CSE - 2018

Q) For an incompressible fluid,

$u = x^2 + 2y^2 + 3z^2$

$v = x^2 y - y^2 z + xz$

Determine w so they satisfy the eqn of continuity. Also find the z component of acceleration.

Soln: Using the continuity equation in Cartesian co-ordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow 2x + x^2 - 2yz + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial w}{\partial z} = 2yz - 2x - x^2$$

Integrating w.r.t 'z':

$$\Rightarrow \boxed{w = yz^2 - 2xz - x^2 z + f(x,y)}$$

The z component of acceleration:

$$a_z = (q \cdot \nabla) w + \frac{\partial w}{\partial t}$$

$$= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$\cancel{= (x^2 + 2y^2 + 3z^2)\left(\frac{\partial f}{\partial x}(x,y)\right)}$$

$$
\boxed{
\begin{aligned}
az = {} & (x^2 + 2y^2 + 3z^2)\left(\frac{\partial f}{\partial x} - 2z - 2xz\right) \\
& + (x^2 y - y^2 z + xz)\left(\frac{\partial f}{\partial y} + z^2\right) \\
& + (yz^2 - 2xz - x^2 z + f(x, y))\,(2yz - 2x - x^2)
\end{aligned}
}
$$

Q) For a 2D flow :

$$\phi = x^2 y - xy^2 + \frac{1}{3}(x^3 - y^3)$$

Determine velocity components along the dir^n x & y. Also determine ψ and check if ϕ is a possible flow or not.

Soln: Let $\vec{q} = u\hat{i} + v\hat{j}$

Then $u = -\frac{\partial \phi}{\partial x}$

$$\Rightarrow u = -(2xy - y^2 + x^2)$$

$$\Rightarrow \boxed{u = y^2 - x^2 - 2xy}$$

$$v = -\frac{\partial \phi}{\partial y}$$

$$\Rightarrow v = -(x^2 - 2xy - y^2)$$

$$\Rightarrow \boxed{v = y^2 - x^2 + 2xy}$$

We know that $\phi + i\psi$ is an analytic function and satisfies Cauchy Riemann equations.

So, $-\frac{\partial \psi}{\partial y} = u$ and $\frac{\partial \psi}{\partial x} = v$

$\Rightarrow \frac{\partial \psi}{\partial x} = y^2 - x^2 + 2xy$

Integrating w.r.t x:

$\Rightarrow \psi = xy^2 - \frac{x^3}{3} + x^2 y + f(y)$

Now, $\frac{\partial \psi}{\partial y} = 2xy + x^2 + f'(y)$

and $\frac{\partial \psi}{\partial y} = -u$

$\Rightarrow x^2 + 2xy + f'(y) = x^2 + 2xy - y^2$

$\Rightarrow f'(y) = -y^2$

$\Rightarrow \frac{df}{dy} = -y^2$

$\Rightarrow f = -\frac{y^3}{3} + C$

So, $\psi = xy^2 - \frac{(x^3 + y^3)}{3} + x^2 y + C$

For $\psi = 0$ at origin $\Rightarrow C = 0$

So, $\boxed{\psi = xy^2 + x^2 y - \frac{(x^3 + y^3)}{3}}$

Checking possible flow:

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -2x - 2y + 2y + 2x = 0$

$\Rightarrow$ Equation of continuity is satisfied

$\Rightarrow$ It is a possible flow.

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Q: $L = \frac{1}{2} m (a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) -$
$\frac{1}{2} K (ax^2 + 2bxy + cy^2)$;

a, b, c, m, K are constants and $b^2 \neq ac$.
Write down Lagrangian equations of motion and identify the system.

Soln: Lagrange's x equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{— (1)}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{m}{2}(2a\dot{x} + 2b\dot{y}) = m(a\dot{x} + b\dot{y})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m(a\ddot{x} + b\ddot{y})$$

$$\frac{\partial L}{\partial x} = \frac{K}{2}(2ax + 2by) = K(ax + by)$$

So, Eq (1) gives us:

$$m(a\ddot{x} + b\ddot{y}) = K(ax + by) \quad \text{— (A)}$$

Lagrange's y equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad \text{— (11)}$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{m}{2}(2b\dot{x} + 2c\dot{y}) = m(b\dot{x} + c\dot{y})$$

$$\cancel{\frac{\partial L}{\partial y}} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m(b\ddot{x} + c\ddot{y})$$

$$\frac{\partial L}{\partial y} = \frac{K}{2}(2bx + 2cy) = K(bx + cy)$$

So, Eq (11) gives us:

$$m(b\ddot{x} + c\ddot{y}) = K(bx + cy) \quad \text{— (B)}$$

Now,

$$a\ddot{x} + b\ddot{y} = \frac{k}{m}(ax + by) \quad \text{— (A)}$$

$$b\ddot{x} + c\ddot{y} = \frac{k}{m}(bx + cy) \quad \text{— (B)}$$

(A) $\times c$ — (B) $\times b$

$$\Rightarrow (ac - b^2)\ddot{x} = \frac{k}{m}(ac - b^2)x$$

$$\Rightarrow \boxed{\ddot{x} = \frac{k}{m}x} \quad \text{— (C)}$$

Putting $\ddot{x}$ in (A):

$$a\frac{k}{m}x + b\ddot{y} = \frac{k}{m}(ax + by)$$

$$\Rightarrow b\ddot{y} = \frac{k}{m}by$$

$$\Rightarrow \boxed{\ddot{y} = \frac{k}{m}y} \quad \text{— (D)}$$

Eq (C): $\left(D^2 - \frac{k}{m}\right)x = 0$

The auxillary eqn $m^2 = \frac{k}{m} \Rightarrow m = \pm\sqrt{\frac{k}{m}}$

So, $x = c_1 e^{\sqrt{k/m}\,x} + c_2 e^{-\sqrt{k/m}\,x}$

and $y = c_1 e^{\sqrt{k/m}\,y} + c_2 e^{-\sqrt{k/m}\,y}$

Q: $H = p_1 q_1 - a q_1^2 + b q_2^2 - p_2 q_2$

Solve this Hamiltonian and show

$$\frac{p_2 - b q_2}{q_1} = \text{constant}$$

Soln: The Hamiltonian equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad ; \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\Rightarrow \dot{p}_1 = -(p_1 - 2a q_1)$$

$$\Rightarrow \frac{dp_1}{dt} + p_1 = 2a q_1$$

So, integrating factor $= e^{\int dt} = e^t$

$$\Rightarrow p_1 e^t = \int 2a q_1 e^t \, dt + C$$

$$\Rightarrow p_1 e^t = 2a \int q_1 e^t \, dt + C_1 \quad \text{— (I)}$$

and $\dot{q}_1 = q_1 \Rightarrow \boxed{q_1 = C_2 e^t}$ — (II)

Putting (II) in (I)

$$p_1 e^t = 2a C_2 \int e^{2t} \, dt + C_1$$

$$\Rightarrow p_1 e^t = a C_2 e^{2t} + C_1$$

$$\Rightarrow \boxed{p_1 = a C_2 e^t + C_1 e^{-t}}$$

Now, $\dot{q}_2 = -q_2 \Rightarrow \boxed{q_2 = C_3 e^{-t}}$

$$\dot{p}_2 = -(2b q_2 - p_2)$$

$$\Rightarrow \frac{dp_2}{dt} - p_2 = -2b q_2$$

$$\Rightarrow \frac{dp_2}{dt} - p_2 = -2b C_3 e^{-t}$$

Integrating factor $= e^{-t}$

$$\Rightarrow p_2 e^{-t} = -2bc_3 \int e^{-2t}\, dt$$

$$\Rightarrow p_2 e^{-t} = bc_3 e^{-2t} + c_4$$

$$\Rightarrow \boxed{p_2 = bc_3 e^{-t} + c_4 e^{t}}$$

Now,

$$\frac{p_2 - bq_2}{q_1} = \frac{bc_3 e^{-t} + c_4 e^{t} - bc_3 e^{-t}}{c_2 e^{t}}$$

$$= \frac{c_4}{c_2} = \text{constant.}$$