Mains Test Series - 2020

COMMON TEST [TEST-18, Batch-I] & [TEST-10, Batch-II]

Answer Key, (Paper-II) full Syllabus

1(a) Suppose $G = \{e, x, x^2, y, yx, yx^2\}$ is a non-abelian group with $|x| = 3$ and $|y| = 2$. show $xy = yx^2$.

Sol^n: observe first that $xy \neq yx$. Otherwise G is abelian

Since $|x| = 3$, we have $x \neq e$,

$x^2 \neq e$ and $y \neq e$.

So the only possibilities for xy are x, x^2, yx^2

If $xy = x$ then $y = e$.

If $xy = x^2$ then $y = x$ this implies $y^2 = x^2 = e$

which is impossible $xy = y$ implies $x = e$.

Hence the only remaining element is yx^2.

1(b). If F is a field of characteristic p, p a prime; then
$$(a+b)^p = a^p + b^p \quad \forall\ a, b \in F.$$

Sol'n: Since char $F = p$; $px = 0 \ \forall\ x \in F$ —— ①

Since F is a field, we can write

$$(a+b)^p = a^p + pa^{p-1}b + \frac{1}{2!}p(p-1)a^{p-2}.b^2 + \cdots + pab^{p-1} + b^p$$

$$= a^p + (pb)a^{p-1} + \frac{1}{2!}(p-1)a^{p-2}.b(pb) + \cdots + (pa)b^{p-1} + b^p$$

$$= a^p + b^p \text{ using ①.}$$

1(C) For $u_1 > 0$, the sequence u_n defined by

$$u_{n+1} = 1 + \frac{1}{u_n} \ \forall n, \text{ Converges to } \left(\frac{\sqrt{5}+1}{2}\right).$$

Sol'n: Here for any $u_1 > 0$, $2 \geq u_n \geq 3/2 \ \forall \ n \geq 3$,

as $2 \geq u_n \geq 3/2$

$\Rightarrow 2 > 5/3 \geq u_{n+1} = 1 + \frac{1}{u_n} \geq 3/2 \ \forall \ n \geq 3$. So that $\forall \ n \geq 3$,

$$|u_{n+1} - u_n| = \frac{|u_n - u_{n-1}|}{u_n u_{n-1}} = \cdots = \frac{|u_4 - u_3|}{u_n u_{n-1}^2 \cdots u_3} \leq \frac{|u_4 - u_3|}{(3/2)^{2(n-3)}}$$

$$\Rightarrow |u_{n+p} - u_n| \leq |u_{n+p} - u_{n+p-1}| + |u_{n+p-1} - u_{n+p-2}| + \cdots + |u_{n+1} - u_n|$$

$$\leq |u_4 - u_3| \left(\frac{2}{3}\right)^{2(n-3)} \left\{1 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{2p}\right\}$$

$$< 2|u_4 - u_3| \left(\frac{2}{3}\right)^{2(n-3)} \to 0 \text{ as } n \to \infty$$

$\Rightarrow u_n$ Converges, say to l, then $u_{n+1} = 1 + \frac{1}{u_n}$, as $n \to \infty$

$\Rightarrow l^2 - l - 1 = 0$, i.e. $l = \frac{\sqrt{5}+1}{2}$, as $l \geq 3/2$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$

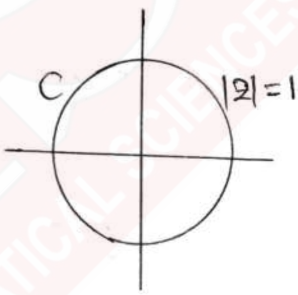
1(d) Prove that if $be^{a+1} < 1$, where a and b are positive real number, then the function $z^n e^{-a} - be^z$ has 'n' zero's in the unit circle.

Sol'n: Using Rouche's theorem that if $f(z)$ & $g(z)$ are analytic function in 'C', then $f(z) + g(z)$ has same number of zeroes as $f(z)$ has if $|g(z)| < |f(z)|$ on C.

Consider, $f(z) = z^n e^{-a}$; $g(z) = -be^z$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be^z}{z^n e^{-a}}\right| \quad \text{using } |z| = 1$$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be}{1^n . e^{-a}}\right| = \left|\frac{be}{e^{-a}}\right|$$



$$\left|\frac{g(z)}{f(z)}\right| = |be^{a+1}| < 1$$

$$\Rightarrow \boxed{|g(z)| < |f(z)| \text{ on } z = 1}$$

Then, function $z^n e^{-a} - be^z$ will have same number of zeros as $f(z) = z^n e^{-a}$ in $|z| = 1$.

Clearly, $f(z)$ has 'n' zeros in unit circle.

1(e) Find all the basic feasible solutions of the equations.

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Show that all the basic solutions are degenerate.

Sol'n: Let $Ax = b$ where.

$$A = \begin{pmatrix} 2 & 6 & 2 & 1 \\ 6 & 4 & 4 & 6 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Rank $(A) = 2$, Therefore no. of basic solutions $= {}^4C_2 = 6$

$\therefore$ Possible submatrices are

$$B_1 = \begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix}$$

$|B_1| = -28$, $|B_2| = 16$, $|B_3| = 8$, $|B_4| = -4$, $|B_5| = 6$, $|B_6| = 32$

$\therefore$ All the values are non-zero. All the submatrices form the basic solutions.

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{-1}{28} \begin{bmatrix} 4 & -6 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 4 & -2 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 6 & -1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 6 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{-1}{4} \begin{bmatrix} 4 & -2 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8/3 \\ -7/3 \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 6 & -1 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

$\therefore$ $(x_1, x_2, x_3, x_4) = (0, \frac{1}{2}, 0, 0)$ is the only basic feasible solution.

Since other basic variable is zero

$\therefore$ It is degenerate.

$\therefore$ All basic feasible solutions in the given problem are degenerate.

2(a) Suppose G is the group defined by the following Cayley table

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

(i) Find the centralizer of 5 in G.

(ii) Find the centralizer of 3 in G.

(iii) Find centre of G.

(iv) Find the order of each element of G.

(v) Is the above group abelian?

(vi) Find a proper normal subgroup of G and verify why it is normal.

Sol^n: (i) $C_G(5) = \{1,2,3,4,5,6,7,8\}$. This shows that $5 \in Z(G)$.

(ii) $C_G(3) = \{1,3,5,7\}$

(iii) Since $Z(G) \leq C_G(3)$ we need only to check that 7 is in the centre or not. But 7 does not commute with 2. Then we have $Z(G) = \{1,5\}$.

(iv) $o(1) = 1$ $\quad o(2) = 2$ $\quad o(3) = 4$ $\quad o(4) = 2$ $\quad o(5) = 2$

$o(6) = 1$ $\quad o(7) = 4$ $\quad o(8) = 2$.

(v) NO, since $Z(G) = \{1,5\}$

(vi) $C_G(3) = \{1,3,5,7\}$ is a normal subgroup of G,

Since $|G| = 8$, and $|C_G(3)| = 4$. Hence $|G : C_G(3)| = 2$.

Every group of index 2 in G is a normal subgroup.

2(b) Show that $\mathbb{Q}^+$ (the set of positive rational numbers) under multiplication is not isomorphic to $\mathbb{Q}$ under addition.

Sol'n: Let $f:(\mathbb{Q},+)\to(\mathbb{Q}^+,\cdot)$

$f(a+b)=f(a)\cdot f(b)$. Then there exists $y\in\mathbb{Q}$ such that $f(y)=2$.

Then we have

$$f(y)=f\left(\frac{y}{2}+\frac{y}{2}\right)=f\left(\frac{y}{2}\right)f\left(\frac{y}{2}\right)$$

$$=a^2$$

$$=2 \text{ for some } a\in\mathbb{Q}$$

But this is impossible in $\mathbb{Q}$.

2(c)(i) Show that $x^{1/5}$ is uniformly continuous on $[0,\infty)$.

Hint: $$|x^{1/5}-y^{1/5}|^5\le|x^{1/5}-y^{1/5}|\,|x^{4/5}+x^{3/5}y^{1/5}+\cdots+y^{4/5}$$
$$-5x^{1/5}y^{1/5}(x^{1/5}-y^{1/5})^2-5x^{2/5}y^{2/5}|\le|x-y|.$$

2(c)(ii) If $x>0$, then $x-\frac{x^2}{2(1+x)} > \log(1+x) > x-\frac{x^2}{2}$.

Sol'n: Let $f(x) = x-\frac{x^2}{2} - \log(1+x)$

$$\therefore f'(x) = 1-x-\frac{1}{1+x} = \frac{1-x^2-1}{1+x}$$

$$= \frac{-x^2}{1+x} < 0 \text{ for } x>0,$$

$\Rightarrow f(x)$ is monotonic decreasing for $x>0$

$\Rightarrow f(x) < f(0)$

But $f(0) = 0-0-\log 1 = 0$

$\therefore f(x) < 0 \Rightarrow x-\frac{x^2}{2} - \log(1+x) < 0$

$\Rightarrow x-\frac{x^2}{2} < \log(1+x)$ —— ①

Now let $g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$

$$\therefore g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\cdot\frac{(1+x).2x - x^2}{(1+x)^2}$$

$$= \frac{1-1-x}{1+x} + \frac{1}{2}\cdot\frac{2x+x^2}{(1+x)^2}$$

$$= -\frac{x}{1+x} + \frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{-2x(1+x)+2x+x^2}{2(1+x)^2}$$

$$= -\frac{x^2}{2(1+x)^2} < 0 \text{ for } x>0$$

$\Rightarrow g(x)$ is monotonic decreasing for $x>0$

$\Rightarrow g(x) < g(0)$

But $g(0) = 0$ $\therefore g(x) < 0$

$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$

$\therefore \log(1+x) < x - \frac{x^2}{2(1+x)}$ —— ②

Combining ① & ②,

$x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$

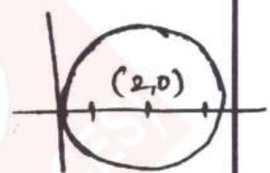
2(d)(I) Use Cauchy's theorem and/or Cauchy's integral formula to evaluate the following integrals.

(i) $\int_{|z-2|=2} \frac{\log(z+1)}{z-3}\,dz$ (ii) $\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$

Sol'n: (i) Given that $\int_{|z-2|} \frac{\log(z+1)}{z-3}\,dz$

Comparing the given integral with $\int_C \frac{f(z)}{z-z_0}\,dz$

where $C: |z-2|=2$
centre $(2,0)$ and $r=2$



Since $f(z)=\log(z+1)$ is analytic in $|z-2|=2$ and $z_0=3$ is a point inside $|z-2|=2$

$\therefore$ we can apply Cauchy's integral formula

$$\int_{|z-2|=2} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \int_{|z-2|=2} \frac{\log(z+1)}{z-3} = 2\pi i \log(3+1)$$

$$= 2\pi i \log 4$$

$$= 4\pi i \log 2.$$

(ii) Given $\int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz$

$$= \frac{1}{5}\int_{|z|=5} \left[\frac{9}{z-4} - \frac{4}{z+1}\right] dz$$

$$= \frac{9}{5}\int_{|z|=5} \frac{1}{z-4}\,dz - \frac{4}{5}\int_{|z|=5} \frac{1}{z+1}\,dz$$

Since $f(z)=1$ is analytic in $|z|=5$ and $z_0=4$, $z_0=-1$ are a points inside $|z|=5$

$\therefore$ By Cauchy's integral formula

$$\int_{|z|=5} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\therefore \int_{|z|=5} \frac{z+5}{z^2-3z-4}\,dz = \frac{9}{5}(2\pi i)\,1 - \frac{4}{5}(2\pi i)(1)$$
$$= 2\pi i$$

2(d)(ii) find the Laurent series expansion of the following.

$$f(z) = \frac{1}{(z^2-1)(z^2-9)}.$$

How many such expansions are there? In which region each of them valid?

Sol^n: Given $f(z) = \frac{1}{(z^2-1)(z^2-9)}$

$$= \frac{1}{8}\left(\frac{1}{z^2-9} - \frac{1}{z^2-1}\right)$$

If $0<1<|z|<3$ (so that $\left|\frac{z}{3}\right|<1$; $\left|\frac{1}{z}\right|<1$)

[i.e., here if $\left|\frac{z^2}{9}\right|<1 \Rightarrow |z|^2<9 \Rightarrow |z|<3$
and $\left|\frac{1}{z^2}\right|<1 \Rightarrow 1<|z|^2 \Rightarrow \boxed{1<|z|}$]

$$f(z) = \frac{1}{8}\left[\frac{1}{-9\left(1-\frac{z^2}{9}\right)} - \frac{1}{z^2\left(1-\frac{1}{z^2}\right)}\right]$$

$$= \frac{1}{8}\left[-\frac{1}{9}\left(1-\frac{z^2}{9}\right)^{-1} - \frac{1}{z^2}\left(1-\frac{1}{z^2}\right)^{-1}\right]$$

$$= -\frac{1}{72}\left[1+\frac{z^2}{9}+\left(\frac{z^2}{9}\right)^2+\cdots\cdots\left(\frac{z^2}{9}\right)^n+\cdots\right]$$

$$-\frac{1}{8z^2}\left(1+\frac{1}{z^2}+\left(\frac{1}{z^2}\right)^2+\cdots+\left(\frac{1}{z^2}\right)^n+\cdots\right)$$

$$= -\frac{1}{72}\sum_{n=0}^{\infty}\left(\frac{z^2}{9}\right)^n - \frac{1}{8z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z^2}\right)^n$$

$$= -\frac{1}{72}\sum_{n=0}^{\infty}\frac{z^{2n}}{9} - \frac{1}{8z^2}\sum_{n=0}^{\infty}\frac{1}{z^{2n+2}}$$

The above series being in positive and negative powers of z represents Laurent's expansion for $f(z)$ in the region $1<|z|<3$.

3(a) If possible, find g.c.d and l.c.m of $10+11i$ and $8+i$ in $Z[i]$.

Sol'n: we have

$$\frac{10+11i}{8+i} = \frac{(10+11i)(8-i)}{(8+i)(8-i)} = \frac{91+78i}{65} = \frac{7}{5}+\frac{6}{5}i.$$

$$\Rightarrow \frac{10+11i}{8+i} = (1+i)+\left(\frac{2}{5}+\frac{1}{5}i\right)$$

$$\Rightarrow 10+11i = (1+i)(8+i)+\left(\frac{2}{5}+\frac{1}{5}i\right)(8+i)$$

$$\Rightarrow 10+11i = (1+i)(8+i)+(3+2i), \text{ where } d(3+2i)<d(8+i)$$

$$[\because d(3+2i)=13, d(8+i)=65]$$

Now we consider

$$\frac{8+i}{3+2i} = \frac{(8+i)(3-2i)}{(3+2i)(3-2i)} = \frac{26-13i}{13} = 2-i$$

$$\Rightarrow (8+i) = (2-i)(3+2i)$$

Hence $3+2i$ is the g.c.d of $10+11i$ and $8+i$

If (a,b) and $[a,b]$, respectively, denote g.c.d and l.c.m of a and b in $Z[i]$, then

$$[a,b] = \frac{ab}{(a,b)}$$ [by theorem If R is a P.I.D & a,b are two non-zero elements of R, then $[a,b](a,b)=abu$, for some unit $u \in R$], [take $u=1$].

Hence l.c.m of $a=10+11i$, $b=8+i$ is

$$= \frac{(10+11i)(8+i)}{3+2i} = \frac{(69+98i)(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{403+156i}{13}$$

$$= 31+12i.$$

3(b) (i) Every bounded infinite set S of real number has at least one limit point.

Sol^n: Let S be an infinite bounded subset of $\mathbb{R}$.

(i) S is bounded $\Rightarrow \exists$ real numbers k and K such that

$$k \leq s \leq K \quad \forall\, s \in S.$$

(ii) Let a set T be defined as follows.

$$T = \{t : t > \text{finitely many elements of } S\}$$

(iii) To Prove that $T \neq \phi$

$k \leq s\ \forall\, s \in S \Rightarrow k$ is greater than no element of S

$$\Rightarrow k \in T \Rightarrow T \neq \phi$$

(iv) To Prove that T is bounded above

For any $\epsilon > 0$, $K + \epsilon > K \geq \ \forall\, s \in S$

$$K + \epsilon \notin T, \quad K \notin T$$

$\forall\, t \in T$, $t < K \Rightarrow T$ is bounded above

T is non-empty bounded above subset of $\mathbb{R}$

T has the l.u.b say u.

(v) To Prove that u is a limit point of S.

Let $(u-\epsilon, u+\epsilon)$ be any nbd of u

u is l.u.b of $T \Rightarrow \exists$ some $t \in T$ such that $t > u-\epsilon$, $\epsilon > 0$.

Now $t \in T \Rightarrow t >$ finitely many elements of S

$\Rightarrow u-\epsilon >$ finitely many elements of S lie to the left of $u-\epsilon$.

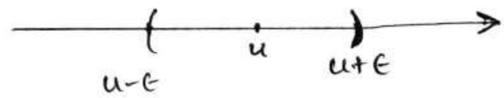
$\Rightarrow$ infinitely many elements of S lie to the right of $u-\epsilon$

Also $u =$ l.u.b of $T \Rightarrow u+\epsilon \notin T$

$\Rightarrow u+\epsilon >$ infinitely many elements of S

$\Rightarrow$ infinitely many elements of S lie to the left of $u+\epsilon$.

(2)



Combining ① and ②, $(u-\epsilon, u+\epsilon)$ has infinitely many elements of S. But $(u-\epsilon, u+\epsilon)$ is any nbd of u

$\therefore$ Every nbd of u has infinitely many elements of S.

Hence u is a limit point of S.

3(b)(ii) If $f(x) = \sin(1/x)$, $\forall$ irrotational $x \in [0,1]$,
$= 0$, $\forall$ rational $x \in [0,1]$

then $f(x)$ is not Riemann integrable on $[0,1]$.

Sol'n: Since every subinterval of $[0,1]$ consists of both irrational and rational numbers x, therefore for $x \in [4/3\pi, 2/\pi] \subset [0,1]$ as $\sin(1/x)$ monotonically increases from $\frac{1}{\sqrt{2}}$ to 1 for any partition P of $[0,1]$, we have

$$U(P,f) > \left(\frac{2}{\pi} - \frac{4}{3\pi}\right)\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3\pi}$$

and $L(P,f) = 0$.

Hence, $U(P,f) - L(P,f) > \frac{\sqrt{2}}{3\pi}$, for any Partition P of $[0,1]$ and so, $U(P,f) - L(P,f)$ cannot be made less than arbitrary $\varepsilon > 0$.

Thus, $f(x)$ is not Riemann integrable on $[0,1]$.

3(c) Find the optimum solution of the following Transportation table.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	1	2	1	4	5	2	30
O_2	3	3	2	1	4	3	50
O_3	4	2	5	9	6	2	75
O_4	3	1	7	3	4	6	20
b_j	20	40	30	10	50	25	

Solution:-

From the given table

Total Demand $= \sum b_j = 20+40+30+10+50+25 = 175$

Total Availability $= \sum a_i = 30+50+75+20 = 175$

Hence, $\boxed{\sum a_i = \sum b_j = 175}$

$\therefore$ Given transportation problem is balanced. Hence, for initial solution using Low cost Entry method:

	D_1	D_2	D_3	D_4	D_5	D_6	
O_1	20 (1)	X (2)	10 (1)	X (4)	X (5)	X (2)	30
O_2	X (3)	X (3)	20 (2)	10 (1)	20 (4)	X (3)	~~50~~ ~~40~~ ~~20~~ 0
O_3	X (4)	40 (2)	X (5)	X (9)	10 (6)	25 (2)	~~75~~ ~~35~~ 10
O_4	X (3)	X (1)	X (7)	X (3)	20 (4)	X (6)	20
	~~20~~ 0	~~40~~ 0	~~30~~ ~~20~~ 0	~~10~~ 0	~~50~~ ~~30~~ ~~20~~ 0	~~25~~ 0	

Here ; the number of positive allocations = 9
and $m+n-1 = 6+4-1 = 9$

$\therefore$ Initial feasible solution $= 20\times1 + 10\times1 + 20\times2 + 10\times1 + 20\times4 + 40\times2 + 10\times6 + 25\times2 + 20\times4$

$\boxed{\text{IBFS} = 430}$

Let's check the optimality

Using MODI method:

For Basic cells ; $\Delta_{ij} = u_i + v_j - c_{ij} = 0$
$= u_i + v_j = c_{ij}$

$u_1 + v_1 = 1$
$u_1 + v_3 = 1$
$u_2 + v_3 = 2$
$u_2 + v_4 = 1$
$u_2 + v_5 = 4$
$u_3 + v_2 = 2$
$u_3 + v_5 = 6$
$u_3 + v_6 = 2$
$u_4 + v_5 = 4$

let $u_2 = 0$
$u_1 = -1$
$u_3 = 2$
$u_4 = 0$

$v_1 = 2$
$v_2 = 0$
$v_3 = 2$
$v_4 = 1$
$v_5 = 4$
$v_6 = 0$

Non-check for non-basic cells (Δ_{ij})

$\Delta_{12} = u_1 + v_2 - 2 = -1 + 0 - 2 = -3$
$\Delta_{14} = u_1 + v_4 - 4 = -1 + 1 - 4 = -4$
$\Delta_{15} = u_1 + v_5 - 5 = -1 + 4 - 5 = -2$
$\Delta_{16} = u_1 + v_6 - 2 = -1 + 0 - 2 = -3$
$\Delta_{21} = u_2 + v_1 - 3 = 0 + 2 - 3 = -1$
$\Delta_{22} = u_2 + v_2 - 3 = 0 + 0 - 3 = -3$
$\Delta_{26} = u_2 + v_6 - 3 = 0 + 0 - 3 = -3$
$\Delta_{31} = u_3 + v_1 - 4 = 2 + 2 - 4 = 0$
$\Delta_{33} = u_3 + v_3 - 5 = 2 + 2 - 5 = -1$
$\Delta_{34} = u_3 + v_4 - 9 = 2 + 1 - 9 = -6$
$\Delta_{41} = u_4 + v_1 - 3 = 0 + 2 - 3 = -1$
$\Delta_{42} = u_4 + v_2 - 1 = 0 + 0 - 1 = -1$
$\Delta_{43} = u_4 + v_3 - 7 = 0 + 2 - 7 = -5$

$\Delta_{44} = u_4 + v_4 - 3 = 0 + 1 - 3 = -2$

$\Delta_{46} = u_4 + v_6 - 6 = 0 + 0 - 6 = -6$

As we observe, there all $\Delta_{ij} \leq 0$ in non-basic cells; Hence optimality obtained.

$\therefore$ optimal Transportation cost at

$x_{11} = 20$, $x_{13} = 10$

$x_{23} = 20$, $x_{24} = 10$, $x_{25} = 20$

$x_{32} = 40$, $x_{35} = 10$, $x_{36} = 25$

$x_{45} = 20$

$\therefore$ Minimum cost of Transportation (i.e optimal value)

$= 1\times20 + 1\times10 + 2\times20 + 1\times10 + 4\times20 + 2\times40$
$\quad + 6\times10 + 2\times25 + 4\times20$

$\Rightarrow 430.$

$\boxed{\therefore \text{ Min. Cost of Transportation} = 430}$

which was our initial solution.

4(a) Let R be the ring of all the real valued continuous functions the closed unit interval. show that $M = \{f \in R : f(\frac{1}{3}) = 0\}$ maximal ideal of R.

Sol'n: Let R denote the set of all real numbers.

The given ring is $R = \{f \mid f : [0,1] \to R \text{ is continuous on } [0,1]\}$

Notice that R is a ring w.r.t the compositions:

$$\left.\begin{array}{l}(f+g)(x) = f(x) + g(x) \\ (fg)(x) = f(x) \cdot g(x)\end{array}\right\} \forall\, x \in [0,1] \text{ and } f, g \in R.$$

we proceed to show that M is an ideal of R.

Let $f, g \in M$. Then $f(\frac{1}{3}) = 0 = g(\frac{1}{3})$.

we have $(f-g)(\frac{1}{3}) = f(\frac{1}{3}) - g(\frac{1}{3}) = 0$ and so $f-g \in M$.

Let $f \in M$ and $h \in R$. Then $f(\frac{1}{3}) = 0$ and

$$(fh)(\tfrac{1}{3}) = f(\tfrac{1}{3})\, h(\tfrac{1}{3}) = 0 \cdot h(\tfrac{1}{3}) = 0 \Rightarrow fh \in M.$$

Similarly, $hf \in M$ and so M is an ideal of R. Finally, we show that M is a maximal ideal of R. Let U be any ideal of R such that

$$M \subset U \subset R \text{ and } M \neq U.$$

we need to show that $U = R$.

Since $M \subset U$ and $M \neq U$, there exists a function $g \in U$ such that $g \notin M$, i.e, $g(\frac{1}{3}) \neq 0$.

i.e, $\alpha \neq 0$, where $\alpha = g(\frac{1}{3})$ —— (1)

[Notice that $g(\frac{1}{3}) = 0 \Rightarrow g \in M$, a contradiction]

we define a function $h : [0,1] \to R$ as

$h(x) = g(x) - \alpha, \forall x \in [0,1]$ — ②

$\Rightarrow h(1/3) = g(1/3) - \alpha = 0$, using ①

$\Rightarrow h \in M \Rightarrow h \in U$, since $M \subset U$

Since U is an ideal of R, therefore

$g \in U$ and $h \in U \Rightarrow g - h \in U \Rightarrow \alpha \in U$, by ②

Since $\alpha \neq 0$, $\alpha^{-1} \in R$ exists. The constant function

$\alpha^{-1} : [0,1] \to R$ defined as $\alpha^{-1}(x) = \alpha^{-1} \; \forall x \in [0,1]$

is a continuous function on $[0,1]$ and as such

$\alpha^{-1} \in R$.

Since U is an ideal of R, so

$\alpha \in U$ and $\alpha^{-1} \in R \Rightarrow \alpha\alpha^{-1} \in U \Rightarrow 1 \in U$

$\Rightarrow 1.f \in U \; \forall f \in R \Rightarrow f \in U \; \forall f \in R \Rightarrow R = U$.

Hence M is a maximal ideal of R.

4(b). If a function f is continuous on $[0,1]$, show that

$$\lim_{x\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \frac{\pi}{2}f(0).$$

Sol'n: Let us put $\displaystyle\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}dx$

By generalised first mean value theorem,

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx = f(\xi)\int_0^{1/\sqrt{n}} \frac{n}{1+n^2x^2}dx, \text{ where } 0\le\xi\le\frac{1}{\sqrt{n}}$$

$$= f(\xi)\tan^{-1}\sqrt{n} \to \frac{\pi}{2}f(0) \text{ as } n\to\infty$$

Again, since f is continuous on $[0,1]$, it is bounded and therefore there exists K such that

$|f(x)|\le K$, $\forall x\in[0,1]$.

$$\left|\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}dx\right| = K\left|\int_{1/\sqrt{n}}^1 \frac{n}{1+n^2x^2}dx\right|$$

$$= K\left|\tan^{-1}n - \tan^{-1}\sqrt{n}\right|$$

$$\to 0 \text{ as } n\to\infty$$

Hence, the result.

4(c) By the method of contour integration, prove that $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\{a-\sqrt{a^2-b^2}\}$, if $a>b>0$

Sol^n:- Write $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$$I = \int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a+b\cos\theta} = \frac{1}{2b}\int_C \frac{(z^2-1)^2\, dz}{z^2\left(z^2+\frac{2a}{b}z+1\right)}$$ where C denotes the unit cir

$$= \frac{i}{2b}\int_C \frac{(z^2-1)^2\, dz}{z^2(z-\alpha)(z-\beta)}.$$

where $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$; $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

obviously $|\beta|>1$, so β lies outside C, and $|\alpha||\beta|=1$.

$\therefore |\alpha|<1$, so α lies inside C.

So inside the contour C there is a simple pole $z=\alpha$ and a pole $z=0$ of order two.

Residue at the simple pole $z=\alpha$ is

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}(z-\alpha)\cdot\frac{i}{2b}\frac{(z^2-1)^2}{z^2(z-\alpha)(z-\beta)}$$

$$= \frac{i}{2b}\frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)} = \frac{i}{2b}\frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{(\alpha-\beta)}$$

$$= \frac{i}{2b}\frac{(\alpha-\beta)^2}{\alpha-\beta} \quad \left(\because \alpha\beta=1 \Rightarrow \frac{1}{\alpha}=\beta\right)$$

$$= \frac{i}{2b}(\alpha-\beta)$$

$$= \frac{i}{2b}\frac{2\sqrt{a^2-b^2}}{b} = \frac{i\sqrt{a^2-b^2}}{b^2}.$$

And residue at the double pole $z=0$ is the coefficient of $\frac{1}{z}$ in $\frac{i}{2b}\frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)}$, where z is small.

Now, $\frac{i}{2b}\frac{(z^2-1)^2}{z^2\left(z^2+\frac{2a}{b}z+1\right)} = \frac{i}{2b}\left(1-\frac{1}{z^2}\right)^2\left[1+\frac{2a}{b}\frac{1}{z}+\frac{1}{z^2}\right]^{-1}$

the coefficient of $\frac{1}{z}$ is

$$\frac{i}{2b}\left[-\frac{2a}{b}\right] = -\frac{ia}{b^2}$$

Hence $I = 2\pi i \times$(sum of the residues at the poles within C)

$$= 2\pi i\left\{\frac{i\sqrt{a^2-b^2}}{b^2} - \frac{ia}{b^2}\right\}$$

$$= \frac{2\pi}{b^2}\left\{a-\sqrt{a^2-b^2}\right\}.$$

5(a)(i) Frame the partial differential equation by eliminating the arbitrary constants a and b from $\log(az-1) = x+ay+b$.

Sol^n Given $\log(az-1) = x+ay+b$. ——— (1)

Differentiating (1) partially w.r.t. 'x'

$$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 \quad \text{——— (2)}$$

Differentiating (1) partially w.r.t 'y',

$$\frac{a}{az-1}\frac{\partial z}{\partial y} = a \quad \text{——— (3)}$$

From (3), $az-1 = \frac{\partial z}{\partial y}$, so that

$$a = \frac{1+(\partial z/\partial y)}{z} \quad \text{——— (4)}$$

Putting the above values of $(az-1)$ and a in (2), we have

$$\frac{1+(\partial z/\partial y)}{z(\partial z/\partial y)}\cdot\frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

5(a)(ii) Find the complete integral of $p^2x+q^2y=z$.

Sol'n: The given equation can be re-written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2+\frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2=1 \text{ (or) } \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2+\left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2=1 \quad \text{—①}$$

Put $\left(\frac{1}{\sqrt{x}}\right)dx = dX$, $\left(\frac{1}{\sqrt{y}}\right)dy = dY$ and $\left(\frac{1}{\sqrt{z}}\right)dz = dZ$ —②

So that $2\sqrt{x}=X$, $2\sqrt{y}=Y$ and $2\sqrt{z}=Z$ —③

Using ②, ① becomes $\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=1$

$$\Rightarrow P^2+Q^2=1 \quad \text{—④}$$

where $P=\frac{\partial Z}{\partial X}$ and $Q=\frac{\partial Z}{\partial Y}$. ④ is of the form $f(P,Q)=0$

$\therefore$ solution of ④ is $Z=aX+bY+c$, —⑤

where $a^2+b^2=1 \Rightarrow b=\sqrt{1-a^2}$, Putting a for P and b for Q in ④

$\therefore$ from ⑤, the required complete integral is

$$Z=aX+Y\sqrt{1-a^2}+c$$

$$\Rightarrow 2\sqrt{z}=2a\sqrt{x}+2\sqrt{y}\sqrt{1-a^2}+c \text{ by ③}$$

where a & c are two arbitrary constants.

5(b) Solve

$$\left(\frac{\partial^2 z}{\partial x^2}\right)-\left(\partial^2 z/\partial y^2\right)+\left(\frac{\partial z}{\partial x}\right)+3\left(\frac{\partial z}{\partial y}\right)-2z=e^{x-y}-x^2y.$$

Solution:-

Given equation:-

$$\left(\frac{\partial^2 z}{\partial x^2}\right)-\left(\frac{\partial^2 z}{\partial y^2}\right)+\left(\frac{\partial z}{\partial x}\right)+3\frac{\partial z}{\partial y}-2z=e^{x-y}-x^2y$$

which can be written as-

$$(D^2-D'^2+D+3D'-2)z=e^{x-y}-x^2y$$

$$=\{(D-D')(D+D')+2(D+D')-(D-D'+2)\}z=e^{x-y}-x^2y$$

$$=\{(D+D')(D-D'+2)-(D-D'+2)\}z=e^{x-y}-x^2y.$$

$$\Rightarrow [(D-D'+2)(D+D'-1)]z=e^{x-y}-x^2y$$

$$\boxed{\therefore\ C.F=e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)} \quad \text{——— (1)}$$

P.I. corresponding to e^{x-y}

$$=\frac{1}{(D-D'+2)(D+D'-1)}e^{x-y}.$$

[from e^{x-y} $\rightarrow$, $D=1$, $D'=-1$] – coefficient of x and y.

$$=\frac{1}{(1+1+2)(\cancel{1}-\cancel{1}-1)}e^{x-y}=\frac{-1}{4}e^{x-y}. \quad \text{——— (2)}$$

Now, P.I corresponding to $(-x^2y)$.

$$=\frac{1}{(D-D'+2)(D+D'-1)}(-x^2y)$$

$$= \frac{1}{2}\left\{1+\frac{D-D'}{2}\right\}^{-1}\{1-(D+D')\}^{-1}\cdot x^2y$$

$$= \frac{1}{2}\left\{1-\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2-\cdots\right\}\left\{1+(D+D')+(D+D')^2+\cdots\right\}x^2y$$

$$= \frac{1}{2}\left[1-\frac{D}{2}+\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}+\frac{3D^2D'}{8}+\cdots\right]\times$$

$$\left[1+D+D'+D^2+2DD'+3D^2D'+\cdots\right]x^2y.$$

$$= \frac{1}{2}\left[1+\frac{1}{2}D+\frac{3}{2}D'+\frac{3}{4}D^2+\frac{3}{2}DD'+\cdots\right]x^2y.$$

$$= \frac{1}{2}\left[x^2y+xy+\frac{3}{2}x^2+\frac{3}{2}y+3x+\frac{21}{4}+\underbrace{\cdots}_{\to 0}\right] \quad \text{——} (3)$$

from (1), (2) and (3)

General solution is $z = C.F + P.I$

$$\boxed{z = e^{2x}\phi_1(y+x)+e^{x}\phi_2(y-x)-\frac{1}{4}e^{x-y}+\frac{1}{2}x^2y+\frac{xy}{2}+\frac{3}{4}x^2+\frac{3}{4}y+3x+\frac{21}{8}+\cdots}$$

5(C) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
velocity in km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car.

Sol^n: we know velocity $v = \frac{ds}{dt}$ —① where s = distance, t = time.

① ⇒ $ds = v\,dt$

∴ so distance covered by car in 12 min. is

$$S = \int_0^S ds = \int_0^{12} v\,dt \quad \text{—②}$$

Here given:

Time	$t_0=0$	$t_1=2$	$t_2=4$	$t_3=6$	$t_4=8$	$t_5=10$	$t_6=12$
velocity	$v_0=0$	$v_1=22$	$v_2=30$	$v_3=27$	$v_4=18$	$v_5=7$	$v_6=0$

② ⇒ $S = \int_0^{12} v\,dt$

Using Simpson's $\frac{1}{3}$rd rule

we get $S = \frac{h}{3}\left[(v_0+v_6)+4(v_1+v_3+v_5)+2(v_2+v_4)\right]$

$= \frac{h}{3}\left[(0+0)+4(22+27+7)+2(30+18)\right]$ —③

Since $h = 2\text{min} = \frac{2}{60} = \frac{1}{30}$ hour.

③ ⇒ $S = \frac{1/30}{3}\left[4(56)+2(48)\right] = 3.556$ km.

Hence distance covered by car is 3.556 km.

5(d) Convert the following to the base indicated against each:
(i) $(266.375)_{10}$ to base 8 (ii) $(341.24)_5$ to base 10
(iii) $(43.3125)_{10}$ to base 2
(iv) $(1011110111111)_2$ to hexadecimal.

Sol'n: (i) $(266.375)_{10}$

Integer part

$$\begin{array}{r|l l}
8 & 266 & \\
8 & 33 & \text{remainder } 2 \\
8 & 4 & \text{remainder } 1 \\
 & 0 & \text{remainder } 4
\end{array}$$

$\therefore (266)_{10} = (412)_8$

Fractional part:

Fraction	Fraction×8	Remainder new fraction	Integer
0.375	3.000	0.000	3
0.000	0.000	0.000	0

$$(0.375)_{10} = (0.3)_8$$

$$\therefore (266.375)_{10} = (412.3)_8$$

(ii) $(341.24)_5 = 3\times5^2 + 4\times5^1 + 1\times5^0 + 2\times5^{-1} + 5\times5^{-2}$

$= 3\times25 + 20 + 1 + 2/5 + 5/25$

$= 75 + 20 + 1 + 0.4 + 0.2$

$= 96.6$

(iii) $(43.3125)_{10}$,

Taking the integer part first.

$$\begin{array}{r|l l}
2 & 43 & \\
2 & 21 & \text{remainder } 1 \\
2 & 10 & \text{remainder } 1 \\
2 & 5 & \text{remainder } 0 \\
2 & 2 & \text{remainder } 1 \\
2 & 1 & \text{remainder } 0 \\
 & 0 & \text{remainder } 1
\end{array}$$

$$(43)_{10} = (101011)_2$$

Taking the fractional part :

Fraction	Fraction ×2	Remainder new fraction	Integer
0.3125	0.625	0.625	0
0.625	1.25	0.25	1
0.25	0.5	0.5	0
0.5	1.0	0.0	1

$$\therefore (0.3125)_{10} = 0101$$

Adding the binary equivalent of 43 and 0.3125

$$\begin{array}{r} 101011.0000 \\ 0.0101 \\ \hline 101011.0101 \end{array}$$

$$\therefore (43.3125)_{10} = (101011.0101)_2$$

(iv) $(101111011111)_2$ to hexadecimal.

Sol'n : Binary number can be converted into equivalent hexa decimal number by making groups of four bits starting from LSB (least significant digit) and moving towards MSB (Most significant digit) for integer part of the number and then replacing each group of four bits by its hexa decimal representation.

$$(101111011111)_2 = (\underbrace{1011}\ \underbrace{1101}\ \underbrace{1111})_2$$

$$= (BDF)_{16}$$

4(d) Solve the following LPP by Simplex Method.

Maximize: $z = 3x_1 - x_2$

subject to the constraints: $2x_1 + x_2 \geq 2$,
$x_1 + 3x_2 \leq 3$
$x_2 \leq 4$ and $x_1, x_2 \geq 0$.

Sol'n: By introducing the surplus variable s_1 and slack variables s_2 and s_3, artificial variable a_1, the problem becomes

Maximize $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1$

subject to the constraints

$$2x_1 + x_2 - s_1 + a_1 = 2$$
$$x_1 + 3x_2 + s_2 = 3$$
$$x_2 + s_3 = 4$$

$x_1, x_2, s_1, s_2, s_3, a_1 \geq 0$.

The initial basic feasible solution is given by

$x_1 = x_2 = s_1 = 0$ (Non-basic)

$a_1 = 2$, $s_2 = 3$, $s_3 = 4$.

Now we put the above information in the simplex tableau.

	c_j	3	−1	0	0	0	−M		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	a_1	b	θ
−M	a_1	(2)	1	−1	0	0	1	2	$\frac{2}{2}=1$ →
0	s_2	1	3	0	1	0	0	3	$\frac{3}{1}=3$
0	s_3	0	1	0	0	1	0	4	
	$z_j = \sum c_B a_{ij}$	−2M	−M	M	0	0	M	2M	
	$C_j = c_j - z_j$	3+2M ↑	−1+M	−M	0	0	−2M		

From the above table, x_1 is the entering variable, a_1 is the outgoing variable and omit the column for this variable in the next table. (2) is the key element make it unity and all other elements in its column equal to zero.

Then the revised simplex table is:

C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
	c_j	3	-1	0	0	0		
3	x_1	1	$1/2$	$-1/2$	0	0	1	—
0	s_2	0	$5/2$	$1/2$	1	0	2	4 $\rightarrow$
0	s_3	0	1	0	0	1	4	—
$Z_j = \sum a_{ij} c_B$		3	$3/2$	$-3/2$	0	0	3	
$C_j = c_j - Z_j$		0	$-5/2$	$3/2$ $\uparrow$	0	0		
3	x_1	1	3	0	1	0	3	
0	s_1	0	5	1	2	0	4	
0	s_3	0	1	0	0	1	4	
$Z_j = \sum a_{ij} c_B$		3	9	0	3	0	9	
$C_j = c_j - Z_j$		0	-10	0	-3	0		

Since all $C_j \le 0$, an optimal solution has been reached.

$\therefore$ The optimum basic feasible solution is $x_1 = 3$, $x_2 = 0$

and $Z_{max} = 9$.

5(e). In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of velocity u, v, w are solutions of Laplace's equation.

Sol: Let $W = \xi i + \eta j + \zeta k$,

$$q = ui + vj + wk.$$

vorticity is constant in magnitude and direction

$\Rightarrow \xi, \eta, \zeta$ are constant.

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta = \text{constant}.$$

$$\therefore \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{constant} \quad \text{——— (1)}$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{constant} \quad \text{——— (2)}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{constant} \quad \text{——— (3)}$$

Differentiating of (2) and (3) w.r.t. z and y

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \cdot \partial z},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\, \partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y\, \partial x} + \frac{\partial^2 w}{\partial x\, \partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}(0) = 0.$$

$\therefore \nabla^2 u = 0$. Similarly we can prove $\nabla^2 v = 0$, $\nabla^2 w = 0$. It means that components of velocity are solutions of Laplace's equation.

6(a)(i) Solve $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$.

Sol^n Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{---(1)}$$

Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of (1).

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2+z-(x^2+z)+x^2-y^2} = \frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$$

so that $\log x + \log y + \log z = \log c_1$

$$\Rightarrow \log(xyz) = \log c_1 \quad \Rightarrow xyz = c_1 \quad \text{---(2)}$$

Choosing $x, y, -1$ as multipliers, each fraction of (1).

$$\frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} = \frac{xdx + ydy - zdz}{0}$$

$\Rightarrow xdx + ydy - zdz = 0$ so that $x^2+y^2-2z = c_2$ ---(3)

From (2) and (3), the required general solution is

$\phi(x^2+y^2-2z, xyz) = 0$, ϕ is being an arbitrary function.

6(a)(ii) Reduce the equation $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + (x-1)p + (y-1)q = 0$ to canonical form and hence solve it.

Sol^n Comparing the given equation with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

Here $R = x(xy-1)$, $S = -(x^2y^2-1)$, $T = y(xy-1)$ —(1)

Now, the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ and (1) give

$$x(xy-1)\lambda^2 - (x^2y^2-1)\lambda + y(xy-1) = 0$$

Or $x\lambda^2 - (xy+1)\lambda + y = 0$

or $(x\lambda - 1)(\lambda - y) = 0$ so that $\lambda = 1/x, y$.

Take $\lambda_1 = 1/x$ and $\lambda_2 = y$.

Hence $\left(\frac{dy}{dx}\right) + \lambda_1 = 0$ and $\left(\frac{dy}{dx}\right) + \lambda_2 = 0$ become

$\frac{dy}{dx} + (1/x) = 0$ and $\left(\frac{dy}{dx}\right) + y = 0$.

$dy + (1/x)dx = 0$ and $(1/y)dy + y\,dx = 0$ —(2)

Integrating (2),

$$y + \log x = \log c_1 \text{ and } \log y + x = \log c_2$$

$\log e^y + \log x = \log c_1$ and $\log y + \log e^x = \log c_2$

$$xe^y = c_1 \text{ and } ye^x = c_2.$$

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u = xe^y \quad \text{and} \quad v = ye^x \quad \text{------} (3)$$

$$\therefore\ p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = e^y\frac{\partial z}{\partial u} + ye^x\frac{\partial z}{\partial v},$$

using (3) ------ (4)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = xe^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v},$$

using (3) ------ (5)

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u} + ye^x\frac{\partial z}{\partial v}\right), \text{ using (4)}$$

$$= e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + ye^x\frac{\partial z}{\partial v} + ye^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + ye^x\frac{\partial z}{\partial v} + ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y\left[\frac{\partial^2 z}{\partial u^2}e^y + \frac{\partial^2 z}{\partial v\partial u}ye^x\right] + ye^x\frac{\partial z}{\partial v} + ye^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y + \frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore\ r = e^{2y}\frac{\partial^2 z}{\partial u^2} + 2ye^{x+y}\frac{\partial^2 z}{\partial u\partial v} + y^2e^{2x}\frac{\partial^2 z}{\partial v^2} + ye^x\frac{\partial z}{\partial v}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(xe^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right), \text{ using (5)}$$

$$= e^y\frac{\partial z}{\partial u} + xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial z}{\partial v} + e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y\frac{\partial z}{\partial u} + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^x\frac{\partial z}{\partial v}$$

$$+ e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y\frac{\partial z}{\partial u} + xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y + \frac{\partial^2 z}{\partial v\partial u}ye^x\right] + e^x\frac{\partial z}{\partial v} + e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y + \frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$= xe^{2y}\frac{\partial^2 z}{\partial u^2} + (xy+1)e^{x+y}\frac{\partial^2 z}{\partial u\partial v} + ye^{2x}\frac{\partial^2 z}{\partial v^2} + e^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(xe^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right), \text{ using } 5$$

$$= xe^y\frac{\partial z}{\partial u} + xe^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= xe^y\frac{\partial z}{\partial u} + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}\right]$$

$$= xe^y\frac{\partial z}{\partial u} + xe^y\left[\frac{\partial^2 z}{\partial u^2}\cdot xe^y + \frac{\partial^2 z}{\partial u\partial v}e^x\right] + e^x\left[\frac{\partial^2 z}{\partial u\partial v}xe^y + \frac{\partial^2 z}{\partial v^2}e^x\right]$$

$$\therefore t = x^2e^{2y}\frac{\partial^2 z}{\partial u^2} + 2xe^{x+y}\frac{\partial^2 z}{\partial u\partial v} + xe^y\frac{\partial z}{\partial u} + e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u\partial v} = 0 \quad \text{or} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0 \qquad \text{——} (6)$$

Integrating (6) w.r.t 'v', $\dfrac{\partial z}{\partial u} = f(u)$. —— (7)

Integrating (7) w.r.t 'u' $\quad z = \int f(u)\,du + \psi(v)$

or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $z = \phi(xe^y) + \psi(ye^x)$, ϕ and ψ being arbitrary functions.

6.(b) Solve the equations:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss - Seidal iteration method.

Solution:

Rewriting the given equations as.

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{—(i)}$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{—(ii)}$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{—(iii)}$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3. \quad \text{—(iv)}$$

First iteration

Putting $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ in (i) we get $x_1 = 0.3$

Putting $x_1 = 03$, $x_3 = 0$, $x_4 = 0$ in (ii) we get $x_2 = 1.56$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_4 = 0$ in (iii) we get $x_3 = 2.886$

Putting $x_1 = 0.3$, $x_2 = 1.56$, $x_3 = 2.886$ in (iv) we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56$, $x_3 = 2.886$, $x_4 = -0.1368$, in (i) $\Rightarrow x_1 = 0.8869$

Putting $x_1 = 0.8869$, $x_3 = 2.886$, $x_4 = -0.1368$, in (ii) $\Rightarrow x_2 = 1.9523$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_4 = -0.1368$; in (iii) $\Rightarrow x_3 = 2.9566$

Putting $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$; in (iv) $\Rightarrow x_4 = -0.0248$

third iteration

Putting ~~$x_1 = 0.8869$~~

$x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (i) $\Rightarrow x_1 = 0.9836$

$x_1 = 0.9836$, $x_3 = 2.9566$, $x_4 = -0.0248$ in (ii) $\Rightarrow x_2 = 1.9899$

$x_1 = 0.9836$, $x_2 = 1.9899$, $x_4 = -0.0248$ in (iii) $\Rightarrow x_3 = 2.9924$

$x_1 = 0.9836$, $x_2 = 1.9899$, $x_3 = 2.9924$ in (iv) $\Rightarrow x_4 = -0.0042$

Fourth iteration:

Putting

$x_2 = 1.9899$, $x_3 = 2.9924$, $x_4 = -0.0042$ in (i) $\Rightarrow x_1 = 0.9968$

$x_1 = 0.9968$, $x_3 = 2.9924$, $x_4 = -0.0042$ in (ii) $\Rightarrow x_2 = 1.9982$

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_4 = -0.0042$ in (iii) $\Rightarrow x_3 = 2.9987$

$x_1 = 0.9968$, $x_2 = 1.9982$, $x_3 = 2.9987$ in (iv) $\Rightarrow x_4 = -0.0008$

fifth iteration

Putting

$x_2 = 1.9982$, $x_3 = 2.9987$, $x_4 = -0.0008$ in (i) $\Rightarrow x_1 = 0.9994$

$x_1 = 0.9994$, $x_3 = 2.9987$, $x_4 = -0.008$ in (ii) $\Rightarrow x_2 = 1.9997$

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_4 = -0.008$ in (iii) $\Rightarrow x_3 = 2.9997$

$x_1 = 0.9994$, $x_2 = 1.9997$, $x_3 = 2.9997$ in (iv) $\Rightarrow x_4 = -0.0001$

Sixth iteration

Putting

$x_2 = 1.9997$, $x_3 = 2.9997$, $x_4 = -0.0001$ in (i) $\Rightarrow x_1 = 0.9999$

$x_1 = 0.9999$, $x_3 = 2.9997$, $x_4 = -0.0001$ in (ii) $\Rightarrow x_2 = 1.9999$

$x_1 = 0.9999$, $x_2 = 1.9999$, $x_4 = -0.0001$ in (iii) $\Rightarrow x_3 = 2.9999$

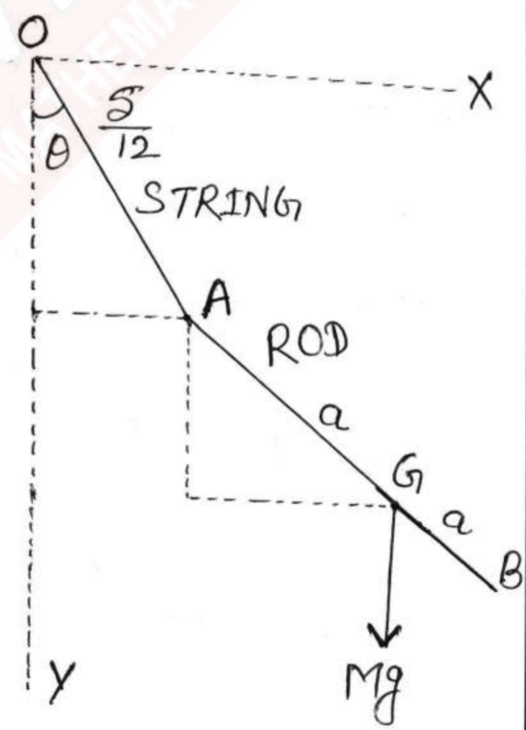
$x_1 = 0.9999$, $x_2 = 1.9999$, $x_3 = 2.9999$ in (iv) $\Rightarrow x_4 = -0.0001$

Hence the solution for the given equations is

$$\boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \\ x_4 = 0 \end{array}}$$

6(c) A uniform rod, of length $2a$, which has one end attached to a fixed point by a light inextensible string of length $5a/12$, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time, and show that the period of its principal oscillations are $2\pi\sqrt{(5a/3g)}$ and $\pi\sqrt{(a/3g)}$.

Sol: Let OA be the string of length $5a/12$ and AB the rod of mass M and length $2a$. Let O be the fixed point. At time t, let the string and the rod make angle θ and ϕ, to the vertical respectively.



Referred to O as origin, horizontal and vertical lines OX and OY as axes, the co-ordinates of the C.G. 'G' of the rod are given by

$$x_G = \frac{5}{12} a \sin\theta + a\sin\phi,$$

$$y_G = \frac{5}{12} a\cos\theta + a\cos\phi$$

$$\therefore\ v_G^2 = \dot{x}_G^2 + \dot{y}_G^2$$

$$= \left(\frac{5}{12} a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi}\right)^2 + \left(-\frac{5}{12} a\sin\theta\dot{\theta} - a\sin\phi\dot{\phi}\right)^2$$

$$= \frac{25}{144} a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6} a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi)$$

$$= \frac{25}{144} a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6} a^2\dot{\theta}\dot{\phi},$$

($\because$ θ and ϕ are small)

If T be the total K.E. and W the work function of the system, then.

$$T = \frac{1}{2} M\cdot\frac{1}{3} a^2\dot{\phi}^2 + \frac{1}{2} M\cdot v_G^2$$

$$= \frac{1}{2} M\left[\frac{1}{3} a^2\dot{\phi}^2 + \frac{25}{144} a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6} a^2\dot{\theta}\dot{\phi}\right]$$

$$\text{or } T = \frac{1}{2} Ma^2\left[\frac{25}{144}\dot{\theta}^2 + \frac{4}{3}\dot{\phi}^2 + \frac{5}{6}\dot{\theta}\dot{\phi}\right]$$

and $W = Mg\, y_G + C$

$$= Mga\left(\frac{5}{12}\cos\theta + \cos\phi\right) + C$$

Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e. $$\frac{d}{dt}\left[\frac{1}{2}Ma^2\left(\frac{25}{72}\dot{\theta} + \frac{5}{6}\dot{\phi}\right)\right] - 0$$

$$= -\frac{5}{12} Mga \sin\theta$$

or $5\ddot{\theta} + 12\ddot{\phi} = -12c\theta$, ——— ①

($\because$ θ is small)

Taking $(g/a) = c$.

And Lagrange's ϕ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

i.e. $$\frac{d}{dt}\left[\frac{1}{2}Ma^2\left(\frac{8}{3}\dot{\phi} + \frac{5}{6}\dot{\theta}\right)\right] - 0$$

$$= -Mga \sin\phi$$

or $5\ddot{\theta} + 16\ddot{\phi} = -12c\phi$, ——— ②

($\because$ ϕ is small and $g/a = c$)

Equations ① and ②, can be written as

$$(5D^2+12c)\theta + 12D^2\phi = 0$$

and $5D^2\theta + (16D^2+12c)\phi = 0$

Eliminating ϕ between these two eq^n, we have

$$[(5D^2+12c)(16D^2+12c) - 60D^4]\theta = 0$$

or $(5D^4 + 63cD^2 + 36c^2)\theta = 0$ ——— ③

Let the solution of ③ be

$\theta = A\cos(pt+B)$

$\therefore D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in ③, we get.

$$(5p^4 - 63cp^2 + 36c^2)\theta = 0$$

or $5p^4 - 63cp^2 + 36c^2 = 0$ $(\because \theta \neq 0)$

or $(5p^2-3c)(p^2-12c) = 0$

$\therefore p_1^2 = \frac{3}{5}c = \frac{3g}{5a}$ and $p_2^2 = 12c = \frac{12g}{a}$

$(\because c = g/a)$

Hence period of oscillations are

$$\frac{2\pi}{p_1} \text{ and } \frac{2\pi}{p_2}$$

i.e. $2\pi\sqrt{(5a/3g)}$ and $2\pi\sqrt{(a/12g)}$

i.e. $2\pi\sqrt{(5a/3g)}$ and $\pi\sqrt{(a/3g)}$.

7(a) Find the characteristics of the equation $xp+yq-pq=0$ and then find the equation of the integral surface through the curve $z=x/2$, $y=0$.

Sol'n: Here $f(x,y,z,p,q)=xp+yq-pq=0$ —— ①.

The integral surface passes through the curve $z=x/2$, $y=0$, whose parametric equation can be written as

$x=f_1(\lambda)=\lambda$, $y=f_2(\lambda)=0$, $z=f_3(\lambda)=\lambda/2$, λ being a parameter.

$\therefore$ Initial values for x,y,z are $x=x_0=\lambda$, $y=y_0=0$, $z=z_0=\lambda/2$, when $t=0$.

And corresponding initial values p_0 and q_0 of p and q are determined by the relations.

$f_3'(\lambda)=p_0 f_1'(\lambda)+q_0 f_2'(\lambda) \Rightarrow \frac{1}{2}=p_0\cdot 1+q_0\cdot 0 \Rightarrow p_0=\frac{1}{2}$

and $f\{f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0)=0 \Rightarrow x_0p_0+y_0q_0-p_0q_0=0$
$\Rightarrow q_0=\lambda$

The characteristic equations of the given partial differential equation ① are.

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-q \quad —— ②.$$

$$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-p \quad —— ③$$

$$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-q)+q(y-p)=px+qy-2pq=-pq$$
using ① —— ④

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-p \quad —— ⑤$$

$$\frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-q \quad —— ⑥$$

from ⑤ and ⑥, we get $p=Ae^{-t}$ and $q=Be^{-t}$

But initially when $t=0$, $p=p_0=\frac{1}{2}$ and $q=q_0=\lambda$

$\therefore A=p_0=\frac{1}{2}$ and $B=q_0=\lambda$

$\Rightarrow p=(\frac{1}{2})e^{-t}$ and $q=\lambda e^{-t}$ —— ⑦

using ⑦, from ②, we have $\frac{dx}{dt}-x=-\lambda e^{-t}$

which is a L.D.E, with I.F $=e^{\int -dt}=e^{-t}$

$\therefore\ x.e^{-t} = c_1 + \int (-\lambda e^{-t})(e^{-t})\,dt = c_1 - \lambda \int e^{-2t}\,dt$
$= c_1 + \frac{1}{2}\lambda e^{-2t}$

But when $t=0$, $x = x_0 = \lambda$

$\therefore\ x_0 = \lambda = c_1 + \frac{1}{2}\lambda \Rightarrow c_1 = \lambda/2$

$\therefore\ xe^{-t} = \lambda/2 + (\lambda/2)e^{-2t}$

$\Rightarrow x = (\lambda/2)(1+e^{-2t})e^{t}$ ——— (8)

Again, using (7), from (3), we have

$$\frac{dy}{dt} - y = -\frac{1}{2}e^{-t}$$

which is L.D.E with I.F $= e^{\int -dt} = e^{-t}$,

$\therefore\ ye^{-t} = c_2 + \int (-\frac{1}{2}e^{-t})e^{-t}\,dt = c_2 - \frac{1}{2}\int e^{-2t}\,dt = c_2 + \frac{1}{4}e^{-2t}$

But when $t=0$, $y = y_0 = 0$, $\therefore\ y_0 = 0 = c_2 + \frac{1}{4} \Rightarrow c_2 = -\frac{1}{4}$

$\therefore\ ye^{-t} = -\frac{1}{4} + \frac{1}{4}e^{-2t} \Rightarrow y = \frac{1}{4}(e^{-2t}-1)e^{t}$ ——— (9)

Now using (7), from (4), we have

$\frac{dz}{dt} = -\frac{\lambda}{2}e^{-2t}$, Integrating $z = \frac{\lambda}{4}e^{-2t} + c_3$

But when $t=0$, $z = z_0 = \lambda/2$, $\therefore\ z_0 = \lambda/2 = \lambda/4 + c_3 \Rightarrow c_3 = \lambda/4$

$\therefore\ z = \frac{\lambda}{4}e^{-2t} + \frac{\lambda}{4} \Rightarrow z = \frac{\lambda}{4}(e^{-2t}+1)$ ——— (10)

Thus, the characteristic strips of the given equation are given by

$x = \frac{\lambda}{2}(1+e^{-2t})e^{t}$, $y = \frac{1}{4}(e^{-2t}-1)e^{t}$ and $z = \frac{\lambda}{4}(e^{-2t}+1)$

where λ and t are two parameters.

The required integral surface is obtained by eliminating λ and t between x, y and z.

we have $\frac{x}{2} = ze^{t} \Rightarrow e^{t} = \frac{x}{2z}$

$$\therefore\ y = \frac{1}{4}(e^{-t} - e^{t}) = \frac{1}{4}\left(\frac{2z}{x} - \frac{x}{2z}\right) = \frac{4z^2 - x^2}{8xz}$$

$$\Rightarrow 4z^2 = x^2 + 8xyz$$

which is the required integral surface.

7(b)(i) From the following data estimate the number of persons having incomes between Rs. 1000 and Rs. 1500.

Income	Below 400	500-1000	1000-2000	2000-3000	3000-4000
No. of Person	6000	4250	3600	1500	650.

Sol'n: First of all we put all the frequencies in the form of cumulative frequencies and form the following table.

Income less than	500	1000	2000	3000	4000
No. of persons	6000	10,250	13850	15350	16,000

Now shift the origin to 500 and divide the scale by 500.

In this way, we have known $f(x)$ for $x = 0, 1, 3, 5$ and 7.

Now we shall find number of persons whose income is less than Rs. 1500.

i.e. $f(x)$ for $x = \frac{1500-500}{500} = \frac{1000}{500} = 2$.

By Lagrange's formula,

$$f(2) = \frac{(2-1)(2-3)(2-5)(2-7)}{(0-1)(0-3)(0-5)(0-7)} \times 6000 + \frac{(2-0)(2-3)(2-5)(2-7)}{(1-0)(1-3)(1-5)(1-7)} \times 10250$$

$$+ \frac{(2-0)(2-1)(2-5)(2-7)}{(3-0)(3-1)(3-5)(3-7)} \times 13850$$

$$+ \frac{(2-0)(2-1)(2-3)(2-7)}{(5-0)(5-1)(5-3)(5-7)} \times 15{,}350 + \frac{(2-0)(2-1)(2-3)(2-5)}{(7-0)(7-1)(7-3)(7-5)} (16000)$$

$$= -\tfrac{1}{7}(6000) + \tfrac{5}{8}(10250) + \tfrac{5}{8}(13850) - \tfrac{1}{3}(15350) + \tfrac{1}{56}(16000)$$

$$= -857.14 + 6406.25 + 8656.25 - 1918.75 + 285.71$$

$$= 12572 \text{ approximately.}$$

i.e, the number of persons having income less than Rs.1500 = 12572

But the no. of persons having income less than Rs 1000 = 10250

$\therefore$ The person having income between Rs. 1000 and Rs.1500 are 2322.

7.(b)(ii) Apply Runge-Kutta method of order 4 to find approximate value of y for $x=0.2$ in steps of 0.1 if $\frac{dy}{dx} = x+y^2$ given that $y=1$ where $x=0$

Sol: Given that $f(x,y) = x+y^2$.

Here we take $h=0.1$ and carry out the calculations in two steps.

Step 1: $x_0=0$, $y_0=1$, $h=0.1$

$k_1 = hf(x_0,y_0) = (0.1)f(0,1) = 0.1$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}) = (0.1)f(0.05, 1.1) = 0.1152$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}) = (0.1)f(0.05, 1.1152) = 0.1168$

$k_4 = hf(x_0+h, y_0+k_3) = (0.1)f(0.1, 1.1168) = 0.1347$

$K = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}(0.1+0.2304+0.2336+0.1347) = 0.1165$

$\therefore y(0.1) = y_0+k = 1.1165$

Step 2: $x_1 = x_0+h = 0.1$, $y_1 = 1.1165$, $h=0.1$

$k_1 = hf(x_1,y_1) = (0.1)f(0.1, 1.1165) = 0.1347$

$k_2 = hf(x_1+\frac{1}{2}h, y_0+\frac{1}{2}k_1) = 0.1 f(0.15, 1.1838) = 0.1551.$

$k_3 = hf(x_1+\frac{1}{2}h, y_0+\frac{k_2}{2}) = (0.1)f(0.15, 1.194) = 0.1576$

$k_4 = hf(x_1+h, y_1+k_3) = (0.1)f(0.2, 1.1576) = 0.1823$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.1571$

Hence $y(0.2) = y_1+k = 1.2736$

7(c) → Show that velocity potential

$$= \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

gives a possible motion. Determine the stream lines.

Sol: Given that,

$$\phi = \frac{1}{2}\log\left[(x+a)^2+y^2\right] - \frac{1}{2}\log\left[(x-a)^2+y^2\right]. \quad \text{——— (1)}$$

$$\frac{\partial\phi}{\partial x} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\left[(x+a)^2+y^2\right]-2(x+a)^2}{\left[(x+a)^2+y^2\right]^2} - \frac{\left[(x-a)^2+y^2\right]-2(x-a)^2}{\left[(x-a)^2+y^2\right]^2}.$$

or $$\frac{\partial^2\phi}{\partial x^2} = \frac{y^2-(x+a)^2}{\left[(x+a)^2+y^2\right]^2} - \frac{y^2-(x-a)^2}{\left[(x-a)^2+y^2\right]^2} \quad \text{——— (2)}$$

By (1), $$\frac{\partial\phi}{\partial y} = \frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{(x+a)^2+y^2-2y^2}{\left[(x+a)^2+y^2\right]^2} - \frac{(x-a)^2+y^2-2y^2}{\left[(x-a)^2+y^2\right]^2} \quad \text{——— (3)}$$

Adding (2) and (3),

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{or} \quad \nabla^2\phi=0.$$

Thus the equation of continuity is satisfied and so (1) gives a possible liquid motion.

<u>Second part:</u> To determine stream lines.

$$-\frac{\partial\phi}{\partial x}=u=-\frac{\partial\psi}{\partial y},$$

$$-\frac{\partial\phi}{\partial y}=v=\frac{\partial\psi}{\partial x}.$$

Hence $\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$

Now $\frac{\partial\psi}{\partial y}=\frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2}$

Integrating w.r.t. y,

$$\psi=\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x-a}+F(x) \quad \text{---(4)}$$

where $F(x)$ is constant of integration. To determine $F(x)$

$$\frac{\partial\psi}{\partial x}=-\frac{\partial\phi}{\partial y}=-\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2} \quad \text{---(5)}$$

By (4), $\dfrac{\partial \psi}{\partial x} = -\dfrac{y}{(x+a)^2+y^2} + \dfrac{y}{(x-a)^2+y^2} + F'(x)$ —— (6)

Equating (5) to (6), $F'(x)=0$. Integrating this $F(x)$ = absolute constant and hence neglected. Since it has no effect on the fluid motion.

Now (4) becomes

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} \quad \text{—— (7)}$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}$$

Stream lines are given by ψ = constant,

i.e., $\tan^{-1}\left[\dfrac{-2ay}{x^2-a^2+y^2}\right] = \text{const.}$

or $\dfrac{y}{x^2-a^2+y^2} = \text{const.}$

If we take const. $=0$, then we get $y=0$, i.e., x-axis.

If we take const. $=\infty$, then we get circle $x^2-a^2+y^2=0$, i.e., $x^2+y^2=a^2$.

Thus stream lines include x axis and circle.

8(a). The points of trisection of a string are pulled aside through a distance h on opposite sides of the position of equilibrium, and the string is released from rest. Derive an expression for the string at any subsequent time and show that the middle point of the string always remains at rest.

Sol^n:

Let the length of the string is $= 3l$.

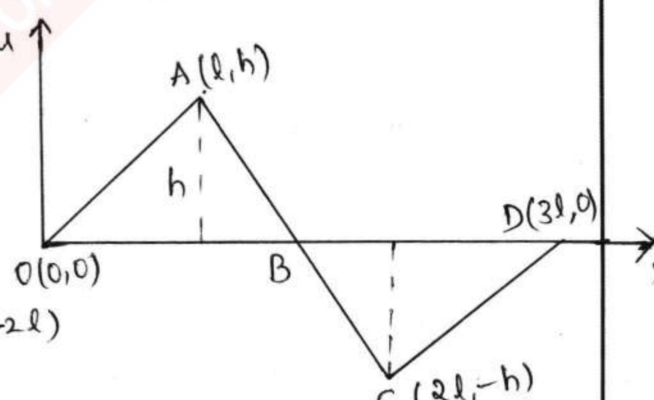
The coordinates of A, C and D are (l, h), $(2l, -h)$, $(3l, 0)$ respectively.

Initial deflection is given by OABCD.

Equation of OA is

$$u - 0 = \frac{h}{l}(x-0) \text{ i.e. } u = \frac{hx}{l}$$

Equations of AC and CD respectively are given by



$$u - h = \frac{-h-h}{2l-l}(x-l)$$

$$\Rightarrow u = \frac{h(3l-2x)}{l}$$

and $$u-(-h) = \frac{0-(-h)}{3l-2l}(x-2l)$$

$$\Rightarrow u = \frac{h(x-3l)}{l}$$

The required deflection is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{3l} \sin\frac{n\pi x}{3l} \quad \text{---} \quad (I)$$

where $E_n = \frac{2}{3l} \cdot \int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$.

The displacement $y(x,t)$ of any point of the string is given by

Boundary conditions $y(0,t) = y(3l,t) = 0 \quad \forall\ t \geq 0$ — ①

Initial conditions
$$y(x,0) = \begin{cases} \frac{hx}{l} & \text{when } 0 \leq x \leq l \\ \frac{h(3l-2x)}{l} & \text{when } l \leq x \leq 2l \\ \frac{h(x-3l)}{l} & \text{when } 2l \leq x \leq 3l \end{cases} \quad \text{— ②}$$

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — ③

we have
$$y(x,t) = \sum_{n=1}^{\infty} \left\{ E_n \cos\frac{n\pi ct}{3l} + F_n \sin\frac{n\pi ct}{3l} \right\} \sin\frac{n\pi x}{3l} \quad \text{— ④}$$

Differentiating ③ partially w.r.t t, we get
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -E_n \frac{n\pi c}{3l} \sin\frac{n\pi ct}{3l} + \frac{n\pi c}{3l} F_n \cos\frac{n\pi ct}{3l} \right\} \sin\frac{n\pi x}{3l} \quad \text{— ⑤}$$

Putting $t=0$ in ④ and ⑤

and using the I.C ② and ③ we get

$$⑤ \equiv \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} F_n \frac{n\pi c}{3l} \sin\frac{n\pi x}{3l} = 0 \quad (\text{by } ③)$$

where $F_n = \frac{2}{n\pi c} \int_0^{3l} (0) \sin\frac{n\pi x}{3l}\, dx = 0$

$$④ \equiv y(x,0) = f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{3l}$$

where $E_n = \frac{2}{3l} \int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$ — ⑥

Now
$$E_n = \frac{2}{3l} \int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$$

$$= \frac{2}{3l}\left[\int_0^l f(x)\sin\frac{n\pi x}{3l}dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}dx\right]$$

$$= \frac{2}{3l}\left[\int_0^l \frac{hx}{l}\sin\frac{n\pi x}{3l}dx + \int_l^{2l}\frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}dx + \int_{2l}^{3l}\frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}dx\right]$$

Continuing in this way we get

$$E_n = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left\{\sin\left(n\pi - \frac{n\pi}{3}\right)\right\}\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left(\sin n\pi\cos\frac{n\pi}{3} - \cos n\pi\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - 0 + \cos n\pi\sin\frac{n\pi}{3}\right] \quad \left[\because \sin n\pi = 0 \ \forall n\right]$$

$$= \frac{18h}{n^2\pi^2}\left[1+\cos n\pi\right]\sin\frac{n\pi}{3} = \frac{18h}{n^2\pi^2}\left[1+(-1)^n\right]\sin\frac{n\pi}{3}.$$

Thus $E_n = 0$ if n is odd.

$E_n = \frac{36h}{n^2\pi^2}\sin\frac{n\pi}{3}$ if n is even put $n=2m$, $m=1,2,\ldots$

i.e, $E_n = \frac{36h}{4m^2\pi^2}\sin\frac{2m\pi}{3}$, $m=1,2,\ldots$

$$= \frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}$$

putting the values of E_n & F_n in (4) the required deflection is given by $y(x,t) = \sum_{m=1}^{\infty}\frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}\sin\frac{n\pi x}{3l}\cos\frac{n\pi ct}{3l}$

$$\Rightarrow y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\frac{2m\pi}{3}\cos\frac{n\pi ct}{3l}\sin\frac{n\pi x}{3l} \quad \text{——— (7)}$$

Putting $x = \frac{3l}{2}$ in (7), we find that the displacement of the midpoint of the string i.e. $y\left(\frac{3l}{2}, t\right) = 0$.

because $\sin m\pi = 0$, for all integral values of m.

This shows that the mid-point of the string always rest.

8(b)(i) Design a logic circuit which has three inputs A, B, C and gives a high output when majority of inputs is high. Also obtain logic circuit using only NAND gates.

Sol'n: Let A, B, C be three inputs. The design of this circuit can be done in the following steps.

- Prepare truth table. The output Y is 1 whenever 2 or more inputs are 1. otherwise output is 0.

A	B	C	Y	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{A}BC$
1	0	0	0	
1	0	1	1	$A\bar{B}C$
1	1	0	1	$AB\bar{C}$
1	1	1	1	ABC

- Write AND terms when Y = 1. These terms have each input variable in either non-inverted (or) inverted form. These terms are shown in truth table.

- The output expression is

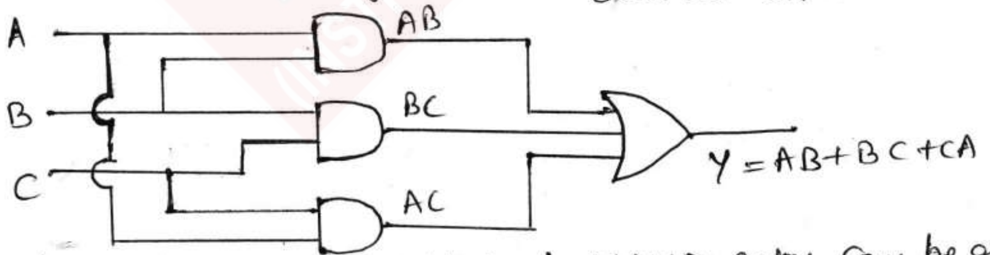
$$Y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

$$= \bar{A}BC + ABC + A\bar{B}C + ABC + AB\bar{C} + ABC$$

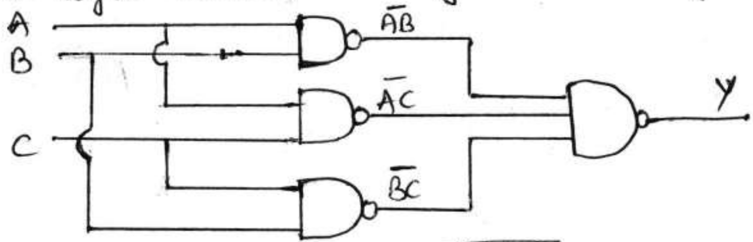
$$= BC(\bar{A}+A) + AC(\bar{B}+B) + AB(\bar{C}+C)$$

$$= BC + AC + AB \qquad (\because x+\bar{x}=1)$$

Now draw the logic circuit which requires 3 AND gates and one OR Gate.



The logic circuit using 4 NAND gates can be given below.



$$Y = BC + AC + AB$$

$$= \overline{\overline{BC + AC + AB}}$$

$$= \overline{\overline{BC} \cdot \overline{AC} \cdot \overline{AB}}$$

8(c) Prove that in a steady motion of a liquid.

$$H = \frac{p}{\rho} + \frac{1}{2}q^2 + V = \text{constant along a Stream line.}$$

If this constant has the same value every-where in the liquid, then prove that the motion must be either irrotational or the vortex lines must coincide with the stream lines.

Sol:

$$\frac{\partial H}{\partial x} = 2(v\zeta - w\eta) \quad \text{——— (1)}$$

$$\frac{\partial H}{\partial y} = 2(w\xi - u\zeta), \quad \text{——— (2)}$$

$$\frac{\partial H}{\partial z} = 2(u\eta - v\xi) \quad \text{——— (3)}$$

where $H = V + \frac{1}{2}q^2 + \int\frac{dp}{\rho} = \frac{p}{\rho} + \frac{1}{2}q^2 + V$

Also $u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + w\frac{\partial H}{\partial z} = 0$ ——— (4)

$$\xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} + \zeta\frac{\partial H}{\partial z} = 0 \quad \text{——— (5)}$$

Equation (4) and (5) show that the surface $H = $ const. contains the stream lines and vortex lines.

If H has the value everywhere, then

$$\frac{\partial H}{\partial x}=0,\ \frac{\partial H}{\partial y}=0,\ \frac{\partial H}{\partial z}=0$$

so that
$$v\zeta-w\eta=0,$$
$$w\xi-u\zeta=0,$$
$$u\eta-v\zeta=0.$$

This $\Rightarrow$ (i) $\dfrac{u}{\xi}=\dfrac{v}{\eta}=\dfrac{w}{\zeta}$

or (ii) $\xi=0,\ \eta=0,\ \zeta=0.$

(iii) $\Rightarrow$ motion is irrotational

(iv) $\Rightarrow$ stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

coincide with vortex lines given by

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}.$$