* MACHINE COMPUTATION

To obtain meaningful results for a given problem using computers, there are five distinct phases.

(i) Choice of a method
(ii) Designing the algorithm
(iii) Flow charting
(iv) programming
(v) Computer execution.

A method is defined to be a mathematical formula for finding the solution of a given problem. There may be more than one method available to solve the same problem. We should choose the method which suits the given problem best. The inherent assumptions and limitations of the method must be studied carefully.

of calculation

Once the method has been decided, we must describe a complete and unambiguous set of computational steps to be followed in a particular sequence to obtain the solution. The description is called an **algorithm**. It may be emphasized that the computer is concerned with the algorithm and not with the method. The algorithm tells the computer where to start, what information to use, what operations to be carried out and in which order, what information to be printed and when to stop

& Draw the flow chart

IFS-2010

Design an algorithm to find the real roots of the equation

$$ax^2 + bx + c = 0,\ a,\ b,\ c \text{ real}$$

for 10 sets of values of a, b, c using the method

$$x_1 = \frac{-b+e}{2a}, \qquad x_2 = \frac{-b-e}{2a} \tag{1.26}$$

where $e = \sqrt{(b^2 - 4ac)}$

The following computational steps are involved:

Step 1: Set $I = 1$.
Step 2: Read a, b, c.
Step 3: Check: is $a = 0$? If yes, print wrong data and go to step 9, otherwise go to next step.
Step 4: Calculate $d = b^2 - 4ac$
Step 5: Check: is $d < 0$? If yes, print complex roots and go to step 9, otherwise go to next step.
Step 6: Calculate $e = \sqrt{(b^2 - 4ac)}$.
Step 7: Calculate x_1 and x_2 using method (1.26).
Step 8: Print x_1 and x_2.
Step 9: Increase I by 1.
Step 10: Check: is $I \le 10$? If yes, go to step 2, otherwise go to next step.
Step 11: Stop

Note: On execution of the above eleven steps or **instructions** in the same order, the problem is completely solved. These eleven steps constitute the algorithm of the method (1.26).

An algorithm has five important features:

1. finiteness: an algorithm must terminate after a finite number of steps.
2. definiteness: each step of an algorithm must be clearly defined or the action to be taken must be unambiguously specified.
3. inputs: an algorithm must specify the quantities which must be read before the algorithm can begin. In the algorithm of Example 1.11 the three input quantities are a, b, c.

5. effectiveness: an algorithm must be effective which means that all operations are executable. For example, in the algorithm of Example 1.11, we must avoid the case $a = 0$, as division by zero is not possible. Similarly, if $b^2 - 4ac < 0$, some alternate path must be defined to avoid finding the square root of a negative number.

A flow chart is a graphical representation of a specific sequence of steps (algorithm) to be followed by the computer to produce the solution of a given problem. It makes use of the flow chart symbols to represent the basic operations to be carried out. The various symbols are connected by arrows to indicate the flow of information and processing. While drawing a flow chart any logical error in the formulation of the problem or in the application of the algorithm can be easily seen and corrected. Some of the symbols used in drawing flow charts are given in Table 1.1.

Flow Chart Symbols

Flow Chart Symbols	*Meaning*
$C = A + B$	A processing symbol such as addition or subtraction of two numbers and movement of data in computer memory.
Is $D < 0$?	A decision taking symbol. Depending on the answer, yes or no, a particular path is chosen.
Read A, B	An input symbol, specifying quantities which are to be read before processing can proceed.
Print A, B	An output symbol, specifying quantities which are to be outputted.
Start or End	A terminating symbol, including start or end of the flow chart. This symbol is also used as a connector.

Draw a flow chart to find real roots of the equation $ax^2 + bx + c = 0$, a, b, c real, using the method

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

for ten sets of values of a, b, c.

The flow chart is given in Fig. 1.1. The flow chart can be easily translated into any high level language, for example C, Fortran, Pascal, Algol, Basic etc. and can be executed on the computer.

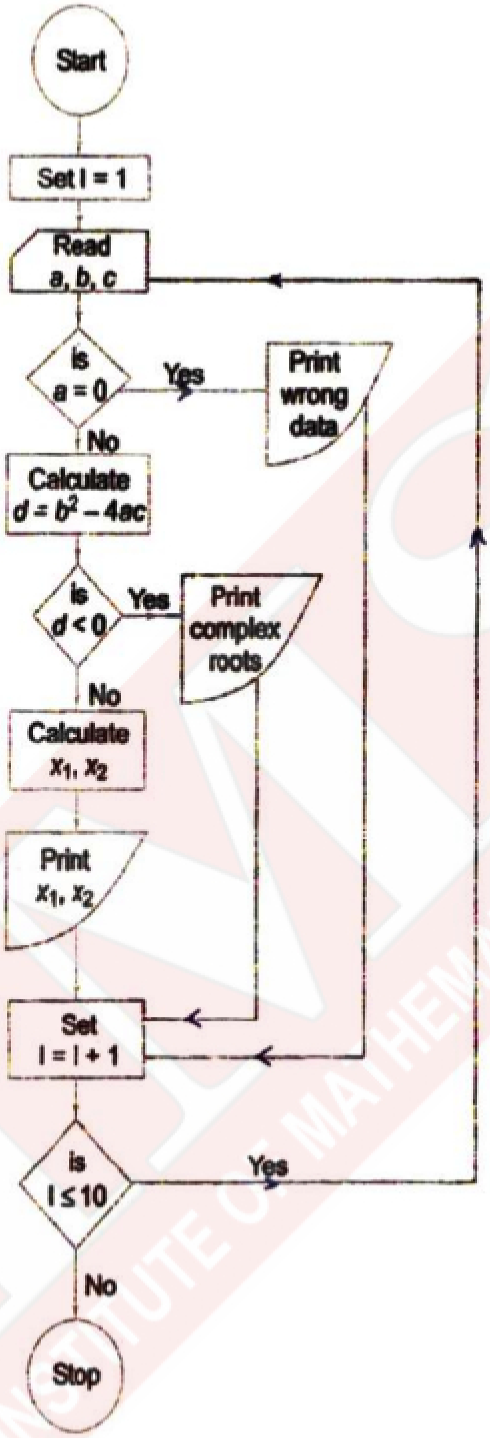


Fig. 1.1. Flow chart for finding real roots of the quadratic equation.

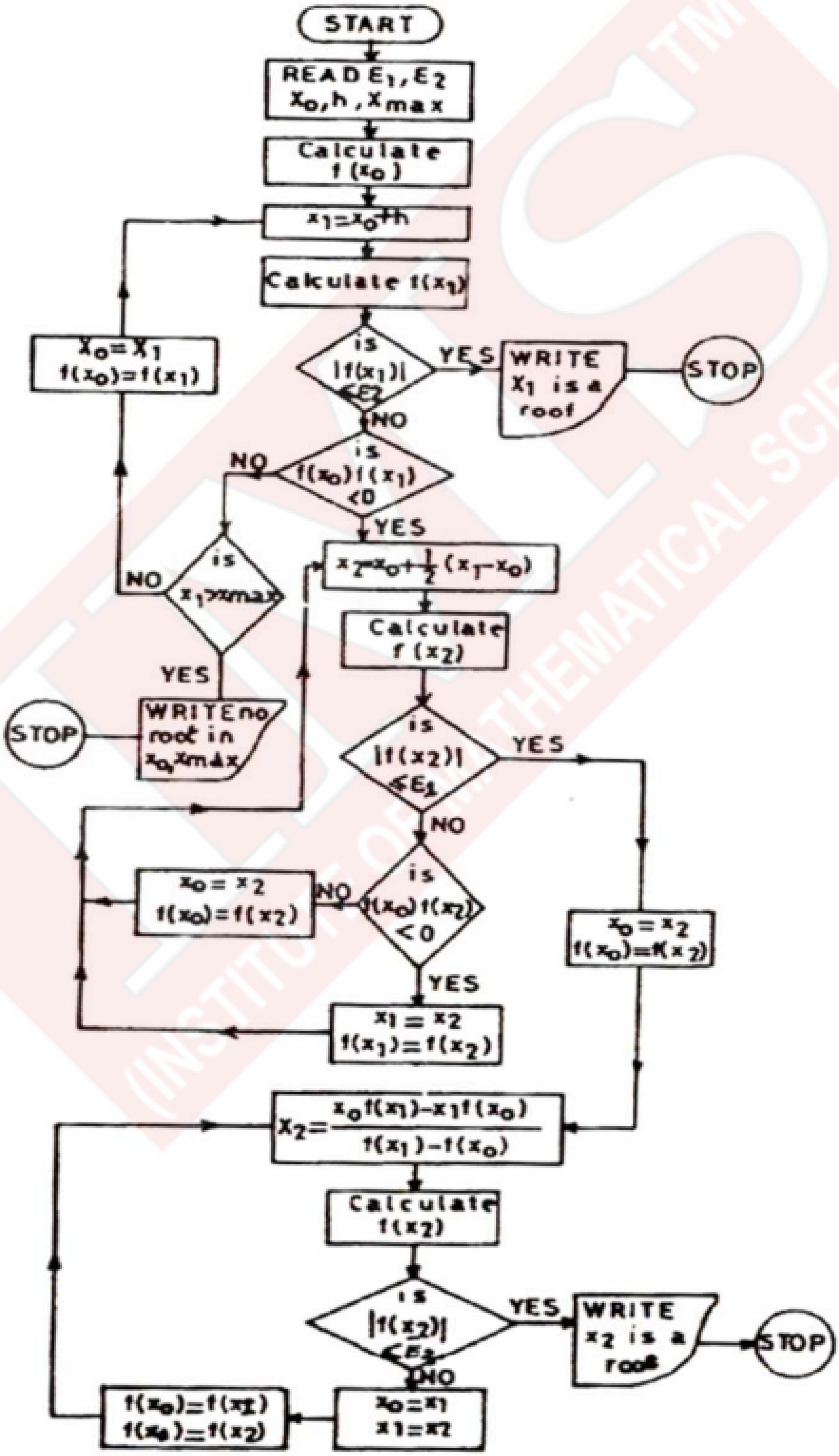
1.5 COMPUTER SOFTWARE

The purpose of computer software is to provide a useful computational tool for users. The writing of computer software requires a good understanding of numerical analysis and art of programming. A good computer software must satisfy certain criteria of **self-starting, accuracy** and **reliability, minimum number of levels, good documentation, ease of use** and **portability**.

Flow Chart for a Zero Finder

The flow chart for the method using incremental search method, bisection method followed by the Secant method is given in Figure 2.8.

The incremental search method searches for an interval $(x_k, x_k + h)$ containing the root in $(x_0, x_{\max})$ starting with x_0 and using the increment h (which may not be small) everytime. ϵ_1 and ϵ_2 are tolerances for the bisection and Secant methods. ϵ_1 is much larger than ϵ_2



Algorithms and Flow Charts for solving Numerical Analysis problems:

MACHINE COMPUTATION/ COMPUTER CALCULATIONS:

To obtain meaningful results for a given problem using computers, there are five distinct phases.

(i) Choice of a method (ii) Designing the algorithm (iii) Flow Charting (iv) programming (v) computer Execution

A method is defined to be a mathematical formula for finding the solution of a given problem. There may be more than one method available to solve the same problem we should choose the method which suits the given problem best. The inherent assumptions and limitations of the method must be studied carefully.

Algorithm: Once the method of calculation has been decided, ~~we and describe a complete~~, we must describe a complete and unambiguous set of computational steps to be followed in a particular sequence to obtain the solution. The description is called an algorithm.

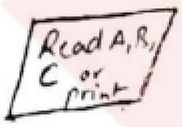
It may be emphasized that the computer is concerned with the algorithm and not with the method

The algorithm tells the computer where to start, what information to use, what operations to be carried out and in which order, what information to be printed and when to stop.

Flow Chart: A flow chart is a graphical (pictorial) [illegible] steps (algorithm) to be followed by the computer to produce the solution of a given problem. It makes use of the flow chart symbols to represent the basic operations to be carried out. The various symbols are connected by arrows to indicate the flow of information and processing. While drawing a flow chart any logical error in the formulation of the problem or in the application of algorithm can be easily seen and corrected.

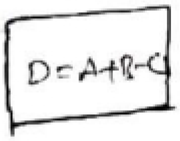
The commonly used symbols and their meanings are given below

A symbol used to indicate Start or Stop/End of a program.

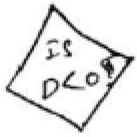


A parallelogram is used to indicate an Input or Output

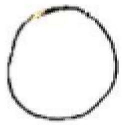
[An input symbol, specifying quantities which are to be read before processing can proceed
An output symbol, specifying quantities which are to be computed]



A rectangle is a processing symbol. for Example addition, subtraction and movement of data to computer memory.



A diamond is a decision making symbol. A particular path is chosen depending on the yes or NO answer.



A small circle with any number or letter in it, used as a connector symbol. it connects various parts of flow chart which are far apart or spread over pages.

IFoS 2010 Design an algorithm and Draw the flow chart to find the real roots of the equation $ax^2+bx+c=0$ a, b, c real; for 10 sets of values of a, b, c using the method $x_1 = \frac{-b+e}{2a}$, $x_2 = \frac{-b+e}{2a}$ ← (1)

where $e = \sqrt{b^2-4ac}$.

Sol^n. The following computational steps are involved

Step 1: Set $I = 1$

Step 2: Read a, b, c

Step 3: Check ; is $a=0$? If yes, print wrong data and go to step 9, otherwise go to next step

Step 4: Calculate $d = b^2-4ac$

Step 5: Check : is $d<0$? if yes, print complex roots and go to step 9, otherwise go to next step.

Step 6: Calculate $e = \sqrt{b^2-4ac}$

Step 7: Calculate x_1 and x_2 using method (1)

Step 8: Print x_1 and x_2

Step 9: Increase I by 1.

Step 10: Check: is $I \leq 10$? If yes, go to step 2, otherwise go to next step.

Step 11: Stop.

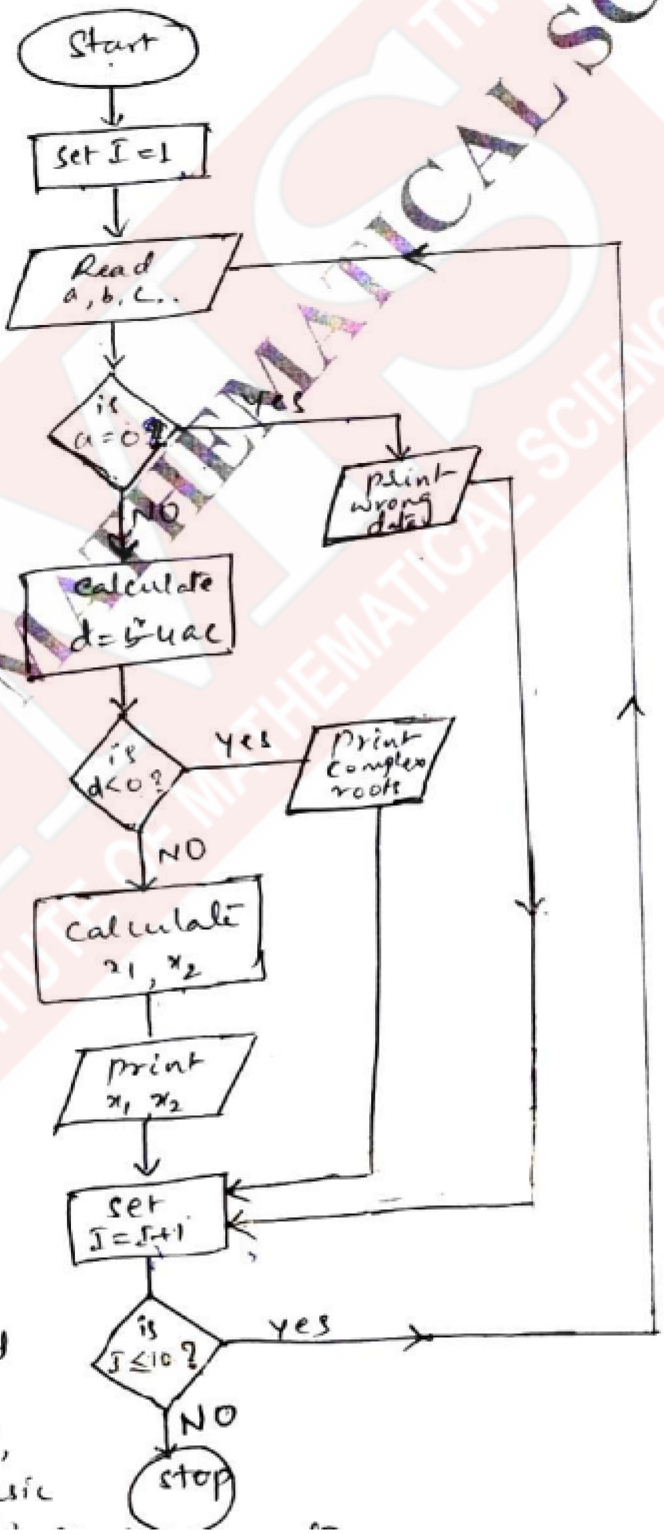
Note: On excution of the above eleven steps or instructions in the same order, the problem is completely solved. These eleven steps constistute the algorithm of the method ①.

An algorithm has five important features:

1. finiteness: An algorithm must after a finite number of steps.
2. definiteness: Each step of an algorithm must be clearly defined or the action to be taken must be unambiguously specified
3. inputs: An algorithm must specify the quantities which must be read before the algorithm can begin. In the above algorithm the three input quantities are a, b, c.
4. outputs: An algorithm must specify the quantities which are to be outputted and their proper place. In the above algorithm the two output quantities are x_1, x_2.
5. Effectiveness: An algorithm must be effective which means that all operations are executable. for example i.e. the above algorithm, we must avoid the case

$a=0$, as division by zero is not possible.
Similarly, if $b^2-4ac<0$, some alternate path must be defined to avoid finding the square root of a negative number.

flow chart



Note: The flow chart can be easily translated into any high level language, for example C, Fortan, pascal, Algol, Basic

15. If S is any set and $P(s)$ is its power set and A and B belong to $P(s)$, prove that $B \cap (A \quad B) = \ldots$
16. If A and B are finite sets then prove that $A \cup B$ and $A \cap B$ are finite sets and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$ (Andhra, 2004)

17. In survey conducted on 200 people, it was found that 140 are smokers while 80 are alcoholic and 40 are both smokers and alocholics. Find how many are neither smokers not alcoholics.
18. How many integers between 1 and 789 are divisible by 5 but not by 7.
19. How many integers are between 1 and 250 while are divisible by any of the integers 3, 5, and 7.
20. Out of a class of 153 students, 54 have taken History, 63 have taken Geography, 62 have taken Economics, and 43 have taken Geography and History, 45 have taken History and Economics, 46 have taken Geography and Economics and 37 have taken all the three subjects. How many of the students have not taken any of these three subjects ? Use a Venn diagram.

II. ALGEBRA OF LOGIC

36.5. INTRODUCTION

(1) Logic is concerned with all types of reasoning such as valid statements, mathematical proofs, valid conclusions etc. Logical reasoning is used to prove theorems, to verify the correctness of computer programs and to draw conclusions from experiments. Later on, we shall observe that the algebra of sets and logic is analogous to the algebra of switching circuits which is similar to Boolean Algebra'.

(2) Propositions and Statements. A *proposition* is a declarative sentence which is either true (1) or false (0). Some authors use T and F respectively for 1 and 0. The truth or falsity of a proposition is defined as its *truth value*.

All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*.

Example 36.9. *Which of the following are statements* . (*a*) *Agra is in India* (*b*) $3 + 4 = 5$ (*c*) *Where do you live ?* (*d*) *Do you speak Hindi ?*

Sol. (*a*) and (*b*) are statements that happen (*a*) is true and (*b*) is false.

(*c*) and (*d*) are questions so they are not statements.

(3) Compound statements. The statement which is composed of sub-statements and logical connectives is called a *compound statement.*

e.g., 'It is raining and it is cold' is a compound statement as it is comprised of two sub-statements 'It is raining' and 'it is cold'.

(4) Truth table. The truth value of a compound statement is completely determined by the truth value of its substatements. *A convenient way to represent a compound statement is by means of the* **truth table** *wherein the values of a compound statement are specified for all possible choices of the values of the sub statements.*

We shall use the numbers 0 and 1 to denote the false and true statements. Also we use letters $p, q, r,$ ...... to represent a proposition or logical variable.

36.6. LOGICAL OPERATORS

(1) Conjunction. *If p and q are two statements then their conjunction p and q written as*

NA

Table 1

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Table 2

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For example, the conjuction of p : it is raining and q : I am cold is $p \wedge q$: It is raining and I am cold.

(2) Disconjunction. *If p and q are two statements, then their disconjunction p or q written as $p \vee q$ is defined by the truth table 2.*

For example, the disjunction of p and q for p : it is raining today ; q : 3 is an odd integer is

$p \vee q$: it is raining today or 3 is an odd integer.

(3) Negation. *If p is a given statement and its negative 'not p', written as $\sim p$ (or Np or $\rceil p$) is defined by the following truth table :*

p	$\sim p$
0	1
1	0

For example, the negation of the following statement

(*a*) $p : 2 + 3 > 1$ is $\sim p : 2 + 3 \leq 1$

(*b*) q : it is hot is $\sim q$: it is cold.

Example 36.10. *If p be 'it is hot' and q be 'it is raining', describe each of the following statements by a sentence :*

(*a*) $q \vee \sim p$ (*b*) $\sim p \wedge \sim q$ (*c*) $\sim (\sim p \vee q)$.

Sol. (*a*) It is raining or it is not hot. (*b*) It is not hot and it is not raining.

(*c*) It is hot but not raining.

(4) Conditional operator. *The conditional statement 'if p then q' written as $p \to q$ is defined by the truth table 4* :

Table 4

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Table 5

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Obs. The contrapositive of conditional statement $p \to q$ is the statement $\sim p \to \sim q$.

(5) Biconditional operator. If p and q be two statements, then the statement 'p if and only if q' denoted by $p \leftrightarrow q$ and abbreviated as 'p if q' is called a *biconditional statement*. The truth table for biconditional statement is table 5.

Example 36.11. *Construct the truth tables for*

(*a*) $p \wedge \sim q$ (*b*) $(p \vee q) \vee \sim p$

Sol. (*a*) The truth table is

p	q	$\sim q$	$p \wedge \sim q$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

(*b*) The truth table is

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
1	1	1	0	1
1	0	1	0	1
0	1	1	1	1
0	0	0	1	1

(*c*) The truth table is

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

(*d*) The truth table in this case is

p	q	$p \to q$	$\sim q$	$p \to \sim q$	$(p \to \sim q)$	$p \to q \vee \sim (p \to \sim q)$
1	1	1	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	1
0	0	1	1	0	1	1

36.7. STATEMENTS GENERATED BY A SET

(1) *If S be a set of statements, then any valid combination of statements in S with conjunction, disconjunction or negation is a statement generated by S.*

A statement generated by a set S need not include each element of S in its expression.

For example, if p, q, r are statements in S then

(*a*) $(p \wedge q) \wedge r$ (*b*) $\sim q \wedge r$ (*c*) $(p \wedge q) \vee (\sim q \wedge r)$

are statements generated by S. Their truth tables are :

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$ (*a*)	$\sim q$	$\sim q \wedge r$ (*b*)	(*c*)
1	1	1	1	1	0	0	1
1	1	0	1	0	0	0	1
1	0	1	0	0	1	1	1
0	1	1	0	0	0	0	0
1	0	0	0	0	1	0	0
0	[illegible]	0	0	0	0	0	0
0	0	1	0	0	1	1	1

(2) **Tautology** *is an expression involving logical variables which is true for all cases in truth table.* It is also called a *logical truth.*

(3) **Contradiction** is an expression involving logical variables which is false for all cases in its truth table. Obviously, the negation of a contradiction is a tautology.

In other words, a statement formula which is a tautology is identically true, while a formula which is a contradiction is identically false.

> **Obs.** The conjunction of two tautologies is also a tautology.

Example 36.12. *Show that (a) $p \vee \sim p$ is a tautology (b) $p \to q \leftrightarrow (\sim p \vee q)$.*

Sol. (*a*) The truth table is

p	$\sim p$	$p \vee \sim p$
1	0	1
0	1	1

Hence $p \vee \sim p$ is a tautology.

(*b*) The truth table is

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Hence $(p \to q) \leftrightarrow (\sim p \vee q)$ is true.

36.8. EQUIVALENCE

(1) *If p and q be statements generated by the set of statements S, then p and q are equivalent if $p \leftrightarrow q$ is a tautology which is denoted by $p \Leftrightarrow q$.*

If $p \to q$ is a tautology, then we say that p *implies* q and write it as $p \Rightarrow q$.

> **Obs.** All tautologies are equivalent to each other and all cntradictions are equivalent to each other.

(2) **Equivalent formulae.** Some basic equivalent formulae are given below which can be proved by using truth tables :

1. $p \vee p \Leftrightarrow p$	$p \wedge p \Leftrightarrow p$	*Idempotent laws*
2. $p \vee q \Leftrightarrow q \wedge p$	$p \wedge q \Leftrightarrow q \wedge p$	*Commutative laws*
3. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	*Associative laws*
4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	*Distributive laws*
5. $p \vee \sim p \Leftrightarrow 1$	$p \wedge \sim p \Leftrightarrow 0$	*Negation laws*
6. $p \vee 0 \Leftrightarrow p$	$p \wedge 1 \Leftrightarrow p$	*Identity laws*
7. $p \vee 1 \Leftrightarrow 1$	$p \wedge 0 \Leftrightarrow 0$	*Null laws*
8. $p \vee (p \wedge q) \Leftrightarrow p$	$p \wedge (p \vee q) \Leftrightarrow p$	*Absorption laws*

10. $p \Rightarrow p \vee q$ $\qquad$ $q \Rightarrow p \vee q$ $\qquad$ *Disjunctive addition*

11. $p \wedge q \Rightarrow p$ $\qquad$ $p \wedge q \Rightarrow q$

12. $(p \vee q) \wedge \sim p \Rightarrow q$ $\qquad$ $(p \vee q) \wedge \sim q \Rightarrow p$

13. $(p \to q) \wedge (q \to r) \Rightarrow (p \to r)$ $\qquad$ *Chain rule*

14. $p \to q \Leftrightarrow \sim p \vee q$ $\qquad$ *Conditional equivalence*

15. $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$ $\qquad$ *Biconditional equivalence*

Example 36.13. *Show that*

(a) $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r) \Leftrightarrow (p \wedge q) \to r$

(b) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

Sol. (*a*) By conditional equivalence $q \to r \Leftrightarrow \sim q \vee r$

Replacing $q \to r$ by $\sim q \vee r$, we get $p \to (\sim q \vee r)$ which is equivalent to $\sim p \vee (\sim q \vee r)$ by the same rule.

Thus $\quad \sim p \vee (\sim q \vee r) \Leftrightarrow (\sim p \vee \sim q) \vee r$ [by (3)

$\Leftrightarrow \sim (p \wedge q) \vee r$ [by (9)

$\Leftrightarrow (p \wedge q) \to r$ [by (14)

(*b*) $\quad [\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow [\sim p \wedge (\sim q \wedge r)] \vee (q \vee p) \wedge r$ [by (4)

$\Leftrightarrow [\sim p \wedge \sim q \wedge r] \vee (q \vee p) \wedge r$ [by (3)

$\Leftrightarrow [\sim (p \vee q) \wedge r] \vee (q \vee p) \wedge r$ [by (9)

$\Leftrightarrow [\sim (p \vee q) \vee (p \vee q)] \wedge r$ [by (4)

$\Leftrightarrow 1 \wedge r$ [by (5)

$\Leftrightarrow r$ [by (6)

36.9. DUALITY LAW

(1) *Two formulae A and A* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.*

If the formula A contains special variables 1 or 0, then its dual A* is obtained by replacing 1 by 0 and 0 by 1.

e.g., (*i*) Dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$

(*ii*) Dual of $(p \wedge q) \vee 0$ is $(p \vee q) \wedge 1$.

(2) Tautology implications. *A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology which is read as "A implies B".*

The implications listed below have important applications which can be proved by truth tables :

1. $p \wedge q \Rightarrow p$ $\qquad$ $p \Rightarrow p \vee q$

2. $\sim p \Rightarrow p \to q$ $\qquad$ $q \Rightarrow p \to q$

3. $\sim (p \to q) \Rightarrow p$ $\qquad$ $\sim (p \to q) \Rightarrow \sim q$

4. $p \wedge (p \to q) \Rightarrow q$ $\qquad$ $\sim p \wedge (p \vee q) \Rightarrow q$

36.10. ARGUMENTS

(1) *An argument is an assertion that a given set of propositions $p_1, p_2, \ldots, p_n$* (called *premises*) *yields another proposition q* (called *conclusion*). The argument is symbolically written as "$p_1, p_2, \ldots\ldots p_n \vdash q$".

An argument $p_1, p_2, \ldots\ldots p_n \vdash q$ is true provided q is true whenever all the premises $p_1, p_2, \ldots\ldots p_n$ are true. An argument which is true is said to be '*valid argument*'. Otherwise it is called a *fallacy*.

Example 36.14. *Show that*

(*a*) *the argument $p \leftrightarrow q, q \vdash p$ is valid.*

(*b*) *the argument $p \to q, q \vdash p$ is a fallacy.*

Sol. (*a*) Let us first prepare the truth table as follows :

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Since $p \leftrightarrow q$ is true in cases (rows) 1 and 4, and q is true in cases 1 and 3, therefore $p \leftrightarrow q$ and q both are true in case 1 only when p is also true. This shows that the given argument is valid.

(*b*) Let us first prepare the truth table below :

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

This table shows that $p \to q$ and q both are true in case 3 only while the conclusion p is false. Hence the given argument is a fallacy.

(2) **Theorem.** *The argument $p_1, p_2, \ldots\ldots p_n \vdash q$ is valid if and only if the proposition $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots \wedge p_n) \to q$ is a tautology.*

The propertions $p_1, p_2, \ldots\ldots p_n$ are simultaneously true if and only if the proposition $p_1 \wedge p_2 \wedge \ldots\ldots \wedge p_n$ is true *i.e.* if the proposition $(p_1 \wedge p_2 \wedge \ldots\ldots \wedge p_n) \to q$ is a tautology.

> Obs. The validity of an argument depends upon the particular form of the argument, not on the truth values of the statement appearing in the argument.

Example 36.15. *Test the validity of the following argument :*

S_1 : *If 5 is less than 3, then 6 is not a prime number*

S_2 : *5 is not less than 6*

Sol. Let '5 is less than 3' be p and '5 is a prime number' be q. Then the given argument is of the form $p \to \sim q, \sim p \vdash q$.

Since in the last line of the truth table, the premises $p \to \sim q$ and $\sim q$ are true but the conclusion q is false, therefore the argument is fallacy.

p	q	$\sim q$	$p \to \sim q$	$\sim p$
1	1	0	0	0
1	0	1	1	0
0	1	0	1	1
0	0	1	1	1

Problems 36.2

1. If p = Sam is a teacher, q = John is an honest boy, then translate the following into logical sentences :
 (a) $\sim (p \wedge q)$, (b) $p \vee \sim q$, (c) $\sim p \Leftrightarrow q$, (d) $p \Rightarrow \sim q$.
2. Change the following sentence into symbols :
 (a) 'If I do not have car or I do not wear good dress then I am not a millionaire'
 (b) Everyone who is healthy can do all kinds of work. (Anna, 2004S)
3. Prepare truth tables for the following statements (a) $(p \Rightarrow q) \wedge \sim q$, (b) $(p \Leftrightarrow q) \wedge (r \vee q)$.
4. Write down the truth table of
 (a) $p \vee q$ (Madras, 1997) (b) $p \wedge (p \wedge q)$ (Madras, 2005S)
5. Verify that the following are tautologies :
 (a) $p \to [q \to (p \wedge q)]$ (Anna, 2005)
 (b) $(p \wedge q \Rightarrow r) \Leftrightarrow (p \Rightarrow r) \vee (q \Rightarrow r)$ (c) $(p \Rightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$.
6. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. (Anna, 2004S ; Madras, 2003S)
7. Over the universe of positive integers
 $p(n)$: n is prime and $n < 32$.
 $q(n)$: n is a power of 3.
 $r(n)$: n is a divisor of 27.
 (i) What are the truth sets of these propositions ?
 (ii) Which of the three propositions implies one of the others ?
8. Given the propositions over the natural numbers
 $p : n < 4$, $q : 2n > 17$ and r : n is a divisor of 18, what are the truth sets of
 (i) q, (ii) $p \wedge q$,
 (iii) r, (iv) $q \to r$. (Madras, 1999)
9. Show that (a) $\sim Q, P \to Q \Rightarrow \sim P$. (Madras, 2003)
 (b) $(P \to R) \wedge (Q \to R) \Leftrightarrow (P \vee Q) \to R$. (Madras, 2001)
10. Construct the truth table for (i) $(\sim p \to q) \wedge (q \rightleftarrows p)$. (Bharthian, Msc. 2001)
 (ii) $\rceil [P \vee (Q \wedge R)] \rightleftarrows (P \vee Q) \wedge (P \vee R)$. (Andhra, 2004)
11. Prove that the following statement is a contradiction :
 $S = [(p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)]$.
12. If p, q, r are three statements then prove that
 (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (b) $(p \Rightarrow q) \vee r \equiv (p \vee r) \Rightarrow (q \vee r)$
 (c) $\sim ($ $q) \equiv \sim p \wedge \sim q$.

14. Show that (*i*) $p \wedge q$ logically implies $p \leftrightarrow q$.
(*ii*) $p \leftrightarrow \sim q$ does not logically imply $p \rightarrow q$. (*U.P.T.U. 2001*)
15. Write the duals of $(p \vee q) \wedge r$ and $(p \wedge q) \vee t$.
16. Show that $s \vee r$ is tautologically implied by $(p \vee q) \wedge (q \rightarrow r) \wedge (q \rightarrow s)$. (*Andhra, 2004 ; Bharathian, 2001*)
17. Let $P(n)$ be '$8^n - 3^n$ is a multiple of 5'. Prove that $P(n)$ is a tautology on n. (*Madras, 1998*)
18. Prove that $P \rightarrow \rceil Q, R \rightarrow Q, R \vdash \rceil P$ is a valid argument. (*Madras, 2003*)
19. Prove that $p \rightarrow (q \vee r) \Leftrightarrow (p \wedge \rceil q) \rightarrow r$. (*Anna, 2005*)
20. Show that $R \vee S$ follows logically from the premises $C \vee D, C \vee D \rightarrow \sim H, H \rightarrow A \wedge \sim B, A \wedge \sim B \rightarrow R \vee S$. (*Madras, 1999S*)

36.11. PREDICATES

Statements involving variables such as '$x > 7$' and '$x = y + 7$' are neither true nor false so long as the values of the variable x, y are not specified. We now, discuss the ways that propositions can be evolved from such statements. The statement '$x > 7$' has two parts : First part— 'the variable x' is the *subject* of the statement ; Second part—'is greater than 7' is the *predicate* which refers to the property that the subject of statement can have. If $P(x)$ denotes the statement '$x > 7$' then P is the *predicate* and x is the *variable*. The statement $P(x)$ is also known as the value of the propositional function P at x. The *predicates* are denoted by capital letters and the *objects* by the variables (denoted by small letters in brackets).

Thus *predicates are simple statements which turn out to be propositions involving variables whose values are not well specifid.*

In other words, *predicate is a variable statement which becomes specific when particular values are assigned to the variables.*

There are statements which involves more than one variable consider the statement '$x = y + 7$' which is denoted by $Q(x, y)$ where Q is the predicate and x, y are the variables. When values are assigned to the variables x, y, the statement $Q(x, y)$ has the truth value.

Similarly $R(x, y, z)$ denotes the statement of the type '$x + y = z$'. When values are assigned to x, z, this statement has a truth value.

For example, consider the statements (i) Ram is fair ; (ii) Sham is fair.

Here in (*i*) and (*ii*) ' is fair' is the predicate while Ram and Sham are the objects. If we denote the predicate by F and the objects by r and s, then the above statements can be symbolically expressed as (*i*) $F(r)$; (*ii*) $F(s)$.

Now consider the statement Ram is fair and the house is pink.

Writing 'the house is pink' as $P(h)$, the given statement can be expressed as $F(r) \wedge P(h)$

36.12. QUANTIFIERS

(1) In a propositional function, when all the variables are assigned values, the resulti statement has a truth value. However, there is another method to create a proposition fr a propositional function which is called *quantification*. It is of two types : *Universal quant. cation* and *Existential quantification*.

(2) Universal quantification. Many statements assert that a property is true for values of a variable in a certain domain. This domain is termed as the universe of discou and such a statement is expressed using universal quantification.

Thus *the universal quantification of P(x) is the proposition 'P(x) is true for all values* normal form of a given form is not unique. In fact, several disjunctive normal forms can be obtained for a given formula by applying the distributive laws in different ways.

A given formula is however, identically false if every elementary product appearing in its disjunctive normal form is identically false.

Example 36.23. *Obtain the disjunctive normal forms of*

(i) $p \wedge (p \to q)$ *(ii)* $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

Sol. *(i)* $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q) \Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$.

which is the desired disjunctive normal form.

$$(ii) \sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow \sim (p \vee q) \wedge (p \wedge q) \vee (p \vee q) \wedge \sim (p \wedge q)$$

$$[\because \; E \leftrightarrow F \;\Leftrightarrow\; (E \wedge F) \vee (\sim E \wedge \sim F)]$$

$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$$

$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$$

$$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$$

which is the desired disjunctive normal form.

(3) Conjunctive normal form *of a given formula is that formula which is equivalent to the given formula and contains the product of elementary sums.*

Example 36.24. *Find a conjunctive normal form of* $\sim (p \vee q) \leftrightarrow (p \wedge q)$.

Sol. $\sim (p \vee q) \leftrightarrow (p \wedge q) \Leftrightarrow [\sim (p \vee q) \to (p \wedge q)] \wedge [p \wedge q \to \sim (p \vee q)]$

$$\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \vee [\sim (p \wedge q) \vee \sim (p \vee q)] \quad \text{[by conditional equivalence}$$

$$\Leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$$

$$\Leftrightarrow p \vee q \vee p \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$$

which is the required conjuctive normal form.

(4) Principal disjunctive normal form. Consider a formula for the propositions p and q using conjunction as $p \wedge q, p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim q$. These terms are called *minterms* or *Boolean conjunction* of p and q.

An equivalent formula for a given formula, consisting of disjunctions of minterms only is called the **principal disjunctive normal form** *(pdnf)* or **sum of products canonical form.**

Procedure *to obtain the principle disjunctive normal form* : *(i)* Replace the conditions and biconditions by their equivalent formulae containing $\wedge, \vee, \sim$ only.

(ii) Using DeMorgans laws, apply negations to the variables.

(iii) Apply the distribution laws.

(iv) Introduce the missing factors to obtain minterms in the disjunctions.

(v) Delete identical minterms appearing in the disjunctions.

Example 36.25. *Obtain the pdnf for*

(i) $p \vee q$ *(ii)* $\sim(p \wedge q)$ *(iii)* $\sim p \vee q$ *i.e,* $p \to q$.

Sol. *(i)* $p \vee q \Leftrightarrow [p \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)]$

$$\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$

$$\Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge \sim p)$$

(ii) $\sim (p \wedge q) \Leftrightarrow (\sim p \vee \sim q) \Leftrightarrow [\sim p \wedge (\sim q \vee q)] \vee [\sim q \wedge (p \vee \sim p)]$

$$\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$$

$$\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim q \wedge p)$$

(iii) $\sim p \vee q \Leftrightarrow \sim p \wedge (q \vee \sim q) \vee [q \wedge (p \vee \sim p)]$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)$.

(5) **Principal conjunctive normal form.** Consider a formula for the propositions p and q using disconjunction as $p \vee q$, $\sim p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$. These terms are called *max terms*.

An equivalent formula for a given formula, consisting of conjunctions of the max term only is called the **principal conjunctive normal form** (*pcnf*) or **product of sum canonical form.**

Procedure *for obtaining pcnf* for a given formula is similar to the one for *pdnf* as all assertions made for *pdnf* can be made for *pcnf* using duality principle.

Example 36.26. *Obtain the principal disjunctive and conjunctive normal forms of*

$$p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]. \qquad \textit{(Bharathiar, 2001)}$$

Sol. (i) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$

$\Leftrightarrow \sim p \wedge [(\sim p \vee q) \wedge (q \wedge p)]$

[using DeMorgan's law and equivalence $p \to q \Leftrightarrow \sim p \vee q$.

$\Leftrightarrow \sim p \vee [\sim p \wedge (q \wedge p)] \vee [q \wedge (q \wedge p)]$

$\Leftrightarrow \sim p \vee (q \wedge p)$

$\Leftrightarrow [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p)$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p)$

This is the desired *pdnf*.

(ii) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$

$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\Leftrightarrow [\sim p \vee (\sim p \vee q)] \wedge [\sim p \vee (q \wedge p)]$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee p)$

$\Leftrightarrow \sim p \vee q$

This is the desired *pcnf*.

Example 36.27. *Obtain the principal conjunctive normal form for* $(Q \to P) \wedge (\sim P \wedge Q)$.

(Andhra, 2004)

Sol. $(Q \to P) \wedge (\sim P \wedge Q) \Leftrightarrow (\sim Q \vee P) \wedge (\sim P \wedge Q)$

$\Leftrightarrow (\sim Q \vee P) \wedge [\sim P \vee (Q \wedge \sim Q) \wedge Q \vee (P \wedge \sim P)]$

$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P) \wedge (Q \vee \sim P)$

$\Leftrightarrow (\sim Q \vee P) \wedge (\sim P \vee Q) \wedge (\sim P \vee \sim Q) \wedge (Q \vee P)$.

36.14. INFERENCE THEORY

Inferring the conclusions from certain premises is known as the *inference theory*. When conclusion is reached from a set of premises by using the accepted rules of reasoning, then this process is called a *deduction* or a *formal proof*. A proof of a theorem is a valid argument.

The criteria for finding 'whether an argument is valid' are called *rules* which are expressed in terms of premises and conclusions or in terms of statement formulae.

The proofs are of two types : Direct or *Indirect*.

(i) If in a proof the truth of the premises directly shows the truth of the conclusions, then it is called a direct proof.

The universal quantification of $P(x)$ is denoted by $\forall\, xP(x)$. The symbol $\forall$ is called the *universal quantifier*.

> **Obs.** When it is possible to list all the elements in the universe of discourse say : $x_1, x_2, \ldots\ldots, x_n$, then the universal quantification $\forall\, x\, P(x)$ is same as the conjunction $P(x_1) \wedge P(x_2) \wedge \ldots\ldots, P(x_n)$ for this conjunction is true if and only if $P(x_1), P(x_2), \ldots\ldots, P(x_n)$ are all true.

Example 36.16. *What is the truth value of the quantification $\forall\, x\, P(x)$ where*

(a) $P(x)$ is the statement $x < 5$' and universe of discourse is the set of real numbers.

(b) $P(x)$ is the statement '$x^2 < 18$' and the universe of discourse consists of positive integers not exceeding 5 ?

Sol. (a) For instance, $P(6)$ is false ; therefore $P(x)$ is not true for all real numbers x.

Thus $\forall\, xP(x)$ is false.

(b) The statement $\forall\, xP(x)$ is same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

Here the universe of discourse is 1, 2, 3, 4, 5 and P(5) is the statement '$5^2 < 18$' which is false Hence $\forall\, xP(x)$ is also false.

(3) Existential quantification. Many statements assert that there is an element with a certain property. To express such statements we use *existential quantification*. In such cases, we form a proposition which is true if and only if $P(x)$ is true for at least one value of x in the universe of discourse.

Thus *the existential quantification of $P(x)$ is the proposition these exists an element x in the universe of discourse such that $P(x)$ is true'*

The notation $\exists xP(x)$ is used for the *existential quantification* wherein $\exists$ is called the *existential quantifier*.

> **Obs.** When it is possible to list all the elements in the universe say : $x_1, x_2, \ldots\ldots, x_n$, then the existential quantification $\exists x\, P(x)$ is same as the disjunction $P(x_1) \vee P(x_2) \vee \ldots\ldots P(x_n)$

Example 36.17. *What is the truth value of the quantification $\exists xP(x)$ where*

(a) $P(x)$ is the statement '$x > 5$' and universe of discourse is the set of real numbers.

(b) $P(x)$ is the statement '$x^2 > 18$' and the universe of discourse consists of positive integers not exceeding 5 ?

Sol. (a) Since '$x > 5$' is true, say for $x = 6, 8$ etc.

$\therefore$ $\exists xP(x)$ is true.

(b) The statement $\exists xP(x)$ is same as the disjunction $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$

The universe of discourse is 1, 2, 3, 4, 5 and $P(5)$ is the statement $5^2 > 18$, which is true. Hence $\exists xP(x)$ is also true.

Example 36.18. *Find the truth value of each of the following statements :*

(a) $\exists x, x^2 = 1$ (b) $\forall\ x, |x| = x$ (c) $\exists x, x + 4 = x$.

Sol. (a) If $x_0 = 1$, then $x_0^2 = 1$, therefore the given statement is true.

(b) If $x_0 = -3$, then $|x_0| \neq x_0$, therefore the given statement is false.

(c) As there is no solution to $x + 4 = x$, the given statement is false.

Example 36.19. *Given $P = \{2, 3, 4, 5, 6\}$, state the truth values of each of the following statements :*

(a) $(\forall x \in P)(x + 3 < 10)$ (b) $(\forall x \in P)(x + 2 < 7)$

(c) $(\exists x \in P)(x + 3 = 10)$ (d) $(\exists x \in P)(x + 2 < 7)$

Sol. (a) True, for each number in P satisfies $x + 3 < 10$.

(b) False, for if $x_0 = 6$, then $x_0 + 2$ is not ≤ 7.

(c) False, for no number in P is a solution to $x + 3 = 10$.

(d) True, for if $x_0 = 2$ then $x_0 + 2 < 7$ is a solution.

Example 36.20. *Negate each of the following statements :*

(a) $\forall x, x^2 = x,$ (b) $\forall x, x + 4 > x$ (c) $\forall x, |x| = x.$

Sol. (a) $\sim \forall x, x^2 = x \equiv \forall x \sim (x^2 = x) \equiv \forall x, x^2 \neq x.$

(b) $\sim \forall x, x + 4 > x \equiv \exists x \sim (x + 4 > x) \equiv \rightarrow x, x + 4 \leq x.$

(c) $\sim \forall x, |x| = x \equiv \exists x \sim (|x| = x) \equiv \exists x \ |x| \neq x.$

Example 36.21. *Symbolise using quantifiers :*

(i) *Every even number is divisible by 2.*

(ii) *There is no prime number between 23 and 29.* (V.T.U., MCA, 2001S)

Sol. (i) $E(x)$: x is even number ; $D(x)$: x is divisible by 2.

$(\forall x)\,[E(x) \rightarrow D(x)]$

(ii) If p denotes the set of prime numbers, then $(\exists_n)_p\ (23 < n < 29)$.

Example 36.22. *Symbolise the expression 'All the world loves a lover.'* (*Madras, 2001*)

Sol. Let $p(x)$: x is a person ,

$L(x)$: x is a lover.

and $Q(x, y)$: x loves y

Then the required expression is

$(\forall x)\,[p(x) \rightarrow (y)\,(p(y) \wedge L(y)] \rightarrow Q(x, y).$

Summary. (i) $\forall\, Q(x)$ means that the predicate $Q(x)$ is true *for all values* in the universe of x.

(ii) $\exists Q(x)$ means that the predicate $Q(x)$ is satisfied if there is *at least one value* in the universe of x.

36.13. NORMAL FORMS

(1) For the given variables $p_1, p_2, \ldots\ldots, p_n$, we may form a statement $S(p_1, p_2, \ldots\ldots, p_n)$ The truth table for S will contain 2^n rows for all possible truth values of the n variables. The expression S may have the truth value 1 in all cases or may have the truth value 0 in all cases or have the truth value 1 for at least one combination of truth values assigned to the n variables. (Here S is said to be satisfiable). The problem of finding in a finite number of steps whether a given expression is a tautology or a contradiction or at least satisfiable is known as a *decision problem*. As the formation of a truth table is quite cumbersome, we go for an alternate approach called *normal form*.

In this approach, we use the word 'sum' in place of disjunction and 'product' in place of conjunction.

A sum of the variables and their negations is called an *elementary sum*. Similarly a product of the variables and their negation is called an *elementary product*.

(2) **Disjunctive normal form** *of a given formula is the formula which is equivalent to the given formula and which contains the sum of the elementary products.* The disjunctive

(*ii*) *An indirect proof* proceeds by assuming that p is true and also C is false, and then deduce a contradiction using p and $\sim C$, along with other premises.

The only difference between assumptions in a direct proof or an indirect proof is the negated conclusion.

Rules for deriving direct and indirect proofs :

(*i*) the proof should have finite steps only ;

(*ii*) each step must be either a premise or a proposition which is implied from previous steps using valid equivalence or implication ;

(*iii*) the last step for a direct proof must be conclusion while for an indirect proof it must be a contradiction.

Example 36.28. *State whether the conclusion C follows logically from the premises R and S*

(*a*) $R : p \to q,\quad S : p,\quad C : \sim q$.

(*b*) $R : p,\quad S : p \leftrightarrow q,\quad C : \sim (p \wedge q)$

(*c*) $R : p \to q,\quad S : p,\quad C : q$

Sol. Let us first form the following truth table :

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim (p \wedge q)$	$p \leftrightarrow q$
1	1	0	0	1	0	1
1	0	0	1	0	1	0
0	1	1	0	1	0	0
0	0	1	1	1	1	1

(*a*) Here only the first row of premises R and S contains 1 but not the conclusion C. Hence C is not valid.

(*b*) Only the first row of both R and S contains 1 but not the conclusion C. Hence C is not valid.

(*c*) Here both R and S contain '1' only in the first row and the conclusion C also has '1' in that row. Hence our conclusion is valid.

Example 36.29. *Find the direct and indirect proofs of* $p \to (q \to r), \sim r \vee p, q \Rightarrow r \to s.$

Sol. *Direct proof*

(*i*) $\sim r \vee p$	(premise)	(*ii*) r	(another premise)
(*iii*) p	[by (*i*) and (*ii*)]	(*iv*) $p \to (q \to r)$	(premise)
(*v*) $q \to r$	[by (*iii*) and (*iv*)]	(*vi*) q	(premise)
(*vii*) r	[by (*v*) and (*vi*)], which is a conclusion.		

Indirect proof.

(*i*) $\sim (s \to r)$		(negative of conclusion)
(*ii*) $s \wedge \sim r$		(by conditional equivalence)
(*iii*) $\sim r \vee p$		(premise)
(*iv*) $\sim r$ and	(*v*) s	[by conjunctive simplification
(*vi*) p		[by disconjuction of (*iii*) and (*iv*)]
(*vii*) $p \to (q \to r)$		(premise)
(*viii*) $q \to r$		[by (*vi*) and (*vii*)]

(ix) q (premise)

(x) r [by (viii) and (ix)]

(xi) $r \wedge \sim r$ [by (x) and (iv)]

This is a contradiction.

Problems 36.8

1. If $A = \{1, 2, 3, 4, 5\}$ be the universal set, determine the truth values of each of the following statements :
   (a) $(\forall x \in A)(x + 2 < 10)$ (b) $\exists x \in A\ (x + 2 = 10)$
2. Negate each of the following statements :
   (a) $\forall x, x^3 = x$; (b) $\forall x,\ x + 5 > x'$
   (c) Some students are 26 or older. (d) All students live in the hostels.
3. What is the truth value of $\forall x P(x)$ where $P(x)$ is a statement '$x^2 < 10$' and the universe of discourse consists of positive integers not exceeding 4.
4. Use universal quantifier to state 'the sum of any two rational numbers is rational'.
5. Over the universe of real numbers, use quantifier to say that the equation $a + x = b$ has a solution for all values of a and b. *(Madras, 1999)*
6. Translate the following statements involving quantifiers, into formulae :
   (a) All rationals are reals. (b) No rationals are reals.
   (c) Some rationals are reals. (d) Some rationals are not reals.
7. Show that $Q \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$ is a tautology. *(Madras, 1998)*
8. Convert $\sim A \wedge (\sim B \to C) \Rightarrow 0$ into CNF. *(V.T.U., MCA, 2001)*
9. Without constructing truth tables, obtain the product of sums canonical form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$. Hence find the sum of products canonicals form. *(Anna, 2004S)*
10. Find the direct proof of $p \to r, q \to s, p \vee q \Rightarrow s \vee r$.
11. Prove that $P \to Q, Q \to R, P \vee R \Rightarrow R$ by using indirect method. *(Anna, 2004)*
12. Using quantifier say $\sqrt{2}$ is not a real number ?
13. State whether the conclusion C follows logically from the premises R and S
    (a) $R : p \to q,\quad S : \sim q,\quad C : q$ (b) $R : p \to q,\quad S : q,\quad C : \sim p$.

III. BOOLEAN ALGEBRA

37.15 INTRODUCTION

(1) The concept of Boolean algebra was first introduced by George Boole* in 1854 through his paper 'An investigation of the laws of thought'. It is basically two values *i.e.* (0, 1) set. Earlier it had applications to statements and sets which are either true or false. In 1938 Claude Shannon showed that basic rules given by Boole could be used to design circuits. These days however, Boolean algebra has wide applications to switching circuits, electrical networks and electronic computers.

Basically there are three operations in the Boolean algebra (i) AND, (ii) OR, and (iii) NOT, which are symbolically represented by $\wedge$, $\vee$ and $/$ respectively. Some authors, use the symbols (+), (.) and (/) for the same operations.

Here ′ denotes the complement of an element and is defined by $0' = 1$ and $1' = 0$.

The operator $\wedge$ (i.e. 'AND') has the following values $1 \wedge 1 = 1, 1 \wedge 0 = 0, 0 \wedge 1 = 0, 0 \wedge 0 = 0$; while the operator $\vee$ (i.e. OR) has the values $1 \vee 1 = 1, 1 \vee 0 = 1, 0 \vee 1 = 1, 0 \vee 0 = 0$.

Def. *Any non-empty set B with the binary operations '$\wedge$' and '$\vee$' and the uninary operation '$'$' is called the Boolean algebra $\{B, \wedge, \vee, '\}$ if the following axioms hold where a, b, c are elements in B :*

1. *Commutative law* : $a \wedge b = b \wedge a$; $a \vee b = b \vee a$
2. *Associative law* : $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   $a \vee (b \vee c) = (a \vee b) \vee c$
3. *Distributive law* : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
   $a \wedge b \vee c = (a \wedge b) \vee (a \wedge c)$
4. *Complement law* : $a \wedge a' = 0, a \vee a' = 1$

The operations $\wedge$, $\vee$ and $/$ are called sum, product, and complement respectively. We shall follow the usual practice that $/$ has precedence over $\vee$ and $\vee$ has precedence over $\wedge$, unless guided by brackets. *e.g.* $a \wedge b \vee c$ means $a \wedge (b \vee c)$ not $(a \wedge b) \vee c$ while $a \vee b'$ implies $a \vee b'$ not $(a \vee b)$

(2) Boolean function. The variable x is called a *Boolean variable* if it assumes values only from B $\{0, 1\}$.

Def. *A function from the set* $\{(x_1, x_2, \ldots\ldots x_n) : x_i \in B, 1 \le i \le n\}$ *is called a Boolean function of degree n.* Boolean functions can be represented by expressions comprised of variables and Boolean operations.

e.g. $0, 1, x_1, x_2, \ldots\ldots x_n$ are Boolean expressions in the variables x_i $(1 \le i \le n)$. If p and q are Boolean expressions then $p \wedge q$, $p \vee q$ and p' are also Boolean expressions and each represents a Boolean function

By substituting 0 and 1 for the variables in the expression, the values of this function can be found.

(3) *If f and g be Boolean functions of degree n, then*

(*i*) *Complement of f* is the function f' where

$$f'(x_1, x_2, \ldots\ldots, x_n) = [f(x_1, x_2, \ldots\ldots, x_n)]'$$

(*ii*) *f and g are equal* if $f(x_1, x_2, \ldots\ldots, x_n) = g(x_1, x_2, \ldots\ldots, x_n)$

(*iii*) *Boolean sum* $f \vee g$ *is*

$$(f \vee g)(x_1, x_2, \ldots\ldots, x_n) = f(x_1, x_2, \ldots\ldots, x_n) \vee g(x_1, x_2, \ldots\ldots, x_n)$$

(*iv*) *Boolean product* $f \wedge g$ *is*

$$(f \wedge g)(x_1, x_2, \ldots\ldots, x_n) = f(x_1, x_2, \ldots\ldots, x_n) \wedge g(x_1, x_2, \ldots\ldots, x_n)$$

(4) Power in a Boolean function : $x^2 = x \vee x = x$

$$x^3 = x^2 \vee x = x \vee x = x, \ldots, x^n = x$$

Similarly $2x = x$, $3x = x$ etc.

Example 36.30. *Find the values of the Boolean function* $f = (x \wedge y') \vee z'$

Sol. f being third degree Boolean function has 2^3 *i.e.* 8 values which are shown in the

x	y	z	y'	z'	$x \wedge y'$	$(x \wedge y') \vee z'$
1	1	1	0	0	0	0
1	1	0	0	1	0	1
0	1	1	0	0	0	0
1	0	1	1	0	1	1
0	0	1	1	0	0	0
1	0	0	1	1	1	1
0	1	0	0	1	0	1
0	0	0	1	1	0	1

36.16. DUALITY

(1) The dual of any Boolean function is obtained by interchanging Boolean sums and Boolean products along with the interchange of zeros and ones.

For example the dual of $x \vee (y \wedge 0)$ is $x \wedge (y \vee 1)$.

The dual of any theorem of a Boolean algebra is also its theorem. This implies that the dual of any theorem in Boolean algebra is always true.

(2) **Principle of duality.** *The dual of any theorem (or property) in Boolean algebra is also a theorem (or property).*

36.17. BOOLEAN IDENTITIES

There are many identities in Boolean algebra which are quite useful in simplifying electrical circuits. Some of the important ones are given below

1. *Identity law* : $x \vee 0 = x$; $x \wedge 1 = x$
2. *Dominance laws* : $x \vee 1 = 1$; $x \wedge 0 = 0$
3. *Complement law* : $x \vee x' = 1$; $x \wedge x' = 0$
4. *Idempotent law* : $x \vee x = x$; $x \wedge x = x$
5. *Double complement law* : $(x')' = x$
6. *Commutative law* : $x \vee y = y \vee x$; $x \wedge y = y \wedge x$
7. *Associative law* : $x \vee (y \vee z) = (x \vee y) \vee z$.
   $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
8. *Distributive law* : $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
   $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
9. *De-Morgan's law* : $x \wedge y = x' \vee y'$; $(x \vee y)' = x' \wedge y'$
10. *Absorption law* : $x \wedge (x \vee y) = x$

Example 36.31. *Let B be a Boolean algebra. Show that for all $a \in B$, there exists a unique complement a'* (*Andhra, 2004*)

Sol. Let b and c be two complements of α.

Then $b = b \wedge 1$ [$\because$ 0 is an additive identity

$= b \wedge (a \vee c)$ [$\because$ c is complement of a

$= (b \wedge a) \vee (b \wedge c) = (a \wedge b) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$ [$\because$ $a \wedge b = a \wedge a = 0$

$= b \wedge c$ ...(*i*)

Similarly $\quad c = c \wedge 1 = c \wedge (a \vee b)$ $\quad$ [$\because$ $a \vee b = a \vee a' = 1$

$= (c \wedge a) \vee (c \wedge b)$

$= (a \wedge c) \vee (b \wedge c)$ $\quad$ [$\because$ $a \wedge c = a \wedge a' = 0$

$= 0 \vee (b \wedge c)$

$= b \wedge c$ $\quad$ ...(*ii*)

From (*i*) and (*ii*), we find that $b = c$.

Thus the complement of a is unique.

Example 36.32. *In a Boolean algebra, show that*

(*i*) $x + (x \,.\, y) = x$ $\qquad$ (*ii*) $x \,.\, (x + y) = x$.

Sol. (*i*) $\quad x + (x \,.\, y) = x \wedge (x \vee y) = (x \vee 0) \wedge (x \vee y)$ $\quad$ [$\because$ $x \vee 0 = x$

$= x \vee (0 \wedge y)$ $\quad$ [by distributive law

$= x \vee (y \wedge 0)$ $\quad$ [by commutative law

$= x \vee 0$ $\quad$ [$\because$ $y \wedge 0 = 0$

$= x$.

(*ii*) $\quad x \,.\, (x + y) = x \vee (x \wedge y)$

$= (x \wedge 1) \vee (x \wedge y)$ $\quad$ [$\because$ $x \wedge 1 = x$

$= x \wedge (1 \vee y)$ $\quad$ [by distributive law

$= x \wedge (y \vee 1)$ $\quad$ [by commutative law

$= x \wedge 1$ $\quad$ [$\because$ $y \vee 1 = 1$

$= x$

Example 36.33. *Simplify the following :*

(*i*) $(x + y) \quad x' \quad y'$ $\qquad$ (*ii*) $x \vee y \wedge y \vee z \wedge y \vee z'$ $\qquad$ (*iii*) $x \vee y \wedge [(x \wedge y') \vee y]'$

Sol. (*i*) $\quad (x \wedge y) \vee x' \vee y' = (x \wedge y) \vee (x' \vee y')$

$= (x \wedge y) \vee (x \wedge y)$ $\quad$ [by De Morgan's law

$= 1.$ $\quad$ [$\because$ $p \vee p' = 1$

(*ii*) $\quad x \vee y \wedge y \vee z \wedge y \vee z' = (y \vee x) \wedge (y \vee z) \wedge (y \vee z')$ $\quad$ [by commutative law

$= [y \vee (x \wedge z)] \wedge (y \vee z')$ $\quad$ [by distributive law

$= y \vee [x \wedge z \wedge z']$

$= y \vee [x \wedge (z \wedge z')]$ $\quad$ [by associative law

$= y \vee (x \wedge 0)$ $\quad$ [$\because$ $z \wedge z' = 0$

$= y \vee 0$ $\quad$ [$\because$ $x \wedge 0 = 0$

$= y$.

(*iii*) $\quad x \vee y \wedge [(x \wedge y') \vee y]' = x \vee y \wedge [y \vee (x \wedge y')]'$ $\quad$ [by commutative law

$= x \vee y \wedge [(y \vee x) \wedge (y \vee y')]'$ $\quad$ [by distributive law

$= (x \vee y) \wedge [(x \vee y) \wedge 1]'$ $\quad$ [$\because$ $y \vee y' = 1$

$= (x \vee y) \wedge (x \vee y)' = 0.$

Example 36.34. *Show that*

(*i*) $x \vee y \wedge y \vee z \wedge z \vee x = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

(*ii*) $(x \wedge y) \vee (x' \wedge z) = (x' \vee y) \wedge (x \vee z)$.

Sol. (*i*) R.H.S. $\quad = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

$= (x \wedge y) \vee (z \wedge y) \vee (z \wedge x)$ $\quad$ [by commutative law

$= (x \wedge y) \vee (z \wedge y \vee x)$ $\quad$ [by distributive law

$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (y \vee x)]$ [$\because$ $x \vee x = x$ etc

$= (x \vee z) \wedge (y \vee z) \wedge [(x \vee y) \wedge (x \vee y)]$

$= (x \vee z) \wedge (y \vee z) \wedge (x \vee y)$ [$\because$ $p \wedge p = p$

$= (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ [by commutative law

$=$ L.H.S.

(*ii*) L.H.S. $= (x \wedge y) \vee (x' \wedge z)$

$= [x \vee (x' \wedge z)] \wedge [y \vee (x' \wedge z)]$ [by distributive law

$= [(x \vee x') \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$

$= [1 \vee (x \vee z)] \wedge [(y \vee x') \wedge (y \vee z)]$

$= (x \vee z) \wedge (y \vee x') \wedge (y \vee z) \vee 1$

$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee (x \wedge x')]$

$= (x \vee z) \wedge (y \vee x') \wedge [(y \vee z) \vee x] \vee (y \vee z) \vee x']$

$= (x \vee z) \wedge [y \vee z \vee x] \wedge (x' \vee y) \vee [x' \wedge (y \vee z)]$

$= (x \vee z) \vee (1 \wedge y) \wedge (x' \vee y) \vee (1 \wedge z)$

$= [(x \vee z) \wedge 1] \wedge [(x' \vee y) \vee 1]$

$= (x \vee z) \wedge (x' \vee y) =$ R.H.S.

Example 36.35. *Show that*

(*i*) $x \vee y \wedge x' \vee y' = (x' \wedge y) \vee (x \wedge y')$ (*ii*) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)] = y$

Sol. (*i*) $(x \vee y) \wedge (x' \vee y') = [(x \vee y) \wedge x'] \vee [(x \vee y) \wedge y']$

$= [(x \wedge x') \vee (y \wedge x')] \vee [(x \wedge y') \vee (y \wedge y')]$ [by distributive law

$= \{0 \vee (x' \wedge y)\} \vee \{(x \wedge y') \vee 0\}$ [$\because$ $x \wedge x' = 0$

$= (x' \wedge y) \vee (x \wedge y')$

(*ii*) $[x \wedge (x' \vee y)] \vee [x' \wedge (x \vee y)]$

$= [(x \wedge x') \vee (x \wedge y)] \vee [(x' \wedge x) \vee (x' \wedge y)]$

$= [0 \vee (x \wedge y)] \vee [0 \vee (x' \wedge y)]$ [$\because$ $x \wedge x' = 0$

$= (x \vee x') \wedge y = y \wedge 1 = y.$

Example 36.36. *If $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$, then prove that $a = b$.*

Sol. Since $a \vee x = b \vee x$ and $a \vee x' = b \vee x'$

$(a \vee x) \wedge (a \vee x') = (b \vee x) \wedge (b \vee x')$

i.e. $a \vee (x \wedge x') = b \vee (x \wedge x')$ [by distributive law

or $a \vee 0 = b \vee 0$ [$\because$ $x \wedge x' = 0$

or $a = b.$

Example 36.37. *In Boolean algebra $[B, +, \cdot, ']$, show that*

$$(x \cdot y' + y \cdot z) \cdot (x \cdot z + y \cdot z') = x \cdot z$$ (*M.P.T.U., 2001*)

Sol. $\{(x \vee y') \wedge (y \vee z)\} \vee \{(x \vee z) \wedge (y \vee z')\}$

$= \{(x \vee y') \wedge y\} \vee \{(x \vee y') \wedge z\} \vee \{(x \wedge z) \wedge y\} \vee \{(x \vee z) \wedge z'\}$

$= \{(x \wedge y) \vee (y' \wedge y)\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee (z \wedge z')\}$

$= \{(x \wedge y) \vee 0\} \vee \{(x \wedge z) \vee (y' \wedge z)\} \vee \{(x \wedge y) \vee (z \wedge y)\} \vee \{(x \wedge z') \vee 0\}$

$= \{(x \wedge y) \vee (x \wedge y)\} \vee \{(x \wedge z) \vee (x \wedge z')\} \vee \{(z \wedge y) \vee (z \wedge y')\}$

$= (x \wedge y) \vee (x \wedge 1) \vee (z \wedge 1) = (x \wedge y) \vee x \vee z$

$= (x \vee x \vee z) \wedge (y \vee x \vee z) = (x \vee z) \wedge (y \vee x \vee z)$

$= (x \vee z) \wedge (1 \vee y) = (x \vee z) \wedge 1$

Problems 36.4

1. Find the truth table for the Boolean function $f(x, y, z) = (x \wedge y) \vee (y \wedge z')$.
2. Write the dual of the Boolean expression $x + x' . y = x + y$. (Andhra, 2004)
3. Simplify the following :
   (i) $(x \wedge y \wedge z)'$ (ii) $(x \vee y \vee z) \wedge (x' \wedge y' \wedge z')$
4. In a Boolean algebra $[B, \wedge, \vee, /]$, prove that
   (i) $(x \wedge y) \vee (x \wedge y') = x$. (Anna, 2005) (ii) $x' \wedge (x \vee y) = x' \wedge y$.
5. If $a \wedge x = b \wedge x$ and $a \wedge x' = b \wedge x'$, then show that $a = b$.
6. In a Boolean algebra $[B, \wedge, \vee, /]$, show that
   (i) $x \wedge (x \wedge y) = x \wedge y$ (ii) $x \vee (x \vee y) = x \vee y$
7. In Boolean algebra, prove that
   (i) $x \wedge (x' \vee y) = x \wedge y$ (ii) $x' \wedge y = x' \wedge (x \vee y)$.
8. Show that $(x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y') = x' \vee y'$
9. If B be a Boolean algebra and $x, y, z \in B$, prove that
$$(x \vee y) \wedge (x \vee y') \wedge (x' \vee y) = x \wedge y$$
10. Prove that $(x \vee y) \wedge [z \vee (x' \wedge y')] = (x \vee y) \wedge z$
11. In any Boolean algebra, show that $a = b$ if and only if $a . b' + a' . b = 0$ (Madras, 2001)
12. In Boolean algebra, show that
    (i) $(a + b) . (a' + c) = a \ \ c + a' \ \ b + b . c$. (Madras 1996)
    (ii) $(a + b') . (b + c') . (c + a') = (a' + b) \ \ (b' + c) . (c' + a)$ (Andhra, 2004)
13. Give the truth table for the Boolean function $f : B_3 \to B$ determined by the polynomial
$$P(x_1, x_2, x_3) = (x_1 \vee x_3) \ \wedge (x_1 \wedge (x_2 \vee x'_3)) \qquad \text{(V.T.U., 2001)}$$

36.18. MINIMAL BOOLEAN FUNCTION

Def. *A minimal Boolean function in n variables is the product of* $x_1, x_2, \ldots\ldots, x_n$. It is also called *minterm*.

If x, y are two variables and x', y' are their complementary variables respectively, then $x \vee y, x' \vee y, x' \wedge y, x \wedge y', x' \vee y'$ are each a minimal Boolean function.

Similarly there are 2^3 *i.e.* 8 minimal Boolean functions in the three variables x, y, z *i.e.* $x \vee y \vee z, x' \vee y \vee z, x \vee y' \vee z, x \vee y \vee z', x \vee y' \vee z, x' \vee y \vee z' \ x' \vee y' \vee z \ x' \vee y' \vee z'$.

In general, *there are* 2^n *minimal Boolean functions (or minterms) in n variables.*

Similarly the join of the variables $x_1, x_2, \ldots x_n$ is called a *maxterm* and there will be 2^n *maxterms.*

36.19. DISJUNCTIVE NORMAL FORM

(1) Def. *A Boolean function which can be expressed as sum of minimal Boolean functions is called a* **Disjunctive normal form** *or* **minterm normal form** or **Canonical form.**

(2) *If the number of distinct terms in a disjunctive normal form of Boolean function in n*

(3) Complement function of a disjunctive normal form *function f is the sum of all those terms of a complete disjunctive normal form which are not present in the disjunctive normal form of f.* The complement of f is denoted by f'.

For example, if $f = (x \vee y) \wedge (x \vee y')$

then its complete disjunctive normal form in variables x and y is

$$(x \vee y) \wedge (x' \vee y) \wedge (x \wedge y') \wedge (x' \vee y')$$

$\therefore$ The complement function of this disjunctve normal form is $f'(x, y) = (x' \wedge y) \wedge (x' \vee y')$.

Example 36.38. *Find the value of the complete disjunctive normal form in three variables x, y, z.*

Sol. The complete disjunctive normal form in three variables x, y, z is

$$f(x, y, z) = (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x' \vee y \vee z)$$
$$\wedge (x \vee y' \vee z') \wedge (x' \vee y \vee z') \wedge (x' \vee y' \vee z) \wedge (x' \vee y' \vee z')$$
$$= [(x \vee y) \vee (z \wedge z')] \wedge [(x \wedge y') \vee (z \wedge z')] \wedge [(x' \wedge y) \vee (z \wedge z')] \wedge [(x' \vee y') \vee (z \wedge z')]$$
$$= [(x \vee y) \vee 0] \wedge [(x \wedge y') \vee 0] \wedge [(x' \wedge y) \vee 0] \wedge [(x' \vee y') \vee 0]$$
$$= [x \vee (y \wedge y')] \wedge [x \vee (y \wedge y')]$$
$$= (x \vee 0) \wedge (x' \vee 0) = x \wedge x' = 0$$

36.20. CONJUNCTIVE NORMAL FORM

Def. *If a Boolean function $f(x_1, x_2, \ldots\ldots, x_n)$ is expressed in the form of factors and each factor is the sum of all the n-variables, then such a function is called a* **conjective normal form or maxterm normal form or dual canonical form**

(2) *If a conjective normal of a function of n variables contains all the 2^n distinct factors then such a function is called a* **complete conjective normal form.**

(3) Complement function of a conjective normal form *function f is a Boolean function which is the product of all those terms of complete conjective normal form which are not present in conjective normal form of f.*

The complement of conjective normal form f is denoted by f'

For example, if $f(x, y) = (x \wedge y) \vee (x \wedge y')$

then its complete conjunctive normal form in x and y is

$$(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

$\therefore$ The complement function of this conjunctive normal form is

$$f'(x, y) = (x' \wedge y) \vee (x' \wedge y')$$

Example 36.39. *Given Boolean expression f, where $f(x_1, x_2, x_3) = (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3)$, simplify this expression stating the laws used and obtain the minterm normal form.*

(Bharathiar, 1997)

Sol. Given $f(x_1, x_2, x_3) = (x_3' \wedge x_2) \vee (x_1' \wedge x_3) \vee (x_2 \wedge x_3)$

$= (x_3' \wedge x_2) \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3)$ [by commutative law

$= (x_2 \wedge x_3') \vee (x_2 \wedge x_3) \vee (x_1' \wedge x_3)$

$= [x_2 \wedge (x_3' \vee x_3)] \vee (x_1' \wedge x_3)$ [by distributive law

$= (x_2 \wedge 1) \vee (x_1' \wedge x_3)$ [$\because p \vee p = 1$

$= x_2 \vee x_1' \wedge x_3$ [by identity law

Minterm normal form of $f(x_1, x_2, x_3)$

$$= [(x_3' \wedge x_2) \wedge (x_1 \vee x_1')] \vee [(x_1' \wedge x_3) \wedge (x_2 \vee x_2')] \vee [(x_2 \wedge x_3) \wedge (x_1 \vee x_1')]$$
$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1) \vee (x_2 \wedge x_3 \wedge x_1')$$
$$= (x_3' \wedge x_2 \wedge x_1) \vee (x_3' \wedge x_2 \wedge x_1') \vee (x_1' \wedge x_3 \wedge x_2) \vee (x_1' \wedge x_3 \wedge x_2') \vee (x_2 \wedge x_3 \wedge x_1)$$

Example 36.40. *Express the following functions into conjunctive normal forms :*

(i) $x' \wedge y$ (ii) $(x \wedge y) \vee (x' \wedge y')$.

Sol. (i) $x' \wedge y = x' \wedge y \wedge (z \vee z') = (x' \wedge y \wedge z) \vee (x' \wedge y \wedge z')$.

(ii) $(x \wedge y) \vee (x' \wedge y') = [x \wedge y \wedge (z \vee z')] \vee [x' \wedge y' \wedge (z \vee z')]$

$$= (x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z').$$

Example 36.41. *The function $f = (x \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$ is in conjunctive normal form. Write its complement ?*

Sol. The complete conjunctive normal form in three variables x, y, z is $(x \wedge y \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \vee y \vee z') \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$

$\therefore$ The complement of the given function F is

$$F' = (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z').$$

Problems 36.5

1. Find the value of a comple disjunctive normal form in
   (i) two variables x, y (ii) three variables x, y, z.
2. Express the following functions into disjunctive normal form :
   (i) $x \vee y$ (ii) $x \wedge (x' \vee y)$.
3. Express the Boolean function $F = A \vee (B' \wedge C)$ in a sum of minterms *(Madras, 1998)*
4. Convert the function $x \wedge y'$ to disjunctive normal form in three variables x, y, z.
5. Express the function $f = (x \vee y') \wedge (x \vee z) \wedge (x \vee y)$ into conjunctive normal form in which maximum number of variables are used.
6. Write the complement of the conjunctive normal form function $(x \wedge y \wedge z') \vee (x \wedge y' \wedge z') \vee (x \wedge y \wedge z')$.

36.21. SWITCHING CIRCUITS

(1) *A switching network is an arrangement of wires and switches* (or gates) *which connect two terminals.* A switch can be either closed or open. A closed switch permits and an open switch stops flow of current

(2) If p denotes a switch, then p denotes that switch which is open when p is closed and p' is closed when p is open.

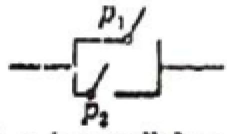
If x denotes the state of the switch p, then x' represents the state of the switch p'. x is called the Boolean variable which is a binary variable.

If $x = 1$ denotes the switch is closed or current flows. then $x = 0$ denotes that the switch is open or current stops.

(3) *Two switches p and p_2 are either connected in series* (represented by $\wedge$) *or connected in parallel* (represented by $\vee$).

(i) 

(p_1 & p_2 in series : $p_1 \wedge p_2$)

(ii) 

(p_1 & p_2 in parallel : $p_1 \vee p_2$)

If B {0, 1} is non-empty set and $\wedge$, $\vee$, / are the operations on B, then the system [{0, 1}, $\wedge$, 1] is usually called *Boolean switching algebra.*

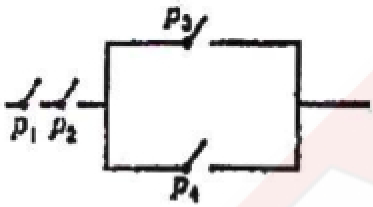
(4) Simplification of circuits. The simplification of a circuit means the least complicated circuit with minimum cost and best results. This depends on the cost of the equipment, number of switches and the type of the material used. Thus the simplification of circuits implies the use of lesser number of switches which can be achieved by using different properties of Boolean algebra. In other words, *the simplification of switching circuits is equivalent to simplification of the corresponding Boolean function.*

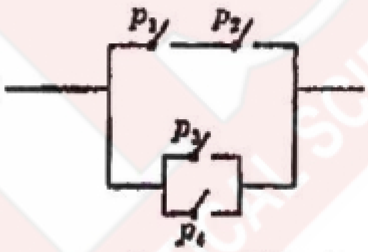
Example 36.42. *Draw the circuit which represents the Boolean function :*

(i) $(p_1 \wedge p_2) \wedge (p_3 \vee p_4)$ (ii) $(p_1 \wedge p_2) \vee (p_3 \vee p_4)$.

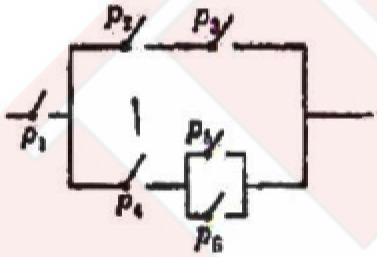
Sol. Here $p_1 \wedge p_2$ is a series circuit while $p_3 \vee p_4$ is a parallel circuit.

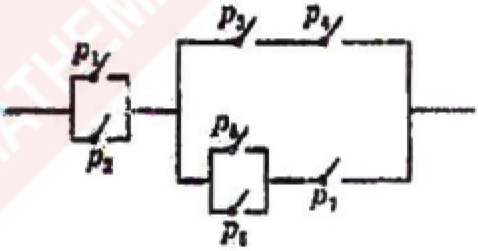
The required circuits are as follows :

(i) 

(ii) 

Example 36.43. *Write the Boolean functions representing the following circuits*

(i) 

(ii) 

Also draw the circuit diagram which would be the complement of the circuit in (ii).

Sol. (i) The given circuit is represented by the Boolean function :

$$f = p_1 \wedge (p_2 \wedge p_3) \vee [p_4 \wedge (p_5 \vee p_6)]$$

(ii) The Boolean function for the given circuit is

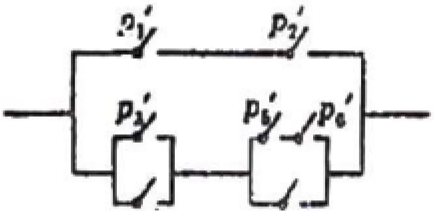
$$f = (p_1 \vee p_2) \wedge [(p_3 \wedge p_4) \vee [(p_5 \vee p_6) \wedge p_7)$$

The complement of f i.e. $f' = (p_1 \vee p_2)' \vee [(p_3 \wedge p_4) \vee ((p_5 \vee p_6) \wedge p_7)]$

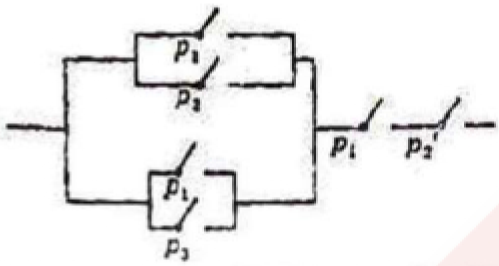
$$= (p_1 \wedge p_2') \vee [(p_3 \wedge p_4)' \wedge ((p_5 \vee p_6) \wedge p_7)']$$

$$= (p_1 \wedge p_2') \vee [(p_3' \vee p_4') \wedge ((p_5' \wedge p_6') \vee p_7')$$

Its circuit diagram is as follows :

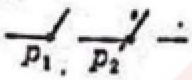


Example 36.44. *Simplify the following circuit and draw the diagram of the resulting circuit :*



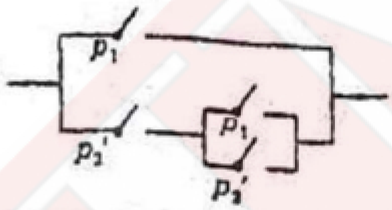
Sol. The given circuit is represented by the Boolean function f

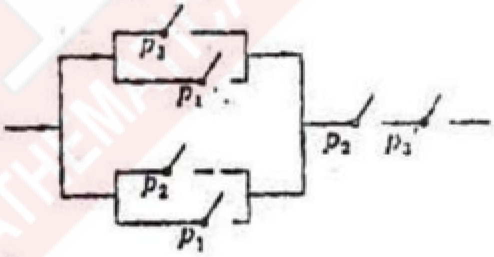
$$= [(p_1 \vee p_2) \vee (p_1 \vee p_3)] \wedge (p_1 \wedge p_2') = (p_1 \vee p_2 \vee p_3) \wedge (p_1 \wedge p_2')$$
$$= (p_1 \wedge p_1 \wedge p_2') \vee (p_2 \wedge p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') \qquad \text{(by distributive law}$$
$$= (p_1 \wedge p_2') \vee (p_3 \wedge p_1 \wedge p_2') = p_1 \wedge [p_2' \vee (p_3 \wedge p_2')]$$
$$= p_1 \wedge p_2' \qquad \text{[by absorption law}$$

Its circuit diagram is 

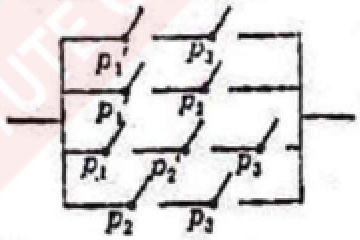
Problems 36.6

1. Draw the circuit diagram represented by the Boolean functions :
   (*i*) $[p_1 \wedge (p_1 \vee p_2)] \vee [p_2 \wedge (p_1' \vee p_2)]$ (*ii*) $p_1 \wedge [(p_2 \vee p_4') \vee (p_3' \wedge (p_1 \vee p_4 \vee p_3'))] \wedge p_2$.
2. Write the Boolean functions representing the following circuits :

(*i*) 

(*ii*) 

3. Simplify the Boolean functions, $p \vee (p' \wedge q) \vee (p \wedge q)$
4. Simplify the following circuit and draw the diagram of the resulting circuit :



5. Draw the simplified network of $f(x, y, z) = (x \vee y \vee z) \wedge (x \vee y' \vee z) \wedge (x' \vee y' \vee z)$ (*M.P.T.U., 2001*)
6. Consider the function $f(x_1, x_2, x_3) = [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee (x_1 \vee x_2')$
   (*a*) Simplify f algebraically. (*b*) Draw the switching circuit of f.
   (*c*) Also find the minterm normal form of f. (*Madras, 1998*)

IV. FUZZY SETS

36.22. FUZZY LOGIC

We have so far dealt with the fundamentals of classical logic. Besides this, we have *crisp* another type of logic which includes not only the crisp values but all the values between true (1) and false (0). But there is some degree of vagueness about the exact value between [0, 1]. *The logic to infer a definite outcome from such vague inputs is called* **fuzzy logic**.

(2) **Fuzzy set.** To provide a mathematical modelling to fuzzy logic, L.A. Zudeh introduced the concept of 'Fuzzy sets' in 1965 on the basis of a *membership function*. The theory of 'fuzzy sets' is now fully developed.

Def. *A fuzzy set F of a non-zero set X(x) is defined as* $F = \{x, \mu_F(x)\} : x \in X$.

Here $\mu_F : X \to [0, 1]$ is a function called the *membership function* of F and $\mu_F(x)$ is the degree of membership of $x \in X$ in F.

In particular $\mu(x) = 1$ implies full membership

$\mu(x) = 0$ implies non-membership

and $0 < \mu(x) < 1$ means intermediate membership.

A fuzzy set F is, therefore, a set of pairs consisting of a particular element of the universe X and its degree of membership *i.e.* each x is assigned a value in the range (0, 1) indicating the extent to which x has the attribute F. It can also be represented as $F = \{[x_1, \mu_F(x_1)], [x_2, \mu_F(x_2)], \ldots\ldots [x_n, \mu_F(x_n)]\}$.

For example, if x is the number of cars in a lane, 'small' may be taken as a particular value of the fuzzy variable x and to each x is assigned a number in the range (0, 1) then $\mu_{small}(x) \in (0, 1)$ is the *membership function*.

Example 36.45. *In a car-race, all the cars complete the race in four time-groups : shortest time, moderate time, long time and longest time. If we note the time taken by each car in a group, it will give rise to a distribution of times. Now let us find the outcome of the race based on engine power, car speed and road conditions. Each of these variables may further be divided into :*

(*i*) *low, medium and high for the variable engine power,*

(*ii*) *slow, moderate and fast for the variable car speed,*

(*iii*) *rough, bumpy and smooth for the variable road conditions.*

Now we try to predict on some basis, in which of the four groups the car will finish, if it has low engine power, moderate speed and rough road.

Then the distribution for the engine power would correspond to the membership function for low, medium and high. Similarly the distribution for the speed would depend on the membership function for slow, moderate and fast, while the distribution for road conditions would depend on membership function for rough, bumpy and smooth.

36.23. FUZZY SET OPERATIONS

(1) A fuzzy set is said to be *normalised* when the largest element of the set (called *supremum*) is unity.

For instance, the set of members {5, 10 , 15, 20, 25} is normalised to {0.2, 0.4, 0.6, 0.8, 1} by dividing each member by 25, the supremum in the set.

The normalization of a fuzzy set F is expressed as $\sup_{x \in X} F(x) = 1$.

(2) **Complement.** *The complement of a fuzzy set F is the set* F^c *with degree of membership of an element in* F^c *equal to one minus degree of membership of this element in F. (Fig. 36.5*

For example, if F = [0.4 Ram, 0.6 Sham, 0.8 Jyoti, 0.9 Ritu]

be a set of intelligent students, then F^c = [0.6 Ram, 0.4 Sham, 0.2 Jyoti, 0.1 Ritu]