the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance $2b$ apart. A weight W is suspended from A. Find the thrust in the rod BC. **(10)**

(*d*) Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. **(10)**

8. (*a*) A solid hemisphere floating in a liquid is completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and tension of the string. **(15)**

(*b*) A snowball of radius $r(t)$ melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process. **(15)**

(*c*) For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = a^2. \qquad \textbf{(10)}$$

PAPER-II

INSTRUCTIONS: *There are* **eight** *questions in all, out of which* **five** *are to be attempted. Question Nos. 1 and 5 are compulsory. Out of the remaining six questions,* **three** *are to be attempted selecting at least* **one** *question from each of the two Sections A and B. Answers must be written in* **English** *only.*

SECTION–A

1. (*a*) Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. **(8)**

(*b*) A function $f: [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$. **(10)**

(*c*) If $u = (x - 1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. **(10)**

(*d*) Solve by simplex method the following Linear Programming Problem: **(12)**

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to the constraints

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0.$$

2. (*a*) Find all the homomorphisms from the group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$. **(10)**

(*b*) Consider the function f defined by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x^2 + y^2 \ne 0 \\ 0, & \text{where } x^2 + y^2 = 0 \end{cases}$$

Show that $f_{xy} \ne f_{yx}$ at (0, 0). **(10)**

(*c*) Prove that

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}. \qquad \textbf{(10)}$$

(*d*) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. **(10)**

3. (*a*) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. **(10)**

(*b*) Show by an example that in a finite commutative ring, every maximal ideal need not be prime. **(10)**

(*c*) Evaluate the integral $\int_0^{2\pi} \cos^{2n}\theta\, d\theta$, where n is a positive integer. **(10)**

(d) Show that the improper integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ is convergent. **(10)**

4. (a) Show that

$$\iint_R x^{m-1}y^{n-1}(1-x-y)^{l-1}\,dx\,dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)};$$

$l, m, n > 0$

taken over R : the triangle bounded by $x = 0$, $y = 0$, $x + y = 1$. **(10)**

(b) Let $f_n(x) = \frac{x}{n+x^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$. **(8)**

(c) Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. **(10)**

(d) The capacities of three production facilities S_1, S_2 and S_3 and the requirements of four destinations D_1, D_2, D_3 and D_4 and transportation costs in rupees are given in the following table:

	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Find the minimum transportation cost using Vogel's Approximation Method (VAM). **(12)**

SECTION–B

5. (a) Find the partial differential equation of all planes which are at a constant distance a from the origin. **(10)**

(b) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x = 0$ and a curve through the points with the following coordinates:

x	y
0.0	1.0
0.25	0.9896
0.50	0.9589
0.75	0.9089
1.00	0.8415
1.25	0.8029
1.50	0.7635

Estimate the volume of the solid formed using Weddle's rule. **(10)**

(c) Write a program in BASIC to multiply two matrices (checking for consistency for multiplication is required). **(10)**

(d) Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t,

then $\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left\{\rho(v^2+k)\right\}$. **(10)**

6. (a) Find the complete integral of the partial differential equation $(p^2 + q^2)x = zp$ and deduce the solution which passes through the curve $x = 0$, $z^2 = 4y$.

Here $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$. **(12)**

(b) Apply fourth-order Runge-Kutta method to compute y at $x = 0.1$ and $x = 0.2$, given that

$\frac{dy}{dx} = x + y^2$, $y = 1$ at $x = 0$. **(12)**

(c) For a particle having charge q and moving in an electromagnetic field, the potential energy is $U = q(\phi - \vec{v}.\vec{A})$, where ϕ and $\vec{A}$

are, respectively, known as the scalar and vector potentials. Derive expression for Hamiltonian for the particle in the electromagnetic field. **(8)**

(*d*) Write a program in BASIC to implement trapezoidal rule to compute $\int_0^{10} e^{-x^2}\,dx$ with 10 subdivisions. **(8)**

7. (*a*) Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$, where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$.

If the solution of the above equation represents a sphere, what will be the coordinates of its centre? **(8)**

(*b*) The velocity v (km/min) of a moped is given at fixed interval of time (min) as below:

t	v
0.1	1.00
0.2	1.104987
0.3	1.219779
0.4	1.34385
0.5	1.476122
0.6	1.615146
0.7	1.758819
0.8	1.904497
0.9	2.049009
1.0	2.18874
1.1	2.31977

Estimate the distance covered during the time (use Simpson's one-third rule). **(10)**

(*c*) Assuming a 16-bit computer representation of signed integers, represent –44 in 2's complement representation. **(10)**

(*d*) In the case of two-dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction, where u is a variable. Show that the maximum value of the velocity at any point of the fluid is $2u$. Prove that the force necessary to hold the disc is $2mu$, where m is the mass of the liquid displaced by the disc. **(12)**

8. (*a*) Find a real function V of x and y, satisfying

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2)$$ and reducing to zero, when $y = 0$. **(10)**

(*b*) The equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between 1.4 and 1.5. Find the root to four places of decimal by regula-falsi method. **(10)**

(*c*) A particle of mass m is constrained to move on the inner surface of a cone of semi-angle α under the action of gravity. Write the equation of constraint and mention the generalized coordinates. Write down the equation of motion. **(10)**

(*d*) Two sources, each of strength m, are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter.

Show also that the fluid speed at any point is $(2ma^2)/(r_1 r_2 r_3)$, where r_1, r_2, r_3 are the distances of the point from the sources and the sink. **(10)**

Paper-II

1.(a) Let $|G| = 2n$, and let $\Omega = \{x \in G \mid x \neq x^{-1}\} \subseteq G$, *i.e.*, Ω is the set of all elements of G which do not have order 1 or 2. Note that if $x \in \Omega$, then $x^{-1} \in \Omega$, and $x \neq x^{-1}$ by the hypothesis that $x \in \Omega$. Hence, Ω has even order, so $G \backslash \Omega$ has even order. Since $e \in G\backslash\Omega$, $G\backslash\Omega$ has at least order 2, whence G must have an element of order 2.

1.(b) Let's apply the IVT to the function g, on the interval $[a, b] = [0, 1]$

We are especially interested in showing that the function g has a zero in its domain. Therefore we apply the IVT with the intention of using a value of k equal to 0 (as per your statement of the IVT).

But $g(0) = f(0) - 0 = f(0)$, with the first equality by the definition of g.

Also $g(1) = f(1) - 1$.

But since f has codomain $[0, 1]$, it must be $f(1) \leq 1$. If the equality stands, then the proof is over: the point $x = 1$ is such that $f(x) = x$.

If the inequality is strict, that is, $f(1) < 1$, then $g(1) < 0$.

Similarly, $g(0) = f(0) \geq 0$, because f has codomain $[0, 1]$. If the equality stands, we are done, just as before: $f(0) = 0$.

Otherwise, $g(0)$ is positive, and finally we can apply the IVT:

The function g is such that $g(0) > 0$ and $g(1) < 0$, and is continuous because it is sum of continuous functions, therefore a number c must exist, such that $0 < c < 1$, and $g(c) = 0$.

But if $g(c) = 0$, then by the definition of g, $f(c) - c = 0$, and rearranging, $f(c) = c$.

1.(c) $u = (x - 1)^3 - 3xy^2 + 3y^2$

By Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$= \frac{\partial}{\partial x}\left\{(x-1)^3 - 3xy^2 + 3y^2\right\}$$

$$= 3(x - 1)^2 - 3y^2$$

$$\therefore \quad \frac{\partial v}{\partial y} = 3(x - 1)^2 - 3y^2$$

$$v = \int\left\{3(x-1)^2 - 3y^2\right\} dy$$

$$v = 3(x - 1)^2.y - y^3 + f(x)$$

$$\frac{\partial v}{\partial x} = 6(x - 1)y + f'(x)$$

$$= 6xy - 6y + f'(x)$$

$$\text{and} \quad -\frac{\partial u}{\partial y} = -(-6xy + 6y) = 6xy - 6y$$

$$\text{From} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$6xy - 6y + f'(x) = 6xy - 6y$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad f(x) = c \text{ (constant)}$$

$$\therefore \quad v = 3(x - 1)^2y - y^3 + c.$$

1.(d) The given problem is ease of maximization all values of b_i's ($i = 1, 2, 3$) are positive. By introducing slack variables convert the problem into standard form and inequality into equality; we have

$$\text{Max } Z = 3x_1 + 2x_2 + 5x_3 + 0S_1 + 0S_2 + 0S_3$$

$$\text{S.T.} \quad x_1 + 2x_2 + x_3 + S_1 = 430$$

$$3x_1 + 0x_2 + 2x_3 + S_2 = 460$$

$$x_1 + 4x_2 + 0x_3 + S_3 = 420$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Initial basic feassible solution is given by

$x_1 = x_2 = x_3 = 0$

$S_1 = 430, S_2 = 460, S_3 = 420$

Now prepare initial simplex table

Initial simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3
0	S_1	430	1	2	1	1	0	0
← 0	S_2	460	3	0	$\boxed{2}$	0	1	0
0	S_3	420	1	4	0	0	0	1
	Z_j	0	0	0	0	0	0	0
		$Z_j - C_j$	–3	–2	–5 ↑	0	0	0

Here all values of $Z_j - C_j$ are not positive. Hence, solution is not optimum. To find optimum solution select the most negative number. Here –5 is the most negative number it will enter in basis. Corresponding column will be treated as key column. To find key row, find

$$\min\left(\frac{X_B}{x_3}\right) = \min = \left(\frac{430}{1}, \frac{460}{2}, \frac{420}{0}\right) = 230$$

∴ The basic variable S_2 will leave the basis.

$\boxed{2}$ is the key element make it unity and other element of key column zero by matrix row transformation. Now we have first simplex table.

First simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3
0	S_1	200	$\frac{-1}{2}$	$\boxed{2}$	0	1	1/2	0
5	x_3	230	$\frac{3}{2}$	0	1	0	1/2	0
0	S_3	420	1	4	0	0	0	1
	Z_j	1150	15/2	0	5	0	5/2	0
		$Z_j - C_j$	$\frac{9}{2}$	–2 ↑	0	0	5/2	0

Further, all values of $Z_j - C_j$ are not positive. Hence, repeat the above process.

$\boxed{2}$ is the key element. Make it unity and other elements of key column zero by applying matrix row transformation, we have the second simplex table.

Second simplex table

C_B	B	C_j / X_B	3 / x_1	2 / x_2	5 / x_3	0 / S_1	0 / S_2	0 / S_3
2	x_2	100	–1/4	1	0	1/2	–1/4	0
5	x_3	230	3/2	0	1	0	1/2	0
0	s_3	20	2	0	0	–2	1	1
	Z_j	1350	7	2	5	1	2	0
		$Z_j - C_j$	4	0	0	1	2	0

Since all values of $Z_j - C_j \geq 0$.

Hence, the solution is optimum. It is given by

$x_1 = 0$,

$x_2 = 100$,

$x_3 = 230$, Max Z = 1350.

2.(*a*) A homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_4$ is determined by $\phi(1)$ since $\phi(n) = n \cdot \phi(1)$ for every $n \in \mathbb{Z}$. Also, for any $a \in \mathbb{Z}_4$, we can get a homomorphism $\mathbb{Z} \to \mathbb{Z}_4$ taking 1 to a by sending n to the reduction mod 4 of an. So, there are four homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_4$, one for each value in $\mathbb{Z}_4$.

If $\phi(1) = 0$, we get the zero map. Its kernel is all of $\mathbb{Z}$ and its image is $\{0\}$.

If $\phi(1) = 1$, our map is just reduction mod 4, which is clearly surjective; that is, its image is all of $\mathbb{Z}_4$. We see that an element is sent to zero if and only if it's a multiple of 4, so $\ker(\phi) = 4\mathbb{Z}$.

If $\phi(1) = 2$, our map takes n to the reduction of $2n$ mod 4. The image is generated by 2, which is $\langle 2 \rangle = \{0, 2\}$. The kernel is the set of elements of n such that $2n$ is a multiple of 4. This is the set of even integers, $2\mathbb{Z}$.

If $\phi(1) = 3$, our map takes n to the reduction of $3n$ mod 4. The image is generated by $\phi(1) = 3$ and so is all of $\mathbb{Z}_4$ (so it's surjective). Since $3n$ is a multiple of 4 if and only if n is a multiple of 4, the kernel is $4\mathbb{Z}$.

2.(b) $f(x, y) = \dfrac{xy(x^2 - y^2)}{(x^2 + y^2)}, = \dfrac{x^3y - xy^3}{(x^2 + y^2)}$

Now,

$$f(x) = \frac{(x^2 + y^2)(3x^2y - y^3) - 2x(x^3y - xy^3)}{(x^2 + y^2)^2}.$$

2.(c)

$$I_2 = \int_0^{\infty} \cos x^2 dx$$

by substituting:

$$t = x^2 \Rightarrow dx = \frac{1}{2\sqrt{|t|}}dt$$

so that:

$$I_2 = \int_0^{\infty} \cos t \frac{1}{2\sqrt{|t|}} dt$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \cos t \frac{1}{2\sqrt{|t|}} dt$$

This is recognized as $\frac{1}{4}$ times the first order Fourier coefficient of:

$$f(t) = \frac{1}{\sqrt{|t|}}$$

with Fourier Transform

$$F(\nu) = \sqrt{\frac{2\pi}{|\nu|}}$$

Since our angular frequency ν equals 1, we obtain

$$I_2 = \frac{1}{4}\sqrt{\frac{2\pi}{1}} = \sqrt{\frac{\pi}{8}}$$

Similarly, $\quad S(z) = \int \sin\left(\dfrac{\pi z^2}{2}\right) dz \quad$ ...(1)

$$\int \sin(x^2)dx = \sqrt{\frac{\pi}{2}} \int \sin\left(\frac{\pi z^2}{2}\right) dz$$

Using expression (1) of the Fresnel integral we get:

$$\sqrt{\frac{\pi}{2}} \int \sin\left(\frac{\pi z^2}{2}\right) dz = \sqrt{\frac{\pi}{2}} S(z)$$

Substituting back:

$$\int \sin(x^2)dx = \sqrt{\frac{\pi}{2}} S\left(\sqrt{\frac{2}{\pi}}x\right) + C$$

On applying the limit from 0 to infinity, we get

$$\int_0^{\infty} \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Hence proved.

2.(d) Suppose that P is a prime ideal. By definition, if $a, b \in R - P$, then $ab \in R - P$. Passing to the quotient ring, we see that $\bar{a}$ and $\bar{b}$ are not $\bar{0}$ in R/P implies that $\overline{ab} \neq 0$ in R/P. Thus, R/P has no zero divisors. The quotient ring R/P contains an identity $\bar{1}$ and inherits commutativity from R. Hence, R/P is an integral domain.

Conversely, suppose that R/P is an integral domain. Let nonzero elements in R/P correspond to $\bar{a}, \bar{b}$ with $a, b \in R - P$. Since R/P is an integral domain, $\overline{ab} \neq \bar{0}$ so $ab \notin P$. We have shown that $a \notin P$ and $b \notin P$ implies that $ab \notin P$. The contrapositive to this is $ab \in P$ implies that $a \in P$ or $b \in P$. This proves that P is a prime ideal.

3.(a) Let $\quad u = x^2 + y^2 + z^2 \quad$ ...(1)

where $\phi(x, y, z) = ax + by + cz - p = 0$ ...(2)

Lagrange's equations for maxima or minima are

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 2x + \lambda a = 0 \quad ...(3)$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 2y + \lambda b = 0 \quad ...(4)$$

$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 2z + \lambda c = 0 \quad ...(5)$$

Multiplying (3) by x, (4) by y, (5) by z and adding, we get

$$2(x^2 + y^2 + z^2) + \lambda(ax + by + cz) = 0$$

or $\qquad 2u + \lambda p = 0$

[Using (1) and (2)]

$$\therefore \qquad \lambda = -\frac{2u}{p}$$

From (3), (4) and (5),

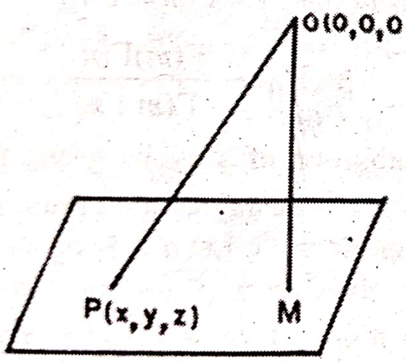
$$x = \frac{au}{p},\ y = \frac{bu}{p},\ z = \frac{cu}{p}$$

$\therefore$ From (1), $\qquad u = \dfrac{(a^2+b^2+c^2)u^2}{p^2}$

or $\qquad u = \dfrac{p^2}{a^2+b^2+c^2}$

This is the maximum or minimum value of u. Now u is the square of the distance of any point P(x, y, z) on the plane (2) from the origin. Also, the length of perpendicular from

O on the plane $= \dfrac{p}{\sqrt{a^2+b^2+c^2}}$



Clearly, OP is least when P coincides with M, the foot of the perpendicular from O on the plane.

Hence minimum value of u

$$= \frac{p^2}{a^2+b^2+c^2}.$$

3.(b) Consider the ring R = {0, 2, 4, 6} under addition and multiplication modulo 8.

Let M = {0, 4} then M is easily seen to be an ideal of R.

Again as $2 \otimes 6 = 4 \in$ M but 2, 6 $\notin$ M, we find M is not a prime ideal. We show M is maximal.

Let M $\subseteq$ N $\subseteq$ R, where N is an ideal of R. Since < M, + > will be a subgroup of <N, +>, by Lagrange's theorem o(M)| o(N). Similarly, o(N)| o(R) = 4

i.e., 2|o(N), o(N)| 4

i.e., $\qquad$ o(N) = 2 or 4

if $\qquad$ o(N) = 2, then M = N as M $\subseteq$ N

if $\qquad$ o(N) = 4, then N = R as N $\subseteq$ R

Hence, M is maximal ideal of R.

Remark: In case the finite commutative ring contains unity, then every prime ideal is maximal.

3.(c)

$$2^n \cos^n x = (e^{ix} + e^{-ix})^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(e^{ix}\right)^k \left(e^{-ix}\right)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)ix} \qquad (*)$$

Since, $\displaystyle\int_0^{2\pi} e^{i\ell x} dx = \begin{cases} 2\pi & (\ell = 0) \\ 0 & (\ell \neq 0) \end{cases}$

at most one term on the right of (*) contributes to the integral $J_n := \int_0^{2\pi} \cos^n x\, dx$.

When n is odd then $2k - n \neq 0$ for all k in (*), therefore $J_n = 0$ in this case. When n is even then $k = n/2$ gives the only contribution to the integral, and we get

$$\int_0^{2\pi} \cos^n x\, dx = \frac{2\pi}{2^n}\binom{n}{n/2}.$$

4.(a) The definitions of Beta and Gamma Functions. We substitute

$x = \cos^2 \theta \Rightarrow dx = -2 \cos \theta \sin \theta\, d\theta$

$$\therefore \qquad \beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$$

$$= -2\int_{0-\pi/2}^{0} \cos^{2m-2} \sin^{2n-2} \theta \cos\theta \sin\theta\, d\theta$$

or $\beta(m, n)$

$$= 2\int_0^{\pi/2} \cos^{2m-2} \theta \sin^{2n-2} \theta\, d\theta \qquad ...(1)$$

Now we substitute

$$t = x^2 \text{ in } G(m)$$

$$= \int_0^\infty t^{m-2} e^{-1} dt$$

$$\therefore \quad \Gamma(m) = \int_0^\infty x^{2m-2} e^{-x^2} .2x\, dx$$

$$= 2\int_0^\infty x^{2m-1} e^{-x^2}\, dx \qquad \text{...(2)}$$

Now, $\Gamma(m)\,\Gamma(n)$

$$= 4\int_0^\infty x^{2m-2} e^{-x^2}\, dx \int_0^\infty y^{2n-1} e^{-y^2}\, dy$$

$$= \lim_{a\to\infty} 4\int_0^a x^{2m-1} e^{-x^2}\, dx \int_0^a y^{2n-1} e^{-y^2}\, dy$$

or $\Gamma(m)\,\Gamma(n)$

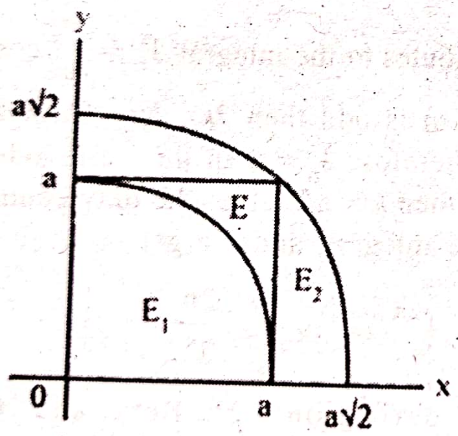
$$= \lim_{a\to\infty} 4\iint_E x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy \qquad \text{...(3)}$$

where E is a square of side a.

It is clear that each diagonal of the square E is of length

$$\sqrt{a^2+a^2} = \sqrt{2}a$$

Let E_1 and E_2 be quarter circles of radii a and $\sqrt{2}a$ respectively.



Then

$$= 4\iint_{E1} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy$$

$$\leq 4\iint_{E} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy$$

$$\leq 4\iint_{E2} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy \qquad \text{...(4)}$$

Putting $x = r\cos\theta$, $y = r\sin\theta$ we obtain

$$= 4\iint_{E1} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy$$

$$= 4\int_0^{\pi/2}\int_0^a r^{2m-1}\cos^{2m-1}\theta\, r^{2n-1}\sin^{2n-1}\theta\, e^{-r^2} r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta \int_0^a r^{2(m+n)-1} e^{-r^2} dr$$

$$= 2\beta(m,n)\int_0^a r^{2(m+n)-1} e^{-r^2}, \text{ using (1)} \qquad \text{...(5)}$$

Similarly,

$$= 4\iint_{E_2} x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy$$

$$= 2\beta(m,n)\int_0^{\sqrt{2}a} r^{2(m+n)-1} e^{-r^2} dr \qquad \text{...(6)}$$

Substituting (5) and (6) in (4), we obtain

$$= 2\beta(m,n)\int_0^a r^{2(m+n)-1} e^{-r^2} dr$$

$$\leq 4\iint_E x^{2m-1} y^{2n-1} e^{-(x^2+y^2)} dx\, dy$$

$$\leq 2\beta\int_0^{\sqrt{2}a} r^{2(m+n)-1} e^{-r^2} dr$$

Letting $a \to \infty$ and using (2) and (3), we get

$$\beta(m, n)\,\Gamma(m+n) \leq \Gamma(m)\,\Gamma(n)$$
$$\leq \beta(m, n)\,\Gamma(m+n)$$

or $\beta(m, n)\Gamma(m+n) = \Gamma(m)n\,\Gamma(n)$

Hence, $$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

4.(c) Every subgroup of a cyclic group is cyclic.

Let $G = \langle g \rangle$ be any cyclic group. Let H be any subgroup of G. Let $a \in H$ be an arbitrary element. Hence $a = g^m$ for some $m \in \mathbb{N}$.

Let $r = \min\{m \mid g^m \in H\}$. Let $x = g^r$ and $K = \langle x \rangle$.

Let $y \in H$ be any element such that $y \notin K$. Again, $y = g^m$ for some $m \in \mathbb{N}$ and $r \leq m$. Hence $m = vr + u$ where $0 \leq u \leq r - 1$.

Hence $g^m = g^{rv} g^u$.

Again, $g^{m-rv} = g^u \in H$, a contradiction. Hence $K = H$.

If N is a cyclic normal subgroup of a finite group G, then show that every subgroup H of N is normal in G.

Define $H = \langle h \rangle$ and $N = \langle k \rangle$ where $h = k^r$. Again, $ghg^{-1} = gk^r g^{-1} = (gkg^{-1})^r = k^{rn} = h^n \in H$ for all $g \in G$ where $n \in \mathbb{N}$ depends on G.

4.(*d*) The following matrix gives data concerning the transportation times t_{ij}

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25	30	20	40	45	37	37
02	30	25	20	30	40	20	22
03	40	20	40	35	45	22	32
04	25	24	50	27	30	25	14
Demand	15	20	15	25	20	10	

We compute an initial basic feasible solution by north west corner rule which is shown in table 1.

Table 1

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25 (15)	30 (20)	20 (2)	40	45	37	37
02	30	25	20 (13)	30 (9)	40	20	22
03	40	20	40	35 (16)	45 (16)	22	32
04	25	24	50	27	30 (4)	25 (10)	14
Demand	15	20	15	25	20	10	

Here $t_{11} = 25$, $t_{12} = 30$, $t_{13} = 20$, $t_{23} = 20$, $t_{24} = 30$, $t_{34} = 35$, $t_{35} = 45$, $t_{45} = 30$, $t_{46} = 25$ Choose maximum from t_{ij}, *i.e.*, $T_1 = 45$. Now, cross out all the unoccupied cells that are $\geq T_1$. The unoccupied cell (O3D6) enters into the basis as shown in table 2.

Table 2

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
02	30	25	20 (13)	30 (6)	40	20	22
03	40	20	40	35 (16)	45 − (16)	+ 22	32
04	25	24	~~50~~	27	30 + (4)	25 − (10)	14
Demand	15	20	15	25	20	10	

Choose the smallest value with a negative position on the closed path, *i.e.*, 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Table 3

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
02	30	25	20 (13)	30 (9)	40	20	22
03	40	20	40	35 (16)	45 (6)	22 (10)	32
04	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

Here, $T_2 = \text{Max}\ (25, 30, 20, 20, 20, 35, 45, 22, 30) = 45$. Now, cross out all the unoccupied cells that are $\geq T_2$.

Table 4

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25 (15)	30 (20)	20 (2)	40	~~45~~	37	37
02	30	25	20 (13)	30 − (9)	+ 40	20	22
03	40	20	40	35 + (16)	45 − (6)	22 (10)	32
04	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

By following the same procedure as explained above, we get the following revised matrix.

Table 5

Origin	D1	D2	D3	D4	D5	D6	Supply
	Destination						
01	25 (15)	30 (20)	20 (2)	~~40~~	~~45~~	37	37
02	30	25	20 (13)	30 (3)	40 (6)	20	22
03	~~40~~	20	~~40~~	35 (22)	~~45~~	22 (10)	32
04	25	24	~~50~~	27	30 (14)	25	14
Demand	15	20	15	25	20	10	

$T_3 = \text{Max}(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40$.

Now, cross out all the unoccupied cells that are $\geq T_3$.

Now we cannot form any other closed loop with T_3. Hence, the solution obtained at this stage is optimal. Thus, all the shipments can be made within 40 units.

5.(*a*) Let the required equation of the plane is

$$z = lx + my + n$$

$$\Rightarrow lx + my - z + n = 0 \quad ...(1)$$

Now the plane (1) is at a constant distance a from the origin

$$\therefore \quad a = \frac{|n|}{\sqrt{l^2+m^2+1}}$$

$$\Rightarrow \quad a = \frac{\pm n}{\sqrt{l^2+m^2+1}}$$

Here, $$p = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \quad n = \pm a\sqrt{l^2+m^2+1}$$

$\therefore$ (1) becomes

$$lx+my-z\pm a\sqrt{l^2+m^2+1}=0 \quad ...(2)$$

Differentiating (2) w.r.t. x and y, we get

$$l-\frac{\partial z}{\partial x}=0,\ m-\frac{\partial z}{\partial y}=0$$

or $$p = l,\ q = m$$

$\therefore$ (2) reduces to

$$px+qy-z\pm a\sqrt{p^2+q^2+1}=0$$

or $$z=px+qy\pm a\sqrt{p^2+q^2+1}$$

is the required differential equation.

5.(*b*) Here $h = 0.25$, $y_0 = 1.0000$, $y_1 = 0.9896$, $y_2 = 0.9589$, $y_3 = 0.9089$, $y_4 = 0.08415$

Volume of the solid of revolution = V}

$$= \pi\int_0^1 y^2dx=\pi\frac{h}{3}\left[(y_0^2+y_4^2)+4(y_1^2+y_3^2)+2y_2^2\right]$$

$$= \frac{0.25\pi}{3}[1+(0.8415)^2+4(0.9896)^2$$

$$+4(0.9089)^2+2(0.9589)^2]$$

$$= \frac{0.25}{3}\times\pi[1.7081+3.9172+3.3044+1.839]$$

$$= \frac{0.25\times\pi}{3}\times 10.7687 = 0.89739\,\pi = 2.81924.$$

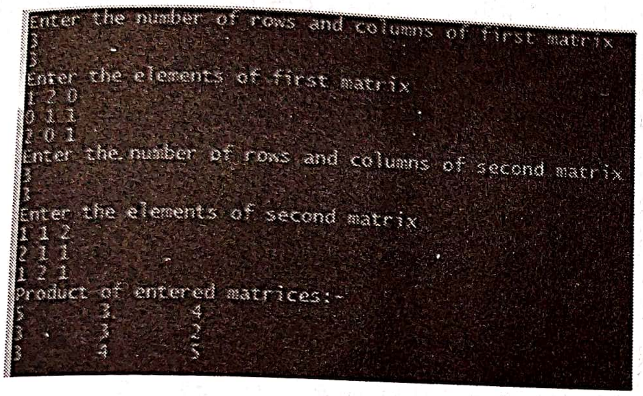
5.(*c*)
```
#include <stdio.h>
int main()
{
  int m, n, p, q, c, d, k, sum = 0;
  int first[10][10], second [10][10], multiply [10][10

  printf("Enter number of rows and columns of fir
  matrix\n");
  scanf ("%d%d", &m, &n);
  printf ("Enter elements of first matrix\n");
  for (c = 0; c < m; c++)
      for (d = 0; d < n; d++)
          scanf ("%d", &first[c][d]);
  printf ("Enter number of rows and columns
  second matrix/n");
  scanf ("%d%d", &p, &q);

  if (n ! = p)
     printf("The multiplicatation isn't possible.\n"
  else
  {
  printf("Enter elements of second matrix\n");
     for (c = 0; c < p; c++)
       for (d = 0; d < q; d++)
          scanf ("%d", &second[c][d]];
     for (c = 0; c < m; c++) {
       for (d = 0; d < q; d++) {
          for (k = 0; k < p; k++) {
          sum = sum + first[c][k]*second[k][d
          }
          multiply[c][d] = sum;
          sum = 0;
       }
  }
  printf("Product of the matrices:\n");
```

```
        for (c = 0; c < m; c++) {
            for (d = 0; d < q; d++)
                printf("%d\t", multiply[c][d]);
            printf("\n");
        }
    }
    return 0;
}
```

An output of 3 × 3 matrix multiplication C program:



5.(*d*) Let p be the pressure and v be the velocity at a distance x from the end of the tube at any time t. The equation of motion and the equation of continuity is given by

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(1)$$

and $$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v) = 0 \qquad ...(2)$$

Since the air obeys Boyle's law, then

$$p = k\rho \Rightarrow dp = kd\rho \qquad ...(3)$$

From (1) and (3), we have

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad ...(4)$$

Differentiating (2) partially with regard to t, we have

$$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}(\rho v)\right\} = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial t}(\rho v)\right\} = 0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}\right\} = 0$$

From (2) and (4), we have

$$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\left(-v\frac{\partial v}{\partial t}-\frac{k}{\rho}\frac{\partial \rho}{\partial x}\right)-v\frac{\partial}{\partial x}(\rho v)\right\}=0$$

or $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left\{\rho v\frac{\partial v}{\partial x}+v\frac{\partial}{\partial x}(\rho v)+k\frac{\partial \rho}{\partial x}\right\}$$

or $$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v.v)+k\frac{\partial \rho}{\partial x}\right\}$$

$$= \frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\} \text{ Proved.}$$

6.(*a*) The given PDE can be written as:

$f(x, y, z, p, q) = 0$ where

$$f(x, y, z, p, q) = (p^2 + q^2)x - pz \qquad ...(1)$$

We find that, $f_x = p^2 + q^2$

$f_y = 0$

$f_z = -p,$

$f_p = 2px - z,$

$f_q = 2qx$

The Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q}$$

$$= \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

i.e., $$\frac{dx}{2px-z} = \frac{dy}{2qx} = \frac{dz}{2(p^2+q^2)x-pz}$$

$$= \frac{dp}{-q^2} = \frac{dq}{pq} \qquad ...(2)$$

Taking the last two fractions of (2), we have

$$\frac{dp}{-q^2} = \frac{dq}{pq} \text{ or } \frac{dp}{-q} = \frac{dq}{p}$$

i.e., $pdp + qdq = 0$

or $2pdp + 2qdq = 0$

Integrating, we get

$$p^2 + q^2 = a^2 \qquad ...(3)$$

where a is an arbitrary constant.

Solving the equations (1) and (3), we get

$$p = \frac{a^2x}{z},$$

$$q = \frac{a\sqrt{z^2 - a^2x^2}}{z} \qquad ...(4)$$

Substituting the values of p and q in the integrable equation

$$dz = pdx + qdy,$$

we get, $$dz = \frac{a^2x}{z}dx + \frac{a\sqrt{z^2 - a^2x^2}}{z}dy$$

i.e., $$zdz = a^2xdx + a\sqrt{z^2 - a^2x^2}\,dy$$

which can be rearranged as

$$\frac{zdz - a^2xdx}{\sqrt{z^2 - a^2x^2}} = ady \qquad ...(5)$$

Substituting $u^2 = z^2 - a^2x^2$

we get $2udu = 2zdz - 2a^2xdx$

or $udu = zdz - a^2xdx$

Thus, the equation (5) can be simplified as

$$du = ady$$

Integrating, we get a complete integral of the given PDE as

$$u = ay + b$$

or $$\sqrt{z^2 - a^2x^2} = ay + b$$

where b is an arbitrary constant.

6.(*b*) Let $h = 0.1$

Here, $x_0 = 0$, $y_0 = 1$

$f(x, y) = x + y^2$

Now, $k_1 = hf(x_0, y_0)$

$= 0.1(0 + 1^2) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$= 0.1f(0 + 0.05, 1 + 0.05)$

$= 0.1[0.05 + (1.05)^2] = 0.11525$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$= 0.1f(0 + 0.05, 1 + 0.05763)$

$= 0.1[0.05 + (1.05763)^2] = 0.11686$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1f(0 + 0.1, 1 + 0.11686)$

$= 0.1[0.1 + (1.11686)^2] = 0.13474$

According to Runge-Kutta (fourth order) formula

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{0.1} = 1 + \frac{1}{6}[0.1 + 2(0.11525) + 2(0.11686) + 0.13474]$$

$$y_{0.1} = 1 + 0.11649 = 1.11649$$

For the second step

$x_1 = 0.1$, $y_1 = 1.11649$

$k_1 = hf(x_1, y_1) = 0.1f[0 + 0.1, 1.11649]$

$= 0.1[0.1 + (1.11649)^2]$

$= 0.1(0.1 + 1.24655) = 0.13466$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 1.11649 + \frac{0.13466}{2}\right)$$

$= 0.1f(0.15, 1.18382)$

$= 0.1[0.15 + (1.18382)^2]$

$= 0.1(0.15 + 1.40143) = 0.15514$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 1.11649 + \frac{0.15514}{2}\right)$$

$= 0.1f(0.15, 1.19406)$

$= 0.1[0.15 + (1.19406)^2]$

$= 0.1(0.15 + 1.42578) = 0.15758$

$k_4 = hf(x_1 + h, y_1 + k_3)$

$= 0.1f(0.1 + 0.1, 1.11649 + 0.15758)$

$= 0.1f(0.2, 1.27407)$

$= 0.1[0.2 + (1.27407)^2]$

$= 0.1(0.2 + 1.62325) = 0.18233$

$$y_{0.2} = y_{0.1} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.11649 + \frac{1}{6}(0.13466 + 2(0.15514) + 2(0.15758) + 0.18233)$$

$$= 1.11649 + \frac{1}{6}(0.94243)$$

$= 1.11649 + 0.15707 = 1.27356.$

6.(*c*) In i e H r c S I

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6.(*c*) In an accelerator beam line, we are interested in charged particles moving through electromagnetic fields. If we are to apply Hamiltonian mechanics to the problem, we need an expression for the Hamiltonian for a charged particle in an electromagnetic field. Since we already know that the equations of motion for such a particle are given by (1) and (2), all we need to do is find a Hamiltonian that, when substituted into Hamilton's equations (3) and (4), gives the correct equation of motion, *i.e.*, equations of motion consistent with the Lorentz force (2) in Newtonian mechanics. In the non-relativistic case, an appropriate Hamiltonian is:

$$H = \frac{(p - qA)^2}{2m} + q\phi \quad \text{...(11)}$$

where ϕ is the scalar potential and A is the vector potential, which are related to the electric and magnetic fields by the usual equations:

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \quad \text{...(12)}$$

$$B = \nabla \times A \quad \text{...(13)}$$

In general, ϕ and A are functions of the co-ordinates x, y and z, and the time t. Note that with the Hamiltonian (11), the relationship between the particle velocity and momentum, given by (3) is:

$$p = mv + qA \quad \text{...(14)}$$

The momentum p defined in this way is known as the *canonical momentum*, to emphasise that it is distinct from the mechanical momentum mv.

Particles in high energy accelerators tend to be moving with relativistic velocities. We therefore need to generalise the Hamiltonian (11) to the relativistic case. In special relativity, the total energy E and momentum p for a particle in free space (*i.e.*, with zero electric and magnetic fields) are related by:

$$E^2 = p^2c^2 + m^2c^4 \quad \text{...(15)}$$

where c is the speed of light in free space. Equation (15) follows from the expressions for the energy and momentum in terms of the mass and velocity of the particle:

$$E = \gamma mc^2 \quad \text{...(16)}$$

$$p = \beta\gamma mc \quad \text{...(17)}$$

where $\beta = v/2$, and:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \text{...(18)}$$

In an electromagnetic field, we simply include the contributions of the scalar and vector potentials in the same way as we would for a non-relativistic particle, so that the total energy and canonical momentum become:

$$E = \gamma mc^2 + q\phi \quad \text{...(19)}$$

$$P = \beta\gamma mc + qA \quad \text{...(20)}$$

It then follows from the relationship (18) between γ and β that the relationship between the total energy and the canonical momentum can be written:

$$(E - q\phi)^2 = (p - qA)^2c^2 + m^2c^4 \quad \text{...(21)}$$

We saw above that, for a particle performing simple harmonic motion, the Hamiltonian could be expressed as the total energy of the particle. If we assume that the same is true for a relativistic particle moving in an electromagnetic field, then we propose the following form for the Hamiltonian:

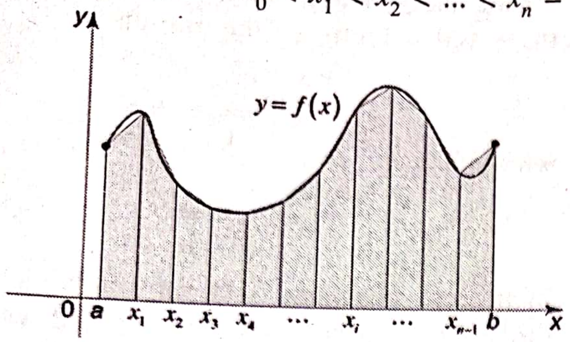
$$H = c\sqrt{(P - qA)^2 + m^2c^2} + q\phi \quad \text{...(22)}$$

Whether or not this is the correct Hamiltonian (that is, whether it gives the correct equations of motion when substituted into Hamilton's equations) must ultimately be tested by experiment. It turns out that (22) is indeed the correct Hamiltonian for a relativistic charged particle moving in an electromagnetic field.

6.(*d*) **Trapezoidal Rule.** Under this rule, the area under a curve is evaluated by dividing the total area into little trapezoids rather than rectangles. Let $f(x)$ be continuous on $[a, b]$. We partition the interval $[a, b]$ into n equal subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

such that, $a = x_0 < x_1 < x_2 < \dots < x_n = b$



The Trapezoidal Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $x_i = a + i\Delta x$

As $n \to \infty$, the right-hand side of the expression approaches the definite integral $\int_a^b f(x)dx$.

7.(a) Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)} \quad ...(1)$$

Taking the last two fractions of (1), we have

$$(y - z)dy = (y + z)dz$$

or $2ydy - 2zdz - 2(zdy + ydz) = 0$

Integrating, $y^2 - z^2 - 2yz = c_1$,

c_1 being an arbitrary constant ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + xy(y+z) + xz(y-z)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow 2xdx + 2ydy + 2zdz = 0$

So that, $x^2 + y^2 + z^2 = c_2$ ...(3)

From (2) and (3), solution is $\phi(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at (0, 0, 0), *i.e.*, origin.

7.(d) The velocity potential of a liquid streaming past a fixed circular disc is given by

$$\phi = u\left(r + \frac{a^2}{r}\right)\cos\theta \quad ...(1)$$

Differentiating (1) w.r. to r and θ, respectively, we have

$$\frac{\partial\phi}{\partial r} = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial\phi}{\partial\theta} = -u\left(r + \frac{a^2}{r}\right)\sin\theta$$

Let q be the velocity at the point P (r, θ), then

$$q^2 = \left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

$$= u^2\left\{1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right\} \quad ...(2)$$

The velocity q will be maximum with regard to θ, if

$$\cos 2\theta = -1 \Rightarrow 2\theta = \pi,$$

then, $q^2 = u^2\left(1 + \dfrac{2a^2}{r^2} + \dfrac{a^4}{r^4}\right)$

$$\Rightarrow \quad q = u\left(1 + \frac{a^2}{r^2}\right) \quad ...(3)$$

Max. value of $(q)_{r=a} = u\left(1 + \dfrac{a^2}{r^2}\right) = 2u$

Hence, the maximum value of the velocity at any point of the fluid is $2u$. Proved.

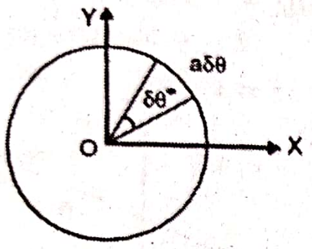
By Bernoulli's equation, we have

$$\frac{p}{\rho} = f(t) - \frac{1}{2}q^2 + \frac{\partial\phi}{\partial t}$$

or $$\frac{p}{\rho} = f(t) - \frac{1}{2}u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + \dot{u}\left(r + \frac{a^2}{r}\right)\cos\theta$$

At the boundary of the disc $r = a$, the pressure becomes

$$\frac{p}{\rho} = f(t) - 2u^2\sin^2\theta + 2a\dot{u}\cos\theta$$



Let $a\delta\theta$ be an element on the surface of the disc. The resultlant pressure on the disc is

$$P = \int_0^{2\pi} (-p\cos\theta) a\, d\theta$$

$$\Rightarrow \quad P = -\rho a \int_0^{2\pi} [f(t) - 2u^2\sin^2\theta + 2a\dot{u}\cos\theta]\cos\theta\, d\theta$$

$$\Rightarrow \quad P = -2\rho a^2 \dot{u} \int_0^{2\pi} \cos^2\theta\, d\theta$$

$$= -2\rho a^2 \dot{u}\pi = -2m\dot{u}$$

where $m = \pi a^2 \rho$

Hence, the force necessary to hold the disc at rest is $2m\dot{u}$.

8.(*a*) A real function V satisfying the given equation can be obtained by finding the particular integral of the given equation.

$$\text{Now, P.I.} = \frac{1}{D^2 + D'^2}\{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2}\left[1 + \frac{D'^2}{D^2}\right]^{-1}\{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2}\left[1 - \frac{D'^2}{D^2}\cdots\right]\{-4\pi(x^2 + y^2)\}$$

$$= \frac{1}{D^2}\{-4\pi(x^2 + y^2)\} - \frac{1}{D^4}D'^2\{-4\pi(x^2 + y^2)\}$$

$$= -4\pi\left\{\frac{x^4}{12} + \frac{x^2y^2}{2}\right\} + 4\pi\left\{2.\frac{x^4}{24}\right\} = -2\pi x^2 y^2$$

which reduces to zero when $y = 0$

$$\therefore \quad V = -2\pi x^2 y^2$$

Note that here the roots of the A.E. are imaginary and hence C.F. does not give a real value of $V(x, y)$.

8.(*b*) Let $f(x) = x^6 - x^4 - x^3 - 1$

$f(1.4) = -0.056$

$f(1.41) = 0.102$

Hence, the root lies between 1.4 and 1.41.

Using method of false position,

$$x_2 = x_0 - \left\{\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right\} f(x_0)$$

$$= 1.4 - \left(\frac{1.41 - 1.4}{0.102 + 0.056}\right)(-0.056)$$

| Let, $x_0 = 1.4$ and $x_1 = 1.41$

$$= 1.4 + \left(\frac{0.01}{0.158}\right)(0.056) = 1.4035$$

Now, $f(x_2) = -0.0016$(–)ve

Hence, the root lies between 1.4035 and 1.41

Using the method of false position,

$$x_3 = x_2 - \left\{\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right\} f(x_2)$$

| Replacing x_0 by x_2

$$= 1.4035 - \left(\frac{1.41 - 1.4035}{0.102 + 0.0016}\right)(-0.0016)$$

$$= 1.4035 + \left(\frac{0.0065}{0.1036}\right)(0.0016) = 1.4036$$

Now, $f(x_3) = -0.00003$(–)ve

Hence, the root lies between 1.4036 and 1.41

Using the method of false position,

$$x_4 = x_3 - \left\{\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right\} f(x_3)$$

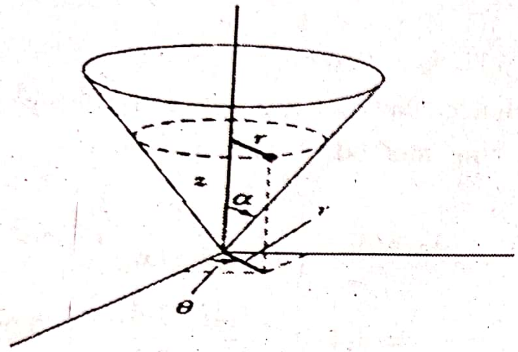
$$= 1.4036 + \left(\frac{1.41 - 1.4036}{0.102 + 0.00003}\right)(0.00003$$

$$= 1.4036 + \left(\frac{0.0064}{0.10203}\right)(0.00003) = 1.4036$$

Since, x_3 and x_4 are approximately the same upto four places of decimal, hence the required root of the given equation is 1.4036.

8.(c) Here we use the cylindrical coordinates r, θ, & z. The equation of constraint is

$$z = r \cot \alpha$$



So we have 2-degrees of freedom and the generalized coordinates area r & θ. Now

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cot^2 \alpha\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + \csc^2 \alpha^2 + r^2\dot{\theta}^2\right)$$

$$U = mgz = mgr \cot \alpha$$

$$\Rightarrow L = \frac{1}{2}m\left(\dot{r}^2 + \csc^2 \alpha + r^2\dot{\theta}^2\right) - mgr\cot \alpha$$

For the r-coordinate we have

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0$$

$$\Rightarrow \quad mr\dot{\theta}^2 - mg\cot\alpha - \frac{d}{dt}\left(m\dot{r}\csc^2\alpha\right) = 0$$

$$\Rightarrow \quad r\dot{\theta}^2 - g\cot\alpha - \ddot{r}\csc^2\alpha = 0$$

For the θ-coordinate we have

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\Rightarrow \quad 0 - \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

$$\Rightarrow \quad mr^2\dot{\theta} = \text{constant}$$

But $\quad L = I\omega = mr^2\dot{\theta} = \text{constant}$

So we recover the conservation of angular momentum about the axis of symmetry of the system.

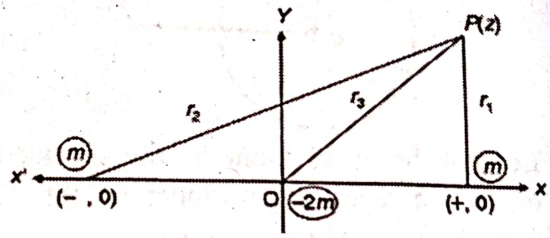
8.(d) First Part: The complex potential w at any point P(z) is given by

$$w = -m \log (z - a) - m \log (z + a) + 2m \log z \quad ...(1)$$

or $\quad w = m[\log z^2 - \log (z^2 - a^2)]$

or $\quad \phi + i\psi = m[\log (x^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)],$

as $\quad z = x + iy$



Equating the imaginary parts, we have

$$\psi = m\left[\tan^{-1}\left\{2xy/(x^2 - y^2)\right\} - \tan^{-1}\left\{2xy/(x^2 - y^2 - a^2)\right\}\right]$$

$$\therefore \ \psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right]$$

on simplification.

The desired streamlines are given by

ψ = constant = $m \tan^{-1}(-2/\lambda)$.

Then we obtain

$(-2/\lambda) = (-2a^2xy)/[(x^2 + y^2)^2 - a^2(x^2 - y^2)]$

or $\quad (x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$

Second Part. From (1), we have

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z}$$

$$= -\frac{2a^2m}{z(z-a)(z+a)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where, $\quad r_1 = |z - a|,$

$\quad r_2 = |z + a|$

and $\quad r_3 = |z|.$