Date : 11 August 2019

Keep Practising! **

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

210/250

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-11: IAS(M)/11-AUG.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: SUJIT SHANKAR

Roll No. —

Test Centre: ORM DELHI

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Sujit Shankar
Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
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	(c)			08	42
	(d)			09	
	(e)			08	
2	(a)			14	
	(b)			10	37
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3	(a)				
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5	(a)			09	
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	(d)			14	
Total Marks					

$\frac{210}{250}$

P.T.O.

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SECTION – A

1. (a) Let T be the linear operator on C^2 which is represented in the standard ordered basis by the matrix.

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the minimal polynomial for T. **[10]**

finding out eigen values of A

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2+1=0$

$\Rightarrow \lambda = \pm i$

Since T is linear operator on c^2 then $\lambda = i, -i$ are two distinct values

and hence minimal polynomial is given by

$(x+i)(x-i) = x^2+1$

08

1. (b) If A be an n-rowed non-singular matrix, **X** be an n × 1 matrix, **B** be an n × 1 matrix, the system of equations **AX** = **B** has a unique solution. **[10]**

Given that A is n-rowed non-singular matrix.
This implies A is inversible matrix.

Let X_1, X_2 are two solutions that satisfies $AX = B$. Then we need to prove that $X_1 = X_2$ to show that solution is unique.

Now, we have

$$AX_1 = B \quad -(1)$$
$$AX_2 = B \quad -(2)$$

Since A is non-singular that means A^{-1} exists
Multiplying post with A^{-1} in both the equation we obtain

$$(A^{-1}A)X_1 = A^{-1}B \quad \Rightarrow X_1 = A^{-1}B \quad -(3)$$
$$(A^{-1}A)X_2 = A^{-1}B \quad \Rightarrow X_2 = A^{-1}B \quad -(4)$$

From (3) and (4),

$$X_1 = X_2$$

09

Hence the system of equations $AX = B$ has a unique solution.

1. (c) Let $f : R^2 \to R$ be defined by setting

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & if\,(x,y) \neq (0,0), \\ 0\ , & if\,(x,y) = (0,0), \end{cases}$$

Show that f possesses partial derivatives at (0, 0) but is not differentiable at the point. **[10]**

Partial derivative of f at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Thus, $f_x(0,0)$ and $f_y(0,0)$ exist

Now Assume that f is differentiable at $(0,0)$
then
$$f(0+h, 0+k) - f(0,0) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$$
where $\lim\limits_{(h,k)} \phi(h,k) \to 0.$

$$\Rightarrow \frac{f(h,k)}{\sqrt{h^2+k^2}} = \phi(h,k)$$

$$\Rightarrow \lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{h^2 k}{(h^4+k^2)(h^2+k^2)^{1/2}}$$

$$= \lim_{h\to 0} \frac{h^2(mh^3)}{(h^4+m^2h^6)(h^2+m^2h^6)^{1/2}} \quad [\text{Put } k = mh^3]$$

$$= \lim_{h\to 0} \frac{m}{(1+m^2h^2)(1+m^2h^4)^{1/2}}$$

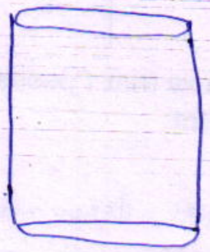
$$= m$$

Since limit depends upon variable m and therefore f is not differentiable at $(0,0)$

08

1. (d) Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base. **[10]**

let the height of cylinder be h and radius be r.



Now

Surface area, S (say) $= 2\pi r h + \pi r^2$

$\Rightarrow dS = 2\pi r\, dh + 2\pi h\, dr + 2\pi r\, dr$

$\Rightarrow 0 = r\, dh + h\, dr + r\, dr$

$\Rightarrow \dfrac{dh}{dr} = -\dfrac{[h+r]}{r}$ —(1)

and volume (V, say) $= \pi r^2 h$

$\dfrac{dV}{dr} = 2\pi r h + \pi r^2 \dfrac{dh}{dr}$ —(2)

Now for volume to be maximum/minimum

$\dfrac{dV}{dr} = 0$

$\Rightarrow 2\pi r h + \pi r^2 \left[-\dfrac{h+r}{r}\right] = 0$ [Using (1)]

$\Rightarrow rh - r^2 = 0$

$\Rightarrow r(h-r) = 0$

$\Rightarrow r \neq 0,\ h = r$

Now $\dfrac{d^2V}{dr^2} = 2\pi h + 2\pi r \dfrac{dh}{dr} + 2\pi r \dfrac{dh}{dr} + \pi r^2 \dfrac{d^2h}{dr^2}$

$= 2\pi\left[h + r[-2r] + r[-2] + \dfrac{r^2}{2} \cdot \dfrac{1}{r}\right] < 0$

Hence volume is maximum when radius is equal to height for a given surface shown

09

1. (e) Show that the straight lines whose direction cosines are given by

$al + bm + cn = 0$, $fmn + gnl + hlm = 0$.

are perpendicular if

$f/a + g/b + h/c = 0$

are parallel if

$\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0.$ **[10]**

Given $al + bm + cn = 0 \quad -(1)$

$fmn + gnl + hlm = 0 \quad -(2)$

Putting the value of l from (1) into (2) we get

$$fmn + (gn + hm)\left(\frac{-bm - cn}{a}\right) = 0$$

$$\Rightarrow (af - bg - hc)mn - bhm^2 - cgn^2 = 0$$

$$\Rightarrow bh\left(\frac{m}{n}\right)^2 + (bg + hc - af)\frac{m}{n} + cg = 0 \quad -(3)$$

$$\Rightarrow \frac{m_1 m_2}{n_1 n_2} = \frac{cg}{bh}$$

Similarly $\frac{l_1 l_2}{n_1 n_2} = \frac{cf}{ah}$

For $\perp$er lines, $m_1m_2 + n_1n_2 + l_1l_2 = 0$

$$\Rightarrow \frac{cg}{bh} + 1 + \frac{cf}{ah} = 0$$

$$\Rightarrow \boxed{\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0}$$ [on multiplying by $\frac{h}{c}$]

shown

For $\parallel$ lines, direction ratios of lines would be equal.

$D = 0$ of (3)

$$\Rightarrow (bg + hc - af)^2 - 4bhcg = 0$$

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$$\Rightarrow bg + hc - af = \pm 2\sqrt{bg}\sqrt{hc}$$

$$\Rightarrow bg + hc \pm 2\sqrt{bg}\sqrt{hc} = af$$

$$\Rightarrow (\sqrt{bg} \pm \sqrt{hc})^2 = (\pm\sqrt{af})^2$$

$$\Rightarrow \boxed{\pm\sqrt{bg} \pm \sqrt{ch} + \sqrt{af} = 0}$$ — shown

2. (a) (i) Define the linear span, L(S), of a subset S of a vector space V(F). What is L(S) if (i) $S = \{0\}$, if (ii) $S = \phi$. Let $V = R^3 (R)$, $S = \{\alpha_1 = (1, 1, 0), \alpha_2 = (0, -1, 1), \alpha_3 = (1, 0, 1)\}$. Prove that $(a, b, c,) \in L(S)$ if and only if $a = b + c$.

(ii) Let W be the subspace of $\mathbf{R}^4$ generated by vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$ and $(3, 8, -3, -5)$. Find a basis and dimension of W. Extend this basis of W to a basis of $\mathbf{R}^4$. **[17]**

2 (a)

Defⁿ of L(S) ??

(i) ~~linear span L(S) is set of vectors formed by~~

??

~~1) these vectors are independent~~
~~2) these vectors form all the vector of set S.~~

vector set of S.

L(S) if $S = \{0\}$ is 0.

L(S) if $S = \phi$ is ϕ

Now

let $(a,b,c) \in L(S)$ and let $k_1, k_2, k_3 \in F$

then $(a,b,c) = k_1(1,1,0) + k_2(0,-1,1) + k_3(1,0,1)$

$\Rightarrow a = k_1 + k_3, \; b = k_1 - k_2, \; c = k_2 + k_3$

$\Rightarrow \boxed{a = b + c}$

(ii) writing given vectors ~~matrices~~ into matrices form

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \underset{R_3 \to R_3 - 3R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix}$$

$$\sim \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of $W = \{(1,-2,5,-3),\ (0,7,-9,2)\}$

and dimension of W is 2.

Extenting the basis to a basis of R^4 gives

$\{(1,-2,5,-3),\ (0,7,-9,2),\ (0,0,1,0),\ (0,0,0,1)\}$

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2. (b) (i) Evaluate : $\iint_D x\sin(x+y)\,dxdy$, where D is the region bounded by $0 \le x \le \pi$ and $0 \le y \le \frac{\pi}{2}$.

(ii) If $w = f[xy / (x^2 + y^2)]$ is a differentiable function of $u = f[xy / (x^2 + y^2)]$ show that $x\left(\frac{\partial w}{\partial x}\right) + y\left(\frac{\partial w}{\partial y}\right) = 0$. **[18]**

(i) Put $x+y = u$, $x = uv$ gives $y = u(1-v)$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & +u \\ 1-v & -u \end{vmatrix} = -u$$

and $0 \le uv \le \pi$ and $0 \le u(1-v) \le \pi/2$

$$\iint_D x[\sin(x+y)]\,dx\,dy = \int\int uv \sin u \cdot u \,du\,dv$$

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(ii) $w = f\left(xy / x^2+y^2\right)$

$$\frac{\partial w}{\partial x} = f'\left(\frac{xy}{x^2+y^2}\right)\cdot \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \frac{-x^2y+y^3}{(x^2+y^2)^2}f'$$

$$\frac{\partial w}{\partial y} = f'\left(\frac{xy}{x^2+y^2}\right)\cdot \frac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \frac{-xy^2+x^3}{(x^2+y^2)^2}f'$$

Now, $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = f'\left[\frac{-x^3y+y^3x-xy^3+x^3y}{(x^2+y^2)^2}\right] = 0$

Hence $\boxed{x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = 0}$

shown

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2. (c) Show that the equation of the plane containing the line

$$\frac{y}{b}+\frac{z}{c}=1, x=0$$

and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$

$$\text{is } \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

and if 2d is the S.D. show that $d^{-2} = a^{-2} + b^{-2} + c^{-2}$. **[15]**

Equation of plane containing the lines $y/b + \frac{z}{c} = 1$, $x=0$ is

$$\lambda x + \frac{y}{b} + \frac{z}{c} - 1 = 0 \quad \text{—(1)}$$

Now plane (1) is parallel to line

$$\frac{x-a}{a} = \frac{y}{0} = \frac{z}{c} \quad \text{— (2)}$$

then,

$$\lambda \cdot a + 0 \cdot 1/b + c \cdot \frac{1}{c} = 0$$

$$\Rightarrow \lambda a = -1$$

$$\Rightarrow \lambda = -1/a$$

Hence equation of plane would be

$$\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$$

Now, the shortest distance between the line and plane is 2d

then $$2d = \frac{\left|\frac{a}{a} - \frac{0}{b} - \frac{0}{c} + 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow d^2 = \frac{1}{1/a^2 + 1/b^2 + 1/c^2}$$

$\Rightarrow \boxed{d^{-2} = a^{-2} + b^{-2} + c^{-2}}$ shown

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3. (a) (i) Show that the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

satisfies Cayley-Hamilton theorem. Hence or otherwise obtain the value of A^{-1} and A^{-2}.

(ii) Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$, defined by $T(x, y, z) = (x + y, 2z)$. Find the matrix of T with respect to the bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u}'_1, \mathbf{u}'_2\}$ of $\mathbf{R}^3$ and $\mathbf{R}^2$, where

$\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 4)$, $\mathbf{u}_3 = (1, 2, 3)$ and $\mathbf{u}'_1 = (1, 0)$, $\mathbf{u}'_2 = (0, 2)$. Use this matrix to find the image of the vector $\mathbf{u} = (2, 3, 5)$. **[20]**

SECTION – B

5. (a) Solve $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$. **[10]**

This is of the form $M\,dx + N\,dy = 0$

Where $M = xy^2 - x^2$ and $N = 3x^2y^2 + x^2y - 2x^3 + y^2$

Here $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^2 - x^2}\left[6xy^2 + 2xy - 6x^2 - 2xy\right] = 6$

thus $I.F. = e^{\int 6\,dy} = e^{6y}$

Thus, multiplying by I.F.

$$\int e^{6y}(xy^2 - x^2)\,dx + \int e^{6y}(y^2)\,dy = C$$

keeping y constant. term not containing x where C is integration constant

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$$\Rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) + \frac{y^2e^{6y}}{6} - \frac{1}{6}\int 2ye^{6y} = C$$

$$\Rightarrow e^{6y}\left(\frac{x^2y^2}{2} - \frac{x^3}{3}\right) + \frac{e^{6y}y^2}{6} - \frac{1}{18}ye^{6y} + \frac{1}{108}e^{6y} = C \text{ Ans}$$

5. (b) Find the orthogonal trajectories of the family of curves $r = c(\cos\theta + \sin\theta)$, where c is the parameter. [10]

Given $r = c(\cos\theta + \sin\theta)$

Taking log both sides

$$\log r = \log c + \log(\cos\theta + \sin\theta)$$

Differentiating the equation w.r.t. θ

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta + \cos\theta}{\cos\theta + \sin\theta}$$

For orthogonal trajectory, replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\Rightarrow \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-\sin\theta + \cos\theta}{\cos\theta + \sin\theta}$$

$$\Rightarrow \frac{\cos\theta + \sin\theta}{\sin\theta - \cos\theta}\, d\theta = \frac{dr}{r}$$

Integrating,

$\log(\sin\theta - \cos\theta) = \log r + \log C$ where $\log C$ is any arbitrary constant

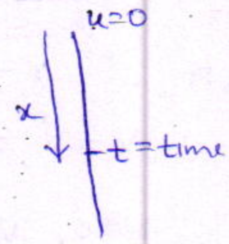
$\Rightarrow \boxed{r = C(\sin\theta - \cos\theta)}$

↳ This is required orthogonal trajectory.

-09-

5. (c) A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t. **[10]**

According to question,



$$\frac{d^2x}{dt^2} = mg - \mu v \quad -①$$

$$\Rightarrow v\frac{dv}{dx} = mg - \mu v \quad \left[\because \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt}\right]$$

$$\Rightarrow \frac{v\,dv}{mg - \mu v} = dx$$

$$\Rightarrow \frac{1}{\mu}\left(\frac{mg}{mg-\mu v} - 1\right)dv = dx$$

Integrating

$$\Rightarrow -\frac{mg}{\mu^2}\log|mg - \mu v| - \frac{v}{\mu} = x + C \quad -②$$

$\Rightarrow$ Now at $v=0,\ x=0$

(2) gives $-\frac{mg}{\mu^2}\log mg = C \quad -③$

Using (3) in (2)

$$x = \frac{mg}{\mu^2}\log\frac{mg}{mg-\mu v} - \frac{v}{\mu} \quad \text{--- (4)}$$

(1) can be written as

$$\frac{dv}{dt} = mg - \mu v \Rightarrow \frac{dv}{mg-\mu v} = dt$$

Integrating, $-\frac{1}{\mu}\log|mg-\mu v| = t + C_1$ where C_1 is constt --- (5)

Now at $t=0$, $v=0$

$C = -\frac{1}{\mu}\log mg$

Putting in (5)

$$\frac{1}{\mu}\log\frac{mg}{mg-\mu v} = t \Rightarrow v = \frac{mg}{\mu}(1-e^{-\mu t}) \quad \text{--- (6)}$$

Using (6) in (4)

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$$x = \frac{mg}{\mu}t - \frac{mg}{\mu^2}(1-e^{-\mu t})$$

$$\boxed{x = \frac{mg}{\mu^2}\left[\mu t - 1 + e^{-\mu t}\right]}$$

5. (d) A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$ where ω is a constant; show that (i) the velocity of the particle is perpendicular to $\mathbf{r}$, (ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin, (iii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$ is a constant vector. **[10]**

(i) $r = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

$v = \dot{r} = \omega(-\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$

Now, $v\cdot r = -\cos\omega t\sin\omega t + \cos\omega t\sin\omega t = 0$

$\Rightarrow$ v and r are perpendicular

(ii) $a = \ddot{r} = \omega^2(-\cos\omega t\,\hat{i} - \sin\omega t\,\hat{j}) = -r\omega^2$

clearly acceleration is directed towards origin and mag. proportional to distance from origin

(iii) $r \times \frac{dr}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{vmatrix}$

$= \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(\omega\cos^2\omega t + \omega^2\sin^2\omega t)$

$= \underline{\omega}$ (a constant vector)

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5. (e) (i) Prove that r^n **r** is an irrotational vector for any value of n but is solenoidal only if $n + 3 = 0$.

(ii) If **a** is constant vector, prove that div $\{r^n(\mathbf{a} \times \mathbf{r})\} = 0$. **[10]**

(i) For irrotational vector $\nabla \times F = 0$ where F is any vector

$\nabla \times (r^n \vec{r}) = \cancel{\hat{i} \times \frac{\partial}{\partial x}}\ \nabla r^n \times \vec{r} + r^n(\nabla \times \vec{r})$ $[\nabla \times (\phi F) = \nabla\phi \times F + \phi(\nabla \times F)]$

$= n r^{n-1} \frac{\vec{r}}{r} \times \vec{r} + r^n(0)$

$= n r^{n-2}(0) + r^n(0)$

$= 0$

Hence $r^n \vec{r}$ is irrotational for any value of n

Now for solenoidal,

$$\nabla \cdot V = 0$$

Now, $\nabla \cdot (r^n \vec{r}) = \nabla r^n \cdot \vec{r} + r^n \nabla \cdot \vec{r}$ $[\because \nabla \cdot (\phi F) = \nabla\phi \cdot F + \phi \nabla \cdot F]$

$$= n r^{n-1} \frac{\vec{r}}{r} \cdot \vec{r} + r^n (3)$$

$$= n r^{n-2+2} + r^n (3)$$

$$= (n+3) r^n$$

Now, it is clear ~~that~~ $r^n \vec{r}$ is solenoidal when

$n+3=0$

(ii) $\text{div}\{r^n (a \times r)\} = \nabla r^n \cdot (a \times r) + r^n \nabla \cdot (a \times r)$ $[\because \nabla \cdot (\phi F) = \nabla\phi \cdot F + \phi \nabla \cdot F]$

$$= n r^{n-1} \frac{\vec{r}}{r} \cdot [a \times r] + r^n \nabla \cdot \left(\sum i (a_2 z - a_3 y)\right)$$ $[\because a = a_1 i + a_2 j + a_3 k]$

$$= 0 + r^n (0)$$

$$= 0$$

08

Hence $\text{div}[r^n (a \times r)] = 0$

6. (a) Solve the differential equation $y = x - 2ap + ap^2$. Find the singular solution and interpret it geometrically. **[13]**

Given $y = x - 2ap + ap^2$ —(1)

$$\frac{dy}{dx} = 1 - 2a\frac{dp}{dx} + 2ap\frac{dp}{dx}$$

$$\Rightarrow p = 1 - 2a(1-p)\frac{dp}{dx}$$

$$\Rightarrow 0 = (1-p)\left[1 - 2a\frac{dp}{dx}\right]$$

$$\Rightarrow 1 - 2a\frac{dp}{dx} = 0$$

$\Rightarrow x - 2ap = c$ where c is any constant

$$\Rightarrow p = \frac{x+c}{2a} \quad \text{—(2)}$$

Put it in (1), $\boxed{y = x - 2ax + \frac{(x+c)^2}{4a}}$ —(2) $y = x - (x+c) + \frac{(x+c)^2}{4a}$

$$\Rightarrow y = -c + \frac{(x+c)^2}{4a} \quad \text{—(3)}$$

Now p-discriminant from ②

is $4a^2 - 4a(x-y) = 0 \Rightarrow a - x + y = 0$

and c-discriminant gives

$a - x + y = 0$

and it also satisfies ①

Hence $a - x + y = 0$ is singular solution

Geometric interpretation, the line $(a - x + y = 0)$ touches the family of curve given by aforementioned differential equations,

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5. (e) (i) Prove that $r^n \mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n + 3 = 0$.

(ii) If $\mathbf{a}$ is constant vector, prove that $\text{div}\,\{r^n(\mathbf{a} \times \mathbf{r})\} = 0$. **[10]**

6. (b) Solve $(d^4y/dx^4) + 6(d^3y/dx^3) + 11(d^2y/dx^2) + 6(dy/dx) = 20\, e^{-2x} \sin x$. **[10]**

Given

$$(D^4 + 6D^3 + 11D^2 + 6D)\, y = 20 e^{-2x} \sin x \quad -①$$

C.F. of ① is given by

$$D(D^3 + 6D^2 + 11D + 6) = 0$$

$$\Rightarrow D = 0, -1, -2, -3$$

$$C.F. = c_1 e^0 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x}$$ where c_1, c_2, c_3, c_4 are constants

and.

$$P.I = 20 \frac{1}{D(D+1)(D+2)(D+3)} e^{-2x} \sin x$$

$$= 20 e^{-2x} \frac{1}{(D-2)(D-2+1)(D-2+2)(D-2+3)} \sin x$$

$$= 20 e^{-2x} \frac{1}{(D-2)(D-1)D(D+1)} \sin x$$

$$= 20 e^{-2x} \frac{1}{D^4 - 2D^3 - D^2 + 2D} \sin x$$

$$= 20 e^{-2x} \frac{1}{(-1)^2 - 2D(-1) + 1 + 2D} \sin x$$

$$= 20 e^{-2x} \frac{1}{2 + 4D} \sin x$$

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$$= 10 e^{-2x} \frac{(1-2D)}{1 - 4D^2} \sin x$$

$$= \frac{10 e^{-2x}}{5} [\sin x - 2\cos x] = 2 e^{-2x} [\sin x - 2\cos x]$$

$$y = C.F. + P.I. = c_1 e^0 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} + 2e^{-2x} [\sin x - 2\cos x]$$ Ans.

6. (b) Solve $(d^4y/dx^4) + 6(d^3y/dx^3) + 11(d^2y/dx^2) + 6(dy/dx) = 20\ e^{-2x} \sin x$. **[10]**

6. (c) Solve y″ + 3y′ + 2y = x + cos x by the method of variation of parameters. **[12]**

Given $y'' + 3y' + 2y = x + \cos x \quad -(1)$

Taking C.F. of (1)

$$(D^2 + 3D + 2)y = 0$$

$$\Rightarrow (D+2)(D+1)y = 0$$

$$\text{C.F. of (1)} = c_1 e^{-2x} + c_2 e^{-x}$$ where c_1 and c_2 are constant.

let $u = e^{-2x}$ and $v = e^{-x}$

P.I. of (1) would be given by.

$$u f(x) + v\, g(x)$$

where

$$f(x) = \int -\frac{vR}{W}dx = \int \frac{-e^{-x}(x+\cos x)}{uv_1 - u_1 v}dx$$

$$= \int -\frac{e^{-x}(x+\cos x)}{e^{-3x}}dx$$

$$= \int -e^{+2x}(x+\cos x)dx$$

$$= \int -x e^{+2x}dx - \int e^{+2x}\cos x\, dx$$

$$= -\frac{x e^{+2x}}{2} + \frac{e^{+2x}}{4} - \frac{e^{+2x}}{5}(+2\cos x + \sin x)$$

and $g(x) = \int \frac{uR}{W}dx = \int \frac{e^{-2x}(x+\cos x)}{e^{-3x}}dx$

$$= \int x e^{+x}dx + \int e^{x}\cos x\, dx$$

$$= \frac{x e^{x}}{1} - \frac{e^{x}}{1} + \frac{e^{x}}{2}(\cos x + \sin x)$$

thus,

$$y = c_1 e^{-2x} + c_2 e^{-x} + \left[-\frac{x}{2} + \frac{1}{4} - \frac{1}{5}(2\cos x + \sin x)\right]$$

$$+ \left[x - 1 + \frac{1}{2}(\cos x + \sin x)\right]$$

$$\therefore y = c_1 e^{-2x} + c_2 e^{-x} + \frac{x}{2} - \frac{3}{4} + \frac{1}{10}\cos x + \frac{3}{10}\sin x$$

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6. (d) (i) Show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$. by using Laplace transform method.

(ii) By using Laplace Transform Method solve the initial value problem $(D^3 - D^2 - D + 1) y = 8t e^{-t}$
$y = D^2 y = 0$, $Dy = 1$ when $t = 0$. **[15]**

(i) $\int_0^\infty e^{-st} \sin t \, dt = \frac{1}{s^2+1} = \mathcal{L}\{\sin t\}$

then $\mathcal{L}\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{u^2+1} du = \tan^{-1}u \Big]_s^\infty$
$= \frac{\pi}{2} - \tan^{-1}s$

Now, Put $s=0$

$\int_0^\infty e^{-(0)t} \frac{\sin t}{t} = \frac{\pi}{2} - \tan^{-1}0$

$\Rightarrow \boxed{\int_0^\infty \frac{\sin t}{t} = \pi/2}$ shown

(ii) Given

$$(D^3 - D^2 - D + 1)\, y = 8te^{-t}$$

Taking Laplace transform

$$L\{D^3 y\} - L\{D^2 y\} - L\{Dy\} + L\{y\} = L\{8te^{-t}\}$$

$$\Rightarrow \left[s^3F(s) - s^2 f(0) - sf'(0) - f''(0)\right] - \left[s^2F(s) - sf(0) - f'(0)\right] - \left[sF(s) - f(0)\right] + F(s) = \frac{8}{(s+1)^2}$$

$$\Rightarrow [s^3 - s^2 - s + 1]F(s) - s + 1 = \frac{8}{(s+1)^2} \qquad [\text{let } L\{y\} = F(s)]$$

$$\Rightarrow (s^3 - s^2 - s + 1)F(s) = s - 1 + \frac{8}{(s+1)^2}$$

$$\Rightarrow F(s) = \frac{1}{(s^2-1)} + \frac{8}{(s+1)^2(s-1)(s^2-1)}$$

$$\Rightarrow F(s) = \frac{1}{s^2-1} + \frac{8}{(s+1)^3(s-1)^2}$$

Taking Laplace inverse

$$\Rightarrow y(t) = L^{-1}\left\{-\frac{1}{2(s+1)} + \frac{1}{2(s-1)}\right\} + L^{-1}\left\{\frac{8}{(s+1)^3(s-1)^2}\right\}$$

$$\Rightarrow y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} + ?$$

Please complete it.

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7. (a) A uniform beam of length 2a rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\alpha > \beta$), show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position. [15]

8. (a) Find the curvature and torsion of the circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$. **[12]**

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + a\theta\cot\alpha\,\hat{k}$

$\Rightarrow \dot{r}\,(=v) = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\alpha\,\hat{k}$

$\Rightarrow \ddot{r}\,(=a) = -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}$

$\Rightarrow \dddot{r}\,(=a') = +a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j}$

and $\dot{r}\times\ddot{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & a\cos\theta & a\cot\alpha \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix} = \hat{i}(a^2\sin\theta\cot\alpha) + \hat{j}(-a^2\cos\theta\cot\alpha) + \hat{k}(a^2)$

Now, K (curvature) $= \dfrac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3}$

$$= \frac{\sqrt{(a^2\sin\theta\cot\alpha)^2 + (a^2\cos\theta\cot\alpha)^2 + (a^2)^2}}{(a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\alpha)^{3/2}}$$

$$= \frac{\sqrt{a^4\cot^2\alpha + a^4}}{(a^2 + a^2\cot^2\alpha)^{3/2}}$$

$$= \frac{1}{a}\,\frac{\operatorname{cosec}\alpha}{\operatorname{cosec}^3\alpha}$$

$$\boxed{K = \frac{\sin^2\alpha}{a}}$$

and τ (torsion) $= \dfrac{(\dot{r}\times\ddot{r})\cdot\dddot{r}}{|\dot{r}\times\ddot{r}|^2}$

$$= \frac{a^3\sin^2\theta\cot\alpha + a^3\cos^2\theta\cot\alpha}{a^4\operatorname{cosec}^2\alpha}$$

$$= \frac{a^3\cot\alpha}{a^4\operatorname{cosec}^2\alpha}$$

$$\tau = \frac{\cos\alpha\sin\alpha}{a} = \frac{\sin 2\alpha}{2a}$$

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8. (b) Evaluate $\oint_C (x^2+y^2)dx + 3xy^2 dy$

where C, a circle of radius two with centre at the origin of the xy plane, is traversed in the positive sense. **[10]**

$C \equiv x^2+y^2=4$

$\oint_C (x^2+y^2)\,dx + 3xy^2\,dy$. This is of the form $M\,dx + N\,dy$

Applying Green's theorem

$$\oint_C (x^2+y^2)\,dx + 3xy^2\,dy = \iint (3y^2-2y)\,dx\,dy$$

Put $x = r\cos\theta$, $y = r\sin\theta$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\sin^2\theta \; r\,dr\,d\theta \quad [\iint y\,dx\,dy = 0 \text{ since it is odd}]$$

$$= \frac{3}{4}\left[r^4\right]_{r=0}^{2} \int_{\theta=0}^{2\pi} \sin^2\theta\,d\theta$$

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8. (d) Verify Stoke's theorem for the vector

$A = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary. **[15]**

Stoke's theorem states

$$\int_C F \cdot dr = \iint_S \text{curl} F \cdot N\, ds$$

$$\int_C A\, dr = \int_C (3y\hat{i} - xz\hat{j} + yz^2\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

Now $C \equiv 4 = x^2 + y^2$

Use $x = 2\cos\theta,\ y = 2\sin\theta,\ z = 2$

$$= -\int_0^{2\pi} 6\sin\theta\,(-2\sin\theta\, d\theta) - (4\cos\theta)(2\cos\theta)\, d\theta$$

[integration would be clockwise]

$$= -\int_0^{2\pi} -12\sin^2\theta\, d\theta - 8\cos^2\theta\, d\theta$$

$$= -4\left[-\int_0^{\pi/2} 12\sin^2\theta\, d\theta - 8\int_0^{\pi/2}\cos^2\theta\, d\theta\right]$$

$$= -4\left[-12\cdot\frac{\pi}{4} - 8\cdot\frac{\pi}{4}\right]$$

$$= +20\pi$$

Now,

$$\text{curl} A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= (z^2 + x)\hat{i} + (0 - 0)\hat{j} + (-z - 3)\hat{k}$$

$$= (z^2 + x)\hat{i} - (z + 3)\hat{k}$$

Now, $\text{div}(\text{curl} A) = 1 - 1 = 0$

By Gauss theorem,

$$\iint_S \text{curl}\, A\cdot N\,ds + \iint_{S'} \text{curl}\, A\cdot N\,ds = 0$$

$$\Rightarrow \iint_S \text{curl}\, A\cdot N\,ds = -\iint (\text{curl}\, A)\cdot\hat{k}\,ds$$

$$= +\iint (z+3)\,dx\,dy$$

$$= 5\iint dx\,dy$$

$$= 5(\pi 2^2)$$

$$= 20\pi$$

Hence $\boxed{\int_C F\cdot dr = \iint_S \text{curl}\, A\cdot N\,ds = 20\pi}$

Verified

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ROUGH SPACE