UPSC Civil Services Main 1979 - Mathematics Calculus

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Question 1(a) *If $f(x) \in C^{n+1}$ for $|x-a| \leq h$, then show that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\, dt$$

Further show that the remainder after $n+1$ terms can be expressed as

$$\frac{f^{(n+1)}(X)(x-X)^n}{n!}(x-a)$$

where $a < X < x$.

Solution. Carrying out integration by parts on the given integral repeatedly, we get

$$\begin{aligned}
\int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\, dt &= \frac{f^{(n)}(t)(x-t)^n}{n!}\Big]_a^x + \int_a^x \frac{f^{(n)}(t)(x-t)^{n-1}}{(n-1)!}\, dt \\
&= -\frac{f^{(n)}(a)(x-a)^n}{n!} + \int_a^x \frac{f^{(n)}(t)(x-t)^{n-1}}{(n-1)!}\, dt \\
&= -\frac{f^{(n)}(a)(x-a)^n}{n!} - \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \int_a^x \frac{f^{(n-1)}(t)(x-t)^{n-2}}{(n-2)!}\, dt \\
&= \ldots \\
&= -\sum_{r=1}^{n} \frac{f^{(r)}(a)(x-a)^r}{r!} + \int_a^x f'(t)\, dt \\
&= -\sum_{r=1}^{n} \frac{f^{(r)}(a)(x-a)^r}{r!} + f(x) - f(a) \\
&= -\sum_{r=0}^{n} \frac{f^{(r)}(a)(x-a)^r}{r!} + f(x)
\end{aligned}$$

Thus

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)}(a)(x-a)^r}{r!} + \int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt$$

as required. Now we use the result — if $f(x)$ is continuous in $[a,b]$, then

$$\int_a^b f(x)\,dx = f(\mu)(b-a) \text{ for some } \mu \in (a,b)$$

This is obvious as $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$, where m, M are respectively the min and max of $f(x)$ over $[a,b]$. and every value between m and M is attained (by the intermediate value theorem).

Since $f \in C^{n+1}$, $f^{(n+1)}$ is continuous and we can use the above result. Therefore

$$\int_a^x \frac{f^{(n+1)}(t)(x-t)^n}{n!}\,dt = \frac{f^{(n+1)}(X)(x-X)^n}{n!}(x-a)$$

where $a < X < x$. ∎

Question 1(b) *If*

$$f(x,y) = \begin{cases} 2xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

show that $\dfrac{\partial^2 f}{\partial x\,\partial y} \neq \dfrac{\partial^2 f}{\partial y\,\partial x}$. *Explain this result.*

Solution.

$$\begin{aligned}
f_y(0,0) &= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0 \\
f_y(h,0) &= \lim_{k\to 0} \frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0} \frac{2hk\frac{h^2-k^2}{h^2+k^2}}{k} = 2h \\
\frac{\partial^2 f}{\partial x\,\partial y}(0,0) &= \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{2h}{h} = 2 \\
f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \\
f_x(0,k) &= \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0} \frac{2hk\frac{h^2-k^2}{h^2+k^2}}{h} = -2k \\
\frac{\partial^2 f}{\partial y\,\partial x}(0,0) &= \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0} \frac{-2k}{k} = -2
\end{aligned}$$

Thus $\dfrac{\partial^2 f}{\partial x\,\partial y} \neq \dfrac{\partial^2 f}{\partial y\,\partial x}$ at $(0,0)$. The reason for this is that neither f_{xy} nor f_{yx} is continuous at $(0,0)$. This implies that neither $f_x(x,y)$ nor $f_y(x,y)$ is differentiable at $(0,0)$. Thus the criteria of Young's or Schwartz theorem are not satisfied. ∎

Question 2(a) *If $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x>0, y>0,$ then show that $B(x,x) = 2^{1-2x}B(x, \frac{1}{2})$.*

Solution. See 1982, question 2(b), where we proved

$$\frac{\Gamma(\frac{1}{2})\Gamma(x)}{\Gamma(x+\frac{1}{2})} = \frac{2^{2x-1}[\Gamma(x)]^2}{\Gamma(2x)}$$

Since $\dfrac{\Gamma(\frac{1}{2})\Gamma(x)}{\Gamma(x+\frac{1}{2})} = B(x, \frac{1}{2})$ and $\dfrac{[\Gamma(x)]^2}{\Gamma(2x)} = B(x,x)$, we get $B(x,x) = 2^{1-2x}B(x,\frac{1}{2})$ as required. ∎

Paper II

Question 3(a) *Find the maximum of $x_1^2 x_2^2 \dots x_n^2$ under the restriction $x_1^2 + x_2^2 + \dots + x_n^2 = 1$. Using the result derive the inequality*

$$(a_1 a_2 \dots a_n)^{\frac{1}{n}} \le \frac{a_1 + \dots + a_n}{n}$$

for positive real numbers $a_1, \dots, a_n$.

Solution. Let $F(x_1,\dots,x_n) = x_1^2 x_2^2 \dots x_n^2 + \lambda(\sum_{i=1}^n x_i^2 - 1)$, where λ is Lagrange's undetermined multiplier. For extreme values,

$$\frac{\partial F}{\partial x_i} = 2x_1^2 x_2^2 \dots x_{i-1}^2 x_i x_{i+1}^2 \dots x_n^2 + 2\lambda x_i = 0, \quad 1 \le i \le n$$

Since $x_1^2 + x_2^2 + \dots + x_n^2 = 1$, $(x_1,\dots,x_n) \neq (0,\dots,0)$, it follows that $\lambda = -x_1^2 x_2^2 \dots x_{i-1}^2 x_{i+1}^2 \dots x_n^2$ for $1 \le i \le n$. Thus $x_1^2 = x_2^2 = \dots = x_n^2 = \frac{1}{n}$, and $\lambda = -\frac{1}{n^{n-1}}$.

$$\begin{aligned}
\frac{\partial^2 F}{\partial x_i^2} &= 2x_1^2 x_2^2 \dots x_{i-1}^2 x_{i+1}^2 \dots x_n^2 + 2\lambda = 0 \quad \text{for } 1 \le i \le n \\
\frac{\partial^2 F}{\partial x_i\,\partial x_j} &= 4x_i x_j \prod_{r=1, r\neq i,j}^{n} x_r^2 = \frac{4}{n^{n-1}} \qquad \text{at } x_1^2 = x_2^2 = \dots = x_n^2 = \frac{1}{n} \\
\Rightarrow d^2F &= \frac{8}{n^{n-1}} \sum_{1 \le i < j \le n} dx_i\,dx_j \text{ (only the cross terms appear)}
\end{aligned}$$

Now $x_1^2 + x_2^2 + \dots + x_n^2 = 1$, so $2\sum_{i=1}^n x_i\,dx_i = 0 \Rightarrow \sum_{i=1}^n dx_i = 0 \Rightarrow dx_n = -(dx_1 + \dots + dx_{n-1})$.

Thus $d^2F = \dfrac{8}{n^{n-1}}\Big[\displaystyle\sum_{1\le i<j\le n-1} dx_i\,dx_j - \Big(\sum_{i=1}^{n-1} dx_i\Big)^2\Big]$. Clearly $d^2F < 0$, so we have a maximum when $x_1^2 = x_2^2 = \dots = x_n^2 = \frac{1}{n}$.

Derivation of inequality:

$$\begin{aligned}
x_1^2x_2^2\dots x_n^2 &\le \left(\frac{1}{n}\right)^n = \left(\frac{x_1^2+x_2^2+\dots+x_n^2}{n}\right)^n \text{ as } x_1^2+x_2^2+\dots+x_n^2=1 \\
\Rightarrow (x_1^2x_2^2\dots x_n^2)^{\frac{1}{n}} &\le \frac{x_1^2+x_2^2+\dots+x_n^2}{n}
\end{aligned}$$

Let $x_i^2 = a_i, 1 \le i \le n$, to get

$$(a_1a_2\dots a_n)^{\frac{1}{n}} \le \frac{a_1+\dots+a_n}{n}$$

as required. ∎

Question 4(a) *Define* $f(x) = \left(\int_0^x e^{-t^2}\,dt\right)^2, g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt$. *Show that* $g'(x) + f'(x) = 0$ *for all* x. *Deduce that (i)* $g(x)+f(x) = \frac{\pi}{4}$ *(ii)* $\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$.

Solution.

$$\begin{aligned}
f'(x) &= 2\left(\int_0^x e^{-t^2}\,dt\right)e^{-x^2} \\
g'(x) &= \int_0^1 \frac{\partial}{\partial x}\left(\frac{e^{-x^2(t^2+1)}}{t^2+1}\right)dt \\
&= \int_0^1 \frac{e^{-x^2(t^2+1)}(-2x(t^2+1))}{t^2+1}dt
\end{aligned}$$

Let $xt = y$ in the second equation, then $dy = x\,dt, g'(x) = -2e^{-x^2}\int_0^x e^{-y^2}\,dy = -f'(x)$, showing that $f'(x)+g'(x) = 0$ for all x.

1. $f'(x)+g'(x) = 0 \Rightarrow f(x)+g(x) = C$, a constant. Clearly $f(0) = 0, g(0) = \int_0^1 \frac{dt}{t^2+1} = \tan^{-1} t\Big]_0^1 = \frac{\pi}{4}$. So $C = f(0)+g(0) = \frac{\pi}{4}$, thus $g(x)+f(x) = \frac{\pi}{4}$.

2. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(\int_0^1 \frac{e^{-x^2t^2}}{t^2+1}\,dt\right)e^{-x^2}$.

   But $\int_0^1 \frac{e^{-x^2t^2}}{t^2+1}\,dt \le \int_0^1 \frac{dt}{1+t^2} = \frac{\pi}{4}, \because e^{-x^2t^2} \le 1$. Thus $0 \le g(x) \le \frac{\pi}{4}e^{-x^2} \Rightarrow \lim_{x\to\infty} g(x) = 0$.

   Thus $\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\int_0^x e^{-t^2}\,dt\right)^2 = \frac{\pi}{4} \Longrightarrow \lim_{x\to\infty}\int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$, as both sides are positive.

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Question 1(a) *Let f be a real continuous function on the interval $[a, b]$ which is differentiable in the open interval (a, b). Prove that there exists at least one point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$.*

Solution. (This is the mean value theorem of calculus.)

Consider the function $\phi(x) = f(x) - f(a) - (x - a)A$, where A is a constant so determined that $\phi(b) = 0 \Rightarrow A = \dfrac{f(b) - f(a)}{b - a}$. Thus $\phi(b) = \phi(a) = 0$, $\phi(x)$ is continuous in $[a, b]$ as it is the sum of functions continuous over $[a, b]$, and $\phi(x)$ is differentiable over (a, b) as $f(x), f(a), (x - a)A$ are all differentiable over (a, b). Thus $\phi(x)$ satisfies the conditions of Rolle's theorem, so there is at least one point $c \in (a, b)$ such that $\phi'(c) = 0$ i.e. $f'(c) - A = 0 \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a}$, thus $f(b) - f(a) = (b - a)f'(c)$. ∎

Question 1(b) *Find the volume of the solid of revolution formed by rotating the area cut off from $y^2 = 4ax$ by the line $y = x$ through 2π radians about the x-axis.*

Solution. The curve $y^2 = 4ax$ intersects $y = x$ at $(0, 0)$ and $(4a, 4a)$. The required volume V = Volume of rotating $y^2 = 4ax$ about the x-axis - Volume of rotating $y = x$ about the x-axis. Thus

$$\begin{aligned}
V &= \int_0^{4a} \pi y_1^2\, dx - \int_0^{4a} \pi y_2^2\, dx \\
&= \int_0^{4a} \pi 4ax\, dx - \int_0^{4a} \pi x^2\, dx \\
&= 4a\pi \frac{x^2}{2}\Big]_0^{4a} - \pi \frac{x^3}{3}\Big]_0^{4a} \\
&= 4a\pi \frac{16a^2}{2} - \pi \frac{64a^3}{3} = \frac{32\pi a^3}{3}
\end{aligned}$$

■

Question 2(a) *A rectangular box open at the top is to have a capacity of* $4m^3$. *Find the dimensions of the box requiring least material for its construction.*

Solution. Let x be the length, y be the breadth, and z the height of the box. Then the capacity of the box is $xyz = 4$. The surface area of the box is $S = 2xz + 2yz + xy = xy + 2(x+y)\frac{4}{xy} = xy + \frac{8}{x} + \frac{8}{y}$. For extreme values

$$\frac{\partial S}{\partial x} = y - \frac{8}{x^2} = 0, \frac{\partial S}{\partial y} = x - \frac{8}{y^2} = 0$$

Thus $y = \frac{8}{x^2}$, substituting in the second equation we get $x = \frac{x^4}{8} \Rightarrow x = 0, 2$. Since $x = 0$ is inadmissible, $x = 2 \Rightarrow y = 2, z = 1$.

$$\frac{\partial^2 S}{\partial x^2} = \frac{16}{x^3}, \frac{\partial^2 S}{\partial y^2} = \frac{16}{y^3}, \frac{\partial^2 S}{\partial x\,\partial y} = 1$$

Thus $\dfrac{\partial^2 S}{\partial x^2}\dfrac{\partial^2 S}{\partial y^2} - \left(\dfrac{\partial^2 S}{\partial x\,\partial y}\right)^2 = 4 - 1 > 0$ when $x = 2, y = 2$, and $\dfrac{\partial^2 S}{\partial x^2} = 2 > 0, \dfrac{\partial^2 S}{\partial y^2} = 2 > 0$. So $x = 2, y = 2$ is a minimum for S.

Thus the dimensions of the required box are 2 meters $\times$ 2 meters $\times$ 1 meter. ■

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Question 1(a) *Let* $f(x) = \dfrac{|x|}{x}$. *Discuss the value of* $\lim_{x\to 0} f(x)$ *and* $f(0)$. *Justify your answer. Discuss the differentiability of* f *and* f^2 *at* $x = 0$.

Solution. Clearly

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Thus $\lim_{x\to 0^+} f(x) = 1, \lim_{x\to 0^-} f(x) = -1$, showing that $\lim_{x\to 0} f(x)$ does not exist. Thus no matter what value is given to $f(x)$ at 0, $f(x)$ cannot be continuous at $x = 0$.

Since $f(x)$ is not continuous at $x = 0$, the question of its differentiability does not arise., as differentiability implies continuity — note that $f(x_0+h)-f(x_0) = \dfrac{f(x_0+h)-f(x_0)}{h}h$, so if $f(x)$ is differentiable at x_0, then $\lim_{h\to 0} f(x_0+h)-f(x_0) = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h} \lim_{h\to 0} h = f'(x_0)\cdot 0 = 0$, so $\lim_{h\to 0} f(x_0+h) = f(x_0)$.

Clearly $f^2(x) = 1$ for every $x \neq 0$, therefore f^2 would become differentiable at $x = 0$ if we define $f(x) = 1$ or -1, making $f^2(x) = 1$ for all x. In any other case $f^2(x)$ would not be differentiable at $x = 0$, as it would not be continuous at $x = 0$. ∎

Question 2(a) *Suppose a manufacturer can sell* x *items at a price* $P = 200 - \dfrac{x}{100}$ *paisa per item, and it costs* $y = 50x + 20000$ *paise to produce the* x *items. What is the production level for maximum profits, and the selling price per item?*

Solution. Selling price of x items $= \left(200 - \dfrac{x}{100}\right)x$

Cost price of x items $= 50x + 20000$.

Profict P on the sale of x items $= \left(200 - \frac{x}{100}\right)x - (50x + 20000)$.
For extreme values,

$$\frac{dP}{dx} = 200 - \frac{x}{100} - \frac{x}{100} - 50 = 150 - \frac{x}{50} = 0 \Longrightarrow x = 7500$$

$$\frac{d^2P}{dx^2} = -\frac{1}{50} < 0$$

showing that the profit is maximum when 7500 items are sold at 125 paisa per item. ∎

Question 2(b) *Evaluate the double integral* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.

Solution. See 2002 question 1(b). ∎

Question 2(c) *State the mean value theorem. If* $f'(x) = \frac{1}{1+x^2}$ *for all* x, *and* $f(0) = 0$, *show that* $0.4 < f(2) < 2.0$.

Solution. If $f(x)$ is a real valued function defined on the closed interval $[a, b]$, such that (i) $f(x)$ is continuous on the closed interval $[a, b]$, (ii) $f(x)$ is differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(c)$.

We apply the mean value theorem to $f(x)$ on $[0, 2]$ and get $f(2) = f(2) - f(0) = (2 - 0)f'(c) = \frac{2}{1+c^2}$ for some $c \in (0, 2)$.

Now $0 < c < 2$ so $\frac{2}{1+4} < \frac{2}{1+c^2} < \frac{2}{1+0}$, thus $0.4 < f(2) < 2$. ∎

Question 3(a) *The region bounded by the curves* $y = x^3$ *and* $y = \sqrt{x}$ *is revolved about the* x*-axis. Find the volume generated.*

Solution. The curves intersect each other at $(0, 0$ and $(1, 1)$. The required volume $V =$ Volume obtained by rotating the arc from $(0, 0)$ to $(1, 1)$ of $y^2 = x$ about the x-axis - Volume obtained by rotating the arc from $(0, 0)$ to $(1, 1)$ of $y = x^3$ about the x-axis. Thus

$$\begin{aligned} V &= \int_0^1 \pi y_1^2\, dx - \int_0^1 \pi y_2^2\, dx \\ &= \pi \int_0^1 x\, dx - \pi \int_0^1 x^6\, dx \\ &= \frac{\pi}{2} - \frac{\pi}{7} = \frac{5\pi}{14} \end{aligned}$$

∎

Question 3(b) *Show that* $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Solution. See 1991 question 2(c). ∎

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Question 1(a) *A function $f(x)$ is defined for $x > 0$ as follows:*

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{2}{q^2}, & x \text{ rational}, x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Show that $f(x)$ has a removable discontinuity at every rational point, but is continuous for irrational values.

Solution.

1. $f(x)$ is continuous at $x = c$, c irrational — Let $\epsilon > 0$ be given. Since the number of positive integers q for which $q^2 \leq \frac{2}{\epsilon}$ is finite, we can find a positive number $\delta > 0$ such that $(c - \delta, c + \delta)$ does not contain any rational number $\frac{p}{q}$ for which $q^2 \leq \frac{2}{\epsilon}$, i.e. $x \in (c - \delta, c + \delta), x = \frac{p}{q}, (p, q) = 1 \Rightarrow q^2 > \frac{2}{\epsilon} \Rightarrow \frac{2}{q^2} < \epsilon$. Thus for $x \in (c - \delta, c + \delta)$,

   - If x is irrational, $|f(x) - f(c)| = 0 < \epsilon$
   - If x is rational, $x = \frac{p}{q}, (p, q) = 1$, then $|f(x) - f(c)| = |\frac{2}{q^2} - 0| < \epsilon$.

   Thus f is continuous at all irrational points.

2. $f(x)$ is discontinuous at $x = c = \frac{p}{q}$, where $(p, q) = 1$. Let $\epsilon > 0, \epsilon < \frac{2}{q^2}$, then whatever $\delta > 0$ we take, we can find an irrational number $x \in (c - \delta, c + \delta)$, and $|f(x) - f(c)| = |0 - \frac{2}{q^2}| > \epsilon$. Thus $f(x)$ is not continuous at any rational point.

3. To show that the discontinuity is removable at $x = c, c \in \mathbb{Q}$, we need to show that $\lim_{x \to c} f(x)$ exists, because if $\lim_{x \to c} f(x) = l$, we can define $f(c) = l$, and make f continuous at c.

Let $\epsilon > 0$, and let $\delta > 0$ be chosen so that $0 < \delta < \dfrac{\epsilon}{2q}$. Let $x \in \mathbb{R}, 0 < |x - \frac{p}{q}| < \delta$. Then

$$|f(x) - 0| = |0 - 0| < \epsilon \quad \text{if } x \notin \mathbb{Q}$$

If $x \in \mathbb{Q}$, let $x = \dfrac{m}{n}, x \neq \dfrac{p}{q}$, then

$$\left|x - \frac{p}{q}\right| = \left|\frac{m}{n} - \frac{p}{q}\right| = \left|\frac{qm - pn}{qn}\right| < \frac{\epsilon}{2q} \Rightarrow \left|\frac{qm - pn}{n}\right| < \frac{\epsilon}{2}$$

Now $|f(x) - 0| = |\frac{2}{n^2} - 0| \leq \dfrac{2}{n}\left|\dfrac{qm - pn}{n}\right| < \dfrac{\epsilon}{n}$ because $|qm - pn| \geq 1$ as $\dfrac{m}{n} \neq \dfrac{p}{q}$.

Hence we have shown that $0 < |x - \frac{p}{q}| < \delta$ with $\delta < \dfrac{\epsilon}{2q} \Rightarrow |f(x) - 0| < \epsilon$, i.e. $\lim_{x \to \frac{p}{q}} f(x) = 0$, showing that $f(x)$ has a removable discontinuity at all rational points.

∎

Question 1(b) *If $f(x)$ is differentiable three times in $(a - h, a + h)$, prove that there is a point c in the interval such that*

$$\frac{f(a+h) + f(a-h)}{2h} - f'(a) = \frac{h^2}{6} f'''(c)$$

Solution. In order to prove this we prove the following particular case.

If $f(x)$ is thrice differentiable and $f(a) = 0, f(a - h) = 0, f(a + h) = 0, f'(a) = 0$ then there exists a point $c \in (a - h, a + h)$ such that $f'''(c) = 0$.

1. $f(a - h) = f(a) = 0$ and $f'(x)$ exists, therefore Rolle's theorem tells us that there exists $\xi_1 \in (a - h, a)$ such that $f'(\xi_1) = 0$.

2. $f(a + h) = f(a) = 0$ and $f'(x)$ exists, therefore Rolle's theorem tells us that there exists $\xi_2 \in (a, a + h)$ such that $f'(\xi_2) = 0$.

3. $f'(\xi_1) = f'(a) = 0$ and $f''(x)$ exists in $[\xi_1, a]$, Rolle's theorem gives a point $\eta_1 \in (\xi_1, a)$ such that $f''(\eta_1) = 0$.

4. $f'(a) = f'(\xi_2) = 0$ and $f''(x)$ exists in $[a, \xi_2]$, Rolle's theorem gives a point $\eta_2 \in (a, \xi_2)$ such that $f''(\eta_2) = 0$.

5. Finally $f''(\eta_1) = f''(\eta_2) = 0$ and $f'''(x)$ exists in $[\eta_1, \eta_2]$, then Rolle's theorem gives a point $c \in (\eta_1, \eta_2)$ such that $f'''(c) = 0$. Clearly, $c \in (\eta_1, \eta_2) \subseteq (a - h, a + h)$.

Now to prove the main result, let

$$Q(x) = f(a) + (x-a)f'(a) + B(x-a)^2 + A(x-a)^3$$

where A, B are so determined that $Q(a+h) = f(a+h), Q(a-h) = f(a-h)$. Thus

$$f(a+h) - Q(a+h) = 0 \;\Rightarrow\; f(a+h) - f(a) - hf'(a) - h^2B - h^3A = 0 \qquad (1)$$
$$f(a-h) - Q(a-h) = 0 \;\Rightarrow\; f(a-h) - f(a) + hf'(a) - h^2B + h^3A = 0 \qquad (2)$$

Subtracting (2) from (1) we get

$$f(a+h) - f(a-h) - 2hf'(a) - 2h^3A = 0 \qquad (3)$$

We now take $\phi(x) = f(x) - Q(x)$, then $\phi(a) = 0, \phi'(a) = 0, \phi(a-h) = 0, \phi(a+h) = 0$, so by the result proved above, we get a point $c \in (a-h, a+h)$ such that $\phi'''(c) = 0$. But $\phi'''(c) = f'''(c) - Q'''(c) = f'''(c) - 6A \Rightarrow A = \frac{f'''(c)}{6}$. Substituting in (3) we have

$$f(a+h) - f(a-h) - 2hf'(a) = \frac{h^3}{3}f'''(c)$$

or

$$\frac{f(a+h) + f(a-h)}{2h} - f'(a) = \frac{h^2}{6}f'''(c)$$

as required. ∎

Note: In such questions the general tendency is to take $\phi(t) = f(a+th) - f(a-th), 0 \leq t \leq 1$ and apply Taylor's theorem to $\phi(t)$ in $[0,1]$. We get

$$\begin{aligned}
\phi(1) &= \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(\theta), 0 < \theta < 1 \\
\phi(1) &= f(a+h) - f(a-h) \\
\phi(0) &= 0 \\
\phi'(0) &= h(f'(a) - (-h)f'(a) = 2hf'(a) \\
\phi''(0) &= h^2f''(a) - h^2f''(a) = 0 \\
\phi'''(\theta) &= h^3f'''(a+\theta h) - (-h^3)f'''(a-\theta h) = h^3[f'''(a+\theta h) + f'''(a-\theta h)]
\end{aligned}$$

Thus

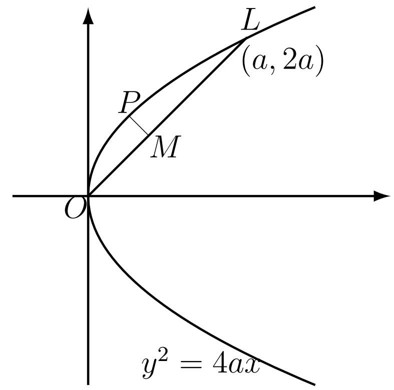
$$\frac{f(a+h) + f(a-h)}{2h} - f'(a) = \frac{h^2}{6}[f'''(a+\theta h) + f'''(a-\theta h)]$$

i.e. we get two points in $(a-h, a+h)$ and not one, so such a solution does not work in our case.

Question 1(c) *The arc of the parabola $y^2 = 4ax$ from the vertex to one extremity of the latus rectum is revolved about the corresponding chord. Prove that the volume of the spindle so formed is $\frac{2\sqrt{5}}{75}\pi a^3$.*

Solution.

Arc OL joining $(0,0)$ to $(a,2a)$ is rotated about the chord OL. Let P be the point (x,y), M the foot of the the perpendicular on OL i.e. $y=2x$. Therefore $PM = \dfrac{|y-2x|}{\sqrt{5}}$.



$$\begin{aligned}
OM^2 &= OP^2 - PM^2 = x^2 + y^2 - \frac{y^2 - 4xy + 4x^2}{5} \\
&= \frac{4y^2 + 4xy + x^2}{5} \\
OM &= \frac{2y+x}{\sqrt{5}} = \frac{4\sqrt{ax}+x}{\sqrt{5}}
\end{aligned}$$

The required volume is V, given by

$$\begin{aligned}
V &= \int_0^a \pi (PM)^2 \, d(OM) \\
(PM)^2 &= \frac{y^2 - 4xy + 4x^2}{5} = \frac{4ax - \sqrt{4ax}\cdot 4x + 4x^2}{5} \\
d(OM) &= \frac{2\frac{\sqrt{a}}{\sqrt{x}} + 1}{\sqrt{5}}\, dx = \frac{2\sqrt{a}+\sqrt{x}}{\sqrt{5x}}\, dx \\
\Rightarrow V &= \pi \int_0^a \frac{4}{5} x\big[a - 2\sqrt{ax} + x\big] \frac{2\sqrt{a}+\sqrt{x}}{\sqrt{5x}}\, dx \\
&= \frac{4\pi}{5\sqrt{5}} \int_0^a \big(x^2 + ax - 2x\sqrt{ax} + 2a\sqrt{ax} - 4ax + 2\sqrt{a}x^{\frac{3}{2}}\big)\, dx \\
&= \frac{4\pi}{5\sqrt{5}} \left[\frac{x^3}{3} - 3a\frac{x^2}{2} + 2a^{\frac{3}{2}}\frac{2}{3}x^{\frac{3}{2}}\right]_0^a \\
&= \frac{4\sqrt{5}\pi}{25}\left[\frac{a^3}{3} - \frac{3a^3}{2} + \frac{4a^3}{3}\right] = \frac{2\sqrt{5}}{75}\pi a^3
\end{aligned}$$

as required. ∎

Question 2(a) *If* $x\cos u + y \sin u = 1$ *and* $v = x\sin u - y\cos u$, *prove that*

$$v^2 \frac{\partial^2 u}{\partial x\, \partial y} + v \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \cos 2u$$

Solution. Differentiate $x\cos u + y\sin u = 1$ with respect to x to get

$$\cos u - x\sin u \frac{\partial u}{\partial x} + y\cos u \frac{\partial u}{\partial x} = 0$$

or

$$\cos u - (x\sin u - y\cos u)\frac{\partial u}{\partial x} = \cos u - v\frac{\partial u}{\partial x} = 0 \;\Rightarrow\; v\frac{\partial u}{\partial x} = \cos u \tag{1}$$

Differentiate $x \cos u + y \sin u = 1$ with respect to y to get

$$-x \sin u \frac{\partial u}{\partial y} + \sin u + y \cos u \frac{\partial u}{\partial y} = 0$$

or

$$\sin u - (x \sin u - y \cos u)\frac{\partial u}{\partial y} = \sin u - v\frac{\partial u}{\partial y} = 0 \;\Rightarrow\; v\frac{\partial u}{\partial y} = \sin u \qquad (2)$$

Differentiate (1) with respect to y to get

$$-\sin u \frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

But

$$\frac{\partial v}{\partial y} = x \cos u \frac{\partial u}{\partial y} - \cos u + y \sin u \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} - \cos u$$

as $x \cos u + y \sin u = 1$. Therefore

$$-\sin u \frac{\partial u}{\partial y} = v\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\cos u$$

Multiplying by v, and substituting (1), (2), we get

$$-\sin u(\sin u) = v^2\frac{\partial^2 u}{\partial x\,\partial y} + v\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \cos u(\cos u)$$

or

$$v^2\frac{\partial^2 u}{\partial x\,\partial y} + v\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = \cos^2 u - \sin^2 u = \cos 2u$$

as required. ∎

Question 2(b) *Prove that*

$$\sqrt{\pi}\, 2^{\frac{1}{3}}\, \Gamma\Big(\frac{1}{6}\Big) = \sqrt{3}\left[\Gamma\Big(\frac{1}{3}\Big)\right]^2$$

Solution. We know that

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

on putting $x = \sin^2\theta$.

1. We take $n = \frac{1}{2}$ to get

$$\frac{\Gamma(\frac{1}{2})\Gamma(m)}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, d\theta$$

2. We take $m = n$ to get

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2n-1}\,d\theta = \frac{1}{2^{2n-2}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2n-1}\,d\theta = \frac{1}{2^{2n-1}}\int_0^{\pi} (\sin\phi)^{2n-1}\,d\phi$$

by putting $2\theta = \phi$. But

$$\int_0^{\pi} (\sin\phi)^{2n-1}\,d\phi = 2\int_0^{\frac{\pi}{2}} (\sin\phi)^{2n-1}\,d\phi$$

so

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{1}{2^{2n-2}}\int_0^{\frac{\pi}{2}} (\sin\theta)^{2n-1}\,d\theta$$

Using both these, we have

$$\frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})} = \frac{2^{2n-1}[\Gamma(n)]^2}{\Gamma(2n)}$$

or

$$\Gamma(n)\Gamma\big(n+\frac{1}{2}\big) = \frac{\Gamma(\frac{1}{2})\Gamma(2n)}{2^{2n-1}} = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

Put $n = \frac{1}{3}$ to get

$$\Gamma\big(\frac{1}{3}\big) = \frac{\sqrt{\pi}\ 2^{\frac{1}{3}}\ \Gamma\big(\frac{2}{3}\big)}{\Gamma\big(\frac{5}{6}\big)} \implies \left(\Gamma\big(\frac{1}{3}\big)\right)^2 = \sqrt{\pi}\ 2^{\frac{1}{3}}\ \frac{\Gamma\big(\frac{2}{3}\big)\Gamma\big(\frac{1}{3}\big)}{\Gamma\big(\frac{5}{6}\big)}$$

Now $\Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$, so $\Gamma\big(\frac{2}{3}\big)\Gamma\big(\frac{1}{3}\big) = \dfrac{2\pi}{\sqrt{3}}$ and $\Gamma\big(\frac{5}{6}\big)\Gamma\big(\frac{1}{6}\big) = 2\pi$. Thus we get

$$\left(\Gamma\big(\frac{1}{3}\big)\right)^2 = \sqrt{\pi}\ 2^{\frac{1}{3}}\ \frac{\frac{2\pi}{\sqrt{3}}\Gamma\big(\frac{1}{6}\big)}{2\pi}$$

so

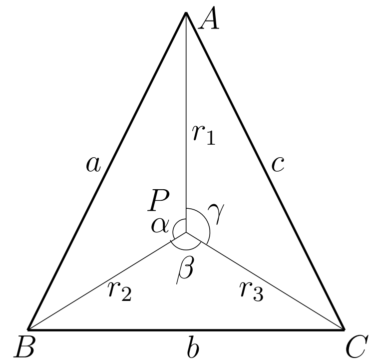
$$\sqrt{3}\left(\Gamma\Big(\frac{1}{3}\Big)\right)^2 = \sqrt{\pi}\,2^{\frac{1}{3}}\,\Gamma\Big(\frac{1}{6}\Big)$$

as required. ∎

Question 2(c) *Find a point within a triangle such that the sum of its distances from the angular points may be a minimum.*

Solution.

Let the point be $P = (x, y)$, and the vertices of the triangle ABC be $(x_i, y_i), i = 1, 2, 3$. Then $u = PA + PB + PC = \sum_{i=1}^{3} \sqrt{(x-x_i)^2 + (y-y_i)^2} = r_1 + r_2 + r_3$, where $r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}, i = 1, 2, 3$. For extreme values



$$\begin{aligned} \frac{\partial u}{\partial x} &= \sum_{i=1}^{3} \frac{x - x_i}{r_i} = 0 \\ \frac{\partial u}{\partial y} &= \sum_{i=1}^{3} \frac{y - y_i}{r_i} = 0 \end{aligned}$$

Thus

$$\frac{x-x_1}{r_1} + \frac{x-x_2}{r_2} = -\frac{x-x_3}{r_3}, \quad \frac{y-y_1}{r_1} + \frac{y-y_2}{r_2} = -\frac{y-y_3}{r_3}$$

Squaring and adding, we get

$$2 + \frac{2}{r_1 r_2}\big((x-x_1)(x-x_2) + (y-y_1)(y-y_2)\big) = 1$$

or

$$\frac{1}{r_1 r_2}\big((x-x_1)(x-x_2) + (y-y_1)(y-y_2)\big) = -\frac{1}{2}$$

Now by the cosine formula, $r_1^2 + r_2^2 - 2r_1 r_2 \cos\alpha = a^2$ or

$$(x-x_1)^2 + (y-y_1)^2 + (x-x_2)^2 + (y-y_2)^2 - 2r_1 r_2 \cos\alpha = (x_1-x_2)^2 + (y_1-y_2)^2$$

Thus

$$\begin{aligned} \cos\alpha &= \frac{1}{2r_1 r_2}(2x^2 - 2xx_1 - 2xx_2 + 2x_1x_2 + 2y^2 - 2yy_1 - 2yy_2 + 2y_1y_2) \\ &= \frac{1}{r_1 r_2}\big((x-x_1)(x-x_2) + (y-y_1)(y-y_2)\big) = -\frac{1}{2} \end{aligned}$$

Thus $\alpha = \frac{2\pi}{3}$. By symmetry, $\beta = \gamma = \frac{2\pi}{3}$, i.e. the sides of the triangle subtend the same angle at P.

Note that this solution is inadmissible when any angle of the triangle is $> \frac{2\pi}{3}$. In this case, the partial derivatives of u do not vanish anywhere inside the triangle, hence the minimum must lie at one of the vertices. A simple check shows that the minimum is at the vertex whose angle is $> \frac{2\pi}{3}$.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \sum_{i=1}^{3}\frac{\partial^2 r_i}{\partial x^2} = \sum_{i=1}^{3}\frac{\partial}{\partial x}\frac{x-x_i}{r_i}\\ &= \sum_{i=1}^{3}\left(\frac{1}{r_i} - \frac{x-x_i}{r_i^2}\,\frac{x-x_i}{r_i}\right)\\ &= \sum_{i=1}^{3}\frac{1}{r_i}\left(1-\frac{(x-x_i)^2}{r_i^2}\right) = \sum_{i=1}^{3}\frac{(y-y_i)^2}{r_i^3} > 0\end{aligned}$$

Similarly $\frac{\partial^2 u}{\partial y^2} > 0$.

$$\frac{\partial^2 u}{\partial x\,\partial y} = \sum_{i=1}^{3}\frac{\partial}{\partial y}\frac{x-x_i}{r_i} = -\sum_{i=1}^{3}\frac{x-x_i}{r_i^2}\frac{\partial r_i}{\partial y} = -\sum_{i=1}^{3}\frac{x-x_i}{r_i^2}\frac{y-y_i}{r_i} = -\sum_{i=1}^{3}\frac{(x-x_i)(y-y_i)}{r_i^3}$$

Now we need to check that $\dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial^2 u}{\partial y^2} - \left(\dfrac{\partial^2 u}{\partial x\,\partial y}\right)^2 > 0$. To make the calculations easier, let $x - x_i = r_i\cos\alpha_i, y - y_i = r_i\sin\alpha_i$. Then

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 &= \left(\sum_{i=1}^{3}\frac{\sin^2\alpha_i}{r_i}\right)\left(\sum_{i=1}^{3}\frac{\cos^2\alpha_i}{r_i}\right) - \left(\sum_{i=1}^{3}\frac{\sin\alpha_i\cos\alpha_i}{r_i}\right)^2\\ &= \frac{1}{r_1r_2}[\sin^2\alpha_1\cos^2\alpha_2 + \cos^2\alpha_1\sin^2\alpha_2 - 2\sin\alpha_1\cos\alpha_2\cos\alpha_1\sin\alpha_2] + \ldots\\ &= \frac{1}{r_1r_2}[\sin\alpha_1\cos\alpha_2 - \cos\alpha_1\sin\alpha_2]^2 + \frac{1}{r_1r_3}[\ldots]^2 + \frac{1}{r_2r_3}[\ldots]^2 > 0\end{aligned}$$

Thus $u = PA + PB + PC$ is at a minimum when P is the point at which the sides subtend the same angle.

Note: This is called the Steiner's problem, and the point P is called the Fermat Point or the Fermat-Torricelli Point of the triangle. ∎

Paper II

Question 3(a) *Find the extreme values of* $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$ *subject to* $\sum_{i,j=1}^{3} a_{ij}x_ix_j = 1$*, where* $a_{ij} = a_{ji}$.

Solution. Let $F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - \lambda\left[\sum_{i,j=1}^{3} a_{ij}x_ix_j - 1\right]$. For extreme values,

$$\frac{\partial F}{\partial x_i} = 2x_i - 2\lambda\sum_{j=1}^{3} a_{ij}x_j = 0, \quad 1 \le i \le 3 \qquad (*)$$

This implies $0 = \sum_{i=1}^{3} x_i \frac{\partial F}{\partial x_i} = 2f - 2\lambda \sum_{i,j=1}^{3} a_{ij} x_i x_j = 2f - 2\lambda \Rightarrow f = \lambda$ at the extreme values.

Now the equations (*) are a set of linear equations, which can be written as $2(\mathbf{I} - \lambda\mathbf{A})\mathbf{x} = \mathbf{0}$, where $\mathbf{I}$ is a 3×3 identity matrix, and $\mathbf{A} = (a_{ij})$ is a 3×3 symmetric matrix, so has real eigenvalues. Since $\mathbf{x} \neq \mathbf{0}$ as $\sum_{i,j=1}^{3} a_{ij} x_i x_j = 1$, $\mathbf{I} - \lambda\mathbf{A}$ must be singular $\Rightarrow \lambda \neq 0$, so $\mathbf{A} - \lambda^{-1}\mathbf{I}$ is singular. Thus λ^{-1} is an eigenvalue of $\mathbf{A}$.

Hence the extreme values of f are λ^{-1}, where λ is an eigenvalue of $\mathbf{A} = (a_{ij})$. ∎

UPSC Civil Services Main 1983 - Mathematics Calculus

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Question 1(a) *Consider the function*

$$f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}$$

in the interval $0 \leq x \leq \frac{\pi}{2}$. *Show that the function does not vanish anywhere in this interval and explain why is so although* $f(0), f(\frac{\pi}{2})$ *differ in sign.*

Solution. Since $x^{2n} \to 0$ as $n \to \infty$ for $0 \leq x < 1$, it follows that

$$f(x) = \log(2 + x), 0 \leq x < 1, f(1) = \frac{\log 3 - \sin 1}{2}$$

For $x > 1$, $\frac{1}{x^{2n}} \to 0$ and $\frac{\log(2+x)}{x^{2n}} \to 0$ as $n \to \infty$, therefore

$$f(x) = \lim_{n \to \infty} \frac{\frac{\log(2+x)}{x^{2n}} - \sin x}{\frac{1}{x^{2n}} + 1} = -\sin x$$

Thus

$$f(x) = \begin{cases} \log(2 + x), & 0 \leq x < 1 \\ \frac{\log 3 - \sin 1)}{2}, & x = 1 \\ -\sin x, & 1 < x \leq \frac{\pi}{2} \end{cases}$$

Clearly $f(0) = \log 2 > 0, f(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1 < 0$, and $f(x) \neq 0$ for any $x \in [0, \frac{\pi}{2}]$. The reason why this can happen is that $f(x)$ is not continuous in the interval $[0, \frac{\pi}{2}]$, so the intermediate value theorem does not apply. The function is discontinuous at $x = 1$ as $\lim_{x \to 1^+} f(x) = -\sin 1, \lim_{x \to 1^-} f(x) = \log 3$ and both of these are different from $f(1)$. ∎

Question 1(b) *Prove that there is a point $a \in (-h, h)$ such that*

$$\int_{-h}^{h} f(x)\,dx = h[f(h) + f(-h)] - \frac{2h^3}{3} f''(a)$$

Solution. Integrating by parts, we get

$$\int_{-h}^{h} f(x)\,dx = xf(x)\Big]_{-h}^{h} - \int_{-h}^{h} xf'(x)\,dx = h[f(h) + f(-h)] - \int_{-h}^{h} xf'(x)\,dx$$

Now put $x = -t$ in $\int_{-h}^{0} xf'(x)\,dx$ to get

$$\int_{-h}^{0} xf'(x)\,dx = \int_{h}^{0} (-t)f'(-t)(-dt) = -\int_{0}^{h} tf'(-t)\,dt$$

Thus

$$\int_{-h}^{h} f(x)\,dx = h[f(h) + f(-h)] - \int_{0}^{h} x\big(f'(x) - f'(-x)\big)\,dx$$

We now note that $\lim_{x\to 0} \dfrac{f'(x) - f'(-x)}{x} = \lim_{x\to 0} \dfrac{f''(x) + f''(-x)}{1} = 2f''(0)$, showing that $\dfrac{f'(x) - f'(-x)}{x}$ is Riemann integrable in $[0, h]$. Since x^2 keeps the same sign, the first mean value theorem of integral calculus gives us

$$\int_{0}^{h} x^2 \frac{f'(x) - f'(-x)}{x}\,dx = \frac{f'(\xi) - f'(-\xi)}{\xi} \int_{0}^{h} x^2\,dx$$

for some $\xi \in (0, h)$. Consequently,

$$\int_{-h}^{h} f(x)\,dx = h[f(h) + f(-h)] - \frac{f'(\xi) - f'(-\xi)}{2\xi}\frac{2h^3}{3}$$

We now apply Lagrange's mean value theorem to the function $f'(x)$ in $[-\xi, \xi]$ and get

$$\frac{f'(\xi) - f'(-\xi)}{2\xi} = f''(a) \text{ for some } a \in (-\xi, \xi)$$

Thus

$$\int_{-h}^{h} f(x)\,dx = h[f(h) + f(-h)] - \frac{2h^3}{3} f''(a)$$

Note that $0 < \xi < h \Rightarrow a \in [-h, h]$.

Corollary: If $f'''(x)$ exists in $(-h, h)$ prove that

$$\frac{f(h) - f(-h)}{2} = h\frac{f'(h) + f'(-h)}{2} - \frac{h^3}{3} f'''(a)$$

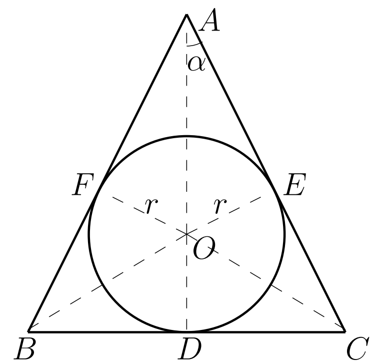
for some $a \in (-h, h)$.

Proof: Use the above result for $f'(x)$ instead of $f(x)$. The left hand side now becomes $f(h) - f(-h)$, and dividing by two, we get the above equation. ∎

Question 1(c) *Find the least perimeter of an isoceles triangle in which a circle of radius r can be inscribed.*

Solution.

Let α be the semi-vertical angle. D is the midpoint of BC. E is the point of contact of AC and the circle or radius r inscribed in ABC. Therefore $\dfrac{OE}{AE} = \tan\alpha$, or $AE = OE\cot\alpha = r\cot\alpha$. Similarly, we find out that $OA = r\csc\alpha$, and $AD = AO + OD = r(1+\csc\alpha)$, consequently $BD = AD\tan\alpha = r(1+\csc\alpha)\tan\alpha$. Since $\triangle ABC$ is isoceles, $BD = DC, DC = CE$ and $AE = AF$.



The perimeter

$$
\begin{aligned}
P &= BD + DC + CE + AE + BF + AF \\
&= 4BD + 2AE = 4r(1+\csc\alpha)\tan\alpha + 2r\cot\alpha \\
&= 4r\tan\alpha + 4r\sec\alpha + 2r\cot\alpha \\
\frac{dP}{d\alpha} &= 4r\sec^2\alpha + 4r\sec\alpha\tan\alpha - 2r\csc^2\alpha = 0 \\
\Rightarrow 0 &= \frac{2}{\cos^2\alpha} + \frac{2\sin\alpha}{\cos^2\alpha} - \frac{1}{sin^2\alpha} \\
\Rightarrow 0 &= 2\sin^2\alpha + 2\sin^3\alpha - \cos^2\alpha \\
\Rightarrow 0 &= 2\sin^3\alpha + 3\sin^2\alpha - 1 \\
\Rightarrow 0 &= (\sin\alpha + 1)^2(2\sin\alpha - 1)
\end{aligned}
$$

Thus $\frac{dP}{d\alpha} = 0 \Rightarrow \sin\alpha = \frac{1}{2}$, since $\sin\alpha = -1$ is not possible as 2α is the angle of a triangle. Thus $\alpha = \frac{\pi}{6}$.

$$\frac{d^2P}{d\alpha^2} = 8r\sec^2\alpha\tan\alpha + 4r(\sec\alpha\tan^2\alpha + \sec^3\alpha) - 4r\csc\alpha(-\csc\alpha\cot\alpha) > 0$$

for $\alpha = \frac{\pi}{6}$. Therefore we have a minimum when $\alpha = \frac{\pi}{6}$. The required perimeter is $4r\frac{1}{\sqrt{3}} + 4r\frac{2}{\sqrt{3}} + 2r\sqrt{3} = 6r\sqrt{3}$. ∎

Question 2(a) *If $f(0) = 0$ and $f'(x) = \frac{1}{1+x^2}$, prove without using the method of integration that*

$$f(x) + f(y) = f\Big(\frac{x+y}{1+xy}\Big)$$

Solution. Let $u = f(x) + f(y), v = \frac{x+y}{1+xy}$. Then

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2} \\
\frac{\partial v}{\partial y} &= \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2} \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix} = 0
\end{aligned}$$

Therefore u, v are functionally dependent. Let $u = \phi(v) \Rightarrow f(x) + f(y) = \phi\Big(\dfrac{x+y}{1-xy}\Big)$. Substituting $y = 0$ gives us $f(x) + f(0) = \phi(x)$ or $\phi(x) = f(x)$ as $f(0) = 0$. Hence $f(x) + f(y) = f\Big(\dfrac{x+y}{1-xy}\Big)$, which was to be proved. ∎

Question 2(b) *Show that under the transformation $u = x^2 - y^2, v = 2xy$ the equation*

$$y^2\frac{\partial^2 H}{\partial x^2} - x^2\frac{\partial^2 H}{\partial y^2} = x\frac{\partial H}{\partial x} - y\frac{\partial H}{\partial y}$$

becomes $\left(u\dfrac{\partial}{\partial v} - v\dfrac{\partial}{\partial u}\right)\dfrac{\partial H}{\partial v} = 0.$

Solution.

$$\begin{aligned}
\frac{\partial H}{\partial x} &= \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x} = 2x\frac{\partial H}{\partial u} + 2y\frac{\partial H}{\partial v} \\
\frac{\partial^2 H}{\partial x^2} &= 2\frac{\partial H}{\partial u} + 2x\left[\frac{\partial^2 H}{\partial u^2}2x + \frac{\partial^2 H}{\partial u\,\partial v}2y\right] + 2y\left[\frac{\partial^2 H}{\partial v\,\partial u}2x + \frac{\partial^2 H}{\partial v^2}(2y)\right] \\
&= 2\frac{\partial H}{\partial u} + 4x^2\frac{\partial^2 H}{\partial u^2} + 8xy\frac{\partial^2 H}{\partial u\,\partial v} + 4y^2\frac{\partial^2 H}{\partial v^2} \qquad (1) \\
\frac{\partial H}{\partial y} &= \frac{\partial H}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial y} = -2y\frac{\partial H}{\partial u} + 2x\frac{\partial H}{\partial v} \\
\frac{\partial^2 H}{\partial y^2} &= -2\frac{\partial H}{\partial u} - 2y\left[\frac{\partial^2 H}{\partial u^2}(-2y) + \frac{\partial^2 H}{\partial u\,\partial v}2x\right] + 2x\left[\frac{\partial^2 H}{\partial v\,\partial u}(-2y) + \frac{\partial^2 H}{\partial v^2}(2x)\right] \\
&= -2\frac{\partial H}{\partial u} + 4y^2\frac{\partial^2 H}{\partial u^2} - 8xy\frac{\partial^2 H}{\partial u\,\partial v} + 4x^2\frac{\partial^2 H}{\partial v^2} \qquad (2) \\
x\frac{\partial H}{\partial x} - y\frac{\partial H}{\partial y} &= 2x^2\frac{\partial H}{\partial u} + 2xy\frac{\partial H}{\partial v} + 2y^2\frac{\partial H}{\partial u} - 2xy\frac{\partial H}{\partial v} \\
&= 2(x^2+y^2)\frac{\partial H}{\partial u} \qquad (3)
\end{aligned}$$

Using (1), (2), (3) we get

$$
\begin{aligned}
0 &= y^2\frac{\partial^2 H}{\partial x^2} - x^2\frac{\partial^2 H}{\partial y^2} - \left(x\frac{\partial H}{\partial x} - y\frac{\partial H}{\partial y}\right)\\
&= 2y^2\frac{\partial H}{\partial u} + 4x^2y^2\frac{\partial^2 H}{\partial u^2} + 8xy^3\frac{\partial^2 H}{\partial u\,\partial v} + 4y^4\frac{\partial^2 H}{\partial v^2}\\
&\quad +2x^2\frac{\partial H}{\partial u} - 4x^2y^2\frac{\partial^2 H}{\partial u^2} + 8x^3y\frac{\partial^2 H}{\partial u\,\partial v} - 4x^4\frac{\partial^2 H}{\partial v^2} - 2(x^2+y^2)\frac{\partial H}{\partial u}\\
&= 8xy(x^2+y^2)\frac{\partial^2 H}{\partial u\,\partial v} + 4(y^4-x^4)\frac{\partial^2 H}{\partial v^2}\\
\Rightarrow 0 &= (y^2-x^2)\frac{\partial^2 H}{\partial v^2} + 2xy\frac{\partial^2 H}{\partial u\,\partial v}\\
\Rightarrow 0 &= -u\frac{\partial^2 H}{\partial v^2} + v\frac{\partial^2 H}{\partial u\,\partial v}\\
\Rightarrow 0 &= \left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right)\frac{\partial H}{\partial v}
\end{aligned}
$$

as required. ∎

Question 2(c) *Prove that the volume of the solid generated by the revolution of the tractrix*

$$x = a\cos t + a\log\tan\frac{t}{2}, \quad y = a\sin t$$

about the asymptote is equal to half the volume of a sphere of radius a.

Solution. The x-axis is the asymptote, as $t \to 0 \Rightarrow x \to \infty, y \to 0$. The volume required is twice the volume in the first quadrant.

$$
\begin{aligned}
V &= 2\int_0^{\frac{\pi}{2}} \pi y^2\frac{dx}{dt}\,dt\\
&= 2\pi\int_0^{\frac{\pi}{2}} a^2\sin^2 t(-a\sin t + \frac{a}{\sin t})\,dt\\
&= 2\pi a^3\int_0^{\frac{\pi}{2}} (\sin t - \sin^3 t)\,dt\\
&= 2\pi a^3\int_0^{\frac{\pi}{2}} \sin t\cos^2 t\,dt\\
&= 2\pi a^3\left[-\frac{\cos^3 t}{3}\right]_0^{\frac{\pi}{2}} = \frac{2\pi a^3}{3}\\
&= \frac{1}{2}\frac{4\pi a^3}{3} = \frac{1}{2}\times \text{ Volume of a sphere of radius } a
\end{aligned}
$$

∎

Paper II

Question 3(a) *Obtain a set of sufficient conditions such that for a function $f(x, y)$*

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$$

Solution. Euler's Theorem: If the partial derivatives f_{xy} and f_{yx} are continuous at (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$.

Proof: Let $\Psi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$ where $(a+h, b+k), (a+h, b), (a, b+k)$ all belong to a neighborhood N of (a, b) — we can take for N an open disc with center (a, b) in which f_{xy} and f_{yx} exist.

Let $G(x) = f(x, b+k) - f(x, b)$ for $x \in I_h$ where $I_h = [a, a+h]$ or $[a+h, a]$ according as $h > 0$ or $h < 0$. Clearly $G'(x) = f_x(x, b+k) - f_x(x, b)$ for $x \in I_h$. We apply Lagrange's Mean Value Theorem to $G(x)$ and obtain:

$$G(a+h) - G(a) = \Psi(h, k) = hG'(a+\theta h) = h[f_x(a+\theta h, b+k) - f_x(a+\theta h, b)] \qquad (*)$$

where $0 < \theta < 1$.

Now we consider $F(t) = f_x(a+\theta h, t)$ for $t \in I_k$ where $I_k = [b, b+k]$ or $[b+k, b]$ according as $k > 0$ or $k < 0$. We apply Lagrange's Mean Value Theorem to $F(t)$ and obtain:

$$F(b+k) - F(b) = f_x(a+\theta h, b+k) - f_x(a+\theta h, b) = kF'(b+\theta_1 k)$$

where $0 < \theta_1 < 1$. But $F'(t) = \frac{\partial}{\partial y} f_x(a+\theta h, t)$, so

$$F(b+k) - F(b) = k\frac{\partial^2 f}{\partial y\,\partial x}(a+\theta h, b+\theta_1 k)$$

Using (*) we get

$$\Psi(h, k) = hk\frac{\partial^2 f}{\partial y\,\partial x}(a+\theta h, b+\theta_1 k)$$

Since f_{yx} is continuous at (a, b), we get

$$\lim_{h\to 0, k\to 0} \frac{\Psi(h, k)}{hk} = \lim_{h\to 0, k\to 0} \frac{\partial^2 f}{\partial y\,\partial x}(a+\theta h, b+\theta_1 k) = \frac{\partial^2 f}{\partial y\,\partial x}(a, b)$$

Now instead of $G(x)$, we start with $H(y) = f(a+h, y) - f(a, y)$ for $y \in I_k$, and proceeding exactly as above, we get

$$\lim_{h\to 0, k\to 0} \frac{\Psi(h, k)}{hk} = \frac{\partial^2 f}{\partial x\,\partial y}(a, b)$$

Hence $\dfrac{\partial^2 f}{\partial x\,\partial y}(a, b) = \dfrac{\partial^2 f}{\partial y\,\partial x}(a, b)$, which completes the proof. ∎

Question 3(b) *Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions $x + y + z = 1, xyz + 1 = 0$.*

Solution. Let $F(x, y, z) = x^2 + y^2 + z^2 + \lambda_1(x + y + z - 1) + \lambda_2(xyz + 1)$ where λ_1, λ_2 are Lagrange's undetermined multipliers. The extreme values are given by

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2x + \lambda_1 + \lambda_2 yz = 0\\ \frac{\partial F}{\partial y} &= 2y + \lambda_1 + \lambda_2 xz = 0\\ \frac{\partial F}{\partial z} &= 2z + \lambda_1 + \lambda_2 xy = 0\end{aligned}$$

Subtracting the first two, $2(x - y) + \lambda_2 z(y - x) = 0 \Rightarrow x = y$ or $\lambda_2 = \frac{2}{z}$ (Note that $x \neq 0, y \neq 0, z \neq 0$ because $xyz + 1 = 0$). Similarly from the other pairs of equations, we get $y = z$ or $\lambda_2 = \frac{2}{x}$, and $x = z$ or $\lambda_2 = \frac{2}{y}$.

Since $xyz = -1, x+y+z = 1$, it follows that x, y, z cannot be all positive or all negative, moreover two must be positive and one negative. In particular, $x = y = z$ is not possible. Suppose $x \neq y$, then $\lambda_2 = \frac{2}{z}$. Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda_1 + 2y = 0$.

Substituting the values of λ_1, λ_2 in $\frac{\partial F}{\partial z} = 0$, we get

$$2z - 2(x + y) + \frac{2xy}{z} = 0$$

But $x + y = 1 - z, xy = -\frac{1}{z}$, so we get $2z - 2(1 - z) - \frac{2}{z^2} = 0 \Rightarrow 2z^3 - z^2 - 1 = 0 \Rightarrow (z - 1)(2z^2 + z + 1) = 0$. But $2z^2 + z + 1$ has no real roots, so the only real root is $z = 1$.

Thus we get $\lambda_2 = 2, z = 1, \lambda_1 = 0(\because x + y + z = 1 \Rightarrow x + y = 0 \Rightarrow \lambda_1 = 0), x = \pm 1, y = \mp 1(\because x + y = 0, xy = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1, y = \mp 1)$. Hence the stationary values are $(1, -1, 1), (-1, 1, 1)$ and $f(x, y, z) = 3$ at these points.

On taking $y \neq z$ we shall get $(1, 1, -1), (1, -1, 1)$ as stationary points (by symmetry), and for $x \neq z$, we get $(1, 1, -1), (-1, 1, 1)$. $f(x, y, z)$ is always equal to 3, hence we cannot say whether it is a maximum or minimum without checking d^2F.

Considering the point $(-1, 1, 1)$,

$$d^2F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 + 4\,dx\,dy + 4\,dx\,dz - 4\,dy\,dz$$

Now $x+y+z = -1 \Rightarrow dx+dy+dz = 0 \Rightarrow dz = -dx-dy$. $xyz = -1 \Rightarrow yz\,dx+zx\,dy+xy\,dz = 0 \Rightarrow dx - dy - dz = 0 \Rightarrow dx = 0, dz = -dy$.

Thus $d^2F = 8(dz)^2 > 0$, so f has a minimum at all these stationary points.

Note: The question can be treated as that of one variable as y, z can be eliminated, but the calculation becomes quite messy. ∎

UPSC Civil Services Main 1984 - Mathematics
Calculus

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Question 1(a) *1. Show that the function*

$$f(x) = \begin{cases} x^2 \sin^2 \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous at 0.

2. $\tan x$ *is not continuous at* $x = \frac{\pi}{2}$

Solution.

1. Given $\epsilon > 0$, let $\delta = \sqrt{\epsilon}$, then $|x| \leq \delta \Longrightarrow |x^2 \sin^2 \frac{1}{x}| \leq |x^2| < \epsilon$, because $|\sin^2 \frac{1}{x}| \leq 1$. Thus $|x - 0| < \delta = \sqrt{\epsilon} \Rightarrow |f(x) - f(0)| < \epsilon$, showing that $f(x)$ is continuous at $x = 0$.

2. Since $\lim_{x \to \frac{\pi}{2}} \sin x = 1$, given $\epsilon = \frac{1}{2}$ there exists $\delta_1 > 0$ such that $0 < |x - \frac{\pi}{2}| < \delta_1 \Rightarrow |\sin x - 1| < \frac{1}{2} \Rightarrow \sin x > \frac{1}{2}$.

   Since $\lim_{x \to \frac{\pi}{2}} \cos x = 0$, given any real number $G > 0$ there exists $\delta_2 > 0$ such that $0 < |x - \frac{\pi}{2}| < \delta_2 \Rightarrow |\cos x| < \frac{1}{2G}$.

   Let $\delta = \min(\delta_1, \delta_2)$. Then $\delta > 0$ and $0 < |x - \frac{\pi}{2}| < \delta \Rightarrow |\tan x| > \frac{1}{2} \cdot 2G = G$. This shows that $\tan x$ is not bounded in any neighborhood of $\frac{\pi}{2}$, therefore $\lim_{x \to \frac{\pi}{2}} \tan x$ does not exist, so $\tan x$ is not continuous at $x = \frac{\pi}{2}$.

Note that if $\lim_{x \to a} f(x) = l$, then $f(x)$ is bounded in a neighborhood of a, because given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$. ∎

Question 1(b) *Find the volume of the torus generated by revolving a disc of radius r about a line at a distance $a > r$ from the center of the circle.*

Solution. Let the line be the x-axis, and let the circle have center $(0, a)$. The circle's equation is $x^2 + (y - a)^2 = r^2$. The upper semicircle is given by $f_1(x) = a + \sqrt{r^2 - x^2}$, and the lower one by $f_2(x) = a - \sqrt{r^2 - x^2}$, and the desired volume is given by

$$\begin{aligned}
V &= \pi \int_{-r}^{r} (f_1^2(x) - f_2^2(x))\, dx \\
&= \pi \int_{-r}^{r} (f_1(x) - f_2(x))(f_1(x) + f_2(x))\, dx \\
&= \pi \int_{-r}^{r} (2\sqrt{r^2 - x^2})(2a)\, dx \\
&= 8a\pi \int_{0}^{r} \sqrt{r^2 - x^2}\, dx \\
&\quad \text{Put } x = r\sin\theta \\
&= 8a\pi \int_{0}^{\frac{\pi}{2}} r\cos\theta \cdot r\cos\theta\, d\theta \\
&= 8ar^2\pi \int_{0}^{\frac{\pi}{2}} \cos^2\theta\, d\theta \\
&= 8ar^2\pi \Big(\frac{1}{2}\frac{\pi}{2}\Big) = 2\pi^2 ar^2
\end{aligned}$$

We could get the same result by applying Pappus' Theorem — the volume of a solid of revolution generated by rotating a plane figure about an external axis is equal to the product of the area of the figure and the distance traveled by its geometric centroid during revolution. Thus $V = \pi r^2 \cdot 2a\pi = 2\pi^2 ar^2$. ∎

Question 1(c) *Let $f(x, y) = (x + y)\sin(\frac{1}{x} + \frac{1}{y})$, when $x \neq 0, y \neq 0$, and $f(x, 0) = f(0, y) = 0$. Examine whether (i) $f(x, y)$ is continuous, and (ii) $\lim_{x\to 0} f(x, y)$ for $y \neq 0$ and $\lim_{y\to 0} f(x, y)$ for $x \neq 0$ exist.*

Solution. (i) Given $\epsilon > 0$, let $\delta_1 = \delta_2 = \frac{\epsilon}{2}$. Then $|x| < \delta_1, |y| < \delta_2 \Rightarrow |f(x, y) - f(0, 0)| = |(x + y)\sin(\frac{1}{x} + \frac{1}{y})| \leq |x| + |y| < \epsilon$, as $|\sin(\frac{1}{x} + \frac{1}{y})| \leq 1$. Thus $f(x, y)$ is continuous at $(0, 0)$.

(ii) Since $|x\sin(\frac{1}{x} + \frac{1}{y})| \leq |x|$, it follows that $\lim_{x\to 0} x\sin(\frac{1}{x} + \frac{1}{y}) = 0$. Thus if $\lim_{x\to 0} f(x, y)$ exists for $y \neq 0$, then $\lim_{x\to 0} y\sin(\frac{1}{x} + \frac{1}{y})$ should also exist, for $y \neq 0$, because $y\sin(\frac{1}{x} + \frac{1}{y}) = (x + y)\sin(\frac{1}{x} + \frac{1}{y}) - x\sin(\frac{1}{x} + \frac{1}{y})$.

Let $g(x) = y\sin(\frac{1}{x} + \frac{1}{y})$. Suppose $\lim_{x\to 0} g(x) = l$ for $y = \frac{2}{\pi}$. Then given $\epsilon > 0, \epsilon < \frac{2}{\pi}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |g(x) - l| < \frac{\epsilon}{2}$. Let $x_1 = \frac{1}{2n\pi}, x_2 = \frac{1}{(2n+1)\pi}$, n large so that $|x_1| < \delta, |x_2| < \delta$. Now

$$\begin{aligned}
|g(x_1) - g(x_2)| &= |g(x_1) - l + l - g(x_2)| \\
&\leq |g(x_1) - l| + |g(x_2) - l| < \epsilon \\
g(x_1) &= \frac{2}{\pi}\sin(2n\pi + \frac{\pi}{2}) = \frac{2}{\pi} \\
g(x_2) &= \frac{2}{\pi}\sin((2n+1)\pi + \frac{\pi}{2}) = -\frac{2}{\pi} \\
\Rightarrow |g(x_1) - g(x_2)| &= \frac{4}{\pi} > \epsilon
\end{aligned}$$

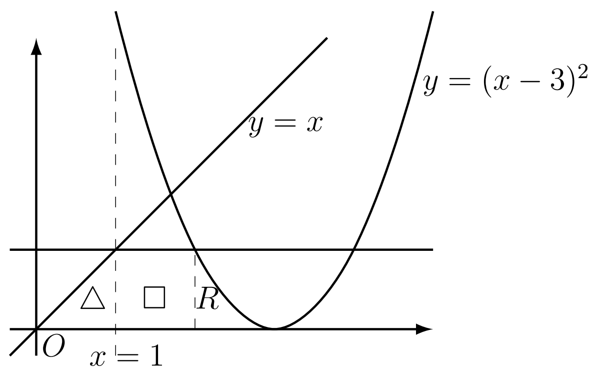
Thus we have a contradiction. Hence $\lim_{x\to 0} y\sin(\frac{1}{x} + \frac{1}{y})$ does not exist for $y \neq 0$, so $\lim_{x\to 0} f(x,y)$ for $y \neq 0$ does not exist. Similarly it can be seen that $\lim_{y\to 0} f(x,y)$ for $x \neq 0$ does not exist, by symmetry. ∎

Question 2(a) *Evaluate* $\iint xy\,dx\,dy$ *over the area given by the boundary* $y = 0 (0 \leq x \leq 3); y = (x-3)^2 (2 \leq x \leq 3); y = 1 (1 \leq x \leq 2); y = x (0 \leq x \leq 1)$.

Solution.

The region of integration consists of three parts

1. The triangle $\triangle$ bounded by $y = 0, x = 1, y = x$.
2. The square $\square$ bounded by $x = 1, x = 2, y = 0, y = 1$.
3. The region R bounded by $x = 2, y = 0, y = (x-3)^2,\ (2 \leq x \leq 3)$.



$$\begin{aligned}
I_1 &= \iint_\triangle xy\,dx\,dy = \int_{x=0}^{1}\int_{y=0}^{x} xy\,dy\,dx = \int_0^1 x\frac{y^2}{2}\Big]_0^x dx = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{8} \\
I_2 &= \iint_\square xy\,dx\,dy = \int_{x=1}^{2}\int_{y=0}^{1} xy\,dy\,dx = \frac{x^2}{2}\Big]_1^2 \cdot \frac{y^2}{2}\Big]_0^1 = \frac{3}{2}\cdot\frac{1}{2} = \frac{3}{4} \\
I_3 &= \iint_R xy\,dx\,dy = \int_{x=2}^{3}\int_0^{(x-3)^2} yx\,dy\,dy = \int_{x=2}^{3} x\frac{y^2}{2}\Big]_0^{(x-3)^2} dx \\
&= \frac{1}{2}\int_2^3 x(x-3)^4\,dx = \frac{1}{2}\int_{-1}^{0} u^4(u+3)\,du = \frac{1}{2}\Big[\frac{u^6}{6} + 3\frac{u^5}{5}\Big]_{-1}^{0} = \frac{1}{2}\Big(\frac{3}{5} - \frac{1}{6}\Big) = \frac{13}{60}
\end{aligned}$$

Thus the given integral $= I_1 + I_2 + I_3 = \dfrac{1}{8} + \dfrac{3}{4} + \dfrac{13}{60} = \dfrac{131}{120}$ ∎

Question 2(b) *If $B(p,q)$ is the Beta function, show that*

$$pB(p,q) = (q-1)B(p+1,q-1)$$

where p, q are real, $p > 0, q > 1$. Hence or otherwise find $B(p,n)$ where n is an integer > 0.

Solution. By definition, $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$. Integrating by parts, we get

$$B(p,q) = \frac{x^p}{p}(1-x)^{q-1}\Big]_0^1 + \int_0^1 \frac{x^p}{p}(q-1)(1-x)^{q-2}\,dx$$

Since $p > 0, (q-1) > 0$, we get

$$pB(p,q) = (q-1)\int_0^1 x^{p+1-1}(1-x)^{(q-1)-1}\,dx = (q-1)B(p+1,q-1)$$

In particular, $pB(p,n) = (n-1)B(p+1,n-1)$. Repeating this formula, we get

$$\begin{aligned}
B(p,n) &= \frac{n-1}{p}B(p+1,n-1) \\
&= \frac{(n-1)(n-2)}{p(p+1)}B(p+2,n-2) \\
&= \frac{(n-1)!}{p(p+1)\ldots(p+n-1)}B(p+n-1,1) \\
&= \frac{(n-1)!}{p(p+1)\ldots(p+n-2)}\int_0^1 x^{p+n-2}(1-x)^{1-1}\,dx \\
&= \frac{(n-1)!}{p(p+1)\ldots(p+n-1)} = \frac{\Gamma(p)\Gamma(n)}{\Gamma(p+n)}
\end{aligned}$$

∎

Question 2(c) *If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$, find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Are u and v functionally related? If so, find the relationship.*

Solution.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{(1-xy)\cdot 1 - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2} \\
\frac{\partial u}{\partial y} &= \frac{(1-xy)\cdot 1 - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2} \\
\frac{\partial v}{\partial x} &= \frac{1}{1+x^2} \\
\frac{\partial v}{\partial y} &= \frac{1}{1+y^2} \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0
\end{aligned}$$

This shows that u, v are functionally related. Let $x = \tan\theta, y = \tan\phi$. Then $v = \theta + \phi$. $u = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \tan(\theta + \phi) = \tan v$.

Alternatively, using$v = \tan^{-1} x + \tan^{-1} y$ we write

$$
\begin{aligned}
x &= \tan(v - \tan^{-1} y) = \frac{\tan v - \tan(\tan^{-1} y)}{1 + \tan v \tan(\tan^{-1} y)} = \frac{\tan v - y}{1 + y \tan v} \\
u &= \frac{\frac{\tan v - y}{1 + y\tan v} + y}{1 - y\frac{\tan v - y}{1 + y \tan v}} = \frac{(\tan v)(1 + y^2)}{1 + y^2} = \tan v
\end{aligned}
$$

as before. ∎

Paper II

Question 3(a) *Show that the maximum and minimum values of the function* $u = x^2 + y^2 + xy$ *where* $ax^2 + by^2 = ab, a > b > 0$ *are given by* $4(u - a)(u - b) = ab$.

Solution. Let $F(x, y) = x^2 + y^2 + xy + \lambda(ax^2 + by^2 - ab)$ where λ is Lagrange's undetermined multiplier. The extreme values are obtained from

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= 2x + y + 2\lambda ax = 0, \frac{\partial F}{\partial y} = 2y + x + 2\lambda by = 0 \\
0 &= x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 2(x^2 + y^2 + xy) + 2\lambda(ax^2 + by^2) = 2u + 2\lambda ab
\end{aligned}
$$

Thus $\lambda = -\frac{u}{ab}$. Consequently $2x + y - \frac{2ux}{b} = 0 \Rightarrow 2x(b - u) + yb = 0$, and $2y + x - \frac{2uy}{a} = 0 \Rightarrow 2y(a - u) + ax = 0$.

Since $ax^2 + by^2 = ab > 0$, $(x, y) \neq (0, 0)$, so the coefficient matrix of the above linear equations must be singular i.e.$\begin{vmatrix} 2(b-u) & b \\ a & 2(a-u) \end{vmatrix} = 0$ or $4(a - u)(b - u) - ab = 0$.

Thus the maximum and minimum values are given by $4(a - u)(b - u) - ab = 0$.

Note: We can substitute $x = \sqrt{b}\cos t, y = \sqrt{a}\sin t$ to make u a function of one variable, and proceed accordingly. ∎

Question 3(b) *Discuss the continuity and differentiability of the function*

$$
f(x, y) = \begin{cases} x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}, & x \neq 0, y \neq 0 \\ 0, & x = y = 0 \end{cases}
$$

Also examine if f_{xy} *and* f_{yx} *are equal at* $(0, 0)$.

Solution.

1. The function is continuous at $(0, 0)$, because $|f(x, y) - f(0, 0)| = 0$ or $\leq \frac{\pi}{2}(x^2 + y^2)$.

2. $f_{xy}(0,0) \neq f_{yx}(0,0)$. For this we calculate the following:

   (a) If $(x,y) \neq (0,0)$,

$$\begin{aligned} f_x(x,y) &= 2x\tan^{-1}\frac{y}{x} + x^2\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - y^2\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y} \\ &= 2x\tan^{-1}\frac{y}{x} - \frac{x^2y}{x^2+y^2} - \frac{y^3}{x^2+y^2} = 2x\tan^{-1}\frac{y}{x} - y \end{aligned}$$

   (b)

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0}\frac{h^2\tan^{-1}\frac{k}{h} - k^2\tan^{-1}\frac{h}{k}}{h}$$

   Now $\lim_{h\to 0} h\tan^{-1}\frac{k}{h} = 0, \lim_{h\to 0}\frac{\tan^{-1}\frac{h}{k}}{h} = \frac{1}{k}$ — if $\theta = \tan^{-1}\frac{h}{k}, \frac{h}{k} = \tan\theta$, then $\frac{\tan^{-1}\frac{h}{k}}{h} = \frac{\theta}{k\tan\theta}$, and $\frac{\theta}{\tan\theta} \to 1$ as $\theta \to 0$.
   Thus $f_x(0,k) = -k$, in particular $f_x(0,0) = 0$.

   (c) By symmetry, $f_y(x,y) = x - 2y\tan^{-1}\frac{x}{y}$ is $x \neq 0, y \neq 0$, and $f_y(h,0) = h$.

$$\begin{aligned} f_{yx}(0,0) &= \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \lim_{k\to 0}\frac{-k-0}{k} = -1 \\ f_{xy}(0,0) &= \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0}\frac{h-0}{h} = 1 \end{aligned}$$

   Thus $f_{xy}(0,0) \neq f_{yx}(0,0)$.

3. The function is differentiable at $(0,0)$ as both $f_x(x,y)$ and $f_y(x,y)$ are continuous at $(0,0)$. Note that $\lim_{x\to 0, y\to 0} f_x(x,y) = \lim_{x\to 0, y\to 0} f_y(x,y) = 0$ and $f_x(0,0) = 0 = f_y(0,0)$.

∎

UPSC Civil Services Main 1985 - Mathematics Calculus

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January 16, 2010

Question 1(a) *If*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show that both the partial derivatives f_x, f_y *exist at* $(0,0)$*, but the function is not continuous there.*

Solution. See 2004, question 2(d). ∎

Question 1(b) *If for all values of the parameter* λ *and some constant* n*,* $F(\lambda x, \lambda y) = \lambda^n F(x,y)$ *identically, where* F *is assumed to be differentiable, prove that* $x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y} = nF(x,y)$*.*

Solution. See 1996, question 2(a). ∎

Question 1(c) *Prove the relation between the beta and gamma functions*

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Solution. See 1991, question 2(c). ∎

Question 2(a) *If a function f defined on $[a,b]$ is continuous on $[a,b]$ and differentiable on (a,b), and $f(a)=f(b)$, then prove that there exists at least one real number $c, a<c<b$ such that $f'(c)=0$.*

Solution. This is Rolle's theorem. Since f is continuous on $[a,b]$, it is bounded and attains its maximum and minimum. Since $f(a)=f(b)$ and we assume that f is not a constant function, it follows that there exists a real number c such that $f(c)=M$, maximum of f (without loss of generality) and $a<c<b$. Then the right hand derivative of f at c is $\lim_{h\to 0^+}\dfrac{f(c+h)-f(c)}{h}$ and clearly it is ≤ 0, because $f(c+h)-f(c)\leq 0$ and $h>0$. However the left hand derivative of f at c is $\lim_{h\to 0^-}\dfrac{f(c+h)-f(c)}{h}\geq 0$ because $f(c+h)-f(c)\leq 0$ and $h<0$. Since $f'(c)$ exists, the right hand derivative of f at c equals the left hand derivative of f at c, which is possible only when both are zero, i.e. $f'(c)=0$. ∎

Question 2(b) *Use Maclaurin's expansion to show that*

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$$

Hence find the value of $\log(1+x+x^2+x^3+x^4)$.

Solution. Since $\log(1+x)$ possesses continuous derivatives of all orders for every value of x for which $1+x>0$ i.e. $x>-1$, we have

$$f(x)=\log(1+x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n$$

where

$$R_n=\frac{x^n}{n!}f^{(n)}(\theta x)=\frac{x^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n},\quad 0<\theta<1$$

as $f^{(n)}(x)$ for $f(x)=\log(1+x)$ is $\dfrac{(-1)^{n-1}(n-1)!}{(x+1)^n}$.

Let $0<x\leq 1$, so that $0<\dfrac{x}{1+\theta x}<1$, then $|R_n|<\frac{1}{n}$ and therefore $R_n\to 0$ as $n\to\infty$.

Let $-1<x\leq 0$, we consider the expansion

$$f(x)=\log(1+x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n$$

where

$$R_n=\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x)=(-1)^n x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\frac{1}{1+\theta x},\quad 0<\theta<1$$

— this is Cauchy's form of the remainder — note that we consider Cauchy's form of the remainder as Lagrange's form of the remainder does not enable us to prove that $R_n \to 0$ as $n \to \infty$. when $-1 < x < 0$.

Since $0 < \theta < 1$ and $|x| < 1$, $0 < \dfrac{1-\theta}{1+\theta x} < 1$. Also, $\dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$. Now since $x^n \to 0$ as $n \to \infty$, we get $R_n \to 0$ as $n \to \infty$.

Now using the fact that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$, we get the required result

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

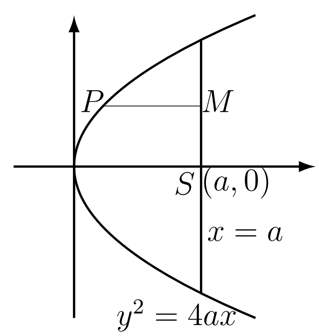
$$\begin{aligned}
\log(1+x+x^2+x^3+x^4) &= \log(\frac{1-x^5}{1-x}) = \log(1-x^5) - \log(1-x) \\
&= -(x^5 + \frac{x^{10}}{2} + \frac{x^{15}}{3} + \ldots) + (x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots) \\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + x^5\Big(\frac{1}{5} - 1\Big) + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{2}\Big(\frac{1}{5} - 1\Big) + \ldots
\end{aligned}$$

which is the required expansion. ∎

Question 2(c) *Find the volume generated by revolving $y^2 = 4ax$ about the latus rectum.*

Solution.

Let P be the point (x, y), M be the foot of the perpendicular on the latus rectum $x = a$. Then $PM = a - x$. $SM = y$, where S is the focus $(a, 0)$. Because of symmetry we can confine ourselves to the first quadrant, $0 \le y \le 2a$.



$$\begin{aligned}
V &= 2\int \pi (PM)^2 d(SM) = 2\pi \int_0^{2a} (a^2 - 2ax + x^2)\, dy \\
&= 2\pi \int_0^{2a} \Big(a^2 - \frac{y^2}{2} + \frac{y^4}{16a^2}\Big)\, dy \\
&= 2\pi \Big[a^2 y - \frac{y^3}{6} + \frac{y^5}{80a^2}\Big]_0^{2a} = 2\pi \Big[2a^3 - \frac{8a^3}{6} + \frac{32a^5}{80a^2}\Big] \\
&= 2\pi a^3 \frac{30 - 20 + 6}{15} = \frac{32\pi a^3}{15}
\end{aligned}$$

∎

Paper II

Question 3(a) *Show that for the function*

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & x=0, y=0 \end{cases}$$

1. *f_x is not differentable at $(0,0)$.*
2. *f_{yx} is not continuous at $(0,0)$.*
3. *$f_{xy}(0,0) = f_{yx}(0,0)$.*

Solution.

1. For $(x,y) \neq (0,0)$,
$$f_x(x,y) = \frac{(x^2+y^2)2xy^2 - x^2y^2(2x)}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2}$$

2. $$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{\frac{h^2y^2}{h^2+y^2} - 0}{h} = \lim_{h\to 0} \frac{hy^2}{h^2+y^2} = 0$$
Thus $f_x(0,y) = 0$ for $y \neq 0$.

3. $$f_x(x,0) = \lim_{h\to 0} \frac{f(x+h,0)-f(x,0)}{h} = 0$$
In particular, $f_x(0,0) = 0$.

4. $f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k} = 0$

5. $f_y(x,0) = \lim_{k\to 0} \frac{f(x,k)-f(x,0)}{k} = \lim_{k\to 0} \frac{x^2k}{x^2+k^2} = 0$ when $x \neq 0$.

6. $f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$.

7. $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = 0$.
Thus we have shown that $f_{xy}(0,0) = f_{yx}(0,0) = 0$.

8. For $(x,y) \neq (0,0)$,
$$\begin{aligned} f_{yx}(x,y) &= \frac{\partial}{\partial y} f_x(x,y) = \frac{\partial}{\partial y} \frac{2xy^4}{(x^2+y^2)^2} \\ &= \frac{8xy^3(x^2+y^2)^2 - 2xy^4 \cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} \\ &= \frac{8xy^3(x^2+y^2) - 8xy^5}{(x^2+y^2)^3} = \frac{8x^3y^3}{(x^2+y^2)^3} = 8\left(\frac{xy}{x^2+y^2}\right)^3 \end{aligned}$$
Now from question 1(a) above, it follows that f_{yx} is not continuous at $(0,0)$.

9. If f_x were differentiable at $(0,0)$, we would have

$$f_x(h,k) = f_x(0,0) + hf_{xx}(0,0) + kf_{yx}(0,0) + \sqrt{h^2+k^2}\phi(h,k)$$

where $\lim_{h\to 0,k\to 0}\phi(h,k) = 0$. Substituting the known values, we get

$$\phi(h,k) = \frac{2hk^4}{(h^2+k^2)^2}\frac{1}{\sqrt{h^2+k^2}}$$

Now we can see that $\lim_{h\to 0,k\to 0}\phi(h,k)$ does not exist — if $k = mh$, $\phi(h,mh) = \frac{2m^4}{(1+m^2)^{\frac{5}{2}}}$, so $\lim_{h\to 0}\phi(h,mh)$ is different for different values of m (say 0, 1).

Thus we have shown all the three results required. ∎

UPSC Civil Services Main 1986 - Mathematics Calculus

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Question 1(a) *1. A function $f(x)$ is defined as follows:*

$$f(x) = \begin{cases} e^{1-\frac{1}{x^2}} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Examine whether or not $f(x)$ is differentiable at $x = 0$.

2. If $f'(x)$ exists and is continuous, find the value of

$$\lim_{x\to 0} \frac{1}{x^2} \int_0^x (x - 3y) f(y)\, dy$$

Solution.

1. By definition, $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x}$, if this limit exists.

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x) - f(0)}{x} &= \lim_{x\to 0} \frac{1}{x} e^{1-\frac{1}{x^2}} \sin \frac{1}{x} \\
&= e \lim_{x\to 0} \frac{1}{x^2} e^{-\frac{1}{x^2}} \lim_{x\to 0} x \sin \frac{1}{x} \\
&= e \lim_{t\to\infty} te^{-t} \cdot 0 \qquad \text{Letting } t = \frac{1}{x^2}, \text{ also } \lim_{x\to 0} x \sin \frac{1}{x} = 0 \\
&= 0 \qquad\qquad \because \lim_{t\to\infty} te^{-t} = 0
\end{aligned}$$

Thus $f(x)$ is differentiable at $x = 0$, and $f'(0) = 0$.

2. Define

$$
\begin{aligned}
F(x) &= \int_0^x (x-3y)f(y)\,dy \\
&= x\int_0^x f(y)\,dy - 3\int_0^x yf(y)\,dy \\
F'(x) &= \int_0^x f(y)\,dy + xf(x) - 3xf(x) = \int_0^x f(y)\,dy - 2xf(x) \\
F''(x) &= f(x) - 2f(x) - 2xf'(x) = -f(x) - 2xf'(x)
\end{aligned}
$$

Note that $F''(x)$ exists because $f'(x)$ exists.

The required limit is $\lim_{x\to 0} \dfrac{F(x)}{x^2}$. Since $\lim_{x\to 0} F(x) = 0, \lim_{x\to 0} x^2 = 0$ and both are differentiable, we can apply L'Hospital's rule to get

$$
\lim_{x\to 0} \frac{F(x)}{x^2} = \lim_{x\to 0} \frac{F'(x)}{2x}
$$

Again L'Hospital's rule applies, so

$$
\lim_{x\to 0} \frac{F(x)}{x^2} = \lim_{x\to 0} \frac{F''(x)}{2} = \lim_{x\to 0} \frac{-f(x) - 2xf'(x)}{2} = -\frac{f(0)}{2}
$$

since $f'(x)$ exists and is continuous, so $\lim_{x\to 0} xf'(x) = 0$.

Thus

$$
\lim_{x\to 0} \frac{1}{x^2} \int_0^x (x-3y)f(y)\,dy = -\frac{f(0)}{2}
$$

∎

Question 1(b) *Use Rolle's theorem to establish that under suitable conditions (to be stated)*

$$
\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}, \quad a < \xi < b
$$

Hence or otherwise deduce the inequality

$$
nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)
$$

where $a > b$ and $n > 1$.

Solution. Let $f(x), g(x)$ be continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b). Let

$$
\begin{aligned}
F(x) &= \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} - \frac{x-a}{b-a} \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} \\
&= [f(a)g(x) - g(a)f(x)] - \frac{x-a}{b-a}[f(a)g(b) - g(a)f(b)]
\end{aligned}
$$

Thus $F(x)$ is

1. continuous in the closed interval $[a, b]$.
2. differentiable in the open interval (a, b)
3. $F(a) = F(b) = 0$.

Thus $F(x)$ satisfies the requirements of Rolle's theorem, consequently, there exists $\xi, a < \xi < b$ such that $F'(\xi) = 0$. But

$$\begin{aligned} F'(\xi) &= [f(a)g'(\xi) - g(a)f'(\xi)] - \frac{1}{b-a}[f(a)g(b) - g(a)f(b)] \\ &= \begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix} - \frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = 0 \end{aligned}$$

therefore

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}, \quad a < \xi < b$$

Let $f(x) = x^n, g(x) = 1$, then the above result implies

$$\begin{vmatrix} b^n & a^n \\ 1 & 1 \end{vmatrix} = (a-b)\begin{vmatrix} b^n & n\xi^{n-1} \\ 1 & 0 \end{vmatrix}, \quad b < \xi < a$$

or $a^n - b^n = (a-b)n\xi^{n-1}$. Now since $b^{n-1} < \xi^{n-1} < a^{n-1}$, we get
$nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$, as required. ∎

Question 1(c) *If* $u = \dfrac{(ax^3 + by^3)^n}{3n(3n-1)} + xf(\frac{y}{x})$, *find the value of* $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Solution. We use the result proved in 2006, question 2(b): If $f(x, y)$ is a homogeneous function of degree n possessing continuous partial derivatives of degree 2,

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Let $v = \dfrac{(ax^3 + by^3)^n}{3n(3n-1)}$, then v is homogeneous of degree $3n$, so

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = 3n(3n-1)\frac{(ax^3 + by^3)^n}{3n(3n-1)} = (ax^3 + by^3)^n$$

Let $w = xf(\frac{y}{x})$, then w is homogeneous of degree 1, so

$$x^2\frac{\partial^2 w}{\partial x^2} + 2xy\frac{\partial^2 w}{\partial x\,\partial y} + y^2\frac{\partial^2 w}{\partial y^2} = 1(1-1)w = 0$$

Now $u = v + w$, so adding the above equations we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (ax^3 + by^3)^n$$

∎

Question 2(a) *1. Without evaluating the involved integrals, show that*

$$\int_1^x \frac{t\,dt}{1+t^2} + \int_1^{\frac{1}{x}} \frac{dt}{t(1+t^2)} = 0$$

2. If $f(x)$ is periodic of period T, show that $\displaystyle\int_a^{a+T} f(t)\,dt$ is independent of a.

Solution.

1. Let $t = \frac{1}{u}$, so that $dt = -\dfrac{du}{u^2}$ and $\dfrac{t}{1+t^2} = \dfrac{1}{u}\dfrac{u^2}{1+u^2} = \dfrac{u}{1+u^2}$, implying that

$$\int_1^x \frac{t\,dt}{1+t^2} = \int_1^{\frac{1}{x}} \frac{u}{1+u^2}\left(-\frac{du}{u^2}\right) = -\int_1^{\frac{1}{x}} \frac{du}{u(1+u^2)}$$

Thus

$$\int_1^x \frac{t\,dt}{1+t^2} + \int_1^{\frac{1}{x}} \frac{dt}{t(1+t^2)} = 0$$

2. Define $F(a) = \displaystyle\int_a^{a+T} f(t)\,dt$. We shall prove that $F(a) = F(b)$ for any b. In particular $F(a) = F(0) = \int_0^T f(t)\,dt$ which is independent of a.

$$\begin{aligned}
F(b) - F(a) &= \int_b^{b+T} f(t)\,dt - \int_a^{a+T} f(t)\,dt \\
&= \int_b^{b+T} f(t)\,dt - \int_b^{a+T} f(t)\,dt - \int_a^{b} f(t)\,dt \\
&= \int_b^{b+T} f(t)\,dt + \int_{a+T}^{b} f(t)\,dt - \int_a^{b} f(t)\,dt \\
&= \int_{a+T}^{b+T} f(t)\,dt - \int_a^{b} f(t)\,dt \\
&= \int_a^{b} f(u+T)\,du - \int_a^{b} f(t)\,dt \qquad \because u+T = t \\
&= \int_a^{b} f(u)\,du - \int_a^{b} f(t)\,dt = 0 \qquad \because f(u+T) = f(u)
\end{aligned}$$

Thus $F(b) = F(a)$ for all b.

∎

Question 2(b) *Find the volume of the solid generated by revolving one arc of $x = a(t - \sin t), y = a(1 - \cos t)$ about its base.*

Solution. One arc is given by $0 \le t \le 2\pi$ and the base is the x-axis. Thus

$$\begin{aligned}
V &= \pi \int_0^{2\pi} y^2 \frac{dx}{dt}\, dt \\
&= \pi \int_0^{2\pi} a^2(1-\cos t)^2 a(1-\cos t)\, dt = \pi a^3 \int_0^{2\pi} \left(2\sin^2 \frac{t}{2}\right)^3 dt \\
&= 8\pi a^3 \int_0^{\pi} 2\sin^6\theta\, d\theta \qquad (\theta = \frac{t}{2}) \\
&= 32\pi a^3 \int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta \qquad (\sin(\pi-\theta) = \sin\theta, \text{so double the integral and half the limit}) \\
&= 32\pi a^3 \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2} = 5\pi^2 a^3
\end{aligned}$$

■

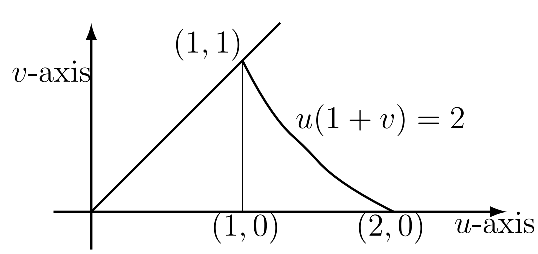
Question 2(c) *Evaluate* $\int_0^2 \int_0^x \left((x-y)^2+2(x+y)+1\right)^{-\frac{1}{2}} dx\, dy$ *by using the transformation* $x = u(1+v), y = v(1+u)$. *Assume u, v are positive in the region concerned.*

Solution.

1. The Jacobian of the transformation is
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = 1+u+v > 0$$
because $u \ge 0, v \ge 0$.

2. The region of integration in the xy-plane is the triangle bounded by the lines $y = 0, y = x, x = 2$.

3. The region of integration in the uv-plane lies in the first quadrant as $u \ge 0, v \ge 0$. Clearly
$$\begin{aligned}
x = y &\iff u = v \\
x = 2 &\iff u(1+v) = 2 \\
y = 0 &\iff v = 0 \quad (\because 1+u > 0)
\end{aligned}$$



The curve $u(1+v) = 2$ meets the u-axis at $(2,0)$ and the line $u = v$ at $(1,1)$. The region of integration is bounded by $v = 0, v = u, u(1+v) = 2$ and therefor consists of the triangle with vertices $(0,0),(1,0),(1,1)$ and the portion bounded by $u = 1, v = 0, u(1+v) = 2$, in which u varies from 1 to 2, and v varies from 0 to $\frac{2-u}{u}$.

4.

$$\begin{aligned}\Big((x-y)^2 + 2(x+y) + 1\Big)^{-\frac{1}{2}} &= \Big((u-v)^2 + 2(2uv + u + v) + 1\Big)^{-\frac{1}{2}} \\ &= (u^2 + v^2 + 2uv + 2u + 2v + 1)^{-\frac{1}{2}} = (1+u+v)^{-1}\end{aligned}$$

Thus the product of the Jacobian and the integrand is 1.

Thus

$$\begin{aligned} I &= \int_0^2 \int_0^x \Big((x-y)^2 + 2(x+y) + 1\Big)^{-\frac{1}{2}} \, dx\, dy \\ &= \int_0^1 \int_0^u dv\, du + \int_1^2 \int_0^{\frac{2}{u}-1} dv\, du \\ &= \frac{1}{2} + \int_1^2 \frac{2}{u}\, du - 1 \\ &= 2\log 2 - \frac{1}{2} \end{aligned}$$

■

Paper II

Question 3(a) *Find the maximum and minimum values of $f(x,y) = 7x^2 + 8xy + y^2$ where x, y are constrained by the relation $x^2 + y^2 = 1$.*

Solution. Let $F(x,y) = 7x^2 + 8xy + y^2 + \lambda(x^2 + y^2 - 1)$, where λ is Lagrange's undetermined multiplier. For extreme values

$$\frac{\partial F}{\partial x} = 14x + 8y + 2\lambda x = 0, \frac{\partial F}{\partial y} = 8x + 2y + 2\lambda y = 0$$

Thus $(7+\lambda)x + 4y = 0, 4x + (1+\lambda)y = 0$. Since $(x,y) \neq (0,0)$ because $x^2 + y^2 = 1$, it follows that

$$\begin{vmatrix} 7+\lambda & 4 \\ 4 & 1+\lambda \end{vmatrix} = (7+\lambda)(1+\lambda) - 16 = \lambda^2 + 8\lambda - 9 = 0 \Rightarrow \lambda = -9, 1$$

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 14x^2 + 8xy + 2\lambda x^2 + 8xy + 2y^2 + 2\lambda y^2 = 2f + 2\lambda(x^2 + y^2) = 0$$

Since $x^2 + y^2 = 1$, we get $f = -\lambda \Rightarrow$ at stationary points $f = 9, -1$. Thus the maximum value of f is 9, minimum value is -1.

Check: We have found the maximum and minimum values without finding the stationary points.

$$\begin{aligned} \lambda = -9 \quad &\Rightarrow \quad x = 2y, x^2 + y^2 = 1 \Rightarrow 5y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{5}}, x = \pm\frac{2}{\sqrt{5}} \\ \lambda = 1 \quad &\Rightarrow \quad y = -2x, x^2 + y^2 = 1 \Rightarrow 5x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{5}}, y = \mp\frac{2}{\sqrt{5}} \end{aligned}$$

In case (1), $f(x, y) = \frac{28}{5} + \frac{16}{5} + \frac{1}{5} = 9$. In case (2) $f(x, y) = \frac{7}{5} - \frac{16}{5} + \frac{4}{5} = -1$, confirming the above.

Note: The question could also be done by substituting $x = \cos t, y = \sin t$, and then $f(x, y) = 7\cos^2 t + 8\cos t \sin t + \sin^2 t$, which is now a function of one variable. Differentiating and letting the derivative be 0, we get $-14\cos t \sin t + 8(\cos^2 t - \sin^2 t) + 2\sin t \cos t = 0$. Let $z = \tan t$, then $2z^2 + 3z - 2 = 0 \Rightarrow z = -2, \frac{1}{2} \Rightarrow (x, y) = (\pm\frac{1}{\sqrt{5}}, \mp\frac{2}{\sqrt{5}}), (\pm\frac{2}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}) \Rightarrow f(x, y) = 9, -1$. ∎

UPSC Civil Services Main 1987 - Mathematics Calculus

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Question 1(a) *If* $x_1 = \dfrac{1}{2}\Big(x + \dfrac{9}{x}\Big)$ *and for* $n > 0, x_{n+1} = \dfrac{1}{2}\Big(x_n + \dfrac{9}{x_n}\Big)$*, find the value of* $\lim_{n\to\infty} x_n$ *(*$x > 0$ *is assumed).*

Solution. $x > 0 \Rightarrow x_1 > 0$. By induction, clearly $x_n > 0$ for all $n \geq 1$.

Now $x_1 = \dfrac{x + \frac{9}{x}}{2} \geq \sqrt{x \cdot \dfrac{9}{x}} = 3$, because the arithmetic mean is always $\geq$ the geometric mean of two positive numbers. Similarly $x_n \geq 3$ for all $n \geq 1$, thus the sequence $\{x_n\}$ is bounded below. Moreover

$$x_{n+1} - x_n = \frac{1}{2}x_n + \frac{9}{2x_n} - x_n = \frac{1}{2x_n}(9 - x_n^2) \leq 0$$

as $x_n^2 \geq 9$. Thus $\{x_n\}$ is a monotonically decreasing sequence. Let l be the greatest lower bound of $\{x_n\}$ — then

$$l = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\Big(x_{n-1} + \frac{9}{x_{n-1}}\Big) = \frac{1}{2}\Big(\lim_{n\to\infty} x_{n-1} + \frac{9}{\lim_{n\to\infty} x_{n-1}}\Big) = \frac{1}{2}\Big(l + \frac{9}{l}\Big)$$

Thus $2l^2 = l^2 + 9 \Rightarrow l = 3$ ($l = -3$ is not admissible as all elements of the sequence are ≥ 3).

Thus $\lim_{n\to\infty} x_n = 3$. ∎

Question 1(b)

1. *If* $x = -a, h = 2a, f(x) = x^{\frac{1}{3}}$*, find* θ *from the mean value theorem:* $f(x + h) = f(x) + hf'(x + \theta h)$.

2. *If* $u = x + y - z, v = x - y + z, w = x^2 + (y - z)^2$*, examine whether or not there is a functional relationship between* u, v, w *and find the relationship, if any.*

Solution.

1. Substituting the given values, we get $f(a) = f(-a) + 2af'(-a + 2a\theta)$. Now $f(x) = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, so

$$\begin{aligned} a^{\frac{1}{3}} &= (-a)^{\frac{1}{3}} + \frac{2a}{3}(-a + 2a\theta)^{-\frac{2}{3}} \\ &= -a^{\frac{1}{3}} + \frac{2a}{3}a^{-\frac{2}{3}}(2\theta - 1)^{-\frac{2}{3}} \\ \Rightarrow 2 &= \frac{2}{3}(2\theta - 1)^{-\frac{2}{3}} \\ \Rightarrow 3 &= (2\theta - 1)^{-\frac{2}{3}} \\ \Rightarrow 27 &= (2\theta - 1)^{-2} \\ \Rightarrow \pm\frac{1}{\sqrt{27}} &= 2\theta - 1 \end{aligned}$$

So $\theta = \frac{1}{2}(1 \pm \frac{1}{3\sqrt{3}})$.

2. To establish that u, v, w have a functional relation, we have to show that the Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$. Clearly

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & -2(y-z) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix} = 0$$

Thus u, v, w are functionally dependent.

Now $u^2 = x^2 + (y-z)^2 + 2x(y-z)$, $v^2 = x^2 + (y-z)^2 - 2x(y-z)$, thus $u^2 + v^2 = 2w$, which is the desired functional relationship.

∎

Question 1(c) *If* $u = \csc^{-1}\Big(\dfrac{x^{\frac{1}{n}+1} + y^{\frac{1}{n}+1}}{x + y}\Big)^{\frac{1}{2}}$, *show that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4n^2}\tan u(2n + \sec^2 u)$$

Solution. Let $v = \csc u$. Then v is a homogeneous function of degree $\frac{1}{2n}$, because

$$v(ax, ay) = \Big(\frac{(ax)^{\frac{1}{n}+1} + (ay)^{\frac{1}{n}+1}}{a(x+y)}\Big)^{\frac{1}{2}} = a^{\frac{1}{2n}}\Big(\frac{x^{\frac{1}{n}+1} + y^{\frac{1}{n}+1}}{x+y}\Big)^{\frac{1}{2}} = a^{\frac{1}{2n}}v(x, y)$$

Applying Euler's theorem to v,

$$\begin{aligned}
x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} &= \frac{1}{2n}v \\
-x\csc u\cot u\frac{\partial u}{\partial x} - y\csc u\cot u\frac{\partial u}{\partial y} &= \frac{1}{2n}\csc u \qquad\qquad \because v = \csc u, \frac{\partial v}{\partial x} = -\csc u\cot u\frac{\partial u}{\partial x} \\
x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= -\frac{1}{2n}\tan u \qquad\qquad (1) \\
\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} &= -\frac{1}{2n}\sec^2 u\frac{\partial u}{\partial x} \qquad \text{Differentiating (1) w.r.t. } x \quad (2) \\
x\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} &= -\frac{1}{2n}\sec^2 u\frac{\partial u}{\partial y} \qquad \text{Differentiating (1) w.r.t. } y \quad (3) \\
x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= (-1 - \frac{1}{2n}\sec^2 u)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \qquad (2)\times x + (3)\times y \\
&= -\frac{1}{2n}(2n + \sec^2 u)(-\frac{1}{2n}\tan u) \qquad \text{From (1)} \\
&= \frac{1}{4n^2}\tan u(2n + \sec^2 u)
\end{aligned}$$

∎

Question 2(a) *1. Show by means of a suitable substitution that*

$$\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta = \frac{1}{2}\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt, \qquad x, y > 0$$

2. Establish the inequality

$$\frac{1}{2} < \int_0^1 \frac{dx}{(4-x^2-x^3)^{\frac{1}{2}}} < \frac{\pi}{6}$$

Solution.

1. By definition $B(y,x) = \displaystyle\int_0^1 z^{y-1}(1-z)^{x-1}\,dz$.

   Put $z = \cos^2\theta, dz = -2\cos\theta\sin\theta\,d\theta$, to get

$$\begin{aligned}
B(y,x) &= \int_{\frac{\pi}{2}}^0 \cos^{2y-2}\theta\sin^{2x-2}\theta\ (-2\cos\theta\sin\theta)\,d\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta
\end{aligned}$$

Now, put $z = \frac{1}{t+1}$ in $B(y,x)$,

$$
\begin{aligned}
B(y,x) &= \int_0^1 z^{y-1}(1-z)^{x-1}\,dz \\
&= \int_\infty^0 \Big(\frac{1}{t+1}\Big)^{y-1}\Big(1-\frac{1}{t+1}\Big)^{x-1}\frac{-dt}{(1+t)^2} \\
&= \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt
\end{aligned}
$$

From these two we have

$$\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta = \frac{1}{2}\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$$

as required.

2. Since $4-x^2+x^3 = 4-x^2(1-x)$, and $x^2(1-x) > 0$ for $0 < x < 1$, it follows that $4-x^2+x^3 < 4 \Rightarrow \dfrac{1}{2} < \dfrac{1}{(4-x^2-x^3)^{\frac{1}{2}}}$.

   On the other hand, $4-x^2+x^3 > 4-x^2$ for $x > 0$, so $(4-x^2+x^3)^{-\frac{1}{2}} < (4-x^2)^{-\frac{1}{2}}$. Hence

$$
\begin{aligned}
\int_0^1 \frac{1}{2}dx &< \int_0^1 \frac{dx}{(4-x^2-x^3)^{\frac{1}{2}}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}} \\
\frac{1}{2} &< \int_0^1 \frac{dx}{(4-x^2-x^3)^{\frac{1}{2}}} < \sin^{-1}\frac{x}{2}\Big]_0^1 = \frac{\pi}{6}
\end{aligned}
$$

$\blacksquare$

Question 2(b) *Find the volume of the solid generated by revolving the curve* $y^2 = \dfrac{x^3}{2a-x}, a > 0$ *about its asymptote* $x = 2a$.

Solution. Consider a thin vertical strip of thickness dx, and rotate it about the asymptote

— its volume is $2\pi rh\,dx = 2\pi(2a-x)2\sqrt{\dfrac{x^3}{2a-x}}\,dx$. Thus

$$\begin{aligned}
V &= 4\pi\int_0^{2a}(2a-x)\sqrt{\frac{x^3}{2a-x}}\,dx \\
&= 4\pi\int_0^{2a}\sqrt{2a-x}\,x^{\frac{3}{2}}\,dx \\
&\ \text{Put } x = 2a\sin^2\theta \Rightarrow dx = 4a\sin\theta\cos\theta\,d\theta \\
&= 4\pi\int_0^{\frac{\pi}{2}}\sqrt{2a}\cos\theta(2a)^{\frac{3}{2}}\sin^3\theta\ 4a\sin\theta\cos\theta\,d\theta \\
&= 64\pi a^3\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta \\
&= 64\pi a^3\,\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\,\frac{\pi}{2} = 2\pi^2a^3
\end{aligned}$$

∎

Question 2(c) *Evaluate* $\iint_D x^{\frac{1}{2}}y^{\frac{1}{3}}(1-x-y)^{\frac{2}{3}}\,dx\,dy$ *where D is the domain bounded by the lines $x=0, y=0, x+y=1$.*

Solution. We convert this to a Dirichlet integral using the standard transformation. Put $x+y=u, y=uv$, so that $dx\,dy = u\,du\,dv$, (see 1989, question 3(a) for example).

$$\begin{aligned}
I &= \int_0^1\int_0^1 u^{\frac{1}{2}}(1-v)^{\frac{1}{2}}u^{\frac{1}{3}}v^{\frac{1}{3}}(1-u)^{\frac{2}{3}}u\,du\,dv \\
&= \int_0^1\int_0^1 u^{\frac{11}{6}}(1-u)^{\frac{2}{3}}v^{\frac{1}{3}}(1-v)^{\frac{1}{2}}\,du\,dv \\
&= \frac{\Gamma(\frac{17}{6})\Gamma(\frac{5}{3})}{\Gamma(\frac{27}{6})}\frac{\Gamma(\frac{4}{3})\Gamma(\frac{3}{2})}{\Gamma(\frac{17}{6})} = \frac{\frac{2}{3}\Gamma(\frac{2}{3})\frac{1}{3}\Gamma(\frac{1}{3})\Gamma(\frac{3}{2})}{\frac{7}{2}\frac{5}{2}\frac{3}{2}\Gamma(\frac{3}{2})} = \frac{16}{945}\Gamma\Big(\frac{2}{3}\Big)\Gamma\Big(\frac{1}{3}\Big)
\end{aligned}$$

But $\Gamma(\frac{2}{3})\Gamma(\frac{1}{3}) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$. Thus $I = \frac{32\pi}{945\sqrt{3}}$. ∎

Paper II

Question 3(a) *Let $f(x) = x$ if x is rational, and $1-x$ if x is irrational. Show that f is continuous only at $x = \frac{1}{2}$.*

Solution. See 2001, question 1(a). ∎

Question 4(a) *Find the maximum and minimum value of $f(x,y) = xy$ subject to the condition that $x^2 + y^2 + xy = a^2$.*

Solution. Let $F(x,y) = xy - \lambda(x^2 + y^2 + xy - a^2)$, where λ is Lagrange's undetermined multiplier. For extreme values,

$$\frac{\partial F}{\partial x} = y - 2\lambda x - \lambda y = 0, \frac{\partial F}{\partial y} = x - 2\lambda y - \lambda x = 0 \Rightarrow \lambda = \frac{y}{2x+y} = \frac{x}{2y+x}$$

as $(x,y) \neq (0,0)$. Thus $2x^2 + xy = 2y^2 + xy \Rightarrow (x+y)(x-y) = 0 \Rightarrow x = y, x = -y$. Using $x^2 + y^2 + xy = a^2$ we get

1. $x = y \Rightarrow 3x^2 = a^2 \Rightarrow x = y = \pm\frac{a}{\sqrt{3}} \Rightarrow f(x,y) = \frac{a^2}{3}$
2. $x = -y \Rightarrow x^2 = a^2 \Rightarrow x = a, y = -a$ or $x = -a, y = a$. In either of these cases $f(x,y) = -a^2$.

Thus the required maximum value is $\frac{a^2}{3}$ and the required minimum is $-a^2$.

Note: In this problem there was no need to check the nature of the critical points, as the maximum amd minimum values occur at these points. If it were required, it could be done as follows.

$$d^2F = -2\lambda(dx)^2 - 2\lambda(dy)^2 + 2(1-\lambda)dx\,dy$$

Now $x^2 + y^2 + xy = a^2 \Rightarrow 2x\,dx + x\,dy + y\,dx + 2y\,dy = 0$, or $dy = -\frac{2x+y}{x+2y}\,dx$. Thus

$$d^2F = -2\lambda(dx)^2 - 2\lambda\left(\frac{2x+y}{x+2y}\right)^2(dx)^2 - 2(1-\lambda)(dx)^2\,\frac{2x+y}{x+2y}$$

Case 1: $x = a, y = -a, \lambda = -1$ or $x = -a, y = a, \lambda = -1$

$$d^2F = 2(dx)^2 - 4(-1)(dx)^2 + 2(-1)^2(dx)^2 > 0$$

so we have a local minimum at $(a,-a)$ or $(-a,a)$.

Case 2: $x = y = \pm\frac{a}{\sqrt{3}}, \lambda = \frac{1}{3}$.

$$d^2F = -\frac{2}{3}(dx)^2 - \frac{4}{3}(dx)^2 - \frac{2}{3}(dx)^2 < 0$$

so we have a local maximum at $x = y = \pm\frac{a}{\sqrt{3}}$. ∎

UPSC Civil Services Main 1988 - Mathematics Calculus

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Question 1(a) *If* $f(x) = \tan x$ *prove that*

$$f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \ldots = \sin \frac{n\pi}{2}$$

Solution. Clearly $\sin x = f(x) \cos x$. Using Leibnitz's formula for the derivative of the product of two functions, we get

$$\frac{d^n \sin x}{dx^n} = \sum_{r=0}^{n} \binom{n}{r} f^{(n-r)}(x) \cos^{(r)}(x)$$

Now $\dfrac{d^r}{dx^r} \cos x = \cos(x + \frac{r\pi}{2})$ and $\dfrac{d^r}{dx^r} \sin x = \sin(x + \frac{r\pi}{2})$. Therefore, $\dfrac{d^r}{dx^r} \cos x = 0$ when $x = 0, n$ odd, and $\dfrac{d^r}{dx^r} \cos x = (-1)^{\frac{n}{2}}$ when $x = 0, n$ even. Also, $\dfrac{d^n}{dx^n} \sin x = \sin \frac{n\pi}{2}$ when $x = 0$.

Thus

$$\sin \frac{n\pi}{2} = f^{(n)}(0) - \binom{n}{2} f^{(n-2)}(0) + \binom{n}{4} f^{(n-4)}(0) - \ldots$$

. ∎

Question 1(b) *Find the minimum value of* $x^2 + y^2 + z^2$ *when* $x + y + z = k$.

Solution. Let $F(x) = x^2 + y^2 + z^2 + \lambda(x + y + z - k)$, where λ is Lagrange's undetermined multiplier. For stationary values,

$$\frac{\partial F}{\partial x} = 2x + \lambda = 0, \frac{\partial F}{\partial y} = 2y + \lambda = 0, \frac{\partial F}{\partial z} = 2z + \lambda = 0$$

Thus $\lambda = -2x = -2y = -2z \Rightarrow x = y = z$. From $x + y + z = k$ we have $x = y = z = \frac{k}{3}$.

Now

$$\begin{aligned} d^2F &= \frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial y^2}(dy)^2 + \frac{\partial^2 F}{\partial z^2}(dz)^2 + \frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 F}{\partial y\,\partial z}\,dy\,dz + \frac{\partial^2 F}{\partial z\,\partial x}\,dz\,dx \\ &= 2(dx)^2 + 2(dy)^2 + 2(dz)^2 \end{aligned}$$

Now $dx + dy + dz = 0$ from $x + y + z = k$, so substituting $dz = -dx - dy$, we get

$$\begin{aligned} d^2F &= 2(dx)^2 + 2(dy)^2 + 2(dx + dy)^2 = 4(dx)^2 + 4(dy)^2 + 4dx\,dy \\ &= 4\left[\left(dx + \frac{dy}{2}\right)^2 + \frac{3}{4}(dy)^2\right] > 0 \end{aligned}$$

Thus $x^2+y^2+z^2$ is minimum when $x = y = z = \frac{k}{3}$ and the minimum value of $\frac{k^2}{9}+\frac{k^2}{9}+\frac{k^2}{9} = \frac{k^2}{3}$. ∎

Question 1(c) *Find the asymptotes of the cubic*

$$x^3 - xy^2 - 2xy + 2x - y = 0$$

and show that they cut the curve again in points which lie on the line $3x - y = 0$.

Solution. Since the coefficient of the highest degree term of y in the equation, i.e.y^2 is $-x$, it follows that $x = 0$ is an asymptote parallel to the y axis.

There is no asymptote parallel to the x-axis, as the coefficient of x^3, the highest degree term of x peresent in the equation, is a constant.

If $y = mx + c$ is an asymptote, then m is given by $\Phi_3(m) = 1 - m^2 = 0 \Rightarrow m = \pm 1$. c is given by $-\dfrac{\Phi_2(m)}{\Phi_3'(m)}$ provided it is not indeterminate, where Φ_r are homogeneous terms of degree r present in the equation. Thus $c = -\frac{-2m}{-2m} = -1$. Thus the asymptotes of the type $y = mx + c$ are $y = x - 1, y + x = -1$.

The joint equation of asymptotes is given by $x(y - x + 1)(y + x + 1) = 0$ or $x(y^2 - x^2 + 2y + 1) = xy^2 - x^3 + 2yx + x = 0$, or $P_3 = x^3 - xy^2 - 2yx - x = 0$.

Thus the point of intersection of the curve and the asymptotes lie on the line $3x - y = 0$ as the equation of the given curve is $P_3 + 3x - y = 0$.

Note: The asymptotes parallel to the y-axis can also be found as follows:

Let $x = my + d$ be an asymptote. Dividing the equation by y^3 and letting $y \to \infty$, we get $m^3 - m = 0$, as $\frac{x}{y} \to m$ when $y \to \infty$. Thus $m = 0, \pm 1$.

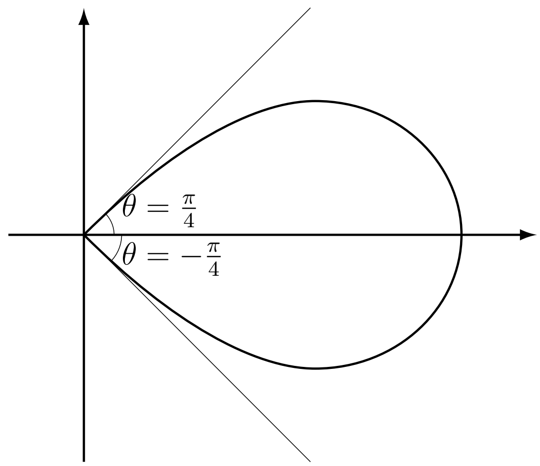
If $m = 0$, then $x = d$, so $d^3 - dy^2 - 2dy + 2d - y = 0$. Dividing by y^2 and letting $y \to \infty$ we get $d = 0$, so $x = 0$ is an asymptote.

Similarly we can find $d = 1$ when $m = \pm 1$. This will determine all three asymptotes. ∎

Question 2(a) *Find the center of gravity of one loop of the lemniscate of Bernoulli* $r^2 = a^2 \cos 2\theta$.

Solution.

Because of symmetry, $\overline{y} = 0$. The area of the lemniscate is



$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\,d\theta = \int_0^{\frac{\pi}{4}} a^2\cos 2\theta\,d\theta = a^2\frac{\sin 2\theta}{2}\Big]_0^{\frac{\pi}{4}} = \frac{a^2}{2}$$

Now

$$\begin{aligned}
\overline{x} &= \frac{2}{3a^2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^3\cos\theta\,d\theta \\
&= \frac{4a}{3}\int_0^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta\,d\theta \\
&\qquad \text{Let } \sqrt{2}\sin\theta = \sin t \Rightarrow \cos\theta\,d\theta = \frac{\cos t\,dt}{\sqrt{2}} \\
&= \frac{4a}{3\sqrt{2}}\int_0^{\frac{\pi}{2}}(1-\sin^2 t)^{\frac{3}{2}}\cos t\,dt = \frac{4a}{3\sqrt{2}}\int_0^{\frac{\pi}{2}}\cos^4 t\,dt = \frac{4a}{3\sqrt{2}}\frac{3\cdot 1\cdot\pi}{4\cdot 2\cdot 2} = \frac{\pi\sqrt{2}a}{8}
\end{aligned}$$

Thus the centroid is $\left(\frac{\pi\sqrt{2}a}{8}, 0\right)$. ∎

Question 2(b) *Evaluate* $\iint x^{\frac{3}{2}}y^2(1-x^2-y^2)\,dx\,dy$ *over the positive quadrant of the circle* $x^2+y^2=1$.

Solution. Substitute $x^2 = X, y^2 = Y \Rightarrow dx\,dy = \dfrac{dX\,dY}{4X^{\frac{1}{2}}Y^{\frac{1}{2}}}$ and the region of integration is transformed to $X \geq 0, Y \geq 0, X+Y \leq 1$. The required integral becomes

$$I = \iint_{\substack{X\geq 0, Y\geq 0\\ X+Y\leq 1}} X^{\frac{3}{4}}Y(1-X-Y)\frac{dX\,dY}{4X^{\frac{1}{2}}Y^{\frac{1}{2}}} = \frac{1}{4}\iint_{\substack{X\geq 0, Y\geq 0\\ X+Y\leq 1}} X^{\frac{1}{4}}Y^{\frac{1}{2}}(1-X-Y)\,dX\,dY$$

Put $X+Y = u, Y = uv \Rightarrow X = u(1-v)$.

$$\frac{\partial(X,Y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$$

Also, $X \geq 0, Y \geq 0, X+Y \leq 1 \Rightarrow 0 \leq u \leq 1, 0 \leq v \leq 1 (\because v = \frac{Y}{X+Y})$.

$$\begin{aligned}
I &= \frac{1}{4}\int_0^1\int_0^1 u^{\frac{1}{4}}(1-v)^{\frac{1}{4}}u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u)u\,du\,dv \\
&= \frac{1}{4}\int_0^1 u^{\frac{7}{4}}(1-u)\,du\int_0^1 v^{\frac{1}{2}}(1-v)^{\frac{1}{4}}\,dv \\
&= \frac{1}{4}B(\tfrac{11}{4},2)B(\tfrac{3}{2},\tfrac{5}{4}) = \frac{1}{4}\frac{\Gamma(\frac{11}{4})\Gamma(2)}{\Gamma(\frac{19}{4})}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{4})}{\Gamma(\frac{11}{4})} = \frac{1}{4}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{5}{4})}{\Gamma(\frac{19}{4})}
\end{aligned}$$

The above integral actually becomes Dirichlet's integral. ∎

Question 2(c) *Show that the volume of the solid obtained by revolving the curve $(a-x)y^2 = a^2x$ about its asymptote is $\dfrac{\pi^2 a^3}{2}$.*

Solution. The asymptote is $x = a$, because $a - x$ is the coefficient of the highest degree term in y present in the equation of the curve. Thus

$$\begin{aligned}
V &= \int_{-\infty}^{\infty} \pi(a-x)^2\, dy \\
&= 2\pi \int_0^{\infty} \Big(a - \frac{ay^2}{a^2+y^2}\Big)^2 dy = 2\pi a^6 \int_0^{\infty} \frac{dy}{(a^2+y^2)^2} \\
&\quad \text{Put } y = a\tan\theta, dy = a\sec^2\theta\, d\theta \\
V &= 2\pi a^6 \int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\, d\theta}{a^4(1+\tan^2\theta)^2} = 2\pi a^3 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 2\pi a^3 \frac{1}{2}\frac{\pi}{2} = \frac{\pi^2 a^3}{2}
\end{aligned}$$

■

Paper II

Question 3(a) *Evaluate*

$$\iint \Big(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\Big)^{\frac{3}{2}} dx\, dy$$

over the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Solution. Let $x = aX, y = bY$, so $dx\, dy = ab\, dX\, dY$. The given integral becomes $ab \iint (1 - X^2 - Y^2)^{\frac{3}{2}}\, dX\, dY$ over the circle $X^2 + Y^2 \leq 1$. Changing to polar coordinates, $X = r\cos\theta, Y = r\sin\theta$, we get

$$\begin{aligned}
I &= 4ab \int_0^{\frac{\pi}{2}} \int_0^1 (1-r^2)^{\frac{3}{2}} r\, dr\, d\theta \\
&= 2\pi ab \int_0^1 (1-r^2)^{\frac{3}{2}} r\, dr \\
&\quad \text{Put } r^2 = t, 2r\, dr = dt \\
&= \pi ab \int_0^1 (1-t)^{\frac{3}{2}} dt = -\pi ab \frac{2}{5}(1-t)^{\frac{5}{2}}\Big]_0^1 = \frac{2\pi ab}{5}
\end{aligned}$$

■

Question 4(a) *Show that a local extreme value of f given by*

$$f(\mathbf{x}) = x_1^k + \ldots + x_n^k, \quad \mathbf{x} = (x_1, \ldots, x_n)$$

subject to the condition $x_1 + \ldots + x_n = a$ is $a^k n^{1-k}$.

Solution. Let $F(\mathbf{x}) = x_1^k + \ldots + x_n^k - \lambda(x_1 + \ldots + x_n - a)$ where λ is Lagrange's undetermined multiplier. For extreme values, for all i

$$\frac{\partial F}{\partial x_i} = kx_i^{k-1} - \lambda = 0 \Rightarrow x_i^{k-1} = \frac{\lambda}{k}$$

Thus $x_1^{k-1} = \ldots = x_n^{k-1} = \dfrac{\lambda}{k}$, or $x_1 = \ldots = x_n$. Using $x_1 + \ldots + x_n = a$ we get $x_i = \dfrac{a}{n}$, so the extreme value is $\displaystyle\sum_{i=0}^{n} \big(\frac{a}{n}\big)^k = \frac{na^k}{n^k} = a^k n^{1-k}$.

Note: To decide the nature of the extreme value, we consider

$$d^2F = k(k-1)x_1^{k-2}[(dx_1)^2 + \ldots + (dx_n)^2]$$

using $x_1 = \ldots = x_n$. Thus $d^2F > 0$ if $k < 0$ or $k > 1$ (assuming $a > 0$), so we have a minimum. If $0 < k < 1$, $d^2F < 0$, and we have a maximum. This is not required for this question. ■

Question 4(b) *The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Prove that at $(0, 0)$ f is continuous and possesses all directional derivatives, but is not differentiable.

Solution. Let $\epsilon > 0$ and $\delta = \epsilon$, then

$$|f(x, y) - f(0, 0)| = \left|\frac{x^2 y}{x^2 + y^2}\right| = \left|\frac{r^2 \cos^2\theta \, r \sin\theta}{r^2}\right| \leq r = \sqrt{x^2 + y^2}$$

Thus for any (x, y) belonging to the open disc with center $(0, 0)$ and radius $\delta = \epsilon$ i.e. for $x^2 + y^2 < \epsilon^2$, we get $|f(x, y) - f(0, 0)| < \epsilon$, so $f(x, y)$ is continuous at $(0, 0)$.

Let $\mathbf{v} = (\cos\theta, \sin\theta)$, then by definition the directional derivative at (a, b) in the direction of $\mathbf{v}$ is given by

$$\lim_{t \to 0} \frac{f(a + t\cos\theta, b + t\sin\theta) - f(a, b)}{t}$$

when this limit exists, and it is denoted by $f_{\mathbf{v}}(a, b)$.

Here $(a, b) = (0, 0)$, so

$$f_{\mathbf{v}}(0, 0) = \lim_{t \to 0} \frac{f(t\cos\theta, t\sin\theta)}{t} = \lim_{t \to 0} \frac{t^2 \cos^2\theta \, t \sin\theta}{t^3} = \cos^2\theta \sin\theta$$

Thus all directional derivatives exist.

However for f to be differentiable, all directional derivatives should be equal — this is clearly not the case here, as the directional derivative for $\theta = 0$ is 0, but for $\theta = \frac{\pi}{4}$ it is $\frac{1}{2\sqrt{2}}$. ■

UPSC Civil Services Main 1989 - Mathematics Calculus

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Question 1(a) *If f is at least thrice continuously differentiable, then show that*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h)$$

where θ lies between 0 and 1, and prove that $\lim_{h\to 0}\theta = \frac{1}{3}$.

Solution. Let

$$\phi(x) = f(a+h) - f(x) - (a+h-x)f'(x) - \frac{(a+h-x)^2}{2!}A$$

where A is so determined that $\phi(a+h) = \phi(a)$. Clearly $\phi(a+h) = 0$, and $\phi(a) = f(a+h) - f(a) - hf'(a) - \frac{h^2}{2!}A$, so $\phi(a) = 0 \Rightarrow \frac{h^2}{2!}A = f(a+h) - f(a) - hf'(a)$.

Now ϕ satisfies the requirements of Rolle's theorem in $[a, a+h]$, therefore there exists $\theta \in (0,1)$ such that $\phi'(a+\theta h) = 0$, note that $a+\theta h \in (a, a+h)$. But

$$\phi'(a+\theta h) = -(h-\theta h)f''(a+\theta h) + (h-\theta h)A = 0 \Rightarrow A = f''(a+\theta h)$$

Now from $\phi(a) = 0$, we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h)$$

as required.

Now consider the equality

$$\frac{f(a+h) - f(a) - hf'(a) - \frac{h^2}{2}f''(a)}{h^3} = \frac{h^2}{2}\frac{f''(a+\theta h) - f''(a)}{h^3}$$

Taking the limit as $h \to 0$ on both sides, we get

$$\begin{aligned}
LHS &= \lim_{h\to 0} \frac{f(a+h) - f(a) - hf'(a) - \frac{h^2}{2} f''(a)}{h^3} \\
&= \lim_{h\to 0} \frac{f'(a+h) - f'(a) - hf''(a)}{3h^2} \quad \text{L'Hospital's rule} \\
&= \lim_{h\to 0} \frac{f''(a+h) - f''(a)}{6h} \quad \text{L'Hospital's rule} \\
&= \frac{1}{6} f'''(a) \\
RHS &= \lim_{h\to 0} \frac{f''(a+\theta h) - f''(a)}{2h} = \lim_{h\to 0} \frac{\theta}{2} \frac{f''(a+\theta h) - f''(a)}{\theta h} \\
&= \frac{\theta}{2} f'''(a)
\end{aligned}$$

Thus $\theta = \frac{1}{3}$ as $h \to 0$, provided $f'''(a) \neq 0$. ∎

The question above is a particular case of the following:

Taylor's Theorem: Let $f^{(n-1)}$, the $(n-1)$-th derivative of a real valued function be continuous in the closed interval $[a, a+h]$, and let $f^{(n)}$ exist in the open interval $(a, a+h)$, then there exists a real number $\theta, 0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

where

$$R_n = \frac{h^n (1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(a+\theta h), \;\; 0 < \theta < 1$$

Proof: The condition that $f^{(n-1)}(x)$ is continuous in $[a, a+h] \Rightarrow f, f', f'', \ldots, f^{(n-2)}$ are continuous in $[a, a+h]$. Let

$$\phi(x) = f(x) + (a+h-x)f'(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + A(a+h-x)^p$$

where A is a constant to be determined so that $\phi(a+h) = \phi(a)$ i.e.

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{(h^{n-1}}{(n-1)!} f^{(n-1)}(a) + Ah^p$$

Now (i) $\phi(x)$ is continuous in $[a, a+h]$ (ii) $\phi(x)$ is differentiable in $(a, a+h)$ and (iii) $\phi(a+h) = \phi(a)$. Thus all requirements of Rolle's theorem are satisfied,so there exists a real number $\theta, 0 < \theta < 1$ such that $\phi'(a+\theta h) = 0$ (note that any real number $c \in (a, a+h)$ can be written as $c = a + \frac{c-a}{h} h$, and since $a < c < a+h$, we get $0 < \theta = \frac{c-a}{h} < 1$). But

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - pA(a+h-x)^{p-1}$$

so

$$\phi'(a+\theta h) = \frac{(h(1-\theta))^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - pAh^{p-1}(1-\theta)^{p-1} = 0 \Rightarrow A = \frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h)$$

Thus

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!p} f^{(n)}(a+\theta h)$$

If we put $n = p$ we get

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

The given question is for $n = 2$, so

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$$

Now we prove the second part: If $f^{(n+1)}(x)$ is continuous from the right at $x = a$, then $\theta \rightarrow \frac{1}{n+1}$ as $h \rightarrow 0$ provided $f^{(n+1)}(a) \neq 0$.

In the given question, $n = 2$, and $f^{(3)}(x)$ is continuous in $[a, a+h]$ so $\theta \rightarrow \frac{1}{3}$.

$f^{(n+1)}(x)$ is continuous from the right at $x = a \Rightarrow$ there exists k, $0 < k < h$, such that $f^{(n+1)}(x)$ exists in $[a, a+k]$. Thus by the above theorem:

$$\begin{aligned} f(a+k) &= f(a) + kf'(a) + \ldots + \frac{(k^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{k^n}{n!} f^{(n)}(a+\theta k) \\ f(a+k) &= f(a) + kf'(a) + \ldots + \frac{(k^n}{n!} f^{(n)}(a) + \frac{k^{n+1}}{(n+1)!} f^{(n+1)}(a+\phi k) \end{aligned}$$

where $0 < \theta < 1, 0 < \phi < 1$. Subtracting the first from the second,

$$f^{(n)}(a+\theta k) - f^{(n)}(a) = \frac{k}{n+1} f^{(n+1)}(a+\phi k)$$

Now we use Lagrange's mean value theorem for $f^{(n)}(x)$ in $[a, a+\theta k]$ and obtain

$$\theta k f^{(n+1)}(a+\theta\theta_1 k) = \frac{k}{n+1} f^{(n+1)}(a+\phi k),\ 0 < \theta_1 < 1$$

Taking limit as $h \rightarrow 0$ i.e. $k \rightarrow 0$, we get $\lim_{h\rightarrow 0} \theta = \frac{1}{n+1}$ as $f^{(n+1)}(a+\theta\theta_1 k)$ and $f^{(n+1)}(a+\phi k)$ both tend to $f^{(n+1)}(a)$ as $k \rightarrow 0$, and $f^{(n+1)}(a) \neq 0$.

Question 1(b) *Prove that the volume of a right circular cylinder of greatest volume which can be inscribed in a sphere is $\frac{\sqrt{3}}{3}$ times that of a sphere.*

Solution. Let the radius of the sphere be a, and the radius and height of the cylinder be r and h respectively. Then $r^2 + (\frac{h}{2})^2 = a^2$. Letting $r = a\cos\theta, h = 2a\sin\theta$, the volume of the cylinder is $2\pi a^3\cos^2\theta\sin\theta$.

For extreme values, $\dfrac{dV}{d\theta} = 2\pi a^3[\cos^3\theta - 2\cos\theta\sin^2\theta] = 0$. Now $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow r = 0$, which is not admissible. So $\cos^2\theta - 2\sin^2\theta = 0 \Rightarrow \tan\theta = \frac{1}{\sqrt{2}}$ as $0 < \theta < \frac{\pi}{2}$. Thus $\sin\theta = \frac{1}{\sqrt{3}}, \cos\theta = \sqrt{\frac{2}{3}}$.

$\dfrac{d^2V}{d\theta^2} = 2\pi a^3[-3\cos^2\theta\sin\theta + 2\sin^3\theta - 4\cos^2\theta\sin\theta] < 0$ for $\sin\theta = \frac{1}{\sqrt{3}}, \cos\theta = \sqrt{\frac{2}{3}}$. Thus V is maximum when $\tan\theta = \frac{1}{\sqrt{2}}$.

$V = 2\pi a^3\frac{2}{3}\cdot\frac{1}{\sqrt{3}} = \frac{4}{3}\pi a^3\cdot\frac{\sqrt{3}}{3}$

Thus V is $\frac{\sqrt{3}}{3}$ times the volume of the sphere. ∎

Question 1(c) *Show that at the point of the surface $x^xy^yz^z = c$ where $x = y = z$, $\dfrac{\partial^2 z}{\partial x\,\partial y} = -[x\log ex]^{-1}$.*

Solution. Taking logs on both sides of $x^xy^yz^z = c$, we get

$$z\log z + y\log y + x\log x = \log c$$

Differentiating with respect to x,

$$(1+\log z)\frac{\partial z}{\partial x} + (1+\log x) = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z} = -\frac{\log ex}{\log ez} \quad (*)$$

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{\log ey}{\log ez}$.

Differentiating (*) w.r.t. y,

$$(1+\log z)\frac{\partial^2 z}{\partial x\,\partial y} + \frac{1}{z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial^2 z}{\partial x\,\partial y} = -\frac{1}{z\log ez}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} = -\frac{\log ex\log ey}{z(\log ez)^3}$$

Letting $x = y = z$, we get $\dfrac{\partial^2 z}{\partial x\,\partial y} = -\dfrac{\log ex\log ex}{x(\log ex)^3} = -[x\log ex]^{-1}$ as required. ∎

Question 2(a) *Find the surface of the solid generated by revolving $x = a\cos^3 t, y = a\sin^3 t$ about the x-axis.*

Solution. We confine ourselves to the first quadrant because of symmetry. The curve is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\begin{aligned}
S &= 2\int_0^{\frac{\pi}{2}} 2\pi y \frac{ds}{dt}\, dt \\
&= 4\pi \int_0^{\frac{\pi}{2}} a\sin^3 t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= 4\pi a \int_0^{\frac{\pi}{2}} \sin^3 t \big[(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2\big]^{\frac{1}{2}}\, dt \\
&= 4\pi a \int_0^{\frac{\pi}{2}} 3a\sin^4 t\cos t\, dt \\
&= 12\pi a^2 \frac{\sin^5 t}{5}\bigg]_0^{\frac{\pi}{2}} = \frac{12}{5}\pi a^2
\end{aligned}$$

■

Question 2(b) *If for a curve* $x\sin\theta + y\cos\theta = f'(\theta)$ *and* $x\cos\theta - y\sin\theta = f''(\theta)$ *then show that* $S = f(\theta) + f''(\theta) + C$.

Solution. Solving for x and y, we get

$$\begin{aligned}
x &= f'(\theta)\sin\theta + f''(\theta)\cos\theta \\
y &= f'(\theta)\cos\theta - f''(\theta)\sin\theta \\
\frac{dx}{d\theta} &= f'(\theta)\cos\theta + f''(\theta)\sin\theta - f''(\theta)\sin\theta + f'''(\theta)\cos\theta \\
&= \big(f'(\theta) + f'''(\theta)\big)\cos\theta \\
\frac{dy}{d\theta} &= -f'(\theta)\sin\theta + f''(\theta)\cos\theta - f''(\theta)\cos\theta - f'''(\theta)\sin\theta \\
&= -\big(f'(\theta) + f'''(\theta)\big)\sin\theta \\
S &= \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int \big[f'(\theta) + f'''(\theta)\big]\, d\theta \\
&= f(\theta) + f''(\theta) + C
\end{aligned}$$

as required. ■

Question 2(c) *Evaluate* $\iiint (1-z)^{\frac{1}{2}}\, dx\, dy\, dz$ *over the interior of the tetrahedron with faces* $x = 0, y = 0, z = 0, x + y + z = 1$.

Solution.

$$\begin{aligned}
I &= \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{1-y-z}(1-z)^{\frac{1}{2}}\,dx\,dy\,dz \\
&= \int_{z=0}^{1}\int_{y=0}^{1-z}(1-z)^{\frac{1}{2}}(1-y-z)\,dy\,dz \\
&= \int_{z=0}^{1}(1-z)^{\frac{1}{2}}\Big[y-\frac{y^2}{2}-zy\Big]_0^{1-z}\,dz \\
&= \int_{z=0}^{1}(1-z)^{\frac{1}{2}}\Big[1-z-\frac{(1-z)^2}{2}-z(1-z)\Big]\,dz \\
&= \int_{z=0}^{1}(1-z)^{\frac{3}{2}}\Big[1-\frac{(1-z)}{2}-z\Big]\,dz \\
&= \int_{z=0}^{1}\frac{1}{2}(1-z)^{\frac{5}{2}}\,dz = \frac{1}{2}\frac{-2}{7}(1-z)^{\frac{7}{2}}\Big]_0^1 = \frac{1}{7}
\end{aligned}$$

∎

Paper II

Question 3(a) *Show that the value of*

$$\iint (1-x-y)^3 x^{\frac{1}{2}} y^{\frac{1}{2}}\,dx\,dy$$

taken over the interior of the triangle whose vertices are the origin and the points $(0,1)$ *and* $(1,0)$ *is* $\frac{\pi}{480}$.

Solution. We will convert this into a Dirichlet integral. Let $x+y=u, y=uv$, so $x=u(1-v)$, and $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u$. u varies from 0 to 1, and $v=\frac{y}{u}=\frac{y}{x+y}$ also varies from 0 to 1.

$$\begin{aligned}
I &= \int_0^1\int_0^1 (1-u)^3 u^{\frac{1}{2}}(1-v)^{\frac{1}{2}}u^{\frac{1}{2}}v^{\frac{1}{2}}u\,du\,dv \\
&= \int_0^1\int_0^1 (1-u)^{4-1}u^{3-1}(1-v)^{\frac{3}{2}-1}v^{\frac{3}{2}-1}\,du\,dv \\
&= \frac{\Gamma(4)\Gamma(3)}{\Gamma(7)}\,\frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} \\
&= \frac{3!}{6!}\frac{\pi}{4} = \frac{\pi}{480}
\end{aligned}$$

as $\Gamma(\frac{1}{2})$. ∎

Question 3(b) *Obtain the largest and the least values of* $2(x+y+z)-xyz$ *on the closed ball* $x^2+y^2+z^2 \le 9$.

Solution. Let $F(x,y,z) = 2(x+y+z) - xyz - \lambda(x^2+y^2+z^2-9)$, where λ is Lagrange's undetermined multiplier. For extreme values:

$$\frac{\partial F}{\partial x} = 2 - yz + 2\lambda x = 0, \frac{\partial F}{\partial y} = 2 - xz + 2\lambda y = 0, \frac{\partial F}{\partial z} = 2 - yx + 2\lambda z = 0$$

Subtracting the first two, $z(x-y) + 2\lambda(x-y) = 0 \Rightarrow x = y$ or $z = -2\lambda$. Similarly from the other pairs, we get $y = z$ or $x = -2\lambda$ and $x = z$ or $y = -2\lambda$.

We now explicitly find extreme vaues to get the greatest and least values of $f(x,y,z) = 2(x+y+z) - xyz$

1. $x = y = z$: This gives us $x^2+x^2+x^2 = 9$, so $x = \pm\sqrt{3}$.
   (a) $x = y = z = \sqrt{3}$: $f(x,y,z) = 6\sqrt{3} - 3\sqrt{3} = 3\sqrt{3}$.
   (b) $x = y = z = -\sqrt{3}$: $f(x,y,z) = -6\sqrt{3} + 3\sqrt{3} = -3\sqrt{3}$.
2. $x = y, y = -2\lambda$: $4\lambda^2 + 4\lambda^2 + z^2 = 9 \Rightarrow z^2 = 9 - 8\lambda^2$. Using $2 - xy + 2\lambda z = 0$, we have $2 - 4\lambda^2 + 2\lambda z = 0 \Rightarrow z = \dfrac{2\lambda^2-1}{\lambda}$. Thus

$$\begin{aligned}
\left(\frac{2\lambda^2-1}{\lambda}\right)^2 &= 9 - 8\lambda^2 \\
\Rightarrow 4\lambda^4 - 4\lambda^2 + 1 &= 9\lambda^2 - 8\lambda^4 \\
\Rightarrow 12\lambda^4 - 13\lambda^2 + 1 &= 0 \\
\Rightarrow \lambda^2 &= 1, \frac{1}{12} \\
\lambda = 1 &\Rightarrow x = y = -2, z^2 = 1 \\
\lambda = -1 &\Rightarrow x = y = 2, z^2 = 1 \\
\lambda = \frac{1}{2\sqrt{3}} &\Rightarrow x = y = -\frac{1}{\sqrt{3}}, z^2 = \frac{25}{3} \\
\lambda = -\frac{1}{2\sqrt{3}} &\Rightarrow x = y = \frac{1}{\sqrt{3}}, z^2 = \frac{25}{3}
\end{aligned}$$

   We examine each of these 8 cases:

   (a) $x = y = -2, z = 1$: $f(x,y,z) = -6 - 4 = -10$.
   (b) $x = y = -2, z = -1$: $f(x,y,z) = -10 + 4 = -6$.
   (c) $x = y = 2, z = 1$: $f(x,y,z) = 10 - 4 = 6$.
   (d) $x = y = 2, z = -1$: $f(x,y,z) = 6 + 4 = 10$.
   (e) $x = y = -\frac{1}{\sqrt{3}}, z = \frac{5}{\sqrt{3}}$: $f(x,y,z) = \frac{6}{\sqrt{3}} - \frac{5}{3\sqrt{3}} = \frac{13}{3\sqrt{3}}$.

(f) $x = y = -\frac{1}{\sqrt{3}}, z = -\frac{5}{\sqrt{3}}$: $f(x, y, z) = -\frac{14}{\sqrt{3}} + \frac{5}{3\sqrt{3}} = -\frac{37}{3\sqrt{3}}$.

(g) $x = y = \frac{1}{\sqrt{3}}, z = \frac{5}{\sqrt{3}}$: $f(x, y, z) = \frac{14}{\sqrt{3}} - \frac{5}{3\sqrt{3}} = \frac{37}{3\sqrt{3}}$.

(h) $x = y = \frac{1}{\sqrt{3}}, z = -\frac{5}{\sqrt{3}}$: $f(x, y, z) = -\frac{6}{\sqrt{3}} + \frac{5}{3\sqrt{3}} = -\frac{13}{3\sqrt{3}}$.

We need not consider the other possibilities like $y = z, z = -2\lambda$ as the situation is symmetric and no new values will result. The greatest value of $f(x, y, z)$ is 10, and the least value is -10.

We don't consider d^2F, as it is not needed. We use the fact the all the maximum and minimum values occur at the extreme values. Calculation of d^2F is a very lengthy process. ∎

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Question 1(a) *If a function $f(x)$ of the real variable x has the first five derivatives 0 at a given value $x = a$, show that it has a maximum or a minimum at $x = a$ according as the sixth derivative is negative or positive. What happens if only the first four derivatives are 0, but not the fifth?*

Solution. We shall take up the general case, which would cover both the cases —

Let $f^{(n)}(a)$ exist and $f^{(n)}(a) \neq 0$. Let $f^{(r)}(a) = 0, 1 \leq r \leq n-1$. Then $f(a)$ is not an extreme value when n is odd, and if n is even, then f has a maximum at $x = a$ if $f^{(n)}(a) < 0$ and a minimum if $f^{(n)}(a) > 0$.

Thus the above result is proved when $n = 6$. When $n = 5$, $f(x)$ has neither maximum nor minimum at $x = a$.

Proof: The existence of $f^{(n)}(a)$ imlies that $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ all exist in a certain neighborhood of a, say $(a - \delta_1, a + \delta_1)$.

Case (1): If $f^{(n)}(a) > 0$, then $f^{(n-1)}(x)$ is increasing at a i.e. there exists $0 < \delta < \delta_1$ such that $f^{(n-1)}(x) < f^{(n-1)}(a) = 0$ for $a - \delta < x < a$, and $f^{(n-1)}(x) > f^{(n-1)}(a) = 0$ for $a < x < a + \delta$.

Case (2): If $f^{(n)}(a) < 0$, then $f^{(n-1)}(x)$ is decreasing at a i.e. there exists $0 < \delta < \delta_1$ such that $f^{(n-1)}(x) > f^{(n-1)}(a) = 0$ for $a - \delta < x < a$, and $f^{(n-1)}(x) < f^{(n-1)}(a) = 0$ for $a < x < a + \delta$.

Let h be such that $|h| < \delta$, then by Taylor's theorem

$$f(a+h) = f(a) + hf'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a + \theta h), \quad 0 < \theta < 1$$

Thus $f(a+h) - f(a) = \dfrac{h^{n-1}}{(n-1)!} f^{(n-1)}(a + \theta h)$ as $f^{(r)}(a) = 0, 1 \leq r \leq n-2$. Since $a + \theta h \in (a - \delta, a + \delta)$, we have the following conclusions:

1. n **even and** $f^{(n)}(a) > 0$: If $h < 0$, then $h^{n-1} < 0$ and $f^{(n-1)}(a+\theta h) < 0$ (by Case (1) above, as $a - \delta < a + \theta h < a$. If $h > 0$, then $h^{n-1} > 0$ and $f^{(n-1)}(a+\theta h) > 0$ (as $a < a + \theta h < a + \delta$.

   In either case $\dfrac{h^{n-1}}{(n-1)!} f^{(n-1)}(a+\theta h) > 0$ for all h with $|h| < \delta$, i.e. $f(a+h) > f(a)$ for $|h| < \delta$, thus $f(x)$ has a minimum at $x = a$.

2. n **even and** $f^{(n)}(a) < 0$: Using Case (2), we get $\dfrac{h^{n-1}}{(n-1)!} f^{(n-1)}(a+\theta h) < 0$ for all h with $|h| < \delta$, i.e. $f(a+h) < f(a)$ for $|h| < \delta$, thus $f(x)$ has a maximum at $x = a$.

3. n **odd and** $f^{(n)}(a) > 0$: If $h < 0$, then $h^{n-1} > 0$ and $f^{(n-1)}(a+\theta h) < 0$, so $f(a+h) < f(a)$ for $h < 0, |h| < \delta$. If $h > 0$, then $h^{n-1} > 0$ and $f^{(n-1)}(a+\theta h) > 0$, so $f(a+h) > f(a)$ for $0 < h < \delta$. Thus f has neither minimum nor maximum at $x = a$.

   The case for n odd and $f^{(n)}(a) < 0$ is similar — $h < 0 \Rightarrow f(a+h) > f(a), h > 0 \Rightarrow f(a+h) < f(a)$, so there is no extreme value at $x = a$.

This completes the proof. ∎

Question 2(a) *Show that*

$$f(x) = (x-2)^2(x^2+2bx+c)(x+3)^3$$

has a critical point at $x = -1 \Leftrightarrow 2b + 5c = 7$.

Solution.

$$\begin{aligned}
f'(x) &= 2(x-2)(x^2+2bx+c)(x+3)^3 + (x-2)^2(2x+2b)(x+3)^3 \\
&\quad + 3(x-2)^2(x^2+2bx+c)(x+3)^2 \\
f'(-1) &= 2(-3)(1-2b+c)8 + 72(-2+2b) + 108(1-2b+c) \\
&= -48 + 96b - 48c - 144 + 144b + 108 - 216b + 108c = -84 + 24b + 60c
\end{aligned}$$

For -1 to be a critical point, $f'(-1) = 0 \Leftrightarrow -84 + 24b + 60c = 0 \Leftrightarrow 2b + 5c = 7$. ∎

Question 2(b) *Assuming that the above condition is satisfied, examine the nature of $f(x)$ at its three critical points.*

Solution.

$f'(x) = 2(x-2)(x^2+2bx+c)(x+3)^3 + (x-2)^2(2x+2b)(x+3)^3 + 3(x-2)^2(x^2+2bx+c)(x+3)^2$

so it follows that the three critical points are $x = 2, x = -3$ and $x = -1$ when $2b + 5c = 7$.

1. $x = 2$.

$$\begin{aligned} f(x) &= (x-2)^2\big[(x^2+2bx+c)(x+3)^3\big] = (x-2)^2 g(x) \\ f'(x) &= 2(x-2)g(x) + (x-2)^2 g'(x) \\ f''(x) &= 2g(x) + 4(x-2)g'(x) + (x-2)^2 g''(x) \\ f''(2) &= 2g(2) = 250(4+4b+c) \\ &= 250(4+14-10c+c) = 250\cdot 9(2-c) \end{aligned}$$

There is a maximum at $x = 2$ if $2 < c$, and a minimum at $x = 2$ if $2 > c$. The test fails when $c = 2$. Considering $f'''(x)$ when $x = 2, c = 2, 2b = -3$,

$$\begin{aligned} f'''(x) &= 2g'(x) + 4g'(x) + 4(x-2)g''(x) + (x-2)^2 g'''(x) + 2(x-2)g''(x) \\ f'''(2) &= 6\big[(4+2b)(2+3)^3 + 3(4+4b+c)(2+3)^2\big] \\ &= 6\big[125 + 75(4-6+2)\big] \neq 0 \end{aligned}$$

so there is no maximum or minimum at $x = 2$ when $c = 2$.

2. $x = -3$

$$\begin{aligned} f(x) &= (x+3)^3\big[(x-2)^2(x^2+2bx+c)\big] = (x+3)^3 h(x) \\ f'(x) &= 3(x+3)^2 h(x) + (x+3)^3 h'(x) \\ f''(x) &= 6(x+3)h(x) + 3(x+3)^2 h'(x) + 3(x+3)^2 h'(x) + (x+3)^3 h''(x) \\ f''(-3) &= 0 \\ f'''(x) &= 6h(x) + 12(x+3)h'(x) + (x+3)^2[\text{some polynomial in } x] \\ f'''(-3) &= 6h(-3) = 150(9-6b+c) = 150(9-21+15c+c) = 600(4c-3) \end{aligned}$$

So if $c \neq \frac{3}{4}$, $f(x)$ does not have a maximum or minimum at $x = -3$. If $c = \frac{3}{4}$, then

$$f^{(4)}(x) = 18h'(x) + (x+3)[\text{some polynomial in } x]$$

so $f^{(4)}(-3) = 18\big[2(-5)(9-6b+c) + 25(-6+2b)\big] = 18(-240+110b-10c) < 0$. Hence if $c = \frac{3}{4}, b = \frac{13}{8}$, $f(x)$ has a maximum at $x = -3$.

3. $x = -1$. Let $y = x+3$. Then we need to consider the value $y = 2$.

$$\begin{aligned} f(y) &= y^3(y-5)^2(y^2+(2b-6)y+9-6b+c) \\ &= y^3(y^2-10y+25)(y^2+(1-5c)y-12+16c) \\ &= y^7 + (1-5c-10)y^6 + (-12+16c+25-10+50c)y^5 \\ &\quad +(120-160c+25-125c)y^4 + (400c-300)y^3 \\ &= y^7 - (9+5c)y^6 + (3+66c)y^5 + (145-285c)y^4 + (400c-300)y^3 \\ f''(y) &= 42y^5 - 30(9+5c)y^4 + 20(3+66c)y^3 + 12(145-285c)y^2 + 6(400c-300)y \\ f''(2) &= 1344 - 480(9+5c) + 160(3+66c) + 48(145-285c) + 12(400c-300) \\ &= 864 - 720c \end{aligned}$$

So if $c > \frac{6}{5}$, then $f''(-1) < 0$ and $f(x)$ has a maximum at $x = -1$. If $c < \frac{6}{5}$, $f(x)$ has a minimum at $x = -1$. If $c = \frac{6}{5}$, then this test fails, we consider higher derivatives at $y = 2$ with $c = \frac{6}{5}$:

$$f'''(y) = 210y^4 - 1800y^3 + 4932y^2 - 4728y + 1080$$

Now it is clear that $f'''(2) \neq 0$, hence there is no maximum or minimum at $x = -1$ when $c = \frac{6}{5}$.

∎

Question 2(c) *Show that $f(x)$ has at least three points of inflection irrespective of the condition $2b + 5c = 7$.*

Solution. From the calculation of $f''(-3) = 0$ whether $2b + 5c = 7$ or not. Since $f(-3) = f(2) = 0$, there exists a real number b between -3 and 2 such that $f'(b) = 0$. Now $f'(2) = f'(b) = f'(-3) = 0$, thus we get $a_1 \in (-3, b), a_2 \in (b, 2)$ where $f''(a_1) = 0, f''(a_2) = 0$. Thus there are least 3 points where $f''(x) = 0$.

Note: This question is not complete: A point of inflection is a point where the curvature of a curve changes sign (the curve goes from convex to concave or vice versa). Any point a is a point of inflection if $f''(a) = 0$ and the lowest order non-zero derivative at a is of odd order, or equivalently, $f''(a + \epsilon)$ and $f''(a - \epsilon)$ have opposite signs in the neighborhood of a.

Thus $f''(x) = 0$ is a necessary condition for a point to be an inflection point, but is not sufficient. $f''(x)$ is a 5th degree curve. Take two of the roots and equate them, and two other roots and equate them too. We can thus get two equations in b, c, which can be solved to get their values. Now $f''(x) = A(x - B)(x - C)^2(x - D)^2$, where B, C, D are distinct. This curve has only one point of inflection, B. For example, the similar curve $g(x) = 35x^3 - 21x^5 + 5x^7$ has only one inflection point, although $g''(x)$ has 3 roots. ∎

Question 3(a) *Prove that the n-th derivative of $f(x)g(x)$ equals*

$$\sum_{i=0}^{n} \binom{n}{i} f^{(n-i)}(x)g^{(i)}(x)$$

where $f^{(m)}$ denotes the m-th derivative of $f(x)$ and $\binom{n}{i}$ are the binomial coefficients.

Solution. The proof is by induction on n. It is true for $n = 1$, as $(fg)' = f'g + fg' = \binom{1}{0} f^{(1)}(x)g^{(0)}(x) + \binom{1}{1} f^{(0)}(x)g^{(1)}(x)$ where $f^{(0)}(x) = f(x)$.

Induction hypothesis: Assume that the result is true for $n = m$.

$$(fg)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)}(x)g^{(i)}(x)$$

Differentiating both sides,

$$\begin{aligned}
(fg)^{(m+1)}(x) &= \sum_{i=0}^{m}\binom{m}{i}\Big(f^{(m-i+1)}(x)g^{(i)}(x)+f^{(m-i)}(x)g^{(i+1)}(x)\Big)\\
&= \binom{m}{0}f^{(m+1)}(x)g^{(0)}(x)+\sum_{i=1}^{m}\left[\binom{m}{i}+\binom{m}{i-1}\right]f^{(m-i+1)}(x)g^{(i)}(x)\\
&\quad +\binom{m}{m}f^{(0)}(x)g^{(m+1)}(x)\\
&= \binom{m+1}{0}f^{(m+1)}(x)g^{(0)}(x)+\sum_{i=1}^{m}\binom{m+1}{i}f^{(m-i+1)}(x)g^{(i)}(x)\\
&\quad +\binom{m+1}{m+1}f^{(0)}(x)g^{(m+1)}(x)\\
&= \sum_{i=0}^{m+1}\binom{m+1}{i}f^{(m+1-i)}(x)g^{(i)}(x)
\end{aligned}$$

We used $\binom{m}{0}=1=\binom{m+1}{0}$, $\binom{m}{m}=1=\binom{m+1}{m+1}$, $\binom{m}{i}+\binom{m}{i-1}=\binom{m+1}{i}$. The result is now established for $n=m+1$, and hence by induction for all n. ∎

Question 3(b) *Show that the m-th derivative $g_m(x)$ of $g(x)=\tan^{-1}x$ satisfies*

$$(1+x^2)g_{m+1}(x)+2mxg_m(x)+m(m-1)g_{m-1}(x)=0$$

and hence is of the form

$$g_m(x)=\frac{(-1)^{m-1}(m-1)!\phi_m(x)}{(1+x^2)^m}$$

where $\phi_m(x)$ is a polynomial of dergree m given by $\phi_1(x)=1, \phi_2(x)=2x$ and the recursion $\phi_{m+1}(x)=2x\phi_m(x)-(1+x^2)\phi_{m-1}(x)$.

Solution. $g_1(x)=\dfrac{1}{1+x^2}$ or $(1+x^2)g_1(x)=1$.

Using Leibnitz's theorem proved above and differentiating the above equation m times, we get

$$\binom{m}{0}(1+x^2)g_{m+1}(x)+\binom{m}{1}2xg_m(x)+\binom{m}{2}2g_{m-1}(x)=0$$

or

$$(1+x^2)g_{m+1}(x)+2mxg_m(x)+m(m-1)g_{m-1}(x)=0$$

which is the required relation.

Now $g_1(x)=\dfrac{1}{1+x^2}=\dfrac{(-1)^{1-1}(1-1)!\phi_1(x)}{(1+x^2)^1}$ as $\phi_1(x)=1$.

$g_2(x) = \dfrac{-2x}{(1+x^2)^2} = \dfrac{(-1)^{2-1}(2-1)!\phi_2(x)}{(1+x^2)^2}$ as $\phi_2(x) = 2x$.

Assume that the result is true for all $n \leq m$. From the relation proved above:

$$(1+x^2)g_{m+1}(x) + 2mx\frac{(-1)^{m-1}(m-1)!\phi_m(x)}{(1+x^2)^m} + m(m-1)\frac{(-1)^{m-2}(m-2)!\phi_{m-1}(x)}{(1+x^2)^{m-1}} = 0$$

or

$$\begin{aligned}
g_{m+1}(x) &= 2mx\frac{(-1)^m(m-1)!\phi_m(x)}{(1+x^2)^{m+1}} + m(m-1)\frac{(-1)^{m-1}(m-2)!\phi_{m-1}(x)}{(1+x^2)^m} \\
&= \frac{(-1)^m m!}{(1+x^2)^{m+1}}\Big[2x\phi_m(x) - (1+x^2)\phi_{m-1}(x)\Big] \\
&= \frac{(-1)^m m!}{(1+x^2)^{m+1}}\phi_{m+1}(x)
\end{aligned}$$

as $\phi_{m+1}(x) = 2x\phi_m(x) - (1+x^2)\phi_{m-1}(x)$. This completes the proof. ∎

Question 3(c) *Find $\phi_m(x)$ for $x \leq 6$. Can you find a general formula?*

Solution.

$$\begin{aligned}
\phi_1(x) &= 1 \\
\phi_2(x) &= 2x \\
\phi_3(x) &= 2x \cdot 2x - (1+x^2) = 3x^2 - 1 \\
\phi_4(x) &= 2x(3x^2-1) - (1+x^2)(2x) = 4x^3 - 4x \\
\phi_5(x) &= 2x(4x^3-4x) - (1+x^2)(3x^2-1) = 5x^4 - 10x^2 + 1 \\
\phi_6(x) &= 2x(5x^4 - 10x^2 + 1) - (1+x^2)(4x^3 - 4x) \\
&= 6x^5 - 20x^3 + 6x
\end{aligned}$$

Writing these down using binomial coefficients:

$$\begin{aligned}
\phi_1(x) &= \binom{1}{0}x^0 \\
\phi_2(x) &= \binom{2}{1}x^1 \\
\phi_3(x) &= \binom{3}{2}x^2 - \binom{3}{0}x^0 \\
\phi_4(x) &= \binom{4}{3}x^3 - \binom{4}{1}x^1
\end{aligned}$$

$$
\begin{aligned}
\phi_5(x) &= \binom{5}{4}x^4 - \binom{5}{2}x^2 + \binom{5}{0}x^0 \\
\phi_6(x) &= \binom{6}{5}x^5 - \binom{6}{3}x^3 + \binom{6}{1}x^1 \\
\phi_n(x) &= \binom{n}{n-1}x^{n-1} - \binom{n}{n-3}x^{n-3} + \binom{n}{n-5}x^{n-5} - \dots \\
&= \sum_{r=0}^{2r+1\leq n} (-1)^r \binom{n}{n-2r-1} x^{n-2r-1} = \sum_{r=0}^{2r+1\leq n} (-1)^r \binom{n}{2r+1} x^{n-2r-1}
\end{aligned}
$$

∎

Question 3(d) *Show that the n-th derivative of $e^x \tan^{-1} x$ at $x = 0$ equals*

$$n - 2!\binom{n}{3} + 4!\binom{n}{5} - 6!\binom{n}{7} + \dots$$

Solution. Let $g(x) = \tan^{-1}(x)$. Using Leibnitz's theorem, we get

$$\frac{d^n}{dx^n}(e^x \tan^{-1} x) = \sum_{r=0}^{n} \binom{n}{r} g_r(x) \frac{d^{n-r} e^x}{dx^{n-r}} = \sum_{r=0}^{n} \binom{n}{r} g_r(x) e^x$$

Thus

$$\frac{d^n}{dx^n}(e^x \tan^{-1} x)\Big)_{x=0} = \sum_{r=0}^{n} \binom{n}{r} g_r(0) = \sum_{r=1}^{n} \binom{n}{r} (-1)^{r-1}(r-1)!\phi_r(0)$$

as $g(0) = 0$ and $g_r(x)$ was computed above in terms of $\phi_r(x)$. Now $\phi_{2r}(0) = 0$, and $\phi_{2r+1} = (-1)^r$. Thus

$$\frac{d^n}{dx^n}(e^x \tan^{-1} x)\Big)_{x=0} = \sum_{r=0}^{2r+1\leq n} \binom{n}{2r+1} (2r)!(-1)^r$$

or

$$\frac{d^n}{dx^n}(e^x \tan^{-1} x)\Big)_{x=0} = 0!\binom{n}{1} - 2!\binom{n}{3} + 4!\binom{n}{5} - 6!\binom{n}{7} + \dots$$

as required. ∎

Paper II

Question 4(a) *Discuss the convergence of the integral $\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$ and if convergent, evaluate it.*

Solution. The integrand $\log \sin x$ has a discontinuity only at $x = 0$. We consider the integral $\int_0^{\frac{\pi}{2}} -\log(\sin x)\, dx$ so that the integrand is positive.

Let $g(x) = x^{-m}, 0 < m < 1$, then

$$\lim_{x \to 0^+} \frac{-\log \sin x}{x^{-m}} = \lim_{x \to 0^+} \frac{-\cos x}{\sin x} \frac{1}{-mx^{-m-1}} = \lim_{x \to 0^+} \frac{x^m \cos x}{m} \frac{x}{\sin x} = 0$$

as L'Hospital's rule applies.

But $\int_0^{\frac{\pi}{2}} g(x)\, dx$ is convergent as $0 < m < 1$, therefore $\int_0^{\frac{\pi}{2}} -\log(\sin x)\, dx$ converges. Now

$$\int_0^{\frac{\pi}{2}} \log(\sin 2x)\, dx = \int_0^{\frac{\pi}{2}} \log 2\, dx + \int_0^{\frac{\pi}{2}} \log(\cos x)\, dx + \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$$

But $\int_0^{\frac{\pi}{2}} \log(\cos x)\, dx = \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$, so

$$2\int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2 + \int_0^{\frac{\pi}{2}} \log(\sin 2x)\, dx$$

By putting $2x = t$.

$$\int_0^{\frac{\pi}{2}} \log(\sin 2x)\, dx = \frac{1}{2} \int_0^{\pi} \log(\sin t)\, dt$$

Now substituting $t = \pi - y$, we have

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \log(\sin t)\, dt &= \int_{\frac{\pi}{2}}^{0} \log(\sin y)\,(-dy) = \int_0^{\frac{\pi}{2}} \log(\sin y)\, dy \\
\therefore \int_0^{\frac{\pi}{2}} \log(\sin 2x)\, dx &= \int_0^{\frac{\pi}{2}} \log(\sin y)\, dy \\
\Rightarrow \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx &= -\frac{\pi}{2} \log 2
\end{aligned}$$

∎

Question 4(b) *Find the point on the parabola $y^2 = 2x, z = 0$ which is nearest to the plane $z = x + 2y + 8$. Show that this distance is $\sqrt{6}$.*

Solution. Let the point $(x, y, 0)$ by on the parabola. Then the distance d from the plane $z = x + 2y + 8$ is given by

$$d = \frac{|x + 2y - 0 + 8|}{\sqrt{1 + 4 + 1}} = \frac{|x + 2y + 8|}{\sqrt{6}}$$

$\Rightarrow 6d^2 = x^2 + 4xy + 4y^2 + 16x + 32y + 64.$

Put $2x = y^2$, and let $F = 6d^2$ to get

$$F = 6d^2 = \frac{y^4}{4} + 2y^3 + 4y^2 + 8y^2 + 32y + 64$$

We need to minimize F. The critical points are given by

$$\begin{aligned} \frac{dF}{dy} &= y^3 + 6y^2 + 24y + 32 = 0 \\ \Rightarrow & \quad (y+2)(y^2 + 4y + 16) = 0 \end{aligned}$$

Thus $y = -2$ is the only real value.

$$\frac{d^2F}{dy^2} = 3y^2 + 12y + 24 = 12 - 24 + 24 > 0$$

at $y = -2$. Hence $y = -2$ is a minimum.

When $y = -2, x = 2$, and $d = \frac{|2-4+8|}{\sqrt{6}} = \sqrt{6}$ as required. ∎

Question 4(c) *Show that*

$$\iiint (a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2)^{\frac{1}{2}}\, dx\, dy\, dz$$

over the volume bounded by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ *is equal to* $\dfrac{\pi^2 a^2 b^2 c^2}{4}$.

Solution. Let

$$\begin{aligned} I &= \iiint_{\{(x,y,z)\ |\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\}} (a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2)^{\frac{1}{2}}\, dx\, dy\, dz \\ &= 8abc \iiint_{\{(x,y,z)\ |\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1, x\ge 0, y\ge 0, z\ge 0\}} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)^{\frac{1}{2}} dx\, dy\, dz \end{aligned}$$

Let $x = aX, y = bY, z = cZ$, so that $dx\, dy\, dz = abc\, dX\, dY\, dZ$

$$I = 8a^2b^2c^2 \iiint_{\substack{X^2+Y^2+Z^2\le 1 \\ X\ge 0, Y\ge 0, Z\ge 0}} (1 - X^2 - Y^2 - Z^2)^{\frac{1}{2}}\, dX\, dY\, dZ$$

Let $X^2 = u, Y^2 = v, Z^2 = w$, so that $dX = \frac{du}{2\sqrt{u}}, dY = \frac{dv}{2\sqrt{v}}, dZ = \frac{dw}{2\sqrt{w}}$ and

$$I = a^2b^2c^2 \iiint_{\substack{u+v+w\le 1 \\ u\ge 0, v\ge 0, w\ge 0}} (1 - u - v - w)^{\frac{1}{2}} u^{-\frac{1}{2}} v^{-\frac{1}{2}} w^{-\frac{1}{2}}\, du\, dv\, dw$$

We now convert this to a Dirichlet integral — let $u + v + w = \alpha, v + w = \alpha\beta, w = \alpha\beta\gamma$, so that $\dfrac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\alpha^2\beta$ and

$$\begin{aligned}
I &= a^2b^2c^2 \int_0^1 \int_0^1 \int_0^1 (1-\alpha)^{\frac{1}{2}} (\alpha - \alpha\beta)^{-\frac{1}{2}} (\alpha\beta - \alpha\beta\gamma)^{-\frac{1}{2}} (\alpha\beta\gamma)^{-\frac{1}{2}} \alpha^2 \beta \, d\alpha \, d\beta \, d\gamma \\
&= a^2b^2c^2 \int_0^1 \int_0^1 \int_0^1 (1-\alpha)^{\frac{1}{2}} \alpha^{\frac{1}{2}} (1-\beta)^{-\frac{1}{2}} (1-\gamma)^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} \, d\alpha \, d\beta \, d\gamma \\
&= a^2b^2c^2 \, \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} \, \frac{\Gamma(\frac{1}{2})\Gamma(1)}{\Gamma(\frac{3}{2})} \, \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \\
&= a^2b^2c^2 \, \frac{\frac{1}{2}\Gamma(\frac{1}{2})^4}{2} = \frac{\pi^2 a^2 b^2 c^2}{4}
\end{aligned}$$

as required, as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. ∎

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Question 1(a) *Sketch the curve* $(x^2 - a^2)(y^2 - b^2) = a^2b^2$.

Solution. We note the following points:

1. The curve is symmetrical about both the coordinate axes. Enough to consider when $x \geq 0, y \geq 0$.

2. $(0, 0)$ lies on the curve.

3. The coefficient of the highest power of x is $y^2 - b^2$. Thus $y = \pm b$ are two asymptotes parallel to the x-axis. The coefficient of the highest power of y is $x^2 - a^2$, thus $x = \pm a$ are two asymptotes parallel to the y-axis. For other asymptotes of the form $y = mx + c$, we know that $\lim_{x\to\infty} \frac{y}{x} = m$. We divide by x^4 and let $x \to \infty$ to obtain $\lim\limits_{x\to\infty} \Big(1 - \frac{a^2}{x^2}\Big)\Big(\frac{y^2}{x^2} - \frac{b^2}{x^2}\Big) - \frac{a^2b^2}{x^4} = m^2 = 0$ so the only asymptotes of the form $y = mx + c$ have $m = 0$, and we have already found those. Thus the curve has only 4 asymptotes.

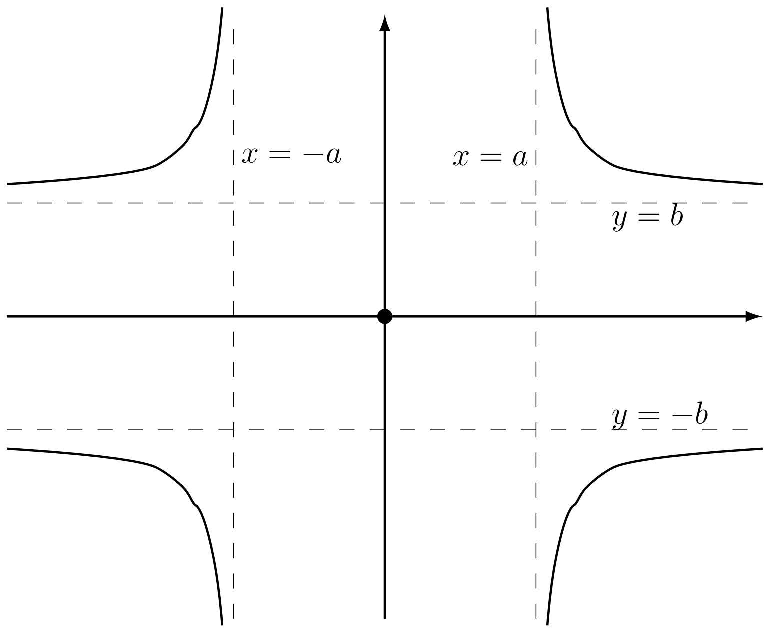
4. $y^2 - b^2 = \dfrac{a^2b^2}{x^2 - a^2} \Rightarrow y^2 = \dfrac{a^2b^2}{x^2 - a^2} + b^2 = \dfrac{b^2x^2}{x^2 - a^2} \Rightarrow y = \pm\dfrac{bx}{\sqrt{x^2 - a^2}}$. Since we need to trace only in the first quadrant, we let $y = \dfrac{bx}{\sqrt{x^2 - a^2}}$.

5. $\dfrac{dy}{dx} = \dfrac{b}{\sqrt{x^2 - a^2}} - \dfrac{bx^2}{(x^2 - a^2)^{\frac{3}{2}}} = \dfrac{-a^2b}{(x^2 - a^2)^{\frac{3}{2}}}$, showing that the curve has no critical points.

6. If $x^2 - a^2 < 0$, then $(x^2 - a^2)y^2 = b^2x^2 \Rightarrow x = 0, y = 0$. Hence $x^2 - a^2 > 0$, so the curve lies beyond the line $x = \pm a$. Similarly it lies beyond the lines $y = \pm b$. $(0, 0)$ is an isolated point.

7. $\dfrac{d^2y}{dx^2} = \frac{3}{2}a^2b(x^2 - a^2)^{-\frac{5}{2}}2x > 0$ for $x > a$, so the curve is downwards convex.

A sketch is shown here:



∎

Question 1(b) *Find the cubic curve which has the same asymptotes as the curve*

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$$

and which passes through the points $(0, 0), (1, 0)$ *and* $(0, 1)$.

Solution. Consider the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + ax + by + c = 0$. Since it has the same terms of degree 3 and 2 as the given curve, it has the same asymptotes. Now since it must pass through $(0, 0)$, $c = 0$. Since it passes through $(1, 0)$, $1 + a = 0 \Rightarrow a = -1$, and since it passes through $(0, 1)$, $-6 + b = 0 \Rightarrow b = 6$. Hence the required curve is $x^3 - 6x^2y + 11xy^2 - 6y^3 - x + 6y = 0$. ∎

Question 1(c) *Show that the function* $f(x, y) = y^2 + x^2y + x^4$ *has* $(0, 0)$ *as the only critical point and* $f(x, y)$ *has a minimum at that point.*

Solution. The critical points are given by $\dfrac{\partial f}{\partial x} = 2xy + 4x^3 = 0 \Rightarrow x = 0, y = -2x^2$, $\dfrac{\partial f}{\partial y} = 2y + x^2 = 0 \Rightarrow y = -\dfrac{x^2}{2}$.

When $x = 0$, $y = -\frac{x^2}{2} \Rightarrow y = 0$.

$y = -2x^2, y = -\frac{x^2}{2} \Rightarrow x = 0, y = 0$. Hence $(0, 0)$ is the only critical point.

Now $\dfrac{\partial^2 f}{\partial x^2} = 2y + 12x^2 = 0$ at $(0,0)$. $\dfrac{\partial^2 f}{\partial y^2} = 2$ and $\dfrac{\partial^2 f}{\partial x\,\partial y} = 2x = 0$ at $(0,0)$. Thus

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \Big(\frac{\partial^2 f}{\partial x\,\partial y}\Big)^2 = 0 \text{ at } (0,0)$$

so we cannot say whether it is maximum or minimum. However $f(x,y) = (x^2 + \frac{y}{2})^2 + \frac{3}{4}y^2 \geq 0$ for all $(x,y) \in \mathbb{R}^2$, so $f(x,y)$ has a minimum at $(0,0)$. ∎

Question 1(d) *Find the percentage error in the volume of a right circular cone when an error of 1% is made in measuring the height and an error of 0.5% is made in measuring the base radius.*

Solution.

$$\begin{aligned} V &= \frac{\pi}{3}r^2 h \\ \log V &= \log\frac{\pi}{3} + 2\log r + \log h \\ \frac{dV}{V} &= 2\frac{dr}{r} + \frac{dh}{h} \end{aligned}$$

Now $\dfrac{dr}{r} = 0.5\% = 0.005, \dfrac{dh}{h} = 1\% = 0.01$. Thus $\dfrac{dV}{V} = 0.02 = 2\%$. Hence the error in measuring the volume is 2%. ∎

Question 2(a) *Evaluate* $\displaystyle\iint_D F(x+y)x^{m-1}y^{n-1}\,dx\,dy$ *with* $F(u) = (1-u)^{l-1}$ *where* D *is the interior of the triangle formed by* $x = 0, y = 0, x + y = 1$ *and* l, m, n *are all positive.*

Solution. (Note: This is Dirichlet's integral.)

Put $u = x + y, uv = x \Rightarrow y = u - uv$, so that $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$.

$$\begin{aligned} I &= \int_0^1\int_0^1 F(u)(uv)^{m-1}u^{n-1}(1-v)^{n-1}u\,du\,dv \\ &= \int_0^1\int_0^1 F(u)u^{m+n-1}v^{m-1}(1-v)^{n-1}\,du\,dv \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\int_0^1 F(u)u^{m+n-1}\,du \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\int_0^1 (1-u)^{l-1}u^{m+n-1}\,du \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\,\frac{\Gamma(l)\Gamma(m+n)}{\Gamma(m+n+l)} = \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)} \end{aligned}$$

∎

Question 2(b) *Find the center of gravity of the solid bounded by the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$ *and lying in the positive octant.*

Solution. Let S be the positive octant of the ellipsoid. Let $M = \displaystyle\iiint_S dx\,dy\,dz$.

Then $\overline{x} = \dfrac{1}{M}\displaystyle\iiint_S x\,dx\,dy\,dz,\ \overline{y} = \dfrac{1}{M}\displaystyle\iiint_S y\,dx\,dy\,dz,\ \overline{z} = \dfrac{1}{M}\displaystyle\iiint_S z\,dx\,dy\,dz.$

Now put $x = aX, y = bY, z = cZ$ so that

$$M = abc\iiint_{\substack{X\geq 0,Y\geq 0,Z\geq 0,\\ X^2+Y^2+Z^2\leq 1}} dX\,dY\,dZ = \frac{\pi}{6}abc$$

because the volume of a sphere of radius 1 is $\frac{4\pi}{3}$.

$$\overline{x} = \frac{6a^2bc}{\pi abc}\iiint_{\substack{X\geq 0,Y\geq 0,Z\geq 0,\\ X^2+Y^2+Z^2\leq 1}} X\,dX\,dY\,dZ$$

Put $X^2 = u, Y^2 = v, Z^2 = w$. Then

$$\begin{aligned}
\overline{x} &= \frac{6a}{\pi}\iiint_{D:\left\{\substack{u\geq 0,v\geq 0,w\geq 0,\\ u+v+w\leq 1}\right\}} u^{\frac{1}{2}}\,\frac{1}{2}u^{-\frac{1}{2}}\,\frac{1}{2}v^{-\frac{1}{2}}\,\frac{1}{2}w^{-\frac{1}{2}}\,du\,dv\,dw \\
&= \frac{3a}{4\pi}\iiint_D u^0 v^{-\frac{1}{2}}w^{-\frac{1}{2}}\,du\,dv\,dw = \frac{3a}{4\pi}\iiint_D u^{1-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw \\
&= \frac{3a}{4\pi}\,\frac{\Gamma(1)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{3a}{4}
\end{aligned}$$

as $\Gamma(2) = \Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$. Note that in evaluating $\iiint_D u^{1-1}v^{\frac{1}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw$ we have used Dirichlet's integral discussed above.

Similarly $\overline{y} = \frac{3b}{4}, \overline{z} = \frac{3c}{4}$, so the center of gravity is $\left(\dfrac{3a}{4}, \dfrac{3b}{4}, \dfrac{3c}{4}\right)$. ∎

Question 2(c) *Prove, by considering the integral*

$$\iint_E x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy$$

where E *is the square* $[0, R, 0, R]$*, or otherwise, that* $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Solution. Let C_1 be the part of the circle $x^2+y^2 \leq R^2$ lying in the first quadrant, and C_2 be the part of the circle $x^2+y^2 \leq 2R^2$ lying in the first quadrant. Then

$$\iint_{C_1} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy \leq \iint_E x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy \leq \iint_{C_2} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy$$

Put $x = r\cos\theta, y = r\sin\theta$, so that $dx\,dy = r\,dr\,d\theta$. Then

$$\begin{aligned}
\iint_{C_1} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy &= \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta \int_0^R r^{2m+2n-1}e^{-r^2}\,dr \\
\iint_{C_2} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy &= \int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta \int_0^{\sqrt{2}R} r^{2m+2n-1}e^{-r^2}\,dr
\end{aligned}$$

Consider $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$. Put $x = \cos^2\theta, dx = -2\cos\theta\sin\theta\,d\theta$, so

$$B(m,n) = \int_{\frac{\pi}{2}}^0 \cos^{2m-2}\theta\sin^{2n-2}\theta(-2\cos\theta\sin\theta)\,d\theta = 2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\sin^{2n-1}\theta\,d\theta$$

Consider $\Gamma(m) = \int_0^\infty t^{m-1}e^{-t}\,dt$. Put $t = r^2$, so that

$$\Gamma(m) = \int_0^\infty r^{2m-2}e^{-r^2}2r\,dr = 2\int_0^\infty r^{2m-1}e^{-r^2}\,dr$$

Thus

$$\lim_{R\longrightarrow\infty} \iint_{C_1} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy = \frac{B(m,n)}{2}\,\frac{\Gamma(m+n)}{2}$$

and

$$\lim_{R\longrightarrow\infty} \iint_{C_2} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy = \frac{B(m,n)}{2}\,\frac{\Gamma(m+n)}{2}$$

Thus

$$\lim_{R\longrightarrow\infty} \iint_{E} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy = \frac{B(m,n)}{2}\,\frac{\Gamma(m+n)}{2}$$

However

$$\lim_{R\longrightarrow\infty} \iint_{E} x^{2m-1}y^{2n-1}e^{-x^2-y^2}\,dx\,dy = \int_0^R x^{2m-1}e^{-x^2}\,dx \int_0^R y^{2n-1}e^{-y^2}\,dy = \frac{\Gamma(m)\Gamma(n)}{4}$$

Hence $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ as required. ∎

Paper II

Question 3(a) *Evaluate* $\displaystyle\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$.

Solution. Let I be the given integral, then regarding I as a function of a, we get

$$\frac{dI}{da} = \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(1+b^2x^2)}\,dx$$

as the conditions for differentiating under the integral are satisfied.

$$\begin{aligned}
\frac{dI}{da} &= \int_0^\infty \frac{2ax^2}{(1+a^2x^2)(1+b^2x^2)}\,dx = \frac{2a}{b^2-a^2}\int_0^\infty \Big(\frac{1}{1+a^2x^2} - \frac{1}{1+b^2x^2}\Big)dx \\
&= \frac{2a}{b^2-a^2}\left[\frac{\tan^{-1} ax}{a} - \frac{\tan^{-1} bx}{b}\right]_0^\infty \\
&= \frac{2a}{b^2-a^2}\frac{\pi}{2}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{\pi}{b(b+a)}
\end{aligned}$$

Thus $I = \dfrac{\pi}{b}\log(b+a) + C$. Since $I = 0$ when $a = 0$, $C = -\dfrac{\pi}{b}\log(b)$ and therefore $I = \dfrac{\pi}{b}\log\Big(\dfrac{b+a}{b}\Big)$ ∎

Question 4(a) *If the rectangular axes are rotated through an angle α about the origin and the new coordinates are $(\overline{x}, \overline{y})$, then show that for any u,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \overline{x}^2} + \frac{\partial^2 u}{\partial \overline{y}^2}$$

Solution. From standard coordinate geometry:

$$\begin{aligned}
x &= \overline{x}\cos\alpha - \overline{y}\sin\alpha \\
y &= \overline{x}\sin\alpha + \overline{y}\cos\alpha
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial u}{\partial \overline{x}} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \overline{x}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \overline{x}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha \\
\frac{\partial^2 u}{\partial \overline{x}^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\Big)\frac{\partial x}{\partial \overline{x}} + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\Big)\frac{\partial y}{\partial \overline{x}} \\
&= \Big(\frac{\partial^2 u}{\partial x^2}\cos\alpha + \frac{\partial^2 u}{\partial x\,\partial y}\sin\alpha\Big)\cos\alpha + \Big(\frac{\partial^2 u}{\partial y\,\partial x}\cos\alpha + \frac{\partial^2 u}{\partial y^2}\sin\alpha\Big)\sin\alpha \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2\alpha + \frac{\partial^2 u}{\partial x\,\partial y}\cos\alpha\sin\alpha + \frac{\partial^2 u}{\partial y\,\partial x}\cos\alpha\sin\alpha + \frac{\partial^2 u}{\partial y^2}\sin^2\alpha \\
\frac{\partial u}{\partial \overline{y}} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \overline{y}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \overline{y}} = \frac{\partial u}{\partial x}(-\sin\alpha) + \frac{\partial u}{\partial y}\cos\alpha \\
\frac{\partial^2 u}{\partial \overline{y}^2} &= \frac{\partial^2 u}{\partial x^2}\sin^2\alpha - \frac{\partial^2 u}{\partial x\,\partial y}\cos\alpha\sin\alpha - \frac{\partial^2 u}{\partial y\,\partial x}\cos\alpha\sin\alpha + \frac{\partial^2 u}{\partial y^2}\cos^2\alpha
\end{aligned}$$

Thus

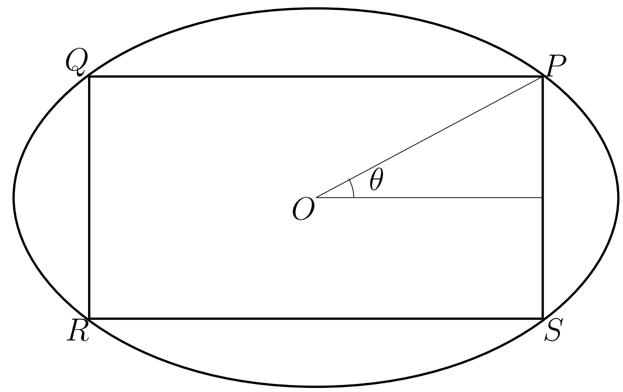
$$\frac{\partial^2 u}{\partial \overline{x}^2} + \frac{\partial^2 u}{\partial \overline{y}^2} = \frac{\partial^2 u}{\partial x^2}(\cos^2\alpha + \sin^2\alpha) + \frac{\partial^2 u}{\partial y^2}(\cos^2\alpha + \sin^2\alpha) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

as required. ∎

Question 4(b) *A rectangle is inscribed in the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$*, what is the maximum possible area of the rectangle?*

Solution.

If P is the point $(a\cos\theta, b\sin\theta)$, the area of the rectangle is $A = 4ab\cos\theta\sin\theta = 2ab\sin 2\theta$, as the sides of the rectangle are $2a\cos\theta, 2b\sin\theta$. Clearly the area is maximum when $\sin 2\theta$ is maximum i.e. $\sin 2\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$. The maximum area is $2ab$.



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UPSC Civil Services Main 1992 - Mathematics Calculus

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Question 1(a) *If $y = e^{ax}\cos bx$, prove that $y_2 - 2ay_1 + (a^2+b^2)y = 0$ and hence expand $e^{ax}\cos bx$ in powers of x. Deduce the expansion of e^{ax} and $\cos bx$.*

Solution.

$$\begin{aligned}
y &= e^{ax}\cos bx \Rightarrow y(0) = 1 \\
y_1 &= e^{ax}(a\cos bx - b\sin bx) \Rightarrow y_1(0) = a \\
y_2 &= ae^{ax}(a\cos bx - b\sin bx) + e^{ax}(-ab\sin bx - b^2\cos bx) \\
&= e^{ax}((a^2-b^2)\cos bx - 2ab\sin bx) \\
\text{Thus} \quad & y_2 - 2ay_1 + (a^2+b^2)y \\
&= e^{ax}((a^2-b^2)\cos bx - 2ab\sin bx - 2a(a\cos bx - b\sin bx) + (a^2+b^2)\cos bx) = 0
\end{aligned}$$

as required.

Thus $y_2 = 2ay_1 - (a^2+b^2)y$. Differentiating $n-2$ times, $y_n = 2ay_{n-1} - (a^2+b^2)y_{n-2}$. Thus we have a recurrence relation for y_n, which we can use to compute $y_n(0)$.

$$\begin{aligned}
y(0) &= 1 \\
y_1(0) &= a \\
y_2(0) &= 2ay_1(0) - (a^2+b^2)y(0) = a^2 - b^2 \\
y_3(0) &= 2ay_2(0) - (a^2+b^2)y_1(0) = 2a^3 - 2ab^2 - a^3 - ab^2 = a^3 - 3ab^2 \\
y_4(0) &= 2a^4 - 6a^2b^2 - a^4 + b^4 = a^4 - 6a^2b^2 + b^4 \\
&\ldots \\
y_n(0) &= \sum_{r=0}^{r\leq\frac{n}{2}} (-1)^r \binom{n}{2r} a^{n-2r}b^{2r}
\end{aligned}$$

The last formula can be proved by induction — the coefficient of $a^{n-2r}b^{2r}$ in $y_n(0)$ from the RHS of the recurrence relation is

$$\begin{aligned}
& 2(-1)^r\binom{n-1}{2r} - (-1)^r\binom{n-2}{2r} - (-1)^{r-1}\binom{n-2}{2r-2} \\
= \quad & (-1)^r\left[2\binom{n-1}{2r} - \binom{n-2}{2r} - \binom{n-2}{2r-1} + \binom{n-2}{2r-1} + \binom{n-2}{2r-2}\right] \\
= \quad & (-1)^r\left[2\binom{n-1}{2r} - \binom{n-1}{2r} + \binom{n-1}{2r-1}\right] \\
= \quad & (-1)^r\left[\binom{n-1}{2r} + \binom{n-1}{2r-1}\right] \\
= \quad & (-1)^r\binom{n}{2r}
\end{aligned}$$

Now by the Taylor-Maclaurin formula

$$y = e^{ax}\cos bx = \sum_{n=0}^{\infty} y_n(0)\frac{x^n}{n!}$$

where the y_n are given as above.

Putting $b = 0$, all terms except the first in each of $y_n(0)$ vanish, so

$$e^{ax} = \sum_{n=0}^{\infty} a^n \frac{x^n}{n!}$$

Putting $a = 0$, all terms in the odd $y_n(0)$ vanish, and all but the last in the even ones vanish:

$$\cos bx = \sum_{r=0}^{\infty}(-1)^r b^{2r}\frac{x^{2r}}{(2r)!}$$

Note: It is possible to expand $e^{ax}\cos bx$ in powers of x directly without using the formula $y_2 = 2ay_1 - (a^2+b^2)y$ as shown below — however in this question we are required to use the formula and proceed as above.

Put $a = r\cos\theta, b = r\sin\theta$, so that

$$\begin{aligned}
y_1 &= e^{ax}r(\cos\theta\cos bx - \sin\theta\sin bx) = re^{ax}\cos(bx+\theta) \\
y_2 &= re^{ax}r(\cos\theta\cos(bx+\theta) - \sin\theta\sin(bx+\theta)) = r^2e^{ax}\cos(bx+2\theta) \\
& \ldots \\
y_n &= r^n e^{ax}\cos(bx+n\theta)
\end{aligned}$$

Thus $y_n(0) = r^n\cos n\theta$. Using the standard formula for $\cos n\theta$, and then using $a = r\cos\theta, b = r\sin\theta$, we see that this is the same as the above result (or alternatively using these results we have derived the standard formula for $\cos n\theta$). ∎

Question 1(b) *If $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$ then prove that*

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

Solution.

$$\begin{aligned} dx &= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \\ dy &= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta \end{aligned}$$

We square and add these three equations.

$$\begin{aligned} \text{Coefficient of } dr^2 &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1 \\ \text{Coefficient of } d\theta^2 &= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2 \\ \text{Coefficient of } d\phi^2 &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta \\ \text{Coefficient of } 2\, dr\, d\theta &= r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - r\sin\theta\cos\theta = 0 \\ \text{Coefficient of } 2\, dr\, d\phi &= -r\sin^2\theta\cos\phi\sin\phi + r\sin^2\theta\cos\phi\sin\phi = 0 \\ \text{Coefficient of } 2\, d\theta\, d\phi &= -r^2\cos\theta\cos\phi\sin\theta\sin\phi + r^2\cos\theta\sin\phi\sin\theta\cos\phi = 0 \end{aligned}$$

Thus $dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$. ∎

Question 1(c) *Find the dimensions of the rectangular parallelopiped inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ that has the greatest volume.*

Solution. See 1997 question 1(b). The dimensions are $\frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}, \frac{2c}{\sqrt{3}}$. ∎

Question 2(a) *Prove that the volume enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{128}{15}a^3$.*

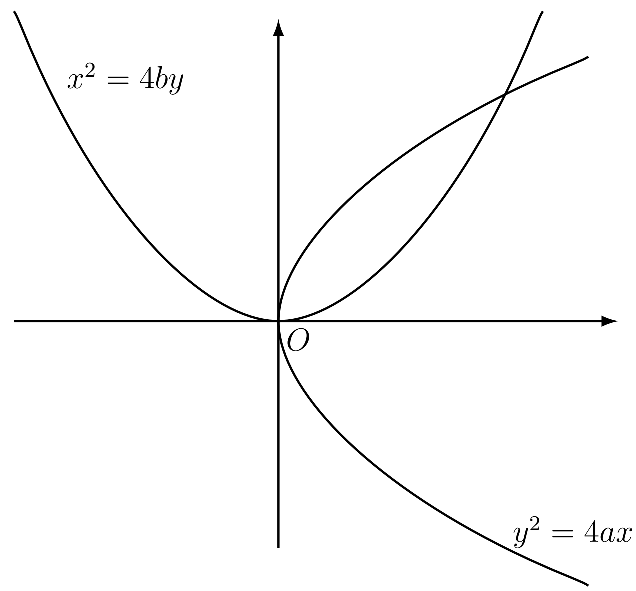
Solution. Clearly on the given volume z varies from $-\sqrt{2ax}$ to $\sqrt{2ax}$, y varies from $-\sqrt{2ax - x^2}$ to $\sqrt{2ax - x^2}$ and x varies from 0 to $2a$ (note that $x^2 + y^2 = 2ax$ is a circle in the XY-plane in which x varies from 0 to $2a$).

$$\begin{aligned} V &= \int_0^{2a} \int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \int_{-\sqrt{2ax}}^{\sqrt{2ax}} dz\, dy\, dx = 2\int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \sqrt{2ax}\, dy\, dx \\ &= 4\int_0^{2a} \sqrt{2ax}\sqrt{2ax - x^2}\, dx = 4\sqrt{2a}\int_0^{2a} x\sqrt{2a - x}\, dx \\ & \quad \text{Put } x = 2a\sin^2\theta \\ &= 4\sqrt{2a}\int_0^{\frac{\pi}{2}} 2a\sin^2\theta\sqrt{2a}\cos\theta\, 4a\sin\theta\cos\theta\, d\theta \\ &= 64a^3\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\, d\theta = 64a^3\frac{2\cdot 1}{5\cdot 3\cdot 1} = \frac{128}{15}a^3 \end{aligned}$$

∎

Question 2(b) *Find the center of gravity of the volume formed by revolving the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4by$ about the x-axis.*

Solution.



To find the intersection points of the parabolas: $x^4 = 16b^2(4ax) \Rightarrow x = 0, x^3 = 64ab^2 \Rightarrow x = 0, 4(ab^2)^{\frac{1}{3}}$ Thus the centroid is given by $(\overline{x}, 0)$, where

$$\begin{aligned}
\overline{x} &= \frac{\int_0^{4(ab^2)^{\frac{1}{3}}} x(y_1^2 - y_2^2)\,dx}{\int_0^{4(ab^2)^{\frac{1}{3}}} (y_1^2 - y_2^2)\,dx} \\
&= \frac{\int_0^{4(ab^2)^{\frac{1}{3}}} x\big(4ax - \frac{x^4}{16b^2}\big)\,dx}{\int_0^{4(ab^2)^{\frac{1}{3}}} \big(4ax - \frac{x^4}{16b^2}\big)\,dx} \\
&= \frac{\left[\frac{4ax^3}{3} - \frac{x^6}{96b^2}\right]_0^{4(ab^2)^{\frac{1}{3}}}}{\left[2ax^2 - \frac{x^5}{80b^2}\right]_0^{4(ab^2)^{\frac{1}{3}}}}
\end{aligned}$$

$$= \frac{\frac{4a \cdot 64ab^2}{3} - \frac{(64ab^2)^2}{96b^2}}{16(ab^2)^{\frac{2}{3}}\left(2a - \frac{64ab^2}{80b^2}\right)} = \frac{\frac{256}{3}a^2b^2 - \frac{256a^2b^4}{6b^2}}{16(ab^2)^{\frac{2}{3}}\left(2a - \frac{4a}{5}\right)} = \frac{128a^2b^2 \cdot 5}{48(ab^2)^{\frac{2}{3}}6a} = \frac{20}{9}(ab^2)^{\frac{1}{3}}$$

Hence the center of gravity is $\left(\frac{20}{9}a^{\frac{1}{3}}b^{\frac{2}{3}}, 0\right)$. ∎

Question 2(c) *Evaluate the following integral in terms of the Gamma function*

$$\int_{-1}^{1} (1+x)^p(1-x)^q\,dx \qquad\qquad (p > -1, q > -1)$$

and prove that $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{2\pi}{\sqrt{3}}$.

Solution. Consider the integral $\displaystyle\int_a^b (x-a)^m(b-x)^n\,dx$. Put

$$\begin{aligned}
x &= a\sin^2\theta + b\cos^2\theta \\
\Rightarrow \quad dx &= (2a\sin\theta\cos\theta - 2b\cos\theta\sin\theta)d\theta = 2(a-b)\sin\theta\cos\theta\,d\theta \\
x - a &= b\cos^2\theta - a(1 - \sin^\theta) = (b-a)\cos^2\theta \\
b - x &= b(1 - \cos^2\theta) - a\sin^2\theta = (b-a)\sin^2\theta
\end{aligned}$$

and $x = a \Rightarrow \theta = \frac{\pi}{2}, x = b \Rightarrow \theta = 0$. Thus

$$\begin{aligned}
I &= 2\int_{\frac{\pi}{2}}^{0} (b-a)^{m+n+1} \cos^{2m}\theta \sin^{2n}\theta(-\sin\theta\cos\theta)d\theta \\
&= 2(b-a)^{m+n+1} \int_0^{\frac{\pi}{2}} \cos^{2m+1}\theta \sin^{2n+1}\theta\, d\theta \\
&= (b-a)^{m+n+1} \frac{\Gamma(\frac{2m+1}{2}+\frac{1}{2})\Gamma(\frac{2n+1}{2}+\frac{1}{2})}{\Gamma(\frac{2m+1}{2}+\frac{2n+1}{2}+1)} \\
&= (b-a)^{m+n+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} = (b-a)^{m+n+1} B(m+1, n+1)
\end{aligned}$$

Substituting $a = -1, b = 1, m = p, n = q$ we have

$$\int_{-1}^{1} (1+x)^p (1-x)^q \, dx = 2^{p+q+1} \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$$

Now put $p = -\frac{2}{3}, q = -\frac{1}{3}$ to get

$$\frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(1)} = \int_{-1}^{1} (1+x)^{-\frac{2}{3}} (1-x)^{-\frac{1}{3}} \, dx = 2\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{1}{3}}\theta}{\cos^{\frac{1}{3}}\theta} \, d\theta$$

Putting $\tan\theta = z$,

$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = 2\int_0^{\infty} z^{\frac{1}{3}} \frac{dz}{1+z^2}$$

Put $z^2 = t$ to get

$$\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = 2\int_0^{\infty} \frac{t^{\frac{1}{6}}}{1+t} \frac{dt}{2t^{\frac{1}{2}}} = \int_0^{\infty} \frac{t^{\frac{2}{3}-1}}{1+t} \, dt = \frac{2\pi}{\sqrt{3}}$$

by proceeding as 1994, question 2(a). ∎

Paper II

Question 3(a) *Examine* $f(x, y, z) = 2xyz - 4xz - 2yz + x^2 + y^2 + z^2 - 2x - 4y - 4z$ *for extreme values.*

Solution. For extreme values,

$$\begin{array}{lclcll}
f_x &=& 2yz - 4z + 2x - 2 = 0 & \Rightarrow & yz - 2z + x - 1 = 0 & (i) \\
f_y &=& 2zx - 2z + 2y - 4 = 0 & \Rightarrow & xz - z + y - 2 = 0 & (ii) \\
f_z &=& 2xy - 4x - 2y + 2z - 4 = 0 & \Rightarrow & xy - 2x - y + z - 2 = 0 & (iii)
\end{array}$$

Subtracting (ii) from (i) we get

$$z(y - x - 1) + x - y + 1 = 0 \Rightarrow (x - y + 1)(1 - z) = 0 \Rightarrow z = 1 \text{ or } x - y + 1 = 0$$

Putting $z = 1$ in (i) and (iii) we get $2y + 2x - 6 = 0 \Rightarrow x = 3 - y$ and $xy - 2x - y - 1 = 0 \Rightarrow y(3-y) - 2(3-y) - y - 1 = 0 \Rightarrow y^2 - 4y + 7 = 0$. This gives imaginary values, showing that $z = 1$ is not possible.

Putting $x = y - 1$ in equation (iii) we get $y(y-1) - 2(y-1) - y + z - 2 = 0 \Rightarrow y^2 - 4y + z = 0 \Rightarrow z = 4y - y^2$. Substituting $x = y - 1, z = 4y - y^2$ in (i) we get $y(4y - y^2) - 2(4y - y^2) + y - 1 - 1 = 0 \Rightarrow 4y^2 - y^3 - 8y + 2y^2 + y - 2 = 0 \Rightarrow y^3 - 6y^2 + 7y + 2 = 0$. Factorizing this, $(y-2)(y^2 - 4y - 1) = 0 \Rightarrow y = 2, 2 \pm \sqrt{5}$.

Case 1: $y = 2, x = 1, z = 4$. $f_{xx} = 2, f_{yy} = 2, f_{zz} = 2, f_{xy} = 2z, f_{xz} = 2y - 4, f_{yz} = 2x - 2$. Thus the matrix associated with the quadratic form

$$d^2 f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{xz}\,dx\,dz + 2f_{yz}\,dy\,dz$$

when $x = 1, y = 2, z = 4$ is given by

$$\begin{pmatrix} 2 & 8 & 0 \\ 8 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

of which the principle minors are $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 8 \\ 8 & 2 \end{vmatrix}$ or $4, -60$, showing that the quadratic form $d^2 f$ is indefinite. Thus the point $(1, 2, 4)$ is neither maximum nor minimum.

Case 2: $x = 1 + \sqrt{5}, y = 2 + \sqrt{5}, z = y(4-y) = (2+\sqrt{5})(2-\sqrt{5}) = -1$. The matrix B associated with $d^2 f$ in this case is

$$\begin{pmatrix} 2 & -2 & 2\sqrt{5} \\ -2 & 2 & 2\sqrt{5} \\ 2\sqrt{5} & 2\sqrt{5} & 2 \end{pmatrix}$$

which again is indefinite as one principle minor is $\begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} = 0$. Thus $(1+\sqrt{5}, 2+\sqrt{5}, -1)$ is also not a maximum or minimum.

Case 3: $x = 1 - \sqrt{5}, y = 2 - \sqrt{5}, z = y(4-y) = (2-\sqrt{5})(2+\sqrt{5}) = -1$. The matrix B associated with $d^2 f$ in this case is

$$\begin{pmatrix} 2 & -2 & -2\sqrt{5} \\ -2 & 2 & -2\sqrt{5} \\ -2\sqrt{5} & -2\sqrt{5} & 2 \end{pmatrix}$$

which again is indefinite for the same reason as Case 2. Thus $(1-\sqrt{5}, 2-\sqrt{5}, -1)$ is also not a maximum or minimum. ∎

Note: Consider the similar question: *Examine* $f(x,y,z) = 2xyz - 4xz - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$ *for extreme values.*

For extreme values,

$$\begin{array}{lllll} f_x & = & 2yz - 4z + 2x - 2 = 0 & \Rightarrow & yz - 2z + x - 1 = 0 & (i) \\ f_y & = & 2zx - 2z + 2y - 4 = 0 & \Rightarrow & xz - z + y - 2 = 0 & (ii) \\ f_z & = & 2xy - 4x - 2y + 2z + 4 = 0 & \Rightarrow & xy - 2x - y + z + 2 = 0 & (iii) \end{array}$$

Adding (ii) and (iii) we get $x(z + y - 2) = 0 \Rightarrow x = 0$ or $y + z = 2$.

Case (1): $x = 0$. Putting $x = 0$ in $(i), (ii)$ we get $yz - 2z - 1 = 0$, and $-z + y - 2 = 0$. Solving these, we get $(z+2)z - 2z - 1 = 0 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1 \Rightarrow y = 3, 1$. Thus $(0,3,1), (0,1,-1)$ are extreme values.

Case (2): $y + z = 2$. From (i) we get $x = 1 + 2z - z(2 - z)$. Using (ii) we get $z(1 + z^2) - z + 2 - z - 2 = 0 \Rightarrow z^3 - z = 0 \Rightarrow z = 0, \pm 1$. Thus $(1,2,0), (2,1,1), (2,3,-1)$ are also extreme values.

Now $f_{xx} = 2, f_{yy} = 2, f_{zz} = 2, f_{xy} = 2z, f_{xz} = 2y - 4, f_{yz} = 2x - 2$.

Case (i): $x = 0, y = 3, z = 1$. The matrix associated with $d^2 f$ is

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$d^2 f$ is indefinite as one principle minor is $\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$, so there is neither a minimum nor maximum at $(0,3,1)$.

Case (ii): $x = 0, y = 1, z = -1$. The matrix associated with $d^2 f$ is

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$$

$d^2 f$ is indefinite as one principle minor is $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$, so there is neither a minimum nor maximum at $(0,1,-1)$.

Case (iii): $x = 1, y = 2, z = 0$. The matrix associated with $d^2 f$ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$d^2 f$ is positive definite so there is a minimum at $(1,2,0)$.

Case (iv): $x = 2, y = 1, z = 1$. The matrix associated with $d^2 f$ is

$$\begin{pmatrix} 2 & 2 & -2 \\ 2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}$$

d^2f is indefinite as one principle minor is $\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 0$, so there is neither a minimum nor maximum at $(2, 1, 1)$.

Case (v): $x = 2, y = 3, z = -1$. The matrix associated with d^2f is

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

d^2f is indefinite as one principle minor is $\begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$, so there is neither a minimum nor maximum at $(2, 3, -1)$.

Question 4(a) *Find the upper and lower Riemann integral for the function defined in the interval* $[0, 1]$ *as follows*

$$f(x) = \begin{cases} \sqrt{1 - x^2}, & x \in \mathbb{Q} \\ 1 - x, & x \notin \mathbb{Q} \end{cases}$$

and show that $f(x)$ *is not Riemann integrable in* $[0, 1]$.

Solution. For $0 \leq x \leq 1, (1-x)^2 = 1-2x+x^2 \leq 1-2x^2+x^2 = 1-x^2$, so $1-x \leq \sqrt{1-x^2}$. For any interval $[a, b] \subseteq [0, 1], 1 - b \leq 1 - x \leq \sqrt{1 - x^2}, a \leq x \leq b$. Thus for any partition $P = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of $[0, 1]$, we have

Lower Riemann Sum $L(f, P) = \sum_{i=1}^{n}(1 - t_i)(t_i - t_{i-1}) = L(g, P)$ where $g(x) = 1 - x, 0 \leq x \leq 1$. Thus

$$\underline{\int_0^1} f(x)\, dx = \sup_P L(f, P) = \sup_P L(g, P) = \int_0^1 g(x)\, dx$$

as g is Riemann integrable. Therefore

$$\underline{\int_0^1} f(x)\, dx = \int_0^1 (1 - x)\, dx = x - \frac{x^2}{2}\Big]_0^1 = \frac{1}{2}$$

.

Similarly, for any $[a, b] \subseteq [0, 1], \sqrt{1 - a^2} \geq \sqrt{1 - x^2}$ and since $\sqrt{1 - x^2}$ is integrable over $[0, 1]$, we get

$$\overline{\int_0^1} f(x)\, dx = \inf_P U(f, P) = \inf_P U(\sqrt{1 - x^2}, P) = \int_0^1 \sqrt{1 - x^2}\, dx = \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{\pi}{4}$$

Thus $\underline{\int_0^1} f(x)\, dx \neq \overline{\int_0^1} f(x)\, dx$, showing that $f(x)$ is not Riemann integrable on $[0, 1]$. ∎

Question 4(b) *Evaluate*

$$\iint \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\, dx\, dy$$

over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

Solution. Put $x = aX, y = bY$ so that the given integral

$$I = ab \iint_{\substack{X^2+Y^2\leq 1\\ X\geq 0, Y\geq 0}} \sqrt{\frac{1 - X^2 - Y^2}{1 + X^2 + Y^2}}\, dX\, dY$$

Transforming to polar coordinates by setting $X = r\cos\theta, Y = r\sin\theta$ so that

$$I = ab \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} r\, dr\, d\theta = \frac{\pi}{2} ab \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}} r\, dr$$

Put $r^2 = \cos t$ so that $2r\, dr = -\sin t$ and

$$\begin{aligned} I &= \frac{\pi}{2} ab \int_{\frac{\pi}{2}}^0 \sqrt{\frac{1-\cos t}{1+\cos t}}\ \frac{(-\sin t)}{2}\, dt = \frac{\pi}{4} ab \int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\cos^2 t}{(1+\cos t)^2}}\ \sin t\, dt \\ &= \frac{\pi ab}{4} \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{1+\cos t}\, dt = \frac{\pi ab}{4} \int_0^{\frac{\pi}{2}} (1 - \cos t)\, dt \\ &= \frac{\pi ab}{4} \Big[t - \sin t \Big]_0^{\frac{\pi}{2}} = \frac{\pi ab}{4} \left(\frac{\pi}{2} - 1 \right) = \frac{\pi ab}{8} (\pi - 2) \end{aligned}$$

which is the required value. ∎

UPSC Civil Services Main 1993 - Mathematics Calculus

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Question 1(a) *Prove that* $f(x) = x^2 \sin\frac{1}{x}, x \neq 0, f(0) = 0$ *is continuous and differentiable at* $x = 0$*, but its derivative is not continuous at* $x = 0$.

Solution. Let $\epsilon > 0$, take $\delta = \sqrt{\epsilon} > 0$. Now $|x| < \delta \Rightarrow |f(x) - f(0)| = |x^2 \sin\frac{1}{x}| \leq |x^2| < \epsilon$, so $f(x)$ is continuous at $x = 0$.

$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ when $x \neq 0$, and $f'(0) = \lim_{x\to 0} \dfrac{x^2 \sin\frac{1}{x} - 0}{x} = 0$, so f is differentiable.

The derivative is not continuous at 0 because $\lim_{x\to 0} f'(x)$ does not exist — the existence of $\lim_{x\to 0} f'(x)$ implies the existence of $\lim_{x\to 0}\cos\frac{1}{x}$, as $\lim_{x\to 0} x\sin\frac{1}{x}$ exists. However $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.

Proof: Let $\lim_{x\to 0}\cos\frac{1}{x} = m$, then given $0 < \epsilon < 1$ there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |\cos\frac{1}{x} - m| < \epsilon/2$. If $0 < |x_1| < \delta, 0 < |x_2| < \delta$ then $|\cos\frac{1}{x_1} - m| < \epsilon/2, |\cos\frac{1}{x_2} - m| < \epsilon/2 \Rightarrow |\cos\frac{1}{x_1} - \cos\frac{1}{x_2}| = |\cos\frac{1}{x_1} - m + m - \cos\frac{1}{x_2}| < \epsilon$.

But for any $\delta > 0$, we can find $x_1 = \frac{2}{(2n+1)\pi}, x_2 = \frac{1}{2n\pi}$ such that $0 < |x_1| < \delta, 0 < |x_2| < \delta$. However $|\cos\frac{1}{x_1} - \cos\frac{1}{x_2}| = 1 \not< \epsilon$. Thus $\lim_{x\to 0}\cos\frac{1}{x}$ and hence $\lim_{x\to 0} f'(x)$ does not exist, so $f'(x)$ is not continuous at $x = 0$. ∎

Question 1(b) *If* $f(x), \phi(x), \psi(x)$ *have derivatives when* $a \leq x \leq b$*, show that there is* $c \in [a, b]$ *such that*

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0$$

Solution. Consider the function

$$F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}, a \leq x \leq b$$

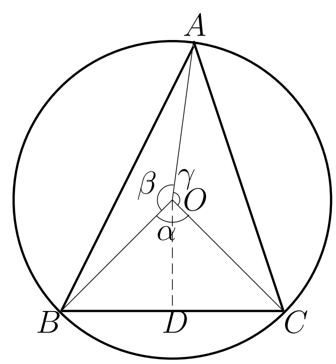
Clearly $F(a) = F(b) = 0$. Also, $F(x)$ is the linear combination of functions $f(x), \phi(x), \psi(x)$, so is differentiable, and therefore continuous in $[a, b]$. Thus $F(x)$ satisfies the requirements of Rolle's theorem, so there is a point $c \in [a, b]$ such that $F'(c) = 0$. Hence

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0$$

Note that if we expand the determinant defining $F(x)$ by the last row, we get $F(x) = Af(x) + B\phi(x) + C\psi(x)$, where A, B, C are constants. Hence $F'(x) = Af'(x) + B\phi'(x) + C\psi'(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(x) & \phi'(x) & \psi'(x) \end{vmatrix}$ ∎

Question 1(c) *Find the triangle of maximum area which can be inscribed in a circle.*

Solution.



Let the triangle be ABC inscribed in the circle of radius r and center O. Let the angles subtended by sides AB, BC, CA at the center be β, α, γ respectively, as shown. Area of $\triangle OBC = 2\triangle OBD = 2(\frac{1}{2}r^2 \cos\frac{\alpha}{2} \sin\frac{\alpha}{2}) = \frac{r^2}{2} \sin\alpha$.

Thus the area of $\triangle ABC = \frac{r^2}{2}[\sin\alpha + \sin\beta + \sin\gamma]$. Since $\alpha + \beta + \gamma = 2\pi$, we get $\triangle ABC = \frac{r^2}{2}[\sin\alpha + \sin\beta - \sin(\alpha + \beta)]$

Thus we want to maximize $F = \sin\alpha + \sin\beta - \sin(\alpha + \beta)$, subject to $0 < \alpha < 2\pi, 0 < \beta < 2\pi, 0 < \alpha + \beta < 2\pi$. For extreme values,

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \cos\alpha - \cos(\alpha + \beta) = 0 \\ \frac{\partial F}{\partial \beta} &= \cos\beta - \cos(\alpha + \beta) = 0 \end{aligned}$$

Thus $\cos\alpha = \cos\beta = \cos(\alpha + \beta)$. From the first equality, $\alpha = \beta$ or $\alpha = 2\pi - \beta$. However the second solution violates $\alpha + \beta < 2\pi$, so $\alpha = \beta$. From the second, $\beta = 2\beta$ or $\beta = 2\pi - 2\beta$. The first gives $\beta = 0$, which is not possible, so $\alpha = \beta = \frac{2\pi}{3}, \gamma = 2\pi - (\alpha + \beta) = \frac{2\pi}{3}$. Thus $\alpha = \beta = \gamma$, so the triangle must be equilateral.

To show that this is a maximum, when $\alpha = \beta = \frac{2\pi}{3}$ —

$$\begin{aligned}
\frac{\partial^2 F}{\partial \alpha^2} &= -\sin\alpha + \sin(\alpha+\beta) = -\sqrt{3} < 0 \\
\frac{\partial^2 F}{\partial \beta^2} &= -\sin\beta + \sin(\alpha+\beta) = -\sqrt{3} < 0 \\
\frac{\partial^2 F}{\partial \alpha\,\partial \beta} &= \sin(\alpha+\beta) = -\frac{\sqrt{3}}{2} \\
&\Rightarrow \frac{\partial^2 F}{\partial \alpha^2}\frac{\partial^2 F}{\partial \beta^2} - \left(\frac{\partial^2 F}{\partial \alpha\,\partial \beta}\right)^2 = 3 + \frac{3}{4} > 0
\end{aligned}$$

Thus the area of the inscribed triangle is maximum when the triangle is equilateral. ∎

Question 2(a) *Prove that* $\displaystyle\int_0^\infty e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{2\sqrt{a}}$ $(a > 0)$ *and deduce that* $\displaystyle\int_0^\infty x^{2n}e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2^{n+1}}(1\cdot 3\cdot 5\cdots(2n-1))$.

Solution. Put $t = \sqrt{a}x, dt = \sqrt{a}\,dx$ and

$$\int_0^\infty e^{-ax^2}\,dx = \int_0^\infty \frac{e^{-t^2}}{\sqrt{a}}\,dt = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

because $\displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ — see 2002, question 1(b).

Let $F(2n) = \displaystyle\int_0^\infty x^{2n}e^{-x^2}\,dx$. Integrating by parts, we get

$$F(2n) = \int_0^\infty x^{2n}e^{-x^2}\,dx = e^{-x^2}\frac{x^{2n+1}}{2n+1}\Big]_0^\infty + \int_0^\infty 2xe^{-x^2}\frac{x^{2n+1}}{2n+1}\,dx = \frac{2}{2n+1}F(2n+2)$$

Hence $F(2n+2) = \frac{2n+1}{2}F(2n)$. Using this formula repeatedly on $F(2n)$, we get

$$F(2n) = \frac{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n}F(0)$$

However, from above $F(0) = \frac{\sqrt{\pi}}{2}$ by letting $a = 1$. Thus

$$F(2n) = \int_0^\infty x^{2n}e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2^{n+1}}(1\cdot 3\cdot 5\cdots(2n-1))$$

as required. ∎

Question 2(b) *Define the Gamma function and prove that*

$$\Gamma(n)\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

Solution. See 1997 question 2(c). ∎

Question 2(c) *Show that the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=ax$ is $\frac{2a^3}{9}(3\pi-4)$.*

Solution. Clearly $V=\iiint dx\,dy\,dz$, where the limits of z are from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$ and x,y vary over the cylinder D given above. Thus

$$V = 2\iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy$$

Because of symmetry, it is enough to compute the integral in the first quadrant. Let $x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$. $x^2+y^2=ax \Rightarrow r^2 = ar\cos\theta \Rightarrow r = a\cos\theta$, so the limits of integration are $0\le\theta\le\frac{\pi}{2}, 0\le r\le a\cos\theta$.

$$\begin{aligned}
V &= 4\int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}(a^2-r^2)^{\frac{1}{2}}r\,dr\,d\theta \\
&= 4\int_0^{\frac{\pi}{2}}\left[\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\cdot\left(-\frac{1}{2}\right)\right]_0^{a\cos\theta}d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}}(a^3-a^3\sin^3\theta)\,d\theta \\
&= \frac{4}{3}a^3\frac{\pi}{2}-\frac{4}{3}a^3\cdot\frac{2}{3} = \frac{2a^3}{9}(3\pi-4)
\end{aligned}$$

as required. ∎

Paper II

Question 3(a) *Find all the maxima and minima of*

$$f(x,y) = x^3+y^3-63(x+y)+12xy$$

Solution. For extreme values

$$\begin{aligned}
f_x &= 3x^2-63+12y=0 \\
f_y &= 3y^2-63+12x=0
\end{aligned}$$

$f_x-f_y=0 \Rightarrow 3(x^2-y^2)+12(y-x)=0 \Rightarrow y-x=0, 3(x+y)-12=0$.

Case 1. $x=y \Rightarrow 3x^2+12x-63=0 \Rightarrow x^2+4x-21=0 \Rightarrow x=-7,3$. Thus in this case the critical points are $(-7,-7)$ and $(3,3)$.

Case 2. $x+y=4 \Rightarrow x^2-21+4(4-x)=0 \Rightarrow x^2-4x-5=0 \Rightarrow x=-1,5$. Thus in this case the critical points are $(-1,5)$ and $(5,-1)$.

Now $f_{xx}=6x, f_{xy}=12, f_{yy}=6y$.

1. $x = -7, y = -7 \Rightarrow f_{xx} < 0, f_{xx}f_{yy} - f_{xy}^2 = (-42)^2 - (12)^2 > 0$. Therefore $(-7, -7)$ is a maximum.

2. $x = 3, y = 3 \Rightarrow f_{xx} > 0, f_{xx}f_{yy} - f_{xy}^2 = (18)^2 - (12)^2 > 0$. Therefore $(3, 3)$ is a minimum.

3. $x = -1, y = 5 \Rightarrow f_{xx} < 0, f_{xx}f_{yy} - f_{xy}^2 = -180 - (12)^2 < 0$. Therefore $(-1, 5)$ is neither a minimum nor a maximum.

4. $x = 5, y = -1 \Rightarrow f_{xx} > 0, f_{xx}f_{yy} - f_{xy}^2 = -180 - (12)^2 < 0$. Therefore $(5, -1)$ is neither a minimum nor a maximum.

So the only maximum is $(-7, -7)$ and the only minimum is $(3, 3)$. ∎

Question 4(a) *Examine the Riemann integrability over* $[0, 2]$ *of the function* f *defined on* $[0, 2]$ *by*

$$f(x) = \begin{cases} x + x^2, & x \in \mathbb{Q} \\ x^2 + x^3, & x \notin \mathbb{Q} \end{cases}$$

Solution. Clearly $(x + x^2) - (x^2 - x^3) = \begin{cases} x - x^3 > 0 \text{ for } 0 < x < 1 \\ x - x^3 < 0 \text{ for } 1 < x < 2 \end{cases}$

$$\begin{aligned}
\overline{\int_0^1} f(x)\,dx &= \int_0^1 (x + x^2)\,dx = \frac{x^2}{2} + \frac{x^3}{3}\Big]_0^1 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \\
\underline{\int_0^1} f(x)\,dx &= \int_0^1 (x^2 + x^3)\,dx = \frac{x^3}{3} + \frac{x^4}{4}\Big]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \\
\overline{\int_1^2} f(x)\,dx &= \int_1^2 (x^2 + x^3)\,dx = \frac{x^3}{3} + \frac{x^4}{4}\Big]_1^2 = \frac{8}{3} + 4 - \frac{1}{3} - \frac{1}{4} = \frac{73}{12} \\
\underline{\int_1^2} f(x)\,dx &= \int_1^2 (x + x^2)\,dx = \frac{x^2}{2} + \frac{x^3}{3}\Big]_1^2 = \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = \frac{23}{6} \\
\underline{\int_0^2} f(x)\,dx &= \frac{7}{12} + \frac{23}{6} = \frac{53}{12}, \quad \overline{\int_0^2} f(x)\,dx = \frac{5}{6} + \frac{73}{12} = \frac{83}{12}
\end{aligned}$$

Thus f in not Riemann integrable over $[0.2]$ because $\underline{\int_0^2} f(x)\,dx \neq \overline{\int_0^2} f(x)\,dx$.

(**Note:** For any $[a, b] \subseteq [0, 1], x^2 + x^3 < x + x^2$ so $\overline{\int_0^1} f(x)\,dx = \inf_P U(f, P) = \inf_P \sum_{i=0}^{n} (t_i - t_{i-1})(t_i + t_i^2) = \int_0^1 (x + x^2)\,dx$.) ∎

Question 4(b) *Evaluate* $\displaystyle\iiint \frac{dx\,dy\,dz}{x+y+z+1}$ *over the volume bounded by the coordinate planes* $x=0, y=0, z=0$ *and the plane* $x+y+z=1$.

Solution. Use the substitution $x+y+z=u, y+z=uv, z=uvw$, as in 1994, question 4(b), the given integral becomes

$$
\begin{aligned}
I &= \int_0^1\int_0^1\int_0^1 \frac{1}{u+1}u^2v\,du\,dv\,dw \\
&= \frac{1}{2}\int_0^1 \frac{u^2}{u+1}\,du \qquad\qquad (\text{Put } u+1=t, du=dt) \\
&= \frac{1}{2}\int_1^2 \frac{t^2-2t+1}{t}\,dt = \frac{1}{2}\int_1^2 t-2+\frac{1}{t}\,dt \\
&= \frac{1}{2}\left[\frac{t^2}{2}-2t+\log t\right]_1^2 = \frac{1}{2}\left[2-\frac{1}{2}-4+2+\log 2\right] \\
&= \frac{1}{2}\log 2-\frac{1}{4} = \frac{1}{4}[2\log 2-1] = \frac{1}{4}\log\frac{4}{e}
\end{aligned}
$$

∎

UPSC Civil Services Main 1994 - Mathematics Calculus

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Question 1(a) *Let $f(x)$ be defined by*

$$f(x) = \begin{cases} \frac{1}{2}(b^2 - a^2), & 0 < x \leq a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x}, & a < x \leq b \\ \frac{1}{3}\frac{b^3 - a^3}{x}, & x > b \end{cases}$$

Prove that $f(x), f'(x)$ are continuous but $f''(x)$ is discontinuous.

Solution. Clearly $f(x)$ is continuous for $0 < x < a, a < x < b$ and $x > b$. The problem points are $x = a, b$.

1. $x = a$:

$$\begin{aligned} \lim_{x \to a^-} f(x) &= \lim_{x \to a} \frac{1}{2}(b^2 - a^2) = \frac{1}{2}(b^2 - a^2) \\ \lim_{x \to a^+} f(x) &= \lim_{x \to a} \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x} = \frac{1}{2}b^2 - \frac{a^2}{6} - \frac{a^3}{3a} = \frac{1}{2}(b^2 - a^2) \end{aligned}$$

Since $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a) = \frac{1}{2}(b^2 - a^2)$, $f(x)$ is continuous at $x = a$.

2. $x = b$:

$$\begin{aligned} \lim_{x \to b^-} f(x) &= \frac{1}{2}b^2 - \frac{b^2}{6} - \frac{a^3}{3b} = \frac{1}{3b}(b^3 - a^3) \\ \lim_{x \to b^+} f(x) &= \frac{1}{3}\frac{b^3 - a^3}{b} \end{aligned}$$

Again $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x) = f(b) = \frac{1}{3b}(b^3 - a^3)$, so $f(x)$ is continuous at $x = b$.

Thus $f(x)$ is continuous everywhere.

Clearly $f(x)$ is differentiable for all $x > 0$ except possibly at $x = a, x = b$, which we examine next:

1. L.H.D at $x = a$ is $\dfrac{d}{dx}\left(\dfrac{1}{2}(b^2 - a^2)\right) = 0.$

   R.H.D at $x = a$ is $\dfrac{d}{dx}\left(\dfrac{1}{2}b^2 - \dfrac{x^2}{6} - \dfrac{a^3}{3x}\right)_{x=a} = -\dfrac{a}{3} + \dfrac{a^3}{3a^2} = 0.$

   Thus the L.H.D = R.H.D at $x = a$, therefore $f'(a)$ exists and $f'(a) = 0$.

2. L.H.D at $x = b$ is $\dfrac{d}{dx}\left(\dfrac{1}{2}b^2 - \dfrac{x^2}{6} - \dfrac{a^3}{3x}\right)_{x=b} = -\dfrac{b}{3} + \dfrac{a^3}{3b^2} = \dfrac{a^3 - b^3}{3b^2}.$

   R.H.D at $x = b$ is $-\dfrac{1}{3}\dfrac{b^3 - a^3}{b^2} = \dfrac{a^3 - b^3}{3b^2}.$

   Thus $f(x)$ is differentiable at $x = b$ also.

Moreover it can easily be seen that

$$f'(x) = \begin{cases} 0, & 0 < x \leq a \\ -\frac{x}{3} + \frac{a^3}{3x^2}, & a < x \leq b \\ \frac{1}{3}\,\frac{a^3 - b^3}{x^2}, & x > b \end{cases}$$

It is obvious that $f'(x)$ is continuous for $0 < x < a, a < x < b$ and $x > b$.

$$\begin{aligned} \lim_{x \longrightarrow a^-} f'(x) &= 0, \lim_{x \longrightarrow a^+} f'(x) = -\frac{a}{3} + \frac{a^3}{3a^2} = 0 = f'(a) \\ \lim_{x \longrightarrow b^-} f'(x) &= -\frac{b}{3} + \frac{a^3}{3b^2} = \frac{a^3 - b^3}{3b^2} = \lim_{x \longrightarrow b^+} f'(x) = f'(b) \end{aligned}$$

Thus $f'(x)$ is continuous at a and b also, so it is continuous everywhere.

$$f''(x) = \begin{cases} 0, & 0 < x < a \\ -\frac{1}{3} - \frac{2a^3}{3x^3}, & a < x < b \\ -\frac{2}{3}\,\frac{a^3 - b^3}{x^3}, & x > b \end{cases}$$

$f''(x)$ does not exist at $x = a$, because LHD of $f'(x)$ at $x = a$ is 0, while RHD of $f'(x)$ at $x = a$ is $-\frac{1}{3} - \frac{2a^3}{3a^3} = -1$.

$f''(x)$ does not exist at $x = b$, because LHD of $f'(x)$ at $x = b$ is $-\frac{1}{3} - \frac{2a^3}{3b^3}$, while RHD of $f'(x)$ at $x = b$ is $-\frac{2}{3}\,\frac{a^3 - b^3}{b^3} = -\frac{2a^3}{3b^3} + \frac{2}{3}$.

Hence $f''(x)$ is not continuous at $x = a$ and $x = b$, since it is not defined at these values. ∎

Question 1(b) *If α, β lie between the greatest and the least values of a, b, c, prove that*

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix} = K \begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \phi(a) & \phi'(\alpha) & \phi''(\beta) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta) \end{vmatrix}$$

where $K = \frac{1}{2}(b-c)(c-a)(a-b)$.

Solution. (Note: The question should have been more clearly worded as : Show that there exist α, β lying between the greatest and the least values of a, b, c, such that ....)

Consider the function

$$F(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix} - \frac{(x-a)(x-b)}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

Clearly, $F(x)$ is the linear combination of functions $f(x), \phi(x), \psi(x)$ and a polynomial, so it is twice differentiable — note that it is implied from the statement of the question that $f(x), \phi(x), \psi(x)$ are twice differentiable. Since $F(a) = F(b) = F(c) = 0$, it follows that $F(x)$ satisfies the requirements of Rolle's theorem in the intervals $[a, c], [c, b]$ (we assume wlog $a < c < b$). Thus there exist x_1, x_2 such that $a < x_1 < c, c < x_2 < b$ and $F'(x_1) = F(x_2) = 0$. Applying Rolle's theorem to $F'(x)$ in the interval $[x_1, x_2]$ we get a real number $\beta \in (x_1, x_2)$ such that $F''(\beta) = 0$.

Now

$$F''(x) = \begin{vmatrix} f(a) & f(b) & f''(x) \\ \phi(a) & \phi(b) & \phi''(x) \\ \psi(a) & \psi(b) & \psi''(x) \end{vmatrix} - \frac{2}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

from which it follows that

$$\begin{vmatrix} f(a) & f(b) & f''(\beta) \\ \phi(a) & \phi(b) & \phi''(\beta) \\ \psi(a) & \psi(b) & \psi''(\beta) \end{vmatrix} = \frac{2}{(c-a)(c-b)} \begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

$\Bigg($Note that if $H(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix} = Af(x) + B\phi(x) + C\psi(x)$ (say), then $H'(x) = Af'(x) + B\phi'(x) + C\psi'(x), H''(x) = Af''(x) + B\phi''(x) + C\psi''(x) \Rightarrow H''(x) = \begin{vmatrix} f(a) & f(b) & f''(x) \\ \phi(a) & \phi(b) & \phi''(x) \\ \psi(a) & \psi(b) & \psi''(x) \end{vmatrix}$.$\Bigg)$

Now we consider

$$g(x) = \begin{vmatrix} f(a) & f(x) & f''(\beta) \\ \phi(a) & \phi(x) & \phi''(\beta) \\ \psi(a) & \psi(x) & \psi''(\beta) \end{vmatrix} - \frac{2(x-a)}{(c-b)(b-a)(c-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

Clearly $g(a) = 0$ and $g(b) = 0$, as proved above. Thus using Rolle's theorem for $g(x)$ in the interval $[a, b]$ we get $\alpha \in (a, b)$ such that $g'(\alpha) = 0$. Thus

$$g'(\alpha) = \begin{vmatrix} f(a) & f(\alpha) & f''(\beta) \\ \phi(a) & \phi(\alpha) & \phi''(\beta) \\ \psi(a) & \psi(\alpha) & \psi''(\beta) \end{vmatrix} - \frac{2}{(c-b)(b-a)(c-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix} = 0$$

or

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \phi(a) & \phi(b) & \phi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix} = \frac{1}{2}(b-c)(c-a)(a-b) \begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \phi(a) & \phi'(\alpha) & \phi''(\beta) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta) \end{vmatrix}$$

as required. ∎

Question 1(c) *Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.*

Solution. Let $V = xyz$, then $S = 2(xy + yz + zx) = 2(xy + \frac{V}{x} + \frac{V}{y})$.

$$\frac{\partial S}{\partial x} = 2\Big(y - \frac{V}{x^2}\Big), \frac{\partial S}{\partial y} = 2\Big(x - \frac{V}{y^2}\Big)$$

For extreme values, $\frac{\partial S}{\partial x} = \frac{\partial S}{\partial y} = 0$. Thus $x^2y = xy^2 = V = xyz \Rightarrow x = y = z$ (as $x \neq 0, y \neq 0$ being the sides of a parallelopiped).

Now $\dfrac{\partial^2 S}{\partial x^2} = \dfrac{4V}{x^3}, \dfrac{\partial^2 S}{\partial y^2} = \dfrac{4V}{y^3}, \dfrac{\partial^2 S}{\partial x\, \partial y} = 2 \Rightarrow \dfrac{\partial^2 S}{\partial x^2}\dfrac{\partial^2 S}{\partial y^2} - \left(\dfrac{\partial^2 S}{\partial x\, \partial y}\right)^2 = \dfrac{16V^2}{x^3y^3} - 4 = 16 - 4 > 0$, because $V^2 = x^2y \cdot xy^2 = x^3y^3$.

Thus S is minimum when $x = y = z, V = x^3$ i.e. when the parallelopiped is a cube. ∎

Question 2(a) *Show by means of the beta function that*

$$I = \int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^\alpha} = \frac{\pi}{\sin \pi\alpha},\ \ 0 < \alpha < 1$$

Solution. Put $x - t = u$ in I.

$$\begin{aligned} I &= \int_0^{z-t} \frac{du}{(z-t-u)^{1-\alpha}u^\alpha}. \qquad\qquad \text{Put } u = (z-t)v. \text{ Then} \\ I &= \int_0^1 \frac{(z-t)\,dv}{(1-v)^{1-\alpha}(z-t)^{1-\alpha}(z-t)^\alpha v^\alpha} = \int_0^1 \frac{dv}{(1-v)^{1-\alpha}v^\alpha} = \int_0^1 v^{-\alpha}(1-v)^{-1+\alpha}\,dv \\ &= B(-\alpha+1, \alpha) = B(\alpha, 1-\alpha) \\ &= \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha+1-\alpha)} = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha} \end{aligned}$$

The last step follows from a standard result, which we prove here.

$$B(m, 1-m) = \Gamma(m)\Gamma(1-m) = \int_0^1 x^{m-1}(1-x)^{(1-m-1)}\,dx$$

Put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

$$B(m, 1-m) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta\cos^{-2m}\theta\ 2\sin\theta\cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \tan^{2m-1}\theta\,d\theta$$

Put $\tan\theta = z \Rightarrow \sec^2\theta\,d\theta = dz \Rightarrow d\theta = \frac{dz}{1+z^2}$.

$$B(m, 1-m) = 2\int_0^\infty z^{2m-1}\frac{dz}{1+z^2}$$

Put $z^2 = t$, so that $2z\,dz = dt$. Thus

$$B(m, 1-m) = \int_0^\infty \frac{t^{m-1}}{1+t}\,dt \qquad 0 < m < 1$$

Consider the integral $\int_1^t \frac{x^{m-1}}{1+x}\,dx$. Put $x = \frac{1}{y}$ so that $dx = -\frac{dy}{y^2}$, and

$$\int_1^t \frac{x^{m-1}}{1+x}\,dx = \int_1^{\frac{1}{t}} \frac{\left(\frac{1}{y}\right)^{m-1}}{1+\frac{1}{y}}\,\frac{-dy}{y^2} = -\int_1^{\frac{1}{t}} \frac{y^{-m}}{1+y}\,dy$$

Letting $t \to \infty$, we get $\int_1^\infty \frac{x^{m-1}}{1+x}\,dx = \int_0^1 \frac{y^{-m}}{1+y}\,dy$. Thus

$$I = \int_0^\infty \frac{x^{m-1}}{1+x}\,dx = \int_0^1 \frac{x^{m-1}}{1+x}\,dx + \int_0^1 \frac{y^{-m}}{1+y}\,dy = \int_0^1 \frac{y^{-m}+y^{m-1}}{1+y}\,dy$$

Taylor's expansion gives us $(1+y)^{-1} = 1 - y + y^2 - \ldots + (-1)^k y^k + \frac{(-1)^{k+1}y^{k+1}}{1+\theta y},\ 0 < \theta < 1$. Since

$$\int_0^1 \frac{y^{k+1-m}+y^{k+1+m-1}}{1+\theta y}\,dy < \int_0^1 (y^{k+m}+y^{k-m+1})\,dy = \frac{1}{k+m+1} + \frac{1}{k+2-m}$$

Clearly $\frac{1}{k+m+1} + \frac{1}{k+2-m} \to 0$ as $k \to \infty$, so we get

$$\begin{aligned}
\int_0^\infty \frac{x^{m-1}}{1+x}\,dx &= \lim_{k\to\infty}\left[\int_0^1 [y^{m-1} - y^m + y^{m+1} - \ldots + (-1)^k y^{k+m-1}]\,dy \right.\\
&\qquad \left. + \int_0^1 [y^{-m} - y^{1-m} + y^{2-m} - \ldots + (-1)^k y^{k-m}]\,dy\right]\\
&= \lim_{k\to\infty}\left[\left(\frac{1}{m} - \frac{1}{m+1} + \ldots + \frac{(-1)^k}{k+m}\right) + \left(\frac{1}{1-m} - \frac{1}{2-m} + \ldots + \frac{(-1)^k}{k+1-m}\right)\right]\\
&= \frac{1}{m} + \sum_{k=1}^\infty (-1)^k\left(\frac{1}{m+k} - \frac{1}{k-m}\right) = \frac{1}{m} + \sum_{k=1}^\infty (-1)^k\frac{2m}{m^2-k^2} = \frac{\pi}{\sin m\pi}
\end{aligned}$$

by using the identity $\csc z = \frac{1}{z} + \sum_{k=1}^\infty (-1)^k \frac{2z}{z^2-k^2\pi^2}$ ∎

Question 2(b) *Prove that the value of* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *taken over the volume bounded by the coordinate planes and the plane* $x+y+z=1$ *is* $\frac{1}{2}\Big(\log 2 - \frac{5}{8}\Big) = \frac{1}{16}\log\frac{256}{e^5}$

Solution.

$$
\begin{aligned}
I &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} \\
&= -\frac{1}{2}\int_0^1 dx \int_0^{1-x} (1+x+y+z)^{-2}\Big]_0^{1-x-y} dy \\
&= \frac{1}{2}\int_0^1 dx \int_0^{1-x} \Big[(1+x+y)^{-2} - \frac{1}{4}\Big] dy \\
&= \frac{1}{2}\int_0^1 \Big[\frac{(1+x+y)^{-1}}{-1} - \frac{y}{4}\Big]_0^{1-x} dx \\
&= \frac{1}{2}\int_0^1 \Big[(1+x)^{-1} - \frac{1}{2} - \frac{1-x}{4}\Big] dx \\
&= \frac{1}{2}\Big[\log(1+x) - \frac{3x}{4} + \frac{x^2}{8}\Big]_0^1 \\
&= \frac{1}{2}\Big[\log 2 - \frac{5}{8}\Big]
\end{aligned}
$$

as required. ∎

Question 2(c) *The sphere* $x^2+y^2+z^2 = a^2$ *is pierced by the cylinder* $(x^2+y^2)^2 = a^2(x^2-y^2)$. *Prove that the volume of the sphere that lies inside the cylinder is* $\frac{8a^3}{3}\Big(\frac{\pi}{4} + \frac{5}{3} - \frac{4\sqrt{2}}{3}\Big)$

Solution. Clearly $V = \iiint dx\,dy\,dz$, where the limits of z are from $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$ and x, y vary over the cylinder D given above. Thus

$$V = 2\iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy$$

Because of symmetry, it is enough to compute the integral in the first octant. Let $x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$. $(x^2+y^2)^2 = a^2(x^2-y^2) \Rightarrow r^4 = a^2r^2\cos 2\theta \Rightarrow r^2 = a^2\cos 2\theta$, so the limits of integration are $0 \le \theta \le \frac{\pi}{4}, 0 \le r \le a\sqrt{\cos 2\theta}$.

$$
\begin{aligned}
V &= 8\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}}(a^2-r^2)^{\frac{1}{2}}r\,dr\,d\theta \\
&= 8\int_0^{\frac{\pi}{4}}\left[\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\cdot\left(-\frac{1}{2}\right)\right]_0^{a\sqrt{\cos 2\theta}}\,d\theta \\
&= \frac{8}{3}\int_0^{\frac{\pi}{4}}\left(a^3-a^3(1-\cos 2\theta)^{\frac{3}{2}}\right)d\theta \\
&= \frac{8a^3}{3}\left[\frac{\pi}{4}-\int_0^{\frac{\pi}{4}}2^{\frac{3}{2}}\sin^3\theta\,d\theta\right] \\
&= \frac{8a^3}{3}\left[\frac{\pi}{4}-2^{\frac{3}{2}}\int_0^{\frac{\pi}{4}}\sin\theta(1-\cos^2\theta)\,d\theta\right] \\
&= \frac{8a^3}{3}\left[\frac{\pi}{4}-2^{\frac{3}{2}}\left[-\cos\theta+\frac{\cos^3\theta}{3}\right]_0^{\frac{\pi}{4}}\right] \\
&= \frac{8a^3}{3}\left[\frac{\pi}{4}-2^{\frac{3}{2}}\left[1-\frac{1}{\sqrt{2}}+\frac{1}{6\sqrt{2}}-\frac{1}{3}\right]\right] \\
&= \frac{8a^3}{3}\left(\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right)
\end{aligned}
$$

as required. ∎

Paper II

Question 3(a) *Let the function f be defined on $[0,1]$ by the condition $f(x) = 2rx$ when $\frac{1}{r+1} < x < \frac{1}{r}, r > 0$, show that f is Riemann integrable in $[0,1]$ and* $\displaystyle\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

Solution. Clearly $f(x) = 2(r-1)x$ when $\frac{1}{r} < x < \frac{1}{r-1}$, therefore

$\lim_{x\to\frac{1}{r}+0} f(x) = \lim_{h\to 0} 2(r-1)(\frac{1}{r}+h) = 2(r-1)\frac{1}{r} = 2-\frac{2}{r}$.

$\lim_{x\to\frac{1}{r}-0} f(x) = \lim_{h\to 0} 2r(\frac{1}{r}+h) = 2r\frac{1}{r} = 2$.

Thus $f(\frac{1}{r}-0) \neq f(\frac{1}{r}+0)$, so f is discontinuous at $x = \frac{1}{r}, r = 2, 3, \ldots$. Since the points of discontinuity have only one limit point, f is Riemann integrable. Now

$$
\begin{aligned}
\int_0^1 f(x)\,dx &= \sum_{r=1}^{\infty}\int_{\frac{1}{r+1}}^{\frac{1}{r}} 2rx\,dx = \sum_{r=1}^{\infty} rx^2\Big]_{\frac{1}{r+1}}^{\frac{1}{r}} \\
&= \sum_{r=1}^{\infty} r\left[\frac{1}{r^2}-\frac{1}{(r+1)^2}\right] \\
&= \left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{2}{2^2}-\frac{2}{3^2}\right)+\left(\frac{3}{3^2}-\frac{3}{4^2}\right)+\ldots \\
&= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{6}
\end{aligned}
$$

∎

Question 3(b) *By means of the substitution* $x+y+z=u, y+z=uv, z=uvw$ *evaluate* $\iiint (x+y+z)^n xyz\,dx\,dy\,dz$ *taken over the volume bounded by* $x=0, y=0, z=0, x+y+z=1$.

Solution. Clearly the region of integration is the interior of the tetrahedron bounded by $x=0, y=0, z=0, x+y+z=1$. The substitution $x+y+z=u, y+z=uv, z=uvw$ implies $0\le u\le 1, 0\le v\le 1, 0\le w\le 1$, and conversely, when (u,v,w) is in the unit cube, then (x,y,z) falls in the interior of the tetrahedron. Moreover

$$\frac{\partial(z,y,x)}{\partial(u,v,w)} = \begin{vmatrix} vw & wu & uv \\ v-vw & u-uw & -uv \\ 1-v & -u & 0 \end{vmatrix} = \begin{vmatrix} vw & wu & uv \\ v & u & 0 \\ 1-v & -u & 0 \end{vmatrix} = -u^2v$$

Hence

$$\begin{aligned}
I &= \iiint_{\substack{x+y+z\le 1\\ x\ge 0, y\ge 0, z\ge 0}} (x+y+z)^n xyz\,dx\,dy\,dz \\
&= \int_0^1\int_0^1\int_0^1 u^n(u-uv)(uvw)(uv-uvw)u^2v\,du\,dv\,dw \\
&= \int_0^1\int_0^1\int_0^1 u^{n+5}(1-v)v^3(1-w)w\,du\,dv\,dw \\
&= \frac{1}{n+6}\int_0^1 v^{4-1}(1-v)^{2-1}\,dv\int_0^1 w^{2-1}(1-w)^{2-1}\,dw \\
&= \frac{1}{n+6}\,\frac{\Gamma(4)\Gamma(2)}{\Gamma(6)}\,\frac{\Gamma(2)\Gamma(2)}{\Gamma(4)} = \frac{1}{n+6}\,\frac{1}{5!}
\end{aligned}$$

■

UPSC Civil Services Main 1995 - Mathematics Calculus

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Question 1(a) *If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$, then prove that $g'(x) = 1+(g(x))^3$.*

Solution. Let $f : D \longrightarrow R, g : R \longrightarrow D$ so that $y = f(x), x = g(y)$, and $(g \circ f)(x) = x$ for $x \in D$, and $(f \circ g)(y) = y$ for $y \in R$. Then by the chain rule

$$\begin{aligned} 1 &= (g \circ f)'(x) = g'(f(x))f'(x) \\ \Rightarrow g'(f(x)) &= \frac{1}{f'(x)} = 1 + x^3 = 1 + (g(f(x)))^3 \\ \Rightarrow g'(y) &= 1 + (g(y))^3 \quad \because x = g(y) \end{aligned}$$

which was to be proved. ∎

Question 1(b) *Taking the n'th derivative of $(x^n)^2$ in two different ways, prove that*

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 \cdot 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2 \cdot 2^2 \cdot 3^2} + \ldots + (n+1 \ \textit{terms}) = \frac{(2n)!}{(n!)^2}$$

Solution.

$$\frac{d^n}{dx^n}(x^{2n}) = 2n \cdot (2n-1) \ldots (2n-n+1)x^n = \frac{(2n)!}{n!}x^n \tag{1}$$

Note that $\dfrac{d^r}{dx^r}(x^n) = n \cdot (n-1) \ldots (n-r+1)x^{n-r} = \dfrac{n!}{(n-r)!}x^{n-r}$. Let $f(x) = g(x) = x^n$. Then $\dfrac{d^n}{dx^n}(x^{2n}) = \dfrac{d^n}{dx^n}(fg)$. Applying Leibnitz's formula

$$\begin{aligned} \frac{d^n}{dx^n}(x^{2n}) &= \frac{d^n}{dx^n}(fg) = \sum_{r=0}^{n} \binom{n}{r} f^{n-r} g^r = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{n!}{r!} x^r \frac{n!}{(n-r)!} x^{n-r} \\ &= n! x^n \sum_{r=0}^{n} \frac{(n!)^2}{(r!)^2((n-r)!)^2} = n! x^n \sum_{r=0}^{n} \frac{n^2(n-1)^2 \ldots (n-r+1)^2}{1^2 \cdot 2^2 \ldots r^2} \end{aligned} \tag{2}$$

Equating the right hand sides in (1), (2) and dividing thoughout by $n!x^n$, we have the desired result. ∎

Question 1(c) *Let $f(x,y)$, which possesses continuous partial derivatives of degree 2, be a homogeneous function of x and y of degree n. Prove that*

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$$

Solution. See 2006 question 2(b). ∎

Question 2(a) *Find the area bounded by the curve*

$$\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2}{4}-\frac{y^2}{9}$$

Solution. Let $X=\frac{x}{2}, Y=\frac{y}{3}$ so that $dx\,dy=6\,dX\,dY$, and the required area $\iint dx\,dy=6\iint dX\,dY$, where the right integral is over the region bounded by $(X^2+Y^2)^2=X^2-Y^2$.

We now switch over to polar coordinates $X=r\cos\theta, Y=r\sin\theta$, so $(X^2+Y^2)^2=X^2-Y^2\Rightarrow r^4=r^2\cos^2\theta-r^2\sin^2\theta\Rightarrow r^2=\cos 2\theta$. Thus the required area $=6\times$ the area bounded by $r^2=\cos 2\theta=24\times$ the area bounded by $r^2=\cos 2\theta$ in the first quadrant, as the curve is symmetrical about both coordinate axes.

The required area $=24\displaystyle\int_0^{\frac{\pi}{4}}\frac{r^2}{2}\,d\theta=24\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta}{2}\,d\theta=24\left[\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}=6.$ ∎

Question 2(b) *Let $f(x), x\geq 1$ be such that the area bounded by the curve $y=f(x)$ and the lines $x=1$ and $x=b$ is $\sqrt{1+b^2}-\sqrt{2}$ for all $b\geq 1$. Does f attain its minimum, if so, what is its value.*

Solution. The given area is $\displaystyle\int_1^b f(x)\,dx=\sqrt{1+b^2}-\sqrt{2}\Rightarrow\int f(x)\,dx=\sqrt{1+x^2}.$

Differentiating, we get $f(x)=\dfrac{x}{\sqrt{1+x^2}}$. Clearly $f'(x)=\dfrac{\sqrt{1+x^2}-x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}=\dfrac{1}{(1+x^2)^{\frac{3}{2}}}>0$. Thus f is monotonically increasing, hence its minimum value is $f(1)=\frac{1}{\sqrt{2}}$. ∎

Question 2(c) *Show that*

$$\Gamma\Big(\frac{1}{n}\Big)\Gamma\Big(\frac{2}{n}\Big)\Gamma\Big(\frac{3}{n}\Big)\dots\Gamma\Big(\frac{n-1}{n}\Big) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

Solution. Let $P = \Gamma\Big(\frac{1}{n}\Big)\Gamma\Big(\frac{2}{n}\Big)\Gamma\Big(\frac{3}{n}\Big)\dots\Gamma\Big(\frac{n-1}{n}\Big)$, so that

$$P^2 = \Gamma\Big(\frac{1}{n}\Big)\Gamma\Big(1-\frac{1}{n}\Big)\Gamma\Big(\frac{2}{n}\Big)\Gamma\Big(1-\frac{2}{n}\Big)\dots\Gamma\Big(\frac{n-1}{n}\Big)\Gamma\Big(1-\frac{n-1}{n}\Big)$$

(We have paired the first term with the last and so on.).

We now use $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin x\pi}$ so that

$$P^2 = \frac{\pi}{\sin\frac{\pi}{n}}\ \frac{\pi}{\sin\frac{2\pi}{n}}\dots\frac{\pi}{\sin\frac{n-1}{n}\pi}$$

We shall now prove that $\sin\dfrac{\pi}{n}\sin\dfrac{2\pi}{n}\dots\sin\dfrac{n-1}{n}\pi = \dfrac{n}{2^{n-1}}$, so that $P^2 = \dfrac{\pi^{n-1}2^{n-1}}{n} \Rightarrow P = \dfrac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$ as required.

Proof of $\sin\dfrac{\pi}{n}\sin\dfrac{2\pi}{n}\dots\sin\dfrac{n-1}{n}\pi = \dfrac{n}{2^{n-1}}$:

Consider $F(x) = x^n - 2\cos n\theta + x^{-n}$. If $x = \cos\theta + i\sin\theta$, then $x^n = \cos n\theta + i\sin n\theta$ for all n, so $x^n + x^{-n} = 2\cos n\theta$. This shows that $x - 2\cos\theta + \frac{1}{x}$ is a factor of $F(x)$. The same is true if we replace θ by $\theta_r = \theta + \frac{2\pi r}{n}, 0 \leq r \leq n-1$. Hence $F(x) = \prod_{r=0}^{n-1}(x - 2\cos\theta_r + \frac{1}{x})$. Put $x = 1$, then $2 - 2\cos n\theta = \prod_{r=0}^{n-1}(2 - 2\cos\theta_r) \Rightarrow 4\sin^2(n\theta/2) = 2^{2n}\prod_{r=0}^{n}\sin^2(\theta_r/2)$. Thus

$$\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} = 2^{2n-2}\prod_{r=1}^{n}\sin^2(\theta_r/2)$$

Letting $\theta \longrightarrow 0$, we have the result. ∎

Paper II

Question 3(a) *Find and classify the extreme values of the function $f(x,y) = x^2 + y^2 + x + y + xy$.*

Solution. The extreme values are given by

$$\frac{\partial f}{\partial x} = 2x + 1 + y = 0,\ \frac{\partial f}{\partial y} = 2y + 1 + x = 0$$

Solving these, we get $x = -\frac{1}{3}, y = -\frac{1}{3}$. Thus there exists only one extreme value at $(-\frac{1}{3}, -\frac{1}{3})$. Now $\dfrac{\partial^2 f}{\partial x^2} = 2, \dfrac{\partial^2 f}{\partial y^2} = 2, \dfrac{\partial^2 f}{\partial x\,\partial y} = 1$. Since $\dfrac{\partial^2 f}{\partial x^2} > 0, \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \Big(\dfrac{\partial^2 f}{\partial x\,\partial y}\Big)^2 = 3 > 0$ at $(-\frac{1}{3}, -\frac{1}{3})$, it follows that f has a minimum at $(-\frac{1}{3}, -\frac{1}{3})$. ∎

Question 3(b) *Suppose α is real and different from $n\pi, n \in \mathbb{Z}$, then prove that*

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+2xy\cos\alpha+y^2)}\,dx\,dy = \frac{\alpha}{2\sin\alpha}$$

Solution. The region of integration is the first quadrant. Switching to polar coordinates, we have

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2(1+2\cos\theta\sin\theta\cos\alpha)} r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2(1+2\cos\theta\sin\theta\cos\alpha)}}{1+2\cos\theta\sin\theta\cos\alpha}\Big(-\frac{1}{2}\Big)\right]_0^\infty d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+2\cos\theta\sin\theta\cos\alpha} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,d\theta}{\sec^2\theta+2\tan\theta\cos\alpha} \qquad (\text{Put } \tan\theta = z) \\
&= \frac{1}{2}\int_0^\infty \frac{dz}{1+z^2+2z\cos\alpha} = \frac{1}{2}\int_0^\infty \frac{dz}{1-\cos^2\alpha+(z+\cos\alpha)^2} = \frac{1}{2}\int_0^\infty \frac{dz}{\sin^2\alpha+(z+\cos\alpha)^2} \\
&= \frac{1}{2}\frac{1}{\sin\alpha}\tan^{-1}\Big(\frac{z+\cos\alpha}{\sin\alpha}\Big)\Big]_0^\infty \\
&= \frac{1}{2}\frac{1}{\sin\alpha}\left[\frac{\pi}{2}-\tan^{-1}\cot\alpha\right] = \frac{1}{2}\frac{1}{\sin\alpha}\left[\frac{\pi}{2}-\Big(\frac{\pi}{2}-\alpha\Big)\right] \qquad \because \cot\alpha = \tan\Big(\frac{\pi}{2}-\alpha\Big) \\
&= \frac{\alpha}{2\sin\alpha} \text{ as required.}
\end{aligned}$$

■

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Question 1(a) *Find the asymptotes of the curve*

$$4(x^4 + y^4) - 17x^2y^2 - 4x(4y^2 - x^2) + 2(x^2 - 2) = 0$$

and show that they pass through the points of intersection of the curve with the ellipse $x^2 + 4y^2 = 4$.

Solution. The curve has no asymptotes parallel to the coordinate axes as the coefficients of x^4 and y^4 are constants, note that these are the highest powers of x, y present in the equation.

If $y = mx + c$ is an asymptote, then m is a root of $\Phi_4(m) = 4(1 + m^4) - 17m^2 = 0$ — this can be obtained by dividing by x^4 and letting $x \longrightarrow \infty$, noting that $\frac{y}{x} \longrightarrow m$.

Thus $4m^4 - 17m^2 + 4 = 0 \Rightarrow m^2 = 4, m^2 = \frac{1}{4} \Rightarrow m = 2, -2, \frac{1}{2}, -\frac{1}{2}$.

Let $\Phi_3(m) = -4(4m^2 - 1)$, then c is given as follows (provided it is determinate):

$$c = -\frac{\Phi_3(m)}{\Phi_4'(m)} = \frac{16m^2 - 4}{16m^3 - 34m}$$

Thus

$$\begin{aligned}
m = 2 \quad &\Rightarrow \quad c = \frac{64 - 4}{128 - 68} = 1 \\
m = -2 \quad &\Rightarrow \quad c = \frac{64 - 4}{-128 + 68} = -1 \\
m = \frac{1}{2} \quad &\Rightarrow \quad c = \frac{4 - 4}{2 - 17} = 0 \\
m = -\frac{1}{2} \quad &\Rightarrow \quad c = \frac{4 - 4}{-2 + 17} = 0
\end{aligned}$$

Hence the asymptotes are $y = 2x + 1, y = -2x - 1, 2y = x, 2y + x = 0$.

Let $P_4 = 0$ be the joint equation of the asymptotes, then

$$\begin{aligned} P_4 &= (y - 2x - 1)(y + 2x + 1)(2y - x)(2y + x) \\ &= (y^2 - (2x + 1)^2)(4y^2 - x^2) \\ &= 4y^4 - 17x^2y^2 - 16xy^2 - 4y^2 + 4x^4 + 4x^3 + x^2 \end{aligned}$$

Thuis the points of intersection of the given curve ($f(x, y) = 0$) and the joint equation of the asymptotes lie on $f(x, y) - P_4 = 0 = x^2 + 4y^2 - 4$. Thus the points of intersection lie on the ellipse $x^2 + 4y^2 = 4$. ∎

Question 1(b) *Show that any continuous function defined for all real x and satisfying $f(x) = f(2x + 1)$ must be a constant function.*

Solution. Clearly $f(x) = f(\frac{x-1}{2}) = f(\frac{x}{2} - \frac{1}{2}) = f(\frac{x}{2^2} - \frac{1}{4} - \frac{1}{2}) = \ldots = f(\frac{x}{2^n} - \frac{2^n-1}{2^n}) = f(\frac{x-1}{2^n} - 1)$.

Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$ for any convergent sequence of real numbers. Thus

$$f(x) = \lim_{n\to\infty} f\Big(\frac{x-1}{2^n} - 1\Big) = f\Big(\lim_{n\to\infty}\Big(\frac{x-1}{2^n} - 1\Big)\Big) = f(-1)$$

Thus f is a constant function. ∎

Question 1(c) *Show that the maximum and minimum of the radii vectors of the sections of the surface*

$$(x^2 + y^2 + z^2)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

by the plane $\lambda x + \mu y + \nu z = 0$ are given by the equation

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0$$

Solution. We have to find extreme values of $r^2 = x^2 + y^2 + z^2$ subject to the constraints

$$\begin{aligned} \phi_1 &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - (x^2 + y^2 + z^2)^2 = 0 \\ \phi_2 &= \lambda x + \mu y + \nu z = 0 \end{aligned}$$

Let $F(x, y, z) = x^2 + y^2 + z^2 + \lambda\phi_1 + \lambda\phi_2$, where λ_1, λ_2 are the undetermined Lagrange multipliers. For extreme values:

$$\begin{aligned} \frac{\partial F}{\partial x} &= 2x + \lambda_1\left[\frac{2x}{a^2} - 4xr^2\right] + \lambda_2\lambda = 0 && (i) \\ \frac{\partial F}{\partial y} &= 2y + \lambda_1\left[\frac{2y}{b^2} - 4yr^2\right] + \lambda_2\mu = 0 && (ii) \\ \frac{\partial F}{\partial z} &= 2z + \lambda_1\left[\frac{2z}{c^2} - 4zr^2\right] + \lambda_2\nu = 0 && (iii) \end{aligned}$$

The operation $x \times (i) + y \times (ii) + z \times (iii)$ gives us

$$2r^2 + 2\lambda_1 \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 2r^4\right] + \lambda_2(\lambda x + \mu y + \nu z) = 0$$

Thus $r^2 + \lambda_1(r^4 - 2r^4) = 0 \Rightarrow \lambda_1 r^4 = r^2 \Rightarrow \lambda_1 = \frac{1}{r^2}$. Now from $(i), (ii), (iii)$, we get

$$x = \frac{\lambda_2 \lambda}{\frac{1}{r^2}\big((4r^2 - \frac{2}{a^2}) - 2r^2\big)} = \frac{\lambda_2\lambda}{2\big(1 - \frac{1}{a^2r^2}\big)}, \quad y = \frac{\lambda_2\mu}{2\big(1 - \frac{1}{b^2r^2}\big)}, \quad z = \frac{\lambda_2\nu}{2\big(1 - \frac{1}{c^2r^2}\big)}$$

Substituting x, y, z in $\lambda x + \mu y + \nu z = 0$, we get that r^2 is given by the equation

$$\frac{a^2\lambda^2r^2}{a^2r^2 - 1} + \frac{b^2\mu^2r^2}{b^2r^2 - 1} + \frac{c^2\nu^2r^2}{c^2r^2 - 1} = 0$$

or

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0$$

as $r^2 \neq 0$. Note that extreme values do exist and are roots of the given equation. ∎

Additional: If the surface under consideration is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ instead on the one above, then $\lambda_1 = -r^2, x = \frac{\lambda_2\lambda}{2\big(\frac{r^2}{a^2} - 1\big)}, y = \frac{\lambda_2\mu}{2\big(\frac{r^2}{b^2} - 1\big)}, z = \frac{\lambda_2\nu}{2\big(\frac{r^2}{c^2} - 1\big)}$ and the equation giving radius vectors of the sections is

$$\frac{a^2\lambda^2}{r^2 - a^2} + \frac{b^2\mu^2}{r^2 - b^2} + \frac{c^2\nu^2}{r^2 - c^2} = 0$$

Particular case: If the surface is $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ then $a^2 = 4, b^2 = 5, c^2 = 25$. If the plane is $z - x - y = 0$, then $\lambda = -1, \mu = -1, \nu = 1$. The above equation becomes

$$\frac{4}{r^2 - 4} + \frac{5}{r^2 - 5} + \frac{25}{r^2 - 25} = 0$$

or

$$4[r^4 - 30r^2 + 125] + 5[r^4 - 29r^2 + 100] + 25[r^4 - 9r^2 + 20] = 0$$

or $34r^4 - 490r^2 + 1500 = 0 \Rightarrow r^2 = 10, \frac{75}{17}$.

Question 2(a) *If* $u = f\big(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\big)$, *prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution. This follows from Euler's theorem on homogeneous functions.

Theorem (Euler): If $f(x, y, z)$ is a homogeneous function of degree n i.e.$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)$$

Proof: Differentiating $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x,y,z)$ partially with respect to λ, we get

$$x\frac{\partial f}{\partial x}(\lambda x, \lambda y, \lambda z) + y\frac{\partial f}{\partial y}(\lambda x, \lambda y, \lambda z) + z\frac{\partial f}{\partial z}(\lambda x, \lambda y, \lambda z) = n\lambda^{n-1}f(x,y,z)$$

(assuming of course that differentiation is possible.) Putting $\lambda = 1$, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

Q.E.D.

In the current case, $n = 0$ because

$$f(\frac{\lambda x}{\lambda y}, \frac{\lambda y}{\lambda z}, \frac{\lambda z}{\lambda x}) = \lambda^0 f(\frac{x}{y}, \frac{y}{z}, \frac{z}{x})$$

Thus $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0u = 0$.

Note: The converse of Euler's theorem is also true — if $f(x,y,z)$ has continuous partial derivatives of the second order, and

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

then f is a homogeneous function of degree n.

Proof: Let $(x_0, y_0, z_0) \in \mathbb{R}^3$. Let $\phi(\lambda) = f(\lambda x_0, \lambda y_0, \lambda z_0)$, defined for $\lambda > 0$. Then

$$\phi'(\lambda) = x_0\frac{\partial f}{\partial x}(\lambda x_0, \lambda y_0, \lambda z_0) + y_0\frac{\partial f}{\partial y}(\lambda x_0, \lambda y_0, \lambda z_0) + z_0\frac{\partial f}{\partial z}(\lambda x_0, \lambda y_0, \lambda z_0) = \frac{n}{\lambda}f(\lambda x_0, \lambda y_0, \lambda z_0)$$

from $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$, which evaluated at the point $\lambda x_0, \lambda y_0, \lambda z_0)$ becomes

$$\lambda x_0\frac{\partial f}{\partial x}(\lambda x_0, \lambda y_0, \lambda z_0) + \lambda y_0\frac{\partial f}{\partial y}(\lambda x_0, \lambda y_0, \lambda z_0) + \lambda z_0\frac{\partial f}{\partial z}(\lambda x_0, \lambda y_0, \lambda z_0) = nf(\lambda x_0, \lambda y_0, \lambda z_0)$$

Thus

$$\lambda\phi'(\lambda) = nf(\lambda x_0, \lambda y_0, \lambda z_0) = n\phi(\lambda)$$

Differentiating $\lambda^{-n}\phi(\lambda)$ with respect to λ, we get

$$\frac{d}{d\lambda}(\lambda^{-n}\phi(\lambda) = \lambda^{-n}\phi'(\lambda) - n\lambda^{-n-1}\phi(\lambda) = \lambda^{-n}[\phi'(\lambda) - n\lambda^{-1}\phi(\lambda)] = 0$$

Thus $\lambda^{-n}\phi(\lambda) = C$ some constant, so $\phi(\lambda) = C\lambda^n$. For $\lambda = 1, \phi(1) = C = f(x_0, y_0, z_0)$. Thus

$$\phi(\lambda) = f(\lambda x_0, \lambda y_0, \lambda z_0) = \lambda^n f(x_0, y_0, z_0)$$

Thus f is homogeneous of degree n. ∎

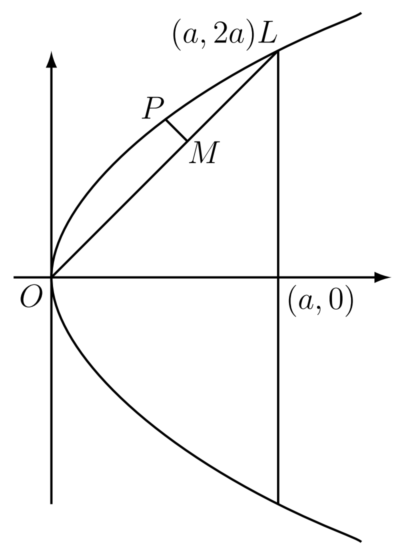
Question 2(b) *Evaluate*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

Solution. See 2006 question 2(c). ∎

Question 2(c) *The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through 4 right angles about the chord. Find the volume of the solid so formed.*

Solution.



The volume of the resulting solid is $V = \int_0^a \pi(PM)^2 d(OM)$ where PM is perpendicular to the chord $y = 2x$.

Let $P = (x, y)$. Then

$$\begin{aligned}
PM &= \frac{|y-2x|}{\sqrt{5}} \\
(OM)^2 &= (OP)^2 - (PM)^2 \\
&= x^2 + y^2 - \frac{y^2 - 4xy + 4x^2}{5} \\
&= \frac{4y^2 + 4xy + x^2}{5} \\
OM &= \frac{2y + x}{\sqrt{5}} = \frac{4\sqrt{ax} + x}{\sqrt{5}} \quad \because y = 2\sqrt{ax}
\end{aligned}$$

$$\begin{aligned}
d(OM) &= \frac{2\frac{\sqrt{a}}{\sqrt{x}} + 1}{\sqrt{5}}\,dx = \frac{2\sqrt{a} + \sqrt{x}}{\sqrt{5x}}\,dx \\
(PM)^2 &= \frac{y^2 - 4xy + 4x^2}{5} = \frac{4ax - 4x\sqrt{4ax} + 4x^2}{5} \\
V &= \frac{4\pi}{5\sqrt{5}} \int_0^a (ax - x\sqrt{4ax} + x^2)\frac{2\sqrt{a} + \sqrt{x}}{\sqrt{x}}\,dx \\
&= \frac{4\pi}{5\sqrt{5}} \int_0^a (a - \sqrt{4ax} + x)(2\sqrt{ax} + x)\,dx \\
&= \frac{4\pi}{5\sqrt{5}} \int_0^a (x^2 - 4ax + ax + 2a\sqrt{ax})\,dx \\
&= \frac{4\pi}{5\sqrt{5}} \Big[\frac{x^3}{3} - \frac{3ax^2}{2} + 2a^{\frac{3}{2}}x^{\frac{3}{2}} \cdot \frac{2}{3}\Big]_0^a \\
&= \frac{4\pi}{5\sqrt{5}} \Big[\frac{a^3}{3} - \frac{3a^3}{2} + \frac{4}{3}a^3\Big] = \frac{4\pi}{5\sqrt{5}}\frac{a^3}{6} = \frac{2\pi a^3}{15\sqrt{5}}
\end{aligned}$$

∎

Paper II

Question 3(a) *A function f is defined in the interval (a, b) as follows:*

$$f(x) = \begin{cases} \frac{1}{q^2}, & \textit{when } x = \frac{p}{q} \\ \frac{1}{q^3}, & \textit{when } x = \sqrt{\frac{p}{q}} \\ 0, & \textit{otherwise} \end{cases}$$

where p, q are relatively prime integers. Is f Riemann integrable? Justify your answer.

Solution. See 2001, question 4(a). ∎

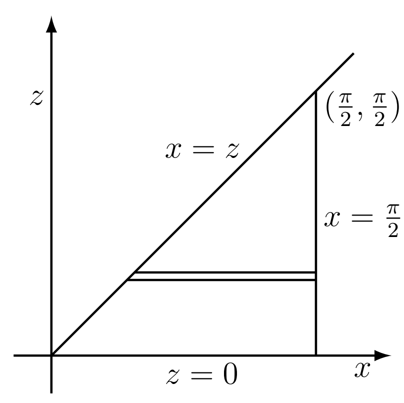
Question 3(b) *Evaluate* $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin x \sin^{-1}(\sin x \sin y)\, dx\, dy$

Solution. We first evaluate $I_1 = \displaystyle\int_0^{\frac{\pi}{2}} \sin x \sin^{-1}(\sin x \sin y)\, dy$.

Let $\sin x \sin y = \sin z$, so that $\sin x \cos y\, dy = \cos z\, dz$, when $y = 0, z = 0$ and when $y = \frac{\pi}{2}$, $\sin z = \sin x \Rightarrow z = x$, as in the given integral $0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}$.

Thus $I_1 = \displaystyle\int_0^x \sin x \frac{z \cos z\, dz}{\sin x \sqrt{1 - \frac{\sin^2 z}{\sin^2 x}}} = \int_0^x \frac{z \sin x \cos z}{\sqrt{\sin^2 x - \sin^2 z}}\, dz$ and the given integral becomes $\displaystyle\int_0^{\frac{\pi}{2}}\int_0^x \frac{z \sin x \cos z}{\sqrt{\sin^2 x - \sin^2 z}}\, dz\, dx$.

We now change the order of integration in the right hand side. Now x varies from z to $\frac{\pi}{2}$ and z varies from 0 to $\frac{\pi}{2}$. Thus the given integral is



$$\begin{aligned}
& \int_0^{\frac{\pi}{2}} z \cos z \int_z^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\sin^2 x - \sin^2 z}}\, dx\, dz \\
= & \int_0^{\frac{\pi}{2}} z \cos z \int_z^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{\cos^2 z - \cos^2 x}}\, dx\, dz \\
= & \int_0^{\frac{\pi}{2}} z \cos z \Big[-\sin^{-1}\Big(\frac{\cos x}{\cos z}\Big)\Big]_z^{\frac{\pi}{2}} dz = \int_0^{\frac{\pi}{2}} z \cos z \sin^{-1}(1)\, dz \\
= & \frac{\pi}{2}\int_0^{\frac{\pi}{2}} z \cos z\, dz = \frac{\pi}{2}\Big[z \sin z + \cos z\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\Big(\frac{\pi}{2} - 1\Big)
\end{aligned}$$

∎

UPSC Civil Services Main 1997 - Mathematics Calculus

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January 9, 2010

Question 1(a) *Suppose* $f(x) = 17x^{12} - 124x^9 + 16x^3 - 129x^2 + x - 1$. *Determine* $\frac{d}{dx}(f^{-1})$ *at* $x = -1$, *if it exists.*

Solution. Clearly $f(0) = -1$ and $f'(x) = 204x^{11} - 1116x^8 + 48x^2 - 258x + 1$. Since $f'(0) = 1$ and f' is continuous, there exists a neighborhood of 0 in which $f'(x) > 0$, so f is strictly increasing in a neighborhood of 0, thus f is one-one in a neighborhood of 0.

Now we use:

Theorem: If $f : (a, b) \longrightarrow \mathbb{R}$ is continuous and one-one, and $f'(c) \neq 0$ for all $c \in (a, b)$, then f^{-1} is differentiable at $f(c)$ and

$$\frac{df^{-1}}{dx} \text{ at } f(c) = \frac{1}{f'(c)}$$

Thus,

$$\frac{df^{-1}}{dx} \text{ at } -1 = \frac{1}{f'(0)} = 1$$

∎

Question 1(b) *Prove that the volume of the greatest parallelopiped that can be inscribed in the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ *is* $\frac{8abc}{3\sqrt{3}}$.

Solution. Let the sides of the parallelopiped be $2x, 2y, 2z$. We have to maximize $V = 8xyz$ subject to the conditions $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $x > 0, y > 0, z > 0$.

Let $F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$, where λ is Lagrange's undetermined multiplier. For extreme values,

$$\begin{aligned} \frac{\partial F}{\partial x} &= 8yz + 2\lambda\frac{x}{a^2} = 0 && (i) \\ \frac{\partial F}{\partial y} &= 8xz + 2\lambda\frac{y}{b^2} = 0 && (ii) \\ \frac{\partial F}{\partial z} &= 8xy + 2\lambda\frac{z}{c^2} = 0 && (iii) \end{aligned}$$

$x \times (i) + y \times (ii) + z \times (iii)$ gives us

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

Since $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ we get $\lambda = -12xyz$. From (i) we now get $8yz - 24\frac{x^2}{a^2}yz = 0 \Rightarrow \frac{x^2}{a^2} = \frac{1}{3}$ as $y > 0, z > 0$. Thus $x = \frac{a}{\sqrt{3}}$. Similarly, $y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$ and $V = \frac{8abc}{3\sqrt{3}}$. This is the maximum volume.

Check:

$$\begin{aligned} F &= 8xyz - \frac{4abc}{\sqrt{3}}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) \\ d^2F &= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)^2 F \\ &= \frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial y^2}(dy)^2 + \frac{\partial^2 F}{\partial z^2}(dz)^2 + 2\frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy + 2\frac{\partial^2 F}{\partial y\,\partial z}\,dy\,dz + 2\frac{\partial^2 F}{\partial x\,\partial z}\,dx\,dz \\ &= \frac{2\lambda}{a^2}(dx)^2 + \frac{2\lambda}{b^2}(dy)^2 + \frac{2\lambda}{c^2}(dz)^2 + 16(z\,dx\,dy + x\,dy\,dz + y\,dx\,dz) \end{aligned}$$

Now since $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0$, $\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}+\frac{z\,dz}{c^2} = 0$, and putting $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$ we get $\frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0$ or $-\frac{dx}{a} - \frac{dy}{b} = \frac{dz}{c}$. Substituting this and the values of λ, x, y, z, we get

$$\begin{aligned} d^2F &= -\frac{8abc}{\sqrt{3}}\left[\frac{(dx)^2}{a^2} + \frac{(dy)^2}{b^2} + \left(\frac{dx}{a} + \frac{dy}{b}\right)^2 - 2\frac{dx\,dy}{ab} + 2\frac{dy}{b}\left(\frac{dx}{a} + \frac{dy}{b}\right) + 2\frac{dx}{a}\left(\frac{dx}{a} + \frac{dy}{b}\right)\right] \\ &= -\frac{8abc}{\sqrt{3}}\left[4\frac{(dx)^2}{a^2} + 4\frac{(dy)^2}{b^2} + 4\frac{dx\,dy}{ab}\right] \\ &= -\frac{32abc}{\sqrt{3}}\left[\left(\frac{dx}{a} + \frac{dy}{2b}\right)^2 + \frac{3}{4}\left(\frac{dy}{b}\right)^2\right] \end{aligned}$$

Clearly $d^2F < 0$ when $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$. Hence $V = \frac{8abc}{3\sqrt{3}}$ is a maximum. ∎

Question 1(c) *Show that the four asymptotes to the curve*

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle of radius 1.

Solution. Here $\phi_4 = (x^2 - y^2)(y^2 - 4x^2)$ and $\phi_3 = 6x^3 - 5x^2y - 3xy^2 + 2y^3$.

Thus the slopes of the asymptotes are given by $\phi_4(m) = (1 - m^2)(m^2 - 4) = 0 \Rightarrow m = \pm 1, \pm 2$.

Now $\phi_3(m) = 6 - 5m - 3m^2 + 2m^3$. $\phi_3(1) = 6 - 5 - 3 + 2 = 0, \phi_3(-1) = 6 + 5 - 3 - 2 = 6, \phi_3(2) = 6 - 10 - 12 + 16 = 0, \phi_3(-2) = 6 + 10 - 12 - 16 = -12$.

$\phi_4'(m) = -2m(m^2 - 4) + 2m(1 - m^2) = -4m^3 + 10m$. Thus $\phi_4'(1) = 6, \phi_4'(-1) = -6, \phi_4'(2) = -12, \phi_4'(-2) = 12$.

For asymptotes $y = mx + c$, the constant term c is given by $c\phi_4'(m) + \phi_3(m) = 0 \Rightarrow c = \dfrac{-\phi_3(m)}{\phi_4'(m)}$.

Thus $m = 1 \Rightarrow c = 0, m = -1 \Rightarrow c = 1, m = 2 \Rightarrow c = 0, m = -2 \Rightarrow c = 1$.

Thus the four asymptotes are $y = x, y = -x + 1, y = 2x, y = -2x + 1$. The combined equation of the asymptotes is

$$\begin{aligned}
& (y - x)(y + x - 1)(y - 2x)(y + 2x - 1) = 0 \\
\Rightarrow\ & (y^2 - x^2 - y + x)(y^2 - 4x^2 - y + 2x) = 0 \\
\Rightarrow\ & (y^2 - x^2)(y^2 - 4x^2) - y^3 + 4x^2y + y^2 - 2xy + xy^2 - 4x^3 - xy \\
& +2x^2 - y^3 + x^2y + 2xy^2 - 2x^3 = 0 \\
\Rightarrow\ & (y^2 - x^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - 3xy + y^2 = 0
\end{aligned}$$

Let $P_n = (y^2 - x^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - 3xy + y^2$. The joint equation of the asymptotes is $P_n = 0$.

Now the equation of the given curve is $P_n + x^2 + y^2 - 1 = 0$, thus the points of intersection of the asymptotes and the curve must satisfy $P_n = 0, P_n + x^2 + y^2 - 1 = 0 \Rightarrow x^2 + y^2 - 1 = 0$, so all the intersection points must lie on the curve $x^2 + y^2 - 1 = 0$, which is clearly a circle of radius 1.

Note that any point where an asymptote cuts the circle satisfies the curve as well, so each point where an asymptote intersects the circle is a point of intersection with the curve. It can be easily checked that each asymptote cuts the circle twice, so there are eight points of intersection (two of these coincide at $(0, 1)$, so there are really only 7 distinct points of intersection). ∎

Question 2(a) *An area bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves around one of the tangents. Find the volume thus generated.*

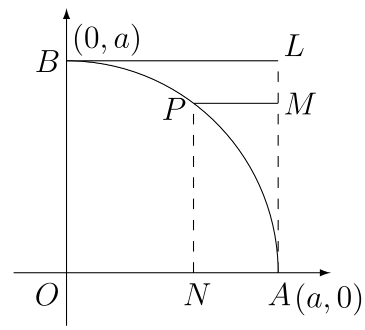
Solution.

If P is (x, y), and the circle is $x^2+y^2 = a^2$, then $PM = a - x$. $PN = y$.

Area $AMLBA$ revolving around AL has volume

$$V = \pi \int (PM)^2 d(AM) = \pi \int_0^a (a - x)^2\, dy$$

Using the parametric equation of $x^2+y^2 = a^2$ i.e. $x = a\cos\theta, y = a\sin\theta$,



$$\begin{aligned}
V &= \pi \int_0^{\frac{\pi}{2}} (a - a\cos\theta)^2 a\cos\theta\, d\theta \\
&= \pi a^3 \int_0^{\frac{\pi}{2}} \left(1 - 2\cos\theta + \cos^2\theta\right)\cos\theta\, d\theta \\
&= \pi a^3 \left[1 - 2\cdot\frac{1}{2}\frac{\pi}{2} + \frac{2}{3}\right] = \pi a^3 \left[\frac{5}{3} - \frac{\pi}{2}\right]
\end{aligned}$$

$\blacksquare$

To find the surface area of the above solid of revolution, we proceed as follows:

$$\begin{aligned}
S &= 2\pi \int PM\, ds = 2\pi \int_0^{\frac{\pi}{2}} (a - a\cos\theta)\frac{ds}{d\theta}\, d\theta \\
\frac{ds}{d\theta} &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a \\
S &= 2\pi \int_0^{\frac{\pi}{2}} (a - a\cos\theta)a\, d\theta \\
&= 2\pi a^2 \Big[\theta - \sin\theta\Big]_0^{\frac{\pi}{2}} = \pi a^2(\pi - 2)
\end{aligned}$$

Question 2(b) *Show how the change of order in the integral* $\int_0^\infty \int_0^\infty e^{-xy} \sin x\, dy\, dx$ *leads to the evaluation of* $\int_0^\infty \frac{\sin x}{x}\, dx$. *Hence evaluate it.*

Solution.

$$\begin{aligned}
I &= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin x\, dy\right) dx = \int_0^\infty \sin x \left[-\frac{e^{-xy}}{x}\right]_0^\infty dx = \int_0^\infty \frac{\sin x}{x}\, dx \\
I &= \int_0^\infty \left(\int_0^\infty e^{-xy}\sin x\, dx\right) dy
\end{aligned}$$

Now

$$\begin{aligned}
\int e^{-xy}\sin x\,dx &= -e^{-xy}\cos x + \int(-y)e^{-xy}\cos x\,dx \\
&= -e^{-xy}\cos x - y\left(e^{-xy}\sin x - \int(-y)e^{-xy}\sin x\,dx\right) \\
\Rightarrow (1+y^2)\int e^{-xy}\sin x\,dx &= -e^{-xy}\cos x - ye^{-xy}\sin x \\
\Rightarrow \int e^{-xy}\sin x\,dx &= -\frac{e^{-xy}}{1+y^2}(\cos x + y\sin x) \\
\Rightarrow \int_0^\infty e^{-xy}\sin x\,dx &= \frac{1}{1+y^2}
\end{aligned}$$

as $e^{-xy}(\cos x + y\sin x) \longrightarrow 0$ as $x \longrightarrow \infty$ and $e^{-xy}(\cos x + y\sin x) \longrightarrow 1$ as $x \longrightarrow 0$.

Thus $I = \displaystyle\int_0^\infty \frac{dy}{1+y^2} = \tan^{-1} y\Big]_0^\infty = \frac{\pi}{2}$, hence $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ ∎

Question 2(c) *Show that*

$$\Gamma(n)\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n), \quad n > 0$$

Solution. We consider the well known identity

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad (1)$$

on substituting $x = \sin^2\theta$.

Put $n = \frac{1}{2}$ in (1). We get

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\,d\theta$$

Put $m = n$ in (1). We get

$$\begin{aligned}
\frac{(\Gamma(m))^2}{\Gamma(2m)} &= 2\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^{2m-1}\,d\theta \\
&= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^{2m-1}\,d\theta \\
&= \frac{1}{2^{2m-2}}\int_0^{\pi} (\sin\phi)^{2m-1}\,d\phi = \frac{1}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (\sin\phi)^{2m-1}\,d\phi
\end{aligned}$$

Thus

$$\frac{\Gamma(m)\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})} = \frac{(\Gamma(m))^2}{\Gamma(2m)}2^{2m-1}$$

or

$$\Gamma(m)\Gamma(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

as required. ∎

Paper II

Question 3(a) *Show that* $\iiint_D xyz\,dx\,dy\,dz = \frac{a^2b^2c^2}{6}$ *where the domain* D *is given by* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$.

Solution. Note: There seems to be a misprint in this question — the integral as given is actually 0. This is because the integrand is an odd function of x, y, z, and the domain of integration is symmetric about the origin. We will compute the integral for the positive octant only.

Put $\frac{x^2}{a^2} = u, \frac{y^2}{b^2} = v, \frac{z^2}{c^2} = w$ so that $dx = \frac{a\,du}{2\sqrt{u}}, dy = \frac{b\,dv}{2\sqrt{v}}, dz = \frac{c\,dw}{2\sqrt{w}}$ and the positive octant of D is transformed to $u+v+w \le 1, u \ge 0, u \ge 0, v \ge 0, w \ge 0$.

The given integral is thus transformed into

$$\begin{aligned}
I &= \iiint_{\substack{u+v+w\le 1\\ u\ge 0, v\ge 0, w\ge 0}} abc\; u^{\frac{1}{2}}v^{\frac{1}{2}}w^{\frac{1}{2}}\;\frac{abc}{8}\; u^{-\frac{1}{2}}v^{-\frac{1}{2}}w^{-\frac{1}{2}}\,du\,dv\,dw \\
&= \frac{a^2b^2c^2}{8}\int_0^1\int_0^{1-u}\int_0^{1-u-v} dw\,dv\,du \\
&= \frac{a^2b^2c^2}{8}\int_0^1\int_0^{1-u}(1-u-v)\,dv\,du \\
&= \frac{a^2b^2c^2}{8}\int_0^1\Big[(1-u)-u(1-u)-\frac{(1-u)^2}{2}\Big]\,du \\
&= \frac{a^2b^2c^2}{16}\int_0^1 (1-u)^2\,du = \frac{a^2b^2c^2}{48}
\end{aligned}$$

In fact, this shows that $\iiint_D |xyz|\,dx\,dy\,dz = \frac{a^2b^2c^2}{6}$. ∎

Question 3(b) *If* $u = \sin^{-1}(x^2+y^2)^{\frac{1}{5}}$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2}{25}\tan u(2\tan^2 u - 3)$$

Solution. If $f(x,y)$ is a homogeneous function of degree n possessing continuous partial derivatives of degree 2, then by the result from 2006, question 2(b), we have

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$$

Let $v=\sin u=(x^2+y^2)^{\frac{1}{5}}$, then v is homogeneous of degree $\frac{2}{5}$ and therefore by the above theorem,

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\,\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=\frac{2}{5}\left(\frac{2}{5}-1\right)v$$

Now $v=\sin u$, $\frac{\partial v}{\partial x}=\cos u\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}=\cos u\frac{\partial u}{\partial y}$ and

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= -\sin u\left(\frac{\partial u}{\partial x}\right)^2+\cos u\frac{\partial^2 u}{\partial x^2}\\ \frac{\partial^2 v}{\partial y^2} &= -\sin u\left(\frac{\partial u}{\partial y}\right)^2+\cos u\frac{\partial^2 u}{\partial y^2}\\ \frac{\partial^2 v}{\partial x\,\partial y} &= -\sin u\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+\cos u\frac{\partial^2 u}{\partial x\,\partial y}\end{aligned}$$

Substituting these in the above result, we have

$$\cos u\left[x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right]-\sin u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right)+y^2\left(\frac{\partial u}{\partial y}\right)^2\right]=-\frac{6}{25}\sin u$$

or

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{6}{25}\tan u+\tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)^2$$

But

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{2}{5}v=\frac{2}{5}\sin u$$

therefore

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{6}{25}\tan u+\frac{4}{25}\tan^3 u=\frac{2}{25}\tan u(2\tan^2 u-3)$$

as required. ∎

UPSC Civil Services Main 1998 - Mathematics Calculus

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January 2, 2010

Question 1(a) *Find the asymptotes of the curve*

$$(2x - 3y + 1)^2(x + y) - 8x + 2y - 9 = 0$$

and show that they intersect the curve again in three points which lie on a straight line.

Solution. The given equation is

$$(2x - 3y)^2(x + y) + 2(2x - 3y)(x + y) + x + y - 8x + 2y - 9 = 0$$

Now $x^3\phi_3(\frac{y}{x})$ = homogeneous terms of degree 3 = $(2x - 3y)^2(x + y)$. If $y = mx + c$ is an asymptote, then m is a root of $\phi_3(m) = (2 - 3m)^2(1 + m) = 0 \Rightarrow m = -1, m = \frac{2}{3}, \frac{2}{3}$, so we may have two parallel asymptotes.

For $m = -1$, $c = -\frac{\phi_2(-1)}{\phi_3'(-1)} = \frac{0}{\phi_3'(-1)} = 0$, as $\phi_3'(-1) \neq 0$. Thus $x + y = 0$ is one asymptote.

For parallel asymptotes $y = \frac{2}{3}x + c$, c is the root of

$$\frac{c^2}{2!}\phi_3''(\frac{2}{3}) + c\phi_2'(\frac{2}{3}) + \phi_1(\frac{2}{3}) = 0$$

$$\begin{aligned}
\phi_3(m) &= (2 - 3m)^2(1 + m) \\
\phi_3'(m) &= (2 - 3m)^2 - 6(2 - 3m)(1 + m) \\
\phi_3''(m) &= -12(2 - 3m) + 18(1 + m) - 6(2 - 3m) \Rightarrow \phi_3''\left(\frac{2}{3}\right) = 30 \\
\phi_2(m) &= 2(1 + m)(2 - 3m) \\
\phi_2'(m) &= -6(1 + m) + 2(2 - 3m) \Rightarrow \phi_2'\left(\frac{2}{3}\right) = -10 \\
\phi_1(m) &= -7 + 3m \Rightarrow \phi_1\left(\frac{2}{3}\right) = -5
\end{aligned}$$

Thus $15c^2 - 10c - 5 = 0 \Rightarrow c = 1, -\frac{1}{3}$. So the asymptotes are $y = \frac{2}{3}x + 1, y = \frac{2}{3}x - \frac{1}{3}$ or $2x - 3y + 3 = 0, 2x - 3y - 1 = 0$.

The joint equation of the asymptotes is $P_n = (x + y)(2x - 3y + 3)(2x - 3y - 1) = 0$ or $P_n = (2x - 3y)^2(x + y) + 2(2x - 3y)(x + y) - 3(x + y) = 0$. The equation of the curve is $f(x, y) = P_n - 4x + 6y - 9 = 0$. This implies that the points of intersection of $f(x, y)$ and P_n lie on $4x - 6y + 9 = 0$, which is a straight line. But this line is parallel to the two parallel asymptotes, which means that the two aymptotes do not cut the curve — this can be verified by solving the simultaneous equations $f(x, y) = 0, 2x - 3y + 3 = 0$ and $f(x, y) = 0, 2x - 3y - 1 = 0$. The asymptote $x + y = 0$ cuts the curve at $(-\frac{9}{10}, \frac{9}{10})$, which lies on the line $4x - 6y + 9 = 0$.

(It seems that there is an error in the question. The following statement however is true — if there are 3 points at which the asymptotes of a cubic curve intersect it, then these must lie in a straight line.) ∎

Question 1(b) *A thin closed rectangular box is to have one edge n times another edge, and the volume of the box is given to be V. Prove that the least surface S is given by $nS^3 = 54(n + 1)^2V^2$.*

Solution. Let the edges be given by x, nx, y, so that $V = nx^2y \Rightarrow y = \frac{V}{nx^2}$. Then $S = 2(nx^2 + xy + nxy) = 2\left(nx^2 + \dfrac{(n + 1)V}{nx}\right)$. For critical points,

$$\begin{aligned}
\frac{dS}{dx} &= 2\left(2nx - \frac{(n + 1)V}{nx^2}\right) = 0 \\
\Rightarrow &\quad 2n^2x^3 - (n + 1)V = 0 \Rightarrow x^3 = \frac{(n + 1)V}{2n^2} \\
\frac{d^2S}{dx^2} &= 2\left(2n + \frac{2(n + 1)V}{nx^3}\right) = 2(2n + 4n) > 0 (\text{when } x^3 = \frac{(n + 1)V}{2n^2})
\end{aligned}$$

Thus S is minimum when $x^3 = \dfrac{(n + 1)V}{2n^2}$.

$S = \dfrac{2}{nx}(n^2x^3 + (n + 1)V) = \dfrac{3(n + 1)V}{nx}$, so $n^3x^3S^3 = 27(n + 1)^3V^3 \Rightarrow n(n + 1)VS^3/2 = 27(n + 1)^3V^3 \Rightarrow nS^3 = 54(n + 1)^2V^2$ as required. ∎

Question 1(c) *If $x + y = 1$, prove that*

$$\frac{d^n}{dx^n}(x^ny^n) = n!\left[y^n - \binom{n}{1}^2 y^{n-1}x + \binom{n}{2}^2 y^{n-2}x^2 + \ldots + (-1)^n x^n\right]$$

Solution. By the Leibnitz formula

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u_{n-r}v_r$$

Let $u = x^n, v = y^n = (1-x)^n$, then

$$\begin{aligned}
u_{n-r} &= n(n-1)\ldots(n-(n-r)+1)x^{n-(n-r)} = \frac{n!}{r!}x^r \\
v_r &= (-1)^r n(n-1)\ldots(n-r+1)(1-x)^{n-r} = (-1)^r \frac{n!}{(n-r)!}y^{n-r} \\
\binom{n}{r}u_{n-r}v_r &= \binom{n}{r}\frac{n!}{r!}x^r(-1)^r\frac{n!}{(n-r)!}y^{n-r} = (-1)^r n!\binom{n}{r}^2 x^r y^{n-r} \\
\Rightarrow \frac{d^n}{dx^n}((x^n y^n) &= n!\sum_{r=0}^{n}(-1)^r\binom{n}{r}^2 x^r y^{n-r} \\
&= n!\Big[y^n - \binom{n}{1}^2 y^{n-1}x + \binom{n}{2}^2 y^{n-2}x^2 + \ldots + (-1)^n x^n\Big]
\end{aligned}$$

∎

Question 2(a) *Show that*

$$\int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}}\,dx = B(p,q)$$

Solution. By definition, $B(p,q) = \int_0^1 x^{q-1}(1-x)^{p-1}\,dx$. Let $x = \frac{1}{1+y}, dx = \frac{-dy}{(1+y)^2}$.

$$\begin{aligned}
B(p,q) &= \int_\infty^0 \frac{1}{(1+y)^{q-1}}\left(1-\frac{1}{1+y}\right)^{p-1}\frac{-dy}{(1+y)^2} \\
&= \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy
\end{aligned}$$

as required. ∎

Question 2(b) *Show that* $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$ *where the integral is over all positive values of x, y, z for which the expression is real.*

Solution. Switching over to polar coordinates, $x = r\sin\theta\cos\phi, y = r\sin\theta\cos\phi, z = r\cos\phi$, $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$, we get

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (1-r^2)^{-\frac{1}{2}}r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{\pi}{2}\int_0^1 (1-r^2)^{-\frac{1}{2}}r^2\,dr \\
&\qquad \text{Put } r^2 = t \\
&= \frac{\pi}{2}\int_0^1 (1-t)^{-\frac{1}{2}}\frac{t^{\frac{1}{2}}}{2}\,dt
\end{aligned}$$

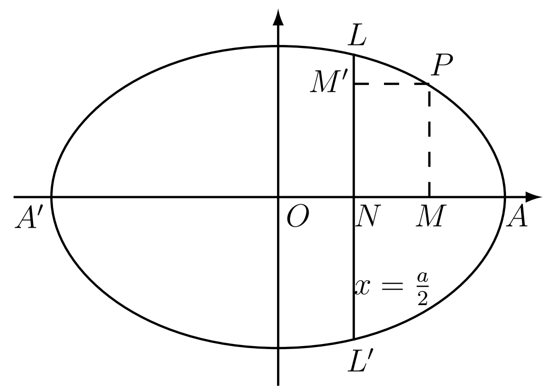
$$= \frac{\pi}{4}\int_0^1 (1-t)^{\frac{1}{2}-1} t^{\frac{3}{2}-1}\, dt$$
$$= \frac{\pi}{4} B\Big(\frac{3}{2}, \frac{1}{2}\Big) = \frac{\pi}{4}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{\pi}{4}\frac{1}{2}\Big(\Gamma\Big(\frac{1}{2}\Big)\Big)^2 = \frac{\pi^2}{8} \because \Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}$$

∎

Question 2(c) *The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $x = a/2$ and the smaller part is rotated through four right angles about this line. Prove that the volume generated is*

$$\pi a^2 b\Big[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\Big]$$

Solution.



The part rotated is $LL'AL$, the coordinates of L are $(\frac{a}{2}, y)$ where y is given by $\frac{a^2}{4a^2} + \frac{y^2}{b^2} = 1$. Thus L is the point $(\frac{a}{2}, \frac{\sqrt{3}}{2}b)$ and L' is the point $(\frac{a}{2}, -\frac{\sqrt{3}}{2}b)$. P is the generic point (x, y) so that $OM = x, NM = x - \frac{a}{2} = PM'$. The required volume

$$V = \int \pi(PM')^2\, d(NM') = 2\int_0^{\frac{\sqrt{3}}{2}b} \pi\Big(x - \frac{a}{2}\Big)^2 dy$$

Put $x = a\cos\theta, y = b\sin\theta$, so that θ varies from 0 to $\frac{\pi}{3}$.

$$\begin{aligned}
V &= 2\pi\int_0^{\frac{\pi}{3}} \Big(a\cos\theta - \frac{a}{2}\Big)^2 b\cos\theta\, d\theta \\
&= \frac{2\pi a^2 b}{4}\int_0^{\frac{\pi}{3}} (2\cos\theta - 1)^2\cos\theta\, d\theta \\
&= \frac{\pi a^2 b}{2}\int_0^{\frac{\pi}{3}} \Big(4\cos\theta(1-\sin^2\theta) - 4\cos^2\theta + \cos\theta\Big)\, d\theta \\
&= \frac{\pi a^2 b}{2}\Big[5\sin\theta - \frac{4\sin^3\theta}{3}\Big]_0^{\frac{\pi}{3}} - 4\frac{\pi a^2 b}{2}\int_0^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2}\, d\theta \\
&= \frac{\pi a^2 b}{2}\Big[5\sin\theta - \frac{4\sin^3\theta}{3} - 2\theta - \sin 2\theta\Big]_0^{\frac{\pi}{3}} \\
&= \frac{\pi a^2 b}{2}\Big[5\frac{\sqrt{3}}{2} - \frac{4}{3}\frac{3\sqrt{3}}{8} - 2\frac{\pi}{3} - \frac{\sqrt{3}}{2}\Big] \\
&= \frac{\pi a^2 b}{2}\Big[\frac{3\sqrt{3}}{2} - 2\frac{\pi}{3}\Big] = \pi a^2 b\Big[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\Big]
\end{aligned}$$

∎

Paper II

Question 3(a) *Show that the function* $f(x,y) = 2x^4 - 3x^2y + y^2$ *has* $(0,0)$ *as the only critical point but the function has neither a minimum nor a maximum at* $(0,0)$*.*

Solution. For critical points, $\frac{\partial f}{\partial x} = 8x^3 - 6xy = 0, \frac{\partial f}{\partial y} = -3x^2 + 2y = 0$. $\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$ or $6y = 8x^2$. But $6y = 8x^2$ is not compatible with the second equation $2y = 3x^2$. Thus $x = 0$, which implies $y = 0$, hence $(0,0)$ is the only critical point.

Now $f(0,0) = 0, f(\delta, 0) = 2\delta^4 > 0$. Let us take $y = \sqrt{3}x^2, f(x,y) = 2x^4 - 3\sqrt{3}x^4 + 3x^4 < 0$. Now whatever neighborhood of $(0,0)$ we take, it has points of the form $(\delta, 0)$ for suitable $\delta > 0$, as well as points that lie on the parabola $y = \sqrt{3}x^2$, thus $f(x,y)$ takes positive as well as negative values in any neighborhood of $(0,0)$, hence has neither maximum nor minimum at $(0,0)$. ∎

UPSC Civil Services Main 1999 - Mathematics Calculus

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January 2, 2010

Question 1(a) *Determine the set of all points where the function* $f(x) = \dfrac{x}{1+|x|}$ *is differentiable.*

Solution. If $x > 0$, then $f(x) = \dfrac{x}{1+x} \Rightarrow f'(x) = \dfrac{(1+x)-x}{(x+1)^2} = \dfrac{1}{(x+1)^2}$, so f is differentiable for all $x > 0$.

If $x < 0$, then $f(x) = \dfrac{x}{1-x} \Rightarrow f'(x) = \dfrac{(1-x)-x(-1)}{(1-x)^2} = \dfrac{1}{(1-x)^2}$, so f is differentiable for all $x < 0$.

If $x = 0, f(x) = 0$. Right hand derivative at $x = 0$ is $\lim_{x\to 0^+} \dfrac{\frac{x}{1+x} - 0}{x} = 1$. The Left hand derivative at $x = 0$ is $\lim_{x\to 0^-} \dfrac{\frac{x}{1-x} - 0}{x} = 1$.

Thus $f(x)$ is differentiable everywhere. ∎

Question 1(b) *Find 3 asymptotes of the curve*

$$x^3 + 2x^2y - 4xy^2 - 8y^3 - 4x + 8y - 10 = 0$$

Also find the intercept of one asymptote between the other two.

Solution. Substituting $y = mx + c$, we get

$$x^3 + 2x^2(mx+c) - 4x(mx+c)^2 - 8(mx+c)^3 - 4x + 8(mx+c) - 10 = 0$$

The coefficient of x^3 is $1 + 2m - 4m^2 - 8m^3 = 0 \Rightarrow (4m^2-1)(2m+1) = 0 \Rightarrow m = -\frac{1}{2}, m = \pm\frac{1}{2}$. So we have two parallel asymptotes.

Setting the coefficient of x^2 to 0 we get $2c - 8mc - 24m^2c = 0$. When $m = \frac{1}{2}$, we have $-8c = 0 \Rightarrow c = 0$, so one asymptote is $y = \frac{1}{2}x$, or $x - 2y = 0$.

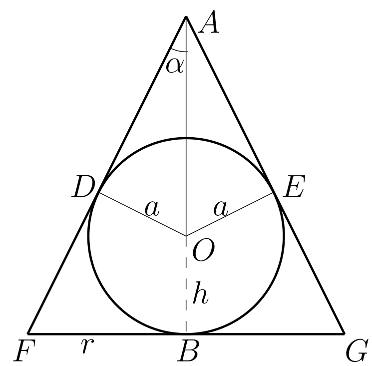
When $m = -\frac{1}{2}$, the coefficient of x^2 is identically 0. So we set the coefficient of x to 0, i.e. $-4c^2 - 24mc^2 + 8m - 4 = 0 \Rightarrow 8c^2 - 8 = 0 \Rightarrow c = \pm 1$. So the asymptotes are $y = -\frac{1}{2}x \pm 1$, or $x + 2y + 2 = 0, x + 2y - 2 = 0$.

To determine the intercept, we first find the points of intersection. $y = \frac{1}{2}x = -\frac{1}{2}x \pm 1 \Rightarrow x = \pm 1 \Rightarrow y = \pm\frac{1}{2}$, so the points of intersection are $(1, \frac{1}{2}), (-1, -\frac{1}{2})$. The distance between them is $\sqrt{(1-(-1))^2 + (\frac{1}{2} - (-\frac{1}{2}))^2} = \sqrt{5}$, which is the required intercept. ∎

Question 1(c) *Find the dimensions of the right circular cone of minimum volume which can be circumscribed about a sphere of radius a.*

Solution.

Let r be the radius of the cone, h be the height and α be the semi-vertical angle. Volume of the cone is $\frac{1}{3}\pi r^2 h$. Let $AO = y$, then $h = a + y$. Now $\sin\alpha = \frac{OD}{OA} = \frac{a}{y}$. Clearly $\frac{BF}{AB} = \tan\alpha$, or $r = BF = AB\tan\alpha = (a+y)\frac{a}{\sqrt{y^2 - a^2}}$. Thus the volume of the cone can be written as



$$V = \frac{\pi}{3}\frac{(a+y)^2a^2(a+y)}{y^2-a^2} = \frac{\pi a^2}{3}\frac{(a+y)^2}{y-a}$$

$$\frac{dV}{dy} = \frac{\pi a^2}{3}\frac{2(y-a)(y+a) - (y+a)^2}{(y-a)^2} = \frac{\pi a^2}{3}\frac{(y-3a)(y+a)}{(y-a)^2}$$

Thus $\frac{dV}{dy} = 0 \Rightarrow y = 3a, y = -a$ (not permissible). Clearly $\frac{dV}{dy} < 0$ when $0 < y < 3a$, and $\frac{dV}{dy} > 0$ when $y > 3a$, so the sign changes from negative to positive, so V is a minimum when $y = 3a$. $h = 4a, r = 4a\frac{a}{\sqrt{8}a} = a\sqrt{2}$. $V = \frac{8}{3}\pi a^3$, and $\alpha = \sin^{-1}\frac{1}{3}$. ∎

Question 2(a) *If $f(x)$ is Riemann integrable over every interval of finite length and $f(x+y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$, shaw that $f(x) = cx$ where $c = f(1)$.*

Solution. $x = y = 0 \Rightarrow f(0) = 0$. $0 = f(1-1) = f(1) + f(-1) \Rightarrow f(-1) = -f(1)$. Thus $f(n) = nf(1)$ for every $n \in \mathbb{Z}$. If $x = \frac{m}{n}$, then $nx = m \Rightarrow f(nx) = nf(x) = f(m) = mf(1) \Rightarrow f(x) = xf(1)$.

Let $\alpha \in \mathbb{R}$. Let $x_n \in \mathbb{Q}, x_n \longrightarrow \alpha$. Then $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} x_n f(1) = \alpha f(1)$. If f is continuous, then $x_n \longrightarrow \alpha \Rightarrow f(x_n) \longrightarrow f(\alpha)$, so $f(\alpha) = cf(1)$. In this case f is Riemann Integrable, so we proceed as follows:

Let $F(x) = \int_0^x f(t)\,dt$, then F is a continuous function. $F(x+y) = \int_0^{x+y} f(t)\,dt$, let $(x+y)z = t$, so $t = 0 \Leftrightarrow z = 0, t = x+y \Leftrightarrow z = 1$. It is enough to consider $x > 0$, because $f(x+(-x)) = f(0) = 0 = f(x) + f(-x)$, so f is an odd function of x.

$$\begin{aligned}
F(x+y) &= \int_0^1 f(xz+yz)(x+y)\,dz \\
\frac{F(x+y)}{x+y} &= \int_0^1 f(xz)\,dz + \int_0^1 f(yz)\,dz \\
&\quad (\text{ Let } xz = u, yz = v) \\
&= \int_0^x f(u)\frac{du}{x} + \int_0^y f(v)\frac{dv}{y} \\
&= \frac{F(x)}{x} + \frac{F(y)}{y}
\end{aligned}$$

Thus the function $G(x) = \frac{F(x)}{x}$ is continuous and satisfies $G(x+y) = G(x) + G(y)$. So $G(x) = xG(1)$, and $F(x) = x^2G(1)$. Differentiating both sides, $f(x) = F'(x) = 2xG(1)$, so $f(x) = cx$, which proves the result. ∎

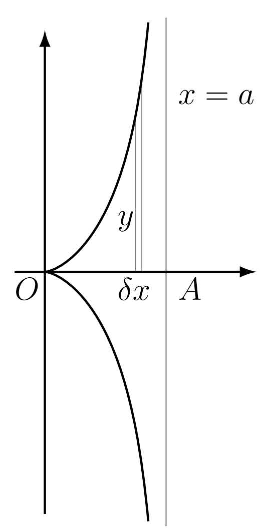
Question 2(b) *Show that the area between the cissoid* $x = a\sin^2 t,\ y = a\dfrac{\sin^3 t}{\cos t}$ *and its asymptote is* $\dfrac{3\pi a^2}{4}$.

Solution.

$$y^2 = \frac{a^2\sin^6 t}{\cos^2 t} = \frac{x^3}{a(1-\sin^2 t)} = \frac{x^3}{a-x}.$$

Thus $y^2(a-x) = x^3$, so $x = a$ is an asymptote.

Required Area $= 2\int_0^a y\,dx$. Substituting for y and x using the parametric equations, the area becomes $2\int_0^{\frac{\pi}{2}} a\dfrac{\sin^3 t}{\cos t}\ 2a\sin t\cos t\,dt = 4a^2\int_0^{\frac{\pi}{2}} \sin^4 t\,dt = 4a^2\,\dfrac{3\cdot 1}{4\cdot 2}\,\dfrac{\pi}{2} = \dfrac{3\pi a^2}{4}$ as required.



∎

Question 2(c) *Show that* $\iint x^{m-1}y^{n-1}\,dxdy$ *over the positive quadrant of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ *is* $\dfrac{a^m b^n}{4}\,\dfrac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma(\frac{m}{2}+\frac{n}{2}+1)}$.

Solution. Put $X = ax, Y = by, \frac{\partial(x,y)}{\partial(X,Y)} = ab$. Thus

$$I = \iint_{\substack{X\geq 0, Y\geq 0\\ X^2+Y^2\leq 1}} a^{m-1}X^{m-1}b^{n-1}Y^{n-1}\ ab\,dX\,dY$$

Put $X = r\cos\theta, Y = r\sin\theta \Rightarrow \dfrac{\partial(X,Y)}{\partial(r,\theta)} = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r$. Then

$$\begin{aligned}
I &= a^m b^n \int_0^{\frac{\pi}{2}}\int_0^1 r^{m+n-2}\sin^{m-1}\theta\cos^{n-1}\theta\ r\,dr\,d\theta\\
&= \frac{a^m b^n}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m-1}\theta\cos^{n-1}\theta\,d\theta\\
&= \frac{a^m b^n}{m+n}\,\frac{\Gamma(\frac{n-1+1}{2})\Gamma(\frac{m-1+1}{2})}{2\Gamma(\frac{n-1}{2}+\frac{m-1}{2}+1)}\\
&= \frac{a^m b^n}{4\frac{m+n}{2}}\,\frac{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{n}{2}+\frac{m}{2})}\\
&= \frac{a^m b^n}{4}\,\frac{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})}{\Gamma(\frac{n}{2}+\frac{m}{2}+1)}
\end{aligned}$$

Here we used $x\Gamma(x) = \Gamma(x+1)$ and

$$\int_0^{\frac{\pi}{2}} \sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{m+1}{2})}{2\Gamma(\frac{n}{2}+\frac{m}{2}+1)}$$

■

Paper II

Question 3(a) *Find the shortest distance from the origin to the hyperbola* $x^2+8xy+7y^2 = 225, z = 0$.

Solution. We have to minimize $f(x,y) = x^2+y^2$ subject to the constraint $x^2+8xy+7y^2-225 = 0$. Let $F(x,y) = x^2+y^2+\lambda(x^2+8xy+7y^2-225)$ where λ is Lagrange's undetermined multiplier. For extreme values

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2x+2\lambda x+8\lambda y = 0\\
\frac{\partial F}{\partial y} &= 2y+8\lambda x+14\lambda y = 0
\end{aligned}$$

From the second equation we get $x = \dfrac{-y-7\lambda y}{4\lambda}$. Substituting the value of x in the first equation, we get $\dfrac{-(1+\lambda)(1+7\lambda)y}{2\lambda} + 8\lambda y = 0$ or $16\lambda^2 - (1+7\lambda)(1+\lambda) = 0$, since $y \neq 0$ — note that $y = 0 \Rightarrow x = 0$ which is not possible because $x^2 + 8xy + y^2 = 225$. Thus we get $9\lambda^2 - 8\lambda - 1 = 0$, so $\lambda = 1, -\frac{1}{9}$.

We shall now show that $\lambda = 1$ is not possible. $\lambda = 1 \Rightarrow 4x + 8y = 0 \Rightarrow x = -2y \Rightarrow 4y^2 - 16y^2 + 7y^2 = 225$, which is not possible, thus $\lambda \neq 1$.

$\lambda = -\frac{1}{9} \Rightarrow 2x - \frac{2}{9}x - \frac{8}{9}y = 0 \Rightarrow 16x - 8y = 0 \Rightarrow y = 2x$. Thus $x^2 + 16x^2 + 28y^2 = 225 \Rightarrow x = \pm\sqrt{5}, y = \pm 2\sqrt{5}$. Thus stationary points are $(\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5})$.

$$\begin{aligned}
F(x,y) &= x^2 + y^2 - \frac{1}{9}(x^2 + 8xy + 7y^2 - 225) \\
&= \frac{1}{9}\Big[8x^2 + 2y^2 - 8xy + 225\Big] \\
&= \frac{1}{9}\Big[2(2x-y)^2 + 225\Big]
\end{aligned}$$

which is minimized when $2x = y$. Thus the shortest distance is $\sqrt{25} = 5$. ∎

Question 3(b) *Show that the double integral* $\displaystyle\iint_R \frac{x-y}{(x+y)^3}\,dx\,dy$ *does not exist over* $R = [0,1:0,1]$.

Solution. For a fixed $x \neq 0$, the function $\frac{x-y}{(x+y)^3}$ is a bounded function of y and

$$\begin{aligned}
\int_0^1 \frac{x-y}{(x+y)^3}\,dy &= \int_0^1 \left[-\frac{x+y}{(x+y)^3} + \frac{2x}{(x+y)^3}\right] dy \\
&= \frac{1}{x+y} + \frac{2x}{-2(x+y)^2}\bigg]_0^1 = \frac{1}{1+x} - \frac{x}{(1+x)^2} = \frac{1}{(1+x)^2}
\end{aligned}$$

Now

$$\int_0^1 \frac{dx}{(1+x)^2} = \lim_{\epsilon\to 0^+}\int_\epsilon^1 (1+x)^{-2}\,dx = \lim_{\epsilon\to 0^+}\left(-\frac{1}{2} + \frac{1}{1+\epsilon}\right) = \frac{1}{2}$$

Thus $\displaystyle\int_0^1\int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx = \frac{1}{2}$.

Similarly, if $y \neq 0$,

$$\begin{aligned}
\int_0^1 \frac{x-y}{(x+y)^3}\,dx &= \int_0^1 \left[\frac{x+y}{(x+y)^3} - \frac{2y}{(x+y)^3}\right] dx \\
&= -\frac{1}{x+y} - \frac{2y}{-2(x+y)^2}\bigg]_0^1 = -\frac{1}{(1+y)^2}
\end{aligned}$$

Thus $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dx\, dy = -\frac{1}{2}$, as $\displaystyle\int_0^1 \frac{dy}{(1+y)^2} = \frac{1}{2}$ as above.

Thus $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dy\, dx \neq \int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\, dx\, dy$ showing that the double integral does not exist, because if the double integral exists and the two repeated integrals exist, these have to be equal. The reason is that the function $\frac{x-y}{(x+y)^3}$ is not bounded in the square $R[0,1:0,1]$. ∎

UPSC Civil Services Main 2000 - Mathematics Calculus

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Question 1(a) *Use the mean value theorem to prove that*

$$\frac{2}{7} \leq \log 1.4 \leq \frac{2}{5}$$

Solution. Let $f(x) = \log(1+x)$. Consider the interval $[0, x]$. Clearly the mean value theorem applies: $\log(1+x) - \log 1 = \frac{1}{1+\zeta}x$ for some $\zeta \in [0, x]$. Take $x = 0.4$.

$$\begin{aligned} 0 &\leq \zeta \leq 0.4 = \frac{2}{5} \\ \Rightarrow 1 &\leq 1 + \zeta \leq \frac{7}{5} \\ \Rightarrow \frac{5}{7}x &\leq \frac{1}{1+\zeta}x \leq x \\ \Rightarrow \frac{2}{7} &\leq \log 1.4 \leq \frac{2}{5} \end{aligned}$$

∎

Question 1(b) *Show that*

$$\iint_A x^{2l-1}y^{2m-1}\,dx\,dy = \frac{1}{4}r^{2(l+m)}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}$$

where A consists of all positive values of x, y lying insode the circle $x^2 + y^2 = r^2$.

Solution. Switching to polar coordinates, $x = R\cos\theta, y = R\sin\theta$ and

$$\begin{aligned}
I &= \int_0^r \int_0^{\frac{\pi}{2}} R^{2l-1}\cos^{2l-1}\theta\ R^{2m-1}\sin^{2m-1}\theta\ R\,d\theta\,dR \\
&= \int_0^r R^{2l+2m-1}\,dR \int_0^{\frac{\pi}{2}} \cos^{2l-1}\theta\ \sin^{2m-1}\theta\ d\theta \\
&= \frac{r^{2l+2m}}{2l+2m}\ \frac{\Gamma(\frac{2l}{2})\Gamma(\frac{2m}{2})}{2\Gamma(\frac{2l-1}{2}+\frac{2m-1}{2}+1)} \\
&= \frac{r^{2(l+m)}}{4(l+m)}\ \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} = \frac{1}{4}r^{2(l+m)}\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}
\end{aligned}$$

∎

Question 2(a) *Find the center of gravity of the positive octant of the ellipsoid* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$, *if the density varies as* xyz.

Solution.

$$Mass = \iiint_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1 \\ x\ge 0, y\ge 0, z\ge 0}} xyz\,dx\,dy\,dz$$

Let $x = aX, y = bY, z = cZ$. We get:

$$\begin{aligned}
Mass &= a^2b^2c^2 \iiint_{\substack{X^2+Y^2+Z^2\le 1 \\ X\ge 0, Y\ge 0, Z\ge 0}} XYZ\,dX\,dY\,dZ \\
&= a^2b^2c^2\int_0^1 x\int_0^{\sqrt{1-x^2}} y\int_0^{\sqrt{1-x^2-y^2}} z\,dz\,dy\,dx \\
&= \frac{a^2b^2c^2}{2}\int_0^1 x\int_0^{\sqrt{1-x^2}} y(1-x^2-y^2)\,dy\,dx \\
&= \frac{a^2b^2c^2}{2}\int_0^1 x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx \\
&= \frac{a^2b^2c^2}{8}\int_0^1 x\big(2(1-x^2)-2x^2(1-x^2)-(1-x^2)^2\big)\,dx \\
&= \frac{a^2b^2c^2}{8}\int_0^1 (x-2x^3+x^5)\,dx \\
&= \frac{a^2b^2c^2}{8}\left[\frac{x^2}{2}-2\frac{x^4}{4}+\frac{x^6}{6}\right]_0^1 \\
&= \frac{a^2b^2c^2}{8}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right] = \frac{a^2b^2c^2}{48}
\end{aligned}$$

If $(\overline{x}, \overline{y}, \overline{z})$ are the coordinates of the center of gravity, then

$$\begin{aligned}
Mass \times \overline{x} &= \iiint_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1 \\ x \geq 0, y \geq 0, z \geq 0}} x^2 yz\, dx\, dy\, dz \\
&= a^3b^2c^2 \iiint_{\substack{X^2+Y^2+Z^2 \leq 1 \\ X \geq 0, Y \geq 0, Z \geq 0}} X^2 YZ\, dX\, dY\, dZ \\
&= \frac{a^3b^2c^2}{8} \int_0^1 (x^2 - 2x^4 + x^6)\, dx \\
&= \frac{a^3b^2c^2}{8} \left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right] = \frac{a^3b^2c^2}{105}
\end{aligned}$$

Thus $\overline{x} = \dfrac{a^3b^2c^2}{105} \times \dfrac{48}{a^2b^2c^2} = \dfrac{16}{35}a$. By symmetry, the center of gravity is $\left(\dfrac{16a}{35}, \dfrac{16b}{35}, \dfrac{16c}{35}\right)$. ∎

Question 2(b) *Let* $f(x) = \begin{cases} 0, & x \textit{ irrational} \\ 1, & x \textit{ rational} \end{cases}$

Show that f is not integrable on $[a, b]$.

Solution. Let $P = \{a = x_0 < x_1 < x_2 < \dots x_n = b\}$ be any partition of $[a, b]$. In any interval $[x_{i-1}, x_i], 1 \leq i \leq n$, there exist rationals as well as irrationals.

Thus $m_i = \min_{x_{i-1} \leq x \leq x_i} f(x) = 0$, $M_i = \max_{x_{i-1} \leq x \leq x_i} f(x) = 1$.

The Lower Riemann Sum $= L(P, f) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = 0$. The Upper Riemann Sum $= U(P, f) = \sum_{i=1}^n M_i(x_i - x_{i-1}) = b - a$.

Thus $\underline{\int_a^b} f(x)\, dx = 0$ and $\overline{\int_a^b} f(x)\, dx = b - a$, showing that f is not Riemann integrable on $[a, b]$. ∎

Question 2(c) *Show that*

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}} \left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right)$$

Solution. We can prove this by induction. It is obviously true for $n = 0$. Suppose it is true for n :

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n \frac{n!}{x^{n+1}} \left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right)$$

Differentiating both sides, we get

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}\left(\frac{\log x}{x}\right) &= (-1)^n n! \frac{d}{dx}\left(\frac{1}{x^{n+1}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right)\right) \\
&= (-1)^n n! \left[\frac{-(n+1)}{x^{n+2}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right) + \frac{1}{x^{n+1}}\left(\frac{1}{x}\right)\right] \\
&= (-1)^{n+1}(n+1)! \left[\frac{1}{x^{n+2}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right) - \frac{1}{x^{n+2}(n+1)}\right] \\
&= (-1)^{n+1}\frac{(n+1)!}{x^{n+2}}\left(\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n} - \frac{1}{n+1}\right)
\end{aligned}$$

which proves the result for $n+1$, and thus by induction for all n.

An alternate method is to use the Leibniz rule:

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} \frac{d^{n-r}u}{dx^{n-r}} \frac{d^r v}{dx^r}$$

and letting $u = \frac{1}{x}, v = \log x$. ∎

Question 2(d) *Find constants a, b which minimize*

$$F(a,b) = \int_0^\pi \big(\sin x - (ax^2 + bx)\big)^2 \, dx$$

Solution.

$$\begin{aligned}
F(a,b) &= \int_0^\pi \big(\sin^2 x - 2(ax^2+bx)\sin x + (ax^2+bx)^2\big)\, dx \\
\int_0^\pi \sin^2 x \, dx &= \int_0^\pi \frac{1-\cos 2x}{2}\, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^\pi = \frac{\pi}{2} \\
\int_0^\pi (ax^2+bx)\sin x \, dx &= -(ax^2+bx)\cos x\Big]_0^\pi + \int_0^\pi (2ax+b)\cos x \, dx \\
&= (a\pi^2 + b\pi) + \Big[(2ax+b)\sin x\Big]_0^\pi - \int_0^\pi 2a \sin x \, dx \\
&= (a\pi^2 + b\pi) + 2a\cos x\Big]_0^\pi = a\pi^2 + b\pi - 4a \\
\int_0^\pi (a^2x^4 + 2abx^3 + b^2x^2)\, dx &= \frac{a^2\pi^5}{5} + \frac{ab\pi^4}{2} + \frac{b^2\pi^3}{3} \\
\Rightarrow \quad F(a,b) &= \frac{\pi}{2} - 2(a\pi^2 + b\pi - 4a) + \frac{a^2\pi^5}{5} + \frac{ab\pi^4}{2} + \frac{b^2\pi^3}{3} \\
&= \frac{1}{30}\Big[6\pi^5 a^2 + 15\pi^4 ab + 10\pi^3 b^2 - (60\pi^2 - 240)a - 60\pi b + 15\pi\Big]
\end{aligned}$$

Let $G(a,b) = 30F(a,b)$. Then at the maximum:

$$\begin{aligned}\frac{\partial G}{\partial x} &= 12\pi^5 + 15\pi^4 b - (60\pi^2 - 240) = 0 \\ \frac{\partial G}{\partial y} &= 15\pi^4 a - 20\pi^3 b - 60\pi = 0\end{aligned}$$

Solving these for a, b we get $b = \dfrac{240 - 12\pi^2}{\pi^4}, a = \dfrac{20\pi^2 - 320}{\pi^5}$.

Now $\dfrac{\partial^2 G}{\partial a^2} = 12\pi^5 > 0, \dfrac{\partial^2 G}{\partial b^2} = 20\pi^3 > 0, \dfrac{\partial^2 G}{\partial a\,\partial b} = 15\pi^4 \Rightarrow \dfrac{\partial^2 G}{\partial a^2}\dfrac{\partial^2 G}{\partial b^2} - \left(\dfrac{\partial^2 G}{\partial a\,\partial b}\right)^2 = 12\pi^5 \times 20\pi^3 - (15\pi^4)^2 = 15\pi^8 > 0$. Thus $G(a,b)$ and hence $F(a,b)$ is minimum when $b = \dfrac{240 - 12\pi^2}{\pi^4}, a = \dfrac{20\pi^2 - 320}{\pi^5}$. ∎

Paper II

Question 3(a) *1. Suppose f is real-valued twice differentiable on $(0,\infty)$ and M_0, M_1, M_2 the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ respectively in $(0,\infty)$. Prove that for each $x > 0, h > 0$*

$$f'(x) = \frac{f(x+2h) - f(x)}{2h} - hf''(u)$$

for some $u \in (x, x+2h)$. Hence show that $M_1^2 \leq 4M_0M_2$.

2. Evaluate $\displaystyle\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$ by transforming it into a triple integral where S is the closed surface formed by the cylinder $x^2 + y^2 = a^2, 0 \leq z \leq b$ and the circular discs $x^2 + y^2 \leq a^2, z = 0$ and $x^2 + y^2 \leq a^2, z = b$.

Solution.

1. Clearly $f(x)$ satisfies the requirements of Taylor's theorem in $[x, x+2h]$. Therefore $f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(u)$ for some $u \in (x, x+2h)$. Thus

$$f'(x) = \frac{f(x+2h) - f(x)}{2h} - hf''(u)$$

as required.

Now

$$|f'(x)| \leq \frac{|f(x+2h)| + |f(x)|}{2h} + h|f''(u)| \leq \frac{M_0 + M_0}{2h} + hM_2 = \frac{M_0}{h} + hM_2$$

for all x and $h > 0$. Therefore

$$\sup_{x>0} |f'(x)| \leq \inf_{h>0}\left(\frac{M_0}{h} + hM_2\right)$$

But $\frac{M_0}{h} + hM_2 = \left(\sqrt{\frac{M_0}{h}} - \sqrt{M_2 h}\right)^2 + 2\sqrt{M_0 M_2}$, so for $h = \sqrt{\frac{M_0}{M_2}}$, we have $\frac{M_0}{h} + hM_2 = 2\sqrt{M_0 M_2} \Rightarrow \inf_{h>0}\left(\frac{M_0}{h} + hM_2\right) \leq 2\sqrt{M_0 M_2}$. Thus

$$M_1 = \sup_{x>0} |f'(x)| \leq \inf_{h>0}\left(\frac{M_0}{h} + hM_2\right) \leq 2\sqrt{M_0 M_2}$$

so $M_1^2 \leq 4M_0 M_2$.

2. By Green's theorem

$$\begin{aligned} & \iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy) \\ = & \iiint_{\substack{x^2+y^2 \leq a^2 \\ 0 \leq z \leq b}} \left(\frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} x^2 y + \frac{\partial}{\partial z} x^2 z\right) dx\, dy\, dz \\ = & \iint_{x^2+y^2 \leq a^2} \left(\int_0^b 5x^2\, dz\right) dy\, dx = 5b \iint_{x^2+y^2 \leq a^2} x^2\, dy\, dx \\ = & 5b \int_0^a \int_0^{2\pi} r^2 \cos^2\theta\, r\, d\theta\, dr = 5b\frac{a^4}{4}\pi = \frac{5}{4}\pi a^4 b \end{aligned}$$

∎

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Question 1(a) *Let f be defined on $\mathbb{R}$ by setting $f(x) = x$ if x is rational, and $f(x) = 1 - x$ if x is irrational. Show that f is continuous at $\frac{1}{2}$ but discontinuous at any other point.*

Solution. f is continuous at $\frac{1}{2}$: Clearly

$$\Big|f(x) - f(\frac{1}{2})\Big| = \Big|f(x) - \frac{1}{2}\Big| = \begin{cases} |x - \frac{1}{2}|, & x \text{ rational} \\ |1 - x - \frac{1}{2}|, & x \text{ irrational} \end{cases} = \Big|x - \frac{1}{2}\Big|$$

Thus for any given $\epsilon > 0$, we can choose $\delta = \epsilon$, and we get $|x - \frac{1}{2}| < \delta \Rightarrow |f(x) - f(\frac{1}{2})| < \epsilon$, so f is continuous at $x = \frac{1}{2}$.

f is discontinuous at $x_0 \neq \frac{1}{2}$: We know that both rationals and irrationals are dense in $\mathbb{R}$, so there exists a sequence of rationals $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ and a sequence of irrationals $\{y_n\}$ such that $\lim_{n\to\infty} y_n = x_0$.

If x_0 is rational, consider the sequence of irrationals $\{y_n\}$. $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}(1 - y_n) = 1 - x_0 \neq x_0 = f(x_0)$, because $x_0 \neq \frac{1}{2}$. Thus f is discontinuous at x_0.

If x_0 is irrational, consider the sequence of rationals $\{x_n\}$. $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x_0 \neq 1 - x_0 = f(x_0)$, because $x_0 \neq \frac{1}{2}$. Thus again f is discontinuous at x_0.

Note: We use the fact that a function f is continuous at x_o if and only if for all sequences $x_n \longrightarrow x_0 \Rightarrow f(x_n) \longrightarrow f(x_0)$. ∎

Question 1(b) *Test the convergence of* $\displaystyle\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$

Solution. Consider $\displaystyle\int_\epsilon^1 \frac{dx}{\sqrt{x}} = 2x^{\frac{1}{2}}\Big]_\epsilon^1 = 2 - 2\sqrt{\epsilon} \Rightarrow \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{dx}{\sqrt{x}} = 2$, so $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ is convergent. Now $\displaystyle\int_0^1 \left|\frac{\sin\frac{1}{x}}{\sqrt{x}}\right| dx \leq \int_0^1 \frac{dx}{\sqrt{x}}$, so the given integral is absolutely convergent, and hence convergent. ∎

Question 2(a) *Find the equation of the cubic curve which has the same asymptotes as*

$$2x(y-3)^2 = 3y(x-1)^2$$

and which passes through the point $(1,1)$ *and has the* x*-axis as tangent at the origin.*

Solution. The equation of the given curve is

$$2xy^2 - 3yx^2 - 6xy + 18x - 3y = 0$$

By dividing throughout by y^2 and letting $y \to \infty$, it is clear that $x = 0$ is an asymptote.

Substituting $y = mx + c$, and equating the coefficients of the two highest powers of x to 0, we get $2m^2 - 3m = 0, 4cm - 3c - 6m = 0$. From the first equation we get $m = 0, \frac{3}{2}$, and the corresponding values of c are $0, 3$. Thus the asymptotes are $x = 0, y = 0, 3x - 2y + 6 = 0$. Any curve with the same asymptotes is of the form $2xy^2 - 3yx^2 - 6xy + ax + by + c = 0$. Since it passes through $(0,0)$, $c = 0$. It passes through $(1,1)$, $a + b - 7 = 0$. The slope of the tangent at $(0,0)$ is $-\frac{f_x(0,0)}{f_y(0,0)} = -\frac{a}{b} = 0 \Rightarrow a = 0, b = 7$. Thus the required curve is $2xy^2 - 3yx^2 - 6xy + 7y = 0$.

Note: For this problem, we did not need to compute the asymptotes — merely ensure that the terms with the highest and second highest powers of x, y be the same — so we could start with the curve $2x(y-3)^2 - 3y(x-1)^2) + ax + by + c = 0$, and then make it pass through $(1,1)$ with $y = 0$ as a tangent at $(0,0)$. ∎

Question 2(b) *Find the minimum and maximum radii vectors of*

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$$

by the plane $lx + my + nz = 0$.

Solution. We have to minimize or maximize

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ \text{subject to } \phi_1 &= a^2x^2 + b^2y^2 + c^2z^2 - (x^2 + y^2 + z^2)^2 = 0 \\ \phi_2 &= lx + my + nz = 0 \end{aligned}$$

Let $F = r^2 + \lambda_1\phi_1 + \lambda_2\phi_2$, where λ_1, λ_2 are Lagrange multipliers. For extreme values:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2x + \lambda_1[2a^2x - 4xr^2] + \lambda_2 l = 0 &(1)\\
\frac{\partial F}{\partial y} &= 2y + \lambda_1[2b^2y - 4yr^2] + \lambda_2 m = 0 &(2)\\
\frac{\partial F}{\partial z} &= 2z + \lambda_1[2c^2z - 4zr^2] + \lambda_2 n = 0 &(3)
\end{aligned}$$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get

$$2r^2 + \lambda_1[2(a^2x^2 + b^2y^2 + c^2z^2) - 4r^4] + \lambda_2\phi_2 = 0$$

which gives us $2r^2+\lambda_1[2r^4-4r^4] = 0$, because $a^2x^2+b^2x^2+c^2z^2 = (x^2+y^2+z^2)^2 = r^4, \phi_2 = 0$. Thus $\lambda_1 = \frac{1}{r^2}$.

Substituting this back, we get $x = \frac{\lambda_2 l}{\frac{1}{r^2}(4r^2-2a^2)-2} = \frac{\lambda_2 lr^2}{2(r^2-a^2)}$, and $y = \frac{\lambda_2 mr^2}{2(r^2-b^2)}, z = \frac{\lambda_2 nr^2}{2(r^2-c^2)}$. Since $lx + my + nz = 0$, we get

$$\frac{l^2}{r^2-a^2} + \frac{m^2}{r^2-b^2} + \frac{n^2}{r^2-c^2} = 0$$

This is a quadratic equation in r^2, its roots give us the required extremum values. Note that the extremum values have to exist. ∎

Question 2(c) *Evaluate* $\iiint (x+y+z+1)^2\,dx\,dy\,dz$ *over the region defined by* $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$.

Solution.

$$\begin{aligned}
I &= \iiint (x+y+z+1)^2\,dx\,dy\,dz\\
&= \int_0^1\int_0^{1-x}\int_0^{1-x-y} (x+y+z+1)^2\,dz\,dy\,dx\\
&= \int_0^1\int_0^{1-x} \left.\frac{(x+y+z+1)^3}{3}\right]_0^{1-x-y} dy\,dx\\
&= \int_0^1\int_0^{1-x}\left(\frac{8}{3} - \frac{(x+y+1)^3}{3}\right) dy\,dx\\
&= \int_0^1 \left[\frac{8}{3}y - \frac{(x+y+1)^4}{12}\right]_0^{1-x} dx\\
&= \int_0^1 \left[\frac{8}{3}(1-x) - \frac{16}{12} + \frac{(x+1)^4}{12}\right] dx\\
&= \left.\frac{16}{12}x - \frac{8}{3}\frac{x^2}{2} - \frac{(x+1)^5}{60}\right]_0^1\\
&= \frac{16}{12} - \frac{8}{6} + \frac{32}{60} - \frac{1}{60} = \frac{31}{60}
\end{aligned}$$

Note: This is actually Dirichlet's Integral :

$$\iiint_A F(x+y+z)x^{p-1}y^{q-1}z^{r-1}\,dx\,dy\,dz = \frac{\Gamma(p)\Gamma(q)\Gamma(r)}{\Gamma(p+q+r)}\int_0^1 F(u)u^{p+q+r-1}\,du$$

where p, q, r are all positive, and A is the volume consisting of all positive values of x, y, z subject to the condition $x+y+z \leq 1$. The result is obtained by substituting $x+y+z = u, y+z = uv, z = uvw$.

In this case, $p = q = r = 1, F(u) = (u+1)^2$, so

$$I = \frac{1}{\Gamma(3)}\int_0^1 (u+1)^2u^2\,du = \frac{1}{2!}\left[\frac{u^5}{5}+\frac{u^4}{2}+\frac{u^3}{3}\right]_0^1 = \frac{1}{2}\left[\frac{1}{5}+\frac{1}{2}+\frac{1}{3}\right] = \frac{31}{60}$$

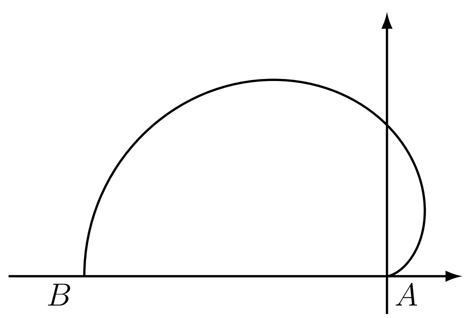
as before. ∎

Question 2(d) *Find the volume of the solid generated by revolving the cardiod $r = a(1-\cos\theta)$ around the initial line.*

Solution.

$$V = \int_B^A \pi y^2\,dx$$

$x = r\cos\theta = a(1-\cos\theta)\cos\theta$, so $\frac{dx}{d\theta} = -a\sin\theta(1-2\cos\theta)$. At A, $\theta = 0$, and at B, $\theta = \pi$. Thus



$$\begin{aligned}
V &= -\int_\pi^0 \pi a^2(1-cos\theta)^2\sin^2\theta\ a\sin\theta(1-2\cos\theta)\,d\theta \\
&= \pi a^3\int_0^\pi (1-2\cos\theta+\cos^2\theta)\sin^3\theta(1-2\cos\theta)\,d\theta \\
&= \pi a^3\int_0^\pi (1-4\cos\theta+5\cos^2\theta-2\cos^3\theta)\sin^3\theta\,d\theta \\
&= \pi a^3\left[2\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta + 10\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^3\theta\,d\theta\right] \quad \text{(The other terms give us 0)} \\
&= 2\pi a^3\left[\frac{2}{3}+5\frac{2}{5\cdot3\cdot1}\right] = \frac{8}{3}\pi a^3
\end{aligned}$$

Note: An easier way is to use the standard formula for the volume:

$$\begin{aligned} V &= \frac{2}{3}\pi \int_\pi^0 r^3 \sin\theta \, d\theta \\ &= \frac{2}{3}\pi \int_0^\pi a^3(1-\cos\theta)^3 \sin\theta \, d\theta \\ &= \frac{2}{3}\pi a^3 \int_0^2 z^3 \, dz \qquad \text{(Substituting } z = 1-\cos\theta) \\ &= \frac{8}{3}\pi a^3 \end{aligned}$$

∎

Paper II

Question 3(a) *Show that* $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x^n}{\sin^m x}\, dx$ *exists if and only if* $m < n+1$.

Solution. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} x^{n-m} \frac{x^m}{\sin^m x}\, dx$.

Case (1): $n-m \geq 0$. In this case the integral is a proper integral because $\displaystyle\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)^m = 1$ and therefore there is no discontinuity.

Case (2): $n-m < 0$. Here 0 is the only point of discontinuity. Note that the integrand is positive when $x > 0$. Since $\frac{x^n}{\sin^m x} = \frac{x^m}{\sin^m x} \times x^{n-m}$, and $\displaystyle\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)^m = 1$, the given integral converges if and only if $\displaystyle\int_0^{\frac{\pi}{2}} x^{n-m}\, dx$ converges.

Now $\displaystyle\int_\epsilon^{\frac{\pi}{2}} x^{n-m}\, dx = \frac{1}{n-m+1}\left[\left(\frac{\pi}{2}\right)^{n-m+1} - \epsilon^{n-m+1}\right]$, which tends to a finite limit as $\epsilon \to 0^+ \Leftrightarrow n-m+1 > 0 \Leftrightarrow m < n+1$.

Thus the given integral converges if and only if $m < n+1$. ∎

Question 4(a) *A function f is defined in the interval (a, b) as follows:*

$$f(x) = \begin{cases} \frac{1}{q^2}, & \textit{when } x = \frac{p}{q} \\ \frac{1}{q^3}, & \textit{when } x = \sqrt{\frac{p}{q}} \\ 0, & \textit{otherwise} \end{cases}$$

where p, q are relatively prime integers. Is f Riemann integrable? Justify your answer.

Solution. (1) Let $\alpha \in \mathbb{Q}$, $f(\alpha) \neq 0$. Since $(\alpha - \frac{1}{n}, \alpha + \frac{1}{n})$ contains uncountably many irrationals not of the form $\sqrt{\frac{p}{q}}$ (there are only countably many of those), we can get a sequence $\{\alpha_n\}$ such that $|\alpha - \alpha_n| < \frac{1}{n}$ i.e. $\alpha_n \longrightarrow \alpha$. But $f(\alpha_n) = 0 \not\longrightarrow f(\alpha) \neq 0$, therefore f is discontinuous at α. The same argument shows that $f(\alpha)$ is discontinuous whenever $\alpha = \sqrt{\frac{p}{q}}$.

(2) f is continuous at all those α for which $f(\alpha) = 0$. Let $\epsilon > 0$. Let n be chosen that $\frac{1}{n} < \epsilon$. Since there are only finitely many $q \leq n$, there are only finitely many numbers $\frac{p}{q}, \sqrt{\frac{p}{q}}$ in $[a, b]$. Let $\delta = \inf\{|\alpha - \frac{p}{q}|, |\alpha - \sqrt{\frac{p}{q}}|, q \leq n\}$, then $\delta > 0$. Now if $x \in (\alpha - \delta, \alpha + \delta)$, we have $f(x) = 0, \frac{1}{q^2}, \frac{1}{q^3}$ with $q > n$. In all cases $|f(x) - f(\alpha)| = 0, \frac{1}{q^2}, \frac{1}{q^3} < \frac{1}{n} < \epsilon$, so f is continuous at $x = \alpha$.

Since the discontinuities of the given function in $[a, b]$ are countable, they form a set of measure zero, it follows that f is Riemann integrable — here we use the result that a bounded function f is Riemann integrable on $[a, b]$ if and only if the discontinuities of f form a set of measure 0.

Alternate solution. Let $\epsilon > 0$. Let us call a number $\alpha \in [a, b]$ exceptional if $\alpha = \frac{p}{q}$ or $\sqrt{\frac{p}{q}}$ and $\frac{1}{q} > \frac{\epsilon}{2(b-a)}$ (here if $(b - a) < 1$, we can always take $\frac{1}{q} > \frac{\epsilon}{2}$, and the rest of the proof goes through). These α are finite in number. We enclose all these points in intervals whose total length is $< \frac{\epsilon}{2(b-a)}$. Since the oscillation of f in these intervals is < 1, therefore the contribution to $U(f, P) - L(f, P)$ =upper Riemann sum - lower Riemann sum $< \frac{\epsilon}{2(b-a)}$, where P is a partition which the finite number of intervals containing the exceptional points give rise to. Now the oscillation of f in any interval not containing the exceptional points $< \frac{\epsilon}{2(b-a)}$, because $\frac{1}{q} < \frac{\epsilon}{2(b-a)}$ for any points $\frac{p}{q}$ that occur in that interval. Since the total length of these intervals is $\leq (b - a)$, the total contribution to $U(f, P) - L(f, P) < \frac{\epsilon}{2(b-a)}(b - a)$. Hence $U(f, P) - L(f, P) < \epsilon$, so f is Riemann integrable. ∎

Question 4(b) *Show that $U = xy + yz + zx$ has a maximum value when the three variables are connected by the relation $ax + by + cz = 1$, and a, b, c are positive constants satisfying the condition $2(ab + bc + ca) > a^2 + b^2 + c^2$.*

Solution. Let $F = U + \lambda(ax + by + cz - 1)$ where λ is Lagrange's undetermined multiplier. For extreme values:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= y + z + \lambda a = 0 \\
\frac{\partial F}{\partial y} &= x + z + \lambda b = 0 \\
\frac{\partial F}{\partial z} &= x + y + \lambda c = 0 \\
\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} - \frac{\partial F}{\partial z} &= 2z + \lambda(a + b - c) = 0 \Rightarrow z = \frac{\lambda}{2}(c - a - b)
\end{aligned}$$

By symmetry, $x = \frac{\lambda}{2}(a-b-c), y = \frac{\lambda}{2}(b-a-c)$. Substituting these values in the constraint, we get $\frac{\lambda}{2}(a^2 - ab - ac + b^2 - ba - bc + c^2 - ca - cb) = 1$ so

$$\lambda = \frac{2}{a^2 + b^2 + c^2 - 2ab - 2bc - 2ca}$$

Moreover, for these values $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 2U + \lambda(ax + by + cz) = 0 \Rightarrow U = -\frac{\lambda}{2} = \frac{1}{2ab+2bc+2ca-(a^2+b^2+c^2)}$.

Now $d^2F = 2\,dx\,dy + 2\,dy\,dz + 2\,dz\,dx$ and $a, dx + b\,dy + c\,dz = 0$ as $ax + by + cz = 1$. Substituting the value of dz in d^2F, we get

$$\begin{aligned} d^2F &= 2\,dx\,dy + 2(dy+dx)\frac{a\,dx + b\,dy}{-c} \\ &= -\frac{2a}{c}(dx)^2 - \frac{2b}{c}(dy)^2 + 2\,dx\,dy\Big[1 - \frac{b}{c} - \frac{a}{c}\Big] \\ &= -\frac{2a}{c}\Big[(dx)^2 + \frac{a+b-c}{a}\,dx\,dy + \frac{b}{a}(dy)^2\Big] \\ &= -\frac{2a}{c}\Big[\Big(dx + \frac{a+b-c}{2a}\,dy\Big)^2 + \Big(\frac{b}{a} - \frac{(a+b-c)^2}{4a^2}\Big)(dy)^2\Big] \end{aligned}$$

Now $\frac{b}{a} - \frac{(a+b-c)^2}{4a^2} = \frac{4ab-a^2-b^2-c^2-2ab+2bc+2ac}{4a^2} > 0$ because $2(ab+bc+ca) > a^2+b^2+c^2$. Hence $d^2F < 0$, so U has a maximum value $\frac{1}{2ab+2bc+2ca-(a^2+b^2+c^2)}$. ∎

Question 4(c) *Evaluate* $\iiint (ax^2 + by^2 + cz^2)\,dx\,dy\,dz$ *taken throughout the region* $x^2 + y^2 + z^2 \leq R^2$.

Solution. Switching to polar coordinates, $x = r\cos\theta, y = r\sin\theta\cos\phi, z = r\sin\theta\sin\phi$ and $dx\,dy\,dz = r^2\sin\theta\,d\phi\,d\theta\,dr$, we get

$$\begin{aligned} I &= \int_0^R\int_0^\pi\int_0^{2\pi}(ar^2\cos^2\theta + br^2\sin^2\theta\cos^2\phi + cr^2\sin^2\theta\sin^2\phi)r^2\sin\theta\,d\phi\,d\theta\,dr \\ &= \int_0^R r^4\,dr\int_0^\pi(2a\pi\cos^2\theta\sin\theta + b\pi\sin^3\theta + c\pi\sin^3\theta)\,d\theta \\ &= \frac{\pi R^5}{5}\int_0^\pi(2a\cos^2\theta\sin\theta + (b+c)\sin^3\theta)\,d\theta \\ &= \frac{\pi R^5}{5}\int_0^\pi((2a-b-c)\cos^2\theta\sin\theta + (b+c)\sin\theta)\,d\theta \\ &= \frac{\pi R^5}{5}\Big[(2a-b-c)\frac{-\cos^3\theta}{3} + (b+c)(-\cos\theta)\Big]_0^\pi \\ &= \frac{\pi R^5}{5}\Big[\frac{2(2a-b-c)+6(b+c)}{3}\Big] = \frac{4\pi R^5(a+b+c)}{15} \end{aligned}$$

∎

UPSC Civil Services Main 2002 - Mathematics Calculus

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Question 1(a) *Show that* $\dfrac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1} b - \sin^{-1} a \leq \dfrac{b-a}{\sqrt{1-b^2}}$ *for* $0 < a < b < 1$

Solution. Consider the function $f(x) = \sin^{-1} x$ on the interval $[a, b]$. Clearly $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$. Thus $f(x)$ satisfies the requirements of Lagrange's Mean Value Theorem on $[a, b]$. Therefore

$$f(b) - f(a) = \sin^{-1} b - \sin^{-1} a = (b-a)\frac{1}{\sqrt{1-c^2}}$$

for some point $a \leq c \leq b$. Since

$$\frac{b-a}{\sqrt{1-a^2}} \leq \frac{b-a}{\sqrt{1-c^2}} \leq \frac{b-a}{\sqrt{1-b^2}}$$

it follows that

$$\frac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1} b - \sin^{-1} a \leq \frac{b-a}{\sqrt{1-b^2}}$$

∎

Question 1(b) *Show that* $\displaystyle\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy = \frac{\pi}{4}$.

Solution. Consider the integral $I_R = \displaystyle\int_0^R \int_0^R e^{-(x^2+y^2)}\, dx\, dy$. Let C_1 be part of the circle $x^2 + y^2 = R^2$ in the first quadrant and let C_2 be part of the circle $x^2 + y^2 = 2R^2$ in the first quadrant. Since the integrand $f(x, y) = e^{-(x^2+y^2)} > 0$, it follows that

$$\iint_{C_1} f(x, y)\, dx\, dy \leq I_R \leq \iint_{C_2} f(x, y)\, dx\, dy$$

Put $x = r\cos\theta, y = r\sin\theta$, we get

$$\iint_{C_1} f(x,y)\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{R} e^{-r^2} r\,dr\,d\theta = \frac{\pi}{2}\frac{e^{-r^2}}{-2}\Big]_0^R = \frac{\pi}{4}\big[1 - e^{-R^2}\big]$$

Similarly

$$\iint_{C_2} f(x,y)\,dx\,dy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\sqrt{2}R} e^{-r^2} r\,dr\,d\theta = \frac{\pi}{4}\big[1 - e^{-2R^2}\big]$$

Thus by the squeeze principle,

$$\lim_{R\to\infty} I_R = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \frac{\pi}{4}$$

Note: Consider the integral $I = \displaystyle\int_0^\infty e^{-x^2}\,dx, I_R = \int_0^R e^{-x^2}\,dx$, so that $(I_R)^2 = \displaystyle\int_0^R \int_0^R e^{-(x^2+y^2)}\,dx\,dy$. As shown above $\displaystyle\lim_{R\to\infty}(I_R)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$, so $I = \displaystyle\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ as $I > 0$. ∎

Question 2(a) *Let* $f(x) = \begin{cases} x^p \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Obtain a condition on p such that (i) f is continuous at $x = 0$ (ii) f is differentiable at $x = 0$.

Solution. (i) $f(x)$ is continuous at $x = 0$ when $p > 0$, because $|x^p \sin\frac{1}{x}| \leq |x^p| < \epsilon$ for $|x| < \delta$, $\delta = \epsilon^{\frac{1}{p}}$ (or $\lim_{x\to 0} |x|^p = 0, p > 0$ as $|\sin\frac{1}{x}| \leq 1$.

When $p \leq 0$, $f(x)$ is not continuous — if $p = 0$, it oscillates between 1 and -1, and when $p < 0$, it oscillates between $-\infty$ and ∞ infinitely often as x approaches 0.

(ii) $f(x)$ is differentiable at $x = 0$ when $\frac{f(x)-f(0)}{x} = x^{p-1}\sin\frac{1}{x}$ has a limit, which is true if $p - 1 > 0 \Rightarrow p > 1$. If $p \leq 1$, this limit does not exist, as shown above. ∎

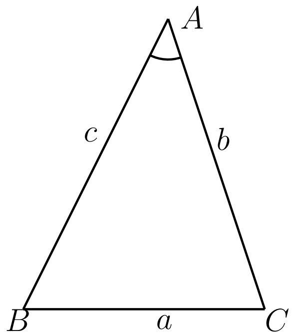
Question 2(b) *Consider the set of triangles having a given base and a given vertex angle. Show that the triangle having maximum area will be isosceles.*

Solution. Given $\angle A$ and a as shown in the figure, the area of $\triangle ABC = y = \frac{1}{2}bc\sin A$. Now

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Therefore $y = \frac{1}{2}\,\frac{a^2 \sin B \sin C}{\sin A}$. But $C = \pi - (A+B) \Rightarrow \sin C = \sin(A+B)$. Thus

$$y = \frac{1}{2}\,\frac{a^2}{\sin A}\sin B \sin(A+B)$$



(Here B is the dependent variable.)

$$\frac{dy}{dB} = \frac{1}{2}\,\frac{a^2}{\sin A}\Big[\cos B\sin(A+B) + \sin B\cos(A+B)\Big] = \frac{1}{2}\,\frac{a^2}{\sin A}\sin(A+2B)$$

Now $\frac{dy}{dB} = 0 \Rightarrow \sin(A+2B) = 0 \Rightarrow A+2B = \pi$ $(A+2B = 0 \Rightarrow B < 0)$. So $B = \frac{\pi}{2} - \frac{A}{2}, C = \pi - (A+B) = \frac{\pi}{2} - \frac{A}{2} = B \Rightarrow$ the triangle is isosceles.

$$\frac{d^2y}{dB^2} = \frac{1}{2}\,\frac{a^2}{\sin A}2\cos(A+2B) = \frac{a^2}{\sin A}\cos(\pi) < 0$$

so y is at a maximum when $B = C = \frac{\pi}{2} - \frac{A}{2}$.

Thus the triangle having maximum area is isosceles. ∎

Question 2(c) *If the roots of the equation*

$$(\lambda - u)^3 + (\lambda - v)^3 + (\lambda - w)^3 = 0$$

in λ are x, y, z, show that

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{2(u-v)(v-w)(w-u)}{(x-y)(y-z)(z-x)}$$

Solution. See 2004, question 2(a). ∎

Question 2(d) *Find the center of gravity of the region bounded by the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ and both axes in the first quadrant, the density being $\rho = kxy$, where k is a constant.*

Solution. If the center of gravity is $(\overline{x}, \overline{y})$, then

$$\overline{x} = \frac{\iint_A x\rho\,dA}{\iint_A \rho\,dA} = \frac{\int_0^a\int_0^y x(kxy)\,dy\,dx}{\int_0^a\int_0^y kxy\,dy\,dx} = \frac{\int_0^a x^2y^2\,dx}{\int_0^a xy^2\,dx}$$

Setting $x = a\cos^3\theta, y = b\sin^3\theta$, we get

$$\overline{x} = \frac{\int_{\frac{\pi}{2}}^0 a^2\cos^6\theta\, b^2\sin^6\theta(-3a\cos^2\theta\sin\theta)\,d\theta}{\int_{\frac{\pi}{2}}^0 a\cos^3\theta\, b^2\sin^6\theta(-3a\cos^2\theta\sin\theta)\,d\theta} = a\frac{\int_0^{\frac{\pi}{2}}\cos^8\theta\sin^7\theta\,d\theta}{\int_0^{\frac{\pi}{2}}\cos^5\theta\sin^7\theta\,d\theta}$$

Using the result $\int_0^{\frac{\pi}{2}} \cos^p \theta \sin^q \theta \, d\theta = \dfrac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$, we have

$$\overline{x} = a\frac{\Gamma(\frac{9}{2})\Gamma(4)\Gamma(7)}{\Gamma(\frac{17}{2})\Gamma(4)\Gamma(3)} = a\frac{7 \cdot 5 \cdot 3 \cdot 1 \cdot 720 \cdot 2^8}{15 \cdot 13 \cdot 11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 \cdot 2 \cdot 2^4} = \frac{128}{429}a$$

If $X = \frac{x}{a}$ and $Y = \frac{y}{b}$, then the curve is $X^{\frac{2}{3}} + Y^{\frac{2}{3}} = 1$, so it is symmetric about $X = Y$, and the density is $kabXY$, which is also symmetric about $X = Y$. Therefore $\overline{Y} = \frac{\overline{y}}{b} = \overline{X} = \frac{\overline{x}}{a}$. Thus $\frac{\overline{y}}{b} = \frac{\overline{x}}{a} = \frac{128}{429}$. Thus $\overline{x} = \frac{128}{429}a, \overline{y} = \frac{128}{429}b$. ∎

Paper II

Question 3(a) *Prove that the integral* $\int_0^\infty x^{m-1}e^{-x}\, dx$ *converges if and only if* $m > 0$.

Solution. See 2005 question 4(b). ∎

Question 4(a) *Obtain the maxima and minima of*

$$x^2 + y^2 + z^2 - yz - zx - xy$$

subject to the condition $x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0$.

Solution. Let $F = x^2 + y^2 + z^2 - yz - zx - xy - \lambda(x^2 + y^2 + z^2 - 2x + 2y + 6z + 9)$. For extreme values:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2x - z - y - 2\lambda x + 2\lambda = 0 \Rightarrow (2-2\lambda)x - y - z = -2\lambda & (1)\\
\frac{\partial F}{\partial y} &= 2y - z - x - 2\lambda y - 2\lambda = 0 \Rightarrow -x + (2-2\lambda)y - z = 2\lambda & (2)\\
\frac{\partial F}{\partial z} &= 2z - y - x - 2\lambda z - 6\lambda = 0 \Rightarrow -x - y + (2-2\lambda)z = 6\lambda & (3)
\end{aligned}$$

The determinant of the coefficient matrix of equations (1), (2) and (3) is

$$\begin{aligned}
& \begin{vmatrix} 2-2\lambda & -1 & -1 \\ -1 & 2-2\lambda & -1 \\ -1 & -1 & 2-2\lambda \end{vmatrix} \\
= \; & (2-2\lambda)^3 - (2-2\lambda) - (2-2\lambda) - 1 - 1 - (2-2\lambda) \\
= \; & (2-2\lambda)[(2-2\lambda)^2 - 3] - 2 \\
= \; & (2-2\lambda)[4\lambda^2 - 8\lambda + 1] - 2 \\
= \; & -8\lambda^3 + 24\lambda^2 - 18\lambda + 2 - 2 = -2\lambda(2\lambda - 3)^2
\end{aligned}$$

Thus the determinant is 0 when $\lambda = 0$, which is not admissible, or when $\lambda = \frac{3}{2}$ — however in this case $-x-y-z = -3$ from (1), and $-x-y-z = 3$ from (2), hence $\lambda = \frac{3}{2}$ is also not admissible. Thus the system of equations is solvable, we use Cramer's rule.

$$\begin{aligned}
x &= \frac{-1}{2\lambda(2\lambda-3)^2}\begin{vmatrix} -2\lambda & -1 & -1 \\ 2\lambda & 2-2\lambda & -1 \\ 6\lambda & -1 & 2-2\lambda \end{vmatrix} = \frac{1}{(2\lambda-3)^2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2-2\lambda & -1 \\ 3 & -1 & 2-2\lambda \end{vmatrix} \\
&= \frac{1}{(2\lambda-3)^2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1-2\lambda & -2 \\ 3 & -4 & -1-2\lambda \end{vmatrix} = \frac{1}{(2\lambda-3)^2}\left[4\lambda^2-1-8\right] = \frac{2\lambda+3}{2\lambda-3} \\
y &= \frac{-1}{2\lambda(2\lambda-3)^2}\begin{vmatrix} 2-2\lambda & -2\lambda & -1 \\ -1 & 2\lambda & -1 \\ -1 & 6\lambda & 2-2\lambda \end{vmatrix} = \frac{-1}{(2\lambda-3)^2}\begin{vmatrix} 2-2\lambda & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 3 & 2-2\lambda \end{vmatrix} \\
&= \frac{-1}{(2\lambda-3)^2}\begin{vmatrix} 2-2\lambda & 1-2\lambda & -3+2\lambda \\ -1 & 0 & 0 \\ -1 & 2 & 3-2\lambda \end{vmatrix} = \frac{-1}{(2\lambda-3)^2}\left[(3-2\lambda)^2\right] = -1 \\
z &= \frac{6\lambda+x+y}{2-2\lambda} = \frac{6\lambda+\frac{2\lambda+3}{2\lambda-3}-1}{2-2\lambda} = \frac{12\lambda^2-18\lambda+6}{(2-2\lambda)(2\lambda-3)} = \frac{3(1-2\lambda)}{2\lambda-3}
\end{aligned}$$

Substituting the values of x, y, z in the constraint $x^2+y^2+z^2-2x+2y+6z+9 = 0 = (x-1)^2+(y+1)^2+(z+3)^2-2$ we get $6^2+6^2-2(2\lambda-3)^2 = 0 \Rightarrow 2\lambda-3 = \pm 6 \Rightarrow \lambda = \frac{9}{2}, -\frac{3}{2}$.

If $\lambda = \frac{9}{2}$, then $x = 2, y = -1, z = -4 \Rightarrow x^2+y^2+z^2-yz-zx-xy = 27$.

If $\lambda = -\frac{3}{2}$, then $x = 0, y = -1, z = -2 \Rightarrow x^2+y^2+z^2-yz-zx-xy = 3$.

Thus the maximum value of $x^2+y^2+z^2-yz-zx-xy$ is 27 and the minimum value is 3.

Note: We need not check d^2F here as extreme values occur when $\lambda = \frac{9}{2}, -\frac{3}{2}$, which give us the maximum and minimum. However it is not difficult to carry out the check.

$$d^2F = (2-2\lambda)(dx)^2 + (2-2\lambda)(dy)^2 + (2-2\lambda)(dz)^2 - 2dx\,dy - 2dy\,dz - 2dz\,dx$$

The constraint gives us $2(x-1)\,dx + 2(y+1)\,dy + 2(z+3)\,dz = 0$. For $\lambda = \frac{9}{2}, x = 2, y = -1, z = -4$ we get $dx = dz$, and $d^2F = -16(dx)^2 - 7(dy)^2 - 4\,dx\,dy < 0$, thus the maximum occurs at $\lambda = \frac{9}{2}$.

For $\lambda = -\frac{3}{2}, x = 0, y = -1, z = -2$ we get $dx = dz$, and $d^2F = 8(dx)^2+5(dy)^2-4\,dx\,dy = (2dx-dy)^2+4(dx)^2+4(dy)^2 > 0$, thus the minimum value occurs at $\lambda = -\frac{3}{2}$. ∎

Question 4(b) *A solid hemisphere H of radius a has density depending on the distance R from the center, given by $\rho = k(2a-R)$, where k is a constant. Find the mass of the hemisphere by the method of multiple integration.*

Solution. In polar coordinates, the hemisphere is given by $r \leq a, 0 \leq \theta \leq \frac{\pi}{2}$. Writing the equation for the mass:

$$\begin{aligned}
M &= \int_0^a \int_0^{\frac{\pi}{2}} \int_0^{2\pi} k(2a-r)r^2 \sin\theta \, d\phi \, d\theta \, dr \\
&= 2k\pi \int_0^a \int_0^{\frac{\pi}{2}} (2a-r)r^2 \sin\theta \, d\theta \, dr \\
&= 2k\pi \int_0^a (2a-r)r^2 \, dr \\
&= 2k\pi \left[\frac{2ar^3}{3} - \frac{r^4}{4}\right]_0^a = 2k\pi \frac{5a^4}{12} = \frac{5\pi a^4 k}{6}
\end{aligned}$$

$\blacksquare$

UPSC Civil Services Main 2003 - Mathematics Calculus

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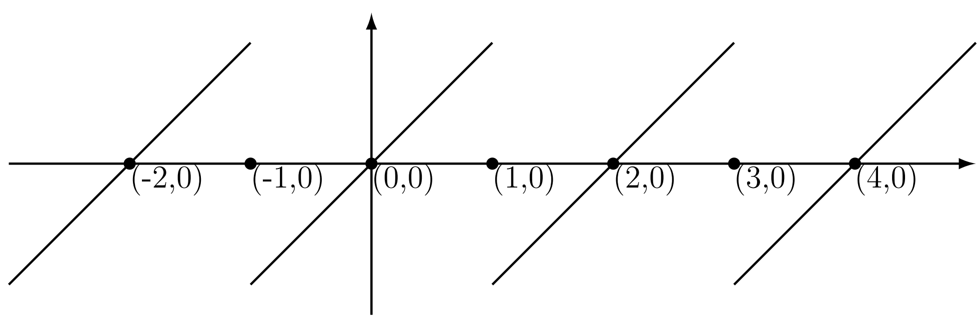
Question 1(a) *Let f be real function defined as follows — $f(x) = x, -1 \leq x < 1, f(x+2) = x, \forall x \in \mathbb{R}$. Show that f is discontinous at every odd integer.*

Solution. The result is obvious from the graph of the function.



Let n be any odd integer, $n = 2m+1$. Then $f(n+h) = f(2m+1+h) = f(-1+h+(2m+2)) = -1+h$. So $\lim_{h\to 0^+} f(n+h) = -1$. $f(n-h) = f(2m+1-h) = 1-h$, so $\lim_{h\to 0^-} f(n+h) = \lim_{h\to 0^+} f(n-h) = 1$. Thus $\lim_{h\to 0^-} f(n+h) \neq \lim_{h\to 0^+} f(n+h)$, showing that f is not continuous at $x = n, n$ odd. ∎

Question 1(b) *For all real numbers $x, f(x)$ is given as*

$$f(x) = \begin{cases} e^x + a\sin x, & x < 0 \\ b(x-1)^2 + x - 2, & x \geq 0 \end{cases}$$

Find values of a, b for which f is differentiable at $x = 0$.

Solution. For $f(x)$ to be differentiable at $x = 0$, it has to be continuous at $x = 0$. Thus

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} b(x-1)^2 + (x-2) = b - 2 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} e^x + a\sin x = 1$$

Thus $b - 2 = 1 \Rightarrow b = 3$. Now the right hand derivative of $f(x)$ at $x = 0$ is the derivative of $b(x-1)^2 + x - 2$ at $x = 0$, which is $2b(x-1) + 1 = -2b + 1 = -5$. The left hand derivative of $f(x)$ at $x = 0$ is derivative of $e^x + a\sin x$, which is $e^x + a\cos x = 1 + a$. Thus for $f(x)$ to be differentiable at $x = 0$, we must have $b = 3, a + 1 = -5 \Rightarrow a = -6$. ∎

Question 2(a) *A rectangular box, open at the top is to have a volume $4m^3$. Using Lagrange's method of multipliers find the dimensions of the box so that the material of a given type required to construct it may be the least.*

Solution. Let x, y, z be the length, breadth and the height of the box respectively. Then we have to minimize $S = xy + 2xz + 2yz$ subject to the constraints $xyz = 4, x > 0, y > 0, z > 0$. Let λ be Lagrange's undetermined multiplier. Then

$$F = xy + 2xz + 2yz + \lambda(xyz - 4)$$

For extreme values,

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2z + y + \lambda yz = 0 \\
\frac{\partial F}{\partial y} &= 2z + x + \lambda xz = 0 \\
\frac{\partial F}{\partial z} &= 2x + 2y + \lambda xy = 0
\end{aligned}$$

$x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y} = 2z(x-y) = 0$. But $z > 0$, so $x = y$. From $\frac{\partial F}{\partial z} = 0$, we get $\lambda x^2 = -4x \Rightarrow \lambda = -\frac{4}{x}$ and using $xyz = 4$, we get $z = \frac{4}{x^2}$. Substituting these values in $\frac{\partial F}{\partial x} = 0$, we get $\frac{8}{x^2} + x - \frac{4}{x}x\frac{4}{x^2} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2$. Thus $x = 2, y = 2, z = 1, \lambda = -2$.

Now

$$\begin{aligned}
d^2F &= 0(dx)^2 + 0(dy)^2 + 0(dz)^2 + 2(1+\lambda z)\,dx\,dy + 2(2+\lambda x)\,dy\,dz + 2(2+\lambda y)\,dz\,dx \\
&= -2\,dx\,dy - 4\,dy\,dz - 4\,dx\,dz
\end{aligned}$$

Also $xyz = 4 \Rightarrow yz\,dx + zx\,dy + xy\,dz = 0 \Rightarrow 2\,dx + 2\,dy + 4\,dz = 0$. Substituting for dz, we get $d^2F = -2\,dx\,dy + dy(2\,dx + 2\,dy) + dx(2\,dx + 2\,dy) = 2(dx)^2 + 2(dy)^2 + 2\,dx\,dy > 0$.

Thus S is minimum when $x = y = 2, z = 1$.

Note: For a simpler solution, see 1980 question 2(a), here we had to give this solution as asked in the question. ∎

Question 2(b) *Test the convergence of the integrals*

$$(1) \quad \int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)} \qquad (2) \quad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

Solution. (1) Let $\phi(x) = x^{-\frac{1}{3}}$, then $\displaystyle\int_\epsilon^1 x^{-\frac{1}{3}}\,dx = \frac{3}{2}x^{\frac{2}{3}}\Big]_\epsilon^1 = \frac{3}{2}(1-\epsilon^{\frac{2}{3}})$. Thus $\displaystyle\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-\frac{1}{3}}\,dx$ exists, so $\displaystyle\int_0^1 x^{-\frac{1}{3}}\,dx$ is convergent. Since $\displaystyle\lim_{x\to 0}\frac{1}{x^{\frac{1}{3}}(1+x^2)}\frac{1}{\phi(x)} = 1$, it follows that the integral $\displaystyle\int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}$ is convergent at $x=0$ if and only if $\displaystyle\int_0^1 \phi(x)\,dx$ is convergent. Since the integral $\displaystyle\int_\delta^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}$ is proper for any $\delta > 0$, it follows that the integral $\displaystyle\int_0^1 \frac{dx}{x^{\frac{1}{3}}(1+x^2)}$ is convergent.

(2) For the integral $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$, the point 0 is not a singularity, since $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1$. Moreover $\displaystyle\int_1^z \frac{dx}{x^2} = -\frac{1}{x}\Big]_1^z = 1 - \frac{1}{z}$, showing that $\displaystyle\int_1^\infty \frac{dx}{x^2}$ is convergent. Since

$$\int_1^\infty \left|\frac{\sin^2 x}{x^2}\right| dx \le \int_1^\infty \frac{dx}{x^2}$$

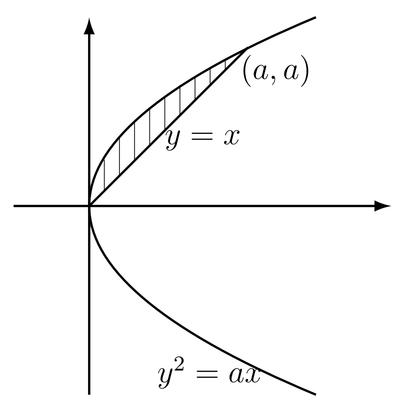
it follows that $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is absolutely convergent, and therefore convergent. Since $\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral, the given integral is convergent. ∎

Question 2(c) *Evaluate the integral*

$$\int_0^a \int_{\frac{y^2}{a}}^y \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

Solution.

The region of integration is the shaded region where x varies from $\frac{y^2}{a}$ to y. In the same region y varies from x to $\sqrt{ax}$ and x varies from 0 to a. Changing the order of integration, we get:



$$
\begin{aligned}
I &= \int_0^a \int_{\frac{y^2}{a}}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}} \\
&= \int_0^a \frac{1}{a-x}\left(\int_x^{\sqrt{ax}} \frac{y\,dy}{\sqrt{ax-y^2}}\right)dx \\
&= \int_0^a \frac{1}{a-x}\left(-\left[\sqrt{ax-y^2}\right]_x^{\sqrt{ax}}\right)dx \\
&= \int_0^a \frac{\sqrt{ax-x^2}}{a-x}\,dx = \int_0^a \frac{\sqrt{x}}{\sqrt{a-x}}\,dx
\end{aligned}
$$

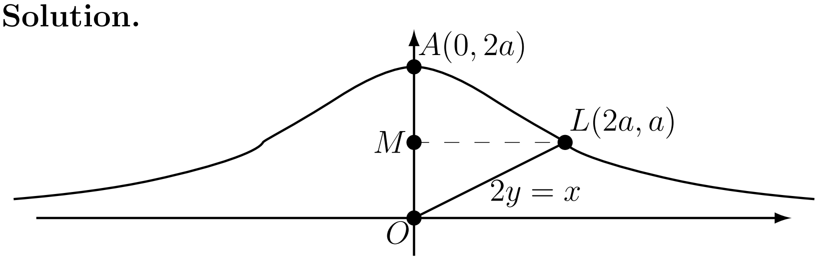
Substitute $x = a\sin^2\theta$ to get

$$
I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{a}\sin\theta\ 2a\sin\theta\cos\theta}{\sqrt{a}\cos\theta}\,d\theta = 2a\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta = 2a\frac{\pi}{4} = \frac{\pi a}{2}
$$

■

Question 2(d) *Find the volume generated by revolving the area bounded by the curves $(x^2 + 4a^2)y = 8a^3$, $2y = x$ and $x = 0$ about the y-axis*

Solution.



The volume generated by the region $OLMO$ is

$$
V_1 = \int_0^a \pi x^2\,dy = \int_0^a \pi\ 4y^2\,dy = \pi\frac{4y^3}{3}\Big]_0^a = \frac{4\pi a^3}{3}
$$

The volume generated by the region $MLAM$ is

$$
\begin{aligned}
V_2 &= \int_a^{2a} \pi x^2\,dy = \int_a^{2a} \pi\left(\frac{8a^3}{y} - 4a^2\right)dy \\
&= \pi\left[8a^3\log y - 4a^2y\right]_a^{2a} \\
&= 8a^3\pi(\log 2a - \log a) - 4\pi a^3 = 8\pi a^3\log 2 - 4\pi a^3
\end{aligned}
$$

The required volume $= V_1 + V_2 = \pi a^3\left(8\log 2 - 4 + \frac{4}{3}\right) = 8\pi a^3\left(\log 2 - \frac{1}{3}\right)$ ■

Paper II

Question 3(a) *Let a be a positive real number and x_n a sequence of rational numbers such that $\lim_{n\to\infty} x_n = 0$, show that $\lim_{n\to\infty} a^{x_n} = 1$.*

Solution. If $f(x)$ is continuous at $x = a$ and x_n is a sequence of real numbers such that $\lim_{n\to\infty} x_n = a$, then $\lim_{n\to\infty} f(x_n) = f(a)$. Let $f(x) = a^x$, f is a continuous function, therefore $\lim_{n\to\infty} a^{x_n} = a^{\lim_{n\to\infty} x_n} = a^0 = 1$. ■

Question 3(b) *If a continuous function of x satisfies the functional equation $f(x+y) = f(x) + f(y)$, then show that $f(x) = \alpha x$ where α is a constant.*

Solution. See 1999, question 2(a). ∎

Question 4(a) *Show that the maximum value of $x^2y^2z^2$ subject to the condition $x^2 + y^2 + z^2 = c^2$ is $c^6/27$. Interpret the result.*

Solution. By symmetry around the origin, we can assume w.l.o.g that x, y, z are all non-negative.

Let $F(x, y, z) = x^2y^2z^2 - \lambda(x^2 + y^2 + z^2 - c^2)$, where λ is Lagrange's undetermined multiplier. For extreme values, we have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2xy^2z^2 - 2\lambda x = 0 \\
\frac{\partial F}{\partial y} &= 2x^2yz^2 - 2\lambda y = 0 \\
\frac{\partial F}{\partial z} &= 2x^2y^2z - 2\lambda z = 0 \\
\Rightarrow \quad x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} &= 3x^2y^2z^2 - \lambda(x^2 + y^2 + z^2) \\
\Rightarrow \quad x^2y^2z^2 &= \frac{\lambda c^2}{3}
\end{aligned}$$

Since x, y, z cannot all be 0, $\lambda = y^2z^2 = z^2x^2 = x^2y^2 \Rightarrow x = \frac{c}{\sqrt{3}} = y = z, \lambda = \frac{c^4}{9}$.

Now

$$d^2F = \frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial y^2}(dy)^2 + \frac{\partial^2 F}{\partial z^2}(dz)^2 + 2\frac{\partial^2 F}{\partial x\,\partial y}\,dx\,dy + 2\frac{\partial^2 F}{\partial y\,\partial z}\,dy\,dz + 2\frac{\partial^2 F}{\partial x\,\partial z}\,dx\,dz$$

and $\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial z^2} = 0$ when $x = y = z = \frac{c}{\sqrt{3}}$. Also, $\frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial^2 F}{\partial y\,\partial z} = \frac{\partial^2 F}{\partial x\,\partial z} = \frac{4c^4}{9}$. Finally, $x^2 + y^2 + z^2 = c^2 \Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$, or $dz = -\frac{2x\,dx + 2y\,dy}{2z} = -dx - dy$ because $x = y = z$.

Thus

$$\begin{aligned}
d^2F &= \frac{8c^4}{9}\big(dx\,dy + dy(-dx - dy) + dx(-dx - dy)\big) \\
&= -\frac{8c^4}{9}\big((dx)^2 + (dy)^2 + dx\,dy\big) \\
&= -\frac{8c^4}{9}\left(\left(dx + \frac{dy}{2}\right)^2 + \frac{3}{4}(dy)^2\right)
\end{aligned}$$

which is negative definite. Thus $x^2y^2z^2$ is maximum subject to the constraint $x^2+y^2+z^2 = c^2$ when $x = y = z = \frac{c}{\sqrt{3}}$ and the maximum value is $\frac{c^6}{27}$ (of course we can change the sign of any of x, y, z and still get the same maximum value).

Interpretation: The maximum volume of a parallelopiped contained in the sphere $x^2 + y^2 + z^2 = c^2$ is $\frac{8c^3}{\sqrt{27}}$, since its volume will be $8xyz$, whose maximum value is computed as above. ∎

Question 4(b) *The axes of two equal cylinders intersect at right angles. If a be their radius, find the volume common to the cylinders by the method of multiple integrals.*

Solution. Let the equations of the cylinders be $x^2 + y^2 = a^2, x^2 + z^2 = a^2$. Then $V = \iiint dx\,dy\,dz$, where z varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$, y varies from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$, and x varies from $-a$ to a.

Thus $V = 2\int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2}\,dy = 4\int_{-a}^{a} (\sqrt{a^2 - x^2})^2\,dx = 4\int_{-a}^{a} (a^2 - x^2)\,dx = 4\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{16}{3}a^3$ ∎

UPSC Civil Services Main 2004 - Mathematics Calculus

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Question 1(a) *Prove that the function f defined on $[0,4]$ by $f(x) = [x]$, the greatest integer $\leq x$, is integrable on $[0,4]$ and that $\int_0^4 f(x)\,dx = 6$.*

Solution.

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & 2 \leq x < 3 \\ 3, & 3 \leq x < 4 \\ 4, & x = 4 \end{cases}$$

Clearly $f(x)$ has only finitely many discontinuities in $[0,4]$, namely at $1,2,3,4$. Therefore f is integrable in $[0,4]$. Note that $f(x)$ is bounded in $[0,4]$. Now

$$\int_0^4 f(x)\,dx = \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_2^4 3\,dx = 0+1+2+3 = 6$$

∎

Question 1(b) *Show that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0$.*

Solution. Let $f(x) = x - \frac{x^2}{2} - \log(1+x)$, then $f(x)$ satisfies the requirements of Lagrange's Mean Value Theorem in the interval $[0,x]$ and therefore

$$\frac{f(x)-f(0)}{x} = \frac{x - \frac{x^2}{2} - \log(1+x)}{x} = f'(\zeta) = 1 - \zeta - \frac{1}{1+\zeta}$$

for some $\zeta \in (0, x)$. Clearly $1 - \zeta - \frac{1}{1+\zeta} = -\frac{\zeta^2}{1+\zeta} < 0$. Therefore for $x > 0, x - \frac{x^2}{2} - \log(1+x) < 0 \Rightarrow x - \frac{x^2}{2} < \log(1 + x)$.

Now let $f(x) = x - \frac{x^2}{2(1+x)} - \log(1 + x)$. Again it satisfies the requirements of Lagrange's Mean Value Theorem in the closed interval $[0, x]$, so

$$\frac{f(x) - f(0)}{x} = \frac{x - \frac{x^2}{2(1+x)} - \log(1 + x)}{x} = f'(\zeta) = 1 - \frac{1}{1 + \zeta} - \frac{\zeta}{1 + \zeta} + \frac{\zeta^2}{2(1 + \zeta)^2}$$

for some $\zeta \in (0, x)$. But $1 - \frac{1}{1+\zeta} - \frac{\zeta}{1+\zeta} + \frac{\zeta^2}{2(1+\zeta)^2} = \frac{\zeta^2}{2(1+\zeta)^2} > 0$, so $x - \frac{x^2}{2(1+x)} - \log(1 + x) > 0 \Rightarrow \log(1 + x) < x - \frac{x^2}{2(1+x)}$. Combining the two inequalities, we get the result. ∎

Question 2(a) *Let the roots of the equation in* λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

be u, v, w. *Prove that*

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y - z)(z - x)(x - y)}{(u - v)(v - w)(w - u)}$$

Solution.

$$\begin{aligned}(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 &= 3\lambda^3 - 3\lambda^2(x + y + z) + 3\lambda(x^2 + y^2 + z^2) - (x^3 + y^3 + z^3) \\ &= 3(\lambda - u)(\lambda - v)(\lambda - w)\end{aligned}$$

Comparing coefficients of like powers, we get

$$\begin{aligned}F_1 &= u + v + w - (x + y + z) = 0 \\ F_2 &= uv + vw + wu - (x^2 + y^2 + z^2) = 0 \\ F_3 &= uvw - \frac{x^3 + y^3 + z^3}{3} = 0\end{aligned}$$

$$\begin{aligned}\text{Now}\quad \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} &= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^2 & y^2 - x^2 & z^2 - x^2 \end{vmatrix} \\ &= -2(y - x)(z - x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y + x & z + x \end{vmatrix} \\ &= -2(x - y)(y - z)(z - x) \\ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} &= \begin{vmatrix} 1 & 1 & 1 \\ v + w & u + w & u + v \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v + w & u - v & u - w \\ vw & w(u - v) & v(u - w) \end{vmatrix} \\ &= (u - v)(u - w)(v - w) = -(u - v)(v - w)(w - u) \\ \text{Thus}\quad \frac{\partial(u, v, w)}{\partial(x, y, z)} &= (-1)^3\frac{\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} = -2\frac{(y - z)(z - x)(x - y)}{(u - v)(v - w)(w - u)}\end{aligned}$$

as required. ∎

Question 2(b) *Prove that an equation of the form $x^n = \alpha$ where $n \in \mathbb{N}$ and $\alpha > 0$ is a real number has a positive root.*

Solution. We divide the set of real numbers $\mathbb{R}$ into two sets X and Y such that $X = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x > 0, x^n \leq \alpha\}$ and $Y = \mathbb{R} - X$. Clearly $X \neq \emptyset$. $Y \neq \emptyset$ as any real number $\beta > \max(1, \alpha)$ belongs to Y. Let $x \in X, y \in Y$, then $x < y$, otherwise $y \leq x \Rightarrow y^n \leq x^n \leq \alpha \Rightarrow y \in X$. Thus the sets X and Y determine a section of real numbers. Dedekind's theorem says that there exists a real number β such that $x < \beta \Rightarrow x \in X$, and $y > \beta \Rightarrow y \in Y$, and β may belong to any set X or Y.

Clearly $\beta \geq 0$. If $\beta = 0$, then for any positive integer m, $\frac{1}{m} > 0$ and therefore belongs to Y, so $(\frac{1}{m})^n > \alpha$. Since it is true for every m, we get $\alpha \leq 0$, which is a contradiction. Thus $\beta > 0$. We now show that $\beta^n = \alpha$. Since $\beta - \frac{1}{m} \in X$, and $\beta + \frac{1}{m} \in Y$, we have

$$\left(\beta - \frac{1}{m}\right)^n \leq \alpha < \left(\beta + \frac{1}{m}\right)^n$$

. Letting $m \to \infty$, we get $\beta^n = \alpha$, so β is the required positive root of $x^n = \alpha$. Note that β is unique. ∎

Question 2(c) *Prove that*

$$\int \frac{x^2 + y^2}{p}\, dx = \frac{\pi ab}{4}\left[4 + (a^2 + b^2)(a^{-2} + b^{-2})\right]$$

when the integral is taken around the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and p is the length of the perpendicular from the center to the tangent.

Solution. There seems to be a misprint in this question.

The tangent at any point (x, y) of the ellipse is $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1 \Rightarrow p = \left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4}\right)^{-\frac{1}{2}} \Rightarrow$ $I = \displaystyle\int (x^2 + y^2)\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}}\, dx$. The parametric representation of the ellipse is $x = a\cos\theta, y = b\sin\theta$, so

$$I = \int_0^{2\pi} (a^2\cos^2\theta + b^2\sin^2\theta)\sqrt{a^{-2}\cos^2\theta + b^{-2}\sin^2\theta}\ (-a\sin\theta)\, d\theta$$

Put $\theta = 2\pi - \phi$, so that

$$\begin{aligned} I &= \int_{2\pi}^{0} (a^2\cos^2\phi + b^2\sin^2\phi)\sqrt{a^{-2}\cos^2\phi + b^{-2}\sin^2\phi}\ (a\sin\phi)\,(-d\phi) \\ &= -\int_0^{2\pi} (a^2\cos^2\phi + b^2\sin^2\phi)\sqrt{a^{-2}\cos^2\phi + b^{-2}\sin^2\phi}\ (-a\sin\phi)\, d\phi = -I \end{aligned}$$

So $2I = 0 \Rightarrow I = 0$. ∎

Question 2(d) *If the function f is defined by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

then show that f possesses both partial derivatives at $(0,0)$ but it is not continuous there at.

Solution. By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$
$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

If f were continuous at $(0,0)$, then $\lim_{(x,y)\to(0,0)} f(x,y)$ would exist. But $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because if we put $y = mx$, we get $\lim_{x\to 0} \dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$, which is different for different values of m, whereas it should be the same for all m if $\lim_{(x,y)\to(0,0)} f(x,y)$ were to exist.

Thus f possesses both partial derivatives at $(0,0)$ but it is not continuous at $(0,0)$. ∎

Paper II

Question 3(a) *Show that the function defined as*

$$f(x) = \begin{cases} \frac{1}{2^n}, & \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0, & x = 0 \end{cases}$$

is integrable in $[0,1]$ although it has infinite number of points of discontinuity. Show that $\displaystyle\int_0^1 f(x)\,dx = \frac{2}{3}$.

Solution. Let $I_n = (\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, then $[0,1] = \{0\} \cup \bigcup_{n=0}^{\infty} I_n$, and $I_n \cap I_m = \emptyset$ if $m \neq n$, so I_n constitute a partition of $(0,1]$. We define a sequence of functions $f_n(x)$ on $[0,1]$ for $n = 0, 1, \ldots$ by

$$f_n(x) = \begin{cases} \frac{1}{2^n}, & x \in I_n \\ 0, & x \notin I_n \end{cases}$$

Clearly each $f_n(x)$ is Riemann integrable, being a step function, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. Note that $0 \notin I_n$ for any n, so $f_n(0)$ for all n, so $f(0) = 0$.

Since $|f_n(x)| \leq \frac{1}{2^n}$ it follows that $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent on $[0,1]$ (Weierstrass M-test). Thus the sum function $f(x)$ is Riemann integrable and

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\,dx = \sum_{n=0}^{\infty} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} \frac{dx}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^n}\left[\frac{1}{2^n} - \frac{1}{2^{n+1}}\right] = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{\frac{1}{2}}{1-\frac{1}{2^2}} = \frac{2}{3}$$

Hence $\displaystyle\int_0^1 f(x)\,dx = \frac{2}{3}$. ∎

Question 3(b) *Show that the function $f(x)$ defined on $\mathbb{R}$ by*

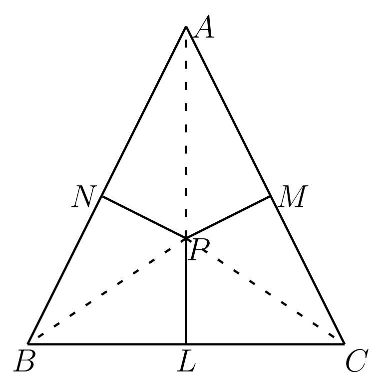
$$f(x) = \begin{cases} -x, & x \text{ rational} \\ x & x \text{ irrational} \end{cases}$$

is continuous only at 0.

Solution. The solution is completely analogous to 2001, question 1(a). ∎

Question 4(a) *If (x, y, z) are the lengths of the perpendiculars drawn from any interior point P of a triangle ABC on the sides BC, CA, AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, when the sides of the triangle are a, b, c.*

Solution.



Area of $\triangle PBC = \frac{1}{2} PL \cdot BC = \frac{1}{2} xa$.
Area of $\triangle PAC = \frac{1}{2} PM \cdot AC = \frac{1}{2} by$.
Area of $\triangle PAB = \frac{1}{2} PN \cdot AB = \frac{1}{2} cz$.
$\triangle ABC = \triangle PBC + \triangle PAC + \triangle PAB$, so area of $\triangle ABC = \frac{1}{2}(ax + by + cz)$

Since $\triangle ABC$ is given, we have to minimize $x^2 + y^2 + z^2$ with the constraint $ax + by + cz = k$ ($k = 2\triangle ABC$).

Let $F(x, y, z) = x^2 + y^2 + z^2 - \lambda(ax + by + cz - k)$, where λ is Lagrange's undetermined multiplier. For extreme values,

$$\frac{\partial F}{\partial x} = 2x - \lambda a = 0, \frac{\partial F}{\partial y} = 2y - \lambda b = 0, \frac{\partial F}{\partial z} = 2z - \lambda c = 0$$

Thus $x = \lambda a/2, y = \lambda b/2, z = \lambda c/2$, so $\frac{\lambda a^2}{2} + \frac{\lambda b^2}{2} + \frac{\lambda c^2}{2} = k$, or $\lambda = \frac{2k}{a^2+b^2+c^2}$.

Now $\frac{\partial^2 F}{\partial x^2} = 2, \frac{\partial^2 F}{\partial y^2} = 2, \frac{\partial^2 F}{\partial z^2} = 2, \frac{\partial^2 F}{\partial x \partial y} = 0 = \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial y \partial z}$ and $d^2 F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 > 0$, so we get that $x^2 + y^2 + z^2$ is minimum when $x = \lambda a/2, y = \lambda b/2, z = \lambda c/2, \lambda = \frac{2k}{a^2+b^2+c^2}$. The minimum value of $x^2 + y^2 + z^2$ is

$$\left(\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}\right)\frac{4k^2}{(a^2 + b^2 + c^2)^2} = \frac{k^2}{a^2 + b^2 + c^2} = \frac{4(\text{area of } \triangle ABC)^2}{a^2 + b^2 + c^2}$$

∎

Question 4(b) *Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.*

Solution. See 2005, question 2(d). In that question the cylinder is $x^2 + y^2 = 2ax$. In our question, the limit of integration for z would be 0 to $\frac{x^2+y^2}{a}$, x varies from $-\sqrt{2ya - y^2}$ to $\sqrt{2ya - y^2}$ and y varies from 0 to $2a$. The roles of x, y have been reversed. ∎

Question 4(c) *Let $f(x) \geq g(x)$ for every $x \in [a,b]$ and f, g be both bounded and Riemann integrable on $[a,b]$. At a point $c \in [a,b]$ if f, g are continuous and $f(c) > g(c)$, then prove that $\displaystyle\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$ and hence show that $\displaystyle -\frac{1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2}\,dx < \frac{1}{2}$.*

Solution. We first prove that if $f(x)$ is continuous at $x = c$ and $f(c) > 0$, then we can find $\delta > 0$ such that

$$|x-c| < \delta \Rightarrow \frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

This follows from the continuity of f at c — choose $\epsilon = \frac{f(c)}{2}$, then there exists $\delta > 0$ such that $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon \Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$, which is the above inequality.

If $f(x)$ is Riemann integrable in $[a,b]$, continuous at c and $f(x) \geq 0$ for $x \in [a,b]$ and $f(c) > 0$, then we will show that $\displaystyle\int_a^b f(x)\,dx > 0$. By the above result, there exists a closed interval $[c_1, d_1]$ containing c such that $\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$ for $x \in [c_1, d_1]$. Now

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{d_1} f(x)\,dx + \int_{d_1}^b f(x)\,dx$$

Clearly $f(x) \geq 0$ implies that the first and third terms are non-negative, and $\displaystyle\int_{c_1}^{d_1} f(x)\,dx \geq \frac{f(c)}{2}(d_1 - c_1) > 0$ — note that $\int_a^b f(x)\,dx \geq m(b-a)$, where $m = \min_{a \leq x \leq b} f(x)$. Thus $\displaystyle\int_a^b f(x)\,dx > 0$.

We get our result by considering the function $F(x) = f(x) - g(x)$ — F satisfies all the conditions above, so

$$\int_a^b F(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx > 0$$

We now consider $\displaystyle\int_0^1 \frac{x^3 \cos 5x}{2+x^2}\,dx$. We know that $|\cos 5x| \leq 1$, so $\displaystyle\left|\frac{x^3 \cos 5x}{2+x^2}\right| < \frac{1}{2}$, or $\displaystyle -\frac{1}{2} < \frac{x^3 \cos 5x}{2+x^2} < \frac{1}{2}$.

Thus $\displaystyle -\frac{1}{2} < \int_0^1 \frac{x^3 \cos 5x}{2+x^2}\,dx < \frac{1}{2}$. ∎

UPSC Civil Services Main 2005 - Mathematics Calculus

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Question 1(a) *Show that the function given below is not continuous at the origin:*

$$f(x,y) = \begin{cases} 0 & \textit{if } xy = 0 \\ 1 & \textit{if } xy \neq 0 \end{cases}$$

Solution. If $f(x,y)$ were continuous at $(0,0)$, then given any $\epsilon > 0$, we should have $\delta > 0$ such that

$$\sqrt{x^2+y^2} < \delta \Longrightarrow |f(x,y) - f(0,0)| < \epsilon \Longrightarrow |f(x,y)| < \epsilon$$

as $f(0,0) = 0$. Choose $\epsilon = \frac{1}{2}$, then for any $\delta > 0$, we see that if $(x,y) = (\frac{\delta}{2}, \frac{\delta}{2})$, then $\sqrt{x^2+y^2} = \sqrt{\frac{\delta^2}{4} + \frac{\delta^2}{4}} = \frac{\delta}{\sqrt{2}} < \delta$, but $|f(x,y)| = f(\frac{\delta}{2}, \frac{\delta}{2}) = 1 \not< \epsilon$.

This shows that $f(x,y)$ is not continuous at $(0,0)$. ∎

Question 1(b) *Let* $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be defined as*

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & x = y = 0 \end{cases}$$

Prove that f_x *and* f_y *exist at* $(0,0)$*, but* f *is not differentiable at* $(0,0)$*.*

Solution. By definition,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$
$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0$$

If $f(x,y)$ were differentiable at $(0,0)$, then

$$f(h,k) = f(0,0) + hf_x(0,0) + kf_y(0,0) + \sqrt{h^2+k^2}\phi(h,k)$$

where $\phi(h,k)$ is a function of h,k such that $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$. Since $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, this would mean that

$$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} f(h,k)\frac{1}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2} = 0$$

But $\displaystyle\lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$ does not exist (take $k = mh$, this becomes $\displaystyle\lim_{(h,k)\to(0,0)} \frac{m}{1+m^2} = \frac{m}{1+m^2}$, which is different for different values of m — if the limit were to exist, it would have been the same for all m).

Thus $f(x,y)$ is not differentiable at $(0,0)$. ∎

Question 2(a) *If* $u = x+y+z, uv = y+z, uvw = z$ *then find* $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution. Clearly $z = uvw, y = uv(1-w), x = u(1-v)$. Therefore

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix} \\
&= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} \quad \text{(operation } R_2+R_3\text{)} \\
&= uv(u - uv + uv) = u^2v
\end{aligned}$$

∎

Question 2(b) *Evaluate*

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

in terms of the Beta function.

Solution. Let $I = \displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$. We consider $\displaystyle\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$, and substitute $x = \frac{1}{y}$ so that $dx = -\frac{1}{y^2}\,dy$. We get

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_\infty^1 \frac{(\frac{1}{y})^{n-1}}{(1+\frac{1}{y})^{m+n}}\left(-\frac{1}{y^2}\right)dy = \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

Thus

$$I = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

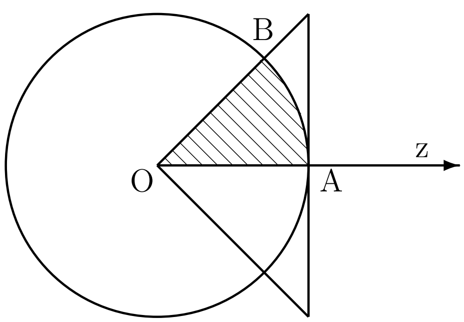
We know that $B(m,n) = B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$. Put $x = \frac{1}{y+1}$ so that $dx = -\frac{1}{(1+y)^2}\,dy$, $1-x = \frac{y}{1+y}$ and when $x = 1, y = 0$, and when $x = 0, y = \infty$. Therefore

$$B(n,m) = \int_\infty^0 \frac{1}{(1+y)^{n-1}}\,\frac{y^{m-1}}{(1+y)^{m-1}}\left(-\frac{1}{(1+y)^2}\right)dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

Hence the given integral $I = \displaystyle\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = B(m,n) = B(n,m)$. ∎

Question 2(c) *Evaluate* $\iiint_V z\,dV$ *where* V *is the volume bounded below by the cone* $x^2+y^2=z^2$*, and above by the sphere* $x^2+y^2+z^2=1$*, lying on the positive side of the y-axis.*

Solution.



$x^2+y^2=z^2$ is the cone with vertex $(0,0,0)$, z-axis as the axis, and semi-vertical angle $\frac{\pi}{4}$. The equation of the circle shown in the figure is $x^2+y^2=1$, or $r=1$ in polar coordinates.

The required volume V is obtained by the revolution of the shaded area OAB about the axis OA.

If an element of the area is $r\,d\theta, dr$, when it is revolved it will generate a ring whose radius is $r\sin\theta$, and therefore the volume of the ring is $2\pi r\sin\theta\; r\,d\theta\,dr$. Thus converting the integral to polar coordinates, we get

$$\iiint_V z\,dV = \int_0^1\int_0^{\frac{\pi}{4}} r\cos\theta\; 2\pi r\sin\theta\; r\,d\theta\,dr = \int_0^1 \pi r^3 dr \int_0^{\frac{\pi}{4}} \sin 2\theta\,d\theta = \frac{\pi}{4}\left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

∎

Question 2(d) *Find the x coordinate of the center of gravity of the solid lying inside the cylinder* $x^2+y^2=2ax$*, between the plane* $z=0$ *and the paraboloid* $x^2+y^2=az$*.*

Solution. The x-coordinate of the center of gravity of a uniform body is $\dfrac{\iiint_V x\,dV}{\iiint_V dV}$.

$$
\begin{aligned}
V=\iiint_V dV &= \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}} dz\,dy\,dx \\
&= \int_0^{2a}\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}} \frac{x^2+y^2}{a} dy\,dx \\
&= \frac{2}{a}\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx \quad \text{(The integrand is an even function of } y\text{)} \\
&= \frac{2}{a}\int_0^{2a}\left[x^2y+\frac{y^3}{3}\right]_0^{\sqrt{2ax-x^2}} dx \\
&= \frac{2}{a}\int_0^{2a}\left[x^2\sqrt{2ax-x^2}+\frac{1}{3}(2ax-x^2)^{\frac{3}{2}}\right] dx \\
&= \frac{2}{a}\int_{-a}^{a}\left[(a+y)^2\sqrt{a^2-y^2}+\frac{1}{3}(a^2-y^2)^{\frac{3}{2}}\right] dy \quad \text{Let } y=x-a, dy=dx \\
&= \frac{2}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(a+a\sin\theta)^2+\frac{1}{3}a^2\cos^2\theta]a^2\cos^2\theta\,d\theta \quad \text{Let } y=a\sin\theta, dy=a\cos\theta\,d\theta \\
&= 2a^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\frac{4}{3}+2\sin\theta+\frac{2}{3}\sin^2\theta]\cos^2\theta\,d\theta \\
&= 2a^3\left(\frac{4}{3}\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\left[-\frac{2}{3}\cos^3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\frac{2}{3}\left[2\frac{1\cdot 1}{4\cdot 2}\frac{\pi}{2}\right]\right) \\
&= 2a^3\left(\frac{2\pi}{3}+\frac{\pi}{12}\right)=\frac{3\pi a^3}{2}
\end{aligned}
$$

We now evaluate the numerator — we will use a different substitution, for illustration. The limits of integration, and the first several steps are similar to the above derivation, so are not repeated here.

$$
\begin{aligned}
\iiint_V x\,dV &= \frac{2}{a}\int_0^{2a}\left[x^3\sqrt{2ax-x^2}+\frac{1}{3}x(2ax-x^2)^{\frac{3}{2}}\right] dx \\
&= \frac{2}{a}\int_0^{\frac{\pi}{2}}\left[(2a)^{\frac{7}{2}}\sin^7\theta(2a)^{\frac{1}{2}}\cos\theta+\frac{1}{3}(2a)^{\frac{5}{2}}\sin^5\theta(2a)^{\frac{3}{2}}\cos^3\theta\right]4a\sin\theta\cos\theta\,d\theta \\
&\qquad \text{(By letting } x=2a\sin^2\theta, dx=4a\sin\theta\cos\theta\,d\theta, 0\le\theta\le\tfrac{\pi}{2}\text{)} \\
&= 128a^4\int_0^{\frac{\pi}{2}}\left[\sin^8\theta\cos^2\theta+\frac{1}{3}\sin^6\theta\cos^4\theta\right] d\theta \\
&= 128a^4\left[\frac{7\cdot 5\cdot 3\cdot 1\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2}+\frac{1}{3}\frac{5\cdot 3\cdot 1\;3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}\frac{\pi}{2}\right]=2\pi a^4
\end{aligned}
$$

Thus the x-coordinate of the centroid $\overline{x}=\frac{4}{3}a$. ∎

Paper II

Question 3(a) *If u, v, w are the roots of the following equation in λ*

$$\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$$

evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Solution. Clearly $(a+\lambda)(b+\lambda)(c+\lambda) - x(b+\lambda)(c+\lambda) - y(a+\lambda)(c+\lambda) - z(a+\lambda)(b+\lambda) = (\lambda-u)(\lambda-v)(\lambda-w)$. Comparing coefficients of like powers of λ, we get

$$\begin{aligned}
F_1 &= u+v+w-(x+y+z)+a+b+c=0 \quad (\text{coefficient of } \lambda^2)\\
F_2 &= uv+vw+wu+x(b+c)+y(c+a)+z(a+b)-(ab+bc+ca)=0 \quad (\text{coeff. of } \lambda)\\
F_3 &= uvw-xbc-yac-zab+abc=0 \quad (\text{constant terms})\\
\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} &= \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}\\
&= -(u-v)(v-w)(w-u)\\
\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} &= \begin{vmatrix} -1 & -1 & -1 \\ b+c & c+a & a+b \\ -bc & -ac & -ab \end{vmatrix} = -(a-b)(b-c)(c-a)\\
\frac{\partial(x,y,z)}{\partial(u,v,w)} &= (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = -\frac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}
\end{aligned}$$

■

Question 3(b) *Evaluate* $\iiint \log(x+y+z, dx\,dy\,dz$ *where the integral is taken over all positive values of x, y, z such that $x+y+z \leq 1$.*

Solution.

$$I = \iiint_{\substack{x+y+z\leq 1\\ x\geq 0, y\geq 0, z\geq 0}} \log(x+y+z)\,dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \log(x+y+z)\,dz\,dy\,dx$$

Now

$$\int \log(x+y+z)\,dz = z\log(x+y+z) - \int \frac{z}{x+y+z}\,dz$$

so

$$\int_0^{1-x-y} \log(x+y+z)\,dz = z\log(x+y+z)\Big]_0^{1-x-y} - \int_0^{1-x-y} \frac{z}{x+y+z}\,dz = -\int_0^{1-x-y} \frac{z}{x+y+z}\,dz$$

Thus $I = -\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{z}{x+y+z}\, dz\, dy\, dx = -\iiint_{\substack{x+y+z\le 1\\ x\ge 0, y\ge 0, z\ge 0}} \frac{z}{x+y+z}\, dz\, dy\, dx.$

Put $u = x+y+z, uv = x+y, uvw = x$, so that $x+y+z \le 1, x \ge 0, y \ge 0, z \ge 0 \Leftrightarrow 0 \le u \le 1, 0 \le v \le 1, 0 \le w \le 1$. Moreover

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} vw & uw & uv \\ v - vw & u - uw & -uv \\ 1-v & -u & 0 \end{vmatrix} = \begin{vmatrix} vw & uw & uv \\ v & u & 0 \\ 1-v & -u & 0 \end{vmatrix} = uv\big[-vu - u + uv\big] = -u^2 v$$

$$\begin{aligned} \therefore I &= -\int_0^1 \int_0^1 \int_0^1 u^{-1}(u - uv)u^2 v\, du\, dv\, dw \\ &= -\int_0^1 \int_0^1 u^2(1-v)v\, du\, dv \\ &= -\frac{1}{3}\left[\frac{v^2}{2} - \frac{v^3}{3}\right]_0^1 = -\frac{1}{18} \end{aligned}$$

Thus $\iiint_{\substack{x+y+z\le 1\\ x\ge 0, y\ge 0, z\ge 0}} \log(x+y+z, dx\, dy\, dz = -\frac{1}{18}$ ∎

Question 4(a) *If f', g' exist for every $x \in [a,b]$ and $g'(x)$ does not vanish anywhere in $[a,b]$, show that there exists $c \in [a,b]$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Solution. Let $\phi(x) = f(x) + Ag(x)$ where A, a constant, is so determined that $\phi(a) = \phi(b)$ i.e.

$$f(a) + Ag(a) = f(b) + Ag(b) \Rightarrow A(g(b) - g(a)) = -(f(b) - f(a))$$

Since $g'(x) \ne 0$ for $x \in [a,b]$, $g(a) \ne g(b)$ — note that if $g(a) = g(b)$, then $g(x)$ satisfies the requirements of Rolle's theorem in $[a,b]$, so there would exists some $\xi \in [a,b]$ such that $g'(\xi) = 0$.

Thus $A = -\dfrac{f(b) - f(a)}{g(b) - g(a)}$.

Now (1) $\phi(x)$ is continuous in the closed interval $[a,b]$, as $f(x), g(x)$ are continuous in $[a,b]$ and A is a constant.

(2) $\phi(x)$ is differentiable in the open interval (a,b) as $f(x), g(x)$ are differentiable in (a,b).

(3) $\phi(a) = \phi(b)$ by choice of A.

Hence ϕ satisfies the requirements of Rolle's theorem in $[a,b]$, so there is a point c in (a,b) such that $\phi'(c) = f'(c) + Ag'(c) = 0$. Hence

$$-A = \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

which was to be proved. This result is known as Cauchy's mean value theorem. ∎

Question 4(b) *Show that* $\int_0^\infty e^{-t}t^{n-1}\,dt$ *is an improper integral which converges for* $n > 0$.

Solution. This integral is an improper integral for two reasons:

1. In a proper integral the range of integration is always a finite closed interval, here it is not.
2. The integrand $e^{-t}t^{n-1}$ has infinite discontinuity at $x = 0$ when $n < 1$, whereas the integrand in a proper integral must be bounded on the interval of integration.

For convergence purposes, we consider two integrals $\int_0^1 e^{-t}t^{n-1}\,dt$ and $\int_1^\infty e^{-t}t^{n-1}\,dt$.

(Instead of 1, we could take any positive real number.)

Convergence of $\int_0^1 e^{-t}t^{n-1}\,dt$: Let $\phi(t) = t^{n-1}$, then $\lim_{t\to 0} \frac{e^{-t}t^{n-1}}{\phi(t)} = 1$, so the integral $\int_0^1 e^{-t}t^{n-1}\,dt$ converges if and only if $\int_0^1 t^{n-1}\,dt$ converges. Clearly $\int_\epsilon^1 t^{n-1}dt = \dfrac{1}{n} - \dfrac{\epsilon^n}{n} \longrightarrow 0$ if and only if $n > 0$ as $\epsilon \longrightarrow 0$. Thus $\int_0^1 e^{-t}t^{n-1}\,dt$ converges for $n > 0$.

Convergence of $\int_1^\infty e^{-t}t^{n-1}\,dt$: It is well known that $e^x > \frac{x^k}{k!}$ for any k when $x > 0$, therefore $x^{n-1}e^{-x} < k\frac{1}{x^2}$ where k is a positive constant. Since $\int_1^\infty \dfrac{dx}{x^2}$ is convergent, it follows that $\int_1^\infty e^{-t}t^{n-1}\,dt$ is convergent.

Hence $\int_0^\infty e^{-t}t^{n-1}\,dt$ is convergent when $n > 0$. ∎

UPSC Civil Services Main 2006 - Mathematics Calculus

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Question 1(a) *Find a and b so that $f'(2)$ exists where*

$$f(x) = \begin{cases} \frac{1}{|x|}, & \text{if } |x| > 2 \\ a + bx^2, & \text{if } |x| \leq 2 \end{cases}$$

Solution. The function $f(x)$ is defined by

$$f(x) = \begin{cases} -\frac{1}{x}, & \text{if } x < -2 \\ a + bx^2, & \text{if } -2 \leq x \leq 2 \\ \frac{1}{x}, & \text{if } x > 2 \end{cases}$$

The right hand derivative (R.H.D.) of f at $x = 2$ is

$$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\frac{1}{2+h} - (a + 4b)}{h}$$

But for the function to have a derivative at $x = 2$, it must be continuous at $x = 2$, therefore $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = f(2) \Rightarrow \frac{1}{2} = a + 4b$. Thus

$$\text{RHD at } x = 2 \quad = \quad \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = -\frac{1}{4}$$

L.H.D. at $x = 2 \quad = \quad$ the derivative of $a + bx^2$ at $x = 2 \quad = 4b$. Thus $f'(2)$ exists if $4b = -\frac{1}{4}, a + 4b = \frac{1}{2} \Rightarrow b = -\frac{1}{16}, a = \frac{3}{4}$. ∎

Question 1(b) *Express* $\int_0^1 x^m(1-x^n)^p\,dx$ *in terms of the Gamma function, and hence evaluate the integral* $\int_0^1 x^6\sqrt{1-x^2}\,dx$.

Solution. Let $x^n = t$, then $dx = \dfrac{1}{n}\dfrac{dt}{x^{n-1}} = \dfrac{1}{n}\dfrac{dt}{t^{\frac{n-1}{n}}} = \dfrac{1}{n}t^{\frac{1}{n}-1}\,dt$ and $x^m = t^{\frac{m}{n}}$. The given integral

$$\begin{aligned}\int_0^1 x^m(1-x^n)^p\,dx &= \int_0^1 t^{\frac{m}{n}}(1-t)^p\frac{1}{n}t^{\frac{1}{n}-1}\,dt \\ &= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{p+1-1}\,dt \\ &= \frac{1}{n}\frac{\Gamma(\frac{m+1}{n})\;\Gamma(p+1)}{\Gamma(\frac{m+1}{n}+p+1)}\end{aligned}$$

since $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

To evaluate $I = \int_0^1 x^6\sqrt{1-x^2}\,dx$, we set $m=6, n=2, p=\frac{1}{2}$ in the above calculated integral and obtain

$$I = \int_0^1 x^6\sqrt{1-x^2}\,dx = \frac{1}{2}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{3}{2})}{\Gamma(5)} = \frac{1}{2}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\;\frac{1}{2}\Gamma(\frac{1}{2})}{4!} = \frac{15}{32\times 24}\pi$$

as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Thus

$$\int_0^1 x^6\sqrt{1-x^2}\,dx = \frac{5\pi}{256}$$

Note: This integral can be easily evaluated by using the substitution $x=\sin\theta$, but the question requires us to use the above integral and Γ functions. ∎

Question 2(a) *Find the value of* a *and* b *such that*

$$\lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4} = \frac{1}{2}$$

Solution. L'Hospital's rule states: Let $f(x)$ and $g(x)$ be real valued functions defined in a deleted neighborhood of a, then if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a}\dfrac{f'(x)}{g'(x)} = l$, then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = l$.

Now L'Hospital's rule is applicable for the evaluation of the given limit. Thus

$$\begin{aligned}\lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4} &= \lim_{x\to 0}\frac{2a\sin x\cos x - b\frac{\sin x}{\cos x}}{4x^3}\\ &= \lim_{x\to 0}\frac{2a\cos^2 x - b}{4x^2}\end{aligned}$$

since $\lim_{x\to 0}\dfrac{\sin x}{x\cos x} = 1$.

The limit on the right hand side would be finite when $\lim_{x\to 0} 2a\cos^2 x - b = 2a - b = 0$. Letting $2a - b = 0$, we use L'Hospital's rule again to get

$$\lim_{x\to 0}\frac{a\sin^2 x + b\log\cos x}{x^4} = \lim_{x\to 0}\frac{-4a\cos x\sin x}{8x} = -\frac{a}{2}$$

Thus for $\lim_{x\to 0}\dfrac{a\sin^2 x + b\log\cos x}{x^4} = \dfrac{1}{2}$, we get $a = -1, b = -2$. ∎

Question 2(b) *If $z = xf(\frac{y}{x}) + g(\frac{y}{x})$, show that*

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

Solution. Let $w_1 = xf(\frac{y}{x})$ and $w_2 = g(\frac{y}{x})$. Then w_1 is a homogeneous function of degree 1, and w_2 is a homogeneous function of degree 0 (a vector function $f : \mathcal{V} \longrightarrow \mathcal{V}$ is homogeneous of degree k if $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$).

We first prove the following result: if $f(x, y)$ is a homogeneous function of degree n possessing continuous partial derivatives of degree 2, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

Proof: Euler's theorem about homogeneous functions states that if $f(x, y)$ is a homogeneous function of degree n possessing continuous partial derivatives of degree 1, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$, and conversely, if $f(x, y)$ has continuous partial derivatives of second order and $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$, then $f(x, y)$ is a homogeneous function of degree n. (For proof, see 1996, question 2(a)).

We first prove that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are homogeneous of degree $n-1$. Since $f(x, y)$ is a homogeneous function of degree n, we have $f(\alpha x, \alpha y) = \alpha^n f(x, y)$. Differentiating this with respect to x and using the chain rule, we get

$$\begin{aligned}\alpha\frac{\partial f}{\partial x}(\alpha x, \alpha y) &= \alpha^n\frac{\partial f}{\partial x}(x, y)\\ \frac{\partial f}{\partial x}(\alpha x, \alpha y) &= \alpha^{n-1}\frac{\partial f}{\partial x}(x, y)\end{aligned}$$

Thus $\frac{\partial f}{\partial x}$ and similarly $\frac{\partial f}{\partial y}$ are homogeneous of degree $n-1$. Now apply Euler's theorem to $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$:

$$\begin{aligned} x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y\,\partial x} &= (n-1)\frac{\partial f}{\partial x} \\ x\frac{\partial^2 f}{\partial x\,\partial y} + y\frac{\partial^2 f}{\partial y^2} &= (n-1)\frac{\partial f}{\partial y} \end{aligned}$$

Multiplying the first by x and the second by y, and remembering that $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$ because f has continuous partial derivatives of second order, we get

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) = n(n-1)f$$

as $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$. Q.E.D.

Applying this result to the current problem,

$$\begin{aligned} x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} &= \left(x^2\frac{\partial^2 w_1}{\partial x^2} + 2xy\frac{\partial^2 w_1}{\partial x\,\partial y} + y^2\frac{\partial^2 w_1}{\partial y^2}\right) \\ &\quad + \left(x^2\frac{\partial^2 w_2}{\partial x^2} + 2xy\frac{\partial^2 w_2}{\partial x\,\partial y} + y^2\frac{\partial^2 w_2}{\partial y^2}\right) \\ &= 1.0.w_1 + 0. - 1.w_2 = 0 \end{aligned}$$

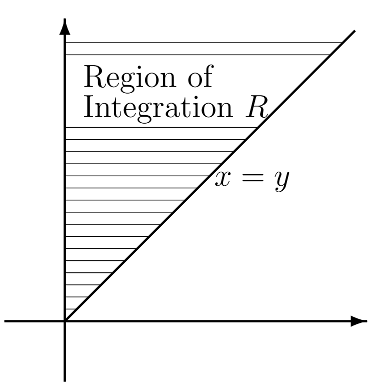
∎

Question 2(c) *Change the order of integration in*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

and hence evaluate it.

Solution. Clearly in the region R, x varies from 0 to y and y varies from 0 to ∞.



Therefore the given integral

$$\begin{aligned} I &= \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx = \int_0^\infty \left(\frac{e^{-y}}{y}\int_0^y dx\right)dy \\ &= \int_0^\infty e^{-y}\,dy = \frac{e^{-y}}{-1}\Big]_0^\infty = 1 \end{aligned}$$

■

Question 2(d) *Find the volume of the uniform ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution. Let $x = aX, y = bY, z = cZ$, so that $dx\,dy\,dz = abc\,dX\,dY\,dZ$ and the ellipsoid is transformed into the sphere $X^2 + Y^2 + Z^2 = 1$.

$$\begin{aligned} V &= \text{Volume of the positive octant of the sphere} \\ &= \int_{X=0}^{1} \int_{Y=0}^{\sqrt{1-X^2}} \int_{Z=0}^{\sqrt{1-X^2-Y^2}} dX\,dY\,dZ \\ &= \int_{X=0}^{1} \int_{Y=0}^{\sqrt{1-X^2}} \sqrt{1-X^2-Y^2}\,dX\,dY \end{aligned}$$

Let $Y = \sqrt{1-X^2}\sin\theta$, so $dY = \sqrt{1-X^2}\cos\theta\,d\theta$.

$$\begin{aligned} V &= \int_{X=0}^{1} \int_{0}^{\frac{\pi}{2}} \sqrt{1-X^2}\cos\theta\sqrt{1-X^2}\cos\theta\,d\theta\,dX \\ &= \int_{X=0}^{1} (1-X^2)\,dX \int_{0}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\ &= X - \frac{X^3}{3}\Big]_0^1 \frac{\pi}{4} = \frac{2}{3}\frac{\pi}{4} = \frac{\pi}{6} \end{aligned}$$

Thus the volume of the ellipsoid is $8abcV = 8abc\frac{\pi}{6} = \frac{4}{3}\pi abc$ ■

Paper II

Question 3(a) *Examine the convergence of*

$$\int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

Solution. The integral has infinite discontinuity at $x = 0$ and $x = 1$. We therefore consider the integrals

$$\int_0^{\frac{1}{2}} \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \text{ and } \int_{\frac{1}{2}}^{1} \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

For the first consider $\phi(x) = x^{-\frac{1}{2}}$, and for the second consider $\psi(x) = (1-x)^{-\frac{1}{2}}$. Clearly

$$\lim_{x\to 0} \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\frac{1}{\phi(x)} = 1, \quad \lim_{x\to 1} \frac{1}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}\frac{1}{\psi(x)} = 1$$

Thus the first integral converges if and only if $\int_0^{\frac{1}{2}} \phi(x)\,dx$ converges, and the second integral converges if and only if $\int_{\frac{1}{2}}^1 \psi(x)\,dx$ converges.

$$\int_\epsilon^{\frac{1}{2}} x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big]_\epsilon^{\frac{1}{2}} = 2\sqrt{\frac{1}{2}} - 2\sqrt{\epsilon}$$

$$\int_{\frac{1}{2}}^{1-\epsilon} (1-x)^{-\frac{1}{2}}\,dx = -2(1-x)^{\frac{1}{2}}\Big]_{\frac{1}{2}}^{1-\epsilon} = 2\frac{1}{\sqrt{2}} - 2\sqrt{\epsilon}$$

showing that $\lim_{\epsilon\to 0}\int_\epsilon^{\frac{1}{2}} \phi(x)\,dx$ and $\lim_{\epsilon\to 0}\int_{\frac{1}{2}}^{1-\epsilon} \psi(x)\,dx$ exist. Thus the integrals $\int_0^{\frac{1}{2}} \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$ and $\int_{\frac{1}{2}}^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$ are convergent, so the given integral is convergent. ∎

Question 3(b) *Prove that the function defined below is nowhere continuous.*

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

Solution. Let $a \in \mathbb{R}$. If f is continuous at a, then given $\epsilon > 0$ there exists $\delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \frac{\epsilon}{2}$. Now for x_1, x_2 with $|x_1 - a| < \delta, |x_2 - a| < \delta$, we get
$|f(x_1) - f(x_2)| \leq |f(x_1) - f(a)| + |f(a) - f(x_2)| < \epsilon$

But given $0 < \epsilon < 1$, whatever $\delta > 0$, we can get x_1 rational and x_2 irrational, $|x_1 - a| < \delta, |x_2 - a| < \delta$, so $|f(x_1) - f(x_2)| = |1 - (-1)| = 2 \not< \epsilon$, therefore $f(x)$ cannot be continuous at a. Since a is arbitrary, f is not continuous anywhere. ∎

Question 4(a) *A twice differentiable function f is such that $f(a) = f(b) = 0$, and $f(c) > 0$ for $a < c < b$. Prove that there is at least one value of $\xi, a < \xi < b$ for which $f''(\xi) < 0$.*

Solution. $f'(a) = \lim_{x\to a+} \frac{f(x)-f(a)}{x-a} \geq 0$ as $x - a > 0, f(x) > 0, f(a) = 0$. Similarly, $f'(b) \leq 0$. If $f''(\xi) \geq 0$ for $a < \xi < b$, then $f'(x)$ is a non-decreasing function in $[a, b]$, but this would mean that $0 \geq f'(b) \geq f'(a) \geq 0 \Rightarrow f'(a) = f'(b) = 0 \Rightarrow f'(x) = 0$ as $f'(x)$ is non-decreasing. Thus $f(x)$ is a constant on $[a, b] \Rightarrow f(x) = 0$ as $f(a) = f(b) = 0$. But this contradicts $f(c) > 0$, so our assumption that $f''(\xi) \geq 0$ must be incorrect — there must be some $\xi, a < \xi < b$ for which $f''(\xi) < 0$. ∎

Question 4(b) *Show that the function given by*

$$f(x, y) = \begin{cases} \frac{x^3+2y^3}{x^2+y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = (0, 0) \end{cases}$$

(i) is continuous at $(0,0)$ (ii) possesses partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

Solution. (i) Let $x = r\cos\theta, y = r\sin\theta$, then

$$|f(x,y) - f(0,0)| = \left|\frac{r^3\cos^3\theta + 2r^3\sin^3\theta}{r^2}\right| \leq 3r = 3\sqrt{x^2+y^2}$$

Thus given $\epsilon > 0$, choose any $\delta, 0 < \delta < \frac{\epsilon}{3}$, such that $\sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - f(0,0)| < \epsilon$. Thus $f(x,y)$ is continuous at $(0,0)$.

(ii) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{h^3}{h^2} - 0}{h} = 1.$

$f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{2h^3}{h^2} - 0}{h} = 2.$

Thus both $f_x(0,0)$ and $f_y(0,0)$ exist. ∎

Question 4(c) *Find the volume of the uniform ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution. See above, question 2(d). ∎