MATHEMATICS

Paper – I

Time Allowed : Three Hours

Maximum Marks : 200

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

*Answers must be written in **ENGLISH** only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y,\ 2x + z,\ x + 2z,\ x + y + z)$.

Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (i.e., (1, 0, 0), (0, 1, 0), etc.). Examine if T is a linear map. *8*

(b) Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$. *8*

(c) Examine if the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at (0, 0). Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin. *8*

(d) If the point (2, 3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord. *8*

(e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$. *8*

Q2. (a) After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin nx \; dx\, dy$, show that $\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$. *10*

(b) A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. *10*

(c) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve. *10*

(d) Let T be a linear map such that $T : V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis $B_1[(1, 1, 0)\ (1, 0, 1)\ (0, 1, 1)]$ and $B_2[(1, 1)\ (1, -1)]$. Obtain the matrix T_1 relative to B_1 and B_2. *10*

Q3. (a) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$. Hence find A^{-1}. *10*

(b) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point $(1, 0, 0)$. *10*

(c) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$. *10*

(d) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle. *10*

Q4. (a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. *10*

(b) Show that the integral $\int_0^\infty e^{-x} x^{\alpha-1}\, dx$, $\alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$. *10*

(c) Prove that $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\, \Gamma\left(z + \frac{1}{2}\right)$. *10*

(d) A plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. *10*

SECTION B

Q5. (a) Obtain the curve which passes through (1, 2) and has a slope $= \frac{-2xy}{x^2+1}$. Obtain one asymptote to the curve. *8*

(b) Solve the dE to get the particular integral of $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$. *8*

(c) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings. *8*

(d) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot \beta = 2 \tan (\alpha - \beta)$. *8*

(e) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \bar{F} \cdot dS$ where S is the surface bounding the volume E and $\bar{F} = (zx \sin yz + x^3)\, \hat{i} + \cos yz\, \hat{j} + (3zy^2 - e^{\lambda^2 + y^2})\, \hat{k}$. *8*

Q6. (a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet ? *10*

(b) Using the method of variation of parameters, solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x .$$ *10*

(c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line. *10*

(d) Evaluate $\iint_S (\nabla \times \bar{f}) \cdot \hat{n}\, dS$ for $\bar{f} = (2x - y)\, \hat{i} - yz^2 \hat{j} - y^2 z\, \hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane. *10*

Q7. (a) State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one. *15*

(b) A uniform rod of weight W is resting against an equally rough horizon and a wall, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W \tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α ? *15*

(c) Obtain the singular solution of the differential equation

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2, \; p = \frac{dy}{dx}.$$ *10*

Q8. (a) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio 3 : 2. *10*

(b) Solve the differential equation

$$\frac{dy}{dx} - y = y^2 (\sin x + \cos x).$$ *10*

(c) Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \times \bar{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$. *10*

(d) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary. *10*