(8) $u(x,y) = 2x - x^3 + 3xy^2$ (2010)

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2 \qquad \frac{\partial^2 u}{\partial x^2} = -6x$$

$$\frac{\partial u}{\partial y} = 6xy \qquad \frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

u satisfies laplace equation.

Also first and second partial derivatives of u are continuous function of x and y. consequently. u is a harmonic function. Let v be harmonic conjugate

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$f'(z) = 2 - 3x^2 + 3y^2 - i(6xy)$$

By milne thompson $x = z$ $y = 0$ gives

$$f'(z) = 2 - 3z^2$$

$$\boxed{f(z) = 2z - z^3 + c}$$

Again

$$f(z) = 2(x+iy) - (x+iy)^3 + c$$

$$= 2x + i2y - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$$

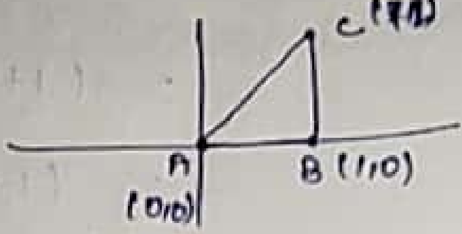
$$= 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3 + c)$$

harmonic conjugate

$$= v(x,y) = \boxed{2y - 3x^2y + y^3 + c}$$

CSE
2010

(2)



(i) $\int_{0} f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$

$$\int_{AB} f(z)\,dz = \int_{AB} z^2\,dz = \int_{AB} (x+iy)^2 (dx+i\,dy)$$

$$= \int_{AB} (x^2-y^2+2ixy)\,(dx+i\,dy)$$

along AB $dy=0$ $y=0$ x varies from 0 to 1

$$= \int_{AB} x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 = 1/3$$

$$\int_{BC} f(z)\,dz = \int_{BC} z^2\,dz = \int_{BC} (x^2-y^2+2ixy)\,(dx+i\,dy)$$

along BC $dx=0$ $x=1$ y varies from 0 to 2

$$= \int_{y=0}^{y=2} (1-y^2+2iy)\,(i\,dy)$$

$$= i\left[y-\frac{y^3}{3}+iy^2\right]_0^2$$

$$= i\left[2-8/3+4i\right] = i\left[-2/3+4i\right]$$

$$= -4-2i/3$$

$$\int_{CA} f(z)\,dz = \int_{CA} (x^2-y^2+2ixy)\,(dx+i\,dy)$$

along CA $\frac{y-0}{x-0}=\frac{2-0}{1-0} \Rightarrow y=2x$ $dy=2dx$

$$= \int^{x=0} (x^2-4x^2+4i(x^2))\,(1+2i)\,dx$$

$$= (1+2i) \int_1^0 \left(x^3/3 - \frac{4x^3}{3} + 4i\frac{x^3}{3} \right)$$

$$= (1+2i) \left[-x^3 + \frac{4i}{3}x^3 \right]_1^0$$

$$= (1+2i)\,(1 - 4i/3)$$

$$= 1 + 8/3 + 2i - 4i/3 = 11/3 + 2i/3$$

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz$$

$$= 1/3 - 4 - 2i/3 + 11/3 + 2i/3$$

$$= 0$$

(ii) let (x_0, y_0) be any point in z plane

as image of this point is under ~~Reverse~~ exponential function, so $w = e^z$

$$= e^{x+iy}$$

$$= e^{x}(\cos y + i \sin y)$$

$\boxed{\text{so image of } (x_0, y_0) \text{ is } (e^{x_0}\cos y_0, e^{x_0}\sin y_0)}$

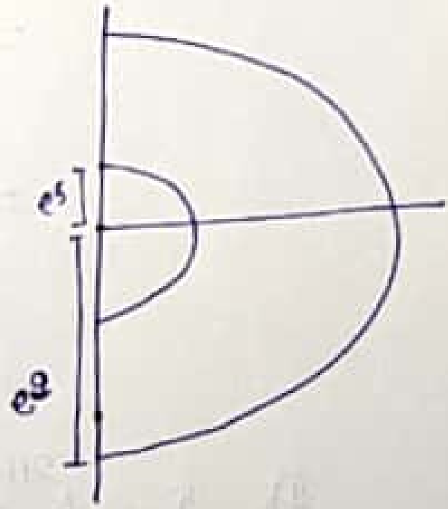
for any vertical strip x_0 is fixed y_0 varies from $-\pi$ to π. so it describes a semicircle in right half plane with centre (0,0) and radius of e^{x_0}

since x_0 varies from 5 to 9, radii of semicircle varies from e^5 to e^9.

required diagram,

Diagram is an annulus having inner radii : e^5

outer radii : e^9



③ $f(z) = e^{\lambda/2(z - 1/z)}$ for $0 < |z| < \infty$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

Here $c_n = \frac{1}{\pi}\int_0^{\pi} (\cos n\phi - \lambda \sin \phi)\, d\phi$ $\quad n = 0, \pm 1, \pm 2 \ldots$

using cauchy's integral formula.

$$f(z) = \sum c_n z^n$$

then $c_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}\, dz$

now given contour is

$$z = e^{i\phi} \quad (-\pi \leq \phi \leq \pi).$$

It can be rewritten as

$$C: \quad |z| = 1$$

so $$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(e^{i\phi})\, d(e^{i\phi})}{(e^{i\phi})^{n+1}}$$

$$= \frac{1}{2\pi i} \int_C \frac{e^{\lambda/2 (e^{i\phi} - e^{-i\phi})} \cdot i e^{i\phi} d\phi}{(e^{i\phi})^{n+1}}$$

$$= \frac{i}{2\pi i} \int_C \frac{e^{\lambda(i \sin\phi)}}{e^{(i\phi)n}} d\phi$$

$$= \frac{1}{2\pi} \int_C e^{i(\lambda \sin\phi - n\phi)} d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(\lambda \sin\phi - n\phi)\, d\phi + \frac{i}{2\pi} \int_0^{2\pi} \sin(\lambda \sin\phi - n\phi)\, d\phi$$

If $$I_1 = \int_0^{2\pi} \sin(\lambda \sin\phi - n\phi)\, d\phi$$

Replacing ϕ by $2\pi - \phi$

$$I_1 = \int_0^{2\pi} \sin(\lambda \sin(2\pi - \phi) - 2\pi n + n\phi)\, d\phi$$

$$I_1 = -\int_0^{2\pi} \sin[2\pi n - (\lambda \sin\phi - n\phi)]\, d\phi$$

$$I_1 = -\int_0^{2\pi} \sin(\lambda \sin\phi - n\phi)\, d\phi$$

$$I_1 = -I_1$$

$$\boxed{I_1 = 0}$$

So, $\boxed{C_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(\lambda \sin\phi - n\phi)\, d\phi}$

Here $\int_0^{2\pi} \cos(\lambda \sin\phi - n\phi)\, d\phi = \int_0^{2\pi} \cos(\lambda \sin(2\pi-\phi) - n(2\pi-\phi))\, d\phi$

So

$$C_n = 2 \times \frac{1}{2\pi} \int_0^{\pi} \cos(\lambda \sin\phi - n\phi)\, d\phi$$

$$\boxed{C_n = \frac{1}{\pi} \int_0^{\pi} \cos(\lambda \sin\phi - n\phi)\, d\phi}$$