Q1 ⇒ If y is a function of x such that the diff. Coefficient $\frac{dy}{dx}$ is equal to $\cos(x+y)+\sin(x+y)$. Find out a relation between x and y which is free from any derivative/differential.

Solⁿ Given $\frac{dy}{dx} = \cos(x+y)+\sin(x+y)$ — ①

let $x+y = z$

$1+\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ replace in eqⁿ ①

$$\frac{dz}{dx} - 1 = \cos z + \sin z$$

$$\Rightarrow \frac{dz}{1+\sin z+\cos z} = dx$$

$$\Rightarrow \frac{dz}{2\cos^2\frac{z}{2} + 2\sin\frac{z}{2}\cos\frac{z}{2}} = dx$$

$$\Rightarrow \frac{1}{2}\frac{\sec^2\frac{z}{2}}{1+\tan\frac{z}{2}} = dx$$

$1+\tan\frac{z}{2} = t$

$\frac{1}{2}\sec^2\frac{z}{2}\,dz = dt$

$$\log\left(1+\tan\frac{z}{2}\right) = x+c$$

$$\boxed{\log\left[1+\tan\frac{(x+y)}{2}\right] = x+c}$$

Q2 ⇒ Obtain the equation of the orthogonal Trajectory of the family of curves represented by $r^n = a\sin n\theta$, (r,θ) being the Polar co-ordinates.

Solⁿ Given $r^n = a\sin n\theta$

$$n\log r = \log a + \log\sin n\theta$$

diff. w.r.t. θ

$$\frac{n}{r}\frac{dr}{d\theta} = n\cot n\theta.$$

for finding orthogonal trajectories replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$n\left(-\frac{r^2 d\theta}{dr}\right) = n\cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = -\frac{dr}{r}$$

$$\frac{\log(\sec n\theta)}{n} = -\log r + c$$

$$\log\sec n\theta = -n\log r + k$$

$$\boxed{r^n = k\cos n\theta} \quad k \to \text{Constant}$$

Q3 ⇒ Solve the diff. Equation $(5x^3+12x^2+6y^2)dx + 6xy\,dy = 0$

Sol^n Given Eq^n

$$(5x^3+12x^2+6y^2)\,dx + 6xy\,dy = 0$$

multiply by 'x'

$$(5x^4 + 12x^3+6xy^2)\,dx + 6x^2y\,dy = 0 \quad \text{--- (1)}$$

Compare with $M\,dx + N\,dy$.

$$M = 5x^4+12x^3+6xy^2 \,/\; N = 6x^2y$$

$$\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x} = 12xy.$$

so Eq^n (1) is Exact

on diff.

$$\int (5x^4+12x^3+6xy^2)\,dx + 0 = c$$

$$\boxed{x^5 + 3x^4 + 3x^2y^2 = c}$$

Q 4 ⇒ Using the method of variation of parameters, solve the differential Equation $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

Sol^n Given Eq^n $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R$

$R = \sec ax$ $\Rightarrow P = 0, \; Q = a^2 \; ; \; R = \sec ax.$ $(D^2+a^2)y = 0$

$\hookrightarrow$ homogenous Eqⁿ.

Auxiliary Eqⁿ Given by.

$$(m^2+a^2) = 0$$

$$m = \pm ai$$

$$\text{C.F. } y_c = c_1 \cos ax + c_2 \sin ax$$

$$y_p = AU + BV \quad ; \quad U = \cos ax \quad V = \sin ax$$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a$$

$$A = \int \frac{-VR}{W} dx = \int -\frac{\sin ax . \sec ax \, dx}{a}$$

$$= \int -\frac{\sin ax}{a \cos ax} dx = \frac{-\log(\sec ax)}{a^2}$$

$$A = \frac{-\log(\sec ax)}{a^2}$$

$$B = \int \frac{UR}{W} dx = \int \frac{\cos ax . \sec ax \, dx}{a} = \frac{x}{a}$$

$$Y_p = \frac{-\log(\sec ax)}{a^2} . \cos ax + \frac{x}{a} \sin ax$$

$$\boxed{Y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a} \sin ax + \frac{\log(\cos ax)\cos ax}{a^2}}$$

Q5⇒ Find the General Solution of the Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \ln x \sin(\ln x)$$

Solⁿ Given eqⁿ $x^2\frac{d^2y}{dx^2} + \frac{x\,dy}{dx} + y = \ln x \sin(\ln x)$

comparing with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$P = \frac{1}{x}$; $Q = \frac{1}{x^2}$; $R = \frac{\ln x . \sin(\ln x)}{x^2}$

$\cancel{u = e^{-\frac{1}{2}\int P dx}} = \cancel{e^{-\frac{1}{2}\ln x}} \Rightarrow \frac{1}{\sqrt{x}}$

$\cancel{I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}}$,

$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{1}{x^2} \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$\boxed{z = \log x}$

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = -\frac{1}{x^2} + \frac{1}{x}\cdot\frac{1}{x} = 0$

$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \ln x \sin(\ln x) = z \sin z$

Putting value of Q_1, P_1, R_1 on

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$\frac{d^2y}{dz^2} + 0 + y = z \sin z$

$\frac{d^2y}{dz^2} + y = z \sin z$

$(D^2+1)y = z \sin z$ / where $D = \frac{dy}{dz}$

Auxiliary Equation

$m^2+1 = 0$; $m = \pm i$

$y_c = c_1 \cos \cancel{\log} z + c_2 \sin z$

$y_c = c_1 \cos(\log x) + c_2 \sin(\log x)$

Particular integral

$y_p = \frac{1}{D^2+1}(z \sin z)$

$\Rightarrow$ Imaginary part of $\frac{1}{D^2+1} z . e^{iz}$

$\Rightarrow e^{iz}\frac{1}{(D+i)^2+1} \cdot z$

$\Rightarrow e^{iz}\frac{1}{(D^2+Di)} z$

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$\rightarrow e^{iz}\left[\frac{1}{2Di}\left(1+\frac{D}{2i}\right)^{-1}\right]z$

$\Rightarrow e^{iz}\left[\frac{1}{2Di}\left[z - \frac{1}{2i} + \cdots\right]\right].$

$\Rightarrow e^{iz}\left[\left(-\frac{z^2 i}{4}+\frac{z}{4}\right)\right]$

$\Rightarrow (\cos z + i\sin z)\left(-\frac{z^2 i}{4}+\frac{z}{4}\right) \rightarrow$ I. P.

$y_p = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$

$y_p = \frac{-(\log x)^2\cos(\log x)}{4} + \frac{\log x}{4}\sin(\log x)$.

$\therefore \; y = y_c + y_p$

$$\boxed{y = c_1\cos(\log x) + c_2\sin(\log x) - \frac{(\log x)^2\cos(\log x)}{4} + \frac{\log}{4}\sin(\log x)}$$

Q6 $\Rightarrow$ By using Laplace transform method, solve the differential equation $(D^2+n^2)x = a\sin(nt+\alpha)$, $D^2 = \frac{d^2}{dt^2}$ subject to initial conditions $x=0$, at $t=0$ In which a, n, α are constants

Sol^n

Given $(D^2+n^2)x = a\sin(nt+\alpha)$

Applying Laplace transform

$L[(D^2+n^2)x] = L[a\sin(nt+\alpha)]$

$p^2 L(x) - p\cdot x(0) - x'(0) + n^2 L(x) = L[a\sin nt\cos\alpha + a\cos nt\cdot\sin\alpha]$

$(p^2+n^2)L(x) = a\cos\alpha\frac{n}{p^2+n^2} + a\sin\alpha\cdot\frac{p}{p^2+n^2}$

let $L(n) = a\cos\alpha\frac{n}{p^2+n^2} + a\sin\alpha\frac{p}{p^2+n^2}$

apply inverse Laplace.

$$x(t) = a\cos\alpha\, L^{-1}\left(\frac{n}{(p^2+n^2)^2}\right) + a\sin\alpha\, L^{-1}\left[\frac{p}{(p^2+n^2)^2}\right] \quad \text{--- (2)}$$

$$\because \quad L^{-1}\left(\frac{n}{p^2+n^2}\right) = \sin nt$$

$$L^{-1}\left[\frac{d}{dp}\left(\frac{n}{p^2+n^2}\right)\right] = (-1)\, t \cdot \sin(nt)$$

$$\therefore \quad L^{-1}\left[\frac{p}{(p^2+n^2)^2}\right] = \int_0^t \sin nt \quad \text{--- (3)}$$

$$L^{-1}\left(\frac{1}{p}\cdot\frac{p}{(p^2+n^2)^n}\right) = \int_0^t \frac{t}{2n}\cdot \sin nt\, dt$$

$$L^{-1}\left[\frac{1}{p}\cdot\frac{p^2}{(p^2+n^2)^2}\right] = \frac{1}{2n}\left[t\cdot\frac{(-\cos nt)}{n} - \frac{(-\sin nt)}{n^2}\right]$$

$$L^{-1}\left[\frac{1}{(p^2+n^2)^2}\right] = \frac{1}{2n^3}\left[-nt\cos nt + \sin nt\right] \quad \text{--- (4)}$$

Using (3) & (4) in (2)

$$x(t) = \frac{a}{2n^2}(\sin nt - nt\cos nt)\cos\alpha + \frac{at}{2n}(\sin nt)\cdot\sin\alpha$$

$$\boxed{x(t) = \frac{a}{2n^2}\sin nt - \frac{at}{2n}\cos(nt+\alpha)}$$

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