Mains Test Series - 2021

Test-10, Paper-II

Answer Key

Batch-I, full Syllabus

1(a) If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G.

Sol^n: Let $g \in G$, so that $g^{-1} \in G$

then $(g^{-1})^2 \in H$ (by hyp)

$\Rightarrow g^{-2} \in H$

and $h^{-1}g^{-2} \in H \;\forall\; h \in H$ ($\because$ H is subgroup & $h^{-1} \in H$)

Since $gh \in G \Rightarrow (gh)^2 \in H$ (by hypothesis)

Now $(gh)^2 \in H$ and $h^{-1}g^{-2} \in H$

$\Rightarrow (gh)^2\, h^{-1}g^{-2} \in H$ ($\because H < G$)

$\Rightarrow gh\,gh\,h^{-1}g^{-1}g^{-1} \in H$

$\Rightarrow gh\,g\,e\,g^{-1}g^{-1} \in H$

$\Rightarrow ghg\,g^{-1}g^{-1} \in H$

$\Rightarrow ghg^{-1} \in H \;\forall\; g \in G, h \in H$ ($\because geg^{-1} = gg^{-1} = e$)

Hence H is normal subgroup of G.

1.(b) → Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f: \mathbb{C} \to S$ defined by $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.)

Ans. TS: $(S, +, \cdot)$ is a field.

1) $(S, +)$ is an abelian group.

Closure: Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S$

then, clearly, $A+B = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$, $a, b, c, d \in \mathbb{R}$
$\Rightarrow a+c, -(b+d), b+d, a+c \in \mathbb{R}$

$\Rightarrow A+B \in S$

Associativity: $\forall A, B, C \in S$

$\Rightarrow (A+B)+C = A+(B+C)$

Since, matrix addition is associative in nature.

Identity: $\forall A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$,
such that $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity of S.

Inverse: $\forall A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element $\begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} \in S$

such that

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} -a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a & b \\ -b & -a \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Commutative: $\forall A, B \in S$ such that,

we have, $A+B = B+A$

as $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} + \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$\forall a, b, c, d \in \mathbb{R}$

ii) $(S,\cdot)$ is an abelian group.

Commutative: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix},\ B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S,\quad a,b,c,d \in \mathbb{R}$

we have, $A\cdot B = \begin{bmatrix} ac & -bd \\ -bd & ac \end{bmatrix} \in S \quad \therefore\ ac, bd \in \mathbb{R}$

Associative: $\forall\ A,B,C \in S$

we have, $(AB)C = A(BC)$

As, Matrix Multiplication is Associative in nature.

Identity: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $\exists$ a unique element $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$

such that $AI = A = I\cdot A$.

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity of S.

Inverse: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S$, $\exists$ a unique element

$B = \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}$, for $a, b \neq 0$ (not both zero) such that

$AB = BA = I$.

$\Rightarrow B = A^{-1}$ is the inverse element.

Commutative: $\forall\ A=\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S,\ B=\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \in S$, we have

$AB = \begin{bmatrix} ac-bd & bc+ad \\ -bc-ad & -bd+ac \end{bmatrix}^T = BA = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & -bd+ac \end{bmatrix}$

$\therefore (S,\cdot)$ is an abelian group.

iii) Distributive laws :-

$\forall\ A, B, C \in S$

we have, $A\cdot(B+C) = A\cdot B + A\cdot C$

$(B+C)\cdot A = B\cdot A + C\cdot A$

Thus, distribution property is satisfied.

Hence $(S,+,\cdot)$ is a field i.e. a commutative division ring with unity.

The additive identity of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and multiplicative identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The inverse of $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

T.S: $f : \mathbb{C} \longrightarrow S$ defined by $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is an isomorphism.

Well Defined: Let $a+ib = c+id$

$$\Rightarrow a=c, \ b=d$$

$$\therefore \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$\Rightarrow f(a+ib) = f(c+id)$$

$\therefore f$ is well defined.

1-1: Let $f(a+ib) = f(c+id)$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$\Rightarrow a=c \text{ and } b=d$$

$$\therefore a+ib = c+id$$

$\Rightarrow f$ is 1-1.

Onto: $\forall \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in S$, $\exists$ a unique preimage $a+ib \in \mathbb{C}$, $a, b \in \mathbb{R}$ such that $f(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Homomorphism: Let $a+ib, c+id \in \mathbb{C}$ be any two arbitrary elements.

$$f((a+ib)+(c+id)) = f(a+c+i(b+d))$$
$$= \begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix}$$
$$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = f(a+ib) + f(c+id)$$

Now, $f((a+ib)\cdot(c+id)) = f((ac-bd)+i(ad+bc))$

$$= \begin{pmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$
$$= f(a+ib) \cdot f(c+id)$$

$\Rightarrow f$ is an homomorphism map.

Thus, f is an isomorphism from $\mathbb{C}$ to S.

1.(C) Let $f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{|x|}{2}+1 & \text{if } 2\le x. \end{cases}$

What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify your answers.

Sol^n: Given that

$$f(x) = \begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{|x|}{2}+1 & \text{if } 2\le x. \end{cases}$$

i.e., $$f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<1 \\ \frac{x}{2}+1 & \text{if } 1\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x. \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -\frac{x}{2}+1 & \text{if } x<0 \\ \frac{x}{2}+1 & \text{if } 0\le x<2 \\ -\frac{x}{2}+1 & \text{if } 2\le x. \end{cases}$$

f is linear function over the various subintervals.

$\Rightarrow$ f is continuous and differentiable over each subinterval.

The only doubtful points are the breaking points $x=0$ and $x=2$

At $x=0$, $f(x) = \frac{0}{2}+1 = 1$

i.e., $f(0)=1$

Now

LHL: $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -\frac{x}{2}+1$

$= 1$

RHL: $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{x}{2}+1$

$= 1$

$$\Rightarrow \lim_{x\to 0-} f(x) = 1 = \lim_{x\to 0+} f(x)$$

$\therefore$ f is continuous at $x=0$.

Also

<u>LHD</u>: $$Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0-} \frac{-\frac{x}{2}+1-1}{x}$$

$$= \lim_{x\to 0-} \frac{-x}{2x}$$

$$= -\frac{1}{2}.$$

<u>RHD</u>:

$$Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0+} \frac{\frac{x}{2}+1-1}{x}$$

$$= \lim_{x\to 0+} \frac{x}{2}/x = \frac{1}{2}.$$

$$\Rightarrow Lf'(0) \neq Rf'(0)$$

$\therefore$ f is not differentiable at $x=0$.

At $x=2$, $f(x) = -\frac{2}{2}+1 = 0$; i.e. $f(2)=0$

Now <u>LHL</u>: $\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \frac{x}{2}+1 = 1+1 = 2$

<u>RHL</u>: $\lim_{x\to 2+} f(x) = \lim_{x\to 2+} -\frac{x}{2}+1 = -1+1 = 0$

$$\Rightarrow \lim_{x\to 2-} f(x) \neq \lim_{x\to 2+} f(x)$$

$\therefore$ f is not continuous at $x=2$

$\therefore$ f is not differentiable at $x=2$

Hence f is continuous for all values of x except at $x=2$.

and also f is differentiable for all values of x except at $x=0$ & $x=2$.

1.(d) If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, find $f(z)$ subject to the condition $f(\pi/2)=\dfrac{3-i}{2}$.

Sol'n: we have $u+iv=f(z)$ $\therefore iu-v=if(z)$

On adding, we have

$u-v+i(u+v)=(1+i)f(z)=F(z)$ say

i.e. $(u-v)+i(u+v)=F(z)$

Let $U=u-v$, and $V=u+v$, then $U+iV=F(z)$ is an analytic function.

Here $U=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}=\dfrac{\cosh y+\sinh y-\cos x+\sin x}{\cosh y-\cos x}$

$$=1+\frac{\sinh y+\sin x}{\cosh y-\cos x}=\left\{1-\frac{\sin x+\sinh y}{\cos x-\cosh y}\right\}$$

$$\therefore \frac{\partial U}{\partial x}=\frac{-1-\sin x\sinh y+\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_1(x,y)$$

and $$\frac{\partial U}{\partial y}=\frac{1-\sin x\sinh y-\cos x\cosh y}{(\cos x-\cosh y)^2}=\phi_2(x,y)$$

By Milne's method we have

$$F'(z)=[\phi_1(z,0)-i\phi_2(z,0)]$$

$$=-\frac{1}{1-\cos z}-i\frac{1}{1-\cos z}$$

$$=-(1+i)\frac{1}{1-\cos z}=-\tfrac{1}{2}(1+i)\operatorname{cosec}^2\frac{z}{2}$$

Integrating we get

$$F(z)=-\tfrac{1}{2}(1+i)\int\operatorname{cosec}^2 z/2+c=(1+i)\cot z/2+c$$

i.e. $(1+i)f(z)=(1+i)\cot z/2+c$

$\Rightarrow f(z)=\cot z/2+c_1$

But when $z=\pi/2$, $f(\pi/2)=\dfrac{3-i}{2}$

$\therefore c_1=\dfrac{3-i}{2}-1=\dfrac{1-i}{2}$

Hence $f(z)=\cot z/2+\tfrac{1}{2}(1-i)$

1.(e) Consider the following LPP.

Max. $Z = x_1 + 2x_2 - 3x_3 + 4x_4$ subject to

$$x_1 + x_2 + 2x_3 + 3x_4 = 12$$
$$x_2 + 2x_3 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Sol: Since, there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero. and then solving the resulting equations. Also the total numbers of basic solution $= 4C_2 = 6$. The Characteristics of the various basic solutions are given below.

No. of basic Solution	Basic Variables	Non-basic Variables	Values of basic variables	Is the Solution feasible? (Are all $x_j \geq 0$)	Value of Z	Is the Solution optimal?
1	x_1, x_2	$x_3=0$ $x_4=0$	$x_1+x_2=12$ $x_2=8$ $\therefore x_1=4$	Yes	20	Yes
2.	x_1, x_3	$x_2=0$ $x_4=0$	$x_1+2x_3=12$ $2x_3=8$ $x_3=4, x_1=4$	Yes	-8	—
3.	x_1, x_4	$x_2=0$ $x_3=0$	$x_1+3x_4=12$ $x_4=8$ $x_1=-12$ $x_4=8$.	NO	-20	No

4	x_2, x_3	$x_1=0$ $x_4=0$	$x_2+2x_3=12$ $x_2+2x_3=8$			No
5.	x_2, x_4	$x_1=0$ $x_3=0$	$x_2+3x_4=12$ $x_2+x_4=8$ $\therefore x_4=2$ $x_2=6$	Yes	6	No
6.	x_3, x_4	$x_1=0$ $x_2=0$	$2x_3+3x_4=12$ $2x_3+x_4=8$ $x_4=2$ $x_3=3$	Yes	-1.	No

Hence, the optimal basic feasible solution is ; $x_1=4,\ x_2=8$
$x_3=0,\ x_4=0$

and the Max. value of $Z = \underline{\underline{20}}$.

2.(a) → (i) How many generators are there of the cylic group G of order 8 ? Explain.

Sol'n :(i) we have $G = \langle a \rangle$, $a^8 = e$.

All the generators of G are a^1, a^3, a^5, a^7

i.e. $G = \langle a \rangle = \langle a^3 \rangle = \langle a^5 \rangle = \langle a^7 \rangle$.

[$\because$ 1, 3, 5, 7 are positive integers less than 8 and prime to 8]

2(a)(ii) Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic.

Sol'n : Consider the Klein 4 group: $V = \{e, a, b, c = ab\}$ whose multiplication table is :

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Here, e is the identity element. Observe that if H is any subgroup containing more than 2 elements, it must be the entire group, and hence not proper.

The square of any element is the identity, therefore we have subgroups containing 2 elements and hence are isomorphic ~~to~~ Z_2 (hence cyclic). A subgroup with 1 element is, of course, the trivial subgroup which is the cyclic group generated by e. Hence every proper subgroup of V is cyclic. But V itself is not cyclic since it does not contain an element of order 4.

2(b) Let $f_n(x) = \begin{cases} 0 & , \text{ if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x} & , \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & ; \text{ if } x > \frac{1}{n} \end{cases}$

Show that $f_n(x)$ converges to a continuous function but not uniformly.

Solution:-

Given; $f_n(x) = \begin{cases} 0 & ; \text{ if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x} & ; \text{ if } \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & ; \text{ if } x > \frac{1}{n} \end{cases}$

By Archemedi's theorem, there exist 'n' such that $xn > 1$ for $x \in R$

i.e. $x > \frac{1}{n}$; this implies $f_n(x) = 0$

So; $\lim\limits_{x\to 0} f_n(x) = f(x) = 0$

So, $f_n(x)$ converges to $f(x) = 0$.

For Uniform Convergence:

Consider;

$$|f_n(x) - f(x)| = \begin{cases} 0 & ; \ x < 1/n+1 \\ \left|\sin\frac{\pi}{x}\right| & ; \ \frac{1}{n+1} < x < \frac{1}{n} \\ 0 & ; \ x > \frac{1}{n} \end{cases}$$

Let ; $M_n = \sup |f_n(x) - f(x)|$

$= \sup \left|\sin\frac{\pi}{x}\right|$; $\frac{1}{n+1} < x < \frac{1}{n}$

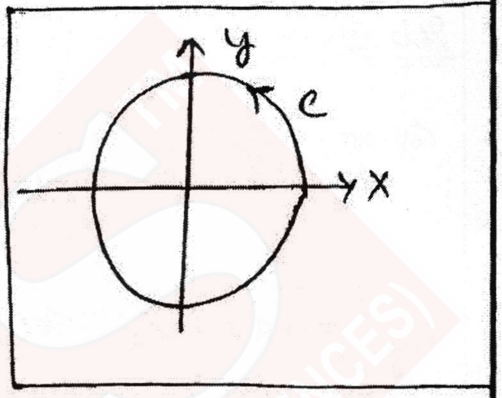
$= 1.$

2(c) → If α, β, γ are real numbers such that $\alpha^2 > \beta^2+\gamma^2$. Show that :

$$\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}}$$

Solution:-

Let $z = e^{i\theta}$, then



$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

$$dz = i.e^{i\theta}\,d\theta$$

$$\boxed{dz = iz\,d\theta}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \oint_C \frac{dz}{iz\left[\alpha+\beta\left(\frac{z+z^{-1}}{2}\right)+\gamma\left[\frac{z-z^{-1}}{2i}\right]\right]}$$

$$= \oint_C \frac{2i\,dz}{\cancel{i}z\left[2\alpha i+\frac{2i\beta}{\cancel{2}}(z+z^{-1})+\gamma(z-z^{-1})\right]}$$

$$= \oint_C \frac{2\,dz}{2\alpha zi+z\left[\frac{z^2\beta i+\beta i}{z}+\frac{\gamma(z^2-1)}{z}\right]}$$

$$= \oint_C \frac{2\,dz}{2\alpha zi+z^2\beta i+\beta i+\gamma z^2-\gamma}$$

$$= \oint_C \frac{2\,dz}{z^2(\beta i+\gamma)+2\alpha zi+(\beta i-\gamma)}$$

$$= \oint_C \frac{2\,dz}{(\gamma+i\beta)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left[\frac{\beta i-\gamma}{\gamma+\beta i}\right]\right]}$$

$$= \oint_c \frac{2\,dz}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\frac{-\gamma+\beta i}{\gamma+\beta i}\right]} \quad \text{--- (1)}$$

where 'c' is the circle of unit radius with centre at the origin.

The poles of $\dfrac{2}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$

are the single poles –

$$z = \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \sqrt{\frac{-4(\alpha^2)}{(\gamma+\beta i)^2} - 4(1)\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)}}{2}$$

$$z = \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \frac{2}{\gamma+\beta i}\sqrt{-\alpha^2-(-\gamma+\beta i)(\gamma+\beta i)}}{2}$$

$$= \frac{-\frac{2\alpha i}{\gamma+\beta i} \pm \frac{2i}{\gamma+\beta i}\sqrt{\alpha^2+(-)(\gamma^2+\beta^2)}}{2}$$

$$\Rightarrow \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-\gamma^2-\beta^2}$$

or

$$z = \frac{-\alpha i}{\gamma+\beta i} \pm \frac{i}{\gamma+\beta i}\sqrt{\alpha^2-(\gamma^2+\beta^2)} \quad \text{--- (2)}$$

Given, $\alpha^2 > \beta^2+\gamma^2$.

$\therefore\ z = \dfrac{-\alpha i + i\sqrt{\alpha^2-(\gamma^2+\beta^2)}}{\gamma+\beta i} = z_1$ (say) lies inside 'c' because $|z|<1$.

Now residue of z_1

$$\Rightarrow \lim_{z\to z_1} (z-z_1)\frac{2}{(\gamma+\beta i)\left[z^2+\left(\frac{2\alpha i}{\gamma+\beta i}\right)z+\left(\frac{-\gamma+\beta i}{\gamma+\beta i}\right)\right]}$$

$$\Rightarrow \lim_{z\to z_1} \frac{2}{\cancel{(\gamma+\beta i)}\left[\frac{z+\alpha i+i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}{\cancel{\gamma+\beta i}}\right]}$$

$$\Rightarrow \frac{1}{i\sqrt{\alpha^2-(\beta^2+\gamma^2)}}.$$

$$\therefore (1) \equiv \oint_0^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = 2\pi i\times\left(\text{residue at } z_1\right)$$

$$\therefore \oint_0^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = 2\pi \cancel{i}\times\frac{1}{\cancel{i}\sqrt{\alpha^2-(\beta^2+\gamma^2)}}$$

$$\therefore \boxed{\oint_c^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2-(\beta^2+\gamma^2)}}}$$

Hence proved

3(a) Let 'S' be the set of all real numbers except -1. Define $a*b = a+b+ab$. Is $(S,*)$ a group? Find the solution of the equation $2*x*3=7$ in S.

Sol^n: (a) Since S is the set of all real numbers except -1 and $*$ is an operation. defined in $S = R - \{-1\}$ such that

$$a*b = a+b+ab \quad \forall\, a,b \in S$$

$\therefore$ $a*b \in S$

$\therefore$ $*$ is a binary operation on S

$\therefore$ $a*b = a+b+ab \quad \forall\, a,b \in S.$ — ①

(b)(i) Closure prop:

$\forall\, a,b \in S$

$a*b = a+b+ab \in S$ by ①

$\therefore$ S is closed under $*$.

If possible let $a*b=-1$
$\Rightarrow a+b+ab=-1$
$\Rightarrow (a+1)+b(a+1)=0$
$\Rightarrow (a+1)(b+1)=0$
$\Rightarrow a+1=0$ (or) $b+1=0$
$\Rightarrow a=-1$ or $b=-1$
clearly which is contradiction to hypo. that $a\neq -1, b\neq -1$ in S.

(ii) Associative prop:

$\forall\, a,b,c \in S \Rightarrow (a*b)*c = (a+b+ab)*c$

$= a+b+ab+c+(a+b+ab)c$

$= a+b+c+ab+bc+ca+abc$

Similarly $a*(b*c) = a+b+c+ab+bc+ca+abc$

$\therefore$ $(a*b)*c = a*(b*c)$

$\therefore$ Associative law holds.

(iii) Existence of left identity:

Let $a \in S$, $e \in S$ then $e*a=a$

Now $e*a=a$

$\Rightarrow e + a + ea = a$

$\Rightarrow e(1+a) = 0$

$\Rightarrow e = 0 \in S \quad (\because a \neq -1)$

$\therefore e * a = 0 * a$

$= 0 + a + 0(a)$

$= a$

$\therefore \forall\, a \in S \;\exists\; 0 \in S$ such that $0 * a = a$.

$\therefore 0$ is the left identity in S.

(iv) Existance of left inverse :

Let $a \in S$, $b \in S$ then $b * a = e$

Now $b * a = a$

$\Rightarrow b + a + ba = 0 \quad (\because e = 0)$

$\Rightarrow b(1+a) = -a$

$\Rightarrow b = \frac{-a}{1+a} \in S \quad (\because a \neq -1)$

$\therefore b * a = \frac{-a}{1+a} * a$

$= \frac{-a}{1+a} + a + \left(\frac{-a}{1+a}\right)a$

$= \frac{-a}{1+a} + a - \frac{a^2}{1+a}$

$= \frac{-a + a(1+a) - a^2}{1+a} = 0$

For each $a \in S$ $\exists$ $b = \frac{-a}{1+a} \in S$ such that $\frac{-a}{1+a} * a = 0$

$\therefore b = \frac{-a}{1+a}$ is left inverse of a in S w.r.t $*$

$\therefore (S, *)$ is a group.

(c) $2 * x * 3 = 7$

$\Rightarrow (2 + x + 2x) * 3 = 7$ by ①

$\Rightarrow (2+3x) * 3 = 7$

$\Rightarrow (2+3x) + 3 + (2+3x)3 = 7$ by ①

$\Rightarrow 5 + 3x + 6 + 9x = 7$

$\Rightarrow 11 + 12x = 7$

$\Rightarrow 12x = -4$

$\Rightarrow x = -\frac{1}{3} \in S$

Now $2 * (-\frac{1}{3}) * 3 = \left[2 + (-\frac{1}{3}) + 2(-\frac{1}{3})\right] * 3$ by ①

$= \left(\frac{5}{3} - \frac{2}{3}\right) * 3$

$= 1 * 3$

$= 1 + 3 + 3$

$= 7$

$\therefore x = -\frac{1}{3}$ is a solution of the equation $2 * x * 3 = 7$ in S.

3(b) Examine the series

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

for uniform convergence. Also, with the help of this example, show that the condition of uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for the sum $S(x)$ of the series to be continuous.

Sol^n: Here $u_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$

$$\therefore\ u_1(x) = \frac{x}{1+x^2} - 0$$

$$u_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$

$$u_3(x) = \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}$$

$$\text{-- -- -- -- -- -- --}$$

$$u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

$$\Rightarrow S_n(x) = \frac{nx}{1+n^2x^2}$$

Here $u_n(x) = \lim\limits_{n\to\infty} u_n(x) = \lim\limits_{n\to\infty} \dfrac{nx}{1+n^2x^2}$

$$= \lim_{n\to\infty} \frac{x/n}{\frac{1}{n^2}+x^2} = 0 \ \forall\ x\in R$$

Let $\varepsilon > 0$ be given. Then

$$|u_n(x) - u(x)| = \left| \frac{nx}{1+n^2x^2} - 0 \right|$$

$$= \frac{n|x|}{1+n^2x^2} < \varepsilon$$

If $n|x| = \varepsilon + n^2x^2\varepsilon$

i.e. if $\varepsilon x^2 n^2 - |x| n + \varepsilon > 0$

i.e. if $n > \dfrac{|x| + \sqrt{x^2 - 4\varepsilon^2 x^2}}{2\varepsilon x^2}$

i.e. if $n > \dfrac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon|x|}$

If we choose a positive integer m just $\geq \dfrac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon|x|}, x \neq 0,$

then $|u_n(x) - u(x)| < \varepsilon \ \forall\ n \geq m$ and $x \neq 0$

Thus the sequence $\langle u_n \rangle$ is uniformly convergent in every interval which doesnot contain 0.

But, when $x \to 0$, $\dfrac{1+\sqrt{1-4\varepsilon^2}}{2\varepsilon|x|} \to \infty$ so that it is not possible to choose a positive integer m such that

$|u_n(x) - u(x)| < \varepsilon \ \forall\ n \geq m$ and $\forall\ x \in R$.

Hence $x = 0$ is a point of non-uniform convergence.

If $u_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$,

then $S_n(x) = nx/(1+n^2x^2)$ is continuous on R and converges to the continuous sum function 0 on R but the convergence of $S_n(x)$ is not uniform on R.

3.(c) Maximize $z = 2x_1 + 3x_2 + 6x_3$

subject to
$$2x_1 + x_2 + x_3 \le 5$$
$$3x_2 + 2x_3 \le 6$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$$

Is the optimal solution unique ? Justify your answer.

Solution:

Max $z = 2x_1 + 3x_2 + 6x_3$

S.T $2x_1 + x_2 + x_3 \le 5$; $3x_2 + 2x_3 \le 6$; $x_i \ge 0$

Soln: Standard form of given problem.

max $z = 2x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2$

$$\left.\begin{aligned} \text{S.T} \quad 2x_1 + x_2 + x_3 + s_1 + 0s_2 &= 5 \\ 0x_1 + 3x_2 + 2x_3 + 0s_1 + s_2 &= 6 \end{aligned}\right] \text{—(i)}$$

$\therefore$ IBFS $= (x_1, x_2, x_3, s_1, s_2) = (0, 0, 0, 5, 6)$

Now,

	C_j	2	3	6	0	0			
CB	Basic	x_1	x_2	x_3	s_1	s_2	B	θ	
0	s_1	2	1	1	1	0	5	5	$\therefore$ 2 = key element
0	s_2	0	3	2	0	1	6	3 ←	$\therefore$ e.v = x_3
$Z_j =$	$\Sigma C_B a_{ij}$	0	0	0	0	0			$\therefore$ O.V = s_2
$C_j =$	$c_j - z_j$	2	3	6 ↑	0	0			
0	s_1	2	$-1/2$	0	1	$-1/2$	2	1 ←	2 = key element
6	x_3	0	$3/2$	1	0	$1/2$	3	—	$\therefore$ e.v = x_1
Z_j	=	0	9	6	0	3			$\therefore$ O.V = s_1
C_j	=	2 ↑	-6	0	0	-3			
2	x_1	1	$-1/4$	0	$1/2$	$-1/4$	1		
6	x_3	0	$3/2$	1	0	$1/2$	3		
Z_j	=	2	$17/2$	6	1	$5/2$			
C_j		0	$-11/2$	0	-1	$-5/2$			

From above table ; C_{ij}'s ≤ 0

And Yes, here optimal Solution is unique.

As, all non-basic variable in the last table are less than zero.

$\therefore$ optimal Basic soln $= (x_1, x_2, x_3, s_1, s_2)$

$= (1, 0, 3, 0, 0)$

$\therefore$ Max $z = 2x_1 + 3x_2 + 6x_3$

$= 2(1) + 3(0) + 6(3)$

$= 2 + 18 = 20$

4(a) If Z is the set of integers then show that $Z[\sqrt{-3}] = \{a+\sqrt{-3}\,b \mid a,b \in Z\}$ is not a unique factorization domain.

Sol^n: we see that $4 = 4 + 0\sqrt{-3} \in Z[\sqrt{-3}]$ and

$$4 = 2.2 \text{ and } 4 = (1+\sqrt{-3})(1-\sqrt{-3}) \quad \text{—(1)}$$

we shall prove that $2,\ 1\pm\sqrt{-3}$ are irreducible elements of $Z[\sqrt{-3}]$. Let $2 = (a+b\sqrt{3}i)(c+d\sqrt{3}i)$ and so $2 = \bar{2} = (a-b\sqrt{3}i)(c-d\sqrt{3}i)$.

(Here a, b, c, d are all integers)

$\Rightarrow 4 = (a^2+3b^2)(c^2+3d^2)$, which gives the following possibilities:

(i) $a^2+3b^2 = 1$ and $c^2+3d^2 = 4$;

(ii) $a^2+3b^2 = 4$ and $c^2+3d^2 = 1$;

(iii) $a^2+3b^2 = 2$ and $c^2+3d^2 = 2$; which is impossible in Z.

The first two possibilities imply

$$[a=\pm 1, b=0] \text{ (or) } [c=\pm 1, d=0]$$

$\Rightarrow a+b\sqrt{3}i = \pm 1$ (or) $c+d\sqrt{3}i = \pm 1$ (± 1 being units in $[Z\sqrt{-3}]$.

Hence 2 is irreducible in $[Z\sqrt{-3}]$

Let $1+\sqrt{-3} = (a+b\sqrt{-3})(c+d\sqrt{-3})$; $a,b,c,d \in Z$

$\Rightarrow 1-\sqrt{-3} = (a-b\sqrt{-3})(c-d\sqrt{-3})$.

On multiplying the respective sides of the above equations, we get

$$4 = (a^2+3b^2)(c^2+3d^2)$$

from above, it follows that $1 \pm \sqrt{-3}$ are irreducible elements of $Z[\sqrt{-3}]$. from ①, we see that $4 \in Z[\sqrt{-3}]$ has two distinct expressions as products of irreducible elements of $Z[\sqrt{-3}]$.

Hence $[Z\sqrt{-3}]$ is not a U.F.D.

4(c) → Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions:

(i) $|z| < 1$ (ii) $1 < |z| < 3$

(iii) $|z| > 3$

Solution:- we have; $f(z) = \frac{1}{(z+1)(z+3)}$

Resolving into partial fraction, we get

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

(i) $|z| < 1$

we have ; $f(z) = \frac{1}{2}(z+1)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$

$$f(z) = \frac{1}{2}\left[1 - z + z^2 - z^3 + \dots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$f(z) = \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{1}{2} - \frac{1}{18}\right)z + \left(\frac{1}{2} - \frac{1}{54}\right)z^2 + \dots$$

$$\boxed{f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \dots}$$

(ii) $1 < |z| < 3$

Then, we have ; $\frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$.

Now, $\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 + 1/z\right)^{-1}$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \dots$$

and $\dfrac{1}{2(z+3)} = \dfrac{1}{6(1+z/3)} = \dfrac{1}{6}\left[1+\dfrac{z}{3}\right]^{-1}$

$$= \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right)$$

$$= \frac{1}{6} - \frac{1}{18}z + \frac{1}{54}z^2 - \frac{1}{162}z^3 + \cdots$$

Thus; the Laurent's series valid for the region $1 < |z| < 3$ is

$$\boxed{\therefore f(z) = \cdots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} + \frac{1}{6} - \frac{1}{18}z + \frac{1}{54}z^2 \cdots}$$

(iii) $|z| > 3$

Then; $3/|z| < 1$

$$\therefore f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z) = \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] - \frac{1}{2z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right]$$

$$f(z) = \left[\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots\right] + \left[\frac{-1}{2z} + \frac{3}{6z^2} - \frac{9}{2z^3} + \frac{27}{2z^4} + \cdots\right]$$

$$\boxed{f(z) = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots}$$

4.(d) A construction Company has to move four large cranes from old construction site to new construction site. The distance in kilometers between the old and new locations are as given in the adjoining table. The at O_3 cannot be used at N_2 but all the cranes can work equally well at any of the other new sites. Determine a plan for moving the cranes that will minimise the total distance involved in the move

	New Const			
	N_1	N_2	N_3	N_4
O_1	15	20	13	40
O_2	38	42	15	20
O_3	25	17	30	18
O_4	18	30	40	35

Sol: Given O_3 cannot be assigned to N_2

So Replace corresponding value in table to very large value

	N_1	N_2	N_3	N_4
O_1	15	20	13	40
O_2	38	42	15	20
O_3	25	∞	30	18
O_4	18	30	40	35

$\Longrightarrow$ Substracting smallest number of each row with whole row values

	N_1	N_2	N_3	N_4
O_1	2	7	0	27
O_2	23	27	0	5
O_3	7	∞	12	0
O_4	0	12	22	17

$\Downarrow$ Substracting smallest Colomn value with the correspondng column values

	N_1	N_2	N_3	N_4
O_1	2	$\boxed{0}$	0	27
O_2	23	20	$\boxed{0}$	5
O_3	7	∞	12	$\boxed{0}$
O_4	$\boxed{0}$	5	22	17

Assigning, $O_1 \to N_2$

$O_2 \to N_3$

$O_3 \to N_4$

$O_4 \to N_1$

minimise the total distance involved in moving the cranes.

5(a) Find complete integral of $(x^2-y^2)pq - xy(p^2-q^2)=1$

Sol'n: Here $f(x,y,z,p,q) = (x^2-y^2)pq - xy(p^2-q^2)-1$ —— (1)

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{2pqx - z(p^2-q^2)} = \frac{dq}{-2pqy - x(p^2-q^2)} = \frac{dx}{-(x^2-y^2)y+2pxy}$$

$$= \frac{dy}{-(x^2-y^2)p-2pxy}$$

Using x, y, p, q as multipliers, each fraction

$$= \frac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \frac{d(xp)+d(yq)}{0}$$

$\Rightarrow d(xp+qy)=0 \Rightarrow xp+yq=0 \Rightarrow p = (a-qy)/x$ —— (2)

Using (2),(1) $\Rightarrow (x^2-y^2)\left(\frac{a-qx}{x}\right)q - xy\left[\left(\frac{a-qy}{x}\right)^2-q^2\right]-1=0$

$\Rightarrow \frac{a-qy}{x}\left\{(x^2-y^2)q-(a-qy)y\right\}+xyq^2-1=0$

$\Rightarrow \{(a-qy)/x\}(x^2q-ay)+xyq^2-1=0$

$\Rightarrow (a-qy)(x^2q-ay)+x^2yq^2-x=0 \Rightarrow aq(x^2+y^2)=a^2y+x$

$\therefore\ q = \frac{a^2y+x}{a(x^2+y^2)}$ and $p = \frac{1}{x}\left[a - \frac{(a^2y+x)y}{a(x^2+y^2)}\right] = \frac{a^2x-y}{a(x^2+y^2)}$

Substituting these values in $dz = p\,dx+q\,dy$, we have

$$dz = \frac{(a^2x-y)dx+(a^2y+x)dy}{a(x^2+y^2)} = a\frac{x\,dx+y\,dy}{x^2+y^2} + \frac{x\,dy-y\,dx}{a(x^2+y^2)}$$

Integrating $z = (a/2)\log(x^2+y^2)+\frac{1}{a}\tan^{-1}(y/x)+b$.

5.(b). Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$.

Sol'n: Here A.E is $m^2 - m - 2 = 0$ so that $m = 2, -1$.

So. $C.F = \phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{(D-2D')}\frac{1}{(D+D')}\{(2x^2+xy-y^2)\sin xy - \cos xy\}$$

$$= \frac{1}{(D-2D')}\frac{1}{D+D'}\{(2x-y)(x+y)\sin xy - \cos xy\}$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin x(c+x) - \cos x(c+x)\}dx$$

(Taking $c = y-x$)

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin(cx+c^2) - \cos((cx+x^2)\}dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos((cx+x^2) + \int\cos(cx+x^2)dx - \int\cos((cx+x^2)dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy \quad \text{as } c = y-x$$

$$= \int(c'-4x)\cos(c'x-2x^2)dx, \text{ where } c' = y+2x$$

$$= \int\cos t\,dt = \sin t, \quad \text{putting } c'x - 2x^2 = t \text{ so that } (c'-4x)dx = dt$$

$$= \sin(cx - 2x^2)$$

$$= \sin xy, \text{ as } c' = y+2x$$

So solution is $\underline{z = \phi_1(y+2x) + \phi_2(y-x) + \sin xy}$.

5(C) Use Newton-Raphson method to find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

Sol:- $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

Both $f(x)$ & $f'(x)$ are continuous

$f(1) = 1.4597$ which is closer to '0' other than nearby values.

Hence, we will take $x_0 = 1$

$x_0 = 1 \qquad f(x_0) = 1.45970$

$f'(x_0) = 3.84147 = 3.8415$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1.45970}{3.8415} = 0.62002$$

Similarly $x_1 = 0.62002$

$f(x_1) = 0.04619$

$f'(x_1) = 3.58105$

$$x_2 = 0.62002 - \frac{0.4619}{3.58105} = 0.60712$$

$\therefore x_2 = 0.60712$

$f(x_2) = 0.00007$

$f'(x_2) = 3.57050$

$$x_3 = 0.60712 - \frac{0.00007}{3.57050} = 0.60710$$

$f(x_3) = 0.00000$

$f'(x_3) = 3.57049$

$$x_4 = 0.60710 - \frac{0.0000}{3.57049} = 0.60710$$

Hence; $\boxed{x_4 = 0.60710}$ is the required result.

5(d) Convert the following decimal numbers to equivalent binary and hexadecimal numbers.

(i) 4096

$\Rightarrow (4096)_{10} \rightarrow (1\,0000\,0000\,0000)_2$

2	4096	
2	2048	0
2	1024	0
2	512	0
2	256	0
2	128	0
2	64	0
2	32	0
2	16	0
2	8	0
2	4	0
2	2	0
	1	0

$(1\underline{0000}\,\underline{0000}\,\underline{0000})_2$

$\downarrow$

$(1\,000)_{16}$.

(ii) $(0.4375)_{10} \leftrightarrow (0.0111)_2$

$\Downarrow$

$0.4375 \times 2 = 0.8750$	0
$0.8750 \times 2 = 1.7500$	1
$0.7500 \times 2 = 1.5000$	1
$0.5000 \times 2 = 1.0000$	1
0.0	

Now to Hexadecimal $\Rightarrow (0.0111)_2 = (0.7)_{16}$.

(iii) $(2048.0625)_{10}$

$(2048)_{10} \rightarrow (1\underline{000}\,\underline{0000}\,\underline{0000})_2$

$\rightarrow (800)_{16}$

2	2048	
2	1024	0
2	512	0
2	256	0
2	128	0
2	64	0
2	32	0
2	16	0
2	8	0
2	4	0
2	2	0
	1	0

$0.0625 \times 2 = 0.1250$	0
$0.1250 \times 2 = 0.2500$	0
$0.2500 \times 2 = 0.5000$	0
$0.5000 \times 2 = 1.0000$	1
0.0000	

$(0.0625)_{10} \rightarrow (0.0001)_2 \rightarrow (0.1)_{16}$

$\Rightarrow \boxed{(2048.0625)_{10} \rightarrow (100000000000.0001)_2 \rightarrow [800.1]_{16}}$ Ans.

5(e) If velocity distribution of an incompressible fluid at point (x,y,z) is given by $\left\{3xz/r^5,\ 3yz/r^5,\ (kz^2-r^2)/r^5\right\}$, determine the parameter k such that it is a possible motion. Hence find its velocity potential.

Sol^n: Here $u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{kz^2-r^2}{r^5} = \dfrac{kz^2}{r^5} - \dfrac{1}{r^3}$ — ①

where $r^2 = x^2+y^2+z^2$ — ②

from ②, $\dfrac{\partial r}{\partial x} = x/r$, $\dfrac{\partial r}{\partial y} = y/r$ and $\dfrac{\partial r}{\partial z} = z/r$ — ③

from ① ② and ③, we have

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15x^2z}{r^7}, \quad \frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2z}{r^7} \quad \text{— ④}$$

and $\dfrac{\partial w}{\partial z} = \dfrac{2kz}{r^5} - 5kz^2r^{-6}\dfrac{\partial r}{\partial z} + 3r^{-4}\dfrac{\partial r}{\partial z}$

$$= \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3}{r^4}\cdot\frac{z}{r} = \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7} \quad \text{— ⑤}$$

Since ① gives a possible liquid motion, the equation of continuity must be satisfied and so

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

$$\Rightarrow \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = 0 \Rightarrow \frac{(2k+9)z}{r^5} - \frac{15z}{r^7}\cdot r^2 = 0$$

using ②, ④ & ⑤

$\Rightarrow (2k-6)z/r^5 = 0$ so that $2k-6=0$ giving $k=3$

Substituting the above value of k in ①, we have

$$u = \frac{3xz}{r^5}, \quad v = \frac{3yz}{r^5}, \quad w = \frac{(3z^2-r^2)}{r^5} \quad \text{— ⑥}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{15z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2)$$

$$= \frac{15z}{r^5} - \frac{15z}{r^7}r^2 = 0$$

Since the equation of continuity is satisfied by the given values of u, v, w, the motion is possible.

Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = -(u\,dx + v\,dy + w\,dz)$$

by definition of ϕ.

$$= -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2 - r^2}{r^5}dz\right]$$

$$= \frac{r^2 dz - 3z(x\,dx + y\,dy + z\,dz)}{r^5}$$

$$= \frac{r^3 dz - 3r^2 z\,dr}{(r^3)^2} = d\left(\frac{z}{r^3}\right), \text{ using } 2$$

Integrating $\phi = z/r^3$

$\therefore$ the required velocity potential ϕ is given by $\frac{z}{r^3}$.

6.(a) Find the equation of the surface satisfying $4yzp+q+2y=0$ and passing through $y^2+z^2=1$ and $x+z=2$.

Sol^n: Given equation can be written as

$$yzp+q=-2y \quad \text{---} (1)$$

Lagrange's auxiliary equations are

$$\frac{dx}{4yz}=\frac{dy}{1}=\frac{dz}{-2y}$$

from first & 3rd fractions we have

$$dx+2z\,dz=0$$

Integrating, we get $x+z^2=c_1$ — (2)

Again from 2nd & 3rd fractions we have

$$2y\,dy+dz=0$$

Integrating we get

$$y^2+z=c_2 \quad \text{---} (3)$$

The surface satisfying (1), are required to pass through $y^2+z^2=1$, $x+z=2$ ——— (4)

Adding (2) & (3), we get $y^2+z^2+x+z=c_1+c_2$

$$\Rightarrow 1+2=c_1+c_2 \text{ (by using 4)}$$

$$\Rightarrow \boxed{c_1+c_2=3}$$

$\therefore$ The required surface is

$$\boxed{y^2+z^2+x+z=3}$$

6(b). Reduce $x\left(\frac{\partial^2 z}{\partial x^2}\right)+\frac{\partial^2 z}{\partial y^2}=x^2$, $(x>0)$ to canonical form.

Solution :-

given equation

$$x\left(\frac{\partial^2 z}{\partial x^2}\right)+\frac{\partial^2 z}{\partial y^2}=x^2 \quad , x>0$$

can be rewritten as

$$xr+t=x^2$$

$$xr+t-x^2=0 \; ; \; x>0 \quad \text{——} \; (1)$$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

Here, $R=x$, $S=0$, $T=1$, $f(x,y,z,p,q)=-x^2$

so that $S^2-4RT=-4x<0$

Showing that (1) is elliptic

The λ- quadratic equation $R\lambda^2+S\lambda+T=0$

reduces to $x\lambda^2+1=0$ or $\lambda^2=\frac{-1}{x}$

$$\boxed{\lambda=\frac{i}{\sqrt{x}}, \frac{-i}{\sqrt{x}}}$$

The corresponding characteristic equations are given by

$$\frac{dy}{dx}+ix^{-1/2}=0 \text{ and } \frac{dy}{dx}-ix^{-1/2}=0$$

Integrating these;

$$y+2ix^{1/2}=c_1 \text{ and } y-2ix^{1/2}=c_2$$

choose; $u=y+2ix^{1/2}=\alpha+i\beta$

$v=y-2ix^{1/2}=\alpha-i\beta$

where $\alpha=y$ & $\beta=2x^{1/2}$ —— (2)

are now two independent variables.

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}=x^{-1/2}\frac{\partial z}{\partial \beta}$ — (3) (By (2))

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \quad \text{- by ②} \quad \text{—— ④}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial z}\left(x^{-1/2}\cdot\frac{\partial z}{\partial \beta}\right), \text{ using ③}$$

$$= -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

$$= -\frac{1}{2}x^{-3/2}\cdot\frac{\partial z}{\partial \beta} + x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right)$$

$$= -\frac{1}{2x^{3/2}}\cdot\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \quad \text{—— ⑤}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2} \quad (\text{using 4}) \quad \text{—— ⑥}$$

using ⑤ & ⑥, in ①, the required canonical form is

$$x\left[-\frac{1}{2x^{3/2}}\cdot\frac{\partial z}{\partial \beta} + \frac{1}{x}\cdot\frac{\partial^2 z}{\partial \beta^2}\right) + \frac{\partial^2 z}{\partial \alpha^2} = x^2$$

$$= -\frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta} + \frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = x^2$$

$$\therefore \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = x^2 + \frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta} \quad \text{—— ⑦}$$

$\Rightarrow$ as $\beta = 2x^{1/2}$.

$$⑦ \approx \boxed{\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\beta^4}{16} + \frac{1}{\beta}\left(\frac{\partial z}{\partial \beta}\right)}$$

which is required canonical form.

6.(c) A Square plate is bounded by the lines $x=0, y=0, x=10$ & $y=10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,10)=x(10-x)$ while the other three faces are kept at 0°C. Find the steady state temperature in the plate.

Sol^n: The steady state temperature, $u(x,y)$ is the solution of

Laplace equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ —— (1)

subject to boundary conditions

$u(0,y)=u(10,y)=0, \quad 0\le y\le 10$ —— (2)

$u(x,0)=0, \quad 0\le x\le 10$ —— (3a)

and $u(x,10)=10x-x^2, \quad 0\le x\le 10$ —— (3b)

Suppose (1) has a solution of the form

$u(x,y)=X(x)\,Y(y)$ —— (4)

Substituting this value of u in (1), we get

$X''Y+XY''=0 \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}$ —— (5)

Since x and t are independent variables, each side of (5) must be equal to the constant, say μ.

Then (5) gives $X''-\mu X=0$ —— (6)

and $Y''+\mu Y=0$ —— (7)

Using (2), (4) gives $X(0)Y(y)=0$ & $X(10)Y(y)=0$

giving $X(0)=0$ and $X(10)=0$ —— (8)

where $Y(y)\neq 0$, since otherwise $u\equiv 0$ which doesnot satisfy 3(b)

We now solve (6) under B.C. (8). Three cases arise.

Case(1): Let $\mu=0$. The solution of (6) is

$X(x)=Ax+B$ —— (9)

Using B.C (8), we get $A=B=0$

so that $X(x)=0$. This leads $u\equiv 0$ which doesnot satisfy 3(b). So we reject $\mu=0$.

Case(2): Let $\mu=\lambda^2, \lambda\neq 0$. Then solution of (6) is

$X(x)=Ae^{x\lambda}+Be^{-\lambda x}$ —— (10)

Using B.C (8), we get $A=B=0$ so that $X(x)=0$ and hence $u\equiv 0$.

So we reject $\mu=\lambda^2$.

Case (3): Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (6) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{— (11)}$$

Using BC (8), (11) gives $0 = A$ and $0 = A\cos\lambda(10) + B\sin\lambda(10)$

$\Rightarrow \sin\lambda(10) = 0$, $B \neq 0$.

Since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which doesnot satisfy 3(b).

Now, $\sin\lambda a = 0 \Rightarrow \lambda(10) = n\pi$, $n = 1, 2, 3 \ldots$

$$\Rightarrow \lambda = \frac{n\pi}{10} ; n = 1, 2, \ldots \quad \text{— (12)}$$

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{10}\right)$ — (13)

Using $\mu = -\lambda^2 = -\frac{n^2\pi^2}{(10)^2}$, (7) becomes $Y'' - \left(\frac{n^2\pi^2}{10^2}\right)Y = 0$ — (14)

whose general solution is

$$Y_n(y) = C_n e^{n\pi y/10} + D_n e^{-\frac{n\pi y}{10}} \quad \text{— (15)}$$

Using 3(a), (9) gives $0 = X(x)\, Y(0)$ so that $Y(0) = 0$, where we have taken $X(x) \neq 0$.

for otherwise we will get $u \equiv 0$ which doesnot satisfy 3(b).

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ — (16)

putting $y = 0$ in (15) and using (16), we have

$0 = C_n + D_n \Rightarrow D_n = -C_n$. Then (6) reduces to

$$Y_n(y) = C_n\left(e^{n\pi y/10} - e^{-n\pi y/10}\right)$$
$$= 2\sinh(n\pi y/10) \quad \text{— (17)}$$

$$\therefore u_n(x,y) = X_n(x)\, Y_n(y) = E_n \sin\left(\frac{n\pi x}{10}\right)\sinh\left(\frac{n\pi y}{10}\right) \quad \text{— (18)}$$

are solutions of (1), satisfying (2) and 3(a).

Here $E_n = 2B_nC_n$.

In order to satisfy condition 3(b), now consider more general solution given by

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) \sinh\left(\frac{n\pi y}{10}\right). \quad \text{(18)}$$

putting $y=10$ in (18) and using 3(b), we get

$$u(x,10) = \underbrace{10x - x^2}_{=f(x)} = \sum_{n=1}^{\infty} E_n \sinh(n\pi) \sin\left(\frac{n\pi x}{10}\right).$$

which is the half range fourier sine series of $f(x)$ in $(0, 10)$.

Hence, we have

$$E_n \sinh(n\pi) = \frac{2}{10}\int_0^{10} (10x - x^2) \sinh\left(\frac{n\pi x}{10}\right) dx$$

$$= \frac{1}{5\sinh n\pi}\left[(10x-x^2)\left(-\frac{10}{n\pi}\cos\frac{n\pi x}{10}\right) - (10-2x)\left(\frac{-100}{n^2\pi^2}\right)\sin\frac{n\pi x}{10} + 2\left(\frac{1000}{n^3\pi^3}\right)\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5\sinh n\pi}\left[-\frac{2000(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3}\right]$$

$$= \frac{400(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}.$$

$$\therefore E_n = \begin{cases} 0, & \text{if } n = 2m \text{ and } m = 1,2,3,\ldots \\ \dfrac{800\,\mathrm{cosech}\,(2m-1)\pi}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1 \\ & m = 1,2,\ldots \end{cases}$$

$\therefore$ from (18), we have

$$u(x,y) = \frac{800}{\pi^3}\sum_{m=1}^{\infty} \frac{\mathrm{cosech}\,(2m-1)\pi}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{10} \sinh\frac{(2m-1)\pi y}{10}.$$

which is the required temperature

6.(d) If the string of length l is initially at rest in equilibrium position and each of the points is given the velocity $v_0 \sin(3\pi x/l) \cdot \cos(2\pi x/l)$ where $0<x<l$ at $t=0$. Find the displacement function.

Sol:- Let the Required displacement function $y(x,t)$ is the the solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{— (1)}$$

Subject to boundary condition.

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t \geq 0 \quad \text{— (2)}$$

Initial conditions

$$y(x,0) = 0 \quad \text{- equilibrium position}$$

$$\frac{\partial y}{\partial t} = 0 \; v_0 \sin(3\pi x/l) \cos(2\pi x/l) \quad \text{— (3)}$$
- initial velocity

Suppose (1) has the solution of the form.

$$y(x,t) = X(x) T(t) \quad \text{— (4)}$$

Putting (4) in (1) $\quad X''T = \frac{1}{c^2} \cdot XT'' \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \mu \quad$ -(5)

Now solve (7) $\quad T'' + (n^2\pi^2 c^2/l^2)T = 0. \quad$ Since $\mu = -\lambda^2 = \frac{n\pi}{l}$

$$T_n = C_n \cos\frac{n\pi c}{l} t + D_n \sin\frac{n\pi c}{l} t$$

$$y_n(x,t) = X_n(x) T_n(t) = \left(E_n \cos\left(\frac{n\pi c t}{l}\right) + F_n \sin\left(\frac{n\pi c t}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos\left(\frac{n\pi c t}{l}\right) + F_n \sin\left(\frac{n\pi c t}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right)$$

from (3) equilibrium position

$$y(x,0) = 0.$$

$$E_n \sin\left(\frac{n\pi x}{l}\right) = 0 \Rightarrow E_n = 0.$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} F_n \cdot \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{n\pi c t}{l}\right) \quad \text{-(8)}$$

differentiating (8) w.r.t t

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} F_n\left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \cdot \cos\left(\frac{n\pi c t}{l}\right).$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} F_n\left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = V_0 \sin\left(\frac{3\pi x}{l}\right) \cos\left(\frac{2\pi x}{l}\right)$$

$$(\pi c F_1/l)\sin\left(\frac{\pi x}{l}\right) + (2\pi c F_2/l)\sin\left(\frac{2\pi x}{l}\right) \dots = \frac{V_0}{2}\left[\sin\left(\frac{5\pi x}{l}\right) + \sin\left(\frac{\pi x}{l}\right)\right]$$

$$\pi c F_1/l = \frac{V_0}{2} \qquad 5\pi c F_5/l = V_0/2$$

$$F_2 = F_3 = F_4 = F_6 = F_7 \dots = 0.$$

$$F_1 = (lV_0/2\pi c) \qquad F_5 = (lV_0/10\pi c)$$

$$\therefore y(x,t) = (lV_0/2\pi c)\sin(\pi x/l)\cdot\sin(\pi c t/l) + \frac{lV_0}{10\pi c}\cdot\sin\left(\frac{5\pi x}{l}\right)\sin(5\pi c t/l)$$

from (5)

$$X'' - \mu X = 0 \quad \text{——(6)}$$

$$T'' - \mu c^2 T = 0 \quad \text{——(7)}$$

from (2) $\quad X(0)T(t) = 0 \quad$ and $\quad X(a)T(t) = 0$.

Suppose $T(t) \neq 0$ for some t.

$$\Rightarrow \underline{X(0) = 0} \quad \underline{X(a) = 0}$$

with these conditions we solve the (6)

$$X'' - \mu X = 0.$$

Case (i) $\mu = 0 \quad \Rightarrow \quad X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$

$X(a) = 0 \Rightarrow A = 0$

$\therefore$ we reject $\mu = 0$

Case (ii) $\mu = \lambda^2 \quad \Rightarrow \quad X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

$X(0) = 0 \Rightarrow A + B = 0$

$X(a) = 0 \Rightarrow Ae^{\lambda a} + Be^{-\lambda a} = 0$

$A(e^{\lambda a} - e^{-\lambda a}) = 0$

$\underline{A = 0} \Rightarrow B = 0$

$\therefore$ we ~~jest~~ reject $\mu = \lambda^2$

Case (iii) $\mu = -\lambda^2 \quad \Rightarrow \quad X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = 0 \quad A = 0.$

$X(a) = 0 \quad \lambda l = n\pi \quad \Rightarrow \lambda = \frac{n\pi}{l}$

$$\therefore X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$$

7(a)) Using Gauss-Seidel iterative method, find the solution of the following system. $4x-y+8z=26$, $5x+2y-z=6$, $x-10y+2z=-13$ upto three iterations.

Ans: $x=1$, $y=2$, $z=3$

7.(b) Obtain the Simpson's rule for the integral $I = \int_a^b f(x)\,dx$ and show that this rule is exact for polynomials of degree $n \leq 3$. In general show that the error of approximation for Simpson's rule is given by $R = \frac{-(b-a)^5}{2880} f^{iv}(\eta)$, $\eta \in (0,2)$. Apply this rule to the integral $\int_0^1 \frac{dx}{1+x}$ and show that $|R| \leq 0.008333$.

Solution:-

Consider the integral

$$I = \int_a^b f(x)\,dx \quad \text{——} \ (1)$$

where $f(x)$ takes the values.

$f(x_0) = y_0$, $f(x_0+h) = y_1$, $f(x_0+2h) = y_2, \ldots$

$f(x_0+nh) = y_n$, when $x = x_0$, $x = x_0+h$

$x = x_0+2h, \ldots \quad x = x_0+nh$

respectively.

Let the interval $[a,b]$ be divided into n sub-interval of width h, so that

$x_0 = a$, $x_1 = x_0+h$, $x_2 = x_1+h = x_0+2h \ldots$

$\ldots\ x_n = x_0+nh = b$.

Approximating $f(x)$ by Newton's forward interpolation formula, we can write the integral (1) as.

$$I = \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx.$$

$$= \int_{x_0}^{x_0+nh}\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots\right]dx \quad \text{— (2)}$$

Since $p = \frac{x - x_0}{h} \Rightarrow x = x_0 + ph$

$\Rightarrow \boxed{dx = h\,dp}$

when $x = x_0$, $p = 0$ and

when $x = x_0 + nh$, $p = n$.

Equation (2) can be written as

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0 + \cdots\right]dp$$

$$I = h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{\frac{p^3}{3} - \frac{p^2}{2}}{2!}\Delta^2 y_0 + \cdots\right]_0^n.$$

$$\therefore I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right] \quad \text{— (3)}$$

This is known as Newton-cotes quadrature formula. from this general formula, we can obtain Different integration formulae by putting $n = 1, 2, 3 \ldots$ etc.

Simpson's 1/3rd Rule:

putting $n=2$ in the quadrative formula and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola; i.e, a polynomial of second order so that differences of order higher than second vanishes, we get

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx$$

$$= h\left\{2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left[\frac{8}{3} - \frac{4}{2}\right]\Delta^2 y_0\right\}$$

$$= h\left\{2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta^2 y_0\right\}$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \quad \left[\because \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0\right]$$

Similarly,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]; \ n \text{ being even.}$$

Adding all these integrals, if n is even +ve integer. i.e., the number of ordinates $y_0, y_1, y_2, \ldots, y_n$ is odd.

we have;

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$= \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right]$$

$$\therefore \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\text{sum of first \& last ordinates} + 2(\text{sum of even ordinates}) + 4(\text{sum of odd ordinates})\right]$$

Which is known as simpson's 1/3 rule.

Now putting $n=3$.

and taking the curve through (x_i, y_i), $i=0,1,2,3$ as a polynomial of third order so that difference above the third vanish, we get

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx.$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right).$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right)..$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right].$$

and so on

Adding all these integrals from x_0 to x_0+nh where 'n' is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots$$

$$= \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\right]$$

$$= \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})\right]$$

which is known as Simpson's $3/8$-rule.

Alternative: you can check whether it will be applicable for $n\le 3$ or $n>3$.

for polynomial of degree 1, we have.

$$I = \int_a^b x\,dx = \frac{b^2-a^2}{2}.$$

using Simpson's rule, we have

$$I = \frac{h}{3}\left[f(x_0)+f(x_2)+4(x_1)\right]$$

$x_0 = a$, $x_2 = b$, $x_1 = \frac{a+b}{2}$ and $h = \frac{b-a}{2}$

$$\therefore\ I = \frac{h}{3}\left[a+b+\frac{4(b+a)}{2}\right] = \frac{b-a}{2\times \cancel{3}}\left[\cancel{3}(b+a)\right]$$

$$\boxed{I = \frac{b^2-a^2}{2}}$$

Similarly, for polynomial of degree 2, we have

$$I = \int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$

By Simpson rule,

$$I = \frac{h}{3}\left[a^2+b^2+4\left(\frac{a+b}{2}\right)^2\right] = \frac{b^3-a^3}{3}$$

for, $n=3$; $\quad I = \int_a^b x^3\,dx = \frac{b^4-a^4}{4}$

using Simpson's rule,

$$I = \frac{h}{3}\left[a^3+b^3+4\left(\frac{a+b}{2}\right)^3\right]$$

$$I = \frac{b-a}{2\cdot 3}\left[a^3+b^3+\frac{a^3+b^3+3ab(a+b)}{2}\right]$$

$$I = \frac{b^4-a^4}{4}.$$

But, for $n=4$; $\quad I = \int_a^b x^4\,dx = \frac{b^5-a^5}{5}$

using Simpson rule; $I = \frac{b-a}{2\cdot 3}\left[a^4+b^4+4\left(\frac{b+a}{2}\right)^4\right]$

$$\neq \frac{b^5-a^5}{5}.$$

Also for $n=0$; zero degree polynomial

$$I = \int_a^b 1 \cdot dx = b-a.$$

By simpson's rule $\Rightarrow I = \frac{b-a}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]$

$$I = \frac{b-a}{6}\left[1+1+4\right] = b-a.$$

Hence, we can say that simpson's rule is exact for polynomial of degree $n \leq 3$, not for $n > 3$.

Now, for Error R;

as per Simpson's rule, we know that

$$R = -\frac{nh^5}{90} y^{iv}(\bar{x}).$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivative.

$$\boxed{R = -\frac{(b-a)}{180} h^4 y^{iv}(\bar{x})} \quad \left[\because (2n)h = b-a \Rightarrow nh = \frac{b-a}{2}\right]$$

for Simpson's 1/3 rule, $n=2$. and

$$h = \frac{b-a}{n} = \frac{b-a}{2}.$$

$$\therefore \quad R = -\frac{(b-a)}{180}\left(\frac{b-a}{2}\right)^4 y^{iv}(\bar{x})$$

$$\boxed{\therefore \; R = \frac{-(b-a)^5}{2880} y^{iv}(\bar{x})}$$ Hence, proved.

For the given function ; $f(x) = \frac{1}{1+x}$

in interval $[0,1]$ for $n=6$

Let $h = \frac{1-0}{6} = 1/6$

Then.

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1	0.8571	0.75	0.6666	0.6	0.5454	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Then by Simpson's 1/3rd rule

$$I = \frac{h}{3}\left[y_0 + y_6 + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$I = \frac{1}{18}\left[1 + 0.5 + 2(0.75+0.6) + 4(0.8571+0.6666+0.5454)\right]$$

$$\boxed{I = \frac{12.4764}{18} = 0.69313}.$$

$$\therefore\ I = \int_0^1 \frac{1}{1+x}\,dx = 0.69313$$

$$R = -\frac{(b-a)^5}{2880}\cdot f^{iv}(\bar{x}) = -\frac{(1-0)^5 \cdot 24}{2880}$$

$$\therefore\ R = -0.00833 \quad \Rightarrow \boxed{|R| \ll +0.00833}$$

proved.

$\Big[\because\ f'(x) = \frac{-1}{(1+x)^2}$, $f''(x) = \frac{2}{(1+x)^3}$, $f'''(x) = \frac{-6}{(1+x)^4}$

$f^{iv}(x) = \frac{+24}{(1+x)^5}$; $f^{iv}(0) = 24\ \Big]$

7.(c) Use Runge-Kutta formula of fourth order to find the value of y at $x=0.8$, where $\frac{dy}{dx} = \sqrt{x+y}$, $y(0.4)=0.41$. Take the step length $h=0.2$.

Sol: Given that $\frac{dy}{dx} = \sqrt{x+y} = f(x,y)$

To find $y(0.6)$:

Here $x_0=0.4$, $y_0=0.41$, $h=0.2$

$$f(x_0, y_0) = \sqrt{0.81}$$

$\therefore k_1 = h f(x_0, y_0) = (0.2)\sqrt{0.81} = 0.18$

$k_2 = h f(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}) = (0.2) f(0.5, 0.5) = 0.2$

$k_3 = h f(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}) = (0.2) f(0.5, 0.51) = 0.20099$

$k_4 = h f(x_0+h, y_0+k_3) = (0.2) f(0.6, 0.61099) = 0.220089$

$$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$= \frac{1}{6}[0.18 + 2(0.2) + 2(0.20099) + 0.220089]$$
$$= \frac{1}{6}(1.202069982)$$
$$= 0.2003449$$

$y_1 = y(0.6) = y_0 + k = 0.41 + 0.2003449 = 0.6103449$

To find $y(0.8)$:

Here $x_1 = 0.6$, $y_1 = 0.6103449$, $h=0.2$

$k_1 = h f(x_1, y_1) = (0.2) f(0.6, 0.6103449) = 0.220031$

$k_2 = h f(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}) = (0.2) f(0.7, 0.72036) = 0.23836$

$k_3 = h f(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}) = (0.2) f(0.7, 0.72952) = 0.23911$

$k_4 = h f(x_1+h, y_1+k_3) = (0.2) f(0.8, 0.852571) = 0.257105$

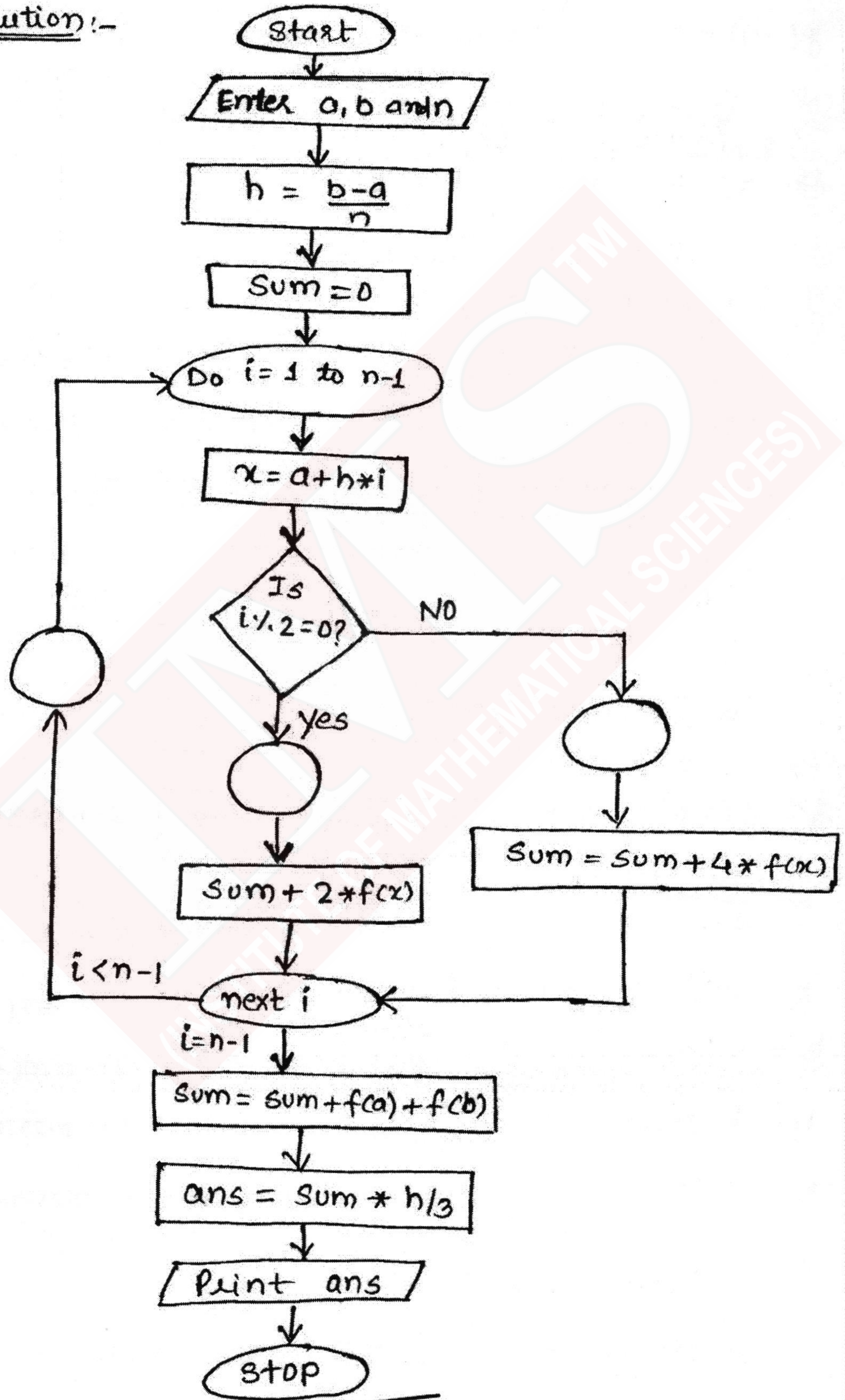
$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.238686$

$y_2 = y(0.8) = y_1 + k = 0.6103449 + 0.238686$

$$= 0.8490309.$$

7(d) Draw a flow chart for Simpson's one third rule?

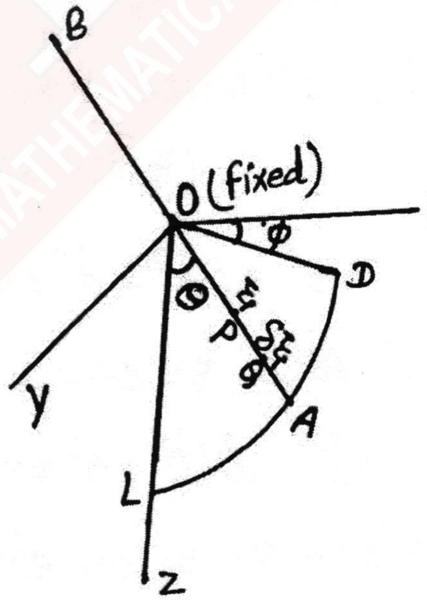
Solution:-



8(a) A uniform rod, of mass $3m$ and length $2l$, has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{(2ng/l)}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is

$$\cos^{-1}\left[\sqrt{(n^2+1)}-n\right], \text{ and will then rise again.}$$

Sol: Let AB be the rod of mass $3m$ and length $2l$. The middle point O of the rod is fixed and a mass m attached at the extremity A.



Initially let the rod rest along OX in the plane of the paper. Let a line OY perpendicular to the plane of the paper and a line OZ perpendicular to OX in the plane of the paper be taken as axes of y and z respectively. At time t, let the rod turn through an angle ϕ to OX i.e., the plane

OAL containing the rod and z axis make an angle ϕ with x-z plane. And let θ be the inclination of the rod with OZ at this time t. If P is a point of the rod at a distance $OP = \xi$, from O then coordinates of P are given by

$$x_p = \xi \sin\theta \cos\phi,$$

$$y_p = \xi \sin\theta \sin\phi, \quad z_p = \xi \cos\theta.$$

$\therefore$ v_p and v_A are the velocity of the point P and A respectively, then

$$v_P^2 = \dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2$$

$$= (\xi \cos\theta \cos\phi \dot{\theta} - \xi \sin\theta \sin\phi \dot{\phi})^2 + (\xi \cos\theta \sin\phi \dot{\theta} + \xi \sin\theta \cos\phi \dot{\phi})^2 + (-\xi \sin\theta \dot{\theta})^2$$

$$= \xi(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

$\therefore$ At A, $\xi = OA = l$,

$$\therefore v_A^2 = l^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

Let $PQ = \delta\xi$ be an element of the rode at P, then mass of this element, $\delta m = \frac{3m}{2l}\delta\xi$.

$\therefore$ K.E. of the element $PQ = \frac{1}{2}\delta m \cdot v_p^2$

$$= \frac{1}{2}\xi^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \cdot \frac{3m}{2l}\delta\xi$$

$\therefore$ K.E. of the rod AB $= \frac{3m}{4l}\int_{-l}^{l} \xi^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)\,d\xi$

$$= \frac{1}{2}m(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)l^2.$$

and K.E. of mass m at A $= \frac{1}{2}mv_A^2$

$$= \frac{1}{2}ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

$\therefore$ The total kinetic energy of the system

$T =$ K.E. of the rod + K.E. of the particle

$$= ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$$

The work function $W = mg\cdot z_A = mgl\cos\theta$.

$\therefore$ Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e., $\frac{d}{dt}(2ml^2\dot{\theta}) - 2ml^2\dot{\phi}^2\sin\theta\cos\theta = -mgl\sin\theta$

or $2l\ddot{\theta} - 2l\dot{\phi}^2\sin\theta\cos\theta = -g\sin\theta$ ——— (1)

And Lagrange's ϕ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

i.e., $\frac{d}{dt}(2ml^2\dot{\phi}\sin^2\theta) = 0$

or $\frac{d}{dt}(\dot{\phi}\sin^2\theta) = 0$ ——— (2)

Integrating (2) we get $\dot{\phi}\sin^2\theta = C$ (constant)

But initially when $\theta = (\pi/2)$

(∵ Rod was horizontal),

$$\dot{\phi} = \sqrt{(2ng/l)}$$

$$\therefore C = \sqrt{(2ng/l)}$$

$$\therefore \dot{\phi}\sin^2\theta = \sqrt{(2ng/l)} \quad \text{——— (3)}$$

Substituting the value of $\dot{\phi}$ from (3) in (1), we get

$$2l\ddot{\theta} - 2l\frac{2ng}{l\sin^4\theta}\sin\theta\cos\theta = -g\sin\theta$$

or $\quad 2l\ddot{\theta} - 4ng\cot\theta\,\mathrm{cosec}^2\theta = -g\sin\theta \quad$ ——— (4)

Multiplying both sides by $\dot{\theta}$ and integrating, we get

$$l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta + D.$$

But initially when $\theta = \pi/2$, $\dot{\theta} = 0$

$$\therefore D = 0$$

$$\therefore l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta \quad \text{——— (5)}$$

The rod will fall till $\dot{\theta} = 0$

i.e., $2ng\cot^2\theta = g\cos\theta$

or $\quad 2n\cos^2\theta - \cos\theta\sin^2\theta = 0$

or $\quad \cos\theta(2n\cos\theta - \sin^2\theta) = 0.$

$\therefore$ either $\cos\theta = 0$. i.e., $\theta = (\pi/2)$

or $2n\cos\theta - \sin^2\theta = 0$

i.e.; $2n\cos\theta - (1-\cos^2\theta) = 0$

or $\cos^2\theta + 2n\cos\theta - 1 = 0$

$\therefore \cos\theta = \dfrac{-2n \pm \sqrt{(4n^2+4)}}{2}$

or $\cos\theta = -n + \sqrt{(n^2+1)}$, Leaving negative sign.

$\because$ negative value of $\cos\theta$ is inadmissible as θ can not be obtuse.

$\therefore \theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right]$.

From ④ we have

$$2l\ddot{\theta} = \frac{g(4n\cos\theta - \sin^4\theta)}{\sin^3\theta} \quad \text{———} ⑥$$

When $\cos\theta = -n + \sqrt{(n^2+1)}$,

$$\cos^2\theta = 2n^2 + 1 - 2n\sqrt{(n^2+1)}$$

$\therefore 4n\cos\theta - \sin^4\theta = 4n\cos\theta - (1-\cos^2\theta)^2$

$= 4n\left[-n + \sqrt{(n^2+1)}\right] - \left[-2n^2 + 2n\sqrt{(n^2+1)}\right]^2$

$= -4n^2 + 4n\sqrt{(n^2+1)} - 4n^4 - 4n^2(n^2+1) + 8n^3\sqrt{(n^2+1)}$

$= -8n^2 - 8n^4 + 4n\sqrt{(n^2+1)} + 8n^3\sqrt{(n^2+1)}$

$= 4n\sqrt{(n^2+1)}\left[-2n\sqrt{(n^2+1)} + 1 + 2n^2\right]$

$= 4n\sqrt{(n^2+1)}\left[-n+\sqrt{(n^2+1)}\right]^2$

which is positive.

$\because$ θ is acute angle.

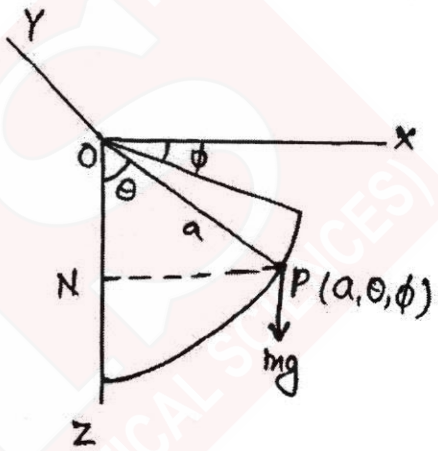
$\therefore$ $\sin^3\theta$ is also positive.

$\therefore$ when $\theta = \cos^{-1}\left[\sqrt{(n^2+1)}-n\right]$, from (6), we see that $\ddot{\theta}$ is positive. Hence from this position the rod will rise again.

8.(b) Determine the motion, of a spherical pendulum, by using Hamilton's equations.

Sol'n: Let m be the mass of the bob suspended by a light rod of length a. In a spherical pendulum of length a, the path of the motion of the bob is the surface of a sphere of radius a and centre at the fixed point O.

At time t, let $P(a,\theta,\phi)$ be the position of the bob. If (x,y,z) are the cartesian coordinates of P then



$x = a\sin\theta\cos\phi, \quad y = a\sin\theta\sin\phi$

$z = a\cos\theta$

$\therefore$ K.E., $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

and potential $V = -mgz = -mga\cos\theta$ (since m is below the fixed point O).

$\therefore L = T-V$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) + mga\cos\theta$

$\therefore p_\theta = -\frac{\partial L}{\partial\theta} = ma^2\dot{\theta}$ and

$p_\phi = -\frac{\partial L}{\partial\phi} = ma^2\dot{\phi}\sin^2\theta$ ——— (1).

Since L does not contain t explicity.

$\therefore H = T+V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - mga\cos\theta$

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations (1), we get

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \operatorname{cosec}^2\theta\, p_\phi^2\right) - mga\cos\theta.$$

Hence the four Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{ma^2}\operatorname{cosec}^2\theta \cot\theta\, p_\phi^2 - mga\sin\theta \quad \text{---} (H_1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2} p_\theta \quad \text{---} (H_2)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{1}{ma^2} p_\theta \quad \text{---} (H_3)$$

and $$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{ma^2}\operatorname{cosec}^2\theta\, p_\phi \quad \text{---} (H_4)$$

$\therefore$ from (H_3), Integrating $p_\phi = c$ (constant)

$\therefore$ from (H_4), we have

$$\dot{\phi} = \frac{1}{ma^2} c \operatorname{cosec}^2\theta = A/\sin^2\theta \quad (\text{where } A = c/ma^2)$$

Also from (H_1) & (H_2), we have

$$\ddot{\theta} = \frac{1}{ma^2}\dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2}\frac{\cos\theta}{\sin^3\theta}p_\phi^2 - mga\sin\theta\right]$$

$$= \frac{1}{(ma^2)^2}c^2\frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta, \quad \because p_\phi = c$$

$$= A^2\frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta \quad (\because A = c/ma^2)$$

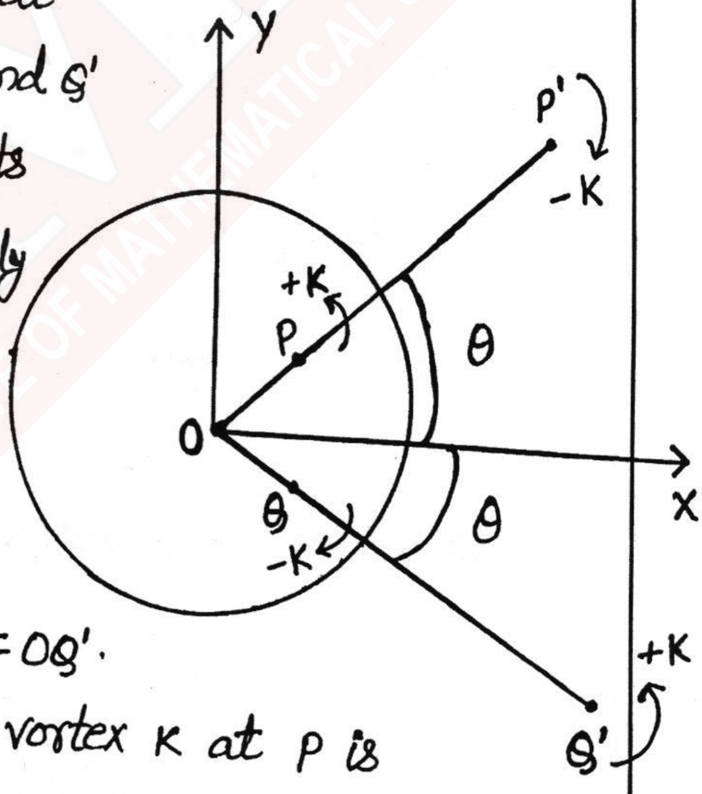
Multiplying bothsides by $2\dot{\theta}$ and integrating, we get

$$\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B, \quad (B \text{ is const}). \quad \text{---} (3)$$

Equations (2) and (3) determine the required motion.

8(c) When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation $(r^2\sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta$, θ being measured from the line through the centre perpendicular to the joint of the vortices.

Sol: Let K be the strength of the vortex at $P(r,\theta)$ and $-K$ at $Q(r,-\theta)$. Let P' and Q' be the inverse points of P and Q respectively with regard to the circular cylinder $|z|=a$.



So that $OP' = a^2/r = OQ'$.

Then the image of vortex K at P is a vortex $-K$ at P' and the image of vortex $-K$ at Q is a vortex K at Q'.

Hence the complex potential of the

System of four vortices is given by

$$w = \frac{ik}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

or $w = (ik/2\pi)\log(z - re^{i\theta}) + w'$,

Since the motion of vortex p is solely due to other vortices, the complex potential of the vortex at p is given by value of w' at $z = re^{i\theta}$.

$$\therefore [w']_{z=re^{i\theta}} = \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]_{z=re^{i\theta}}$$

$$\therefore \phi + i\psi = -\frac{ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta} - re^{-i\theta}) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$\therefore \psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \right.$$

$$\frac{1}{2}\log\left\{\left(r-\frac{a^2}{r}\right)^2\cos^2\theta+\left(r+\frac{a^2}{r}\right)^2\sin^2\theta\right\}\Bigg]$$

$$=-\frac{K}{2\pi}\left[\log\left(r-\frac{a^2}{r}\right)+\log(2r\sin\theta)-\frac{1}{2}\log\left\{\left(r^2+\frac{a^4}{r^2}-2r\cdot\frac{a^2}{r}\cos\right.\right.\right.$$

Thus, $$\psi=-\frac{K}{4\pi}\log\frac{(r-a^2/r)^2\,(2r\sin\theta)^2}{r^2+a^4/r^2-2a^2\cos 2\theta}$$

So the required streamlines are given by $\psi=$ constant. i.e.,

$$\frac{(r^2-a^2)^2\,r^2\sin^2\theta}{r^4+a^4-2a^2r^2\cos 2\theta}=b^2,\ \text{say}$$

i.e; $b^2(r^4+a^4-2a^2r^2\cos 2\theta)=r^2(r^2-a^2)^2\sin^2\theta$

i.e. $b^2\{(r^2-a^2)^2+2a^2r^2(1-\cos 2\theta)\}=r^2(r^2-a^2)^2\sin^2\theta$

i.e. $2a^2b^2r^2(1-\cos 2\theta)=(r^2-a^2)^2(r^2\sin^2\theta-b^2)$

or $4a^2b^2r^2\sin^2\theta=(r^2-a^2)^2(r^2\sin^2\theta-b^2)$.