2019 CSE

1(d) Suppose $f(z)$ is analytic function on a domain D in $\mathbb{C}$ and satisfies the equation
$$\operatorname{Im} f(z) = [\operatorname{Re} f(z)]^2, \quad z \in D.$$
Show that $f(z)$ is constant in D.

Let $f(z) = u + iv$

Given $v = u^2$

$$\therefore \left.\begin{aligned} v_x &= 2u \cdot u_x \\ v_y &= 2u \cdot u_y \end{aligned}\right] \text{ — } ①$$

Also, As $f(z)$ is analytic it satisfies C-R equations

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right] \text{ — } ②$$

Let us eliminate u_y and v_y from ① using ②

$$v_x = 2u \cdot u_x$$
$$u_x = 2u \cdot (-v_x)$$
ie. $u_x = -2u(2u \cdot u_x)$

ie. $u_x(1+2u^2) = 0$.

$\Rightarrow u_x = 0 \quad \therefore v_x = 0$

$[\because 1+2u^2 \neq 0]$

$$\therefore f'(z) = u_x + i v_x$$
$$= 0$$

By integrating, $f(z) = K$ (constant)

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2(d) Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form
$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$
where $\phi(z)$ is analytic and non-zero at z_0.
Moreover,
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \quad \text{if } m \geq 1.$$

First, let $f(z)$ has a pole of order m, then by definition, for $0<|z-z_0|<R$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}, \quad b_m \neq 0.$$

$$= \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \dots + b_m\right]$$

$$\therefore \quad f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

Clearly, $\phi(z_0) = b_m \neq 0$ and is analytic at z_0, as it has Taylor series expansion about z_0.

Conversely,
Suppose $f(z)$ can be written in the form
$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \quad \text{then}$$

$$\phi(z) = \phi(z_0) + \phi'(z_0)(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \dots$$

($\because$ $\phi(z)$ is analytic, hence Tayor expansion is possible)

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for $0 < |z-z_0| < R$

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)}{2!\,(z-z_0)^{m-2}}$$

$$+ \dots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \cdot \frac{1}{(z-z_0)}$$

$$+ \frac{\phi^{(m)}(z_0)}{m!} + \frac{\phi^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots$$

3)

Since, $\phi(z_0) \neq 0$, $f(z)$ has a pole of order m.

with residue,

$$b_1 = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

Also, in case of simple pole, ie. $m=1$

$$\operatorname*{Res}_{z=z_0} f(z) = \phi(z_0).$$

Hence Proved.

Definition of Residue at a Pole of order m.

Let $z=a$ be a pole of order m of $f(z)$

If

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

is the Laurent's series expansion of $f(z)$ in a region $0 < |z-a| < \delta$, then The cofficient b_1 is called the residue of $f(z)$ at $z=a$.

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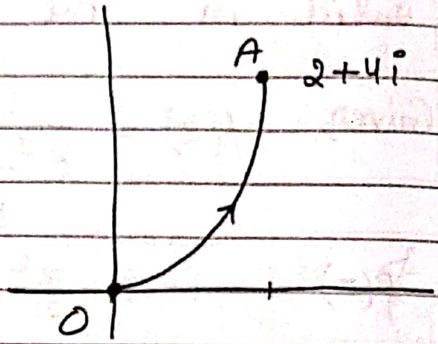
Evaluate the integral $\int_C Re(z^2)\,dz$ ~~fro~~ from 0 to $2+4i$ along the curve C where C is a parabola $y = x^2$.

C: Parabola, $y = x^2$

$\Rightarrow \quad dy = 2x\,dx$



$$I = \int_C Re(z^2)\,dz$$

$$= \int_C Re(x+iy)^2\,dz$$

$$= \int_0^{2+4i} (x^2-y^2)(dx+i\,dy)$$

$[\because\ z = x+iy \Rightarrow dz = dx+i\,dy]$

$$= \int_0^2 [x^2-(x^2)^2](dx + i\,2x\,dx)$$

$$= \int_0^2 (x^2-x^4)(1+2ix)\,dx$$

$$= \int_0^2 [x^2+2x^3 i - x^4 - 2x^5 i]\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{2}i - \frac{x^5}{5} - \frac{x^6}{3}i\right]_0^2$$

$$= -\frac{56}{15} - i\,\frac{40}{3}$$

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(4b) Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{e^z - 1}$ about the point $z=0$ valid in the region, $0 < |z| < 2\pi$.

Given, $f(z) = \frac{1}{e^z - 1}$ about the point $z=0$ in region $0<|z|<2\pi$.

$$f(z) = \frac{1}{\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\right)-1}$$

$$= \frac{1}{z\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)}$$

$$= \frac{1}{z}\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)\right]^{-1}$$

$$= \frac{1}{z}\left[1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\dots\right)+\frac{z^2}{(2!)^2}+O(z^3)\right]$$

[using binomial expansion, $(1+x)^{-1} = 1-x+x^2-x^3+x^4+\dots, \; |x|<1$]

$$= \frac{1}{z}\left[1-\frac{z}{2}+\frac{z^2}{12}+O(z^3)\right]$$

$$= \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+O(z^2) \quad \text{—— } (A)$$

Since, Laurent series expansion is about $z=0$ ie. $|z|<1$, hence (A) is valid.

Hence, first three terms are $\frac{1}{z}, \frac{-1}{2}, \frac{z}{12}$.

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