A COURSE
OF
MATHEMATICAL ANALYSIS

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Structure. The interrelations between these structures as also the laws of each structure will also be obtained.

The satisfactory accounts of the theory of real numbers have only recently been given by Dedekind, Cantor and Weierstrass. The account, as given by Dedekind, with some modifications of details, will be considered here.

The treatment of complex numbers is not included in the scope of this book, dealing as it does with functions of real variables only.

In the following section, we shall give the fundamental properties of the aggregate of rational numbers.

3. Fundamental properties of the aggregate of rational numbers. We shall now describe the fundamental laws of the

Order Structure and Algebraic Structure

possessed by the aggregate of rational numbers.

I. **Order structure.** If a, b are two different rational numbers, then

$$\text{either } a > b \text{ or } a < b.$$

The order relation satisfies the following two properties :

(*i*) If a, b, c denote three rational numbers such that

$$a > b \text{ and } b > c,$$

then

$$a > c.$$

This property is expressed by saying that **Order relation is transitive.**

(*ii*) If a, b be two different rational numbers and $a < b$, then there is always a third rational number, c, such that

$$a < c < b.$$

This property is expressed by saying that the system of rational number is *dense*.

The number, c, is said to lie *between* a and b.

From this it follows that *between two different rational numbers, there lie an infinite number of rational numbers.*

II. **Algebraic structure.** This refers to the operations of addition and multiplication and the inverse operations of subtraction and division.

Addition. If a, b be two rational numbers, then $a+b$ is also a rational number. Addition obeys the following laws :

(*i*) *Commutative Law* :

$$a+b = b+a.$$

(*ii*) *Associative Law* :

$$(a+b)+c = a+(b+c).$$

(*iii*) *Existence of Additive identity called zero* :

$$a+0 = a.$$

(*iv*) *Existence of Additive Inverse called negative.* To each rational number, a, their corresponds another, $-a$, such that

$$a+(-a) = 0.$$

Multiplication. If a, b be two rational numbers, then, ab, is also a rational number. Multiplication obeys the following laws :

(*i*) *Commutative Law* :

$$ab = ba.$$

(*ii*) *Associative Law* :

$$(ab)c = a(bc).$$

(*iii*) *Existence of multiplicative identity, called unity* :

$$a\,1 = a.$$

(*iv*) *Existence of multiplicative inverse* :

To each non-zero rational number, a, there corresponds another, a^{-1}, such that

$$a\,a^{-1} = 1.$$

(*v*) *Distributive Law* :

$$a(b+c) = ab+ac.$$

This law relates the two operations of addition and multiplication.

Compatibility of the two structures. The following laws relate the two structures :

Order and addition.

If $a < b$, then $a+c < b+c$.

Order and multiplication.

If $a < b$ and $c > 0$, then $ac < bc$.

In view of the two above laws, we say that the order structure of the set of rational numbers is *compatible* with its algebraic structure.

There is another property of rational numbers ; *viz.*, that if a, b are rational numbers such that $a > b$, and $b \neq 0$, then there exists a positive integer n such that $nb > a$.

This property is expressed by saying that the set of Rational numbers is *Archimedean.*

Ex. 1. Employing strictly the properties of the set of rational numbers listed above, derive the following additional properties :—

(*i*) $a(-b) = -(ab)$. (*ii*) $(-a)(-b) = ab$.

(*iii*) $a(b-c) = ab-ac$.

2. There exists one and only one rational number x such that

$$a+x=b\,;$$

a, b being two given rational numbers.

3. There exists one and only one rational number x such that

$$ax=b\,;$$

a, b being two given rational numbers and $b \neq 0$.

4. Sections of Rational Numbers. *Let the set of rational numbers be divided into two classes L and R, in such a way that*

(*i*) *each class exists, i.e.*, the numbers do not all belong to the same class so that the other class contains no number and is void ;

(*ii*) *each number has a class, i.e.*, no number escapes classification;

(*iii*) *every member of L is less than every number of R.*

Such a division of rational numbers into two classes L and R is called a **section** of rational numbers and is denoted as (L, R); L is called the *lower*, and R, the *upper class* of the section.

Ex. Show that if (L, R) is a section, then any rational number which is less than a member of L is also a member of L and any rational number greater than a member of R is also a member of R.

4·1. Three types of sections. Illustrations.

(*i*) Let every rational number less than any rational number, say, 3 belong to L and every rational number $\geqq 3$ belong to R. Clearly the two classes L and R constitute a section, satisfying as they do the three characteristics of a section given above in § 4.

The class L has no greatest member but the class R has a least member, *viz.*, 3.

(*ii*) Let every rational number $\leqq$ any rational number, say, 3 belong to L and every rational number > 3 belong to R.

The class L of this section has a greatest member, *viz.*, 3, but the class R has no least member.

(*iii*) Let every negative rational number, zero, and every positive rational number whose square is less than 2 belong to L and every positive rational number whose square is > 2 belong to R.

In order to be sure that no number escapes classification, it is necessary to prove that there is no rational number whose square is equal to 2.

If possible, let p/q be a rational number such that

$$(p/q)^2 = 2 \text{ or } p^2 = 2q^2. \qquad .. (i)$$

We suppose that p, q have no common factor, for, such factors, if any, can be cancelled to begin with.

From (*i*), we see that p^2 is an even number. Therefore p must also be even. Let, then, $p = 2m$, where m is an integer. Therefore

$$4m^2 = 2q^2 \text{ or } q^2 = 2m^2.$$

Thus q^2 is also even and so q is even.

Hence p, q have a common factor 2 and this conclusion contradicts the hypothesis that p, q have no common factor.

It will now be shown that L has no greatest member and R no least.

If possible, let k be the greatest member of L so that

$$0 < k \text{ and } k^2 < 2.$$

Consider, now, the positive number $(4+3k)/(3+2k)$. We have

$$2 - \left(\frac{4+3k}{3+2k}\right)^2 = \frac{2-k^2}{(3+2k)^2} > 0, \text{ so that } \left(\frac{4+3k}{3+2k}\right)^2 < 2;$$

$$\frac{4+3k}{3+2k} - k = \frac{2(2-k^2)}{3+2k} > 0, \text{ so that } \frac{4+3k}{3+2k} > k.$$

Thus the positive number $(4+3k)/(3+2k)$ belongs to L and is greater than k so that we have a contradiction.

As above, it may also be shown that if k is the least member of R so that $k^2 > 2$, then $(4+3k)/(3+2k)$ is a still smaller member of R so that we again have a contradiction.

Conclusion. Thus we see that a section (L, R) may be such that

(*i*) L has no greatest member, but R has a least ;

(*ii*) L has a greatest member, but R has no least ;

(*iii*) L has no greatest member, and R has no least.

In order to see that these are the only *three* possibilities, it is necessary to show that for no section (L, R) can L have a greatest member *and* also R a least. This fact is easily seen to be true, if we observe that in such a case the infinite number of numbers lying between the greatest member of L and the least member of R will neither belong to L nor to R and thus escape classification so that the classes L, R will not constitute a section. [§ 4, (*ii*), page 5].

4·2. Modification in the definition of a section. It will be seen that to each rational number there correspond two sections according as it is the greatest member of the lower class or the least member of the upper class. The presentation of the theory of real numbers is a good deal simplified, if we so modify the definition of a section that to each rational number there corresponds only one section. Accordingly we modify the definition by further insisting *that the lower class must not have a greatest member.* Thus we now say that any division of rational numbers into two classes L and R is a section if

(*i*) *each class exists*; (*ii*) *each number has a class*; (*iii*) *every member of L is less than every member of R*; (*iv*) *L has no greatest member.*

Note. An equally suitable modification could have been that R has no least member.

4·3. Criteria for any set of rational numbers to constitute the lower class of a section. The following simple theorem which can be easily established will prove very convenient for the later developments :—

Any *given set of rational numbers can form the lower class of a section if, and only if, it is such that*

(*i*) *all the rational numbers do not belong to it*;

(*ii*) *it has no greatest member*;

(*iii*) *a rational number which is less than any member of the set is also a member of the set.*

If these conditions are satisfied, then all those rational numbers which do *not* belong to this set form the upper class of the section in question.

Ex. What are the conditions which must be satisfied by a set so that it may form the upper class of a section ?

4·4. A property of sections. *For any section (L, R), corresponding to any positive number, k, however small, there exists a member, x, of L and a member, y, of R, such that*

$$y-x=k.$$

Let a and b be any two members of the classes L and R respectively. There exists a positive integer n such that (§ 3, page 4)

$$nk > b-a, \text{ i.e., } a+nk > b.$$

Consider the set of numbers

$$a, a+k, a+2k, \ldots\ldots, a+nk.$$

Thus the set of real numbers is the set of sections of rational numbers.

The number, a, belongs to L and, $a+nk$, to R. There must exist, therefore, two consecutive members, say,

$$a+rk, a+(r+1)k$$

of this set such that $x=a+rk$ belongs to L and $y=a+(r+1)k$ to R.

These are, then, the required numbers x and y.

5. Set of real numbers.

Real number. Def. *A section of rational numbers is called a real number.*

Real rational number, irrational number. *A section is said to be a real rational, or an irrational number according as the upper class of the section has or does not have a least member.*

If, a, be the least member of the upper class R of the real rational number (L, R), then we say that the rational number, a, **corresponds** *to the real rational number (L, R).*

Notation. *The real rational number corresponding to the rational number, a, will always be denoted as, $\bar{a}$.*

5·1. Correspondence between rational and real rational numbers. In view of the modified definition of a section, we see that to each rational number, a, there corresponds only one real rational number, *viz.*, (L, R), where L consists of all those rational numbers which are $< a$ and R of those which are $\geqslant a$; also to each real rational number (L, R) corresponds only one rational number, *viz.*, the least member of R. Thus *there is a one-to-one correspondence between the set of rational numbers and the set of real rational numbers* which is the set of all those sections for which the upper class has a least member.

Note. To a beginner it might appear strange to call a *section*, which is only a division of rational numbers into two classes, a *real number*. There may be several reasons for this attitude on his part. One reason may be that the definition of a real number as a section is abstract and the idea of magnitude is foreign to it and the reader finds it difficult to disassociate *number* from any idea of magnitude which is the very basis of the manner in which he is introduced to the notion of number in Elementary Mathematics. Another reason for his difficulties may be that he is at a loss to understand as to how the notion of order and the four operations of addition, etc., can be extended to the new set. This latter difficulty is only temporary and in the following sections it is shown as to how this can be removed. The properties of the set of rational numbers given in § 3 will constitute the basis of all the developments that will be undertaken in this connection in the succeeding sections.

6. Order structure of the set of real numbers. Let

$$\alpha_1 \equiv (L_1, R_1,) \text{ and } \alpha_2 \equiv (L_2, R_2),$$

be two real numbers.

The following are the three mutually exclusive possibilities :—

(*i*) *L_1 is a proper part of L_2, i.e.*, every member of L_1 is also a member of L_2 but every member of L_2 is not a member of L_1 ;

(*ii*) *L_2 is a proper part of L_1* ;

(*iii*) *L_1 and L_2 are identical i.e.*, every member of L_1 is a member of L_2 and every member of L_2 is a member of L_1.

Def. *A real number $\alpha_1 \equiv (L_1, R_1)$ is said to be less than, greater than or equal to another real number $\alpha_2 \equiv (L_2, R_2)$ according as L_1 is a proper part of L_2 or L_2 is a proper part of L_1 or L_1 and L_2 are identical.*

These relations between α_1 and α_2 are respectively denoted by the symbols

$$\alpha_1 < \alpha_2, \quad \alpha_1 > \alpha_2, \quad \alpha_1 = \alpha_2.$$

Thus

$$\alpha_1 < \alpha_2 \text{ if and only if } L_1 \text{ is a proper part of } L_2.$$

Note. Since L_2 has no greatest member, we see that if L_1 is a proper part of L_2, there exist an *infinite* number of members of L_2 which do not belong to L_1, and which, therefore, belong to R_1.

Ex. 1. Show that $\alpha_1 < \alpha_2$ if $\alpha_2 > \alpha_1$.

2. Show that $\alpha_1 > \alpha_2$ if, and only if, L_1 and R_2 have an infinite number of common members.

7. Zero, positive and negative real numbers.

Def. *The real number, $\bar{0}$, corresponding to the rational number zero is called the real number zero.*

Thus the real number (L, R) is the real number zero if, 0, is the least member of R.

Def. *A real number is said to be positive or negative according as it is greater or smaller than the real number zero, $\bar{0}$.*

It is easy to see that the real number (L, R) is positive, if the class, L, contains at least one and, therefore, an infinite number of positive members; also it is negative if R contains at least one and, therefore, an infinite number of negative members.

8. Fundamental Laws of Order.

8·1. Transitivity of the order relation. *If α, β, γ, are three real numbers such that*

$$\alpha < \beta \text{ and } \beta < \gamma,$$

then

$$\alpha < \gamma.$$

Let $\qquad \alpha \equiv (L_1, R_1), \ \beta \equiv (L_2, R_2), \ \gamma \equiv (L_3, R_3).$

Since every member of L_1 is a member of L_2 and every member of L_2 is a member of L_3, therefore, every member of L_1 is a member of L_3.

Also, there exists a member of L_3 which does not belong to L_2 and this, again, cannot belong to L_1, for, if it did belong to L_1, it will also have to belong to L_2.

Hence L_1 is a proper part of L_3. Thus $\alpha < \gamma$.

Ex. 1. If $\alpha=\beta$ and $\beta > \gamma$; show that $\alpha > \gamma$.

2. Show that every positive real number is greater than every negative real number.

8·2. *If a, b, be any two members of the classes L, R respectively of a real number $\alpha \equiv (L, R)$,*

then $$\bar{a} < \alpha \text{ and } \bar{b} \geqslant \alpha.$$

Let $$\bar{a} \equiv (L_1, R_1) \text{ and } \bar{b} \equiv (L_2, R_2),$$

so that a, b are the least members of the classes R_1 and R_2 respectively.

Every member of L_1 which consists of the rational numbers $<a$ is also a member of L which contains a and, therefore, also every number $<a$; also, a is a member of L but not of L_1. Thus L_1 is a proper part of L and, accordingly $\bar{a} < \alpha$,

If b be the least member of R, then L and L_2 are identical and, therefore, $\alpha=\bar{b}$.

If b be not the least member of R, then it may also be easily shown that L is a proper part of L_2 so that $\alpha < \bar{b}$ or $\bar{b} > \alpha$.

8·3. *Between two different real numbers, there lie an infinite number of real rational numbers.*

Let $$\alpha_1 \equiv (L_1, R_1) \text{ and } \alpha_2 \equiv (L_2, R_2)$$

be two *different* real numbers. For the sake of definiteness, suppose that $\alpha_1 < \alpha_2$, so that L_1 is a proper part of L_2.

There exist an *infinite* number of rational numbers which belong to L_2 but not to L_1 and, therefore, belong to R_1.

If, a, be any one of these *infinite* rational numbers (other than the least member of R_1, if there be any), we have, by § 8·2 above
$$\alpha_1 < \bar{a} < \alpha_2.$$

Hence the result.

8·4. *If a and b are two rational numbers, then*
$$\bar{a} < \bar{b} \text{ if } a < b \,;\; \bar{a} = \bar{b} \text{ if } a = b \,;\; \bar{a} > \bar{b} \text{ if } a > b.$$

Let $$\bar{a} \equiv (L_1, R_1)\ \bar{b} \equiv (L_2, R_2)\,;$$

so that a, b are the least members of R_1 and R_2 respectively.

Let $$a < b.$$

A rational member which is $<a$ is also $<b$ so that every member of L_1 is a member of L_2.

Also the rational numbers which lie between the different rational numbers a and b belong to L_2 but not to L_1.

Thus L_1 is a proper part of L_2 and accordingly

$$\bar{a} < \bar{b},$$

Let, now $$a > b,$$
so that $$b < a.$$

Then, from above,

$$\bar{b} < \bar{a}$$

and therefore

$$\bar{a} > \bar{b}$$

The case of equality is obvious.

Note. The result proved above shows that the relationship of order between two real rational numbers (in terms of the definition of order given in § 6) is the same as that between the corresponding rational numbers (in terms of the definition of order between rational numbers).

Ex. Show that $\bar{a}$ is positive or negative according as the rational number a is positive or negative.

Algebraic structure of the set of real numbers

9. Sum of two real numbers. Let $\alpha_1 \equiv (L_1, R_1)$ and $\alpha_2 \equiv (L_2, R_2)$, be any two real numbers.

We shall now set up a section (L, R) of rational numbers to be called the sum of the real numbers α_1 and α_2.

Let a class L be formed of numbers obtained by adding every member of L_1 to every member of L_2.

Clearly, the class L exists and does not contain all the rational numbers.

In order to show that L can be the lower class of a section, we shall prove (§ 4·3 page 6) that any rational number which is less than any member of L is also a member of L.

Let, b, be a rational number less than any member a, of L which is obtained by adding the members a_1, a_2 of L_1, L_2 respectively so that

$$a = a_1 + a_2,$$

We write

$$b = a - x = a_1 + a_2 - x = (a_1 - x) + a_2,$$

where, x, is a positive rational number.

The number, $a_1 - x$, which is less than the member, a_1, of L_1, must also be a member of L_1. Thus we see that the number, b, can be obtained by adding the member $a_1 - x$ of L_1 to the member a_2 of L_2 and accordingly it must belong to L.

Since L_1 and L_2 have no greatest member, L also, can have no greatest member. (*Prove*)

Thus we find that L can be the lower class of a section (§ 4 3, page 6). The section (L, R), where R consists of all those rational numbers which do not belong to L, is called the sum of (L_1, R_1) and (L_2, R_2) and this relationship is exhibited as

$$(L, R) = (L_1, R_1) + (L_2, R_2) \equiv \alpha_1 + \alpha_2.$$

Def. *The sum of two real numbers* $\alpha_1 \equiv (L_1, R_1)$ *and* $\alpha_2 \equiv (L_2, R_2)$ *is defined to be the real number* (L, R) *where the class* L *is formed of all those rational numbers which are obtained by adding members of* L_1, *to members of* L_2.

9·1. The negative of a real number $\alpha \equiv (\mathbf{L}, \mathbf{R})$. We form a class L_1, consisting of the negatives of all the members of R excepting that of the least member of R, if there be any,

Clearly the class L_1 exists and has no greatest member.

In order to prove that L_1 can be the lower class of a section, it will be shown that a rational number less than any member of L_1 is also a member of L_1.

Let a_1' be a rational number less than any member a_1 of L_1.

Since $-(-a_1)=a_1$ *i.e.*, a_1 is the negative of $-a_1$, therefore $-a_1$ belongs to R_1. Also

$$\because \quad a_1' < a_1, \qquad \therefore \quad -a_1' > -a_1.$$

Thus, $-a_1'$, is a member of R, and accordingly $-(-a_1')=a_1'$ is a member of L_1.

The section (L_1, R_1) where R_1 consists of all those rational numbers which do not belong to L_1 is called the negative of (L, R) and is denoted by $-\alpha$ or by $-(L, R)$.

It is easy to show that R_1 will consist of the negatives of the members of L and the negative of the least member of R, if there be any.

Def. *The negative,* $-\alpha$, *of a real number*

$$\alpha \equiv (L, R)$$

is defined to be the real number (L_1, R_1) *where* L_1 *consists of the negatives of all the members of* R *excepting that of its least member, if there be any.*

9·2. The difference of two real numbers. The difference, $\alpha-\beta$, of two real members α and β is defined by the equality

$$\alpha-\beta=\alpha+(-\beta)$$

so that to obtain $\alpha-\beta$ we add the negative of β to α.

10. Fundamental laws of Addition. In what follows, α, β, γ will denote real numbers. Also let

$$\alpha \equiv (L_1, R_1),\ \beta \equiv (L_2, R_2),\ \gamma \equiv (L_3, R_3).$$

A·1. Commutative Law.

$$\alpha+\beta=\beta+\alpha.$$

Let a_1, a_2 denote any two members of L_1, L_2 respectively.

The result now follows from the fact that

$$a_1+a_2=a_2+a_1 \qquad (\S\ 3, \text{page } 3)$$

so that the two classes formed of numbers obtained by adding the members of L_1 to the members of L_2 and the members of L_2 to the members of L_1 are identical.

A·2. Associative Law.

$$\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma.$$

Let a_1, a_2, a_3 be any three members of L_1, L_2, L_3 respectively,

The result now follows from the fact that the associative law holds for the addition of rational numbers, *i.e.*,

$$a_1+(a_2+a_3)=(a_1+a_2)+a_3.$$

A·3. Existence of Additive identity.

$$\alpha+\bar{0}=\alpha.$$

Let

$$\bar{0}\equiv(L', R')$$

so that the rational number 0, is the least member of R', and let

$$\alpha \equiv (L_1, R_1) \text{ and } \alpha+\bar{0} \equiv (L, R).$$

We have to show that the classes L, L_1 are identical.

A member of L, obtained, as it is, by adding some member a_1 of L_1 to a member of L' which is negative is essentially less than a_1 and must, therefore, belong to L_1. Thus every member of L is a member of L_1.

Let b_1 be *any* member of L_1. Since L_1 has no greatest member, there exists a member, say, b_1+k, of L_1 greater than b_1, $(k>0)$.

The number b_1 which can be obtained by adding the member, b_1+k of L_1 to the member $-k$ of L' is also a member of L. Thus every member of L_1 is a member of L.

Therefore the classes L and L_1 are identical. Hence the result.

Note. This property is expressed by saying that *the real number* $\bar{0}$, *is the identity for addition.* The uniqueness property is established in cor. 2 after § 11·1.

A·4. Existence of Additive inverse.

$$\alpha+(-\alpha)=\bar{0}.$$

Let

$$\alpha \equiv (L_1, R_1),\ -\alpha \equiv (L, R)$$

$$\alpha+(-\alpha) \equiv (L', R').$$

We shall show that, 0, is the least member of R'.

The class L' is formed of numbers obtained by adding to the members of L_1 the negatives of the members of R_1 excepting that of its least if there be any.

Now, if a_1 be any member of L_1 and b_1 any member of R_1, (not the least), then L' consists of the numbers of the type $a_1+(-b_1)$, *i e.*, a_1-b_1, which are necessarily negative.

Thus every member of L' is negative.

Again, if k, be any negative rational number, there exists a member b_2 of R_1 and a member a_2 of L_1 such that

$$b_2-a_2=-k \qquad (\S\,4{\cdot}4)$$

i.e.,

$$a_2+(-b_2)=k,$$

so that, k, belongs to L'.

Thus every negative rational number belongs to L'.

From above we deduce that the rational number, 0, is the least member of R'. Hence the result.

Note. This property is expressed by saying *that, $-\alpha$, is the additive inverse of α.*

A·5. Compatibility of order with addition.

If
$$\alpha > \beta$$
then
$$\alpha+\gamma > \beta+\gamma.$$

We have
$$\alpha \equiv (L_1, R_1),\ \beta \equiv (L_2, R_2),\ \gamma \equiv (L_3, R_3).$$

Let
$$\alpha+\gamma \equiv (L', R'),\ \beta+\gamma \equiv (L'', R''),$$
so that the two classes L', L'' are formed of numbers obtained by adding the members of L_1 to the members of L_3 and the members of L_2 to the members of L_3 respectively.

Since every member of L_2 is a member of L_1, therefore, every member of L'' is also a member of L'.

Let, a_1, be any member of L_1 which does not belong to L_2. Thus, a_1, belongs to R_2. Let, a_1', be any member of L_1 which is $>a_1$.

Let
$$a_1'-a_1=\varepsilon,$$
so that, ε, is a positive rational number.

There exists a member a_3 of L_3 and a member b_3 of R_3 such that
$$b_3-a_3=\varepsilon \qquad (\S\ 4{\cdot}4).$$

We have
$$a_1'+a_3=a_1+\varepsilon+b_3-\varepsilon=a_1+b_3.$$

Now, $a_1'+a_3$, which is obtained by adding the member, a_1', of L_1 to the member, a_3 of L_3 is a member of L' ; also a_1+b_3 which is obtained by adding the member a_1 of R_2 to the member b_3 of R_3 is a member of R''.

Thus there exists a common member of L' and R'', *i.e*, there exists a member of L' which does not belong to L''.

Thus L'' is a proper part of L'. Hence the result.

Ex. 1. Show that the sum of two positive real numbers is positive and that of two negative numbers is negative.

2. If $\alpha > \beta$ and $\gamma > \delta$, show that $(\alpha+\gamma) > (\beta+\delta)$.

A·6. *If a, b be two rational numbers then*
$$\bar{a}+\bar{b}=\overline{a+b},$$
i.e., the sum of two real rational numbers is a real rational number corresponding to the sum of the corresponding rational numbers.

Let
$$\bar{a}=(L_1, R_1),\ \bar{b}=(L_2, R_2),\ \bar{a}+\bar{b}=(L, R).$$

The rational numbers a, b are the least members of R_1 and R_2 respectively. It has, now, to be shown that $a+b$ is the least member of R.

Let x, y be any members of L_1 and L_2 respectively so that $x+y$ is any member of L. We have

$$x < a \text{ and } y < b.$$

$$\therefore \qquad x+y < a+b.$$

Thus *every member of L is $< a+b$.*

Now consider any rational number, $a+b-k$, $(k > 0)$, which is $< a+b$. We write

$$a+b-k=(a-\tfrac{1}{2}k)+(b-\tfrac{1}{2}k).$$

Since $a-\frac{1}{2}k$ belongs to L_1 and $b-\frac{1}{2}k$ to L_2, therefore $a+b-k$ belongs to L.

Thus *every rational number $< (a+b)$ is a member of L.*

Hence $(a+b)$ is the least member of R.

Thus,

$$\bar{a}+\bar{b}=\overline{a+b}.$$

Ex. Show that the sum of a real rational and an irrational number is necessarily irrational.

Note. The six fundamental properties of addition proved above will be referred to as **A·1, A·2,..... , A·6** respectively.

11. Some derived properties of addition. We shall now derive some properties of addition from the fundamental ones proved in the preceding section.

11·1. *There exists one and only one real number, x, such that*

$$\alpha+x=\beta,$$

where α, β are two given real numbers.

Suppose that there exists a real number, x, such that

$$\alpha+x=\beta.$$

$$\begin{aligned}
\therefore \qquad (-\alpha)+(\alpha+x)&=(-\alpha)+\beta \\
[(-\alpha)+\alpha]+x&=(-\alpha)+\beta, && \text{(A·2)}\\
[(-\alpha)+\alpha]+x&=\beta+(-\alpha), && \text{(A·1)}\\
[\alpha+(-\alpha)]+x&=\beta+(-\alpha), && \text{(A·1)}\\
\bar{0}+x&=\beta+(-\alpha) && \text{(A·4)}\\
x&=\beta+(-\alpha), && \text{(A·3)}\\
&=\beta-\alpha, && \text{[By def.]}
\end{aligned}$$

Also we may now show that $x=\beta-\alpha$ satisfies the given equation. In fact, we have

$$\begin{aligned}
\alpha+(\beta-\alpha)&=\alpha+[\beta+(-\alpha)], && \text{[By def.]}\\
&=\alpha+[(-\alpha)+\beta], && \text{(A·1)}\\
&=[\alpha+(-\alpha)]+\beta, && \text{(A·2)}\\
&=\bar{0}+\beta, && \text{(A·4)}\\
&=\beta+\bar{0}, && \text{(A·1)}\\
&=\beta. && \text{(A·3)}
\end{aligned}$$

Cor. 1. $-(-\alpha)=\alpha$. This follows from the fact that α as well as $-(-\alpha)$ is the solution of

$$x+(-\alpha)=\bar{0}.$$

Cor. 2. We may show that

if $\alpha+\beta=\alpha$, then $\beta=\bar{0}$ and if $\alpha+\beta=\bar{0}$, then $\beta=-\alpha$.

Ex. Show that

$$-(\alpha+\beta)=-\alpha-\beta, \quad -(\alpha-\beta)=\beta-\alpha$$

11·2. *The number, $-\alpha$, is positive, negative or zero according as, α, is negative, positive or zero.*

Let, α, be negative, *i.e.*,

$$\alpha < \bar{0}.$$

$$\therefore \quad \alpha+(-\alpha) < \bar{0}+(-\alpha), \qquad \text{(A·5)}$$
$$\bar{0} < \bar{0}+(-\alpha), \qquad \text{(A·4)}$$
$$\bar{0} < (-\alpha)+\bar{0}, \qquad \text{(A·1)}$$
$$\bar{0} < -\alpha, \qquad \text{(A·3)}$$

i.e., $-\alpha$, is positive.

The remaining cases can be similarly disposed of.

11·3. *If* $\alpha > \beta$, *then* $-\alpha < -\beta$.

We have

$$\alpha > \beta$$

$$\therefore \quad \alpha+(-\alpha) > \beta+(-\alpha) \qquad \text{(A·5)}$$
$$\bar{0} > \beta+(-\alpha), \qquad \text{(A·4)}$$
$$(-\beta)+\bar{0} > (-\beta)+[\beta+(-\alpha)]$$
$$(-\beta)+\bar{0} > [(-\beta)+\beta]+(-\alpha), \qquad \text{(A·2)}$$
$$(-\beta)+\bar{0} > [\beta+(-\beta)]+(-\alpha), \qquad \text{(A·1)}$$
$$(-\beta)+\bar{0} > \bar{0}+(-\alpha), \qquad \text{(A·4)}$$
$$(-\beta)+\bar{0} > (-\alpha)+\bar{0}, \qquad \text{(A·1)}$$
$$-\beta > -\alpha. \qquad \text{(A·3)}$$

11·4. $\alpha-\beta$ *is positive or negative according as* $\alpha > \beta$ *or* $\alpha < \beta$.

Let $\alpha > \beta$.

$$\therefore \quad \alpha+(-\beta) > \beta+(-\beta) \qquad \text{(A·5)}$$
$$\alpha+(-\beta) > \bar{0}, \qquad \text{(A·4)}$$
$$\alpha-\beta > \bar{0}, \qquad \text{[By def.]}$$

Thus $(\alpha-\beta)$ is positive.

11·5. *The negative of any real rational number, $\bar{a}$, is also real rational and*

$$-(\bar{a})=\overline{(-a)}.$$

We have

$$a+(-a)=0.$$

$$\therefore \qquad \bar{a}+\overline{(-a)}=\bar{0}. \qquad \text{(A·6)}$$

so that $\qquad \overline{(-a)}=-(\bar{a}).$

Hence the result.

11·6. *If a, b be two rational numbers, then*

$$\bar{a}-\bar{b}=\overline{a-b},$$

i.e., the difference of two real rational numbers is a real rational number corresponding to the difference of the corresponding rational numbers.

We have

$$\begin{aligned}\bar{a}-\bar{b}&=\bar{a}+(-\bar{b}) &&\text{[By def.]}\\ &=\bar{a}+\overline{(-b)}, &&\text{(Proved above)}\\ &=\overline{a+(-b)}, &&\text{(A·6)}\\ &=\overline{(a-b)}.\end{aligned}$$

11·7. *Between two different real numbers, there lie an infinite number of irrational numbers.*

Let, α, β be two real numbers and let $\alpha < \beta$. Take γ any irrational number.

$$\because \qquad \alpha < \beta,$$

$$\therefore \qquad \alpha+(-\gamma) < \beta+(-\gamma), \text{ i.e., } \alpha-\gamma < \beta-\gamma. \qquad \text{(A·5)}$$

Let $\bar{a}$, be *any* one of the infinite number of real rational numbers lying between $\alpha-\gamma$ and $\beta-\gamma$, (§ 8·3). We have

$$\alpha-\gamma < \bar{a} < \beta-\gamma.$$

or $\qquad [\alpha+(-\gamma)]+\gamma < \bar{a}+\gamma < [\beta+(-\gamma)]+\gamma \qquad$ (A·5)

or $\qquad \alpha+\bar{0} < \bar{a}+\gamma < \beta+\bar{0} \qquad$ (A·2, A·4)

or $\qquad \alpha < \bar{a}+\gamma < \beta, \qquad$ (A·5)

so that the irrational number $\bar{a}+\gamma$ lies between α and β.

12. A property of positive sections. We shall now obtain a property of sections analogous to that of §4·4. This property will prove useful in the treatment of products of real numbers.

Corresponding to any rational number $k > 1$ and any positive real number (L, R), there exist positive members x, y, of L, R respectively such that $y/x=k$.

Let a, b be *any* positive members of L, R respectively.

We write $k=1+l$, so that l is positive.

There exists a positive integer n such that

$$n.al > (b-a), \text{ i.e., } a(1+nl) > b. \qquad \text{(§ 3, page 3)}$$

Consider the set of numbers

$$a, ak, ak^2, \ldots \ldots, ak^n$$

Since $ak^n = a(1+l)^n > a(1+nl) > b$, we see that ak^n belongs to R.

There must exist two consecutive members ak^r, ak^{r+1} of this set such that ak^r belongs to L and ak^{r+1} to R. These, then, are the required numbers x *and* y.

13. The product of two real numbers. Let

$$\alpha_1 \equiv (L_1, R_1) \text{ and } \alpha_2 \equiv (L_2, R_2),$$

be two real numbers.

Firstly we suppose that α_1, α_2 are both positive, so that L_1 and L_2 contain some positive rational numbers also.

We have to set up a section (L, R) to be called the product of α_1 and α_2.

Let a class L be formed of (i) all the negative rational numbers, (ii) the rational number zero, (iii) all those positive rational numbers which can be obtained by multiplying a positive member of L_1 with a positive member of L_2.

It will now be shown that L can be the lower class of a section. Clearly the class L exists and does not contain all the rational numbers.

Let, b, be any positive rational number which is smaller than a positive member, a, of L Let, a, be obtained by multiplying the positive members a_1, a_2 of L_1, L_2 respectively.

Let

$$b/a = x \text{ so that } 0 < x < 1.$$

We have

$$b = ax = (a_1a_2)x = a_1(a_2x).$$

The number, a_2x, which is smaller than, a_2, belongs to L_2. Thus we see that, b, is the product of the members a_1 and a_2x of L_1 and L_2 respectively and accordingly it is a member of L.

Since L_1 and L_2 have no greatest members, we easily see that L also cannot have a greatest member.

Thus we see that L can be the lower class of a section, say, (L, R).

The section (L, R) is called the product of (L_1, R_1) and (L_2, R_2) and this relationship is exhibited as

$$(L, R) = (L_1, R_1).(L_2, R_2) \equiv \alpha_1.\alpha_2,$$

or $$(L, R) = (L_1, R_1).(L_2, R_2) \equiv \alpha_1\alpha_2, \text{ omitting the dot.}$$

Def. The product of two positive real numbers $\alpha_1 \equiv (L_1, R_1)$ *and* $\alpha_2 \equiv (L_2, R_2)$ *is defined to be the real number* (L, R) *where the class* L *consists of* (i) *all the negative rational numbers,* (ii) *the rational number zero, and* (iii) *all those positive rational numbers which can be obtained on multiplying a positive member of* L_1 *with a positive member of* L_2.

Let, now, α be positive and β negative so that $-\beta$ is positive. (§ 11·2). Then, by def.

$$\alpha.\beta=-[\alpha.(-\beta)].$$

Let β be positive and α negative so that $-\alpha$ is positive. (§ 11·2). Then by def.,

$$\alpha\,\beta=-[(-\alpha).\beta].$$

Let α, β be both negative so that $-\alpha$, $-\beta$ are both positive. (§ 11·2). Then, by def.,

$$\alpha.\beta=(-\alpha).(-\beta).$$

Let either α or β or both be zero. Then, by def.,

$$\alpha\beta=\overline{0}.$$

Ex. *Show that for all the real numbers* α, β,

$$(-\alpha)(-\beta)=\alpha\beta,\ \alpha(-\beta),=-(\alpha\beta),\ (-\alpha)\beta=-(\alpha\beta).$$

By def.,

$(-\alpha)(-\beta)=\alpha\beta$, if α, β be both negative.

Let α, β be both positive so that $-\alpha$, $-\beta$ are both negative. Therefore, by def.

$$(-\alpha)(-\beta)=[-(-\alpha)][-(-\beta)],$$
$$=\alpha\beta.\quad (\S\ 11{\cdot}1,\ \text{Cor.})$$

Let α be positive and β negative so that $-\beta$ is positive and $-\alpha$ negative. By def.,

$$(-\alpha)(-\beta)=-\{[-(-\alpha)][-\beta]\}$$
$$=-[\alpha(-\beta)]$$
$$=\alpha\beta.\quad [\text{By def.}]$$

Let α be negative and β positive so that $-\alpha$ is positive and $-\beta$ negative. Then

$$(-\alpha)(-\beta)=(-\beta)(-\alpha)=\beta\alpha=\alpha\beta.$$

The remaining two results may be similarly proved.

13·1. The reciprocal of a non-zero real number.

Let $\alpha\equiv(L_1, R_1)$ be any positive real number.

Let a class L be formed of

(*i*) all the negative rational numbers, (*ii*) zero, and (*iii*) the reciprocals of all the members of the class R_1 excepting that of its least if it exists.

If a_1 be any positive member of L, then $1/a_1$ must be a member of R_1. Let $b_1 < a_1$ be any positive rational number where a_1 is a positive member of L.

We have $1/b_1 > 1/a_1$ so that $1/b_1$ belongs to R_1 and accordingly $1/(1/b_1)$, *i.e.*, b_1 belongs to L.

It is now easy to see that L can be the lower class of section, say, (L, R).

This section (L, R) is said to be the **reciprocal** of (L_1, R_1), *i.e.*, α and is denoted by α^*.

If α be negative so that $-\alpha$ is positive, then, by def.,

$$\alpha^* = -[(-\alpha)^*].$$

Def. The reciprocal of a positive real number $\alpha \equiv (L_1, R_1)$ *is defined to be the real number* (L, R) *where* L *is formed of (i) all the negative rational numbers, (ii) zero, and (iii) the reciprocals of all the members of* R_1 *excepting that of its least if there be any.*

Ex. Show that α^* is positive or negative, according as α is positive or negative.

13·2. The quotient of two real numbers. If α, β be two real numbers and $\beta \neq \bar{0}$, then the real number $\alpha.\beta^*$ which is the product of α and the reciprocal of β, is said to be obtained on dividing α and β and we write

$$\alpha \div \beta = \alpha\beta^*.$$

14. Fundamental Laws of Multiplication.

M·1. Commutative Law.

$$\alpha\beta = \beta\alpha.$$

The proof is simple and depends upon the fact that the multiplication of rational numbers follows the commutative law.

M·2. Associative Law.

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

The proof is simple.

M·3. Existence of Multiplicative Identity. *To prove that*

$$\alpha \cdot \bar{1} = \alpha.$$

Let

$$\alpha \equiv (L_1, R_1), \ \bar{1} \equiv (L_2, R_2), \ \alpha.\bar{1} = (L, R).$$

Since 1 is the least number of R_2, L_2 is comprised of all those rational numbers which are < 1.

Let α be positive.

If a_1 be any positive member of L_1 and a_2 of L_2 then a_1a_2 is a member of L.

Since $a_1a_2 < a_1.1 = a_1$, we see that a_1a_2 is also a member of L_1. Thus every member of L is a member of L_1.

Let, a_1, be any positive member of L_1. Let $a_2 > a_1$ be also a member of L_1. Since $a_1/a_2 < 1$, therefore, a_1/a_2 belongs to L_2. We have

$$a_1 = a_2.(a_1/a_2),$$

so that a_1 appears as the product of a positive member of L_1 and of a positive member of L_2 and accordingly a_1 is a member of L. Thus every member of L_1 is a member of L.

Therefore, L, L_1 are identical. Hence the result when α is positive.

Let, now, α be negative so that $-\alpha$ is positive. As proved above, we have

$$(-\alpha).\bar{1}=(-\alpha).$$

We have

$$\alpha.\bar{1}=-[(-\alpha).\bar{1}] \qquad \text{[By def.]}$$
$$=-(-\alpha)$$
$$=\alpha.$$

Note. This property is expressed by saying that the real number, $\bar{1}$, is the identity for multiplication.

M·4. Existence of Multiplicative Inverse.

$$\alpha.\alpha^*=\bar{1}, \textit{ where } \alpha\neq\bar{0}.$$

To start with, we suppose that α is positive.

Let

$$\alpha \equiv (L_1, R_1),\ \alpha^* \equiv (L, R),\ \alpha\,\alpha^* \equiv (L', R').$$

Let a_1 be any positive member of L_1 and b_1 any, but not the least member of R_1 so that $1/b_1$ is a positive member of L.

The positive members of L' are, therefore, of the type a_1/b_1.

Since $a_1/b_1 < 1$, we see that every member of L' is < 1.

Again, let k be any positive rational number < 1.

There exist positive members a_2, b_2 of L_1, R_1 such that $a_2/b_2=k$.

Since $k=a_2\left(\dfrac{1}{b_2}\right)$, we see that *every rational number* < 1 *is a member of* L'.

Hence, 1, is the least member of R' and accordingly

$$(L', R') \equiv \bar{1}.$$

Let, now, α be negative so that α^* is also negative.

We have, by the def. of the product of two negative numbers,

$$\alpha.\alpha^*=(-\alpha)[-(\alpha^*)]$$
$$=\bar{1}, \text{ as proved above.}$$

Note. This property is expressed by saying *that α^* is the multiplicative inverse of α.*

Notation for inverse. In view of the property proved above, we shall write

$$\alpha^*=\frac{\bar{1}}{\alpha}.$$

Also since

$$(\beta\alpha^*)\alpha=(\beta\alpha)\alpha^*=\beta(\alpha\alpha^*)=\beta\bar{1}=\beta.$$

We write

$$\beta\div\alpha=\beta\alpha^*=\frac{\beta}{\alpha}.$$

M·5. Compatibility of order with multiplication. *If* $\alpha < \beta$ *and* γ *is positive, then* $\alpha\gamma < \beta\gamma$.

Firstly suppose that α, β are both positive.

Let
$$\alpha \equiv (L_1, R_1),\ \beta \equiv (L_2, R_2),\ \gamma \equiv (L_3, R_3);$$
$$\alpha\gamma \equiv (L, R),\ \beta\gamma \equiv (L', R').$$

All the negative rational numbers and the rational number zero are necessarily contained in L as well as L'.

Since every member of L_1 is a member of L_2, therefore, every member of L is also a member of L'.

Let a_2 be any positive number which is a member of L_2 but not of L_1. Since L_2 has no greatest member, there exists a member of L_2, say a_2' which is $> a_2$.

The numbers a_2, a_2' both belong to R_1.

Let $a_2'/a_2 = k$, which is greater than 1.

There exist positive numbers a_3, b_3 of L_3, R_3, respectively such that
$$b_3/a_3 = k.$$

We have
$$a_2'a_3 = a_2ka_3 = a_2b_3.$$

Since a_2' is a positive member of L_2 and a_3 a positive member of L_3, therefore, $a_2'a_3$ *i.e.*, a_2b_3 belongs to L'. Also since a_2 is a member of R_1 and b_3 of R_3, therefore, a_2b_3 belongs to R and not to L. Thus each member L' is not a member of L. Therefore L is a proper part of L'. Hence the result.

Let, now, α, β be both negative so that $-\alpha$, $-\beta$ are both positive.

Since $\quad \alpha < \beta$, therefore $-\alpha > -\beta$ or $-\beta < -\alpha$. (§ 11·3)

$\therefore \quad -\beta\gamma < -\alpha\gamma$, as proved above,

or $\quad -(-\beta\gamma) > -(-\alpha\gamma)$, § 11·3

or $\quad \beta\gamma > \alpha\gamma$.

If α be negative and β positive then $\alpha\gamma$ is negative and $\beta\gamma$ positive and accordingly $\alpha\gamma < \beta\gamma$.

Cor. *If* $\alpha < \beta$ *and* γ *is negative, then* $\alpha\gamma > \beta\gamma$.

Since γ is negative, therefore, $-\gamma$ is positive.

$\therefore \quad \alpha(-\gamma) < \beta(-\gamma) \quad$ or $-(\alpha\gamma) < -(\beta\gamma)$,

or $\quad -[-(\alpha\gamma)] > -[-(\beta\gamma)]$, *i.e.*, $\quad \alpha\gamma > \beta\gamma$.

Ex. 1. Show that the product of two members, both positive or both negative is positive but if one number is positive and the other negative, then the product is negative.

2. If $\alpha > \beta$ and $\gamma > \delta$ and β, γ are positive, show that $\alpha\gamma > \beta\delta$.

M·6. *If* a, b *be two rational numbers, then*
$$\bar{a} \,.\, \bar{b} = \overline{ab},$$
i.e., the product of two real rational numbers is also a real rational number which corresponds to the product of the corresponding rational numbers.

Let

$$\bar{a} \equiv (L_1, R_1), \bar{b} \equiv (L_2, R_2), \overline{ab} \equiv (L, R).$$

Let a, b be both positive.

Let x, y be any positive members of L_1, L_2 respectively so that xy is a positive member of L.

$$\because \quad x < a \text{ and } y < b, \quad \therefore \ xy < ab$$

Thus every member of L is $< ab$.

Again consider any positive rational number abk which is $<ab$; so that $k < 1$.

We write

$$k = \frac{1+k}{2} \cdot \frac{2k}{1+k}.$$

Since $k < 1$, therefore, $(1+k)/2$ and $2k/(1+k)$ are both less than 1.

We have $$abk = \left(a \cdot \frac{1+k}{2}\right) \cdot \left(b \cdot \frac{2k}{1+k}\right).$$

Since $a.(1+k)/2 < a$ and $b.[2k/(1+k)] < b$, therefore, $a.(1+k)/2$ belongs to L_1 and $b.[2k/(1+k)]$ belongs to L_2 and accordingly their product abk belongs to L.

Thus every rational number $< ab$ is a member of L.

Hence ab is the least member of R so that

$$\bar{a}.\bar{b} = \overline{ab}.$$

The case where a, b are both negative or only one of them is negative can now be easily disposed of.

M·7. Distributive Law.

$$\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma.$$

Let

$$\alpha \equiv (L_1, R_1), \beta \equiv (L_2, R_2), \gamma \equiv (L_3, R_3).$$

Let

$$\alpha(\beta+\gamma) \equiv (L, R) \text{ and } \alpha\beta+\alpha\gamma \equiv (L', R').$$

Firstly we consider the case, when α, β, γ are all positive.

All the negative rational numbers and the rational number zero are necessarily members of L as well as of L'.

The positive members of L are of the type

$$a_1(a_2+a_3),$$

and the positive members of L' are of the type

$$a_1a_2+a_1'a_3,$$

where a_1, a_1' are any positive members of L_1 and a_2, a_3 are any positive members of L_2, L_3 respectively.

Since

$$a_1(a_2+a_3) = a_1a_2+a_1a_3,$$

therefore, on taking $a_1' = a_1$, we see that every positive member of L is also a member of L'.

Any member $a_1a_2+a_1'a_3$ of L' is clearly a member of L, if $a_1'=a_1$. In general, let $a_1 > a_1'$ so that $a_1'/a_1 < 1$.

We write

$$a_1'a_3=a_1[(a_1'/a_1)a_3]=a_1a_3', \text{ say}$$

Now

$$a_3'=(a_1'/a_1).a_3 < 1.a_3=a_3,$$

and therefore a_3' belongs to L_3.

Since

$$a_1a_2+a_1'a_3=a_1a_2+a_1a_3'=a_1(a_2+a_3'),$$

we see that every positive member of L' is also a member of L.

Thus L, L' are identical. Hence the result.

Before proceeding to consider the other cases, we prove that

$$\alpha(\beta-\gamma)=\alpha\beta-\alpha\gamma,$$

where α, β, γ are all positive.

Case I. Let $\beta=\gamma$.

$$\text{L. H. S.}=\alpha(\beta-\beta)=\alpha.\bar{0}=\bar{0}. \qquad \S\ 13$$

$$\text{R. H. S.}=\alpha\beta-\alpha\beta=\bar{0} \qquad \S\ 9{\cdot}1.$$

Case II. Let $\beta > \gamma$ so that $\beta-\gamma$ is positive. § 11·4, page 15.

We have

$$\begin{aligned}\alpha\beta&=\alpha[\gamma+(\beta-\gamma)]\\&=\alpha\gamma+\alpha(\beta-\gamma), && \text{Proved above.}\end{aligned}$$

$$\begin{aligned}\therefore\quad \alpha\beta-\alpha\gamma&=\alpha\gamma+[\alpha(\beta-\gamma)-\alpha\gamma]\\&=\alpha(\beta-\gamma).\end{aligned}$$

Case III. Let $\beta < \gamma$ so that $\beta-\gamma$ is negative and $\gamma-\beta$ positive.

We have proved in case II that

$$\alpha(\gamma-\beta)=\alpha\gamma-\alpha\beta$$

$$\begin{aligned}\text{Now}\quad \alpha(\beta-\gamma)&=-\alpha\{[-(\beta-\gamma)]\} && \text{by def.}\\&=-[\alpha(\gamma-\beta)] && \text{Ex. after § 11·1.}\\&=-[\alpha\gamma-\alpha\beta] && \text{Proved above.}\\&=\alpha\beta-\alpha\gamma. && \text{Ex. after § 11·1.}\end{aligned}$$

We now return to the main result.

Let α, β be negative and γ positive and let $\beta+\gamma$ be positive.

$$\begin{aligned}\text{We have}\quad \alpha(\beta+\gamma)&=-[(-\alpha)(\beta+\gamma)]\\&=-\{(-\alpha)[-(-\beta)+\gamma)]\} && \text{§ 11·1, cor.}\\&=-\{(-\alpha)[\gamma-(-\beta)]\}\\&=-[(-\alpha)\gamma-(-\alpha)(-\beta)] && \text{Proved above.}\\&=-[-\alpha\gamma-\alpha\beta]\\&=-(-\alpha\gamma)-(-\alpha\beta)\\&=\alpha\gamma+\alpha\beta\\&=\alpha\beta+\alpha\gamma.\end{aligned}$$

Let α, β, be negative and γ positive and let $\beta+\gamma$ be negative.

We have

$$\alpha(\beta+\gamma)=(-\alpha)[-(\beta+\gamma)]$$
$$=(-\alpha)(-\beta-\gamma), \quad \text{Ex. after § 11·1.}$$
$$=(-\alpha)(-\beta)-(-\alpha)(\gamma), \text{ proved above}$$
$$=\alpha\beta+\alpha\gamma.$$

The other possibilities may be similarly discussed.

15. Some derived properties of Multiplication.

15·1. *There exists one and only one real number x such that*

$$\alpha x=\beta,$$

where α, β are two given numbers and $\alpha \neq \bar{0}$.

Suppose that there exists a real number x such that

$$\alpha x=\beta,$$

$$\therefore \quad \alpha^*(\alpha x)=\alpha^*\beta$$
$$(\alpha^*\alpha)x=\alpha^*\beta, \qquad (\text{M}\cdot 2)$$
$$\bar{1}x=\alpha^*\beta \qquad (\text{M}\cdot 1, \text{M}\cdot 4)$$
$$x=\alpha^*\beta. \qquad (\text{M}\cdot 3)$$

Also we may easily show that $\alpha^*\beta$ satisfies the given equation

Cor. 1. $(\alpha^*)^*=\alpha$.

Cor. 2. It may be shown that

if $\alpha\beta=\alpha$ and $\alpha \neq \bar{0}$, then $\beta=\bar{1}$ and if $\alpha\beta=\bar{1}$, then $\beta=\alpha^*$.

Ex. Show that $\dfrac{\bar{1}}{\alpha\beta}=\dfrac{\bar{1}}{\alpha}\cdot\dfrac{\bar{1}}{\beta}$.

15·2. *If α, β are two positive real numbers such that*

$$\alpha < \beta,$$

then

$$\bar{1}/\alpha > \bar{1}/\beta.$$

We have $\qquad \alpha < \beta.$

$$\therefore \quad \alpha^*\alpha < \alpha^*\beta \qquad (\text{M}\cdot 5)$$
$$\bar{1} < \alpha^*\beta, \qquad (\text{M}\cdot 1, \text{M}\cdot 4)$$
$$\bar{1}\beta^* < (\alpha^*\beta)\beta^*=\alpha^*(\beta\beta^*), \qquad (\text{M}\cdot 2)$$
$$=\alpha^*\bar{1}. \qquad (\text{M}\cdot 4)$$
$$\therefore \quad \beta^* < \alpha^* \qquad (\text{M}\cdot 3)$$

i.e., $\qquad \bar{1}/\alpha > \bar{1}/\beta.$

15·3. *The reciprocal of non-zero real rational number, $\bar{a}$,* is real *rational and*

$$(\bar{a})^*=\overline{\left(\frac{1}{a}\right)}, \text{ i.e., } \frac{\bar{1}}{\bar{a}}=\overline{\left(\frac{1}{a}\right)}.$$

e have

$$a\,\frac{1}{a}=\bar{1}.$$

$\therefore$
$$\bar{a}\,\overline{\left(\frac{1}{a}\right)}=\bar{1}. \qquad \text{(M·6)}$$

Thus

$$\frac{\bar{1}}{\bar{a}}=\overline{\left(\frac{1}{a}\right)}.$$

15·4. *The quotient of two real rational numbers is also a real rational number which corresponds to the quotient of the corresponding rational numbers.*

Let a, b be two rational numbers and $b \neq 0$.

We have

$$\begin{aligned}\frac{\bar{a}}{\bar{b}} &=\bar{a}\left(\frac{\bar{1}}{\bar{b}}\right)\\ &=\bar{a}\,\overline{\left(\frac{1}{b}\right)}. && \text{(Proved above)}\\ &=\overline{\left(a\frac{1}{b}\right)}, && \text{(M·6)}\\ &=\overline{\left(\frac{a}{b}\right)}.\end{aligned}$$

16. The modulus of a real number. Def. *By the modulus of a real number, α, is meant the real number α, $-\alpha$ or $\bar{0}$ according as, α, is positive, negative or zero and is written as* $|\alpha|$. Thus

$$|\alpha| = \begin{cases} \alpha, \text{ if } \alpha \text{ is positive,}\\ \bar{0}, \text{ if } \alpha \text{ is zero,}\\ -\alpha, \text{ if } \alpha \text{ is negative.}\end{cases}$$

It will be seen that the modulus of a real number cannot be negative. Also, we have, for every α,

$$\alpha \leqslant |\alpha| \text{ as also } -\alpha \leqslant |\alpha|$$

Ex. 1. Show that $|\alpha-\beta| = |\beta-\alpha|$.

Ex. 2. If $|\alpha| < k$, show that $-k < \alpha < k$.

17. Some results involving moduli.

In the following, the reader should supply the reference to each result employed.

17·1. *To prove that*

$$|\alpha+\beta| \leqslant |\alpha|+|\beta|.$$

Case I. Let $\alpha+\beta$ be positive.

We have

$$|\alpha|+|\beta| \geqslant \alpha+\beta = |\alpha+\beta|,$$

for

$$|\alpha| \geqslant \alpha,\ |\beta| \geqslant \beta$$

Case II. Let $\alpha+\beta$ be negative.

We have

$$|\alpha| + |\beta| \geqslant (-\alpha)+(-\beta) = -(\alpha+\beta) = |\alpha+\beta|$$

for

$$|\alpha| \geqslant -\alpha, \quad |\beta| \geqslant -\beta.$$

Thus in either case, we have

$$|\alpha+\beta| \leqslant |\alpha| + |\beta|.$$

17·2. *To prove that*

$$|\alpha-\beta| \geqslant \big| |\alpha| - |\beta| \big|.$$

We have

$$\alpha=\alpha-\beta+\beta.$$

$$\therefore \quad |\alpha| = |\alpha-\beta+\beta| \leqslant |\alpha-\beta| + |\beta|,$$

or

$$|\alpha| - |\beta| \leqslant |\alpha-\beta| + |\beta| - |\beta| = |\alpha-\beta| + 0$$
$$= |\alpha-\beta|.$$

$$\therefore \quad |\alpha-\beta| \geqslant |\alpha| - |\beta|.$$

Also, $\therefore \quad |\alpha-\beta| = |\beta-\alpha| \geqslant |\beta| - |\alpha|.$

Finally, we have

$$\big| |\alpha| - |\beta| \big| = \begin{cases} |\alpha| - |\beta| \\ \quad or \\ |\beta| - |\alpha| \end{cases}$$

$$\therefore \quad |\alpha-\beta| \geqslant \big| |\alpha| - |\beta| \big|.$$

17·3. *If* $\quad |\alpha-\beta| < \gamma,$

then $\quad \beta-\gamma < \alpha < \beta+\gamma.$

We have

$$\alpha-\beta \leqslant |\alpha-\beta| < \gamma$$

so that $\quad \alpha-\beta < \gamma$, *i.e.*, $\alpha < \beta+\gamma.$

Also

$$\beta-\alpha \leqslant |\alpha-\beta| < \gamma$$

so that $\quad \beta-\alpha < \gamma$, *i.e.*, $\beta-\gamma < \gamma.$

Thus we have

$$\beta-\gamma < \alpha < \beta+\gamma.$$

17·4. *To prove that*

$$|\alpha\beta| = |\alpha| . |\beta|.$$

Case I. Let α, β be both positive so that $\alpha\beta$ is also positive.

We have

$$|\alpha\beta| = \alpha\beta = |\alpha| . |\beta|.$$

Case II. Let α, β be both negative so that $\alpha\beta$ is positive.

We have

$$|\alpha\beta| = \alpha\beta = (-\alpha).(-\beta) = |\alpha|.|\beta|.$$

Case III. Let α be positive and β negative so that $\alpha\beta$ is negative.

We have

$$\begin{aligned} |\alpha\beta| &= -(\alpha\beta) \\ &= -\{-[(\alpha).(-\beta)]\} \\ &= (\alpha).(-\beta) = |\alpha|.|\beta|. \end{aligned}$$

Case IV. Let α be negative and β positive.

We have

$$\begin{aligned} |\alpha\beta| &= |\beta\alpha| \\ &= |\beta|.|\alpha|. \text{ Case III above.} \\ &= |\alpha|.|\beta|. \end{aligned}$$

17·5. *To prove that*

$$\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|}, \quad \text{if } \beta \neq \bar{0}.$$

If β be positive, then $\bar{1}/\beta$ is also positive, so that we have

$$\left|\frac{\bar{1}}{\beta}\right| = \frac{\bar{1}}{\beta} = \frac{\bar{1}}{|\beta|}.$$

If β be negative, then $\bar{1}/\beta$ is also negative, so that we have

$$\left|\frac{\bar{1}}{\beta}\right| = -\frac{\bar{1}}{\beta} = \frac{\bar{1}}{-\beta} = \frac{\bar{1}}{|\beta|}$$

Finally, we have

$$\left|\frac{\alpha}{\beta}\right| = \left|\alpha\frac{\bar{1}}{\beta}\right| = |\alpha|\left|\frac{\bar{1}}{\beta}\right| = |\alpha|.\frac{\bar{1}}{|\beta|} = \frac{|\alpha|}{|\beta|}.$$

18. Sections of real numbers. Dedekind's theorem. *If all the* **real** *numbers be divided into two classes L and R such that*

(i) each class exists, (ii) each real number has a class, (iii) every member of L is less than every member of R, then

either the class L has a greatest member, or the class R has a least member.

We form two classes L_1, R_1 consisting of rational numbers which correspond to the real rational members of L, R respectively.

Let the class L_1 have a greatest member say, a. The real rational number $\bar{a}$ will belong to L. It will be shown that $\bar{a}$, is the greatest member of L.

If not, let α be a greater member of L.

Let $\bar{b}$ be any one of the infinite number of real rational numbers lying between $\bar{a}$ and α so that we have

$$\bar{a} < \bar{b} < \alpha.$$

Since $\bar{b}$ is less than a member α of L it must itself belong to L and accordingly b *is a member of* L_1.

Also, $\therefore$ $\bar{a} < \bar{b}$,

$\therefore$ $a < b$. [§ 8·4, page 9)

Thus b, a member of L_1 is greater than the greatest member a of L_1.

This conclusion is absurd.

Hence in this case L must have a greatest member.

Let the class R_1 have a least member, say, b. In this case $\bar{b}$ will be the least member of R. This may be proved as above.

Let neither L_1 have a greatest member nor R_1 a least. In this case the section (L_1, R_1) is an irrational number, α, which must either belong to L or to R.

Let α belong to L. It will be shown that α is the greatest member of L. If not, let β be a greater member of L.

Let, $\bar{a}$, be any real rational number such that

$$\alpha < \bar{a} < \beta. \qquad \text{(§ 8·3, page 9)}$$

The number, $\bar{a}$, belongs to L and, therefore, a, belongs to L_1.

Thus there exists a member, a, of the lower class L_1, of a real number $\alpha \equiv (L_1, R_1)$ such that

$$\bar{a} > \alpha$$

and this is absurd. (§ 8·2, page 8)

It may similarly be shown that if, α, belongs to R, then it is the least member of R.

Note 1. The theorem is sometimes stated in the following equivalent form :—

If all the real numbers are divided into two classes L, R such that, (*i*) each class exists; (*ii*) each number has a class, (*iii*) every member of L is less than every member of R, then *there exists a real number α such that every real number less than α belongs to L and every real number greater than α belongs to R ; α itself may belong to either class.*

This number, α, is the greatest number of L or the least of R whichever may exist. Also this number, α, is said to determine the section.

Note 2. The theorem discussed above indicates a fundamental difference between the sections of rational numbers and the sections of real numbers inasmuch as we have seen, that if (L, R) be a section of rational numbers, it is possible that neither L may have a greatest member nor R may have a least, but if it were a section of real numbers, then this cannot be the case. This difference is generally described by saying that there may be a gap between the classes L, R of rational numbers but there is no gap between the classes L, R of real numbers ; the system of rational numbers has gaps while the system of real numbers has none.

Note 3. It is easy to show that any given set of real numbers will form the lower class of a section, if and only if (*i*) all the numbers do not belong to it, (*ii*) a number which is less than any member of the set is also a member of the set.

Note 4. It is interesting to note that Dedekind's theorem proved above has reference purely to the *Order structure* of the set of real numbers and *algebraic structure* comes nowhere in the picture.

19. Representation of real numbers by points along a straight line. Every thinking person possesses an intuitive idea of a straight line which, further, he can easily conceive as composed of points. Even though this physical notion of a straight line and that of points on it has nothing to do with Analysis as such, yet it provides a very convenient and helpful *picture* of the set of real numbers and is often employed in the course of study of Analysis to provide suitable language and suggest ideas. One danger, which is inherent in this use should, however, be avoided ; it may be that we accept a proposition suggested by this picture, obvious as it may seem, as obviously true and this obviousness may blind us to the necessity of a rigorous proof.

We now proceed to see how a straight line can be employed to provide a picture of the set of real numbers.

We consider any straight line and mark any two points O and A on it. The point O divides the line into two parts ; the part containing the point A will be termed positive and the other negative.

According to the usual convention, the line in question is always drawn parallel to the printed lines of the page and the point A taken on the right of O. Representing the rational numbers 0 and 1 by the points O and A respectively, we find a point P of the line representing any rational number p/q, $(q>0)$ by marking from O, $|p|$ steps each equal to the qth part of OA to the right or to the left of O according as p is positive or negative.

It is easy to see that if a, b be two rational numbers and $a<b$, then the point representing, b, lies to the right of the point representing a.

If we call the points which represent rational numbers as rational points, we see that, since the set of rational numbers is dense, an infinite number of rational points lie between every two different rational points.

Insufficiency of rational numbers to provide a picture of straight line. Even though, as we have seen above, a line can be covered with rational points as closely as we like, there exist points of the line which are not rational. For example, a point P such that OP is equal to the diagonal of the square with side OA is one such point (§ 4 1). Also a point L on the line such that OL is any rational multiple p/q of OP cannot be a rational point. For, if possible, let L represent a rational number m/n, so that we have

$$\frac{p}{q} \cdot OP = OL = \frac{m}{n} \quad \text{or} \quad OP = \frac{mq}{np},$$

which shows that OP is a rational, *i.e.*, P is rational point and this is a contradiction.

Thus we see that the set of rational numbers is not sufficient to provide us with a picture of complete straight line.

Real numbers. Let

$$\alpha \equiv (L, R)$$

be any real number. The section (L, R) of rational numbers determines a section of the rational points of the line into two classes A and B such that A consists of rational points corresponding to the members of L and B of rational points corresponding to the members of R. Every point of the class A will lie to the left of every point of the class B.

From our intuitive picture of a straight line and its *continuity*, we can convince ourselves that there will exist a point P of the line separating the two classes in the sense that every point of the line lying to the left of P belongs to the class A and every point lying to the right of P belongs to the class B. *This point P*, we say, *denotes the real number* (L, R). Thus to every real number there corresponds a point of the line.

Conversely, let P be any point of the line. The point P divides the rational points of the line into two classes A and B such that the points lying to the left of P belong to A and those to the right of P belong to B; the point P, if rational, belongs to B. The classes A, B of rational points determine a section (L, R) of rational numbers, which corresponds to the point P.

Thus to every point there corresponds a real number.

The set of real numbers is called the *Arithmetical Continuum* and the set of points on a straight line is called the *linear Geometric Continuum*. In view of what has been shown above, we see that there is a one-to-one correspondence between the two sets or continua and accordingly it may be found convenient to use the word 'point' for 'real number.'

20. Notation for real rational numbers. From above it will be seen that if (L, R) is a real rational number, then the point P which denotes this real number also denotes the rational number which is the least member of R and which, we know, is the rational number corresponding to (L, R). Thus we see that, according to the manner of representation explained above, a real rational number and the corresponding rational number are denoted by the same point of the line.

Also it has been seen that

$$\begin{aligned} &\text{if} \quad a>b, \text{ then } \quad \bar{a}>\bar{b} \ ; \\ &\text{if } a\pm b=c, \text{ then } \bar{a}\pm\bar{b}=\bar{c} \ ; \\ &\text{if} \quad ab=c, \text{ then } \bar{a}\,.\,\bar{b}=\bar{c} \ ; \\ &\text{if} \quad a/b=c, \text{ then } \quad \bar{a}/\bar{b}=\bar{c} \ ; \end{aligned}$$

where a, b, c denote rational numbers.

Thus, for example, the proposition

$$2+3=5,$$

where 2, 3, 5 denote rational numbers, remains true even when they denote corresponding real rational numbers which have so far been denoted by the symbols $\bar{2}$, $\bar{3}$, $\bar{5}$.

In view of this, *we agree to denote*, for future developments, *a real rational number by the same symbol which denotes the corresponding rational number* so that if 'a' is a rational number, then the same symbol 'a' will also, now, be used to denote the corresponding real rational number which has so far been denoted by $\bar{a}$.

The context in which the symbol may appear will fix the interpretation.

This use of a single symbol to denote two different concepts leads to no confusion, but is helpful, inasmuch as we have seen that a statement, which describes some relation between rational numbers, remains true when the symbols for rational numbers are interpreted as symbols for the corresponding real rational numbers.

An important note. In the following chapters, the word 'number' will always mean 'real number' and the word 'rational number' will mean 'real rational number'.

Exercises

1. Give a careful outline (without proofs) of Dedekind's introduction of real numbers by means of sections of rational numbers.

2. Give an account of Dedekind's theory of real numbers. Show that there are gaps between rational numbers, but the continuum of real numbers, as postulated by Dedekind, is free from gaps.

3. In order to generalize the conception of number, what are the essential requisites which must be satisfied. Develop Dedekind's theory of real numbers, and show how far this theory satisfies these requisites.

4. State and prove 'Dedekind's theorem' on real numbers.

5. Explain briefly the theory of real numbers and establish their correspondence with the points of a line continuum.

Define addition of real numbers by the use of a section of rational numbers and show that the Associative law of addition holds.

6. What is Arithmetic and what is Geometric continuum. Explain under which conditions they are equivalent.

7. Give Dedekind's definition of a real number as a section of rational numbers.

If α is any real number, different from zero, define $-\alpha$ and $1/\alpha$.

If α and β are any two real numbers, define $\alpha+\beta$, $-\alpha\beta$, $\alpha\beta$ and α/β.

8. Prove that, between any two real numbers, there is an unlimited number of real numbers.

Applying Dedekind's definition of a real number, explain briefly how you would show that real numbers obey the fundamental laws of arithmetic.

9. Explain Dedekind's extension of the system of rational numbers by means of sections of them and show that the system of 'real numbers' so obtained cannot be further extended by the same process.

10. Describe Dedekind's method of introducing *irrationals* by the sections of the rationals. Prove that no new numbers are obtained by considering sections of the *reals*.

11. Define the operation of multiplication for two numbers defined by sections of rationals.

Show that if every rational number whose square is less than 2 is put into the lower class and every rational number whose square is greater than 2 is put into the upper class, this division is not a section.

12. If d is a positive integer but not the square of an integer, show that, d, is not the square of a rational number.

13. A law of composition denoted by the symbol $\odot$ is defined on every pair of real numbers a, b as follows :—

$$a \odot b = a + b + ab.$$

Show that this law of composition is commutative and associative. Show also that multiplication is not distributive *w. r.* to this law, *i.e.*,

$$a[b \odot c] \odot ab + ac.$$

14. Show that while multiplication distributes addition, addition does not distribute multiplication.

15. A law of combination denoted by the symbol $\odot$ is defined on every pair of positive integers a, b as follows :—

$$a \odot b = a^b \quad \text{[The usual exponentiation]}.$$

Show that this law is neither commutative nor associative.

CHAPTER II

BOUNDS AND LIMITING POINTS

21. A set, S, of real numbers is defined, when there is given a law or laws which determine, without ambiguity, whether any given real number does or does not belong to it. A set of numbers may also be spoken of as a set of points.

Intervals, closed and open. The set of numbers, x, such that $a \leqslant x \leqslant b$, where a, b are two numbers, is called a *closed interval* $[a, b]$. The set of numbers, x, such that $a < x < b$ is called an open interval $]a, b[$. The sets $a < x \leqslant b$, $a \leqslant x < b$ are called intervals open on the left and open on the right respectively and are denoted as $]a, b]$, and $[a, b[$.

We write

$$[a, b]=\{x : a \leqslant x \leqslant b\}.$$

This means that $[a, b]$ denotes the set of points x such that $a \leqslant x \leqslant b$. Similarly we have

$$]a, b[=\{x : a < x < b\},$$

$$]a, b]=\{x : a < x \leqslant b\},$$

$$[a, b[=\{x : a \leqslant x < b\}.$$

Neighbourhood of a point. If I is an open interval and ξ a real number, we say that I is a neighbourhood of ξ if ξ is a member of I.

Sub-sets. A set, S_1, is said to be a sub-set of another, S, if every member of S_1 is also a member of S.

Finite and infinite sets. A set is *finite*, if there exists a positive integer, n, such that it contains just, n, members and otherwise the set is *infinite*.

For example, the set of all the integers between 20 and 30 is finite, but the set of all the rational numbers between 20 and 30 is infinite.

22. Greatest and least members of a set. A number M is the greatest member of a set S, if

(*i*) M is a member of S;

(*ii*) no member of S is greater than M.

Again, a number m is the least member of a set S, if

(*i*) m is a member of S;

(*ii*) no member of S is less than m.

The greatest and least members of a set are also respectively called the *maximum* and *minimum* members of the set.

Every finite set has necessarily a greatest and a least member, but an infinite set may or may not have a greatest or a least member.

The set of all the integers has neither a greatest nor a least member; the set of positive integers has no greatest member but has a least *viz.*, 1; the set of negative integers has no least member but has a greatest *viz.*, -1.

1 is the greatest and 0 is the least member of the closed interval $[0, 1]$; the open interval $]0, 1[$ has neither a greatest nor a least member; the semi-closed interval $[0, 1[$ has no greatest member but has a least *viz.*, 0; the semi-closed interval $]0, 1]$ has no least member but has a greatest member, *viz.*, 1.

23. Bounded and unbounded sets. Rough bounds. If there exists a number, K, such that every member of a set S is $\leqslant K$, then we say that S is *bounded above* or that it is *bounded on the right* and further say that K is a *rough upper bound* of S.

Similarly, if there exists a number, k, such that every member of a set S is $\geqslant k$, then we say that S is *bounded below* or that it is *bounded on the left* and further say that, k, is a *rough lower bound* of S.

A set is said to be *bounded*, if it is bounded above as well as below.

The set of all the integers is neither bounded above nor below; the set of all the positive integers is bounded below but not above; the set of all the negative integers is bounded above but not below.

The intervals $[0, 1]$, $]0, 1[$, $[0, 1[$, $]0, 1]$ are all bounded.

Ex. 1. Show that if K is a rough upper bound of a set S and $K' > K$, then K' is also a rough upper bound of S; also give examples to show that if $K' < K$, then K' may or may not be a rough upper bound.

State a similar result concerning sets which are bounded below.

2. Show that a set with a greatest member is bound above but that the converse is not necessarily true. State a similar result for sets which are bounded below.

3. Show that every sub-set of a bounded set is bounded.

4. Show that the set of rough upper bounds of a set bounded above is bounded below. State a similar result for sets which are bounded below.

5. *Show that for a bounded set, S, there exists a positive number A such that $|x| < A$, where x is any member of S.*

Since S is bounded, there exist numbers k and K such that

$$k \leqslant x \leqslant K, \qquad \dots (i)$$

Let A be any number greater than both $|k|$ and $|K|$, so that we have

$$|k| < A, \quad |K| < A.$$

These give

$$-A < k \text{ and } K < A. \qquad \dots (ii)$$

From (i) and (ii), we have

$$-A < k \leqslant x \leqslant K < A, \text{ i.e., } -A < x < A,$$

or

$$|x| < A.$$

Conversely, if $|x| < A$, then $-A < x < A$ so that the set is bounded.

6. x, y are any two members of the bounded sets S_1 and S_2 respectively; show that the sets of numbers, $x+y$, $x-y$, xy are also bounded.

24. The upper and lower bounds. A set bounded above may or may not have a greatest member. For example, b, is the greatest member of the closed interval $[a, b]$ but the open interval $]a, b[$ has no greatest member. The number, b, being not a member of $]a, b[$ is not the greatest member thereof and also there exists a member of $]a, b[$ greater than any member of the same. Similar remarks apply to the case of the smallest members of sets bounded below.

The following two theorems introduce the concepts of *the upper and lower bounds* which constitute generalisations of the concepts of *greatest and least members* respectively.

24·1. Theorem on upper bounds. *For every set S bounded above, there* **exists** *a number B such that*

(*i*) *every member of the set is less than or equal to B;*

(*ii*) *every member less than B is smaller than at least one member of the set, i.e., however small the positive number, ε, may be, there is a member of S greater than $B-\varepsilon$.*

Divide all the real numbers into two classes L and R, putting a number in L if it is smaller than at least one member of S and otherwise in R.

Clearly each number has a class. Since a number less than any member of S belongs to L and any rough upper bound K of S, belongs to R, we see that each class exists. Finally, a number which is less than a member of L is necessarily less than a member of the set, and accordingly it belongs to L. Thus the two classes L, R determine a section of the real numbers.

There exists, therefore, a number B separating the two classes. Every number less than B belongs to L and every number greater than B belongs to R. It will now be shown that this is the number B of the theorem.

Any number $B-\varepsilon$, $(\varepsilon > 0)$, being less than B, belongs to L and is, therefore, smaller than at least one member of the set.

Also no member of the set is greater than B. For, if possible, let there be a member B', of the set which is greater than B.

The members of the open interval $]B, B'[$ all belong to R, for each of them is greater than B; also they all belong to L, for each of them is smaller than a member B' of the set. This is a contradiction.

Thus we have proved that the number, B, which separates the two classes, possesses the two properties stated in the theorem.

This number B is said to be the **upper bound** of the set S.

Remarks. 1. The property (*i*) implies that the upper bound B is a rough upper bound of S and (*ii*) implies that no number less than B is a rough upper bound, *i.e.*, B is the least of all the rough upper bounds. In other words, therefore, the theorem states that the *set of rough upper bounds of a set bounded above possesses a least member.* Also further we may thus say that the *upper bound of a set bounded above is the least of all its rough upper bounds.*

2. The maximum, *i.e.*, the greatest member of the set in case it exists, is also the upper bound, and we then say that the *set attains its upper bound.*

24·2. Theorem on lower bounds. *For every set S bounded below, there exists a number, b, such that*

(i) every member of the set is greater than or equal to b;

(ii) every number greater than b is greater than at least one member of the set, i.e., however small the positive number, ε, may be there is a member of S less than $b+\varepsilon$.

Its proof is similar to that of the previous theorem on upper bounds. To prove it, the real numbers will have to be divided into classes L and R such that a number will belong to R if it is greater than at least one member of the set and otherwise to L.

This number, b, is said to be **the lower bound** of S

Remarks. 1. The theorem actually states that the lower bound, b, is the greatest of all the rough lower bounds of S so that, in other words, the theorem states that *the set of rough lower bounds of a set bounded below possesses a greatest member. The lower bound of a set bounded below is the greatest of all its rough lower bounds.*

2. The minimum, if it exists, is the lower bound, and we then say that the *set attains its lower bound.*

3. Obviously $B \geqslant b$.

4. What we have called rough upper bound here is also often called simply *upper bound* and what we have called the upper bound is then called the *least upper bound* (*l.u.b.*). Similarly what we have called rough lower bound and lower bound are often referred to as lower bound and *greatest lower bound* (*g. l. b.*) respectively.

The least upper bound and the greatest lower bound are also described in literature as *Supremum* and *Infinimum* respectively.

Ex. 1. 1 is the upper bound and 0 is the lower bound of each of the four intervals $[0, 1]$, $]0, 1[$, $[0, 1[$, $]0, 1]$.

2. The upper bound of the set of numbers

$$1, \frac{1}{10}, \frac{1}{10^2}, \ldots\ldots\ldots, \frac{1}{10^n}, \ldots\ldots$$

is 1; what is the lower bound ?

3. Construct examples to show that the bounds of a set may not themselves be members of the set.

4. Show that the greatest member of a set, in case it exists, is the upper bound and the least member, if it exists, is the lower bound.

5. The members of a bounded set are all positive; show that the bounds cannot be negative.

24·3. Oscillation of a bounded set. *The difference, $B-b$, of the bounds B, b of a bounded set is called its oscillation.*

Examples

1. *B, b are the bounds of a set S and B_1, b_1 are bounds of a sub-set S_1 of S; show that*

$$b \leqslant b_1 \leqslant B_1 \leqslant B.$$

Every member of S_1 is a member of S and, accordingly, it must be $\leqslant B$ and thus B is a rough upper bound of S_1. The upper bound B_1 of S_1 being the least of its rough upper bounds we have

$$B_1 \leqslant B.$$

In a similar manner, it may be proved that

$$b \leqslant b_1.$$

Also, obviously

$$b_1 \leqslant B_1.$$

2. *x is any member of a bounded set S_1 whose bounds are B_1, b_1, show that the bounds of the set, S, of numbers, $-x$, are $-b_1$, $-B_1$.*

As $x \geqslant b_1$, we have

$$-x \leqslant -b_1 \qquad \dots (i)$$

where, $-x$, is any member of S.

Let ε be any positive number, however small. There exists a member, x, of S_1 such that

$$x < b_1+\varepsilon,$$

which shows that there exists a member, $-x$, of S such that

$$-x > -b_1-\varepsilon. \qquad \dots (ii)$$

Hence, from (i) and (ii), $-b_1$, is the upper bound of S.

It may similarly be shown that, $-B_1$, is the lower bound of S.

(*In examples* 3—6 *below x, y denote any two members of the bounded sets S_1, S_2 and B_1, b_1; B_2, b_2, are respectively their bounds*).

3. *Show that the bounds af the set S, of numbers $x+y$ are B_1+B_2, b_1+b_2.*

Since $\qquad x \leqslant B_1$ and $y \leqslant B_2$

therefore $\qquad x+y \leqslant B_1+B_2 \qquad \dots (i)$

where $x+y$ is *any* member of S.

Let ε be any positive number. There exist members x, y of S_1, S_2 respectively such that

$$x > B_1-\tfrac{1}{2}\varepsilon,\ y > B_2-\tfrac{1}{2}\varepsilon$$

which show that there exists a member $x+y$ of S such that

$$(x+y) > (B_1+B_2)-\varepsilon. \qquad \dots (ii)$$

Thus (B_1+B_2) is the upper bound of S.

It may similarly be shown that, b_1+b_2, is the lower bound of S.

4. Show that the bounds of the set S of numbers $x-y$ are B_1-b_2, b_1-B_2.

5. *If the members of the sets S_1, S_2 are all positive, then show that the bounds of the set S of numbers, xy, are B_1B_2, b_1b_2.*

The numbers B_1, B_2, b_1, b_2 must all be non-negative.

Since $$x \leqslant B_1,\ y \leqslant B_2,$$
therefore $$xy \leqslant B_1 B_2, \qquad \dots (i)$$
where, xy is *any* member of S.

Let, ε, be any *given* positive number, however small.

If ε_1, ε_2, are any two positive numbers, then there exist members x, y of S_1, S_2 respectively such that

$$x > B_1-\varepsilon_1,\ y > B_2-\varepsilon_2$$

whence we have

$$xy > (B_1-\varepsilon_1)(B_2-\varepsilon_2). \qquad \dots (ii)$$

It will now be shown that it is possible to choose $\varepsilon_1, \varepsilon_2$ in terms of, ε, such that

$$(B_1-\varepsilon_1)(B_2-\varepsilon_2) > B_1B_2-\varepsilon, \qquad \dots (iii)$$

so that it will be deduced from (*ii*) and (*iii*) that there exists a member, xy, of S such that

$$xy > B_1B_2-\varepsilon. \qquad \dots (iv)$$

Now $$(B_1-\varepsilon_1)(B_2-\varepsilon_2) > B_1B_2-\varepsilon$$
if $$\varepsilon_1 B_2+\varepsilon_2 B_1 < \varepsilon+\varepsilon_1\varepsilon_2$$
or if $$\varepsilon_1 B_2+\varepsilon_2 B_1 < \varepsilon.$$

Taking $\varepsilon_1=(\frac{1}{3}\varepsilon)/B_2$, $\varepsilon_2=(\frac{1}{3}\varepsilon)/B_1$, if $B_1\neq 0$, $B_2\neq 0$, we see that

$$\varepsilon_1 B_2+\varepsilon_2 B_1=\tfrac{2}{3}\varepsilon < \varepsilon.$$

The argument can be easily modified if either or both of B_1, B_2 are zero.

The case of lower bound may be similarly discussed.

6. If b_1, B_1 are the lower and upper bounds of a bounded set S_1 with positive members and $b_1\neq 0$, show that the lower and upper bounds of the set S consisting of the reciprocals of the members of S_1 are $\dfrac{1}{B_1}$ and $\dfrac{1}{b_1}$ respectively.

Let x be any member of S_1.

Since $\quad b_1 \leqslant x \leqslant B_1, \quad \therefore \quad \dfrac{1}{B_1} \leqslant \dfrac{1}{x} \leqslant \dfrac{1}{b_1}.$

For a given positive number ε, there exists a member x of S_1 such that

$$x > B_1-\varepsilon \quad \text{or} \quad \frac{1}{x} < \frac{1}{B_1-\varepsilon}.$$

Let ε' be any given positive number. We shall have

$$\frac{1}{B_1}+\varepsilon' < \frac{1}{B_1-\varepsilon} \text{ if } \frac{B_1}{1+\varepsilon' B_1} > B_1-\varepsilon \ \textit{i.e.,} \qquad \text{if } \varepsilon < B_1+\frac{B_1}{1+\varepsilon' B_1}$$

With a choice of ε satisfying the condition above we see that there exists a member $\dfrac{1}{x}$ of S such that

$$\frac{1}{x} < \frac{1}{B_1}+\varepsilon'.$$

Thus $\dfrac{1}{B_1}$ is the lower bound of S. Similarly it may be shown that $\dfrac{1}{b_1}$ is the upper bound of S.

7. *If the members of S_1, S_2 are all positive and $b_1 \neq 0$, show that the bounds of the set S of numbers y/x are B_2/b_1, b_2/B_1.*

8. Show that the bounds of the set S, consisting of the members of both S_1 and S_2 are Max. $\{B_1, B_2\}$, min. $\{b_1, b_2\}$.

9. *x, y are any two members of a bounded set S; show that the upper bound of the set S_1 of numbers*

$$(i)\ \ x-y, \quad (ii)\ |x-y|$$

is the oscillation of S.

Let B, b be the bounds of S.

(*i*) Since $\qquad x \leqslant B$ and $y \geqslant b$ or $-y \leqslant -b$

therefore $\qquad x-y \leqslant B-b$

where, $x-y$, is any member of the set S_1.

Let ε be any given positive number.

There exist two members x and y of S such that

$$x > B-\tfrac{1}{2}\varepsilon,\ y < b+\tfrac{1}{2}\varepsilon \text{ or } -y > -b-\tfrac{1}{2}\varepsilon$$

whence we see that there exists a member, $x-y$, of S_1 such that

$$x-y > B-b-\varepsilon.$$

Thus the oscillation, $B-b$, is the upper bound of the set of numbers $x-y$.

(*ii*) It is now obvious.

25. Limiting point of a set. We shall now introduce an important concept of the *Limiting point* of a set.

Def. *A number, ξ is said to be a limiting point of a set, if* **every** *neighbourhood of ξ, contains an infinite number of members of the set.*

A limiting point is also often called *Accumulation point, Condensation point* or *Cluster point.*

Clearly a real number will **not** be a limiting point if **some** neighbourhood around the same includes at the most only **a** finite number of members of the set.

A limiting point of a set may or may not itself be a member of the set.

Obviously, a finite set cannot have a limiting point; it is only infinite sets which may have one, more or even an infinite number of limiting points. Of course, even some infinite sets may have no limiting point. For example, the set of integers has no limiting point. It will be shown below that an infinite set which is bounded must have at least one limiting point.

The exercises below illustrate the various possibilities.

Ex. 1. Every real number is a limiting point of the set of rational numbers (Refer § 8·3, page 9).

Rational limiting points are members of the set but irrational limiting points are not.

2. Every real number is a limiting point of the set of irrational numbers. (Refer § 11·6, page 16).

3. The set of integers, even though infinite, has no limiting point.

4. The set of numbers

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots, \frac{1}{n}, \ldots\ldots,$$

has only one limiting point *viz.*, 0, and it does not belong to the set.

5. The set

$$1+\tfrac{1}{2}, -1-\tfrac{1}{2}, 1+\tfrac{1}{3}, -1-\tfrac{1}{3}, 1+\tfrac{1}{4}, -1-\tfrac{1}{4} \ldots\ldots$$

has two limiting points, *viz*, 1 and -1.

6 The set of numbers

$$\frac{1}{m}+\frac{1}{n}+\frac{1}{p},$$

where m, n, p take up all integral values is the set of numbers

$$\frac{1}{m}+\frac{1}{n},$$

and the number 0.

7. Limiting point of any sub-set of a set S, is also a limiting point of S.

8. S_1 S_2 are two sets and a set S, consists of the members of S_1 and S_2; show that a limiting point of S must be a limiting point of either S_1 or S_2 and conversely.

9. Show that the upper bound of a set S which does not have a maximum is a limiting point of S; state and prove a similar result for lower bounds.

10. Show that ξ is a limiting point of a set S if and only if every neighbourhood of ξ contains a member of S other than ξ.

25·1. Bolzano Weierstrass's theorem on the existence of limiting points.

Theorem. *Every infinite bounded set has at least one limiting point.*

Let, S, be any infinite bounded set.

Since S is bounded there exist real numbers, k, K such that every member of S belongs to the interval $[k, K]$.

Divide all the real numbers into two classes L, R, putting a number in L if it exceeds only a finite number of members of S and otherwise in R.

Clearly each number has a class. Also k belongs to L and, the set being infinite, K belongs to R and, therefore, each class exists. Finally, any number which is less than a member of L can exceed only a finite number of members of S and accordingly it must belong to L. Thus the two classes L, R determine a section of real numbers. There exists, therefore, a number, g, separating the two classes. It will now be shown that, g, is a limiting point of S.

Consider *any* neighbourhood $]g-\varepsilon, g+\varepsilon'[$ of g.

Now, $g-\varepsilon$, which belongs to L can exceed only a finite number of members of S, while $g+\varepsilon'$, which belongs to R, must exceed an infinite number of members of S and accordingly there must belong an infinite number of members of S to the neighbourhood $]g-\varepsilon, g+\varepsilon'[$.

Hence, g, is a limiting point of S.

Ex. A number, c, is the *only* limiting point of a bounded set S ; I, is *any* neighbourhood of c, show that there can exist at the most, a finite number of members of S not belonging to I. Construct an example to show that the result may not necessarily be true if S is not bounded.

26. Derived Sets. The set consisting of the limiting points, if any, of a set S is called the *first derivative* or simply the *derivative* of S and is denoted by S'. The derivative of S' is called the *second derivative* of S and is denoted by S''. Proceeding thus, we may have a number of sets

$$S, S', S'', \ldots \ldots, S^n \ldots$$

which are the successive derivatives of S.

If the *nth derivative* S^n contains only a finite number of members, then it has no limiting point and this chain of successive derivatives ceases at S^n. In such a case *when a set possesses only a finite number of derivatives, we say that the set is of first species and otherwise of second species.*

Ex. 1. Show that the set of numbers $1/m+1/n+1/p$, where m, n, p, take up all integral values is of the first species.

2. Show that the set of rational numbers is of second species.

26·1. Theorem. *The derived set of a bounded set is bounded and attains its bounds.*

Let S be any bounded set and let every member of the same belong to an interval $[k, K]$.

Clearly, no limiting point of S, *i.e.*, no member of the derivative S' can be less than k or greater than K, and accordingly S' is bounded. Let g, G be the bounds of S'. It will now be shown that g, G are themselves members of S' *i.e.*, limiting points of S.

Let $]G-\varepsilon, G+\varepsilon'[$ be any neighbourhood of G.

Since G is upper bound of S', therefore, there must exist a member ξ of S' such that

$$G-\varepsilon < \xi \leq G,$$

so that $]G-\varepsilon, G+\varepsilon', [$ is a neighbourhood of a limiting point ξ of S and as such contains an infinite number of members of S. Thus G must be a limiting point of S.

It may similarly be shown that, g, is also a limiting point of S.

Note. The theorem may also be stated thus : *The derived set of a bounded set is bounded and possesses greatest and least members.* Thus we can always talk of the greatest and least limiting points of an infinite bounded set.

Here, G, g are the greatest and least limiting points of S and are respectively called the *upper and lower limits.*

27·1. The two characteristic properties of the upper limit G. The following characteristic properties of the upper and lower limits can be easily deduced from the fact that they are respectively the greatest and least limiting points.

If ε be any positive number, however small, then

(i) an infinite number of members of the set are greater than $G-\varepsilon$;

(ii) a finite number, at the most, of members of the set are greater than $G+\varepsilon$.

27·2. The two characteristic properties of the lower limit g.

If ε be any positive number, however small, then

(i) an infinite number of members of the set are smaller than $g+\varepsilon$;

(ii) a finite number, at the most, of members of the set are smaller than $g-\varepsilon$.

Note . It is important to remember that for a bounded set

$$b \leqslant g \leqslant G \leqslant B,$$

as may be easily seen.

2. Throughout this book, the *maximum*, the *upper bound* and the *upper limit* of a set will be denoted by M, B, G, respectively. Similarly the minimum, the lower bound and the lower limit will be denoted by m, b, g respectively.

28. Theorem. *For a bounded set*

(i) M, B, G cannot all be different numbers;

and (ii) m, b, g cannot all be different numbers.

(*i*) Let S be any bounded set.

In case S has a maximum, *i.e.*, if M exists, then

$$M = B.$$

Now, suppose M does not exist so that B is not a member of S.. In this case, as will be shown,

$$G \quad B$$

Consider any neighbourhood $]B-\varepsilon, B+\varepsilon'[$ of B.

No member of S is greater than B.

If possible, let only a *finite* number of members of S lie between $B-\varepsilon$ and B so that there will be a greatest of them, say, ξ.

Since $\xi \neq B$, we see that no member of S is greater than ξ which is less than the upper bound B and this is impossible. (§ 24·1, page 35)

Thus there belong an infinite number of members of S to $]B-\varepsilon, B[$ and consequently to $]B-\varepsilon, B+\varepsilon'[$ and, accordingly B, is a limiting point of S.

Thus $\qquad B \leqslant G.$

Now, if possible let $\quad B < G.$

Since B is the upper bound, no member of S is greater than B and accordingly G cannot be a limiting point. This is a contradiction.

Thus $$B=G.$$

It may similarly be shown that m, b, g cannot all be different.

Exercises

1. Construct bounded sets for which

(*i*) $b<g<G<B$. (*ii*) $b<g<G=B$.

(*iii*) $b<g=G<B$. (*iv*) $b<g=G=B$.

(*v*) $b=g<G<B$. (*vi*) $b=g<G=B$.

(*vii*) $b=g=G<B$. (*viii*) $b=g=G=B$.

[The following sets exhibit the above possibilities :—

(*i*) $1+1/n, -1-1/n$. (*ii*) $-1-1/n$, $-1/n$.

(*iii*) -2, 2, $-1/n$, $1/n$. (*iv*) -2, $-1/n$.

(*v*) $1/n$, $1+1/n$. (*vi*) $-1+1/n$, $1-1/n$.

(*vii*) 2, $-1+1/n$. (*viii*) This case is not possible.

Here, n, takes up different positive integral values.

The student may construct other sets exhibiting the various possibilities.]

2. Construct a set whereof no element lies between its upper and lower limits.

3. Examine the existence and the values of M, B, G ; m, b, g for the following sets ; n taking up all positive integral values.

(*i*) $3, 2, (2^{n-1}+1)/2^n, (2^n-1)/2^n$.

(*ii*) $8, -3, 3, 4, 2\frac{1}{3}, 4\frac{1}{2}, 2\frac{1}{9}, 4\frac{3}{4}, 2\frac{1}{27}, 4\frac{5}{8}, 2\frac{1}{81}, 4\frac{7}{16}, \ldots\ldots$

(*iii*) $\frac{4}{3}, -\frac{1}{2}, \frac{6}{5}, -\frac{3}{4}, \frac{8}{7}, -\frac{5}{6}, \frac{10}{9}, -\frac{7}{8}, \frac{12}{11}, \ldots\ldots$

(*iv*) $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7} \ldots\ldots\ldots$

(*v*) $a+n^{(-1)^n}$. (*vi*) $a-n^{(-1)^n}$.

(*vii*) $(-1)^n\left(1+\dfrac{1}{n}\right)$. (*viii*) $(-1)^n\left(\dfrac{1}{4}-\dfrac{8}{n}\right)$.

4. Give an example of a set

(*i*) which is identical with its limiting points,

(*ii*) whose limiting points do not belong to the set.

CHAPTER III

SEQUENCES

29. Sequences. If there be given a law which associates to each positive integer n, a real number a_n, we say that $\{a_n\}$ is a sequence of real numbers and that

$$a_1, a_2, a_3 \ldots\ldots\ldots, a_n, \ldots\ldots\ldots,$$

are its members. The integer n is known as the suffix.

Examples.

(*i*) Taking $a_n = \frac{1}{n}$, we have a sequence with members

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\ldots$$

(*ii*) Taking $a_n = (-1)^n n$, we have a sequence with members

$$-1, 2, -3, 4, \ldots\ldots\ldots$$

(*iii*) Taking $a_n = (-1)^n$, we have a sequence with members

$$-1, 1, -1, 1, \ldots\ldots$$

(*iv*) If a_n denotes the nth prime, there arises a sequence with members

$$1, 2, 3, 5, 7 \ldots\ldots$$

(*v*) If $a_n = \left(1 + \frac{1}{n}\right)$ if n is even and $\left(-1 - \frac{1}{n}\right)$ if n is odd we have a sequence with members

$$-2, \tfrac{3}{2}, -\tfrac{4}{3}, \tfrac{5}{4}, \ldots\ldots\ldots$$

(*vi*) If $a_n = \frac{1}{n+1} + \frac{1}{n+2} \ldots\ldots\ldots + \frac{1}{n+n}$, we have a sequence with members

$$\frac{1}{2}, \frac{7}{12}, \frac{37}{60}, \ldots\ldots\ldots$$

Bounded Sequences. A sequence $\{a_n\}$ is said to be bounded above (below) if there exists a real number K (k) such that

$$a_n \leqslant K \ (a_n \geqslant k) \text{ for all } n.$$

A sequence is said to be bounded if it is both bounded above as well as below.

A sequence is bounded if and only if the set determined by its members is bounded and the bounds of the latter set are known as the bounds of the sequence.

Limiting point of a sequence. A member, ξ, is said to be a limiting point of a sequence $\{a_n\}$, if given any neighbourhood whatsoever of ξ, a_n belongs to the same for an infinite number of values of n.

Ex. 1. Show that if ξ is a limiting point of a sequence $\{a_n\}$ and $]\xi-\varepsilon, \xi+\varepsilon[$ is any given neighbourhood of ξ and m is any given integer, then there exists an integer $n \geqslant m$ such that a_n belongs to $]\xi-\varepsilon, \xi+\varepsilon[$.

Note. A limiting point of a sequence $\{a_n\}$ may not necessarily be a limiting point of the set determined by its members. Thus, for example, 1 as well as -1 are the limiting points of the sequence considered in Example (*iii*) above but none of these numbers is a limiting point of the set determined by its members inasmuch it is a finite set consisting of just two members. Of course every limiting point of the set consisting of the members of a sequence is a limiting point of the sequence.

In this connection it may be noted that ξ is a limiting point of a sequence $\{a_n\}$ if $a_n=\xi$ for an infinite number of values of n.

Theorem. *Every bounded sequence has at least one limiting point.*

Let $\{a_n\}$ be a bounded sequence. If there exists a number ξ such that $a_n=\xi$ for an infinite number of values of n, then ξ is a limiting point. In the alternating case, the set consisting of members of the sequence is infinite and bounded and as such has a limiting point by Bolzano-Weierstrass Theorem. This limiting point of the set is also a limiting point of the sequence.

30. Upper and Lower limits of a bounded sequence. As in § 26·1, it may be seen that the set of limiting points of a bounded sequence is bounded and attains its bounds. These bounds are known as the Upper and Lower limits of the corresponding sequence.

We shall now obtain characteristic properties of the Upper and Lower limits of a bounded sequence.

Let, G, be the upper limit of a bounded sequence $\{a_n\}$ and let, ε, be any positive number. Consider the interval

$$[G-\varepsilon, G+\varepsilon].$$

Since, G, is the greatest limiting point of $\{a_n\}$ there can exist, at the most, only a finite number of members of the sequence $> G+\varepsilon$ for, otherwise, the sequence would have at least one limiting point $\geqslant G+\varepsilon$.

If, m, is any integer greater than the suffixes of all those members of the sequence (which are finite in number) which are greater than $G+\varepsilon$, we have

$$a_n < G+\varepsilon, \text{ for } n \geqslant m. \qquad \ldots(1)$$

This relation is also often expressed by saying that

$$a_n < G+\varepsilon,$$

for almost a values of n, or for all sufficiently large values of n.

Again since G is a limiting point, a_n belongs to $[G-\varepsilon, G+\varepsilon]$ for an infinite number of values of n so that we have

$$a_n > G-\varepsilon, \textit{ for an infinity of values of } n. \qquad \dots (2)$$

Also we may see that a number, G, satisfying the two properties (1) and (2) in relation to a sequence $\{a_n\}$ must be the upper limit of $\{a_n\}$.

Thus we have the following two *characteristic properties* of the upper limit G of $\{a_n\}$.

If ε be any positive number, however small, then

(*i*) $G-\varepsilon < a_n$ *for an infinite number of values of* n;

(*ii*) *there exists a positive integer* m *such that* $a_n < G+\varepsilon$, *for every positive integral value of* $n \geq m$.

The lower limit, g, of a bounded sequence $\{a_n\}$ possesses the following two *characteristic properties* :—

If ε be any positive number, however small, then

(*i*) $a_n < g+\varepsilon$ *for an infinite number of values of* n;

(*ii*) *there exists a positive integer* m *such that* $g-\varepsilon < a_n$, *for every positive integral value of* $n \geq m$.

The upper and lower limits of a sequence are also sometimes known as *upper and lower limits of indetermination.*

It is usual to denote the upper and lower limits of a *sequence* $\{a_n\}$ by the symbols $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$ respectively.

† **Ex. 1.** Find M, m; B, b; G, g whichever may exist, for the following sequences,

(*i*) $a_n=(-1)^n/n$. (*ii*) $a_n=1+(-1)^n$. (*iii*) $a_n=(-1)^n(1+1/n)$.

$$(iv)\ a_n=\begin{cases}(n+1)/n, & \text{when } n=3m,\\ (n+2)/2n, & \text{when } n=3m+1,\\ 1/(n^2+1), & \text{when } n=3m+2\,;\end{cases}$$

m being a positive integer.

(*v*) $a_n=(4^n+1)/4^n$ or $(1-4^n)/4^n$ according as n is even or odd.

(*vi*) $a_n=[n+(-1)^n]n$.

2. Construct a sequence with ± 2 for its bounds and ± 1 for its upper and lower limits such that no member lies between ± 1.

31. Convergent sequences. Def. *A bounded sequence $\{a_n\}$ is said to be* **convergent,** *if it has only one limiting point and this unique limiting point is called the limit of the sequence.*

If, l, be the limit of a convergent sequence $\{a_n\}$, then we say that $\{a_n\}$ converges to the limit, l, and symbolically write

$$\lim_{n\to\infty} a_n=l, \text{ or } a_n \to l \text{ as } n\to\infty.$$

It will be seen that a *bounded sequence $\{a_n\}$ is convergent if, and only if, its upper and lower limits are equal.*

An important note. It will be well to emphasise that the symbolic statement: $\lim\limits_{n \to \infty} a_n = l$ is equivalent to the following two assertions :—

(*i*) *The sequence* $\{a_n\}$ *is convergent.*

(*ii*) *The limit of the convergent sequence* $\{a_n\}$ *is* l.

31·1. Condition for Convergence. Theorem. *The necessary and sufficient condition for a* **sequence** $\{a_n\}$ **to converge to a limit,** l, *is that to every positive number* ε, *however small, there corresponds a positive integer,* m, *such that*

$$|a_n - l| < \varepsilon, \text{ when } n \geqslant m.$$

Remarks. Before proceeding to prove this theorem, we observe that the condition implies that if I be *any* interval $]l-\varepsilon, l+\varepsilon[$ enclosing l, then a finite number of members of the sequence, at the most, can be outside I, *i.e.*, all the members excepting, at the most a finite number of them belong to I. Here, m, denotes any integer greater than the suffixes of all those members which do not belong to I.

The condition is necessary. Let the sequence $\{a_n\}$ converge to a limit, l, so that it is bounded and, l, is its only limiting point.

Let ε be any positive number, however small.

There lie, at the most, only a *finite* number of members of $\{a_n\}$ outside $]l-\varepsilon, l+\varepsilon[$, for, if they were infinite, then the sequence, which is bounded, will have at least one more limiting point which is different from l.

Let m be any positive integer greater than the suffix of every member which lies outside $[l-\varepsilon, l+\varepsilon]$. Then we have

$$l-\varepsilon < a_n < l+\varepsilon, \text{ i.e., } |a_n - l| < \varepsilon, \text{ when } n \geqslant m.$$

The condition is sufficient. It will firstly be shown that under this condition the sequence is bounded.

Consider any interval say, $[l-1, l+1]$, which encloses l. There exists a positive integer, p, such that every member of the sequence excepting at the most $a_1, a_2, \ldots\ldots\ldots\ldots, a_{p-1}$ belong to this interval. If k be the least and K the greatest of the finite set of numbers

$$a_1, a_2, \ldots\ldots\ldots\ldots, a_{p-1}, l-1, l+1$$

we see that

$$k \leqslant a_n \leqslant K, \text{ for every value of } n,$$

so that $\{a_n\}$ is bounded.

Clearly, the given condition implies that, l, is a limiting point of $\{a_n\}$ and we have now to show that this is the *only* limiting point.

If possible, let $l' \neq l$, be any other limiting point of $\{a_n\}$.

We enclose, l, in an interval, I, so small that, l', does not belong to it. According to the given condition, a finite number of members, at the most, can lie outside I so that l' cannot be a limiting point.

Thus the condition is sufficient also.

Note. It should be carefully noted that a *convergent sequence is necessarily bounded.*

32. Non-convergent bounded sequence. *A bounded sequence $\{a_n\}$ which does not converge is said to* **oscillate finitely.**

33. Unbounded sequences. In the case of sequences, which are not bounded, we distinguish the following *three* behaviours.

33·1. Divergence to ∞. If to every positive number $\triangle$, however large, there corresponds a positive integer m such that

$$a_n > \triangle, \text{ when } n \geq m,$$

then we say that $\{a_n\}$ is divergent and that it tends (or diverges) to ∞ as n tends to infinity and, in symbols, write

$$a_n \to \infty \text{ as } n \to \infty.$$

33·2. Divergence to $-\infty$. If to every positive number $\triangle$, however large, there corresponds a positive integer m such that

$$a_n < -\triangle, \text{ when } n \geq m,$$

then we say that $\{a_n\}$ is divergent and that it tends (or diverges) to $-\infty$ as n tends to infinity and, in symbols, write

$$a_n \to -\infty \text{ as } n \to \infty.$$

33·3. Infinite Oscillation. If an unbounded sequence does not diverge, *i.e.*, when it neither tends to ∞ nor to $-\infty$, then we say that it *oscillates infinitely.*

Ex. 1. Show that a divergent sequence cannot have a limiting point.

2. $\{a_n\}$ and $\{b_n\}$ are two sequences; $\{a_n\}$ diverges to ∞ and $b_n > a_n$ for every n; show that b_n also diverges to infinity.

3. The sequence $\{a_n\}$ is divergent and the sequence $\{b_n\}$ is convergent; show that the sequence $\{a_n+b_n\}$ is also divergent.

4. Show that a sequence obtained on re-arranging the members of another convergent sequence is also convergent and the limits are the same.

[A sequence $\{b_n\}$ is said to be obtained on re arranging the members of the sequence $\{a_n\}$ if every member of either sequence is some member of the other.]

[The result follows from the fact that the sets consisting of the members of such sequences are identical.]

5. Prove that $\lim \left(\frac{1}{n^2}+\frac{2}{n^2}+\dots\dots+\frac{n}{n^2}\right)=\frac{1}{2}$.

6. Show that the sequence $\{a_n\}$, where $a_n=(-1)^n$, does not converge.

7. *Determine the least value of m for which it is true that*

$$\left|\frac{n^2+n+1}{3n^2+1}-\frac{1}{3}\right| < \varepsilon, \text{ when } n \geqslant m,$$

ε being any positive number.

Now,
$$\left|\frac{n^2+n+1}{3n^2+1}-\frac{1}{3}\right|=\frac{3n+2}{3(3n^2+1)}$$
$$<\frac{3n+n}{9n^2}, \text{ if } n>2$$
$$=\frac{4}{9n}<\varepsilon, \text{ if } n>4/9\varepsilon.$$

The integer just greater than 2 and $4/9\varepsilon$ is the required value of m.

For example $m=5$ if $\varepsilon=\frac{1}{10}$, $m=45$, if $\varepsilon=\frac{1}{100}$.

It shows that $\lim\,[(n^2+n+1)/(3n^2+1)]=1/3$.

8. Show that

(*i*) $\lim \frac{n}{n+1}=1$. (*ii*) $\lim \frac{n^2+2}{n^3-1}=0$. (*iii*) $\lim \frac{4n^3+6n-7}{n^3+2n^2+1}=4$.

9. Show that

(*i*) $\lim\,(n^2-2n)=\infty$. (*ii*) $\lim\,[(n^2+1)/(n+1)]=\infty$.

(*iii*) $\lim\,[n+(-1)^n]=\infty$. (*iv*) $\lim\,[(2-n^2)/(n+1)]=-\infty$.

(*v*) $\{n(-1)^n\}$ oscillates infinitely.

34. Intrinsic tests of convergence. The condition, as obtained in § 31·1, answers the question "*Is any given number, l, the limit of a sequence $\{a_n\}$, or is it not?*" If, with the help of this condition, it be shown that a given number, l, is not the limit of $\{a_n\}$, then it will not follow that the sequence does not converge; there being another possibility also, *viz.*, that $\{a_n\}$ may converge to a number *different* from l. Thus this condition examines the question of '*Convergence to a number l*' and not that of '*Essential convergence*' which question, as will be seen, is of far more frequent occurrence in the theoretical parts of the subject. To this purpose of examining the question of essential convergence are directed two tests which are developed in the following two sub-sections.

Cauchy's general principle of convergence which is more general of the two tests to be considered is a development and refinement of the rough idea that *if a sequence $\{a_n\}$ is convergent, then for large values of n, the members of the sequence, being near the limit, are also near each other.*

34·1. Cauchy's general principle of convergence. *The necessary and sufficient condition for the convergence of a sequence $\{a_n\}$ is that to every positive number ε,* however small, there corresponds a positive integer m such that*
$$|a_{n+p}-a_n|<\varepsilon,$$
when $n \geqslant m$ and p has any positive integral value.

Remarks. The theorem may also be stated as follows :—

The necessary and sufficient condition for the convergence of $\{a_n\}$ is that to every positive number ε, however small, there corresponds a member,

* The reader would do well to understand and remember that the words 'however small' are really redundant.

a_m, of the sequence such that the modulus of the difference between any two members (not necessarily consecutive) which come after a_m in the succession

$$a_1, a_2, \ldots\ldots, a_{m+1}, \ldots\ldots$$

is less than ε.

The condition is necessary. Let the sequence converge and let its limit be l. If ε be any positive number, then there exists a positive integer m such that

$$|a_n - l| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m.$$

From this we deduce that

$$|a_{n+p} - l| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m \text{ and } p \geqslant 0.$$

$$\begin{aligned} \therefore \quad |a_{n+p} - a_n| &= |a_{n+p} - l + l - a_n| \\ &\leqslant |a_{n+p} - l| + |l - a_n| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \end{aligned}$$

i.e., $$|a_{n+p} - a_n| < \varepsilon, \text{ when } n \geqslant m \text{ and } p \geqslant 0.$$

The condition is sufficient. It will firstly be shown that under the given condition, the sequence is bounded.

We give any particular value to ε, say, 1. There exists a positive integer, r, such that

$$|a_{n+p} - a_n| < 1, \text{ when } n \geqslant r \text{ and } p \geqslant 0.$$

From this, taking $n = r$, we say that

$$|a_{r+p} - a_r| < 1, \text{ when } p \geqslant 0,$$

i.e., $$a_r - 1 < a_{r+p} < a_r + 1, \text{ when } p \geqslant 0.$$

This means that all the members of the sequence $\{a_n\}$ except, perhaps, the finite set of members

$$a_1, a_2, a_3, \ldots\ldots, a_{r-1},$$

lie between two fixed numbers, $a_r - 1$ and $a_r + 1$.

If, k, be the least and, K, the greatest of the finite set of numbers

$$a_1, a_2, a_3, \ldots\ldots, a_{r-1}, a_r - 1, a_r + 1,$$

we see that

$$k \leqslant a_n \leqslant K,$$

for every value of n, and accordingly $\{a_n\}$ is bounded.

The sequence $\{a_n\}$ has, therefore, at least one limiting point, say, l.

If possible, let there also be another limiting point, l'.

Let ε be any positive number, however small. There exists a positive integer m such that

$$|a_{n+p} - a_n| < \tfrac{1}{3}\varepsilon \text{ when } n \geqslant m \text{ and } p \geqslant 0 \qquad \ldots(1)$$

Also, since l, l' are the limiting points, there exist positive integers s, t both $\geqslant m$, such that

$$|a_s - l| < \tfrac{1}{3}\varepsilon, \; |a_t - l'| < \tfrac{1}{3}\varepsilon. \qquad \ldots(2)$$

From (1), we have, in particular

$$| a_s - a_t | < \tfrac{1}{3}\varepsilon. \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\begin{aligned} | l' - l | &= | l' - a_t + a_t - a_s + a_s - l | \\ &\leqslant | l' - a_t | + | a_t - a_s | + | a_s - l | \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon, \end{aligned}$$

i.e.,

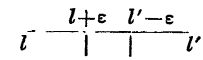
$$| l' - l | < \varepsilon.$$

Thus a non-negative number, $| l' - l |$, is less than every positive number ε and accordingly it must be 0 so that $l' = l$.

Thus we prove that $\{a_n\}$ is convergent.

[**Alternatively**, the sufficiency may also be seen in another way as follows :

Let $l' > l$ and let $l' - l = 3\varepsilon$. The numbers $l + \varepsilon$ and $l' - \varepsilon$ lie between l and l' and $l + \varepsilon < l' - \varepsilon$.



There exists a member a_m such that every two members of the sequence which appear after a_m differ from each other by a number which is less than ε. Since l, l' are the limiting points, there exist members which appear after a_m, and lie in the intervals $]l - \varepsilon, l + \varepsilon[$ and $]l' - \varepsilon, l' + \varepsilon[$ and such members, obviously, differ from each other by a number greater than ε. This is a contradiction.]

34·2. Monotonic Sequences and their convergence.

Def. *A sequence $\{a_n\}$ is said to be monotonically increasing if*

$$a_{n+1} \geqslant a_n, \text{ for every value of } n \text{ ;}$$

it is said to be monotonically decreasing if

$$a_{n+1} \leqslant a_n, \text{ for every value of } n.$$

A sequence which is monotonically increasing or decreasing is known as a monotonic sequence.

A monotonic sequence no two members of which are equal is called a *strictly monotonic* sequence.

The sequence $\{a_n\}$, where

(*i*) $a_n = 1/n$ is monotonically decreasing. (*ii*) $a_n = -1/n$, is monotonically increasing. (*iii*) $a_n = \dfrac{(-1)^n}{n}$, is not monotonic.

34·21. Convergence of monotonic sequences. Theorem. *The necessary and sufficient condition for the convergence of a monotonic sequence is that it is bounded.*

The condition is necessary. This is obvious, for every convergent sequence is necessarily bounded.

The condition is sufficient. Let a monotonic sequence $\{a_n\}$ be bounded.

Firstly, let it be monotonically increasing and let B be the upper bound of the set determined by its members. It will be shown that B is the limit.

Let ε be any positive number, however small.

There exists a member say, a_m, of the sequence such that

$$B-\varepsilon < a_m. \qquad (\S\ 24\cdot1, \text{ page } 35) \quad \ldots(1)$$

Also, since, $\{a_n\}$ is monotonically increasing, we have

$$a_m \leqslant a_n, \text{ when } n \geqslant m. \quad \ldots(2)$$

From (1) and (2),

$$B-\varepsilon < a_n, \text{ when } n \geqslant m. \quad \ldots(3)$$

Also, since $a_n \leqslant B$, we have $a_n < B+\varepsilon$, for every value of n, and, in particular

$$a_n < B+\varepsilon, \text{ when } n \geqslant m. \quad \ldots(4)$$

From (3) and (4),

$$B-\varepsilon < a_n < B+\varepsilon, \text{ when } n \geqslant m,$$

so that the sequence $\{a_n\}$ converges ; its limit being the upper bound B.

It may similarly be shown that a bounded monotonically decreasing sequence is also convergent ; the limit, in this case, being the lower bound of the set determined by the members of the sequence.

Cor. If K be a rough upper bound of a monotonically increasing sequence $\{a_n\}$, then

$$\lim a_n \leqslant K,$$

if k be a rough lower bound of a monotonically decreasing sequence, $\{b_n\}$, then

$$\lim b_n \geqslant k.$$

Note. Of the two tests for convergence considered above, the former is more general inasmuch as it is applicable to any sequence whatsoever, whereas the latter is applicable to only monotonic sequences. It would be a good and simple exercise for the reader to deduce from the General principle of convergence the fact of convergence of a bounded monotonic sequence. It should, however, be noted that the condition of being bounded is a sufficient condition for the convergence of monotonic sequences only and a bounded non-monotonic sequence may not be convergent. For example, the bounded sequence $\{(-1)^n\}$ is not convergent.

The discussion of the convergence of sequences known to be monotonic, is comparatively simpler inasmuch we have only to examine whether the sequence is bounded or not.

Ex. 1. Show directly from definition (§ 31, page 45) that a bounded monotonic sequence is convergent.

2. Deduce from Cauchy's principle of convergence that a bounded monotonic sequence is convergent.

EXAMPLES

1. *Show that a monotonic sequence which is not bounded diverges to* ∞ *or* $-\infty$ *according as it is increasing or decreasing.*

Let $\{a_n\}$ be a monotonically increasing sequence which is not bounded.

Let $\triangle$ be any positive number, however large.

Since $\{a_n\}$ is not bounded, there exists a member a_m of the sequence such that

$$a_m > \triangle. \qquad \ldots(1)$$

As $\{a_n\}$ is monotonically increasing,

$$a_n \geqslant a_m, \text{ when } n \geqslant m. \qquad \ldots(2)$$

From (1) and (2), we deduce that

$$a_n > \triangle, \text{ when } n \geqslant m,$$

so that $\{a_n\}$ diverges to ∞.

The second part may be similarly proved.

2. $\{a_n\}$ *and* $\{b_n\}$ *are two convergent sequences ; deduce from Cauchy's general principle of convergence that the sequence* $\{a_n+b_n\}$ *is also convergent.*

Let ε be any positive number.

Since $\{a_n\}$ and $\{b_n\}$ are convergent, there exist positive integers m_1, m_2 such that

$$|\, a_{n+p}-a_n \,| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m_1 \text{ and } p \geqslant 0\,; \qquad \ldots(i)$$

$$|\, b_{n+p}-b_n \,| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m_2 \text{ and } p \geqslant 0, \qquad \ldots(ii)$$

Let $m = Max\ (m_1, m_2)$. From (*i*) and (*ii*), we deduce that for every $n \geqslant m$ and $p \geqslant 0$,

$$|\, \overline{a_{n+p}+b_{n+p}} - \overline{a_n+b_n} \,| \leqslant |\, a_{n+p}-a_n \,| + |\, b_{n+p}-b_n]$$
$$< \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.$$

Hence $\{a_n+b_n\}$ is convergent

3. *Show, with the help of Cauchy's general principle of convergence, that the sequence* $\{a_n\}$, *where*

$$a_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n},$$

is **not** *convergent.*

Suppose that $\{a_n\}$ is convergent. Taking $\varepsilon=\frac{1}{4}$, we see that there exists a positive integer m such that when $n \geqslant m$ and $p \geqslant 0$,

$$\left| a_{n+p}-a_n \right| = \left| \frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{n+p} \right| < \frac{1}{4}.$$

In particular, taking $n=m$, we see that for *every* value of p, we must have

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{m+p} < \frac{1}{4}.$$

Taking now $p=m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\dots\dots+\frac{1}{m+m}>\frac{m}{m+m}=\frac{1}{2},$$

so that we arrive at a contradiction.

4. *Show that the sequence $\{a_n\}$, where*

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\dots\dots+\frac{1}{n+n},$$

is convergent.

We have,

$$a_{n+1}=\frac{1}{n+2}+\frac{1}{n+3}+\dots\dots+\frac{1}{2n+2},$$

$$\therefore \qquad a_{n+1}-a_n=\frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1}=\frac{1}{(2n+1)(2n+2)}>0,$$

so that $\{a_n\}$ is monotonically increasing.

Also we have, for every n,

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\dots\dots+\frac{1}{n+n}$$

$$<\frac{1}{n+1}+\frac{1}{n+1}+\dots+\frac{1}{n+1}=\frac{n}{n+1}=1-\frac{1}{n+1}<1.$$

Thus the monotonically increasing sequence $\{a_n\}$ is bounded and accordingly it is convergent.

5. *Show that the sequence $\{a_n\}$, where*

$$a_n=1+\frac{1}{1\,!}+\frac{1}{2\,!}+\frac{1}{3\,!}+\dots\dots+\frac{1}{n\,!}$$

is convergent and that

$$2<\lim a_n\leqslant 3.$$

Clearly $\{a_n\}$ is monotonically increasing.

Also
$$a_n<1+1+\frac{1}{2}+\frac{1}{2^2}+\dots\dots+\frac{1}{2^{n-1}}$$
$$=1+2[1-(\tfrac{1}{2})^n]$$
$$=3-(\tfrac{1}{2})^{n-1}<3, \text{ for all } n$$

so that $\{a_n\}$ is bounded above.

Hence the result. The second part is obvious.

35. Invertibility of the algebraic operations and the limiting operation. *If $\{a_n\}$, $\{b_n\}$ be two sequences such that, when $n\to\infty$,*

$$lim\ a_n=A,\ lim\ b_n=B,$$

then

(1) $lim\ (a_n+b_n)=A+B$;

(2) $lim\ (a_n-b_n)=A-B$;

(3) $lim\ (a_n\,.\,b_n)=AB$;

(4) $lim\ (a_n\,/\,b_n)=A/B$, *if* $B\neq 0$.

Proof. (1), (2). Let ε be any positive number.

There exist positive integers m_1, m_2 such that

$$|a_n - A| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m_1,$$
$$|b_n - B| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant m_2.$$

Let $m = \text{Max}(m_1, m_2)$. Then we see that for every $n \geqslant m$,

$$|a_n - A| < \tfrac{1}{2}\varepsilon, \quad |b_n - B| < \tfrac{1}{2}\varepsilon.$$

Thus for every $n \geqslant m$,

$$|\overline{a_n + b_n} - \overline{A+B}| \leqslant |a_n - A| + |b_n - B| \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

and

$$|\overline{a_n - b_n} - \overline{A-B}| \leqslant |a_n - A| + |B - b_n| \leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

which show that

$$(a_n + b_n) \rightarrow (A+B) \text{ and } (a_n - b_n) \rightarrow (A-B).$$

(3) We have, for every value of n,

$$|a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)|$$
$$\leqslant |a_n| \; |b_n - B| + |B| \; |a_n - A|.$$

Since $\{a_n\}$ is convergent, there exists a number K such that

$$|a_n| \leqslant K, \text{ for every value of } n.$$

Thus

$$|a_n b_n - AB| \leqslant K |b_n - B| + \{|B| + 1\} |a_n - A|, \quad \ldots (i)$$

for every n.

Let ε be any positive number. There exist positive integers m_1, m_2 such that

$$|b_n - B| < \varepsilon/2K, \text{ for } n \geqslant m_1, \quad \ldots (ii)$$

and

$$|a_n - A| < \varepsilon/2\{|B| + 1\}, \text{ for } n \geqslant m_2 \quad \ldots (iii)$$

Let

$$m = \text{Max.}(m_1, m_2).$$

From (i), (ii) and (iii), we deduce that for every $n \geqslant m$,

$$|a_n b_n - AB| < \varepsilon,$$

so that

$$(a_n b_n) \rightarrow AB.$$

Note. If in (i), we had not introduced $|B| + 1$ in place of $|B|$, then in (iii), we would have to render $|a_n - A| < \varepsilon/2|B|$, which will fall if $B = 0$ and thus the proof, as given, will hold only if $B \neq 0$. It is just to include this case in the proof that we had introduced this artifice, for $|B| + 1$ can never be 0.

(4) We have, for every value of n,

$$\left| \frac{a_n}{b_n} - \frac{A}{B} \right| = \left| \frac{B(a_n - A) - A(b_n - B)}{Bb_n} \right|$$
$$\leqslant \frac{|B| \; |a_n - A| + |A| \; |b_n - B|}{|B| \; |b_n|} \quad (iv)$$

Since $\{b_n\} \to B \neq 0$, there exists a positive integer m_1 such that when $n \geqslant m_1$,

$$|b_n - B| < \tfrac{1}{2} |B|,$$

or

$$|B| - |b_n| \leqslant |b_n - B| < \tfrac{1}{2} |B|,$$

i.e.,

$$\tfrac{1}{2} |B| < |b_n|. \qquad \ldots (v)$$

From (*iv*) and (*v*), we see that for every $a \geqslant m_1$,

$$\left| \frac{a_n}{b_n} - \frac{A}{B} \right| \leqslant \frac{|B|\,|a_n - A| + |A|\,|b_n - B|}{\tfrac{1}{2} |B|^2}$$

$$= \frac{2}{|B|} |a_n - A| + \frac{2|A|}{|B|^2} |b_n - B|$$

$$\leqslant \frac{2}{|B|} |a_n - A| + 2\frac{|A|+1}{|B|^2} |b_n - B| \quad \ldots (vi)$$

Let ε be any positive number, however small. There exist positive integers m_2 and m_3 such that for every $n \geqslant m_2$,

$$|a_n - A| < \tfrac{1}{4} |B| \varepsilon, \text{ i.e., } \frac{2}{|B|} |a_n - A| < \frac{\varepsilon}{2}, \quad \ldots (vii)$$

for every $n \geqslant m_3$,

$$|b_n - B| < \tfrac{1}{4} \frac{|B|^2 \varepsilon}{|A|+1},$$

i.e.,

$$\frac{2|A|+1}{|B|^2} |b_n - B| < \frac{\varepsilon}{2}. \quad \ldots (viii)$$

Let $m = \text{Max.}(m_1, m_2, m_3)$.

From (*vi*), (*vii*) and (*viii*), we deduce that for every $n \geqslant m$,

$$\left| \frac{a_n}{b_n} - \frac{A}{B} \right| < \varepsilon,$$

so that $(a_n/b_n) \to A/B$ as $n \to \infty$.

Note. As a particular case of the theorems proved above, we note that if the sequences $\{a_n\}$, $\{b_n\}$ be convergent, then the sequences $\{a_n \pm b_n\}$, $\{a_n \,.\, b_n\}$ are also convergent; further if $\lim\ b_n \neq 0$, then $\{a_n/b_n\}$ is also convergent.

The converse, however, may not be true as the following examples show:

(*i*) Taking $a_n = (-1)^{n+1}$ and $b_n = (-1)^n$, we see that $\{a_n + b_n\}$ is convergent but $\{a_n\}$ and $\{b_n\}$ are not.

(*ii*) For $a_n = (-1)^n$, $b_n = (-1)^n$, $\{a_n - b_n\}$, $\{a_n \,.\, b_n\}$, $\{a_n/b_n\}$ all converge, but $\{a_n\}$ and $\{b_n\}$ do not.

Note. The result established above amounts to saying that the *operation of taking limit is invertible with each of the operations of addition, subtraction, multiplication and division performed on two sequences and also on any finite number of sequences.* [This last fact can easily be established by induction].

36. Theorem. *$\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are three sequences such that*

(i) $a_n \leqslant b_n \leqslant c_n$, *for every value of n,* ... *(i)*

(ii) $\lim a_n = \lim c_n = l$.

Then

$$\lim b_n = l$$

Let ε be any positive number. There exists a positive integer m such that for every value of $n \geqslant m$,

$$l-\varepsilon < a_n < l+\varepsilon, \quad \text{... (ii)}$$

$$l-\varepsilon < c_n < l+\varepsilon. \quad \text{...(iii)}$$

From (*i*), (*ii*) and (*iii*) we deduce that

$$l-\varepsilon < b_n < l+\varepsilon, \text{ for } n \geqslant m.$$

Hence

$$\lim b_n = l.$$

37. Sub-sequences. A sequence $\{b_n\}$ is said to be a sub-sequence of a sequence $\{a_n\}$ if there exists a strictly monotonically increasing sequence $\{i_n\}$ of positive integers such that

$$b_n = a_{i_n} \text{ for all } n.$$

The following results about sub-sequences are capable of simple proofs.

(*i*) Every sub-sequence of a convergent sequence is convergent and the limits are the same.

(*ii*) If ξ is a limiting point of a sequence $\{a_n\}$, then there exists a sub-sequence $\{b_n\}$ of $\{a_n\}$ such that the sequence $\{b_n\}$ is convergent with limit ξ.

Exercises

1. $\{a_n\} \to 0$ and $\{b_n\}$ oscillates finitely ; show that $\{a_n b_n\} \to 0$.
2. $\{a_n\}$ is convergent and $\{b_n\}$ divergent ; show that $\{a_n/b_n\} \to 0$.
3. If $\{a_n\}$ is convergent and $\{b_n\}$ divergent, then $\{a_n+b_n\}$ is divergent.
4. If $\{a_n\} \to a$, then $\{\, |a_n| \,\} \to |a|$.
   (It follows from the inequality, $|\, |a_n| - |a| \,| \leqslant |a_n - a|$).
5. If $\{a_n\} \to \infty$ and $b_n \geqslant a_n$ for all n, then $b_n \to \infty$.

6. (*i*) $\{a_n\}$ and $\{b_n\}$ are two sequences such that $a_n < b_n$, for every value of n. If $a_n \to a$ and $b_n \to b$, show that $a \leqslant b$.

Give an example to show that the equality is possible

(*ii*) If $a_n \to a$ and $b_n \to b$ and if $c_n = \max(a_n, b_n)$, $d_n = \min(a_n, b_n)$ show that

$$c_n \to \max(a, b), \; d_n \to \min(a, b).$$

7. $\lim a_n = a$ and b is a number such that $a_n < b$, for all n, show that $a \leqslant b$.

8. $\lim a_n = a$ and c is a number such that $c < a_n$, for all n, show that $c \leqslant a$.

9. If $\{a_n\}$ and $\{b_n\}$ are two convergent sequences, such that $a_n \leqslant b_n$ for all n, then $\lim a_n \leqslant b_n$.

38. An important limit. The number e. *To show that the sequence*

$$\left(1+\frac{1}{n}\right)^n$$

is convergent and

$$\lim\left(1+\frac{1}{n}\right)^n=\lim\left(1+\frac{1}{1\,!}+\frac{1}{2\,!}+\ldots+\frac{1}{n\,!}\right).$$

We have, by the Binomial theorem.*

$$a_n=\left(1+\frac{1}{n}\right)^n$$

$$=1+n.\frac{1}{n}+\frac{n(n-1)}{2\,!}\cdot\frac{1}{n^2}+\ldots\ldots+\frac{n(n-1)(n-2)\ldots1}{n\,!}\,\frac{1}{n^n}$$

$$=1+1+\frac{1}{2\,!}\left(1-\frac{1}{n}\right)+\ldots\ldots+$$

$$\frac{1}{n\,!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{n-1}{n}\right) \quad \ldots (1)$$

From this we easily deduce that $\{a_n\}$ is a monotonically increasing sequence. From (1), we also have

$$a_n \leqslant 1+1+\frac{1}{2\,!}+\ldots\frac{1}{n\,!}=b_n, \text{ say} \quad \ldots (2)$$

As shown in Ex. 5 on page 54.

$$b_n \leqslant 3, \text{ for all } n.$$

Thus $\{a_n\}$ is a bounded monotonically increasing sequence and is, therefore, convergent.

The limit of this convergent sequence $(1+1/n)^n$ *is denoted by e.*

From (2), we have

$$e=\lim a_n \leqslant \lim b_n=b, \text{ say} \quad \ldots (3)$$

Again, if m is any integer $\geqslant n$, we deduce from (1),

$$a_m>1+1+\frac{1}{2\,!}\left(1-\frac{1}{m}\right)+\ldots\ldots+\left[\frac{1}{n\,!}\left(1-\frac{1}{m}\right)\ldots\left(1-\frac{n-1}{m}\right)\right]\ldots(4)$$

Keeping n fixed and letting $m\to\infty$, we obtain from (4),

$$e \geqslant b_n$$

so that

$$b=\lim b_n \leqslant e \quad \ldots (5)$$

From (3) and (5),

$$e=b.$$

---

*The Binomial Theorem for a positive integral index is a simple consequence of the commutative, associative and distributive laws of addition and multiplication.

39. Infinite series. Its convergence and sum. If $\{a_n\}$ be any given sequence, then a symbol of the form

$$\sum_{n=1}^{\infty} a_n,$$

i.e.,
$$a_1+a_2+a_3+\ldots\ldots+a_n+\ldots$$
is called an *infinite series.*

This infinite series is said to be convergent, if the sequence $\{S_n\}$, where, S_n, denotes the sum

$$a_1+a_2+a_3+\ldots\ldots+a_n$$

is convergent, and lim S_n, in case it exists, is said to be the sum of the series.

This sequence $\{S_n\}$ is known as the sequence of *partial sums* of the given series.

The series is said to be divergent (or oscillatory) if the sequence $\{S_n\}$ is divergent (or oscillatory). The question of the sum of such a series does not arise.

Note. If we add the first two terms of an infinite series, and then add the sum so obtained to the third, and thus go on adding each term to the sum of the previous terms, we see that, as there is no last term of the series, the process will never be completed. In the case of a finite series, this process of addition will be completed at some stage, however large a number of terms the series may consist of. Thus, in the ordinary sense, the expression "*Sum of an infinite series*" has no meaning. The notion of limit has, therefore, been employed to give a meaning to this expression.

Exercises

1. Show that the infinite geometrical series

$$\sum_{n=0}^{\infty} x^n$$

is convergent, if and only if $|x|<1$.

2. Show that the series $\Sigma\, a_n$ where $a_n=(-1)^n$ is not convergent.

3. Show that the following series are convergent; also find their sum :—

(*i*) $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots\ldots\ldots+\dfrac{1}{n(n+1)}+\ldots\ldots$

(*ii*) $\dfrac{1}{1.2.3}+\dfrac{1}{2.3.4}+\dfrac{1}{3.4.5}+\ldots\ldots\ldots+\dfrac{1}{n(n+1)(n+2)}+\ldots\ldots$

(*iii*) $1+2x+3x^2+\ldots\ldots+nx^{n-1}+\ldots\ldots,\ |x|<1.$

4. Show that

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2n+3)}{(n+1)(n+2)}=1.$$

5. Show that

$$\sum_{1}^{\infty}\frac{n^2+9n+3}{(n+1)(2n+3)(2n+5)(n+4)}=\frac{5}{36}.$$

6. Show that the arithmetic series

$$a+(a+d)+(a+2d)+\dots\dots+(a+nd)+\dots\dots$$

is always divergent, except when a, d are both zero.

7. A series Σa_n is convergent and k is a constant ; show that the series $\Sigma k a_n$ is also convergent.

8. A series Σa_n is given : a sequence $\{b_n\}$ is defined such that

$$b_n = a_{m+n} ; \ m \text{ being a given positive integer ;}$$

show that the series Σa_n and Σb_n have the same behaviour in relation to convergence or otherwise.

9. Σa_n Σb_n are two convergent series, S_1, S_2 being their sums ; show that the series $\Sigma(a_n+b_n)$ is also convergent and its sum is equal to S_1+S_2.

10. Show that a sequence $\{a_n\}$ is convergent if, and only if, the series $\Sigma(a_{n+1}-a_n)$ is convergent.

40. Tests for the Convergence of infinite series. Corresponding to the two tests of convergence of a sequence obtained in §§ 34·1, 34·2, page 49, page 51 we obtain herebelow two tests of convergence for an infinite series.

40·1. Cauchy's general principle of convergence of a series. *The necessary and sufficient condition for the convergence of an infinite series Σa_n is that to every positive number ε, however small, there corresponds a positive integer, m, such that*

$$| a_{n+1}+a_{n+2}+ \dots +a_{n+p} | < \varepsilon,$$

for every $n \geqslant m$ and every $p \geqslant 0$.

We write

$$S_n = a_1+a_2+\dots\dots+a_n.$$

By §34·1, page 49, the necessary and sufficient condition for the convergence of $\{S_n\}$, *i.e.*, of Σa_n is that to every positive number ε, however small, there corresponds a positive integer m such that, for every $n \geqslant m$ and every $p \geqslant 0$,

$$| S_{n+p}-S_n | < \varepsilon,$$

i.e., $$| a_{n+1}+a_{n+2}+\dots\dots+a_{n+p} | < \varepsilon.$$

40·2. Convergence of a positive term series. *The necessary and sufficient condition for the convergence of a series Σa_n, whose terms a_n are all $\geqslant 0$, is that there exists a positive constant K such that*

$$S_n = a_1+a_2+\dots\dots+a_n \leqslant K,$$

for every n, i.e., S_n is bounded above.

The result follows from the fact that if every $a_n \geqslant 0$, then the sequence $\{S_n\}$ is monotonically increasing and will, therefore, be convergent if, and only if, it is bounded above. (§ 34·2, page 51).

Ex. 1. *Show that the series*

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots\dots+\frac{(-1)^{n+1}}{n}+\dots\dots,$$

is convergent.

This is not a positive term series.

It is easy to see that

$$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}$$

is positive and less than $1/(n+1)$.

Let ε be any positive number. We have

$$|S_{n+p}-S_n| = \left|\frac{1}{n+1}-\frac{1}{n+2}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}\right|$$

$$=\frac{1}{n+1}-\frac{1}{n+2}+\ldots\ldots+\frac{(-1)^{p+1}}{n+p}<\frac{1}{n+1}<\varepsilon,$$

if

$$n > (1/\varepsilon-1).$$

Let m be any integer greater than $(1/\varepsilon-1)$. Then we have

$$|S_{n+p}-S_n| < \varepsilon, \text{ for } n \geqslant m \text{ and } p \geqslant 0.$$

Hence the series converges.

Ex. 2. *Show that the series*

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots+\frac{1}{n}+\ldots\ldots$$

does not converge.

This is a positive term series.

Suppose that it converges. There exists a positive integer m such that for every $n \geqslant m$, and every $p \geqslant 0$,

$$\left|\frac{1}{n+1}+\frac{1}{n+2}+\ldots\ldots+\frac{1}{n+p}\right| < \frac{1}{4}. \quad \left(\text{Taking } \varepsilon=\frac{1}{4}\right)$$

But taking $n=m$ and $p=m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots\ldots+\frac{1}{2m} > \frac{m}{2m}=\frac{1}{2}.$$

Thus we have a contradiction. Hence the series does not converge. (Refer Ex. 3, page 53).

41. Representation of real numbers as Decimals. Firstly, we show that the *infinite series*

$$a+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots\ldots+\frac{a_n}{10^n}+\ldots\ldots \quad \ldots(1)$$

is convergent, if, a, is any integer and $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$, are integers such that

$$0 \leqslant a_n \leqslant 9, \text{ for all } n \geqslant 1.$$

Clearly

$$S_n=a+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots\ldots+\frac{a_n}{10^n}$$

is monotonically increasing.

Also, we have

$$S_n \leqslant a+\frac{9}{10}+\frac{9}{10^2}+\ldots\ldots+\frac{9}{10^n}$$
$$=a+\frac{9}{10}\left[\frac{1-(\frac{1}{10})^n}{1-\frac{1}{10}}\right]$$
$$=a+1-\frac{1}{10^n}<a+1, \text{ for all } n,$$

so that S_n is bounded above.

Thus the series is convergent.

This infinite series (1) is often written as

$$a\ .\ a_1\ a_2\ a_3\ a_4\ldots\ldots a_n\ldots\ldots \qquad \ldots(2)$$

The symbol (2) denotes the real number which is the sum of the infinite series (1).

We shall now show that every real number can be written in the form (2), *i.e.*, can be expressed as the sum of an infinite series of the form (1).

Consider any real number x. There exist two integers $a, a+1$ such that

$$a \leqslant x < a+1.$$

Divide now the interval $[a, a+1]$ into 10 equal parts. We then obtain an integer a_1 between 0 and 9 such that

$$a+\frac{a_1}{10} \leqslant x < a+\frac{a_1+1}{10}.$$

i.e.,

$$a\ .\ a_1 \leqslant x < a\ .\ (a_1+1).$$

Again divide the interval

$$\left[a+\frac{a_1}{10},\ a+\frac{a_1+1}{10}\right]$$

into 10 equal parts. We shall then obtain an integer, a_2, between 0 and 9, such that

$$a+\frac{a_1}{10}+\frac{a_2}{10^2} \leqslant x < a+\frac{a_1}{10}+\frac{a_2+1}{10^2}$$

i.e.,

$$a.a_1a_2 \leqslant x < a.a_1(a_2+1).$$

Proceeding in this manner, we shall obtain a sequence $\{a_n\}$ of integers between 0 and 9 such that

$$a.a_1a_2\ldots\ldots a_n \leqslant x < a.a_1a_2\ldots\ldots(a_n+1).$$

We write

$$b_n=a.a_1a_2\ldots\ldots a_n,\ c_n=a.a_1a_2\ldots\ldots(a_n+1).$$

As seen before, $\{b_n\}$ is convergent. Also since

$$c_n-b_n=\frac{1}{10^n} \quad \text{or} \quad c_n=b_n+\frac{1}{10^n},$$

we see that $\{c_n\}$ is convergent and

$$\lim c_n = \lim b_n.$$

From the inequalities

$$b_n \leqslant x < c_n,$$

we now conclude that

$$x = \lim b_n,$$

$$x = a.a_1a_2a_3 \ldots\ldots a_n \ldots\ldots$$

Note. In case x is a rational number, the decimal will be terminating or recurring.

42 The meaning of a^x, when $a > 0$, and x is any rational number. When x is a positive integer, the symbol a^x denotes the product

$$a.a.a.a\ldots\ .\ a, \ (x \text{ times})$$

and when x is a negative integer so that, $-x$, is a positive integer, we have

$$a^x = 1/a^{-x}.$$

Thus the concepts of multiplication and division (§13, page 17) are all that we require in order to define a^x, when x is any integer.

The theorem below is fundamental for giving a meaning to the symbol, a^x, when x is any rational number, and a is positive.

42·1. Theorem. *If m is a positive integer, and a, any given positive number, then the equation*

$$x^m = a, \qquad \ldots (i)$$

in x, has one and only one positive root.

Divide all the real numbers into two classes L and R, putting (*i*) all the negative numbers, (*ii*) zero, (*iii*) all the positive numbers x such that $x^m \leqslant a$, in L and all the others in R.

Clearly every number has a class. The class R exists, for any number k which is greater than a as well as 1 belongs to R. Also if y be any positive number less than a member x of L, we have $y^m < x^m \leqslant a$ and accordingly y belongs to L. Thus the classes L, R determine a section of real numbers. Let B be the number which divides the two classes.

Clearly B cannot be negative. If possible, let $B=0$ so that $1/n$ which is $> B=0$, belongs to R, and accordingly

$$(1/n)^m > a \qquad \ldots (ii)$$

n, being any positive integer.

If $n \to \infty$, we obtain from (*ii*), $0 \geqslant a$, so that we have a contradiction.

Thus B is necessarily positive.

It will, now, be shown that

$$B^m = a.$$

For every positive integral value of n, $B-1/n$ belongs to L and $B+1/n$ to R so that we have

$$(B-1/n)^m \leqslant a < (B+1/n)^m. \quad \ldots(iii)$$

Let $n \to \infty$. We obtain, from (iii),

$$B^m \leqslant a \leqslant B^m$$

so that

$$B^m = a.$$

Thus B is a positive root of $x^m = a$.

If possible, let B' be another positive root. From

$$B' \gtrless B,$$

we deduce

$$B'^m \gtrless B^m,$$

and so B^m, B'^m cannot both be equal to the same number.

Def. *The unique positive root of the equation*

(*i*) $\qquad x^m = a, \qquad (a > 0,\ m,$ *any positive integer*)

is symbolically written as

$$a^{1/m} \quad \text{or} \quad \sqrt[m]{a},$$

and is called the mth root of a.

Note. If $a > 0$, and m is even, the equation (*i*) possesses a negative root $-B$ also; but if m is odd, it cannot obviously, have any negative root.

If $a < 0$, and m is even, the equation cannot obviously, have any root, positive or negative, but if m is odd, it has no positive root but has a negative root $-B$, where B is the positive root of $x^m = -a$.

To avoid this ambiguity and indefiniteness, we will always take the base, a, as positive, and the symbol $a^{1/m}$ will, then, always denote the *unique positive* root of $x^m = a$.

Def. *If n/m is a rational number where m is positive, then by def.,*

$$a^{n/m} = (\sqrt[m]{a})^n$$

a, being positive.

Exercises

(The following exercises are to be considered as a part of the text).

1. Prove that

$$(\sqrt[m]{a})^n = (\sqrt[m]{a^n}).$$

2. x, y are any rational numbers and a is positive; show that

(*i*) $a^x \cdot a^y = a^{x+y}$. $\qquad$ (*ii*) $(a^x)^y = a^{xy}$.

(Reduce x, y to a common denominator).

3. a, b are two positive real numbers and x is a rational number; show that

$$(ab)^x = a^x b^x.$$

4. x, y are two rational numbers such that $x > y$; show that

$$a^x \gtreqless a^y, \text{ according as } a \gtreqless 1.$$

5. x is any positive rational number; show that

$$a^x \gtreqless 1, \text{ according as } a \gtreqless 1.$$

6. *a, b* are positive numbers, and x is any positive rational number show that

$$a^x \gtrless b^x \text{ according as } a \gtrless b.$$

7. *Show that*

$$\lim \sqrt[n]{a}=1, \textit{ when } n \to \infty, \ (a > 0).$$

For $a = 1$, the result is obvious.

Let $a > 1$. We write

$$\sqrt[n]{a}=1+h_n,$$

so that
$$h_n > 0.$$

We know that

$$(1+h_n)^n > 1 + nh_n.$$

$$\therefore \qquad a=(1+h)^n > 1 + nh_n$$

or
$$0 < h_n < (a-1)/n.$$

Let ε be any positive number. There exists a positive integer m such that $(a-1)/n < \varepsilon$, for $n \geqslant m$. Thus, we have

$$-\varepsilon < 0 < h_n < (a-1)/n < \varepsilon, \text{ for } n \geqslant m$$

or
$$|\sqrt[n]{a}-1| = |h_n| < \varepsilon, \text{ for } n \geqslant m.$$

Hence the result.

If $a < 1$, we write $a=1/b$ so that $b > 1$.

$\therefore \quad \sqrt[n]{a}=1/\sqrt[n]{b}$ and the result now follows.

8. Show that $\lim a^{-1/n}=1$.

9. *$\{x_n\}$ is any sequence of rational numbers such that lim $x_n=0$, when $n \to \infty$; show that*

$$a^{x_n} \to 1, \textit{ when } n \to \infty. \quad (a > 0).$$

Since
$$\lim_{n \to \infty} a^{\frac{1}{n}} = 1 \text{ and } \lim_{n \to \infty} a^{-\frac{1}{n}} = 1,$$

therefore, there exists a positive integer m such that, for $n \geqslant m$,

$$1-\varepsilon < a^{\frac{1}{n}} < 1-\varepsilon,$$

$$1-\varepsilon < a^{-\frac{1}{n}} < 1+\varepsilon.$$

In particular, therefore,

$$1-\varepsilon < a^{\frac{1}{m}} < 1+\varepsilon,$$

$$1-\varepsilon < a^{-\frac{1}{m}} < 1+\varepsilon.$$

Also since $\lim x_n=0$, there exists a positive integer m_1, such that for $n \geqslant m_1$,

$$-1/m < x_n < 1/m.$$

$\therefore \qquad a^{x_n}$ lies between $a^{-1/m}$ and $a^{1/m}$ for $n \geqslant m_1$.

Thus we see that there exists a positive integer m_1 such that for $n \geqslant m_1$,

$$1-\varepsilon < a^{x_n} < 1+\varepsilon.$$

Hence the result.

42·2. Powers with arbitrary real indices. *To define a^x, when, a, is any positive real number and, x, any real number.*

Let $\{x_n\}$ be* a monotonically increasing sequence of rational numbers such that

$$\lim x_n = x.$$

If $a > 1$, the sequence a^{x_n} is monotonically increasing and bounded above inasmuch as $a^{x_n} < a^k$, where k is any rational number greater than x. Thus the sequence $\{a^{x_n}\}$ is convergent.

If $a < 1$, the sequence a^{x_n} is monotonically decreasing and bounded below inasmuch as $a^{x_n} > 0$. Thus, again, the sequence $\{a^{x_n}\}$ is convergent.

Let, now, $\{x'_n\}$ be *any* convergent sequence such that

$$\lim x'_n = x$$

The sequence $\{x'_n - x_n\} \rightarrow 0$ and, therefore, the sequence

$$a^{(x'_n - x_n)} \rightarrow 1. \qquad \text{(Ex. 9, page 65).}$$

Now, $$a^{x'_n} = a^{x'_n - x_n} . a^{x_n}.$$

We have

$$\lim a^{x'_n} = \lim a^{x'_n - x_n} . \lim a^{x_n} = 1 . \lim a^{x_n} = \lim a^{x_n}.$$

Thus we see that $\{a^{x'_n}\}$ is convergent and its limit is the same as that of the convergent sequence $\{a^{x_n}\}$.

This discussion justifies the following definition of a^x :—

If $a > 0$, and x is any real number, then a^x is defined as the limit of $\{a^{x_n}\}$, where $\{x_n\}$ is any sequence of rational numbers with x as its limit.

Note 1. To justify ehe above definition, it is necessary to show that (*i*) $\lim a^{x_n}$ exists, and (*ii*) that it is the same whatever be the sequence, $\{x_n\}$, provided $\lim x_n = x$. Both these facts have been established above in § 42·2.

Note 2. The laws of indices, *viz.*,

$$a^x a^y = a^{x+y}, \quad (ab)^x = a^x b^x, \quad (a^x)^y = a^{xy},$$

may easily be shown to remain valid when x, y are any real numbers and $a > 0$.

For example, let $\{x_n\}$, $\{y_n\}$ be any two sequences such that

$$\lim x_n = x, \ \lim y_n = y.$$

so that

$$\lim (x_n + y_n) = x + y, \ \lim (x_n y_n) = xy.$$

We have

$$a^{x_n} a^{y_n} = a^{x_n + y_n}$$

---

*One such sequence arises if we take x_n as any rational number such that

$$x - \frac{1}{n} < x_n < x - \frac{1}{n+1}.$$

Taking limits, when $n \to \infty$, we obtain

$$a^x \cdot a^y = a^{x+y}. \qquad \text{[§35, page 54]}$$

43. Theorem. *If x be any real number and, r, s, two rational numbers such that*

$$r < x < s,$$

then

$$a^r \lessgtr a^x \lessgtr a^s, \textit{ if, } a \gtrless 1.$$

Let $a > 1$. Let $\{x_n\}$ be any sequence of rational numbers such that $\{x_n\} \to x$. Consider any pair r', s' of rational numbers such that

$$r < r' < x < s' < s.$$

As $]r', s'[$ is a neighbourhood of x and $\lim x_n = x$, there exists a positive integer m such that

$$r' < x_n < s', \text{ when } n \geq m.$$

$\therefore \quad a^{r'} < a^{x_n} < a^{s'}$ when $n \geq m$. (Ex. 7 and Ex. 8, page 57)

Let $n \to \infty$. We obtain

$$a^{r'} \leq a^x \leq a^{s'}.$$

But we know that

$$a^r < a^{r'} \text{ and } a^{s'} < a^s.$$

$$\therefore \qquad a^r < a^x < a^s.$$

The case, when $a < 1$, may be similarly discussed.

Cor. *If x, y are two real numbers such that*

$$x < y,$$

then

$$a^x \lessgtr a^y, \textit{ if } a \gtrless 1,$$

Let r be any rational number such that

$$x < r < y.$$

Then

$$a^x \lessgtr a^r \lessgtr a^y, \text{ according as } a \gtrless 1,$$

or

$$a^x \lessgtr a^y, \text{ according as } a \gtrless 1.$$

44. Theorem. *If $\{\alpha_n\}$ is any convergent sequence of real numbers such that*

$$\lim \{\alpha_n\} = \alpha,$$

then

$$\lim a^{\alpha_n} = a^{\alpha}, \ (a > 0).$$

To each number α_n, we can associate a pair of rational numbers r_n and $r_n + 1/n$ such that

$$r_n < \alpha_n < r_n + 1/n \qquad \dots (1)$$

or

$$0 < \alpha_n - r_n < 1/n.$$

$\therefore$ $\lim (\alpha_n - r_n) = 0,$

or $\lim r_n = \lim \alpha_n = \alpha.$

Also $\lim (r_n + 1/n) = \alpha.$

From (1),

$$a^{r_n} \lesseqgtr a^{\alpha_n} \lesseqgtr a^{r_n + 1/n}, \text{ according as } a \gtrless 1.$$

Taking limits, we obtain

$$\lim a^{\alpha_n} = a^{\alpha}. \qquad [\S 36, \text{ page } 57]$$

45. Logarithms.

Theorem. *If a, b are any two real and positive numbers and $a \neq 1$, then there exists one and only one real number x such that*

$$a^x = b,$$

Let $a > 1$. We divide all the real numbers into two classes L and R putting any number x in L if $a^x < b$ and otherwise in R

Clearly each number has a class; also each class has a number, for a negative integer $-k$ such that $a^{-k} < b$ belongs to L; and a positive integer m such that $a^m > b$ belongs to R. (The existence of k and m follow from the fact $a^{-n} \to 0$ and $a^n \to \infty$, as $n \to \infty$.). Also from Cor. to §43 it follows that each member of L is less than each member of R. Thus L, R determine a section of real numbers.

Let ξ be the number which separates the two classes.

It will be known that $a^\xi = b$.

Now $(\xi - 1/n)$ belongs to L and $(\xi + 1/n)$ to R; n being any positive integer. We have

$$a^{\xi - 1/n} \leq b < a^{\xi + 1/n}.$$

Let $n \to \infty$. We obtain

$$a^\xi \leq b \leq a^\xi, \textit{ i.e., } a^\xi = b.$$

Thus ξ satisfies the equation $a^x = b$.

If possible, let η be another root. We have

$$a^\eta \gtrless a^\xi, \text{ according as } \eta \gtrless \xi,$$

and, accordingly, we cannot have $a^\eta = b = a^\xi$.

If $a < 1$, we take $\alpha = 1/a$ so that $\alpha > 1$. The number ξ is then obtained from $\alpha^x = 1/b$.

Hence the theorem.

Def. *If a and b are any real positive numbers, then the number x, which is uniquely determined by $a^x = b$, is called the logarithm of b to the base a, and written as $\log_a b$.*

Ex. 1. a, x, y are any real positive numbers; $x > y$; show that

$$\log_a x \lesseqgtr \log_a y,$$

according as $a \lesseqgtr 1$.

2. a, x, y are any real positive numbers: show that

(*i*) $\log_a (xy) = \log_a x + \log_a y$.

(*ii*) $\log_a (x/y) = \log_a x - \log_a y$.

(*iii*) $\log_a (x^y) = y \log_a x$.

Examples

1. *Show that the sequence* $\{x^n\}$ *is convergent if, and only if,* $-1 < x \leqslant 1$.

(*i*) *Let* $x > 1$. We write $x = 1+h$ so that h is positive.

By mathematical induction, it may easily be shown that

$$x^n = (1+h)^n > 1+nh.$$

Let $\triangle$ be any positive number, however large. We have

$$1+nh > \triangle, \text{ if } n > (\triangle - 1)h.$$

Taking m as any positive integer $> (\triangle - 1)/h$, we see that

$$x^n > \triangle \text{ for } n \geqslant m, \text{ so that } \lim x^n = \infty.$$

(*ii*) *Let* $x=1$. Clearly, in this case, $\lim x^n = 1$.

(*iii*) *Let* $0 < x < 1$. We write $x = 1/(1+h)$ so that h is positive.

We have $\quad 0 < x^n = 1/(1+h)^n < 1/(1+nh)$.

Let ε be any positive number, however small. We have

$$1/(1+nh) < \varepsilon, \text{ if } n > (1/\varepsilon - 1)/h.$$

Taking m as any integer $> (1/\varepsilon - 1)/h$, we see that

$$0 < x^n < \varepsilon \text{ or } |x^n| < \varepsilon, \text{ for } n \geqslant m,$$

so that
$$\lim x^n = 0.$$

(*iv*) *Let* $x=0$. Clearly, $\lim x^n = 0$.

(*v*) *Let* $-1 < x < 0$. We write $x = -\alpha$ so that $0 < \alpha < 1$.

We have
$$|x^n| = \alpha^n.$$

It now follows from (*iii*) that $\lim x^n = 0$.

(*vi*) *Let* $x = -1$. Obviously x^n oscillates finitely.

(*vii*) *Let* $x < -1$. We write $x = -\alpha$ so that $\alpha > 1$.

As $n \to \infty$, x^n takes values, both positive and negative greater than any assigned number. Hence x^n oscillates infinitely.

2. *Show that* $\lim \sqrt[n]{n} = 1$.

We write
$$a_n = \sqrt[n]{n}.$$

Let
$$a_n = 1 + h_n \quad \text{where } h_n > 0.$$

We have
$$\begin{aligned} n &= a_n{}^n \\ &= (1+h_n)^n \\ &= 1 + nh_n + \tfrac{1}{2}n(n-1)h_n{}^2 + \ldots\ldots + h_n{}^n \\ &> \tfrac{1}{2}n(n-1)h_n{}^2. \end{aligned}$$

$$\therefore \qquad h_n{}^2 < 2/(n-1),$$

or
$$0 < h_n < \sqrt{[2/(n-1)]}.$$

The result now follows.

To be rigorous, let ε be any positive number. Now

$$\sqrt{[2/(n-1)]} < \varepsilon, \text{ if } n > 1 + 2/\varepsilon^2.$$

If m be any integer $> 1+2/\varepsilon^2$, we see that, for $n \geqslant m$,

$$-\varepsilon < 0 < h_n < \sqrt{[2/(n-1)]} < \varepsilon$$

ence $h_n \to 0$.

3. *$\{a_n\}$ is a sequence such that*

$$lim \{a_{n+1}/a_n\} = l,$$

where $|l| < 1$; show that lim $a_n = 0$.

Since $|l| < 1$, we can choose a positive number, ε, so small that

$$|l|+\varepsilon < 1.$$

There exists a positive integer m such that for $n \geqslant m$,

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon,$$

or
$$\left|\frac{a_{n+1}}{a_n}\right| - |l| \leqslant \left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon,$$

or
$$\left|\frac{a_{n+1}}{a_n}\right| < |l| + \varepsilon = k, \text{ say, where } k < 1.$$

Changing n to m, $m+1$, $m+2, \ldots, (n-1)$ and multiplying, we get

$$\left|\frac{a_n}{a_m}\right| < k^{n-m} \text{ or } |a_n| < k^n.(|a_m|/k^m).$$

Since $k^n \to 0$, we have the required result.

Note. The general result obtained here enables us to prove the following particular but important results on limits :—

(*i*) lim $(x^n/n!) = 0$, x any number.

(*ii*) lim $(n^r/x^n) = 0$, $|x| > 1$.

(*iii*) lim $\dfrac{m(m-1)\ldots(m-n+1)x^n}{n!} = 0$, $|x| < 1$.

4. *If $\{a_n\}$ is a sequence such that $a_n > 0$ and*

$$lim \{a_{n+1}/a_n\} = l > 1,$$

then
$$lim\ a_n = \infty.$$

We choose a positive number ε such that $l - \varepsilon > 1$.

There exists a positive integer m such that for $n \geqslant m$

$$l - \varepsilon < a_{n+1}/a_n < l + \varepsilon.$$

Thus for $n \geqslant m$.

$$a_{n+1}/a_n > l - \varepsilon = k, \text{ say, } (k > 1),$$

From this we deduce [As in Ex. 3 above] that

$$a_n > k^n. (a_m/k^m).$$

Since $k^n \to \infty$, we have the required result.

Note. Compare Ex. 3 and Ex. 4 above.

5. *If the sequences $\{a_n\}$, $\{b_n\}$ tend to 0 and if $\{b_n\}$ is a strictly monotonically decreasing sequence so that $b_n > b_{n+1} > 0$, then*

$$\lim \frac{a_n}{b_n} = \lim \frac{a_n - a_{n+1}}{b_n - b_{n+1}},$$

provided that the limit on the right exists, whether finite or infinite.

Case I. Let $\lim \dfrac{a_n - a_{n+1}}{b_n - b_{n+1}} = l$, where l is finite ... (*i*)

Let ε be any positive number. By virtue of (*i*), there exists a positive integer m such that

$$l - \varepsilon < \frac{a_n - a_{n+1}}{b_n - b_{n+1}} < l + \varepsilon, \text{ when } n \geqslant m,$$

i.e.,

$$(l-\varepsilon)(b_n - b_{n+1}) < (a_n - a_{n+1}) < (l+\varepsilon)(b_n - b_{n+1}), \text{ when } n \geqq m,$$

for $(b_n - b_{n+1})$ is positive.

Changing n to $n, n+1, n+2, \ldots, (n+p-1)$, in turn and adding, we see that

$$(l-\varepsilon)(b_n - b_{n+p}) < (a_n - a_{n+p}) < (l+\varepsilon)(b_n - b_{n+p}),$$

for every $n \geqslant m$ and every $p > 0$.

Keep n fixed and let $p \to \infty$. Since $a_{n+p} \to 0$, and $b_{n+p} \to 0$, therefore we obtain

$$(l-\varepsilon)\, b_n \leqslant a_n \leqslant (l+\varepsilon)\, b_n,$$

or

$$(l-\varepsilon) \leqslant (a_n/b_n) \leqslant l+\varepsilon, \text{ for every } n \geqq m$$

Hence $a_n/b_n \to l$ as $n \to \infty$.

Case II. Let $\lim \dfrac{a_n - a_{n+1}}{b_n - b_{n+1}} = \infty$. ... (*i*)

Let $\triangle$ be any positive number. By virtue of (*i*), there exists a positive integer m such that

$$\frac{a_n - a_{n+1}}{b_n - b_{n+1}} > \triangle, \text{ when } n \geqslant m,$$

i.e., $(a_n - a_{n+1}) > \triangle\, (b_n - b_{n+1})$ when $n \geqslant m$, for $(b_n - b_{n+1})$ is positive.

As in Case I, we obtain

$$a_n - a_{n+p} > \triangle\, (b_n - b_{n+p})$$

or

$$a_n \geqslant \triangle\, b_n, \quad i.e., \quad a_n/b_n \geqslant \triangle, \text{ when } n \geqslant m.$$

Hence $a_n/b_n \to \infty$ as $n \to \infty$.

6. *If $\{b_n\}$ is a strictly monotonically increasing sequence so that $b_{n+1} > b_n$ and if $b_n \to \infty$ and $\{a_n\}$ be any sequence, then*

$$\lim \frac{a_n}{b_n} = \lim \frac{a_{n+1} - a_n}{b_{n+1} - b_n},$$

provided that the limit on the right exists, whether finite or infinite.

Case I. Let $\lim \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}=l$, where l is finite. ... (*i*)

Let ε be any positive number. By virtue of (*i*), there exists a positive integer m_1 such that

$$(l-\tfrac{1}{2}\varepsilon)(b_{n+1}-b_n) < (a_{n+1}-a_n) < (l+\tfrac{1}{2}\varepsilon)(b_{n+1}-b_n), \text{ when } n \leqq m_1.$$

Changing n to n, $n+1$, $n+2,\ldots,n+p-1$, in turn and adding, we see that

$$(l-\tfrac{1}{2}\varepsilon)(b_{n+p}-b_n) < (a_{n+p}-a_n) < (l+\tfrac{1}{2}\varepsilon)(b_{n+p}-b_n).$$

Dividing by b_{n+p} and adding a_n/b_{n+p}, we obtain

$$(l-\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}<\frac{a_{n+p}}{b_{n+p}} < (l+\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} \quad \ldots (ii)$$

for every $n \geqslant m_1$, and $p \geqslant 0$.

We keep n fixed and let $p \to \infty$.

Since

$$(l-\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} \to l-\tfrac{1}{2}\varepsilon,$$

and

$$(l+\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} \to l+\tfrac{1}{2}\varepsilon,$$

we see that there exists a positive integer m_2 such that for every $p \geqslant m_2$ we have

$$l-\tfrac{1}{2}\varepsilon-\tfrac{1}{2}\varepsilon< (l-\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} < l-\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon. \quad \ldots (iii)$$

and $$l+\tfrac{1}{2}\varepsilon-\tfrac{1}{2}\varepsilon < (l+\tfrac{1}{2}\varepsilon)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} < l+\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon \quad \ldots (iv)$$

From (*ii*), (*iii*) and (*iv*), we obtain

$$l-\varepsilon < \frac{a_{n+p}}{b_{n+p}} < l+\varepsilon,$$

for every $n \geqq m_1$ and $p \geqslant m_2$,

i.e., $$l-\varepsilon < \frac{a_n}{b_n} < l+\varepsilon, \text{ for every } n \geqslant (m_1+m_2)$$

Hence $a_n/b_n \to l$, as $n \to \infty$.

Case II. Let $\lim \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n}=\infty$. ... (*i*)

Let k be any positive number, however large.

By virtue of (*i*), there exists a positive integer m_1 such that for every $n \geqslant m_1$,

$$(a_{n+1}-a_n) > (k+1)(b_{n+1}-b_n).$$

As in case I, we obtain

$$a_{n+p}-a_n > (k+1)(b_{n+p}-b_n),$$

or $$\frac{a_{n+p}}{b_{n+p}} > (k+1)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}}. \quad \ldots (ii)$$

Keeping n fixed and letting $p \to \infty$, we see that

$$(k+1)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} \to k+1,$$

so that there exists a positive integer m_2 such that for every $p \geqslant m_2$,

$$(k+1)-1 < (k+1)\left(1-\frac{b_n}{b_{n+p}}\right)+\frac{a_n}{b_{n+p}} < (k+1)+1. \quad \dots (iii)$$

From (ii) and (iii), we obtain

$$\frac{a_{n+p}}{b_{n+p}} > k, \text{ when } n \geqslant m_1 \text{ and } p \geqslant m_2,$$

i.e.,

$$\frac{a_n}{b_n} > k, \text{ when } n \geqslant (m_1+m_2).$$

Hence $a_n/b_n \to \infty$ as $n \to \infty$.

Note. Compare Ex. 5 and Ex. 6 above.

7. *If* $\qquad \lim a_n = l,$

then $$\lim \frac{a_1+a_2+a_3+\dots+a_n}{n} = l.$$

We write

$$a_n - l = b_n,$$

so that the sequence $\{b_n\} \to 0$.

We have

$$\frac{a_1+a_2+\dots+a_n}{n} = l + \frac{b_1+b_2+\dots+b_n}{n},$$

so that we have to prove that $(b_1+b_2+\dots+b_n)/n \to 0$, when $b_n \to 0$.

Let ε be any positive number. There exists a positive integer p, such that

$$|b_n| < \tfrac{1}{2}\varepsilon, \text{ when } n \geqslant p$$

Also since $\{b_n\}$ is convergent, it is bounded and, therefore, there exists a positive number k such that

$$|b_n| < k, \text{ for all } n.$$

We write

$$\left|\frac{b_1+b_2+\dots+b_n}{n}\right| = \left|\frac{b_1+b_2+\dots+b_p}{n}+\frac{b_{p+1}+b_{p+2}+\dots+b_n}{n}\right|$$

$$\leqslant \frac{|b_1|+|b_2|\dots+|b_p|}{n}+\frac{|b_{p+1}|+\dots+|b_n|}{n}$$

$$\leqslant \frac{kp}{n}+\frac{\varepsilon(n-p)}{2n} < \frac{kp}{n}+\frac{\varepsilon}{2}.$$

We keep p fixed and see that

$$\frac{kp}{n} < \frac{\varepsilon}{2}, \text{ if } n > \frac{2kp}{\varepsilon}.$$

Let ν be any positive integer greater than $2kp/\varepsilon$ so that for $n \geqslant \nu$ we have $kp/n < \varepsilon/2$.

Let

$$m = \text{Max.}\ (p, \nu).$$

Thus, for every $n \geqslant m$, we have

$$\left| \frac{b_1 + b_2 + \dots b_n}{n} \right| < \varepsilon.$$

Hence the result.

(This is known as *Cauchy's first theorem on limits.*)

Note. This result could also be deduced from Ex. 6, Page 71, by putting $(a_1 + a_2 + \dots + a_n)$ for a_n and n for b_n.

Note. The converse of the result obtained above is not true as may be seen on considering $a_n = (-1)^n$.

8. *If* $\lim\ (a_{n+1} - a_n) = l$, *then* $\lim\ (a_n/n) = l$.

9. *If* $a_n > 0$ *and* $\lim a_n = l$, *show that*

$$\lim\ (a_1 a_2 \dots a_n)^{\frac{1}{n}} = l.$$

We have

$$\lim\ (\log a_n) = \log l,$$

so that by the result of Ex. 7, above

$$\frac{\log a_1 + \log a_2 + \dots + \log a_n}{n} \to \log l. \qquad \dots (1)$$

Also $$\frac{\log a_1 + \log a_2 + \dots + \log a_n}{n} = \log\ (a_1 a_2 \dots a_n)^{\frac{1}{n}}. \qquad \dots (2)$$

From (1) and (2), we deduce that

$$\lim\ (a_1 a_2 \dots a_n)^{\frac{1}{n}} = l.$$

10. *The sequence* $\{x_n\}$ *tends to the limit* l, *finite or infinite* ; $\{a_n\}$ *is another sequence of positive members such that the sequence* $\{a_1 + a_2 + \dots + a_n\}$ *diverges to* ∞ ; *prove that*

$$\frac{a_1 x_1 + a_2 x_2 + \dots + a_n x_n}{a_1 + a_2 + \dots + a_n} \to l \text{ as } n \to \infty.$$

Let $x_n - l = y_n$. Then we have to prove that

$$\left(\sum_{r=1}^{n} a_r y_r \right) \div \left(\sum_{r=1}^{n} a_r \right) \to 0,$$

when $\{y_n\} \to 0$ and $\sum_{r=1}^{n} a_r \to \infty$.

Let ε be any positive number. There exists a positive integer μ such that $|\ y_n\ | < \frac{1}{2}\varepsilon$ for every $n \geqslant \mu$. Also there exists a number k such that $|\ y_n\ | < k$, for all n.

We have

$$\left|\frac{\sum\limits_{r=1}^{n} a_r y_r}{\sum\limits_{r=1}^{n} a_r}\right| \leqslant \frac{\sum\limits_{r=1}^{\mu} a_r \mid y_r \mid}{\sum\limits_{r=1}^{n} a_r} + \frac{\sum\limits_{r=\mu+1}^{n} a_r \mid y_r \mid}{\sum\limits_{r=1}^{n} a_r}$$

$$\leqslant \frac{kc}{\sum\limits_{r=1}^{n} a_r} + \frac{\varepsilon}{2}$$

where c is the constant $\sum\limits_{r=1}^{\mu} a_r$; μ having been fixed.

There exists a number ν such that for $n \geqq \nu$,

$$\sum_{r=1}^{n} a_r > \frac{2kc}{\varepsilon}, \textit{ i.e.}, \frac{kc}{\sum\limits_{r=1}^{n} a_r} < \frac{\varepsilon}{2}.$$

If $m = \text{Max}\ (\mu, \nu)$, then for $n \geqq m$,

$$\left|\frac{\sum\limits_{r=1}^{n} a_r y_r}{\sum\limits_{r=1}^{n} a_r}\right| < \varepsilon.$$

Hence the result.

11. *If $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, then*

$$\frac{a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1}{n} \rightarrow AB.$$

12. *If $\{a_n\}$ be a sequence of positive terms, prove that*

$$lim \quad [\sqrt[n]{a_n}] = lim \quad [a_{n+1}/a_n],$$

provided that the limit on the right exists, whether finite or infinite.

(This is known as *Cauchy's second theorem on limits.*)

Case I. Let lim $[a_{n+1}/a_n] = l$, be finite.

Let ε be any positive number. There exists a positive integer m_1 such that for every $n \geqslant m_1$,

$$l - \tfrac{1}{2}\varepsilon < \frac{a_{n+1}}{a_n} < l + \tfrac{1}{2}\varepsilon.$$

Changing n to $n, n+1, n+2, \dots, (n+p-1)$ and multiplying, we get

$$(l - \tfrac{1}{2}\varepsilon)^p < \frac{a_{n+p}}{a_n} < (l + \tfrac{1}{2}\varepsilon)^p$$

or $$(a_n)^{\frac{1}{n+p}}(l-\tfrac{1}{2}\varepsilon)^{\frac{p}{n+p}} < (a_{n+p})^{\frac{1}{n+p}} < (l+\tfrac{1}{2}\varepsilon)^{\frac{p}{n+p}}(a_n)^{\frac{1}{n+p}}.$$

Keeping n fixed, we let $p \to \infty$. Since

$$(a_n)^{\frac{1}{n+p}}(l-\tfrac{1}{2}\varepsilon)^{\frac{p}{n+p}} \to (l-\tfrac{1}{2}\varepsilon),$$

and $$(a_n)^{\frac{1}{n+p}}(l+\tfrac{1}{2}\varepsilon)^{\frac{p}{n+p}} \to (l+\tfrac{1}{2}\varepsilon),$$

we see, as in Ex. 6, Page 71, that there exists a positive integer m_2 such that

$$l-\varepsilon < (a_{n+p})^{\frac{1}{n+p}} < l+\varepsilon, \text{ when } n \geqslant m_1 \text{ and } p \geqq m_2$$

or

$$l-\varepsilon < \sqrt[n]{a_n} < l+\varepsilon, \text{ when } n \geqq (m_1+m_2).$$

Hence the result.

Case II. Let $\lim [a_{n+1}/a_n]$ be infinite.

Let $\triangle$ be any positive number. There exists a positive integer m_1 such that for every $n \geqslant m_1$,

$$a_{n+1}/a_n > (\triangle+1).$$

Changing n to n, $(n+1)$, $(n+2)$,......, $(n+p-1)$ and multiplying, we get

$$(a_{n+p})^{\frac{1}{n+p}} > (a_n)^{\frac{1}{n+p}}(\triangle+1)^{\frac{p}{n+p}}.$$

The right hand expression $\to (\triangle+1)$, as $p \to \infty$, keeping n fixed. There exists, therefore, a positive integer m_2 such that for every $p \geqq m_2$

$$\triangle=\triangle+1-1 < (a_n)^{\frac{1}{n+p}}(\triangle+1)^{\frac{p}{n+p}} < \triangle+1+1=\triangle+2.$$

From above, we deduce that, for every $n \geqslant (m_1+m_2)$,

$$\sqrt[n]{a_n} > \triangle,$$

i.e., $$\sqrt[n]{a_n} \to \infty.$$

13. *If* x_1, x_2 *are positive and* $x_{n+1}=\frac{1}{2}(x_n+x_{n-1})$, *then the sequences*

$$x_1, x_3, x_5, \ldots\ldots ; \textit{ and } x_2, x_4, x_6, \ldots\ldots$$

are one a decreasing and the other an increasing sequence, and they converge to the same limit $\frac{1}{3}(x_1+2x_2)$.

Let $x_1 > x_2$. On this account, we have

$$x_2 < x_3 < x_1.$$

Also, since $x_2 < x_3$, we have

$$x_2 < x_4 < x_3.$$

In this manner, we may easily see that

$$x_4 < x_5 < x_3 ;\ x_4 < x_6 < x_5 ;\ x_6 < x_7 < x_5 ; \text{ and so on.}$$

Thus

$$x_2 < x_4 < x_6 < \ldots < x_5 < x_3 < x_1.$$

Hence $x_1, x_3, x_5 \ldots$ is decreasing and $x_2, x_4, x_6 \ldots$ increasing and being bounded, they are convergent.

We have

$$x_3 - x_2 = \tfrac{1}{2}(x_1 - x_2).$$

$$x_4 - x_2 = x_4 - x_3 + x_3 - x_2 = -\tfrac{1}{4}(x_1 - x_2) + \tfrac{1}{2}(x_1 - x_2)$$
$$= (x_1 - x_2)(-\tfrac{1}{4} + \tfrac{1}{2}).$$

$$x_5 - x_2 = x_5 - x_4 + x_4 - x_2 = (x_1 - x_2)(\tfrac{1}{8} - \tfrac{1}{4} + \tfrac{1}{2}).$$

$$x_6 - x_2 = x_6 - x_5 + x_5 - x_2 = (x_1 - x_2)(-\tfrac{1}{16} + \tfrac{1}{8} - \tfrac{1}{4} + \tfrac{1}{2}).$$

In general, we get

$$x_n - x_2 = (x_1 - x_2)\left(\frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^4} \ldots\ldots\ldots n-2 \text{ terms}\right)$$
$$= \frac{1}{3}(x_1 - x_2)\,[1 - (-\tfrac{1}{2})^{n-2}] \to \frac{1}{3}\,(x_1 - x_2),$$

as $n \to \infty$.

$\therefore \quad x_n \to \tfrac{1}{3}(x_1 + 2x_2)$, whether $n \to \infty$ through even or through odd integral values.

14. *If*

$$a_{n+1} = \sqrt{(k + a_n)}, \qquad \ldots(1)$$

where k, a_1 are positive, prove that the sequence $\{a_n\}$ is increasing or decreasing according as a_1 is less than or greater than the positive root of the equation

$$x^2 = x + k$$

and has in either case this root as its limit.

We have

$$a_{n+1}^2 - a_n^2 = (k + a_n) - (k + a_{n-1}) = a_n - a_{n-1},$$

so that $a_{n+1} \gtrless a_n$ according as $a_n \gtrless a_{n-1}$ and thus $\{a_n\}$ is a monotonic sequence; it is an increasing or decreasing sequence according as

$$a_2 > \text{ or } < a_1.$$

The equation

$$x^2 - x - k = 0, \qquad k > 0$$

has one root positive and the other negative. Let α, $-\beta$ be these roots where α and β are positive. Now

$$\because \quad x^2 - x - k \equiv (x - \alpha)(x + \beta), \text{ for every } x \qquad \ldots(2)$$

$$\therefore \quad a_1^2 - a_1 - k = (a_1 - \alpha)(a_1 + \beta). \qquad \ldots(3)$$

Let $a_1 > \alpha$. Then, by (3),

$$a_1^2 - a_1 - k > 0$$

so that

$$a_2 = \sqrt{(a_1 + k)} < a_1.$$

Thus $\{a_n\}$ is a decreasing sequence in this case.

Now,

$$a_n^2 = a_{n-1} + k > a_n + k,$$

i.e.,

$$a_n^2 - a_n - k > 0.$$

$\therefore$ from (2) we deduce that

$$a_n > \alpha, \text{ for every } n.$$

Thus $\{a_n\}$ is a monotonically decreasing sequence which is bounded below.

Hence $\lim \{a_n\}$ exists. Let it be l. We have

$$l \geq \alpha.$$

Taking limits in (1), we get

$$l=\sqrt{(k+l)}$$

or

$$l^2=l+k$$

so that l is equal to the positive root, α, of the equation

$$x^2=x+k.$$

The case when $a_1 < \alpha$ may be similarly treated.

If $a_1=\alpha$, then, as may be easily seen

$$a_n=\alpha \text{ for all } n.$$

15. *If $\{a_n\}$ and $\{b_n\}$ are two bounded sequences, show that*

(*i*) $\overline{\lim}\ a_n + \overline{\lim}\ b_n \geq \overline{\lim}\ (a_n+b_n)$,

(*ii*) $\underline{\lim}\ a_n + \underline{\lim}\ b_n \leq \underline{\lim}\ (a_n+b_n)$.

We write

$$\overline{\lim}\ a_n=G_1,\ \overline{\lim}\ b_n=G_2,\ \overline{\lim}\ (a_n+b_n)=G.$$

Let ε, be any positive number.

There exist positive integers m_1 and m_2 such that

$$a_n < G_1+\tfrac{1}{2}\varepsilon \qquad \text{for } n \geqslant m_1$$

and

$$b_n < G_2+\tfrac{1}{2}\varepsilon. \qquad \text{for } n \geqq m_2$$

From these we deduce that

$$a_n+b_n < G_1+G_2+\varepsilon \qquad \text{for } n \geqq \max.\ (m_1, m_2).$$

Thus

$$G \leq G_1+G_2+\varepsilon.$$

As ε is arbitrary, we obtain

$$G \leq G_1+G_2$$

which is (*i*).

We may similarly prove (*ii*).

Exercises

1. Find the limit of

$$\frac{2^n-1}{2^n+1}+\frac{(\frac{1}{2})^n-1}{(\frac{1}{2})^n+1},$$

as $n \to \infty$.

Find n such that for all values greater than this, the given sum differs from the limit by a number less than 1/1000.

2. Prove that

(*i*) $\lim \dfrac{1}{n}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\dots\dots\dots+\dfrac{1}{n}\right)=0.$

(*ii*) $\lim \dfrac{1+\sqrt{2}+\sqrt[3]{3}+\dots\dots\dots+\sqrt[n]{n}}{n}=1.$

3. Show that

$$\lim \frac{n}{\sqrt[n]{(n\,!)}}=e.$$

4. Find the limit to which the following sequence tends as $n \to \infty$;

$$\left[\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots\cdots\cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$$

5. Prove that, when $n \to \infty$,

(i) $\lim \left[\frac{1}{\sqrt{(n^2+1)}}+\frac{1}{\sqrt{(n^2+2)}}+\cdots\cdots\cdots+\frac{1}{\sqrt{(n^2+n)}}\right]=1.$

(ii) $\lim \left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\cdots\cdots+\frac{1}{(2n)^2}\right]=0$

(iii) $\lim \left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{(n+1)}}+\cdots\quad\cdots+\frac{1}{\sqrt{(2n)}}\right]=\infty.$

6. Prove that the sequence

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \cdots\cdots\cdots\cdots$$

converges to 2.

7. Prove that the sequence

$$\sqrt{7},\ \sqrt{7+\sqrt{7}},\ \sqrt{7+\sqrt{7+\sqrt{7}}},\ldots$$

converges to the positive root of $x^2-x-7=0$.

8. If $a_n=a/(1+a_{n-1})$ where a, a_1 are positive, show that the sequence $\{a_n\}$ is convergent and tends to the positive root of the equation $x^2+x=a$ as its limit.

9. A sequenee $\{a_n\}$ of positive terms is defined by the conditions

$$a_1=k>0,\ a_{n+1}(2+a_n)=(3+2a_n),\ (n \leq 1).$$

Show that the sequence converges to a limit independent of k and find this limit.

10. If $\{a_n\}$ is defined by

$$a_n \geq 0,\ 2a_{n+1}^2=a_n^3+a_n,\ n \geq 1,$$

show that $\{a_n\}$ tends to 0, 1 or infinity as n tends to infinity according to the value of a_1.

11. If $\{v_n\}$ is a sequence of positive numbers such that

$$v_{n+1}^2=\frac{2v_n}{v_n+1},$$

show that

$$v_n \to 1, \text{ as } n \to \infty.$$

12. A sequence $\{S_n\}$ is defined as follows :—

$$S_1=a>0,\ S_{n+1}=\sqrt{\left(\frac{ab^2+S_n^2}{a+1}\right)},\ b>a.$$

Show that S_n is an increasing bounded sequence and $\lim S_n=b$.

13. If k is positive and α, $-\beta$ are the positive and negative roots of

$$x^2-x-k=0.$$

prove that if

$$v_n=\sqrt{(k-v_{n-1})} \text{ and } v_1>k,$$

then

$$v_n \to \beta.$$

14. If S_1, S_2 are positive and

$$S_{n+2}=\sqrt{(S_{n+1}S_n)},$$

prove that the sequences

$$S_1,\ S_3,\ S_5,\cdots\cdots\cdots\ ;\ S_2,\ S_4,\ S_6,\cdots\cdots\cdots$$

are one an increasing and the other a decreasing sequence and show that the common limit is

$$\sqrt[3]{(S_1S_2^2)}.$$

15. If $0 < S_1 < S_2$ and

$$S_n = \frac{2S_{n-1} S_{n-2}}{S_{n-1}+S_{n-1}},$$

so that S_n is the harmonic mean between S_{n-1} and S_{n-2}, show that

$$S_n \to 3S_1 S_2/(2S_1+S_2).$$

16. $a_{n+1}=\frac{1}{2}(a_n+b_n)$ and $b_{n+1}=\sqrt{a_n b_n}$ show that the sequences $\{a_n\}$, $\{b_n\}$ converge to a common limit where

$$a > b > 0 \text{ and } a_1=\tfrac{1}{2}(a+b) \text{ and } b_1=\sqrt{ab}.$$

17. If x_1, y_1, are positive and if, for $n \geqslant 1$,

$$x_{n+1}=\frac{x_n+y_n}{2}, \quad \frac{2}{y_{n+1}}=\frac{1}{x_n}+\frac{1}{y_n},$$

show that $\{x_n\}$ and $\{y_n\}$ are monotonic sequences and approach a common limit l where $l^2=x_1y_1$.

18. If $\{a_n\}$ is a decreasing sequence and $\{b_n\}$ an increasing sequence and if

$$(b_n-a_n) \to 0 \text{ as } n \to \infty,$$

prove that both the sequences tend to the same finite limit as $n \to \infty$.

19. If a sequence of intervals $[a_n, b_n]$, any one of which is entirely contained within the preceding one, is such that

$$\lim (b_n-a_n)=0,$$

show that there is one and only one point common to all the intervals of the sequence. [Refer § 50, page 84.]

Show that the sequence of intervals

$$\left[\frac{2^{n-1}-1}{2^n}, 1-\frac{2^{n-1}-1}{2^n}\right]$$

satisfies the conditions of the above theorem and determine the point common to all of them.

20. If $k \neq 0$ and $\{a_n\}$ is a sequence such that

$$(a_{n+1}-a_n) \to k.$$

then

$$a_n \to \infty \text{ or } -\infty,$$

according as k is positive or negative.

21. If λ, μ are two given numbers and $\{a_n\}$ is a given sequence such that

(*i*) $|\lambda| < 1$, (*ii*) $(a_{n+1}+\lambda a_n+\mu) \to 0$,

then show that

$$a_n \to -\mu/(1+\lambda).$$

22. (a_n) and (b_n) are two sequences such that

(*i*) $b_n > 0$,

(*ii*) $S_n=(b_1+b_2+\ldots\ldots+b_n)$ is divergent,

(*iii*) $a_n/b_n \to s$;

show that

$$\lim \frac{a_1+a_2+\ldots\ldots+a_n}{b_1+b_2+\ldots\ldots+b_n}=s.$$

23. Show that

$$\overline{\lim}(-a_n)=-\underline{\lim}\,(a_n).$$

when $\{a_n\}$, is a bounded sequence.

24. If $\{a_n\}$, $\{b_n\}$ are two bounded sequences of positive terms, show that

(*i*) $(\underline{\lim}\, a_n)\,(\underline{\lim}\, b_n) \leqslant \underline{\lim}\,(a_n b_n)$.

(*ii*) $(\overline{\lim}\, a_n)\,(\overline{\lim}\, b_n) \geqslant \overline{\lim}\,(a_n b_n)$.

CHAPTER IV

REAL VALUED FUNCTIONS OF A REAL VARIABLE

Limit and Continuity

46. Real valued functions of a real variable. Domain and Range of a function. Let S be any given set of real numbers. A law which may be denoted by, f, which associates to each number x of S a real number y is called a real valued function of a real variable. The number y which the law f associates to x is denoted by $f(x)$ and is called the value of the function f for x. The set S is called the *Domain* of the function and the set of numbers $f(x)$ when x varies over S is called the *Range* of the function. Also a function is said to be defined in its domain. The chapters IV to IX will be concerned with real valued functions of a real variable only and as such throughout these chapters we shall refer to them only as functions.

The domain of a function will, in general, be an interval, open or closed.

While it is f which denotes a function and $f(x)$ a value of the function, it has become customary to denote a function itself by $f(x)$ and this usage will be adopted throughout this book.

Illustrations

1. If $y=0$, when x is rational and $y=1$, when x is irrational, then y is a function of x defined for $]-\infty, \infty[$, *i.e.*, for the entire continuum. Its domain is the continuum $]-\infty, \infty[$ and range consists of only 2 numbers, *viz.*, 0 and 1.

2. If $y=x!$ then y is a function of x defined for the set of positive integers. The range of this function consists of the set of numbers 1, 2, 6, 24 etc.

3. If $y=[x]$, where $[x]$ denotes the greatest integer not greater than x, then y is a function of x defined for $]-\infty, \infty[$.

4. If $y=1/(1+x)$, then y is a function of x defined for the entire continuum excepting -1, since the determination of y for $x=-1$ involves the meaningless operation of division by 0.

5. If $y=\left[\begin{array}{ll} 1/(1+x), & \text{when } x\neq -1, \\ 1 & \text{, when } x=-1, \end{array}\right.$

then the function y of x is defined for the entire continuum.

6. A sequence is a function; the domain of the function being the set of positive integers.

Ex. Compare the domains of the functions $(x^2-1)/(x-1)$ and $x+1$.

46·1. Classification of functions:

(*i*) *Algebraic.* (*ii*) *Transcendental.*

Before defining an algebraic function, we note that a function of the form

$$a_0+a_1x+a_2x^2+\ldots+a_mx^m,$$

where $a_0, a_1, \ldots, a_m$ are constants and m is an integer $\geqslant 0$, is called a *polynomial.*

(*i*) *A function $f(x)$ is called an algebraic function, if it satisfies an equation of the form*

$$P_0[f(x)]^n+P_1[f(x)]^{n-1}+\ldots+P_n=0,$$

when $P_0, P_1, \ldots, P_n$ are polynomials.

A polynomial itself is a particular case of an algebraic function, as we may see on taking $n=1$ and P_0=a constant.

The rational function, *i.e.*, a function of the form

$$\frac{a_0+a_1x+a_2x^2+\ldots+a_nx^n}{b_0+b_1x+b_2x^2+\ldots+b_mx^m},$$

is also an algebraic function. It is defined in every interval which does not contain a number for which the denominator becomes zero.

(*ii*) *A function which is not algebraic is called a transcendental function.*

46·2. Bounded and unbounded functions. A function is said to be bounded, if its range *i.e.*, the set of its values is bounded; the bounds of the range, in case they exist, are said to be the bounds of the function.

Ex. 1. Show that the function

$$f(x)=\begin{cases}1/x, & \text{when } x\neq 0,\\ 0, & \text{when } x=0,\end{cases}$$

is not bounded.

2. Show that the function $x/(x+1)$ is bounded in $[0, \infty[$. Find its bounds and show that it attains its lower bound but not the upper bound.

47. Limit of a function. *A function $f(x)$ is said to tend to the limit, l, as x approaches, a, or, symbolically*

$$\lim_{x\to a} f(x)=l \text{ or } f(x)\to l, \text{ as } x\to a, \qquad \ldots (1)$$

if, to every positive number ε, however small, there corresponds a positive number δ, such that

$$|f(x)-l|<\varepsilon \text{ when } 0<|x-a|\leqslant\delta, \qquad \ldots (2)$$

i.e., $f(x)$ lies between $l-\varepsilon$ and $l+\varepsilon$, for all those values of x, (except possibly a), which belong to the interval $(a-\delta, a+\delta)$.

Note 1. The part

$$0<|x-a|$$

of the inequality (2) only excludes the possibility

$$0=|x-a|$$

i.e., $$x=a.$$

The fact that the inequality

$$|f(x)-l| < \varepsilon,$$

does or does not hold for $x=a$ is irrelevant to the existence of

$$\lim_{x \to a} f(x).$$

The function may *not* even be defined for $x=a$.

Note 2. In order that $f(x)$ may tend to a limit as $x \to a$, it is *necessary* that $f(x)$ should be defined in a certain neighbourhood $]a-h, a+h[$ of a except possibly at a.

Note 3. In terms of neigh bourhood concept. we may say that $\lim_{x \to a} f(x)=l$, if to any given neighbourhood N of l there corresponds a neighbourhood N' of a such that for every $x \neq a$ which belongs to N', $f(x)$ belongs to N.

Note 4. The statement

$$\lim_{x \to a} f(x)=l,$$

means two things :—(*i*) the limit of $f(x)$, as $x \to a$, exists ; (*ii*) the limit is equal to l.

Ex. 1. *Criticise the following statements* :

A function $f(x)$ is said to tend to the limit, l, as x tends to a,

(*i*) if $f(x)$ is nearly equal to, l, when x is nearly equal to a.

(*ii*) if as x approaches nearer and nearer a, then $f(x)$ approaches nearer and nearer l.

(*iii*) if the difference between $f(x)$ and l can be made as small as we like by taking x sufficiently near a.

2. Show that a function $f(x)$ cannot tend to two different limits.

3. Show that if the inequality (2) of § 47, holds even for $x=a$, then, l, is necessarily equal to $f(a)$.

48. One sided limits.

48·1. Right handed limit. If, to every positive number ε, there corresponds a positive number δ, such that

$$|f(x)-l| < \varepsilon, \text{ when } a < x \leqslant a+\delta,$$

we say that $f(x) \to l$ as $x \to a$ through values greater than a, and symbolically write

$$\lim_{x \to (a+0)} f(x)=l, \text{ or } f(a+0)=l.$$

48·2. Left handed limit. If, to every positive number ε, there corresponds a positive number δ such that

$$|f(x)-l| < \varepsilon, \text{ when } a-\delta \leqslant x < a,$$

we say that $f(x) \to l$ as $x \to a$ through values less than a, and symbolically write

$$\lim_{x \to (a-0)} f(x)=l \text{ or } f(a-0)=l.$$

Note. It is easy to see that

$$\lim_{x \to a} f(x)=l,$$

if, and only if,

$$\lim_{x \to (a+0)} f(x)=l=\lim_{x \to (a-0)} f(x).$$

In case either or both the limits, *viz.* ;

$$\lim_{x \to (a+0)} f(x) \text{ and } \lim_{x \to (a-0)} f(x)$$

do not exist, or exist but are not equal, then, $\lim_{x \to a} f(x)$ does not exist.

Ex. 1. If $y=[x]$, show that

$$\lim_{x \to (2+0)} y=2, \quad \lim_{x \to (2-0)} y=1, \text{ but } \lim_{x \to 2} y \text{ does not exist.}$$

2. Show that $\lim_{x \to 0} [\,|x|/x]$ does not exist.

3. A function $f(x) \to l$ as $x \to a$ and a sequence $\{x_n\} \to a$, show that the sequence $\{f(x_n)\} \to l$.

Conversely, show that $\lim_{x \to a} f(x)=l$ if for every convergent sequence with limit a, the sequence $\{f(x_n)\}$ converges to l.

4. If $f(x) \to l$ as $x \to a$, then there exists a neighbourhood of a in which $f(x)$ is bounded.

5. Show that

(*i*) $\lim (x^3+3x)=4$, as $x \to 1$.

(*ii*) $\lim (2x^2+3)/(x+1)=3$ as $x \to 0$.

49·1. $\lim_{x \to \infty} f(x)=l$; $\lim_{x \to -\infty} f(x)=l$.

If, to every positive number, ε, there corresponds a positive number $\triangle$ such that

$$|f(x)-l| < \varepsilon, \text{ when } x \geqslant \triangle,$$

then we say that $f(x) \to l$ as $x \to \infty$, and write

$$\lim_{x \to \infty} f(x)=l.$$

Similarly, if, to every positive number ε, there corresponds a positive number $\triangle$ such that

$$|f(x)-l| < \varepsilon, \text{ when } x \leqslant -\triangle,$$

then we say that $f(x) \to l$, as $x \to -\infty$.

49·2. Infinite limits. The definitions, as given in §47 and §49·1 above, may be easily modified to give precise meanings to the following :—

$$\lim_{x \to a} f(x)=\pm\infty, \quad \lim_{x \to \infty} f(x)=\pm\infty, \quad \lim_{x \to -\infty} f(x)=\pm\infty$$

49·3. Finite and infinite oscillation. If a function $f(x)$ neither tends to a finite nor to an infinite limit as $x \to a$, (∞ or $-\infty$), we say that it *oscillates* ; the oscillation is said to be finite or infinite according as the function is bounded or not in a certain neighbourhood of a, (in a certain interval $[X, \infty[$, or, $]-\infty, X]$).

50. A theorem concerning a sequence of intervals. We shall now state and prove a theorem concerning a sequence of intervals. We may not often actually employ this theorem in the following pages of the book but the *idea* in the theorem will play a very important part.

If a sequence of closed intervals $[a_n, b_n]$, *is such that each member* $[a_{n+1}, b_{n+1}]$ *is contained in the preceding one* $[a_n, b_n]$ *and*

$$\lim (b_n - a_n) = 0,$$

then there is one and only one point common to all the intervals of the sequence.

As each interval member of the sequence is contained in the preceding one, we have

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant \ldots\ldots$$
$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n \geqslant \ldots\ldots$$

so that $\{a_n\}$ is a monotonically increasing and $\{b_n\}$ a monotonically decreasing sequence. Also both these sequences are bounded since

$$a_n < b_1 \text{ and } b_n > a_1 \text{ for all } n.$$

Thus the two sequences are convergent.

Let

$$\lim a_n = \xi, \quad \lim b_n = \eta.$$

Now we have

$$0 = \lim (b_n - a_n) = \lim b_n - \lim a_n = \xi - \eta$$

so that

$$\xi = \eta.$$

As ξ is the upper bound of the sequence $\{a_n\}$ and lower bound of the sequence $\{b_n\}$, we have

$$a_n \leqslant \xi \leqslant b_n$$

so that ξ belongs to all the intervals.

Let, if possible, ξ_1, ξ_2 be two different points common to all the intervals. Also let $\xi_1 < \xi_2$. We then have

$$a_n \leqslant \xi_1 < \xi_2 \leqslant b_n$$

i.e.,

$$b_n - a_n > \xi_2 - \xi_1 \neq 0, \text{ for all } n,$$

which is in contradiction to the fact that $\lim (b_n - a_n) = 0$.

Hence the result.

Ex. By means of an example show that the conclusion of the above theorem may not be true if we have a sequence of open intervals.

51. Condition for the existence of finite limit. *The necessary and sufficient condition that* $f(x)$ *may tend to a finite limit, as* x *tends to* a, *is that, to every positive number* ε, *however small, there corresponds a positive number* δ, *such that*

$$| f(x_2) - f(x_1) | < \varepsilon,$$

for every pair x_1, x_2 *of values of* x *which satisfy the inequalities*

$$0 < | x_1 - a | \leqslant \delta, \; 0 < | x_2 - a | \leqslant \delta,$$

i.e., for every pair x_1, x_2 *of values, other than* a, *which belong to the interval* $[a - \delta, a + \delta]$.

The condition is necessary. Let $f(x) \to l$, as $x \to a$.

Let ε be any positive number. There exists a positive number δ, such that

$$| f(x) - l | < \tfrac{1}{2}\varepsilon, \text{ when } 0 < | x - a | \leqslant \delta,$$

so that if x_1, x_2 be *any* two numbers such that,

$$0 < |x_1 - a| \leqslant \delta,\ 0 < |x_2 - a| \leqslant \delta.$$

we have

$$|f(x_1) - l| < \tfrac{1}{2}\varepsilon,\ |f(x_2) - l| < \tfrac{1}{2}\varepsilon.$$

and accordingly

$$\begin{aligned} |f(x_2) - f(x_1)| &= |f(x_2) - l + l - f(x_1)| \\ &\leqslant |f(x_2) - l| + |l - f(x_1)| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

The condition is sufficient. Let

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots\ldots\ldots, \varepsilon_n, \ldots\ldots\ldots \qquad \ldots (1)$$

be any monotonically decreasing sequence of positive numbers converging to 0.

To each positive number ε_n there corresponds a positive number δ_n, such that

$$|f(x_2) - f(x_1)| < \varepsilon_n \text{ when } 0 < |x_1 - a| \leqslant \delta_n,\ 0 < |x_2 - a| \leqslant \delta_n.$$

Thus we obtain another sequence

$$\delta_1, \delta_2, \delta_3 \ldots\ldots\ldots, \delta_n, \ldots\ldots\ldots \qquad \ldots (2)$$

of positive numbers corresponding to the sequence (1).

Obviously, we may suppose that this sequence $\{\delta_n\}$ is also monotonically decreasing.

Writing $a + \delta_n$ for x_1 and x for x_2, we see that

$$|f(x) - f(a + \delta_n)| < \varepsilon_n, \text{ when } 0 < |x - a| \leqslant \delta_n.$$

Thus for all values of x, other than a, which belong to the interval $[a - \delta_n, a + \delta_n]$, $f(x)$ belongs to the interval

$$[f(a + \delta_n) - \varepsilon_n, f(a + \delta_n) + \varepsilon_n]$$

which we call A_n and whose length is $2\varepsilon_n$.

Since $\delta_{n+1} < \delta_n$, we may suppose that the interval A_{n+1} is contained in A_n.

Thus we obtain a sequence of intervals

$$A_1, A_2, A_3 \ldots\ldots\ldots, A_n \ldots\ldots \qquad \ldots (3)$$

such that each member of the sequence is contained in the preceding one. Also the length $2\varepsilon_n$, of $A_n \to 0$ as $n \to \infty$.

There exists, therefore, one and only one point, say, l, common to all the intervals of the sequence (3).

Let ε be any positive number.

We choose n so large that $2\varepsilon_n < \varepsilon$. We have

$$f(a + \delta_n) - \varepsilon_1 \leqslant l \leqslant f(a + \delta_n) + \varepsilon_n,$$

and

$$f(a + \delta_n) - \varepsilon_n < f(x) < f(a + \delta_n) + \varepsilon_n,$$

when

$$0 < |x - a| \leqslant \delta_n.$$

$$\therefore \qquad |f(x) - l| < 2\varepsilon_n < \varepsilon, \text{ when } 0 < |x - a| \leqslant \delta_n.$$

Hence

$$f(x) \to l, \text{ as } x \to a.$$

Ex. 1. State and prove the corresponding theorems for the existence of the right handed and left handed limits.

2. Prove that the necessary and sufficient condition that $f(x)$ may tend to a finite limit as $x \to \infty$ is that to every positive number ε, there corresponds a positive number $\triangle$ such that

$$|f(x_2)-f(x_1)| < \varepsilon,$$

for every pair x_1, x_2 of numbers which are both greater than or equal to $\triangle$.

State and prove a similar condition for $f(x)$ to tend to a finite limit as $x \to -\infty$.

Heine-Borel Theorem. *Let a closed interval $[a, b]$ and an infinite set, S, of open intervals be given such that each point of $[a, b]$ is a point of at least one member of S.*

Then there exists a finite sub-set S_1 of S with the same property, i.e., there exists a set S_1 consisting of a finite number of members of S such that every point of $[a, b]$ is a point of atleast one member of S_1.

The system, S, of open intervals may be referred to as an *open cover* of the closed interval $[a, b]$ inasmuch every member of S is an open interval and each point of $[a, b]$ is contained [or covered] by some member of S. The finite sub-set S_1 of S such that every member of $[a. b]$ is contained in some member of S_1 may be referred to as a finite sub-cover. Employing this notion of cover and sub-cover we may restate the theorem as follows.

Every open cover of a closed interval admits of a finite sub-cover.

Proof. Suppose that the theorem is not true. We take

$$c=\tfrac{1}{2}(a+b).$$

Then atleast one of the two closed sub-intervals $[a, c]$ and $[c, b]$ must not admit of a finite sub-cover. For, otherwise, there would exist two finite sub-sets S' and S'' of S covering $[a, c]$ and $[c, b]$ respectively and accordingly $[a, b]$ would be covered by the finite sub-set of S consisting of the members of S' and S'' and as such the theorem would be true.

Let $[a_1, b_1]$ denote a sub-interval not admitting of a finite sub-cover.

Again we take

$$c_1=\tfrac{1}{2}(a_1+b_1)$$

and arrive at a sub-interval $[a_2, b_2]$ not admitting of a finite sub-cover.

Thus proceeding we arrive at a sequence of intervals

$$[a_1, b_1], \ldots\ldots\ldots, [a_n, b_n], \ldots \ldots$$

such that no member of the same admits of a finite sub-cover.

It may be easily seen that each member of the sequence is contained in the preceding one and the length $(\frac{1}{2})^n (b-a)$ of $[a_n, b_n]$ tends to 0. Thus there exists a point ξ belonging to every member of the sequence. We have

$$a \leqslant \xi \leqslant b.$$

By hypothesis, there exists a member say, $]a', b'[$ of S such that ξ belongs to the same. Also there exists a positive integer m

such that the interval member $[a_m, b_m]$ of the sequence is contained in $]a', b']$. Thus $]a_m, b_m[$ admits of a finite cover and we arrive at a contradiction.

Hence the theorem is true.

52. Monotonic Functions. Let a function $f(x)$ be defined in an interval $[a, b]$ and let x_1, x_2 be any two points of this interval such that $x_1 < x_2$. Then the function is said to be monotonically increasing if $f(x_1) \leqslant f(x_2)$ and monotonically decreasing if $f(x_1) \geqslant f(x_2)$.

A function $f(x)$ is said to be *strictly* monotonically increasing, if

$$f(x_2) > f(x_1), \text{ when } x_2 > x_1,$$

so that the sign of equality is not admissible.

For a strictly monotonically decreasing function $f(x)$,

$$f(x_2) < f(x_1) \text{ when } x_2 > x_1.$$

The properties of monotonic functions in regard to the existence of limits are quite similar to those of monotonic sequences and may bê similarly proved.

We have the following results for monotonically increasing functions :—

(*i*) If $f(x)$ is a monotonically increasing function in $[a, \infty[$, and there exists a number k, such that $f(x) \leqslant k$ when $x \geqq a$, then

$$\lim_{x \to \infty} f(x) \text{ exists and is } \leqslant k.$$

(*ii*) If (x) is monotonically increasing in $]-\infty, a]$ and there exists a number k, such that $f(x) \geqslant k$ when $x \leqslant a$, then

$$\lim_{x \to -\infty} f(x) \text{ exists and is } \geqslant k.$$

(*iii*) If $f(x)$ is a monotonically increasing function in the open interval $]a, b[$ and there exists a number k such that

(*a*) $f(x) \leqslant k$ in $]a, b[$ then $f(b-0)$, *i.e.*, $\lim f(x)$, when $x \to (b-0)$, exists.

(*b*) $f(x) \geqslant k$ in $]a, b[$ then $f(a+0)$, *i.e.*, $\lim f(x)$, when $x \to (a+0)$ exists.

Similar results are easily obtained for monotonically decreasing functions.

53. Invertibility of the Algebraic operation, and the limiting operations. *If $f_1(x)$, $f_2(x)$ be two functions such that, when $x \to a$,*

$$\lim f_1(x) = l_1, \lim f_2(x) = l_2$$

then

(*i*) $\lim [f_1(x) \pm f_2(x)] = \lim f_1(x) \pm \lim f_2(x) = l_1 \pm l_2$.

(*ii*) $\lim [f_1(x) . f_2(x)] = \lim f_1(x) . \lim f_2(x) = l_1 . l_2$.

(*iii*) $\lim [f_1(x)/f_2(x)] = \lim f_1(x)/\lim f_2(x) = l_1/l_2$, when $l_2 \neq 0$.

The proofs are similar to those of the corresponding results on sequences 35, p. 54.

Proof. (*i*) The proof is simple and is, therefore, left to the student.

(*ii*) Let ε be any positive number, however small.

We have

$$|f_1(x)f_2(x)-l_1l_2| = |f_2(x)[f_1(x)-l_1]+l_1[f_2(x)-l_2]|$$
$$\leqslant |f_2(x)|\ |f_1(x)-l_1| + |l_1|\ |f_2(x)-l_2|.$$

There exists a positive number δ such that

$$|f_1(x)-l_1| < \varepsilon',\ |f_2(x)-l_2| < \varepsilon'$$

when

$$0 < |x-a| \leqslant \delta,$$

where ε' is *any* given positive number.

Since

$$|f_2(x)| - |l_2| \leqslant |f_2(x)-l_2| < \varepsilon'$$
$$\therefore \quad |f_2(x)| < |l_2| + \varepsilon'.$$

Therefore when $0 < |x-a| \leqslant \delta$, we have

$$|f_1(x)f_2(x)-l_1l_2| \leqslant (|l_2|+\varepsilon')\varepsilon' + |l_1|\varepsilon' = [|l_2|+|l_1|+\varepsilon']\varepsilon'.$$

Choosing ε' any positive number less than 1 and

$$< \varepsilon\,|\,[|l_2|+|l_1|+1],$$

where ε is the given positive number we see that

$$|f_1(x)f_2(x)-l_1l_2| < \varepsilon, \text{ when } 0 < |x-a| \leqslant \delta.$$

Hence the result.

(*iii*) Let ε be any positive number.

We have

$$\left|\frac{f_1(x)}{f_2(x)}-\frac{l_1}{l_2}\right| = \left|\frac{l_2[f_1(x)-l_1]-l_1[f_2(x)-l_2]}{l_2 f_2(x)}\right|$$
$$\leqslant \frac{|l_2|\ |f_1(x)-l_1| + |l_1|\ |f_2(x)-l_2|}{|l_2|\ |f_2(x)|} \quad \dots (i)$$

There exists a positive number δ_1, such that

$$|f_2(x)-l_2| < \tfrac{1}{2}|l_2|, \text{ when } 0 < |x-a| \leqslant \delta_1,\ l_2 \neq 0.$$
$$\therefore \quad |l_2| - |f_2(x)| \leqslant |f_2(x)-l_2| < \tfrac{1}{2}|l_2|,$$

or

$$|f_2(x)| > \tfrac{1}{2}|l_2| \text{ when } 0 < |x-a| \leqslant \delta_1. \quad \dots (ii)$$

There exists a positive number δ_2 such that

$$|f_1(x)-l_1| < \varepsilon',\ |f_2(x)-l_2| < \varepsilon', \text{ when } 0 < |x-a| \leqslant \delta_2 \quad \dots (iii)$$

where ε' is any given positive number.

If $\delta = \min(\delta_1, \delta_2)$, we deduce from (*i*), (*ii*), (*iii*), that when $0 < |x-a| \leqslant \delta$,

$$\left|\frac{f_1(x)}{f_2(x)}-\frac{l_1}{l_2}\right| \leqslant \frac{2}{|l_2|^2}[|l_2|+|l_1|]\varepsilon'.$$

Choosing ε' any positive number less than

$$\varepsilon\,|l_2|^2/2\,[|l_2|+|l_1|],$$

where ε is the given positive number, we see that

$$\left|\frac{f_1(x)}{f_2(x)}-\frac{l_1}{l_2}\right| < \varepsilon, \text{ when } 0 < | x-a | \leqslant \delta.$$

Hence the result.

Exercises

1. $f(x) \to l$ as $x \to a$ and k is a constant, show that

$$\lim k\, f(x)=kl, \text{ as } x \to a.$$

2. If the functions $f_1(x), f_2(x), \ldots \ldots f_n(x)$ approach finite limits, when x approaches a and $k_1, k_2, k_3, \ldots \ldots, k_n$ are constants, then

$$(i) \lim \sum_{r=1}^{n} k_r f_r(x) = \sum k_r \lim f_r(x)$$

$$(ii) \lim \prod_{r=1}^{n} f_r(x) = \prod_{r=1}^{n} \lim f_r(x).$$

3. If $\lim f(x)=0$, show, by giving examples, that $1/f(x)$ may tend to $+\infty$, $-\infty$ or may oscillate infinitely.

4. If $\lim\limits_{x \to a} f(x)=0$ and $f(x)$ is positive for values of x in a certain neighbourhood of a, show that $1/f(x) \to \infty$ as $x \to a$.

5. $f(x)$, $\varphi(x)$, $\psi(x)$ are three functions such that for all values of x, (excepting possibly a), which lie in a certain neighbourhood of a,

$$f(x) \leqslant \varphi(x) \leqslant \psi(x)$$

and

$$\lim f(x)=\lim \psi(x)=l, \text{ as } x \to a\,;$$

show that

$$\varphi(x) \to l \text{ as } x \to a.$$

6. $f(x) \to l$ as $x \to a$; show that $| f(x) | \to | l |$, but the converse is not necessarily true except when $l=0$.

54. An important limit. *To show that*

$$e = \lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = \lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x = \lim_{z \to 0} (1+z)^{1/z}$$

(*i*) Let x be any real number greater than 1, and let the positive integer n be so chosen that

$$n \leqslant x < n+1.$$

$$\therefore \qquad 1+\frac{1}{n} \geqslant 1+\frac{1}{x} > 1+\frac{1}{n+1},$$

or

$$\left(1+\frac{1}{n}\right)^{n+1} \geqslant \left(1+\frac{1}{x}\right)^{n} > \left(1+\frac{1}{n+1}\right)^{n}.$$

Let $x \to \infty$; then n also $\to \infty$. We have

$$\lim \left(1+\frac{1}{n}\right)^{n+1} = \lim \left(1+\frac{1}{n}\right)^{n} . \lim \left(1+\frac{1}{n}\right) = e.\ 1 = e.$$

$$\lim \left(1+\frac{1}{n+1}\right)^{n} = \lim \left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \lim \left(1+\frac{1}{n+1}\right) = e/1 = e$$

Hence

$$\lim (1+1/x)^x = e.$$

(*ii*) Let $x=-y$ so that $y \to +\infty$ as $x \to -\infty$. We have

$$\left(1+\frac{1}{x}\right)^x=\left(1-\frac{1}{y}\right)^{-y}=\left(1+\frac{1}{y-1}\right)^{y-1}\cdot\left(1+\frac{1}{y-1}\right)$$

$$\therefore \quad \lim_{x \to -\infty}\left(1+\frac{1}{x}\right)^x=e.1=e.$$

(*iii*) Putting $z=1/x$ so that $z \to (0+0)$ or $(0-0)$ according as $x \to +\infty$ or $-\infty$, we obtain

$$\lim_{z \to 0}(1+z)^{\frac{1}{z}}=e.$$

55. Continuous Functions.

Let $f(x)$ be defined in an interval $[a, b]$.

Continuity at an interior point. *The function $f(x)$ is said to be continuous at any interior point c, $a < c < b$, if*

$$lim\, f(x)=f(c), \text{ when } x \to c,$$

i.e., if lim $f(x)$ exists and is equal to $f(c)$.

Continuity at an end point. *$f(x)$ is said to be continuous at the left end a, if*

$$lim\, f(x)=f(a), \text{ when } x \to (a+0),$$

and is said to be continuous at the right end b, if

$$lim\, f(x)=f(b), \text{ when } x \to (b-0).$$

Continuity in an interval. *$f(x)$ is said to be continuous in an interval $[a, b]$, if it is continuous at every point of the interval.*

In case $f(x)$ is continuous at every x such that $a < x < b$ but not at the end points a, b, we say that it is continuous in the open interval $]a, b[$.

Remark. The definition of continuity, as given, is *motivated* by the fact, that, *in ordinary parlance, continuity implies absence of the suddenness of change* so that if x changes by a small amount, then $f(x)$ must also change by a small amount *i.e.*, the change $| f(x)-f(a) |$ will be small if only the change $| x-a |$ is small.

Ex. *Criticise the following statements* :—

1. "A function $f(x)$ is said to be a continuous function of x between the limits a and b, when, to each value of x between these limits, there corresponds a finite value of the function and when an infinitely small change in the value of x produces only an infinitely small change in the function."

2. "The continuous function of a variable is a quantity that changes gradually and passes through every intermediate value from an initial to a final value as the variable that enters it passes through every intermediate value from its initial to its final value." (See Cor. to §58·1, p. 96.)

3. "$f(x)$ is continuous between $x=a$ and $x=b$, when the locus of $y=f(x)$ between the points $[a, f(a)]$ and $[b, f(b)]$ is an unbroken line, straight or curved."

Ex. Formally prove that the functions

(*i*) k, a constant; (*ii*) x,

are continuous for every value of x.

56. Classification of discontinuities. Let, c, be any point of the interval of definition $]a, b[$ of $f(x)$. For continuity at c, it is necessary and sufficient that $\lim f(x)$ should exist and be equal to $f(c)$.

Let, now, $f(x)$ be discontinuous at c. It is, in the first instance, clear that the question of the continuity or otherwise of $f(x)$ for $x=c$, where, c, is any number, can arise only if $f(x)$ is defined for $x=c$. Now supposing that $f(x)$ is defined for $x=c$, we see that the discontinuity of $f(x)$ for $x=c$ can arise only in any one of the two following ways:

(i) $\lim\limits_{x \to c} f(x)$ does not exist.

(ii) $\lim\limits_{x \to c} f(x)$ exists but is different from $f(c)$.

In case (ii), we say that *Discontinuity is of the first kind and in case (i) of the second kind.*

Thus the discontinuity at c is of the **First kind,** *if $\lim f(x)$ exists but $\neq f(c)$, and is of the* **second kind** *if $\lim f(x)$ does not exist finitely.*

Also we say that, c, is a point of *infinite discontinuity* if $f(x)$ is not bounded in some neighbourhood of c.

Note. We may also sometimes distinguish between the two sides of c. Thus if $\lim\limits_{x \to (c+0)} f(x)$ exists and $=f(c)$ but $\lim\limits_{x \to (c-0)} f(x)$ does not exist finitely, then c is a point of continuity on the right and of discontinuity of the second kind on the left.

Similarly we may describe the nature of the discontinuity at the point, c, in other cases.

Exercises

1. Show that $3x^2+2x-1$ is continuous for $x=2$.

2. Show that the function $f(x)$ where

$$f(x)=\begin{cases} x, \text{ when } 0 \leqslant x < \tfrac{1}{2}; \\ 1, \text{ when } x=\tfrac{1}{2}; \\ 1-x, \text{ when } \tfrac{1}{2} < x < 1, \end{cases}$$

is discontinuous for $x=\frac{1}{2}$. What is the nature of the discontinuity ?

3. What is the nature of discontinuity at

(i) $x=1$ of $f(x)=[x]$.

(ii) $x=0$ of $f(x)$ where $f(x)=1/x$ when $x \neq 0$ and $f(0)=0$.

4. Obtain the points of discontinuity of $f(x)$ defined in $(0, 1)$ as follows :—

$f(x)=0$ when $x=0$, to $\frac{1}{2}-x$ in $0 < x < \frac{1}{2}$, to $\frac{1}{2}$ when $x=\frac{1}{2}$, to $\frac{3}{2}-x$ when $\frac{1}{2} < x < 1$ and 1 when $x=1$.

Examine also the nature of the points of discontinuity.

Functions defined means of limits.

5. Examine the nature of the points of discontinuity of the function $\varphi(x)$ defined as follows, n tending to ∞ ;

(i) $\varphi(x)=\lim\, [(x^n-1)/(x^n+1)]$.

(It is easy to say that $\varphi(x)=1$ when $|x| > 1$, $\varphi(x)=-1$ when $|x| < 1$, $\varphi(x)=0$, when $x=1$ and $\varphi(x)$ is not defined when $x=-1$).

(ii) $\lim\, [1/(1+x^{2n})]$. (iii) $\lim\, [x^n/(x^n+1)]$.

(iv) $\lim\, [nx/(1+nx)]$. (v) $\lim\, [(x^{2n}+3x^n+1)/(x^{2n}+x^n+1)]$.

57. Continuity of functions which are combinations of continuous functions. The following theorems follow easily from the definition of continuity and the theorems on limits proved in §53.

57·1. (*i*) *The sum, the difference, the product of two functions which are continuous at a point, (in an interval) are continuous at that point, (in that interval).*

(*ii*) *The quotient of two functions which are continuous at a point (in an interval) is continuous at that point (in that interval) provided that the denominator does not vanish at the point (at any point of the interval).*

As an illustration of the proof, we consider the case of product. Let $f_1(x), f_2(x)$ be continuous at a point, c, so that

$$\lim_{x \to c} f_1(x)=f_1(c), \quad \lim_{x \to c} f_2(x)=f_2(c).$$

By § 53, p. 88,

$$\lim_{x \to c} [f_1(x)f_2(x)]=\lim_{x \to c} f_1(x) \lim_{x \to c} f_2(x)=f_1(c)\, f_2(c)$$

which is the value of $f_1(x)\, f_2(x)$ at c.

Hence $f_1(x)\, f_2(x)$ is continuous at c.

57·2. Continuity of a function of a function. Let $f(x)$ be a function defined in an interval $[a, b]$ and $\varphi(t)$ be a function defined in $[\alpha, \beta]$; and let every value of $\varphi(t)$ belong to the interval $[a, b]$.

Writing

$$y=f(x), \quad x=\varphi(t),$$

we see that y is a function of t defined in $[\alpha, \beta]$ and which we may write as

$$y=f[\varphi(t)].$$

Here y is a function of a function of t.

Theorem. *If $x=\varphi(t)$ be a continuous function of t, at a point t_0, of $[\alpha, \beta]$ and $y=f(x)$, a continuous function of x at the corresponding point $x_0=\varphi(t_0)$, then $y=f[\varphi(t)]$, is a continuous function of t at t_0.*

Let ε be *any* positive number. Let $y_0=f(x_0)$.

Since $y=f(x)$ is continuous at x_0, there exists a positive number δ_0 such that

$$| y-y_0 | < \varepsilon, \text{ when } | x-x_0 | \leqslant \delta_0. \qquad \dots (1)$$

Again, since $x=\varphi(t)$ is continuous at, t_0, there exists a positive number δ such that

$$| x-x_0 | < \delta_0, \text{ when } | t-t_0 | \leqslant \delta. \qquad \dots (2)$$

From (1) and (2), we see that there exists a positive number δ such that

$$| y-y_0 | < \varepsilon, \text{ when } | t-t_0 | \leqslant \delta.$$

Hence the result.

Exercises

1. Show that a polynomial is continuous for every value of x.

2. Show that an algebraic rational function of x is continuous for every value of x which is not a zero of the denominator.

3. Show that

$$f(x)=(x^3-7x^2+3x-1)/(x^2-3x)$$

is continuous at $x=2$ and hence find lim $f(x)$ when $x \to 2$.

58. Properties of functions which are continuous in any closed finite interval. We shall now obtain a system of properties of functions which are continuous in any *closed finite* interval. These properties play a very important part in the studies of Differentiation and Integration.

58·1. Theorem. *If $f(x)$ is continuous in a closed interval $[a, b]$ and $f(a), f(b)$ have opposite signs, then $f(x)$ vanishes for at least one point of the interval.*

Lemma I. If $f(x)$ is continuous at any interior point c of $[a, b]$ and $f(c)\neq 0$, then there exists an interval $[c-\delta, c+\delta]$ enclosing c, such that for every point x of this interval, $f(x)$ has the sign of $f(c)$.

If ε be *any* positive number, however small, there exists a positive number δ such that

$$|f(x)-f(c)| < \varepsilon, \textit{ i.e., } f(c)-\varepsilon < f(x) < f(c)+\varepsilon,$$

for every point, x, of the interval $[c-\delta, c+\delta]$.

Let $f(c)$ be positive. If we take for ε, any positive number less than $f(c)$, we see that for every point x of $[c-\delta, c+\delta]$, $f(x)$ is positive, lying as it does between the two positive numbers $f(c)-\varepsilon$ and $f(c)+\varepsilon$.

Let $f(c)$ be negative. If we take for, ε, any positive number less than the positive number, $-f(c)$, we see that for every point x of $[c-\delta, c+\delta]$, $f(x)$, is negative, lying as it does between the two negative numbers $f(c)-\varepsilon$ and $f(c)+\varepsilon$.

Lemma II. If $f(x)$ is continuous at the end point, a, of $[a, b]$ and $f(a)\neq 0$, then there exists an interval $[a, a+\delta]$ such that for every point x of this interval, $f(x)$ has the sign of $f(a)$.

A similar result holds for continuity at b.

The proof is exactly similar to that of Lemma I.

Proof of the main theorem. Take the number $c=\frac{1}{2}(a+b)$, the mid-point of $[a, b]$. In case $f(c)=0$, we have finished.

If $f(c)\neq 0$, then either $f(a), f(c)$ or $f(c), f(b)$ have opposite signs. Of the two intervals $[a, c]$ and $[c, b]$, the one at the ends of which $f(x)$ has opposite sign, we re-name as $[a_1, b_1]$.

Thus in either case we have

$$a \leqslant a_1 < b_1 \leqslant b\,;$$
$$b_1-a_1=\tfrac{1}{2}(b-a)\,;$$
$$f(a_1), f(b_1) \text{ have opposite signs.}$$

We now bisect $[a_1, b_1]$ and, proceeding as above, see that either $f(x)=0$ at the mid-point $c_1=\frac{1}{2}(a_1+b_1)$ of $[a_1, b_1]$ or otherwise, we obtain an interval $[a_2, b_2]$ such that

$$a_1 \leqslant a_2 < b_2 \leqslant b_1;$$
$$b_2-a_2=\tfrac{1}{2}(b_1-a_1)=(\tfrac{1}{2})^2(b-a);$$
$$f(a_2), f(b_2) \text{ have opposite signs.}$$

Proceeding as above we see that either, after a finite number of steps, we shall arrive at a point at which the function vanishes or we shall obtain an infinite sequence of intervals.

$$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots\ldots$$

such that

$$a \leqslant a_1 \leqslant a_2 \ldots \leqslant a_n < b_n \leqslant \ldots \leqslant b_2 \leqslant b_1 < b; \quad \ldots (i)$$
$$b_n-a_n=(\tfrac{1}{2})^n(b-a); \quad \ldots (ii)$$
$$f(a_n), f(b_n) \text{ have opposite signs.} \quad \ldots (iii)$$

From (i), we see that the sequence $\{a_n\}$ is monotonically increasing and bounded above and the sequence $\{b_n\}$ is monotonically decreasing and bounded below and accordingly we see that the sequences $\{a_n\}$, $\{b_n\}$ are both convergent.

From (ii), we see that

$$\lim (b_n-a_n)=0.$$

Now, since $\{a_n\}$ and $\{b_n\}$ are convergent, therefore

$$0=\lim (b_n-a_n)=\lim b_n-\lim a_n$$

or

$$\lim b_n=\lim a_n=\xi.$$

The point, ξ, may be either an interior or an end-point of $[a, b]$. It will, now, be shown that $f(\xi)=0$. If possible, let $f(\xi)\neq 0$.

Case I. *Let ξ be an interior point.* There exists, by lemma I an interval $[\xi-\delta, \xi+\delta]$ such that for every point, x, of this interval $f(x)$ and $f(\xi)$ have the same sign.

Also, since $a_n \to \xi$ from below and $b_n \to \xi$ from above, there exists an integer m such that $[a_m, b_m]$ lies within $[\xi-\delta, \xi+\delta]$ and accordingly $f(a_m)$, $f(b_m)$ have the same sign so that we arrive at a contradiction of (iii). Hence $f(\xi)=0$.

Case II. *Let ξ coincide with a.* In this case $a_n=a$, for every n. Since $f(a)\neq 0$, there exists an interval $[a, a+\delta]$ such that for every point x of this interval $f(x)$ and $f(a)$ have the same sign.

Also, since $b_n\to\xi=a$, there exists a positive integer m such that $a < b_m < a+\delta$, and accordingly $f(a_m)=f(a)$ and $f(b_m)$ have the same sign so that again we have a contradiction.

Hence ξ cannot coincide with a.

It may similarly be shown that ξ cannot coincide with b.

Another proof. For the sake of definiteness, we suppose that

$$f(a) > 0 \text{ and } f(b) < 0.$$

Since $f(x)$ is continuous at a and $f(a) > 0$, there exists an interval $[a, a+\delta]$, $(\delta > 0)$, such that for every point x of this interval $f(x)$ is positive. (Lemma II.)

Consider a set, S, defined as follows :—

Any point x of $[a, b]$ belongs to S, if $f(x)$ is positive for every point of the closed interval $[a, x]$.

Clearly S exists inasmuch as $a+\delta$ belongs to it. Also S is bounded above ; being a rough upper bound.

Let, c, be the upper bound of S. Suppose that, x, is an interior point of $[a, b]$. It will be shown that $f(c)=0$.

If possible, let $f(c)\neq 0$.

There exists an interval $[c-h, c+h]$ such that for every point x of this interval $f(x)$ and $f(c)$ have the same sign. (Lemma I.)

Since c is the upper bound of S, there exists a member η of S such that

$$c-h < \eta \leqslant c$$

As η belongs to S, $f(x)$ is positive for every point x of $[a, \eta]$ and, in particular, $f(\eta)$ is positive. Also since η is a member of $[c-h, c+h]$, we deduce that for every point x of $[c-h, c+h]$, $f(x)$ is positive.

Thus we see that $f(x)$ is positive for every point x of $[a, c+h]$ so that $c+h$ belongs to S and this contradicts the fact that, c, is the upper bound of S.

Hence

$$f(c)=0.$$

If possible, let c coincide with b. Since $f(b) < 0$, there exists an interval $[b-k, b]$ for every point x of which $f(x)$ is negative. Also since b is the upper bound of S, there exists a member α of $[b-k, b]$ such that $f(x)$ is positive in $[a, \alpha]$ and in particular $f(x)$ is positive. Thus we have a contradiction so that, c cannot coincide with b.

Cor. *If $f(x)$ is continuous in a closed interval $[a, b]$ and $f(a)\neq f(b)$, then, as x changes from a to b, $f(x)$ assumes at least once every value between $f(a)$ and $f(b)$ i.e., $f(x)$ assumes every given value between $f(a)$ and $f(b)$ for at least one value of x between a and b.*

Let k be any number between $f(a)$ and $f(b)$. The function

$$\varphi(x)=f(x)-k$$

is continuous in $[a, b]$ and $\varphi(a)=f(a)-k$ and $\varphi(b)=f(b)-k$ have opposite signs. There exists, therefore, a point, c, of $[a, b]$ such that

$$0=\varphi(c)=f(c)-k, \text{ or } f(c)=k.$$

58·2. Theorem. *If $f(x)$ is continuous in a closed interval $[a, b]$ and ε is any positive number, however small, then there exists a division of $[a, b]$ into a* **finite** *number of sub-intervals such that*

$$| f(x_2)-f(x_1) | < \varepsilon,$$

where x_1, x_2, are any two numbers belonging to the same sub-interval.

We *assume* that the theorem is false *i.e.*, we suppose that it is not possible to divide $[a, b]$ into finite number of sub-intervals which possess the required property.

Take

$$c=\tfrac{1}{2}(a+b).$$

The theorem must be false for at least one of the two sub-intervals $[a, c]$ or $[c, b]$ for, otherwise the theorem would as well as true for $[a, b]$.

Let the sub-interval in which the theorem is false be re-named as $[a_1, b_1]$. If there be a choice, which will happen if the theorem is false for both, we may, for the sake of definiteness, consider the left-hand interval $[a, c]$. In either case, we have

$$a \leqslant a_1 < b_1 \leqslant b,$$
$$b_1-a_1=\tfrac{1}{2}(b-a),$$
the theorem is false in $[a_1, b_1]$.

We now bisect $[a_1, b_1]$ and proceeding, as above, obtain another interval $[a_2, b_2]$ such that

$$a_1 \leqslant a_2 < b_2 \leqslant b_1,$$
$$b_2-a_2=\tfrac{1}{2}(b_1-a_1)=(\tfrac{1}{2})^2(b-a),$$
the theorem is false in $[a_2, b_2]$.

Proceeding as above, we shall obtain an infinite sequence of intervals

$$[a_1, b_1], [a_2, b_2]\ldots\ldots, [a_n, b_n], \ldots\ldots.$$

such that

(*i*) $a \leqslant a_1 \leqslant a_2 \ldots\ldots \leqslant a_n < b_n \leqslant b_{n-1} \ldots\ldots \leqslant b_2 \leqslant b_1 \leqslant b,$

(*ii*) $b_n-a_n=(\tfrac{1}{2})^n(b-a),$

(*iii*) the theorem is false in $[a_n, b_n]$.

From (*i*) and (*ii*) we easily deduce, as in § 58·1 that the sequences $\{a_n\}$, $\{b_n\}$ converge to the same limit. Let this common limit be c. The point c may be an interior or an end point of (a, b).

Let c, be an interior point.

Since $f(x)$ is continuous at, c, there exists a positive number δ such that

$$| f(x)-f(c) | < \tfrac{1}{2}\varepsilon, \text{ when } | x-c | \leqslant \delta.$$

If x_1, x_2 be any two members of $[c-\delta, c+\delta,]$, we have

$$| f(x_1)-f(c) | < \tfrac{1}{2}\varepsilon, \; | f(x_2)-f(c) | < \tfrac{1}{2}\varepsilon,$$

and, accordingly,

$$\begin{aligned} | f(x_2)-f(x_1) | &= | f(x_2)-c+c-f(x_1) | \\ &\leqslant | f(x_2)-c | + | c-f(x_1) | < \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon, \end{aligned}$$

so that the theorem is true for $[c-\delta, c+\delta]$.

Let m be a positive integer so chosen that $[a_m, b_m]$ lies within $[c-\delta, c+\delta]$. The theorem being true in $[c-\delta, c+\delta]$, we see that it is also true in $[a_m, b_m]$ which is a part of $[c-\delta, c+\delta]$ and thus we arrive at a contradiction. Hence the assumption that the theorem is false for $[a, b]$ is not true. Thus the theorem is true.

Let, c, coincide with the end point, a. There exists a positive number δ such that

$$| f(x)-f(a) | < \tfrac{1}{2}\varepsilon, \text{ when } a \leqslant x \leqslant a+\delta.$$

Now, if x_1, x_2 be any two members of $[a, a+\delta]$ we prove, as above, that

$$| f(x_2)-f(x_1) | < \varepsilon,$$

so that the theorem is true for $[a, a+\delta]$.

If m be a positive integer so chosen that $b_m < a+\delta$, then $[a_m, b_m]$ lies within $[a, a+\delta]$ and accordingly the theorem is true in $[a_m, b_m]$ which is only a part of $[a, a+\delta]$.

Here, again, we have a contradiction and so on.

The case when, c, coincides with, b, may be similarly disposed of.

58·3. Boundedness of Continuous functions. Theorem. *If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in the interval.*

Let ε be any positive number.

We can divide $[a, b]$ into a *finite* number of sub-intervals such that

$$| f(x_2)-f(x_1) | < \varepsilon,$$

when x_1, x_2 are any two members of the same sub-interval. Let

$$a, t_1, t_2, \ldots\ldots, t_{r-1}, t_r, t_{r+1}, \ldots\ldots, t_{n-1}, b$$

be the points of division.

If x be any point of $[a, t_1]$, we have

$$| f(x)-f(a) | < \varepsilon, \text{ i.e., } f(a)-\varepsilon < f(x) < f(a)+\varepsilon;$$

If x be any point of $[t_1, t_2]$, we have

$$| f(x)-f(t_1) | < \varepsilon, \text{ i.e., } f(t_1)-\varepsilon < f(x) < f(t_1)+\varepsilon;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

if x be any point of $[t_r, t_{r+1}]$, we have

$$| f(x)-f(t_r) | < \varepsilon, \text{ i.e., } f(t_r)-\varepsilon < f(x) < f(t_r)+\varepsilon;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

finally, if x be any point of $[t_{n-1}, b]$, we have

$$| f(x)-f(t_{n-1}) | < \varepsilon, \text{ i.e., } f(t_{n-1})-\varepsilon < f(x) < f(t_{n-1})+\varepsilon.$$

If k be the least member of the finite set of numbers

$$f(a)-\varepsilon, f(t_1)-\varepsilon, \ldots, f(t_r)-\varepsilon, \ldots\ldots, f(t_{n-1})-\varepsilon,$$

and K be the greatest member of the finite set of numbers

$$f(a)+\varepsilon, f(t_1)+\varepsilon, \ldots, f(t_r)+\varepsilon, \ldots\ldots, f(t_{n-1})+\varepsilon,$$

we see that for every point x of (a, b),

$$k < f(x) < K,$$

i.e., $f(x)$ is bounded.

Another proof. If ε be any positive number, then, because of the continuity of $f(x)$ at a, there exists a positive number δ such that for every point x of $[a, a+\delta]$,

$$|f(x)-f(a)| < \varepsilon, \text{ i.e., } f(a)-\varepsilon < f(x) < f(a)+\varepsilon,$$

so that we see that $f(x)$ is bounded in $[a, a+\delta]$.

Consider, now, a set S defined as follows :—

Any point x of $[a, b]$ belongs to S, if $f(x)$ is bounded in $[a, x]$. The set S exists inasmuch as $a+\delta$ belongs to the same and, as no number of S is $> b$, it is bounded also.

Let, c, be the upper bound of S. Now $c \leqslant b$.

If possible, let c be an interior point of S.

There exists an interval $[c-h, c+h]$ such that for every point x of this interval, $f(x)$ lies between $f(c)-\varepsilon$ and $f(c)+\varepsilon$ so that we see that $f(x)$ is bounded in $[c-h, c+h]$.

Since, c, is the upper bound of S, there exists a member η of S such that

$$c-h < \eta \leqslant c.$$

As η belongs to S, $f(x)$ is bounded in $[a, \eta]$.

As η is an interior point of $[c-h, c+h]$, we deduce from above that $f(x)$ is bounded in $[a, c+h]$ and accordingly $c+h$ is a member of S and this plainly is a contradiction. Thus, c, cannot be an interior point so that we have $c=b$.

As $f(x)$ is continuous at b, there exists an interval $[b-k, b]$ such that for every point x of this interval $f(x)$ lies between $f(b)-\varepsilon$ and $f(b)+\varepsilon$ *i.e.*, $f(x)$ is bounded in this interval.

Also there exists a member μ of S such that

$$b-k < \mu < b.$$

Now, $f(x)$ is bounded in $[a, \mu]$ and, in $[b-k, b]$ and, therefore, we deduce that it is bounded in $[a, b]$.

Hence the result.

Another proof. Suppose that $f(x)$ is not bounded. On this account, to every positive integer n, there corresponds a point t_n of $[a, b]$ such that

$$|f(t_n)| > n.$$

Now $\{t_n\}$ is a bounded infinite sequence and has, therefore, at least one limiting point. Let, t, be any limiting point of the sequence $\{t_n\}$.

As $f(x)$ is continuous at t, there exists a positive number δ such that

$$|f(x)-f(t)| < 1, \text{ when } |x-t| \leqslant \delta \qquad [\text{taking } \varepsilon=1].$$

or

$$|f(x)| < 1+|f(t)|,$$

for any member x of the interval $[t-\delta, t+\delta]$.

The interval $[t-\delta, t+\delta]$ contains an infinite number of members of the sequence $\{t_n\}$. Thus for an infinite number of values of n,

$$n < |f(t_n)| < 1 + |f(t)|,$$

which is clearly not true.

Hence $f(x)$ must be bounded in $[a, b]$.

Note. A function $f(x)$ which is continuous in *only* an open interval $]a, b[$ and is not continuous at any of the end points a or b *may not* be bounded. For example

$$f(x)=1/x \text{ when } x \neq 0 \text{ and } f(0)=0$$

is not bounded in $]0, 1[$. This function is not continuous at the end point 0. This example illustrates the fact that for the truth of the theorem it is necessary that the interval of continuity be closed.

Ex Show, by a process of continued bisection, that if a function $f(x)$ is defined in any interval $]a, b[$.

(*i*) and is not bounded, then there exists a point c of $]a, b[$ such that $f(x)$ is not bounded in any neighbourhood of c. Deduce the theorem above.

(*ii*) and is bounded, then there exists a point c of $]a, b[$ such that in any neighbourhood of c, the upper bound of $f(x)$ is the same as the upper bound of $f(x)$ in the whole interval. (Weierstrass's Theorem).

58·4. Theorem. *If a function $f(x)$ is continuous in a closed interval $[a, b]$ then it has greatest and least values,* i.e., *it attains its bounds at least once in the interval.*

It has been shown above in §58·3, that $f(x)$ is bounded in $[a, b]$. Let M, m be the bounds of $f(x)$.

We have to show that there exist members α, β of $[a, b]$ such that

$$f(\alpha)=M, \ f(\beta)=m.$$

We consider the case of upper bound. Suppose that $f(x)$ does not attain the value M for any value of x so that, $M-f(x)$, does not vanish for any point x of $[a, b]$.

From §57·1, we deduce that $M-f(x)$ and therefore,

$$1/[M-f(x)]$$

is continuous in $[a, b]$ and accordingly $1/[M-f(x)]$ is bounded.

Let k be any positive number, however large.

Since M is the upper bound of $f(x)$, there exists a value $f(c)$ of $f(x)$ such that

$$f(c) > M-1/k \text{ or } M-f(c) < 1/k \text{ or } 1/[M-f(c)] > k.$$

There exists, therefore, a value of the function $1/[M-f(x)]$ which is greater than any positive number k, however large, *i.e.*, $1/[M-f(x)]$ is not bounded. This is a contradiction.

Hence the theorem.

The case of lower bound may be similarly disposed of.

Cor. 1. The function $f(x)$ which is continuous in $[a, b]$ must also be continuous in $[\alpha, \beta]$ and so must assume every value between $f(\alpha)=M$ and $f(\beta)=m$.

Thus *a function which is continuous in a closed interval must assume every value between its upper and lower bounds.*

Note. The results established in Cor. to §58·1 and that in Cor. 1 to §58·4 amount to the following.

The range of a function $f(x)$ continuous in a closed interval is itself a closed interval.

Let $[m, M]$ be the range of a function $f(x)$ continuous in a closed interval $[a, b]$. To each number y belonging to the closed interval $[m, M]$ there corresponds *at least one* x belonging to $[a, b]$ such that $f(x)=y$. The result in §59 gives a condition which guarantees the *uniqueness* of x such that $f(x)=y$.

Cor. 2. If a function $f(x)$ is continuous in a closed interval $[a, b]$, and ε is any given positive number, then there exists a division of $[a, b]$ into a finite number of sub-intervals such that the oscillation of $f(x)$ in every sub-interval is less than ε.

This follows from § 58·2 and the Cor. 1, above.

58·5. Uniform continuity.

Theorem. *If a function $f(x)$ is continuous in a closed interval $[a, b]$ and ε is any given positive number, then there exists a positive number δ such that the oscillation of $f(x)$ in every sub-interval of length less than δ is less than ε.*

We can divide $[a, b]$ into a finite number of sub-intervals such that

$$|f(x_2)-f(x_1)| < \tfrac{1}{2}\varepsilon,$$

where x_1, x_2 are any two members of the same sub-interval.

Now every one of these sub-intervals has a certain length. Let δ be the least of these lengths. Clearly, δ, is positive.

Consider any pair of points x_1, x_2 such that $|x_2-x_1| \leqslant \delta$. The points x_1, x_2 either belong to the same or to two consecutive sub-intervals of the division obtained above.

In case they belong to the same sub-interval, we have

$$|f(x_2)-f(x_1)| < \tfrac{1}{2}\varepsilon < \varepsilon.$$

If they belong to two consecutive sub-intervals, let t_r be the common end point. We have

$$\begin{aligned} |f(x_2)-f(x_1)| &= |f(x_2)-f(t_r)+f(t_r)-f(x_1)| \\ &\leqslant |f(x_2)-f(t_r)| + |f(t_r)-f(x_1)| \\ &< \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon. \end{aligned}$$

Now, consider any arbitrary sub-interval δ_r of $[a, b]$ whose length $\leqslant \delta$. If M_r, m_r be the bounds of $f(x)$ in δ_r, there exist points ξ_r, η_r of this interval such that

$$f(\xi_r)=M_r,\ f(\eta_r)=m_r.$$

Since $|\xi_r-\eta_r| \leqslant \delta$, we have

$$\begin{aligned} M_r-m_r &= f(\xi_r)-f(\eta_r) \\ &= |f(\xi_r)-f(\eta_r)| < \varepsilon. \end{aligned}$$

Hence the theorem.

Note. The property of continuous functions proved above is known by the name of **uniform continuity** so that the theorem may be restated as follows :—

A *function which is continuous in a closed interval is also uniformly continuous in that interval.*

We now consider the *justification* for the name 'Uniform Continuity'.

The continuity of $f(x)$ at a point x' implies that there exists a positive number δ such that

$$|f(x)-f(x')| < \varepsilon, \text{ when } |x-x'| \leqslant \delta,$$

ε being any given positive number. Corresponding to the same ε, there exists a number δ for every point x of $[a, b]$. The question arises, "*Does there exist a positive number* δ *which holds uniformly for every point of* $[a, b]$?" The theorem proved above shows, that such a choice of δ is possible. In fact it has been shown there that corresponding to any positive number ε, there exists a positive number δ, such that

$$|f(x_2)-f(x_1)| < \varepsilon,$$

where x_1, x_2 are any two numbers such that

$$|x_2-x_1| \leqslant \delta.$$

Note. The reader is strongly advised to prove the above result by means of the Heine-Borel Theorem.

Ex. 1. *If* $f(x)=x^2+3x$ *in* $[-1, 1]$, *find* δ, *so that*

$$|f(x_2)-f(x_1)| < \varepsilon \textit{ whenever } |x_2-x_1| < \delta.$$

We have

$$\begin{aligned} |f(x_2)-f(x_1)| &= |x_2-x_1| \; |x_2+x_1+3| \\ &\leqslant |x_2-x_1| \, [\, |x_2|+|x_1|+3] \\ &\leqslant 5\,|x_2-x_1| < \varepsilon, \text{ if } |x_2-x_1| < \varepsilon/5. \end{aligned}$$

Thus $\delta=\varepsilon/5$.

In particular $\delta=\frac{1}{50}$ if $\varepsilon=\frac{1}{10}$: $\delta=\frac{1}{500}$, if $\varepsilon=\frac{1}{100}$, etc.

2. Do the same, as in the Ex. above for

(*i*) $f(x)=x^3+3x^2-2x+7$ in $[-3, 2]$.

(*ii*) $f(x)=(x+2)/(2x+3)$ in $[-1, 5]$.

59. Inverse functions. Theorem. *If a function* $f(x)$ *be continuous and* **strictly** *monotonic in* (a, b), *then there exists* **one and only one** *value of* x *which satisfies the equation*

$$f(x)=y \qquad \ldots(i)$$

where y *is any number lying between* $f(a)$ *and* $f(b)$,

The existence of at least one value of x follows from the fact of continuity of $f(x)$, (Cor. to § 58·1). Also since the function is strictly monotonic there cannot exist more than one such value. Hence the result.

59·1. Continuity of Inverse Functions.

To each value of y, lying between $f(a)$ and $f(b)$, there corresponds one and only one value of x, as determined from (i), and thus (i) determines x as a function of y, say $\varphi(y)$, defined in the interval $[f(a), f(b)]$ or $[f(b), f(a)]$ as the case may be.

This function $\varphi(y)$ is said to be the *inverse* of $f(x)$. We now show that $\varphi(y)$ is continuous in its interval of definition.

Let y_1 be any value of y and let $x_1=\varphi(y_1)$ so that, we have

$$f(x_1)=y_1.$$

Let ε be any positive number.

Firstly suppose that $f(x)$ is strictly *increasing.*

Let

$$f(x_1-\varepsilon)=y_1-\delta_1,\ f(x_1+\varepsilon)=y_1+\delta_2\ ;$$

δ_1, δ_2 being necessarily positive.

Since $f(x)$ is strictly monotonic, y will lie in the interval $[y_1-\delta_1, y_1+\delta_2]$ when x lies in $[x_2-\varepsilon,\ x_1+\varepsilon]$.

Thus if $\delta=\min(\delta_1, \delta_2)$, we see that

$$|\ x-x_1\ | < \varepsilon, \textit{i.e.},\ |\ \varphi(y)-\varphi(y_1)\ | < \varepsilon, \text{ when } |\ y-y_1\ | \leqslant \delta.$$

Hence $\varphi(y)$ is continuous at y_1 and, therefore, in $[f(a), f(b)]$.

The continuity of $\varphi(y)$ can be similarly established if $f(x)$ is strictly *decreasing.*

It should be understood that the *inverse function* $\varphi(y)$ *exists and is continuous if, and only if,* $f(x)$ *is continuous and strictly monotonic.*

60. Continuity of a^x, $\log_a x$, x^n.

60·1. *To show that a^x, $(a>0)$, is continuous for every value of x.*

Let, c, be any value of x.

Suppose that, a^x, is not continuous for $x=c$.

This implies that there exists a positive number, ε, such that whatever be the interval $[c-\delta, c+\delta]$ around c, for at least one point x of this interval,

$$|\ a^x-a^c\ | > \varepsilon.$$

Consider a sequence of intervals $[c-1/n,\ c+1/n]$ around c. There exists a point x_n of $[c-1/n,\ c+1/n]$ such that

$$|\ a^{x_n}-a^c\ | > \varepsilon. \qquad ...(1)$$

Since

$$c-1/n < x_n < c+1/n,$$

we see that the sequence $[x_n]$ converges to c, and hence

$$\lim a^{x_n} = a^c \qquad (\S\ 44, \text{ page } 67)$$

and this conclusion contradicts the statement (1).

Hence the result.

Cor. 1. The exponential function e^x is continuous for every value of x.

Cor. 2. Since a^x is continuous and strictly monotonic, its inverse function, *viz.*, the logarithmic function $\log_a y$ is continuous for every positive value of y.

It should be remembered that the logarithmic function $\log_a x$ is defined for positive values of x only.

Cor. 3. (*i*) If $f(x)$ is continuous in any interval, then $e^{f(x)}$ is also continuous in the same interval. In particular, $x^n = e^{n \log x}$ is continuous for every positive value of x.

(*ii*) If $f(x)$ is positive and continuous in any interval, then $\log f(x)$ is also continuous in the same.

These results follow from § 57·2, page 92.

Cor. 4. If $f(x)$, $\varphi(x)$ are continuous in any interval and $f(x)$ is also positive, then $f(x)^{\varphi(x)}$ is continuous in the same interval.

This result is seen to be true on writing

$$[f(x)]^{\varphi(x)} = e^{\varphi(x) \log f(x)}.$$

61·1. *To prove that*

(*a*) $\lim\limits_{x \to \infty} a^x = \infty$ or 0 *according as* $a >$ *or* < 1 ;

(*b*) $\lim\limits_{x \to -\infty} a^x = 0$ *or* ∞ *according as* $a >$ *or* < 1.

The result (*a*) follows from the facts that (*i*) the sequence a^x tends to ∞ or 0 according as $a > 1$ or $0 < a < 1$. (*ii*) a^x is positive and monotonically increasing or decreasing according as $a > 1$ or $0 < a < 1$.

The result (*b*) can be deduced from (*a*) by putting

$$x = -y \text{ so that } y \to \infty \text{ as } x \to -\infty.$$

61·2. Theorem. *To prove that*

$$\lim_{x \to \infty} \log_a x = \infty \text{ or } -\infty \text{ according as } a > \text{ or } < 1,$$

$$\lim_{x \to (0+0)} \log_a x = -\infty \text{ or } \infty \text{ according as } a > \text{ or } < 1.$$

This can be deduced from above on writing

$$y = \log_a x, \quad i.e., \; x = a^y.$$

62. Infinitesimals. *A variable which tends to zero is called an infinitesimal.* In order to know whether a function is infinitesimal or not, we must know the independent variable and its limit. For example $(x^2 - a^2)$, $e^{-1/(x-a)^2}$ are infinitesimals only when $x \to a$ and $1/x$ is an infinitesimal when $x \to \infty$. An independent variable which tends to zero is also an infinitesimal.

Comparison of infinitesimals. Let $f(x)$, $\varphi(x)$ be two infinitesimals. The following cases arise :—

(*i*) $f/\varphi \to l \neq 0$. In this case we say that $f(x)$ is an infinitesimal of the same order as $\varphi(x)$.

In case $l = 1$, we say that the two infinitesimals are equivalent and write

$$f(x) \sim \varphi(x).$$

If $f(x)/\varphi^r(x) \to$ a finite non-zero limit, we say that $f(x)$ is an infinitesimal of order r with respect to $\varphi(x)$.

(*ii*) $f/\varphi \to 0$. In this case we say that $f(x)$ is an infinitesimal of higher order and $\varphi(x)$ of lower order.

(*iii*) $f/\varphi \to \infty$ or $-\infty$ so that $\varphi/f \to 0$. In this case $\phi(x)$ is an infinitesimal of higher order and $f(x)$ of lower order.

Principal part of an infinitesimal. If an infinitesimal be expressible as sum of a number of infinitesimals of different orders, then the one of lowest order is called the *principal part.*

Note. It should be noted that a constant number, however small, is certainly not an infinitesimal. A great deal of confusion has arisen because of the assumption in the older forms of presontation of analysis that there exist numbers so small that they can be neglected.

Examples

1. *Consider the continuity for $x=0$ of*

$$f(x)=1/(1-e^{1/x}) \text{ when } x\neq 0 \text{ and } f(0)=0.$$

Now

$1/x \to \infty$ or $-\infty$ according as $x\to(0+0)$ or $(0-0)$.

Thus

$$\lim_{x\to(0+0)} f(x)=0,$$

and

$$\lim_{x\to(0-0)} f(x)=1, \text{ for } \lim_{x\to(0-0)} e^{1/x}=0.$$

Since these two limits are different, $\lim f(x)$ as $x\to 0$, does not exist. Thus $f(x)$ is discontinuous for $x=0$ and the point of discontinuity is of the second kind.

Of course the function is continuous on the right and has a discontinuity of the second kind on the left of $x=0$.

2. *If $f(x)$ is a continuous function of x satisfying the functional equation*

$$f(x+y)=f(x)+f(y),$$

show that $f(x)=ax$ where, a, is a constant.

Taking $x=0=y$, we obtain

$$f(0)=0 \qquad \ldots(1)$$

and taking $y=-x$, we obtain

$$f(-x)=-f(x). \qquad \ldots(2)$$

If, x, be a positive integer, we have

$$\begin{aligned} f(x)&=f(1+1+\ldots\ldots+1)\\ &=f(1)+f(1)\ldots\ldots+f(1)\\ &=xf(1)=ax, \text{ say,}\end{aligned}$$

where $f(1)=a$.

Let x, now, be a negative integer. We write $x=-y$ so that y is a positive integer. We have

$$f(x)=f(-y)=-f(y)=-ay=ax.$$

Again let $x=p/q$ be a rational number ; q being positive. We have

$$f(p)=f\left(\frac{p}{q}\right)=f\left(\frac{p}{q}\right)+f\left(\frac{p}{q}\right)+\dots\dots\ q \text{ times}=qf\left(\frac{p}{q}\right)$$

$$\therefore \qquad ap=q\left(\frac{p}{q}\right)$$

or

$$f(x)=f\left(\frac{p}{q}\right)=a\,\frac{p}{q}=ax,$$

where x is any rational number.

Suppose, now, that x is any real number.

Let $\{x_n\}$ be any sequence of rational numbers such that

$$\lim x_n=x.$$

We have, x_n, being rational,

$$f(x_n)=ax_n. \qquad \text{.. (3)}$$

Let $n \to \infty$. As $f(x)$ is a continuous function, we obtain from (3)

$$f(x)=ax.$$

Hence the result.

3. *Discuss the function*

$$f(x)=\begin{cases} 1/q, & \textit{when } x \textit{ is rational } p/q \textit{ in its lowest terms,} \\ 0, & \textit{when } x \textit{ is irrational,} \end{cases}$$

as regards its continuity.

It will be shown that $f(x)$ is continuous for irrational values of x and discontinuous for rational values of x.

Let p/q be any rational number so that

$$f\left(\frac{p}{q}\right)=\frac{1}{q}.$$

We know that in every interval there lie an infinite number of irrational numbers. Accordingly in every interval around p/q, there exist numbers x, *viz.*, irrational numbers, such that for these numbers x, we have

$$\left|f(x)-f\left(\frac{p}{q}\right)\right| = \left|0-\frac{1}{q}\right| = \frac{1}{q}.$$

Thus, if, ε, be any positive number $< 1/q$, we see that there cannot exist any interval around p/q for every point x of which

$$\left|f(x)-f\left(\frac{p}{q}\right)\right| < \varepsilon.$$

Hence $f(x)$ is discontinuous for $x=p/q$.

Let, now, c, be any irrational number so that $f(c)=0$.

Suppose that we arrange the fractions $1/q$ in the order of increasing q.

Let, ε, be any pre-assigned positive number. Then there exist only a finite number of fractions $1/q$ such that $q < (1/\varepsilon)$. We can thus enclose, c, in an interval which does not enclose any rational number p/q for which $q < (1/\varepsilon)$. Then for irrational x in this interval

$$|f(x)-f(c)| = 0 < \varepsilon$$

and for any rational $x=p/q$ in this interval

$$\left|f(x)-f(c)\right| = \frac{1}{q} < \varepsilon.$$

Thus $f(x)$ is continuous for $x=c$.

Exercises

1. Show that

$$f(x)=(x-1)/[1+e^{1/(x-1)}],\ x \neq 1\ ;\ f(1)=0.$$

is continuous for every value of x.

2. Show that

$$f(x)=(e^{1/x}-1)\ /\ (e^{1/x}+1),\text{ when } x \neq 0\ ;\ f(0)=0.$$

is continuous for every value of x except $x=0$. What is the nature of discontinuity at $x=0$?

3. Consider the continuity of

$$f(x)=e^{1/x^2}\ /\ (e^{1/x^2}-1),\text{ when } x \neq 0\ ;\ f(0)=1.$$

4. State with reasons for your conclusion which of the following sets of circumstances are sufficient and which of them are not to determine the value of $f(0)$ for a function $f(x)$, giving the value of $f(0)$ where possible :—

(*i*) $f(x)$ is continuous at $x=0$ and takes in any neighbourhood of $x=0$ both positive and negative values.

(*ii*) For any arbitrary positive ε, there is a δ such that

$$|f(x)| < \varepsilon \text{ for } 0 < |x| < \delta.$$

(*iii*) $[f(h)+f(-h)-2f(0)]/h \to l$ and $f(h) \to a$ as $h \to 0$.

5. Show that the sum-function $S(x)$ of the infinite series

$$x^2+x^2(1-x^2)+x^2(1-x^2)^2+\ldots\ldots+x^2(1-x^2)^n+\ldots\ldots$$

is not continuous for $x=0$ even though every term of the series is continuous for $x=0$.

6. Examine the continuity for $x=0$ of the sum-function $S(x)$ of the following series :

(*i*) $$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\ldots\ldots$$

(*ii*) $$\sum_{n=0}^{\infty}\left(e^{-n|x|}-e^{-2n|x|}\right)$$

(*iii*) $$\sum_{n=1}^{\infty}\frac{x^2}{[1+nx^2][1+(n+1)x^2]}.$$

7. Let $f(x)$ be continuous in $[-1, 1]$ and assume rational values only and let $f(0)=0$. Prove that $f(x)=0$ everywhere.

8. A function $f(x)$ is continuous in the interval $[0, 1]$ and assumes only rational values in the entire interval. If $f(x)=\frac{1}{2}$ when $x=\frac{1}{2}$, prove that $f(x)=\frac{1}{2}$ everywhere.

9. Show that the function $\varphi(x)$ where

$\varphi(x)=x$ when x is rational and $\varphi(x)=1-x$ when x is irrational assumes every value between 0 and 1 once and only once as x increases from 0 to 1, but is discontinuous for every value of x except $x=\frac{1}{2}$.

10. Discuss the continuity of $f(x)$ where

$$f(x)=\begin{cases} 0, & \text{when } x \text{ is irrational or zero.} \\ \dfrac{1}{q^3}, & \text{when } x=\dfrac{p}{q}, \text{ a fraction in its lowest terms.} \end{cases}$$

11. If a continuous function $f(x)$ satisfies the equation

$$f(x+y)=f(x)\,f(y),$$

then either $f(x)=0$ or else $f(x)=e^{ax}$.

12. If a continuous function $f(x)$ satisfies the equation

$$f(xy)=f(x)+f(y),$$

and $x \neq 0$, show that $f(x)=a \log x$.

13. Prove that a function $f(x)$ satisfying the condition

$$f(x') < f(x)+\varepsilon, \quad (\,|\,x'-x\,| < \delta),$$

at every point x of $[a, b]$ is bounded above and attains its upper bound in $[a, b]$.

[A function satisfying the condition above is called *upper semi-continuous.*]

CHAPTER V

THE DERIVATIVE

63. Derivability. Derivative. Let $f(x)$ be a function defined in an interval $[a, b]$.

63·1. *Derivability at an interior point.* Let, c, be any interior point of $[a, b]$. We take $c+h$, any other point of the same interval. Then $f(x)$ is said to be derivable at $x=c$, if

$$\lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\right]$$

exists ; the limit, called the derivative of $f(x)$ at $x=c$, is denoted by the symbol $f'(c)$.

One sided derivatives. The

$$\lim_{h\to(0-0)} [f(c+h)-f(c)]/h,$$

in case it exists, is called the **right hand derivative** *of $f(x)$ at $x=c$: and is denoted by $Rf'(c)$.*

Similarly

$$\lim_{h\to(0-0)} [f(c+h)-f(c)]/h,$$

in case it exists, is called the **left hand derivative** *of $f(x)$ at $x=c$ and is denoted by $Lf'(c)$.*

63·2. *Derivability at end points.* $f(x)$ is said to be derivable at a, if $Rf'(a)$ exists and $f'(a)$, then, means $Rf'(a)$; it is said to be derivable at b, if $Lf'(b)$ exists and $f'(b)$, then, means $Lf'(b)$.

Note. It is obvious that for an interior point c, $f'(c)$ exists if and only if $Rf'(c)$, $Lf'(c)$ both exist and are equal, and conversely ; also, then

$$f'(c)=Lf'(c)=Rf'(c).$$

63·3. *Finitely derivable functions.* A function is said to be finitely derivable at a point, if its derivative at that point exists and is finite.

63·4. *Derivability in an interval. Derived function.* A function is said to be derivable in an interval, if it is derivable at every point thereof. The function determined by the values of the derivatives of $f(x)$ for points of $[a, b]$ is called the derived function of $f(x)$ and is denoted by

$$f'(x), \quad df/dx \quad \text{or} \quad D_x f(x),$$

Ex. Show that the functions

(*i*) k, a constant, (*ii*) x,

arederivable for every value of x.

64. Continuity of a derivable function. Theorem. *If a function is finitely derivable at a point, it is also continuous at that point.*

Let $f(x)$ be finitely derivable for $x=c$ so that $\{[f(c+h)-f(c)]/h\}$ tends to a finite limit denoted by $f'(c)$ as $h \to 0$.

We have

$$f(c+h)-f(c)=\frac{f(c+h)-f(c)}{h}h.$$

$$\therefore \quad \lim [f(c+h)-f(c)]=\lim\frac{f(c+h)-f(c)}{h} . \lim h=f'(c)0=0.$$

$$\therefore \qquad \lim f(c+h)=f(c), \text{ when } h \to 0,$$

i.e.,

$$\lim f(x)=f(c), \text{ when } x \to c.$$

Hence $f(x)$ is continuous at $x=c$.

Remarks. This theorem may also be stated as follows: *The necessary condition for a function to be finitely derivable at a point is that it is continuous at that point.*

The converse of this theorem is not necessarily true, *i.e.*, *the condition of continuity is not sufficient for derivability.*

Consider the derivability of

$$f(x)= | x |$$

for $x=0$. We have

$$\frac{f(0+h)-f(0)}{h}=\frac{f(h)}{h}=\frac{| h |}{h}=\begin{cases} 1, \text{ if } h > 0 \\ -1, \text{ if } h < 0, \end{cases}$$

$$\therefore \qquad Rf'(0)=1 \text{ and } Lf'(0)=-1.$$

Hence $f'(0)$ does not exist.

To examine the continuity, we take any positive number ε. We have

$$| f(x)-f(0) | = | x | < \varepsilon, \text{ when } | x | \leqslant \delta,$$

δ being any number $< \varepsilon$, and thus $f(x)$ is continuous for $x=0$.

Hence the result.

Ex. 1. Show that $| x+1 | + | x | + | x-1 |$ is continuous but not derivable for $x=-1$, 0 and 1.

2. Show that the function

$$y=\text{degree of } (u^{x^2}+u^2+2u-3)$$

is continuous but not derivable for $x=2$.

Note 1. There exist functions which are continuous in an interval but are not derivable for *any* point thereof. Example of such a function is given in an appendix at the end of the book.

2. The student would do well to remember that the statement

$$f'(c)=l.$$

is equivalent to two distinct statements, *viz.*,

(*i*) that $f(x)$ is derivable at c and

(*ii*) that teh derivative is l.

3. The existence of the derivative of $f(x)$ for $x=c$ implies that

(*i*) $f(x)$ is defined in a certain neighbourhood of c;

(*ii*) $f(x)$ is continuous at c.

65. Differentiability and differentials. *A function $f(x)$ is said to be differentiable at a point x, of the interval of definition $[a, b]$ of $f(x)$ if the change, $f(x+\delta x)-f(x)$ in the function which corresponds to the change δx in x is capable of being expressed in the form*

$$\delta f=f(x+\delta x)-f(x)=A\delta x+\varepsilon\delta x.$$

where A is independent of δx and ε is a function of δx which $\to 0$ as $\delta x \to 0$.

Taking δx as the principal infinitesimal, we see that the principal part of the infinitesimal δf is $A\delta x$. This principal part is called the *differential* of $f(x)$ and is denoted by $df(x)$ or simply df. If y represents $f(x)$, then the differential is also denoted by dy. It may also be noticed that, ε, is an infinitesimal of order higher than that of δx.

Condition for differentiability. Theorem. *The necessary and sufficient condition for $f(x)$ to be differentiable at a given point is that it possesses a finite derivative at that point.*

Let $f(x)$ be differentiable at x. We have

$$f(x+\delta x)-f(x)=A\delta x+\varepsilon\delta x$$

or

$$[f(x+\delta x)-f(x)]/\delta x=A+\varepsilon.$$

Let $\delta x \to 0$. In the limit, we see that

$$f'(x)=A,$$

so that $f(x)$ is finitely derivable at x; the derivative being A.

Let $f(x)$ *possess a finite derivative $f'(x)$ at x so that*

$$\lim\{[f(x+\delta x)-f(x)]/\delta x\}=f'(x) \text{ as } \delta x \to 0.$$

We write

$\dfrac{[f(x+\delta x)-f(x)]}{\delta x}-f'(x)=\varepsilon$, so that $\varepsilon \to 0$ as $\delta x \to 0$, and obtain

$$f(x+\delta x)-f(x)=\delta x f'(x)+\varepsilon\delta x.$$

Thus $f(x)$ is differentiable at x.

By definition, we have

$$dy=A\delta x=f'(x)\delta x, \text{ for } A=f'(x).$$

Taking $y=x$, we see that

$$dx=dy=1.\delta x=\delta x,$$

i.e., the differential of an independent variable may be taken equal to the arbitrary increment δx in x. Thus we have

$$dy=f'(x)dx.$$

Since the derivative $f'(x)$ appears as the co-efficient of the differential dx, the derivative $f'(x)$ is also often called the *differential coefficient of $f(x)$* and the process of obtaining the derivative is called *differentiation.*

66. Fundamental theorems on Derivation. *If $f_1(x), f_2(x)$ are derivable for $x=c$, then*

(*i*) $\varphi'(c)=f_1'(c)\pm f'_2(c)$, *where* $\varphi(x)=f_1(x)\pm f_2(x)$,

(*ii*) $\varphi'(c)=f_1'(c)\, f_2(c)+f_2'(c) f_1(c)$, *where* $\varphi(x)=f_1(x).f_2(x)$,

(*iii*) $\varphi'(c)=[f_1'(c)f_2(c)-f_2'(c)f_1(c)]/[f_2(c)]^2$, *where* $\varphi(x)=f_1(x)/f_2(x)$ and $f_2(c)\neq 0$.

(*i*) We have

$$\frac{\varphi(c+h)-\varphi(c)}{h}=\frac{[f_1(c+h)\pm f_2(c+h)]-[f_1(c)\pm f_2(c)]}{h}$$

$$=\frac{f_1(c+h)-f_1(c)}{h}\pm\frac{f_2(c+h)-f_2(c)}{h}.$$

The result now follows from § 53, page 88.

(*ii*) We have

$$\frac{\varphi(c+h)-\varphi(c)}{h}=\frac{f_1(c+h)f_2(c+h)-f_1(c)\,f_2(c)}{h}$$

$$=f_2(c+h)\,\frac{f_1(c+h)-f_1(c)}{h}+f_1(c)\,\frac{f_2(c+h)-f_2(c)}{h}.$$

Since $f_2'(c)$ exists, $f_2(x)$ is continuous for $x=c$, *i.e.*,

$$f_2(c+h)\to f_2(c) \text{ when } h\to 0.$$

The result now follows from § 53, p. 88.

(*iii*) As $f_2'(c)$ exists, therefore $f_2(x)$ is continuous at c. Also $f_2(c)\neq 0$. There exists, therefore, an interval $[c-\delta, c+\delta]$, such that $f_2(x)\neq 0$ for any point x of this interval.

Let $(c+h)$ be any point of this interval so that $f_2(c+h)\neq 0$.

We have

$$\frac{\varphi(c+h)-\varphi(c)}{h}=\left[\frac{f_1(c+h)}{f_2(c+h)}-\frac{f_1(c)}{f_2(c)}\right]\Big/h$$

$$=\frac{1}{f_2(c)\,f_2(c+h)}\left[f_2(c)\frac{f_1(c+h)-f_1(c)}{h}-\frac{f_2(c+h)-f_2(c)}{h}f_1(c)\right]$$

The result now follows from § 53, page 82.

Remarks. As a particular case of the above, we see that if two functions be derivable at a point (or in an interval), then (*i*) their sum, difference and product are also derivable at that point (or in that interval), (*ii*) their quotient is also derivable at that point (or in that interval) provided that the denominator of the quotient is not zero at that point (or at any point of that interval).

67·1. Derivative of function of a function. *If $\varphi(t)$ possesses a finite derivative $\varphi'(t)$ at a certain point, t, and $f(x)$ possesses a finite derivative $f'(x)$ at the corresponding point $x=\varphi(t)$, then the function $\psi(t)=f[\varphi(t)]$ also possesses a derivative at t, and*

$$\psi'(t)=f'(x).\varphi'(t).$$

We write

$$y=\psi(t)=f[\varphi(t)].$$

With usual notation, since $x=\varphi(t)$ and $y=f(x)$ possess finite derivatives, therefore

$$\delta x=\varphi'(t)\delta t+\varepsilon_1\delta t, \qquad \text{where } \varepsilon_1\to 0 \text{ as } \delta t\to 0;$$

$$\delta y=f'(x)\delta x+\varepsilon_2\delta x, \qquad \text{where } \varepsilon_2\to 0 \text{ as } \delta x\to 0$$

From these, we obtain

$$\delta y=f'(x)\varphi'(t)\delta t+[\varepsilon_1 f'(x)+\varepsilon_2\varphi'(t)+\varepsilon_1\varepsilon_2]\delta t$$

$$=f'(x)\varphi'(t)\delta t+\varepsilon_3\delta t, \text{ where } \varepsilon_3\to 0 \text{ as } \delta t\to 0.$$

Thus y is a differentiable function of t and

$$\psi'(t)=f'(x).\varphi'(t).$$

Note. The proof given in elementary text books, which is based on the equality

$$\frac{\delta y}{\delta t}=\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta t},$$

is incomplete inasmuch as no heed is paid to the case which arises when $\delta x=0$ for *some* point in *every* neighbourhood of the point t. For examples of such functions refer to Chapter VII.

67·2. Derivatives of inverse functions. Let $y=f(x)$ be continuous and strictly monotonic and let $x=g(y)$ be its inverse and let $f(x)$ possess a finite non-zero derivative $f'(x)$, at a point x. Then $g(y)$ also possesses a finite derivative at the corresponding point y such that

$$g'(y)=1/f'(x).$$

With usual notation, we have

$$\frac{\delta x}{\delta y}=1\bigg/\left(\frac{\delta y}{\delta x}\right).$$

Proceeding to the limit, we obtain the required result.

68. Derivatives of $\log_a x$, a^x, x^n.

68·1. *The function $\log_a x$ is derivable for every value of x for which it is defined,* i.e., *for positive values of x and its derivative is*

$$\frac{1}{x}\log_a e \qquad (a>0).$$

Let

$$y=\log_a x.$$

$$\therefore \qquad \frac{\delta y}{\delta x}=\frac{1}{x}\cdot\log_a\left(1+\frac{\delta x}{x}\right)^{(x/\delta x)}.$$

When $\delta x\to 0, \left(1+\dfrac{\delta x}{x}\right)^{x/\delta x}\to e$ and, since $\log_a t$ is a continuous function of t, therefore

$$\lim\log_a\left(1+\frac{\delta x}{x}\right)^{(x/\delta x)}=\log_a e.$$

Thus $\lim (\delta y/\delta x)$ exists and we have

$$\frac{dy}{dx}=\frac{1}{x}\log_a e.$$

If $y=\log x=\log_e x$, then $dy/dx=1/x$.

Cor. If $f(x)$ is positive and derivable, then $\log f(x)$ is also derivable and its derivative is $f'(x) \,/\, f(x)$.

Follows from § 67·1.

68·2. *The function a^x is derivable for every value of x and its derivative is $a^x \log_e a$. $(a > 0)$.*

The function $y=a^x$ is the inverse of the derivable function $x=\log_a y$. Hence y is derivable (§ 67·2). Also

$$\frac{dy}{dx}=1\bigg/\left(\frac{dx}{dy}\right)=1\bigg/(\log_a e)/y=y\log_e a=a^x\log_e a.$$

Cor. 1. If $f(x)$ is derivable, then $a^{f(x)}$ is also derivable, $(a>0)$, and its derivative is

$$a^{f(x)}\,.\,f'(x)\log_e a. \quad (\text{§ } 67·1).$$

Cor. 2. If $f(x)$, $\varphi(x)$ are derivable and $f(x)$ is positive, then $f(x)^{\varphi(x)}$ is also derivable.

Write $f(x)^{\varphi(x)}=e^{[\varphi(x)\log f(x)]}$

68·3. *The function x^n is derivable for every positive value of x and its derivative is nx^{n-1}, n, being any real number.*

We have

$$y=x^n=e^{n\log x}$$

and, therefore, by § 67·2, y is derivable and

$$\frac{dy}{dx}=e^{n\log x}\,.\,\frac{n}{x}=nx^{n-1}.$$

Cor. If $f(x)$ is positive and derivable, then $[f(x)]^n$ is also derivable and its derivative is $n[f(x)]^{n-1}f'(x)$

Note. If n is a positive integer, then x^n is derivable for every value of x and if n is a negative integer, x^n is derivable for every value of x except zero. The proof based on the binomial theorem for a positive integer index, as given in elementary books, is satisfactory for this case.

69. Meaning of the sign of derivative at a point.

Let, c, be any *interior* point of the interval of definition $[a, b]$ of a function $f(x)$.

Let $f'(c) > 0$.

To each positive number ε, there corresponds a positive number δ such that

$$\left|\frac{f(c+h)-f(c)}{h}-f'(c)\right| < \varepsilon, \text{ when } 0 < |h| \leqslant \delta,$$

i.e.,
$$f'(c)-\varepsilon < \frac{f(c+h)-f(c)}{h} < f'(c)+\varepsilon \text{ when } 0 < |h| \leqslant \delta.$$

Giving to ε any positive value smaller than the positive number $f'(c)$, we find that

$$\frac{f(c+h)-f(c)}{h} > 0 \text{ when } 0 < |h| \leqslant \delta,$$

i.e.,
$$f(c+h) > f(c) \text{ when } 0 < h \leqslant \delta,$$

and
$$f(c+h) < f(c) \text{ when } -\delta \leqslant h < 0.$$

Thus we conclude that if $f'(c)$ be positive, then *there exists a neighbourhood* $[c-\delta, c+\delta]$ *of* c *such that*

$$f(x) > f(c), \textit{ for any point } x \textit{ of }]c, c+\delta],$$

$$f(x) < f(c), \textit{ for any point } x \textit{ of } [c-\delta, c[.$$

Let $f'(c) < 0$. We write

$$\varphi(x) = -f(x) \text{ so that } \varphi'(c) = -f'(c) > 0.$$

From above, we now deduce that *there exists a neighbourhood* $[c-\delta, c+\delta]$ *of, c, such that*

$$\varphi(x) > \varphi(c), \textit{ i.e., } f(x) < f(c) \textit{ for any point } x \textit{ of }]c, c+\delta],$$

$$\varphi(x) < \varphi(c), \textit{ i.e., } f(x) > f(c) \textit{ for any point } x \textit{ of } [c-\delta, c[.$$

These results may also be obtained independently exactly in the manner in which the first case has been treated.

We now consider *end-points*. It is easy to show that if

(*i*) $f'(a)$ is positive, (negative), there exists an interval $]a, a+\delta]$ such that $f(x) > f(a)$, $[f(x) < f(a)]$, for any point x of $]a, a+\delta]$.

(*ii*) $f'(b)$ is positive, (negative), there exists an interval $[b-\delta, b[$ such that $f(x) < f(b)$, $[f(x) > f(b)]$, for any point x of $[b-\delta, b[$.

70. **Darboux's theorem.** *If $f(x)$ is derivable in a closed interval $[a, b]$ and $f'(a)$, $f'(b)$ are of opposite signs, then there exists at least one point, c, of the interval $[a, b]$ such that $f'(c)=0$.*

For the sake of definiteness, we suppose that $f'(a)$ is positive and $f'(b)$ negative. On this account there exist intervals $]a, a+\delta]$, $[b-\delta, b[$, $(\delta > 0)$, such that

for every point x of $]a, a+\delta]$, $f(x) > f(a)$; ... (*i*)

and

for every point x of $[b-\delta, b[$, $f(x) > f(b)$. ...(*ii*)

Again $f(x)$, being derivable, is continuous in $[a, b]$ (§ 64). Therefore it is bounded and attains its bounds (§ 58·3, 58·4). Thus if M be the upper bound, there exists a point, c, of $[a, b]$ such that

$$f(c)=M.$$

From (i) and (ii), we see that the upper bound is not attained at the end points a and b so that c is an interior point of $[a, b]$.

If $f'(c)$ be positive, then there exists an interval $]c, c+\eta]$, $(\eta > 0)$ such that for every point x of this interval $f(x) > f(c)=M$ and this is a contradiction ;

If $f'(c)$ be negative, then there exists an interval $[c-\eta, c[$, $(\eta > 0)$ such that for every point x of this interval $f(x) > f(c)=M$ and this is, again, a contradiction.

Hence

$$f'(c)=0$$

Cor. If $f(x)$ is derivable in a closed interval $[a, b]$ and $f'(a)\neq f'(b)$ and k is any number lying between $f'(a)$ and $f'(b)$, then there exists at least one point, c, of the interval such that

$$f'(c)=k.$$

We write

$$\varphi(x)=f(x)-kx.$$

The function $\varphi(x)$ is derivable in $[a, b]$ and

$$\varphi'(a)=f'(a)-k,$$
$$\varphi'(b)=f'(b)-k$$

are of opposite signs. Therefore there exists at least one point, c, of $[a, b]$ such that $\varphi'(c)=0$, *i.e.*, $f'(c)-k=0$.

Note. We have seen in § 58·1, that if $\varphi(a)$, $\varphi(b)$ are of opposite signs, then $\varphi(x)$ vanishes for at least one value, c, in $[a, b]$ if $\varphi(x)$ is continuous in $[a, b]$. In this context, the importance of the Darboux's theorem above lies in the fact that if $\varphi(x)$ *is the derivative of a function,* then the conclusion of the vanishing of $\varphi(x)$ remains valid even when $\varphi(x)$ *is not continuous.*

71. Rolle's theorem: *If a function $f(x)$ is*

(i) *continuous in a closed interval $[a, b]$;*

(ii) *derivable in the open interval $]a, b[$;*

and (iii) $f(a)=f(b)$,

then there exists at least one point, c, of the open interval $]a, b[$ such that

$$\mathbf{f'(c)=0.}$$

As $f(x)$ is continuous in $[a, b]$, it is bounded and attains its bounds, so that if M, m be the bounds, there exist points c, d, such that

$$f(c)=M, \ f(d)=m.$$

Now,

$$\text{either } M=m \text{ or } M\neq m.$$

If $M=m$, the function $f(x)$ is clearly constant throughout $[a, b]$ and its derivative $f'(x)$, therefore, is equal to 0 for every value of x in $[a, b]$. Hence the theorem is proved for this case.

If $M \neq m$, then at least one of them must be different from the equal values $f(a), f(b)$. Let $M = f(c)$ be different from them. The number, c, being different from a and b belongs to the open interval $]a, b[$.

The function $f(x)$ which is derivable in the open interval $]a, b[$ is, in particular, derivable at $x = c$, *i.e.*, $f'(c)$ exists.

If $f'(c)$ be positive, then there exists an interval $]c, c+\delta]$ such that for every point x of this interval $f(x) > f(c) = M$ and this is a contradiction.

If $f'(c)$ be negative, then there exists an interval $[c-\delta, c[$ such that for every point x of this interval $f(x) > f(c) = M$ and this is, again, a contradiction.

Hence

$$f'(c) = 0.$$

[The vanishing of $f'(c)$ may also be shown as follows :

We have $$Rf'(c) = Lf'(c) = f'(c).$$

Also $$f(c+h) - f(c) \leqslant 0,$$

for every point $(c+h)$ of $[a, b]$.

$$\therefore \quad [f(c+h) - f(c)]/h \leqslant 0, \text{ when } h > 0 \qquad \dots \text{(I)}$$

and $$[f(c+h) - f(c)]/h \geqslant 0, \text{ when } h < 0. \qquad \dots \text{(II)}$$

From I, we have, $Rf'(c) \leqslant 0$, *i.e.*, $f'(c) \leqslant 0$.

From II, we have $Lf'(c) \geqslant 0$, *i.e.*, $f'(c) \geqslant 0$.

Hence $$f'(c) = 0.]$$

72. Lagrange's mean value theorem. *If a function* $f(x)$ *is*

(*i*) *continuous in a closed interval* $[a, b]$;

and (*ii*) *derivable in the open interval* $]a, b[$;

then there exists at least one point, c, of the open interval $]a, b[$ *such that*

$$\frac{\mathbf{f(b) - f(a)}}{\mathbf{b - a}} = \mathbf{f'(c)}.$$

Let a function $\varphi(x)$ be defined by

$$\varphi(x) = f(x) + Ax,$$

where A is a constant to be determined such that

$$\varphi(a) = \varphi(b).$$

This requires

$$f(a) + Aa = f(b) + Ab.$$

$$\therefore \quad A = -[f(b) - f(a)]/(b-a).$$

The function $\varphi(x)$ is continuous in $[a, b]$, derivable in $]a, b[$, and $\varphi(a) = \varphi(b)$. Hence, by Rolle's theorem, there exists at least one point, c, of $]a, b[$ such that $\varphi'(c) = 0$.

But

$$\varphi'(x)=f'(x)+A.$$

$\therefore$
$$0=\varphi'(c)=f'(c)+A,$$

or
$$f'(c)=-A=\frac{f(b)-f(a)}{b-a}.$$

Another form of statement. If a function $f(x)$ is, (*i*) continuous in $[a, a+h]$. (*ii*) derivable in $]a, a+h[$, then there exists at least one number, θ, between 0 and 1 such that

$$f(a+h)-f(a)=hf'(a+\theta h).$$

Note. The reader should note that the conditions as stated for the validity of the conclusion of Lagrange's theorem are only sufficient but not necessary. (See Example 1 at the end of this Chapter).

73. Theorem. *If* $f(x)$ *is continuous at a point c, and*

$$\lim_{x\to c} f'(x)=l,$$

then

$$f'(c)=l.$$

The condition that $f'(x)\to l$ as $x\to(c+0)$ implies that there exists an interval $]c, c+h]$, $(h>0)$ for every point x of which $f'(x)$ exists and therefore, $f(x)$ is continuous. Since $f(x)$ is given to be continuous at c also, we see that $f(x)$ is continuous in the closed interval $[c, c+h]$. If, x, be any point of this interval, we have

$$f(x)-f(c)=(x-c)f'(\xi), \qquad c<\xi<x$$

or
$$\frac{f(x)-f(c)}{x-c}=f'(\xi)$$

Let $x\to(c+0)$. Then we have

$$Rf'(c)=\lim_{x\to(c+0)} f'(\xi)=\lim_{x\to(c+0)} f'(x)=l.$$

It may similarly be shown that

$$Lf'(c)=l.$$

Thus $f'(c)$ exists and is equal to l.

Note. The theorem above is equivalent to saying that a derived function $f'(x)$ cannot have discontinuity of the first kind.

74. Some elementary deductions from the mean value theorem. Meaning of the constancy of the sign of the derivative in an interval. It will be assumed that the function $f(x)$ is continuous in the closed interval $[a, b]$ and derivable in the open interval $]a, b[$ so that the mean value theorem is applicable to the interval $[a, b]$ and, therefore, also to any sub-interval thereof.

74·1. Theorem. *If* $f'(x)=0$ *for every point* x *of* $]a, b[$, *then* $f(x)$ *is constant in* $[a, b]$.

Let x be any number such that

$$a<x\leqslant b.$$

Applying the mean value theorem to $[a, x]$, we get

$$f(x)-f(a)=(x-a)f'(\xi)=0. \qquad \text{where } a<\xi<x$$

or

$$f(x)=f(a).$$

Hence the result.

Cor. 1. If $f'(x)=k$, for every point x of $]a, b[$; k being a constant, then $f(x)$ is of the form $kx+l$, where l is a constant.

If x be any number such that $a<x\leqslant b$, then we have

$$f(x)-f(a)=(x-a)\,f'(\xi), \text{ where } a<\xi<x$$
$$=k(x-a)$$

or

$$f(x)=kx+l, \text{ where } l=f(a)-ak.$$

As $f(x)$ is continuous at a,

$$f(a)=\lim_{x\to a} f(x)=ka+l,$$

so that the result is true for $x=a$ also.

Cor. 2. If two functions $f(x)$, $F(x)$ are (*i*) continuous in $[a, b]$ (*ii*) derivable in $]a, b[$ and (*iii*) $f'(x)=F'(x)$ in $]a, b[$; then $f(x)$ and $F(x)$ differ by a constant.

Let

$$\varphi(x)=f(x)-F(x).$$

Then

$$\varphi'(x)=f'(x)-F'(x)=0.$$

$\therefore$ $$\varphi(x)=\text{a constant}.$$

74·2. Derivative constantly positive. Theorem. *If $f'(x)>0$ for every point x of $]a, b[$, then $f(x)$ is strictly increasing in $[a, b]$.*

Let x_1, x_2 be any two numbers such that

$$a\leqslant x_1<x_2\leqslant b.$$

We have

$$f(x_2)-f(x_1)=(x_2-x_1)\,f'(\xi)>0. \qquad x_1<\xi<x_2$$

$\therefore$ $$f(x_2)>f(x_1).$$

Hence the result.

Cor. If $f'(x)\geqslant 0$ in $]a, b[$ and does not vanish throughout any sub-interval of $]a, b[$, then $f(x)$ is *strictly* increasing in $[a, b]$.

If x_1, x_2 be *any* two numbers such that

$$a\leqslant x_1<x_2\leqslant b,$$

we have

$$f(x_2)-f(x_1)=(x_2-x_1)\,f'(\xi)\geqslant 0.$$

$\therefore$ $$f(x_2)\geqslant f(x_1),$$

so that $f(x)$ is a *monotonically* increasing function. We have now to show that no two values of the function can be equal. If possible, let

$$f(\alpha)=f(\beta),$$

where

$$a \leqslant \alpha < \beta \leqslant b.$$

For any x in $[\alpha, \beta]$, we have

$$f(\alpha) \leqslant f(x) \leqslant f(\beta)=f(\alpha), \text{ or } f(x)=f(\alpha)$$

i.e., $f(x)$ remains constant in $[\alpha, \beta]$. Therefore $f'(x)$ vanishes throughout $[\alpha, \beta]$ and this is a contradiction.

74·3. Derivative constantly negative. Theorem. *If* $f'(x)<0$ *for every point* x *of* $]a, b[$, *or if* $f'(x) \leq 0$ *in* $]a, b[$ *and does not vanish in any sub-interval of* $]a, b[$, *then* $f(x)$ *is strictly decreasing in* $[a, b]$.

The proof is similar to that of the last case.

75. Cauchy's mean value theorem. *If two functions* $f(x)$ *and* $F(x)$ *are*

(*i*) *continuous in a closed interval* $[a, b]$;

(*ii*) *derivable in the open interval* $]a, b[$;

and (*iii*) $F'(x)\neq 0$ *for any point* x *in the open interval* $]a, b[$,

then there exists at least one point, c, *of the open interval* $]a, b[$ *such that*

$$\frac{\mathbf{f(b)-f(a)}}{\mathbf{F(b)-F(a)}}=\frac{\mathbf{f'(c)}}{\mathbf{F'(c)}}.$$

Let a function $\varphi(x)$ be defined by

$$\varphi(x)=f(x)+AF(x),$$

where, A, is a constant to be determined such that

$$\varphi(a)=\varphi(b).$$

This requires

$$[F(b)-F(a)]A=-[f(b)-f(a)]. \qquad \ldots(1)$$

Now, $[F(b)-F(a)]\neq 0$, for, if it were 0, then $F(x)$ would satisfy all the conditions of Rolle's theorem, and its derivative would, therefore, vanish at least once in $]a, b[$ and the condition (*iii*) would be contradicted. On this account, we have from (1),

$$A=-[f(b)-f(a)]/[F(b)-F(a)].$$

The function $\varphi(x)$ is continuous in $[a, b]$, derivable in $]a, b[$, and $\varphi(a)=\varphi(b)$. Hence, by Rolle's theorem, there exists at least one point c, of $]a, b[$ such that $\varphi'(c)=0$.

But

$$\varphi'(x)=f'(x)+AF'(x).$$

$$\therefore \qquad 0=\varphi'(c)=f''(c)+AF''(c),$$

or

$$\frac{f'(c)}{F'(c)}=-A=\frac{f(b)-f(a)}{F(b)-F(a)}, \qquad \because F'(c)\neq 0.$$

Another form of statement. If two functions $f(x)$, $F(x)$ are continuous in $[a, a+h]$, derivable in $]a, a+h[$ and $F(x)\neq 0$ in $]a, a+h[$, then there exists at least one number, θ, between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{F(a+h)-F(a)}=\frac{f'(a+\theta h)}{F'(a+\theta h)}$$

Note. Taking $F(x)=x$, we may see that Lagrange's mean value theorem is only a particular case of Cauchy's theorem.

76. Higher derivatives. Let $f(x)$ be derivable, *i.e.*, let $f'(x)$ exist in a certain neighbourhood of c. This implies that $f(x)$ is defined and continuous in a neighbourhood of c. If the function $f'(x)$ has derivative at c, then this derivative is called the second derivative of $f(x)$ at c, and is denoted by $f''(c)$. In this case $f'(x)$ is necessarily continuous at c.

In general, if $f^{n-1}(x)$ exists in a neighbourhood of c, then the derivative of $f^{n-1}(x)$ at c, in case it exists, is called the nth derivative of $f(x)$ at c and is denoted by $f^n(c)$.

77. Generalised Mean Value Theorem: Taylor's theorem. *If a function $f(x)$ is such that*

(*ii*) *the $(n-1)$th derivative $f^{n-1}(x)$ is continuous in a closed interval $[a, a+h]$.*

(*ii*) *the nth derivative $f^n(x)$ exists in the open interval $]a, a+h[$,*

and (*iii*) *p is any given positive integer,*

then there exists at least one number, θ, between 0 and 1 such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta h). \qquad \ldots(1)$$

The condition (*i*) implies the continuity of

$$f(x), f'(x), f''(x), \ldots\ldots, f^{n-2}(x) \text{ in } [a, a+h]$$

Let a function $\varphi(x)$ be defined by

$$\varphi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\ldots\ldots\ldots\ldots\ldots+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+A(a+h-x)^p,$$

where A is constant to be determined such that

$$\varphi(a)=\varphi(a+h).$$

Thus A is given by

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p \quad \dots (2)$$

The function $\varphi(x)$ is continuous in $[a, a+h]$, derivable in $]a, a+h[$ and $\varphi(a)=\varphi(a+h)$. Hence, by Rolle's theorem, there exists atleast one number, θ, between 0 and 1 such that

$$\varphi'(a+\theta h)=0.$$

But

$$\varphi'(x)=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-pA(a+h-x)^{p-1}.$$

$$\therefore \quad 0=\varphi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)-pA(1-\theta)^{p-1}h^{p-1}$$

or $$A=\frac{h^{n-p}(1-\theta)^{n-p}}{p.(n-1)!}.f^n(a+\theta h), \text{ for } (1-\theta)\neq 0 \text{ and } h\neq 0.$$

Substituting the value of A in (2), we get the required result (1).

(*i*) *Remainder after n terms*. The term

$$R_n=\frac{h^n(1-\theta)^{n-p}}{p.(n-1)!}f^n(a+\theta h),$$

is known as the Taylor's remainder R_n after n terms and is due to *Schlomilch and Roche*.

(*ii*) Putting $p=1$, we obtain

$$R_n=\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h),$$

which form of remainder is due to *Cauchy*.

(*iii*) Putting $p=n$, we obtain

$$R_n=\frac{h^n}{n!}f^n(a+\theta h),$$

which is due to *Lagrange*.

Cor. 1. Let x be *any* point of the interval $[a, a+h]$.

Let $f(x)$ satisfy the conditions of Taylor's theorem for $[a, a+h]$. Then it satisfies the conditions for $[a, x]$ also.

Changing $a+h$ to x *i.e.*, h to $x-a$, in (1), we obtain

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots\dots\dots+$$
$$\frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{(x-a)^n(1-\theta)^{n-p}}{p.(n-1)!}f^n[a+\theta(x-a)],$$
$$0<\theta<1.$$

This result holds for every point x of $[a, a+h]$. Of course, θ, may be different for different points x.

Cor. 2. Maclaurin's theorem. Putting $a=0$, we see that if x is any point of the interval $[0, h]$, then

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\frac{x^3}{3\,!}f'''(0)+\dots\dots\dots+\frac{x^{n-1}}{(n-1)\,!}f^{n-1}(0)+\frac{x^n(1-\theta)^{n-p}}{p(n-1)\,!}f^n(\theta x)$$

which holds when

(*i*) $f^{n-1}(x)$ is continuous in $[0, h]$,

and (*ii*) $f^n(x)$ exists in $]0, h[$.

Putting $n=1$ and $n=p$, respectively in the Schlomilch form of remainder

$$\frac{x^n(1-\theta)^{n-p}f^n(\theta x)}{p.(n-1)\,!}$$

we see that Cauchy's and Lagrange's forms are respectively

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)\,!}f^n(\theta x) \text{ and } \frac{x^n}{n\,!}f^n(\theta x).$$

78. Taylor's infinite series. Suppose that *a given function $f(x)$ possesses a continuous derivative of* **every** *order in* $[a, a+h]$.

Then to every positive integer n, however large it may be, there corresponds an equality of the form

$$f(a+h)=f(a)+hf'(a)+\dots\dots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+R_n,$$

when R_n denotes Taylor's remainder after n terms.

We write

$$S_n=f(a)+hf'(a)+\dots\dots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a),$$

so that

$$f(a+h)=S_n+R_n.$$

If $R_n \to 0$, as $n \to \infty$, we have

$$\lim S_n=f(a+h),$$

so that the infinite series

$$f(a)+hf'(a)+\dots\dots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+\frac{h^n}{n\,!}f^n(a)+\dots\dots$$

converges and its sum is equal to $f(a+h)$.

Thus we have proved that if

(*i*) *$f(x)$ possesses continuous derivatives of every order in* $[a, a+h]$ *and* (*ii*) *Taylor's remainder $R_n \to 0$ as $n \to \infty$,*

then

$$f(a+h)=f(a)+hf'(a)+\dots\dots+\frac{h^n}{n\,!}f^n(a)+\dots\dots$$

The infinite series

$$f(a)+hf'(a)+\dots\dots+\frac{h^n}{n!}f^n(a)+\dots\dots$$

is known as *Taylor's series*.

It should be clearly understood that the *mere convergence of this series does not mean that its sum is equal to* $f(a+h)$. (See § 86)

78·1. Maclaurin's infinite series. From above we deduce that *if* $f(x)$ *possesses continuous derivatives of every order in* $[0, x]$ *and* $R_n \to 0$ *as* $n \to \infty$, *then*

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots\dots+\frac{x^n}{n!}f^n(0)+\dots\dots$$

Note. The student should note that there is no fundamental distinction between the Taylor's and Maclaurin's infinite series each of which seeks to express the value of a function at any point in terms of the various derivatives of the function at any other point and the distance between the two points.

In this connection it is also interesting to see 'a priori' why the Taylor's or Maclaurin's expansions are not valid for any arbitrary function. Clearly the system of derivatives of a function at any point takes account of the values of the function in a neighbourhood of the point only and accordingly the value of any arbitrary function at a point cannot possibly be given by those in a neighbourhood of another point. The conditions imposed on the function for the validity of the Taylor's or Maclaurin's expansions provide, so to say for the *linking up* of the values of the function at the two points.

79. Maclaurin's expansions of e^x, log $(1+x)$, $(1+x)^m$.

79·1. *Let* $f(x)=e^x$. We know that, e^x, possesses continuous derivatives of every order for every value of x. In fact, $f^n(x)=e^x$.

If R_n denotes Lagrange's remainder, we have

$$R_n=\frac{x^n}{n!}e^{\theta x}.$$

If x be positive, then $e^{\theta x}<e^x$; and if x be negative, $e^{\theta x}<1$. Thus

$$\left|R_n\right|=\left|\frac{x^n}{n!}\right|.e^{\theta x}<\begin{cases}|x^n/n!|.e^x, \text{ if } x>0.\\ |x^n/n!|, \text{ if } x<0.\end{cases}$$

Since $|x^x/n!| \to 0$ as $n\to\infty$, we see that $R_n\to 0$ as $n\to\infty$, for every value of x.

The validity of Maclaurin's infinite expansion for e^x has thus been established for every value of x. Making substitutions, we obtain

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots+\frac{x^n}{n!}+\dots\dots$$

79·2. *Let* $f(x)=\log(1+x)$.

We know that $\log(1+x)$ possesses continuous derivatives of every order for every value of x such that $(1+x)$ is positive, *i.e.*,

$x > -1$. In fact, when $x > -1$, we have

$$f^n(x)=\frac{(-1)^{n-1}(n-1)\,!}{(1+x)^n}.$$

If R_n denotes Lagrange's form of remainder, we have

$$R_n=\frac{x^n}{n\,!}f^n(\theta x)=(-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

Let $0 \leqslant x \leqslant 1$ so that $x/(1+\theta x)$ is positive and < 1. We have

$$|\,R_n\,| < \frac{1}{n}.$$

Therefore $R_n \to 0$, when $0 \leqslant x \leqslant 1$.

Let $-1 < x < 0$. In this case $x/(1+\theta x)$ may not be numerically less than unity so that we fail to draw any conclusion from Lagrange's R_n in this case. Taking Cauchy's form of remainder, we have

$$R_n=\frac{x^n}{(n-1)\,!}(1-\theta)^{n-1}f^n(\theta x)=(-1)^{n-1}x^n.\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.\frac{1}{1+\theta x}.$$

Since $|\,x\,| < 1$, $[(1-\theta)/(1+\theta x)]$ is positive and less than unity and hence

$$0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1.$$

Also
$$\frac{1}{1+\theta x} < \frac{1}{1-|\,x\,|}.$$

Finally
$$x^n \to 0 \text{ as } n \to \infty.$$

Thus $R_n \to 0$ as $n \to \infty$.

We have thus established that the Maclaurin's expansion for $\log\,(1+x)$ is valid when $-1 < x \leqslant 1$. Making substitutions, we obtain

$$(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots$$

79·3. *Let* $f(x)=(1+x)^m$.

If m is a positive integer, then we know that $f^{m+1}(x)$ and the following successive derivatives are identically zero for every value of x, so that we may easily show that, *when m is positive integer*

$$(1+x)^m=1+mx+\frac{m(m-1)}{2\,!}x^2+\;\ldots..+x^m,$$

for every value of x.

If m is any real number, not necessarily a positive integer, then we know that $(1+x)^m$ possesses continuous derivative of every order for values of x such that $(1+x)$ is positive, *i.e.*, $x > -1$.

We have

$$f^n(x)=m(m-1)\ldots(m-n+1)\,(1+x)^{m-n}.$$

If R_n denotes Cauchy's form of remainder, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}$$

If $|x| < 1$.

$$\frac{m(m-1)\ldots(m-n+1)}{(n-1)!} x^n \to 0,$$

$$0 < \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} < 1,$$

$$(1+\theta x)^{m-1} < \begin{cases}(1+1)^{m-1} = 2^{m-1}, \text{ if } (m-1) \text{ is positive.} \\ (1-|x|)^{m-1}, \text{ if } (m-1) \text{ is negative.}\end{cases}$$

Thus $R_n \to 0$ as $n \to \infty$, when $|x| < 1$.

Maclaurin's infinite expansion for $(1+x)^m$ being thus valid, we see that

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots\ldots,$$

when $|x| < 1$.

Note. It is easy to see that we cannot directly prove that the *Lagrange's form of* $R_n \to 0$, when $-1 < x < 0$.

80. Young's form of Taylor's theorem. *If a function $f(x)$ be such that $f^n(a)$ exists and M be defined as a function of h, by the equation*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} M,$$

then

$$M \to f^n(a) \text{ as } h \to 0.$$

The fact that $f^n(a)$ exists implies that $f(x)$, $f'(x)$, $f''(x)$,...$f^{n-1}(x)$ all exist in a certain neighbourhood $[a-\delta, a+\delta]$ of a.

The result holds good whether $f^n(a)$ be finite or infinite.

Case I. *Let $f^n(a)$ be finite.*

Let ε be any positive number.

Firstly we take $h \geqslant 0$. We define a function $\varphi(h)$ of h as follows :—

$$\varphi(h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!}[f^n(a) + \varepsilon] - f(a+h).$$

We easily see that

$$\varphi(0) = 0,\ \varphi'(0) = 0, \ldots\ldots, \varphi^{n-1}(0) = 0 \text{ and } \varphi^n(0) = \varepsilon > 0.$$

Since $\varphi^n(0)$ is positive and $\varphi^{n-1}(0) = 0$, we see that there exists an interval $]0, \delta_1]$ such that for every point h of this interval $\varphi^{n-1}(h)$ is positive (§ 69, page 114).

Since $\varphi^{n-1}(h)$ is positive in $]0, \delta_1]$ and $\varphi^{n-2}(0)=0$, we see that $\varphi^{n-2}(h)$ is positive in $]0, \delta_1]$ (§ 74·2, page 119).

Now successively applying the theorem of §74·2, we deduce that

$$\varphi(h) \text{ is positive in } [0, \delta_1].$$

Thus we see that there exists a positive number δ_1 such that for

$$0 < h < \delta_1$$

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}[f^n(a)+\varepsilon]- f(a+h) > 0.$$

Similarly we may prove that there exists a positive number δ_2 such that for

$$0 < h < \delta_2$$

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}[f^n(a)-\varepsilon]- f(a+h) < 0.$$

But, as given,

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}M-f(a+h)=0.$$

Let $\eta=\min(\delta_1, \delta_2)$.

From above we deduce that corresponding to every positive number ε there exists a positive number η, such that

$$f^n(a)-\varepsilon < M < f^n(a)+\varepsilon, \text{ i.e., } |M-f^n(a)| < \varepsilon, \text{ when } 0 < h < \eta.$$

$$\therefore \lim_{h \to (0+0)} M = f^n(a).$$

Taking negative values of h, we may similarly show that

$$\lim_{h \to (0-0)} M = f^n(a).$$

Hence

$$M \to f^n(a) \text{ as } h \to 0.$$

Case II. *Let $f^n(a)$ be infinite.*

Let $\triangle$ be any positive number, however large. Firstly we take $h > 0$, and define a function $\varphi(h)$ as follows :—

$$\varphi(h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}\triangle-f(a+h).$$

We easily see that

$$\varphi(0)=0,\ \varphi'(0)=0,\dots\dots\dots,\varphi^{n-1}(0)=0.$$

Also since $\varphi^{n-1}(h)=f^{n-1}(a)+h\,\Delta-f^{n-1}(a+h),$

$$=h\left[\Delta-\frac{f^{n-1}(a+h)-f^{n-1}(a)}{h}\right]$$

and $[f^{n-1}(a+h)-f^{n-1}(a)]/h\to\infty$ as $h\to 0$, we see that there exists a positive number δ such that for $0<h\leqslant\delta$, $\varphi^{n-1}(h)$ is negative.

Now, proceeding as in case I, we prove that there exists a positive number δ_1 such that for every point h of $[0, \delta_1]$,

$$\varphi(h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+\frac{h^n}{n\,!}\Delta-f(a+h)$$

is less than zero.

Also, as given,

$$f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(a)+\frac{h^n}{n\,!}M-f(a+h)=0.$$

From above we deduce that corresponding to every positive number Δ, however large, there exists a positive number δ_1 such that for every point h of $]0, \delta_1]$.

$$M>\Delta \text{ i.e., } \lim_{h\to(0+0)} M=\infty.$$

It may similarly be shown that

$$\lim_{h\to(0-0)} M=\infty.$$

Hence

$$M\to\infty \text{ as } h\to 0.$$

81. Applications of Taylor's theorem.

Applications of Taylor's theorem in a finite form to the problem of *Extreme* values of a function and to the problem of the evaluation of certain special kinds of limits popularly known as *Indeterminate forms* is given in the following three sections.

82. Extreme values of a function. Maxima and minima.

If c, be any *interior* point of the interval of definition $[a, b]$ of a function $f(x)$, then

(*i*) $f(c)$ is said to be a *maximum* value of $f(x)$, if there exists some neighbourhood $[c-\delta, c+\delta]$ of, c, such that for every point x of this neighbourhood, other than c,

$$f(c)>f(x)\;;$$

(*ii*) $f(c)$ is said to be a *minimum* value of $f(x)$, if for every point x, other than c, of some neighbourhood $[c-\delta, c+\delta]$ of c,

$$f(c)<f(x)\;;$$

(*iii*) $f(c)$ is said to be an *extreme* value of $f(x)$, if it is either a maximum or a minimum value.

For $f(c)$ to be an extreme value,

$$f(c)-f(x)$$

must keep the same sign for every point x, other than c, of some neighbourhood $[c-\delta, c+\delta]$ of c.

82·1 Theorem. *If $f(c)$ be an extreme value of a function $f(x)$, then $f'(c)$, in case it exists, is zero.*

If $f'(c)$ be not 0, then in every neighbourhood of c there exist points x for which $f(x) > f(c)$ and points x for which $f(x) < f(c)$, (§69, page 114) so that $f(c)$ cannot be an extreme value.

Note 1. The theorem may also be stated a little differently as follows :—

The *necessary condition for $f(c)$ to be an extreme value is that $f'(c)=0$, in case it exists.*

To show that this condition is only necessary and not sufficient, we consider the function $f(x)=x^3$ when $x=0$.

Clearly

$$f'(0)=0.$$

Also, when $x>0$, $f(x)>f(0)$, and when $x<0$, $f(x)<f(0)$, so that $f(0)$ is not an extreme value even though $f'(0)=0$.

2. If $f(x)=|x|$, then clearly $f(0)$ is a minimum value and $f'(0)$ does not exist (Remarks after §64, page 110).

This example shows that $f(c)$ may be an extreme value even when $f'(c)$ does not exist.

82·2. Sufficient Criteria for extreme values. *Let, c, be interior point of the interval of definition $[a, b]$ of a function $f(x)$. Let*

(i) $f^n(c)$ exist and be not zero,

and (ii) $f'(c)=f''(c)=f'''(c)=\ldots\ldots=f^{n-1}(c)=0$;

then $f(c)$ is not an extreme value if n is odd ; and if n be even, $f(c)$ is a maximum or a minimum value according as $f^n(c)$ is negative or positive.

The condition (*i*) implies that $f'(x)$, $f''(x)\ldots\ldots\ldots\ldots$, $f^{n-1}(x)$ all exist in a certain neighbourhood, $[c-\delta_1, c+\delta_1]$, of c. ... I

As $f^n(c)$ exist and $\neq 0$, there exists a neighbourhood $[c-\delta, c+\delta]$, of c, $(0<\delta<\delta_1)$ such that

$$\left.\begin{array}{l} f^{n-1}(x) < f^{n-1}(c)=0, \text{ when } c-\delta \leqslant x < c, \\ f^{n-1}(x) > f^{n-1}(c)=0, \text{ when } c < x \leqslant c+\delta, \end{array}\right\} \text{(§69, page 114)}\ldots\text{II}$$

in case

$$f^n(c) \text{ is positive ;}$$

and

$$\left.\begin{array}{l} f^{n-1}(x) > f^{n-1}(c)=0, \text{ when } c-\delta \leqslant x < c \\ f^{n-1}(x) < f^{n-1}(c)=0, \text{ when } c < x \leqslant c+\delta \end{array}\right\} \text{(§69, page 114)}\ \ldots\text{III}$$

in case

$$f^n(c) \text{ is negative.}$$

Because of I, we have by Taylor's theorem, when $|h| \leqslant \delta$,

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(c+\theta h),$$

which, by virtue of (*ii*), gives

$$f(c+h)-f(c)=\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(c+\theta h), \qquad \ldots \text{ IV}$$

where, $c+\theta h$, belongs to the interval $[c-\delta, c+\delta]$.

Let n be even. From II and IV, we deduce that if $f^n(c)$ be > 0, then for every point $x=c+h$ of $[c-\delta, c+\delta]$, other than c.

$$f(c+h) > f(c),$$

i.e., $f(c)$ is a minimum.

From III and IV, it may similarly be shown that $f(c)$ is a maximum if $f^n(c)$ be < 0.

Let n be odd. From II and IV, we deduce that if $f^n(c) > 0$, then

$$f(c+h) > f(c), \text{ when } c < x=c+h \leqslant c+\delta,$$

and

$$f(c+h) < f(c), \text{ when } c-\delta \leqslant x=c+h < c$$

so that $f(c)$ is no extreme value.

It may similarly be shown that $f(c)$ is not an extreme value when $f^n(c) < 0$.

Another Proof. We now give another proof which is dependent upon the Young's form of Taylor's theorem.

We have

$$f(c+h)=f(c)+hf'(c)+\frac{h^2}{2\,!}f''(c)+\dots\dots+\frac{h^{n-1}}{(n-1)\,!}f^{n-1}(c)+\frac{h^n}{n\,!}M,$$

where

$$M \to f^n(c) \text{ as } h \to 0.$$

With the help of (*ii*), it becomes

$$f(c+h)-f(c)=\frac{h^n}{n\,!}M. \qquad \dots \text{ V}$$

Since $M \to f^n(c)$ as $h \to 0$, there exist a positive number δ, such that for $0 < |\,h\,| \leqslant \delta$, M has the same sign as $f^n(c)$.

From V, we now deduce that

when n is even, $f(c+h)-f(c)$ has the same sign as $f^n(c)$ for $0 < |\,h\,| \leqslant \delta$, so that $f(c)$ is a maximum or minimum according as $f^n(c)$ is negative or positive, and

when n is odd, $f(c+h)-f(c)$ changes sign with the change in the sign of h so that $f(c)$ is not an extreme value.

Ex. If, c be an interior point of the interval of definition of a function $f(x)$, then

$f(c)$ is a maximum value of $f(x)$, if

$$Rf'(c)=-\infty,\ Lf'(c)=\infty,$$

and $f(c)$ is a minimum value of $f(x)$, if

$$Rf'(c)=\infty,\ Lf'(c)=-\infty.$$

83. The Indeterminate form (o/o.)

83·1. Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

$$(i) \qquad \lim_{x \to c} f(x)=0, \qquad \lim_{x \to c} g(x)=0, \qquad \dots (1)$$

and (ii) $f'(c)$, $g'(c)$ *both exist and* $g'(c)\neq 0$;

then $$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}.$$

Since $f(x)$, $g(x)$ are derivable at c, therefore they are continuous at c and accordingly, by (1), $f(c)=g(c)=0$.

We have

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}\frac{f(c+h)}{h},$$

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=\lim_{h\to 0}\frac{g(c+h)}{h}.$$

$$\therefore\quad \frac{f'(c)}{g'(c)}=\lim_{h\to 0}\frac{[f(c+h)/h]}{[g(c+h)/h]}=\lim_{h\to 0}\frac{f(c+h)}{g(c+h)}=\lim_{x\to c}\frac{f(x)}{g(x)}.$$

Note. This result may also be stated as follows :—

If $f(c)=g(c)=0$, and $f'(c)$, $g'(c)$ exist but $g'(c)\neq 0$, then $\lim\,[f(x)/g(x)]=f'(c)/g'(c)$ when, $x\to c$.

83·2. Theorem. *Let* $f(x)$, $g(x)$ *be two functions such that*

(i) $\lim\limits_{x\to c} f(x)=0$, $\lim\limits_{x\to c} g(x)=0$

and (ii) $\lim\,[f'(x)/g'(x)]=l$ *when* $x\to c$,

then $\lim\,[f(x)/g(x)]=l$ *when* $x\to c$.

The condition (*ii*) implies that $f'(x)$ and $g'(x)$ exist and $g'(x)\neq 0$ at every point x, other than c, of a certain neighbourhood $[c-\delta, c+\delta]$ of c.

We *suppose* that $f(c)=g(c)=0$, for, this change in the definition of $f(x)$ and $g(x)$ influences neither the hypothesis nor the conclusion of the theorem.

If x be any point of $[c-\delta, c+\delta]$, we have, by Cauchy's theorem.

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(c)}{g(x)-g(c)}=\frac{f'(\xi)}{g'(\xi)},$$

when ξ lies between c and x and also depends upon x.

By virtue of (*ii*), $f'(\xi)/g'(\xi)\to l$ as $x\to c$.

$\therefore$ $f(x)/g(x)\to l$ as $x\to c$.

Note. The theorem may also be stated in another form as follows :

If $$\lim_{x\to c} f(x)=0=\lim_{x\to c} g(x),$$

then $$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)},$$

provided that the limit on the right exists.

It will later on be seen by means of an example that $\lim\,[f(x)/g(x)]$ may exist without $\lim\,[f'(x)/g'(x)]$ existing.

Note. The reader will find it useful to compare the hypotheses and the conclusions of the two preceding theorems. In § 83·1, the derivability of $f(x)$ and $g(x)$ is assumed at $x=c$ *only* whereas in § 83·2, while assuming the derivability of $f(x)$, $g(x)$ in the neighbourhood of $x=c$, it is exactly at $x=c$ where we have not needed it. (For illustrations of each of these theorems, see examples at the end of Chap. VII).

83·3. Theorem. *Let $f(x)$, $g(x)$ be two functions such that, when $x \to c$,*

$$(i)\quad \begin{cases} \lim f(x)=\lim f'(x)=\ldots\ldots=\lim f^{n-1}(x)=0, \\ \lim g(x)=\lim g'(x)=\ldots\ldots=\lim g^{n-1}(x)=0. \end{cases}$$

and $\quad (ii)\quad \lim\ [f^n(x)/g^n(x)]=l,$

then

$$\lim\ [f(x)/g(x)]=l, \text{ when } x \to c.$$

Since $\quad \lim f^{n-1}(x)=\lim g^{n-1}(x)=0$ and $\lim\ [f^n(x)/g^n(x)]=l$,

therefore $\quad \lim\ [f^{n-1}(x)/g^{n-1}(x)]=l$, § 83·2 above.

Again, since $\quad \lim f^{n-2}(x)=\lim g^{n-2}(x)=0$

and $\quad \lim\ [f^{n-1}(x)/g^{n-1}(x)]=l$,

therefore $\quad \lim\ [f^{n-2}(x)/g^{n-2}(x)]=l$.

Proceeding in this manner, we finally prove that

$$\lim\ [f(x)/g(x)]=l,$$

when $x \to c$.

83·4. Theorem. *Let $f(x)$, $g(x)$ be two functions such that when $x \to c$,*

$$(i)\quad \begin{cases} \lim f(x)=\lim f'(x)=\ldots\ldots=\lim f^{n-1}(x)=0 \\ \lim g(x)=\lim g'(x)=\ldots\ldots=\lim g^{n-1}(x)=0 \end{cases}$$

and $\quad (ii)\quad f^n(c),\ g^n(c)$ *exist* and $g^n(c)\neq 0$,

then $\quad \lim\ [f(x)/g(x)]=f^n(c)/g^n(c)$, *when* $x \to c$.

By virtue of (*ii*), on applying the theorem of § 83·1, we see that

$$\lim \frac{f^{n-1}(x)}{g^{n-1}(x)}=\frac{f^n(c)}{g^n(c)}.$$

Also from the theorem of § 83·3, changing n to $(n-1)$, we see that

$$\lim \frac{f(x)}{g(x)}=\lim \frac{f^{n-1}(x)}{g^{n-1}(x)}=\frac{f^n(c)}{g^n(c)}.$$

83·5. Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

$$(i)\quad \begin{cases} f(c)=f'(c)=f''(c)=\ldots\ldots=f^{n-1}(c)=0, \\ g(c)=g'(c)=g''(c)=\ldots\ldots=g^{n-1}(c)=0 \end{cases}$$

and $\quad (ii)\quad f^n(c),\ g^n(c)$ *exist but* $g^n(c)\neq 0$,

then $\quad \lim\ [f(x)/g(x)]=f^n(c)/g^n(c)$, *when* $x \to c$.

Employing Young's form of Taylor's theorem, we have

$$f(c+h)=f(c)+hf'(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(c)+\frac{h^n}{n!}M=\frac{h^n}{n!}M,$$

$$g(c+h)=g(c)+hg'(c)+\ldots\ldots+\frac{h^{n-1}}{(n-1)!}g^{n-1}(c)+\frac{h^n}{n!}M'=\frac{h^n}{n!}M'.$$

$$\therefore \qquad \frac{f(c+h)}{g(c+h)}=\frac{M}{M'}.$$

$$\therefore \qquad \lim_{x \to c} \frac{f(x)}{g(x)}=\lim_{h \to 0} \frac{f(c+h)}{g(c+h)}=\lim_{h \to 0} \frac{M}{M'}=\frac{f^n(c)}{g^n(c)}.$$

Note. The theorems of this and the preceding sub-section are really identical even though they appear to have been stated differently.

83·6. Theorem. *Let $f(x)$, $g(x)$ be two functions such that*

$$(i) \qquad \lim_{x \to \infty} f(x)=0, \quad \lim_{x \to \infty} g(x)=0,$$

and
$$(ii) \qquad \lim_{x \to \infty} [f'(x)/g'(x)]=l,$$

then
$$\lim_{x \to \infty} [f(x)/g(x)]=l.$$

We write $x=1/z$ so that $x \to \infty$, when $z \to (0+0)$.

Let $\qquad F(z)=f(1/z),\ G(z)=g(1/z),$

so that

$$\lim_{z \to (0+0)} F(z)=0. \qquad \lim_{z \to (0+0)} G(z)=0. \qquad \text{... (1)}$$

We have

$$F'(z)=-f'\left(\frac{1}{z}\right)\cdot\frac{1}{z^2},\quad G'(z)=-g'\left(\frac{1}{z}\right)\frac{1}{z^2}.$$

$$\therefore \qquad \frac{F'(z)}{G'(z)}=\frac{f'(1/z)}{g'(1/z)},$$

so that, by (*ii*), $F'(z)/G'(z) \to l$, when $z \to (0+0)$. ... (2)

Hence, from (1) and (2),

$$\lim_{z \to (0+0)} [F(z)/G(z)]=l,$$

i.e.,
$$\lim_{x \to \infty} [f(x)/g(x)]=l.$$

84. The indeterminate form (∞/∞).

84·1. Theorem. *If $f(x)$, $g(x)$ be two functions such that when $x \to c$,*

$$(i) \qquad \lim |g(x)| = \infty,$$

and
$$(ii) \qquad \lim \frac{f'(x)}{g'(x)}=0,$$

then
$$\lim \frac{f(x)}{g(x)}=0, \text{ when } x \to c.$$

The condition (*ii*) implies that there exists a neighbourhood $[c-\delta, c+\delta]$ of c such that for every point x, other than c, of this neighbourhood, $f'(x)$, $g'(x)$ exist and $g'(x) \neq 0$.

From Darboux's theorem of § 70, page 115 it follows that $g'(x)$ keeps the same sign, positive or negative for every point x of $]c, c+\delta[$, and the same thing is true for $]c-\delta, c[$ also.

Firstly we consider $]c, c+\delta[$. Let $g'(x)$ remain positive in it.

Let ε by any positive number, however small. Their exists, by virtue of (*ii*), a positive number $\delta_1 < \delta$ such that for every point of $]c, c+\delta_1[$,

$$\left|\frac{f'(x)}{g'(x)}-0\right| < \frac{\varepsilon}{2}, \text{ i.e., } \left|f'(x)\right| < \tfrac{1}{2}\varepsilon \left|g'(x)\right| = \tfrac{1}{2}\varepsilon g'(x),$$

i.e.,
$$-\tfrac{1}{2}\varepsilon g'(x) < f'(x) < \tfrac{1}{2}\varepsilon g'(x).$$

By virtue of the theorem of § 74, page 118, we have

$$-\tfrac{1}{2}\varepsilon[g(c+\delta)-g(x)] < [f(c+\delta)-f(x)] < \tfrac{1}{2}\varepsilon[g(c+\delta)-g(x)]$$

i.e.,
$$|f(c+\delta)-f(x)| < \tfrac{1}{2}\varepsilon\, |g(c+\delta)-g(x)|$$

or
$$|f(x)| - |f(c+\delta)| < \tfrac{1}{2}\varepsilon\, |g(c+\delta)| + \tfrac{1}{2}\varepsilon\, |g(x)|$$

or
$$|f(x)| < \tfrac{1}{2}\varepsilon\, |g(x)| + \tfrac{1}{2}\varepsilon\, |g(c+\delta)| + |f(c+\delta)|$$

or $|f(x)| < \tfrac{1}{2}\varepsilon\, |g(x)| + k$, for any point x of $]c, c+\delta_1[$,

k being free of x,

or
$$\left|\frac{f(x)}{g(x)}\right| < \tfrac{1}{2}\varepsilon + \frac{k}{|g(x)|}.$$

There exists a positive number $\delta_2 < \delta_1$, such that

$$\frac{k}{|g(x)|} < \tfrac{1}{2}\varepsilon,$$

for any point x of $]c, c+\delta_2[$.

Thus we see that corresponding to every positive number ε, there exists a positive number δ_2. such that for every point x of $]c. c+\delta_2[$,

$$\left|\frac{f(x)}{g(x)}\right| < \varepsilon.$$

Hence
$$\lim_{x \to (c+o)} \frac{f(x)}{g(x)} = 0.$$

It may similarly be shown that

$$\lim_{x \to (c-o)} \frac{f(x)}{g(x)} = 0.$$

Thus
$$f(x)/g(x) \to 0 \text{ as } x \to c.$$

84·2. Theorem.

If $f(x)$, $g(x)$ be two functions such that when $x \to c$,

(i) $\lim |g(x)| = \infty$, (ii) $\lim [f'(x)/g'(x)] = l$,

then

$$\lim f(x)/g(x) = l, \text{ when } x \to c.$$

We write
$$\varphi(x) = f(x) - lg(x).$$

We have

$$\lim_{\to c} \frac{\varphi'(x)}{g'(x)} = \lim_{x \to 0}\left[\frac{f'(x)}{g'(x)} - l\right] = 0, \text{ (by } ii)$$

$$\therefore \qquad \lim_{x \to 0} \frac{\varphi(x)}{g(x)} = 0, \ (\S\ 84 \cdot 1)$$

i.e.,
$$\lim_{x \to c} \left[\frac{f(x)}{g(x)} - l \right] = 0$$

or
$$\lim_{x \to c} \frac{f(x)}{\varphi(x)} = l$$

Note. As in § 83·6, it may be shown that the result remains true even when $x \to \infty$.

Note. It should be especially noted that in this preceding theorem nothing whatsoever has been said about the limit of $f(x)$ as $x \to c$ so that the result holds good independently of the behaviour of $f(x)$. In particular, therefore, the result holds good when $f(x) \to \infty$ as $x \to c$; this being the form in which the theorem is usually stated.

85. Two very important special cases of limits.

85·1. Theorem. *To prove that*

$$\lim_{x \to \infty} \left(\frac{x^m}{e^{\alpha x}} \right) = 0$$

where α, m are any positive numbers whatsoever.

Let $\quad f(x) = x^m$, $g(x) = e^{\alpha x}$,
so that $f(x) \to \infty$ and $g(x) \to \infty$, as $x \to \infty$.

We can write

$$m = n + p,$$

where n is some positive integer and p is a number such that $0 \leqslant p < 1$.

It is easy to see that $f'(x)$, $f''(x)$,..., $f^n(x)$ all tend to ∞ and $f^{n+1}(x) \to 0$ as $x \to \infty$.

Also $g'(x)$, $g''(x)$, $g'''(x)$,......all tend to ∞.

Thus $\quad \lim_{x \to \infty} \{ f^{n+1}(x) / g^{n+1}(x) \} = 0$

Therefore from the theorem of § 84·2, we deduce that

$$\lim \left[f(x)/g(x) \right] = 0, \text{ when } x \to \infty.$$

Note. The result of this theorem is roughly expressed by saying that $e^{\alpha x}$ tends to ∞ more rapidly than any positive power of x, when $x \to \infty$.

85·2. Theorem. *To prove that*

$$\lim_{x \to \infty} \frac{(\log x)^m}{x^\alpha} = 0,$$

where α, m are any positive numbers whatsoever.

The independent proof is similar to that of the preceding result. Otherwise if we write $\log x = y$ we see that

$$\lim_{x \to \infty} \frac{(\log x)^m}{x^\alpha} = \lim_{y \to \infty} \frac{y^m}{e^{\alpha y}} = 0$$

Note. This shows that *every positive power of x tends to infinity more rapidly than any positive power of $\log x$ as, $x \to \infty$.*

Cor. $\lim\limits_{x\to(0+0)} [x^{\alpha} (\log x)^{m}]=0$; α, m being any positive numbers.

Putting $x=1/y$, we may easily obtain it.

86. Note on a special function. *The function $f(x)$ defined as*

$$f(x)=e^{-1/x^2} \text{ when } x\neq 0, f(0)=0 ;$$

possesses a remarkable property, viz., that $f^n(0)=0$ for every value of n.

We have

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{e^{-1/x^2}}{x}=\lim_{y\to\infty}\frac{y}{e^{y^2}}=0,$$

$$(y=1/x)$$

$$f'(x)=(2/x^3)\, e^{-1/x^2}, \text{ when } x\neq 0.$$

$$\therefore\ f''(0)=\lim_{x\to 0}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0}\frac{2e^{-1/x^2}}{x^4}$$

$$=\lim_{y\to\infty}\frac{2y^4}{e^{y^2}}=0.$$

$$f''(x)=\left(\frac{4}{x^6}-\frac{6}{x^4}\right)e^{-1/x^2}, \text{ when } x\neq 0,$$

$$\therefore\ f'''(0)=\lim_{x\to 0}\left(\frac{4}{x^6}-\frac{6}{x^4}\right)e^{-1/x^2}=\lim_{y\to\infty}\frac{4y^6-6y^4}{e^{y^2}}=0.$$

If we form the higher derivative for $x\neq 0$, we shall obviously always obtain the product of e^{-1/x^2} and a polynomial in $1/x$. Thus we see that all the higher derivatives will vanish at the point $x=0$.

It is thus seen that this function possesses a continuous derivative for every value of x. Maclaurin's infinite series for this function is

$$0+x.0+\frac{x^2}{2\,!}.0+\frac{x^3}{3\,!}.+\dots\,..+\frac{x^n}{n\,!}.0+\dots\dots$$

which is certainly not equal to $f(x)$ for any value of x, other than 0.

Thus we see that there exist functions which possess continuous derivatives for every value of x and yet cannot be expanded in terms of Maclaurin's series.

Examples

1. *If $a=-1$, $b\geqslant 1$ and $f(x)=1/\mid x\mid$, show that the conditions of Lagrange's mean value theorem are not satisfied but that the conclu sion is true if and only if $b>1+\sqrt{2}$ for the interval $[a, b]$.*

We have

for $x<0$, $\qquad f(x)=1/\mid x\mid=-1/x$ so that $f'(x)=1/x^2$;

for $x > 0$, $\quad f(x)=1/\mid x \mid =1/x$ so that $f'(x)=-1/x^2$;
for $x=0$, $\quad f(x)$ is not derivable.

Thus the conditions of the Lagrange's mean value theorem are not satisfied in respect of the interval $[a, b]$ in as much as $f(x)$ is not derivable at the point, '0', of the open interval $]a, b[$.

Again,

$$\frac{f(b)-f(a)}{b-a}=\frac{1/\mid b \mid -1/\mid a \mid}{b-a}$$

$$=\frac{(1/b)-1}{b+1}=\frac{1-b}{b(b+1)}, \text{ for } b \geqslant 1 \text{ and } a=-1.$$

As $[f(b)-f(a)]/(b-a)$ is negative, it cannot equal the derivative of $f(x)$ for any negative value of x; the derivative being necessarily positive for such values of x. Let, if possible, there exist a positive number, c, such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

This requires

$$-\frac{1}{c^2}=\frac{1-b}{b(b+1)}=-\frac{b-1}{b(b+1)}$$

or

$$c^2=\frac{b(b+1)}{b-1}.$$

For the conclusion of the theorem to be true, must have $c < b$. This requires

$$\frac{b(b+1)}{b-1} < b^2$$

or

$$b > 1+\sqrt{2}.$$

2. *A function $f(x)$ is twice differentiable and satisfies the inequalities*

$$\mid f(x) \mid < A, \mid f''(x) \mid < B,$$

in the range $x > a$, where A and B are constants. Prove that, in this range,

$$\mid f'(x) \mid < 2\sqrt{(AB)}.$$

Let $x > a$ and h any positive number.

There exists a number, θ, $(0 < \theta < 1)$ such that

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h).$$

$$\therefore \quad \mid hf'(x) \mid = \mid f(x+h)-f(x)-\frac{h^2}{2!}f''(x+\theta h) \mid$$

$$\leqslant \mid f(x+h) \mid + \mid f(x) \mid +\frac{h^2}{2!} \mid f''(x+\theta h) \mid$$

$$< 2A+\frac{Bh^2}{2}.$$

or $$|f'(x)| < \frac{2A}{h}+\frac{Bh}{2}. \quad \dots (i)$$

Now $|f'(x)|$ is independent of h and also, by (*i*) it is less than $(2A/h+Bh/2)$ for all positive values of h. Thus $|f'(x)|$ must be less than the least value of $(2A/h+Bh/2)$. Also

$$\frac{2A}{h}+\frac{Bh}{2}=\left(\sqrt{\frac{2A}{h}}-\sqrt{\frac{Bh}{2}}\right)^2+2\sqrt{AB}$$

so that

$$2\sqrt{AB}\leqslant \frac{2A}{h}+\frac{Bh}{2} \text{ for all } h>0.$$

$$\therefore \quad |f'(x)| < 2\sqrt{(AB)}.$$

3. *Show that θ which occurs in the Lagrange's from of remainder viz., $(h^n/n!)f^n(a+\theta h)$, tends to the limit, $1/(n+1)$ when $h\to 0$, provided that $f^{n+1}(x)$ is continuous at, a, and $f^{n+1}(a)\neq 0$.*

Since $f^{n+1}(x)$ is continuous at a, there exists an interval $[a-\delta, a+\delta]$ at every point of which $f^{n+1}(x)$ exists. Also, therefore, $f(x), f'(x), \dots, f^n((x)$ are all continuous in $[a-\delta, a+\delta]$.

If $(a+h)$ be any point of this interval, we obtain the necessary conditions being satisfied.

$$f(a+h)=f(a)+hf'(a)+\dots\dots+\frac{h^n}{n!}f^n(a+\theta h)$$

and $$f(a+h)=f(a)+hf'(o)+\dots\dots+\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{n+1}(a+\theta' h).$$

$$\therefore \quad f^n(a+\theta h)-f^n(a)=\frac{h}{n+1}f^{n+1}(a+\theta' h).$$

Again applying Lagrange's mean value theorem to the expression on the left, we see that

$$\theta h f^{n+1}(a+\theta\theta'' h)=\frac{h}{n+1}f^{n+1}(a+\theta' h),$$

i.e., $$\theta f^{n+1}(a+\theta\theta'' h)=\frac{1}{n+1}f^{n+1}(a+\theta' h).$$

Let $h\to 0$. Then, we obtain

$$\lim \theta . f^{n+1}(a)=\frac{1}{n+1}f^{n+1}(a),$$

$\because \; f^{n+1}(x)$ is continuous.

or $$\lim \theta=\frac{1}{n+1} \qquad \because \; f^{n+1}(a)\neq 0.$$

4. *Prove that e is an irrational number.*

If possible, let $e=p/q$, where p, q are integers.

Let n be any integer greater than q.

By Maclaurin's theorem with Langrange's form of remainder after n terms, we have

$$e=2+\frac{1}{2!}+\ldots\ldots+\frac{1}{n!}+\frac{1}{(n+1)!}e^{\theta}, \text{ where } 0<\theta<1.$$

$$\therefore \quad n!\,e=2n!+\frac{n!}{2!}+\ldots\ldots1+\frac{1}{n+1}e^{\theta}.$$

Now, $n!\,e=n!\,(p/q)$ must be an integer, and since $e^{\theta}<e<3$ we have

$$0<e^{\theta}/(n+1)<1.$$

Hence, we see that

an integer $=$ an integer $+$ a non-zero proper fraction,

and this is impossible.

Hence the result.

5. *Prove that e^x is not a rational function.*

If possible, let

$$e^x=\frac{a_0x^m+a_1x^{m-1}+\ldots\ldots+a_m}{b_0x^n+b_1x^{n-1}+\ldots\ldots+b_n}=f(x), \text{ say}$$

where $a_0\neq0$, $b_0\neq0$.

Obviously $f(x)/x^{m-n}\to a_0/b_0$ as $x\to\infty$ and this is impossible if $f(x)=e^x$. (Refer theorem of § 85·1, page 135).

Exercises

1. If

$$f(x)=x\,\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, \text{ when } x\neq0, \text{ and } f(0)=0,$$

show that $f(x)$ is continuous but not derivable for $x=0$; show that

$$Rf'(0)=+1,\ Lf'(0)=-1.$$

2. Verify the Rolle's theorem in $[a, b]$ for

(i) $\log[(x^2+ab)/(a+b)x]$.

(ii) $(x-a)^m(x-b)^n$; m, n being positive integers.

3. Verify Lagrange's mean values theorem for

(i) lx^2+mx+n in $[a, b]$, (ii) $x(x-1)(x-2)$ in $[0, \frac{1}{2}]$.

4. Discuss the applicability of Rolle's theorem to

(i) $f(x)=|x|$ in $[-1, 1]$.

(ii) $f(x)=2+(x-1)^{\frac{2}{3}}$ in $[0, 2]$.

5. Applying Lagrange's mean value theorem in turn to the functions $\log x$ and e^x, determine the corresponding values of θ in terms of a and h.

Deduce that

(i) $0<[\log(1+x)]^{-1}-x^{-1}<1$, (ii) $0<x^{-1}\log[(e^x-1)/x]<1$.

6. Show that

(*i*) $x^2 > (1+x)[\log(1+x)]^2$ for $x > 0$.

(*ii*) $x < \log[1/(1-x)] < x/(1-x)$, when $0 < x < 1$.

7. Examine the validity of the

(*i*) hypothesis, and

(*ii*) conclusion of the Lagrange's mean value theorem in the case

(*a*) $f(x)=1/x$, (*b*) $f(x)=x^{\frac{1}{3}}$

for the interval $[-1, 1]$.

8. Prove that, if $f'(c)$ exists and $\neq 0$, then, as $h \to 0$,

$$\lim \frac{f(c+h)+f(c-h)-2f(c)}{h} \text{ and } \lim \frac{f(c+h)-f(c)}{f(c-h)-f(c)}$$

exist and find their values. State the theorems on limits you require in the proof.

Examine the existence of these limits if

$$f(x) = |x|.$$

9. If $f(x)$, $\varphi(x)$ and $\psi(x)$ are continuous in the closed interval $[a, b]$ and differentiable in the open interval $]a, b[$ then show that there is a value, ξ, lying between a and b such that

$$\begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \\ f(\xi) & \varphi'(\xi) & \psi'(\xi) \end{vmatrix} = 0.$$

[Apply Rolle's theorem to the function

$$\begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \\ f(x) & \varphi(x) & \psi(x) \end{vmatrix}.]$$

10. If $f(0)=0$ and $f''(x)$ exists in $[0, \infty[$, show that

$$f'(x) - \frac{f(x)}{x} = \tfrac{1}{2}xf''(\xi),\ 0 < \xi < x$$

and deduce that if $f''(x)$ is positive for positive values of x, then $f(x)/x$ is a strictly increasing function of x.

11. Prove that, if $0 \leqslant x \leqslant 1$,

$$|\log(1+x) - x + \tfrac{1}{2}x^2| \leqslant \tfrac{1}{3}x^3.$$

12. Show that $a^x > x^a$ *if* $x > a \geqslant e$.

[Let $f(x)=x\log a - a\log x$. Show that $f(a)=0$ and $f'(x) > 0$].

13. If $\varphi''(x) \geqslant 0$ for every value of x, then

$$\varphi[\tfrac{1}{2}(x_1+x_2)] \leqslant \tfrac{1}{2}[\varphi(x_1)+\varphi(x_2)],$$

$$\varphi[\frac{1}{n}(x_1+x_2+\dots+x_n)] \leqslant \frac{1}{n}[\varphi(x_1)+\dots+\varphi(x_n)]$$

where $x_1, x_2, \dots\dots, x_n$ lie in $[a, b]$.

14. Show by means of an example that the mere convergence of

$$f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ldots\ldots$$

does not mean that it converges to $f(x+h)$.

15. Assuming $f''(x)$ continuous in $[a, b]$, show that

$$f(c)-f(a)\,\frac{b-c}{b-a}-f(b)\,\frac{c-a}{b-a}=\tfrac{1}{2}\,(c-a)\,(c-b)\,f''(\xi),$$

where c and ξ both lie in $[a, b]$,

16. If $f^n(x)=0$ for every x in $[a, b]$, then there are numbers

$$a_0,\ a_1,\ a_2,\ldots\ldots a_{n-1}$$

such that

$$f(x)=\sum_{r=0}^{n-1} a_r x^r,$$

in $[a, b]$.

17. Show that for the infinite series

$$\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right],$$

the derivative of the sum is not equal to the sum of the derivatives for $x=0$.

CHAPTER VI

RIEMANN THEORY OF INTEGRATION

87. Introduction. In elementary treatment of Integral Calculus the subject of integration is treated from the point of view of the inverse of Differentiation so that a function $\varphi(x)$ is called an integral of a given function $f(x)$ if $\varphi'(x)=f(x)$.

Historically, however, the subject arose in connection with the determination of areas of plane regions and was based on the notion of the limit of a type of sum when the number of terms in the sum tends to infinity and each term tends to zero.

In fact the name Integral Calculus has its origin in this process of summation and the words '*To integrate*' literally mean '*To give the sum of*'. It was only afterwards that it was seen that the subject of integration can also be viewed from the point of the inverse of differentiation.

In elementary works the reference to integration from summation point of view is always associated with intuitively perceived geometrical concepts.

Consistent with our general purpose, we shall, in this book, give a purely arithmetic treatment of the subject based on the aggregate of real numbers and the same will, moreover, basically be independent of the notion of differentiation. A function $\varphi(x)$ such that $\varphi'(x)=f(x)$ will be called a *primitive* of $f(x)$ and the relation between *Integrals and primitives* will be given later on by means of what is known as the *Fundamental Theorem of Integral Calculus.*

The first satisfactory rigorous arithmetic treatment of *Integration* was first given by Riemann and we shall be following the definition given by him. Many refinements and generalisations of the subject have appeared since then. The most noteworthy of these is the theory of integration by *Lebesgue* (1902).

88. Riemann integrability and integral of a bounded function over a finite range. Let $f(x)$ be a *bounded* function defined in some *finite* interval $[a, b]$

Divide $[a, b]$ into a finite number of sub-intervals by means of any *arbitrary* set of points $x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots, x_n$, where

$$a=x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots\ldots < x_{n-1} < x_n=b.$$

Then

$$[x_0, x_1], [x_1, x_2], \ldots\ldots\ldots, [x_{r-1}, x_r], \ldots\ldots, [x_{n-1}, x_n],$$

are the sub-intervals in which $[a, b]$ is divided.

Let the sub-interval $[x_{r-1}, x_r]$ and also its length, x_r-x_{r-1}, be both denoted by the same symbol, δ_r.

The function $f(x)$ which is bounded in $[a, b]$ is also necessarily bounded in each sub-interval δ_r.

Let M_r, m_r, be the bounds of $f(x)$ in δ_r.

Set up the two sums S and s defined as follows :—

$$S = M_1\delta_1 + \dots\dots + M_r\delta_r + \dots\dots + M_n\delta_n = \sum_{r=1}^{r=n} M_r\delta_r,$$

$$s = m_1\delta_1 + \dots\dots + m_r\delta_r + \dots\dots + m_n\delta_n = \sum_{r=1}^{r=n} m_r\delta_r.$$

If M, m be the bounds of $f(x)$ in $[a, b]$, we have, for every value of r,

$$m \leqslant m_r \leqslant M_r \leqslant M,$$

or

$$m\delta_r \leqslant m_r\delta_r \leqslant M_r\delta_r \leqslant M\delta_r.$$

Putting $r = 1, 2, \dots\dots, n$, and adding, we deduce that

$$m(b-a) \leqslant s \leqslant S \leqslant M(b-a). \quad \dots (1)$$

Now, a pair of sums S, s corresponds to each mode of division of $[a, b]$ into sub-intervals and, from (1) we see that the sets of the sums S, s, obtained by considering all possible modes of division, are bounded.

Upper and lower Integrals. Def. The lower bound *of the set of the sums, S, is called the* **upper integral** *of $f(x)$ over $[a, b]$ and is denoted by*

$$U = \overline{\int_a^b} f(x)dx.$$

The upper bound *of the set of the sums, s, is called the* **lower integral** *of $f(x)$ over $[a, b]$ and is denoted by*

$$L = \underline{\int_a^b} f(x)\,dx.$$

[Clearly the upper bound of the sums, S, is $M(b-a)$ and the lower bound of the sums, s, is $m(b-a)$ and they are attained.]

A bounded function $f(x)$ is said to be Riemann integrable, or simply integrable, (for the purposes of this book), *over $[a, b]$, if its upper and lower integrals are equal; the common value of these integrals which is called the* **Riemann integral** *or simply the* **integral** *is denoted by the symbol*

$$I = \int_a^b f(x)\,dx.$$

Notes.

1. The number a, b are respectively called the *lower* and the *upper limits of integration.*

2. The definition of integrability as given above is based on the *notion of bounds.* Another equivalent approach based on the notion of limits is given in § 90.

3. It should be clearly understood that *every bounded function is not necessarily integrable, i.e.*, there may exist a bounded function $f(x)$ for which

$$\overline{\int_a^b} f(x)\, dx \neq \underline{\int_a^b} f(x)\, dx.$$

(Refer Ex. 1, page 145).

The necessary and sufficient condition for the existence of the symbol

$$\int_a^b f(x)\, dy$$

i.e., for the integrability of $f(x)$ over $[a, b]$ is obtained in § 91.

4. The concept of integrability of a function over an interval, as introduced here, is subject to two very important limitations, *viz.*, (*i*) *the function is bounded* (*ii*) *the interval is finite* so that neither of the end points is infinite.

In Chapter VIII, we shall see how these limitations can be removed and the concept generalised so as to be applicable sometimes even to cases where the function is not bounded or where one or both the limits of integration are infinite.

5. The statement that, $\int_a^b f(x)\, dx$, exists, implies that the function $f(x)$ is bounded and integrable over $[a, b]$.

6. The symbol

$$D(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

will be used to denote the division obtained by inserting the points

$$x_0, x_1 \ldots\ldots, x_{r-1}, x_r \ldots\ldots, x_n$$

between a and b. The numbers

$$x_0, x_1, x_2 \ldots\ldots, x_{r-1}, x_r \ldots\ldots, x_n$$

will be called the points of the division, D, and the sub-intervals

$$[x_0, x_1] \ldots\ldots, [x_{r-1}, x_r], \ldots\ldots, [x_{n-1}, x_n]$$

the intervals of the division D.

7. Norm. The length of the greatest of all the intervals $[x_{r-1}, x_r]$ of D will be called the **norm** of the division D.

8. The sums S, s corresponding to a division D are sometimes written as S_D, s_D respectively. Clearly

$$S_D \geqslant s_D.$$

9. Oscillatory Sum. We have

$$S_D - s_D = \Sigma M_r\, \delta_r - \Sigma m_r\, \delta_r = \Sigma (M_r - m_r)\, \delta_r = \Sigma O_r\, \delta_r,$$

where O_r denotes the oscillation of the function in δ_r. The sum $\Sigma O_r \delta_r$ is called the *oscillatory* sum and is denoted by ω_D.

As $O_r \geqslant 0$, each oscillatory sum consists of the sum of a finite number of non-negative terms.

Exercises

1. *If*

$$f(x) = \begin{cases} 0, \text{ where } x \text{ is rational} \\ 1, \text{ where } x \text{ is irrational,} \end{cases}$$

show that $f(x)$ is not integrable in any interval.

2. Show that

$$\overline{\int_a^b} k\, dx = \underline{\int_a^b} k\, dx = k\,(b-a),$$

where k is a constant.

(This proves that every function which is a constant is integrable.)

3. A function $f(x)$ is bounded in $[a, b]$; show that

$$(i)\ \overline{\int_a^b} kf(x)\, dx = k \underline{\int_a^b} f(x)\, dx,\quad \underline{\int_a^b} kf(x)\, dx = k \overline{\int_a^b} f(x)\, dx,$$

where k is a positive constant.

$$(ii)\ \overline{\int_a^b} kf(x)\, ax = k \underline{\int_a^b} f(x)\, dx,\quad \underline{\int_a^b} kf(x)\, dx = k \int_a^b f(x)\, dx,$$

where k is a negative constant.

Deduce that if $f(x)$ is bounded and integrable over $[a, b]$ then so is $kf(x)$, where k is any constant, and that

$$\int_a^b kf(x)\, dx = k \int_a^b f(x)\, dx.$$

[If M_r, m_r be the bounds of $f(x)$ in δ_r, then kM_r, km_r, (km_r, kM_r) are the bounds of $kf(x)$ in δ_r, where k is positive, (k is negative).

4. A bounded function $f(x)$ is integrable over $[a, b]$ and M, m are the bounds of $f(x)$, show that

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a).$$

89. Darboux's theorem I. *To every positive number* ε, *there corresponds a positive number* δ *such that*

$$S_D < \int_a^{\bar{b}} f(x)\,dx + \varepsilon$$

for every division, D, whose norm is less than or equal to δ.

Lemma. *Let* $|f(x)| \leqslant k$ *in* $[a, b]$ *and* D_1 *any division of* $[a, b]$ *and let* δ *be a positive number such that the norm of* D_1 *is* $< \delta$. *Then if* D_2 *be any other division of* $[a, b]$ *consisting of, as its points of division, all the points of* D_1 *and at the most, p, more, then*

$$S_{D_1} \geqslant S_{D_2} \geqslant S_{D_1} - 2pk\delta.$$

Firstly suppose that $p=1$ so that only one interval, say δ_r, of D_1 is divided into two intervals, say δ'_r and δ''_r. Let M_r, M'_r, M''_r be the upper bounds of $f(x)$ in δ_r, δ'_r, δ''_r respectively.

We have

$$S_{D_1} - S_{D_2} = M_r\delta_r - (M'_r\delta'_r + M''_r\delta''_r)$$

$$= (M_r - M'_r)\delta'_r + (M_r - M''_r)\delta''_r,$$

for $\delta_r = \delta'_r + \delta''_r$.

Now, since $|f(x)| \leqslant k$, therefore

$$-k \leqslant M'_r \leqslant M_r \leqslant k,$$

i.e.,

$$0 \leqslant M_r - M'_r \leqslant 2k.$$

Similarly

$$0 \leqslant M_r - M''_r \leqslant 2k.$$

$$\therefore \quad 0 \leqslant S_{D_1} - S_{D_2} \leqslant 2k(\delta'_r + \delta''_r) = 2k\delta_r \leqslant 2k\delta.$$

Now supposing that each additional point is introduced one by one, we obtain the result.

We now prove the *main theorem.*

As $f(x)$ is bounded, there exists a positive number, k, such that

$$|f(x)| \leqslant k \text{ in } [a, b],$$

Since

$$\int_a^{\bar{b}} f(x)\,dx$$

is the lower bound of the set of sums S, there exists a division.

$$D_1(a=x_0,\ x_1,\ x_2, \ldots\ldots\ldots,\ x_{p-1},\ x_p=b)$$

such that for the corresponding sum S_{D_1}, we have

$$S_{D_1} < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2}.$$

The points of D_1 are $(p+1)$ in number.

We determine a positive number δ such that

$$2k(p-1)\delta = \tfrac{1}{2}\varepsilon.$$

Consider, now any division D whose norm is less than or equal to δ. Consider a division D_2 consisting of, as its points of division, the points of D_1 as well as of D.

We have, by the lemma above,

$$\frac{\varepsilon}{2} + \int_a^{\bar{b}} f(x)\,dx > S_{D_1} \geqslant S_{D_2} \geqslant S_D - 2(p-1)k\delta$$

or
$$S_D < 2(p-1)k\delta + \frac{\varepsilon}{2} + \int_a^{\bar{b}} f(x)\,dx,$$

or
$$S_D < \int_a^{\bar{b}} f(x)\,dx + \varepsilon.$$

Darboux's theorem II. *To every positive number* ε, *there corresponds a positive number* δ, *such that*

$$s_D > \int_{\underline{a}}^{b} f(x)\,dx - \varepsilon$$

for every division, D, whose norm is less than or equal to δ.

This proof is similar to that of the corresponding result on the upper integral.

Note. Darboux's theorem may be symbolically exhibited as follow

$$\lim S_D = \overline{\int_a^b} f(x)\, dx, \quad \lim s_D = \underline{\int_a^b} f(x)\, dx,$$

when δ, the norm of division D, tends to zero.

Cor.

$$\overline{\int_a^b} f(x)\, dx \geqslant \underline{\int_a^b} f(x)\, dx,$$

i.e.,

the upper integral $\geqslant$ the lower integral.

If possible, let

$$\overline{\int_a^b} f(x)\, dx < \underline{\int_a^b} f(x)\, dx.$$

Let, k, be any number lying between the upper and lower integrals.

There exists, by Darboux's theorem, a positive number δ_1 such that for every division whose norm is $\leqslant \delta_1$,

$$S < k.$$

Also, there exists a positive number δ_2 such that for every division whose norm is $\leqslant \delta_2$,

$$s > k.$$

If, δ, be any positive number smaller than both δ_1 and δ_2, then for every division whose norm is $\leqslant \delta$,

$$S < k < s, \quad i.e., \quad S < s,$$

which is absurd.

Hence the result.

Ex. Show that

$$S_{D_1} \geqslant s_{D_2},$$

even when D_1, D_2 are two *different* divisions.

(This at once follows from the cor. above.)

90. Another equivalent definition of integrability and integral. We shall firstly state and prove a theorem which will suggest an alternative definition of the integrability and integral of a function. Let $f(x)$ be a function defined in $[a, b]$.

Let

$$D(a = x_0 < x_1 < x_2 < \ldots\ldots < x_{r-1} < x_r < \ldots\ldots < x_n = b)$$

be any division of $[a, b]$ and ξ_r any arbitrary point of

$$\delta_r=[x_{r-1}, x_r].$$

Form the sum

$$\sum_{r=1}^{r=n} f(\xi_r)\delta_r=f(\xi_1)\delta_1+\ldots+f(\xi_r)\delta_r+\ldots+f(\xi_n)\delta_n.$$

90·1. Theorem. *If $f(x)$ is bounded and integrable over $[a, b]$, then to every positive number ε, however small, there corresponds a positive number, δ, such that for every division*

$$D(a=x_0, x_1 \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

of norm $\leqslant \delta$, and for every arbitrary choice of ξ_r in $[x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{r=n} f(\xi_r)(x_r-x_{r-1})-\int_a^b f(x)\, dx \right| < \varepsilon.$$

In a more concise but less precise manner the result may be stated a little differently as follows :—*If a function $f(x)$ is bounded and integrable, then*

$$\lim_{\delta \to 0} \sum_{r=1}^{r=n} f(\xi_r)\delta_r,$$

exists and is the integral of $f(x)$ over $[a, b]$.

Since $f(x)$ is a bounded and integrable functiod, therefore

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx.$$

Let ε be any positive number.

By Darboux's theorem, there exists a positive number δ such that for every division D whose norm is $\leqslant \delta$,

$$S_D < \overline{\int_a^b} f(x)\, dx+\varepsilon= \int_a^b f(x)\, dx+\varepsilon, \qquad \ldots(1)$$

and

$$s_D > \underline{\int_a^b} f(x)\, dx-\varepsilon= \int_a^b f(x)\, dx-\varepsilon. \qquad \ldots(2)$$

If ξ_r be *any* point of the interval δ_r of D, we have

$$s_D \leqslant \sum_{r=1}^{r=n} f(\xi_r)\delta_r \leqslant S_D . \qquad \ldots(3)$$

From (1), (2) and (3), we deduce that for every division D, whose norm is $\leqslant \delta$,

$$\int_a^b f(x)\,dx-\varepsilon < \sum_{r=1}^{r=n} f(\xi_r)\delta_r < \int_a^b f(x)\,dx+\varepsilon$$

i.e.,

$$\left| \sum_{r=1}^{r=n} f(\xi_r)\delta_r - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Hence the theorem.

90·2. A second definition. We shall now prove that the definition of integrability as given in § 88 is equivalent to the following :

A function $f(x)$ is said to be integrable over $[a, b]$, if there exists a number I such that to every positive number ε, there corresponds a positive number δ, such that for every division

$$D(a=x_0, x_1 \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

of norm $\leqslant \delta$ and for every arbitrary choice of ξ_r in $[x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{r=n} f(\xi_r)(x_r-x_{r-1})-I \right| < \varepsilon.$$

Also then I is said to be the integral of $f(x)$ over $[a, b]$.

In a more concise but less precise manner, this definition may be stated as follows :

A function $f(x)$ is integrable, if

$$\lim_{\delta \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

exists and is independent of the choice of the interval δ_r and of the point ξ_r of δ_r ; the limit I, if it exists is called the integral of $f(x)$ over $[a, b]$.

The equivalence of the two definitions will now be established.

As a consequence of the theorem proved in § 90·1 above, we see that if a function $f(x)$ be integrable according to the former definition, then it is so according to the latter also.

Now, *let $f(x)$ be integrable according to the* **latter** *definition so that*

$$lim\ \Sigma\ f(\xi_r)\delta_r,$$

exists, as the norm $\delta \to 0$.

It will firstly be deduced that $f(x)$ is bounded in $[a, b]$.

If possible, let $f(x)$ be not bounded.

There exists a division D such that for every choice of ξ_r in δ_r,

$$| \Sigma f(\xi_r)\delta_r - I | < 1,$$

or

$$| \Sigma f(\xi_r)\delta_r | < | I | + 1.$$

As $f(x)$ is not bounded in (a, b), it must also be so in at least one δ_r say in δ_m.

We take $\xi_r = x_r$, when $r \neq m$ so that every number ξ_r, except ξ_m, is fixed and, accordingly, every term of $\Sigma f(\xi_r)\delta_r$ except the term $f(\xi_m)\delta_m$ is also fixed. Since $f(x)$ is not bounded in δ_m, we can now choose a point ξ_m in δ_m such that

$$| \Sigma f(\xi_r)\delta_r | > | I | + 1,$$

and thus we arrive at a contradiction. Hence $f(x)$ is bounded in $[a, b]$.

Now, let ε be any positive number. There exists a positive number δ such that for every division whose norm is $\leqslant \delta$,

$$| \Sigma f(\xi_r)\delta_r - I | < \tfrac{1}{2}\varepsilon,$$

i.e.,

$$I - \tfrac{1}{2}\varepsilon < \Sigma f(\xi_r)\delta_r < I + \tfrac{1}{2}\varepsilon, \qquad \ldots(1)$$

for every choice of the point ξ_r, in δ_r.

If M_r, m_r be the bounds of $f(x)$ in δ_r, there exist points α_r, β_r of δ_r such that

$$f(\alpha_r) > M_r - \varepsilon/2(b-a),$$

$$f(\beta_r) < m_r + \varepsilon/2(b-a).$$

From these we deduce that

$$\Sigma f(\alpha_r)\delta_r > S - \tfrac{1}{2}\varepsilon \text{ or } S < \Sigma f(\alpha_r)\delta_r + \tfrac{1}{2}\varepsilon, \qquad \ldots(2)$$

$$\Sigma f(\beta_r)\delta_r < s + \tfrac{1}{2}\varepsilon \text{ or } s > \Sigma f(\beta_r)\delta_r - \tfrac{1}{2}\varepsilon. \qquad \ldots(3)$$

From (1), (2) and (3), we deduce, taking $\xi_r = \alpha_r$ and β_r, that

$$I - \varepsilon < s \leqslant S < I + \varepsilon, \qquad \ldots(4)$$

for every division whose norm is $\leqslant \delta$.

Also we know that

$$s \leqslant \underline{\int_a^b} f(x)\, dx \leqslant \overline{\int_a^b} f(x)\, dx \leqslant S. \qquad \ldots(5)$$

From (4) and (5), we have

$$I - \varepsilon < \underline{\int_a^b} f(x)\, dx \leqslant \overline{\int_a^b} f(x)\, dx < I + \varepsilon, \qquad \ldots(6)$$

or
$$0 \leqslant \overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx < 2\varepsilon,$$

so that the non-negative number

$$\overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx$$

is less than every positive number ; ε being arbitrary, and hence

$$\overline{\int_a^b} f(x)\, dx - \underline{\int_a^b} f(x)\, dx = 0, \qquad \ldots(7)$$

so that the function is integrable according to the former definition also

From (6) and (7), we have

$$I - \varepsilon < \int_a^b f(x)\, dx < I + \varepsilon,$$

and since ε is arbitrary, this gives

$$I = \int_a^b f(x)\, dx.$$

Thus the equivalence is completely established.

91. Conditions for integrability.

91·1. First form. *The necessary and sufficient condition for the integrability of a bounded function $f(x)$ is, that to every positive number ε, there corresponds a positive number δ, such that for every division D whose norm is $\leqslant \delta$, the oscillatory sum, w_D, is $< \varepsilon$.*

The condition is necessary. The bounded function $f(x)$ being integrable,

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx.$$

Let ε be any positive number. By Darboux's theorem, there exists a positive number δ such that for every division D whose norm is $\leqslant \delta$,

$$S_D < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \frac{\varepsilon}{2},$$

and
$$s_D > \underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} = \int_a^b f(x)\,dx - \frac{\varepsilon}{2}.$$

$$\therefore \quad \int_a^b f(x)\,dx - \frac{\varepsilon}{2} < s_D \leqslant S_D < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

or

$$w_D = S_D - s_D < \varepsilon.$$

for every division D whose norm is $\leqslant \delta$.

The condition is sufficient. Let ε be any positive number. There exists a division D such that

$$S_D - s_D = \left[S_D - \overline{\int_a^b} f(x)\,dx \right] + \left[\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \right] +$$
$$\left[\underline{\int_a^b} f(x)\,dx - s_D \right] < \varepsilon.$$

Since each one of the three numbers

$$S_D - \overline{\int_a^b} f(x)\,dx, \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx, \quad \underline{\int_a^b} f(x)\,dx - s_D$$

is non-negative, we see that

$$0 \leqslant \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon.$$

As ε is an arbitrary positive number, we see that the non-negative number

$$\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx,$$

is less than *every* positive number, and hence

$$\int_a^{\bar{b}} f(x)\,dx-\int_{\underline{a}}^{b} f(x)\,dx=0 \text{ or } \int_a^{\bar{b}} f(x)\,dx=\int_{\underline{a}}^{b} f(x)\,dx,$$

so that $f(x)$ is integrable,

Thus the theorem is established.

Note. This theorem is sometimes stated differently as follows :

The necessary and sufficient condition for a bounded function $f(x)$ to be integrable in $[a, b]$ is that

$$\lim w_D = 0,$$

when δ, the norm of the division D, tends to 0.

91·2. The condition of integrability. Second form. *The necessary and sufficient condition for the integrability of a bounded function $f(x)$ is that to every positive number ε, there corresponds a division D such that the corresponding oscillatory sum $w_D < \varepsilon$.*

That this condition is necessary follows at once from the first part of § 91·1.

The condition is sufficient. As in the second part of § 91·1, we write

$$S_D - s_D = \left[S_D - \int_a^{\bar{b}} f(x)\,dx \right] + \left[\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx \right] + \left[\int_{\underline{a}}^{b} f(x)\,dx - s_D \right] < \varepsilon,$$

and see that

$$0 \leqslant \int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx < \varepsilon.$$

From this relation, which holds for every positive ε, we deduce that

$$\int_a^{\bar{b}} f(x)\,dx - \int_{\underline{a}}^{b} f(x)\,dx=0, \text{ i.e., } \int_a^{\bar{b}} f(x)\,dx=\int_{\underline{a}}^{b} f(x)\,dx.$$

Note. On comparing the two forms of conditions, the reader will easily see that, from the point of view of *necessity*, the first form is more valuable than the second but, from the point of view of sufficiency, the second form is more valuable than the first.

92. Particular classes of bounded integrable functions.

92·1. *Every continuous function is integrable.*

Suppose that $f(x)$ is continuous in $[a, b]$.

Since $f(x)$ is continuous, it is bounded. (§ 58·3, page 98).

Let ε be any positive number.

We divide $[a, b]$ into a finite number of sub-intervals, by δ_r, such that the oscillation of $f(x)$ in each of such sub-intervals is $< \varepsilon/(b-a)$; this being possible as proved in § 58·5, page 101.

If ω_D be the oscillatory sum for this division,

$$\begin{aligned}\omega_D &= \Sigma(M_r - m_r)\delta_r\\ &< \Sigma[\varepsilon/(b-a)]\delta_r\\ &= [\varepsilon/(b-a)]\Sigma\delta_r\\ &= [\varepsilon/(b-a)] \,.\, (b-a) = \varepsilon,\end{aligned}$$

i.e.,

$$\omega_D < \varepsilon.$$

Hence $f(x)$ is integrable in $[a, b]$. [§ 91·2].

92·2. *A bounded function $f(x)$ which has only a* **finite** *number of points of discontinuity in $[a, b]$ is integrable in $[a, b]$.*

Let

$$a_1, a_2, a_3, \ldots\ldots\ldots, a_p,$$

be the *finite* number of points of discontinuity of $f(x)$.

Let ε be any positive number.

We enclose the points $a_1, a_2, \ldots\ldots\ldots a_p$, in p non-overlapping intervals

$$[a_1', a_1''], [a_2', a_2''] \ldots\ldots, [a_p', a_p'']$$

such that the sum of their lengths is $< \varepsilon/2(M-m)$; M, m being the bounds of $f(x)$ in $[a, b]$. The oscillation of $f(x)$ in each of these intervals is $\leqslant (M-m)$ and accordingly the oscillatory sum for these is

$$< [\varepsilon/2(M-m)] \,.\, (M-m) = \varepsilon/2.$$

Now, $f(x)$ is continuous in the $(p+1)$ sub-intervals

$$[a, a_1'], [a_1'', a_2'], \ldots\ldots, [a_p'', b].$$

As in § 92·1 above each of these $(p+1)$ sub-intervals can be further sub-divided so that the part of the oscillatory sum arising from the sub-intervals of each of them separately is $< \varepsilon/2(p+1)$.

Thus there exists a division of $[a, b]$ such that the corresponding oscillatory sum is

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(p+1)}(p+1) = \varepsilon.$$

Hence $f(x)$ is integrable in $[a, b]$. [§ 91·2]

92·3. *If a function $f(x)$ is bounded in $[a, b]$ and the set of its points of discontinuity has only a* **finite number of limiting points**, *then $f(x)$ is integrable in $[a, b]$.*

Let

$$a_1, a_2, a_3, \ldots\ldots, a_p,$$

be the limiting points of the set of the points of discontinuity of $f(x)$.

We enclose them in, p, non-overlapping intervals

$$[a_1', a_1''], [a_2', a_2''], \ldots\ldots, [a_p', a_p'']$$

such that the sum of their lengths is $< \varepsilon/2(M-m)$; M, m being the bounds of $f(x)$ in $[a, b]$. The oscillatory sum for these intervals is

$$< \varepsilon/2.$$

Only a *finite* number of points of discontinuity of $f(x)$ can lie in each of the $(p+1)$ intervals

$$[a, a_1'], [a_1'', a_2'], \ldots, [a_p'', b],$$

so that, as in § 92·2, they can be so sub-divided that the part of the oscillatory sum arising from the sub-intervals of each of these $(p+1)$ intervals is separately $< \varepsilon/2(p+1)$.

Thus there exists a division of $[a, b]$ such that the corresponding oscillatory sum is

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2(p+1)}(p+1) = \varepsilon.$$

Hence $f(x)$ is integrable in $[a, b]$. [§ 91·2].

92·4. *If $f(x)$ is monotonic in $[a, b]$, then it is integrable in $[a, b]$.*

Clearly $f(x)$ is bounded and $f(a)$, $f(b)$ are its two bounds.

Let ε be any positive number.

For the sake of definiteness, we suppose that $f(x)$ is monotonically increasing.

We divide $[a, b]$ so that the length of each sub-interval is

$$< \varepsilon/[f(b)-f(a)+1].$$

Let

$$D(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be any one such division.

Let

$$\delta_r = x_r - x_{r-1}.$$

Here

$$M_r = f(x_r),\ m_r = f(x_{r-1})$$

$$\omega_D = \Sigma(M_r - m_r)\delta_r$$

$$= \Sigma[f(x_r) - f(x_{r-1})]\delta_r$$

$$< \frac{\varepsilon}{f(b)-f(a)+1} \cdot \Sigma[f(x_r) - f(x_{r-1})]$$

$$= \frac{\varepsilon}{f(b)-f(a)+1} \cdot [f(b)-f(a)] < \varepsilon.$$

Hence $f(x)$ is integrable in $[a, b]$.

Note. If we had taken $[f(b)-f(a)]$ instead of $[f(b)-f(a)+1]$, the proof would not have been valid for the case when $f(b)-f(a)=0$, *i.e.*, when $f(x)$ is a constant. The artifice of taking, $[f(b)-f(a)+k]$, where k is positive, or, in particular, $f(b)-f(a)+1$, serves to make the proof applicable even in this case.

Ex. 1. *Show that the function defined as follows*

$$f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \leqslant \frac{1}{2^n},\ (n=0, 1, 2, 3\ldots)$$

$$f(0) = 0$$

is integrable in $[0, 1]$, although it has an infinite number of points of discontinuity.

We have, as given,

$$f(0) = 0,$$

$$f(x) = 1, \text{ when } \tfrac{1}{2} < x \leqslant 1,$$

$$f(x) = \tfrac{1}{2}, \text{ when } (\tfrac{1}{2})^2 < x \leqslant \tfrac{1}{2},$$

$$\cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots$$

$$f(x) = (\tfrac{1}{2})^{n-1}, \text{ when } (\tfrac{1}{2})^n < x \leqslant (\tfrac{1}{2})^{n-1},$$

$$\cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots$$

$$\cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots$$

Since $f(x)$ is bounded and monotonically increasing in $[0, 1]$, it is integrable (§ 92·4).

Or we notice that $f(x)$ is continuous in $[0, 1]$ except at the set of points

$$0,\ \tfrac{1}{2},\ (\tfrac{1}{2})^2,\ (\tfrac{1}{2})^3, \ldots\ldots, (\tfrac{1}{2})^n \ldots\ldots$$

which has only one limiting point, *viz.*, 0 and hence $f(x)$ is integrable (§ 92·3).

2. *A function $f(x)$ is defined in $(0, 1)$ as follaws :*

$f(x) = 1/q$ when x is any rational number p/q in its lowest terms, and $f(x) = 0$ when x is irrational.

Show that $f(x)$ is integrable in $[0, 1]$ and the value of the integral is 0.

Let, ε, be any positive number.

There exist only a finite number of integers q such that

$$\frac{1}{q} > \frac{\varepsilon}{2}, \; i.e., \; q < \frac{2}{\varepsilon}.$$

We call points p/q in $[0, 1]$ for which $1/q > \varepsilon/2$ exceptional points. Surely they are *finite* in number.

Also we know that every interval encloses rational as well as irrational points.

Clearly the oscillation of $f(x)$ in any interval which does not include any exceptional point is $< \varepsilon/2$ and the oscillation of $f(x)$ in any interval which includes any exceptional point is at the most 1.

We now enclose the exceptional points of $[0, 1]$ in intervals the sum of whose length is $< \varepsilon/2$ so that the part of the oscillatory sum arising from these $< \varepsilon/2$. Also, howsoever we may divide the remaining part of $[0, 1]$, the part of the oscillatory sum arising from the same will be $< \varepsilon/2$.

Thus we have a division of $[0, 1]$ such that the oscillatory sum of $f(x)$ for the same is less than any given pre-assigned positive number ε. Hence, by § 91, the function is integrable.

Also since for every division D, $S_D = 0$,

we have

$$\overline{\int_0^1} f(x)\, dx = \underline{\int_0^1} f(x)\, dx = 0.$$

Note. It has been seen in Ex. 3, page 106, that the function $f(x)$ is discontinuous for every rational value of x so that the set of the limiting points of the set of points of discontinuity of $f(x)$ is the entire interval $[0, 1]$. This shows that the condition obtained for integrability in § 92·3 is only sufficient but not necessary.

2. A function $f(x)$ is integrable in $[a, b]$; show that

(*i*) $$\lim \sum_{r=1}^{r=n} hf(a+rh) = \int_a^b f(x)\, dx,$$

when $h \to 0$, $n \to \infty$, $nh = b-a$.

(*ii*) $$\lim \sum_{p=1}^{p=n} f(ar^{p-1})\,(ar^p - ar^{p-1}) = \int_a^b f(x)\, dx,$$

when $r \to 1$, $n \to \infty$, $r^n = b/a$.

93. Properties of integrable function.

93·1. *If a bounded function $f(x)$ is integrable in $[a, b]$, then it is also integrable in $[a, c]$, and $[c, b]$, where c is a point $[a, b]$.*

Conversely, if $f(x)$ is bounded and integrable in $[a, c]$, $[c, b]$, then it is also integrable in $[a, b]$.

Also in either case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \qquad a < c < b.$$

Suppose that $f(x)$ is bounded and integrable in $[a, b]$.

Let ε be any positive number.

There exists a positive number δ such that for each division of $[a, b]$ whose norm is $\leqslant \delta$, the oscillatory sum is $< \varepsilon$. Let D be a division of $[a, b]$ such that, c, is a point of D and that the norm of D is $\leqslant \delta$.

The oscillatory sum for the division D breaks itself into two parts, respectively consisting of the terms which arise from the sub-intervals of $[a, c]$, and $[c, b]$. Since the terms of an oscillatory sum are all positive, each part must itself be $< \varepsilon$. Hence $f(x)$ is integrable both in $[a, c]$ and $[c, b]$.

Let, now, $f(x)$ be bounded and integrable in $[a, c]$ and $[c, b]$. Let ε be any positive number. There exist divisions of $[a, c]$ and $[c, b]$ such that the corresponding oscillatory sums are $< \varepsilon/2$. The divisions of $[a, c]$ and $[c, b]$ give rise to a division of $[a, b]$ for which the oscillatory sum is $< (\varepsilon/2 + \varepsilon/2) = \varepsilon$. Hence $f(x)$ is integrable in $[a, b]$.

The relationship of equality is to be proved now. Let ε be any positive number.

As $f(x)$ is simultaneously integrable in $[a, c]$, $[c, b]$, and $[a, b]$ there exists a positive number δ such that for divisions of norm $\leqslant \delta$, and of which, c, is a point, we have

$$\left| \sum_{(a,\,c)} f(\xi_r)\delta_r - \int_a^c f(x)\,dx \right| < \frac{\varepsilon}{3}, \quad \left| \sum_{(c,\,b)} f(\xi_r)\delta_r - \int_c^b f(x)\,dx \right| < \frac{\varepsilon}{3}$$

$$\left| \sum_{(a,\,b)} f(\xi_r)\delta_r - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{3};$$

where the meanings of the symbols $\sum\limits_{(a,\,c)} f(\xi_r)\delta_r$, etc., are obvious.

Since

$$\sum_{(a,\,c)} f(\xi_r)\delta_r + \sum_{(c,\,b)} f(\xi_r)\delta_r = \sum_{(a,\,b)} f(\xi_r)\delta_r,$$

we deduce that

$$\left| \int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_a^b f(x)\,dx \right| < \varepsilon,$$

from which it follows that

$$\int_a^b f(x)\,dx - \int_a^c f(x)\,dx - \int_c^b f(x)\,dx = 0$$

ε being an arbitrary positive number.

Cor. *If $f(x)$ is bounded and integrable in $[a, b]$, then it is also bounded and integrable in $[\alpha, \beta]$ where $a < \alpha < \beta < b$.*

As $f(x)$ is integrable in $[a, b]$, therefore it is integrable in (a, β) and hence also in $[\alpha, \beta]$.

93·2. Integrability of the sum, difference, product and quotient of integrable functions. Before taking up the main question, we state and prove a simple lemma.

Lemma. *The oscillation of a bounded function $f(x)$ in an interval $[a, b]$ is the upper bound of the set of numbers*

$$| f(\alpha) - f(\beta) |,$$

where α, β are any arbitrary numbers belonging to $[a, b]$.

Let m, M be the bounds of $f(x)$ in $[a, b]$. If α, β be any arbitrary numbers of $[a, b]$, we have

$$m \leqslant f(\alpha), f(\beta) \leqslant M$$

so that

$$| f(\alpha) - f(\beta) | \leqslant M - m. \qquad \dots (1)$$

Let $\varepsilon > 0$ be given.

Since M is the upper bound of $f(x)$, there exists α_1 belonging to $[a, b]$ such that

$$f(\alpha_1) > M - \tfrac{1}{2}\varepsilon. \qquad \dots (2)$$

Since m is the lower bound of $f(x)$, there exists β_1 belonging to $[a, b]$ such that

$$f(\beta_1) < m + \tfrac{1}{2}\varepsilon. \qquad \dots (3)$$

From (2) and (3), we have

$$f(\alpha_1) - f(\beta_1) > M - m - \varepsilon,$$

$$\therefore \quad | f(\alpha_1) - f(\beta_1) | \geqslant f(\alpha_1) - f(\beta_1) > M - m - \varepsilon.$$

There exists therefore a pair of numbers α_1, β_1 such that

$$| f(\alpha_1) - f(\beta_1) | > M - m - \varepsilon, \qquad \dots (4)$$

where $\varepsilon > 0$ is arbitrary.

From (1) and (4), it follows that $M - m$ is the upper bound of the set of numbers $| f(\alpha) - f(\beta) |$.

93·22. Integrability of sum and difference. *If $f(x)$ and $g(x)$ are two functions, both bounded and integrable in $[a, b]$, then $f(x) \pm g(x)$ are also bounded and integrable in $[a, b]$, and*

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

Let

$$D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be *any* division of $[a, b]$.

Let

$$M_r', m_r' ; \; M_r'', m_r'' ; \; M_r, m_r$$

be the bounds of $f(x)$, $g(x)$ and $f(x)+g(x)$ in $\delta_r \equiv [x_{r-1}, x_r]$. If α_1, α_2 be any two points of δ_r, we have

$$\begin{aligned} | [f(\alpha_2)+g(\alpha_2)]-[f(\alpha_1)+g(\alpha_1)] | &\leqslant | f(\alpha_2)-f(\alpha_1) | + | g(\alpha_2)-g(\alpha_2) | \\ &\leqslant (M'_r-m'_r)+(M''_r-m''_r). \end{aligned}$$

$$\therefore \qquad M_r-m_r \leqslant (M'_r-m'_r)+(M''_r-m''_r). \qquad \ldots (1)$$

Let $\varepsilon > 0$ be any given number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division of norm $\leqslant \delta$, the oscillatory sums of $f(x)$ and $g(x)$ are both less than $\frac{1}{2}\varepsilon$.

We now suppose that D is a division of norm $\leqslant \delta$, so that for D, we have from (1),

$$\Sigma(M_r-m_r)\delta_r \leqslant \Sigma(M'_r-m'_r)\delta_r+\Sigma(M''_r-m''_r)\delta_r < \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon,$$

i.e., the oscillatory sum $\Sigma(M_r-m_r)\delta_r$ of $f(x)+g(x)$ for the division D is less than ε. Thus $f(x)+g(x)$ is integrable in $[a, b]$.

Now to prove that

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Let ε be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division of norm $\leqslant \delta$ and for every choice of ξ_r in δ_r,

$$\left| \Sigma f(\xi_r)\delta_r - \int_a^b f(x)\,dx \right| < \frac{\varepsilon}{2},$$

$$\left| \Sigma g(\xi_r)\delta_r - \int_a^b g(x)\,dx \right| < \frac{\varepsilon}{2}.$$

Therefore

$$\left| \Sigma\Big[f(\xi_r)+g(\xi_r) \Big]\delta_r - \Big[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \Big] \right| < \varepsilon.$$

Thus we obtain

$$\int_a^b [f(x)+g(x)]\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx \text{ (§ 90).}$$

The case of difference may be similarly discussed.

Cor. The result can be extended, by Mathematical induction, to the case of algebraic sum of any **finite** number of functions.

93·23. Integrability of product. *If $f(x)$, $g(x)$ are two functions, both bounded and integrable in $[a, b]$, then their product $f(x)\,g(x)$ is also bounded and integrable in $[a, b]$.*

Since $f(x)$, $g(x)$ are bounded, there exists a number k, such that for every x in $[a, b]$,

$$|f(x)| \leqslant k, \ |g(x)| \leqslant k.$$

Therefore $|f(x)\,g(x)| \leqslant k^2$ in $[a, b]$ so that $f(x)\,g(x)$ is bounded.

Let

$$D(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be any division of $[a, b]$.

Let

$$M'_r, m'_r;\ M''_r, m''_r;\ M_r, m_r$$

be the bounds of $f(x)$, $g(x)$ and $f(x)\,g(x)$ in $\delta_r \equiv [x_{r-1}, x_r]$. If α_1, α_2 be any two points of δ_r, we have

$$f(\alpha_2)\,g(\alpha_2)-f(\alpha_1)\,g(\alpha_1)=g(\alpha_2)\,[f(\alpha_2)-f(\alpha_1)]+f(\alpha_1)\,[g(\alpha_2)-g(\alpha_1)]$$

or $$|f(\alpha_2)\,g(\alpha_2)-f(\alpha_1)\,g(\alpha_1)| \leqslant |g(\alpha_2)|\ |f(\alpha_2)-f(\alpha_1)| + |f(\alpha_1)|\ |g(\alpha_2)-g(\alpha_1)|$$

$$\leqslant k(M'_r-m'_r)+k(M''_r-m''_r).$$

$$\therefore \quad (M_r-m_r) \leqslant k(M'_r-m'_r)+k(M''_r-m''_r). \qquad \ldots (1)$$

Now let ε be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division of norm $\leqslant \delta$, the oscillatory sums of $f(x)$ and $g(x)$ are both $< \varepsilon/2k$.

We now suppose that D is a division of norm $\leqslant \delta$, so that for D, we have, from (1),

$$\Sigma(M_r-m_r)\delta_r < k\Sigma(M'_r-m'_r)\delta_r+k\Sigma(M''_r-m''_r)\delta_r$$
$$< k.(\varepsilon/2k)+k(\varepsilon/2k)=\varepsilon,$$

i.e., the oscillatory sum, $\Sigma(M_r-m_r)\delta_r < \varepsilon$.

Hence $f(x)\,g(x)$ is integrable in $[a, b]$.

Cor. The result can, by Mathematical induction, be extended to the product of a **finite** number of bounded and integrable functions.

Ex. Show by means of an example that a product of two non-integrable functions may be integrable.

93·24. Integrability of Quotient. *If $f(x)$, $g(x)$ are two functions, both bounded and integrable in $[a, b]$, and there exists a positive number, t, such that $|g(x)| \geqslant t$ in $[a, b]$, then $f(x)/g(x)$ is bounded and integrable in $[a, b]$.*

Since there exist positive numbers k and t such that in $[a, b]$

$$|f(x)| \leqslant k,\ |g(x)| \leqslant k,\ |g(x)| \geqslant t,$$

therefore

$$|f(x)/g(x)| \leqslant k/t,$$

in $[a, b]$. Hence $f(x)/g(x)$ is bounded.

Let

$$D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be any division of $[a, b]$ and let M'_r, m'_r; M''_r, m''_r; M_r, m_r be the bounds of $f(x)$, $g(x)$, $f(x)/g(x)$ in $\delta_r \equiv [x_{r-1}, x_r]$. If α_1, α_2 be any two points of δ_r,

$$\left|\frac{f(\alpha_2)}{g(\alpha_2)} - \frac{f(\alpha_1)}{g(\alpha_1)}\right| = \left|\frac{g(\alpha_1)[f(\alpha_2)-f(\alpha_1)]-f(\alpha_1)[g(\alpha_2)-g(\alpha_1)]}{g(\alpha_1)g(\alpha_2)}\right|$$

$$\leqslant (k/t^2)\,|f(\alpha_2)-f(\alpha_1)| + (k/t^2)\,|g(\alpha_2)-g(\alpha_1)|$$

$$\leqslant (k/t^2)(M'_r-m'_r)+(k/t^2)(M''_r-m''_r)$$

$$\therefore\ (M_r-m_r) \leqslant (k/t^2)(M'_r-m'_r)+(k/t^2)(M''_r-m''_r). \qquad \ldots (1)$$

Let, now ε be *any* positive number.

Since $f(x)$, $g(x)$ are integrable, there exists a positive number δ such that for every division D of norm $\leqslant \delta$, the oscillatory sums for $f(x)$, $g(x)$ are both less than $t^2\varepsilon/2k$.

We now suppose that D is any division of norm $\leqslant \delta$ so that for D, we have, from (1).

$$\Sigma(M_r-m_r)\delta_r \leqslant (k/t^2)\Sigma(M'_r-m'_r)\delta_r+(k/t^2)\Sigma(M''_r-m''_r)\delta_r$$

$$< (k/t^2)(t^2\varepsilon/2k)+(k/t^2)(t^2\varepsilon/2k)=\varepsilon.$$

Hence $f(x)/g(x)$ is bounded and integrable in $[a, b]$.

93·25. Integrability of the modulus of an integrable function. *If $f(x)$ is bounded and integrable in $[a, b]$, then $|f(x)|$ is also bounded and integrable in $[a, b]$.*

There exists a positive number k such that $|f(x)| \leqslant k$ so that $|f(x)|$ is bounded.

Let ε be *any* positive number.

Since $f(x)$ is integrable, there exists a division

$$D(a=x_0, x_1, x_2, \ldots\ldots\ldots, x_{r-1}, x_r, \ldots\ldots\ldots, x_n=b)$$

such that the corresponding oscillatory sum for $f(x)$ is $< \varepsilon$.

Let M'_r, m'_r; M_r, m_r, be the bounds of $f(x)$ and $|f(x)|$ in

$$\delta_r \equiv [x_{r-1}, x_r].$$

If α_1, α_2 be any two points of δ_r,

$$\big|\, [\, |f(\alpha_2)| - |f(\alpha_1)|\,]\, \big| \leqslant |f(\alpha_2)-f(\alpha_1)|$$
$$\leqslant M'_r - m'_r$$

$$\therefore \qquad M_r - m_r \leqslant M'_r - m'_r.$$

This gives

$$\Sigma(M_r - m_r)\delta_r \leqslant \Sigma(M'_r - m'_r)\delta_r < \varepsilon.$$

i.e., $$\Sigma(M_r - m_r)\delta_r < \varepsilon$$

Hence $|f(x)|$ is integrable in $[a, b]$.

Remarks. The *converse* of this theorem is *not* true. If we take

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational,} \\ -1, & \text{when } x \text{ is irrational,} \end{cases}$$

then

$$\overline{\int_a^b} f(x)\, dx = (b-a), \quad \underline{\int_a^b} f(x)\, dx = -(b-a),$$

so that

$$\int_a^b f(x)\, dx$$

does not exist.

But since $|f(x)| = 1$, for all x, therefore

$$\int_a^b |f(x)|\, dx \text{ exists and is equal to } (b-a).$$

94. Definition. *The meaning of*

$$\int_a^b f(x)\, dx,$$

when $b \leqslant a$.

If $f(x)$ be bounded and integrable in $[b, a]$ where $a > b$, then, by def.,

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

Also, by def., $\displaystyle\int_a^a f(x)\, dx = 0.$

It is easy to show that the results about integrals obtained in §§ 92,93 hold true when the upper limit is less than or equal to the lower limit.

Note. The reader may carefully note that the statement :

$$\text{“}\int_a^b f(x)\,dx \text{ exists,”}$$

means that $f(x)$ is bounded and integrable in $[a, b]$.

95. Inequalities for an integral.

Theorem. *If*

$$\int_a^b f(x)\,dx$$

exists and M, m are the bounds of $f(x)$ in $[a, b]$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a), \qquad \textit{if } b \geqslant a.$$

$$m(b-a) \geqslant \int_a^b f(x)\,dx \geqslant M(b-a), \qquad \textit{if } b \leqslant a.$$

For $a=b$, the result is trivial.

If $b > a$, then for any division D, we have

$$m(b-a) \leqslant s_D \leqslant \int_a^b f(x)\,dx \leqslant S_D \leqslant M(b-a), \qquad (\S 88)$$

i.e.,

$$m(b-a) < \int_a^b f(x)\,dx < M(b-a).$$

If $b < a$, *i.e.*, $a > b$, then, as proved above,

$$m(a-b) \leqslant \int_b^a f(x)\,dx \leqslant M(a-b).$$

$$\therefore \quad -m(a-b) \geqslant -\int_b^a f(x)\,dx \geqslant -M(a-b)$$

i.e., $$m(b-a) \geqslant \int_a^b f(x)\,dx \geqslant M(b-a).$$

Hence the result.

Cor. I. *If* $\int_a^b f(x)\,dx$ *exists, then there exists a number,* μ, *lying between the bounds of* $f(x)$ *such that*

$$\int_a^b f(x)\,dx = \mu(b-a).$$

Cor. 2. *If* $f(x)$ *is continuous in* $[a, b]$, *then there exists a number,* c, *lying between* a *and* b *such that*

$$\int_a^b f(x)\,dx = (b-a)\,f(c).$$

Note. We may write $c=a+\theta(b-a)$, where $0 \leqslant \theta \leqslant 1$.

Cor. 3. *If* $\int_a^b f(x)dx$ *exists, and,* k, *is a number such that for ali* x,

$$|f(x)| \leqslant k,$$

then $$\left|\int_a^b f(x)\,dx\right| \leqslant k\,|b-a|\,.$$

For $a=b$, the result is trivial.

We have

$$-k \leqslant f(x) \leqslant k,$$

so that if M, m be the bounds of $f(x)$ in $[a, b]$,

$$-k \leqslant m \leqslant f(x) \leqslant M \leqslant k, \qquad \dots (1)$$

Let $b > a$. Therefore, from 1,

$$-k(b-a) \leqslant m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a) \leqslant k(b-a)$$

or
$$\left|\int_a^b f(x)\,dx\right| \leqslant k\,|\,b-a\,|\,.$$

Let $b < a$. We have, from above,
$$\left|\int_b^a f(x)\,dx\right| \leqslant k\,|\,a-b\,|$$

i.e.,
$$\left|\int_a^b f(x)\,dx\right| \leqslant k\,|\,b-a\,|\,.$$

Cor. 4. *If* $\int_a^b f(x)\,dx$ *exists and* $f(x) \geqslant 0$, *then*
$$\int_a^b f(x)\,dx \begin{cases} \geqslant 0, & \text{when } b \geqslant a\,; \\ \leqslant 0, & \text{when } b \leqslant a. \end{cases}$$

For $b=a$, the result is trivial.

Since $f(x) \geqslant 0$, therefore, $m \geqslant 0$.

Let $b > a$. We have
$$\int_a^b f(x)\,dx \geqslant m(b-a) \geqslant 0. \qquad \because \quad (b-a) \geqslant 0$$

Let $b < a$. We have, as proved above,
$$\int_b^a f(x)\,dx \geqslant 0.$$

$$\therefore \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \leqslant 0.$$

Cor. 5. *If*
$$\int_a^b f(x)\,dx, \qquad \int_a^b g(x)\,dx$$

both exist and $f(x) \geqslant g(x)$, *then*

$$\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx, \textit{ when } b \geqslant a,$$

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx, \textit{ when } b \leqslant a.$$

Under the given condition $[f(x)-g(x)]$ is integrable and $\geqslant 0$. Therefore

$$\int_a^b [f(x)-g(x)]\,dx \geqslant 0 \text{ or } \leqslant 0,$$

according as $b \geqslant a$ or $b \leqslant a$,

i.e.,
$$\left[\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right] \geqslant 0 \text{ or } \leqslant 0,$$

according as $b \geqslant a$ or $b \leqslant a$.

Hence the result.

Cor. 6. *If*

$$\int_a^b f(x)\,dx,$$

exists, then

$$\left|\int_a^b f(x)\,dx\right| \leqslant \left|\int_a^b |f(x)|\,dx\right|.$$

It has been shown in § 93·25, page 163 that

$$\int_a^b |f(x)|\,dx$$

exists. We have

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|.$$

If $b \geqslant a$, we have

$$-\int_a^b |f(x)|\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b |f(x)|\,dx, \qquad \text{(cor. 5)}$$

or

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx = \left|\int_a^b |f(x)|\,dx\right|$$

If $b \leqslant a$, we have, as proved above,

$$\left|\int_b^a f(x)\,dx\right| \leqslant \left|\int_b^a |f(x)|\,dx\right|,$$

i.e.,

$$\left|\int_a^b f(x)\,dx\right| \leqslant \left|\int_a^b |f(x)|\,dx\right|.$$

96. Functions defined by definite integrals. If

$$\int_a^b f(x)\,dx$$

exists, then the function $\varphi(t)$ of t, $a < t < b$, where

$$\varphi(t) = \int_a^t f(x)\,dx,$$

is defined in $[a, b]$. It is, now, proposed to study the properties of the function $\varphi(t)$ of t, in relation to continuity and derivability.

The function $\varphi(t)$ may be called the *integral function* of $f(x)$.

96·1. Continuity of the integral function.

If

$$\int_a^b f(x)\,dx \textit{ exists,}$$

then

$$\varphi(t) = \int_a^t f(x)\, dx$$

is continuous in $[a, b]$.

There exists a number k such that $|f(x)| \leqslant k$ in $[a, b]$.

Let, c, be any point of $[a, b]$.

Let ε be *any* positive number. We have

$$\varphi(c) = \int_a^c f(x)\, dx, \quad \varphi(c+h) = \int_a^{c+h} f(x)\, dx.$$

$$\therefore \qquad |\varphi(c+h) - \varphi(c)| = \left| \int_c^{c+h} f(x)\, dx \right|$$

$$\leqslant |h|\, k. \qquad \text{(cor. 3, § 95)}$$

$$< \varepsilon, \text{ if } |h| < \varepsilon/k.$$

Hence $\varphi(t)$ is continuous at any point, c, of $[a, b]$ and so in the interval $[a, b]$.

Ex. *If* $\int_a^b f(x)\, dx$ exists, prove that

$$\varphi(t) = \int_t^b f(x)\, dx$$

is continuous in (a, b).

96·2. Derivability of the integral function.

If $f(x)$ *is continuous in* $[a, b]$, *then*

$$\varphi(t) = \int_a^t f(x)\, dx,$$

is derivable in $[a, b]$ *and*

$$\varphi'(t) = f(t).$$

Let, c, be any point of $[a, b]$. We have

$$\varphi(c+h)-\varphi(c)=\int_{c}^{c+h} f(x)\, dx=hf(c+\theta h) \text{ where } (0 \leqslant \theta \leqslant 1).$$

(cor. 2 § 95)

Since $f(x)$ is continuous at c, therefore, when $h \to 0$,

$$\lim f(c+\theta h)=f(c).$$

Hence

$$\lim_{h \to 0} \frac{\varphi(c+h)-\varphi(c)}{h} = \lim_{h \to 0} f(c+\theta h)=f(c).$$

Thus

$$\varphi'(c)=f(c).$$

As, c, is any point of $[a, b]$, we have in $[a, b]$,

$$\varphi'(t)=f(t).$$

Ex. If $f(x)$ is continuous, prove that

$$\psi(t)=\int_{t}^{b} f(x)\, dx$$

is derivable in $[a, b]$, and

$$\psi'(t)=-f(t).$$

Remarks. Primitive. If there exists a derivable function $\varphi(x)$ such that $\varphi'(x)$ is equal to a given function $f(x)$ in $[a, b]$, then we say that $\varphi(x)$ is a primitive of $f(x)$. The theorem above shows that *every continuous function possesses a primitive.*

Cor. *If a function $\varphi(x)$ possesses a continuous derivative $\varphi'(x)$, then*

$$\int_{a}^{b} \varphi'(x)\, dx=\varphi(b)-\varphi(a).$$

The continuous function $\varphi'(x)$ is integrable. Let

$$\psi(t)=\int_{a}^{t} \varphi'(x)\, dx.$$

Since $\varphi'(x)$ is continuous, we have $\psi'(t)=\varphi'(t)$. Therefore the functions $\varphi(t)$, $\psi(t)$ differ by a constant. Let

$$\varphi(t)=\psi(t)+k=\int_{a}^{t}\varphi'(x)\, dx+k,$$

where k is a constant. Therefore

$$\varphi(a)=0+k=k,$$

and
$$\varphi(b)=\int_a^b \varphi'(x)\,dx+k.$$

Hence
$$\varphi(b)-\varphi(a)=\int_a^b \varphi'(x)\,dx.$$

97. Fundamental theorem of Integral Calculus. *If*

$$\int_a^b f(x)\,dx$$

exists and there exists a function $\varphi(x)$ *such that*

$$\varphi'(x)=f(x) \text{ in } [a,\ b].$$

then
$$\int_a^b f(x)\,dx=\varphi(b)-\varphi(a).$$

Let ε be any positive number. Since $\varphi'(x)=f(x)$ is integrable in $[a, b]$, there exists a division

$$D(a=x_0,\ x_1,\ x_2,\ldots\ldots,\ x_{r-1},\ x_r,\ldots\ldots,\ x_n=b)$$

such that

$$\left|\sum_{r=1}^{r=n} \varphi'(\xi_r)\delta_r-\int_a^b \varphi'(x)dx\right|<\varepsilon.$$

We particularise the *arbitrary* point ξ_r of $\delta_r\equiv[x_{r-1},\ x_r]$, in the following manner:—

By the Lagrange's mean value theorem of Differential Calculus, there exists a point, ξ_r, of δ_r such that

$$\varphi(x_r)-\varphi(x_{r-1})=\varphi'(\xi_r)\,\delta_r.$$

$$\therefore \qquad \Sigma\varphi'(\xi_r)\delta_r=\Sigma[\varphi(x_r)-\varphi(x_{r-1})]=\varphi(b)-\varphi(a).$$

$$\therefore \qquad \left|\varphi(b)-\varphi(a)-\int_a^b \varphi'(x)\,dx\right|<\varepsilon.$$

As ε is an arbitrary positive number, we have

$$\varphi(b)-\varphi(a)-\int_a^b \varphi'(x)\,dx=0.$$

Hence the result.

Remarks. In the Cor. to the preceding section and in the present section, we have established the truth of the same equality, *viz.*,

$$\int_a^b \varphi'(x)\,dx=\varphi(b)-\varphi(a),$$

but the proofs are different, depending as they do upon the different conditions imposed upon the function in question.

For the validity of the proof given in the Cor., $\varphi'(x)$ has to be assumed *continuous* but for the validity of the proof in the present section we require $\varphi'(x)$ to be *merely bounded and integrable*. Thus the present proof seems to be more valuable, but, the reader would do well to notice that the value of the theorem in the preceding section lies in the fact that it sets down a *sufficient* condition for the *existence* of a function $\varphi(x)$ whose derivative is a given function $f(x)$; such a function $\varphi(x)$ is usually known as the primitive or the indefinite integral of $f(x)$. Thus the theorem of the preceding section may be stated a little differently as follows :—

A sufficient condition for a function $f(x)$ to possess a primitive is that $f(x)$ is continuous and the primitive is, then, given by

$$\int_a^x f(x)\,dx+k$$

where, k, is any constant whatsoever.

The present theorem is quite unconcerned with the question of the *existence* of a primitive of $f(x)$: it simply states that if a bounded and integrable function $f(x)$ does possess a primitive $\varphi(x)$, then

$$\int_a^b f(x)\,dx=\int_a^b \varphi'(x)\,dx=\varphi(b)-\varphi(a).$$

98. Mean value theorem of Integral Calculus.

98·1. First mean value theorem. *If*

$$\int_a^b f(x)\,dx \text{ and } \int_a^b \varphi(x)\,dx$$

both exist and $\varphi(x)$ keeps the same sign, positive or negative, throughout the interval of integration, then there exists a number, μ, lying between the upper and lower bounds of $f(x)$ such that

$$\int_a^b f(x)\,\varphi(x)\,dx = \mu \int_a^b \varphi(x)\,dx \qquad \ldots(1)$$

Firstly suppose that $\varphi(x)$ is positive. If M, m be the bounds of $f(x)$,

$$m \leqslant f(x) \leqslant M.$$

$$\therefore \qquad m\varphi(x) \leqslant f(x)\,\varphi(x) \leqslant M\varphi(x), \text{ for } \varphi(x) \geqslant 0$$

$$i.e., \quad \left.\begin{aligned} m\int_a^b \varphi(x)\,dx &\leqslant \int_a^b f(x)\,\varphi(x)\,dx \leqslant M\int_a^b \varphi(x)\,dx, \\ m\int_a^b \varphi(x)\,dx &\geqslant \int_a^b f(x)\,\varphi(x)\,dx \geqslant M\int_a^b \varphi(x)\,dx, \end{aligned}\right\} \begin{aligned} &\text{if } b \geqslant a\,; \\ &\text{if } b \leqslant a. \\ &(\S\ 95 \text{ cor. } 5) \end{aligned}$$

In either case we see that there exists a number, μ, lying between M and m, such that (1) is true. Hence the result.

The case when $\varphi(x)$ is negative may be similarly disposed of.

Cor. In addition to the conditions of the theorem, if $f(x)$ is continuous also, then there exists a number, ξ, belonging to the range of integration such that

$$\int_a^b f(x)\,\varphi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx.$$

93·2. Second mean value theorem. *If*

$$\int_a^b f(x)\,dx \quad \textit{and} \quad \int_a^b \varphi(x)\,dx$$

both exist and $\varphi(x)$ is monotonic in $[a, b]$, then there exists a point, ξ, of $[a, b]$ such that

$$\int_a^b f(x)\,\varphi(x)\,dx = \varphi(a)\int_a^\xi f(x)\,dx + \varphi(b)\int_\xi^b f(x)\,dx.$$

(This theorem is due to *Weierstrass*.)

99. Abel's Lemma The proof of the theorem depends upon a lemma which is due to Abel which we now state and prove.

If

(i) $a_1, a_2, \ldots\ldots, a_n$ *is a monotonically decreasing set of n positive numbers,*

(ii) $v_1, v_2, \ldots\ldots, v_n$ *is a set of any, n, numbers,*

and (iii) k, K *are two numbers such that*

$$k < v_1+v_2+\ldots\ldots+v_p < K. \qquad \textit{for } 1 \leqslant p \leqslant n.$$

then

$$a_1 k < \sum_{r=1}^{r=n} a_r v_r < a_1 K.$$

We write

$$S_p = v_1+v_2+\ldots\ldots+v_p.$$

We have

$$\sum_{r=1}^{r=n} a_r v_r = a_1 S_1 + a_2(S_2-S_1)+\ldots+a_r(S_r-S_{r-1})+\ldots+a_n(S_n-S_{n-1})$$

$$=(a_1-a_2)S_1+(a_2-a_3)S_2+\ldots+(a_{n-1}-a_n)S_{n-1}+a_n S_n.$$

Now, by *(i)*,

$$(a_1-a_2),\ (a_2-a_3), \ldots,\ (a_{n-1}-a_n),\ a_n$$

are all positive. Also, by *(iii)*,

$$k < S_p < K, \text{ for all } p \leqslant n.$$

Therefore

$$\sum_{r=1}^{r=n} a_r v_r < (a_1-a_2)K+(a_2-a_3)K+\ldots+(a_{n-1}-a_n)K+a_n K = a_1 K,$$

$$\underset{r=1}{\overset{r=n}{K}} a_r v_r > (a_1-a_2)k+(a_2-a_3)k+\ldots+(a_{n-1}-a_n)k+a_n k = a_1 k.$$

Hence the lemma.

Proof of the theorem. Firstly, we prove the following :—

If $\int_a^b f(x)\,dx$ *and* $\int_a^b \psi(x)\,dx$ *both exist and* $\psi(x)$ *is monotonically*

decreasing *and* **positive** *in* $[a, b]$, *then there exists a point*, ξ, *of* $[a, b]$ *such that*

$$\int_a^b f(x)\,\psi(x)\,dx = \psi(a)\int_a^\xi f(x)\,dx.$$

(This result is due to Bonnett.)

Let

$$D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b)$$

be any division of $[a, b]$. Let M_r, m_r be the bounds of $f(x)$ in $\delta_r \equiv [x_{r-1}, x_r]$. Let $\xi_1 = a$ and ξ_r, when $r \neq 1$, be any arbitrary point of δ_r.

We have

$$\left.\begin{aligned} m_r\delta_r \leqslant \int_{x_{r-1}}^{x_r} f(x)\,dx \leqslant M_r\delta_r \\ m_r\delta_r \leqslant f(\xi_r)\delta_r \leqslant M_r\delta_r \end{aligned}\right\}$$

Putting $r=1, 2, 3, \ldots\ldots, p$ where $p \leqslant n$, and adding we obtain

$$\left.\begin{aligned} \sum_{r=1}^{r=p} m_r\delta_r \leqslant \int_a^{x_p} f(x)\,dx \leqslant \sum_{r=1}^{r=p} M_r\delta_r \\ \sum_{r=1}^{r=p} m_r\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\delta_r \leqslant \sum_{r=1}^{r=p} M_r\delta_r \end{aligned}\right\}$$

This gives

$$\left|\int_a^{x_p} f(x)\,dx - \sum_{r=1}^{r=p} f(\xi_r)\delta_r\right| \leqslant \sum_{r=1}^{r=p} (M_r - m_r)\delta_r \leqslant \sum_{r=1}^{r=n} (M_r - m_r)\delta_r,$$

or $$\int_a^{x_p} f(x)\,dx - \sum_{r=1}^{r=n} O_r\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\delta_r \leqslant \int_a^{x_p} f(x)\,dx + \sum_{r=1}^{r=n} O_r\delta_r.$$

where $O_r = (M_r - m_r)$ is the oscillation of $f(x)$ in δ_r.

Now, $\int^t f(x)\,dx$, being a continuous function of t, (§§96·1, 58·3) is bounded. Let C, D be the bounds. Therefore we have

$$C-\sum_{r=1}^{r=n} O_r\,\delta_r \leqslant \sum_{r=1}^{r=p} f(\xi_r)\delta_r \leqslant D+\sum_{r=1}^{r=n} O_r\,\delta_r.$$

In the Abel's lemma, we put, as is justifiable,

$$v_r=f(\xi_r)\delta_r,\ a_r=\psi(\xi_r)\ ;$$
$$k=C-\Sigma\ O_r\delta_r, \qquad K=D+\Sigma\ O_r\delta_r,$$

and obtain

$$\psi(a)\left[C-\sum_{r=1}^{r=n} O_r\delta_r\right] \leqslant \sum_{r=1}^{r=n} f(\xi_r)\psi(\xi_r)\delta_r \leqslant \psi\ (a)\left[D+\sum_{r=1}^{r=n} O_r\delta_r\right].$$

Let the norm of the division tend to 0. We then obtain, in the limit,

$$C\psi(a) \leqslant \int_a^b f(x)\ \psi(x)\ dx \leqslant D\psi(a).$$

or
$$\int_a^b f(x)\ \psi(x)\ dx=\mu\psi(a),$$

where μ is some number between C and D.

The continuous function

$$\int_a^t f(x)\ dx$$

of, t, must assume, for some value ξ of t, the value μ which lies between its bounds C, D. (Cor. 1 to § 58·4). Thus we obtain

$$\int_a^b f(x)\ \psi(x)\ dx=\psi(a)\int_a^\xi f(x)\ dx.$$

We now turn to the theorem proper.

Let $\varphi(x)$ be monotonically decreasing so that

$$\psi(x)=\varphi(x)-\varphi(b)$$

is monotonically decreasing and positive.

There exists, therefore, a number, ξ, between a and b, such that

$$\int_a^b f(x)\ [\varphi(x)-\varphi(b)]\ dx=[\varphi(a)-\varphi(b)]\int_a^\xi f(x)\ dx$$

$$\text{or} \quad \int_a^b f(x)\,\varphi(x)\,dx=\varphi(a)\int_a^\xi f(x)\,dx+\varphi(b)\left\{\int_a^b f(x)\,dx-\int_a^\xi f(x)\,dx\right\}$$

$$=\varphi(a)\int_a^\xi f(x)\,dx+\varphi(b)\int_\xi^b f(x)\,dx.$$

Let $\varphi(x)$ *be monotonically increasing* so that, $-\varphi(x)$, is monotonically decreasing.

There exists, therefore, by the preceding, a number between a and b, such that

$$\int_a^b f(x)[-\varphi(x)]\,dx=-\varphi(a)\int_a^\xi f(x)\,dx-\varphi(b)\int_\xi^b f(x)\,dx,$$

i.e.,
$$\int_a^b f(x)\,\varphi(x)\,dx=\varphi(a)\int_a^\xi f(x)\,dx+\varphi(b)\int_\xi^b f(x)\,dx.$$

Thus we have completely established the theorem.

Note. The reader may easily show that the theorem holds good even if $a > b$.

Ex. 1. Taking $f(x)=x$ and $\varphi(x)=e^x$, verify the two mean value theorems for the interval $[-1, 1]$.

Ex. 2. Show that the second mean value theorem does not hold good in the interval $[-1, 1]$ for $f(x)=\varphi(x)=x^2$. What about the validity of the first mean value theorem in the same case ?

100. Change of variable in an integral. *If*

(i) $\int_a^b f(x)\,dx$ *exists,*

(ii) $x=\varphi(t)$ *is a derivable function in* $[\alpha, \beta]$ *and* $\varphi'(t) \neq 0$ *for any value of* t, *and* $\varphi(\alpha)=a$, $\varphi(\beta)=b$,

(iii) $f[\varphi(t)]$ *and* $\varphi'(t)$ *are bounded and integrable in* $[\alpha, \beta]$.

then
$$\int_a^b f(x)\,dx=\int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt.$$

Since $\varphi'(t) \neq 0$, it follows by Darboux's theorem (§70, page 115), that $\varphi'(t)$ must always have the same sign and therefore $\varphi(t)$ must be strictly monotonic in $[\alpha, \beta]$ (§ 74, page 118).

Let

$$D(\alpha=t_0, t_1, t_2, \ldots\ldots, t_{r-1}, t_r, \ldots\ldots, t_{n-1}, t_n=\beta)$$

be any division of $[\alpha, \beta]$ and let

$$D'(a=x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_{n-1}, x_n=b)$$

be the corresponding division of $[a, b]$; $\varphi(t_r)$ being equal to x_r.

By the Lagrange's mean value theorem, we have

$$x_r - x_{r-1} = \varphi(t_r) - \varphi(t_{r-1}) = (t_r - t_{r-1})\varphi'(\eta_r),$$

where η_r lies between t_{r-1} and t_r.

Let

$$\varphi(\eta_r) = \xi_r.$$

We have

$$\sum_{r=1}^{r=n} f(\xi_r)\,(x_r - x_{r-1}) = \sum_{r=1}^{r=n} f[\varphi(\eta_r)]\,\varphi'(\eta_r)(t_r - t_{r-1}). \qquad \ldots(1)$$

Now

$f(x)$ is integrable in $[a, b]$.

Further $f[\varphi(t)]$, $\varphi'(t)$ are integrable in $[\alpha, \beta]$ so that $f[\varphi(t)]\varphi'(t)$ is also integrable in $[\alpha, \beta]$.

As the norm of the division $D \to 0$, the norm of D' also $\to 0$. From (1), therefore, we obtain in the limit

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\varphi(t)]\,\varphi'(t)\,dt.$$

Note. *The theorem holds even if* $\varphi'(t)=0$ *for a* **finite** *number of values of* t *belonging to* $[\alpha, \beta]$. In this case we can divide the range $[\alpha, \beta]$ into a finite number of ranges in each of which $\varphi(t)$ is strictly increasing or decreasing and repeat the argument for each interval in turn and add the results.

101. Integration by parts. *If*

$$\int_a^b f(x)\,dx, \quad \int_a^b g(x)\,dx$$

both exist and

$$F(x) = A + \int_a^x f(x)\,dx, \quad G(x) = B + \int_a^x g(x)\,dx$$

where A, B, are two constants, then,

$$\int_a^b F(x)\,g(x)\,dx = \Big| F(x)\;G(x) \Big|_a^b - \int_a^b G(x)\,f(x)\,dx.$$

[Here $\Big| F(x)\,G(x) \Big|_a^b$ denotes the difference $[F(b)\,G(b)-F(a)G(a)]$.

Proof. Let

$$D(a=x_0,\, x_1,\, x_2,\, x_3,\, \ldots\ldots x_{r-1},\, x_r, \ldots\ldots,\, x_n=b)$$

be any division of $[a, b]$.

We have

$$\Big| F(x)\,G(x) \Big|_a^b = \sum_{r=1}^{r=n} [F(x_r)\,G(x_r)-F(x_{r-1})\,G(x_{r-1})]$$

$$= \Sigma\, F(x_r)[G(x_r)-G(x_{r-1})]+\Sigma\, G(x_{r-1})[F(x_r)-F(x_{r-1})]$$

$$= \Sigma\, F(x_r)\int_{x_{r-1}}^{x_r} g(x)\,dx + \Sigma\, G(x_{r-1})\int_{x_{r-1}}^{x_r} f(x)\,dx. \quad \ldots (1)$$

Let M_r, m_r, O_r denote the bounds and oscillation of $f(x)$ and M'_r, m'_r, O'_r those of $g(x)$ in $\delta_r \equiv [x_{r-1}, x_r]$.

For every x in δ_r,

$$| g(x)-g(x_r) | \leqslant O'_r,\; | f(x)-f(x_{r-1}) | \leqslant O_r,$$

i e.,

$$\left.\begin{aligned} g(x_r)-O'_r \leqslant g(x) \leqslant g(x_r)+O'_r\,; \\ f(x_{r-1})-O_r \leqslant f(x) \leqslant f(x_{r-1})+O_r, \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\quad [g(x_r)-O'_r]\,\delta_r \leqslant \int_{x_{r-1}}^{x_r} g(x)\,dx \leqslant [g(x_r)+O'_r]\,\delta_r; \\ [f(x_{r-1})-O_r]\,\delta_r \leqslant \int_{x_{r-1}}^{x_r} f(x)\,dx \leqslant [f(x_{r-1})+O_r]\,\delta_r \end{aligned}\right\}.$$

These give

$$\left.\begin{aligned} \int_{x_{r-1}}^{x_r} g(x)\, dx &= [g(x_r)+\theta'_r\, O'_r]\delta_r \\ \int_{x_{r-1}}^{x_r} f(x)\, dx &= [f(x_{r-1})+\theta_r O_r]\delta_r \end{aligned}\right\} \quad \begin{aligned} &\ldots(2) \\ &\ldots(3) \end{aligned}$$

where, $-1 \leqslant \theta_r, \theta'_r \leqslant 1$.

From (1), (2) and (3) we obtain

$$\Big| F(x)\, G(x) \Big|_a^b = \Sigma\, F(x_r)\, g(x_r)\delta_r + \Sigma\, G(x_{r-1})\, f(x_{r-1})\delta_r + \sigma, \quad \ldots(4)$$

where

$$\sigma = \Sigma[(F(x_r)\theta'_r\, O'_r + G(x_{r-1})\, \theta_r\, O_r)]\, \delta_r.$$

Since $F(x)$, $G(x)$ are continuous, therefore they are bounded. Let k be a number such that

$$| F(x) | \leqslant k, \quad | G(x) | \leqslant k.$$

$$\therefore \qquad | \sigma | \leqslant k(\Sigma\, O_r + \Sigma\, O'_r)\delta_r.$$

Let the norm of the division $D \to 0$. Then $\sigma \to 0$.

From (4), we now obtain

$$\Big| F(x)\, G(x) \Big|_a^b = \int_a^b F(x)\, g(x)\, dx + \int_a^b G(x)\, f(x)\, dx.$$

Hence the result.

Cor. If a function $g(x)$ is bounded and integrable in $[a, b]$ and a function $f(x)$ is derivable in $[a, b]$ and the derivative $f'(x)$ is bounded and integrable, then

$$\int_a^b f(x)\, g(x)\, dx = \left| f(x) \int_a^x g(x)\, dx \right|_a^b - \int_a^b \left\{ f'(x) \int_a^x g(x)\, dx \right\} dx$$

$$= f(b) \int_a^b g(x)\, dx - \int_a^b \left\{ f'(x) \int_a^b g(x)\, dx \right\} dx.$$

Examples

1. *If* $G(x, \xi) = \begin{cases} x(\xi-1), & \text{when } x \leqslant \xi \\ \xi(x-1), & \text{when } \xi < x, \end{cases}$

and if $f(x)$ is a continuous function of x in $0 \leqslant x \leqslant 1$,

and if
$$g(x) = \int_0^1 f(\xi)\, G(x, \xi)\, d\xi,$$

show that
$$g''(x) = f(x),$$
and find $g(0)$ and $g(1)$.

We have

$$g(x) = \int_0^x f(\xi)\, \xi(x-1)\, d\xi + \int_x^1 f(\xi)\, x(\xi-1)\, d\xi$$

$$= (x-1) \int_0^x \xi f(\xi)\, d\xi + x\int_x^1 (\xi-1) f(\xi)\, d\xi$$

$$= x \int_0^1 \xi f(\xi)\, d\xi - \int_0^x \xi f(\xi)\, d\xi - x\int_x^1 f(\xi)\, d\xi.$$

$$\therefore \qquad g'(x) = \int_0^1 \xi f(\xi)\, d\xi - xf(x) + xf(x) - \int_x^1 f(\xi)\, d\xi.$$

$$\therefore \qquad g''(x) = f(x).$$

We may easily see that $g(0)$ and $g(1)$ are both zero.

2. *Prove that if the functions $f(x)$ and $\phi(x)$ are bounded and integrable in $[a, b]$, then*

$$\left[\int_a^b f(x)\, \phi(x)\, dx\right]^2 \leqslant \int_a^b [f(x)]^2\, dx . \int_a^b [\phi(x)]^2\, dx.$$

Under what conditions does the sign of equality hold?

We have

$$\left[\int_a^b f(x)\phi(x)\,dx\right]^2 = [\lim \Sigma(x_r - x_{r-1})\, f(\xi_r)\phi(\xi_r)]^2$$

$$\int_a^b [f(x)]^2\,dx = \lim \Sigma[\sqrt{(x_r - x_{r-1})}\, f(\xi_r)]^2$$

$$\int_a^b [\varphi(x)]^2\,dx = \lim \Sigma[\sqrt{(x_r - x_{r-1})}\phi(\xi_r)]^2.$$

Now putting

$$a_r = \sqrt{(x_r - x_{r-1})}\, f(\xi_r),\ b_r = \sqrt{(x - x_{r-1})}\varphi(\xi_r).$$

in the Cauchy's inequality

$$\Sigma a^2{}_r\ \Sigma b^2{}_r \geqslant (\Sigma a_r\, b_r)^2,$$

we get the required result.

The sign of equality holds only when

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots\ldots$$

i.e.,
$$\frac{f(\xi_1)}{\phi(\xi_1)} = \frac{f(\xi_2)}{\phi(\xi_2)} = \ldots\ldots$$

i.e., when $f(x)$, $\phi(x)$ are both constants.

3. *If $f(x)$ is positive and monotonically decreasing in* $[1, \infty[$, *show that the sequence $\{A_n\}$, where*

$$A_n = \left\{ f(1) + f(2) + \ldots\ldots + f(n) - \int_1^n f(x)\,dx \right\},$$

is convergent.

Deduce the convergence of $(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \dfrac{1}{n} - \log n)$.

We have

$$A_n = \left[f(1) - \int_1^2 f(x)\,dx\right] + \left[f(2) - \int_2^3 f(x)\,dx\right] + \ldots\ldots$$

$$+ \ldots\ldots + \left[f(n-1) - \int_{n-1}^n f(x)\,dx\right] + f(n).$$

Now, because of the montonic character of $f(x)$, each of the expressions within brackets is positive. Also $f(n)$, is positive. Thus A_n is positive. Again

$$A_{n+1}-A_n=f(n+1)-\int_1^{n+1} f(x)\,dx+\int_1^n f(x)\,dx$$

$$=f(n+1)-\int_n^{n+1} f(x)\,dx < 0.$$

$\therefore \quad A_{n+1} < A_n.$

Thus, A_n is monotonically decreasing. Also being positive, A_n is bounded below. Hence $\{A_n\}$ is convergent.

Taking $f(x)=1/x$, we can now deduce the required result.

$$\lim_{n\to\infty} \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots+\frac{1}{n}-\log n\right)$$

is known as **Euler's constant.**

4. *Show that the function $F(x)$ defined in the interval* $[0, 1]$ *by the condition that if, r, is a positive integer*

$F(x)=2rx$ *when* $1/(r+1) < x < 1/r$,

is integrable over $[0, 1]$ *and that*

$$\int_0^1 F(x)\,dx=\frac{\pi^2}{6}.$$

The function $F(x)$, as given, is not defined at the set of points

$$0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ldots\ldots\ldots,\ \frac{1}{r},\ldots\ \ldots\ldots \qquad \ldots (1)$$

We may, however, define $F(x)$ at these points in any manner we please provided $F(x)$ remains bounded.

Now, the only points of discontinuity of $F(x)$ are those given above in (1). The aggregate formed by these points is infinite having only one limiting point *viz.*, 0. Thus the function is integrable.

Consider

$$\psi(\varepsilon)=\int_\varepsilon^1 F(x)\,dx.$$

We know that $\psi(\varepsilon)$ is a continuous function of ε, so that

$$\psi(0)=\int_0^1 F(x)\,dx=\lim_{\varepsilon\to 0}\psi(\varepsilon).$$

We, now, find $\psi(\varepsilon)$. We take $\varepsilon=1/n$ so that $\varepsilon\to 0$ as $n\to\infty$. We have

$$\int_{\frac{1}{n}}^{1} F(x)\,dx=\int_{\frac{1}{2}}^{1} F(x)\,dx+\int_{\frac{1}{3}}^{\frac{1}{2}} F(x)\,dx+\ldots+\int_{1/(r+1)}^{1/r} F(x)\,dx+\ldots\ldots+\int_{\frac{1}{n}}^{\frac{1}{n-1}} F(x)\,dx$$

Now $$\int_{1/(r+1)}^{1/r} F(x)\,dx=\int_{1/(r+1)}^{1/r} 2\,rx\,dx=\frac{2r+1}{r(r+1)^2}.$$

$$\therefore\quad \int_{\frac{1}{n}}^{1} F(x)\,dx=\sum_{r=1}^{n-1}\frac{2r+1}{r(r+1)^2}=\sum_{r=1}^{n-1}\left(\frac{1}{r}-\frac{1}{r+1}+\frac{1}{(r+1)^2}\right).$$

$$=\sum_{r=1}^{n-1}\left(\frac{1}{r}-\frac{1}{r+1}\right)+\sum_{r=1}^{n-1}\frac{1}{(r+1)^2}$$

$$=1-\frac{1}{n}+\sum_{r=1}^{n-1}\frac{1}{(r+1)^2}.$$

We know that the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots$$

is convergent and its sum is $\pi^2/6$.

$$\therefore\quad \lim_{n\to\infty}\int_{\frac{1}{n}}^{1} F(x)\,dx=1+\frac{\pi^2}{6}-1=\frac{\pi^2}{6}.$$

Thus $$\int_0^1 F(x)\,dx=\frac{\pi^2}{6}.$$

5. *Show that, when* $-1<x\leqslant 1$,

$$\lim_{m\to\infty}\int_0^x \frac{t^m}{t+1}\,dt=0.$$

Let $0\leqslant x\leqslant 1$. Then

$$0\leqslant\int_0^x \frac{t^m dt}{1+t}\leqslant\int_0^x t^m dt=\frac{x^{m+1}}{m+1}<\frac{1}{m+1}.$$

Let $-1 < x < 0$. Putting $t=-u$, we obtain

$$\left|\int_0^x \frac{t^m}{1+t}\,dt\right| = \left|\int_0^{-x} \frac{u^m}{1-u}\,du\right| < \frac{1}{1+x}\int_0^{-x} u^m du < \frac{1}{(m+1)(x+1)}.$$

Hence the result.

6. *Show that, when* $|x| \leq 1$,

$$\int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \frac{1}{13}x^{13} + \ldots\ldots\ldots$$

We have

$$\frac{1}{1+t^4} = 1 - t^4 + t^8 - t^{12} + \ldots\ldots\ldots + (-1)^{n-1}t^{4n-4} + \frac{(-1)^n t^{4n}}{1+t^4}.$$

$$\therefore \int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \frac{1}{13}x^{13} + \ldots\ldots + \frac{(-1)^{n-1}x^{4n-3}}{4n-3} +$$

$$(-1)^n \int_0^x \frac{t^{4n}}{1+t^4}\,dt.$$

Now, we have

$$0 \leq \left|\int_0^x \frac{t^{4n}}{1+t^4}\,dt\right| < \left|\int_0^x t^{4n}\,dt\right| = \left|\frac{x^{4n+1}}{4n+1}\right| < \frac{1}{4n+1},$$

so that

$$\lim_{n\to\infty} \int_0^x \frac{t^{4n}}{1+t^n}\,dt = 0.$$

Hence the result.

Exercises

1. If $f(y, x) = 1+2x$, for y rational

$f(y, x) = 0$, for y irrational;

calculate

$$F(y) = \int_0^1 f(y, x)\,dx.$$

2. Integrate in $[0, 2]$ the function $x[x]$, where $[x]$ denotes the greatest integer not greater than x.

3. Evaluate

$$\int_0^2 f(x)\,dx,$$

where

$f(x)=0$, when $x=n/(n+1)$, $(n+1)/n$, $(n=1, 2, 3, \ldots)$.
$f(x)=1$, elsewhere.

Is $f(x)$ integrable in $[0, 2]$?

Examine for continuity the function $f(x)$ so defined at the point $x=1$.

4. If a, b are positive and p is a positive integer, show that, as $n \to \infty$,

(i) $\sum_{r=1}^{pn} \dfrac{1}{na+r} \to \log\left(1+\dfrac{p}{a}\right)$,

(ii) $\sum_{r=1}^{pn} \dfrac{1}{na+rb} \to \dfrac{1}{b}\log\left(1+\dfrac{pb}{a}\right)$.

5. A function $f(x)$ is defined, for $x \geqslant 0$, by

$$f(x)=\int_{-1}^{1} \frac{dt}{\sqrt{(1-2tx+x^2)}}.$$

Prove that if $0 \leqslant x \leqslant 1$, $f(x)=2$. What is the value of $f(x)$ if $x>1$? Has the function $f(x)$ a differential co-efficient for $x=1$?

[For $x>1$, $f(x)=2/x$; $f(x)$ is not derivable for $x=1$ even though it is continuous thereat].

6. If, for $x \geqslant 0$, $\varphi(x)$ is defined as

$$\lim_{n\to\infty} \frac{x^n+2}{x^n+1},$$

and

$$f(x)=\int_0^x \varphi(t)\,dt,$$

prove that $f(x)$ is continuous but not differentiable for $x=1$.

7. If $f(x, y)=xy^2e^{-xy}+x^2y/(1+y)$ and a, b, are positive, show that

$$\lim_{y\to\infty}\int_a^b f(x, y)\,dx=\int_a^b\left[\lim_{y\to\infty} f(x, y)\right]dx.$$

Also show that the equality does not hold for $a=0$.

8. $f(x)$ is bounded and integrable in $[a, b]$; show that

$$\int_a^b [f(x)]^2\,dx=0,$$

if, and only if, $f(c)=0$ at every point, c, of continuity of $f(x)$.

9. If $a>0$, $n>0$, show that

$$0<n\int_0^1 \frac{x^{n-1}}{(1+x^2)^a}\,dx<1.$$

10. The functions $f(x)$ and $g(x)$ are bounded and integrable in $[a, b]$. If, further

$$F(x) = \int_a^x f(t)\, dt \text{ and } H(x) = \int_a^x f(t)\, g(t)\, dt, \text{ where } a \leqslant x \leqslant b$$

and if $F'(x) = f(x)$ and $g(x)$ is continuous, show that

$$H'(x) = f(x)\, g(x).$$

11. A function $f(x)$ is integrable in $[a-c, a+c]$ and
$|f(x)| \leqslant M$ in $[a-c, a+c]$,

$$\int_{a-c}^{a+c} f(x)\, dx = 0 \text{ and } F(x) = \int_{a-c}^{x} f(t)\, dt.$$

Prove that

$$\left| \int_{a-c}^{a+c} F(x)\, dx \right| \leqslant Mc^2.$$

12. Prove that

$$\lim_{x \to \infty} e^{-x^2} \int_0^x e^{t^2} dt = 0.$$

13. If $f(x)$ be defined in the interval $[0, 1]$ by the condition that if, r, is a positive integer, $r = 1, 2, 3, \ldots\ldots\ldots$

$$f(x) = (-1)^{r-1}, \text{ when } 1/(r+1) < x < 1/r,$$

prove that

$$\int_0^1 f(x)\, dx = \log 4 - 1.$$

14. Show that the function $f(x)$ defined in the interval $[0, 1]$ by the conditions that

$$f(x) = \frac{1}{2^n} \text{ when } \frac{1}{2^{n+1}} < x \leqslant \frac{1}{2^n},\quad n = 0, 1, 2, \ldots\ldots$$

and $\quad f(0) = 0,$

is integrable over the interval $[0, 1]$ and find the value of

$$\int_0^1 f(x)\, dx.$$

15. A function $f(x)$ is defined in $[0, 1]$ as follows :

$$f(x)=\frac{1}{a^{r-1}} \text{ where } \frac{1}{a^r}<x\leqslant\frac{1}{a^{r-1}} \text{ for } r=1, 2, 3, \ldots\ldots$$

where a is an integer greater than 2.

Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$.

16. If $f(x)=0$ for all values in the closed interval $[0, 1]$ except at a sequence of points

$$x_1, x_2, \ldots\ldots\ldots, x_n, \ldots\ldots\ldots$$

and $f(x_n)=1/\sqrt{n}$; show that $f(x)$ is integrable in $[0, 1]$.

17. By repeatedly employing the method of integration by parts to the integral

$$\int_0^x e^{-t}\,dt,$$

show that

$$e^x=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots\ldots+\frac{x^n}{n\,!}+e^x\int_0^x \frac{t^n}{n\,!}\,e^{-t}\,dt,$$

and deduce the Maclaurin's infinite series for e^x.

18. Obtain, by integration, from the identity,

$$\frac{1}{1+t}=1-t+t^2-t^3+\ldots\ldots+(-1)^{n-1}\,t^{n-1}+(-1)^n\,\frac{t^n}{1+t},$$

the Maclaurin's infinite series for $\log(1+x)$ in $[-1, 1]$.

19. Show that if $a, b>0$,

$$\frac{t^{a-1}}{1+t^b}=t^{a-1}-t^{a+b-1}+\ldots+(-1)^{n-1}\,t^{a+\overline{n-1}\,b-1}+(-1)^n\,\frac{t^{a+nb-1}}{1+t^b},$$

and justify the equation

$$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt=\frac{1}{a}-\frac{1}{a+b}+\frac{1}{a+2b}-\frac{1}{a+3b}+\ldots\ldots,$$

Deduce that $\frac{1}{1}-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\ldots=\frac{1}{3}\left(\frac{\pi}{\sqrt{3}}+\log 2\right)$.

20. Prove Bonnet's form of the second mean value theorem that if $f'(x)$ is continuous and of constant sign and $f(b)$ has the same sign as $f(a)-f(a)$, then

$$\int_a^b f(x)\,\varphi(x)dx=f(a)\int_a^\xi \varphi(x)\,dx$$

where ξ lies between a and b.

CHAPTER VII

UNIFORM CONVERGENCE

Analytical Theory of Trigonometric Functions

102. Limit function of a convergent sequence of Functions. Let $\{S_n(x)\}$, *i.e.*,

$$S_1(x),\ S_2(x), \ldots\ldots, S_n(x), \ldots\ldots$$

be a sequence, every member of which is a function of x defined in some interval $[a, b]$.

To each point, c, of $[a, b]$, there corresponds a sequence

$$S_1(c),\ S_2(c), \ldots\ldots\ ,\ S_n(c), \ldots\ldots$$

of constant numbers. We suppose that all such sequences obtained by taking different points of $[a, b]$ are convergent. The limiting values of these sequences define a function, say, $S(x)$, such that the value $S(c)$ of this function for a value, c, of x is the limiting value of the convergent sequence $\{S_n(c)\}$. This function $S(x)$ is said to be the *limit function* or *the limit of the convergent sequence* $\{S_n(x)\}$.

Again, let $\{f_n(x)\}$ be any given sequence of functions.

Consider the infinite series

$$f_1(x)+f_2(x)+\ \ldots\ldots+f_n(x)+\ldots\ldots \qquad \ldots(1)$$

This series gives rise to a sequence $\{S_n(x)\}$, of function where

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x).$$

The series (1) is said to be a convergent, if the sequence $\{S_n(x)\}$ is convergent and the limit $S(x)$ of the sequence is said to be the sum of the series.

103. In chapters, IV, V, VI, it has been shown that the algebraic *sum* of **a finite** number of continuous (derivable, integrable) functions is itself continuous (derivable, integrable). Also if

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x),$$

then

$$S_n'(x)=f'_1(x)+f'_2(x)+\ldots\ldots+f'_n(x),$$

i.e.,

$$\frac{d}{dx}\,\Sigma\, f_r(x)=\Sigma\,\frac{d}{dx}\,f_r(x),$$

where each, $f_r(x)$, is derivable,

and

$$\int_a^b S_n(x)\ dx=\int_a^b f_1(x)\,dx+\int_a^b f_2(x)\ dx+\ldots+\int_a^b f_n(x)\,dx.$$

i.e., $$\int_a^b \Sigma f_r(x)\, dx = \Sigma \int_a^b f_r(x)\, dx,$$

where each $f_r(x)$ is integrable in $[a, b]$.

We now consider some example which illustrate that these results may *not* hold good in case the number of functions is *infinite*.

103·1. Let $f_n(x)=x^2(1-x^2)^{n-1}$.

The series $\Sigma f_n(x)$, *i.e.*,

$$x^2+x^2(1-x^2)+x^2(1-x^2)^2+\ldots\ldots+x^2(1-x^2)^{n-1}+\ldots\ldots$$

is a geometrical progression whose common ratio is $(1-x^2)$. The sum $S_n(x)$ of the first n terms of this series is given by

$$\left.\begin{aligned} S_n(x)=\frac{x^2[1-(1-x^2)^n]}{1-(1-x^2)}=1-(1-x^2)^n \text{ ; if } x\neq 0, \end{aligned}\right\}$$

and $$S_n(x)=0, \text{ if } x=0.$$

We know that if $x\neq 0$, then $\lim (1-x^2)^n$ exists finitely and $=0$, if and only if

$$-1 < 1-x^2 < 1, \text{ i.e., if } |x| < \sqrt{2}.$$

Thus we see that if $S(x)$ denotes the sum to infinity of the given series, we have

$$S(x)=\begin{cases} 0, \text{ when } x=0 \\ 1, \text{ when } 0 < |x| < \sqrt{2}. \end{cases}$$

This shows that *the sum function $S(x)$ is not continuous for $x=0$ even though every term $f_n(x)$ is continuous for $x=0$.*

103·2. Let

$$f_n(x)=nxe^{-nx^2} -(n-1)xe^{-(n-1)x^2}.$$

It will now be shown that

$$\int_0^1 [f_1(x)+f_2(x)+\ldots\ldots+f_n(x)+\ldots\ldots]\, dx$$

$$\neq \int_0^1 f_1(x)dx+\int_0^1 f_2(x)\, dx+\ldots+\int_0^1 f_n(x)dx+\ldots, \quad \ldots(1)$$

i.e., the integral of the sum $\neq$ the sum of the integrals.

The sum $S_n(x)$ of the first n terms of the finite series is given by

$$S_n(x)=nxe^{-nx^2}.$$

We know from § 79·1, page 124 that for every value of x,

$$e^{nx^2} > (nx^2)^2/2 = n^2x^4/2,$$

so that

$$\left| nxe^{-nx^2} \right| < \frac{n\,|\,x\,|}{n^2x^4/2} = \frac{2}{n\,|\,x\,|^3}, \text{ if } x \neq 0.$$

Thus we see that as $n \to \infty$, $S_n(x) \to 0$ for every value of x, *i e*,

$$S(x)=0, \qquad \text{for every value of } x.$$

Therefore the left hand side of $(1)=0$.

The sum, ρ, of the infinite series on the right hand side of (1) is the limit of ρ_n where ρ_n denotes the sum of the first n integrals.

Since

$$\int_0^1 nxe^{-nx^2}\,dx = -\tfrac{1}{2}\left| e^{-nx^2} \right|_0^1 = \tfrac{1}{2}(1-e^{-n}),$$

we have

$$\int_0^1 f_n(x)\,dx = \tfrac{1}{2}\left[1-e^{-n}\right] - \tfrac{1}{2}\left[1-e^{-(n-1)}\right] = \tfrac{1}{2}\left[e^{-(n-1)} - e^{-n}\right]$$

and accordingly

$$\rho_n = \tfrac{1}{2}(1-e^{-n}).$$

$$\therefore \qquad \rho = \lim \rho_n = \tfrac{1}{2}, \text{ as } n \to \infty.$$

Since $0 \neq \frac{1}{2}$, we have the required result.

103·3. Let $\qquad f_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}.$

It will be shown that

the derivative for $x=0$ of the sum $S(x)$, i.e.,

$$S'(0) \neq \textit{the sum of the derivatives, i.e., } \Sigma f'_n(0).$$

We have

$$S_n(x) = \frac{nx}{1+n^2x^2}.$$

$$\therefore \qquad S'(x) = 0, \text{ for every value of } x.$$

so that

$$S'(0) = 0.$$

Also it is easy to see that $f'_n(0)=1$, for every value of n, so that $\Sigma f'_n(0)$ is a divergent series. Hence

$$\frac{d}{dx}\Sigma f_n(x) \neq \Sigma \frac{d}{dx} f_n(x),$$

for $x=0$.

Note. It will thus be seen that the *inversion of the operations of addition and integration* as implied by the equality

$$\int_a^b \Sigma f_n(x)dx = \Sigma \int_a^b f_n(x)\,dx$$

(integral of the sum = the sum of the integrals).

and the *inversion of the operations of addition and differentiation* as implied by the equality

$$\frac{d}{dx}\Sigma f_n(x)=\Sigma\frac{d}{dx}f_n(x)$$

(derivative of the sum = the sum of the derivatives)

(which are certainly valid when the summation extends only to a finite number of terms) may not be valid when the summation extends to an infinite number of terms. The concept of *Uniform convergence* which is introduced in the following section enables us to obtain sufficient conditions for the validity of the inversions in the case of an infinite number of terms.

104. The uniform convergence. The condition for $S(x)$ to be the limit of a convergent sequence of functions $\{S_n(x)\}$ in $[a, b]$, is that to each positive number ε, there corresponds a positive integer m such that

$$|S_n(x)-S(x)| < \varepsilon, \text{ when } n \geqslant m.$$

Obviously the value of m will depend upon ε as well as x and as such it may be written as $m(\varepsilon, x)$.

Suppose, now, that we fix ε and vary x. To each value of x will correspond a value of m. The infinite aggregate of these values of m may or may not be bounded above. In case this aggregate is bounded above, there exists a value m_0, (the upper bound of the aggregate of the values of m) such that

$$|S_n(x)-S(x)| < \varepsilon,$$

when $n \geqq m_0$ and *x has any value whatsoever.* In such a case we say that the sequence uniformly converges.

Def. Uniform convergence. *A sequence $\{S_n(x)\}$ is said to* **converge uniformly** *to a function $S(x)$ in $[a, b]$, if, given any positive number ε, there exists a positive integer m such that*

$$|S_n(x)-S(x)| < \varepsilon,$$

for every value of $n \geqslant m$ and every value of x in $[a, b]$.

Also a series $\Sigma f_n(x)$ is said to converge uniformly, if the squence $\{S_n(x)\}$, where

$$S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x),$$

uniformly converges.

We now illustrate the notion of uniform convergence by considering some simple examples.

1. *Let* $S_n(x)=n/(x+n)$ $\quad (x \geqslant 0)$.

We have, when $n \to \infty$,

$$S(x)=\lim\,[n/(x+n)]=1, \text{ for every value of } x.$$

If ε be any given positive number, we have

$$|S_n(x)-S(x)| = x/(x+n) < \varepsilon, \text{ if } n > x(1/\varepsilon-1).$$

Here $m(\varepsilon, x)$ = the integer just greater than $x(1/\varepsilon-1)$.

Obviously $m(\varepsilon, x)$ increases as x increases and $\to \infty$ as $x \to \infty$ so that it is not possible to choose a number m_0 such that

$$| S_n(x) - S(x) | < \varepsilon,$$

for every $n \geqslant m_0$ and every value of x in $[0, \infty[$.

Thus the convergence is not uniform in $[0, \infty[$.

If, however, we consider the interval $[0, k]$, where k is any fixed number, however large, we see that the maximum value of $x(1/\varepsilon - 1)$ is $k\,(1/\varepsilon - 1)$ so that, taking m_0=any integer greater than $k(1/\varepsilon - 1)$, we have

$$| S_n(x) - S(x) | < \varepsilon,$$

for every $n \geqq m_0$ and every x in $[0, k]$.

Thus we see that the sequence $\{S_n(x)\}$ converges uniformly in the interval $[0, k]$ where k is any fixed positive number, however large.

Note. It may similarly be shown that the sequence converges uniformly in $[-k, 0]$, where k is any fixed positive number, however large.

2. *Let* $S_n(x) = x^n$. $\qquad (0 \leqslant x \leqslant 1)$.

We have, when $n \to \infty$,

$$S(x) = \lim S_n(x) = \begin{cases} 0, & \text{when } 0 \leqslant x < 1, \\ 1, & \text{when } x = 1. \end{cases}$$

We consider

$$0 \leqslant x < 1, \text{ i.e., the interval } [0, 1[.$$

Let ε be any positive number. We have

$$| S_n(x) - S(x) | = x^n < \varepsilon,$$

if

$$(1/x)^n > 1/\varepsilon, \text{ i.e., if } n \log (1/x) > \log (1/\varepsilon),$$

or if

$$n > \log (1/\varepsilon)/\log (1/x). \qquad [\text{when } x \neq 0].$$

Thus, when $x \neq 0$, $m(\varepsilon, x)$ = the integer just greater than

$$\log (1/\varepsilon)/\log (1/x).$$

Also obviously $m(\varepsilon, x) = 1$, when $x = 0$.

We now see that $m(\varepsilon, x)$ increases and $\to \infty$ when x, starting from 0, increases and tends to, 1, so that it is not possible to choose m_0 such that

$$| S_n(x) - S(x) | < \varepsilon,$$

for *every* $n \geqslant m_0$ and every value of x in $[0, 1[$. Thus the convergence is not uniform in the interval $[0, 1[$.

If, however, we consider any number k such that $0 \leqslant k < 1$, we see that the maximum value of $\log (1/\varepsilon)/\log (1/x)$ is $\log (1/\varepsilon) / \log (1/k)$

so that if we take m_0 = any integer greater than this maximum value, we have

$$| S_n(x) - S(x) | < \varepsilon,$$

for every $n \geqslant m_0$ and every x in $[0, k]$.

Thus the convergence is uniform in $[0, k]$.

Note. The point $x=1$, which is such that the sequence does not converge uniformly in any neighbourhood of $x=1$, however small it may be, is said to be **a point of non-uniform convergence** of the sequence.

3. *Let* $\quad S_n(x) = e^{-nx}. \quad x \geqslant 0.$

We have, when $n \to \infty$,

$$S(x) = \lim S_n(x) = \begin{cases} 1, \text{ when } x = 0, \\ 0, \text{ when } x > 0. \end{cases}$$

We consider $x > 0$.

Let ε be any positive number. We have

$$| S_n(x) - S(x) | = e^{-nx} < \varepsilon,$$

if

$$e^{nx} > 1/\varepsilon, \text{ i.e., if } nx > \log (1/\varepsilon),$$

or if

$$n > \log (1/\varepsilon)/x.$$

Here $m(\varepsilon, x)$ = the integer just greater than $\log (1/\varepsilon)/x$. Arguing as before, we see that the convergence is not uniform in $]0, \infty$, but it is uniform in $[k, \infty[$, where k is any positive number whatsoever.

The point $x=0$ is a point of non-uniform convergence of the sequence.

Ex. 1. *Show that, 0, is a point of non-uniform convergence of the sequence* $\{S_n(x)\}$, *where* $S_n(x) = 1-(1-x^2)^n$.

It is easily seen that

$$S(x) = \begin{cases} 0, \text{ when } x=0, \\ 1, \text{ when } 0 < | x | < \sqrt{2}. \end{cases}$$

Let, if possible, the sequence be uniformly convergent in a neighbourhood $]0, k]$ of, 0, where k is a number such that $0 < | k | < \sqrt{2}$. There exists, therefore, a positive integer m such that taking $\varepsilon = \frac{1}{2}$,

$$| S_m(x) - S(x) | = (1-x^2)^m < \tfrac{1}{2}, \qquad \ldots(1)$$

for every value of x is $]0, k]$.

Since $(1-x^2)^m \to 1$ as $x \to 0$, we see that the inequality (1) cannot hold true in the neighbourhood of $x=0$. Hence the supposition is wrong.

2. *Show that, 0, is a point of non-uniform convergence of the sequence $\{S_n(x)\}$, where $\{S_n(x)\}=nxe^{-nx^2}$.*

It is easily seen that $S(x)=0$ for every value of x.

If possible, let the sequence be uniformly convergent in a neighbourhood $[0, k]$ of 0, where k is any positive number.

There exists, therefore, a positive integer m such that, taking $\varepsilon=1$,

$$| S_n(x)-S(x) | = nxe^{-nx^2} < 1, \qquad ...(1)$$

for every value of x in $[0, k]$ and for $n \geqslant m$.

In particular, the inequality (1) must be true for $x=1/\sqrt{n}$ where n is any integer $> 1/k^2$ as well as m, so that we have

$$nxe^{-nx^2} = \sqrt{n}/e < 1. \qquad ...(2)$$

Since $x \to 0$ when $n \to \infty$, we see that on taking x sufficiently near 0, we can take n so large that $\sqrt{n}/e > 1$ and thus we have a contradiction of (2).

3. Show that $x=0$ is a point of non-uniform convergence of the sequence

$$[nx/(1+n^2x^2)].$$

4. Show that the series

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\cdots$$

is uniformly convergent in $]k, \infty]$, where k is any positive number. Show also that the series is non-uniformly convergent near the point $x=0$.

5. Show that the series

$$(1-x)^2+x(1-x)^2+x^2(1-x)^2+\cdots\cdots$$

is not uniformly convergent in $[0, 1]$.

6. Show that the series

$$\Sigma \frac{x}{n(n+1)}$$

is uniformly convergent in $[0, k]$ where k is any positive number whatsoever but that it does not converge uniformly in $[0, \infty[$.

105. Test for the uniform convergence of a series.

105·1. General principle of convergence. *The necessary and sufficient condition for the uniform convergence in $[a, b]$ of a series $\Sigma f_n(x)$ is that to every positive number, ε, there corresponds a positive integer m such that*

$$| f_{n+1}(x)+f_{n+2}(x)+\cdots\cdots+f_{n+p}(x) | < \varepsilon,$$

for every $n \geqslant m$, every positive integer p and every value of x in $[a, b]$.

The condition is necessary. Let ε be any positive number.

Let

$$S_n(x)=f_1(x)+f_2(x)+\cdots\cdots+f_n(x),$$

and

$$S(x)=\lim S_n(x), \text{ when } n \to \infty.$$

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \tfrac{1}{2} \varepsilon, \qquad \ldots(1)$$

for every $n \geqslant m$ and for every value of x in $[a, b]$.

Also, therefore,

$$| S_{n+p}(x)-S(x) | < \tfrac{1}{2} \varepsilon, \qquad \ldots(2)$$

for every $n \geqslant m$, every positive integer p and every x in $[a, b]$.

From (1) and (2), we see that

$$| f_{n+1}(x)+f_{n+2}(x)+\ldots\ldots+f_{n+p}(x) | = | S_{n+p}(x)-S_n(x) | < \varepsilon,$$

for every $n \geqslant m$, every positive integer p and every x in $[a, b]$.

The condition is sufficient. We know that when this condition is satisfied the series is convergent. (§40·1, page 60). All that we have now to show is that the convergence is uniform. Let $S(x)$ be the sum of the series.

Let ε be any positive number. There exists a positive integer m such that

$$| S_{n+p}(x)-S_n(x) | < \tfrac{1}{2}\varepsilon,$$

i.e.,

$$S_n(x)-\tfrac{1}{2}\varepsilon < S_{n+p}(x) < S_n(x)+\tfrac{1}{2}\varepsilon, \qquad \ldots(3)$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in $[a, b]$.

Keeping n fixed, let $p \to \infty$ so that $S_{n+p}(x) \to S(x)$.

Therefore we have, from (3),

$$S_n(x)-\tfrac{1}{2}\varepsilon \leqslant S(x) \leqslant S_n(x)+\tfrac{1}{2}\varepsilon.$$

i.e.,

$$| S_n(x)-S(x) | \leqslant \tfrac{1}{2}\varepsilon < \varepsilon,$$

for every $n \geqslant m$ and every x in $[a, b]$.

Hence the series is uniformly convergent.

105·2. Weierstrass's M-test for uniform convergence. *A series $\Sigma f_n(x)$ will converge uniformly in $[a, b]$, if there exists a convergent series ΣM_n of positive constants such that*

$$| f_n(x) | < M_n, \qquad \ldots(i)$$

for every value of n and every value of x in $[a, b]$.

Let ε be any positive number. Since ΣM_n is convergent, there exists a positive integer m such that

$$| M_{n+1}+M_{n+2}+\ldots\ldots+M_{n+p} | < \varepsilon \qquad \ldots(ii)$$

for every $n \geqslant m$ and every $p \geqslant 0$. (§40·1, page 60)

From (*i*) and (*ii*), we obtain

$$| f_{n+1}(x)+f_{n+2}(x)+\dots\dots+f_{n+p}(x) |$$
$$< [M_{n+1}+M_{n+2}+\dots\dots+M_{n+p}] < \varepsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in $[a, b]$.

Hence $\Sigma f_n(x)$ is uniformly convergent in $[a, b]$.

106. Properties of uniformly convergent series.

106·1. Continuity of the sum. *The sum function of a uniformly convergent series of continuous functions is itself continuous.*

Let $\Sigma f_n(x)$ be a series which is uniformly convergent in $[a, b]$ and let each term $f_n(x)$ be continuous in $[a, b]$. It will be shown that the sum function $S(x)$ of the series is also continuous in $[a, b]$.

Let

$$S_n(x)=f_1(x)+f_2(x)+\dots\dots+f_n(x).$$

Let, c be *any* point of $[a, b]$. Let, ε, be any arbitrarily assigned positive number.

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \tfrac{1}{3}\varepsilon,$$

for every $n \geqslant m$ and every x in $[a, b]$.

In particular, this gives

$$| S_m(x)-S(x) | < \tfrac{1}{3}\varepsilon, \text{ for every } x \text{ in } [a, b] \qquad \dots(i)$$

and

$$| S_m(c)-S(c) | < \tfrac{1}{3}\varepsilon. \qquad \dots(ii)$$

Now $S_m(x)$, being the sum of a finite number m of continuous functions, is also continuous. (§57·1, page 93).

There exists, therefore, a positive number δ such that

$$| S_m(x)-S_m(c) | < \tfrac{1}{3}\varepsilon, \text{ when } | x-c | \leqslant \delta. \qquad \dots(iii)$$

From (*i*), (*ii*), (*iii*), we deduce that when $| x-c | \leqslant \delta$,

$$| S(x)-S(c) | = | S(x)-S_m(x)+S_m(x)-S_m(c)+S_m(c)-S(c) | < \varepsilon.$$

Hence $S(x)$ is continuous at any point, c, and, therefore, in $[a, b]$.

Remarks. This theorem shows that if the sum function of a series of continuous functions is discontinuous at any point, then that point must necessarily be a point of non-uniform convergence of the series. On the other hand, as the following example shows, the condition of uniform convergence is *only sufficient but not necessary* for the continuity of the sum function.

Let

$$f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}.$$

We have

$$S_n(x)=nx/(1+n^2x^2).$$

$\therefore$ $$S(x)=0, \text{ for every value of } x,$$

so that $S(x)$ is continuous at $x=0$ even though, as may be easily shown, $x=0$ is a point of non-uniform convergence of the series.

106·2. Term-by-term integration. *If*

(*i*) *$\Sigma f_n(x)$ is uniformly convergent in $[a, b]$,*

and (ii) each $f_n(x)$ is bounded and integrable in $[a, b]$,

then the sum function $S(x)$ of the series is also bounded and integrable in $[a, b]$ and

$$\int_a^b \Sigma f_n(x)dx=\Sigma \int_a^b f_n(x)dx.$$

Firstly, we will show that the sum function $S(x)$ is integrable in $[a, b]$. Let ε be any positive number.

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \varepsilon/4(b-a),$$

for every $n \geqslant m$ and every x in $[a, b]$.

In particular, we have

$$| S_m(x)-S(x) | < \varepsilon/4(b-a),$$

for every x in $[a, b]$.

We write

$$S(x)-S_m(x)=R_m(x), \text{ i.e., } S(x)=S_m(x)+R_m(x).$$

Now the function $S_m(x)$, being the sum of a finite number m of integrable functions, is itself integrable. (§93·2, page 160). There exists, therefore, a division D of $[a, b]$ such that the corresponding *oscillatory sum* for $S_m(x)$ is $< \frac{1}{2}\varepsilon$.

Since for every x in $[a, b]$, $| R_m(x) | \leqslant \varepsilon/4(b-a)$, the oscillation of $R_m(x)$ in every sub-interval of $[a, b]$ is $< 2.\varepsilon/4(b-a)=\varepsilon/2(b-a)$. Thus the oscillatory sum of $R_m(x)$ corresponding to any division and, in particular, the division D is

$$< (b-a).\varepsilon/2(b-a)=\tfrac{1}{2}\varepsilon.$$

Now corresponding to any division, the oscillatory sum for the sum of two functions is $\leqslant$ the sum of the oscillatory sums for the two functions. Thus there exists a division D such that the corresponding oscillatory sum for $S(x)$ is $< (\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon)=\varepsilon$. Hence $S(x)$ is integrable in $[a, b]$. This proves the first part.

Again, let ε be any positive number.

Since the series is uniformly convergent, there exists a positive integer m such that

$$| S_n(x)-S(x) | < \varepsilon/(b-a),$$

i.e.,

$$-\varepsilon/(b-a) < S_n(x)-S(x) < \varepsilon/(b-a), \qquad ...(1)$$

for every $n \geqslant m$ and for every x in $[a, b]$.

Now $S_n(x)$, being the sum of a finite number n of integrable functions, is itself integrable. Also $S(x)$ is integrable. Therefore $S_n(x)-S(x)$ is integrable. From (1), we have, therefore,

$$-\varepsilon < \int_a^b S_n(x)\, dx - \int_a^b S(x)\, dx < \varepsilon$$

i.e.,

$$\left| \int_a^b S_n(x)\, dx - \int_a^b S(x)\, dx \right| < \varepsilon, \qquad ...(2)$$

for every $n \geqslant m$.

Also we have

$$\int_a^b S_n(x)\, dx = \int_a^b \sum_{r=1}^{r=n} f_r(x) dx = \sum_{r=1}^{r=n} \int_a^b f_r(x) dx,$$

(Cor. to §93·2, page 160)

so that the relation (2) means that

$$\lim_{n\to\infty} \sum_{r=1}^{r=n} \int_a^b f_r(x)\, dx = \int_a^b S(x)\, dx = \int_a^b \lim_{n\to\infty} \sum_{r=1}^{r=n} f_r(x)\, dx,$$

i.e.,

$$\Sigma \int_a^b f_n(x)\, dx = \int_a^b \Sigma f_n(x)\, dx.$$

Remarks 1. From (1), we have

$$\left| \int_a^x S_n(x)\, dx - \int_a^x S(x)\, dx \right| < [\varepsilon/(b-a)]\ (b-a) < \varepsilon,$$

i.e., $$\left| \sum_{r=1}^{r=n} \int_a^x f_r(x)\,dx - \int_a^x S(x)\,dx \right| < \varepsilon,$$

for every $n \geqslant m$ and every x in $[a, b]$.

This shows that the series $\Sigma \int_a^x f_n(x)\,dx$, *i.e.*,

$$\int_a^x f_1(x)\,dx + \int_a^x f_2(x)\,dx + \ldots + \int_a^x f_n(x)\,dx + \ldots\ldots$$

is also uniformly convergent in $[a, b]$. Also since each $\int_a^x f_r(x)\,dx$ is integrable (in fact continuous), we deduce that the operation of successive integration may be carried out any number of times.

2. *The condition of uniform convergence is only sufficient but not necessary for the validity of term by term integration* as is shown by considering the series $\Sigma f_n(x)$, for which the sum $S_n(x)$ of the first n terms is given by

$$S_n(x) = nx/(1+n^2x^2) \qquad \text{(See remarks after § 106, page 198)}$$

We have

$$\int_0^1 \Sigma f_n(x)dx = \int_0^1 S(x)\,dx = \int_0^1 0\,dx = 0,$$

and $$\sum_{r=1}^{r=n} \int_0^1 f_r(x)\,dx = \frac{1}{2n} \log(1+n^2) \text{ which} \to 0 \text{ as } n \to \infty.$$

Thus $$\int_0^1 \Sigma f_n(x)\,dx = \Sigma \int_0^1 f_n(x)\,dx,$$

even though, 0, is a point of non-uniform convergence of the series.

106·3. Term-by-term differentiation. *If*

(i) the series $S(x) = \Sigma f_n(x)$ is convergent in $[a, b]$,

(ii) each term $f_n(x)$ is derivable in $[a, b]$,

and (iii) the series $\sigma(x) = \Sigma f'_n(x)$ is uniformly convergent in $[a, b]$ then

$$\sigma(x) = S'(x).$$

Let c be any *fixed* point of $[a, b]$.

Let y be a variable which varies in the interval $[a-c, b-c]$ so that $c+y$ is also a variable which varies in the interval $[a, b]$.

We define, as follows, a sequence $\{\phi_n(y)\}$ of functions of y.

$$\phi_n(y)=\begin{cases} \dfrac{f_n(c+y)-f_n(c)}{y}, & \text{when } y\neq 0 \\ f'_n(c), & \text{when } y=0 \end{cases}$$

By the Lagrange's mean value theorem, there exists a number η lying between c and $c+y$ and accordingly, between a and b such that

$$[f_n(c+y)-f_n(c)]/y=f'_n(\eta).$$

Thus we see that for *any* value of y in $[a-c, b-c]$, there exists a value of x in $[a, b]$ such that

$$\phi_n(y)=f'_n(x).$$

Since the series $\Sigma f'_n(x)$ is uniformly convergent in $[a, b]$, there exists a positive integer m such that

$$|f'_{n+1}(x)+f'_{n+2}(x)+\ldots\ldots+f'_{n+p}(x)| < \varepsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every x in $[a, b]$.

Therefore we have

$$|\phi_{n+1}(y)+\phi_{n+2}(y)+\ldots\ldots+\phi_{n+p}(y)| < \varepsilon,$$

for every $n \geqslant m$, every $p \geqslant 0$ and every y in $[a-c, b-c]$, so that the series $\Sigma\phi_n(y)$ is uniformly convergent in $[a-c, b-c]$. From the definition of $\phi_n(y)$, we see that $\phi_n(y)$ is continuous for $y=0$. Therefore from § 106·1, we deduce that the sum function $G(y)=\Sigma\phi_n(y)$ is also continuous for $y=0$.

We have

$$\begin{aligned}\sigma(c)&=\Sigma f'_n(c)=\Sigma \phi_n(0)=G(0)\\ &= \lim G(y)= \lim_{y\to 0} \Sigma \phi_n(y)\\ &= \lim_{y\to 0} \Sigma \frac{f_n(c+y)-f_n(c)}{y}\\ &= \lim_{y\to 0} \frac{S(c+y)-S(c)}{y}=S'(c)\end{aligned}$$

Since c is any point of $[a, b]$, we have

$$\sigma(x)=S'(x).$$

A simple case of term by term differentiation. If we assume *each function $f'_n(x)$ is continuous* in $[a, b]$, then the proof becomes much simplified.

Since $\Sigma f'_n(x)$ is a uniformly convergent series of continuous and, therefore, integrable functions, the term by term integration is valid, so that we have

$$\int_a^x \sigma(x)\,dx = \int_a^x f'_1(x)\,dx + \int_a^x f'_2(x)\,dx + \ldots + \int_a^x f'_n(x)\,dx + \ldots\ldots$$

$$= \sum_{r=1}^{\infty} [f_r(x) - f_r(a)]$$

$$= \sum_{r=1}^{\infty} f_r(x) - \sum_{r=1}^{\infty} f_r(a)$$

$$= S(x) - S(a).$$

Since $\sigma(x)$ is continuous, (§ 106·1), we have, on differentiating with respect to x, (§ 96·2, page 170),

$$\sigma(x) = S'(x).$$

This is the form in which the theorem is generally proved.

Note. The reader should note that for the validity of term by term differentiation, it is the derived series which must be uniformly convergent and that the original series need only be simply convergent.

Exercises

1. Show that

$$\sum_{n=-\infty}^{\infty} e^{-(x-n)^2}$$

converges uniformly in any fixed interval $[a, b]$.

2. Show that the series

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

is uniformly convergent for all values of x; and that $S'(x)$ is given by term by term differentiation.

3. Show tha

$$S(x) = \sum_{n=0}^{\infty} \frac{1}{1 + n^2 + n^4 x^2}$$

converges uniformly for all values of x; examine whether $S'(0)$ can be found by term by term differentiation.

4. Show that the series Σn^{-x} is uniformly convergent in $[1+\delta, \infty[$ where δ is any positive number. Show also that term by term differentiation is valid in the same interval.

5. Show that the following series converge uniformly in the intervals indicated :—

$$x - x^2 + x^3 - x^4 + x^5 - x^6 + \ldots\ldots (-\tfrac{1}{2} \leqslant x \leqslant \tfrac{1}{2})$$

$$e^{x} + e^{2x} + e^{3x} + e^{4x} + \ldots\ldots, (-2 \leqslant x \leqslant -\tfrac{1}{2}).$$

6. Show that

$$\sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$$

is uniformly convergent in $\delta \leqslant x \leqslant 1$ for any $\delta > 0$ but is not uniformly convergent in $0 \leqslant x \leqslant 1$.

7. Show that the series

$$\sum_{n=0}^{\infty} \left[e^{-n|x|} - e^{-2n|x|} \right],$$

though convergent for all values of x, is not uniformly convergent in any interval which contains $x=0$.

8. Show that, in the interval, $|x| \leqslant 1$, each of the functions

$$\frac{nx}{1+nx^2}, \quad \frac{nx}{1+n^2x^2}, \quad \frac{nx}{1+n^3x^3}$$

tends to a limit as $n \to \infty$ and discuss the uniformity of convergence.

9. Discuss the uniform convergence of

$$1+\frac{e^{-2x}}{2^2-1}-\frac{e^{-4x}}{4^2-1}+\frac{e^{-6x}}{6^2-1}\cdots\cdots$$

where $x \geqslant 0$.

107. Analytical theory of Trigonometrical Functions. The functions sin x and cos x.

107·1. Theorem. *The two series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,!}, \text{ i.e., } x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\cdots\cdots \quad \ldots(A)$$

$$\sum_{n=}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,!}, \text{ i.e., } 1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\cdots\cdots \quad \ldots(B)$$

are uniformly convergent in every interval $[a, b]$.

Let M be any positive number greater than $|a|$ as well as $|b|$ so that if x denotes any number in $[a, b]$, we have

$$\left| \frac{(-1)^n x^{2n+1}}{(2n+1)\,!} \right| \leqslant \frac{M^{2n+1}}{(2n+1)\,!},$$

and

$$\left| \frac{(-1)^n x^{2n}}{(2n)\,!} \right| \leqslant \frac{M^{2n}}{(2n)\,!}.$$

Consider now the two series

$$M+\frac{M^3}{3\,!}+\frac{M^5}{5\,!}+\cdots\cdots \quad \ldots(C)$$

$$1+\frac{M^2}{2\,!}+\frac{M^4}{4\,!}+\cdots\cdots \quad \ldots(D)$$

We know that (§ 79·1, page 124)

$$e^M=1+M+\frac{M^2}{2\,!}+\frac{M^3}{3\,!}+\cdots\cdots$$

$$e^{-M}=1-M+\frac{M^2}{2\,!}-\frac{M^3}{3\,!}+\ldots\ldots$$

$$\therefore\ \frac{1}{2}\left\{e^{M}+e^{-M}\right\}=1+\frac{M^2}{2\,!}+\frac{M^4}{4\,!}+\ldots\ldots$$

$$\frac{1}{2}\left\{e^{M}-e^{-M}\right\}=M+\frac{M^3}{3\,!}+\frac{M^5}{5\,!}+\ldots\ldots$$

This shows that the series (C), (D) are convergent, whatever the positive constant M may be.

Therefore, by Weierstrass's M-test, we prove that the series (A), (B) are uniformly convergent in $[a, b]$.

Definitions. This theorem justifies the following two definitions :—

$$(A)\qquad \sin x=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\ldots\ldots+\frac{(-1)^n\,x^{2n+1}}{(2n+1)\,!}+\ldots\ldots$$

$$(B)\qquad \cos x=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots\ldots+\frac{(-1)^n\,x^{2n}}{(2n)\,!}+\ldots\ldots$$

107·2. *The functions* $\sin x$, $\cos x$ *are defined and continuous for every value of* x. This follows from § 106·1.

107·3. *The functions* $\sin x$, $\cos x$ *are derivable for every value of* x *and*

$$\frac{\mathbf{d(\sin x)}}{\mathbf{dx}}=\mathbf{\cos x}\ \text{ and }\ \frac{\mathbf{d(\cos x)}}{\mathbf{dx}}=-\mathbf{\sin x}.$$

If we differentiate term by term the convergent series (A), we get the uniformly convergent series (B) and hence, (§ 106·3), $\sin x$ is derivable and $(\sin x)'=\cos x$.

Again, on differentiating term by term the convergent series (B), we get the uniformly convergent series

$$-x+\frac{x^3}{3\,!}-\frac{x^5}{5\,!}+\ldots\ldots$$

and hence $\cos x$ is derivable and $(\cos x)'=-\sin x$.

107·4. $\sin 0=0$ *and* $\cos 0=1$.

The proof is obvious.

107·5. $\sin(-x)=-\sin x$ *and* $\cos(-x)=\cos x$.

The proof is obvious.

108. The addition theorems. *If* x, y *are any numbers, then*

$\sin(x+y)=\sin x\cos y+\cos x\sin y$;

$\cos(x+y)=\cos x\cos y-\sin x\sin y$.

Giving any value to y and then keeping it fixed, we write

$f(x)=\sin(x+y)-\sin x\cos y-\cos x\sin y$;

$g(x)=\cos(x+y)-\cos x\cos y+\sin x\sin y$.

We have

$$f'(x)=\cos(x+y)-\cos x \cos y+\sin x \sin y=g(x),$$
$$g'(x)=-\sin(x+y)+\sin x \cos y+\cos x \sin y=-f(x),$$

$$\therefore \quad \frac{d[f^2(x)+g^2(x)]}{dx}=2f(x)f'(x)+2g(x)g'(x)$$
$$=2f(x)g(x)-2g(x)f(x)=0$$

so that

$$f^2(x)+g^2(x)$$

is a constant.

Hence for every value of x,

$$f^2(x)+g^2(x)=f^2(0)+g^2(0)=0.$$
$$\therefore \quad f(x)=0,\ g(x)=0.$$

Hence the theorems.

Cor. 1. *$\cos^2 x+\sin^2 x=1$.* We have

$$\begin{aligned} 1&=\cos 0\\ &=\cos(x-x)\\ &=\cos x\cos(-x)-\sin x \sin(-x)\\ &=\cos x \cos x+\sin x \sin x=\cos^2 x+\sin^2 x. \end{aligned}$$

Cor. 2. $\quad |\sin x| \leqslant 1,\ |\cos x| \leqslant 1.$

This follows from the preceding corollary.

Cor. 3. Changing y to x, we have

$$\sin 2x=2\sin x \cos x,\ \cos 2x=\cos^2 x-\sin^2 x.$$

Cor. 4. Changing y to $-y$, we have

$$\sin(x-y)=\sin x \cos y-\cos x \sin y,$$
$$\cos(x-y)=\cos x \cos y+\sin x \sin y.$$

109. The number π. The smallest positive root of the equation

$$\mathbf{\cos x=0.}$$

Theorem. *To prove that there exists a positive number, π, such that*

$$\cos(\pi/2)=0,$$

and $\quad \cos x > 0,\ \text{for } 0 \leqslant x < \pi/2.$

Consider the interval [0, 2].

We know that $\cos 0(=1)$, is positive and we will now show that $\cos 2$ is negative. We have

$$\cos 2=1-\frac{2^2}{2!}+\frac{2^4}{4!}-\frac{2^6}{6!}+\ldots\ldots$$
$$=1-\frac{2^2}{2!}\left(1-\frac{2^2}{3\cdot 4}\right)-\frac{2^6}{6!}\left(1-\frac{2^2}{7\cdot 8}\right)\ldots\ldots$$

Since the brackets are all positive, we have

$$\cos 2 < 1-\frac{2^2}{2\,!}\left(1-\frac{2^2}{3\cdot 4}\right)=-\frac{1}{3},$$

so that cos 2 is negative.

There exists, therefore, at least, one number, α, between 0 and 2, such that $\cos \alpha=0$. Also there cannot exist more than one such value ; for, if possible, let β be another so that

$$\cos \alpha=0, \cos \beta=0 \qquad 0<\alpha, \beta<2.$$

By Rolle's theorem, their exists at least, one number, λ, between α and β such that the derivative, $\sin x$, of $\cos x$ vanishes for $x=\lambda$,

i.e., $$\sin \lambda=0. \qquad 0<\lambda<2.$$

But

$$\sin \lambda=\frac{\lambda}{1\,!}\left(1-\frac{\lambda^2}{2\cdot 3}\right)+\frac{\lambda^5}{5\,!}\left(1-\frac{\lambda^2}{6\cdot 7}\right)+\ldots\ldots$$

which is clearly positive.

Thus there exists one and only one root of the equation $\cos x=0$ lying between 0 and 2. Denoting twice this value by π, we see that $\pi/2$ is the *least positive* root of the equation $\cos x=0$. Also, therefore, we have $\cos x>0$, when $\leqslant x<\pi/2$.

110. *$\sin x>0$, when $0<x\leqslant \frac{1}{2}\pi$.*

Since the derivative $\cos x$ of $\sin x$ is positive in $[0, \frac{1}{2}\pi]$ therefore by § 74·2, page 119, $\sin x$ is strictly increasing. Also, since $\sin 0=0$, we see that $\sin x$ is positive when $0<x\leqslant \frac{1}{2}\pi$.

111·1. *$\sin (\pi/2)=1$.*

Since

$$\sin^2(\pi/2)+\cos^2(\pi/2)=1,$$

therefore

$$\sin^2(\pi/2)=1, \textit{ i.e.}, \sin (\pi/2)=\pm 1.$$

But, by the Lagrange's mean value theorem,

$$\sin (\pi/2)=\sin (\pi/2)-\sin 0=(\pi/2) \cos \alpha>0. \qquad 0<\alpha<\pi/2.$$

$$\therefore \qquad \sin (\pi/2)=1.$$

111·2. *$\cos \pi=-1$, $\sin \pi=0$.*

$$\cos \pi=2 \cos^2(\pi/2)-1=-1 \text{ and } \sin \pi=2 \sin \pi/2 \cos \pi/2=0.$$

111·3. *$\cos 2\pi=1$, $\sin 2\pi=0$.*

$$\cos 2\pi=2 \cos^2\pi-1=1 \text{ and } \sin 2\pi=2 \sin \pi \cos \pi=0.$$

111·4. *$\cos (\pi/4)=\sin (\pi/4)=1/\sqrt{2}$.*

We have

$$0=\cos(\pi/2)=2 \cos^2(\pi/4)-1$$

$$\therefore \qquad \cos(\pi/4)=1/\sqrt{2},$$

rejecting the negative sign as $\cos \pi/4$ is necessarily positive. (§ 109).

Also

$$1=\sin(\pi/2)=2\sin(\pi/4)\cos(\pi/4)$$

$\therefore$ $\sin(\pi/4)=1/\sqrt{2}.$

111·5. $\sin(\tfrac{1}{2}\pi+x)=\cos x,\quad \cos(\tfrac{1}{2}\pi-x)=\ \sin x.$ (*i*)

$\sin(\tfrac{1}{2}\pi+x)=\cos x,\quad \cos(\tfrac{1}{2}\pi+x)=-\sin x.$ (*ii*)

$\sin(\ \pi+x)=-\sin x,\quad \cos(\ \pi+x)=-\cos x.$ (*iii*)

$\sin(\ \pi-x)=\sin x,\quad \cos(\ \pi-x)=-\cos x.$ (*iv*)

$\sin(2\pi+x)=\sin x,\quad \cos(2\pi+x)=\ \cos x.$ (*v*)

These are easily proved with the help of the addition formulae.

Note. Because of formulae (*v*), $\sin x$, $\cos x$ are said to be *periodic functions* with 2π as their *period*.

112·1. *$\cos x > 0$, when $0 \leqslant x < \tfrac{1}{2}\pi$ or $\tfrac{3}{2}\pi < x \leqslant 2\pi$;*
$\cos x < 0$, when $\tfrac{1}{2}\pi < x \leqslant \pi$ or $\pi \leqslant x < \tfrac{3}{2}\pi$.

When $0 \leqslant x < \tfrac{1}{2}\pi$, we know from § 109 that $\cos x > 0$.

When $\tfrac{1}{2}\pi < x \leqslant \pi$, we write $x=\tfrac{1}{2}\pi+y$ so that $0 < y \leqslant \tfrac{1}{2}\pi$.

$\therefore$ $\cos x=\cos(\tfrac{1}{2}\pi+y)=-\sin y < 0.$ (§ 110).

When $\pi \leqslant x < \tfrac{3}{2}\pi$, we write $x=\pi+y$ so that $0 \leqslant y < \tfrac{1}{2}\pi$.

$\therefore$ $\cos x=\cos(\pi+y)=-\cos y < 0.$ (§ 109).

When $\tfrac{3}{2}\pi < x \leqslant 2\pi$, we write $x=\pi+y$ so that $\tfrac{1}{2}\pi < y \leqslant \pi$.

$\therefore$ $\cos x=\cos(\pi+y)=-\cos y > 0.$

112·2 *$\sin x > 0$, when $0 < x < \pi$; $\sin x < 0$, when $\pi < x < 2\pi$.*

The proof is similar to that of the preceding theorem.

Ex. 1. *Discuss how $\sin x$ and $\cos x$ vary, (monotonically increase or decrease), as x varies in the interval* $[0, 2\pi]$.

2. Show that when n is any integer,

$$\sin n\pi=0. \qquad \cos\tfrac{1}{2}(2n+1)\pi=0;$$

and $$\sin\tfrac{1}{2}(2n+1)\pi=(-1)^n, \qquad \cos n\pi=(-1)^n.$$

113. The function tan x. $\tan x$ is defined by the relation

$$\tan x=\frac{\sin x}{\cos x}.$$

Clearly $\tan x$ is defined, continuous and derivable for all values of x except those for which the denominator, $\cos x$, vanishes which is the case for $x=\tfrac{1}{2}(2n+1)\pi$; n being any integer, positive, negative or zero.

From the formula (*iii*) of § 111·5, we have

$$\tan(\pi+x)=\tan x,$$

so that we see that, $\tan x$, is a periodic function whose period is π.

Also we may easily show that when $x\neq\tfrac{1}{2}(2n+1)\pi$,

$$\frac{d(\tan x)}{dx}=\frac{d(\sin x/\cos x)}{dx}=\frac{1}{\cos^2 x}.$$

113·1. *To show that*

$$\lim_{x\to(\frac{1}{2}\pi-0)} \tan x=\infty, \qquad \lim_{x\to(\frac{1}{2}\pi+0)} \tan x=-\infty.$$

Let G be any positive number, however large.

Since $\sin x \to \sin \frac{1}{2}\pi=1$ as $x \to \frac{1}{2}\pi$, there exists a positive number δ_1 such that, (taking $\varepsilon=\frac{1}{2}$),

$$\tfrac{1}{2} < \sin x, \text{ when } \tfrac{1}{2}\pi-\delta_1 \leqslant x \leqslant \tfrac{1}{2}\pi+\delta_1. \qquad \ldots(1)$$

Since $\cos x \to 0$ as $x \to \frac{1}{2}\pi$, there exists a positive number δ_2 such that,

$$-1/2G < \cos x < 1/2G, \text{ when } \tfrac{1}{2}\pi-\delta_2 \leqslant x \leqslant \tfrac{1}{2}\pi+\delta_2.$$

As $\cos x$ is positive in $[0, \frac{1}{2}\pi[$ and negative in $]\frac{1}{2}\pi, \pi]$, we have

$$0 < \cos x < 1/2G, \text{ when } \tfrac{1}{2}\pi-\delta_2 \leqslant x < \tfrac{1}{2}\pi \qquad \ldots(2)$$

and

$$-1/2G < \cos x < 0, \text{ when } \tfrac{1}{2}\pi < x \leqslant \pi+\delta_2. \qquad \ldots(3)$$

From (1) and (2), we have, if $\delta=\min(\delta_1, \delta_2)$,

$$\tan x=\frac{\sin x}{\cos x} > G, \text{ when } \tfrac{1}{2}\pi-\delta \leqslant x < \tfrac{1}{2}\pi$$

and from (1) and (3), we have

$$\tan x=\frac{\sin x}{\cos x} < -G, \text{ when } \tfrac{1}{2}\pi < x \leqslant \tfrac{1}{2}\pi+\delta.$$

Hence the results.

114. The inverse trigonometrical functions. $\sin^{-1} y$, $\cos^{-1} y$, $\tan^{-1} y$.

114·1. $\sin^{-1} y$. Since, as may easily be seen, $y=\sin x$ strictly increases from -1 to 1 as x increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have x as an inverse function of y, (§ 59 page 102), known as inverse sine of y. In symbols, we write

$$x=\sin^{-1} y.$$

Thus $\sin^{-1}y$ is the number, x, lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is y.

Clearly $\sin^{-1}y$ is defined in the interval $[-1, 1]$ only.

To prove that $\sin^{-1}y$ is derivable in the open interval $]-1, 1[$ and that its derivative with respect to y is $1/\sqrt{(1-y^2)}$.

Let $x=\sin^{-1}y$ so that $y=\sin x$.

We have $dy/dx=\cos x$, which is 0 only when $x=-\frac{1}{2}\pi$ or $\frac{1}{2}\pi$, *i.e.*, when $y=-1$ or 1.

$\therefore$ when $y\neq 1$, or -1, we have

$$\frac{dx}{dy}=\frac{1}{\cos x}=\frac{1}{\sqrt{(1-\sin^2 x)}}=\frac{1}{\sqrt{(1-y^2)}},$$

as $\cos x$ is necessarily positive in $]-\frac{1}{2}\pi, \frac{1}{2}\pi[$.

114·2. $\cos^{-1} y$. Since $y = \cos x$ strictly decreases from 1 to -1 as x increases from 0 to π, we have x as an inverse function of y and write

$$x = \cos^{-1} y.$$

Thus $\cos^{-1} y$ is the number, x, lying between 0 and π, whose cosine is x and which is defined in the interval $[-1, 1]$ only.

It is easy to show that $\cos^{-1} y$ is derivable in the open interval $]-1, 1[$ and that its derivative is $-1/\sqrt{(1-y^2)}$.

114·3. $\tan^{-1} y$. Since, as may easily be seen, $y = \tan x$ strictly increases from $-\infty$ to ∞ as x increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we have x as an inverse function of y and write

$$x = \tan^{-1} y.$$

Thus $\tan^{-1} y$ is the number, x, lying between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is y and which is defined for the entire aggregate of real numbers.

It may easily be shown that $\tan^{-1} y$ is derivable for every value of y and that its derivative is $1/(1+y^2)$.

Examples

1. *Discuss the nature of the discontinuity of the function*

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}},$$

at $x = 1$.

Show that $f(0)$ and $f(\pi/2)$ differ in sign, and explain still why $f(x)$ does not vanish in $[0, \pi/2]$.

We know that

$$\lim_{n \to \infty} x^{2n} = \lim_{n \to \infty} (x^2)^n = \begin{cases} \infty & \text{if } x^2 > 1, \\ 1 & \text{if } x^2 = 1, \\ 0 & \text{if } x^2 < 1, \end{cases} \qquad (\S\ 34, \text{ page } 49)$$

Thus,

for $x > 1$,

$$f(x) = \lim_{n \to \infty} \frac{(1/x^{2n}) \log(2+x) - \sin x}{(1/x^{2n}) + 1}$$

$$= -\sin x, \qquad \text{.. } (i)$$

for $-1 < x < 1$,

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}}$$

$$= \log(2+x), \qquad \text{.. } (ii)$$

for $x = 1$,

$$f(x) = f(1) = \frac{\log 3 - \sin 1}{2}.$$

$$\therefore \lim_{x \to (1+0)} f(x) = -\sin 1, \quad \lim_{x \to (1-0)} f(x) = \log 3.$$

As these two right and left handed limits $-\sin 1$ and $\log 3$ [being at least of opposite sign] are unequal, we see that

$$\lim_{x \to 1} f(x)$$

does not exist. Thus the function $f(x)$ is discontinuous at $x=1$ and the discontinuity is of the *second* kind.

Now, since $\pi/2 > 1$, we have, from (*i*),

$$f(\pi/2) = -1 < 0.$$

Also, since $-1 < 0 < 1$, we have, from (*ii*),

$$f(0) = \log 2 > 0.$$

$\therefore$ $f(0)$ and $f(\pi/2)$ are of opposite signs.

The point, 1, of discontinuity of $f(x)$ belongs to the interval $[0, \pi/2]$ so that the *function is not continuous in this closed interval.* This explains why $f(x)$ may not vanish anywhere in $[0, \pi/2]$ in spite of the fact that $f(0)$ and $f(\pi/2)$ are of opposite signs.

It will be seen that in the interval $[0, \pi/2]$, the function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} \log(2+x) \text{ when } 0 \leqslant x < 1, \\ \frac{1}{2}(\log 3 - \sin 1) \text{ when } x=1, \\ -\sin x \text{ when } 1 < x \leqslant \pi/2, \end{cases}$$

so that it is nowhere zero in $[0, \pi/2]$.

2. *If*

$$f(x) = e^{-1/x^2} \sin(1/x), \textit{ for } x \neq 0, f(0) = 0,$$

show that

(*i*) *this function has at every point a differential co-efficient and this is continuous at $x=0$.*

(*ii*) *the differential co-efficient vanishes at $x=0$ and at an infinite number of points in every neighbourhood of $x=0$.*

We have $f'(x) = \dfrac{e^{-1/x^2}}{x^3}\left(2 \sin \dfrac{1}{x} - x \cos \dfrac{1}{x}\right)$, when $x \neq 0$

and $$f'(0) = \lim_{x \to 0} \frac{e^{-1/x^2} \sin(1/x) - 0}{x}.$$

Since $e^{1/x^2} > 1/x^2$,

we see that $\left|\dfrac{e^{-1/x^2}\sin(1/x)}{x}\right| < \dfrac{x^2.1}{|x|} = |x|$.

$$\therefore \qquad f'(0)=0.$$

Also, when $x \neq 0$,

$$\left| f'(x)-f'(0) \right| = \left| \frac{e^{-1/x^2}}{x^3}\left(2\sin\frac{1}{x} - x\cos\frac{1}{x}\right)\right|$$

$$\leqslant \frac{e^{-1/x^2}}{|x|^3}(2+|x|).$$

Since

$$e^{1/x^4} > 1/2x^4,$$

we see that

$$|f'(x)-f'(0)| \leqslant 2|x|(2+|x|),$$

so that

$$[f'(x)-f'(0)] \to 0, \text{ as } x \to 0,$$

i.e., $f'(x)$ is continuous for $x=0$.

Also clearly $f(x)$ is continuous for every non-zero value of x.

We have now to show that $f'(x)$ vanishes at a point in *every* neighbourhood of $x=0$.

Let δ be any positive number however small. There surely exists a positive integer x such that

$$0 < \frac{2}{(2n+1)\pi} < \frac{1}{n\pi} < \delta.$$

It is easy to see that,

for $x=1/n\pi$, $f'(x)$ is $-$ve or $+$ve according as n is even or odd;

for $x=2/(2n+1)\pi$, $f'(x)$ is $+$ve or $-$ve according as n is even or odd.

Therefore $f'(x)$, which is continuous, must vanish at least once between

$$2/(2n+1)\pi \text{ and } 1/(n\pi$$

so that

$$[f'(x)-f'(0)] \to 0, \text{ as } x \to 0,$$

i.e., $f'(x)$ is continuous for $x=0$.

We write

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx.$$

By the first mean value theorem, we have

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx = f(\alpha_n)\int_0^{1/\sqrt{n}} \frac{ndx}{1+n^2x^2}, \text{ where } 0 \leqslant \alpha_n \leqslant 1/\sqrt{n}$$

$$= f(\alpha_n)\tan^{-1}\sqrt{n}, \text{ which} \to f(0).\tfrac{1}{2}\pi \text{ as } n \to \infty.$$

Again

$$\left|\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx\right| = \left| f(\beta_n)\int_{1/\sqrt{n}}^1 \frac{ndx}{1+n^2x^2}\right|, \text{ where } 1/\sqrt{n} \leqslant \beta_n \leqslant 1$$

$$= f(\beta_n)(\tan^{-1} n - \tan^{-1}\sqrt{n}\,|$$

$$\leqslant M(\tan^{-1} n - \tan^{-1}\sqrt{n}), \text{ which} \to 0, \text{ as } n \to \infty,$$

M, being the upper bound of $|f(x)|$.

Hence the result.

4. *If $f(x)$ is bounded and integrable in the interval $[a, b]$, show that*

$$\lim_{n \to \infty} \int_a^b f(x)\cos nx\,dx = 0.$$

We write

$$I_n = \int_a^b f(x)\cos nx\,dx.$$

Let ε be any positive number. Since $f(x)$ is bounded and integrable in $[a, b]$ there exists a division

$$D(a = x_0 < x_1 < x_2 < \ldots\ldots < x_{r-1} < x_r < \ldots\ldots < x_p = b)$$

such that the corresponding oscillatory sum

$$\Sigma(x_r - x_{r-1})\,O_r < \tfrac{1}{2}\varepsilon\,;$$

O_r being the oscillation of $f(x)$ in $[x_{r-1}, x_r]$.

We have

$$I_n = \Sigma \int_{x_{r-1}}^{x_r} f(x) \cos nx \, dx$$

$$= \Sigma f(x_{r-1}) \int_{x_{r-1}}^{x_r} \cos nx \, dx + \Sigma \int_{x_{r-1}}^{x_r} [f(x) - f(x_{r-1})] \cos nx \, dx,$$

$$\therefore \quad \left| I_n \right| < \Sigma \left| f(x_{r-1}) \right| \left| \int_{x_{r-1}}^{x_r} \cos nx \, dx \right| +$$

$$\Sigma \left| \int_{x_{r-1}}^{x_r} \left\{ f(x) - f(x_{r-1}) \right\} \cos nx \, dx \right|.$$

As x varies in $[x_{r-1}, x_r]$, we have

$$\left| f(x) - f(x_{r-1}) \right| \leqslant O_r$$

so that

$$\left| [f(x) - f(x_{r-1})] \cos nx \right| \leqslant O_r.$$

Also

$$\left| \int_{x_{r-1}}^{x_r} \cos nx \, dx \right| \leqslant \frac{1}{n} \left\{ \left| \sin nx_r \right| + \left| \sin nx_{r-1} \right| \right\} < \frac{2}{n},$$

$$\therefore \quad \left| I_n \right| \leqslant \frac{2}{n} \Sigma \left| f(x_{r-1}) \right| + \Sigma (x_r - x_{r-1}) O_r$$

$$\leqslant \frac{2}{n} \Sigma \left| f(x_{r-1}) \right| + \frac{\varepsilon}{2}.$$

Keeping the division D fixed, we see that $\Sigma \left| f(x_{r-1}) \right|$ is fixed. We now choose a positive integer m such that

$$\frac{2}{n} \Sigma \left| f(x_{r-1}) \right| < \frac{\varepsilon}{2} \text{ where } n \geqslant m.$$

Thus for $n \geqslant m$,

$$\left| I_n \right| < \varepsilon.$$

Hence the result.

It may similarly be shown that

$$\lim_{n \to \infty} \int_a^b f(x) \sin nx \, dx = 0.$$

5. *Show that $\lim I_n$, where*

$$I_n = \int_0^h \frac{\sin nx}{x} dx,$$

exists when $n \to \infty$ through positive integral values and that the limit is equal to $\pi/2$.

The integral becomes continuous for every value of x, if we assign to it the value n for $x=0$. The result will be proved in *three* steps.

I. Firstly, it will be proved that $\{I_n\}$ is convergent. Putting

$$nx = t,$$

we have

$$I_n = \int_0^n \frac{\sin t}{t} dt.$$

$$\therefore \qquad \left| I_{n+p} - I_n \right| = \left| \int_{nh}^{(n+p)h} \frac{\sin t}{t} dt \right|.$$

As $1/t$ is positive and monotonically decreasing in $]nh, (n+p)h[$, we have, by the Bonnett's form of the second mean value theorem.

$$\left| I_{n+p} - I_n \right| = \frac{1}{nh} \left| \int_{nh}^{\alpha} \sin t \, dt \right| \leqslant \frac{2}{nh} < \varepsilon, \text{ if } n > 2/\varepsilon h.$$

Hence, by Cauchy's principle of convergence, $\{I_n\}$ converges. (§34 1, page 49).

II. It will now be proved that, when $n \to \infty$,

$$\lim I_n = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x} dx.$$

We write

$$\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx = \int_0^h \frac{\sin nx}{x} dx + \int_h^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx.$$

As proved in the preceding example,

$$\int_h^{\frac{1}{2}\pi} \frac{\sin nx}{x}\, dx \to 0 \text{ as } n \to \infty.$$

$$\therefore \qquad \lim I_n = m \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx.$$

Again, taking $f(x)=(1/x-1/\sin x)$ in the preceding example, we have,

$$\lim \int_0^{\frac{1}{2}\pi} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \sin nx\, dx = 0,$$

for $f(x)$ is continuous in $[0, \frac{1}{2}\pi]$, if we set $f(0)=0$.

$$\therefore \qquad \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}\, dx.$$

$$\therefore \quad \lim I_n = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{x} dx = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x} dx.$$

To determine the actual value of the limit, we proceed by making $n \to \infty$ through odd integral values.

III. We have, as may be easily shown,

$$\frac{\sin (2n+1)x}{\sin x} = 2[\tfrac{1}{2} + \cos 2x + \cos 4x + \ldots\ldots + \cos 2nx]$$

so that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin (2n+1)x}{\sin x}\, dx = \frac{\pi}{2}.$$

Hence the result.

Exercises

I

Continuity and differentiation)

1. (*i*) Show that $\lim \sin (1/x)$, as $x \to 0$, does not exist.

(*ii*) If

$$f(x) = x^2 \sin (1/x), \text{ when } x \neq 0 \text{ and } f(0) = 0,$$

show that $f'(x)$ exists for every value of x but is not continuous for $x=0$

2. If

$$f(x)=\begin{cases} x^2 \sin (1/x), & \text{when } x\neq 0. \\ 0 & \text{when } x=0, \end{cases}$$

and

$$g(x)=x,$$

show that

$$\lim_{x\to 0} [f'(x)/g'(x)] \text{ does not exist,}$$

but

$$\lim_{x\to 0} [f(x)/g(x)] \text{ exists,}$$

and is equal to $f'(0)/g'(0)$.

3. If $f(x) = |x|$, $g(x)=2|x|$, show that $f'(0)$ and $g'(0)$ do not exist but $\lim [f(x)/g(x)]$ exists and is equal to $\lim [f'(x)/g'(x)]$, when $x \to 0$.

4. Characterise the discontinuity of

$$f(x)=\frac{1}{(x-a)} \operatorname{cosec} \frac{1}{x-a},$$

at $x=a$.

5. If

$$f(x)=\sqrt{x}(1+x \sin 1/x), \text{ for } x>0,$$
$$f(x)=-\sqrt{-x}(1+x \sin 1/x), \text{ for } x<0,$$
$$f(0)=0,$$

show that $f'(x)$ exists everywhere and is finite except at $x=0$, in the neighbourhood of which it oscillates between indefinitely great positive and negative values.

6. Show that the function

$$f(x)=x[1+\tfrac{1}{3} \sin (\log x^2)], \text{ when } x\neq 0,$$
$$f(0)=0,$$

is everywhere continuous but has no differential co-efficient at $x=0$.

7. Show that the function

$$f(x)=4+7x+x^2(8+x \sin 1/x),$$

where $x \sin (1/x)$ is zero for $x=0$ has a first derivative but no second derivative at the origin.

8. Find the points of discontinuity of

$$f(x)= \lim_{m\to\infty} (\cos \pi x)^{2m} \; ; \; \varphi(x)= \lim_{n\to\infty} \left[\lim_{m\to\infty} (\cos \pi n\,!\, x)^{2m} \right]$$

[$f(x)=1$ when x is an integer and otherwise $f(x)=0$. Since when x is rational $n\,!\,x$ is an integer for sufficiently large value of n, therefore $\varphi(x)=1$ when x is rational and 0 when x is irrational]

9. Find the points of discontinuity of

(i) $f(x)=\lim\limits_{t\to\infty} \dfrac{(1+\sin \pi x)^t-1}{(1+\sin \pi x)^t+1}$. (ii) $f(x)= \lim\limits_{n\to\infty} \lim\limits_{t\to 0} \dfrac{\sin^2(n\,!\,\pi x)}{\sin^2(n\,!\,\pi x)+t^2}$

10. If

$$f(x)=\sin x \sin (1/\sin x), \text{ when } 0<x<\pi<x<2\pi,$$

and

$$f(x)=0 \text{ when } x=0, \pi, 2\pi\,;$$

show that

$f(x)$ is continuous but not derivable for $x=0, \pi, 2\pi$.

11. If $a_0, a_1, a_2, \ldots\ldots, a_n$ are real and

$$|a_0| + |a_1| + |a_2| + \ldots\ldots + |a_{n-1}| < a_n,$$

show that

$$u(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + a_n \cos nx$$

has at least $2n$ zeros in the interval $0 < x < 2\pi$. Show also that $u'(x)$ has at least $2n$ zeros in the interval $\alpha \leqslant x < 2\pi + \alpha$ for every real α.

[$u(0)$, $u(\pi/n)$, $u(2\pi/n)\ldots\ldots$, $u(2n\pi/n)$ have positive and negative signs alternately.]

12. Determine the differential co-efficient, if any, of

$$f(x) = (x-a)\frac{Ae^{1/(x-a)} + Be^{-1/(x-a)}}{e^{1/(x-a)} + e^{-1/(x-a)}} \sin \frac{\pi}{2(x-a)} \quad \text{when } x \neq a$$

$$f(a) = 0,$$

when $x = a$.

13. A function $f(x)$ is defined as follows :—

$$f(x) = (x-w_1)(x-w_2)^2 (x-w_3)^3 \sin \frac{1}{(x-w_1)} \sin \frac{1}{(x-w_2)} \sin \frac{1}{(x-w_3)},$$

for all values of x except w_1, w_2, w_3, in the domain $[a, b]$ and $f(x) = 0$ when $x = w_1$ or w_2, or w_3. Show that

(*i*) df/dx does not exist at the point $x = w_1$,

(*ii*) df/dx exists but has a discontinuity of the second kind at $x = w_2$,

(*iii*) df/dx exists and is continuous at $x = w_3$.

II

(*Integration*)

1. By applying the first mean value theorem of Integral Calculus, show that

$$\frac{\pi}{6} \leqslant \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]}} \leqslant \frac{\pi}{6} \cdot \frac{1}{\sqrt{(1-\frac{1}{4}k^2)}}.$$

2. Prove that

$$\frac{1}{2}\left(1 - \frac{1}{e}\right) < \int_0^1 e^{-x^2}\, dx < 1 \text{ and } 0 < \int_1^\infty e^{-x^2}\, dx < \frac{1}{2e},$$

and deduce that

$$\frac{1}{2}\left(1 - \frac{1}{e}\right) < \int_0^\infty e^{-x^2}\, dx < 1 + \frac{1}{2e}.$$

3. Show that

$$(i) \quad \frac{\pi}{6} > \int_0^1 \frac{dx}{\sqrt{(4-x^2+x^3)}} > \frac{1}{2}.$$

(ii) $$\frac{\pi}{6} > \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^{2n})}} > \frac{1}{2}$$

for $n > 1$.

4. By the use of the inequality $\sin x < x$, or otherwise prove that, if $0 < c < 1$, then

$$2\int_0^1 \sqrt{(1-c^2\sin^2 x)}\,dx > \sqrt{(1-c^2)}+1.$$

5. Show that

(i) $$\left|\int_p^q \frac{\sin x}{x}\,dx\right| \leqslant \frac{2}{p}, \text{ if } q > p > 0.$$

(ii) $$\left|\int_a^b \sin x^2\,dx\right| \leqslant \frac{1}{a}.$$

6. If

$$f(t, x) = \pi t \sin \pi tx \text{ when } 0 \leqslant x \leqslant 1/t$$

and

$$f(t, x) = 0 \text{ when } 1/t < x \leqslant 1,$$

show that

$$\int_0^1 \left\{\lim_{t\to\infty} f(t, x)\right\} dx = 0 \neq 2 = \lim_{t\to\infty}\left\{\int_0^1 f(t, x)\,dx\right\}$$

7. If $f(t, x) = tx/(1+t^2x^4)$, show that

$$\int_0^1 \left\{\lim_{t\to\infty} f(t, x)\right\} dx = 0 \neq \frac{\pi}{4} = \lim_{t\to\infty}\left\{\int_0^1 f(t, x)\,dx\right\}.$$

8. If a is a fixed positive number, prove that

$$\lim_{h\to 0} \int_{-a}^{a} \frac{h\,dx}{h^2+x^2} = \pi.$$

9. If $f(x)$ is continuous in the interval $[-1, 1]$, prove that

$$\lim_{h\to 0} \int_{-1}^{1} \frac{hf(x)}{1+h^2x^2}\,dx = \pi f(0).$$

[Split up the range $[-1, 1]$ into three ranges

$[-1, -\sqrt{h}]$, $[-\sqrt{h}, \sqrt{h}]$, $[\sqrt{h}, 1)$].

10. If $f(x)$ is bounded and integrable in $[a, b]$, show that

$$\lim_{n \to \infty} \int_a^b f(x) \frac{\sin nx}{x} dx = 0$$

when

$$0 < a < b.$$

11. Assuming that $|n| \neq |m|$, prove that

$$\lim_{y \to \infty} \frac{1}{y} \int_0^y \sin nx \sin mx \, dx = 0.$$

12. In the second mean value theorem of integral Calculus, show that $\varphi(x)$ must be necessarily monotonic by showing that the theorem does not hold if

$$\varphi(x) = \cos x, \; f(x) = x^2.$$

13. Illustrate the advantage of the second mean value theorem over the first by considering the integral

$$I = \int_1^{10} \frac{\sin x}{x} dx$$

and showing that

(*i*) the application of the first mean value theorem gives

$$1 < 10 \log 10 < 24 ;$$

(*ii*) the application of the second mean value theorem gives

$$1 < 2.$$

14. If

$$f(x) = \lim_{p \to \infty} \frac{\left(1 + \sin \frac{\pi}{x}\right)^p - 1}{\left(1 + \sin \frac{\pi}{x}\right)^p + 1},$$

draw the graph of $f(x)$ in $0 < x < 1$ and show that $f(x)$ is integrable in $[0, 1]$.

CHAPTER VIII

IMPROPER INTEGRALS

115. The theory of Riemann integration, as developed in Chapter VI, expressly requires that the *range of integration is finite* and that the *integrand is bounded* in that range. It is possible, however, to so extend the theory that the symbol

$$\int_a^b f(x)\,dx$$

may sometimes have a meaning (*i.e.*, denote a number), even when $f(x)$ is not bounded or when either a or b or both are infinite. In case $f(x)$ is unbounded or the limits a or b are infinite, the symbol

$$\int_a^b f(x)\,dx$$

is called an **improper** (or **generalised** or **infinite**) integral. Thus

$$\int_0^1 \frac{dx}{x^3}, \quad \int_1^2 \frac{dx}{(1-x)(2-x)}, \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

are examples of improper integrals.

For the sake of distinction an integral which is not improper will be called a **proper** integral.

We know (Ex. after, § 58·3, page 97) that if a function $f(x)$ is not bounded in a finite interval $[a, b]$, then *there exists at least one point, c, of the interval such that in every neighbourhood of c, however small it may be, $f(x)$ is not bounded.* Such a point, c, is a point of **infinite discontinuity** of the function $f(x)$. It will always be assumed that the function $f(x)$ is such that its points of infinite discontinuity, which lie in any interval, finite or infinite, are finite in number; the consideration of functions having an infinite number of points of infinite discontinuity being beyond the scope of the book.

In a finite interval which encloses no point of infinite discontinuity, the function is always bounded and we *assume*, once for all, in order to avoid tedious repetition, that it is also integrable in such an interval.

116. Definitions.

116·1. Convergence at the left-end. *Let, a, be the only point of infinite discontinuity of a* function $f(x)$ in a finite interval $[a, b]$

so that, according to the assumption made in the last paragraph, the integral

$$\int_{a+\varepsilon}^{b} f(x)\,dx, \qquad \text{where } 0 < \varepsilon < (b-a),$$

exists and is a function of ε say $\varphi(\varepsilon)$.

If, when $\varepsilon \to (0+0)$, $\varphi(\varepsilon)$ tends to a finite limit, say I, we say that the improper integral

$$\int_{a}^{b} f(x)\,dx, \qquad \ldots(1)$$

exists, or converges at a, and use the symbol (1) to denote the number I. Thus

$$\int^{b} \quad x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit on the right exists. In case the limit does not exist, we say that the improper integral $\int_{a}^{b} f(x)\,dx$ does not exist or that it does not converge.

Ex. 1. Show that if $\int_{a}^{b} f(x)\,dx$ converges at a, then $\int_{a}^{c} f(x)\,dx$, $(a<c<b)$ also converges at a.

2. The improper integral $\int_{a}^{b} f(x)\,dx$ converges at a, and k. is any constant; show that $\int_{a}^{b} kf(x)\,dx$ also converges, at, a, and conversely.

116·2. Convergence at the right-end. *Let, b be the only point of infinite discontinuity of $f(x)$ in a finite* interval $[a, b]$. If then the proper integral

$$\int_{a}^{b-\varepsilon} f(x)\,dx, \text{ where } 0 < \varepsilon < (b-a) \qquad \ldots(1)$$

which is a function of ε, tends to a finite limit, as $\varepsilon \to (0+0)$, we say that the improper integra l

$$\int_a^b f(x)\,dx \qquad \ldots(2)$$

exists or converges at b and use the symbol (2) to denote the limit of (1).

Ex. *Examine the convergence of the improper integrals*

$$(i) \quad \int_0^1 \frac{dx}{x^2}, \qquad (ii) \quad \int_0^1 \frac{dx}{\sqrt{(1-x^2)}}.$$

(i) The left-end point, 0, is the only point of infinite discontinuity of the integrand $1/x^2$ in $[0, 1]$. We have, when $0 < \varepsilon \leqslant 1$,

$$\varphi(\varepsilon) = \int_\varepsilon^1 \frac{dx}{x^2} = \left| -\frac{1}{x} \right|_\varepsilon^1 = \frac{1}{\varepsilon} - 1, \text{ which} \to +\infty \text{ as } \varepsilon \to (0+0).$$

Thus we see that $\varphi(\varepsilon)$ does not tend to a finite limit. Hence the improper integral, in question, does not converge.

(ii) The right-end point, 1, is the only point of infinite discontinuity of $1/\sqrt{(1-x^2)}$ in $[0, 1]$. We have

$$\varphi(\varepsilon) = \int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x^2)}} = \sin^{-1}(1-\varepsilon), \text{ which } \to \sin^{-1} 1 = \tfrac{1}{2}\pi, \text{ as } \varepsilon \to 0.$$

Thus the improper integral converges and is equal to $\frac{1}{2}\pi$.

Note. The reader will note that the integrand $1/x^2$ is not defined for $x=0$ and $1/\sqrt{(1-x^2)}$ is not defined for $x=1$. We may assign to the integrand at such points any value we please without affecting the convergence and the value of the corresponding improper integral.

11·6 3. Convergence at both the end points. *Let the end points a and b be the only points of infinite discontinuity of $f(x)$.*

We take any point, c, within $[a, b]$.

If the improper integrals

$$\int_a^c f(x)\,dx \text{ and } \int_c^b f(x)\,dx$$

converge at the left-end a and at the right-end b respectively, we say that the improper integral $\int_a^b f(x)\,dx$ exists and write

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

It is easy to show that the convergence and the value of the improper integral, in question, is independent of the position of c. If d be any point of $[a, b]$, we have

$$\int_{a+\varepsilon}^d f(x)\,dx=\int_{a+\varepsilon}^c f(x)\,dx+\int_c^d f(x)\,dx.$$

If $\varepsilon \to 0$, we see that $\int_{a+\varepsilon}^d f(x)dx$ tends to a finite limit, *i.e.*, $\int_a^d f(x)\,d$

converges if, and only if, $\int_{a+\varepsilon}^c f(x)\,dx$ tends to a finite limit, *i.e*,

$$\int_a^c f(x)\,dx \text{ converges.}$$

Also, in case they exist finitely,

$$\int_a^d f(x)\,dx=\int_a^c f(x)\,dx+\int_c^d f(x)\,dx. \qquad ...(1)$$

It may similarly be shown that

$\int_d^b f(x)\,dx$ **converges if,** and only if, $\int_c^b f(x)\,dx$ converges,

and in case they converge, we have

$$\int_d^b f(x)\,dx=\int_d^c f(x)\,dx+\int_c^b f(x)\,dx. \qquad ...(2)$$

Adding (1) and (2), we get

$$\int_a^d f(x)\,dx+\int_d^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx.$$

Ex. Examine the convergence of the improper integrals :

$$(i) \int_0^1 \frac{dx}{\sqrt{(x-x^2)}}, \quad (ii) \int_0^2 \frac{dx}{x(2-x)}, \quad (iii) \int_0^\pi \frac{dx}{\sin x}.$$

116·4. General case. Any finite number of points of infinite discontinuity. *Let $c_1, c_2, c_3, \ldots\ldots, c_n$ be any* **finite** *number of points of infinite discontinuity of $f(x)$ lying in $[a, b]$, where*

$$a \leqslant c_1 < c_2 < \ldots\ldots < c_{n-1} < c_n \leqslant b.$$

If the improper integrals

$$\int_a^{c_1} f(x)\,dx, \int_{c_1}^{c_2} f(x)\,dx, \ldots\ldots, \int_{c_{n-1}}^{c_n} f(x)\,dx, \int_{c_n}^b f(x)\,dx$$

all exist in accordance with the definitions given above and we regard

$$\int_a^{c_1} f(x)\,dx=0 \text{ if } c_1=a \text{ and } \int_{c_n}^b f(x)\,dx=0 \text{ if } b=c_n,$$

then we say that $\int_a^b f(x)\,dx$ exists and write

$$\int_a^b f(x)\,dx=\int_a^{c_1} f(x)\,dx+\int_{c_1}^{c_2} f(x)\,dx+\ldots\ldots+\int_{c_n}^b f(x)\,dx.$$

116·5. Infinite Range of Integration. Convergence at ∞. Let $f(x)$ be bounded and integrable in $[a, X]$ where X is any number $\geqslant a$ so that the proper integral

$$\int_a^X f(x)\,dx$$

exists and is a function of X, say, $\varphi(X)$.

If $\varphi(X)$ tends to a finite limit, I, as $X \to \infty$, we say that the improper integral

$$\int_a^\infty f(x)\, dx \qquad .. (1)$$

exists or that it converges at ∞ and regard the symbol (1) as denoting the number I. Thus

$$\int_a^\infty f(x)\, dx = \lim_{X \to \infty} \int_a^X f(x)\, dx,$$

provided the limit exists.

Ex. Examine the convergence of

$$(i) \int_0^\infty \frac{dx}{1+x^2}, \qquad (ii) \int_1^\infty \frac{dx}{x^2}.$$

116·6. Convergence at, $-\infty$. If $f(x)$ be bounded and integrable in $[X, b]$ where $X \leqslant b$, and

$$\int_X^b f(x)\, dx \qquad .. (1)$$

tends to a finite limit as $X \to -\infty$, we say that

$$\int_{-\infty}^b f(x)\, dx \qquad .. (2)$$

converges or exists and regard the symbol (2) as denoting the limit of (1).

116·7. *Let the range of integration be* $]-\infty, \infty[$.

If, c, is any number and

$$\int_{-\infty}^c f(x)\, dx, \quad \int_c^\infty f(x)\, dx,$$

both exist in accordance with the definitions already given, then we say that $\int^\infty f(x)\, dx$ exists and write

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{c} f(x)\,dx+\int_{c}^{\infty} f(x)\,dx.$$

It is easy to show that the existence and the value of

$$\int_{-\infty}^{\infty} f(x)\,dx$$

is independent of the choice of c.

116·8. If an infinite range of integration includes a finite number of points of infinite discontinuity, then we arbitrarily consider an interval $[a, b]$ which embraces all the points of infinite discontinuity of $f(x)$ and examine the existence of the three improper integrals

$$\int_{-\infty}^{a} f(x)\,dx,\quad \int_{a}^{b} f(x)\,dx,\quad \int_{b}^{\infty} f(x)\,dx,$$

in accordance with the definitions given above, and in case they all exist, we say that

$$\int_{-\infty}^{\infty} f(x)\,dx$$

exists and write

$$\int_{-\infty}^{\infty} f(x)\,dx=\int_{-\infty}^{a} f(x)\,dx+\int_{a}^{b} f(x)\,dx+\int_{b}^{\infty} f(x)\,dx.$$

Note. It will be seen that in order to examine the existence of any given improper integral, we have to examine the convergence of a system of improper integrals of the four types which have been considered in the sub-sections 116·1, 116·2, 116·5, 116·6. Also we see that the case § 116·2 is analogous to that of § 116·1 and the case of § 116·6 is analogous to that of § 116·5.

117. From the foregoing, it will thus be seen that it is enough to consider tests for the convergence of

(i) $\displaystyle\int_{a}^{b} f(x)\,dx$ *at* a, *where* $f(x)$ *is bounded and integrable in*

$$[a+\varepsilon, b],\ \ 0 < \varepsilon \leqslant (b-a).$$

(ii) $\int_a^\infty f(x)\,dx$ *at* ∞, *where* $f(x)$ *is bounded and integrable in*

$$[a, X],\ X \geqslant a.$$

Ex. Examine the existence of the following improper integrals and evaluate those which exist :

$$(i)\ \int_1^\infty \frac{dx}{x(1+x)}.\quad (ii)\ \int_0^\infty \frac{dx}{(1+x)\sqrt{x}}.\quad (iii)\ \int_0^\pi \frac{dx}{1+\cos x}.\quad (iv)\ \int_{-1}^\infty \frac{dx}{\sqrt{[x(1+x)]}}.$$

$$(v)\ \int_{-\infty}^\infty \frac{dx}{x(1+x^2)}.\quad (vi)\ \int_0^1 \log x\,dx.\quad (vii)\ \int_{-2}^2 \frac{dx}{x^2(1-x)}.\quad (viii)\ \int_0^\infty x^2 e^{-x}\,dx.$$

Note. In the examples given above, the improper integrals are such that the integrands admit of primitives in terms of elementary functions. In such cases the examination of the existence is generally easy, but more advanced methods are necessary when the integrand does not possess a primitive in terms of elementary functions.

117. Test for convergence at 'a'. Positive integrand. Let a be the only point of infinite discontinuity of $f(x)$ in $[a, b]$. The case where the integrand $f(x)$ is positive in a certain neighbourhood $]a, c[$ of, a, is particularly simple and important and covers a large class of improper integrals.

Since

$$\int_{a+\varepsilon}^b f(x)\,dx = \int_{a+\varepsilon}^c f(x)\,dx + \int_c^b f(x)\,dx,$$

it follows that either

$$\int_a^c f(x)\,dx \text{ and } \int_a^b f(x)\,dx$$

are both convergent at, a, or both non-convergent. It is, therefore, no loss of generality to suppose that $f(x)$ is positive in $[a, b]$.

The question of the existence of the integral is, in such a case, decided by comparison with another suitably chosen integral whose existence or otherwise is already known.

117·1. The necessary and sufficient condition *for the convergence of the improper integral*

$$\int_a^b f(x)\,dx$$

at, a, where $f(x)$ is positive in $[a, b]$, is that there exists a positive number, M, independent of ε, such that

$$\int_{a+\varepsilon}^{b} f(x)\, dx < M, \text{ where } 0 < \varepsilon < (b-a).$$

The proof follows from the fact that, since $f(x)$ is positive in $]a, b]$, the integral

$$\int_{a+\varepsilon}^{b} f(x)\, dx$$

monotonically increases as ε decreases and will, therefore, tend to a finite limit if, and only if, it is bounded above.

Note. In case $\varphi(\varepsilon) = \int_{a+\varepsilon}^{b} f(x)\, dx$ is not bounded above, then $\varphi(\varepsilon) \to +\infty$ as $\varepsilon \to (0+0)$ and we say that the improper integral $\int_{a}^{b} f(x)\, dx$ diverges to ∞.

117·2. Comparison of two integrals. *Let $f(x)$ and $\varphi(x)$ be two functions such that in $]a, b]$, they are both positive and $f(x) \leqslant \varphi(x)$; then*

(i) $\int_{a}^{b} f(x)\, dx$ converges if $\int_{a}^{b} \varphi(x)\, dx$ converges,

and (ii) $\int_{a}^{b} \varphi(x)\, dx$ does not converge if $\int_{a}^{b} f(x)\, dx$ does not.

It is assumed that $f(x)$ and $\varphi(x)$ are both bounded and integrable in $[a+\varepsilon, b]$, $0 < \varepsilon \leqslant (b-a)$, and, a, is the *only* point of infinite discontinuity.

We have

$$\int_{a+\varepsilon}^{b} f(x)\, dx \leqslant \int_{a+\varepsilon}^{b} \varphi(x)\, dx. \qquad \text{.. (1)}$$

Let $\int_a^b \varphi(x)\, dx$ converge so that there exists a number M such that

$$\int_{a+\varepsilon}^{b} \varphi(x)\, dx < M \text{ for } 0<\varepsilon\leqslant(b-a). \qquad \ldots(2)$$

From (1) and (2),

$$\int_{a+\varepsilon}^{b} f(x)\, dx < M \text{ for } 0<\varepsilon\leqslant(b-a).$$

Therefore $\int_a^b f(x)\, dx$ converges at a.

For the second part, we see that if $\int_a^b f(x)\, dx$ does not converge at a, then $\int_{a+\varepsilon}^{b} f(x)\, dx$ is not bounded above and consequently, from (1), $\int_{a+\varepsilon}^{b} \varphi(x)\, dx$ is also not bounded above so that $\int_a^b \varphi(x)\, dx$ does not converge.

117·3. Practical Comparison Test. *If $f(x)/\phi(x) \to l$ when $x \to a$, and, l, is neither 0 nor infinite, then the two integrals*

$$\int_a^b f(x)\, dx \text{ and } \int_a^b \varphi(x)\, dx$$

either both converge or both do not converge.

Since $f(x)/\varphi(x)$ is positive, l cannot be negative.

Let δ be any positive number less than l,

There exists a number c, $(a<c<b)$, such that

$$l-\delta<f(x)/\varphi(x)<l+\delta$$

for $a < x \leqslant c$,

i.e., in $]a, c]$, we have

$$(l-\delta)\varphi(x) < f(x), \qquad \text{... } (i)$$

and

$$f(x) < (l+\delta)\varphi(x). \qquad \text{...}(ii)$$

Let $\int_a^b f(x)\,dx$ converge ;

$\therefore$ $\int_a^c f(x)\,dx$ converges ;

$\therefore$ from (i), $(l-\delta)\int_a^c \varphi(x)\,dx$ converges ;

$\therefore$ $\int_a^c \varphi(x)\,dx$ converges.

Hence $\int_a^b \varphi(x)\,dx$ converges.

From (i), it may similarly be shown that if $\int_a^b \varphi(x)\,dx$ does not converge, then $\int_a^b f(x)\,dx$, also, does not.

Also, from (ii), we may prove that if $\int_a^b \varphi(x)\,dx$, converges, then $\int_a^b f(x)\,dx$ converges, and if, $\int_a^b f(x)\,dx$ does not converge, then $\int_a^b \varphi(x)\,dx$ also does not.

Ex. Prove that if $f(x)/\varphi(x) \to 0$, then $\int_a^b f(x)\,dx$ converges if $\int_a^b \varphi(x)\,dx$ converges and that if $f(x)/\varphi(x) \to \infty$, then $\int_a^b \varphi(x)\,dx$ converges if $\int_a^b f(x)\,dx$ converges.

117·4. A useful comparison integral $\int_a^b \frac{dx}{(x-a)^n}$.

The improper integral

$$\int_a^b \frac{dx}{(x-a)^n} \quad \textit{converges if, and only if } n < 1.$$

We have, if $n \neq 1$,

$$\int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \left| \frac{1}{(1-n)(x-a)^{n-1}} \right|_{a+\varepsilon}^{b}$$
$$= \frac{1}{(1-n)}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right],$$

which tends to $1/(1-n)(b-a)^{n-1}$ or $+\infty$ according as $n < 1$ or >1.

Again, if $n=1$,

$$\int_{a+\varepsilon}^b \frac{dx}{x-a} = \log(b-a) - \log\varepsilon, \text{ which } \to +\infty \text{ as } \varepsilon \to (0+0).$$

Hence the theorem.

Note. The integral $$\int_a^b \frac{dx}{(x-a)^n} \qquad ..(1)$$

is proper if $n \leqslant 0$. The notion of the convergence of improper integrals has enabled us to give a meaning to the symbol (1) even for those values of n which are positive but <1. For $n \geqslant 1$, the symbol does not represent any number.

118. With the help of § 117, we now deduce two important *practical* tests for the convergence at, a, of

$$\int_a^b f(x)\,dx, \qquad ...(1)$$

which arise on comparison with the integral of

$$\varphi(x)=\frac{1}{(x-a)^n}.$$

I. *Let $f(x)$ be positive in $[a, b]$. Then the integral (1) converges at, a, if there exists a positive number n less than 1 and a fixed positive number M such that $f(x) \leqslant M/(x-a)^n$ in the interval $a < x \leqslant b$.*

Also, the integral (1) does not converge if, there exists a number $n \geqslant 1$ and a fixed positive number M such that $f(x) \geqslant M/(x-a)^n$ in the interval $a < x \leqslant b$.

II. *If $\lim\limits_{x \to a} [f(x)(x-a)^n]$ exists and is equal to 'l' and 'l' is neither 0 nor infinite, then the integral* (1) *converges if, and only if, $n < 1$.*

Examples

1. *Examine the convergence of*

$$(i) \int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}, \qquad (ii) \int_0^1 \frac{dx}{x^2(1+x)^2}, \qquad (iii) \int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}}.$$

The integrands are all positive.

(*i*) Here, 0, is the only point of infinite discontinuity of the integrand.

We have

$$f(x)=\frac{1}{x^{\frac{1}{3}}(1+x^2)}.$$

Take $\qquad \varphi(x)=\dfrac{1}{x^{\frac{1}{3}}}.$

Then $\qquad \lim\limits_{x \to 0} \dfrac{f(x)}{\varphi(x)} = \lim\limits_{x \to 0} \dfrac{1}{1+x^2} = 1.$

$$\therefore \qquad \int_0^1 f(x)\,dx \text{ and } \int_0^1 \varphi(x)\,dx$$

have identical behaviours. But $n=\frac{1}{3}$ being less than 1, the latter integral converges by § 117·4. Hence the given integral also converges.

(*ii*) Here, 0, is the only point of infinite discontinuity of the given integrand.

We have

$$(x) = \frac{1}{x^2(1+x)^2}.$$

Take $$\varphi(x) = \frac{1}{x^2}.$$

Then $$\lim_{x \to 0} \frac{f(x)}{\varphi(x)} = 1.$$

$\therefore$ $$\int_0^1 f(x)\,dx \text{ and } \int_0^1 \varphi(x)\,dx$$

have identical behaviours.

But $n=2$ being greater than 1, the latter integral does not converge by §117·4. Hence the given integral also does not converge.

(*ii*) Here, 0, and, 1, are the two points of infinite discontinuity of the integrand.

We have

$$f(x) = \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{3}}}.$$

We take any number between 0 and 1, say, $\frac{1}{2}$, and examine the convergence of the improper integrals,

$$\int_0^{\frac{1}{2}} f(x)\,dx \text{ and } \int_{\frac{1}{2}}^1 f(x)\,dx$$

at, 0, and, 1, respectively.

To examine the convergence of the former integral, we take

$$\varphi(x) = \frac{1}{x^{\frac{1}{2}}}.$$

Then $$f(x)/\varphi(x) \to 1 \text{ as } x \to 0.$$

$\therefore$ by § 117·4, $$\int_0^{\frac{1}{2}} f(x)\,dx \text{ converges.}$$

For the latter, we take

$$\varphi(x) = \frac{1}{(1-x)^{\frac{1}{3}}}.$$

Then $$f(x)/\varphi(x) \to 1 \text{ as } x \to 1.$$

$\therefore$ by § 117·4, $$\int_{\frac{1}{2}}^{1} f(x)\, dx \text{ converges.}$$

Hence $$\int_{0}^{1} f(x)\, dx \text{ converges.}$$

2. Beta Function. *Show that* $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ *exists if, and only if, m, n, are both positive.*

The integral is proper if $m \geqslant 1$ and $n \geqslant 1$.

The number, 0, is a point of infinite discontinuity if $m < 1$ and the number, 1, is a point of infinite discontinuity if $n < 1$.

Let

$$m < 1 \text{ and } n < 1.$$

We take any number, say $\frac{1}{2}$, between 0 and 1 and examine the convergence of the improper integrals

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\, dx \text{ and } \int_{\frac{1}{2}}^1 x^{m-1}(1-x)^{n-1}\, dx$$

at, 0, and, 1, respectively.

Convergence at 0. We write

$$f(x) = x^{m-1}(1-x)^{n-1} = (1-x)^{n-1}/x^{1-m},$$

and take

$$\varphi(x) = 1/x^{1-m}.$$

Then

$$f(x)/\varphi(x) \to 1 \text{ as } x \to 0.$$

As $$\int_0^{\frac{1}{2}} \varphi(x)\, dx = \int_0^{\frac{1}{2}} \frac{1}{x^{1-m}}\, dx$$

is convergent at, 0, if and only if,

$$1-m < 1,\ i.e.,\ 0 < m,$$

we deduce *that the integral*

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$$

is convergent at 0, if and only if, m is positive.

Convergence at 1. We write

$$f(x)=x^{m-1}(1-x)^{n-1}=x^{m-1}/(1-x)^{1-n},$$

and take

$$\varphi(x)=1/(1-x)^{1-n}.$$

Then

$$f(x)/\varphi(x) \to 1 \text{ as } x \to 1$$

As $$\int_{\frac{1}{2}}^{1} \varphi(x)\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{(1-x)^{1-n}}\,dx$$

is convergent, if and only if,

$$1-n < 1, \text{ i.e., } 0 < n,$$

we deduce that the integral

$$\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

converges if, and only if,

$$n > 0.$$

Thus $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ *exists for positive values of m, n only.*

It is a function of *two* variables m, n and is called **Beta function** denoted by **B(m, n)**. It is defined for all positive values of m and n.

3. *Show that*

$$\int_0^{\frac{1}{2}\pi} x^m \operatorname{cosec}^n x\,dx \qquad \ldots(i)$$

exists if, and only if, $n < (m+1)$.

Writing,

$$f(x)=\frac{x^m}{(\sin x)^n}=\left(\frac{x}{\sin x}\right)^n\cdot x^{m-n}=\left(\frac{x}{\sin x}\right)^n\frac{1}{x^{n-m}},$$

we see that when $x \to 0$ then, $f(x) \to 0$ if $(m-n) > 0$, $\to 1$ if $m-n=0$, and $\to \infty$ if $(m-n) < 0$.

Thus (i) is a proper integral if $(m-n) \geqslant 0$; and improper if $(m-n) < 0$; 0 being the only point of infinite discontinuity of the integrand in this case.

Let $$(m-n) < 0.$$

Take

$$\varphi(x)=\frac{1}{x^{n-m}}.$$

Then

$$f(x)/\varphi(x) \to 1 \text{ as } x \to 0.$$

Also $$\int_0^{\frac{1}{2}\pi} \varphi(x)dx=\int_0^{\frac{1}{2}\pi}\frac{1}{x^{n-m}}\,dx$$

converges if, and only if,

$$(n-m) < 1, \text{ i.e., } n < (m+1).$$

Therefore the integral converges if, and only if, $n < (m+1)$, which also includes the case $n \leqslant m$ when the integral is proper.

4. *Examine the convergence of*

$$\int_0^1 x^{n-1} \log x \, dx.$$

The integrand is negative in the interval $]0, 1]$ and we, therefore, consider

$$\int_0^1 -x^{n-1}\log x\,dx=\int_0^1 x^{n-1}\log\left(\frac{1}{x}\right)dx.$$

The integrand is proper if $(n-1) > 0$, in as much as the integrand, in that case, $\to 0$ as $x \to 0$ and accordingly, 0, is not a point of infinite discontinuity in this case. (Cor. to § 85, page 135).

Let $(n-1) \leqslant 0$ so that we have now to examine the convergence at 0. Let m be a positive number such that

$$(m+n-1) > 0.$$

We have

$$\lim_{x \to 0} [-x^{m+n-1} \log x] = 0,$$

so that for values of x, sufficiently near 0,

$$-x^{m+n-1} \log x < \varepsilon,$$

where ε is a given positive number,

or
$$-x^{n-1} \log x < \varepsilon/x^m.$$

Now the integral of ε/x^m converges at 0, if, and only if, $m < 1$.

It is possible to choose a number $m < 1$ such that

$$(m+n-1) = [(m-1)+n] > 0$$

if, and only if, $n > 0$.

Thus the integral converges if $n > 0$.

When $n=0$, the integrand becomes $x^{-1} \log x$. We have

$$\int_\varepsilon^1 x^{-1} \log x \, dx = -\frac{(\log \varepsilon)^2}{2} \text{ which } \to -\infty \text{ as } \varepsilon \to 0.$$

When $n < 0$, we have

$$x^{-1} < x^{n-1} \text{ for } x \text{ in }]0, 1],$$

so that in this case also the integral does not converge.

Thus the integral converges if, and only if, $n > 0$.

5. *Show that*

$$\int_0^{\frac{1}{2}\pi} \sin x \log \sin x \, dx,$$

is convergent and its value is $\log (2/e)$.

Integrating by parts,

$$\int_\varepsilon^{\frac{1}{2}\pi} \sin x \log \sin x \, dx = \Big| -\cos x \log \sin x + \log \tan \tfrac{1}{2}x + \cos x \Big|_\varepsilon^{\frac{1}{2}\pi}$$

$$= -(1-\cos \varepsilon) \log \sin \tfrac{1}{2}\varepsilon + \cos \varepsilon \log 2 \cos \tfrac{1}{2}\varepsilon - \cos \varepsilon + \log \cos \tfrac{1}{2}\varepsilon.$$

Now, when $\varepsilon \to 0$,

$$\lim (1-\cos \varepsilon) \log \sin \tfrac{1}{2}\varepsilon = \lim 2t^2 \log t \text{ ; when } t \to 0, \ (t = \sin \tfrac{1}{2}\varepsilon)$$
$$= 0$$

$$\therefore \quad \int_0^{\frac{1}{2}\pi} \sin x \log \sin x \, dx = \log 2 - 1 = \log (2/e).$$

Exercises

1. Test the convergence of the following infinite integrals :—

(i) $\int_0^{\pi/2} \frac{\cos x}{x^n}\, dx.$ (ii) $\int_0^{\pi/2} \frac{\sin x}{x^n}\, dx.$ (iii) $\int_0^1 \frac{x^{a-1}}{1+x}\, dx.$

(iv) $\int_0^1 \frac{x^{a-1}}{1-x}\, dx.$ (v) $\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\, dx.$ (vi) $\int_0^{\pi/4} \frac{dx}{\sqrt{(\tan x)}}.$

(vii) $\int_0^{\pi/2} \sin^{m-1} x \cos^{n-1} x\, dx.$ (viii) $\int_0^1 x^{a-1} e^{-x}\, dx.$

(ix) $\int_0^3 \frac{dx}{[(x-1)^2(x-2)^3]^{\frac{1}{5}}}.$ (x) $\int_0^1 \frac{x^p \log x\, dx}{(1+x)^2}.$

(xi) $\int_0^{\pi} \frac{\sqrt{x}}{\sin x}\, dx.$ (xii) $\int_0^{\pi} \frac{dx}{\cos \alpha - \cos x}.$

2. Discuss the convergence of the integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} \log\left(\frac{1}{x}\right) dx.$$

3. Show that

$$\int_0^{\frac{1}{2}\pi} \log \sin x\, dx$$

converges.

118. f(x), not necessarily positive. General Test for Convergence. We now obtain a *general* test for convergence at, a, of the infinite integral

$$\int_a^b f(x)\, dx. \qquad \ldots(1)$$

The necessary and sufficient condition for the convergence of the improper integral (1) at, a, is that to every positive number η, there corresponds a positive number δ, such that

$$\left| \int_{a+\varepsilon_1}^{a+\varepsilon_2} f(x)\, dx \right| < \eta,$$

where ε_1, ε_2 *are any two positive numbers less than or equal to* δ.

We write

$$\varphi(\varepsilon)=\int_{a+\varepsilon}^{b} f(x)\,dx.$$

From § 51, page 85, we know that the necessary and sufficient condition for $\lim \varphi(\varepsilon)$ to exist finitely is that to every positive number η there corresponds a number $\delta>0$ such that when $0<\varepsilon_1$, $\varepsilon_2<\delta$, then

$$|\,\varphi(\varepsilon_1)-\varphi(\varepsilon_2)\,|<\eta,$$

i.e.,
$$\left|\int_{a+\varepsilon_1}^{b} f(x)dx-\int_{a+\varepsilon_2}^{b} f(x)dx\right|<\eta \text{ or } \left|\int_{a+\varepsilon_1}^{a+\varepsilon_2} f(x)dx\right|<\eta.$$

118·1. *From above, we deduce an important* **sufficient** *test for convergence, viz.*

If $\int_a^b |\,f(x)\,|\,dx$ *exists, then* $\int_a^b f(x)\,dx$ *also exists.*

(It is assumed that the proper integral $\int_{a+\varepsilon}^{b} f(x)\,dx$ *exists.*)

This theorem follows from the general condition of convergence above and the inequality

$$\left|\int_{a+\varepsilon_1}^{a+\varepsilon_2} f(x)dx\right| \leqslant \left|\int_{a+\varepsilon_1}^{a+\varepsilon_2} |\,f(x)\,|\,dx\right|. \qquad \text{(Cor. 6 to §95, page 168)}$$

Def. Absolute convergence. *The improper integral* $\int_a^b f(x)\,dx$ *is said to be absolutely convergent if* $\int_a^b |\,f(x)\,|\,dx$ *is convergent.*

From the result proved above, it follows that *every absolutely convergent integral is also convergent.*

Ex. 1. *Test the convergence of* $\int_0^1 \frac{\sin (1/x)}{\sqrt{x}}\,dx.$

Let $f(x)=\sin(1/x)/\sqrt{x}$. Here there is no neighbourhood of the point, 0, in which $f(x)$ constantly keeps the same sign.

In]0, 1], we have

$$\left|\frac{\sin 1/x}{\sqrt{x}}\right|=\frac{|\sin 1/x|}{\sqrt{x}}\leqslant\frac{1}{\sqrt{x}}.$$

Also
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx,$$
is convergent

$\therefore \quad \displaystyle\int_0^1 \left|\frac{\sin 1/x}{\sqrt{x}}\right| dx$ is convergent so that $\displaystyle\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$ is absolutely convergent.

119. Convergence at ∞. *Convergence of*

$$\int_a^\infty f(x)\,dx,$$

where $f(x)$ is bounded and integrable in $[a, X]$ for every $X \geqslant a$.

Positive integrand. *Let $f(x)$ be positive in $[a, X]$. The necessary and sufficient condition for $\displaystyle\int_a^\infty f(x)\,dx$ to be convergent is that there exists a positive number, M, independent of X, such that*

$$\int_a^X f(x)\,dx < M \text{ for every } X \geqslant a.$$

The proof follows immediately from the fact that, since $f(x)$ is positive, the integral $\displaystyle\int_a^X f(x)\,dx$ *monotonically increases as X increases* and will, therefore, tend to a finite limit or to ∞ according as it is bounded above or not.

From this we immediately deduce that *if $f(x)$ and $\varphi(x)$ are both positive and $\varphi(x) \leqslant f(x)$ in (a, X), then*

$$\int_a^\infty \varphi(x)\,dx \textit{ converges if } \int_a^\infty f(x)\,dx \textit{ converges.}$$

Also it may be shown that *if $f(x)$ and $\varphi(x)$ are positive and when $x \to \infty$, lim (f/φ) exists and is equal to, l, and, l, is neither 0 nor ∞, then*

$$\int_a^\infty f(x)\,dx \textit{ and } \int_a^\infty \varphi(x)\,dx$$

either both converge or both do not converge

Ex. What conclusion can be drawn if, l, is zero or infinite.

119·1. A useful comparison integral. *To prove that*

$$\int_a^\infty \frac{dx}{x^n},\ (a > 0) \qquad \dots (1)$$

converges if, and only if, $n > 1$.

We have, if $n \neq 1$,

$$\int_a^X \frac{dx}{x^n} = \frac{1}{1-n}\left|\frac{1}{x^{n-1}}\right|_a^X = \frac{1}{1-n}\left[\frac{1}{X^{n-1}} - \frac{1}{a^{n-1}}\right],$$

which $\to 1/(n-1)a^{n-1}$ or ∞ according as $n > 1$ or $n < 1$ when $X \to \infty$.

For $n = 1$, we have

$$\int_a^X \frac{dx}{x} = \log \frac{X}{a}, \text{ which} \to \infty \text{ as } X \to \infty.$$

Hence the result.

Adopting (1) as the comparison integral and employing the test of § 119, we may now easily obtain the following practical tests for convergence at ∞.

119·2. *If $f(x)$ is positive in $[a, X]$, then the integral converges, if there exists a positive number n greater than 1 and a fixed positive number M such that*

$$f(x) \leqslant M/x^n \textit{ for every } x \geqslant a.$$

Again, the integral does not converge, if there exists a positive number $n \leqslant 1$ and a fixed positive number M such that

$$f(x) \geqslant M/x^n \textit{ for every } x \geqslant a.$$

119·3. *If, when $x \to \infty$, lim $[f(x)x^n]$ exists finitely, the limit being neither 0 nor infinite, then the integral converges if, and only if, $n > 1$.*

Examples

1. *Examine the convergence of*

$$(i)\ \int_0^{\infty} \frac{x\,dx}{(1+x)^3}. \qquad (ii)\ \int_1^{\infty} \frac{dx}{(1+x)\sqrt{x}}.$$

$$(iii)\ \int_1^{\infty} \frac{dx}{x^{\frac{1}{3}}\,(1+x)^{\frac{1}{2}}}. \qquad (iv)\ \int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx.$$

(*i*) Let

$$f(x)=\frac{x}{(1+x)^3}.$$

We take

$$\varphi(x)=\frac{x}{x^3}=\frac{1}{x^2}.$$

As $$\lim_{x\to\infty} \frac{f(x)}{\varphi(x)}=\lim_{x\to\infty} \frac{x^3}{(1+x)^3}=1,$$

the two integrals

$$\int_1^{\infty} \frac{x\,dx}{(1+x)^3} \text{ and } \int_1^{\infty} \frac{1}{x^2}\,dx$$

have identical behaviours for convergence at ∞.

By § 119·1, the latter integral is convergent. Accordingly, the given integral is also convergent.

$$\therefore \qquad \int_0^{\infty} \frac{x\,dx}{(1+x)^3} \text{ is convergent.}$$

(*ii*) Let $$f(x)=\frac{1}{(1+x)\sqrt{x}}.$$

We take

$$\varphi(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{\frac{3}{2}}}.$$

We have

$$\lim_{x\to\infty} \frac{f(x)}{\varphi(x)}=1.$$

and
$$\int_1^\infty \frac{1}{x^2}\,dx$$

is convergent. Thus
$$\int_1^\infty f(x)\,dx \text{ is convergent.}$$

(*iii*) Let
$$f(x)=\frac{1}{x^{\frac{1}{3}}(1+x)^{\frac{1}{2}}}.$$

We take
$$\varphi(x)=\frac{1}{x^{\frac{1}{3}}\,x^{\frac{1}{2}}}=\frac{1}{x^{\frac{5}{6}}}.$$

We have $\lim \dfrac{f(x)}{\varphi(x)}=1$, when $x\to\infty$,

and
$$\int_1^\infty \varphi(x)\,dx=\int_1^\infty \frac{1}{x^{\frac{5}{6}}}\,dx,\ n=\tfrac{5}{6}<1$$

is not convergent.
$$\therefore \qquad \int_1^\infty f(x)\,dx \text{ is divergent.}$$

(*iv*) Let $\qquad f(x)=\sin^2 x/x^2.$

Here $x^2 f(x)=\sin^2 x$, which does not tend to a limit but is bounded in $[1, \infty[$.

Now, $\qquad \dfrac{\sin^2 x}{x^2}\leqslant\dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges.

$\therefore \quad \displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges. Also $\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral for $\lim(\sin^2 x/x^2)=1$ when $x\to 0$, so that, 0, is not a point of infinite discontinuity. Therefore $\displaystyle\int_0^\infty f(x)\,dx$ is convergent.

2. Gamma function. *Show that* $\int_0^\infty x^{a-1} e^{-x}\, dx$ *is convergent if, and only if,* $a > 0$.

Let

$$f(x) = x^{a-1} e^{-x}.$$

Here, 0, is a point of infinite discontinuity of $f(x)$ if $a < 1$. Thus we have to examine the convergence at ∞ as well as at, 0.

We consider any positive number > 0, say 1, and examine the convergence of

$$\int_0^1 f(x)\, dx \text{ and } \int_1^\infty f(x)\, dx,$$

at, 0, and ∞, respectively.

(*i*) *Convergence at 0.* Let $a < 1$. We take

$$\phi(x) = \frac{1}{x^{1-a}},$$

so that

$$\lim_{x \to 0} \frac{f(x)}{\phi(x)} = 1.$$

Also

$$\int_0^1 \frac{1}{x^{1-a}}\, dx$$

converges, if and only if, $(1-a) < 1$,

i.e.,

$$0 < a.$$

(*ii*) *Convergence at* ∞.

We know that $e^x > x^{a+1}$ whatever value, a, may have.

$$\therefore \qquad x^{a-1} e^{-x} < 1/x^2.$$

But $\int_1^\infty \frac{1}{x^2}\, dx$ converges. Therefore $\int_1^\infty x^{a-1} e^{-x}\, dx$ also converges for *every* value of a.

Thus $\int_0^\infty x^{a-1} e^{-x}\, dx$ *converges if, and only if,* $a > 0$.

The integral $\int_0^\infty x^{a-1} e^{-x}\, dx$, which is a function of a, is called **Gamma Function** and is denoted by $\Gamma(a)$. This function is defined only for positive values of a.

3. *Show that*

$$\int_0^\infty \left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{dx}{x},$$

is convergent.

The point, 0, is not a point of infinite discontinuity of the integrand in as much as the integrand $\to \frac{1}{6}$ as $x \to 0$. We have, therefore, to examine convergence at ∞ only. We have

$$\left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{}{x}=\frac{e^x-e^{-x}-2x}{x^2\,(e^x-e^{-x})}$$

$$< \frac{e^x}{x^2(e^x-e^{-x})}, \text{ for } x > 0,$$

$$= \frac{e^{2x}}{e^{2x}-1} \cdot \frac{1}{x^2}.$$

Since $e^{2x}/(e^{2x}-1) \to 1$ as $x \to \infty$, we can find a number X such that for $x \geqslant X$,

$$e^{2x}/(e^{2x}-1) < \tfrac{3}{2}.$$

Thus for $x \geqslant X$,

$$\left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{1}{x} < \frac{3}{2} \cdot \frac{1}{x^2},$$

so that

$$\int_X^\infty \left(\frac{1}{x}-\frac{1}{\sinh x}\right)\frac{1}{x}\, dx,$$

is convergent, and, also, therefore the given integral is convergent.

4. *Show that the integral*

$$\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}$$

is convergent.

The integrand, $\dfrac{e^x-1-x}{x(1+x)e^x}$, is clearly positive, when $x > 0$.

Also, 0, is not a point of infinite discontinuity, for the integrand tends to 0 as $x \to 0$.

For convergence at ∞, we rewrite the integrand as

$$\left(\frac{1}{1+x}-e^{-x}\right)\frac{1}{x}=\frac{e^x-(1+x)}{e^x}\cdot\frac{x}{1+x}\cdot\frac{1}{x^2}$$

and note that $[e^x-(1+x)]/e^x \to 1$ as $x \to \infty$. Taking $\phi(x)=1/x^2$, we may see that the given integral is convergent.

Exercises

1. Discuss the convergence of the following :—

(i) $\displaystyle\int_0^\infty \frac{x^m(1+x^n)}{1+x^p}\,dx.$ (ii) $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx.$ $(m>0,\ n>0).$

(iii) $\displaystyle\int_0^\infty \frac{x^{m-1}-x^{n-1}}{1-x}\,dx.$ (iv) $\displaystyle\int_1^\infty x^{m-1}e^{-nx}\,dx.$

(v) $\displaystyle\int_2^\infty x^m(\log x)^n\,dx.$ (vi) $\displaystyle\int_0^\infty \frac{x^a}{(1+x)^b\,[1+(\log x)^2]}\,dx.$

2. Show that the improper integral

$$\int_0^\infty \log(1+2\operatorname{sech} x)\,dx,$$

converges.

[$\log(1+2\operatorname{sech} x) < 2\operatorname{sech} x$, for $\log(1+x) < x$, if $x > 0$.

$$=\frac{2\cdot 2}{e^x+e^{-x}}<\frac{4}{e^x}=4e^{-x}.]$$

3. Show that

$$\int_0^\infty \frac{\cosh bt}{\cosh at}\,dt, \qquad a>0,\ b>0,$$

converges if, and only if, $b<a$.

[If $b<a$, we have

$$\frac{\cosh bt}{\cosh at}=\frac{e^{bt}+e^{-bt}}{e^{at}+e^{-at}}<\frac{e^{bt}+e^{bt}}{e^{at}}=2e^{-(a-b)t},$$

and if $b>a$, we have

$$\frac{\cosh bt}{\cosh at}=\frac{e^{bt}+e^{-bt}}{e^{at}+e^{-at}}>\frac{e^{bt}}{e^{at}+e^{at}}=\tfrac{1}{2}e^{(b-a)t}.]$$

4. Show that

$$\int_0^\infty \frac{\sinh bx}{\sinh ax}\, dx, \qquad a>0,\ b>0;$$

converges if, and only if $a>b$.

[If $a>b$, we write

$$\frac{\sinh bx}{\sinh ax}=\frac{e^{bx}-e^{-bx}}{e^{ax}-e^{-ax}}<\frac{e^{bx}}{e^{ax}-1};$$

and if $a<b$, we write

$$\frac{\sinh bx}{\sinh ax}=\frac{e^{bx}-e^{-bx}}{e^{ax}-e^{-ax}}>\frac{e^{bx}-1}{e^{ax}}.\ ]$$

5. Discuss the convergence of the improper integral

$$\int_0^\infty \left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}\right)\frac{e^{-kx}}{x}\, dx.$$

6. Show that the integrals

$$\int_{-\infty}^\infty e^{-x^2}\, dx \quad\text{and}\quad \int_{-\infty}^\infty e^{-(x-a/x)^2}\, dx$$

converge.

120. Convergence of $\int_a^\infty f(x)\,dx$ at ∞ when $f(x)$ is not necessarily positive. General test for convergence.

The necessary and sufficient condition for the convergence of

$$\int_a^\infty f(x)\, dx$$

at ∞ is that, corresponding to every positive number η, there exists a number X, such that

$$\left|\int_{X_1}^{X_2} f(x)\, dx\right| < \eta,$$

when X_1, X_2 are any two numbers $\geqslant X$.

We write

$$\varphi(X)=\int_a^X f(x)\, dx.$$

From § 51, page 85, we know that the necessary and sufficient condition for $\lim \varphi(x)$ to exist finitely is that to every positive number, η, there corresponds a number X_0 such that, when X_1 X_2 are $\geqslant X_0$,

$$| \varphi(X_2) - \varphi(X_1) | < \eta,$$

i.e., $$\left| \int_a^{X_2} f(x)\, dx - \int_a^{X_1} f(x) dx \right| < \eta \text{ or } \left| \int_{X_1}^{X_2} f(x)\, dx \right| < \eta.$$

121. Theorem. *If* $\int_a^{\infty} | f(x) |\, dx$ *converges, then* $\int_a^{\infty} f(x)\, dx$ *also converges.*

Let η be any positive number. Since $\int_a^{\infty} | f(x) |\, dx$ converges, there exists a number X, such that,

$$\left| \int_{X_1}^{X_2} | f(x) |\, dx \right| < \eta, \text{ where } X_1, X_2 \text{ are both} \geqslant X. \quad .. (1)$$

Also, $$\left| \int_{X_1}^{X_2} f(x)\, dx \right| \leqslant \left| \int_{X_1}^{X_2} | f(x) |\, dx \right|. \quad ...(2)$$

From (1) and (2), the result follows.

Def. Absolute convergence. *The improper integral* $\int_a^{\infty} f(x)\, dx$ *is said to be absolutely convergent, if* $\int_a^{\infty} | f(x) |\, dx$ *is convergent.*

From the theorem above it follows that *every absolutely convergent improper integral is convergent.*

It will later on be seen that the converse is not necessarily true. (See solved Ex. 1, page 257, at the end of this chapter).

121·1. Test for the absolute convergence of the integral of a product. *Let* $\varphi(x)$ *be bounded in* $[a, \infty[$ *and integrable in* $[a, X]$

where X is any number $\geqslant a$. *Let* $\int_a^\infty f(x)\,dx$ *converge absoutely at* ∞.

Then

$$\int_a^\infty f(x)\varphi(x)\,dx,$$

is absolutely convergent.

Since $\varphi(x)$ is bounded in $[a, \infty[$ there exists a positive constant A such that

$$|\,\varphi(x)\,| \leqslant A, \text{ for every } x \geqslant a. \qquad \ldots(1)$$

Since $\int_a^\infty |\,f(x)\,|\,dx$ is convergent, there exists a positive number, B, such that

$$\int_a^X |\,f(x)\,|\,dx \leqslant B, \text{ for every } X \geqslant a, \qquad \ldots(2)$$

the integrand, $|\,f(x)\,|$, being positive. (§ 119, page 241)

We have, from (1),

$$|\,f(x)\varphi(x)\,| \leqslant A\,|\,f(x)\,|, \text{ for every } x \geqslant a.$$

$$\therefore \qquad \int_a^X |\,f(x)\varphi(x)\,|\,dx \leqslant A\int_a^X |\,f(x)\,|\,dx \leqslant AB, \text{ for every}$$

$X \geqslant a$, so that

$$\int_a^X |\,f(x)\varphi(x)\,|\,dx \text{ is bounded above for } X \geqslant a.$$

Hence
$$\int_a^\infty |\,f(x)\varphi(x)\,|\,dx$$

is convergent,

i.e.,
$$\int_a^\infty f(x)\varphi(x)\,dx$$
is absolutely convergent.

Ex. Discuss the convergence of the following integrals :

(*i*) $\displaystyle\int_1^\infty \frac{\sin x}{x^2}dx.$ (*ii*) $\displaystyle\int_0^\infty e^{-ax}\cos x\,dx.$

122. Tests for conditional convergence.

122·1. Abel's theorem for the convergence of the integral of a product. *Let* $\varphi(x)$ *be bounded and monotonic in* $[a, \infty[$ *and let*
$$\int_a^\infty f(x)dx \text{ be convergent.}$$

Then $\displaystyle\int_a^\infty f(x)\,\varphi(x)\,dx$ *is convergent.*

The bounded function $\varphi(x)$, which is monotonic in $x \geqslant a$, is integrable in $[a, X]$ where X is any number $\geqslant a$.

Applying the second mean value theorem, we have
$$\int_{X_1}^{X_2} f(x)\,\varphi(x)\,dx=\varphi(X_1)\int_{X_1}^{\xi} f(x)\,dx+\varphi(X_2)\int_{\xi}^{X_2} f(x)\,dx, \qquad ...(1)$$
where $a < X_1 \leqslant \xi \leqslant X_2$.

Let η be any positive number.

Since $\varphi(x)$ is bounded in $[a, \infty[$, there exists a positive number A such that
$$|\,\varphi(x)\,| \leqslant A, \text{ for every } x \geqslant a.$$

In particular, therefore,
$$|\,\varphi(X_1)\,| \leqslant A, \; |\,\varphi(X_2)\,| \leqslant A. \qquad ...(2)$$

Also, since $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, there exists, by § 120, a number X_0 such that,

$$\left| \int_{X_1}^{X_2} f(x)\, dx \right| < \frac{\eta}{2A}, \text{ for } X_1, X_2 \geqslant X_0.$$

We now suppose that, in (1), X_1, X_2 are numbers $\geqslant X_0$ so that ξ which lies between X_1 and X_2 is also $\geqslant X_0$.

$$\therefore \quad \left| \int_{X_1}^{\xi} f(x)\, dx \right| < \frac{\eta}{2A} \text{ and } \left| \int_{\xi}^{X_2} f(x) dx \right| < \frac{\eta}{2A}. \qquad \ldots(3)$$

From (1), (2) and (3), we deduce that there exists a number X_0 such that for any pair of numbers X_1, $X_2 \geqslant X_0$,

$$\left| \int_{X_1}^{X_2} f(x)\, \varphi(x) dx \right| \leqslant |\, \varphi(X_1)\, | \left| \int_{X_1}^{\xi} f(x)\, dx \right| + |\, \varphi(X_2)\, | \left| \int_{\xi}^{X_2} f(x)\, dx \right|$$

$$< A . \frac{\eta}{2A} + A . \frac{\eta}{2A} = \eta,$$

where η is any positive number assigned arbitrarily.

Hence $\displaystyle\int^{\infty} f(x)\, \varphi(x)\, dx$ converges at ∞.

122·2. Dirichilet's theorem for the convergence of the integral of a product. *Let $\varphi(x)$ be bounded and monotonic in $[a, \infty[$ and let $\varphi(x) \to 0$, when $x \to \infty$. Also let $\displaystyle\int_a^X f(x)\, dx$ be bounded when $X \geqslant a$.*

Then
$$\int_a^{\infty} f(x)\, \varphi(x)\, dx$$
is convergent.

The function $\varphi(x)$, which is monotonic in $[a, \infty[$, is integrable in $[a, X]$ where X is any number $\geqslant a$.

Applying the second mean value theorem, we have

$$\int_{X_1}^{X_2} f(x)\, \varphi(x)\, dx = \varphi(X_1) \int_{X_1}^{\xi} f(x)\, dx + \varphi(X_2) \int_{\xi}^{X_2} f(x)\, dx, \quad \ldots(1)$$

where $a < X_1 \leqslant \xi \leqslant X_2$.

Since $\int_a^X f(x)\,dx$ is bounded when $X \geqslant a$, there exists a number A such that

$$\left| \int_a^X f(x)\,dx \right| \leqslant A, \text{ for every } X \geqslant a$$

$$\therefore \quad \left| \int_{X_1}^{\xi} f(x)\,dx \right| = \left| \int_a^{\xi} f(x)\,dx - \int_a^{X_1} f(x)\,dx \right|$$

$$\leqslant \left| \int_a^{\xi} f(x)\,dx \right| + \left| \int_a^{X_1} f(x)\,dx \right|$$

$$\leqslant A + A = 2A. \qquad \text{..(2)}$$

Similarly $$\left| \int_{\xi}^{X_2} f(x)\,dx \right| \leqslant 2A. \qquad \text{...(3)}$$

Let η be any positive number.

Since $\varphi(x) \to 0$ as $x \to \infty$, there exists a number X_0, such that

$$|\varphi(x)| < \eta/4A, \text{ when } x \geqslant X_0. \qquad \text{...(4)}$$

We now suppose that X_1, X_2 are any two numbers $\geqslant X_0$, so that from (4),

$$|\varphi(X_1)| < \eta/4A, \quad |\varphi(X_2)| < \eta/4A. \qquad \text{...(5)}$$

From (1), (2), (3) and (5), we deduce that there exists a number X_0, such that for any pair of numbers $X_1, X_2 \geqslant X_0$.

$$\left| \int_{X_1}^{X_2} f(x)\,\varphi(x)\,dx \right| \leqslant |\varphi(X_1)| \left| \int_{X_1}^{\xi} f(x)\,dx \right| + |\varphi(X_2)| \left| \int_{\xi}^{X_2} f(x)\,dx \right|$$

$$\leqslant (\eta/4A).2A + (\eta/4A).2A = \eta,$$

where η is any positive number arbitrarily assigned.

Hence $\int_a^\infty f(x)\,\varphi(x)\,dx$ converges at ∞.

Examples

1. *Show that*

$$\int_0^\infty \frac{\sin x}{x}\,dx \qquad \ldots(1)$$

is convergent.

Since the integrand $\to 1$, as $x \to 0$, therefore, 0, is not a point of infinite discontinuity.

Now consider the improper integral

$$\int_1^\infty \frac{\sin x}{x}\,dx. \qquad \ldots(2)$$

The factor $1/x$ of the integrand is monotonic and $\to 0$ as $x \to \infty$. Also

$$\left|\int_1^X \sin x\,dx\right| = \left|-\cos X + \cos 1\right| \leqslant \left|\cos X\right| + \left|\cos 1\right| < 2,$$

so that

$$\int_1^X \sin x\,dx$$

is bounded above for every $X \geqslant 1$.

Here by § 122·2, the integral (2) is convergent. Now since

$$\int_0^1 \frac{\sin x}{x}\,dx$$

is only a proper integral, we see that the given integral (1) is convergent.

2. *Show that* $\int_0^\infty \sin x^2\, dx$ *is convergent.*

We write

$$\sin x^2 = \frac{1}{2x}\, 2x \sin x^2.$$

and consider the improper integral

$$\int_1^\infty \sin x^2\, dx, \text{ i.e., } \int_1^\infty \frac{1}{2x} \cdot 2x \sin x^2\, dx.$$

Now, $1/2x$ is monotonic and $\to 0$ as $x \to \infty$. Also.

$$\left| \int_1^X 2x \sin x^2\, dx \right| = \left| -\cos X^2 + \cos 1 \right| < 2,$$

so that

$$\int_1^X 2x \sin x^2\, dx$$

is bounded for $X \geqslant 1$.

Hence

$$\int_1^\infty \frac{1}{2x} \cdot 2x \sin x^2\, dx, \quad \text{i.e.,} \int_1^\infty \sin x^2\, dx$$

is convergent.

Since $\int_0^1 \sin x^2\, dx$ is only a proper integral, we see that the given integral is convergent.

3. *Show that*

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, \; a \geqslant 0,$$

is convergent.

Here e^{-ax} is monotonic and bounded and $\int_0^\infty (\sin x/x)\,dx$ is convergent.

Hence we obtain the result by applying Abels' test.

Exercises

1. Test the following for convergence :

(i) $\int_0^\infty \frac{\sin kx}{x}\,dx.$ (ii) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx.$

(iii) $\int_0^\infty e^{-a^2x^2} \sin 2bx \frac{dx}{x}.$ (iv) $\int_0^\infty \frac{\cos ax \cos bx}{x}\,dx.$

(v) $\int_0^\infty \frac{\sin x^m}{x^n}\,dx.$ (vi) $\int_0^\infty \frac{\sin x(1-\cos x)}{x^a}\,dx.$

(vii) $\int_0^\infty \frac{\sin (x+x^2)}{x^n}\,dx.$ (viii) $\int_0^\infty \frac{x^m \cos ax}{1+x^n}\,dx.$

2. Discuss the convergence or divergence of the following :—

(i) $\int_0^\infty \frac{\cos x}{\sqrt{(x^2+x)}}\,dx.$ (ii) $\int_0^\infty e^{-a^2x^2} \cos bx\,dx.$

(iii) $\int_1^\infty \sin x^p\,dx$ (iv) $\int_0^\infty \frac{x^p \sin^2 x}{1+x^2}\,dx.$

3. Show that

$$\int_1^\infty \frac{\sin x}{x^p}\,dx \quad \text{and} \quad \int_1^\infty \cos (x^{1+p})\,dx$$

are convergent if $p > 0$.

Examples

1. *Show that the improper integral*

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

is not absolutely convergent.

We have to show that

$$\int_0^\infty \frac{|\sin x|}{x}\, dx$$

is not convergent.

Consider the proper integral

$$\int_0^{n\pi} \frac{|\sin x|\, dx}{x},$$

where n is a positive integer. We have

$$\int_0^{n\pi} \frac{|\sin x|}{x}\, dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx.$$

We put $x=(r-1)\,\pi+y$ so that y varies in $[0, \pi]$. We have

$$|\sin[(r-1)\pi+y]| = |(-1)^{r-1}\sin y| = \sin y.$$

$$\therefore \quad \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\, dx = \int_0^{\pi} \frac{\sin y}{(r-1)\pi+y}\, dy.$$

Since, $r\pi$ is the max. value of $[(r-1)\pi+y]$ in $[0, \pi]$,

we have

$$\int_0^{\pi} \frac{\sin y}{(r-1)\pi+y}\, dy \geqslant \frac{1}{r\pi}\int_0^{\pi} \sin y\, dy = \frac{2}{r\pi}.$$

$$\therefore \quad \int_0^{n\pi} \frac{|\sin x|}{x}\, dx \geqslant \sum_1^n \frac{2}{r\pi} = \frac{2}{\pi}\sum_1^n \frac{1}{r}.$$

Since $\sum\limits_{1}^{n} \frac{1}{r} \to \infty$ as $n \to \infty$, we see that

$$\int_0^{n\pi} \frac{|\sin x|}{x}\,dx \to \infty,\ \text{as } n \to \infty.$$

Let, now, X be any real number. There exists a positive integer n such that

$$n\pi \leqslant X < (n+1)\pi.$$

We have

$$\int_0^{X} \frac{|\sin x|}{x}\,dx \geqslant \int_0^{n\pi} \frac{|\sin x|}{x}\,dx.$$

Let $X \to \infty$ so that n also $\to \infty$. Thus we see that

$$\int_0^{X} \frac{|\sin x|}{x}\,dx \to \infty,$$

so that $$\int_0^{\infty} \frac{|\sin x|}{x}\,dx \text{ does not converge.}$$

2. *Show that*

$$\int_0^{\infty} \frac{x\,dx}{1+x^6 \sin^2 x},$$

is convergent.

The integrand is positive for positive values of x but the test obtained in § 119, page 241 does not enable us to establish the convergence.

In order to show that the integral converges, we proceed as follows. Consider the proper integral

$$\int_0^{n\pi} \frac{x\,dx}{1+x^6 \sin^2 x},$$

and write

$$\int_0^{n\pi} \frac{x\,dx}{1+x^6 \sin^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^6 \sin^2 x}.$$

Now, if x varies in $[(r-1)\pi, r\pi]$, we have

$$\frac{x}{1+x^6 \sin^2 x} \leqslant \frac{r\pi}{1+(r-1)^6\pi^6 \sin^2 x}.$$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^6 \sin^2 x} \leqslant \int_{(r-1)\pi}^{r\pi} \frac{r\pi\,dx}{1+(r-1)^6\,\pi^6 \sin^2 x} = a_r, \text{ say.}$$

Putting $x=(r-1)\pi+y$, we see that

$$a_r = \int_0^{\pi} \frac{r\pi\,dy}{1+(r-1)^6\,\pi^6 \sin^2 y} = 2\int_0^{\pi/2} \frac{r\pi\,dy}{1+(r-1)^6\,\pi^6 \sin^2 y}.$$

If, now, A is any positive number, we have

$$\int_0^{\pi/2} \frac{dy}{1+A \sin^2 y} = \int_0^{\pi/2} \frac{\operatorname{cosec}^2 y\,dy}{A+1+\cot^2 y} = -\frac{1}{\sqrt{(A+1)}} \left| \tan^{-1} \frac{\cot y}{\sqrt{(A+1)}} \right|_0^{\pi/2}$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sqrt{(A+1)}},$$

$$\therefore \quad a_r = 2.r\pi.\frac{\pi}{2} \cdot \frac{1}{\sqrt{[(r-1)^6\,\pi^6+1]}} = \frac{r\pi^2}{\sqrt{[(r-1)^6\,\pi^6+1]}}.$$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{x}{1+x^6 \sin^2 x}\,dx \leqslant \frac{r\pi^2}{\sqrt{[1+(r-1)^6\pi^6]}} < \frac{r}{(r-1)^3} \cdot \frac{1}{\pi}. (r \neq 1).$$

Now $$\frac{r}{(r-1)^3} = \frac{1}{(r-1)^2} + \frac{1}{(r-1)^3},$$

and, $$\sum_{r=2}^{\infty} \frac{1}{(r-1)^2}, \quad \sum_{r=2}^{\infty} \frac{1}{(r-1)^3}$$

are both convergent.

$$\therefore \quad \sum_{r=2}^{\infty} \frac{r}{(r-1)^3} \cdot \frac{1}{\pi} \text{ is convergent.}$$

$$\therefore \quad \int_0^{n\pi} \frac{x\,dx}{1+x^6 \sin^2 x} \to \text{a finite limit as } n \to \infty.$$

$$\therefore \quad \int_0^{\infty} \frac{x\,dx}{1+x^6 \sin^2 x} \text{ converges.}$$

3. *Show that* $\int_0^\infty \frac{x\,dx}{1+x^4 \sin^2 x}$ *is divergent.*

We write

$$\int_0^{n\pi} \frac{x\,dx}{1+x^4 \sin^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^4 \sin^2 x}.$$

If x varies in $[(r-1)\pi, r\pi]$, we have

$$\frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x} \leqslant \frac{x}{1+x^4 \sin^2 x},$$

or

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x}\,dx \leqslant \int_{(r-1)\pi}^{r\pi} \frac{x}{1+x^4 \sin^2 x}\,dx.$$

Now,

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi}{1+(r\pi)^4 \sin^2 x}\,dx = (r-1)\pi \int_0^{\pi} \frac{dy}{1+(r\pi)^4 \sin^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{dy}{1+(r\pi)^4 \sin^2 y}$$

$$= 2(r-1)\pi \cdot \frac{\pi}{2} \frac{1}{\sqrt{[1+(r\pi)^4]}}$$

$$= \frac{(r-1)\pi^2}{\sqrt{[1+(r\pi)^4]}}.$$

The infinite series $\Sigma \frac{(r-1)\pi^2}{\sqrt{[1+(r\pi)^4]}}$ diverges, as we may see on comparison with the series $\Sigma\,(1/r)$. Hence the given integral does not converge.

4. **Frullani's Integral.** *If* $\varphi(x)$ *is continuous in* $[0, \infty[$ *and*

$$\lim_{x \to 0} \varphi(x) = \varphi_0 \text{ and } \lim_{x \to \infty} \varphi(x) = \varphi_1,$$

then

$$\int_0^\infty \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = (\varphi_0-\varphi_1) \log \frac{b}{a}.$$

We consider the proper integral

$$\int_{\varepsilon} \frac{\varphi(ax)-\varphi(bx)}{x}\,dx,$$

where ultimately we shall make, ε, tend to, 0, and, X, to ∞.

We have

$$\int_{\varepsilon}^{X} \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = \int_{\varepsilon}^{X} \frac{\varphi(ax)}{x}\,dx - \int_{\varepsilon}^{X} \frac{\varphi(bx)}{x}\,dx.$$

Putting $ax=t$ and $bx=t$ separately for the two integrals on the right, we obtain

$$\int_{\varepsilon}^{X} \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = \int_{a\varepsilon}^{aX} \frac{\varphi(t)}{t}\,dt - \int_{b\varepsilon}^{X} \frac{\varphi(t)}{t}\,dt$$

$$= \int_{a\varepsilon}^{b\varepsilon} \frac{\varphi(t)}{t}\,dt - \int_{aX}^{bX} \frac{\varphi(t)}{t}\,dt. \qquad \ldots(1)$$

Applying first mean value theorem to the two integrals on the right, we see that there exist two numbers ξ, η belonging to the intervals $[a\varepsilon, b\varepsilon]$, $[aX, bX]$ respectively such that

$$\int_{a\varepsilon}^{b\varepsilon} \frac{\varphi(t)}{t}\,dt = \varphi(\xi) \int_{a\varepsilon}^{b\varepsilon} \frac{dt}{t} = \varphi(\xi) \log \frac{b}{a},$$

$$\int_{aX}^{bX} \frac{\varphi(t)}{t}\,dt = \varphi(\eta) \int_{aX}^{bX} \frac{dt}{t} = \varphi(\eta) \log \frac{b}{a}.$$

$$\therefore \qquad \int_{\varepsilon}^{X} \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = [\varphi(\xi)-\varphi(\eta)] \log \frac{b}{a}.$$

Let, now, $\varepsilon \to 0$ and $X \to \infty$. Then, by the given conditions $\varphi(\xi) \to \varphi_0$ and $\varphi(\eta) \to \varphi_1$ when $\xi \to 0$ and $\eta \to \infty$. Thus we see that

$$\int^{\infty} \frac{\varphi(ax)-\varphi(bx)}{x}\,dx = (\varphi_0-\varphi_1) \log \frac{b}{a}.$$

Remarks. The following points are worthy of notice :

If $\int_0^1 \frac{(\varphi x)}{x} dx$ is convergent at, 0, then by the general principle of convergence (§ 118, page 239)

$$\lim_{\varepsilon \to 0} \int_{a\varepsilon}^{b\varepsilon} \frac{\varphi(x)}{x} dx = 0.$$

Also if $\int_1^\infty \frac{\varphi(x)}{x} dx$ is convergent at, ∞, then by the general principle of convergence (§ 120, page 248)

$$\lim_{X \to \infty} \int_{aX}^{bX} \frac{\varphi(x)}{x} dx = 0.$$

Then we obtain from (1), page 261 the following results :

(1) If $\int_0 \frac{\varphi(x)}{x} dx$ is convergent at, 0, and $\lim_{x \to \infty} \varphi(x) = \varphi_1$

then

$$\int_0^\infty \frac{\varphi(ax) - \varphi(bx)}{x} dx = \varphi_1 \log \frac{a}{b}.$$

(2) If $\int_1^\infty \frac{\varphi(x)}{x} dx$ is convergent at ∞ and $\lim_{x \to 0} \varphi(x) = \varphi_0$,

then

$$\int_0^\infty \frac{\varphi(ax) - \varphi(bx)}{x} dx = \varphi_0 \log \frac{b}{a}.$$

(3) If $\int_0^1 \frac{\varphi(x)}{x} dx$ and $\int_1^\infty \frac{\varphi(x)}{x} dx$ are both convergent at 0 and

∞ respectively, then

$$\int_0^\infty \frac{\varphi(ax)-\varphi(bx)}{x}\,dx=0.$$

Ex. Show that

(*i*) $\displaystyle\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx=\log\frac{b}{a}.$ $\qquad (a>0,\ b>0)$

(*ii*) $\displaystyle\int_0^\infty \frac{\cos ax-\cos bx}{x}\,dx=\log\frac{b}{a}.$

5. *Prove that*

$$\lim_{k\to\infty}\int_0^\infty \frac{dx}{1+kx^{10}}=0.$$

We write

$$\int_0^\infty \frac{dx}{1+kx^{10}}=\int_0^\varepsilon \frac{dx}{1+kx^{10}}+\int_\varepsilon^\infty \frac{dx}{1+kx^{10}},$$

where ε is any given positive number. Now, we have

$$\int_0^\varepsilon \frac{dx}{1+kx^{10}}<\varepsilon,$$

and
$$\int_\varepsilon^\infty \frac{dx}{1+kx^{10}}<\int_\varepsilon \frac{dx}{kx^{10}}=\frac{1}{9k\varepsilon^9}<\varepsilon,\ \text{if } k>\frac{1}{9\varepsilon^{10}}.$$

$$\therefore \qquad 0<\int_0^\infty \frac{dx}{1+kx^{10}}<2\varepsilon,\ \text{ if } k>\frac{1}{9\varepsilon^{10}}.$$

Hence the result.

6. *Prove that, as $p\to 0$ through positive values, then*

$$p\sum_{r=1}^{\infty}\frac{1}{r^{1+p}}\to 1.$$

Since $(1+p)>1$, therefore the infinite series $\Sigma(1/r^{1+p})$ is convergent.

Let $$S_n = \sum_{r=1}^{n} \frac{1}{r^{1+p}}; \quad S = \sum_{r=1}^{\infty} \frac{1}{r^{1+p}}.$$

We have

$$\frac{1}{2^{1+p}} \leqslant \int_1^2 \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{1^{1+p}}.$$

$$\frac{1}{3^{1+p}} \leqslant \int_2^3 \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{2^{1+p}};$$

....

....

$$\frac{1}{n^{1+p}} \leqslant \int_{n-1}^{n} \frac{1}{x^{1+p}}\, dx \leqslant \frac{1}{(n-1)^{1+p}}.$$

$$\therefore \quad \left(S_n - 1\right) \leqslant \int_1^n \frac{dx}{x^{1+p}} = \frac{1}{p}\left(1 - \frac{1}{n^p}\right) \leqslant \left(S_n - \frac{1}{n^{1+p}}\right).$$

Let $n \to \infty$.

$$\therefore \quad (S-1) \leqslant 1/p \leqslant S \quad \text{or} \quad 1 \leqslant pS \leqslant (p+1).$$

$$\therefore \quad \lim pS = 1 \text{ as } p \to (0+0).$$

Exercises

1. Show that the following improper integrals converge :—

(*i*) $\int_0 \log x \log (1+x)\, dx.$ (*ii*) $\int_0^1 \frac{\log x}{\sqrt[3]{(1-x)}}\, dx.$

(*iii*) $\int_0^\infty \frac{x\, dx}{(1+x^2)\sqrt[5]{\sin x}}.$ (*iv*) $\int_0^\infty e^{-a^2 x} \log \cos^2 x\, dx. \quad (a \neq 0)$

(*v*) $\int_0^\infty \frac{e^{-ax}}{\sqrt[3]{\sin x}}\, dx. \quad (a > 0).$ (*vi*) $\int_0 \frac{x^m + x^{-m}}{x} \log(1+x)\, dx.\ |m| < 1.$

(vii) $\displaystyle\int_0^{\infty} x^n\, e^{-a^2x^2-b^2/x^2}\,dx.$ $\quad (n>-1)$

(viii) $\displaystyle\int_0^{\infty} e^{-x}\, x^{n-}\,(\log x)^m\,dx.$ ($n>0$ and m is a positive integer).

(ix) $\displaystyle\int_0^{\infty} \frac{x\cosh ax}{\sinh x}\,dx.$ $|a|<1.$

(x) $\displaystyle\int_0^{\infty} \frac{e^x}{e^{4x}\sin^2 x+\cos^2 x}\,x.$

(xi) $\displaystyle\int_0^{\infty} \frac{\cosh ax\cosh bx}{\cosh x}\,dx.$ $\quad |a|+|b|<1.$

2. Show that

$$\int_0^{\infty} \frac{dx}{1+x^4\sin^2 x} \text{ converges but } \int_0^{\infty} \frac{dx}{1+x^2\sin^2 x} \text{ does not.}$$

3. Evaluate

$$\int_0^{\infty} \frac{\tan^{-1} ax-\tan^{-1} bx}{x}\,dx.$$

4. Show that a, b being positive,

$$\int_0^{\infty} \frac{\cos ax-\cos bx}{x}\,dx=\log\frac{b}{a},$$

and deduce that

$$\int_0^{\infty} \frac{\sin ax\sin bx}{x}\,dx=\tfrac{1}{2}\log\frac{a+b}{a-b}; \qquad (a>b>0).$$

5. Show that

$$\int_0^{\infty} \frac{\sin x}{x}\,dx=\frac{\pi}{}.$$

[(i) $\int_0^{\infty} \frac{\sin x}{x}\, dx = \lim_{n\to\infty} \int_0^{\frac{1}{2}(2n+1)\pi} \frac{\sin x}{x}\, dx = \lim_{n\to\infty} \int_0^{\frac{1}{2}} \frac{\sin(2n+1)x}{x}\, dx.$

(ii) Integrating by parts or otherwise. (See Ex. 4, page 213 and Ex. 5, page 215) show that

$$\lim_{n\to\infty} \int_0^{\frac{1}{2}} \sin(2n+1)x \left(\frac{1}{\sin x} - \frac{1}{x}\right) dx = 0.\ ]$$

6. Show that

$$\int_0^{\frac{1}{2}\pi} \frac{\sin(2n+1)x}{\sin x}\, dx = \frac{\pi}{2}.$$

7. Show that

$$\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2},\ 0 \text{ or } -\frac{\pi}{2}$$

according as, a, is positive, zero or negative.

8. Show that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

(Integrate by parts).

9. Show that

$$\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a). \qquad (a>0, b>0)$$

(Integrate by parts).

10. Show that

(i) $\int_0^{\infty} \frac{\sin ax \sin x}{x^2}\, dx = \frac{1}{2}\pi a$ when $0 \leqslant a < 1$ and $\frac{1}{2}\pi$ when $a \geqslant 1$.

(ii) $\int_0^{\infty} \frac{\sin ax \sin^2 x}{x^3}\, dx = \frac{1}{8}\pi a\,(4-a)$ when $0 \leqslant a \leqslant 2$ and $\frac{1}{2}\pi$ when $a \geqslant 2$.

11. Find the value of the definite integral

$$\int_0^{\pi} \frac{\sin x \, dx}{\sqrt{(1-2a\cos x+a^2)}},$$

where, a, is positive.

12. Evaluate the integral

$$\int_{-1}^{+1} \frac{\sin a \, dx}{1-2x\cos a+x^2}.$$

For what value of, a, is the integral a discontinuous function of a ?

13. Show that

$$\int_0^1 \frac{dx}{x^2+2x\cos a+1} = \frac{}{2\sin a},$$

if $-\pi < a < \pi$, except when $a=0$, when the value of the integral is $\frac{1}{2}$.

14. Prove that

$$\int_0^{\infty} \log(1+a^2x^{-2})\, dx = \pi a, \text{ if } a > 0,$$

15. Discuss the convergence of

$$\int_0^{\frac{1}{2}\pi} \cos 2nx \log \sin x \, dx,$$

and evaluate it when n is a positive integer.

[0 is the only point of infinite discontinuity. Now, since, when $x \to 0$, $\lim (\sqrt{x} \log \sin x \cos 2nx)=0$, therefore, for values of x sufficiently near 0,

$$|\cos 2nx \log \sin x| < \varepsilon/\sqrt{x}.$$

For evaluation, proceed integrating by parts.]

16. From the preceding example, deduce that

(*i*) $$\int_0^{\frac{1}{2}\pi} \cos 2nx \log \cos x \, dx = (-1)^{n+1} \frac{\pi}{4n}.$$

(*ii*) $$\int_0^{\pi} \cos nx \log 2(1-\cos x)\, dx = -\frac{\pi}{n}.$$

(*iii*) $$\int_0^{\pi} \cos nx \log 2(1+\cos x)\, dx = (-1)^{n+1} \frac{\pi}{n}.$$

17. Prove that if $g(x)$ is bounded and integrable, then

$$\int_a^b g(x) \sin nx \, dx \to 0 \text{ as } n \to \infty,$$

where $[a, b]$ is any finite interval.

If further

$$\int_{-\infty}^{+\infty} | g(x) | \, dx$$

is convergent, prove that

$$\int_{-\infty}^{+\infty} g(x) \sin nx \, dx \to 0 \text{ as } n \to \infty.$$

18. Prove that

$$\lim_{p \to (0+0)} \left[\lim_{n \to \infty} \left(p \sum_1^n \frac{1}{r^{1+p}} \right) \right] = 1.$$

19. If $f(x)$ is positive and decreases for $x \leqslant 1$, prove that

$$u_n = \sum_{r=1}^{n} f(r) - \int_1^n f(x) \, dx$$

tends to a limit, l, as $n \to \infty$ and that $0 \leqslant l \leqslant f(1)$.

[Show that (u_n) is a monotonic bounded sequence.]

20. Prove that if $s > 1$, then

$$\sum_{r=1}^{n} \left(\frac{s}{r} - \frac{1}{r^s} \right) - s \log n,$$

tends to a limit as $n \to \infty$, (s being fixed), and that if this limit is $\varphi(s)$, then

$$0 \leqslant [\varphi(s) + (s-1)^{-1}] \leqslant (s-1).$$

CHAPTER IX

FOURIER SERIES

123. Fourier Series. A trigonometric series of the form

$$\tfrac{1}{2}a_0+(a_1 \cos x+b_1 \sin x)+\ldots+(a_n \cos nx+b_n \sin nx)+\ldots$$

is said to be a Fourier series. Here a_n, b_n are constants independent of x and are known as Fourier constants. In this chapter, we propose to find a set of sufficient conditions for a function $f(x)$ to be represented as a Fourier Series and to find the constants a_n and b_n, if $f(x)$ can be so represented.

Since every term of a Fourier series is periodic with the period 2π, it is obvious that the sum function $f(x)$ of a convergent Fourier series of the above form must also be necessarily periodic with period 2π. It is not necessary, however, that $f(x)$ should be a trigonometric function. A function, with 2π as its period, will arise, if it is *arbitrarily* defined in an interval of length 2π and then periodically extended beyond this interval to the left and to the right so as to satisfy the functional equation $f(x\pm 2\pi)=f(x)$.

Firstly, we proceed to show that the constants a_n, b_n can be easily determined, if we *assume* that

(*i*) a given function $f(x)$ can actually be represented as a Fourier series and that

(*ii*) the series is uniformly convergent.

It should, however, be clearly understood that since there is nothing to prove *a priori*, that these two assumptions are justifiable, this determination is *purely formal.*

This determination depends upon the following simple integrals,

$$\int_{-\pi}^{+\pi} \sin nx\, dx=0=\int_{-\pi}^{+\pi} \cos nx\, dx\,;\ \int_{-\pi}^{+\pi} \sin mx \cos nx\, dx=0.$$

$$\int_{-\pi}^{+\pi} \cos mx \cos nx\, dx=\begin{cases}0, \text{ if } m\neq n\\ \pi, \text{ if } m=n\end{cases};\ \int_{-\pi}^{+\pi} \sin mx \sin nx\, dx=\begin{cases}0, \text{ if } m\neq n,\\ \pi, \text{ if } m=n.\end{cases}$$

Integrating term by term from, $-\pi$ to $+\pi$, the hypothetical equality,

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx) \qquad \ldots(1)$$

we obtain
$$a_0=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx.$$

Multiplying the equality (1) by cos nx, and integrating term by term from, $-\pi$ to $+\pi$, we obtain
$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx, \qquad \ldots(2)$$
which is seen to be true for $n=0$ also.

Finally, on multiplying with sin nx and integrating term by term from, $-\pi$ to $+\pi$, we obtain
$$b_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx. \qquad \ldots(3)$$

Note. The series (1), with the co-efficients (2) and (3), will be called the *Fourier series* **corresponding** *to $f(x)$ in the interval* $[-\pi, \pi]$.

Of course, the mere fact that a series *corresponding* to a function can be written, does not ensure its convergence, or if convergent, that its sum will be $f(x)$. Complete result in regard to this is given in § 126, page 274.

124. Theorem. *Let $f(x)$ be bounded and integrable in* $[0, a]$, $a > 0$, *and let it be monotonic in some interval* $[0, \alpha]$ *where*, α, *is any positive number less than* a. *Then, as* $n \to \infty$,
$$\int_0^a f(x)\frac{\sin nx}{x}dx \to f(+0)\int_0^\infty \frac{\sin x}{x}dx,$$
where $f(+0)$ denotes the limit of $f(x)$ as x tends to zero through positive values.

[Since $f(x)$ is monotonic and bounded in $[0, \alpha]$, therefore $f(+0)$ must exist].

Firstly, we suppose that $f(+0)=0$. Also, without affecting the result in any way whatsoever, we can suppose that $f(0)=0$.

Let h be any positive number less than α.

By the second mean value theorem, we have
$$\int_0^h f(x)\frac{\sin nx}{x}\,dx=f(0)\int_0^{h_1}\frac{\sin nx}{x}dx+f(h)\int_{h_1}^{h}\frac{\sin nx}{x}\,dx,$$
$$(0 \leqslant h_1 \leqslant h).$$

$$=f(h)\int_{h_1}^{h}\frac{\sin nx}{x}dx$$

$$=f(h)\int_{nh_1}^{nh}\frac{\sin y}{y}dy=f(h)\int_{nh_1}^{nh}\frac{\sin x}{x}\,dx. \qquad \ldots(1)$$

Since

$$\int_0^{\infty}\frac{\sin x}{x}dx$$

is convergent, there exists a positive constant, k, such that

$$\left|\int_0^X\frac{\sin x}{x}\,dx\right| \leqslant k, \text{ for all } X \geqslant 0.$$

$$\therefore \qquad \left|\int_{nh_1}^{nh}\frac{\sin x}{x}\,dx\right|=\left|\int_0^{nh}\frac{\sin x}{x}\,dx-\int_0^{nh_1}\frac{\sin x}{x}dx\right| \leqslant 2k. \qquad \ldots(2)$$

Also

$$f(h) \to 0 \text{ as } h \to (0+0). \qquad \ldots(3)$$

From (1), (2) and (3), we deduce that there exists a positive number δ such that

$$\left|\int_0^h f(x)\frac{\sin nx}{x}dx\right| \leqslant 2k\mid f(h)\mid < \tfrac{1}{2}\varepsilon, \text{ when } 0 < h \leqslant \delta.$$

Now, we have

$$\int_0^a f(x)\,\frac{\sin nx}{x}\,dx=\int_0^{\delta} f(x)\,\frac{\sin nx}{x}\,dx+\int_{\delta}^{a} f(x)\frac{\sin nx}{x}\,dx.$$

*Since, as proved in Ex. 4, page 213, the second integral on the right $\to 0$, as $n \to \infty$, we see that there exists a positive integer m such that for $n \geqslant m$,

$$\left|\int_0^a f(x)\,\frac{\sin nx}{x}\,dx\right| < \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.$$

* $f(x)/x$ being bounded and integrable in $[\delta, a]$.

Thus the theorem is proved in this case.

In the general case, changing $f(x)$ to $[f(x)-f(+0)]$, we see that as $n \to \infty$,

$$\int_0^a [f(x)-f(+0)]\frac{\sin nx}{x}\,dx \to 0.$$

Now

$$\int_0^a \frac{\sin nx}{x}\,dx = \int_0^{na} \frac{\sin y}{y}\,dy \to \int_0^\infty \frac{\sin y}{y}\,dy, \text{ as } n \to \infty.$$

$$\therefore \quad \int_0^a f(x)\frac{\sin nx}{x}\,dx \to f(+0)\int_0^\infty \frac{\sin x}{x}\,dx.$$

Cor. *If $f(x)$ satisfies the conditions of the theorem above, and $0 < a < \pi$, then as $n \to \infty$,*

$$\int_0^a f(x)\frac{\sin nx}{\sin x}\,dx \to f(+0)\int_0^\infty \frac{\sin x}{x}\,dx.$$

We write

$$f(x)\frac{\sin nx}{\sin x} = f(x)\frac{x}{\sin x}\cdot\frac{\sin nx}{x} = G(x)\frac{\sin nx}{x},$$

where we assign the values 1 and n to $(x/\sin x)$ and $(\sin nx/x)$ respectively for $x=0$. Now we know that $(x/\sin x)$ is positive and monotonically increasing in $[0, \frac{1}{2}\pi]$. If $f(x)$ be monotonically *increasing* in $[0, \alpha]$, $(\alpha < \frac{1}{2}\pi)$, then $G(x)$ is also monotonically increasing in $[0, \alpha]$.

Therefore, as $n \to \infty$,

$$\int_0^a f(x)\frac{\sin nx}{\sin x}\,dx = \int_0^a G(x)\frac{\sin nx}{x}\,dx \to G(+0)\int_0^\infty \frac{\sin x}{x}\,dx$$

$$= f(+0)\int_0^\infty \frac{\sin x}{x}\,dx.$$

If $f(x)$ be decreasing, so that $-f(x)$ is increasing, we see that, as $n \to \infty$

$$\int_0^a [-f(x)] \frac{\sin nx}{\sin x}\, dx \to -f(+0) \int_0^\infty \frac{\sin x}{x}\, dr,$$

i.e ,
$$\int_0^a f(x) \frac{\sin nx}{\sin x}\, dx \to f(+0) \int_0^\infty \frac{\sin x}{x}\, dx.$$

125. Theorem. *Let $f(x)$ be bounded and integrable in $[-\pi, \pi]$ and let it be monotonic in $[-\alpha, 0]$ and $[0, \alpha]$ (not necessarily in the same sense), where α is some positive number less than π. Then*

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n = \frac{f(+0)+f(-0)}{2},$$

where $f(+0)$, $f(-0)$ denote the limits of $f(x)$ as x tends to 0 through positive and negative values respectively.

We have

$$\frac{1}{2} a_0 + \sum_{n=1}^{m} a_n$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)dx + \frac{1}{\pi} \int_{-\pi}^{-\pi} f(x) \sum_{n=1}^{m} \cos nx\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)[1 + 2 \sum_{n=1}^{m} \cos nx]\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) \frac{\sin (m+\frac{1}{2})r}{\sin \frac{1}{2}x}\, dx$$

$$= \frac{1}{2\pi} \int_0^\pi f(-x) \frac{\sin (m+\frac{1}{2})x}{\sin \frac{1}{2}x}\, dx + \frac{1}{2\pi} \int_0^\pi f(x) \frac{\sin (m+\frac{1}{2})x}{\sin \frac{1}{2} x}\, dx$$

$$= \frac{1}{2\pi} \cdot 2 \int_0^{\frac{1}{2}\pi} f(-2x) \frac{\sin (2m+1)x}{\sin x}\, dx + \frac{1}{2\pi} \cdot 2 \int_0^{\frac{1}{2}\pi} f(2x) \frac{\sin (2m+1)x}{\sin x}\, dx,$$

$$= \frac{1}{\pi} [f(-0) + f(+0)] \int_0^\infty \frac{\sin x}{x}\, dx \text{ as } m \to \infty.$$

Taking $f(x)=1$, we see that

$$\frac{1}{2}a_0=\frac{1}{2\pi}\int_{-\pi}^{+\pi}1.\,dx=1 \text{ and } a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi}\cos nx\,dx=0.$$

$$\therefore \qquad 1=\frac{1+1}{\pi}\int_0^\infty \frac{\sin x}{x}\,dx, \text{ i.e., } \int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

Hence the theorem.

Note. The value of the integral of $(\sin x/x)$ over $[0, \infty[$ has also been already obtained in another way in Ex. 5, page 265. Still another method will be given later on in Ch. XII.

126. Main Theorem. Let f(x) be bounded and integrable in $[-\pi, \pi]$, and let it be possible to divide $[-\pi, \pi]$ into a finite number of open sub-intervals, in each of which f(x) is monotonic. Then

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\xi+b_n\sin n\xi)=\begin{cases}\tfrac{1}{2}[f(\xi-0)+f(\xi+0)], & \textit{for } -\pi<\xi<\pi\\ \tfrac{1}{2}[f(\pi-0)+f(-\pi+0)], & \textit{for } \xi=-\pi \text{ or } \xi=\pi.\end{cases}$$

Here $f(\xi-0)$ and $f(\xi+0)$ stand for the limits of $f(x)$ as x tends to ξ from values smaller than and greater than ξ respectively. Under the given conditions $f(\xi-0)$ and $f(\xi+0)$ necessarily exist.

Lemma. If $f(x)$ is bounded and integrable in every interval and is periodic with 2π as its period, then

$$\int_{-\pi}^{+\pi}f(x)\,dx=\int_{-\pi}^{+\pi}f(a+x)\,dx,$$

a, being any number whatsoever.

Putting $a+x=y$, we have

$$\int_{-\pi}^{+\pi}f(a+x)\,dx=\int_{a-\pi}^{a+\pi}f(y)\,dy$$

$$=\int_{a-\pi}^{-\pi}f(y)\,dy+\int_{-\pi}^{+\pi}f(y)\,dy+\int_{+\pi}^{a+\pi}f(y)\,dy.$$

Putting $y=z-2\pi$, we see that

$$\int_{a-\pi}^{-\pi}f(y)\,dy=\int_{a+\pi}^{\pi}f(z-2\pi)\,dz=-\int_{\pi}^{a+\pi}f(z)\,dz=-\int_{\pi}^{a+\pi}f(y)\,dy.$$

Hence the result.

Proof of the Main Theorem. We have

$$\tfrac{1}{2}a_0+\sum_{n=1}^{m}(a_n\cos n\xi+b_n\sin n\xi)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x)dx+\sum_{n=1}^{m}\frac{1}{\pi}\int_{-\pi}^{+\pi}f(x)(\cos nx\cos n\xi+\sin nx\sin n\xi)\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x)[1+2\sum_{n=1}^{m}\cos n(x-\xi)]\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x+\xi)[1+2\sum_{n=1}^{m}\cos nx]\,dx \qquad \text{(lemma)}$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi}f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0}f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx+\frac{1}{2\pi}\int_{0}^{\pi}f(x+\xi)\,\frac{\sin(m+\frac{1}{2})x}{\sin\frac{1}{2}x}\,dx$$

$$=\frac{1}{\pi}\int_{0}^{\frac{1}{2}\pi}f(-2y+\xi)\frac{\sin(2m+1)y}{\sin y}\,dy+\frac{1}{\pi}\int_{0}^{\frac{1}{2}\pi}f(2y+\xi)\frac{\sin(2m+1)y}{\sin y}\,dy$$

$$\to\frac{1}{\pi}\left\{\frac{\pi}{2}\cdot f(\xi-0)+\frac{\pi}{2}\cdot f(\xi+0)\right\}=\frac{f(\xi+0)+f(\xi-0)}{2},\ \text{as } m\to\infty.$$

Note. The result arrived at above may be re-stated as follows:—*If $f(x)$ be bounded and integrable in $[-\pi, \pi]$ and if it be possible to divide $[-\pi, \pi]$ into a finite number of open sub-intervals in each of which $f(x)$ is monotonic, then the Fourier series, corresponding to $f(x)$, converges for every value of x, and if $S(x)$ denotes the sum function of series, then*

$$S(x)=\tfrac{1}{2}[f(x+0)+f(x-0)] \text{ when } -\pi<x<\pi$$

$$S(x)=\tfrac{1}{2}[f(\pi-0)+f(-\pi+0)] \text{ when } x=\pm\pi,$$

and $$S(x+2\pi)=S(x).$$

The relation $S(x+2\pi)=S(x)$ enables us to determine the value of the sum function at a point which does not belong to $[-\pi, \pi]$.

Cor. At a point of continuity ξ of $f(x)$,

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\xi+b_n\sin n\xi)=\frac{f(\xi+0)+f(\xi-0)}{2}$$

$$=\frac{f(\xi)+f(\xi)}{2}=f(\xi).$$

Thus we see that if $f(x)$ satisfies the conditions of the theorem of § 126, in $[-\pi, \pi]$, then the sum of the Fourier series *corresponding* to $f(x)$ is actually $f(x)$ at all such points x of $[-\pi, \pi]$, where $f(x)$ is continuous ; at points of discontinuity the sum of the series is

$$\tfrac{1}{2}[f(x+0)+f(x-0)].$$

Example

Expand in a series of sines and cosines of multiples of x, the function

$$f(x)=x-\pi, \text{ when } -\pi<x<0;\ f(x)=\pi-x, \text{ when } 0<x<\pi.$$

What is the sum of the series for $x=\pm\pi$ and $x=0$?

The given function $f(x)$ satisfies the conditions of the theorem of § 126. We have

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx.$$

Integrating by parts, we obtain

$$a_n=\frac{2[1-(-1)^n]}{n^2\pi},\ n\neq 0.$$

Also $a_0=-\pi$.

Similarly

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx=\frac{2[1-(-1)^n]}{n}.$$

The co-efficients a_n and b_n are zero when n is even.

In $[-\pi, \pi]$ the points $x=0$ and $x=\pm\pi$ are the only points of discontinuity of $f(x)$. Therefore when x is different from 0 and $\pm\pi$, we have

$$f(x)=-\frac{1}{2}\pi+\frac{4}{\pi}\cdot\frac{1}{1^2}\cos x+\frac{4}{1}\sin x+\frac{4}{\pi}\cdot\frac{1}{3^2}\cos 3x+\frac{4}{3}\sin 3x+\ldots\ldots$$

$$=-\frac{1}{2}\pi+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\ldots\right]+4\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots\ldots\right].$$

Since $x=0$ is a point of discontinuity of $f(x)$, therefore, the sum of the series for $x=0$

$$=\tfrac{1}{2}[f(+0)+f(-0)]=\tfrac{1}{2}[(\pi)+(-\pi)]=0.$$

Thus we obtain a well-known result, *viz.*,

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots \qquad \ldots(1)$$

For $x=\pm\pi$, the sum of the series

$$=\tfrac{1}{2}[f(\pi-0)+f(-\pi+0)]=\tfrac{1}{2}[0+(-2\pi)]=-\pi.$$

Putting $x=\pm\pi$, we obtain the same result, *viz.*, (1)

127. Fourier Series for Odd and Even Functions.

The Fourier series with sines only. If $f(x)$ be an odd function, *i.e.*, if $f(-x)=-f(x)$, then $f(x)\cos nx$ is an odd function and $f(x)\sin nx$ is even and hence

$$a_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_{0}^{+\pi} f(x)\sin nx\,dx, \qquad \ldots(2)$$

so that we see that Fourier series corresponding to an odd function, consists of terms with sines only and the co-efficients b_n may be computed by (2).

The Fourier series with cosines only. If $f(x)$ is an even function, *i.e.*, if $f(-x)=f(x)$, then $f(x)\cos nx$ is even and $f(x)\sin nx$ is odd and hence

$$a_n=\frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx,$$

$$b_n=0,$$

so that we see that the Fourier series corresponding to an even function consists of terms with cosines only.

128. Half range series. From the preceding, we will now deduce the following two results :—

128·1. *If $f(x)$ satisfies the conditions of the theorem of § 126 in $[0, \pi]$, then the sum af the sine series*

$$\Sigma b_n \sin nx, \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx,$$

is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and π and is equal to 0 when $x=0$ and $x=\pi$.

To see the truth of this result, we define an odd function $F(x)$ in $[-\pi, \pi]$ which is identical with $f(x)$ in $[0, \pi]$. Thus

$$F(x)=f(x) \text{ in } [0, \pi] \text{ and } F(x)=-F(-x)=-f(-x) \text{ in } [-\pi, 0].$$

Clearly $F(x)$ will satisfy the conditions of the theorem of § 126 in $[-\pi, \pi]$ if $f(x)$ does so in $[0, \pi]$. Thus we see that the sum of the series

$$\Sigma b_n \sin nx, \textit{ where } b_n = \frac{2}{\pi}\int_0^{\pi} F(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx,$$

is equal to

$$\tfrac{1}{2}[F(x+0)+F(x-0)]=\tfrac{1}{2}[f(x+0)+f(x-0)],$$

at every point x between 0 and π.

At $x=0$, the sum of the series$=\frac{1}{2}[F(+0)+F(-0)]=0$, for $F(x)$ is odd. Similarly we see that the sum of the series is 0 for $x=\pi$.

128·2. *If $f(x)$ satisfies the conditions of the theorem of § 126, in $[0, \pi]$, then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma a_n \cos nx, \textit{ where } a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx,$$

is equal to $\frac{1}{2}[f(x+0)+f(x-0)]$ at every point x between 0 and π and is equal to $f(+0)$ for $x=0$ and $f(\pi-0)$ for $x=\pi$.

To prove this result, we have to consider an even function $F(x)$ defined in $[-\pi, \pi]$ which is identical with $f(x)$ in $[0, \pi]$.

Note. The sum functions of the Half-range sine and cosine series are periodic with period 2π.

Example

Find (a) the Fourier sine series, and (b) the Fourier cosine series which represents $f(x)=(\pi-x)$ in $0 < x < \pi$.

To find sine series. We have

$$b_n=\frac{2}{\pi}\int_0^\pi (\pi-x)\sin nx\,dx=\frac{2}{n}.$$

Since $f(x)$ is continuous in $0 < x < \pi$, we have

$$\pi-x=2\Sigma\frac{1}{n}\sin nx=2\left(\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\frac{\sin 4x}{4}+\dots\dots\right).$$

According to § 128·1, the sum of the series must be 0 for $x=0$ and $x=\pi$ and this fact can also be directly verified. The representation holds for $x=\pi$ but not for $x=0$.

To find cosine series. We have

$$a_n=\frac{2}{\pi}\int_0^\pi (\pi-x)\cos nx\,dx=\frac{2[1-(-1)^n]}{\pi n^2},\quad n\neq 0.$$

Also, $a_0=\pi$.

Since $f(x)$ is continuous in $0 < x < \pi$, we have

$$\pi-x=\frac{1}{2}\pi+\Sigma\frac{2[1-(+1)^n]}{\pi n^2}\cos nx$$

$$=\frac{1}{2}\pi+\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\dots\dots\right).$$

According to the theorem of § 128·2, the sum of the series must be $f(+0)=\pi$ for $x=0$ and $f(\pi-0)=0$ for $x=\pi$. This gives the result

$$\frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots\dots,$$

which was obtained on page 277 also.

Observation. The series in the Ex. on page 276 and the two series obtained above have identical sums for values of x which belong to the open interval $]0, \pi[$ but for values of x belonging to the interval $]-\pi, 0[$; their sums are different. Thus for $-\pi < x < 0$, the sum of the series in the example on page 276, is $(x-\pi)$, whereas the sums of series above are

$$-[\pi-(-x)]=-(\pi+x) \text{ and } [\pi-(-x)]=(\pi+x)$$

respectively.

Similar differences in the sums exist for values of x outside the interval $[-\pi, \pi]$.

The sum functions of all the three series are periodic with period 2π.

129. Other forms of Fourier series. The particular interval $[-\pi, \pi]$ which we have so far considered had been introduced only as a matter of convenience. We shall now see that it is easy to change to any other finite interval.

129·1. The Interval $[0, 2\pi]$. We write $x=y+\pi$ so that y varies in $[-\pi, \pi]$ as x varies in $[0, 2\pi]$. Let

$$f(x)=f(y+\pi)=F(y).$$

Let $f(x)$ satisfy the conditions of the theorem of § 126 in $[0, 2\pi]$ so that $F(y)$ satisfies the same conditions in $[-\pi, \pi]$. Thus we see that the sum of the series

$$\tfrac{1}{2}a'_0+\Sigma(a'_n \cos ny+b'_n \sin ny),$$

where $$a'_n=\frac{1}{\pi}\int_{-\pi}^{+\pi} F(y) \cos ny\, dy,\ b'_n=\int_{-\pi}^{+\pi} F(y) \sin ny\, dy,$$

is $\frac{1}{2}[F(y+0)+F(y-0)]$ at any point y between $-\pi$ and π and is $\frac{1}{2}[F(\pi-0)+F(-\pi+0)]$ at $y=\pm\pi$ and is periodic with period 2π.

Changing the variables, we see that

$$a'_n=\frac{1}{\pi}\int_0^{2\pi} F(x-\pi) \cos n(x-\pi)dx =\frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx,$$

$$b'_n=\frac{1}{\pi}\int_0^{2\pi} F(x-\pi) \sin n(x-\pi)\, dx =\frac{(-1)^n}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx.$$

Also $\cos ny=(-1)^n \cos nx$, $\sin ny=(-1)^n \sin nx$.

Finally, we have

$$\tfrac{1}{2}F[(y+0)+F(y-0)]=\tfrac{1}{2}[f(y+\pi+0)+f(y+\pi-0)]$$
$$=\tfrac{1}{2}[f(x+0)+f(x-0)]$$

and $$\tfrac{1}{2}[F(\pi-0)+F(-\pi+0)]=\tfrac{1}{2}[f(2\pi-0)+f(+0)].$$

Thus we see that *if $f(x)$ satisfies the conditions of the theorem of § 126, in $[0, 2\pi]$, then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx),$$

where $$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx,\ b_n=\frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point* x *between* 0 *and* 2π *and is* $\frac{1}{2}[f(2\pi-0)+f(+0)]$ *at* $x=0$ *and* $x=2\pi$ *and is periodic with period* 2π.

129·2. The Interval $[-l, l]$, *where* l *is any real number.* By means of the substitution $y=\pi x/l$ and considering a function $F(y)$ such that

$$f(x)=f(ly/\pi)=F(y),$$

so that y varies in $[-\pi, \pi]$ as x varies in $[-l, l]$, we can show as before that *if* $f(x)$ *satisfies the conditions of the theorem of* § 126 *in* $[-l, l]$, *then the sum of the sreies*

$$\tfrac{1}{2}a_0+\Sigma[a_n \cos(n\pi x/l)+b_n \sin(n\pi x/l)],$$

where

$$a_n=\frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx,\quad b_n=\frac{1}{l}\int_{-l}^{+l} f(x)\frac{\sin n\pi x}{l}\,dx$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point* x *between* $-l$ *and* l *and is*

$$\tfrac{1}{2}[f(l-0)+f(-l+0)],$$

for $x=-l$ *and* $x=l$ *and is periodic with period* $2l$.

Note. As in § 128, we can have half range sine series at will for a function $f(x)$ given in $[0, l]$.

129·3. Any Interval $[a, b]$. *If* $f(x)$ *satisfies the conditions of the theorem of* § *126 in* $[a, b]$, *then the sum of the series*

$$\tfrac{1}{2}a_0+\Sigma\{a_n \cos[2n\pi x/(b-a)]+b_n \sin[2n\pi x/(b-a)]\}$$

$$\textit{where } a_n=\frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx,\ b_n=\frac{2}{b-a}\int_a^b f(x)\sin\left(\frac{2n\pi x}{b-a}\right)dx,$$

is $\frac{1}{2}[f(x+0)+f(x-0)]$ *at every point* x *between* a *and* b *and is* $\frac{1}{2}[f(a+0)+f(b-0)]$ *at* $x=a$ *and at* $x=b$ *and is periodic with period* $(b-a)$.

This result follows on writing

$$y^*=\frac{2\pi}{b-a}x-\frac{b+}{b-a}\pi,$$

so that y varies in $[-\pi, \pi]$ as x varies in $[a, b]$.

Example

Show that when $0 < x < \pi$,

$$\pi-x=\frac{1}{2}\pi+\frac{\sin 2x}{1}+\frac{\sin 4x}{2}+\frac{\sin 6x}{3}+\ldots\ldots$$

Each term of the series is periodic with period π. Taking $a=0$ and $b=\pi$ in § 129·3, we see that $a_0=\pi$,

*This transformation is obtained, if we determine the two constants l and m such that the relation $y=lx+m$ gives $y=-\pi$ when $x=a$ and $y=\pi$ when $x=b$.

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\cos 2nx\,dx = 0,\ (n \neq 1),$$

$$b_n = \frac{2}{\pi}\int_0^\pi (\pi - x)\sin 2nx\,dx = \frac{1}{n}.$$

Hence the result.

Note. It will now be seen that we have obtained *four* different series which represent $(\pi - x)$ in $0 < x < \pi$ and it will naturally prove of great interest to examine the differences in the sums of these series for values of x other than those which belong to $]0, \pi[$. To do so, the reader would do well to draw the graphs of the four sum functions.

One important difference which must be emphasised is that the sum function of the series obtained above in § 129·3, is periodic with period π, whereas the former three sum functions were periodic with period 2π.

Exercises

1. If $f(x) = -\frac{1}{4}\pi$ when $-\pi < x < 0$, $f(x) = \frac{1}{4}\pi$ when $0 < x < \pi$, $f(-\pi) = f(0) = f(\pi) = 0$ and $f(x + 2\pi) = f(x)$ for all x, show that

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\dots\dots$$

for all values of x. Deduce that

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\dots$$

2. If $f(x) = cx$ when $-\pi < x < \pi$, $f(-\pi) = f(\pi) = 0$, and $f(x + 2\pi) = f(x)$ for all x, show that

$$f(x) = 2c[\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \dots\dots]$$

for all values of x. Draw the graph of $f(x)$.

3. Show that the Fourier series which converges to $f(x)$ in $-\pi \leqslant x \leqslant \pi$, where

$$f(x) = x + x^2 \text{ when } -\pi < x < \pi \text{ and } f(x) = \pi^2 \text{ when } x = \pm\pi, \text{ is}$$

$$\frac{\pi^2}{3} + 4\Sigma\,(-1)^n\left(\frac{\cos nx}{n^2} - \frac{\sin nx}{2n}\right).$$

Deduce that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots$$

4. Obtain Fourier series which will be equal to $-\pi - x$ when $-\pi \leqslant x < -\frac{1}{2}\pi$, equal to x when $-\frac{1}{2}\pi \leqslant x < \frac{1}{2}\pi$ and equal to $\pi - x$ when $\frac{1}{2}\pi \leqslant x \leqslant \pi$. Explain graphically how to obtain the sum of the series for any value of x.

5. Obtain Fourier series whose sum is equal to $f(x)$, where
$f(x) = 0$ when $-\pi \leqslant x < -\frac{1}{2}\pi$, $f(-\frac{1}{2}\pi) = -\frac{1}{4}\pi$,
$f(x) = x$ when $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, $f(\frac{1}{2}\pi) = \frac{1}{4}\pi$,
$f(x) = 0$ when $\frac{1}{2}\pi < x \leqslant \pi$.

6. $f(x)=\cos x$ for $0<x<\pi$ and $f(x)=-\cos x$ for $-\pi<x<0$; show that the Fourier series which converges to $f(x)$ is

$$\frac{4}{\pi}\left(\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x+\ldots\ldots\right).$$

Draw the graph of the sum function of the series for $-2\pi\leqslant x\leqslant 2\pi$.

7. Find the Fourier series which represents $|\sin x|$ in $-\pi\leqslant x\leqslant\pi$.

8. Find a series of sines of multiples of x, which will represent $f(x)$ in the interval $0<x<\pi$, where

$$f(x)=\tfrac{1}{3}\pi, \text{ when } 0<x<\tfrac{1}{3}\pi;$$
$$f(x)=0, \text{ when } \tfrac{1}{3}\pi<x<\tfrac{2}{3}\pi;$$
$$f(x)=-\tfrac{1}{3}\pi, \text{ when } \tfrac{2}{3}\pi<x<\pi.$$

Find the sums of the series at the points $2\pi/3$ and π.

9. (a) Show that for $-\pi<x<\pi$,

$$e^x=\frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2}+\sum_{1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx-n\sin nx)\right].$$

What is the sum of the series for $x=\pm\pi$.

(b) Show that for $0<x<\pi$,

$$e^x=\begin{cases}\dfrac{2}{\pi}\displaystyle\sum_{1}^{\infty}\left(1-(-1)^n e^{\pi}\right)\dfrac{n\sin nx}{n^2+1}\\[2ex] \dfrac{e^{\pi}-1}{\pi}-\dfrac{2}{\pi}\displaystyle\sum_{1}^{\infty}\left(1-(-1)^n e^{\pi}\right)\dfrac{\cos nx}{n^2+1}.\end{cases}$$

What are the sums of these series for $-\pi\leqslant x\leqslant 0$.

10. Show that when $-\pi\leqslant x\leqslant\pi$,

$$\cos kx=\frac{\sin k\pi}{\pi}\left(\frac{1}{k}-\frac{2k\cos x}{k^2-1^2}+\frac{2k\cos 2x}{k^2-2^2}-\ldots\ldots\right)$$

k being non-integral. Deduce that

$$\pi\cot k\pi=\frac{1}{k}+2k\Sigma\frac{1}{k^2-n^2},$$

and

$$\frac{\pi}{\sin k\pi}=\Sigma(-1)^n\left(\frac{1}{n+k}+\frac{1}{n+1-k}\right).$$

11. Obtain a sine series which will be equal to x^2 for $0\leqslant x<\pi$. Draw a graph of the sum function of the series for $-2\pi\leqslant x\leqslant 2\pi$.

12. Show that in $[0,\pi]$,

$$f(x)=\frac{\pi^2}{16}-2\sum_{1}^{\infty}\frac{\cos(4n-2)x}{(4n-2)^2},$$

where $f(x)=\frac{1}{4}\pi x$ when $0\leqslant x\leqslant\frac{1}{2}\pi$ and $f(x)=\frac{1}{4}\pi(\pi-x)$, when $\frac{1}{2}\pi<x\leqslant\pi$.

13. Prove that when $0<\theta<2\pi$,

$$\tfrac{1}{2}(\pi-\theta)=\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}.$$

What is the sum of the series for $\theta=0$ and $\theta=2\pi$?

14. Prove that when $-1<x<1$,

$$x+x^2=\frac{1}{3}+\frac{2}{\pi}\sum_{1}^{\infty}(-1)^n\left[\frac{2\cos n\pi x}{n^2\pi}-\frac{\sin n\pi x}{n}\right].$$

15. Find (*i*) the series of sines, (*ii*) the series of cosines, which will represent $f(x)$ in $(0, l)$, where

$$f(x)=kx, \text{ when } 0\leqslant x\leqslant \tfrac{1}{2}l,\ f(x)=k(l-x) \text{ when } \tfrac{1}{2}l<x\leqslant l.$$

16. If $\varphi(x)$ be a periodic function of period 4 and such that $\varphi(x)=0$ when $0<x<1$ and $\varphi(x)=x-1$ when $1\leqslant x<2$, and $\varphi(2)=0$, express $\varphi(x)$ as a series of sines of multiples of x. Draw the graph of $\varphi(x)$.

17. Show that for $0\leqslant x\leqslant 1$,

$$x-\frac{1}{2}=-\frac{1}{\pi}\left\{\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}\right\},\quad x^2-x+\frac{1}{6}=\frac{1}{\pi^2}\left\{\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}\right\}.$$

18. If $f(x)$ be a periodic function of period $\frac{1}{2}\pi$ and such that $f(x)=\sin x$ for $0\leqslant x\leqslant \frac{1}{4}\pi$ and $f(x)=\cos x$ for $\frac{1}{4}\pi\leqslant x\leqslant \frac{1}{2}\pi$, express $f(x)$ as a Fourier series.

19. If $f(x)=1$ when $0<x<1$, $f(x)=2$ when $1<x<3$ and $f(x)=\frac{5}{3}$ when $x=0$, 1 and 3 and $f(x+3)=f(x)$ for all x; show that

$$f(x)=\frac{5}{3}-\frac{1}{\pi}\sum_{1}^{\infty}\frac{2\sin\frac{1}{3}n\pi}{n}\cos\frac{n\pi(2x-1)}{3}$$

for all x.

20. If $f(x)=\frac{1}{4}a-x$ when $0\leqslant x\leqslant \frac{1}{2}a$ and $f(x)=x-\frac{3}{4}a$ when $\frac{1}{2}a\leqslant x\leqslant a$, show that for all x in $0\leqslant x\leqslant a$,

$$f(x)=\frac{2a}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(4n-2)\pi x}{a}.$$

21. Show that if a function $f(x)$ is bounded in $[0, \frac{1}{2}\pi]$ and is such that it is possible to divide $[0, \frac{1}{2}\pi]$ into a finite number of open sub-intervals in each of which $f(x)$ is monotonic, then $f(x)$ can be expressed in the form

$$b_1\sin x+b_3\sin 3x+b_5\sin 5x+\cdots\cdots\cdots$$

for $0<x<\frac{1}{2}\pi$ and obtain the co-efficients b_n and the sum of the series outside $]0, \frac{1}{2}\pi[$.

If $f(x)=\sin x$ when $0\leqslant x\leqslant \frac{1}{4}\pi$ and $f(x)=\cos x$ when $\frac{1}{4}\pi\leqslant x\leqslant \frac{1}{2}\pi$, show that

$$f(x)=\frac{\sin x}{2}+\frac{2}{\pi}\left(\frac{\sin 3x}{1\cdot 2}-\frac{\sin 5x}{2\cdot 3}+\frac{\sin 11x}{5\cdot 6}-\frac{\sin 13x}{6\cdot 7}+\cdots\cdots\right).$$

22. Expand in a Fourier series of the form $a_n\sin n\pi x$ a function $f(x)$ given by

$$f(x)=\sinh \pi x, \text{ for } 0\leqslant x<\tfrac{1}{2},\ f(0)=0, \text{ for } \tfrac{1}{2}<x\leqslant 1.$$

Deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2+(n+1)^2}=\frac{\pi}{2}\tanh\frac{\pi}{2}.$$

23. Show that the graph of the equation

$$y^2=\frac{a^2}{3}+\frac{16a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\frac{(2n-1)\pi x}{2a}+\frac{8a^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)^2}\cos\frac{n\pi x}{a}$$

is a series of circles of radius a connected by straight lines of length $2a$, the origin being the centre of one of the circles.

[Expand $f(x)$ in $[-2a, 2a]$ where $f(x)=0$ when $-2a\leqslant x\leqslant -a$, $f(x)=a^2-x^2$ when $-a\leqslant x\leqslant a$; $f(x)=0$ when $a\leqslant x\leqslant 2a$.]

APPENDIX A

FUNCTIONS OF BOUNDED VARIATION

A·1. This appendix will be devoted to a discussion about the important concept of *the functions of bounded variation.* This concept is closely related to that of monotonic functions and plays an important part in various parts of Analysis.

Let $f(x)$ be a function defined in an interval $[a, b]$ and let

$$D(a=x_0 < x_1 \ldots < x_{r-1} < x_r < \ldots\ldots < x_n=b)$$

be any division of $[a, b]$.

Consider the sum

$$\sum_{r=1}^{r=n} \mid f(x_r)-f(x_{r-1}) \mid \qquad \ldots(1)$$

of the moduli of the differences of the values of $f(x)$ at the end points of the sub-intervals of D.

There corresponds a sum (1) to each division D of $[a, b]$. Consider the aggregate of all such sums corresponding to the aggregate of divisions of $[a, b]$. *Then $f(x)$ is said to be of bounded variation over $[a, b]$ if this aggregate of sums is bounded above and the upper bound of the aggregate is known as the* **total variation** *of $f(x)$ over $[a, b]$.*

The total variation of the function $f(x)$ over $[a, b]$ will be denoted by the symbol

$$V_f(a, b).$$

Illustrations.

1. *A bounded monotonic function is a function of bounded variation.*

If $f(x)$ is monotonically increasing over $[a, b]$ and

$$D(a=x_0,\ldots\ldots, x_{r-1}, x_r,\ldots\ldots, b)$$

is *any* division of $[a, b]$, we have

$$\sum_{r=1}^{r=n} \mid f(x_r)-f(x_{r-1}) \mid = \sum_{r=1}^{r=n} [f(x_r)-f(x_{r-1})]$$

$$=f(b)-f(a).$$

Thus the monotonically increasing bounded function, $f(x)$, is of bounded variation over $[a, b]$ and

$$V_f(a, b)=f(b)-f(a).$$

Similarly we may show that a monotonically decreasing bounded function $g(x)$ is also of bounded variation and

$$V_g(a, b) = g(a) - g(b).$$

2. *A continuous function may* **not** *be a function of bounded variation.* refer to (Ex 1 164 Apostal)

Consider

$$f(x) = x \sin \frac{\pi}{x}, \text{ when } x \neq 0,$$

$$f(0) = 0.$$

Clearly $f(x)$ is continuous in $[0, 1]$.

Consider the division

$$D\left(0, \frac{2}{2n+1}, \ldots\ldots, \frac{2}{7}, \frac{2}{5}, \frac{2}{3}, 1\right);$$

n being any positive integer.

We have

$$\left|f(1) - f\left(\frac{2}{3}\right)\right| + \left|f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right)\right| + \ldots\ldots + \left|f\left(\frac{2}{2n+1}\right) - f(0)\right|$$

$$= \frac{2}{3} + \left(\frac{2}{3} + \frac{2}{5}\right) + \left(\frac{2}{5} + \frac{2}{7}\right) + \ldots\ldots + \left(\frac{2}{2n+3} + \frac{2}{2n+1}\right) + \frac{2}{2n+1}$$

$$= 4\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots + \frac{1}{2n+1}\right].$$

The infinite series

$$\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \ldots\ldots\ldots$$

being divergent, the sum

$$\frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n+1},$$

as a function of n, is not bounded above. Thus there exists a suitable value on n, such that for the corresponding division the sum of the moduli of the differences of the values of $f(x)$ for the end point of sub-intervals exceeds any given number.

Thus the function $f(x)$ in question, even though continuous, is not of bounded variation over $[0, 1]$.

Also it may be seen that *a function of bounded variation may not be continuous.* For example $f(x) = [x]$ where $[x]$ denotes the greatest integer not greater than x is of bounded variation in $[0, 2]$ even though it is not continuous.

3. *If $f(x)$ is derivable with bounded derivative in $[a, b]$, then $f(x)$ is of bounded variation over $[a, b]$ i.e.,* a derivable function with bounded derivative is of bounded variation.

There exists a number k such that

$$| f'(x) | \leqslant k$$

for each x belonging to $[a, b]$.

By Lagrange's mean value theorem, there exists ξ_r belonging to $[x_{r-1}, x_r]$ such that

$$f(x_r)-f(x_{r-1})=(x_r-x_{r-1})\, f'(\xi_r).$$

$$\therefore \qquad | f(x_r)-f(x_{r-1}) | = | x_r-x_{r-1} | \; | f'(\xi_r) | \leqslant k(x_r-x_{r-1}).$$

$$\therefore \qquad \sum_{r=1}^{n} | f(x_r)-f(x_{r-1}) | \leqslant k \sum_{r=1}^{n} (x_r-x_{r-1})=k(b-a).$$

Hence the result.

A·2. Some properties of functions of bounded variation.

1. *A function of bounded variation is necessarily bounded.*

If $f(x)$ be of bounded variation over $[a, b]$ and x be any point of $[a, b]$, we have, considering the division

$$D(a, x, b)$$

consisting of three points of division a, x, b only,

$$| f(x)-f(a) | + | f(b)-f(x) | \leqslant V_f(a, b).$$

This gives

$$| f(x)-f(a) | \leqslant V_f(a, b).$$

Now, we have

$$| f(x) | - | f(a) | \leqslant | f(x)-f(a) | \leqslant V_f(a, b)$$

or

$$| f(x) | \leqslant | f(a) | + V_f(a, b)$$

so that $f(x)$ is bounded in $[a, b]$.

2. *The sum, difference and product of functions of bounded variation are also of bounded variation.*

Let $f(x)$ and $g(x)$ be two functions of bounded variation over $[a, b]$. We have for any division

$$D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, b)$$

$$\sum_{r=1}^{n} | [f(x_r)+g(x_r)]-[f(x_{r-1})+g(x_{r-1})] |$$

$$\leqslant \sum_{r=1}^{n} | f(x_r)-f(x_{r-1}) | + \sum_{r=1}^{n} | g(x_r)-g(x_{r-1}) |$$

$$\leqslant V_f(a, b)+V_g(a, b),$$

so that $f(x)+g(x)$ is of bounded variation over $[a, b]$ and

$$V_{f+g}(a, b) \leqslant V_f(a, b)+V_g(a, b).$$

Similarly it may be shown that $f(x)-g(x)$ is of bounded variation over $[a, b]$ and

$$V_{f-g}(a, b) \leqslant V_f(a, b)+V_g(a, b).$$

We consider now the case of product.

The functions $f(x)$ and $g(x)$ being of bounded variation over $[a, b]$, they are both bounded and accordingly there exists a number k such that

$$| f(x) | \leqslant k, \; | g(x) | \leqslant k$$

for each x belonging to $[a, b]$.

We have for any division

$$D(a=x_0, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b),$$

$$\sum_{r=1}^{n} | f(x_r)g(x_r)-f(x_{r-1})g(x_{r-1}) |$$

$$= \sum_{r=1}^{n} | f(x_r)[g(x_r)-g(x_{r-1})]-g(x_{r-1})[f(x_{r-1})-f(x_r)] |$$

$$\leqslant \sum_{r=1}^{n} \{ | f(x_r) | \; | g(x_r)-g(x_{r-1}) | + | g(x_{r-1}) | \; | f(x_{r-1})-f(x_r) | \}$$

$$\leqslant k\{ \sum_{r=1}^{n} | g(x_r)-g(x_{r-1}) | + \sum_{r=1}^{n} | f(x_r)-f(x_{r-1}) | \}$$

$$\leqslant kV_g(a, b)+kV_f(a, b).$$

Thus $f(x)g(x)$ is of bounded variation over $[a, b]$.

3. *If $f(x)$ is of bounded variation over $[a, b]$ and is such that there exists a number k such that $| f(x) | \geqslant k$ for each x belonging to $[a, b]$, then $1/f(x)$ is also of bounded variation over $[a, b]$.*

For any division

$$D(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots, x_n=b),$$

we have

$$\sum_{r=1}^{n} \left| \frac{1}{f(x_r)}-\frac{1}{f(x_{r-1})} \right| = \sum_{r=1}^{n} \left| \frac{f(x_{r-1})-f(x_r)}{f(x_r)\, f(x_{r-1})} \right|$$

$$\leqslant \frac{1}{k^2} \sum_{r=1}^{n} | f(x_r)-f(x_{r-1}) |$$

$$\leqslant \frac{V_f(a, b)}{k^4}.$$

Thus, $1/f(x)$, is of bounded variation over $[a, b]$.

4. *If $f(x)$ is of bounded variation over $[a, b]$ and c is any point of $[a, b]$, then $f(x)$ is also of bounded variation over $[a, c]$ and $[c, b]$, and conversely. Also*

$$V_f(a, b)=V_f(a, c)+V_f(c, b).$$

Let $f(x)$ be of bounded variation over $[a, b]$.

Let

$$D_1(a=x_0, x_1, \ldots \ldots, x_{p-1}, x_p \ldots \ldots x_m=c)$$

and

$$D_2(c=y_0, y_1, \ldots \ldots, y_{q-1}, y_q, \ldots \ldots y_n=b)$$

be *any* two divisions of $[a, c]$, $[c, b]$ respectively.

They give rise to a division

$$D(a=x_0, \ldots \ldots, x_m=y_0, y_1, \ldots \ldots, y_{q-1}, y_q, \ldots \ldots, y_n=b)$$

of $[a, b]$.

We have

$$\sum_{p=1}^{m} | f(x_p)-f(x_{p-1}) | + \sum_{q=1}^{n} | f(y_q)-f(y_{q-1}) | \leqslant V_f(a, b)$$

so that

$$\sum_{p=1}^{m} | f(x_p)-f(x_{p-1}) | \leqslant V_f(a, b), \sum_{q=1}^{n} | f(y_q)-f(y_{q-1}) | \leqslant V_f(a, b),$$

and accordingly $f(x)$ is of bounded variation over $[a, c]$ as well as $[c, b]$.

Now suppose that $f(x)$ is of bounded variation over $[a, c]$ and $[c, b]$.

Let

$$(a=z_0 \ldots \ldots, z_{r-1}, z_r, \ldots \ldots z_t=b)$$

be any division of $[a, b]$ and let

$$z_{s-1} < c \leqslant z_s, \qquad \text{where } s < t.$$

We have

$$\sum_{r=1}^{t} | f(z_r)-f(z_{r-1}) | = \sum_{r=1}^{s-1} | f(z_r)-f(z_{r-1}) | + | f(z_s)-f(z_{s-1}) | + \sum_{r=s+1}^{t} | f(z_r)-f(z_{r-1}) |$$

$$\leqslant \sum_{r=1}^{s-1} | f(z_r)-f(z_{r-1}) | + | f(c)-f(z_{s-1}) | + | f(z_s)-f(c) | + \sum_{r=s+1}^{t} | f(z_r)-f(z_{r-1}) |$$

$$\leqslant V_f(a, c)+V_f(c, b). \qquad \ldots(1)$$

Thus $f(x)$ is of bounded variation over $[a, b]$ if the same is of bounded variation over $[a, c]$ and $[c, b]$. Also then, we deduce from (1),

$$V_f(a, b) \leqslant V_f(a, c)+V_f(c, b). \qquad \ldots (2)$$

Let, now, ε be any arbitrary positive number.

Since $f(x)$ is of bounded variation over $[a, c]$ as well as $[c, b]$, there exist divisions

$$D_1(a=x_0, x_1, \ldots\ldots, x_{p-1}, x_p, \ldots\ldots, x_m=c)$$

$$D_2(c=y_0, y_1, \ldots\ldots, y_{q-1}, y_q, \ldots\ldots, y_n=b)$$

of $[a, c]$ and $[c, b]$ respectively such that

$$\sum_{p=1}^{m} |f(x_p)-f(x_{p-1})| > V_f(a, c)-\tfrac{1}{2}\varepsilon, \qquad \ldots (3)$$

$$\sum_{q=1}^{n} |f(y_q)-f(y_{q-1})| > V_f(c, b)-\tfrac{1}{2}\varepsilon. \qquad \ldots (4)$$

From (3) and (4), we have

$$\sum_{p=1}^{m} |f(x_p)-f(x_{p-1})| + \sum_{q=1}^{n} |f(y_q)-f(y_{q-1})| > V_f(a, c)+V_f(c, b)-\varepsilon,$$

so that

$$V_f(a, b) \geqslant V_f(a, c)+V_f(c, b)-\varepsilon. \qquad \ldots (5)$$

As ε is any arbitrary positive numbers, we deduce from (5) that

$$V_f(a, b) \geqslant V_f(a, c)+V_f(c, b) \qquad \ldots (6)$$

Finally, from (2) and (6), we have

$$V_f(a, b)=V_f(a, c)+V_f(c, b),$$

as was to be proved.

A·3. Variation function of a function of bounded variation. We shall now define a function associated with every function of bounded variation.

Let $f(x)$ be a function of bounded variation over an interval $[a, b]$.

If x be any point of $[a, b]$, then we denote by $V(x)$ the total variation of $f(x)$ over $[a, x]$ so that $V(x)$ is a function of x defined over $[a, b]$. This function will be referred to as the variation function of $f(x)$ over $[a, b]$.

A·4. Monotonically increasing character of the variation function. Let x_1, x_2 be two points of $[a, b]$ such that $x_2 > x_1$.

We have

$$0 \leqslant |f(x_2)-f(x_1)| \leqslant V_f(x_1, x_2)=V_f(a, x_2)-V_f(a, x_1)$$
$$=V(x_2)-V(x_1)$$

so that

$$V(x_2) \geqslant V(x_1)$$

and hence we have the monotonically increasing character of the variation function $V(x)$.

A·5. Characterisation of functions of bounded variation. *A function of boudned variation is expressible as the difference of two monotonically increasing functions.*

We ave

$$f(x) = \tfrac{1}{2}[V(x)+f(x)] - \tfrac{1}{2}[V(x)-f(x)]$$
$$= G(x) - H(x), \text{ say,}$$

and shall show that $G(x)$ and $H(x)$ are monotonically increasing in $[a, b]$.

Now if $x_2 > x_1$, we have

$$G(x_2) - G(x_1) = \tfrac{1}{2}[V(x_2) - V(x_1) + f(x_2) - f(x_1)]$$
$$= \tfrac{1}{2}\{V_f(x_1, x_2) - [f(x_1) - f(x_2)]\}.$$

Since

$$V_f(x_1, x_2) \geqslant | f(x_1) - f(x_2) |,$$

we deduce that

$$G(x_2) - G(x_1) \geqslant 0, \textit{ i.e., } G(x_2) \geqslant G(x_1),$$

so that $G(x)$ is monotonically increasing in $[a, b]$.

Again, we have

$$H(x_2) - H(x_1) = \tfrac{1}{2}\{[V(x_2) - V(x_1)] - [f(x_2) - f(x_1)]\}$$
$$= \tfrac{1}{2}\{V_f(x_1, x_2) - [f(x_2) - f(x_1)]\}$$

so that as before

$$H(x_2) - H(x_1) \geqslant 0, \textit{ i.e., } H(x_2) \geqslant H(x_1).$$

Hence the result.

With the help of the results proved here and in A·1 and A·2 we may state that *a function $f(x)$ is of bounded variation over $[a, b]$ if and only if it can be expressed as the difference of two monotonically increasing functions.*

Integrability of functions of bounded variation.

Cor. *If $f(x)$ is of bounded variation over $[a, b]$, then it is integrable over $[a, b]$.*

A·6. Variation function of a continuous function of bounded variation. We shall now show that the variation function $V(x)$ of a continuous function $f(x)$ of bounded variation is itself continuous.

Theorem. *If c be any point of $[a, b]$, then $V(x)$ is continuous at c if, and only if, $f(x)$ is continuous at c,* i.e., *a point of continuity of $f(x)$ is also a point of continuity of $V(x)$ and conversely.*

Firstly suppose that $V(x)$ is continuous at c.

Let $\varepsilon > 0$ be given. Since $V(x)$ is continuous at c, there exists a positive number δ such that

$$| V(x)-V(c) | < \varepsilon \text{ for } | x-c | \leqslant \delta. \qquad \text{.. (1)}$$

Also we have

$$| f(x)-f(c) | \leqslant V(x)-V(c) \text{ if } x > c, \qquad \text{.. (2)}$$

and

$$| f(x)-f(c) | \leqslant V(c)-V(x) \text{ if } x < c. \qquad \text{...(3)}$$

From (1), (2) and (3), we deduce that

$$| f(x)-f(c) | < \varepsilon \text{ for } | x-c | \leqslant \delta,$$

so that c is a point of continuity of $f(x)$.

Now suppose that c is a point of continuity of $f(x)$. Let $\varepsilon > 0$ be given.

There exists a $\delta > 0$ such that

$$| f(x)-f(c) | < \tfrac{1}{2}\varepsilon \text{ for } | x-c | \leqslant \delta.$$

Also there exists a division

$$D_1(c=y_0, y_1 \ldots\ldots, y_{q-1}, y_q, \ldots\ldots, y_n=b)$$

of $[c, b]$ such that

$$\sum_{q=1}^{n} | f(y_q)-f(y_{q-1}) | > V_f(c, b)-\tfrac{1}{2}\varepsilon. \qquad \text{...(4)}$$

Since as a result of introducing additional points to D_1, the corresponding sum of the moduli of the differences of the function values at end points will not be decreased, we may assume that $0 < (y_1-c) \leqslant \delta$ so that

$$| f(y_1)-f(c) | < \tfrac{1}{2}\varepsilon. \qquad \text{...(5)}$$

Thus (4) becomes

$$V_f(c, b) - \tfrac{1}{2}\varepsilon < \tfrac{1}{2}\varepsilon + \sum_{q=2}^{n} | f(y_q)-f(y_{q-1}) |$$

$$\leqslant \tfrac{1}{2}\varepsilon + V_f f(y_1, b)$$

or

$$V_f(c, b)-V_f(y_1, b) < \varepsilon,$$

or

$$V(y_1)-V(c)=V_f(c, y_1) < \varepsilon.$$

Thus for $0 < (y_1-c) \leqslant \delta$, we have

$$-\varepsilon < 0 < V(y_1)-V(c) < \varepsilon$$

Thus

$$\lim_{x \to (c+0)} V(x)=V(c)$$

Similarly

$$\lim_{x \to (c-0)} V(x) = V(c).$$

Thus $V(x)$ is continuous at c.

From the preceding we deduce *that $V(x)$ is continuous in* $[a, b]$ *if and only if $f(x)$ is continuous in* $[a, b]$.

Cor. A continuous function is of bounded variation if and only if it can be expressed as the difference of two continuous monotonically increasing functions. 8-15 (proved By Apostol

Ex. 1. Show that a polynomial is of bounded variation over any finite interval.

Ex. 2. Show that $\sin x$ and $\cos x$ are of bounded variation over any finite interval.

Ex. 3. Give an example of a function of bounded variation which is derivable but whose derived function is not bounded.

Ex. 4. Show that a function of bounded variation is integrable.

CHAPTER X

REAL VALUED FUNCTIONS OF SEVERAL VARIABLES

Differentiation

130. Upto this point we have been concerned with real valued functions of one variable only, but, in the following part of the book, the real valued functions of *several variables* will be considered.

It is usually sufficient to consider the case of *two* independent variables only, for, the extension to three or more variables can, in general, be made without introducing any essentially new ideas. In order to avoid complicating our statements, notions and proofs, we shall, therefore, mainly confine ourselves to functions of two variables only.

131. Real valued function of two variables defined in a certain domain. The set of all ordered pairs (x, y) of real numbers x, y will be called '*Space of two dimensions*' and each ordered pair a *point* of the space. The space of two dimensions will be denoted by the symbol E_2.

Suppose now that S is a sub-set of E_2.

Let there be given a rule f, which associates to each point (x, y) of S a real number z. We then say that, f, is a function defined in S. As in the case of real valued functions of a real variable we write

$$z=f(x, y)$$

and by an abuse of language refer to $f(x, y)$ rather than f as a function.

The set S is said to be the *domain* of the function. Further the set of numbers $f(x, y)$ where (x, y) varies over S is called the *Range* of the function.

131·1. Open and closed Rectangles. The set of points (x, y) such that

$$a \leqslant x \leqslant b, \; c \leqslant y \leqslant d$$

is called a *closed* rectangle denoted by the symbol

$$R[a, b \; ; c, d],$$

or simply by the symbol

$$[a, b \; ; c, d]$$

a, b, c, d being given arbitrary real numbers.

The set of points (x, y) such that

$$a < x < b,\ c < y < d$$

is called an *open* rectangle denoted by the symbol

$$R]\, a, b\ ;\ c, d\, [$$

or simply by the symbol

$$]\, a, b\ ;\ c, d\, [.$$

131·2. Neighbourhood of a point. A rectangle

$$]a-\delta,\ a+\delta\ ;\ b-\delta,\ b+\delta[,$$

where δ has any positive value whatsoever, is said to be a neighbourhood of the point (a, b).

Note 1. The use of the word 'point' is suggested by 'Plane analytical geometry' where, by choosing a pair of co-ordinate axes, a point is represented by an ordered pair of numbers. Obviously, a domain $R\,[a, b\ ;\ c, d]$ corresponds, when geometrically interpreted, to the geometrical rectangle bounded by the lines

$$x=a,\ x=b\ ;\ y=c,\ y=d.$$

Illustration. For the domain defined by the inequalities $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 1$, the variable x varies in the interval $[0, 1]$ and to each value of x in this interval, y varies in the variable interval $[0, 1-x]$. Geometrically, this domain consists of the points lying in the interior and on the boundary of the triangle formed by the co-ordinate axes and the line $x+y=1$.

A domain is generally given by means of relationships of inequality between x and y

Thus the domain of a function of two variables may be regarded as a part of a plane to every point of which there corresponds a value of the function.

Ex. Locate geometrically the domains of the following functions:

(*i*) $z=\sqrt{(1-x^2-y^2)}$. (*ii*) $z=\sqrt{[(x-y)/(x+y)]}$ (*iii*) $z=[\log(xy)]^{-1}$.

Note 2. Local and Global Concepts. The reader would do well in respect of a better grasp of the subject if he learns to distinguish between *local and global concepts and properties*. A concept or property is said to be local if it is describable just in terms of *any* neighbourhood of a point and is global if it relates to an interval or rectangle. Thus the concepts of $\lim f(x)$ as $x \to a$, continuity at a point, derivability at a point, a function being maximum or minimum at a point are all local, whereas the theorems considered in § 58 describing the properties of functions which are continuous in closed finite intervals, Rolles' theorem, Taylor's theorem and the concept of Riemann integrability are all global.

In this chapter, we shall be mostly concerned with Local concepts in respect of functions of two or more variables.

132. Simultaneous limit.

$$\lim_{(x,\, y) \to (a,\, b)} f(x, y)$$

A function $f(x, y)$ is said to tend to the limit, l, as a point (x, y) tends to the point (a, b), if, to every positive number ε, there corresponds a positive number δ, such that

$$|f(x, y) - l| < \varepsilon,$$

for every point (x, y), [different possibly from (a, b) itself], which belongs to the domain of the function and to the rectangle

$$[a-\delta,\, a+\delta\,;\; b-\delta,\, b+\delta].$$

Note. Instead of saying that '$(x, y) \to (a, b)$', we also sometimes say that '$x \to a$ and $y \to b$' and write

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l,$$

even though the first form is more appropriate.

Ex. If $\lim\limits_{(x,\, y) \to (a,\, b)} f(x, y) = l$, then $\lim\limits_{x \to a} f(x, b) = l = \lim\limits_{y \to b} f(a, y)$.

132·1. Non-Existence of limit. In general, to determine whether a simultaneous limits exist or not, is a difficult matter but a simple consideration, as we now describe, sometimes enables us to show that the limit does *not* exist.

It is easy to see that *if*

$$\lim_{(x,\, y) \to (a,\, b)} f(x, y) = l,$$

and if $y = \varphi(x)$ is any function whatsoever such that

$$\varphi(x) \to b, \text{ when } x \to a,$$

then,

$$\lim_{x \to a} f[x, \varphi(x)]$$

must exist and be equal to l.

Thus if we can determine two functions $\varphi_1(x)$, $\varphi_2(x)$ such that the limits of $f[x, \varphi_1(x)]$ and $f[x, \varphi_2(x)]$ are different, then we can certainly say that the simultaneous limit, in question, does not exist.

(The reader is advised to geometrically interpret the consideration outlined here).

Ex. 1. *Show that*

$$\lim\, [2xy/(x^2+y^2)], \text{ when } (x, y) \to (0, 0)$$

does not exist.

Taking $y = mx$, we see that, when $x \to 0$,

$$\lim \frac{2x.mx}{x^2+m^2x^2} = \frac{2m}{1+m^2},$$

which is different for different values of m. Hence the limit does not exist.

2. Show that

$$\lim \frac{xy^3}{x^2+y^6}, \text{ when } (x, y) \to (0, 0)$$

does not exist.

[Consider the relation $x=my^3$]

3. Evaluate the following limits or show that the limits do not exist :—

(*i*) $\lim \dfrac{xy^2}{x^2+y^4}$, (*ii*) $\lim\ (x+y)\dfrac{y+(x+y)^2}{y-(x+y)^2}$,

(*iii*) $\lim \dfrac{xy}{\sqrt{(x^2+y^2)}}$, (*iv*) $\lim\ (y \sin 1/x + x \sin 1/y)$,

when, in each case, $(x, y) \to (0, 0)$

133. Calculation with simultaneous limits. Theorem. *If, when* $(x, y) \to (a, b)$,

$$\lim f(x, y) = l_1,\ \lim g(x, y) = l_2,$$

then

(*i*) $\lim\ [f(x, y) \pm g(x, y)] = l_1 \pm l_2$,

(*ii*) $\lim\ [f(x, y) g(x, y)] = l_1 l_2$,

(*iii*) $\lim\ [f(x, y)/g(x, y)] = l_1/l_2$, *if* $l_2 \neq 0$.

The proofs are exactly similar to those of the corresponding results in the case of functions of one variable.

134. Repeated limits. Let $f(x, y)$ be defined in a certain neighbourhood of (a, b). Then

$$\lim_{x \to a} f(x, y),$$

if it exists, is a function of y, say $\varphi(y)$. If, then,

$$\lim_{y \to b} \varphi(y),$$

exists and is equal to λ, we write

$$\lim_{y \to b}\ \lim_{x \to a} f(x, y) = \lambda,$$

and say that λ is a *repeated limit* of (x, y) as $x \to a$ and $y \to b$. A change in the order of passing to limits may produce a change in the final result. Thus

$$\lim_{x \to a}\ \lim_{y \to b} f(x, y),$$

where first $y \to b$ and then $x \to a$, may be different from λ.

Ex. We have

$$\lim_{y \to 0}\ \lim_{x \to 0} \frac{x-y}{x+y} = \lim_{y \to 0} \frac{-y}{y} = -1,$$

$$\lim_{x \to 0}\ \lim_{y \to 0} \frac{x-y}{x+y} = \lim_{x \to 0} \frac{x}{x} = 1,$$

so that the two repeated limits are different. The simultaneous limit, as may easily be seen, does not even exist.

Ex. Show that for the function

$$f(x, y)=\frac{x^2y^2}{x^2 y^2+(x-y)^2}$$. مثال

the repeated limits exist but the simultaneous limit does not when $(x, y) \to (0, 0)$.

135. Continuity. *A function $f(x, y)$ is said to be continuous at a point (a, b) of its domain, if*

$$\lim_{(x, y) \to (a, b)} f(x, y)=f(a, b).$$

Again $f(x, y)$ is said to be continuous in a domain, if it is continuous at every point of the same.

It can now be easily shown that (*i*) the sum, difference, and product of two continuous functions are also continuous. (*ii*) The quotient of two continuous functions is continuous except where the denominator vanishes. (*iii*) Continuous functions of continuous functions are themselves continuous.

Note. It is easy to show that if $f(x, y)$ is a continuous function of two variables at (a, b), then $f(x, b)$ is a continuous function of one variable x for $x=a$ and $f(a, y)$ is a continuous function of one variable y for $y=b$.

The converse of this result is not necessarily true, as may be seen by considering a function $f(x, y)$ which $=0$ when x is 0 or when y is 0 and $=1$ elsewhere.

Note 2. It is also easy to show *that if $f(x, y)$ is continuous at (a, b) and $f(a, b)\neq 0$, then there exists a neighbourhood of (a, b) such that for every point of this neighbourhood, $f(x, y)$ has the sign of $f(a, b)$.*

Ex. 1. Discuss the continuity and discontinuity of the following functions :—

(*i*) $f(x, y)=2xy^2/(x^3+y^3)$ when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$.

(*ii*) $\varphi(x, y)=2xy/\sqrt{(x^2+y^2)}$ when $(x, y)\neq(0, 0)$ and $\varphi(0, 0)=0$.

2. Show that the following function is continuous at $(0, 0)$:

$$f(x, y)=e^{-|x-y|/(x^2-2xy+y^2)}, \text{ when } (x, y)\neq(x, x) \text{ and } f(x, x)=0.$$

3. Show that

$$f(x, y)=xy\,\frac{x^2-y^2}{x^2+y^2} \text{ if } (x, y)\neq(0, 0),\ f(0, 0)=0$$

is continuous at $(0, 0)$.

4. Can $f(0, 0)$ be defined so that

$$f(x, y)=\frac{x^2y}{x^4+y^2}$$

is continuous at $(0,0)$?

5. Show that the functions

(*i*) $f(x, y)=x^4y^4/(x^2+y^4)^3$, when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$.

(*ii*) $\varphi(x, y)=x^2/(x^2+y^2-x)$, when $(x, y)\neq(0, 0)$ and $\varphi(0, 0)=0$

tend to 0 if (x, y) approaches the origin along any straight line, but that they are discontinuous at the origin.

[**Hint**. $\lim f(x, y)=\frac{1}{8}$, $\lim \varphi(x, y)=1$ if $(x, y)\to(0, 0)$ along $y^2=x$.]

6. The functions $f(x)$ and $g(y)$ are continuous in the range $a\leqslant x\leqslant b$ and $c\leqslant y\leqslant d$ respectively and $h(x, y)=\max\ \{f(x), g(y)\}$; prove that $h(x, y)$ is a continuous function of (x, y) in the rectangle $a\leqslant x\leqslant b$, $c\leqslant y\leqslant d$

136. Properties of continuous functions of two variables. We shall now obtain properties of real valued continuous functions of two variables similar to those of the real valued continuous functions of a single variable proved in § 58, page 94.

136 1. Theorem. *If $f(x, y)$ is continuous in a rectangle*

$$R[a, b\ ;\ c, d]$$

and ε is any positive number, then there exists a division of R into a finite number of sub-rectangles such that

$$|\ f(x_2, y_2)-f(x_1, y_1)\ |<\varepsilon,$$

where (x_1, y_1) and (x_2, y_2) are two points belonging to the same sub-rectangle.

We assume that the theorem is false in R. By drawing lines parallel to the co-ordinate axes, divide R into four sub-rectangles

$$[a, \tfrac{1}{2}(a+b)\ ;\ c, \tfrac{1}{2}(c+d)],\ [\tfrac{1}{2}(a+b), b\ ;\ c, \tfrac{1}{2}(c+d)],$$

$$[a, \tfrac{1}{2}(a+b)\ ;\ \tfrac{1}{2}(c+d), d],\ [\tfrac{1}{2}(a+b), b\ ;\ \tfrac{1}{2}(c+d), d].$$

Because of the assumption, the theorem must be false in atleast one of these sub-rectangles, say R_1, which we would re-name as $[a_1, b_1\ ;\ c_1, d_1]$. Sub-dividing R_1 in a similar manner and continuing the process indefinitely, we would obtain a sequence of rectangles

$$R_1, R_2, R_3, \ldots\ldots, R_n, \ldots\ldots \tag{1}$$

where R_n is the rectangle $[a_n, b_n\ ;\ c_n, d_n]$.

The theorem is false in every rectangle R_n. As in § 58·1, page 94, it can be shown that the sequences $\{a_n\}$ and $\{b_n\}$ approach a common limit ξ and $\{c_n\}$ and $\{d_n\}$ approach a common limit η. This point (ξ, η) belongs to every R_n.

Since $f(x, y)$ is continuous at (ξ, η), there exists a positive number δ such that

$$|\ f(x, y)-f(\xi, \eta)\ |<\tfrac{1}{2}\varepsilon, \tag{2}$$

for every point (x, y) of R which lies in the rectangle

$$[\xi-\delta, \xi+\delta\ ;\ \eta-\delta, \eta+\delta]. \tag{3}$$

If (x_1, y_1) and (x_2, y_2) be any two points of (3), we deduce from (2) that

$$| f(x_2, y_2) - f(x_1, y_1) | < \varepsilon.$$

Now, since there exists a positive integer m such that a rectangle R_m of the sequence (1) lies wholly within the rectangle (3), we see that the theorem is true for R_m. Thus we arrive at a contradiction. Hence the theorem.

Cor. 1. *If $f(x, y)$ is continuous in R, then it is necessarily bounded in R.* We divide R into a finite number, say, p, of sub-rectangles such that for every pair of points (x_1, y_1) and (x_2, y_2) of the same sub-rectangle,

$$| f(x_2, y_2) - f(x_1, y_1) | < \varepsilon,$$

so that if (α_r, β_r) be any *fixed* point of the rth sub-rectangle and (x, y) any *variable* point of the same, we have

$$f(\alpha_r, \beta_r) - \varepsilon < f(x, y) < f(\alpha_r, \beta_r) + \varepsilon. \tag{4}$$

From these, p, inequalities, similar to (4), obtained for the, p, sub-rectangles, we deduce as in (§ 58·3, page 98) that $f(x, y)$ is bounded.

Cor. 2. *If $f(x, y)$ is continuous in R, then it must attain its bounds.*

The proof is exactly similar to that of the corresponding theorem for functions of a single variable. (§ 58·4, page 100).

Cor. 3. *If $f(x, y)$ is continuous in R, it is possible to divide R into a finite number of sub-rectangles such that the oscillation of $f(x, y)$ in every sub-rectangle is less than a given positive number.*

Follows from the main theorem and the Cor. 2 above.

Cor. 4. Uniform Continuity. *If $f(x, y)$ is continuous in R, and ε is any given positive number, then there exists a positive number δ such that*

$$| f(x_2, y_2) - f(x_1, y_1) | < \varepsilon,$$

when (x_1, y_1), (x_2, y_2) are any two points such that

$$| x_2 - x_1 | \leqslant \delta, \; | y_2 - y_1 | \leqslant \delta.$$

We divide R into a finite number of sub-rectangles such that for every pair of points (x_1, y_1), (x_2, y_2) of the *same* sub-rectangle,

$$| f(x_2, y_2) - f(x_1, y_1) | < \tfrac{1}{2}\varepsilon.$$

It will now be shown that a positive number δ which is smaller than every side of every sub-rectangle is the requisite one. Now, a pair of points (x_1, y_1) and (x_2, y_2) such that

$$| x_2 - x_1 | \leqslant \delta, \; | y_2 - y_1 | \leqslant \delta,$$

either belong to the same sub-rectangle or to two such sub-rectangles as have a side or a part of side in common. In the former case,

$$|f(x_2, y_2)-f(x_1, y_1)| < \tfrac{1}{2}\varepsilon < \varepsilon,$$

and in the latter, if (x, y) be a point of the common side, we have

$$|f(x_2, y_2)-f(x_1, y_1)| \leqslant |f(x_2, y_2)-f(x, y)| + |f(x, y)-f(x_1, y_1)|$$
$$< \tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon=\varepsilon.$$

Hence the result.

Cor. 5. *If $f(x, y)$ is continuous in R, then it must assume every value between its upper and lower bounds.*

Let M, m be the bounds and let

$$f(\alpha, \beta)=M, f(\gamma, \delta)=m.$$

We write

$$x=\alpha+t(\gamma-\alpha), \; y=\beta+t(\delta-\beta)$$

where $0 \leqslant t \leqslant 1$.

It may be easily seen that (x, y) belongs to the rectangle R for every value of t belonging to the interval $[0, 1]$.

[In fact *geometrically* interpreted the point (x, y) describes the line segment joining (α, β) to (γ, δ) as t varies from 0 to 1].

Writing $z=f(x, y)$, where

$$x=\alpha+t(\gamma-\alpha), \; y=\beta+t(\delta-\beta)$$

we see that z is a continuous function of *one* variable t in the interval $[0, 1]$ assuming the values M and m for $t=0$ and $t=1$ respectively.

Thus z assumes every value between m and M, for some value of t between 0 and 1.

Hence the result.

137. Partial Derivation. Let (a, b) be any point of the domain of definition of a function $f(x, y)$. Then

$$\lim_{h \to 0} \frac{f(a+h, b)-f(a, b)}{h},$$

if it exists, is called the partial derivative of $f(x, y)$ with respect to x at (a, b) and is symbolically denoted by $f_x(a, b)$ or $\partial f(a, b)/\partial x$.

Similarly

$$\lim_{k \to 0} \frac{f(a, b+k)-f(a, b)}{k},$$

if it exists, is called the partial derivative of $f(x, y)$ with respect to y at (a, b) and is denoted by $f_y(a, b)$ or $\partial f(a, b)/\partial y$.

Now, if $f(x, y)$ possesses a partial derivative w.r. to x at every point of its domain of definition, then the function determined by the aggregate of the values of the derivative, is called the partial derivative of $f(x, y)$ w.r. to x and is denoted by $f_x(x, y)$ or $\partial f/\partial x$.

Similarly the partial derivative w.r. to y may be defined.

Note If $]a-\delta, a+\delta\ ;\ b-\delta, b+\delta[$ be a neighbourhood of (a, b) then the question of the existence and the values of the partial derivatives of $f(x, y)$ at (a, b) depends only on the values of the function at those points of the neighbourhood which lie along the lines $x=a$, $y=b$ and is absolutely independent of the values of the function at the remaining points of the neighbourhood. The question of continuity at (a, b), however, takes into account the values of the function at *every* point of a neighbourhood. There is, therefore, nothing surprising in the fact, which we will now illustrate by means of an example that the *partial derivatives of a function may exist at a point at which the function is not even continuous.*

Consider a function $f(x, y)$ such that

$f(x, y)=0$ when either x or y is 0 *i.e.*, along the lines $x=0$, $y=0$ and $f(x, y)=1$ elsewhere.

Clearly $f(x, y)$ is not continuous at $(0, 0)$ even though $f_x(0, 0), f_y(0, 0)$ both exist and are zero.

Ex. 1. *If*

$$\varphi(x, y)=(x^3+y^3)/(x-y), \qquad \textit{when } x\neq y,$$
$$\varphi(x, y)=0, \qquad \textit{when } x=y,$$

show that this function is discontinuous at the origin, but that the partial derivatives exist at that point.

Putting $y=x-mx^3$, we see that

$$\lim_{x\to 0} \varphi(x, y)=2/m,$$

so that this limit is different for different values of m.

Thus $\lim \varphi(x, y)$ when $(x, y)\to(0, 0)$ does not exist and therefore the function is necessarily discontinuous at $(0, 0)$.

Again, we have

$$\varphi_x(0, 0)=\lim_{h\to 0} \frac{\varphi(0+h, 0)-\varphi(0, 0)}{h}=\lim_{h\to 0}\frac{h^3}{h}=0\ ;$$

$$\varphi_y(0, 0)=\lim_{k\to 0} \frac{\varphi(0, 0+k)-\varphi(0, 0)}{k}=\lim_{k\to 0}\frac{k^3}{k}=0.$$

2. If $(fx, y)=xy(x^2-y^2)/(x^2+y^2)$ when x and y are not simultaneously zero and $f(0, 0)=0$, show that $f_x(x, 0)=0$ and $f_y(0, y)=0$.

3. Show that the function

$$f(x, y)=xy/\sqrt{(x^2+y^2)} \text{ if } (x^2+y^2)\neq 0 \text{ and } f(0, 0)=0,$$

possesses partial derivatives f_y, f_x at every point (x, y) but they are not continuous as $(0, 0)$

4. Show that $f(x, y)=\sqrt{(x^2+y^2)}$ possesses partial derivaties f_x and f_y at all points different from the origin.

138. Differentiability and Differentials. Let (a, b) be any point of the domain of definition of a function $f(x, y)$ and let $(a+h, b+k)$ be any point in a neighbourhood of (a, b) and in the domain of definition of the function. Now,

$$f(a+h, b+k)-f(a, b),$$

is the change in the function, as the point (x, y) changes from

$$(a, b) \text{ to } (a+b, b+k).$$

The function $f(x, y)$ is said to be differentiable at (a, b) if, as (x, y) changes from (a, b) to $(a+h, b+k)$, the change in the value of the function can be expressed in the form

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\varphi(h, k)+k\psi(h, k),$$

where A, B are constants independent of h and k and $\varphi(h, k)$, $\psi(h, k)$ are functions of h and k such that

$$\lim_{(h, k)\to(0, 0)} \varphi(h, k)=0=\lim_{(h, k)\to(0, 0)} \psi(h, k).$$

Also then, $Ah+Bk$, is called the differential of $f(x, y)$ at (a, b) and is denoted as $df(a, b)$.

139. Necessary conditions for differentiability.

139·1. Theorem. *If $f(x, y)$ is differentiable at (a, b), then it is also necessarily continuous at (a, b).*

Since

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\varphi(h, k)+k\psi(h, k),$$

we have, when $(h, k)\to(0, 0)$,

$$[f(a+h, b+k)-f(a, b)]\to 0,$$

i.e., $f(x, y)$ is continuous at (a, b).

The converse of this result is not true. (See Ex. 1, below.)

139·2. Theorem. *If $f(x, y)$ is differentiable at (a, b), then it also necessarily possesses the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ at (a, b).*

Putting $k=0$ in the relation

$$f(a+h, b+k)-f(a, b)=Ah+Bk+h\varphi(h, k)+k\psi(h, k),$$

we have

$$[f(a+h, b)-f(a, b)]/h=A+\varphi(h, 0),$$

so that when $h \to 0$, we obtain

$$f_x(a, b) = A.$$

Similarly

$$f_y(a, b) = B.$$

The proof above also shows that if $f(x, y)$ is differentiable at (a, b) then A, B are uniquely defined in as much as they are the partial derivatives of $f(x, y)$ at (a, b).

The converse of this theorem is also not true. (See Ex. 1, below.)

Ex. 1. *Given*

$$f(x, y) = xy/\sqrt{(x^2+y^2)}, \text{ when } (x, y) \neq (0, 0),$$
$$f(0, 0) = 0,$$

show that $f(x, y)$ is continuous, possesses partial derivatives but is not differentiable at $(0, 0)$.

We have

$$\left| \frac{xy}{\sqrt{(x^2+y^2)}} - 0 \right| = \frac{|xy|}{\sqrt{(x^2+y^2)}}.$$

Also

$$x^2+y^2 - |xy| = (|x| - \tfrac{1}{2}|y|)^2 + \tfrac{3}{4}|y^2| \geqslant 0$$

$$\therefore \qquad \frac{|xy|}{\sqrt{(x^2+y^2)}} \leqslant \sqrt{(x^2+y^2)}.$$

Again

$\sqrt{(x^2+y^2)} < \varepsilon$, if $x^2 < \frac{1}{2}\varepsilon^2$, $y^2 < \frac{1}{2}\varepsilon^2$. *i.e.*, if $|x| < \varepsilon/\sqrt{2}$, $|y| < \varepsilon/\sqrt{2}$

Thus

$$\left| \frac{xy}{\sqrt{(x^2+y^2)}} - 0 \right| < \varepsilon \text{ if } |x-0| < \varepsilon/\sqrt{2},\ |y-0| < \varepsilon/\sqrt{2},$$

and accordingly $f(x, y)$ is continuous at $(0, 0)$.

Also it is easy to see that $f_x(0, 0) = 0 = f_y(0, 0)$ so that if $f(x, y)$ were differentiable at $(0, 0)$, we should have, by definition,

$$\frac{hk}{\sqrt{(h^2+k^2)}} = 0h + 0k + h\varphi(h, k) + k\psi(h, k), \qquad \ldots(1)$$

where $\varphi(h, k)$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Putting $k = mh$ and letting $h \to 0$, we obtain from (1),

$$m/\sqrt{(1+m^2)} = 0$$

which is absurd, as m may have any value whatsoever. Hence $f(x, y)$ is not differentiable at $(0, 0)$.

2. Show that $f(x, y) = |x| + |y|$ is continuous but not differentiable at $(0, 0)$.

3. Discuss the differentiability of the following functions at (0, 0) :

(*i*) $f(x, y)=\sqrt{|xy|}$.

(*ii*) $f(x, y)=xy^2/(x^2+y^2)$ when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$.

Examine them for continuity also.

4. If $f(x, y)=x \sin 1/x+y \sin 1/y$ when $xy\neq 0$, $f(x, 0)=x \sin 1/x$ when $x\neq 0$, $f(0, y)=y \sin 1/y$ when $y\neq 0$; $f(0, 0)=0$, show that $f(x, y)$ is continuous but not differentiable at (0, 0).

5. Investigate the differentiability of the function $f(x, y)=(xy)^{\frac{1}{3}}$ at all points of the (xy) plane.

6. Find the values of k for which the function

$$\{|x+y|+(x+y)\}^k$$

is everywhere differentiable

139·3. A sufficient condition for differentiability. Theorem. *If* (a, b) *be a point of the domain of definition of a function* $f(x, y)$ *such that*

$$(i)\ f_x(a, b) \textit{ exists},$$

$$(ii)\ f_y(x, y) \textit{ is continuous at } (a, b),$$

then

$$f(x, y) \textit{ is differentiable at } (a, b).$$

The condition (*ii*) implies that $f_y(x, y)$ exists in a certain neighbourhood $(a-\delta, a+\delta\,;\, b-\delta, b+\delta)$ of (a, b). Let $(a+h, b+k)$ be any point of this neighbourhood. We have

$$f(a+h, b+k)-f(a, b)=f(a+h, b+k)-f(a+h, b)+f(a+h, b)-f(a, b). \quad \ldots(1)$$

The function $f(a+h, y)$ of y is derivable w.r. to y in the interval $(b, b+k)$. Therefore, by the mean value theorem,

$$f(a+h, b+k)-f(a+h, b)=kf_y(a+h, b+\theta k), \quad \ldots(2)$$

where θ, which lies between 0 and 1, depends upon h and k.

Now, if we write,

$$f_y(a+h, b+\theta k)-f_y(a, b)=\psi(h, k), \quad \ldots(3)$$

we see that, because of the condition (*ii*), $\psi(h, k)\to 0$ as

$$(h, k)\to(0, 0).$$

Again, because of the condition, (*i*), we have, when $h\to 0$,

$$\lim \frac{f(a+h, b)-f(a, b)}{h}=f_x(a, b),$$

so that if we write

$$[f(a+h, b)-f(a, b)]/h-f_x(a, b)=\varphi(h), \quad \ldots(4)$$

then $\varphi(h)\to 0$ as $h\to 0$.

From (1), (2), (3), (4), we obtain

$$f(a+h, b+k)-f(a, b)=hf_x(a, b)+kf_y(a, b)+h\varphi(h)+k\psi(h, k).$$

Hence the result.

Note. It may similarly be shown that if $f_y(a, b)$ exists and $f_x(x, y)$ is continuous at (a, b), then $f(x, y)$ is differentiable at (a, b).

Remarks. We have shown above that the *mere existence of one* partial derivative and the *continuity of the other is sufficient* for the differentiability of the function but, by considering an example, we now show that the condition of continuity is not *necessary* so that a function may be differentiable even though neither partial derivative is continuous.

Let

$$f(x, y)=x^2 \sin(1/x)+y^2 \sin(1/y), \text{ when } xy\neq 0.$$
$$f(x, 0)=x^2 \sin(1/x) \text{ when } x\neq 0,$$
$$f(0, y)=y^2 \sin(1/y) \text{ when } y\neq 0,$$
$$f(0, 0)=0.$$

It is easy to see that

$$f_x(x, y)=2x \sin(1/x)-\cos(1/x) \text{ when } x\neq 0 \text{ and } f_x(0, y)=0.$$
$$f_y(x, y)=2y \sin(1/y)-\cos(1/y) \text{ when } y\neq 0 \text{ and } f_y(x, 0)=0,$$

so that neither $f_x(x, y)$ nor $f_y(x, y)$ is continuous at the origin.

Again, we have

$$f(h, k)-f(0, 0)=h^2 \sin(1/h)+k^2 \sin(1/k)$$
$$=0h+0k+h\,.\,(h \sin 1/h)+k\,.\,(k \sin 1/k),$$

so that, since $(h \sin 1/h)$ and $(k \sin 1/k)$ both $\to 0$ as $(k, k)\to(0, 0)$, we see that $f(x, y)$ is differentiable at $(0, 0)$.

140. Differentiation and algebraic operations. Theorem. *If $f(x, y)$ and $g(x, y)$ are differentiable at (a, b), then*

$f(x, y)\pm g(x, y)$, $f(x, y)\, g(x, y)$ are also differentiable at (a, b)

and $$d(f\pm g)=df\pm dg, \quad d(fg)=fdg+gdf;$$

also if $g(a, b)\neq 0$, then $f(x, y)/g(x, y)$ is differentiable at (a, b)

and $$d(f/g)=(gdf-fdg)/g^2.$$

The proofs are simple.

141. Partial derivatives of the second and higher orders. Suppose that a function $f(x, y)$ possesses partial derivatives $f_x(x, y)$, $f_y(x, y)$ of the first order in a certain neighbourhood of (a, b).

Then we write

$$\lim_{h\to 0}\frac{f_x(a+h, b)-f_x(a, b)}{h}=f_{xx}(a, b),$$

$$\lim_{k \to 0} \frac{f_x(a, b+k)-f_x(a, b)}{k}=f_{yx}(a, b);$$

$$\lim_{h \to 0} \frac{f_y(a+h, b)-f_y(a, b)}{h}=f_{xy}(a, b),$$

$$\lim_{k \to 0} \frac{f_y(a, b+k)-f_y(a, b)}{k}=f_{yy}(a, b),$$

in case the limits exist.

$$f_{xx}(a, b), f_{yx}(a, b), f_{xy}(a, b), f_{yy}(a, b)$$

are known as partial derivatives of the second order at (a, b) and are also sometimes written as

$$(\partial^2 f/\partial x^2)(a, b)\ (\partial^2 f/\partial y \partial x)(a, b),\ (\partial^2 f/\partial x \partial y)(a, b),\ (\partial^2 f/\partial y^2)(a, b),$$

respectively. We may also write

$$f_{x^2}(a, b) \quad \text{and} \quad f_{y^2}(a, b)$$

in place of

$$f_{xx}(a, b) \text{ and } f_{yy}(a, b)$$

respectively.

The reader should carefully note the difference in the meanings of

$$f_{yx}(a, b) \text{ and } f_{xy}(a, b).$$

Partial derivatives of the third and higher orders can be similarly defined.

141·1. Change in the order of derivation. In general, a partial derivative has the same value in whatever order the different operations are performed. Thus, for example, in general, we have

$$f_{xy}=f_{yx}, f_{x^2y}=f_{xyx}, f_{x^2y^3}=f_{xy^2xy}.$$

That this is not always the case is shown below by considering an example.

141·2. It is easy to see *a priori* why $f_{yx}(a, b)$ may be different from $f_{xy}(a, b)$.

We have

$$f_{xy}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b)-f_y(a, b)}{h}.$$

Also $f_y(a+h, b)= \lim_{k \to 0} \dfrac{f(a+h, b+k)-f(a+h, b)}{k}$.

and $$f_y(a, b)= \lim_{k \to 0} \frac{f(a, b+k)-f(a, b)}{k}.$$

$$\therefore\ f_{xy}(a, b) = \lim_{h \to 0} \lim_{k \to 0} \frac{f(a+h, b+k)-f(a+h, b)-f(a, b+k)+f(a, b)}{hk}.$$

$$= \lim_{h \to 0} \lim_{k \to 0} \frac{\phi(h, k)}{hk}, \text{ say.}$$

It may similarly be shown that

$$f_{yx}(a, b) = \lim_{k \to 0} \lim_{h \to 0} \frac{\phi(h, k)}{hk}.$$

Thus we see that $f_{yx}(a, b)$ and $f_{xy}(a, b)$ are repeated limits of the same expression taken in different orders.

Ex. 1. *If* $f(x, y) = xy(x^2-y^2)/(x^2+y^2)$ when $(x, y) \neq (0, 0)$ *and*

$$f(0, 0) = 0,$$

show that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

We have

$$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(0+h, 0)-f_y(0, 0)}{h}.$$

But $\quad f_y(0, 0) = \lim\limits_{k \to 0} \dfrac{f(0, 0+k)-f(0, 0)}{k} = \lim\limits_{k \to 0} \dfrac{0}{k} = 0,$

and $\quad f_y(h, 0) = \lim\limits_{k \to 0} \dfrac{f(h, 0+k)-f(h, 0)}{k} = \lim\limits_{k \to 0} \dfrac{hk(h^2-k^2)}{k(h^2+k^2)} = h.$

$$\therefore\quad f_{xy}(0, 0) = \lim_{h \to 0} \frac{h-0}{h} = 1.$$

Again $f_{yx}(0, 0) = \lim\limits_{k \to 0} \dfrac{f_x(0, 0+k)-f_x(0, 0)}{k}.$

But $\quad f_x(0, 0) = \lim\limits_{h \to 0} \dfrac{f(0+h, 0)-f(0, 0)}{h} = 0,$

and $\quad f_x(0, k) = \lim\limits_{h \to 0} \dfrac{f(0+h, k)-f(0, k)}{h} = \lim\limits_{h \to 0} \dfrac{hk(h^2-k^2)}{h(h^2+k^2)} = -k.$

$$\therefore\quad f_{yx}(0, 0) = \lim_{k \to 0} \frac{-k-0}{k} = -1.$$

Thus $\quad f_{xy}(0, 0) \neq f_{yx}(0, 0).$

2. Given that

$f(x, y) = xy$ if $|y| \leqslant |x|$ and $f(x, y) = -xy$ if $|y| > |x|$;

show that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

3. Examine for change in the order of derivation at the origin for

(*i*) $f(x, y) = |x^2 - y^2|$.

(*ii*) $f(x, y) = \begin{cases} x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y), & x \neq 0,\ y \neq 0. \\ 0 \text{ elsewhere} \end{cases}$

4. Examine the equality of $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function $f(x, y)=\sqrt{(x^2+y^2)} \sin 2\varphi$, where $f(0, 0)=0$ and $\varphi=\tan^{-1}(y/x)$.

5. If $f(x, y)=(x^2+y^2)\tan^{-1}(y/x)$ when $x \neq 0$

and $f(0, y)=\pi y^2/2$,

show that

$$f_{xy}(0, 0) \neq f_{yx}(0, 0).$$

142. Sufficient condition for the validity of reversal in the order of derivation. We now prove two theorems which lay down *sufficient* conditions for the equality of f_{xy} and f_{yx}.

142·1. Schawarz's theorem. *If (a, b) be a point of the domain of a function $f(x, y)$ such that*

(i) $f_x(x, y)$ exists in a certain neighbourhood of (a, b) ;

(ii) $f_{xy}(x, y)$ is continuous at (a, b).

then

$$f_{yx}(a, b) \text{ exists and is equal to } f_{xy}(a, b).$$

The given conditions imply that there exists a certain neighbourhood of (a, b) at every point (x, y) of which $f_x(x, y)$, $f_y(x, y)$ and $f_{xy}(x, y)$ exist. Let $(a+h, b+k)$ be any point of this neighbourhood. We write

$$\varphi(h, k)=f(a+h, b+k)-f(a+h, b)-f(a, b+k)+f(a, b),$$
$$g(y)=f(a+h, y)-f(a, y),$$

so that

$$\varphi(h, k)=g(b+k)-g(b). \qquad \text{...(1)}$$

Since f_y exists in a neighbourhood of (a, b) the function $g(y)$ is derivable in $[b, b+k]$, and, therefore, by applying the mean value theorem to the expression on the right of (1), we have

$$\varphi(h, k)=kg'(b+\theta k) \qquad (0<\theta<1)$$
$$=k[f_y(a+h, b+\theta k)-f_y(a, b+\theta k)]. \qquad \text{...(2)}$$

Again since f_{xy} exists in a neighbourhood of (a, b), the function $f_y(x, b+\theta k)$ of x is derivable w.r. to x in $(a, a+h)$ and, therefore, by applying the mean value theorem to the right of (2), we have

$$\varphi(h, k)=hk\, f_{xy}(a+\theta' h, b+\theta k), \qquad (0<\theta'<1)$$

or

$$\frac{1}{k}\left[\frac{f(a+h, b+k)-f(a, b+k)}{h}-\frac{f(a+h, b)-f(a, b)}{h}\right]$$
$$=f_{xy}(a+\theta' h, b+\theta k).$$

Since $f_x(x, y)$ exists in a neighbourhood of (a, b), this gives when $h \to 0$,

$$\frac{f_x(a, b+k)-f_x(a, b)}{k}=\lim_{h\to 0} f_{xy}(a+\theta' h, b+\theta k).$$

Let, now, $k \to 0$. Since $f_{xy}(x, y)$ is continuous at (a, b), we obtain

$$f_{yx}(a, b) = \lim_{k \to 0} \lim_{h \to 0} f_{xy}(a+\theta' h, b+\theta k) = f_{xy}(a, b).$$

Cor. 1. *If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are both continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.*

142·2. Young's theorem. *If (a, b) be a point of the domain of definition of a function $f(x, y)$ such that $f_x(x, y)$ and $f_y(x, y)$ are both differentiable at $(a, b,)$ then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

The differentiability of f_x and f_y at (a, b) implies that they exist in a certain neighbourhood of (a, b) and that

$$f_{xx}, f_{yx}, f_{xy}, f_{yy}$$

exist at (a, b). Let $(a+h, b+h)$ be a point of this neighbourhood.

We write

$$\varphi(h, h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a, b),$$

$$g(y) = f(a+h, y) - f(a, y),$$

so that

$$\varphi(h, h) = g(b+h) - g(b). \qquad \ldots(1)$$

Since f_y exists in a neighbourhood of (a, b), the function $g(y)$ is derivable in $(b, b+h)$ and, therefore, by applying the mean value theorem to the expression on the right of (1), we obtain

$$\varphi(h, h) = hg'(b+\theta h) \qquad (0 < \theta < 1)$$

$$= h[f_y(a+h, b+\theta h) - f_y(b, b+\theta h)]. \qquad \ldots(2)$$

Since $f_y(x, y)$ is differentiable at (a, b), we have, by definition,

$$f_y(a+h, b+\theta h) - f_y(a, b) = hf_{xy}(a, b) + \theta h f_{yy}(a, b) + h\varphi_1(h, h) + \theta h\psi_1(h, h), \qquad \ldots(3)$$

and
$$f_y(a, b+\theta h) - f_y(a, b) = \theta h f_{yy}(a, b) + \theta h\psi_2(h, h) \qquad \ldots(4)$$

where

$$\varphi_1, \psi_1, \psi_2 \text{ all} \to 0 \text{ as } h \to 0.$$

From (2), (3) and (4), we obtain

$$\varphi(h, h)/h^2 = f_{xy}(a, b) + \varphi_1(h, h) + \theta\psi_1(h, h) - \theta\psi_2(h, h). \qquad \ldots(5)$$

By a similar argument and on considering

$$\phi(x) = f(x, b+h) - f(x, b),$$

we can show that

$$\varphi(h, h)/h^2 = f_{yx}(a, b) + \psi_3(h, h) + \theta'\varphi_2(h, h) - \theta'\varphi_3(h, h) \qquad \ldots(6)$$

where

$$\varphi_2, \varphi_3, \psi_3 \text{ all} \to 0 \text{ as } h \to 0.$$

Equating the right hand sides of (5) and (6) and making $h \to 0$, we obtain

$$f_{xy}(a, b)=f_{yx}(a, b).$$

Ex. 1. In view of the Schwarz's and Young's theorems, explain the inequality $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function considered in Ex. 1, after § 141, page 308.

[Show that neither $f_{xy}(x, y)$ nor $f_{yx}(x, y)$ is continuous at (0, 0) and that $f_p(x, y)$ and $f_y(x, y)$ are not differentiable at (0, 0)].

Explain the same inequality for the functions considered in Ex. 2, Ex. 3 and Ex. 4, pages 308-9 also.

2 If $f(x, y)=x^2y^2/(x^2+y^2)$ for $x^2+y^2\neq 0$ and is equal to zero otherwise, verify that $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ exist in a neighbourhood of (0, 0) but are not continuous at (0, 0) and yet are equal at (0, 0).

3. $f(x, y)=(x^2+y^2) \log (x^2+y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0)=0$; show that f_{xy} and f_{yx} are not continuous at (0, 0) but $f_{xy}(0, 0)=f_{yx}(0, 0)$.

143. Generalised Reversal Theorem. *If $f(x, y)$ possesses continuous partial derivatives of the nth order at a point (a, b), then*

$$f_{p_n p_{n-1}\cdots\cdots p_2 p_1}(a, b)=f_{q_n q_{n-1}\cdots\cdots q_2 q_1}(a, b),$$

where each p and q is either x or y and the number of x's among p's is the same as the number of x's among q's and similarly about the y's.

The proof which is simple, may be obtained by the principle of mathematical induction from the cor. of § 142·1, page 309.

For example, the theorem shows that when $n=3$,

$$f_{xxy}=f_{xyx}=f_{yxx}.$$

144. Differentials of second and higher orders. Let $z=f(x, y)$ be defined in a domain E and let it be differentiable at a point (x, y) of the domain. The differential, to be denoted by dz, of the function z at (x, y) is given by

$$dz=\frac{\partial z}{\partial x}\,\delta x+\frac{\partial z}{\partial y}\,\delta y, \qquad \ldots(1)$$

where we have taken δx, δy for h, k respectively and $\partial z/\partial x$, $\partial z/\partial y$ denote the partial derivatives of the function z at (x, y).

Taking $z=x$, we obtain from (1)

$$dx=dz=1 \,.\, \delta x+0 \,.\, \delta y=\delta x.$$

Similarly taking $z=y$, we obtain $dy=\delta y$.

Thus we see that the differentials dx, dy of the independent variables x, y are δx and δy respectively so that we can write

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy. \qquad \ldots(2)$$

Regarding dx and dy as constants, we see that the differential dz is a function of two variables x and y and is itself differentiable at (x, y) if $\partial z/\partial x$, $\partial z/\partial y$ are differentiable thereat, (§ 140, page. 306) and also then

$$d(dz)=d\left(\frac{\partial z}{\partial x}\right)dx+d\left(\frac{\partial z}{\partial y}\right)dy.$$

Replacing z by $\partial z/\partial x$ and $\partial z/\partial y$ in (2), we obtain

$$d\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}dx+\frac{\partial^2 z}{\partial y\,\partial x}dy,$$

$$d\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}dx+\frac{\partial^2 z}{\partial y^2}dy.$$

Since $\partial z/\partial x$, $\partial z/\partial y$ are differentiable at (x, y), we have, by Young's theorem,

$$\partial^2 z/\partial x\partial y=\partial^2 z/\partial y\partial x,$$

at (x, y).

Thus denoting $d(dz)$ by d^2z, we obtain

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\,\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2,$$

where

$$dx^2=(dx)^2,\ dy^2=(dy)^2.$$

For the sake of brevity, this is usually written as

$$d^2z=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^2 z,$$

whose meaning is self-evident.

Again d^2z is differentiable at (x, y), if the second order partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ are all differentiable thereat. This condition ensures the legitimacy of the inversion of the order of derivation w.r. to x and w.r. to y in the partial derivatives of the third order. Thus we have

$$d^3z=\frac{\partial^3 z}{\partial x^3}dx^3+3\frac{\partial^3 z}{\partial x^2\partial y}dx^2dy+3\frac{\partial^3 z}{\partial x\partial y^2}dxdy^2+\frac{\partial^3 z}{\partial y^3}dy^3$$

$$=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^3 z.$$

Proceeding in this manner, we see that d^nz exists if $d^{n-1}z$ is differentiable which is the case when all the partial derivatives of the $(n-1)$th order are differentiable. This condition also ensures the legitimacy of inverting the order of derivation w.r. to x and w.r. to y in the partial derivatives of the nth order. The expression for d^nz

in terms of partial derivatives of the nth order, as may be shown by *Mathematical induction*, is given by

$$d^n z = \frac{\partial^n z}{\partial x^n} dx^n + n \frac{\partial^n z}{\partial y \partial x^{n-1}} dy dx^{n-1} + \frac{n(n-1)}{2!} \frac{\partial^n z}{\partial y^2 \partial x^{n-2}} dy^2 dx^{n-2} + \ldots\ldots$$
$$\ldots\ldots + \frac{\partial^n z}{\partial y^n} dy^n,$$

i.e.,
$$d^n z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} \right)^n z.$$

145. Functions of Functions.

145·1. Differentials of first order.

If

$$x = \phi(u, v),\ y = \psi(u, v)$$

be two functions of (u, v) defined in a domain E of the point (u, v), then the domain E_1 of the point (x, y) as (u, v) varies in E may be referred to as the *image* of E.

Theorem. *If*

(*i*) *$x = \phi(u, v)$, $y = \psi(u, v)$ are two functions of (u, v) defined in a domain E and differentiable at a point (u, v),*

(*ii*) *$z = f(x, y)$ is a function of (x, y) defined in a domain E_1, and differentiable at a point (x, y),*

(*iii*) *E_1 is the image of E,*

then

z regarded as a function of u, v, is differentiable at (u, v) and

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

[This theorem shows that the expression for dz is the same whether the variables x, y are independent or dependent upon some other variables. Of course dx, dy are not constants when x, y are dependent variables.]

Let (u, v) and $(u+\delta u,\ v+\delta v)$ be two points of E, and let (x, y) and $(x+\delta x,\ y+\delta y)$ be the two corresponding points of E_1 so that

$$\delta x = \phi(u+\delta u,\ v+\delta v) - \phi(u, v),\ \delta y = \psi(u+\delta u,\ v+\delta v) - \psi(u, v).$$

Because of the differentiability, $\phi(u, v)$ and $\psi(u, v)$ are continuous functions of u and v, *i.e.*, $\delta x \to 0$ and $\delta y \to 0$ as $(\delta u, \delta v) \to (0, 0)$.

Since $x = \phi(u, v)$, $y = \psi(u, v)$ are differentiable at (u, v), therefore

$$\delta x = \phi_u \delta u + \phi_v \delta v + \phi_1 \delta u + \phi_2 \delta v, \qquad \ldots(1)$$
$$\delta y = \psi_u \delta u + \psi_v \delta v + \psi_1 \delta u + \psi_2 \delta v, \qquad \ldots(2)$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are functions of δu, δv and $\to 0$ as $(\delta u, \delta v) \to (0, 0)$.

Also since $z = f(x, y)$ is differentiable at (x, y), therefore

$$\delta z = f_x \delta x + f_y \delta y + f_1 \delta x + f_2 \delta y, \qquad \ldots(3)$$

where f_1, f_2 are functions of δx, δy and $\to 0$, as $(\delta x, \delta y) \to (0, 0)$.

From (1), (2), (3), we obtain

$$\delta z=(f_x\phi_u+f_y\psi_u)\delta u+(f_x\phi_v+f_y\psi_v)\delta v+F_1\delta u+F_2\delta v,$$

where

$$F_1=(f_x\phi_1+f_y\psi_1+f_1\phi_u+f_1\phi_1+f_2\psi_u+f_2\psi_1),$$

and

$$F_2=(f_x\phi_2+f_y\psi_2+f_1\phi_v+f_1\phi_2+f_2\psi_v+f_2\psi_2).$$

Since the co-efficients F_1 and F_2 of δu and $\delta v \to 0$ as

$$(\delta u, \delta v) \to (0, 0),$$

we see that z is a differentiable function of u, v at (u, v) and

$$\begin{aligned} dz &= (f_x\phi_u+f_y\psi_u)du+(f_x\phi_v+f_y\psi_v)dv \\ &=f_x(\phi_u du+\phi_v dv)+f_y(\psi_u du+\psi_v dv) \\ &=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy. \end{aligned}$$

145·2. Differentials of higher orders. From the preceding theorem, we deduce that if $\partial z/\partial x$, $\partial z/\partial y$ are differentiable functions of x, y at (x, y) so that they are also differentiable functions of u, v at (u, v) and dx, dy are differentiable functions of u, v at (u, v) (*i.e.*, d^2x, d^2y exist), then dz is a differentiable function of u, v at (u, v) and we have

$$d^2z=d(dz)=d\left(\frac{\partial z}{\partial x}\right)dx+\frac{\partial z}{\partial x}d(dx)+d\left(\frac{\partial z}{\partial y}\right)dy+\frac{\partial z}{\partial y}d(dy)$$

(§ 140, page 306)

$$=\frac{\partial^2 z}{\partial x^2}\,dx^2+2\,\frac{\partial^2 z}{\partial x\,dy}dxdy+\frac{\partial^2 z}{\partial y^2}\,dy^2+\frac{\partial z}{\partial x}\,d^2x+\frac{\partial z}{\partial x}\,d^2y.$$

Proceeding in this manner, we see that $d^{n-1}z$ is a differentiable function of u, v at (u, v), *i e.*, d^nz exists, if the $(n-1)$th order partial derivatives of z are differentiable functions of x and y at (x, y) and the nth differentials d^nx, d^ny of x, y exist at (x, y). But the formation of differentials of higher orders becomes more and more complicated and no simple general formula exists for d^nz in this case.

Note. In case x, y are *linear functions of u and v, i.e.*, x, y are of the form $x=a+bu+cv$, $y=d+eu+fv$, then d^2x, d^2y and all higher differentials of x and y are 0 and, therefore, we have

$$d^nz=\left(dx\,\frac{\partial}{\partial x}+dy\,\frac{\partial}{\partial y}\right)^n z,$$

so that the form is the same as if x and y are independent.

145·3. Derivation. *If*

(i) $x=\varphi(u, v)$, $y=\psi(u, v)$ possess continuous first order partial derviatives at a point (u, v) of the domain E,

(ii) $z=f(x, y)$ possesses continuous first order partial derivatives at a point (x, y) of the domain E_1,

and (iii) E_1 is the image of E,

then, z, possesses continuous first order partial derivatives w.r. to u and v at (u, v) ; and also

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u},$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

Because of (*i*), x, y are differentiable functions of u, v and because of (*ii*), z is a differentiable function of x, y. Hence, from § 145·1, z is a differentiable function of (u, v). Therefore $\partial z/\partial u$, $\partial z/\partial v$ exist and

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv. \qquad ...(1)$$

Also from § 145·1, we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

$$=\frac{\partial z}{\partial x}\cdot\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)$$

$$=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv. \quad ..(2)$$

$\therefore$ From (1) and (2), we obtain

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u},\quad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Again, because of the conditions (*i*) and (*ii*), we see that $\partial z/\partial u$ and $\partial z/\partial v$ are continuous functions of (u, v).

Cor. 1. If (*i*) $x=\varphi(u, v)$, $y=\psi(u, v)$ possess continuous nth order partial derivatives at a point (u, v) of the domain E, (*ii*) $z=f(x, y)$ possesses continuous nth order partial derivatives at a point (x, y) of the domain E_1 and (*iii*) E_1 is the image of E, then z possesses continuous nth order partial derivatives w.r. to u and v at the point (u, v) of the domain E.

Because of (*i*) and (*ii*), $d^{n-1}x$, $d^{n-1}y$ are differentiable at (x, y) in E and $d^{n-1}z$ at (u, v) in E_1 so that the result now follows from § 145·2.

Cor. 2. A particular case. *If $z=f(x, y)$ possesses nth order partial derivatives and $x=a+ht$, $y=b+kt$, where a, b, h, k are constants then*

$$\frac{d^nz}{dt^n}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n z.$$

We have

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=h\frac{\partial z}{\partial x}+k\frac{\partial z}{\partial y},$$

$$\frac{d^2z}{dt^2}=h\left[\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y\partial x}\frac{dy}{dt}\right]+k\left[\frac{\partial^2 z}{\partial x dy}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right]$$

$$=h_2\frac{\partial^2 z}{\partial x^2}+2hk\frac{\partial^2 z}{\partial x\partial y}+k^2\frac{\partial^2 z}{\partial y^2}=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 z.$$

By induction, we may now obtain the value of d^nz/dt^n.

146. Changes of Variables. Calculation of Second Order partial derivatives. *If $z=f(x, y)$ be a function of two variables x and y and if u, v be two new variables connected with x and y by the relations*

$$x=\varphi(u, v),\ y=\psi(u, v);$$

to express the first and second order partial derivatives of z with respect to x and y in terms of u, v and the partial derivatives of z with respect to u and v.

Here it is understood that f, φ, ψ possess continuous partial derivatives w.r. to the corresponding variables.

By the rule of § 145·3, we have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v},$$

whence we obtain

$$\frac{\partial z}{\partial x}=A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v},$$

$$\frac{\partial z}{\partial y}=C\frac{\partial z}{\partial u}+D\frac{\partial z}{\partial v},$$

where

$$A=\frac{\partial y}{\partial v}\bigg|\frac{\partial(x, y)}{\partial(u, v)},\quad B=-\frac{\partial y}{\partial u}\bigg|\frac{\partial(x, y)}{\partial(u, v)};$$

$$C=-\frac{\partial x}{\partial u}\bigg|\frac{\partial(x, y)}{\partial(u, v)},\quad D=\frac{\partial x}{\partial v}\bigg|\frac{\partial(x, y)}{\partial(u, v)},$$

are known functions of u and v.

Here $\dfrac{\partial(x, y)}{\partial(u, v)}$ stands for $\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$.*

*A general study of such determinants known as *Jacobians* is made late on in this chapter.

Thus we see that the *derivatives of z with respect to x is the sum of the two products formed by multiplying the derivatives of z with respect to u and v by the functions A, B respectively* which depends upon the equations of transformation only. Similar rule holds good for $\partial z/\partial y$. These rules may be expressed as :

$$\frac{\partial}{\partial x}=A\frac{\partial}{\partial u}+B\frac{\partial}{\partial v}, \quad \frac{\partial}{\partial y}=C\frac{\partial}{\partial u}+D\frac{\partial}{\partial v}.$$

In order to calculate the second derivatives, we apply the above rule to the first derivatives and obtain

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)\\ &=\left(A\frac{\partial}{\partial u}+B\frac{\partial}{\partial v}\right)\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)\\ &=A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)+B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u}+B\frac{\partial z}{\partial v}\right)\\ &=A^2\frac{\partial^2 z}{\partial u^2}+2AB\frac{\partial^2 z}{\partial u\partial v}+B^2\frac{\partial^2 z}{\partial v^2}+\\ &\quad\left(A\frac{\partial A}{\partial u}+B\frac{\partial A}{\partial v}\right)\frac{\partial z}{\partial u}+\left(A\frac{\partial B}{\partial u}+B\frac{\partial B}{\partial v}\right)\frac{\partial z}{\partial v}.\end{aligned}$$

The derivatives $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ can be similarly obtained.

Example

If z be a function of the two variables x, y and

$$x=r\cos\theta,\ y=r\sin\theta,$$

prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}+\frac{1}{r}\frac{\partial z}{\partial r},$$

We have

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}=\cos\theta\frac{\partial z}{dx}+\sin\theta\frac{\partial z}{\partial y},$$

$$\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta}=-r\sin\theta\frac{\partial z}{\partial x}+r\cos\theta\frac{\partial z}{\partial y}.$$

These give

$$\frac{\partial z}{\partial x}=\sin\theta\,.\,\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\,.\,\frac{\partial z}{\partial\theta},$$

$$i.e.,\ \frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}.$$

$$\frac{\partial z}{\partial y}=\sin\theta\,.\,\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\,.\,\frac{\partial z}{\partial\theta},$$

$$i.e.,\quad \frac{\partial}{\partial y}=\sin\theta\,\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\,.\,\frac{\partial}{\partial\theta}.$$

Hence

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$=\left(\cos\theta\,\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right)\left(\cos\theta\,\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\,\frac{\partial z}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 z}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\,\frac{\partial^2 z}{\partial r\partial y}+\frac{\sin^2\theta}{r^2}\,\frac{\partial^2 z}{\partial\theta^2}+$$

$$\frac{2\cos\theta\sin\theta}{r^2}\,.\,\frac{\partial z}{\partial\theta}+\frac{\sin^2\theta}{r}\,.\,\frac{\partial z}{\partial r}.$$

Similary

$$\frac{\partial^2 z}{\partial y^2}=\sin^2\theta\,\frac{\partial^2 z}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\,.\,\frac{\partial^2 z}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\,.\,\frac{\partial z}{\partial\theta^2}-$$

$$\frac{2\cos\theta\sin\theta}{r^2}\,.\,\frac{\partial z}{\partial\theta}+\frac{\cos^2\theta}{r}\,.\,\frac{\partial z}{\partial r}.$$

On adding, we get the result as given.

Exercises

1. If z be a function of two variables x and y and

$$x=c\cosh u\cos v,\ y=c\sinh u\sin v,$$

prove that

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\tfrac{1}{2}c^2\left(\cosh 2u-\cos 2v\right)\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)$$

2. Express $(x^2+y^2)^{-1}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)$ in terms of the derivatives of, f, with respect to u and v where $u=x^2-y^2$, $v=2xy$, $f(u,v)=\varphi(x,y)$.

Deduce that the most general function of x, y satisfying

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0,$$

is $axy+b$, where a and b are constants and find the most general function of $xy(x-y^2)$ satisfying the same equation.

3. A function $f(x,y)$, when expressed in terms of the new variables u, v defined by the equations

$$x=\tfrac{1}{2}(u+v),\ y=uv,$$

becomes $g(u,v)$; prove that

$$\frac{\partial^2 g}{\partial u\partial v}=\frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2}+2\frac{x}{y}\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial^2 f}{\partial y^2}+\frac{1}{y}\frac{\partial f}{\partial y}\right).$$

4. If $F(u,v)$ is a twice differentiable function of (u,v) and if $u=x^2-y^2$, $v=2xy$, prove that

$$4(u^2+v^2)F_{uv}+2uF_v+2vF_u=xy(f_{xx}-f_{yy})+(x^2-y^2)f_{xy}.$$

5. Show that if $x=u^2v$, $y=v^2u$, then

$$2x^2\frac{\partial^2 f}{\partial x^2}+2y^2\frac{\partial^2 f}{\partial y^2}+5xy\frac{\partial^2 f}{\partial x\partial y}=uv\frac{\partial^2 f}{\partial u\partial v}-\frac{2}{3}\left(u\frac{\partial f}{\partial u}+v\frac{\partial f}{\partial v}\right).$$

6. Given that z is a function of x and y and that

$$x=e^u+e^{-v},\quad y=e^v+e^{-u},$$

prove that

$$\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

147. Taylor's theorem for a function of two variables. *If $f(x, y)$ possesses continuous partial derivatives of the nth order in any neighbourhood of a point (a, b) and if $(a+h, b+k)$ be any point of this neighbourhood then, there exists a position number, θ, which is less than 1 such that*

$$f(a+h,b+k)=f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b)+$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+$$

$$\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,b+\theta k),$$

$$(0<\theta<1)$$

or equivalently

we have

$$g^n(t)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(x,y).$$

Applying Maclaurin's theorem to $g(t)$, we obtain

$$g(t)=g(0)+tg'(0)+\frac{t^2}{2!}g''(0)+\ldots\ldots+\frac{t^{n-1}}{(n-1)!}g^{n-1}(0)+\frac{t^n}{n!}g^n(\theta t),$$

$$(0<\theta<1).$$

For $t=1$, this becomes

$$g(1)=g(0)+g'(0)+\frac{1}{2!}g''(0)+\ldots\ldots+\frac{1}{(n-1)!}g^{n-1}(0)+\frac{1}{n!}g^n(\theta).$$

Since

$$g(1)=f(a+h,\ b+k), \qquad g(0)=f(a,\ b),$$

$$g'(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,\ b),\ g''(0)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,\ b)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$g^n(\theta)=\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h,\ b+\theta k),$$

we get the result as stated.

It is easy to see how Taylor's theorem can be written more compactly as

$$f(a+h,\ b+k)=f(a,\ b)+df(a,\ b)+\frac{1}{2\,!}\,d^2 f(a,\ b)+\ldots$$

$$+\ldots\ldots+\frac{1}{(n-1)\,!}\,d^{n-1} f(a,\ b)+\frac{1}{n\,!}\,d^n f(a+\theta h,\ b+\theta k).$$

Ex. 1. Show that

(*i*) $\sin x \sin y = xy-\frac{1}{6}[(x^3+xy^2)\cos\theta x\sin\theta y+(y^3+3x^2 y)\sin\theta x\sin\theta y]$
$(0<\theta<1)$.

(*ii*) $e^{ax}\sin by=by+abxy+\frac{1}{6}[(a^3x^3-3ab^2xy^2)\sin v+(3a^2bx^2y-b^3y^3)\cos v]$
where $u=a\theta x,\ v=b\theta y,\ (0<\theta<1)$.

2. *If* $f(x,\ y)=\sqrt{|xy|}$, *show that*

$$f_x(x,\ y)=\begin{cases}\dfrac{1}{2}\dfrac{\sqrt{|y|}}{\sqrt{|x|}} & \text{if } x>0.\\[2ex] -\dfrac{1}{2}\dfrac{\sqrt{|y|}}{\sqrt{|x|}} & \text{if } x<0;\end{cases}$$

$$f_y(x,\ y)=\begin{cases}\dfrac{1}{2}\dfrac{\sqrt{|x|}}{\sqrt{|y|}} & \text{if } y>0.\\[2ex] -\dfrac{1}{2}\dfrac{\sqrt{|x|}}{\sqrt{|y|}} & \text{if } y<0;\end{cases}$$

and prove that Taylor's expansion of $f(x,\ y)$ *is not valid about the point* $(x,\ x)$ *in any region enclosing the origin.*

We have

$$f_x(x,\ y)=\lim_{h\to 0}\frac{f(x+h,\ y)-f(x,\ y)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{|x+h|}\sqrt{|y|}-\sqrt{|x|}\sqrt{|y|}}{h}$$

$$=\lim_{h\to 0}\frac{|x+h|\ |y|-|x|\ |y|}{h[\sqrt{|x+h|}\sqrt{|y|}+\sqrt{|x|}\sqrt{|y|}]}.$$

Suppose that, x, is positive. We can then take, h, so small that $x+h$ is also positive. Thus, for $x > 0$,

$$f_x(x, y) = \lim_{h \to 0} \frac{(x+h) \mid y \mid - x \mid y \mid}{h[\sqrt{\mid x+h \mid}\sqrt{\mid y \mid} - \sqrt{\mid x \mid}\sqrt{\mid y \mid}]}$$

$$= \lim_{h \to 0} \frac{\mid y \mid}{\sqrt{\mid x+h \mid}\sqrt{\mid y \mid} + \sqrt{\mid x \mid}\sqrt{\mid y \mid}}$$

$$= \frac{1}{2}\frac{\sqrt{\mid y \mid}}{\sqrt{\mid x \mid}}.$$

Similarly other cases may be disposed of.

Now, by Taylor's theorem, with $n=1$, we have

$$f(x+h, x+h) = f(x, x) + h[f_x(x+\theta h, x+\theta h) + f_y(x+\theta h, x+\theta h)],$$

or

$$\mid x+h \mid = \begin{cases} \mid x \mid + h, & \text{if } x+\theta h > 0 \\ \mid x \mid - h, & \text{if } x+\theta h < 0 \\ \mid x \mid, & \text{if } x+\theta h = 0. \end{cases} \qquad \ldots(1)$$

If the rectangle $(x, x;\ x+h, x+h)$ is to enclose the origin, then x and $x+h$ must have opposite signs so that either

$$\mid x+h \mid = x+h,\ \mid x \mid = -x,$$

or

$$\mid x+h \mid = -(x+h),\ \mid x \mid = x.$$

It may now be seen that none of the inequalities (1) can be true.

Extreme values.

148. Maxima and Minima for functions of two variables. Let (a, b) be any* inner point of the domain of a function $f(x, y)$. *Then* $f(a, b)$ *is said to be an extreme value of* $f(x, y)$, *if, for every point* (x, y), *[other than* (a, b)*], of some neighbourhood of* (a, b), *the difference*

$$f(x, y) - f(a, b)$$

is of the same sign; the extreme value of $f(a, b)$ *being called a maximum or a minimum value according as this difference is negative or positive.*

148·1. The necessary conditions *for* $f(a, b)$ *to be an extreme value of* $f(x, y)$ *are that*

$$f_x(a, b) = 0 = f_y(a, b),$$

provided that these partial derivatives exist.

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*A point (a, b) is said to be an *inner point* of a domain every point of some neighbourhood of (a, b) is a point of the domain.

If, $f(a, b)$, is an extreme value of the function $f(x, y)$ of two variables then, clearly, it is also an extreme value of the function $f(x, b)$ of one variable x for $x=a$ and therefore its derivative $f_x(a, b)$ for $x=a$, in case it exists, must necessarily be 0. Similarly we have $f_y(a, b)=0$.

Note. If $f(x, y)=0$. when $x=0$ or $y=0$ and $f(x, y)=1$ elsewhere, then $f_x(0, 0)=0=f_y(0, 0)$ but $f(0, 0)$ is not an extreme value so that we see that the conditions obtained above are only necessary and not sufficient.

Ex. Show that $f(x, y)=|x|+|y|$ has an extreme value at (0, 0) even though $f_x(0, 0)$, $f_y(0, 0)$ do not exist.

148·2. Sufficient conditions *for $f(a, b)$ to be an extreme value of $f(x, y)$.*

We suppose that

$$f_x(a, b)=0=f_y(a, b).$$

Also we suppose that $f(x, y)$ possesses continuous second order partial derivatives in a certain neighbourhood of (a, b) and that these derivatives at (a, b) *viz.*,

$$f_{xx}(a, b),\ f_{xy}(a, b),\ f_{yy}(a, b)$$

are not all zero.

We write

$$A=f_{xx}(a, b),\ B=f_{xy}(a, b),\ C=f_{yy}(a, b).$$

By Taylor's theorem, we have

$$f(a+h, b+k)=f(a, b)+[hf_x(a, b)+kf_y(a, b)]+$$
$$\frac{1}{2!}[h^2 f_{xx}(\alpha, \beta)+2hkf_{xy}(\alpha, \beta)+k^2f_{yy}(\alpha, \beta)],$$

where $\alpha=a+\theta h$, $\beta=b+\theta k$ and $0<\theta<1$.

We write

$$f_{xx}(a+\theta h, b+\theta k)-f_{xx}(a, b)=\rho_1,$$
$$f_{xy}(a+\theta h, b+\theta k)-f_{xy}(a, b)=\rho_2,$$
$$f_{yy}(a+\theta h, b+\theta k)-f_{yy}(a, b)=\rho_3,$$

so that ρ_1, ρ_2, ρ_3, are functions of h and k and $\to 0$ as (h, k), $\to (0, 0)$.

$$\therefore\quad f(a+h, b+k)-f(a, b)=\tfrac{1}{2}[Ah^2+2Bhk+Ck^2+\rho(h^2+k^2)],$$

where ρ is a function of h, k defined by

$$\rho(h^2+k^2)=h^2\rho_1+2hk\rho_2+k^2\rho_3.$$

Since

$$|\rho| \leqslant \frac{h^2}{h^2+k^2}|\rho_1|+2\frac{|hk|}{h^2+k^2}|\rho_2|+\frac{k^2}{h^2+k^2}|\rho_3|$$
$$\leqslant |\rho_1|+|\rho_2|+|\rho_3|,$$

we see that $\rho \to 0$ as $(h, k)\to(0, 0)$.

Writing

$$h=r\cos\varphi,\ k=r\sin\varphi,$$

we obtain

$$f(a+h,\ b+k)-f(a,\ b)=\tfrac{1}{2}r^2[A\cos^2\varphi+2B\sin\varphi\cos\varphi+C\sin^2\varphi+\rho). \quad \text{...(1)}$$

It will be seen that the sign of the difference

$$f(a+h,\ b+k)-f(a,\ b)$$

in general, depends upon the nature of the function

$$G(\varphi)=A\cos^2\varphi+2B\sin\varphi\cos\varphi+C\sin^2\varphi.$$

Accordingly, we first determine conditions for the function

$$G(\phi)=A\cos^2\varphi+2B\sin\varphi\cos\phi+C\sin^2\phi$$

of ϕ to be *definite, semi-definite, or indefinite.*

(The definitions of definite, semi-definite and indefinite functions are given in a note at the end of this section.)

I. *Let* $A\neq 0$. Then

$$G(\varphi)=[(A\cos\varphi+B\sin\varphi)^2+(AC-B^2)\sin^2\varphi]/A.$$

If $(AC-B^2)>0$, *then* $G(\varphi)$ *has always the sign of* A *and is, therefore, definite.*

The only possibility of the vanishing of $G(\varphi)$ arises when $\sin\varphi=0$ as well as $A\cos\varphi+B\sin\varphi=0$, *i.e.*, when $\sin\varphi=0$ and $\cos\varphi=0$ but that is impossible.

If $(AC-B^2)=0$ *then the function is semi-definite,* for it vanishes when $A\cos\varphi+B\sin\varphi=0$ and has for every other value of φ, the sign of A.

If $(AC-B^2)<0$, *then the function is indefinite,* for it assumes values of different signs

when $\sin\varphi=0$ and when $A\cos\varphi+B\sin\varphi=0$.

II. *Let* $C\neq 0$. As in the preceding case, we may see that

if $(AC-B^2)>0$, then $G(\varphi)$ has the sign of C and is definite;

if $(AC-B^2)=0$, $G(\varphi)$ is semi-definite.

if $(AC-B^2)<0$, $G(\phi)$ is indefinite.

We may also see that if $AC-B^2>0$, then A and C must both be of the same sign.

III. *Let* $A=0, C=0, B\neq 0$. Then

$$G(\varphi)=2B\sin\varphi\cos\varphi=B\sin 2\varphi$$

is clearly indefinite.

Thus we see that $G(\varphi)$ *is definite if, and only if,* $(AC-B^2)>0$; *also, then, it is positively or negatively definite according as* A *(or* C*) is positive or negative.*

We now return to the equality (1) *viz.*,

$$f(a+h,\ b+k)-f(a,\ b)=\tfrac{1}{2}r^2[G(\varphi)+\rho].$$

Let $G(\varphi)$ be positively definite. There exists, in this case, a positive number, m, the lower bound of the continuous function $G(\varphi)$, such that for every value of φ,

$$m \leqslant G(\varphi).$$

Also, since $\rho \to 0$ as $(h,k) \to (0,0)$, there exists a positive number δ such that, when $|h| \leqslant \delta$, $|k| \leqslant \delta$, we have

$$|\rho| \leqslant \tfrac{1}{2}m, \text{ i.e., } -\tfrac{1}{2}m \leqslant \rho \leqslant \tfrac{1}{2}m.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$, we have

$$G(\varphi)+\rho \geqslant m-\tfrac{1}{2}m=\tfrac{1}{2}m > 0.$$

Hence $f(a,\ b)$ is a minimum value of $f(x,\ y)$ in this case.

Let $G(\varphi)$ be negatively definite. There exists in this case a negative number, M, the upper bound of the continuous function $G(\varphi)$, such that for every value of φ,

$$G(\varphi) \leqslant M.$$

Also there exists a positive number δ such that when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$|\rho| \leqslant -\tfrac{1}{2}M, \text{ i.e., } \tfrac{1}{2}M \leqslant \rho \leqslant -\tfrac{1}{2}M.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$G(\varphi)+\rho \leqslant \tfrac{1}{2}M < 0.$$

Hence $f(a,\ b)$ is a maximum value of $f(x,\ y)$ in this case.

Let $G(\varphi)$ be semi-definite. This case is doubtful, for the sign of $f(a+h,\ b+k)-f(a,\ b)$ depends upon ρ.

Let $G(\varphi)$ be indefinite. Let φ_1 and φ_2 be any two numbers such that

$$G(\varphi_1) > 0,\ G(\varphi_2) < 0.$$

There surely exists a positive number m such that

$$G(\varphi_1) > m,\ G(\varphi_2) < -m,$$

We choose a positive number δ such that when

$$|h| \leqslant \delta \text{ and } |k| \leqslant \delta,$$

$$|\rho| < \tfrac{1}{2}m, \text{ i.e.,} -\tfrac{1}{2}m < \rho < \tfrac{1}{2}m.$$

Thus when $|h| \leqslant \delta$, $|k| \leqslant \delta$,

$$G(\varphi_1)+\rho > \tfrac{1}{2}m > 0 \text{ and } G(\varphi_2)+\rho < -\tfrac{1}{2}m < 0,$$

so that in every neighbourhood of $(a,\ b)$, there exist points $(a+h,\ b+k)$ for which the difference $f(a+h,\ b+k)-f(a,\ b)$ has different signs. Hence $f(a,\ b)$ is not an extreme value in this case.

Rule. *If, at a point* (a, b),

$$f_x(a, b)=0=f_y(a, b)$$

and

$$f_{xx}(a, b)\, f_{yy}(a, b)-[f_{xy}(a, b)]^2 > 0,$$

then $f(a, b)$ *is an extreme value which is a maximum or a minimum according as* $f_{xx}(a, b) < 0$ *or* > 0 *(and consequently* $f_{yy}(a, b) < 0$ *or* > 0*).*

Note. Any function $A=B=C=0$ or the doubtful case which arises when $AC-B^2=0$ or when $A=B=0$ and $C \neq 0$ require more elaborate consideration where we have to employ Taylor's theorem with remainder after three or more terms but this consideration is beyond the scope of this book.

Note. The function $G(\varphi)$ of φ is said to be *definite* if for every value of φ, it assumes values with the same sign ; also a definite function is said to be positively or negatively definite according as the sign is positive or negative.

2. The function is said to be *semi-definite* if it vanishes for some value or values of φ and when not 0, has always one sign.

3. The function is said to be *indefinite* if it can assume values which are of different signs.

Note. A point (a, b) is said to be a stationary point of $f(x, y)$, if

$$f_x(a, b)=0=f_y(a, b).$$

148·3. Another presentation of the rule.

Since

$$df(a, b)=hf_x(a, b)+kf_y(a, b),$$

and

$$\begin{aligned} d^2f(a, b) &= h^2 f_{xx}(a, b)+2hkf_{xy}(a, b)+k^2 f_{yy}(a, b) \\ &= Ah^2+2Bhk+Ck^2 \\ &= \rho[A\cos^2\varphi+2B\sin\varphi\cos\varphi+C\sin^2\phi], \end{aligned}$$

we see that $f(a, b)$ *will be an extreme value of* $f(x, y)$, *if at* (a, b) *the first differential* $df=0$ *and the second differential* d^2f *is of invariable sign (i.e., is definite) for all values of* $(h, k) \neq (0, 0)$, *where of course it is zero.*

Consider the two by two square array, (2 × 2 square matrix),

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

which is said to be the array or matrix of the quadratic form

$$d^2f=Ah^2+2Bhk+Ck^2.$$

Thus we have shown that d^2f will be positive definite if and only if the principal minors

$$A, \quad \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC-B^2,$$

of the array are both positive and it will be negatively definite if these principal minors are respectively negative and positive.

Examples

1. *Find all the maxima and minima of the function*

$$x^3+y^3-63(x+y)+12xy.$$

Writing

$$f(x, y)=x^3+y^3-63(x+y)+12xy,$$

we have

$$f_x=3x^2-63+12y,\ f_y=3y^2-63+12x\ ;$$
$$f_{xx}=6x,\ f_{xy}=12,\ f_{yy}=6y.$$

The equations

$$f_x=3x^2-63+12y=0,\ f_y=3y^2-63+12x=0,$$

are equivalent to

$$\left.\begin{aligned} f_x=3x^2-63+12y=0 \\ f_x-f_y=3(x^2-y^2)+12(y-x)=0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x^2-21+4y=0 \\ (y-x)(4-y-x)=0 \end{aligned}\right\}$$

which are further equivalent to

$$\left.\begin{aligned} x^2-21+4y=0 \\ y-x=0 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} x^2-21+4y=0 \\ 4-y-x=0 \end{aligned}\right\}$$

These give

$$(-7, -7),\ (3, 3),\ (5, -1),\ (-1, 5)$$

as 4 pairs of solutions. Thus the function has four stationary points.

At $(-7, -7)$,

$$f_{xx}=-42 < 0 \text{ and } f_{xx}f_{yy}-(f_{xy})^2=1620 > 0,$$

so that the function is maximum at $(-7, -7)$.

At $(3, 3)$,

$$f_{xx}=18 > 0 \text{ and } f_{xx}\, f_{yy}-(f_{xy})^2=180 > 0,$$

so that the function is minimum at $(3, 3)$.

At each of the other points $(5, -1)$ and $(-1, 5)$,

$$f_{xx}\, f_{yy}-(f_{xy})^2=-324 < 0$$

so that the function is neither a maximum nor a minimum.

2. *Prove that the function*

$$f(x, y)=x^2-2xy+y^2+x^3-y^3+x^5$$

has neither a maximum nor a minimum at the origin.

It may be easily seen that (0, 0),

$$f_x=0, \quad f_y=0,$$

$$f_{xx}=2, f_{xy}=-2, f_{yy}=2 \text{ so that } f_{xx} f_{yy}-(f_{xy})^2=0.$$

Thus we need to further examine the question. We have

$$f(0, 0)=0.$$

Since

$$f(x, y)=(x-y)^2+(x-y)(x^2+xy+y^2)+x^5,$$

we see that for points along the line $y-x=0$, we have

$$f(x, y)=x^5$$

and accordingly in every neighbourhood of (0, 0) there are points wherefor $f(x, y)$ is positive and there are points wherefor $f(x, y)$ is negative.

Hence $f(0, 0)$ is neither a maximum nor a minimum value.

Exercises

1. Examine the following for extreme values :

(i) $y^2+4xy+3x^2+x^3$. (ii) $3x^4y^2-5x^2y^2+3x^4+2y^3-6x^2-3y^2+1$.

(iii) $y^2+x^2y+ax^4$. (iv) $x^2+xy+y^2+ax+by$.

(v) $x^3y^3(12-3x-4y)$. (vi) $xy(a-x-y)$.

(vii) $x^3-y^3+3x^2+3y^2-9x$. (viii) $(x^2+y^2)^2-a^2(x^2-y^2)$.

(ix) $(x^2+y^2-4)^2-x^2$. (x) $(x^2+y^2-1)(xy+4)$.

(xi) $3x^4y^2-6x^2y^2+3x^4+2y^3-6x^2-3y^2+1$.

(xii) $(x^2+y^2)\, e^{6x+2x^2}$. (xiii) $(x-y)^2(x^2+y^2-2)$.

2. Show that

$$f(x, y)=(y-x)^4+(x-2)^4,$$

has a minimum at (2, 2).

3. Show that

$$f(x, y)=y^2+x^2y+x^4,$$

has a minimum at (0, 0) where $f_{xx} f_{yy}-(f_{xy})^2=0$.

4 Show that

$$f(x, y)=2x^4-3x^2y+y^2,$$

has neither a maximum nor a minimum at (0, 0) where $f_{xx} f_{yy}-(f_{xy})^2=0$.

5. Determine for each of the following functions whether it has a maximum or a minimum at the origin :

(i) $x^2+3xy+y^2+x^3+y^3$.

(ii) $x^2+2xy+y^2+x^2+x^2y-xy^2-y^3+x^4$.

(iii) $x^2+4xy+4y^2+x^3+2x^2y+y^4$.

6. Show that there are five sets of real values of x and y for which the polynomial

$$x^4+y^4-6(x^2+y^2)+8xy$$

is stationary and determine the nature of each of the stationary values.

7. If r is the distance between two points P and Q situated respectively on the curves

$$y^2=4ax \text{ and } (x-3a)^2+y^2=12a^2,$$

find the positions of P and Q when r is a maximum or a minimum and interpret the results geometrically.

149. Functions of several variables. We shall now briefly refer to the various points concerning functions of several variables.

A point in a space of, n, dimensions : E_n. The set of all ordered n-tuples

$$(a_1, a_2, \ldots\ldots, a_n)$$

of real numbers is called an *n dimensional space* and denoted by the symbol E_n. Every n-tuple is called a point of E_n or simply a point if no confusion be possible.

The set of points

$$(x_1, \ldots\ldots\ldots, x_n)$$

where

$$a_1 \leqslant x_1 \leqslant b_1, \ldots\ldots\ldots, a_n \leqslant x_n \leqslant b_n$$

is called a closed rectangle denoted by the symbol

$$[a_1, b_1 ; \ldots\ldots\ldots ; a_n, b_n].$$

Also the set of points

$$(x_1, \ldots\ldots\ldots, x_n)$$

where

$$a_1 < x_1 < b_1 ; \ldots\ldots\ldots ; a_n < x_n < b_n$$

is called an open rectangle denoted by the symbol

$$]a_1, b_1 ; \ldots\ldots\ldots ; a_n, b_n[.$$

Neighbourhood of a point. An open rectangle is called a neighbourhood of each of its points.

Real valued function of n real variables. Let S be any sub-set of E_n and let there be given a rule f which associates to each point of S a real number u, we say that f is a function with S as its *domain* and write

$$u=f(x_1, \ldots\ldots\ldots, x_n)$$

where $(x_1, \ldots\ldots\ldots, x_n)$ is any point of S.

Continuity. A function $f(x_1, x_2, \ldots\ldots, x_n)$ is said to be continuous at a point $P(a_1, a_2 \ldots\ldots, a_n)$, if to every positive number ε, there corresponds a neighbourhood of P such that for every point $(x_1, x_2, \ldots\ldots, x_n)$ of this neighbourhood, we have

$$| f(x_1, x_2, \ldots\ldots, x_n) - f(a_1, a_2, \ldots\ldots, a_n) | < \varepsilon.$$

Partial Derivatives.

$$\lim_{h_1 \to 0} \frac{f(a_1 + h_1, a_2, \ldots\ldots, a_n) - f(a_1, a_2, \ldots\ldots, a_n)}{h_1},$$

if it exists, is called the partial derivative of the function, f, w.r. to x_1, at $(a_1, a_2, \ldots\ldots, a_n)$ and is denoted as

$$f_{x_1}(a_1, a_2, \ldots\ldots, a_n).$$

We may similarly define other partial derivatives of the first and of the second and higher orders.

Differentiability. The function $f(x_1, x_2, \ldots\ldots, x_n)$ is said to be differentiable at $(a_1, a_2, \ldots\ldots, a_n)$ if, as the point changes from

$$(a_1, a_2, \ldots\ldots, a_n) \text{ to } (a_1 + h_1, a_2 + h_2, \ldots\ldots, a_n + h_n),$$

the change in the value of the function can be expressed in the form

$$A_1 h_1 + A_2 h_2 + \ldots\ldots + A_n h_n + h_1 \varphi_1 + h_2 \varphi_2 + \ldots\ldots h_n \varphi_n,$$

where $\varphi_1, \varphi_2 \ldots\ldots, \varphi_n$ are functions of $h_1, h_2, \ldots\ldots, h_n$ and tend to zero when

$$(h_1, h_2 \ldots\ldots, h_n) \to (0, 0, \ldots\ldots, 0).$$

The differentials df, $d^2 f$ etc. may now be easily defined as in the case of functions of two variables.

Note. There hold results for real valued functions of n variables analogous to those for real valued functions of two variables and the proofs are also quite similar. The reader is advised to state and prove these results. We state below without proof Taylor's theorem for real valued functions of n real variables.

Taylor's theorem for a function of n variables. If $f(x_1, \ldots\ldots, x_n)$ possesses continuous partial derivatives of the nth order in a neighbourhood of $(a_1, \ldots\ldots, a_n)$ and if $(a_1 + h_1, \ldots\ldots, a_n + h_n)$ be any point of this neighbourhood, then there exists a number θ, $0 < \theta < 1$ such that

$$f(a_1 + h_1, \ldots\ldots, a_n + h_n) = f(a_1, \ldots\ldots, a_n) + df(a_1, \ldots\ldots, a_n) + \frac{1}{2!} d^2 f(a_1, \ldots\ldots, a_n) + \ldots\ldots + \frac{1}{(n-1)!} d^{n-1} f(a_1, \ldots\ldots, a_n) + \frac{1}{n!} d^n f(a_1 + \theta h_1, \ldots\ldots, a_n + \theta h_n)$$

Ex. 1. Show that the function

$$f(x_1, x_2, \ldots\ldots, x_n)$$

is differentiable at a point

$$(a_1, a_2, \ldots\ldots, a_n),$$

if the function possesses continuous first order partial derivatives at the point.

Show that differentiability holds even if one of the partial derivatives just exists at the point whereas all others are continuous at the point.

Ex. 2. State and prove results for functions of n variables corresponding to those given on pages 303—325 for functions of two variables.

Ex. 3. If $(x_1, x_2, \ldots\ldots, x_n)$ possesses continuous first order partial derivatives at each point in a certain neighbourhood of $(a_1, \ldots\ldots, a_n)$ and

$$(a_1+h_1 \ldots\ldots, a_n+h_n)$$

is any other point of this neighbourhood, then there exists a number θ, $0 < \theta < 1$ such that

$$f(a_1+h_1, \ldots\ldots, a_n+h_n) = f(a_1, \ldots\ldots, a_n) + \sum_{i=1}^{n} h_i f_{x_i}(a_1+\theta h_1, \ldots\ldots, a_n+\theta h_n).$$

150. Extreme values of a function of, n, variables. Stationary points and stationary values. As in the case of functions of one and two variables, we say that a function $f(x_1, x_2, \ldots\ldots, x_n)$ has a maximum (or a minimum) at a point $(a_1, a_2, \ldots\ldots, a_n)$, if at every point in a certain neighbourhood of $(a_1, a_2, \ldots \ldots a_n)$, the function assumes a smaller value (or a larger value) than at the point itself; in either case we say that $(a_1, a_2, \ldots\ldots, a_n)$ is an extreme point and that $f(a_1, a_2, \ldots\ldots, a_n)$ is an extreme value of the function.

If $f(a_1, a_2, \ldots, a_n)$ is an extreme value of the function $f(x_1, x_2 \ldots x_n)$, then it is also an extreme value of the function $f(x_1, a_2, \ldots\ldots, a_n)$ of one variable x_1 for $x_1 = a_1$ and therefore the derivative $f_{x_1}(a_1, a_2, \ldots\ldots, a_n)$, in case it exists, must be zero.

Thus we see that **necessary conditions** *for* $f(a_1, a_2, \ldots\ldots, a_n)$ *to be an extreme value of the function* $f(x_1, x_2, \ldots\ldots, x_n)$ *are* that each of the n first order partial derivatives of the function is zero at the point $(a_1, a_2, \ldots\ldots, a_n)$ *in case the function possesses partial derivatives, in question.*

To find the extreme point of a function $f(x_1, x_2, \ldots\ldots, x_n)$ which possesses partial derivatives at every point of its domain of definition, we have to find the points $(x_1, x_2, \ldots\ldots, x_n)$ which satisfy the, n, equations, obtained by equating to zero the partial derivatives of the function. But a point obtained in this way may not necessarily be an extreme point; further investigation is necessary to decide whether it is really an extreme point or not. (See note on next page.)

It proves useful to have a special name for the point at which the necessary conditions for the extreme position are satisfied, irrespective of whether it is really an extreme point or not. Thus we say *that a point* $(a_1, a_2, a_3, \ldots\ldots, a_n)$ *is a* **stationary point** *of the function, if each of the first order partial derivatives of the function vanishes at that point.*

The conditions for a stationary point may also be given in a more compact form. Thus if $(a_1, a_2, \ldots\ldots, a_n)$ is a stationary point, then

$$df(a_1, a_2, \ldots\ldots, a_n)=0,$$

i.e., the first differential of a function vanishes at a stationary point, no matter what values are assigned to the differentials $dx_1, dx_2 \ldots\ldots, dx_n$, of the independent variables. For, we have

$$df(a_1, a_2 \ldots, a_n)=f_{x_1}(a_1, a_2 \ldots, a_n)\, dx_1+f_{x_2}(a_1, a_2 \ldots\ldots, a_n)dx_2+\ldots=0.$$

Conversely, if the differential df is 0 for arbitrary values of the differentials dx_1, dx_2, etc., of the independent variables, then separately taking all but one of these differentials equal to zero, we can show that all the partial derivatives are zero.

Note. It is easy to see that the results obtained in § 148, page 321 concerning the extreme values of a function $f(x, y)$ of two variables may be restated in the following form :—

If at (a, b), the first differential $df=0$, then

(i) $f(a, b)$ is a minimum or a maximum according as d^2f is a positive or a negative quadratic form ; (ii) $f(a, b)$ is not an extreme value if d^2f is indefinite ; and finally (iii) the case is doubtful if d^2f is a semi-definite form.

Similar results hold true for a function of any number of variables also, but the details of the proof will not be given here.

Thus in the case of a function $f(x, y, z)$ of three variables sufficient conditions for (a, b, c) to be an extreme value are that

$$df(a, b, c) = f_x dx+f_y dy + f_z dz=0$$

for arbitrary values of dx, dy, dz, *i.e.*,

$$f_x = f_y = f_z=0,$$

and

$$d^2f(a, b, c)=f_x{}^2(dx)^2+f_y{}^2(dy)^2+f_z{}^2(dz)^2+$$
$$2f_{xy}\, dxdy+2f_{zy}\, dydz+2f_{zx}\, dzdx$$

is definite, *i.e.*, assumes values of the same sign for arbitrary values of dx, dy and dz.

Also, if the conditions are satisfied, $f(a, b, c)$ is a maximum or a minimum according as d^2f is a negative or a positive definite form.

The conditions for d^2f to be positive or negative definite may be stated as follows. (The proof will not be given here).

Consider the 3×3 matrix

$$\begin{Bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{Bmatrix}.$$

The result, then, is that d^2f will be positive definite if and only if the three principal minors

$$f_{xx}, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

are all positive and $d^2 f$ will be negatively definite if, and only if, their signs are alternately negative and positive. The corresponding result for a function of n variables may now be easily stated.

A function, f, of any number of variables will be maximum if at a stationary point, d^2f is negatively definite and it will be minimum if d^2f is positively definite. The function will be neither maximum nor minimum if d^2f is indefinite and the case requires further investigation if d^2f is semi-definite.

Example

1. *Examine*

$$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z,$$

for extreme values.

We have

$$f_x = 2yz - 4z + 2x - 2,$$
$$f_y = 2zx - 2z + 2y - 4,$$
$$f_z = 2xy - 4x - 2y + 2z + 4.$$

The stationary points (x, y, z) are thus given by the system of equations

$$\left.\begin{aligned} yz - 2z + x - 1 &= 0 \\ zx - z + y - 2 &= 0 \\ xy - 2x - y + z + 2 &= 0 \end{aligned}\right\}.$$

Adding the last two equations, we see that the system is equivalent to

$$yz-2z+x-1=0,$$
$$zx-\ z+y-2=0,$$
$$zx+xy-2x=0, \text{ i.e., } x(z+y-2)=0.$$

Thus stationary points are given by the two systems of equations

$$\left.\begin{array}{r} yz-2z+x-1=0 \\ zx-\ z+y-2=0 \\ x=0 \end{array}\right\}, \quad \left.\begin{array}{r} yz-2z+x-1=0 \\ zx-\ z+y-2=0 \\ z+y-2=0 \end{array}\right\}.$$

These give

$$(0, 3, 1), (0, 1, -1), (1, 2, 0), (2, 1, 1), (2, 3, -1)$$

as the stationary points of the function.

Again, we have

$$f_{xx}=2, f_{yy}=2, f_{zz}=2\ ;\ f_{xy}=2z, f_{yz}=2x-2, f_{zx}=2y-4.$$

For (0, 3, 1), we have

$$d^2f=2h^2+2k^2+2l^2+4hk-4kl+4lh.$$

Consider the matrix of the quadratic form d^2f, *viz.*,

$$\begin{Bmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{Bmatrix}$$

Its principal minors

$$2, \quad \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}, \quad \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix}$$

are 2, 0, and -32 and as such d^2f is indefinite.

Thus the function is neither a maximum nor a minimum at (0, 3, 1).

It may similarly be shown that the function is neither a maximum nor a minimum at the stationary points $(0, 1, -1)$, $(2, 1, 1)$ and $(2, 3, -1)$.

At (1, 2, 0) we have

$$d^2f = 2h^2 + 2k^2 + 2l^2$$

which is clearly positive definite. Thus the function is a minimum at this point.

Exercises

1. Examine for extreme values

   (*i*) $2xyz + x^2 + y^2 + z^2$, (*ii*) $2x^2 + 3y^2 + 4z^2 - 3xy + 8z$.

2. Prove that the maximum value of

$$f(x, y, z) = (ax + by + cz)e^{-\alpha^2x^2 - \beta^2y^2 - \gamma^2z^2}$$

is

$$\sqrt{\tfrac{1}{2}[(a^2\alpha^{-2} + b^2\beta^{-2} + c^3\gamma^{-2})(e^{-1})]}.$$

3. Show that the function

$$3 \log (x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3$$

has only one extreme value, *viz.*, $\log (3/e^2)$.

4. Show that

$$u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$$

has min. at (1, 1, 1) and max. at (−1, −1, −1).

151. Jacobians. We shall now introduce the important notion of Jacobians and prove also an important property concerning the same. If $u_1, u_2, \ldots, u_n$ be n functions of n variable $x_1, x_2, \ldots, x_n$ possessing partial derivatives of the first order at every point of the common domain of definition of the functions, then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1}, & \dfrac{\partial u_1}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1}, & \dfrac{\partial u_2}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial u_2}{\partial u_n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ \dfrac{\partial u_n}{\partial x_1}, & \dfrac{\partial u_n}{\partial x_2}, & \ldots\ldots, & \dfrac{\partial x_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots\ldots, u_n$ w.r. to $x_1, x_2, \ldots\ldots, x_n$ and is denoted by

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} \text{ or } J\left(\frac{u_1, u_2, \ldots, u_n}{x_1, x_2, \ldots, x_n}\right).$$

The Jacobian is itself a function of the n variables

$$(x_1, x_2, \ldots\ldots, x_n).$$

151·1. An important property of Jacobians.

Theorem. *If*

$$u_i = f_i(y_1, \ldots\ldots, y_n)\ ;\ i = 1, 2, \ldots\ldots, n$$

and

$$y_r = \varphi_r(x_1, \ldots\ldots, x_n)\ ;\ r=1, 2, \ldots\ldots, n.$$

then

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}. \quad \ldots (1)$$

If i and j denote any numbers from 1 to n, we have, by § 145·3, page 314.

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_j} + \frac{\partial u_i}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_j} + \ldots\ldots + \frac{\partial u_i}{\partial y_n} \cdot \frac{\partial y_n}{\partial x_j}. \quad \ldots (2)$$

The theorem, now follows from the fact that $\partial u_i/\partial x_j$ is by def., the element in the ith row and jth column of the left hand expression of (1), and, by the theorem on the multiplication of determinants, the right hand expression of (2) is the element in the ith row and jth column of the right hand expression of (1).

Ex. 1. *If* λ, μ, ν *are the roots of the equation in* k,

$$\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1,$$

prove that

$$J\left(\frac{x, y, z}{\lambda, \mu, \nu}\right) = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

We assume that the given equation has all the three roots real.

We have

$$(a+k)(b+k)(c+k) - x(b+k)(c+k) - y(c+k)(a+k) - z(a+k)(b+k)$$
$$\equiv (k-\lambda)(k-\mu)(k-\nu).$$

$$\therefore\quad 1 - \frac{x}{a+k} - \frac{y}{b+k} - \frac{z}{c+k} \equiv \frac{(k-\lambda)(k-\mu)(k-\nu)}{(a+k)(b+k)(c+k)} \quad \ldots (1)$$

$$= 1 - \frac{(a+\lambda)(a+\mu)(a+\nu)}{(a+k)(b-a)(c-a)} -$$

$$\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b+k)(a-b)(c-b)} - \frac{(c+\lambda)(c+\mu)(c+\nu)}{(c+k)(b-c)(a-c)},$$

by splitting the right hand side of (1) into partial fractions. Thus

$$x = -(a+\lambda)(a+\mu)(a+\nu)/(a-b)(c-a),$$
$$y = -(b+\lambda)(b+\mu)(b+\nu)/(a-b)(b-c),$$
$$z = -(c+\lambda)(c+\mu)(c+\nu)/(b-c)(c-a).$$

$$\therefore\quad \frac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)} =$$

$$= -\frac{1}{(a-b)^2(b-c)^2(c-a)^2} \times \begin{vmatrix} (a+\mu)(a+\nu), & (a+\nu)(a+\lambda), & (a+\lambda)(a+\mu) \\ (b+\mu)(b+\nu), & (b+\nu)(b+\lambda), & (b+\lambda)(b+\mu) \\ (c+\mu)(c+\nu), & (c+\nu)(c+\lambda), & (c+\lambda)(c+\mu) \end{vmatrix}$$

$$= -\frac{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{(a-b)(b-c)(c-a)}.$$

Another Proof. The given equation is

$$k^3-k^2(\Sigma x-\Sigma a)-k[\Sigma(b+c)x-\Sigma ab]-\Sigma bcx+abc=0.$$

We write

$$u=\Sigma x-\Sigma a,$$
$$v=-\Sigma(b+c)x+\Sigma ab$$
$$w=\Sigma bcx-abc$$

so that

$$u=\Sigma\lambda,$$
$$v=\Sigma\lambda\mu,$$
$$w=\lambda\mu\nu.$$

Then,

$$\frac{\partial(u,\ v,\ w)}{\partial(\lambda,\ \mu,\ \nu)}=\begin{vmatrix} 1 & & 1 \\ \mu+\nu & \nu+\lambda & \lambda+\mu \\ \mu\nu & \nu\lambda & \lambda\mu \end{vmatrix}$$

$$=-\ (\lambda-\mu)(\mu-\nu)(\nu-\lambda)$$

$$\frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)}=-\begin{vmatrix} & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$

$$=(b-c)(c-a)(a-b)$$

$$\therefore\quad J\left(\frac{x,\ y,\ z}{\lambda,\ \mu,\ \nu}\right)=J\left(\frac{x,\ y,\ z}{u,\ v,\ w}\right).\ J\left(\frac{u,\ v,\ w}{\lambda,\ \mu,\ \nu}\right)$$

$$=-\ \frac{(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{(b-c)(c-a)(a-b)}.$$

2. Find $\partial(x,\ y,\ z)/\partial(r,\ \theta,\ \varphi)$, if

$$x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta.$$

3. The roots of

$$(\lambda-u)^3+(\lambda-v)^3+(\lambda-w)^3=0,$$

regarded as an equation in λ, are $x,\ y,\ z$; prove that

$$\frac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}=-2\,\frac{(v-w)(w-u)(u-v)}{(y-z)(z-x)(x-y)}.$$

4. Find $\partial(y_1,\ldots\ldots,\ y_n)/\partial(x_1,\ldots\ldots,\ x_n)$ where

$$y_1=x_1(1-x_2),\ y_2=x_1\,x_2(1-x_3),\ldots\ldots,$$
$$y_{n-1}=x_1\,x_2\ldots\ldots\,x_{n-1}\,(1-x_n),\ y_n=x_1x_2\ldots\ldots x_n.$$

152. One-one Transformations. Inverse transformation of a one-one transformation. Many-one transformations.

Consider the transformation given by

$$u_i = f_i(x_1, \ldots\ldots, x_n),\ 1 \leqslant i \leqslant n$$

of R on S. To each point of R corresponds a point of S. It is possible, however, that in any given case, two different points of R are transformed to the same point of S. This leads us to distinguish between what may be described as

(*i*) *one-one transformations,*

(*ii*) *many-one transformations.*

A transformation of R to S is said to be one-one if while to each point of R corresponds a point of S there is one and only one point of R which corresponds to a given point of S so that no two different points of R correspond to the same point of S.

The transformation is said to be many one, if there exist atleast some two points of R corresponding to the same point of S.

If the transformation A of R to S is one-one, then we have a natural way of defining a transformation B of S to R by the condition that if Q be any point of S, then B transforms Q to that unique point P of R which is transformed by A to Q

This transformation B of S on R is said to be the inverse of the one-one transformation A of R on S.

A one-one transformation is also known as an **Invertible** transformation.

153. Locally invertible transformations.

Consider the transformation defined by the system of functions

$$u_i = f_i(x_1, \ldots\ldots, x_n),\ 1 \leqslant i \leqslant n$$

and suppose that it transforms R to S.

Let $P(a_1 \ldots\ldots, a_n)$ be a point of R and $Q(b_1, \ldots\ldots, b_n)$ be the corresponding point of S.

We say that the transformation is locally invertible at $P(a_1, \ldots\ldots, a_n)$ if there exists a neighbourhood N of P and a neighbourhood N_1 of Q such that the transformation in question, maps N on N_1 one-one, *i e.*,

(*i*) each point of N is mapped on a point of N_1,

(*ii*) each point of N_1 is the image of one and only one point of N.

It may be noticed that if a transformation is locally invertible at P, then there may also exist a point other than that belonging to N which is also mapped on a given point of N_1.

154. Globally invertible transformations.

If the transformation

$$u_i = f_i(x_1, \ldots\ldots, x_n),\ 1 \leqslant i \leqslant n$$

maps R one-one on S, then we, for the sake of distinction, say that the transformation is globally invertible.

Illustrations. 1. Consider the transformation defined by

$$u = x^2 - y^2,$$
$$v = xy.$$

This transformation is not globally invertible inasmuch as the two points (x, y) and $(-x, -y)$ are mapped on the same point.

This transformation is, however, locally invertible at each point other than $(0, 0)$.

Howsoever small a neighbourhood of $(0, 0)$ in the xy-plane we may take, some two points thereof must necessarily be mapped on the same point in the uv-plane.

If, however, we take any point

$$(h, k) \neq (0, 0),$$

we can mark a neighbourhood of the same such that no two points thereof are mapped on the same point in the uv-plane.

2. The transformation defined by

$$u = e^x \cos y,$$
$$v = e^x \sin y,$$

is not globally invertible even though it is locally invertible in respect of each point in the xy-plane.

Note. In the following chapter we shall find sufficient conditions for a given transformation to be locally invertible at any given point.

155. Transformations. Resultants of Transformations. Consider a system of n functions

$$u_i = f_i(x_1, x_2, \ldots\ldots, x_n),\ 1 \leqslant i \leqslant n$$

of n variables $(x_1, x_2, \ldots\ldots, x_n)$.

Let R denote the set of points $(x_1, \ldots\ldots, x_n)$ to each of which corresponds a point $(u_1, u_2, \ldots\ldots, u_n)$ so that R denotes what may be called the *domain of definition* of the system of functions.

Let, again, S denote the set of points $(u_1, u_2, \ldots\ldots, u_n)$ arising when the point $(x_1, \ldots\ldots, x_n)$ varies over R so that S denotes what may be called the *Range* of the system of functions.

Both R and S are sub-sets of the n-dimensional space E_n.

It is found useful to think of the system of n functions as defining a *transformation* of R to S *or a mapping of* R *into* S as a result of which each point of R is transformed to or mapped on a point of S.

Sometimes the system of functions is itself thought of as a Transformation or a mapping.

Consider now the two systems :

$$u_i = f_i(y_1, y_2, \ldots, y_n), \qquad \ldots(1)$$

$$y_j = \varphi_j(x_1, x_2, \ldots, x_n), \qquad \ldots(2)$$

where i and j both vary from 1 to n.

Suppose that

(i) R is the domain and S the range of the system (2).

(ii) S is the domain and T the range of the system (1).

Then the two systems of functions can be thought of as defining a transformation of R to T via S.

This transformation of R to T is known as the *resultant* or the *composite* of the two transformations of R to S and of S to T which are then known as the components of the resultant transformation.

155·1. Jacobian of a transformation. Consider the transformation

$$u_i = f_i(x_1, x_2, \ldots, x_n), \; 1 \leq i \leq n$$

of R to S. Suppose that each of the function f_i possesses a partial derivative of the first order w. r. to each of the n variables at a point of R. Then the Jacobian

$$\partial(u_1, \ldots, u_n)/\partial(x_1, \ldots, x_n),$$

is known as the Jacobian of the transformation at the point in question.

The result proved in § 151·1 may now be stated as follows :

Jacobian of the resultant of two transformations is the product of the Jacobians of the component transformations.

It will later on be seen that the Jacobian of a transformation is very intimately related to the nature of the transformation.

CHAPTER XI

IMPLICITLY DEFINED FUNCTIONS

Functional Dependence

156. Introduction. If $f(x, y)$ be a function of two variables and $y=\varphi(x)$ be a function of x such that for every value of x for which $\varphi(x)$ is defined, $f[x, \varphi(x)]$ vanishes identically, then we say that $y=\varphi(x)$ is an *implicit function* defined by the functional equation $f(x, y)=0$.

A functional equation may not define any implicit function or it may define one or more than one such function ; for example, the functional equation

$$x^2+y^2-1=0,$$

determines two implicit functions, *viz.*,

$$y=\sqrt{(1-x^2)} \text{ and } y=-\sqrt{(1-x^2)},$$

whereas the equation

$$x^2+y^2+1=0$$

determines no such function.

It is only in elementary cases, such as those given above, that we may be able to *determine* explicitly the implicit functions, (in case they exist), defined by a functional equation. For more complicated functional equations such as $x^2y+\sin y+\log x=0$, no such explicit *determination* may be possible ; this impossibility of explicit determination, however, does not rule out the possibility of the *existence* of the implicit function or functions defined by an equation. The question of the *existence* of implicit functions, (apart from their actual explicit determination) will be investigated in this chapter.

If $f(x, y)$ be any function of two variables defined in some domain S, then

$$f(x, y)=0 \qquad \ldots(1)$$

determines a sub-set of S consisting of those points (x, y) wherefor (1) holds.

It does not follow, however, that this aggregate of points (x, y) wherefor $f(x, y)=0$ can be thought of as determining y as a function of x. The theorem below which is *local* in character seeks to show that under certain assumptions in respect of continuity and derivability satisfied by $f(x, y)$, this is actually the case.

157. Implicit function determined by a single functional equation.

157·1. Case of two variables. *Let $f(x, y)$ be a function of two variables and let (a, b) be a point of its domain of definition such that*

(*i*) $f(a, b)=0$,

(*ii*) *the function possesses continuous partial derivatives f_x and f_y in a certain neighbourhood of (a, b) and*

(*iii*) $f_y(a, b) \neq 0$;

then their exists a rectangle $[a-h, a+h \,;\, b-k, b+k]$ about (a, b) such that for every value of x in the interval $[a-h, a+h]$, the equation $f(x, y)=0$ determines one and only one value $y=\varphi(x)$, lying in the interval $[b-k, b+k]$, with the following properties :

(1) $b=\varphi(a)$,

(2) $f[x, \varphi(x)]=0$ *for every value of x in $[a-h, a+h]$*,

(3) $\varphi(x)$ *is derivable and both $\varphi(x)$ as well as $\varphi'(x)$ are continuous in $[a-h, a+h]$.*

(The reader is advised to follow the following line of argument by means of the diagram attached).

Without any loss of generality, we suppose that $f_y(a, b)>0$, for, otherwise, we should only have to replace $f(x, y)$ by $-f(x, y)$ and this change would leave the equation $f(x, y)=0$ unaltered.

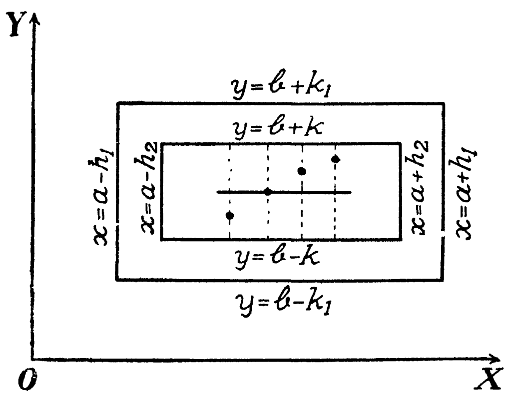


Fig. 1.

Unique Existence. Let f_x, f_y be continuous in a neighbourhood

$$R_1[a-h_1, a+h_1 \,;\, b-k_1, b+k_1]$$

of (a, b).

Since f_x, f_y are continuous in R_1, therefore $f(x, y)$ is also continuous in R_1. (§ 139).

Since $f_y(x, y)$ is continuous at (a, b) and $f_y(a, b) > 0$, there exists a rectangle

$$R_2[a-h_2, a+h_2 ; b-k, b+k], \qquad h_2 < h_1, k < k_1$$

such that for every point (x, y) of this rectangle R_2, $f_y(x, y) > 0$.

(Note 2 after § 135, page 298).

Now, since $f_y(x, y) > 0$ in R_2, therefore, for every value of x in $[a-h_2, a+h_2]$, the function $f(x, y)$ of y strictly increases as y increases from $b-k$ to $b+k$, (§ 74·2, page 119). In particular, since $f(a, b)=0$, we have

$$f(a, b-k) < 0, \; f(a, b+k) > 0.$$

In view of this and the fact that $f(x, y)$ is continuous, there exists an interval $[a-h, a+h]$, $(h < h_2)$ such that for every x of this interval, we have

$$f(x, b-k) < 0, \; f(x, b+k) > 0. \quad \text{(Lemma 1, § 58·1, page 94)}$$

Now for every fixed value of x in $[a-h, a+h]$, the continuous function $f(x, y)$ of y *strictly* increases from a negative to a positive value as y increases from $b-k$ to $b+k$ and therefore there exists *one and only one* value of y for which the function $f(x, y)$ vanishes.

(§ 58·1, page 94)

Hence for each value of x in $[a-h, a+h]$, there is a uniquely determined value of y for which $f(x, y)=0$, this value of y is a function of x, say $\varphi(x)$, such that the properties (1) and (2) are true.

This completes the proof of the *existence* and the *uniqueness* of the implicit function $\varphi(x)$.

Continuity. We now prove that $\varphi(x)$ is continuous in $[a-h, a+h]$. Let x_0 be any point of this interval and let $y_0=\varphi(x_0)$. Let ε be any given positive number. Let

$$R'[x_0-\delta_1, x_0+\delta_1 \; ; \; y_0-\varepsilon, y_0+\varepsilon]$$

be a rectangle entirely lying within the rectangle

$$R[a-h, a+h \; ; \; b-k, b+k]$$

found above. For this rectangle, we can carry out exactly the same process as before in order to obtain a solution of $f(x, y)=0$. Since the solution was uniquely determined in R which encloses R', we see that the same function *viz.*, $y=\varphi(x)$ is the solution in R' also. Thus there exists an interval $(x_0-\delta, x_0+\delta)$, $(\delta \leqslant \delta_1)$ such that for every value of x in this interval, $y=\varphi(x)$ lies between $y_0-\varepsilon$ and $y_0+\varepsilon$, *i.e.*,

$$| y-y_0 | = | \varphi(x)-\varphi(x_0) | < \varepsilon, \text{ when } | x-x_0 | \leqslant \delta.$$

Hence $\varphi(x)$ is *continuous* at x_0 and, therefore, in $[a-h, a+h]$.

Derivability. Let x be a point of the interval $[a-h, a+h]$ and let $x+\delta x$ be another point of the same interval. Let

$$y=\varphi(x),\ y+\delta y=\varphi(x+\delta x),$$

so that

$$f(x, y)=0,\ f(x+\delta x, y+\delta y)=0.$$

$$\begin{aligned}\therefore\quad 0&=f(x+\delta x, y+\delta y)-f(x, y)\\ &=f(x+\delta x, y+\delta y)-f(x+\delta x, y)+f(x+\delta x, y)-f(x, y)\\ &=\delta y f_y(x+\delta x, y+\theta_1\delta y)+\delta x f_x(x+\theta_2\delta x, y).\end{aligned}$$

(By mean value theorem)

Since $f_y \neq 0$ in R and $(x+\delta x, y+\theta_1\delta y)$ is a point of R, we have

$$\frac{\delta y}{\delta x}=-\frac{f_x(x+\theta_2\delta x, y)}{f_y(x+\delta x, y+\theta_1\delta y)}.$$

Since $\varphi(x)$ is continuous, $\delta y \to 0$ as $\delta x \to 0$. Therefore f_x and f_y being continuous, we obtain from above, when $(\delta x, \delta y) \to (0, 0)$,

$$\varphi'(x)=\frac{dy}{dx}=-\frac{f_x(x, y)}{f_y(x, y)}.$$

Thus $\varphi(x)$ is *derivable* and $\varphi'(x)=-f_x(x, y)/f_y(x, y)$. Also this formula for $\varphi'(x)$ shows that it is continuous.

Note. The function $y=\varphi(x)$ is said to be the unique solution of $f(x, y)=0$ *near* (a, b) or the unique implicit function determined by $f(x, y)=0$ *near* (a, b).

Note. It is very important to bear in mind the fact that the theorem just proved is essentially of a *Local Character.* It states that if $f(x, y)$ is a function of two variables satisfying certain assumptions in respect of continuity and derivability in a neighbourhood of a point (a, b) and if $f(a, b)=0$, then there exists a neighbourhood

$$[a-h, a+h\ ;\ b-k, b+k] \text{ of } (a, b)$$

such that for each x belonging to $[a-h, a+h]$ there exists a unique y belonging to $[b-k, b+k]$ such that $f(x, y)=0$, *i.e.*, $f(x, y)=0$ defines a function $y=\varphi(x)$ in $[a-h, a+h]$; y lying in $[b-k, b+k]$ and this function $\varphi(x)$ of x is derivable.

Ex. Consider $f(x, y)=x^2+y^2-1$ and a point $(0, 1)$ so that $f(0, 1)=0, f_y(0, 1) \neq 0$. How do you reconcile the conclusions of the theorem with the fact that $x^2+y^2-1=0$ defines *two* functions, *viz.*

$$y=+\sqrt{(1-x^2)},\ y=-\sqrt{(1-x^2)}.$$

157·2. General Case. *If $f(x_1, x_2, \ldots\ldots, x_n, y)$ be a function of $(n+1)$ variables and $(a_1, a_2, \ldots\ldots, a_n, b)$ be any point of its domain of definition such that*

(*i*) $f(a_1, a_2, \ldots\ldots, a_n, b)=0$,

(*ii*) *the function possesses continuous first order partial derivatives w.r. to the $(n+1)$ variables in a certain neighbourhood of $(a_1, a_2, \ldots\ldots, a_n, b)$, and*

(*iii*) $f_y(a_1, a_2, \ldots\ldots, a_n, b) \neq 0$, *then there exists a rectangle*

$$[a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ \dots\ ,\ a_n-h_n,\ a_n+h_n\ ;\ b-k,\ b+k]$$

such that for every point $(x_1, x_2, \dots, x_n)$ *of the rectangle*

$$R[a_1-h_1,\ a_1+h_1\ ;\ a_2-h_2,\ a_2+h_2\ ;\ \dots\ ;\ a_n-h_n,\ a_n+h_n]$$

the equation $f(x_1, x_2, \dots, x_n, y)=0$ *determines one and only one value* $y=\varphi(x_1, x_2, \dots, x_n)$ *lying in* $[b-k, b+k]$ *with the following properties* :—

(1) $b=\varphi(a_1, a_2, a_3, \dots, a_n)$.

(2) $f(x_1, x_2, \dots, x_n, \varphi)=0$ *for every point* $(x_1, x_2, \dots, x_n)$ *in* R.

(3) φ *is continuous and possesses continuous partial derivatives of the first order w.r. to* $x_1, x_2, \dots, x_n$ *in* R.

The proof of this general theorem follows exactly on the same lines as the proof of the preceding theorem (§ 157·1) and offers no fresh difficulties.

Exercises

1. If $f(x, y)$ is a continuous function of each variable x and y separately in a certain neighbourhood of (a, b), $f(a, b)=0$, $f_y(x, y)$ is continuous at (a, b) and $f_y(a, b)\neq 0$, then the equation $f(x, y)=0$ determines a *unique* continuous implicit function $y=\varphi(x)$ near (a, b).

2. If $f(x, y)$ is a continuous function of each variable x and y separately in a neighbourhood of (a, b), $f(a, b)=0$ and $f_y'(a, b)$ exists and $\neq 0$, then the equation $f(x, y)=0$ determines *at least one* continuous implicit function $y=\varphi(x)$ near (a, b).

3. $f(x, y)$ is a continuous function of x and y in the neighbourhood of (a, b) such that $f(a, b)=0$; and further $f(x, y)$ is, for all x in the neighbourhood of (a, b), strictly increasing function of y. Prove that there exists a unique function $y=\varphi(x)$, which, when substituted in the equation $f(x, y)=0$, satisfies it identically for all values of x in the neighbourhood of a, and that $\varphi(x)$ is continuous for all values of x in the neighbourhood of a.

If $f(x, y)=y^4-y^2+x^2$, discuss the existence of the function $y=\varphi(x)$ in the neighbourhood of $x=0$, $y=0$.

4. Show that the following equations determine unique implicit functions near the points indicated ; find also the first derivatives of the solutions :

(*i*) $x^3+y^3-3xy+y=0$, $(0, 0)$. (*ii*) xy sin$+x\cos y=0$, $(0, \frac{1}{2}\pi)$.
(*iii*) $y^3\cos x+y^2\sin^2 x=7$, $(\frac{1}{3}\pi, 2)$. (*iv*) $2xy-\log xy=2$, $(1, 1)$.

5. Examine the following equations for the existence of unique implicit functions near the points indicated and verify by direct calculation.

(*i*) $y^2-yx^2-2x^5=0$ near $(0, 0)$ and $(1, -1)$.
(*ii*) $y^4+y^2x^2-2x^5=0$ near $(1, 1)$.
(*iii*) $y^2+2x^2y+x^5=0$ near $(1, -1)$.
(*iv*) $y^2+yx^3+x^2=0$ near $(0, 0)$.

6. Show that the least positive root of $xy=\tan y$ is a continuous function of x throughout the interval $[1, \infty[$ and increases steadily from 0 to $\frac{1}{2}\pi$ as y increases from 1 towards ∞. (Use § 59, page 102).

158. Implicit functions determined by a system of functional Equations. Theorem. *Let $f_1(x, y, z, u, v)$ and $f_2(x, y, z, u, v)$ be two functions of five variables and let $(a_1, a_2, a_3, b_1, b_2)$ be a point of their domain of definition such that*

(*i*) $f_1(a_1, a_2, a_3, b_1, b_2)=0=f_2(a_1, a_2, a_3, b_1, b_2)$,

(*ii*) *the functions possess continuous first order partial derivatives w.r. to the five variables in a certain neighbourhood of the point $(a_1, a_2, a_3, b_1, b_2)$, and*

(*iii*) $\partial(f_1, f_2)/\partial(u, v)\neq 0$ *at* $(a_1, a_2, a_3, b_1, b_2)$,

then there exists one ond only one pair of functions

$$u=\varphi_1(x, y, z), \qquad v=\varphi_2(x, y, z)$$

of x, y, z, defined in a certain neighbourhood of (a_1, a_2, a_3) such that

(1) $\varphi_1(a_1, a_2, a_3)=b_1, \qquad \varphi_2(a_1, a_2, a_3)=b_2$,

(2) $f_1(x, y, z, \varphi_1, \varphi_2)=0=f_2(x, y, z, \varphi_1, \varphi_2)$ *for every point x, y, z of the domain of definition of φ_1* and φ_2,

(3) φ_1, φ_2, *are continuous and possess continuous first order partial derivatives w.r. to x, y, z.*

Proof. Since $\partial(f_1, f_2)/\partial(u, v)\neq 0$ at $(a_1, a_2, a_3, b_1, b_2)$, one at least of the partial derivatives $\partial f_1/\partial v$, $\partial f_2/\partial v$ must not vanish at this point. Suppose that $\partial f_1/\partial v\neq 0$.

Then, by the theorem of (§ 157·2, page 342), the equation

$$f_1(x, y, z, u, v)=0,$$

is satisfied by one and only one function

$$v=g(x, y, z, u),$$

defined in a certain neighbourhood of (a_1, a_2, a_3, b_1) and such that $b_2=g(a_1, a_2, a_3, b_1)$; the function, g, is continuous and possesses continuous first order partial derivatives. Replacing v by $g(x, y, z, u)$ in the function $f_2(x, y, z, u, v)$, we write

$$h(x, y, z, u)=f_2(x, y, z, u, g),$$

so that

$$h(a_1, a_2, a_3, b_1)=0.$$

We have at (a_1, a_2, a_3, b_1)

$$\frac{\partial h}{\partial u}=\frac{\partial f_2}{\partial u}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial g}{\partial u}.$$

Also, since $f_1(x, y, z, u, g)=0$, we have at (a_1, a_2, a_3, b_1),

$$\frac{\partial f_1}{\partial u}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial g}{\partial u}=0.$$

From these, we obtain

$$\frac{\partial h}{\partial u}=-\frac{\partial(f_1, f_2)/\partial(u, v)}{\partial f_1/\partial v},$$

so that

$$\partial h/\partial u\neq 0,$$

at (a_1, a_2, a_3, b_1).

Thus, again, by preceding theorem, the equation

$$h(x, y, z, u)=0,$$

is satisfied by one and only one function

$$u=\varphi_1(x, y, z),$$

defined in a certain neighbourhood of (a_1, a_2, a_3) and such that $b_1=\varphi_1(a_1, a_2, a_3)$. Replacing u by $\varphi_1(x, y, z)$ in $g(x, y, z, u)$, we obtain

$$v=g(x, y, z, \varphi_1)=\varphi_2(x, y, z)$$

where

$$\varphi_2(a_1, a_2, a_3)=g(a_1, a_2, a_3, b_1)=b_2.$$

It is easy to see that φ_1, φ_2 are continuous and possess continuous first order partial derivatives.

Hence the theorem.

Cor. *To determine*

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}; \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}.$$

Applying the theorem on the derivation of functions of functions to $f_1(x, y, z, u, v)$ and $f_2(x, y, z, u, v)$ where $u=\varphi_1(x, y, z)$ and $v=\varphi_2(x, y, z)$, we have, f_1, f_2 being identically zero,

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x}=0,$$

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x}=0,$$

which give

$$\frac{\partial u}{\partial x}=-\frac{\partial(f_1, f_2)/\partial(x, v)}{\partial(f_1, f_2)/\partial(u, v)}, \quad \frac{\partial v}{\partial x}=\frac{\partial(f_1, f_2)/\partial(x, u)}{\partial(f_1, f_2)/\partial(u, v)}.$$

We may similarly determine

$$\partial u/\partial y,\ \partial v/\partial y \text{ and } \partial u/\partial z,\ \partial v/\partial z.$$

Note. The most general form of the above result may be stated without proof as follows :

If

$$f_i(x_1,\ldots, x_m, u_1, \ldots, u_n),\ 1 \leqslant i \leqslant n$$

be a system of n functions of m+n variables and

$$(a_1, a_2,\ldots, a_m ; b_1, b_2,\ldots, b_n) \qquad \ldots(\text{I})$$

be a point of their domain of definition such that

(*i*) $f_i(a_1,\ldots, a_m, b_1,\ldots, b_n)=0, \quad 1 \leqslant i \leqslant n,$

(*ii*) *each of f_i possesses a continuous first order partial derivative w r. to each of the m+n variables in a certain neighbourhood of the point* (I),

(*iii*) $\dfrac{\partial(f_1, f_2,\ldots, f_n)}{\partial(u_1, u_2,\ldots, u_n)} \neq 0$ *at the point* (I),

then there exist one and only one system of n functions

$$u_i=\phi_i(x_1,\ldots, x_m), \quad 1 \leqslant i \leqslant n$$

defined in a certain neighbourhood of $(a_1, a_2 \ldots, a_m)$ *such that*

(1) $\qquad \phi_i(a_1, a_2,\ldots, a_m)=b_i, \quad 1 \leqslant i \leqslant n$

(2) $\qquad f_i(x_1,\ldots, x_m, \phi_1,\ldots, \phi_n)=0$

for every point of the domain of definition of $\phi_1\ldots, \phi_n$.

(3) *each of* φ_i *is continuous and possesses continuous first order partial derivatives w r to each of*

$$x_1,\ldots\ldots\ldots,x_m.$$

159. Sufficient conditions for locally invertible transformations.

159·1. Case of Two Variables. If the two functions

$$u=f(x, y),\ v=\varphi(x, y),$$

possesses continuous first order partial derivatives at and in a neighbourhood of a point (a, b) *and*

$$\partial(u, v)/\partial(x, y)\neq 0 \text{ at } (a, b),$$

then the transformation is locally invertible at (a, b).

Let

$$\alpha=f(a, b),\ \beta=\varphi(a, b).$$

Consider now the two functions

$$f_1(u, v, x, y)\equiv u-f(x, y)$$
$$\varphi_1(u, v, x, y)\equiv v-\varphi(x, y),$$

of four variables u, v, x, y. These two functions possess continuous first order partial derivatives w.r. to each of the four variables in a nighbourhood of (a, b, α, β) and

$$f_1(\alpha, \beta, a, b) = \alpha - f(a, b) = 0$$

$$\varphi_1(\alpha, \beta, a, b) = \beta - \varphi(a, b) = 0$$

$$\frac{\partial(f_1, \varphi_1)}{\partial(x, y)} = -\frac{\partial(f, \varphi)}{\partial(x, y)} \neq 0 \text{ at } (a, b, \alpha, \beta).$$

Thus by the theorem of § 158 page 345 there exists one and only one pair of functions

$$x = g(u, v) \; y = h(u, v)$$

defined in a certain neighbourhood of (α, β) such that

(*i*) $a = g(\alpha, \beta)$, $b = h(\alpha, \beta)$,

(*ii*) $f_1[u, v, g(u, v), h(u, v)] = u - f[g(u, v), h(u, v)] = 0$,

$f_2[u, v, g(u, v), h(u, v)] = v - \varphi[g(u, v), h(u, v)] = 0$,

for every point (u, v) of the domain of definition of $g(u, v)$ and $h(u, v)$,

(*iii*) $g(u, v)$ and $h(u, v)$ are continuous and possess continuous first order partial derivatives w.r. to u and v.

From above, we deduce that there exist neighbourhoods of (a, b) and (α, β) such that the neighbourhood of (a, b) is mapped one-one on the neighbourhood of (α, β) by

$$u = f(x, y), \; v = \varphi(x, y).$$

The inverse transformation is given by

$$x = g(u, v), \; y = h(u, v).$$

The two functions g and h possess continuous first order partial derivatives in a neighbourhood of (α, β).

Hence the result.

Illustrations. **1.** Consider

$$u = x^2 - y^2, \; v = xy$$

We have

$$\frac{\partial(u, v)}{\partial(x, y)} = 2(x^2 + y^2),$$

so that the Jacobian is non-zero except at $(0, 0)$. The conditions in respect of continuity and partial derivability being clearly satisfied, we see that the transformation is locally invertible at each point other than $(0, 0)$.

2. For

$$u = e^x \cos y, \; v = e^x \sin y,$$

$$\frac{\partial(u, v)}{\partial(x, y)} = e^x \neq 0 \text{ at any point.}$$

Thus the transformation is locally invertible at each point.

159·2. Case of several variables. The following result is again a consequence of the theorem proved in §158, page 345.

If the functions

$$u_i = f_i(x_1, \ldots\ldots, x_n), \qquad 1 \leqslant i \leqslant n$$

possess continuous first order partial derivatives at and in a certain neighbourhood of

$$(a_1, \ldots\ldots, a_n)$$

and

$$\frac{\partial(u_1, \ldots\ldots, u_n)}{\partial(x_1, \ldots\ldots, x_n)} \neq 0,$$

at $(a_1, \ldots\ldots, a_n)$, *then the transformation given by*

$$u_i = f_i(x_1, \ldots\ldots, x_n) \qquad 1 \leqslant i \leqslant n$$

is locally invertible at $(a_1, \ldots\ldots, a_n)$.

Ex. Examine the nature of the following transformations from the point of view of invertibility :

(*i*) $x = r\cos\theta,\ y = r\sin\theta$. (*ii*) $u = x+y,\ v = x/(x+y)$.
(*iii*) $x = \sin\varphi\cos\theta,\ y = \sin\varphi\sin\theta$.
(*iv*) $x = u(1+v),\ y = v(1+u)$.

160. Case of Vanishing Jacobian. At a point where the Jacobian vanishes, the transformation may or may not be invertible.

The transformation

$$u = x^2 - y^2,\ v = xy$$

wherefor Jacobian is zero at (0, 0) is not invertible thereat but the transformation

$$u = x^3,\ v = y^3$$

is invertible at (0, 0) even though the Jacobian is zero there.

If, however, the Jacobian is such that it vanishes not only at one point but at every point in a neighbourhood of a certain point, then it is possible to draw some important conclusions in respect of the nature of the transformation. This point is considered in the following theorem.

161. Dependence of functions or functionally related functions. Let $u_1, u_2 \ldots, u_n$ be n functions of n variables $x_1 \ldots\ldots, x_n$. These functions are said to be *dependent or functionally related,* if there exist one or more relations between them such that $x_1, \ldots\ldots, x_n$ do not appear explicitly. In the alternative case, the functions are said to be independent.

Thus if

$$u = x+y-z,\ v = x-y+z,\ w = x^2+y^2+z^2-2yz,$$

then u, v, w are functionally related inasmuch we have the relation

$$u^2 + v^2 = 2w.$$

The theorem below gives necessary and sufficient condition for a system of functions to be functionally related. We consider the case of a system of three functions only even though the result and the line of proof is quite general.

Theorem. *The necessary and sufficient condition that three functions u, v, w of x, y, z be functionally related is that*

$$\frac{\partial(u, v, w)}{\partial(x, y, z)},$$

vanishes identically, i.e., vanishes at **every** *point in a certain neighbourhood of some point.*

The condition is necessary.

Suppose that

$$w=\varphi(u, v),$$

is the relation satisfied by

$$u=f_1(x, y, z),\ v=f_2(x, y, z),\ w=f_3(x, y, z),$$

for every (x, y, z) in a certain neighbourhood of some point (a, b, c).

Thus taking partial derivatives w.r. to x, y and z, we obtain

$$\frac{\partial w}{\partial x}=\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial w}{\partial y}=\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y},$$

$$\frac{\partial w}{\partial z}=\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial z}.$$

In order that these relations be consistent, we have

$$\begin{vmatrix} \frac{\partial w}{\partial x} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial y} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} & \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{vmatrix}=0$$

i.e.,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=0,$$

identically.

Thus the condition is necessary.

The same condition may be shown to be true, if the functional relation be of the form

$$u=\varphi(v, w) \text{ or } v=\varphi(u, w).$$

The condition is sufficient.

Let

$$u=f_1(x, y, z),\ v=f_2(x, y, z),\ w=f_3(x, y, z). \qquad \ldots(1)$$

We have

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=0,$$

i.e.,

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}=0, \qquad \ldots(2)$$

identically.

Suppose that at least one of the first minors of the Jacobian (2) is not zero at (a, b, c). Let

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \neq 0 \text{ at } (a, b, c).$$

By the theorem of § 158, page 345 the functional equations

$$u-f_1(x, y, z)=0,\ v-f_2(x, y, z)=0, \qquad \ldots(3)$$

determine implicit functions

$$x=g_1(u, v, z),\ y=g_2(u, v, z). \qquad \ldots(4)$$

These give

$$w=f_3(x, y, z)=f_3(g_1, g_2, z)=h(u, v, z), \qquad \ldots(5)$$

It will now be proved that

$$\partial h/\partial z=0.$$

In the following discussion,

$$u, v, z,$$

will be looked upon as independent and

$$x, y, w,$$

as dependent variables.

From (3), differentiating with respect to z, we obtain

$$0=\frac{\partial f_1}{\partial x}\frac{\partial g_1}{\partial z}+\frac{\partial f_1}{\partial y}\frac{\partial g_2}{\partial z}+\frac{\partial f_1}{\partial z}, \qquad \ldots(6)$$

$$0=\frac{\partial f_2}{\partial x}\frac{\partial g_1}{\partial z}+\frac{\partial f_2}{\partial y}\frac{\partial g_2}{\partial z}+\frac{\partial f_2}{\partial z}, \qquad \ldots(7)$$

As shown in Note 2, page 298, for function of two variables, there exists, a certain neighbourhood of (a, b, c) at every point of which

$$\frac{\partial(f_1, f_2)}{\partial(x, y)}\neq 0.$$

For the following, we shall be working in this neighbourhood of (a, b, c) and the corresponding neighbourhood of (α, β, γ) where

$$\alpha=f_1(a, b, c),\ \beta=f_2(a, b, c),\ \gamma=f_3(a, b, c).$$

Also, from (5), differentiating with respect to z,

$$\frac{\partial h}{\partial z}=\frac{\partial f_3}{\partial x}\frac{\partial g_1}{\partial z}+\frac{\partial f_3}{\partial y}\frac{\partial g_2}{\partial z}+\frac{\partial f_3}{\partial z}. \qquad \text{.. (8)}$$

Multiplying the elements of the first and second columns by $\partial g_1/\partial z$ and $\partial g_2/\partial z$ respectively and adding to those of the third in the determinant (2), we obtain, with the help of (6), (7) and (8),

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & 0 \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & 0 \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial h}{\partial z} \end{vmatrix}=0$$

i.e.,

$$\frac{\partial h}{\partial z}\cdot\frac{\partial(f_1, f_2)}{\partial(x, y)}=0.$$

As $\partial(f_1, f_2)/\partial(x, y)\neq 0$, we obtain

$$\frac{\partial h}{\partial z}=0$$

so that h does not depend upon z and we have

$$w=h(u, v),$$

as the relation sought for.

Suppose now that the first minors of the Jacobian (2) are all zero but one of the second minors is not zero. Let

$$\partial f_1/\partial x\neq 0 \text{ at } (a, b, c).$$

By the theorem of § 157·2, page 342 the functional equation

$$u-f_1(x, y, z)=0, \qquad \text{...(9)}$$

determines an implicit function

$$x=g(u, y, z), \qquad \text{...(10)}$$

so that we get

$$v=f_2(x, y, z)=f_2(g, y, z)=h_1(u, y, z), \qquad \text{..(11)}$$

$$w=f_3(x, y, z)=f_3(g, y, z)=h_2(u, y, z). \qquad \text{.. (12)}$$

It will now be proved that

$$\frac{\partial h_1}{\partial y}=0,\ \frac{\partial h_1}{\partial z}=0,\ \frac{\partial h_2}{\partial y}=0,\ \frac{\partial h_2}{\partial z}=0.$$

In the subsequent discussion,

$$u, y, z,$$

will be looked upon as independent and

$$x, v, w,$$

as dependent variables.

There exists again, a neighbourhood of (a, b, c) at each point of which

$$\partial f_1/\partial x \neq 0.$$

From (9), (11) and (12), differentiating partially w. r. to y, we obtain

$$0=\frac{\partial f_1}{\partial x}\frac{\partial g}{\partial y}+\frac{\partial f_1}{\partial y}, \qquad \ldots(13)$$

$$\frac{\partial h_1}{\partial y}=\frac{\partial f_2}{\partial x}\frac{\partial g}{\partial y}+\frac{\partial f_2}{\partial y}, \quad \frac{\partial h_2}{\partial y}=\frac{\partial f_3}{\partial x}\cdot\frac{\partial g}{\partial x}+\frac{\partial f_3}{\partial y}. \qquad \ldots(14)$$

Since

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}=0, \quad \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{vmatrix}=0,$$

we obtain, on making use of the relations (13), (14),

$$\frac{\partial h_1}{\partial y}=0, \quad \frac{\partial h_2}{\partial y}=0.$$

so that h_1 and h_2 do not contain y. It may similarly be shown that

$$\frac{\partial h_1}{\partial z}=0, \qquad \frac{\partial h_2}{\partial z}=0,$$

so that h_1 and h_2 do not contain z also.

$$v=h_1(u), \; w=h_2(u)$$

are two relations which exist between u, v and w in the present case.

Note 1. Geometrically interpreted the theorem states that if the Jocobian

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}$$

vanishes at each point in some neighbourhood of a point, then this three dimensional neighbourhood is mapped on a surface or a curve and not on a three dimensional neighbourhood as would be the case when the Jacobian is not zero.

Note 2. The theorem above is capable of generalisation to n functions of n variables.

Example

If $f(0)=0$, and $f'(x)=1/(1+x^2)$, prove, without using the method of integration that

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

We write

$$u=f(x)+f(y),\ v=\frac{x+y}{1-xy}.$$

Now

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{1}{1+x^2} & \dfrac{1}{1+y^2}\\[2mm] \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2}\end{vmatrix}\equiv 0.$$

Hence u and v are functionally related. Let

$$u=\phi(v),$$

i e.,

$$f(x)+f(y)=\phi\left(\frac{x+y}{1-xy}\right).$$

Putting $y=0$, we obtain

$$f(x)=\phi(x), \qquad \text{for } f(0)=0.$$

Thus

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

Exercises

1. If

$$u=(ayz+by+bz+c)/(y-z),$$
$$v=(azx+bz+bx+c)/(z-x),$$
$$w=(axy+bx+by+c)/(x-y),$$

show that u, v, w are connected by a functional relation and find it.

2. Show that

$$u=3x+2y-z,\ v=x-2y+z,\ w=x(x+2y-z)$$

are connected by a functional equation and find that equation.

3. Prove that the three functions, u, v, w are connected by an identical functional relation, if

$$u=\frac{x}{y-z},\ v=\frac{y}{z-x},\ w=\frac{z}{x-y}.$$

Find the functional relation.

4. Prove that the functions

$$F_1=x+y+z+t, \qquad F_2=x^2+y^2+z^2+t^2,$$
$$F_3=x^3+y^3+z^3+t^3, \qquad F_4=xyz+xyt+xzt+yzt,$$

are dependent

(*i*) by evaluating the Jocobian of the system,

(*ii*) by establishing a relation between the four functions.

5. Show that the quadratic forms

$$ax^2+2hxy+by^2,\ Ax^2+2Hxy+By^2$$

are independent unless

$$\frac{a}{A}=\frac{h}{H}=\frac{b}{B}.$$

6. If $z=f(x, y)$, $p=\partial z/\partial x$, $q=\partial z/\partial y$, show that

$$p,\ q,\ px+qy-z$$

can be expressed in terms of one of them, if

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2=0.$$

162. Stationary points under subsidiary conditions. *To find the stationary points of the function*

$$f(x_1, x_2, \ldots\ldots, x_n\,;\, u_1, u_2, \ldots\ldots, u_m), \qquad \ldots(1)$$

of $(n+m)$ variables which are connected by the, m, equations

$$\varphi_r(x_1, x_2, \ldots\ldots, x_n\,;\, u_1, u_2, \ldots\ldots, u_m)=0,\ (r=1, 2, \ldots\ldots, m), \qquad \ldots(2)$$

If we assume that the system of equations (2) is such so as to determine the m variables $u_1, u_2 \ldots\ldots, u_m$ as functions of $x_1, x_2, \ldots, x_n$, then the function f in (1) is essentially a function of n independent variables $x_1, x_2, \ldots\ldots, x_n$.

For a stationary point of this function, we must have $df=0$

$$0=df=\frac{\partial f}{\partial x_1}\,dx_1+\ldots\ldots+\frac{\partial f}{\partial x_n}\,dx_n+\frac{\partial f}{\partial u_1}\,du_1+\ldots\frac{\partial f}{\partial u_m}\,du_m$$

(§ 150, page 330), .. (3)

Again differentiating the system of equations (2), we obtain

$$\left.\begin{array}{l}
\dfrac{\partial \varphi_1}{\partial x_1}dx_1+\ldots\ldots+\dfrac{\partial \varphi_1}{\partial x_n}dx_n+\dfrac{\partial \varphi_1}{\partial u_1}du_1+\ldots\ldots+\dfrac{\partial \varphi_1}{\partial u_m}\,du_m=0, \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\dfrac{\partial \varphi_m}{\partial x_1}dx_1+\ldots\ldots+\dfrac{\partial \varphi_m}{\partial x_n}dx_n+\dfrac{\partial \varphi_m}{\partial u_1}du_1+\ldots\ldots+\dfrac{\partial \varphi_m}{\partial u_m}\,du_m=0
\end{array}\right\} \quad .. (4)$$

Solving the system (4) of m equations for the m differentials $du_1, du_2, \ldots\ldots, du_m$ of the dependent variables in terms of the n differentials $dx_1, dx_2, \ldots\ldots, dx_n$ of the independent variables and substituting their values in (3), we will express, df, in terms of the differentials of the independent variables only. Since $df=0$, the co-efficients of each of these n differentials must separately vanish. These n equations together with the system of equations (2) constitute a system of $(m+n)$ equations for determining the $(m+n)$ co-ordinates of the stationary points.

Ex. 1. Find the stationary points of the function, xy, where x, y are connected by the relation $x^2+y^2-a^2=0$.

2. Find the stationary points of the function $x^2y^2z^2$, where x, y, z are connected by the relation $x^2+y^2+z^2-a^2=0$.

162·1. Lagrange's method of multipliers. Lagrange has given a method of forming the system of equations for the determination of stationary points, which is often useful. It rests on the introduction of certain *undetermined multipliers*

$$\lambda_1, \lambda_2, \ldots\ldots, \lambda_m.$$

Multiplying the system of equations (4) of § 162 above by λ_1, $\lambda_2,\ldots\ldots, \lambda_m$ and adding the results to the equation (3), we obtain

$$0=df=\left(\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial x_1}\right)dx_1+\ldots\ldots+\left(\frac{\partial f}{\partial u_1}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial u_1}\right)du_1+\ldots\ldots \qquad \ldots(5)$$

We now assume that the m multipliers $\lambda_1\ldots\ldots, \lambda_m$ have been so chosen that the, m, co-officients of the differentials $du_1, du_2,\ldots, du_m$ all vanish, *i.e.*,

$$\frac{\partial f}{\partial u_1}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial u_1}=0,\ldots\ldots,\frac{\partial f}{\partial u_m}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial u_m}=0. \qquad \ldots(6)$$

Then (5) gives

$$0=df=\left(\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial x_1}\right)dx_1+\ldots\ldots+\left(\frac{\partial f}{\partial x_n}+\Sigma\lambda_r\frac{\partial \phi_r}{\partial x_n}\right)dx_n,$$

so that the differential df which vanishes is expressed in terms of the differentials of the independent variables only. Hence

$$\frac{\partial f}{\partial x_1}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial x_1}=0,\ldots\ldots,\frac{\partial f}{\partial x_n}+\Sigma\lambda_r\frac{\partial \varphi_r}{\partial x_n}=0. \qquad \ldots(7)$$

The systems (2), (6) and (7) of $(m+m+n)$, *i.e.*, $(n+2m)$ equations will determine the values of the m multipliers and of the $(m+n)$ co-ordinates of the stationary points of the function f.

An important note. We define a function

$$g=f+\lambda_1\varphi_1+\lambda_2\varphi_2+\ldots\ldots+\lambda_m\varphi_m,$$

and observe that, considering $x_1, x_2,\ldots\ldots, x_n, u_1, u_2,\ldots\ldots, u_m$ as independent variables,

$$\frac{\partial g}{\partial x_1}=0,\ \ldots..,\ \frac{\partial g}{\partial x_n}=0,\ \frac{}{\partial u_1}=0,\ldots\ldots,\ \frac{\partial g}{\partial u_m}=0,$$

are exactly the systems of equations (6) and (7).

In practice, therefore, the system of equations (6) and (7) may be conveniently obtained by first forming the function, g, and then equating its first order partial derivatives to zero, considering all the variables as independent.

To determine whether a stationary point is really an extreme point or not, it is necessary to consider the second order differential d^2F where, F, denotes the function, f, considered as a function of x_1, $x_2, \ldots\ldots, x_n$ only. In this connection it is generally found convenient to make use of the fact *that at a stationary point $d^2F=d^2g$, where the differential d^2g is calculated on the supposition that* **all** *the variables are independent.* Of course, in this expression for d^2g, we have to replace the differentials of dependent vaiables by their values in terms of the differentials of independent variables as obtained from

$$d\varphi_1=0, \ldots\ldots, d\varphi_m=0.$$

Let f and g considered as functions of $x_1, x_2, \ldots\ldots, x_n$ alone be denoted as F and G. Since, at the stationary points, φ's, vanish, we have

$$F(x_1, x_2, \ldots\ldots, x_n)=G(x_1, x_2, \ldots\ldots, x_n).$$

and

$$d^2F=d^2G$$

But

$$d^2G=d^2g+\Sigma\left(\frac{\partial g}{\partial x_1}d^2x_1+\ldots\ldots+\frac{\partial g}{\partial u_1}d^2u_1\ldots\right)=d^2g$$

(§ 164, page 311)

Hence the result.

Examples

1. *If $u=a^3x^2+b^3y^2+c^3z^2$ where $1/x+1/y+1/z=1$, show that a stationary value is given by $ax=by=cz$ and this gives a maximum or a minimum if $abc(a+b+c)$ is positive.*

Let

$$g(x, y, z)=a^3x^2+b^3y^2+c^3z^2+\lambda(1/x+1/y+1/z-1).$$

Equating to zero, the first order partial derivatives of the function g, we obtain

$$2a^3x-\lambda/x^2=0,\ 2b^3y-\lambda/y^2=0,\ 2c^3z-\lambda/z^2=0, \qquad \ldots (i)$$

which give

$$ax=by=cz. \qquad \ldots (ii)$$

It may be seen that for the stationary point, we have

$$x=(a+b+c)/a,\ y=(a+b+c)/b,\ z=(a+b+c)/c,$$

and

$$\lambda=2(a+b+c)^3.$$

To determine whether the stationary value, in question, is an extreme value or not, we find d^2g. We have

$$d^2g=\left(\frac{\partial}{\partial x}\,dx+\frac{\partial}{\partial y}\,dy+\frac{\partial}{\partial z}\,dz\right)^2 g$$

$$=\left(2a^3+\frac{2\lambda}{x^3}\right)\left(dx\right)^2+\left(2b^3+\frac{2\lambda}{y^3}\right)\left(dy\right)^2+\left(2c^3+\frac{2\lambda}{z^3}\right)\left(dz\right)^2$$

$$=6[a^3(dx)^2+b^3(dy)^2+c^3(dz)^2], \text{ from } (i).$$

There being only two independent variables, we shall express d^2g in terms of dx and dy alone. From the relation $\Sigma 1/x=1$, we have

$$\Sigma dx/x^2=0 \text{ or } \Sigma a^2dx=0, \text{ from } (ii).$$

$$\therefore\quad d^2g=\frac{6}{c}\,[(c+a)a^3(dx)^2+2a^2b^2dxdy+(c+b)b^3(dy)^2].$$

We have seen in § 148·3, page 325, that $Ah^2+2Bhk+Ck^2$ is definite if B^2-AC is negative. Thus d^2g is definite if

$$\frac{a^4b^4}{c^4}-\frac{(c+a)(c+b)a^3b^3}{c^2}=\frac{-(a+b+c)a^3b^3}{c}=\frac{-(a+b+c)(abc)^3}{c^4},$$

is negative, *i.e.*, $(a+b+c)\,abc$ is positive.

2. *Prove that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is $8\,abc/3\sqrt{3}$.*

The problem is to find the greatest value of $V=8xyz$ where x, y, z are all positive and subject to the condition

$$x^2/a^2+y^2/b^2+z^2/c^2=1. \qquad \ldots(1)$$

Let

$$g(x,\ y,\ z)=8xyz+\lambda(x^2/a^2+y^2/b^2+z^2/c^2-1).$$

For stationary points, we have

$$g_x=8yz+2\lambda x/a^2=0, \qquad \ldots(2)$$

$$g_y=8zx+2\lambda y/b^2=0, \qquad \ldots(3)$$

$$g_z=8xy+2\lambda z/c^2=0. \qquad \ldots(4)$$

Multiplying (2), (3), (4) by x, y, z respectively and adding, we get

$$24xyz+2\lambda=0, \text{ or } \lambda=-12xyz, \text{ using } (1). \qquad \ldots(5)$$

Thus we obtain, from (2) and (5), $x=a/\sqrt{3}$.

Similarly

$$y=b/\sqrt{3},\ z=c/\sqrt{3}.$$

Also,

$$\therefore\qquad \lambda=-4abc/\sqrt{3}.$$

Thus the stationary value of $V=8abc/3\sqrt{3}$.

Again

$$d^2g=2\lambda\Sigma(dx)^2/a^2+16\Sigma ydzdx$$

$$=-(8abc/\sqrt{3})\Sigma(dx)^2/a^2+(16/\sqrt{3})\Sigma bdzdx \qquad \ldots(6)$$

Also we have

$$xdx/a^2+ydy/b^2+zdz/c^2=0,$$

i.e., $$dx/a+dy/b+dz/c=0,$$

for the stationary point in question.

These give

$$2dxdy/ab=(dz)^2/c^2-(dx)^2/a^2-(dy)^2/b^2,$$

and two similar results.

Substituting these values of $dxdy$, etc., in (6), we see that

$$d^2g=(-16abc/\sqrt{3})[\Sigma(dx)^2/a^2],$$

which is a definite negative form.

Hence the stationary value, in question, is a maximum.

Note. As in the preceding example, the question could also be completed by expressing d^2g in terms of dx and dy alone.

3. *If* $\varphi(a)=k\neq 0$, $\varphi'(a)\neq 0$ *and* x, y, z, *satisfy the relation*

$$\varphi(x)\varphi(y)\varphi(z)=k^3,$$

prove that the function

$$f(x)+f(y)+f(z)$$

has a maximum when $x=y=z=a$, *provided that*

$$f'(a)\left\{\frac{\varphi''(a)}{\varphi'(a)}-\frac{\varphi'(a)}{\varphi(a)}\right\}>f''(a).$$

Let

$$g(x, y, z)=f(x)+f(y)+f(z)+\lambda[\varphi(x)\varphi(y)\varphi(z)-k^3].$$

For stationary points, we have

$$g_x=f'(x)+\lambda\varphi'(x)\varphi(y)\varphi(z)=0,$$
$$g_y=f'(y)+\lambda\varphi(x)\varphi'(y)\varphi(z)=0,$$
$$g_z=f'(z)+\lambda\varphi(x)\varphi(y)\varphi'(z)=0.$$

For the function to be a minimum at (a, a, a), we must thus necessarily have

$$f'(a)+\lambda\varphi'(a)\varphi(a)\varphi(a)=0,$$

i.e., $$\lambda=-f'(a)/\varphi'(a)\varphi^2(a), \text{ for } \varphi'(a)\neq 0, \varphi(a)\neq 0.$$

Now, $$d^2g=\Sigma[f''(x)+\lambda\varphi''(x)\varphi(y)\varphi(z)](dx)^2+2\lambda\Sigma\varphi'(x)\varphi'(y)\varphi(z)\,dxdy$$
$$=[f''(a)+\lambda k^2\varphi''(a)]\,\Sigma(dx)^2+2\lambda k[\varphi'(a)^2]\,\Sigma\,dxdy, \text{ at } (a, a, a).$$

From the given equation, we have

$$\Sigma\varphi'(x)\varphi(y)\varphi(z)\,dx=0$$

so that for (a, a, a), we have $\Sigma dx=0$.

This gives

$$2\Sigma\,dxdy=-\Sigma(dx)^2.$$

Thus we have

$$d^2g=[f''(a)-f'(a)\varphi''(a)/\varphi'(a)]\,\Sigma(dx)^2+[f'(a)\varphi'(a)/\varphi(a)]\,\Sigma(dx)^2$$
$$=\left\{f''(a)-f'(a)\left[\frac{\varphi''(a)}{\varphi(a)}-\frac{\varphi'(a)}{\varphi(a)}\right]\right\}\Sigma\,(dx)^2,$$

which, under the condition to be proved, is a definite negative form.

Hence the result.

Exercises

1. Find the minimum value of $x^2+y^2+z^2$, when
(*i*) $x+y+z=3a$. (*ii*) $xy+yz+zx=3a^2$. (*iii*) $xyz=a^3$.

2. Find the stationary values of $ayz+bzx+cxy$ where $x+y+z=1$.

3. Find the extreme values of xy when $x^2+xy+y^2=a^2$.

4. Determine the maximum and minimum values of $7x^2+8xy+y^2$ when
$$x^2+y^2=1.$$

5. Find the shortest distance of the point (a, b, c) from the plane
$$lx+my+nz=0.$$

6. Find the shortest distance between the lines
$$(x-x_1)/l_1=(y-y_1)/m_1=(z-z_1)/n_1\ ;\ (x-x_2)/l_2=(y-y_2)/m_2=(z-z_2)/n_2.$$

7. Find the point of the ellipse $5x^2-6x+5y^2=4$ for which the tangent is a the greatest distance from the origin.

8. Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point (2, 1, 3).

9. Determine the maximum and minimum values of
$$u=(x+1)(y+1)(z+1)$$
where x, y, z are connected by the relation $a^xb^yc^z=k$.

10. Given x_0, y_0, z_0 are constants and x_1, y_1, z_1 are variables such that $\Sigma x_0=\Sigma x_1=1$, $lx_1+my_1+nz_1=1$; find the minimum value of the function
$$a^2(y_0-y_1)(z_0-z_1)+b^2(z_0-z_1)(x_0-x_1)+c^2(x_0-x_1)(y_0-y_1),$$
where a, b, c and l, m, n are constants.

11. Show that the length of the axes of the section of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ by the plane $lx+my+nz=0$ are the roots of the equation
$$l^2a^2/(r^2-a^2)+m^2b^2/(r^2-b^2)+n^2c^2/(r^2-c^2)=0.$$

12. If $lx+my+nz=1$, where l, m, n are positive constants, prove that the only stationary value of
$$yx+zx+xy,$$
is
$$(2mn+2nl+2lm-l^2-m^2-n^2)^{-1},$$
and that this value is a maximum when it is positive.

13. Show that
$$u=yz+zx+xy,$$
has no extreme value when considered as a function of three independent variables x, y, z but has a maximum value when the three variables are connected by the relation
$$ax+by+cz=1,$$
and a, b, c are positive constants satisfying the condition
$$2(ab+bc+ca)>(a^2+b^2+c^2).$$

14. A point P moves on the plane $x+y+z=k$ and another point Q on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Assuming that the plane does not cut the ellipsoid, find the shortest distance between P and Q.

15. Find the maximum and minimum radii vectors of the section of the surface

$$(x^2+y^2+z^2)^2=l^2x^2+m^2y^2+n^2z^2,$$

made by the plane

$$ax+by+cz=0.$$

16. If $u=f(x, y)$ where x and y are connected by a relation $\varphi(x, y)=0$ attains a stationary value, show that it does so when x and y satisfy the relations

$$f_x+\lambda\varphi_x=0,\ f_y+\lambda\varphi_y=0,\ \varphi=0,$$

where λ is a Lagrange's multiplier.

Show also that this stationary value is certainly a maximum when the values of x, y, λ make

$$\begin{vmatrix} f_{xx}+\lambda\varphi_{xx} & f_{xy}+\lambda\varphi_{xy} & \varphi_x \\ f_{xy}+\lambda\varphi_{xy} & f_{xy}+\lambda\varphi_{xy} & \varphi_y \\ \varphi_x & \varphi_y & 0 \end{vmatrix}$$

positive.

17. Show that if

$$x_1^2+y_1^2+z_1^2=a^2,\ x_2^2+y_2^2+z_2^2=b^2,\ x_3^2+y_3^2+z_3^2=c^2,$$

then the maximum and minimum value of the determinant

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$

are respectively $+abc$ and $-abc$.

18. If x, y, z are subject to the condition $ax+by+cz=1$, show that, in general $x^3+y^3+z^3-3xyz$ has the two stationary values 0 and $(a^3+b^3+c^3-3abc)^{-1}$ of which the first is a maximum or a minimum according as $a+b+c \lesseqgtr 0$, but the second is not an extreme value. Discuss, in particular, the cases in which (*i*) $a+b+c=0$, (*ii*) $a=b=c$.

19. Obtain the stationary values of

$$x^3+y^3+z^3+3mxyz, \qquad (m \neq 2)$$

when x, y, z are subject to the condition

$$x+y+z=1,$$

and show that the symmetrical stationary value is a maximum or minimum according as $m \lesseqgtr 2$. Show also that the other stationary values are not extreme values.

Show that $x^3+y^3+z^3+6xyz$ has only one stationary value and no extreme value.

20. Establish Lagrange's method of undetermined multipliers for finding the stationary values of a function $f(x, y, z, \ldots)$ of n variables connected by $m(<n)$ independent equations $\varphi_r(x, y, z, \ldots)=0$, $(r=1, 2, \ldots m)$, f and φ being supposed to have continuous derivatives for all values of the variables concerned.

The sum of 12 edges of a rectangular block is a; the sum of the areas of the 6 faces is $a^2/25$. Prove that, when the excess of the volume of the block over that of a cube whose edge is equal to least edge of the block is greatest, the least edge is $a/20$ and find the other edges.

21. If x, y and z are connected by the relation

$$f(x)+\varphi(y)+\psi(z)=f(a)+\varphi(b)+\psi(c),$$

show that the following conditions are sufficient for $x^2+y^2+z^2$ to have a maximum value at (a, b, c);

$$A\psi_1^2(c)+Cf_1^2(a) < 0 < AB\psi_1^2(c)+BCf_1^2(a)+CA\varphi_1^2(b).$$

where

$$A=1-af_2(a)/f_1(a),\ B=1-b\varphi_2(b)/\varphi_1(b),\ C=1-c\psi_2(c)/\psi_1(c),$$

provided none of the derivatives $f_1(a)$, $\varphi_1(b)$, $\psi_1(c)$ is zero and

$$a/f_1(a)=b/\varphi_1(b)=c/\psi_1(c).$$

CHAPTER XII

DEFINITE INTEGRALS AS FUNCTIONS OF A PARAMETER

163. Definite integral as function of a parameter. Let $f(x, y)$ be a continuous function of two variables defined in

$$R[a, b\ ;\ c, d].$$

For a *fixed* value of y in $[c, d]$, the function $f(x, y)$ of x is continuous and therefore the integral

$$\int_a^b f(x, y)\, dx,$$

exists and defines a function of y, say $\phi(y)$, in $[c, d]$.

It is now proposed to investigate the nature of the function $\phi(y)$ in relation to continuity and derivability.

163·1. Theorem. *The function* $\phi(y)$ *is continuous in* $[c, d]$.

Let y, $y+k$ be any two points of $[c, d]$. We have

$$\phi(y+k)-\phi(y)=\int_a^b [f(x, y+k)-f(x, y)]\, dx.$$

Let ε be any positive number.

Since $f(x, y)$ is continuous in R, therefore there exists a positive number δ such that

$$|f(x_2, y_2)-f(x_1, y_1)| < \varepsilon/(b-a), \text{ when } |x_2-x_1| \leqslant \delta,\ |y_2-y_1| \leqslant \delta.$$

(Cor. 4, § 130 page 300)

In particular, we see that *when* $|k| \leqslant \delta$ *and* x *has any value*, we have

$$|f(x, y+k)-f(x, y)| < \varepsilon/(b-a).$$

We have

$$|\phi(y+k)-\phi(y)| = \left|\int_a^b [f(x, y+k)-f(x, y)]\, dx\right|$$

$$\leqslant \int_a^b |f(x, y+k)-f(x, y)|\, dx$$

(Cor. 6, page 168)

$$\leqslant [\varepsilon/(b-a)]\,(b-a)=\varepsilon, \text{ when } |k| \leqslant \delta.$$

Hence $\phi(y)$ is a continuous function of y in $[c, d]$.

Ex. If $f(x, y)$ is continuous in $R[a, b\,;\, c, d]$ and $F(x)$ is a function of x which is bounded and integrable in $[a, b]$, then $\int_a^b f(x, y)\, F(x)\, dx$ is a cotinuous function of y in $[c, d]$.

163·2. Theorem. *If, in addition to the continuity of $f(x, y)$, $f_y(x, y)$ also exists and is continuous in $R[a, b\,;\, c, d]$, then $\phi(y)$ is derivable in $[c, d]$ and*

$$\phi'(y) = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx,$$

i.e.

$$\frac{d}{dy}\left\{\int_a^b f(x, y)\, dx\right\} = \int_a^b \frac{\partial f(x, y)}{\partial y}\, dx,$$

so that the inversion of the operations of differentiation and integration is valid.

We have

$$\phi(y+k) - \phi(y) = \int_a^b [f(x, y+k) - f(x, y)]\, dx$$

By Lagrange's mean value theorem, we have

$$\begin{aligned} f(x, y+k) - f(x, y) &= kf_y(x, y+\theta k) \\ &= k[f_y(x, y+\theta k) - f_y(x, y) + f_y(x, y)]. \end{aligned}$$

Thus

$$\frac{\phi(y+k) - \phi(y)}{k} - \int_a^b f_y(x, y)\, dx = \int_a^b [f_y(x, y+\theta k) - f_y(x, y)]\, dx.$$

Let ε be any positive number. Because of the continuity and consequent uniform continuity of $f_y(x, y)$, there exists a positive number δ, such that when $|\, k \,| \leqslant \delta$ and x has any value, then

$$|\, f_y(x, y+k) - f_y(x, y) \,| < \varepsilon/(b-a).$$

Thus there exists a positive number δ such that

$$\left| \frac{\phi(y+k) - \phi(y)}{k} - \int_a^b f_y(x, y)\, dx \right| < \varepsilon \text{ when } |\, k \,| \leqslant \delta.$$

Hence the result.

Ex. If $f(x, y)$ and $f_y'(x, y)$ be continuous in R and $F(x)$ be bounded and integrable in $[a, b]$, then $\int_a^b f(x, y) F(x)\, dx$ is derivable in $[c, d]$ and the derivative is

$$\int_a^b f_y'(x, y) F(x)\, dx.$$

Note. In the investigation above, the limits a, b of integration were considered independent of y. The case where they are themselves functions of y is considered below.

163·3. *Let $f(x, y), f_y'(x, y)$ be continuous in $R[a, b\,;\, c, d]$; and let $g_1(y), g_2(y)$ be two functions of y derivable in $[c, d]$ such that the points*

$$[g_1(y), y] \text{ and } [g_2(y), y]$$

belong to the rectangle R for every value of y in $[c, d]$, then

$$\varphi(y) = \int_{g_1(y)}^{g_2(y)} f(x, y)\, dx,$$

is derivable is $[c, d]$ and

$$\varphi'(y) = \left\{ \int_{g_1(y)}^{g_2(y)} f_y(x, y)\, dx \right\} - g_1'(y) f[g_1(y), y] + g_2'(y) f[g_2(y), y].$$

We have

$$\varphi(y+k) - \varphi(y) = \int_{g_1(y)}^{g_2(y)} [f(x, y+k) - f(x, y)]dx -$$

$$\int_{g_1(y)}^{g_1(y+k)} f(x, y+k)\, dx + \int_{g_2(y)}^{g_2(y+k)} f(x, y+k)\, dx.$$

Applying the result of Cor. 2 to § 95, page 166 to each of the last two integrals and dividing by k, we get

$$\frac{\varphi(y-k) - \varphi(y)}{k} = \int_{g_1(y)}^{g_2(y)} f_y(x, y+\theta k)\, dx - \frac{g_1(y+k) - g_1(y)}{y} f(\xi, y+k) +$$

$$\frac{g_2(y+k) - g_2(y)}{k} f(\eta, y+k),$$

where ξ, η lie between $g_1(y), g_1(y+k)$, and $g_2(y), g_2(y+k)$ respectively. As in the previous section, it can be shown that, when $k \to 0$, then

$$\int_{g_1(y)}^{g_2(y)} f_y(x, y+\theta k)\, dx \to \int_{g_1(y)}^{g_2(y)} f_y(x, y)\, dx.$$

Thus on taking limits, when $k \to 0$, we deduce that

$$\varphi'(y)=\int_{g_1(y)}^{g_2(y)} \left[f_y(x, y)\, dx \right] - g_1'(y) f[g_1(y), y] + g_2'(y) f[g_2(y), y].$$

163·4. Inversion of the order of Integration. *If $f(x, y)$ be continuous in $R[a, b\,;\, c, d]$, then*

$$\int_c^d \left\{ \int_a^b f(x, y)\, dx \right\} dy = \int_a^b \left\{ \int_c^d f(x, y)\, dy \right\} dx,$$

i.e., the two repeated integrals are equal.

Because of the continuity of $f(x, y)$, the integrals of $f(x, y)$ over $[a, b]$ with respect to x and over $[c, d]$ with respect to y exist (§ 92 1, page 155) and are continuous functions of y and x respectively (§163·1, page 363) and therefore both the repeated integrals exist.

We consider two functions of t defined as follows :—

$$\varphi(t)=\int_c^t \left\{ \int_a^b f(x, y) dx \right\} dy \,;\; \psi(t)=\int_a^b \left\{ \int_c^t f(x, y)\, dy \right\} dx,$$

so that

$$\varphi(c)=\psi(c)=0$$

Now

$$\varphi'(t)=\int_a^b f(x, t)\, dx \qquad \text{(§ 96·2, page 170)}$$

Also
$$\psi'(t)=\int_a^b \left\{ \frac{\partial}{\partial t} \int_c^t f(x, y)\, dy \right\} dx, \qquad \text{(§ 163 2, page 364)}$$

$$=\int_a^b f(x, t)\, dx. \qquad \text{(§ 96·2, page 170)}$$

Since, $\varphi'(t)=\psi'(t)$, therefore $\varphi(t)$ and $\psi(t)$ differ by a constant. Also, since $\varphi(c)=\psi(c)$, we see that for every value of t, $\varphi(t)=\psi(t)$. Putting $t=d$, we obtain the required equality.

Ex. If $f(x, y)$ is continuous in $R[a, b\,;\, c, d]$ and

$$F(x, y)=\int_c^y \left\{ \int_a^x f(x,y)\, dx \right\} dy,$$

then show that

$$F_{xy}=F_{yx}=f(x, y).$$

Deduce that

$$\int_a^b \left\{ \int_c^d f(x, y)dy \right\} dx = F(a, c) + F(b, d) - F(a, d) - F(b, c)$$

$$= \int_c^d \left\{ \int_a^b f(x, y)\, dx \right\} dy.$$

Note. The theorems above enable us to evaluate certain definite integrals without the knowledge of the corresponding primitives. A few examples of such evaluation are given below.

Example

1. *If* $|a| < 1$, *show that*

$$\int_0^\pi \frac{\log(1+a \cos x)}{\cos x}\, dx = \pi \sin^{-1} a.$$

We write

$$f(a, x) = \frac{\log(1+a \cos x)}{\cos x}.$$

It may be seen that $x = \pi/2$ is only a removable discontinuity of the integrand inasmuch as

$$\lim_{x \to \pi/2} \frac{\log(1+a \cos x)}{\cos x} = a.$$

Also

$$\frac{\partial f(a, x)}{\partial a} = \frac{1}{1+a \cos x}.$$

We see that $f(a, x)$ and $\partial f(a, x)/\partial a$ are both continuous functions of a, x for $0 \leqslant x \leqslant \pi$, $|a| < 1$, if we assign $f(a, x)$ the value a for $x = \pi/2$.

We write

$$\phi(a) = \int_0^\pi \frac{\log(1+a \cos x)}{\cos x}\, dx.$$

$$\phi'(a) = \int_0^\pi \frac{\partial}{\partial a}\left[\frac{\log(1+a \cos x)}{\cos x} \right] dx. \quad \text{.. (1)}$$

$$= \int_0^\pi \frac{1}{1+a \cos x}\, dx.$$

Putting $\tan(x/2)=t$, we may see that

$$\int_0^\pi \frac{dx}{1+a\cos x}=\frac{\pi}{\sqrt{(1-a^2)}}.$$

$$\therefore \qquad \phi'(a)=\frac{\pi}{\sqrt{(1-a^2)}}.$$

Integrating, we get

$$\phi(a)=\sin^{-1} a+c, \qquad \ldots(2)$$

where, c, is any arbitrary constant.

From (1), we have

$$\phi(0)=0.$$

Putting $a=0$ in (2), we obtain $c=0$.

Hence,

$$\phi(a)=\pi\sin^{-1} a.$$

Exercises

1. Starting from a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}.$$

2. Find the value of

$$\int_0^\pi \frac{dx}{a+b\cos x}, \qquad a>0 \ |b|<a$$

and deduce that

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2}=\frac{\pi a}{(a^2-b^2)^{3/2}} \text{ and } \int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^2}=-\frac{\pi b}{(a^2-b^2)^{3/2}}.$$

3. Show that

$$\int_0^{\frac{1}{2}\pi} \log(1-x^2\cos^2\theta)\,d\theta=\pi\log[1+\sqrt{(1-x^2)}]-\pi\log 2,$$

if $x^2 \leqslant 1$.

4. If $|a| \leqslant 1$, show that

$$\int_0^\pi \log(1+a\cos x)dx=\pi\log[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-a^2)}].$$

5. Show that

$$\int_0^a \frac{\log(1+ax)}{1+x^2}\,dx = \tfrac{1}{2}\log(1+a^2)\tan^{-1} a.$$

Hence or otherwise show that

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi \log 2}{8}.$$

6. Show that

$$\int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta = \pi\log[\tfrac{1}{2}(a+b)]. \qquad (a > 0,\ b > 0)$$

7. Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx = \frac{\pi^2 - 4\alpha^2}{8}.$$

8. Show that, if $a > b$,

$$\int_0^{\pi} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\,d\theta = \pi\sin^{-1}\frac{b}{a}.$$

9. Prove that

$$\int_{\frac{1}{2}\pi-\alpha}^{\pi} \sin\theta\cos^{-1}(\cos\alpha\operatorname{cosec}\theta)\,d\theta = \frac{\pi}{2}(1-\cos\alpha).$$

10. Show that

$$\int_0^{\frac{1}{2}\pi} \frac{\log(1+y\sin^2 x)}{\sin^2 x}\,dx = \pi[\sqrt{(1+y)} - 1].$$

11. Verify that

$$y = \frac{1}{k}\int_0^x f(t)\sin k(x-t)\,dt,$$

satisfies the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = f(x),$$

where k is a constant.

12. Show that the theorem of § 163·2, p. 364 does not hold for $y=0$ when, $a=0$, $b=1$ and $f(x, y) = xy^3/(x+y^2)^3$.

13. If

$$f(x, y)=y^3/(x^2+y^2) \text{ and } g(y)=\int_0^1 f(x, y)\, dx,$$

show that $g'(+0)$ and $g'(-0)$, the right and left hand derivatives of $g(y)$ at $y=0$ differ from each other and from

$$\int_0^1 f_y(x, 0)\, dx.$$

14. Show that

$$\int_0^1 \frac{x^y-1}{\log x}\, dx=\log(1+y), \int_0^1 \frac{x-1-\log x}{(\log x)^2}\, dx=2 \log 2-1.$$

164. Uniform Convergence of Improper Integrals. The operations of differentiation and integration may not be invertible when the integrals, in question, are improper so that it should be carefully understood that while proving various theorems in the preceding § 163, we have constantly regarded the integral

$$\int_a^b f(x, y)\, dx,$$

as proper. The situation here is analogous to that for infinite series in which case derivative (integral) of the sum may not be equal to the sum of the derivatives (integrals). It has been seen in § 106, page 198, that with the help of the notion of uniform convergence of a series, it is possible to lay down sufficient conditions for the invertibility of the operation of derivation (integration) and the summation of a series. It will now be seen that the corresponding notion of the uniform convergence of improper integrals can do the same job for improper integrals.

164·1. Definitions 1. Range Infinite. *The convergent improper integral*

$$\varphi(y)=\int_a^\infty f(x, y)\, dx, \qquad \dots(1)$$

is said to converge uniformly w. r. to y in the interval, $c \leqslant y \leqslant d$, if corresponding to any positive number, η, there exists a positive number

X_1 **which does not depend on y,** *such that*

$$\left| \int_a^X f(x, y)dx - \varphi(y) \right| = \left| \int_X^\infty f(x, y)\, dx \right| < \eta,$$

for $X \geqslant X_1$.

164·2. The integrand unbounded. Let $f(x, y)$ tend to ∞ as $x \to a$. *The convergent improper integral*

$$\varphi(y) = \int_a^b f(x, y)\, dx, \qquad \text{.. (2)}$$

is said to converge uniformly in the interval $[c, d]$, *if corresponding to any positive number,* η, *there exists a positive number* δ, *independent of* y, *such that*

$$\left| \int_{a+\varepsilon} f(x, y)\, dx - \varphi(y) \right| < \eta,$$

for $0 < \varepsilon \leqslant \delta$.

165. The tests for uniform convergence. The following are the straightforward analogues of the tests for the uniform convergence of series and may be proved in a similar manner.

165·1. General test. *The necessary and sufficient condition for the uniform convergence of the improper integral* (1) *is that corresponding to any positive number* η, *there exists a positive number* X, **independent of y,** *such that*

$$\left| \int_{X_1}^{X_2} f(x, y)\, dx \right| < \eta, \text{ where } X_1, X_2 \geqslant X.$$

165·2. Weierstrass's 'M' Test. *If a function* $M(x)$ *of* x *is positive and is such that*

$$\int_a^\infty M(x)\, dx,$$

converges and if

$$| f(x, y) | \leqslant M(x),$$

for $c \leqslant y \leqslant d$ *and every value of* x *belonging to the interval under consideration, then the integral,* (1) *of* § 164·1 *is uniformly convergent.*

Note. Similar theorem holds for the uniform convergence of

$$\int_a^b f(x, y)\, dx.$$

Examples

1. Since $\mid e^{-x^2} \cos yx \mid \leqslant e^{-x^2}$ for every value of y, and

$$\int_0^\infty e^{-x^2}\, dx,$$

is convergent, therefore,

$$\int_0^\infty e^{-x^2} \cos yx\, dx,$$

is uniformly convergent in y in the interval $]-\infty, \infty[$

2. Since $\mid \cos yx/\sqrt{(1-x^2)} \mid \leqslant 1/\sqrt{(1-x^2)}$,

and

$$\int_{-1}^{+1} \frac{1}{\sqrt{(1-x^2)}}\, dx$$

is convergent, therefore

$$\int_{-1}^{+1} \frac{\cos yx}{\sqrt{(1-x^2)}}\, dx,$$

is uniformly convergent in y in the interval $]-\infty, \infty[$.

3. The uniformity of convergence of

$$\int_1^\infty \frac{x \sin xy}{1+x^2}\, dx,$$

cannot be established by M-test.

For $x \geqslant 1$, $x/(1+x^2)$ is monotonically decreasing and tends to 0 as $x \to \infty$ so that employing Bonnett's form of second mean value theorem, we have

$$\left| \int_{X_1}^{X_2} \frac{x \sin xy}{1+x^2}\, dx \right| = \left| \frac{X_1}{1+X_1^2} \int_{X_1}^{\xi} \sin xy\, dx \right| \leqslant \frac{2X_1}{k(1+X_1^2)},$$

if $y > k$, where k is any fixed positive number.

Thus we see that the integral is uniformly convergent for $y \geqslant k > 0$. (§ 165·1).

Exercises

1. Establish the uniform convergence of the following integrals :—

(*i*) $\int_0^\infty \frac{y\,dx}{x^2+y^2}$, $(0 < c \leqslant y \leqslant d)$, (*ii*) $\int_0^\infty e^{-yx} \sin x\,dx$, $(y \geqslant k > 0)$.

(*iii*) $\int_0^\infty e^{-xy} \frac{\sin x}{x}\,dx$, $(y \geqslant 0)$, (*iv*) $\int_0^\infty e^{(-x^2-y^2/x^2)}\,dx$.

(*v*) $\int_0^\infty e^{-xy}\,dx$, $(y \geqslant k > 0)$.

166·1. Theorem. *Uniformly convergent improper integral of a continuous function is itself a continuous function.*

Consider the improper integral (1) of § 164·1, page 370. We choose a number X such that

$$\left| \int_X^\infty f(x, y)dx \right| = \left| \int_a^X f(x, y)\,dx - \varphi(y) \right| < \frac{\varepsilon}{3}.$$

We have

$$|\ \varphi(y+k) - \varphi(y)\ | \leqslant \left| \int_a^X [f(x, y+k) - f(x, y)]\,dx \right| + \frac{2\varepsilon}{3}.$$

The range $[a, X]$ being finite and $f(x, y)$ being continuous, we can choose a positive number δ, as in § 163·1, page 363, so that the finite integral on the right is less than $\varepsilon/3$ when $|\ k\ | \leqslant \delta$. Thus $\varphi(y)$ is continuous.

166·2. *If $f(x, y)$ be a continuous function of x, y when $c \leqslant y \leqslant d$ and $x \geqslant a$ and the integral*

$$\varphi(y) = \int_a^\infty f(x, y)\,dx,$$

is uniformly convergent, then $\varphi(y)$ can be integrated under the integral sign, i.e.,

$$\int_c^d \left\{ \int_a^\infty f(x, y)\,dx \right\} dy = \int_c^d \varphi(y)dy = \int_a^\infty \left\{ \int_c^d f(x, y)\,dy \right\} dx.$$

We choose a number X_1 such that

$$\left| \int_X^\infty f(x, y)\, dx \right| < \varepsilon/(d-c) \text{ when } X \geqslant X_1,$$

so that

$$\int_c^d \left\{ \left| \int_X^\infty f(x, y)\, dx \right| \right\} dy < \varepsilon.$$

We have

$$\int_c^d \varphi(y) dy = \int_c^d \left\{ \int_a^X f(x, y)\, dx \right\} dy + \int_c^d \left\{ \int_X^\infty f(x, y)\, dx \right\} dy.$$

In the first integral on the right, the order can be interchanged. We, therefore, have

$$\left| \int_c^d \varphi(y)\, dy - \int_a^X \left\{ \int_c^d f(x, y)\, dy \right\} dx \right| \leqslant \varepsilon, \text{ when } X \geqslant X_1.$$

Hence the result.

166·3. Theorem. *If, in addition to the conditions of the preceding theorem, the integral*

$$\int_a^\infty f_y(x, y)\, dx,$$

also converges uniformly in y in $[c, d]$, then

$$\phi'(y) = \int_a^\infty f_y(x, y)\, dx.$$

Let

$$\psi(y) = \int_a^\infty f_y(x, y)\, dx.$$

As proved above, the order of integration can be changed so that we have

$$\int_c^y \psi(y)\,dy = \int_c^y dy \int_a^\infty f_y(x, y)\,dx$$

$$= \int_a^\infty dx \int_c^y f_y(x, y)\,dy$$

$$= \int_a^\infty [f(x, y) - f(x, c)]\,dx = \phi(y) - \phi(c).$$

Differentiating, we get

$$\psi(y) = \phi'(y),$$

as was to be proved.

Note. Results similar to those above hold also when the range of integration is finite but the integrand has a point of finite discontinuity.

Examples

1. *Evaluate*

$$f(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x}\,dx, \text{ where } \alpha \geqslant 0$$

and deduce that

$$\int_0^\infty \frac{\sin \beta x}{x}\,dx = \begin{cases} +\tfrac{1}{2}\pi, & \text{if } \beta > 0, \\ 0, & \text{if } \beta = 0, \\ -\tfrac{1}{2}\pi, & \text{if } \beta < 0. \end{cases}$$

When $x > 0$, we have

$$\left| e^{-\alpha x} (\sin \beta x)/x \right| \leqslant e^{-\alpha x}/x,$$

and the integral of $(e^{-\alpha x}/x)$ is convergent at ∞ if $\alpha > 0$. Thus we see that the integral is uniformly convergent with respect to β as parameter varying in $]-\infty, \infty[$; α being any fixed positive number.

Again, the derivative of the integrand with respect to β is

$$e^{-\alpha x} \cos \beta x.$$

For a fixed $\alpha > 0$, we have

$$\left| e^{-\alpha x} \cos \beta x \right| \leqslant e^{-\alpha x},$$

which is independent of β.

Since, when $\alpha > 0$, the integral of $e^{-\alpha x}$ converges over the range $[0, \infty]$, therefore, for fixed positive x, the integral

$$\int_0^\infty e^{-\alpha x} \cos \beta x \, dx,$$

is uniformly convergent w.r. to β varying in $]-\infty, \infty[$.

Thus the differentiation under the integral sign is valid and we have

$$f_\beta\,(\alpha, \beta) = \int_0^\infty e^{-\alpha x} \cos \beta x \, dx$$

$$= \left| e^{-\alpha x} \frac{\beta \sin \beta x - \alpha \cos \beta x}{\alpha^2 + \beta^2} \right|_0^\infty = \frac{x}{(\alpha^2 + \beta^2)}.$$

$\therefore \quad f(\alpha, \beta) = \tan^{-1}(\beta/\alpha) + c$, where c is a constant.

To determine, c, we have, putting $\beta = 0$ in the given integral

$$0 = f(x, 0) = 0 + c, \text{ i.e., } c = 0.$$

It will now be shown that $f(\alpha, \beta)$ is a continuous function of α also (for $\alpha > 0$ and for a fixed β). We have

$$\left| \int_{X_1}^{X_2} e^{-\alpha x} \frac{\sin \beta x}{x} dx \right| = \left| e^{-\alpha X}, \int_X^\xi \frac{\sin \beta x}{x} dx \right| \leqslant \left| \int_{X_1}^\xi \frac{\sin \beta x}{x} dx \right|.$$

Now, since
$$\int_0^\infty \frac{\sin \beta x}{x} dx,$$

is known to be convergent, we deduce that the integral is uniformly convergent with α as parameter; α being $\geqslant 0$.

$$\therefore \qquad f(0, \beta) = \lim_{\alpha \to 0} f(\alpha, \beta) = \tfrac{1}{2}\pi, \text{ when } \alpha \to (0+0). \quad (\beta > 0)$$

or
$$\int_0^\infty \frac{\sin \beta x}{x} dx = \tfrac{1}{2}\pi. \qquad (\beta > 0)$$

The other results, now, at once follow.

2. *Evaluate*

$$f(\alpha)=\int_0^\infty e^{-x^2}\cos\alpha x\,dx.$$

Since

$$\left| xe^{-x^2}\sin\alpha x\right| \leqslant xe^{-x^2},$$

we see that the differentiation under integral sign is justified.

$$\therefore \quad f'(\alpha)=-\int_0^\infty xe^{-x^2}\sin\alpha x\,dx.$$

Integrating by parts, we have

$$f'(\alpha)=\left|\tfrac{1}{2}e^{-x^2}\sin\alpha x\right|_0^\infty-\frac{\alpha}{2}\int_0^\infty e^{-x^2}\cos\alpha x\,dx=-\frac{\alpha}{2}f(\alpha).$$

$$\therefore \quad \log f(\alpha)=-\tfrac{1}{4}\alpha^2+c_1 \quad \text{or} \quad f(\alpha)=ce^{-\frac{1}{4}\alpha^2}$$

Now, $f(0)=\displaystyle\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$ (See Chapter XIII)

$$\therefore \quad c=f(0)=\sqrt{\pi}/2.$$

$$\therefore \quad f(\alpha)=\tfrac{1}{2}\sqrt{\pi}e^{-\frac{1}{4}\alpha^2}$$

3. *Evaluate*

$$f(y)=\int_0^\infty\frac{\cos xy}{1+x^2}\,dx. \qquad \ldots(i)$$

It is easy to see that differentiation and integration under integral sign are valid. We, therefore, have

$$f'(y)=-\int_0^\infty\frac{x\sin xy}{1+x^2}dx=-\int_0^\infty\frac{\sin xy}{x}\,dx+\int_0^\infty\frac{\sin xy}{x(1+x^2)}dx.$$

Again

$$\int_0^y f(y)\,dy=\int_0^y dy\int_0^\infty\frac{\cos xy}{1+x^2}\,dx=\int_0^\infty dx\int_0^y\frac{\cos xy}{1+x^2}\,dy=\int_0^\infty\frac{\sin xy}{x(1+x^2)}dx.$$

$$\therefore \qquad f'(y) = -\frac{\pi}{2} + \int_0^y f(y)\, dy. \qquad (y > 0) \qquad \ldots(ii)$$

Since $f(y)$ is continuous, we have, on differentiation,

$$f''(y) = f(y).$$

$\therefore \quad f(y) = Ae^y + Be^{-y}$, where A, B are constants.

Since $f(0)$ is easily seen to be $\pi/2$, we have, on making $y \to 0$,

$$\tfrac{1}{2}\pi = A + B.$$

Also, since from (ii), $f'(0) = -\pi/2$, we see that

$$-\tfrac{1}{2}\pi = A - B.$$

Thus $A = 0$, and $B = \frac{1}{2}\pi$ so that we have, when $y > 0$,

$$\int_0^\infty \frac{\cos xy}{1+x^2}\, dx = \tfrac{1}{2}\pi\, e^{-y} \text{ and } \int_0^\infty \frac{\sin xy}{x(1+x^2)}\, dx = \tfrac{1}{2}\pi(1 - e^{-y}).$$

The necessary modification can easily be made when y is negative.

Exercises

1. Show, by differentiating under integral sign, that

$$\int_0^\infty e^{-x^2 - a^2/x^2} dx = \tfrac{1}{2}\sqrt{\pi}\, e^{-2\,|\,a\,|}.$$

2. Establish the right to integrate under integral sign

$$\int_0^\infty e^{-xy} \cos mx\, dx,$$

and deduce that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \cos mx\, dx = \tfrac{1}{2} \log \frac{b^2 + m^2}{a^2 + m^2} \qquad a,\, b > 0.$$

3. By considering the identity

$$\frac{1}{y} = \int_0^\infty e^{-xy} dx, \qquad y > 0,$$

deduce that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log \frac{b}{a}. \qquad a > 0,\ b > 0.$$

4. Show that

$$\Bigg[\int_0^\infty \frac{\operatorname{sech} ax - \operatorname{sech} bx}{x}\, dx = \log \frac{b}{a} \qquad (a > 0\ ;\ b > 0)$$

[Change the order of integration in

$$\int^b dy \int_0^\infty \operatorname{sech} xy \tanh xy \, dx. \Bigg]$$

5. Under certain conditions, show that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = f(0) \log \frac{b}{a}.$$

CHAPTER XIII

INTEGRATION IN E_2
LINE INTEGRALS, DOUBLE INTEGRALS

167. Introduction. This chapter will deal with two types of integrals, *viz.*,

(*i*) Line integrals in plane,

(*ii*) Double integrals in the 2-dimensional space E_2, also known as plane.

The following chapter will deal with the three types of integrals in E_3 *viz.*,

(*i*) Line integrals in the 3-dimensional space,

(*ii*) Surface integrals,

(*iii*) Volume integrals.

/**168. The concept of a plane curve.** Let $x=\varphi(t)$, $y=\psi(t)$ be two functions of t defined in an interval $[\alpha, \beta]$. Then the set of points (x, y) obtained by giving different values to t is called a curve. The curve is said to be closed, if the points corresponding to the end points α, β of the interval of t coincide, *i.e.*, if

$$\varphi(\alpha)=\varphi(\beta)\ ;\ \psi(\alpha)=\psi(\beta).$$

If $f(x)$ be a function defined in an interval $[a, b]$, then the set of points (x, y) where $y=f(x)$ is also a curve, as we may see on setting

$$x=t,\ y=f(t).$$

Similarly $x=\psi(y)$ is a curve.

In respect of integration, it is appropriate to think of a curve as *oriented* so that we say that a curve $x=\phi(t)$, $y=\psi(t)$ is oriented one way or the other according as t varies from α to β or from β to α. Thus in a way we think of the set of points constituting a curve as *ordered,* the order being induced by the order of the set of numbers t. If the curve as oriented in one way is denoted by C, then the curve oriented in the second way is denoted by $-C$.

169. Line Integral. Let

$$x=\phi(t),\ y=\psi(t)$$

be a curve C, where $\varphi(t)$ and $\psi(t)$ are functions of t defined in an interval $[\alpha, \beta]$.

Let $f(x, y)$ be a function of x and y defined in a region containing the curve C.

To define the line integral, $\int_C f(x, y)\, dx.$

Let

$D(\alpha=t_0 < t_1 < t_2 < \ldots\ldots < t_{r-1} < t_r < \ldots\ldots < t_{n-1} < t_n=\beta)$
be any division of $[\alpha, \beta]$.

Let

$$x_r=\varphi(t_r),\ y_r=\psi(t_r).$$

Let ξ_r be any value of t belonging to the interval $[t_{r-1}, t_r]$.

We form the sum

$$S=\Sigma f[\varphi(\xi_r), \psi(\xi_r)]\ (x_r-x_{r-1}). \qquad \ldots(1)$$

If, as the norm of D tends to zero, this sum, S, tends to a finite limit which is independent of the choice of the points ξ_r, then we denote the limit by the symbol

$$\int_C f(x, y)\, dx$$

and call it a *line integral* of the function $f(x, y)$ along the curve C.

169·1. A sufficient condition for the existence of the integral. *Assuming* that $f(x, y)$, $\varphi(t)$ and $\psi(t)$ are continuous and $\varphi(t)$ possesses a continuous derivative $\varphi'(t)$, we now show that this limit does exist.

There exists a point η_r of $[t_{r-1}, t_r]$ such that

$$(x_r-x_{r-1})=\varphi(t_r)-\varphi(t_{r-1})=(t_r-t_{r-1})\phi'(\eta_r)=\phi'(\eta_r)\delta_r.$$

We have

$$\begin{aligned} S&=\Sigma f[\varphi(\xi_r), \psi(\xi_r)](x_r-x_{r-1})\\ &=\Sigma f[\varphi(\xi_r), \psi(\xi_r)]\varphi'(\xi_r)\delta_r+\Sigma f[\varphi(\xi_r), \psi(\xi_r)][\varphi'(\eta_r)-\varphi'(\xi_r)]\delta_r\\ &=S_1 \qquad\qquad\qquad\qquad +S_2.\end{aligned}$$

Since $\varphi'(t)$ and $f[\varphi(t), \psi(t)]$ are continuous, it follows that the sum $S_1 \to$ a finite limit, *viz.*,

$$\int_\alpha^\beta f[\varphi(t), \psi(t)]\ \varphi'(t)\, dt,$$

as the norm of D tends to zero.

Since $f[\varphi(t), \psi(t)]$ is continuous, it is bounded. Let A be a positive number such that

$$|\ f[\varphi(t), \psi(t)]\ | \leqslant A, \text{ for } t \text{ in } [\alpha, \beta].$$

Since, $\varphi'(t)$ is continuous it is integrable and accordingly there exists a positive number δ such that for every division D $[\alpha, \beta]$ of norm less than or equal to δ, the oscillatory sum of $\varphi'(t)$ is less than the arbitrarily small positive number ε/A.

$$\therefore \; | S_2 | \leqslant A\Sigma \; | \varphi'(\eta_r)-\varphi'(\xi_r) | \; \delta_r \leqslant A\Sigma(M_r-m_r)\delta_r < A(\varepsilon/A)=\varepsilon,$$

where M_r, m_r are the bounds of $\varphi'(t)$ in $\delta_r \equiv [t_{r-1}, t_r]$.

Thus $S_2 \to 0$ as the norm of D tends to zero.

Hence we see that the line integral, in question, does exist and further we have the equality

$$\int_C f(x, y)dx = \int_\alpha^\beta f[\varphi(t), \psi(t)] \, \varphi'(t) \, dt,$$

where we have an ordinary integral on the right.

Note. We may similarly define and examine the existence of the line integral

$$\int_C g(x, y) \, dy.$$

and the line integral

$$\int_C [f(x, y) \, dx + g(x, y) \, dy]$$

Note. It is easy to see that the line integral

$$\int_C g(x, y) \, dy$$

along the curve C, $y=\varphi(x)$, $a \leqslant x \leqslant b$, is equal to the ordinary integral

$$\int^b g[x, \varphi(x)] \, \varphi'(x) \, dx.$$

Note. It may be seen that

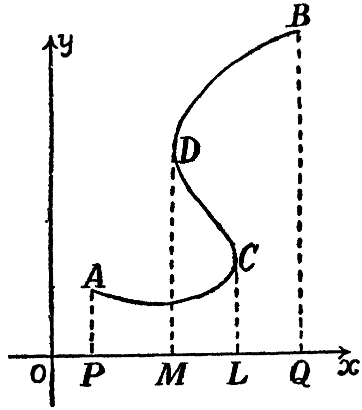
$$\int_{-C} f\,dx + g\,dy = -\int_C f\,dx + g\,dy$$

Note. The ordinary integral is a special case of a line integral, where we take an interval of the x or y axis as the path of integration.

169·2. *It can be easily proved that*

(*i*) *line integrals are additive for arcs, i.e.,*

$$\int_{AB} f(x, y)\, dx = \int_{AC} f(x, y)\, dx + \int_{CD} f(x, y)\, dx + \int_{DB} f(x, y)\, dx.$$



and (*ii*)
$$\int_{AC} f(x, y)\, dx = -\int_{CA} f(x, y)\, dx.$$

Exercises

1. Evaluate

$$\int_C (x^2+y^2)\, dx \quad \text{and} \quad \int_C (x^2+y^2)\, dy$$

where C is the arc of the parabola $y^2=4ax$ between $(0, 0)$ and $(a, 2a)$.

2. Show that

$$\int_C [(x-y)^3 dx+(x-y)^3 dy]=3\pi a^4,$$

taken along the circle $x^2+y^2=a^2$ in the counter clockwise sense.

3. Evaluate

$$\int_C \frac{dx}{x+y},$$

where C is the curve $x=at^2$, $y=2at$. $\qquad 0 \leqslant t \leqslant 2.$

4. Show that the value of the integral

$$\int (xy^2 dy - x^2 y dx),$$

taken in the counter-clockwise sense along the Cardioide $r=a(1+\cos\theta)$ is $35a^4\pi/16$.

5. Find the value of

$$\int (x+y^2)\,dx+(x^2-y)\,dy,$$

taken in the clockwise sense along the closed curve C formed by $y^3=x^2$ and $y=x$ between $(0, 0)$ and $(1, 1)$.

6. Find the value of

$$\int (x^2y\,dx+y^2x\,dy),$$

taken in the clockwise sense along the hexagons whose veritices are

(*i*) $(\pm 3a, 0)$; $(\pm 2a, \pm\sqrt{3}a)$, (*ii*) $(0, \pm 3a)$, $(\pm\sqrt{3}a, \pm 2a)$.

7. Show that

$$\int_C \frac{y\,dx-x\,dy}{x^2+y^2}=-2\pi,$$

round the circle C:

$$x^2+y^2=1$$

or any simple closed curve containing the origin in its interior.

170. The area of a plane region.

The area of a rectangle $R[a, b\,;\,c, d]$. As suggested by intuitive considerations, we *define* the area of a rectangle $R[a, b\,;\,c, d]$ to be the number $(b-a)(d-c)$.

The area of any bounded plane region E. Since E is bounded, there exists a rectangle R which completely encloses E. We divide R into sub-rectangles by drawing parallels to the sides of R. Let

(*i*) s_D *denote the sum of the areas of the sub-rectangles which consist* **entirely** *of points of* E *and*

(*ii*) *let* S_D *denote the sum of the areas of the sub-rectangle which have at least one point in common with* E. Clearly

$$s_D \leqslant S_D.$$

To each mode of division D of R will correspond a pair of sums S_D and s_D. Clearly the sets of these sums are both bounded.

Inner area and outer area. The area.

The upper bound of the sums s_D will be called the *inner area* of E and the lower bound of the sums S_D will be called the *outer area* of E.

The region E will be said to possess an area, if the inner and outer areas are equal and the common value will be called the area of E.

In § 177, page 394 will be obtained a sufficient condition for a region to possess an area.

171. Integrability of a bounded function over a rectangle. Let $f(x, y)$ be a bounded function of (x, y) defined in a rectangle $R[a, b\ ;\ c, d]$.

Let

$$D_1(a=x_0 < x_1 < x_2 < \dots < x_{r-1} < x_r < \dots < x_n = b),$$

and

$$D_2(c=y_0 < y_1 < y_2 < \dots < y_{s-1} < y_s < \dots < y_m = d),$$

be any divisions of the intervals $[a, b]$ and $[c, d]$ respectively. These divisions of the intervals give rise to a division of the rectangle R into mn sub-rectangles $[x_{r-1}, x_r\ ;\ y_{s-1}, y_s]$ where r, s take up all positive integral values from 1 to n and 1 to m respectively. We will denote the rectangle $[x_{r-1}, x_r\ ;\ y_{s-1}, y_s]$ as well as its area

$$(x_r - x_{r-1})\ (y_s - y_{s-1})$$

by the same symbol, w_{rs}. Let M_{rs}, m_{rs} denote the bounds of $f(x, y)$ in w_{rs}. Consider the two sums

$$S = \sum_{s=1}^{s=m} \sum_{r=1}^{r=n} M_{rs}\, w_{rs} \qquad s = \sum_{s=1}^{s=m} \sum_{r=1}^{r=n} m_{rs}\, w_{rs}.$$

It is easy to see that for every mode of division of R into sub-rectangles, we have

$$m(b-a)\ (d-c) \leqslant s \leqslant S \leqslant M\ (b-a)\ (d-c),$$

where M, m are the bounds of $f(x, y)$ in $[a, b\ ;\ c, d]$.

Thus the two aggregates of the sums S and s are bounded.

Upper and lower integrals. Def. *The lower bound of the set of the upper sums, S, is called the upper integral and the upper bound of that of lower sums, s, is called the lower integral of $f(x, y)$ over R and are denoted by the symbols*

$$U = \overline{\int\int_R} f(x, y)dxdy, \quad L = \underline{\int\int_R} f(x, y)dxdy,$$

respectively.

In case these two upper and lower integrals are equal, then $f(x, y)$ is said to be integrable and the common value, which is denoted by the symbol

$$I = \int\int_R f(x, y)\ dxdy,$$

is said to be the *double integral of $f(x, y)$* over R.

Note. Norm of a division of a rectangle. By the norm of a division of a rectangle into sub-rectangles $[x_{r-1}, x_r\ ;\ y_{s-1}, y_s]$ is meant the greatest member of the set of numbers $(x_r - x_{r-1})$ and $(y_s - y_{s-1})$; $r=1, 2, \ldots\ldots n$; $s=1, 2, \ldots\ldots, m$.

172. Darboux's theorem. To every pre-assigned positive number ε, there corresponds a positive number δ, such that for every division whose norm is $\leqslant \delta$,

$$S < U+\varepsilon\,;\ s > L-\varepsilon.$$

The proof is exactly similar to the corresponding proof for functions of a single variable.

Cor. *The upper integral $\geqslant$ the lower integral.*

173. Conditions for Integrability.

173·1. First form. *The necessary and sufficient condition for the integrability of a bounded function $f(x, y)$ over a rectangle R is that to every positive number ε, there corresponds a positive number δ, such that for every division of R whose norm $\leqslant \delta$, the oscillatory sum $(S-s)$ is less than ε.*

The condition is necessary. The bounded function $f(x, y)$ being integrable,

$$U=L=I.$$

If ε be any positive number, then by Darboux's theorem there exists a positive number δ such that for every division of norm $\leqslant \delta$,

$$S < U+\tfrac{1}{2}\varepsilon = I+\tfrac{1}{2}\varepsilon\,;\ s > L-\tfrac{1}{2}\varepsilon = I-\tfrac{1}{2}\varepsilon,$$

i.e.,

$$I-\tfrac{1}{2}\varepsilon < s \leqslant S < I+\tfrac{1}{2}\varepsilon,$$

or

$$S-s < \varepsilon.$$

The condition is sufficient. There exists a division such that if S, s be the corresponding upper and lower sums, then

$$S-s \equiv (S-U)+(U-L)+(L-s) < \varepsilon.$$

Since each of the three numbers $(S-U)$, $(U-L)$, $(L-s)$, is non-negative, it follows that

$$0 \leqslant (U-L) < \varepsilon.$$

As ε is an arbitrary positive number, we deduce that

$$U-L=0,\ i.e.,\ U=L.$$

Therefore the function is integrable.

173·2. Second form. *The necessary and sufficient condition for the integrability of a bounded function $f(x, y)$ is that to every pre-assigned positive number ε, there corresponds a division for which the oscillatory sum is less than ε.*

The proof which is quite similar to that of the first form is left to the reader.

174. Particular classes of bounded integrable functions.

174·1. Every continuous function is integrable. Let $f(x, y)$ be continuous in a rectangle $R[a, b \,;\, c, d]$. Let ε be any positive number. There exists (Cor. 3 to § 136·1, page 300), a division such that the oscillation of $f(x, y)$ in every sub rectangle of the division is

$$< \varepsilon/(b-a)(d-c).$$

For such a division, the oscillatory sum

$$S-s=\Sigma\Sigma(M_{rs}-m_{rs})w_{rs} \leqslant [\varepsilon/(b-a)(d-c)].\ \Sigma\Sigma w_{rs}=\varepsilon.$$

Hence $f(x, y)$ is integrable in the rectangle R.

174·2. *If a function $f(x, y)$ is bounded in $R[a, b \,;\, c, d]$ and is such that its points of discontinuity can be enclosed in a finite number of rectangles the sum of whose areas is less than a given positive number, then $f(x, y)$ is integrable in R.*

Let ε be any positive number. We enclose the points of discontinuity in rectangles the sum of whose areas is $< \varepsilon/2(M-m)$. The part of the oscillatory sum $(S-s)$, arising from these rectangles or from the sub-rectangles into which they may be further divided is $< \varepsilon/2$.

On producing the sides of the rectangles which enclose the points of discontinuity, we obtain a division of R into sub-rectangles. These sub-rectangles are of two types :

(*i*) those which include points of discontinuity and

(*ii*) those which do not include any point of discontinuity.

The sub-rectangles of the latter types can be further sub divided such that the part of $(S-s)$ arising from them is $< \varepsilon/2$.

Thus we have a division of R and that the corresponding oscillatory sum is less than the given positive number ε.

Hence the result. (§ 173·2)

Cor. *If a function $f(x, y)$ is bounded in $R[a, b \,;\, c, d]$ and its points of discontinuity lie on a finite number of curves of the form $y=\phi(x)$, $x=\psi(y)$, etc., where $\varphi(x)$, $\psi(y)$, etc., are continuous, then $f(x, y)$ is integrable in R.*

Let, p, be the number of the curves in question. Let ε by any positive number.

Since $\varphi(x)$ is continuous in $[a, b]$, there exists a division

$D(a=x_0 < x_1 < x_2 < x_3 \ldots < x_{r-1} < x_r \ldots\ldots < x_n=b)$ of $[a, b]$ such that the oscillation of $\varphi(x)$ in every sub-interval is $< \varepsilon/p(b-a)$

The points of $y=\varphi(x)$ which correspond to the values of x, the sub-interval $[x_{r-1}, x_r]$ of the division D clearly belong to the rectangle $[x_{r-1}, x_r ; m_r, M_r]$ of area $(x_r-x_{r-1})(M_r-m_r)$; M_r, m_r being the bounds of $\varphi(x)$ in $[x_{r-1}, x_r]$. Thus all the points of $y=\varphi(x)$ have been enclosed in a system of rectangles of total area.

$$\Sigma(x_r-x_{r-1})\,(M_r-m_r)$$

which is clearly less than ε/p.

Applying the same process to the other curves, we see that they can all be enclosed in a finite number of rectangles of total area less than any given positive number ε. Hence the result. (§ 173·2)

175. The calculation of a double integral. Equivalence of a double with repeated integrals. Theorem. *If the double integral*

$$\int\int_R f(x, y)\, dxdy$$

exists where R is the rectangle $[a, b ; c, d]$ and if also

$$\int_a^b f(x, y)\, dx$$

exists for each value of y in $[c, d]$, then the repeated integral

$$\int_c^d \left\{ \int_a^b f(x, y)\, dx \right\} dy$$

exists and is equal to the double integral.

[**Observation.** The proof depends upon a simple consideration, *viz.*, that if D be any divsion of $[a, b]$ and K_r, k_r be a rough upper and a rough lower bound of a function $\varphi(x)$ in any sub-interval δ_r of the division, then

$$\bar{\int_a^b} \varphi(x)\, dx \leqslant S_D \leqslant \Sigma K_r\delta_r \text{ and } \underline{\int_a^b} \varphi(x)\, dx \geqslant s_D \geqslant \Sigma k_r\delta_r.\Big]$$

Let U and L denote upper and lower integrals of $f(x, y)$ over R. Let ε be any positive number.

There exists a division of R into sub-rectangles $[x_{r-1}, x_r ; y_{s-1}, y_s]$ such that

$$\Sigma\,\Sigma\, M_{rs}(x_r-x_{r-1})(y_s-y_{s-1}) < U+\varepsilon. \qquad ...(1)$$

Since for every fixed value of y in $[y_{s-1}, y_s]$, M_{rs} is a rough bound of $f(x, y)$ in $[x_{r-1}, x_r]$, therefore

$$\int_a^{\bar{b}} f(x, y)\, dx \leqslant \sum_{r=1}^{r=n} M_{rs}(x_r - x_{r-1}), \text{ when } y_{s-1} \leqslant y \leqslant y_s. \quad \ldots(2)$$

Since, from (2), $\displaystyle\sum_{r=1}^{r=n} M_{rs}(x_r - x_{r-1})$

is a rough upper bound of the function

$$\int_a^{\bar{b}} f(x, y)\, dx, \text{ of } y \text{ in } [y_{s-1}, y_s],$$

we have by an application of the same reasoning,

$$\int_c^{\bar{d}} \left\{ \int_a^{\bar{b}} f(x, y)\, dx \right\} dy \leqslant \sum_{s=1}^{m} \sum_{r=1}^{n} M_{rs}(x_r - x_{r-1})(y_s - y_{s-1}) < U + \varepsilon, \quad \ldots(3)$$

As ε is an arbitrary positive number, and, by hypothesis,

$$\int_a^{\bar{b}} f(x, y)\, dx = \int_{\underline{a}}^{b} f(x, y)\, dx = \int_a^b f(x, y)\, dx,$$

therefore we have

$$\int_c^{\bar{d}} \left\{ \int_a^{b} f(x, y)\, dx \right\} dy = \int_c^{\bar{d}} \left\{ \int_a^{\bar{b}} f(x, y)\, dx \right\} dy \leqslant U = \overline{\iint_R} f(x, y)\, dxdy. \quad \ldots(4)$$

We can similarly prove that

$$\int_{\underline{c}}^{d} \left\{ \int_a^{b} f(x, y)\, dx \right\} dy = \int_{\underline{c}}^{d} \left\{ \int_{\underline{a}}^{b} f(x, y)\, dx \right\} dy \geqslant L = \iint_{\underline{R}} f(x, y)\, dxdy. \quad \ldots(5)$$

Therefore

$$\iint_{\underline{R}} f(x, y)\, dxdy \leqslant \int_{\underline{c}}^{d} \left\{ \int_a^{b} f(x, y)\, dx \right\} dy$$

$$\leqslant \int_c^{\bar{d}} \left\{ \int_a^{b} f(x, y)\, dx \right\} dy \leqslant \overline{\iint_R} f(x, y)\, dxdy \quad \ldots(6)$$

But by hypothesis

$$\int\int_R f(x, y)\, dxdy = \underline{\int\int}_R f(x, y)\, dxdy. \qquad \ldots(7)$$

Now, from (6), (7), we have

$$\int\int_R f(x, y)\, dxdy = \int_c^{\bar{d}} \left\{ \int_a^b f(x, y)\, dx \right\} dy$$

$$= \int_{\underline{c}}^{d} \left\{ \int_a^b f(x, y)\, dx \right\} dy = \int_c^d \left\{ \int_a^b f(x, y)\, dx \right\} dy.$$

Cor. *If a double integral exists then, the two repeated integrals cannot exist without being equal.*

Ex. A function $f(x, y)$ is defined in $R[0, 1 ; 0, 1]$ as follows :—

$f(x, y) = \frac{1}{2}$, when y is rational, $f(x, y) = x$, when y is irrational ;

show that

$$\int_0^1 \left\{ \int_0^1 f(x, y)\, dx \right\} dy$$

exists and is equal to $\frac{1}{2}$, but the double integral and the second repeated integral do not exist.

Example

Prove that

$$\int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\, dy \right\} dx = \tfrac{1}{2} \neq -\tfrac{1}{2} = \int_0^1 \left\{ \int_0^1 \frac{x-y}{(x+y)^3}\, dx \right\} dy,$$

and show that this result is compatible with the theorem proved.

We suppose (x, y) to be any point of the rectangle $[0, 1 ; 0, 1]$. For any fixed value of $x \neq 0$, the function $(x-y)/(x+y)^3$ is a bounded function of y and if $x=0$ then $y=0$ is a point of infinite discontinuity.

$$\text{If } x \neq 0,\ \varphi(x) = \int_0^1 \frac{x-y}{(x+y)^3}\, dy = \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy = \frac{1}{(1+x)^2},$$

and $\varphi(0)$ does not exist.

Again, $\displaystyle\int_0^1 \varphi(x)\,dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{(1+x)^2}dx = \lim_{\varepsilon\to 0}\left(-\frac{1}{2}+\frac{1}{1+\varepsilon}\right)=\frac{1}{2}.$

Thus
$$\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right\}dx = \frac{1}{2}.$$

If $y\neq 0$, $\displaystyle\psi(y)=\int_0^1 \frac{x-y}{(x+y)^3}dx = \int_0^1 \left\{\frac{1}{(x+y)^2}-\frac{2y}{(x+y)^3}\right\}dx = -\frac{1}{(1+y)^2},$

and $\psi(0)$ does not exist.

As before
$$\int_0^1 \psi(y)dy = -\frac{1}{2}.$$

The function $(x-y)/(x+y)^3$ is not bounded in the neighbourhood of the origin (0. 0) and therefore the double integral does not exist.

Exercises

1. Evaluate the following integrals :—

(i) $\displaystyle\iint xy(x^2+y^2)\,dxdy$ over $R[0, a\,;\,0, b)]$.

(ii) $\displaystyle\iint ye^{xy}\,dxdy$ over $R[0, a\,;\,0, b]$.

(iii) $\displaystyle\iint \frac{x-y}{x+y}\,dxdy$ over $R[0, 1\,;\,0, 1]$.

[The integrand is bounded and (0, 0) is its only point of discontinuity in the square [0, 1 ; 0, 1].]

(iv) $\displaystyle\iint \frac{dxdy}{\sqrt{[c^2+(x-y^2)]}}$ over the square $[0, a\,;\,0, a]$.

2. Show that
$$\int_0^1\left\{\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dy\right\}dx = \int_0^1\left\{\int_0^1 \frac{x^2-y^2}{x^2+y^2}\,dx\right\}dy.$$

3. Show that if $0<h<1$,
$$\int_h^1\left\{\int_0^1 f(x, y)\,dx\right\}dy = \int_h^1\left\{\int_h^1 f(x, y)\,dy\right\}dx = 0,$$

$$\text{but} \int_0^1 \left\{ \int_0^1 f(x, y)\, dx \right\} dy \neq \int_0^1 \left\{ \int_0^1 f(x, y)\, dy \right\} dx,$$

where

$$f(x, y) = (y^2 - x^2)/(y^2 + x^2)^2.$$

4. Show that $\varphi(h) = \int_h^1 \left\{ \int_h^1 \frac{x-y}{(x+y)^3} dy \right\} dx$ is not continuous for $h=0$.

5. *Is it true that*

$$\int_1^\infty dx \int_1^\infty \frac{x-y}{(x+y)^3} dy = \int_1^\infty dy \int_1^\infty \frac{x-y}{(x+y)^3} dx \text{ ?}$$

If not, why so ?

6. Show that

$$\iint_R \varphi(x)\psi(y)\, dxdy = \left\{ \int_a^b \varphi(x)\, dx \right\} \left\{ \int_c^d \psi(y)\, dy \right\}$$

where R is the rectangle $[a, b\ ;\ c, d]$.

176. Integrability and integral of a bounded function f(x, y) over any finite region E. Since the given region E is finite, there must exist a rectangle R which completely encloses E. We define a function $F(x, y)$ over R as follows :—

$$F(x, y) = \begin{cases} f(x, y), \text{ at all points of } E, \\ 0, \text{ elsewhere.} \end{cases}$$

Def. *A function $f(x, y)$ is said to be integrable over E, if $F(x, y)$ is integrable over the rectangle R. Also then*

$$\iint_E f(x, y)\, dxdy = \iint_R F(x, y)\, dxdy.$$

176·1. An important case of integrability. *If $f(x, y)$ is continuous in a region E which is bounded by a finite number of continuous curves of the form $y=\varphi(x)$, $x=\psi(y)$, etc., then*

$$\iint_E f(x, y)\, dxdy$$

exists.

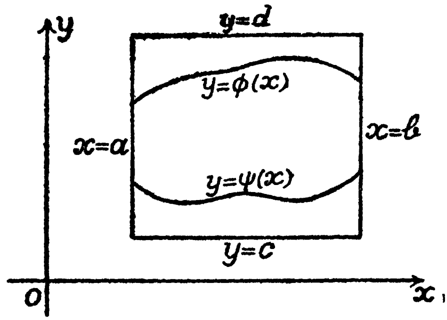
The result follows from the fact that the only possible points of discontinuity of $F(x, y)$, defined as in § 176 above, are the points of the curves $y=\varphi(x)$, etc., (Cor. to § 174·2).

176·2. Calculation of double integrals. *If $f(x, y)$ is continuous in a region E which is enclosed by the curves*

$$y=\varphi(x),\ y=\psi(x)\ ;\ x=a,\ x=b\ ;$$

where $\phi(x)$, $\psi(x)$ are continuous and $\psi(x) \leqslant \phi(x)$ in $[a, b]$, then

$$\iint_E f(x, y)\, dxdy = \int_a^b \left\{ \int_{\psi(x)}^{\phi(x)} f(x, y)\, dy \right\} dx.$$



Let $R[a, b\ ;\ c, d]$ enclose the region E and let $F(x, y)$ be defined over R as in § 176. We have

$$\iint_E f(x, y)\, dxdy = \iint_R F(x, y)\, dxdy$$

$$= \int_a^b \left\{ \int_c^d F(x, y)\, dy \right\} dx$$

$$= \int_a^b \left\{ \int_c^{\psi(x)} F dy + \int_{\psi(x)}^{\varphi(x)} F dy + \int_{\varphi(x)}^{d} F dy \right\} dx$$

$$= \int_a^b \left\{ \int_{\psi(x)}^{\varphi(x)} F dy \right\} dx\ ;$$

each of the remaining two integrals being equal to zero.

Note. If a function $f(x, y)$ is continuous in a region E which is bounded by the curves $x=\varphi(y)$, $x=\psi(y)$; $y=c$, $y=d$, where $\varphi(y)$, $\psi(y)$ are continuous and $\psi(y) \leqslant \varphi(y)$, then

$$\iint_E f(x, y)\, dxdy = \int_c^d \left\{ \int_{\psi(y)}^{\varphi(y)} f(x, y)\, dx \right\} dy.$$

177. Area of a region. *A sufficient condition for a region E to possess an area is that it is bounded by a finite number of continuous curves of the form $y=\varphi(x)$, $x=\psi(y)$, etc, and the area is given by the double integral*

$$\iint_E dxdy$$

We enclose E in a rectangle R and define a function $F(x, y)$ in R as follows :—

$$F(x, y)=1 \text{ at points of } E \text{ and } F(x, y)=0 \text{ elsewhere.}$$

Corresponding to any division D of R into sub-rectangles, the upper sum S_D of $F(x, y)$ is the sum of the areas of those sub-rectangles which contain at least one point of E and the lower sum s_D of $F(x, y)$ is the sum of the areas of those sub-rectangles which consist entirely of points of E. From this we conclude that the outer and inner areas of E are upper and lower integrals of $F(x, y)$ over R.

Since these two integrals are equal, we see that E possesses an area and the same is given by

$$\iint_R F(x, y)\, dxdy.$$

But, by def.,

$$\iint_E dxdy = \iint_E 1.\, dxdy = \iint_R F(x, y) dxdy.$$

Hence the result.

Examples

1. The double integral

$$\iint f(x, y)\, dxdy$$

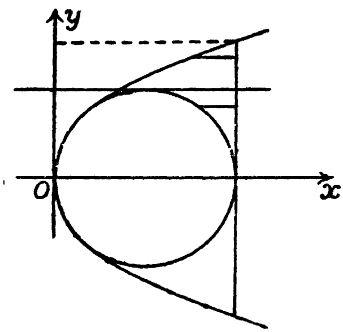
where the field of integration is the circle $x^2+y^2=a^2$, is equivalent to the repeated integral

$$\int_{-a}^{+a} dx \int_{-\sqrt{(a^2-x^2)}}^{+\sqrt{(a^2-x^2)}} f(x, y)\, dy.$$

2. *Change the order of integration in the double integral*

$$\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x, y)\, dxdy.$$

The given domain of integration is described by a line which starts from $x=0$ and, moving parallel to itself, goes over to $x=2a$; the extremities of the moving line lying on the parts of the parabola $y^2=2ax$ and the circle $x^2+y^2=2ax$ in the first quadrant.



We have now to regard the same region as described by a line moving parallel to x-axis instead of y-axis.

In this way the region of integration is sub divided into three sub-regions to each of which corresponds a double integral. Thus we have

$$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{2ax}} f\,dx\,dy=\int_0^{a}\int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} f\,dy\,dx+\int_0^{a}\int_{a+\sqrt{(a^2-y^2)}}^{2a} f\,dy\,dx+\int_a^{2a}\int_{y^2/2a}^{2a} f\,dy\,dx.$$

Exercises

1. Express, as a repeated integral, the double integral

$$\int\int f(x, y)\,dx\,dy,$$

taken over the quadrilateral bounded by the lines

$$x+y=0,\ x-y=0,\ 2x-y=1,\ 2x-3y+5=0$$

taken in order.

2. Evaluate $\int\int x^2y^2\,dx\,dy$ over the region defined by

$$x \geqslant 0,\ y \geqslant 0,\ (x^2+y^2) \leqslant 1.$$

3. Show that the value of

$$\int\int \sqrt{(4y-x^2)}\,dx\,dy,$$

taken over in interior of the circle $x^2+y^2=2y$ is $\pi+\frac{8}{3}$.

4. Change the order of integration in

$$\int_0^{2a}\int_{x^2/4a}^{3a-x} \varphi(x, y)\,dx\,dy.$$

5. Show that

$$\int_a^b \int_{a^2/x}^{x} F\,dx\,dy = \int_{a^2/b}^{a} \int_{a^2/y}^{b} F\,dx\,dy + \int_a^b \int_y^b F\,dx\,dy.$$

6. In the integral

$$\int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (4-y)\,dydx,$$

change the order of integration, and evaluate the integral.

7. By changing the order of integration, prove that

$$\int_0^{2a} \int_0^{\sqrt{(2ax-x^2)}} \frac{(x^2+y^2)x\varphi'(y)\,dxdy}{\sqrt{[4a^2x^2-(x^2+y^2)^2]}} = \pi a^2[\varphi(a)-\varphi(0)].$$

8. Change the order of integration in the integral

$$\int_0^1 dx \int_0^{\sqrt{(1-x^2)}} \frac{dy}{(1+e^y)\sqrt{(1-x^2-y^2)}}.$$

and hence evaluate.

9. Prove that

$$\int_0^1 dx \int_x^{1/x} \frac{ydy}{(1+xy)^2(1+y^2)} = \frac{\pi-1}{4}.$$

10. Evalute $\int\int (x^2+y^2)\,dxdy$ over the region bounded by

$$xy=1,\ y=0,\ y=x,\ x=2.$$

11. Evaluate $\int\int x^2\,dxdy$ over the area bounded by $x^2-y^2=1$, $x^2+y^2=4$ which contains the origin.

12. Evaluate

$$\int\int (x+y+a)\,dxdy,$$

taken over the circular area $x^2+y^2 \leqslant a^2$.

13. Prove that

$$\int_0^a dx \int_0^x f(x,y)dy = \int_0^a dy \int_y^a f(x,y)\,dx.$$

Deduce Dirichdet's formula

$$\int_0^t dx \int_0^x f(y)dy = \int_0^t (t-y)f(y)\,dy.$$

Def. A domain E will be said to be *Quadratic* with respect to y-axis, if it is bounded by the curves of the form

$$y=\varphi(x),\ y=\psi(x)\ ;\ x=a,\ x=b,$$

where $\varphi(x)$, $\psi(x)$ are continuous and $\varphi(x) \geqslant \psi(x)$ in $[a, b]$.

Thus a domain which is quadratic with respect to y-axis is such that a line parallel to y-axis and lying between $x=a$, $x=b$ meets the boundary of E in just two points.

Similarly we may have regions which are quadratic with respect to x-axis.

178. Green's Theorem. Connection between double and line integrals. Let $f(x, y)$ and $f_y(x, y)$ be continuous in any region E which is quadratic with respect to y-axis, and let C denote the contour of E. Let the region E be bounded by the curves

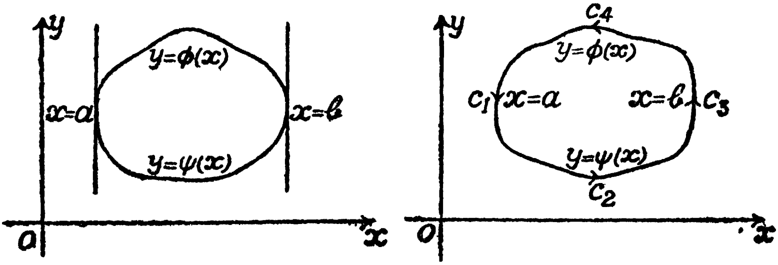
$$x=a,\ x=b\ ;\ y=\phi(x),\ y=\psi(x).$$

We have

$$\iint\limits_E f_y(x, y)\, dx\, dy=\int_a^b \left\{ \int_{\psi(x)}^{\varphi(x)} f_y(x, y) dy \right\} dx$$

$$=\int_a^b f[x, \varphi(x)]dx-\int_a^b f[x, \psi(x)]dx.$$

Let C_1, C_2, C_3, C_4 denote the four parts of C taken in the positive sense, *i.e.*, in such a way that the interior of the region lies to the left as the contour is described in the counter-clockwise sense.



We have

$$\int_C f(x, y)\, dx=\int_{C_1} f(x, y)\, dx+\int_{C_2} f(x, y)\, dx+\int_{C_3} f(x, y)\, dx+\int_{C_4} f(x, y) dx.$$

But $\displaystyle\int_{C_1} f(x, y)\, dx=0=\int_{C_3} f(x, y)\, dx,$

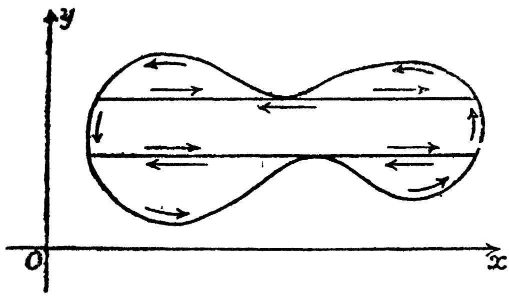
$$\int_{C_2} f(x, y)\, dx=\int_a^b f[x, \psi(x)]dx,$$

and
$$\int_{C_4} f(x, y)\, dx = \int_b^a f[x, \varphi(x)]\, dx = -\int_a^b f[x, \varphi(x)]\, dx.$$

$$\therefore \quad \int\!\!\int_E f_y(x, y)\, dxdy = -\int_C f(x, y)\, dx.$$

We, now, *generalise* the above result.

Let a region E be such as can be divided into a finite number of sub-regions $E_1, E_2, \ldots, E_n$ each of which is quadratic with respect to y-axis.



The contour C of the region E and the contours $C_1, C_2, C_3, \cdots, C_n$ of $E_1, E_2, E_3, \cdots, E_n$ are to be so described that the corresponding regions constantly lie on the left hand side. The theorem holds for each separate region, and on addition, the parts of the line integral along the connecting lines cancel one another, since each of these is described twice, once in each direction, and we arrive at the theorem for the whole region.

We have thus proved that *if $f(x, y)$ and $f_y(x, y)$ are continuous in a region E which can be split up into a finite number of regions quadratic with respect to y-axis, and C is the contour of E, described in the positive sense, then*

$$\int\!\!\int_E f_y(x, y)\, dx\, dy = -\int_C f(x, y)\, dx. \qquad \ldots(i)$$

It may similarly be proved that *if $g(x, y)$ and $g_x(x, y)$ are continuous in a region E which can be split up into finite number of regions quadratic with respect to x-axis and C is the contour of E, then*

$$\int\!\!\int_E g_x(x, y)\, dx\, dy = \int_C g(x, y)\, dy. \qquad \ldots(ii)$$

Finally, on subtracting (i) from (ii), we see that

If we suppose that $f(x, y)$, $g(x, y)$, $f_y'(x, y)$ and $g_x(x, y)$ are continuous in a domain E which can be split up in finite number of regions quadratic with respect to either axis, then

$$\int_C [f(x, y)\,dx + g(x, y)\,dy] = \int\int_E [g_x'(x, y) - f_y(x, y)]\,dxdy,$$

where the integral on the left is a line integral round the contour C of the region taken in such a way that the interior of the region remains on the left as the boundary is described.

This may be written as

$$\int_C (fdx + gdy) = \int\int_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy.$$

A particular case. Taking $f(x, y) = y$, we see from (*i*) that the area of a domain which can be split up into a finite number of domains quadratic with respect to Y-axis $= -\int_C ydx$.

Similarly, putting $g(x, y) = x$, we see from (*ii*) that the area of a domain which can be split up into a finite number of domains quadratic with respect to X-axis $= \int_C xdy$.

Thus the area of a region E which can be divided into a finite number of regions quadratic w.r. to either axis, is given by

$$\frac{1}{2}\int_C (xdy - ydx),$$

where the contour C of E is described in the positive sense.

Ex. 1. Verify the Green's theorem by evaluating in two ways the following line integrals :—

(*i*) $\int (x^2ydx + xy^2dy)$,

taken along the closed path formed by $y = x$ and $x^3 = y^2$ in the first quadrant.

(*ii*) $\int [(x^3 + y^2)dx + (x^2 + y^3)\,dy]$,

taken along the boundary of the pentagon whose vertices are

(0, 0), (1, 0), (2, 1), (1, 2), (0, 1).

2. Evaluate

$$\int (xydx + xy^2dy)$$

taken round the positively oriented square with vertices $(\pm 1, 0)$, $(0, \pm 1)$. Verify the result by using Green's Theorem.

3. Evaluate

$$\int_C (x^4 dy - y^4 dx),$$

where C is the semi-circumference of the ellipse

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which runs from the origin to $(2a, 0)$ in the first quadrant.

179. Double integral as a limit. Let $f(x, y)$ be continuous in a region E. Let the region E be divided into sub-regions E_1, $E_2 \ldots$, E_n with areas $A_1, A_2, \ldots\ldots, A_n$. Let (ξ_r, η_r) be any point of the region E_r. Form the sum

$$\Sigma f(\xi_r, \eta_r) A_r. \qquad \ldots(1)$$

By the diameter of a sub-region E_r will be meant the upper bound of the set of the distances between pairs of points on its boundary and the greatest of the diameters of all the sub-regions will be called the *norm* of the division.

It can new be proved that *as the norm* $\delta \to 0$, *the limit of the sum* (1) *is the integral of* $f(x, y)$ *over* E, *i e.*,

$$\lim_{\delta \to 0} \Sigma f(\xi_r, \eta_r) A_r = \iint_E f(x, y)\, dx dy.$$

The details of the proof will not be given here.

Cor. *If* $f(x, y)$ *be continuous in a region* E *with area* A, *then there exists a point* (ξ, η) *of* E *such that* .

$$\iint_E f(x, y)\, dx dy = A f(\xi, \eta).$$

180. Change of variable in a double integral.

First Proof.

Lemma. Let

$$x = \phi(u, v), \quad y = \psi(u, v) \qquad \ldots(1)$$

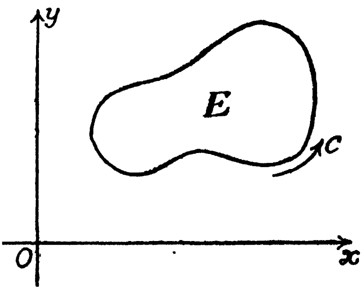
be two functions of u, v defined in a certain region E_1 of the u, v plane bounded by a curve C_1. We suppose that

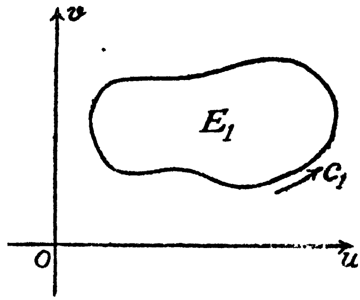
(*i*) the two functions possess continuous first order partial derivatives at all points of E_1 and C_1.

(*ii*) the equations (1) transform the region E_1 bounded by C_1, into a region E of the xy plane bounded by a curve C in such a way that a *one-to-one* correspondence exists between the two regions and their contours.

(*iii*) the Jacobian, $\partial(x, y)/\partial(u, v)$, does not change sign at any point of E_1, though it may vanish at certain points of C_1.

As the point (u, v) describes the contour C_1 in the positive sense, then the point (x, y) may describe C in the positive or else in the negative sense without ever changing the direction of motion. The transformation will be said to be *direct* or *inverse* respectively in the two cases.





We shall now obtain a formula connecting the areas A and A_1 of the regions E, E_1.

Let C_1 be given by

$$u=u(t),\ v=v(t)$$

where t varies from α to β, say.

Then C is given by

$$x=\phi[u(t),\ v(t)]=x(t), \text{ say}$$
$$y=\psi[u(t)\ \ v(t)]=y(t), \text{ say.}$$

We have

$$A=\int_C x\,dy,$$

taken along C in the positive sense.

Expressing the line integral as an ordinary integral we have

$$A=\int_\alpha^\beta x(t)\frac{dy}{dt}dt$$

$$=\int_\alpha^\beta x(t)\left[\frac{\partial y}{\partial u}\,\frac{du}{dt}+\frac{\partial y}{\partial v}\,\frac{dv}{dt}\right]dt.$$

Expressing this again as a line integral, we see that

$$A=\pm\int\left(x\frac{\partial y}{\partial u}du+x\frac{\partial y}{\partial v}dv\right)$$

$$=\pm\int_{C_1}\varphi(u,v)\left(\frac{\partial\psi}{\partial u}du+\frac{\partial\psi}{\partial v}dv\right),$$

where the new integral is to be taken along the positive sense of C_1 and the sign is + or − according as the transformation is direct or inverse.

Applying Green's theorem, we have

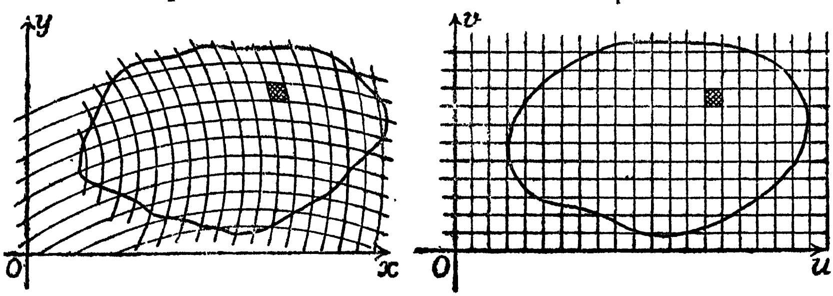
$$\int_{C_1}(\varphi\psi_u\,du+\varphi\psi_v\,dv)=\iint_{E_1}\left(\frac{\partial(\varphi\psi_v)}{\partial u}-\frac{\partial(\varphi\psi_u)}{\partial v}\right)dudv$$

$$=^{*}\iint_{E_1}\frac{\partial(\varphi,\psi)}{\partial(u,v)}dudv=A_1\left[\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right]_{\xi,\eta}, \text{ (Cor. to § 179)}$$

where (ξ, η) is a point of E_1.

$$\therefore\quad A=\pm A_1\left[\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right]\quad=\pm A_1[J]_{\xi,\eta}\quad=A_1\,|\,J\,|_{\xi,\eta}.$$

Since A, A_1 are essentially positive, we see that the sign, +, or, −, should be taken according as J is positive or negative. This shows that *the transformation is direct or inverse according as the Jacobian is positive or negative.*



Main Theorem. Let $f(x, y)$ be continuous in the region E. We divide the region E_1 by lines parallel to the u-axis and v-axis. This division of E_1 gives rise to a curvilinear division of the region E. Let E'_{rs} be any sub-region of E_1 and E_{rs} the corresponding sub-region of E and let w'_{rs}, w_{rs} be their areas.

By the lemma, we have

$$w_{rs}=w'_{rs}\,|\,J\,|_{\xi_{rs},\,\eta_{rs}},$$

*It is assumed that $\psi_{uv}=\psi_{vu}$.

where (ξ_{rs}, η_{rs}) is some point of E'_{rs}. Let (x_{rs}, y_{rs}) be the corresponding point of E_{rs}. We have

$$f(x_{rs}, y_{rs})w_{rs} = f[\varphi(\xi_{rs}, \eta_{rs}), \psi(\xi_{rs}, \eta_{rs})] \mid J \mid_{(\xi_{rs}, \eta_{rs})} w'_{rs}.$$

A similar equality will be obtained for each pair of corresponding sub-regions. Adding them and letting the norm of the divisions tend to zero, we see that

$$\int\int_E f(x, y)\, dxdy = \int\int_{E_1} f[\varphi(u, v), \psi(u, v)] \mid J \mid dudv.$$

Another proof. We shall now give another proof which is independent of the lemma.

We have supposed that $f(x, y)$ is continuous in E.

If we define

$$g(x, y) = \int_a^x f(x, y)\, dy,$$

we have

$$\frac{\partial g}{\partial x} = f(x, y).$$

We suppose that C_1 is given by

$$u = u(t),\ v = v(t)\ ;$$

and C by

$$x = x(t),\ y = y(t)\ ;$$

t varying from α to β.

We have

$$\int\int_E f(x, y)\, dxdy = \int\int_E \frac{\partial g}{\partial x} dxdy$$

$$= \int_C g\, dy, \text{ by Green's theorem}$$

$$= \int_\alpha^\beta g \frac{dy}{dt} dt$$

$$= \int_\alpha^\beta g \left(\frac{\partial y}{\partial u} \frac{du}{dt} + \frac{\partial y}{\partial v} \frac{dv}{dt} \right) dt$$

$$= \pm \int_{C_1} \left(g \frac{\partial y}{\partial u}\, du + g \frac{\partial y}{\partial v} dv \right),$$

where we take + or − sign according as the transformation is direct or inverse.

Again, applying Green's theorem, we see that

$$\int_C \left(g\frac{\partial y}{\partial u}\,du + g\frac{\partial y}{\partial v}\,dv \right)$$

$$= \int\int_E \left[\frac{\partial}{\partial u}\left(g\frac{\partial y}{\partial v} \right) - \frac{\partial}{\partial v}\left(g\frac{\partial y}{\partial u} \right) \right] du\,dv.$$

Now

$$\frac{\partial}{\partial u}\left(g\frac{\partial y}{\partial v} \right) - \frac{\partial}{\partial v}\left(g\frac{\partial y}{\partial u} \right)$$

$$= \frac{\partial g}{\partial u}\frac{\partial y}{\partial v} + g\frac{\partial^2 y}{\partial u \partial v} - \frac{\partial g}{\partial v}\frac{\partial y}{\partial u} - g\frac{\partial^2 y}{\partial v \partial u}$$

$$= \left(\frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u} \right)\frac{\partial y}{\partial v} - \left(\frac{\partial g}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial v} \right)\frac{\partial y}{\partial u}$$

$$= \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right)$$

$$= f\,\frac{\partial(x, y)}{\partial(u, v)}.$$

Thus we obtain

$$\int\int_E f\,dx\,dy = \pm \int\int_{E_1} f\,\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv$$

$$= \int\int_{E_1} f \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\,dv.$$

Note. The principle of the second proof is contained in the following steps :

I. By Green's theorem, express the given double integral over, E, as a line integral around C.

II. Express the line integral around, C, as a line integral around C_1.

III. By Green's theorem, express the line integral, C_1, as a double integral over E_1.

Ex. 1. $x = \frac{u}{1+v^2}$ and $y = \frac{uv}{1+v^2}$; find the region in the (u, v) plane which corresponds to the region defined by

$$x^2 - x + y^2 \leqslant 0 \leqslant y.$$

2. If $x = \frac{1}{2}u(1+v)$, $y = \frac{1}{2}u(1-v)$, find the region in the (u, v) plane which corresponds to the first quadrant in the (x, y) plane.

Examples

1. *Evaluate*

$$\iint \frac{\sqrt{(a^2b^2-b^2x^2-a^2y^2)}}{\sqrt{(a^2b^2+b^2x^2+a^2y^2)}}\,dxdy,$$

the field of integration being the positive quadrant of the ellipse

$$x^2/a^2+y^2/b^2=1.$$

Changing the variables x, y to X, Y where

$$x=aX,\ y=bY$$

we see that, since $\partial(x, y)/\partial(X, Y)=ab$, the integral

$$=ab\iint\sqrt{\left(\frac{1-X^2-Y^2}{1+X^2+Y^2}\right)}\,dXdY,$$

the new field of integration being the positive quadrant of the circle

$$X^2+Y^2=1.$$

Changing X, Y to r, θ where

$$X=r\cos\theta,\ Y=r\sin\theta,$$

so that $\partial(X, Y)/\partial(r, \theta)=r$, we see that the integral

$$=ab\iint\frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,drd\theta.$$

It is easily seen that the positive quadrant of the circle $X^2+Y^2=1$, will be described if θ varies from 0 to $\pi/2$ and corresponding to each value of θ between 0 and $\pi/2$, r varies from 0 to 1. This new field of integration, therefore, is the rectangle $[0, 1\,;\,0, \tfrac{1}{2}\pi]$. Thus,

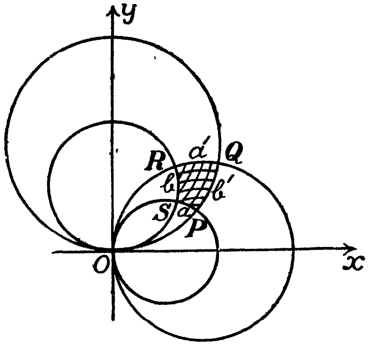
$$\text{the integral}=ab\int_0^{\frac{1}{2}\pi}d\theta\int_0^1\frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,dr$$

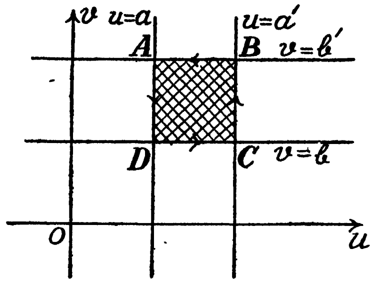
$$=\frac{\pi}{2}\,ab\int_0^1\frac{\sqrt{(1-r^2)}}{\sqrt{(1+r^2)}}\,r\,dr$$

$$=\tfrac{1}{4}\pi(\pi-2)ab,$$

where the integral has been evaluated by putting $r^2=\cos t$.

2. *Integrate the function $1/xy$ over the area bounded by the four circles $x^2+y^2=ax, a'x, by, b'y$ where a, a', b, b' are all positive.*





The integration is to be carried over the shaded area shown in figure. We have supposed $a' > a$ and $b' > b$.

The region of integration is defined by

$$ax \leqslant x^2+y^2 \leqslant a'x \; ; \; by \leqslant x^2+y^2 \leqslant b'y.$$

We change the variables to u, v, where

$$u=(x^2+y^2)/x, \; v=(x^2+y^2)/y.$$

$$\therefore \qquad x=\frac{uv^2}{u^2+v^2},$$

$$y=\frac{u^2v}{\;}$$

It is easy to see that

$$\partial(u, v)/\partial(x, y)=-(x^2+y^2)^2/x^2y^2.$$

$$\therefore \qquad {}^*\partial(x, y)/\partial(u, v)=-x^2y^2/(x^2+y^2)^2$$

Since the Jacobian is negative, the transformation is inverse. This fact may also be directly verified. The new field of integration is determined by the boundaries $u=a$, $u=a'$, $v=b$, $v=b'$, and is, therefore, the rectangle $[a, a' \,;\, b, b']$. Thus we see that

---

*If $x=\varphi(u, v)$ and $y=\psi(u, v)$ then looking upon u, v as functions of x and y we have

$$1=\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad 0=\frac{}{}\;\frac{\partial u}{\partial}+\frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial y},$$

$$0=\frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} \qquad 1=\frac{\partial y}{\partial u}\cdot\frac{}{\partial y}+\frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\therefore \qquad \frac{\partial(x, y)}{\partial(u, v)}\cdot\frac{\partial(u, v)}{\partial(x, y)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1.$$

the integral $= \int\int \frac{1}{xy} \mid J \mid dudv$

$$= \int\int \frac{xy}{(x^2+y^2)^2}\, dudv$$

$$= \int\int \frac{1}{uv}\, dudv = \int_a^{a'} \frac{1}{u}\, du \int_b^{b'} \frac{1}{v}\, dv = \log \frac{a'}{a} . \log \frac{b'}{b}.$$

3. *Prove that*

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \qquad (m > 0,\ n > 0)$$

We have

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx = 2\int_0^{\frac{1}{2}\pi} \cos^{2m-1}\theta \sin^{2n-1}\theta d\theta, \quad \ldots(1)$$

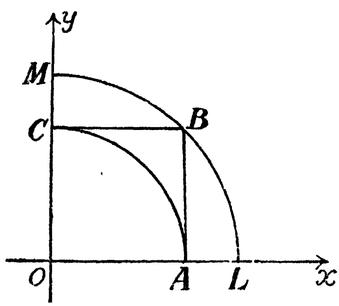
(Putting $x = \cos^2\theta$)

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t}\, dt = 2\int_0^\infty r^{2m-1} e^{-r^2} dr, \text{ (Putting } t = r^2) \quad \ldots(2)$$

From (2), we see that the integral

$$\int\int_E x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dxdy$$

where E is the square $[0, R\ ;\ 0, R]$ tends as its limit, to $\Gamma(m)\,\Gamma(n)$ as $R \to \infty$.



The positive quadrant of the circle $x^2+y^2=R^2$ is a part of the square E which, again, is a part of the positive quadrant of the circle $x^2+y^2=2R^2$. We denote these positive quadrants by E_1, E_2 respectively. The integrand being positive, we have

$$4\int\int_{E_1} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dxdy \leqslant 4\int\int_E x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dxdy$$

$$\leqslant 4\int\int_{E_2} x^{2m-1} y^{2n-1} e^{-x^2-y^2}\, dxdy$$

But changing the variables to r, θ, where $x=r\cos\theta$, $y=r\sin\theta$, we have

$$4\iint_{E_1} x^{2m-1}\,y^{2n-1}\,e^{-x^2-y^2}\,dxdy$$

$$=4\int_0^{\frac{1}{2}\pi}\cos^{2m-1}\theta\,\sin^{2n-1}\theta d\theta\int_0^R e^{-r^2}r^{m+2n-1}\,dr$$

$$=2\beta(m,n)\int_0^R e^{-r^2}\,r^{2m+2n-1}\,dr.$$

Similarly

$$4\iint_{E_2} x^{2m-1}\,y^{2n-1}e^{-x^2-y^2}\,dxdy=2\beta(\mathrm{m},n)\int_0^{\sqrt{2}R} e^{-r^2}\,r^{2m+2n-1}\,dr.$$

$$\therefore\ 2\beta(m,n)\int_0^R e^{-r^2}\,r^{2m+2n-1}\,dr\leqslant 4\iint_{E} x^{2m-1}\,y^{2n-1}\,e^{-x^2-y^2}\,dxdy$$

$$\leqslant 2\beta(m,n)\int_0^{\sqrt{2}R} e^{-r^2}\,r^{2m+2n-1}\,dr.$$

Letting $R\to\infty$, we obtain the required result.

Ex. 1. Show that

(*i*) $\Gamma(\frac{1}{2})=\sqrt{\pi}$.

(*ii*) $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(\sin\theta)}}\cdot\int_0^{\pi}\sqrt{(\sin\theta)}\,d\theta=\pi.$

Ex. 2. If $\displaystyle I=\int_0^a e^{-x^2}\,dx,$

show that

$$\frac{\pi}{4}\left(1-e^{-a^2}\right)<I^2<\frac{\pi}{4}\left(1-e^{-2a^2}\right)$$

and deduce that

$$\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$$

4. *By substituting $x+y=u$, $x=uv$, prove that the value of*

$$\iint\sqrt{[xy(1-x-y)]}\,dxdy,$$

taken over the area of the triangle bounded by the lines $x=0$, $y=0$, $x+y-1=0$ is $2\pi/105$.

Since

$$x=uv, \quad y=u(1-v)$$

we have

$$\partial(x, y)/\partial(u, v)=-u,$$

which is negative.

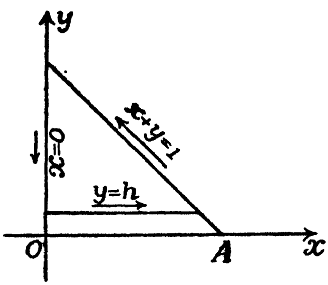
The Jacobian vanishes when $u=0$, *i.e.*, when $x=y=0$, but not otherwise. It is easy to see that to the origin of the xy plane corresponds the whole line $u=0$ of the uv plane so that the correspondence ceases to be *one to one*. In order to exclude $x=0$, $y=0$, we look upon the given integral, which certainly exists, as the limit, when $h \to 0$, of the integral over the region bounded by

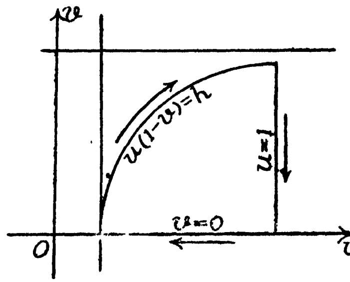
$$x+y=1, \quad x=0, \quad y=h \qquad (h>0)$$

The transformed region is, then, bounded by the lines

$$u=1, \quad v=0, \quad u(1-v)=h,$$

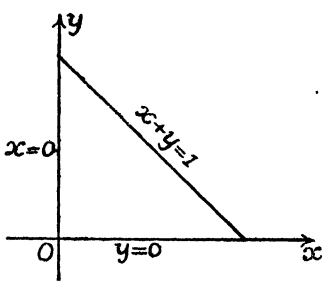
which correspond to the three boundaries of the region in the xy plane.

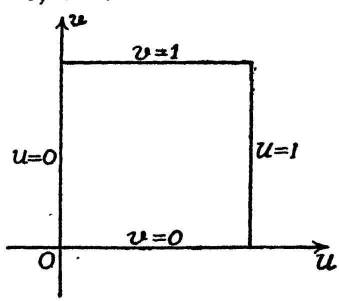




When $h \to 0$, this new region of the uv plane tends, as its limit, to the square bounded by the lines

$$u=1, \quad v=1, \quad u=0, \quad v=0$$





Thus the integral

$$=\int\int \sqrt{[uv.\, u(1-v)(1-u)]}\; u\, du dv$$

$$=\int_0^1 u^2\sqrt{(1-u)}\, du \int_0^1 \sqrt{[v(1-v)]}\, dv.$$

Putting $u=\sin^2\theta$ and $v=\sin^2\psi$, we see that

$$\int_0^1 u^2\sqrt{(1-u)}\,du=2\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^2\theta\,d\theta=\frac{2.4.2.1}{7.5\,3.1}=\frac{16}{105},$$

$$\int_0^1 \sqrt{[v(1-v)]}\,dv=2\int_0^{\frac{1}{2}\pi}\sin^2\psi\cos^2\psi\,dx=\frac{2.1.1}{4.2}\cdot\frac{\pi}{2}=\frac{\pi}{8}$$

Hence the result.

Note. We have

$$u=x+y,\ v=x/(x+y),$$

and

$$x=uv,\quad y=u(1-v).$$

Thus when x, y are both positive and $(x+y)$ is less than unity, then u, v both lie between 0 and 1. Conversely, when u, v both lie between 0 and 1, then x, y are both positive and $x+y$ is less than 1. Arguing in this manner also, we can obtain the transform of the given region in the (x, y) plane.

5. *Evaluate*

$$\int\int (1-x-y)^{l-1}x^{m-1}y^{n-1}\,dx\,dy,$$

taken over the area of the triangle formed by the lines

$$x=0,\ y=0,\ x+y=1\ ;$$

l, m, n being all positive.

Employing the transformation

$$x+y=u,\ x=uv,$$

as in Ex. 4, we see that the given integral

$$=\int_0^1\int_0^1 (1-u)^{l-1}u^{m+n-1}v^{m-1}(1-v)^{n-1}\,du\,dv$$

$$=\left\{\int_0^1 (1-u)^{l-1}u^{m+n-1}\,du\right\}\left\{\int_0^1 v^{m-1}(1-v)^{n-1}\,dv\right\}$$

$$=\beta(l, m+n)\,\beta(m, n)$$

$$=\frac{\Gamma(l)\,\Gamma(m+n)}{\Gamma(l+m+n)}\cdot\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}.$$

Exercises 1

1. Evaluate

$$\int\int\sqrt{[x(2a-x)+y(2b-y)]}\,dxdy,$$

over the circle $x^2+y^2-2ax-2by=0$.

2. Evaluate

$$\int\int \sqrt{(a^2-x^2-y^2)}\, dx\, dy,$$

over the circle $x^2+y^2=ax$.

3. Show that the value of the integral

$$\int\int (1-x-y)^3\, x^{\frac{1}{2}}\, y^{\frac{1}{2}}\, dxdy,$$

taken over the triangle whose vertices are the origin and the points (0, 1) and (1, 0) is $\pi/480$.

4. Show that

$$\int\int x^{\frac{1}{2}}\, y^{\frac{1}{3}}(1-x-y)\, dxdy=\beta\left(\frac{17}{6}, \frac{5}{4}\right)\beta\left(\frac{3}{2}, \frac{4}{3}\right),$$

over the triangle bounded by the lines $x=0$, $y=0$, $x+y=1$.

5. Show that

$$\int\int \sqrt{[x^3\, y^4(1-x^2-y^2)]}\ \ dxdy=\tfrac{1}{4}\beta\left(\frac{11}{4}, \frac{3}{2}\right)\beta\left(\frac{5}{4}, \frac{3}{2}\right)$$

over the positive quadrant of the circle $x^2+y^2=1$.

6. Show that

$$\int_0^2\int_0^x \left\{ (x+y+1)^2-4xy \right\}^{-\frac{1}{2}} dxdy=\tfrac{1}{2}\log(16/e),$$

by means of the transformation

$$x=u(1+v),\ y=v(1+u).$$

[$\partial(x, y)/\partial(u, v)=1+u+v$, which is positive, for non-negative values of u, v. The field of integration in the xy plane is bounded by $y=0$, $x=2$, $y=x$. Taking into consideration only non-negative value of u and v, we see that the corresponding region of the uv plane lies in the positive quadrant and is bounded by

$$v=0,\ u(1+v)=2,\ u=v.$$

Therefore, the integral

$$=\int_0^1\int_0^u dudv+\int_1^2\int_0^{(2-u)/u} dudv.]$$

7. Transform the integral

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sqrt{\frac{\sin\varphi}{\sin\theta}}\, d\varphi d\theta,$$

by the substitution $x=\sin\varphi\cos\theta$, $y=\sin\varphi\sin\theta$, and find its value.

8. Change the variables in the integral

$$\int_0^{2a} dx \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(4ax-x^2)}} \left(1+\frac{y^2}{x^2}\right) dy,$$

to r and θ, where $x=r\cos^2\theta$, $y=r\sin\theta\cos\theta$ and show that the value of the integral is $(\pi+\frac{8}{3})\, a^2$.

9. Transform the integral

$$\int_0^c \int_0^{c-x} V(x, v)\, dxdy,$$

by the substitution $x+y=u$, $y=uv$.

10. In the double integral $\int \int f'x, y)\, dxdy$, change the variables from x, y to u, v, where $x^2+y^2=u$, $xy=v$ and find the limits in the new integral when the integration is performed over the positive quadrant of the circle $x^2+y^2=a^2$.

11. By using the transformation

$$x=u^2-v^2, \quad y=2uv$$

or otherwise, evaluate

$$\int \int \frac{dxdy}{\sqrt{(x^2+y^2)}},$$

taken over the region enclosed by arcs of the confocal parabolas

$$y^2=4a_r(x+a_r), \quad (r=1, 2, 3)$$

where $a_1 > a_2 > 0$, $a_3 < 0$.

12. Find the area of the curvilinear quadrilateral bounded by the four confocal conics of the system

$$x^2/\lambda+y^2/(\lambda-c^2)=1,$$

which are determined by giving, λ, the values $\frac{1}{4}c^2$, $\frac{3}{4}c^2$, $\frac{4}{3}c^2$, $\frac{5}{3}c^2$, respectively.

[Transform into confocal co-ordinates *i e.*, express x and y in terms of λ, μ the parameters of the two confocals which pass through (x, y).]

13. Express the integral

$$\int_0^3 dx \int_0^{\sqrt{(5-5x^2/9)}} dy$$

in terms of the variables λ and μ defined by the equations

$$\lambda+\mu=\sqrt{[(x+2)^2+y^2]}, \quad \lambda-\mu=\sqrt{[(x-2)^2+y^2]}$$

and thus verify that the value of the integral is $\frac{3}{4}\pi\sqrt{5}$.

14. Evaluate $\int \int xy\, dxdy$; the field of integration being the area common to the circles $x^2+y^2=x$, $x^2+y^2=y$.

[Change x, y to u, v where $u=(x^2+y^2)/x$, $v=(x^2+y^2)/y$. The new field in the uv plane is the square [0, 1 ; 0, 1].]

15. Evaluate

$$\int \int (xy^3+x^3y)\, dxdy,$$

where the field of integration lies in the first quadrant and is bounded by the central conics.

$$ax^2+by^2=l,\ ax^2+by^2=m,\ ax^2-by^2=n.\ ax^2-by^2=p,\ (l > m > 0,\ n > p > 0).$$

16. Evaluate $\int \int (x^4-y^4)\, dxdy$ over the part of the positive quadrant in which

$$1 \leqslant (x^2-y^2) \leqslant 2, \quad 1 \leqslant xy \leqslant 2.$$

Exercises II

1. Show in a diagram the field of integration of the integral

$$\int_0^1 dx \int_x^{1/x} \frac{y^2 dy}{(x+y)^2\sqrt{(1+y^2)}} .$$

and, by changing the order of integration, show that the value of the integral is $\sqrt{2}-\frac{1}{2}$.

2. Change the order of integration in the integral

$$\int_0^1 dy \int_{1-\sqrt{y-1}}^{1+\sqrt{1-y}} \frac{dx}{(x^2-2x+y-3)^2}$$

and hence evaluate it.

3. Evaluate

$$\int \int (e^x+e^{2y})\, dxdy$$

over the square

$$0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1$$

(*i*) directly ;

(*ii*) as the limit of a sum, first dividing the square into n^2 equal squares and then making $n \to \infty$.

4. Evaluate

$$\int\int_D x^2 y\, dxdy \text{ and } \int\int_D xy^2\, dxdy$$

where D is the region

$$x^2+y^2 \leqslant x.$$

5. Evaluate

$$\int\int_D \frac{y^2}{x} dxdy,$$

where, D, is determined by

$$x^2+y^2 \leqslant x.$$

6. Evaluate

$$\int \int (x+y)^n\, xy\,dxdy,$$

over all non-negative values of x and y for which $x+y \leqslant 1$, n being positive.

7. Evaluate

$$\int\int_R (2x+3y)^3\, dxdy,$$

(*i*) where R is the triangular region bounded by the lines

$$x=0,\ y=0,\ 2x+3y=1,$$

(*ii*) where R is the region of the half plane $y \geqslant 0$ lying inside the circle

$$x^2+y^2=1.$$

8. Evaluate

$$\iint_D \sin \pi x \sin \pi y \, dx dy,$$

where the region D is defined by

$$0 \leqslant x, \ 0 \leqslant y, \ x+y \leqslant 1.$$

9. Evaluate

$$\iint_A y^{-1} dx dy,$$

where A is the part of the (x, y) plane that lies between the lines $x=1$, $x=2$ and also between the line $y=4x$ and the curve $y=x^2$.

10. Evaluate

$$\iint \frac{xy(x+y)^2}{x^2+y^2} dx dy$$

over the sector in the first quadrant bounded by the straight lines

$$y=0, \ y=x \text{ and the circle } x^2+y^2=1.$$

11. Evaluate

$$\iint \sin \pi(x^2+y^2) dx dy$$

over the interior of the circle

$$x^2+y^2=1.$$

12. Evaluate

$$\iint_D \sqrt{xy} \, dx dy$$

where D is defined by

$$0 \leqslant x, \ 0 \leqslant y, \ x+y \leqslant 1.$$

13. Show that

$$\iint \sqrt{(4a^2-x^2-y^2)} \, dx dy = \tfrac{4}{9} \, (3\pi-4)a^3,$$

taken over the upper half of the circle

$$x^2+y^2=2ax.$$

14. Evaluate

$$\iint_S (x^2+y^2) \, dx dy$$

where S is the area of the xy-plane defined by the relations

$$2a \geqslant x^2+y^2 \geqslant a, \ x \geqslant 0, \ a \geqslant x^2-y^2 \geqslant -a, \ y \geqslant 0,$$

by means of the transformation

$$x^2+y^2=u, \ x^2-y^2=v.$$

15. By means of the substitution

$$x=u(1-v), \ y=uv,$$

prove that

$$\iint \varphi'(x+y) \frac{(xy)^{n-1}}{(x+y)^{2n-1}} dx dy, \ (n \geqslant 1)$$

over the region defined by

$$x \geqslant 0,\ y \geqslant 0,\ 1 \leqslant x+y \leqslant 2$$

within which $\varphi'(x+y)$ is given to be continuous, is equal to

$$\{\varphi(2)-\varphi(1)\}\ I,$$

where

$$I=2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta.$$

16. Calculate the integral

$$\int\int \frac{(x-y)^2}{x^2+y^2}\, dxdy,$$

both over the triangle

$$0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x$$

and over the circle

$$x^2+y^2 \leqslant 1.$$

17. Evaluate

$$\int\int_D (x^2+y^2)\, dxdy$$

where D is defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1.$$

18. Evaluate

$$\int\int_D \sin x \sin y \sin (x+y)\, dxdy$$

where D is defined by

$$x \geqslant 0,\ y \geqslant 0,\ x+y \leqslant \pi/2.$$

19. Find the area in the first quadrant between the curves

$$x^2+y^2=a^2,\ x^2+y^2=2a^2,$$
$$x^2-y^2=a^2,\ x^2-y^2=-a^2.$$

20. By means of a suitable substitution, or otherwise, evaluate the integral

$$\int_0^1 dx \int_0^{1-x} e^{(y-x)/(y+x)}\, dy.$$

21. Evaluate

$$\int\int \frac{dxdy}{a^2+x^2+y^2},$$

taken over the circle $x^2+y^2 \leqslant 1$ and hence, by equating the double integral to an equivalent line integral, show that

$$\int_0^{2\pi} \frac{\sin\theta}{\sqrt{(1-k^2\sin^2\theta)}} \sin^{-1}[\sin(k\sin\theta)]d\theta = -\frac{\pi}{4}\log(1-k^2),\ 0 < k < 1.$$

22. Prove that

$$\int\int_D x^{m-1}y^{n-1}f(x+y)\,dxdy, \quad = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\int_0^1 t^{m+n-1}f(t)\,dt,$$

where D is defined by

$$x, y \geqslant 0,\ x+y \leqslant 1.$$

23. Prove that if $a, b, h, ab-h^2$ are all positive,

$$\int_0^\infty\int_0^\infty e^{-(ax^2+2hxy+by^2)}\,dxdy = \frac{\pi}{2\sqrt{(ab-h^2)}}.$$

24. Prove that

$$\int\int_D x^n\,dxdy = \frac{2}{(n+1)(n+2)} H_n \| D \|,$$

where D is the triangle whose vertices are

$$(x_1, y_1), (x_2, y_2), (x_3, y_3),$$

$\| D \|$ is its area and H_n is the sum of the homogeneous products of degree n in x_1, x_2, x_3.

25. Prove that

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{m^2\cos^2\phi+n^2\cos^2\theta}{[(1-m^2\cos^2\phi)(1-n^2\cos^2\theta)]}\,d\theta d\phi = \frac{\pi}{2}.$$

CHAPTER XIV

CURVE LENGTHS. SURFACE AREAS

181. Rectifiability of a curve. Let

$$x=f(t),\ y=g(t),\ z=h(t)$$

be three functions of t defined and continuous in an interval $[a, b]$. Then the set of points (x, y, z) obtained when t ranges over $[a, b]$ is called a continuous curve in the three dimensional space E_3. We shall be concerned with continuous curves only.

Let

$$D(a=t_0,\ t_1,\ldots\ldots,\ t_{r-1},\ t_r\ldots\ldots,\ t_n=b), \qquad \ldots(1)$$

be any division of $[a, b]$.

We write

$$x_r=f(t_r),\ y_r=g(t_r),\ z_r=h(t_r)$$

so that corresponding to a division (1) D of $[a, b]$ we get an ordered system of points

$$(x_0,\ y_0,\ z_0),\ldots\ldots,\ (x_r,\ y_r,\ z_r),\ldots\ldots,\ (x_n,\ y_n,\ z_n)$$

on the curve. This system of points may be referred to as a polygon inscribed in the curve. We write

$$\pi_D = \sum_{r=1}^{n} \sqrt{\left[\ (x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{r-1})^2\ \right]},$$

so that π_D denotes the length of the inscribed polygon in question.

Def. *The curve is said to be rectifiable, if the set of numbers π_D obtained on considering all possible divisions D of $[a, b]$ is bounded and, in the event of boundedness, the upper bound is known as the length of the curve.*

Characterisation of Rectifiable curves. Theorem. *Necessary and sufficient conditions for the curve*

$$x=f(t),\ y=g(t),\ z=h(t), \qquad a \leqslant t \leqslant b$$

to be rectifiable are that the functions

$$f(t),\ g(t),\ h(t)$$

are all of bounded variation over $[a, b]$.

The conditions are necessary. Let the curve be rectifiable and let, l, denote the length of the curve.

We have, for any division D.

$$|x_r - x_{r-1}| \leqslant \sqrt{[(x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{r-1})^2]},$$

so that summing over r, we obtain

$$\sum_{r=1}^{n} |x_r - x_{r-1}| \leqslant \sum_{r=1}^{n} [(x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{-r1})^2] \leqslant l$$

i.e.,

$$\sum_{r=1}^{n} |f(t_r)-f(t_{r-1})| \leqslant l, \qquad \ldots(i)$$

for each division D of $[a, b]$. Thus $f(t)$ is of bounded variation over $[a, b]$.

Similarly $g(t)$, $h(t)$ are also of bounded variation over $[a, b]$.

The conditions are sufficient. Let the three functions be of bounded variation.

We have, as may be easily shown,

$$\sqrt{[(x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{r-1})^2]} \leqslant |x_r-x_{r-1}| + |y_r-y_{r-1}| + |z_r-z_{r-1}|$$

so that on summing up we obtain

$$\pi_D = \sum_{r=1}^{n} \sqrt{[(x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{r-1})^2]}$$

$$\leqslant \sum_{r=1}^{n} |x_r-x_{r-1}| + \sum_{r=1}^{n} |y_r-y_{r-1}| + \sum_{r=1}^{n} |z_r-z_{r-1}|$$

$$\leqslant V_f[a, b]+V_g[a, b]+V_h[a, b] \qquad \ldots(ii)$$

where $V_f[a, b]$, $V_g[a, b]$, $V_h[a, b]$ denote the variations of the functions $f(t)$, $g(t)$, $h(t)$ over $[z, b]$.

Thus π_D is bounded above and the curve is rectifiable.

Cor. 1. If a curve

$$x=f(t),\ y=g(t),\ z=h(t). \qquad a \leqslant t \leqslant b$$

is rectifiable and, l, denotes the length thereof, then from (i) and (ii), it follows that

$$V_f[a, b] \leqslant l,\ V_g[a, b] \leqslant l,\ V_h[\ \ , b] \leqslant l \qquad \ldots(1)$$

$$l \leqslant V_f[a, b]+V_g[a, b]+V_h[a, b] \qquad \ldots(2)$$

Cor. 2. *The given curve*

$$x=f(t), \quad y=g(t), \quad z=h(t),$$

is rectifiable if the functions

$$f(t), g(t), h(t)$$

are derivable with bounded derivatives.

This follows from Illustration 3 on page 286 and the preceding theorem.

182. Properties of rectifiable curves

Arc of a curve. If

$$x=f(t), y=g(t), z=h(t), \qquad a \leqslant t \leqslant b$$

is a curve C and $[c, d]$ is any sub-interval of $[a, b]$, then the curve given by

$$x=f(t), y=g(t), z=h(t)$$

when, t, varies over $[c, d]$ is said to be an *arc* of the curve C.

Theorem. *Each arc of a rectifiable curve is rectifiable.*

This follows from the fact that if $f(t)$ is of bounded variation over $[a, b]$, then it is also of bounded variation over each sub-interval thereof.

[Refer 4 of A·2, page 288].

Theorem. *If the curve C*

$$x=f(t), y=g(t), z=h(t), \qquad a \leqslant t \leqslant b$$

is rectifiable and, c, is any point of $[a, b]$, then

$$l[a, b]=l[a, c]+l[c, b],$$

where $l[a, c]$ denotes the length of the arc of the curve arising for t lying in $[a, c]$ and similar meanings are given to $l[c, b]$ and $l\,[a, b]$.

Let D be *any* division of $[a, b]$ and let D' denote the division of $[a, b]$ consecutive to D with c as the only additional point, if it were not already a point of D. The division D' of $[a, b]$ gives rise to divisions of $[a, c]$ and $[c, b]$ which we may denote by D_1 and D_2 respectively. We have then

$$\pi_D \leqslant \pi_{D'}=\pi_{D_1}+\pi_{D_2}.$$

Also

$$\pi_{D_1} \leqslant l[a, c], \qquad \pi_{D_2} \leqslant l[c, b],$$

$$\therefore \qquad \pi_D \leqslant l[a, c]+l[c, b].$$

This gives

$$l[a, b] \leqslant l[a, c]+l[c, b]. \qquad \ldots 1)$$

Now suppose that $\varepsilon > 0$ is any given number.

There exist divisions D_3 and D_4 of $[a, c]$ and $[c, b]$ respectively such that

$$\pi_{D_3} > l[a, c]-\tfrac{1}{2}\varepsilon,$$

$$\pi_{D_4} > l[c, b]-\tfrac{1}{2}\varepsilon.$$

Let D'' denote the division of $[a, b]$ arising from the divisions D_3 and D_4 of $[a, c]$ and $[c, b]$ respectively. We have then

$$\pi_{D''} = \pi_{D_3} + \pi_{D_4} > l[a, c]+l[c, b]-\varepsilon.$$

$$\therefore \qquad l[a, b] \geqslant l[a, c]+l[c, b]-\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we obtain

$$l[a, b] \geqslant l[a, c]+l[c, b]. \qquad \ldots(2)$$

From (1) and (2), we obtain

$$l[a, b] = l[a, c]+l[c, b]$$

as was to be proved.

Theorem. *The function*

$$l[a, t], \qquad a \leqslant t \leqslant b$$

is a monotonically increasing continuous function of t in $[a, b]$.

Let t_1, t_2 be two numbers such that

$$a \leqslant t_1 < t_2 \leqslant b.$$

We have

$$l[a, t_2]=l[a, t_1]+l[t_1, t_2],$$

so that

$$l[a, t_2] \geqslant l[a, t_1], \text{ for } l[t_1, t_2] \geqslant 0.$$

Thus $l[a, t]$ is a monotonically increasing function of t.

Suppose, now, that c, $c+h$ are any two points of $[a, b]$.

We have, by cor. 1 of § 181, page 418.

(*i*) if $h > 0$,

$$\begin{aligned} 0 \leqslant l[c, c+h] &\leqslant V_f[c, c+h]+V_g[c, c+h]+V_h[c, c+h] \\ &=[V_f[a, c+h]-V_f[a, c]]+[V_g[a, c+h]-V_g[a, c]]+ \\ &\qquad [V_h[a, c+h]-V_h[a, c]] \\ &=[V_f(c+h)-V_f(c)]+[V_g(c+h)-V_g(c)]+ \\ &\qquad [V_h(c+h)-V_h(c)] \end{aligned}$$

(*ii*) if $h < 0$,

$$\begin{aligned} 0 \leqslant l[c+h, c] \leqslant [V_f(c)-V_f(c+h)]+[V_g(c)-V_g(c+h)]+ \\ [V_h(c)-V_h(c+h)] \end{aligned}$$

From these, we see that

$$|\, l[a, c+h]-l[a, c] \,| \leqslant |\, V_f(c+h)-V_f(c) \,| + |\, V_g(c+h)-V_g(c) \,| + |\, V_h(c+h)-V_h(c) \,|.$$

Now, as shown in A·6, page 291, $V_f(t)$, $V_g(t)$ and $V_h(t)$ are continuous at, c, and accordingly for given $\varepsilon > 0$, there exists a $\delta > 0$ such that for $|\, h \,| \leqslant \delta$,

$$|\, V_f(c+h)-V_f(c) \,| < \tfrac{1}{3}\varepsilon, \quad |\, V_g(c+h)-V_g(c) \,| < \tfrac{1}{3}\varepsilon,$$
$$|\, V_h((c+h)-V_h(c) \,| < \tfrac{1}{3}\varepsilon.$$

Thus for $|\, h \,| \leqslant \delta$,

$$|\, l[a, c+h]-l[a, c] \,| < \varepsilon,$$

so that $l[a, t]$ is continuous for $t=c$.

Hence the theorem.

Note 1. It can be seen that *the function $l[a, t]$ is a strictly monotonically increasing function of t, unless $f(t)$, $g(t)$, $h(t)$ are all constant over some sub-interval of* $[a, b]$. Let, if possible,

$$l[a, t_1]=l[a, t_2], \qquad t_2 > t_1.$$

$\therefore$ it follows that

$$l[t_1, t_2]=0.$$

$$\therefore \quad V_f[t_1, t_2]=0, \; V_g[t_1, t_2]=0, \; V_h[t_1, t_2]=0.$$

[Cor. 1 to § 181]

$\therefore$ $f(t)$, $g(t)$, $h(t)$ are constant over $[t_1, t_2]$.

Note 2. It is usual to write

$$s(t)=l[a, t],$$

so that $s(t)$ denotes the length of the arc corresponding to the sub-interval $[a, t]$ of $[a, b]$.

Supposing that $f(t)$, $g(t)$, $h(t)$ are such that there is no sub-interval of $[a, b]$ for which they are all constant, $s(t)$ is a *stric ly* monotonically increasing continuous function of t, and increases from 0 to l as t increases from a to b. By § 59, page 102, we see that, t, can itself be regarded as a continuous function of s. If we write

$$t=\varphi(s), \qquad 0 \leqslant s \leqslant l,$$

we see that

$$x=f(t)=f[\varphi(s)]=F(s), \text{ say}$$
$$y=g(t)=g[\varphi(s)]=G(s),$$
$$z=h(t)=h[\varphi(s)]=H(s),$$

so that the given rectifiable curve is as well given by

$$x=F(s), \; y=G(s), \; z=H(s),$$

where the *parameter is the arc length s.*

183. Integral expression for the length of a curve. Theorem. *The length of the curve*

$$x=f(t),\ y=g(t),\ z=h(t),\ a \leqslant t \leqslant b$$

is

$$\int_a^b \sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}dt,$$

if $f(t)$, $g(t)$, $h(t)$ are derivable with continuous derivatives and for no value of t,

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2=0.$$

Clearly $f(t)$, $g(t)$, $h(t)$ are, under the given conditions, functions with bounded derivatives, and accordingly they are functions, with bounded variation (Illustration 3, page 286). Thus the curve is rectifiable.

We shall now proceed to find an integral expression for the length of the curve.

Let

$$D(a=t_0 \ldots\ldots, t_{r-1},\ t_r, \ldots\ldots, t_n=b),$$

be any division of $[a, b]$.

We have, by Lagrange's mean value theorem,

$$\begin{aligned} x_r-x_{r-1} &= f(t_r)-f(t_{r-1}) \\ &= (t_r-t_{r-1})\, f'(\alpha_r), \\ y_r-y_{r-1} &= (t_r-t_{r-1})\, g'(\beta_r), \\ z_r-z_{r-1} &= (t_r-t_{r-1})\, h'(\gamma_r), \end{aligned}$$

where α_r, β_r, γ_r belong to the sub-interval $[t_{r-1}, t_r]$.

$$\therefore \quad \pi_D = \sum_{r=1}^{n} \sqrt{[(x_r-x_{r-1})^2+(y_r-y_{r-1})^2+(z_r-z_{r-1})^2]}$$

$$= \sum_{r=1}^{n} (t_r-t_{r-1})\sqrt{[f'^2(\alpha_r)+g'^2(\beta_r)+h'^2(\gamma_r)]}.$$

Let ξ_r be any point of $[t_{r-1}, t_r]$.

We write

$$a_r=f'(\alpha_r),\quad b_r=g'(\beta_r),\quad c_r=h'(\gamma_r)\ ;$$

$$a'_r=f'(\xi_r),\quad b'_r=g'(\xi_r),\quad c'_r=h'(\xi_r).$$

We have

$$\sqrt{(a_r^2+b_r^2+c_r^2)}-\sqrt{(a'^2_r+b'^2_r+c'^2_r)}$$

$$=\frac{(a_r^2+b_r^2+c_r^2)-(a_r'^2+b_r'^2+c_r'^2)}{\sqrt{(a_r^2+b_r^2+c_r^2)}+\sqrt{(a_r'^2+b_r'^2+c_r'^2)}}$$

$$=\frac{(a_r-a_r')(a_r+a_r')+(b_r-b_r')(b_r+b_r')+(c_r-c_r')(c_r+c_r')}{\sqrt{(a_r^2+b_r^2+c_r^2)}+\sqrt{(a_r'^2+b_r'^2+c_r'^2)}}.$$

Suppose that k is a number such that

$$|f'(t)| \leqslant k, \; |g'(t)| \leqslant k, \; |h'(t)| \leqslant k$$

for each t belonging to $[a, b]$.

Also since

$$[f'(t)]^2+[g'(t)]^2+[h'(t)]^2 \neq 0,$$

for any value of t, and is continuous, there exists a positive number K^2, *viz.*, the lower bound of the same, such that

$$[f'(t)]^2+[g'(t)]^2+[h'(t)]^2 \geqslant K^2,$$

for every t.

Thus we obtain

$$\left| \sqrt{(a_r^2+b_r^2+c_r^2)}-\sqrt{(a_r'^2+b_r'^2+c_r'^2)} \right|$$

$$\leqslant \frac{2k}{K}(\,|a_r-a_r'| + |b_r-b_r'| + |c_r-c_r'|\,).$$

We write

$$S_D=\sum_{r=1}^{n}(t_r-t_{r-1})\sqrt{[f'^2(\xi_r)+g'^2(\xi_r)+h'^2(\xi_r)]}.$$

$$\therefore \quad |\pi_D - S_D| \leqslant \sum_{r=1}^{n} |t_r-t_{r-1}| \frac{2k}{K}\{\,|f'(\alpha_r)-f'(\xi_r)| +$$

$$|g'(\beta_r)-g'(\xi_r)| + |h'(\gamma_r)-h'(\xi_r)|\,\}.$$

Let $\varepsilon > 0$ be given. Because of the uniform continuity of $f'(t)$, $g'(t)$, $h'(t)$ in $[a, b]$, there exists $\delta > 0$ such that

$$|f'(t')-f'(t'')| < \varepsilon, \; |g'(t')-g'(t'')| < \varepsilon, \; |h'(t')-h'(t'')| < \varepsilon$$

when $|t''-t'| < \delta$.

Now choosing D as any division of norm $\leqslant \delta$, we see that

$$\left|\pi_D - S_D\right| \leqslant \sum_{r=1}^{n}\left| t_r-t_{r-1}\right|\frac{2k}{K}.3\varepsilon = \frac{6k\varepsilon}{K}(b-a).$$

Thus

$$\lim\,(\pi_D - S_D) = 0,$$

as the norm of D tends to 0.

Also we know that

$$\lim S_D = \int_a^b \sqrt{[f'^2(t)+g'^2(t)+h'^2(h)]}\, dt,$$

as the norm of D tends to zero. Thus we have

$$\lim \pi_D = \int_a^b \sqrt{[f'^2(t)+g'^2(t)+h'^2(t)]}\, dt, \qquad \ldots(1)$$

as the norm of D tends to zero.

We denote the integral in (1) by I, and show that I is the upper bound of the aggregate of numbers π_D.

Firstly the upper bound cannot be less than I, for we can choose π_D as near as I as we like.

Secondly the upper bound cannot exceed I, for, while on the one hand π_D increases on making D finer, it can also be made as near I as we like by choosing sufficiently fine divisions.

[The reader may also work out a *straight* proof.]

Smooth curves. Def. *A curve*

$$x=f(t),\ y=g(t),\ z=h(t),\ a \leqslant t \leqslant b$$

is said to be smooth, if the three functions $f(t)$, $g(t)$, $h(t)$ possess continuous derivatives in $[a, b]$ and at **no** *point thereof*

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2,$$

is zero.

Note. The condition of the non-vanishing of

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2,$$

is equivalent to saying that for no value t, the three derivatives

$$\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}$$

simultaneously vanish.

The reader having some knowledge of differential geometry of curves may recall that geometrically interpreted this condition means that the curve has a definite direction at each point thereof and is free from multiple points.

Ex. Find the lengths of the following curves :—

(1) $x=a\cos\theta,\ y=a\sin\theta,\ z=a\theta$; $\qquad 0 \leqslant \theta \leqslant 2\pi.$

(2) $x=at^2,\ y=2at,\ z=at$; $\qquad 0 \leqslant t \leqslant 1.$

(3) $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta),\ z=a\theta$; $\qquad -\pi \leqslant \theta \leqslant \pi.$

184. Surface areas. Let

$$x=f(u, v),\ y=g(u, v),\ z=h(u, v)$$

be three continuous functions of (u, v) defined in some domain D which is a sub-set of $\mathbf{E}_2$. Then the set of points (x, y, z) of $\mathbf{E}_3$ obtained as (u, v) ranges over D is known as a surface in $\mathbf{E}_3$.

It is usual to say that while a curve in $\mathbf{E}_3$ is a set with *one* degree of freedom, a surface in $\mathbf{E}_3$ is a set with *two* degrees of freedom.

Associated with the concept of surface is that of surface area just as curve length is associated with a curve. The subject of surface area is, however, beset with difficulties and it is not possible to adopt a straight generalisation of the method employed for defining the lengths of curves by means of inscribed polygons. Instead of adopting an elementary but rather difficult approach to the concept of surface area, we shall take to the more sophisticated one as given below.

184·1. Surface area. Def. *The area of the surface*

$$x=f(u, v),\ y=g(u, v),\ z=h(u, v)$$

where (u, v) ranges over a domain D, is the double integral

$$\iint_D \sqrt{\left\{\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2+\left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2+\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2\right\}}\, dudv,$$

if the functions $f(u, v)$, $g(u, v)$, $h(u, v)$ possess continuous first order partial derivatives over D and at no point of D,

$$\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2+\left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2+\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2=0.$$

(Compare with the expression for curve length in § 183, page 422.

Note. Here, we do not propose to *derive* the expression as given above by basing the derivation on some *elementary* notion of surface area. Lest the definition as given should appear very arbitrary, we outline some considerations to call forth reader's *faith* in the same.

I. Let the surface be plane. We take

$$x=u,\ y=v,\ z=0,$$

where (u, v) ranges over a domain D in the XY-plane. In fact, the surface coincides with D in the present case.

We have

$$\frac{\partial(x, y)}{\partial(u, v)}=\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}=1,\ \frac{\partial(y, z)}{\partial(u, v)}=0,\ \frac{\partial(z, x)}{\partial(u, v)}=0,$$

so that

$$\iint_D \sqrt{\left\{\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2+\left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2+\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2\right\}}\, dudv=\iint_D dudv$$

$$= \text{area of } D.$$

Refer to an article by T. Rado '*what is an area of the surface*' in American mathematical monthly, vol. 50, pp. 139—141, 1943 and to an article by J. W. T. Youngs, '*Curves and Surfaces*' in American mathematical monthly, vol. 51, pp. 1—11, 1944.

Thus the definition of surface area as now given agrees with that of areas of plane regions as given in § 170, page 384. See also § 177, page 394.

II. Consider the surface given by

$$z=h(x, y).$$

We take it as given by

$$x=u,\ y=v,\ z=h(u, v)$$

where $(u, v)=(x, y)$ ranges over a domain D of the XY-plane.

We have

$$\frac{\partial(y, z)}{\partial(u, v)}=-\frac{\partial z}{\partial u}=-\frac{\partial z}{\partial x},$$

$$\frac{\partial(z, x)}{\partial(u, v)}=-\frac{\partial z}{\partial v}=-\frac{\partial z}{\partial y},$$

$$\frac{\partial(x, y)}{\partial(u, v)}=1.$$

$$\therefore\ \text{surface area}=\iint_D\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dxdy. \qquad ...(1)$$

The reader acquainted with the elements of Differential Geometry knows that

$$-\frac{\partial z}{\partial x},\ -\frac{\partial z}{\partial y},\ 1$$

are the direction ratios of the normal to the surface at any point (x, y, z) thereof. So that if we suppose $\delta\sigma$ to be an element of surface lying on the tangent plane at the point, the projection area δA of $\delta\sigma$ on the XY-plane is given by

$$\delta A=\frac{\delta\sigma}{\sqrt{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]}}$$

or

$$\delta\sigma=\sqrt{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]}\,\delta A$$

$$=\sqrt{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]}\,\delta x\,\delta y.$$

This *suggests*, on summation, the truth of (1).

Note. The reader may easily show that

$$\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2+\left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2+\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2=EG-F^2,$$

where

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2,$$

$$F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{dv}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2,$$

Smooth surfaces. Def. *A surface as given by*

$$x=f(u, v),\ y=g(u, v),\ z=h(u, v)$$

where (u, v) *ranges over* D *is said to be* **smooth,** *if* f, g, h *possess continuous first order partial derivatives at each point of* D *and at no point thereof*

$$\frac{\partial(y, z)}{\partial(u, v)},\ \frac{\partial(z, x)}{\partial(u, v)},\ \frac{\partial(x, y)}{\partial(u, v)}$$

vanish simultaneously.

Illustrations.

I. *Surface area of the sphere given by*

$$x=a \sin\theta \cos\varphi,$$
$$y=a \sin\theta \sin\varphi,$$
$$z=a \cos\theta,$$

where

$$0 \leqslant \theta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi.$$

It may be easily seen that

$$E=\left(\frac{\partial x}{\partial\theta}\right)^2+\left(\frac{\partial y}{\partial\theta}\right)^2+\left(\frac{\partial z}{\partial\theta}\right)^2=a^2,$$

$$F=\frac{\partial x}{\partial\theta}\cdot\frac{\partial x}{\partial\varphi}+\frac{\partial y}{\partial\theta}\,\frac{\partial y}{\partial\varphi}+\frac{\partial z}{\partial\theta}\,\frac{\partial z}{\partial\varphi}=0,$$

$$G=\left(\frac{\partial x}{\partial\varphi}\right)^2+\left(\frac{\partial y}{\partial\varphi}\right)^2+\left(\frac{\partial z}{\partial\varphi}\right)^2=a^2 \sin^2\theta.$$

$$\therefore\quad \text{surface area} = \int_0^{2\pi}\int_0^{\pi} \sqrt{(EG-F^2)}\, d\theta\, d\varphi$$

$$=a^2\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\varphi=4a^2\pi.$$

Exercises

1. Find the surface area of the portion of the sphere

$$x^2+y^2+z^2=a^2,$$

inside the cylinder

$$x^2+y^2=ax.$$

2. Find the area of the part of the surface of the paraboloid

$$4ax=y^2+z^2,$$

enclosed by the cylinder

$$y^2=ax,$$

and the plane

$$x=3a.$$

3. Prove that the surface area of the cylinder

$$x^2+y^2-y=0,$$

contained between $z=0$ and the surface

$$z=y^2/(x^2+y^2),$$

is $\pi/2$.

4. Find the area of the part of the spherical surface

$$x^2+y^2+z^2=4a^2$$

enclosed by the cylinder

$$(x^2+y^2)^2=2a^2(2x^2+y^2).$$

5. Find the area of the surface cut from the cylinder

$$x^2+z^2=a^2,$$

by the cylinder

$$x^2+y^2=a^2.$$

6. Find the area of that part of the cone

$$z^2=x^2+y^2.$$

which lies inside the cylinder

$$z^2+y^2=2x.$$

7. Show that the surface area of the torus

$$x=(r-\cos v)\cos u,\ y=(r-\cos v)\sin u,\ z=\sin v,$$
$$\text{where } -\pi \leqslant u \leqslant \pi,\ -\pi \leqslant v \leqslant \pi,\ r > 1$$

is

$$4\pi^2 r.$$

8. Show that the surface area of

$$z=2-x^2-y,$$

where (x, y) ranges over the triangle formed by the lines

$$x=0,\ y=1,\ y=x$$

is

$$\tfrac{1}{2}\log(\surd 2+\surd 3)+\surd 2/6.$$

9. Show that the area of the surface of the sphere

$$x^2+y^2+z^2=1$$

that lies inside the cylinder

$$2x^2(x^2+y^2)=3(x^2-y^2)$$

is

$$2\pi-4\surd 2[\surd 3\log(\surd 3+\surd 2)-2\log(1+\surd 2)]$$

10. If a surface is given by the equations

$$x=c\sin u,\ y=c\cos v,\ z=c(\cos u+\cos v),$$

prove that its area bounded by

$$u=0,\ u=\pi/2,\ v=0,\ v=\pi/2$$

is

$$\frac{\pi c^2}{4}\left[1-\sum_{n=1}^{\infty}\frac{p^3_{2n}}{2n-1}\right] \text{ where}$$

$$p_{2n}=\frac{(2n-1)(2n-3)\ldots\ldots\ldots 1}{2n(2n-2)\ldots\ldots\ldots 2}$$

11. Show that if x and y co-ordinates of any point on the paraboloid

$$2z=x^2/a+y^2/b$$

are expressed in the form

$$x=a\tan\theta\cos\phi,\ y=b\tan\theta\sin\phi$$

the angle θ is the inclination of the normal at any point to the axis of z, prove that the area of the cap of the surface cut off by the curve $\theta=\lambda$ is

$$\frac{2\pi ab}{3}(\sec^3\lambda-1).$$

CHAPTER XV

INTEGRATION IN E_3. GAUSS'S AND STOKE'S THEOREMS

Introduction. This chapter will be devoted to developing the notions of line, surface and volume integrals in the three dimensional Euclidean space.

Integral transformation theorems due to Gauss and Stokes will also be taken up.

185. Line Integral. Consider a curve C

$$x=f(t),\ y=g(t),\ z=h(t),$$

t varying from a to b.

Also let

$$P(x, y, z),\ Q(x, y, z),\ R(x, y, z)$$

be three functions of (x, y, z) defined in a domain containing the curve C.

We shall now define the symbol

$$\int_C (P\ dx+Q\ dy+R\ dz).$$

Let

$$D(a=t_0, t_1, \ldots\ldots, t_{r-1}, t_r, \ldots\ldots, t_n=b)$$

be any division of $[a, b]$.

Also let ξ_r be any value of t belonging to the sub-interval $[t_{r-1}, t_r]$.

We write

$$x_r=f(t_r),\ y_r=g(t_r),\ z_r=h(t_r), \qquad 1 \leqslant r \leqslant n$$
$$\alpha_r=f(\xi_r),\ \beta_r=g(\xi_r),\ \gamma_r=h(\xi_r).$$

Consider the sum

$$\sum_{r=1}^{n} [P(\alpha_r, \beta_r, \gamma_r)\ (x_r-x_{r-1})+Q(\alpha_r, \beta_r, \gamma_r)\ (y_r-y_{r-1})+ R(\alpha_r, \beta_r, \gamma_r)\ (z_r-z_{r-1})]$$

If, as the norm of D tends to zero, the sum tends to a finite limit which is independent of the choice of points ξ_r, then we denote the limit by the symbol

$$\int_C (P\ dx+Q\ dy+R\ dz).$$

185·1. Sufficient condition for the existence of

$$\int_C (P\,dx+Q\,dy+R\,dz). \qquad \ldots(1)$$

As in § 169·1, page 38, *it can be shown that if*

(*i*) *$f(t)$, $g(t)$, $h(t)$ possess continuous derivatives in* $[a, b]$; *and*

(*ii*) *$P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are continuous,*

then the line integral (1) exists and is equal to the ordinary integral

$$\int_a^b \left(P\frac{dx}{dt}+Q\frac{dy}{dt}+R\frac{dz}{dt}\right)dt.$$

Here $P=P[f(t), g(t), h(t)]$, etc.

185·2. Line integral w.r. to arc length.

Suppose that the curve C given by

$$x=f(t),\ y=g(t),\ z=h(t), \qquad a \leqslant t \leqslant b$$

is smooth. Let, l, denote the length of the curve. As seen in Note 2, page 421 we can express x, y, z as functions of s where s varies from 0 to l. Let

$$x=x(s),\ y=y(s),\ z=z(s).$$

Also let $P(x, y, z)$ be continuous functions defined in a domain containing C.

Let

$$D(0, s_1, s_2, \ldots\ldots, s_{r-1}, s_r, \ldots\ldots\ s_n=l),$$

be any division of $[0, l]$. Let ξ_r be any number between s_{r-1} and s_r.

We write

$$\alpha_r=x(\xi_r),\ \beta_r=y(\xi_r),\ \gamma_r=z(\xi_r).$$

Consider the sum

$$\sum_{r=1}^{n} (s_r-s_{r-1})\, P(\alpha_r, \beta_r, \gamma_r).$$

It can be shown that under the given conditions, this sum tends to a limit when the norm of D tends to zero. This limit is denoted by the symbol

$$\int_C P\,ds.$$

This is equal to the ordinary integral

$$\int_a^b P\,[x(s), y(s), z(s)]\frac{ds}{dt}\,dt.$$

Note. Vectorial formulation. Assuming knowledge of the Elements of Vector Calculus, we give a vectorial formulation of the concept of Line integral.

We write

$$\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$$
$$=\mathbf{i}f(t)+\mathbf{j}g(t)+\mathbf{k}h(t),$$

so that $\mathbf{r}$ denotes the position vector of any point x, y, z on the curve and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vector along the three co-ordinate axes.

Also, we consider the vector function

$$F(x, y, z)=\mathbf{i}P(x, y, z)+\mathbf{j}Q(x, y, z)+\mathbf{k}R(x, y, z).$$

Considering the scalar product, we have

$$\mathbf{F}.\ \frac{dx}{dt}=P\frac{dx}{dt}+Q\frac{dy}{dt}+R\frac{dz}{dt}.$$

$$\therefore\quad \int_a^b \left(P\frac{dx}{dt}+Q\frac{dy}{dt}+R\frac{dz}{dt}\right)dt=\int_a^b\left(\mathbf{F}.\ \frac{d\mathbf{r}}{dt}\right)dt$$

and we write

$$\int_a^b\left(\mathbf{F}.\ \frac{d\mathbf{r}}{dt}\right)dt=\int_C \mathbf{F}.\ d\mathbf{r}.$$

Note. It is known from Differential Geometry, that if θ, φ, ψ denote the angles which the tangent at any point of the curve drawn in the direction of, s, increasing makes with the three co ordinate axes, then

$$\cos\theta=\frac{dx}{ds},\ \cos\varphi=\frac{dy}{ds},\ \cos\psi=\frac{dz}{ds}.$$

Thus we have

$$\int_C (Pdx+Qdy+Rdz)=\int_0^l\left(P\frac{dx}{ds}+Q\frac{dy}{ds}+R\frac{dz}{ds}\right)ds$$

$$=\int_0^l (P\cos\theta+Q\cos\varphi+R\cos\psi)\,ds.$$

Exercises

1. Prove that

$$\int\frac{x^2+y^2}{p}ds$$

taken round the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is equal to

$$\tfrac{1}{4}\pi ab\{4+(a^2+b^2)(a^{-2}+b^{-2})\},$$

where, p, is the perpendicular from the centre to the tangent.

2. Evaluate

$$\int_C \frac{x^4}{p}dx,$$

where C is the ellipse,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

and, p, is the perpendicular from the origin to the tangent.

3. Show that the value of

$$\int \frac{dx}{y^2},$$

taken along the whole curve

$$x^2+y^2=y^3,$$

is 2π.

[The curve is given by

$$x=\frac{1+t^2}{t^3},\ y=\frac{1+t^2}{t^2}$$

and t varies from $-\infty$ to $+\infty$].

4. Evaluate

$$\int_C (xy\,dx+yz\,dy+zx\,dz)$$

where the curve C is given by

$$x=t,\ y=t^2,\ z=t^3;$$

t varying from -1 to $+1$.

5. Show that

$$\int_C (y^2+z^2)dx+(z^2+x^2)dy+(x^2+y^2)dz=-2\pi ab^2,$$

where the curve C is the part for which $z \geqslant 0$ of the intersection of the surfaces

$$x^2+y^2+z^2=2ax,\ x^2+y^2=2bx,\quad a>b>0;$$

the curve begins at that origin and runs at first in the positive octant

$$[x=b(1+\cos\theta),\ y=b\sin\theta,\ z=2\sqrt{(a-b)b}\cos\tfrac{1}{2}\theta,\ -\pi \leqslant \theta \leqslant \pi].$$

186. Oriented curves and surfaces. Associated with each *type* of integral, there is that of '*Path*' *or* '*Region*' of integration. Thus the path of integration associated with an ordinary integral

$$\int_a^b f(x)\,dx,$$

is the interval $[a, b]$. It is, however, clear that the mere mention of the interval is not enough. We have to further mention whether the path extends from a to b or from b to a so that we get two integrals, *viz.*,

$$\int_a^b f(x)dx,\quad \int_b^a f(x)dx$$

in the two cases. Thus there arises what we may say the concept of an *oriented* path. We have two possible orientations and the reversal of the orientation is equivalent to multiplication with -1.

In the case of line integral along any curve C, we may again distinguish between the two possible oriented paths; one of which corresponds to the direction in which the parameter increases and the other to the direction in which the parameter decreases.

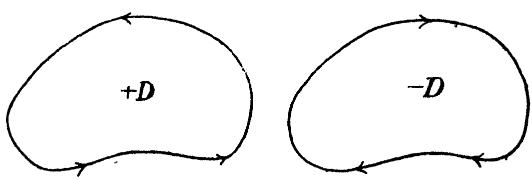
Also if C denotes any one of the two oriented paths, then $-C$ denotes the same path with its orientation reversed and the integral along, $-C$, is equal to that along, C, multiplied with -1. While considering a line integral along any curve, we must thus necessarily specify the orientation also.

We now consider the concept of an *oriented surface* prior to considering surface integrals wherefor the *region* of integration is a surface.

A surface is referred to as *Bilateral*, if it is possible to distinguish one of its sides from the other. It is interesting to note that *not* all surfaces are bilateral and we have instances of *unilateral* surfaces. One example of a unilateral surface is provided by what is known as a *Möbius strip* which may be constructed by taking a rectangular sheet $ABCD$ of paper, and by pasting the side BC to the side AD in such a way that the point C coincides with the point A and the point B with D.

In the following, we shall be exclusively concerned with bilateral surfaces only.

Oriented plane regions. A plane region will be considered oriented in respect of the orientation of its bounding curve so that orientation of a region will be reversed on reversing the orientation of its boundary curve. It is *usual* to consider a plane region as positively (negatively) oriented if the orientation of its boundary curve is such that it lies to the left (right) while the curve is being described.



Also if D be any plane region and $f(x, y)$ any continuous function over the same so that

$$\iint_D f(x,y)\,dxdy$$

denotes the double integral of $f(x, y)$ over D, we shall find it *convenient* to write

$$\iint_{+D} f(x, y)\, dxdy = \iint_{D} f(x, y)\, dxdy,$$

and
$$\iint_{-D} f(x, y)\, dxdy = -\iint_{D} f(x, y)\, dxdy,$$

If D_1 be any oriented region in the uv plane and we transform the same one-one by some relations

$$x = x(u, v),\ y = y(u, v),$$

then the new region D in the xy-plane which is the transform of D_1 acquires what we may term an **Induced** orientation.

The orientation D will be the same as that of D_1 or different according as the Jacobian.

$$\frac{\partial(x, y)}{\partial(u, v)},$$

is everywhere positive or negative.

Adopting the above convention in respect of oriented plane regions and double integrals over the same, we have the relation.

$$\iint_{D} f(x, y)\, dxdy = \iint_{D_1} f(x, y)\, \frac{\partial(x, y)}{\partial(u, v)}\, dudv,$$

irrespective of the sign of the Jacobian.

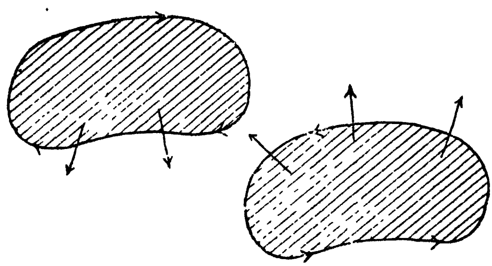
Oriented surfaces. We shall consider here only those surfaces which have as their boundary some closed curve or which are themselves closed.

A surface having some closed curve as a boundary thereof will be given an orientation as determined by the orientation of the closed bounding curve.

The two sides of an oriented surface bounded by a closed curve can be intuitively distinguished as follows :

Consider any one side of the surface and draw at each point thereof a *half line* normal to the surface and lying on the side of the surface under consideration. Then the orientation of the bounded curve is either right handed or left handed with respect to *each* of these semi-lines. (See figures on page 435).

Note. The reader may see that here we have adopted only an *intuitive* manner of introducing the notion of oriented surfaces and have avoided the rigorous and analytical way for the purpose.



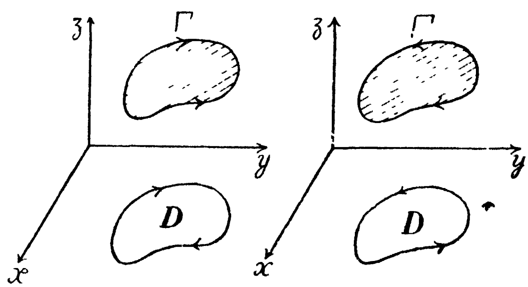
187. Surface Integrals. Firstly we consider the case of a surface S of the form

$$z=\varphi(x, y),$$

where (x, y) ranges over some domain D of the xy-plane.

Let $h(x, y, z)$ be any function which is continuous in some region containing the given surface.

Let, C, denote the boundary of D and Γ the boundary curve of the surface.



The orientation of Γ and the consequent orientation of S determines that of C and D.

Then, by def.,

$$\iint\limits_{S} h(x, y, z)\, dxdy = \iint\limits_{D} h(x, y, \varphi(x, y))\, dxdy,$$

where the integral on the right denotes a double integral on the oriented plane region D.

Now suppose that the surface S is given by

$$x=x(u, v),\ y=y(u, v),\ z=z(u, v)$$

where (u, v) ranges over some domain D_1 of the uv-plane. Then clearly the domain D of the xy-plane is the image of the domain D_1 of the uv-plane. We suppose that D_1 is itself oriented in a manner that the orientation of D corresponds to that of D_1. We thus have

$$\iint_D h(x, y, \varphi)\, dxdy = \iint_{D_1} h(x, y, \varphi) \frac{\partial(x, y)}{\partial(u, v)}\, dudv.$$

Thus we have

$$\iint_S h(x, y, z)\, dxdy = \iint_{D_1} h(x, y, z) \frac{\partial(x, y)}{\partial(u, v)} dudv.$$

Now we consider the case of any oriented surface S which can be split up into a finite number of orientable sub-surfaces each of which is given by an equation of the form $z=\varphi(x, y)$. Then, by def., the surface integral over S is the sum of the surface integrals over the different sub-surfaces.

Then if the *entire* surface S is given by

$$x=x(u, v),\ y=y(u, v),\ z=z(u, v)$$

where (u, v) ranges over a domain D_1 whose orientation corresponds to that of S, the result

$$\iint_S h(x, y, z)\, dxdy = \iint_{D_1} h(x, y, z) \frac{\partial(x, y)}{\partial(u, v)} dudv,$$

holds in respect of the entire surface and we do not need to split up S into sub-surfaces of the requisite type.

We may similarly define the surface integrals

$$\iint_S f(x, y, z)\, dy\, dz, \quad \iint_S g(x, y, z)\, dz\, dx,$$

and show that

$$\iint_S f(x, y, z)\, dy\, dz = \iint_{D_1} f(x, y, z) \frac{\partial(y, z)}{\partial(u, v)} du\, dv,$$

$$\iint_S g(x, y, z)\, dz\, dx = \iint_{D_1} g(x, y, z) \frac{\partial(z, x)}{\partial(u, v)} du\, dv.$$

Also, we have the general result as follows:

$$\iint_S [f(x, y, z)\, dydz + g(x, y, z) dzdx + h(x, y, z)\, dxdy]$$

$$= \iint_{D_1} \left[f \frac{\partial(y, z)}{\partial(u, v)} + g \frac{\partial(z, x)}{\partial(u, v)} + h \frac{\partial(x, y)}{\partial(u, v)} \right] dudv,$$

whe,er on the right hand side, we have an ordinary double integral.

Note 1. If θ, ϕ, ψ denote the angles which the semi-normal drawn at any point of the surface and lying on the side of the surface under consideration makes with the co-ordinate axes, we have

$$\int\int_S h(x, y, z)\, dxdy = \int\int_S h(x, y, z) \cos \psi \, dS,$$

$$\int\int_S g(x, y, z)\, dzdx = \int\int_S g(x, y, z) \cos \phi \, dS,$$

$$\int\int_S f(x, y, z)\, dydz = \int\int_S f(x, y, z) \cos \theta \, dS,$$

where dS denotes *an element* of the surface.

Note 2. Vectorial formulation.

Let

$$\mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z,$$

be the position vector of any point of the surface and

$$\mathbf{F}(x, y, z)=\mathbf{i}\, f(x, y, z)+\mathbf{j}g(x, y, z)+\mathbf{k}h(x, y, z),$$

be a vector function with f, g, h as its components.

Also let, $\mathbf{n}$, denote the unit normal vector at any point of the surface on the side of the surface under consideration so that

$$\mathbf{n}=\mathbf{i} \cos \theta+\mathbf{j} \cos \varphi+\mathbf{k} \cos \psi.$$

$$\therefore \quad \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS=\int\int_S (f \cos \theta+g \cos \varphi+h \cos \psi)dS$$

$$=\int\int_S (f\, dy\, dz+g\, dz\, dx+h\, dx\, dy)$$

We thus see that the surface integral

$$\int\int_S (f\, dydz+g\, dzdx+h\, dxdy)$$

can be compactly written as $\int\int_S \mathbf{F}.\mathbf{n}\, dS$

Exercises

1. Show that

$$\iint_S (yz\, dydz + zx\, dzdx + xy\, dxdy) = \frac{3}{8},$$

where S is the surface of the sphere

$$x^2+y^2+z^2=1,$$

in the first constant.

2. Evaluate the surface integra l

$$\iint_S (x^3\, dydz + y^3\, dzdx + z^3\, dxdy)$$

where S is the outer surface of the sphere

$$x^2+y^2+z^2=1.$$

3. Evaluate

$$\iint_S (x\, dydz + y\, dzdx + z\, dxdy)$$

where S is the outer surface of the part of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

lying above the xy plane.

4. If S is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

prove that

$$\iint_S \frac{x^2}{p}\, dS = \frac{4\pi abc}{15}\left[6+(\Sigma a^2)(\Sigma a^{-2})\right],$$

$$\int p\,ds = 4\pi abc$$

$$\int \frac{ds}{p} = \frac{4}{3}\pi abc$$

$$I_n = \int \frac{dS}{p^n} = \frac{1}{3}\left(\Sigma \frac{1}{a^2}\right) I_{n-2}$$

where p is the length of the perpendicular from the origin to the tangent plane at the point (x, y, z).

5. Find the value of

$$\iint_S p(x^4+y^4+z^4)\, dS,$$

where S is the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and, p, is the perpendicular from the origin to the tangent plane at the point (x, y, z),

6. Evaluate

$$\int\int_{\Sigma} (x+y+z)(lx+my+nz)\, dS,$$

where Σ is the bonndary of the region

$$x^2+y^2 \leqslant a^2,\ 0 \leqslant z \leqslant h.$$

188. First generalisation of Green's theorem. Stoke's theorem. It is possible to generalise in *two* ways the Green's theorem which expresses a double integral over a plane region in terms of a line integral over the boundary curve of the same. One generalisation arises when we replace the plane region by any surface and express a surface integral taken over the same by a line integral taken over the boundary curve of the surface. This generalisation will now be stated and proved in the form of *Stoke's theorem*. A second generalisation arises when we replace double integral by a triple integral and express a triple integral taken over a three dimensional region in terms of a surface integral taken over the bounding surface of the region. This will be given as Gauss's theorem after we have introduced the concept of Triple integration.

Stoke's theorem. *If, S, denotes an oriented surface relative to the sense of description of its boundary curve C and*

$$f(x, y, z),\ g(x, y, z),\ h(x, y, z)$$

are three functions of x, y, z which are continuous and which possess continuous first order partial derivatives at each point of a region containing S, then

$$\int_C (f\, dx+g\, dy+h\, dz)$$

$$=\int\int_S \left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right) dydz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right) dz\, dx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right) dxdy\right].$$

Suppose that the oriented surface is given by

$$x=x(u, v),\ y=y(u, v),\ z=z(u, v),$$

where (u, v) ranges over a certain oriented domain D of the uv-plane. Also let Γ denote the oriented boundary curve of the domain D. We suppose that Γ is given by

$$u=u(t),\ v=v(t),$$

t varying from a to b.

The principle of the proof is contained in the following steps:

I. Express the line integral along C in terms of an ordinary integral.

II. Express the ordinary integral in I as a line integral along Γ.

III. Express, by Green's theorem, the line integral along Γ, in terms of a double integral over D.

IV. Express the double integral over D in terms of a surface integral over the surface S.

Now we carry out the programme indicated above.

We have

$$\int_C (f\,dx+g\,dy+h\,dz)$$

$$=\int_a^b\left[f\left(\frac{\partial x}{\partial u}\frac{du}{dt}+\frac{\partial x}{\partial v}\frac{dv}{dt}\right)+g\left(\frac{\partial y}{\partial u}\frac{du}{dt}+\frac{\partial y}{\partial v}\frac{dv}{dt}\right)+h\left(\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\right)\right]dt$$

$$=\int_\Gamma\left[\left(f\frac{\partial x}{\partial u}+g\frac{\partial y}{\partial u}+h\frac{\partial z}{\partial u}\right)du+\left(f\frac{\partial x}{\partial v}+g\frac{\partial y}{\partial v}+h\frac{\partial z}{\partial v}\right)dv\right]$$

$$=\int\int_D\left[\frac{\partial}{\partial u}\left(f\frac{\partial x}{\partial v}+g\frac{\partial y}{\partial v}+h\frac{\partial z}{\partial v}\right)-\frac{\partial}{\partial v}\left(f\frac{\partial x}{\partial u}+g\frac{\partial y}{\partial u}+h\frac{\partial z}{\partial u}\right)\right]dudv. \quad ..\ (1)$$

We now have

$$\frac{\partial}{\partial u}\left(f\frac{\partial x}{\partial v}\right)=\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v}+f\frac{\partial^2 x}{\partial u\partial v},$$

and also we may compute the other expressions contained in the integrand.

Carrying out these straightforward steps, we see that the double integral over D on the right hand side of (1) is

$$=\int\int_D\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\frac{\partial(z,x)}{\partial(u,v)}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}\right]dudv$$

which again is equal to the following surface integral over S,

$$\int\int_S\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dy\,dz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dz\,dx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dx\,dy\right].$$

Thus we have proved that

$$\int_C \left(f\,dx+g\,dy+h\,dz \right)=\int_S\int\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dydz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dzdx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy\right].$$

Hence the theorem.

Vectorial Formulation. We write

$$\mathbf{F}=\mathbf{i}f+\mathbf{j}g+\mathbf{k}h,$$

so that,

$$\text{curl }\mathbf{F}=\mathbf{i}\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)+\mathbf{j}\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)+\mathbf{k}\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right).$$

Thus Stoke's theorem can be re-written as

$$\int_C \mathbf{F}\,.\,d\mathbf{r}=\int_S\int \text{curl }\mathbf{F}\,.\,\mathbf{n}\,dS.$$

Ex. 1. Verify Stoke's theorem for the line integral

$$\int_C (x^2\,dx+y\,x\,dy),$$

where C denotes the square in the plane $z=0$ with sides along the lines

$$x=0,\ y=0,\ x=a,\ y=a.$$

Ex. 2. Apply Stoke's theorem to prove that

$$\int_C (y\,dx+z\,dy+x\,dz)=-2\sqrt{2}\pi\,a^2,$$

where C is the curve given by

$$x^2+y^2+z^2-2ax-2ay=0,\ x+y=2a$$

and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.

Ex. 3 Show that

$$\int_S\int [(y-z)\,dydz+(z-x)\,dzdx+(x-y)\,dxdy]=a^3\pi$$

where S is the portion of the surface

$$x^2+y^2-2ax+az=0$$

for which $z\geqslant 0$.

189. Volume integrals. The treatment of volume integrals, also known as triple integrals, over three dimensional regions, is a simple and straight extension of the ideas in respect of double integrals. Thus we shall only briefly indicate the various stages of develop ment of the theory of triple integrals.

I. Volume of a three dimensional rectangle

$$R[a, b\,;\, c, d\,;\, e, f\,]$$

is, by definition,

$$(b-a)(d-c)(f-e).$$

Note. A three dimensional rectangle is also often referred to as a rectangular parallelopiped.

II. Integrability and integral of a bounded function f(x, y, z) over a rectangle

$$R[a, b\,;\, c, d\,;\, e, f].$$

Let

$$D_1(a=x_0, x_1, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots x_m=b)$$
$$D_2(c=y_0, y_1, \ldots\ldots, y_{s-1}, y_s, \ldots\ldots y_n=d)$$
$$D_3(e=z_0, z_1, \ldots\ldots, z_{t-1}, z_t, \ldots\ldots, z_p=f)$$

be divisions of the intervals

$$[a, b], [c, d], [e, f]$$

respectively. These three divisions D_1, D_2, D_3 give rise to a division say, D of the rectangle R into, mnp, sub-rectangles

$$[x_{r-1}, x_r\,;\, y_{s-1}, y_s\,;\, z_{t-1}, z_t], \qquad \ldots(1)$$

r, s, t varying from 1 to m, 1 to n and 1 to p respectively.

We write

$$\omega_{rst}=(x_r-x_{r-1})(y_s-y_{s-1})(z_t-z_{t-1}),$$

which is the volume of the rectangle (1).

Also let

$$M_{rst}, m_{rst}$$

denote the upper and lower bounds of $f(x, y, z)$ in the rectangle (1).

We write

$$S_D = \sum_{t=1}^{p} \sum_{s=1}^{n} \sum_{r=1}^{m} M_{rst}\, \omega_{rst},$$

$$s_D = \sum_{t=1}^{p} \sum_{s=1}^{n} \sum_{r=1}^{m} M_{rst}\, \omega_{rst}.$$

Then the lower bound of the aggregate of sums S_D and the upper bound of the aggregate of sums s_D are known respectively as the upper and lower integrals of $f(x, y, z)$ over R and donoted by

$$\overline{\iiint_R} f(x, y, z)\, dxdydz, \quad \underline{\iiint_R} f(x, y, z)\, dxdydz.$$

In case these two are equal, we say that $f(x, y, z)$ is integrable over R and the common value, called the integral of (x, y, z) over R is denoted by

$$\iiint_R f(x, y, z)\, dxdydz.$$

Norm of a division. By the norm of a division D is meant the max. of the norms of D_1, D_2 and D_3.

III. **Darboux's theorem and necessary and sufficient conditions for integrability.**

The statements and proofs are exactly the same as for double integrals (page 386) except that we have to replace $f(x, y)$ by $f(x, y, z)$.

IV. **Particular classes of bounded integrable functions.** We have the following *three* results in this connection :

(*i*) *Every continuous function is integrable.*

(*ii*) *A bounded function whose points of discontinuity can be enclosed in a finite number of rectangles the sum of whose volumes is less than a given positive number is integrable.*

(*iii*) *A bounded function whose points of discontinuity lie on a finite number of surfaces of the form $z=\varphi(x, y)$, etc., where $\varphi(x, y)$ is continuous, is integrable.*

The proofs of the corresponding results for double integrals are easily adaptable except that we now need properties of continuous functions of three variables.

V. **Calculation of Triple integrals. Reduction to Ordinary integrals.**

If

(*i*) $$\iiint_R f(x, y, z)\, dx\, dy\, dz,$$

and (*ii*) $$\iint_S f(x, y, z)\, dx\, dy,$$

where $e \leqslant z \leqslant f$ *and* $S \equiv [a, b\,;\, c, d]$,
both exist, then

$$\int_e^f \left[\iint_S f(x, y, z)\, dxdy \right] dz,$$

also exists and is equal to the triple integral.

Let U and L denote the upper and lower integrals of $f(x, y, z)$ over R. Let $\varepsilon > 0$ be given.

There exists a division of R into, mnp, sub-rectangles

$$[x_{r-1}, x_r \,;\, y_{s-1}, y_s \,;\, z_{t-1}, z_t],$$

such that

$$\sum_{t=1}^{p} \sum_{s=1}^{n} \sum_{r=1}^{m} M_{rst}\, w_{rst} < U+\varepsilon, \qquad \ldots(1)$$

the symbols having their usual meanings.

For each fixed value of z in $[z_{t-1}, z_t]$, we have

$$\overline{\int\int_S} f(x, y, z)\, dxdy = \underline{\int\int_S} f(x, y, z)\, dxdy$$

$$\leqslant \sum_{s=1}^{n} \sum_{r=1}^{m} M_{rst}(x_r - x_{r-1})(y_s - y_{s-1}). \qquad \ldots(2)$$

$\therefore$ from (2),

$$\int_e^{\overline{f}} \left[\int\int_S f(x, y, z)\, dxdy \right] dz$$

$$\leqslant \sum_{t=1}^{p} \sum_{s=1}^{n} \sum_{r=1}^{m} M_{rst}\, w_{rst} < U+\varepsilon, \text{ by (1)}$$

As ε is an arbitrary positive number, we have

$$\int_e^{\overline{f}} \left[\int\int_S f(x, y, z)\, dxdy \right] dz \leqslant U.$$

Similarly, we may see that

$$\int_{\underline{e}}^{f} \left[\int\int_S f(x, y, z)\, dxdy \right] dz \geqslant L.$$

As

$$U = L,$$

the result follows.

Cor. If $f(x, y, z)$ is continuous over R, we have

$$\int\int\int_R f(x, y, z)\, dxdydz = \int_e^f \left[\int_c^d \left[\int_a^b f(x, y, z)\, dx \right] dy \right] dz,$$

where we can interchange the order of x, y, z on the right hand side in any manner we like.

VI. **Integrability and integral over any bounded region.** If $f(x, y, z)$ be defined in a bounded region E, we enclose E in a rectangle R and define a function $F(x, y, z)$ over the same as follows :

$$F(x, y, z) = \begin{cases} f(x, y, z) \text{ at all points of } E, \\ 0, \text{ elsewhere.} \end{cases}$$

Then $f(x, y, z)$ is defined to be integrable over E, if $F(x, y, z)$ is integrable over R and we write

$$\iiint_E f(x, y, z)\, dxdydz = \iint_R F(x, y, z)\, dxdydz.$$

It can be shown that a continuous function $f(x, y, z)$ is integrable over E, if E is bounded by a finite number of surfaces of the form $z=\varphi(x, y)$, $\psi(x, y)$ etc., where $\varphi(x, y)$, $\psi(x, y)$ etc., are continuous.

VII. **Calculation of Triple integrals.** If $f(x, y, z)$ is continuous in a region E bounded by surfaces

$$z=\varphi(x, y),\ z=\psi(x, y)\ ;\ y=g(x),\ y=h(x)\ ;\ x=a,\ x=b$$

then

$$\iiint_E f(x, y, z)\, dxdydz = \int_a^b \left[\int_{g(x)}^{h(x)} \left\{ \int_{\varphi(x, y)}^{\psi(x, y)} f(x, y, z)\, dz \right\} dy \right] dx.$$

The proof may be constructed as in the case of double integrals.

190. Gauss's theorem. Second Generalisation of Green's theorem. *If S be any closed surface enclosing a three dimensional region E and if*

$$f(x, y, z),\ g(x, y, z),\ h(x, y, z)$$

are three functions which are continuous and which possess continuous first order partial derivatives

$$\frac{\partial f}{\partial x},\ \frac{\partial g}{\partial y},\ \frac{\partial h}{dz}$$

at each point of E and S, then

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial g}{dy} + \frac{\partial h}{\partial z} \right) dx\, dy\, dz = \iint_S (f\, dydz + g\, dzdx + h\, dxdy),$$

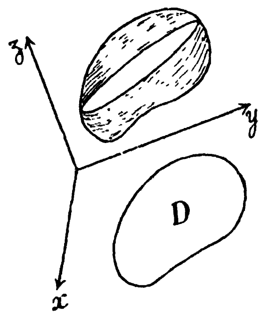
where the surface integral on the right is taken over the exterior of the surface S.

Firstly we consider the case of a surface S which is such that a line parallel to each axis and having points in common with the interior of S meets the same in at the most two points.

Then it is possible to split up the external surface S into two sub-surfaces S_1 and S_2 both of which project onto the same region of the xy-plane. These two sub-surfaces have equations of the form

$$z=\varphi(x, y), \quad z=\psi(x, y)$$

where (x, y) varies over the same region of the xy-plane.



These two sub-surfaces S_1 and S_2 have the same bounding curve. Relative to the external surface S under consideration and the consequently determined sides of S_1 and S_2, we see that the boundary curve of S_1 differs from that of S_2 in orientation only. Accordingly, therefore, if, D denotes the region of the xy-plane on which S_1 projects, then, $-D$, is the region on which S_2 projects.

Thus we have

$$\iiint_E \frac{\partial h}{\partial z}\, dxdydz = \iint_D \left[\int_{\psi(x, y)}^{\varphi(x, y)} \frac{\partial h}{\partial z}\, dz \right] dx\, dy$$

$$= \iint_D [h(x, y, \varphi)-h(x, y, \psi)]\, dxdy$$

$$= \iint_D h(x, y, \varphi)\, dxdy - \iint_D h(x, y, \psi)\, dxdy$$

$$= \iint_D h(x, y, \varphi)\, dxdy + \iint_{-D} h(x, y, \psi)\, dxdy$$

$$= \iint_{S_1} h(x, y, z)\, dxdy + \iint_{S_2} h(x, y, z)\, dxdy$$

$$= \iint_{S} h(x, y, z)\, dxdy.$$

We may similarly show that

$$\iiint_E \frac{\partial f}{\partial x}\, dxdydz = \iint_S f\, dy\, dz,$$

$$\iiint_E \frac{\partial g}{\partial y}\, dxdydz = \iint_S g\, dz\, dx.$$

Adding we get

$$\iiint_E \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dxdydz = \iint_S (f\, dydz + g\, dzdx + h\, dxdy).$$

To generalise, we suppose that the surface S is such that it can be divided by a finite number of smooth curves into a finite number of portions in such a way that each portion satisfies the assumptions made above. We then apply the theorem proved to each portion and add the results. On adding, we shall obtain on the left a triple integral over E and, on the right, some of the surface integrals combine to form the surface integral over S and others cancel one another in pairs. Hence the theorem.

$$\iiint_E \text{div. } \mathbf{F}\, dv = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where, dv, denotes element of volume.

Exercises

1. Show that using Gauss's theorem that the surface integral

$$\int\int_S \left[(x^3 - yz)\, dydz - 2x^2 y\, dzdx + 2dxdy \right]$$

taken over the outer surface of the cube bounded by the planes

$$x=0,\ x=a\ ;\ y=0,\ y=a\ ;\ z=0,\ z=a$$

is $\frac{1}{3}a^5$.

Also verify the result by direct evaluation of the surface integral.

2. Evaluate

$$\int\int_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

taken over the outer surface of the cube

$$[0, a\ ;\ 0, a\ ;\ 0, a].$$

3. Evaluate

$$\int\int_S (y^2 z^2\, dydz + z^2 x^2\, dzdx + x^2 y^2\, dxdy)$$

where S is the part of the sphere

$$x^2 + y^2 + z^2 = 1,$$

above the XY-plane.

4. Show that

$$\int\int_S (x^2\, dydz + y^2\, dzdx + z^2\, dxdy) = 0$$

where S denotes the surface of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

5. Show that

$$\int\int_S (ax\, dy\, dz + by\, dz\, dx + cz\, dx\, dy) = \frac{4}{3}\pi(a+b+c),$$

where S is the surface of the sphere

$$x^2 + y^2 + z^2 = 1.$$

6. Evaluate

$$\int\int_S (x\, dy\, dz + y\, dzdx + z^2\, dxdy)$$

where S denotes the closed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1$.

191. Change of variables in a triple integral.

Let $f(x, y, z)$ be continuous in a region E bounded by a surface S in xyz-space.

Also let

$$x=x(X, Y, Z),\ y=y(X, Y, Z),\ z=z(X, Y, Z)$$

be three functions of X, Y, Z defined in a region E_1 bounded by a surface S_1 in XYZ-space.

We suppose that these three functions

$$x(X, Y, Z),\ y(X, Y, Z),\ z(X, Y, Z)$$

(i) possess continuous first order partial derivatives at each point of E_1 and S_1,

(ii) transform E_1 into E and S_1 into S,

(iii) the transformation is one-one,

(iv) the Jacobian

$$\frac{\partial(x, y, z)}{\partial(X, Y, Z)},$$

does not change sign at any point of E_1 even though it may vanish at some points of S_1.

It will then be proved that

$$\iiint_E f(x, y, z)\, dxdydz = \iiint_{E_1} f(x, y, z) \left| \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \right| dX\, dY\, dZ,$$

where x, y, z have to be replaced by their values in terms of X, Y, Z in $f(x, y, z)$ in the integral on the right.

There exists a continuous function $F(x, y, z)$ such that at every point of E,

$$\frac{\partial F}{\partial z} = f(x, y, z).$$

Also we suppose that the surface S is given by

$$x=x(u, v),\ y=y(u, v),\ z=z(u, v)$$

where (u, v) ranges over some domain D of the uv-plane.

We have

$$\iiint_E f(x, y, z)\ dxdy\, dz = \iiint_E \frac{\partial F}{\partial z}\ dx\, dy\, dz$$

$$= \iint_S F(x, y, z)\, dx\, dy. \qquad \ldots(1)$$

(By Gauss's theorem)

Expressing the surface integral on the right of (1) as a double integral on the domain D of the uv plane, we have

$$\iint_S F(x, y, z)\,dxdy = \iint_D F(x, y, z)\frac{\partial(x, y)}{\partial(u, v)}\,dudv, \qquad \ldots(2)$$

where x, y, z have to be replaced by their values in terms of u, v.

Now, as may be easily shown,

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(X, Y)}\frac{\partial(X, Y)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(Y, Z)}\frac{\partial(Y, Z)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(Z, X)}\frac{\partial(Z, X)}{\partial(u, v)}.$$

$$\therefore \quad \iint_D F(x, y, z)\frac{\partial(x, y)}{\partial(u, v)}\,dudv$$

$$= \iint_D F(x, y, z)\left[\frac{\partial(x, y)}{\partial(X, Y)}\frac{\partial(X, Y)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(Y, Z)}\frac{\partial(Y, Z)}{\partial(u, v)} + \frac{\partial(x, y)}{\partial(Z, X)}\frac{\partial(Z, X)}{\partial(u, v)}\right]dudv$$

$$= \pm \iint_{S_1} F\frac{\partial(x, y)}{\partial(X, Y)}dXdY + F\frac{\partial(x, y)}{\partial(Y, Z)}dYdZ + F\frac{\partial(x, y)}{\partial(Z, X)}dZdX, \ldots(3)$$

where on the right we have a surface integral on S_1 and the sign to be taken is $+$ or $-$ according as the transformation is direct or inverse.

Applying Gauss's theorem to the integral on the right of (3), we see that the same is equal to

$$\pm \iiint_{E_1}\left\{\frac{\partial}{\partial Z}\left[F\frac{\partial(x, y)}{\partial(X, Y)}\right] + \frac{\partial}{\partial X}\left[F\frac{\partial(x, y)}{\partial(Y, Z)}\right] + \frac{\partial}{\partial Y}\left[\frac{\partial(x, y)}{\partial(Z, X)}\right]\right\} dX\,dY\,dZ \quad \ldots(4)$$

Now

$$\frac{\partial}{\partial Z}\left[F\frac{\partial(x, y)}{\partial(X, Y)}\right] = \left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial Z} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial Z} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial Z}\right)\frac{\partial(x, y)}{\partial(X, Y)} +$$

$$F\left(\frac{\partial^2 x}{\partial Z\partial X}\frac{\partial y}{\partial Y} + \frac{\partial x}{\partial X}\frac{\partial^2 y}{\partial Z\partial Y} - \frac{\partial^2 x}{\partial Z\partial Y}\frac{\partial y}{\partial X} - \frac{\partial x}{\partial Y}\frac{\partial^2 y}{\partial Z\partial X}\right).$$

Similarly carrying out the computation of

$$\frac{\partial}{\partial X}\left[F\frac{\partial(x, y)}{\partial(Y, Z)}\right], \quad \frac{\partial}{\partial Y}\left[F\frac{\partial(x, y)}{\partial(Z, X)}\right]$$

and some straightforward simplifications, we may see that the integral on the right of (4) becomes

$$\pm \iiint\limits_{E_1} \frac{\partial F}{\partial z} \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \, dX \, dY \, dZ$$

$$= \pm \iiint\limits_{E_1} f \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \, dX \, dY \, dZ.$$

Now assuming that the transformation is direct or inverse according as the Jacobian $\partial(x, y, z)/\partial(X, Y, Z)$ is positive or negative we see from above that

$$\iiint\limits_{E} f \, dx \, dy \, dz = \iiint\limits_{E_1} f \left| \frac{\partial(x, y, z)}{\partial(X, Y, Z)} \right| dXdYdZ,$$

as was to be proved.

Examples

1. The triple integral

$$\iiint f(x, y, z) \, dx \, dy \, dz,$$

over the region defined by

$$x^2 + y^2 + z^2 \leqslant a^2,$$

is equivalent to the repeated integral

$$\int_{-a}^{+a} dx \int_{-\sqrt{(a^2-x^2)}}^{+\sqrt{(a^2-x^2)}} dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{+\sqrt{(a^2-x^2-y^2)}} f(x, y, z) \, dz.$$

2. *Evaluate*

$$\iiint (x+y+z+1)^2 \, dxdydz,$$

over the region defined by

$$x \geqslant 0, \; y \geqslant 0, \; z \geqslant 0, \; x+y+z \leqslant 1.$$

The region of integration is the tetrahedron bounded by the three co-ordinate planes and the plane

$$x+y+z=1.$$

The given integral is equal to the repeated integral

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z+1)^2 \, dz = \frac{31}{60}.$$

3. *Evaluate*

$$\iiint (1-x-y-z)^{l-1}\, x^{m-1}\, y^{n-1}\, z^{p-1}\, dx\, dy\, dz,$$

over the tetrahedron bounded by the planes

$$x=0,\ y=0,\ z=0,\ x+y+z=1.$$

We employ the transformation defined by

$$x+y+z=u,\ x+y=uv,\ x=uvw.$$

These are equivalent to

$$x=uvw,\ y=uv\,(1-w),\ z=u(1-v),$$

and

$$u=x+y+z,\ v=(x+y)/(x+y+z),\ w=x/(x+y).$$

When x, y, z are all positive and $x+y+z$ is less than 1, then u, v, w all lie between 0 and 1.

Conversely, when u, v, w all lie between 0 and 1, the x, y, z are all positive and $x+y+z$ is less than 1. Thus the given region transforms to the region bounded by the planes

$$u=0,\ u=1\ ;\ v=0,\ v=1\ ;\ w=0,\ w=1.$$

Also

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}=-u^2v.$$

$\therefore$ the given integral

$$=\left[\int_0^1 (1-u)^{l-1}u^{m+n+p-1}du\right]\left[\int_0^1 v^{m+n-1}\,(1-v)^{p-1}\,dv\right]\times \left[\int_0^1 w^{m-1}\,(1-w)^{n-1}\,dw\right]$$

$$=\beta(l,\ m+n+p)\ \beta(m+n,\ p)\ \beta(m,\ n)$$

$$=\frac{\Gamma(l)\ \Gamma(m)\ \Gamma(u)\ \Gamma(p)}{\Gamma(l+m+n+p)}.$$

4. *Evalute*

$$I=\iiint\sqrt{(a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2)}\ dx\ dy\ dz,$$

taken throughout the region

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1.$$

Change the variables x, y, z to X, Y, Z where

$$x=aX,\ y=bY,\ z=cZ$$

so that

$$\partial(x, y, z)/\partial(X, Y, Z)=abc.$$

Thus

$$I=a^2b^2c^2 \int\int\int \sqrt{(1-X^2-Y^2-Z^2)}\, dX\, dY\, dZ$$

taken throughout

$$X^2+Y^2+Z^2 \leqslant 1.$$

Changing X, Y, Z to polar co-ordinates r, θ, φ so that

$$X=r \sin\theta \cos\varphi,\ Y=r\sin\theta\sin\varphi,\ Z=r\cos\theta,$$

we have, since

$$\partial(X, Y, Z)/\partial(r, \theta, \varphi)=r^2 \sin\theta,$$

$$I=a^2b^2c^2 \int\int\int \sqrt{(1-r^2)}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi.$$

It is easily seen that to describe the whole region

$$X^2+Y^2+Z^2 \leqslant 1,$$

r varies from 0 to 1 ; θ varies from 0 to π ; φ varies from 0 to 2π.

Thus (r, θ, φ) varies in the rectangular parallelopiped

$$[0, 1 ; 0, \pi ; 0, 2\pi]$$

$$\therefore\quad I=a^2b^2c^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^1 \sqrt{(1-r^2)}r^2\, dr=\frac{1}{4}a^2b^2c^2\pi^2.$$

Exercises

1. Show that

$$\int\int\int \frac{dx\, dy\, dz}{(x+y+z+1)^3}=\frac{1}{16}\log\frac{226}{e^5}$$

taken throughout the tetrahedron bouded by the planes

$$x=0,\ y=0,\ z=0,\ x+y+z=1.$$

2. Show that

$$\int\int\int (lx+my+nz)^2\, dx\, dy\, dz=\frac{4}{15}\pi(l^2+m^2+n^2)$$

taken throughout the region

$$x^2+y^2+z^2 \leqslant 1.$$

3. Prove that the value of

$$\int\int\int \frac{xyz}{\sqrt{(x^2+y^2+z^2)}} dx\, dy\, dz,$$

taken through the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is

$$a^2b^2c^2(bc+ca+ab)/15(b+c)(c+a)(a+b).$$

4 Show that

$$\int\int\int z\, dx\, dy\, dz = \frac{\pi}{4} h^4 \cot\theta \cot\varphi,$$

taken throughout the volume bounded by the cone

$$z^2=x^2 \tan^2\theta + y^2 \tan^2\varphi,$$

and the planes

$$z=0,\ z=h.$$

5. Show that

$$\int\int\int (ax^2+by^2+cz^2)\, dx\, dy\, dz = \frac{4}{15}\pi(a+b+c)R .$$

taken throughout

$$x^2+y^2+z^2 \leqslant R^2.$$

6. Show that

$$\int\int\int z^2\, dx\, dy\, dz = \frac{30\pi-32}{225} a^5,$$

the field of integration being the region common to the surfaces

$$x^2+y^2+z^2=a^2,\ x^2+y^2=ax.$$

7. Show that the integral of the function

$$e^{\sqrt{(x^2/a^2+y^2/b^2+z^2/c^2)}}$$

taken throughout the region

$$x^2/a^2+y^2/b^2+z^2/c^2 \leqslant 1$$

is $4\pi\, abc(e-2)$.

8. Evalute

$$\int\int\int \sqrt{\left(\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}\, dx\, dy\, dz$$

integral being taken over all positive values of x, y, z such that $x^2+y^2+z^2 \leqslant 1$.

9. Prove that

$$\int\int\int \frac{dx\, dy}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8}$$

integral being extended to all positive values of the variables for which the expression is real.

10. Integrate $1/xyz$ throughout the volume bounded by the six spheres.

$$x^2+y^2+z^2=ax,\ a'x,\ by,\ b'y,\ cz,\ c'z,$$

where a, a', b, b', c, c', are positive.

(Take $u=(x^2+y^2+z^2)/x,\ v=(x^2+y^2+z^2)/y,\ w=(x^2+y^2+z^2)/z$.)

11. Prove that $\iiint xyz\,dx\,dy\,dz$ taken through the volume common to three spheres

$$x^2+y^2+z^2=2ax,\ x^2+y^2+z^2=2by,\ x^2+y^2+z^2=2cz$$

is

$$\frac{2}{15}\left(a^{-2}+b^{-2}+c^{-2}\right)^3.$$

12. Evalute

$$\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$$

when the variables are all positive and

(i) $x+y+z \leqslant 1$

(ii) $(x/a)^p+(y/b)^q+(z/c)^r \leqslant 1.$

13. Show that

$$\iiint (x+y+z)\,x^2y^2z^2\,dx\,dy\,dz=\tfrac{1}{50400}$$

throughout the region

$$x+y+z \leqslant 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0.$$

14. Evalute

$\iiint xyz^2\,dx\,dy\,dz$ for $0 \leqslant 4(x-2)^2+(y-1)^2+9z^2 \leqslant 36.$

15. Evalute

$$\iiint (y^2z^2+z^2x^2+x^2y^2)\,dx\,dy\,dz$$

taken through the volume of the cylinder $x^2+y^2-2ax=0$ between the sheets of the cone $z^2=k^2(x^2+y^2)$.

16. Prove that

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{dx\,dy\,dz}{(a^2+x^2+y^2+z^2)^{\frac{9}{2}}}=\frac{4\pi}{105a^6}.$$

17. Find

$$\iiint z\,dx\,dy\,dz,$$

throughout the region

$$x^2+y^2+z^2 \leqslant 1,\ x^2+y^2 \leqslant z^2,\ z \geqslant 0.$$

18. Calculate

$$\iiint x^\alpha\,y^\beta\,z^\gamma\,(1-x-y-z)^\delta\,dx\,dy\,dz,$$

over the interior of the tetrahedron formed by the co-ordinate planes and the plane

$$px+qy+rz=1.$$

19. Find the value of

$$\int\int\int F\left[\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r\right]x^{l-1}\,y^{m-1}\,z^{n-1}\,dxdydz$$

for all positive values of x, y, z subject to the condition that

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r< h.$$

20. Prove that

$$\int\int\int\frac{1}{z^3}\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)}\,dxdydz=\frac{\pi}{lm}$$

taken over the smaller region bounded by the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

and the half cone

$$z^2=l^2x^2+m^2y^2,\ z>0.$$

21. Evaluate

$$\int\int\int\sqrt{(1-z)}\,dxdydz$$

over the interior of the tetrahedron with faces

$$x=0,\ y=0,\ z=0,\ x+y+z=1,$$

and

$$\int\int\int\frac{dxdydz}{a^2+x^2+y^2+z^2}$$

over the interior of the sphere

$$x^2+y^2+z^2=a^2.$$

22. Show that

$$\int\int\int_{\Omega} e^{a(2x+3y+4z)}\,dxdydz=\frac{(e^a-1)^3\,(3e^a+1)}{24a^3},$$

where Ω is the region bounded by

$$x=0,\ y=0,\ z=0,\ x+y+z=1.$$

Deduce that

$$\int\int\int_{\Omega}(2x+3y+4z)\,e^{2x+3y+4z}\,dxdydz=\frac{(e^2-1)^3}{8}.$$

23. If A is the smaller of the two regions into which the interior of the ellipsoid

$$\frac{x^2}{16}+\frac{y^2}{9}+\frac{z^2}{4}=1$$

is divided by the plane

$$x+y+z=1,$$

determine the volume of A and evaluate

$$\int\int\int_A z^3\,dxdydz.$$

24. Evaluate

$$\iiint_V z^2\, dxdydz,$$

where V is the sphere

$$x^2+y^2+z^2 \leqslant 1.$$

25. If

$$I=\iiiint_\Omega e^{-x^2-y^2-z^2-u^2}\, dxdydzdu,$$

where Ω is the finite region in 4 dimensions bounded by

$$x^2+y^2+z^2+u^2=a^2,$$

prove that

$$I=\pi\iint_C \left(e^{-x^2-y^2} - e^{-a^2}\right) dxdy$$

where C is the interior of the circle $x^2+y^2=a^2$.

Deduce that

$$I=\pi^2\left(1-e^{-a^2} - a^2 e^{-a^2}\right).$$

26. Prove that

$$\iiint dxdydz=\frac{\frac{4}{3}\pi}{(1-m)\sqrt{(1+2m)}}$$

where the integration extends over all values of

$$x, y, z$$

such that

$$x^2+y^2+z^2+2m(xy+yz+zx)$$

does not exceed unity and

$$-\tfrac{1}{2} < m < 1.$$

192. Evaluation of volumes.

192·1. By triple integration. The triple integral

$$\iiint_E dxdydz,$$

carried throughout a region E in space of three dimensions gives the volume of E.

192·2. By double integration.

Let C be the boundary of a region E of the xy plane and let a cylinder be constructed by lines through the points of C parallel to z-axis, Then the volume of the cylinder enclosed between the surfaces

$$z=\varphi(x, y),\ z=\psi(x, y),\ [\varphi(x, y) \geqslant \psi(x, y)]$$

is, as can be easily seen, given by the double integral

$$\iint_E (\varphi-\psi)\, dxdy.$$

Exercises

1. Find the volume included between the co-ordinate planes, the part of the right cylinder standing on the quadrant $y=\sqrt{(9-x^2)}$ for which x and y are both positive and the plane $z=3-\frac{1}{2}x-\frac{1}{3}y$.

2 Show that the volume included between the elliptical paraboloid $2z=x^2/p+y^2/q$, the cylinder $x^2+y^2=a^2$ and the $x\,y$ plane is $\pi a^4(p+q)/8\,pq$.

3. Calculate the volume bounded by the surface $z=kxy$, the plane OXY and the first quarter of the cylinder $x^2+y^2=a^2$.

4. Show that the volume of the solid bounded by the cylinder $x^2+y^2=2ax$ and the paraboloid $y^2+z^2=4ax$ is $\frac{2}{3}a^3(3\pi+8)$.

5. Find the volume of the region bounded by the plane $z=x+y$ and the paraboloid $cz=x^2+y^2$.

6. Show that the volume of the region bounded by the hyperboloid of one sheet $x^2/a^2+y^2/b^2-z^2/c^2=1$, its asymptotic cone $x^2/a^2+y^2/b^2-z^2/c^2=0$ and the planes $z=z_1$, $z=z_2$, $(z_2>z_1)$, is $\pi ab(z_2-z_1)$.

7. Prove that the volume in the positive octant bounded by the planes $x=0$, $y=0$, $z=h$ and the surface $z/c=(x/a)^m+(y/b)^m$ is equal to

$$\frac{\frac{1}{2}abh(h/c)^{2/m}\ \Gamma(1/m)\,\Gamma(1/m)}{(m+2)\ \Gamma(2/m)}.$$

8. Show that the volume enclosed by the surfaces defined by the equations $x^2+y^2=cz$, $x^2+y^2=ax$, $z=0$ is $3\pi a^4/32c$.

9. Show that the entire volume bounded by the positive side of the three co-ordinate planes and the surface $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}+(z/c)^{\frac{1}{2}}=1$, is $abc/90$.

[Change the variables to r, θ, φ where

$$x/a=r^4\sin^4\theta\cos^4\varphi,\ y/b=r^4\sin^4\theta\sin^4\varphi,\ z/c=r^4\cos^4\theta\,;$$

r varies from 0 to 1, θ from 0 to $\frac{1}{2}\pi$ and φ from 0 to $\frac{1}{2}\pi$.]

10. Show that the entire volume of the solid $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=1$ is $4\pi abc/35$.

11. Show that the volume of the solid bounded by the cylinders

$$bz^2=c^2y,\ bz^2=2c^2y,\ cx^2=a^2z,\ cx^2=2a^2z,\ ay^2=b^2x,\ ay^2=2b^2x \text{ is } \tfrac{1}{7}\,abc.$$

12. The area lying in the first quadrant which is enclosed by the curves

$$y=ax^3,\ y=bx^3\,;\ x=cy^3,\ x=dy^3,\ (a>b,\ c>d)$$

revolves about X axis; obtain the volume of the solid generated.

[The required volume is

$$\int\int 2\pi\ y\,dx\,dy\,;$$

change the variables to u, v where

$$u=y/x^3,\ v=x/y^3.]$$

13. A plane lamina of thickness, k, is bounded by the lines

$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x}{a}+\frac{y}{b}=2,$$

and the rectangular axes $x=0$, $y=0$. The density at each point (x, y) is proportional to the nth power of the distance from the line $x/a+y/b=1$.

Find the mass of the lamina.

14. Find the ratio in which the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k,\ k>1$$

is divided by the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

15. Prove that the volume of that part of the sphere $x^2+y^2+z^2=1$ lying inside the cone formed by joining the origin to the points of intersection of the sphere with the surface

$$36x^2+25y^2-16z=16$$

is $16\pi/45$.

16. Prove that the volume common to the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$ is $16a^3/3$.

17. The space enclosed by the planes $x=0, y=0, x+y=1$ and the surface $xz=c^{x+y}$ is filled with matter whose density is $\delta=\left(\frac{x}{y}\right)^{2/3}$ at the point (x, y, z); show that the whole mass is $\frac{2\pi}{\sqrt{3}}(e-1)$.

$$\left[\text{Use } \Gamma(a)\,\Gamma(1-a)=\frac{\pi}{\sin a\pi}.\right]$$

18. Find the volume of that part of the cylinder

$$x^2+y^2-2ax=0$$

which is cut off by the solid

$$z^2=2ax$$

19. Find the volume of the part of the solid sphere $x^2+y^2+z^2 \leqslant 4a^2$ enclosed by the cylinder

$$(x^2+y^2)^2=2a^2(2x^2+y^2).$$

20. Prove that the 'volume' and 'area' of n-dimensional sphere of radius a are respectively

$$\frac{1}{2^n}a^n[\Gamma(\tfrac{1}{2})]^n/\Gamma(\tfrac{1}{2}n+1)$$

and

$$2a^{n-1}[\Gamma(\tfrac{1}{2})]^n/\Gamma(\tfrac{1}{2}n)$$

21. Prove that the volume of a cone which extends from the origin to the surface

$$x=f(u, v),\ y=g(u, v),\ z=h(u, v)$$

is given by

$$\tfrac{1}{3}\int\int \Delta\, du\, dv$$

where

$$\Delta=\begin{vmatrix} x & y & z \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}$$

and suffixes denote partial derivation

$$x_u=\frac{\partial x}{\partial u}, \ldots\ldots .$$

Hence prove that the volume of one octant of the cone

$$x^4+y^4=z^4\tan^2\alpha$$

between its vertex and the surface

$$x^4+y^4+z^4=1$$

is

$$\frac{[\Gamma(\frac{1}{4})]^2}{24\,\Gamma(\frac{1}{2})}\int_0^{\alpha}\frac{du}{\sqrt{(\cos u)}}.$$

Miscellaneous Exercises

1. If d is a positive integer but not the square of an integer and

$$y = x(x^2+3d)/(3x^2+d),$$

where x is a positive rational number, shew that

$$y-x = \frac{2x(d-x^2)}{3x^2+d} \text{ and } y^2-d = \frac{(x^2-d)^3}{(3x^2+d)^2}.$$

Hence shew that the section of the positive rational numbers determined by assigning to the upper class all rational numbers whose square is greater than, d, and to the lower class all the other rational numbers, is not generated by a rational number.

2. Show that an algebraic equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0,$$

with integral co-efficients, cannot have a rational but non-integral root.

Deduce that $\sqrt{3}$ and $\sqrt[3]{2}$ are not rational numbers.

3. If m/n is a good approximation to $\sqrt{2}$, prove that

$$(m+2n)/(m+n)$$

is a better one, and that the errors in the two cases are in opposite directions. Apply this result to show that the limit of the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408}, \ldots\ldots$$

is $\sqrt{2}$.

4. Prove that, as $n \to \infty$,

$$[(2n)!/(n!)^2]^{\frac{1}{n}} \to 4. \qquad \text{(Use Ex. 11, P. 75)}$$

5. Prove that, as $n \to \infty$,

$$(n!)(a/n)^n \to 0 \text{ or } +\infty,$$

according as $a < \text{or} > e$. (Use Ex. 2, Ex. 3, P. 69)

6. Show that, as $n \to \infty$,

(i) $(n^5/2^n) \to 0$. (ii) $(n^2)^{\frac{1}{n}} \to 1$.

7. Denoting

$$\lim \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right),$$

by γ, known as Euler's constant, show that

$$\lim \frac{1}{n}\left[\frac{n}{1} + \frac{n-1}{2} + \frac{n-2}{3} + \ldots + \frac{1}{n} - \log(n!)\right] = \gamma.$$

8. Prove that if a and b are positive, then

$$\tfrac{1}{2}\{(a^{\frac{1}{n}} + b^{\frac{1}{n}})\}^n \to \sqrt{(ab)}, \text{ as } n \to \infty.$$

9. If $S_n \to S$ as n tends to infinity, prove that the following sequences also tend to S.

(i) $\left[\frac{S_n}{1} + \frac{S_{n-1}}{2} + \frac{S_{n-2}}{3} + \ldots + \frac{S_1}{n}\right] \Big/ \log n$.

(ii) $\{S_1 + 2S_2 + 2^2S_3 + \ldots\ldots + 2^{n-1}S_n\}2^{-n}$.

10. Prove that if $x>0$. then

$$n\log\{\tfrac{1}{2}(1+x^{\frac{1}{n}})\}\to-\tfrac{1}{2}\log x,\text{ when } n\to\infty.$$

11. If

$$f(x,y)=(xy^5-x^2y^3)/(x^2+xy^2+y^4),\text{ when }(x,y)\neq(0,0),$$

and $$f(0,0)=0,$$

show that at the origin $f_{yx}\neq f_{xy}$ but that f_x is continuous and f_y differentiable at the origin. Discuss the applicability of the conditions of § 142, page 309.

12. Discuss the existence and the equality at the origin of f_{yx} and f_{xy} for

$$f(x,y)=(ax^2+2bxy+cy^2)^3/(x^2+y^2)^2\text{ when }(x,y)\neq(0,0),$$

and $f(0,0)=0$.

13. If at all points of the plane, $f(x,y)$ is continuous with respect to x and has partial derivative $f_y(x,y)$ and if $f_y(x,y)$ is bounded in the whole plane, prove that $f(x,y)$ is continuous with respect to two variables at all points of the plane.

14. The number θ is defined by the Taylor's formula

$$f(a+h,b+k)=f(a,b)+hf_x(a+\theta h,b+\theta k)+kf_y(a+\theta h,b+\theta k).$$

Prove that if f_x and f_y are differentiable at (a,b) and if $(h,k)\to(0,0)$ so that

$$(h^2+k^2)/\{h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\}$$

is bounded, then $\theta\to\frac{1}{2}$

15. Discuss at $(0,0)$ the existence of the first order partial derivatives and the continuity and differentiability of $f(x,y)$ itself when $f(x,y)$ has the following forms :—

$$(i)\ x^2y^2\log(x^2+y^2),\qquad (ii)\ xy\log(x^2+y^2),$$

$f(0,0)$ being equal to zero in either case.

16. Show that, if $f_x(x,y)$, $f_y(x,y)$ are differentiable at a point, then $f(x,y)$ is also differentiable at that point.

17. Prove that the functions

$$ax^2+by^2+cz^2,\ Ax+By+Cz,$$

$$a^2x^2(B^2c+C^2b)+b^2y^2(C^2a+A^2c)+c^2z^2(A^2b+B^2a)-2abc(BCyz+CAzx+ABxy)$$

are not independent and find the relation between them.

18. If u and v are given in terms of x and y by the relations

$$u^2+v^2=x^3+y^3,\ u^3+v^3=x^2+y^2,$$

show that

$$\frac{\partial(u,v)}{\partial(x,y)}=-\frac{xy(x-y)}{uv(u-v)}.$$

19. Find the values of x, y, z for which the function

$$e^{-u}(x-y+2z)$$

is a maximum, where

$$u=x^2+y^2+2z^2-2yz+2zx-xy.$$

20. Prove that if $x+y+z=3c$, then $f(x)f(y)f(z)$ will be a maximum or minimum for $x=y=z=c$, according as

$$[f(c)]f''(c)<\text{ or }>[f'(c)]^2.$$

21. From the point $B(0, -b)$, of the ellipse $x^2/a^2+y^2/b^2=1$, a chord BP of the ellipse is drawn. Show that if $a^2 \geqslant 2b^2$, the greatest length the chord can have is

$$a^2/\sqrt{(a^2-b^2)}.$$

What is the corresponding result when $a^2 < 2b^2$.

22. Find all the stationary values of the function

$$\frac{x}{y}+\frac{y}{x}-\frac{(x-y)^2}{a^2}$$

of the variables (x, y); discuss how their nature depends on the value of the parameter a. Consider in particular the points at which $x=y=a$.

23. P and Q are any two points on the parabolas

$$y^2=4x,\ y^2=2x-6,$$

respectively. Show that the minimum distance between P and Q is $\sqrt{5}$.

24. Find the stationary points of the function

$$x^2+y^2+z^2-yz-zx-xy$$

where x, y, z are subjected to the condition

$$x^2+y^2+z^2-2x+2y+6z+9=0.$$

25. Discuss the maximum and minimum values of

$$7x^2+8xy+y^2$$

where $x^2+y^2=1$.

26. Show that

$$u=yz+zx+xy$$

has no maximum and minimum, when considered as a function of the three independent variables x, y, z but has a minimum value when the three variables are connected by the relation

$$ax+by+cz=1,$$

and a, b, c are positive constants satisfying the condition

$$2(ab+bc+ca) > (a^2+b^2+c^2).$$

27. Prove that the function

$$y=x_1^2+x_2^2+\ldots\ldots+x_n^2$$

has one and only one critical value, which is a minimum, when the variables $x_1, \ldots\ldots, x_n$ are subject to

(*i*) the single condition $\Sigma a_r x_r=1$.

(*ii*) the two conditions $\Sigma a_r x_r=0$, $\Sigma b_r x_r=1$,

and the a_r, b_r are constants.

Find these minimum values in the forms

(*i*) $(\Sigma a_r^2)^{-1}$; (*ii*) $(\Sigma a_r^2)/\Sigma(a_r b_s-a_s b_r)^2$.

28. Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

and

$$lx+my+nz=0;$$

and interpret the result geometrically.

29. Prove that if all the symbols denote positive quantities, the stationary value of $lx+my+nz$, subject to the condition

$$x^p+y^p+z^p=c^p$$

is given by

$$c(l^q+m^q+n^q)^{1/q}$$

where $q=p/(p-1)$, Show further that this value is a maximum or a minimum according as $p >$ or < 1.

30. Find the volume of the largest rectangular parallelopiped which has three faces in the co-ordinate planes and one vertex in the plane

$$x/a+y/b+z/c=1.$$

31. Find the dimensions of the rectangular box, without a top, of maximum capacity whose surface is a^2.

32. Given n points P_i whose co-ordinates are

$$(x_i, y_i, z_i), (i=1, 2, \ldots, n),$$

show that the co-ordinates of the points $P(x, y, z)$ such that the sum of the squares of the distances from P to the points P_i is a minimum are given by

$$[\Sigma x_i/n, \Sigma y_i/n, \Sigma z_i/n].$$

33. Show that the maximum value of $x^2y^2z^2$ subject to the condition $x^2+y^2+z^2=c^2$ is $c^6/27$. Interpret the result.

34. If $u=x^2+y^2+z^2$ where $x^p+y^p+z^p=3^p$, show that a stationary value of the function u is equal to $3c^p$ and prove that this is minimum or a maximum recording as $p <$ or > 2.

35. Find the greatest and least distances from the origin of a point of the surface,

$$(x/a)^p+(y/b)^p+(z/c)^p=1\,;$$

where a, b, c are fixed positive numbers and p is a fixed even integer greater than 2.

36. If $m \geqslant 0$, prove that

$$\iint (1-x^2-y^2)^m f(Ax+By)\, dxdy$$
$$=\beta(\tfrac{1}{2}, m+1) \int_{-1}^{1} (1-x^2)^{m+\frac{1}{2}} f(kx)dx,\ k^2=A^2+B^2,$$

where the integral on the left is taken over the circle $x^2+y^2=1$.

37. By using the transformation

$$x=uv,\ y=u(1-v),$$

prove that

$$\iint \frac{f(x+y)}{\sqrt{(xy)}}\, dxdy = \pi \int_0^\alpha f(u)du,$$

where the double integral extends over all positive values of x and y subject to

$$x+y < \alpha.$$

38. Show that volume of the wedge intercepted between the cylinder $x^2+y^2=2ax$ and the planes $z=x \tan\alpha$, $z=x\tan\beta$ is $\pi(\tan\beta-\tan\alpha)a^3$.

39. Find the volume contained between the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

and the cylinder $x^2/a^2+y^2/b^2=x/a$.

40. Show that the volume obtained by one revolution of the loop of the Folium $x^3+y^3=3axy$ about OX is $4\pi^2a^3/3\sqrt{3}$.

[Change the variables to u, v where $u=x^2/y$, $v=y^2/x$.]

41. If $x=r\cos\theta$, $y=r\sin\theta$, find the volume included by the surfaces whose equations are $r=a$, $z=0$, $\theta=0$, $z=mr\cos\theta$.

42. Using the transformation

$$x=c\cosh u\cos v,\ y=c\sinh u\sin v,$$

or otherwise, where $c^2=a^2-b^2$, show that the mean value of the product of the focal distances of a point inside the ellipse $x^2/a^2+y^2/b^2=1$ is

$$\frac{1}{4}\left[a^2+b^2+\frac{(a^2-b^2)^2}{ab}\log\frac{a+b}{a-b}\right].$$

[The mean value$=[\int\int(SP.S'P\,dxdy)]/\int\int dxdy$].

43. Evaluate

$$\int\int[(2x^2+y^2)/xy]\,dxdy$$

taken over the area in the positive quadrant of the xy plane bounded by the curves

$$x^2+y^2=h^2,\ x^2+y^2=k^2,\ y^2=4ax,\ y^2=4bx.$$

44. The axes of two equal cylinders intersect at right angles. If, a, be their radius, show that the volume common to the cylinders is $(16/3)\,a^3$.

[If $x^2+y^2=a^2$ and $y^2+z^2=a^2$ be the two cylinders, then the required volume

$$=8\int_0^a dx\int_0^{\sqrt{(a^2-x^2)}} dy\int_0^{\sqrt{(a^2-y^2)}} dz.$$

45 Prove that the graph of

$$y=f(x)=\frac{2}{\pi}\int_0^\infty\frac{\sin^2 t\cos xt}{t}\,dt,$$

consists of parts of the lines $y=-\frac{1}{2}$, $y=0$, $y=\frac{1}{2}$ together with four isolated points.

[Use the result obtained in Ex. 1, Page 375.]

46. Show that, when $n\to\infty$,

$$\int_a^b f(x)\,|\sin nx|\,dx\to\frac{2}{\pi}\int_a^b f(x)dx.$$

47. Evaluate the limit of the sequence $\{u_n\}$, where

$$u_n=\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots\dots+\right),\text{ when } n\to\infty.$$

48. Show that, as $n \to \infty$,

$$\frac{2.2\ 4.4\ 6.6..\ (2n)(2n)}{1.3\ 3.5.5.7...(2n-1)(2n+1)} \to \frac{\pi}{2}.$$

Deduce that (Walli's formula)

$$\{[(n\,!)^2\, 2^{2n}]/(2n)\,!\,\sqrt{n}\} \to \sqrt{\pi} \text{ as } n \to \infty.$$

49. $f(x)$ is a non-negative function admitting an elementary infinite integral and another function $f_n(x)$ is defined thus :—

$$f_n(x) = \begin{cases} f(x), \text{ if } f(x) < n, \\ n, \text{ if } f(x) \geqslant n, \end{cases}$$

show that

(*i*) if the interval of integration is bounded, then as $n \to \infty$

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

(*ii*) if the interval of integration is infinite, then as $n \to \infty$

$$\int_{-n}^{+n} f_n(x)\,dx \to \int_{-\infty}^{+\infty} f(x)\,dx.$$

50. If $f(x)$, $f'(x)$ are continuous in $[a, b]$ and

$$S_n = \sum_{r=1}^{n} hf(a+rh),\quad I = \int_a^b f(x)\,dx,$$

where $h=(b-a)/n$, then as $\to \infty$,

$$n(S_n - I) \to \tfrac{1}{2}(b-a)[f(b)-f(a)].$$

51. If I, h have the same meaning as in the Ex. above, but if now $f''(x)$ is also continuous in $[a, b]$ and

$$S_n = \sum_{r=1}^{n} hf[a+\tfrac{1}{2}(2r-1)h],$$

show that, as $n \to \infty$,

$$n^2(I - S_n) \to \tfrac{1}{24}(b-a)^2[f'(b)-f'(a)].$$

52. If $f(t, x) = \pi\, t \sin \pi t x$ when $0 \leqslant x \leqslant 1/t$ and $f(t, x)=0$ when $1/t \leqslant x \leqslant 1$, show that

$$\int_0^1 \left\{ \lim_{t \to \infty} f(t, x) \right\} dx = 0 \neq 2 = \lim_{t \to \infty} \left\{ \int_0^1 f(t, x)\,dx \right\}.$$

53. If $f(t, x) = tx/(1+t^2x^4)$, show that

$$\int_0^1 \left\{ \lim_{t \to \infty} f(t, x) \right\} dx = 0 \neq \frac{\pi}{4} = \lim_{t \to \infty} \left\{ \int_0^1 f(t, x)\,dx \right\}.$$

54. Prove that if

(*i*) a is positive ;

(*ii*) $f(x)$ is continuous except perhaps at the origin ;

(*iii*) $\int_0^a f(x)dx = \lim_{\varepsilon \to 0} \int_\varepsilon^a f(x)\,dx$, where the limit exists

(*iv*) $g(x) = \int_x^a \frac{f(t)}{t}\,dt$,

then

$$\int_0^a g(x)\,dx = \int_0^a f(x)\,dt.$$

[**Hint :** Change the order of integration.]

55. If $0 < a < b$ and $f(x)$ is continuous in $[0, b]$, prove that

$$\lim_{h \to (0+0)} \int_h^1 \frac{f(at)-f(bt)}{t}\,dt\,.$$

exists and is equal to

$$f(0)\log \frac{b}{a} - \int_a^b \frac{f(x)}{x}\,dx.$$

56. Show that

$$\int_{-1}^{+1} \frac{\sqrt{(1-x^2)}\,dx}{(1+\alpha^2+2\alpha x)(1+\beta^2+2\beta x)} = \begin{cases} \pi/2(1-\alpha\beta), \text{ if } \alpha^2 < 1,\ \beta^2 < 1, \\ \pi/2\alpha(\alpha-\beta), \text{ if } \alpha^2 > 1 > \beta^2. \end{cases}$$

57. A is a fixed point on the axis OX and an ellipse is drawn with OA as its minor axis. Calculate the value of $\int(x^2\,dy - y^2\,dx)$ taken along the elliptic arc from O to A, y negative and show that if it has its max. value, the eccentricity is $\sqrt{(1-64/9\pi^2)}$.

58. Show that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{n-1}\,x^{2n-1}}{2n-1} + (-1)^n \int_0^x \frac{t^{2n}}{1+t^2}\,dt.$$

59. *Show that, when* $n \to \infty$

$$\int_0^1 \frac{x^{n+1}}{1-x} \log x\,dx \to 0.$$

Sol. We write,

$$\int_0^1 \frac{x^{n+1}}{1-x} \log x \, dx = \int_0^a \frac{x^{n+1}}{1-x} \log x \, dx + \int_a^1 \frac{x^{n+1}}{1-x} \log x \, dx = I_1 + I_2.$$

$$(0 < a < 1)$$

Now, $| I_1 | \leqslant a^{n+1} \displaystyle\int_0^a \left| \frac{\log x}{1-x} \right| dx$, which $\to 0$ as $n \to \infty$: a being < 1.

Again, if we assign to the function $\log x/(1-x)$, the value 1 for $x=1$, it becomes continuous in $[a, 1]$. Thus $\log x/(1-x)$ is bounded in $[a, 1]$. Therefore

$$| I_2 | \leqslant M \int_a^1 x^{n+1} \, dx \text{ which} \to 0, \text{ as } n \to \infty.$$

Hence the result.

60. From the identity, $\dfrac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \dfrac{x^{n+1}}{1-x}$,

show, by integration in the interval $[0, 1]$, that

$$\int_0^1 \frac{\log x}{1-x} dx = -\sum_{n=1}^{\infty} \frac{1}{n^2},$$

61. If p, q are positive, prove that

$$\int_0^\infty \left\{ e^{-px}[1+(p+r)x] - c^{-qx}[1+(q+r)x] \right\} \frac{dx}{x^2} = q - p + r \log \frac{q}{p}.$$

62. *Show that* $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$. $(0 < p < 1)$.

Sol. It is easy to see that the given improper integral converges, when $0 < p < 1$. We have

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \int_0^1 \frac{x^{p-1}}{1+x} dx + \int_1^\infty \frac{x^{p-1}}{1+x} dx.$$

By means of the substitution $y = 1/x$, we see that

$$\int_1^\infty \frac{x^{p-1}}{1+x} dx = \int_0^1 \frac{y^{-p}}{1+y} dy = \int_0^1 \frac{x^{-p}}{1+x} dx.$$

$$\therefore \quad \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \int_0^1 \left\{ \frac{x^{p-1}+x^{-p}}{1+x} \right\} dx$$

$$= \int_0^1 (x^{p-1}+x^{-p})(1-x+x^2-\ldots+(-1)^n x^n+(-1)^{n+1}\frac{x^{n+1}}{1+x}\Big)\,dx$$

$$= \sum_{k=0}^{n} (-1)^k \left(\frac{1}{k+p} + \frac{1}{k+1-p} \right) + R_n,$$

where $|R_n| = \int_0^1 (x^{p-1}+x^{-p})\frac{x^{n+1}}{1+x}\,dx < \int_0^1 (x^{p-1}+x^{-p})x^{n+1}\,dx$

$$= \left\{ \frac{1}{n+p+1} + \frac{1}{n+2-p} \right\} \to 0 \text{ as } n \to \infty .$$

$$\therefore \quad \int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{k+p} + \frac{1}{k+1-p} \right) = \frac{\pi}{\sin p\pi}.$$

(See Ex. 10, page 283.)

63. Prove that $\Gamma(x+1) = x\,\Gamma(x)$.

64. *Show that* $\Gamma(a)\,\Gamma(1-a) = \pi/\sin a\pi$,

As proved in Ex. 3, page 407, we have

$$\beta(a, 1-a) = \frac{\Gamma(a)\,\Gamma(1-a)}{\Gamma(1)} = \Gamma(a)\,\Gamma(1-a).$$

Also

$$\beta(a, 1-a) = \int_0^1 x^{a-1}(1-x)^{-a}\,dx.$$

By means of the substitution $x = y/(1+y)$, we may now show that

$$\int_0^1 x^{a-1}(1-x)^{-a}\,dx = \int_0^\infty \frac{y^{a-1}}{1+y}\,dy = \frac{\pi}{\sin a\pi}.$$

Hence the result.

65. Show that

$$\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{2}{a}\right)\Gamma\left(\frac{3}{a}\right) \ldots\ldots \Gamma\left(\frac{a-1}{a}\right) = \left(\frac{(2\pi)^{a-1}}{a}\right)^{1/2}.$$

66. Prove that

$$\sum_{k=1}^{n} \log \Gamma\left(\frac{k}{n}\right) = \tfrac{1}{2}(n-1)\log(2\pi)\ \tfrac{1}{2}\log n.$$

67. *Show that* $\int_0^1 \log \Gamma(x)\, dx = \frac{1}{2} \log(2\pi)$.

Sol. The point '0' is a point of infinite discontinuity of the integral. We have

$$\Gamma(x) = \Gamma(x+1)/x,$$

so that

$$\log \Gamma(x) = \log \Gamma(x+1) - \log x.$$

We know that the integral of $\log x$ converges at 0 and the integral of $\log \Gamma(x+1)$ is proper, and hence the integral of $\log \Gamma(x)$ is convergent. By means of the substitution $x = 1 - y$, we see that

$$\int_0^1 \log \Gamma(x)\, dx = \int_0^1 \log \Gamma(1-y)\, dy$$

$$= \int_0^1 \log \Gamma(1-x)\, dx$$

$$= \frac{1}{2} \int_0^1 \log [\Gamma(x)\, \Gamma(1-x)]\, dx$$

$$= \frac{1}{2} \int_0^1 \log \frac{\pi}{\sin \pi x} = \frac{1}{2} \log(2\pi).$$

68. *Prove that* $(\sqrt{\pi})\, \Gamma(2x) = 2^{2x-1}\, \Gamma(x)\, \Gamma(x+\frac{1}{2})$.

[*Duplication formula*]

Sol. We have $$\frac{\Gamma(x)\, \Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})} = \beta(x, \tfrac{1}{2}) = \int_0^1 \frac{t^{x-1}\, dt}{\sqrt{(1+t)}},$$

and $$\frac{\Gamma(x)\, \Gamma(x)}{\Gamma(2x)} = \beta(x, x) = \int_0^1 t^{x-1}(1-t)^{x-1}\, dt.$$

In the latter integral, put $v = 2t - 1$ so that we have

$$\beta(x, x) = \frac{1}{2^{2x-1}} \int_{-1}^{+1} (1-v^2)^{x-1}\, dy$$

$$= \frac{1}{2^{2x-1}} \int_0^1 (1-w)^{x-1}\, w^{-\frac{1}{2}}\, dw, \quad (v^2 = w)$$

$$= \frac{1}{2^{2x-1}}\, \beta(\tfrac{1}{2}, x) = \frac{1}{2^{2x-1}}\, \beta(x, \tfrac{1}{2}).$$

Since $T(\frac{1}{2}) = \sqrt{\pi}$, we can now obtain the given result.

ANSWERS

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4. (*i*) $f(0)=0$, (*ii*) $f(0)$ is not determinable. (*iii*) $f(0)=a$.

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8. $0, -1$. For $f(x)=|x|$ and $c=0$, the first limit does not exist but the second exists and is equal to 1. For $c \neq 0$, both the limits exist.

10. By Taylor's theorem, $f(0)-f(x)=-xf'(x)+(1/2\,!)x^2f''(\xi)$ so that $f'(x)-f(x)/x > 0$ and $\therefore$ $d[f(x)/(x)]/dx$ is positive.

Page 186

1. $F(y)$ is 2 for y rational and is 0 for y irrational.

2. $\frac{3}{2}$. **3.** $f(x)$ is integrable; 1 being the only limiting point of the aggregate of the points of discontinuity; the integral is 2. The function $f(x)$ is not continuous for $x=1$.

12. Use § 84·2, page 134.

14. $\frac{2}{3}$.

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9. (*i*) $x=n$; n being any integer. (*ii*) Every point x.

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12. Does not exist.

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(*i*) $\log 2$. (*ii*) π. (*vi*) -1. (*viii*) 2. The integrals (*iii*), (*iv*), (*v*) and (*vii*) do not exist.

Page 239

1. C stands for *Convergent* and $N.C.$ for *Not convergent.*

(*i*) C for $n > 1$. (*ii*) C for $n < 2$. (*iii*) C for $a > 0$.
(*iv*) $N.C.$ (*v*) C. (*vi*) C.
(*vii*) C for $m>0$ and $n>0$, (*viii*) C for $a>0$. (*ix*) C.
(*x*) C for $p > -1$. (*xi*) $N.C.$ (*xii*) $N.C.$

2. C for $n > -1$ and $m > 0$.

Page 247

1. (*i*) C for $p > (1+m+n)$; m, n, being > 0. Consider also the case when either m or n or both are negative.

(*ii*) C for $(n-m) > \frac{1}{2}$. (*iii*) C for $0 < m, n, < 1$.
(*iv*) C for $n > 0$. (*v*) C for $m < -1$.
(*vi*) C for $0 < (a+1) < b$.

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5. Convergent.

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(*i*) C. (*ii*) C for $a > 0$.

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1. (*i*) C. (*ii*) C. (*iii*) C. (*iv*) $N.C$.
(*v*) C for $(1-m) < n < (1+m)$. (*vi*) C for $0 < a < 4$.
(*vii*) C for $-1 < n < 2$.
(*viii*) C for $n > 0$, $-1 < m < n$ and for $n < 0$, $0 > m > (n-1)$.

2. (*i*) C. (*ii*) C. (*iii*) C for $p > 1$. (*iv*) C for $-3 < p < 2$.

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3. $\dfrac{\pi}{2} \log \dfrac{a}{b}$.

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11. The integral is $2/a$, or $-2/a$ acccording as $a > 1$, $a < -1$ or, $-1 < a < 1$.

12. $\pi/2$ if $2m\pi < a < (2m+1)\pi$;
$-\pi/2$ if $(2m+1)\pi < a < (2m+2)\pi$
and 0 if $a = 2m\pi$ or $(2m+1)\pi$; m being any integer.

15 $-\pi/4n$.

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4. $\dfrac{4}{\pi} \Sigma \dfrac{\sin \frac{1}{2} n\pi \sin nx}{n^2}$.

5. $\dfrac{2}{\pi} - \Sigma\left(\dfrac{\sin \frac{1}{2} n\pi}{n^2} - \dfrac{\pi \cos \frac{1}{2} n\pi}{2n}\right) \sin nx$.

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7. $\dfrac{2}{\pi} \Sigma \dfrac{4 \cos 2nx}{(4n^2-1)\pi}$.

8. $\dfrac{\sin 2x}{1} + \dfrac{\sin 4x}{2} + \dfrac{\sin 8x}{4} + \dfrac{\sin 10x}{5} + \dfrac{\sin 14x}{7} + \ldots\ldots$

9. (*a*) $\cosh \pi$.
(*b*) $-e^{-x}$ for $-\pi < x < 0$ and 0 for $x = 0, \pm\pi$ and e^{-x} for $-\pi \leqslant x \leqslant 0$.

11. $\Sigma \left[\dfrac{-2(-1)^n \pi}{n} - \dfrac{4}{\pi n^3} [1-(-1)^n] \right] \sin nx$. 13. 0.

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15. (*i*) $\dfrac{4lk}{\pi^2}\Sigma \dfrac{1}{n^2} \sin \dfrac{n\pi}{2} \sin \dfrac{n\pi x}{l}$.

(*ii*) $\dfrac{kl}{4}+\dfrac{4kl}{n^2}\Sigma \dfrac{1}{n^2}\left(\cos \dfrac{n\pi}{2} - \cos^2 \dfrac{n\pi}{2}\right) \cos \dfrac{n\pi x}{l}$.

16. $\Sigma \left[\dfrac{2(-1)^{n-1}}{n\pi} - \dfrac{4}{n^2\pi^2} \sin \dfrac{n\pi}{2}\right] \sin \dfrac{n\pi x}{2}$.

18. $\dfrac{4-2\sqrt{2}}{\pi}+\dfrac{4\sqrt{2}}{\pi}\Sigma \left[\dfrac{(-1)^n-\sqrt{2}}{16n^2-1}\right] \cos 4nx$.

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3. (*i*) Does not exist. (*ii*) does not exist. (*iii*) 0. (*iv*) 0.

Page 298

1. (*i*) Discontinuous. (*ii*) continuous.
4. No.

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3. (*i*) Not differentiable. (*ii*) Not differentiable.
5. Differentiable at all points except (0, 0).
6. $k > 0$.

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3. (*i*) Equal. (*ii*) Unequal.

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4. Equal.

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2 $4(f_u^2+f_v^2)$, $axy(x^2-y^2)$.

Page 327

(*i*) min. at $(\frac{2}{3}-\frac{4}{3})$,

(*ii*) min. at (1, 2) and (−1, 2) ; max. at (0, 0).

(*iii*) min. at (0, 0) if $a > \frac{1}{4}$.

(*iv*) min. at $[\frac{1}{3}(b-2a), \frac{1}{3}(a-2b)]$. (*v*) max. at (2, 1).

(*vi*) max. or min. at $(\frac{1}{3}a, \frac{1}{3}a)$ according as $a > 0$ or $a < 0$.

(*vii*) min. at (1, 0) and max. at (−3, 2).

(*viii*) min. at $(\pm a/\sqrt{2}, 0)$.

(*ix*) min. at $(\pm 3/\sqrt{2}, 0)$.

(*x*) min. at every point of the line $y=x$.

(*xi*) max. at (0, 0) and min. at $(\pm 1, 2)$.

(*xii*) Extremes at (0, 0), (−1, 0), $(-\frac{1}{2}, 0)$.

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6. max. at (0, 0) ; min. at $(\pm\sqrt{5}, \mp\sqrt{5})$. Neither max. nor min. at $(\pm 1, \mp 1)$.

7. The positions of the corresponding points P, Q, are

$(3a, \pm\sqrt{12}a)$, $(3a, \pm\sqrt{12}a)$; $(0, 0)$, $(\pm\sqrt{12}a+3a, 0)$;

$(a, 3a)$, $(3a \mp\sqrt{6}a, \pm\sqrt{6}a)$; $(a, -2a)$, $(3a\pm\sqrt{6}a, \pm\sqrt{6}a)$.

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1. (*i*) Min. at (0, 0, 0). (*ii*) min. at (0, 0 -1).

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2. $r^2 \sin\theta$. **4.** $x_1^{n-1}\, x_2^{n-2} \ldots\ldots x_{n-1}\, x^0_n$.

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3. $\varphi(x)$ is not unique.

4. (*i*) 0. (*ii*) 0. (*iii*) $2\sqrt{3}/9$. (*iv*) -1.

5. (*i*) $\varphi(x)$ unique near $(1, -1)$ but not near $(0, 0)$. (*ii*) $\phi(x)$ unique near $(1, 1)$. (*iii*) $\varphi(x)$ not unique but the equation possesses a unique continuous solution $x=\psi(y)$ near $(1, 1)$.

(*iv*) Does not exist.

Page 354

1. $uv+vw+wu=ac-b^2$. **2.** $u^2-v^2=8w$.

3. $uv+vw+wu+1=0$.

Page 359

1. (*i*) $3a^2$ for (a, a, a). (*ii*) $3a^2$ for (a, a, a), $(-a, -a, -a)$.

(*iii*) $3a^2$ for (a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$,

2. $abc/(2\Sigma ab-\Sigma a^2)$.

3. $-a^2$ (min.) at $(a, -a)$ and $(-a, a)$; $\frac{1}{3}a^2$ (max.) at $(a/\sqrt{3}, a/\sqrt{3})$ and $(-a/\sqrt{3}, -a/\sqrt{3})$ **4.** 9, -1.

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7. (1, 1) and $(-1, -1)$. **8.** $(-2/\sqrt{14}, -1/\sqrt{14}, -3/\sqrt{14})$.

9. The extreme value is

$$\tfrac{1}{27}\,(\log abck)^3/(\log a \log b \log c)$$

and is a max. or min. according as $\log\,(abck)/(\log a \log b \log c)$ is positive or negative.

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14. $(\sqrt{\Sigma a^2}-k)/\sqrt{3}$.

15. They are the roots of

$$\Sigma \frac{a^2}{l^2-r^2}=0.$$

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19. The symmetrical stationary value is $\frac{1}{9}$ $(1+m)$ and there are three unsymmetrical values, each being equal to

$$(m^2-m+1)/(m+1)^2.$$

20. $(a/10,\ a/10)$.

Page 383

1. $\frac{7}{3}a^3$, $\frac{46}{15}a^3$. 3. $2 \log 2$.

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5. $\frac{1}{84}$. 6. (*i*) $38a^4/\sqrt{3}$. 7. -2π.

Page 391

1. (*i*) $\frac{1}{8}a^2b^2(a^2+b^2)$. (*ii*) $(e^{ab}-1)/(a-b)$.
(*iii*) 0. (*iv*) $2[a \sinh^{-1}(a/c)+c-\sqrt{(a^2+c^2)}]$.

Pages 395—96

1. $\int_{-1}^{0} dx \int_{-x}^{\frac{1}{3}(2x+5)} f dy + \int_{0}^{1} dx \int_{x}^{\frac{1}{3}(2x+5)} f dy + \int_{1}^{2} dx \int_{(2x-1)}^{\frac{1}{3}(2x+5)} f dy.$

2. $\pi/96$. 4. $\int_{0}^{a} dy \int_{0}^{\sqrt{(4ay)}} \varphi dx + \int_{a}^{3a} dy \int_{0}^{3a-y} \varphi dx.$

6. $12-16 \log 2$. 8. $\frac{1}{2}\pi \log [2e/(1+e)]$. 10. $47/24$.

11. $8 \sin^{-1} (\sqrt{10}/4)+\frac{1}{2} \log [(\sqrt{3}+\sqrt{5})/\sqrt{2}]-3\sqrt{15}/4$. 12. $a^3\pi$

Pages 410—412

1. $\frac{2}{3}\pi(a^2+b^2)^{\frac{3}{2}}$. 2. $\frac{1}{9}a^3(3\pi-4)$. 7. $\frac{1}{4}\beta(\frac{3}{4}, \frac{1}{2})\, \beta(\frac{1}{2}, \frac{1}{4})$

8. In the $r\theta$ plane the field of integration is bounded by the lines $r=2a$, $r=4a$; $a=\frac{1}{2}\pi$, $r=2a \sec^2 \theta$.

9. $\int_{0}^{1} \int_{0}^{c} V[u(1-v),\ uv]u\, du dv.$

10. To the two parts of the given region in the (x, y) plane determined by the line $y=x$, corresponds the same region of the (u, v) plane bounded by the lines $u=0$, $u=a^2$, $u=2v$. The transformations for the two parts are given by

$$\left.\begin{aligned} x+y&=(u+2v)^{\frac{1}{2}} \\ x-y&=(u-2v)^{\frac{1}{2}} \end{aligned}\right\} \quad \begin{aligned} x+y&=(u+2v)^{\frac{1}{2}} \\ x-y&=-(u-2v)^{\frac{1}{2}}. \end{aligned}$$

11. $8(\sqrt{a_1}-\sqrt{a_2})\sqrt{-a_3}$.

12. If λ, μ denote the values of the parameter for the confocal ellipse and hyperbola respectively through (x, y), then

$$\lambda+\mu=(x^2+y^2+c^2),\ \lambda\mu=c^2x^2,$$

so that $\quad x=\sqrt{(\lambda\mu)}/c, \qquad y=\sqrt{(\lambda-c^2)(c^2-\mu)}/c.$

$\therefore$ the required area is

$$\iint dxdy=\iint\frac{\partial(x, y)}{\partial(\lambda, \mu)}d\lambda d\mu=\tfrac{1}{4}\iint\frac{\mu-\lambda}{\sqrt{\lambda\mu(\lambda-c^2)(c^2-\mu)}}d\lambda\, d\mu,$$

$$=\tfrac{1}{6}c^2(\sqrt{10}-2)\sin^{-1}\tfrac{1}{3}.$$

where for the double integral, $\frac{1}{3}c^2\leqslant\mu\leqslant\frac{2}{3}c^2$, $\frac{4}{3}c^2\leqslant\lambda\leqslant\frac{5}{3}c^2$.

13. The field of integration in the xy plane in the positive quadrant of the ellipse $x^2/9+y^2/5=1$. As (x, y) moves along x-axis from $(0, 0)$ to $(2, 0)$, the point (λ, μ) moves in the $\lambda\mu$ plane along the line $\lambda=2$ from $(2,0)$ to $(2, 2)$: as (x, y) moves along x-axis from $(2, 0)$ to $(3, 0)$ (λ, μ) moves along $\mu=2$ from $(2, 2)$ to $(3, 2)$; as (x, y) moves along the arc of the ellipse from $(3, 0)$ to $(0, \sqrt{5})$, (λ, μ) moves along $\lambda=3$ from $(3, 2)$ to $(3, 0)$; finally as (x, y) moves along $x=0$ from $(0, \sqrt{5})$ to $(0, 0)$, (λ, μ) moves along $\mu=0$ from $(3, 0)$ to $(2, 0)$. Thus the region in $\lambda\mu$ plane is the rectangle $[2, 3; 0, 1]$.

Since, as may easily be seen,

$$x=\tfrac{1}{2}\lambda\mu,\ y=\tfrac{1}{2}\sqrt{(\lambda^2-4)(4-\mu^2)},$$

$\partial(x, y)/\partial(\lambda, \mu)$ can be calculated.

14. $\dfrac{1}{96}$. This equation may be solved by the substitution $x=r\cos\theta$, $y=r\sin\theta$ also.

15. $(n-p)(l-m)[(l+m)(a+b)-(n+p)(a-b)]/32a^2b^2$. **16.** $\frac{3}{4}$.

Pages 413—416

2. $\frac{2}{3}+\frac{1}{4}\log\frac{1}{9}$. **3.** $\frac{1}{2}(e^2+2e-3)$.

4. $0, \pi/128$. **5.** $\pi/24$.

6. $1/6(n+4)$. **7.** $\frac{1}{30}, 12$. **8.** $2\pi^{-2}$.

9. 1. **10.** $(\pi+4)/64$.

11. 2. **12.** $\pi/24$. **14.** $\pi a^2/24+\sqrt{3}a^2/4$.

16. $\frac{1}{2}\log\dfrac{e}{2}, \pi$. **17.** $\dfrac{\pi ab(a^2+b^2)}{4}$.

18. $\pi/16$. **19.** $\dfrac{a^2}{2}\left[-\dfrac{\pi}{6}+\log(2+\sqrt{3})\right]$.

20. $(e^2-1)/4e$. **21.** $\pi\log\dfrac{1+a^2}{a^2}$.

Page 424

1. $2\sqrt{2}a\pi$. **2.** $a(12+5\log 5)/8$.

3. $2a \int_0^{\pi} \sqrt{(3-2\cos\theta)}\, d\theta.$

Pages 427—428

1. $2a^2\pi - 4a^2$. 2. $56\pi a^2/9$. 4. $16a^2(\pi-\sqrt{2})$.
5. $4a^2\pi$, 6. $2\sqrt{2}\pi$.

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2. $12\pi/5$. 3. $2\pi abc$.
5. $\frac{4}{5}\pi abc(a^4+b^4+c^4)$. 6. $\frac{1}{3}\pi ah[3(l+m)a^2+3nah+2nh^2]$.

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2. $\pi a^3(a^2+5b^2)/8b$. 4. $10/7$.

Page 448

2. $3a^3$. 3. $\pi/24$. 6. $5\pi/6$.

Pages 454—457

8. $\frac{1}{8}\pi[\beta(\frac{3}{4},\frac{1}{2})-\beta(\frac{5}{4},\frac{1}{2})]$. 10. $\log(a'/a)\log(b'/b)\log(c'/c)$.
12. $\Gamma(l)\,\Gamma(m)\,\Gamma(n)/\Gamma(l+m+n+1)$. 14. $384\pi/5$.
15. $\frac{8192}{735}a^7 k\left(k^2+\frac{8}{33}\right)$. 17. $\pi/8$.
19. $$\frac{a^l b^m c^n}{pqr}(h)^{\frac{l}{p}+\frac{m}{q}+\frac{n}{r}}\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}\right)}\int_0^1 F(hu)\,u^{\Sigma\frac{l}{p}-1}\,du.$$
21. $\frac{1}{7}.a\pi(4-\pi)$. 24. $4\pi/3$.

Pages 458—459

1. $9(3\pi-4)/4$. 3. $\frac{1}{8}ka^4$. 5. $\frac{1}{8}\pi c^3$.
12. $\frac{16}{35}\pi(b^{-\frac{5}{8}}-a^{-\frac{5}{8}})(d^{-\frac{7}{8}}-c^{-\frac{7}{8}})$.
13. $\frac{(2n+3)k}{(n+1)(n+2)}\cdot\frac{a^{n+1}b^{n+1}}{(a^2-b^2)^{\frac{n}{2}}}$. 14. $\frac{4k\sqrt{(2k)}-(k+5)\sqrt{(k-1)}}{(k+5)\sqrt{(k-1)}}$.
18. $128a^3/15$. 19. $32a^3(3\pi-\sqrt{2})/9$.

Pages 461—465

11. $f_{yx}(0,0)=1\neq 0=f_{xy}(0,0)$.
12. $f_{xy}(0,0)=6a^2b\neq 6c^2b=f_{yx}(0,0)$.

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