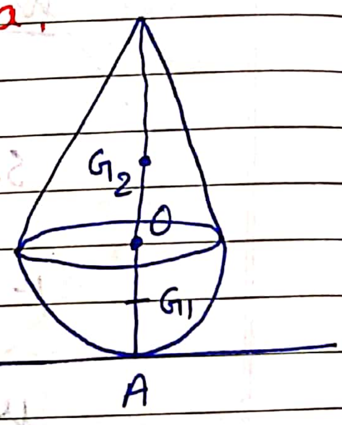
IFoS 2014 7(a)

2019 / 6(a) / 15m

2. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius 'a'. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.



Let us first try to find out the C.G. of the whole body:

As we know C.G. of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its flat base, where 'a' is radius of sphere.

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

w_1 = weight of hemisphere

$$= \frac{2}{3}\pi a^3 \rho g$$

x_2 = distance of centre of gravity of cone from table

$$= AO + OG_2 = a + \frac{H}{4}$$

w_2 = weight of cone

$$= \frac{1}{3}\pi a^2 H \rho g, \quad H - \text{Height of cone}$$

h = distance of C.G. of combined body from horizontal plane.

$$= \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$$

$$= \frac{\frac{2}{3}\pi a^3 \rho g \cdot \frac{5a}{8} + \frac{1}{3}\pi a^2 H \rho g \left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H \rho g}$$

$$= \frac{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)}{2a + H}$$

$$h = \frac{5a^2 + H(4a + H)}{4(2a + H)}$$

Let R = radius of lower surface $= \infty$
r = radius of upper surface $= a$

For stable equilibrium, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$\frac{4(2a+H)}{5a^2 + H(4a+H)} > \frac{1}{a} + \frac{1}{\infty}$$

$$a(8a + \cancel{4H}) > 5a^2 + \cancel{4aH} + H^2$$

$$3a^2 > H^2$$

or $\boxed{H < \sqrt{3}a}$

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7(c) Solve the D.E.

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

(10)

$$(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$$

Auxiliary Eqn:

$$D^3 - 3D^2 + 4D - 2 = 0$$

$$(D-1)(D^2 - 2D + 2) = 0$$

$$D = 1, \frac{2 \pm \sqrt{4-8}}{2} \quad \text{i.e. } D = 1, 1 \pm i$$

$$C.F. = c_1 e^x + e^x(c_2' \cos x + c_3' \sin x)$$

$$= c_1 e^x + c_2 e^x \cos(x + c_3)$$

$$P.I. = \frac{1}{D^3 - 3D^2 + 4D - 2}(e^x + \cos x)$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} e^x + \frac{1}{D(-1^2) - 3(-1^2) + 4D - 2} \cos x$$

$$= \frac{x \cdot e^x}{3 - 6 + 4} + \frac{1}{(3D+1)} \cdot \frac{3D-1}{3D-1} \cos x$$

$$= x e^x + \frac{1}{9D^2 - 1}(3D-1)(\cos x)$$

$$= x e^x + \frac{1}{9(-1^2) - 1}(-3\sin x - \cos x)$$

$$= x e^x + \frac{1}{10}(3\sin x + \cos x)$$

$\therefore$ Complete Solution: $y = CF + PI$

$$\boxed{y = c_1 e^x + c_2 e^x \cos(x + c_3) + x e^x + \frac{1}{10}(3\sin x + \cos x)}$$