Set - I

* Groups *

practice problems

1. Let $(G, *)$ be a group and a be an element of G such that $o(a) = n$. (i) If $a^m = e$ for some positive integer m, then n divides m.

(ii) For every positive integer t,

$$o(a^t) = \frac{n}{\gcd(t, n)}$$

2. Which of the following groupoids are semigroups? Which are groups?

(i) $(\mathbf{N}, *)$ where $a * b = ab$ for all $a, b \in \mathbf{N}$.

(ii) $(\mathbf{N}, *)$ where $a * b = b$ for all $a, b \in \mathbf{N}$

(iii) $(\mathbf{Z}, *)$ where $a * b = a + b + 2$ for all $a, b \in \mathbf{Z}$

(iv) $(\mathbf{Z}, *)$ where $a * b = a - b$ for all $a, b \in \mathbf{Z}$.

(v) $(\mathbf{Z}, *)$ where $a * b = a + b + ab$ for all $a, b \in \mathbf{Z}$.

(vi) $(\mathbf{R}, *)$ where $a * b = a|b|$ for all $a, b \in \mathbf{R}$.

(vii) $(\mathbf{R}, *)$ where $a * b = 2^a b$ for all $a, b \in \mathbf{R}$

(viii) $(\mathbf{R} \setminus \{-1\}, *)$ where $a * b = a + b + ab$ for all $a, b \in \mathbf{R} \setminus \{-1\}$.

3. Write all complex roots of $x^6 = 1$. Show that they form a group under the usual complex multiplication.

4. Let $G = \{a \in \mathbf{R} : -1 < a < 1\}$. Define $*$ on G by $a * b = \dfrac{a+b}{1+ab}$ for all $a, b \in G$. Show that $*$ is a binary operation on G. Hence prove that $(G, *)$ is a group.

5. Write down the Cayley table for the group operation of the group $\mathbf{Z}_5$.

6. Consider the group $\mathbf{Z}_{30}$. Find the smallest positive integer n such that $n[5]=[0]$ in $\mathbf{Z}_{30}$.

7. write down all elements of the group U_{10}. write the Cayley table for this group.

8. Let $G = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbf{Z} \right\}$. show that G becomes a group under usual matrix multiplication.

9. Find the order of $[6]$ in the group $\mathbf{Z}_{14}$ and the order of $[3]$ in U_{14}.

10. Let $(G,*)$ be a group and $a,b \in G$. Suppose that $a^2=e$ and $a*b*a = b^7$. Prove that $b^{48}=e$.

11. which of the following groupoids are semigroups? which are groups?

(a) $(\mathbf{N},*)$, where $a*b = a+b$ for all $a,b \in \mathbf{N}$.

(b) $(\mathbf{N},*)$, where $a*b = a$ for all $a,b \in \mathbf{N}$.

(c) $(\mathbf{Z},*)$, where $a*b = a+b+1$ for all $a,b \in \mathbf{Z}$.

(d) $(\mathbf{Z},*)$, where $a*b = a+b-1$ for all $a,b \in \mathbf{Z}$.

(e) $(\mathbf{Z},*)$, where $a*b = a+2b$ for all $a,b \in \mathbf{Z}$.

(f) $(\mathbf{Z},*)$, where $a*b = a+b-ab$ for all $a,b \in \mathbf{Z}$.

(g) $(\mathbf{R},*)$, where $a*b = |a|b$ for all $a,b \in \mathbf{R}$.

(h) $(\mathbf{R},*)$, where $a*b = a^2b^2$ for all $a,b \in \mathbf{R}$.

(i) $(\mathbf{R},*)$, where $a*b = a+b+ab$ for all $a,b \in \mathbf{R}$.

(j) $(\mathbf{Q}^+,*)$, where $a*b = ab$ for all $a,b \in \mathbf{Q}^+$.

(k) $(\mathbf{Q}\setminus\{0\},*)$, where $a*b = ab$ for all $a,b \in \mathbf{Q}\setminus\{0\}$.

12 Let $P(x)$ be the Power set of a set x. Consider the operation Δ (symmetric difference) on $P(x)$. Then for all $A, B \in P(x)$,

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

$(P(x), \Delta)$ is a Commutative group. The empty set ϕ is the identity of $(P(x), \Delta)$ and every element of $P(x)$ is its own inverse. We warn the reader that verification of the associative law is tedious.

13. Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in R, ad - bc \neq 0 \right\}$, the set of all 2×2 real matrices having a non-zero determinant. Define a binary operation $*$ on G by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} u & v \\ w & s \end{bmatrix} = \begin{bmatrix} au+bw & av+bs \\ cu+dw & cv+ds \end{bmatrix}$$

for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} u & v \\ w & s \end{bmatrix} \in G$. This binary operation is the usual matrix multiplication. Since matrix multiplication is associative, we have $*$ is associative. The element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ is the identity element for the above operation. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$. Then $ad - bc \neq 0$. Consider the matrix $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$. Since

$$\frac{d}{ad-bc}\cdot\frac{a}{ad-bc}-\frac{-b}{ad-bc}\cdot\frac{-c}{ad-bc}=\frac{1}{ad-bc}\neq 0,$$

we have

$$\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}\in G.$$

Now,

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}*\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and

$$\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}*\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Thus, $\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$ is the inverse of $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Hence,

G is a group. Now

$$\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix},\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\in G$$

and

$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}*\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\neq\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}*\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

Hence, G is a noncommutative group.

This group is known as the general linear group of degree 2 over $\mathbf{R}$ and is denoted by $GL(2,\mathbf{R})$.

14 Let R^- denote the set of all negative real numbers. Can you define a binary operation $*$ on R^- so that the system $(R^-, *)$ becomes a group?

15 write all complex roots of $x^7 = 1$. show that they form a group under the usual complex multiplication.

16. show that the set of all complex numbers $a+ib$ such that $a^2+b^2=1$ is a group under the usual multiplication of complex numbers.

17. Let $G = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} : a, b \in R, a \neq 0 \right\}$. show that G becomes a group under the usual matrix multiplication.

18. Let $G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in R, \text{ a and b not both zero} \right\}$. show that $(G, *)$ is a commutative group, where $*$ denotes the usual matrix multiplication.

19. Consider the group Z_{15}. Find the smallest positive integer n such that $n[5] = [0]$ in Z_{15}.

20. Consider the group Z_{20}. Find the smallest positive integer n such that $n[5] = [0]$ in Z_{20}.

21. write down all elements of the group U_{10}. write the cayley table for this group.

22 Let $(G, *)$ be a group and $a, b, c \in G$. show that there exists a unique element x in G such that $a * x * b = c$.

23. Let $(G,*)$ be a finite abelian group and $G=\{a_1,a_2,\cdots a_n\}$. Let $a_1a_2\cdots a_n=x$. Prove that $x*x=e$.

24. Let $(G,*)$ be a group and $a,b\in G$. Suppose that $a*b^3*a^{-1}=b^2$ and $b^{-1}*a^2*b=a^3$. Show that $a=b=e$.

25. Let $(G,*)$ be a group and $a,b\in G$. Suppose that $a^2=e$ and $a*b^4*a=b^7$. Prove that $b^{33}=e$.

26. Let $(G,*)$ be a group and $a,b\in G$. Show that $(a*b*a^{-1})^n=a*b^n*a^{-1}$ for all positive integers n.

27. In a group G, if $a^5=e$ and $a*b*a^{-1}=b^m$ for some positive integer m, and some $a,b\in G$, then Prove that $b^{m^5-1}=e$.

28. In $GL(2,R)$, show that $A=\begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$ are elements of finite order, whereas AB is of infinite order.

29. Let $(G,*)$ be a group. If for $a,b\in G$, $(a*b)^3=a^3*b^3$ and $(a*b)^5=a^5*b^5$, then Prove that $a*b=b*a$.

30. Show that a group $(G,*)$ is Commutative if and only if $(a*b)^5=a^5*b^5$, $(a*b)^6=a^6*b^6$ and $(a*b)^7=a^7*b^7$ for all $a,b\in G$.

31. In the group Z_{15}, find the order of the following elements: $[5]$, $[8]$, and $[10]$.

32. Let G be a group and $a\in G$. If $o(a)=24$, then find $o(a^4)$, $o(a^7)$ and $o(a^{10})$.

33. Let G be a group and $a,b\in G$, Such that $ab=ba$ and $o(a)$ and $o(b)$ are relatively Prime. Then Prove that $o(ab)=o(a)o(b)$

34. Find the smallest positive integer n such that $[7]^n = [1]$ in U_{10} and in U_{12}.

35. Find the order of $[6]$ in the group Z_{10} and the order $[3]$ in U_{10}.

36. Show that $\{1,2,3\}$ under multiplication modulo 4 is not a group but that $\{1,2,3,4\}$ under multiplication modulo 5 is a group.

37. Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, Z_{11})$.

38. Give an example of group elements a and b with the property that $a^{-1}ba \neq b$.

39. Let p and q be distinct primes. Suppose that H is a proper subset of the integers and H is a group under addition that contains exactly three elements of the set $\{p, p+q, pq, p^q, q^p\}$. Determine which of the following are the three elements in H.

a. pq, p^q, q^p

b. $p+q, pq, p^q$

c. $p, p+q, pq$

d. p, p^q, q^p

e. p, pq, p^q

40. Prove that the set of all 2×2 matrices with entries from R and determinant $+1$ is a group under matrix multiplication.

41. Let G be a group with the following property: If a, b and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian.

42. An Abstract Algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out? (This really happened!).

43. (Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-abelian groups?

44. (Socks-shoes Property) In a group, Prove that $(ab)^{-1} = b^{-1}a^{-1}$. Find an example that shows it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. Find a non-abelian example that shows it is possible to have $(ab)^{-1} = a^{-1}b^{-1}$ for some distinct non identity elements a and b. Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes.

45. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?

46. If $a_1, a_2, ---, a_n$ belong to a group, what is the inverse of $a_1 a_2 -- a_n$?

47 Suppose the table below is a group table. Fill in the blank entries.

	e	a	b	c	d
e	e	–	–	–	–
a	–	b	–	–	e
b	–	c	d	e	–
c	–	d	–	a	b
d	–	–	–	–	–

48. Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$.

49. Let G be a finite group. show that the number of elements x of G such that $x^3 = e$ is odd. show that the number of elements x of G such that $x^2 \neq e$ is even.

50. Prove that the set of all 3×3 matrices with real entries of the form.

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$ is a group.

51. In a finite group, show that the number of nonidentity elements that satisfy the equation $x^5 = e$ is a multiple of 4.

52. Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} / a \in \mathbb{R}, a \neq 0 \right\}$. show that G is a group under matrix multiplication.

<u>Note</u>: The group $GL(2, \mathbb{R})$ is known as the general linear group of degree 2 over $\mathbb{R}$

* Groups *

Answers

11 Ans.

(a) Semi group but not group, (b) Semigroup but not group

(c) Semi group as well as a group (d) Semigroup as well as a group.

(e) not a Semigroup (f) Semi group but not group.

(g) Semigroup but not group (h) Not a Semigroup

(i) Semigroup but not group (j) Semigroup as well as a group

(k) Semigroup as well as a group

17 Yes; [Hint: For some $c \in R^-$; define $a*b = acb$ for all $a, b \in R^-$]

19. $n = 3$.

20 $n = 4$

21. $U_{10} = \{[1], [3], [7], [9]\}$ and the Cayley table is given by the following.

*	[1]	[3]	[7]	[9]
[1]	[1]	[3]	[7]	[9]
[3]	[3]	[9]	[1]	[7]
[7]	[7]	[1]	[9]	[3]
[9]	[9]	[7]	[3]	[1]

31. 3, 15, 3.

32. 6, 24, 12

34 $n=4$ in U_{10}, $n=2$ in U_{12}.

35 $o([6]) = 5$, $o([3]) = 4$

36. Under modulo 4, 2 does not have an inverse. Under modulo 5, each element has an inverse.

37. $\begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix}$

39 Ans : (e)

40. Use the fact that $\det(AB) = (\det A)(\det B)$.

42. 29

43. $(ab)^n$ need not equal $a^n b^n$ in a non-abelian group.

45. The identity is 25

49. If $x^3 = e$ and $x \neq e$, then $(x^{-1})^3 = e$ and $x \neq x^{-1}$. So, nonidentity solutions come in pairs. If $x^2 \neq e$, then $x^{-1} \neq x$ and $(x^{-1})^2 \neq e$. So solutions to $x^2 \neq e$ come in pairs.

52. closure follows from the definition of multiplication. The identity is $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. The inverse of $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ is $\begin{bmatrix} 1/4a & 1/4a \\ 1/4a & 1/4a \end{bmatrix}$

→ Let $P(X)$ be the powerset of a set X. consider operation Δ (symmetric difference) on $P(X)$. Then for all $A, B \in P(X)$, $A \Delta B = (A \setminus B) \cup (B \setminus A)$.
Show that $(P(X), \Delta)$ is a commutative group.

Sol: Closure prop:
Let $A, B \in P(X)$

Then $A \subset X$, $B \subset X$.

Now $A \Delta B = (A-B) \cup (B-A)$
which is also a subset of X.

$\therefore$ $A \Delta B$ is also a member of $P(X)$

i.e, $A \Delta B \in P(X)$.

$\therefore$ $P(X)$ is closed w.r.t operation 'Δ'.

($\because$ Difference and union of sets are binary operation on $P(X)$.
because: $A, B \in P(X)$
$\Rightarrow A-B,\ B-A \in P(X)$
$\Rightarrow (A-B) \cup (B-A) \in P(X)$
i.e, $A \Delta B \in P(X)$]

Associative prop:
The verification of the associative law is tedious.
we are enough to show through an example.

Existence of left identity:

The empty set ϕ is a subset of X.

$\therefore \phi$ is a member of $P(X)$.

If A is any member of $P(X)$, we have

$$\phi \Delta A = (\phi - A) \cup (A - \phi)$$
$$= \phi \cup A$$
$$= A$$
$\therefore \phi$ is the left identity.

Existence of left inverse:

Every element of $P(X)$ is its own inverse.

Since $A \Delta A = (A-A) \cup (A-A)$
$= \phi \cup \phi$
$= \phi$
which is a member of $P(X)$

$\therefore (P(X), \Delta)$ is a group.

Commutative prop:

Let $A, B \in P(X)$

$A \Delta B = (A-B) \cup (B-A)$
$= (B-A) \cup (A-B)$
$= B \Delta A$

$\therefore A \Delta B = B \Delta A$.
$\therefore$ commutative prop is satisfied

$\therefore (P(X), \Delta)$ is a commutative group.

Set-II * Permutation Groups *

Practice Problems

1. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 6 & 7 & 3 & 5 & 2 \end{pmatrix}$

be elements of S_7.

(i) write α as a product of disjoint cycles.

(ii) write β as a product of 2 cycles.

(iii) Is β an even permutation?

(iv) Is α^{-1} an even permutation?

2. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$. Find the smallest positive integer k such that $\alpha^k = e$ in S_4.

3. Compute each of the following and express it in two-row notation in S_7.

(i) $(1\ 3\ 4\ 7)(5\ 4\ 2)$ (ii) $(1\ 2\ 5\ 4)^2(1\ 2\ 3)(2\ 5)$.

4. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \in S_4$. Find the smallest positive integer k such that $\alpha^k = e$.

5. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ in S_5. Find a permutation γ in S_5 such that $\alpha\gamma = \beta$.

6. If $\beta \in S_7$ and $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$ then find β.

7. If $\beta = (1\ 2\ 3)(1\ 4\ 5)$, write β^{99} in cycle notation.

8. Let $\beta = (1\ 3\ 5\ 7\ 9\ 8\ 6)(2\ 4\ 10)$ in S_{10}. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

9. In S_3, find elements α and β so that $|\alpha| = 2$, $|\beta| = 2$, and $|\alpha\beta| = 3$.

* Permutation Groups *

Answers

4. $K = 4$

5. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$

6. $\beta = (1\ 3\ 6\ 2\ 4\ 5\ 7)$

7. $\beta^{99} = (1\ 3\ 2\ 5\ 4)$

8. $n = 16.$

Set-iii

* Subgroups *

Practice Problems

1. Let $GL(2,\mathbb{R})$ be the group of all non-singular 2×2 matrices over $\mathbb{R}$. Show that each of the following sets is a subgroup of $GL(2,\mathbb{R})$.

(i) $H = \left\{\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in GL(2,\mathbb{R}) \,\middle|\, ad \neq 0\right\}$

(ii) $H = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in GL(2,\mathbb{R}) \text{ either } a \text{ or } b \neq 0\right\}$.

2. Find all subgroups of the group $\mathbb{Z}$ of all integers under usual addition.

3. In each case, determine whether H is a subgroup of the group G under usual operation.

(a) $H = \{3n \mid n \in \mathbb{Z}\}$, $G = \mathbb{Z}$

(b) $H = \{n \mid n \in \mathbb{Z} \text{ and } n \geq 0\}$, $G = \mathbb{Z}$

(c) $H = \{n \mid n \in \mathbb{Z} \text{ and } |n| \geq 1\}$, $G = \mathbb{Z}$

(d) $H = \{(m,n) \mid m,n \in \mathbb{Z} \text{ and } m+n \text{ is even}\}$, $G = \mathbb{Z}\times\mathbb{Z}$

(e) $H = \{1, -1, 0\}$, $G = \mathbb{Z}$.

(f) $H = \{[0], [2], [4], [6]\}$, $G = \mathbb{Z}_8$

4. In each case, determine whether H is a subgroup of the group $\mathbb{R}^* = (\mathbb{R}\setminus\{0\}, \cdot)$.

(a) $H = \{1, -1\}$

(b) H = the set of all positive real numbers.

(c) H = the set of all positive integers.

(d) $H = \{a + b\sqrt{3} \in \mathbb{R}^* \mid a, b \in \mathbb{Q}\}$.

5. Let $GL(2, \mathbb{R})$ denote the group of all nonsingular 2×2 matrices with real entries. In each case, determine whether S is a subgroup of the group $GL(2, \mathbb{R})$.

(a) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, ad - bc = 1 \right\}$.

(b) $S = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, n \in \mathbb{Z} \right\}$.

(c) $S = \left\{ \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, b \text{ is nonzero} \right\}$.

(d) $S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, ad > 0 \right\}$.

(e) $S = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, a^2 + b^2 \neq 0 \right\}$

(f) $S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \in GL(2, \mathbb{R}) \,/\, a \neq 0 \right\}$.

6. Show that the set $H = \{a + ib \in \mathbb{C}^* \,/\, a^2 + b^2 = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, where $\cdot$ is the usual multiplication of complex numbers.

7. Let G be a group. Prove that a nonempty subset H is a subgroup of G if and only if for $a, b \in H$, ab in H and a^{-1} in H.

8. Let G be a group and $a \in G$. $C(a) = \{x \in G \mid ax = xa\}$. Show that $C(a)$ is a subgroup of G and $Z(G)$ is contained in $C(a)$.

9. If G is a Commutative group, then Prove that $H = \{a^2 \mid a \in G\}$ is a Subgroup of G.

10. If G is a Commutative group, then prove that $H = \{a \in G / a^2 = e\}$ is a Subgroup of G.

11. Let K be a subgroup of a group G and H be a subgroup of K. Is it true that H is a subgroup of G? Justify.

12. Let G be a group and $a \in G$. show that $H = \{a^{2n} : n \in Z\}$ is a Subgroup of G.

13. In the group S_3, show that the subset $H = \{a \in S_3 \mid o(a) \text{ divides } 2\}$ is not a Subgroup.

14. In the Symmetric group S_3, show that $H = \{e, (2\ 3)\}$ and $K = \{e, (1\ 2)\}$ are subgroups but $H \cup K$ is not a subgroup of S_3.

15. If H and K are Subgroups of a group G, then Prove that $H \cup K$ is a Subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$.

16. Let G be a group and H be a nonempty subset of G.

(a) show that if H is a Subgroup of G, then $HH = H$.

(b) If H is finite and $HH \subseteq H$, then prove that H is a subgroup of G.

(c) Give an example of a group G and a nonempty subset H such that $HH \subseteq H$, but H is not a subgroup of G.

17. Let G be a Commutative group. Prove that the set H of all elements of finite order in G is a subgroup of G.

18. Let G be a commutative group. Prove that the subset $H = \{a \in G \mid o(a) \text{ divides } 10\}$ is a subgroup of G.

19. Let $G = \{(a,b) : a, b \in \mathbb{R} \text{ and } b \neq 0\}$. Show that $(G, *)$ is a non commutative group under the binary operation $(a,b)*(c,d) = (a+bc, bd)$ for all $(a,b), (c,d) \in G$.

(a) Show that $H = \{(a,b) \in G \mid a = 0\}$ is a subgroup of G.

(b) Show that $K = \{(a,b) \in G \mid b > 0\}$ is a subgroup of G.

(c) Show that $T = \{(a,b) \in G \mid b = 1\}$ is a subgroup of G.

(d) Does G contain a finite subgroup of order 2?

20. Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5.

21. Let G be a group. Prove or disprove that $H = \{g^2 \mid g \in G\}$ is a subgroup of G.

22. For each divisor k of n, let $U_k(n) = \{x \in U(n) \mid x \equiv 1 \bmod k\}$. For example, $U_3(21) = \{1, 4, 10, 13, 16, 19\}$ and $U_7(21) = \{1, 8\}$.) List the elements of $U_4(20)$, $U_5(20)$, $U_5(30)$, and $U_{10}(30)$. Prove that $U_k(n)$ is a subgroup of $U(n)$.

23. Suppose that H is a proper subgroup of $\mathbb{Z}$ under addition and H contains 18, 30, and 40. Determine H.

24. Let G be a group. Show that $Z(G) = \cap_{a \in G} C(a)$. [This means the intersection of all subgroups of the form $C(a)$.]

25. Let G be a group, and let $a \in G$. Prove that $C(a) = C(a^{-1})$.

26. Let $H = \{x \in U(20) \mid x \equiv 1 \bmod 3\}$. Is H a subgroup of $U(20)$?

27. Suppose G is the group defined by the following Cayley table.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

(a) find the Centralizer of each member of G.

(b) Find $Z(G)$

(c). Find the order of each element of G. How are these orders arithmetically related to the order of the group?

28. Consider the elements $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}$ from $SL(2,R)$. Find $|A|$, $|B|$, and $|AB|$. Does your answer suprise you?

29. Consider the element $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ in $SL(2,R)$. What is the order of A? If we view $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ as a member of $SL(2,Z_p)$ (p is a prime), what is the order of A?

30. For any positive integer n and any angle θ, show that in the group $SL(2,R)$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Use this formula to find the order of

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \text{ and } \begin{bmatrix} \cos\sqrt{2}^\circ & -\sin\sqrt{2}^\circ \\ \sin\sqrt{2}^\circ & \cos\sqrt{2}^\circ \end{bmatrix}.$$

31. Compute the orders of the following.

a. $U(3)$, $U(4)$, $U(12)$ b. $U(5)$, $U(7)$, $U(35)$

(c) $U(4)$, $U(5)$, $U(20)$ d. $U(3)$, $U(5)$, $U(15)$

32. Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in Z \right\}$ under addition. Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,\ a+b+c+d = 0 \right\}$. Prove that H is a subgroup of G. what if 0 is replaced by 1 ?

33. Let $G = GL(2, R)$. Let $H = \{ A \in G \mid \det A \text{ is a power of } 2 \}$. Show that H is a subgroup of G.

34. Let H be a subgroup of R under addition. Let $K = \{ 2^a \mid a \in H \}$. Prove that K is a subgroup of R^* under multiplication.

35. Let G be a group of functions from R to R^* under multiplication. Let $H = \{ f \in G \mid f(1) = 1 \}$. Prove that H is a subgroup of G.

36. Let $G = GL(2, R)$ and $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a \text{ and } b \text{ are non-zero integers} \right\}$. Prove or disprove that H is a subgroup of G.

37. Let $G = GL(2, R)$

(a) find $C\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)$ (b). find $C\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$

(c). find $Z(G)$.

* Subgroups *

Answers.

3. (a) yes (b) no (c) no (d) yes (e) no (f) yes.

4. (a) yes (b) yes (c) no (d) yes

5. (a) yes (b) yes (c) no (d) yes (e) no (f) yes.

16. (c). consider $G = (Z, +)$ and $H = \{n \in Z / n \geq 1\}$.

24 $\langle 2 \rangle$

24 If $x \in Z(G)$, then $x \in C(a)$ for all a, so $x \in \bigcap_{a \in G} C(a)$. If $x \in \bigcap_{a \in G} C(a)$, then $xa = ax$ for all a in G so $x \in Z(G)$.

26. No. $7 \in H$ but $7 \cdot 7 \notin H$.

27. a. $C(5) = G$; $C(7) = \{1, 3, 5, 7\}$

b. $Z(G) = \{1, 5\}$

c. $|2| = 2$, $|3| = 4$. They divide order of the group.

29. Note that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

32. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ belong to H. It suffices to show that $a - a' + b - b' + c - c' + d - d' = 0$. This follows from $a + b + c + d = 0 = a' + b' + c' + d'$. If 0 is replaced by 1, H is not a subgroup.

34. If 2^a and $2^b \in K$, then $2^a (2^b)^{-1} = 2^{a-b} \in K$ since $a - b \in H$.

36. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ is not in H.

37. a. $\left\{ \begin{bmatrix} a+b & a \\ a & b \end{bmatrix} \Big/ ab + b^2 \neq a^2 ;\ a, b \in R \right\}$

b. $\left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \Big/ a^2 \neq b^2 ;\ a, b \in R \right\}$

c. $\left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \Big/ a \neq 0 ;\ a \in R \right\}$

Set-IV * Cosets and Lagrange's Theorem *

* Practice Problems *

1. Let H be a subgroup of a group G. Then $|L| = |R|$, where L (resp. R) denotes the set of all left (resp. right) cosets of H in G.

2. Find all subgroups of S_3. Show that union of any two non-trivial distinct subgroups of S_3 is not a subgroup of S_3.

3. Let H be a subgroup of a group G. Denote by L_H, the relation on G defined by

$$L_H = \{(a,b) \in G \times G : a^{-1}b \in H\}.$$ Prove that

(i) L_H is an equivalence relation

(ii) Every equivalence class is a left coset of H in G.

(iii) Every left coset of H is an equivalence class of the relation L_H.

4. Find all distinct left cosets of the subgroup H in the group G.

(a) $H = \{1, -1\}$, $G = (\mathbb{R} \setminus \{0\}, \cdot)$ (b) $H = 7\mathbb{Z}$, $G = \mathbb{Z}$

(c) $H = \{e, (2\ 3)\}$, $G = S_3$. (d) $H = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$, $G = S_3$.

5. Show that the set L of all left cosets of $\mathbb{Z}$ in the additive group $(\mathbb{R}, +)$ of all real numbers is given by.

$$L = \{x + 8\mathbb{Z} \mid x = 0, 1, 2, \ldots, 7\}.$$

6. Let H be a subgroup of G and suppose that $g_1, g_2 \in G$. Prove that the following conditions are equivalent.
(a) $g_1 H = g_2 H$ (b) $Hg_1^{-1} = Hg_2^{-1}$ (c) $g_2 \in g_1 H$ (d) $g_1^{-1} g_2 \in H$.

7. Determine whether or not the following cosets and the subgroup $H = 5\mathbb{Z}$ in the group $(\mathbb{Z}, +)$ are equal.
(a) $-1+H$ and $5+H$ (b) $3+H$ and $2+H$.

8. Find all subgroups of Klien's four group.

9. Prove that every group of order 4 is a commutative group.

10. Prove that every group of order 49 contains a subgroup of order 7.

11. Let G be a group such that $|G|<320$. Suppose G has subgroups of order 35 and 45. Find the order of G.

12. Let G be a group of order 15 and A and B subgroups of G of order 5 and 3, respectively. Show that $G = AB$.

13. Let A and B be two subgroups of a group G. If $|A| = p$, a prime integer, then show that either $A \cap B = \{e\}$ or $A \subseteq B$

14. Let $H = \{(1), (12)(34), (13)(24), (14)(23)\}$. Find the left cosets of H in A_4.

15. Let $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$. Find all left cosets of H in $\mathbb{Z}$.

16. Find all of the cosets of $\{1, 11\}$ in $U(30)$.

Cosets and Lagrange's Theorem

Answers.

1. Proof: To establish this, we need to show the existence of a bijective function from $\mathcal{L}$ onto $\mathcal{R}$. Define $f : \mathcal{L} \to \mathcal{R}$ by $f(aH) = Ha^{-1}$ for all $aH \in \mathcal{L}$. Observe that Ha^{-1} is a right coset of H in G and hence $Ha^{-1} \in \mathcal{R}$.

Now, we show that $aH = bH$ if and only if $Ha^{-1} = Hb^{-1}$.

Suppose $aH = bH$. Then $a^{-1}b \in H$. Hence $b^{-1}(a^{-1})^{-1} \in H$ and so by known theorem [Let H be a subgroup of a group G and let $a, b \in G$, $Ha = Hb$ if and only if $ba^{-1} \in H$] we have $Ha^{-1} = Hb^{-1}$.

Conversely, assume that $Ha^{-1} = Hb^{-1}$. Then by known theorem [Let H be a subgroup of a group G and let $a, b \in G$, $Ha = Hb$ if and only if $ba^{-1} \in H$], $b^{-1}(a^{-1})^{-1} \in H$, i.e., $b^{-1}a \in H$ and so $a^{-1}b = (b^{-1}a)^{-1} \in H$. Then by theorem [Let H be a subgroup of a group G and let $a, b \in G$, $aH = bH$ if and only if $a^{-1}b \in H$], $aH = bH$. Thus we find that f is well-defined and one-one. Since for all $Ha \in \mathcal{R}$, $Ha = H(a^{-1})^{-1} = f(a^{-1}H)$ and $a^{-1}H \in \mathcal{L}$, f is onto. Thus f is a one-one and onto mapping.

2 Sol'n: $S_3 = \{e, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}$,

$o(1\,2) = o(1\,3) = o(2\,3) = 2$, $o(1\,2\,3) = o(1\,3\,2) = 3$. Now

$\{e\}, \{e, (1\,2)\}, \{e, (1\,3)\}, \{e, (2\,3)\}, \{e, (1\,2\,3), (1\,3\,2)\}$

and S_3 itself are the nontrivial subgroups of S_3.
Let H be a subgroup of S_3. Now $|H|$ divides $|G|$. Thus, $|H| = 1, 2, 3,$ or 6. If $|H| = 1$, then $H = \{e\}$. If $|H| = 6$, then $H = S_3$. If $|H| = 2$, then H is a cyclic group of order 2. Hence H is one of $\{e, (1\,2)\}, \{e, (1\,3)\}, \{e, (2\,3)\}$.
Suppose $|H| = 3$. Then by Lagrange's theorem, H has no subgroup of order 2. Thus, $(1\,2), (1\,3), (2\,3) \notin H$. Hence $e, (1\,2\,3), (1\,3\,2) \in H$. Also $\{e, (1\,2\,3), (1\,3\,2)\}$ is a subgroup and so $H = \{e, (1\,2\,3), (1\,3\,2)\}$. Hence $H_0 = \{e\}$, $H_1 = \{e, (1\,2)\}$, $H_2 = \{e, (1\,3)\}$, $H_3 = \{e, (2\,3)\}$, $H_4 = \{e, (1\,2\,3), (1\,3\,2)\}$, and S_3 are the only subgroups of S_3.

Let H and K be two nontrivial distinct subgroups of S_3. Then $|H| = 2$ or 3 and $|K| = 2$ or 3. Also we note that $H \cap K = \{e\}$. Now $|H \cup K| = 3$ or 4. But there exists only one subgroup of order 3 in S_3 and a subgroup of order 3 cannot contain any subgroup of order 2. Also 4 does not divide $|S_3|$. Hence we find that $H \cup K$ is not a subgroup of S_3.

3. Sol^n: (i) Let $a \in G$. Since $a^{-1}a = e \in H$, we find that $(a, a) \in L_H$ for all $a \in G$. Let $a, b \in G$ such that $(a, b) \in L_H$. Then $a^{-1}b \in H$ and so $b^{-1}a = (a^{-1}b)^{-1} \in H$. Hence $(b, a) \in L_H$. Suppose now $(a, b) \in L_H$ and $(b, c) \in L_H$. Hence $a^{-1}b \in H$ and $b^{-1}c \in H$. Then $a^{-1}c = (a^{-1}b)(b^{-1}c) \in H$. Consequently,

$(a,c) \in L_H$. So it follows that L_H is an equivalence relation. 3

(ii) Let $[a]$ be an equivalence class of the relation L_H. Now,

$[a] = \{x \in G \mid (a,x) \in L_H\} = \{x \in G \mid a^{-1}x \in H\} = \{x \in G \mid x \in aH\} \subseteq aH$.

Again for any $ah \in aH$, $a^{-1}(ah) = h \in H$ implies that $(a, ah) \in L_H$. Hence $ah \in [a]$ and then $aH \subseteq [a]$. Consequently $[a] = aH$.

(iii) Let aH be a left coset. proceeding as in (ii), show that $[a] = aH$.

4. Let S be the set of all left cosets of H in G.

(a) $S = \{\{x, -x\} \mid x \in \mathbb{R}^+\}$

(b) $S = \{\{7n \mid n \in \mathbb{Z}\}, \{7n+1 \mid n \in \mathbb{Z}\}, \{7n+2 \mid n \in \mathbb{Z}\}, \{7n+3 \mid n \in \mathbb{Z}\}, \{7n+4 \mid n \in \mathbb{Z}\}, \{7n+5 \mid n \in \mathbb{Z}\}, \{7n+6 \mid n \in \mathbb{Z}\}\}$

(c) $S = \{H, \{(1\ 2), (1\ 2\ 3)\}, \{(1\ 3), (1\ 3\ 2)\}\}$

(d) $S = \{H, \{(1\ 2)\ (2\ 3)\ (1\ 3)\}\}$

7. (i) no (ii) no

8. Let $K_4 = \{e, a, b, c\}$; then $\{e\}$, $\{e,a\}$, $\{e,b\}$, $\{e,c\}$, K_4 are the only subgroups of it.

11. $|G| = 315$

14. $H = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\alpha_5 H = \{\alpha_5, \alpha_8, \alpha_6, \alpha_7\}$,

$\alpha_9 H = \{\alpha_9, \alpha_{11}, \alpha_{12}, \alpha_{10}\}$

15. H, $(1+H)$, $2+H$

16. $8/2 = 4$ so there are four cosets. Let $H = \{1, H\}$. The cosets are H, $7H$, $13H$, $19H$.

* Cyclic Groups *

Set-V

Practice Problems

1. Show that the 8th roots of unity form a cyclic group. Find all generators of this group.

2. Show that Z_{10}, the additive group of all integers modulo 10 is a cyclic group. Find all generators of Z_{10}.

3. The group $(Q,+)$ is not cyclic.

4. Prove that any finitely generated subgroup of $(Q,+)$ is cyclic.

5. Let G be a group of order 28. Show that G has a nontrivial subgroup.

6. If $G = \langle a \rangle$ is a cyclic group of order 30, then find all distinct elements of the subgroups (i) $\langle a^5 \rangle$ (ii) $\langle a^6 \rangle$.

7. Show that the 7th roots of unity form a cyclic group. Find all generators of this group.

8. Show that the cyclic group $(Z,+)$ has only two generators.

9. Is the group $(Z_{10},+)$ a cyclic group? If so, find all generators of this group and also find all its subgroups.

10. Show that for every positive integer n, the nth roots of unity form a cyclic group.

11. Show that $(Q^+,\cdot)$, $(Q^*,\cdot)$, $(R^+,\cdot)$, $(R^*,\cdot)$, $(C^*,\cdot)$ are not cyclic groups.

12. If a group G has only two subgroups, then prove that G is a cyclic group.

13. Let G be a cyclic group of order 42. Find the number of elements of order 6 and the number of elements of order 7 in G.

14. Let $G = \langle a \rangle$ be a cyclic group of order 20. Find all distinct elements of the subgroups (i) $\langle a^4 \rangle$ (ii) $\langle a^7 \rangle$.

15. Prove that every noncommutative group has a nontrivial cyclic group.

16. Let $G = \{a, b, c, d, e\}$ be a group. Complete the following Cayley table for this group.

*	e	a	b	c	d
e	e	a	b	c	d
a	a				
b	b		c	d	
c	c				
d	d				

17. Prove that any finite subgroup of the group of non-zero complex numbers is a cyclic group.

18. Let $G \neq \{e\}$ be a group of order p^n, p is a prime. Show that G contains an element of order p.

19. Prove that every proper subgroup of S_3 is cyclic.

20. Find all generators of Z_6, Z_8 and Z_{10}.

21. Suppose that $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$ are cyclic groups of order 6, 8 and 20 respectively. Find all generators of $\langle a \rangle$, $\langle b \rangle$ and $\langle c \rangle$.

22. List the elements of the subgroups $\langle 20 \rangle$ and $\langle 10 \rangle$ in Z_{30}.

23. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 15 \rangle$ in Z_{18}.

24. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 7 \rangle$ in $U(20)$.

25. List the cyclic subgroups of $U(30)$.

26. Let Z denote the group of integers under addition. Is every subgroup of Z cyclic? why? Describe all the subgroups of Z.

27. Find all generators of Z.

28. List all the elements of order 8 in $Z_{8000000}$. How do you know your list is complete.

29. Consider the set $\{4, 8, 12, 16\}$. show that this set is a group under multiplication modulo 20 by constructing its Cayley table. what is the identity element? Is the group cyclic? If so, find all of its generators.

30. List all the elements of Z_{40} that have order 10.

31. Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10.

32. Let a and b belong to a group. If $|a| = 24$ and $|b| = 10$, what are the possibilities for $|\langle a \rangle \cap \langle b \rangle|$?

33. Prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \middle/ n \in Z \right\}$ is a cyclic subgroup of $GL(2, R)$.

34. Let a and b belong to a group. If $|a| = 12$, $|b| = 22$, and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$, Prove that $a^6 = b^{11}$.

2. Sol: The group Z_{10} consists of all the following 10 distinct elements, viz., $[0], [1], [2], \ldots [9]$. Since $[m] = m[1]$ for $m = 0, 1, \ldots 9$, it follows that Z_{10} is generated by $[1]$. Hence Z_{10} is a cyclic group. Now an element $m[1]$, $(m = 1, 2, \ldots 9)$ is a generator of Z_{10} if and only if $\gcd(m, 10) = 1$. Hence $1[1], 3[1], 7[1]$, and $9[1]$ are the generators of Z_{10}, i.e., $[1], [3], [7]$ and $[9]$ are the generators of Z_{10}.

3. Sol'n: Suppose $(Q, +)$ is cyclic. Then $Q = \langle x \rangle$ for some $x \in Q$. clearly $x \neq 0$. Hence $x = \frac{p}{q}$, where p and q are integers prime to each other and $q \neq 0$. Since $\frac{p}{2q} \in Q$, there exists $n \in Z$, $n \neq 0$ such that $\frac{p}{2q} = n\frac{p}{q}$. This implies that $\frac{1}{2} = n \in Z$, which is a contradiction. Hence $(Q, +)$ is not cyclic.

4. Let H be any finitely generated subgroup of $(Q, +)$ and suppose $H = \left\langle \left\{ \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n} \right\} \right\rangle$. Let $x \in H$. Then $x = k_1\frac{p_1}{q_1} + k_2\frac{p_2}{q_2} + \cdots + k_n\frac{p_n}{q_n}$ for some $k_1, k_2 \ldots k_n \in Z$. Now,

$$x = \frac{\sum_{i=1}^{n} k_i p_i \bar{q}_i}{q_1 q_2 \cdots q_n} \quad \text{where } \bar{q}_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} q_j$$

Then it is easy to see that $x \in \left\langle \frac{1}{q_1 q_2 \cdots q_n} \right\rangle$ since $\sum_{i=1}^{n} k_i p_i \bar{q}_i \in Z$.

Thus $H \subseteq \left\langle \frac{1}{q_1 q_2 \cdots q_n} \right\rangle$, hence H become a subgroup of a cyclic group $\left\langle \frac{1}{q_1 q_2 \cdots q_n} \right\rangle$ and consequently H is cyclic.

Hence the result.

Cyclic Groups

Answers.

(1) sol'n: The 8^{th} roots of unity are

$$\alpha_K = \cos\frac{2K\pi}{8} + i\sin\frac{2K\pi}{8}, \quad K = 0, 1, 2, \cdots 7$$

Let $G = \{\alpha_0, \alpha_1, \cdots \alpha_7\}$.

Here we can easily show that G is a group of order 8.

Now

$$\alpha_K = \cos\frac{2K\pi}{8} + i\sin\frac{2K\pi}{8}$$

$$= \left(\cos\frac{2\pi}{8} + i\sin\frac{2\pi}{8}\right)^k = \alpha_1^k \text{ for } k = 0, 1, 2, \cdots 7$$

Hence we find that $G = \langle \alpha_1 \rangle$ and so, G is a cyclic group of order 8. Now for any integer $1 \le t < 8$, α_1^t is a generator of G if and only if $\gcd(t, 8) = 1$. Hence $\alpha_1^1, \alpha_1^3, \alpha_1^5$ and α_1^7 are generators of this cyclic group.

5. Sol'n: First suppose that G is cyclic. Then by theorem

[Let $G = \langle a \rangle$ be a cyclic group of order n.

(i) If H is a subgroup of G, then $|H|$ divides $|G|$.

(ii) If m is a positive integer such that m divides n, then there exists a unique subgroup of G of order m.]

for every positive divisor m of $|G|$, G has a subgroup of order m. Now 4 is a divisor of 28. So G has a subgroup of order 4. Hence there is a nontrivial subgroup of G. Now suppose that G is not cyclic. Let $e \neq a \in G$ and let H be the subgroup $\langle a \rangle$ generated by a. Then $H \neq \{e\}$. Also $G \neq H = \langle a \rangle$, as otherwise G becomes cyclic. Hence H is a proper subgroup of G.

6. Sol'n: (i) Here $\langle a^5 \rangle = \{(a^5)^n / n \in \mathbb{Z}\}$. Now $o(a) = |\langle a \rangle| = |G| = 30$. Hence $a^{30} = e$. Then $(a^5)^6 = e$ implies that $o(a^5) = 6$. Observe that the divisors of 6 are 1, 2, 3 and 6. Since $(a^5)^1 \neq e$, $(a^5)^2 \neq e$, $(a^5)^3 \neq e$ it follows that $o(a^5) = 6$. Hence,

$$\langle a^5 \rangle = \{(a^5)^0, (a^5)^1, (a^5)^2, (a^5)^3, (a^5)^4, (a^5)^5\}$$

$$= \{e, a^5, a^{10}, a^{15}, a^{20}, a^{25}\}$$

(ii) The order of a^6 is 5. Hence,

$$\langle a^6 \rangle = \{(a^6)^0, (a^6)^1, (a^6)^2, (a^6)^3, (a^6)^4\}$$

$$= \{e, a^6, a^{12}, a^{18}, a^{24}\}.$$

7. All non-identity elements.

9. Yes; [1], [3], [7], [9] are the generators. {[0]}, {[0], [5]}, {[0], [2], [4], [6], [8]} and Z_{10} are the only subgroups of Z_{10}.

13. 2 and 6

14. (i) $\{e, a^4, a^8, a^{12}, a^{16}\}$
(ii) G.

16.

*	e	a	b	c	d
e	e	a	b	c	d
a	a	d	e	b	c
b	b	e	c	d	a
c	c	b	d	a	e
d	d	c	a	e	b

17 Sol'n: Let H be a finite subgroup of $C^* = C\setminus\{0\}$. Let $|H| = n$ and $a \in H$. Then by the Knocon theorem [Let G be a group of finite order n and $a^n = e$], $a^n = 1$. Hence any element of H is a root of $x^n = 1$. On the otherhand, $x^n = 1$ has only n distinct roots. So it follows that $H = \{w \in C^* : w \text{ is a root of } x^n = 1\}$. we know that the set of nth roots of unity forms a cyclic group. Hence H is a cyclic subgroup of C^*.

18. Sol'n: Let $a \in G$, $a \neq e$. then $H = \langle a \rangle$ is a cyclic subgroup of G. Now $|H|$ divides $|G| = p^n$ and so $|H| = p^m$ for some $m \in Z$, $0 < m \leq n$. Now in a cyclic group of order p^m, for every

divisor d of p^m, there exists a subgroup of order d. Since p divides $|\langle a\rangle|$, there exists a subgroup T of H such that $|T| = p$. Let $T = \langle b\rangle$. Then $o(b) = p$. Hence the result.

20. For Z_6, generators are 1 and 5; for Z_8, generators are 1, 3, 5 and 7; for Z_{20}, generators are 1, 3, 7, 9, 11, 13, 17 and 19.

22. $\langle 20\rangle = \{20, 10, 0\}$
$\langle 10\rangle = \{10, 20, 0\}$

24. $\langle 3\rangle = \{3, 9, 7, 1\}$, $\langle 7\rangle = \langle 7, 9, 3, 1\rangle$

25. $\langle 1\rangle, \langle 7\rangle, \langle 11\rangle, \langle 17\rangle, \langle 19\rangle, \langle 29\rangle$.

26. Yes, by the known theorem [Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k-namely, $\langle a^{n/k}\rangle$], the subgroups of Z are of the form $\{0, \pm n, \pm 2n, \pm 3n, \ldots\}$ where n is any integer.

28. 1000000, 3000000, 5000000, 7000000
by the known theorem [Every subgroup of a cyclic group is a cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k- namely, $\langle a^{n/k}\rangle$], $\langle 1000000\rangle$ is the unique subgroup of order 8 and only those on the list are generators.

30. $4, 3\cdot 4, 7\cdot 4, 9\cdot 4$.

32. 1 and 2.

34. Use the fact the a cyclic group of even order has a unique element of order 2.

Set - VI Practice Probems.

* Normal Subgroups *

(1) Let H be a proper subgroup of a group G such that for all $x, y \in G \setminus H$, $xy \in H$. Prove that H is a normal subgroup of G.

(2)(a) Let H be a subgroup of a group G. Show that for any $g \in G$, $K = gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G and $|K| = |H|$.

(b) If H is the only subgroup of order n in a group G, then prove that H is a normal subgroup.

(3) Show that $K = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$ is a normal subgroup of A_4.

(4) Let $GL(2, \mathbb{R})$ denote the set of all non singular 2×2 matrices with real entries. Show that $SL(2, \mathbb{R}) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) : ad - bc = 1\right\}$ is a normal subgroup of the group $GL(2, \mathbb{R})$.

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(5) Let T denote the group of all non singular upper triangular 2×2 matrices with real entries, i.e, the matrices of the form, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Show that $H = \left\{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in T\right\}$ is a normal subgroup of T.

(6) In the symmetric group S_3, show that $H = \{e, (2\,3)\}$ is a subgroup but not a normal subgroup.

(7) Show that $H = \{e, (1\,2)(3\,4)\}$ is not a normal subgroup of A_4.

(8) In A_4, find subgroups H and K such that H is normal in K and K is normal in A_4, but H is not normal in A_4.

(9) Show that A_3 is a normal subgroup of S_3.

(10) Let G be a group and H a subgroup of G. If for all $a, b \in G$, $ab \in H$ implies $ba \in H$, then prove that H is a normal subgroup of G.

(11) Show that $12\mathbb{Z}$ is a normal subgroup of the group $(3\mathbb{Z}, +)$. Write the Cayley table for the factor group $3\mathbb{Z}/12\mathbb{Z}$.

(12) Write down the Cayley table for the quotient group $\mathbb{Z}/15\mathbb{Z}$

(13) Let $G = \langle a \rangle$ be the cyclic group such that $o(a) = 12$. Let $H = \langle a^4 \rangle$. Find the order of a^3H in G/H.

(14) Write down the Cayley table for the quotient group A_4/K, where $K = \{e, (1\ 2)(3\ 4), (1\ 4)(3\ 2), (1\ 3)(2\ 4)\}$. Is the group A_4/K a commutative group?

(15) Let $\mathbb{R}^*$ be the group of all non-zero real numbers under usual multiplication. Show that the set $\mathbb{R}^+$ of all positive real numbers is a subgroup of R^*. What is the index of R^+ in R^*?

(16) Let G be a group and $a \in Z(G)$. Prove that $H = \langle a \rangle$ is a normal subgroup.

(17) Let H be a normal subgroup of a group G. Prove that

(i) if G is commutative, then so is the quotient group G/H.

(ii) if G is cyclic, then so is G/H.

(18) Let G be a group. Let H be a subgroup of G such that $H \subseteq Z(G)$. Show that if G/H is cyclic, then $G = Z(G)$, i.e, G is abelian.

→(19). Let K be a normal subgroup of a group G such that $[G:K]=m$. If n is +ve integer such that $\gcd(m,n)=1$, then show that $K \supseteq \{g\in G \mid o(g)=n\}$

→(20). Let K be a normal subgroup of a finite group. If G/K has an element of order n, then show that G has an element of order n.

→(21) Let H be a subset of a group G and let the set $N(H)$, called the normalizer of H in G, be defined by $N(H)=\{a\in G \mid aHa^{-1}=H\}$. Prove that $N(H)$ is a subgroup of G. If in addition H be a subgroup of G, then prove that

(a) H is normal in $N(H)$

(b) N is normal in G if and only if $N(H)=G$.

(c) $N(H)$ is the largest subgroup of G in which H is normal, i.e, if H is normal in a subgroup K of G, then $K\subseteq N(H)$.

→(22). Let G be a group. Let H be a normal subgroup of G. Define the relation ρ_H on G by, for all $a,b\in G$, $a\rho_H b$ if and only if $a^{-1}b\in H$. Prove that (i) ρ_H is an equivalence relation on G.

[Note: An equivalence relation ρ on a group G is called a congruence relation if for all $a,b,c\in G$, $a\rho b$ implies that $ca\rho cb$ and $ac\rho bc$]

(ii) the ρ_H class $a\rho_H=\{b\in G \mid a\rho_H b\}$ is the left coset aH.

(iii) $H=e\rho_H$

(23). Let H be a subgroup of a group G. Define a relation ρ_H on G by $\rho_H = \{(a,b) \in G \times G \mid a^{-1}b \in H\}$. Show that if ρ_H is a congruence relation, then H is a normal subgroup of G.

(24). Let ρ be a congruence relation on a group G. Show that there exists a normal subgroup H of G such that

$$\rho = \{(a,b) \in G \times G \mid a^{-1}b \in H\}$$

(25). Prove that a non-empty subset H of a group G is normal subgroup of $G \Leftrightarrow$ for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$.

(26) If G is the union of proper normal subgroups such that any two of them have only e in common, then G is abelian.

(27) Show that a subgroup N of G is normal iff $xy \in N \Rightarrow yx \in N$.

(28). Let H be a subgroup of G and let $N = \bigcap_{x \in G} xHx^{-1}$ then show that N is a normal subgroup of G.

(29) Let H be a subset of a group G. Let $N(H) = \{x \in G \mid Hx = xH\}$ be the normalizer of H in G.

(i) If H is a subgroup of G then $N(H)$ is the largest subgroup of G in which H is normal.

(ii) If H is a subgroup of G then H is normal in G iff $N(H) = G$.

(iii) Show by an example, the converse of (ii) fails (if H is only a subset of G

(iv) If H is a subgroup of G and K is a subgroup of $N(H)$ then H is normal subgroup of HK

(30) Let H be normal in G such that $O(H)$ and $\frac{O(G)}{O(H)}$ are co prime. Show that H is unique subgroup of G of given order.

→(31) —

→(32) Let $\langle Z, + \rangle$ be the group of integers and let $N = \{3n \mid n \in Z\}$ then N is a normal subgroup of Z.

→(33) Let N be a normal subgroup of a group G. show that $o(Na) \mid o(a)$ for any $a \in G$.

→(34). If G is a group such that $\frac{G}{Z(G)}$ is cyclic, where $Z(G)$ is centre of G then show that G is abelian.

→(35) Give an example of an infinite group in which every element is of finite order

Answers

(1) Let $x \in G/H$. then $x^{-1} \in G \setminus H$. Let $y \in H$. Then $xy \in G \setminus H$. (for otherwise, $x = xyy^{-1} \in H$). Thus $xy, x^{-1} \in G \setminus H$. Hence $xyx^{-1} \in H$. Also for any $x \in H$, we have $xyx^{-1} \in H$. Thus H is a normal subgroup of G.

(2)(a) Let $a = ghg^{-1}$ and $b = gh_1g^{-1}$ be two elements of K. Then

$$ab^{-1} = ghg^{-1}(gh_1g^{-1})^{-1}$$

$$= ghg^{-1}\left((g^{-1})^{-1}h_1^{-1}g^{-1}\right)$$

$$= ghg^{-1}gh_1^{-1}g^{-1}$$

$$= ghh_1^{-1}g^{-1} \longrightarrow (*)$$

Now, $h, h_1 \in H$ and H is a subgroup of G. Hence $hh_1^{-1} \in H$. Then from $(*)$ above,

$$ab^{-1} = g(hh_1^{-1})g^{-1} \in gHg^{-1}$$

Hence K is a subgroup of G.

To show that $|K| = |H|$, we prove that there exists a bijective function from H onto gHg^{-1}. Define $f: H \to gHg^{-1}$ by $f(h) = ghg^{-1}$ for all $h \in H$. Let h_1 and $h_2 \in H$, such that $f(h_1) = f(h_2)$. Then $gh_1g^{-1} = gh_2g^{-1}$. By Cancellation, we obtain $h_1 = h_2$. Hence f is injective. Let $a \in gHg^{-1}$. Then $a = ghg^{-1}$ for some $h \in H$ and $f(h) = ghg^{-1} = a$. This implies f is surjective and so $|H| = |K|$.

(b) Let $g \in G$. from the above Problem, gHg^{-1} is a subgroup of G and $|H| = |gHg^{-1}|$.

Hence $|gHg^{-1}| = n$ and so by the given condition $gHg^{-1} = H$. This is true for all $g \in G$. thus we find that H is a normal subgroup of G.

(3). A_4 has 12 elements. these elements are $e, (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,3\,4), (1\,4\,3), (2\,3\,4), (2\,4\,3), (1\,2)(3\,4), (1\,4)(2\,3)$. Hence A_4 has no element of order 4. the only elements of order 2 are $a = (1\,2)(3\,4)$, $b = (1\,3)(2\,4)$, $c = (1\,4)(2\,3)$. Now $a^2 = b^2 = e$ and $ab = ba = c$. Hence $K = \{e, a, b, ab = c\}$ is a subgroup of order 4 and this is the only subgroup of order 4 in G.

$\therefore$ we conclude that K is a normal subgroup of G.

($\because$ we know that if H is the only subgroup of order n in a group G, then prove that H is a normal subgroup.)

(25) <u>sol'n</u>: Let H be normal subgroup of G.

Let $x, y \in H$, $g \in G$ be any elements,

then $(gx)(gy)^{-1} = (gx)(y^{-1}g^{-1}) = g(xy^{-1})g^{-1} \in H$

as $xy^{-1} \in H$, $g \in G$, H is normal in G.

Conversely, we show H is normal subgroup of G.

Let $x, y \in H$ be any elements,

then $xy^{-1} = exy^{-1}e = (ex)(ey)^{-1} \in H$ as $e \in G$

i.e., H is a subgroup of G.

Again let $h \in H$, $g \in G$ be any elements

then as $(gh)(ge)^{-1} \in H$

we get $(gh)(eg^{-1}) \in H$

$\Rightarrow ghg^{-1} \in H$

$\Rightarrow H$ is normal.

(26). Sol'n: Let $G = H_1 \cup H_2 \cup \cdots \cup H_k$

Let $x, y \in G$ be any elements, then $x \in H_i$, $y \in H_j$ for some i, j

Case(i): If $i \neq j$ then $xy = yx$

Case(ii): $i = j$, then $x, y \in H_i$.

Now since H_i is a proper subgroup of G, $\exists$ some $g \in G$ such that, $g \notin H_i$ (and $g \in H_t$ for some $t \neq i$).

We know that g commutes with both x and y.

i.e. $xg = gx$ and $yg = gy$

Now $g \notin H_i \Rightarrow gx \notin H_i$

$\therefore gx$ also commutes with x, y and $xy \in H_i$

Also $(xy)g = g(xy)$

$= (gx)y = y(gx)$

$= y(xg) = (yx)g$

$\Rightarrow xy = yx$ (Cancellation)

Hence G is abelian.

(27) Let N be normal in G and let $xy \in N$

Since $yx = y(xy)y^{-1}$

and $xy \in N$, $y \in G$, N is normal in G we find

$y(xy)y^{-1} \in N \Rightarrow yx \in N$

Conversely, let $n \in N$, $g \in G$ be any elements

then $n \in N \Rightarrow (ng)g^{-1} \in N$

$\Rightarrow g^{-1}(ng) \in N$ (given Condition)

$\Longrightarrow$ N is normal in G.

→(28) <u>sol'n</u>: we know that intersection of subgroups is a subgroup and also subsets of the type xHx^{-1} are subgroups.

Hence $\bigcap\limits_{x\in G} xHx^{-1}$ is a subgroup of G.

Let $g\in G$ be any element, then

$gNg^{-1} = g(\cap xHx^{-1})g^{-1} = \cap(gxHx^{-1}g^{-1}) = \cap(yHy^{-1}) = N$

showing thereby that N is normal.

we have used above the result $g(H\cap K) = gH\cap gK$ for subgroups H, K and $g\in G$, It is true as.

$x\in g(H\cap K) \Rightarrow x = ga,\ a\in H\cap K$

$a\in H \Rightarrow ga\in gH \Rightarrow x\in gH \Rightarrow x\in gH\cap gK$

$a\in K \Rightarrow ga\in gK \Rightarrow x\in gK$

Also $y\in gH\cap gK \Rightarrow y\in gH,\ y\in gK$

$\Rightarrow y = gh,\ y = gk \quad h\in H, k\in K$

$\Rightarrow gh = gk$

$\Rightarrow h = k \Rightarrow h, k\in H\cap K.$

$\therefore y = gh\in g(H\cap K)$ proving the result.

→(29) <u>sol'n</u>: (i) we show H is normal in N(H)

Since $Hh = hH$ for all $h\in H$, we find

$h\in N(H)$ for all $h\in H$

Thus $H\leq N(H)$.

Again by definition of N(H), $Hx = xH$ for all $x\in N(H)$

$\Rightarrow$ H is normal in N(H)

To show that N(H) is the largest subgroup of G in which H is normal. Suppose K is any subgroup of G such that H is normal in K.

then $k^{-1}Hk = H$ for all $k \in K$

$\Rightarrow Hk = kH$ for all $k \in K$

$\Rightarrow k \in N(H)$ for all $k \in K$

$\Rightarrow K \subseteq N(H)$

(ii) Let H be a normal subgroup of G

then $N(H) \subseteq G$ (by definition)

Let $x \in G$ be any element,

then $xH = Hx$ as H normal in G.

$\Rightarrow x \in N(H) \Rightarrow G \subseteq N(H)$

hence $G = N(H)$

Conversely, let $G = N(H)$

H is a subgroup of G (given)

Let $h \in H$, $g \in G$ be any elements

Then $g \in N(H)$ as $N(H) = G$

$\Rightarrow gH = Hg$

$\Rightarrow H$ is normal in G.

(iii) Consider $G = \langle a \rangle = \{e, a, a^2, a^3\}$

then G being cyclic is abelian group.

Take $H = \{a\}$

then H is a subset and not a subgroup of G ($e \notin H$)

Also $N(H) = G$ as G is abelian.

(iv) Let K be a subgroup of $N(H)$

then $k \in K \Rightarrow k \in N(H) \Rightarrow Hk = kH$

i.e. $Hk = kH$ for all $k \in K$

$\Rightarrow HK = KH$

$\Rightarrow HK$ is subgroup of $N(H)$

Note, $h \in H \Rightarrow Hh = hH \; (=H)$

$\Rightarrow H \subseteq N(H)$ Also $K \subseteq N(H)$

Again $H \subseteq HK \subseteq N(H)$

hence H is a subgroup of HK

$\Rightarrow H$ is a subgroup of HK

$[a \in HK \Rightarrow a \in N(H) \Rightarrow Ha = aH]$

(30) Sol'n: Let $o(H) = m$, $\frac{o(G)}{o(H)} = n$. Suppose K is a subgroup of G of order m.

Then $o(HK) = \frac{m.m}{d}$; where $d = o(H \cap K)$

Since H is normal, $HK \leq G$.

Thus $o(HK) \backslash o(G)$

$\Rightarrow m \cdot \frac{m}{d} \Big| m.n \Rightarrow \frac{m}{d} \Big| n$

$\Rightarrow d \frac{m}{d} \Big| dn \Rightarrow m | dn$

$\Rightarrow m | d$ as $(m, n) = 1$

But $d | m$ as $H \cap K \leq H$

Thus $d = m$ and hence

$o(H \cap K) = o(H) = o(K)$

$\Rightarrow H = K$

(31) Let G be the set of 2×2 matrices over reals of the type $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where $ad \neq 0$. Then it is easy to see that G will form a group under matrix multiplication. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ will be identity, $\begin{bmatrix} 1/a & -b/ad \\ 0 & 1/d \end{bmatrix}$ will be inverse of any element $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. Also G is not abelian.

Let N be the subset containing members of the type $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$.

Then N is a subgroup of G. (Prove!) Also it is normal as the product of the type.

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} \\ 0 & \frac{1}{d} \end{bmatrix} = \begin{bmatrix} 1 & akd + bd - b/d \\ 0 & 1 \end{bmatrix} \in N.$$

So we get the quotient group $\frac{G}{N}$. we show $\frac{G}{N}$ is abelian.

Let $Nx, Ny \in \frac{G}{N}$ be any elements, then $x, y \in G$

Let $x = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, $y = \begin{bmatrix} c & e \\ 0 & f \end{bmatrix}$

$\frac{G}{N}$ will be abelian iff $NxNy = NyNx$

$$\Leftrightarrow Nxy = Nyx$$

$$\Leftrightarrow xy(yx)^{-1} \in N$$

$$\Leftrightarrow xyx^{-1}y^{-1} \in N$$

All we need check now is that the product

$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} c & e \\ 0 & f \end{bmatrix}\begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} \\ 0 & \frac{1}{d} \end{bmatrix}\begin{bmatrix} \frac{1}{c} & -\frac{e}{cf} \\ 0 & \frac{1}{f} \end{bmatrix}$ is a matrix of the type $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$.

thus we can have an abelian quotient group, without the 'parent' group being abelian

→(32) sol'n $\frac{Z}{N}$ will consist of members of the type $N+a, a \in Z$.

we show $\frac{Z}{N}$ contains only three elements. Let $a \in Z$ be any element, where $a \neq 0,1,2$ then we can write, by division algorithm.

$$a = 3q + r \text{ wher } 0 \le r \le 2.$$

$$\Rightarrow N+a = N+(3q+r) = (N+3q)+r = N+r \text{ as } 3q \in N.$$

but r can take values $0,1,2$.

Hence $N+a$ will be one of

$$N, N+1, N+2$$

or that $\frac{Z}{N}$ contains only these three members.

Remarks: (i) This example also tells us that in case of cosets,
Ha=Hb may not necessarily mean a=b.
(ii) This serves as an example of an infinite group which has a subgroup N having finite index in G.

→ (83) Sol^n: Let $o(a)=n$
then n is the least +ve integer such that $a^n=e$.
This gives $Na^n = Ne$

$$\Rightarrow N\underbrace{a.a \ldots a}_{(n\text{ times})} = N$$

$$\Rightarrow \underbrace{Na.Na \ldots Na}_{(n\text{ times})} = N$$

$$\Rightarrow (Na)^n = N; \; Na \in \frac{G}{N} \text{ and } N \text{ is identity of } \frac{G}{N}$$

$$\Rightarrow o(Na)|n \text{ or } o(Na)|o(a)$$

→ (84). Sol^n: Let us write $Z(G)=N$. Then $\frac{G}{N}$ is cyclic, suppose it is generated by Ng.
Let $a,b \in G$ be any two elements.
then $Na, Nb \in \frac{G}{N}$

$$\Rightarrow Na = (Ng)^n, \; Nb = (Ng)^m \text{ for some } n,m$$

$$\Rightarrow Na = Ng.Ng \ldots Ng = Ng^n$$

$$Nb = Ng^m$$

$$\Rightarrow ag^{-n} \in N, \; bg^{-m} \in N$$

$$\Rightarrow ag^{-n} = x, \; bg^{-m} = y \text{ for some } x,y \in N$$

$$\Rightarrow a = xg^n, \; b = yg^m$$

$$\Rightarrow ab = (xg^n)(yg^m) = x(g^n y)g^m$$

$$= x(yg^n)g^m \text{ as } y \in N = Z(G)$$

$= xyg^n g^m$

$= xyg^{n+m}$

similarly $ba = (yg^m)(xg^n) = y(g^m x)g^n = y(xg^m)g^n$

$= (yx)g^{m+n}$

$\Rightarrow ab = ba$ as $xy = yx$ as $x, y \in Z(G)$

showing that G is abelian.

Remarks (i): (i) we are talking about $\frac{G}{Z(G)}$ assuming, therefore, that $Z(G)$ is a normal subgroup of G.

(ii) one can, moving on same lines as in the above solution prove that if G/H is cyclic, where H is a subgroup of $Z(G)$ then G is abelian

(iii) If G is a non-abelian group then $G/Z(G)$ is not cyclic.

→(35) sol'n: Let $\langle Z, + \rangle$ be the group of integers under addition.

Let $G = \left\{ Z + \frac{m}{p^n} \,\middle|\, \begin{matrix} m \\ n \geq 0 \end{matrix}, n \text{ are integers}, p = \text{fixed prime} \right\}$

Then G is a subgroup of $\frac{Q}{Z}$ where $\langle Q, + \rangle$ is the group of rationals under addition.

Now $p^n \left(Z + \frac{m}{p^n} \right) = Z + \frac{m}{p^n} p^n = Z + m = Z =$ zero of G

$\Rightarrow$ order of $Z + \frac{m}{p^n}$ divides p^n.

$\Rightarrow$ order of $Z + \frac{m}{p^n}$ is $p^r, r \leq n$

$\Rightarrow$ order of every element in G is finite.

Here G is an infinite group.

In fact, one can also show that every subgroup $H \neq G$ is of finite order. So, this also gives an example of an infinite group in which every proper subgroup is of finite order.