IFoS P-2 2020

G-20 Maths

2(a) Let R be a non-zero commutative ring with unity. Show that M is a maximal ideal in a ring R if and only if R/M is a field. (10)

Proof: Let the ideal M be maximal.
Since R is a commutative ring with unity, R/M is also a commutative ring with unity.
Let $M+a \in R/M$ be any non-zero element.

Then to show that R/M is a field, all we need to show is that $(M+a)$ is invertible and $(M+a)^{-1} \in R/M$.
Now, $M+a \in R/M$

$\Rightarrow \quad M+a \neq M \quad \Rightarrow \quad a \notin M$

Let $aR = \{ar : r \in R\}$

Here $aR = \langle a \rangle$ is principal ideal of R generated by a.
Since sum of two ideals is an ideal, $M+aR$ will be an ideal of R.
Also,
$$a = 0 + a\cdot 1 \in M+aR \text{ and } a \notin M$$
$$\therefore \quad M \subsetneq M+aR \subseteq R$$

M is maximal $\Rightarrow$ $M+aR = R$

$\therefore \quad 1 \in R \Rightarrow 1 \in M+aR$

$\Rightarrow 1 = m + ar$ for some $m \in M, r \in R$

$\Rightarrow \quad M+1 = M+(m+ar)$

$= (M+m)+(M+ar) = M+ar$

$= (M+a)(M+r)$

$\Rightarrow$ $(M+r)$ is a multiplicative inverse of $(M+a)$

$\therefore$ R/M is a field.

Conversely,
Let R/M be a field.

Let M' be any ideal of R such that
$$M \subset M' \subseteq R$$

$\therefore$ There exist $m \in M'$, such that $m \notin M$.
Now, $m \notin M \Rightarrow M+m \neq M$

$\Rightarrow M+m$ is a nonzero element of R/M.

Since R/M is a field, so $M+m$ has a multiplicative inverse.
Let $(M+t)$ be its inverse.

$$\therefore \quad (M+m)(M+t) = M+1$$
$$\Rightarrow \quad M+mt = M+1$$
$$\Rightarrow \quad mt-1 \in M$$
$$\Rightarrow \quad mt-1 = m' \text{ for some } m' \in M$$
$$\Rightarrow \quad 1 = mt - m' \in M' \quad [\because M \subset M']$$

Thus M' is an ideal of R with unity element.
Hence M is a maximal ideal of R.

2(b) Show that the sequence of functions $\langle f_n(x) \rangle$
$f_n(x) = nx(1-x)^n$,
does not converge uniformly on $[0,1]$.
(15)

Sol: When $x = 0$

$f_n(x) = 0 \quad \forall n \in N$

When $x = 1$

$f_n(x) = 0 \quad \forall n \in N$

Hence, $f(x) = \lim_{n\to\infty} f_n(x) = 0$ when $x=0$ and $x=1$

Again for, $0 < x < 1$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} \qquad [\text{L-Hopital}]$$

$$= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)}$$

$= 0$ as $0 < x < 1$ so that $(1-x)^n \to 0$ as $n \to \infty$

Thus we have,

$$f(x) = 0 \quad \forall x \in [0,1]$$

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$\therefore |f_n(x) - f(x)| = |nx(1-x)^n - 0|$

$= nx(1-x)^n = y$, say

Where, $y = nx(1-x)^n$

$$\frac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$$

$$= n(1-x)^{n-1}\{(1-x) - nx\}$$

$$= n(1-x)^{n-1}\{1 - x(n+1)\}$$

For max or min value of y, we have

$\frac{dy}{dx} = 0 \Rightarrow \boxed{x = \frac{1}{n+1}}$

Again,

$$\frac{d^2y}{dx^2} = -n(n-1)(1-x)^{n-2} \times \{1-(n+1)x\} - n(1-x)^{n-1}(n+1)$$

When $x = \frac{1}{n+1}$

$$\frac{d^2y}{dx^2} = -n(n+1)\left(\frac{n}{n+1}\right)^{n-1} < 0$$

It shows that y is maximum at $x = \frac{1}{n+1}$ & max value of $y = \left(\frac{n}{n+1}\right)^{n+1}$

$\therefore M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$

$= \sup\{|y| : x \in [0,1]\} = \left(\frac{n}{n+1}\right)^{n+1}$

$\therefore M_n \rightarrow e^{-1}$ as $n \rightarrow \infty$.

Since M_n does not tend to 0 as $n \rightarrow \infty$, the sequence $\langle f_n \rangle$ is not uniformly convergent on $[0, 1]$.

Here 0 is a point of non-uniform convergence because $x = \frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

2(c) Using Cauchy's Theorem and Cauchy Integral formula, evaluate the integral

$$\oint_C \frac{e^z}{z^2(z+1)^3}\,dz$$

where C is $|z| = 2$. (15)

Sol: Cauchy Integral formula : If $f(z)$ is analytic within and on a closed curve C and if a is any point within C, then

$$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

or $$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = (2\pi i)\,\frac{f^n(a)}{n!} \quad -(1)$$

Let $$\frac{1}{z^2(z+1)^3} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1} + \frac{D}{(z+1)^2} + \frac{E}{(z+1)^3}$$

$$1 = Az(z+1)^3 + B(z+1)^3 + Cz^2(z+1)^2 + Dz^2(z+1) + Ez^2$$

Put $z=0 \Rightarrow 1 = B$

$z=-1 \Rightarrow 1 = E$

$$1 = Az(z+1)^3 + (z+1)^3 + Cz^2(z+1)^2 + Dz^2(z+1) + z^2$$

Comparing coefficients on both sides.

$0 = A + C$ Coeff of z^4

$0 = 3A + 1 + 2C + D$ Coeff of z^3

$0 = 3A + 3 + C + D + 1$ Coeff of z^2

$0 = A + 3$ Coeff of z.

$\Rightarrow A = -3$ & $C = 3$ & $D = 2$

$$\therefore \frac{1}{z^2(z+1)^3} = \frac{-3}{z} + \frac{1}{z^2} + \frac{3}{z+1} + \frac{2}{(z+1)^2} + \frac{1}{(z+1)^3} \quad -(2)$$

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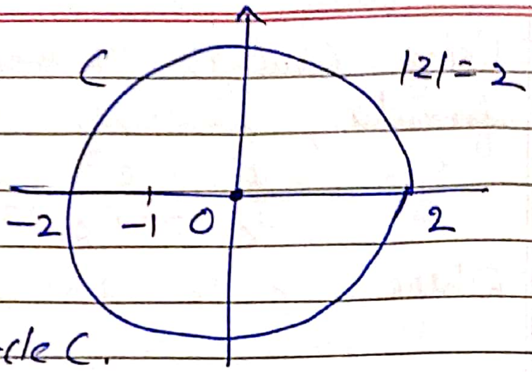
$C: |z| = 2$



Points of singularity

$z=0$ and $z-1$

both lies insid circle C.

Using ① and ②

$$\oint_C \frac{e^z}{z^2(z+1)^3}\,dz = \oint_C e^z \cdot \frac{1}{z^2(z+1)^3}\,dz$$

$$= \oint_C \frac{-3e^z}{z} + \frac{e^z}{z^2} + \frac{3e^z}{z+1} + \frac{2e^z}{(z+1)^2} + \frac{e^z}{(z+1)^3}\,dz$$

$$= 2\pi i \left[\left(-3e^z\right)_{z=0} + \left(\frac{d}{dz}e^z\right)_{z=0} + \left(3e^z\right)_{z=-1} + \left(\frac{d}{dz}2e^z\right)_{z=-1} + \frac{1}{2!}\left(\frac{d^2}{dz^2}e^z\right)_{z=-1} \right]$$

$$= 2\pi i \left[-3 + 1 + 3e^{-1} + 2e^{-1} + \frac{1}{2}e^{-1} \right]$$

$$= 2\pi i \left[-2 + \frac{11}{2}e^{-1} \right]$$