IFoS-2011 → paper II

5) (b) For the data

x :	0	1	2	5
$f(x)$:	2	3	12	147

Find the cubic function of x.

⇒ There exists unequal interval so we use Lagrange interpolation.

Lagrange interpolation formula is,

$$L(x) = \omega(x) \sum_{r=0}^{n} \frac{f(x)}{(x-x_r)\,\omega'(x_r)} = \omega(x) \sum_{r=0}^{n} \frac{y_r}{D_r}$$

where, $\omega(x) = (x-x_0)(x-x_1) \cdots (x-x_r) \cdots (x-x_n)$

and $D_r = (x-x_r)(x_r-x_0)(x_r-x_1) \cdots (x_r-x_{r-1})(x_r-x_{r+1}) \cdots (x_r-x_n)$

Here, $x_0=0$, $x_1=1$, $x_2=2$, $x_3=5$

$f(x_0)=2$, $f(x_1)=3$, $f(x_2)=12$, $f(x_3)=147$

				D_r	y_r	y_r/D_r
$(x-x_0)=x$	$x_0-x_1=-1$	$x_0-x_2=-2$	$x_0-x_3=-5$	$-10x$	2	$-1/5x$
$(x_1-x_0)=1$	$x-x_1=x-1$	$x_1-x_2=-1$	$x_1-x_3=-4$	$4(x-1)$	3	$3/4(x-1)$
$(x_2-x_0)=2$	$x_2-x_1=1$	$x-x_2=x-2$	$x_2-x_3=-3$	$-6(x-2)$	12	$-2/(x-2)$
$(x_3-x_0)=5$	$x_3-x_1=4$	$x_3-x_2=3$	$x-x_3=x-5$	$60(x-5)$	147	$49/20(x-5)$

and $\omega(x) = x(x-1)(x-2)(x-5)$

$$\therefore L(x) = x(x-1)(x-2)(x-5)\left[-\frac{1}{5x} + \frac{3}{4(x-1)} - \frac{2}{(x-2)} + \frac{49}{20(x-5)}\right]$$

$$= \frac{1}{20}\big[-4(x-1)(x-2)(x-5) + 15(x-1)x(x-5) - 40x(x-1)(x-5) + 49(x-1)x(x-2)\big]$$

$$= \frac{1}{20}\left[20x^3 + 20x^2 - 20x + 40\right]$$

$$= x^3 + x^2 - x + 2$$

5) (c) Solve by Gauss-Jacobi method of iteration the equations,

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

(Correct upto two decimal places)

⇒ The given system is diagonally dominant.
Now, we write the iteration formula for Gauss-Jacobi method as:

$$x^{(K+1)} = \frac{1}{27}\left[85 - 6y^{(K)} + z^{(K)}\right]$$
$$y^{(K+1)} = \frac{1}{15}\left[72 - 6x^{(K)} - 2z^{(K)}\right]$$
$$z^{(K+1)} = \frac{1}{54}\left[110 - x^{(K)} - y^{(K)}\right]$$

we now consider an initial arbitrary solution as,

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0$$

$$\therefore x^{(1)} = \frac{1}{27}\left[85 - 6\times 0 + 0\right] = 3.148$$
$$y^{(1)} = \frac{1}{15}\left[72 - 6\times 0 - 2\times 0\right] = 4.800$$
$$z^{(1)} = \frac{1}{54}\left[110 - 0 - 0\right] = 2.037$$

$$x^{(2)} = \frac{1}{27}\left[85 - 6\times 4.8 + 2.037\right] = 2.157$$
$$y^{(2)} = \frac{1}{15}\left[72 - 6\times 3.148 - 2\times 2.037\right] = 3.269$$
$$z^{(2)} = \frac{1}{54}\left[110 - 3.148 - 4.8\right] = 1.890$$

$$x^{(3)} = \frac{1}{27}\left[85 - 6\times 3.269 + 1.890\right] = 2.492$$
$$y^{(3)} = \frac{1}{15}\left[72 - 6\times 2.157 - 2\times 1.890\right] = 3.685$$
$$z^{3} = \frac{1}{54}\left[110 - 2.157 - 3.269\right] = 1.937$$

$x^{(4)} = \frac{1}{27}[85 - 6\times3.685 + 1.937] = 2.401$

$y^{(4)} = \frac{1}{15}[72 - 6\times2.492 - 2\times1.937] = 3.545$

$z^{(4)} = \frac{1}{54}[110 - 2.492 - 3.685] = 1.923$

$x^{(5)} = \frac{1}{27}[85 - 6\times3.545 + 1.923] = 2.432$

$y^{(6)} = \frac{1}{15}[72 - 6\times2.401 - 2\times1.923] = 3.583$

$z^{(5)} = \frac{1}{54}[110 - 2.401 - 3.545] = 1.927$

$x^{(6)} = \frac{1}{27}[85 - 6\times3.583 + 1.927] = 2.423$

$y^{(6)} = \frac{1}{15}[72 - 6\times2.432 - 2\times1.927] = 3.571$

$z^{(6)} = \frac{1}{54}[110 - 2.432 - 3.583] = 1.926$

$x^{(7)} = \frac{1}{27}[85 - 6\times3.571 + 1.926] = 2.426$

$y^{(7)} = \frac{1}{15}[72 - 6\times2.423 - 2\times1.926] = 3.574$

$z^{(8)} = \frac{1}{54}[110 - 2.423 - 3.571] = 1.926$

$\therefore$ $x = 2.42$, $y = 3.57$, $z = 1.92$, correct upto two decimal places.

6)(c) Find the smallest positive root of the equation, $x^3 - 6x + 4 = 0$ correct to four decimal places using Newton-Raphson method. From this root, determine the positive square root of 3 Correct upto 4 decimal places.

$\Rightarrow$ Let $f(x) = x^3 - 6x + 4$, clearly there are 2 (+ve) roots.

$f(-1) = 9 > 0$, $f(0) = 4 > 0$,

$f(1) = -1 < 0$, $f(2) = 0$ & $f(3) = 13 > 0$

so, there ~~are~~ exists two roots between 0, 1 and 1, 3. but here we find smallest positive root so, we take the points 0 and 1.

Now taking $x_0 = 0$, we compute the successive iteration:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0	4	−6	0.66667	0.66667
1	0.66667	0.29628	−4.66667	0.06349	0.73016
2	0.73016	0.00831	−4.40060	0.00188	0.73204
3	0.73204	0.00005	−4.39235	0.00001	0.73205

∴ 0.7320 is a root of $x^3 - 6x + 4 = 0$, correct upto four decimal places.

2nd part

let $x = \sqrt{3} \Rightarrow x^2 - 3 = 0$

let $g(x) = x^2 - 3$

Then, we get the iteration formula,

$$x_{n+1} = \frac{x_n^2 + 3}{2x_3}$$

Taking $x_0 = 0.7320$ we have,

n	x_n	x_{n+1}
0	0.7320	2.41518
1	2.41518	1.82866
2	1.82866	1.73460
3	1.73460	1.73205
4	1.73205	1.73205

∴ $\sqrt{3} = 1.7320$, correct upto 4 decimal places.

7) (c) The velocity of a particle at time t is as follows:

t (seconds) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	36	60	94	136

Find its displacement at the 12th second and acceleration at the 2nd seconds.

$\Rightarrow$ $v = \frac{ds}{dt} \Rightarrow \int ds = \int v\,dt \Rightarrow s = \int_0^{12} v\,dt$

here $h = 2$ seconds

t_i $i=0$ to 6	v_i $i=0$ to 6	v_i $i=0,6$	v_i $i=1,3,5$	v_i $i=2,4$
$t_0=0$	4	4	—	—
$t_1=2$	6	—	6	—
$t_2=4$	16	—	—	16
$t_3=6$	36	—	36	—
$t_4=8$	60	—	—	60
$t_5=10$	94	—	94	—
$t_6=12$	136	136	—	—
		$\sum v_i = 140 (= Y_0)$	$\sum v_i = 136 (= Y_1)$	$\sum v_i = 76 (= Y_2)$

Now by Simpson $1/3$rd rule,

$$S = \int_0^{12} v\,dt = \frac{h}{3}\left[v_0 + v_6 + 4(v_1+v_3+v_5) + 2(v_2+v_4)\right]$$

$$= \frac{h}{3}\left[Y_0 + 4Y_1 + 2Y_2\right]$$

$$= \frac{2}{3}\left[140 + 4\times 136 + 2\times 76\right]$$

$$= 557.33 \text{ m}$$