8(a) Find the general solution of the DE
$(x-2)y'' - (4x-7)y' + (4x-6)y = 0.$

(10)

$$y'' - \left(\frac{4x-7}{x-2}\right)y' + \left(\frac{4x-6}{x-2}\right)y = 0 \quad \text{—} (A)$$

Comparing with: $y'' + Py' + Qy = 0.$

$$P = \frac{-(4x-7)}{x-2}, \quad Q = \frac{4x-6}{x-2}$$

Let e^{ax} be a solution, then

$$a^2 + aP + Q = 0$$

$$a^2 - \frac{a(4x-7)}{x-2} + \frac{4x-6}{x-2} = 0$$

$\Rightarrow \quad a^2(x-2) - 4ax + 7a + 4x - 6 = 0$

i.e. $\quad x(a^2 - 4a + 4) - 2a^2 + 7a - 6 = 0$

$\Rightarrow \quad a^2 - 4a + 4 = 0 \quad \Rightarrow (a-2)^2 = 0 \Rightarrow a = 2$

& $\quad 2a^2 - 7a + 6 = 0 \quad \Rightarrow (2a-3)(a-2) = 0$

i.e. $a = 3/2, 2$

$\therefore \quad a = 2$ is common root

$\Rightarrow e^{2x}$ is one solution.

Consider,

$$y = v \cdot e^{2x}$$

$$\frac{dy}{dx} = \frac{dv}{dx}e^{2x} + 2ve^{2x}$$

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \cdot e^{2x} + 4\frac{dv}{dx}e^{2x} + 4ve^{2x}$$

Putting these values in (A)

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$$\left(\frac{d^2v}{dx^2}e^{2x}+4\frac{dv}{dx}e^{2x}+4ve^{2x}\right)-\left(\frac{4x+7}{x-2}\right)\left(\frac{dv}{dx}e^{2x}+2ve^{2x}\right)$$

$$+\frac{4x-6}{x-2}\cdot ve^{2x}=0.$$

$$\frac{d^2v}{dx^2}+\frac{dv}{dx}\left[4-\frac{(4x+7)}{x-2}\right)+4v-\frac{2(4x+7)v}{x-2}+\frac{(4x-6)v}{x-2}$$

$$=0$$

$$\frac{d^2v}{dx^2}-\frac{1}{x-2}\cdot\frac{dv}{dx}=0$$

Let $\frac{dv}{dx}=p \Rightarrow \frac{dp}{dx}-\frac{1}{x-2}\cdot p=0$

$$\frac{dp}{p}=\frac{dx}{x-2}$$

$$\log p=\log(x-2)+\log C_1$$

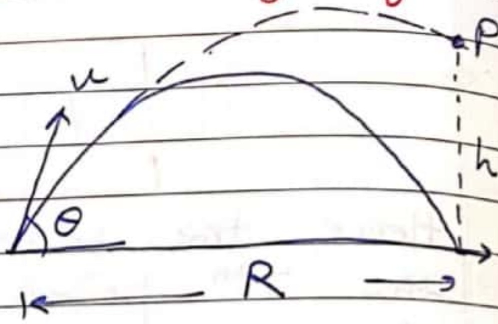
$$\Rightarrow p=C_1(x-2)$$

i.e. $\frac{dv}{dx}=C_1(x-2)$

$$v=C_1\left(\frac{x^2}{2}-2x\right)+C_2\ )$$

$$\therefore y=e^{2x}\cdot v=e^{2x}\left[C_1\left(\frac{x^2}{2}-2x\right)+C_2\right]$$

8(b) A shot projected with a velocity 'u' can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h meter above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection.



We know that

$$x = (u\cos\theta)t$$

$$y = (u\sin\theta)t - \frac{1}{2}gt^2$$

Equation of trajectory,

$$y = x\tan\theta - \frac{g}{2}\cdot\frac{x^2}{u^2\cos^2\theta}$$

When velocity is u,

Range, $R = \dfrac{u^2\sin 2\theta}{g}$

With new velocity (say v), point P(R, h) lies on the eqn of trajectory

$$h = R\tan\theta - \frac{g}{2}\cdot\frac{R^2}{v^2\cos^2\theta}$$

$$= \frac{u^2\sin 2\theta}{g}\cdot\tan\theta - \frac{g}{2}\left(\frac{u^2\sin 2\theta}{g}\right)^2\frac{1}{v^2\cos^2\theta}$$

$$= \frac{2u^2\sin^2\theta}{g} - \frac{2u^4\sin^2\theta}{g\cdot v^2}$$

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$$h = \frac{2u^2 \sin^2\theta}{g}\left(1 - \frac{u^2}{v^2}\right)$$

$$\Rightarrow \quad 1 - \frac{u^2}{v^2} = \frac{gh}{2u^2\sin^2\theta}$$

$$\frac{u}{v} = \left[1 - \frac{gh}{2u^2\sin^2\theta}\right]^{1/2}$$

i.e. $$v = u\left(1 - \frac{gh}{2u^2\sin^2\theta}\right)^{-1/2}$$

$$\simeq u\left(1 + \frac{1}{2}\cdot\frac{gh}{2u^2\sin^2\theta}\right)$$ (Binomial Approximation)

$$\therefore \quad \boxed{v - u = \frac{gh}{4u\sin^2\theta}}$$

Which is the required increase in the velocity of projection with same elevation θ°.

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8(c) Derive $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in spherical coordinates and compute $\nabla^2\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$ in spherical coordinates. (15)

$$\nabla^2 F = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial F}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial F}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial F}{\partial u_3}\right)\right]$$

Here,

$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi$

$h_1 = h_r = 1, \quad h_2 = h_\theta = r$

$h_3 = h_\phi = r\sin\theta$

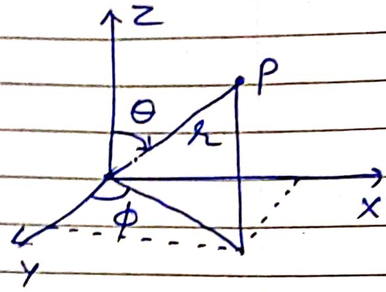
$$\nabla^2 F = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r \cdot r\sin\theta}{1}\frac{\partial F}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{\cancel{r}\sin\theta \cdot 1}{\cancel{r}}\frac{\partial F}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1 \cdot \cancel{r}}{\cancel{r}\sin\theta}\frac{\partial F}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial F}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial F}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot \frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}$$

$$F = \frac{x}{(x^2+y^2+z^2)^{3/2}} = \frac{r\sin\theta\cos\phi}{r^3}$$

$$= \frac{\sin\theta\cos\phi}{r^2}$$



$$\frac{\partial F}{\partial r} = \frac{-2\sin\theta\cos\phi}{r^3}$$

$$\frac{\partial F}{\partial \theta} = \frac{\cos\theta\cos\phi}{r^2}$$

$$\frac{\partial F}{\partial \phi} = \frac{-\sin\theta\sin\phi}{r^2}$$

$$\frac{\partial^2 F}{\partial \phi^2} = \frac{-\sin\theta\cos\phi}{r^2}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{-2\cancel{r^2}\sin\theta\cos\phi}{r^{\cancel{3}}}\right)$$

$$= \frac{-2\sin\theta\cos\phi}{r^2}$$

$$\therefore \nabla^2 F = \frac{1}{r^2}\left(\frac{2\sin\theta\cos\phi}{r^2}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta\cos\theta\cos\phi}{r^2}$$

$$- \frac{1}{r^2\sin^{\cancel{2}}\theta}\cdot\frac{\cancel{\sin}\theta\cos\phi}{r^2}$$

$$= \frac{2\sin\theta\cos\phi}{r^4} + \frac{\cos2\theta\cdot\cos\phi}{r^4\sin\theta} - \frac{\cos\phi}{r^4\sin\theta}$$

$$= \frac{\cos\phi}{r^4}\left[2\sin\theta + \frac{\cos2\theta - 1}{\sin\theta}\right]$$

$$= \frac{\cos\phi}{r^4}\left[\frac{\cancel{2\sin^2\theta} + (\cancel{1} - \cancel{2\sin^2\theta}) - \cancel{1}}{\sin\theta}\right]$$

$$= 0.$$