- Inside Earth

  ↳ $\frac{d^2x}{dt^2} = -\lambda x.$

- $\int \frac{dx}{\sqrt{\log \frac{a}{x}}}$ Put $\sqrt{\log \frac{a}{x}} = u$

  $\Rightarrow \frac{a}{x} = e^{u^2} \Rightarrow \boxed{ae^{-u^2} = x}$

- $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

DYNAMICS

① h be height due to velocity v at earth's surface (at constant g) and H be height taking variation of g. Show $\frac{1}{h} - \frac{1}{H} = \frac{1}{r}$ r = radius earth.

$$\Rightarrow v^2 = u^2 + 2gs \Rightarrow u^2 = 2gs \Rightarrow \boxed{v^2 = 2gh}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2} \qquad \text{at } x = r,\ \frac{d^2x}{dt^2} = -g \Rightarrow g = +\frac{k}{r^2} \Rightarrow k = gr^2.$$

$$\frac{d^2x}{dt^2} = -\frac{gr^2}{x^2} \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{2gr^2}{x} + A.$$

$$v^2 = \frac{2gr^2}{r} + A \Rightarrow A = v^2 - 2gr$$

$\boxed{\text{* } r+H \text{ Not } H}$

$$0^2 = \frac{2gr^2}{r+H} + v^2 - 2gr \Rightarrow \frac{2gr - v^2}{2gr^2} = \frac{1}{r+H}.$$

$$\cancel{\frac{1}{h} - \frac{1}{H} = \frac{2g}{v^2} - \frac{2gr - v^2}{2gr^2}}$$

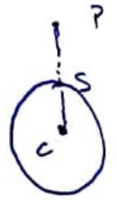
$$\frac{2gr - 2gh}{2gr^2} = \frac{1}{r+H} \Rightarrow \frac{1}{r} - \frac{h}{r^2} = \frac{1}{r+H}.$$

$$+\frac{h}{r^2} = \frac{1}{r} - \frac{1}{r+H} = \frac{\cancel{r} + H - \cancel{r}}{r(r+H)} \Rightarrow \frac{H}{r+H} = \frac{h}{r}.$$

$$\Rightarrow \frac{r}{h} = \frac{r}{H} + 1. \Rightarrow \frac{1}{h} - \frac{1}{H} = \frac{1}{r} \quad \text{Ans}$$

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

(2) A particle falling under gravity can penetrate earth freely. Show a particle from rest from $(b > a)$ from centre of earth would acquire velocity $\sqrt{ga(3b-2a)/b}$ and time to travel from surface to centre is $\sqrt{\frac{a}{g}}\sin^{-1}\left(\sqrt{\frac{b}{3b-2a}}\right)$



$\Rightarrow$ PS

Acceleration $\equiv \frac{d^2x}{dt^2} = -\frac{k}{x^2}$ $\qquad -g = -\frac{k}{a^2} \Rightarrow$

$k = a^2 g$

$$\frac{d^2x}{dt^2}\left(2\frac{dx}{dt}\right) = -\frac{ga^2}{x^2}\left(\frac{2dx}{dt}\right)$$

$$\left(\frac{dx}{dt}\right)^2 = +\frac{2ga^2}{x} + C. \Rightarrow C = -\frac{2ga^2}{b}$$

$$\left(\frac{dx}{dt}\right)^2 = 2ga^2\left(\frac{1}{x} - \frac{1}{b}\right).$$

V at surface $\Rightarrow$ $\boxed{V^2 = 2ga^2\left(\frac{1}{a} - \frac{1}{b}\right).}$

$\Rightarrow$ CS $\equiv$ Acceleration $= \frac{d^2x}{dt^2} = -\lambda x. \Rightarrow \frac{g}{a} = \lambda.$

$$\left(\frac{d^2x}{dt^2}\right)2.\frac{dx}{dt} = -\frac{gx}{a}.\frac{2dx}{dt}$$

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\cancel{2}g}{a}\frac{x^2}{\cancel{2}} + \beta. \Rightarrow 2ga^2\left(\frac{1}{a} - \frac{1}{b}\right) = -ga + \beta.$$

$$\beta = ga^2\left(\frac{3b-2a}{ab}\right).$$

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + \frac{ga}{b}\left(\frac{3b-2a}{\cancel{a}}\right)$$

At centre $x=0 \Rightarrow \frac{dx}{dt} = \sqrt{\frac{ga}{b}(3b-2a)}$ ✓

$$\frac{dx}{dt} = -\sqrt{\frac{g}{a}}\sqrt{\underbrace{\frac{a^2}{b}(3b-2a)}_{k^2} - x^2}\,.$$

$$-\sqrt{\frac{a}{g}}\int_a^0 \frac{dx}{\sqrt{k^2-x^2}} = \int_0^t dt\,.$$

$$\sqrt{\frac{a}{g}}\,\sin^{-1}\left(\frac{x}{k}\right)\Bigg]_0^a = t \Rightarrow \sqrt{\frac{a}{g}}\,\sin^{-1}\left(\frac{a\sqrt{b}}{a\sqrt{(3b-2a)}}\right) = t$$

DONE

(Q) A particle moves in a line with acc towards a fixed pt $\frac{\mu}{x^2} - \frac{\lambda}{x^3}$ at a distance x from given point. Particle starts from rest at a distance a. Show that it oscillates between this distance and $\frac{\lambda a}{2\mu a - \lambda}$ and time period is $\frac{2\pi \mu a^3}{(2a\mu-\lambda)^{3/2}}$

(2)

$$\frac{d^2x}{dt^2} = -\left[\frac{\mu}{x^2} - \frac{\lambda}{x^3}\right].$$

B . O a

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{\lambda}{x^2} + A \Rightarrow A = \frac{\lambda}{a^2} - \frac{2\mu}{a}.$$

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right) - \lambda\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \left(\frac{1}{x} - \frac{1}{a}\right)\left[2\mu - \lambda\left[\frac{1}{x} + \frac{1}{a}\right]\right] \quad (1)$$

Particle comes to rest at other end pt $\Rightarrow$

$$x = a \text{ or } \frac{2\mu}{\lambda} = \frac{1}{x} + \frac{1}{a} \Rightarrow \frac{2\mu}{\lambda} - \frac{1}{a} = \frac{1}{x} \Rightarrow \frac{2a\mu - \lambda}{a\lambda} = \frac{1}{x}.$$

$$\boxed{x = \frac{a\lambda}{2a\mu - \lambda}}$$

To prove other $\Rightarrow$ put $\boxed{\frac{\lambda a}{2a\mu - \lambda} = b}$ in (1).

$$\left(\frac{dx}{dt}\right)^2 = \left(\frac{1}{x} - \frac{1}{a}\right)\lambda\left(\frac{1}{b} - \frac{1}{x}\right).$$

$$dt = -\sqrt{\frac{ab}{\lambda}}\frac{x\,dx}{\sqrt{(a-x)(x-b)}} = \sqrt{\frac{ab}{\lambda}}\int_a^b \frac{x\,dx}{\sqrt{\{-ab - [x^2 - (a+b)x]\}}}$$

$$= \sqrt{\frac{ab}{\lambda}}\int_a^b \frac{x\,dx}{\sqrt{\frac{1}{4}(a-b)^2 - \{x - \frac{1}{2}(a+b)\}^2}}$$

Put $x - \frac{1}{2}(a+b) = y$

$$= \sqrt{\frac{ab}{\lambda}}\int_{-\frac{(a-b)}{2}}^{\frac{(a-b)}{2}} \frac{\{\frac{1}{2}(a+b) + y\}\,dy}{\sqrt{\frac{1}{4}(a-b)^2 - y^2}}$$

(2) SHM

Centre of force.

$$\frac{d^2x}{dt^2} = -\mu(x-\alpha)$$

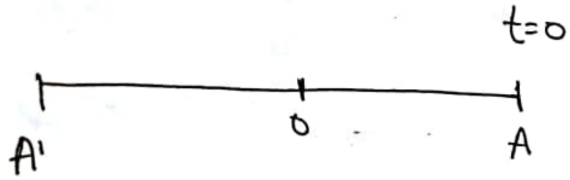
- $T = \frac{2\pi}{\sqrt{\mu}}$, $f = \frac{1}{T}$.

- Amplitude $\equiv$ where velocity $= 0$

$$v^2 = \mu(a^2 - x^2)$$

ampli.

$$x = a\cos\sqrt{\mu}t$$



Q. Body is attached to one end of inelastic string, the other end moves in a vertical line with S.H.M of amplitude a, making n oscillations per second. Show string is not tight unless $n^2 < g/(4\pi^2 a)$.

E

$$f = n = \frac{\sqrt{\mu}}{2\pi} \Rightarrow \boxed{\mu = 4\pi^2 n^2}$$

For SHM $\Rightarrow \frac{d^2x}{dt^2} = -\mu x.$

Mass $\equiv m\frac{d^2x}{dt^2} = T - mg. \Rightarrow T = mg + m\frac{d^2x}{dt^2}.$

~~T=0~~ $\Rightarrow$ T is least, when $\underbrace{\frac{d^2x}{dt^2}}_{-\mu a.}$ is least

String is tight if $\Rightarrow \cancel{m}g - \cancel{m}\mu a > 0$

$$\Rightarrow g > 4\pi^2 a n^2 \Rightarrow \boxed{g/4\pi^2 a > n^2} \text{ Ans}$$

E

Q. m is attached to wire of tension T, a & b distance from ends. Prove period of small oscillation (transverse) $2\pi\sqrt{\frac{mab}{T(a+b)}}$

$$\equiv m\frac{d^2x}{dt^2} = -\left(T\frac{x}{\sqrt{a^2+x^2}} + T\frac{x}{\sqrt{b^2+x^2}}\right)$$

$$= -Tx\left(\frac{(1+x^2/a^2)^{-1/2}}{a} + \frac{(1+x^2/b^2)^{-1/2}}{b}\right)$$

$$-Tx\left(\frac{(1-\frac{1}{2}\frac{x^2}{a^2}+..)^0}{a} + \frac{(1-\frac{1}{2}\frac{x^2}{b^2}+...)^0}{b}\right)$$

$$\frac{d^2x}{dt^2} = -\frac{Tx}{m}\left(\frac{a+b}{ab}\right) = -\mu x \Rightarrow \mu = \frac{T(a+b)}{mab} \Rightarrow$$

$$T = 2\pi\sqrt{\frac{mab}{T(a+b)}} \text{ Ans}$$

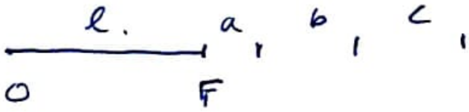
UPSC MWST.

8] If in SHM, U, V, W be velocities at a, b, c from a fixed point on path ($\neq$ centre), Show period T is

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} U^2 & V^2 & W^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Amplitude.

$$U^2 = \mu\left(A^2 - (l+a)^2\right)$$
$$V^2 = \mu\left(A^2 - (l+b)^2\right)$$
$$W^2 = \mu\left(A^2 - (l+c)^2\right)$$



$$\frac{U^2}{\mu} = (A^2 - l^2) - a^2 - 2al$$

$$\Rightarrow \left(\frac{U^2}{\mu} + a^2\right) + 2al + (l^2 - A^2) = 0$$
$$\left(\frac{V^2}{\mu} + b^2\right) + 2bl + (l^2 - A^2) = 0$$
$$\left(\frac{W^2}{\mu} + c^2\right) + 2cl + (l^2 - A^2) = 0.$$

Eliminating $2l$, $l^2 - A^2$.

$$\begin{vmatrix} \frac{U^2}{\mu} + a^2 & a & 1 \\ \frac{V^2}{\mu} + b^2 & b & 1 \\ \frac{W^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0 \Rightarrow \frac{1}{\mu}\begin{vmatrix} U^2 & a & 1 \\ V^2 & b & 1 \\ W^2 & c & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$\Rightarrow \mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} U^2 & V^2 & W^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

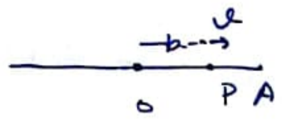
$$\Rightarrow \boxed{\mu(a-b)(b-c)(c-a) = \begin{vmatrix} U^2 & V^2 & W^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}}$$

⑨ Particle is performing SHM of period T about centre O and it passes through point P where $OP = b$ with velocity v_0 in direction OP. Prove that elapsed time before return to P is

ME

$$\frac{T}{\pi} \tan^{-1}\left(\frac{v_0 T}{2\pi b}\right).$$

$v^2 = \mu(a^2 - x^2)$ $\qquad T = \frac{2\pi}{\sqrt{\mu}}$



$v_0^2 = \mu(a^2 - b^2)$.

Let A be extremity. In SHM $t_{PA} = t_{AP}$

So, we only need compute t_{PA}

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) \Rightarrow \frac{dx}{dt} = -\sqrt{\mu(a^2 - x^2)}$$

(As $x \uparrow$ but v decreases)

$$-\frac{1}{\sqrt{\mu}} \frac{dx}{\sqrt{a^2 - x^2}} = dt \Rightarrow \frac{1}{\sqrt{\mu}} \cos^{-1}\left(\frac{x}{a}\right)\Big]_b^a = t\Big]_0^t$$

$$\Rightarrow \frac{1}{\sqrt{\mu}} \cos^{-1}\left(\frac{b}{a}\right) = t_1 \Rightarrow t_1 = \frac{1}{\sqrt{\mu}} \tan^{-1}\left(\frac{\sqrt{a^2 - b^2}}{b}\right)$$

$\sqrt{\mu} = \frac{2\pi}{T}$; $+ b^2 = a^2 \Rightarrow \sqrt{a^2 - b^2} = \frac{2\pi v}{T}$

$t_1 = \frac{2\pi}{T}$ $\qquad t_1 = \frac{T}{2\pi} \tan^{-1}\left(2\pi v\right.$

$$\sqrt{a^2 - b^2} = \sqrt{\frac{v^2}{\mu}} = \frac{Tv}{2\pi}$$

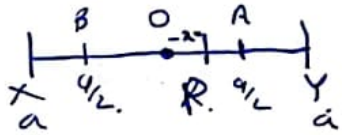
$$T_m = 2 \times t_1 = 2 \times \frac{T}{2\pi} \tan^{-1}\left(\frac{Tv}{2\pi b}\right)$$ ✓

Elastic Strings

$$T = \lambda \cdot \frac{x - l}{l} \leftarrow \text{natural length}$$

$$\text{Work Done} = \text{Extension} \times \frac{[\text{Initial } T + \text{Final } T]}{2}.$$

Q) A string λ is stretched to double length and tied to fixed points 2a apart. m is tied to mid-point and displaced to half distance from fixed points. Find Time Period and v_{max}.



$XY = 2a.$

$$T = \lambda \frac{(2a - a)}{a} = \lambda.$$

$$\text{Tension in } PY \Rightarrow \lambda \frac{(a - x - a/2)}{a/2}$$

$$PX \Rightarrow \lambda \frac{(a + x - a/2)}{a/2}$$

$$\text{Net } T \Rightarrow PX - PY \Rightarrow \frac{2\lambda}{a}\left(\cancel{a} + x - \cancel{\frac{a}{2}} - \cancel{a} + x + \cancel{a/2}\right) = \frac{4\lambda x}{a}.$$

$$+ m\frac{d^2x}{dt^2} = -\frac{4\lambda}{a}x \Rightarrow \frac{d^2x}{dt^2} = -\frac{4\lambda}{am}x.$$

$$\mu = \frac{4\lambda}{am} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{am}{4\lambda}}} \quad \underline{\text{Ans}}$$

$$v_{max}^2 = \mu\,(\text{Ampli})^2 = \frac{4\lambda}{am} \cdot \frac{a^2}{4} \Rightarrow \boxed{v_{max} = \sqrt{\frac{a\lambda}{m}}} \quad \underline{\text{Ans}}$$

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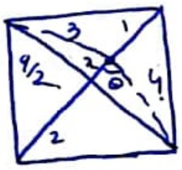
9 Particle tied by 4 strings (l, λ) to corners of square. If displaced small towards one corner, find Time period. a = diagonal of square.



$$T_2 - T_1 = \lambda \left[\frac{\frac{a}{2}+x-l}{l} - \frac{\frac{a}{2}-x-l}{l}\right]$$

$$= \frac{2\lambda}{l} x.$$

$$T_3\cos\theta + T_4\cos\theta = 2\left[\lambda \frac{\sqrt{\frac{a^2}{4}+x^2}-l}{l} \cdot \frac{x}{\sqrt{\frac{a^2}{4}+x^2}}\right]$$

$$= 2\lambda x\left[\frac{1}{l} - \frac{2}{a}\left(1+\frac{4x^2}{a^2}\right)^{-\frac{1}{2}}\right]$$

$$= \cancel{2\lambda x\left[\frac{1-al}{2l}\right]}\ 2\lambda x\left[\frac{a-2l}{al}\right]$$

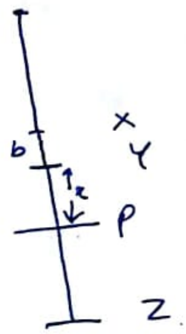
$$(2-m)\frac{d^2x}{dt^2} = -\left(2\lambda x\left(\frac{a+a-2l}{la}\right)\right) = -\frac{4\lambda(a-l)}{al}x$$

$$\mu = \frac{4\lambda(a-l)}{al} \Rightarrow T = \frac{2\pi}{\sqrt{\mu}}$$ Ans

(8) Particle at one end of string, which is drawn vertically down till it is four times its natural length and let go. Show particle returns to this pt in $t_0 = \sqrt{\frac{a}{g}}\left(\frac{4\pi}{3} + 2\sqrt{3}\right)$ $a \equiv$ natural length

$\lambda = mg$

$\Rightarrow$ As string has tension from $X \leftrightarrow Z$ it is in SHM o.w no tension acts on body.



$$\text{At } P \equiv \quad T = mg\left[\frac{b+x-\cancel{a}}{a}\right] \qquad mg\left(\frac{b}{a}\right) = mg$$

$$\Rightarrow b = a$$

$$T = mg\left[\frac{a+x}{a}\right].$$

$$\cancel{m}\frac{d^2x}{dt^2} = \cancel{m}g - \cancel{m}g\left(\frac{a+x}{a}\right) \Rightarrow \frac{d^2x}{dt^2} = \frac{-gx}{a} \quad -(1)$$

$$\mu = \frac{g}{a} \Rightarrow \boxed{T = \frac{2\pi\sqrt{a}}{\sqrt{g}}}$$

$$(1) \Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{gx^2}{a} + A. \quad \Rightarrow \quad 0 = -\frac{g\,4a^2}{a} + A \Rightarrow A = 4ag.$$

$$\left(\frac{dx}{dt}\right)^2 = -\frac{gx^2}{a} + 4ag.$$

So, velocity at $x = a$ (at X) $\Rightarrow$ $v^2 = -ga + 4ga \Rightarrow V = \sqrt{3ag}$.

$\lfloor$ Time above $X \equiv 2\sqrt{\frac{3a}{g}}$

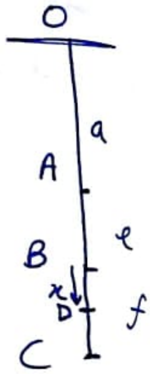
$$\frac{dx}{dt} = -\sqrt{\frac{4a^2g - gx^2}{a}} \Rightarrow -\sqrt{\frac{g}{a}}\sqrt{4a^2 - x^2}$$

$$\sqrt{\frac{a}{g}}\frac{dx}{\sqrt{4a^2 - x^2}} = dt \Rightarrow \sqrt{\frac{a}{g}}\cos^{-1}\left(\frac{x}{2a}\right)\Big]_{2a}^{-a} = t\Big]_0^t$$

$$\boxed{\sqrt{\frac{a}{g}}\,\frac{2\pi}{3} = t_1} \equiv Z \text{ to } X.$$

$$\text{Total Time} \equiv 2\left(\sqrt{\frac{3a}{g}} + \sqrt{\frac{a}{g}}\,\frac{2\pi}{3}\right) \text{ Ans}$$

Q Heavy particle is attached by elastic string, stretching the string by quantity e. It is drawn by additional distance f and let go; determine the height to which it will arise if $f^2 - e^2 = 4ae$, a being unstretched length of string.



m, $(\lambda, a) \equiv$ coefficient, length

At $D \equiv T_D = \lambda \frac{e+x}{a}$.

$$m\frac{d^2x}{dt^2} = mg - \lambda\frac{e+x}{a} = -\frac{\lambda x}{a}.$$

$$\boxed{\begin{array}{l}\frac{\lambda e}{a} = mg \\ \rightarrow \frac{\lambda}{am} = \frac{g}{e}\end{array}}$$

$$\rightarrow \quad \frac{d^2x}{dt^2}\cdot 2\frac{dx}{dt} = -\frac{\lambda}{m}\frac{x}{a}\,2\left(\frac{dx}{dt}\right).$$

$$\left(\frac{dx}{dt}\right)^2 = A - \frac{g}{e}x^2.$$

$$\left(\frac{dx}{dt}\right)^2 = \frac{g}{e}(f^2 - x^2)$$

At A, let velocity be v_1 $(x = -e)$

$$v_1^2 = \frac{g}{e}(f^2 - e^2) = 4ag.$$

$$\cancel{2}\,4a\cancel{g} = \cancel{2}\cancel{g}H \Rightarrow \boxed{H = 2a} \quad \text{Ans}$$

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* 2-D motion

$$\omega = \frac{d\theta}{dt} = \frac{vP}{r^2} \Big/ \frac{v_{B\,rel\,A} \perp AB}{AB}$$

2-Dimensions

(Q) A particle is acted on by a force parallel to y-axis with acc towards x-axis μy^{-2}. At $y=a$ projected // to x axis with $v=\sqrt{2\mu/a}$. Find its path

$$\frac{d^2y}{dt^2} = -\frac{\mu}{y^2} \qquad \frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}.$$

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + A \quad \Rightarrow \text{At } y=a,\ \frac{dy}{dt}=0 \Rightarrow A = -\frac{2\mu}{a}.$$

$$\frac{dy}{dt} = -\sqrt{2\mu}\sqrt{\left(\frac{1}{y} - \frac{1}{a}\right)}$$

$$\frac{dy}{dx} = \frac{-\sqrt{a}\sqrt{\frac{1}{y} - \frac{1}{a}}}{} \Rightarrow \frac{-dy\ \cancel{(-\sqrt{a})}}{\sqrt{\frac{a-y}{ay}}\ (\sqrt{a})} = dx.$$

$$\Rightarrow -\sqrt{\frac{y}{a-y}}\,dy = dx.$$

$$y = a\cos^2\theta \Rightarrow dy = -2a\cos\theta\sin\theta\,d\theta$$

$$\frac{\cancel{\sqrt{a}}\cos\theta \cdot 2a\cos\theta\,\cancel{\sin\theta}\,d\theta}{\cancel{\sqrt{a}}\,\cancel{\sin\theta}} = dx$$

$$2a\int\cos^2\theta\,d\theta = dx.$$

$$x = a\int 1+\cos 2\theta\,d\theta = a\left(\theta + \frac{\sin 2\theta}{2}\right) + C.$$

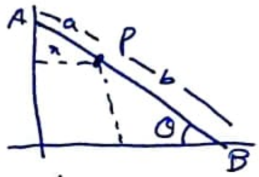
Let $x=0$, when $y=a \Rightarrow \theta=0. \Rightarrow C=0$

$$\left.\begin{aligned} x &= \frac{a}{2}(2\theta + \sin 2\theta) \\ y &= \frac{a}{2}(1+\cos 2\theta) \end{aligned}\right\} \text{Cycloid}$$

Boxed

(3) A rod moves with ends sliding on OX and OY. $P(x,y)$ and angular velocity ω of rod is constant, show acceleration of P along axes are $-x\omega^2$, $-y\omega^2$ and result acc. $= OP.\omega^2$ towards O.

$\Rightarrow \frac{d\theta}{dt} = \omega$ (constant) $\Rightarrow \frac{d^2\theta}{dt^2} = 0$.



P is fixed $\equiv$ AP be a, PB be b.

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt} \qquad \frac{dy}{dt} = b\cos\theta\frac{d\theta}{dt}$.

$\frac{d^2x}{dt^2} = -a\sin\theta\frac{d^2\theta}{dt^2} - a\cos\theta\left(\frac{d\theta}{dt}\right)^2$; $\frac{d^2y}{dt^2} = -b\sin\theta\left(\frac{d\theta}{dt}\right)^2 + b\cos\theta\frac{d^2\theta}{dt^2}$

$= -b\sin\theta\,\omega^2$

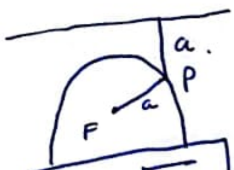
$= -a\cos\theta\,\omega^2$

$= -y\omega^2$.

$= -x\omega^2$

Resultant $= -\omega^2 x\hat{i} - \omega^2 y\hat{j} = -\omega^2(x\hat{i} + y\hat{j}) = -\omega^2\vec{OP}$

$= \boxed{\omega^2\vec{PO}}$ Ans

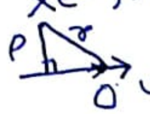


$\boxed{v_P = \sqrt{2ga}}$

* Projectile $\equiv p^2 = ar$.

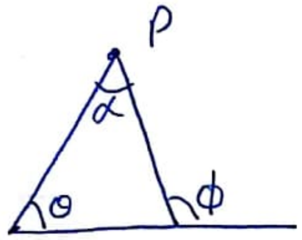
Angular velocity $\equiv$

- B to rel to A $\Rightarrow \frac{v_B \text{ rel } A \perp \text{ to } AB}{|AB|}$.



- w.r.t fixed pt $\Rightarrow \frac{vP}{r^2}$

(18) If points moves such that its angular velocity about two fixed pts is same, show it describes a circle.



So, $\frac{d\theta}{dt} = \frac{d\phi}{dt} \Rightarrow \frac{d(\theta-\phi)}{dt} = 0$

$\Rightarrow \theta - \phi$ is Constant

$\theta - \phi = \alpha$

So, P subtends a constant angle at A, B.
Thus a circle

* $v = \frac{dr}{dt} e_r + r\frac{d\theta}{dt} e_\theta$

radial

transverse

$$\boxed{\begin{aligned} \frac{de_r}{dt} &= \frac{d\theta}{dt} e_\theta \\ \frac{de_\theta}{dt} &= -\frac{d\theta}{dt} e_r \end{aligned}}$$

$$a = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\} e_r + \left\{\frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]\right\} e_\theta$$

5) Particle moves along $r = 2a\cos\theta$ s.t acc towards Origin is O. Find transverse acceleration.

$$a_r = 0 \Rightarrow \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0. \quad \text{— (1)}$$

$$\frac{dr}{dt} = -2a\sin\theta \frac{d\theta}{dt} \Rightarrow \frac{d^2 r}{dt^2} = -2a\sin\theta \frac{d^2\theta}{dt^2} - 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$-2a\sin\theta \frac{d^2\theta}{dt^2} = 2a\cos\theta\left(\frac{d\theta}{dt}\right)^2 + r\left(\frac{d\theta}{dt}\right)^2. \text{ Putting in (1)}$$

$$\frac{+\cancel{\sin\theta}\, d^2\theta/dt^2}{d\theta/dt} = \frac{d\theta}{dt}\,\frac{(-2\cos\theta)}{\sin\theta}$$

$$\Rightarrow \log\left(\frac{d\theta}{dt}\right) = -2\log\sin\theta + \log C$$

$$\frac{d\theta}{dt} = C\operatorname{cosec}^2\theta$$

$$a_t \Rightarrow \frac{1}{r}\frac{d}{dt}\left(r^2 C\operatorname{cosec}^2\theta\right) = \frac{1}{r}\left[\cancel{2Cr\cos^2\theta}\ \frac{d}{dt}\left(4a^2 C\cot^2\theta\right)\right]$$

$$= \frac{1}{r}\left(4a^2 C \cdot 2\cot\theta\,(-\operatorname{cosec}^2\theta)\frac{d\theta}{dt}\right) = \boxed{-4aC^2\operatorname{cosec}^2\theta}$$

Q. Velocities along and $\perp r$ to radius vector are λr^2, $\mu\theta^2$. Find path, and both accelerations

$\Rightarrow$ $\frac{dr}{dt} = \lambda r^2$ $\qquad r\frac{d\theta}{dt} = \mu\theta^2$.

$$\frac{dr}{r\,d\theta} = \frac{\lambda r^2}{\mu\theta^2} \Rightarrow \mu\frac{dr}{\lambda r^3} = \frac{d\theta}{\theta^2} \Rightarrow \boxed{\frac{\mu}{\lambda(-2)r^2} = \frac{-1}{\theta} + c}$$

path.

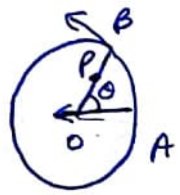
$$\frac{d^2r}{dt^2} = 2\lambda r.\frac{dr}{dt} = 2\lambda r.\lambda r^2.$$

$$\frac{d^2\theta}{dt^2} = \frac{2\mu\theta}{r}.\frac{\mu\theta^2}{r} - \frac{\mu\theta^2}{r^2}.\lambda r^2 \quad \times \text{ No need.}$$

Put in formula $\boxed{r\frac{\partial}{dt}\left(r^2\frac{d\theta}{dt}\right)} = a_{cc}$

Q. Insect crawls at constant rate u along spoke of cart of radius a. Cart is moving at v. Find acc along & $\perp r$ to spoke.

Initial ≡ Spoke at OA, Ant at O.



After t ≡ $OP = ut$. — $r = ut$

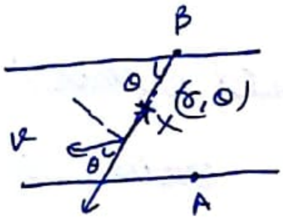
Let $P \equiv (r, \theta)$. — $\theta = \angle AOP = \frac{v}{a}$

Angular velocity ≡ B rel to O $\Rightarrow \frac{v}{OB} = \frac{v}{a} = \frac{d\theta}{dt}$.

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -ut\frac{v^2}{a^2}$$

$$a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{1}{ut}\frac{d}{dt}\left(\frac{v}{a}u^2t^2\right) = \frac{2uv}{a}$$

A boat is rowed with constant u starts from A, river vel $= v$ and it always points to B (exactly opposite A). Find path



$$\frac{dr}{dt} = v\cos\theta - u$$

$$\frac{r\,d\theta}{dt} = -v\sin\theta.$$

$$\frac{dr}{r\,d\theta} = \frac{v\cos\theta - u}{-v\sin\theta} \Rightarrow \frac{dr}{r} = \frac{u - v\cos\theta}{v\sin\theta}\,d\theta.$$

$$\Rightarrow \frac{dr}{r} = \left(\frac{u}{v}\operatorname{cosec}\theta - \cot\theta\right)d\theta$$

$$\Rightarrow \int\frac{dr}{r} = +\frac{u}{v}\log\left(\tan\frac{\theta}{2}\right) - \log\sin\theta + \log c$$

$$\frac{r}{c} = \frac{(\tan\theta/2)^{u/v}}{\sin\theta}$$

---

* A straight smooth tube revolves with ω in horizontal plane about an end which is fixed.
At $t=0$ particle is at a from fixed end and moving with V along tube. Find distance at time t. (Show $a\cosh\omega t + \frac{V}{\omega}\sinh\omega t$)

· No external force in radial direction as tube is smooth.

$$a_c = \frac{d^2r}{dt^2} - \left(\frac{d\theta}{dt}\right)^2 r = 0 \Rightarrow \frac{d^2r}{dt^2} = \omega^2 r.$$

$$\frac{d\theta}{dt} = \omega$$

$$\left(\frac{dr}{dt}\right)^2 = A + \omega^2 r^2$$

If Show

$(D^2 - \omega^2)r = 0.$

$V^2 - \omega^2 a^2 = A^2.$

$$\frac{dr}{dt} = \sqrt{V^2 - \omega^2 a^2 + \omega^2 r^2}$$

$$\frac{dr}{\sqrt{\frac{V^2}{\omega^2} - a^2 + r^2}} = dt$$

$t=0$, $r=a$, $v=V$.

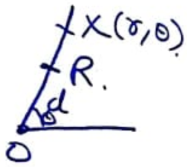
$r = A\cosh\omega t + B\sinh\omega t \Rightarrow A = a$

$V = A\omega\sinh\omega t + B\omega\cosh\omega t \Rightarrow B = V/\omega$

$$\boxed{r = a\cosh\omega t + \frac{V}{\omega}\sinh\omega t}$$ Ans

Scanned by CamScanner

⑧ A sliding ring on a rod distant d from end O. Rod revolves horizontally about O. Show ring $\equiv r = d\cosh\theta$.



As rod is smooth, no force on ring along OX.

So, radial acc $= 0$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0.$$

As uniform rotation $\frac{d\theta}{dt} = \omega$ (const).

$$\left(\frac{dr}{dt}\right)^2 = r^2\omega^2 + A.$$

at $r = d$, $\frac{dr}{dt} = 0$

$\Rightarrow A = -d^2\omega^2$

$$\frac{dr}{dt} = \omega\sqrt{r^2 - d^2}.$$

$$\frac{dr}{d\theta}\cdot\cancel{\frac{d\theta}{dt}} = \cancel{\omega}\sqrt{r^2-d^2} \Rightarrow d\theta = \frac{dr}{\sqrt{r^2-d^2}}.$$

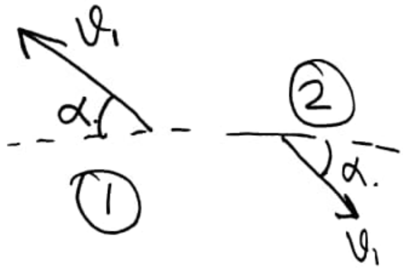
$$\theta + B = \cosh^{-1}\left(\frac{r}{d}\right).$$

when $r = d$, $\theta = 0 \Rightarrow B = 0$

$$r = d\cosh(\theta).$$

IMP * Finding Latus rectum

$$\boxed{LR = \frac{2}{g} v_1^2 \cos^2\alpha.}$$



* When left ~~from A~~

$$m\frac{d^2s}{dt^2} = mg\sin\theta \quad (+ve)$$

* Equation of Projectile

$$\boxed{y = x\tan(-\alpha) - \frac{1}{2}\frac{gx^2}{v_1^2\cos^2(-\alpha)}}$$

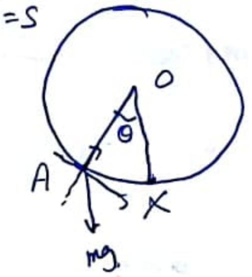
Constrained Motion

* Path of projectile $\equiv$ $\boxed{y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{V_1^2\cos^2\alpha}}$

(2) Q. Particle inside and at lowest point of circular wire (a) with velocity just sufficient to carry it to highest point. Show Reaction is zero after time t. Find t.

Let m be projected from X with u, $\widehat{AX} = s$

At any time t,



$$\frac{mV^2}{a} = R - mg\cos\theta$$

$$s = a\theta$$

$$\frac{md^2s}{dt^2} = -mg\sin\theta$$

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\frac{d\theta}{dt}\, 2a^2\frac{d^2\theta}{dt^2} = -2ag\sin\theta\frac{d\theta}{dt} \Rightarrow V^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

$$\theta = \pi, V = 0 \rightarrow A = +2ag.$$

$$V^2 = 2ag(\cos\theta + 1).$$

$$R = m\left(g\cos\theta + 2g(\cos\theta + 1)\right) \Rightarrow.$$

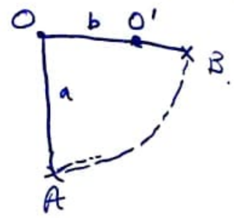
R is 0 at $\Rightarrow$ $3\cancel{g}\cos\theta = -2\cancel{g} \Rightarrow \cos\theta = -\frac{2}{3} \Rightarrow \theta_1 = \cos^{-1}\left(-\frac{2}{3}\right)$

$$a\frac{d\theta}{dt} = \sqrt{2ag(1+\cos\theta)} \Rightarrow \frac{\sqrt{a}\,d\theta}{\sqrt{2g(2\cdot\cos^2\frac{\theta}{2})}} = dt \Rightarrow \frac{\sqrt{a}\,d\theta}{2\cos\frac{\theta}{2}\sqrt{g}} = dt.$$

$$t = \frac{1}{2}\sqrt{\frac{a}{g}}\int_0^{\theta_1}\sec\frac{\theta}{2}\,d\theta \Rightarrow \frac{1}{\cancel{2}}\sqrt{\frac{a}{g}}\cdot\cancel{2}\left[\log\left\{\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right\}\right]_0^{\theta_1}$$

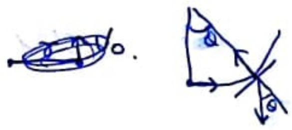
$$= \sqrt{\frac{a}{g}}\log\left(\sqrt{6} + \sqrt{5}\right) \text{ Ans}$$

Q. Hanging from O. a nail at O' ($b < a$)
Find min v such that complete revolution round the nail.



$\Rightarrow$ Condition for complete rev.

$\quad L\ v_0$ at $\theta = 0$



$$T - mg\cos\theta = \frac{mv^2}{r} \quad \text{—— (2)}$$

$$mg\sin\theta = -m\frac{d^2s}{dt^2} = -ma\frac{d^2\theta}{dt^2}$$

$$-2ag\sin\theta\frac{d\theta}{dt} = a^2 2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}$$

$$\left(a\frac{d\theta}{dt}\right)^2 = v^2 = 2ga\cos\theta + A \quad \Rightarrow \quad A = v_0^2 - 2ag.$$

$$v^2 = 2ga\cos\theta - 2ga + v_0^2 \quad \text{—— (1)}$$

$$T = mg\cos\theta + m(2g(\cos\theta - 1)) + \frac{mv_0^2}{a} \quad \text{— from (2)}$$

$$T = 3mg\cos\theta - 2mg + \frac{mv_0^2}{a}$$

To complete $T > 0\ \forall\theta \Rightarrow$ -ve most value at $\theta = \pi$.

$$-3mg - 2mg + \frac{mv_0^2}{a} > 0 \Rightarrow \boxed{v_0^2 > 5ag}$$

v at $\theta = \pi/2 \Rightarrow \boxed{v^2 = 3ag}$ — from (1)

If at A, $v_1 \Rightarrow$

$$v_B^2 = -2ag + v_1^2. \qquad \text{Min to encircle} \Rightarrow v_B^2 > 3(a-b)g.$$

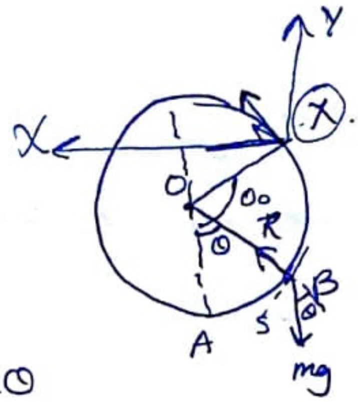
$$-2ag + v_1^2 > 3(a-b)g. \Rightarrow \boxed{v_1^2 > 2ag + 3(a-b)g}$$

$$\boxed{U^2 = \sqrt{\frac{ag}{2}}(\sqrt{3}+1)} \Leftarrow \frac{U^2 - 2ag}{3ag} = \frac{1}{\sqrt{3}} \Leftarrow \cos^2\alpha = \frac{1}{3}$$

Q. Particle along inside of smooth ring (a) from lowest point. Projection velocity such after leaving circle, it may pass through centre is $\left(\sqrt{\frac{1}{2}ag}\right)(\sqrt{3}+1)$

Let particle of m be projected with v_0 from A at t=0.

After t, let m be at B, arc AB = S.



$$R - mg\cos\theta = \frac{mv^2}{a} \Rightarrow (*)$$

$$S = a\theta$$

$$mg\sin\theta = -\frac{md^2s}{dt^2} \qquad \frac{d^2s}{dt^2} = \frac{ad^2\theta}{dt^2}$$

$$\Rightarrow -2ag\sin\theta\frac{d\theta}{dt} = 2a^2\frac{d^2\theta}{dt}\frac{d\theta}{dt} \Rightarrow \left(\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

$$A = v_0^2 - 2ag$$

$$v^2 = 2ag(\cos\theta - 1) + v_0^2.$$

$$R = mg\cos\theta + 2mg(\cos\theta - 1) + \frac{mv_0^2}{a}$$

$$\hookrightarrow R = 0 \Rightarrow 3g\cos\theta = 2g - \frac{v_0^2}{a} \Rightarrow \boxed{\cos\theta_0 = \frac{2ga - v_0^2}{3ag}}$$

pt of leaving

Angle of projection $= \alpha = (\pi - \theta_0)$

Speed $\Rightarrow$ $\cancel{2ag\left(\frac{2ag - v_0^2 - 3ag}{3ag}\right) + v_0^2 \Rightarrow v_0^2 + \frac{2}{3}(-ag - v_0^2) \Rightarrow \frac{v_0^2}{3} - \frac{2ag}{3}}$ $= v_1^2$

Use (*) instead

$\boxed{v_1^2 = -ag\cos\theta_0}$

Consider parabolic motion from (X), setting origin at X $\Rightarrow$

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_1^2\cos^2\alpha}.$$

Centre (O) $\equiv (a\sin(\pi-\theta_0), -a\cos(\pi-\theta_0))$

$\downarrow$

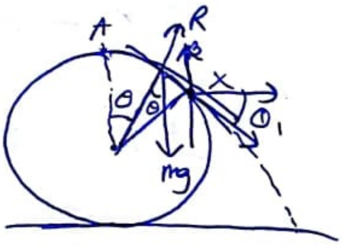
$(a\sin\alpha, -a\cos\alpha) \equiv (a\sin\theta_0, a\cos\theta_0)$

$$a\frac{\cos^2\theta_0}{\cos\theta_0} = -a\frac{\sin^2\theta_0}{\cos\theta_0} - \frac{1}{2}\frac{g a^2\sin^2\theta_0}{v_1^2\cos^2\theta_0}. \quad \text{Let it be in } \alpha$$

$$\frac{-2a}{ga^2\sin^2\theta} = -\frac{1}{v_1^2\cos^2\theta_0} \Rightarrow v_1^2 = \frac{a\tan^2\theta_0}{2g} \Rightarrow \frac{v_0^2}{3} - \frac{2ag}{3} = -\frac{a\tan^2\theta_0}{2g}$$

$$= -\frac{a}{2g}\left(\left(\frac{3ag}{2ag - v_0^2}\right)^2 - 1\right)$$

$\boxed{8}$ A particle slides down the arc of a smooth vertical circle of radius a, slightly displaced at highest point. Find where it leaves circle, prove it strikes horizontal plane at a distance $\frac{5}{27}(\sqrt{5}+4\sqrt{2})a$ from vertical diameter.



Let it start from A, mass m.
At t,
arc $\widehat{AB} = S$.

$$mg\cos\theta - R = \frac{mv^2}{a}. \qquad S = a\theta$$

$$mg\sin\theta = \frac{md^2s}{dt^2} \qquad \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

$$2ag\sin\theta\frac{d\theta}{dt} = 2a^2\frac{d^2\theta}{dt^2}\frac{d\theta}{dt}$$

$$-2ag\cos\theta + A = \left(\frac{ad\theta}{dt}\right)^2 = v^2. \Rightarrow \quad \theta=0,\ v=0 \quad A = 2ag$$

$$v^2 = 2ag(1-\cos\theta)$$

$$R = mg\cos\theta - 2mg(1-\cos\theta) = 3mg\cos\theta - 2mg.$$

$$R=0 \Rightarrow \frac{2mg}{3mg} = \cos\theta \Rightarrow \boxed{\cancel{\theta = 60°}} \quad \cos\theta_1 = \frac{2}{3} \quad \text{at } X$$

$$\boxed{v_1^2 = \frac{2}{3}ag}$$

Let X be origin

Angle of projection $= -\theta_1$

$$y = x\tan(-\theta_1) - \frac{1}{2}\frac{gx^2}{v_1^2\cos^2(-\theta_1)}$$

$$= -\frac{x\sqrt{5}}{2} - \frac{1\,gx^2}{2\cdot\frac{2}{3}ag\cdot\frac{4}{9}} \Rightarrow -\frac{\sqrt{5}x}{2} - \frac{27x^2}{16a} = y.$$

y of ground $= -a - a\cos\theta_1 = -\frac{5}{3}a.$

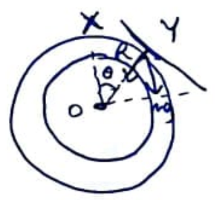
$$-\frac{5}{3}a = -\frac{\sqrt{5}x}{2} - \frac{27}{16}\frac{x^2}{a}$$

$$\Rightarrow -80a^2 + 24\sqrt{5}ax + 81x^2 = 0.$$

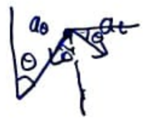
$$\frac{-24\sqrt{5}a \pm \sqrt{2880a^2 + 4.81.80a^2}}{162.} = X$$

Ans $\Rightarrow \boxed{X + a\sin\theta_1}$ Ans

Q. Starts from X with $v_0 = \sqrt{2ag}$.
Find Pressure on curve when vertical component of acceleration is maximum.



Vertical acceleration $(A_v) = \boxed{a_0\cos\theta + a_t\sin\theta}$



$$\frac{mv^2}{a} = R + mg\cos\theta \qquad S = a\theta.$$

$$mg\sin\theta = \frac{md^2s}{dt^2}$$

$$A - 2ag\cos\theta = \left(\frac{a\,d\theta}{dt}\right)^2 = v^2 \Rightarrow \quad 2ag + 2ag = A \Rightarrow A = 4ag.$$

$$R = 2mg\left(2 - \cos\theta - \frac{1}{2}\cos\theta\right)$$

$$2ag(2 - \cos\theta) = v^2.$$

$$\boxed{A_v = \frac{v^2}{a}\cos\theta + g\sin^2\theta}$$

$$= (4g - 2g\cos\theta)\cos\theta + g\sin^2\theta.$$

Max $A_v \Rightarrow \frac{d\left(4g\cos\theta - 3g\cos^2\theta + g\sin^2\theta\right)}{d\theta} \Rightarrow -4g\sin\theta + 6g\cos\theta\sin\theta = 0$

$$2g\sin\theta(3\cos\theta - 2) = 0.$$

$\boxed{\cos\theta_0 = \frac{2}{3}} \Rightarrow \frac{d^2A_v}{d\theta} = -4g\cos\theta + 6g\cos^2\theta - 6g\sin^2\theta < 0$ at θ_0

$$R = 2mg\left(2 - \frac{2}{3} - \frac{1}{3}\right) = \boxed{2mg} \text{ Ans}$$

Cycloid

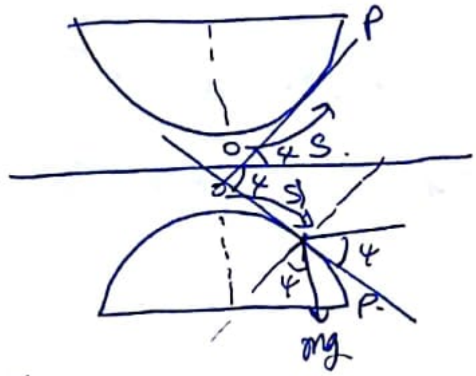
$$\boxed{\begin{aligned} x &= a(\theta+\sin\theta) \\ y &= a(1-\cos\theta) \\ \theta &= 2\psi \end{aligned}}$$

$S = 4a\sin\psi$ — $\pi/2$ at top [cusp], 0 at base.

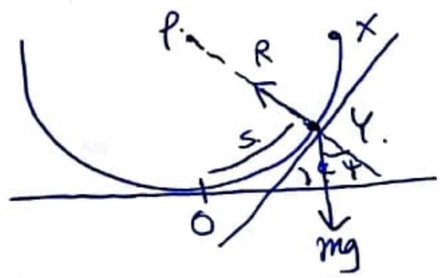
$\frac{mv^2}{\rho} \longrightarrow \frac{ds}{d\psi} = 4a\cos\psi$

S measured always from O.

$S^2 = 8ay$



Q Show when it has fallen through half distance measured along arc to the vertex, 2/3 of time of descent would have elapsed.



$$R - mg\cos\psi = \frac{mv^2}{\rho}$$

$$-mg\sin\psi = m\frac{d^2s}{dt^2}$$

$S = 4a\sin\psi$

~~$\frac{d^2s}{dt^2} = -4a\cos\psi\frac{d^2\psi}{dt^2}$~~

$\boxed{\sin\psi = \frac{S}{4a}}$

$\rho = \frac{ds}{d\psi} = 4a\cos\psi$

~~$-g\sin\psi \cdot 2\frac{ds}{dt} = -4a\sin\psi\frac{d^2\psi}{dt^2}$~~

$$2\frac{ds}{dt}\frac{d^2s}{dt^2} = -\frac{gs}{4a}\cdot 2\frac{ds}{dt} \Rightarrow \left(\frac{ds}{dt}\right)^2 = -\frac{gs^2}{4a} + A.$$

$$0^2 = -g\frac{(4a)^2}{4a} + A \Rightarrow A = 4ag$$

$$\left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\left(16a^2 - s^2\right)$$

$$\frac{ds}{dt} = -\frac{1}{2}\sqrt{\frac{g}{a}}\sqrt{16a^2-s^2} \Rightarrow -\frac{ds}{\sqrt{16a^2-s^2}} = \frac{1}{2}\sqrt{\frac{g}{a}}\,dt.$$

$$\cos^{-1}\frac{S}{4a}\Big]_{4a}^{2a} = \frac{1}{2}\sqrt{\frac{g}{a}}\Big]_0^t$$

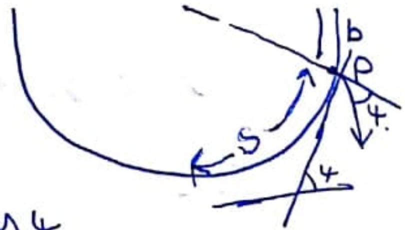
Total $= \sqrt{\frac{a}{g}}\pi$

$$2\sqrt{\frac{a}{g}}\left(\cos^{-1}\left(\tfrac{1}{2}\right) - \cos^{-1}(1)\right) = t = \boxed{2\sqrt{\frac{a}{g}}\cdot\frac{\pi}{3}}$$ Ans.

2/3 time

Q Particle starts from any point P of $s = 4a \sin\psi$. Prove time of descent is $\pi\sqrt{\frac{a}{g}}$ [independent of point]

Let particle start from $P \equiv S_0 = (4a - b)$



$$R - mg\cos\psi = \frac{mv^2}{\rho} \qquad s = 4a\sin\psi$$

$$\cancel{m}g\sin\psi = -\cancel{m}\frac{d^2s}{dt^2} \qquad \rho = \frac{ds}{d\psi} = 4a\cos\psi$$

$$-\frac{gs}{4a}\cdot 2\frac{ds}{dt} = \frac{d^2s}{dt^2}\cdot 2\frac{ds}{dt} \Rightarrow \left(\frac{ds}{dt}\right)^2 = A - \frac{gs^2}{4a}$$

At $s = 4a - b$, $\frac{ds}{dt} = 0 \Rightarrow \quad A = \frac{g(4a-b)^2}{4a}$

$$\left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}\left((4a-b)^2 - s^2\right)$$

$$\frac{ds}{dt} = -\frac{\sqrt{g}}{\sqrt{4a}}\sqrt{(4a-b)^2 - s^2} \Rightarrow \frac{-ds}{\sqrt{(4a-b)^2 - s^2}} = \sqrt{\frac{g}{4a}}\,dt$$

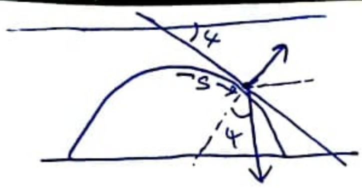
$$\cos^{-1}\frac{s}{(4a-b)}\Bigg]_{4a-b}^{0} = \sqrt{\frac{g}{4a}}\,t$$

$$\frac{\pi}{\cancel{2}} - 0 = \frac{1}{\cancel{2}}\sqrt{\frac{g}{a}}\,t \Rightarrow \boxed{t = \pi\sqrt{\frac{a}{g}}} \text{ Ans}$$

Use

$\left(\frac{ds}{dt}\right)^2 = v^2 \leftarrow$ in other Qs where reqd

8. Particle slides from vertex.
Prove it leaves curves after it has fallen through half vertical height.



At time t,

$$mg\cos\psi - R = \frac{mv^2}{\rho}$$

$$mg\sin\psi = m\frac{d^2s}{dt^2}$$

$s = 4a\sin\psi$

$\frac{ds}{d\psi} = \rho = 4a\cos\psi.$

$$\frac{d^2s}{dt^2}\cdot\frac{2ds}{dt} = \frac{gs}{4a}\cdot 2\frac{ds}{dt}$$

$s=0,\ \frac{ds}{dt}=0.$

$A=0$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \frac{gs^2}{4a} + \cancel{A} 0.$$

$$v^2 = \frac{gs^2}{4a}.$$

$$R = mg\cos\psi - \frac{mgs^2}{4a\cdot 4a\cos\psi}.$$

$$= mg\cos\psi - \frac{mg\cdot 16a^2\sin^2\psi}{16a^2\cos\psi} = \frac{mg}{\cos\psi}(\cos^2\psi - \sin^2\psi)$$

$R=0 \Rightarrow \sin\psi = \cos\psi. \Rightarrow \tan^2\psi = 1$ or $\psi = 45°.$

$$s = \frac{4a}{\sqrt{2}} = 2\sqrt{2}a$$

Now $s^2 = 8ay \Rightarrow 8a^2 = 8ay. \Rightarrow y = a = \frac{1}{2}h$ Ans

Total height $\Rightarrow 16a^2 = 8ay \Rightarrow 2a = h$

Central Orbits

- Path under a central force

* A central orbit is a plane curve.

As O is centre of force, only acceleration is towards O.

So, $\frac{d^2\bar{r}}{dt^2} = k\bar{r} \Rightarrow \left|\frac{d^2\bar{r}}{dt^2} \times \bar{r}\right| = 0.$

$$\frac{d^2 r}{dt^2} \times r + \frac{dr}{dt} \times \frac{dr}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\cancel{r}\frac{d\bar{r}}{dt} \times \bar{r}\right) = 0$$

$$\frac{d\bar{r}}{dt} \times \bar{r} = \hat{h} \text{ (constant)} \Rightarrow \bar{r}\cdot\left(\frac{d\bar{r}}{dt} \times \bar{r}\right) = r \cdot \hat{h}$$

$$\downarrow$$

$$0$$

$$\Rightarrow \bar{r}\cdot\bar{h} = 0.$$

So, $\bar{r}$ is always $\perp$r to a constant vector.

- $r^2 \frac{d\theta}{dt} = h \equiv$ Angular momentum [Conserved in central orbit]

(*) DE of central orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \qquad v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$P = \frac{h^2}{p^3}\frac{dp}{dr}$$

- Area description rate $\equiv h/2$.

To find Time Period $= T \cdot \frac{h}{2} = \pi ab \Rightarrow T = \frac{2\pi ab}{h}$

$$= \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$h^2 = \mu l \quad l = b^2/a$

Given orbit, find force

[Generally log in polar form when r^n]

Q $r^n = a^n \cos n\theta$

$u^{-n} = a^n \cos n\theta$

$-n \log u = n \log a + \log \cos n\theta$

$+\frac{n}{u}\frac{du}{d\theta} = \frac{+n \sin n\theta}{\cos n\theta}$

$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + u\, n \sec^2 n\theta = u \tan^2 n\theta + nu \sec^2 n\theta$

$P = h^2u^3\left[\tan^2 n\theta + n\sec^2 n\theta + 1\right] = h^2u^3\left[\sec^2 n\theta + n\sec^2 n\theta\right]$

$= h^2u^3(1+n)a^{2n}u^{2n}$

*

$= h^2a^{2n}(1+n)\frac{1}{r^{2n+3}}$ Ans

Q Velocity at any pt of a central orbit is $1/n$ th of what it would be for a circular orbit at same distance. Show central force $\propto \frac{1}{r^{2n^2+1}}$ and orbit is $r^{n^2-1} = a^{n^2-1}\cdot\cos(n^2-1)\theta$.

[... orbit]

→ Force P, distance r.

$P = \frac{V_c^2}{r} \Rightarrow$ so $v^2 = \frac{V_c^2}{n^2} \Rightarrow v^2 = \frac{Pr}{n^2}$

$v^2 = \underbrace{h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{P}{n^2u}}_{d/d\theta}$

$2h^2u\frac{du}{d\theta} + 2h^2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = \frac{-P}{n^2u^2}\frac{du}{d\theta} + \frac{1}{n^2u}\cancel{\left(\frac{dP}{d\theta}\right)}\frac{dP}{du}\frac{du}{d\theta}$.

$2h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{dP}{du}\frac{1}{n^2u} - \frac{P}{n^2u^2}$

$\frac{P}{n^2u^2} + \frac{2n^2P}{n^2u^2} = \frac{dP}{du}\frac{1}{n^2u} \Rightarrow \frac{dP}{P} = du\left(\frac{u(1+2n^2)}{u^2}\right)$

$\log P = \log u^{2n^2+1} + \log C \Rightarrow P \propto u^{2n^2+1} \Rightarrow P \propto \frac{1}{r^{2n^2+1}}$

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Q. Show that only law for central attraction for which velocity in a circle at any distance is equal to velocity acquired in falling from ∞ to the distance is that of inverse cube.

Central acc. $\Rightarrow$ $P = f'(r)$.

From ∞ $\Rightarrow$ $v\frac{dv}{dr} = -f'(r) \Rightarrow v^2 = -2\int f'(r) + A$.

$v^2 = -2f(r) + A$

$\frac{V_c^2}{r} = f'(r) \Rightarrow V_c^2 = rf'(r)$.

So, $rf'(r) = -2f(r) + A \Rightarrow r^2f'(r) + 2rf(r) = Ar$.

$d(r^2 f(r)) = d\left(\frac{Ar^2}{2}\right)$

$r^2 f(r) = \frac{Ar^2}{2} + B$.

$f(r) = \frac{A}{2} + \frac{B}{r^2}$.

$$\boxed{f'(r) = \frac{-2B}{r^3}}$$

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* Apse

- A point on central orbit where r has min or max value
- At a apse radius vector is $\perp r$ to tangent

Features ≡ $\frac{du}{d\theta} = 0$ $\qquad \phi = 90°$ $\qquad p = r$.

ALITER: $u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^2 \sin^2\phi}$

- When central acc varies as some integral power of distance there are at most 2 apsidal distances.

Let $P = \mu r^n = \mu u^{-n}$

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} \Rightarrow \quad h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \mu u^{-(n+2)}.$$

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^{-(n+1)}}{-(n+1)} + A$$

Put $du/d\theta = 0$

$$h^2u^2 = -\mu \frac{u^{-(n+1)}}{n+1} + A \Rightarrow r^{n+3} - \frac{(n+1)}{\mu} A r^2 + \frac{(n+1)}{\mu} h^2 = 0$$

This equation cannot have more than 2 sign changes
So, Max 2 roots.

- After integrating $\frac{du}{d\theta}$ (Final one)

① Take constant $\theta + \beta = \ldots$
② Take $\theta = 0$ at apse ($r = a$) [even if project at angle & not apse]

TRICK TO INTEGRATE

- $\frac{du}{d\theta} = g(u) \Rightarrow \frac{du}{d\theta} = u^{\alpha} f(u)$ where $\alpha > 0$ then put $u = \frac{1}{r}$

= Ar

Given force, find orbit

Q. $P = +\mu\left(\frac{5}{r^3} + \frac{8c^2}{r^5}\right)$ and m is projected from apse at a distance c with velocity $3\sqrt{\mu}/c$. Find orbit.

no -ve sign

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = +\mu\left[5u + 8c^2u^3\right]$$

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left[\frac{5}{2}u^2 + 2c^2u^4\right] + A$$

$$\frac{9\mu}{c^2} = \frac{h^2}{c^2} = 2\mu\left[\frac{5}{2c^2} + \frac{2}{c^2}\right] + A.$$

$\boxed{h^2 = 9\mu}$ $\qquad A = \cancel{\frac{9\mu}{c^2}} - \left(\cancel{\frac{5\mu}{2c^2}} \cancel{\frac{4\mu}{c^2}}\right) = \cancel{\frac{25\mu}{2c^2}} = 0.$

$$9\mu\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = 2\mu\left(\frac{5}{2}u^2 + 2c^2u^4\right)$$

$$9\left(\frac{du}{d\theta}\right)^2 = -4u^2 + 4c^2u^4. \qquad \boxed{\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}}$$

$$\frac{9}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{4c^2}{r^4} - \frac{4}{r^2}. = \frac{1}{9}(4c^2 - 4r^2)$$

IMP $\Rightarrow$ $\frac{dr}{d\theta} = -\frac{2}{3}\sqrt{c^2 - r^2} \Rightarrow \frac{2}{3}d\theta = \frac{-dr}{\sqrt{c^2 - r^2}}$

$$\cos^{-1}\left(\frac{r}{c}\right) = \frac{2\theta}{3} + B. \qquad \text{when } r = c,\ \theta = 0 \Rightarrow B = 0.$$

$\boxed{r = c\cos\frac{2\theta}{3}}$ Ans

. Find time of arrival at origin.

$$r^2 \frac{d\theta}{dt} = h$$

$\theta = 0$ at pt of projection

At origin $\Rightarrow c\cos\frac{2\theta}{3} = 0.$

$\frac{2\theta}{3} = \frac{\pi}{2} \Rightarrow \theta = \frac{3\pi}{4}.$

$$c^2\cos^2\frac{2\theta}{3}\, d\theta = ht.$$

$$\frac{c^2}{2h}\left(1 + \cos\frac{4\theta}{3}\right) = t\, dt.$$

$$t_1 = \int_0^{3\pi/4} \frac{c^2}{2.3\sqrt{\mu}} \left(1 + \cos\frac{4\theta}{3}\right) d\theta$$

$$= \frac{c^2}{6\sqrt{\mu}}\left[\frac{3\pi}{4} + \cancel{\sin\frac{4\theta}{3}}\Big|_0^{3\pi/4}\right] = \boxed{\frac{\pi c^2}{8\sqrt{\mu}} = t_1}$$

[9] Particle is attached to fixed point on plane by elastic string of length a. It is projected horizontally $\perp r$ to the string with KE = PE of string when extension is $3a/\sqrt{2}$. Find second apsidal distance

- Final Tension $= \lambda \frac{3a/\sqrt{2}}{a} = \frac{3\lambda}{\sqrt{2}}$.

Initial Tension $= 0$.

$PE = \frac{1}{2} \frac{3\lambda}{\sqrt{2}} \cdot \frac{3a}{\sqrt{2}} = \frac{9a\lambda}{4}$.

$\frac{9a\lambda}{4_2} = \frac{1}{2} mV^2 \Rightarrow V^2 = \frac{9a\lambda}{2m}$.

$T = \lambda \frac{r-a}{a}$.

$P = \frac{\lambda}{am}\left(\frac{1}{u} - a\right)$

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{P}{u^2}.$$

$$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \frac{2\lambda}{am}\left[\frac{-1}{2u^2} + \frac{a}{u}\right] + A$$

$$\frac{9a\lambda}{2m} = \frac{h^2}{a^2} = \frac{\lambda}{am}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + A.$$

$$\boxed{h^2 = \frac{9a^3\lambda}{2m}} \Rightarrow \frac{9a\lambda}{2m} + \frac{\lambda a}{m} - \frac{2\lambda a}{m} = \boxed{A = \frac{7a\lambda}{2m}}$$

$$\frac{9a^3\lambda}{2m}(u^2) = \frac{2\lambda}{am}\left(\frac{-a}{2u^2} + \frac{2a^2}{u}\right) + \frac{7a^2\lambda}{2m}.$$

$$9a^4u^4 = -2a + 4a^2u + 7a^2u^2.$$

Put in $r \Rightarrow 2r^4 - 4ar^3 - 7a^2r^2 + 9a^4 = 0$

$r = a$, $r = 3a$.

apsidal distance

No $-$ve sign in P

$-$ve sign for repulsive force

Pg-358 $\Rightarrow x^4 + y^4 = c^4$

[8] $P = \mu u^3$ is projected with velocity $\frac{\sqrt{\mu}}{a}$ at angle $\pi/4$ with initial distance a from centre of force. Show orbit is $r = ae^{-\theta}$.

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right)\cdot 2\frac{du}{d\theta} = 2\frac{\mu u^3}{u^2}\cdot\frac{du}{d\theta}.$$

$$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \mu u^2 + A.$$

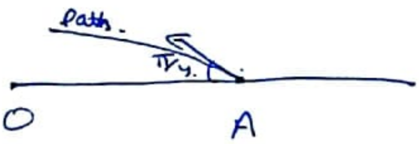
$v = \frac{\sqrt{\mu}}{a}$ at angle $\frac{\pi}{4}$ where $r = a$.

$\phi \equiv$ Angle $= \frac{\pi}{4}$,

$p = r\sin\phi = \frac{a}{\sqrt{2}} \Rightarrow \boxed{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \Rightarrow \frac{2}{a^2}}$ VVI

$$\frac{\mu}{a^2} = h^2\cdot\frac{2}{a^2} = \mu u^2 + A.$$

$\Rightarrow h^2 = \frac{\mu}{2}$; $A = 0$.



$$\frac{\cancel{\mu}}{2}\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \cancel{\mu} u^2$$

$$\left(\frac{du}{d\theta}\right)^2 = u^2 \Rightarrow \cancel{\frac{du}{d\theta} = -u}.$$

$\cancel{\log u = -\theta + c}$

$\cancel{u = Ke^{-\theta} \Rightarrow \frac{1}{r} = Ke^{-\theta}}$

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2}.$$

$\Rightarrow \frac{dr}{d\theta} = -r$ (-ve sign as r decreases with $\theta \uparrow$)

$\log r = -\theta + B.$ $\boxed{r = a,\ \theta = 0.}$ $\theta \neq \frac{\pi}{4}$

$\log r = -\theta + \log a \Rightarrow \boxed{r = ae^{-\theta}}$ Ans

Always taken 0

Planetary Motion

$$\boxed{P = \mu/r^2}$$

- $v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$. Ellipse $\quad$ l: b^2/a

$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]$. Hyperbola $\quad$ b^2/a

$v^2 = \mu\left(\frac{2}{r}\right)$. Parabola. $\quad$ $2a$.

- Kepler's Laws

I → Each planet describes an ellipse with sun at focus

$$\frac{l}{r} = 1 + e\cos\theta$$

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \qquad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{h^2u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2}$$

∴ Hence acc of each planet $\propto \frac{1}{r^2}$ towards sun.

II → Radius vector from Sun sweeps equal areas in equal times

III → $T^2 \propto a^3$ (cube of semi-major axis)

$$T = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{\sqrt{\mu l}} = \frac{2\pi a\cancel{b}\sqrt{a}}{\sqrt{\mu}\,\cancel{b}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$T^2 = \frac{4\pi^2a^3}{\mu}$$

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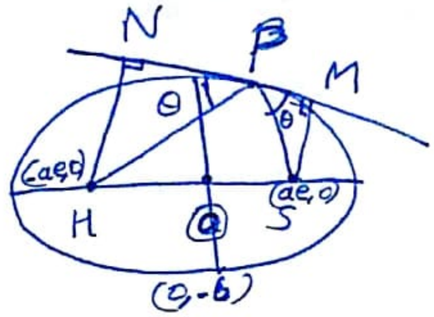
- Ellipse Properties.

  - $SP + HP = 2a$
  - $l = \frac{b^2}{a}$

    $b^2 = a^2(1-e^2)$
  - $SM \cdot HN = b^2$.

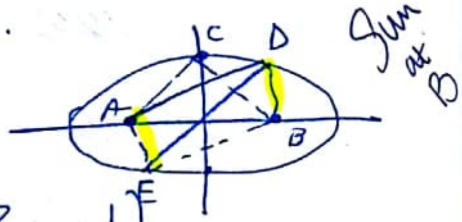


Q. A particle moving in ellipse. Prove velocity at end of minor axis is GM of velocities at ends of any diameter

$AC + BC = 2a$.

$AC = BC \Rightarrow$

$AC = a$



$$v_C^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right)$$

$$v_E^2 = \mu\left(\frac{2}{BE} - \frac{1}{a}\right) \qquad v_D^2 = \mu\left(\frac{2}{BD} - \frac{1}{a}\right)$$

$BE + AE = 2a$

$BD + AD = 2a$

$\boxed{BD = AE}$

$BE + BD = 2a$.

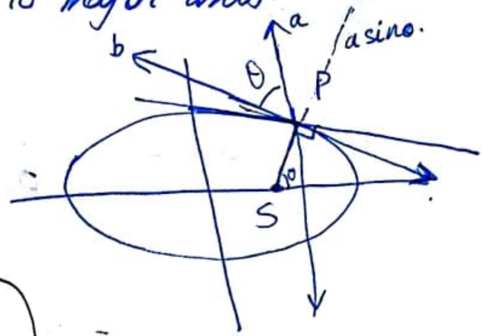
$$v_E^2 v_D^2 = \mu^2\left(\frac{4}{BE \cdot BD} - \frac{2}{a}\left(\frac{BE+BD}{BE \cdot BD}\right) + \frac{1}{a^2}\right)$$

$$= \mu^2\left(\frac{\cancel{4}}{BE \cdot BD} - \frac{2 \cdot \cancel{2a}}{\cancel{a}\, BE \cdot BD} + \frac{1}{a^2}\right)$$

$$v_E^2 v_D^2 = \frac{\mu^2}{a^2} = \left(v_c^2\right)^2 \Rightarrow v_E v_D = v_c^2$$

$$\Rightarrow \boxed{v_c = \sqrt{v_E v_D}} \text{ Ans}$$

[8] Show that velocity of particle moving in an ellipse about a centre of force at focus, has two components μ/h $\perp r$ to radius and $\mu e/h$ $\perp r$ to major axis.



$$\boxed{\frac{l}{r} = 1 + e\cos\theta} \quad (1)$$

$$+\frac{l}{r^2}\frac{dr}{dt} = +e\sin\theta\frac{d\theta}{dt}$$

Radial velocity $\equiv \frac{dr}{dt} = a\sin\theta$

Tangential $\equiv r\frac{d\theta}{dt} = b + a\cos\theta.$

$$-\frac{l}{r^2}a\sin\theta = -\frac{e\sin\theta}{r}(b+a\cos\theta)$$

$$\boxed{r^2\frac{d\theta}{dt} = h} \Rightarrow +l\frac{dr}{dt} = e\sin\theta\, h \Rightarrow \boxed{\frac{dr}{dt} = \frac{eh\sin\theta}{l}.}$$

$h^2 = \mu l$

$$a = \frac{eh}{l} = \frac{e h\mu}{h^2} = \boxed{\frac{e\mu}{h}} \text{ Ans}$$

$$b = \frac{r\,d\theta}{dt} - a\cos\theta = \frac{h}{r} - a\cos\theta = \frac{h}{r} - \frac{e\mu\cos\theta}{h}$$

$$= \frac{h}{r} - \frac{\mu}{h}\underbrace{\left(\frac{l}{r} - 1\right)}_{\text{from (1)}} = \frac{h}{r} - \frac{\mu l}{hr} + \frac{\mu}{h} = \boxed{\frac{\mu}{h}} \text{ Ans}$$

Ans

$\frac{dr}{dt} = a\sin\theta$	$\frac{r\,d\theta}{dt} = b + a\cos\theta$	$+\frac{l}{r^2}\frac{dr}{dt} = e\sin\theta\frac{d\theta}{dt}.$	$r^2\frac{d\theta}{dt} = h$

$$\frac{l}{r^2}a\sin\theta = \frac{e\sin\theta}{r}(b+a\cos\theta) = e\sin\theta\frac{d\theta}{dt} \Rightarrow l a\sin\theta = e\sin\theta\, r^2\frac{d\theta}{dt}\, h$$

$$a = \frac{eh}{l} = \frac{eh\cdot\mu}{h^2} = \boxed{\frac{e\mu}{h}}$$

$$r^2\frac{d\theta}{dt} = r\left(b + \frac{e\mu}{h}\cos\theta\right) \Rightarrow \frac{h}{r} - \frac{e\mu}{h}\cos\theta = b$$

$$\Rightarrow \frac{h}{r} - \frac{\mu}{h}\left[\frac{l}{r} - 1\right] \Rightarrow \frac{h}{r} - \frac{\mu}{h}\left[\frac{h^2}{\mu r} - 1\right] = \boxed{\frac{\mu}{h}} \quad (l = h^2/\mu)$$

(8) Particle is projected from earth with v. Taking change in gravity into account, path is an ellipse of major axis $\frac{2ga^2}{(2ga-v^2)}$, $a \equiv R_{Earth}$.

Let $2p$ be major axis of ellipse

$$V_0^2 = \mu\left(\frac{2}{r} - \frac{1}{p}\right)$$

$$v^2 = \mu\left(\frac{2}{a} - \frac{1}{p}\right) \quad -1$$

$$P = -\frac{\mu}{r^2} \qquad . \text{acc} = g \text{ at earth's surface.}$$

$$+\frac{\mu}{a^2} = +g \Rightarrow \mu = ga^2.$$

$$(1) \Rightarrow v^2 = ga^2\left(\frac{2}{a} - \frac{1}{p}\right) \Rightarrow \frac{v^2}{ga^2} = \frac{2}{a} - \frac{1}{p}.$$

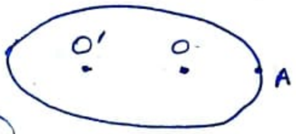
$$\Rightarrow \frac{1}{p} = \frac{2ag - v^2}{ga^2} \Rightarrow \boxed{p = \frac{ga^2}{2ag - v^2}}$$

$$\text{Major axis} \equiv 2p = \boxed{\frac{2ga^2}{2ag - v^2}} \text{ Ans}$$

5. A body in an ellipse e under force tending to a focus and when at nearer apse, the centre of force is transferred to other focus. Prove that the eccentricity of new orbit is $\boxed{e(3+e)/(1-e)}$.

Let force be at O.

When body at A



$$V_A^2 = \mu\left[\frac{2}{a-ae} - \frac{1}{a}\right] \quad -(1)$$

Let force be shifted at O'

A is an apse

$$V_A^2 = \mu\left[\frac{2}{a+ae} - \frac{1}{a'}\right] \quad -(2)$$

where a' is new major axis

$$\frac{2}{a-ae} - \frac{1}{a} = \frac{2}{a+ae} - \frac{1}{a'} \Rightarrow \frac{2}{a+ae} - \frac{2}{a-ae} + \frac{1}{a} = \frac{1}{a'}$$

$$\frac{1}{a'} = \frac{1}{a}\left[\frac{2}{1+e} - \frac{2}{1-e} + 1\right] = \frac{1}{a}\left(\frac{\cancel{2}-2e-\cancel{2}-2e+1-e^2}{1-e^2}\right)$$

$$a' = \frac{a(1-e^2)}{1-4e-e^2}$$

Now for new ellipse O'A is nearer apse.

$$O'A = a'(1-e') = a(1+e)$$

e': eccentricity of new ellipse

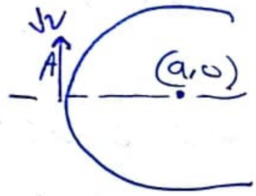
$$1-e' = \frac{\cancel{a}\,\cancel{(1+e)}(1-4e-e^2)}{\cancel{a}(1-e^2)}$$

$$e' = \frac{\cancel{1}-e-\cancel{1}+4e+e^2}{1-e} = \boxed{\frac{e(3+e)}{1-e}}$$

Q Comet in parabola around Sun, when nearest breaks into two equal w/o loss of energy. 1 part in a circle. Find path of other

2001

- (m)



$$v^2 = \frac{2\mu}{r}$$

$A \Rightarrow m, \; v_A^2 = \frac{2\mu}{a}.$

$$2.\cancel{2}\frac{1}{\cancel{2}}\cancel{m}v_A^2 = \frac{1}{2}\frac{\cancel{m}}{2}v_1^2 + \frac{1}{2}\frac{\cancel{m}}{2}v_2^2$$

Let $1 \Rightarrow$ be in circle

$$\frac{v_1^2}{a} = \frac{\mu}{a^2} \Rightarrow v_1^2 = \frac{\mu}{a}.$$

$$2.\frac{2\mu}{a} - \frac{\mu}{a} = v_2^2 \Rightarrow v_2^2 = \frac{3\mu}{a}$$

$$= \mu\left[\frac{2}{a} + \frac{1}{a}\right]$$

Path of hyperbola.

finding eccentricity

(a) ← transverse axis

Let conjugate axis b, eccentricity e

If hyperbolic orbit $\Rightarrow$ $vp = h = \sqrt{\mu l}$ $\qquad l = b^2/a$

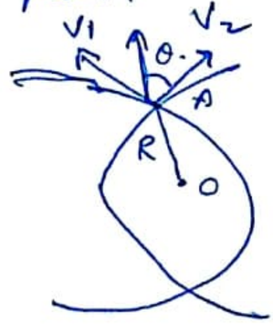
At $A \Rightarrow$ $v_2 a = \sqrt{\mu l} = \sqrt{\mu b^2/a}$ $\qquad b^2 = a^2(e^2-1)$

$$\sqrt{\frac{3\mu}{\cancel{a}}}.\cancel{a} = \sqrt{\frac{\mu a^2}{\cancel{a}}(e^2-1)}$$

$$3 = e^2 - 1 \Rightarrow \boxed{e = 2} \text{ Ans}$$

(8) Two particles m_1, m_2 move in coplanar parabolas around Sun. They collide at $90°$ and coalesce when common distance is R. Find subsequent path.

Let m_1, m_2 be at velocity v_1, v_2 at A. $OA = R$ $O \equiv$ Sun



Since, parabolic path $\Rightarrow$

$$v_1^2 = \frac{2\mu}{R} \qquad v_2^2 = \frac{2\mu}{R}.$$

Let particles collide and common velocity be V at θ to v_2.

$$\left.\begin{aligned}(m_1+m_2)\,V\cos\theta &= m_2 V_2 \\ (m_1+m_2)\,V\sin\theta &= m_1 V_1\end{aligned}\right\} \text{Square and add.}$$

$$(m_1+m_2)^2 V^2 = m_1^2 v_1^2 + m_2^2 V_2^2$$

$$\Rightarrow V^2 = \frac{m_1^2 v_1^2 + m_2^2 V_2^2}{(m_1+m_2)^2}$$

As $v_1 = V_2$ and $m_1^2 + m_2^2 < (m_1+m_2)^2$. So, $V^2 < \frac{2\mu}{R}$.

Path is an ellipse.

$$\frac{m_1^2 V_1^2 + m_2^2 V_2^2}{\mu (m_1+m_2)^2} = \left(\frac{2}{R} - \frac{1}{a}\right) \qquad \frac{2}{R} = \frac{v_1^2}{\mu}$$

$$\frac{1}{a} = \frac{2}{R} - \frac{v_1^2 (m_1^2+m_2^2)}{\mu (m_1+m_2)^2} = \frac{v_1^2}{\mu}\left(1 - \frac{m_1^2+m_2^2}{(m_1+m_2)^2}\right)$$

$$\frac{1}{a} = \frac{v_1^2}{\mu}\left(\frac{2m_1 m_2}{(m_1+m_2)^2}\right) \Rightarrow \boxed{a = \frac{(m_1+m_2)^2 \mu}{2 m_1 m_2 v_1^2}} = \frac{(m_1+m_2)^2 R}{4 m_1 m_2}$$

Major axis $= 2a = \frac{(m_1+m_2)^2 R}{2 m_1 m_2}$ Ans