2012 - ODE - IFoS

Q 1 ⇒ Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Sol^n Given $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

$\Rightarrow \cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$ let $\sin y = t$

$\cos y \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - \frac{t}{1+x} = e^x(1+x)$

Comparing eq^n with $\frac{dt}{dx} + P(x)t + Q(x)$

$P(x) = \frac{-1}{1+x}$; $Q(x) = e^x(1+x)$

$I.F. = e^{\int P dx} = e^{-\int \frac{1}{1+x}dx} = \frac{1}{1+x}$

$t \cdot \frac{1}{1+x} = \int e^x (1+x) \frac{1}{(1+x)} dx + c$

$\frac{t}{1+x} = e^x + c$

$$\boxed{\frac{\sin y}{1+x} = e^x + c}$$

Q 2 ⇒ Solve $x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = (1-x)^{-2}$

Sol^n Given eq^n $(x^2D^2 + 3xD + 1)y = (1-x)^{-2}$

let $x = e^z \Rightarrow z = \log x$; $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$

$x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow xD = D_1$

similarly $x^2D^2 = D_1(D_1 - 1)$

so, $\{D_1(D_1 - 1) + 3D_1 + 1\} y = \left(\frac{1}{1-x}\right)^2$

$(D_1^2 + 2D_1 + 1)y = \frac{1}{(1-x)^2}$

Auxiliary Eq^n is.

$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$m = -1, -1$

C.F., $y_c = (c_1 + c_2 z) e^{-z} = (c_1 + c_2 \log x) \cdot \frac{1}{x}$

P.I.

$$y_p = \frac{1}{(D_1+1)^2} (1-x)^{-2}$$

$$y_p = \frac{1}{(D_1+1)(D_1+1)} (1-x)^{-2} \Rightarrow \frac{1}{(D_1+1)} x^{-1} \int (1-x)^{-2} dx$$

$$\Rightarrow \frac{x^{-1}}{(D_1+1)} (1-x)^{-1}$$

$$= x^{-1} \int x^{-1}(1-x^{-1}) \Rightarrow x^{-1} \int \frac{dx}{x(1-x)}$$

$$y_p = \cancel{x^{-1}\left[\frac{\log x}{(1-x)}\right]} \frac{1}{x} \left[\log\left(\frac{x}{1-x}\right)\right]$$

$$y = y_c + y_p$$

$$\boxed{y = \left[c_1 + c_2 \log x + \log\left(\frac{x}{1-x}\right)\right] \frac{1}{x}}$$

Q3 $\Rightarrow$ Solve $x = y \frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$

Sol^n let $\frac{dy}{dx} = p$, $x = py - p^2$ — ① ; Deff. w.r.t. y.

$$\frac{1}{p} = p + y\left(\frac{dp}{dy}\right) - 2p\left(\frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1-p^2}{p} = (y-2p)\frac{dp}{dy}$$

$$\frac{dy}{dp} = \frac{p(y-2p)}{1-p^2}$$

$$\frac{dy}{dp} - \frac{py}{(1-p^2)} = \frac{-2p^2}{1-p^2} \quad \text{— ②}$$

so ~~eq~~ eq^n is Linear

$$\text{I.F.} = e^{\int P dp} = e^{\int \frac{p}{1-p^2} dp} = e^{\log(1-p^2)^{1/2}} = (1-p^2)^{1/2}$$

$\therefore$ sol^n of ② is

$$y(1-p^2)^{1/2} = -\int \frac{2p^2}{(1-p^2)} (1-p^2)^{1/2} dp + c$$

$$y(1-p^2)^{1/2} = -2\int \frac{p^2}{(1-p^2)^{1/2}}\,dp + c$$

$$= +2\left[\int \frac{(1-p^2)}{(1-p^2)^{1/2}}\,dp - \int \frac{1\,dp}{(1-p^2)^{1/2}}\right] + c.$$

$$= 2\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] - 2\sin^{-1}p + c$$

$$= p(1-p^2)^{1/2} + \sin^{-1}p - 2\sin^{-1}p + c$$

$$\Rightarrow y(1-p^2)^{1/2} = p(1-p^2)^{1/2} - \sin^{-1}(p) + c$$

$$y = (c-\sin^{-1}p)(1-p^2)^{-1/2}$$

Put value of y in eq^n ①.

$$x = p\left[(c-\sin^{-1}p)(1-p^2)^{-1/2} + p\right] - p^2$$

$$\boxed{x. = p[(c-\sin^{-1}p)(1-p^2)^{-1/2}}$$

Q 4 ⇒ Solve $\frac{d^4y}{dx^4} + \frac{2d^2y}{dx^2} + y = x^2\cos x$

Sol^n

$$(D^4+2D^2+1)y = x^2\cos x.$$

Auxillary Eq^n

$$m^4+2m^2+1=0$$

$$(m^2+1)^2=0 \qquad m_1, m_2 = \pm i.$$

$$y_c = (c_1+c_2)\cos x + (c_3+c_4x)\sin x$$

P.I.

$$y_p = \frac{1}{D^4+2D^2+1}\,x^2\cos x.$$

$$\text{R.P. of } \frac{1}{(D^2+1)^2}\,x^2 e^{ix}$$

$$\text{R.P. of } e^{ix}\frac{1}{[(D+i)^2+1]^2}\,x^2$$

$$= e^{ix}\frac{1}{(D^2-1+2Di+1)^2}$$

R.P. of $e^{ix}\frac{1}{-4D^2\left(1+\frac{D}{2i}\right)^2}x^2$

$\Rightarrow e^{ix}\frac{1}{-4D^2}\left(1-\frac{Di}{2}\right)^{-2}x^2$

$\Rightarrow \frac{e^{ix}}{-4D^2}\left(1+iD+\frac{3i^2D^2}{4}+\cdots\right)x^2$

$\Rightarrow \frac{e^{ix}}{-4D^2}\left[1+iD-\frac{3}{4}D^2\right)x^2$

$\Rightarrow \frac{e^{ix}}{-4D^2}\left[x^2+2ix-\frac{3}{4}\cdot 2\right]$

R.P. of $\frac{e^{ix}}{-4}\left[\frac{x^4}{12}+i\frac{x^3}{3}-\frac{3x^2}{4}\right]$

R.P. of $-\frac{1}{4}(\cos x+i\sin x)\left(\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3x^2}{4}\right)$

$= -\frac{1}{4}\left[\left(\frac{x^4}{12}-\frac{3x^4}{4}\right)\cos x-\frac{x^3}{3}\sin x\right]$

$$\boxed{y=(C_1+C_2x)\cos x+(C_3+C_4x)\sin x-\frac{1}{4}\left[\left(\frac{x^4}{12}-\frac{3x^2}{4}\right)\cos x-\frac{x^3}{3}\sin x\right]}$$

Q5 $\Rightarrow$ Solve:- $x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0$

Sol^n

$x^2y\frac{d^2y}{dx^2}+x^2\left(\frac{dy}{dx}\right)^2-2xy\frac{dy}{dx}+y^2=0$

$x^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\left[2xy\frac{dy}{dx}-y^2\right]=0$

$\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{\left[2xy\frac{dy}{dx}-y^2\right]}{x^2}=0$

$\frac{d}{dx}\left(y\frac{dy}{dx}\right)-\frac{d}{dx}\left(\frac{y^2}{x}\right)=0$

integrating.

$y\frac{dy}{dx}-\frac{y^2}{x}=c$ let $y^2=v \Rightarrow 2y\frac{dy}{dx}=\frac{dv}{dx}$

ve

$$\frac{1}{2}\frac{dv}{dx} - \frac{v}{x} = c$$

$$\frac{dv}{dx} - \frac{2v}{x} = 2c$$

$$I.F. = e^{-\int 2/x\,dx} = e^{-2\log x} = \frac{1}{x^2}$$

$$v.\left(\frac{1}{x^2}\right) = 2c\int \frac{1}{x^2}\,dx + c,$$

$$\frac{y^2}{x^2} = -\frac{2c}{x} + c$$

$$\boxed{y^2 = x(c_2 x - 2c_1)}$$

Q6 ⇒ Solve and find the singular solution of

$$x^3p^2 + x^2py + a^3 = 0 \quad \text{—①}$$

Solⁿ

$$y + xp + \frac{a^3}{x^2p} = 0$$

$$y = -xp - \frac{a^3}{x^2p}$$

Diff. w.r.t. x.

$$\frac{dy}{dx} = -p - x\frac{dp}{dx} - a^3\left(\frac{-2}{x^3p} - \frac{1}{x^2p^2}\frac{dp}{dx}\right)$$

$$\Rightarrow \quad 2p + x\frac{dp}{dx} - \frac{2a^3}{x^3p} - \frac{a^3}{x^2p^2}\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \cancel{2p + x\frac{dp}{dx}}$$

$$\Rightarrow \quad 2p\left(1 - \frac{a^3}{x^3p^2}\right) + x\frac{dp}{dx}\left(1 - \frac{a^3}{x^3p^2}\right) = 0$$

$$\left(1 - \frac{a^3}{x^3p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0$$

Ommit the first factor, ∵ it does not involve $\frac{dp}{dx}$,

$$2p + x\frac{dp}{dx} = 0$$

$$\frac{1}{p}dp + \frac{2}{x}dx = 0$$

$$\log(px^2) = \log c$$

$$px^2 = c$$

$$P = \frac{c}{x^2}$$

$\therefore$ from Eq^n (1)

$$x^3 \frac{c^4}{x^4} + x^2y\left(\frac{c}{x^2}\right) + a^3 = 0$$

$$\frac{c^2}{x} + cy + a^3 = 0$$

$$c^2 + xyc + a^3x = 0$$

c - Discriminant relation is

$$(xy)^2 - 4(1)\,a^3x = 0$$

$$x(xy^2 - 4a^3) = 0$$

$x = 0$ & $(xy^2 - 4a^3) = 0$ both satisfy Eq^n (1). Hence both are singular Sol^n.