Mains Test Series - 2021

Test-4, Paper-II

Answer Key

PDE, NA & COMPUTER PROG. & MECHANICS AND FLUID DYNAMICS

1(a) Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

Sol^n: Let $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ be the equation of the plane in intercept form, so that $a+b+c=1$.

Thus we have

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{1-a-b}=1 \quad \text{——} \quad (1)$$

Differentiating equation (1) w.r.t x & y, we have

$$\frac{1}{a}+\frac{p}{1-a-b}=0$$

$$\Rightarrow \frac{p}{1-a-b}=-\frac{1}{a} \quad \text{——} \quad (2)$$

and $$\frac{1}{b}+\frac{q}{1-a-b}=0$$

$$\Rightarrow \frac{q}{1-a-b}=-\frac{1}{b} \quad \text{——} \quad (3)$$

from equ'ns (2) & (3), we get

$$\frac{p}{q}=\frac{b}{a} \quad \text{——} \quad (4)$$

Also from equ'ns (2) & (4), we get

$$pa=a+b-1=a+\frac{p}{q}a-1$$

$$\Rightarrow a\left(1+\frac{p}{q}-p\right)=1$$

$$\therefore \quad a = q/(p+q-pq) \quad \text{——— (5)}$$

Similarly, from equ'ns (3) & (4), we find

$$b = p/(p+q-pq) \quad \text{——— (6)}$$

Substituting the values of a & b from equ'ns (5) & (6) respectively to equ'n (1), we have

$$\frac{p+q-pq}{q}x + \frac{p+q-pq}{p}y + \frac{p+q-pq}{-pq}z = 1$$

$$\Rightarrow \frac{x}{q}+\frac{y}{p}-\frac{z}{pq} = \frac{1}{p+q-pq}$$

i.e.

$$px+qy-z = \frac{pq}{p+q-pq} \quad \text{— (7)}$$

which is the required PDE.

1.(b) Solve $[D^3 - 4D^2D' + 5D'^2D - 2D'^3]z = e^{y+2x} + (y+x)^{1/2}$

Solution :-

Given Partial D.E.

$$[D^3 - 4D^2D' + 5DD'^2 - 2D'^3]z = e^{y+2x} + (y+x)^{1/2}$$

Here, auxillary equation is → putting $D = m$, $D' = 1$.

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$(m-1)^2(m-2) = 0$$

$$m = 1, 1, 2.$$

$$\therefore \text{C.F} = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x).$$

Now; P.I. corresponding to

$$e^{y+2x}$$

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} \cdot e^{y+2x}$$

$$= \frac{1}{D-2D'}\left\{\frac{1}{(D-D')^2} e^{y+2x}\right\}$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{(2-1)^2} \iint e^v \, dv\, dv \; ; \text{ where } v = y+2x.$$

$$= \frac{1}{D-2D'} \int e^v dv = \frac{1}{D-2D'} \cdot e^v$$

$$= \frac{1}{(1 \cdot D - 2 \cdot D')^1} e^{y+2x} = \frac{x}{1 \cdot 1!} e^{y+2x} = x e^{y+2x} \quad \text{②}$$

[Using formula; $a=2$, $b=1$, $m=1$].

Finally P.I corresponding to $(y+x)^{1/2}$

$$= \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3}(y+x)^{1/2}$$

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'}(y+x)^{1/2}\right\}$$

$\Rightarrow$ let $y+x=u$ then

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{1-2.1}\int u^{1/2}du\right.$$

$$= -\frac{1}{(D-D')^2}\cdot\frac{2}{3}u^{3/2} = -\frac{2}{3}\cdot\frac{1}{(D-D')^2}\cdot(y+x)^{3/2}.$$

$$= -\frac{\cancel{2}}{3}\cdot\frac{x^2}{1^2\cdot\cancel{2}!}(y+x)^{3/2} = -\frac{x^2}{3}(y+x)^{3/2}$$

[where, $a=b=1$, $m=2$] —— (3)

From (1), (2) and (3), the required general solution is

$$\boxed{z = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y+2x) + xe^{y+x} - \left(\frac{x^2}{3}\right)(y+x)^{3/2}}$$

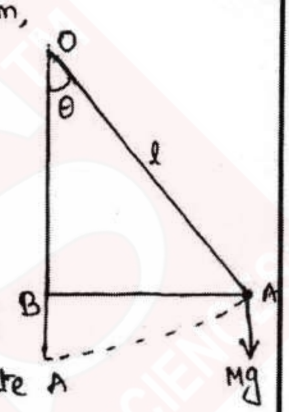
which is required.

1.(c) Find the positive root of $\log_e x = \cos x$ nearest to five places of decimal by Newton-Raphson method.

Ans: 1.3030.
(Try yourself.)

1.(d) Use Hamilton's equations to find the equation of motion of the simple pendulum.

Sol'n: Let l be the length of the pendulum and M the mass of the bob. At time t, let θ be the inclination of the string to the downward vertical. Then, if T and V are the kinetic and potential energies of pendulum,



then $T = \frac{1}{2}M(l\dot{\theta})^2 = \frac{1}{2}Ml^2\dot{\theta}^2$

and V = workdone against $Mg = Mg\,A'B$

$= Mgl(1-\cos\theta)$

$\therefore L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$ — ①

Here θ is the only generalised coordinate

$$\therefore p_\theta = \frac{\partial L}{\partial \dot\theta} = Ml^2\dot{\theta} \quad \text{——— ②}$$

Since L does not contain t explicitly,

$\therefore H = T + V = \frac{1}{2}Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta)$

$\Rightarrow H = p_\theta^2/(2Ml^2) + Mgl(1-\cos\theta)$, (from ②)

Here the Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ i.e } \dot{p}_\theta = -Mgl\sin\theta$$

$$\text{and } \dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ i.e. } \dot{\theta} = p_\theta/(Ml^2)$$

Differentiating ④, we get

$$\ddot{\theta} = \dot{p}_\theta/(Ml^2) = -(Mgl\sin\theta)/(Ml^2) \text{ from ③}$$

$$\Rightarrow \ddot{\theta} = -(g/l)\sin\theta$$

which is the equation of motion of a simple pendulum.

1.(c) Find the stream lines and paths of the particles for the two dimensional velocity field:

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Solution:

We have

$$u = \frac{x}{1+t}, \quad v = y, \quad w = 0.$$

Step I: To determine stream lines.

Stream lines are the solution of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Putting the value $\frac{(1+t)}{x} dx = \frac{dy}{y} = \frac{dz}{0}$

$$\Rightarrow \left(\frac{1+t}{x}\right) dx = \frac{dy}{y}, \quad \frac{dy}{y} = \frac{dz}{0}$$

$$\Rightarrow (1+t)\log x = \log y + \log a, \quad dz = 0$$

$$\Rightarrow \boxed{x^{1+t} = ay, \quad z = b} \quad \text{——(i)}$$

These two equations represent stream lines.

Step II: To determine path lines:

Path lines are solutions of

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0$$

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = dt, \quad dz = 0$$

Integrating,

$$\log x = \log(1+t) + \log a,$$

$$\log y = t - \log b, \quad z = c$$

$$\Rightarrow x = a(1+t), \quad (y/b) = e^t, \quad z = c$$

$$\Rightarrow \boxed{y = b e^{[(x/a)-1]}, \quad z = c} \quad \text{——(ii)}$$

These two equations represent path lines.

Hence, the result.

2.(a)(i). Find the surface which is orthogonal to the one parameter system $z = cxy(x^2+y^2)$ which passes through the hyperbola $x^2-y^2=a^2$, $z=0$.

Sol^n: The given system of surfaces

$$f(x,y,z) = \frac{z}{x^3y+xy^3} = c$$

$$\frac{\partial f}{\partial x} = \frac{-z(3x^2y+y^3)}{(x^3y+xy^3)^2}, \quad \frac{\partial f}{\partial y} = \frac{-z(3y^2x+x^3)}{(x^3y+xy^3)^2}$$

$$\frac{\partial f}{\partial z} = \frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

$$\frac{-z(3x^2y+y^3)}{(x^3y+xy^3)^2}p - \frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q = \frac{1}{(x^3y+xy^3)^2}$$

$$\left\{\frac{(3x^2+y^2)}{x}\right\}p + \left\{\frac{3y^2+x^2}{y}\right\}q = -\frac{(x^2+y^2)}{z} \quad \text{—— (2)}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{\left(\frac{3x^2+y^2}{x}\right)} = \frac{dy}{\left(\frac{3y^2+x^2}{y}\right)} = \frac{dz}{-\frac{(x^2+y^2)}{z}} \quad \text{—— (3)}$$

Taking the first two fractions of (3)

$2xdx - 2ydy = 0$ So that $x^2-y^2=c_1$

Choosing $x, y, 4z$ as multipliers, each fraction of (3) $= \dfrac{xdx+ydy+4zdz}{0}$

$\therefore 2xdx + 2ydy + 8z\,dz = 0$

$\Rightarrow x^2+y^2+4z^2 = c_2$

Hence any surface which is orthogonal to (1) is of the form $x^2+y^2+4z^2 = \phi(x^2-y^2)$, ϕ being an arbitrary function. —— (4)

For the particular surface passing through the hyperbola $x^2-y^2=a^2$, $z=0$

we must take $\phi(x^2-y^2) = \dfrac{a^4(x^2+y^2)}{(x^2-y^2)^2}$

Hence the required surface is given by $(x^2+y^2+4z^2)^2(x^2-y^2)^2 = a^4(x^2+y^2)$

2(a)(iii) Solve $(x^2+y^2)(p^2+q^2)=1$.

Sol^n: To reduce the given equation to the standard form

Let $x=r\cos\theta$, $y=r\sin\theta$

so that $x^2+y^2=r^2$, $\theta=\tan^{-1}(y/x)$

Differentiating $x^2+y^2=r^2$ w.r.t x and partially,

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta, \quad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta$$

Differentiating $\theta=\tan^{-1}(y/x)$, w.r.t x & y partially.

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{1}{r}\sin\theta$$

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)=-\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{1}{r}\cos\theta$$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\frac{\partial z}{\partial r}\cos\theta-\frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}$$

$$\& \; q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial z}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}=\frac{\partial z}{\partial r}\cdot\sin\theta+\frac{\cos\theta}{r}\frac{\partial z}{\partial\theta}$$

putting in the given equation, it reduces to

$$r^2\left\{\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2\right\}=1 \Rightarrow \left(r\frac{\partial z}{\partial r}\right)^2+\left(\frac{\partial z}{\partial\theta}\right)^2=1 \quad \text{--- (1)}$$

Again putting $\left(\frac{1}{r}\right)dr=dR$, so that $R=\log r$, (1) reduces to

$$\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{\partial\theta}\right)^2=1 \Rightarrow P^2+Q^2=1 \quad \text{--- (2)}$$

where $P=\frac{\partial z}{\partial R}$ and $Q=\frac{\partial z}{\partial\theta}$

Equation (2) is of the form $f(P,Q)=0$

$\therefore$ its solution can be taken as

$$z = aR + b\theta + c \quad \text{——} \quad (3)$$

$\therefore\ P = \frac{\partial z}{\partial R} = a\ ,\ Q = \frac{\partial z}{\partial \theta} = b$

Putting in (2), we get $a^2 + b^2 = 1 \Rightarrow b = \sqrt{(1-a^2)}$

Putting in (3), the complete integral of (2) is

$$z = aR + \sqrt{(1-a^2)}\theta + c$$

$$\Rightarrow z = a\log r + \sqrt{(1-a^2)}\tan^{-1}(y/x) + c$$

$$\Rightarrow z = \tfrac{1}{2}a\log(x^2+y^2) + \sqrt{(1-a^2)}\tan^{-1}(y/x) + c$$

which contains two arbitrary constants a and c and is the complete integral of the given equation.

2.(b) Obtain the Simpson's rule for the integral $I=\int_a^b f(x)\,dx$ and show that this rule is exact for polynomials of degree $n\leq 3$. In general show that the error of approximation for Simpson's rule is given by $R=\frac{-(b-a)^5}{2880}f^{iv}(\eta)$, $\eta\in(0,2)$. Apply this rule to the integral $\int_0^1 \frac{dx}{1+x}$ and show that $|R|\leq 0.008333$.

Solution:-

Consider the integral

$$I=\int_a^b f(x)\,dx \quad \text{---} \quad (1)$$

where $f(x)$ takes the values.

$f(x_0)=y_0$, $f(x_0+h)=y_1$, $f(x_0+2h)=y_2, \ldots$

$f(x_0+nh)=y_n$, when $x=x_0$, $x=x_0+h$, $x=x_0+2h, \ldots$ $x=x_0+nh$ respectively.

Let the interval $[a,b]$ be divided into n sub-interval of width h, so that

$x_0=a$, $x_1=x_0+h$, $x_2=x_1+h=x_0+2h \ldots\ldots x_n=x_0+nh=b$.

Approximating $f(x)$ by Newton's forward interpolation formula, we can write the integral (1) as.

$$I = \int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx.$$

$$= \int_{x_0}^{x_0+nh}\left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots\right]dx \quad \text{--- (2)}$$

Since $p = \frac{x-x_0}{h} \Rightarrow x = x_0 + ph$

$\Rightarrow \boxed{dx = h\,dp}$

when $x = x_0$, $p = 0$ and

when $x = x_0 + nh$, $p = n$.

Equation (2) can be written as

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0 + \cdots\right]dp$$

$$I = h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{\frac{p^3}{3}-\frac{p^2}{2}}{2!}\Delta^2 y_0 + \cdots\right]_0^n.$$

$$\therefore I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\right] \quad \text{--- (3)}$$

This is known as Newton-cotes quadrature formula. from this general formula, we can obtain Different integration formulae by putting $n = 1, 2, 3 \ldots$ etc.

Simpson's 1/3rd Rule:

putting $n=2$ in the quadrative formula and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola; i.e., a polynomial of second order so that differences of order higher than second vanishes, we get

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx$$

$$= h\left\{2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left[\frac{8}{3} - \frac{4}{2}\right]\Delta^2 y_0\right\}$$

$$= h\left\{2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta^2 y_0\right\}$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \quad \left[\because \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0\right]$$

Similarly,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

$$\vdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]; \ n \text{ being even.}$$

Adding all these integrals, if n is even +ve integer. i.e., the number of ordinates $y_0, y_1, y_2, \ldots y_n$ is odd.

we have;

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

$$= \frac{h}{3}\left[y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})\right]$$

$$\therefore \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\text{sum of first \& last ordinates} + 2(\text{sum of even ordinates}) + 4(\text{sum of odd ordinates})\right]$$

Which is known as simpson's $1/3$ rule.

Now putting $n=3$.
and taking the curve through (x_i, y_i), $i=0,1,2,3$ as a polynomial of third order so that difference above the third vanish, we get

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx.$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right).$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right)..$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right].$$

and so on

Adding all these integrals from x_0 to x_0+nh where "n" is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots\cdots$$

$$= \frac{3h}{8}\Big[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \cdots\cdots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\Big]$$

$$= \frac{3h}{8}\Big[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots\cdots + y_{n-3})\Big]$$

which is known as Simpson's $3/8$-rule.

Alternative: you can check whether it will be applicable for $n \le 3$ or $n > 3$.

for polynomial of degree ①, we have.

$$I = \int_a^b x\,dx = \frac{b^2 - a^2}{2}.$$

using Simpson's rule, we have

$$I = \frac{h}{3}\left[f(x_0) + f(x_2) + 4(x_1)\right]$$

$$x_0 = a,\ x_2 = b,\ x_1 = \frac{a+b}{2} \text{ and } h = \frac{b-a}{2}$$

$$\therefore\ I = \frac{h}{3}\left[a+b+\frac{4(b+a)}{2}\right] = \frac{b-a}{2\times\cancel{3}}\left[\cancel{3}(b+a)\right]$$

$$\boxed{I = \frac{b^2 - a^2}{2}}$$

Similarly, for polynomial of degree 2, we have

$$I = \int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$$

By Simpson rule,

$$I = \frac{h}{3}\left[a^2 + b^2 + 4\left(\frac{a+b}{2}\right)^2\right] = \frac{b^3 - a^3}{3}$$

for, $n = 3$; $\quad I = \int_a^b x^3\,dx = \frac{b^4 - a^4}{4}$

using Simpson's rule,

$$I = \frac{h}{3}\left[a^3 + b^3 + 4\left(\frac{a+b}{2}\right)^3\right]$$

$$I = \frac{b-a}{2\cdot 3}\left[a^3 + b^3 + \frac{a^3 + b^3 + 3ab(a+b)}{2}\right]$$

$$I = \frac{b^4 - a^4}{4}.$$

But, for $n = 4$; $\quad I = \int_a^b x^4\,dx = \frac{b^5 - a^5}{5}$

using Simpson rule ; $I = \frac{b-a}{2\cdot 3}\left[a^4 + b^4 + 4\left(\frac{b+a}{2}\right)^4\right]$

$$\neq \frac{b^5 - a^5}{5}.$$

Also for $n=0$; zero degree polynomial

$$I = \int_a^b 1 \cdot dx = b-a.$$

By simpson's rule $\Rightarrow I = \frac{b-a}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]$

$$I = \frac{b-a}{6}\left[1+1+4\right] = b-a.$$

Hence, we can say that simpson's rule is exact for polynomial of degree $n \leq 3$, not for $n > 3$.

Now, for Error R;

as per simpson's rule, we know that

$$R = -\frac{nh^5}{90} y^{iv}(\bar{x}).$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivative.

$$\boxed{R = -\frac{(b-a)}{180} h^4 y^{iv}(\bar{x})} \quad \left[\because (2n)h = b-a \Rightarrow nh = \frac{b-a}{2}\right]$$

for Simpson's 1/3 rule, $n=2$. and

$$h = \frac{b-a}{n} = \frac{b-a}{2}.$$

$$\therefore R = -\frac{(b-a)}{180}\left(\frac{b-a}{2}\right)^4 y^{iv}(\bar{x})$$

$$\boxed{\therefore R = \frac{-(b-a)^5}{2880} y^{iv}(\bar{x})}$$ Hence, proved.

For the given function ; $f(x) = \frac{1}{1+x}$

in interval $[0, 1]$ for $n = 6$

Let $h = \frac{1-0}{6} = 1/6$

Then.

x	0	1/6	2/6	3/6	4/6	5/6	1
$f(x)$	1	0.8571	0.75	0.6666	0.6	0.5454	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Then by Simpson's 1/3rd rule

$$I = \frac{h}{3}\left[y_0 + y_6 + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$I = \frac{1}{18}\left[1 + 0.5 + 2(0.75 + 0.6) + 4(0.8571 + 0.6666 + 0.5454)\right]$$

$$\boxed{I = \frac{12.4764}{18} = 0.69313}$$

$$\therefore \; I = \int_0^1 \frac{1}{1+x}\,dx = 0.69313$$

$$R = -\frac{(b-a)^5}{2880} \cdot f^{iv}(\bar{x}) = -\frac{(1-0)^5 \cdot 24}{2880}$$

$$\therefore \; R = -0.00833 \quad \Rightarrow \boxed{|R| \leq +0.00833}$$

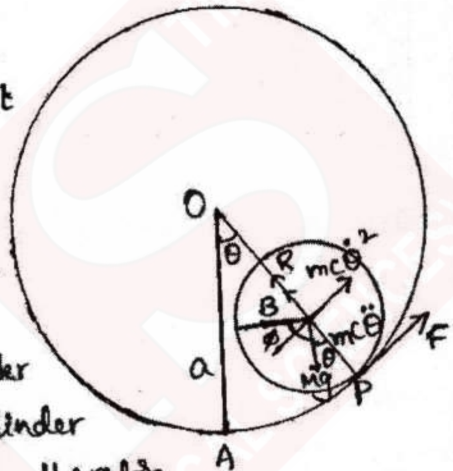
proved.

$\Big[\because f'(x) = \frac{-1}{(1+x)^2}, \; f''(x) = \frac{2}{(1+x)^3}, \; f'''(x) = \frac{-6}{(1+x)^4}$

$f^{iv}(x) = \frac{+24}{(1+x)^5} \; ; \; f^{iv}(0) = 24 \Big]$

2.(C). A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through axes moves in a circular pendulum of length $(b-a)\left(1+\frac{k^2}{a^2}\right)$.

Sol^n:



Let P be the point of contact of the two cylinders at time t such that $\angle AOP = \theta$. Let ϕ the angle which the line CB fixed in moving cylinder make with the vertical at time t.

Here radius of fixed cylinder is a and that of moving cylinder is a. Since there is pure rolling therefore

Arc AP = Arc BP

$\Rightarrow b\theta = a(\phi+\theta)$

$\Rightarrow a\phi = (b-a)\theta$

$\therefore \ddot{\phi} = c\ddot{\theta}$ ——— (1)

where $c = (b-a)$

Let R be the normal reaction and F the friction at the point P.

$\therefore$ The centre C describes a circle of radius $OC = b-a = c$

$\therefore$ Its accelerations along and perpendicular to CO are $c\dot{\theta}^2$ and $c\ddot{\theta}$ respectively

$\therefore$ The equations of motion of the moving cylinder are

$$Mc\dot{\theta}^2 = R - Mg\cos\theta \quad \text{——— (2)}$$

and $$Mc\ddot{\theta} = F - Mg\sin\theta \quad \text{——— (3)}$$

Also for the motion relative to the centre of inertia C,

$Mk^2\ddot{\phi}$ = Moment of the forces about $C = -Fa$ —— (4)

$$Mk^2 \frac{c}{a}\ddot{\theta} = -Fa$$

$$\Rightarrow F = -Mk^2 \frac{c}{a^2}\ddot{\theta}$$

Substituting in (3), we get

$$Mc\ddot{\theta} = -Mk^2\frac{c}{a^2}\ddot{\theta} - Mg\sin\theta$$

$$\Rightarrow c\left(1+\frac{k^2}{a^2}\right)\ddot{\theta} = -g\sin\theta$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{c(1+k^2/a^2)}\theta$$

$$= -\mu\theta$$

$\because$ θ is very small

$\therefore$ Length of the simple equivalent pendulum is

$$g/\mu = c\left(1+\frac{k^2}{a^2}\right) = (b-a)\left(1+k^2/a^2\right).$$

3(a) Find the characteristics of the equation $xp+yq-pq=0$ and then find the equation of the integral surface through the curve $z=x/2, y=0$.

Sol^n: Here $f(x,y,z,p,q)=xp+yq-pq=0$ —— ①

The integral surface passes through the curve $z=x/2$, $y=0$, whose parametric equation can be written as

$$x=f_1(\lambda)=\lambda,\ y=f_2(\lambda)=0,\ z=f_3(\lambda)=\lambda/2$$

λ being parameter

$\therefore$ Initial values for x,y,z are $x=x_0=\lambda$, $y=y_0=0$, $z=z_0=\lambda/2$, when $t=0$

And corresponding initial values p_0 and q_0 of p and q are determined by the relations.

$$f_3'(\lambda)=p_0f_1'(\lambda)+q_0f_2'(\lambda) \Rightarrow \tfrac{1}{2}=p_0\cdot 1+q_0\cdot 0 \Rightarrow p_0=\tfrac{1}{2}$$

and $f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0)=0$

$$\Rightarrow x_0p_0+y_0q_0-p_0q_0=0 \Rightarrow q_0=\lambda$$

The characteristic equations of the given partial differential equation ① are

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-q \quad —— ②$$

$$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-p \quad —— ③$$

$$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-q)+q(y-p)=px+qy-2pq=-pq \quad —— ④$$

using ①

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-p \quad —— ⑤$$

$$\frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-q \quad —— ⑥$$

from ⑤ & ⑥, we get $p = Ae^{-t}$ and $q = Be^{-t}$

But initially when $t=0$, $p = p_0 = \frac{1}{2}$ & $q = q_0 = \lambda$

$\therefore A = p_0 = \frac{1}{2}$ and $B = q_0 = \lambda$

$\Rightarrow p = \frac{1}{2}e^{-t}$ and $q = \lambda e^{-t}$ —— ⑦

Using ⑦, from ②, we have $\frac{dx}{dt} - x = -\lambda e^{-t}$

which is a L.D.E with I.F $= e^{\int -dt} = e^{-t}$

$\therefore xe^{-t} = c_1 + \int (-\lambda e^{-t})(e^{-t})dt = c_1 - \lambda \int e^{-2t}dt$

$= c_1 + \frac{1}{2}\lambda e^{-2t}$

But when $t=0$, $x = x_0 = \lambda$

$\therefore x_0 = \lambda = c_1 + \frac{1}{2}\lambda \Rightarrow c_1 = \lambda/2$

$\therefore xe^{-t} = \lambda/2 + \lambda/2\, e^{-2t}$

$\Rightarrow x = \lambda/2\,(1 + e^{-2t})e^{t}$ —— ⑧

again using ⑦, from ③, we have

$$\frac{dy}{dt} - y = -\frac{1}{2}e^{-t}$$

which is L.D.E with I.F $= e^{\int -dt} = e^{-t}$,

$\therefore ye^{-t} = c_2 + \int \left(-\frac{1}{2}e^{-t}\right)e^{-t}dt = c_2 - \frac{1}{2}\int e^{-2t}dt$

$= c_2 + \frac{1}{4}e^{-2t}$

But when $t=0$, $y = y_0 = 0$,

$\therefore y_0 = 0 = c_2 + \frac{1}{4} \Rightarrow c_2 = -\frac{1}{4}$

$\therefore ye^{-t} = -\frac{1}{4} + \frac{1}{4}e^{-2t} \Rightarrow y = \frac{1}{4}(e^{-2t} - 1)e^{t}$ —— ⑨

Now using ⑦, from ④, we have

$$\frac{dz}{dt} = -\lambda/2\, e^{-2t}$$

Integrating $z = \lambda/4\, e^{-2t} + c_3$

But when $t=0$, $z=z_0=\lambda/2$,

$$\therefore \; z_0 = \lambda/2 = \lambda/4 + C_3 \Rightarrow C_3 = \lambda/4$$

$$\therefore \; z = \lambda/4\, e^{-2t} + \lambda/4 \Rightarrow z = \lambda/4\,(e^{-2t}+1) \quad \text{---} \; (10)$$

Thus, the characteristic strips of the given equation are given by

$$x = \lambda/2\,(1+e^{-2t})e^{t}, \quad y = 1/4\,(e^{-2t}-1)e^{t} \text{ and}$$

$$z = \lambda/4\,(e^{-2t}+1)$$

where λ and t are two parameters.

The required integral surface is obtained by eliminating λ and t between x, y and z.

We have $\quad x/z = 2e^{t} \Rightarrow e^{t} = \frac{x}{2z}$

$$\therefore \; y = 1/4\,(e^{-t}-e^{t})$$

$$= 1/4\left(\frac{2z}{x} - \frac{x}{2z}\right) = \frac{4z^2-x^2}{8xz}$$

$$\therefore \; 4z^2 = x^2 + 8xyz$$

which is the required integral surface.

3.(b)(i) → Apply Lagrange's interpolation formula to find $f(5)$ and $f(6)$; given that $f(1)=2$, $f(2)=4$, $f(3)=8$, $f(7)=128$.

Solution:- From above, we can draw a table.

x	1^{x_0}	2^{x_1}	3^{x_2}	7^{x_3}
$f(x)$	2 $f(x_0)$	4 $f(x_1)$	8 $f(x_2)$	128 $f(x_3)$

To find $f(5)$ and $f(6)$.

Lagranges's formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot f(x_3)$$

$$\therefore f(5) = \frac{(5-2)(5-3)(5-7)}{(1-2)(1-3)(1-7)} \times 2 + \frac{(5-1)(5-3)(5-7)}{(2-1)(2-3)(2-7)} \times 4$$

$$+ \frac{(5-1)(5-2)(5-7)}{(3-1)(3-2)(3-7)} \times 8 + \frac{(5-1)(5-2)(5-3)}{(7-1)(7-2)(7-3)} \times 128$$

$$f(5) = \frac{\cancel{3} \times \cancel{2} \times -\cancel{2}}{-\cancel{1} \times -\cancel{2} \times -\cancel{6}} \times 2 + \frac{4 \times 2 \times -2}{1 \times -1 \times -5} \times 4$$

$$+ \frac{\cancel{4} \times 3 \times -\cancel{2}}{\cancel{2} \times 1 \times -\cancel{4}} \times 8 + \frac{\cancel{4} \times \cancel{3} \times \cancel{2}}{\cancel{6} \times 5 \times \cancel{4}} \times 128$$

$$f(5) = 2 - \frac{64}{5} + 24 + \frac{128}{5} = 26 + \frac{64}{5} = \frac{130}{5} + \frac{64}{5}$$

$$\boxed{f(5) = \frac{194}{5}}$$ required value.

Now; for f(6).

$$\therefore f(6) = \frac{(6-2)(6-3)(6-7)}{(1-2)(1-3)(1-7)} \times 2 + \frac{(6-1)(6-3)(6-7)}{(2-1)(2-3)(2-7)} \times 4$$

$$\frac{(6-1)(6-2)(6-7)}{(3-1)(3-2)(3-7)} \times 8 + \frac{(6-1)(6-2)(6-3)}{(7-1)(7-2)(7-3)} \times 128$$

$$\therefore f(6) = \frac{\cancel{4} \times \cancel{3} \times \cancel{-1}}{\cancel{-1} \times \cancel{-2} \times \cancel{-6}} \times 2 + \frac{\cancel{5} \times 3 \times -1}{1 \times \cancel{-1} \times \cancel{-5}} \times 4$$

$$+ \frac{5 \times \cancel{4} \times \cancel{-1}}{\cancel{2} \times 1 \times \cancel{-4}} \times \cancel{8}\,4 + \frac{\cancel{5} \times \cancel{4} \times \cancel{3}}{\cancel{6}_{2} \times \cancel{5} \times \cancel{4}} \times \cancel{128}\,64$$

$$f(6) = 2 - 12 + 20 + 64$$

$$f(6) = 86 - 12$$

$$\boxed{f(6) = 74}$$

$$\therefore \boxed{f(5) = \frac{194}{5} \text{ and } f(6) = 74}$$

Required solution.

3.(b)(ii) Using Newton's forward formula find the number of men getting wages between Rs.10 and Rs.15 from the following data:

Wages in Rs. :	0–10	10–20	20–30	30–40
Frequency :	9	30	35	42

Solution:

The above wage-Frequency table can be redrawn as

Wages less than	10	20	30	40
Frequency	9	39	74	116

Now, make the difference table:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

Here ; $x_0 = 10$, $x = 15$, $h = 10$

$$p = \frac{x - x_0}{h} = \frac{15-10}{10} = \frac{5}{10} = 0.5$$

By Newton's forward formula.

$$y_{15} = y_{10} + p\Delta y_{10} + \frac{p(p-1)}{2!}\Delta^2 y_{10} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{10}$$

$$y_{15} = 9 + 0.5 \times 30 + \frac{0.5(-0.5)}{2} \times 5 + \frac{0.5(-0.5)(-1.5)\cancel{2}}{3 \times \cancel{2}}$$

$$y_{15} = 9 + 15 + (-0.625) + 0.125$$

$$y_{15} = 24 - 0.5 = \underline{23.5}$$

The number of men getting wages less than $15 = 23.5$
$< 15 = 24$ i.e

But, the no. of men with wages less than $10 = 9$

Hence,

$$\boxed{\begin{gathered}\text{Number of men with wages b/w Rs. 10 to Rs. 15} \\ = 24 - 9 = 15\end{gathered}}$$

which is required result.

3.(C) If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right), \quad r^2 = x^2+y^2+z^2,$$

then prove that the liquid motion is possible and that the velocity potential is $\frac{z}{r^3}$.

Further, determine the streamlines.

Solution:

Given $u = \frac{3xz}{r^5}$, $v = \frac{3yz}{r^5}$, $w = \frac{3z^2-r^2}{r^5}$

Since $r^2 = x^2+y^2+z^2$ hence $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Step I: To prove that liquid motion is possible.

i.e. To prove that the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ is satisfied.}$$

$$\frac{\partial u}{\partial x} = \frac{3z}{r^{10}}(r^5 - 5r^3x^2),$$

$$\frac{\partial v}{\partial y} = \frac{3z}{r^{10}}(r^5 - 5r^3y^2),$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}\left[(6z-2z)r^5 - 5r^3(3z^2-r^2)z\right]$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3z}{r^{10}}\left[2r^5 - 5r^3(r^2-z^2)\right] + \frac{1}{r^{10}}(9zr^5 - 15r^3z^3) = 0$$

Hence, the result.

Step II: To show that $\phi = \frac{z}{r^3}$.

$$\partial\phi = \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$$

$$= -u\,dx - v\,dy - w\,dz$$

$$= -\frac{1}{r^5}\left[3xz\,dx + 3yz\,dy + (3z^2 - r^2)\,dz\right]$$

$$= -\frac{1}{r^5}\left[3z\,(x\,dx + y\,dy + z\,dz) - r^2\,dz\right]$$

$$= -\frac{1}{r^5}\left[3z\,d\left(\frac{r^2}{2}\right) - r^2\,dz\right]$$

$$= -\frac{3z}{r^4}\,dr + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right)$$

Integrating,

$$\boxed{\phi = \frac{z}{r^3}}$$ (neglecting the constant of integration).

Hence, the result.

* Alternatively,

$$\frac{\partial\phi}{\partial x} = -u = -\frac{3xz}{r^5}$$

Integrating w.r.t. 'x', we have,

$$\phi = -\frac{3z}{2}\int (2x)\,(x^2+y^2+z^2)^{-5/2}\,dx$$

$$= \left(-\frac{3z}{2}\right)\left(-\frac{2}{3}\right)(x^2+y^2+z^2)^{-3/2}$$

$$\phi = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

i.e. $$\boxed{\phi = \frac{z}{r^3}}$$

Hence, the result.

Step III :

Streamlines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Putting the values of respective terms,

$$\underset{(1)}{\frac{dx}{3xz}} = \underset{(2)}{\frac{dy}{3yz}} = \underset{(3)}{\frac{dz}{3z^2-r^2}} = \underset{(4)}{\frac{x\,dx+y\,dy+z\,dz}{3z(x^2+y^2+z^2)-r^2z}}$$

Taking the ratios (1) and (2), $\frac{dx}{x} = \frac{dy}{y}$

Integration yields the result,

$$\log x = \log y + \log a$$

$$\Rightarrow \boxed{x = ay} \quad \text{—— (5)}$$

By the ratios (1) and (4),

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2r^2}$$

$$\Rightarrow \frac{4\,dx}{x} = 3\left(\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}\right)$$

Integrating, $4\log x = 3\log(x^2+y^2+z^2) + \log b$

$$\Rightarrow \boxed{x^4 = b(x^2+y^2+z^2)^3} \quad \text{—— (6)}$$

The equations (5) and (6) represent stream lines.

Hence, the result.

4.(a) The temperature at one end of a bar, 50cm long with insulated sides, is kept at 0°c and that the other end is kept at 100°c until steady state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

Sol^n: Let the rod lie along the axis of x with one end which is kept at 0°c be at the origin. The temperature distribution $u(x,t)$ in the bar at any time t at any distance x is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{———} \quad (1)$$

Initially $u=0$ at $x=0$ and $u=100°c$ at $x=50$.

In the steady state, u is independent of t.

$$\therefore \frac{\partial u}{\partial t} = 0$$

$\therefore$ from (1), we have $\frac{\partial^2 u}{\partial x^2} = 0$

$$\Rightarrow u = Ax + B$$

Since at $x=0, u=0$ and $x=50$, $u=100°c$

$\therefore 0 = 0 + B$ and $100 = 50A + B \Rightarrow B=0$ and $A=2$.

i.e, initial temperature distribution in the bar is given by

$u=2x$ i.e. $u(x,0)=2x$.

Now let $u(x,t)$ represent the temperature distribution in the rod at time t measured from the instant the two ends of the bar are suddenly insulated after the rod reaches to steady state.

Thus, we are required to solve (1) under the following conditions.

B.C. $u_x(0,t)=0$ and $u_x(50,t)=0$

I.C $u(x,0)=2x$.

we can prove that the solution of the heat equation

$$K\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \quad \text{---} (1)$$

subject to boundary conditions are

$$u_x(0,t)=u_x(50,t)=0 \quad \forall\, t\geq 0 \quad \text{---} (2)$$

and the initial condition $u(x,0)=2x,\ 0<x\leq 50$ (3)

Suppose (1) has solution of the form

$$u(x,t)=X(x)T(t) \quad \text{---} (A)$$

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$KX''T=XT' \Rightarrow \frac{X''}{X}=\frac{T'}{KT}$$

Since x and t are independent variables.

$$X''-\mu X=0 \quad \text{---} (4) \quad \text{and} \quad T'-\mu KT=0 \quad \text{---} (5)$$

Differentiating (A) partially w.r.t x, we get

$$u_x(x,t)=X'(x)T(t) \quad \text{---} (6)$$

using (2), (6) gives $X'(0)T(t)=0$ & $X'(50)T(t)=0$ --- (7)

Since $T(t)=0$ leads to $u=0$ so assume that $T(t)\neq 0$.

Hence from (7), we get

$$X'(0)=0 \text{ and } X'(50)=0 \quad \text{---} (8)$$

Three cases arise:

Case 1: Let $\mu=0$. Then solution of (4) is

$X(x)=Ax+B$ --- (9) which yields $X'(x)=A$ --- (10)

using B.C (8), (10) gives $A=0$.

Then (9) reduces to $X(x)=B$.

Again, corresponding to $\mu=0$, (5) yields

$$\frac{dT}{dt}=0 \Rightarrow T=\text{constant}=\frac{E_0}{2B} \text{ (say)}$$

$\therefore$ corresponding to $\mu=0$ a solution of the given boundary value problem from (A) is given by $u(x,t) = B\times\frac{E_0}{2B} = \frac{E_0}{2}$ —— (11)

Case(2): Let $\mu=\lambda^2$, $\lambda\neq 0$. The solution of (4) is

$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

which yields $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ —— (12)

using B.C. (8), (12) gives $A=B=0$ so that $X(x)=0$ and hence $u\equiv 0$. which does not satisfy (3). so reject $\mu=\lambda^2$.

Case 3: let $\mu=-\lambda^2$, $\lambda\neq 0$. Then the solution of (4) is

$X(x) = A\cos\lambda x + B\sin\lambda x$.

which yields $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ —— (13)

using B.C. (8); (13) gives $0 = B\lambda$

and $0 = -A\lambda\sin\lambda(50) + B\lambda\cos\lambda(50)$

$\Rightarrow B=0$ and $A\lambda\sin(\lambda 50)=0$

$\Rightarrow \sin(\lambda 50)=0$ ($\because A\neq 0$).

$\Rightarrow \lambda(50) = n\pi \Rightarrow \lambda = \frac{n\pi}{50}$, $n=1,2,3,\ldots$ —— (14)

Hence non-zero solutions $X_n(x)$ of (4) are given by

$X_n(x) = A_n\cos\left(\frac{n\pi x}{50}\right)$ —— (15)

using (14), (5) reduces to $\frac{dT}{T} = -\frac{n^2\pi^2 k}{(50)^2}dt$ $\left(\because \mu=-\lambda^2=\frac{n^2\pi^2}{(50)^2}\right)$

$\Rightarrow \frac{1}{T}dT = -C_n^2\,dt$ where $C_n^2 = \frac{n^2\pi^2 k}{(50)^2}$ —— (16)

solving (16), $T_n(t) = D_n e^{-C_n^2 t}$ —— (17)

from (15) & (17) $u_n(x,t) = X_n(t)\,T_n(t)$

$= E_n\cos\left(\frac{n\pi x}{50}\right)e^{-C_n^2 t}$ —— (18)

are solutions of (1). For $n=1,2,3\ldots$ each one of these satisfy the boundary conditions (2).

Here $E_n = A_n D_n$

thus (11) and (18) constitute a set of infinite solutions of (1). To obtain a solution also satisfy the initial condition (3), we consider a linear combination of these solutions. Hence complete solution of ① may be taken in the following form.

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{50}\right) e^{-c_n^2 t} \quad \text{—— (19)}$$

which is Fourier Cosine Series.

substituting $t=0$ in (19), and using ③,

we have $2x = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{50}\right)$

where $E_0 = \frac{2}{50}\int_0^{50} 2x\,dx = \frac{2}{50}\left[x^2\right]_0^{50} = \frac{2}{50}[2500] = 100$

and $E_n = \frac{2}{50}\int_0^{50} (2x)\cos\frac{n\pi x}{50}\,dx$ —— (20)

$$= \frac{1}{25}\int_0^{50} 2x\cos\left(\frac{n\pi x}{50}\right)dx$$

$$= \frac{1}{25}\left[2x\,\frac{50}{n\pi}\sin\left(\frac{n\pi x}{50}\right) - 2\left(\frac{50}{n\pi}\right)^2\cos\left(\frac{n\pi x}{50}\right)\right]_0^{50}$$

$$= \frac{2}{25}\left(\frac{50}{n\pi}\right)^2(\cos n\pi - 1)$$

$$= \frac{200}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$E_n = \begin{cases} 0 & \text{if } n = 2m \text{ (even)} \\ \frac{-400}{n^2\pi^2} & \text{when } n = 2m-1 \text{ (odd)} \end{cases}$$

i.e, $E_{2m} = 0$ and $E_{2m-1} = \frac{-400}{(2m-1)^2\pi^2}$ —— (21)

Now using (20) and (21) the required solution from (19) is given by

$$u(x,t) = \frac{100}{2} + \sum_{m=1}^{\infty} \frac{-400}{(2m-1)^2\pi^2}\cos\frac{(2m-1)\pi x}{50}\, e^{-c_{2m-1}^2 t}$$

$$= 50 - \frac{400}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{50}\, e^{-\frac{(2m-1)^2\pi^2 k t}{2500}}$$

4.(b) Provide a computer Algorithm to solve an ordinary differential equation $\frac{dy}{dx} = f(x,y)$ in the interval $[a,b]$ for n number of discrete points, where the initial value is $y(a) = \alpha$. using Euler's method?

Solution:-

Given; $\frac{dy}{dx} = f(x,y)$

have interval $[a,b]$.

$h = b-a$

$y(a) = \alpha$ i.e y_a] initial values condition.

$\therefore$ x_a

x_b is the required value.

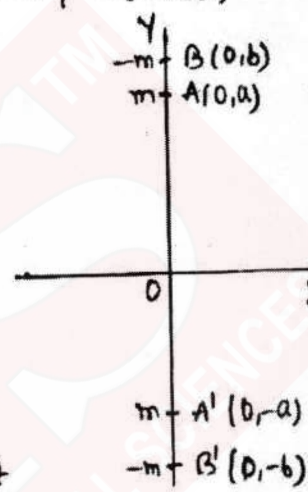
n – number of discrete points

where $n = \frac{x_b - x_a}{h} + 1$

Hence, now the Algorithm for the solution of above ordinary differential equation using Euler's method:

1. Start
2. Define function [i.e $\frac{dy}{dx} = f(x,y)$]
3. Get the values of x_a, y_a, h and x_b
4. $n = \frac{(x_b - x_a)}{h} + 1$.
5. Start loop from $i=1$ to n
6. $y = y_a + h * f(x_a, y_a)$
   $x = x + h$
7. Print values of y_a and x_a
8. Check if $x < x_b$
   If yes, assign $x_a = x$ and $y_a = y$
   If no, go to 9.
9. End loop i
10. Stop.

4.(c) If the fluid fills the region of space on the +ve side of x-axis, is a rigid boundary, and if there be a source $+m$ at the point $(0,a)$, and an equal sink at $(0,b)$, and if the pressure on the -ve side of the boundary be the same as the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m^2(a-b)^2/ab(a+b)$, where ρ is the density of the fluid.



Sol'n: The object system consists of source $+m$ at $A(0,a)$, i.e. at $z=ia$ and sink $-m$ at $z=ib$. The image system consists of source $+m$ at $A'(z=-ia)$ and sink $-m$ at $B'(z=-ib)$ w.r.t the +ve line OX which is rigid boundary. The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary

$$\therefore\ w=-m\log(z-ia)+m\log(z-ib)$$
$$-m\log(z+ia)+m\log(z+ib)$$
$$\Rightarrow w=-m\log(z^2+a^2)+m\log(z^2+b^2)$$
$$\frac{dw}{dz}=-2mz\left[\frac{1}{z^2+a^2}-\frac{1}{z^2+b^2}\right]=\frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$
$$\therefore\ q=\left|\frac{dw}{dz}\right|=\frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

For any point on x-axis, we have $z=x$ so that

$$q=\frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x=\infty$. By Bernoulli's equation for steady motion

$$\therefore\ \frac{p}{\rho}+\frac{1}{2}q^2=C$$

In view of $p = p_0$, $q = 0$ when $x = \infty$, we get $c = p_0/\rho$

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundary is given by

$$P = \int_{-\infty}^{\infty} (p_0 - p)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\rho q^2 dx$$

$$= \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx$$

$$= 4\rho m^2 (a^2-b^2)^2 \int_0^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

$$= 4m^2\rho \int_0^{\infty} \left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right\} - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right] dx$$

$$= 4m^2\rho \left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b} - \frac{\pi}{2a}\right\} - \frac{\pi}{4a} - \frac{\pi}{4b}\right]$$

$$= \frac{\pi \rho m^2 (a-b)^2}{ab(a+b)}$$

5.(a) Find a complete integral of $p^2+q^2-2px-2qy+2xy=0$

Sol'n: Given equation is

$$f(x,y,z,p,q) \equiv p^2+q^2-2px-2qy+2xy=0 \quad \text{——} (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}$$

which gives $\dfrac{dp+dq}{2(x+y-p-q)} = \dfrac{dx+dy}{2(x+y-p-q)}$

$\Rightarrow dp+dq = dx+dy$ i.e, $dp-dx+dq-dy=0$

Integrating $\quad (p-x)+(q-y)=a$ —— (2).

Rewriting (1) $\quad (p-x)^2+(q-y)^2=(x-y)^2$ —— (3)

putting the value of $(q-y)$ from (2) in (3), we get

$(p-x)^2+[a-(p-x)]^2=(x-y)^2$

$\Rightarrow 2(p-x)^2-2a(p-x)+\{a^2-(x-y)^2\}=0$

$\therefore \; p-x = \dfrac{2a \pm \sqrt{[4a^2-4.2.\{a^2-(x-y)^2\}]}}{4}$

$\Rightarrow p = x+\frac{1}{2}[a \pm \sqrt{\{2(x-y)^2-a^2\}}]$

$\therefore$ (2) gives $\quad q=a+y-p+x$

$\Rightarrow q = y+\frac{1}{2}[a \pm \sqrt{\{2(x-y)^2-a^2\}}]$

Putting these values of p and q in $dz=pdx+qdy$, we get

$dz = x\,dx+y\,dy+\frac{a}{2}(dx+dy) \pm \frac{1}{2}\sqrt{\{2(x-y)^2-a^2\}}\,(dx-dy)$

$\Rightarrow dz = x\,dx+y\,dy+\frac{a}{2}(dx+dy) \pm \frac{1}{\sqrt{2}}\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}(dx-dy)$

Integrating the desired Complete integral is

$$z = \frac{x^2}{2}+\frac{y^2}{2}+\frac{a}{2}(x+y) \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}-\frac{a^2}{4}\log\left[(x-y)+\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}\right]\right)$$

$$\Rightarrow 2z = x^2+y^2+ax+ay \pm \frac{1}{\sqrt{2}}\left((x-y)\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}-\frac{a^2}{2}\log\left[(x-y)+\sqrt{\left\{(x-y)^2-\frac{a^2}{2}\right\}}\right]\right).$$

5.(b) Solve $r+s-6t = y\cos x$.

Solution:- Given equation can be re-written as –

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$$

$$(D^2+DD'-6D'^2)z = y\cos x \quad \text{——} \quad (1)$$

Its auxillary equation $\Rightarrow$ $m^2+m-6=0$

$$\boxed{m=2,-3}$$

$$\boxed{\therefore C.F = \phi_1(y+2x)+\phi_2(y-3x)}$$

ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{D^2+DD'-6D'^2}y\cos x = \frac{1}{(D-2D')(D+3D')}y\cos x$$

$$= \frac{1}{D-2D'}\int(3x+c)\cos x\,dx \quad ; \text{ where } c=y-3x$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\,dx\right]$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right] \text{ as } c=y-3x$$

$$= \int\left[(c'-2x)\sin x + 3\cos x\right]dx \quad ; \quad c'=y+2x$$

$$= (c'-2x)(-\cos x) - \int(-2)(-\cos x)dx + 3\sin x.$$

$$= y(-\cos x) - 2\sin x + 3\sin x \quad [\because c'=y+2x)$$

$$\boxed{P.I = \sin x - y\cos x}$$

General solution $= C.F + PI$

$$\boxed{z = \phi_1(y+2x)+\phi_2(y-3x)+\sin x - y\cos x}$$

5.(c) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t(sec)	0	10	20	30	40	50	60	70	80
f(cm/sec²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67.

Sol'n: Since acceleration is defined as the rate of change of velocity, we have

$$\frac{dv}{dt} = a \quad (\text{or}) \quad v = \int_0^{80} a\,dt$$

using Simpson's $\frac{1}{3}$rd rule.

we have

$$v = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$= \frac{10}{3}\left[(30+50.67)+4(31.63+35.47+40.33+46.69) + 2(33.34+37.75+43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

$\therefore$ The required velocity is given by

$$v = 3.0861 \text{ km/sec}$$

5.(d) (i) Simplify the expression $A = xy + \bar{x}\bar{z} + x\bar{y}z(xy+z)$

(ii) Simplify the Boolean expression $Y = \overline{A\cdot B} + \overline{\bar{A}+B}$

Prepare truth table to show that the simplified expression is correct.

Sol^n: (i) $A = xy + \overline{xz} + x\bar{y}z(xy+z)$

$= xy + \overline{xz} + xxy\bar{y}z + x\bar{y}zz$

$= xy + \overline{xz} + x\bar{y}z$ $\quad (\because zz = z$ and $y\bar{y} = 0)$

$= xy + \bar{x} + \bar{z} + x\bar{y}z$

$= xy + \bar{z} + \bar{x} + x\bar{y}z$ (by commutative law)

$= xy + \bar{z} + \bar{x} + \bar{y}z$

$= \bar{x} + y + \bar{z} + \bar{y}$

$= 1$

(ii) $Y = \overline{A\cdot B} + \overline{\bar{A}+B}$

$= \bar{A} + \bar{B} + \bar{\bar{A}}\cdot\bar{B}$

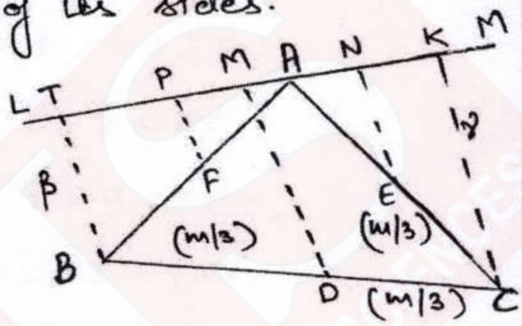
$= \bar{A} + \bar{B}$

$= \overline{A\cdot B}$ (using Demorgan's theorem)

A	B	A·B	$\overline{A\cdot B}$	$\bar{A}$	$\bar{A}+B$	$\overline{\bar{A}+B}$	$\overline{A\cdot B}+\overline{\bar{A}+B}$
0	0	0	1	1	1	0	1
0	1	0	1	1	1	0	1
1	0	0	1	0	0	1	1
1	1	1	0	0	1	0	0

5.(e) Prove that the moment of inertia of a triangular lamina ABC about any axis through A in its plane is $\frac{M}{6}(\beta^2+\beta\gamma+\gamma^2)$ where M is the mass of the lamina and β, γ are respectively the length of perpendicular from B and C on the axis.

Sol^n Let m be the mass of the triangle ABC, then the triangle is equimomental to the three particles each of mass m/3 placed at the middle points D, E, F of its sides.



Let LM be any line through the vertex A and in the plane of the triangle ABC.
Let β and γ be the distance of the vertex B and C from the lines LM i.e

$BT = \beta$ and $CK = \gamma$

$\therefore$ Perpendicular distances of D, E, F from LM are as follows.

$DM = \frac{1}{2}(\beta+\gamma)$, $EN + \frac{1}{2}CK = \frac{1}{2}\gamma$ and

$FP = \frac{1}{2}BT = \frac{1}{2}\beta$

$\therefore$ M.I of the triangle ABC about L.M

= Sum of M.I of the masses m/3 each at D, E, F about LM.

$= \frac{m}{3}.DM^2 + \frac{m}{3}.EN^2 + \frac{m}{3}FP^2$

$= \frac{m}{3}\left[\frac{1}{4}(\beta+\gamma)^2 + \frac{1}{4}\gamma^2 + \frac{1}{4}\beta^2\right]$

$= \frac{m}{6}(\beta^2+\gamma^2+\beta\gamma)$

6.(a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x+y+z, x^2+y^2-z^2)=0$ what is the order of this partial differential equation?

Sol'n: Let $u = x+y+z$, $v = x^2+y^2-z^2$

$$\therefore \phi(u,v) = 0 \quad \text{— (1)}$$

Differentiating (1) partially w.r.t x

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}(1+0+p) + \frac{\partial\phi}{\partial v}(2x+0+(-2z)p) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{-(2x-2zp)}{1+p} \quad \text{— (2)}$$

Differentiating (1) partially w.r.t y

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}(0+1+q) + \frac{\partial\phi}{\partial v}(0+2y+(-2z)q) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{-(2y-2zq)}{1+q} \quad \text{— (3)}$$

from (2) & (3) we have

$$\frac{2x-2zp}{1+p} = \frac{2y-2zq}{1+q}$$

$$\Rightarrow x - zp + xq - \cancel{zpq} = y - zq + yp - \cancel{zpq}$$

$$\Rightarrow x - y = p(z+y) + q(-z-x)$$

$$\Rightarrow p(z+y) - q(x+z) = x - y$$

which is the required partial differential equation is of order 1.

6.(b) Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Sol^n: Here Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{——} (1)$$

Choosing 1,−1,0 and 0,1,−1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

so that $\dfrac{dx-dy}{(x-y)(x+y+z)}=\dfrac{dy-dz}{(y-z)(y+z+x)} \Rightarrow \dfrac{d(x-y)}{x-y}-\dfrac{d(y-z)}{y-z}=0$

Integrating, $\log(x-y)-\log(y-z)=\log c_2$

$$\Rightarrow (x-y)/(y-z)=c_2 \quad \text{——} (2)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{——} (3)$$

Again, choosing 1,1,1 as multipliers, each fraction of (1)

$$=\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{——} (4)$$

from (3) & (4), $\dfrac{xdx+ydy+zdz}{x+y+z}=dx+dy+dz$

$$\Rightarrow 2(x+y+z)\,d(x+y+z)-(2xdx+2ydy+2zdz)=0$$

Integrating, $(x+y+z)^2-(x^2+y^2+z^2)=2c_2$

$$\Rightarrow (x^2+y^2+z^2+2xy+2yz+2zx)-(x^2+y^2+z^2)=2c_2$$

$$\Rightarrow xy+yz+zx=c_2 \quad \text{——} (5)$$

from (2) & (5), the required general solution is given by
$\phi\,[xy+yz+zx,\ (x-y)/(y-z)]=0$, ϕ being an arbitrary function.

6(C) → Reduce $x^2r+2xys+y^2t=0$ to canonical form and hence solve

Sol'n: Given $x^2r+2xys+y^2t=0$ —— ①

comparing ① with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=x^2$, $S=2xy$ and $T=y^2$ so that

$S^2-4RT=0$ showing that ① is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$

reduces to $x^2\lambda^2+2xy\lambda+y^2=0$

$\Rightarrow (x\lambda+y)^2=0$

$\Rightarrow \lambda=-y/x, -y/x.$

The corresponding characteristic equation is

$\frac{dy}{dx}-\frac{y}{x}=0 \Rightarrow \frac{1}{y}dy-\frac{dx}{x}=0$

$\Rightarrow \log y-\log x=\log c$

$\Rightarrow y/x=c.$

Choose $u=y/x$ and $v=y$ —— ②

where we have chosen $v=y$ in such a manner that u and v are independent functions.

Now $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=-\frac{y}{x^2}\frac{\partial z}{\partial u}$ —— ③

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$ —— ④

$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u}\right)$

$=\frac{2y}{x^3}\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$

$=\frac{2y}{x^3}\frac{\partial z}{\partial u}-\frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]$

$=\frac{2y}{x^3}\frac{\partial z}{\partial u}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2}$ —— ⑤

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{-1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial z}{\partial v} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} - \frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 z}{\partial u \partial v} \quad \text{—— (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$+ \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{2}{x}\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{—— (7)}$$

Using (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \text{which is the required Canonical form} \quad \text{—— (8)}$$

Integrating (8) partially w.r.t v, $\frac{\partial z}{\partial v} = \phi(u)$ —— (9)

Integrating (9) partially w.r.t v, $z = v\phi(u) + \psi(u)$

$\therefore z = y\,\phi(y/x) + \psi(y/x)$, ϕ, ψ being arbitrary functions

6(d) A tightly stretched elastic string of length l, with fixed end points $x=0$ and $x=l$ is initially in the position given by $y = c\sin^3(\pi x/l)$, c being constant. It is released from the position of rest. Find the displacement $y(x,t)$.

Sol: The partial differential equation of the transverse vibrations of the given elastic string is given by

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{1}{a^2}\right)\left(\frac{\partial^2 y}{\partial t^2}\right) \rightarrow (1)$$

where $y(x,t)$ is the deflection of the string and a is a constant. Given boundary and initial conditions are:

Boundary conditions (B.C): $y(0,t) = y(l,t) = 0$ for all $t \rightarrow (2)$

Initial conditions (I.C): $y(x,0) = c\sin^3(\pi x/l)$

(Initial deflection) $\rightarrow 3(a)$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = y_t(x,0) = 0 \text{ (initial velocity)} \rightarrow 3(b)$$

Let solution of (1) be of the form $y(x,t) = X(x)T(t)$ $\rightarrow (4)$

Substituting this value of y in (1) we have

$$X''T = \frac{1}{a^2} X T'' \quad (\text{or}) \quad \frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} \rightarrow (5)$$

Since x and t are independent (5) can only be true if each side is equal to the same constant, say μ

Then (5) gives $\rightarrow (6)$

$$X'' - \mu X = 0 \longrightarrow (6)$$

and

$$T'' - \mu a^2 T = 0 \longrightarrow (7)$$

Using (2) and (4) gives $X(0)T(t) = 0 \longrightarrow (8)$

Since $T(t)=0$ leads to $y=0$, hence we assume that $T(t)\neq 0$

Then (8) gives $X(0)=0$ and $X(l)=0 \longrightarrow (9)$

we now solve (6) under B.C (9). Three cases arise:

case(i): let $\mu=0$ then solution of (6) is given by

$$X(x) = Ax+B \longrightarrow (10)$$

using B.C (9), (10) gives $0=B$ and $0=Al+B$.

These give $A=B=0$ so that $X(x)=0$ with the help of (4) this leads to $y\equiv 0$ which does not satisfy 3(a) and 3(b). So we reject $\mu=0$

case(ii): let $\mu=\lambda^2$, where $\lambda\neq 0$ then solution of (6) is

$$X(x) = Ae^{\lambda x}+Be^{-\lambda x} \longrightarrow (11)$$

using (B.C) (9) (11) given $A+B=0$ and $Ae^{\lambda l}+Be^{-\lambda l}=0$ $\longrightarrow (12)$

Solving (12), $A=B=0$ and so $X(x)=0$ and

$A\cos\lambda l + B\sin\lambda l = 0$ so that $A=0$ and $\sin\lambda l=0$

where we have taken $B\neq 0$, since otherwise.

$X\equiv 0$ so that $y=0$ which does not satisfy 3(a) and 3(b).

Now $\sin\lambda l=0 \Rightarrow \lambda l = n\pi \Rightarrow \lambda = n\pi/l$, $n=1,2,3,\dots \to (14)$

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) \longrightarrow (15)$$

using (14), (7) reduces to

$$\left(d^2T/dt^2\right) + \left(n^2\pi^2a^2/l^2\right)T = 0 \quad \left[\because \mu = -\lambda^2 = -(n^2\pi^2/l^2)\right]$$

whose general solution is $T_n(t) = C_n\cos\frac{n\pi at}{l} + D_n\sin\frac{n\pi at}{l} \to (16)$

$\therefore y_n(x,t) = X_n(t)\,T_n(t) = \left(E_n\cos\frac{n\pi at}{l} + F_n\sin\frac{n\pi at}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$

are solutions of (1) satisfying (2) for $n=1,2,3,\dots$

Here $E_n (= C_n B_n)$ and $F_n (= B_n D_n)$ are newly arbitrary constants. In order to obtain a solution also $F_n (= B_n D_n)$ are new arbitrary constants. In order to obtain a solution also satisfying 8(a) and 8(b) we consider most general solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi at}{l} + F_n \sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (17)$$

Differentiating (17) partially w.r.t to 't' we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{n\pi a E_n}{l}\sin\frac{n\pi at}{l} + \frac{n\pi a F_n}{l}\cos\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l} \rightarrow (18)$$

putting $t=0$ in (17) and (18) and using (8a) and (8b) we get

$$c\sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \text{ and } 0 = \sum_{n=1}^{\infty}\left(\frac{n\pi a F_n}{l}\right)\sin\frac{n\pi x}{l}.$$

which are fourier sine series. Accordingly we have.

$$E_n = \frac{2}{l}\int_0^l c\sin^3\left(\frac{\pi x}{l}\right)\sin\frac{n\pi x}{l}\,dx \rightarrow (19)$$

and $$\frac{n\pi a F_n}{l} = \frac{2}{l}\int_0^l 0\cdot\sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow F_n = 0 \rightarrow (20)$$

Now $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = (3\sin\theta - \sin 3\theta)/4.$

$$\therefore \sin^3\left(\frac{\pi x}{l}\right) = \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] \rightarrow (21)$$

$$\therefore (19) \Rightarrow E_n = \frac{2c}{l}\int_0^l \frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\sin\frac{n\pi x}{l}\,dx$$

(or) $$E_n = \frac{3c}{2l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx - \frac{c}{2l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{n\pi x}{l}\,dx \rightarrow (22)$$

we now show that

$$I = \int_0^l \sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx = \begin{cases} l/2, & \text{if } m=n \\ 0, & \text{if } m\neq n \end{cases} \rightarrow (23)$$

If $m=n$ then, we have

$$I=\int_0^l \sin^2\frac{n\pi x}{l}\,dx=\int_0^l \frac{1}{2}\left(1-\cos\frac{2n\pi x}{l}\right)dx$$

$$=\frac{1}{2}\left[x-\frac{1}{2n\pi}\sin\frac{2n\pi x}{l}\right]_0^l$$

$\therefore\ I=l/2$ when $m=n$

if $m\neq n$ then we have

$$I=\frac{1}{2}\int_0^l 2\sin\frac{m\pi x}{l}\sin\frac{n\pi x}{l}\,dx$$

$$=\int_0^l\left(\cos\frac{(n-m)\pi x}{l}-\cos\frac{(n+m)\pi x}{l}\right)dx$$

$$=\frac{1}{2}\left[\frac{1}{(n-m)\pi}\sin\frac{(n-m)\pi x}{l}-\frac{1}{(n+m)\pi}\sin\frac{(n+m)\pi x}{l}\right]_0^l=0$$

using (23) from (22) we have.

$$E_1=\left(\frac{3c}{2l}\right)\left(l/2\right)=\frac{3c}{4},\quad E_3=-\left(\frac{c}{2l}\right)\left(\frac{l}{2}\right)=\frac{-c}{4}$$

Also $E_n=0$ for $n\neq 1$ and $n\neq 3$

using (20) and (24), (17) reduces to

$$y(x,t)=E_1\cos\frac{\pi at}{l}\sin\frac{\pi x}{l}+E_3\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}.$$

(or)

$$y(x,t)=\frac{c}{4}\left[3\cos\frac{\pi at}{l}\sin\frac{\pi x}{l}-\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}\right] \quad (25)$$

7(a) Solve the following system of linear equations correct to two decimal places by Gauss – Seidel method.

$10x+2y+z=9, \quad 2x+20y-2z=-44, \quad -2x+3y+10z=22$

Ans: $x=1.0, \ y=-2.0, \ z=3.0.$

(Try yourself)

7(b) Using fourth order Runge-kutta method find the solution of the initial value problem.

$$y' = \frac{1}{(x+y)}, \; y(0)=1 \text{ in the range}$$

$0.5 \le x \le 2.0$, by taking $h=0.5$.

Sol'n: Runge kutta 4th order

$$f(x,y) = \frac{1}{x+y}, \quad y(0)=1 \Rightarrow x_0=0, y_0=1$$
$$h=0.5$$

$$y_1 = y_0 + \tfrac{1}{6}(K_1+2K_2+2K_3+K_4)$$

$K_1 = hf(x_0, y_0) = 0.5(1) = 0.5$

$K_2 = hf(x_0+h/2, y_0+K_1/2) = 0.5f(0.25, 1.25) = 0.333$

$K_3 = hf(x_0+h/2, y_0+K_2/2) = 0.5f(0.25, 1.1666) = 0.35294$

$K_4 = hf(x_0+h, y_0+K_3) = 0.5f(0.5, 1.35294) = 0.2698$

$$y(0.5) = 1 + \tfrac{1}{6}[0.5 + 0.666 + 0.70588 + 0.2698]$$

$$\boxed{y(0.5) = 1.3569}$$

Now, $y_0 = 1.3569$, $x_0 = 0.5$, $h = 0.5$

$K_1 = 0.5f(0.5, 1.3569) = 0.2692$

$K_2 = 0.5f(0.75, 1.4915) = 0.2230$

$K_3 = 0.5f(0.75, 1.4684) = 0.2253$

$K_4 = 0.5f(1, 1.5822) = 0.1936$

$$y(1) = 1.3569 + \tfrac{1}{6}[0.2692 + 0.446 + 0.4506 + 0.1936]$$

$$\boxed{y(1) = 1.5834}$$

Now, $y_0 = 1.5834$, $x_0 = 1$, $h = 0.5$

$K_1 = 0.5f(1, 1.5834) = 0.1935$

$K_2 = 0.5f(1.25, 1.6801) = 0.1706$

$k_3 = 0.5f\ (1.25, 1.6687) = 0.1713$

$k_4 = 0.5f\ (1.5, 1.7547) = 0.1536$

$y(1.5) = 1.5834 + \frac{1}{6}[0.1935 + 0.3412 + 0.3426 + 0.1536]$

$\boxed{y(1.5) = 1.7552}$

Now $y_0 = 1.7554, x_0 = 1.5, h = 0.5$

$K_1 = 0.5f\ (1.5, 1.7554) = 0.1536$

$K_2 = 0.1396$

$K_3 = 0.1399$

$K_4 = 0.1283$

$y(1.75) = 1.89555$

7.(C) Simplify the boolean expression: $(a+b)(\bar{b}+c)+b.(\bar{a}+\bar{c})$ by using the laws of boolean algebra. From its truth table write it in minterm normal form.

Sol^n: $(a+b)(\bar{b}+c)+b(\bar{a}+\bar{c})$

$$= a\bar{b}+ac+b\bar{b}+bc+b\bar{a}+b\bar{c}$$

$$= a\bar{b}+ac+0+b(c+\bar{c})+b\bar{a} \quad [\because b\bar{b}=0]$$

$$= a\bar{b}+ac+b[0]+b\bar{a} \quad [\because c+\bar{c}=1]$$

$$= a\bar{b}+ac+b[1+\bar{a}]$$

$$= a\bar{b}+ac+b \quad [\because 1+\bar{a}=1]$$

a	b	c	$\bar{b}$	$a.\bar{b}$	ac	$a\bar{b}+ac$	$z=a\bar{b}+ac+b$	minterm.
0	0	0	1	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
0	0	1	1	0	0	0	0	$\bar{a}\bar{b}c$
0	1	0	0	0	0	0	1	$\bar{a}b\bar{c}$
0	1	1	0	0	0	0	1	$\bar{a}bc$
1	0	0	1	1	0	1	1	$a\bar{b}\bar{c}$
1	0	1	1	1	1	1	1	$a\bar{b}c$
1	1	0	0	0	0	0	1	$ab\bar{c}$
1	1	1	0	0	1	1	1	abc

From truth table, minterm normal form is

$$\bar{a}b\bar{c}+\bar{a}bc+a\bar{b}\bar{c}+a\bar{b}c+ab\bar{c}+abc$$

7(d). (i) Convert 1011101.1011 to octal and then to hexadecimal.
(ii) Convert hexadecimal number 2647 to octal.
(iii) Convert hexadecimal number 4A.67 to binary.

Sol^n: (i) Binary number can be converted into equivalent octal number by making groups of three bits starting from LSB (least significant digit) and moving towards MSB (most significant digit) for integer part of the number and then replacing each group of three bits by its octal representation.

For fractional part, the grouping of three bits are made starting from the binary point.

$$(1011101.1011)_2 = (\underset{1}{\underline{001}}\ \underset{3}{\underline{011}}\ \underset{5}{\underline{101}}\ .\ \underset{5}{\underline{101}}\ \underset{4}{\underline{100}})_2$$

$$= (135.54)_8$$

Now, octal number can be converted to equivalent hex number by converting it to equivalent binary and then to hex number.

$$(135.54) = (001\ 011\ 101\ .\ 101100)_2$$

$$= (0010\ 1101\ .\ 1011)_2$$

$$= (5D.B)_{16}$$

(ii) $$(2647)_{16} = (0010\ 0110\ 0100\ 0111)_2$$

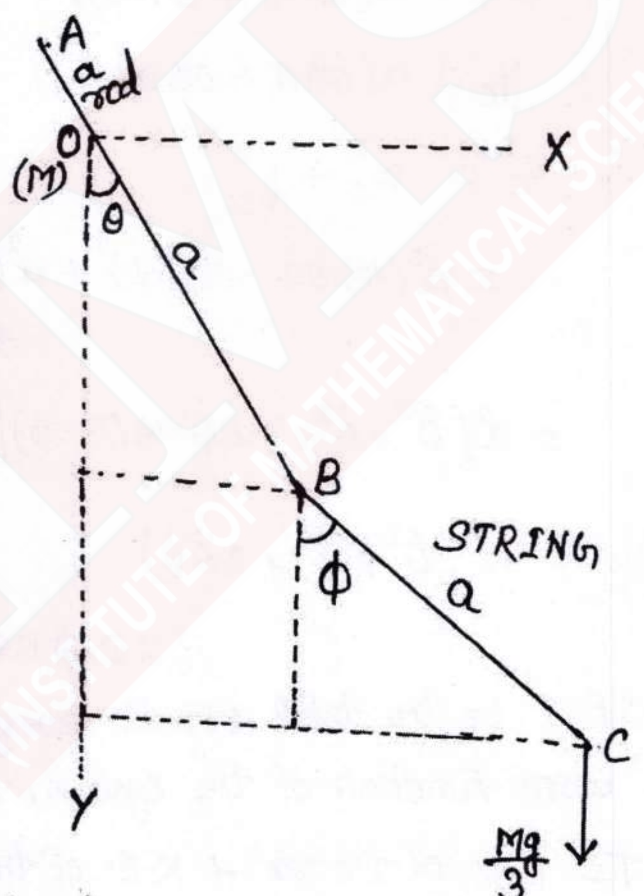
$$= (\underset{2}{\underline{010}}\ \underset{3}{\underline{011}}\ \underset{1}{\underline{001}}\ \underset{0}{\underline{000}}\ \underset{7}{\underline{111}})_2$$

$$= (23107)_8$$

(iii) $$(4A.67)_{16} = (01001010\ .\ 0110\ 0111)_2$$

8.(a) A uniform straight rod of length $2a$ is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of length a to one end of the rod; show that one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{(a/g)}$.

Sol:



Let M be the mass of the rod AB of length $2a$, BC the string and $M/3$ the mass at C.

At time t, let the rod and the

String make angle θ and ϕ to the vertical respectively.

Referred to the middle point O of the rod AB as origin. horizontal and vertical lines OX and OY through O as axes, the co-ordinates of C are given by

$$x_C = a(\sin\theta + \sin\phi),$$

$$y_C = a(\cos\theta + \cos\phi).$$

$$\therefore \; v_C^2 = \dot{x}_C^2 + \dot{y}_C^2$$

$$= a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$$

$$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)]$$

$$= a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

($\because \theta, \phi$ are small)

If T be the total kinetic energy and W the work function of the system, then

T = K.E. of the rod + K.E. of the particle at C

$$= \left[\frac{1}{2}M\cdot\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}Mv_0^2\right] + \frac{1}{2}\left(\frac{1}{3}M\right)v_C^2$$

$$= \frac{1}{6} Ma^2\dot{\theta}^2 + \frac{1}{6} Ma^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$$

$$= \frac{1}{6} Ma^2(2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) \qquad (\because v_0 = 0)$$

and $W = mg \cdot 0 = \frac{1}{3} Mg \cdot y_c + C$

$$= \frac{1}{3} Mga(\cos\theta + \cos\phi) + C$$

Lagranges θ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\text{down30 dedel}\,\dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e. $\frac{d}{dt}\left[\frac{1}{6} Ma^2(4\dot{\theta} + 2\dot{\phi})\right] - 0 =$

$$\frac{1}{3} Mga(-\sin\theta)$$

$$= -\frac{1}{3} Mga\theta, \qquad (\because \theta \text{ is small})$$

or $2\ddot{\theta} + \ddot{\phi} = -c\theta,$ ——— ①

(where $c = g/a$)

And Lagranges ϕ-equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$$

i.e. $\frac{d}{dt}\left[\frac{1}{6}Ma^2(2\dot{\phi}+2\dot{\theta})\right]-0=\frac{1}{3}Mga(-\cos\phi)$

$=-\frac{1}{3}Mga\phi$

$\because \phi$ is small

or $\ddot{\theta}+\ddot{\phi}=-c\phi,$ ——— ②

(where $c=g/a$)

Equations ① and ② can be written as

$(2D^2+c)\theta+D^2\phi=0$ and

$D^2\theta+(D^2+c)\phi=0$

eliminating ϕ between these two equations, we get

$$\left[(D^2+c)(2D^2+c)-D^4\right]\theta=0$$

or $(D^4+3cD^2+c^2)\theta=0$ ——— ③

Let the solution of ③ be given by

$\theta=A\cos(pt+B)$

$\therefore D^2\theta=-p^2\theta$ and $D^4\theta=p^4\theta$.

Substituting in ②, we get

$$(p^4 - 3cp^2 + c^2)\theta = 0$$

or $p^4 - 3cp^2 + c^2 = 0 \qquad (\because \theta \neq 0)$

$$\therefore\ p^2 = \frac{3c \pm \sqrt{(9c^2 - 4c^2)}}{2}$$

$$= \left(\frac{3 \pm \sqrt{5}}{2}\right)c$$

$$= \left(\frac{3 \pm \sqrt{5}}{2}\right)\frac{g}{a}$$

$\therefore$ one value of p^2 is $p_1^2 = \left(\frac{3-\sqrt{5}}{2}\right)\frac{g}{a}$

$\therefore$ one period of principal oscillation

$$= \frac{2\pi}{p_1} = 2\pi\sqrt{\left[\frac{2}{3-\sqrt{5}} \cdot \frac{a}{g}\right]}$$

$$= 2\pi\sqrt{\left[\frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \cdot \frac{a}{g}\right]}$$

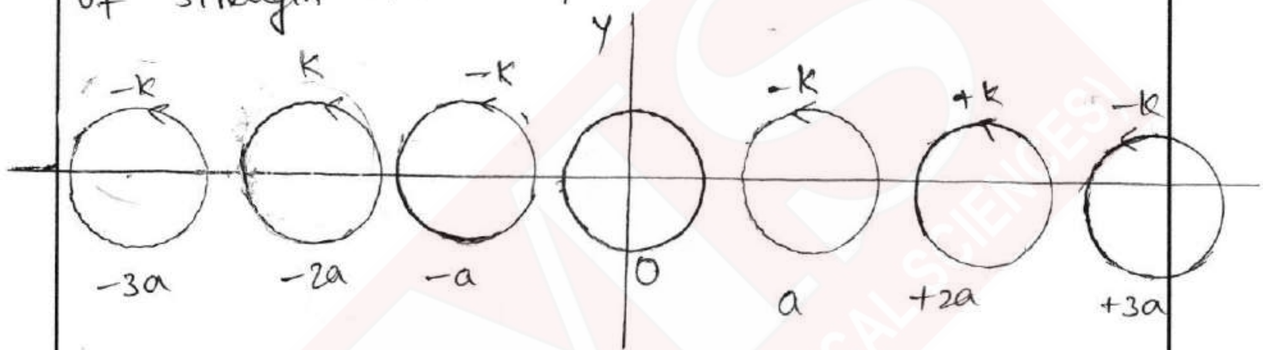
$$= 2\pi\sqrt{\left[\frac{6+2\sqrt{5}}{4} \cdot \frac{a}{g}\right]}$$

$$= 2\pi\sqrt{\left[\left(\frac{\sqrt{5}+1}{2}\right)^2 \frac{a}{g}\right]}$$

$$= (\sqrt{5}+1)\pi\sqrt{(a/g)}.$$

8(b) An infinite row of equidistant rectilinear vortices is at a distance a part. The velocities are of the same numerical strength K but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function.

Sol'n: Let the vortices each of strength k be placed at (0,0), ($\pm$2a,0), ($\pm$4a,0) --- and vortices each of strength $-k$ be placed at ($\pm$a,0), ($\pm$3a,0), ($\pm$5a,0)...



The complex potential at any point P(z) is given by

$$W = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\left[\log(z-2a)+\log(z+2a)+\log(z-4a)+\log(z+4a)\cdots\right] - \frac{ik}{2\pi}\left[\log(z-a)+\log(z+a)+\log(z-3a)+\log(z+3a)+\cdots\right]$$

$$= \frac{ik}{2\pi}\log\left[\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\cdots}{(z^2-a^2)(z^2-3^2a^2)(z^2-5^2a^2)\cdots}\right]$$

$$= \frac{ik}{2\pi}\log\left[\frac{\frac{z}{2a}\left\{1-\left(\frac{z}{2a}\right)^2\right\}\left\{1-\left(\frac{z}{4a}\right)^2\right\}\cdots}{\left\{1-\left(\frac{z}{a}\right)^2\right\}\left\{1-\left(\frac{z}{3a}\right)^2\right\}\cdots}\right] + \text{Const}$$

$$= \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right)$$

$$W = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \quad \text{——— (1)}$$

$$\phi + i\psi = \frac{ik}{2\pi} \log \tan\left(\frac{\pi z}{2a}\right) \quad \text{--- (2)}$$

$$\phi - i\psi = -\frac{ik}{2\pi} \log \tan\left(\frac{\pi \bar{z}}{2a}\right) \quad \text{--- (3)}$$

(2) - (3) gives, $2i\psi = \frac{ik}{2\pi} \log\left(\tan\frac{\pi z}{2a}\right)\left(\tan\pi\frac{\bar{z}}{2a}\right)$

$$\Rightarrow \psi = \frac{k}{4\pi} \log\left[\sin\left(\frac{\pi z}{2a}\right)\sin\left(\frac{\pi \bar{z}}{2a}\right) \Big/ \cos\left(\frac{\pi z}{2a}\right)\cos\left(\frac{\pi \bar{z}}{2a}\right)\right]$$

$$\psi = \frac{k}{4\pi} \log\left[\frac{\cosh\frac{\pi y}{a} - \cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a} + \cos\frac{\pi x}{a}}\right] \quad \text{--- (4)}$$

Stream lines are given by $\psi = \text{Const}$, i.e.

$$\cosh\frac{\pi y}{a} = b\cos\frac{\pi x}{a}.$$

(2) + (3) gives $2\phi = \frac{ik}{2\pi} \log \frac{\tan(\pi z/2a)}{\tan(\pi \bar{z}/2a)}$

$$\phi = \frac{ik}{4\pi} \log\left[\frac{\sin(\pi z/2a)\cos(\pi \bar{z}/2a)}{\sin(\pi \bar{z}/2a)\cos(\pi z/2a)}\right]$$

$$= \frac{ik}{4\pi} \log \frac{\sin(\pi x/a) + i\sinh(\pi y/a)}{\sin(\pi x/a) - i\sinh(\pi y/a)}$$

$$\phi = -\frac{k}{4\pi}\left[\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} + \tan^{-1}\frac{\sinh(\pi y/2a)}{\sin(\pi x/a)}\right]$$

$$\phi = -\frac{k}{2\pi} \tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} \quad \text{--- (5)}$$

Required velocity potential and stream function are given by (4) & (5).

8.(c) Given the velocity potential $\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$ determine the streamlines.

Solution:

The given velocity potential

$$\phi = \frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$

To determine the stream lines, we proceed as follows:

$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y}, \quad -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}.$$

Hence,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Now, $\frac{\partial \psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$

Integrating w.r.t. y,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + F(x) \quad \text{---} (4)$$

where $F(x)$ is constant of integration.

To determine $F(x)$.

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \quad \text{---} (5)$$

By (4),

$$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} + F'(x) \quad \text{---} (6)$$

Equating (5) to (6), $F'(x) = 0$.

Integrating this,

$F(x)$ = absolute constant and hence neglected

Since it has no effect on the fluid motion

Now (4) becomes,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} \quad \text{——} (7)$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}$$

Stream lines are given by ψ = constant

i.e., $\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right]$ = constant

or $\frac{y}{x^2-a^2+y^2}$ = constant.

If we take constant = 0, then we get $y=0$,
i.e. x-axis.

If we take constant = ∞, then we get circle
$x^2-a^2+y^2=0$,
i.e. $x^2+y^2=a^2$.

$\boxed{\text{Thus, stream lines include } x\text{-axis and circle.}}$

Hence, the result.