oint_r \frac{z^2}{(z^2+1)(z-1)^2}\,dz = 2\pi i\left[-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\right] = \boxed{\text{Zero}}$$

③ $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = ?$.

Let us define a curve



$\int_{-\infty}^{\infty} f(x)\,dx = \int_{BOA} f(x)\,dx + \lim_{R\to\infty} \int_{\gamma\,-R}^{R} f(x)\,dx$.

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{BOA} f(x)\,dx + \lim_{R\to\infty} \gamma \int_{-R}^{R} f(x)\,dx$.

$\int_{BOA} f(x)\,dx = zero$.

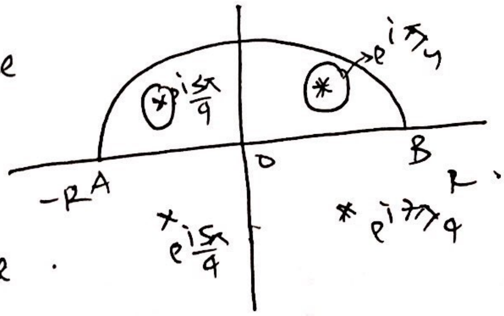
Singularities of $f(x)$ are . $1+x^4=0$

$x^4=-1$

$x=(-1)^{1/4} \Rightarrow e^{\frac{(2n+1)\pi i}{4}}$. $n=0,1,2,3$.

Singularities are $\Rightarrow e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{\frac{i7\pi}{4}}$.

In the given curve



only $e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}$ are inside the curve .

Aspn Cauchy's Residue theorem .

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{z^2}{1+z^4}\,dz = 2\pi i \left[\lim_{z\to e^{i\pi/4}} \frac{d}{dz} \frac{(z-e^{i\pi/4})\,z^2}{1+z^4} + \lim_{z\to e^{\frac{i3\pi}{4}}} \frac{d}{dz} \frac{(z-e^{\frac{i3\pi}{4}})\,z^2}{1+z^4} \right]$$

$$= 2\pi i \left[\lim_{z\to e^{i\pi/4}} \frac{z^2}{4z^3} + \lim_{z\to e^{\frac{i3\pi}{4}}} \frac{z^2}{4z^3} \right]$$

$$= \frac{\pi i}{2}\left[e^{-i\frac{\pi}{4}} + e^{-i\frac{3\pi}{4}}\right]$$

$$= \frac{\pi i}{2}\left[\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi}{\sqrt{2}}.$$