Mains Test Series - 2020

COMMON TEST [TEST-17, Batch-I] & [TEST-9, Batch-II]

Answer Key, [Paper-I], full Syllabus.

1(a). If A is both real symmetric and orthogonal, Prove that all its eigenvalues are +1 (or) −1.

Sol'n: If A is a real symmetric matrix, then all its eigenvalues are real.

Further, if A is orthogonal, then all its eigen values must be of unit modulus.

Now ±1 are the only real numbers of unit modulus.

Hence, if A is both real symmetric and orthogonal,

Then all its eigen values are +1 (or) −1.

1(c) → If $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$ Show that the function is discontinuous at the origin but possesses partial derivatives f_x and f_y at every point, including the origin.

Sol Let $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y} ; & x \neq y \\ 0 ; & x = y. \end{cases}$

let us approach $(0,0)$ along the path $y = x - mx^3$ then we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{x - (x - mx^3)}$$

$$= \lim_{x\to 0} \frac{x^3 + x^3 - m^3x^9 - 3x^2 m x^3 + 3x m^2 x^6}{m x^3}$$

$$= \lim_{x\to 0} \frac{2x^3 - m^3x^9 - 3x^5 m + 3x^7 m^2}{m x^3}$$

$$= \lim_{x\to 0} \frac{2 - m^3x^6 - 3x^2 m + 3x^4 m^2}{m}$$

$= \frac{2}{m}$ does not

$\therefore f(x,y)$ is not a continuous at $(0,0)$.

Now we have

$$f_x(0,0) = \lim_{\delta x\to 0} \frac{f(0+\delta x, 0) - f(0,0)}{\delta x}$$

$$= \lim_{\delta x\to 0} \frac{(\delta x)^3 - 0}{\delta x} = 0.$$

$$f_y(0,0) = \lim_{\delta y\to 0} \frac{f(0, 0+\delta y) - f(0,0)}{\delta y}$$

$$= \lim_{\delta y\to 0} \frac{(\delta y)^3}{(\delta y)}$$

$$= 0.$$

$f_x(0,0)$ & $f_y(0,0)$ exist

1(d). If $V = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$, find the value $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$ when $x=0,\ y=1,\ z=2$.

Sol'n: Given

$$V = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow e^V = \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow \sin^{-1}e^V = \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}} \quad (=u) \text{ say} \quad \text{——} (1)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu \quad \text{——} (2).$$

where $n = 1-\frac{2}{3} = \frac{1}{3}$

But from (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial y} \text{ and } \frac{\partial u}{\partial z} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial z}$$

$\therefore$ from (2)

$$\frac{e^V}{\sqrt{1-e^{2V}}}\left[x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}\right] = \frac{1}{3}(\sin^{-1}e^V)$$

$$\Rightarrow x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \frac{1}{3}(\sin^{-1}e^V)\frac{\sqrt{1-e^{2V}}}{e^V} \quad \text{——} (3)$$

when $(x,y,z) = (0,1,2)$

$$V = \log_e \sin\left\{\frac{\pi(1)^{1/2}}{2(4+4)^{1/3}}\right\}$$

$$= \log_e \sin\left[\frac{\pi}{2(8)^{1/3}}\right]$$

$$= \log_e \sin\left(\pi/4\right)$$

$$v = \log_e\left(1/\sqrt{2}\right)$$

$$\Rightarrow e^v = 1/\sqrt{2}$$

and $u = \sin^{-1} e^v = \sin^{-1}\left(1/\sqrt{2}\right) = \pi/4$

$\therefore$ from (3)

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3}\left(\pi/4\right)\frac{\sqrt{1-1/2}}{1/\sqrt{2}}$$

$$= \frac{\pi}{12}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right)$$

$$= \pi/12$$

1(e) If the plane $2x-y+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines, find the value of c.

Sol: Let the plane $2x-y+cz=0$ cut the cone $yz+zx+xy=0$ in a line

$$x/l = y/m = z/n.$$

Then $2l-m+cn=0$ and $mn+nl+lm=0$ ——— (1)

Eliminating m between these relations, we get

$$(2l+cn)n+nl+l(2l+cn)=0$$

or $$2l^2+(c+3)ln+cn^2=0$$

or $$2(l/n)^2+(c+3)(l/n)+c=0 \quad \text{———} \quad (2)$$

If the roots of this equation are (l_1/n_1) and (l_2/n_2), then

$$\frac{l_1}{n_1}\cdot\frac{l_2}{n_2} = \text{Product of the roots} = \frac{c}{2}$$

or $$\frac{l_1 l_2}{c} = \frac{n_1 n_2}{2} \quad \text{———} \quad (3)$$

Eliminating l between the relations (1) we get

$$2nm + n(m - cn) + m(m - cn) = 0$$

or $\quad m^2 + (3-c)mn - cn^2 = 0$

or $\quad c(n/m)^2 + (c-3)(n/m) - 1 = 0$

$\therefore$ If the roots of this equation are (n_1/m_1) and (n_2/m_2) then

$$\frac{n_1}{m_1} \cdot \frac{n_2}{m_2} = \text{product of the roots} = -\frac{1}{c}.$$

or $\quad \dfrac{n_1 n_2}{1} = \dfrac{m_1 m_2}{-c}$

or $\quad \dfrac{n_1 n_2}{2} = \dfrac{m_1 m_2}{-2c}$ ——— (4)

From (3) and (4), we get

$$\frac{l_1 l_2}{1} = \frac{m_1 m_2}{-2c} = \frac{n_1 n_2}{2} = K \text{ (say)}.$$

If the angle between the lines is a right angle, then we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

or $\quad c + (-2c) + 2 = 0$

or $\quad \underline{\underline{c = 2.}}$

2(a) Let F be a field and let n be a +ve integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A and V are subspaces of V?

(i) All invertible A.
(ii) All non-invertible A
(iii) All A such that $AB = BA$, where B is some fixed matrix in V
(iv) All A such that $A^2 = A$.

Soln: V be the vector space of all 2×2 matrices over the field F. (Here $n = 2$)

(i) $W_1 = \{A \in V / |A| \neq 0\}$.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in W$

Then $A + B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

we see that $|A| = 1 \neq 0$ & $|B| = -1 \neq 0$

but $|A + B| = 0$

$\Rightarrow A + B \notin W$.

Hence W is not a subspace of V.

(ii) $W_2 = \{A \in V / |A| = 0\}$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in W$.

Since $|A| = 0$ and $|B| = 0$.

But $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin W$

Since $|A + B| = 1 \neq 0$

$\therefore W_2$ is not a subspace of V.

(iii) $W_3 = \{A \in V / AB = BA\}$.

i.e, W_3 consists of all matrices which commute with a given matrix B.

Now $O \in W$, since $OB = O = BO$.

Suppose $A_1, A_2 \in W_3 \Rightarrow A_1B = BA_1$ and $A_2B = BA_2$.

for any scalars, $a, b \in F$,

$$(aA_1 + bA_2)B = (aA_1)B + (bA_2)B$$
$$= a(A_1B) + b(A_2B)$$
$$= a(BA_1) + b(BA_2)$$
$$= B(aA_1) + B(bA_2)$$
$$= B(aA_1 + bA_2)$$

$\therefore aA_1 + bA_2$ commutes with B.

$\Rightarrow aA_1 + bA_2 \in W_3$.

Hence W_3 is a subspace of V.

(iv) $W_4 = \{A \in V / A^2 = A\}$

We know that $I^2 = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in W_4$

for $2 \in R$, $2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $(2I)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$
$$\neq 2I.$$

$\therefore (2I)^2 \neq 2I$, $2I \notin W_4$.

Hence W_4 is not a subspace of V.

2(b)(i) Examine the convergence of $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Sol^n let $f(x) = \frac{1}{x\sqrt{x^2+1}}$, (behaves like x^{-2} at ∞) and

$g(x) = 1/x^2$

So that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2+1}} = 1$ (Non-zero finite)

Hence, the two integrals $\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ behaves alike

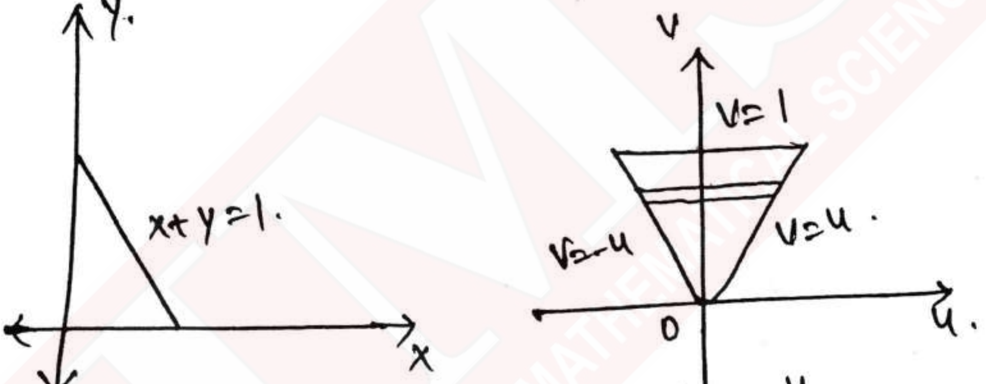
As $\int_1^\infty \frac{dx}{x^2}$ converges therefore $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ also converges.

2(b)(ii) Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\,dy$, where E is the region bounded by the co-ordinate axes and $x+y=1$ in the first quadrant.

Sol^n. Taking $x-y=u$, $x+y=v$ so that $x=\frac{(u+v)}{2}$

$y=\frac{(v-u)}{2}$. and the Jacobian is $\frac{1}{2}$.

$$\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\,dy = \iint_{E_{uv}} \sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv. \rightarrow (1)$$



Now $\iint_{E_{uv}} \sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^1 dv \int_{-v}^{v} \sin\frac{u}{v}\,du.$

$$= \frac{1}{2}\int_0^1 v.\{-\cos 1 + \cos(-1)\}\,dv$$

$$= 0. \longrightarrow (2)$$

Hence from (1) and (2) the required integral is zero.

2(c)(i) The plane $lx+my=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Prove that equation to the plane in its new position is $lx+my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0$.

Sol: The equation of any plane through the line of intersection of the plane $lx+my=0$ and $z=0$ is $lx+my+\lambda z=0$. ——— (1)

It is given that the angle between the plane $lx+my=0$ and (1) is α, so the angle between their normal is $(\pi-\alpha)$.

Also the d.c's of their normals are $l, m, 0$ and l, m, λ respectively.

$$\therefore \tan(\pi-\alpha) = \pm \frac{\sqrt{[\Sigma(m_1n_2-m_2n_1)]}}{\Sigma l_1 l_2}$$

$$= \pm \frac{\sqrt{[(m\lambda-0)^2+(0-l\lambda)^2+(lm-ml)^2]}}{l^2+m^2+0}$$

$$= \pm \frac{\sqrt{[\lambda^2(l^2+m^2)]}}{(l^2+m^2)} = \pm\frac{\lambda}{\sqrt{l^2+m^2}} \Rightarrow -\tan\alpha = \pm\frac{\lambda}{\sqrt{l^2+m^2}}$$

$\Rightarrow \lambda = \pm\sqrt{l^2+m^2}\tan\alpha$

$\therefore$ from (1) the required equation is

$lx+my \pm z\sqrt{l^2+m^2}\tan\alpha = 0$

2(c)(ii) Show that the line $x+2y-z=3,\ 3x-y+2z=1$ is coplanar with the line $2x-2y+3z=2,\ x-y+z+1=0$ and find the plane in which these two lines lie.

Sol^n: Let l, m, n be the d.c's of the line

$$x+2y-z=3,\quad 3x-y+2z=1$$

Then we must have $l+2m-n=0,\ 3l-m+2n=1$

$$\frac{l}{4-1}=\frac{-m}{2-3}=\frac{n}{-1-6}$$

$$\frac{l}{3}=\frac{m}{5}=\frac{n}{-7}$$

which gives the direction ratio's of the first line.

Let $(x_1, y_1, 0)$ be any point on this line, then we have.

$$x_1+2y_1=3,\quad 3x_1-y_1=1$$

solving these we get $x_1 = 5/7,\ y_1 = 8/7$

$\therefore$ Any point on the first line is $(5/7, 8/7, 0)$, so the equation first line in the symmetric form can be written as

$$\frac{x-5/7}{-3}=\frac{y-8/7}{5}=\frac{z-0}{+7} \quad \text{—— (1)}$$

Similarly for other line:

$2l-2m+3n=0;\ l-m+n=0 \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{0}$

Let $(x_1, 0, z_1)$ be any point on this line, then we have $2x_1+3z_1=2$ and $x_1+z_1=-1$

$$\Rightarrow x_1=-5,\ y_1=0,\ z_1=4.$$

The symmetric form of other line can be written as $\frac{x+5}{1}=\frac{y}{1}=\frac{z-4}{0}$ —— (2)

Any point on the line (1) is $\left(\frac{5}{7}-3r, \frac{8}{7}+5r, 7r\right)$ —— (3)

Any point on the line (2) is $(r'-5, r', 4)$ —— (4)

If these two lines meet in a point, then for some values of r and r', the points given by (3) & (4) must be identical.

i.e., $\frac{5}{7}-3r=-5+r'$ | $\frac{8}{7}+5r=r'$ || $7r=4 \Rightarrow \boxed{r=\frac{4}{7}}$

$\therefore\; r' = \frac{8}{7}+\frac{20}{7}=4$

$\because$ we have $r=\frac{4}{7}$, $r'=4$.

which satisfy $\frac{5}{7}-3r=-5+r'$.

Hence the two given lines intersect. i.e, coplanar.

The equation of the plane in which these lines lie is

$$\begin{vmatrix} x+5 & y & z-5 \\ -3 & 5 & 7 \\ 1 & 1 & 0 \end{vmatrix}=0$$

$$\Rightarrow \boxed{7x-7y+8z+3}$$

2(c)(iii) Find the equation of the sphere which passes through the points (1,0,0), (0,1,0) and (0,0,1) and has its radius as small as possible.

Sol: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad \text{--- (1)}$$

If it passes through (1,0,0), (0,1,0) and (0,0,1) then

$$1+2u+c=0,$$
$$1+2v+c=0,$$
$$1+2w+c=0.$$

or $\quad u=v=w=-\frac{1}{2}(1+c) \quad \text{--- (2)}$

$\therefore$ If r be the radius of the sphere (1), then

$$r^2=u^2+v^2+w^2+c=R \text{ (say)}$$

or $\quad R=\frac{3}{4}(1+c)^2-c$, from (2)

If r is least then R is least.

Now $\frac{dR}{dc}=\frac{3}{2}(1+c)-1$ and $\frac{d^2R}{dc^2}=\frac{3}{2}$
$=$ positive.

Equating $\frac{dR}{dc}$ to zero, we get

$$\frac{3}{2}c+\frac{1}{2}=0$$

or $c=-\frac{1}{3}$ and $\frac{d^2R}{dc^2}$ being positive R is least when $c=-\frac{1}{3}$.

$\therefore$ From (2) when R i.e. r^2 is least we have $u=v=w=-\frac{1}{2}\left(1-\frac{1}{3}\right)=-\frac{1}{3}$.

$\therefore$ From (1), the required equation is

$$x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$$

or $3(x^2+y^2+z^2)-2(x+y+z)-1=0$.

3(a)(i) Show that the vectors $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (0,0,1,1)$, $\alpha_3 = (1,0,0,4)$; $\alpha_4 = (0,0,0,2)$ form a basis for R^4. Find the co-ordinates of each of the standard basis vectors in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$.

Sol^n:- Reduce to echelon form the matrix whose rows are the given vectors:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

which is in echelon form.

$\therefore$ The vectors $\alpha_1, \alpha_2, \alpha_3$ and α_4 are LI. and also form a basis of W.

Let $(a,b,c,d) \in \mathbb{R}^4$.

$(a,b,c,d) = p(1,1,0,0) + q(0,0,1,1) + r(1,0,0,4) + s(0,0,0,2)$.

$= (p+r, p, q, q+4r+2s)$

$\Rightarrow p = b$, $q = c$, $r = a-b$, $s = \frac{1}{2}(d-c-4a+4b)$

$(a,b,c,d) = b(1,1,0,0) + c(0,0,1,1) + (a-b)(1,0,0,4) + \frac{1}{2}(d-c-4a+4b)(0,0,0,2)$

$\therefore$ The co-ordinates of standard basis vectors are $(0,0,1,-2)$, $(1,0,-1,2)$, $(0,1,0,-\frac{1}{2})$ and $(0,0,0,\frac{1}{2})$

3(a)(ii) Let $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$. Is A similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix?

Sol^n :- The characteristic polynomial of A is

$$|xI - A| = \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix} = (x-2)(x^2+1)$$

$\therefore$ The characteristic values of A are $2, \pm i$

Let A be similar to a diagonal matrix over R. Then there exists an invertible matrix P such that $P^{-1}AP = \text{diag}(a, b, c)$, where a, b, c are eigenvalues of A and $a, b, c \in R$.

But the characteristic values of A are $2, \pm i \in C$. So we arrive at a contradiction.

Hence A is not similar over the field R to a diagonal matrix. Since the characteristic values of A are $2, \pm i$, which are all distinct.

So A is similar over C to a diagonal matrix.

3(b) By using Lagrange Multipliers method find the maximum and minimum values of $f(x,y,z) = xyz$ subject to the constraint $x+9y^2+z^2=4$. Assume that $x \geq 0$ for this problem. Why is this assumption needed?

Sol^n: Given that $f(x,y,z) = xyz$

Subject to the $x+9y^2+z^2=4$.

Let us consider a function F of independent variables x, y, z

where $F = xyz + \lambda(x+9y^2+z^2)$

$dF = (yz+\lambda)dx + (xz+18y\lambda)dy + (xy+2z\lambda)dz$. $(f_x dx + f_y dy + f_z dz = df)$

for stationary points, $dF = 0$

$F_x = 0 \Rightarrow yz+\lambda = 0 \Rightarrow \lambda = -yz$

$f_y = 0 \Rightarrow xz+18y\lambda = 0 \Rightarrow xz = -18y\lambda$
$= -18y(-yz)$
$= 18y^2z$
$\Rightarrow xz = 18y^2z$
$\Rightarrow z(x-18y^2) = 0$
$\Rightarrow z = 0$ (or) $x = 18y^2$ — (ii)

$f_z = 0 \Rightarrow xy+2z\lambda = 0$
$\Rightarrow xy = -2z\lambda$
$\Rightarrow xy = -2z(-yz)$
$\Rightarrow xy = 2z^2y$
$\Rightarrow y(x-2z^2) = 0$
$\Rightarrow y = 0$ (or) $x = 2z^2$ — (i)

from (i) & (ii) $\left.\begin{array}{l} x = 18y^2 \\ \text{and } z^2 = 9y^2 \end{array}\right\}$ — (iii)

If $z=0$ then $y=0$, $x=0$ and $\lambda=0$

If $y=0$ then $x=0$, $z=0$ and $\lambda=0$

we cannot have all three of the variables be zero but we could have two of them be zero. So, this leads to the following two cases that we can plug into the constraint to find the value of the third variable.

$y=0,\ x=0 \Rightarrow z^2=4 \Rightarrow z=\pm 2$

$y=0,\ z=0 \Rightarrow x=4$

$x=0,\ z=0 \Rightarrow 9y^2=4 \Rightarrow y=\pm\frac{2}{3}$

$\therefore$ we get a total of five possible absolute extrema.

$(0,0,\pm 2),\ (0,\pm\frac{2}{3},0),\ (4,0,0).$

Another possibility from (iii) : $x=18y^2$ & $z^2=9y^2$.

$\therefore$ we have $18y^2+9y^2+9y^2=4$

$\Rightarrow 36y^2=4$

$\Rightarrow \boxed{y=\pm\frac{1}{3}}$

$\Rightarrow \boxed{x=2}$ and $\boxed{z=\pm 1}$

$\therefore$ The other four possible absolute extrema:

$(2,-\frac{1}{3},-1),\ (2,-\frac{1}{3},1),\ (2,\frac{1}{3},-1),\ (2,\frac{1}{3},1)$

Now at $f(0,0,\pm 2)=f(0,\pm\frac{2}{3},0)=f(4,0,0)=0$

$f(2,-\frac{1}{3},1)=f(2,\frac{1}{3},-1)=-\frac{2}{3}$

$f(2,\frac{1}{3},-1)=f(2,\frac{1}{3},1)=\frac{2}{3}$

$\therefore$ The absolute maximum is then $\frac{2}{3}$ which occurs at $\left(2, -\frac{1}{3}, -1\right)$ & $\left(2, \frac{1}{3}, 1\right)$

The absolute minimum is then $-\frac{2}{3}$ which occurs at $\left(2, -\frac{1}{3}, 1\right)$, and $\left(2, \frac{1}{3}, -1\right)$

The assumption that $x \geq 0$. is needed for this problem. without that assumption this function will not have absolute extrema.

If there are no restrictions on x then we could make x as large and $-$ve as we wanted to and we could still meet the constraint simply by chose a very large y and/or z. Note as well that because y and z are both squared we could chose them to be either +ve or -ve.

If we took our choices for x, y and z and plugged them into the function then the function would be similarly large. Also, the larger we chose x the larger we need to choose appropriate y and/or z. and hence the larger our function would become.

In other words, if we have no restriction on x, we can make the function arbitrarily large in a +ve and -ve sense and so this function would not have absolute extrema

(or)

if $x \geq 0$ assumption is not given i.e, $x \leq 0$, then y and z cannot be taken real values.

3(c) Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$ is given by

$$ab(x^2+y^2)-2(a+b)z-1=0$$

Sol^n: Enveloping cone of the paraboid $ax^2+by^2=2cz$ with vertex at the point (α,β,γ).

The Equations of a line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \rightarrow (i)$$

Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr) \rightarrow$ (ii)

If the line (i) meets the given paraboloid at a distance r from the point (α,β,γ) then the point given by (ii) must lie on the given paraboloid and so we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

(or) $r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \rightarrow$ (iii)

If the line (i) is a tangent of the given paraboloid then the line (i) should meet the paraboloid in two coincident points, the condition for the same is that the roots of (iii) are equal i.e "$B^2=4AC$"

(or) $4(al\alpha+bm\beta-cn)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma) \rightarrow$ (iv)

The locus of line (i) which is tangent to the given paraboloid is obtained by eliminating l,m,n between (i) and (iv) and is

$$(a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma))^2$$
$$=\left[a(x-\alpha)^2+b(y-\beta)^2\right]\left[(a\alpha^2+b\beta^2-2c\gamma)\right] \rightarrow \text{(V)}$$

If $S \equiv ax^2+by^2-2cz$, $S_1 = a\alpha^2+b\beta^2-2c\gamma$ and $P \equiv a\alpha x + b\beta y - c(z+\gamma)$ then eqn (V) can be written as

$$(P-S_1)^2 = (S+S_1-2P)S_1$$
$$P^2+S_1^2-2PS_1 = SS_1+S_1^2-2PS_1 \quad \text{(or)} \quad SS_1 = P^2$$
$$(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma) = \left[a(\alpha x)+b\beta y + c(z+\gamma)\right]^2 \rightarrow \text{(VI)}$$

the required equation of the enveloping cone of the given paraboloid.

Cor. To find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2cz$.

Here we are to apply the condition that the enveloping cone, of the given paraboloid with vertex at (α,β,γ) may have three mutually perpendicular generators and we know that the condition for the same is that the sum of the co-efficients of x^2, y^2 and z^2 in the equation of the cone is zero.

$\therefore$ from (vi) above we get

$$\left[a(a\alpha^2+b\beta^2-2c\gamma)-a^2\alpha^2\right]+\left[b(a\alpha^2+b\beta^2-2c\gamma)-b^2\beta^2\right]-c^2=0$$

$$ab\beta^2-2ca\gamma+ba\alpha^2-2cb\gamma-c^2=0$$

$$ab(\alpha^2+\beta^2)-2c(a+b)\gamma-c^2=0$$

Hence the required locus of the point (α,β,γ) is

$$ab(x^2+y^2)-2c(a+b)z-c^2=0 \rightarrow (vii)$$

putting '$c=1$' in the above equation we get the required answer

$$ab(x^2+y^2)-2(a+b)z-1=0$$

4(a) Let T be the linear transformation from R^3 into R^2 defined by $T(x_1, x_2, x_3) = (x_1+x_2, 2x_3-x_1)$.

(i) If β is the standard ordered basis for R^3 and β' is the standard ordered basis for R^2, what is the matrix of T relative to the pair β, β'.

(ii) If $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\beta' = \{\beta_1, \beta_2\}$ where $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,1,1)$, $\alpha_3 = (1,0,0)$, $\beta_1=(0,1)$ $\beta_2=(1,0)$. what is the matrix of T relative to the pair β, β'.

Sol^n: (i) Let $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$.

and $\beta' = \{(1,0), (0,1)\}$ be the standard basis of $\mathbb{R}^2$.

Then $T(1,0,0) = (1,-1) = 1(1,0) + (-1)(0,1)$

$T(0,1,0) = (1,0) = 1(1,0) + 0(0,1)$

$T(0,0,1) = (0,2) = 0(1,0) + 2(0,1)$

$\therefore$ The matrix of T is

$$[T : \beta, \beta'] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(ii) Given $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$

$\beta' = \{(0,1), (1,0)\}$

Hence $T(1,0,-1) = (1,-3) = -3(0,1) + 1(1,0)$

$T(1,1,1) = (2,1) = 1(0,1) + 2(1,0)$

$T(1,0,0) = (1,-1) = -1(0,1) + 1(1,0)$

$\therefore$ The matrix of T is $[T : \beta, \beta'] = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

4(b) i) Show that $\frac{2}{\pi} < \frac{\sin x}{x} < 1,\ 0 < x < \pi/2$

Sol'n: Let $f(x) = \begin{cases} \frac{\sin x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

f is continuous in $[0, \pi/2]$ and derivable in $]0, \pi/2[$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

Let $F(x) = x\cos x - \sin x,\ x \in [0, \pi/2]$

$$F'(x) = \cos x - x\sin x - \cos x$$

$$= -x\sin x < 0,\ x \in]0, \pi/2[$$

$\Rightarrow F$ is strictly decreasing in $[0, \pi/2]$

$\Rightarrow F(x) < F(0) = 0,\ x \in [0, \pi/2]$

$\Rightarrow f'(x) < 0,\ x \in]0, \pi/2[$

$\Rightarrow f$ is strictly decreasing in $[0, \pi/2]$

$\Rightarrow f(0) > f(x) > f(\pi/2)$ for $0 < x < \pi/2$

$\Rightarrow 1 > \frac{\sin x}{x} > \frac{1}{\pi/2}$

$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $0 < x < \pi/2$.

4(b)(ii) Determine $\lim \left(\frac{\pi}{2}-x\right)^{\tan x}$ as $x \to \left(\frac{\pi}{2}-0\right)$

Sol'n: Let $y = \left(\frac{\pi}{2}-x\right)^{\tan x}$

$$\log y = \tan x \log\left(\frac{\pi}{2}-x\right)$$

$$\lim_{x\to(\pi/2-0)} \log y = \lim_{x\to(\pi/2-0)} \tan x \log\left(\frac{\pi}{2}-x\right)$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{\log\left(\frac{\pi}{2}-x\right)}{\cot x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{-\sin^2 x}{-\left(\frac{\pi}{2}-x\right)}$$

$$= \lim_{x\to\left(\frac{\pi}{2}-0\right)} \frac{-2\sin x \cos x}{1}$$

$$= 0$$

$$\log \lim_{x\to(\pi/2-0)} y = 0$$

$$\therefore \lim_{x\to(\pi/2-0)} y = e^0 = 1$$

4(b)(iii) Find the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$.

Sol'n: Volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$ is

$$V = \iiint_V dx\,dy\,dz$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-y} dx\,dy\,dz$$

$$= 2\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} \left(4y-\frac{y^2}{2}\right)_0^{\sqrt{4-x^2}} dx$$

$$= 8\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2} - 2\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 8\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right]$$

$$= 8\left[2\left(\frac{\pi}{2}\right) - 2\left(-\frac{\pi}{2}\right)\right]$$

$$= 8\left[\pi+\pi\right]$$

$$= 16\pi$$

4(c). Prove that the projections of the generators of a hyperboloid on coordinate plane are tangents to the section of the hyperboloid by that plane.

Sol: Let the equation of the hyperboloid be

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{——} (1)$$

we know that a generator of the hyperboloide (1) is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c} \quad \text{——} (2)$$

Now consider the coordinate plane $z=0$. The section of the hyperboloid (1) by this plane $z=0$ is given by

$$(x^2/a^2)+(y^2/b^2)=1,\ z=0 \quad \text{——} (3)$$

The projection of the generator (2) on the plane $z=0$ is given by

$$\frac{x-a}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta},\ z=0$$

Which is a plane through the generator Perpendicular to the plane $z=0$.

On simplifying it reduces to

$$\frac{x}{a\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{y}{-b\cos\theta} + \frac{\sin\theta}{\cos\theta}, \ z=0.$$

i.e. $$\frac{x}{a\sin\theta} + \frac{y}{b\cos\theta} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}, \ z=0.$$

i.e. $$\frac{x}{a\sin\theta} + \frac{y}{b\cos\theta} = \frac{1}{\sin\theta\cos\theta}, \ z=0$$

i.e. $$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \ z=0$$

which is evidently a tangent to the section ③ of the hyperboloid ① by the plane $z=0$ at the point $(a\cos\theta, b\sin\theta, 0)$

Again consider the coordinate plane $x=0$. The section of the hyperboloid ① by this plane $x=0$ is given by

$$(y^2/b^2)-(z^2/c^2)=1, \ x=0 \quad \text{——— ④}$$

The projection of the generator ② on the plane $x=0$ is given by

$$\frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{c}, \ x=0$$

which is a plane through the generator perpendicular to the plane $x=0$.

on simplifying it reduces to

$$\frac{y}{-b\cos\theta} + \frac{\sin\theta}{\cos\theta} = \frac{z}{c}, \quad x=0$$

or $$\frac{y}{b} + \frac{z\cos\theta}{c} = \sin\theta, \quad x=0$$

or $$\frac{y}{b}\operatorname{cosec}\theta + \frac{z}{c}\cot\theta = 1, \quad x=0$$

which is evidently a tangent to the section ④ of the hyperboloid ① by the plane $x=0$ at the point $(0, b\operatorname{cosec}\theta, -c\cot\theta)$.

Similarly we can prove the result by considering the plane $y=0$.

5(a)(i) Solve $(2\sqrt{xy}-x)dy + y\,dx = 0$.

Sol: Given that

$$\frac{dy}{dx} = -\frac{y}{2\sqrt{xy}-x} = \frac{y/x}{1-2(\sqrt{y/x})} \quad \text{——— (1)}$$

Putting $y/x = v$ or $y = xv$, we have

$$\frac{dy}{dx} = v + x(dv/dx) \quad \text{——— (2)}$$

From (1) and (2),

$$v + x\frac{dv}{dx} = \frac{v}{1-2\sqrt{v}}$$

or $$x\frac{dv}{dx} = \frac{2v\sqrt{v}}{1-2\sqrt{v}}$$

or $$\frac{dx}{x} = \frac{1-2\sqrt{v}}{2v\sqrt{v}}\,dv$$

or $$\frac{dx}{x} = \left(\frac{1}{2}v^{-3/2} - \frac{1}{v}\right)dv$$

Integrating, $\log x = -v^{-1/2} - \log v + \log c$

or $\log\left(\frac{xv}{c}\right) = -\frac{1}{\sqrt{v}}$ (or) $\log\left(\frac{y}{c}\right) = -\sqrt{(x/y)}$

or $\frac{y}{c} = e^{-\sqrt{(x/y)}}$

so that $y = ce^{-\sqrt{(x/y)}}$.

5(a)(ii) Solve $(y + y^3/3 + x^2/2)dx + (1/4)\times(x + xy^2)dy = 0$

Sol: Given

$$(y + y^3/3 + x^2/2)dx + (1/4)\times(x+xy^2)dy = 0 \quad \text{---(1)}$$

comparing (1) with $Mdx + Ndy = 0$,

$M = y + y^3/3 + x^2/2$ and $N = (1/4)\times(x + xy^2)$.

Here $\frac{\partial M}{\partial y} = 1 + y^2$ and $\frac{\partial N}{\partial x} = (1/4)\times(1+y^2)$.

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x(1+y^2)}\left\{(1+y^2) - \frac{1}{4}(1+y^2)\right\}$$

$$= \frac{4}{x}\left(1 - \frac{1}{4}\right) = \frac{3}{x},$$

which is a function of x alone. so

$$\text{I.F.} = e^{\int (3/x)dx} = e^{3\log x} = e^{\log x^3} = x^3.$$

Multiplying (1) with x^3, we have

$$\{x^3y + (1/3)x^3y^3 + (1/2)x^5\}dx + \tfrac{1}{4}(x^4 + x^4y^2)dy = 0$$

whose solution as usual is

$$\int \{x^3y + \tfrac{1}{3}x^3y^3 + \tfrac{1}{2}x^5\}dx = c/12 \quad (\text{Treating } y \text{ as constant})$$

$$\Rightarrow \tfrac{1}{4}x^4y + \tfrac{1}{12}x^4y^3 + \tfrac{1}{12}x^6 = c/6$$

$$\Rightarrow 3x^4y + x^4y^3 + x^6 = c$$

where c is an arbitrary constant.

5(b)(i) Prove that $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

Sol^n: Here $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$

$\therefore L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s)$, say

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$= \int_s^\infty \left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\} ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{1+a^2/s^2}{1+b^2/s^2} + \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= 0 + \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

5(b) (iii) Evaluate $\int_0^\infty t^3 e^{-t} \sin t \, dt$.

Sol^n: Here $L\{\sin t\} = \frac{1}{(s^2+1)} = f(s)$, say ——— ①

$\therefore L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} f(s) = -\frac{d^3}{ds^3}(s^2+1)^{-1}$, by ①

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}(s^2+1)^{-1}\right] = -\frac{d^2}{ds^2}\left[-(s^2+1)^{-2} 2s\right]$$

$$= 2\frac{d}{ds}\left[\frac{d}{ds}\frac{s}{(s^2+1)^2}\right]$$

$$= 2\frac{d}{ds}\frac{1.(s^2+1)^2 - s.2(s^2+1).2s}{(s^2+1)^4}$$

$$= 2\frac{d}{ds}\frac{1-3s^2}{(1+s^2)^3} = 2\frac{(-6s)(1+s^2)^3 - 3(1+s^2)^2.2s(1-3s^2)}{(1+s^2)^6}$$

$$= -12s\frac{1+s^2+1-3s^2}{(1+s^2)^4} = \frac{24s(s^2-1)}{(1+s^2)^4}$$

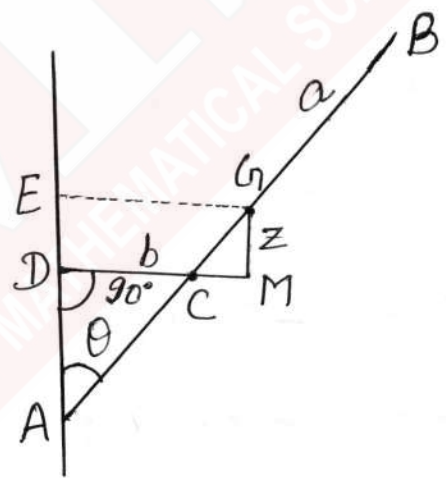
$\Rightarrow \int_0^\infty e^{-st}\{t^3 \sin t\} dt = \frac{24s(s^2-1)}{(1+s^2)^4}$, by definition ——— ②

Taking limit as $s \to 1$ on bothsides of ②, we get

$$\int_0^\infty t^3 e^{-t} \sin t \, dt = 0.$$

5(c) → A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; Find the position of equilibrium and show that its is unstable.

Sol: Let AB be a uniform rod of length 2a. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e., CD = b.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C i.e., GM = z.

We shall express z in terms of θ.
We have

$$z = GM = ED = AE - AD$$

$$= AG\cos\theta - CD\cot\theta$$

$$= a\cos\theta - b\cot\theta.$$

$$\therefore\ dz/d\theta = -a\sin\theta + b\,\mathrm{cosec}^2\theta$$

and $d^2z/d\theta^2 = -a\cos\theta - 2b\,\mathrm{cosec}^2\theta\cot\theta.$

For equilibrium of the rod, we have

$dz/d\theta = 0$

i.e., $-a\sin\theta + b\,\mathrm{cosec}^2\theta = 0$

or $a\sin\theta = b\,\mathrm{cosec}^2\theta$

or $\sin^3\theta = b/a$

or $\sin\theta = (b/a)^{1/3}$ or $\theta = \sin^{-1}(b/a)^{1/3}.$

This gives position of the equilibrium of rods.

Again $d^2z/d\theta^2 = -(a\cos\theta + 2b\,\mathrm{cosec}^2\theta\cot\theta)$

$= -\text{ve}$ for all acute values of θ.

Thus $d^2z/d\theta^2$ is $-$ve in the position of equilibrium and so z is maximum. Hence, the equilibrium is unstable.

5(d) → A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$

Sol'n: Here the central acceleration varies inversely as the cube of the distance i.e., $P = \mu/r^3 = \mu u^3$, where μ is a constant. If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

$$V = \sqrt{(\mu/a^2)}$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying bothsides by $2(du/d\theta)$ and integrating, we get

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \quad \text{— (1)}$$

where A is a constant.

But initially when $r = a$, i.e. $u = 1/a$, $du/d\theta = 0$ (at an apse), and $v = v_1 = \sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$\therefore h^2 = 2\mu$ and $A = \mu/a^2$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

$$\Rightarrow \sqrt{2}\,a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)} \Rightarrow d\theta/\sqrt{2} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}$$

Integrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$, where B is a constant.

But initially, when $u = 1/a$, $\theta = 0$, $\therefore B = \sin^{-1} 1 = \frac{1}{2}\pi$

$$\therefore (\theta/\sqrt{2}) + \tfrac{1}{2}\pi = \tfrac{1}{2}\sin^{-1}(au) \Rightarrow au = a/r = \sin\left\{\tfrac{1}{2}\pi + (\theta/\sqrt{2})\right\}$$

$\Rightarrow a = r\cos(\theta/\sqrt{2})$, which is required equation of the path.

5(e) Verify Green's theorem in the plane for

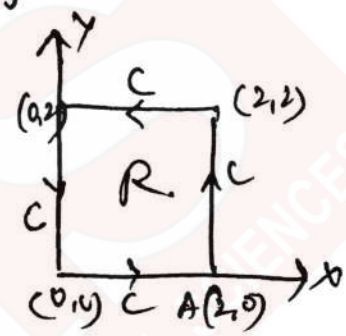
$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy,$$ where C is the square with vertices $(0,0), (2,0), (2,2), (0,2)$

Sol^n: By Green's theorem, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

Here $M = x^2 - xy^3$, $N = y^2 - 2xy$.

The closed curve C consists of the straight lines OA, AB, BD and DO. The positive direction in traversing C is as shown in the figure and R is the region bounded by C.



We have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(y^2 - 2xy) - \frac{\partial}{\partial y}(x^2 - xy^3)\right] dx\,dy$$

$$= \iint_R (-2y + 3xy^2)\,dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2)\,dx\,dy$$

$$= \int_{x=0}^{2} \left(-y^2 + xy^3\right)_{y=0}^{2} dx$$

$$= \int_0^2 (-4 + 8x)\,dx$$

$$= \left(-4x + 4x^2\right)_0^2 = -8 + 16 = 8. \quad \text{--- (1)}$$

Now let us evaluate the line integral along the closed curve C.

Along OA, $y=0$, $dy=0$ and x varies from 0 to 2

Along AB, $x=2$, $dx=0$ and y varies from 0 to 2

Along BD, $y=2$, $dy=0$ and x varies from 2 to 0

Along DO, $x=0$, $dx=0$ and y varies from 2 to 0

We have $\int_C M\,dx+N\,dy = \int_{OA} M\,dx+N\,dy + \int_{AB} + \int_{BD} + \int_{DO}$

$$= \int_{x=0}^{2} x^2\,dx + \int_{y=0}^{2} (y^2-4y)\,dy + \int_{x=2}^{0} (x^2-8x)\,dx + \int_{y=2}^{0} y^2\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left(\frac{y^3}{3} - 2y^2\right)_0^2 + \left(\frac{x^3}{3} - 4x^2\right)_2^0 + \left(\frac{y^3}{3}\right)_2^0$$

$$= \frac{8}{3} + \frac{8}{3} - 8 - \frac{8}{3} + 16 - \frac{8}{3}$$

$$= 8 \text{ ——— } (2)$$

$\therefore$ from (1) & (2) we see that

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

This verifies the Green's theorem

6(a)(i). Find the equation of the family of oblique trajectories which cut the family of concentric circle at 30°.

Sol: Let the equation of the given family of concentric circles, having (0,0) as common centre be $x^2+y^2=a^2$, where a, is parameter. —— ①

Differentiating ①,

$$2x+2y(dy/dx)=0$$

or $\quad x+yp=0$, —— ②

where $p=dy/dx$.

Replacing p by $\dfrac{p+\tan 30^\circ}{1-p\tan 30^\circ}$,

i.e., $\dfrac{p+(1/\sqrt{3})}{1-p(1/\sqrt{3})}$, i.e., $\dfrac{\sqrt{3}p+1}{\sqrt{3}-p}$ in ②,

the differential equation of the desired family of curve is

$$x+y\{(\sqrt{3}p+1)/(\sqrt{3}-p)\}=0$$

or $\quad x(\sqrt{3}-p)+y(\sqrt{3}p+1)=0$

or $\quad p=\dfrac{x\sqrt{3}+y}{x-y\sqrt{3}}$

or $\dfrac{dy}{dx} = \dfrac{\sqrt{3}+(y/x)}{1-\sqrt{3}(y/x)}$. ———— (3)

putting $y/x = v$ or $y = xv$ so that

$dy/dx = v + x(dv/dx)$, (3) gives

$$v + x\frac{dv}{dx} = \frac{\sqrt{3}+v}{1-v\sqrt{3}}$$

or $x\dfrac{dv}{dx} = \dfrac{\sqrt{3}+v}{1-v\sqrt{3}} - v = \dfrac{\sqrt{3}(v^2+1)}{1-v\sqrt{3}}$

or $\sqrt{3}\dfrac{dx}{x} = \dfrac{1-v\sqrt{3}}{v^2+1}$

or $\sqrt{3}\dfrac{dx}{x} - \dfrac{dv}{v^2+1} + \dfrac{\sqrt{3}}{2}\cdot\dfrac{2v\,dv}{v^2+1} = 0$.

Integrating,

$$\sqrt{3}\log x - \tan^{-1}v + (\sqrt{3}/2)\log(v^2+1) = (\sqrt{3}/2)\log c$$

or $\log\{x^2(v^2+1)/c\} = (2/\sqrt{3})\tan^{-1}v$

or $x^2(v^2+1)/c = e^{(2/\sqrt{3})\tan^{-1}v}$

or $x^2[(y/x)^2+1] = ce^{(2/\sqrt{3})\tan^{-1}(y/x)}$

or $y^2+x^2 = ce^{(2/\sqrt{3})\tan^{-1}(y/x)}$,

which is the required family of curves, c being a parameter.

6(a)(ii) Reduce the equation $x^2p^2 + py(2x+y) + y^2 = 0$ where $p = \frac{dy}{dx}$ to clairaut's form and find its complete primitive and its singular solution.

Sol'n: The given equation is $x^2p^2 + py(2x+y) + y^2 = 0$ —— (1)

Given $y = u$ and $xy = v$ —— (2)

Differentiation (2), $dy = du$ and $x\,dy + y\,dx = dv$

$$\therefore \frac{x\,dy + y\,dx}{dy} = \frac{dv}{du} \Rightarrow x + y\frac{dx}{dy} = \frac{dv}{du}$$

$$\Rightarrow x + \frac{y}{p} = P$$

$$\Rightarrow \frac{y}{p} = P - x \Rightarrow p = \frac{y}{(P-x)}, \text{ where } p = \frac{dy}{dx}, \; P = \frac{dv}{du}$$

Putting $p = \frac{y}{(P-x)}$ in (1), we have

$$\frac{x^2y^2}{(P-x)^2} + \frac{y^2}{P-x}(2x+y) + y^2 = 0$$

$$\Rightarrow x^2 + (P-x)(2x+y) + (P-x)^2 = 0$$

$$\Rightarrow Py - xy + P^2 = 0 \Rightarrow v = uP + P^2, \text{ using (2)}.$$ —— (3)

(3) is in clairaut's form. So replacing P by c its general solution is

$$v = uc + c^2 \Rightarrow xy = yc + c^2, \text{ c being an arbitrary constant}$$

$$\Rightarrow c^2 + yc - xy = 0.$$

which is a quadratic equation in c and hence c-discriminant relation is

$$y^2 - 4.1.(-xy) = 0$$

$$y(y + 4x) = 0$$

Since $y = 0$ and $y + 4x = 0$ both satisfy (1), so there are both singular solutions.

6(b) Solve $x^2(d^3y/dx^3) + 2x(d^2y/dx^2) + 2(y/x) = 10(1 + 1/x^2)$.

Sol: Multiplying both sides by x, the given equation becomes

$$x^3(d^3y/dx^3) + 2x^2(d^2y/dx^2) + 2y = 10(x + 1/x)$$

or $$(x^3D^3 + 2x^2D^2 + 2)y = 10(x + x^{-1}), \quad \text{——— (1)}$$

where $D \equiv d/dx$.

Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$. Then (1) becomes

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2]y = 10(e^z + e^{-z})$$

or $$(D_1^3 - D_1^2 + 2)y = 10e^z + 10e^{-z} \quad \text{——— (2)}$$

A.E. of (2) is

$$D_1^3 - D_1^2 + 2 = 0$$

or $$(D_1+1)(D_1^2 - 2D_1 + 2) = 0$$

giving $D_1 = -1, 1 \pm i$.

$$\text{C.F.} = c_1e^{-z} + e^z(c_1 \cos z + c_2 \sin z) = c_1x^{-1} + x(c_2 \cos \log x + c_3 \sin \log x)$$

P.I. corresponding to $10e^z = 10\dfrac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z$

$$= 10 \frac{1}{2(1-2+2)} e^{z}$$

$$= 5x$$

and P.I. corresponding to $10e^{-z}$ =

$$10 \frac{1}{(D_1+1)(D_1^2-2D_1+2)} e^{-z}$$

$$= 10 \frac{1}{D_1+1} \cdot \frac{1}{1+2+2} e^{-z}$$

$$= 2 \frac{1}{D_1+1} e^{-z} \cdot 1$$

$$= 2e^{-z} \frac{1}{D_1-1+1} \cdot 1$$

$$= 2e^{-z} \frac{1}{D_1} \cdot 1$$

$$= 2e^{-z} z = 2x^{-1} \log x.$$

$\therefore$ Required solution is

$$y = c_1 x^{-1} + x(c_2 \cos \log x + c_3 \sin \log x) + 5x + 2x^{-1} \log x.$$

6(c) → Using method of variation of parameters, solve $\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right) + y = xe^x \sin x$ with $y(0)=0$ and $\left(\frac{dy}{dx}\right)_{x=0} = 0$.

Sol'n : Given $(D^2-2D+1)y = xe^x \sin x$, where $D \equiv \frac{d}{dx}$ —— ①

comparing ① with $y_2 + Py_1 + Qy = R$, here $R = xe^x \sin x$

consider $(D^2-2D+1)y = 0 \Rightarrow (D-1)^2 y = 0$, $D \equiv \frac{d}{dx}$ —— ②

Auxiliary equation of ② is $(D-1)^2 = 0 \Rightarrow D = 1,1$.

$\therefore$ C.F of ① $= (c_1 + c_2 x)e^x = c_1 e^x + c_2 xe^x$, c_1 and c_2 being arbitrary const. —— ③.

let $u = e^x$, $v = xe^x$.

Also here $R = xe^x \sin x$ —— ④

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0$ —— ⑤

Then, P.I of ① $= uf(x) + vg(x)$ —— ⑥

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{xe^x(xe^x \sin x)}{e^{2x}} dx$

$= -\int x^2 \sin x \, dx$ by ④ & ⑤

$= -\{x^2(-\cos x) - (2x)(-\sin x) + 2\cos x\}$

and $g(x) = \int \frac{uR}{W} dx = \int \frac{e^x (xe^x \sin x)}{e^{2x}} dx = \int x \sin x \, dx$, by ④ & ⑤

$= (x)(-\cos x) - (1)(-\sin x) = \sin x - x\cos x$

$\therefore$ P.I of ① $= e^x(x^2\cos x - 2x\sin x - 2\cos x) + xe^x(\sin x - x\cos x)$ by ⑥

$= -xe^x \sin x - 2e^x \cos x$

Hence the general solution of ① is

$$y = C.F + P.I$$

i.e. $y = c_1 e^x + c_2 x e^x - x e^x \sin x - 2e^x \cos x$

$$= e^x (c_1 + c_2 x - x\sin x - 2\cos x) \quad \text{--- ⑦}$$

Given that $y=0$ when $x=0$.

Hence ⑦ gives $0 = c_1 - 2 \Rightarrow c_1 = 2$

putting $c_1 = 2$ in ⑦

$$y = e^x (2 + c_2 x - x\sin x - 2\cos x) \quad \text{--- ⑧}$$

⑧ $\Rightarrow dy/dx = e^x(2 + c_2 x - x\sin x - 2\cos x) + e^x\{c_2 - (\sin x + x\cos x) + 2\sin x\}$

Given that $dy/dx = 0$ when $x=0$.

So above equation gives $0 = c_2$

putting $c_1 = 2$ & $c_2 = 0$ in ⑧, the required solution is $y = e^x(2 - x\sin x - 2\cos x)$.

6(d). Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t, \quad y(0)=0,\ y'(0)=0.$$

by using Laplace-transform.

Sol^n: Given equation is $\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t$

$$\Rightarrow y'' + y = 8e^{-2t}\sin t \quad \text{——— (1)}$$

Taking Laplace transform of bothsides of (1)

we get $L(y'') + L(y) = 8L(e^{-2t}\sin t)$

$$\Rightarrow p^2 L\{y(t)\} - py(0) - y'(0) + L\{y(t)\} = \frac{8}{(p+2)^2+1}$$

$$\Rightarrow p^2 L\{y(t)\} + L\{y(t)\} = \frac{8}{p^2+4p+5}$$

$$\Rightarrow L\{y(t)\}(p^2+1) = \frac{8}{p^2+4p+5}$$

$$\Rightarrow L\{y(t)\} = \frac{8}{(p^2+1)(p^2+4p+5)}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{8}{(p^2+1)(p^2+4p+5)}\right\}$$

$$y(t) = L^{-1}\left[\frac{-p+1}{p^2+1} + \frac{p+3}{p^2+4p+5}\right]$$

$$= L^{-1}\left(\frac{-p}{p^2+1}\right) + L^{-1}\left(\frac{1}{p^2+1}\right) + L^{-1}\left(\frac{(p+2)+1}{(p+2)^2+1}\right)$$

$$= -\cos t + \sin t + e^{-2t}L^{-1}\left(\frac{p+1}{p^2+1}\right)$$

$$= -\cos t + \sin t + e^{-2t}\left\{L^{-1}\left(\frac{p}{p^2+1}\right) + L^{-1}\left(\frac{1}{p^2+1}\right)\right\}$$

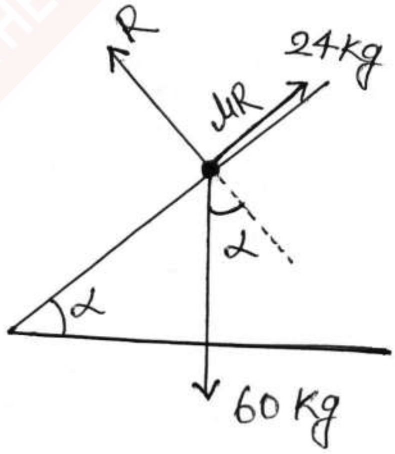
$$= -\cos t + \sin t + e^{-2t}\cos t + e^{-2t}\sin t$$

$$= (e^{-2t}-1)\cos t + (e^{-2t}+1)\sin t$$

which is the required solution.

7(a) A weight of 60 kg is on the point of motion down a rough inclined plane when supported by a force of 24 kg wt. acting parallel to the plane along a line of greatest slope, and is on the point of motion up the plane when pulled in the same direction by a force of 36 kg wt. Find the co-efficient of friction and the inclination of the plane.

Sol: Let μ be the co-efficient of friction and α the inclination of the plane with the horizontal. As the body is on the point of moving down the plane, the force of friction μR acts up the plane.



Resolving perpendicular to the plane,

$$R = 60 \cos\alpha \text{ ——— (1)}$$

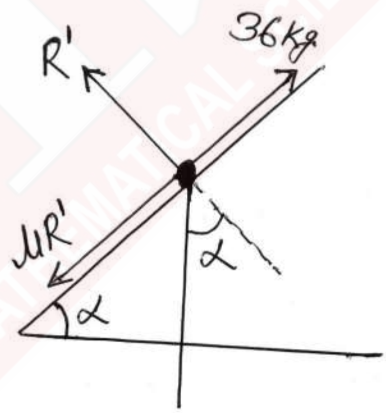
Resolving along the plane,

$$\mu R + 24 = 60 \sin\alpha$$

or $60\mu \cos\alpha + 24 = 60 \sin\alpha$ [$\because$ of 1]

or $\mu = \dfrac{5\sin\alpha - 2}{5\cos\alpha}$ ——— (2)

In the second case, the body is on the point of motion up the plane, therefore the force of friction $\mu R'$ acts down the plane.



Resolving perpendicular to the plane,

$R' = 60\cos\alpha$ ——— (3)

Resolving along the plane, we have

$$\mu R' + 60\sin\alpha = 36$$

or $\dfrac{5\sin\alpha - 2}{5\cos\alpha} \cdot 60\cos\alpha + 60\sin\alpha = 36$

[$\because$ of (2) and (3)]

or $5\sin\alpha - 2 + 5\sin\alpha = 3$

or $10\sin\alpha = 5$

Or $\sin\alpha = \frac{1}{2}$

Hence $\alpha = 30^\circ$

Also from (2)

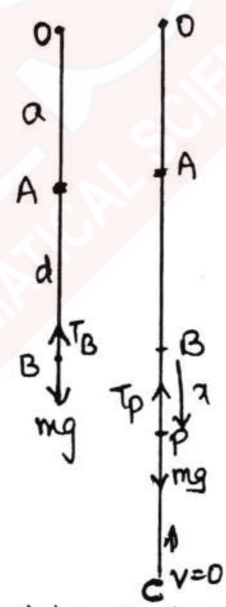
$$\mu = \frac{5\sin 30^\circ - 2}{5\cos 30^\circ}$$

$$= \frac{5\cdot\frac{1}{2} - 2}{5\cdot\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{5\sqrt{3}}$$

7(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\frac{a}{g}}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, where a is the natural length of the string.

<u>Sol'n</u>: Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law, $T_B = \lambda \frac{OB-OA}{OA} = \lambda \frac{d}{a}$



where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B we have

$$mg = T_B = \lambda\frac{d}{a} = mg\frac{d}{a} \quad [\because \lambda = mg, \text{ as given}]$$

$$\therefore d = a$$

Now the particle is pulled down to a point C such that $OC = 4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP = x$

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P = \lambda\frac{a+x}{a} = \frac{mg}{a}(a+x)$ in the string OP acting in the direction PO, i.e. in the direction of x decreasing

(ii) The weight mg of the particle acting vertically downwards i.e. in the direction of x increasing.

Hence by Newton's second law of motion ($P = mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = mg - \frac{mg}{a}(a+x) = -\frac{mgx}{a}$$

Thus $$\frac{d^2x}{dt^2} = -\frac{g}{a}x \quad \text{——— (1)}$$

which is the equation of S.H.M with centre at the origin B and the amplitude $BC = 2a$ which is greater than $AB = a$. Multiplying bothsides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + k, \quad \text{where } k \text{ is a constant.}$$

At the point C, $x = BC = 2a$, and the velocity $dx/dt = 0$

$$\therefore k = \frac{g}{a}4a^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2) \quad \text{——— (2)}$$

Taking square root of (2), we have

$$dx/dt = -\sqrt{(g/a)}\sqrt{4a^2 - x^2}$$

the $-$ve sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\sqrt{a/g}\frac{dx}{\sqrt{(4a^2-x^2)}} \quad \text{——— (3)}$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1} dt = -\sqrt{(a/g)}\int_{2a}^{-a}\frac{dx}{\sqrt{4a^2-x^2}}$$

$$\Rightarrow t_1 = \sqrt{\frac{a}{g}}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a} = \sqrt{\frac{a}{g}}\left[\cos^{-1}\left(-\tfrac{1}{2}\right) - \cos^{-1}(1)\right] = \sqrt{\frac{a}{g}}\cdot\frac{2\pi}{3}$$

Let v_1 be the velocity of the particle at A, Then at A
$x = -a$ and $(dx/dt)^2 = v_1^2$

So from (2), we have $v_1^2 = (g/a)(4a^2 - a^2)$

$\Rightarrow v_1 = \sqrt{3ag}$, the direction of v_1 being vertically upwards. Thus the velocity at A is $\sqrt{3ag}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{3ag}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by $0 = \sqrt{3ag} - gt_2$. so that $t_2 = \sqrt{\frac{3a}{g}}$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A
Thus the whole time taken by the particle return to C $= 2(t_1 + t_2)$

$$= 2\left[\sqrt{a/g}\cdot\frac{2\pi}{3} + \sqrt{\frac{3a}{g}}\right] = \sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right].$$

7(c) A particle is projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β, and strikes it at right angles. show that

(i) the time of flight is $\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$

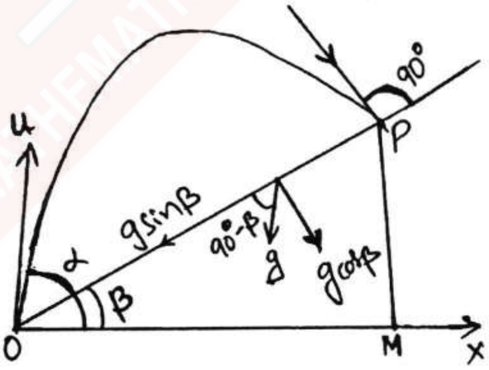
(ii) the range on the inclined plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$

and (iii) the vertical height of the point struck, above the point of projection is $\dfrac{2u^2\sin^2\beta}{g\cdot(1+3\sin^2\beta)}$

Sol'n: Let 'O' be the point of projection, u the velocity of projection, α the angle of projection and P the point where the particle strikes the plane at right angles.

Let T be the time of flight from O to P. Then by the time of flight on inclined plane, we have

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$



Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particles at P along the inclined plane is zero. Also the resolved part of the acceleration g along the inclined plane and using the formula

$$v = u + ft,$$

$$0 = u\cos(\alpha-\beta) - g\sin\beta T$$

$$\Rightarrow T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad \text{——— (2)}$$

Equating the values of T from (1) & (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\Rightarrow \tan(\alpha-\beta) = \tfrac{1}{2}\cot\beta \quad \text{——— (3)}$$

as the condition for striking the plane at right angles,

(i) from (2)

$$T = \frac{u}{g\sin\beta\sec(\alpha-\beta)} = \frac{u}{g\sin\beta\sqrt{\{1+\tan^2(\alpha-\beta)\}}}$$

$$= \frac{u}{g\sin\beta\sqrt{(1+\frac{1}{4}\cot^2\beta)}} \qquad \left(\text{from eqn (3). } \tan(\alpha-\beta)=\tfrac{1}{2}\cot\beta\right).$$

$$= \frac{2u\sin\beta}{g\sin\beta\sqrt{(4\sin^2\beta+\cos^2\beta)}} = \frac{2u}{g\sqrt{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}$$

$$= \frac{2u}{g\sqrt{(1+3\sin^2\beta)}}$$

(ii) Let R be the range on the inclined plane; then $R=OP$. Considering the motion from O to P the inclined plane and using the formula $v^2=u^2+2fs$

we have $\quad 0 = u^2\cos^2(\alpha-\beta) - 2g\sin\beta R$

$$\Rightarrow R = \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} = \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$= \frac{u^2}{2g\sin\beta\{1+\tan^2(\alpha-\beta)\}}$$

$$= \frac{u^2}{2g\sin\beta\{1+\frac{1}{4}\cot^2\beta\}} \qquad [\text{from (3)}]$$

$$= \frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)} = \frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$$

(iii) The vertical height of P above $O = PM$

$$= OP\sin\beta = R\sin\beta = \frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

Ques: 8 (a) (i) In what direction the directional derivative of $\phi = x^2y^2z^2$ from (1,1,2) will be maximum and what is its magnitude? Also find a unit normal vector to the surface $x^2y^2z = 2$ at the point (1,1,2).

Solution:

We know that the directional derivative of ϕ at the point (x, y, z) is maximum in the direction of the normal to the surface ϕ = constant; i.e in the direction of the vector grad ϕ.

Now; $$\text{grad } \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\text{grad } \phi = 2xy^2z\,\hat{i} + 2x^2yz\,\hat{j} + x^2y^2\hat{k}$$

$$\text{grad } \phi = 4\hat{i} + 4\hat{j} + \hat{k}, \text{ at point } (1,1,2)$$

Hence, the directional derivative of ϕ at the point (1, 1, 2) will be maximum in the direction of the vector $4\hat{i}+4\hat{j}+\hat{k}$.

Also, the magnitude of this maximum directional derivative = modulus of grad ϕ at (1,1,2)

$$= |4\hat{i}+4\hat{j}+\hat{k}| = \sqrt{16+16+1} = \sqrt{33}.$$

The unit vector along the normal to the surface $x^2y^2z = 2$ at point (1,1,2).

$$= \frac{\text{grad } \phi}{|\text{grad } \phi|}, \text{ at } (1,1,2).$$

$$= \frac{4i+4j+k}{|4i+4j+k|} = \frac{4i+4j+k}{\sqrt{16+16+1}}.$$

The unit vector along the normal to the surface $x^2y^2z = 2$ at point $(1,1,2)$

$$= \frac{4i+4j+k}{\sqrt{33}}.$$

8(a)(ii) Prove that $\text{curl}[r^n(a\times r)] = (n+2)r^n a - nr^{n-2}(r\cdot a)r$, where a is a constant vector.

Sol'n: we know that

$$\text{Curl}(\phi\vec{A}) = (\text{grad}\,\phi)\vec{A} + \phi\,\text{curl}\vec{A}$$

putting $\phi = r^n$ and $\vec{A} = \vec{a}\times\vec{r}$

$$\therefore \text{curl}[r^n(\vec{a}\times\vec{r})] = \nabla r^n \times (\vec{a}\times\vec{r}) + r^n\,\text{curl}(\vec{a}\times\vec{r}) \quad \text{--- (1)}$$

Now $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\left(\frac{1}{r}\right)\vec{r} = nr^{n-2}\vec{r}$

$$\therefore \nabla r^n \times (\vec{a}\times\vec{r}) = (nr^{n-2}\vec{r})\times(\vec{a}\times\vec{r})$$

$$= nr^{n-2}\{\vec{r}\times(\vec{a}\times\vec{r})\}$$

$$= nr^{n-2}[(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}]$$

$$= nr^{n-2}[r^2\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}]$$

$$= nr^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r} \quad \text{--- (2)}$$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

where the scalars a_1, a_2, a_3 are all constants.

then $$\vec{a}\times\vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i(a_2z - a_3y) + j(a_3x - a_1z) + k(a_1y - a_2x)$$

$$\therefore \text{curl}(\vec{a}\times\vec{r}) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_2z - a_3y & a_3x - a_1z & a_1y - a_2x \end{vmatrix}$$

$$= (a_1+a_1)\hat{i} + (a_2+a_2)\hat{j} + (a_3+a_3)\hat{k} = 2(a_1i + a_2j + a_3k)$$
$$= 2\vec{a} \quad \text{--- (3)}$$

Substituting (2) & (3) in (1), we get

$$\text{Curl}[r^n(\vec{a}\times\vec{r})] = nr^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r} + r^n(2\vec{a})$$
$$= (n+2)r^n\vec{a} - nr^{n-2}(\vec{r}\cdot\vec{a})\vec{r}$$

8(b) Find κ and τ for the space curve $x=t,\ y=t^2,\ z=t^3$.

Sol'n: The position vector $\vec{r}$, of any point on the given curve is $\vec{r}=t\hat{i}+t^2\hat{j}+t^3\hat{k}$.

$$\frac{d\vec{r}}{dt}=i+2tj+3t^2k$$

$$\frac{d^2\vec{r}}{dt^2}=2\hat{j}+6t\hat{k}$$

$$\frac{d^3\vec{r}}{dt^3}=6\hat{k}$$

Now $\left|\frac{dr}{dt}\right|=\sqrt{1+4t^2+9t^4}$

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}=\begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}=i(12t^2-6t^2)-j(6t-0)+k(2-0)$$

$$=6t^2\hat{i}-6t\hat{j}+2\hat{k}$$

$$\kappa=\frac{\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}=\frac{\sqrt{36t^4+36t^2+4}}{\left(\sqrt{1+4t^2+9t^4}\right)^3}=\frac{2\sqrt{9t^4+9t^2+1}}{\left[\sqrt{9t^4+4t^2+1}\right]^3}$$

$$\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt}\ \ \frac{d^3\vec{r}}{dt^3}\right]=\begin{vmatrix} 1 & 2t & 3t^2 \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix}$$

$$=1(2)(6)=12$$

$$\therefore\ \tau=\frac{\left[\frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\ \ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|^2}$$

$$=\frac{12}{\left[2\sqrt{9t^4+9t^2+1}\right]^2}=\frac{3}{9t^4+9t^2+1}$$

8(c) → If $F=(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}$, evaluate $\iint_S (\nabla\times F)\cdot n\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=16$ above the xy-plane.

Sol'n: the boundary C of S is the circle $x^2+y^2=16, z=0$ lying in the xy-plane. Suppose $x=4\cos t, y=4\sin t, z=0$, $0<t<2\pi$ are parametric equations of C. Then

$$\oint_C F\cdot dr=\oint_C\left[(x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$=\oint_C\left[(x^2+y-4)dx+3xy\,dy+(2xz+z^2)dz\right]$$

$$=\oint_C (x^2+y-4)dx+3xy\,dy \quad (\because \text{ on } C\ z=0 \Rightarrow dz=0)$$

$$=\int_0^{2\pi}\left[(x^2+y-4)\frac{dx}{dt}+3xy\frac{dy}{dt}\right]dt$$

$$=\int_0^{2\pi}\left[(16\cos^2 t+4\sin t-4)(-4\sin t)+3.16\sin t\cos t.4\cos t\right]dt$$

$$=128\int_0^{2\pi}\cos^2 t\sin t\,dt-16\int_0^{\pi}\sin^2 t\,dt+\int_0^{2\pi}\sin t\,dt$$

$$=128.(0)-16.(4)\int_0^{\pi/2}\sin^2 t\,dt+16(0)=-64\left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right)$$

$$=-16\pi \quad \text{——(1)}$$

Now let us evaluate $\iint_S \text{Curl}\,F\cdot\hat{n}\,ds$

$$\text{Curl}\,F=\nabla\times F=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=-z\hat{j}+(3y-1)\hat{k}$$

If S_1 is the plane region bounded by the circle C, then by an application of Gauss divergence theorem, we have

$$\iint_S \text{Curl}\,F\cdot\hat{n}\,ds=\iiint_V \text{div}(\text{curl}\,F)\,dV=0$$

[Here 's' is the surface consisting of S and S_1, the s' is closed surface and let V be the volume bounded by s'.]

$$\therefore \iint_S \text{Curl}\,F\cdot\hat{n}\,ds+\iint_{S_1}(\text{Curl}\,F\cdot\hat{n})ds=0 \quad [\because \text{div}(\text{curl}\,F)=0]$$

$$\Rightarrow \iint_S \text{curl}\,F\cdot\hat{n}\,ds=\iint_{S_1}\text{Curl}\,F\cdot k\,ds \quad [\because \text{ on } S_1,\ \hat{n}=-\hat{k}]$$

$$=\iint_{S_1}(3y-1)\,ds=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}(3r\sin\theta-1)\,r\,d\theta\,dr$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}3r^2\sin\theta\,d\theta-\int_0^{2\pi}\int_0^4 r\,d\theta\,dr=0-\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^4 d\theta \quad \left[\because \int_0^{2\pi}\sin\theta=0\right]$$

$$=-8\left[\theta\right]_0^{2\pi}=-8(2\pi)=-16\pi$$

Q(d) → Verify stoke's theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Sol^n The boundary C of S is a circle in the xy-plane of radius unity and centre origine. Suppose $x = \cos t$, $y = \sin t$, $z=0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C \left[(2x-y)\,dx - yz^2\,dy - y^2z\,dz\right]$$

$$= \oint_C (2x-y)\,dx, \text{ since } z=0 \text{ and } dz=0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} \left[\sin 2t - \frac{1}{2}(1-\cos 2t)\right]dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\,\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\left[\left(-\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}(\pi - 0) + \frac{1}{4}(0-0)\right]$$

$$= \pi \qquad \text{———} \quad (1)$$

Also find $(\nabla \times \vec{F})$

and $(\nabla\times\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$

$= (-2yz+2yz)\hat{i} - (0-0)\hat{j} + (0+1)\hat{k}$

$= \hat{k}$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

$\therefore$ by an application of Gauss divergence theorem, we have,

$$\iint_{S'} \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = 0$$

or $\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds + \iint_{S_1} \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = 0$

($\because$ S' consists of S and S_1)

or $\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds - \iint_{S_1} \text{Curl}\,\vec{F}\cdot\hat{k}\,ds = 0$

[$\because$ on S_1, $\hat{n} = -\hat{k}$]

or $\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \text{Curl}\,\vec{F}\cdot\hat{k}\,ds$

$\therefore \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \text{Curl}\,\vec{F}\cdot\hat{k}\,ds$

$= \iint_{S_1} \hat{k}\cdot\hat{k}\,ds = \iint_{S_1} ds = S_1 = \pi$ —— (2)

Note that S_1 = area of a circle of radius 1
$= \pi(1)^2 = \pi$,

Hence from (1) and (2) Stoke's theorem verified.