Set - I

1

Vector Analysis

Paper - I

Section - B

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Syllabus:

Scalar and vector fields, differentiation of vector field of scalar variable; Gradient divergence and curl in cartesian and cylindrical co-ordinates; Higher order derivatives; vector identities and vector equations.

Application to geometry; curves in space, curvature and torsion; Serret - frenet's formulae.

Gauss and Stoke's theorems, Greens identities.

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Some basic concepts:

→ Many physical quantities can be divided into broadly two types (i) Scalars (ii) vectors

— A scalar is a quantity that is determined by its magnitude, its number of units measured on a suitable scale.

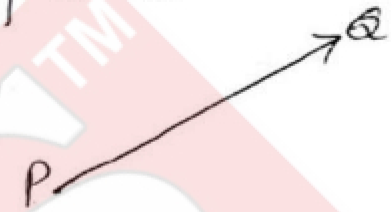
for example: length, temperature and voltage are scalars.

– vector is a quantity that is determined by both its magnitude and its direction.

For example: displacement, velocity, acceleration and force are vectors.

→ A vector is represented by a directed line segment.

i.e, $\overrightarrow{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P to Q.



The point P is called initial (tail) point of vector $\overrightarrow{PQ}$ and Q is called the terminal point (or) head (or) tip.

→ vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ etc.

→ The magnitude of a vector $\vec{a}$ is the +ve number which is the measure of its length and is denoted by $|\vec{a}|$ or a.

→ A vector of unit magnitude is called unit vector.

→ A vector of zero magnitude (which can have no direction) is called zero (null) vector.

→ The vector $\overrightarrow{QP}$ represents the negative of $\overrightarrow{PQ}$

→ Two vectors $\vec{a}$ & $\vec{b}$ having the same magnitude and same (or parallel) directions said to be equal and we write $\vec{a} = \vec{b}$.

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Product of two vectors:

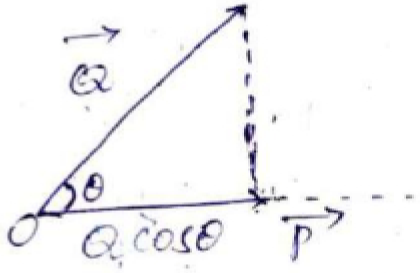
As a vector quantity involves magnitude and direction both, therefore, it is difficult to assign a definite meaning to the product of two vectors, whether the product is a scalar or a vector quantity. Accordingly, there are two types of products of two vectors.

(i) Scalar product (or) dot product:

The scalar product or dot product of any two vectors is the product of the magnitude of the first vector and the component of the second vector in the direction of the first vector.

(or)

The scalar product or dot product of any two vectors is the product of the magnitude of the two vectors and the cosine of the angle between them.



If $\vec{P}$ and $\vec{Q}$ are any two vectors, then the scalar product or dot product

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of these vectors is given by

$$\vec{P}\cdot\vec{Q} = |\vec{P}||\vec{Q}|\cos\theta$$
$$= PQ\cos\theta.$$

<u>Note</u>: By convention, we take 'θ' to be the angle smaller than or equal to π so that $0 \le \theta \le \pi$.

The dot product of two vectors is a scalar quantity.

<u>Properties of Scalar Product</u>:

If $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are any three vectors and K is a scalar quantity, then from

$$\vec{P}\cdot\vec{Q} = PQ\cos\theta$$

(1) $\vec{P}\cdot\vec{Q} = \vec{Q}\cdot\vec{P}$, i.e, the scalar product is commutative.

(2) $\vec{P}\cdot(\vec{Q}+\vec{R}) = \vec{P}\cdot\vec{Q} + \vec{P}\cdot\vec{R}$.
i.e, the scalar product is distributive over addition.

(3) $(k\vec{P})\cdot\vec{Q} = k(\vec{P}\cdot\vec{Q}) = \vec{P}\cdot(k\vec{Q})$

(4) If $\vec{P}\cdot\vec{Q}=0$ and $\vec{P}$ & $\vec{Q}$ are not zero vectors then $\vec{P}$ is $\perp$lar to $\vec{Q}$.

(5) $\vec{P}\cdot\vec{P} = |\vec{P}||\vec{P}|\cos 0^\circ = |\vec{P}|^2$ $\quad(\because \cos 0^\circ = 1)$

(6) $\vec{P}\cdot\vec{P} > 0$ for any non-zero vector $\vec{P}$.

(7) $\vec{P}\cdot\vec{P} = 0$ only if $\vec{P} = 0$.

(8) $\vec{P}, \vec{Q}$ are vectors in opposite direction

$$(\vec{P},\vec{Q}) = 180^\circ \Rightarrow \cos(\vec{P},\vec{Q}) = -1$$
$$\Rightarrow \vec{P}\cdot\vec{Q} = -|\vec{P}||\vec{Q}|.$$

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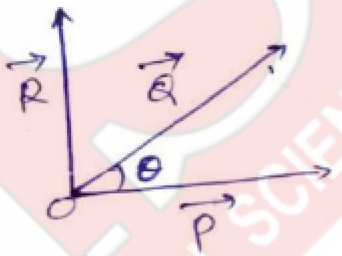
(ii) Vector product or cross product of two vectors:

→ The vector product of any two vectors is another vector which is ⊥lar to the plane formed by these two vectors. Its magnitude is equal to the product of the magnitudes of the two vectors and the sine of the angle between them.



→ If $\vec{P}$ and $\vec{Q}$ are two vectors then the cross product of $\vec{P}$ and $\vec{Q}$ is given by

$$\vec{P}\times\vec{Q} = PQ\sin\theta\,\hat{n}\ ; \quad 0\leq\theta\leq\pi$$
$$= \vec{R}$$

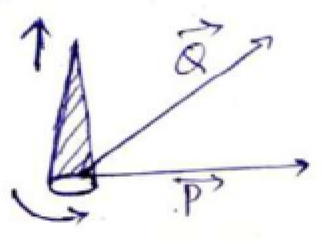
where P and Q are magnitudes of $\vec{P}$ & $\vec{Q}$;

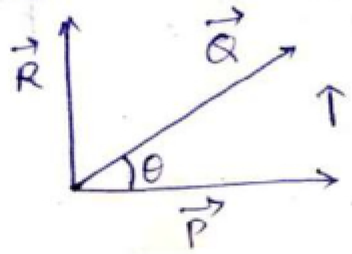
θ is the angle between $\vec{P}$ & $\vec{Q}$;

$\hat{n}$ is a unit vector ⊥lar to the plane containing $\vec{P}$ & $\vec{Q}$.

→ $|\vec{P}\times\vec{Q}| = PQ\sin\theta.$

→ The direction of the vector $\vec{R}$ can be obtained from the right-handed screw.





The axis of the screw is $\perp$lar to the plane containing the vectors $\vec{P}$ & $\vec{Q}$ be turned from $\vec{P}$ to $\vec{Q}$ through an angle θ between them, the direction of the advancement of the screw gives the direction of the vector. i.e, $\vec{P}\times\vec{Q}$.

Properties of vector product or cross product:

$\vec{P}, \vec{Q}$ and $\vec{R}$ are any three vectors and K is a scalar quantity, then $|\vec{P}\times\vec{Q}| = PQ\sin\theta$.

(1) $\vec{P}\times\vec{Q}$ is a vector

(2) $\vec{P}\times\vec{Q} = -\vec{Q}\times\vec{P}$

(3) If $\vec{P}$ and $\vec{Q}$ are non-zero vectors, and $\vec{P}\times\vec{Q} = 0$, then $\vec{P}$ is parallel to $\vec{Q}$

(4) $\vec{P}\times\vec{P} = 0$ for any vector $\vec{P}$.

i.e, $\vec{P}\times\vec{P} = PP\sin 0^\circ = 0$

(5) $\vec{P}\times(\vec{Q}+\vec{R}) = (\vec{P}\times\vec{Q}) + (\vec{P}\times\vec{R})$

(6) $(\vec{P}+\vec{Q})\times\vec{R} = (\vec{P}\times\vec{R}) + (\vec{Q}\times\vec{R})$

i.e, the vector product is distributive over addition.

(7) $(K\vec{P})\times\vec{Q} = K(\vec{P}\times\vec{Q}) = \vec{P}\times(K\vec{Q})$

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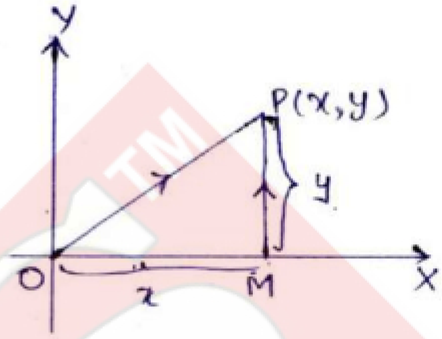
Components of Vectors :—

→ Let $P(x, y)$ be a point in a plane with reference OX & OY as the coordinate axes.

Then $OM = x$ & $MP = y$



→ Let $\hat{i}, \hat{j}$ be unit vectors along OX and OY respectively.

Then $\vec{OM} = x\hat{i}$ and $\vec{MP} = y\hat{j}$.

→ Vectors $\vec{OM}$ and $\vec{MP}$ are known as the components of $\vec{OP}$ along x-axis & y-axis respectively.

$$\text{Now} \quad \vec{OP} = \vec{OM} + \vec{MP} \quad [\text{By triangle law of addition}]$$

$$\Rightarrow \vec{OP} = x\hat{i} + y\hat{j}$$

→ Let $\vec{OP} = \vec{r}$ then $\vec{r} = x\hat{i} + y\hat{j}$

$$\text{Now} \quad OP^2 = OM^2 + MP^2$$

$$\Rightarrow OP^2 = x^2 + y^2$$

$$\Rightarrow OP = \sqrt{x^2 + y^2}$$

$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$$

Hence if a point 'P' in a plane has Coordinates (x, y) then

(i) $\vec{OP} = x\hat{i} + y\hat{j}$

(ii) $|\vec{OP}| = \sqrt{x^2 + y^2}$

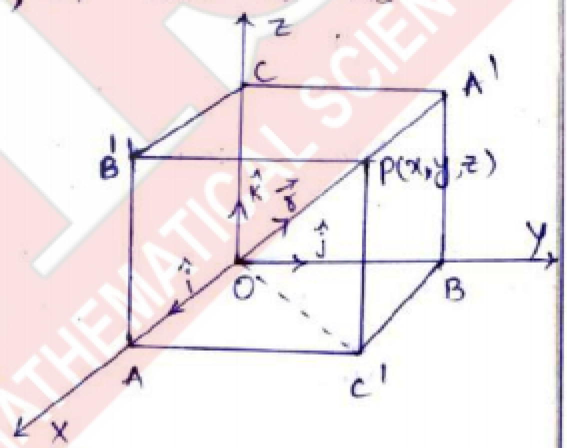
(iii) the component of $\vec{OP}$ along x-axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and whose direction is along ox (or) ox' according as x is positive (or) negative.

(iv) The component of $\vec{OP}$ along y-axis is a vector $y\hat{j}$ whose magnitude is $|y|$ and whose direction is along oy (or) oy' according as y is +ve (or) −ve.

Components of a vector in three Dimensions :-

Let $P(x, y, z)$ be a point in three dimensional space with reference to ox, oy and oz as coordinate axes.

Then $OA = x$, $OB = y$ and $OC = z$



Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along ox, oy and oz respectively.

Then $\vec{OA} = x\hat{i}$, $\vec{OB} = y\hat{j}$ and $\vec{OC} = z\hat{k}$.

we have $\vec{BC'} = \vec{OA} = x\hat{i}$

$\vec{C'P} = \vec{OC} = z\hat{k}$

$$\begin{aligned}
\vec{OP} &= \vec{OC'} + \vec{C'P} \\
&= \vec{OB} + \vec{BC'} + \vec{C'P} \\
&= \vec{OB} + \vec{OA} + \vec{OC} \\
&= x\hat{i} + y\hat{j} + z\hat{k}.
\end{aligned}$$

Let $\vec{OP} = \vec{r}$. Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

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Hence a position vector of a point $P(x,y,z)$ in space is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now $OP^2 = OC'^2 + C'P^2$

$$= (OB^2 + BC'^2) + C'P^2$$

$$= (OB^2 + OA^2) + OC^2$$

$$= OA^2 + OB^2 + OC^2$$

$$OP^2 = x^2 + y^2 + z^2$$

$$\therefore \quad OP = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

$\therefore$ If a point in space has coordinates (x,y,z), then its position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The vectors $x\hat{i}$, $y\hat{j}$, $z\hat{k}$ are known as the component vectors of $\vec{r}$ along x, y and z axes respectively.

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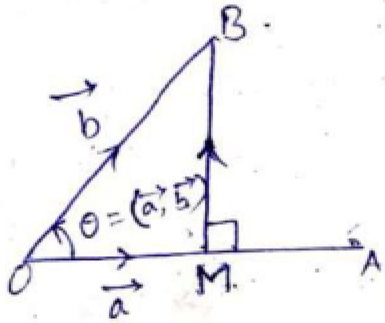
Projection of a vector along another vector :-

Let $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$. choose points O, A and B s.t $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $\angle AOB = \theta$

From B draw BM perpendicular to $\overleftrightarrow{OA}$. then OM is the projection of $\vec{b}$ on $\vec{a}$.



From definition,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= |\vec{a}||\vec{OB}|\cos\theta$$
$$= |\vec{a}|\cdot OB\cos\theta.$$
$$= |\vec{a}|\cdot OM.$$
$$= |\vec{a}|\,(\text{projection of } \vec{b} \text{ on } \vec{a})$$

Similarly $\vec{b}\cdot\vec{a} = |\vec{b}|\,(\text{projection of } \vec{a} \text{ on } \vec{b})$

$\Rightarrow$ projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$= \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}$$
$$= \hat{a}\cdot\vec{b}.$$

Similarly projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$$= \vec{a}\cdot\hat{b}.$$

→ If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors and $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ then $\vec{a} = \vec{b}$ (or) $(\vec{a}-\vec{b})$ is perpendicular to $\vec{c}$.

Solⁿ: $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c}) - (\vec{b}\cdot\vec{c}) = 0$

$\Rightarrow (\vec{a}\cdot\vec{c}) + (-\vec{b}\cdot\vec{c}) = 0$

$\Rightarrow (\vec{a}-\vec{b})\cdot\vec{c} = 0$

$\Rightarrow (\vec{a}-\vec{b}) = 0$ (or) $(\vec{a}-\vec{b}) \perp \vec{c}$

$\Rightarrow \vec{a} = \vec{b}$ (or) $(\vec{a}-\vec{b}) \perp \vec{c}$

→ We know that $\vec{a}\cdot\vec{b} = ab\cos\theta$.

where 'θ' be the angle between the vectors $\vec{a}$ & $\vec{b}$.

For two mutually perpendicular vectors

$$\vec{a}\cdot\vec{b} = ab\cos\pi/2 = 0;$$

therefore if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually $\perp$lar axes,

then $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Obviously two given vectors are orthogonal iff $\vec{a}\cdot\vec{b} = 0$.

→ The scalar product of two equal vectors is given by $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0$

$= a\cdot a\cos 0 = a^2 \quad (\because |\vec{a}| = a)$

i.e., the square of any vector is equal to the square of its modulus,

thus, if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1.$$

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

$$\left(\because \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0 \right.$$
$$\left. \& \ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1\right)$$

※ Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$:

Let $(\vec{a}, \vec{b}) = \theta$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$$

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are non-zero vectors and $(\vec{a}, \vec{b}) = \theta$

then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\,\sqrt{b_1^2+b_2^2+b_3^2}}$$

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Note:

(1) If $\vec{a}$ and $\vec{b}$ are perpendicular then
$a_1b_1 + a_2b_2 + a_3b_3 = 0$

(2) If $\vec{a}$ & $\vec{b}$ are parallel, then $\exists$ a scalar 't' such that $a_1 = tb_1$, $a_2 = tb_2$, & $a_3 = tb_3$

→ For any two vectors $\vec{a}$ and $\vec{b}$, the following are true.

(i) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b}$

(ii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$

(iii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(iv) $|\vec{a}\cdot\vec{b}| \le |\vec{a}|\,|\vec{b}|$

Sol^n: (iv) $|\vec{a}\cdot\vec{b}| = \left|\,|\vec{a}|\,|\vec{b}|\cos(\vec{a},\vec{b})\right|$

$= |\vec{a}||\vec{b}|\,|\cos(\vec{a},\vec{b})|$

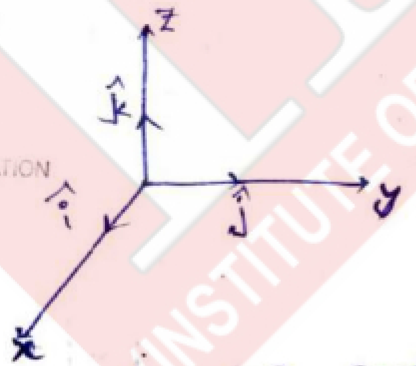
$\le |\vec{a}|\,|\vec{b}| \quad (\because |\cos\theta| \le 1)$

Vector product in component form:

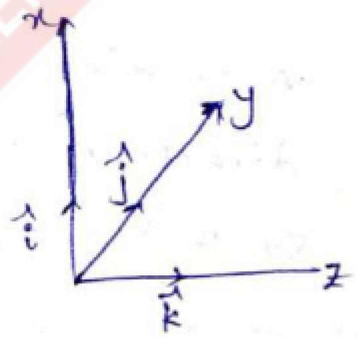
Let us express the vector product in component form w.r.t a cartesian coordinate system. But before we do so we would like to note that there are two types of such systems. Depending on the orientation of axes, they are termed as right-handed (or) left-handed.

By definition, a cartesian co-ordinate system is called right handed if the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in the directions of positive x, y, z axes form a right handed triple.

A cartesian co-ordinate system is called left-handed if $\hat{i}, \hat{j}, \hat{k}$ form a left-handed triple.



Right-handed cartesian co-ordinate system. fig (i)



Left-handed cartesian co-ordinate system. fig (ii)

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Now let two vectors $\vec{a}$ and $\vec{b}$ be given as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

where a_1, a_2, a_3 and b_1, b_2, b_3 are their components w.r.t a right-handed cartesian coordinate system.

Then

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1\,\hat{i}\times\hat{i} + a_1b_2\,\hat{i}\times\hat{j} + a_1b_3\,\hat{i}\times\hat{k}$$
$$+ a_2b_1\,\hat{j}\times\hat{i} + a_2b_2\,\hat{j}\times\hat{j} + a_2b_3\,\hat{j}\times\hat{k}$$
$$+ a_3b_1\,\hat{k}\times\hat{i} + a_3b_2\,\hat{k}\times\hat{j} + a_3b_3\,\hat{k}\times\hat{k}.$$

Now we have to know the cross product of the unit vectors $\hat{i}, \hat{j}, \hat{k}$ with themselves and each other to determine $\vec{a}\times\vec{b}$.

Let us consider the products $\hat{i}\times\hat{i}$ and $\hat{i}\times\hat{j}$.

From the definition of cross product,

$$|\hat{i}\times\hat{i}| = 1.1(\sin 0^\circ) = 0$$

so $\hat{i}\times\hat{i} = \vec{0}$

And $|\vec{i}\times\vec{j}| = 1.1(\sin 90^\circ) = 1$

According to the right hand rule the direction of $\hat{i}\times\hat{j}$ is along $\hat{k}$ (fig(i))

So that $\hat{i}\times\hat{j} = \hat{k}$.

Similarly, $\hat{j}\times\hat{j} = 0$ and $\hat{k}\times\hat{k} = 0$;

$\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times i = \hat{j}$;

$\hat{j}\times\hat{i} = -k$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$

From above we can see a cyclic pattern in the cross-products $\hat{i}\times\hat{j}$, $\hat{j}\times\hat{k}$, $\hat{k}\times\hat{i}$ etc. It is a good way to remember these cross products.

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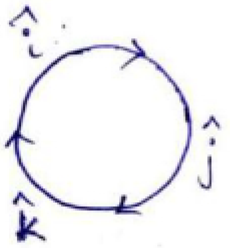
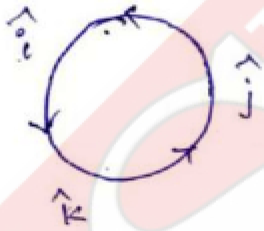
If we go around the circle clockwise all vector products are positive, i.e, $\hat{i}\times\hat{j}=\hat{k}$ and so on.

If we go in an anti-clockwise direction. the cross products are negative. i.e, $\hat{j}\times\hat{i}=-\hat{k}$. and so on.





Using the above results we can write the vector product in its component form:

$$\vec{a}\times\vec{b}=(a_2b_3-a_3b_2)\hat{i}+(a_3b_1-a_1b_3)\hat{j}+(a_1b_2-a_2b_1)\hat{k}.$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_2\\ b_1 & b_2 & b_3\end{vmatrix}$$

→ $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2.$

Sol :- $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2$

$$=\left|\,|\vec{a}||\vec{b}|\sin\theta\,.\hat{n}\right|^2+\left|\,|\vec{a}||\vec{b}|\cos\theta\right|^2$$

$$=|\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$=|\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta)$$

$$=|\vec{a}|^2|\vec{b}|^2$$

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which is known as Lagranges Identity.

→ $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$ where $\theta = (\vec{a}\cdot\vec{b})$

$$\Rightarrow \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

Scalar Tripple Product

→ Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors. Then we call $(\vec{a}\times\vec{b})\cdot\vec{c}$, the scalar triple product of $\vec{a}, \vec{b}$ & $\vec{c}$. this is a real number.

→ If $(\vec{a}\times\vec{b})\cdot\vec{c} = 0$, the following cases arise

(i) Atleast one of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ should be zero vector.

(ii) $\vec{a}\neq\vec{0}$, $\vec{b}\neq\vec{0}$, $\vec{c}\neq\vec{0}$ and $\vec{c}$ is $\perp$lar to $\vec{a}\times\vec{b}$.

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:— 1. we write the scalar triple product $(\vec{a}\times\vec{b})\cdot\vec{c}$ as $[\vec{a}\ \vec{b}\ \vec{c}]$ (or) $(\vec{a}, \vec{b}, \vec{c})$.

Properties of Scalar Tripple Product :-

→ If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same.

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

(or)

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

(or)

$[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}]$.

→ The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not in magnitude.

i.e. $[\vec{a} \ \vec{b} \ \vec{c}] = -[\vec{b} \ \vec{a} \ \vec{c}] = -[\vec{c} \ \vec{b} \ \vec{a}]$
$= -[\vec{a} \ \vec{c} \ \vec{b}]$

→ In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same.

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

→ The scalar triple product of three vectors is zero if any two of them

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are equal.

i.e. if two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ,

$[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$.

→ The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

i.e. If two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel (or) collinear then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ The necessary and sufficient condition for three non-zero non collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

i.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ For points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ will be coplanar if

$[\vec{d}\ \vec{b}\ \vec{c}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]$.

→ If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, then

$[\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

→ If $\vec{a}, \vec{b}, \vec{c}$ are not coplanar and they are adjacent sides of a parallelopiped, then the volume of parallelopiped is $[\vec{a}\ \vec{b}\ \vec{c}]$.

→ If $A = (x_1, y_1, z_1)$; $B = (x_2, y_2, z_2)$; $C = (x_3, y_3, z_3)$ and $D = (x_4, y_4, z_4)$ are the vertices of a tetrahedron, then the volume of the tetrahedron

$$= \frac{1}{6}\left|[\vec{AB}\ \vec{AC}\ \vec{AD}]\right|$$

$$= \frac{1}{6}\left|[(x_2-x_1, y_2-y_1, z_2-z_1)(x_3-x_1, y_3-y_1, z_3-z_1)(x_4-x_1, y_4-y_1, z_4-z_1)]\right|$$

$$= \frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}$$

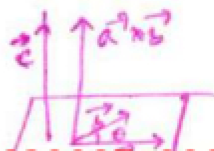
Vector Triple Product

→ Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are the vectors called vector triple products of $\vec{a}, \vec{b}, \vec{c}$.

Note:— (i) If any one of $\vec{a}, \vec{b}$ and $\vec{c}$ is the zero vector, then $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$.

(ii) If $\vec{a}$ is parallel to $\vec{b}$, then $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$. $(\because \vec{a} \times \vec{b} = \vec{0})$

(iii) If $\vec{c}$ is a perpendicular to both $\vec{a}$ & $\vec{b}$, then $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$.



→ For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}.$$

Note :— 1. $\vec{a}\times(\vec{b}\times\vec{c}) = -((\vec{b}\times\vec{c})\times\vec{a})$

$$= -((\vec{b}\cdot\vec{a})\vec{c} - (\vec{c}\cdot\vec{a})\vec{b})$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

2. $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c}).$

→ For any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Sol :- $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\cdot\vec{r}$ where $\vec{r} = \vec{c}\times\vec{d}$

$$= \vec{a}\cdot(\vec{b}\times\vec{r})$$

$$= \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}]$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

→ : For any vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Sol : $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{m}$

where $\vec{m} = \vec{c}\times\vec{d}$

$$= (\vec{a}\cdot\vec{m})\vec{b} - (\vec{b}\cdot\vec{m})\vec{a}$$

$$= (\vec{a}\cdot(\vec{c}\times\vec{d}))\vec{b} - (\vec{b}\cdot(\vec{c}\times\vec{d}))\vec{a}$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} \quad \text{--- (1)}$$

Now $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})$

$$= -\{[\vec{c}\ \vec{a}\ \vec{b}]\vec{d} - [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}\}$$

$$= [\vec{d}\ \vec{a}\ \vec{b}]\vec{c} - [\vec{c}\ \vec{a}\ \vec{b}]\vec{d}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} \quad \text{--- (2)}$$

From (1) & (2) we have

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

→ **Reciprocal system of vectors:**

The sets of vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are called reciprocal sets (or) system of vectors if $\vec{a}\cdot\vec{a'} = \vec{b}\cdot\vec{b'} = \vec{c}\cdot\vec{c'} = 1$

$\vec{a'}\cdot\vec{b} = \vec{a'}\cdot\vec{c} = \vec{b'}\cdot\vec{a} = \vec{c'}\cdot\vec{a} = \vec{c'}\cdot\vec{b} = \vec{b'}\cdot\vec{c} = 0$

— The sets $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ are reciprocal of vectors iff

$$\vec{a'} = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}, \quad \vec{b'} = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$$

$$\vec{c'} = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot(\vec{b}\times\vec{c})} \quad ; \text{ where } \vec{a}\cdot(\vec{b}\times\vec{c}) \neq 0$$

→ prove that $(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = (\vec{a}\cdot\vec{b}\times\vec{c})^2$

Sol: W.K.T $\vec{x}\times(\vec{c}\times\vec{a}) = \vec{c}\,(\vec{x}\cdot\vec{a}) - \vec{a}\,(\vec{x}\cdot\vec{c})$.

Let $\vec{x} = \vec{b}\times\vec{c}$

then $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{c}\,(\vec{b}\times\vec{c}\cdot\vec{a}) - \vec{a}\,(\vec{b}\times\vec{c}\cdot\vec{c})$

$= \vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{c}) - \vec{a}\,(\vec{b}\cdot\vec{c}\times\vec{c})$

$[\because (\vec{b}\times\vec{c})\cdot\vec{c} = \vec{b}\cdot(\vec{c}\times\vec{c})]$

$= \vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{c})$

$(\because \vec{c}\times\vec{c} = 0)$

$\therefore (\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = (\vec{a}\times\vec{b})\cdot\vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{c})$

$= (\vec{a}\times\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{b}\times\vec{c})$

$= (\vec{a}\cdot\vec{b}\times\vec{c})(\vec{a}\cdot\vec{b}\times\vec{c})$

$(\because \vec{a}\times\vec{b}\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}))$

$= (\vec{a}\cdot\vec{b}\times\vec{c})^2.$

→ Given the vectors $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}$, show that if $\vec{a}\cdot\vec{b}\times\vec{c} \neq 0$.

(i) $\vec{a}'\cdot\vec{a} = \vec{b}'\cdot\vec{b} = \vec{c}'\cdot\vec{c} = 1$,

(ii) $\vec{a}'\cdot\vec{b} = \vec{a}'\cdot\vec{c} = 0$, $\vec{b}'\cdot\vec{a} = \vec{b}'\cdot\vec{c} = 0$, $\vec{c}'\cdot\vec{a} = \vec{c}'\cdot\vec{b} = 0$,

(iii) if $\vec{a}\cdot\vec{b}\times\vec{c} = V$ then $\vec{a}'\cdot\vec{b}'\times\vec{c}' = 1/V$,

(iv) $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are non-coplanar if $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar.

(i) $\vec{a}'\cdot\vec{a} = \vec{a}\cdot\vec{a}' = \vec{a}\cdot\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

$\vec{b}'\cdot\vec{b} = \vec{b}\cdot\vec{b}' = \vec{b}\cdot\dfrac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\cdot\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

$\vec{c}'\cdot\vec{c} = \vec{c}\cdot\vec{c}' = \vec{c}\cdot\dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{c}\cdot\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

(ii) $\vec{a}'\cdot\vec{b} = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\times\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 0$

Similarly $\vec{a}'\cdot\vec{c} = \vec{c}\cdot\vec{a}' = 0$

$\vec{b}'\cdot\vec{a} = \vec{a}\cdot\vec{b}' = 0$, $\vec{b}'\cdot\vec{c} = \vec{c}\cdot\vec{b}' = 0$

$\vec{c}'\cdot\vec{a} = \vec{a}\cdot\vec{c}' = 0$, $\vec{c}'\cdot\vec{b} = \vec{b}\cdot\vec{c}' = 0$

(iii) $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{V}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{V}$, $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{V}$

Then $\vec{a}'\cdot\vec{b}'\times\vec{c}' = \dfrac{(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{V^3}$

$= \dfrac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{V^3}$

$$= \frac{(\vec{a}\cdot\vec{b}\times\vec{c})^2}{V^3} = \frac{V^2}{V^3} = \frac{1}{V}$$

(iv) If $\vec{a}, \vec{b}$, and $\vec{c}$ are non-coplanar $\vec{a}\cdot\vec{b}\times\vec{c} \neq 0$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$. then from part (iii) it follows that $\vec{a}'\cdot\vec{b}'\times\vec{c}' \neq 0$, so that $\vec{a}', \vec{b}'$ and $\vec{c}'$ are also non-coplanar.

12M 2003 → show that if $\vec{a}', \vec{b}'$ and $\vec{c}'$ are the reciprocals of the non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then any vector $\vec{r}$ may be expressed as

$$\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}.$$

Sol We know that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{b}\,(\vec{a}\cdot\vec{c}\times\vec{d}) - \vec{a}\,(\vec{b}\cdot\vec{c}\times\vec{d})$$
$$= \vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{d}) - \vec{d}\,(\vec{a}\cdot\vec{b}\times\vec{c}).$$

we take $\vec{b}\,(\vec{a}\cdot\vec{c}\times\vec{d}) - \vec{a}\,(\vec{b}\cdot\vec{c}\times\vec{d}) = \vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{d}) - \vec{d}\,(\vec{a}\cdot\vec{b}\times\vec{c})$.

$$\Rightarrow \vec{d}\,(\vec{a}\cdot\vec{b}\times\vec{c}) = \vec{a}\,(\vec{b}\cdot\vec{c}\times\vec{d}) - \vec{b}\,(\vec{a}\cdot\vec{c}\times\vec{d}) + \vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{d}).$$

$$\Rightarrow \vec{d} = \frac{\vec{a}\,(\vec{b}\cdot\vec{c}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}} - \frac{\vec{b}\,(\vec{a}\cdot\vec{c}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}} + \frac{\vec{c}\,(\vec{a}\cdot\vec{b}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}}$$

$(\because \vec{a}\cdot\vec{b}\times\vec{c} = [\vec{a}\,\vec{b}\,\vec{c}] \neq 0)$.

Let $\vec{d} = \vec{r}$ then

$$\vec{r} = \frac{\vec{r}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{a} + \frac{\vec{r}\cdot\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{b} + \frac{\vec{r}\cdot\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{c}$$

$$= \vec{r}\cdot\left(\frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{a} + \vec{r}\cdot\left(\frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{b} + \vec{r}\cdot\left(\frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{c}$$

$$= (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}$$

2000 → 7M.

Simplify $(A+B)\cdot(B+C)\times(C+A)$.

Sol $(A+B)\cdot(B+C)\times(C+A) = (A+B)\cdot[(B+C)\times(C+A)]$

$= (A+B)\cdot[(B+C)\times C + (B+C)\times A]$

$= (A+B)\cdot[B\times C + C\times C + B\times A + C\times A]$

$= (A+B)\cdot[B\times C + B\times A + C\times A]$ $(\because C\times C = 0)$.

$= A\cdot(B\times C) + A\cdot(B\times A) + A\cdot(C\times A) + B\cdot(B\times C) + B\cdot(B\times A) + B\cdot(C\times A)$

$= A\cdot(B\times C) + B\cdot(A\times A) + C\cdot(A\times A) + C\cdot(B\times B) + A\cdot(B\times B) + A\cdot(B\times C)$.

$= 2A\cdot(B\times C)$

$= 2A\cdot B\times C$.

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Vector Differential Calculus (14)

This is the beginning of vector calculus, which involves two kinds of functions, vector functions and scalar functions.

Scalar function:

Let $S \subset \mathbb{R}$, if for each scalar (element) $t \in S$, $\exists$ a unique real number $f(t)$, then $f(t)$ is said to scalar function of the scalar variable 't'. Here 'S' is called domain of $f(t)$ and $f(t)$ is a scalar quantity, so f is scalar function.

Vector function of a Scalar variable:

Let $S \subset \mathbb{R}$, if for each scalar $t \in S$, $\exists$ a unique vector $\overrightarrow{f(t)}$ then $\overrightarrow{f(t)}$ is said

to be vector function of the scalar variable 't'.
Here $\vec{f}(t)$ is a vector quantity. So $\vec{f}$ is a vector function.

— Let $\vec{i}, \vec{j}, \vec{k}$ be the three mutually perpendicular unit vectors in three dimensional space then the vector function $\vec{f}(t)$ may be expressed in the form

$$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}.$$

Here $f_1(t)$, $f_2(t)$, $f_3(t)$ are real valued functions and are called the components of $\vec{f}(t)$.

Scalar field:

If to each point (x, y, z) of a region R in space, there corresponds a unique number or scalar $\phi(x, y, z)$, then ϕ is called a scalar function of position or scalar point function and we say that a scalar field ϕ has been defined in R.

Ex: 1) The temperature at any point within or on the earth's surface at a certain time defines a scalar field.

2) $\phi(x, y, z) = x^3y - z^2$ defines a scalar field.

vector field: If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{V}(x, y, z)$ then $\vec{V}$ is called a vector function of position or vector point function and we say that a vector field $\vec{V}$ has been defined in R.

examples: 1) If the velocity at any point (x, y, z) within a moving fluid is known at a certain time, then a vector field is defined.

(2) $\vec{V}(x, y, z) = xy^2\vec{i} - 2yz^3\vec{j} + x^2z\vec{k}$ defines a vector field.

Limit of a vector function:–

A vector function $\vec{f}(t)$ said to have limit 'L' when t tends to t_0, if given $\epsilon>0$, $\exists$ a $\delta>0$ (depending on ϵ) such that

$|\vec{f}(t)-L|<\epsilon$ whenever $0<|t-t_0|<\delta$.

i.e, $\vec{f}(t)\longrightarrow L$ as $t\to t_0$

i.e, $\underset{t\to t_0}{Lt}\vec{f}(t)=L$

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Note: Let $\underset{t\to t_0}{Lt}\vec{f}(t)=L$ and $\underset{t\to t_0}{Lt}\vec{g}(t)=M$ and λ is a constant

then (1) $\underset{t\to t_0}{Lt}\left[\vec{f}(t)+\vec{g}(t)\right]=L+M$ (3) $\underset{t\to t_0}{Lt}\left(\vec{f}(t)\times\vec{g}(t)\right)=L\times M$.

(2) $\underset{t\to t_0}{Lt}\left[\vec{f}(t)\cdot\vec{g}(t)\right]=LM$ (4) $\underset{t\to t_0}{Lt}\left[\lambda\vec{f}(t)\right]=\lambda L$

Continuity of vector function:

A vector function $\vec{f}(t)$ is said to be continuous at $t=t_0$ if (i) $f(t_0)$ is defined.

(ii) given any $\epsilon>0$ (however small) $\exists$ a $\delta>0$ (depending on ϵ) such that $|\vec{f}(t)-\vec{f}(t_0)|<\epsilon$ whenever $|t-t_0|<\delta$.

i.e, $\vec{f}(t)\to\vec{f}(t_0)$ as $t\to t_0$.

i.e, $\underset{t\to t_0}{Lt}\vec{f}(t)=\vec{f}(t_0)$.

Note 1. The function f is said to be continuous on I if f is continuous at each point of I.

2. If f and g are continuous then $f\pm g$, $f\cdot g$ and $f\times g$ are also continuous.

Derivative of a vector function with respect to a scalar:

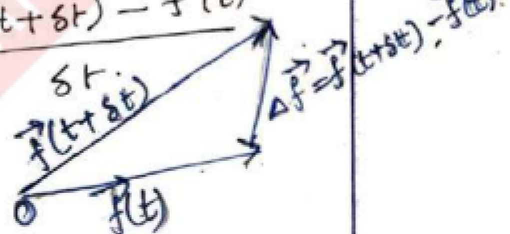
Let $\vec{f}(t)$ be a vector function on an interval I and $t_0 \in I$ then $\underset{t \to t_0}{Lt} \dfrac{\vec{f}(t) - \vec{f}(t_0)}{t - t_0}$ if it exists, it is called the derivative of $\vec{f}(t)$ at t_0. and is denoted by $\vec{f}'(t_0)$ or $\left(\dfrac{d\vec{f}}{dt}\right)_{t=t_0}$.

Also it is said that $\vec{f}(t)$ is differentiable at $t = t_0$

Note: 1. If $\vec{f}(t)$ is differentiable on I and $t_0 \in I$ then the derivative of $\vec{f}(t)$ at t is denoted by $\dfrac{d\vec{f}}{dt}$.

2. If the changes in t, $\vec{f}(t)$ are denoted by δt and $\delta \vec{f}(t)$ respectively then we have

$$\frac{d\vec{f}}{dt} = \underset{\delta t \to 0}{Lt} \frac{\delta \vec{f}}{\delta t} = \underset{\delta t \to 0}{Lt} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$



Higher order Derivatives:

Let $\vec{f}(t)$ be differentiable on an Interval I and $\vec{f}' = \dfrac{d\vec{f}}{dt}$ be the derivative of $\vec{f}$.

If $\underset{t \to t_0}{Lt} \dfrac{\vec{f}'(t) - \vec{f}'(t_0)}{t - t_0}$ exists for each $t \in I_1 \subset I$

then $\vec{f}'$ is said to be differentiable on I_1.

Also $\vec{f}(t)$ is said to possess second derivative on I_1 and is denoted by $\vec{f}''(t)$ or $\dfrac{d^2\vec{f}}{dt^2}$.

Similarly the derivative of $\dfrac{d^2\vec{f}}{dt^2}$ is denoted by $\dfrac{d^3\vec{f}}{dt^3}$ and is called the third derivative of $\vec{f}(t)$ and so on.

(16)

→ $\frac{d\vec{f}}{dt}, \frac{d^2\vec{f}}{dt^2}, \ldots\ldots$ are also denoted by $\dot{\vec{f}}, \ddot{\vec{f}}, \ldots$ respectively.

→ Let $\vec{A}, \vec{B}$ and $\vec{C}$ be three differentiable vector functions of scalar variable 't' and ϕ is a differentiable scalar function of the same variable t, then

(1) $\frac{d}{dt}(\vec{A} \pm \vec{B}) = \frac{d\vec{A}}{dt} \pm \frac{d\vec{B}}{dt}$

(2) $\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B}$

(3) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

(4) $\frac{d}{dt}(\phi\vec{A}) = \phi\frac{d\vec{A}}{dt} + \vec{A}\frac{d\phi}{dt}$

(5) $\frac{d}{dt}[\vec{A}\ \vec{B}\ \vec{C}] = \left[\frac{d\vec{A}}{dt}\ \vec{B}\ \vec{C}\right] + \left[\vec{A}\ \frac{d\vec{B}}{dt}\ \vec{C}\right] + \left[\vec{A}\ \vec{B}\ \frac{d\vec{C}}{dt}\right]$

(6) $\frac{d}{dt}\{\vec{A} \times (\vec{B} \times \vec{C})\} = \frac{d\vec{A}}{dt} \times (\vec{B} \times \vec{C}) + \vec{A} \times \left(\frac{d\vec{B}}{dt} \times \vec{C}\right) + \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{dt}\right)$

Derivative of a constant vector:

A vector is said to be constant only if both its magnitude and direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

→ Let $\vec{A}$ be a constant vector function in the interval I and $t_0 \in I$ then $f'(t_0) = 0$

Sol^n: Let $\vec{f}(t) = c$, where c is a constant vector.

then $\lim_{t \to t_0} \frac{\vec{f}(t) - \vec{f}(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{c - c}{t - t_0}$

$= \lim_{t \to t_0} (0) = 0$

$\therefore \left(\frac{d\vec{f}}{dt}\right)_{t=t_0} = \vec{f}'(t_0) = 0$

→ Derivative of a vector function in terms of its Components.

Let $\vec{r}$ be a vector function of the scalar variable t.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where the components x, y, z are scalar functions of the scalar variable t and $\vec{i}, \vec{j}, \vec{k}$ are fixed unit vectors.

we have $\vec{r} + \delta\vec{r} = (x+\delta x)\vec{i} + (y+\delta y)\vec{j} + (z+\delta z)\vec{k}$

$$\therefore \delta\vec{r} = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta x\,\vec{i} + \delta y\,\vec{j} + \delta z\,\vec{k}.$$

$$\therefore \frac{\delta\vec{r}}{\delta t} = \frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t}\vec{j} + \frac{\delta z}{\delta t}\vec{k}.$$

$$\therefore \lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0} \left\{ \frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t}\vec{j} + \frac{\delta z}{\delta t}\vec{k} \right\}.$$

$$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

Thus in order to differentiate a vector we should differentiate its components.

<u>Note</u>: If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then some times we also write it as $\vec{r} = (x, y, z)$.

In this notation

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right), \text{ and so on.}$$

Alternative method.

we have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ are constant vectors and so their derivatives will be zero.

(17)

Now $\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\vec{i} + y\vec{j} + z\vec{k})$

$= \frac{d}{dt}(x\vec{i}) + \frac{d}{dt}(y\vec{j}) + \frac{d}{dt}(z\vec{k})$

$= \frac{dx}{dt}\vec{i} + x\frac{d\vec{i}}{dt} + \frac{dy}{dt}\vec{j} + y\frac{d\vec{j}}{dt} + \frac{dz}{dt}\vec{k} + z\frac{d\vec{k}}{dt}$.

$= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$.

$\left(\because \frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0\right)$

Note:— If f_1, f_2, f_3 are constant functions then $\vec{f} = f_1 i + f_2 j + f_3 k$ is called a constant vector function.

*** Some important results:**

→ A vector function $\vec{f}$ is constant iff $\frac{d\vec{f}}{dt} = 0$.

Proof Suppose $\vec{f}$ is constant. Then $\frac{d\vec{f}}{dt} = 0$.

Conversely suppose that $\frac{d\vec{f}}{dt} = 0$.

Let $\vec{f} = f_1 i + f_2 j + f_3 k$

Since $\frac{d\vec{f}}{dt} = 0 \Rightarrow \frac{df_1}{dt} i + \frac{df_2}{dt} j + \frac{df_3}{dt} k = 0$

$\Rightarrow \frac{df_1}{dt} = 0, \frac{df_2}{dt} = 0, \frac{df_3}{dt} = 0$.

$\Rightarrow f_1, f_2, f_3$ are constants.

$\Rightarrow \vec{f}$ is a constant vector function.

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→ A vector function $\vec{f}$ is of constant magnitude iff $\vec{f}\cdot\frac{d\vec{f}}{dt} = 0$.

Proof Let $\vec{f}$ be a vector of constant magnitude.

Then $\vec{f}\cdot\vec{f} = |\vec{f}|^2 = \text{constant}$.

Now diff. w.r.t 't', we get $\vec{f}\cdot\frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt}\cdot\vec{f} = 0$

$\Rightarrow 2\left(\vec{f}\cdot\frac{d\vec{f}}{dt}\right) = 0 \quad (\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$.

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$$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{dt}=0.$$

Conversely suppose that

$$\vec{f}\cdot\frac{d\vec{f}}{dt}=0.$$

then $\frac{1}{2}\frac{d}{dt}(\vec{f}\cdot\vec{f})=0$

$$\Rightarrow \vec{f}\cdot\vec{f}=\text{const.}$$

$$\Rightarrow |\vec{f}|^2=\text{const.}$$

$$\Rightarrow |\vec{f}|=\text{const.}$$

→ If $\vec{a}$ is a differentiable vector function of the scalar variable 't' and if $|\vec{a}|=a$, then (i) $\frac{d}{dt}(\vec{a}^2)=2a\frac{da}{dt}$ and (ii) $\vec{a}\cdot\frac{d\vec{a}}{dt}=a\frac{da}{dt}$.

Sol: (i) we have $\vec{a}^2=\vec{a}\cdot\vec{a}$

$$=(a)(a)\cos 0$$

$$=a^2$$

$$\therefore \frac{d}{dt}(\vec{a}^2)=\frac{d}{dt}(a^2)=2a\frac{da}{dt}.$$

(ii) we have $\frac{d}{dt}(\vec{a}^2)=\frac{d}{dt}(\vec{a}\cdot\vec{a})$

$$=\frac{d\vec{a}}{dt}\cdot\vec{a}+\vec{a}\cdot\frac{d\vec{a}}{dt}$$

$$=2\vec{a}\cdot\frac{d\vec{a}}{dt} \quad \text{—(1)}$$

Also $\frac{d}{dt}(\vec{a}^2)=\frac{d}{dt}(a^2)$

$$=2a\frac{da}{dt} \quad \text{—(2)}$$

$\therefore$ from (1) & (2), we have

$$2\vec{a}\cdot\frac{d\vec{a}}{dt}=2a\frac{da}{dt}$$

$$\Rightarrow \boxed{\vec{a}\cdot\frac{d\vec{a}}{dt}=a\frac{da}{dt}.}$$

→ If $\vec{a}$ has constant length (fixed magnitude) then $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$. (18)

Sol Let $|\vec{a}| = a$ (constant).

Then $\vec{a}\cdot\vec{a} = a^2$ (constant)

$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

$\Rightarrow \frac{d\vec{a}}{dt}\cdot\vec{a} + \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

$\Rightarrow 2\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

$\Rightarrow \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$.

$\therefore$ The scalar product of two vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ is zero.

$\therefore \vec{a}$ is $\perp^{lar}$ to $\frac{d\vec{a}}{dt}$, provided $\frac{d\vec{a}}{dt}$ is not null vector

i.e provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$.

Thus the derivative of a vector of constant length is perpendicular to the vector provided the vector itself is not constant.

→ If $\vec{a}$ is a differentiable vector function of the scalar variable 't', then

$$\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = \vec{a}\times\frac{d^2\vec{a}}{dt^2}.$$

Sol we have $\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = \frac{d\vec{a}}{dt}\times\frac{d\vec{a}}{dt} + \vec{a}\times\frac{d^2\vec{a}}{dt^2}$

$= 0 + \vec{a}\times\frac{d^2\vec{a}}{dt^2}$

$= \vec{a}\times\frac{d^2\vec{a}}{dt^2}$

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→ A vector function $\vec{a}(t)$ has constant direction iff $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

<u>proof</u> Let $\vec{a}(t) = a(t)A(t)$

where $a(t) = |\vec{a}(t)|$

and $A(t)$ is a vector function with unit magnitude, for every t in ~~the~~ domain of $\vec{a}(t)$.

$A(t) = \cos\theta\, i + \cos\beta\, j + \cos\gamma\, k$

$|A| = 1$.

$$\frac{d\vec{a}}{dt} = \frac{d}{dt}(aA)$$

$$= a\frac{dA}{dt} + \frac{da}{dt}A$$

Now $$\vec{a} \times \frac{d\vec{a}}{dt} = aA \times \left(a\frac{dA}{dt} + \frac{da}{dt}A\right)$$

$$= \left(aA \times a\frac{dA}{dt}\right) + aA \times \frac{da}{dt}A$$

$a\frac{da}{dt}(A \times A) = 0$

$$= a^2\left(A \times \frac{dA}{dt}\right) + 0.$$

$$= a^2\left(A \times \frac{dA}{dt}\right). \quad \text{——(1)}$$

suppose $\vec{a}(t)$ has constant direction.

Then A is constant

$$\Rightarrow \frac{dA}{dt} = 0$$

$$\Rightarrow A \times \frac{dA}{dt} = 0 \quad \text{——(2)}.$$

$\therefore \vec{a} \times \frac{d\vec{a}}{dt} = a^2\left(A \times \frac{dA}{dt}\right)$ (by (1).)

$= a^2(0)$ (by (2))

$= 0.$

$\therefore \boxed{\vec{a} \times \frac{d\vec{a}}{dt} = 0}$

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Conversely suppose that

$$\vec{a} \times \frac{d\vec{a}}{dt} = 0.$$

Then $a^2 \left(A \times \frac{dA}{dt}\right) = 0$

$$\Rightarrow A \times \frac{dA}{dt} = 0.$$

Since A is of unit length

$$\therefore A \cdot \frac{dA}{dt} = 0$$

Now $A \times \frac{dA}{dt} = 0$, $A \cdot \frac{dA}{dt} = 0$

$$\Rightarrow \frac{dA}{dt} = 0$$

$\Rightarrow A$ is constant.

$\Rightarrow \vec{a}$ has constant direction.

Ex: If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, m a constant, differentiate the following w.r.t t:

(i) $\vec{r} \cdot \vec{a}$ (ii) $\vec{r} \times \vec{a}$ (iii) $\vec{r} \times \frac{d\vec{r}}{dt}$ (iv) $\vec{r} \cdot \frac{d\vec{r}}{dt}$.

(v) $\vec{r}^2 + \frac{1}{\vec{r}^2}$ (vi) $m\left(\frac{d\vec{r}}{dt}\right)^2$ (vii) $\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ (viii) $\frac{\vec{r} \times \vec{a}}{\vec{r} \cdot \vec{a}}$.

Sol^n: (i) Let $R = \vec{r} \cdot \vec{a}$

Then $\frac{dR}{dt} = \frac{d}{dt}(\vec{r} \cdot \vec{a})$

$$= \frac{d\vec{r}}{dt} \cdot \vec{a} + \vec{r} \cdot \frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt} \cdot \vec{a} + \vec{r} \cdot 0 \qquad \left(\because \frac{d\vec{a}}{dt} = 0, \text{ as } \vec{a} \text{ is constant vector}\right)$$

$$= \frac{d\vec{r}}{dt} \cdot \vec{a}$$

(iv) Let $R = \vec{r} \cdot \frac{d\vec{r}}{dt}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\vec{r} \cdot \frac{d\vec{r}}{dt}\right]$

$$= \vec{r} \cdot \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}$$

$$= \vec{r} \cdot \frac{d^2\vec{r}}{dt^2} + \left(\frac{d\vec{r}}{dt}\right)^2$$

(v) Let $R = \vec{r}^2 + \frac{1}{\vec{r}^2}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left\{\vec{r}^2 + \frac{1}{\vec{r}^2}\right\}$

$$= \frac{d}{dt}\left\{r^2 + \frac{1}{r^2}\right\} \quad \left(\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r^2 = x^2+y^2+z^2 = \vec{r}\cdot\vec{r} = (\vec{r})^2\right)$$

$$= 2r\frac{dr}{dt} - \frac{2}{r^3}\frac{dr}{dt}$$

$$= 2\left(r - \frac{1}{r^3}\right)\frac{dr}{dt}$$

(vi) Let $R = m\left(\frac{d\vec{r}}{dt}\right)^2$. Then $\frac{dR}{dt} = m\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2$

$$= m\frac{d}{dt}\left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right]$$

$$= m\left[\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

$$= m\left[2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

$$= 2m\,\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}$$

(vii) Let $R = \frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}\right]$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt} + \frac{d\vec{a}}{dt}\right) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}^2+\vec{a}^2}\right)\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) + \left\{\frac{(-1)}{(\vec{r}^2+\vec{a}^2)^2}\frac{d}{dt}(\vec{r}^2+\vec{a}^2)\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) + \left\{\frac{(-1)}{(\vec{r}^2+\vec{a}^2)}\,2\vec{r}\cdot\frac{d\vec{r}}{dt}\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) - \frac{2\vec{r}\cdot\frac{d\vec{r}}{dt}}{\vec{r}^2+\vec{a}^2}(\vec{r}+\vec{a})$$

(viii) Let $R = \frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}\right]$

$$= \frac{1}{\vec{r}\cdot\vec{a}}\frac{d}{dt}(\vec{r}\times\vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}\cdot\vec{a}}\right)\right\}(\vec{r}\times\vec{a})$$

$$= \frac{1}{\vec{r}\cdot\vec{a}}\left(\frac{d\vec{r}}{dt}\times\vec{a}\right) + \left[\frac{-1}{(\vec{r}\cdot\vec{a})^2}\left(\frac{d\vec{r}}{dt}\cdot\vec{a}\right)\right](\vec{r}\times\vec{a})$$

$$= \frac{\frac{d\vec{r}}{dt}\times\vec{a}}{\vec{r}\cdot\vec{a}} - \left\{\frac{\frac{d\vec{r}}{dt}\cdot\vec{a}}{(\vec{r}\cdot\vec{a})^2}(\vec{r}\times\vec{a})\right\}$$

Problems:

→ If $\vec{r} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$, find

(i) $\frac{d\vec{r}}{dt}$ (ii) $\frac{d^2\vec{r}}{dt^2}$ (iii) $\left|\frac{d\vec{r}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|$

(i) Given $\vec{r} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}\right)$$

$$= \frac{d}{dt}(\sin t)\vec{i} + \frac{d}{dt}(\cos t)\vec{j} + \frac{d}{dt}(t)\vec{k}$$

$$= \cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}.$$

(ii) $$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}\right)$$

$$= \frac{d}{dt}(\cos t)\vec{i} - \frac{d}{dt}(\sin t)\vec{j} + \frac{d}{dt}(1)\vec{k}$$

$$= -\sin t\,\vec{i} - \cos t\,\vec{j} + 0$$

$$= -\sin t\,\vec{i} - \cos t\,\vec{j}$$

(iii) $$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{\cos^2 t + \sin^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}.$$

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(iv) $$\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= 1$$

→ If $\vec{a}, \vec{b}$ are constant vectors, ω is a constant and $\vec{r}$ is a vector function of the scalar variable t given by $\vec{r} = \cos\omega t\,\vec{a} + \sin\omega t\,\vec{a}$

Show that (i) $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ (ii) $\vec{r} \times \frac{d\vec{r}}{dt} = \omega\,\vec{a} \times \vec{b}$.

Solⁿ: Since $\vec{a}, \vec{b}$ are constant vectors.

$$\therefore \frac{d\vec{a}}{dt} = 0, \quad \frac{d\vec{b}}{dt} = 0.$$

(i) $$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b}\right)$$

$$= \frac{d}{dt}(\cos\omega t)\vec{a} + \frac{d}{dt}(\sin\omega t)\vec{b}$$

$$\frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \frac{d}{dt}\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right)$$

$$= \frac{d}{dt}(-\omega\sin\omega t)\vec{a} + \frac{d}{dt}(\omega\cos\omega t)\vec{b}$$

$$= -\omega\frac{d}{dt}(\sin\omega t)\vec{a} + \omega\frac{d}{dt}(\cos\omega t)\vec{b}$$

$$= -\omega(\omega\cos\omega t)\vec{a} + \omega(-\omega\sin\omega t)\vec{b}$$

$$= -\omega^2\cos\omega t\,\vec{a} - \omega^2\sin\omega t\,\vec{b}$$

$$= -\omega^2(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \qquad (\because \vec{r} = \cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})$$

$$\therefore \frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0.$$

(ii) $$\vec{r}\times\frac{d\vec{r}}{dt} = (\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})\times(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b})$$

$$= -\omega\cos\omega t\sin\omega t\,\vec{a}\times\vec{a} + \omega\cos^2\omega t\,\vec{a}\times\vec{b} - \omega\sin^2\omega t\,\vec{b}\times\vec{a} + \omega\sin\omega t\cos\omega t\,\vec{b}\times\vec{b}$$

$$= \omega\cos^2\omega t\,\vec{a}\times\vec{b} - \omega\sin^2\omega t\,\vec{b}\times\vec{a} \qquad (\because \vec{a}\times\vec{a} = 0 \;\&\; \vec{b}\times\vec{b} = 0)$$

$$= \omega\cos^2\omega t\,\vec{a}\times\vec{b} + \omega\sin^2\omega t\,\vec{a}\times\vec{b} \qquad \because \vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$$

$$= \omega(\cos^2\omega t + \sin^2\omega t)\vec{a}\times\vec{b}$$

$$= \omega\,\vec{a}\times\vec{b}.$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = \omega\,\vec{a}\times\vec{b}.$$

→ If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$,
find $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$

Sol^n: Given $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$.

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}.$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\,\vec{i} - a\cos t\,\vec{j}$$

$$\therefore \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= a^2\sin t\tan\alpha\,\vec{i} - a^2\cos t\tan\alpha\,\vec{j} + a^2\vec{k}$$

$$\therefore \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(a^2\sin t\tan\alpha)^2 + (-a^2\cos t\tan\alpha)^2 + (a^2)^2}$$

$$= \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4}$$

$$= \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^4\tan^2\alpha + a^4}$$

$$= \sqrt{a^4(1+\tan^2\alpha)} = \sqrt{a^4(\sec^2\alpha)}$$

$$= a^2\sec\alpha.$$

Also $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right] = \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d^3\vec{r}}{dt^3}$.

$\left\{\because [\vec{a}\,\vec{b}\,\vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c}\right\}$

$$= (a^2\sin t\tan\alpha\,\vec{i} - a^2\cos t\tan\alpha\,\vec{j} + a^2\vec{k})\cdot(a\sin t\,\vec{i} - a\cos t\,\vec{j})$$

$$= a^3\sin^2 t\tan\alpha\,\vec{i}\cdot\vec{i} + a^3\cos^2 t\tan\alpha\,\vec{j}\cdot\vec{j}$$

$\left[\because \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{k}\cdot\vec{i} = \vec{k}\cdot\vec{j} = 0\right]$

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$= a^3 \sin^2 t \tan\alpha + a^3 \cos^2 t \tan\alpha$

$(\because \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = 1)$

$= a^3 \tan\alpha (\sin^2 t + \cos^2 t)$

$= a^3 \tan\alpha.$ $\quad \therefore \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = a^3 \tan\alpha$

→ If $\frac{d\vec{u}}{dt} = \vec{w}\times\vec{u}$, $\frac{d\vec{v}}{dt} = \vec{w}\times\vec{v}$, show that

$\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{w}\times(\vec{u}\times\vec{v})$

Soln: W.K.T $\frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$

$= (\vec{w}\times\vec{u})\times\vec{v} + \vec{u}\times(\vec{w}\times\vec{v})$

$\left[\because \frac{d\vec{u}}{dt} = \vec{w}\times\vec{u} \ \& \ \frac{d\vec{v}}{dt} = \vec{w}\times\vec{v}\right]$

$= (\vec{v}\cdot\vec{w})\vec{u} - (\vec{v}\cdot\vec{u})\vec{w} + (\vec{u}\cdot\vec{v})\vec{w} - (\vec{u}\cdot\vec{w})\vec{v}$

$= (\vec{v}\cdot\vec{w})\vec{u} - (\vec{u}\cdot\vec{w})\vec{v}$ $\quad (\because \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u})$

$= \vec{w}\times(\vec{u}\times\vec{v})$

2002 → If $\vec{R}$ be a unit vector in the direction of $\vec{r}$. Prove that $\vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$, where $r = |\vec{r}|$

Soln: We have $\vec{r} = r\vec{R}$;

$\Rightarrow \vec{R} = \frac{1}{r}\vec{r}$

$\therefore \frac{d\vec{R}}{dt} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}.$

Hence $\vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r}\vec{r}\times\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)$

$= \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\vec{r}\times\vec{r}$

$= \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$ $\quad (\because \vec{r}\times\vec{r} = 0)$

$\therefore \vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}.$

HW → If $\vec{A} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$ and $\vec{B} = \sin t\,\vec{i} - \cos t\,\vec{j}$, find (a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$, (b) $\frac{d}{dt}(\vec{A}\times\vec{B})$, (c) $\frac{d}{dt}(\vec{A}\cdot\vec{A})$.

HW → If $\vec{A} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{B} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$, find (a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$ (b) $\frac{d}{dt}(\vec{A}\times\vec{B})$ (c) $\frac{d}{dt}|A+B|$, (d) $\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt}\right)$ at $t=1$

HW → If $\vec{r} = e^{-t}\vec{i} + \log(t^2+1)\vec{j} - \tan t\,\vec{k}$ then find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$, $\left|\frac{d\vec{r}}{dt}\right|$, $\left|\frac{d^2\vec{r}}{dt^2}\right|$ at $t=0$.

HW → If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\,\vec{k}$ then find $\left|\frac{d\vec{r}}{dt}\right|$

HW → If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ then show that $|\vec{r}'| = 10$ and $|\vec{r}''| = 18$

HW → If $\vec{A} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$, $\vec{B} = \cos t\,\vec{i} - \sin t\,\vec{j} - 3\vec{k}$ and $\vec{C} = 2\vec{i} + 3\vec{j} - \vec{k}$ then find $\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right]$ at $t=0$

HW → If $\vec{A} = 3t^2\vec{i} - (t+4)\vec{j} + (t^2 - 2t\vec{k})$ and $\vec{B} = \sin t\,\vec{i} + 3e^{-t}\vec{j} - 3\cos t\,\vec{k}$, find $\frac{d^2}{dt^2}(\vec{A}\times\vec{B})$ at $t=0$.

→ If $\vec{r} = e^t(\vec{c}\cos 2t + \vec{d}\sin 2t)$ where c and d are constant vectors, then show that $\frac{d^2\vec{r}}{dt^2} - 2\frac{d\vec{r}}{dt} + 5\vec{r} = 0$

→ If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $x = \frac{2t+1}{t-1}$, $y = \frac{t^2}{t-1}$, $z = t+2$ then find $\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]$

→ If $\vec{r} = 2t\vec{i} + t^2\vec{j} + \frac{t^3}{3}\vec{k}$ then show that

$$\frac{\left[\vec{r}'\ \vec{r}''\ \vec{r}'''\right]}{\left|r'\times r''\right|^2} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} \text{ at } t=1$$

Differential Geometry

— Differential Geometry involves a study of space curves and surfaces. It differs essentially from algebraic geometry which deals with a much narrower and restricted class of curves and surfaces and employs algebra as its principal tool.

For example:

The theory of conic sections or quadric surfaces come under the purview of algebraic geometry whereas the study of curvature of a general curve or the tangent plane to a general surface pertains to differential geometry.

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→ Parametric representation of space curves:

There are two ways of representing a curve analytically.

— One of them is to represent it as the intersection of two surfaces represented by two equations of the form

$$F_1(x,y,z)=0,\ F_2(x,y,z)=0. \quad (1)$$

— The other one is the parametric representation of the form

$$x=f_1(t),\ y=f_2(t),\ z=f_3(t). \quad (2)$$

where x,y,z are scalar functions of the scalar 't', also represents a curve in three dimensional space.

Here (x, y, z) are co-ordinates of a current point of the curve.
The scalar variable 't' may range over a set of values $a \le t \le b$.

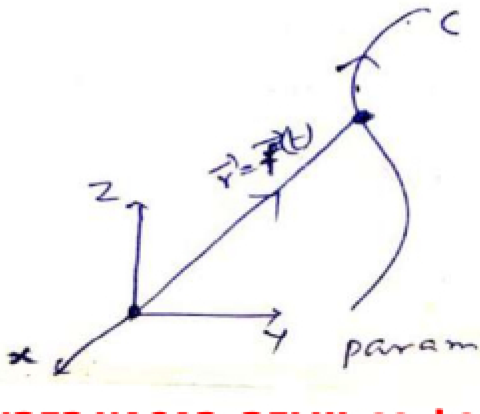
In vector notation, an equation of the form $\vec{r} = \vec{f}(t)$, represents a curve in three dimensional space if $\vec{r}$ is the position vector of a current point on the curve; As 't' changes, $\vec{r}$ will give position vectors of different points on the curve.

The vector $\vec{f}(t)$ can be expressed as $f_1(t)i + f_2(t)j + f_3(t)k$.
Also if (x, y, z) are the coordinates of a current point on the curve whose position vector is $\vec{r}$, then $\vec{r} = xi + yj + zk$.
Therefore the single vector equation

$$\vec{r} = \vec{f}(t).$$

$$\text{i.e. } xi + yj + zk = f_1(t)i + f_2(t)j + f_3(t)k.$$

is equivalent to the three parametric equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$.
Thus a curve in space may be defined as the locus of a point whose co-ordinates may be expressed as a function of a single parameter.



parametric representation of a curve.

* Typical Examples

Kinds of curves.

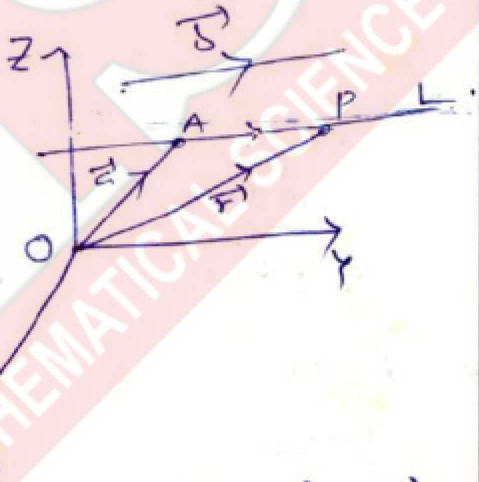
→ straight line:

A straight line 'L' through a point with position vector $\vec{a}$ in the direction of a constant vector $\vec{b}$ can be represented in the form $\vec{r} = \vec{a} + t\vec{b}$

$$= [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$$

where $\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$; $t \in \mathbb{R}$.



Sol

Let O be the origin and $\overrightarrow{OA} = \vec{a}$.

Let 'L' be the straight line parallel to the given vector $\vec{b}$ passing through 'A'.

Let 'P' be the point on L and $\overrightarrow{OP} = \vec{r}$.

then $\overrightarrow{AP} \parallel \vec{b}$

$\therefore \overrightarrow{AP} = t\vec{b}$; $t \in \mathbb{R}$. —— ①

Since $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$\therefore \overrightarrow{OP} = \overrightarrow{AP} + \overrightarrow{OA}$

$\overrightarrow{OP} = \vec{a} + t\vec{b}$.

$\therefore \boxed{\vec{r} = \vec{a} + t\vec{b}}$; $t \in \mathbb{R}$.

This is the required vector equation of the straight line.

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→ **Ellipse, circle:**

The vector function $\vec{r}(t) = [a\cos t, b\sin t, 0]$
$= a\cos t\,\hat{i} + b\sin t\,\hat{j}$ — (1)

represents an ellipse in the xy-plane with centre at the origin and principal axes in the direction of the x and y axes.

since $\cos^2 t + \sin^2 t = 1$ (by taking $x = a\cos t$, $y = b\sin t$, $z = 0$ $\Rightarrow \cos t = \frac{x}{a}$, $\sin t = \frac{y}{b}$)

$\therefore$ (1) $\equiv$ we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$

If $b = a$, then (1) represents a circle of radius 'a'.

→ A **plane curve** is a curve that lies in a plane in space.

→ A curve that is not plane is called a **twisted curve**.

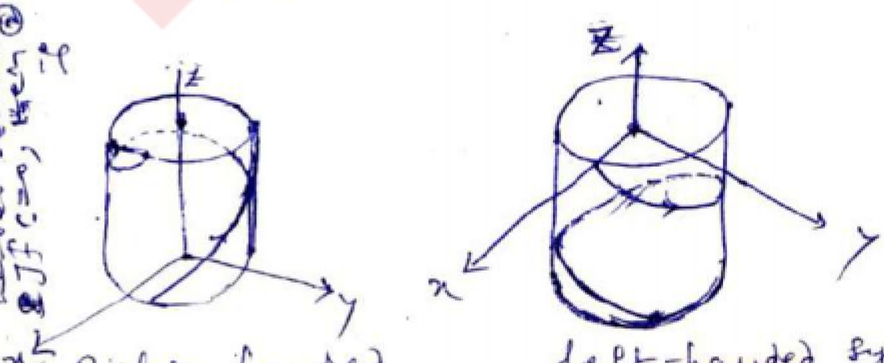
For example:

Circular helix

The twisted curve C represented by the vector

$\vec{r}(t) = [a\cos t, a\sin t, ct]$
$= a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\hat{k}$; $c \neq 0$ — (2)

is called a circular helix.
It lies on the cylinder $x^2+y^2=a^2$.
If $c > 0$, the helix is shaped like a right-handed screw.
If $c < 0$, it looks like a left-handed screw.
If $c = 0$, then (2) is circle.

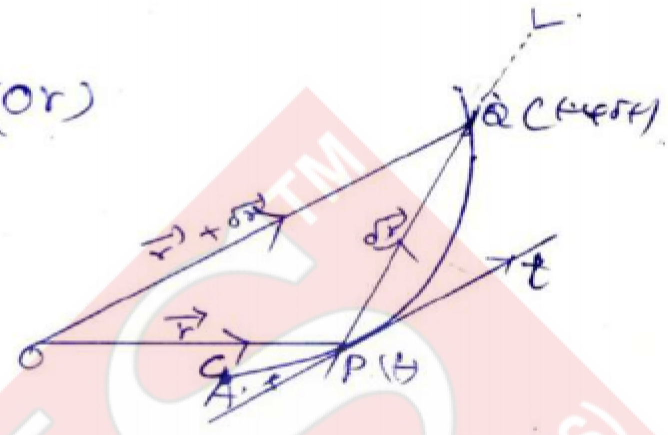


Right-handed system.　　left-handed system

Tangent to a curve:

The tangent to a curve 'C' at a point 'P' of C is the limiting position of a straight line 'L' through P and a point Q of curve 'C' approaches P along 'C'.

(Or)



Let $\vec{r}=\vec{f}(t)$ be a curve 'C' and 'P' be a point on the curve.

If Q ($\neq$P) is a point on the curve then $\overleftrightarrow{PQ}$ is called a secant line. If the secant line $\overleftrightarrow{PQ}$ approaches the same limiting position as Q moves along the curve and approaches to 'P' from either sides, then the limiting position is called a tangent line to the curve at 'P'.

A vector parallel to the tangent at 'P' is called a tangent vector to the curve at 'P'.

Theorem → If $\vec{r}=\vec{f}(t)$ be a differentiable vector function represents a curve 'C' and 'P' is a point on the curve then the tangent vector to the curve at 'P' is $\frac{d\vec{r}}{dt}$.

proof

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Let $\vec{r}$, $\vec{r}+\delta\vec{r}$ be the position vectors of two neighbouring points P and Q on the curve 'C'.

Thus we have

$$\vec{OP} = \vec{r} = \vec{f}(t)$$

and
$$\vec{OQ} = \vec{r}+\delta\vec{r} = \vec{f}(t+\delta t).$$

$$\therefore \vec{PQ} = \vec{OQ} - \vec{OP}$$
$$= (\vec{r}+\delta\vec{r}) - \vec{r}$$
$$= \delta\vec{r}$$

$$\therefore \delta\vec{r} = \vec{OQ} - \vec{OP}$$

$$\boxed{\delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)}.$$

$$\Rightarrow \frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}.$$

Thus, $\frac{\delta\vec{r}}{\delta t}$ is a vector parallel to the chord PQ.

As $Q \to P$

i.e. as $\delta t \to 0$, chord $PQ \to$ tangent at P to the curve

$\therefore \underset{\delta t \to 0}{Lt} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$ is a vector parallel to the tangent at 'P' to the curve $\vec{r} = \vec{f}(t)$.

Unit tangent vector:

Suppose in place of the scalar parameter 't', we take the parameter as 's' where 's' denotes the arc length measured along the curve from some fixed point A on the curve. Thus arc AP = s and arc AQ = s+δs.

Then $\frac{d\vec{r}}{ds}$ will be a vector along the tangent at P to the curve and in the direction of s increasing.

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Also

$$\left|\frac{d\vec{r}}{ds}\right| = \lim_{\delta s\to 0}\left|\frac{\delta\vec{r}}{\delta s}\right|$$

$$= \lim_{Q\to P}\frac{|\delta\vec{r}|}{\text{arc}PQ}$$

$$= \lim_{Q\to P}\frac{\text{Chord }PQ}{\text{arc}PQ}$$

$$= 1.$$

Thus $\frac{d\vec{r}}{ds}$ is a unit vector along the tangent at P in the direction of s increasing. We denote it by 'T'.

i.e $\boxed{T = \frac{d\vec{r}}{ds}}$.

→ If x, y, z are cartesian co-ordinates of P we have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\Rightarrow \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}.$$

$$\Rightarrow T = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \quad \left(\because T = \frac{d\vec{r}}{ds}\right)$$

$$\Rightarrow |T| = \left|\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right|$$

$$\Rightarrow 1 = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} \quad (\because |T| = 1)$$

$$\Rightarrow 1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

$$\Rightarrow 1 = \left(\frac{dx}{dt}\right)^2\left(\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\right)^2\left(\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\right)^2\left(\frac{dt}{ds}\right)^2$$

where 't' is any parameter.

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left|\frac{d\vec{r}}{dt}\right|^2$$

$$\Rightarrow \boxed{\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|}.$$

<u>Note</u>: we shall denote differentiation w.r.t arc length 's' by using primes (i.e, dashes) and differentiation w.r.t any other parameter 't' with dots.

i.e, r' for $\frac{dr}{ds}$, r'' for $\frac{d^2r}{ds^2}$

and $\dot{r}$ for $\frac{dr}{dt}$, $\ddot{r}$ for $\frac{d^2r}{dt^2}$ etc.

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* Serret - frenet formulae :-

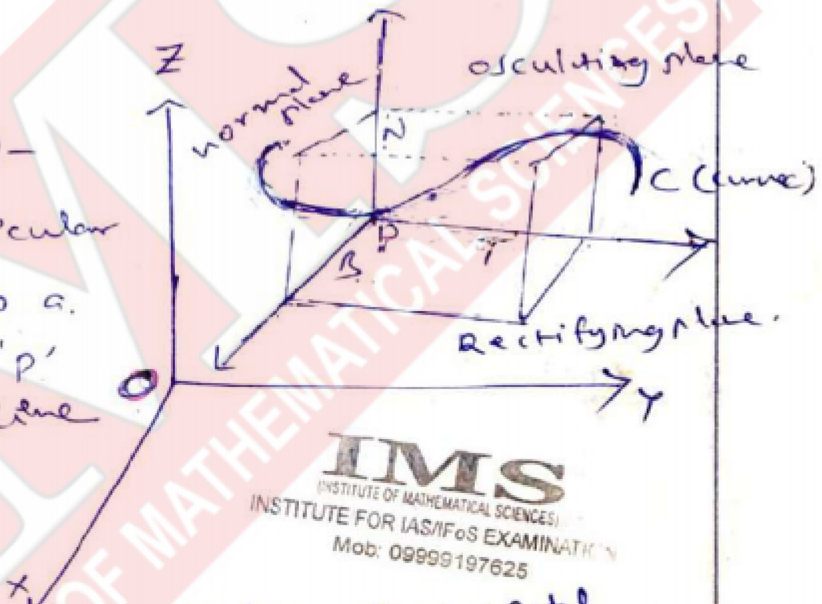
The set of relations involving the derivatives of the fundamental vectors T, N, B is known collectively as the serret-frenet formulae given by

① $\frac{dT}{ds} = kN$, ② $\frac{dB}{ds} = -\tau N$ ③ $\frac{dN}{ds} = \tau B - kT$.

where τ is a scalar called the torsion.

The quantity $\sigma = \frac{1}{\tau}$ is called the radius of the torsion.

* principal Normal vector :-



- Any line perpendicular to the tangent to a curve at a point 'p' is called a normal line at 'p'.

The normal line lying in the osculating plane is called the principal normal at 'p'.

- The unit principal normal is denoted by N.

- The osculating plane to a curve at a point is the plane containing the tangent and principal normal at 'p'.

Normal plane :-

The plane through the point 'p' perpendicular to the tangent at 'p' is called the normal plane at 'p'.

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Rectifying plane:-

The plane through the point 'p' perpendicular to the principal normal is called rectifying plane.

Binormal:

Let T be the unit tangent vector, N be the unit principal normal vector to the curve at a point 'p'.
If B is the unit vector perpendicular to both T and N such that T, N, B form a right handed system then B is called binormal vector to the curve at 'p'.

Thus the binormal is the perpendicular to the osculating plane.

* Right-handed system of T, N, B:-

The vectors T, N, B form a right handed system of unit vectors

$\therefore T\cdot T = 1,\ N\cdot N = 1,\ B\cdot B = 1;\ T\cdot N = N\cdot B = B\cdot T = 0.$

and $T\times N = B,\ N\times B = T,\ B\times T = N$,

$T\times T = N\times N = B\times B = 0.$

Proof of Serret-Frenet formulae:

① $\frac{dT}{ds} = kN.$

Let $\vec{r}(t)$ be the position vector of the point 'p' on the curve $(\vec{r}(t) = \vec{r}(t))$, then the unit vector T at p is given by $\frac{d\vec{r}}{ds} = T$.

Since $|T| = 1$ i.e T is of constant magnitude.

we have $T\cdot\frac{dT}{ds} = 0.$

$\therefore \frac{dT}{ds}$ is perpendicular to T.

But we know that $\frac{dT}{ds}$ lies in the osculating plane.

$\therefore \frac{dT}{ds}$ is parallel to N. $\Rightarrow \frac{dT}{ds} = \pm kN$ for some scalar k.

By convention, we take +ve sign

$\Rightarrow \frac{dT}{ds} = kN$. for some scalar 'k'. —— (1)

<u>curvature</u>:- If T is the unit tangent vector to the curve $\vec{r}(s)$ at a point 'p' then the rate of change of 'T' w.r.t 's' is called curvature of the curve at 'p'. It is denoted by k.

The reciprocal of k is called radius of curvature of the curve at 'p'. It is denoted by ρ i.e $\rho = \frac{1}{k}$.

<u>Note</u>: $\left|\frac{dT}{ds}\right| = k$. (from (1), $|N|=1$).

(2) $\frac{dB}{ds} = -\tau N$.

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since $|B|=1$

i.e B is of constant magnitude.

$\therefore B \cdot \frac{dB}{ds} = 0$.

$\therefore \frac{dB}{ds}$ is perpendicular to B

we know that $\frac{dB}{ds}$ lies in the osculating plane

Now we have

$B \cdot T = 0$

$\Rightarrow B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T = 0$ (by diff w.r.t s).

$\Rightarrow B \cdot (kN) + \frac{dB}{ds} \cdot T = 0$. ($\because \frac{dT}{ds} = kN$).

$$\Rightarrow \frac{dB}{ds}\cdot T + (B\cdot N)k = 0.$$

$$\Rightarrow \frac{dB}{ds}\cdot T = 0 \quad (\because B\cdot N = 0)$$

$$\Rightarrow T\cdot \frac{dB}{ds} = 0.$$

$$\Rightarrow \frac{dB}{ds} \text{ is } \perp \text{ to } T.$$

since $\frac{dB}{ds}$ lies in the osculating plane

so it must be parallel to N.

$$\therefore \frac{dB}{ds} = \pm\tau N.$$

By convention, $\frac{dB}{ds} = -\tau N$. (ii)

Torsion:- If B is the binormal vector to the curve $\vec{r}(s)$ at a point 'p' then the rate of change of B w.r.t. 's' is called torsion of the curve at 'p'. It is denoted by τ.

The reciprocal of τ is called the radius of torsion and is denoted by σ

i.e $\sigma = \frac{1}{\tau}$.

Note:- $\left|\frac{dB}{ds}\right| = \tau$ (from (ii), $|N|=1$).

(3) $\frac{dN}{ds} = \tau B - kT$

Now we have $B\times T = N$.

$$\Rightarrow B\times\frac{dT}{ds} + \frac{dB}{ds}\times T = \frac{dN}{ds} \quad (\text{by diff. w.r.t } s).$$

$$\Rightarrow B\times(kN) + (-\tau N)\times T = \frac{dN}{ds}$$

$$\Rightarrow \frac{dN}{ds} = k(B\times N) - \tau(N\times T)$$
$$= k(-T) - \tau(-B)$$
$$= \tau B - kT.$$

Hence the theorem

IFoS-2009

→ If k is the curvature and τ is the torsion of a curve $\vec{r}(s)$ then $k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$ and

$$\tau = \left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] \Big/ \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2$$

Sol: We know that $T = \frac{d\vec{r}}{ds}$

and $kN = \frac{d^2\vec{r}}{ds^2}$ $\left(\because \underbrace{\frac{dT}{ds}}_{(kN)} = \frac{d^2r}{ds^2}\right)$.

Now $\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} = T \times kN$

$= k(T \times N)$

$= kB$ $\quad (\because T \times N = B)$

$\therefore k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$ ⟵ (1)

$$\frac{d^3\vec{r}}{ds^3} = \frac{d}{ds}\left(\frac{d^2\vec{r}}{ds^2}\right) = \frac{d}{ds}(kN)$$

$= k\frac{dN}{ds} + \frac{dk}{ds}N$

$= k(\tau B - kT) + \frac{dk}{ds}N$

$\left(\because \frac{dN}{ds} = \tau B - kT\right)$

$= k\tau B - k^2T + \frac{dk}{ds}N$

$$\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] = \left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right) \cdot \frac{d^3\vec{r}}{ds^3} \quad \left(\because [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}\right)$$

$= kB \cdot \left(k\tau B - k^2T + \frac{dk}{ds}N\right)$

$= k^2\tau\, B \cdot B - k^3(B \cdot T) + k\frac{dk}{ds}(B \cdot N)$

$= k^2\tau$ $\quad (\because B \cdot B = 1;\ B \cdot T = 0$ and $B \cdot N = 0)$.

$$\Rightarrow \tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{k^2}$$

$$\therefore \tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2} \quad (\text{by } ①)$$

→ If k is the curvature and τ is the torsion of a curve $\vec{r}(t)$ then $k = \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left|\frac{d\vec{r}}{dt}\right|^3$

and $\tau = \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] \Big/ \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2$

Sol^n: W.K.T $\left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$ ——①

Let $\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds}$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)$$

$$= \frac{d}{ds}\left(\frac{d\vec{r}}{dt}\frac{dt}{ds}\right)$$

$$= \frac{d\vec{r}}{dt}\frac{d^2t}{ds^2} + \frac{d^2\vec{r}}{dt^2}\left(\frac{dt}{ds}\right)^2$$

$$\frac{d^3\vec{r}}{ds^3} = \frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + \frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + 2\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3$$

$$= \frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + 3\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3$$

Now $\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} = \frac{d\vec{r}}{dt}\frac{dt}{ds} \times \left[\frac{d\vec{r}}{dt}\cdot\frac{d^2t}{ds^2} + \frac{d^2\vec{r}}{dt^2}\left(\frac{dt}{ds}\right)^2\right]$

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$$= \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\left(\frac{dt}{ds}\right)^3$$

$$\therefore \kappa = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$$

$$= \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left(\frac{ds}{dt}\right)^3$$

$$= \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left|\frac{d\vec{r}}{dt}\right|^3$$

$$\left[\frac{d\vec{r}}{ds} \; \frac{d^2\vec{r}}{ds^2} \; \frac{d^3\vec{r}}{ds^3}\right] = \left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right) \cdot \frac{d^3\vec{r}}{ds^3}$$

$$= \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\left(\frac{dt}{ds}\right)^3 \cdot \left[\frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + 3\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3\right]$$

$$= \left[\left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) \cdot \frac{d^3\vec{r}}{dt^3}\right]\left(\frac{dt}{ds}\right)^6$$

$$= \left[\frac{d\vec{r}}{dt} \; \frac{d^2\vec{r}}{dt^2} \; \frac{d^3\vec{r}}{dt^3}\right] \cdot \left(\frac{dt}{ds}\right)^6$$

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$$\tau = \frac{\left[\frac{d\vec{r}}{ds} \; \frac{d^2\vec{r}}{ds^2} \; \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2}$$

$$= \frac{\left[\frac{d\vec{r}}{dt} \; \frac{d^2\vec{r}}{dt^2} \; \frac{d^3\vec{r}}{dt^3}\right]\left(\frac{dt}{ds}\right)^6}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2 \left(\frac{dt}{ds}\right)^6}$$

$$= \frac{\left[\frac{d\vec{r}}{dt} \; \frac{d^2\vec{r}}{dt^2} \; \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$$

2004 → Show that the Frenet-Serret formulae can be written in the form $\frac{d\vec{T}}{ds} = \vec{\omega} \times \vec{T}$, $\frac{d\vec{N}}{ds} = \vec{\omega} \times \vec{N}$ and $\frac{d\vec{B}}{ds} = \vec{\omega} \times \vec{B}$

where $\vec{\omega} = \tau\vec{T} + k\vec{B}$.

Solⁿ: $\vec{\omega} \times \vec{T} = (\tau\vec{T} + k\vec{B}) \times \vec{T}$

$= \tau\vec{T}\times\vec{T} + k\vec{B}\times\vec{T}$

$= 0 + k\vec{N} \quad (\because \vec{B}\times\vec{T} = \vec{N})$

$= k\vec{N}$

$= \frac{d\vec{T}}{ds}$.

$\vec{\omega} \times \vec{N} = (\tau\vec{T} + k\vec{B}) \times \vec{N}$

$= \tau\vec{T}\times\vec{N} + k\vec{B}\times\vec{N}$

$= \tau\vec{B} - k\vec{T}$

$= \frac{d\vec{N}}{ds}$

T N B

$\vec{\omega} \times \vec{N} = (\tau\vec{T} + k\vec{B}) \times \vec{B}$

$= \tau\vec{T}\times\vec{B} + k\vec{B}\times\vec{B}$

$= -\tau\vec{N} + 0$

$= -\tau B$

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Problems

→ for the curve $x = 3\cos t,\ y = 3\sin t,\ z = 4t$, find T, N, B and k, τ.

Sol. The position vector for any point on the curve is $\vec{r} = 3\cos t\, i + 3\sin t\, j + (4t)k$.

$$\therefore \frac{d\vec{r}}{dt} = -3\sin t\, i + 3\cos t\, j + 4k \quad \text{--- (1)}$$

Now $$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$$
$$= \sqrt{9\sin^2 t + 9\cos^2 t + 16}$$
$$= \sqrt{9(1) + 16}$$
$$= 5$$

$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt}$

Now $$T = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{1}{5}\left(-3\sin t\, i + 3\cos t\, j + 4k\right).$$

$\frac{dT}{ds} = ?$

$$\therefore \frac{dT}{dt} = \frac{1}{5}\left(-3\cos t\, i - 3\sin t\, j\right)$$

$$\therefore \frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{1}{25}\left(-3\cos t\, i - 3\sin t\, j\right).$$

$$\therefore \text{curvature } k = \left|\frac{dT}{ds}\right|$$
$$= \sqrt{\left(\frac{-3}{25}\cos t\right)^2 + \left(\frac{-3}{25}\sin t\right)^2}$$

$$\boxed{k = \frac{3}{25}}$$

we know that
$$\frac{dT}{ds} = kN$$
$$\Rightarrow N = \frac{1}{k}\frac{dT}{ds}$$
$$= \frac{25}{3}\left(\frac{1}{25}\right)\left(-3\cos t\, i - 3\sin t\, j\right)$$
$$= \frac{1}{3}\left(-3\cos t\, i - 3\sin t\, j\right).$$

$$= -\cos t\, i - \sin t\, j$$

$B = T \times N$

$$= \begin{vmatrix} i & j & k \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{5}\left(4\sin t\, i - 4\cos t\, j + 3k\right)$$

$$\therefore \frac{dB}{dt} = \frac{1}{5}\left(4\cos t\, i + 4\sin t\, j\right)$$

Now $\frac{dB}{ds} = \frac{dB/dt}{ds/dt} = \frac{1}{25}\left(4\cos t\, i + 4\sin t\, j\right)$.

$$= \frac{4}{25}(\cos t\, i + \sin t\, j).$$

we know that

$$\frac{dB}{ds} = -\tau N.$$

$$\Rightarrow \frac{4}{25}(\cos t\, i + \sin t\, j) = -\tau(-\cos t\, i - \sin t\, j)$$

$$\Rightarrow \boxed{\frac{4}{25} = \tau.}$$

Note :- whenever only finding k and τ, you can better to use the formulas $k = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$ &

$$\tau = \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$$

HW IFS-2005 IAS-2010 12M → For the curve $x = a\cos t,\ y = a\sin t,\ z = bt$. Show that $K = \frac{a}{a^2+b^2}$, $\tau = \frac{b}{a^2+b^2}$.

3.

→ Show that the torsion τ for the space curve $x = \frac{2t+1}{t-1},\ y = \frac{t^2}{t-1},\ z = t+2$ is zero.

→ Find the curvature K and torsion τ for the space curve $x = t - \sin t$, $y = 1-\cos t$, $z = t$.

→ Find the curvature K and torsion τ for the space curve $\vec{r} = e^t\hat{i} - e^{-t}\hat{j} + \sqrt{2}t\hat{k}$.

2002 → Find the curvature K for the space curve: $x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\tan\alpha$.

IAS 2005 → Find the curvature and the torsion of the space curve $x = a(3u-u^3)$, $y = 3au^2$, $z = a(3u+u^3)$.

2007 → Find the curvature and torsion at any point of the curve $x = a\cos 2t,\ y = a\sin 2t$, $z = 2a\sin t$.

→ Find T, N, B, Curvature K, torsion τ for the space curve

(i) $x = t,\ \ y = t^2,\ z = \frac{2}{3}t^3$ at $t=1$.

IFS 2003 → (ii) $x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\cot\alpha$.

→ Find the curvature k and τ for the space curve.

(i) $r = 3t\,i + 3t^2 j + 2t^3 k$ (ii) $r = (3t - t^3)i + 3t^2 j + (3t + t^3)k$

(iii) $r = t\,i + t^2 j + t^3 k$ at $t=1$. (iv) $x = \frac{t}{2}$, $y = \frac{t^2}{4}$, $z = \frac{t^3}{12}$

(v) $x = t - \frac{t^3}{3}$, $y = t^2$, $z = t + \frac{t^3}{3}$

(vi) $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, $z = 4\sin\theta/2$

(vii) $x = t$, $y = t^2$, $z = 2t^3/3$ at $(\sqrt{3}, 3, 2\sqrt{3})$.

→ Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$. Also determine the unit tangent vector at $t = 2$.

IFS-2002 → IAS-2012 For the space curve $x = t - \frac{1}{3}t^3$, $y = t^2$, $z = t + \frac{1}{3}t^3$ show that

$$k = \tau = \frac{1}{(1+t^2)^2}.$$

→ For the curve $\vec{r} = e^u i - e^{-u} j + \sqrt{2}\,u k$, show that $k = +\tau = \frac{\sqrt{2}}{(e^u + e^{-u})^2}$.

IFS-2001 → IAS-2008 IAS-2012 For the space curve $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$ — find the values of k and τ at $t = 1$.

IAS 2005 → Find a unit tangent vector to any point on the curve $x = a\cos\omega t$, $y = a\sin\omega t$, $z = bt$ where a, b, ω are constants.

Ans: $\dfrac{-a\omega\sin\omega t\,\hat{i} + a\omega\cos\omega t\,\hat{j} + b\hat{k}}{\sqrt{a^2\omega^2 + b^2}}$