IAS
PREVIOUS YEARS QUESTIONS (2019-1983)
SEGMENT-WISE

COMPLEX ANALYSIS

2019

- Suppose f(z) is analytic function on a domain D in $\not\subset$ and satisfies the equation

  Im f(z) = (Re f(z))^2, Z $\in$ D. Show that f(z) is constant in D. **[10]**

- Show that an isolated singular point z_0 of a function f(z) is a pole of order m if and only if f(z) can be written in the form $f(z) = \dfrac{\phi(z)}{(z - z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

  Moreover $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if m ≥ 1. **[15]**

- Evaluate the integral $\int_C \text{Re}(z^2)\,dz$ from 0 to 2 + 4i along the curve C where C is a parabola y = x^2. **[10]**

- Obtain the first three terms of Laurent series expansion of the function $f(z) = \dfrac{1}{(e^z - 1)}$ about the point z = 0 valid int he region 0 < |z| < 2 π. **[10]**

2018

1. Prove that the function: $u(x,y)=(x-1)^3-3xy^2+3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z. **(10)**

3. Show by applying the residue theorem that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0.$ **(15)**

4. Find the Laurent's series which represent the function $\dfrac{1}{(1+z^2)(z+2)}$ when
   (i) $|z|<1$
   (ii) $1<|z|<2$
   (iii) $|z|>2$ **(15)**

2017

- Using contour integral method, prove that $\displaystyle\int_0^\infty \frac{x \sin mx}{a^2+x^2}\,dx = \frac{\pi}{2}e^{-ma}$ **(15)**

- For a function f : $\mathbb{C}\to\mathbb{C}$ and n ≥ 1, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)}$ = f. Let f be an entire function such that for some $n \geq 1,\ f^{(n)}\left(\dfrac{1}{k}\right) = 0$ for all k = 1, 2, 3,....... Show that f is a polynomial. **(15)**

- Let f = u + iv be an analytic function on the unit disc D = {z ∈ $\mathbb{C}$: |z| < 1}. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}$ at all points of D. **(15)**

- Determine all entire functions f(z) such that 0 is a removable singularity of $f\left(\dfrac{1}{z}\right)$. **(10)**

2016

- Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function ? Prove your claim. If yes, find its conjugate harmonic function u(x, y) and hence obtain the analytic function whose real and imaginary parts are u and v respectively. **(10)**

- Let $\gamma : [0,1] \to \mathbb{C}$ be the curve

  $\gamma(t) = e^{2\pi i t}, 0 \leq t \leq 1$.

  Find, giving justifications, the value of the contour integral

  $\displaystyle\int_\gamma \frac{dz}{4z^2-1}$ **(15)**

- Prove that every power series represents an analytic function inside its circle of convergence. **(20)**

2015

- Show that the function $v(x, y) = \ln(x^2+y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also find the corresponding analytic function $f(z) = u + iv$ in terms of z.

- Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about the point $z = 0$.
- State Cauchy's residue theorem. Using it, evaluate the integral $\int_C \dfrac{e^z+1}{z(z+1)(z-i)^2}dz; C: |z| = 2$.

2014

- Prove that the function $f(z) = u + iv$, where
$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, z \neq 0; f(0) = 0$$
satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist.
- Expand in Laurent series the function
$f(z) = \dfrac{1}{z^2(z-1)}$ about z = 0 and z = 1.
- Evaluate the integral $\int_0^{\pi} \dfrac{d\theta}{\left(1+\frac{1}{2}\cos ,\right)^2}$ using residues.

2013

- Prove that if $b\,e^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - b\,e^z$ has n zeroes in the unit circle.
- Using Cauchy's residue theorem, evaluate the integral
$$I = \int_0^{\pi} \sin^4\theta\, d\theta$$

2012

- Show that the function defined by
$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$
is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. **(12)**
- Use Cauchy integral formula to evaluate $\int_C \dfrac{e^{3z}}{(z+1)^4}dz$, where C is the circle $|z| = 2$. **(15)**
- Expand the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for
  (i) $1 < |z| < 3$
  (ii) $|z| > 3$
  (iii) $0 < |z+1| < 2$
  (iv) $|z| < 1$ **(15)**
- Evaluate by Contour integration
$$I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}, a^2 < 1.$$ **(15)**

2011

- Evaluate by Contour integration, $\int_0^1 \dfrac{dx}{\left(x^2-x^3\right)^{1/3}}$. **(15)**
- Find the Laurent Series for the function $f(z) = \dfrac{1}{1-z^2}$ with centre z=1. **(15)**
- Show that the series for which the sum of first n terms $f_n(x) = \dfrac{nx}{1+n^2x^2}, 0 \le x \le 1$ cannot be differentiated term-by-term at x=0. What happens at $x \neq 0$? **(15)**
- If f(z)=u+iv is an analytic function of z=x+iy and $u - v = \dfrac{e^y - \cos x + \sin x}{\cos hy - \cos x}$, find f(z) subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$. **(12)**

2010

- Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of u(x, y). Hence find the analytic function f for which u(x, y) is the real part. **(12)**
- (i) Evaluate the line integral $\int_c f(z)dz.$ Where $f(z) = z^2$, c is the boundary of the triangle with vertices A (0, 0), B (1, 0), C (1, 2) in that order.
  (ii) Find the image of the finite vertical strip R : x = 5 to x =9, $-\pi \le y \le \pi$ of z – plane under the exponential function. **(15)**

- Find the Laurent series of the function

$$f(z)=\exp\left[\frac{\lambda}{2}\left(z-\frac{1}{z}\right)\right] \text{ as } \sum_{n=-\infty}^{\infty} c_n z^n$$

$for\ 0<|z|<\infty$

Where $C_n=\frac{1}{\pi}\int_0^{\pi}\cos(n\phi-\lambda\sin\phi)d\phi, n=0,\pm1,\pm2,\ldots,$

with λ a given complex number and taking the unit circle C given by $z=e^{i\phi}(-\pi\le\phi\le\pi)$ as contour in this region. **(15)**

2009

- Let $f(z)=\frac{a_0+a_1z+.......+a_{n-1}z^{n-1}}{b_0+b_1z+.......+b_nz^n}, b_n\neq 0.,$

Assume that the zeroes of the denominator are simple. Show that the sum of the residues of f(z) at its poles is equal to $\frac{a_{n-1}}{b_n}$. **(12)**

- If α, β, γ are real numbers such that $\alpha^2>\beta^2+\gamma^2$

Show that:

$$\int_0^{2\pi}\frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta}=\frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}} \quad \textbf{(30)}$$

2008

- Find the residue of $\frac{\cot z\cot hz}{z^3}$ at z = 0. **(12)**

- Evaluate

$$\int_C\left[\frac{e^{2z}}{z^2(z^2+2z+2)}+\log(z-6)+\frac{1}{(z-4)^2}\right]dz$$

where c is the circle |Z|=3. State the theorem you use in evaluating above integral. **(15)**

- Let f(z) be entire function satisfying $f(z)\ "\ k\,|z|^2$ for some +ve constant *k* and all *z*. show that $f(z)=az^2$ for some constant a. **(15)**

2007

- Prove that the function f defined by

$$f(z)=\begin{cases}\frac{z^5}{|z|^4}, z\neq 0\\ 0,\ z=0\end{cases}$$ is not differentiable at z = 0

(12)

- Evaluate (by using residue theorem)

$$\int_0^{2\pi}\frac{d\theta}{1+8\cos^2\theta}. \quad \textbf{(15)}$$

2006

- With the aid of residues, evaluate

$$\int_0^{\pi}\frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2};-1<a<1. \quad \textbf{(15)}$$

- If f (z) = u + iv is an analytic function of the complex variable z and $u-v=e^x(\cos y-\sin y)$ determine f(z) in terms of z. **(12)**

- Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for (i) 1< |z| <3 (ii) |z| >3 (iii) |z| < 1. **(30)**

2004

- If all zeros of a polynomial p(z) lie in a half plane then show that zeros of the derivative p'(z) also lie in the same half plane. **(15)**

- Using contour integration , evaluate

$$\int_0^{2\pi}\frac{\cos^2 3\theta d\theta}{1-2p\cos 2\theta+p^2}, 0<p<1. \quad \textbf{(15)}$$

2003

- Use the method of contour integration to prove that

$$\int_0^{\pi}\frac{a d\theta}{a^2+\sin^2\theta}=\frac{\pi}{\sqrt{1+a^2}};(a>0). \quad \textbf{(15)}$$

2002

- Suppose that f and g are two analytic functions on the set ℂ of all complex numbers with $f\left(\frac{1}{n}\right)=g\left(\frac{1}{n}\right)$ for n=1,2,3,….. then show that f(z)=g(z) for each *z* in ℂ . **(12)**

- Show that when 0 < | z-1 | < 2, the function $f(z)=\frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of z-1 as

$$-\frac{1}{2(z-1)}-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}. \quad \textbf{(15)}$$

2001

- Prove that the Riemann Zeta function ζ defined by $\xi(z)=\sum_{n=1}^{\infty} n^{-z}$ converges for Re z > 1 and converges uniformly for $\operatorname{Re} z \geq 1+\in$ where $\in>0$ is arbitrary small. **(12)**
- Show that $\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx=\frac{\pi}{\sqrt{2}}$.

2000

- Suppose $f(\xi)$ is continuous on a circle C. show that $\int_c \frac{f(\xi)}{(\xi-z)}d\xi$ as z varies inside of 'C', is differentiable under the integral sign. Find the derivative hence or otherwise derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C. **(30)**

1999

- Examine the nature of the function $f(z)=\frac{x^2y^5(x+iy)}{x^4+y^{10}}, z\neq 0$ $f(0)=0$ in a region including the origin and hence show that Cauchy – Riemann equations are satisfied at the origin but f(z) is not analytic there.
- For the function $f(z)=\frac{-1}{z^2-3z+2}$, find Laurent series for the domain (i) $1<|z|<2$ (ii) $|z|>2$ show further that $\oint_c f(z)dz=0$ where 'c' is any closed contour enclosing the points z=1 and z=2 .
- Using residue theorem show that $\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}dx=\frac{\pi}{2}e^{-a}\sin a;(a>0)$ ***(1984, 1998)***
- The function f(z) has a double pole at z=0 with residue 2, a simple pole at z=1 with residue 2, is analytic at all other finite points of the plane and is bounded as $|z|\to\infty$. If f(2)=5 and f(-1) = 2, find f(z).
- What kind of singularities the following functions have?

  (i) $\frac{1}{1-e^z}$ $at\ z=2\pi i$

  (ii) $\frac{1}{\sin z-\cos z}$ $at\ z=\frac{\pi}{4}$

  (iii) $\frac{\cot \pi z}{(z-a)^2}$ $at\ z=a\ and\ z=\infty$.

  In case (iii) above what happens when 'a' is an integer.(including a = 0)?

1998

- Show that the function $f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, z\neq 0$
- $f(0)=0$ is continuous and C-R conditions are satisfied at z=0, but $f'(z)$ does not exist at z=0.
- Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z=-2$. Specify the region of convergence and the nature of singularity at z = -2
- By using the integral representation of $f^n(0)$, prove that $\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi i}\oint_c \frac{x^n e^{xz}}{n!z^{n+1}}dz$, where 'c' is any closed contour surrounding the origin. Hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2=\frac{1}{2\pi}\int_0^{2\pi}e^{2x\cos\theta}d\theta.$
- Using residue theorem $\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$.

1997

- If $f(z)=\frac{A_1}{z-a}+\frac{A_2}{(z-a)^2}+-----+\frac{A_n}{(z-a)^n}$ find the residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2,..........A_n$, a & b are constant. What is the residue at infinity.
- Find the Laurent series for the function $e^{1/z}$ in $0<|z|\leq\infty$.

Deduce that $\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$

($n = 0, 1, 2, ...$) **(2001)**

- Find the function f(z) analytic with in the unit circle which takes the values $\frac{a-\cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}, 0 \le \theta \le 2\pi$ on the circle.

- Integrating e^{-z^2} along a suitable rectangular contour. Show that $\int_0^{\infty} e^{-x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$.

1996

- Evaluate $\underset{z\to 0}{Lt}\frac{1-\cos z}{\sin(z^2)}$

- Show that z = 0 is not a branch point for the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity ?

- Prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = 0, a_n \ne 0, n \ge 1$ has exactly 'n' roots.

- By using residue theorem, evaluate $\int_0^{\infty}\frac{\log_e(x^2+1)}{x^2+1}dx$

- About the singularity z =-2, find the Laurent expansion of $(z-3)\sin\frac{1}{z+2}$. Specify the region of convergence and nature of singularity at $z = -2$.

1995

- Let $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that 'u' is a harmonic function. Find a harmonic function v such that u+iv is an analytic function of z.

- Find the Taylor series expansion of the function $f(z) = \frac{z}{z^4+9}$ around $z = 0$. Find also the radius of convergence of the obtained series.

- Let 'C' be the circle | Z |=2 described contour clockwise .Evaluate the integral $\int_c \frac{\cosh \pi z}{z(z^2+1)}dz$

- Let $a \ge 0$. Evaluate the integral $\int_0^{\infty}\frac{\cos ax}{x^2+1}dx$ with the aid of residues. **(2006)**

- Let f be analytic in the entire complex plane. Suppose that there exists a constant A > 0 such that $|f(z)| \le A|z|$ for all z. Prove that there exists a complex number 'a' such that f(z) = az for all z.

- Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \ne 0$.

  Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \ne 0$

  $|z_1| < |z_0|$ and $z_1 \ne 0$ show that the series converges uniformly in the disc $\{z : |z| \le |z_1|\}$.

1994

- How many zeros does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$. Posses in (i) the first quadrant (ii) the fourth quadrant.

- Test for uniform convergence in the region $|Z| \le 1$ the series $\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}$.

- Find Laurent series for (i) $\frac{e^{2z}}{(z-1)^3}$ about z=1.

  (ii) $\frac{1}{z^2(z-3)^2}$ about z = 3.

- Find the residues of $f(z) = e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane.

- By means of contour integration, evaluate $\int_0^{\infty}\frac{(\log_e u)^2}{u^2+1}du$.

1993

- In the finite Z- plane show that the function $f(z) = \sec\frac{1}{z}$.

  has infinitely many isolated singularities in a finite interval which includes '0'.

- Prove that (by applying Cauchy integral formula or otherwise)

$$\int_0^{2\pi}\cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}2\pi,$$

where $n = 1, 2, 3, \ldots$

- If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points (1,1) and (2,3) find the value of $\int_C (12z^2 - 4iz)\,dz$
- Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges absolutely for $|z| \le 1$.
- Evaluate $\int_0^{\infty} \frac{dx}{x^6+1}$ by choosing an appropriate contour.

1992

- If $u = e^{-x}$ (x siny-ycosy), find 'v' such that f(z)= u + iv is analytic. Also find f(z) explicitly as a function of z. **(1997)**
- Let f(z) be analytic inside and on the circle C defined by |z| = R and let $z = re^{i\theta}$ be any point inside C. prove that
$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(\mathrm{Re}^{i\phi}\right)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} d\phi .$$
- Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lies between the circles $|z| = 1$ and $|z| = 2$. **(1998,2006)**
- Find the region of convergence of the series whose n-th term is $\frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$
- Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for
$(i)\,|z| > 3 \quad (ii)\,1 < |z| < 3 \quad (iii)\,|z| < 1$ **(2005)**
- By integrating along a suitable contour evaluate $\int_0^{\infty} \frac{\cos mx}{x^2+1} dx$.

1991

- **A** function f(z) is defined for finite values of z by f(0) = 0 and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy Riemann equation are satisfied at the origin. Show also that f(z) is not analytic at the origin.
- If $|a| \ne R$ show that $\int_{|z|=R} \frac{|dz|}{|z-a||z+a|} < \frac{2\pi R}{\left|R^2 - |a|^2\right|}$
- If $J_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta.$ show that
$$e^{\frac{1}{2}\left(z - \frac{1}{z}\right)} = J_0(t) + zJ_1(t) + z^2 J_2(t) + --- - \frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + ---$$
- Examine the nature of the singularity of e^z at infinity
- Evaluate the residues of the function $\frac{Z^3}{(Z-2)(Z-3)(Z-5)}$ at all singularities and show that their sum is zero.
- By integrating along a suitable contour show that $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin a\pi}$ where 0 < a < 1.

1990

- Let f be regular for |Z| < R , prove that , if 0<r<R, $f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta) e^{-i\theta} d\theta$; where $u(\theta) = \mathrm{Re}\ f(re^{i\theta})$
- Prove that the distance from the origin to the nearest zero of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is at least $\frac{r|a_0|}{M + |a_0|}$. where r is any number not exceeding the radius of the convergence of the series and $M = M(r) = \sup_{|z|=r} |f(z)|$.
- Prove that $\int_{-\infty}^{\infty} \frac{x^4}{1+x^8} dx = \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{8}$ using residue calculus.
- Prove that if f = u + iv is regular through out the complex plane and au + bv-c ≥ 0 for suitable constants a,b,c then f is constant.
- Derive a series expansion of $\log(1+e^z)$ in powers of z.
- Determine the nature of singular points $\sin\left(\frac{1}{\cos 1/z}\right)$ and investigate its behaviour at $z = \infty$.

1989

- Find the singularities of $\sin\left(\frac{1}{1-z}\right)$ in the complex plane.

1988

- By evaluating $\int \frac{dz}{z+2}$ over a suitable contour C, Prove that $\int_0^\pi \frac{1+2\cos\theta}{5+4\cos\theta}d\theta$ **(1997)**
- If f is analytic in $|Z| \leq R$ and x, y lie inside the disc, evaluate the integral $\int_{|z|=R} \frac{f(z)dz}{(z-x)(z-y)}$ and deduce that a function analytic and bounded for all finite z is a constant.
- If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R and prove that $\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$
- Evaluate $\int_C \frac{Ze^z}{(z-a)^3}$, if a lies inside the closed contour C.
- Prove that $\int_0^\infty e^{-x^2}\cos(2bx)\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2};(b>0)$ by the integrating e^{-z^2} along the boundary of the rectangle |x| d" R,0 d" y d"b. **(1997)**
- Prove that the coefficients C_n of the expansion $\frac{1}{1-z-z^2}=\sum_{n=0}^{\infty} C_n z^n$ satisfy $C_n = C_{n-1}+C_{n-2}$, $n \geq 2$ Determine C_n.

1987

- By considering the Laurent series for $f(z)=\frac{1}{(1-z)(z-2)}$ prove that if 'C' be a closed contour oriented in the contour clockwise direction, then $\int_C f(z)\,dx = 2\pi i$
- State and prove Cauchy's residue theorem.
- By the method of contour integration , show that $\int_0^\infty \frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-x}}{2a}, a>0.$

1986

- Let f(z) be single valued and analytic with in and on a closed curve C. If z_0 is any point interior to C, then show that $f(z_0)=\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}dz$,where the integral is taken in the +ve sense around C.
- By contour integration method show that
  (i) $\int_0^\infty \frac{dx}{x^4+a^4}=\frac{\pi\sqrt{2}}{4a^3}$, where a > 0.
  (ii) $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

1985

- Prove that every power series represents an analytic function within its circle of convergence.
- Prove that the derivative of a function analytic in a domain is itself an analytic function.
- Evaluate , by the method of contour integration $\int_0^\infty \frac{x\sin ax}{x^2-b^2}dx$.

1984

- Evaluate by contour integration method :
  (i) $\int_0^\infty \frac{x\sin mx}{x^4+a^4}dx$
  (ii) $\int_0^\infty \frac{x^{a-1}\log x}{1+x^2}dx$ **(1998, 1999)**
- Distinguish clearly between a pole and an essential singularity. If z=a is an essential singularity of a function f(z) , then for an arbitrary positive integers $\eta, \in and\ \rho$, prove that $\exists$ a point z , such that $0<|z-a|<\rho\ for\ which \left|f(z)-\eta\right|<\in$.

1983

- Obtain the Taylor and Laurent series expansions which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions (i) |z| < 2 (ii) 2 < |z| < 3 (iii) |z| > 3.
- Use the method of contour integration to evaluate $\int_0^\infty \frac{x^{a-1}}{1+x^2}dx, 0<a<2$

IFoS
PREVIOUS YEARS QUESTIONS (2019-2000) SEGMENT-WISE

COMPLEX ANALYSIS
(ACCORDING TO THE NEW SYLLABUS PATTERN) PAPER - II

2019

- Using Cauchy's Integral formula, evaluate the integral $\oint_c \frac{dz}{\left(z^2+4\right)^2}$ where c : $|z-i|=2$. **(08)**
- If f(z) is analytic in a domain D and |f(z)| is a non-zero constant in D, then show that f(z) is constant in D. **(15)**
- Classify the singular point z = 0 of the function $f(z)=\frac{e^z}{z+\sin z}$ and obtain the principal part of the Laurent series expansion of f(z). **(15)**

2018

- If $u=(x-1)^3-3xy^2+3y^2$, determine v so that $u+iv$ is a regular function of $x+iy$ **(10)**
- Evaluate the integral $\int_0^{2\pi}\cos^{2n}\theta\, d\theta$, where n is a positive integer. **(10)**

2017

- If f(z) = u(x, y) + iv(x, y) is an analytic function of z = x+ iy and $u+2v=x^3-2y^3+3xy(2x-y)$ then find f(z) in terms of z. **(8)**
- Prove by the method of contour integration that $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}d\theta=0$. **(10)**
- Find the sum of residues of $f(z)=\frac{\sin z}{\cos z}$ at its poles inside the circle |z| = 2. **[8]**

2016

- Find the analytic function of which the real part is $e^{-x}\left\{\left(x^2-y^2\right)\cos y+2xy\sin y\right\}$. **(8)**
- Find the Laurent series for the function $f(z)=\frac{1}{1-z^2}$ with centre z = 1. **(10)**
- Evaluate by Contour integration $\int_0^{\pi}\frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$. **(10)**

2015

- Let u(x, y) = cos x sinh y. Find the harmonic conjugate v(x, y) of u and express u(x, y) + i v (x, y) as a function of z = x + iy. **(8)**
- Evaluate the integral $\int_r \frac{z^2}{(z^2+1)(z-1)^2}dz$, where r is the circle |z| = 2 **(12)**
- Show that $\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx=\frac{\pi}{\sqrt{2}}$ by using contour integration and the residue theorem. **(15)**

2014

- Using Cauchy integral formula, evaluate $\int_C \frac{z+2}{(z+1)^2(z-2)}dz$ where C is the circle $|z-i|=2$
- Find the constants a, b, c such that the function $f(z)=2x^2-2xy-y^2+i(ax^2-bxy+cy^2)$ is analytic for all $z(=x+iy)$ and express $f(z)$ in terms of z. **(8)**
- Evaluate : $\int_C \frac{z}{z^4-6z^2+1}dz$ when C is the circle $|z-i|=2$ **(8)**
- Find the bilinear transformation which map the points -1, ∞, i into the points–
  (i) i, 1, 1 + i
  (ii) ∞, i, 1
  (iii) 0, ∞, 1 **(15)**
- Find the Laurent series expansion at z=0 for the function $f(z)=\frac{1}{z^2(z^2+2z-3)}$ in the regions *(i)* 1<|z|<3 and *(ii)* |z| > 3. **(15)**

2013

- Construct an analytic function $f(z) = u(x, y) + iv(x, y)$, where $v(x, y) = 6xy - 5x + 3$. Express the result as a function of z.
- Evaluate $\oint_c \frac{e^{2z}}{(z+1)^4} dz$ where c is the circle $|z| = 3$.
- Find Laurent series about the indicated singularity. Name the singularity and give the region of convergence.

  $\frac{z - \sin z}{z^3}; z = 0.$

2012

- Evaluate the integral

  $$\int_{2-i}^{4+i} (x + y^2 - ixy)dz$$

  along the line segment AB joining the points A(2,–1) and B(4, 1). **(10)**
- Showthatthefunction $u(x, y) = e^{-x}(x\cos y + y\sin y)$

  is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function f(z). **(13)**
- Using the Residue Theorem, evaluate the integral

  $$\int_c \frac{e^z - 1}{z(z-1)(z+i)^2} dz,$$

  where C is the circle $|z| = 2$ **(13)**

2011

- Expand the function

  $$f(z) = \frac{2z^2 + 11z}{(z+1)(z+4)}$$

  in a Laurent's series valid for $2 < z < 3$. **(10)**
- Examine the convergence of

  $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ and evaluate, if possible. **(10)**
- State Cauchy's residue theorem. Using it, evaluate the integral

  $$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)} dz$$

  Counterclockwise around the circle C:|z+1|=4. **(13)**

2010

- Determine the analytic function

  $f(z) = u + iv$ if $v = e^x(x\sin y + y\cos y)$ **(10)**
- Using the method of contour integration, evaluate

  $$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$$ **(14)**
- Obtain Laurent's series expansion of the function

  $f(z) = \frac{1}{(z+1)(z+3)}$ in the region 0 < |z + 1| Z< 2

  (13)

2009

- Evaluate

  $$\int_C \frac{2z+1}{z^2+z} dz$$

  By Cauchy's integral formula, where C is $|z| = \frac{1}{2}$

  (10)
- Determine the analytic function $w = u + iv$, is

  $$u = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$ **(13)**
- Evaluate by contour integration

  $$\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}, 0 < a < 1$$ **(13)**

2008

- Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + zi$. Along

  the curve given by $z = t^2 + it$. **(10)**
- Expand in a Laurent's series the function

  $f(z) = \frac{1}{(z-1)z^2}$ about $z = 0$. **(13)**
- Find the residue of $f(z) = \tan z$ at $\pi/2$. **(13)**

2007

- If$f(z)$=u+ivisanalyticand $u = e^{-x}(x\sin y - y\cos y)$

  then find v and f(z). **(10)**
- Applying Cauchy's criterion for convergence, show that the sequence (S_n) defined by $S_n = 1 + 1/2 + 1/3 + ... +$ is not convergent. **(13)**
- Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent

  series valid for $(i)\ 1 <| z |> 3$. $(ii)\ |z| > 3$. **(13)**

❖ Using residue theorem, evaluate

$$\int_0^{2\pi} \frac{d\theta}{(3-2\cos\theta+\sin\theta)}$$

2005

❖ If f is analytic, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ **(10/2006)**

❖ Show that the transformation $w=\frac{5-4z}{4z-2}$ maps unit circle $|z| = 1$ onto a circle of radius unity and centre at $-1/2$ **(13/2006)**

❖ Use contour integration technique to find the value of $\int_0^{2p} \frac{d\theta}{2+\cos\theta}$ **(14/2006)**

2004

❖ Investigate the continuity at (0, 0) of the function

$$f(x,y)=\begin{cases}\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$ (10)

❖ Find the analytic function $f(z)=u(x,y)+iv(x,y)$ for which $u-ve^x(\cos y-\sin y)$.

❖ Find the bilinear transformation that maps $z=1,0,\infty$ to $w=0,-\infty,1$ respectively. (13)

❖ Find the singular points with their nature and the residues there at of $f(z)=\frac{\cot\pi z}{\left(z-1/3\right)^2}$ (13)

❖ Prove that a function analytic for all finite values of z and bounded, is a constant. (13)

2003

❖ If $w=f(z)=u(x,y)+iv(x,y), z=x+iy,$ is analytic in a domain, show that $\frac{\partial w}{\partial \bar{z}}=0$. Hence or otherwise, show that $\sin(x+i3y)$ cannot be analytic

❖ Discuss the transformation $w=z+\frac{1}{z}$ and hence, show that

(1) a circle in z-plane is mapped on an ellipse in the w-plane

(2) a line in the z-plane is mapped into a hyperbola in the w-plane. **(13)**

❖ Find the Laurent series expansion of the function $f(z)=\frac{z^2-1}{(z+2)(z+3)}$ Valid in the region $2<|z|<3$. **(13)**

2002

❖ If $f(z)$ has a simple pole with residue K at the origin and is analytic on $0<|z|\leq|$ Show that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)(z-b)}dz=\frac{f(a)-f(b)}{a-b}+\frac{K}{ab}$$

Where $0 < a, b, < 1$ and C is the circle $|z| = 1$.

❖ If $f(a)=\oint_C \frac{3z^2+7z+1}{z-a}dz$ Where C is the circle $|z| = 2$; Find

$(i)\ f(1-i)$; $(ii)\ f''(1-i)$; $(iii)\ f(1+i)$ **(12)**

❖ Under the bilinear transformation $w=\frac{3-z}{z-2}$

Find the images of

(1) $\left|z-\frac{5}{2}\right|=1/2$ *and*

(2) $\left|z-\frac{5}{2}\right|<1/2$ *in the W – plane.*

2001

❖ Compute the Taylor series around z = 0 and give the radius of convergence for $\frac{z}{z-1}$

❖ Show that the function $f(z)=\sqrt{xy}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied **(13)**

- By using the Residue Theorem evaluate the integral
$$\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2} \quad \textit{Where} \quad 0<a<1. \qquad \textbf{(14)}$$

2000

- Expand the function $f(z)=\log(z+2)$ in a power series and determine its radius of convergence.
- Prove that the function $f(z)=u+iv$

Where $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$

$f(0)=0$
Satisfies Cauchy-Riemann equations at the origin, but $f'(0)$ does not exist.

❖❖❖