Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x-h)^2+(y-k)^2+z^2=\lambda^2$. **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $(x-h)^2+(y-k)^2+z^2=\lambda^2$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$2(x-h)+2z(\partial z/\partial x)=0$ or $(x-h)=-z(\partial z/\partial x)$ ...(2)

and $2(y-k)+2z(\partial z/\partial y)=0$ or $(y-k)=-z(\partial z/\partial y)$. ...(3)

Substituting the values of $(x-h)$ and $(y-k)$ from (2) and (3) in (1) gives

$z^2(\partial z/\partial x)^2+z^2(\partial z/\partial y)^2+z^2=\lambda^2$ or $z^2[(\partial z/\partial x)^2+(\partial z/\partial y)^2+1]=\lambda^2$,

which is the required partial differential equation.

Ex. 5(*b*). *Find the differential equation of all spheres of radius* λ, *having centre in the xy-plane.* **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$(x-h)^2+(y-k)^2+(z-0)^2=\lambda^2$ or $(x-h)^2+(y-k)^2+z^2=\lambda^2$, ...(1)

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z=(x^2+a)(y^2+b)$.

[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $z=(x^2+a)(y^2+b)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x=2x(y^2+b)$ or $(y^2+b)=(1/2x)\times(\partial z/\partial x)$ ...(2)

and $\partial z/\partial y=2y(x^2+a)$ or $(x^2+a)=(1/2y)\times(\partial z/\partial y)$. ...(3)

Substituting the values of (y^2+b) and (x^2+a) from (2) and (3) in (1) gives

$z=(1/2y)\times(\partial z/\partial y)\times(1/2x)\times(\partial z/\partial x)$ or $4xyz=(\partial z/\partial x)(\partial z/\partial y)$,

which the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2+y^2=(z-c)^2\tan^2\alpha, \quad \text{... (1)}$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$2x=2(z-c)(\partial z/\partial x)\tan^2\alpha$ and $2y=2(z-c)(\partial z/\partial y)\tan^2\alpha$

$\Rightarrow$ $y(z-c)(\partial z/\partial x)\tan^2\alpha=xy$ and $x(z-c)(\partial z/\partial y)\tan^2\alpha=xy$

$\Rightarrow$ $y(z-c)(\partial z/\partial x)\tan^2\alpha=x(z-c)(\partial z/\partial y)\tan^2\alpha$

Thus, $y(\partial z/\partial x)=x(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation.*

[I.A.S. 1979, 2009]

Sol. The equation of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \quad \text{...(1)}$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p = \partial z/\partial x$ and $q = \partial z/\partial y$)

$$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0 \quad \text{or} \quad ax + gz + hy + p(cz + gx + fy) = 0 \quad \text{...(2)}$$

and $$2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0 \quad \text{or} \quad by + fz + hx + q(cz + fy + gx) = 0. \quad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding, we have

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0.$$

$$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0, \text{ using (1)}$$

or $$(cz + fy + gx)(px + qy - z) = 0 \quad \text{or} \quad px + qy - z = 0, \quad \text{...(4)}$$

which is required partial differential equation.

Second Part : Given surface is $$yz + zx + xy = 0 \quad \text{...(5)}$$

Differentiating (5) partially w.r.t. 'x' and 'y' by turn, we get

$$yp + px + z + y = 0 \quad \text{and} \quad z + qy + xq + x = 0. \quad \text{...(6)}$$

Solving (6) for p and q, $\quad p = -(z + y)/(x + y) \quad$ and $\quad q = -(z + x)/(x + y).$

$$\therefore \quad px + qy - z = -\frac{x(z+y)}{x+y} - \frac{y(z+x)}{x+y} - z = -\frac{2(xy+yz+zx)}{x+y} = 0, \text{ using (5)}$$

Hence (5) is a surface satisfying (4).

Ex. 11. *Eliminate a, b and c from $z = a(x + y) + b(x - y) + abt + c$* **[I.A.S. 1998]**

Sol. Given $$z = a(x + y) + b(x - y) + abt + c \quad \text{... (1)}$$

Differentiating (1) partially w.r.t. 'x', 'y' and 't', we get

$$\partial z/\partial x = a + b \quad \text{... (2)} \qquad \partial z/\partial y = a - b \quad \text{... (3)} \qquad \partial z/\partial t = ab \quad \text{... (4)}$$

We have the identity: $$(a + b)^2 - (a - b)^2 = 4ab$$

$$\therefore \quad (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t), \text{ using (2), (3) and (4)}$$

Ex. 12. *Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$.* **[I.A.S. 2002]**

Sol. (*a*) Given $$\log(az - 1) = x + ay + b \quad \text{... (1)}$$

Differentiating (1) partially w.r.t. 'x', we get $$\frac{a}{az-1}\frac{\partial z}{\partial x} = 1 \quad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\frac{a}{az-1}\frac{\partial z}{\partial y} = a \quad \text{... (3)}$$

From (3), $\quad az - 1 = \dfrac{\partial z}{\partial y} \quad$ so that $$a = \frac{1+(\partial z/\partial y)}{z} \quad \text{... (4)}$$

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1+(\partial z/\partial y)}{z(\partial z/\partial y)}\frac{\partial z}{\partial x} = 1 \quad \text{or} \quad \left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$. ...(2)

Then, (1) becomes $\phi(u, v) = 0$. ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \quad ...(4)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x$, $\partial u/\partial y = 2y$, $\partial u/\partial z = 2z$, $\partial v/\partial x = -2y$, $\partial v/\partial y = -2x$, $\partial v/\partial z = 2z$. ...(5)

Using (5), (4) reduces to $(\partial \phi/\partial u)(2x + 2pz) + (\partial \phi/\partial v)(-2y + 2pz) = 0$

or $(x + pz)(\partial \phi/\partial u) = (y - pz)(\partial \phi/\partial v)$. ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $(\partial \phi/\partial u)(2y + 2qz) + (\partial \phi/\partial v)(-2x + 2qz) = 0$, by (5)

or $(y + qz)(\partial \phi/\partial u) = (x - qz)(\partial \phi/\partial v)$. ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x$.

Ex. 9. *Form partial differential equation by eliminating arbitrary functions f and g from* $z = f(x^2 - y) + g(x^2 + y)$. **[Nagpur 1996 ; I.A.S. 1996; Kanpur 2011]**

Sol. Given $z = f(x^2 - y) + g(x^2 + y)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}. \quad ...(2)$$

and $$\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y). \quad ...(3)$$

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \quad ...(4)$$

and $$\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y). \quad ...(5)$$

Again, (2) $\Rightarrow$ $f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x)$. ...(6)

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2},$$

which is the required partial differential equation.

Ex. 10. *Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation.* **[I.A.S. 1997]**

Sol. From coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$z = \phi[(x^2 + y^2)^{1/2}]$, where ϕ is an arbitrary function. ...(1)

Differentating (1) partially with respect to x and y, we get

$$\partial z/\partial x = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2x \quad ...(2)$$

and $$\partial z/\partial y = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2y. \quad ...(3)$$

Dividing (2) by (3), $$\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{x}{y} \quad \text{or} \quad y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}.$$

Ex. 11. *Form a partial differential equation by eliminating the arbitrary functions f and g from* $z = y\ f(x) + x\, g(y)$. (*Guwahati 2007*)

Sol. Given $z = y\, f(x) + x\, g(y)$. ...(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\partial z/\partial x = y\, f'(x) + g(y)$ ...(2) $\partial z/\partial y = f(x) + x\, g'(y)$. ...(3)

Differentiating (3) with respect to x, $\partial^2 z/\partial x\partial y = f'(x) + g'(y)$. ...(4)

From (2) and (3), $f'(x) = \dfrac{1}{y}\left[\dfrac{\partial z}{\partial x} - g(y)\right]$ and $g'(y) = \dfrac{1}{x}\left[\dfrac{\partial z}{\partial y} - f(x)\right]$.

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $xy\dfrac{\partial^2 z}{\partial x\partial y} = x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - \{x\, g(y) + y\, f(x)\}$ or $xy\dfrac{\partial^2 z}{\partial x\partial y} = x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - z$, by (2)

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$. **[Patna 2003; Madras 2005; Delhi Maths Hons. 1 9 9 1 ; Bhopal 2004; Meerut 2008, 10; Sagar 2002; I.A.S. 1977; Kanpur 2005, 06]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad ...(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + n\, dz}{l(mx - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + n\, dz}{0}.$$

$\therefore\ ldx + mdy + n\, dz = 0$ so that $l\, x + m\, y + n\, z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$\phi\,(x^2 + y^2 + z^2,\ l\, x + m\, y + n\, z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$. **[I.A.S. 2004; Agra 2005 ; Delhi Maths (H) 2006; M.S. Univ. T.N. 2007; Indore 2003; Meerut 2009; Purvanchal 2007]**

Sol. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}. \qquad ...(1)$$

Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx+(1/y)dy+(1/z)dz}{y^2+z-(x^2+z)+x^2-y^2} = \frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\Rightarrow \log(xyz) = \log c_1 \Rightarrow xyz = c_1$. ...(2)

Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx+ydy-dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)} = \frac{xdx+ydy-dz}{0}$$

$\Rightarrow x\,dx + y\,dy - z\,dz = 0$ so that $x^2+y^2-2z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(x^2+y^2-2z, xyz) = 0$, ϕ is being an arbitrary function.

Ex. 17. *Solve* $x(y-z)\,p + y\,(z-x)q = z(x-y)$, *i.e.*, $\{(y-z)/(yz)\}p + \{(z-x)/(zx)\}q = (x-y)/(xy)$. **[Delhi B.A (Prog) II 2010; I.A.S. 2005, M.S. Univ. T.N. 2007; Vikram 2003]**

Sol. Given $x(y-z)p + y(z-x)q = z(x-y)$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ ... (2)

Choosing $1/x$, $1/y$, $1/z$ as multipliers each fraction of (1)

$$= \frac{(1/x)dx+(1/y)dy+(1/z)dz}{(y-z)+(z-x)+(x-y)} = \frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore \log(xyz) = c_1$ or $xyz = c_1$ ... (3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(xy-xz)+(yz-yx)+(zx-zy)} = \frac{dx+dy+dz}{0}$$

$\Rightarrow dx + dy + dz = 0$ so that $x+y+z = c_2$ ... (4)

From (3) and (4), solution is $\phi(x+y+z, xyz) = 0, \phi$ being an arbitrary function.

Ex. 3. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$ or $(y^2 + z^2 - x^2)p - 2xyq = -2xz$.

[Bangalore 1993, I.A.S. 1973; P.C.S. (U.P.) 1991; Bhopal 2010]

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2} = \frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dz}{-2xz} \quad \text{or} \quad \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log(x^2 + y^2 + z^2) - \log z = \log c_2$ or $(x^2 + y^2 + z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi\,(y/z, (x^2 + y^2 + z^2)/z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. **Delhi Math (H) 2005, 11, M.D.U. Rohtak 2005; Agra 2008, 09; Guwahati 2007; Meerut 2006; Sagar 2000; Ravishankar 2000; Lucknow 2010]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-zx} = \dfrac{dz}{z^2-xy}$. ...(1)

Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$= \frac{dx - dy}{x^2 - y^2 + z(x-y)} = \frac{dy - dz}{(y-z)(y+z+x)}$$

so that $\dfrac{dx-dy}{(x-y)(x+y+z)} = \dfrac{dy-dz}{(y-z)(y+z+x)}$ or $\dfrac{d(x-y)}{x-y} - \dfrac{d(y-z)}{y-z} = 0.$

Integrating, $\log(x - y) - \log(y - z) = \log c_2$ or $(x - y)/(y - z) = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{xdx + ydy + zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}. \qquad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx}. \qquad ...(4)$$

From (3) and (4), $\dfrac{xdx + ydy + zdz}{x+y+z} = dx + dy + dz$

or $\quad 2(x + y + z)\,d(x + y + z) - (2xdx + 2ydy + 2zdz) = 0.$

Integrating, $\quad (x + y + z)^2 - (x^2 + y^2 + z^2) = 2c_2$

or $\quad (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (x^2 + y^2 + z^2) = 2c_2$

or $\quad xy + yz + zx = c_2$, c_2 being an arbitrary constant. ...(5)

From (2) and (5), the required general solution is given by

$$\phi\,[xy + yz + zx, (x - y)/(y - z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 9. *Solve* $\cos(x+y)p + \sin(x+y)q = z$. **[Garhwal 2010, Vikram 1998; Meerut 2007; Delhi Maths (H) 2007; Rajasthan 1994; Delhi B.A./B.Sc. (Prog.) Maths 2007]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{\cos(x+y)} = \dfrac{dy}{\sin(x+y)} = \dfrac{dz}{z}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \quad ...(2)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ...(3)

From (1), (2) and (3), $\dfrac{dz}{z} = \dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ...(4)

Taking the first two fractions of (4), $\dfrac{dz}{z} = \dfrac{d(x+y)}{\cos(x+y)+\sin(x+y)}$. ...(5)

Putting $x+y=t$ so that $d(x+y)=dt$, (5) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{(1/\sqrt{2})\cos t+(1/\sqrt{2})\sin t\right\}} = \frac{dt}{\sqrt{2}\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $(\sqrt{2}/z)dz = \operatorname{cosec}(t+\pi/4)\ dt$.

Integrating, $\sqrt{2}\log z = \log\tan\dfrac{1}{2}\left(t+\dfrac{\pi}{4}\right)+\log c_1$, or $z^{\sqrt{2}} = c_1\tan\left(\dfrac{t}{2}+\dfrac{\pi}{8}\right)$

or $z^{\sqrt{2}}\cot\left(\dfrac{x+y}{2}+\dfrac{\pi}{8}\right) = c_1$. as $t=x+y$ ...(6)

Taking the last two fraction of (4), $dx-dy = \dfrac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y)$. ...(7)

On R.H.S. of (7), putting $x+y=t$, so that $d(x+y)=dt$, (7) reduces to

$dx-dy = \dfrac{\cos t-\sin t}{\cos t+\sin t}dt$. so that $x-y = \log(\sin t+\cos t)-\log c_2$

or $(\sin t+\cos t)/c_2 = e^{x-y}$ or $e^{-(x-y)}(\sin t+\cos t) = c_2$

or $e^{y-x}[\sin(x+y)+\cos(x+y)] = c_2$, as $t=x+y$. ...(8)

From (6) and (8), the required general solution is

$\phi\left[z^{\sqrt{2}}\cot\left(\dfrac{x+y}{2}+\dfrac{\pi}{8}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right] = 0$, where ϕ is an arbitrary function.

Ex. 12. *Solve* $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x^3+3xy^2} = \dfrac{dy}{y^3+3x^2y} = \dfrac{dz}{2z(x^2+y^2)}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{x^3+3xy^2+3x^2y+y^3} = \dfrac{d(x+y)}{(x+y)^3}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{x^3+3xy^2-y^3-3x^2y} = \dfrac{d(x-y)}{(x-y)^3}$. ...(3)

From (2) and (3), $(x + y)^{-3}\, d(x + y) = (x - y)^{-3}\, d(x - y)$

or $u^{-3}du - v^{-3}dv = 0$, on putting $u = x + y$ and $v = x - y$.

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx+(1/y)\,dy}{(1/x)\times(x^3+3xy^2)+(1/y)\times(y^3+3x^2y)} = \frac{(1/x)dx+(1/y)dy}{4(x^2+y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)dx+(1/y)dy}{4(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x}+\frac{dy}{y}-2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

From (4) and (6), the required general solution is given by

$\phi\, [(x - y)^{-2} - (x + y)^{-2}, (xy)/z^2] = 0$, ϕ being an arbitrary function.

Ex. 13. *Solve* $p+q=x+y+z$. **[Bhopal 2010, Bilaspur 2000, 02; I.A.S. 1975; Gulberge 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{x+y+z}$. ...(1)

Taking the first two fractions of (1), $dx - dy = 0$ so that $x - y = c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $= \dfrac{dx+dy+dz}{1+1+(x+y+z)} = \dfrac{d(2+x+y+z)}{2+x+y+z}$ ... (3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z) = dx$.

Integrating, $\log\,(2 + x + y + z) - \log c_2 = x$ or $(2 + x + y + z)/c_2 = e^x$

or $e^{-x}(2 + x + y + z) = c_2$, c_2 being arbitrary function ...(4)

From (2) and (4), the required general solution is

$\phi[x - y, e^{-x}(2 + x + y + z)] = 0$, ϕ being an arbitrary function.

Ex. 14. *Solve* $(2x^2 + y^2 + z^2 - 2yz - zx - xy)p + (x^2 + 2y^2 + z^2 - yz - 2zx - xy)q = x^2 + y^2 + 2z^2 - yz - zx - 2xy$. **[Meerut 1996 ; I.A.S. 1992]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy} = \frac{dy}{x^2+2y^2+z^2-yz-2zx-xy} = \frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}. \quad ...(1)$$

Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$= \frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy} = \frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore \quad \frac{dx-dy}{(x-y)\,(x+y+z)} = \frac{dy-dz}{(y-z)\,(x+y+z)} = \frac{dz-dx}{(z-x)\,(x+y+z)}. \quad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx - dy)/(x - y) - (dy - dz)/(y - z) = 0.$$

Integrating, $\log (x - y) - \log (y - z) = \log c_1$ or $(x - y)/(y - z) = c_1$. ...(3)

Taking the last two fractions of (2), $(dy - dz)/(y - z) - (dz - dx)/(z - x) = 0$.

Integrating, $\log (y - z) - \log (z - x) = \log c_2$ or $(y - z)/(z - x) = c_2$. ...(4)

From (3) and (4), the required general solution is

$$\phi[(x - y)/(y - z),\ (y - z)/(z - x)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 16. *Solve* $\{my(x+y)-nz^2\}\,(\partial z/\partial x) - \{lx(x+y)-nz^2\}\,(\partial z/\partial y) = (lx-my)z$ **[I.A.S. 2001]**

Sol. Re-writing the given equation, $\{my(x+y)-nz^2\}\ p - \{lx(x+y)-nz^2\}q = (lx-my)z$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{my(x+y)-nz^2} = \dfrac{dy}{-lx(x+y)+nz^2} = \dfrac{dz}{(lx-my)z}$ ...(2)

Each fraction of (2) = $\dfrac{dx+dy}{(my-lx)(x+y)} = \dfrac{dz}{-(my-lx)z}$ so that $\dfrac{d(x+y)}{x+y} = -\dfrac{dz}{z}$

Integrating, $\log (x + y) = -\log z + \log C_1$ or $(x + y)z = C_1$ ... (3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lxdx+mydy+nzdz}{lx\,my\,(x+y)-lx\,nz^2-mylx(x+y)+mynz^2+nz^2\,(lx-my)} = \frac{lx\,dx+my\,dy+nz\,dz}{0}$$

$\therefore\ 2lx\,dx + 2my\,dy + 2nzdz = 0$ so that $lx^2 + my^2 + nz^2 = C_2$ ...(4)

From (3) and (4), solution is $\Phi(xz+yz,\ lx^2+my^2+nz^2) = 0$, Φ being an arbitrary function.

Ex. 2 (*a*). *Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.*

(*b*) *Formulate partial differential equation for surfaces whose tangent planes form a tetrahedrom of constant volume with the coordinate planes.* **[I.A.S. 2005]**

Sol. (*a*) We know that the equation of the tangent plane at point (x, y, z) to a surface is given by

$$p(X - x) + q(Y - y) = Z - z, \qquad ...(1)$$

where X, Y, Z denote current coodrinates of any point on the plane (1). Since (1) cuts an intercept k on the z–axis, it follows that (1) must pass through the point $(0, 0, k)$. Hence putting $X = 0$, $Y = 0$ and $Z = k$ in (1), we obtain

$$px + qy = z - k, \qquad ...(2)$$

which is well known Lagrange's linear equation. For (2), the Lagrange's auxiliary equations are

$$(dx)/x = (dy)/y = (dz)/(z - x). \qquad ...(3)$$

Taking the first two fractions of (3), $(1/x)dx - (1/y)dy = 0$. so that $x/y = c_1$. ...(4)

Again, taking the first and third fraction of (3), $[1/(z - k)]dz - (1/x)dx = 0$

Integrating, $\log (z - k) - \log x = \log c_2$ or $(z - k)/x = c_2$. ...(5)

From (4) and (5), the required surface (solution) is given by

$$\phi[y/x, (z - k)/x] = 0, \quad \phi \text{ being an arbitrary function.}$$

(*b*) Left as an exercise.

Ex. 3. *Find the integral surface of the partial differential equation* $(x - y)p + (y - x - z)q = z$ through the circle $z = 1$, $x^2 + y^2 = 1$. **(Nagpur 2002)**

Sol. Given $(x - y)p + (y - x - z)q = z$. ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x-y} = \dfrac{dy}{y-x-z} = \dfrac{dz}{z}$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) $= (dx + dy + dz)/0$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(3)

Taking the last two fractions of (2) and using (3) we get

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z} \qquad \text{or} \qquad \frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0.$$

Integrating it, $\log \ (2y - c_1) - 2 \log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$. ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1$. ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1$. ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2$. ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0$. ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

or

$$z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0.$$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1$, $x + z = 2$. **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y$. ...(1)

Given curve is given by $y^2 + z^2 = 1$, and $x + z = 2$. ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}$. ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1$. ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2$. ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2$, using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0$.

Ex. 6. *Find the general integral of the partial differential equation* $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ *and also the particular integral which passes through the line* $x = 1$, $y = 0$. **[I.A.S. 2008]**

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$. ...(1)

Given line is given by $x = 1$ and $y = 0$. ...(2)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2x-2yz}$. ...(3)

Taking z, 1, x as multipliers, each fraction of (3) $= (zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1$. ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) $= \{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2$. ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1$.

Ex. 7. *Find the integral surface of $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ which passes through the hyperbola $xy = x + y$, $z = 1$.* **[I.A.S. 1994, 2009]**

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2$. ...(1)

Given curve is given by $xy = x + y$ and $z = 1$. ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2)$. ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0$.

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1$. ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0$.

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2$. ...(5)

Adding (4) and (5), $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{2}{z} = c_1 + c_2$ or $\dfrac{x+y}{xy}+\dfrac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2$, using (2) or $c_1 + c_2 = 3$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz$.

Ex. 7. *Find the integral surface of the partial differential equation $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ passing through the curve $xz = a^3$, $y = 0$.*

Sol. Given equation is $(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$ ... (1)

and the given curve is given by $xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $$\frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}=\frac{dz}{(x^2+y^2)z} \quad \text{... (3)}$$

Each fraction of (3) $= \dfrac{dx-dy}{(x-y)\,(y^2+x^2)}=\dfrac{dz}{(x^2+y^2)z}$ so that $\dfrac{d(x-y)}{x-y}-\dfrac{dz}{z}=0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary contant ... (4)

Taking the first two fractions, $3x^2dx+3y^2dy=0$

Integrating it, $x^3+y^3=C_2$, C^2 being an arbitrary constant. ... (5)

The parameteric form of the given curve (2) is $z = t$, $x = a^3/t$, $y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$a^3/t^2 = C_1$ so that $t^2 = a^3/C_1$ ... (7)

and $(a^3/t)^3 = C_2$ so that $t^3 = a^9/C_2$ ... (8)

Squaring both sides of (8), $t^6 = a^{18}/C_2^2$ or $(t^2)^3 = a^{18}/C_2^2$

or $(a^3/C_1)^3 = a^{18}/C_2^2$, since $t^2 = a^3/C_1$, by (7)

or $a^9/C_1^3 = a^{18}/C_2^2$, or $C_2^2 = a^9\ C_1^3$ ... (9)

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$(x^3+y^3)^2 = a^9\ (x-y)^3/z^3$ or $z^3(x^3+y^3)^2 = a^9\ (x-y)^3$.

Ex. 1. *Find the surface which intersects the surfaces of the system* $z(x + y) = c(3z + 1)$ *orthogonally and which passes through the circle* $x^2 + y^2 = 1$, $z = 1$. **[I.A.S. 1999]**

Sol. The given system of surfaces is $\quad f(x, y, z) \equiv \{z(x+y\} / (3z+1) = C. \quad$ ...(1)

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \qquad \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \qquad \frac{\partial f}{\partial z} = (x+y)\frac{(3z+1)-z\times 3}{(3z+1)^2} = \frac{x+y}{(3z+1)^2}.$$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \qquad \text{or} \qquad \frac{z}{3z+1}p + \frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $\qquad z(3z + 1)p + z(3z + 1)q = x + y. \qquad$ ...(2)

Lagrange's auxiliary equations for (2) are $\qquad \dfrac{dx}{z(3z+1)} = \dfrac{dy}{z(3z+1)} = \dfrac{dz}{x+y}. \qquad$ ...(3)

Taking the first two fractions of (3), we get $\quad dx - dy = 0 \quad$ so that $\quad x - y = C_1. \qquad$ ...(4)

Choosing x, y, $-z(3z + 1)$ as multipliers, each fraction of (3) = $[xdx + ydy - z(3z + 1)dz]/0$

$\therefore \qquad xdx + ydy - 3z^2dz - zdz = 0 \qquad$ or $\qquad 2xdx + 2ydy - 6z^2dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z^2 = C_2$, C_2 being an arbitrary constant. ...(5)

Hence any surface which is orthogonal to (I) has equation of the form

$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y)$, ϕ being an arbitrary function ...(6)

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, $z = 1$ we must choose $\phi(x - y) = -2$. Thus, the required particular surface is $\quad x^2 + y^2 - 2z^3 - z^2 = -2.$

Ex. 3. *Find the surface which is orthogonal to the one parameter system* $z = cxy(x^2 + y^2)$ *which passes through the hyperbola* $x^2 - y^2 = a^2$, $z = 0$

Sol. The given system of surfaces is $\quad f(x, y, z) = z/(x^3y + xy^3) = C \quad$ ... (1)

$$\frac{\partial f}{\partial x} = -\frac{z(3x^2y + y^3)}{(x^3y + xy^3)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{z(3y^2x + x^3)}{(x^3y + xy^3)^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{x^3y + xy^3}$$

The required orthogonal surface is solution of $\qquad p(\partial f / \partial x) + q(\partial f / \partial y) = \partial f / \partial z$

or
$$-\frac{z(3x^2y + y^3)}{(x^3y + xy^3)^2}p - \frac{z(3y^2x + x^3)}{(x^3y + xy^3)^2}q = \frac{1}{x^3y + xy^3}$$

or $\qquad \{(3x^2 + y^2)/x\}p + \{(3y^2 + x^2)/y\}q = -(x^2 + y^2)/z \qquad$ ... (2)

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{(3x^2 + y^2)/x} = \frac{dy}{(3y^2 + x^2)/y} = \frac{dz}{-(x^2 + y^2)/z} \qquad \text{... (3)}$$

Taking the first two fractions of (3), $\quad 2xdx - 2ydy = 0 \quad$ so that $\quad x^2 - y^2 = C_1$

Choosing x, y, $4z$ as multipliers, each fraction of (3) = $(xdx + ydy + 4zdz)/0$

$\therefore \qquad 2xdx + 2ydy + 8zdz = 0 \qquad$ so that $\qquad x^2 + y^2 + 4z^2 = C_2$

Hence any surface which is orthogonal to (1) is of the form

$x^2 + y^2 + 4z^2 = \Phi\,(x^2 - y^2)$, Φ being an arbitrary function. ... (4)

For the particular surface passing through the hyperbola $x^2 - y^2 = a^2$, $z = 0$ we must take $\Phi(x^2 - y^2) = a^4(x^2 + y^2)/(x^2 - y^2)^2$. Hence, the required surface is given by

$$(x^2 + y^2 + 4z^2)^2\ (x^2 - y^2)^2 = a^4(x^2 + y^2)$$

Ex. 2. *Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by* $x(x^2+y^2+z^2)=C_1y^2$. **[I.A.S. 2001]**

Sol. Given family of surfaces is $\qquad x(x^2+y^2+z^2)/y^2=C_1$

Let $\qquad f(x,y,z)=x(x^2+y^2+z^2)/y^2=C_1 \qquad \text{... (1)}$

Then the surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x}=\frac{dy}{\partial f/\partial y}=\frac{dz}{\partial f/\partial z} \quad \text{or} \quad \frac{dx}{(3x^2+y^2+z^2)/y^2}=\frac{dy}{-2x(x^2+z^2)/y^3}=\frac{dz}{2x/y^2z}$$

or

$$\frac{dx}{y(3x^2+y^2+z^2)}=\frac{dy}{-2x(x^2+z^2)}=\frac{dz}{2xyz} \qquad \text{... (2)}$$

Taking x, y, z as multipliers, each fraction of (2)

$$=\frac{xdx+ydy+zdz}{xy(3x^2+y^2+z^2)-2xy(x^2+z^2)+2xyz}=\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} \qquad \text{... (3)}$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)}=\frac{dz}{2xyz} \quad \text{or} \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{\mathrm{dz}}{\mathrm{z}}$$

Integrating, $\quad \log(x^2+y^2+z^2)=\log z+\log C_2 \quad$ or $\quad (x^2+y^2+z^2)/z=C_2$ ...(4)

Taking $4x$, $2y$, 0 as multipliers, each fraction of (2)

$$=\frac{4xdx+2ydy}{4xy(3x^2+y^2+z^2)-4xy(x^2+y^2)}=\frac{4xdx+2ydy}{4xy(2x^2+y^2)} \qquad \text{... (5)}$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx+2ydy}{4xy(2x^2+y^2)}=\frac{dz}{2xyz} \quad \text{or} \quad \frac{4xdx+2ydy}{2x^2+y^2}=\frac{2dz}{z}$$

Integrating, $\quad \log(2x^2+y^2)=2\log z+\log C_3 \quad$ or $\quad (2x^2+y^2)/y^2=C_3 \qquad$ ... (6)

From (4) and (5), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form $(x^2+y^2+z^2)/z=\phi\,\{(2x^2+y^2)/z^2\}$, *i.e.*,

or $\quad x^2+y^2+z^2=z\,\phi\,\{(2x^2+y^2)/z^2\}$, where ϕ is an arbitrary function.

2.18 (*a*). Geometrical description of the solutions of *Pp* + *Qq* = *R* and of the system of equations *dx*/*P* = *dy*/*Q* = *dz*/*R* and to establish relationship between the two.

[G.N.D.U. Amritsar 1998; Meerut 1997; Kanpur 1996]

Proof. Consider $$Pp + Qq = R. \quad \text{....(1)}$$

and $$(dx)/P = (dy)/Q = (dz)/R, \quad \text{...(2)}$$

where P, Q and R are functions of x.

Let $$z = \phi(x, y) \quad \text{...(3)}$$

represent the solution of (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Also we know that the simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp + Qq + R(-1) = 0, \quad \text{...(4)}$$

showing that the normal to surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence the member must touch the surface at that point. Since this holds for each point on (3), we conclude that the curves (2) lie completely on the surface (3) whose differential equation is (1).

2.18 (b). Another geomertical interpretation of Lagrange's equation Pp + Qq = R.

To show that the surfaces represented by $Pp + Qq = R$ *are orthogonal to the surfaces represented by* $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solutions of

$$(dx)/P = (dy)/Q = (dz)/R \quad \text{...(1)}$$

are orthogonal to the system of the surfaces whose equation satisfies

$$Pdx + Qdy + Rdz = 0. \quad \text{...(2)}$$

Again from Art 2.18 (*a*) the curves of (1) lie completely on the surface represented by

$$Pp + Qq = R. \quad \text{...(3)}$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

Ex. 1. *Find the family orthogonal to* $\phi\,[z(x+y)^2, x^2-y^2] = 0$.

Sol. Given $\qquad \phi[z(x+y)^2, x^2-y^2] = 0. \qquad$ ...(1)

Let $\qquad u = z(x+y)^2 \qquad$ and $\qquad v = x^2 - y^2 \qquad$ ...(2)

Then (1) becomes $\qquad \phi(u, v) = 0. \qquad$ ...(3)

Differentiating (3) w.r.t. x and y partially by turn, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \text{...(4)}$$

and
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0. \qquad \text{...(5)}$$

From (2), $\quad (\partial u/\partial x) = 2z(x+y), \quad (\partial u/\partial y) = 2z(x+y), \quad (\partial u/\partial z) = (x+y)^2,$
$(\partial v/\partial x) = 2x, \quad (\partial v/\partial y) = -2y, \quad (\partial v/\partial z) = 0.$

Putting these values in (4) and (5), we get

$$(\partial\phi/\partial u)\left[2z(x+y) + p(x+y)^2\right] + (\partial\phi/\partial v)\,(2x+0) = 0 \qquad \text{...(6)}$$

and
$$(\partial\phi/\partial u)\left[2z(x+y) + q(x+y)^2\right] + (\partial\phi/\partial v)\,(-2y+0) = 0 \qquad \text{...(7)}$$

Evaluating the values of $-\dfrac{\partial\phi/\partial u}{\partial\phi/\partial v}$ from (6) and (7) and then equating these, we get

$$-\frac{\partial\phi/\partial u}{\partial\phi/\partial v} = \frac{2x}{2z(x+y) + p(x+y)^2} = \frac{-2y}{2z(x+y) + q(x+y)^2}$$

or $x(x+y)[2z + q(x+y)] = -y(x+y)[2z + p(x+y)]$ $\quad$ or $\quad 2xz + qx(x+y) + 2yz + py(x+y) = 0$

or $\quad py(x+y) + qx(x+y) = -2z(x+y) \quad$ or $\quad py + qx = -2z \qquad$ ...(8)

which is differential equation of the family of surfaces given by (1). So the differential equation of the family of surfaces orthogonal to (8) is given by [use Art. 2.18 (b)]

$ydx + xdy - 2zdz = 0 \qquad$ or $\qquad d(xy) - 2zdz = 0. \qquad$ ...(9)

Integrating (9), $\qquad xy - z^2 = C,$

which is the desired family of orthogonal surfaces, C being parameter

Ex. 2. *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation* $(y+z)p + (z+x)q = x + y$.

Sol. Let $\qquad P = y + z, \qquad Q = z + x \qquad$ and $\qquad R = x + y. \qquad$ ...(1)

Then, the given differential equation can be written as $\qquad Pp + Qq = R. \qquad$ ...(2)

Now, the differential equation of the family of surfaces orthogonal to the given family is

$Pdx + Qdy + Rdz = 0 \qquad$ or $\qquad (y+z)dx + (z+x)dy + (x+y)dz = 0$

or $\qquad (ydx + xdy) + (ydz + zdy) + (zdx + xdz) = 0.$

Integrating, $\qquad xy + yz + zx = C,$

which is the required family of surfaces, C being a parameter.

Ex. 1. *Find a complete integral of* $z = px + qy + p^2 + q^2$.

[Bilaspur 2000l; Bhopal 1996, I.A.S. 1996; Indore 2000; Jabalpur 2000; K.U. Kurukshetra 2005; Ravishankar 2000; 04; Meerut 2010; Garhwal 2010]

Sol. Let $\qquad f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \qquad \dots (1)$

Charpit's auxiliary equations are $\qquad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q} \qquad \dots (2)$

From (1), $\quad f_x = -p, \quad f_y = -q, \quad f_z = 0, \quad f_p = -x - 2p \quad$ and $\quad f_q = -y - 2q \qquad \dots (3)$

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \qquad \dots (4)$$

Taking the first fraction of (4), $\quad dp = 0 \quad$ so that $\quad p = a \qquad \dots (5)$

Taking the second fraction of (4), $\quad dq = 0 \quad$ so that $\quad q = b \qquad \dots (6)$

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$z = ax + by + a^2 + b^2$, a, b being arbitrary constants.

Ex. 5. *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997; Meerut 2007]**

Sol. Here given equation is $\qquad f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0. \qquad \dots(1)$

Charpit's auxiliary equations are $\qquad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $\qquad \dfrac{dp}{p(4p^2z^3 + 2zq^2)} = \dfrac{dq}{q(4p^2z^3 + 2zq^2)} = \dfrac{dz}{-2p^2z^4 - 2q^2z^2} = \dfrac{dx}{-2pz^4} = \dfrac{dy}{-2qz^2}$, by (1) $\quad \dots (2)$

Taking the first two fractions, $\qquad (1/p)dp = (1/q)dq \qquad$ so that $\qquad p = aq.$

Solving (1) and (2) for p and q, $\qquad p = \dfrac{a}{z(a^2z^2+1)^{1/2}}, \qquad q = \dfrac{1}{z(a^2z^2+1)^{1/2}}.$

$\therefore \quad dz = pdx + qdy = (a\,dx + dy)/z\,(a^2z^2 + 1)^{1/2} \qquad$ or $\qquad adx + dy = z(a^2z^2 + 1)^{1/2}dz.$

Integrating, $\qquad ax + y = \displaystyle\int (a^2z^2+1)^{1/2}.\,zdz. \qquad \dots(3)$

Putting $a^2z^2 + 1 = t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$ax + y = \displaystyle\int (1/a^2)\,t.\,t\,dt \qquad$ or $\qquad ax + y + b = (1/3a^2)t^3$, where $t = (a^2z^2 + 1)^{1/2}$

or $\qquad ax + y + b = (1/3a^2) \times (a^2z^2 + 1)^{3/2} \qquad$ or $\qquad 9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3,$

which is a complete integral, a and b being arbitrary constants.

Ex. 6. *Find a complete integral of $px + qy = pq$.* **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+p.0} = \frac{dq}{q+q.0}, \text{ by (1)} \qquad ...(2)$$

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq.$

Integrating, $\log p = \log q + \log a$ or $p = aq.$ ...(3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y$, as $q \neq 0$ ...(4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y$. ...(5)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\, dy$ or $adz = (ax + y)\,(adx + dy)$

or $adz = (ax + y)\, d(ax + y) = udu$, where $u = ax + y$.

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 10(a). *Find a complete integral of $(p^2 + q^2)x = pz$.*

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation $(p^2 + q^2)x = pz$ and deduce the solution which passes through the curve $x = 0$, $z^2 = 4y$.* **[Meerut 2007]**

Sol. Let $f(x, y, q, p, q) = (p^2 + q^2)x - pz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or $2pdp + 2qdq = 0.$

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant. ...(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$. ...(3)

$\therefore$ $dz = pdx + qdy = \dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2 - a^2x^2)}\, dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2 - a^2x^2)}} = ady.$

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2xdx) = dt$, we get

$(1/2\sqrt{t})dt = ady$ or $(1/2) \times t^{-1/2} = ady.$

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2 - a^2x^2)} = ay + b$, as $t = \sqrt{z^2 - a^2x^2}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$. ... (4)

(b) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve $x = 0$, $z^2 = 4y$ are given by

$$x = 0, \qquad y = t^2, \qquad z = 2t \qquad \ldots (5)$$

Therefore the intersections of (1) and (2) are determined by

$$4t^2 = (at^2 + b)^2 \qquad \text{or} \qquad a^2t^4 + 2(ab - 2)t^2 + b^2 = 0 \qquad \ldots (6)$$

Equation (6) has equal roots if its discriminant = 0, *i.e.*, if

$$4(ab - 2)^2 - 4a^2b^2 = 0 \qquad \text{or} \qquad a^2b^2 = 1 \qquad \text{so that} \qquad b = 1/a$$

Hence from (4), the appropriate one paremeter sub–system is given by

$$z^2 = a^2x^2 + (ay + 1/a)^2 \qquad \text{or} \qquad a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0,$$

which is a quatratic equation in parameter 'a'. Therefore, this has for its envelope surface

$$(2y - z^2)^2 - 4(x^2 + y^2) = 0 \qquad \text{or} \qquad (2y - z^2)^2 = 4(x^2 + y^2) \qquad \ldots (7)$$

The desired solution is given by the function z defined by equation (7).

Ex. 10(c). *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.
[Guwahati 2007; Agra 2001; Bilaspur 1998; Delhi Maths (H) 2003, 05; Garhwal 2005; Meerut 2010, 11; K.V. Kurukshetra 2004; Kanpur 2005; Rohilkhand 2001; Pune 2010]

Sol. Here the given equation is $\qquad f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0. \qquad \ldots(1)$

Charpit's auxiliary equations are $\qquad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or
$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}, \text{ by (1)} \qquad \ldots(2)$$

Taking the first two fractions, we get $\quad 2pdp + 2qdq = 0 \quad$ so that $\quad p^2 + q^2 = a \quad \ldots (3)$

Using (3), (1) gives $\qquad a^2y = qz \qquad$ or $\qquad q = a^2y/z$.

Putting this value of q in (3), we get

$$p = \sqrt{(a^2 - q^2)} = \sqrt{a^2 - (a^4y^2/z^2)} = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}.$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\, dx + \frac{a^2 y\, dy}{z} dy \qquad \text{or} \qquad \frac{zdz - a^2 y\, dy}{\sqrt{(z^2 - a^2y^2)}} = a\ dx.$$

Integrating, $\qquad (z^2 - a^2y^2)^{1/2} = ax + b \qquad$ or $\qquad z^2 - a^2y^2 = (ax + b)^2, \qquad \ldots(4)$

which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2\,(ax + b)x \qquad \ldots(5)$$

and
$$0 = 2(ax + b). \qquad \ldots(6)$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \qquad \ldots(7)$$

Differentiating (7) partially w.r.t. a, $\qquad -2ay^2 = 2[ax + \phi(a)]\,.\,[x + \phi'(a)]. \qquad \ldots(8)$

General integral is obtained by eliminating a from (7) and (8).

Ex. 12. *Find a complete and singular integrals of* $2xz - px^2 - 2q\,xy + pq = 0$. **[I.A.S. 1991, 93, 2007, 2008; Delhi Hons. 2001, 01, 05; Kanpur 2001, 03; Meerut 2005; Bhopal 2004, 10; Indore 1999; M.D.U. Rohtak 2004, Ravishanker 2004; Rajasthan 2000, 03, 05, 10]**

Sol. Here given equation is $\quad f(x, y, z, p, q) = 2xz - px - 2qxy + pq = 0.\quad$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}, \text{ by (1)}$$

The second fraction gives $\quad dq = 0 \quad$ so that $\quad q = a$

Putting $q = a$ in (1), we get $\quad p = 2x(z - ay)/(x^2 - a)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy \quad \text{or} \quad \frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}.$$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a) \quad$ or $\quad z = ay + b(x^2 - a),\quad$ ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$0 = y - b \quad$ and $\quad 0 = x^2 - a.\quad$ ...(3)

Solving (3) for a and b, $\quad a = x^2 \quad$ and $\quad b = y.\quad$ ...(4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2y$, which is the required singular integral.

Ex. 15. *Find a complete integral* $p^2 + q^2 - 2px - 2qy + 1 = 0$.

[Patna 2003; Meerut 99, 2003; Delhi Maths Hons 91; Ravishanker 2010]

Sol. Given $\quad f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 1 = 0.\quad$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-pf_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-2p}=\frac{dq}{-2q}=\frac{dz}{-p(2p-2x)-q(2q-2y)}=\frac{dx}{-(2p-2y)}=\frac{dy}{-(2q-2y)}, \text{ by (1)}$$

The first two fractions give $\quad (1/p)dp = (1/q)dq \quad$ so that $\quad p = aq.$

Putting $p = aq$ in (1), $a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$ or $(a^2 + 1)q^2 - 2(ax - y)q + 1 = 0.$

$$\Rightarrow\ q=\frac{2(ax+y)\pm\sqrt{\{4(ax+y)^2-4(a^2+1)\}}}{2(a^2+1)},\ p=aq=a\,\frac{2(ax+y)\pm\sqrt{\{4(ax+y)^2-4(a^2+1)\}}}{2(a^2+1)}.$$

Putting these values of p and q in $dz = p\,dx + y\,dy$, we get

$$dz = \frac{(ax+y)\pm\sqrt{\{(ax+y)^2-(a^2+1)\}}}{(a^2+1)}(adx+dy). \quad ...(2)$$

Put $ax + y = v$ so that $a\,dx + dy = dv$. Then (2) gives

$$(a^2 + 1)dz = [v \pm \sqrt{\{v^2 - (a^2+1)\}}]dv.$$

Integrating, $$(a^2 + 1)\, z = v^2/2 \pm [\,(v/2) \times \sqrt{\{v^2 - (a^2+1)\}}$$

$$- (1/2) \times (a^2 + 1) \log (v + \sqrt{\{v^2 - (a^2+1)\}}\,)] + b$$

is the complete integral, where $v = ax + b$ and a, b are arbitrary constants.

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $\quad f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}}$$

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives $$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $\quad dp + dq = dx + dy \quad$ *i.e.*, $\quad dp - dx + dq - dy = 0.$

Integrating, $\quad (p - x) + (q - y) = a$ ...(2)

Re–writing (1), $\quad (p - x)^2 + (q - y)^2 = (x - y)^2.$ ...(3)

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2 \quad$ or $\quad 2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0.$

$$\therefore\ p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \quad \Rightarrow \quad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}],$$

$\therefore$ (2) gives $\quad q = a + y - p + x \quad$ or $\quad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}]$.

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or $$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2}.(dx - dy).$$

Integrating, the desired complete integral is

$$z = \frac{x^2 + y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} \;\; - \frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $f(x, y, z, p, q) = p^2x + q^2y - z = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} - \dfrac{dy}{-f_q}$

or $$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2) $= \dfrac{2px\,dp + p^2dx}{2px(-p+p^2)+p^2(-2px)} = \dfrac{2qy\,dq+q^2dy}{2qy(-q+q^2)+q^2(-2qy)}$

or $$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\log(p^2x) = \log(q^2y) + \log a$ or $p^2x = q^2ya$. ...(3)

Form (1) and (3), $aq^2y + q^2y = z$ or $q = [z/(1+a)]^{1/2}$. ...(4)

Form (3) and (4), $$p = q\left(\frac{ya}{x}\right)^{1/2} = \left\{\frac{za}{(1+a)x}\right\}^{1/2}.$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy \qquad \text{or} \qquad (1+a)^{1/2}z^{-1/2}\,dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $(1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 18. *Find a complete integral of* $2z + p^2 + qy + 2y^2 = 0$. **[I.F.S. 2005; Meerut 2000; Rohilkhand 1993; Bilaspur 2004, M.D.U Rohtak 2005; Rawa 1999; Ranchi 2010]**

Sol. Given equation is $f(x, y, z, p, q) = 2z + p^2 + qy^2 + 2y^2 = 0$. ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dp}{-f_p} = \dfrac{dq}{-f_q}$

or $$\frac{dp}{0+2p} = \frac{dq}{(q+4y)+2q} = \frac{dz}{-p\times(2p)-qy} = \frac{dx}{-2p} = \frac{dy}{-y}, \text{ by (1)}$$

Taking the first and fourth fractions, $dp = -dx$.

Integrating, $p = a - x$ or $p = -(x-a)$. ...(2)

Using (2), (1) becomes $2z + (a-x)^2 + qy + 2y^2 = 0$

$\therefore$ $q = -[2z + (x-a)^2 + 2y^2]/y$. ...(3)

$\therefore$ $dz = p\,dx + q\,dy = -(x-a)\,dx - [\{2z + (x-a)^2 + 2y^2\}/y]dy$, by (2) and (3)

Multiplying both sides by $2y^2$ and re–writing, we have

$$2y^2dz = -2(x-a)y^2\,dx - 4zydy - 2y(x-a)^2dy - 4y^3dy$$

or $$2(y^2dz + 2zy\,dy) + [2(x-a)^2y^2dx + 2y(x-a)^2dy] + 4y^3dy = 0$$

or $$2d(y^2z) + d[y^2(x-a)^2] + 4y^3dy = 0.$$

Integrating, $2y^2z + y^2(x-a)^2 + y^4 = b$, a, b being arbitrary constants

Ex. 27. *Find a complete integral of* $px + qy = z(1 + pq)^{1/2}$

[Meerut 2001, 02; Kanpur 1995, I.A.S. 1992]

Sol. Given $\qquad f(x, y, z, p, q) = px + qy - z(1 + pq)^{1/2} = 0. \qquad$ ...(1)

Charpit's auxiliary equation are $\qquad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\qquad \dfrac{dp}{p - p(1+pq)^{1/2}} = \dfrac{dq}{q - q(1+pq)^{1/2}} = ...$ so that $\qquad \dfrac{dp}{p} = \dfrac{dq}{q}$, by (1)

$\Rightarrow \qquad \log p = \log a + \log q \qquad \Rightarrow \qquad p = aq. \qquad$ ...(2)

Using (2), $\quad$ (1) $\Rightarrow$ $q(ax + y) = z\,(1 + aq^2)^{1/2}$ $\qquad$ or $\qquad q^2\,[(ax + y)^2 - az^2] = z^2$.

$\therefore \quad q = \dfrac{z}{[(ax+y)^2 - az^2]^{1/2}} \qquad$ and $\qquad p = aq = \dfrac{az}{[(ax+y)^2 - az^2]^{1/2}}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$dz = \dfrac{z(a\,dx + dy)}{\sqrt{\{(ax+y)^2 - az^2\}}} \qquad$ or $\qquad \dfrac{dz}{z} = \dfrac{a\,dx + dy}{\sqrt{\{(ax+y)^2 - az^2\}}}$. $\qquad$ ... (3)

Let $\qquad ax + y = \sqrt{a}\,u \qquad$ so that $\qquad a\,dx + dy = \sqrt{a}\,du$.

$\therefore \quad$ (3) $\Rightarrow \quad \dfrac{dz}{z} = \dfrac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \qquad$ or $\qquad \dfrac{du}{dz} = \dfrac{\sqrt{(u^2 - z^2)}}{z} = \sqrt{\left\{\left(\dfrac{u}{z}\right)^2 - 1\right\}}$, $\qquad$ ... (4)

which is linear homogeneus equation. To solve it, we put

$\dfrac{u}{z} = v \qquad$ or $\qquad u = vz \qquad$ so that $\qquad \dfrac{du}{dz} = v + z\dfrac{dv}{dz}$.

$\therefore \quad$ (4) yields $\qquad v + z\dfrac{dv}{dz} = (v^2 - 1)^{1/2}$. $\qquad$ or $\qquad \dfrac{dz}{z} = \dfrac{dv}{(v^2-1)^{1/2} - v}$

or $\qquad (1/z)dz = -\left[(v^2 - 1)^{1/2} + v\right]dv$, on rationalization.

Integrating, $\log z = -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\}\right] - \dfrac{v^2}{2} + b$, where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex. 30. *Solve* $z = (1/2) \times (p^2 + q^2) + (p - x)\,(q - y)$ **[I.A.S. 2002]**

Sol. Given $\qquad z = (1/2) \times (p^2 + q^2) + (p - x)(q - y)$

Re–writing (1), $\qquad f(x, y, z, p, q) = (1/2) \times (p^2 + q^2) + pq - xq - yp + xy - z = 0$ ... (2)

Charpit's auxiliary equations are $\qquad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\qquad \dfrac{dp}{-q + z - p} = \dfrac{dq}{-p + x - q} = \dfrac{dz}{-p(p + q - y) - q(p + q - x)} = \dfrac{dx}{-(p + q - y)} = \dfrac{dy}{-(p + q - x)}$

Taking the first and the fourth fractions, we have

$dp = dx \qquad$ so that $\qquad p = x + a$, a being an arbitratry constant. $\qquad$ ... (3)

Taking the second and the fifth fractions, we have

$dq = dy \qquad$ so that $\qquad q = y + b,\ b$ being an arbitrary constant $\qquad$ ... (4)

Putting $p = x + a$ and $q = y + b$ in (1), the required solution is

$z = (1/2) \times \{(x + a)^2 + (y + b)^2\} + ab$, a and b being arbitrary constants.

Ex. 32. *Use Charpit's method to find the complete integral of* $2x\{z^2(\partial z/\partial y)^2+1\}=z(\partial z/\partial x)$.

[I.A.S. 1998]

Sol. Given $\qquad 2x(z\partial z/\partial y)^2+2x-(z\partial z/\partial x)=0 \qquad$ ... (1)

Let $\qquad z\,dz=dZ \qquad$ so that $\qquad z^2=2Z \qquad$ ... (2)

Then (1) becomes $\quad 2x(\partial Z/\partial y)^2+2x-(\partial Z/\partial x)=0 \quad$ or $\quad 2xQ^2+2x-P=0$

where $\qquad P=\partial Z/\partial x \qquad$ and $\qquad Q=\partial Z/\partial y \qquad$ ... (3)

Let $\qquad f(x,y,Z,P,Q)=2x\,Q^2+2x-P=0 \qquad$ ... (4)

Charpit's auxiliary equations are $\quad \dfrac{dP}{f_x+P\,f_Z}=\dfrac{dQ}{f_y+Q\,f_Z}=\dfrac{dZ}{-Pf_P-Q\,f_Q}=\dfrac{dx}{-f_P}=\dfrac{dy}{-f_Q}$

giving $\qquad \dfrac{dP}{2Q^2+2}=\dfrac{dQ}{O}=\ldots$, by (4) $\qquad$ so that $\qquad dQ=0.$

Integrating, $\qquad Q=a$, a being an arbitrars constant $\qquad$ ... (5)

Using $Q=a$, (4) gives $\qquad P=2x\,(a^2+1), \qquad Q=a \qquad$ ... (6)

$\therefore \qquad dZ=P\,dx+Q\,dy=2x(a^2+1)dx+a\,dy$, by (5) and (6)

Integrating, $Z=x^2(a^2+1)+ay+b/2$, $\qquad$ or $\qquad z^2/2=x^2(a^2+1)+ay+b/2$, using (2)

or $\qquad z^2=2x^2(a^2+1)+2ay+b$, which is complete integral of (1)

Ex. 33. *Solve by Charpit's method the partial differential equation.*

$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0$. $\qquad$ **[I.A.S. 2000]**

Sol. Let $\quad f(x,y,z,p,q)=p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0 \qquad$ ... (1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q} \quad$ ... (2)

From (1), $\qquad f_x=p^2(2x-1)+2pqy-2pz, \qquad f_y=2pqx+q^2(2y-1)-2qz$,

$f_z=-2px-2qy+2z, \qquad f_p=2px(x-1)+2qxy-2xz\,; \qquad f_q=2pxy+2qy(y-1)-2yz$

and so $\quad f_x+p\,f_z=-p^2, \qquad f_y+q\,f_z=-q^2$. Then (2) becomes

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)} \qquad \ldots (3)$$

$$\text{Each fraction of (3)} = \frac{(1/p)\,dp}{-p}=\frac{(1/q)\;dq}{-q}=\frac{(1/p)\,dp-(1/q)dq}{-p+q} \qquad \ldots (4)$$

Also, each fraction of (3) = $\dfrac{(1/x)\ dx - (1/y)\ dy}{-2px + 2p - 2qy + 2z + 2px + 2qy - 2q - 2z}$ ... (5)

$\therefore$ (4) and (5) $\Rightarrow$ $\dfrac{(1/p)dp - (1/q)\ dq}{-(p-q)} = \dfrac{(1/x)\ dx - (1/y)dy}{2(p-q)}$

or $(1/2) \times \{(1/x)\ dx - (1/y)dy\} = (1/q)\ dq - (1/p)\ dp$

Integrating, $(1/2)\times\{\log x - \log y\} = \log q - \log p + \log a$ or $(x/y)^{1/2} = aq/p$

or $p = (ay^{1/2}q)/x^{1/2}$, a being an arbitrary constant. ... (5)

Re–writing (1), $(px + qy - z)^2 = p^2 x + q^2 y$ or $px + qy - z = \pm(p^2 x + qy^2)^{1/2}$ ... (6)

Taking + ive sign in (7), $px + qy - z = (p^2 x + q^2 y)^{1/2}$ ... (7)

[The case of – ve sign in (7) can be discussed similarly]

Substituting the value of p given by (6) in (8), $aqy^{1/2}x^{1/2} + qy - z = (a^2 q^2 y + q^2 y)^{1/2}$

or $q\{y + a(xy)^{1/2} - (1+a^2)^{1/2} y^{1/2}\} = z$ so that $q = z/y^{1/2}\{y^{1/2} + a\ x^{1/2} - (1+a^2)^{1/2}\}$ ... (9)

Then (6) gives $p = az/x^{1/2}\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}$ ... (10)

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{az\ dx}{x^{1/2}\{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}} + \frac{z\ dy}{y^{1/2}\{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}}$$

Integrating, $\log z = 2\log\{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\} + \log b$

or $z = b\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}^2$, a and b being an arbitrary constants.

Ex. 34. *Find the complete integral of* $(p+q)\ (px+qy) = 1$.

[Meerut 2007; Delhi Maths (H) 2007, Purvanchal 2007]

Sol. Let $f(x, y, z, p, q) = (p+q)\ (px+qy) - 1 = 0$ ... (1)

Charpit's auxiliary equations $\dfrac{dp}{f_x + p\ f_z} = \dfrac{dq}{f_y + q\ f_z} = \dfrac{dz}{-p\,f_p - q\ f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

give $\dfrac{dp}{p(p+q)} = \dfrac{dq}{q(p+q)} = \ldots$ so that $\dfrac{dp}{p} = \dfrac{dq}{q}$, using (1)

Integrating, $p = aq$, a being an arbitrary constant ... (2)

Putting $p = aq$ in (2) gives $(aq+q)\ (aqx+qy) - 1 = 0$ or $q^2\ (1+a)\ (ax+y) = 1$ ... (3)

$\therefore$ From (2) and (3), $q = 1/(1+a)^{1/2}\ (ax+y)^{1/2}$, $p = a/(1+a)^{1/2}(ax+y)^{1/2}$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a\ dx}{(1+a)^{1/2}(ax+y)^{1/2}} + \frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{d(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b$, a, b being arbitrary constants.

Ex. 4. *Find the complete integral of*

(*i*) $x^2p^2 + y^2q^2 = z$ **[Delhi Maths (H) 2004]**

(*ii*) $p^2x + q^2y = z.$ **[Meerut 1994]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(\frac{x\,\partial z}{\sqrt{z}\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{\sqrt{z}\,\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $(1/x)dx = dX, \quad (1/y)dy = dY \quad$ and $\quad (1/\sqrt{z})dz = dZ$ ...(2)

so that $\log x = X, \quad \log y = Y \quad$ and $\quad 2\sqrt{z} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1,$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c,$ ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = a \log x + \log y \,.\, \sqrt{1-a^2} + c$, by (3)

or $\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}$, taking $c = -\log c'$

or $\log \{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z}$ or $x^a\, y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$

where a and c' are two arbitrary constants.

(*ii*) The given equation can be re–written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $\left(1/\sqrt{x}\right)dx = dX, \quad \left(1/\sqrt{y}\right)dy = dY \quad$ and $\quad \left(1/\sqrt{z}\right)dz = dZ$ ...(2)

so that $2\sqrt{x} = X, \quad 2\sqrt{y} = Y \quad$ and $\quad 2\sqrt{z} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1,$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aX + bY + c,$ ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c$, by (3)

where a and c are two arbitrary constants.

Ex. 6. *Find a complete integral of* (*i*) $pq = x^m y^n z^{2l}$ **[Delhi B.Sc. (Prog) II 2007]**

(*ii*) $pq = x^m y^n z^l$ **[I.A.S. 1989, 94]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{z^{-l}z^{-l}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad \text{...(1)}$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l}dz = dZ$ ...(2)

so that $\dfrac{x^{m+1}}{m+1} = X$, $\dfrac{y^{n+1}}{n+1} = Y$ and $\dfrac{z^{1-l}}{1-l} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)(\partial Z/\partial Y) = 1$ or $PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $ab = 1$ or $b = 1/a$, on putting a from P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-l}}{1-l} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

(*ii*) The given equation can be rewritten as

$$\frac{z^{-l/2}z^{-l/2}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{z^{-l/2}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l/2}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad \text{...(1)}$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l/2}dz = dZ$ ...(2)

so that $\dfrac{x^{m+1}}{m+1} = X$, $\dfrac{y^{n+1}}{n+1} = Y$ and $\dfrac{z^{1-(l/2)}}{1-(l/2)} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)(\partial Z/\partial Y) = 1$ or $PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $ab = 1$ or $b = 1/a$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-(l/2)}}{1-(l/2)} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

Ex. 9. *Find the complete integral of* $(y-x)(qy-px)=(p-q)^2$. **[Delhi Maths (H) 2005; Ravishankar 2010; Meerut 1995, 97; Agra 1999; Kanpur 2001, 04, 07, 08]**

Sol. Let X and Y be two new variables such that

$$X = x + y \qquad \text{and} \qquad Y = xy. \qquad \text{...(1)}$$

Given equation is $$(y-x)(qy-px) = (p-q)^2. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of p and q in (2), we have

$$(y-x)\left[y\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right) - x\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)\right] = \left[\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right) - \left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $$(y-x)^2\frac{\partial z}{\partial X} = (y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2 \qquad \text{or} \qquad \frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2 \qquad \text{or} \qquad P = Q^2, \qquad \text{...(5)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad \text{...(6)}$$

where $a = b^2$, on putting a for P and b for Q in (5).

$\therefore$ from (6), the required complete integral is

$$z = b^2X + bY + c \qquad \text{or} \qquad z = b^2(x+y) + bxy + c, \text{ by (1).}$$

Ex. 10. *Find the complete integral of* $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$.

[I.A.S. 1991; Kanpur 2006; Meerut 1997]

Sol. Let X and Y be two new variables such that

$$X^2 = x + y \qquad \text{and} \qquad Y^2 = x - y. \qquad \text{...(1)}$$

Given equation is $$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1/2X$ and $\partial Y/\partial x = 1/2Y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1/2X$ and $\partial Y/\partial y = -1/2Y$]

(3) and (4) $\Rightarrow$ $$p + q = \frac{1}{X}\frac{\partial z}{\partial X} \qquad \text{and} \qquad p - q = \frac{1}{Y}\frac{\partial z}{\partial Y}. \qquad \text{...(5)}$$

Using (1) and (5), (2) reduces to

$$X^2 \times \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \times \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1 \qquad \text{or} \qquad P^2 + Q^2 = 1, \qquad \text{...(6)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

Ex. 11. *Find a complete integral of* $(x^2 + y^2)(p^2 + q^2) = 1$.

[Agra 2008; Indore 2004; Vikram 2000; Meerut 1995; Rohitkhand 1994]

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$. ...(1)

Then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. ...(2)

Differentiating (2) partially with respect to x and y, we get

$$2r(\partial r/\partial x) = 2x \quad \text{and} \quad 2r(\partial r/\partial y) = 2y$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{r\sin\theta}{r} = \sin\theta. \quad ...(3)$$

and
$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\times\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \quad ...(4)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\times\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad ...(5)$$

Given equation is $(x^2 + y^2)(p^2 + q^2) = 1$. ....(6)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (4)

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \sin\theta\dfrac{\partial z}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (5).

Hence $p^2 + q^2 = (\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2$. ...(7)

$\therefore$ (6) becomes $r^2[(\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2] = 1$, using (2) and (7)

or
$$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1. \quad ...(8)$$

Let R be a new variable such that $(1/r)dr = dR$ so that $\log r = R$. ...(9)

Then (8) becomes $(\partial z/\partial R)^2 + (\partial z/\partial\theta)^2 = 1$ or $P^2 + Q^2 = 1$, ...(10)

where $P = \partial z/\partial R$ and $Q = \partial z/\partial\theta$. (10) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aR + b\theta + c$, ...(11)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (10)

$\therefore$ from (11), the required complete integral is

$$z = aR + \theta\sqrt{1-a^2} + c \quad \text{or} \quad z = a\log r + \theta\sqrt{1-a^2} + c,$$

or $z = a\log(x^2+y^2)^{1/2} + \tan^{-1}(y/x)\,.\,\sqrt{1-a^2} + c$, by (2)

or $z = (a/2)\times\log(x^2+y^2) + \sqrt{1-a^2}\tan^{-1}(y/x) + c$, a and c being arbitrary constants.

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial z}{x\partial x} + \frac{\partial z}{y\partial y} = \left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \quad ...(1)$$

Put $x\,dx = dX$, $y\,dy = dY$ so that $x^2/2 = X$, $y^2/2 = Y$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)(\partial z/\partial Y)$ or $P + Q = PQ$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, where $a + b = ab$ so that $b = a/(a-1)$ ... (4)

or $z = a(x^2/2) + a(a-1)^{-1}(y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \qquad \text{or} \qquad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \qquad \text{... (1)}$$

Putting $2x\ dx = dX$, $2y\ dy = dY$ so that $x^2 = X$, $y^2 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = 1$ or $PQ = 1$ whose solution is

$z = aX + bY + d$, where $ab = 1$ so that $b = 1/a$.

$$\therefore \qquad z = ax^2 + (1/a)y^2 + d \qquad \text{... (2)}$$

is complete integral of (1) containing two arbitrary constants a and d.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\ y^2 + \phi(a) \qquad \text{... (3)}$$

Differenting (3) partially w.r.t. 'a', $\qquad 0 = x^2 - (1/a^2)\ y^2 + \phi'(a) \qquad$ ... (4)

Then general integral is obtained by eliminating a from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$0 = x^2 + (-1/a^2)\ y^2 \qquad$ ... (5) $\qquad\qquad 0 = 1 \qquad$ ... (6)

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$$\boldsymbol{z = 2xy + C,\ C \text{ being an arbitrary constant}} \qquad \text{... (7)}$$

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.*, if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$$z = (y/x)\ x^2 + (x/y)\ y^2 + C \qquad \text{or} \qquad z = 2xy + C,$$

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write $\qquad z = ax^2 + (1/a)\ y^2 + C \qquad$ ... (8)

Differenting (8) partially w.r.t. 'a', we get

$$0 = x^2 - (1/a^2)y^2 \qquad \text{or} \qquad a = y/x \qquad \text{... (9)}$$

Eliminating a from (8) and (9), we get $\qquad z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$ **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \qquad \text{... (1)}$$

Let X, Y and Z be new variables such that

$dX = dx$, $\quad dY = dy \quad$ and $\quad dZ = z^{-1/2}dz \quad$ so that $\quad X = x$, $\quad Y = y$, $\quad Z = 2z^{1/2}$ ... (2)

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \qquad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $\quad Z = aX + by + c$, ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$$Z = aX \pm (a^2 - 1)^{1/2}Y + c \qquad \text{or} \qquad 2z^{1/2} = ax \pm (a^2 - 1)^{1/2}y + c,$$

which is the complete integral, a and c being arbitrary constants and $|a| \geq 1$.

Ex. 1. *Solve* $z = px + qy + pq$. **[Ravishanker 1997; Bangalore 2005; Sagar 1995, 96]**

Sol. The complete integral is $\quad z = ax + by + ab$, a, b being arbitrary constants ...(1)

Singular integral. Differentiating (1) partially w.r.t. a and b, we have

$$a = x + b \qquad \text{and} \qquad 0 = y + a. \qquad \text{...(2)}$$

Eliminating a and b between (1) and (2), we get $\quad z = -xy - xy + xy \quad$ *i.e.*, $\quad z = -xy$, which is the required singular solution, for it satisfies the given equation.

General Integral. Take $b = \phi(a)$, where ϕ denotes an arbitrary function.

Then (1) becomes $\quad z = a\,x + \phi(a)\,y + a\,\phi\,(a)$. ...(3)

Differentiating (3) partially w.r.t. a, $\quad 0 = x + \phi'(a)y + \phi(a) - a\,\phi'(a)$. ...(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 2. *Prove that complete integral of the equations* $\quad (px + qy - z)^2 = 1 + p^2 + q^2$ is $\; ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$. **[I.A.S. 1989]**

Sol. Re–writting the given equation, we have

$$px + qy - z = \pm\sqrt{(1 + p^2 + q^2)} \qquad \text{or} \qquad z = px + qy \pm \sqrt{(1 + p^2 + q^2)}$$

which is of standard form II and so its complete integral is

$$z = Ax + By \pm (1 + A^2 + B^2)^{1/2}. \qquad \text{...(1)}$$

To get the desired form of solution we take +ve sign in (1) and set $A = -a/c$ and $B = -b/c$.

Then (1) becomes $\quad z = -(ax + by)/c + (c^2 + a^2 + b^2)^{1/2}/c$

or $\quad ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$.

Ex. 3. *Solve* $z = px + qy + c\sqrt{(1+p^2+q^2)}$. **[I.A.S. 1989; Meerut 1998]**

Sol. The complete integral of the given equation is

$$z = ax + by + c\sqrt{(1+a^2+b^2)},\ a,\ b \text{ being arbitrary constants.} \qquad ...(1)$$

Singular Integral. Differentiating (1) partially w.r.t. a and b, we get

$$0 = x + ac/\sqrt{(1+a^2+b^2)} \qquad ...(2) \qquad\qquad 0 = y + bc/\sqrt{(1+a^2+b^2)}. \qquad ...(3)$$

$\therefore$ From (2) and (3), $\quad x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2)$.

$$\therefore \qquad c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2 + b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

so that $\quad 1 + a^2 + b^2 = c^2/(c^2 - x^2 - y^2)$. ...(4)

From (2), $\quad a = -\dfrac{x\sqrt{(1+a^2+b^2)}}{c} = -\dfrac{x}{\sqrt{(c^2-x^2-y^2)}}$, by (4)

Similarly from (3) and (4), we obtain $\quad b = -y/\sqrt{c^2-x^2-y^2}$.

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}$$

or $\quad z = (c^2 - x^2 - y^2)^{1/2} \quad$ or $\quad z^2 = c^2 - x^2 - y^2 \quad$ or $\quad x^2 + y^2 + z^2 = c^2$. ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take $b = \phi(a)$, where ϕ is an arbitrary function.

Then, (1) yeilds $\quad z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$. ...(6)

Differentiating both sides of (6) partially w.r.t. 'a', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\,\phi'(a)]. \qquad ...(7)$$

Eliminating a from (6) and (7), we get the general integral.

Ex. 10. *Find the compelte integral of* $p^2x + q^2y = (z - 2px - 2qy)^2$.

Sol. Taking positive root, the given equation reduces to

$$z - 2px - 2qy = (p^2x + q^2y)^{1/2} \qquad \text{or} \qquad z = 2px + 2qy + (p^2x + q^2y)^{1/2}$$

or
$$z = \sqrt{x}\,\frac{\partial z}{(1/2\sqrt{x})\partial x} + \sqrt{y}\,\frac{\partial z}{(1/2\sqrt{y})\partial y} + \frac{1}{2}\left[\left(\frac{\partial z}{(1/2\sqrt{x})\partial x}\right)^2 + \left(\frac{\partial z}{(1/2\sqrt{y})\partial y}\right)^2\right]^{1/2} \qquad ...\ (1)$$

Put $\quad (1/2\sqrt{x})\,dx = dX \quad$ and $\quad (1/2\sqrt{y})dy = dY \quad$ so that $\quad \sqrt{x} = X \quad$ and $\quad \sqrt{y} = Y$ ... (2)

Using (2), (1) gives $\quad z = (\partial z/\partial X)X + (\partial z/\partial Y)Y + (1/2) \times \{(\partial z/\partial X)^2 + (\partial z/\partial Y)^2\}^{1/2}$

or $\quad z = PX + QY + (1/2) \times (P^2 + Q^2)^{1/2}, \quad$ where $\quad P = \partial z/\partial X \quad$ and $\quad Q = \partial z/\partial Y$.

It is of the Clairaut's form $z = Px + Qy + f(P, Q)$ and so its complete integral is given by

$$z = aX + bY + (1/2) \times (a^2 + b^2)^{1/2} \qquad \text{or} \qquad z = a\sqrt{x} + b\sqrt{y} + (1/2) \times (a^2 + b^2)^{1/2}$$

Ex. 4. *Find the complete and singular integrals of the following equations:*

(*i*) $z = px + qy + \log(pq)$ **[Indore 2004; K.U. Kurukshetra 2006]**

(*ii*) $z = px + qy - 2\sqrt{pq}$. **[Bangalore 1993; Lucknow 2010]**

Sol. (*i*) The complete integral is $z = ax + by + \log(ab)$

or $z = ax + by + \log a + \log b$, a, b being arbitrary constants ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x + (1/a)$ and $0 = y + (1/b)$ so that $a = -1/x$ and $b = -1/y$. ...(2)

Eliminating a and b from (1) and (2), the required singular integral is

$$z = -1 - 1 + \log(1/xy) \quad \text{or} \quad z = -2 - \log(xy).$$

(*ii*) The complete integral is $z = ax + by - 2\sqrt{ab}$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$$0 = x - \frac{2b}{2\sqrt{ab}} \quad \text{and} \quad 0 = y - \frac{2a}{2\sqrt{ab}} \quad \text{so that} \quad x = \sqrt{\frac{b}{a}} \quad \text{and} \quad y = \sqrt{\frac{a}{b}}. \quad ...(2)$$

Now, using (1) $x - z = x - (ax + by - 2\sqrt{ab}) = \sqrt{\frac{b}{a}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$, using (2)

$$\therefore \quad x - z = \sqrt{(b/a)}. \quad ...(3)$$

Similarly, using (1) $y - z = y - (ax + by - 2\sqrt{ab})$, $= \sqrt{\frac{a}{b}} - a\sqrt{\frac{b}{a}} - b\sqrt{\frac{a}{b}} + 2\sqrt{ab}$

$$\therefore \quad y - z = \sqrt{(a/b)}. \quad ...(4)$$

From (3) and (4), $(x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 11. *Find a complete and the singular integral of* $4xyz = pq + 2px^2y + 2qxy^2$

Sol. The given equation can be rewritten as

$$z = \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right). \quad ...(1)$$

Put $2x\,dx = dX$ and $2y\,dy = dY$ ... (2)

so that $x^2 = X$ and $y^2 = Y$. ... (3)

Using (2), (1) becomes $z = (\partial z/\partial X)(\partial z/\partial Y) + X(\partial z/\partial X) + Y(\partial z/\partial Y)$

or $z = XP + YQ + PQ$, ... (4)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $z = XP + YQ + f(P, Q)$.

$\therefore$ Solution of (4) is $z = aX + bY + ab$, a, b being arbitrary constants.

or $z = ax^2 + by^2 + ab$, which is complete integral. ... (5)

Differentiating (5) partially w.r.t a and b, we have

$0 = x^2 + b$ and $0 = y^2 + b$ so that $b = -x^2$ and $a = -y^2$ .... (6)

Eliminating a and b between (5) and (6), the required singular integral is

$$z = -x^2y^2 - x^2y^2 + x^2y^2 \quad \text{or} \quad z = -x^2y^2.$$

Ex. 12. *Find the complete and singular solutions of* $z = px + qy + p^2q^2$. **[Jabalpur 2000; Sagar 1995; Rewa 2003; Ravishankar 2004]**

Sol. Given $$z = px + qy + p^2q^2 \qquad \text{... (1)}$$

Since (1) is in Clairaut's form, its complete solution is

$$z = ax + by + a^2b^2,\ a,\ b \text{ being arbitrary constants} \qquad \text{... (2)}$$

To find singular solution of (1). Differentiating (2) partially w.r.t. 'a' and 'b' successively,

$$0 = x + 2ab^2 \qquad \text{and} \qquad 0 = y + 2a^2b \qquad \text{... (3)}$$

From (3), $$a = -(y^2/2x)^{1/3} \qquad \text{and} \qquad b = -(x^2/2y)^{1/3} \qquad \text{... (4)}$$

Substituting the values of a and b given by (4) in (2), we get

$$z = -x(y^2/2x)^{1/3} - y(x^2/2y)^{1/3} + (x^2y^2/16)^{1/3} \qquad \text{or} \qquad z = -(3/4) \times 4^{1/3}\, x^{2/3}\, y^{2/3},$$

which is the required singular solution of (1)

Ex. 7. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$. **[I.A.S. 1991]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation,

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0 \quad \text{or} \quad (1 + a^3)\frac{dz}{du} = 3az \quad \text{or} \quad \frac{1+a^3}{z}dz = 3au.$$

Integrating $$(1 + a^3) \log z = 3au + b \qquad \text{or} \qquad (1 + a^3) \log z = 3a(x + ay) + b.$$

Ex. 13. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

[Delhi Maths Hons 2005; Meerut 2003]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \qquad \text{or} \qquad z^2(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

or $$du = \pm\, z(z^2 + a^2)^{1/2}\, dz = \pm\,(1/2) \times (z^2 + a^2)^{1/2}\,(2zdz)$$

Integrating, $$u + b = \pm\,(1/2) \times [(z^2 + a^2)^{3/2}/(3/2)]$$

or $$9(u + b)^2 = (z^2 + a^2)^3 \qquad \text{or} \qquad 9(x + ay + b)^2 = (z^2 + a^2)^3, \quad \text{...(1)}$$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially, w.r.t. 'a' and 'b', we get

$$18(x + ay + b)y = 3(z^2 + a^2) \times 2a \qquad \text{...(2)}$$

and $$18(x + ay + b) = 0. \qquad \text{...(3)}$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ do not satisfy the given equation. Hence $z = 0$ is not a singular solution of the given equation.

Ex. 14. (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

[Jabalpur 2004; Banglore 1993; I.A.S. 1996; Meerut 1997]

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$. **[I.A.S. 1979]**

Sol. (*i*) The given equation is of the form $f(p, q\, z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = k^2 \quad \text{or} \quad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

or $$\pm (1 + a^2)^{1/2}\frac{z}{(k^2 - z^2)^{1/2}}\, dz = du \quad \text{or} \quad \pm \frac{1}{2}(1 + a^2)(k^2 - z^2)^{-1/2}(-2z dz) = du.$$

Integrating, $\pm (1 + a^2)^{1/2}(k^2 - z^2)^{1/2} = u + b$ or $(1 + a^2)(k^2 - z^2) = (u + b)^2$

or $$(1 + a^2)(k^2 - z^2) = (x + ay + b)^2.$$

(*ii*) Here $k = 1$. Proceed as in part (*i*) and get complete integral

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2. \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. a and b, we get

$$2a(1 - z^2) = 2(x + ay + b) \times y \qquad \text{...(2)}$$

and $$0 = 2(x + ay + b). \qquad \text{...(3)}$$

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Now, $p = 0$, $q = 0$ and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z^2 = 1$.

Ex. 6. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

[Agra 2006; Jabalpur 2004; Rewa 2002 Sagar 1999; Vikram 1996 Delhi Maths Hons 1990; I.A.S. 1989; Kanpur 1994; Meerut 2003]

Sol. Given $$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{or} \quad \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \qquad \text{...(1)}$$

Let $z\, dz = dz$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$, where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side of the above equation to an arbitrary constant a^2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, we have

$$dZ = (a^2 + x^2)^{1/2}\, dx + (y^2 - a^2)^{1/2}\, dy.$$

Integrating, $$Z = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\} + (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$$

or $$z^2 = x^2(a^2 + x^2)^{1/2} + a^2 \log [x + (a^2 + x^2)^{1/2}\} + y(y^2 - a^2)^{1/2} - a^2\log\{y + (y^2 - a^2)^{1/2}\} + b$$

$[\because$ From (2), $Z = z^2/2]$

Ex. 7. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

[Bilaspur 2003; Indore 2002, 02; Jiwaji 2000; Bangalore 1995; I.A.S 1989]

Sol. Re–writting the given equation, $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y$. ...(1)

Let $\sqrt{z}\, dz = dZ$ so that $(2/3) \times z^{3/2} = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial y)^2 = x - y$ or $P^2 - Q^2 = x - y$, where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y. \quad ...(3)$$

Equating each side to an arbitrary constant a, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of P and Q in $dZ = P\, dx + Q\, dy$, $dZ = (x + a)^{1/2} dx + (y + a)^{1/2}\, dy$.

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$, as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$, a, b being arbitrary constants.

Ex. 14. *Find the complete integral of the partial differential equation*

$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2 + y^2)$ **[I.A.S. 2001]**

Sol. Re–writing the given equation, we have

$2q^2(p^2 - 4x^4) = x^2y^2(8q^2 - 3)$ or $(p^2 - 4x^4)/x^2 = y^2\ (8q^2 - 3)/2q^2 = 4a^2$, say

where a is an arbitrary constant. Then, $p^2 = 4x^2(a^2 + x^2)$ and $8q^2(y^2 - a^2) = 3y^2$

so that $p = 2x(a^2 + x^2)^{1/2}$ and $q = (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}$

Substituting these values in $dz = p\, dx + q\, dy$, we get

$$dz = 2x(a^2 + x^2)^{1/2}\, dx + (3/2)^{1/2} \times (y/2) \times (y^2 - a^2)^{-1/2}\, dy$$

Integrating, $z = 2\int x(a^2 + x^2)^{1/2} dx + (3/2)^{1/2} \times (1/2) \times \int y(y^2 - a^2)^{-1/2} dy + b$ ... (1)

Put $x^2 + a^2 = u$ and $y^2 - a^2 = v$ so that $2xdx = du$ and $2ydy = dv$ ... (2)

i.e., $xdx = (1/2) \times du$ and $ydy = (1/2) \times dv$. Then (1) reduces to

$$z = \int u^{1/2} du + (3/2)^{1/2} \times (1/4) \times \int v^{-1/2} dv + b$$

or $$z = (2/3) \times u^{3/2} + (3/2)^{1/2} \times (1/4) \times 2v^{1/2} + b$$

or $$z = (2/3) \times (x^2 + a^2)^{3/2} + (3/2)^{1/2} \times (1/2) \times (y^2 - a^2)^{1/2} + b,$$

which is the required complete integral containing a and b as arbitrary constants.

Ex. 1. *Find the characteristics of the equation* $pq = z$, *and determine the integral surface which passes through the parabola* $x = 0$, $y^2 = z$. **[Meerut 2005; I.A.S. 1999]**

Sol. Given equation is $$pq = z \qquad \text{... (1)}$$

We are to find its integral surface which passes through the given parabola given by

$$x = 0, \qquad \text{and} \qquad y^2 = z \qquad \text{... (2)}$$

Re-writing (2) in parametric form, we have

$$x = 0, \qquad y = \lambda, \qquad z = \lambda^2, \qquad \lambda \text{ being a parameter} \qquad \text{... (3)}$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = 0, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = \lambda^2 \qquad \text{... (4A)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$$p_0 q_0 = z_0, \qquad \text{or} \qquad p_0\, q_0 = \lambda^2, \text{ by (4A)} \qquad \text{... (5)}$$

Also, we have $$z_0'(\lambda) = p_0\, x_0'(\lambda) + q_0\, y_0'(\lambda)$$

so that $$2\lambda = p_0 \times 0 + q_0 \times 1 \qquad \text{or} \qquad q_0 = 2\lambda, \text{ by (4A)} \qquad \text{... (6)}$$

Solving (5) and (6), $$p_0 = \lambda/2 \qquad \text{and} \qquad q_0 = 2\lambda \qquad \text{... (4B)}$$

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = 0, \quad y_0 = \lambda, \quad z_0 = \lambda^2, \quad p_0 = \lambda/2, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Re-writing (1), let $$f(x, y, z, p, q) = pq - z = 0 \qquad \text{... (8)}$$

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = p \qquad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \qquad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q \qquad \text{... (13)}$$

From (9) and (13), $$(dx/dt) - (dq/dt) = 0, \qquad \text{so that} \qquad x - q = C_1, \qquad \text{.... (14)}$$

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$$x_0 - q_0 = C_1 \qquad \text{or} \qquad 0 - 2\lambda = C_1 \qquad \text{or} \qquad C_1 = -2\lambda, \text{ Then (14) becomes}$$

$$x - q = -2\lambda \qquad \text{or} \qquad x = q - 2\lambda, \qquad \text{... (15)}$$

From (10) and (12), $$(dy/dt) - (dp/dt) = 0 \qquad \text{so that} \qquad y - p = C_2, \qquad \text{... (16)}$$

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2$ or $\lambda - (\lambda/2) = C_2$ or $C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2$ or $y = p + (\lambda/2)$ ... (17)

From (12), $(1/p)\ dp = dt$ so that $\log p - \log C_3 = t$ or $p = C_3\ e^t$ ... (18)

Using initial values (7), (18) gives $p_0 = C_3 e^0$ or $\lambda/2 = C_3$

Hence (18) reduces to $p = (\lambda/2) \times e^t$ ... (19)

From (13), $(1/q)\ dq = dt$ so that $\log q - \log C_4 = t$ or $q = C_4\ e^t$ ... (20)

Using initial values (7), (20) gives $q_0 = C_4\ e^0$ or $2\lambda = C_4$

Hence (20) reduces to $q = 2\lambda\, e^t$ ... (21)

Using (21), (15) becomes $x = 2\lambda\ e^t - 2\lambda$ or $x = 2\lambda\ (e^t - 1)$ ... (22)

Using (19), (17) becomes $y = (\lambda/2)\ e^t + \lambda/2$ or $y = (\lambda/2) \times (e^t + 1)$ ... (23)

Substituting values of p and q from (19) and (21) in (11), we get

$dz/dt = 2\{(\lambda/2) \times e^t\} \times \{2\lambda\ e^t\}$ or $dz = 2\lambda^2\ e^{2t}\ dt$.

Integrating, $z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant .... (24)

Using initial values (7), (24) gives $z_0 = \lambda^2 e^0 + C_5$ or $\lambda^2 = \lambda^2 + C_5$ or $C_5 = 0$

Then, (24) gives $z = \lambda^2\ e^{2t}$ or $z = \lambda^2 (e^t)^2$ ... (25)

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$e^t = (x+4y)/(4y-x)$ and $\lambda = (4y-x)/4$

Substituting these values of e^t and λ in (25), we have

$z = \{(4y-x)^2/16\} \times \{(x+4y)/(4y-x)\}^2$ or $16z = (4y+x)^2$,

which is the required integral surface of (1) passing through (2).

Ex. 2. *Find the solution of the equation* $z = (p^2+q^2)/2 + (p-x)(q-y)$ *which passes through the x-axis.* **[Himachal 1996; 2004; I.A.S. 2002]**

Sol. Given equation is $z = (p^2+q^2)/2 + (p-x)(q-y)$ ... (1)

We are to find its integral surface which passes through x-axis which is given by equations

$y = 0$ and $z = 0$ ... (2)

Re-writing (2) in parametric form, $x = \lambda$, $y = 0$, $z = 0$, λ being the parameter ... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0 = x_0(\lambda) = \lambda$, $y_0 = y_0(\lambda) = 0$, $z = z_0(\lambda) = 0$ ... (4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$$z_0 = (p_0^2 + q_0^2)/2 + (p_0 - x_0)\ (q_0 - x_0) \qquad \text{or} \qquad 0 = (p_0^2 + q_0^2)/2 + q_0(p_0 - \lambda), \text{ by (4A)}$$

or $$p_0^2 + q_0^2 + 2q_0\ p_0 - 2q_0\lambda = 0 \qquad \text{... (5)}$$

Also, we have $$z_0'(\lambda) = p_0\ x_0'(\lambda) + q_0\ y_0'(\lambda)$$

so that $0 = p_0 \times 1 + q_0 \times 0$ or $p_0 = 0$, by (4A) ... (6)

Solving (5) and (6), $p_0 = 0$ and $q_0 = 2\lambda$ ... (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Let $$f(x, y, z, p, q) = (p^2 + q^2)/2 + pq - py - qx + xy - z = 0 \qquad \text{... (8)}$$

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = p + q - y \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = q + p - x \qquad \text{... (10)}$$

$$dz/dt = p\ (\partial f/\partial p) + q(\partial f/\partial q) = p(p + q - y) + q(q + p - x), \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y \qquad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q\ (\partial f/\partial z) = p + q - x \qquad \text{... (13)}$$

From (9) and (12), $(dx/dt) - (dp/dt) = 0$ so that $x - p = C_1$ ... (14)

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $x - p = \lambda$ or $x = p + \lambda$ ... (15)

From (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = C_2$, ... (16)

where C_2 is an arbitrary constant.

Using initial conditions (7), (16) gives $0 - 2\lambda = C_2$ or $C_2 = -2\lambda$.

Hence (16) reduces to $y - q = -2\lambda$ or $y = q - 2\lambda$ ... (17)

$$\therefore \quad \frac{d(p+q-x)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - y + p + q - x - (p + q - y), \text{ using (9), (12) and (13)}$$

or $$\frac{d(p+q-x)}{dt} = p + q - x \qquad \text{or} \qquad \frac{d(p+q-x)}{p+q-x} = dt.$$

Integrating, $\log(p + q - x) - \log C_3 = t$ or $p + q - x = C_3\ e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = C_3$ or $C_3 = \lambda$.

Hence (18) reduces to $$p + q - x = \lambda e^t \qquad \text{... (19)}$$

Now, $$\frac{d(p+q-y)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p + q - y + p + q - x - (q + p - x), \text{ by (10), (12) and (13)}$$

or $$\frac{d(p+q-y)}{dt} = p + q - y \qquad \text{or} \qquad \frac{d(p+q-y)}{b+q-y} = dt.$$

Integrating, $\log(p+q-x)-\log C_3 = t$ or $p+q-x = C_3\ e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0+2\lambda-\lambda = C_3$ or $C_3 = \lambda$.

Hence (18) reduces to $p+q-x = \lambda e^t$ ... (19)

Now, $\dfrac{d(p+q-y)}{dt} = \dfrac{dp}{dt}+\dfrac{dq}{dt}-\dfrac{dy}{dt} = p+q-y+p+q-x-(q+p-x)$, by (10), (12) and (13)

or $\dfrac{d(p+q-y)}{dt} = p+q-y$ or $\dfrac{d(p+q-y)}{b+q-y} = dt$.

Integrating, $\log(p+q-y)-\log C_4 = t$ or $p+q-y-C_4\,e^t$ ... (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0+2\lambda-0 = C_4$ or $C_4 = 2\lambda$.

Hence (20) reduces to $p+q-y = 2\lambda\ e^t$ ... (21)

From (9) and (21), $dx/dt = 2\lambda\ e^t$ so that $x = 2\lambda\ e^t + C_5$ ... (22)

where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda = 2\lambda + C_5$ or $C_5 = -\lambda$.

Hence (22) reduces to $x = 2\lambda\ e^t - \lambda$ or $x = \lambda\ (2e^t - 1)$ ... (23)

From (10) and (19), $dy/dt = \lambda\ e^t$ so that $y = \lambda\ e^t + C_6$ ... (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0 = \lambda + C_6$ or $C_6 = -\lambda$.

$p\ e^{-t} = \int (2\lambda)\ e^{-t} dt + C_7 = -2\lambda\, e^{-t} + C_3$ or $p = -2\lambda + C_3\, e^t$ ... (27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7$ or $C_7 = 2\lambda$.

Hence (27) reduces to $p = -2\lambda + 2\lambda e^t$ or $p = 2\lambda\ (e^t - 1)$ ... (28)

Substituting value of x from (15) in (13), we get

$dq/dt = p+q-(p+\lambda)$ or $dq/dt - q = -\lambda$, ... (29)

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$q e^{-t} = \int (-\lambda)\ e^{-t} dt + C_8 = \lambda e^{-t} + C_8$ or $q = \lambda + C_8\ e^t$ ... (30)

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.

Hence (30) reduces to $q = \lambda + \lambda\ e^t$ or $q = \lambda\ (1+e^t)$ ... (31)

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda\ e^t) + q(\lambda\ e^t) = 2\lambda\ (e^t-1)\ (2\lambda\ e^t) + \lambda(1+e^t)\ (\lambda\ e^t)$$

[on putting values of p and q with help of (28) and (31)]

or $dz/dt = 5\lambda^2 e^{2t} - 3\lambda^2 e^t$ or $dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^t)dt$.

Integrating, $z = (5/2)\times\lambda^2 e^{2t} - 3\lambda^2 e^t + C_9$ ... (32)

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives $0 = (5/2) \times \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to

$$z = (5/2) \times \lambda^2 (e^{2t} - 1) - 3\lambda^2 (e^t - 1) \qquad \text{... (33)}$$

Solving (23) and (25) for λ and e^t, $\quad \lambda = x - 2y \quad$ and $\quad e^t = (x-y)/(x-2y)$ ... (34)

Eliminating λ and e^t from (33) and (34), we have

$(p^2 + q^2)x = pz$

$$z = \frac{5}{2}(x-2y)^2 \left\{ \left(\frac{x-y}{x-2y} \right)^2 - 1 \right\} - 3(x-2y)^2 \left(\frac{x-y}{x-2y} - 1 \right)$$

or $$z = (5/2) \times \{(x-y)^2 - (x-2y)^2\} - 3 \ \{(x-2y)\,(x-y) - (x-2y)^2\}$$

or $$z = (y/2) \times (4x - 3y), \text{ on simplification.}$$

Ex. 3. *Determine the characteristics of the equation* $z = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4z + x^2 = 0,\ y = 0$. **[Himachal 2000, 05]**

Sol. Do yourself, the required characteristics are $x = 2\lambda(2 - e^{-t})$, $y = 2\sqrt{2}\lambda\ (e^{-t} - 1)$, $z = -\lambda^2 e^{-2t}$, λ being parameter. Solution is $4z + (x + y\sqrt{2})^2 = 0$.

Ex. 4. *Determine the characteristics of the equation* $p^2 + q^2 = 4z$ *and find the solution of this equation which reduces to* $z = x^2 + 1$ *when* $y = 0$.

Ex. 6. *Solve the following partial differential equations :*

(*a*) $(D^2 - 2DD' + D'^2)z = \tan(y + x)$ or $(D - D')^2 z = \tan(y + x)$ **[Jiwaji 1996]**

(*b*) $(D^2 - 2aDD' + a^2D'^2)z = f(y + ax)$ or $(D - aD')^2 z = f(y + ax)$.

(*c*) $4r - 4s + t = 16 \log(x + 2y)$. **[Agra 2009; Meerut 2009; Ravishankar 2000]**

Sol. (*a*) Here auxiliary equation is $(m - 1)^2 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(D - D')^2}\tan(y + x) = \frac{x^2}{1^2 \times 2!}\tan(y + x) = \frac{x^2}{2}\tan(y + x)$$

[Using formula (*ii*) of working rule with $a = 1$, $b = 1$, $m = 2$]

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times \tan(y + x)$.

(*b*) Here auxiliary equation is $(m - a)^2 = 0$ so that $m = a, a$.

$\therefore$ C.F. $= \phi_1(y + ax) + x\phi_2(y + ax)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - aD')^2} f(y + ax) + \frac{x^2}{1^2 \times 2!} f(y + ax) = \frac{x^2}{2} f(y + ax).$$

[using formula (*ii*) of working rule with $a = a$, $b = 1$, $m = 2$]

$\therefore$ General solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times f(y + ax)$.

(*c*) Since $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$, the given equation becomes

$4(\partial^2 z/\partial x^2) - 4(\partial^2 z/\partial x \partial y) + (\partial^2 z/\partial y^2) = 16 \log(x + 2y)$ or $(4D^2 - 4DD' + D'^2)z = 16 \log(x + 2y)$

Its auxiliary equation is $4m^2 - 4m + 1 = 0$ so that $m = 1/2, 1/2$.

$\therefore$ C.F. $= \phi_1(2y + x) + x\phi_2(2y + x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x + 2y) = 16 \times \frac{x^2}{2^2 \times 2!} \log(x + 2y) = 2x^2 \log(x + 2y)$$

[using formula (*ii*) of working rule with $a = 1$, $b = 2$, $m = 2$]

$\therefore$ The required solution is $z = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2 \log(x + 2y)$.

Ex. 11. *Solve* $(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos(x-y)$ **[K.U. Kurukshetra, 2004]**

Sol. Given $$(D^3 - 7DD'^2 - 6D'^3)\, z = x^2 + xy^2 + y^3 + \cos(x-y) \qquad ...(1)$$

Here auxiliary equation is $m^3 - 7m - 6 = 0$ so that $m = -1, -2, 3.$

$\therefore\ C.F. = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

P.I. Corresponding to $(x^2 + xy^2 + y^3)$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}(x^2 + xy^2 + y^3) = \frac{1}{D^3}\left\{1 - \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right)\right\}^{-1}(x^2 + xy^2 + y^3)$$

$$= \frac{1}{D^3}\left\{1 + \left(7\frac{D'^2}{D^2} + 6\frac{D'^3}{D^3}\right) + \ldots\right\}(x^2 + xy^2 + y^3) = \frac{1}{D^3}(x^2 + xy^2 + y^3) + \frac{7}{D^5}(2x + 6y) + \frac{36}{D^6}1$$

$$= (x^5/60 + x^4y^2/24 + x^3y^3/6) + 7(x^6/360 + x^5y/20) + 36 \times (x^6/720)$$

$$= 5x^6/72 + x^5/60 + 7x^5y/20 + x^4y^2/24 + x^3y^3/6$$

P.I. Corresponding to $\cos(x-y)$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}\cos(x-y) = \frac{1}{(D+D')}\,\frac{1}{(D^2 - DD' - 6D'^2)}\cos(x-y)$$

$$= \frac{1}{D+D'}\,\frac{1}{1^2 - 4\times 1\times(-1) - 6\times(-1)^2}\iint \cos v\,dv\,dv, \text{ where } v = x-y$$

$$= \frac{1}{D+D'} \times \frac{1}{(-4)} \times (-\cos v) = \frac{1}{4}\,\frac{1}{D+D'}\cos(x-y) = \frac{1}{4}\,\frac{1}{(-1)\times D - 1\times D'}\cos(x-y) = -\frac{1}{4}\,\frac{x}{(-1)^1 \times 1!}\cos(x-y)$$

$$= (x/4) \times \cos(x-y)$$

Hence the required general solution is $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

$$+(5/72)\times x^6 + x^5/60 + (7/20)\times x^5 y + (1/24)\times x^4y^2 + (1/6)\times(x^3y^3) + (x/4)\times\cos(x-y),$$

Ex. 4. *Solve* $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy.$

[I.A.S 2009; Meerut 1994; Delhi Maths (Hons.) 2007]

Sol. Here auxiliary equation is $m^2 - m - 2 = 0$ so that $m = 2, -1.$

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - 2D')} \frac{1}{D + D'}\{(2x^2 + xy - y) \sin xy - \cos xy\}$$

$$= \frac{1}{D - 2D'} \frac{1}{D + D'}\{(2x - y)(x + y) \sin xy - \cos xy\}$$

$$= \frac{1}{D - 2D'} \int \{(x - c)(2x + c) \sin x (c + x) - \cos x (c + x)\}\, dx \text{, taking } c = y - x$$

$$= \frac{1}{D - 2D'} \int \{(x - c)(2x + c) \sin (cx + x^2) - \cos (cx + x^2)\}\, dx$$

$$= \frac{1}{D - 2D'}\left[-(x - c) \cos (cx + x^2) + \int \cos (cx + x^2)\, dx - \int \cos (cx + x^2)\, dx\right]$$

$$= \frac{1}{D - 2D'}(y - 2x) \cos xy, \text{ as } c = y - x$$

$$= \int (c' - 4x) \cos (c'x - 2x^2)\, dx \text{, where } c' = y + 2x$$

$$= \int \cos t\, dt = \sin t, \text{ putting } c'x - 2x^2 = t \text{ so that } (c' - 4x)dx = dt$$

$$= \sin (c'x - 2x^2) = \sin xy, \text{ as } c' = y + 2x.$$

$\therefore$ Required solution is $z = \phi_1(y + 2x) + \phi_2(y - x) + \sin xy.$

Ex. 5. *Solve* (*a*) $r + s - 6t = y \cos x.$ or $(D^2 + DD' - 6D'^2)z = y \cos x$

[Bilaspur 2002, Indore 2002, Jabalpur 1999 Meerut 2000, 02, Ravishankar 1994, Jiwaji 1999, Garhwal 2005, 10; I.A.S.1992, 2008; Vikram 1999, Delhi Maths (Hons) 2007, Purvanchal 2007; Kanpur 2011]

(*b*) $(D^2 + DD' - 6D'^2)z = y \sin x.$

Sol. (*a*) Since $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$, the given equation becomes

$$\partial^2 z/\partial x^2 + \partial^2 z/\partial x\, \partial y - 6(\partial^2 z/\partial y^2) = y \cos x \quad \text{or} \quad (D^2 + DD' - 6D'^2)z = y \cos x \quad \ldots(1)$$

Its auxiliary equation is $m^2 + m - 6 = 0$ so that $m = 2, -3.$

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y - 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} y \cos x = \frac{1}{(D - 2D')(D + 3D')} y \cos x$$

$$= \frac{1}{D - 2D'} \int (3x + c) \cos x\, dx \text{, where } c = y - 3x$$

$$= \frac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\,dx\right], \text{ integrating by parts}$$

$$= \frac{1}{D-2D'}[y\sin x + 3\cos x], \text{ as } c = y-3x$$

$$= \int [(c'-2x)\sin x + 3\cos x]\,dx, \text{ where } c' = y+2x$$

$$= (c'-2x)(-\cos x) - \int (-2)(-\cos x)\,dx + 3\sin x, \text{ integrating by parts}$$

$$= y(-\cos x) - 2\sin x + 3\sin x, \text{ as } c' = y+2x$$

$$= \sin x - y\cos x.$$

$\therefore$ General solution is $\qquad z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x.$

(*b*) Proceed as in part (*a*). $\qquad$ **Ans.** $z = \phi_1(y+2x) + \phi_2(y-3x) - y\sin x - \cos x$

Ex. 6. *Solve* $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y.$ **[Agra 2009; Meerut 1999; Bilaspur 2002; Indore 2004; Jabalpur 1999; Rewa 2002; Ranchi 2010]**

Sol. Given equation is $\qquad (D+D')^2 z = 2\cos y - x\sin y. \qquad$ ...(1)

Its auxiliary equation is $\qquad (m+1)^2 = 0 \qquad$ so that $\qquad m = -1, -1.$

$\therefore$ $\quad$ C.F. $= \phi_1(y-x) = x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{D+D'}\dfrac{1}{D+D'}(2\cos y - x\sin y) = \dfrac{1}{D+D'}\int[2\cos(x+c) - x\sin(x+c)]\,dx$, where $c = y-x$

$$= \frac{1}{D+D'}\left[2\int\cos(x+c)\,dx - \int x\sin(x+c)\,dx\right] = \frac{1}{D+D'}\left[2\sin(x+c) - \left\{-x\cos(x+c) + \int\cos(x+c)\,dx\right\}\right]$$

$$= \frac{1}{D+D'}[2\sin(x+c) + x\cos(x+c) - \sin(x+c)] = \frac{1}{D+D'}(\sin y + x\cos y), \text{ as } c = y-x$$

$$= \int[\sin(x+c') + x\cos(x+c')]\,dx = -\cos(x+c') + x\sin(x+c') - \int\{1.\sin(x+c')\}\,dx, \text{ where } c' = y-x$$

$$= -\cos(x+c') + x\sin(x+c') + \cos(x+c') = x\sin y, \text{ as } c' = y-x.$$

So the required solution is $\qquad z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y.$

Ex. 4. *A surface is drawn satisfying $r + t = 0$ and touching $x^2 + z^2 = 1$ along its section by $y = 0$. Obtain its equation in the form $x^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$.* **[Meerut 1998]**

Sol. Given equation $(D^2 + D'^2)z = 0$ i.e. $(D + iD')(D - iD')z = 0$.

$\therefore$ Its solution is $z = \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

The given surface is $x^2 + z^2 = 1$ or $z = (1 - x^2)^{1/2}$. ...(2)

Since (1) and (2) touch along their common section by $y = 0$, ...(3)

the values of p and q from (1) and (2) must be the same.

$$\therefore \quad p = i\,\phi_1'(y + ix) - i\,\phi_2'(y - ix) = -\frac{x}{(1-x^2)^{1/2}} \quad \text{and} \quad q = \phi_1'(y + ix) + \phi_2'(y - ix) = 0.$$

Using (3), these reduce to

$$\phi_1'(ix) - \phi_2'(-ix) = \frac{ix}{(1 + x^2 i^2)^{1/2}} \quad \text{and} \quad \phi_1'(ix) + \phi_2'(-ix) = 0, \quad \text{noting that } i^2 = -1$$

Solving these for $\phi_1'(ix)$ and $\phi_2'(-ix)$, $\phi'_1(ix) = \dfrac{ix}{2(1+x^2i^2)^{1/2}}$, $\phi'_2(-ix) = \dfrac{-ix}{2(1+x^2i^2)^{1/2}}$.

Writing $ix = X$ and $-ix = Y$, these give $\phi_1'(X) = \dfrac{X}{2(1+X^2)^{1/2}}$ $\phi_2'(Y) = \dfrac{Y}{2(1+Y^2)^{1/2}}$

Integrating, $\phi_1(X) = (1/2) \times (1 + X^2)^{1/2} + c_1$, $\phi_2(Y) = (1/2) \times (1 + Y^2)^{1/2} + c_2$

These give $\phi_1(y + ix) = (1/2) \times \{1 + (y + ix)^2\}^{1/2} + c_1$, $\phi_2(y - ix) = (1/2) \times \{1 + (y - ix)^2\}^{1/2} + c_2$.

Putting these in (1) and writing $c_1 + c_2 = c$, $z = (1/2) \times [\sqrt{\{1 + (y + ix)^2\}} + \sqrt{\{1 + (y - ix)^2\}}] + c$ ...(4)

Now equating two values of z from (2) and (4) at $y = 0$, we get

$(1/2) \times [\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)}] + c = \sqrt{(1 - x^2)}$ so that $c = 0$.

Then, (4) gives $2z = \sqrt{\{1 + (y + ix)^2\}} + \sqrt{\{1 + (y - ix)^2\}}$. Squaring its both sides gives

$$\text{or} \quad 4z^2 = \{1 + (y + ix)^2\} + (1 + (y - ix)^2\} + 2\sqrt{[\{1 + (y + ix)^2\}\{1 + (y - ix)^2\}]},$$

$$\text{or} \quad 2z^2 = (1 + y^2 - x^2) + \sqrt{[\{1 + (y + ix)^2\}\{1 + (y - ix)^2\}]}. \qquad ...(5)$$

Squaring both sides of (5), we get

$$4z^4 = (1 + y^2 - x^2)^2 + \{1 + (y + ix)^2\}\{1 + (y - ix)^2\} + 2(1 + y^2 - x^2)\sqrt{[\{1 + (y + ix)^2\}\{1 + (y - ix)^2\}]}$$

$$\text{or} \quad 4z^4 = (1 + y^2 - x^2)^2 + \{(1 + y^2 - x^2) + 2ixy\}\{(1 + y^2 - x^2) - 2ixy\}$$
$$+ 2(1 + y^2 - x^2)\{2z^2 - (1 + y^2 - x^2)\}, \text{ using (5)}$$

$$\text{or} \quad 4z^4 = (1 + y^2 - x^2)^2 + (1 + y^2 - x^2)^2 + 4x^2y^2 + 4z^2(1 + y^2 - x^2) - 2(1 + y^2 - x^2)^2$$

$$\text{or} \quad 4z^4 = 4x^2y^2 + 4z^2(1 + y^2 - x^2) \quad \text{or} \quad z^2(x^2 + z^2 - 1) = y^2(x^2 + z^2).$$

Ex.5. *Find a surface satisfying the equation $D^2z = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.* **Ans.** $z = x^3 + y^3 + (x + y + 1)^2$

Ex. 4. *Solve* $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. **[KU Kurukshetra 2005]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $= \dfrac{1}{(D - D' - 1)(D - D' - 2)}\sin(2x + 3y) = \dfrac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}\sin(2x + 3y)$

$$= \frac{1}{-2^2 + 2 \times (2 \times 3) - 3^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1}\sin(2x + 3y) = D\frac{1}{-3D^2 + 3DD' + D}\sin(2x + 3y)$$

$$= D\frac{1}{-3 \times (-2^2) + 3 \times (2 \times 3) + D}\sin(2x + 3y) = D\frac{1}{D - 6}\sin(2x + 3y)$$

$$= D(D+6)\frac{1}{D^2 - 36}\sin(2x + 3y) = (D^2 + 6D)\frac{1}{-2^2 - 36}\sin(3x + 2y)$$

$= -(1/40) \times [D^2\sin(2x + 3y) + 6D\sin(2x + 3y)] = -(1/40) \times [-4\sin(2x + 3y) + 12\cos(2x + 3y)]$

$\therefore$ Solution is $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/10) \times [\sin(2x + 3y) - 3\cos(2x + 3y)]$.

Ex. 5. *Solve* (*a*) $(D - D'^2)z = \cos(x - 3y)$ **[Delhi Maths (Hons.) 1998, 2007, 2009, 2011]**

(*b*) $(D^2 - D')z = \cos(3x - y)$.

Sol. (*a*) Here $(D - D'^2)$ cannot be resolved into linear factors in D and D'. Hence in order to find C.F. of the given equation, consider the equation

$$(D - D'^2)z = 0. \qquad ...(1)$$

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2z = Ak^2e^{hx + ky}$. Then (1) gives

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

$\therefore$ C.F. $= \Sigma Ae^{k(kx+y)}$, where A, k are arbitrary constants.

Now, P.I. $= \dfrac{1}{D - D'^2}\cos(x - 3y) = \dfrac{1}{D - (-3^2)}\cos(x - 3y)$

$= (D-9)\dfrac{1}{(D+9)(D-9)}\cos(x - 3y) = (D-9)\dfrac{1}{D^2 - 81}\cos(x - 3y) = \dfrac{(D-9)}{-1^2 - 81}\cos(x - 3y)$

$= -(1/82) \times [D\cos(x - 3y) - 9\cos(x - 3y)] = -(1/82) \times [-\sin(x - 3y) - 9\cos(x - 3y)]$.

$\therefore$ General solution is $z = \Sigma Ae^{k(kx + y)} + (1/82) \times [\sin(x - 3y) + 9\cos(x - 3y)]$

Ex. 7. *Solve* $(D^2 - D'^2 - 3D + 3D')z = xy.$

Sol. Re–writing, given equation is $(D - D')(D + D' - 3)z = xy.$

Its C.F. $= \phi_1(y + x) + e^{3x}\phi_2(x - y)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+...\right)\left\{1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+...\right\}xy = -\frac{1}{3D}\left(1+\frac{D'}{D}+...\right)\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+...\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\frac{D'}{D}+\frac{1}{3}D'+...\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right) = -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right)$$

$\therefore$ solution is $z = \phi_1(y+x) + e^{3x}\phi_2(y-x) - (1/6)\times x^2y - (x^2/9) - (2x/27) - (x^3/18)$

Ex. 4. *Solve* $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x + y)$. **[Delhi Maths (H) 1999, 2008]**

Sol. Since $(3D^2 - 2D'^2 + D - 1)$ cannot be resolved into linear factors in D and D', hence C.F. $= \sum Ae^{hx+ky}$, where A, h are arbitrary constants connected by $3h^2 - 2k^2 + h - 1 = 0$.

$$\text{P.I.} = \frac{1}{3D^2 - 2D'^2 + D - 1}4e^{x+y}\cos(x+y) = 4e^{x+y}\frac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{3D^2 + 7D - 2D'^2 - 4D' + 1}\cos(x+y) = 4e^{x+y}\frac{1}{3(-1^2) + 7D - 2(-1^2) - 4D' + 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{7D - 4D'}\cos(x+y) = 4e^{x+y}(7D + 4D')\frac{1}{49D^2 - 16D'^2}\cos(x+y)$$

$$= 4e^{x+y}\frac{7D + 4D'}{49(-1^2) - 16(-1^2)}\cos(x+y)$$

$$= -(4/33)\times e^{x+y}(7D + 4D')\cos(x+y) = -(4/33)e^{x+y}\times[7D\cos(x+y) + 4D'\cos(x+y)]$$

$$= -(4/33)\times e^{x+y}[-7\sin(x+y) - 4\sin(x+y)] = (4/3)\times e^{x+y}\sin(x+y).$$

Hence general solution is $z = \Sigma Ae^{hx+ky} + (4/3)\times e^{x+y}\sin(x+y).$

Ex. 1. *Solve* (*a*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$. **[Jabalpur 2004; I.A.S. 1992]**

(*b*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^x$.

Sol. (*a*) From given equation $\quad (D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y$. ...(1)

$\therefore \quad$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\times\sin(x+2y).$$

P.I. corresponding to e^y

$$= \frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{D-D'+1}\cdot\frac{1}{D-1}e^{0\cdot x+1\cdot y} = \frac{1}{D-D'+1}\cdot\frac{1}{0-1}e^{0\cdot x+1\cdot y}$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D(1-D'/D)}1 = -e^y\frac{1}{D}\left(1-\frac{D'}{D}+\ldots\right)^{-1}1$$

$$= -e^y(1/D)(1+\ldots).1 = -e^y x.$$

$\therefore$ The general solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2)\times\sin(x + 2y) - xe^y$.

Ex. 8. *Solve* $(D^2 - D')(D - 2D')z = e^{2x+y} + xy$.

Sol. C.F. corresponding to linear factor $(D - 2D')$ is $\phi(y + 2x)$. Now, $(D^2 - D')$ cannot be resolved into linear factor in D and D'. To find C.F. corresponding to it, we consider the equation

$$(D^2 - D')z = 0. \qquad \text{...(1)}$$

Let a trial solution of (1) be $\quad z = Ae^{hx+ky}$. ...(2)

$\therefore$ $D^2z = Ah^2e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) becomes

$A(h^2 - k)e^{hx+ky} = 0$ so that $h^2 - k = 0$ or $k = h^2$.

So from (2), C.F. corresponding to $(D^2 - D')$ is $\sum Ae^{hx+h^2y}$.

Now, P.I. corresponding to e^{2x+y}

$$= \frac{1}{D-2D'} \cdot \frac{1}{D^2 - D'} e^{2x+y} = \frac{1}{D-2D'} \frac{1}{2^2 - 1} e^{2x+y} = \frac{1}{3} \frac{1}{D-2D'} e^{2x+y} \cdot 1$$

$$= \frac{1}{3} e^{2x+y} \frac{1}{(D+2) - 2(D'+1)} 1 = \frac{1}{3} e^{2x+y} \frac{1}{D(1 - 2D'/D)} 1 = \frac{1}{3} e^{2x+y} \frac{1}{D}\left(1 - \frac{2D'}{D}\right)^{-1} 1$$

$= (1/3) \times e^{2x+y} \times (1/D)\,(1 + \ldots)\,1 = (x/3) \times e^{2x+y}$

and P.I. corresponding to xy

$$= \frac{1}{(D-2D')(D^2 - D')} xy = \frac{1}{(-2D')(1 - D/2D')(-D')(1 - D^2/D')} xy$$

$$= \frac{1}{2D'^2}\left(1 - \frac{D}{2D'}\right)^{-1}\left(1 - \frac{D^2}{D'}\right)^{-1} xy = \frac{1}{2D'^2}\left(1 + \frac{D}{2D'} + \ldots\right)(1 + \ldots)xy$$

$$= \frac{1}{2D'^2}\left(1 + \frac{D}{2D'} + \ldots\right)xy = \frac{1}{2D'^2}\left(xy + \frac{1}{2D'}y\right) = \frac{1}{2D'^2}\left(xy + \frac{y^2}{4}\right) = \frac{1}{2}\left(\frac{xy^3}{6} + \frac{y^4}{3 \times 4 \times 4}\right).$$

$$\therefore \text{ General solution } z = \phi(y + 2x) + \sum Ae^{hx+h^2y} + (x/3) \times e^{2x+y} + (xy^3)/12 + y^4/96,$$

where ϕ is an arbitrary function and A and h are arbitrary constants.

Example. *Find a surface satisfying $r + s = 0$, i.e., $(D^2 + DD')z = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$.* **[I.A.S. 1994]**

Sol. Given $(D^2 + DD')z = 0.$ or $D(D + D') = 0$ ...(1)

$\therefore$ Solution of (1) is $z = \text{C.F.} = \phi_1(y) + \phi_2(y - x),$ ...(2)

where ϕ_1 and ϕ_2 are arbitrary functions.

Since (2) touches the curve given by $z = 4x^2 + y^2$ ...(3)

and $y = 2x + 1,$ ...(4)

values of p $(= \partial z/\partial x)$ and q $(= \partial z/\partial y)$ obtained from (2) and (3) must be equal for any point on (4).

$\therefore$ $-\phi_2'(y - x) = 8x$ for $y = 2x + 1$ or $\phi_2'(x + 1) = -8x.$ ...(5)

and $\phi_1'(y) + \phi_2'(y - x) = 2y$ for $y = 2x + 1$ or $\phi_1'(2x + 1) + \phi_2'(x + 1) = 4x + 2$ ...(6)

From (5), $\phi_2'(x) = 8 - 8x$

Integrating it, $\phi_2(x) = 8x - 4x^2 + c_1$, c_1 being an arbitrary constant ...(7)

Subtracting (5) from (6), $\phi_1'(2x + 1) = 12x + 2 = 6(2x + 1) - 4$

so that $\phi_1'(x) = 6x - 4.$

Integrating it, $\phi_1(x) = 3x^2 - 4x + c_2$, c_2 being an arbitrary constant ...(8)

From (8), $\phi_1(y) = 3y^2 - 4y + c_2.$

and from (7), $\phi_2(y - x) = 8(y - x) - 4(y - x)^2 + c_1.$

Putting the above values of $\phi_1(y)$ and $\phi_2(y - x)$ in (2), we get

$$z = 3y^2 - 4y + c_2 + 8(y - x) - 4(y - x)^2 + c_1.$$

or $z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $c_3 = c_1 + c_2.$ ...(9)

Equating the values of z from (3) and (9), we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $y = 2x + 1.$

$\therefore$ $c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy = 8x^2 + 2(2x + 1)^2 - 4(2x + 1) + 8x - 8x(2x + 1) = -2$

Hence, from (9), the required surface is $4x^2 - 8xy + y^2 - 4y + z + 2 = 0.$

Illustrative example. *Find a surface satisfying equation* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$ *and touching the hyperbolic paraboloid* $z = x^2 - y^2$ *along its section by the plane* $y = 1$.

[Meerut 1998]

Sol. Re–writing given equation, $2x^2\dfrac{\partial^2 z}{\partial x^2} - 5xy\dfrac{\partial^2 z}{\partial x\partial y} + 2y^2\dfrac{\partial^2 z}{\partial y^2} + 2\left(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}\right) = 0.$

or $\{2x^2D^2 - 5xyDD' + 2y^2D'^2 + 2(xD + yD')\}z = 0$ ...(1)

Put $x = e^u$, $y = e^v$ so that $u = \log x$ and $v = \log y$.

If $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$, then (1) reduces to

$$[2D_1(D_1 - 1) - 5D_1D_1' + 2D_1'(D_1' - 1) + 2(D_1 + D_1')]z = 0$$

or $(2D_1^2 - 5D_1D_1' + 2D_1'^2) = 0$ or $(2D_1 - D_1')(D_1 - 2D_1') = 0.$

$\therefore$ solution is $z =$ C.F. $= \phi_1(2v + u) + \phi_2(u + 2v)$, ϕ_1, ϕ_2 being arbitrary function

or $z = \phi_1(2\log y + \log x) + \phi_2(\log y + 2\log x) = \phi_1(\log y^2x) + \phi_2(\log yx^2)$

or $z = f_1(y^2x) + f_2(yx^2)$, f_1 and f_2 being arbitrary functions ...(2)

The given surface is $z = x^2 - y^2$. ...(3)

Now (2) and (3) are to touch each other along the section by the plane

$y = 1$. ...(4)

Therefore the values of p and q for (2) and (3) must be equal at $y = 1$. Equating values of p and q from (2) and (3), we get

$y^2f_1'(y^2x) + 2xyf_2'(x^2y) = 2x$ ...(5)

and $2xyf_1'(y^2x) + x^2f_2'(x^2y) = -2y$. ...(6)

Putting $y = 1$, (5) and (6) reduce to

$f_1'(x) + 2xf_2'(x^2) = 2x$ and $2xf_1'(x) + x^2f_2'(x^2) = -2$.

Solving these, $f_1'(x) = -(2/3) \times x - (4/3) \times x^{-1}$ ...(7)

and $f_2'(x^2) = (2/3) \times x^{-2} + (4/3)$ ...(8)

Integrating (7), $f_1(x) = -(1/3) \times x^2 - (4/3) \times \log x + c_1$

which gives $f_1(y^2x) = -(1/3) \times y^4x^2 - (4/3) \times \log(y^2x) + c_1$. ...(9)

Writing X for x^2 in (8), $f_2'(X) = (2/3) \times (1/X) + (4/3)$

Integrating it, $f_2(X) = (2/3) \times \log X + (4/3) \times X + c_2$

which gives $f_2(yx^2) = (2/3) \times \log(yx^2) + (4/3) \times (yx^2) + c_2$ ...(10)

Putting the values of $f_1(y^2x)$ and $f_2(yx^2)$ from (9) and (10) in (2) and writing $c_1 + c_2 = c/3$, the complete solution is

$$z = -(1/3) \times y^4x^2 - (4/3) \times \log(y^2x) + (2/3) \times \log(yx^2) + (4/3) \times (yx^2) + c/3$$

or $3z = -y^4x^2 - 4(\log x + 2\log y) + 2(\log y + 2\log x) + 4yx^2 + c$

or $3z = -y^4x^2 - 6\log y + 4yx^2 + c$.

Now equating values of z from (3) and (11) and putting $y = 1$, we have

$$x^2 - 1 = (1/3)[-x^2 - 6\log 1 + 4x^2 + c], \text{ giving } c = -3.$$

So the required surface is $3z = 4yx^2 - y^4x^2 - 6\log y - 3$.

Ex. 1. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) - y(\partial z/\partial y) + x(\partial z/\partial x) = 0.$ **(Jabalpur 1996)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1) - D_1' + D_1]z = 0$$

or $(D^2_1 - D'^2_1)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0.$

Hence the required general solution is $z = \text{C.F.} = \phi_1(v + u) + \phi_2(v - u)$

or $z = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (xy) + \phi_2 \log (y/x)$ or $z = f_1(xy) + f_2(y/x)$, where f_1, f_2 are arbitrary functions.

Ex.2. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 0.$ **[Delhi Maths (H) 1994, CDLU 2004]**

or Solve $x^2r + 2xys + y^2t = 0$ **(Purvanchal 2007)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation can be written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = 0$ which reduces to

$$[D_1(D_1 - 1) + 2DD' + D'(D' - 1)]z = 0$$

$[(D_1 + D_1')^2 - (D_1 + D_1')]z = 0$ or $(D_1 + D_1')(D_1 + D_1' - 1)z = 0.$

Hence the required general solution is $z = \text{C.F.} = \phi_1(v - u) + e^u\phi_2(v - u)$

or $z = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (y/x) + x\phi_2 \log (y/x)$ or $z = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Ex. 19. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) - y(\partial z/\partial y) = x^2y^4$ *by reducing it to the equation with constant coefficients.* **[I.A.S. 2001]**

Sol. Re-writing, the given equation $(x^2D^2 - y^2D'^2 + xD - yD')z = x^2y^4$ ... (1)

Let $x = e^u$ and $y = e^v$ so that $u = \log x$ and $v = \log y$ ... (2)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$. Then (1) becomes

$\{D_1\ (D_1 - 1) - D_1'(D_1' - 1) + D_1 - D_1'\}z = e^{2u}e^{4v}$ or $(D_1^2 - D_1'^2)z = e^{2u+4v}$

or $$(D_1 - D_1')\ (D_1 + D_1')z = e^{2u+4v} \quad \text{... (3)}$$

C.F. $= \phi_1(v + u) + \phi_2(v - u) = \phi_1(\log y + \log x) + \phi_2(\log y - \log x) = \phi_1(\log xy) + \phi_2(\log(y/x)\}$

or C.F. $= \psi_1(xy) + \psi_2(y/x)$, ψ_1, ψ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}\ e^{2u+4v} = \frac{1}{(2^2 - 4^2)}e^{2u+4v} = -\frac{1}{12}\ (e^u)^2\ (e^v)^4 = -\frac{1}{12}x^2y^4$$

$\therefore$ $z = \psi_1\ (xy) + \psi_2(y/x) - (1/12) \times x^2y^4$ is the required solution.

Ex. 14. *Solve* $(x^2D^2 - 4y^2D'^2 - 4yD' - 1)z = x^2y^2 \log y$. **[Delhi Maths (H) 2006]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to $[D_1(D_1 - 1) - 4D_1'(D_1' - 1) - 4D_1' - 1]z = e^{2u}e^{2v}v$

or $$(D_1^2 - D_1 - 4D_1'^2 - 1)z = e^{2u+2v}v. \quad ...(2)$$

Here $(D_1^2 - D_1 - 4D_1'^2 - 1)$ cannot be resolved into linear factors in D_1 and D_1'. To find C.F. corresponding to it, we consider the equation.

$$(D_1^2 - D_1 - 4D_1'^2 - 1)z = 0. \quad ...(3)$$

Let a trial solution of (3) be $z = Ae^{hu+kv}$ ...(4)

$\therefore$ $D_1^2 z = Ah^2e^{hu+kv}$, $D_1 z = Ahe^{hu+kv}$, and $D_1'^2 z = Ak^2e^{hu+kv}$.

Then, (3) $\Rightarrow$ $A(h^2 - h - 4k^2 - 1)e^{hu+hv} = 0$ $\Rightarrow$ $h^2 - h - 4k^2 - 1 = 0$. ...(5)

$\therefore$ C.F. of (2) $= \Sigma Ae^{hu+kv} = \Sigma A(e^u)^h(e^v)^k = \Sigma Ax^hy^k$

$$\text{P.I. of (2)} = \frac{1}{D_1^2 - D_1 - 4D_1'^2 - 1}e^{2u+2v}v = e^{2u+2v}\frac{1}{(D_1+2)^2 - (D_1+2) - 4(D_1'+2)^2 - 1}v$$

$$= e^{2u+2v}\frac{1}{D_1^2 + 3D_1 - 4D_1'^2 - 16D_1' - 15}v$$

$$= e^{2u+2v}\frac{1}{(-15)\times[1 + (16/15)\times D_1' + (4/15)\times D_1'^2 - (1/5)\times D_1 - (1/15)\times D_1^2]}v$$

$$= e^{2u+2v}\frac{1}{(-15)}\left[1 + \left\{\frac{16}{15}D_1' + \frac{4}{15}D_1'^2 - \frac{1}{5}D_1 - \frac{1}{15}D_1^2\right\}\right]^{-1}v$$

$$= \frac{e^{2u+2v}}{(-15)}\left(1 - \frac{16}{15}D_1' + \dots\right)v = \frac{e^{2u+2v}}{(-15)}\left(v - \frac{16}{15}\right) = \frac{(e^u)^2 \times (e^v)^2(16 - 15v)}{225}$$

$= (1/225) \times x^2y^2(16 - 15\log y)$, using (1).

The required general solution is $z = \Sigma Ax^hy^k + (1/225) \times x^2y^2(16 - 15\log y)$

where $h^2 - h - 4k^2 - 1 = 0$, and A, h and k are arbitrary constants.

Ex. 9. *Solve the following partial differential equations :*

(*a*) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x + 2y)$.

(*b*) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + \cos(x + 2y)$ **[Delhi Maths (H) 2006]**

(*c*) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y + x)^{1/2}$.

(*d*) $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x + 2y) + e^{3x+y}$. **[I.A.S. 1995]**

Sol. (*a*) Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, P.I. corresponds to e^{2x-y}

$$= \frac{1}{D^2 - 3DD' + 2D'^2}e^{2x-y} = \frac{1}{2^2 - 3\times 2\times(-1) + 2\times(-1)^2}\iint e^v\,dv\,dv, \text{ where } v = 2x - y$$

$$= (1/12)\times\int e^v dv = (1/12)\times e^v = (1/12)\times e^{2x-y}. \quad ...(2)$$

Ex. 1. *Find the characteristics of $y^2r - x^2t = 0$* **[I.A.S. 2009]**

Sol. Given $$y^2r - x^2t = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$ and $T = -x^2$. Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolaic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad y^2\lambda^2 - x^2 = 0 \quad$ ... (2)

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots). Corresponding characteristic equations are

$(dy/dx) + (x/y) = 0 \quad$ and $\quad (dy/dx) - (x/y) = 0$

or $\quad x\,dx + y\,dx = 0 \quad$ and $\quad xdx - y\,dy = 0$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Ex. 2. *Find the characteristics of $x^2r + 2xys + y^2t = 0$.*

Sol. Given $$x^2r + 2xys + y^2t = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$. Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ and hence (1) is parabolic everywhere.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad x^2\lambda^2 + 2xy\lambda + y^2 = 0 \quad$ ... (2)

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots). The characteristic equation is $(dy/dx) - (y/x) = 0$ or $(1/y)\,dy - (1/x)\,dx = 0$ giving $y/x = c_1$ or $y = c_1 x$, which is the required family of characteristics. Here it represents a family of straight lines passing through the origin.

Ex. 6. *Reduce $\partial^2 z/\partial x^2 = x^2(\partial^2 z/\partial y^2)$ to canonical form.*

[Agra 2005; Himanchal 2005; Delhi B.Sc. (Prog) II 2002, 07; Kurukshetra 2004; Ravishankar 2004; Nagpur 2010, Kanpur 2011]

Sol. Re-writing the given equation becomes $$r - x^2t = 0. \quad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = 1$, $S = 0$, $T = -x^2$.

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ gives $\lambda^2 - x^2 = 0$ so that $\lambda = \pm x$.

$\therefore$ Here $\lambda_1 = x$ and $\lambda_2 = -x$ (Real and disticnt roots)

Hence characterstic equations $dy/dx + \lambda_1 = 0$ and $dy/dx + \lambda_2 = 0$

become $dy/dx + x = 0$ and $dy/dx - x = 0$.

Integrating these, $y + (x^2/2) = c_1$ and $y - (x^2/2) = c_2$.

Hence in order to reduce (1) to canonical form, we change x, y, to u, v by taking

$u = y + (x^2/2) \quad$ and $\quad v = y - (x^2/2) \quad$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \quad \text{...(4)}$$

$$\therefore\ r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = x\,\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1.\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial^2 v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = \dfrac{\partial^2 z}{\partial u^2} + 2\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial^2 z}{\partial v^2}$, using (4)

Putting the above values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right) \quad$ or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, by (2)

which is the required canonical form of the given equation.

Ex. 13. *(a) Reduce* $x^2 \, (\partial^2 z / \partial x^2) - y^2 \, (\partial^2 z / \partial y^2) = 0$ *to canonical form and hence solve it.*

(b) Reduce $y^2 (\partial^2 z / \partial x^2) - x^2 (\partial^2 z / \partial y^2) = 0$ *to canonical form.*

Sol. (*a*) Re-writing the given equation, $\quad x^2 r - y^2 t = 0 \quad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2 y^2 > 0$ for $x \neq 0, y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x, \; -y/x$ and hence the corresponding characteristic equations become $\quad (dy/dx) + (y/x) = 0 \quad$ and $\quad (dy/dx) - (y/x) = 0$

Integrating these, $\quad xy = c_1 \quad$ and $\quad x/y = c_2$

In order to reduce (1) to its canonical form, we choose $\quad u = xy$ and $v = x/y$ ... (2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \quad \text{and} \quad t = x^2 \frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2 \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2 \frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\quad 4x^2 \dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{2x}{y}\dfrac{\partial z}{\partial v} = 0 \quad$ or $\quad 2xy \dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{\partial z}{\partial v} = 0$

or $\quad 2u \, (\partial^2 z / \partial u \partial v) - (\partial z / \partial v) = 0$, using (2). $\quad$ ... (3)

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\,\frac{\partial^2 z}{\partial u\,\partial v} - v\,\frac{\partial z}{\partial v} = 0 \qquad \text{or} \qquad (2uv\,DD' - vD')z = 0 \qquad \text{... (4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $\quad u = e^X \quad$ and $\quad v = e^Y \quad$ so that $\quad X = \log u \quad$ and $\quad y = \log v \quad$ ... (5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (4) reduces to

$$(2D_1D_1' - D_1')z = 0 \qquad \text{or} \qquad D_1'\,(2D_1 - 1)\,z = 0$$

Its general solution is given by (use Art. 5.6 of chapter 5)

$$z = e^{X/2}\,\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u), \text{ using (5)}$$

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\,\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions

(*b*) Try yourself. Choose $u = (y^2 - x^2)/2$, $v = (y^2 + x^2)/2$.

Ans. $\dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{2(u^2 - v^2)}\left(v\dfrac{\partial z}{\partial u} - u\dfrac{\partial z}{\partial v}\right)$.

Ex. 14. *Reduce the equation* $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + (x-1)p + (y-1)q = 0$ *to canonical form and hence solve it.*

Sol. Comparing the given equation with $\quad Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

here, $\quad R = x(xy-1), \quad S = -(x^2y^2-1), \quad T = y(xy-1)$. ...(1)

Now, the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ and (1) give

$$x(xy-1)\lambda^2 - (x^2y^2-1)\lambda + y(xy-1) = 0 \qquad \text{or} \qquad x\lambda^2 - (xy+1)\lambda + y = 0$$

or $\quad (x\lambda - 1)(\lambda - y) = 0 \quad$ so that $\quad \lambda = 1/x, \quad y. \quad$ Take $\quad \lambda_1 = 1/x \quad$ and $\quad \lambda_2 = y$.

Hence characterstic equations $\quad (dy/dx) + \lambda_1 = 0 \quad$ and $\quad (dy/dx) + \lambda_2 = 0$

become $\quad (dy/dx) + (1/x) = 0 \quad$ and $\quad (dy/dx) + y = 0$

or $\quad dy + (1/x)dx = 0 \quad$ and $\quad (1/y)dy + dx = 0$. ...(2)

Intergrating (2), $\quad y + \log x = \log c_1 \quad$ and $\quad \log y + x = \log c_2$

or $\quad \log e^y + \log x = \log c_1 \quad$ and $\quad \log y + \log e^x = \log c_2$

$\quad x\,e^y = c_1 \quad$ and $\quad y\,e^x = c_2$.

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$$u = x\,e^y \qquad \text{and} \qquad v = y\,e^x. \qquad \text{...(3)}$$

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = e^y\frac{\partial z}{\partial u} + y\,e^x\frac{\partial z}{\partial v}, \text{ using (3)} \qquad \text{...(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = xe^y\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}, \text{ using (3)}. \qquad \text{...(5)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(e^y \frac{\partial z}{\partial u} + y e^x \frac{\partial z}{\partial v}\right) = e^y \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + ye^x \frac{\partial z}{\partial v} + y e^x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + y e^x \frac{\partial z}{\partial v} + ye^x \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y \left[\frac{\partial^2 z}{\partial u^2} e^y + \frac{\partial^2 z}{\partial v \partial u} y e^x\right] + y e^x \frac{\partial z}{\partial v} + y e^x \left[\frac{\partial^2 z}{\partial u \partial v} e^y + \frac{\partial^2 z}{\partial v^2} y e^x\right]$$

$$\therefore \qquad r = e^{2y} \frac{\partial^2 z}{\partial u^2} + 2y\, e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + y^2 e^{2x} \frac{\partial^2 z}{\partial v^2} + ye^x \frac{\partial z}{\partial v}.$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right) = e^y \frac{\partial z}{\partial u} + x e^y \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^x \frac{\partial z}{\partial v} + e^x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^x \frac{\partial z}{\partial v} + e^x \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= e^y \frac{\partial z}{\partial u} + xe^y \left[\frac{\partial^2 z}{\partial u^2} e^y + \frac{\partial^2 z}{\partial v \partial u} ye^x\right] + e^x \frac{\partial z}{\partial v} + e^x \left[\frac{\partial^2 z}{\partial u \partial v} e^y + \frac{\partial^2 z}{\partial v^2} ye^x\right]$$

$$= x\, e^{2y} \frac{\partial^2 z}{\partial u^2} + (xy+1)\, e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + y\, e^{2x} \frac{\partial^2 z}{\partial v^2} + e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}\right) = x e^y \frac{\partial z}{\partial u} + x e^y \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + e^x \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + e^x \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial^2 z}{\partial u^2} x e^y + \frac{\partial^2 z}{\partial u \partial v} e^x\right] + e^x \left[\frac{\partial^2 z}{\partial u \partial v} x e^y + \frac{\partial^2 z}{\partial v^2} e^x\right],$$

$$\therefore \qquad t = x^2 e^{2y} \frac{\partial^2 z}{\partial u^2} + 2x\, e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + x e^y \frac{\partial z}{\partial u} + e^{2x} \frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0. \qquad ...(6)$$

Integrating (6) w.r.t. 'v', $\quad \partial z/\partial u = f(u)$, f being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $\quad z = \int f(u)\,du + \psi(v)$ or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $\quad z = \phi(xe^y) + \psi(ye^x)$, ϕ and ψ being arbitrary functions.

Ex. 15. (*a*) *Reduce the one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, $(c > 0)$ *to canonical form and hence find its general solution.*

Sol. (*a*) Given $\partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$, $c > 0$. ... (1)

To re-write (1), put $y = ct$, ... (2)

Then, (1) reduces to $\partial^2 z/\partial x^2 - (\partial^2 z/\partial y^2) = 0$ or $r - t = 0$ ... (3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad \text{... (4)}$$

where $u = y + x$, $v = y - x$ or $u = ct + x$ and $v = ct - x$. ... (5)

Integrating (4) w.r.t. 'u', $\partial z/\partial v = f(v)$, where f is an arbitrary function ... (6)

Integrating (6) w.r.t. 'v', $z = \int f\ (v)\ dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)\,dv$

or $$z(x, t) = F\,(ct - x) + \psi\ (ct + x), \text{ using (5)}$$

or $$z(x, t) = \phi(x - ct) + \psi(x + ct), \qquad \text{... (7)}$$

where we take $\phi\ (x - ct) = F\ (ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

Ex. 3. *Find a complete integral of*

$$z^2\,(p^2 + q^2) = x^2 + y^2. \qquad \textbf{(Rohilkhand 1992; I.C.S. 89)}$$

Sol. Put $z dz = dZ$; *i.e.*, $Z = \frac{1}{2} z^2$, so that

$$z\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P \text{ (say)}, \; z\frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q \text{ (say)}.$$

Putting these values the given equation becomes

$$P^2 + Q^2 = x^2 + y^2 \text{ or } P^2 - x^2 = y^2 - Q^2 = a^2 \text{ (say)}.$$

$\therefore$ $$P = \sqrt{(a^2 + x^2)},\; Q = \sqrt{(y^2 - a^2)}.$$

Now $dZ = Pdx + Qdy = \sqrt{(a^2 + x^2)}\,dx + \sqrt{(y^2 - a^2)}\,dy$.

Integrating ,

$$Z = \frac{1}{2} x\sqrt{(a^2 + x^2)} + \frac{a^2}{2}\log\{x + \sqrt{(a^2 + x^2)}\}$$
$$+ \frac{1}{2} y\sqrt{(y^2 - a^2)} - \frac{a^2}{2}\log\{y + \sqrt{(y^2 - a^2)}\} + b$$

or $$z^2 = x\sqrt{(a^2 + x^2)} + a^2\log\{x + \sqrt{(a^2 + x^2)}\} + y\sqrt{(y^2 - a^2)}$$
$$- a^2\log\{y + \sqrt{(y^2 - a^2)}\} + b,$$

which is a complete integral of the given equation.

Ex. 4. *Reduce* $x\,(\partial^2 z/\partial x^2) + \partial^2 z/\partial y^2 = x^2$ *(x > 0) to canonical form.* **[Delhi Maths(H) 2007, 11]**

Sol. Re-writing the given equation, we get $xr + t - x^2 = 0$, $(x > 0)$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x$, $S = 0$ and $T = 1$ so that $S^2 - 4RT = -4x < 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$x\lambda^2 + 1 = 0$ or $\lambda^2 = -(1/x^2)$ so that $\lambda = i/x^{1/2},\ -i/x^{1/2}$

The corresponding characteristic equations are given by

$dy/dx + i\ x^{-1/2} = 0$ and $dy/dx - i\ x^{-1/2} = 0$.

Integrating these, $y + 2i\ x^{1/2} = C_1$ and $y - 2i\ x^{1/2} = C_2$

Choose $u = y + 2i\ x^{1/2} = \alpha + i\beta$ and $v = y - 2i\,x^{1/2} = \alpha - i\beta$,

where $\alpha = y$ and $\beta = 2x^{1/2}$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right) = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

or $$r = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right) = -\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ using (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right) + \frac{\partial^2 z}{\partial \alpha^2} = x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = x^2 + \frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

Ex. 11. *Solve* $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$. ... (1)

Sol. The auxiliary equations of (1) are

$$\frac{dx}{x^2(y-x)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}.$$

Each fraction $=\dfrac{\dfrac{dx}{x^2}+\dfrac{dy}{y^2}+\dfrac{dz}{z^2}}{0}$ and also $=\dfrac{\dfrac{dx}{x}+\dfrac{dy}{y}+\dfrac{dz}{z}}{0}$.

$$\therefore \qquad \frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}=0 \quad \text{and} \quad \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0.$$

Integrating, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=c_1$ and $xyz=c_2$.

Hence the general solution of (1) is given by

$$f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0.$$

Ex. 12. *Solve* $x(y^2+z)p-y(x^2+z)q=z(x^2-y^2)$. ... (1)

(Meerut 92, 93, 94, 95; Kanpur 1980)

Sol. The auxiliary equations of (1) are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}.$$

Choosing $1/x, 1/y, 1/z$ as multipliers, we get

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0.$$

Integrating, $\log x+\log y+\log z=\log c_1$ or $xyz=c_1$.

Again choosing $x, y, -1$ as multipliers, we get

$$x\,dx+y\,dy-dz=0.$$

Integrating, $x^2+y^2-2z=c_2$.

Hence the general solution of (1) is given by $f(xyz,\ x^2+y^2-2z)=0$.

Ex. 17. *Find the family orthogonal to* $f(z(x+y)^2,\ x^2-y^2)=0$.

Sol. Let $z(x+y)^2=u$, $x^2-y^2=v$. Then $f(u,v)=0$.

Differentiating partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial u}\{z.2(x+y)+(x+y)^2.\,p\}+\frac{\partial f}{\partial v}\{2x-0\}=0 \qquad \ldots (1)$$

and $$\frac{\partial f}{\partial u}\{z.2(x+y)+(x+y)^2.\,q\}+\frac{\partial f}{\partial v}\{0-2y\}=0. \qquad \ldots (2)$$

Eliminating f between (1) and (2), we get

$$\frac{2z(x+y)+p(x+y)^2}{2z(x+y)+q(x+y)^2}=\frac{-2x}{2y}$$

or $$y(x+y)^2p+x(x+y)^2q=-2z(x+y)^2$$

or $yp+xq=-2z$, which is of the form $Pp+Qq=R$.

Here $P=y$, $Q=x$, $R=-2z$.

Hence the differential equation of the family of surfaces orthogonal to the given family is

$$P\,dx+Q\,dy+R\,dz=0 \quad \text{or} \quad y\,dx+x\,dy-2z\,dz=0.$$

Integrating, $xy-z^2=c$, which is the required family of surfaces.

Ex. 4. *Find the singular integral of*

$$z=px+qy+c\sqrt{(1+p^2+q^2)}.$$ **(Meerut 1989; I.C.S. 89)**

Sol. The complete integral of the given equation is

$$z=ax+by+c\sqrt{(1+a^2+b^2)}. \qquad \ldots (1)$$

Singular integral. Differentiating (1) partially w.r.t. a and b, we get

$$0=x+\frac{ac}{\sqrt{(1+a^2+b^2)}},\quad 0=y+\frac{bc}{\sqrt{(1+a^2+b^2)}}, \qquad \ldots (2)$$

so that $$x^2+y^2=\frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

i.e., $$c^2-x^2-y^2=\frac{c^2}{1+a^2+b^2}$$

i.e., $$1+a^2+b^2=\frac{c^2}{c^2-x^2-y^2}. \qquad \ldots (3)$$

Also, from (2)

$$a = -\frac{x\sqrt{(1+a^2+b^2)}}{c} = \frac{-x}{\sqrt{(c^2-x^2-y^2)}} \qquad \dots (4)$$

and
$$b = -\frac{y\sqrt{(1+a^2+b^2)}}{c} = \frac{-y}{\sqrt{(c^2-x^2-y^2)}} \qquad \dots (5)$$

Putting the values from (3), (4) and (5) in (1), we get the singular solution as

$$\therefore z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}}$$

or
$$z = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}} \text{ i.e., } z^2 = c^2-x^2-y^2 \quad \text{i.e., } \quad x^2+y^2+z^2=c^2.$$

$$\Rightarrow \qquad dz = \frac{a}{r}\,dr + \sqrt{(1-a^2)}\,d\theta.$$

Integrating, $z = a\log r + \sqrt{(1-a^2)}\,\theta + b$

or
$$z = \frac{1}{2}a\log(x^2+y^2) + \sqrt{(1-a^2)}\tan^{-1}(y/x) + b.$$

Ex. 9. *Find a complete integral of* $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0.$ **[I.A.S. 1994]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0. \qquad \dots(1)$

Charpit's auxiliary equations are
$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or
$$\frac{dp}{-p(32p^2z+18q^2z+8z)} = \frac{dq}{q(32p^2z+18q^2z+8z)} = \frac{dz}{-p(32pz^2)-q(18qz^2)} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}.$$

Taking the first and second fractions, $\quad (1/p)dp = (1/q)dq \quad$ so that $\quad p = aq \qquad \dots(2)$

Solving (1) and (2) for p and q, we have

$$q = \frac{2(1-z^2)^{1/2}}{z(16a^2+9)^{1/2}} \qquad \text{and} \qquad p = \frac{2a(1-z^2)^{1/2}}{z(16a^2+9)^{1/2}}. \qquad \dots(3)$$

Hence,
$$dz = pdx + qdy = \frac{2(1-z^2)^{1/2}}{z(16a^2+9)^{1/2}}\,(adx+dy), \text{ using (3)}$$

or
$$(1/2)\times(16a^2+9)^{1/2}(1-z^2)^{-1/2}(-2zdz) = -2(adx+dy). \qquad \dots(4)$$

Putting $\quad 1 - z^2 = t \quad$ so that $\quad -2zdz = dt$, (4) becomes

or
$$(1/2)\times(16a^2+9)^{1/2}\,t^{-1/2}\,dt = -2\,(adx+dy).$$

Integrating, $\quad (16a^2+9)^{1/2}\,t^{1/2} = -2(ax+y)+b$, a, b being arbitrary constants.

or
$$(16a^2+9)^{1/2}\sqrt{(1-z^2)} + 2(ax+y) = b, \text{ as } t = 1-z^2.$$

Ex. 10(a). *Find a complete integral of* $(p^2+q^2)x = pz.$

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation* $(p^2+q^2)x = pz$ *and deduce the solution which passes through the curve* $x = 0$, $z^2 = 4y$. **[Meerut 2007]**

Sol. Let $\qquad f(x, y, q, p, q) = (p^2+q^2)x - pz = 0. \qquad \dots(1)$

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $\qquad f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0.$ ...(1)

Charpit's auxiliary equations are
$$\frac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \frac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \frac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \frac{dx}{-\dfrac{\partial f}{\partial p}} = \frac{dy}{-\dfrac{\partial f}{\partial q}}$$

or
$$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives
$$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $\qquad dp + dq = dx + dy \qquad$ i.e., $\qquad dp - dx + dq - dy = 0.$

Integrating, $\qquad (p - x) + (q - y) = a$ ...(2)

Re–writing (1), $\qquad (p - x)^2 + (q - y)^2 = (x - y)^2.$ ...(3)

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2 \qquad$ or $\qquad 2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0.$

$$\therefore\ p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \quad \Rightarrow \quad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}],$$

$\therefore$ (2) gives $\qquad q = a + y - p + x \qquad$ or $\qquad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}]$.

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get
$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or
$$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2}.(dx - dy).$$

Integrating, the desired complete integral is
$$z = \frac{x^2 + y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} - \frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \qquad \text{or} \qquad \frac{\partial z}{x\partial x} + \frac{\partial z}{y\partial y} = \left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \qquad \text{... (1)}$$

Put $\quad x\,dx = dX, \quad y\,dy = dY \quad$ so that $\quad x^2/2 = X, \quad y^2/2 = Y$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)\ (\partial z/\partial Y) \qquad$ or $\qquad P + Q = PQ$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, $\qquad$ where $\qquad a + b = ab \qquad$ so that $\qquad b = a/(a-1)$ ... (4)

or $\qquad z = a(x^2/2) + a(a-1)^{-1}\ (y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $f(x, y, z, p, q) = p^2x + q^2y - z = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} - \frac{dy}{-f_q}$$

$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2) $$= \frac{2px\,dp + p^2dx}{2px(-p+p^2)+p^2(-2px)} = \frac{2qy\,dq + q^2dy}{2qy(-q+q^2)+q^2(-2qy)}$$

$$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\log(p^2x) = \log(q^2y) + \log a$ or $p^2x = q^2ya$. ...(3)

Form (1) and (3), $aq^2y + q^2y = z$ or $q = [z/(1+a)]^{1/2}$. ...(4)

Form (3) and (4), $$p = q\left(\frac{ya}{x}\right)^{1/2} = \left\{\frac{za}{(1+a)x}\right\}^{1/2}.$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2}dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2}dy \quad \text{or} \quad (1+a)^{1/2}z^{-1/2}\,dz = \sqrt{a}\,x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $(1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \quad \text{or} \quad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \quad \text{or} \quad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \qquad ...\ (1)$$

Putting $2x\,dx = dX$, $2y\,dy = dY$ so that $x^2 = X$, $y^2 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = 1$ or $PQ = 1$ whose solution is

$z = aX + bY + d$, where $ab = 1$ so that $b = 1/a$.

$$\therefore \qquad z = ax^2 + (1/a)y^2 + d \qquad ...\ (2)$$

is complete integral of (1) containing two arbitrary constants a and d.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\ y^2 + \phi(a) \qquad ...\ (3)$$

Differenting (3) partially w.r.t. 'a', $0 = x^2 - (1/a^2)\ y^2 + \phi'(a)$ ... (4)

Then general integral is obtained by eliminating a from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$$0 = x^2 + (-1/a^2)\, y^2 \qquad \text{... (5)} \qquad\qquad 0 = 1 \qquad \text{... (6)}$$

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$$z = 2xy + C, \ C \text{ being an arbitrary constant} \qquad \text{... (7)}$$

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.*, if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$$z = (y/x)\, x^2 + (x/y)\, y^2 + C \qquad \text{or} \qquad z = 2xy + C,$$

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write

$$z = ax^2 + (1/a)\, y^2 + C \qquad \text{... (8)}$$

Differenting (8) partially w.r.t. 'a', we get

$$0 = x^2 - (1/a^2) y^2 \qquad \text{or} \qquad a = y/x \qquad \text{... (9)}$$

Eliminating a from (8) and (9), we get $\quad z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$ **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \qquad \text{... (1)}$$

Let X, Y and Z be new variables such that

$dX = dx, \quad dY = dy \quad$ and $\quad dZ = z^{-1/2} dz \quad$ so that $\quad X = x, \quad Y = y, \quad Z = 2z^{1/2}$ ... (2)

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \qquad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $\quad Z = aX + by + c, \quad$ ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$$Z = aX \pm (a^2 - 1)^{1/2}\, Y + c \qquad \text{or} \qquad 2z^{1/2} = ax \pm (a^2 - 1)^{1/2}\, y + c,$$

which is the complete integral, a and c being arbitrary constants and $|a| \geq 1$.

Example 2. *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.*

Find also the temperature if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.

Sol. Initial temperature distribution in the rod is

$$u_1 = 0 + \left(\frac{100 - 0}{l}\right) x = \frac{100}{l} x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 0 + \left(\frac{0 - 0}{l}\right) x = 0$$

To get u in the intermediate period,

$$u = u_2(x) + u_1(x, t)$$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$u_1(x, t)$ satisfies one dimensional heat flow equation

$$\therefore \qquad u(x, t) = \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2 p^2 t} \qquad \ldots(1)$$

In steady state, $\quad u(0, t) = 0 = u(l, t) \qquad \ldots(2)$

$$\therefore \text{ From (1),} \qquad u(0, t) = 0 = \sum_{n=1}^{\infty} a_n e^{-c^2 p^2 t} \quad \Rightarrow \quad a_n = 0 \qquad \ldots(3)$$

Also,

$$u(l, t) = 0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2 p^2 t} \qquad | \text{ Using (3)}$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi, \; n \in \text{I}$$

or

$$p = \frac{n\pi}{l} \qquad \ldots(4)$$

$$\therefore \text{ From (1), (3) and (4),} \quad u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}\, e^{-\left(\frac{n\pi}{l}\right)^2 c^2 t} \qquad \ldots(5)$$

Using initial condition $u(x, 0) = \dfrac{100}{l} x$ in eqn. (5), we get

$$\frac{100}{l} x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

which is half-range sine series for $\dfrac{100}{l} x$.

$$\therefore \qquad b_n = \frac{2}{l} \int_0^l \frac{100}{l} x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2} \left[\left\{ x \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_0^l - \int_0^l 1 \cdot \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) dx \right]$$

$$= \frac{200}{l^2} \left[\frac{-l^2}{n\pi} \cos n\pi + \frac{l}{n\pi} \left(\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_0^l \right] = - \frac{-200}{n\pi} (-1)^n$$

Hence the temperature function

$$u(x, t) = -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2c^2t}{l^2}}$$

In the second part, the initial condition remains the same as in first part *i.e.,*

$$u(x, 0) = \frac{100}{l} x.$$

Boundary conditions are $u(0, t) = 20$ and $u(l, t) = 80$ for all values of t then, final temperature distribution is

$$u_2 = 20 + \left(\frac{80 - 20}{l}\right) x = 20 + \frac{60}{l} x$$

Then, $$u = u_2(x) + u_1(x, t)$$

$$u = 20 + \frac{60}{l} x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-c^2p^2t} \quad \text{...(6)}$$

$$u(0, t) = 20 = 20 + \sum_{n=1}^{\infty} a_n e^{-c^2p^2t} \quad \text{| From (6)}$$

$\Rightarrow$ $$a_n = 0$$

$\therefore$ From (6), $$u = 20 + \frac{60}{l} x + \sum_{n=1}^{\infty} b_n \sin px\, e^{-c^2p^2t} \quad \text{...(7)}$$

$$u(l, t) = 80 = 20 + \frac{60}{l} l + \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t} \quad \text{| From (7)}$$

$\Rightarrow$ $$0 = \sum_{n=1}^{\infty} b_n \sin pl\, e^{-c^2p^2t}$$

$$\sin pl = 0 = \sin n\pi,\ n \in \mathrm{I}$$

$\therefore$ $$p = \frac{n\pi}{l} \quad \text{...(8)}$$

From (7) and (8), $$u = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t} \qquad \ldots(9)$$

Using initial condition,

$$u(x, 0) = \frac{100}{l}x = 20 + \frac{60}{l}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{40}{l}x - 20 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l \left(\frac{40}{l}x - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\left\{\left(\frac{40}{l}x - 20\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l \frac{40}{l}\cdot\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{2}{l}\left[\frac{-20l}{n\pi}\cos n\pi - \frac{20l}{n\pi} + \frac{40}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l\right]$$

$$= \frac{-40}{n\pi}(1 + \cos n\pi) = \begin{cases} 0 & , \text{ when } n \text{ is odd} \\ \frac{-80}{n\pi} & , \text{ when } n \text{ is even} \end{cases}$$

Hence equation (9) becomes,

$$u(x, t) = 20 + \frac{60}{l}x - \frac{80}{\pi}\sum_{\substack{n=2,4,\ldots \\ (n \text{ is even})}}^{\infty} \frac{1}{n}\sin\frac{n\pi x}{l}\, e^{-\left(\frac{n\pi c}{l}\right)^2 t}$$

$$= 20 + \frac{60}{l}x - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l}\, e^{-\frac{4c^2m^2\pi^2 t}{l^2}} \qquad \text{(Taking } n = 2m\text{)}$$

Example 3. *The ends A and B of a rod of length 20 cm are at temperatures 30°C and 80°C until steady state prevails. Then the temperature of the rod ends are changed to 40°C and 60°C respectively. Find the temperature distribution function u(x, t). The specific heat, density and the thermal conductivity of the material of the rod are such that the combination* $\frac{k}{\rho\sigma} = c^2 = 1$.

Sol. Initial temperature distribution in the rod is

$$u_1 = 30 + \left(\frac{80-30}{20}\right)x = 30 + \frac{5}{2}x$$

Final temperature distribution (*i.e.*, in steady state) is

$$u_2 = 40 + \left(\frac{60-40}{20}\right)x = 40 + x$$

To get u in the intermediate period,

$$u = u_1(x, t) + u_2(x)$$

where $u_2(x)$ is the steady state temperature distribution in the rod. $u_1(x, t)$ is the transient temperature distribution which tends to zero as t increases.

$\because$ $u_1(x, t)$ satisfies one dimensional heat flow equation.

$$\therefore \quad u = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2 t} \quad \quad ...(1)$$

In steady state,

$$u(0, t) = 40 \quad \quad ...(2)$$

$$u(20, t) = 60 \quad \quad ...(3)$$

From (1), $\quad u(0, t) = 40 = 40 + \sum_{1}^{\infty} a_n\, e^{-p^2 t}$ $\quad$ | From (2)

$$\Rightarrow \quad 0 = \sum_{1}^{\infty} a_n\, e^{-p^2 t} \quad \Rightarrow \quad a_n = 0 \quad \quad ...(4)$$

$\therefore$ From (1) and (4),

$$u = 40 + x + \sum_{n=1}^{\infty} b_n \sin px\, e^{-p^2 t}$$

Again, $\quad u(20, t) = 60 = 60 + \sum_{n=1}^{\infty} b_n \sin 20\, p\, e^{-p^2 t}$

$$\Rightarrow \qquad 0 = \sum_{1}^{\infty} b_n \sin 20\, p\, e^{-p^2 t}$$

$$\sin 20p = 0 = \sin n\pi,\ n \in \text{I}$$

$$\Rightarrow \qquad p = \frac{n\pi}{20}$$

$$\therefore \qquad u = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-\left(\frac{n\pi}{20}\right)^2 t} \qquad \text{...(5)}$$

Using initial condition,

$$u(x, 0) = 30 + \frac{5}{2}x \quad \text{in eqn. (5), we get}$$

$$\Rightarrow \qquad 30 + \frac{5}{2}x = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

$$\Rightarrow \qquad \frac{3}{2}x - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $$b_n = \frac{2}{20}\int_0^{20}\left(\frac{3}{2}x - 10\right)\sin \frac{n\pi x}{20}\, dx = -\frac{20}{n\pi}\,[2(-1)^n + 1]$$

From (5), $$u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n + 1}{n}\right\}\sin \frac{n\pi x}{20}\, e^{-\left(\frac{n\pi}{20}\right)^2 t}$$

Example 2. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$.*

Sol. The equation of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{...(1)}$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad \text{...(2)}$$

| Refer Sol. of Ex. 1

Boundary conditions are

$$y(0, t) = 0 \qquad \text{...(3)}$$

$$y(l, t) = 0 \qquad \text{...(4)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...(5)}$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} \qquad \text{...(6)}$$

Applying boundary condition in (2),

$$y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$$\Rightarrow \quad c_3 = 0$$

$\therefore$ From (2), $\quad y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(7)}$

$\therefore$ From (2), $\quad y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad \text{...(7)}$

Again, $\quad y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow \quad \sin pl = 0 = \sin n\pi \quad (n \in \mathrm{I})$

$$\therefore \quad p = \frac{n\pi}{l}$$

Hence, from (7),

$$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \qquad \text{...(8)}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[- c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $c_2 = 0$.

$\therefore$ From (8),

$$y(x, t) = c_1 c_4 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad ...(9)$$

$$y(x, 0) = y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \quad y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + ...$$

Comparing, we get

$$b_1 = \frac{3y_0}{4},\ b_2 = 0,\ b_3 = \frac{-y_0}{4},\ b_4 = b_5 = ... = 0$$

Hence, from (9),

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}$$

which is the required displacement.

Example 3. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$ the string is given a shape defined by $F(x) = \mu x (l - x)$, μ is a constant and then released. Find the displacement $y(x, t)$ of any point x of the string at any time $t > 0$.*

(P.T.U. May 2007)

Sol. The wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{dx^2} \qquad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

(Refer sol. of Ex. 1)

Boundary conditions are

$$y(0, t) = 0 \qquad ...(3)$$

$$y(l, t) = 0 \qquad ...(4)$$

$$\frac{\partial y}{\partial t} = 0 \text{ at } t = 0 \qquad ...(5)$$

and

$$y(x, 0) = \mu x(l - x) \qquad ...(6)$$

From (2), $y(0, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$\Rightarrow$ $c_3 = 0$.

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad ...(7)$

$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \ (n \in \mathrm{I})$

$$p = \frac{n\pi}{l}.$$

From (7), $$y(x, t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l} \quad \ldots(8)$$

Now from (7), $$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right] . c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l} c_2 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$c_2 = 0.$$

$\therefore$ From (8), $$y(x, t) = c_1 c_4 \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$\Rightarrow$ $$y(x, t) = b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ where } c_1 c_4 = b_n.$$

The most general solution is

$$y(x, t) = \sum_1^\infty b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \quad \ldots(9)$$

$$y(x, 0) = \sum_1^\infty b_n \sin \frac{n\pi x}{l}$$

$$\mu(lx - x^2) = \sum_1^\infty b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l \mu\,(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\mu}{l}\left[\left\{(lx - x^2)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_0^l - \int_0^l (l - 2x)\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) dx\right]$$

$$= \frac{2\mu}{l}\left[\frac{1}{n\pi}\int_0^l (l - 2x) \cos \frac{n\pi x}{l}\, dx\right]$$

$$= \frac{2\mu}{n\pi}\left[\left\{(l - 2x)\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\}_0^l - \int_0^l (-2)\frac{\sin \frac{n\pi x}{l}}{\frac{n\pi}{l}}\, dx\right] = \frac{2\mu}{n\pi} . \frac{2l}{n\pi}\int_0^l \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{4\mu l}{n^2\pi^2}\left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^l = \frac{4\mu l^2}{n^3\pi^3}(-\cos n\pi + 1) = \frac{4\mu l^2}{n^3\pi^3}[1 - (-1)^n].$$

$\therefore$ From (9), $$y(x, t) = \frac{4\mu l^2}{\pi^3}\sum_1^\infty \frac{[1 - (-1)^n]}{n^3} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

$$= \frac{8\mu l^2}{\pi^3}\sum_{n=1}^\infty \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)\pi ct}{l}$$

Example 4. *A string is stretched between two fixed points (0, 0) and (l, 0) and released at rest from the initial deflection given by*

and $$f(x) = \begin{cases} \left(\dfrac{2k}{l}\right)x & , \ 0 < x < \dfrac{l}{2} \\ \left(\dfrac{2k}{l}\right)(l-x), & \dfrac{l}{2} < x < l \end{cases}$$

Find the deflection of the string at any time. (P.T.U. May 2009)

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{dx^2} \qquad ...(1)$$

The solution of eqn. (1) is

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \qquad ...(2)$$

[Refer Sol. of Ex. 1]

Boundary conditions are, $y(0, t) = 0$

$$y(l, t) = 0$$

$$\frac{\partial y}{\partial t} = 0 \quad \text{at} \quad t = 0$$

$$y(x, 0) = \begin{cases} \dfrac{2k}{l}x & , \quad 0 < x < \dfrac{l}{2} \\ \dfrac{2k}{l}(l-x), & \dfrac{l}{2} < x < l \end{cases}$$

From (2), $y(0, t) = (c_1 \cos cpt + c_2 \sin cpt)c_3$

$$0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_3$$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x, t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \qquad ...(3)$

$$y(l, t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$\Rightarrow$ $\sin pl = 0 = \sin n\pi \;;\; n \in \mathrm{I}$

$$p = \frac{n\pi}{l}.$$

$\therefore$ From (3), $y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l} \qquad ...(4)$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left[-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$$

At $t = 0$, $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0 = \dfrac{n\pi c}{l}\left[c_2 c_4 \sin \dfrac{n\pi x}{l}\right]$

$\Rightarrow$ $c_2 = 0.$

$\therefore$ From (4), $y(x, t) = c_1 c_4 \cos \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$

$$= b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \text{(where } c_1 c_4 = b_n\text{)} \quad ...(5)$$

The most general solution is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad ...(6)$$

$$y(x, 0) = \sum_{1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{[From (6)]}$$

where $$b_n = \frac{2}{l}\int_0^l y(x,0)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2k}{l}x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{2k}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4k}{l^2}\left[\int_0^{l/2}x\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{4k}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} - \int_0^{l/2}1.\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right.$$

$$\left. + \left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)\right\}_{l/2}^{l} - \int_{l/2}^{l}(-1).\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)dx\right]$$

$$= \frac{4k}{l^2}\left[-\frac{l}{n\pi}.\frac{l}{2}\cos\frac{n\pi}{2} + \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_0^{l/2} + \frac{l}{2}.\frac{l}{n\pi}\cos\frac{n\pi}{2} - \frac{l}{n\pi}\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)_{l/2}^{l}\right]$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{4k}{l^2}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$\therefore$ From (6), $$y(x,t) = \frac{8k}{\pi^2}\sum_1^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}.$$

Example 6. *If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}$, find the displacement $y(x,t)$.*

Sol. The equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

The solution of equation (1) is

$y(x,t) = (c_1\cos cpt + c_2\sin cpt)(c_3\cos px + c_4\sin px)$ ...(2) [Refer Sol. of Ex. **1**]

Boundary conditions are,
$$y(0,t) = 0 \qquad ...(3)$$
$$y(l,t) = 0 \qquad ...(4)$$
$$y(x,0) = 0 \qquad ...(5)$$
$$\left(\frac{\partial y}{\partial t}\right) = b\sin^3\frac{\pi x}{l} \text{ at } t = 0 \qquad ...(6)$$

From eqn. (2), $y(0,t) = 0 = (c_1\cos cpt + c_2\sin cpt)c_3$

$\Rightarrow$ $c_3 = 0.$

$\therefore$ From (2), $y(x,t) = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin px$ ...(7)

$y(l,t) = 0 = (c_1\cos cpt + c_2\sin cpt)\,c_4\sin pl$

$\Rightarrow \qquad \sin pl = 0 = \sin n\pi \ (n \in \mathrm{I})$

$\therefore \qquad p = \dfrac{n\pi}{l}.$

$\therefore$ From (7), $\quad y(x, t) = \left(c_1 \cos \dfrac{n\pi ct}{l} + c_2 \sin \dfrac{n\pi ct}{l}\right) c_4 \sin \dfrac{n\pi x}{l} \qquad \ldots(8)$

$$y(x, 0) = 0 = c_1 c_4 \sin \frac{n\pi x}{l}$$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (8), $\quad y(x, t) = c_2 c_4 \sin \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l}$

$$= b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \text{ where } c_2 c_4 = b_n$$

The general solution is $y(x, t) = \sum_1^{\infty} b_n \sin \dfrac{n\pi ct}{l} \sin \dfrac{n\pi x}{l} \qquad \ldots(9)$

$$\frac{\partial y}{\partial t} = \sum_1^{\infty} b_n \cdot \frac{n\pi c}{l} \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

At $t = 0$, $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \sum_1^{\infty} b_n \cdot \dfrac{n\pi c}{l} \sin \dfrac{n\pi x}{l}$

$$b \sin^3 \frac{\pi x}{l} = \sum_1^{\infty} b_n \cdot \frac{n\pi c}{l} \sin \frac{n\pi x}{l}$$

$$\frac{b}{4}\left[3 \sin \frac{\pi x}{l} - \frac{3\pi x}{l}\right] = b_1 \frac{\pi c}{l} \sin \frac{\pi x}{l} + \frac{2b_2 \pi c}{l} \sin \frac{2\pi x}{l} + 3b_3 \frac{\pi c}{l} \sin \frac{3\pi x}{l} + \ldots$$

$\Rightarrow \qquad b_1 \dfrac{\pi c}{l} = \dfrac{3b}{4} \quad \Rightarrow \quad b_1 = \dfrac{3bl}{4\pi c}$

$$b_2 = 0 \quad \text{and} \quad \frac{3b_3 \pi c}{l} = -\frac{b}{4} \quad \Rightarrow \quad b_3 = -\frac{bl}{12\pi c}$$

Also, $\qquad b_4 = 0 = b_5 = \ldots$ etc.

Hence from (9), $\quad y(x, t) = \dfrac{3bl}{4\pi c} \sin \dfrac{\pi ct}{l} \sin \dfrac{\pi x}{l} - \dfrac{bl}{12\pi c} \sin \dfrac{3\pi ct}{l} \sin \dfrac{3\pi x}{l}$

$$= \frac{bl}{12\pi c}\left[9 \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} - \sin \frac{3\pi x}{l} \sin \frac{3\pi ct}{l}\right].$$

Example 7. *A tightly stretched string with fixed end points x = 0 and x = l is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points an initial velocity λx(l – x), find the displacement of the string at any distance x from one end at any time t.*

Sol. Here the boundary conditions are $y(0, t) = y(l, t) = 0$

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi ct}{l} + b_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \qquad \ldots(1)$$

| Refer Sol. of Ex. 2

Since the string was at rest initially, $y(x, 0) = 0$

$\therefore$ From (1), $$0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad \Rightarrow \quad a_n = 0$$

$$\therefore \qquad y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \qquad \ldots(2)$$

and $$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l}$$

But $$\frac{\partial y}{\partial t} = \lambda x(l - x) \quad \text{when } t = 0$$

$$\therefore \qquad \lambda x(l - x) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n b_n \sin \frac{n\pi x}{l}$$

$$\Rightarrow \qquad \frac{\pi c}{l} n b_n = \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2\lambda}{l} \left[x(l - x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right]_0^l$$

$$= \frac{2\lambda}{l} \left[x(l - x)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l} \right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l} \right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right]_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3} (1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n]$$

$$= \begin{cases} 0 \quad , & \text{when } n \text{ is even} \\ \dfrac{8\lambda l^2}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases} \quad i.e., \ \frac{8\lambda l^2}{\pi^3 (2m - 1)^3}, \text{ taking } n = 2m - 1$$

$$\Rightarrow b_n = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4}$$

$\therefore$ From (2), the required solution is

$$y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi ct}{l} \sin \frac{(2m - 1)\pi x}{l}.$$

Example 8. *Transform the equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

to its normal form using the transformation $u = x + ct$, $v = x - ct$ and hence solve it. Show that the solution may be put in the form $y = \frac{1}{2}[f(x + ct) + f(x - ct)]$.

Assume initial conditions $y = f(x)$ and $(\partial y/\partial t) = 0$ at $t = 0$.

Sol. One Dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce two new independent variables

$$u = x + ct \qquad ...(2)$$

and

$$v = x - ct \qquad ...(3)$$

so that y becomes a function of u and v.

Then,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \qquad ...(4) \quad \text{[Using (2) and (3)]}$$

Also,

$$\frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad ...(5)$$

$\therefore$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \qquad ...(6)$$

Also,

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial t} = c\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}(-c) = c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right) \qquad ...(7)$$

$\Rightarrow$

$$\frac{\partial}{\partial t} \equiv c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \qquad ...(8)$$

$\therefore$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = c\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)c\left(\frac{\partial y}{\partial u} - \frac{\partial y}{\partial v}\right)$$

$$= c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) \qquad ...(9)$$

From (1), (6) and (9), we have

$$c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right) = c^2\left(\frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

$$\Rightarrow \quad -4c^2\frac{\partial^2 y}{\partial u \partial v} = 0$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial u \partial v} = 0 \qquad \text{...(10)} \quad (\because \quad c^2 \neq 0)$$

Integrating eqn. (10) partially, w.r.t. u, we get

$$\frac{\partial y}{\partial v} = f_1(v).$$

Integrating again w.r.t. v partially, we get

$$y = \int f_1(v)\, \partial v + \psi(u) = \phi(v) + \psi(u)$$

$$\Rightarrow \quad y(x, t) = \phi(x - ct) + \psi(x + ct) \qquad \text{...(11)}$$

which is d' Alembert's solution of wave equation.

Applying initial conditions $y = f(x)$ and $\frac{\partial y}{\partial t} = 0$ at $t = 0$ in (11), we get

$$f(x) = \phi(x) + \psi(x) \text{ and } 0 = -\phi'(x) + \psi'(x)$$

Hence, $\phi(x) = \psi(x) = \frac{1}{2}f(x)$

$$\therefore \quad y = \frac{1}{2}[f(x + ct) + f(x - ct)] \qquad \text{...(12)}$$

Example 1. *Use separation of variables method to solve the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin\frac{n\pi x}{l}$.

Sol. The given equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$

Let $\quad u = XY \qquad$ ...(2)

where X is a function of x only and Y is a function of y only then,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(XY) = X\frac{d^2Y}{dy^2}$$

$\therefore$ From (1), $\quad YX'' + XY'' = 0$

$$\Rightarrow \quad \frac{X''}{X} + \frac{Y''}{Y} = 0$$

Case I. When $\frac{X''}{X} = -\frac{Y''}{Y} = p^2$ (say)

(*i*) $\frac{X''}{X} = p^2$

$$X'' - p^2 X = 0$$

Auxiliary eqn. is $m^2 - p^2 = 0$

$$m = \pm p$$

$\therefore$ C.F. $= c_1 e^{px} + c_2 e^{-px}$

P.I. $= 0$

$\therefore$ $X = c_1 e^{px} + c_2 e^{-px}$

(*ii*) $\frac{-Y''}{Y} = p^2 \quad \Rightarrow \quad Y'' + p^2 Y = 0$

Auxiliary eqn. is $m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$\therefore$ C.F. $= c_3 \cos py + c_4 \sin py$

P.I. $= 0$

$\therefore$ $y = c_3 \cos py + c_4 \sin py$

Now, $X(0) = 0$

$\Rightarrow$ $c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1$

$X(l) = 0$

$\Rightarrow$ $c_1 e^{pl} + c_2 e^{-pl} = 0 \quad \Rightarrow \quad c_1 (e^{pl} - e^{-pl}) = 0$

$\Rightarrow$ $c_1 = 0$ | Since $e^{pl} - e^{-pl} \neq 0$ (as $p \neq 0 \neq l$)

$\therefore$ $c_2 = 0$

$\therefore$ $X = 0 \quad \Rightarrow \quad u = XY = 0$ which is impossible

Hence we reject case I.

Case II. When $\frac{X''}{X} = -\frac{Y''}{Y} = 0$ (say)

(*i*) $\frac{X''}{X} = 0$

$\Rightarrow$ $X'' = 0 \quad \Rightarrow \quad X = c_5 x + c_6$

(*ii*) $\frac{-Y''}{Y} = 0$

$\Rightarrow$ $Y'' = 0 \quad \Rightarrow \quad Y = c_7 y + c_8$

Now, $X(0) = 0 \quad \Rightarrow \quad c_6 = 0$

$X(l) = 0$

$\Rightarrow$ $c_5 l + c_6 = 0 \quad \Rightarrow \quad c_5 l = 0$

$\Rightarrow$ $c_5 = 0$ (Since $l \neq 0$)

$\therefore$ $X = 0$

$\therefore$ $u = XY = 0$ which is impossible

Hence we also reject case II.

Case III. When $\frac{X''}{X} = -\frac{-Y''}{Y} = -p^2$ (say)

(*i*) $\frac{X''}{X} = -p^2$

$$\Rightarrow \qquad X'' + p^2 X = 0 \quad \Rightarrow \quad \frac{d^2X}{dx^2} + p^2X = 0.$$

Auxiliary eqn. is $\quad m^2 + p^2 = 0 \quad \Rightarrow \quad m = \pm pi$

$$\text{C.F.} = c_9 \cos px + c_{10} \sin px$$

$$\text{P.I.} = 0$$

$$X = c_9 \cos px + c_{10} \sin px$$

(*ii*) $$-\frac{Y''}{Y} = -p^2$$

$$\Rightarrow \qquad \frac{Y''}{Y} = p^2 \quad \Rightarrow \quad \frac{d^2Y}{dy^2} - p^2Y = 0.$$

Auxiliary equation is

$$m^2 - p^2 = 0$$

$$m = \pm p.$$

$$\therefore \qquad \text{C.F.} = c_{11}e^{py} + c_{12}e^{-py}$$

$$\text{P.I.} = 0$$

Hence, $\quad Y = c_{11}\, e^{py} + c_{12}\, e^{-py}.$

Now, $\quad X(0) = 0 \quad \Rightarrow \quad c_9 = 0$

$$\therefore \qquad X = c_{10} \sin px$$

$$X(l) = 0$$

$$c_{10} \sin pl = 0$$

$$\Rightarrow \qquad \sin pl = 0 = \sin n\pi,\ n \in I$$

$$\therefore \qquad p = \frac{n\pi}{l}$$

$$\therefore \qquad X = c_{10} \sin \frac{n\pi x}{l} \qquad \ldots(3)$$

Again, $\quad Y(0) = 0$

$$\Rightarrow \qquad c_{11} + c_{12} = 0 \quad \Rightarrow \quad c_{12} = -c_{11}$$

$$Y = c_{11}(e^{py} - e^{-py}) = c_{11}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right) \qquad \ldots(4)$$

$$\therefore \qquad u = XY = c_{10}c_{11} \sin \frac{n\pi x}{l}\ [e^{(n\pi y/l)} - e^{(-n\pi y/l)}]$$

or $$u(x, y) = b_n \sin \frac{n\pi x}{l}\ [e^{(n\pi y/l)} - e^{(-n\pi y/l)}] \qquad \ldots(5)$$

Now, $$u(x, a) = \sin \frac{n\pi x}{l} = b_n \sin \frac{n\pi x}{l}\ [e^{(n\pi a/l)} - e^{-(n\pi a/l)}]$$

$$\Rightarrow \qquad b_n = \frac{1}{e^{\frac{n\pi a}{l}} - e^{-\frac{n\pi a}{l}}} = \frac{1}{2 \sin h\left(\frac{n\pi a}{l}\right)}$$

$$\therefore \qquad u(x, y) = \frac{e^{(n\pi y/l)} - e^{-(n\pi y/l)}}{2 \sin h\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l} = \frac{\sin h\left(\frac{n\pi y}{l}\right)}{\sin h\left(\frac{n\pi a}{l}\right)} \sin \frac{n\pi x}{l}$$

which is the required result.

Example 2. *A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by*

$$u(x, 0) = 100 \sin \frac{\pi x}{8}, \; 0 < x < 8$$

while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0°C, show that the steady state temperature at any point of the plate is given by

$$u(x, y) = 100 e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}.$$

Sol. Let $u(x, y)$ be the temperature at any point P of the plate.

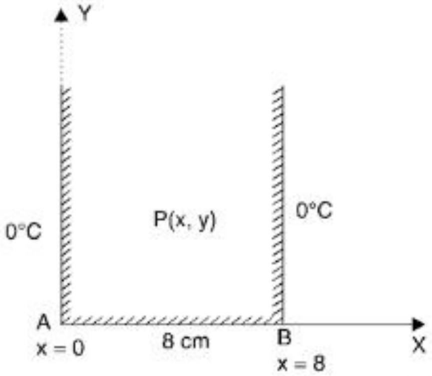
Two dimensional heat flow equation in steady state is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$

Its solution is $\quad u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(2)$

Boundary conditions are

$$u(0, y) = 0$$
$$u(8, y) = 0$$
$$\underset{y \to \infty}{\text{Lt}} \; u(x, y) = 0$$
$$u(x, 0) = 100 \sin \frac{\pi x}{8}, \quad 0 < x < 8$$



From (2),

$$u(0, y) = 0 = c_1 (c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad c_1 = 0.$

$\therefore$ From (2),

$$u(x, y) = c_2 \sin px \, (c_3 e^{py} + c_4 e^{-py}) \qquad \ldots(3)$$
$$u(8, y) = 0 = c_2 \sin 8p (c_3 e^{py} + c_3 e^{-py})$$

$\Rightarrow \qquad \sin 8p = 0 = \sin n\pi$

$\Rightarrow \qquad p = \frac{n\pi}{8} \quad (n \in \text{I})$

$\therefore$ From (3),

$$u(x, y) = c_2 \sin \frac{n\pi x}{8} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}}) \qquad \ldots(4)$$

$$\underset{y \to \infty}{\text{Lt}} \; u(x, y) = 0 = c_2 \sin \frac{n\pi x}{8} \lim_{y \to \infty} (c_3 e^{\frac{n\pi y}{8}} + c_4 e^{-\frac{n\pi y}{8}})$$

which is satisfied only when

$$c_3 = 0.$$

$\therefore$ From (4), $\quad u(x, y) = c_2 c_4 \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} = b_n \sin \frac{n\pi x}{8} e^{-\frac{n\pi y}{8}} \qquad \ldots(5)$

From (5), $\quad u(x, 0) = b_n \sin \frac{n\pi x}{8}$

$$100 \sin \frac{\pi x}{8} = b_n \sin \frac{n\pi x}{8}$$

$$\Rightarrow \qquad b_n = 100,\ n = 1.$$

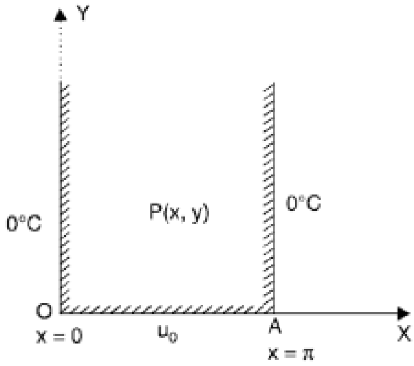
$$\therefore \quad \text{From (5),} \quad u(x, y) = 100 \sin\left(\frac{\pi x}{8}\right) e^{-(\pi y/8)}$$

which is the required steady state temperature at any point of the plate.

Example 3. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state.*

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...(1)}$$



Boundary conditions are,

$$u(0, y) = 0 = u(\pi, y)$$

$$\lim_{y \to \infty} u(x, y) = 0 \ (0 < x < \pi)$$

and $$u(x, 0) = u_0 \ (0 < x < \pi)$$

Solution to equation (1) is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad \text{...(2)}$$

From (2), $$u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \qquad c_1 = 0.$$

From (2), $$u(x, y) = c_2 \sin px(c_3 e^{py} + c_4 e^{-py})$$

$$u(\pi, y) = 0 = c_2 \sin p\pi(c_3 e^{py} + c_4 e^{-py})$$

$$\Rightarrow \qquad \sin p\pi = 0 = \sin n\pi \ (n \in \mathrm{I}) \qquad \text{...(3)}$$

$$\therefore \qquad p = n.$$

$$\therefore \quad \text{From (3),} \quad u(x, y) = c_2 \sin nx\ (c_3 e^{ny} + c_4 e^{-ny}) \qquad \text{...(4)}$$

$$\lim_{y \to \infty} u(x, y) = 0 = c_2 \sin nx \lim_{y \to \infty} (c_3 e^{ny} + c_4 e^{-ny})$$

which is satisfied only when $c_3 = 0$.

$\therefore$ From (4), $u(x, y) = c_2 c_4 e^{-ny} \sin nx = b_n e^{-ny} \sin nx$, where $c_2 c_4 = b_n$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx \qquad \text{...(5)}$$

$$u(x, 0) = u_0 = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx$$

$$= \frac{2u_0}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{2u_0}{n\pi}\{1 - (-1)^n\}$$

$$= \begin{cases} \dfrac{4u_0}{n\pi}; & \text{if } n \text{ is odd} \\ 0 \quad ; & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad \text{From (5),} \quad u(x, y) = \frac{4u_0}{\pi} \sum_{n = 1, 3, 5, \ldots}^{\infty} \frac{\sin nx}{n} e^{-ny} \qquad (n \text{ is odd})$$

or $$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin (2n-1)\, x\, e^{-(2n-1)y}$$

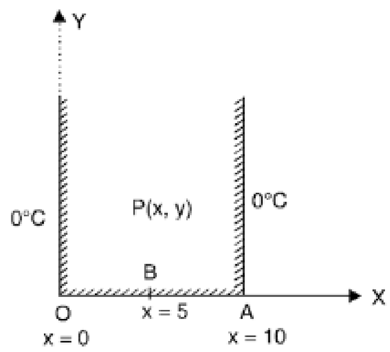
Example 4. *A rectangular plate with insulated surfaces is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge y = 0 is given by*

and $$u(x, y) = \begin{cases} 20x & , \quad 0 < x \le 5 \\ 20(10 - x), & 5 < x < 10 \end{cases}$$

and the two long edges x = 0 and x = 10 as well as other short edge are kept at 0°C. Find the temperature u at any point P(x, y).

Sol. In steady state, two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \ldots(1)$$



Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3\, e^{py} + c_4\, e^{-py}) \qquad \ldots(2)$$

Boundary conditions are $u(0, y) = 0$

$$u(10, y) = 0$$

$$\lim_{y \to \infty} u(x, y) = u(x, \infty) = 0$$

and $$u(x, 0) = \begin{rcases} 20x, & 0 < x \le 5 \\ 20(10 - x), & 5 < x \le 10 \end{rcases}$$

From (2), $\quad u(x, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py}) \quad \Rightarrow \quad c_1 = 0$

From (2), $\quad u(x, y) = c_2 \sin px\, (c_3\, e^{py} + c_4\, e^{-py}) \qquad \ldots(3)$

$$u(10, y) = 0 = c_2 \sin 10p\, (c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow \qquad \sin 10p = 0 = \sin n\pi$

or $\qquad 10p = n\pi \; (n \in \text{I})$

$\Rightarrow \qquad p = \dfrac{n\pi}{10}.$

$\therefore$ From (3), $\quad u(x, y) = c_2 \sin \dfrac{n\pi x}{10} \left(c_3\, e^{\frac{n\pi y}{10}} + c_4\, e^{-\frac{n\pi y}{10}}\right) \qquad \ldots(4)$

$$\lim_{y \to \infty} u(x, y) = c_2 \sin \frac{n\pi x}{10} \lim_{y \to \infty} \left(c_3\, e^{\frac{n\pi y}{10}} + c_4\, e^{-\frac{n\pi y}{10}}\right)$$

which is satisfied only when $c_3 = 0$.

Hence from (4), $u(x, y) = c_2 c_4 \sin \dfrac{n\pi x}{10} e^{-\frac{n\pi y}{10}} = b_n \sin \dfrac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \qquad \ldots(5)$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} \qquad \ldots(6)$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}, \text{ where}$$

$$b_n = \frac{2}{10}\int_0^{10} u(x, 0) \sin \frac{n\pi x}{10}\, dx$$

$$= \frac{1}{5}\left[\int_0^5 20x \sin \frac{n\pi x}{10} dx + \int_5^{10} 20\,(10 - x) \sin \frac{n\pi x}{10}\, dx\right]$$

$$= 4\left[\left\{x\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_0^5 - \int_0^5 1\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) dx + \left\{(10 - x)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)\right\}_5^{10} - \int_5^{10} (-1)\left(\frac{-\cos \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) dx\right]$$

$$= 4\left[\frac{10}{n\pi}(-5) \cos \frac{n\pi}{2} + \frac{10}{n\pi}\left(\frac{\sin \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_0^5 + \frac{50}{n\pi} \cos \frac{n\pi}{2} - \frac{10}{n\pi}\left(\frac{\sin \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right)_5^{10}\right]$$

$$= 4\left[\frac{-50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{50}{n\pi} \cos \frac{n\pi}{2} - \frac{100}{n^2\pi^2}\left(0 - \sin \frac{n\pi}{2}\right)\right]$$

$$= \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

From (6), $$u(x, y) = \frac{800}{\pi^2} \sum_1^{\infty} \frac{\sin n\pi/2}{n^2} \sin \frac{n\pi x}{10}\, e^{-\frac{n\pi y}{10}}$$

which is the required temperature u at any point $P(x, y)$.

Example 5. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < \pi$, $0 < y < \pi$, *which satisfies the conditions :*

$$u(0, y) = u(\pi, y) = u(x, \pi) = 0 \text{ and } u(x, 0) = \sin^2 x.$$

Sol. The given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ...(1)

Its solution consistent with boundary conditions is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3\, e^{py} + c_4\, e^{-py}) \quad ...(2)$$

From (2), $u(0, y) = 0 = c_1(c_3 e^{py} + c_4\, e^{-py})$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (2), $u(x, y) = c_2 \sin px\,(c_3\, e^{py} + c_4\, e^{-py})$ ...(3)

$u(\pi, y) = 0 = c_2 \sin p\pi(c_3\, e^{py} + c_4\, e^{-py})$

$\Rightarrow$ $\sin p\pi = 0 = \sin n\pi \; (n \in I)$

$\therefore$ $p = n.$

Hence from (3), $u(x, y) = c_2 \sin nx\,(c_3 e^{ny} + c_4 e^{-ny}) = \sin nx\,(Ae^{ny} + Be^{-ny})$ ...(4)

where $c_2c_3 = A$ and $c_2c_4 = B$.

From (4), $u(x, \pi) = \sin nx(Ae^{n\pi} + Be^{-n\pi})$

$0 = \sin nx(Ae^{n\pi} + Be^{-n\pi})$

$\Rightarrow \quad 0 = Ae^{n\pi} + Be^{-n\pi}$

$\Rightarrow \quad Ae^{n\pi} = -Be^{-n\pi} = -\frac{1}{2}B_n$ (say)

then (4) becomes,
$$u(x, y) = \sin nx\left[-\frac{1}{2}B_n e^{-n\pi}\, e^{ny} + \frac{1}{2}B_n\, e^{n\pi}\, e^{-ny}\right]$$
$$= \frac{1}{2}B_n[e^{n(\pi-y)} - e^{-n(\pi-y)}]\sin nx = B_n \sinh n\,(\pi - y)\sin nx.$$

The most general solution is
$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh n\,(\pi - y)\sin nx \qquad ...(5)$$
$$u(x, 0) = \sin^2 x = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin nx$$

where
$$B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}(1 - \cos 2x)\sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin nx - \frac{1}{2}\{\sin(n+2)x + \sin(n-2)x\}\right]dx$$
$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n} + \frac{\cos(n+2)x}{2(n+2)} + \frac{\cos(n-2)x}{2(n-2)}\right]_0^{\pi}$$
$$= \frac{1}{2\pi}\left[\left(\frac{1}{n+2} + \frac{1}{n-2} - \frac{2}{n}\right)\{(-1)^n - 1\}\right], \text{ when } n \neq 2$$
$$B_n \sinh n\pi = \begin{cases} \dfrac{-8}{\pi n(n^2-4)}, & \text{when } n \text{ is odd} \\ 0, & \text{when } n \text{ is even and} \neq 2 \end{cases}$$

when $n = 2$,
$$B_2 \sinh 2\pi = \frac{2}{\pi}\int_0^{\pi}\sin^2 x \sin 2x\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}(1 - \cos 2x)\sin 2x\, dx = \frac{1}{\pi}\int_0^{\pi}\left(\sin 2x - \frac{1}{2}\sin 4x\right)dx$$
$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2} + \frac{1}{8}\cos 4x\right)_0^{\pi} = \frac{1}{\pi}\left[\frac{-1}{2}(1-1) + \frac{1}{8}(1-1)\right] = 0$$

$\therefore \quad B_2 = 0.$

Hence the solution (5) becomes,
$$u(x, y) = \frac{-8}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin nx \sinh n(\pi - y)}{n(n^2-4)\sinh n\pi}$$

or $$u(x, y) = -\frac{8}{\pi} \sum_{m=1,2,3,...}^{\infty} \frac{\sin(2m-1)x \sinh(2m-1)(\pi-y)}{(2m-1)\{(2m-1)^2-4\}\sinh(2m-1)\pi}.$$

Example 6. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$*, with the rectangle* $0 \le x \le a$, $0 \le y \le b$ *; given that*

$u(x, b) = u(0, y) = u(a, y) = 0$ *and* $u(x, 0) = x(a-x)$. (P.T.U. May 2008)

Sol. The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

Its solution is

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(2)$$

$$u(0, y) = 0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $c_1 = 0.$

$\therefore$ From (2), $u(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad ...(3)$

$$u(a, y) = 0 = c_2 \sin ap\,(c_3 e^{py} + c_4 e^{-py})$$

$\Rightarrow$ $\sin ap = 0 = \sin n\pi \; (n \in I)$

$\Rightarrow$ $ap = n\pi$ or $p = \frac{n\pi}{a}$.

$\therefore$ From (3), $u(x, y) = c_2 \sin \frac{n\pi x}{a}(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}})$

$$u(x, y) = \sin \frac{n\pi x}{a}(A\, e^{\frac{n\pi y}{a}} + B\, e^{-\frac{n\pi y}{a}}) \qquad ...(4)$$

where $c_2 c_3 = A$ and $c_2 c_4 = B$

$$u(x, b) = \sin \frac{n\pi x}{a}(A\, e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}})$$

$$0 = \sin \frac{n\pi x}{a}(A\, e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}})$$

$\Rightarrow$ $A\, e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}} = 0$

$$A\, e^{\frac{n\pi b}{a}} = -B e^{-\frac{n\pi b}{a}} = -\frac{1}{2} B_n \text{ (say)}.$$

Then (4) becomes,

$$u(x, y) = \sin \frac{n\pi x}{a}\left[\frac{-1}{2} B_n e^{-\frac{n\pi b}{a}} e^{\frac{n\pi y}{a}} + \frac{1}{2} B_n e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}}\right]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} [e^{\frac{n\pi}{a}(b-y)} - e^{-\frac{n\pi}{a}(b-y)}]$$

$$= \frac{1}{2} B_n \sin \frac{n\pi x}{a} . 2 \sin h \frac{n\pi}{a}(b-y) = B_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi}{a}(b-y).$$

The most general solution is

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{a} \sin h \frac{n\pi}{a}(b-y) \qquad ...(5)$$

Applying to this the condition $u(x, 0) = x(a-x)$, we get

From (5), $$u(x, 0) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

$$\Rightarrow \qquad x(a-x) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $$B_n \sin \frac{n\pi}{a} b = \frac{2}{a} \int_0^a x(a-x) \sin \frac{n\pi}{a} x \, dx$$

$$= \frac{2}{a}\left[\left\{(ax - x^2)\left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}}\right)\right\}_0^a - \int_0^a (a-2x) . \left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}}\right) dx\right]$$

$$= \frac{2}{a} . \frac{a}{n\pi} \int_0^a (a-2x) . \cos \frac{n\pi}{a} x \, dx$$

$$= \frac{2}{n\pi}\left[\left\{(a-2x)\frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}}\right\}_0^a - \int_0^a (-2)\left(\frac{\sin \frac{n\pi}{a} x}{\frac{n\pi}{a}}\right) dx\right]$$

$$= \frac{4}{n\pi} . \frac{a}{n\pi}\left(\frac{-\cos \frac{n\pi}{a} x}{\frac{n\pi}{a}}\right)_0^a = \frac{4a}{n^2\pi^2} . \frac{a}{n\pi} (1 - \cos n\pi)$$

$$= \frac{4a^2}{n^3\pi^3} [1 - (-1)^n] = \begin{cases} \dfrac{8a^2}{n^3\pi^3} & , \quad \text{when } n \text{ is odd} \\ 0 & , \quad \text{when } n \text{ is even} \end{cases}$$

$$\therefore \qquad B_n = \begin{cases} \dfrac{8a^2}{\sinh\left(\frac{n\pi}{a} b\right)(n^3\pi^3)} & , \quad \text{when } n \text{ is odd} \\ 0 & , \quad \text{when } n \text{ is even} \end{cases}$$

$\therefore$ From (5), $$u(x, y) = \frac{8a^2}{\pi^3} \sum_{\substack{n=1,3,5,\ldots \\ (n \text{ is odd})}}^{\infty} \frac{\sin \frac{n\pi x}{a}}{n^3 \sinh \frac{n\pi}{a} b} . \sinh \frac{n\pi}{a}(b-y)$$

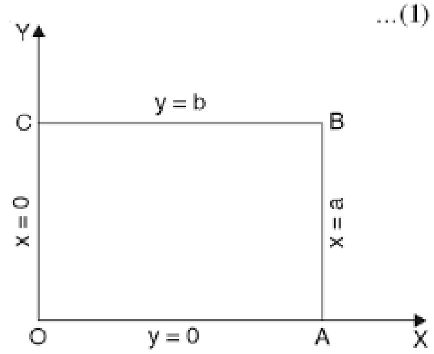
or $$u(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin (2n+1) \frac{\pi x}{a} . \frac{\sinh \frac{(2n+1)\pi}{a}(b-y)}{\sinh \frac{(2n+1)\pi}{a} b}$$

which is the required solution.

Example 7. *A thin rectangular plate whose surface is impervious to heat flow has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. Its four edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature. Determine the temperature at a point of the plate as t increases.*

Sol. Two dimensional heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}. \quad \ldots(1)$$



Boundary conditions are

$$u(0, y, t) = 0$$
$$u(a, y, t) = 0$$
$$u(x, 0, t) = 0$$
$$u(x, b, t) = 0$$

and $\quad y(x, y, t) = f(x, y)$ at $t = 0$.

Let the solution be $\quad u = \text{XYT}$

where X is a function of x only, Y is a function of y only and T is a function of t only.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\text{XYT}) = \text{XY}\frac{d\text{T}}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(\text{XYT}) = \text{YT}\frac{d^2\text{X}}{dx^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial y^2}(\text{XYT}) = \text{XT}\frac{d^2\text{Y}}{dy^2}.$$

From (1), $\text{YT X}'' + \text{XTY}'' = \frac{1}{c^2}(\text{XYT}')$

$$\Rightarrow \quad \frac{\text{X}''}{\text{X}} + \frac{\text{Y}''}{\text{Y}} = \frac{\text{T}'}{c^2\text{T}} \quad \ldots(2)$$

There are three possibilities :

(*i*) $\quad \frac{\text{X}''}{\text{X}} = 0, \quad \frac{\text{Y}''}{\text{Y}} = 0, \quad \frac{\text{T}'}{c^2\text{T}} = 0$

(*ii*) $\quad \frac{\text{X}''}{\text{X}} = \text{K}_1^{\ 2}, \quad \frac{\text{Y}''}{\text{Y}} = \text{K}_2^{\ 2}, \quad \frac{\text{T}'}{c^2\text{T}} = \text{K}^2$

(*iii*) $\quad \frac{\text{X}''}{\text{X}} = -\text{K}_1^{\ 2}, \quad \frac{\text{Y}''}{\text{Y}} = -\text{K}_2^{\ 2}, \quad \frac{\text{T}'}{c^2\text{T}} = -\text{K}^2$

where $\quad \text{K}^2 = \text{K}_1^{\ 2} + \text{K}_2^{\ 2}$.

Of these three solutions, we have to select the solution which is consistent with the physical nature of the problem.

The solution satisfying the given boundary conditions will be given by (*iii*).

Then, $\quad \text{X} = c_1 \cos \text{K}_1 x + c_2 \sin \text{K}_1 x$

$$\text{Y} = c_3 \cos \text{K}_2 y + c_4 \sin \text{K}_2 y$$

$$\text{T} = c_5 e^{-c^2\text{K}^2 t}$$

$\therefore \quad u = \text{XYT}$

EXAMPLE 3.5 A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

Solution The initial boundary value problem IBVP of heat conduction is given by

$$\text{PDE: } \frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2}, \qquad 0 \le x \le L,\ t > 0$$

$$\text{BCs: } \theta(0, t) = 0, \qquad t \ge 0$$

$$\frac{\partial \theta}{\partial x}(L, t) = 0, \qquad t > 0$$

$$\text{IC: } \theta(x, 0) = \theta_0, \qquad 0 \le x \le L$$

From Section 3.5, it can be noted that the physically meaningful and non-trivial solution is

$$\theta(x, t) = e^{-\alpha\lambda^2 t}(A \cos \lambda x + B \sin \lambda x)$$

Using the first boundary condition, we obtain $A = 0$. Thus the acceptable solution is

$$\theta = Be^{-\alpha\lambda^2 t} \sin \lambda x$$

$$\frac{\partial \theta}{\partial x} = \lambda Be^{-\alpha\lambda^2 t} \cos \lambda x$$

Using the second boundary condition, we have

$$0 = \lambda Be^{-\alpha\lambda^2 t} \cos \lambda L$$

implying $\cos \lambda L = 0$. Therefore,

The eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$$

Thus, the acceptable solution is of the form

$$\theta = B \exp\,[-\alpha\{(2n+1)/2L\}^2 \pi^2 t] \sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using the principle of superposition, we obtain

$$\theta(x,t) = \sum_{n=0}^{\infty} B_n \exp[-\alpha\{(2n+1)/2L\}^2 \pi^2 t] \sin\left(\frac{2n+1}{2L}\pi x\right)$$

Finally, using the initial condition, we have

$$\theta_0 = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2L}\pi x\right)$$

which is a half-range Fourier-sine series and, thus,

$$B_n = \frac{2}{L}\int_0^L \theta_0 \sin\left(\frac{2n+1}{2L}\pi x\right) dx$$

$$= \frac{2}{L}\left[-\theta_0 \frac{2L}{(2n+1)\pi}\left\{\cos\left(\frac{2n+1}{2L}\pi x\right)\right\}_0^L\right]$$

$$= -\frac{4\theta_0}{(2n+1)\pi}[\cos\{(2n+1)\pi/2\} - \cos 0] = \frac{4\theta_0}{(2n+1)\pi}$$

Thus, the required temperature distribution is

$$\theta(x,t) = \sum_{n=0}^{\infty} \frac{4\theta_0}{(2n+1)\pi} \exp[-\alpha\{(2n+1)/2L\}^2 \pi^2 t] \sin\left(\frac{2n+1}{2L}\pi x\right)$$

EXAMPLE 3.4 Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi, t \ge 0$, subject to the conditions

(i) T remains finite as $t \to \infty$

(ii) $T = 0$, if $x = 0$ and π for all t

(iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \frac{\pi}{2} \le x \le \pi. \end{cases}$

Solution Since T should satisfy the diffusion equation, the three possible solutions are:

$$T(x,t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x}) e^{\alpha\lambda^2 t}$$

$$T(x,t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha\lambda^2 t}$$

$$T(x,t) = (c_1 x + c_2)$$

The first condition demands that T should remain finite as $t \to \infty$. We therefore reject the first solution. In view of BC (ii), the third solution gives

$$0 = c_1 \cdot 0 + c_2, \qquad 0 = c_1 \cdot \pi + c_2$$

implying thereby that both c_1 and c_2 are zero and hence $T = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the third solution also. Thus, the only possible solution satisfying the first condition is

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha\lambda^2 t}$$

Using the BC (ii), we have

$$0 = (c_1 \cos \lambda x + c_2 \sin \lambda x)\Big|_{x=0}$$

implying $c_1 = 0$. Therefore, the possible solution is

$$T(x, t) = c_2 e^{-\alpha\lambda^2 t} \sin \lambda x$$

Applying the BC: $T = 0$ when $x = \pi$, we get

$$\sin \lambda\pi = 0 \Rightarrow \lambda\pi = n\pi$$

where n is an integer. Therefore,

$$\lambda = n$$

Hence the solution is found to be of the form

$$T(x, t) = c e^{-\alpha n^2 t} \sin nx$$

Noting that the heat conduction equation is linear, its most general solution is obtained by applying the principle of superposition. Thus,

$$T(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha n^2 t} \sin nx$$

Using the third condition, we get

$$T(x, 0) = \sum_{n=1}^{\infty} c_n \sin nx$$

which is a half-range Fourier-sine series and, therefore,

$$c_n = \frac{2}{\pi}\int_0^{\pi} T(x, 0) \sin nx \, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

Integrating by parts, we obtain

$$c_n = \frac{2}{\pi}\left[\left(-x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_0^{\pi/2} + \left\{-(\pi - x)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

or

$$c_n = \frac{4 \sin (n\pi/2)}{n^2\pi}$$

Thus, the required solution is

$$T(x, t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2 t} \sin (n\pi/2)}{n^2} \sin nx$$

Ex. 4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1, y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. ...(1)

Proceed as in solved Ex. 6, Art. 2.12 and show that

$(x - y)/(y - z) = c_1$ ...(2)

and $xy + yz + zx = c_2$. ...(3)

The given curve is represented by $x = 1$ and $y = 0$. ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1 c_2$ or $c_1 c_2 + 1 = 0$. ...(5)

Putting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\,(xy + yz + zx) + 1 = 0$ or $(x - y)\,(xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1, x + z = 2$. **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y$. ...(1)

Given curve is given by $y^2 + z^2 = 1$, and $x + z = 2$. ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}$. ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1$. ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2$. ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

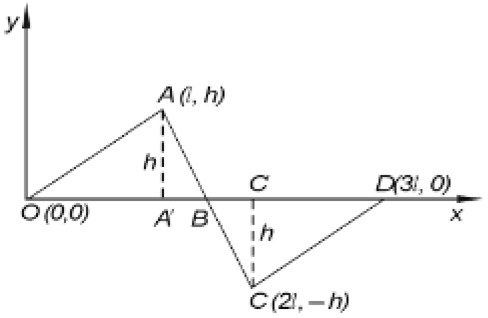
or $1 + 2 = c_1 + c_2$, using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0$.

Exercise 2. *The points of trisection of a string are pulled aside through a distance h on opposite sides of the position of equilibrium, and the string is released from rest. Derive an expresion for the string at any subsequent time and show that the middel point of the sting always remains at rest.* [*Meerut 2008*]

Sol. The displacement function $y(x, t)$ is the solution of the one-dimensional wave equation of string (of length $3l$, say) $\quad \partial^2 y/\partial x^2 = (1/c^2)(\partial^2 y/\partial t^2)$ ... (1)

subject to the boundary conditions $y(0, t) = y(3l, 0) = 0$, for all t ... (2)



Let $OA' = A'C' = C'D = l$ so that the A' and C' are the points of trisection of the string of length AD (= $3l$). According to given problems, A' and C' are pulled in the direction of y through a distance h as shown in figure and hence the coordinates of A, C and D are (l, h), $(2l, -h)$ and $(3l, 0)$ respectively.

Initial deflection is given by $OABCD$. Equation of OA is

$$y - 0 = (h/l)(x - 0) \qquad i.e., \qquad y = hx/l.$$

Equations of AC and CD respectively are given by

$$y - h = \frac{-h-h}{2l-l}(x-1) \qquad \text{or} \qquad y = \frac{h(3l-2x)}{l}$$

and $$y-(-h) = \frac{0-(-h)}{3l-2l}(x-2l) \qquad \text{or} \qquad y = \frac{h(x-3l)}{l}$$

Hence, the initial displacement is given by

$$y(x,0) = f(x) = \begin{cases} hx/l, \text{ where } 0 \le x \le l \\ \{h(3l-2x)\}/l, \text{ where } l \le x \le 2l \\ \{h(x-3l)\}/l, \text{ where } 2l \le x \le 3l \end{cases} \qquad \text{... 3}(a)$$

Since initially the string was in the position of equilibrium, Therefore

the initial velocity $= y_t(x, 0) = 0, \; 0 \le x \le 3l$ ... 3(b)

As explained in working rule of Art 2.10C, prove that the solution of one dimensional wave equation (1) subject to the

boundary condition $\quad y(0, t) = y(a, t) = 0$, for all t ... (4)

and initial conditions $\quad y(x, 0) = f(x), \qquad y_t(x, 0) = 0, \; 0 \le x \le a$ ... (5)

is $$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a}\cos\frac{n\pi ct}{a} \qquad \text{... (6)}$$

where $$E_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx \qquad \text{... (7)}$$

Comparing the given problem given by (1), (2), 3(*a*) and 3(*b*) with the boundary value problem gievn by (1), (4) and (5) we have $c = c$, $a = 3l$ and $f(x)$ is given by 3(*a*).

From (7), $E_n = \dfrac{2}{3l}\displaystyle\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx$

$$= \frac{2}{3l}\left[\int_0^{l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx\right]$$

$$= \frac{2}{3l}\left[\int_0^{l} \frac{hx}{l}\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l} \frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l} \frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}\,dx\right], \text{ using } 3(a)$$

$$= \frac{2h}{3l^2}\left[x\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\}-(1)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_0^l + \frac{2h}{3l^2}\left[(3l-2x)\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\}-(-2)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_l^{2l}$$

$$+\frac{2h}{3l^2}\left[(x-3l)\left\{-\frac{\cos(n\pi x/3l)}{n\pi/3l}\right\}-(1)\left\{-\frac{\sin(n\pi x/3l)}{n^2\pi^2/9l^2}\right\}\right]_{2l}^{3l}, \quad \text{by chain rule of integration by parts}$$

$$= \frac{2h}{3l^2}\left[-\frac{3l^2}{n\pi}\cos\frac{n\pi}{3} + \frac{9l^2}{n^2\pi^2}\sin\frac{n\pi}{3} + \frac{3l^2}{n\pi}\cos\frac{2n\pi}{3} - \frac{18l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{3l^2}{n\pi}\cos\frac{n\pi}{3}\right.$$
$$\left. + \frac{18l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \frac{3l^2}{n\pi}\cos\frac{2n\pi}{3} - \frac{9l}{n^2\pi^2}\sin\frac{2n\pi}{3}\right]$$

$$= \frac{2h^2}{3l^2}\times\frac{27l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3}-\sin\frac{2n\pi}{3}\right) = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\sin\left(n\pi-\frac{n\pi}{3}\right)\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3}-\left\{\sin n\pi\cos\frac{n\pi}{3}-\cos n\pi\sin\frac{n\pi}{3}\right\}\right] = [1+(-1)^n]\left(18\times\frac{h}{n^2\pi^2}\right)\sin\frac{n\pi}{3}$$

$\therefore$ If n is odd and equal to $2m-1$, then $\qquad E_n = 0$

and if n is even and equal to $2m$, then

$$E_n = E_{2m} = 2(18h/4m^2\pi^2)\sin(2m\pi/3) = (9h/m^2\pi^2)\sin(2m\pi/3)$$

Putting these value in (6), the required deflection is given by

$$\therefore \qquad y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2mkct}{3l}\sin\frac{2m\pi x}{3l} \qquad \text{... (8)}$$

In order to get the displacement of the mid-point of the string, we put $x = 3l/2$ in (8) and obtain $y(3l/2, t) = 0$. This shows that the middle point of the string is always at rest.

Verify that the pfaffian differential equation yz dx+ (x^2y-zx) dy +(x^2z-xy) dz = 0 is integrable and hence find its integral.

Solution

We have

$$yzdx + (x^2y - zx)dy + (x^2z - xy)dz = 0$$

General form of the Pfafffian equation is

$$Pdx + Qdy + Rdz = 0$$

The integrability condition for the Pfaffian equation is [1, page 384]

$$(\text{curl}F, F) = 0$$

where $F = (P, Q, R)$, or

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Verify this condition for the given equation. We get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) =$$

$= yz(-x + x) + (x^2y - zx)(2xz - y - y) + (x^2z - xy)(z - 2xy + z) =$

$= 2x(xy - z)(xz - y) + x(xz - y)2(z - xy) = 0$

The integrability condition for this equation hold.

If the Pfaffian equation is multiplied by a certain function $\mu(x, y, z)$ then one can obtain in the left-hand side the total differential

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0$$

That gives the solution of the Pfaffian equation $u = const$

multiply the original equation by $\frac{1}{x^2}$. We get

$$\frac{yz}{x^2}dx + \left(y - \frac{z}{x}\right)dy + \left(z - \frac{y}{x}\right)dz = 0$$

$$\left(\frac{yz}{x^2}dx - \frac{z}{x}dy - \frac{y}{x}dz\right) + ydy + zdz$$

$$d\left(-\frac{yz}{x}\right) + d\left(\frac{y^2}{2}\right) + d\left(\frac{z^2}{2}\right) = 0$$

$$d\left(-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2}\right) = 0$$

Finally we get solution

$$-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Answer: The differential equation is integrable. The integral of the original equation is

$$-\frac{yz}{x} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Example 6. *Verify that the differential equation*

$$(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$$

is integrable and find its primitive.

First of all to verify the integrability we note that in this case

$$\mathbf{X} = (y^2 + yz, xz + z^2, y^2 - xy)$$

so that

$$\text{curl } \mathbf{X} = 2(-x + y - z, y, -y)$$

and it is readily verified that

$$\mathbf{X} \cdot \text{curl } \mathbf{X} = 0$$

If we treat z as a constant, the equation reduces to

$$\frac{dx}{x + z} + \frac{dy}{y} - \frac{dy}{y + z} = 0$$

which has solution $U(x,y,z) = c_1$, where

$$U(x,y,z) = \frac{y(x + z)}{y + z}$$

Now

$$\mu = \frac{1}{P}\frac{\partial U}{\partial x} = \frac{1}{y(y + z)}\frac{y}{y + z} = \frac{1}{(y + z)^2}$$

and, in the notation of equation (12),

$$K = \frac{1}{(y + z)^2}\, y(y - x) - \frac{y}{y + z} + \frac{y(x + z)}{(y + z)^2} = 0$$

Since $K = 0$, equation (11) reduces to the simple form $dU = 0$ with solution $U = c$; i.e., the solution of the original equation is

$$y(x + z) = c(y + z)$$

where c is a constant.

Ex. 3. *Find the deflection $u(x, t)$ of the vibrating string (length) $l=\pi$, ends fixed, and $c^2=1$) corresponding to zero initial velocity and initial deflection*

$$f(x)=k(\sin x-\sin 2x).$$

Solution. Deflection $u(x, t)$ of the string is given by

$$u(x, t)=\sum_{n=1}^{\infty} c_n \cos nct \sin nx \qquad \text{since } l=\pi$$

$$=\sum_{n=1}^{\infty} c_n \cos nt \sin nx \qquad \text{as } c=1$$

where $$c_n=\frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

$$=\frac{2}{\pi}\int_0^{\pi} k(\sin x-\sin 2x) \sin nx\, dx$$

$$=\frac{2k}{\pi}\int_0^{\pi} \sin x \sin nx\, dx-\frac{2k}{\pi}\int_0^{\pi} \sin 2x \sin nx\, dx.$$

It is obvious that $c_3=c_4=\ldots=0$ (each)

and $$c_1=\frac{2k}{\pi}\int_0^{\pi} \sin^2 x\, dx-\frac{2k}{\pi}\int_0^{\pi} \sin 2x \sin x\, dx$$

$$=\frac{2k}{\pi}\int_0^{\pi} \tfrac{1}{2}(1-\cos 2x)\, dx-\frac{k}{\pi}\int_0^{\pi} (\cos x-\cos 3x)\, dx$$

$$=\frac{k}{\pi}.\pi=k.$$

Also $$c_2=\frac{2k}{\pi}\int_0^{\pi} \sin x \sin 2x\, dx-\frac{2k}{\pi}\int_0^{\pi} \sin^2 2x\, dx$$

$$=\frac{k}{\pi}\int_0^{\pi} (\cos x-\cos 3x)\, dx-\frac{k}{\pi}\int_0^{\pi} (1-\cos 4x)\, dx$$

$$=0-\frac{k}{\pi}.\pi=-k.$$

Hence $u(x, t)=c_1 \cos t \sin x+c_2 \cos 2t \sin 2x$
$=k(\cos t \sin x-\cos 2t \sin 2x)$. **Ans.**

Ex. 18. *Show that a surface of revolution satisfying the differential equation* $r = 12x^2 + 4y^2$ *and touching the plane* $z = 0$ *is* $z = (x^2 + y^2)^2$.

[Agra 88; Meerut 93]

Sol. The given differential eqn. is

$$r = \frac{\partial^2 z}{\partial x^2} = 12x^2 + 4y^2$$

Integrating partially w.r.t. 'x', $\dfrac{\partial z}{\partial x} = p = 4x^3 + 4xy^2 + f(y)$ ....(1)

Again integrating partially w.r.t. 'x',

$$z = x^4 + 2x^2y^2 + xf(y) + F(y) \qquad \text{........(2)}$$

$\therefore$ The surface touches the plane $z = 0$

$\therefore$ for $z = 0$, $\dfrac{\partial z}{\partial x} = p = 0 \qquad \therefore \; 4x^3 + 4xy^2 + f(y) = 0$

But $4x^3 + 4xy^2$ is not a function of y only,

$\therefore$ we have $f(y) = 0$.

$\therefore$ $4x^3 + 4xy^2 = 0$ or $y^2 = -x^2$ ....(3)

Putting $z = 0$ and $x^2 = -y^2$ in (2), we get

$$0 = y^4 - 2y^4 + 0 + F(y). \qquad \therefore F(y) = y^4.$$

Hence from (2), the required surface is

$$z = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2.$$

Ex. 27. *Solve* $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$.

[Meerut 2005, 07, 11; Garhwal 1996; Karnataka 2001, 06: Rajasthan 2005; Vikram 2001]

Sol. We have, $y^2 + yz + z^2 = (y + z)^2 - yz = (y + z)^2 + x(y + z) - (xy + yz + zx)$

$$= (y + z)(x + y + z) - (xy + yz + zx).$$

Similarly, $z^2 + zx + x^2 = (z + x)(x + y + z) - (xy + yz + zx)$

and $x^2 + xy + y^2 = (x + y)(x + y + z) - (xy + yz + zx)$.

Using these new forms of $y^2 + yz + z^2$ etc, the given equation becomes

$(x + y + z)\{(y + z)dx + (z + x)dy + (x + y)dz\} - (xy + yz + zx)(dx + dy + dz) = 0$

or $(x + y + z)\{(ydx + xdy) + (ydz + zdy) + (zdx + xdz)\} - (xy + yz + zx)(dx + dy + dz) = 0$

or $(x + y + z)\, d(xy + yz + zx) = (xy + yz + zx)\, d(x + y + z)$

or $$\frac{d(xy + yz + zx)}{xy + yz + zx} = \frac{d(x + y + z)}{x + y + z}.$$

Integrating, $\log(xy + yz + zx) = \log(x + y + z) + \log c$

or $xy + yz + xz = c(x + y + z)$, c being an arbitrary constant.

Ex. 26. Find complete integral of $xp - yq = xq\, f(z - px - qy)$.

Sol. Let $\quad F(x, y, z, p, q) = xp - yq - xq\, f(z - px - qy) = 0$. ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x + p(\partial F/\partial z)} = \frac{dq}{\partial F/\partial y + q(\partial F/\partial z)} = \frac{dz}{-p(\partial F/\partial p) - q(\partial F/\partial q)} = \frac{dx}{-(\partial F/\partial p)} = \frac{dy}{-(\partial F/\partial q)}$$

or
$$\frac{dp}{p - qf + xqpf' - pqxf'} = \frac{dq}{-q + xq^2 f' - xq^2 f'} = \cdots, \text{ by (2)} \qquad ...(3)$$

Each ratio of (3) $= \dfrac{x\,dp + y\,dq}{xp - yq - qxf} = \dfrac{x\,dp + y\,dq}{0}$, by (2)

$\Rightarrow \quad x\,dp + y\,dq = 0 \quad \Rightarrow \quad x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow \quad dz - d(xp) - d(yq) = 0$, as $dz = pdx + qdy$

Integrating, $\quad z - xp - yq = \text{constant} = a$, say ...(4)

$\therefore \quad xp + yq = z - a$. ...(5)

Using (4), (1) becomes $\quad x\,p - y\,q = x\,q\,f(a)$. ...(6)

Subtracting (6) from (5), $2yq = z - a - xqf(a) \quad \Rightarrow \quad q = (z - a)/\{2y + xf(a)\}$ ...(7)

Using (7), $\quad$ (5) $\Rightarrow \quad p = \dfrac{(z-a)\{y + xf(a)\}}{x\{2y + xf(a)\}}$. ...(8)

Using (7) and (8), $\quad dz = p\,dx + q\,dy$ reduces to

$$dz = (z - a)\left[\frac{\{y + xf(a)\}\,dx}{x\{2y + xf(a)\}} + \frac{dy}{2y + xf(a)}\right]$$

or
$$\frac{2dz}{z-a} = \frac{2y\,dx + 2xf(a)dx + 2x\,dy}{x\{2y + xf(a)\}} = \frac{2d(xy) + 2xf(a)\,dx}{2xy + x^2 f(a)}.$$

Integrating, $2\log(z - a) = \log\{2xy + x^2 f(a)\} + \log b \quad$ or $\quad (z - a)^2 = b\,\{2xy + x^2 f(a)\}$.