Mains Test Series – 2020

COMMON TEST [TEST–16, Batch-I] & [TEST–8, Batch-II]

ANSWER KEY, Paper-II [full syllabus]

1(a) Show that A_4 does not contain a subgroup of order 6.

Sol^n: Let H be a subgroup of A_4 of order 6.

Since $[A_4 : H] = 2$

[Let H be a subgroup of a group G such that $[G:H]=2$. thus H is a normal subgroup of G].

we conclude that H is normal in A_4.

Now since $\text{Ord}(H) = 6 = (3)(2)$

let K be a sylow-3-subgroup of H (observe that K is also a sylow-3-subgroup of A_4).

Then by theorem [Let G be a finite group, H be a normal subgroup of G, and let K be a sylow p-subgroup of H. Then $G = HN_G(K)$ and $[G:H]$ divides $\text{Ord}(N_G(K))$, where $N_G(K) = \{g \in G : g^{-1}Kg = K\}$ (the normalizer of K in G)].

we conclude that $A_4 = HN_{A_4}(K)$ (note that $N_{A_4}(K)$ is the normalizer of K in A_4).

Since $[H:K] = 2$, once again K is normal in H.

Thus $H \subset N_{A_4}(K)$. Hence by theorem [Let H and K be finite subgroups of a group G. Then $\text{Ord}(HK) = \dfrac{\text{ord}(H)\,\text{ord}(K)}{\text{Ord}(H \cap K)}$]

we have $\text{Ord}(A4) = \dfrac{\text{ord}(H)\,\text{Ord}(N_{A_4}(K))}{\text{Ord}(H \cap N_{A_4}(K))}$

$$= 6 \operatorname{ord}(N_{A_4}(K))/6$$
$$= \operatorname{ord}(N_{A_4}(K)).$$

Hence $N_{A_4}(K) = A_4$.

Thus K is normal in A_4. Hence K is unique by theorem [A Sylow p-subgroup of a finite group G is a normal subgroup of G if and only if it is the only Sylow p-subgroup of G].

Thus there are exactly two elements of order 3 in A_4.

But $(1,2,3)$, $(1,3,2)$, $(1,2,4)$ are elements in A_4 and each of order 3. Thus A_4 has at least 3 elements of order 3, a contradiction.

Hence A_4 does not contain a subgroup of order 6.

1(b) Prove that x^3-9 is irreducible over the integers mod 31.

Sol'n: $F = Z_{31} = \{0, 1, 2, 3, \ldots 30\}$.

Since $9+22 = 0$ in Z_{31}

so $-9 = 22$ in Z_{31}

Let $f(x) = x^3 - 9 = x^3 + 22 = (x+a)(x^2+bx+c)$

$= x^3 + x^2(a+b) + x(ab+c) + ac$

where $a, b, c \in F$.

Comparing the coefficients of x^2, x and constant terms on both sides, we get

$a+b = 0$ —— ①

$ab+c = 0$ —— ②

$ac = 22$ —— ③

⇒ ① is satisfied when

$(a,b) = (1,30), (2,29), (3,28)$ ------ $(14,17), (15,16)$

$(0,0), (16,15), (17,14), (18,13)$ ------ $(30,1)$

but there are no values to satisfy ② & ③.

i.e., $ab+c = 0$ but $ac \neq 22$.

$1\times30+1 = 0$ but $1\times1 = 1 \neq 22$

$2\times29+4 = 0$ but $2\times4 = 8 \neq 22$

$3\times28+9 = 0$ but $3\times9 = 27 \neq 22$

$4\times27+15 = 0$ but $4\times15 = 60 = 29 \neq 22$

and so on ......

∴ x^3-9 irreducible over Z_{31}.

(OR)

$f(0) = -9 = 22$, $f(1) = -8 = 23$, $f(2) = -1 = 30$; $f(3) = 18$

$f(4) = 24$, $f(5) = 23$, $f(6) = 21$, $f(7) = 24$, $f(8) = 7$

$f(9) = 7$, $f(10) = 30$, $f(11) = 21$, $f(12) = 14$, $f(13) = 18$

$f(14) = 7$, $f(15) = 18$, $f(16) = 26$, $f(17) = 6$, $f(18) = 26$

$f(19) = 30$, $f(20) = 24$, $f(21) = 14$, $f(22) = 6$, $f(23) = 6$

$f(24) = 20$, $f(25) = 23$, $f(26) = 21$, $f(27) = 20$, $f(28) = 26$, $f(29) = 14$, $f(30) = 22$

Hence $f(x) = x^3-9$ has no root in Z_{31}. ∴ x^3-9 is irreducible over Z_{31}.

1(C) Define an open set. Prove that the union of an arbitrary family of open sets is open. Show also that the intersection of a finite family of open sets is open. Does it hold for an arbitrary family of open sets? Explain the reason for your answer by example.

Sol'n: Open set: A subset S of $\mathbb{R}$ is said to be an open set if S is a nbd of each of its points i.e, if for each $p \in S$ $\exists$ an $\epsilon > 0$ such that

$$(p-\epsilon, p+\epsilon) \subset S.$$

(or) If S is a subset of $\mathbb{R}$ is said to be open if every point of S is an interior point of S

i.e, S is open $\Leftrightarrow S^0 = S$.

→ The union of an arbitrary family of opensets is an open set.

Sol'n: Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of open sets.

Let x be any element of $S = \bigcup_{\lambda\in\Lambda} A_\lambda$.

$$x \in S$$

$$\Rightarrow x \in \bigcup_{\lambda\in\Lambda} A_\lambda$$

$\Rightarrow x \in A_\lambda$ for at least one $\lambda \in \Lambda$

$\Rightarrow$ at least on A_λ is a nbd of x [$\because$ each A_λ is open]

But $A_\lambda \subset S$ for all $\lambda \in \Lambda$

$\therefore$ S is a nbd of x.

$\Rightarrow S$ is a nbd of each of its points ($\because x$ is arbitrary)

Hence S is an open set.

1(d). Find the regular function $w = u + iv$ where

$u = e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$

Sol'n: Here $u = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$$\therefore \frac{\partial u}{\partial x} = e^{-x}\{-(x^2-y^2)\cos y + 2xy\sin y\} + e^{-x}\{2x\cos y + 2y\sin y\}$$
$$= \phi_1(x,y) \text{ say}$$

and $$\frac{\partial u}{\partial y} = e^{-x}\{-(x^2-y^2)\sin y - 2y\cos y + 2x\sin y + 2x\cos y\}$$
$$= \phi_2(x,y) \text{ say.}$$

Hence by Milens method, we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) = e^{-z}(-z^2+2z)$$

Integrating, we get

$$f(z) = \int e^{-z}(2z - z^2)\,dz + C$$
$$= -\int e^{-z} z^2 + 2\int z e^{-z}\,dz + C$$
$$= e^{-z} z^2 - 2\int \cancel{z e^{-z} dz} + 2\int \cancel{z e^{-z}}\,dz + C$$
$$= z^2 e^{-z} + C$$

$\therefore f(z) = z^2 e^{-z}$ where $z = x + iy$.

→ The intersection of a finite number of open sets is an open set.

Sol^n: Let $A_1, A_2, A_3, \ldots, A_n$ be n open sets and

$$S = \bigcap_{i=1}^{n} A_i$$

If $S=\phi$, then S is open because ϕ is open

If $S \neq \phi$, let x be any element of S,

$$x \in S \Rightarrow x \in \bigcap_{i=1}^{n} A_i$$

$$\Rightarrow x \in A_i \text{ for each } i = 1,2,3,\ldots n.$$

But A_i is an open set for each $i = 1,2,3,\ldots n$.

$\Rightarrow$ Each A_i is a nbd of x

$\Rightarrow$ For each i, $\exists\, \epsilon_i > 0$ such that $(x-\epsilon_i, x+\epsilon_i) \subset A_i$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots \epsilon_n\}$, then

$(x-\epsilon, x+\epsilon) \subset (x-\epsilon_i, x+\epsilon_i) \subset A_i$ for each i

$$\Rightarrow (x-\epsilon, x+\epsilon) \subset \bigcap_{i=1}^{n} A_i$$

$\Rightarrow (x-\epsilon, x+\epsilon) \subset S$

$\Rightarrow S$ is a nbd of x

$\Rightarrow S$ is an open set ($\because x$ is arbitrary)

* The intersection of an infinite family of open sets may or may not be open set

for example: (i) Let $I_n = (0,n)$, $n \in N$. Then $\{I_n\}_{n\in N}$ is an infinite family of open sets.

$$\bigcap_{n\in N} I_n = (0,1) \text{ which is an open set.}$$

(ii) Let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in N$. Then $\{I_n\}_{n\in N}$ is an infinite family of open sets.

$\bigcap_{n\in N} I_n = \{0\}$, which being a non-empty finite set is not an open set.

1.(e) Two products are manufactured sequentially on two machines. The time available on each machine is 8 hours per day and may be increased by upto 4 hours of overtime, if necessary at an additional cost of Rs. 100 per hour. The table below gives the production rate on the two machines as well as the price per unit of the two products. Determine the optimum production schedule and the recomended use of overtime if any

	Production 1	Production 2	
Machine 1:	5	5	production rate (units/hr)
Machine 2:	8	4	
Price per unit (Rs):	110	118	

Sol Mathematical formulation:

In this model it is required to determine the number of units of each product (variables) that maximises net profit (objective) provided that the maximum allowance machine hours are increased only on an overtime basis (constraints).

Let x_j = number of units project j, $j=1,2$

In absence of overtime option:

$$\frac{x_1}{5}+\frac{x_2}{5}\le 8 \ (M1) \text{ machine 1}$$

$$\frac{x_1}{8}+\frac{x_2}{4}\le 8 \ (M2).$$

To include overtime option:

$$\frac{x_1}{5}+\frac{x_2}{5}-y_1\le 8$$

$$\frac{x_1}{8}+\frac{x_2}{4}-y_2\le 8.$$

where y_j ($j=1,2$) are variables unrestricted in sign due to the following reasons:

If $y_j < 0$, the 8 hour limit on the capacity of the machine is not increased and no overtime is needed.

If $y_j > 0$, the used machine hours will increase the daily limit and y_j will then represent overtime hours.

Next, we need to limit the daily use overtime to 4 hours and also to include the cost of overtime in the objective function.

Since $y_j > 0$ only when overtime is used, the constraints $y_j \le 4$, $j=1,2$ which will provide the restriction on overtime.

Our goal is to maximise the net profit that equals the total profit from the two products less than the additional cost of overtime.

$\therefore$ Overtime cost $=$ (cost/hr) $\times$ (overtime hrs)

$$= 100\{\max(0, y_j)\}.$$

where $\max(0, y_j) = 0$, $y_j < 0$ yields zero overtime cost.

$$\therefore \text{Max } Z = 110x_1 + 118x_2 - 100\{\max(0, y_1) + \max(0, y_2)\}.$$

s.c.
$$\frac{x_1}{5} + \frac{x_2}{5} - y_1 = 8$$
$$\frac{x_1}{8} + \frac{x_2}{4} - y_2 = 8.$$
$$y_1 \le 4$$
$$y_2 \le 4$$
$x_1, x_2 \ge 0$, y_1, y_2 unrestricted in sign.

2(a). Consider the set $G = \{0,1,2,3,4,5,6,7\}$. Suppose there is a group operation $*$ on G that satisfies the following two conditions.

(i) $a*b \le a+b$ for all a,b in G.

(ii) $a*a = 0$ for all a in G.

Construct the multiplication table for G.

Sol^n: Hint:

*	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

2(b) If R is a division ring, prove that $Z(R)$ is a field.

Sol^n: Let R be a division ring. The centre of R is defined as $Z(R) = \{a \in R : xa = ax \ \forall x \in R\}$ —— ①
($Z(R)$ = Centre of ring)

we know that $Z(R)$ is a subring of R.

In other words, $Z(R)$ is a ring. we have to show that $Z(R)$ is a field.

Let $a, b \in Z(R)$ be arbitrary.

Using ①, $ax = xa \ \forall x \in R$

In particular, $ab = ba \ \forall a, b \in Z(R)$

$\Rightarrow Z(R)$ is a commutative ring.

Since R is a division ring, $1 \in R$ and $1x = x1 \ \forall x \in R$.

Thus $1 \in Z(R)$

Finally, we show that each non-zero element of $Z(R)$ has its multiplicative inverse in $Z(R)$.

Let $a \neq 0 \in Z(R)$ be arbitrary $\Rightarrow a \neq 0 \in R$

$\Rightarrow a^{-1} \in R$., Since R is a division ring.

Let $x \neq 0 \in R$ be arbitrary, so that $x^{-1} \in R$ exists.

we have $a^{-1}x = (x^{-1}a)^{-1} = (ax^{-1})^{-1}$, since $a \in Z(R)$

$\Rightarrow ax^{-1} = x^{-1}a$

$\therefore a^{-1}x = xa^{-1} \ \forall x \neq 0 \in R$.

obviously, $a^{-1}0 = 0a^{-1}$.

Thus $a^{-1}x = xa^{-1} \ \forall x \in R$

It means that $a^{-1} \in Z(R) \ \forall a \neq 0 \in Z(R)$

Hence $Z(R)$ is a field.

2(c) Show that the sequence $\{f_n\}$ where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2x + 2n & 1/n \le x \le 2/n \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$

is not uniformly convergent on $[0,1]$.

Sol^n The sequence converges to f, where $f(x)=0$, for all $x \in [0,1]$. Each function f_n and f are continuous on $[0,1]$.

Also $$\int_0^1 f_n\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (-n^2x + 2n)\,dx + \int_{2/n}^{1} 0\,dx = 1$$

But $$\int_0^1 f\,dx = 0$$

$$\lim_{n\to\infty} \int_0^1 f_n\,dx \ne \int_0^1 f\,dx$$

So by the theorem

(i) If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable, then f is integrable on $[a,b]$ and the sequence $\left\{\int_a^x f_n\,dt\right\}$ converges.

uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt \quad \forall x \in [a,b]$$

(B) If a series Σf_n converges uniformly to f on $[a,b]$ and each term $f_n(x)$ is integrable then 'f' is integrable on $[a,b]$ and the series $\Sigma\left(\int_a^x f_n\,dt\right)$ converges uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right), \forall x \in [a,b]$$

$\therefore$ from the above theorem the sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

2(d) Use the method of contour integration to prove that $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}$, $a^2 > (b^2+c^2)$.

Sol^n: $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2}$.

$$= \int_0^{2\pi} \frac{d\theta}{\left[a+\frac{b}{2}(e^{i\theta}+e^{-i\theta})+\frac{c}{2i}(e^{i\theta}-e^{-i\theta})\right]^2}$$

write $e^{i\theta} = z \to d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{dz/iz}{\left[a+\frac{b}{2}(z+\frac{1}{z})+\frac{c}{2i}(z-\frac{1}{z})\right]^2}$$

$$= \int_C \frac{4izdz}{[2iza+bi(z^2+1)+c(z^2-1)]^2}$$

$$= \int_C \frac{4izdz}{[z^2(bi+c)+2iza+(bi-c)]^2}$$

$$= \frac{4i}{(bi+c)^2}\int_C \frac{zdz}{\left[z^2+\frac{2iaz}{bi+c}+\frac{bi-c}{bi+c}\right]^2} = \frac{4i}{(bi+c)^2}\int_C f(z)\,dz \quad \text{--- (2)}$$

where C is the unit circle.

$f(z)$ has poles of order 2 given by

$$z^2+\frac{2ia}{bi+c}z+\frac{bi-c}{bi+c} = 0.$$

$$\Rightarrow z = \frac{-\frac{2ia}{bi+c} \pm \sqrt{\frac{-4a^2}{(bi+c)^2}-\frac{4(bi-c)}{bi+c}}}{2}$$

$$\Rightarrow z = \frac{1}{2}\left[\frac{-2ia}{bi+c} \pm \frac{2i\sqrt{a^2-b^2-c^2}}{bi+c}\right]$$

$$\Rightarrow z = \frac{i}{bi+c}\left[-a \pm \sqrt{a^2-b^2-c^2}\right]$$

$z = \frac{i}{bi+c}\left(-a+\sqrt{a^2-b^2-c^2}\right)$ & $z = \frac{i}{(bi+c)}\left(-a-\sqrt{a^2-b^2-c^2}\right)$

$= \alpha$ (say) $= \beta$ (say)

only $z=\alpha$ lies inside C of order 2.

Residue at $z=\alpha$ is

$$\lim_{z\to\alpha} \frac{d}{dz}\left\{(z-\alpha)^2 \frac{z}{(z-\alpha)^2(z-\beta)^2}\right\}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{(z-\beta)^2}\right]$$

$$= \lim_{z\to\alpha} \frac{-(z+\beta)}{(z-\beta)^3}$$

$$= \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{\frac{-i}{(bi+c)}(-2a)}{\left(\frac{2i}{bi+c}\right)^3\left(\sqrt{a^2-b^2-c^2}\right)^3} = \frac{a}{\frac{-4}{(bi+c)^2}\left(\sqrt{a^2-b^2-c^2}\right)^3}$$

Then from (1).

$$I = \frac{4i}{(bi+c)^2}[2\pi i](\text{Residue at } z=\alpha)$$

$$= \frac{4i}{(bi+c)^2}(2\pi i)\frac{-a}{\frac{4}{(bi+c)^2}(a^2-b^2-c^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2-b^2-c^2)^{3/2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{[a+b\cos\theta+c\sin\theta]^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}$$

3(a) (i) Let G be a group of order 105. Prove that it is impossible that $\text{ord}(Z(G)) = 7$.

(ii) Let p be prime number in $\mathbb{Z}$. Suppose that H is a subgroup $\mathbb{Q}^*$ under multiplication such that $p \in H$. Prove that there is no group homomorphism from $\mathbb{Q}$ under addition onto H. Hence $\mathbb{Q} \not\cong H$.

Sol'n: If possible suppose that $O(Z(G)) = 7$.

Then $O(G/Z(G)) = 15$.

Since $15 = 3 \times 5$ and 3 doesnot divide $5-1=4$.

$\Rightarrow G/Z(G)$ is cyclic.

[Let G be a group such that $O(G) = pq$, for some primes $p<q$, and p doesnot divide $q-1$, then $G \cong \mathbb{Z}_{pq}$ is cyclic]

[WKT, if G be a group and $Z(G)$ be the centre of G.
If $Z/Z(G)$ is cyclic then G is abelian]

Hence, G is abelian

Hence, $Z(G) = G$, a contradiction.

Thus, it is impossible that $O(Z(G)) \neq 7$.

(ii) Then there is a group homomorphism ϕ from $\mathbb{Q}$ onto H. Since $p \in H$, there is an element $x \in \mathbb{Q}$ such that $\phi(x) = p$. Hence

$$p = \phi(x) = \phi\left(\tfrac{x}{2} + \tfrac{x}{2}\right)$$
$$= \phi\left(\tfrac{x}{2}\right)\phi\left(\tfrac{x}{2}\right) = \phi\left(\tfrac{x}{2}\right)^2$$

Since $\phi(x/2)^2 = p$

we conclude $\phi(x/2) = \sqrt{p}$

A contradiction, since p is prime and

$\phi(x/2) \in H \subset \mathbb{Q}^*$ and $\sqrt{p} \notin \mathbb{Q}$.

3(b)ii) Test for convergence the series

$$1+\frac{1!}{2}x+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\cdots$$

Sol: Here $u_n = \frac{(n-1)!}{n^{n-1}} x^{n-1}$

$\therefore u_{n+1} = \frac{n!}{(n+1)^n} x^n$

$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \frac{(n-1)!}{n^{n-1}} \frac{(n+1)^n}{n!} = \frac{1}{x} \frac{(n+1)^n}{n^n}$

$= \frac{1}{x}\left(1+\frac{1}{n}\right)^n$

$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{e}{x}$. $\left(\because \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e\right)$

$\therefore$ By ratio test Σu_n is convergent if $\frac{e}{x}>1$ i.e. if $x<e$. and Σu_n is divergent if $\frac{e}{x}<1$ i.e. if $x>e$.

for $\frac{e}{x}=1$ i.e. $x=e$, the ratio test fails and we proceed to apply logarithmic test in this case.

when $x=e$, we have $\frac{u_n}{u_{n+1}} = \frac{1}{e}\left(1+\frac{1}{n}\right)^n$

$\therefore n \log \frac{u_n}{u_{n+1}} = n\left[-1+n\log\left(1+\frac{1}{n}\right)\right]$ $(\because \log_e e = 1)$

$= n\left[-1+n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right)\right]$

$= -\frac{1}{2}+\frac{1}{3n}-\frac{1}{4n^2}+\cdots$

$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = -\frac{1}{2} < 1$

$\therefore$ by log test, Σu_n is divergent for $x=e$.

Hence the given series is convergent if $x<e$ and is divergent if $x \geq e$.

3(c) Solve minimize $Z = 7.5x_1 - 3x_2$

subject to the constraints

$3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

$x_1, x_2, x_3 \geq 0$.

Sol^n: The objective function of the given LPP is of maximization type:

So we convert it into maximization type

we have

$$\text{Max } Z_1 = \text{Min}(-Z)$$

$$\therefore \text{Max } Z_1 = -7.5x_1 + 3x_2$$

we write the given LPP in the standard form:

$$\text{Max } Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to

$$\left.\begin{aligned} 3x_1 - x_2 - x_3 - S_1 + 0S_2 + A_1 + 0A_2 &= 3 \\ x_1 - x_2 + x_3 + 0S_1 - S_2 + 0A_1 + A_2 &= 2 \end{aligned}\right\} \text{---} (1)$$

$$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0.$$

where S_1, S_2 are surplus variables

and A_1, A_2 are artificial varibles.

There are two methods for solving the problem involving of the artificial variables.

(i) Big-M-method (ii) Two phase method.

Here we apply the two phase method:

~~Now~~ Phase-I: we formulate an artificial objective function Z_1^* by assigning (-1) cost to an artificial variables A_1 and A_2 and zero cost to all other variables.

∴ we have

$$\text{Max } Z_1^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - A_1 - A_2$$

Subject to

$$\left.\begin{array}{l} 3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3 \\ x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2 \end{array}\right\} \text{—(2)}$$

All $x_i,\ s_i,\ A_i \geq 0$

Now the IBFS is given by

Setting $x_1 = x_2 = s_1 = s_2 = 0$ (non-basic)

$A_1 = 3,\ A_2 = 2$ (basic)

and $Z_1^* = -5 < 0$

∴ Initial ~~Simplex~~ Simplex table is:

C_B	c_j Basis	0 x_1	0 x_2	0 x_3	0 s_1	0 s_2	−1 A_1	−1 A_2	b	θ
−1	A_1	(3)	−1	−1	−1	0	1	0	3	1 →
−1	A_2	1	−1	1	0	−1	0	1	2	2
$Z_j^* = \sum C_B a_{ij}$		−4	2	0	1	1	−1	−1	−5	
$C_j = c_j - Z_j^*$		4 ↑	−2	0	−1	−1	0	0		

from the above, x_1 is the entering variable

A_1 is the outgoing variable

Here (3) is the key element and make it into unity and make all other elements in its column to zero.

∴ The new simplex table is :

	C_j	0	0	0	0	0	-1	-1		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b	θ
0	x_1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	1	-3
-1	A_2	0	$-2/3$	$(4/3)$	$1/3$	-1	$-1/3$	1	1	$\frac{3}{4}$ →
$Z_j^* = \Sigma C_B a_{ij}$		0	$2/3$	$-4/3$	$-1/3$	1	$1/3$	-1	-1	0
$C_j = c_j - Z_j^*$		0	$-2/3$	$4/3$ ↑	$1/3$	-1	$-1/3$	0		

from the above table, x_3 is the entering variable, A_2 is the outgoing variable.
Here $(4/3)$ is the key element and we convert it into unity and all other elements in its column equal to zero.
∴ the revised simplex table is

	C_j	0	0	0	0	0	-1	-1	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	A_1	A_2	b
0	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	$3/4$
$Z_j^* = \Sigma C_B a_{ij}$		0	0	0	0	0	0	0	
$C_j = c_j - Z_j^*$		0	0	0	0	0	-1	-1	

From the above table, all C_j's ≤ 0,
this gives the optimal solution.
Also $\text{Max } Z_1^* = 0$
and no artificial variable appears in the basis. Therefore $x_1 = 5/4$, $x_3 = 3/4$ is an optimal basic feasible solution (OBFS) to the auxiliary LPP.
To find the OBFS to the original problem we

proceed to phase-II.

phase II:
Considering the actual costs associated with the original variables, the objective function is

$\text{Max} Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2 - 0A_1 - 0A_2$

subject to

$3x_1 - x_2 - x_3 - s_1 + 0s_2 + A_1 + 0A_2 = 3$

$x_1 - x_2 + x_3 + 0s_1 - s_2 + 0A_1 + A_2 = 2$

All $x_i, s_i, A_i \geq 0$.

Using final table of phase-I, the initial simplex table of phase-II is as follows

	C_j	$-15/2$	3	0	0	0	
C_B	Basis	x_1	x_2	x_3	s_1	s_2	b
$-\frac{15}{2}$	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$3/4$
$Z_j = \Sigma C_B a_{ij}$		$-\frac{15}{2}$	$15/4$	0	$15/8$	$-15/8$	$-75/8$
$C_j = C_j - Z_j$		0	$-3/4$	0	$-15/8$	$-15/8$	

From the above table, we have

all C_j's ≤ 0

This gives the optimal solution.

Hence an OBFS to the given LPP is

$x_1 = 5/4, \quad x_2 = 0, \quad x_3 = 3/4$

and $\text{Max} Z_1 = -75/8$.

Hence $\text{Min} Z = \text{Max}(-Z)$

$= -\text{Max} Z$

$= 75/8$

4(a) If R is ring, let $Z(R) = \{x \in R \mid xy = yx \text{ all } y \in R\}$.
Prove that $Z(R)$ is a subring of R.
Is $Z(R)$ an ideal? If not, Justify your answer.

Sol'n: The centre of a ring R, denoted by $Z(R)$, is defined as $Z(R) = \{a \in R : xa = ax \text{ for all } x \in R\}$.

Clearly, $Z(R)$ is non-empty;

Since $0x = x0 \; \forall x \in R \Rightarrow 0 \in Z(R)$.

Let $a, b \in Z(R)$, Then $xa = ax$ and $xb = bx \; \forall x \in R$ —(1)

we shall show that $a-b$ and ab are in $Z(R)$.

Consider $(a-b)x = ax - bx = xa - xb$, by (1)

Thus $(a-b)x = x(a-b) \; \forall x \in R \Rightarrow a-b \in Z(R)$

Again $(ab)x = a(bx) = a(xb)$, by (1)
$= (ax)b = (xa)b$, by (1)

Thus $(ab)x = x(ab) \; \forall x \in R \Rightarrow ab \in Z(R)$

Hence $Z(R)$ is a subring of R.

→ $Z(R)$ is need not be an ideal of R.

for example

Let M_2 be the ring of all 2×2 matrices over the integers.

For any $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2$ and $A = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \in M_2$,

we see that

$$AX = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ap & bp \\ cp & dp \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = XA$$

Hence $Z(M_2) = \left\{\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} : p \text{ is an integer}\right\}$.

we proceed to show that $Z(M_2)$ is not an ideal of

For $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M_2$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \in Z(M_2)$,

we have

$$SA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \notin Z(M_2).$$

Hence $Z(M_2)$ is not an ideal of M_2.

4(b) Let $f: \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{if } x<0 \\ c & \text{if } x=0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{if } x>0 \end{cases}$$

Determine the values of a, b, c for which the function is continuous at $x=0$.

Sol'n: At $x=0$, $f(0)=c$

LHL: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{\sin(a+1)x + \sin x}{x}$

$$= \lim_{x\to 0^-}\left[\frac{\sin(a+1)x}{x} + \frac{\sin x}{x}\right]$$

$$= \lim_{x\to 0^-}\left[\frac{(a+1)\sin(a+1)x}{(a+1)x}\right] + \lim_{x\to 0^-}\frac{\sin x}{x}$$

$$= (a+1)\lim_{x\to 0^-}\frac{\sin(a+1)x}{(a+1)x} + \lim_{x\to 0^-}\frac{\sin x}{x}$$

$$= (a+2)(1) + 1$$

$$= a+2 \qquad \left(\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

RHL: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$

$$= \lim_{x\to 0^+}\frac{x^{1/2}(1+bx)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0+} \frac{(1+bx)^{1/2}-1}{bx}$$

$$= \lim_{x\to 0+} \frac{\sqrt{1+bx}-1}{bx} \cdot \frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}$$

$$= \lim_{x\to 0+} \frac{1+bx-1}{bx\left[\sqrt{1+bx}+1\right]}$$

$$= \lim_{x\to 0+} \frac{bx}{bx\left[\sqrt{1+bx}+1\right]}$$

$$= \lim_{x\to 0+} \frac{1}{\sqrt{1+bx}+1}$$

$$= \frac{1}{2}$$

This is the independent of b. so that b can have any non-zero real value

Since f is continuous at $x=0$

we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$\Rightarrow a+2 = \frac{1}{2} = c$$

$$\Rightarrow a+2 = \frac{1}{2} \ \& \ c = \frac{1}{2}$$

$$\Rightarrow a = -\frac{3}{2} \ \& \ c = \frac{1}{2}$$

$$\therefore a = -\frac{3}{2},\ b \neq 0 \ \& \ c = \frac{1}{2}$$

(any real number)

4(C) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{z+3}{z^4+az^3}$; $|a|>1$ (ii) $\int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz$.

Sol: (i) Given that

$$\int_{|z|=1} \frac{z+3}{z^4+az^3} = \int_{|z|=1} \frac{z+3}{z^3(z+a)}\,dz$$

Comparing the given integral with $\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \int \frac{f(z)}{z^{2+1}}dz = \frac{2\pi i\, f''(z_0)}{2!}$

since $f(z) = \frac{z+3}{z+a}$ is analytic in $|z|=1$.

Here $z_0=0$. and $z_0=0$ is a point inside $|z|=1$.

$\therefore$ we can apply the Cauchy's integral formula

$$\int_{|z|=1} \frac{f(z)}{(z-z_0)^3} = \frac{2\pi i}{2!} f''(z_0) \quad \text{---}(1)$$

$f(z) = \frac{z+3}{(z+a)} \Rightarrow f'(z) = \frac{z+a-z-3}{(z+a)^2} = \frac{a-3}{(z+a)^2}$

$\Rightarrow f'(0) = \frac{a-3}{a^2}$.

and $f''(z) = (a-3)\left[\frac{-2}{(z+a)^3}\right] = \frac{-2(a-3)}{(z+a)^3}$

$\Rightarrow f''(z_0) = f''(0) = \frac{-2(a-3)}{(z+a)^3}$

$\therefore$ from (1) $\int_{|z|=1} \frac{f(z)}{(z+a)^3}\,dz = \frac{2\pi i}{2!}\,\frac{(-2)(a-3)}{a^3}$

$$= -\frac{2\pi i(a-3)}{a^3} = \frac{2\pi i(3-a)}{a^3}$$

(ii) Given that $\int\limits_{|z|=4} \frac{z^4}{(z-i)^3} dz$.

Here $f(z) = z^4$ is analytic in $|z| = 4$.

Comparing the given integral with $\int_C \frac{f(z)}{(z-z_0)^n} dz$

since $f(z) = z^4$ is analytic in $|z|=4$.

$\therefore$ we can apply the Cauchy's integral formula

$$\int\limits_{|z|=4} \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i f''(z_0)}{2!}$$

$$f(z) = z^4 \Rightarrow f'(z) = 4z^3.$$

$$f''(z) = 12z^2$$

$$\therefore \int \frac{z^4}{(z-i)^3} dz = \frac{2\pi i}{2!} f''(z_0)$$

$$= \frac{2\pi i}{2!} (12)(i)^2 \quad (\because z_0 = i)$$

$$= -12\pi i.$$

4(d) Solve the following transportation problem:

Destinations

FACTORIES	D_1	D_2	D_3	D_4	D_5	D_6	Availability
F_1	2	1	3	3	2	5	50
F_2	3	2	2	4	3	4	40
F_3	3	5	4	2	4	1	60
F_4	4	2	2	1	2	2	30
Demand	30	50	20	40	30	10	

Finding the initial solution by Matrix Minima Method.

Solution:-

From the given table

Total Demand $= 30+50+20+40+30+10$
$= 180$.

Total Availability $= 50+40+60+30 = 180$

Hence; $\boxed{\text{Demand} = \text{Availability}}$

$\therefore$ Given Transportation problem is balanced.

Hence, for initial solution - by Matrix Minima Mtd.

	D_1	D_2	D_3	D_4	D_5	D_6	Availability
F_1	× 2	50 1	× 3	× 3	× 2	× 5	50 ✓
F_2	20 3	0 2	20 2	× 4	× 3	× 4	40 ✓
F_3	10 3	× 5	× 4	10 2	30 4	10 1	60
F_4	× 4	× 2	× 2	30 1	× 2	× 2	30
Demand	30	50 ✓	20 ✓	40 ✓	30	10	

Here ; number of positive allocations = 8

and $m+n-1 = 6+4-1 = 9$

$\therefore$ Solution degenerates.

Let , cell $a_{22} = 0$ be a zero basic cell in the initial Basic feasible solution.

$\therefore$ Initial feasible solution $= 2\times0 + 1\times50 + 3\times20 + 2\times20 + 3\times10 + 2\times10 + 4\times30 + 1\times10 + 1\times30$

$IBFS = 0+50+60+40+30+20+120+10+30$

$\boxed{IBFS = 360}$ using Matrix minima method.

Now, for OBFS

By using MODI method:

For Basic cells ; $\Delta_{ij} = u_i + v_j - c_{ij} = 0$

$u_2 + v_2 - 2 = 0$

$\Rightarrow u_2 + v_2 = 2$

$\Rightarrow u_2 + v_1 = 3$

$u_3 + v_1 = 3$

$u_1 + v_2 = 1$

$u_2 + v_3 = 2$

$u_3 + v_4 = 2$

$u_4 + v_4 = 1$

$u_3 + v_5 = 4$

$u_3 + v_6 = 1$

Let $u_3 = 0$, $u_1 = -1$, $u_2 = 0$, $u_4 = -1$

$v_1 = 3$, $v_2 = 2$, $v_3 = 2$, $v_4 = 2$, $v_5 = 4$, $v_6 = 1$

Let us, calculate Δ_{ij} for Non-Basic Cells :-

$\Delta_{41} = u_4 + v_1 - 4 = -1+3-4 = -2$

$\Delta_{11} = u_1 + v_1 - 2 = -1+3-2 = 0$

$\Delta_{32} = u_3 + v_2 - 5 = 0+2-5 = -3$

$\Delta_{42} = u_4 + v_2 - 2 = -1+2-2 = -1.$

$\Delta_{13} = u_1 + v_3 - 3 = -1+2-3 = -2$

$\Delta_{33} = u_3 + v_3 - 4 = 0+2-4 = -2$

$\Delta_{43} = u_4 + v_3 - 2 = -1+2-2 = -1$

$\Delta_{15} = u_1 + v_5 - 2 = -1+4-2 = 1 > 0$

$\Delta_{25} = u_2 + v_5 - 3 = 0+4-3 = 1 > 0$

$\Delta_{45} = u_4 + v_5 - 2 = -1+4-2 = 1 > 0$

$\Delta_{16} = u_1 + v_6 - 5 = -1+1-5 = -5$

$\Delta_{26} = u_2 + v_6 - 4 = 0+1+(-4) = -3$

$\Delta_{46} = u_4 + v_6 - 2 = -1+1-2 = -2$

As we observe, there are some positive values in non-basic cells; hence,

	D_1	D_2	D_3	D_4	D_5	D_6
F_1		50				
F_2	20	0	20			
F_3	10			10 $+\theta$	30 $-\theta$	10
F_4				30 $-\theta$	$+\theta$	

$+\theta$ 10	30 $-\theta$
$-\theta$ 30	$+\theta$

Here $\theta = 30$

$\Rightarrow$

40	0
0	30

$\therefore$ New feasible solution table is given by –

	D_1	D_2	D_3	D_4	D_5	D_6
F_1		50_1				
F_2	20_3	0_2	20_2			
F_3	10_3			40_2	4	10_1
F_4				0_1	30_2	

$\because$ No. of allocations $= 9 = m+n-1$

$\therefore$ Using MODI Method

For Basic cells

$u_1+v_2 = 1$ $\quad$ $u_3+v_4 = 2$

$u_2+v_1 = 3$ $\quad$ $u_4+v_4 = 1$

$u_3+v_1 = 3$ $\quad$ $u_3+v_6 = 1$

$u_2+v_2 = 2$ $\quad$ $u_4+v_5 = 2$

$u_2+v_3 = 2$

Put

$u_3 = 0$ $\quad$ $v_1 = 3$

$u_4 = -1$ $\quad$ $v_4 = 2$

$u_1 = -1$ $\quad$ $v_5 = 3$

$u_2 = 0$ $\quad$ $v_6 = 1$

$v_2 = 2$

$v_3 = 2$

for non basic cells :- (Δ_{ij})

$\Delta_{11} = u_1+v_1 - 2 = -1+3-2 = 0$

$\Delta_{13} = u_1+v_3 - 3 = -1+2-3 = -2$

$\Delta_{14} = u_1+v_4 - 3 = -1+2-3 = -2$

$\Delta_{15} = u_1+v_5 - 2 = -1+3-2 = 0$

$\Delta_{16} = u_1+v_6 - 5 = -1+1-5 = -5$

$\Delta_{24} = u_2+v_4 - 4 = 0+2-4 = -2$

$\Delta_{25} = u_2+v_5 - 3 = 0+3-3 = 0$

$\Delta_{26} = u_2+v_6 - 4 = 0+1-4 = -3$

$\Delta_{32} = u_3+v_2 - 5 = 0+2-5 = -3$

$\Delta_{33} = u_3+v_3 - 4 = 0+2-4 = -2$

$\Delta_{41} = u_4+v_1 - 4 = -1+3-4 = -2$

$\Delta_{42} = u_4+v_2 - 2 = -1+2-2 = -1$

$\Delta_{43} = u_4+v_3 - 2 = -1+2-2 = -1$

$\Delta_{35} = u_3+v_5 - 4 = 0+3-4 = -1$

$\Delta_{46} = u_4+v_6 - 2 = -1+1-2 = -2$

By observing all the values of Δ_{ij} we get all

$\boxed{\Delta_{ij} \leq 0}$ Hence optimality obtained

$\therefore$ optimal Transportation cost $=$

$1\times50 + 3\times20 + 0\times2 + 2\times20 + 3\times10 + 2\times40 + 1\times10 + 0\times1 + 2\times30$

$= 50+60+0+40+30+80+10+0+60$

$= 110+150+70 = \underline{\underline{330}}$

$\boxed{\therefore \text{ optimal cost for given Transportation} = \text{Rs: } 330}$

Alternatively:

	D_1	D_2	D_3	D_4	D_5	D_6
F_1	[2] 0	[1] 50	[3]	[3]	[2]	[5]
F_2	[3]	[2]	[2] 20	[4]	[3] 20	[4]
F_3	[3] 30	[5]	[4]	[2] 10	[4] 10	[1] 10
F_4	[4]	[2]	[2]	[1] 30	[2]	[2]

$$\text{IBFS} = 2\times 0 + 1\times 50 + 2\times 20 + 3\times 20 + 3\times 30 + 2\times 10 + 4\times 10 + 1\times 10 + 1\times 30$$

$$\text{IBFS} = 0+50+40+60+90+20+40+10+30$$

$$= 340 \text{ units.}$$

It is basic solution. Now check for optimality using U-V mtd / MODI Mtd. is the table, taking $u_3 = 0$, $\Delta_{ij} = 0$ for basic cells

	D_1	D_2	D_3	D_4	D_5	D_6	
F_1	[2] 0, (0)	[1] 0, (50)	[3] −1	[3] −2	[2] $\boxed{1}$	[5] −5	$u_1=-1$
F_2	[3] −1	[2] −1	[2] 0, (20)	[4] −3	[3] 0, (20)	[4] −4	$u_2=-1$
F_3	[3] 0, (30)	[5] −3	[4] −1	[2] 0, (10)	[4] 0, (10)	[1] 0, (10)	$u_3=0$
F_4	[4] −2	[2] −1	[2] 0	[1] 0, (30)	[2] $\boxed{1}$	[2] −2	$u_4=-1$
	$v_1=3$	$v_2=2$	$v_3=3$	$v_4=2$	$v_5=4$	$v_6=1$	

clearly, we can see that $\Delta_{ij} \nleq 0$ for all non-basic cells. i.e $\Delta_{15} = \Delta_{45} = 1 > 0$.
so, we choose Δ_{45} for reallocation using close ring.

$\theta = 10$

$+\theta$ 10	10 $-\theta$
$-\theta$ 30	$+\theta$

$\Rightarrow$

20	0
20	10

Hence, the transformed table will be given below, also use the u-v / MODI method to check the optimality of the allocation; taking $\Delta_{ij} = 0$ for all basic cells and $u_3 = 0$, we get the following table.

	D_1	D_2	D_3	D_4	D_5	D_6	
F_1	2, 0, **0**	1, 0, **50**	3, -2	3, -2	2, 0	5, -5	$u_1 = -1$
F_2	3, 0	2, 0	2, 0, **20**	4, -2	3, 0, **20**	4, -3	$u_2 = 0$
F_3	3, 0, **30**	5, -3	4, -2	2, 0, **20**	4, -1	1, 0, **10**	$u_3 = 0$
F_4	4, -2	2, -1	2, -1	1, 0, **20**	2, 0, **10**	2, 2	$u_4 = -1$
	$v_1 = 3$	$v_2 = 2$	$v_3 = 2$	$v_4 = 2$	$v_5 = 3$	$v_6 = 1$.	

clearly, we can se optimality acheived as $\Delta_{ij} \le 0$ for all non basic cells.

$\boxed{\text{OBFS} = 330 \text{ units}}$

Hence the result.

5.(a) Find the family orthogonal to $\phi[z(x+y)^2, x^2-y^2]=0$.

Solⁿ Given $\phi[z(x+y)^2, x^2-y^2]=0.$ ——— (1)

Let $u = z(x+y)^2$ and $v = x^2-y^2$ ——— (2)

Then (1) becomes $\phi(u,v)=0$ ——— (3)

Diff. (3) w.r.t. x and y partially by turn, we get,

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{——— (4)}$$

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad \text{——— (5).}$$

From (2), $\left(\frac{\partial u}{\partial x}\right)=2z(x+y)$, $\left(\frac{\partial u}{\partial y}\right)=2z(x+y)$

and $\frac{\partial u}{\partial z}=(x+y)^2$,

$\left(\frac{\partial v}{\partial x}\right)=2x$, $\left(\frac{\partial v}{\partial y}\right)=-2y$, $\left(\frac{\partial v}{\partial z}\right)=0.$

Putting these values in (4) and (5), we get

$$\left(\frac{\partial\phi}{\partial u}\right)\left[2z(x+y)+p(x+y)^2\right]+\left(\frac{\partial\phi}{\partial v}\right)(2x+0)=0 \quad \text{——— (6)}$$

and $$\left(\frac{\partial\phi}{\partial u}\right)\left[2z(x+y)+q(x+y)^2\right]+\left(\frac{\partial\phi}{\partial v}\right)(-2y+0)=0 \quad \text{——— (7)}$$

Evaluating the values of $-\frac{\partial\phi/\partial u}{\partial\phi/\partial v}$ from (6) and (7) and then equating these, we get

$$\frac{-\partial\phi/\partial u}{\partial\phi/\partial v} = \frac{2x}{2z(x+y)+p(x+y)^2} = \frac{-2y}{2z(x+y)+q(x+y)^2}$$

Or $\quad x(x+y)[2z+q(x+y)] = -y(x+y)[2z+p(x+y)]$

Or $\quad 2xz + qx(x+y) + 2yz + py(x+y) = 0$

Or $\quad py(x+y) + qx(x+y) = -2z(x+y)$

Or $\quad py + qx = -2z \qquad \text{——— (8)}$

which is diff. eqn. of the family of surfaces given by (1). So the diff. eqn. of the family of surfaces orthogonal to (8) is given by

$$y\,dx + x\,dy - 2z\,dz = 0 \quad \text{or} \quad d(xy) - 2z\,dz = 0 \qquad \text{—— (9)}$$

Integrating (9), $\quad xy - z^2 = C,$

where C being a parameter.

which is the desired family of orthogonal surfaces.

5.(b) Find complete integral of $xp - yq = xqf(z-px-qy)$.

Sol^n Given $xp - yq = xqf(z-px-qy)$. ——— (1)

Let $F(x,y,z,p,q) = xp - yq - xqf(z-px-qy) = 0$ ——(2)

Now Charpit's auxiliary equations are.

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{\partial x}{-\frac{\partial F}{\partial p}} = \frac{\partial y}{-\frac{\partial F}{\partial q}}$$

Or

$$\frac{dp}{p - qf + xqpf' - pqxf'} = \frac{dq}{-q + xq^2f' - xq^2f'} = \cdots\cdots$$

Each ratio of (3) ——— (3)

$$= \frac{x\,dp + y\,dq}{xp - yq - qxf} = \frac{x\,dp + y\,dq}{0}, \text{ by } (2)$$

$\Rightarrow x\,dp + y\,dq = 0 \Rightarrow x\,dp + y\,dq + p\,dx + q\,dy$
$= p\,dx + q\,dy.$

$\Rightarrow dz - d(xp) - d(yq) = 0 \qquad (\because dz = p\,dx + q\,dy)$

Integrating,
$$z - xp - yq = \text{Constant} = a, \text{ say} \quad \text{—(4)}$$
$$xp + yq = z - a \quad \text{———— (5)}$$

Using (4), (1) $\Rightarrow xp - yq = xqf(a)$ ———— (6).

Subtracting (6) from (5) $2yq = z - a - xqf(a)$

Or $$q = \frac{z-a}{2y + xf(a)} \quad \text{———— (7)}$$

Using (7), (5) $\Rightarrow$ $p = \dfrac{(z-a)\{y+xf(a)\}}{x\{2y+xf(a)\}}$ ——(8)

Using (7) and (8), $dz = p\,dx + q\,dy$ reduces to

$$dz = (z-a)\left[\frac{\{y+xf(a)\}dx}{x\{2y+xf(a)\}} + \frac{dy}{2y+xf(a)}\right]$$

or $$\frac{2dz}{z-a} = \frac{2y\,dx + 2xf(a)dx + 2x\,dy}{x\{2y+xf(a)\}}$$

$$\frac{2dz}{z-a} = \frac{2d(xy) + 2xf(a)dx}{2xy + x^2 f(a)}$$

Integrating,

$$2\log(z-a) = \log\{2xy + x^2 f(a)\} + \log b$$

$$(z-a)^2 = b\{2xy + x^2 f(a)\}.$$

5(c) The following are the number of deaths in four successive ten year age groups. By using Newton's forward formula find the number of deaths at 45–50 and 50–55.

Age group	25–35	35–45	45–55	55–65
Deaths	13229	18139	24225	31496

Sol:- first we prepare the cumulative frequency table, as follows.

Age group under	35	45	55	65
Deaths	13229	31368	55593	87089

The difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
35	13229			
		18139		
45	31368		6086	
		24225		1185
55	55593		7271	
		31496		
65	87089			

we have to obtain $f(50)$ i.e. the no. of deaths above the age 25 and below 50.

Taking $x_0 = 35$, $x = 50$, $h = 10$

Here $p = \frac{50-35}{10} = \frac{15}{10} = 1.5$

$p = \frac{x - x_0}{h}$

By Newton's forward formula.

$$y_{50} = y_{35} + p\Delta y_{35} + \frac{p(p-1)}{2!}\Delta^2 y_{35} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{35}$$

$\Rightarrow f(50) = 13229 + (1.5)(18139) + \frac{(1.5)(0.5)}{2!}(6086) + \frac{(1.5)(0.5)(-0.5)}{3!}(1185)$

$= 13229 + 27208.5 + 2282.25 - 74.0625$

$\simeq 42646$

Therefore the required no. of deaths between 45 and 50 $= 42646 - 31368 = 11278.$

Hence the no. of deaths between 50 and 55

$= 24225 - 11278$

$= 12947.$

5(d) Draw a switching circuit that realizes the following switching function. If possible, draw a simpler switching circuit.

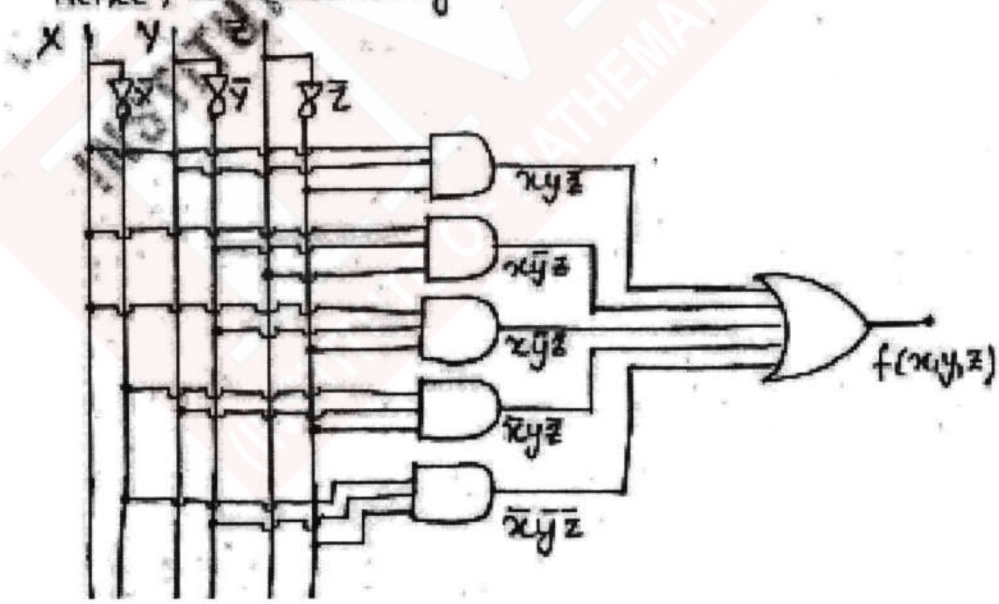
x	y	z	$f(x,y,z)$
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

Solution:-

From the above truth table

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

Hence, the switching circuit will be.



$f(x,y,z)$ can be simplied into simpler form so as the switching circuit.

Using Boolean Expression.

$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$

$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}\bar{z}(y+\bar{y})$

$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}\bar{z}$ $[\because y+\bar{y}=1]$

$f(x,y,z) = x\bar{z}[y+\bar{y}] + \bar{x}\bar{z} + x\bar{y}z$

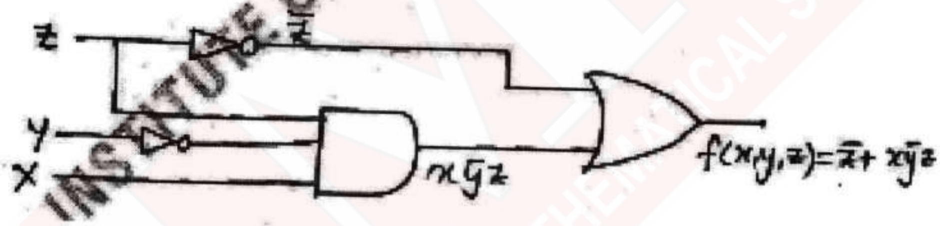
$f(x,y,z) = x\bar{z} + \bar{x}\bar{z} + x\bar{y}z$ $[\because y+\bar{y}=1]$

$f(x,y,z) = \bar{z}(x+\bar{x}) + x\bar{y}z$

$\boxed{f(x,y,z) = \bar{z} + x\bar{y}z}$ $[\because x+\bar{x}=1]$

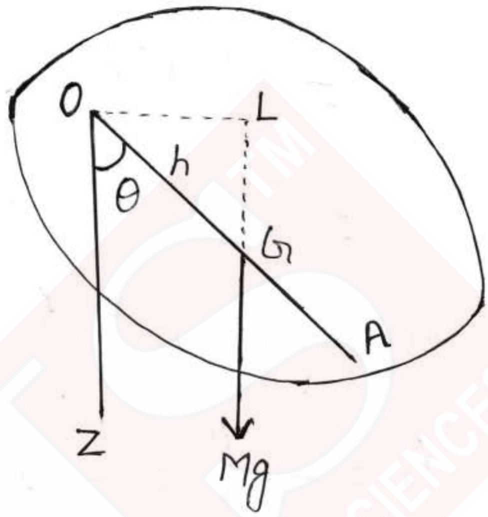
which is simplified $f(x,y,z)$.

Hence, simpler switching circuit is shown—



5(e) Write the Hamiltonian function and equation of motion a compound pendulum.

Sol: At time t, let θ be the angle between the vertical plane through the fixed axis (plane fixed in space) and the plane through the C.G. 'G' and the fixed axis (plane fixed in the body). Let $OG = h$.



If T and V are the kinetic and potential energies of the pendulum then

$$T = \frac{1}{2} MK^2 \dot{\theta}^2$$

and $V = -Mgh\cos\theta$.

(Negative sign is taken because G is below the fixed axis.)

$$\therefore L = T - V$$

$$= \frac{1}{2} MK^2 \dot{\theta}^2 + Mgh\cos\theta.$$

Here θ is the only generalised co-ordinate

$$\therefore P_\theta = \frac{\partial L}{\partial \theta} = \frac{1}{2} MK^2 \dot{\theta} \quad \text{——— (1)}$$

Since L does not t explicitly,

$$\therefore \; H = T + V$$

$$= \frac{1}{2} MK^2 \dot{\theta}^2 - Mgh\cos\theta$$

$$= \frac{1}{2MK^2} p_\theta^2 - Mgh\cos\theta, \quad \text{from } (1)$$

Hence the two Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -Mgh\sin\theta \quad \text{——— } (H_1)$$

and $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{MK^2} p_\theta \quad \text{——— } (H_2)$$

Differentiating (H_2) and substituting from (H_1), we get

$$\ddot{\theta} = \frac{1}{MK^2} \dot{p}_\theta$$

$$= \frac{1}{MK^2}(-Mgh\sin\theta)$$

or $$\ddot{\theta} = -\frac{gh}{K^2}\sin\theta$$

which is the equation of motion of a compound pendulum.

6(a) (i) Form a partial differential equation by eliminating the function ϕ from $lx+my+nz = \phi(x^2+y^2+z^2)$.

Sol^n Given

$$lx+my+nz = \phi(x^2+y^2+z^2). \quad \text{——} (1)$$

Differentiating (1) partially with respect to x and y, we get

$$l+n\left(\frac{\partial z}{\partial x}\right) = \phi'(x^2+y^2+z^2).\left\{2x+2z\left(\frac{\partial z}{\partial x}\right)\right\} \text{——}(2)$$

and $$m+n\left(\frac{\partial z}{\partial y}\right) = \phi'(x^2+y^2+z^2)\cdot\left\{2y+2z\left(\frac{\partial z}{\partial y}\right)\right\} \text{——}(3)$$

Dividing (2) by (3), $\phi'(x^2+y^2+z^2)$ is eliminated and we get

$$\frac{l+n\left(\frac{\partial z}{\partial x}\right)}{m+n\left(\frac{\partial z}{\partial y}\right)} = \frac{2\left\{x+z\left(\frac{\partial z}{\partial x}\right)\right\}}{2\left\{y+z\left(\frac{\partial z}{\partial y}\right)\right\}}$$

Or $$(ny-mz)\left(\frac{\partial z}{\partial x}\right)+(lz-nx)\left(\frac{\partial z}{\partial y}\right) = mx-ly,$$

which is the required partial differential equation.

6(a)(ii) Solve $(x^3+3xy^2)p + (y^3+3x^2y)q = 2z(x^2+y^2)$.

Sol^n Here the Lagrange's subsidiary eq^n are

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} \quad \text{---(1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{x^3+3xy^2+3x^2y+y^3} = \frac{d(x+y)}{(x+y)^3} \quad \text{---(2)}$$

Choosing 1, -1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{x^3+3xy^2-y^3-3x^2y} = \frac{d(x-y)}{(x-y)^3} \quad \text{---(3)}$$

From (2) and (3), $(x+y)^{-3}d(x+y) = (x+y)^{-3}d(x-y)$

or $u^{-3}du - v^{-3}dv = 0$ or putting $u = x+y$ and $v = x-y$.

Integrating $u^{-2}/(-2) - v^{-2}/(-2) = C_1/2$

or $v^{-2} - u^{-2} = C_1$

or $(x-y)^{-2} - (x+y)^{-2} = C_1$ as $u = x+y$, $v = x-y$ ---(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx + (1/y)\,dy}{(1/x)(x^3+3xy^2)+(1/y)(y^3+3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2+y^2)} \quad \text{---(5)}$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)\,dx + (1/y)\,dy}{4(x^2+y^2)}$$

Or $\frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$

Integrating,

$$\log x + \log y - 2\log z$$

$$= \log c_2 \quad \text{or} \quad (xy)/z^2 = c_2 \qquad \text{—(6)}$$

From (4) and (6), the required general solution is given by

$$\phi[(x-y)^{-2} - (x+y)^{-2}, (xy)/z^2] = 0$$

where, ϕ being an arbitrary function.

6(b) Solve $[D^2 - DD' + D' - 1]z = \cos(x+2y) + e^y$.

Sol^n The given equation can be re-written as

$$(D-1)(D-D'+1)z = \cos(x+2y) + e^y \quad \text{---(1)}$$

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$. ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x+2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y)$$

$$= \frac{1}{-1^2 - (-1.2) + D' - 1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\sin(x+2y).$$

P.I. corresponding to e^y

$$\frac{1}{(D-1)(D-D'+1)}e^y \Rightarrow \frac{1}{D-D'+1}\cdot\frac{1}{D-1}e^{0\cdot x+1\cdot y}$$

$$= \frac{1}{D-D'+1}\cdot\frac{1}{0-1}e^{0\cdot x+1\cdot y}$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D\left(1-\frac{D'}{D}\right)}\cdot 1$$

$$= -e^y\frac{1}{D}\left(1-\frac{D'}{D}+ \cdots\right)^{-1}\cdot 1 \Rightarrow -e^y\frac{1}{D}(1+\cdots)\cdot 1$$

$$= -e^y x.$$

$\therefore$ the required general sol^n is z = C.F. + total P.I.

i.e. $z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) + (1/2)\sin(x+2y) - xe^y$.

6(c). Reduce the equation $yr+(x+y)s+xt=0$ to canonical form and hence find its general solution.

Soln: Given $yr+(x+y)s+xt=0 \rightarrow (1)$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=y, S=x+y$ and $T=x$ So that

$$S^2-4RT=(x+y)^2-4xy=(x-y)^2>0 \text{ for } x\neq y.$$

ans so (1) is hyperbolic. Its λ-quadratic equation

$R\lambda^2+S\lambda+T=0$ reduces to $y\lambda^2+(x+y)\lambda+x=0$

(or) $(y\lambda+x)(\lambda+1)=0$

So that $\lambda=-1, -x/y$. Then the corresponding characteristic equations are given by

$$\frac{dy}{dx}-1=0 \text{ and } \frac{dy}{dx}-(x/y)=0$$

Integrating these $y-x=c_1$ and $\frac{y^2}{2}-x^2/2=c_2$

In order to reduce one (1) to its canonical form, we choose

$$u=y-x \text{ and } v=y^2/2-x^2/2 \longrightarrow (2)$$

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right), \text{ using (2)} \rightarrow (3)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}), \text{ using (2)} \rightarrow (4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right) \text{ using } ③$$

$$=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

$$\Rightarrow -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right)-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]-\frac{\partial z}{\partial v}$$

$$=-\left(-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}\right)-x\left(-\frac{\partial^2 z}{\partial u\partial v}-x\frac{\partial^2 z}{\partial v^2}\right)-\frac{\partial z}{\partial v} \text{ using } ②$$

$$r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}$$

Now, $t=\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial v}\right) \text{ by } ④$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}.$$

$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}$$

Also $s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$ using (4)

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial}\left(y\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$\Rightarrow\ -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial u\partial u} - y\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2}.\ \text{using (2)}$$

$$\therefore\ s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2} \to (7)$$

using (5), (6) and (7) in (1) we get

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u\partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right) + (x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u\partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\} + x\left\{\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right\} = 0$$

(or) $\left\{4xy - (x+y)^2\right\}\frac{\partial^2 z}{\partial u\partial v} - y\frac{\partial z}{\partial v} + x\frac{\partial z}{\partial v} = 0$

(or) $(y-x)^2\frac{\partial^2 z}{\partial u\partial v} + (y-x)\frac{\partial z}{\partial v} = 0$

(or) $u^2 \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial v} = 0$ (or) $u \frac{\partial^2 z}{\partial v \partial v} + \frac{\partial z}{\partial v} = 0 \rightarrow (8)$

[$\because$ $u \neq 0$ and $y - x = u$, by (2)]

(8) is the required Canonical form of (1).

Solution of (8) multiplying both sides of (8) by v we get

$uv\left(\frac{\partial^2 z}{\partial u \partial v}\right) + v\left(\partial z/\partial v\right) = 0$ (or) $(uvDD' + vD')z = 0$ $\rightarrow (9)$

where $D \equiv \frac{\partial}{\partial u}$ and $D' \equiv \partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables x and y as follows

let $u = e^x$ and $v = e^y$ so that $x = \log u$, $y = \log v$ $\rightarrow (10)$

let $D_1 \equiv \partial/\partial x$ and $D_1' \equiv \partial/\partial y$ then (9) reduces to

$(D_1 D_1' + D_1')z = 0$ (or) $D_1'(D+1)z = 0$

Its general solution is

$z = e^{-x}\phi_1(y) + \phi_2(x) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$

(or) $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$,

where ψ_1 and ψ_2 are arbitrary functions.

6(d) A tightly stretched elastic string of length l, with fixed end points $x=0$ and $x=l$ is initially in the position given by $y = c\sin^3(\pi x/l)$, c being constant. It is released from the position of rest. Find the displacement $y(x,t)$.

Sol: The partial differential equation of the transverse vibrations of the given elastic string is given by

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{a^2}\right)\left(\frac{\partial^2 y}{\partial t^2}\right) \rightarrow (1)$$

where $y(x,t)$ is the deflection of the string and a is a constant. Given boundary and initial conditions are:

Boundary conditions (B.C): $y(0,t) = y(l,t) = 0$ for all $t \rightarrow (2)$

Initial conditions (I.C): $y(x,0) = c\sin^3(\pi x/l)$ (Initial deflection) $\rightarrow 3(a)$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = y_t(x,0) = 0 \text{ (Initial Velocity)} \rightarrow 3(b)$$

Let solution of (1) be of the form $y(x,t) = X(x)T(t) \rightarrow (4)$

Substituting this value of y in (1) we have

$$X''T = \frac{1}{a^2} \times T'' \quad (\text{or}) \quad \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} \rightarrow (5)$$

Since x and t are independent (5) can only be true if each side is equal to the same constant, say μ

Then (5) gives $\rightarrow (6')$

$$X'' - \mu X = 0 \longrightarrow (6)$$

and

$$T'' - \mu a^2 T = 0 \longrightarrow (7)$$

using (2) and (4) gives $X(0)T(t) = 0 \longrightarrow (8')$

Since $T(t)=0$ leads to $y=0$, hence we assume that $T(t)\neq 0$

$x(0)=0$ and $x(l)=0 \rightarrow (9)$

Then (8) gives

we now solve (6) under B.C (9). Three cases arise:

case i): let $\mu=0$ then solution of (6) is given by

$$x(x)=Ax+B \longrightarrow (10)$$

using B.C (9), (10) gives $0=B$ and $0=Al+B$. These give $A=B=0$ so that $x(x)=0$ with the help of (4) this leads to $y\equiv 0$ which does not satisfy 3(a) and 3(b). So we reject $\mu=0$

case(ii) let $\mu=\lambda^2$, where $\lambda\neq 0$ then solution of (6) is

$$x(x)=Ae^{\lambda x}+Be^{-\lambda x} \longrightarrow (11)$$

using (B.C) (9) (11) given $A+B=0$ and $Ae^{\lambda l}+Be^{-\lambda l}=0$ $\rightarrow (12)$

Solving (12), $A=B=0$ and so $x(x)=0$ and

$A\cos\lambda l+B\sin\lambda l=0$ so that $A=0$ and $\sin\lambda l=0$ where we have taken $B\neq 0$, since otherwise $x\equiv 0$ so that $y=0$ which does not satisfy 3(a) and 3(b).

Now $\sin\lambda l=0 \Rightarrow \lambda l=n\pi \Rightarrow \lambda=n\pi/l$, $n=1,2,3\ldots \rightarrow (14)$

Hence non-zero solutions $x_n(x)$ of (6) are given by

$$x_n(x)=B_n\sin\left(\frac{n\pi x}{l}\right) \longrightarrow (15)$$

using (14), (7) reduces to

$$\left(d^2T/dt^2\right)+\left(n^2\pi^2a^2/l^2\right)T=0 \quad \left[\because \mu=-\lambda^2=-\left(n^2\pi^2/l^2\right)\right]$$

whose general solution is $T_n(t)=C_n\cos\frac{n\pi at}{l}+D_n\sin\frac{n\pi at}{l} \rightarrow (16)$

$\therefore y_n(x,t)=X_n(t)\,T_n(t)=\left(E_n\cos\frac{n\pi at}{l}+F_n\sin\frac{n\pi at}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$

are solutions of (1) satisfying (2) for $n=1,2,3,\ldots$

Here $E_n (= C_n B_n)$ and $F_n (= B_n D_n)$ are newly arbitrary constants. In order to obtain a solution also $F_n (= B_n D_n)$ are new arbitrary constants. In order to obtain a solution also satisfying 8(a) and 8(b) we consider most general solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi at}{l} + F_n \sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (17)$$

Differentiating (17) partially w.r.t to 't' we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{n\pi aE_n}{l}\sin\frac{n\pi at}{l} + \frac{n\pi aF_n}{l}\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (18)$$

putting $t=0$ in (17) and (18) and using (8a) and (8b) we get

$$c\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \text{ and } 0 = \sum_{n=1}^{\infty}\left(\frac{n\pi aF_n}{l}\right)\sin\frac{n\pi x}{l}.$$

which are fourier sine series. Accordingly we have.

$$E_n = \frac{2}{l}\int_0^l c\sin^3\left(\frac{\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx \rightarrow (19)$$

and $\frac{n\pi aF_n}{l} = \frac{2}{l}\int_0^l 0\cdot\sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow F_n = 0 \rightarrow (20)$

Now $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = (3\sin\theta - \sin 3\theta)/4.$

$$\therefore \sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] \rightarrow (21)$$

$$\therefore (19) \Rightarrow E_n = \frac{2c}{l}\int_0^l \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\sin\frac{n\pi x}{l}\,dx$$

(or) $$E_n = \frac{3c}{2l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx - \frac{c}{2l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\,dx \rightarrow (22)$$

we now show that

$$I = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} l/2, & \text{if } m=n \\ 0, & \text{if } m\neq n \end{cases} \rightarrow (23)$$

If $m=n$ then, we have

$$I=\int_0^l \sin^2\frac{n\pi x}{l}\,dx=\int_0^l \frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)dx$$

$$=\frac{1}{2}\left[x-\frac{1}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l$$

$\therefore$ $I=l/2$ when $m=n$

if $m\neq n$ then we have

$$I=\frac{1}{2}\int_0^l 2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$=\int_0^l\left(\cos\frac{(n-m)\pi x}{l}-\cos\frac{(n+m)\pi x}{l}\right)dx$$

$$=\frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}-\frac{1}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]_0^l=0$$

using (23) from (22) we have.

$$E_1=\left(\frac{3c}{2l}\right)\left(l/2\right)=\frac{3c}{4},\quad E_3=-\left(\frac{c}{2l}\right)\left(\frac{l}{2}\right)=\frac{-c}{4}$$

Also $E_n=0$ for $n\neq 1$ and $n\neq 3$

using (20) and (24), (17) reduces to

$$y(x,t)=E_1\cos\frac{\pi at}{l}\sin\frac{\pi x}{l}+E_3\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}.$$

(or)

$$y(x,t)=\frac{c}{4}\left[3\cos\frac{\pi at}{l}\sin\frac{\pi x}{l}-\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}\right]\longrightarrow (25)$$

7(a) The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5.

Sol: Given that $C = 4e^{-2t} + e^{-0.1t}$.

Now the bacteria concentration to be 0.5 $C = 0.5$

i.e., $0.5 = 4e^{-2t} + e^{-0.1t}$

i.e. $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$

Now we have to find the time required for the bacteria concentration to be 0.5

$\therefore$ we want to find the roots of the equation $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$.

By Newton Raphson method,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$f(t) = 4e^{-2t} + e^{-0.1t} - 0.5$$

$$f'(t) = -8e^{-2t} - 0.1e^{-0.1t}$$

Let the initial time be $t_0 = 0$.

Then $t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$

$= 0 - \frac{4.5}{-8.1}$

$= 0.5555$

$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = 0.5555 - \frac{1.7628}{-2.72843}$

$= 1.20168$

Continue in this way we get $t = 6.932$.

7(b) Solve $20x+y-2z=17$; $3x+20y-z=-18$; $2x-3y+20z=25$ by Gauss Seidal method.

Sol'n: The Given equations $x=\frac{1}{20}(17-y+2z)$

$$\left.\begin{aligned} x &= \tfrac{1}{20}(17-y+2z)\\ y &= \tfrac{1}{20}(-18-3x+z)\\ z &= \tfrac{1}{20}(25-2x+3y) \end{aligned}\right\} \text{—} (1).$$

By Gauss - Seidal method System (1) can be written as

$$x^{k+1} = \tfrac{1}{20}(17-y^k+2z^k)$$

$$y^{k+1} = \tfrac{1}{20}(-18-3x^{k+1}+z^k)$$

$$z^{k+1} = \tfrac{1}{20}(25-2x^{k+1}+3y^{k+1})$$

Now taking $x^{(0)}=0$. (i.e. $x=y=z=0$) we obtain the following iterations.

First Iteration: put $k=0$

$$x^{(1)} = \tfrac{1}{20}(17-y^{(0)}+2z^{(0)}) = 0.8500$$

$$y^{(1)} = \tfrac{1}{20}(-18-3x^{(1)}+z^{(0)}) = -1.0275$$

$$z^{(1)} = \tfrac{1}{20}(25-2x^{(1)}+3y^{(1)}) = 1.0109$$

Second Iteration: Put $k=1$

$$x^{(2)} = \tfrac{1}{20}(17-y^{(1)}+2z^{(1)}) = 1.0025$$

$$y^{(2)} = \tfrac{1}{20}(-18-3x^{(2)}+z^{(1)}) = -0.9998$$

$$z^{(2)} = \tfrac{1}{20}(25-2x^{(2)}+3y^{(2)}) = 0.9998$$

Third Iteration: put $k=2$

$$x^{(3)} = \tfrac{1}{20}(17-y^{(2)}+2z^{(2)}) = 1.0000$$

$$y^{(3)} = \tfrac{1}{20}(-18-3x^{(3)}+z^{(2)}) = -1.0000$$

$$z^{(3)} = \tfrac{1}{20}(25-2x^{(3)}+3y^{(3)}) = 1.0000$$

$\therefore$ The solution is given by $x=1,\ y=-1,\ z=1$.

7(c) A reservoir discharging water through sluices at a depth 'h' below the water surface has a surface area 'A' for various values of 'h' as given below:

h(ft.)	10	11	12	13	14
A(sq.ft)	950	1070	1200	1350	1530

If 't' denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h}/A$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

Solution:-

Given; the rate of fall of the surface is given by

$$\frac{dh}{dt} = -48\sqrt{h}/A$$

$$\Rightarrow dt = -\frac{1}{48}\frac{A}{\sqrt{h}}dh$$

$$\Rightarrow t = -\frac{1}{48}\int_{14}^{10}\frac{A}{\sqrt{h}}dh.$$

$$t = \frac{1}{48}\int_{10}^{14}\frac{A}{\sqrt{h}}dh.$$

Given table

height(h)(ft)	10	11	12	13	14
Area(A)-Sq.ft	950	1070	1200	1350	1530

which improves into given table

h	10	11	12	13	14
$\sqrt{h}$	3.1623	3.3166	3.464	3.6056	3.7417
A	950	1070	1200	1350	1530
$\frac{A}{48\sqrt{h}}$	6.2587 (y_0)	6.7212 (y_1)	7.2169 (y_2)	7.8005 (y_3)	8.519 (y_4)

Using Simpson's $\frac{1}{3}$rd rule

$$t = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$

$$t = \frac{1}{3}\left[(6.2587+8.519)+4(6.7212+7.8005)+2(7.2169)\right]$$

$$t = \frac{1}{3}[87.3195]$$

$$t = 29.1 \approx 29 \text{ minutes.}$$

$$\boxed{\therefore \text{Time required} = 29 \text{ minutes}}$$

7(c) → A reserviour discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below:

h(ft)	10	11	12	13	14
A (sq. ft)	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h/A}$. Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

Sol'n :
$$\frac{dh}{dt} = -48\frac{\sqrt{h}}{A}$$

$$\Rightarrow dt = -\frac{1}{48}\frac{A}{\sqrt{h}}\,dh$$

$$\Rightarrow t = \frac{1}{48}\int_{10}^{14}\frac{A}{\sqrt{h}}\,dh$$

By using Simpson's $\frac{1}{3}$rd rule

$t = 29$ minutes (approximately).

7(d)(ii) Find the decimal equivalent of $(357.32)_8$.

Sol'n: $(357.32)_8 = 3\times 8^2 + 5\times 8^1 + 7\times 8^0 + 3\times 8^{-1} + 2\times 8^{-2}$

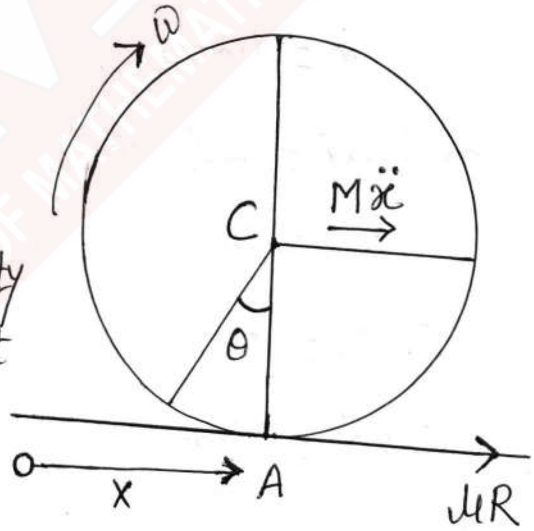
$= 3\times 64 + 40 + 7 + 3/8 + 2/8^2$

$= 192 + 40 + 7 + 0.375 + 0.03125$

$= (239.40625)_{10}$.

8(a) A homogeneous sphere of radius, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $(2a\omega/7\mu g)$, and that then the sphere will roll with angular velocity $(2\omega/7)$.

Sol: As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.



Initial velocity of the point of contact = Initial velocity of the centre C + Initial velocity of the point of contact with respect to the centre C = 0 + $a\omega$ in the direction from right to left, i.e. the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t the equations of motion are

$$M\ddot{x} = \mu R, \text{ (where } R = Mg). \quad \text{---} (1)$$

and $$MK^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu Ra \quad \text{---} (2)$$

from (1), we have $\ddot{x} = \mu g$ and from (2), we have $a\ddot{\theta} = -\frac{5}{2}\mu g$.

Integrating these equations, we have

$$\dot{x} = \mu g t + C_1$$

and $$a\dot{\theta} = -\frac{5}{2}\mu g t + C_2$$

Since initially when $t = 0$, $\dot{x} = 0$, $\dot{\theta} = \omega$.

$\therefore C_1 = 0$ and $C_2 = a\omega$.

$$\therefore \dot{x} = \mu g t, \quad \text{---} (3)$$

and $$a\dot{\theta} = -\frac{5}{2}\mu g t + a\omega. \quad \text{---} (4)$$

Velocity of the point of contact $= \dot{x} - a\dot{\theta}$

$\therefore$ The point of contact will come to rest when $\dot{x} - a\dot{\theta} = 0$, i.e. when

$\mu g t - \left(-\frac{5}{2}\mu g t + a\omega\right) = 0$ or when

$t = (2a\omega / 7\mu g)$.

Therefore after time $(2a\omega/7\mu g)$ the slipping will stop and pure rolling will commence.

Putting this value of t in (4), we get $\dot{\theta} = (2\omega/7)$.

When rolling commences, let F be the frictional force. Therefore the equations of motion are

$$M\ddot{x} = F, \quad \text{——— (5)}$$

$$M \cdot \frac{2}{5} a^2 \ddot{\theta} = -Fa, \quad \text{——— (6)}$$

and $\quad x - a\dot{\theta} = 0 \quad$ ——— (7)

From (7) $\dot{x} - a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.

Now from (5) and (6), we get

$$M\ddot{x} = F = -\frac{2}{5} M a\ddot{\theta}$$

or $a\ddot{\theta} = -\frac{2}{5} a\ddot{\theta} \quad (\because \ddot{x} = a\ddot{\theta})$

or $\frac{7}{5} a\ddot{\theta} = 0$ or $\ddot{\theta} = 0$

Integrating $\dot{\theta} = \text{constant} = \frac{2}{7}\omega$.

8(b) Show that $\phi = x f(r)$ is a possible form for the velocity potential of an incompressible liquid motion. Given that the liquid speed $q \to 0$ as $r \to \infty$, deduce that the surfaces of constant speed are $(r^2+3x^2) r^{-8} =$ constant.

Sol: Given that

$$\phi = x f(r) \quad \text{--- (1)}$$

$$\therefore \quad q = -\nabla\phi = -\nabla[x f(r)]$$

$$= -[f(r)\nabla x + x\nabla f(r)] \quad \text{--- (2)}$$

Now, $r^2 = x^2+y^2+z^2$

$$\Rightarrow 2r(\partial r/\partial x) = 2x$$

$$\Rightarrow \partial r/\partial x = x/r \quad \text{--- (3)}$$

Similarly,

$$\partial r/\partial y = y/r \text{ and } \partial r/\partial z = z/r \quad \text{--- (4)}$$

Also,

$$\nabla x = [i(\partial/\partial x) + j(\partial/\partial y) + k(\partial/\partial z)]x = i$$

and $\nabla f(r) = [i(\partial/\partial x) + j(\partial/\partial y) + k(\partial/\partial z)] f(r)$

$$= i f'(r)(\partial r/\partial x) + j f'(r)(\partial r/\partial y) + k f'(r)(\partial r/\partial z)$$

$$= i f'(r)(x/r) + j f'(r)(y/r) + k f'(r)(z/r),$$
(by ③ and ④)

$$= (1/r) f'(r)(ix + jy + kz)$$

$$= (1/r) f'(r) \mathbf{r}$$

$$\therefore ② \Rightarrow \quad q = -f(r) i - (x/r) f'(r) \mathbf{r}. \quad \text{——— ⑤}$$

for a possible motion of an incompressible fluid, we have

$$\nabla \cdot q = 0 \quad \text{or} \quad \nabla \cdot (-\nabla \phi) = 0 \quad \text{or} \quad \nabla^2 \phi = 0.$$

$$\text{or} \ (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)[x f(r)] = 0, \quad \text{——— ⑥}$$
(using ①)

Now, $$\frac{\partial^2}{\partial x^2}[x f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x f(r)\}\right]$$

$$= \frac{\partial}{\partial x}\left[f(r) + x \frac{\partial f(r)}{\partial x}\right]$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x \frac{\partial^2 f}{\partial x^2}$$

$$= 2 \frac{\partial f}{\partial x} + x \frac{\partial^2 f}{\partial x^2}$$

Also $$\frac{\partial^2}{\partial y^2}[x f(r)] = x \frac{\partial^2 f}{\partial y^2}.$$

and $\frac{\partial^2}{\partial z^2}\left[x f(r)\right] = x\frac{\partial^2 f}{\partial z^2}$

$\therefore$ (6) becomes

$$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 \quad \text{——— (7)}$$

Now, $\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} = f'\frac{x}{r}$, using (3). ——— (8)

and $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right)$

$$= \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$$

$$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\cdot\frac{\partial r}{\partial x}$$

$$= \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \quad \text{——— (9)}$$

Similarly, $\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f'$ ——— (10)

and $\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f'$ ——— (11)

Adding (9), (10) and (11), we get

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

$$=\frac{3f'}{r}+\frac{x^2+y^2+z^2}{r^2}f''-\frac{x^2+y^2+z^2}{r^3}f'$$

$$=\frac{3f'}{r}+f''-\frac{f'}{r}, \text{ as } x^2+y^2+z^2=r^2.$$

$$\therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=\frac{2f'}{r}+f'' \quad \text{——— (12)}$$

using (8) and (12), (7) reduces to

$$\frac{2f'x}{r}+x\left(\frac{2f'}{r}+f''\right)=0$$

or $f''+\frac{4f'}{r}=0$

or $f''/f'+4/r=0$

Integrating,

$$\log f'+4\log r=\log c_1$$

So that $f'=c_1 r^{-4}$, ——— (13)

Integrating (13),

$$f = -(c_1/3)\cdot r^{-3} + c_2, \quad \text{---(14)}$$

(c_2 being an arbitrary constant.)

Substituting the values of f' and f from (13) and (14) in (5), we get

$$q = -\{(c_1/3r^2) - c_2\} i - (c_1 x/r^5) r \quad \text{---(15)}$$

Given that $q \to 0$ as $r \to \infty$, hence (15) shows that $c_2 = 0$.

$\therefore$ from (15),

$$q = \frac{c_1}{3r^3}\left(i - \frac{3xr}{r^2}\right) \quad \text{---(16)}$$

Now, $q^2 = q \cdot q$

$$= \frac{c_1^2}{9r^6}\left(i - \frac{3xr}{r^2}\right)\cdot\left(i - \frac{3xr}{r^2}\right)$$

$$= \frac{c_1^2}{9r^6}\cdot\left[i\cdot i - \frac{6x}{r^2} r\cdot i + \frac{9x^2}{r^4} r\cdot r\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2 r^2}{r^4}\right)$$

as $\mathbf{r}\cdot\mathbf{r} = r^2$ and $\mathbf{r}\cdot\mathbf{i} = x$

$$= \frac{c_1}{9r^6}\left(1 + \frac{3x^2}{r^2}\right)$$

$$= \frac{c_1^2}{9r^8}(r^2 + 3x^2).$$

Hence the required surfaces of constant speed are

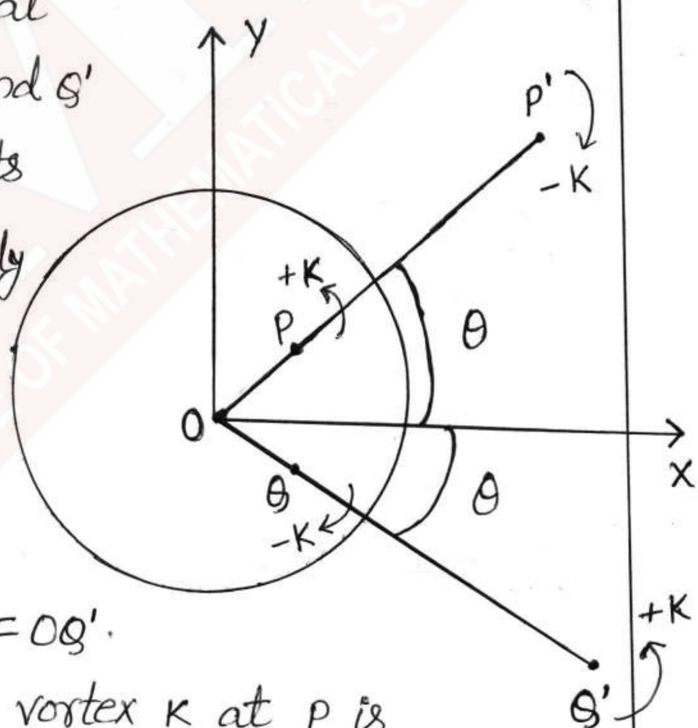
$$q^2 = \text{constant}$$

or $(c_1^2/9r^8)(r^2+3x^2) = \text{constant}.$

or $(r^2+3x^2)r^{-8} = \text{constant}.$

8(C). When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation $(r^2\sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta$, θ being measured from the line through the centre perpendicular to the joint of the vortices.

Sol: Let K be the strength of the vortex at $P(r,\theta)$ and $-K$ at $Q(r,-\theta)$. Let P' and Q' be the inverse points of P and Q respectively with regard to the circular cylinder $|z|=a$.



So that $OP' = a^2/r = OQ'$.

Then the image of vortex K at P is a vortex $-K$ at P' and the image of vortex $-K$ at Q is a vortex K at Q'.

Hence the complex potential of the

System of four vortices is given by

$$w = \frac{ik}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

or $w = (ik/2\pi)\log(z - re^{i\theta}) + w'$,

Since the motion of vortex P is solely due to other vortices, the complex potential of the vortex at P is given by value of w' at $z = re^{i\theta}$.

$$\therefore [w']_{z=re^{i\theta}} = \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]_{z=re^{i\theta}}$$

$$\therefore \phi + i\psi = -\frac{ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta} - re^{-i\theta}) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$\therefore \psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \right.$$

$$\frac{1}{2}\log\left\{\left(r-\frac{a^2}{r}\right)^2\cos^2\theta+\left(r+\frac{a^2}{r}\right)^2\sin^2\theta\right\}\Bigg]$$

$$=-\frac{K}{2\pi}\left[\log\left(r-\frac{a^2}{r}\right)+\log(2r\sin\theta)-\frac{1}{2}\log\left\{\left(r^2+\frac{a^4}{r^2}-2r\cdot\frac{a^2}{r}\cos\right.\right.\right.$$

Thus, $$\psi=-\frac{K}{4\pi}\log\frac{(r-a^2/r)^2(2r\sin\theta)^2}{r^2+a^4/r^2-2a^2\cos 2\theta}$$

So the required streamlines are given by $\psi=$ constant. i.e.,

$$\frac{(r^2-a^2)^2\,r^2\sin^2\theta}{r^4+a^4-2a^2r^2\cos 2\theta}=b^2,\ \text{say}$$

i.e; $b^2(r^4+a^4-2a^2r^2\cos 2\theta)=r^2(r^2-a^2)^2\sin^2\theta$

i.e. $b^2\{(r^2-a^2)^2+2a^2r^2(1-\cos 2\theta)\}=r^2(r^2-a^2)^2\sin^2\theta$

i.e. $2a^2b^2r^2(1-\cos 2\theta)=(r^2-a^2)^2(r^2\sin^2\theta-b^2)$

or $4a^2b^2r^2\sin^2\theta=(r^2-a^2)^2(r^2\sin^2\theta-b^2)$.