**Q3 (c)** State Stokes' theorem. Verify this theorem for $\vec{F}(x, y, z) = xz\hat{i} - y\hat{j} + x^2y\hat{k}$ where the surface $S$ is the surface of the region bounded by $x = 0$, $y = 0$, $z = 0$, $2x + y + 2z = 8$ which is not included on the $xz$-plane.

*Solution:* Stokes Theorem

Let $S$ be a surface bounded by a closer curve $c$ then

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS = \int_C \vec{F} \cdot d\vec{R}$$

where $\vec{F}$ is vector point function having continuous first order partial derivatives.

Verification of Stokes theorem on the example:

$\int_C (xz\,\hat{i} - y\,\hat{j} + x^2 y\,\hat{k}) \cdot d\vec{R}$ where $C$ is curve consisting of straight lines $AO$, $OD$ and $DA$ given below

$$\int_{AO+OD+DA} (xzdx - ydy + x^2 ydz)$$

On $AO$, $z = 0$ and $y = 0$

$$\therefore \quad \int_{AO} xzdx - ydy + x^2 ydz = 0$$

On $OD$; $x = 0$, $y = 0$

$$\therefore \quad \int_{OD} = 0$$

On $DA$; $x + z = 4$ and $y = 0$

$$\therefore \quad \int_{DA} xzdx = \int_4^4 x(4 - x)dx = \left(\frac{4x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^4 = \frac{32}{3}$$

So total integral $= \frac{32}{3}$

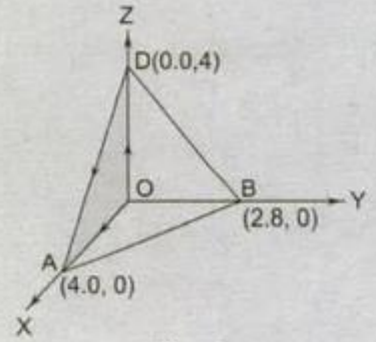



**Fig. 3**

Now, using Stokes Theorem: $S$ contains three planes, i.e., $OAB$, $OBD$ and $ABD$.

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS = \iint_{OAB} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS$$

$$+ I_1 \iint_{OBD} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS + \iint_{ABD} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, dS$$

$$I_1 = I_2 \iint (x^3\, \hat{\imath} + x(1-2y)\, \hat{\jmath}) \cdot I_3 - \hat{k}\ dx\, dy = 0$$

$$I_2 = \iint (x^2\, \hat{\imath} + x(1-2y)\, \hat{\jmath}) \cdot (-\hat{\imath}) dy\ dz \text{ and}$$

put $x = 0$

$= 0$

$$I_3 = \iint (x^2\, \hat{\imath} + x(1-2y)\, \hat{\jmath}) \cdot \frac{\vec{\nabla}(2x + y + 2z)}{|\vec{\nabla}(2x + y + 2z)|} dS$$

$$= \iint \frac{1}{3} (2x^2 + x(1-2y)) dS\,)$$

$$dS = \frac{dx\, dy}{\hat{n} \cdot (-\hat{\imath})}$$

$$= \frac{1}{2} \int_0^4 \int_0^{8-2x} \left(2x^2 + x(1-2y)\right) dx\, dy = \frac{3}{2} dx\, dy$$

$$= \frac{32}{3}$$ Hence the result is verified.

***Example 43:***

*Show that*

$$I = \iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy) = \tfrac{3}{8}$$

*where S is the outer surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.*

***Solution:***

Let we have

$$\left.\begin{aligned} x &= \cos\theta\cos\phi \\ y &= \cos\theta\sin\phi \\ z &= \sin\theta \end{aligned}\right\}, \quad \begin{aligned} 0 \le \theta \le \pi/2 \\ 0 \le \phi \le \pi/2 \end{aligned}$$

So that

$$\frac{\partial(y, z)}{\partial(\theta, \phi)} = -\cos^2\theta\cos\phi, \quad \frac{\partial(yz, x)}{\partial(\theta, \phi)} = -\cos^2\theta\sin\phi,$$

$$\frac{\partial(x, y)}{\partial(\theta, \phi)} = -\sin\theta\cos\theta.$$

The negative signs show that the correspondence is inverse and so the double integrals are to be taken with negative signs.

Thus, we get

$$I = 3\int_0^{\pi/2} \cos^3\theta\sin\theta\,d\theta \int_0^{\pi/2} \sin\phi\cos\phi\,d\phi = \tfrac{3}{8}. \qquad \textbf{Ans.}$$

## § 12. Invariance.

**Theorem 1.** *Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator* $\nabla$ *remains invariant.*

**Proof.** Let $O$ be the fixed origin. Let $Ox, Oy, Oz$ be one system of rectangular axes and $Ox', Oy', Oz'$ be the other system of rectangular axes. Take $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as unit vectors along $Ox, Oy, Oz$ and $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ as unit vectors along $Ox', Oy', Oz'$. Let $P$ be any point in space whose co-ordinates are $(x, y, z)$ or $(x', y', z')$ with respect to the two systems of axes. Let $l_1, m_1, n_1$ ; $l_2, m_2, n_2$ ; $l_3, m_3, n_3$ be the direction cosines of the lines $Ox', Oy', Oz'$ with respect to the co-ordinate axes $Ox, Oy, Oz$.

The scheme of transformation will be as follows ;

$$\left.\begin{aligned} x' &= l_1x + m_1y + n_1z \\ y' &= l_2x + m_2y + n_2z \\ z' &= l_3x + m_3y + n_3z \end{aligned}\right\} \qquad \ldots(1)$$

Also we know that if $l, m, n$ are the direction cosines of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along co-ordinate axes. Therefore

$$\left.\begin{aligned} \mathbf{i}' &= l_1\mathbf{i} + m_1\mathbf{j} + n_1\mathbf{k} \\ \mathbf{j}' &= l_2\mathbf{i} + m_2\mathbf{j} + n_2\mathbf{k} \\ \mathbf{k}' &= l_3\mathbf{i} + m_3\mathbf{j} + n_3\mathbf{k} \end{aligned}\right\} \qquad \ldots(2)$$

If $V$ is any function (vector or scalar) of $x, y, z$, then

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial V}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial V}{\partial z'}\frac{\partial z'}{\partial x}.$$

$$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

But from (1), $\dfrac{\partial x'}{\partial x} = l_1$, $\dfrac{\partial y'}{\partial x} = l_2$, $\dfrac{\partial z'}{\partial x} = l_3$.

$$\left.\begin{aligned} &\therefore & \frac{\partial}{\partial x} &= l_1\frac{\partial}{\partial x'} + l_2\frac{\partial}{\partial y'} + l_3\frac{\partial}{\partial z'} \\ &\text{Similarly} & \frac{\partial}{\partial y} &= m_1\frac{\partial}{\partial x'} + m_2\frac{\partial}{\partial y'} + m_3\frac{\partial}{\partial z'} \\ && \frac{\partial}{\partial z} &= n_1\frac{\partial}{\partial x'} + n_2\frac{\partial}{\partial y'} + n_3\frac{\partial}{\partial z'} \end{aligned}\right\} \qquad \ldots(3)$$

Multiplying the equations (3) by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively, adding and using the results (2), we get

If V(x, y, z) is vector function invariant with respect to a rotation of axes, then div V is a scalar invariant under this transformation.

$$\left.\begin{aligned} \therefore \quad & \frac{\partial \mathbf{V}}{\partial x}=l_1 \frac{\partial \mathbf{V}'}{\partial x'}+l_2 \frac{\partial \mathbf{V}'}{\partial y'}+l_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{Similarly} \quad & \frac{\partial \mathbf{V}}{\partial y}=m_1 \frac{\partial \mathbf{V}'}{\partial x'}+m_2 \frac{\partial \mathbf{V}'}{\partial y'}+m_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{and} \quad & \frac{\partial \mathbf{V}}{\partial z}=n_1 \frac{\partial \mathbf{V}'}{\partial x'}+n_2 \frac{\partial \mathbf{V}'}{\partial y'}+n_3 \frac{\partial \mathbf{V}'}{\partial z'} \end{aligned}\right\} \quad \ldots(3)$$

Taking dot product of these three equations by **i, j, k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $\quad \text{div } \mathbf{V}=\text{div } \mathbf{V}'.$

**Theorem 4.** *If* **V** *(x, y, z) is a vector function invariant under a rotation of axes, then curl* **V** *is a vector invariant under this rotation.* **[Punjab 1966]**

**Proof.** Proceed exactly in the same manner as in theorem 3.

In place of taking dot product of equations (3), take cross product, We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

or $\quad \text{curl } \mathbf{V}=\text{curl } \mathbf{V}'.$

Prove

$$\begin{aligned} \mathbf{i}' &= l_{11}\mathbf{i}+l_{12}\mathbf{j}+l_{13}\mathbf{k} \\ \mathbf{j}' &= l_{21}\mathbf{i}+l_{22}\mathbf{j}+l_{23}\mathbf{k} \\ \mathbf{k}' &= l_{31}\mathbf{i}+l_{32}\mathbf{j}+l_{33}\mathbf{k} \end{aligned}$$

**Solution**

For any vector **A**, we have $\mathbf{A}=(\mathbf{A}\cdot\mathbf{i})\mathbf{i}+(\mathbf{A}\cdot\mathbf{j})\mathbf{j}+(\mathbf{A}\cdot\mathbf{k})\mathbf{k}$.
Then, letting $\mathbf{A}=\mathbf{i}', \mathbf{j}'$, and $\mathbf{k}'$ in succession,

$$\begin{aligned} \mathbf{i}' &= (\mathbf{i}'\cdot\mathbf{i})\mathbf{i}+(\mathbf{i}'\cdot\mathbf{j})\mathbf{j}+(\mathbf{i}'\cdot\mathbf{k})\mathbf{k}=l_{11}\mathbf{i}+l_{12}\mathbf{j}+l_{13}\mathbf{k} \\ \mathbf{j}' &= (\mathbf{j}'\cdot\mathbf{i})\mathbf{i}+(\mathbf{j}'\cdot\mathbf{j})\mathbf{j}+(\mathbf{j}'\cdot\mathbf{k})\mathbf{k}=l_{21}\mathbf{i}+l_{22}\mathbf{j}+l_{23}\mathbf{k} \\ \mathbf{k}' &= (\mathbf{k}'\cdot\mathbf{i})\mathbf{i}+(\mathbf{k}'\cdot\mathbf{j})\mathbf{j}+(\mathbf{k}'\cdot\mathbf{k})\mathbf{k}=l_{31}\mathbf{i}+l_{32}\mathbf{j}+l_{33}\mathbf{k} \end{aligned}$$

Suppose $\phi(x, y, z)$ is a scalar invariant with respect to a rotation of axes. Prove that grad $\phi$ is a vector invariant under this transformation.

***Solution***

By hypothesis, $\phi(x, y, z) = \phi'(x', y', z')$. To establish the desired result, we must prove that

$$\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} = \frac{\partial \phi'}{\partial x'}\mathbf{i}' + \frac{\partial \phi'}{\partial y'}\mathbf{j}' + \frac{\partial \phi'}{\partial z'}\mathbf{k}'$$

Using the chain rule and the transformation equations (3) of Problem 4.38, we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \phi'}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial \phi'}{\partial z'}\frac{\partial z'}{\partial x} = \frac{\partial \phi'}{\partial x'}l_{11} + \frac{\partial \phi'}{\partial y'}l_{21} + \frac{\partial \phi'}{\partial z'}l_{31}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial \phi'}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial \phi'}{\partial z'}\frac{\partial z'}{\partial y} = \frac{\partial \phi'}{\partial x'}l_{12} + \frac{\partial \phi'}{\partial y'}l_{22} + \frac{\partial \phi'}{\partial z'}l_{32}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi'}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial \phi'}{\partial y'}\frac{\partial y'}{\partial z} + \frac{\partial \phi'}{\partial z'}\frac{\partial z'}{\partial z} = \frac{\partial \phi'}{\partial x'}l_{13} + \frac{\partial \phi'}{\partial y'}l_{23} + \frac{\partial \phi'}{\partial z'}l_{33}$$

Multiplying these equations by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively, adding, and using Problem 4.39, the required result follows.

Prove $\iiint\limits_V \nabla\phi \, dV = \iint\limits_S \phi\mathbf{n} \, dS$.

In the divergence theorem, let $\mathbf{A} = \phi\mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then

$$\iiint\limits_V \nabla\cdot(\phi\mathbf{C}) \, dV = \iint\limits_S \phi\mathbf{C}\cdot\mathbf{n} \, dS$$

Since $\nabla\cdot(\phi\mathbf{C}) = (\nabla\phi)\cdot\mathbf{C} = \mathbf{C}\cdot\nabla\phi$ and $\phi\mathbf{C}\cdot\mathbf{n} = \mathbf{C}\cdot(\phi\mathbf{n})$,

$$\iiint\limits_V \mathbf{C}\cdot\nabla\phi \, dV = \iint\limits_S \mathbf{C}\cdot(\phi\mathbf{n}) \, dS$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C}\cdot\iiint\limits_V \nabla\phi \, dV = \mathbf{C}\cdot\iint\limits_S \phi\mathbf{n} \, dS$$

and since $\mathbf{C}$ is an arbitrary constant vector,

$$\iiint\limits_V \nabla\phi \, dV = \iint\limits_S \phi\mathbf{n} \, dS$$

Prove $\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS$.

In the divergence theorem, let $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is a constant vector. Then

$$\iiint_V \nabla \cdot (\mathbf{B} \times \mathbf{C})\, dV = \iint_S (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{n}\, dS$$

Since $\nabla \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\nabla \times \mathbf{B})$ and $(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{n} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{n}) = (\mathbf{C} \times \mathbf{n}) \cdot \mathbf{B} = \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})$,

$$\iiint_V \mathbf{C} \cdot (\nabla \times \mathbf{B})\, dV = \iint_S \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})\, dS$$

Taking $\mathbf{C}$ outside the integrals,

$$\mathbf{C} \cdot \iiint_V \nabla \times \mathbf{B}\, dV = \mathbf{C} \cdot \iint_S \mathbf{n} \times \mathbf{B}\, dS$$

and since $\mathbf{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS$$

$$\iiint_V [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)]\, dV = \iint_S (\phi \nabla \psi) \cdot d\mathbf{S}$$

This is called *Green's first identity or theorem.*

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S}$$

This is called *Green's second identity or symmetrical theorem.* See Problem 21.

$$\iiint_V \nabla \times \mathbf{A}\, dV = \iint_S (\mathbf{n} \times \mathbf{A})\, dS = \iint_S d\mathbf{S} \times \mathbf{A}$$

Note that here the dot product of Gauss' divergence theorem is replaced by the cross product. See Problem 23.

$$\oint_C \phi\, d\mathbf{r} = \iint_S (\mathbf{n} \times \nabla \phi)\, dS = \iint_S d\mathbf{S} \times \nabla \phi$$

Prove $\iiint\limits_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint\limits_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$.

Let $\mathbf{A} = \phi\nabla\psi$ in the divergence theorem. Then

$$\iiint\limits_V \nabla\cdot(\phi\nabla\psi)\, dV = \iint\limits_S (\phi\nabla\psi)\cdot\mathbf{n}\, dS = \iint\limits_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

But $\quad \nabla\cdot(\phi\nabla\psi) = \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi) = \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)$

Thus $\quad \iiint\limits_V \nabla\cdot(\phi\nabla\psi)\, dV = \iiint\limits_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\, dV$

or

$$(1) \qquad \iiint\limits_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\, dV = \iint\limits_S (\phi\nabla\psi)\cdot d\mathbf{S}$$

which proves *Green's first identity*. Interchanging $\phi$ and $\psi$ in *(1)*,

$$(2) \qquad \iiint\limits_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]\, dV = \iint\limits_S (\psi\nabla\phi)\cdot d\mathbf{S}$$

Subtracting *(2)* from *(1)*, we have

$$(3) \qquad \iiint\limits_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint\limits_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{S}$$

Prove $\oint d\mathbf{r} \times \mathbf{B} = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{B}\, dS$.

In Stokes' theorem, let $\mathbf{A} = \mathbf{B} \times \mathbf{C}$ where $\mathbf{C}$ is a constant vector, Then

$$\oint d\mathbf{r} \cdot (\mathbf{B} \times \mathbf{C}) = \iint_S [\nabla \times (\mathbf{B} \times \mathbf{C})] \cdot \mathbf{n}\, dS$$

$$\oint \mathbf{C} \cdot (d\mathbf{r} \times \mathbf{B}) = \iint_S [(\mathbf{C} \cdot \nabla)\mathbf{B} - \mathbf{C}(\nabla \cdot \mathbf{B})] \cdot \mathbf{n}\, dS$$

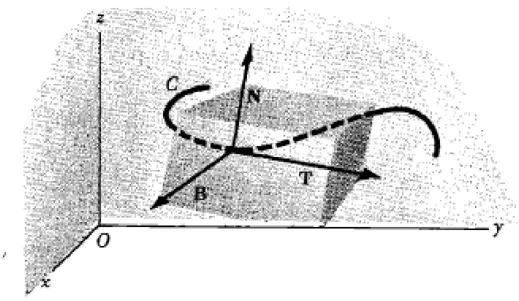
$$\mathbf{C} \cdot \oint d\mathbf{r} \times \mathbf{B} = \iint_S [(\mathbf{C} \cdot \nabla)\mathbf{B}] \cdot \mathbf{n}\, \mathrm{dS} - \iint_S [\mathbf{C}(\nabla \cdot \mathbf{B})] \cdot \mathbf{n}\, dS$$

$$= \iint_S \mathbf{C} \cdot [\nabla(\mathbf{B} \cdot \mathbf{n})]\, dS - \iint_S \mathbf{C} \cdot [\mathbf{n}(\nabla \cdot \mathbf{B})]\, dS$$

$$= \mathbf{C} \cdot \iint_S [\nabla(\mathbf{B} \cdot \mathbf{n}) - \mathbf{n}(\nabla \cdot \mathbf{B})]\, dS = \mathbf{C} \cdot \iint_S (\mathbf{n} \times \nabla) \times \mathbf{B}\, dS$$

Since $\mathbf{C}$ is an arbitrary constant vector $\oint d\mathbf{r} \times \mathbf{B} = \iint_S (\mathbf{n} \times \nabla) \times \mathbf{B}\, dS$

**DIFFERENTIAL GEOMETRY** involves a study of space curves and surfaces. If $C$ is a space curve defined by the function $\mathbf{r}(u)$, then we have seen that $\frac{d\mathbf{r}}{du}$ is a vector in the direction of the tangent to $C$. If the scalar $u$ is taken as the arc length $s$ measured from some fixed point on $C$, then $\frac{d\mathbf{r}}{ds}$ is a unit tangent vector to $C$ and is denoted by $\mathbf{T}$ (see diagram below). The rate at which $\mathbf{T}$ changes with respect to $s$ is a measure of the curvature of $C$ and is given by $\frac{d\mathbf{T}}{ds}$. The direction of $\frac{d\mathbf{T}}{ds}$ at any given point on $C$ is normal to the curve at that point (see Problem 9). If $\mathbf{N}$ is a unit vector in this normal direction, it is called the *principal normal* to the curve. Then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, where $\kappa$ is called the *curvature* of $C$ at the specified point. The quantity $\rho = 1/\kappa$ is called the *radius of curvature*.




A unit vector $\mathbf{B}$ perpendicular to the plane of $\mathbf{T}$ and $\mathbf{N}$ and such that $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, is called the *binormal* to the curve. It follows that directions $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a localized right-handed rectangular coordinate system at any specified point of $C$. This coordinate system is called the *trihedral* or *triad* at the point. As $s$ changes, the coordinate system moves and is known as the *moving trihedral.*

A set of relations involving derivatives of the fundamental vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ is known collectively as the *Frenet-Serret formulas* given by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

where $\tau$ is a scalar called the *torsion*. The quantity $\sigma = 1/\tau$ is called the *radius of torsion.*

The *osculating plane* to a curve at a point $P$ is the plane containing the tangent and principal normal at $P$. The *normal plane* is the plane through $P$ perpendicular to the tangent. The *rectifying plane* is the plane through $P$ which is perpendicular to the principal normal.

Prove the Frenet-Serret formulas (*a*) $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, (*b*) $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, (*c*) $\frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}$.

(*a*) Since $\mathbf{T}\cdot\mathbf{T} = 1$, it follows from Problem 9 that $\mathbf{T}\cdot\frac{d\mathbf{T}}{ds} = 0$, i.e. $\frac{d\mathbf{T}}{ds}$ is perpendicular to $\mathbf{T}$.

If $\mathbf{N}$ is a unit vector in the direction $\frac{d\mathbf{T}}{ds}$, then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$. We call $\mathbf{N}$ the *principal normal*, $\kappa$ the *curvature* and $\rho = 1/\kappa$ the *radius of curvature*.

(*b*) Let $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, so that $\frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds} + \frac{d\mathbf{T}}{ds}\times\mathbf{N} = \mathbf{T}\times\frac{d\mathbf{N}}{ds} + \kappa\mathbf{N}\times\mathbf{N} = \mathbf{T}\times\frac{d\mathbf{N}}{ds}$.

Then $\mathbf{T}\cdot\frac{d\mathbf{B}}{ds} = \mathbf{T}\cdot\mathbf{T}\times\frac{d\mathbf{N}}{ds} = 0$, so that $\mathbf{T}$ is perpendicular to $\frac{d\mathbf{B}}{ds}$.

But from $\mathbf{B}\cdot\mathbf{B} = 1$ it follows that $\mathbf{B}\cdot\frac{d\mathbf{B}}{ds} = 0$ (Problem 9), so that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$ and is thus in the plane of $\mathbf{T}$ and $\mathbf{N}$.

Since $\frac{d\mathbf{B}}{ds}$ is in the plane of $\mathbf{T}$ and $\mathbf{N}$ and is perpendicular to $\mathbf{T}$, it must be parallel to $\mathbf{N}$; then $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. We call $\mathbf{B}$ the *binormal*, $\tau$ the *torsion*, and $\sigma = 1/\tau$ the *radius of torsion*.

(*c*) Since $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form a right-handed system, so do $\mathbf{N}, \mathbf{B}$ and $\mathbf{T}$, i.e. $\mathbf{N} = \mathbf{B}\times\mathbf{T}$.

Then $\frac{d\mathbf{N}}{ds} = \mathbf{B}\times\frac{d\mathbf{T}}{ds} + \frac{d\mathbf{B}}{ds}\times\mathbf{T} = \mathbf{B}\times\kappa\mathbf{N} - \tau\mathbf{N}\times\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B} = \tau\mathbf{B} - \kappa\mathbf{T}$.

**20.** Prove that the radius of curvature of the curve with parametric equations $x = x(s)$, $y = y(s)$, $z = z(s)$ is given by $\rho = \left[\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right]^{-1/2}$.

The position vector of any point on the curve is $\mathbf{r} = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$.

Then $\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}$ and $\frac{d\mathbf{T}}{ds} = \frac{d^2x}{ds^2}\mathbf{i} + \frac{d^2y}{ds^2}\mathbf{j} + \frac{d^2z}{ds^2}\mathbf{k}$.

But $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ so that $\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}$ and the result follows since $\rho = \frac{1}{\kappa}$.

**21.** Show that $\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \frac{\tau}{\rho^2}$.

$\frac{d\mathbf{r}}{ds} = \mathbf{T}$, $\frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\frac{d^3\mathbf{r}}{ds^3} = \kappa\frac{d\mathbf{N}}{ds} + \frac{d\kappa}{ds}\mathbf{N} = \kappa(\tau\mathbf{B} - \kappa\mathbf{T}) + \frac{d\kappa}{ds}\mathbf{N} = \kappa\tau\mathbf{B} - \kappa^2\mathbf{T} + \frac{d\kappa}{ds}\mathbf{N}$

$$\frac{d\mathbf{r}}{ds} \cdot \frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} = \mathbf{T} \cdot \kappa\mathbf{N} \times \left(\kappa\tau\mathbf{B} - \kappa^2\mathbf{T} + \frac{d\kappa}{ds}\mathbf{N}\right)$$

$$= \mathbf{T} \cdot \left(\kappa^2\tau\,\mathbf{N}\times\mathbf{B} - \kappa^3\mathbf{N}\times\mathbf{T} + \kappa\frac{d\kappa}{ds}\mathbf{N}\times\mathbf{N}\right) = \mathbf{T} \cdot (\kappa^2\tau\,\mathbf{T} + \kappa^3\mathbf{B}) = \kappa^2\tau = \frac{\tau}{\rho^2}$$

The result can be written

$$\tau = \left[(x'')^2 + (y'')^2 + (z'')^2\right]^{-1} \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

where primes denote derivatives with respect to $s$, by using the result of Problem 20.

**22.** Given the space curve $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$, find (*a*) the curvature $\kappa$, (*b*) the torsion $\tau$.

(*a*) The position vector is $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + \frac{2}{3}t^3\,\mathbf{k}$.

Then
$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j} + 2t^2\,\mathbf{k}$$

$$\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}} = \sqrt{(1)^2 + (2t)^2 + (2t^2)^2} = 1 + 2t^2$$

and
$$\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{i} + 2t\,\mathbf{j} + 2t^2\,\mathbf{k}}{1 + 2t^2}.$$

$$\frac{d\mathbf{T}}{dt} = \frac{(1 + 2t^2)(2\mathbf{j} + 4t\,\mathbf{k}) - (\mathbf{i} + 2t\,\mathbf{j} + 2t^2\,\mathbf{k})(4t)}{(1 + 2t^2)^2} = \frac{-4t\,\mathbf{i} + (2 - 4t^2)\mathbf{j} + 4t\,\mathbf{k}}{(1 + 2t^2)^2}$$

Then $\dfrac{d\mathbf{T}}{ds} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{-4t\,\mathbf{i} + (2-4t^2)\mathbf{j} + 4t\,\mathbf{k}}{(1+2t^2)^3}$ .

Since $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{\sqrt{(-4t)^2 + (2-4t^2)^2 + (4t)^2}}{(1+2t^2)^3} = \dfrac{2}{(1+2t^2)^2}$

(*b*) From (*a*), $\mathbf{N} = \dfrac{1}{\kappa}\dfrac{d\mathbf{T}}{ds} = \dfrac{-2t\,\mathbf{i} + (1-2t^2)\mathbf{j} + 2t\,\mathbf{k}}{1+2t^2}$

$$\text{Then } \mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{1}{1+2t^2} & \dfrac{2t}{1+2t^2} & \dfrac{2t^2}{1+2t^2} \\ \dfrac{-2t}{1+2t^2} & \dfrac{1-2t^2}{1+2t^2} & \dfrac{2t}{1+2t^2} \end{vmatrix} = \dfrac{2t^2\,\mathbf{i} - 2t\,\mathbf{j} + \mathbf{k}}{1+2t^2}$$

Now $\dfrac{d\mathbf{B}}{dt} = \dfrac{4t\,\mathbf{i} + (4t^2-2)\mathbf{j} - 4t\,\mathbf{k}}{(1+2t^2)^2}$ and $\dfrac{d\mathbf{B}}{ds} = \dfrac{d\mathbf{B}/dt}{ds/dt} = \dfrac{4t\,\mathbf{i} + (4t^2-2)\mathbf{j} - 4t\,\mathbf{k}}{(1+2t^2)^3}$

Also, $-\tau\mathbf{N} = -\tau\left[\dfrac{-2t\,\mathbf{i} + (1-2t^2)\mathbf{j} + 2t\,\mathbf{k}}{1+2t^2}\right]$. Since $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$, we find $\tau = \dfrac{2}{(1+2t^2)^2}$

Note that $\kappa = \tau$ for this curve.