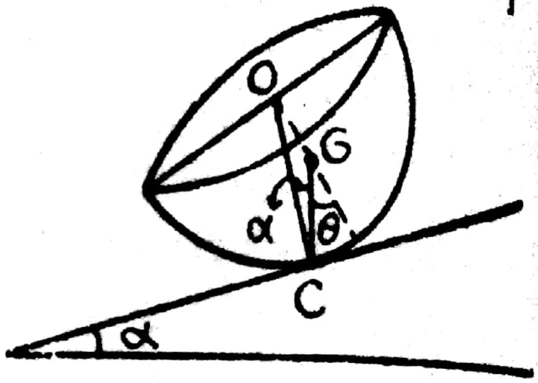
6.(c) एक एकसमान अर्धगोला, क्षैतिज से ϕ कोण पर झुके हुए रुक्ष समतल पर अपने वक्रीय पृष्ठ को समतल से स्पर्श करते हुए रखा है। सन्तुलन के लिए ϕ का अधिकतम ग्राह्य मान निकालिए। यदि ϕ का मान इससे कम है तब क्या यह साम्यावस्था स्थिर है ?

A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable ? 17

i.e., $\overline{h} > \overline{ab}$ ... 3 $\quad a+b \quad\quad a+b$

Ex. 14. *A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1} \frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.* **[Meerut 90; Lucknow 79]**

Sol. Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC=r$. Let G be the centre of gravity of the hemisphere.



Then $OG=3r/8$. In the position of equilibrium the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle OCG=\alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From $\triangle OGC$, we have

$$\frac{OG}{\sin \alpha}=\frac{OC}{\sin \theta} \text{ i.e., } \frac{3r/8}{\sin \alpha}=\frac{r}{\sin \theta}.$$

$\therefore \quad \sin \theta=\frac{8}{3} \sin \alpha$, or $\theta=\sin^{-1} (\frac{8}{3} \sin \alpha)$,

giving the position of equilibrium of the hemisphere.

Since $\sin \theta<1$, therefore $\frac{8}{3} \sin \alpha<1$

i.e.,

$$\sin \alpha < \frac{8}{3} \text{ i.e., } \alpha < \sin^{-1} \frac{3}{8}.$$

Thus for the equilibrium to exist, we must have

$$\alpha < \sin^{-1} \frac{3}{8}.$$

Now let $CG=h$. Then

$$\frac{h}{\sin (\theta-\alpha)}=\frac{3r/8}{\sin \alpha}, \text{ so that } h=\frac{3r \sin (\theta-\alpha)}{8 \sin \alpha}.$$

Here $\rho_1=r$ and $\rho_2=\infty$.

The equilibrium will be stable if

$$h<\frac{\rho_1\rho_2 \cos \alpha}{\rho_1+\rho_2}$$

[See § 7]

i.e., $\frac{1}{h} > \frac{\rho_1+\rho_2}{\rho_1\rho_2} \sec\alpha$ i.e., $\frac{1}{h} > \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right) \sec\alpha$

i.e., $\frac{1}{h} > \frac{1}{r} \sec\alpha$

i.e., $h < r\cos\alpha$ $\quad [\because \rho_1 = r,\ \rho_2 = \infty]$

i.e., $\frac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$ $\quad$ [substituting for h]

or, $3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$

or, $3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$

or, $8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{(1-\frac{64}{9}\sin^2\alpha)} < 8\sin\alpha\cos\alpha$

$[\because \sin\theta = \frac{8}{3}\sin\alpha]$

or, $-\sin\alpha\sqrt{(9-64\sin^2\alpha)} < 0$

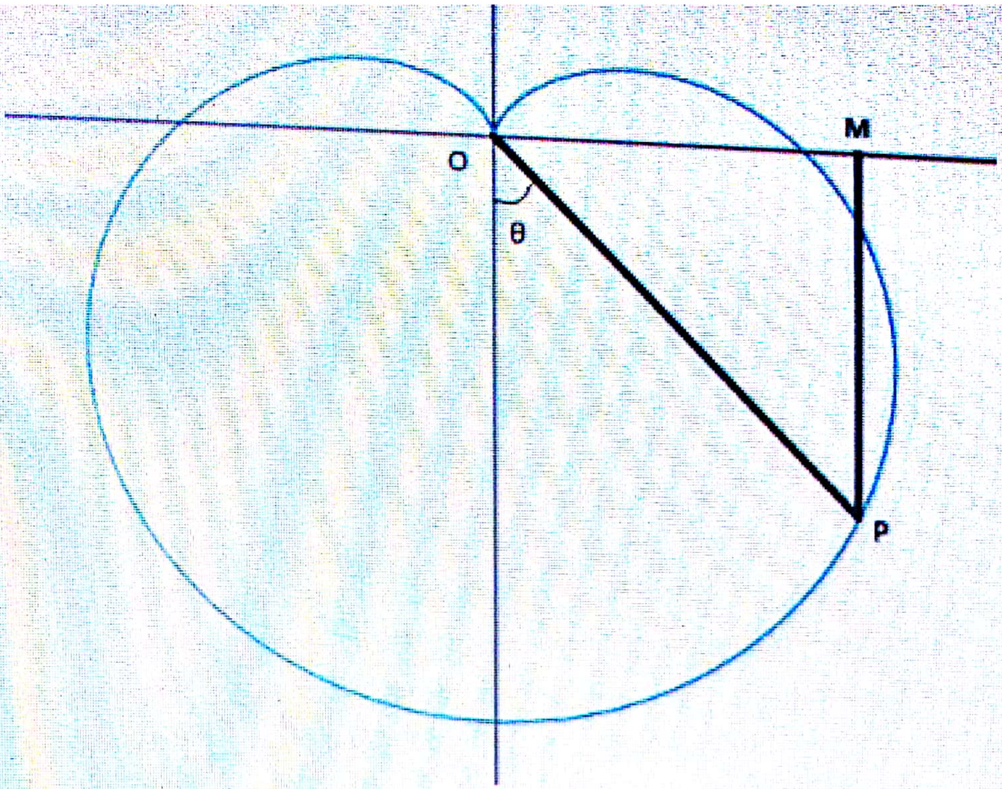
or, $\sin\alpha\sqrt{(9-64\sin^2\alpha)} > 0.$ $\quad \ldots(2)$

But from (1),

$\sin\alpha < \frac{3}{8}$ *i.e.*, $64\sin^2\alpha < 9$ *i.e.*, $\sqrt{(9-64\sin^2\alpha)}$ is a positive real number. Therefore the relation (2) is true. Hence the equilibrium is stable.

5.(c) एक तार, कार्डिआॅइड $r = a(1 + \cos\theta)$, जिसकी प्रारम्भिक रेखा अधोगत ऊर्ध्वाधर है, के आकार में स्थिर है । m द्रव्यमान का एक छोटा छल्ला तार पर फिसल सकता है तथा कार्डिआॅइड के बिन्दु $r = 0$ से स्वाभाविक लम्बाई a की प्रत्यास्थ (elastic) डोरी से, जिसका प्रत्यास्थता गुणांक 4 mg है, बंधा है । छल्ले को विरामावस्था से, जबकि डोरी क्षैतिज है, छोड़ा जाता है । ऊर्जा-संरक्षण के नियमों का प्रयोग कर दर्शाइये कि $a\dot{\theta}^2(1 + \cos\theta) - g\cos\theta\,(1 - \cos\theta) = 0$, g गुरुत्वाकर्षण के कारण त्वरण है ।

A fixed wire is in the shape of the cardiod $r = a(1 + \cos\theta)$, the initial line being the downward vertical. A small ring of mass m can slide on the wire and is attached to the point $r = 0$ of the cardiod by an elastic string of natural length a and modulus of elasticity 4 mg. The string is released from rest when the string is horizontal. Show by using the laws of conservation of energy that
$a\dot{\theta}^2(1 + \cos\theta) - g\cos\theta\,(1 - \cos\theta) = 0$, g being the acceleration due to gravity. 10



Here, $OP = r, PM = r\cos\theta = a(1+\cos\theta)\cos\theta$

KE = $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$

$\frac{1}{2}m(a^2\sin^2\theta + a^2(1+\cos\theta)^2)\dot{\theta}^2$

$ma^2(1+\cos\theta)\dot{\theta}^2$

PE due to gravity $= -mg(PM) = -mga(1+\cos\theta)\cos\theta$

PE due to extension in string $= \frac{1}{2}\frac{4mg}{l}(r-a)^2$

$= \frac{1}{2}\frac{4mg}{a}(a\cos\theta)^2$

$= 2mga\cos^2\theta$

Thus, by conservation of energy,

KE + PE due to gravity + PE due to extension $= 0$

$\implies ma^2(1+\cos\theta)\dot{\theta}^2 - mga(1+\cos\theta)\cos\theta + 2mga\cos^2\theta = 0$

$\implies a(1+\cos\theta)\dot{\theta}^2 - g(\cos\theta + \cos^2\theta - 2\cos^2\theta) = 0$

$\implies a(1+\cos\theta)\dot{\theta}^2 - g\cos\theta(1-\cos\theta) = 0$

Hence, proved.

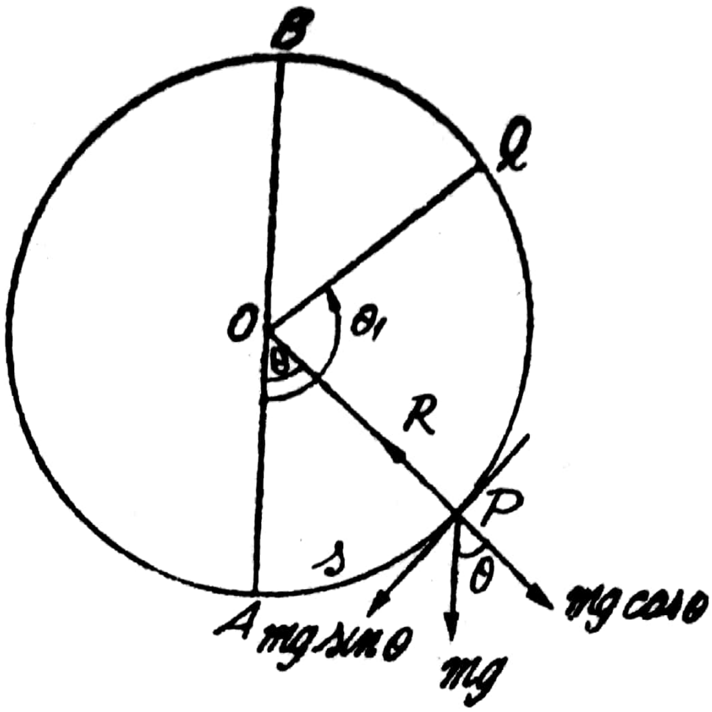
7.(c) एक कण त्रिज्या a के चिकनी ऊर्ध्वाधर वृत्ताकार तार पर चलने में स्वतंत्र है। प्रारम्भिक समय $t = 0$ पर इसे वृत्त के निम्नतम बिन्दु A से ऐसे वेग से, वृत्त के अनुदिश फेंका जाता है जो इसे मात्र उच्चतम बिन्दु B तक ले जाने में ही सक्षम है। समय T को ज्ञात कीजिए जिस पर कण व तार के बीच प्रतिक्रिया शून्य हो।

A particle is free to move on a smooth vertical circular wire of radius a. At time $t = 0$ it is projected along the circle from its lowest point A with velocity just sufficient to carry it to the highest point B. Find the time T at which the reaction between the particle and the wire is zero. 17

Ex. 15. *A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time*

$$\sqrt{(a/g)}.\log(\sqrt{5}+\sqrt{6}).$$

[Agra 1980, 86; Kanpur 79, 81; Meerut 86P, 87P, 90]



Sol. Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and
$$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$$

Also
$$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question $v=0$ at the highest point B, where $\theta=\pi$. $\therefore\ 0=2ag\cos\pi+A$ or $A=2ag$.

$$\therefore\ v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+2ag. \qquad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\ (v^2+ag\cos\theta)=\frac{m}{a}\ (2ag+3ag\cos\theta). \qquad \ldots(5)$$

If the reaction $R=0$ at the point Q where $\theta=\theta_1$, then from (5), we have

$$0=\frac{m}{a}\ (2ag+3ag\cos\theta_1)$$

or $\cos\theta_1=-2/3. \qquad \ldots(6)$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2=2ag\ (\cos\theta+1)=2ag.2\cos^2\tfrac{1}{2}\theta=2ag\cos^2\tfrac{1}{2}\theta.$$

$\therefore\ \dfrac{d\theta}{dt}=2\sqrt{(g/a)}\cos\tfrac{1}{2}\theta$, the positive sign being taken before the radical sign because θ increases as t increases

or $dt=\tfrac{1}{2}\sqrt{(a/g)}\sec\tfrac{1}{2}\theta\ d\theta.$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\ d\theta$$

or $t=\sqrt{(a/g)}\left[\log(\sec\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\theta)\right]_0^{\theta_1}$

or $t=\sqrt{(a/g)}\log(\sec\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_1). \qquad \ldots(7)$

From (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1-1=-\tfrac{2}{3}$$

or $2\cos^2\tfrac{1}{2}\theta_1=1-\tfrac{2}{3}=\tfrac{1}{3}$

or $\cos^2\tfrac{1}{2}\theta_1=\tfrac{1}{6}$ or $\sec^2\tfrac{1}{2}\theta_1=6.$

$\therefore\ \sec\tfrac{1}{2}\theta_1=\sqrt{6}$

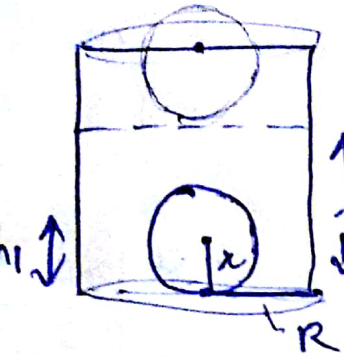
and $\tan\tfrac{1}{2}\theta_1=\sqrt{(\sec^2\tfrac{1}{2}\theta_1-1)}=\sqrt{(6-1)}=\sqrt{5}.$

Substituting in (7), the required time is given by

$$t=\sqrt{(a/g)}\log(\sqrt{6}+\sqrt{5}).$$

8.(a) W ग्राम भार व r त्रिज्या का एक गोला R से.मी. त्रिज्या की बेलनाकार बाल्टी की तली पर स्थित है। बाल्टी में h से.मी. $(h > 2r)$ की गहराई तक पानी भरा हुआ है। दर्शाइये कि गोले को पानी की सतह के ठीक ऊपर लाने में किया गया न्यूनतम कार्य $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ से.मी. ग्राम होना चाहिए। W' ग्राम गोले द्वारा विस्थापित पानी का भार है।

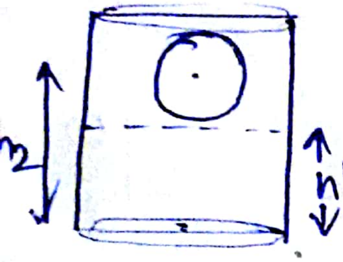
A spherical shot of W gm weight and radius r cm, lies at the bottom of cylindrical bucket of radius R cm. The bucket is filled with water up to a depth of h cm $(h > 2r)$. Show that the minimum amount of work done in lifting the shot just clear of the water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm gm. W' gm is the weight of water displaced by the shot. 16



$$\Delta PE_s = W(h_2 - h_1)$$

$$= \cancel{W((h+r) - r)}$$

$$= \cancel{Wh}$$



h' is the final height of water

Volume of water $= \pi R^2 h - \frac{4}{3}\pi r^3$

$$\therefore \pi R^2 h' = \pi R^2 h - \frac{4}{3}\pi r^3$$

$$\Rightarrow h' = h - \frac{4r^3}{3R^2} \quad \text{— (1)}$$

$$\Delta PE_w = -W'\left(\left(\frac{h+h'}{2}\right) - r\right)$$ [Final centre of mass and initial centre of mass]

$$= -W'\left[h - \frac{2r^3}{3R^2} - r\right]$$

$$\Delta PE_s = W[(h'+r) - r] = Wh' = W\left[h - \frac{4r^3}{3R^2}\right]$$

$\therefore$ Workdone $= \Delta PE_s + \Delta PE_w$

$$= W\left[h - \frac{4r^3}{3R^2}\right] - W'\left[h - \frac{2r^3}{3R^2} - r\right]$$

$$= W\left[h - \frac{4r^3}{3R^2}\right] + W'\left[r - h + \frac{2r^3}{3R^2}\right]$$