CSE
2019

1 $\text{Im } f(z) = (\text{Re } f(z))^2$, $z \in D$; $f(z)$ is analytic. T.S.T $f(z)$ is constant in D.

let $f(z) = u + iv$, where u, v are real functions

given $f(z)$ is analytic so it satisfies CR equations ie.

$u_x = v_y$; $u_y = -v_x$ —— (1)

Also, $f'(z) = u_x + i v_x$ —— (2)

given $v = u^2$ ie $v_x = 2uu_x$
$v_y = 2uu_y$] (3)

ie. $f'(z) = u_x + i 2uu_x$ (using (3))

Also, $u_x = v_y = 2uu_y = 2u(-v_x)$
$= 2u(-2uu_x)$
$= -4u^2 u_x$

ie $u_x(1 + 4u^2) = 0$

since u, v are real functions

so $1 + 4u^2 \neq 0 \Rightarrow \boxed{u_x = 0}$

$\boxed{u_x = 0} \Rightarrow v_y = u_x = 0 \Rightarrow v = g(x) + c'$

$\Downarrow$ $u = f(y) + c$

By (3) $v_x = 2uu_x = 0$

ie $u_x = 0$, $v_y = 0$, $v_x = 0$, $u_y = 0$

$\Rightarrow u = C_1$; $v = C_2$ $\Rightarrow$

$f(z) = C_1 + iC_2$ = constant.

Hence $f(z)$ is constant in D.

2. Show that isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

Moreover $\underset{z=z_0}{\text{Res}} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$ if $m \geq 1$

$\Leftarrow$ let $f(z) = \frac{\phi(z)}{(z-z_0)^m}$, $\phi(z)$ is analytic & non-zero.

since $\phi(z)$ is analytic at z_0 so by Taylor series expansion of $\phi(z)$ at $z=z_0$

$$\phi(z) = \phi(z_0) + (z-z_0)\frac{\phi'(z_0)}{1!} + (z-z_0)^2\frac{\phi''(z_0)}{2!}$$

$$+ \frac{(z-z_0)^{m-1}}{(m-1)!}\phi^{(m-1)}(z_0) + \sum_{n=m}^{\infty}\frac{(z-z_0)^n}{n!}\phi^{(n)}(z_0)$$

given $\phi(z_0) \neq 0$

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)}{2!(z-z_0)^{m-2}}$$

$$+ \frac{\phi^{(m-1)}(z_0)}{(m-1)!(z-z_0)} + \sum_{n=m}^{\infty}\frac{(z-z_0)^{n-m}\phi^{(n)}(z_0)}{n!} \quad (*)$$

$\Rightarrow f(z)$ has most negative power of $z-z_0$ is m i.e.

$f(z)$ has a pole of order m, only z_0.

$\Rightarrow$ conversely, let us assume $f(z)$ has a pole of order m. Then define

$$\phi(z) = \begin{cases} (z-z_0)^m f(z), & z \neq z_0 \\ b_m, & z = z_0 \end{cases}$$

since $f(z)$ has pole of order m at $z=z_0$

i.e. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}$

i.e. $\phi(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n}$

i.e. $\phi(z)$ has power series expansion so by result that $f(z)$ is analytic $\Leftrightarrow$ it has power series expansion $\phi(z)$ is analytic and $\phi(z_0) = b_m \neq 0$

Also, $\operatorname{Res}_{z=z_0} f(z)$ = coefficient of term $\frac{1}{z-z_0}$

which by eq (*) is

$$\operatorname{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$ Hence proved

3 Evaluate $\int_C Re(z^2)dz$ from 0 to 2+4i along $C: y=x^2$.

We know that $z = x+iy$

$Re\, z^2 = Re\,(x+iy)^2 = Re(x^2-y^2 \dots)$

$= x^2-y^2$

$dz = dx + i\,dy$

Along $C: y=x^2$, $dy = 2x\,dx$

As z varies from 0 to 2+4i, x varies from 0 to 2 so

$$I = \int_C Re(z^2)\,dz = \int_C (x^2-y^2)(dx+i\,dy)$$

$$= \int_C (x^2-x^4)(dx + i\,2x\,dx)$$

$$= \int_{x=0}^{2} (x^2 - x^4 + 2x^3 i - 2x^5 i)\,dx$$

$$= \left. \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^4}{2}i - \frac{x^6}{3}i \right|_0^2$$

$$= \left(\frac{8}{3} - \frac{32}{5}\right) + i\left(8 - \frac{64}{3}\right)$$

$$I = \boxed{-\frac{56}{15} - \frac{40}{3}i}$$

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4) obtain the first three terms of Laurent series expansion of $f(z) = \frac{1}{e^z - 1}$ about point $z=0$ valid in region $0<|z|<2\pi$.

we know that $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

so, $e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$

ie. $f(z) = \frac{1}{e^z - 1} = \frac{1}{z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \dots\right)}$

$= \frac{1}{z}\left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!}\right)\right]^{-1}$

$= \frac{1}{z}\left[1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \dots\right) + \frac{z^2}{(2!)^2} + o(z^3)\right]$

using binomial expansion of $(1+z)^{-1} = 1 - z + z^2 - z^3 + \dots$

$= \frac{1}{z}\left\{1 - \frac{z}{2} + \left(\frac{z^2}{4} - \frac{z^2}{6}\right) + o(z^3)\right\}$

$= \frac{1}{z} - \frac{1}{2} + \frac{z}{12} + o(z^2) \dots$

so, first 3 terms are $\boxed{\frac{1}{z}, -\frac{1}{2}, \frac{z}{12}}$