CSE

2010

error in statement

1\. $y' = \alpha x$, $x > 0$, α is constant. T.S.T

(i) if $\phi(x)$ is any sol & $\psi(x) = \phi(x)e^{-\alpha x}$ then $\psi(x)$ is a const

(ii) if $\alpha < 0$, then every sol. tends to zero as $x \to \infty$.

(i) ϕ is a sol. $\Rightarrow \phi'(x) = \alpha\phi(x)$ — (1)

$\psi(x) = \phi(x)e^{-\alpha x}$

$\Rightarrow \psi'(x) = -\alpha\phi(x)e^{-\alpha x} + e^{-\alpha x}\phi'(x)$

using (1), $\psi'(x) = -\alpha\phi(x)e^{-\alpha x} + \alpha\phi(x)e^{-\alpha x}$
$= 0$

$\Rightarrow \psi(x) = $ const.

(ii) ~~$y' = \alpha x \Rightarrow \frac{dy}{dx} = \alpha x \Rightarrow dy = \alpha x dx$~~

~~$y = \frac{\alpha x^2}{2} + c$~~

by part (i) $\phi(x) = \frac{k}{e^{-\alpha x}} = ke^{\alpha x}$

where k = constant.

$\Rightarrow$ as $\alpha < 0$, as $x \to \infty$

$\alpha x \to -\infty \Rightarrow e^{\alpha x} \to 0$

$\Rightarrow \phi(x) \to 0$.

so every sol. tends to zero as $x \to \infty$

2\. TST $(3y^2 - x)dx + 2y(y^2 - 3x)dy = 0$ admits an IF which is fun of $(x+y^2)$ Hence solve the eqn.

let IF $= f(x+y^2)$ then

$f(x+y^2)(3y^2 - x)dx + 2y(y^2-3x)f(x+y^2)dy = 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow 6y f(x+y^2) + 6y^3 f'(x+y^2) - 2xy f'(x+y^2)$

$= -6y f(x+y^2) + 2y^3 f'(x+y^2) - 6xy f'(x+y^2)$

$\Rightarrow 12y f(x+y^2) = -4y^3 - 4xy f'(x+y^2)$

$\frac{f(x+y^2)}{f'(x+y^2)} = \frac{(-y^2 - x)}{3}$

Page No.
Date

let $\boxed{x+y^2=z}$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{-3}{z}$$

Integrating wrt z;

$$\int \frac{f'(z)}{f(z)}dz = \int \frac{-3}{z}dz$$

$$\log f(z) = -3\log z = \log \frac{1}{z^3} \Rightarrow f(z) = \frac{1}{z^3}$$

$$f(x+y^2) = \frac{1}{(x+y^2)^3} = IF$$

So, sol. is

$$\int_{y\,const} \frac{3y^2-x}{(x+y^2)^3}dx + \int 0\,dx = C$$

$$\frac{4y^2 \times 1}{-2\,(x+y^2)^2} - \int \frac{(x+y^2)}{(x+y^2)^3}dx = C$$

$$\boxed{-\frac{2y^2}{(x+y^2)^2} + \frac{1}{(x+y^2)} = C}$$

verify $\frac{1}{2}(Mx+Ny)\,d(\log_e(xy)) + \frac{1}{2}(Mx-Ny)\,d\left(\log_e\left(\frac{x}{y}\right)\right) = Mdx+Ndy$

Consider LHS

$$\frac{1}{2}(Mx+Ny)\left[\frac{1}{xy}(xdy+ydx)\right] + \frac{1}{2}(Mx-Ny)\left[\frac{y}{x} \times \frac{ydx-xdy}{y^2}\right]$$

$$= \frac{1}{2}(Mx+Ny)\left[\frac{dy}{y}+\frac{dx}{x}\right] + \frac{1}{2}(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)$$

$= Mdx+Ndy = RHS$ Hence verified.

(ii) TST if $Mdx + Ndy = 0$ is homogenous then $\frac{1}{Mx+Ny}$ is an IF unless $Mx+Ny \equiv 0$.

$$Mdx + Ndy = \frac{1}{2}(Mx+Ny)\left[\frac{dx}{x}+\frac{dy}{y}\right] + \frac{1}{2}(Mx-Ny)\left[\frac{dx}{x}-\frac{dy}{y}\right]$$

dividing by $Mx+Ny$ assuming $Mx+Ny \neq 0$.

gives, $$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}\left[\frac{dx}{x}+\frac{dy}{y}\right] + \frac{1}{2}\left(\frac{Mx-Ny}{Mx+Ny}\right)\left[\frac{dx}{x}-\frac{dy}{y}\right]$$

or $$\frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}d\left(\log_e(xy)\right) + \frac{1}{2}\frac{Mx-Ny}{Mx+Ny}d\left(\log_e\frac{x}{y}\right)$$

since ~~Mx+Ny is~~ M, N are homogeneous so,

$$\frac{Mx-Ny}{Mx+Ny} = f\left(\frac{x}{y}\right) = f\left(e^{\log_e\left(\frac{x}{y}\right)}\right) = g\left(\log_e\frac{x}{y}\right)$$

i.e. $$Mdx+Ndy = \frac{1}{2}d\log_e(xy) + \frac{1}{2}g\left(\log_e\left(\frac{x}{y}\right)\right)\times d\log_e\left[\frac{x}{y}\right]$$

$$= d\left[\frac{1}{2}\log_e(xy) + \frac{1}{2}\int g\left(\log_e\frac{x}{y}\right)d\left(\log_e\left(\frac{x}{y}\right)\right)\right]$$

$$= d\phi \quad \text{for } \phi = \frac{1}{2}\log_e xy + \frac{1}{2}\int g\left(\log_e\frac{x}{y}\right)d\left(\log_e\frac{x}{y}\right)$$

so, $Mdx+Ndy$ is exact differential. hence $\frac{1}{Mx+Ny}$ is an I.F. if $Mx+Ny \neq 0$.

(vi) If $Mdx + Ndy = 0$ is not exact but of form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$ then $(Mx - Ny)^{-1}$ is an IF unless $Mx - Ny \equiv 0$

$$Mdx + Ndy = \frac{1}{2}(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{1}{2}(Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right) \quad (1)$$

given:-

~~Mdx~~ Let $Mx - Ny \not\equiv 0$.

and $M = yf_1(xy)$, $N = xf_2(xy)$

then.
$$\frac{Mx + Ny}{Mx - Ny} = \frac{xyf_1(xy) + xyf_2(xy)}{xyf_1(xy) - xyf_2(xy)}$$
$$= \frac{f_1(xy) + f_2(xy)}{f_1(xy) - f_2(xy)}$$

So (1) becomes,

$$\frac{1}{Mx - Ny}(Mdx + Ndy) = \frac{f_1 + f_2}{f_1 - f_2}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{1}{2}\left(\frac{dx}{x} - \frac{dy}{y}\right)$$
$$= \frac{1}{2}f(xy)\,d\log_e(xy) + \frac{1}{2}d\left(\log_e\frac{x}{y}\right)$$
$$= d\left[\frac{1}{2}\cancel{\int f(xy)\,d\log}\right. \text{ using.}$$

$$\frac{1}{2}f(xy) = \frac{1}{2}f\left(e^{\log_e(xy)}\right) = \frac{1}{2}g\left(\log_e(xy)\right)$$

$$\frac{1}{Mx - Ny}(Mdx + Ndy) = d\left[\frac{1}{2}\int g(\log_e xy)\,d(\log_e xy) + \frac{1}{2}d\left(\log_e\frac{x}{y}\right)\right]$$
$$= d\phi$$

So, $\dfrac{Mdx + Ndy}{Mx - Ny}$ is exact differential

So, $\dfrac{1}{Mx - Ny}$ is an IF.

Page No.
Date

Doubt? check answer

4\. show that set of solutions of homogeneous differential equation $y' + p(x)y = 0$ on $I = [a,b]$ forms a vector subspace W of real vector space of continuous functions on I. What is dimension of W?

Let V = vsp of continuous functions on $I = [a,b]$

let $W = \{ y \in V \mid y' + p(x)y = 0 \}$.

T.S.T. W is a vector subspace of W.

step 1 → $y = 0 \in V$ such that $0' + p(x) \times 0 = 0$
i.e. $f(x) \equiv 0 \in W$ so,
W is non-empty.

step 2 → Closure → let $y, z \in W$ then
$y' + p(x)y = 0$; $z' + p(x)z = 0$ — (i)
Then $(y+z)' + p(x)(y+z) = y' + z' + p(x)y + p(x)z = (y' + p(x)y) + (z' + p(x)z)$
$= 0 + 0 = 0$ (by (i))
so closure holds.

step 3 → Additive identity : $f(x) \equiv 0 \in W$
such that for $g(x) \in W$,
$g(x) + f(x) = g(x) + 0 = g(x)$.

step 4 → Additive inverse → for each $g(x) \in W$
$\exists$ $(-g)(x) = -g(x) \in W$ such
[$\because$ if $g'(x) + p(x)g(x) = 0 \Rightarrow -g'(x) - p(x)g(x) = 0$]
that $g(x) + (-g(x)) \equiv 0 \in W$.
so inverse exists.

Page No.
Date

~~step 5 → Associativity → Let f(x), g(x)~~
~~∈ W then~~
~~$(f(x)+g(x)) + h(x)$~~

If $g(x) \in W$ then for $\alpha \in \mathbb{R}$
$\alpha g(x) \in W$ as $(\alpha g(x))' + p(x)[\alpha g(x)]$
$= \alpha g'(x) + \alpha p(x) g(x) = \alpha[g'(x) + p(x)g(x)]$
$= \alpha \cdot 0 = 0$
⇒ closed under scalar multiplication

since W is closed under vector addition and scalar multiplication so, W is a subspace of V.

Also, to find dimension of W i.e. solution space of $y' + p(x)y = 0$
$y' + p(x)y = 0$ ⇒ linear differential equation of first order.
Integrating factor $= e^{\int P dx}$.
i.e. solution is $y e^{\int p(x) dx} = C$

$\Rightarrow y = C e^{-\int p(x) dx}$ where C is constant

So, the solution space are ~~mult~~ generated by $w = e^{-\int p(x) dx}$ so dimension of $W = 1$. as
$S = \{e^{-\int p(x)}\}$ is linearly independent being single element and spans W
i.e. solution space of $y' + p(x)y = 0$.
Hence $\boxed{\dim W = 1}$

~~Q-5~~ (Doubt)

$y'' + y = \sin x + (1+x^2)e^x$

$y_c = c_1\cos x + c_2\sin x$

$y_p = A\sin x + B\cos x + (Cx^2 + Dx + E)e^x$

Since $\sin x, \cos x$ are present in CF, so multiply by x;

$y_p = Ax\sin x + Bx\cos x + (Cx^3 + Dx^2 + Ex)e^x$

$y_p' = Ax\cos x + A\sin x - Bx\sin x + B\cos x + (Cx^3 + Dx^2 + Ex + 3Cx^2 + 2Dx + E)e^x$

$y_p'' = -Ax\sin x + A\cos x + A\cos x - Bx\cos x - B\sin x - B\sin x + \big((Cx^3 + (D+3C)x^2 + (2D+E)x + E + 3Cx^2 + 2(D+3C)x + (2D+E)\big)e^x$

$y_p'' + y_p = \sin x + (1+x^2)e^x$

$= 2A\cos x - 2B\sin x + \big[2Cx^3 + (6C+2D)x^2 + (4D+2E+6C)x + (2D+2E)\big]e^x$

$A = 0,\ B = \frac{-1}{2},\ C = 0,\ D = \frac{1}{2};\ E = -1;$

Then $2D + 2E = -1 \neq$ coeff of $e^x = 1$