* Improper Integrals *

Introduction:—

The Concept of Riemann integrals requires that the range of integration is finite and the integrand 'f' remains bounded in that domain.

If either (or both) of these assumptions is not satisfied, it is necessary to attach a new interpretation to the integral.

In Case the integrand 'f' becomes infinite in the interval $a \le x \le b$; i.e. f has points of infinite discontinuity (Singular points) in $[a,b]$ (or) the limits of integration a or b (or both) become infinite, the Symbol $\int_a^b f(x)dx$ is Called an improper integral or (infinite (or) generalised) integral.

Ex:-

$$\int_1^\infty \frac{1}{x^2}dx,\ \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx,\ \int_0^1 \frac{1}{x(1-x)}dx,\ \int_{-1}^{\infty}\frac{1}{x^2}dx$$

are all improper integrals.

— The integrals which are not improper are Called proper integrals.

Ex:- $\int_0^1 \frac{\sin x}{x}dx$ is a Proper integral. ($\because$ As $x \to 0$ $f(x) = \frac{\sin x}{x} \to 1$)

→ ~~In the definite~~ integral $\int_a^b f(x)dx$ if either a or b (or both) are infinite So that the interval of integration is unbounded (ie the range of the integration is unbounded) but f is bounded then $\int_a^b f(x)dx$ is Called an improper integral of the first kind.

Ex:- $\int_1^\infty \frac{dx}{x^2}$, $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$, $\int_{-1}^{\infty}\frac{1}{x^2}dx$, $\int_{-\infty}^{0} e^{2x}dx$ are improper integrals of the first kind.

→ In the definite integral $\int_a^b f(x)dx$ if both a and b are finite So that the interval of integration is finite but f has one or more points of infinite discontinuity. i.e. f is not bounded on $[a,b]$ then $\int_a^b f(x)dx$ is Called an improper integral of the Second kind

Ex: $\int_0^1 \frac{1}{x(1-x)}dx$, $\int_0^1 \frac{1}{x^2}dx$, $\int_1^2 \frac{1}{2-x}dx$, $\int_1^4 \frac{1}{(x-1)(4-x)}dx$ are improper integrals of the Second kind.

→ In the definite integral $\int_\alpha^b f(x)dx$ if the interval of the integration is unbounded (so that 'a' or 'b' or both are infinite) and f is also unbounded then $\int_a^b f(x)\,dx$ is called an improper integral of the third kind

Ex:- $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is an improper integral of the third kind.

* Improper Integral as the limit of a proper Integral:

→ when the improper integral is of the first kind, either a (or) b (or) both a and b are infinite but f is bounded.

we define (i) $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx \quad (t>a)$

The improper integral $\int_a^\infty f(x)\,dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to divergent if the limit is $+\infty$ (or) $-\infty$.

— If the integral is neither convergent nor divergent then it is said to be oscillating.

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx \quad (t<b)$

The improper integral $\int_{-\infty}^b f(x)\,dx$ is said to be convergent if the limit on the right handside exists finitely and the integral is said to be divergent if the limit is $+\infty$ (or) $-\infty$

(iii) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

where c is any real numbers.

$= \lim_{t_1\to-\infty} \int_{t_1}^c f(x)\,dx + \lim_{t_2\to\infty} \int_c^{t_2} f(x)\,dx$

The improper integral $\int_{-\infty}^\infty f(x)\,dx$ is said to be convergent if both the limits on the right handside exist finitely and independent of eachother. otherwise, it is said to be divergent.

Note: $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{t\to\infty}\left[\int_{-t}^c f(x)\,dx + \int_c^t f(x)\,dx\right]$

(b) when the improper integral is of the second kind, both a and b are finite but f has one (or more) points of infinite discontinuity on $[a,b]$.

(i) If $f(x)$ becomes infinite at $x=a$ only;

we define $\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_{a+\epsilon}^b f(x)\,dx$

$0<\epsilon<b-a$

The improper integral $\int_a^b f(x)\,dx$ converges if the limit on the right handside exists finitely, otherwise it is said to be divergent.

(ii) If $f(x)$ becomes infinite at $x=b$ only; we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0+} \int_a^{b-\epsilon} f(x)\,dx \qquad 0<\epsilon<b-a$$

the improper integral $\int_a^b f(x)\,dx$ is said to be Convergent if the limit on the right hand exists finitely and the integral is said to be divergent if the limit is $+\infty$ (or) $-\infty$.

(iii) If $f(x)$ becomes infinite at $x=a$ & $x=b$ only.

we define $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$= \lim_{\epsilon_1\to 0+} \int_{a+\epsilon_1}^{c} f(x)\,dx + \lim_{\epsilon_2\to 0+} \int_c^{b-\epsilon_2} f(x)\,dx$$

The improper integral $\int_a^b f(x)\,dx$ is said to be Convergent if both the limits on the right hand exist finitely and independent of each other, otherwise it is said to be divergent.

Note: The improper integral is also defined as $\int_a^b f(x)\,dx = \lim_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \int_{a+\epsilon_1}^{b-\epsilon_2} f(x)\,dx.$

The improper integral exists if the limit exists.

(iv) If $f(x)$ becomes infinite at $x=c$ only where $a<c<b$ and c is an interior point.

we define $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$= \lim_{\epsilon_1\to 0} \int_a^{c-\epsilon_1} f(x)\,dx + \lim_{\epsilon_2\to 0+} \int_{c+\epsilon_2}^{b} f(x)\,dx.$$

The improper integral $\int_a^b f(x)\,dx$ is said to be Convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Similarly,

if the function has a finite number of points of infinite discontinuity, $c_1, c_2, c_3, \ldots, c_m$ with in $[a,b]$, where $a\le c_1 < c_2 < c_3 < \ldots c_m \le b$.

we define the improper integral $\int_a^b f(x)\,dx$ as

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots + \int_{c_m}^{b} f(x)\,dx.$$

and is said to be convergent if all the integrals on the R.H.S are convergent, otherwise it is divergent.

Note(1): If f has infinite discontinuity at an end point of the interval of the integration then the point of infinite discontinuity is approached from within the interval.

i.e., if the interval of integration is $[a,b]$ and

i, f has infinite discontinuity at 'a' then we consider $[a+\epsilon, b]$ as $\epsilon \to 0+$.

ii, f has infinite discontinuity at 'b' then we consider $[a, b-\epsilon]$ as $\epsilon \to 0+$.

Note(2):

A Proper integral is always Convergent.

Note(3):

If $\int_a^b f(x)\,dx$ is convergent then

(i) $\int_a^b k f(x)\,dx$ is convergent ; $k \in \mathbb{R}$.

(ii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where $a<c<b$. and each integral on right hand side is convergent.

Note(4):

For any point c between a & b i.e. $a<c<b$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $\int_c^b f(x)\,dx$ is a proper integral then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^c f(x)\,dx$ converge or diverge together i.e, while testing the integral $\int_a^b f(x)\,dx$ for convergence at 'a' it may be replaced by $\int_a^c f(x)\,dx$ for any convenient 'c' such that $a<c<b$.

Problems:-

Examine the convergence of the improper integral.

(i) $\int_1^\infty \frac{1}{x}\,dx$ (ii) $\int_0^\infty \frac{1}{1+x^2}\,dx$

Sol'n: i) By definition

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} [\log x]_{x=1}^{t} = \lim_{t\to\infty} (\log t - \log 1)$$

$$= \lim_{t\to\infty} (\log t) = \infty$$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

ii, By definition,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} [\tan^{-1}x]_0^t$$

$$= \lim_{t\to\infty} [\tan^{-1}t - \tan^{-1}(0)]$$

$$= \lim_{t\to\infty} [\tan^{-1}t - 0]$$

$= \pi/2$ which is finite.

$\therefore \int_0^\infty \frac{1}{1+x^2}\,dx$ is Convergent.

5. (i) $\int_0^\infty e^{-mx}\,dx \quad (m>0)$

(ii) $\int_a^\infty \frac{x}{1+x^2}\,dx$ (iii) $\int_0^\infty \sin x\,dx$

(iv) $\int_0^\infty \frac{1}{x^2+4a^2}\,dx$ (v) $\int_0^\infty e^{2x}\,dx$

(ii) By definition

$$\int_a^\infty \frac{x}{1+x^2}\,dx = \underset{t\to\infty}{lt}\int_a^t \frac{x}{1+x^2}\,dx \quad (a<t)$$

$$= \underset{t\to\infty}{lt}\ \tfrac{1}{2}\int_a^t \frac{2x}{1+x^2}\,dx$$

$$= \underset{t\to\infty}{lt}\ \tfrac{1}{2}\left[\log(1+x^2)\right]_a^t$$

$$= \underset{t\to\infty}{lt}\ \tfrac{1}{2}\left[\log(1+t^2)-\log(1+a^2)\right]$$

$$= \tfrac{1}{2}\left[\infty-\log(1+a^2)\right]=\infty$$

$\therefore \int_a^\infty \frac{x}{1+x^2}\,dx$ is divergent.

(iii) $\int_0^\infty \sin x\,dx = \underset{t\to\infty}{lt}\int_0^t \sin x\,dx = \underset{t\to\infty}{lt}\left[-\cos x\right]_0^t$

$$= \underset{t\to\infty}{lt}\left[-\cos t+\cos 0\right]$$

$$= \underset{t\to\infty}{lt}\left[-\cos t+1\right]$$

$$= l \quad (\because -1\le \cos t\le 1)$$

which does not exist uniquely.

Since l is finite but not fixed because $\cos t$ oscillates between -1 & $+1$.

$\therefore \int_0^\infty \sin x\,dx$ oscillates.

(iv) $\int_0^\infty \frac{1}{x^2+4a^2}\,dx = \underset{t\to\infty}{lt}\int_0^t \frac{dx}{x^2+(2a)^2}$

$$= \underset{t\to\infty}{lt}\left[\frac{1}{2a}\tan^{-1}\left(\frac{x}{2a}\right)\right]_0^t$$

→ (i) $\int_{\sqrt2}^\infty \frac{dx}{x\sqrt{x^2-1}}$ (ii) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx$

(iii) $\int_1^\infty \frac{x}{(1+2x)^3}\,dx$ (iv) $\int_1^\infty \frac{x}{(1+x)^3}\,dx$

sol'n: (i) $\int_{\sqrt2}^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \underset{t\to\infty}{lt}\int_{\sqrt2}^t \frac{1}{x\sqrt{x^2-1}}\,dx$

$$= \underset{t\to\infty}{lt}\left[\sec^{-1}x\right]_{\sqrt2}^t$$

(ii) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \underset{t\to\infty}{lt}\int_2^t \frac{2x^2}{x^4-1}\,dx$

$$= \underset{t\to\infty}{lt}\int_2^t \frac{2x^2}{(x^2)^2-1}\,dx = \underset{t\to\infty}{lt}\int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2-1)(x^2+1)}\,dx$$

$$= \underset{t\to\infty}{lt}\int_2^t \left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx$$

$$= \underset{t\to\infty}{lt}\left[\tfrac{1}{2}\log\left(\frac{x-1}{x+1}\right)+\tan^{-1}x\right]_2^t$$

$$= \underset{t\to\infty}{lt}\left[\tfrac{1}{2}\left(\log\left(\frac{t-1}{t+1}\right)\right)+\tan^{-1}t-\tfrac{1}{2}\log\left(\tfrac{1}{3}\right)-\tan^{-1}(2)\right]$$

$$= \underset{t\to\infty}{lt}\left[\tfrac{1}{2}\log\left(\frac{1-1/t}{1+1/t}\right)+\tan^{-1}t-\tfrac{1}{2}\log\left(\tfrac{1}{3}\right)-\tan^{-1}(2)\right]$$

$$= \tfrac{1}{2}(0)+\tfrac{\pi}{2}-\tfrac{1}{2}\log\tfrac{1}{3}-\tan^{-1}(2)$$

$$= \frac{\pi}{2}-\tfrac{1}{2}\log\left(\tfrac{1}{3}\right)-\tan^{-1}(2)$$

which is finite.

$\therefore \int_2^\infty \frac{2x^2}{x^4-1}\,dx$ is convergent.

(iii) $\int_1^\infty \frac{x}{(1+2x)^3}dx = \lim_{t\to\infty}\int_1^t \frac{x}{(1+2x)^3}dx$

$= \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(1+2x)-\frac{1}{2}}{(1+2x)^3}dx$

(iv) $\int_1^\infty \frac{x}{(1+x)^3}dx = \lim_{t\to\infty}\int_1^t \frac{(1+x)-1}{(1+x)^3}dx$

→ (i) $\int_1^\infty xe^{-x}dx$ (ii) $\int_0^\infty x^2e^{-x}dx$

(iii) $\int_0^\infty xe^{-x^2}dx$ (iv) $\int_0^\infty x^3e^{-x^2}dx$ (v) $\int_0^\infty x\sin x\,dx$

Sol^n: (ii) $\int_0^\infty x^2e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}dx$

(Integrating by parts)

$= \lim_{t\to\infty}\left[x^2(-e^{-x}) - \int 2x(e^{-x})dx\right]_0^t$

$= \lim_{t\to\infty}\left[-x^2e^{-x}+2(-xe^{-x}-e^{-x})\right]_0^t$

$= \lim_{t\to\infty}\left[-x^2e^{-x}-2e^{-x}(x+1)\right]_0^t$

$= \lim_{t\to\infty}\left[-t^2e^{-t}-2e^{-t}t-2e^{-t}+2\right]$

$= \lim_{t\to\infty}(-t^2e^{-t}) - 2\lim_{t\to\infty}e^{-t}t - \lim_{t\to\infty}e^{-t}$

$+\lim_{t\to\infty}(2)$

$= -\lim_{t\to\infty}\frac{t^2}{e^t} - 2\lim_{t\to\infty}\frac{t}{e^t} - 0 + 2$

(Applying L-Hospitals rule)

$= -\lim_{t\to\infty}\frac{2t}{e^t} - 2\lim_{t\to\infty}\frac{1}{e^t}+2$

$= -\lim_{t\to\infty}\frac{2(1)}{e^t} - 2(0)+2$

$= -2(0)+2 = 2$ which is finite.

∴ $\int_0^\infty x^2e^{-x}dx$ is convergent.

(iii) $\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_0^t xe^{-x^2}dx$ —— (1)

put $x^2 = z$, $2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

and if $x=0$ then $z=0$ of $x=t$ then $z=t^2$

∴ (1) ≡

$\int_0^\infty xe^{-x^2}dx = \lim_{t\to\infty}\int_{z=0}^{t^2} e^{-z}\frac{dz}{2}$

(iv) $\int_0^\infty x^3e^{-x^2}dx = \lim_{t\to\infty}\int_0^\infty x.x^2e^{-x^2}dx$

put $x^2=z$; $2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

→ (i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_0^\infty e^{-x}\sin x\,dx$

(iii) $\int_0^\infty e^{-ax}\cos bx\,dx$

Sol^n: (i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty}\int_1^t \frac{1}{(1+x)\sqrt{x}}dx$ —— (1)

put $\sqrt{x}=z \Rightarrow \frac{1}{2\sqrt{x}}dx = dz$

$\Rightarrow \frac{1}{\sqrt{x}}dx = 2dz$

when $x=1 \Rightarrow z=1$; when $x=t \Rightarrow z=\sqrt{t}$

∴ (1) ≡

$\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{1}{(1+z^2)}(2dz)$

$= \lim_{t\to\infty}\left[2\tan^{-1}z\right]_{z=1}^{\sqrt{t}}$

(iii) $L \int_0^\infty e^{-x} \sin x\, dx = \lim_{t\to\infty} \int_0^t e^{-x}\sin x\, dx$

$$= \lim_{t\to\infty}\left[\frac{e^{-x}}{(-1)^2+1^2}(-1.\sin x - 1.\cos x)\right]_0^t$$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

iii, by using the formula

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

→ (i) $\int_1^\infty \frac{dx}{x(x+1)}$ ii, $\int_1^\infty \frac{dx}{x^2(x+1)}$

(By using partial fractions)

(iii) $\int_1^\infty \frac{\tan^{-1}x}{x^2}dx$ (iv) $\int_0^\infty e^{-\sqrt{x}}dx$

Solⁿ (iii): $\int_1^\infty \frac{\tan^{-1}x}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}dx$ ——— ①

put $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\, d\theta$

Now $\int \frac{\tan^{-1}x}{x^2}dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\, d\theta$

$= \int \theta\, \mathrm{cosec}^2\theta\, d\theta$

$= \theta(-\cot\theta) - \int 1\,(-\cot\theta)d\theta$

$= -\theta\cot\theta + \log\sin\theta$

$$= \frac{-\tan^{-1}x}{x} + \log\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$\left[\because \tan\theta = \frac{x}{1}, \sin\theta = \frac{x}{\sqrt{1+x^2}}\right]$

$$\therefore ① \equiv \int_1^\infty \frac{\tan^{-1}x}{x^2}dx$$

$$\lim_{t\to\infty}\left[\frac{-\tan^{-1}x}{x} + \log\frac{x}{\sqrt{1+x^2}}\right]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-\tan^{-1}t}{t} + \log\frac{t}{\sqrt{1+t^2}} + \frac{\tan^{-1}(1)}{1} - \log\left(\tfrac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{-\pi/2}{\infty} + \log\left(\frac{1}{\sqrt{1+\frac{1}{\infty}}}\right) + \frac{\pi}{4} - \log\left(\tfrac{1}{\sqrt{2}}\right)$$

$= 0 + 0 + \pi/4 + \log(\sqrt{2})$

$= \pi/4 + \tfrac{1}{2}\log 2$ which is finite.

$\therefore \int_1^\infty \frac{\tan^{-1}x}{x^2}$ is convergent.

(iv) $\int_0^\infty e^{-\sqrt{x}}dx = \lim_{t\to\infty}\int_0^t e^{-\sqrt{x}}dx$,

put $\sqrt{x} = z$

$\Rightarrow x = z^2$

$\Rightarrow dx = 2z\,dz$

when $x = 0 \Rightarrow z = 0$

$x = t \Rightarrow z = \sqrt{t}$

$$\therefore \int_0^\infty e^{-\sqrt{x}}dx = \lim_{t\to\infty}\int_{z=0}^{\sqrt{t}} e^{-z}(2z\,dz)$$

→ (i) $\int_{-\infty}^0 e^{2x}dx$ (ii) $\int_{-\infty}^{1} \frac{dx}{1+x^2}$

(iii) $\int_{-\infty}^0 \cosh x\, dx$ $\left[\text{using } \cosh x = \frac{e^x+e^{-x}}{2}\right]$

(iv) $\int_{-\infty}^0 \sinh x\, dx$ $\left[\text{using } \sinh x = \frac{e^x - e^{-x}}{2}\right]$

Solⁿ:- $\int_{-\infty}^0 e^{2x}dx = \lim_{t\to-\infty}\int_t^0 e^{2x}dx.$

→ (i) $\int_{-\infty}^\infty e^{-x}dx$ (ii) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ (iii) $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

(iv) $\int_{-\infty}^\infty \frac{1}{x^2+2x+2}dx$ (v) $\int_{-\infty}^\infty \frac{1}{(1+x^2)^2}dx$

put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$ etc.

Sol'n :- $\int_{-\infty}^{\infty} e^{-x}dx = \int_{-\infty}^{0} e^{-x}dx + \int_{0}^{\infty} e^{-x}\,dx$

$= \lim_{t_1\to-\infty} \int_{t_1}^{0} e^{-x}dx + \lim_{t_2\to\infty} \int_{0}^{t_2} e^{-x}\,dx$

(iv) $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx = \lim_{t_1\to-\infty} \int_{t_1}^{0} \frac{1}{(x+1)^2+1}\,dx$

$+ \lim_{t_2\to\infty} \int_{0}^{t_2} \frac{1}{(x+1)^2+1}\,dx$

$= \lim_{t_1\to-\infty} \left[\tan^{-1}(x+1)\right]_{t_1}^{0} + \lim_{t_2\to\infty} \left[\tan^{-1}(x+1)\right]_{0}^{t_2}$

$= \lim_{t_1\to-\infty} \left[\frac{\pi}{4} - \tan^{-1}(t_1+1)\right]$

$+ \lim_{t_2\to\infty} \left[\tan^{-1}(t_2+1) - \frac{\pi}{4}\right]$

$= \frac{\pi}{4} - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \pi/4$

$= 2\tan^{-1}(\infty)$

$= 2\,\pi/2 = \pi$ (finite)

$\therefore \int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx$ is Convergent.

→ (i) $\int_{0}^{1} \log x\,dx$ (ii) $\int_{0}^{1/e} \frac{dx}{x(\log x)^2}$

(iii) $\int_{0}^{e} \frac{1}{x(\log x)^3}\,dx$ (iv) $\int_{1}^{2} \frac{dx}{x(\log x)}$

Sol'n: (i) 0 is the only point of infinite discontinuity of the integrand 'f' on $[0,1]$.

$\therefore \int_{0}^{1} \log x\,dx = \lim_{\epsilon\to0+} \int_{0+\epsilon}^{1} (\log x).1\,dx$

$= \lim_{\epsilon\to0+} \left[x\log x - x\right]_{\epsilon}^{1}$

$= \lim_{\epsilon\to0+} \left[1(0) - 1 - \epsilon\log\epsilon + \epsilon\right]$

$= \lim_{\epsilon\to0+} \left[-1 - \epsilon\log\epsilon + \epsilon\right]$

$= -1$ $\left(\because \lim_{x\to0+} x^n\log x = 0;\ n>0\right)$

$\therefore \int_{0}^{1} \log x\,dx$ is Convergent.

(ii) Since $\lim_{x\to0+} x(\log x)^n = 0;\ n>0$

$\therefore$ 0 is the only point of infinite discontinuity of the integrand on $[0,1/e]$

$\therefore \int_{0}^{1/e} \frac{1}{(x\log x)^2}\,dx = \lim_{\epsilon\to0+} \int_{0+\epsilon}^{1/e} \frac{1}{x(\log x)^2}\,dx$

$= \lim_{\epsilon\to0+} \int_{\epsilon}^{1/e} (\log x)^{-2}.\frac{1}{x}\,dx$

$= \lim_{\epsilon\to0+} \left[\frac{(\log x)^{-1}}{-1}\right]_{\epsilon}^{1/e}$

$= \lim_{\epsilon\to0+} \left(-\left[\frac{1}{\log 1/e} - \frac{1}{\log\epsilon}\right]\right)$

$= \left(-\left[\frac{1}{\log e} - 0\right]\right)$

$= -[-1-0]$

$= 1$

$\therefore \int_{0}^{1/e} \frac{dx}{x(\log x)^2}$ is Convergent.

6 (i) $\displaystyle\int_0^a \frac{dx}{\sqrt{a-x}}$ (ii) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$

(iii) $\displaystyle\int_0^{\pi/2} \tan\theta\, d\theta$.

(i) 'a' is the only point of infinite discontinuity of the integrand f on $[0,a]$

$$\therefore \int_0^a \frac{dx}{\sqrt{a-x}} = \mathop{Lt}_{\epsilon\to 0+} \int_0^{a-\epsilon} (a-x)^{-1/2}\,dx$$

$$= \mathop{Lt}_{\epsilon\to 0+} \left[\frac{-(a-x)^{-1/2+1}}{-\frac{1}{2}+1}\right]_0^{a-\epsilon}$$

$$= \mathop{Lt}_{\epsilon\to 0+} \left[-2(a-x)^{1/2}\right]_0^{a-\epsilon}$$

$$= \mathop{Lt}_{\epsilon\to 0+} -\left[2(\epsilon)^{1/2} - 2(a)^{1/2}\right]$$

$$= -\left[0-2\sqrt{a}\right]$$

$$= 2\sqrt{a} \quad (\text{finite})$$

$\therefore \displaystyle\int_0^a \frac{dx}{\sqrt{a-x}}$ is Convergent.

(i) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$ (ii) $\displaystyle\int_a^{3a} \frac{dx}{(x-2a)^2}$

(iii) $\displaystyle\int_0^{2a} \frac{dx}{(x-a)^2}$

Sol^n:- (i) The integrand f becomes infinite at $x=0$ and $-1<a<1$.

$\therefore$ 0 is the only point of infinite discontinuity of the integrand f on $[-1,1]$

$$\therefore \int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_0^1 \frac{1}{x^2}\,dx$$

$$= \mathop{Lt}_{\epsilon_1\to 0+} \int_{-1}^{0-\epsilon_1} \frac{1}{x^2}\,dx + \mathop{Lt}_{\epsilon_2\to 0+} \int_{0+\epsilon_2}^{1} \frac{1}{x^2}\,dx$$

$$= \mathop{Lt}_{\epsilon_1\to 0+} \left[-\frac{1}{x}\right]_{-1}^{-\epsilon_1} + \mathop{Lt}_{\epsilon_2\to 0+} \left[-\frac{1}{x}\right]_{\epsilon_2}^{1}$$

$$= \mathop{Lt}_{\epsilon_1\to 0+} \left(\frac{1}{\epsilon_1}-1\right) + \mathop{Lt}_{\epsilon_2\to 0+} \left(-1+\frac{1}{\epsilon_2}\right)$$

$$= (\infty-1) + (-1+\infty) = \infty.$$

$\therefore \displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$ is divergent.

(ii) The integrand f becomes infinite at $x=2a$ and $a<2a<3a$.

(i) $\displaystyle\int_0^4 \frac{1}{x(4-x)}\,dx$ (ii) $\displaystyle\int_0^2 \frac{1}{2x-x^2}\,dx$

(iii) $\displaystyle\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\,dx$ (iv) $\displaystyle\int_0^{\pi} \frac{1}{\sin x}\,dx$ (v) $\displaystyle\int_0^{\pi} \frac{1}{1+\cos x}\,dx$

(i) Both the end points 0 & 4 are the points of infinite discontinuity of the integrand f on $[0,4]$.

$$\therefore \int_0^4 \frac{1}{x(4-x)}\,dx = \mathop{Lt}_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \int_{0+\epsilon_1}^{4-\epsilon_2} \frac{1}{x(4-x)}\,dx \quad \left[(\text{or}) \int_0^1 + \int_1^4 \right.$$

$$\left. = \mathop{Lt}_{\epsilon_1\to 0} \int_{0+\epsilon_1}^{1} + \mathop{Lt}_{\epsilon_2\to 0} \int_1^{4-\epsilon_2}\right]$$

$$= \mathop{Lt}_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \int_{\epsilon_1}^{4-\epsilon_2} \frac{1}{4}\left(\frac{1}{x}+\frac{1}{4-x}\right)dx$$

$$= \mathop{Lt}_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \frac{1}{4}\left[\log x - \log(4-x)\right]_{\epsilon_1}^{4-\epsilon_2}$$

$$= \mathop{Lt}_{\substack{\epsilon_1\to 0+\\ \epsilon_2\to 0+}} \frac{1}{4}\left[\log(4-\epsilon_2) - \log\epsilon_2 - \log\epsilon_1 + \log(4-\epsilon_1)\right]$$

$$= 2\log 4 - (-\infty) - (-\infty) = \infty$$

* Comparison Tests for Convergence at 'a' of $\int_a^b f(x)\,dx$:-

→ Let 'a' be the only point of infinite discontinuity of 'f' on $[a,b]$

The case when 'b' is the only point of infinite discontinuity can be dealt with in the same way.

When the integrand 'f' keeps the same sign, +ve (or) −ve in a small neighbourhood of 'a', we may suppose that f is non-negative there in, for, if negative it can be replaced by $(-f)$, for testing the convergence of $\int_a^b f(x)dx$.

The case $f=0$ being trivial so there is no loss of generality to suppose that f is +ve throughout.

Theorem

A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ at the point 'a', where 'f' is +ve on $[a,b]$, is that there exists a +ve number M (i.e. $M>0$), independent of $\epsilon>0$, such that

$$\int_{a+\epsilon}^{b} f(x)\,dx < M \;;\; 0<\epsilon<b-a.$$

i.e. the improper integral $\int_a^b f(x)dx$ converges iff $\exists$ a $M>0$ and independent of $\epsilon>0$ such that $\int_{a+\epsilon}^{b} f(x)dx < M$ $\forall\, \epsilon$ in $(0, b-a)$

Note:- If for every $M>0$ and some ϵ in $(0, b-a)$,

$\int_{a+\epsilon}^{b} f(x)\,dx > M$, then $\int_{a+\epsilon}^{b} f(x)dx$ is not bounded above.

$\therefore \int_{a+\epsilon}^{b} f(x)dx \longrightarrow \infty$ as $\epsilon \to 0_+$ and hence the improper integral $\int_a^b f(x)dx$ diverges to $+\infty$.

Comparison Test-I :-

If f and g are two +ve functions with $f(x) \le g(x)$ $\forall\, x\in[a,b]$ and 'a' is the only point of infinite discontinuity on $[a,b]$ then

(i) $\int_a^b g(x)\,dx$ is convergent $\Rightarrow \int_a^b f(x)dx$ is convergent

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow \int_a^b g(x)dx$ is divergent.

Comparison Test II (Limit Form):

→ If f and g be two +ve functions on $(a,b]$, 'a' is the only point of infinite discontinuity and

$\lim_{x\to a+} \frac{f(x)}{g(x)} = l$ where 'l' is a non-zero finite number, then the two integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ converge (or) diverge together.

→ Let f and g be two +ve functions on $(a,b]$, 'a' is the point of infinite discontinuity, then

(i) $\lim_{x \to a+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ Converges

$\Rightarrow \int_a^b f(x)\,dx$ converges.

(ii) $\lim_{x \to a+} \frac{f(x)}{g(x)} = +\infty$ and $\int_a^b g(x)\,dx$ diverges.

$\Rightarrow \int_a^b f(x)\,dx$ diverges.

→ Useful Comparison Integral

The improper integral $\int_a^b \frac{1}{(x-a)^n}\,dx$ is Convergent if and only if $n<1$.

Proof: If $n \le 0$ then the integral $\int_a^b \frac{1}{(x-a)^n}$ is proper.

If $n>0$, the integral is improper and 'a' is the only point of infinite discontinuity of the integrand on $[a,b]$.

Case (i): when $n=1$

$$\int_a^b \frac{1}{(x-a)^n}\,dx = \int_a^b \frac{1}{(x-a)}\,dx$$

$$= \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\epsilon \to 0+} \left[\log(x-a)\right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon \to 0+} \left[\log(b-a) - \log\epsilon\right]$$

$$= \log(b-a) - \infty$$

$$= \infty$$

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n=1$.

Case-II: when $n \neq 1$.

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon \to 0+} \int_{a+\epsilon}^b (x-a)^{-n}\,dx$$

$$= \lim_{\epsilon \to 0+} \left[\frac{(x-a)^{1-n}}{1-n}\right]_{a+\epsilon}^b$$

$$= \lim_{\epsilon \to 0+} \left(\frac{1}{1-n}\right)\left[(b-a)^{1-n} - \epsilon^{1-n}\right] \qquad \text{(i)}$$

Subcase I when $n>1$

$\Rightarrow (n-1)>0$

$$\therefore \text{(1)} \equiv \int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon \to 0+} \left(\frac{1}{1-n}\right)\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\epsilon^{n-1}}\right]$$

$$= \left(\frac{1}{1-n}\right)\left[\frac{1}{(b-a)^{n-1}} - \infty\right]$$

$$= \left(\frac{1}{1-n}\right)(-\infty) = \infty \quad (\because 1-n<0)$$

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ diverges if $n>1$.

$\boxed{\therefore \int_a^b \frac{dx}{(x-a)^n} \text{ diverges if } n \ge 1}$

Subcase-II when $0<n<1$

$\Rightarrow 1-n>0$

$$\therefore \text{(1)} \equiv \int_a^b \frac{dx}{(x-a)^n} = \lim_{\epsilon \to 0+} \frac{1}{1-n}\left[(b-a)^{1-n} - \epsilon^{1-n}\right]$$

$$= \left(\frac{1}{1-n}\right)(b-a)^{1-n}$$

which is finite.

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ Converges if $n<1$.

$\therefore \int_a^b \frac{dx}{(x-a)^n}$ is convergent iff $n<1$.

Note: — The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent iff $n<1$.

→ (I) If a is the only point of infinite discontinuity of f on $[a,b]$ and $\lim_{x\to a+}(x-a)^n f(x)$ exists and is non-zero finite then $\int_a^b f(x)\,dx$ Converges iff $n<1$.

(II) If b is the only point of infinite discontinuity of f on $[a,b]$ and $\lim_{x\to b-}(b-x)^n f(x)$ exists and is non-zero finite then $\int_a^b f(x)\,dx$ Converges iff $n<1$.

III) If f is +ve on $(a,b]$, a is the only point of infinite discontinuity then the integral $\int_a^b f(x)\,dx$ converges at a.

If $\exists$ +ve number $n<1$ and a fixed +ve number M such that

$$f(x)\le \frac{M}{(x-a)^n} \quad \forall\, x\in(a,b].$$

Also $\int_a^b f(x)\,dx$ diverges if $\exists$ a number $n\ge 1$ and a fixed +ve number G such that $f(x)\ge \frac{G}{(x-a)^n}$ $\forall\, x\in(a,b]$

Problems:

* Examine the convergence of the integrals (i) $\int_0^1 \frac{1}{\sqrt{x^2+x}}\,dx$ (ii) $\int_1^2 \frac{1}{(1+x)\sqrt{2-x}}\,dx$

(iii) $\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$ (iv) $\int_0^1 \frac{1}{x^{1/2}(1+x^2)}\,dx$

(v) $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$.

Sol^n (i) :- Method ①:

Let $f(x) = \frac{1}{\sqrt{x^2+x}}$

$= \frac{1}{\sqrt{x}\,(\sqrt{x+1})}$

Let $g(x) = \frac{1}{\sqrt{x}}$

$\therefore f,g$ are +ve on $(0,1]$ and 0 is the only point of infinite discontinuity

Now $\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+}\frac{1}{\sqrt{x+1}} = 1$ (non-zero finite number)

$\therefore$ By Comparison Test $\int_0^1 f(x)\,dx$ & $\int_0^1 g(x)\,dx$ are convergent (or) divergent together.

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{\sqrt{x}}\,dx$

$= \int_0^1 \frac{1}{(x-0)^{1/2}}\,dx$ is of the form $\int_a^b \frac{1}{(x-a)^m}\,dx$.

Here $n=\frac{1}{2}<1$

$\therefore \int_0^1 g(x)\,dx$ is convergent.

$\int_0^1 f(x)\,dx$ is Convergent.

Method ②: Let $f(x) = \frac{1}{\sqrt{x^2+x}}$

$\therefore f$ is +ve on $(0,1]$.

and '0' is the only point of infinite discontinuity of 'f' on $[0,1]$.

Now $f(x) = \dfrac{1}{\sqrt{x}\,(\sqrt{x+1})}$

Clearly $\dfrac{1}{\sqrt{x+1}}$ is bounded function on $[0,1]$.

$\therefore \exists$ a +ve number M as an upperbound such that $\dfrac{1}{\sqrt{x+1}} \leq M \ \forall\, x \in [0,1]$

$\therefore f(x) \leq \dfrac{M}{\sqrt{x}} \ \forall\, x \in (0,1]$

$\Rightarrow f(x) \leq \dfrac{M}{(x-0)^{1/2}} \ \forall\, x \in (0,1]$.

Also $\displaystyle\int_0^1 \frac{1}{(x-0)^{1/2}}$ is convergent $(\because n = \tfrac{1}{2} < 1)$

$\therefore$ By Comparison test.

$\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+x}}\,dx$ is convergent.

(ii) Let $f(x) = \dfrac{1}{(1+x)(\sqrt{2-x})}$

$\therefore f$ is +ve on $[1,2)$.

and 2 is the only point of infinite discontinuity of 'f' on $[1,2]$

Now $f(x) = \dfrac{1}{(1+x)(\sqrt{2-x})}$

clearly $\dfrac{1}{1+x}$ is bounded on $[1,2]$

Let M be the upperbound

$\therefore \dfrac{1}{1+x} \leq M \ \forall\, x \in [1,2]$.

$\therefore f(x) \leq \dfrac{M}{\sqrt{2-x}} \ \forall\, x \in [1,2)$

$\Rightarrow f(x) \leq \dfrac{M}{(2-x)^{1/2}} \ \forall\, x \in [1,2)$

Also $\displaystyle\int_1^2 \frac{1}{(2-x)^{1/2}}\,dx$ is convergent. $(\because n = \tfrac{1}{2} < 1)$.

$\therefore$ By Comparison test

$\displaystyle\int_1^2 \frac{1}{(1+x)(\sqrt{2-x})}\,dx$ is Convergent.

(iii) Let $f(x) = \dfrac{1}{\sqrt{1-x^3}}$

$= \dfrac{1}{\sqrt{(1-x)(1+x+x^2)}}$

$= \dfrac{1}{(\sqrt{1-x})(\sqrt{1+x+x^2})}$

$\therefore$ 1 is the only point of infinite discontinuity of 'f' on $[0,1)$.

Clearly $\dfrac{1}{\sqrt{1+x+x^2}}$ is bounded on $[0,1]$

Let M be the upper bound

$\therefore \dfrac{1}{\sqrt{1+x+x^2}} \leq M \ \forall\, x \in [0,1]$

$\therefore f(x) \leq \dfrac{M}{(1-x)^{1/2}} \ \forall\, x \in [0,1)$.

Also $\displaystyle\int_0^1 \frac{1}{(1-x)^{1/2}}\,dx$ is convergent $(\because n = \tfrac{1}{2} < 1)$

$\therefore \displaystyle\int_0^1 \frac{1}{\sqrt{1-x^3}}\,dx$ is Convergent.

(v) For $P \leq 1$ it is a proper integral.

For $P > 1$, it is an improper integral and '0' is the only point of infinite discontinuity.

Now let $f(x) = \dfrac{\sin x}{x^P}$

$$= \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x}$$

Let $g(x) = \frac{1}{x^{p-1}} \ \forall x \in (0, \pi/2]$

$\therefore \lim\limits_{x\to 0^+} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0^+}\frac{\sin x}{x} = 1$ (a non-zero finite number)

$\therefore$ By Comparison Test $\int_0^{\pi/2} f(x)\,dx$ & $\int_0^{\pi/2} g(x)\,dx$ are convergent (or) divergent together.

Since $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2}\frac{1}{(x-0)^{p-1}}$ is Convergent if $p-1<1$ i.e. $p<2$

$\therefore \int_0^{\pi/2}\frac{\sin x}{x^p}\,dx$ is Convergent for $p<2$

and $\int_0^{\pi/2}\frac{\sin x}{x^p}\,dx$ divergent $p\geq 2$.

→ (i) $\int_0^1 \frac{dx}{x^3(2+x^2)^5}$ (ii) $\int_0^1 \frac{dx}{\sqrt{x}(1+x)^2}$

(iii) $\int_0^1\frac{dx}{(1+x)^2(1-x)^3}$ (iv) $\int_0^1\frac{dx}{\sqrt{x(1-x)}}$

Sol'n (iv) :-

Let $f(x) = \frac{1}{(\sqrt{x})(\sqrt{1-x})}$

Both the end points 0 & 1 are the points of infinite discontinuity of f on $[0,1]$.

Now $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx = \int_0^a\frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$ —(i)

where $0<a<1$.

To examine the convergence at $x=0$.

Let $I_1 = \int_0^a \frac{dx}{\sqrt{x(1-x)}}$

0 is the only point of infinite discontinuity of f on $[0,a]$.

Let $g(x)=\frac{1}{\sqrt{x}} \ \forall x\in(0,a]$

Then $\lim\limits_{x\to0^+}\frac{f(x)}{g(x)} = \lim\limits_{x\to0^+}\frac{1}{\sqrt{1-x}}$

$= 1$ (a non-zero finite number)

By Comparison Test $I_1 = \int_0^a f(x)\,dx$ & $\int_0^a g(x)\,dx$ are Convergent (or) divergent together.

But $\int_0^a g(x)\,dx = \int_0^a \frac{dx}{(x-0)^{1/2}}$ is convergent ($\because n=\frac{1}{2}<1$)

$\therefore I_1$ is convergent.

To examine the convergence at $x=1$

Let $I_2 = \int_a^1 \frac{1}{\sqrt{x(1-x)}}\,dx$

1 is the only point of discontinuity of f on $[a,1]$

Let $g(x)=\frac{1}{\sqrt{1-x}} \ \forall x\in[a,1)$

then $\lim\limits_{x\to1^-}\frac{f(x)}{g(x)} = \lim\limits_{x\to1^-}\frac{1}{\sqrt{x}}=1$

(which is finite and non-zero)

$\therefore$ By Comparison test I_2 & $\int_a^1 g(x)\,dx$ convergent (or) divergent together.

But $\int_a^1 g(x)\,dx = \int_a^1\frac{1}{\sqrt{1-x}}\,dx$ is Convergent ($\because n=\frac{1}{2}<1$)

$\therefore I_2$ is convergent.

Since I_1 & I_2 are both convergent

$\therefore$ from ①,

$\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$ is convergent.

Note: If I_1 or I_2 is divergent then

$\int_0^1 f(x)\,dx$ is divergent.

$\Longrightarrow$ (i) $\int_2^3 \frac{dx}{(x-2)^{1/4}(3-x)^2}$

(ii) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$ (iii) $\int_a^b \frac{dx}{(x-a)\sqrt{b-x}}$

(iv) $\int_0^1 \frac{x^n}{1-x}\,dx$ (v) $\int_0^1 \frac{x^n}{1+x}\,dx$

(vi) $\int_1^2 \frac{x^\lambda}{x-1}\,dx$ (vii) $\int_2^3 \frac{x^2+1}{x^2-4}\,dx$

Sol'n: (iv) Let $f(x) = \frac{x^n}{1-x}$

Case (i): if $n \geq 0$ then 1 is the only point of infinite discontinuity on $[0,1]$

Let $g(x) = \frac{1}{1-x}\ \forall x \in [0,1)$.

Then $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} x^n = 1$ (a non-zero & finite number)

$\therefore$ By comparison test

$\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ are convergent (or) divergent together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{(1-x)}\,dx$ is divergent ($\because n=1$)

$\therefore \int_0^1 f(x)\,dx$ is divergent.

Case (ii) If $n<0$;

Let $n=-m$ where $m>0$.

then $f(x) = \frac{1}{x^m(1-x)}$

$\therefore$ 0 & 1 both are the points of infinite discontinuity of f on $[0,1]$. (Note)

Now $\int_0^1 f(x)\,dx = \int_0^a f(x)\,dx + \int_a^1 f(x)\,dx$ ——— ①

follow this step [for upper & lower bounds show discontinuity]

where $0<a<1$

Next please try yourself.

(v) Here $f(x) = \frac{x^n}{1+x}$ if $n \geq 0$ then

$\int_0^1 f(x)$ is a proper integral and hence it is convergent.

if $n<0$ then let $n=-m$; where $m>0$.

$\therefore f(x) = \frac{1}{x^m(1+x)}$

Here '0' is the point of infinite discontinuity of on $[0,1]$

Let $g(x) = \frac{1}{x^m}$ proceed Next.

$\longrightarrow$ (i) $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$ (ii) $\int_0^1 \frac{\log x}{\sqrt{x}}\,dx$

(iii) $\int_1^2 \frac{\sqrt{x}}{\log x}\,dx$

Sol'n: (i) Let $f(x) = \frac{\log x}{\sqrt{2-x}}$

Clearly 0 & 2 are only the points of infinite discontinuity of f on $[0,2]$ $\{\because \log 0$ is not defined$\}$

Now

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \quad \text{——— (1)}$$

To test the Convergence of $\int_0^1 f(x)\,dx$ at $x=0$:

Since $f(x)$ is $-$ve on $[0,1]$

we consider $-f(x)$

Take $g(x) = \dfrac{1}{x^n}$

Now $\lim\limits_{x\to 0+} \dfrac{-f(x)}{g(x)} = \lim\limits_{x\to 0+} \dfrac{-x^n \log x}{\sqrt{2-x}}$

$= 0$ if $n>0$

$(\because \lim\limits_{x\to 0+} x^n \log x = 0$ if $n>0)$

$\therefore$ Taking n b/w 0 & 1

$\therefore \int_0^1 g(x)\,dx$ is Convergent.

$\therefore$ By Comparison Test

$\int_0^1 -f(x)\,dx$ is Convergent.

To test the Convergence of $\int_1^2 f(x)\,dx$ at $x=2$:-

Take $g(x) = \dfrac{1}{\sqrt{2-x}}$ $\forall x \in [1,2)$

$\therefore \lim\limits_{x\to 2-} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 2-} \log x$

(a non-zero finite number).

$\therefore$ By Comparison test

$\int_1^2 f(x)\,dx$ & $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \dfrac{1}{(2-x)^{1/2}}$ is Convergent

$(\because n = \frac{1}{2} < 1)$

$\therefore \int_1^2 f(x)\,dx$ is also Convergent.

$\therefore$ from (1)

$\int_0^2 f(x)\,dx$ is convergent.

(iii) Let $f(x) = \dfrac{\sqrt{x}}{\log x}$

1 is the only point of infinite discontinuity of f on $[1,2]$

Take $g(x) = \dfrac{1}{(x-1)^n}$

$\therefore \lim\limits_{x\to 1+} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 1+} \dfrac{(x-1)^n \sqrt{x}}{\log x}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim\limits_{x\to 1+} \dfrac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x}$

$= \lim\limits_{x\to 1+} (x-1)^{n-1}\left[n x^{3/2} + \left(\dfrac{x-1}{2}\right)\cdot\sqrt{x}\right]$

$= 1$ if $n=1$

$(\because$ a non-zero finite number$)$

$\therefore$ By Comparison test

$\int_1^2 f(x)\,dx$ & $\int_1^2 g(x)\,dx$ are convergent (or) divergent together. But $\int_1^2 g(x)\,dx$ diverges. $(\because n=1)$

$\therefore \int_1^2 f(x)\,dx$ diverges.

→ Ex: (i) $\int_0^1 \frac{\log x}{1+x}dx$ (ii) $\int_0^1 \frac{\log x}{1+x^2}dx$

(iii) $\int_0^2 \frac{\log x}{2-x}dx$ (iv) $\int_0^1 \frac{\log x}{1-x^2}dx$

Sol'n: (iii) Let $f(x) = \frac{\log x}{2-x}$

clearly 0 & 2 are only the points of infinite discontinuity of 'f' on $[0,2]$.

$\therefore \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)dx$ ——— (I)

To test the convergence of $\int_0^1 f(x)dx$

at $x=0$:-

Since $f(x)$ is −ve in $(0,1]$.

we consider $-f(x)$ which is +ve in $(0,1]$.

Take $g(x) = \frac{1}{x^n}$ $\forall x \in (0,1]$

$\therefore \lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{2-x}$

$= 0$ if $n>0$.

($\because \lim_{x\to 0+} x^n \log x = 0$ if $n>0$)

Taking n b/w 0 & 1,

$\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 f(x)dx$ is Convergent.

To test the convergent of $\int_1^2 f(x)dx$

at $x=2$:

Take $g(x) = \frac{1}{2-x}$

$\therefore \lim_{x\to 2^-} \frac{f(x)}{g(x)} = \lim_{x\to 2^-} \log(x)$

$= \log 2$ (a non-zero finite number)

By Comparison Test,

$\int_1^2 f(x)dx$ & $\int_1^2 g(x)dx$ are convergent

(or) divergent together.

but $\int_1^2 g(x)\,dx = \int_1^2 \frac{1}{(2-x)}dx$ is divergent. ($\because n=1$)

$\therefore \int_1^2 f(x)dx$ is divergent.

$\therefore$ from (I),

$\int_0^2 f(x)\,dx$ is divergent.

(iv) Since $\frac{\log x}{1-x^2}$ is −ve in $(0,1]$ then

Let $f(x) = \frac{-\log x}{1-x^2}$

Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{-\log x}{1-x^2}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 1^-} \frac{-1/x}{-2x} = \frac{1}{2}$

$\therefore$ 0 is the only point of infinite discontinuity of 'f' on $[0,1]$.

Take $g(x) = \frac{1}{x^n}$

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-x^n \log x}{1-x^2}$

$= 0$ if $n>0$.

Take n b/w 0 & 1, the integral $\int_0^1 g(x)dx$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 f(x)\,dx$

is convergent.

$\therefore \int_0^1 \frac{\log x}{1-x^2}\,dx$ is convergent.

$\longrightarrow$ (i) $\int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ (ii) $\int_0^1 \frac{x^p \log x}{1+x^2}\,dx$

(iii) $\int_0^1 \frac{(x^p + x^{-p})\log(1+x)}{x}\,dx$

(iv) $\int_0^1 x^{n-1} \log x\,dx$

Sol'n (i) Let $f(x) = \frac{x^n \log x}{(1+x)^2}$

$\underset{x\to 0_+}{Lt} \frac{x^n \log x}{(1+x)^2} = 0$ if $n>0$

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is a proper integral and hence it is convergent.

If $n=0$: Let $f(x) = \frac{-\log x}{(1+x)^2}$

0 is the only point of infinite discontinuity.

Take $g(x) = \frac{1}{x^p}$

$\underset{x\to 0_+}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 0_+}{Lt} \frac{-x^p \log x}{(1+x)^2}$

$= 0$ if $p>0$

Taking p b/w 0 & 1,

$\int_0^1 g(x)\,dx$ is convergent.

$\Rightarrow \int_0^1 f(x)\,dx$ is convergent.

$\Rightarrow \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is convergent.

If $n<0$, let $n=-m$, $m>0$

Let $f(x) = \frac{-x^n \log x}{(1+x)^2}$

$= \frac{-\log x}{x^m (1+x)^2}$

Take $g(x) = \frac{1}{x^q}$

$\underset{x\to 0_+}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 0_+}{Lt} \frac{-x^{q-m}\log x}{(1+x)^2}$

$= 0$ if $q-m>0$

Taking $0<q<1$ and also $q-m>0$.

i.e. $q>m$.

$\Rightarrow 0<m<q<1$

$\Rightarrow m<1$

$\Rightarrow -n<1$

$\Rightarrow n>-1$

$\therefore \int_0^1 g(x)\,dx$ is convergent and hence

$\int_0^1 f(x)\,dx$ is convergent.

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2}\,dx$ is convergent for all $n>-1$.

(iii) Let $P>0$ and

$f(x) = \left(x^P + \frac{1}{x^P}\right)\frac{\log(1+x)}{x}$

Here '0' is the point of infinite discontinuity.

Take $g(x) = \frac{1}{x^P}$

Now

$\underset{x\to 0_+}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 0_+}{Lt} (x^{2P}+1)\frac{\log(1+x)}{x}$

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$= \lim_{x\to 0+}(x^{2P}+1) \lim_{x\to 0+} \frac{\log(1+x)}{x}$

$= 2\lim_{x\to 0+} \frac{\frac{1}{1+x}}{2}$

$= (1)(1) = 1$ (a non-zero finite number)

Since $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^P}dx$ is convergent if $0<P<1$.

$\therefore \int_0^1 f(x)dx$ is convergent if $0<P<1$.

If $P=0$.

$f(x) = \frac{2\log(1+x)}{x}$

Since $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{2\log(1+x)}{x}$ $\left|\frac{0}{0}\right.$ form

$= \lim_{x\to 0+} \frac{2\left(\frac{1}{1+x}\right)}{(1)}$

$= 2$

$\therefore \int_0^1 f(x)dx$ is a proper integral and hence convergent.

If $P<0$:

Let $g(x) = \frac{1}{x^{-P}}$

$\therefore \lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+}\left(1+\frac{1}{x^{2P}}\right)\frac{\log(1+x)}{x}$

$= \lim_{x\to 0+}\left(1+\frac{1}{x^{2P}}\right)$ $\left[\because \lim_{x\to 0+}\frac{\log(1+x)}{x} = 1\right]$

$= 1$ (since $P<0$).

which is non-zero and finite.

Since $\int_0^1 g(x)dx = \int_0^1 \frac{1}{x^{-P}}dx$ is convergent if $-P<1$ i.e. if $P>-1$

$\therefore \int_0^1 f(x)dx$ is convergent if $P>-1$

$\therefore \int_0^1 f(x)dx$ is convergent if $P>-1$.

$\therefore \int_0^1 f(x)$ is convergent if $-1<P<1$.

iv) we know that $\lim_{x\to 0+} x^r\log x = 0$ when $r>0$.

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is a proper integral when $(n-1)>0$ i.e. when $n>1$.

when $n=1$ 0 is the point of infinite discontinuity.

$\therefore \int_0^1 \log x\,dx = \lim_{\epsilon\to 0+}\int_{0+\epsilon}^1 \log x\,dx$

$= \lim_{\epsilon\to 0+}\left[x\log x - x\right]_\epsilon^1$

$= \lim_{\epsilon\to 0+}\left[0-1-\epsilon\log\epsilon+\epsilon\right]$

$= -1$ $(\because \lim_{\epsilon\to 0+}\epsilon\log\epsilon = 0)$

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is convergent if $n=1$.

when $n<1$:

Let $f(x) = -x^{n-1}\log x$ $(\because x^{n-1}\log x$ is $-$ve in $(0,1])$

Here 0 is the point of infinite discontinuity.

Take $g(x) = \frac{1}{x^P}$

$\therefore \lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+}\frac{-x^{(n-1)}\log x}{1/x^P}$

$= \lim_{x\to 0+} -x^{P+n-1}\log x$

$= 0$ if $p+n-1 > 0$

$= \infty$ if $p+n-1 \le 0$

Taking $0<p<1$ and $p > 1-n$

$\Rightarrow 1-n < p < 1$

$\Rightarrow 1-n < 1$

$\Rightarrow n > 0$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^p}$ is convergent. ($\because 0<p<1$)

$\therefore \int_0^1 f(x)\,dx$ is also convergent if $n>0$ (i.e. $0<n<1$)

When $p=1$ and $p \le 1-n$

$\Rightarrow 1 \le 1-n$

$\Rightarrow n \le 0$.

Since $\int_0^1 g(x)\,dx$ is divergent ($\because p=1$)

$\therefore \int_0^1 f(x)\,dx$ is divergent

$\therefore \int_0^1 x^{n-1}\log x\,dx$ is divergent $\forall n \le 0$.

$\rightarrow$ (i) $\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ (ii) $\int_0^1 \frac{\operatorname{cosec} x}{x}\,dx$

(iii) $\int_0^1 \frac{\sec x}{x}\,dx$

Sol^n: (i) Let $f(x) = \frac{\cos x}{x^n}$

If $n \le 0$ then the integral $\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$ is a proper integral

If $n>0$ then 0 is the only point of infinite discontinuity of f on $(0, \pi/2]$

Let $g(x) = \frac{1}{x^n}$ $\forall x \in (0, \pi/2]$

$\therefore \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \cos x$

$= 1$ (a non-zero finite number)

$\therefore$ By comparison test $\int_0^{\pi/2} f(x)\,dx$ & $\int_0^{\pi/2} g(x)\,dx$ are convergent or divergent together.

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{1}{x^n}\,dx$ is convergent if $n<1$.

$\therefore \int_0^{\pi/2} f(x)$ is convergent.

and $\int_0^{\pi/2} g(x)\,dx$ is divergent if $n \ge 1$

$\therefore \int_0^{\pi/2} f(x)\,dx$ is divergent if $n \ge 1$.

(ii) Let $f(x) = \frac{\operatorname{cosec} x}{x}$

0 is the only point of infinite discontinuity of f on $[0,1]$.

Since $|\sin x| \le 1$ $\forall x \in \mathbb{R}$.

$\Rightarrow \left|\frac{1}{\sin x}\right| \ge 1$

$\Rightarrow |\operatorname{cosec} x| \ge 1$

$\Rightarrow \left|\frac{\operatorname{cosec} x}{x}\right| \ge \frac{1}{|x|}$ $\forall x \in (0,1]$

$= \frac{1}{x}$

$\Rightarrow f(x) \ge \frac{1}{x}$ $\forall x \in (0,1]$

Since $\int_0^1 \frac{1}{(x-0)^1}\,dx$ is divergent ($\because n=1$)

$\therefore$ By comparison test $\int_0^1 f(x)\,dx$ is divergent.

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Show that $\int_0^{\pi/2} x^m \operatorname{cosec}^n x \, dx$ exists iff $n < m+1$.

Let $f(x) = x^m \operatorname{Cosec}^n x$

$= \dfrac{x^m}{\sin^n x}$

$= \left(\dfrac{x}{\sin x}\right)^n x^{(m-n)}$

$\therefore \lim_{x \to 0+} f(x) = \begin{cases} 0 & \text{if } m-n>0 \\ 1 & \text{if } m-n=0 \\ \infty & \text{if } m-n<0 \end{cases}$

$\Rightarrow$ The given integral is proper integral if $m-n \geq 0$ i.e. if $m \geq n$.

AND

The given integral is improper integral if $m-n<0$ i.e. if $m<n$.

$\therefore$ 0 is the only the point of infinite discontinuity of f on $[0, \pi/2]$.

when $m-n<0 \Rightarrow n-m>0$.

$\therefore f(x) = \left(\dfrac{x}{\sin x}\right)^n \dfrac{1}{x^{n-m}} \; \forall x \in (0, \pi/2]$

Let $g(x) = \dfrac{1}{x^{n-m}} \; \forall x \in (0, \pi/2)$

Now $\lim_{x \to 0+} \dfrac{f(x)}{g(x)} = 1$ (a non-zero finite number)

By Comparison test

$\int_0^{\pi/2} f(x) dx$ & $\int_0^{\pi/2} g(x) dx$ are convergent or divergent together.

But $\int_0^{\pi/2} g(x) dx = \int_0^{\pi/2} \dfrac{1}{x^{n-m}}$ is convergent iff $n-m<1$ i.e. iff $n<m+1$.

$\therefore \int_0^{\pi/2} f(x) dx$ is Convergent iff $n<m+1$.

$\rightarrow$ Show that $\int_0^{\pi/2} \dfrac{\sin^m x}{x^n} dx$ exists iff $n<m+1$.

Sol^n: Let $f(x) = \dfrac{\sin^m x}{x^n}$

$= \left(\dfrac{\sin x}{x}\right)^m \cdot x^{m-n}$

$\rightarrow$ (i) $\int_0^{\pi/4} \dfrac{1}{\sqrt{\tan x}} dx$ (ii) $\int_0^1 \left(\log \frac{1}{x}\right)^n dx$

Sol^n:- (i) 0 is the only point of infinite discontinuity of f on $[0, \pi/4]$

Let $f(x) = \dfrac{1}{\sqrt{\tan x}}$

$= \sqrt{\dfrac{\cos x}{\sin x}}$

Let $g(x) = \dfrac{1}{\sqrt{x}} \; \forall x \in (0, \pi/4]$

then $\lim_{x \to 0+} \dfrac{f(x)}{g(x)} = \lim_{x \to 0+} \dfrac{\sqrt{x}}{\sqrt{\sin x}} \cdot \sqrt{\cos x}$

$= \lim_{x \to 0+} \sqrt{\dfrac{x}{\sin x}} \cdot \lim_{x \to 0+} \sqrt{\cos x}$

$= 1$

$\therefore$ By Comparison test $\int_0^{\pi/4} f(x) dx$ & $\int_0^{\pi/4} g(x) dx$ are convergent or divergent together.

But $\int_0^{\pi/4} g(x) dx = \int_0^{\pi/4} \dfrac{1}{x^{1/2}} dx$ is convergent ($\because n = \frac{1}{2} < 1$).

$\therefore \int_0^{\pi/4} f(x)$ is also convergent.

(ii) Let $f(x) = \left(\log \frac{1}{x}\right)$

Since 0 & 1 are the only points of infinite discontinuity on $[0,1]$.

Now we write

$$\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx \quad \text{——— (1)}$$

where $0<a<1$

To test the convergence of $\int_0^a f(x)dx$ at $x=0$.

Now $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\log \frac{1}{x}\right)^n$

$$= \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n<0 \end{cases}$$

$\therefore$ The integral is proper if $n \le 0$.

If $n>0$: 0 is the only point of infinite discontinuity.

Let $g(x) = \frac{1}{x^p}$ $\forall x \in (0,a]$

Now $\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \left(\log \frac{1}{x}\right)^n \cdot x^p$

$= 0$

Since $\int_0^a f(x)dx = \int_0^a \frac{1}{x^p}$ is convergent if $0<p<1$.

$\therefore$ By Comparison test —

$\int_0^a f(x)dx$ is convergent.

To test the convergence of $\int_a^1 f(x)dx$ at $x=1$;

The integral is proper if $x \ge 0$.

If $n<0$ then 1 is the only point of infinite discontinuity.

For $n<0$:

Let $g(x) = \frac{1}{(1-x)^{-n}}$

Now $\lim_{x\to 1^-} \frac{f(x)}{g(x)} = \lim_{x\to 1^-} \left(\frac{\log 1/x}{1-x}\right)^n$

$= \left[\lim_{x\to 1^-} \frac{\log 1/x}{1-x}\right]^n$

$= \lim_{x\to 1^-} \frac{x(-1/x^2)}{-1}$

$= 1$ (which is non-zero and finite)

But $\int_a^1 g(x)dx = \int_a^1 \frac{1}{(1-x)^{-n}}$ is convergent if $-n<1$

i.e. if $n>-1$

$\therefore$ By Comparison test,

$\int_a^1 f(x)dx = \int_a^1 \left(\log \frac{1}{x}\right)^n dx$ is convergent if $-1<n<0$.

$\therefore$ From (1),

$\int_0^1 \left(\log 1/x\right)^n dx$ is convergent if $-1<n<0$.

→ Find the values of m & n for which the integral $\int_0^\infty e^{-mx} \cdot x^n dx$ converges

Sol^n: Irrespective of the values of 'm', when $n \ge 0$.

$\therefore$ the given integral is proper and hence it is convergent.

when $n<0$:

whatever 'n' may be, '0' is the only point of infinite discontinuity.

Let $f(x) = e^{-mx} \cdot x^n$

Let $g(x) = x^n = \frac{1}{x^{-n}}$

$$\therefore \lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-mx} = 1$$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{-n}}$ converges if $-n<1$ i.e. if $n>-1$

$\therefore$ By comparison test, $\int_0^1 f(x)\,dx$ also converges if $-1<x<0$.

$\therefore \int_0^1 e^{-mx} x^n\,dx$ converges only for $-1<n<0$.

irrespective of the value of m.

$\rightarrow$ show that $\int_0^{\pi/2} \log \sin x\,dx$ is convergent.

Sol'n:- Let $f(x) = \log \sin x$

0 is the point of infinite discontinuity.

Since f is -ve on $[0, \pi/2]$

we consider

Take $g(x) = \frac{1}{x^n}$; $n>0$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} -x^n \log \sin x$$

$$= \lim_{x\to 0+} \frac{-\log \sin x}{\frac{1}{x^n}} \quad \left| \frac{\infty}{\infty} \right.$$

$$= \lim_{x\to 0+} \frac{\frac{\cot x}{n}}{x^{n+1}}$$

$$= \lim_{x\to 0+} \frac{x^n}{n} \cdot \frac{x}{\tan x}$$

$$= 0$$

Taking n b/w 0 & 1,

$\int_0^{\pi/2} g(x)\,dx$ is convergent

By Comparison test.

$\int_0^{\pi/2} -f(x)\,dx$ is convergent.

$\Rightarrow \int_0^{\pi/2} f(x)\,dx$ is convergent.

$\rightarrow$ Show that $\int_0^1 \frac{\operatorname{cosec} x}{x^n}\,dx$ is divergent if $n\geq 1$.

Sol'n: Let $f(x) = \frac{\operatorname{cosec} x}{x^n}$

Since $|\sin x| \leq 1 \;\forall x \in \mathbb{R}$

$\Rightarrow |\operatorname{cosec} x| \geq 1 \;\forall x \in \mathbb{R}$

$\Rightarrow \left|\frac{\operatorname{cosec} x}{x^n}\right| \geq \frac{1}{|x^n|}$ for all $x \in (0,1]$.

$\therefore f(x) \geq \frac{1}{x^n} \;\forall x \in (0,1]$

Since $\int_0^1 \frac{1}{x^n}\,dx$ is divergent if $n\geq 1$.

$\therefore$ By Comparison Test

$$\int_0^1 \frac{\operatorname{Cosec} x}{x^n}\,dx \text{ is divergent if } n \geq 1.$$

$\rightarrow$ Test for Convergence the integral $\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$.

Sol^n: Let $f(x) = \frac{\sin x}{x^{3/2}}$

$$= \left(\frac{\sin x}{x}\right) \cdot \frac{1}{\sqrt{x}}$$

$$\forall x \in (0,1]$$

Let $g(x) = \frac{1}{\sqrt{x}}$ $\forall x \in [0,1]$.

[0 is the only point of infinite discontinuity of f on $[0,1]$]

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\sin x}{x}$$

$$= 1$$

Since $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x^{1/2}}\,dx$ is

Convergent $(\because n = \frac{1}{2} < 1)$

$\therefore$ By Comparison test

$\int_0^1 f(x)\,dx$ is also Convergent.

General Test for Convergence (Integrand may change sign):

This test for convergence of an improper integral (finite limits of integration but discontinuous integrand) hold whether or not the integrand keeps the same sign.

Cauchy's Test:

The improper integral $\int_a^b f(x)\,dx$, 'a' is the only the point of infinite discontinuity, converges at 'a' iff to each $\epsilon>0$, $\exists$ a $\delta>0$ such that $\left|\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx\right| < \epsilon \ \forall\, 0<\lambda_1,\lambda_2<\delta$

Note: $\int_{a+\lambda_1}^{a+\lambda_2} f(x)\,dx \to 0$ as $\lambda_1,\lambda_2 \to 0$.

Definition:

Absolute Convergence: The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent if $\int_a^b |f(x)|\,dx$ is convergent.

→ Every absolutely convergent integrand is convergent.

i.e $\int_a^b |f(x)|\,dx$ exists.

$\Rightarrow \int_a^b f(x)\,dx$ exists.

Note: (1) The converse of the above is not true.

i.e. Every convergent integral need not be absolutely convergent.

→ A convergent integral which is not absolutely convergent is called a conditional convergent integral.

Problems:

2000 P-I → Test the convergence of

$$\int_0^1 \frac{\sin 1/x}{\sqrt{x}}\,dx$$

Sol'n: Let $f(x) = \dfrac{\sin 1/x}{\sqrt{x}}$

clearly f does not keep the same sign in a ~~bounded~~ nbd of '0'.

Now $|f(x)| = \left|\dfrac{\sin 1/x}{\sqrt{x}}\right|$

$$= \frac{|\sin 1/x|}{|\sqrt{x}|} \le \frac{1}{\sqrt{x}} \ \forall x \in [0,1]$$

$[\because |\sin 1/x| \le 1]$

Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent at '0' $(\because n = \tfrac{1}{2} < 1)$

$\therefore$ By Comparison Test

$\int_0^1 |f(x)|\,dx$ is convergent at '0'.

since absolutely $\Rightarrow$ convergence $\Rightarrow$ convergence

$\therefore \int_0^1 f(x)\,dx$ is convergent.

→ Show that $\int_0^1 \frac{\sin 1/x}{x^p}$; $p>0$ converges absolutely for $p<1$.

Sol'n: Let $f(x) = \dfrac{\sin 1/x}{x^p}$; $p>0$

clearly f doesnot keep the same sign in a neighbourhood of '0'.

Now $|f(x)| = \left|\dfrac{\sin 1/x}{x^p}\right|$

$= \dfrac{|\sin 1/x|}{x^p} \le \dfrac{1}{x^p}$

$\forall x \in (0,1]$.

Since $\int_0^1 \frac{1}{x^p}dx$ is convergent iff $p<1$

By Comparison test $\int_0^1 |f(x)|dx$ is convergent if $p<1$.

$\therefore \int_0^1 f(x)\,dx$ Converges absolutely for $p<1$.

* Convergence at ∞, the integrand f being +ve :

→ A necessary and sufficient Condition for the Convergence of $\int_a^\infty f(x)\,dx$, where $f(x)>0$ $\forall x \in [a,t]$ is that there exists a +ve number M, independent of t, Such that $\int_a^t f(x)\,dx < M \;\forall\; t \ge a$.

→ Comparison test I :-

If f & g are two functions such that $0 < f(x) \le g(x) \forall x \in [a,\infty)$

then (i) $\int_a^\infty g(x)\,dx$ is convergent

$\Rightarrow \int_a^\infty f(x)\,dx$ is Convergent.

ii) $\int_a^\infty f(x)\,dx$ is divergent

$\Rightarrow \int_a^\infty g(x)\,dx$ is divergent.

Comparison test - II :-

If f and g are +ve functions on $[a,\infty)$ and $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = l$ then

if (i) 'l' is non-zero and finite. then the two integrals $\int_a^\infty f(x)\,dx$ & $\int_a^\infty g(x)\,dx$ converge (or) diverge together.

ii) If $l=0$ and $\int_a^\infty g(x)\,dx$ Converges then $\int_a^\infty f(x)\,dx$ Converges.

iii) If $l=\infty$ and $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ diverges.

A useful Comparison integrals

→ The improper integral $\int_a^\infty \dfrac{dx}{x^n}$, $(a>0)$ is Convergent iff $n>1$.

→ $\int_a^\infty \dfrac{dx}{x^n}$ $(a>0)$ is divergent iff $n \le 1$.

Problems:

Examine the Convergence of the following:

(i) $\displaystyle\int_1^\infty \frac{x^3}{(1+x)^5}\,dx$ (ii) $\displaystyle\int_1^\infty \frac{dx}{(2+x)\sqrt{x}}$

(iii) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ (iv) $\displaystyle\int_1^\infty \frac{x^3+1}{x^4}\,dx$

Sol'n:- (i) Let $f(x) = \dfrac{x^3}{(1+x)^5}$

$= \dfrac{x^3}{x^5(1+\frac{1}{x})^5} = \dfrac{1}{x^2(1+\frac{1}{x})^5}$

Take $g(x) = \frac{1}{x^2}$

$\therefore \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{1}{(1+\frac{1}{x})^5}$

$= 1$ (finite & non-zero)

$\therefore$ By Comparison test

$\int_0^\infty f(x)\,dx$ & $\int_0^\infty g(x)\,dx$ are convergent (or) divergent together.

But $\int_0^\infty g(x)\,dx = \int_0^\infty \frac{1}{x^2}\,dx$ is Convergent.

$(\because n = 2 > 1)$

$\therefore \int_0^\infty f(x)\,dx$ is convergent.

(i) $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx \quad m,n>0$

(ii) $\displaystyle\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$

Sol'n:- (i) $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \int_0^a \frac{x^{2m}}{1+x^{2n}}\,dx + \int_a^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$ —— (1)

where $0<a<\infty$

Since $\displaystyle\int_0^a \frac{x^{2m}}{1+x^{2n}}\,dx$ is a proper integral.

hence it is a Convergent.

The given integral $\displaystyle\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$ is Convergent or divergent according as

$\displaystyle\int_a^\infty \frac{x^{2m}}{1+x^{2n}}\,dx$ is Convergent or divergent

Let $f(x) = \dfrac{x^{2m}}{1+x^{2n}}$

$= \dfrac{x^{2m}}{x^{2n}\left(1+\frac{1}{x^{2n}}\right)}$

$= \dfrac{x^{2m-2n}}{\left(1+\frac{1}{x^{2n}}\right)}$

Let $g(x) = x^{2m-2n} = \dfrac{1}{x^{2n-2m}}$

$\therefore \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim_{x\to\infty} \dfrac{1}{1+\frac{1}{x^{2n}}} = 1 \quad (\because n>0)$

By Comparison test

$\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ Convergent or divergent together.

But $\displaystyle\int_a^\infty g(x)\,dx = \int_a^\infty \frac{dx}{x^{2n-2m}}$ converges

iff $2n-2m>1$ i.e. iff $n-m>\frac{1}{2}$

$\therefore \int_a^\infty f(x)\,dx$ converges iff $n-m>\frac{1}{2}$

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(i) $\displaystyle\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ (ii) $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

(iii) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$ (iv) $\displaystyle\int_e^\infty \frac{dx}{x(\log x)^{n+1}}$

Sol'n: (i) $\displaystyle\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx =$

$\displaystyle\int_0^a \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx + \int_a^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ —— (1)

where $0<a<\infty$

Since $\displaystyle\int_0^a \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ is a proper integral.

and hence it is Convergent.

The given integral is $\displaystyle\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$ convergent (or) divergent according as $\displaystyle\int_a^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$

is Convergent (or) divergent.

Let $f(x) = \dfrac{x\tan^{-1}x}{(1+x^4)^{1/3}}$

$= \dfrac{\tan^{-1}x}{x^{4/3-1}\left(1+\frac{1}{x^4}\right)}$

$= \dfrac{\tan^{-1}x}{x^{1/3}\left(1+\frac{1}{x^4}\right)}$

Let $g(x) = 1/x^{1/3}$

(iii) Put $\log x = t$

So that $\frac{1}{x}dx = dt$

when $x = e \Rightarrow t = 1$

when $x \to \infty \Rightarrow t \to \infty$

$\therefore \displaystyle\int_e^\infty \frac{dx}{x(\log x)^{3/2}} = \int_1^\infty \frac{dt}{t^{3/2}}$ is Convergent

$(\because n = 3/2 > 1)$

$\therefore \displaystyle\int_e^\infty \frac{dx}{x(\log x)^{3/2}}$ is convergent.

$\rightarrow$ (i) $\displaystyle\int_0^\infty e^{-x^2}dx$ (ii) $\displaystyle\int_1^\infty x^n e^{-x}dx$

(iii) $\displaystyle\int_1^\infty \frac{\log x}{x^2}dx$ (iv) $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}dx$

Sol'n: (i) $\displaystyle\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$ —— (1)

Since $\displaystyle\int_0^1 e^{-x^2}dx$ is a proper integral,

hence it is convergent.

Let us consider $\displaystyle\int_1^\infty e^{-x^2}dx$

We know that $e^{x^2} > x^2 \ \forall x \in \mathbb{R}$

$\Rightarrow e^{-x^2} < \frac{1}{x^2}$

Since $\displaystyle\int_1^\infty \frac{1}{x^2}dx$ Converges at ∞

$\therefore \displaystyle\int_1^\infty e^{-x^2}dx$ also Convergent

$\therefore$ from (1), $\displaystyle\int_0^\infty e^{-x^2}dx$ is Convergent.

Note: $\displaystyle\int_0^\infty e^{-x^2}dx$ is called the Euler-poison integral and its value is $\frac{\sqrt{\pi}}{2}$.

(ii) Let $f(x) = x^n e^{-x}$

Take $g(x) = \frac{1}{x^2}$

$\therefore \displaystyle Lt_{x\to\infty} \frac{f(x)}{g(x)} = Lt_{x\to\infty} \frac{x^{n+2}}{e^x} = 0 \ \forall n$

Since $\displaystyle\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}dx$ is convergent.

$(\because n = 2 > 1)$

$\therefore$ By Comparison test

$\displaystyle\int_1^\infty f(x)\,dx = \int_1^\infty x^n e^{-x}dx$ is Convergent.

(iv) Since $|\cos x| \le 1 \ \forall x \in \mathbb{R}$

$\Rightarrow \left|\dfrac{\cos x}{1+x^2}\right| \le \dfrac{1}{1+x^2} \ \forall x \in [0,\infty)$

Since $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = Lt_{t\to\infty} \int_0^t \frac{1}{1+x^2}dx;\ 0<t$

$= \displaystyle Lt_{t\to\infty} \left[\tan^{-1}x\right]_0^t$

$= \displaystyle Lt_{t\to\infty} (\tan^{-1}t)$

$= \pi/2$

$\therefore \displaystyle\int_0^\infty \frac{1}{1+x^2}dx$ is Convergent

$\therefore$ By Comparison test.

$\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}$ is convergent.

$\longrightarrow$ Show that

$\displaystyle\int_0^\infty \left(\frac{1}{1+x} - e^{-x}\right)\frac{dx}{x}$ is Convergent.

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Sol'n: Let $f(x) = \left(\frac{1}{1+x} - e^{-x}\right)\frac{1}{x}$

$$= \left(\frac{1}{1+x} - \frac{1}{e^x}\right)\frac{1}{x}$$

$$= \frac{e^x - 1 - x}{e^x(1+x)x}$$

$$= \frac{\left(\cancel{1}+\cancel{x}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)-\cancel{1}-\cancel{x}}{x(1+x)e^x}$$

$$= \frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}{x(1+x)e^x} > 0 \ \forall x>0.$$

Now $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\frac{x}{2!}+\frac{x^2}{3!}+\cdots}{(1+x)e^x}$

$$= \underline{\underline{0}}$$

$\therefore$ 0 is not point of infinite discontinuity.

$$\therefore \int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx \quad \text{--- (1)}$$

Here

$\int_0^1 f(x)dx$ is proper integral and hence it is convergent.

Now $f(x) = \frac{e^x-1-x}{e^x(1+x)x}$

$$= \frac{e^x-1-x}{e^x(\cancel{1+x})\cancel{x}} \cdot \frac{x}{1+x} \cdot \frac{1}{x^2}$$

Let $g(x) = \frac{1}{x^2}$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty}\left(\frac{e^x-1-x}{e^x}\right)\cdot\frac{x}{x+1}$$

$$= \lim_{x\to\infty}\frac{e^x-1-x}{e^x} \cdot \lim_{x\to\infty}\frac{x}{1+x}$$

$$= \lim_{x\to\infty}\frac{e^x-1}{e^x} \quad \lim_{x\to\infty}\left(\frac{1}{1}\right)$$

$$= \lim_{x\to\infty}(1-e^{-x})$$

$$= 1$$

Since $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x^2}dx$ is convergent $(\because n = 2 > 1)$

$\therefore$ By comparison test $\int_1^\infty f(x)\,dx$ is also convergent

$\therefore$ The given integral $\int_0^\infty f(x)\,dx$ is Convergent.

* General Test For Convergence at ∞ (Integrand may Change sign):—

Cauchy's Test: The improper integral $\int_a^\infty f(x)\,dx$ Converges at ∞ iff to each $\epsilon>0$, $\exists$ a +ve real number K such that $\left|\int_{t_1}^{t_2} f(x)\,dx\right| < \epsilon \ \forall t_1, t_2 \geq K$.

Absolute Convergence:

Definition: The improper integral

$\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if $\int_a^\infty |f(x)|\,dx$ is convergent.

→ Every absolutely convergent integral is convergent.

i.e. $\int_a^\infty |f(x)|\,dx$ exists

$\Rightarrow \int_a^\infty f(x)\,dx$ exists.

Note (1): The converse of above is not true.

(2) A convergent integral which is not absolutely convergent is called a conditionally convergent integral.

* Tests for Convergence of the integral of a product of two functions:

→ Abel's Test :-

If $\int_a^\infty f(x)\,dx$ is convergent at ∞ and $g(x)$ is bounded and monotonic for $x \geq a$ then

$\int_a^\infty f(x)\,g(x)\,dx$ converges at ∞.

→ Dirichlet's Test :

If $\int_a^t f(x)\,dx$ is bounded for all $t \geq a$ and $g(x)$ is a bounded and monotonic function for $x > a$, tending to '0' as $x \to \infty$, then

$\int_a^\infty f(x)\,g(x)\,dx$ is convergent at ∞.

Problems

→ Examine the convergence of the integrals:

(i) $\int_0^\infty \frac{\sin x}{x}\,dx$ (ii) $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx$

(iii) $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$ $\left[\int_0^\infty = \int_0^1 + \int_1^\infty\right]$

[∵ 0 is the point of infinite discontinuity]

(iv) $\int_a^\infty \frac{\sin x}{x^m}\,dx$ where a & m are +ve.

(v) $\int_0^\infty \frac{\sin kx}{x}\,dx$ (vi) $\int_1^\infty \frac{\sin x^m}{x^n}\,dx$.

Sol'n: (i) Let $f(x) = \frac{\sin x}{x}$

Clearly f doesnot keep the same sign in $(0, \infty)$

Since $\lim_{x \to 0+} \frac{\sin x}{x} = 1$

∴ 0 is not a point of infinite discontinuity.

Now $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$ —— (1)

Since $\int_0^1 \frac{\sin x}{x}\,dx$ is a proper integral and hence it is convergent.

To test the convergence of $\int_1^\infty \frac{\sin x}{x}\,dx$ at ∞:

Let $f(x) = \sin x$; $g(x) = 1/x$

Now

$$\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t \sin x\,dx\right|$$

$$= |\cos 1 - \cos t|$$

$$\leq |\cos 1| + |\cos t|$$

$$\leq 1+1 \quad (\because |\cos x| \leq 1)$$

$$= 2$$

$\therefore \left|\int_1^t f(x)\,dx\right| \leq 2.$

$\therefore \int_1^t f(x)\,dx$ is bounded for all $t \geq 1$.

Now $g(x)$ is bounded and Monotonically decreasing function and tending to '0' as $x \to \infty$

$\therefore$ By Dirichlet's test

$\int_1^\infty f(x) g(x)\,dx = \int_1^\infty \frac{\sin x}{x}\,dx$ is Convergent.

$\therefore$ From ①, $\int_0^\infty \frac{\sin x}{x}\,dx$ is Convergent.

(v) Since $\lim_{x\to 0} \frac{K\sin Kx}{Kx}$

$$= \lim_{x\to 0} K\left(\frac{\sin Kx}{Kx}\right) = K$$

$\therefore 0$ is not Point of infinite discontinuity.

(vi) For $m=0$ the given integral $\int_1^\infty \frac{\sin x^m}{x^n}\,dx$ reduces to $\sin 1 \int_1^\infty \frac{1}{x^n}\,dx$ is Convergent at ∞ if $n>1$.

For $m \neq 0$

Put $x^m = t \Rightarrow x = t^{1/m}$

$dx = \frac{1}{m} t^{1/m - 1}\,dt$

$$\therefore \int_1^\infty \frac{\sin x^m}{x^n}\,dx = \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{n/m}} \cdot t^{1/m - 1}\,dt$$

$$= \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{n/m - 1/m + 1}}\,dt$$

$$= \frac{1}{m}\int_1^\infty \frac{\sin t}{t^{\frac{n-1}{m}+1}}\,dt$$

Let $f(t) = \sin t$; $g(t) = \frac{1}{t^{\frac{n-1}{m}+1}}$

$\rightarrow$ (i) $\int_0^\infty \sin x^2\,dx$ (ii) $\int_0^\infty \frac{x}{1+x^2}\sin x\,dx$

(iii) $\int_0^\infty \cos x^3\,dx$ (iv) $\int_0^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$

Sol^n:- we have

(i) $$\int_0^\infty \sin x^2\,dx = \int_0^1 \sin x^2\,dx + \int_1^\infty \sin x^2\,dx \quad ①$$

Since $\int_0^1 \sin x^2\,dx$ is a proper integral.

$\therefore$ It is convergent.

To test the Convergence of $\int_1^\infty \sin x^2\,dx$ at ∞:-

Let $f(x) = 2x\sin x^2$ & $g(x) = \frac{1}{2x}$

Since $\left|\int_1^t f(x)\,dx\right| = \left|\int_1^t 2x\sin x^2\,dx\right|$

$$= \left| (\cos x^2)\Big|_1^t \right|$$

$$= |\cos 1 - \cos t^2|$$

$$\leq 2$$

$\therefore \int_1^t f(x)\,dx$ is bounded for all $t \geq 1$.

(iv) Since $x=0$ is a point of infinite discontinuity.

$\therefore$ we have to test the convergence of the given integral both at '0' & ∞

Now $\int_0^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx = \int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx + \int_a^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$ $a>0$. (1)

To test the convergence of

$\int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx$ at '0' :—

Let $f(x) = \frac{\cos x}{\sqrt{x+x^2}} = \frac{\cos x}{\sqrt{x}\sqrt{1+x}}$

$\forall x \in (0,a]$

Let $g(x) = \frac{1}{\sqrt{x}}$ $\forall x \in (0,a]$

$\therefore \lim_{x\to 0} \frac{f(x)}{g(x)} = 1$

Since $\int_0^a g(x)\,dx = \int_0^a \frac{1}{\sqrt{x}}\,dx$ is convergent.

$(\because n = 1/2 < 1)$

$\cdot$ By Comparison test.

$\int_0^a f(x)\,dx = \int_0^a \frac{\cos x}{\sqrt{x+x^2}}\,dx$ is

Convergent.

To test the convergence of

$\int_a^\infty \frac{\cos x}{\sqrt{x+x^2}}\,dx$ at '∞'

Let $f(x) = \cos x$

$g(x) = \frac{1}{\sqrt{x+x^2}}$

$\rightarrow$ (i) $\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx$; $a \geq 0$

(ii) $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\,dx$; $a>0$

Sol^n (i) : Let $f(x) = \frac{\sin x}{x}$ and

$g(x) = e^{-ax}$; $a \geq 0$.

Since $\int_0^\infty f(x)\,dx$ is Convergent.

(By known method).

and $g(x)$ is bounded and monotonically $\downarrow$ function for $x>0$.

$\therefore$ By abels test

$\int_0^\infty f(x)\,g(x)\,dx = \int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx$

is convergent.

Rustem madari

Beta and Gamma Functions

12·1. BETA FUNCTION (M.D.U. 1981 ; K.U. 1982 ; G.N.D.U. 1981 S, 82 S ; Kanpur 1987 ; Meerut 1988, 90)

Definition. If $m>0$, $n>0$ then the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, which is obviously a function of m and n, is called a **Beta function** and is denoted by $B(m, n)$.

Thus $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $\forall m>0, n>0$. Beta function is also called the **First Eulerian Integral.**

For example,

(i) $$\int_0^1 x^3(1-x)^5\,dx=B(3+1, 5+1)$$
$$=B(4, 6)$$

(ii) $$\int_0^1 \sqrt{x}\,(1-x)^3\,dx=B\left(\frac{1}{2}+1, 3+1\right)$$
$$=B\left(\frac{3}{2}, 4\right)$$

(iii) $$\int_0^1 x^{-\frac{2}{3}}(1-x)^{-\frac{1}{2}}\,dx=B\left(-\frac{2}{3}+1, -\frac{1}{2}+1\right)$$
$$=B\left(\frac{1}{3}, \frac{1}{2}\right)$$

(iv) $\int_0^1 x^{-3}(1-x)^5\,dx$ is not a Beta function since $m=-3+1$
$$=-2<0.$$

12·2. CONVERGENCE OF BETA FUNCTION

Theorem. *Show that* $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ *exists if and only if m and n are both positive.* (M.D.U. 1991)

Proof. The integral is proper if $m\geq 1$ and $n\geq 1$. 0 is the only point of infinite discontinuity if $m<1$ and 1 is the only point of infinite discontinuity if $n<1$.

For m < 1 and n < 1.

Take a number, $\frac{1}{2}$ (say), between 0 and 1 and examine the convergence of the improper integrals

$$\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx, \qquad \int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

at 0 and 1 respectively.

Convergence at 0, when m < 1

Let $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{(1-x)^{n-1}}{x^{1-m}}$.

Take $g(x)=\dfrac{1}{x^{1-m}}$

Then $\underset{x\to 0+}{\text{Lt}}\dfrac{f(x)}{g(x)}=\underset{x\to 0+}{\text{Lt}}(1-x)^{n-1}=1$ which is non-zero, finite.

Also $\int_0^{\frac{1}{2}} g(x)\,dx=\int_0^{\frac{1}{2}}\dfrac{dx}{x^{1-m}}$

is convergent if and only if $1-m<1$ i.e., $m>0$.

$\left[\because \int_a^b \dfrac{dx}{(x-a)^n} \text{ is convergent iff } n<1\right]$

$\therefore$ By comparison test,

$$\int_0^{\frac{1}{2}} f(x)\,dx=\int_0^{\frac{1}{2}} x^{m-1}(1-x)^{n-1}\,dx$$

is convergent at $x=0$ if $m>0$.

Convergence at 1, when n < 1

Let $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$

Take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\underset{x\to 1-}{\text{Lt}}\dfrac{f(x)}{g(x)}=\underset{x\to 1-}{\text{Lt}}x^{m-1}=1$ which is non-zero, finite.

Also $\int_{\frac{1}{2}}^{1} g(x)\,dx=\int_{\frac{1}{2}}^{1}\dfrac{dx}{(1-x)^{1-n}}$ is convergent if and only if $1-n<1$, i.e., $n>0$

$\left[\because \int_a^b \dfrac{dx}{(b-x)^n} \text{ is convergent iff } n<1\right]$

$\therefore$ By comparison test,

$$\int_{\frac{1}{2}}^{1} f(x)\,dx=\int_{\frac{1}{2}}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

is convergent at $x=1$ if $n>0$.

Hence $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges iff $m>0$, $n>0$.

12·3. PROPERTIES OF BETA FUNCTION

Property I. Symmetry of Beta function i.e. B(m, n)= B(n, m). *(M.D.U. 1983 ; K.U. 1982 ; G.N.D.U. 1981)*

Proof. By definition,

$$B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ m>0,\ n>0$$

Changing x to $1-x$ $\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$

$$B(m, n)=\int_0^1 (1-x)^{m-1}\,[1-(1-x)]^{n-1}\,dx$$

$$=\int_0^1 (1-x)^{m-1}x^{n-1}\,dx$$

$$=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx=B(n, m)$$

Hence $B(m, n)=B(n, m)$.

Property II. If m, n are positive integers, then

$$B(m, n)=\frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)\,!}$$

(M.D.U. 1983 S, 84 ; K.U. 1981 ; G.N.D.U. 1982 S)

Proof. $B(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Integrating by parts

$$=\left[x^{m-1}\cdot\frac{(1-x)^n}{n(-1)}\right]_0^1-\int_0^1 (m-1)x^{m-2}\cdot\frac{(1-x)^n}{n(-1)}\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^n\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}(1-x)\,dx$$

$$=\frac{m-1}{n}\int_0^1 [x^{m-2}(1-x)^{n-1}-x^{m-1}(1-x)^{n-1}]\,dx$$

$$=\frac{m-1}{n}\int_0^1 x^{m-2}(1-x)^{n-1}dx-\frac{m-1}{n}\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$=\frac{m-1}{n}B(m-1, n)-\frac{m-1}{n}B(m, n)$$

$$\Rightarrow \left(1+\frac{m-1}{n}\right)B(m, n)=\frac{m-1}{n}B(m-1, n)$$

$$\Rightarrow B(m, n)=\frac{m-1}{m+n-1}B(m-1, n) \qquad ...(1)$$

Changing m to $(m-1)$, we have

$$B(m-1, n)=\frac{m-2}{m+n-2} B(m-2, n)$$

Putting this value of $B(m-1, n)$ in (1), we have

$$B(m, n)=\frac{(m-1)(m-2)}{(m+n-1)(m+n-2)} B(m-2, n) \quad ...(2)$$

Generalising from (1) and (2)

$$B(m, n)=\frac{(m-1)(m-2)......1}{(m+n-1)(m+n-2)......(n+1)} B(1, n) \quad ...(3)$$

But $B(1, n)=\int_0^1 x^0(1-x)^{n-1}\, dx=\left[\frac{(1-x)^n}{n(-1)}\right]_0^1=\frac{1}{n}$

$\therefore$ From (3), we get

$$B(m, n)=\frac{(m-1)(m-2)......1}{(m+n-1)(m+n-2)......(n+1)n}$$

$$=\frac{(m-1)\,!}{(m+n-1)(m+n-2)......(n+1)n}$$

Multiplying the num. and denom. by $(n-1)\,!$, we have

$$B(m, n)=\frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)(m+n-2)...(n+1)n.(n-1)\,!}$$

$$=\frac{(m-1)\,!\,(n-1)\,!}{(m+n-1)\,!}$$

Property III.

$$\mathbf{B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx,\ m>0,\ n>0.}$$

(M.D.U. 1982, Rohilkhand 1984)

Proof. $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Note 1. Put $x=\frac{z}{1+z}$

then $dx=\frac{(1+z).1-z.1}{(1+z)^2}\, dz=\frac{dz}{(1+z)^2}$

$$1-x=1-\frac{z}{1+z}=\frac{1}{1+z}$$

Also $x(1+z)=z \Rightarrow x=z(1-x)$

or $z=\frac{x}{1-x}$

when $x=0,\ z=0$

When $x\to 1,\ z\to\infty$

$$\therefore \quad B(m, n)=\int_0^\infty \left(\frac{z}{1+z}\right)^{m-1}\left(\frac{1}{1+z}\right)^{n-1}\frac{dz}{(1+z)^2}$$

$$=\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

...(2)

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

[Second Method]

...(3)

Note!.

Put $\frac{x}{1+x}=z$ then $x=z(1+x)$

or $x(1-z)=z$

$$\therefore \quad x=\frac{z}{1-z}$$

$$dx=\frac{(1-z).1-z(-1)}{(1-z)^2}\,dz=\frac{dz}{(1-z)^2}$$

$$1+x=1+\frac{z}{1-z}=\frac{1}{1-z}$$

When $x=0$, $z=0$

When $x\to\infty$,

$$z=\underset{x\to\infty}{\text{Lt}}\frac{x}{1+x} \qquad \text{Form } \frac{\infty}{\infty}$$

$$=\underset{x\to\infty}{\text{Lt}}\frac{1}{1}=1$$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

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$$=\int_0^1 \left(\frac{z}{1-z}\right)^{m-1}.(1-z)^{m+n}.\frac{dz}{(1-z)^2}$$

$$=\int_0^1 z^{m-1}(1-z)^{n-1}\,dz=B(m, n)$$

$z = \frac{x}{1-x}$

$dz = \frac{(1-x)-(x)(-1)}{(1-x)^2}dx = \frac{1}{(1-x)^2}dx$

Cor. We have proved that

$$B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \quad B(n, m)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

But $B(m, n)=B(n, m)$

$$\therefore \quad B(m, n)=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Hence $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Express the following integrals in terms of Beta functions :*

(i) $\int_0^1 x^m(1-x^2)^n\,dx$ if $m>-1,\ n>-1$

(ii) $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ (G.N.D.U. 1982 S)

(iii) $\int_0^2 (8-x^3)^{-1/3}\,dx.$

Sol. (i) Put $x^2=z$ i.e. $x=z^{1/2}$ so that $dx=\frac{1}{2}z^{-1/2}\,dz$

When $x=0,\quad z=0$; when $x=1,\ z=1$

$$\therefore \int_0^1 x^m(1-x^2)^n\,dx=\int_0^1 z^{\frac{m}{2}}(1-z)^n\cdot\frac{1}{2}z^{-1/2}\,dz$$

$$=\frac{1}{2}\int_0^1 z^{\frac{m-1}{2}}(1-z)^n\,dz$$

$$=\frac{1}{2}B\left(\frac{m-1}{2}+1,\ n+1\right)$$

$$=\frac{1}{2}B\left(\frac{m+1}{2},\ n+1\right).$$

(ii) Put $x^5=z$, i.e. $x=z^{1/5}$ so that $dx=\frac{1}{5}z^{-4/5}\,dz$

When $x=0,\ z=0$; when $x=1,\ z=1.$

$$\therefore \int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx=\int_0^1 x^2(1-x^5)^{-1/2}\,dx$$

$$=\int_0^1 z^{2/5}(1-z)^{-1/2}\cdot\frac{1}{5}z^{-4/5}\,dz$$

$$=\frac{1}{5}\int_0^1 z^{-2/5}(1-z)^{-1/2}\,dz$$

$$=\frac{1}{5}B\left(-\frac{2}{5}+1,\ -\frac{1}{2}+1\right)$$

$$=\frac{1}{5}B\left(\frac{3}{5},\ \frac{1}{2}\right).$$

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(*iii*) Put $x^3=8z$, *i.e.* $x=2z^{1/3}$ so that $dx=\frac{2}{3}z^{-2/3}\,dz$

When $x=0$, $z=0$; when $x=2$, $z=1$

$$\therefore \int_0^2 (8-x^3)^{-1/3}\,dx=\int_0^1 (8-8z)^{-1/3}\cdot\frac{2}{3}\,z^{-2/3}\,dz$$

$$=\int_0^1 \frac{2}{3}\,z^{-2/3}\,\frac{1}{2}\,(1-z)^{-1/3}\,dz$$

$$=\frac{1}{3}\int_0^1 z^{-2/3}\,(1-z)^{-1/3}\,dz$$

$$=\frac{1}{3}\,B\left(-\frac{2}{3}+1,\ -\frac{1}{3}+1\right)$$

$$=\frac{1}{3}\,B\left(\frac{1}{3},\ \frac{2}{3}\right).$$

Example 2. *Express the following as Beta functions* :

(*i*) $\int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx$

(*ii*) $\int_0^1 x^3\,(1-x^2)^{3/2}\,dx$

(*iii*) $\int_0^2 x^3\,(8-x^3)^{-1/3}\,dx$

(*iv*) $\int_0^1 x^{m-1}\,(1-x^2)^{n-1}\,dx.$

Sol. (*i*) Put $x^2=4z$ *i.e.* $x=2z^{1/2}$

so that $dx=z^{-1/2}\,dz$

when $x=0$, $z=0$; when $x=2$, $z=1$

$$\therefore \int_0^2 \sqrt{x}\,(4-x^2)^{-1/4}\,dx$$

$$=\int_0^1 2^{1/2}\,z^{1/4}\,(4-4z)^{-1/4}\,z^{-1/2}\,dz$$

$$=\int_0^1 2^{1/2}\,z^{-1/4}\,.\,4^{-1/4}\,(1-z)^{-1/4}\,dz$$

$$=\int_0^1 z^{-1/4}\,(1-z)^{-1/4}\,dz$$

$$=B\left(-\frac{1}{4}+1,\ -\frac{1}{4}+1\right)$$

$$=B\left(\frac{3}{4},\ \frac{3}{4}\right)$$

(*ii*) Please try yourself. $\left[\textbf{Ans. } B\left(4,\ \frac{5}{2}\right)\right]$

(iii) Please try yourself. $\left[\text{Ans. } \frac{8}{3} \text{ B}\left(\frac{4}{3}, \frac{2}{3}\right)\right]$

(iv) Please try yourself. $\left[\text{Ans. } \frac{1}{2} \text{ B}\left(\frac{1}{2} m, n\right)\right]$

Example 3. *Show that*

$$\int_0^p x^m (p^q - x^q)^n \, dx = \frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

if $p>0, q>0, m>-1, n>-1.$ (K.U. 1981 S)

Sol. Put $x^q = p^q \cdot z$, *i.e.* $x = pz^{\frac{1}{q}}$

so that $dx = \frac{p}{q} z^{\frac{1}{q}-1} dz$

When $x=0, z=0$; when $x=p, z=1$

$$\therefore \int_0^p x^m (p^q - x^q)^n \, dx = \int_0^1 p^m z^{\frac{m}{q}} (p^q - p^q z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1} dz$$

$$= \int_0^1 p^m \cdot z^{\frac{m}{q}} \cdot p^{qn} (1-z)^n \cdot \frac{p}{q} z^{\frac{1}{q}-1} dz$$

$$= \frac{p^{qn+m+1}}{q} \int_0^1 z^{\frac{m+1}{q}-1} (1-z)^n \, dz$$

$$= \frac{p^{qn+m+1}}{q} \text{B}\left(\frac{m+1}{q}, n+1\right)$$

$$= \frac{p^{qn+m+1}}{q} \text{B}\left(n+1, \frac{m+1}{q}\right)$$

$\left| \because \text{B}(m, n) = \text{B}(n, m)\right.$

Example 4. *Prove that*

$$\int_0^a (a-x)^{m-1} \cdot x^{n-1} \, dx = a^{m+n-1} B(m, n).$$

(M.D.U. 1984)

Sol. Please try yourself. (Put $x = az$)

Example 5. *Show that*

$$\int_0^n \left(1 - \frac{x}{n}\right)^n \cdot x^{t-1} \, dx = n^t \, B(t, n+1) \text{ when } t>0, n>-1.$$

Sol. Put $\frac{x}{n} = z$ so that $dx = n\,dz$

When $x=0, z=0$; when $x=n, z=1$

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or

$$\therefore \int_0^n \left(1-\frac{x}{n}\right)^n . x^{t-1}\, dx=\int_0^1 (1-z)^n .(nz)^{t-1}. n\, dz$$

$$=n^t\int_0^1 z^{t-1}(1-z)^n\, dz$$

$$=n^t\, B(t, n+1).$$

Example 6. *Show that if $m>0$, $n>0$, then*

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\, dx=(b-a)^{m+n-1}\, B(m, n).$$

Sol. Put $x=a+(b-a)\,z$; so that $dx=(b-a)\,dz$

When $x=a$, $z=0$; when $x=b$, $z=1$

$$\therefore \int_a^b (x-a)^{m-1}(b-x)^{n-1}\, dx$$

$$=\int_0^1 [(b-a)z]^{m-1}\,[b-a-(b-a)z]^{n-1}\,(b-a)\, dz$$

$$=\int_0^1 (b-a)^{m-1}.\, z^{m-1}.\, (b-a)^{n-1}.\, (1-z)^{n-1}.\, (b-a)\, dz$$

$$=(b-a)^{m+n-1}\int_0^1 z^{m-1}(1-z)^{n-1}\, dz$$

$$=(b-a)^{m+n-1}\, B(m, n).$$

Example 7. *Show that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx=\frac{1}{(a+b)^m .\, a^n}B(m, n).$$

(K.U. 1981)

Sol. Put $\dfrac{x}{a+bx}=\dfrac{z}{a+b\,.\,1}$

so that $\dfrac{(a+bx).1-x.b}{(a+bx)^2}\,dx=\dfrac{dz}{a+b}$

or $\dfrac{a}{(a+bx)^2}\,dx=\dfrac{dz}{a+b}$ $\quad\therefore\quad \dfrac{dx}{(a+bx)^2}=\dfrac{dz}{a(a+b)}$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx$$

$$=\int_0^1 \left(\frac{x}{a+bx}\right)^{m-1}\cdot\left(\frac{1-x}{a+bx}\right)^{n-1}\cdot\frac{1}{(a+bx)^2}\, dx$$

$$=\int_0^1 \left(\frac{z}{a+b}\right)^{m-1}\cdot\left(\frac{1-z}{a}\right)^{n-1}\cdot\frac{dz}{a(a+b)}$$

$$\left[\because\ (a+b)x=az+bxz \text{ or } x=\frac{az}{a+b-bz}\ \therefore\ \frac{1-x}{a+bx}=\frac{1-z}{a}\right]$$

$$= \frac{1}{(a+b)^m . a^n} \int_0^1 z^{m-1} (1-z)^{n-1} dz$$

$$= \frac{1}{(a+b)^m . a^n} B(m, n).$$

Example 8. *Show that*

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{B(m, n)}{a^n (1+a)^m}$$

Sol. Please try yourself (same as Ex. 7 with $b=1$).

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Example 9. *Express* $\int_0^1 x^m (1-x^n)^p dx$ *in terms of Beta function and hence evaluate* $\int_0^1 x^5 (1-x^3)^3 dx$. (G.N.D.U. 1981)

Sol. Put $x^n = z$, *i.e.* $x = z^{\frac{1}{n}}$

so that $dx = \frac{1}{n} z^{\frac{1}{n}-1} dz$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore \int_0^1 x^m (1-x^n)^p dx = \int_0^1 z^{\frac{m}{n}} (1-z)^p . \frac{1}{n} z^{\frac{1}{n}-1} dz$$

$$= \frac{1}{n} \int_0^1 z^{\frac{m+1}{n}-1} (1-z)^p dz$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right) \qquad ...(1)$$

Comparing $\int_0^1 x^5 (1-x^3)^3 dx$ with $\int_0^1 x^m (1-x^n)^p dx$,

we have $m=5$, $n=3$, $p=3$

$$\therefore \text{ From (1), } \int_0^1 x^5 (1-x^3)^3 dx = \frac{1}{3} B\left(\frac{5+1}{3}, 3+1\right)$$

$$= \frac{1}{3} B(2, 4) = \frac{1}{3} \int_0^1 x^1 (1-x)^3 dx$$

$$= \frac{1}{3} \int_0^1 (1-x) [1-(1-x)]^3 dx = \frac{1}{3} \int_0^1 (1-x)^3 x dx$$

$$= \frac{1}{3} \int_0^1 (x^3 - x^4) dx = \frac{1}{3} \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{3} \left[\frac{1}{4} - \frac{1}{5}\right] = \frac{1}{60}$$

Example 10. *Prove that*

$$\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta=\frac{B(m,n)}{2a^m b^n}.$$

Sol. Let $I=\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta$

$$=\int_0^{\pi/2} \frac{\cos^{2m-2}\theta \sin^{2n-2}\theta\,.\cos\theta\sin\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta$$

$$=\int_0^{\pi/2} \frac{(\cos^2\theta)^{m-1}(\sin^2\theta)^{n-1}\cos\theta\sin\theta}{(a\cos^2\theta+b\sin^2\theta)^{m+n}}\,d\theta$$

Put $\sin^2\theta=x$ so that $2\sin\theta\cos\theta\,d\theta=dx$

and $\cos^2\theta=1-\sin^2\theta=1-x$

When $\theta=0$, $x=0$; when $\theta=\pi/2$, $x=1$

$$\therefore\quad I=\int_0^1 \frac{(1-x)^{m-1}x^{n-1}\,.\,\frac{1}{2}\,dx}{[a(1-x)+bx]^{m+n}}$$

$$=\frac{1}{2}\int_0^1 \frac{(1-x)^{m-1}x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

Put $\frac{x}{a+(b-a)x}=\frac{z}{a+(b-a)\,.\,1}=\frac{z}{b}$

$$\therefore\quad \frac{[a+(b-a)x]\,.\,1-x\,.\,(b-a)}{[a+(b-a)x]^2}\,dx=\frac{dz}{b}$$

$$\Rightarrow\quad \frac{dx}{[a+(b-a)x]^2}=\frac{dz}{ab}$$

When $x=0$, $z=0$; when $x=1$, $z=1$

Also $\frac{x}{a+(b-a)x}=\frac{z}{b}\quad\Rightarrow\quad bx=az+(b-a)xz$

$$\Rightarrow\quad [b-(b-a)z]=azx\quad\Rightarrow\quad x=\frac{az}{b-(b-a)z}$$

$$1-x=1-\frac{az}{b-(b-a)z}=\frac{b(1-z)}{b-(b-a)z}$$

$$a+(b-a)x=\frac{bx}{z}=\frac{abz}{b-(b-a)z}$$

so that $\frac{1-x}{a+(b-a)x}=\frac{1-z}{az}$

$$\therefore\quad I=\frac{1}{2}\int_0^1 \frac{(1-x)^{m-1}x^{n-1}}{[a+(b-a)x]^{m+n}}\,dx$$

$$=\frac{1}{2}\int_0^1\left[\frac{1-x}{a+(b-a)x}\right]^{m-1}\left[\frac{x}{a+(b-a)x}\right]^{n-1}\,.\,\frac{dx}{[a+(b-a)x]^2}$$

$$=\frac{1}{2}\int_0^1\left(\frac{1-z}{az}\right)^{m-1}\left(\frac{z}{b}\right)^{n-1}\frac{dz}{ab}$$

$$=\frac{1}{2a^m b^n}\int_0^1 z^{n-1}(1-z)^{m-1}dz=\frac{B(n,m)}{2a^m b^n}$$

$$=\frac{B(m,n)}{2a^m b^n} \qquad [\because \; B(n,m)=B(m,n)]$$

Example 11. *Prove that if p, q are positive then*

(i) $\dfrac{B(p,q+1)}{q}=\dfrac{B(p+1,q)}{p}=\dfrac{B(p,q)}{p+q}$

(M.D.U. 1982 S ; K.U. 1983)

(ii) $B(p,q)=B(p+1,q)+B(p,q+1)$.

Sol. (i) $\dfrac{B(p,q+1)}{q}=\dfrac{1}{q}\displaystyle\int_0^1 x^{p-1}(1-x)^q\,dx$

$$=\frac{1}{q}\int_0^1 (1-x)^q.\,x^{p-1}\,dx$$

Integrating by parts

$$=\frac{1}{q}\left[\left\{(1-x)^q.\frac{x^p}{p}\right\}_0^1-\int_0^1 q(1-x)^{q-1}(-1).\frac{x^p}{p}\,dx\right]$$

$$=\frac{1}{q}\cdot\frac{q}{p}\int_0^1 x^p(1-x)^{q-1}\,dx \qquad \ldots(\mathrm{I})$$

$$=\frac{B(p+1,q)}{p} \qquad \ldots(\mathrm{II})$$

Also from (I)

$$\frac{B(p,q+1)}{q}=\frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\,dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}.\,x(1-x)^{q-1}\,dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}[1-(1-x)]\,(1-x)^{q-1}\,dx$$

$$=\frac{1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\,dx-\frac{1}{p}\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$=\frac{1}{p}B(p,q)-\frac{1}{p}B(p,q+1)$$

or $\dfrac{B(p,q+1)}{q}+\dfrac{B(p,q+1)}{p}=\dfrac{B(p,q)}{p}$

or $\dfrac{p+q}{pq}B(p,q+1)=\dfrac{1}{p}B(p,q)$

$\therefore$ $\dfrac{B(p,q+1)}{q}=\dfrac{B(p,q)}{p+q}$ ...(III)

From (II) and (III)

$$\frac{B(p,q+1)}{q}=\frac{B(p+1,q)}{p}=\frac{B(p,q)}{p+q}$$

Note. For another method, see Gamma function.

(ii) R.H.S. $=B(p+1, q)+B(p, q+1)$

$$=\int_0^1 x^p(1-x)^{q-1}\,dx+\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$=\int_0^1 [x^p(1-x)^{q-1}+x^{p-1}(1-x)^q]\,dx$$

$$=\int_0^1 x^{p-1}(1-x)^{q-1}[x+(1-x)]\,dx$$

$$=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

$=B(p, q)=$ L.H.S.

Example 12. *Prove that*

$$\frac{B(m+1, n)}{B(m, n)}=\frac{m}{m+n},\quad m>0,\ n>0.$$

Sol. $B(m+1, n)=\int_0^1 x^m(1-x)^{n-1}\,dx$

Integrating by parts

$$=\left[x^m\cdot\frac{(1-x)^n}{-n}\right]_0^1-\int_0^1 mx^{m-1}\cdot\frac{(1-x)^n}{-n}\,dx$$

$$=\frac{m}{n}\int_0^1 x^{m-1}(1-x)^n\,dx$$

$$=\frac{m}{n}\int_0^1 x^{m-1}(1-x)^{n-1}(1-x)\,dx$$ $= (1-x)^{n-1}\cdot(x^{m-1}-x^m)$

$$=\frac{m}{n}\left[\int_0^1 x^{m-1}(1-x)^{n-1}\,dx-\int_0^1 x^m(1-x)^{n-1}\,dx\right]$$

$$=\frac{m}{n}[B(m, n)-B(m+1, n)]$$

$$\Rightarrow\quad \left(1+\frac{m}{n}\right)B(m+1, n)=\frac{m}{n}B(m, n)$$

$$\Rightarrow\quad \frac{B(m+1, n)}{B(m, n)}=\frac{m}{m+n}.$$

Note. For another method, see Gamma function.

Example 13. *Using the property $B(m, n)=B(n, m)$, evaluate*

$$\int_0^1 x^3(1-x)^{4/3}\,dx.$$

Sol. $\int_0^1 x^3(1-x)^{4/3}\,dx = B\left(3+1, \frac{4}{3}+1\right)$

$$= B\left(4, \frac{7}{3}\right) = B\left(\frac{7}{3}, 4\right)$$

$$= \int_0^1 x^{4/3}(1-x)^3\,dx$$

$$= \int_0^1 x^{4/3}(1-3x+3x^2-x^3)\,dx$$

$$= \int_0^1 (x^{4/3}-3x^{7/3}+3x^{10/3}-x^{13/3})\,dx$$

$$= \left[\frac{x^{7/3}}{\frac{7}{3}} - 3 \cdot \frac{x^{10/3}}{\frac{10}{3}} + 3 \cdot \frac{x^{13/3}}{\frac{13}{3}} - \frac{x^{16/3}}{\frac{16}{3}}\right]_0^1$$

$$= \frac{3}{7} - \frac{9}{10} + \frac{9}{13} - \frac{3}{16} = \frac{243}{7280}.$$

Example 14. *Prove that*

$$B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta. \qquad \textit{(M.D.U. 1983)}$$

Sol. $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$

$$\therefore \quad B(m, n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot \sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

Example 15. *Show that*

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

where $p>-1$, $q>-1$. Deduce that

$$\int_0^2 x^4(8-x^3)^{-1/3}\,dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right).$$

Sol. Put $\sin^2\theta = z$ so that $2\sin\theta\cos\theta\, d\theta = dz$

When $\theta=0$, $z=0$; when $\theta=\frac{\pi}{2}$, $z=1$

Also $\cos^2\theta=1-\sin^2\theta=1-z$

$$\therefore \int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\int_0^{\pi/2}(\sin^{p-1}\theta\cos^{q-1}\theta)\sin\theta\cos\theta\,d\theta$$

$$=\int_0^{\pi/2}(\sin^2\theta)^{\frac{p-1}{2}}(\cos^2\theta)^{\frac{q-1}{2}}.\sin\theta\,\cancel{\cos\theta\,d\theta}$$

$$=\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}.\tfrac{1}{2}\,dz$$

$$=\frac{1}{2}\int_0^1 z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$=\frac{1}{2}\,B\left(\frac{p-1}{2}+1,\ \frac{q-1}{2}+1\right)$$

$$=\frac{1}{2}\,B\left(\frac{p+1}{2},\ \frac{q+1}{2}\right) \qquad \ldots(1)$$

Second Part. Put $x^3=8z$ *i.e.*, $x=2z^{1/3}$

so that $dx=\frac{2}{3}z^{-2/3}\,dz$

When $x=0,\ z=0$; when $x=2,\ z=1$

$$\therefore \int_0^2 x^4(8-x^3)^{-1/3}\,dx=\int_0^1 16z^{4/3}(8-8z)^{-1/3}.\frac{2}{3}z^{-2/3}\,dz$$

$$=\int_0^1\frac{32}{3}\times 8^{-1/3}z^{2/3}(1-z)^{-1/3}\,dz$$

$$=\frac{32}{3\times 2}\int_0^1 z^{2/3}(1-z)^{-1/3}\,dz$$

$$=\frac{16}{3}\int_0^{\pi/2}\sin^{4/3}\theta\,(\cos^2\theta)^{-1/3}\times 2\sin\theta\cos\theta\,d\theta$$

where $z=\sin^2\theta$

$$=\frac{32}{3}\int_0^{\pi/2}\sin^{7/3}\theta\cos^{1/3}\theta\,d\theta$$

$$=\frac{32}{3}.\frac{1}{2}\,B\left(\frac{\frac{7}{3}+1}{2},\ \frac{\frac{1}{3}+1}{2}\right)$$

Here $p=\frac{7}{3}$, $q=\frac{1}{3}$ [using 1]

$$=\frac{16}{3}\,B\left(\frac{5}{3},\ \frac{2}{3}\right)$$

Example 16. *By putting* $\frac{x}{1-x}=\frac{at}{1-t}$, *where the constant a is suitably selected, show that*

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx=\frac{1}{9^{1/3}}\,B\left(\frac{2}{3},\frac{1}{3}\right).$$

Sol. $\frac{x}{1-x}=\frac{at}{1-t}$

$\Rightarrow \quad x-tx=at-atx$

$\Rightarrow \quad x[1-(1-a)t]=at$

$\Rightarrow \quad x=\frac{at}{1-(1-a)t}$

$\therefore \quad 1-x=1-\frac{at}{1-(1-a)t}=\frac{1-t}{1-(1-a)t}$

$1+2x=1+\frac{2at}{1-(1-a)t}=\frac{1-(1-3a)t}{1-(1-a)t}$

Also $dx=\frac{[1-(1-a)t]\,.\,a-at[-(1-a)]}{[1-(1-a)t]^2}\,dt$

$=\frac{adt}{[1-(1-a)t]^2}$

when $x=0$, $t=0$; when $x=1$, $\frac{at}{1-(1-a)t}=1$ so that $t=1$

$$\therefore \int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$=\int_0^1\left[\frac{at}{1-(1-a)t}\right]^{-\frac{1}{3}}\cdot\left[\frac{1-t}{1-(1-a)t}\right]^{-\frac{2}{3}}\times\left[\frac{1-(1-3a)t}{1-(1-a)t}\right]^{-1}\cdot\frac{adt}{[1-(1-a)t]^2}$$

$$=a^{2/3}\int_0^1 t^{-\frac{1}{3}}(1-t)^{-\frac{2}{3}}[1-(1-3a)t]^{-1}\,dt$$

Choosing $1-3a=0$ i.e. $a=\frac{1}{3}$, we get

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\,dx$$

$$=\left(\frac{1}{3}\right)^{2/3}\int_0^1 t^{-1/3}(1-t)^{-2/3}\,dt$$

$$=\frac{1}{9^{1/3}}\,B\left(-\frac{1}{3}+1,-\frac{2}{3}+1\right)$$

$$=\frac{1}{9^{1/3}}\,B\left(\frac{2}{3},\frac{1}{3}\right).$$

stant a is

$\frac{1}{3}\Big)$.

$\dfrac{dt}{1-a)t]^2}$

Example 17. *Show that*

$$\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx=\frac{1}{4(2)^{1/4}}\,B\left(\frac{7}{4},\frac{1}{4}\right).$$

Sol. Put $\dfrac{1-x^4}{1+x^4}=z$ so that $x^4=\dfrac{1-z}{1+z}$

i.e.
$$x=\left(\frac{1-z}{1+z}\right)^{1/4}$$

$$dx=\frac{1}{4}\left(\frac{1-z}{1+z}\right)^{-3/4}\times\frac{(1+z)(-1)-(1-z)\cdot 1}{(1+z)^2}\,dz$$

$$=\frac{1}{4}\left(\frac{1+z}{1-z}\right)^{3/4}\cdot\frac{-2}{(1+z)^2}\,dz$$

$$=\frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

Also $1-x^4=1-\dfrac{1-z}{1+z}=\dfrac{2z}{1+z}$

$$1+x^4=1+\frac{1-z}{1+z}=\frac{2}{1+z}$$

When $x=0$, $z=1$;

When $x=1$, $z=0$

$$\therefore\quad \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2}\,dx=\int_1^0 \frac{\left(\dfrac{2z}{1+z}\right)^{3/4}}{\left(\dfrac{2}{1+z}\right)^2}\times\frac{-dz}{2(1-z)^{3/4}(1+z)^{5/4}}$$

$$=\int_0^1 \frac{1}{4(2)^{1/4}}\,z^{3/4}(1-z)^{-3/4}\,dz$$

$$=\frac{1}{4(2)^{1/4}}\,B\left(\frac{7}{4},\frac{1}{4}\right).$$

Example 18. *Show that*

$$\int_0^\pi \frac{\sin^{n-1} x}{(a+b\cos x)^n}\,dx=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}\,B\left(\frac{1}{2}n,\frac{1}{2}n\right),$$

if $a^2>b^2$

Sol. Let $I=\displaystyle\int_0^\pi \frac{\sin^{n-1} x}{(a+b\cos x)^n}\,dx$

$$=\int_0^{\pi}\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^{n-1}dx}{\left[a+b\left(1-2\sin^2\frac{x}{2}\right)\right]^n}$$

$$=2^{n-1}\int_0^{\pi}\frac{\sin^{n-1}\frac{x}{2}\cos^{n-1}\frac{x}{2}}{\left(a+b-2b\sin^2\frac{x}{2}\right)^n}dx$$

Put $\frac{x}{2}=\theta$ then $dx=2d\theta$

when $x=0,\ \theta=0$; when $x=\pi,\ \theta=\frac{\pi}{2}$

$$\therefore\ I=2^{n-1}\int_0^{\pi/2}\frac{\sin^{n-1}\theta\cos^{n-1}\theta}{(a+b-2b\sin^2\theta)^n}\cdot 2d\theta$$

$$=2^{n-1}\int_0^{\pi/2}\frac{\sin^{n-2}\theta\cos^{n-2}\theta . 2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}d\theta$$

Put $\sin^2\theta=t$ so that $2\sin\theta\cos\theta\, d\theta=dt$

when $\theta=0,\ t=0$, when $\theta=\frac{\pi}{2},\ t=1$

$$\therefore\ I=2^{n-1}\int_0^{\pi/2}\frac{(\sin^2\theta)^{\frac{n-2}{2}}(1-\sin^2\theta)^{\frac{n-2}{2}} . 2\sin\theta\cos\theta}{(a+b-2b\sin^2\theta)^n}d\theta$$

$$=2^{n-1}\int_0^1\frac{t^{\frac{n-2}{2}}(1-t)^{\frac{n-2}{2}}}{(a+b-2bt)^n}dt$$

Put $\frac{1-t}{a+b-2bt}=\frac{z}{a+b}$ i.e. $t=\frac{(a+b)(1-z)}{a+b-2bz}$

$$dt=\frac{a+b}{(a+b-2bz)^2}[(a+b-2bz)(-1)-(1-z)(-2b)]\,dz$$

$$=\frac{(a+b)(-a+b)}{(a+b-2bz)^2}dz=-\frac{a^2-b^2}{(a+b-2bz)^2}dz$$

Also $1-t=1-\frac{(a+b)(1-z)}{a+b-2bz}=\frac{(a-b)z}{a+b-2bz}$

and $a+b-2bt=a+b-\frac{2b(a+b)(1-z)}{a+b-2bz}$

$$=\frac{a^2-b^2}{a+b-2bz}$$

When $t=0,\ z=1$, when $t=1,\ z=0$

$$\therefore\ I=-2^{n-1}\int_1^0 \frac{\left[\frac{(a+b)(1-z)}{a+b-2bz}\right]^{\frac{n-2}{2}}\left[\frac{(a-b)z}{a+b-2bz}\right]^{\frac{n-2}{2}}}{\left[\frac{a^2-b^2}{a+b-2bz}\right]^n}\times\frac{a^2-b^2}{(a+b-2bz)^2}\,dz$$

$$=2^{n-1}\int_0^1 \frac{(a^2-b^2)^{\frac{n-2}{2}}(1-z)^{\frac{n-2}{2}}\cdot z^{\frac{n-2}{2}}}{(a^2-b^2)^{n-1}}\,dz$$

$$=2^{n-1}\int_0^1 \frac{z^{n/2-1}(1-z)^{n/2-1}}{(a^2-b^2)^{n/2}}\,dz$$

$$=\frac{2^{n-1}}{(a^2-b^2)^{n/2}}\,B\left(\frac{n}{2},\frac{n}{2}\right).$$

Example 19. *Prove that* $\int_0^\infty \frac{t^3}{(1+t)^7}\,dt=\frac{1}{60}$

Sol. $\int_0^\infty \frac{t^3}{(1+t)^7}\,dt=\int_0^\infty \frac{t^{4-1}}{(1+t)^{4+3}}\,dt$

$$\left[\text{Form}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt=B(m,n)\right]$$

Remember if possible optional

$$=B(4,3)=\int_0^1 t^3(1-t)^2\,dt$$

$$=\int_0^1 t^3(1-2t+t^2)dt=\int_0^1 (t^3-2t^4+t^5)\,dt$$

$$=\left[\frac{t^4}{4}-\frac{2t^5}{5}+\frac{t^6}{6}\right]_0^1$$

$$=\frac{1}{4}-\frac{2}{5}+\frac{1}{6}=\frac{1}{60}.$$

Example 20. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ *in terms of Beta function, where* $m>0,\ n>0\ ;\ a>0,\ b>0.$

Sol. Put $bx=az$ or $x=\frac{az}{b}$ so that $dx=\frac{a}{b}\,dz$

When $x=0,\ z=0$ and when $x\to\infty,\ z\to\infty$

$$\therefore\ \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx=\int_0^\infty \left(\frac{az}{b}\right)^{m-1}\cdot\frac{1}{(a+az)^{m+n}}\cdot\frac{a}{b}\,dz$$

$$=\int_0^\infty \frac{a^{m-1}\cdot z^{m-1}\cdot a}{b^{m-1}\cdot a^{m+n}(1+z)^{m+n}\cdot b}\,dz$$

$$=\frac{1}{a^n b^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$=\frac{1}{a^n b^m}\,B(m, n).$$

Example 21. *Show that* $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=2B\,(m, n),$ *where* $m>0,\ n>0.$ (M.D U. 1981 S)

Sol. $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=B(m, n)$

Also $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx=B(m, n)$

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Adding $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=2B\,(m, n).$

Example 22. *Prove that* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=B(m, n),$ *where* m, n *are both positive.*

Sol. $B(m, n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

Note!

$$=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \ldots(i)$$

In the second integral on R.H.S. of (*i*), put $x=\frac{1}{t}$, so that

$$dx=-\frac{1}{t^2}\,dt$$

When $x=1,\ t=1$; when $x\to\infty,\ t=0$

$$\therefore\ \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx=\int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right)dt$$

$$=\int_0^1 \frac{1}{t^{m-1}}\cdot\frac{t^{m+n}}{(1+t)^{m+n}}\cdot\frac{1}{t^2}\,dt$$

$$=\int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx=\int_a^b f(z)\,dz\right]$$

$\therefore$ From (*i*),

$$B(m, n)=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx+\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

Example 23. *For $m>0$, $n>0$, show that*

$$\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx=0.$$

Sol. $\displaystyle\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx-\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$=B(m,n)-B(n,m)=0.$$

12·4. GAMMA FUNCTION

(*M.D.U. 1981 ; G.N.D.U. 1981 S ; Kanpur 1987 ; Meerut 1988, 90*)

Definition. If $n>0$, then the integral $\int_0^\infty x^{n-1}e^{-x}\,dx$, which (i-e) is obviously a function of n, is called a **Gamma function** and is denoted by $\Gamma(n)$.

Beta function

Thus $\boxed{\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx,\ \forall\ n>0}$ $\quad B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Gamma function is also called the **Second Eulerian Integral.**

For example,

(*i*) $\displaystyle\int_0^\infty x^3 e^{-x}\,dx=\Gamma(3+1)=\Gamma(4)$

(*ii*) $\displaystyle\int_0^\infty x^{2/3} e^{-x}\,dx=\Gamma\left(\frac{2}{3}+1\right)=\Gamma\left(\frac{5}{3}\right).$

12·5. CONVERGENCE OF GAMMA FUNCTION

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Theorem. *Show that $\int_0^\infty x^{n-1}e^{-x}\,dx$ converges iff $n>0$.*

(M.D.U. 1990 ; Meerut 1981)

Proof. If $n\geq 1$, the integrand $x^{n-1}e^{-x}$ is continuous at $x=0$.

If $n<1$, the integrand $\dfrac{e^{-x}}{x^{1-n}}$ has infinite discontinuity at $x=0$.

Thus we have to examine the convergence at 0 and ∞ both. Consider any positive number, say 1, and examine the convergence of

$$\int_0^1 x^{n-1}e^{-x}\,dx \quad\text{and}\quad \int_1^\infty x^{n-1}e^{-x}\,dx$$

at 0 and ∞ respectively.

Convergence at 0, when $n<1$

Let $\quad f(x)=\dfrac{e^{-x}}{x^{1-n}}$

Take $g(x)=\dfrac{1}{x^{1-n}}$

Then $\displaystyle \operatorname*{Lt}_{x\to 0+} \frac{f(x)}{g(x)} = \operatorname*{Lt}_{x\to 0+} e^{-x}=1$

which is non-zero, finite.

Also $\displaystyle \int_0^1 g(x)\,dx=\int_0^1 \frac{dx}{x^{1-n}}$

is convergent iff $1-n<1$ i.e. $n>0$

$\therefore$ By comparison test

$$\int_0^1 f(x)dx=\int_0^1 \frac{e^{-x}}{x^{1-n}}\,dx=\int_0^1 x^{n-1}e^{-x}\,dx$$

is convergent at $x=0$ if $n>0$.

Convergence at ∞

We know that $e^x>x^{n+1}$ whatever value n may have

For given n we have for sufficiently large value $e^x > x^{n+1}$

$\therefore$ $e^{-x}<x^{-n-1}$

and $x^{n-1}e^{-x}<x^{n-1}\cdot x^{-n-1}=\dfrac{1}{x^2}$

Since $\displaystyle \int_1^\infty \frac{1}{x^2}\,dx$ is convergent at ∞.

$\therefore$ $\displaystyle \int_1^\infty x^{n-1}e^{-x}\,dx$ is convergent at ∞ for every value of n.

Now $\displaystyle \int_0^\infty x^{n-1}e^{-x}\,dx=\int_0^1 x^{n-1}e^{-x}\,dx+\int_1^\infty x^{n-1}e^{-x}\,dx$

$\therefore$ $\displaystyle \int_0^\infty x^{n-1}e^{-x}\,dx$ converges iff $n>0$.

12·6. RECURRENCE FORMULA FOR GAMMA FUNCTION

Prove that $\Gamma(n)=(n-1)\,\Gamma(n-1)$, when $n>1$

Proof. By def. $\displaystyle \Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$

Integrating by parts

$$=\left[x^{n-1}\cdot\frac{e^{-x}}{-1}\right]_0^\infty-\int_0^\infty (n-1)\,x^{n-2}\cdot\left(\frac{e^{-x}}{-1}\right)dx$$

$$=-\left[\operatorname*{Lt}_{x\to\infty}\frac{x^{n-1}}{e^x}-0\right]+(n-1)\int_0^\infty e^{-x}\cdot x^{n-2}\,dx$$

$$=(n-1)\int_0^\infty e^{-x}\cdot x^{n-2}\,dx\quad\left[\because \operatorname*{Lt}_{x\to\infty}\frac{x^{n-1}}{e^x}=0 \text{ for } n>0\right]$$

$$=(n-1)\,\Gamma(n-1)$$

Hence $\Gamma(n)=(n-1)\,\Gamma(n-1)$.

Show that $e^{-x}x^n$ is bdd on $[0,\infty)$ for all +ve integral values of n, using this result show that $\int_0^\infty e^{-x}x^n$ and ... is finite.

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Cor. If n is a positive integer, then

$$\Gamma(n)=(n-1)\,!$$

When n is a +ve integer, then by repeated application of above formula, we get

$$\Gamma(n)=(n-1)\,\Gamma(n-1)$$
$$=(n-1)\,.\,(n-2)\,\Gamma(n-2)$$
$$=(n-1)(n-2)\,.\,(n-3)\,\Gamma(n-3)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$=(n-1)(n-2)\ldots\ldots 1\,\Gamma(1)$$
$$=(n-1)\,!\,\Gamma(1)$$

But $$\Gamma(1)=\int_0^\infty x^0 e^{-x}\,dx \qquad \text{(By def.)}$$

$$=\int_0^\infty e^{-x}\,dx=\left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$=-\left[\underset{x\to\infty}{\text{Lt}}\frac{1}{e^x}-e^0\right]=-[0-1]=1$$

Hence $\Gamma(n)=(n-1)\,!$ when n is a +ve integer.

Note. (i) If n is a +ve fraction, then

$\Gamma(n)=(n-1)\times$ go on decreasing by 1..........,

the series of factors being continued so long as the factors remain positive, the last factor being Γ (last factor).

For example, $\Gamma\left(\frac{7}{2}\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)=\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$$=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

(ii) If n is a +ve integer, $\Gamma(n)=(n-1)\,!$

12·7. RELATION BETWEEN BETA AND GAMMA FUNCTIONS

To show that

$$\boxed{B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \text{ where } m>0,\ n>0.}$$

(Agra 1984 ; Meerut 1986, 87, 88 ; Kanpur 1986)

Proof. We know that for $n>0$,

$$\Gamma(n)=\int_0^\infty x^{n-1} e^{-x}\,dx$$

Putting $x=az$ so that $dx=a\,dz$, we have

$$\Gamma(n)=\int_0^\infty (az)^{n-1} e^{-az}\,.\,a\,dz$$

$$=\int_0^\infty a^n z^{n-1} e^{-az}\,dz$$

Replacing z by x,

$$=\int_0^\infty a^n x^{n-1} e^{-ax}\, dx$$

Replacing ~~a by~~ z, we have

$$\Gamma(n)=\int_0^\infty z^n x^{n-1} e^{-zx}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma(n)\,.\, e^{-z} z^{m-1}=\int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx$$

Integrating both sides w.r.t. z between the limits 0 to ∞, we have

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1}\, dz=\int_0^\infty\int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx\, dz$$

$$=\int_0^\infty\int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dz\, dx$$

$$\Rightarrow \quad \Gamma(n)\Gamma(m)=\int_0^\infty x^{n-1}\left[\int_0^\infty z^{m+n-1} e^{-z(1+x)}\, dz\right] dx$$

Putting $\quad z(1+x)=y$

so that $\quad dz=\dfrac{dy}{1+x}$

$$\Gamma(n)\Gamma(m)=\int_0^\infty x^{n-1}\left[\int_0^\infty \left(\frac{y}{1+x}\right)^{m+n-1} e^{-y}\,\frac{dy}{1+x}\right] dx$$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\int_0^\infty y^{m+n-1} e^{-y}\, dy\right] dx$$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,[\Gamma(m+n)]\, dx$$

$$=\Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$=\Gamma(m+n)\, B(m, n)$$

$$\left[\because \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx=B(m, n)\right]$$

Hence $B(m, n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

12·8. PROVE THAT $\Gamma(\frac{1}{2})=\sqrt{\pi}$

(Meerut 1986 ; Kanpur 1985, 87 ; K.U. 1983 ; G.N.D.U. 1982)

Proof. We know that

$$B(m, n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

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Taking $m=n=\frac{1}{2}$,

$$B(\tfrac{1}{2},\tfrac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}=\frac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)}$$

or $$B(\tfrac{1}{2},\tfrac{1}{2})=[\Gamma(\tfrac{1}{2})]^2 \qquad [\because \Gamma(1)=1] \qquad ...(i)$$

Now $$B(\tfrac{1}{2},\tfrac{1}{2})=\int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$=\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

Putting $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\frac{\pi}{2}$

$$\therefore \quad B(\tfrac{1}{2},\tfrac{1}{2})=\int_0^{\pi/2}\frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}d\theta=2\Big[\theta\Big]_0^{\pi/2}=2\left(\frac{\pi}{2}-0\right)=\pi$$

$\therefore$ From (i), $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2=\pi$$

or $$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

12·9. PROVE THAT $\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$.

(M.D.U. 1983)

Proof. Put $x^2=z$ so that $2x\,dx=dz$ or $dx=\frac{dz}{2\sqrt{z}}$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty e^{-x^2}\,dx=\int_0^\infty e^{-z}\frac{dz}{2\sqrt{z}}=\frac{1}{2}\int_0^\infty e^{-z}z^{-\frac{1}{2}}\,dz$$

$$=\frac{1}{2}\int_0^\infty e^{-z}z^{\frac{1}{2}-1}\,dz$$

$$=\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \left[\because \Gamma(n)=\int_0^\infty e^{-x}x^{n-1}\,dx,\text{ here } n=\frac{1}{2}\right]$$

$$=\frac{1}{2}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

Cor. 1. Prove that $\int_{-\infty}^0 e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

Put $x=-z$ so that $dx=-dz$

When $x\to-\infty$, $z\to\infty$; when $x=0$, $z=0$

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$$\therefore \int_{-\infty}^{0} e^{-x^2}\,dx=\int_{\infty}^{0} e^{-z^2}(-dz)=-\int_{\infty}^{0} e^{-z^2}\,dz$$

$$=\int_{0}^{\infty} e^{-z^2}\,dz=\int_{0}^{\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

[By ~~Art. 12·9~~]

Cor. 2. Prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$.

$$\int_{-\infty}^{\infty} e^{-x^2}dx=2\int_{0}^{\infty} e^{-x^2}\,dx$$

$\left[\because\ e^{-x^2}\right.$ is an even function of x and

$\left.\int_{-a}^{a} f(x)\,dx=2\int_{0}^{a} f(x)\,dx \text{ if } f(x) \text{ is an even function of } x\right]$

$$=2\cdot\frac{\sqrt{\pi}}{2}$$

[By Cor. 1]

$$=\sqrt{\pi}.$$

12·10. TO EVALUATE $\int_{0}^{\pi/2} \sin^p x \cos^q x\, dx$

where p>−1, q>−1. *(K.U. 1980, 82 S)*

Put $\sin^2 x=z$ so that $2\sin x\cos x\,dx=dz$

When $x=0,\ z=0$; when $x=\frac{\pi}{2},\ z=1$

Also $\cos^2 x=1-\sin^2 x=1-z$

$$\therefore \int_{0}^{\pi/2} \sin^p x \cos^q x\,dx$$

$$=\int_{0}^{\pi/2}(\sin^{p-1} x\cos^{q-1} x)\sin x\cos x\,dx$$

$$=\int_{0}^{\pi/2}(\sin^2 x)^{\frac{p-1}{2}}(\cos^2 x)^{\frac{q-1}{2}}\cdot\sin x\cos x\,dx$$

$$=\int_{0}^{1} z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\cdot\tfrac{1}{2}\,dz$$

$$=\frac{1}{2}\int_{0}^{1} z^{\frac{p-1}{2}}(1-z)^{\frac{q-1}{2}}\,dz$$

$$=\frac{1}{2}\,B\left(\frac{p-1}{2}+1,\ \frac{q-1}{2}+1\right)$$

$$=\frac{1}{2}\cdot B\left(\frac{p+1}{2},\ \frac{q+1}{2}\right)$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

$$\left[\because\ B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

Hence $\displaystyle\int_0^{\pi/2}\sin^p x\cos^q x\,dx=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}.$

Example 1. *Express the following in terms of Gamma functions :*

2006 P-I (i) $\displaystyle\int_0^1 x^p(1-x^q)^n\,dx$ where $p>0,\ q>0,\ n>0$

(ii) $\displaystyle\int_0^1 x^{p-1}(1-x^2)^{q-1}\,dx$ where $p>0,\ q>0$

(iii) $\displaystyle\int_0^a x^{p-1}(a-x)^{q-1}\,dx$ where $p>0,\ q>0.$

Sol. (*i*) Put $x^q=z$ or $x=z^{\frac{1}{q}}$

so that $\displaystyle dx=\frac{1}{q}\,z^{\frac{1}{q}-1}\,dz=\frac{1}{q}\,z^{\frac{1-q}{q}}\,dz$

When $x=0,\ z=0$ and when $x=1,\ z=1$

$$\therefore\ \int_0^1 x^p(1-x^q)^n\,dx=\int_0^1 z^{\frac{p}{q}}(1-z)^n\cdot\frac{1}{q}\,z^{\frac{1-q}{q}}\,dz$$

$$=\frac{1}{q}\int_0^1 z^{\frac{p+1-q}{q}}(1-z)^n\,dz$$

$$=\frac{1}{q}\,B\left(\frac{p+1-q}{q}+1,\ n+1\right)$$

$$=\frac{1}{q}\,B\left(\frac{p+1}{q},\ n+1\right)$$

$$=\frac{1}{q}\,\frac{\Gamma\left(\frac{p+1}{q}\right)\Gamma(n+1)}{\Gamma\left(\frac{p+1}{q}+n+1\right)}.$$

(*ii*) Please try yourself. (Put $x^2=z$)

$$\left[\textbf{Ans. }\frac{\Gamma(p/2)\,\Gamma(q)}{2\,\Gamma(p/2+q)}\right]$$

(*iii*) Please try yourself. (Put $x = az$)

$$\left[\text{Ans. } a^{p+q-1} \cdot \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right]$$

Example 2. *Show that* $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$.

(*Meerut 1989*)

Sol. Put $x^n = z$ i.e. $x = z^{\frac{1}{n}}$

so that $$dx = \frac{1}{n} z^{\frac{1}{n}-1} dz = \frac{1}{n} \cdot z^{\frac{1-n}{n}} dz$$

When $x=0,\ z=0$; when $x=1,\ z=1$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^n}} = \int_0^1 \frac{\frac{1}{n} z^{\frac{1-n}{n}}}{\sqrt{1-z}} dz$$

$$= \frac{1}{n}\int_0^1 z^{\frac{1}{n}-1} (1-z)^{-1/2}\, dz$$

$$= \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{2}\right) = \frac{1}{n} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{n\,\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

Example 3. *Show that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Sol. $$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{B(m,n)}{a^n(1+a)^m} \text{ [See Beta functions]}$$

$$= \frac{\Gamma(m)\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Example 4. *Evaluate*

(*i*) $\displaystyle\int_0^\infty e^{-x^3}\, dx$ (*ii*) $\displaystyle\int_0^\infty x^4 e^{-x^2}\, dx$.

(G.N.D.U. 1981 S)

(*iii*) $\displaystyle\int_0^\infty \sqrt{x}.e^{-x^3}\, dx$ (*iv*) $\displaystyle\int_0^\infty e^{-a^2x^2}\, dx,\ a>0.$

Sol. (*i*) Put $x^3 = z$ or $x = z^{1/3}$

so that $dx = \frac{1}{3} z^{-\frac{2}{3}} dz$

When $x = 0, z = 0$; when $x \to \infty, z \to \infty$

$$\therefore \int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-z} . \tfrac{1}{3} z^{-\frac{2}{3}} dz$$

$$= \frac{1}{3} \int_0^\infty e^{-z} . z^{\frac{1}{3}-1} dz = \tfrac{1}{3}\Gamma\left(\tfrac{1}{3}\right)$$

Note. $\int_0^\infty e^{-x^3} dx = \frac{1}{3} \Gamma\left(\frac{1}{3}\right) = \Gamma\left(\frac{4}{3}\right)$

(*G.N.D.U. 1980 S*)

$[\because (n-1)\, \Gamma(n-1) = \Gamma(n)]$

(*ii*) Proceeding as in part (*i*)

$$\int_0^\infty x^3 e^{-x^3} dx = \int_0^\infty z e^{-z} . \tfrac{1}{3} z^{-\frac{2}{3}} dz$$

$$= \frac{1}{3} \int_0^\infty e^{-z} . z^{1/3} dz = \frac{1}{3} \int_0^\infty e^{-z} . z^{\frac{4}{3}-1} dz = \frac{1}{3} \Gamma\left(\frac{4}{3}\right)$$

$$= \frac{1}{3} \cdot \frac{1}{3} \Gamma\left(\frac{1}{3}\right) \qquad [\because \Gamma(n) = (n-1)\, \Gamma(n-1)]$$

$$= \frac{1}{9} \Gamma\left(\frac{1}{3}\right).$$

(*iii*) Proceeding as in part (*i*)

$$\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \int_0^\infty z^{1/6} e^{-z} . \tfrac{1}{3} z^{-\frac{2}{3}} dz$$

$$= \frac{1}{3} \int_0^\infty e^{-z} . z^{-\frac{1}{2}} dz$$

$$= \frac{1}{3} \int_0^\infty e^{-z} . z^{\frac{1}{2}-1} dz = \frac{1}{3} \Gamma\left(\frac{1}{2}\right) = \frac{1}{3} \sqrt{\pi}$$

(*iv*) Put $a^2 x^2 = z$

i.e. $x = \dfrac{\sqrt{z}}{a}$

so that $dz = \dfrac{z^{-\frac{1}{2}}}{2a} dz.$

When $x=0,\ z=0$; when $x\to\infty,\ z\to\infty$

$$\therefore \int_0^\infty e^{-a^2x^2}\,dx=\int_0^\infty e^{-t}.\frac{z^{-\frac{1}{2}}}{2a}\,dz=\frac{1}{2a}\int_0^\infty e^{-t}.z^{\frac{1}{2}-1}\,dz$$

$$=\frac{1}{2a}\,\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2a} \qquad \left[\because\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\,\right]$$

Example 5. *Show that*

$$\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}=\pi. \qquad \text{(M.D.U. 1981 S)}$$

Sol. $\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$

$$=\int_0^{\pi/2}\sin^{1/2}\theta\cos^\circ\theta\,d\theta\times\int_0^{\pi/2}\sin^{-\frac{1}{2}}\theta\cos^\circ\theta\,d\theta$$

$$=\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}\times\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{2\Gamma(5/4)}\times\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{3}{4})}$$

$$=\frac{1}{4}\cdot\frac{[\Gamma(\frac{1}{2})]^2\,\Gamma(\frac{1}{4})}{\Gamma(5/4)}=\frac{1}{4}\cdot\frac{(\sqrt{\pi})^2\,.\,\Gamma(\frac{1}{4})}{\frac{1}{4}\,\Gamma(\frac{1}{4})}=\pi.$$

Example 6. *Prove that if $n>-1$,*

$$\int_0^\infty x^n e^{-a^2x^2}\,dx=\frac{1}{2a^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right) \qquad \text{(Agra 1984)}$$

Hence or otherwise show that $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx=\frac{\sqrt{\pi}}{a}$.

(M.D.U. 1982)

Sol. Put $a^2x^2=z$; i.e. $x=\frac{\sqrt{z}}{a}$

so that $dx=\frac{z^{-1/2}}{2a}$

When $x=0,\quad z=0$; When $x\to\infty,\ z\to\infty$.

$$\therefore \int_0^\infty x^n e^{-a^2x^2}\,dx=\int_0^\infty\frac{z^{n/2}}{a^n}\,.\,e^{-t}.\frac{z^{-1/2}}{2a}\,dz$$

$$=\frac{1}{2a^{n+1}}\int_0^\infty e^{-t}.\,z^{\frac{n-1}{2}}\,dz$$

$$=\frac{1}{2a^{n+1}}\,\Gamma\left(\frac{n-1}{2}+1\right)=\frac{1}{2a^{n+1}}\,\Gamma\left(\frac{n+1}{2}\right) \qquad ...(1)$$

are po

where

Putting $n=0$, $\int_0^\infty e^{-a^2x^2}\,dx=\frac{1}{2a}\,\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2a}$

$$\therefore \int_{-\infty}^{\infty} e^{-a^2x^2}\,dx=2\int_0^\infty e^{-a^2x^2}\,dx=2\,.\,\frac{\sqrt{\pi}}{2a}=\frac{\sqrt{\pi}}{a}$$

[$\because$ $e^{-a^2x^2}$ is an even function of x and

$\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ if $f(x)$ is an even function.]

Example 7. *Show that if $a>1$,*

$$\int_0^\infty \frac{x^a}{a^x}\,dx=\frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$ (K.U. 1983 S)

Sol. $\because$ $a=e^{\log a}$ $\quad\therefore\quad a^x=e^{x\log a}$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx=\int_0^\infty \frac{x^a}{e^{x\log a}}\,dx$$

$$=\int_0^\infty e^{-x\log a}\,.\,x^a\,dx$$

Put $x\log a=z$, *i.e.* $x=\frac{z}{\log a}$

so that $dx=\frac{dz}{\log a}$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\,dx=\int_0^\infty e^{-z}\,.\,\frac{z^a}{(\log a)^a}\,.\,\frac{dz}{\log a}$$

$$=\frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}\,.\,z^{(a+1)-1}\,dz$$

$$=\frac{\Gamma(a+1)}{(\log a)^{a+1}}.$$

Example 8. *Prove that* $\int_0^\infty e^{-ax}x^{n-1}\,dx=\frac{\Gamma(n)}{a^n}$ *where a, n are positive. Hence show that*

(i) $\int_0^\infty e^{-ax}\,x^{n-1}\cos bx\,dx=\frac{\Gamma(n)}{r^n}\cos n\theta$

(ii) $\int_0^\infty e^{-ax}\,x^{n-1}\sin bx\,dx=\frac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2=a^2+b^2$ *and* $\theta=\tan^{-1}\frac{b}{a}$.

Sol. Put $ax=z$ so that $dx=\frac{dz}{a}$

When $x=0$, $z=0$, when $x \to \infty$, $z \to \infty$

$$\therefore \int_0^\infty e^{-ax} x^{n-1}\, dx = \int_0^\infty e^{-z} \cdot \left(\frac{z}{a}\right)^{n-1} \cdot \frac{dz}{a}$$

$$= \frac{1}{a^n} \int_0^\infty e^{-z} z^{n-1}\, dz = \frac{\Gamma(n)}{a^n}$$

Replacing a by $a+ib$, we have

$$\int_0^\infty e^{-(a+ib)x}\, x^{n-1}\, dx = \frac{\Gamma(n)}{(a+ib)^n} \quad \ldots(1)$$

Now $e^{-(a+ib)x} = e^{-ax-ibx} = e^{-ax}(\cos bx - i \sin bx)$

$[\because \; e^{p-iq} = e^p (\cos q - i \sin q)]$

Also putting $a = r\cos\theta$ and $b = r \sin\theta$

$$a^2+b^2=r^2 \text{ and } \frac{b}{a} = \tan\theta \text{ i.e. } \theta = \tan^{-1}\frac{b}{a}$$

and $$(a+ib)^n = (r\cos\theta + i \cdot r\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$$

$$= r^n(\cos n\theta + i \sin n\theta)$$

[De Moivre's Theorem]

From (1),

$$\int_0^\infty e^{-ax}(\cos bx - i \sin bx)\, x^{n-1}\, dx$$

$$= \frac{\Gamma(n)}{r^n(\cos n\theta + i\sin n\theta)}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta + i \sin n\theta)^{-1}$$

$$= \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Equating real parts

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta$$

Equating imaginary parts

$$\int_0^\infty e^{-ax} x^{n-1} \sin bx\, dx = \frac{\Gamma(n)}{r^n} \sin n\theta.$$

Example 9. *Evaluate*

(i) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx$ (ii) $\int_0^{\pi/2} \sin^7 x\, dx$

(iii) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta.$

Sol. (*i*) $$\int_0^{\pi/2} \sin^3 x \cos^{5/2} x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5/2+1}{2}\right)}{\Gamma\left(\frac{3+1}{2}+\frac{5/2+1}{2}\right)}$$

$$\left[\because \int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}\right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma(2)\,\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{15}{4}\right)} = \frac{1}{2} \cdot \frac{1!\,.\,\Gamma\left(\frac{7}{4}\right)}{\frac{11}{4}\cdot\frac{7}{4}\,\Gamma\left(\frac{7}{4}\right)}$$

$$= \frac{8}{77}$$

[$\because$ $\Gamma(n)=(n-1)!$ if n is a +ve integer

and $\Gamma(n)=(n-1)\,\Gamma(n-1)$

$$\therefore \quad \Gamma\left(\frac{15}{4}\right) = \frac{11}{4}\,\Gamma\left(\frac{11}{4}\right) = \frac{11}{4}\cdot\frac{7}{4}\,\Gamma\left(\frac{7}{4}\right)\Bigg]$$

(*ii*) $$\int_0^{\pi/2} \sin^7 x \, dx = \int_0^{\pi/2} \sin^7 x \cos^0 x \, dx$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{0+1}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma(4)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{(4-1)!\,.\,\Gamma\left(\frac{1}{2}\right)}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{6}{\frac{105}{8}} = \frac{16}{35}.$$

(*iii*) $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{\sin\theta}{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{-\frac{1}{2}+1}{2}\right)}$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$=\frac{1}{2}\cdot\frac{\sqrt{2}\pi}{1}\quad\left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\sqrt{2}\pi\right.$$

[See Cor. with Duplication Formula]

$$=\frac{\pi}{\sqrt{2}}.$$

Example 10. *Show that*

$$\int_0^{\pi/2}\sin^n\theta\,d\theta=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\text{ where } n>-1.$$

(M.D.U. 1981)

Sol. $\displaystyle\int_0^{\pi/2}\sin^n\theta\,d\theta=\int_0^{\pi/2}\sin^n\theta\cos^0\theta\,d\theta$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+2}{2}\right)}\qquad\left[\because\ \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}\right]$$

$$=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

Example 11. *Prove that*

$$\int_0^1\frac{1}{\sqrt{1-x^4}}\,dx=\frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.\qquad\text{(K.U. 1981 S)}$$

Sol. Put $x^4=z$ *i.e.* $x=z^{1/4}$ so that $dx=\frac{1}{4}z^{-3/4}\,dz$

When $x=0$, $z=0$; when $x=1$, $z=1$

$$\therefore\quad\int_0^1\frac{1}{\sqrt{1-x^4}}\,dx=\int_0^1\frac{1}{\sqrt{1-z}}\cdot\frac{1}{4}z^{-3/4}\,dz$$

$$=\frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{-1/2}\,dz=\frac{1}{4}\,B\left(\frac{1}{4},\frac{1}{2}\right)$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma(\frac{1}{4})\,.\,\sqrt{\pi}}{\Gamma(\frac{3}{4})}$$

$$=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi/\Gamma(\frac{1}{4})}\left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)=\sqrt{2}\pi\right]$$

$$=\frac{1}{8}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Example 12. *Prove that* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

Sol. $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx=B(m,n)$

[See examples with Beta Function]

$$=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 13. *Evaluate* $\int_0^1 x^3(1-x)^{4/3}\,dx$. (K.U. 1982)

Sol. Put $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\,d\theta$

When $x=0,\ \theta=0$; when $x=1,\ \theta=\pi/2$.

$$\therefore\ \int_0^1 x^3(1-x)^{4/3}\,dx=\int_0^{\pi/2}\sin^6\theta\cos^{8/3}\theta\,.\,2\sin\theta\cos\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^7\theta\cos^{11/3}\theta\,d\theta$$

$$=2\,.\,\frac{1}{2}\,\frac{\Gamma\left(\frac{7+1}{2}\right)\Gamma\left(\frac{11/3+1}{2}\right)}{\Gamma\left(\frac{7+1}{2}+\frac{11/3+1}{2}\right)}$$

$$=\frac{\Gamma(4)\,\Gamma\left(\frac{7}{3}\right)}{\Gamma\left(\frac{19}{3}\right)}$$

$$=\frac{(4-1)\,!\,.\,\Gamma\left(\frac{7}{3}\right)}{\frac{16}{3}\cdot\frac{13}{3}\cdot\frac{10}{3}\cdot\frac{7}{3}\,\Gamma\left(\frac{7}{3}\right)}$$

$$=\frac{6\times 81}{16\times 13\times 10\times 7}=\frac{243}{7280}.$$

Example 14. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ *in terms of Beta and Gamma functions ; where* $m>0,\ n>0,\ a>0,\ b>0$.

and

Sol. Put $bx=az$ so that $dx=\frac{a}{b}\,dz$.

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx=\int_0^\infty \frac{\left(\frac{az}{b}\right)^{m-1}}{(a+az)^{m+n}}\cdot\frac{a}{b}\,dz$$

$$=\int_0^\infty \frac{a^{m-1}\,.\,a}{b^{m-1}\,.\,b.a^{m+n}}\cdot\frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$=\frac{1}{a^nb^m}\int_0^\infty \frac{z^{m-1}}{(1+z)^{m+n}}\,dz$$

$$=\frac{1}{a^nb^m}\,B(m,n) \qquad |\text{ By def.}$$

$$=\frac{1}{a^nb^m}\cdot\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Example 15. *Prove that* $\Gamma(n)=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy.$

(Meerut 1986 ; Kanpur 1985, 86 ; Agra 1981)

Sol. Put $\log\frac{1}{y}=z$, *i.e.* $\frac{1}{y}=e^z$

or $y=e^{-z}$ so that $dy=-e^{-z}\,dz$

When $y=0$, $z\to\infty$; when $y=1$, $z=0$

$$\therefore \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy=\int_\infty^0 z^{n-1}(-e^{-z})\,dz$$

$$=\int_0^\infty z^{n-1}e^{-z}\,dz=\Gamma(n).$$

Example 16. *Show that*

(i) $$\int_0^\infty \sqrt{x}\,e^{-x^2}\,dx\times\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\frac{\pi}{2\sqrt{2}}$$

(ii) $$\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy\times\int_0^\infty y^2e^{-y^4}\,dy=\frac{\pi}{4\sqrt{2}}.$$

Sol. (*i*) Put $x^2=z$ *i.e.* $x=z^{\frac{1}{2}}$ so that $dx=\frac{1}{2}z^{-\frac{1}{2}}\,dz$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty \sqrt{x}\,e^{-x^2}\,dx=\int_0^\infty z^{\frac{1}{4}}e^{-z}\,.\,\tfrac{1}{2}z^{-\frac{1}{2}}\,dz$$

$$=\tfrac{1}{2}\int_0^\infty z^{-\frac{1}{4}}e^{-z}\,dz=\tfrac{1}{2}\,\Gamma\left(\tfrac{3}{4}\right)$$

and $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\int_0^\infty \frac{e^{-z}}{z^{1/4}}\cdot\frac{1}{4}z^{-\frac{1}{4}}\,dz$

$$=\frac{1}{4}\int_0^\infty z^{-\frac{1}{2}}e^{-z}\,dz=\frac{1}{4}\Gamma(\tfrac{1}{2})$$

$$\therefore\ \int_0^\infty \sqrt{x}\,e^{-x^4}\,dx\times\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\,dx=\frac{1}{4}\Gamma\left(\frac{3}{4}\right)\cdot\frac{1}{4}\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{1}{4}\cdot\sqrt{2}\,\pi \qquad \left[\because\ \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)=\sqrt{2}\pi\right]$$

$$=\frac{\pi}{2\sqrt{2}}.$$

| By def.

(ii) Put $y^2=z$, $\int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$ [As in part (i)]

Put $y^4=z$ i.e. $y=z^{\frac{1}{4}}$ so that $dy=\frac{1}{4}z^{-\frac{3}{4}}\,dz$

When $y=0, z=0$; when $y\to\infty, z\to\infty$

gra 1981)

$$\therefore\ \int_0^\infty y^2 e^{-y^4}\,dy=\int_0^\infty z^{\frac{1}{2}}e^{-z}\cdot\frac{1}{4}z^{-\frac{3}{4}}\,dz$$

$$=\frac{1}{4}\int_0^\infty z^{-\frac{1}{4}}e^{-z}\,dz$$

$$\therefore\ \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}}\,dy\times\int_0^\infty y^2e^{-y^4}\,dy=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{1}{8}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{8}\times\sqrt{2}\pi=\frac{\pi}{4\sqrt{2}}.$$

Example 17. *Prove that*

$$\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}\qquad\text{(Kanpur 1987)}$$

Sol. Please try yourself. [See Art. 12·10]

Example 18. *Show that*

$$2^n\Gamma\left(n+\frac{1}{2}\right)=1.3.5.......(2n-1)\sqrt{\pi},$$

where n is a positive integer.

Sol. $\Gamma\left(n+\frac{1}{2}\right)=\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\ldots\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\ldots\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$=\frac{1}{2^n}(2n-1)(2n-3)\ldots\ldots3.1\sqrt{\pi}$$

$$\Rightarrow 2^n\,\Gamma\left(n+\frac{1}{2}\right)=1.3.5.\ \ldots\ldots(2n-1)\sqrt{\pi}$$

(writing the factors in reverse order)

Example 19. *Show that*

IFoS 2010 (i) $\displaystyle\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}\times\int_0^1\frac{dx}{\sqrt{1+x^4}}=\frac{\pi}{4\sqrt{2}}$

(Kanpur 1980)

(ii) $\displaystyle\int_0^\infty x e^{-x^8}\,dx\times\int_0^\infty x^2e^{-x^4}\,dx=\frac{\pi}{16\sqrt{2}}$

(iii) $\displaystyle\int_0^{\pi/2}\sin^p x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\,dx=\frac{\pi}{2(p+1)}$.

Sol. (*i*) Put $x^2=\sin\theta$ i.e. $x=\sqrt{\sin\theta}$

so that $dx=\dfrac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$

When $x=0$, $\theta=0$; when $x=1$, $\theta=\dfrac{\pi}{2}$

$$\therefore\ \int_0^1\frac{x^2}{\sqrt{1-x^4}}\,dx=\int_0^{\pi/2}\frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)}=\frac{\Gamma\left(\frac{3}{4}\right)\sqrt{\pi}}{4\times\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$=\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

$\Gamma\left(\frac{1}{2}\right)$

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Now put $x^2=\tan\phi$ i.e. $x=\sqrt{\tan\phi}$

so that $dx=\dfrac{\sec^2\phi}{2\sqrt{\tan\phi}}\,d\phi$

When $x=0$, $\phi=0$; when $x=1$, $\phi=\dfrac{\pi}{4}$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1+x^4}}=\int_0^{\pi/4}\frac{1}{\sec\phi}\cdot\frac{\sec^2\phi}{2\sqrt{\tan\phi}}\,d\phi$$

$$=\frac{1}{2}\int_0^{\pi/4}\frac{d\phi}{\sqrt{\sin\phi\cos\phi}}=\frac{\sqrt{2}}{2}\int_0^{\pi/4}\frac{d\phi}{\sqrt{2\sin\phi\cos\phi}}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\phi}{\sqrt{\sin 2\phi}}=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{dt}{\sqrt{\sin t}}$$

where $t=2\phi$

$$=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\sin^{-1/2}t\cos^{0}t\,dt$$

$$=\frac{1}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{1}{4\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}=\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \int_0^1\frac{x^2}{\sqrt{1-x^4}}\,dx\times\int_0^1\frac{dx}{\sqrt{1+x^4}}$$

$$=\sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\cdot\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$=\frac{\pi}{4\sqrt{2}}$$

(ii) Put $x^8=z$ i.e. $x=z^{1/8}$

so that $dx=\dfrac{1}{8}z^{-7/8}\,dz$

When $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty xe^{-x^8}\,dx=\int_0^\infty z^{1/8}e^{-z}\cdot\frac{1}{8}z^{-7/8}\,dz$$

$$=\frac{1}{8}\int_0^\infty z^{-3/4}e^{-z}\,dz=\frac{1}{8}\Gamma\left(\frac{1}{4}\right)$$

Now put $x^4=t$ i.e. $x=t^{1/4}$

so that $dx=\frac{1}{4}t^{-3/4}\,dt$

When $x=0$, $t=0$; when $x\to\infty$, $t\to\infty$

$$\therefore \int_0^\infty x^2 e^{-x^4}\,dx=\int_0^\infty t^{1/2} e^{-t}\cdot\frac{1}{4}e^{-3/4}\,dt$$

$$=\frac{1}{4}\int_0^\infty t^{-1/4}e^{-t}\,dt=\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$\therefore \int_0^\infty xe^{-x^8}\,dx\times\int_0^\infty x^2e^{-x^4}\,dx$$

$$=\frac{1}{8}\Gamma\left(\frac{1}{4}\right)\times\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$$

$$=\frac{1}{32}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{32}\times\sqrt{2}\pi$$

[See cor. with Art. 12·11]

$$=\frac{\pi}{16\sqrt{2}}$$

(iii) $\int_0^{\pi/2}\sin^p x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\,dx$

$$=\int_0^{\pi/2}\sin^p x\cos^\circ x\,dx\times\int_0^{\pi/2}\sin^{p+1}x\cos^\circ x\,dx$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}+\frac{0+1}{2}\right)}\times\frac{\Gamma\left(\frac{p+1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+1+1}{2}+\frac{0+1}{2}\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{p+3}{2}\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma\left(\frac{p+1}{2}+1\right)}$$

$$=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)(\sqrt{\pi})^2}{\frac{p+1}{2}\Gamma\left(\frac{p+1}{2}\right)}=\frac{\pi}{2(p+1)}.$$

Example 20. *Show that*

$$\int_0^1\sqrt{1-x^4}\,dx=\frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Sol. Put $x^4 = z$ i.e. $x = z^{1/4}$

so that $dx = \frac{1}{4} z^{-3/4}\, dz$

When $x=0$, $z=0$, when $x=1$, $z=1$

$$\therefore \int_0^1 \sqrt{1-x^4}\, dx = \int_0^1 (1-z)^{1/2} \cdot \tfrac{1}{4} z^{-3/4}\, dz$$

$$= \frac{1}{4}\int_0^1 z^{-3/4}(1-z)^{1/2}\, dz$$

$$= \frac{1}{4} B\left(\frac{1}{4}, \frac{3}{2}\right) = \frac{1}{4} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{2}\right)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)} = \frac{1}{8} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\frac{3}{4}\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{6} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{6} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\dfrac{\sqrt{2}\,\pi}{\Gamma\left(\frac{1}{4}\right)}}$$

$$\left[\because \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi\right]$$

$$= \frac{1}{6\sqrt{2\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2 = \frac{1}{12}\sqrt{\frac{2}{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2$$

Example 21. *Prove that*

(i) $B(p, q) = B(p+1, q) + B(p, q+1)$

(ii) $B(p, q)B(p+q, r) = B(q, r)B(q+r, p)$
$= B(r, p)B(r+p, q)$

(iii) $B(p, q)\, B(p+q, r)\, B(p+q+r, s) = \dfrac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{\Gamma(p+q+r+s)}$

Sol. (i) R.H.S. $= B(p+1, q) + B(p, q+1)$

$$= \frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \frac{p\,\Gamma(p)\,\Gamma(q) + \Gamma(p).\,q\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}$$

$$=\frac{(p+q)\,\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

$$=\mathrm{B}(p,q)=\text{L.H.S.}$$

(*ii*) $$\mathrm{B}(p,q)\,\mathrm{B}(p+q,r)=\frac{\mathrm{B}(p)\,\mathrm{B}(q)}{\mathrm{B}(p+q)}\cdot\frac{\mathrm{B}(p+q)\,\mathrm{B}(r)}{\mathrm{B}(p+q+r)}$$

$$=\frac{\mathrm{B}(p)\,\mathrm{B}(q)\,\mathrm{B}(r)}{\mathrm{B}(p+q+r)}$$

Similarly for others.

(*iii*) Please try yourself.

Example 22. *Prove that*

(*i*) $$\frac{B(p,q+1)}{q}=\frac{B(p+1,q)}{p}=\frac{B(p,q)}{p+q}$$

(*ii*) $$\frac{B(m+1,n)}{B(m,n)}=\frac{m}{m+n}$$

(*iii*) $$\frac{B(m+2,n-2)}{B(m,n)}=\frac{m(m+1)}{(n-1)(n-2)}.$$

Sol. (*i*) $$\frac{\mathrm{B}(p,q+1)}{q}=\frac{\Gamma(p)\,\Gamma(q+1)}{q\,\Gamma(p+q+1)}$$

$$=\frac{\Gamma(p)\,q\Gamma(q)}{q(p+q)\,\Gamma(p+q)}$$

$$=\frac{\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)}=\frac{\mathrm{B}(p,q)}{p+q}$$

Similarly $$\frac{\mathrm{B}(p+1,q)}{q}=\frac{\mathrm{B}(p,q)}{p+q}$$

Hence the result.

(*ii*) $$\mathrm{B}(m+1,n)=\frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+n+1)}$$

$$=\frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$=\frac{m}{m+n}\,\mathrm{B}(m,n)$$

$$\Rightarrow\quad\frac{\mathrm{B}(m+1,n)}{\mathrm{B}(m,n)}=\frac{m}{m+n}$$

(*iii*) $$\mathrm{B}(m+2,n-2)=\frac{\Gamma(m+2)\,\Gamma(n-2)}{\Gamma(m+n)}$$

$$=\frac{(m+1)m\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)}$$

$$B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \quad \frac{B(m+2, n-2)}{B(m, n)} = \frac{m(m+1)\,\Gamma(m)\,\Gamma(n-2)}{\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)}$$

$$= \frac{m(m+1)\,\Gamma(n-2)}{\Gamma(n)}$$

$$= \frac{m(m+1)\,\Gamma(n-2)}{(n-1)(n-2)\Gamma(n-2)}$$

$$= \frac{m(m+1)}{(n-1)(n-2)}.$$

Example 23. *Evaluate the following integrals :*

(i) $\int_0^\infty x^6 e^{-2x}\, dx$ (ii) $\int_0^\infty e^{-4x} x^{3/2}\, dx$

(iii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$ (iv) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

(v) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx$ (vi) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx.$

(Meerut 1989 ; Kanpur 1985)

Sol. (i) Put $2x=z$ i.e., $x=\frac{1}{2}z$

then $dx=\frac{1}{2}dz$

when $x=0$, $z=0$; when $x\to\infty$, $z\to\infty$

$$\therefore \int_0^\infty x^6 e^{-2x}\, dx = \int_0^\infty (\tfrac{1}{2}z)^6 e^{-z} \cdot \tfrac{1}{2}\, dz$$

$$= \frac{1}{128}\int_0^\infty z^6 e^{-z}\, dz = \frac{1}{128}\,\Gamma(7)$$

$$= \frac{1}{128}\,(6\,!) \qquad \Big|\ \because \Gamma(n)=(n-1)! \text{ where } n\in N$$

$$= \frac{6\times5\times4\times3\times2}{128} = \frac{45}{8}$$

(ii) Please try yourself. $\left[\text{Ans. } \frac{3\sqrt{\pi}}{128}\right]$

(iii) Put $x=2z$ then $dx=2dz$

When $x=0$, $z=0$; when $x=2$, $z=1$

$$\therefore \int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx = \int_0^1 \frac{4z^2}{\sqrt{2(1-z)}} \cdot 2dz$$

$$=4\sqrt{2}\int_0^1 z^2(1-z)^{-1/2}\,dz$$

$$=4\sqrt{2}B\left(3,\frac{1}{2}\right)$$

$$=4\sqrt{2}\,\frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}$$

$$=4\sqrt{2}\,.\,\frac{(2\,!)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}$$

$$=8\sqrt{2}\,.\,\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}=\frac{64\sqrt{2}}{15}$$

(iv) $I=\int_0^3\frac{dx}{\sqrt{3x-x^2}}=\int_0^3\frac{dx}{\sqrt{x}\sqrt{3-x}}$

Put $x=3z$ then $dx=3dz$

When $x=0,\ z=0$; when $x=3,\ z=1$

$\therefore$ $$I=\int_0^1\frac{3dz}{\sqrt{3z}\sqrt{3(1-z)}}=\int_0^1 z^{-1/2}(1-z)^{-1/2}\,dz$$

$$=B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\left[\Gamma\left(\frac{1}{2}\right)\right]^2$$

$[\because\ \Gamma(1)=1]$

$$=(\sqrt{\pi})^2=\pi$$

(v) $$I=\int_0^\infty\frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$$

$$=\int_0^\infty\frac{x^8}{(1+x)^{24}}\,dx-\int_0^\infty\frac{x^{14}}{(1+x)^{24}}\,dx$$

$$=\int_0^\infty\frac{x^{9-1}}{(1+x)^{9+15}}\,dx-\int_0^\infty\frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$=B(9,15)-B(15,9)$$

$$\left[\because\ \int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}\,dx=B(m,n)\right]$$

$=0$ $\qquad[\because\ B(m,n)=B(n,m)]$