Complex Analysis

- $z = x + iy$ $\longrightarrow$ $r = (x^2+y^2)^{1/2}$, $\theta = \tan^{-1}(y/x)$ $\quad [-\pi < \theta \le \pi]^*$

(1) TS: $|z_1+z_2| \le |z_1| + |z_2|$.

$$|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) = |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + \bar{z}_1 z_2)$$

$$\left[z_1\bar{z}_2 + z_2\bar{z}_1 = (x+iy)(x_1-iy_1) + (x-iy)(x_1+iy_1) = 2(xx_1+yy_1) = 2\,\mathrm{Re}(z_1\bar{z}_2) \right]$$

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) \qquad [2\,\mathrm{Re}(z_1\bar{z}_2) \le 2|z_1\bar{z}_2|]$$

$$|z_1+z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = (|z_1| + |z_2|)^2$$

$\Rightarrow |z_1+z_2| \le |z_1| + |z_2|$ Proved.

[IIly $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$]

[Alternatively $\rightarrow$ Use polar coordinates easier.]

(2) Modulus and argument of $1 + i\tan\alpha$ $\quad (-\pi < \alpha < \pi \quad \alpha \ne \pm\frac{\pi}{2})$

$r\cos\theta = 1 \qquad r\sin\theta = \tan\alpha. \quad \Rightarrow r = \sqrt{\sec^2\alpha} = |\sec\alpha| = \frac{1}{|\cos\alpha|}$

$\cos\theta = |\cos\alpha| \qquad \sin\theta = \tan\alpha\,|\cos\alpha|$.

Case I $\Rightarrow \cos\alpha > 0 \Rightarrow$ $\cos\theta = \cos\alpha$, $\sin\theta = \sin\alpha$. $\quad [\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})]$

$$\boxed{1 + i\tan\alpha = \sec\alpha(\cos\alpha + i\sin\alpha)}$$

Case II $\Rightarrow \cos\theta = -\cos\alpha$, $\sin\theta = -\sin\alpha$ $\quad [\alpha \in (-\pi, -\frac{\pi}{2})$ or $(\frac{\pi}{2}, \pi)]$

$\theta = \pi + \alpha$.

$$\boxed{1 + i\tan\alpha = -\sec\alpha\,[\cos(\pi+\alpha) + i\sin(\pi+\alpha)]}$$

· Holomorphic function ⇒ Satisfies C-R equations

$$\boxed{u_r = \frac{1}{r} v_\theta \quad v_r = -\frac{1}{r} u_\theta}$$

* Find f(z)

① $f(z) = u(z,0) + i\,v(z,0)$

② $f'(z) = u_x(z,0) - i\,u_y(z,0) \quad (u_x + i v_x)$

$v_y(z,0) + i\,v_x(z,0)$

③ $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$ [This is exact form]

$$v = \underbrace{\int \frac{\partial v}{\partial x}dx}_{\text{y constant}} + \underbrace{\int \frac{\partial v}{\partial y}dy}_{\text{not x terms.}}$$

④ If $u + v = \alpha$ is given

$(1+i)\,f(z) = \int \frac{\partial \alpha}{\partial x} + i\frac{\partial \alpha}{\partial y} \quad \begin{pmatrix} x = z \\ y = 0 \end{pmatrix}$

If $u - v = \beta$

$(1+i)\,f(z) = \int \frac{\partial \beta}{\partial x} - i\frac{\partial \beta}{\partial y}$

* $\frac{1}{R} = \frac{u_{n+1}}{u_n}$

Analytic Functions

① Show $f(z) = z^n$ is analytic.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} = \frac{nz^{n-1}\Delta z + {}^nC_2 z^{n-1}\Delta z^2 + \ldots \Delta z^n}{\Delta z}$$

$$= nz^{n-1} + {}^nC_2 z^{n-1}\Delta z + \ldots \Delta z^{n-1} = nz^{n-1}$$

Thus $f'(z)$ exists for all finite values of z.
Hence $f(z)$ is analytic.

② Show $|z|^2$ is nowhere differentiable except at origin

Consider a point $a, \neq 0$.

$$\lim_{\Delta z \to 0} \frac{|a+\Delta z|^2 - |a|^2}{\Delta z} = \frac{(a+\Delta z)\overline{(a+\Delta z)} - a\bar{a}}{\Delta z}$$

$$\lim_{\Delta z \to 0} = \frac{a\overline{\Delta z} + \bar{a}\Delta z + \Delta z . \overline{\Delta z}}{\Delta z} = a + a\frac{\overline{\Delta z}}{\Delta z}$$

At $a = 0$, $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = 0$, Hence $f'(0) = 0$.

$a \neq 0$, let $\Delta z = r(\cos\theta + i\sin\theta) \Rightarrow \frac{\overline{\Delta z}}{\Delta z} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta}$.

$= \cos 2\theta - i\sin 2\theta$.

(Not unique as limit depends on θ)

So, not differentiable.

$f(z)$ is analytic $\equiv$ A single valued function defined and differentiable at each point of domain D.

- Isolated singularity at $z_0 \equiv$ if analytic at each pt in nbd except at z_0
- Removable singularity at $z_0 \equiv$ If assigning suitable value to $f(z_0)$ removes singularity.

(*) C-R equations (only necessary). $\left[f(z) = u(x,y) + iv(x,y)\right]$

- If $f(z)$ is differentiable/analytic at any pt z, then u_x, v_x, u_y, v_y exist and $\boxed{u_x = v_y,\ u_y = -v_x}$

→ Sufficient Condition $\equiv$ Apart from C-R if u_x, u_y, v_x, v_y are also continuous on domain.

→ $f(z) = u + iv$ is analytic, then u, v satisfy Laplace ($\nabla^2 u = 0$). u, v are thus harmonic functions / harmonic conjugates

IMP: $f(z)$ analytic then ⇒.

Q. $u, v \Rightarrow u = c_1,\ v = c_2$ intersect at right angles.

$\frac{\partial u}{\partial x} = v_y,\ u_y = -v_x \Rightarrow$ Multiply both to get (1)

Q. Show harmonic u satisfies $\partial^2 u / \partial z \partial \bar{z} = 0$.

$x = \frac{1}{2}(z+\bar{z})$
$y = \frac{1}{2i}(z-\bar{z})$.

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial z} + \frac{1}{2}\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial y}{\partial z} - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial z} + \frac{\partial^2 u}{\partial x \partial y}\frac{\partial x}{\partial z}\right]$$

$$= 0 = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$u(x,y) = c_1 \qquad v(x,y) = c_2$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\underbrace{\frac{dy}{dx}}_{m_1} = 0 \qquad \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\underbrace{\frac{dy}{dx}}_{m_2} = 0$

$\perp$ if $m_1 m_2 = -1$

$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} = 0$ — (1)

(3) FINDING F(z) $\qquad f'(z) = U_x + iV_x$

- *Milne's Method.* [Milne - Thomson]

Let $f'(z) = \phi_1(x,y) - i\,\phi_2(x,y)$ $\qquad \boxed{\phi_1 = U_x \\ \phi_2 = U_y}$

Then $\underline{f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)} \Rightarrow$ Now integrate with z.

Also for $v \Rightarrow V_y = \psi_1(x,y)$, $V_x = \psi_2(x,y)$.

$\hookrightarrow \underline{f'(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz}$

(*) C-R in Polar form. $\left[\begin{array}{l} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r} \end{array}\right\} \boxed{\dfrac{df}{dz} = e^{-i\theta}\dfrac{df}{dr}}$

$\boxed{Q} \Rightarrow U = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$. Find $f(z)$.

$U_x - iU_y = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$
$\qquad - i(\sin x \sinh y + 2\cos x \cosh y - 2y + 4x)$

Put $x = z$, $y = 0$

$f'(z) = (\cos z + 2z) - i(2\cos z + 4z)$

$f(z) = (\sin z + z^2) - i(2\sin z + 2z^2) + c$ Ans

$\boxed{Q}$ Find real part of $\sinh(e^z)$

$\rightarrow \sinh(e^{x+iy}) = \sinh(e^x \cdot e^{iy}) = \sinh(e^x\cos y + i e^x \sin y)$

$= \sinh(e^x\cos y)[\cosh(ie^x\sin y)] + \sinh(ie^x\sin y)\cosh(e^x\cos y)$

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$\boxed{9}$ $U + V = \dfrac{2\sin 2x}{e^{2y} - e^{-2y} + 2\cos 2x}$. Find $f(z)$.

$$\frac{\partial (U+V)}{\partial x} \Rightarrow \frac{4\cos 2x}{e^{2y} - e^{-2y} + 2\cos 2x} + \frac{2\sin 2x \cdot 4\sin 2x}{(e^{2y} - e^{-2y} + 2\cos 2x)^2} \quad (1)$$

$[U_x + V_x]$
$(U_x - U_y)$

$$\frac{\partial}{\partial y}(U+V) \Rightarrow \frac{-2\sin 2x\,[2e^{2y} + 2e^{-2y}]}{(e^{2y} - e^{-2y} + 2\cos 2x)^2} \quad -(2).$$

$U_y + V_y$
$U_y + U_x$.

Add (1), (2) $\rightarrow$ $2u_x = (1) + (2)$

$2u_y = (2) - (1)$

$f'(z) = U_x - iU_y$

$= U_x(z,0) - i(U_y(z,0))$

$$2U_x(z,0) = 2 + \frac{2\sin 2z \cdot \cancel{4}\sin 2z}{\cancel{4}(\cos 2z)^2} = 2 + 2\tan^2 2z - 4\tan 2z$$

$$2U_y(z,0) = \frac{-2\sin 2z \cdot 4}{2\cos 2z} - \left[2 + 2\tan^2 2z\right]$$

$$f'(z) = (1 + \tan^2 2z - 2\tan 2z) - i(-2\tan 2z - 1 - \tan^2 2z)$$

$$f(z) = \tan 2z - 2\int \tan 2z\, dz - i\left[-\int 2\tan 2z\, dz - \tan 2z\right] + C.$$

$$= \tan 2z + \log(\cos 2z) - i\left[\log(\cos 2z) - \tan 2z\right] + C$$

IMP: U, V harmonic $\Leftrightarrow$ U_x, V_x, U_y, V_{xy} continuous

TS: Analyticity

FOR SUCH QUESTIONS: $u - iv = U$, $u + iv = V$

$$(1+i)f(z) = \int \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}\, dz \;/\; \int \frac{\partial V}{\partial x} + i\frac{\partial V}{\partial y}\, dz$$

Putting $x = z, y = 0$

Q Show $\sqrt{xy}$ is not analytic at origin.

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \boxed{\lim_{z\to 0} \frac{\sqrt{|xy|}-0}{x+iy}}$$

$x+iy$ in D in all such Qs for differentiability

Along $y = mx$, $z \to 0 \Rightarrow$

$$\lim_{x\to 0} \frac{\sqrt{mx^2}}{x(1+im)} = \frac{\sqrt{m}}{1+im}$$ dependent $\Rightarrow$ Hence not analytic.

*

Q Show $e^{-z^{-4}}$ $(z\neq 0)$ and $f(0)=0$ is not analytic $z=0$. though C-R equations are satisfied there.

$$\Rightarrow f(z) = e^{-\frac{1}{(x+iy)^4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2-4ix^3y+4ixy^3)}$$

$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)} \times \left[\cos\left\{\frac{4xy(x^2-y^2)}{r^8}\right\} + i\sin\left\{\frac{4xy}{r^8}(x^2-y^2)\right\}\right]$$

$$U_x(0,0) = \lim_{x\to 0} \frac{U(x,0)-U(0,0)}{x} = \lim_{x\to 0} \frac{1}{x \cdot e^{1/x^4}}$$

$$= \lim_{x\to 0} \frac{1}{x\left[1+\frac{1}{x^4}+\frac{1}{2x^8}+\cdots\right]} = \frac{1}{\infty} = 0.$$

||ly $U_y(0,0) = 0$; $V_x = V_y = 0$.

So, C-R are satisfied

Not Analytic $\Rightarrow$ $\boxed{z = re^{(1/4)/\pi}}$ Let

$$f(z) = \exp\left(+r^{-4}e^{-i\pi}\right) = e^{r^{-4}} \Rightarrow \infty \text{ as } r \to 0$$

* UNSOLVED (Page 89) [Also 89-95]

For what values of z do f^n defined ceases to be analytic $\Rightarrow$ $\boxed{z = \log \rho + i\phi}$ where $\boxed{w = \rho e^{i\phi}}$

$\boxed{Q}$ If arg $f(z)$ = constant, then $f(z)$ is constant

$(\arg f(z)) = k \Rightarrow \tan^{-1}\frac{v}{u} = k \Rightarrow v = u \tan k.$

$\frac{\partial v}{\partial x} = c\frac{\partial u}{\partial x}$; $\frac{\partial v}{\partial y} = c\frac{\partial u}{\partial y}$ $(c = \tan k)$

Using C-R $\Rightarrow$

$-\frac{\partial u}{\partial y} = c\frac{\partial u}{\partial x}$; $\frac{\partial u}{\partial x} = c\frac{\partial u}{\partial y}$

$-\frac{\partial u}{\partial y} = c^2\frac{\partial u}{\partial y}$ $\qquad$ $\frac{\partial u}{\partial x} = c\cdot(-c)\frac{\partial u}{\partial x}$

$\frac{\partial u}{\partial y}(1+c^2) = 0$ $\qquad$ $\frac{\partial u}{\partial x}(1+c^2) = 0$

$\tan^2 k = -1 \Rightarrow \tan k = \pm i$ Not possible

~~Shown~~ $\Rightarrow$ So, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0 \Rightarrow f'(z) = u_x + iv_x = 0$

$\Rightarrow f(z)$ is constant.

- If $|f(z)|$ is constant, then $f(z)$ is constant.

$|f(z)|^2 = u^2+v^2 = c^2 \Rightarrow u\frac{du}{dx} + v\frac{dv}{dx} = 0$, $u\frac{du}{dy} + \frac{v\partial v}{\partial y} = 0 \rightarrow \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$

$\Rightarrow \underbrace{f'(z) = 0}_{\boxed{f(z) = k}}$ if $u^2+v^2 \neq 0$ OR if $u^2+v^2 = 0$, then $\boxed{f(z) = 0}$

Zeroes, Poles $\rightarrow$ $P(z)$ at finite plane

$\rightarrow$ $P(1/z)$ at infinity

Line $\equiv$ $z = a + bt$ $\qquad \alpha = \arg\left(\frac{b}{b'}\right) \equiv$ Angle btw 2 lines

└ Two ⊥r lines $\Rightarrow$ $\frac{b}{b'}$ is purely imaginary

Circle $\equiv$ $|z - z_0| = r$

• Zeros and Poles $\qquad$ [ORDER of function = # of zeroes or poles] ← max(n, m)

$$R(z) = \frac{a_0 \dots + a_n z^n}{b_0 + \dots b_m z^m}$$

(# of zeroes = # of poles)

Zeroes $\Rightarrow$ n [Finite part], $m-n$ [At $\infty \Rightarrow$ Use $R(\frac{1}{z}) \leftarrow$ coeff of z]

└ Only if $(m > n)$

Poles $\Rightarrow$ m [Finite plane], $n-m$ [if $m < n$, at ∞]

└ Use $R(\frac{1}{z}) \leftarrow$ coeff z.

POWER Series

→ Radius of Convergence of complex series.

$$\boxed{\begin{array}{l} ① \ \frac{1}{R} = \overline{\lim} |a_n|^{1/n} \\ ② \ \frac{1}{R} = \lim\limits_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| \end{array}}$$

} No info on cvgce on $|z| = R$

$\boxed{\text{Convgt only in } |z| < R}$

• Radius of cvgce of $\underline{\sum_{n=0}^{\infty} a_n z^n}$ and $\underline{\sum_{n=1}^{\infty} n a_n z^{n-1}}$ $\boxed{\text{are equal.}}$

Q Find radius of convergence $\sum \left(1 + \frac{1}{n^2}\right)^{n^2} z^n$

$$\frac{1}{R} = \lim_{n \to \infty} \left|\left(1 + \frac{1}{n^2}\right)^{n^2}\right|^{1/n} = \lim_{n \to \infty} \left|\left(1 + \frac{1}{n}\right)^n\right| = e \Rightarrow \boxed{R = 1/e}$$

NOT DONE

└ Luca Theorem (98) → Zeroes of $P(z)$ lie in half plane then so are zero of $P'(z)$ lying in same half plane.

$\boxed{Q}$ Radius of Convergence $\equiv \sum \frac{n\sqrt{2}+i}{1+2in} z^n$

$$|a_n| = \left|\frac{(n\sqrt{2}+i)(1-2in)}{1+4n^2}\right| = \left|\frac{n\sqrt{2}+2n+i(1-2\sqrt{2}n^2)}{1+4n^2}\right|$$

$$= \left|\frac{(n\sqrt{2}+2n)^2+(1-2\sqrt{2}n^2)^2}{1+4n^2}\right| = \frac{\sqrt{2n^2+4n^2+4\sqrt{2}n^2+1+8n^4-4\sqrt{2}n^2}}{1+4n^2}$$

$$= \frac{\sqrt{8n^4+6n^2+1}}{1+4n^2}$$

$$R = \left|\frac{a_n}{a_{n+1}}\right| = 1. \quad \underline{\underline{An}}$$

$\boxed{Q}$ Find domain of Convergence

$$\sum \frac{1.3.5\ldots(2n-1)}{n!}\left[\frac{1-z}{z}\right]^n$$

Put $\left(\frac{1}{z}\right) = \xi$

$$\sum_n \frac{1.3.5\ldots 2n-1}{n!}\left[\xi - 1\right]^n$$

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim\left|\frac{n+1}{2n+1}\right| = \frac{1}{2}$$

Domain $\equiv |\xi - 1| < \frac{1}{2} \Rightarrow \left|\frac{1}{z} - 1\right| < \frac{1}{2}$

$$\left|\frac{1-z}{z}\right|^2 < \frac{1}{4} \Rightarrow (1-z)(1-\bar{z}) < \frac{1}{4}z\bar{z}$$

$$\Rightarrow 3z\bar{z} - \frac{4}{3}(z+\bar{z}) + \frac{4}{3} < 0 \Rightarrow z\left(\bar{z}-\frac{4}{3}\right) - \frac{4}{3}\left(\bar{z}-\frac{4}{3}\right) < \frac{4}{9}$$

$$\Rightarrow \left(z-\frac{4}{3}\right)\left(\bar{z}-\frac{4}{3}\right) < \frac{4}{9} \Rightarrow \underline{\left|z-\frac{4}{3}\right|^2 < \frac{4}{9}} \Rightarrow \boxed{\left|z-\frac{4}{3}\right| < \frac{2}{3}}$$

MUG

Q $\sum n^2\left[\frac{z^2+1}{1+i}\right]^n$.

Put $z^2 = \xi$

$$\sum \frac{n^2}{(1+i)^n}(\xi+1)^n \rightarrow R = \sqrt{2}$$

$$|\xi+1| < \sqrt{2}$$

$\frac{18}{3} \quad + \frac{4}{3} + \frac{8}{12}$

$\boxed{8}$ Examine convergence on circle of convergence

$$\sum_{n=2}^{\infty} \frac{z^{4n}}{4n+1}. \qquad \frac{1}{R} = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4n+5}{4n+1}\right| = 1.$$

$$\boxed{R=1}$$

At $z = \pm 1, \pm i \Rightarrow$ Series $\equiv \sum_{2}^{\infty} \frac{1}{4n+1}$ is dvgt.

For other z, we look at $\boxed{\text{dirichlet test}}$ ⇐ IMP.

$a_n = z^{4n} \qquad U_n = \frac{1}{4n+1}.$

$$\left|1+z^4+z^8+ \cdots z^{4n}\right| = \left|\frac{1-z^{4n+4}}{1-z^4}\right| \le \left|\frac{1+|z|^{4n+4}}{1-z^4}\right| \quad (z \ne \pm 1, \pm i)$$

$$\le \frac{2}{|1-z^4|}$$ Hence not

So $|1+z^4+\cdots z^{4n}|$ is bounded.

$U_n - U_{n+1} = \frac{4}{(4n+1)(4n+5)}$ $\therefore \Sigma |U_n - U_{n+1}|$ is convergent.

$\lim U_n = \lim \frac{1}{4n+1} = 0$ Ans

* Power Series

Period

$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$ $2n\pi i$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$ $2n\pi$

$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$ $2n\pi$

$\cosh z = \frac{e^{z} + e^{-z}}{2} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

$\sinh z = \frac{e^{z} - e^{-z}}{2} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

$\sinh(iz) = i \sin z.$

$\cosh(iz) = \cos z.$

• Log

$\log w = \log|w| + i\arg(w)$ $\quad w = Re^{i\phi}$ ($R \to |w|$, $\phi \to \arg$)

Also, $\log w = \log|w| + i(\arg w + \underline{2n\pi})$

Principal value of Log ≡ Restrict $0 \le \phi < 2\pi$ $= (\arg w)$

• Power ⇒

IMP $\Longrightarrow$ $z^a = \exp(a\log z) = \exp(a\log|z| + ia\arg(z))$

$= \exp\{a\log|z|\}.\exp(i a\alpha).\exp(2n\pi a i)$

$$\frac{d}{dz}(z^a) = \frac{d}{dz}(\exp(a\log z)) = \exp(a\log z)\cdot\frac{a}{z}$$

$$= \frac{a \times \exp(a\log z)}{\exp(\log z)} = a\exp[(a-1)\log z] = az^{a-1}$$

Inverse Trigo Left.
Multiply Connected Regions Left

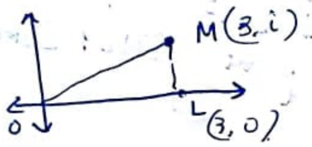
Complex Integration

- Smooth / Regular Curve ≡ iff $z'(t) \neq 0$
- Rectifiable curve ≡ A smooth arc is rectifiable

  L Length $(l) = \int_a^b \left[\sqrt{(x'(t))^2 + (y'(t))^2}\right] dt$

$$\boxed{\int f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt}$$

$\boxed{Q} \Rightarrow$ $\int_0^M z^2\, dz$ $\quad$ Evaluate over OM and OL+ML



Line ≡ $x = 3y$

$y = t,\ x = 3t.$

$\int_{OM} (x^2 - y^2) + 2xy\, i\, dz \qquad [\because dz = dx + i\,dy]$

$\int_0^1 (8t^2 + 6t^2 i)(3 + i)\, dt \Rightarrow (8 + 6i)(3 + i)\frac{t^3}{3} - - -$

Curve C of length l.

– Estimation of Integral : given $|f(z)| \leq M$ on l

Then $\left|\int_L f(z)\,dz\right| \leq Ml$ $\quad$ [ONLY if $f(z)$ is continuous on C]

– Exact differential (Integrand) $\Leftrightarrow \int_C$ closed curve $= 0 \Leftrightarrow \int$ only on end points

* Cauchy's Theorem

D is simply connected region, C is any closed contour in D.

$$\boxed{\text{If } f'(z) \text{ exists and is continuous at each pt of } D \text{ Then } \int_C f(z)\,dz = 0}$$

EXTRA: Cauchy-Goursat ≡ If f(z) is analytic in D, then $\int_C f(z)dz = 0$

└ Cor. ≡ If f is analytic $\int_A^B f$ independent of path.

ONLY SUFFICIENT : NOT NECESSARY

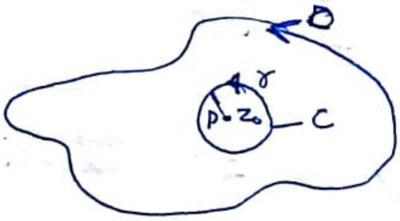
Says that continuity of $f'(z)$ is not needed

* Cauchy's Integral Formula [CIF]

Let f(z) be analytic in D, enclosed by a rectifiable Jordan Curve C and f(z) be continuous on C

Then, $$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$ Proof in 1486

└ any pt of D



Gauss' MVT : $f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} f(z_0+\rho e^{i\theta})\,d\theta \qquad |z-z_0| \le \rho$

└ Pf ≡ Let Curve be $(z = z_0 + \rho e^{i\theta})$ Put in (CIF)

Left: Poisson Integral Formula
Multiply Connected Domain

Pt 307

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Need $f(z)$ analytic within and on C [For CIF need only within]

- Derivatives at z_0 (Just differentiate CIF w.r.t z_0)

$$\boxed{f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

- Morera's Thm: $f(z)$ is continuous in D, $C \equiv$ (Any) closed Jordan curve

$$\boxed{\int_C f(z)\,dz = 0 \Rightarrow f(z) \text{ is analytic}}$$ ← Converse of Cauchy

IMP

→ Cauchy's Inequality Theorem

$\gamma : |z-z_0| = \rho$

$|f(z)| \le M$

$dz = \rho i e^{i\theta} d\theta$

$$f^n(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$|f^n(z_0)| = \left|\frac{n!}{2\pi i}\right| \int_\gamma \frac{|f(z)|}{|z-z_0|^{n+1}}\,dz$$

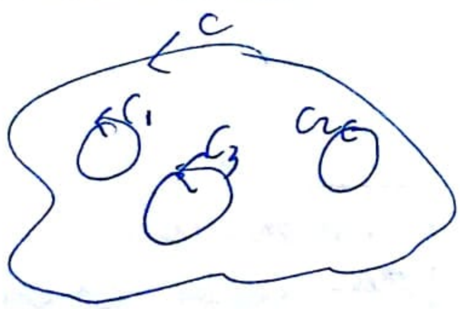
$$\le \frac{Mn!}{2\pi}\int_\gamma \frac{dz}{|z-z_0|^{n+1}} \le \frac{Mn!}{2\pi\rho^n}\int_0^{2\pi} d\theta$$

$$\boxed{|f^n(z_0)| \le \frac{Mn!}{\rho^n}}$$

Use to prove Liouville's Thm — $|f(z)| \le M \Rightarrow f(z) = k$

Proof of Cauchy's theorem: Use Green's Theorem $\int P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$\int f(z)\,dz = \int (u+iv)(dx+idy)$ (Now apply Green's) — Easy Proof
+ Use C-R equation



If $f(z)$ is analytic then

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$$

* Taylor's Theorem

$f(z)$ be analytic at all pts within Circle(z_0, ρ_0) C_0

Then for all z in C_0

$$\boxed{f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{(z-z_0)^n}{n!} f^n(z_0)}$$

Q. Expand $\log(1+z)$ about $z=0$, determine region of convergence.

$f(z) = \log(1+z)$

$f'(z_0) = 1/1+z_0$

$f''(z_0) = -\frac{1}{(1+z_0)^2}$

$f'''(z_0) = 2/(1+z_0)^3$

$f^{IV}(z_0) = (-1)^n 3!/(1+z_0)^4$

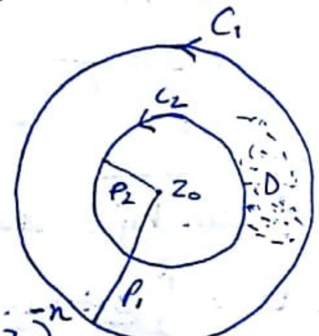
$$f(z) = \log(1) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+z_0)^n}$$

$$= \sum_{1}^{\infty} (-1)^{n-1} \frac{z^n}{n(1+z_0)^n}$$

$$\lim_{n\to\infty} \left|\frac{U_n}{U_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{nz}\right| = \frac{1}{|z|}$$. Series converges for $|z|<1$

$z=-1$ is a singularity. For all other $|z|=1 \to$ Convergent

* Laurent's Theorem



Let $f(z)$ be analytic in the region bdd by C_1, C_2 with centre z_0, ρ_1, ρ_2

Then, $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$

any pt in D

$$\boxed{a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \quad b_n = \frac{1}{2\pi i} \int_{C_2} (\xi - z_0)^{n-1} f(\xi)\, d\xi}$$

$$\boxed{b_n = a_{-n}}$$

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when only coeff deg ≥ 0 then Taylor else Laurent

Find Series expansions

① $\dfrac{1}{z(z^2-3z+2)}$ (i) $0<|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$

$\dfrac{1}{z(z-1)(z-2)} \longrightarrow$ Into Partial Fraction. [Just put $z=2$ in given removing $z-2$ from D^r to get C]

$\dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$.

$\dfrac{1}{2z} - \dfrac{1}{(z-1)} + \dfrac{1}{2(z-2)}$

(i) $\boxed{\dfrac{1}{2z} + \left[1+z+z^2+\dots\right] - \dfrac{1}{4}\left[1+\dfrac{z}{2}+\left(\dfrac{z}{2}\right)^2+\dots\right]}$

~~(ii) $\dfrac{1}{2z} + \dfrac{1}{2(1-\frac{z}{2})}$~~

(ii) $\dfrac{1}{2z} - \dfrac{1}{z(1-\frac{1}{z})} - \dfrac{1}{4(1-\frac{z}{2})}$

$\boxed{\dfrac{1}{2z} - \dfrac{1}{z}\left[1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dfrac{1}{z^3}\dots\right] - \dfrac{1}{4}\left[1+\dfrac{z}{2}+\dfrac{z^2}{4}+\dots\right]}$

(iii). $\dfrac{1}{2z} - \dfrac{1}{z}\left[1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dots\right] + \dfrac{1}{2z(1-\frac{2}{z})}$

$\boxed{\dfrac{1}{2z} - \dfrac{1}{z}\left[1+\dfrac{1}{z}+\dfrac{1}{z^2}+\dots\right] + \dfrac{1}{2z}\left(1+\dfrac{2}{z}+\dfrac{4}{z^2}+\dots\right)}$

② $\dfrac{1}{2(z+1)} - \dfrac{1}{2(z+3)}$ in $(1<|z+1|<2)$

Put $\boxed{z+1=u}$

$\dfrac{1}{2u} - \dfrac{1}{4(\frac{u}{2}+1)} = \dfrac{1}{2u} - \dfrac{1}{4}\left[1-\dfrac{u}{2}+\dfrac{u^2}{4}-\dfrac{u^3}{8}+\dots\right]$

$= \dfrac{1}{2(z+1)} - \dfrac{1}{4} + \dfrac{(z+1)}{8} - \dots$

③. Find Taylor series about nbd of $z = i$ for :

$$f(z) = \frac{2z^3+1}{z^2+z}$$

$$f(z) = 2(z-1) + \frac{1}{z} + \frac{1}{z+1}$$

└ NOTHING OBVIOUS — GO LONG WAY.

$$\equiv 2(z-1) \Rightarrow 2(i-1) + \frac{(z-i).2}{1} + 0 \ldots$$

$$\equiv \frac{1}{z} \Rightarrow \frac{1}{i} + \frac{(z-i)(-1)}{i^2} + \frac{(z-i)^2(-1)^2 2!}{2!\ i^3}$$

$$= \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n (z-i)^n}{i^{n+1}}$$

$$\equiv \frac{1}{z+1} \Rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$$

$$\boxed{\begin{array}{l} |z| < 1 \\ 1 < |z| < 2 \\ |z| > 2 \end{array}}$$

④ $$\frac{1}{(1+z^2)(z+2)}$$

$$\frac{Az+B}{1+z^2} + \frac{C}{z+2}$$

$$(C+A)z^2 + z(A.2+B) + 2B + C = 1 = 0.$$

$C+A=0$ $\quad 2A+B=0$ $\quad 2B+C=1$

$A=-C$ $\quad B=2C \Rightarrow$ $C = 1/5$

$B = 2/5$

$A = -1/5$.

$$\frac{1}{5}\left[\frac{2-z}{1+z^2} + \frac{1}{z+2}\right]$$

(i) $$\frac{1}{5}\left[\frac{1}{2}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right) + (2-z)\left[1 - z^2 + z^4 + \cdots\right)\right]$$

(iii) $$\frac{1}{5}\left[\frac{1}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} + \cdots\right) + \frac{2-z}{z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \cdots\right)\right]$$

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⑤ E Prove $\cosh\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=\phi}^{\infty} a_n \left[z^n + \frac{1}{z^n}\right]$

$\cosh\left[z+\frac{1}{z}\right]$ is analytic everywhere except at $z=0$.

So, it is analytic in $r < |z| < R$ (r is small, R is large).

We can expand $\cosh\left[z+\frac{1}{z}\right]$ as Laurent series

$$\cosh\left[z+\frac{1}{z}\right] = \sum_0^{\infty} a_n z^n + \sum_1^{\infty} b_n z^{-n}$$

$$\boxed{a_n = \frac{1}{2\pi i}\int_C \cosh\left[z+\frac{1}{z}\right] \times \frac{1}{z^{n+1}}\,dz}$$

Take $C \equiv |z| = 1. \Rightarrow z = e^{i\theta},\ dz = ie^{i\theta}d\theta$.

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \cosh\left[e^{i\theta}+e^{-i\theta}\right]\frac{ie^{i\theta}\,d\theta}{e^{(n+1)i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\left(\cos n\theta - i\sin n\theta\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta)\cos n\theta\, d\theta \qquad \int_0^{2\pi}\cosh(2\cos\theta)\sin n\theta\, d\theta =$$

Put $a_{-n} = b_n$ gives other coefficients $(\cos(-n\theta) = \cos n\theta)$

$\hookrightarrow a_n = b_n$

$= a_0 + \sum a_n (z^n + z^{-n})$ Ans.

(6) $e^{z+c^3/2z^2} = \sum_{-\infty}^{\infty} a_n z^n$

Analytic, except at $z=0$.
So, consider $r<z<R$, r is very small, R is large

Consider $|z|=c \Rightarrow z=ce^{i\theta}$, $dz = c\,i\,e^{i\theta}$

$$a_n = \frac{1}{2\pi c^n}\int_0^{2\pi} \exp\left\{ce^{i\theta} + \frac{c}{2}e^{2i\theta} - ni\theta\right\} d\theta$$

$$= \frac{1}{2\pi c^n}\int_0^{2\pi} \exp\left(c\cos\theta + \frac{c}{2}\cos 2\theta\right) \times \exp\left(i\left(c\sin\theta - \frac{c}{2}\sin 2\theta - n\theta\right)\right)$$

$$\underbrace{= \frac{1}{2\pi c^n}\int_0^{2\pi} \exp\left(c\cos\theta + \frac{c}{2}\cos 2\theta\right) \times \left[\cos\left(c\sin\theta - \frac{c}{2}\sin 2\theta - n\theta\right) + i\,\cancel{\sin}\ldots\right]}_{a_n \text{ Proved}}$$

(7) Find function regular within $|z|=1$ and value at circumference

$$\frac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4 - 2a^2\cos 2\theta + 1} \qquad a^2>1$$

As analytic $f(z) = \sum_0^{\infty} a_n z^n$ $\qquad a_n = \frac{1}{2\pi i}\int \frac{f(z)}{z^{n+1}}dz$

On circle,

$$f(z) = \frac{a^2e^{i\theta} - e^{-i\theta}}{a^4 - a^2(e^{2i\theta}+e^{-2i\theta})+1} = \frac{1}{(a^2e^{-i\theta} - e^{i\theta})}$$

$$a_n = \frac{1}{2\pi i}\int \frac{1}{(a^2e^{-i\theta}-e^{i\theta})}\frac{ie^{i\theta}d\theta}{e^{(n+1)i\theta}} = \frac{1}{2\pi}\int \frac{e^{-i(n-1)\theta}}{a^2\cancel{e^{-i\theta}}}\left(1-\frac{1}{a^2}e^{2i\theta}\right)^{-1} d\theta$$

$$= \frac{1}{2\pi a^2}\int_0^{2\pi} e^{-(n-1)i\theta}\left[1 - \frac{e^{2i\theta}}{a^2} + \frac{e^{4i\theta}}{a^4} - \cdots\right] d\theta.$$

$= 0$ if n is even, If odd $\Rightarrow a_n = \frac{1}{2\pi a^2}\int_0^{2\pi} e^{-(n-1)i\theta}\cdot\frac{e^{(n-1)i\theta}}{a^{n-1}}d\theta$

$$= \boxed{\frac{1}{a^{n+1}}}$$

Left = Examples to find functions with given value on circumference

Do. 14, 15 — 347 → 1997
16 — 1998 UPSC

$$f(z) = a_1 z + a_3 z^3 + a_5 z^5 + \dots$$

$$= \frac{z}{a^2} + \frac{z^3}{a^4} + \frac{z^5}{a^5} + \dots = \boxed{\frac{z}{a^2 - z^2}} \text{ Ans}$$

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(8) Using integral representation of $f^n(0)$ prove

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$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int \frac{x^n e^{xz}}{n!\, z^{n+1}}\,dz. \quad \text{Also Show } \sum\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}\,d\theta.$$

Taking $f(z) = e^{xz}$ in Cauchy's formula for derivative.

$$f^n(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z - z_0)^{n+1}}$$

$z_0 = 0$, $f(z) = e^{xz}$.

$$(x^n e^{xz})_{z_0} = \frac{n!}{2\pi i}\int \frac{e^{xz}}{z^{n+1}}$$

$$\frac{x^n}{n!} = \frac{1}{2\pi i}\int \frac{e^{xz}}{z^{n+1}}\,dz$$

$$\boxed{\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int \frac{x^n e^{xz}}{n!\, z^{n+1}}}$$

Put ~~$z = e^{i\theta}$~~ $\Rightarrow \sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \sum_0^{\infty}\frac{1}{2\pi i}$ $\int \frac{x^n}{n!}\frac{e^{xz}}{z^{n+1}}dz$ and Sum

$$\sum\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\int e^{xz}\boxed{\left(\sum \frac{1}{n!}\left(\frac{x}{z}\right)^n\right)}\frac{dz}{z} = \frac{1}{2\pi i}\int e^{xz}\cdot e^{x/z}\cdot\frac{dz}{z}$$

$z = e^{i\theta}$ $= \boxed{\frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}\,d\theta}$ Ans

355 * $|f(z)| \le A|z|^k$ where $f(z)$ is an integral function. A, k are +ve constant. Prove $f(z)$ is a polynomial of degree $\le k$.

$\Rightarrow$ Let $f(z) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} z^n$

As $|f(z)| \le A|z|^k$.

Cauchy's Inequality Theorem gives $\Rightarrow |f^n(0)| \le \frac{n!\, A|z|^k}{\rho^n}$

For $n > k$, let $\rho \to \infty \Rightarrow |f^n(0)| = 0$.

So, $f(z) = \sum_0^k \frac{f^{(n)}(0) z^n}{n!}$

- $f(z)$ analytic in $|z-a| < R$, when $0 < r < R$

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta} d\theta \quad , \ P(\theta) \text{ is } Re\{f(a+re^{i\theta})\}$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2} dz.$$

$f(z) = \sum_0^\infty a_m (z-a)^m \Rightarrow$ Put $z = a + re^{i\theta} \to f(a+re^{i\theta}) = \sum a_m r^m e^{im\theta}$.

$\bar{f}(z) = \sum_0^\infty \bar{a}_m r^m e^{-im\theta} \to \frac{1}{2\pi i}\int_C \frac{\overline{f(z)}}{(z-a)^2} dz = \frac{1}{2\pi}\sum \bar{a}_m r^{m-1}\int_0^{2\pi} \cancel{e^{-(m+1)i\theta}} d\theta$

$= 0$

So, adding

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z) + \overline{f(z)}}{(z-a)^2} dz.$$

- $f(z)$ be analytic in D. Then $f(z)$ can be expanded about $z = z_0$ in form $\sum_0^\infty a_n (z-z_0)^n$

  If $a_0 = a_1 = \dots a_{m-1} = 0$, $a_m \neq 0 \Rightarrow$ Then $f(z)$ has zero of order m at $z = z_0$

- Zeros of analytic function are ISOLATED.

$$f(z) = \sum a_n (z-z_0)^n + \underbrace{\sum_1^\infty b_n (z-z_0)^{-n}}_{\text{Principal part}} \text{ of } f(z) \text{ at } (z = z_0)$$

→ If Principal part contains :

1. m terms = Pole of order m [$m = 1$ - Simple pole]
2. ∞ terms ≡ Isolated essential pole
3. 0 terms ≡ Removable singularity

- Entire function ≡ If $f(z)$ has 0 singularity in finite part of plane.

  ↓

  [$f(1/z)$ may be singular]. $\boxed{\text{Only poles as singularity} \equiv \text{Meromorph}}$

* Residue at pole ≡ Coefficient of $(z-z_0)^{-1}$ in Laurent Series

  Cor. ⇒ If simple pole at $z_0 \Rightarrow$ Residue $\Rightarrow \boxed{\lim_{z \to z_0} (z-z_0) f(z)}$

  If $f(z)$ has pole of order m at $z = z_0$, then $\phi(z) = (z-z_0)^m f(z)$ has removable discont.

  (+) Residue at $z_0 \Rightarrow \dfrac{\phi^{(m-1)}(z_0)}{(m-1)!}$

OR $B(z)$ has pole of order m at a.

$B(z) = \dfrac{F(z)}{(z-a)^m} \Rightarrow \boxed{\operatorname{Res}_{\text{at } z=a} (B(z)) \equiv \dfrac{F^{m-1}(a)}{(m-1)!}}$

Q. $\frac{1}{z(e^z - 1)}$ Find sing...

Singularities $\equiv z(e^z - 1) = 0 \Rightarrow z = 0$ or $e^z = 1 \Rightarrow z = \pm 2n\pi i$
$\quad$ ↳ 0 also

⇒ So, 0 is pole of order 2, rest of order 1 (simple poles)

$$z\left[z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots\right] \Big) \; 1 \; \Big(\frac{1}{z^2} - \frac{1}{2z} + \frac{1}{12}$$

$$\begin{array}{l} 1 \quad z/2 \quad z^2/3! \\ \hline -z/2 \quad -z^2/3! \\ -z/2 \quad -z^2/4 \\ \hline z^2/12 \\ z^2/12 . \end{array}$$

OR

Better ⇒ $\frac{1}{z(e^z-1)} = \frac{1}{z(1+z+z^2/2! \cdots - 1)}$

$$= \frac{1}{z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}$$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \cdots\right) + \left(\frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24}\right)^2 - \left(\frac{z}{2} + \frac{z^2}{6} + \cdot\right)^3 \cdots\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{z}{2} - \frac{z^2}{6} + \frac{z^2}{4} - \frac{z^3}{24} + \frac{z^3}{6} - \frac{z^3}{8} + z^4(\cdots)\right]$$

$$= \frac{1}{z^2} - \frac{1}{2z} + \frac{1}{12} + \frac{1}{360}z^2 \cdots$$ Ans

Q. Show $e^{1/z}$ takes every value except zero an ∞ # of times

(*) Thm: Limit Pt of seq. of poles is non-isolated essential singularity $\boxed{\text{Lt pt of zeroes is isolated essential}}$

Q Singularities of $\sin\frac{1}{1-z} = f(z)$

Zeroes of $\sin\frac{1}{1-z} \Rightarrow \frac{1}{1-z} = n\pi \Rightarrow z = 1 - \frac{1}{n\pi}$

Lt. pt of zeroes is at 1. So 1 is isolated essential singularity.

Singularities

└ ISOLATED ESSENTIAL ─┬ Laurent Series has ∞ terms in Principal part.
OR
└ Lt point of zeroes.

NON-ISOLATED ESSENTIAL → Lt pt of poles

TIP: Always look at poles first, if none then zeroes.

If neither available, then Find Laurent Series

Q $\frac{1-e^z}{1+e^z}$ at $z=\infty$.

Poles: $e^z = -1 \Rightarrow z = (2n+1)\pi i \Rightarrow z = \pm\pi i, \pm 3\pi i, \dots$

$z=\infty$ is a lt point of sequence

So, $z=\infty$ has non-isolated essential singularity

Q Show $z=a$ is I.E. Singularity $\frac{e^{c/z-a}}{e^{z/a}-1}$.

$\boxed{e^{1+x} = e[1+x+\frac{x^2}{2!}+)}$

$$f(z) = \frac{e^{c/(z-a)}}{e^{1+(z-a)/a}-1} = \frac{1+\frac{c}{z-a}+\dots\frac{c^n}{n!(z-a)^n}+\dots}{e\left[1+\frac{z-a}{a}+\left(\frac{z-a}{a}\right)^2+\right]-1}$$

This shows ∞ terms in -ve powers of $(z-a)$

Thus, $z=a$ is isolated singularity of $f(z)$.

Ex-7,8,9 - left 322, ← Solve

[illegible] $f(z)$ has [illegible] with residue 2, [illegible]
simple pole at $z=1$ with residue 2, [illegible]
finite plane, bounded as $|z| \to \infty$.
If $f(2)=5$, $f(-1)=2$, find $f(z)$.

$$\equiv \sum \frac{2}{z-1} + \frac{2}{z} + \frac{k}{z^2} + a_n z^n = f(z).$$

As $f(z)$ is bdd $\Rightarrow$ $|f(z)| \le M \;\forall z$.

$\hookrightarrow$ $f(z)$ has no singularity at $z=\infty$.

$\hookrightarrow$ $f(\frac{1}{z})$ has no singularity at $z=0$

$$f(1/z) = \sum a_n z^{-n} + kz^2 + 2z + \left(\frac{2}{\frac{1}{z}-1}\right) = \frac{2z}{1-z}.$$

$\Downarrow$

$a_n,\ n \neq 0 = 0$ for $f(1/z)$ to have no singularity at $z=0$.

$$\boxed{f(z) = a_0 + \frac{2}{z-1} + \frac{2}{z} + \frac{k}{z^2}} \Rightarrow \frac{z^3 + 3z^2 + 2z - 4}{z^2(z-1)}$$

Using values given

INFO: $(1-x)^{-n} = 1 + nx + \frac{1}{2}(n)(n+1)x^2 + \frac{n(n+1)(n+2)}{6}x^3$

Q. $\phi(z), \psi(z) \equiv$ Analytic. If $z=a$ is once repeated root of $\psi(z)=0$, $\phi(a)\neq 0$. Find residue of ϕ/ψ at $z=a$ in terms of ϕ, ψ

$$\psi(z) = (z-a)^2 g(z) \quad \text{s.t } g(a) \neq 0.$$

$$\frac{\phi}{\psi} = \frac{\phi/g}{(z-a)^2} \Rightarrow z=a \text{ is double pole of } \phi/\psi.$$

So, Residue at $z=a$.

$$\left(\frac{\phi(z)}{g(z)}\right)'_{z=a} = \frac{\phi'(z)\,g(z) - \phi(z)\,g'(z)}{(g(z))^2}$$

$$= \frac{\phi'(a)\,g(a) - \phi(a)\,g'(a)}{g(a)^2} \quad -(1).$$

Find $g'(a)$, $g(a)$.

$$\psi(z) = (z-a)^2 g(z).$$

$$\psi'(z) = 2(z-a)g(z) + (z-a)^2 g'(z) \quad ; \quad \psi'' = 2g(z) + 4(z-a)g'(z) + (z-a)^2 g''(z).$$

$$\psi'''(z) = 2g'(z) + 4g'(z) + 4(z-a)g''(z) + (z-a)^2 g'''(z)$$

$$\left(\psi'''(a) = 6g'(a) \quad ; \quad \psi''(a) = 2g(a)\right) - \text{Put in } (1)$$

$$\boxed{\frac{6\,\phi'(a)\,\psi''(a) - 2\,\phi(a)\,\psi'''(a)}{3\,[\psi''(a)]^2}}$$

(1) f has zero of order m at $z=0$

$$f(z) = \sum_{n=m}^{\infty} a_n z^n$$

(2) f has pole of order m at $z=0$

$$f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{1}^{m} \frac{b_n}{z^n}$$

(3) f has no singularity in finite plane

$$f(z) = \sum_{0}^{\infty} a_n z^n$$

* Pole is a special case of singularity

L a non-essential singularity [of finite order]

Thm : Show a f^n that has no singularity in finite part and pole of order n at ∞ is a polynomial of degree n.

$\rightarrow$ $f(z)$ has no singularity in finite $\quad \left[f(1/z) = \sum_{0}^{\infty} a_n z^{-n} \right]$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Pole of order n at $\infty \Rightarrow f(1/z)$ has pole of order n at 0

$$\cancel{f(1/z) = \sum_{0}^{\infty} a_n z^{-n} = \sum^{n} a_n z^{n}}$$

$$f(1/z) = F(z) + \sum_{1}^{n} B_m z^{-m}$$

$$\sum_{0}^{\infty} a_n z^{-n} = F(z) + \sum_{1}^{n} B_m z^{-m}$$

$F(z) = a_0$, $B_m = a_n$ .. $a_{n+1} = a_{n+2} = \dots 0$

Hence $f(z)$ is poly of degree n.

* Rational functions
  └ No singularities other than poles.

— Maximum Modulus Principle

[$f(z)$ analytic within and on simple, closed C.
$|f(z)|$ attains maximum value on C, unless it is constant]

— Min. Modulus Principle.

[$f(z)$ analytic.. , $f(z) \neq 0$ inside C. Then $|f(z)|$ must reach its minimum value on C.]

Meromorphic function ≡ Has poles as its only singularities in finite plane.

$f(z)$ is meromorphic, with no zeroes on C,

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \cdot dz = \underset{\text{zeros}}{N} - \underset{\text{poles}}{P} \text{ (inside } C)$$

$\boxed{N - P = \frac{1}{2\pi} \Delta_C \arg(f)}$

* Rouche's Theorem

① f, g be analytic inside and on C.
② $|g(z)| < |f(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ have same number of zeroes inside C.

To look $f(z)$ at $z=\infty$, look $f(1/z)$ at $z=0$

Left = Schwarz Lemma

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* Fundamental Theorem of Algebra

$P(z)$ a polynomial of degree ≥ 1. $P(z)=0$ has atleast one root.

L (Proof left.)

Q. Show $P(z)=a_0+a_1z+\ldots a_nz^n=0$ has n roots exactly.

L By FTA, it has atleast one root α_1

$P(\alpha_1)=0$.

~~$P(z)=(z-\alpha)$~~

$\therefore P(z)=P(z)-P(\alpha_1)=\cancel{a_0}\, a_1(z-\alpha_1)+a_2(z^2-\alpha_1^2)+\ldots$

$=(z-\alpha_1)\,\cancel{P_1(\alpha)}\;P_1(z)$ of degree $(n-1)$

Apply FTA, recursively $\Rightarrow$ n roots.

Q

① If $a > e$, Show $e^z = az^n$ has n roots in $|z|=1$

$\underbrace{-e^z, az^n \text{ are analytic}}_{\text{in } |z| \le 1.}$. Consider $\boxed{az^n - e^z = 0}$

Let $f(z) = az^n$, $g(z) = -e^z$.

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|e^z|}{a|z|^n} = \frac{|1 + z + z^2/2! + z^3/3! + \cdots|}{a|z|^n} \le \frac{1+|z|+\cdots}{a|z|^n}$$

$$\le \frac{1 + 1 + 1/2! + 1/3! + \cdots}{a} \quad \text{as } |z| = 1 \text{ on } C.$$

$$= \frac{e}{a} < 1.$$

So, $g(z) + f(z)$ has same # of roots within C as $f(z)$.
(By Rouche's Thm)

az^n has n roots all at origin $\Rightarrow$ $az^n - e^z$ has n roots in $|z|=1$

② Show all roots $z^7 - 5z^3 + 12 = 0$ lie in $|z|=1, |z|=2$.

TS: No roots in $|z|=1$, all roots in $|z|=2$.

$|z|=2$; Let $f(z) = z^7$ $g(z) = 12 - 5z^3$

$$\left|\frac{g(z)}{f(z)}\right| \le \frac{12 + 5|z|^3}{|z|^7} = \frac{12+40}{128} < 1 \quad \text{So all in } |z|=2$$

$|z|=1$; $f(z) = z^7 - 5z^3$ $g(z) = 12$

$$\left|\frac{g(z)}{f(z)}\right| \le \frac{12}{2} > 1 \Rightarrow$$

$g(z)+f(z)$ Same roots as $g(z)$
$g(z)$ 0 roots in $|z|=1$

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Argument principle example.

1994 : No. of zeroes of $p(z) = z^4 + 2z^3 + 3z + 4$ in 1st Quadrant, 4th quadrant

$\equiv$ Arg Principle $= \frac{1}{2\pi} \Delta\Theta = N - P$ ← zero for $p(z)$.

To find $\Delta\Theta$

$1 \Rightarrow z = x \Rightarrow (x^4 + 2x^3 + 3x + 4)$

$\Downarrow$

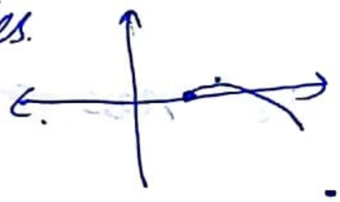
Always on X-axis $\equiv \Delta\Theta_1 = 0$

$2 \Rightarrow z = Re^{i\theta} \Rightarrow R^4 e^{i4\theta} + 2R^3 e^{3i\theta} + 3Re^{i\theta} + 4 = R^4 e^{i4\theta}\left[1 + \frac{2}{Re^{i\theta}} + \frac{3}{R^2 e^{2i\theta}} + \frac{4}{R^4 e^{i4\theta}}\right]$

$\Downarrow$

As Θ from $0 \to \frac{\pi}{2}$, R^4 term dominates.

So, $\Delta\Theta_2 = 4\Theta = 4\frac{\pi}{2} = 2\pi$



$3 \Rightarrow z = iy \Rightarrow y^4 - 2iy^3 + 3iy + 4 = y^4 + 4 + i(-2y^3 + 3y)$.

$\Theta = \tan^{-1}\left(\frac{-2y^3 + 3y}{y^4 + 4}\right)$ $\quad y = \infty \quad \Theta = 0$

$y = 0 \quad \Theta = 0$

$\Delta\Theta_3 = 0$

$\boxed{N = 1}$ Ans

- Cauchy's Theorem

  - D is simply connected - C is a closed contour in D
  - f'(z) is exists and continuous in D.

Then $\boxed{\int_C f(z)\, dz = 0}$

* Cauchy-Goursat ≡ f'(z) continuity need not be checked.

- Cauchy Integral Formula

  - f(z) is analytic in D - D is simply connected
  - C is rectifiable, Jordan
  - f(z) is continuous on C.

$\boxed{f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0}\, dz}$ where z_0 is any point of D.

* f(z) be analytic within and on C

- C lies within D - z_0 be within C.

$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz}$

* Cauchy's Inequality Theorem

- f(z) be analytic - $|z-z_0| = \rho : \gamma$ - $|f(z)| \leq M$ on γ

$\boxed{|f^n(z_0)| \leq n! \cdot \frac{M}{\rho^n}}$

* Liouville's Theorem $|f(z)| \leq M \Rightarrow f(z) = C$

- Rouche's Theorem

  - f(z), g(z) be analytic inside and on simple C
  - $|g(z)| < |f(z)|$ on C

Then f(z) and f(z)+g(z) have same number of zeroes inside C.

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Residue Calculation

① $\dfrac{\phi(z)}{(z-a)^m}$ at $z=a \Rightarrow \dfrac{\phi^{m-1}(a)}{(m-1)!}$

② At ∞

$\longrightarrow \dfrac{(-1)}{2\pi i}\int_C f(z)\,dz$ - (C anti-clock

$\quad L(e^{i\theta})$

$\longrightarrow \lim_{z\to\infty}\left[-zf(z)\right]$

③ Simple Pole $\rightarrow f(z) = \dfrac{\phi(z)}{\psi(z)}$, $\psi(z) = (z-a)F(z)$

$$\boxed{\operatorname{Res}_{z=a} f(z) = \frac{\phi(a)}{\psi'(a)}}$$

④ Aliter : For Residue at ∞.

Find Res $\{z=0\}$ for $f\left(\frac{1}{z}\right)$

⑤ Coefficient of $\frac{1}{z-a}$ in Laurent Series

New Method to FIND Residue

Residue of $f(z)$ at $z = z_0$

$\hookrightarrow$ Coeff $\frac{1}{t}$ in $f(t+z_0)$

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Contour Integration.

* Cauchy's Residue Theorem

If $f(z)$ is regular except at finite points within C and cont. on C; then

$$\boxed{\int_C f(z)\,dz = 2\pi i \underbrace{\sum R}}$$ Sum of residues of $f(z)$ at its poles

Cor $\Rightarrow$ Sum of residues at all singularities is 0

[NOTE: Only if finite singularities]

$$\left[\int_C f(z)\,dz = 2\pi i \sum R \;;\; \frac{-1}{2\pi i}\int_C f(z)\,dz = \text{Res at } \infty \right]$$

Theorems:

- Around $|z-a| = r$.

  If $\lim_{z\to a}(z-a)f(z) = A$ and C is the arc $\theta_1 \le \theta \le \theta_2$ of circle

  $$\lim_{r\to 0} \int_C f(z)\,dz = iA(\theta_2 - \theta_1)$$

- Around $|z| = r$., $\lim_{R\to\infty} zf(z) = A$ then

  $$\lim_{R\to\infty} \int_C f(z)\,dz = i(\theta_2 - \theta_1)A$$

$\boxed{Q}$ $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ $(a>b>0)$ $\boxed{\text{TYPE 1}}$

$C \equiv |z|=1.$

$z = e^{i\theta}$; $dz = ie^{i\theta}d\theta$; $\cos\theta = \frac{1}{2}(z+1/z)$

$$\frac{1}{i}\int_C \frac{dz}{z(a+\frac{b}{2}(z+1/z))} = \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b}$$

$$bz^2+2az+b=0 \Rightarrow \frac{-\cancel{2}a \pm \sqrt{\cancel{4}a^2-\cancel{4}b^2}}{\cancel{2}b} = \frac{-a \pm \sqrt{a^2-b^2}}{b} \ (\alpha,\beta)$$

$$\left|\frac{-a-\sqrt{a^2-b^2}}{b}\right| > 1,\ \alpha\beta = 1 \Rightarrow |\alpha| < 1.$$

So only pole at $z=\alpha$. ; $f(z) = \frac{1}{b(z-\alpha)(z-\beta)}$

Residue $\equiv$ at $z=\alpha \Rightarrow \frac{2}{i}\times\frac{1}{b(\alpha-\beta)} = \frac{1}{i\sqrt{a^2-b^2}}$

So, $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = 2\pi \cancel{i} \times \frac{1}{\cancel{i}\sqrt{a^2-b^2}}$

$\boxed{Q}$ $\int_0^{2\pi}\frac{d\theta}{(a+b\cos\theta)^2}$ $a>b>0.$

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$$\int_C \frac{dz/(iz)}{(a+\frac{b}{2}(z+\frac{1}{z}))^2} = \int_C \frac{4\,dz}{\frac{(iz)}{z^2}(bz^2+2az+b)^2} \quad dz/iz = d\theta$$

$$= \int_C \frac{4z\,dz}{i(bz^2+2az+b)^2}$$

$$bz^2+2az+b=0 \Rightarrow \frac{-2a\pm\sqrt{4a^2-4b^2}}{2b} \Rightarrow \frac{-a\pm\sqrt{a^2-b^2}}{\cancel{2}b} = \alpha,\beta.$$

$\left|\frac{-a-\sqrt{a^2-b^2}}{\cancel{2}b}\right| > 1 \Rightarrow \beta$ lies outside $|z|=1.$

So, residue only at $\alpha \Rightarrow \frac{\phi'(\alpha)}{1!} \equiv \frac{d}{dz}\left(\frac{4}{i}\frac{z}{(z-\beta)^2}\right)$

$$= \frac{4}{i}\left[\frac{1}{(z-\beta)^2} - \frac{2z}{(z-\beta)^3}\right] = \frac{4}{i}\left(\frac{\alpha-\beta-2\alpha}{(\alpha-\beta)^3}\right) = \frac{-\cancel{2}4}{\cancel{2}i}\left(\frac{\alpha+\beta}{(\alpha-\beta)^3}\right)$$

$$= 2\pi\cancel{i}\times\left(\frac{-4}{\cancel{i}}\right)\left[\frac{-2a/b}{\left(\frac{\cancel{2}\sqrt{a^2-b^2}}{\cancel{2}b}\right)^3}\right] = +8\pi\left[\frac{2a\cdot b^{3}}{b(a^2-b^2)^{3/2}}\right] \Rightarrow \frac{2\pi ab^2}{(a^2-b^2)^{3/2}}$$

$\boxed{Q8}$ Correct Version.

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} \qquad a>b>0$$

$$\int_C \frac{4z\ dz}{i\,(bz^2+2az+b)^2}$$

[Here make coff $z^2 = 1$ by taking out b^2]

$$\int_C \frac{(-4iz)\,dz}{b^2\left(z^2+\frac{2a}{b}z+1\right)^2}$$

$$\alpha, \beta \Rightarrow \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2} \Rightarrow \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\phi(z) = \frac{-4iz}{b^2(z-\beta)^2}$$

$$\phi'(\alpha) = -\frac{4i}{b^2}\left(\frac{\alpha-\beta-2\alpha}{(\alpha-\beta)^3}\right) = \frac{4i}{b^2}\frac{(\alpha+\beta)}{(\alpha-\beta)^2} = \frac{-ai}{(a^2-b^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}.$$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}.$$

Whenever only $\cos\theta$ in N^r, D^r then can do so.

- General Pointers for TYPE 1
  - → Take out coeff z^2 from D^r
  - → Make limits $0-2\pi$ before anything $-\left[\begin{array}{l} 2\theta \to \sin\theta \\ \frac{1}{2}\int_0^{2\pi} \to \cos\theta \end{array}\right.$
  - → $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$
  - → If $\cos 2\theta, 3\theta$ w/o $(0-\pi)$ limits use $Re\{e^{i2\theta}\}$

$$\int_0^{\pi} f(\cos\theta) = \frac{1}{2}\int^{2\pi} f \quad \leftarrow \text{DIRECT}$$

$\boxed{8}$

$$\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta}$$ π ← make 2π here!

IMPORTANT

$$\boxed{2\theta = \phi \Rightarrow 2d\theta = d\phi}$$

$$\int_0^{\pi} \frac{2a\, d\theta}{2a^2+(1-\cos 2\theta)}$$

$$\int_0^{2\pi} \frac{a\, d\phi}{2a^2+(1-\cos\phi)} \qquad z = e^{i\phi}\,;\ \frac{dz}{iz} = d\phi$$

$$\int_0^{2\pi} \frac{a\, d\phi}{\left(2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right)\times iz} = \int_0^{2\pi} \frac{a\, d\phi}{(iz)\left[-\frac{z}{2}-\frac{1}{2z}+1+2a^2\right]}$$

$$\int_0^{2\pi} \frac{a\, dz}{\frac{iz}{(-2z)}\left[z^2+1-2z(1+2a^2)\right]} = \int_C \frac{(2ai)\, dz}{z^2+1-2z(1+2a^2)}$$

$$\frac{2(1+2a^2) \pm \sqrt{4(1+2a^2)^2-4}}{2} \Rightarrow (1+2a^2) \pm \sqrt{4a^4+4a^2} \equiv \alpha, \beta.$$

$$\alpha, \beta \equiv (1+2a^2) \pm (2a)\sqrt{1+\cancel{2}a^2}$$

$$1+2a^2+2a\sqrt{1+2a^2} > 1 \Rightarrow \alpha\beta = 1 \Rightarrow \beta < 1.$$

Only only Residue at β is calculated.

$$\int_C \frac{2ai\, dz}{(z-\alpha)(z-\beta)} = \operatorname{Res}(z=\beta) \Rightarrow \frac{2ai}{\beta-\alpha} = \frac{\cancel{2}ai}{-2\cdot\cancel{2a}\sqrt{1+a^2}}$$

$$= \frac{1}{2i\sqrt{1+a^2}}$$

$$\int_0^{\pi} \frac{a\, d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+\cancel{2}a^2}}$$

$\boxed{8}$ $$\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\, d\theta$$

Put $z = e^{i\theta} \Rightarrow d\theta = dz/(iz)$

$$I = \frac{1}{i}\int_C \frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{a+\frac{b}{2}\left[z+\frac{1}{z}\right]} \cdot \frac{dz}{z}$$

$$= \frac{1}{i \times \cancel{2}i}\int_C \frac{\cancel{2}(z^2-1)^2}{\left(\frac{b}{\cancel{2}}z^2+2az+\frac{b}{\cancel{2}}\right)} \cdot \frac{dz}{z^{\cancel{2}} \times z} \times \frac{1}{2i}$$

$$= \int_C \frac{(1-z^2)^2\, dz}{b z^2\left(z^2+\frac{2a}{b}z+1\right)} \times \frac{1}{(-2i)}$$

$$= \frac{(-1)}{2ib}\int_C \underbrace{\frac{(z^2-1)^2\, dz}{z^2\left(z^2+\frac{2a}{b}z+1\right)}}_{f(z)}$$

Double Pole at $z=0$

Pole at $z=\alpha, \beta$.

$$\int_C f(z)\, dz = (2\pi \sum \text{Res})$$

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Q. 2006 $\displaystyle\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$ $(a^2<1)$ $z=e^{i\theta}$

$dz/(iz) = d\theta$

$$\int_0^{\pi} \frac{\cos 2\theta \left(1+a^2+2a\cos\theta\right) d\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} = \int_0^{\pi} \frac{(1+a^2)\cos 2\theta}{(1+a^2)^2-4a^2\cos^2\theta} + \underbrace{\frac{2a\cos 2\theta\cos\theta}{(1+a^2)^2-4a^2\cos^2\theta}}_{\to 0} d\theta$$

$$\int_0^{\pi} \frac{(1+a^2)\cos 2\theta}{(1+a^2)^2-4a^2\cos^2\theta} = \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos 2\theta \, d\theta}{(1+a^2)^2 - 2a^2[1+\cos 2\theta]}$$

As $(\pi+\theta - \theta) \to \theta$ gives $-f(z)$

Put $2\theta = \phi \Rightarrow 2\,d\theta = d\phi$

$z = e^{i\phi}$

$dz/iz = d\theta.$

$$\boxed{\frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos\phi \, d\phi}{1+a^4-2a^2\cos\phi}}$$

$$\frac{(1+a^2)}{4i}\int_c \frac{(z^2+1)\,dz}{z\left((1+a^4)z - a^2z^2 - a^2\right)}$$

$$\frac{(1+a^2)i}{4a^2}\int_c \frac{(z^2+1)\,dz}{z(z-a^2)(z-1/a^2)}$$

($1/a^2$ outside $|z|=1$)

$$\text{Sum} = \frac{(1+a^2)i}{4a^2}\left[1 + \frac{a^4+1}{a^2(a^2-1/a^2)}\right]$$

- Different $n\theta$ in $N^r, D^r \rightarrow$ Rationalise if limits $\neq 2\pi$ | OR If limits 0 to 2π Use $Re\{e^{i2\theta}\}$

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$\boxed{Q}$ UPSC 2004

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} \qquad 0 < p < 1$$

$$\frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta}{1-2p\cos 2\theta + p^2} = Re\left\{\frac{1}{2}\int_0^{2\pi} \frac{1+e^{i6\theta}\, d\theta}{1-2p\cos 2\theta+p^2}\right.$$

$$z = e^{i\theta} \Rightarrow dz/iz = d\theta.$$

$$Re\left\{\frac{1}{2iz}\int_C \frac{1+z^6 \quad dz}{1-\frac{2p}{2}\left[z^2+\frac{1}{z^2}\right]+p^2}\right\} = Re\left\{\frac{z^2}{2iz}\int_C \frac{1+z^6\, dz}{-pz^4+(p^2+1)z^2-p}\right.$$

$$Re\left\{\frac{(-1)}{2pi}\int_C \frac{z(1+z^6) \quad dz.}{z^4-\frac{(p^2+1)}{p}z^2+1}\right\} = Re\left\{\frac{(-1)}{2pi}\int_C \frac{z(1+z^6)\,dz}{(z^2-p)(z^2-1/p)}\right.$$

Only at $z = \pm\sqrt{p}$. lie with $|z|=1$.

$$\text{Residues} = \frac{\sqrt{p}(1+p^3)}{2\sqrt{p}\,(p-1/p)}, \quad \frac{+\sqrt{p}\,(1+p^3)}{+2\sqrt{p}\,(p-1/p)}.$$

$$2\pi i \times \frac{(-1)}{2pi}\left[\frac{(1+p^3)p}{(p^2-1)}\right] \Rightarrow \frac{\pi(1+p^3)}{(1-p^2)} \quad -(1)$$

Equating real parts $\Rightarrow$ we get integral to be (1)

$$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = 2\pi/n!$$

$$\int_0^{2\pi} e^{\cos\theta}\cdot e^{-i(n\theta-\sin\theta)}\, d\theta = \int_0^{2\pi} e^{\cos\theta+i\sin\theta}\cdot e^{-ni\theta}\, d\theta$$

Pg-446

$\int_{-\infty}^{\infty} f(x)\,dx$ $\boxed{\text{TYPE 2}}$

* $f(z)$ analytic in upper half, with finite poles, no poles on Real axis (+) $zf(z) \to 0$ as $|z| \to \infty$

$$\boxed{\int_{-\infty}^{\infty} f(x) = 2\pi i \sum R^+}$$

Take Contour as C of large semi-circle Γ of radius R and real axis $x = -R$ to R

$\boxed{Q}$ $\int_{0}^{\infty} \frac{dx}{x^4 + a^4}$ $(a > 0)$.

$C \equiv$ 

Consider $f(z) = \frac{1}{z^4 + a^4}$

$$\int_C f(z)\,dz \Rightarrow \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \sum R^+ \quad -①$$

$$\lim_{z \to \infty} zf(z) = \frac{z}{z^4 + a^4} = \frac{1}{4z^3} = 0$$

So, $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$

So, ① $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$

Poles of $\frac{1}{z^4 + a^4} \Rightarrow z = ae^{i\pi/4}, ae^{i3\pi/4}, ae^{i5\pi/4}, ae^{i7\pi/4}$

L Only $ae^{i\pi/4}, ae^{i3\pi/4}$ lie in C

Let α be any one of poles $\equiv \alpha^4 = -a^4$.

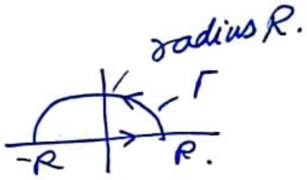
$$\operatorname*{Res}_{z=\alpha} (z-\alpha) f(z) = \lim_{z \to \alpha} \frac{z-\alpha}{z^4 - \alpha^4} \Rightarrow \lim_{z \to \alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \frac{-\alpha}{4a^4}$$

$$\text{Sum of Residues} \equiv -\frac{1}{4a^4}\left(ae^{i\pi/4} + a^{i3\pi/4}\right) = -\frac{1}{4a^3}\left(e^{i\pi/4} - e^{-i\pi/4}\right)$$

$$= \frac{-\sqrt{2}\,i}{4a^3}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4 - a^4}\,dx = \frac{2.\sqrt{2}\pi}{4a^3}$$

Q $\int_0^\infty \frac{x^2}{x^6+1}\,dx. = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx$



$f(z) = \frac{z^2}{z^6+1}$

$\int_{-R}^{R} \frac{x^2}{x^6+1}\,dx. + \int_\Gamma \frac{z^2}{z^6+1}\,dz = 2\pi i \sum R^+$

$\lim_{z\to\infty} z f(z) = 0$ So, $\int_\Gamma \frac{z^2}{z^6+1} \to 0$ as $R\to\infty$.

$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx + \int_\Gamma \overbrace{\frac{z^2}{z^6+1}\,dz}^{0} = 2\pi i \sum R^+.$

Poles of $\frac{z^2}{z^6+1} \Rightarrow z^6 = -1 \Rightarrow z^6 = e^{(2n\pi+1)\pi i}$

$\Rightarrow z = e^{\frac{(2n+1)}{6}\pi i}$

$z = \underbrace{e^{\frac{\pi}{6}i}, e^{\frac{\pi}{2}i}, e^{\frac{5\pi}{6}i}}_{\text{lie in upper half}}, e^{\frac{7\pi}{6}i}, e^{\frac{9\pi}{6}i}, e^{\frac{11\pi}{6}i}.$

$\boxed{\text{* Another residue approach for simple pole}}$

& Residue at $z=\alpha \equiv \left[\frac{z^2}{\frac{d}{dz}(z^6+1)}\right]_{z=\alpha}$

$= \left[\frac{\alpha^2}{6\alpha^5}\right] = \left[\frac{-\alpha^3}{6}\right]$

Sum of residues at poles $\equiv -\frac{1}{6}\left[e^{\pi i/2} + e^{3\pi i/2} + e^{5\pi i/2}\right]$

$= -i/6$

$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{6}$

$\int_0^\infty f(x) = \frac{\pi}{6}$ $\underline{\underline{\text{An}}}$

$\boxed{\text{Ex-7.8 (457)}}$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx / \cos mx \, dx.$$ $\boxed{\text{TYPE - 3}}$

(i) P, Q are polynomials

(ii) $\deg(Q) > \deg(P)$

(iii) $Q(x) = 0$ has no real roots.

$\Rightarrow$ **Jordan's Lemma** $\equiv$

(i) $f(z) \to 0$ uniformly as $z \to \infty$

(ii) $f(z)$ is meromorphic in upper half plane

then $\lim\limits_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$ $(m > 0)$

where Γ denotes semi-circle $|z| = R$

Consider

$$\int_C e^{imz} f(z)\, dz \quad \begin{cases} Re - \cos mx \\ Im - \sin mx. \end{cases}$$

$$\Rightarrow \int_{-R}^{R} e^{imx} f(x)\, dx + \underbrace{\int_{\Gamma} e^{imz} f(z)\, dz}_{\Downarrow} = 2\pi i \sum R^{+}$$

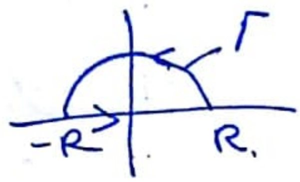
(Jordan's Lemma) $\to 0$ as $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} f(x) \cos mx \, dx + i \int_{-\infty}^{\infty} f(x) \sin mx \, dx = 2\pi i \sum R^{+}$$

Equating real and imaginary parts.

(8) Find $\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx$ by $\int \frac{e^{iz}}{(z-ai)}$ around c.

Let $\int_c \frac{e^{iz}\,dz}{(z-ai)} = \int_c f(z)\,dz$



By Residue Thm $\Rightarrow$.

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \sum R^+$$

By Jordan's Lemma $\Rightarrow \lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0$

So, as $\lim_{R\to\infty} \int_c f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum R^+$

Poles $\Rightarrow z = ai$; Residue at $ai \Rightarrow \lim_{z\to ai} e^{ixai} = e^{-a}$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-ai)}\,dx = 2\pi i\, e^{-a}.$$ (Just rationalise LHS)

$$\int_{-\infty}^{\infty} \frac{e^{ix}(x+ai)}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \frac{(x\cos x - a\sin x) + i(x\sin x + a\cos x)}{x^2+a^2}$$

$= \underline{2\pi i e^{-a}}$

Equate Im parts

(Q) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+x^2+1}\,dx.$

$$\frac{e^{imz}}{z^4+z^2+1}$$

Jordan's $\equiv \int_{\Gamma} f(z)\,dz = 0$

CRT $\equiv \displaystyle\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+x^2+1}\,dx = 2\pi i \sum R^+$

Poles of $\dfrac{e^{imz}}{z^4+z^2+1}$

$(1-z^2)(z^4+z^2+1)$

$= 1-z^6$

Six roots of unity $\equiv$ 4 of which are roots of given

$e^{i\pi/3}$, $e^{i2\pi/3}$ lie in C.
(α) (α^2)

Residue at $z=\alpha \Rightarrow \dfrac{e^{imz}}{\frac{d}{dz}(z^4+z^2+1)} = \dfrac{e^{im\alpha}}{4\alpha^3+2\alpha}, \dfrac{e^{im\alpha^2}}{4\alpha^6+2\alpha}.$

Sum of residues $\Rightarrow$

$$\frac{e^{im\alpha}}{4\alpha^3+2\alpha} + \frac{e^{im\alpha^2}}{4+2\alpha}.$$

Add then, rationalise to simplify

Put $\alpha = e^{i\pi/3} = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right).$

$$= \boxed{\frac{-i}{\sqrt{3}}\, e^{-m(\sqrt{3}/2)} \sin\left(\frac{m}{2}+\frac{\pi}{6}\right)}$$

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$\boxed{\Theta}$ $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2$

Consider $\boxed{\displaystyle\int_c \frac{\log(z+i)}{1+z^2}\,dz} = \displaystyle\int_c f(z)\,dz.$

$$\int_\Gamma f(z)\,dz \le \int_0^\pi \left| \frac{\log(Re^{i\theta}+i)}{1+R^2e^{i2\theta}}\, iRe^{i\theta} \right| d\theta.$$

$\Downarrow$

$$\le \int_0^\infty |\log Re^{i\theta}|$$

No idea

Pg-423

Can use

$(zf(z) \to 0$ as $z \to \infty.)$

TS: $\displaystyle\int_\Gamma f(z)\,dz = 0.$

$$\int_{-\infty}^{\infty} \frac{\log(x+i)}{1+x^2}\,dx = 2\pi i \sum \operatorname{Res}_{x=i} \frac{\log(z+i)}{(z+i)}$$

$$= \cancel{2\pi i}\left(\frac{\log 2i}{\cancel{2i}}\right) = \pi \log 2 + i\frac{\pi^2}{2}$$

Real part $= \pi \log 2$ Ans

$\boxed{9}$ $\int_0^{\infty} \frac{\cos x^2 + \sin x^2 - 1}{x^2} dx = 0$

Consider $\int_C \frac{e^{iz^2} - 1}{z^2} dz$

$\left(zf(z) \Rightarrow \lim_{z\to\infty} \frac{z(\cos z^2 + \sin z^2 - 1)}{z^2} = 0. \right) = \int_\Gamma = 0$

$\int_{-\infty}^{\infty} \frac{e^{ix^2} - 1}{x^2} = 2\pi i \sum \text{Res}(0)$ *

$= 0.$

$\frac{i2xe^{ix^2}}{2x} \Rightarrow i$

No pole at $z=0$

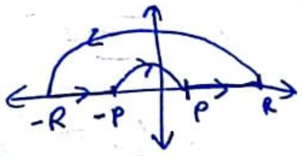
Equate Real, Im parts $= 0$ Ans

TYPE-4.

Poles on Real Axis

Q. $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$

$$\boxed{\int_C \frac{e^{iz}}{z}\,dz} \ast$$

$C \equiv$ 

As no singularity within $C \Rightarrow$

$$\int_\rho^R + \int_\Gamma + \int_{-R}^{-\rho} + \int_\gamma = 0 \quad \left(\text{else } 2\pi i \Sigma R^+\right)$$

$\int_\Gamma \Downarrow$ (Jordan's $\rightarrow 0$)

$\int_\gamma \hookrightarrow \lim_{z\to 0} z f(z) = 1 \cdot \{\theta_1 - \theta_2\}$

So, $\int_\gamma f(z)\,dz = i \times 1 \times (0 - \pi)$ clockwise

$= -\pi i$

As $\rho \to 0$, $R \to \infty \Rightarrow$

$$\int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx = \pi i \Rightarrow \boxed{\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx = \pi i}$$

Equating $\Rightarrow \int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$

When is Jordan's Applicable

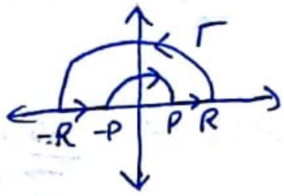
$\hookrightarrow \int_\Gamma e^{imz} f(z)\,dz$

$\ast$ (Remove $e^{i\beta}$)

$\boxed{\text{If } f(z) \to 0 \text{ as } z \to \infty}$

Else $(z = Re^{i\theta})$

10 Q $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$



$$\int_C \frac{1-e^{iz}}{z^2}\,dz$$

$$\int_{-R}^{-\rho} + \int_\gamma + \int_\rho^R + \int_\Gamma = 0$$

As singularity only $z=0$

$$\int_\Gamma \le \int_0^\pi \left|\frac{1-e^{iRe^{i\theta}}}{R^2e^{2i\theta}}\cdot Re^{i\theta}\right| d\theta$$

$$\le \int_0^\pi \frac{1+\exp(-R\sin\theta)}{R} \to 0 \text{ as } R\to\infty$$

$$\int_\gamma = \int_\pi^0 \frac{1-\exp(i\rho e^{i\theta})}{\rho e^{i\theta}}(i)\,d\theta$$

$$= \int_\pi^0 \frac{ie^{-i\theta}}{\rho}\left[1-1-i\rho e^{i\theta}-(\ldots)^2-\right]$$

$$= \int_\pi^0 [1+O(\rho)]\,d\theta \quad \text{as } \lim_{\rho\to 0}$$

So $\Rightarrow \boxed{\int_\gamma = -\pi}$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \pi.}$$

$R\to\infty$
$\rho\to 0$

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$\boxed{Q}$ $\displaystyle\int_{-\infty}^{\infty} \frac{\cancel{x}\sin x}{x(x^2+a^2)} = \frac{\pi}{2a^2}(1-e^{-a})$ $a>0.$

Consider $\Rightarrow$ $\dfrac{\cancel{z}\,e^{iz}}{z(z^2+a^2)}$

$$\int_{-R}^{-\rho} + \int_{\gamma} \frac{\cancel{z}\,e^{iz}}{z(z^2+a^2)}\,dz + \int_{\rho}^{R} + \int_{\Gamma} = 2\pi i \operatorname*{Res}_{(z=ai)}$$

$$\lim_{z\to 0} \frac{\cancel{z}\,e^{iz}}{\cancel{z}(z^2+a^2)} = \frac{1}{a^2}$$

$$\int_{\gamma} = \frac{i}{a^2}(-\pi)$$

$$\lim_{z\to\infty} \frac{1}{z(z^2+a^2)} = 0$$

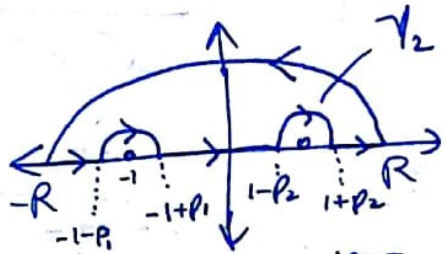
(Jordan's Lemma $\to 0$)

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)}\,dx - \frac{\pi i}{a^2} = -\frac{\cancel{2}\pi i}{a^2}\cdot\frac{e^{-a}}{\cancel{(ai)}(\cancel{2}ai)}$$

Equation Im parts $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)} = \frac{\pi}{a^2}(1-e^{-a})$$ Ans

$\boxed{8}$ $P\int_0^\infty \frac{x^4}{x^6-1}\,dx = \frac{\pi\sqrt{3}}{6}.$



Consider $\int_C \frac{z^4}{z^6-1}\,dz$

$$\int_{-R}^{-1-\rho_1} + \int_{\gamma_1} + \int_{-1+\rho_1}^{1-\rho_2} + \int_{\gamma_2} + \int_{1+\rho_2}^{R} + \int_{\Gamma} = 2\pi i \,\Sigma R$$

$z^6 = e^{i2n\pi}$ $\Rightarrow z = 1, e^{i2\pi/6}, e^{i4\pi/6}, -1, e^{i8\pi/6}, e^{i10\pi/6}$

γ_1: $\lim_{z\to -1}(z+1)f(z) = -1/6$

$\Downarrow$

$\int = -\frac{1}{6}\times i \times(-\pi) = \frac{\pi i}{6}$

γ_2: $\lim_{z\to 1}(z-1)f(z) = \frac{z^4(z-1)}{z^6-1} = \frac{5z^4-4z^3}{6z^5} = \frac{1}{6}$

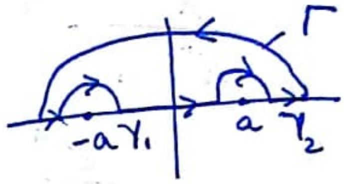
$\lim_{\rho_2\to 0}\int_{\gamma_2} = \frac{1}{6}\times i\,(-\pi) = \frac{-\pi i}{6}$

Γ: $|I| \le \int_0^\pi \left|\frac{R^5 i e^{5i\theta}}{R^6 e^{6i\theta}-1}\right| d\theta \le \int_0^\pi \frac{R^5}{R^6-1}\,d\theta \to 0$ as $R\to\infty$

$$\int_{-R}^{-1-\rho_1} + \int_{-1+\rho_1}^{1-\rho_2} + \int_{1+\rho_2}^{R} + 0 = 2\pi i\left[\frac{\alpha^4}{6\alpha^5} + \frac{\alpha_2^4}{6\alpha_2^5}\right] = \frac{\pi}{\sqrt{3}}.$$

$$\boxed{P\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{3}}} \Rightarrow \int_0^\infty \frac{x^4}{x^6-1}\,dx = \frac{\pi}{2\sqrt{3}}$$ Ans

Q) $P\int_{-\infty}^{\infty} \frac{\cos x}{a^2-x^2}\,dx.$



$$f(z) = \frac{e^{iz}}{a^2-z^2}$$

$$\int_{-R}^{-a-\rho_1} + \int_{\gamma_1} + \int_{-a+\rho_1}^{a-\rho_2} + \int_{\gamma_2} + \int_{a+\rho_2}^{R} + \int_{\Gamma} = 0$$

(No singula within C)

γ_1: $\lim_{z\to -a}\left(\frac{e^{-ia}}{a+a}\right)$ ↳ $\int = \frac{ie^{-ia}}{2a}(-\pi)$

γ_2: $\lim_{z\to a} \frac{e^{ia}}{(-2a)}$ ↳ $\frac{ie^{ia}\pi}{2a}$

Γ: Jordan's Lemm $\lim_{z\to\infty} \frac{1}{a^2-z^2} = 0.$

$$P\int_{-\infty}^{\infty} f(x)\,dx = -\frac{\pi}{2a}\, i\left(e^{ia} - e^{-ia}\right) = \frac{\pi}{a}\sin a.$$

Equate real part ⇒ $\underline{\underline{\ \ }}$

Left Over Problems

① Find z where w defined ceases to be analytic

(i) Case I : $w = u + iv$ $\quad \frac{dz}{dw} = \frac{\partial z}{\partial u}$. $\boxed{\text{Look } \frac{dw}{dz}}$

• $z = e^{-v}(+\cos u + i\sin u)$, where $w = u + iv$

$$\frac{dz}{dw} = \frac{\partial z}{\partial u} = e^{-v}(-\sin u + i\cos u) = ie^{-v}(\cos u + i\sin u)$$
$$= iz.$$

$$\frac{dw}{dz} = \frac{1}{iz} \Rightarrow \text{Not analytic at } z = 0$$

(w is not analytic at $z=0$)

(ii) Case II : $w = \rho(\cos\phi + i\sin\phi)$ $\quad \frac{dz}{dw} = e^{-i\phi}\frac{\partial z}{\partial \rho}$.

• $z = \log \rho + i\phi$

$$\frac{dz}{dw} = (\cos\phi - i\sin\phi).\frac{1}{\rho} \Rightarrow \frac{dw}{dz} = \rho(\cos\phi + i\sin\phi) = w.$$

So w is analytic in finite domain

② Show analytic function with constant modulus is constant.

$f(z) = u + iv$

As $|f(z)| = u^2 + v^2 = c^2 \Rightarrow 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0.$

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

$$\left.\begin{array}{l} u u_x - v u_y = 0 \\ u u_y + v u_x = 0 \end{array}\right\} (u^2+v^2)u_x = 0 \Rightarrow u_x = 0; \text{ Ily } u_y = v_x = v_y = 0$$

So, u, v are constant.

③ A function with no singularity in finite part of plane or at infinity is constant.

- If no singularity in finite
$$f(z) = \sum_0^\infty a_n z^n$$
- If no singularity at ∞
$f\left(\frac{1}{z}\right)$ is analytic $\Rightarrow f\left(\frac{1}{z}\right) = \sum_{n=0}^\infty b_n z^n$

$$\sum_0^\infty \frac{a_n}{z^n} = \sum_0^\infty b_n z^n \Rightarrow b_n = a_n = 0 \quad \forall n \geq 1$$
$$a_0 = b_0 \text{ for } n = 0$$

So $f(z) = a_0 \leftarrow$ constant.

④ f is analytic in finite part and as $|z| \to \infty$ $|f(z)| = a|z|^k$
Show $f(z)$ is poly deg $\leq k$.

- $f(z) = \sum_0^\infty a_n z^n$ for $|z| < R$.

Max of $f(z)$ is on $|z| = R \equiv M(r)$

$\max|f(z)| = M(r)$ $\quad a_n = \frac{f^{(n)}}{n!}$

By Cauchy's Inequality $\Rightarrow |a_n| \leq \frac{M(r)}{r^n} = \frac{A|z|^k}{r^n} = Ar^{k-n}$

$\boxed{\frac{n!\,M}{r^n}}$

As $r \to \infty$, RHS $\to 0$ as $n > k$.

So, $\boxed{a_n = 0 \text{ for } n > k.}$

(5) Integration of many valued functions [DO NOT ATTEMPT]

- x^{a-1} etc.
- $Log(1+x^2)$

* Make indentation around O.
* Integrate $-R \to -\rho$ as $\boxed{\int_R^\rho f(xe^{i\pi}) e^{i\pi} dx}$

(6) $f(z)$ is analytic and E is a set of zeroes of $f(z)$ with a lt point, Then $f(z)=0$

→ $f(z)$ analytic ⇒ Zeroes isolated.
α is lt point ⇒ $f(z)=0$ at ∞ pts near α. } $f(z)=0$

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* $f(z)$ be analytic in $|z| < \rho$ and $z = re^{i\theta}$ be any pt in region.

Show $f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$

where $0 < R < \rho$

Let C be $|z| = R$ s.t $r < R < \rho$

$z = re^{i\theta}$

$$f(z) = \frac{1}{2\pi i} \int \frac{f(w)}{w - z}\, dw. \quad -(1)$$

Inverse of z is $\frac{R^2}{\bar{z}} \Rightarrow \frac{f(w)}{w - R^2/\bar{z}}$ is analytic within, on C.

$$\rightarrow \int \frac{f(w)}{w - R^2/\bar{z}} = 0 \quad -(2).$$

$$(1) - (2) \Rightarrow f(z) = \frac{1}{2\pi i} \int \left(\frac{1}{w - z} - \frac{1}{w - R^2/\bar{z}} \right) f(w)\, dw$$

Now $z = re^{i\theta}$, $w = Re^{i\phi}$, $\bar{z} = re^{-i\theta}$,

$$f(re^{i\theta}) = \frac{1 \cdot i}{2\pi i} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{R^2 - Rr\{e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}\} + r^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})\, d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

* Luca's Theorem

If all zeroes of a polynomial $P_n(z)$ lie in a half plane, show zeroes of $P_n'(z)$ will lie in same half plane.

$$\boxed{\begin{aligned} &P(z) = a_n(z-z_1)\dots(z-z_n) \\ &\frac{P'(z)}{P(z)} = \frac{1}{z-z_1} + \dots \frac{1}{z-z_n} \end{aligned}}$$

– Let $z_1, \dots z_n$ be zeroes of $P_n(z)$

Let half plane be, $\boxed{\operatorname{Im}\left(\frac{z-a}{b}\right) < 0}$

If z_k lies in this half plane then $\operatorname{Im}\left(\frac{z_k - a}{b}\right) < 0$ — (1)

Let α be any zero of $P'(z)$

TS: α's belong to same plane

Suppose if not $\Rightarrow \operatorname{Im}\left(\frac{\alpha - a}{b}\right) > 0$. — (2)

$$\operatorname{Im}\left[\frac{\alpha - z_k}{b}\right] = \operatorname{Im}\left[\frac{(\alpha - a) - (z_k - a)}{b}\right] = \operatorname{Im}\left(\frac{\alpha - a}{b}\right) - \operatorname{Im}\left(\frac{z_k - a}{b}\right)$$

(> 0) from (1), (2)

$\boxed{\text{FACT: } \operatorname{Im} z = -|z|^2 \operatorname{Im} z^{-1}}$

$$\Rightarrow -\left|\frac{b}{\alpha - z_k}\right|^2 \operatorname{Im}\left(\frac{b}{\alpha - z_k}\right) > 0$$

$$\Rightarrow \operatorname{Im}\left[\frac{b}{\alpha - z_k}\right] < 0$$ = As it is true for all z_k

$$\operatorname{Im}\left[b\frac{P'(\alpha)}{P(\alpha)}\right] = \sum_1^n \operatorname{Im}\left(\frac{b}{\alpha - z_k}\right) < 0. \text{ — (3)}$$

This implies $P'(\alpha) \neq 0$

$\Downarrow$

So, α is not a zero of $P'(z)$

Or if α is zero $\Rightarrow P'(\alpha) = 0$

$\Rightarrow$ (3) must be 0

but (3) < 0

Thus all zeroes of $P'(z)$ lie in same half-plane

① Finding Residue

- Coeff of $1/t$ in $f(a+t)$.

Ex: Res $\frac{1}{(z^2+1)^3}$ at $z=i$

$$\frac{1}{((i+t)^2+1)^3} = \frac{1}{(t^2+2it)^3} = \frac{1}{(2it)^3\left(1+\frac{t}{2i}\right)^3}$$

$$\frac{-1\left(1+\frac{t}{2i}\right)^{-3}}{8t^3 i} = \frac{-1}{8t^3 i}\left(1-\frac{3t}{2i}+6\left(\frac{t}{2i}\right)^2 - \cdots\right)$$

$$\text{Coeff } \frac{1}{t} \equiv +\frac{1}{8i}\cdot\frac{6}{4} = \boxed{\frac{3}{16i}} \Leftarrow \text{Residue.}$$