IMS
MATHS
BOOK-03

* REAL NUMBER SYSTEM *

→ The set of natural numbers
$N = \{1, 2, 3, ----\}$

→ The set of whole numbers
$W = \{0, 1, 2, 3 ----\}$

→ The set of integers
$I = \{ ---- -3, -2, -1, 0, 1, 2, 3 ---\}$

→ The set of +ve integers
$I^+ = \{1, 2, 3 ----\}$

→ The set of -ve integers
$I^- = \{----, -3, -2, -1\}$

→ The set of rational numbers
$Q = \left\{\frac{p}{q} \,/\, p, q \in I,\ q \neq 0\right\}$

→ The set of irrational numbers
Q' = the numbers which cannot be expressed in the form of p/q ($q \neq 0$) are known as irrational numbers.

Ex: $\sqrt{2}, \sqrt{5}, e, \pi$ etc.

Note:

→ (1). The rational number can be expressed either as a terminating decimal (or) non-terminating recurring decimal.

→ (2). An irrational number can be expressed as non-terminating non-recurring decimal.

The set of real numbers $\mathbb{R} = Q \cup Q'$ i.e. the set of real numbers $\mathbb{R}$ which contains the set of rational and irrational numbers.

Note:

(1). $N \subset W \subset I \subset Q \subset \mathbb{R}$ and $Q' \subset \mathbb{R}$.

(2). Between any two distinct consequtive integers, there exists no integer.

(3) Between any two distinct rational numbers, there lie infinitely many rational numbers.

(4) Between any two rational numbers there lie infinitely many irrational numbers.

(5) Between any two irrational numbers there lie infinitely many irrational numbers as well as infinitely many rational numbers.

(6) Between any two real numbers there lie infinitely many real numbers.

Note: The symbols $\exists$ and $\forall$ are known as Quantifies and the symbols $\Rightarrow, \Leftrightarrow$ as connectives.

→ Some important Properties of real numbers in the form of axioms. These axioms can be divided into three types:

1. Field axioms 2. order axioms

3. Completeness axiom.

(1) Field Axioms:

Let IR be the set of real numbers then the algebraic structure (IR, +, ·) satisfies the following axioms.

(I) (IR, +) is an abelian group.

i.e. (i) Closure Property:

$\forall\ a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$

(ii) Associative Property:

$\forall\ a, b, c \in \mathbb{R} \Rightarrow a+(b+c) = (a+b)+c$

(iii) Existence of identity:

$\forall\ a \in \mathbb{R}, \exists\ 0 \in \mathbb{R}$ such that

$$a+0 = 0+a = a$$

the real number '0' is Called The additive identity of IR.

(iv) Existence of Inverse:

$\forall\ a \in \mathbb{R}, \exists\ b \in \mathbb{R}$ such that

$$a+b = 0 = b+a$$

The real number 'b' is called the additive inverse of 'a'.

(V) Commutative Property:

$\forall\ a, b \in \mathbb{R} \Rightarrow a+b = b+a$

(II) (IR, ·) is an abelian group.

i.e. (i) Closure Property:

$\forall\ a, b \in \mathbb{R} \Rightarrow a \cdot b \in \mathbb{R}$

(ii) Associative Property:

$\forall\ a, b, c \in \mathbb{R} \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(iii) Existence of Identity:

$\forall\ a \in \mathbb{R}, \exists\ 1 \in \mathbb{R}$ such that

$$a \cdot 1 = 1 \cdot a = a$$

The real number '1' is Called the multiplicative identity of IR.

(iv) Existence of inverse:

$\forall\ a \in \mathbb{R}, a \neq 0, \exists\ b \in \mathbb{R}$ such that $a \cdot b = b \cdot a = 1$

The real number 'b' is Called the multiplicative inverse of 'a' and is denoted by a^{-1}.

(V) Commutative Property:

$\forall\ a, b \in \mathbb{R} \Rightarrow a \cdot b = b \cdot a$.

III. Distributivity :-

Multiplication is distributive with respect to addition in $\mathbb{R}$.

i.e. $\forall\ a, b, c \in \mathbb{R}$

$\Rightarrow a\cdot(b+c) = a\cdot b + a\cdot c$ (L.D.L)

and

$(b+c)\cdot a = b\cdot a + c\cdot a$ (R.D.L)

→ A non-empty set 'S' is said to be field if it possesses the two compositions $+^n$ & $\times^n$ and satisfied all the above axioms.

Ex:- $(\mathbb{Q}, +, \cdot)$ is a field but $(\mathbb{Z}, +, \cdot)$ & $(\mathbb{N}, +, \cdot)$ are not fields.

2. Order axioms :-

The order relation '>' between pairs of real numbers $\mathbb{R}$ satisfies the following axioms.

Let $a, b, c \in \mathbb{R}$; then

O_1 : for $a, b \in \mathbb{R}$, exactly one of the following holds

(i) $a>b$ (ii) $a=b$ and

(iii) $b>a$

which is known as the law of trichotomy.

O_2 : For $a, b, c \in \mathbb{R}$;

$a>b,\ b>c \Rightarrow a>c$

which is known as the law of transitivity.

O_3 :- $\forall\ a, b, c \in \mathbb{R}$;

$a>b \Rightarrow a+c > b+c$

which is known as the monotone property for $+^n$.

O_4 : $\forall\ a, b, c \in \mathbb{R}$;

$a>b$ and $c>0 \Rightarrow ac > bc$

which is known as the monotone property for $\times^n$.

→ A field satisfying the above properties, is called an ordered field.

Hence $(\mathbb{R}, +, \cdot)$ is an ordered field.

Note: $(\mathbb{Q}, +, \cdot)$ is an ordered field.

* Some more definitions :-

→ Less than relation: For $a, b \in \mathbb{R}$

$a<b \Leftrightarrow b>a$.

→ +ve real numbers:

$a \in \mathbb{R}$ is said to be +ve if $a>0$ and is denoted by $\mathbb{R}^+$.

→ **$-$ve real numbers:**

$a \in \mathbb{R}$ is said to be $-$ve if $a<0$ and is denoted by $\mathbb{R}^-$

$\therefore \mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$.

→ If $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^-$ then $a>b$.

→ A real number 'a' is said to be greater than (or) equal to 'b' (i.e. $a \geq b$) if either $a>b$ (or) $a=b$.

→ A real number 'a' is said to be less than (or) equal to 'b' (i.e. $a \leq b$) if either $a<b$ (or) $a=b$.

* Some Properties of order relation: —

→ $a \in \mathbb{R}^+ \Leftrightarrow a>0$ and $a \in \mathbb{R}^- \Leftrightarrow a<0$

→ $\forall a, b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$ i.e. $a>0, b>0$ $\Rightarrow a+b>0$ & $ab>0$

→ $\forall a, b \in \mathbb{R}^- \Rightarrow a+b \in \mathbb{R}^-$ and $ab \in \mathbb{R}^+$.

i.e. $a<0,$ $a+b<0$ & $ab>0$.

→ $a<b$ and $b<c \Rightarrow a<c$

(law of transitivity)

→ $a<b \Leftrightarrow a+c < b+c$ & $a<b$ and $c<0 \Rightarrow ac > bc$.

→ $a<0 \Leftrightarrow -a>0$ &

$a>0 \Leftrightarrow -a<0$.

→ $a>b \Leftrightarrow (a-b)>0$ &

$a<b \Leftrightarrow (a-b)<0$.

→ $a>b \Leftrightarrow -a<-b$

→ $a>0 \Leftrightarrow 1/a>0$.

→ $a>b>0 \Rightarrow \frac{1}{b} > \frac{1}{a} > 0$.

→ $a \neq 0 \Rightarrow a^2 > 0$.

→ $a>b>0 \Rightarrow a^2>b^2$ and

$a<b<0 \Rightarrow a^2>b^2$.

→ The relations $\geq$ and $\leq$ are known as the weak inequalities and the relations $>$ and $<$ are known as the strict inequalities.

* Intervals :—

Intervals are two types:

① finite intervals

② Infinite intervals.

1. ~~Finite~~ Intervals :

~~Let~~ $a, b \in \mathbb{R}$ with $a<b$ then

(i) the set $\{x / x \in \mathbb{R}, a \le x \le b\}$ is called a closed interval and is denoted by $[a,b]$, a and b are called the end ~~points~~ of the interval.

a is called the left end point while b is called the right end point.

Here both the end points a & b belong to the interval.

(ii) the set $\{x / x \in \mathbb{R}; a < x < b\}$ is called an open interval and is denoted by (a,b) or $]a,b[$.

Here both the end points donot belong to the interval.

(iii) the set $\{x / x \in \mathbb{R}, a \le x < b\}$ is called left-half closed interval (or right-half open interval). and is denoted by $[a,b)$ or $[a,b[$.

Here the left end point a belon[gs] to the interval and right end po[int] b doesnot belong to the interv[al]

(iv) the set $\{x / x \in \mathbb{R}, a < x \le b\}$ is called right-half closed interv[al] (or left-half open interval) a[nd] is denoted by $(a,b]$ or $]a,b]$

Note:— If $a=b$

$(a,a) = \phi$ and $[a,a] = \{a\}$.

2. Infinite intervals:

Let $a \in \mathbb{R}$ then

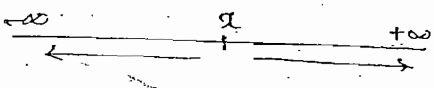
(i) the set $\{x / x \in \mathbb{R}, x \ge a$ i.e. $a \le x\}$ is called closed right ray a[nd] is denoted by $[a, \infty)$.

(ii) The set $\{x / x \in \mathbb{R}, a < x\}$ is called open right ray and is denoted by (a, ∞).

(iii) The set $\{x / x \in \mathbb{R}, x \le a\}$ is calle[d] closed left ray and is denoted $(-\infty, a]$.

(iv) The set $\{x / x \in \mathbb{R}, x < a\}$ is called open left ray and is denoted by $(-\infty, a)$.

(v) The set $\{x \mid x \in \mathbb{R}\}$ is also called an interval and has no end points. It is denoted by $(-\infty, \infty)$.



→ Length of an interval :-

For each interval whose end points are any real numbers a & b such that $a<b$, the length of the interval is $b-a$.

Obviously the length of each of the intervals $[a,b]$, $]a,b[$, $]a,b]$ and $[a,b[$ is $b-a$. These intervals are called finite intervals because the length of each of them is finite.

The intervals $[a,\infty[$, $]a,\infty[$, $]-\infty,a]$, $]-\infty,a[$ and $]-\infty,\infty[$ are called infinite intervals because the length of each of them is infinite.

Note :- (i) Every interval is an infinite set but every infinite set need not be an interval.

Ex :- ① $\mathbb{N}$ is not an interval
② $\mathbb{Z}$ is not an interval
③ $\mathbb{Q}$ is not an interval
④ $\mathbb{R}-\mathbb{Q}$ is not an interval
⑤ ϕ, $\mathbb{R}$ sets are intervals.

2. A finite interval is also an infinite set because the word finite only signifies that the length of the interval is finite.

3. A ray is an infinite interval.

System ($\mathbb{R}^*$):—

To extend the real number system by adjoining two "ideal points" denoted by $+\infty$ and $-\infty$. The enlarged set is called the set of extended real numbers.

Note: $\mathbb{R}$ is denoted by $(-\infty, \infty)$ and $\mathbb{R}^*$ by $[-\infty, \infty]$

If $x \in \mathbb{R}$ then $-\infty < x < \infty$,

$x + \infty = \infty + x = -x + \infty = \infty - x = \infty$;

$x - \infty = -\infty + x = -\infty - x = -x - \infty = -\infty$;

$\frac{x}{\infty} = 0$

$\frac{\infty}{x} = \infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$

Further $\infty \times \infty = (-\infty) \times (-\infty) = \infty + \infty = \infty$

$\infty \times (-\infty) = (-\infty) \times \infty = -\infty - \infty = -\infty$

The following combinations are meaningless.

$\infty - \infty$, $-\infty + \infty$, $0 \times \infty$, $\infty \times 0$, $\frac{\infty}{\infty}$

Bounds of Set:-

Lower bound of a subset of $\mathbb{R}$:

Let S be a non-empty subset of $\mathbb{R}$. If there exists a number $u \in \mathbb{R}$ such that $u \leq x$ $\forall x \in S$ then u is called a lower bound of S.

Ex: (1) $N = \{1, 2, 3, \dots\} \subseteq \mathbb{R}$.

$1 \leq x$ $\forall x \in N$.

$\therefore$ 1 is called the lower bound of N.

(2) The set $S = \{0, 1, 2, 3, \dots\} \subseteq$

$0 \leq x$ $\forall x \in S$

$\therefore$ 0 is the lower bound

Bounded below set:

A non-empty subset S of $\mathbb{R}$ (i.e, $S \subseteq \mathbb{R}$) is said to be bounded below if it has lower bound.

Ex: (1) $S = \{1^2, 2^2, 3^2, 4^2, \dots\} \subseteq \mathbb{R}$ is bounded below.

Since 1 is lower bound.

(2) $\mathbb{R}^+ = \{x / x > 0\} = (0, \infty)$ is bounded below.

Since 0 is lower bound & of

(3) $S = \{x / x \geq 0\} = [0, \infty)$ is bounded below.

Since 0 is lower bound & of

Note: If u is lower bound of S then every real number smaller than u is also a lower bound of S. i.e, if a set S is bounded below then the set of all such bounds of S is infinite.

Greatest lower bound (glb) or infimum:

Let S be a non-empty subset of $\mathbb{R}$. If a set 'S' is bounded below and if the set of all lower bounds of 'S' has a greatest member, say 't' then 't' is called greatest lower bound or infimum of 'S'.

(or)

If 't' is a lower bound of S and any real number greater than t is not lower bound of S then t is called the greatest lower bound or infimum of S.

(or)

If S is bounded below, then a number t is said to be greatest lower bound or infimum of S if it satisfies the conditions

1. t is lower bound of S and
2. if w is any lower bound of S then $w \leq t$.

Ex:- $S = \{\underbrace{1, 2, 3, \ldots}_{x}\} \subseteq \mathbb{R}$.

Since $-1 < x \ \forall \ x \in S$.

$\therefore -1$ is lower bound of S but -1 is not greatest lower bound of S.

Since $0 < x \ \forall \ x \in S$

$\therefore 0$ is a lower bound of S, but 0 is not greatest lower bound of S.

Since $0.9 < x \ \forall \ x \in S$

$\therefore 0.9$ is a lower bound of S.
but $.9$ is not greatest lower bound of S.

$1 \leq x \ \forall \ x \in S$

$\therefore 1$ is a lower bound of S. and is greatest lower bound of S.

because, the greatest of all lower bounds of S is 1.

Note :- If t is infimum of S then for each $\epsilon > 0$ (however small), the number $t + \epsilon$ is not a lower bound of S, there exists atleast one member $x \in S$ such that $t \leq x < t + \epsilon$.

Upper bound :- Let S be a non-empty subset of $\mathbb{R}$. If there exists a number $v \in \mathbb{R}$ such that

$x \leq v\ \forall x \in S$ then v is called an upper bound of S.

Ex:- $S = \{\ldots -3, -2, -1\} \subseteq \mathbb{R}$

$x \leq -1\ \forall x \in S$

$\therefore -1$ is called the upper bound of S.

Bounded Above set:-

A non-empty subset S of $\mathbb{R}$ (i.e. $S \subseteq \mathbb{R}$) is said to be bounded above if it has an upper bound.

Ex:- (1) $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$ is bounded above

Since 0 is an upper bound and $0 \notin \mathbb{R}^-$

(2). $S = \{x \in \mathbb{R} : x \leq 0\} = (-\infty, 0]$ is bounded above.

Since 0 is an upper bound and $0 \in S$.

Note:- If v is an upper bound of a set 'S' then every real number greater than v is also an upper bound of S. i.e. If a set 'S' is bounded above then set of all such numbers that are upper bounds of S is infinite.

Least ...

Supremum :-

Let S be a non-empty subset of $\mathbb{R}$. If a set 'S' is bounded above and if the set of all upperbounds of S has a least member say 't_1' then 't_1' is called least upper bound (or) supremum of S.

(or)

If t_1 is an upper bound of S and any real number less than t_1 is not an upper bound of S then t_1 is called least upper bound (or) supremum of S.

(or)

If S is bounded above then a number t_1 is said to be least upper bound supremum of S if it satisfies the following conditions.

(1) t_1 is an upper bound of S and

(2) If ω_1 is any upper bound of S then $t_1 \leq \omega_1$.

Ex:- $S = \{\ldots 48, 49, 50\} \subseteq \mathbb{R}^-$

Since $x < 51\ \forall x \in S$

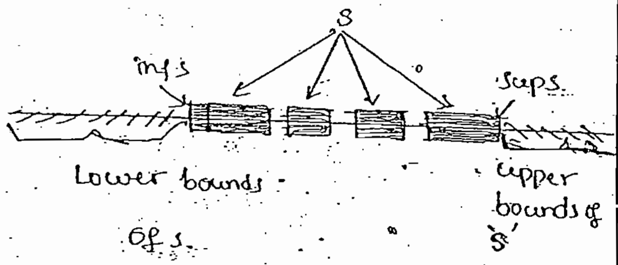
$\therefore 51$ is an upper bound of S but is not supremum of S.

Since $x < 50.5\ \forall x \in S$

$\therefore 50.5$ is an upper bound of S but is not supremum of S.

$x \leq 50\ \forall x \in S$

Note :— If t_1 is supremum of S then for each $\epsilon > 0$ (however small) the number $t_1 - \epsilon$ is not an upper bound of S, there exists at least one member $x \in S$ such that $t_1 - \epsilon < x \leq t_1$



→ Find the infimum & supremum of the following sets and also find whether they are belong to set or not.

(1). $S = \{3, 4, 7\} \subseteq \mathbb{R}$

$\therefore \inf S = 3 \in S$, $\sup S = 7 \in S$.

(2) $S = \{\frac{1}{n} / n \in N\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \mathbb{R}$

Since $n \in N$, $n > 0$

$\Rightarrow 0 < \frac{1}{n} \leq 1 \quad \forall n \in N.$

$\Rightarrow 0 < x \leq 1, \forall x \in S$

$\therefore \inf S = 0 \notin S$ &

$\sup S = 1 \in S$.

(3). $S = \{-\frac{1}{n} / n \in N\} = \{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\} \subseteq \mathbb{R}$

Since $n \in N$; $n > 0$

$\Rightarrow 0 < \frac{1}{n} \leq 1$

$\Rightarrow -1 \leq -\frac{1}{n} < 0.$

$\therefore \inf = -1 \in S$ & $\sup = 0 \notin S$.

(4) $S = \{\frac{1}{3^n} / n \in N\} = \{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\}$

~~$\sup = \frac{1}{3} \in S$, $\inf = 0 \notin S$.~~

(5). $S = \{\frac{(-1)^n}{n} / n \in N\}$

$= \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\}$

$= \{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots, \frac{1}{2}, \frac{1}{4}, \ldots\}$

$\inf = -1 \in S$; $\sup = \frac{1}{2} \in S$.

(6) $S = \{a + \frac{1}{n} / n \in N\} = \{a+1, a+\frac{1}{2}, \ldots\}$

Since $n \in N$, $n > 0$

$\Rightarrow 0 < \frac{1}{n} \leq 1$

$\Rightarrow a < a + \frac{1}{n} \leq a+1$

$\therefore \inf = a \notin S$; $\sup = a+1 \in S$

(7) $S = \{1\}$

$\inf = \sup = 1 \in S$

(8) $S = \{\frac{3n+2}{2n+1} / n \in N\} \subseteq \mathbb{R}$

$= \{5/3, 8/5, \ldots\}$

$\sup = 5/3 \in S$; $\inf = \text{Lt } S$

$= 3/2 \notin S$.

(9). $S = \{\frac{1}{5n} / n \in Z ; n \neq 0\}$

$= \{\pm\frac{1}{5}, \pm\frac{1}{10}, \pm\frac{1}{15}, \ldots\}$

$\inf = -\frac{1}{5} \in S$; $\sup = \frac{1}{5} \in S$.

(10). $S = \{2^n / n \in N\} = \{2^1, 2^2, 2^3, ----\}$

$\inf = 2 \in S$; $\sup = \underset{n\to\infty}{Lt}\ 2^n = \infty$.

$\therefore$ Supremum doesnot exist.

(11). $S = \{1 - 1/n \,/\, n \in N\}$

(12). $S = \{x / -5 < x < 3\}$

(13). $S = \{x / x = (-1)^n ; n \in N\}$

(14) $S = \{x / x = (-1)^n . n ;\ \forall n \in N\}$

$= \{-1, 2, -3, 4, -5, 6, -7, ----\}$

$= \{-1, -3, -5, ---; 2, 4, 6, ---\}$

$\therefore$ inf S = does not exist, &

sup S = doesnot exist.

* **Bounded Subset 'S' of real Numbers :—**

A subset 'S' of IR is said to be bdd if it is bdd below as well as bdd above.

i.e. A set 'S' is bdd iff there exist two real numbers u, v such that $u \le x \le v\ \forall x \in S$.

i.e. $x \in [u, v]$; $x \in S$

i.e. $S \subseteq [u, v]$

i.e. S is a subset of [u, v].

Ex:- (1). Every finite set is bdd and it has inf & sup.

(2). ...

inf & Sup of ϕ does not exist

Because :

The null set ϕ is bdd above. 'u' is any real number then 'u' i[s] an upper bound for ϕ obviously. condition $x \le u$ for all $x \in \phi$ is vaccuously satisfied because ϕ h[as] no elements.

Thus every real number is an upper bound for ϕ. Since the s[et] of all real numbers has no small[est] member.

$\therefore$ Sup ϕ does not exist

Similarly inf ϕ doesnot exist.

(3). $N = \{1, 2, 3, ----\}$ is bdd below but not bdd above.

(4). $IR^+ = \{x \in IR / x > 0\}$ is bdd below but not bdd above.

(5). $IR^- = \{x \in IR / x < 0\}$ is bdd above but not bdd below.

* **Greatest & Least members of a subset of IR :—**

If the supremum of a subset 'S' of IR is a member of S (i.e. 'S' attains its supremum) th[en] this supremum is called greatest

→ If the inf of a subset 's' of $\mathbb{R}$ is a member of 's' (i.e. 's' attains its infimum) then this infimum is least member of 's'.

→ Ex:- (1) $S =]2,3[$

i.e. $S = \{x / 2 < x < 3\}$

$\sup S = 3 \notin S$ & $\inf S = 2 \notin S$.

∴ 3 is not greatest member & 2 is not least member.

(2.) $S = [2,3]$

i.e. $S = \{x \mid 2 \le x \le 3\}$

(3). $S = [1,2)$ i.e. $S = \{x / 1 \le x < 2\}$

(4). $S = (1,2]$ i.e. $S = \{x / 1 < x \le 2\}$

(5). The unbounded intervals are

$[a, \infty)$, $]a, \infty[$

$]-\infty, a]$, $]-\infty, a[$, $]-\infty, \infty[$

Now suppose

$S = [a, \infty) = \{x / a \le x\}$

inf $= a \in S$ & sup does not exist.

∴ least member of $S = a$.

* Note :- (1). Every finite set has two bounds

(2) Every infinite set may or may not have bounds.

(3). The bounds of a set may or may not belong to the set.

(4). Supremum & infimum of a bounded set need not belong to the set.

(5). Every greatest member of a set 's' is the supremum of 's' but every sup. of S need not be the greatest member of S.

(6). Every least member of a set 's' is the infimum of 's' but every inf of 'S' need not be the least member. s.

* Completeness property of $\mathbb{R}$ (or Completeness axiom):—

Every non-empty [illegible] real [illegible] above [illegible] supremum. [illegible] bounded)

i.e. If S is any non-empty subset of $\mathbb{R}$ which is bounded above, then the set of all upper bounds of S must have smallest member i.e. S must possess the least upper bound which is a member of $\mathbb{R}$.

This property of real numbers is known as completeness.
(This property is also called the supremum property of $\mathbb{R}$).

(or)

→ Every non-empty subset of real numbers which is bounded below has the infimum (or glb) in $\mathbb{R}$. This property of real numbers is known as completeness. This property is also called infimum property of $\mathbb{R}$.

* Complete Ordered Field:-

An ordered field F is said to be a complete ordered field if every non-empty subset S of F (i.e. $S \subseteq F$) which is bounded above has the supremum (or least upper bound) in F.

Ex:- The set $\mathbb{R}$ of real numbers is complete ordered field.

Because $\mathbb{R}$ satisfies
(1) field axioms
(2) order axioms and
(3) completeness axiom.

Ex:-(2). The set Q of rational numbers is an ordered field but not completeness.

→ Now we shall show that the ordered field of rational numbers is not a complete ordered field.

For this we are enough to show that there exists a non-empty subset of Q which is bounded above but which does not have a supremum in Q.

i.e. no rational numbers exists which can be the supremum.

Let us consider the set of all those +ve ~~rational~~ numbers whose squares are less than

i.e. let $S = \{x : x \in Q^{+} \text{ and } 0<x$

Since $1 \in S$ ...

$\therefore S \neq \phi$

i.e. S is non-empty.

Clearly 2 is an upper bound of S.

$\therefore$ S is bounded above.

$\therefore$ S is a non-empty subset of Q and is bounded above.

If possible suppose that the rational number k be its least upper bound.

Clearly k is +ve.

By law of trichotomy, which holds good in Q one and only one of (i) $k^2<2$ (ii) $k^2=2$ (iii) $k^2>2$ holds.

(i) $k^2<2$

Let us consider the +ve rational number $y = \dfrac{4+3k}{3+2k}$

then $k-y = k-\left(\dfrac{4+3k}{3+2k}\right)$

$[S=\{1/n \mid n\in N\} = \{x \mid x\in Q^+, 0<x\le 1\} \subseteq Q]$

$$= \frac{2(k^2-2)}{3+2k}$$

<0 $(\because k^2<2$ i.e. $k^2-2<0)$

$\therefore \boxed{k<y}$

Also $2-y^2 = 2-\left(\dfrac{4+3k}{3+2k}\right)^2$

$$= \frac{2-k^2}{(3+2k)^2}$$

>0 $(\because k^2<2$ i.e. $2-k^2>0)$

$\therefore 2-y^2>0$

$\Rightarrow y^2<2$

$\Rightarrow \boxed{y\in S}$

$\therefore$ The member y of S is greater than k so that k cannot be an upper bound

$\therefore$ which is contradiction.

(ii) $k^2=2$, we know that there exists no rational number whose square is equal to 2.

$\therefore$ This case is not ~~possible~~.

(iii) $k^2>2$

Let us consider the +ve rational number.

$$y = \frac{4+3k}{3+2k} \quad (>0)$$

then $k-y = k-\left(\dfrac{4+3k}{3+2k}\right)$

$$= \frac{2(k^2-2)}{3+2k}$$

>0 $(\because k^2>2 \Rightarrow k^2-2>0)$

$\therefore k-y>0$

$\Rightarrow \boxed{k>y}$

Also $2-y^2 = 2-\left(\dfrac{4+3k}{3+2k}\right)^2$

$$= \frac{2-k^2}{(3+2k)^2} < 0 \quad (\because 2-k^2<0)$$

$\therefore 2-y^2<0$

$\Rightarrow 2<y^2$

$\Rightarrow \boxed{y^2>2}$.

$\therefore y<k$ & $y^2>2 \Rightarrow y^2<k^2$ & $y^2>2$

$\Rightarrow 2<y^2<k^2$

If x is any member of S then

$0<x^2<2<y^2<k^2$

$\Rightarrow 0<x<y<k$.

which shows k & y are ... upper bounds of S.

But $y < k$

$\therefore$ k cannot be the supremum.

$\therefore$ Since k is any rational number, we conclude that no rational number can be the supremum of S.

* The Archimedian Property :-

If a & b be any two real numbers and if $a > 0$ then there exists a +ve integer n such that $na > b$.

Proof :- Let a & b be any two real numbers and $a > 0$.

Now if possible suppose that for all +ve integers n

i.e. $n \in I^+$, $na \leq b$

Let $S = \{na / n \in I^+\}$ then S is bounded above by b (i.e. b is an upper bound of S).

$\therefore$ By Completeness Property of the ordered field of real numbers, the set S must have a supremum M (say).

$\therefore na \leq M \quad \forall \; n \in I^+$

$\Rightarrow (n+1)a \leq M$

$\Rightarrow na \leq M - a \quad \forall \; n \in I^+$

$\therefore$ $M - a$ is an upper bound of S

The number $M - a$ is less than supremum M (least upper bound) is an upper bound of S.

$\therefore$ which is a contradiction.

$\therefore$ Our supposition is wrong.

Hence theorem.

* Absolute Value (modulus of a real number) :-

If $x \in \mathbb{R}$ then the modulus (or absolute value or numerical value) of x is denoted by $|x|$ and defined as
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties :

Prove that (i) $|x| = \max\{x, -x\}$

(ii) $|x|^2 = x^2$ (iii) $x \leq |x|$ and $-x \leq |x|$

(iv) $|x| = |-x|$.

Proof :- (i) Since $x \in \mathbb{R}$, either $x \geq 0$ or $x < 0$.

If $x \geq 0$ then $|x| = x$ and $x \geq -x$

and if $x \leq 0$, then $|x| = -x$ and $-x > x$

$\therefore$ $|x|$ is greater of two numbers x & $-x$.

$\therefore |x| = \max\{x, -x\}$

(ii) Since $|x| = x$ if $x \geq 0$

$= -x$ if $x < 0$

$\therefore |x|^2 = x^2$ (or) $(-x)^2$

$= x^2$

$\therefore \boxed{|x|^2 = x^2}$

(iii) Since $|x| = \text{Max}\{x, -x\}$

$\therefore |x| \geq x$ or $-x$

$\therefore x \leq |x|$ and $-x \leq |x|$.

(iv) Since $|x| = \text{Max}\{x, -x\}$,

and $|-x| = \text{Max}\{-x, x\} = \text{Max}\{x, -x\}$

$\therefore \boxed{|x| = |-x|}$

Note: $|x|^2 = x^2$

$\Rightarrow |x| = \pm\sqrt{x^2}$

Since $|x| \geq 0$.

$\therefore$ rejecting the $-$ve sign,

we have $\boxed{|x| = \sqrt{x^2}}$

(iv) $|x| = \sqrt{x^2}$

$\therefore |(-x)| = \sqrt{(-x)^2}$

$= \sqrt{x^2}$

$= |x|$

$\therefore \boxed{|x| = |-x|}$

$\rightarrow$ If x & y are any two real numbers

then (a) $|x+y| \leq |x| + |y|$

(b) $|x-y| \geq ||x| - |y||$

(c) $|xy| = |x|\,|y|$

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ if $y \neq 0$.

(a) $|x+y| = \sqrt{(x+y)^2}$

$= \sqrt{x^2+y^2+2xy}$

$\leq \sqrt{x^2+y^2+2|x||y|}$ $(\because x \leq |x|$ & $y \leq |y|)$

$= \sqrt{|x|^2+|y|^2+2|x||y|}$

$= \sqrt{[|x|+|y|]^2}$

$= \big||x|+|y|\big|$ $(\because |x| = \sqrt{x^2})$

$= |x|+|y|$ $(\because |x|+|y| \geq 0$ i.e., $|a| = a$ if $a > 0)$

$\therefore \boxed{|x+y| \leq |x|+|y|}$

(b) $|x-y| = \sqrt{(x-y)^2}$

$= \sqrt{x^2+y^2-2xy}$

$\geq \sqrt{x^2+y^2-2|x||y|}$

$(\because x \leq |x|$ & $y \leq |y|$
$\Rightarrow xy \leq |x||y|$
$\Rightarrow -xy \geq -|x||y|)$

$= \sqrt{|x|^2+|y|^2-2|x||y|}$

$= \sqrt{(|x|-|y|)^2}$

$= \big||x|-|y|\big|$ $(\because |x| = \sqrt{x^2})$

$\therefore \boxed{|x-y| \geq \big||x|-|y|\big|}$

(c) $|xy| = \sqrt{(xy)^2}$

$= \sqrt{x^2y^2}$

$= \sqrt{x^2}\cdot\sqrt{y^2}$

$= |x|\cdot|y|$.

(d) $\left|\frac{x}{y}\right| = \sqrt{\left(\frac{x}{y}\right)^2}$

$= \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$ provided $y \neq 0$.

→ If $r>0$ then Prove that

(a) $|x|<r \Leftrightarrow -r<x<r$

(b) $|x-a|<r \Leftrightarrow a-r<x<a+r$.

Solⁿ:- (a) $|x|<r \Leftrightarrow \text{Max}\{x,-x\}<r$

$\cancel{\Leftrightarrow x<r \text{ and } -x<r}$

$\Leftrightarrow +x<+r \text{ and } x>-r$

$\Leftrightarrow -r<x \text{ and } x<r$

$\Leftrightarrow -r<x<r$

(b) $|x-a|<r \Leftrightarrow \text{Max}\{x-a,-(x-a)\}<r$

$\Leftrightarrow x-a<r \text{ and } -(x-a)<r$

$\Leftrightarrow x<a+r \text{ and } -x<r-a$

$\Leftrightarrow x<a+r \text{ and } x>a-r$

$\Leftrightarrow a-r<x \text{ and } x<a+r$

$\Leftrightarrow a-r<x<a+r$

→ Prove that (i) $|x-y| \le |x|+|y|$

(ii) $a<x<b \Leftrightarrow \left|x-\left(\frac{a+b}{2}\right)\right|<\frac{b-a}{2}$

Proof:- (i) $|x-y| = |x+(-y)|$

$\le |x|+|-y|$

$= |x|+|y|$

$(\because |-y|=|y|)$

$\therefore |x-y| \le |x|+|y|$.

(ii) $a<x<b$

Adding throughout $-\left(\frac{a+b}{2}\right)$, we get

$\Leftrightarrow a-\left(\frac{a+b}{2}\right)<x-\left(\frac{a+b}{2}\right)<b-\left(\frac{a+b}{2}\right)$

$\Leftrightarrow \frac{a-b}{2}<x-\left(\frac{a+b}{2}\right)<\frac{b-a}{2}$

$\Leftrightarrow -\left(\frac{b-a}{2}\right)<x-\left(\frac{a+b}{2}\right)<\left(\frac{b-a}{2}\right)$

$\Leftrightarrow \left|x-\left(\frac{a+b}{2}\right)\right|<\frac{b-a}{2}$

$(\because |x|<r \Leftrightarrow -r<x<r)$

* Neighbourhood of a point:

If a is any real number and $\delta>0$ (however small), then the open interval $(a-\delta, a+\delta)$ is called a δ-neighbourhood of 'a' and is denoted by $N_\delta(a)$ or $N(\delta, a)$ i.e. $N_\delta(a) = (a-\delta, a+\delta)$

— shortly written as ~~neighbourhood~~ nbd.

$a-\delta \qquad a \qquad a+\delta$

$x \in (a-\delta, a+\delta)$

→ If from the neighbourhood of a point, the point itself is excluded we get the deleted neighbourhood of that point.

i.e. $N_\delta(a)-\{a\}$ is a deleted neighbourhood of a point 'a'. and is denoted by $N_\delta d(a)$

i.e. $N_\delta d(a) = N_\delta(a)-\{a\}$.

Ex:- If $a=5$, $\delta=0.2>0$ then $(4.8, 5.2)$ is a neighbourhood of 5.

Now $x\in(4.8, 5.2)-\{5\} \Rightarrow x\in(4.8, 5.2)$, $x\ne 5$ is a deleted

Note:- $x \in N_\delta(a)$

$\Leftrightarrow x \in (a-\delta, a+\delta)$

$\Leftrightarrow a-\delta < x < a+\delta$

$\Leftrightarrow -\delta < x-a < \delta$

$\Leftrightarrow |x-a| < \delta$

($\because |x| < r \Leftrightarrow -r < x < r$)

and $x \in N_\delta d(a)$

$\Leftrightarrow x \in (a-\delta, a+\delta) - \{a\}$

$\Leftrightarrow x \in (a-\delta, a+\delta)$; $x \neq a$

$\Leftrightarrow a-\delta < x < a+\delta$; $x \neq a$

$\Leftrightarrow |x-a| < \delta$; $x \neq a$

$\Leftrightarrow 0 < |x-a| < \delta$

* Neighbourhood of a set:-

→ A subset S of IR (i.e. $S \subseteq IR$) is said be neighbourhood of a point $a \in R$ if there exists a $\delta > 0$ (however small) such that $(a-\delta, a+\delta) \subset S$

Note:- If S is a neighbourhood of a point 'a' then $S - \{a\}$ is called deleted neighbourhood of 'a'.

Ex:- (1) If $a \in IR \subseteq IR$ then IR is a neighbourhood of 'a' because $a \in (a-\delta, a+\delta) \subset IR$.

(2) If $a \in Q \subseteq IR$ then Q is not neighbour hood of a because $a \in (a-\delta, a+\delta) \not\subset Q$.

... is not neighbourhood of 'a' because $a \in (a-\delta, a+\delta) \not\subset Z$

(4). If $a \in N \subset IR$ then N is not neighbourhood of a because $a \in (a-\delta, a+\delta) \not\subset N$.

Problems

→ Any open interval is a neighbourhood of each of its points.

Sol'n :- Let $S = (a,b)$

Let P be any point of (a,b)

i.e. $P \in (a,b)$

$\Rightarrow a < P < b$

Let $\epsilon = \mathrm{Min}\{P-a, b-P\} > 0$

$\Rightarrow \epsilon \leq P-a$; $\epsilon \leq b-P$

$\Rightarrow a \leq P-\epsilon$; $b \geq P+\epsilon$

$\Rightarrow a \leq P-\epsilon < P < P+\epsilon \leq b$

$\Rightarrow P \in (P-\epsilon, P+\epsilon) \subset (a,b)$

$\therefore (a,b)$ is a neighbourhood of P.

→ A closed interval $[a,b]$ is a neighbourhood of each of its points except the two end points a & b.

Sol'n :- Let $S = [a,b]$

Let $P \in [a,b]$

$\Rightarrow a \le P \le b$

$\Rightarrow$ (i) $a < P < b$

(ii) $P = a$ &

(iii) $P = b$

Let $\epsilon = \text{Min}\{P-a,\ b-P\} > 0$

(i) $P \in (P-\epsilon, P+\epsilon) \subset (a,b) \subset [a,b]$

$\therefore P \in (P-\epsilon, P+\epsilon) \subset [a,b]$

$\therefore [a,b]$ is a neighbourhood of P.

i.e. $[a,b]$ is a neighbourhood of each $P \in (a,b)$.

(ii) $P = a$

$\Rightarrow (P-\epsilon, P+\epsilon) = (a-\epsilon, a+\epsilon) \not\subset [a,b]$

$\therefore [a,b]$ is not neighbourhood of a.

(iii) $P = b$

$\Rightarrow (P-\epsilon, P+\epsilon) = (b-\epsilon, b+\epsilon) \not\subset [a,b]$

$\therefore [a,b]$ is not a neighbourhood of b.

$\rightarrow$ $[a,b)$ is a neighbourhood of each of its points except 'a'

$\rightarrow$ $(a,b]$ is a neighbourhood of each of its points except

$\rightarrow$ A non-empty finite set cann be a neighbourhood any of its points

Sol'n :- Let S be any non-empty finite set.

Let P be any point of S

Let $\epsilon > 0$ (however small)

then $(P-\epsilon, P+\epsilon)$ is an infinite set.

$\therefore (P-\epsilon, P+\epsilon) \not\subset S$

$\therefore$ S is not a neighbourhood of

$\rightarrow$ Empty set ϕ is a neighbourhood of each of its points.

Sol'n :- The empty set ϕ is a neighbourhood of each of its poi because there is no point at all in ϕ

and so there is no point in ϕ o which it is not a neighbourhood.

$\rightarrow$ Show that the set 'N' of all natural is not a neighbourhood of any of its points.

Sol'n :- Let $P \in N$

and let $\epsilon > 0$.

then $(p-\epsilon, p+\epsilon)$ contains infinitely many rational and irrational numbers.

$\therefore (p-\epsilon, p+\epsilon) \not\subset N$

$\therefore N$ is not a neighbourhood of any point $p \in N$.

Similarly, the set 'ω' of all whole numbers is not a neighbourhood of any of its points.

- and the set 'I' of integers is not a neighbourhood of any of its points.

→ show that the set Q of all rational numbers is not a neighbourhood of any of its points.

Solⁿ:- Let $p \in Q$ and let $\epsilon > 0$ (however small)

then $(p-\epsilon, p+\epsilon)$ contains infinitely many irrational numbers which are not members of Q.

$\therefore (p-\epsilon, p+\epsilon) \not\subset Q$.

$\therefore$ 'Q' is not a neighbourhood of any point $p \in Q$.

Similarly, the set Q' of all irrational numbers is not nbd of any of its points.

is a nbd of each of its points.

Solⁿ: Let $p \in \mathbb{R}$ and let $\epsilon > 0$.

then $(p-\epsilon, p+\epsilon)$ contains infinitely many real numbers

$p \in (p-\epsilon, p+\epsilon) \subset \mathbb{R}$

$\therefore$ The set $\mathbb{R}$ of real numbers is a nbd of its points.

→ Any set 'S' cannot be a nbd of any point of the set $\mathbb{R}-S$.

Solⁿ: Let $p \in \mathbb{R}-S$

then $p \notin S$.

let $\epsilon > 0$.

$\Rightarrow (p-\epsilon, p+\epsilon) \not\subset S$.

$\therefore S$ is not a nbd of any point $p \in \mathbb{R}-S$.

→ Every superset of nbd of a point 'p' is also nbd of p.

Solⁿ: Let S be a nbd of p.

$\Rightarrow (p-\epsilon, p+\epsilon) \subset S$.

if T is any superset of S, then $S \subset T$.

$\therefore (p-\epsilon, p+\epsilon) \subset S \subset T$

$\Rightarrow (p-\epsilon, p+\epsilon) \subset T$.

$\therefore T$ is a nbd of p.

→ The intersection of two nbds of a point is also a nbd of that point.

Sol^n: Let M_1 and M_2 be two nbds of p.

$\therefore \exists\ \epsilon_1$ and $\epsilon_2 > 0$ (however small) such that

$p \in (p-\epsilon_1, p+\epsilon_1) \subset M_1$ and $p \in (p-\epsilon_2, p+\epsilon_2) \subset M_2$

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

$\therefore (p-\epsilon, p+\epsilon) \subset (p-\epsilon_1, p+\epsilon_1) \subset M_1$

and $(p-\epsilon, p+\epsilon) \subset (p-\epsilon_2, p+\epsilon_2) \subset M_2$

$\therefore p \in (p-\epsilon, p+\epsilon) \subset M_1 \cap M_2$

$\therefore M_1 \cap M_2$ is also a nbd of p.

→ If M_1 is a nbd of p (or) M_2 is a nbd of p then $M_1 \cup M_2$ is also a nbd of p.

Interior point of a set:-

Let $S \subset \mathbb{R}$, $p \in S$ is called an interior point of a set S if S is a nbd of p.

i.e., if $\exists\ \epsilon > 0$ (however small) such that

$(p-\epsilon, p+\epsilon) \subset S$

Ex: (1) Every point of an open interval (a,b) is an interior point of the interval.

(2) Every point of a closed interval $[a,b]$ is an interior point of the interval except the end points a and b.

(3) Every point of a semi closed interval $[a,b)$ is an interior point of the interval

except the left end point 'a'.

→ Every point of a semi open interval $(a, b]$ is an interior point of the interval except the right end point 'b'.

→ Every point of the empty set is an interior point.

→ Every non-empty finite set has no interior point.

→ N has no interior point.

Similarly $W, Z, Q, R-Q$.

→ Every point of a real number set is an interior point of R.

Interior of a set:

→ The set of all interior points of a set 'S' is called interior of a set 'S' and is denoted by S^o (or) Int(S).

Ex: (1) If $S=(a,b)$ then $S^o = S$ because every point of S is an interior point of S.

(2) If $S=[a,b]$ then $S^o=(a,b)$ because every point of S is an interior point of S except the end points a & b.

(3) If $S=[a,b)$ then $S^o=(a,b)$

(4) If $S=(a,b]$ then $S^o=(a,b)$.

(5) $R^o=R$ because every point of R is an interior point.

(6) If S is a non-empty finite set then $S^o=\phi$.

(7) $N^o=\phi$, $Z^o=\phi$, $Q^o=\phi$, $(R-Q)^o=\phi$ & $W^o=\phi$. because $N, Z, Q, R-Q, W$ are not nbd of any points and therefore no point is an interior point of $N, Z, Q, R-Q$ or W.

(8) If $S=\phi$ then $S^o=\phi$.

Find the interior of the sets

(i) $\{1,2,3,4,5\}$ (ii) $[0,1]$ (iii) $[0,1]\cup[3,5]$ (iv) $\{\frac{1}{n}/n\in N\}$

Sol^n: (i) Let $A=\{1,2,3,4,5\}$.

then A is a non-empty finite set.

$\Rightarrow$ A is not nbd of any point.

$\therefore$ no point is an interior point of A.

$\Rightarrow$ $A^0=\phi$.

(iii) Let $A=[0,1]\cup[3,5]$.

then A is nbd of each point of $(0,1)\cup(3,5)$

$\therefore$ $A^0=(0,1)\cup(3,5)$.

(iv) Let $A=\{\frac{1}{n}/n\in N\}$

$=\{1,\frac{1}{2},\frac{1}{3},\ldots\}$

Let p be any point of A

i.e, $p\in A$,

then $\nexists$ no $\epsilon>0$ such that

$(p-\epsilon,p+\epsilon)\subseteq A$.

$\Rightarrow$ p is not an interior point of A.

i.e, A has no interior point

$\therefore$ $A^0=\phi$.

Open set: A subset S of $\mathbb{R}$ is said to be an open set if S is a nbd of each of its points i.e., if for each $p \in S$ $\exists$ an $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subset S$.

(or)

If 'S' is a subset of $\mathbb{R}$ is said to be open if every point of S is an interior point of S

i.e, S is open $\Leftrightarrow$ $S^0 = S$.

Ex: (1) Every open interval is an openset.

Sol: Let $S = (a, b)$

then $S^0 = (a, b)$

$\therefore S^0 = S$.

$\Rightarrow S$ is openset

(2) $S = [a, b]$, then $S^0 = (a, b)$.

$\therefore S \neq S^0$

$\Rightarrow S$ is not an open set.

Similarly $[a, b)$, $(a, b]$ are not opensets.

(3) $S = N$ then $S^0 = \phi$.

$\therefore S \neq S^0$

$\Rightarrow S$ is not open set.

Similarly W, Z, Q and $\mathbb{R} - \bar{Q}$ are not opensets.

(4) $S = \mathbb{R}$. then $S^0 = \mathbb{R}$.

$\therefore S = S^0$

$\Rightarrow S$ is open.

(5) $S = \mathbb{R}^+ = (0, \infty)$ is an open set.

because let $x \in S$

$\exists$ an $\epsilon > 0$ (however small) such

$x \in (x-\epsilon, x+\epsilon) \subset S$

$\therefore S = S^{\circ}$.

(6) $S = \mathbb{R}^{-} = (-\infty, 0)$ is an open set.

(7) Every non-empty-finite set is not an open set.

because for every nbd of a point contains infinitely many points.

(8) $S = \{\ \}$ is an open set.

because $S^{\circ} = \phi$

$\Rightarrow S^{\circ} = S$.

(9) $S = \{\frac{1}{n} \mid n \in N\}$ is not an open set

since $S \neq S^{\circ}$

→ Union of two open sets is an open set.

Sol^n: Let S_1 and S_2 be two open sets.

Let $S = S_1 \cup S_2$.

Let $x \in S \Rightarrow x \in S_1 \cup S_2$

$\Rightarrow x \in S_1$ or $x \in S_2$

If $x \in S_1$ then $\exists$ $\epsilon > 0$ such that

$x \in (x-\epsilon, x+\epsilon) \subset S_1 \subset S_1 \cup S_2$ ($\because S_1$ is open)

If $x \in S_2$ then $\exists$ $\epsilon > 0$ such that

$x \in (x-\epsilon, x+\epsilon) \subset S_2 \subset S_1 \cup S_2$ ($\because S_2$ is open)

$\therefore x \in (x-\epsilon, x+\epsilon) \subset S_1 \cup S_2 = S$

$\therefore x$ is an interior point of $S_1 \cup S_2 = S$.

$\Rightarrow S = S_1 \cup S_2$ is an open set.

→ The union of an arbitrary family of open sets is an open set.

The intersection of two open sets is an open set.

∴ Let S_1 & S_2 are two open sets.

To S.T $S_1 \cap S_2$ is also an openset.

Let $S = S_1 \cap S_2$.

Let $x \in S \Rightarrow x \in S_1 \cap S_2$

$\Rightarrow x \in S_1$ and $x \in S_2$

$\Rightarrow x \in (x-\epsilon_1, x+\epsilon_1) \subseteq S_1$ and

$x \in (x-\epsilon_2, x+\epsilon_2) \subset S_2$

($\because S_1$ & S_2 are two open sets)

Choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$

$x \in (x-\epsilon, x+\epsilon) \subset (x-\epsilon_1, x+\epsilon_1) \subset S_1$ and

$x \in (x-\epsilon, x+\epsilon) \subseteq (x-\epsilon_2, x+\epsilon_2) \subset S_2$

$\Rightarrow x \in (x-\epsilon, x+\epsilon) \subseteq S_1 \cap S_2 = S$

$\therefore S = S_1 \cap S_2$ is an open set.

- The intersection of a finite collection of open sets is an openset.

- The intersection of an infinite collection of opensets need not be an open set.

Sol: (i) Let $S_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$

$\Rightarrow S_1 \cap S_2 \cap S_3 \cap \cdots = (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \cdots$

$= \{0\}$

which is not an openset.

Because $(0-\epsilon, 0+\epsilon) \not\subset \{0\}$.

∴ The intersection of an infinite collection of opensets is not an open set.

(ii) Let $S_n = (0, 1/n)$ ...

Then $S_1 \cap S_2 \cap \cdots = (0,1) \cap (0,2) \cap \cdots$

$= (0,1)$

which is an open set.

$\therefore$ The intersection of an infinite collection of open sets need not be an open set.

Note: Every open interval is an open set. but every open set need not be an open interval.

for example: Let $S_1 = (1,2)$; $S_2 = (3,4)$ are two open sets.

$S_1 \cup S_2 = (1,2) \cup (3,4)$ is an open set

but $(1,2) \cup (3,4)$ is not an open interval.

* Limit point of a subset S of $\mathbb{R}$:-

A point $p \in \mathbb{R}$ is said to be a limit point of a subset S of $\mathbb{R}$, if every nbd of p has a point of S other than p itself.

(or)

A point $p \in \mathbb{R}$ is said to be a limit point of a subset S of $\mathbb{R}$ if every nbd of p has an infinite number of points of 'S'

(or)

A point $p \in \mathbb{R}$ is said to be a limit point of subset 'S' of $\mathbb{R}$ if every nbd of p contains atleast one point of S other than p.

i.e, p is a limit point of $S \Leftrightarrow$

$$(p-\epsilon, p+\epsilon) \cap S - \{p\} \neq \phi.$$

Note: (1) Limit point is also called cluster point (or) condensation point (or) accumulation point.

(2) A limit point of 'S' may or may not belong to the set 'S'.

(3) A set may have no limit point, a unique limit point or a finite or infinite number of limit points.

(4) $p \in \mathbb{R}$ is not a limit point of a subset 'S' of $\mathbb{R}$ if there exists a nbd of 'p' which doesnot contain any point of 'S'.

(5) p is not a limit point of 'S' if for some $\epsilon > 0$, $(p-\epsilon, p+\epsilon) \cap S = \phi$ (or)

$$(p-\epsilon, p+\epsilon) \cap S = \{p\}.$$

A finite set has no limit points.

Sol: Let 'A' be a finite set.

If possible suppose ~~suppose~~ p is a limit point of 'A' also let $\epsilon > 0$.

Then $(p-\epsilon, p+\epsilon)$ contains infinite number of points of 'A'.

$\therefore$ A is infinite.

It is a contradiction.

$\therefore$ 'A' has no limit points

$\therefore$ A finite set 'A' has no limit points

Closure of a set :-

The set of all adherent points of a set 'S' is called the closure of S and is denoted by Cl S or $\bar{S}$

Thus $\bar{S} = S \cup D(S)$.

Dense set:

A subset 'S' of $\mathbb{R}$ is said to be dense (or dense in $\mathbb{R}$ or everywhere dense) if every point of $\mathbb{R}$ is a point of S or a limit point of S or both.

(or)

Let $S \subset \mathbb{R}$ then 'S' is said to be dense if $\bar{S} = \mathbb{R}$

Dense in itself:

A set 'S' is said to be dense-in-itself if every point of S is a limit point of 'S'.

(or)

A subset 'S' of $\mathbb{R}$ is said to be dense-in-itself if $S \subseteq D(S)$

(or)

A subset 'S' of $\mathbb{R}$ is said to be dense-in-itself if it possesses no isolated points.

perfect set:

A set 'S' is said to be perfect set if $S = D(S)$.

(or)

A set 'S' is said to be perfect set if it is dense-in-itself and if it contains all its limit points.

→ The set $\mathbb{Q}$ of rational numbers.

Let $S = \mathbb{Q} \subseteq \mathbb{R}$

Let x be any real number. Then for each $\epsilon > 0$ (however small),

$(x-\epsilon, x+\epsilon)$ is a nbd of x

and it contains infinitely many rational numbers other than 'x'.

i.e $(x-\epsilon, x+\epsilon) \cap \mathbb{Q} - \{x\} \neq \phi$.

$\Rightarrow$ x is a limit point of $S = \mathbb{Q}$

$\Rightarrow$ every real number is a limit point of $\mathbb{Q}$.

Hence the set of the limit points of $\mathbb{Q}$ is the set of all real numbers $\mathbb{R}$

$\therefore \boxed{D(\mathbb{Q}) = \mathbb{R}}$.

also $\bar{S} = S \cup D(S)$

$= \mathbb{Q} \cup \mathbb{R}$

$\boxed{\bar{S} = \mathbb{R}}$

clearly 'S' is dense in $\mathbb{R}$ and $S \subseteq D(S)$

$\therefore S = \mathbb{Q}$ is dense-in-itself

since $S \neq D(S)$

i.e $\mathbb{Q} \neq D(\mathbb{Q})$

$\therefore S = \mathbb{Q}$ is not a perfect set.

→ The set $\mathbb{R} - \mathbb{Q}$ of irrational numbers.

Let $S = \mathbb{R} - \mathbb{Q} \subseteq \mathbb{R}$ then

$D(S) = \mathbb{R}$.

→ The set $\mathbb{N}$ of natural numbers.

Let $S = \mathbb{N} \subseteq \mathbb{R}$

Let $x \in \mathbb{R}$

* Adherent point :-

A real number 'p' is called an adherent point of a set $S \subseteq \mathbb{R}$ if every nbd of p contains a point of S.

i.e point $p \in \mathbb{R}$ is an adherent point of $S \subseteq \mathbb{R}$
$\Leftrightarrow$ for each nbd N of p, $N \cap S \neq \phi$.

Note:- Due to a close resemblance between the definitions of an adherent point of a set and a limit point of a set, the distinction between the two should be carefully noted.

— For a point 'p' to be a limit of a set S, every nbd N of 'p' must contain a point of S other than p.

i.e. $N \cap S - \{p\} \neq \phi$.

— For a point 'p' to be an adherent point of a set S, every nbd of p must contain a point of S which can be 'p' itself.

i.e $N \cap S \neq \phi$.

— If $p \in S$, ~~then~~ 'p' is an adherent point of S, since every nbd of p contains p which belongs to S.

— If $p \in D(S)$ then p is a limit point of S and, therefore, every nbd of 'p' contains a point of S other than p.

Thus p is also an adherent point of S.

Clearly, a real number p is an adherent point $\Leftrightarrow$ either $p \in S$, or $p \in D(S)$.

— Every point of S is an adherent of S.
— Every limit point of S is an adherent point
But an adherent point of S need not be a limit point of S.

Derived set :

The set of all limit points of a subset 'S' of $\mathbb{R}$ is called the derived set of S and is denoted by S' or $D(S)$.

i.e, $D(S)$ or $S' = \{ \cancel{x \in \mathbb{R} / x \text{ is a}} \text{ limit point of } S \}$

- Again the derived set of $D(S)$ is called the second derived set of S and is denoted by $D^2(S)$, or S''.

In general, the n^{th} derived set of S denoted by $D^{(n)}(S)$ or $S^{(n)}$

→ A set is said to be of first species if it has only a finite number of derived sets.
- the set is said to be of second species if the number of derived sets is infinite.

Note: ① If the set S is finite, then 'S' has no limit point and consequently, $D(S) = \phi$.

② If a set 'S' is of first species, then its last derived set must be empty.

③ A set whose n^{th} derived set is a finite set; so that its $(n+1)^{th}$ derived set is empty is called a set of n^{th} order.

→ $N = \{1, 2, 3, \ldots\} \subseteq \mathbb{R}$ has no limit points.

→ $A = \{\ldots -3, -2, -1\}$ has no limit points.

→ Every point of the set $\mathbb{R}$ of all real numbers is a limit point of $\mathbb{R}$.

→ Let $S = \mathbb{R}$

of $p \in \mathbb{R}$; $\epsilon > 0$.

$(p-\epsilon, p+\epsilon) \cap \mathbb{R}$ = infinite number of real numbers

→ Every real number is a limit point of the $\mathbb{Q}$ of all rational numbers

Let $S = \mathbb{Q}$

let $p \in \mathbb{R}$: $\epsilon > 0$.

$(p-\epsilon, p+\epsilon) \cap S$ = infinite number of rational numbers

Similarly ; $S = \mathbb{Q}'$ or $\mathbb{R} - \mathbb{Q}$.

→ The empty set ϕ has no limit points

i.e. let $S = \phi$

$p \in \mathbb{R}$; $\epsilon > 0$

$(p-\epsilon, p+\epsilon) \cap S = \phi$
not an infinite set.

→ $S = (a, b)$.

Every point of S is a limit point of S.

→ $S = [a, b]$; $(a, b]$, $[a, b)$.

Every point of S is a limit point of S

→ $S = \{\frac{1}{n} \mid n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \mathbb{R}$

Let $0 \in \mathbb{R}$; $\epsilon > 0$

$(0-\epsilon, 0+\epsilon) \cap S$ = infinite set.

$\therefore 0$ is a limit point of S.

(or)

$\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0 \notin S$. (i.e, 0 is not a member of S)

→ $S = \left\{ \frac{n}{n+1} \middle/ n \in N \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots \right\}$

(or)

$= \left\{ 1 - \frac{1}{n+1} \middle/ n \in N \right\}$.

Since $1 \in \mathbb{R}$, $\epsilon > 0$ such that $(1-\epsilon, 1+\epsilon)$ contains infinitely many points of S.

$\therefore$ 1 is a limit point of S and $1 \notin S$.

(or)

$\lim_{n\to\infty} S = 1 \notin S$ is a limit point.

→ $S = \left\{ 1 - \frac{1}{n} \middle/ n \in N \right\}$

$\lim_{n\to\infty} S = 1 \notin S$ is a limit point of S.

→ $S = \{ (-1)^n / n \in N \} = \{ -1, +1, -1, +1, \cdots \}$

Since $\lim_{n\to\infty} S = \lim_{n\to\infty} (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ +1 & \text{if } n \text{ is even} \end{cases}$

$\therefore$ S has two limit points -1 and $+1$ which are members of S.

→ $S = \{ (-1)^n n / n \in N \} = \{ -1, +2, -3, +4, -5, \cdots \}$ has no limit points.

Since $\lim_{n\to\infty} S = \lim_{n\to\infty} (-1)^n n = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{cases}$

Note: (1) Every finite set has no limit points.

(2) Every infinite set may or may not have limit points.

(3) Every interior point is a limit point, but every limit point need not be an interior point.

ex: Let $S = (a, b)$, then $D(S) = [a, b]$ & $S^0 = (a, b)$.

$\therefore$ a and b are limit points but not interior points.

→ If the supremum of a set does not belong to the set, then it is a limiting point of the set.

Sol Let S be the non-empty subset of real number set R.

and has supremum but not belong to the set S.

Let it be 'u'

i.e u = supremum of S but $u \notin S$.

Now we have to prove that 'u' is a limiting point of a set 'S'.

For this we have to prove that every nbd. of the point u contains a point of S other than u.

Let $(u-\epsilon, u+\epsilon)$ be any nbd of the point 'u' where $\epsilon > 0$.

Since u = l.u.b (supremum) of S.

$\therefore u-\epsilon$ is not an upper bound of S.

$\therefore \exists$ some $x \in S$ s.t $x > u-\epsilon$ —— (1)

$(\because x \in S$ and $u \notin S$ $\therefore x \neq u)$

Also $x \leq u < u+\epsilon$ —— (2).

from (1) and (2), we have

$u-\epsilon < x < u+\epsilon$ where $x \neq u$.

$\Rightarrow (u-\epsilon, u+\epsilon)$ contains a point $x \in S$

$\Rightarrow$ 'u' is a limiting point of the set S.

※ If the infimum of a set doesnot belong to the set, then it is a limit point of a set.

For example:

① $S = (-\infty, a) \subseteq \mathbb{R}$

$\therefore$ S is bdd above

and $\sup S = a \notin S$

$\therefore$ 'a' is a limiting point of S.

② $S = (a, \infty)$

$\therefore$ S is bdd below by 'a'

and $\inf S = a \notin S$

$\therefore$ 'a' is a limiting point of S.

Isolated point:-

A point $p \in S$ is called an isolated point of S if p is not a limit point of S.

i.e if $\exists$ a nbd of 'p' which contains no points of S other than 'p' it self.

A set 'S' is called a discrete set if all its points are isolated points.

For example: Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

since all the points of the set 'S' are its isolated points and so it is a discrete set.

If $x \in N$, the nbd of x (i.e $x-\epsilon, x+\epsilon$)

contains no point of N other than x

$\therefore$ x is not limit point of N of natural numbers.

If $x \notin N$, then the nbd of 'x' does not contain any point of 'S'

$\therefore$ x is not limit point of the set N of natural numbers.

$\therefore$ N has no limit points.

$\therefore D(N) = \phi$.

Since no point of N is a limit point of N,

$\therefore$ All the points of N are isolated points.

Hence $\boxed{N \text{ is descrete set}}$

Also $\boxed{N \text{ is of first species}}$ $(\because D(N) = \phi)$

→ The set of all whole numbers.

→ The set Z of all integers.

→ Let $S = \phi \subseteq \mathbb{R}$.

Let $x \in \mathbb{R}$, then for each $\epsilon > 0$, (however small)

$\therefore (x-\epsilon, x+\epsilon) \cap \phi = \phi$

$\therefore$ x is not a limit point of $S = \phi$.

$\Rightarrow$ No real number is a limit of ϕ.

$\therefore \boxed{D(\phi) = \phi}$

$\bar{S} = S \cup D(S)$

$= \phi \cup D(\phi)$

$\boxed{\bar{S} = \phi}$

Since $\phi \subseteq D(\phi)$

$\therefore$ S is dense-in-itself.

Also $S = D(S)$ i.e $\phi = D(S)$

$\therefore$ ϕ is perfect set.

→ One set. ...

Let $x \in \mathbb{R} \subseteq \mathbb{R}$

Then for each $\epsilon > 0$, (however small)

the nbd of x (i.e $(x-\epsilon, x+\epsilon)$)

contains infinitely many real numbers other than 'x'.

$\therefore$ x is a limit point of $\mathbb{R}$

$\Rightarrow$ Every real number is a limit point of $\mathbb{R}$.

$\therefore$ $\boxed{D(\mathbb{R}) = \mathbb{R}}$.

$\overline{\mathbb{R}} = \mathbb{R} \cup D(\mathbb{R})$

$\boxed{\overline{\mathbb{R}} = \mathbb{R}}$.

$\therefore$ $\mathbb{R}$ is dense set

and it is dense-in-itself

Also it is perfect set ($\because \mathbb{R} = D(\mathbb{R})$).

Note:- we have $D(\mathbb{R}) = \mathbb{R}$, $D^2(\mathbb{R}) = D(\mathbb{R}) = \mathbb{R}$

$D^3(\mathbb{R}) = \mathbb{R}$ and so on.

$\therefore$ for every +ve integer 'n', $D^{(n)}(\mathbb{R}) = \mathbb{R}$

$\therefore$ The number of derived sets of $\mathbb{R}$ is infinite

$\therefore$ $\mathbb{R}$ is of the second species.

→ $S = (a, b)$.

Sol: If $x \in [a, b]$

then $x = a$ or $x = b$ or $x \in (a, b)$

If $x = a$, then for every $\epsilon > 0$,

$(x-\epsilon, x+\epsilon) = (a-\epsilon, a+\epsilon)$ contains infinitely many points of (a, b) to the right of 'a'.

If $x = b$, then for every $\epsilon > 0$,

$(x-\epsilon, x+\epsilon) = (b-\epsilon, b+\epsilon)$ contains infinitely many points of (a, b) to the left of 'b'.

If $x \in (a, b)$, then for every $\epsilon > 0$,

$(x-\epsilon, x+\epsilon)$ contains infinitely many points of (a, b).

Thus, if $x \in [a, b]$, then for every $\epsilon > 0$,

$(x-\epsilon, x+\epsilon)$ is a nbd of 'x' containing infinitely many points of (a, b).

$\Rightarrow$ every point of $[a, b]$ is a limit point of (a, b).

$$\boxed{D\big((a,b)\big) = [a, b]}.$$

Since $\overline{S} = S \cup D(S)$

$= (a, b) \cup [a, b]$

$$\boxed{\overline{S} = [a, b] \subseteq \mathbb{R}}$$

Since $S \subseteq D(S)$ i.e. $(a, b) \subseteq [a, b]$.

$\therefore$ S is dense-in-itself.

Since $S \neq D(S)$

$\therefore$ S is not a perfect set.

$\longrightarrow$ $S = [a, b)$

$\longrightarrow$ $S = (a, b]$

$\longrightarrow$ $S = [a, b]$.

$\longrightarrow$ $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R}$.

Here: $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset (0, 1] \subseteq \mathbb{R}$

Let $p \in \mathbb{R}$, then for each $\epsilon > 0$ (however small)

If $p = 0$, then $(0-\epsilon, 0+\epsilon)$ is a nbd of '0'.

and it contains infinitely many points of S other than '0'

$\therefore$ 0 is the limit point of S

Now we shall show that no other real number p other than '0' can be a limit point of S.

The following cases arise:

Case (i) If $p < 0$ then $(-\infty, 0)$ is a nbd of p which contains no point of S i.e, $(-\infty, 0) \cap S = \phi$

$\therefore$ p is not a limit point of S.

Case (ii): If $p > 1$ then $(1, \infty)$ is a nbd of p which doesnot contain any point of S. i.e, $(1, \infty) \cap S = \phi$

$\therefore$ p is not a limit point of S

Case (iii): If $p = 1$, then $(\frac{1}{2}, \infty)$ is a nbd of p which contains no point of 'S' other than p. i.e, $(\frac{1}{2}, \infty) \cap S - \{1\} = \phi$

$\therefore$ p is not limit point of 'S'.

Case (iv):

If $0 < p < 1$, then $\frac{1}{p} > 0$.

$\therefore$ $\exists$ a unique natural number 'n' such that $n \leq \frac{1}{p} < n+1$.

$$\Rightarrow \frac{1}{n} \geq p > \frac{1}{n+1}$$

$$\Rightarrow \frac{1}{n+1} < p \leq \frac{1}{n} < \frac{1}{n-1}$$

$\Rightarrow$ The nbd $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ of p contains only one point $\frac{1}{n}$ of S.

$\therefore$ p is not limit point of S.

Hence 0 is the only limit point of 'S'

$\therefore$ $\boxed{D(S) = \{0\}}$

Also $D(S') = D(0) = \phi$.

$\therefore$ The set S is of the first species and of first order.

and $\bar{S} = S \cup D(S)$
$= \{\frac{1}{n} \mid n \in N\} \cup \{0\}$

$\rightarrow$ find S' i.e $D(S)$.
where $S = \{\frac{1}{n} \mid n \in Z,\ n \neq 0\}$.

Let $S = \{\frac{1}{n} \mid n \in Z_{Z \neq 0}\} \subseteq [-1, 1] \subseteq \mathbb{R}$

Let $p = 0 \in \mathbb{R}$ then the nbd of '0' contains infinitely many numbers.

$\therefore$ 0 is a limit of S.

Now we shall show that no real number p other than 0 can be a limit point of S.

The following cases will arise!

case (i) If $p < -1$ then $(-\infty, -1)$ is a nbd of p which contains no point of S

i.e $(-\infty, -1) \cap S = \phi$.

$\therefore$ p is not a limit point of S.

case (ii) If $p > 1$ then $(1, \infty)$ is a nbd of p which contains no point of S.

i.e $(1, \infty) \cap S = \phi$

case (iii): If $p = 1$ then $(\frac{1}{2}, \infty)$ is a nbd of p which does not contain any point of S other than p.

i.e $(\frac{1}{2}, \infty) \cap S - \{1\} = \phi$.

$\therefore p$ is not a limit point of S.

case (iv) If $p = -1$ then $(-\infty, -\frac{1}{2})$ is a nbd of p which does not contain any point of S other than p.

i.e $(-\infty, -\frac{1}{2}) \cap S - \{-1\} = \phi$

of S.

Case (v) If $0 < p < 1$, then

$\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ is a nbd of p

which contains only one point

~~$\frac{1}{n}$ of S~~

i.e. a finite number of points of S

$\therefore$ p is not a limit point of S

$p > 0$, $\frac{1}{p} > 0$

$p > 0 \Rightarrow \frac{1}{p} > 0$

$\exists$ $n \in \mathbb{N}$

$n \le \frac{1}{p} < n+1$

$n \le \frac{1}{p} < n+1$

$n \le \frac{1}{p}$, $\frac{1}{p} < n+1$

$\frac{1}{n} \ge p$, $p > \frac{1}{n+1}$

Case (vi) If $-1 < p < 0$, so that $0 < -p < 1$ and

$-\frac{1}{p} > 0$, $\exists$ a unique $n \in \mathbb{N}$ $\ni$

$n \le -\frac{1}{p} < n+1$

~~$\Rightarrow -\frac{1}{n} \le p < -\frac{1}{n+1}$~~

$\Rightarrow -\frac{1}{n-1} < -\frac{1}{n} \le p < -\frac{1}{n+1}$

$\Rightarrow$ The nbd $\left(-\frac{1}{n-1}, -\frac{1}{n+1}\right)$ of p

contains only one point $-\frac{1}{n}$ of S.

$\therefore$ p is not a limit point of S.

Hence '0' is the only limit point of S.

$\therefore D(S) = \{0\}$.

and $\bar{S} = S \cup \{0\}$.

$\rightarrow$ Find the derived set of each of the following:

(i) $(1, \infty)$ (ii) $(-\infty, -1)$ (iii) $\left\{\frac{n}{n+1} / n \in \mathbb{N}\right\}$.

(iv) $\left\{a + \frac{1}{n} / a \in \mathbb{R}, n \in \mathbb{N}\right\}$

(v) $\left\{\frac{1 + (-1)^n}{n} / n \in \mathbb{N}\right\}$, (vi) $\left\{\frac{1}{2^n} / n \in \mathbb{N}\right\}$ (vii) $\left\{\frac{1}{3^n} / n \in \mathbb{N}\right\}$.

Sol

(i) Let $S = (1, \infty)$

Let x be any real number.

If $x<1$, then for $0<\epsilon<1-x$,

$$(x-\epsilon, x+\epsilon) \cap (1,\infty) = \phi.$$

$\Rightarrow$ Any real number <1 is not a limit point of $(1,\infty)$.

If $x \in [1,\infty)$, then for every $\epsilon>0$, $(x-\epsilon, x+\epsilon)$ contains infinitely many points of $(1,\infty)$ to the right of 1.

$\Rightarrow$ Every elt of $[1,\infty)$ is a limit point of $(1,\infty)$

$\therefore (1,\infty)' = [1,\infty)$.

(ii) Ans: $(-\infty, -1)' = (-\infty, -1]$.

(iii) Let $S = \left\{ \frac{1+(-1)^n}{n} / n \in N \right\} \subseteq \mathbb{R}$.

when 'n' is odd,

$$\frac{1+(-1)^n}{n} = \frac{1-1}{n} = 0.$$

when 'n' is even,

$$\frac{1+(-1)^n}{n} = \frac{1+1}{n} = \frac{2}{n}.$$

$$\therefore S = \{0\} \cup \left\{ \frac{2}{n} / n \in N \text{ and } n \text{ is even} \right\}$$

$$= \{0\} \cup \left\{ \frac{2}{2}, \frac{2}{4}, \frac{2}{6}, \dots \right\}$$

$$= \{0\} \cup \left\{ 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \right\} \subset [0,1] \subseteq \mathbb{R}.$$

Let $p \in \mathbb{R}$

If $p=0$ then for each $\epsilon>0$ (however small), $(0-\epsilon, 0+\epsilon)$ is a nbd of '0' and it contains infinitely many points of S other than '0'.

$\therefore$ 0 is the limit point of S.

Now we shall show that no other real number p other than '0' can be a limit point of S:

The following cases arise:

Case(i) If $p<0$ then $(-\infty,0)$ is a nbd of 'p' which contains no point of S

i.e $(-\infty,0)\cap S=\phi$.

$\therefore$ p is not a limit point of S.

Case(ii) If $p>1$ then $(1,\infty)$ is a nbd of p which does not contain any point of S

i.e $(1,\infty)\cap S=\phi$

$\therefore$ p is not a limit point of S.

Case(iii): If $p=1$, then $(1/2,\infty)$ is a nbd of p which contains no point of S other than p

i.e $(1/2,\infty)\cap S-\{1\}=\phi$

$\therefore$ 'p' is not a limit.

Case(iv) If $0<p<1$, then $\frac{1}{p}>0$.

$\therefore$ $\exists$ a unique natural number 'n' such that $n\leq\frac{1}{p}<n+1$

$$\Rightarrow \frac{1}{n}\geq p>\frac{1}{n+1}$$

$$\Rightarrow \frac{1}{n+1}<p\leq\frac{1}{n}<\frac{1}{n-1}$$

$\Rightarrow$ The nbd $\left(\frac{1}{n+1},\frac{1}{n-1}\right)$ of p contains only one point $\frac{1}{n}$ of S.

$\therefore$ p is not limit point of S.

Hence '0' is the only limit point of S.

$\therefore D(S)=\{0\}$.

$\rightarrow S = \{ \cos(\frac{n\pi}{2}) / n \in \mathbb{Z} \} = \ldots$

$= \{ \ldots -1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots \}$

clearly $-1, 0, 1$ are limit points of S

$\therefore D(S) = \{-1, 0, 1\}$.

$\rightarrow$ Let $S = \{ \sin(\frac{n\pi}{2}) / n \in \mathbb{N} \} \subseteq \mathbb{R}$

then $D(S) = \{-1, 0, 1\}$.

$\rightarrow$ Let $S = \{ \cos(\frac{n\pi}{3}) / n \in \mathbb{N} \} \subseteq \mathbb{R}$.

then $D(S) = \{ -\frac{1}{2}, -1, \frac{1}{2}, 1 \}$.

$\rightarrow$ Let $S = \{ \sin\frac{n\pi}{3} / n \in \mathbb{N} \} \subseteq \mathbb{R}$ then

$D(S) = \{ -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \}$.

* Existence of limit points of a set:
Bolzano - Weierstrass theorem:

We have seen ~~that a finite set has no~~ ~~we have observed~~ that ~~limit points~~. Also we have ~~observed that~~ a finite set has no limit points. Also an infinite set may or may not have a limit point.

for example:

The infinite set $\mathbb{N}$ of natural numbers has no limit point whereas the infinite set $S = \{ \frac{1}{n} / n \in \mathbb{N} \}$ has '0' as its limit point.

We observe that the set S is bdd.

Now we shall give a theorem which gives us a set of sufficient conditions for a set to have a limit point. ~~This theorem is known as Bolzano-Weierstrass theorem.~~

Statement:

Every infinite bounded set of real numbers has a limit point.

Note: The converse of the above need not be true.

i.e, An infinite set has a limit point, then the set is not bounded.

for example:

Ex-1 $[0, \infty)$ is an infinite set and has limit points but it is not bounded.

Ex:2 $S = \mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{R}$

Some results on Derived sets:

→ If A and B be any two subsets of $\mathbb{R}$, then

(1) $A \subset B \Rightarrow D(A) \subseteq D(B)$

(2) $D(A \cup B) = D(A) \cup D(B)$

(3) $D(A \cap B) \subseteq D(A) \cap D(B)$

(4) $D(D(A)) \subseteq D(A)$

→ $D\left(\bigcup_{i \in N} A_i\right) = D(A_1) \cup D(A_2) \cup \cdots$

→ $D\left(\bigcap_{i \in N} A_i\right) \subseteq D(A_1) \cap D(A_2) \cap \cdots$

Note:

(1) The derived set of any bounded set is bounded.

(2) Every infinite bounded set has the greatest and the smallest limit points.

i.e, the derived set of any infinite bounded set attains its bounds.

(3). The smallest and the greatest member ... u of the derived set $D(S)$ of an infinite and bounded set S always exist.

They are usually denoted by $\underline{\lim}\, S$ and $\overline{\lim}\, S$ respectively and are called the inferior (or lower) limit of S and the superior (or upper) limit of S.

Also $\underline{\lim}\, S \leq \overline{\lim}\, S$

(4) The supremum (or infimum) of a bounded set S is always members of $\bar{S}$

(5) If S is bounded then $\bar{S}$ is also bounded.

Closed Set:

A subset S of $\mathbb{R}$ is said to be closed if its complement (i.e., $S^c = \mathbb{R} - S$) is an open set.

(or)

A set $S \subset \mathbb{R}$ is said to be closed if every limit point of the set S is a member of the set S.

(or)

A subset S of $\mathbb{R}$ is said to be closed if $D(S) \subseteq S$

(i) S is closed set $\Leftrightarrow D(S) \subseteq S$

(ii) S is closed set $\Leftrightarrow S^c$ is open.

(iii) S is open set $\Leftrightarrow S^c$ is closed

(iv) S is closed set $\Leftrightarrow \bar{S} = S$

If S is a closed set then every limit point of S is a member of S, but every point of S is not limit point.

Ex:

(1) If $S=\{a\}$ then $S^c = \mathbb{R}-S$
$= \mathbb{R}-\{a\}$
$= (-\infty, a) \cup (a,\infty)$

Since union of two open sets is again open

$\therefore S^c$ is open.

$\Rightarrow S$ is closed.

(or)

$S=\{a\}$ then $D(S) = \phi \subseteq S$

i.e, $D(S) \subseteq S$

$\therefore S$ is closed.

(or)

$S=\{a\}$ then $D(S) = \phi$.

$\therefore \bar{S} = S \cup D(S)$
$= \{a\} \cup \phi = \{a\}$
$= S$.

$\therefore \bar{S} = S$

$\therefore S$ is closed.

(2) If S be a non-empty finite set.

then $D(S) = \phi \subseteq S$

i.e, $D(S) \subseteq S$

$\therefore S$ is closed.

(3) If $S=N$ then $D(S) = \phi \subseteq S$.

$\Rightarrow D(S) \subseteq S$

$\therefore S$ is closed

Similarly, $S = W, I$.

(4) $S = \mathbb{Q}$

then $D(S) = \mathbb{R} \not\subseteq S$.

$\Rightarrow S$ is not closed

(5) $S = \mathbb{R} - \mathbb{Q}$

then $D(S) = \mathbb{R} \not\subseteq S$

$\Rightarrow S$ is not closed.

(6) $S = (a, b)$ then $D(S) = [a, b] \not\subseteq S$

$\therefore S$ is not closed.

(7) $S = [a, b), (a, b]$

then $D(S) = [a, b] \not\subseteq S$

$\therefore S$ is not closed

(8) $S = [a, b]$.

then $D(S) = [a, b] \subseteq S$

$\therefore S$ is closed.

(9) $S = \mathbb{R}$, then $D(S) = \mathbb{R}$

$\therefore D(S) \subseteq \mathbb{R}$

$\therefore S$ is closed.

(10) $S = \mathbb{R}^{+}$

i.e., $S = \{x \mid x > 0\}$

$= (0, \infty)$

$\Rightarrow D(S) = [0, \infty)$

$\therefore D(S) \not\subseteq S$

$\therefore S$ is not closed.

(11) $S = \mathbb{R}^{-}$

$= (-\infty, 0) \Rightarrow D(S) = (-\infty, 0] \not\subseteq S$

$\therefore S$ is not closed.

(12) $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$

then $D(S) = \{0\} \not\subseteq S$ $(\because 0 \notin S)$

$\therefore S$ is not closed.

(13) $S = \{\frac{1}{n} \mid n \in \mathbb{Z}\}$ then $D(S) = \{0\} \not\subseteq S$

$\therefore S$ is not closed.

Note: If a set has no limit point then $\bar{S} = S$.

The intersection of an arbitrary family of closed sets is a closed set.

Sol^n: Let $S_1, S_2, S_3, \ldots$ be closed sets.

then $S_1^c, S_2^c, S_3^c \ldots$ be the open sets.

Let $S = S_1 \cap S_2 \cap S_3 \cap \ldots$

$\Rightarrow S^c = (S_1 \cap S_2 \cap S_3 \cap \ldots)^c$

$= S_1^c \cup S_2^c \cup S_3^c \cup \ldots$

Since the union of arbitrary family of open sets is open.

$\Rightarrow S^c$ is open

$\Rightarrow S$ is closed set

→ The union of a finite collection of closed sets is a closed set.

Sol^n: Let $S_1, S_2, \ldots S_n$ be closed sets.

then $S_1^c, S_2^c, S_3^c, \ldots S_n^c$ be the opensets.

Let $S = S_1 \cup S_2 \cup \ldots \cup S_n$

$\Rightarrow S^c = (S_1 \cup S_2 \cup \ldots \cup S_n)^c$

$= S_1^c \cap S_2^c \cap \ldots \cap S_n^c$

Since intersection of finite collection of open sets is open.

$\therefore S^c$ is an open set

$\Rightarrow S$ is closed set

→ The union of an infinite collection of closed sets need not be a closed set.

Sol^n: Let $S_n = [\frac{1}{n}, 1]$ $\forall n \in N$.

Then each S_n is a closed set.

Now $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cup \ldots$

$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \ldots$

$= (0, 1] = S$ (say)

which is not a closed set ($\because D(S)$ ...)

$\therefore$ The union of an infinite collection of closed sets need not be a closed set.

$\longrightarrow$ Let 'A' be a closed set and B be an open set. then (i) $A - B$ is closed (ii) $B - A$ is open.

Soln: Since A is closed $\Rightarrow A^c$ is open

B is open $\Rightarrow B^c$ is closed.

(i) $B - A = B \cap A^c$.

Since B and A^c are open,

$\Rightarrow B \cap A^c$ open.

$\therefore B - A$ is open

(ii) $A - B = A \cap B^c$.

Since A and B^c are closed.

$\Rightarrow A \cap B^c$ is closed

$\therefore A - B$ is closed.

Compact sets:

A non-empty subset of R is said to be compact if it is closed and bounded.

Ex: (1) $S = \phi$.

$D(S) = \phi \subseteq S$

$\Rightarrow S$ is closed and bounded

$\Rightarrow \therefore S$ is compact.

$D(S) = [a, b] \subseteq S$

$\therefore$ S is closed and bounded.

$\therefore$ S is compact.

(3) $S = [-1, 1] \cup [2, 3]$

~~Since the union of~~ two closed sets is closed

and bounded.

$\therefore$ S is compact.

(4) $S = N$,

$D(S) = \phi \subseteq N$

$\therefore$ S is closed but not bounded.

$\therefore$ S is not compact.

Similarly, $S = W, Z$.

(5) $S = Q \Rightarrow D(S) = \mathbb{R} \nsubseteq Q$.

i.e., $D(S) \nsubseteq S$.

$\therefore$ S is not closed and not bound.

$\therefore$ S is not compact.

(6) $S = \mathbb{R} - Q$.

$\Rightarrow D(S) = \mathbb{R} \nsubseteq \mathbb{R} - Q$

i.e, $D(S) \nsubseteq S$.

$\therefore$ S is not closed and bounded

$\therefore$ S is not compact.

(7) $S = \mathbb{R}$; $D(S) = \mathbb{R}$.

$\because D(S) \subseteq \mathbb{R}$.

$\therefore$ S is closed but not bounded.

$\therefore$ S is not ~~closed~~ compact.

(8) $S = (a, b) \Rightarrow D(S) = [a, b] \nsubseteq S$

i.e, $D(S) \nsubseteq S$

$\therefore$ S is not closed but S is bounded

$\therefore$ S is not compact.

Similarly $S = [a, b)$, $(a, b]$.

(v) $S = \{x : a \le x\}$
$= [a, \infty)$
$\Rightarrow D(S) = [a, \infty) \subseteq S$
$\therefore D(S) \subseteq S$
$\therefore S$ is closed but is not bounded.
$\therefore S$ is not compact.

(iv) $S = \{1^2, 2^2, 3^2, \ldots\ldots (123)^2\}$.

Since S is finite.
$\Rightarrow D(S) = \phi \subseteq S$
$\Rightarrow D(S) \subseteq S$
$\therefore S$ is closed and bounded
$\therefore S$ is compact.

(c) → The union of finite family of compact sets is compact.

Soln: Let $S_1, S_2, \ldots S_n$ be compact sets.
Then $S_1, S_2, S_3, \ldots S_n$ be closed and bounded

Let $S = \bigcup_{i=1}^{n} S_i$

Since the union of finite collection of closed sets is a closed.
$\therefore S$ is closed.

~~Now we have to show~~ that S is bounded.

Also $S_i \subset [a_i, b_i]$, $1 \le i \le n$.

If $a = \min\{a_1, a_2, \ldots a_n\}$
and $b = \max\{b_1, b_2, \ldots b_n\}$
then $S = \bigcup_{i=1}^{n} S_i \subset [a, b]$
$\therefore S$ is bounded.

$\therefore$ S is closed and bounded.
$\therefore S$ is compact.

→ The intersection of an arbitrary family of Compact sets, Containing atleast one point in common, is compact.

Sol: Let $S_1, S_2, \ldots S_n, \ldots$ be arbitrary family of compact sets

Then $S_1, S_2, \ldots S_n, \ldots$ be closed and bounded.

Let $S = \bigcap_{i=1}^{\infty} S_i$

Since the intersection of arbitrary family of closed sets is closed.

$\therefore$ S is closed.

Also $S \subset S_i$ for each i.

and each S_i bounded.

$\therefore$ S is bounded.

$\therefore$ S is closed and bounded.

$\therefore$ S is compact.

Cover of a set:

→ Let 'S' be a set and $\{G_\alpha\}$ be a family of sets.

We say that $\{G_\alpha\}$ is a cover of S, if the union of members of $\{G_\alpha\}$ contains S as a subset.

i.e., if every point of S belongs to some member of the family $\{G_\alpha\}$.

→ We say that $\{G_\alpha\}$ is an open cover if every member of $\{G_\alpha\}$ is an open set.

(or)

Let S be a set and $\{G_\alpha\}$ be a collection of Some open subsets of $\mathbb{R}$ such that $S \subseteq \cup G_\alpha$. Then $\{G_\alpha\}$ is called an open cover of S.

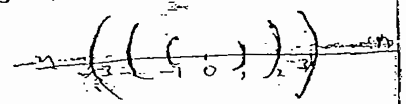
→ S.T $G = \{(-n, n) / n \in N\}$ is an open cover of the set $\mathbb{R}$.

Soln: Given $G = \{(-n, n) / n \in N\}$

$= \{\underset{G_1}{(-1,1)}, \underset{G_2}{(-2,2)}, \underset{G_3}{(-3,3)}, \ldots\ldots\}$

Since every $x \in \mathbb{R}$ belongs to atleast one of the open interval in G.

$\therefore G$ is an open cover of $\mathbb{R}$.



Also $\mathbb{R} = \bigcup_{n=1}^{\infty} G_n$, where $G_n = (-n, n)$

Similarly:

→ $G_1 = \{(-2n, 2n) / n \in N\}$

$G_2 = \{(n, n+2) / n \in Z\}$ is an

$G_3 = \{(n, n+1) / n \in Z\}$ are open covers of $\mathbb{R}$.

→ Show that $G_1 = \left\{\left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{3}, \frac{9}{4}\right)\right\}$ is open cover of interval $[1,2]$ whereas $G_2 = \left\{\left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{3}{2}, \frac{9}{4}\right)\right\}$ is not an open cover of the interval $[1,2]$.

Soln: Given $G_1 = \left\{\left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{3}, \frac{9}{4}\right)\right\}$

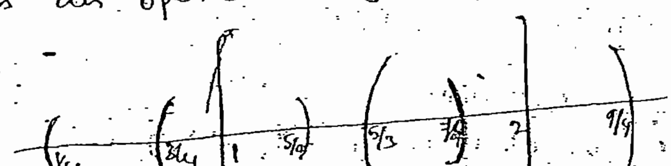
is an open cover of the interval $[1,2]$

Since every element of the set $S = [1,2] = \{x / 1 \le x \le 2\}$

belongs to atleast one of the subsets of G_1

and each of the subsets of G_1 is an open set.

$\therefore G_1$ is an open cover of $S = [1,2]$



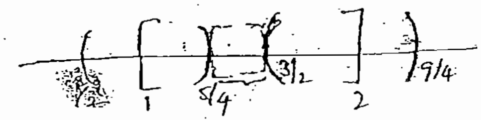
i.e. $[1,2] \subset \left(\frac{1}{4}, \frac{5}{4}\right) \cup \left(\frac{3}{4}, \frac{7}{4}\right) \cup \left(\frac{5}{3}, \frac{9}{4}\right)$

$G_2 = \{(\frac{1}{2}, \frac{5}{4}), (\frac{3}{2}, \frac{9}{4})\}$ is not an open cover of the interval $S = [1,2] = \{x / 1 \le x \le 2\}$.

because none of points in the interval $[\frac{5}{4}, \frac{3}{2}]$ belongs to any of the subsets of G_2.

i.e., $S = [1,2]$ is not covered by union of opensets $(\frac{1}{2}, \frac{5}{4})$ & $(\frac{3}{2}, \frac{9}{4})$

$\therefore G_2$ is not an open cover of S.



Subcover and finite subcover of a set:

Let G be an open cover of a set S. A subcollection E of G is called a subcover of S if E too is a cover of S.

further, if there are only a finite number of sets in E, then we (~~set in E is a finite number of~~) say that E is a finite subcover of the open cover G of S.

Thus if G is an open cover of a set S, then a collection E is a finite subcover of the open cover G of S provided the following three conditions hold:

(i) E is contained in G.

(ii) E is a finite collection

(iii) E is itself a cover of S.

Heine-Borel property:

A subset S of $\mathbb{R}$ is said to have the Heine-Borel property if every open cover of S has a finite sub-cover.

Sequence:- A function whose domain is the set N of all natural numbers and the range is a subset of real numbers is called a sequence. (or) Real sequence.

The sequence is denoted by $x : N \to \mathbb{R}$ (or) $S : N \to \mathbb{R}$

— A set of numbers which are 1-1 correspondence of natural numbers is called a sequence.

NOTE

— The domain for a sequence is always natural numbers

— A sequence is specified by the values $x(n)$ (or) x_n $\forall n \in N$

— A sequence may be denoted by $\{x_n : n \in N\}$ (or) $(x_n : n \in N)$ (or) x (or) $\{x_1, x_2, \ldots x_n, \ldots\}$

The values $x_1, x_2, \ldots x_n, \ldots$ are called first, second, third.... terms of the sequence.

— The m^{th} & n^{th} terms x_m & x_n for $m \neq n$ are treated as distinct terms even if $x_m = x_n$.

i.e, the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occuring at different positions are treated as distinct terms even if they have the same value.

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Range of a sequence:- The set of all distinct terms in a sequence is called its range.

Notes: In a sequence $\{x_n : n \in N\}$ and N is infinite, the number of terms in a sequence is always infinite

— The range of a sequence may be finite set.

Ex: If $x_n = (-1)^n$; $n \in N$ then $\{x_n\} = \{-1, +1, -1, +1, \ldots\}$

∴ There are only two distinct elements

∴ The range of a sequence $\{x_n\} = \{-1, +1\}$. which is finite.

Ex: $\{x_n\} = \{\frac{1}{n}\}_{n \in N}$

$= \{1, \frac{1}{2}, \cdots\}$

All the elements of the sequence are distinct.

∴ The range of a sequence is infinite.

Constant sequence :- A sequence $\{x_n\}$ is defined by $x_n = c \in R \ \forall n \in N$ is called constant sequence.

i.e, $\{x_n\} = \{c, c, c, \ldots\}$ is constant sequence

— with a range $= \{c\}$ which is a singleton set.

Ex $\{x_n\}_{\forall n \in N} = \{1\}$

problems :-

→ The sequence $\{x_n\}$ is defined by the following formulas for the n^{th} term. write the first five terms in each case.

(a) $x_n = 1 + (-1)^n$ (b) $x_n = \frac{(-1)^n}{n}$ (c) $x_n = \frac{1}{n(n+1)}$ (d) $x_n = \frac{1}{n^2+2}$

→ The first few terms of a sequence (x_n) are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the n^{th} term x_n.

(a) 5, 7, 9, 11, ... (b) $\frac{1}{2}, \frac{-1}{4}, \frac{1}{8}, \frac{-1}{16}, \cdots$

(c) $1/2, 2/3, 3/4, 4/5, \cdots$ (d) 1, 4, 9, 16, ...

Sol: Let $x = (5, 7, 9, 11, \cdots) = (2n+3 / n \in N)$.

are two sequences in $\mathbb{R}$ then $X+Y=(x_n+y_n)$ in $\mathbb{R}$ is called sum of two sequences.

→ **Difference of sequences :-** If $X=(x_n)$ and $Y=(y_n)$ are two sequences in $\mathbb{R}$ then $X-Y=(x_n-y_n)$ in $\mathbb{R}$ is called difference of two sequences.

→ **Product of sequences :-** If $X=(x_n)$ and $Y=(y_n)$ are two sequences then $XY=(x_n y_n)$ in $\mathbb{R}$ is called product of two sequences.

→ **Quotient :** If $X=(x_n)$, $Y=(y_n)$ are two sequences in $\mathbb{R}$ then $\frac{X}{Y}=\left(\frac{x_n}{y_n}\right)$ $(y_n \neq 0)$ is called quotient.

Bounds of a sequence :- If the range of a sequence is bdd below, then the sequence is said to be bdd below.

i.e, A sequence $\{x_n\}$ is said to be bdd below if $\exists\, k \in \mathbb{R}$ s.t $x_n \geq k$ $\forall$

→ If the range of a sequence is bdd above then the sequence is said to be bdd above.

i.e, A sequence $\{x_n\}$ is said to be bdd above if $\exists\, K \in \mathbb{R}$ s.t $x_n \leq K$ $\forall n \in N$

→ If the range of a sequence is bdd, the sequence is said to be bdd.

i.e, A sequence $\{x_n\}$ is bdd, if $\exists$ two real numbers k, K s.t $k \leq x_n \leq K$ $\forall n \in N$.

→ A sequence is said to be unbdd if it is not bdd.

→ If k is a lowerbound of the sequence $\{x_n\}$, every real number less than k is also lower bound of seq $\{x_n\}$. The greatest of all lower bounds is called glb (or) inf of $\{x_n\}$.

→ If K is an upperbound of the seq $\{x_n\}$, every real number greater than K is also an upperbound of a seq $\{x_n\}$; The least of all upper bounds ... sup ...

→ (1) $X = (x_n)$ or $\{x_n\}$ where $x_n = n \ \forall n \in N$

$\{x_n\} = \{n / n \in N\} = \{1, 2, 3, \ldots\}$ is not bdd sequence.

Since L·B = 1 ; U·B is not defined

∴ sequence $\{x_n\}$ is bdd below.

→ (2) $X = \{-n / n \in N\}$.

→ (3) $X = \{(-1)^n / n \in N\} = \{-1, +1, -1, +1, \ldots\} = \{-1, +1\}$ is bdd seque

L·B = −1 , U·B = +1

→ (4) $X = \{(-1)^n n / n \in N\}$

→ (5) $X = \{\sqrt{n} / n \in N\}$.

→ **Theorem:-** A sequence $\{x_n\}$ is bounded if and only if $\exists$ +ve real number M (i.e, $M > 0$) s.t $|x_n| \le M$ $\forall n \in N$.

N·C : Let $\{x_n\}$ be a bdd seq.

∴ By defn $\exists$ two real numbers h, k st

$h \le x_n \le k \quad \forall n \in N$ ——(1)

Let $M = \max\{|h|, |k|\}$

$\Rightarrow |h| \le M , |k| \le M$

$\Rightarrow -M \le h \le M$ ——(2), $-M \le k \le M$ ——(3)

from (1), (2) & (3)

$-M \le h \le x_n \le k \le M \quad \forall n \in N$

$\Rightarrow -M \le x_n \le M \quad \forall n \in N$

$\Rightarrow |x_n| \le M \quad \forall n \in N$.

S·C $|x_n| \le M \quad \forall n \in N$

$\Rightarrow -M \le x_n \le M \quad \forall n \in N$

∴ $\{x_n\}$ is bdd.

* **Limit of a sequence:-** Let $X = (x_n)$ be a sequence and $x \in R$, the real number x is said to be the limit of the sequence $\{x_n\}$ if to each $\epsilon > 0$ (however small) $\exists K \in N$ (K depending on ϵ; i.e, $K(\epsilon)$) s.t $|x_n - x| < \epsilon \ \forall n \ge K(\epsilon)$

$x_n \to x$ as $n \to \infty$ (or) $\lim_{n \to \infty} x_n = x$.

convergence $\Rightarrow -\epsilon < x_n - x < \epsilon \quad \forall n \geq k$

$\Rightarrow x-\epsilon < x_n < x+\epsilon \quad \forall n \geq k$

$\Rightarrow x_n \in (x-\epsilon, x+\epsilon) \quad \forall n \geq k.$

* Cgt of a sequences:- Let (x_n) be a sequence ~~if $\lim_{n\to\infty} x_n = x$ then the sequence (x_n) is said to be cgs to x.~~

If a sequence (x_n) has a limit then the sequence (x_n) is called cgt sequence.

(or)

→ A sequence (x_n) is said to be cgs to x, if for given $\epsilon>0$ (however small), $\exists$ a +ve integer k (k depending on ϵ, i.e, $k(\epsilon)$) s.t $|x_n - x| < \epsilon \; \forall n$

Here the real number x is limit of the sequence (x_n)

* Divergence of a sequences:- Let (x_n) be a sequence, if $\lim_{n\to\infty} x_n = +\infty$ (or) $-\infty$. then the sequence (x_n) is called dgt sequence.

(or)

→ If a sequence has no limit then the sequence is called dgt sequence.

(or)

(i) A sequence (x_n) is said to dgs to $+\infty$. If given any +ve real number k (however large) $k>0$ $\exists$ a +ve integer m (depending on k) s.t $x_n \geq k \; \forall n \geq m$.

i.e, $\lim_{n\to\infty} x_n = +\infty$ (or) $x_n \to \infty$ as $n \to \infty$.

(ii) A sequence (x_n) is said to dgs to $-\infty$. If given any ~~+ve~~ real number k (however large) $\exists$ a +ve integer m (depending on k) s.t $x_n < k \; \forall n \geq m$.

$n\to\infty$

i.e., $x_n \to \infty$ as $n \to \infty$

Oscillatory sequences:- If a sequence (x_n) neither cgs to a finite number nor diverges to $+\infty$ (or) $-\infty$, then the sequence (x_n) is called an oscillatory sequence.

→ If the oscillatory sequence is bdd then the sequence is called finite oscillatory sequence.

→ If the oscillatory sequence is unbdd then the sequence is called an infinite oscillatory seq.

Ex: (1) $(x_n) = \left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)$

$\because$ U.B $= 1$; L.B $= 0$

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

$\therefore (x_n)$ is cgt.

(2) $(x_n) = \frac{1}{3^n}$ (3) $(x_n) = n^2$ (4) $(x_n) = -n$

(5) $(x_n) = (-1)^n = (-1, +1, -1, +1, \dots)$

L.B $= -1$; U.B $= +1$

$\lim\limits_{n\to\infty} x_n = -1$ if n is odd
$= +1$ if n is even

$\therefore (x_n)$ is neither cgt nor dgt

$\therefore$ It is oscillatory sequence and it is bdd seq.

$\therefore$ finite oscillatory sequence.

(6) $(x_n) = \left((-1)^n . n\right) = (-1, +2, -3, +4, \dots)$

U.B = not defined ; L.B = not defined.

$\lim\limits_{n\to\infty} x_n = +\infty$ if n is even
$= -\infty$ if n is odd

$\therefore$ It is neither cgt nor dgt.

$\therefore$ It is oscillatory sequence and it is unbdd.

$\therefore$ It is infinite oscillatory sequence.

~~Null sequences: A sequence (x_n) is said to be a null sequence if it cgs to zero i.e., $\lim x_n = 0$. Ex: The sequences $\left(\frac{1}{n}\right), \left(\frac{1}{n^2}\right), \left(\frac{1}{2^n}\right)$ are null sequence~~

i.e., A sequence cannot converge to more than ~~one limit~~.

Proof: If possible let a sequence (a_n) converge to two distinct limits x' & x''

since $x' \neq x'' \Rightarrow |x' - x''| > 0$.

let $\epsilon = \frac{1}{2}|x' - x''|$

since the sequence (a_n) cgs to x'.

$\therefore$ Given $\epsilon > 0$, $\exists$ a +ve integer k' (depending on ϵ) s.t $|a_n - x'| < \epsilon/2 \quad \forall n \geq k'$.

and also the sequence (a_n) cgs to x''.

$\therefore$ Given $\epsilon > 0$, $\exists$ a +ve integer k'' (depending on ϵ) s.t $|a_n - x''| < \epsilon/2 \quad \forall n \geq k''$.

Let $K = \max\{k', k''\}$.

then $|a_n - x'| < \epsilon/2$ & $|a_n - x''| < \epsilon/2 \quad \forall n \geq K$

Now $|x' - x''| = |x' - a_n + a_n - x''|$

$\leq |x' - a_n| + |a_n - x''|$

$< \epsilon/2 + \epsilon/2 = \epsilon$

$\therefore |x' - x''| < \epsilon \quad \forall n \geq K$.

which is a contradiction to $\epsilon = \frac{1}{2}|x' - x''|$

$\therefore$ our assumption that a sequence cgs to two distinct limits x', x'' is wrong.

$\therefore x' = x''$.

---

* ~~Theorem~~: ~~Every~~ cgt sequence is bdd.

Pf: Let $X = (x_n)$ be a cgt sequence. Et cgs to x (say)

$\therefore$ Given $\epsilon > 0$, $\exists$ a natural number K s.t $|x_n - x| < \quad \forall n \geq K$

$\leq |a_n - a| + |a|$

$< \epsilon + |a|$.

Let $M = \sup\{|a_1|, |a_2|, \ldots |a_{K-1}|, \epsilon + |a|\}$

$\therefore |a_n| \leq M \;\forall n \in \mathbb{N}$

$\therefore (a_n)$ is bdd

Note: The converse of above theorem need not be true.

i.e, Every bdd sequence need not be cgt.

Ex: $(a_n) = ((-1)^n)$ is bdd.

but $\lim\limits_{n\to\infty} a_n = -1$ if n is odd
$= +1$ if n is even.

$\therefore$ it is an oscillatory sequence.

→ Use the defn of the limit of a sequence to S.T

(1) $\lim\left(\frac{1}{n}\right) = 0$

Sol) Given $\epsilon > 0$

We have $\left|\frac{1}{n} - 0\right| = \frac{1}{n}$ —— (1)

for given $\epsilon > 0$, by Archemedian property $\exists K \in I^+$

s.t $K\epsilon > 1$

$\Rightarrow \frac{1}{K} < \epsilon$ —— (2)

Now we have, $\forall n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K}$

$\Rightarrow \frac{1}{n} \leq \frac{1}{K} < \epsilon$ (by (2)) —— (3)

$\therefore$ (1) $\Rightarrow \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$ (by (3)) $\forall n \geq K$.

$\therefore \left|\frac{1}{n} - 0\right| < \epsilon \;\forall n \geq K$

i.e, $\frac{1}{n} \to 0$ as $n \to \infty$ i.e, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

(2) $\lim \frac{1}{n^2+1} = 0$ (3) $\lim\left(\frac{n}{n^2+1}\right) = 0$ (4) $\lim \frac{2n}{n+1} = 2$ (5) $\lim \frac{3n+1}{2n+5} = 3/2$ (6) $\lim \frac{n^2-1}{2n^2+3} = \frac{1}{2}$ (7) $\lim \frac{1}{\sqrt{n+7}} = 0$ (8) $\lim \frac{2n}{n+2} = 2$ (9) $\lim \frac{\sqrt{n}}{n+1} = 0$ (10) $\lim \frac{(-1)^n n}{n^2+1} = 0$ (11) for any $b \in I$ P.T $\lim\left(\frac{b}{n}\right) = 0$

and let $x \in \mathbb{R}$. if (a_n) is a sequence of +ve real numbers with $\lim_{n\to\infty} a_n = 0$ and if for some consta $c > 0$ and some $m \in N$. we have $|x_n - x| \le c a_n \; \forall n \ge m$ then it follows that $\lim x_n = x$.

proof: Since $\lim_{n\to\infty} a_n = 0$

i.e, $a_n \to 0$ as $n \to \infty$

Given $\epsilon > 0$, $\Rightarrow \frac{\epsilon}{c} > 0$, $\exists \; K \in I^+$ $(K = K(\epsilon/c))$

s.t $n \ge K \Rightarrow |a_n - 0| < \epsilon/c$

$\Rightarrow |a_n| < \epsilon/c$

$\Rightarrow a_n < \epsilon/c$ —— (1)

Since $|x_n - x| \le c a_n \; \forall n \ge m$

i.e, $n \ge m \Rightarrow |x_n - x| \le c a_n < c(\epsilon/c)$

$\Rightarrow |x_n - x| < \epsilon$ (from (1))

$\therefore$ we have $n \ge m \Rightarrow |x_n - x| < \epsilon$.

i.e, $x_n \to x$ as $n \to \infty$

$\therefore \lim_{n\to\infty} x_n = x$.

Corresponding problems in pg no 283 — Bernoulli's Inequality: If $x > -1$ then $(1+x)^n \ge 1 + nx \; \forall n \in N$

Theorem:

(a) Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that cgs to x & y respectively and $c \in \mathbb{R}$ then the sequence $X+Y$, $X-Y$, $X \cdot Y$ and cX converge to $x+y$, $x-y$, xy and cx respectively.

(b) If $X = (x_n)$ cgs to x and $Z = (z_n)$ is sequence of non zero real numbers that cgs to z and if $z \ne 0$ then the quotient sequence $\frac{X}{Z}$ cgs to $\frac{x}{z}$.

→ [illegible] ..., (z_n) are cgt sequences

Then $A+B+\cdots+Z = (a_n+b_n+\cdots+z_n)$ is also cgt.

and $\text{Lt}(a_n+b_n+\cdots+z_n) = \text{Lt}\, a_n + \text{Lt}\, b_n + \cdots + \text{Lt}\, z_n$

→ (2) $A \cdot B \cdots Z = (a_n \cdot b_n \cdots z_n)$ is cgt sequence.

and $\text{Lt}(a_n \cdot b_n \cdots z_n) = \text{Lt}\, a_n \cdot \text{Lt}\, b_n \cdots \text{Lt}\, z_n$

→ (3) If $k \in N$ and if $A = (a_n)$ is a cgt sequence then

$\text{Lt}\, a_n^k = \left(\text{Lt}\,(a_n)\right)^k$

Theorem: If $X = (x_n)$ is cgt to x and if $x_n \geq 0\ \forall n \in N$

then $x = \text{Lt}\, x_n \geq 0$ (i.e. $x \geq 0$)

proof: If possible suppose that $x < 0$

— Since the sequence (x_n) cgt to 'x'.

$\therefore \exists\, k \in I^+$ s.t $|x_n - x| < \epsilon\ \forall n \geq k$.

$\Rightarrow x - \epsilon < x_n < x + \epsilon\ \forall n \geq k$.

Taking $\epsilon = -x/2 > 0$ $(\because x < 0)$

$\therefore x + \frac{x}{2} < x_n < x - \frac{x}{2}\ \forall n \geq k$.

$x_n < x/2 < 0\ \forall n \geq k$.

$\Rightarrow x_n < 0\ \forall n \geq k$

But which is contradiction to the hypothesis that $x_n \geq 0\ \forall n \in N$.

$\therefore$ Our supposition that $x < 0$ is wrong.

$\therefore x \geq 0$.

Theorem: If $X = (x_n)$ and $Y = (y_n)$ are cgt and if $x_n \leq y_n\ \forall n \in N$ then $\text{Lt}\, x_n \leq \text{Lt}\, y_n$.

proof: Since (x_n) & (y_n) cgt sequences and converge to x & y (say).

$\therefore \text{Lt}\, x_n = x$; $\text{Lt}\, y_n = y$

Let $z_n = y_n - x_n$; then $z_n \geq 0\ \forall n$ $(\because y_n \geq x_n)$.

Now $\underset{n\to\infty}{\text{Lt}}\, z_n = \text{Lt}\, y_n - \text{Lt}\, x_n = y - x$. $\Rightarrow y - x \geq 0 \Rightarrow y \geq x$

$\Rightarrow x \leq y \Rightarrow \text{Lt}\, x_n \leq \text{Lt}\, y_n$.

Theorem:- If $X = (x_n)$ is a Convergent sequence and if $a \le x_n \le b \ \forall n \in N$ then $a \le Lt\, x_n \le b$.

Proof:- Let $y_n = b - x_n$ then

$$y_n \ge 0 \ \forall n \ (\because b \ge x_n)$$

$$\therefore Lt\, y_n = Lt(b - x_n)$$

$$= b - Lt\, x_n$$

$$\Rightarrow b - Lt\, x_n \ge 0 \quad (\because y_n \ge 0)$$

$$\Rightarrow b \ge Lt\, x_n$$

$$\Rightarrow Lt\, x_n \le b$$

Similarly $Lt\, x_n \ge a$ (Let $y_n = x_n - a$)

$$\therefore a \le Lt\, x_n \le b$$

Squeeze Theorem :-

Suppose that $X = (x_n)$, $Y = (y_n)$ and $Z = (z_n)$ are sequences of real numbers such that $x_n \le y_n \le z_n \ \forall n \in N$ and that $Lt\, x_n = Lt\, z_n$ then $Y = (y_n)$ is Convergent and

$$Lt(x_n) = Lt(y_n) = Lt(z_n).$$

Proof:- Let $Lt(x_n) = Lt\, z_n = \omega$

i.e. the sequences (x_n) & (z_n) are convergent to ω.

$\therefore$ Given $\epsilon > 0$, $\exists K \in I^+$ such that

$n \ge K \Rightarrow |x_n - \omega| < \epsilon$; $|z_n - \omega| < \epsilon$

$n \ge K \Rightarrow \omega - \epsilon < x_n < \omega + \epsilon$

and $\omega - \epsilon < z_n < \omega + \epsilon$

Since $x_n \le y_n \le z_n \ \forall n \in N$.

$\therefore$ we have $\omega - \epsilon < x_n \le y_n \le z_n < \omega + \epsilon \ \forall n \ge K$

$$\Rightarrow \omega - \epsilon < y_n < \omega + \epsilon \quad \forall n \ge K.$$

$$\Rightarrow |y_n - \omega| < \epsilon \quad \forall n \ge K.$$

$$\Rightarrow \lim_{n \to \infty} y_n = \omega$$

$\therefore (y_n)$ Converges to ω.

and also $Lt\, x_n = Lt\, y_n = Lt\, z_n$.

Theorem :- Let the sequence $X = (x_n)$ converge to x then the sequence $(|x_n|)$ of absolute values converge to $|x|$ i.e. if $Lt\, x_n = x$ then $Lt(|x_n|) = |x|$.

proof :- Since $X = (x_n)$ is convergent to x.

$\therefore$ Given $\epsilon > 0$, $\exists K \in I^+$ such that $|x_n - x| < \epsilon \ \forall n \ge K$.

Now we have

$$\big| |x_n| - |x| \big| \le |x_n - x| < \epsilon \quad \forall n \ge K$$

$$\therefore \big| |x_n| - |x| \big| < \epsilon \quad \forall n \ge K$$

$\therefore |x_n|$ Converges to $|x|$.

Converse part

$$Ex :- (x_n) = ((-1)^n) \ \forall n \in N$$

Theorem :- Let (x_n) be a sequence of positive real numbers such that $L = Lt\left(\frac{x_{n+1}}{x_n}\right)$ exists.

If $L < 1$ then (x_n) Converges and $\lim_{n \to \infty}(x_n) = 0$.

Proof :- Since $x_n > 0 \ \forall n$

$\Rightarrow x_{n+1} > 0$

$\therefore \frac{x_{n+1}}{x_n} > 0 \quad \forall n$

$\therefore \lim_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = L \geq 0$

i.e. $L \geq 0$.

Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$

$\therefore$ Given $\epsilon > 0$, $\exists\, K \in N$ such that

$\left|\frac{x_{n+1}}{x_n} - L\right| < \epsilon \;\forall\, n \geq K.$

$\Rightarrow -\epsilon < \frac{x_{n+1}}{x_n} - L < \epsilon \;\forall\, n \geq K$

$\Rightarrow L-\epsilon < \frac{x_{n+1}}{x_n} < L+\epsilon \;\forall\, n\geq K$ —— (1)

Now replacing n by $K, K+1, K+2, \ldots n-1$ in (1) we get

$L-\epsilon < \frac{x_{K+1}}{x_K} < L+\epsilon$

$L-\epsilon < \frac{x_{K+2}}{x_{K+1}} < L+\epsilon$

$L-\epsilon < \frac{x_{K+3}}{x_{K+2}} < L+\epsilon$

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$L-\epsilon < \frac{x_n}{x_{n-1}} < L+\epsilon$

Now multiplying the above $(n-K)$ inequalities, we have

$(L-\epsilon)^{n-K} < \frac{x_n}{x_K} < (L+\epsilon)^{n-K}$ —— (2)

Since $L < 1$

Since $L \geq 0$

$\therefore 0 \leq L < L+\epsilon < 1$

$\Rightarrow 0 < L+\epsilon < 1$ —— (3)

$\therefore$ from (2), we have

$\frac{x_n}{x_K} < (L+\epsilon)^{n-K}$

$\Rightarrow x_n < x_K (L+\epsilon)^{n-K}$

$\Rightarrow x_n < x_K (L+\epsilon)^n \frac{1}{(L+\epsilon)^K}$

since $x_n > 0 \;\forall n$

$\therefore 0 < x_n < x_K (L+\epsilon)^n \frac{1}{(L+\epsilon)^K}$ —— (4)

Let $m = x_K \frac{1}{(L+\epsilon)^K} > 0$

$\therefore 0 < x_n < m(L+\epsilon)^n$ (by (4)) —— (5)

since $0 < L+\epsilon < 1$ (by (3))

$\therefore \lim_{n\to\infty} (L+\epsilon)^n = 0$

Since the eqn (5) is of the form $y_n < x_n < z_n \;\forall n$

with $\lim y_n = \lim z_n = 0$

$\therefore$ By squeeze theorem, $\lim x_n = 0$ and (x_n) is convergent to zero.

$\frac{x_n}{x_K} = x_{n-K}$

$70 \geq 50$

$n \geq K.$

$n = 50, 50+1, 50+2, \ldots 69$

$K, K+1, K+2, \ldots n-1$

$69 \to 50 = 20$ terms

$n-K = 20$

Problems

→ Apply above theorem

(i.e. let (x_n) be a sequence of +ve real numbers such that $L = \underset{}{Lt}\left(\frac{x_{n+1}}{x_n}\right)$ exists. If $L<1$, then (x_n) converges and $\underset{n\to\infty}{Lt}(x_n)=0$) to the following sequences, where a, b satisfy $0<a<1$, $b>1$.

(a) $\left(\frac{n}{b^n}\right)$, (b) $\left(\frac{2^{3n}}{3^{2n}}\right)$, (c) $\left(\frac{b^n}{2^n}\right)$

Sol'n :- a) $\left(\frac{n}{b^n}\right)$

Let $x_n = \frac{n}{b^n}$ then $x_{n+1} = \frac{n+1}{b^{n+1}}$

Now $\frac{x_{n+1}}{x_n} = \frac{n+1}{b^{n+1}} \times \frac{b^n}{n}$

$= \frac{n+1}{bn}$

$= \frac{1+1/n}{b}$

$Lt\left(\frac{x_{n+1}}{x_n}\right) = Lt\left(\frac{1+1/n}{b}\right)$

$= \frac{1+0}{b} = 1/b < 1$

$(\because b>1)$

$\therefore Lt \frac{x_{n+1}}{x_n} = 1/b < 1$ (i.e. $L<1$)

$\therefore (x_n)$ converges & $Lt\, x_n = 0$.

(b) $\left(\frac{2^{3n}}{3^{2n}}\right)$

Let $x_n = \frac{2^{3n}}{3^{2n}}$ then $x_{n+1} = \frac{2^{3n+3}}{3^{2n+2}}$

Now $\underset{n\to\infty}{Lt}\left(\frac{x_{n+1}}{x_n}\right) = Lt\left[\frac{2^{3n+3}}{3^{2n+2}} \times \frac{3^{2n}}{2^{3n}}\right]$

$= \underset{n\to\infty}{Lt}\left(\frac{2^3}{3^2}\right) = 8/9 < 1$

$\therefore (x_n)$ is convergent & $Lt\, x_n = 0$

(c) $\left(\frac{b^n}{2^n}\right)$

Let $x_n = \frac{b^n}{2^n}$ then $x_{n+1} = \frac{b^{n+1}}{2^{n+1}}$

Now $\frac{x_{n+1}}{x_n} = \frac{b^{n+1}}{2^{n+1}} \cdot \frac{2^n}{b^n}$

$= b/2$

$\therefore Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2$

if $1<b<2$ then $Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2 < 1$

$\therefore (x_n)$ is convergent & $Lt\, x_n = 0$.

If $b>2$ then $Lt\left(\frac{x_{n+1}}{x_n}\right) = b/2 > 1$

$\therefore (x_n)$ is not convergent & $Lt\, x_n \neq$

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Cauchy's First theorem on limits :-

If $\{a_n\}$ converges to l then the sequence $\{x_n\}$ where $x_n = \frac{a_1+a_2+\dots+a_n}{n}$ also converges to l (or)

$\underset{n\to\infty}{Lt}\, a_n = l \Rightarrow \underset{n\to\infty}{Lt} \frac{a_1+a_2+\dots+a_n}{n}$

Proof :- Let $b_n = a_n - l$ then

$\underset{n\to\infty}{Lt}\, b_n = \underset{n\to\infty}{Lt}\, a_n - l$

$= l - l$

$= 0$

$\therefore \lim_{n\to\infty} b_n = 0$

i.e. $b_n \to 0$ as $n \to \infty$ —— (1).

Since $a_n = b_n + l \;\; \forall n$

$\therefore x_n = \dfrac{(b_1+l)+(b_2+l)+\cdots+(b_n+l)}{n}$

$= \dfrac{(b_1+b_2+\cdots+b_n)+nl}{n}$

$= \dfrac{b_1+b_2+\cdots+b_n}{n} + l$

$\therefore$ In order to prove that $x_n \to l$,

For this we are enough to show that

$\dfrac{b_1+b_2+\cdots+b_n}{n} \to 0$. —— (2)

From (1), $\lim b_n = 0$ i.e. $b_n \to 0$ as $n\to\infty$.

$\therefore$ Given $\epsilon > 0$, $\exists\; m \in I^+$ such that

$|b_n - 0| < \epsilon/2 \;\; \forall\, n \geq m$.

$\Rightarrow |b_n| < \epsilon/2 \;\; \forall\, n \geq m$. —— (3)

Also the sequence $\{b_n\}$ is convergent

$\therefore \{b_n\}$ is bounded.

$\therefore \exists\; M > 0$ such that $|b_n| \leq M \;\forall n$ —— (4)

Now let us prove (2).

we have

$\left|\dfrac{b_1+b_2+\cdots+b_n}{n} - 0\right| = \left|\dfrac{b_1+b_2+\cdots+b_n}{n}\right|$

$= \dfrac{|b_1+b_2+\cdots+b_n|}{|n|}$

$= \dfrac{1}{n}\left[|b_1+b_2+\cdots+b_m+b_{m+1}+b_{m+2}+\cdots+b_n|\right]$

$(\because n \in N \Rightarrow |n| = n)$

$\leq \dfrac{1}{n}\Big[\big(|b_1|+|b_2|+\cdots+|b_m|\big)+\big(|b_{m+1}|+|b_{m+2}|+\cdots+|b_n|\big)\Big]$

$< \dfrac{1}{n}\Big[\big(M+M+\cdots+M(m\text{ times})\big) + \big(\epsilon/2+\epsilon/2+\cdots+\epsilon/2\,(n-m\text{ times})\big)\Big]$.

$\forall\, n \geq m$.

(using (3) & (4))

$\Rightarrow \left|\dfrac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \dfrac{1}{n}\left[mM + (n-m)\dfrac{\epsilon}{2}\right]$

$\forall n \geq m$

$\Rightarrow \left|\dfrac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \dfrac{mM}{n} + \dfrac{(n-m)\epsilon}{2n}$

$\forall\, n \geq m$

$< \dfrac{mM}{n} + \dfrac{\epsilon}{2}$

$\left(\because \dfrac{n-m}{n} = 1 - \dfrac{m}{n} < 1\right) \forall n \geq m$

—— (5)

Now $\dfrac{mM}{n} < \epsilon/2$ if $\dfrac{n}{mM} > \dfrac{2}{\epsilon}$

$\Rightarrow$ if $n > \dfrac{2mM}{\epsilon}$

If P is a natural number $> \dfrac{2mM}{\epsilon}$

then $n \geq P$.

Let $q = \text{Max}\{P, m\}$

$\therefore$ From (5),

$\left|\dfrac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \dfrac{\epsilon}{2}+\dfrac{\epsilon}{2}$

$\forall n \geq q$

$= \epsilon$.

$\therefore \left|\dfrac{b_1+b_2+\cdots+b_n}{n} - 0\right| < \epsilon \;\; \forall n \geq q$.

$\therefore \dfrac{b_1+b_2+\cdots+b_n}{n} \to 0$ as $n\to\infty$

$\therefore x_n \to l$ as $n\to\infty$

i.e. $\lim_{n\to\infty} x_n = l$

$\therefore$ Hence the theorem.

Note:- The converse of the above theorem need not be true.

i.e. If the sequence $\{x_n\}$ converges to l then the sequence $\{a_n\}$ need not be converge to l.

where $x_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$

Ex:- Let $\{a_n\} = \{(-1)^n\}$

$= \{-1, +1, -1, +1, \cdots\}$

then $x_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$

$= 0$ if n is even

$= -\frac{1}{n}$ if n is odd

if n is even i.e., $n=1$ to 6 then $x_n = \dfrac{-1+1-1+1-1+1}{n} = 0$

$\therefore \lim_{n\to\infty} x_n = 0$ i.e. the sequence $\{x_n\}$ convergent to 0.

But $\{a_n\}$ is not convergent.

because $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = +1$ if n is even.

$= -1$ if n is odd.

$\therefore \{a_n\}$ is oscillatory sequence.

$\therefore$ It is not convergent.

Problems

→ Show that $\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) =$

Sol^n:- Let $a_n = 1/n$ then $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore$ By Cauchy's first theorem on lim

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n} = 0$$

→ Show that

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots+\frac{(n+1)}{n}\right) =$$

Let $a_n = \dfrac{n+1}{n}$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} ($

$= 1$

$\therefore$ By Cauchy's first theorem on lim

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{\frac{2}{1}+\frac{3}{2}+\cdots+\left(\frac{n+1}{n}\right)}{n} = 1$$

→ Show that

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right] = 1$$

L.H.S. $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+}}\right.$

$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\sqrt{1+1/n^2}}+\frac{1}{\sqrt{1+2/n^2}}+\cdots+\frac{1}{\sqrt{1+}}\right.$

$$= \lim_{n\to\infty} \frac{\left[\frac{1}{\sqrt{1+1/n^2}}+\frac{1}{\sqrt{1+2/n^2}}+\cdots+\frac{1}{\sqrt{1+}}\right.}{n}$$

Let $a_n = \frac{1}{\sqrt{1+1/n}}$ then $\underset{n\to\infty}{Lt}\, a_n = 1$

$\therefore$ By Cauchy's first theorem on limits

$$\underset{n\to\infty}{Lt} \frac{a_1+a_2+\cdots+a_n}{n} = 1$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{\left[\frac{1}{\sqrt{1+1/n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \cdots + \frac{1}{\sqrt{1+1/n^2}}\right]}{n} = 1$$

i.e. $$\underset{n\to\infty}{Lt} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} = 1$$

$\rightarrow$ show that

$$\underset{n\to\infty}{Lt}\left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \cdots + \frac{1}{\sqrt{2n}}\right] = \infty$$

L.H.S. $$Lt\, \frac{1}{n}\left[\frac{n}{\sqrt{n}} + \frac{n}{\sqrt{n+1}} + \cdots + \frac{n}{\sqrt{2n}}\right] \quad (i)$$

Let $a_n = \frac{n}{\sqrt{2n}}$ then $a_n = \frac{1}{\sqrt{2}}\sqrt{n}$

$\therefore \underset{n\to\infty}{Lt}\, a_n = \infty$

$\therefore$ By Cauchy's first theorem on limits

$$Lt\, \frac{\left[\frac{n}{\sqrt{n}} + \frac{n}{\sqrt{n+1}} + \cdots + \frac{n}{\sqrt{2n}}\right]}{n} = \infty$$

i.e. $$\underset{n\to\infty}{Lt}\left[\frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{2n}}\right] = \infty$$

H.W. $\rightarrow$ show that

$$\underset{n\to\infty}{Lt}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right] = 0.$$

$\rightarrow$ Theorem:-

If $\{a_n\}$ is a sequence of +ve terms for all n and $\underset{n\to\infty}{Lt}\, a_n = l$ then $\underset{n\to\infty}{Lt}\,(a_1 \cdot a_2 \cdots a_n)^{1/n} = l$

Proof :- Let $b_n = \log a_n \ \forall n \ (\because a_n > 0)$ — (1)

Since $\underset{n\to\infty}{Lt}\, a_n = l$

$\therefore \underset{n\to\infty}{Lt}\, b_n = \log l$ ($\because l > 0$ because $a_n > 0 \ \forall n$)

$\therefore$ By Cauchy's first theorem on limits.

$$\underset{n\to\infty}{Lt} \frac{b_1+b_2+\cdots+b_n}{n} = \log l$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n} = \log l \quad (\text{by } (1))$$

$$\Rightarrow \underset{n\to\infty}{Lt} \frac{\log[a_1 \cdot a_2 \cdots a_n]}{n} = \log l$$

$$\Rightarrow \underset{n\to\infty}{Lt} \log(a_1 \cdots a_n)^{1/n} = \log l$$

$$\Rightarrow \log\left[\underset{n\to\infty}{Lt}(a_1 a_2 \cdots a_n)^{1/n}\right] = \log l$$

$$\Rightarrow \underset{n\to\infty}{Lt}(a_1 \cdot a_2 \cdots a_n)^{1/n} = l$$

Theorem $\rightarrow$ If (a_n) is a sequence such that $a_n > 0 \ \forall n$ and $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = l$ then $\underset{n\to\infty}{Lt}(a_n)^{1/n} = l$.

Proof :- Let us define the sequence $\{b_n\}$ such that

$$b_1 = a_1,\ b_2 = \frac{a_2}{a_1},\ b_3 = \frac{a_3}{a_2}, \cdots b_n = \frac{a_n}{a_{n-1}}$$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = l$

$\Rightarrow \underset{n\to\infty}{Lt} \frac{a_n}{a_{n-1}} = l \Rightarrow \underset{n\to\infty}{Lt} b_n = l$ — (1)

Since $a_n > 0 \ \forall n$

$\therefore b_n > 0 \ \forall n$

Now we have a sequence $\{b_n\}$ such that $b_n > 0 \ \forall n$ and $\underset{n\to\infty}{Lt} b_n = l$

$\therefore \underset{n\to\infty}{Lt} (b_1 . b_2 \cdots b_n)^{1/n} = l$

(By previous theorem)

$\Rightarrow \underset{n\to\infty}{Lt} (a_n)^{1/n} = l$

Note :- The converse of above theorem need not be true.

Ex :- Let $a_n = 2^{-n+(-1)^n}$

then $a_n^{1/n} = 2^{-1+\frac{(-1)^n}{n}}$

$\therefore \underset{n\to\infty}{Lt} a_n^{1/n} = 2^{-1}$ $\quad \Big/ \ \underset{n\to\infty}{Lt} \frac{(-1)^n}{n} = 0$

$= 1/2$

But $\frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)+(-1)^{n+1}}}{2^{-n}+(-1)^n}$

$= 2^{-(n+1)+(-1)^{n+1}+n-(-1)^n}$

$= 2^{-1+(-1)^{n+1}-(-1)^n}$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = 2^{-1-1-1} = 2^{-3}$ if n is even.

$= 2^{-1+1+1} = 2^1$ if n is odd

$\therefore \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n}$ — 1/8

$= 2$ if n is odd.

$\therefore \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n}$ does not exist.

Note :- The above theorem known Cauchy's second theorem on limit

Problems :

→ Show that $\{n^{1/n}\}$ converges to 1.

Sol'n : Let $a_n = n$ then

$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{n}$

$= (1 + 1/n)$

Now $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = 1$

$\therefore$ By Cauchy's second theorem on limits.

$\underset{n\to\infty}{Lt} a_n^{1/n} = 1$

→ Show that

$\frac{1}{n}\left(1+2^{1/2}+3^{1/3}+\cdots+n^{1/n}\right) = 1$

~~First theorem~~.

→ Find $\underset{n\to\infty}{Lt} (n!)^{1/n}$

Let $a_n = n!$ then $a_{n+1} = (n+1)!$

Now $\frac{a_{n+1}}{a_n} = n+1$

$\therefore \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \infty$

$\therefore$ By Cauchy's second theorem on limits

$\underset{n\to\infty}{Lt} (a_n)^{1/n} = \infty$

i.e. $\underset{n\to\infty}{Lt} (n!)^{1/n} = \infty$

$n^n \qquad n\to\infty$

Sol'n :- Now $x_n = \dfrac{n!}{n^n}$

$$\Rightarrow x_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Now $\dfrac{x_n}{x_{n+1}} = \dfrac{n!}{n^n} \times \dfrac{(n+1)^{n+1}}{(n+1)!}$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\therefore \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{e} < 1$$

$$\therefore \lim x_n = 0$$

$\rightarrow$ If $x_n = \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n\right]^{1/n}$

then show that $\lim_{n\to\infty} x_n = e$.

Let $a_n = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n$

then $a_{n+1} = \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)^3 \cdots \cdots$

$$\cdots \left(\frac{n+1}{n}\right)^n \left(\frac{n+2}{n+1}\right)^{n+1}$$

Now $\dfrac{a_{n+1}}{a_n} = \left(\dfrac{n+2}{n+1}\right)^{n+1}$

$$= \left(1+\frac{1}{n+1}\right)^{n+1}$$

$$\therefore \lim \frac{a_{n+1}}{a_n} = e$$

$\therefore$ Cauchy's second theorem on limits.

$$\lim_{n\to\infty} (a_n)^{1/n} = e$$

i.e. $\lim x_n = e$

H.W. $\rightarrow$ Prove that $\lim_{n\to\infty} \left(\dfrac{n^n}{n!}\right)^{1/n} = e$

(Cauchy's second theorem)

H.W. $\rightarrow$ Show that $\lim_{n\to\infty} \dfrac{(n!)^{1/n}}{n} = 1/e$

(Cauchy's second theorem)

Sol'n :- $\lim_{n\to\infty} \left(\dfrac{n!}{n^n}\right)^{1/n}$

Sequence (x_n) be Convergent sequence.

Then we have to Prove that (x_n) is bounded.

~~Sufficient Condition :-~~

Let the sequence (x_n) be monotone bounded sequence.

Then we have to Prove that the sequence (x_n) is Convergent.

Since (x_n) is monotone bounded sequence.

$\therefore$ it is either $M\uparrow$ sequence or $M\downarrow$ sequence.

Also it is bounded above as well as bounded below.

(i) Suppose that the sequence (x_n) is bounded $M\uparrow$ sequence then (x_n) is bounded above.

(ii) Suppose that the sequence (x_n) is bounded $M\downarrow$ sequence, then (x_n) is bounded below.

* Limit Points of a Sequence :-

~~A real number l is said to be limit point of a sequence (x_n) if every neighbourhood of l contains infinitely many terms of the sequence.~~

i.e. $l\in\mathbb{R}$ is limit point of the sequence $(x_n) \Leftrightarrow$ every neighbourhood of l contains infinitely many terms of the sequence.

$\Leftrightarrow \forall\ \epsilon>0,\ x_n\in(l-\epsilon, l+\epsilon)$ for infinitely many values of n.

$\Leftrightarrow \forall\ \epsilon>0,\ |x_n-l|<\epsilon$ for infinitely many values of n.

Ex :- (1) $(x_n)=((-1)^n)$

$=(-1,+1,-1,+1,-1,\ldots)$

has two limit points -1 & $+1$.

Let $x_n=(-1)^n\ \forall n$

then $x_n=-1$ if n is odd.

and $x_n=+1$ if n is even.

$\therefore$ Every neighbourhood of -1 Contains all the odd terms of sequence (x_n).

$\therefore -1$ is a limit point.

Similarly every neighbourhood $+1$ Contains all even terms the sequence (x_n).

$\therefore +1$ is a limit point.

Ex:(2). $(x_n)=\left(\frac{1}{n}\right)$

$=\left(\frac{1}{1},\frac{1}{2},\frac{1}{3},\ldots\right)$ h

a limit point '0'.

Because the neighbourhood of 0 Cont infinitely many terms of the sequence.

where $x_n = 1$ $\forall$ $n \in N$ has the only limit point 1.

Note :- (1) A limit point of a sequence is also called cluster point (or) an ~~accumulation point (or)~~ condensation point of the sequence

(2) Limit point of a sequence is different from limit of a sequence i.e. if $l \in \mathbb{R}$ is the limit of a sequence (x_n) then for $\epsilon > 0$, $\exists$ $m \in N$ such that $|x_n - l| < \epsilon$ $\forall n \geq m$.

$\Leftrightarrow x_n \in (l-\epsilon, l+\epsilon)$ $\forall n \geq m$.

i.e. Every neighbourhood of l contains all except a finite number of terms of the sequence.

where as if $l \in \mathbb{R}$ is a limit point of the sequence (x_n) then every neighbourhood of l contains infinitely many terms of the sequence (x_n) doesnot exclude the possibility of an infinite number of terms of the sequence lying outside that neighbourhood.

Hence limit of a sequence is a limit point of the sequence, but a limit point of a sequence need not be the limit of the

(3) If $x_n = l$ for infinitely many values of n then l is a limit point of (x_n).

(4) If for $\epsilon > 0$, $x_n \in (l-\epsilon, l+\epsilon)$ for finitely many values of n then l is not a limit point of the sequence (x_n).

(5) Limit point of a sequence need not be a term of the sequence.

→ Bolzano - Weierstrass Theorem for Sequences:

Every bounded sequence has at least one limit point.

Cauchy's General principle of Convergence :-

A necessary and sufficient condition for the convergence of a sequence (x_n) is that, for each $\epsilon > 0$, $\exists$ $m \in I^+$ such that $|x_{n+p} - x_n| < \epsilon$. $\forall n \geq m$ and $p \geq 1$.

Necessary Condition :-

Let the sequence (x_n) be convergent and let it be convergent to l.

$\therefore \underset{n\to\infty}{Lt}\, x_n = l$

i.e. $x_n \to l$ as $n \to \infty$

∴ Given $\epsilon > 0$, $\exists\ m \in I^+$ such that

$|x_n - l| < \epsilon/2 \ \forall\ n \geq m$. —— (1)

Since $p \geq 1 \Rightarrow n+p \geq n+1 > n \geq m$.

∴ $|x_{n+p} - l| < \epsilon/2 \ \forall\ n \geq m\ \&\ p \geq 1$ —— (2)

Now we have

$$|x_{n+p} - x_n| = |x_{n+p} - l + l - x_n|$$
$$\leq |x_{n+p} - l| + |x_n - l|$$
$$< \epsilon/2 + \epsilon/2 \ \forall\ n \geq m \text{ and } p \geq 1.$$

$\Rightarrow |x_{n+p} - x_n| < \epsilon \ \forall\ n \geq m$ and $p \geq 1$.

Sufficient Condition:—

Given that for each $\epsilon > 0$, $\exists\ m \in I^+$ such that $|x_{n+p} - x_n| < \epsilon$ $\forall\ n \geq m$ and $p \geq 1$

In particular $n = m$.

∴ $|x_{m+p} - x_m| < \epsilon \ \forall\ p \geq 1$.

$\Rightarrow -\epsilon < x_{m+p} - x_m < \epsilon \ \forall\ p \geq 1$.

$\Rightarrow x_m - \epsilon < x_{m+p} < x_m + \epsilon \ \forall\ p \geq 1$

$\{x_n\} = \{\frac{1}{n}\}$
$x_m = \frac{1}{m}$
$x_{m+1} = \frac{1}{m+1}$
$\frac{1}{m} - \epsilon < x_{m+1} < \frac{1}{m} + \epsilon$
$\frac{1}{2} - \epsilon < \frac{1}{2+1} < \frac{1}{2} + \epsilon$
$0 < 0.33 < 0.6$

Let $h = \text{Min}\{x_1, x_2, \ldots, x_{m-1}, x_m - \epsilon\}$.

$K = \text{Max}\{x_1, x_2, \ldots, x_{m-1}, x_m + \epsilon\}$

∴ $h \leq x_n \leq K \ \forall\ n$.

∴ (x_n) is bounded.

∴ By Bolzano–Weierstrass theorem every bounded sequence has at least [illegible]

i.e. the sequence (x_n) has a limit point say l.

we shall show that the sequence (x_n) converges to l:

~~i.e. $\lim x_n = l$~~ —— (3)

Given that, for each $\epsilon > 0$, $\exists\ m \in$ such that

$|x_{n+p} - x_n| < \epsilon/3 \ \forall\ n \geq m\ \&\ p \geq 1$ —— (4)

In particular $n = m$

∴ $|x_{m+p} - x_m| < \epsilon/3 \ \forall\ p \geq 1$ —— (5)

Since l is a limit point:

∴ $\exists\ m_1 > m$ such that

$|x_{m_1} - l| < \epsilon/3$ —— (6)

since $m_1 > m$

∴ from (5),

$|x_{m_1} - x_m| < \epsilon/3$ —— (7)

Now we have

$$|x_{m+p} - l| = |x_{m+p} - x_m + x_m - x_{m_1} + x_{m_1} - l|$$
$$\leq |x_{m+p} - x_m| + |x_m - x_{m_1}| + |x_{m_1} - l|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 \ \forall\ p \geq 1$$
$$= \epsilon$$

∴ $|x_{m+p} - l| < \epsilon \ \forall\ p \geq 1$

$\Rightarrow |x_n - l| < \epsilon \ \forall\ n \geq m$.

∴ (x_n) is convergent to l.

Cauchy sequence.

A sequence (x_n) is said to be Cauchy sequence (or) fundamental sequence.

if for each $\epsilon > 0$, $\exists\, m \in I^+$ such that

$|x_{n+p} - x_n| < \epsilon \;\forall\, n \geq m$ and $p \geq 1$.

(or) $|s_p - s_q| < \epsilon \;\forall\, p, q \geq m$.

Theorem → Every Cauchy's sequence is bounded.

Theorem If $X = (x_n)$ is a convergent sequence of real numbers then X is a Cauchy sequence.

Note: A sequence cannot converge if for each $\epsilon > 0$, $\exists\, m \in I^+$ such that

$|x_{n+p} - x_n| \not< \epsilon \;\forall\, n \geq m$ & $p \geq 1$.

sequence which is bounded above ... the least upper bound ... sequence (x_n).

$\leq x \;\forall n$.

is given then $x-\epsilon$ is not ... bound of the sequence (x_n)

it one term of the sequence x_m in the interval $(x-\epsilon, x]$.

$\in (x-\epsilon, x]$

$-\epsilon < x_m \leq x < x+\epsilon$ — (1)

) is monotonically increasing

$x_n \leq x_{n+1} \;\forall n \in N$.

$< x_m \leq x_{m+1} \leq x_{m+2} \leq \cdots$

$\leq x < x+\epsilon$

$\epsilon < x_n < x+\epsilon \;\forall n \geq m$

$< x_n - x < \epsilon \quad \forall n \geq m$

$x_n - x| < \epsilon \quad \forall n \geq m$.

$\lim_{n\to\infty} x_n = x$

the sequence (x_n) converges

Let the sequence (x_n) be ...ded ↑ and let (x_n) be ... and which is not bounded

Prove that it diverges to ∞.

(x_n) is M↑ and which is

Now ∃ atleast one term of the sequence (x_n) is x_m such that

$x_m > K$, $K > 0$ (however large) — (1)

Since the sequence (x_n) is M↑ sequence.

$\therefore x_n \leq x_{n+1} \;\forall x_n$

(1) ≡ $K < x_m \leq x_{m+1} \leq \cdots$

$\Rightarrow K < x_n \;\forall n \geq m$.

$\therefore (x_n)$ diverges to $+\infty$

i.e. $\lim x_n = \infty$

Theorem :- If $Y = (y_n)$ is a bounded decreasing sequence then

$\lim (y_n) = \inf \{y_n : n \in N\}$

further if (y_n) is an unbounded decreasing then $\lim y_n = -\infty$.

(OR)

Every Monotonically decreasing sequence, which is bounded below converges to its greatest lower bound.

further, Every Monotonically decreasing sequence which is not bounded below diverges to $-\infty$.

Monotone Convergence theorem:

A monotone sequence of real numbers convergent iff it is bounded

Necessary Condition :- Let the Monotone

$) = 0$

$n \in N$

$n \in N$

$+ n \in N$

$n \in N$

$-①$

0

Monotonic Sequences :-

→ A sequence (x_n) is said to be monotonically increasing if $x_{n+1} \geq x_n$ $\forall n \in N$; i.e. $x_n \leq x_{n+1}$ $\forall n$

i.e. $x_1 \leq x_2 \leq x_3 \cdots \leq x_n \leq x_{n+1}$

→ A sequence (x_n) is said to be monotonically decreasing if $x_{n+1} \leq x_n$ $\forall n \in N$ i.e. $x_n \geq x_{n+1}$ $\forall n$

i.e. $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$

→ A sequence (x_n) is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

→ A sequence is said to be strictly monotonically increasing if $x_n < x_{n+1}$ $\forall n$.

→ A sequence (x_n) is said to be strictly monotonically decreasing if $x_n > x_{n+1}$ $\forall n$.

→ A sequence (x_n) is said to be strictly monotonic if it is either strictly increasing or strictly decreasing.

Ex-① :-

$(1, 2, 3, 4, \ldots n, \ldots)$, $(1, 2, 2, 3, 3, 3, 4, 4, 4, \ldots)$

$(a, a^2, a^3, \ldots a^n, \ldots)$ if $a > 1$ are increasing sequences

$\left(1 - \frac{1}{n} : n \in N\right) = \left(0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots\right)$

② $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}\right)$, $\left(1, \frac{1}{2}, \frac{1}{2^2}, \ldots \frac{1}{2^{n-1}}, \ldots\right)$

$(b, b^2, b^3, \ldots b^n, \ldots)$ if $0 < b < 1$.

$\left(1 + \frac{1}{n} : n \in N\right) = \left(1+1, 1+\frac{1}{2}, 1+\frac{1}{3}, \ldots\right)$ are decreasing sequences.

③ $(+1, -1, +1, \ldots (-1)^{n+1}, \ldots)$; $(-1, +2, -3, \ldots (-1)^n n, \ldots)$ are not monotonic sequences. Because which are neither increasing nor decreasing.

Theorem : If $X = (x_n)$ is a bounded increasing sequence, then $\lim_{n \to \infty} (x_n) = \sup \{ x_n : n \in N \}$. Further if (x_n) is unbounded increasing sequence then $\lim_{n \to \infty} x_n = \infty$.

(OR) —

Every monotonically increasing sequence which is bounded above converge to its least upperbound. Further every monotonically increasing sequence which is not bounded above diverges to ∞.

Proof :- Case I :

Let $X = (x_n)$ be a bounded increasing sequence.

and let $X = (x_n)$ be a monotonically increasing which is bounded above.

To prove that the (x_n) converges to its least upper bound.

Since (x_n) is monotonically increasing

) = 0

$n \in N$

$n \in N$

$\forall n \in N$

$n \in N$.

(1)

0

Problems

→ Use the definition of the limit of a sequence to establish the following limits.

(1) $\underset{}{Lt}\left(\frac{3n+1}{2n+5}\right) = 3/2$

Sol^n:- Let $\epsilon>0$ be given

Now $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{-13}{4n+10}\right|$

$= \frac{13}{4n+10} < \frac{13}{4n}$

$< \epsilon$

if $\frac{4}{13} n > \frac{1}{\epsilon}$

i.e. $n > \frac{13}{4\epsilon}$

Now if m is a +ve integer $> \frac{13}{4\epsilon}$

then $\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| < \epsilon \ \forall\ n \geq m.$

$\therefore \frac{3n+1}{2n+5} \to 3/2$ as $n \to \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{3n+1}{2n+5} = 3/2$

(or)

Now we have

$\left|\frac{3n+1}{2n+5} - 3/2\right| = \left|\frac{-13}{4n+10}\right|$

$= \frac{13}{4n+10} < \frac{13}{4n}$ —— ①

For $\epsilon>0$, by Archemedian property

$\exists\ K \in I^+$ Such that $K\epsilon > \frac{13}{4}$.

$\Rightarrow \frac{1}{K} < \frac{4}{13}\epsilon$ —— ②

Now we have

$n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K}$

$\Rightarrow \frac{1}{n} \leq \frac{1}{K} < \frac{4}{13}\epsilon$ (by ② —— ③

$\therefore$ ① $= \left|\frac{3n+1}{2n+5} - 3/2\right| < \frac{13}{4}\left(\frac{4}{13}\right)\epsilon\ \forall\ n \geq K$ (by ③)

$\therefore \left|\frac{3n+1}{2n+5} - 3/2\right| < \epsilon \ \forall\ n \geq K.$

i.e. $\frac{3n+1}{2n+5} \to 3/2$ as $n \to \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{3n+1}{2n+5} = 3/2$

② $\underset{n\to\infty}{Lt} \frac{1}{\sqrt{n+7}} = 0$

③ $\underset{n\to\infty}{Lt} \left[(-1)^n \cdot \frac{n}{n^2+1}\right] = 0$

④ $\underset{n\to\infty}{Lt} \frac{1}{3^n} = 0$

Sol^n:- Let $\epsilon>0$ be given

Now we have

$\left|\frac{1}{3^n} - 0\right| = \left|\frac{1}{3^n}\right|$

$= \frac{1}{3^n} < \epsilon$ if $3^n > \frac{1}{\epsilon}$

i.e. if $n \log 3 > \log\left(\frac{1}{\epsilon}\right)$

i.e. if $n > \frac{\log(1/\epsilon)}{\log 3}$ ($\because$ log

Now if m is +ve integer $> \frac{\log(1/\epsilon)}{\log 3}$

then $\left|\frac{1}{3^n} - 0\right| < \epsilon \ \forall\ n \geq m$

i.e. $\frac{1}{3^n} \to 0$ as $n \to \infty$

i.e. $\underset{n\to\infty}{Lt} \frac{1}{3^n} = 0$

Ex: If $x_n = 1 + \frac{(-1)^n}{2n}$, find the least +ve integer m such that

$|x_n - 1| < \frac{1}{10^3} \ \forall \ n > m.$

Sol'n :- Now $|x_n - 1| = \left| 1 + \frac{(-1)^n}{2n} - 1 \right|$

$$= \left| \frac{(-1)^n}{2n} \right|$$

$$= \frac{1}{2n} \quad \text{--- (1)}$$

Since $|x_n - 1| \leq \frac{1}{10^3}$

$\Rightarrow \frac{1}{2n} < \frac{1}{10^3}$ (by (1))

$\Rightarrow 2n > 10^3$

$\Rightarrow n > 500$

$\therefore$ Taking $m = 500$,

we have

$|x_n - 1| < \frac{1}{10^3} \ \forall \ n > m$ where $m = 500$

H.W: If $x_n = 2 + \frac{(-1)^n}{n^2}$,

find the least +ve integer m such that $|x_n - 2| < \frac{1}{10^4} \ \forall \ n > m$.

Theorem Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$, if (a_n) is a sequence of +ve real numbers with $\lim_{n\to\infty}(a_n) = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq c a_n \ \forall \ n \geq m$, then it follows that

$\lim x_n = x$.

[illegible]

If $x > -1$ then $(1+x)^n \geq 1 + nx \ \forall n \in \mathbb{N}$.

Problems

(1) If $a > 0$ then $\lim \left(\frac{1}{1+na} \right) = 0$

Sol'n: Since $a > 0$

$\Rightarrow na > 0 \ \forall n \in \mathbb{N}$

$\Rightarrow 0 < na < 1 + na \ \forall n \in \mathbb{N}$

$\Rightarrow 0 < \frac{1}{1+na} < \frac{1}{na} \ \forall n \in \mathbb{N}$

Now we have

$$\left| \frac{1}{1+na} - 0 \right| = \left| \frac{1}{1+na} \right|$$

$$= \frac{1}{1+na} < \frac{1}{na} \ \forall n \in \mathbb{N}$$

$$= \left(\frac{1}{a}\right)\left(\frac{1}{n}\right) \ \forall n \in \mathbb{N}.$$

$\therefore \left| \frac{1}{1+na} - 0 \right| < \left(\frac{1}{a}\right)\left(\frac{1}{n}\right) \ \forall n \in \mathbb{N}$ —— (1)

Since $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$

and $a > 0$

$\Rightarrow \frac{1}{a} > 0$.

$\therefore \lim_{n\to\infty}\left(\frac{1}{1+na}\right) = 0.$

(2) If $0 < b < 1$ then $\lim(b^n) = 0$

Sol'n :- Since $0 < b < 1$

Take $b = \frac{1}{1+a}$

where $a = \left(\frac{1}{b}\right) - 1$

$\Rightarrow a > 0$

By Bernoulli's inequality,

we have $(1+a)^n \geq 1 + na \ \forall n$.

$\Rightarrow \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \ \forall n \in \mathbb{N}$ —— (1)

Now, $0 < b^n = \frac{1}{(1+a)^n}$

$\leq \frac{1}{1+na}$ (by ①)

$< \frac{1}{na} \; \forall n \in N$ ——— ②

Now we have

$|b^n - 0| = \left| \frac{1}{(1+a)^n} - 0 \right|$

$= \frac{1}{1+na}$

$< \frac{1}{na}$ (by ②) $\forall n \in N$

$= \left(\frac{1}{a}\right)\left(\frac{1}{n}\right)$

Since $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$

and $a > 0$

$\Rightarrow \frac{1}{a} > 0$.

$\therefore \lim\limits_{n\to\infty} b^n = 0$.

$n \to \infty$

Sol'n :- Case(i):

Let $c = 1$ then

$(c^{1/n}) = (1, 1, 1, \ldots)$

$\therefore \lim_{n\to\infty} (c^{1/n}) = 1.$

Case(ii): Let $c > 1$;

then $c^{1/n} = 1 + d_n$ for some $d_n > 0$

$\Rightarrow \boxed{c^{1/n} - 1 = d_n}$

and $c = (1+d_n)^n \geq (1+nd_n) \forall n$ (by Bernoullis inequality)

$\Rightarrow c \geq 1 + nd_n \;\forall n$

$\Rightarrow \frac{c-1}{n} \geq d_n \;\forall n$ ——— ①

Now we have,

$|c^{1/n} - 1| = d_n$

$\leq \frac{c-1}{n} \;\forall n$ (by ①)

$= (c-1)\left(\frac{1}{n}\right)$ ——— ②

Since $\lim (1/n) = 0$

and $(c-1) > 0$ $(\because c > 1)$

$\therefore \lim_{n\to\infty} (c^{1/n}) = 1$

Case(iii):

Let $0 < c < 1$

then $c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$

$\Rightarrow c = \frac{1}{(1+h_n)^n}$ ——— ③

By Bernoulli's inequality,

$(1+h_n)^n \geq 1 + nh_n \;\forall n \in N$

Rough Idea

If $c = 2$ then $c^{1/n} = 2^{1/n}$

$= 2^1, 2^{1/2}, 2^{1/3}, \ldots$

$= 2, \sqrt{2}, \sqrt[3]{2}, \ldots$

$= 2, 1.414, \ldots$

$= 1+1, 1+0.414, \ldots \longrightarrow 1$

$= 1 + d_n; \; d_n > 0$

If $c = 3$ then $c^{1/n} = 3^{1/n}$

$= 3^1, 3^{1/2}, 3^{1/3}, \ldots$

$= 3, \sqrt{3}, \sqrt[3]{3}, \ldots$

$= 3, 1.732, \ldots \longrightarrow 1$

$= 1+2, 1+0.732, \ldots$

$= 1 + d_n; \; d_n > 0.$

Rough Idea

$c = 0.5$

$= 1/2$

$c^{1/n} = (1/2)^{1/n}$

$= (1/2)^1, (1/2)^{1/2}, (1/2)^{1/3} \ldots$

$= \frac{1}{1+1}, \frac{1}{1+0.414}, \ldots \longrightarrow 1$

$= \frac{1}{1+h_n}; \; h_n > 0.$

$\therefore \frac{1}{1+h_n} \leq \frac{1}{1+nh_n} \quad \forall n$ ——— (4)

From (3) & (4)

$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} \quad \forall n$

$\Rightarrow c \leq \frac{1}{1+nh_n} < \frac{1}{nh_n} \quad \forall n$ ——— (5)

Now we have,

$0 < c < \frac{1}{nh_n} \quad \forall n$

$\Rightarrow \quad 0 < ch_n < \frac{1}{n} \quad \forall n$

$\Rightarrow \quad 0 < h_n < \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \quad \forall n$ ——— (6)

Now we have,

$\left|c^{x_n} - 1\right| = \left|\frac{1}{1+h_n} - 1\right|$

$= \left|\frac{-h_n}{1+h_n}\right|$

$= \frac{h_n}{1+h_n} < h_n \quad \forall n$

$< \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \quad \forall n$

(by (6))

Since $\lim_{n\to\infty} \frac{1}{n} = 0$

& $c > 0$.

$\Rightarrow \frac{1}{c} > 0$

$\therefore \lim_{n\to\infty} c^{x_n} = 1.$

2003 → Let 'a' be a +ve real number (i.e $a>0$) and $\{x_n\}$ a sequence of rational numbers such that $\lim_{n\to\infty} x_n = $

Show that

$\lim_{n\to\infty} a^{x_n} = 1.$

Sol^n: Given that $\{x_n\}$ a sequence of rational numbers such that $\lim_{n\to\infty} x_n$

Let the sequence

$\{x_n\} = \{\frac{1}{n}\}$

then we show that

$\lim_{n\to\infty} a^{x_n} = 1$

(proceed as in the above problem.)

→ Prove that the sequences whose n^{th} terms are given below, are monotonic. Find out whether they are increasing (or) decreasing

(i) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$.

Sol^n: Let $x_n = \frac{1}{2} + \frac{1}{2^2} + \cdots +$

and $x_{n+1} = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}+\frac{1}{2^{n+1}}$

Now $x_{n+1}-x_n = \frac{1}{2^{n+1}} > 0 \ \forall n$

$\Rightarrow x_{n+1} > x_n \ \forall n$

$\Rightarrow x_n < x_{n+1} \ \forall n$

$\therefore (x_n)$ is an increasing sequence.

$\therefore (x_n)$ is monotonic sequence

(ii) $\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{2n-1}$

(iii) $\frac{3n+7}{4n+8}$ (iv) $\frac{2n+7}{3n+8}$

(v) $1+\frac{1}{1!}+\frac{1}{2!}+\cdots\cdots+\frac{1}{(n-1)!}$

(vi) $\frac{1}{2n+5}$ (vii) $\frac{-1}{2n+1}$

(viii) $1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}$ (ix) $\left(1+\frac{1}{n}\right)^n$

Solⁿ: Let $x_n = \left(1+\frac{1}{n}\right)^n$

$$= {}^nC_0(1)^n\left(\frac{1}{n}\right)^0 + {}^nC_1(1)^{n-1}\left(\frac{1}{n}\right)^1 + {}^nC_2(1)^{n-2}\left(\frac{1}{n}\right)^2+\cdots\cdots+{}^nC_n(1)^{n-n}\left(\frac{1}{n}\right)^n.$$

$$= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right)+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right)+\cdots\cdots+\frac{n(n-1)(n-2)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left[1-\frac{(n-1)}{n}\right] \quad (1)$$

Now

$$x_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\cdots\cdots+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n}\right)\left(1-\frac{n}{n+1}\right) \quad (2)$$

Now we have,

$\frac{1}{n+1} < \frac{1}{n} \ \forall n \in \mathbb{N}$

$\Rightarrow \frac{K}{n+1} < \frac{K}{n} \ \forall n \in \mathbb{N}$

$\Rightarrow \frac{-K}{n+1} > -\frac{K}{n} \ \forall n \in \mathbb{N}$

$\Rightarrow 1-\frac{K}{n+1} > 1-\frac{K}{n}$; $K=1,2\ldots n$

From (1) & (2) we have

$x_{n+1} > x_n \ \forall n$

$\therefore (x_n)$ is M ↑

$\therefore (x_n)$ is monotonic sequence.

(x) $a_1=1$ and $a_n=\sqrt{2+a_{n-1}}$ $\forall n \geq 2$

Solⁿ: Given that

$a_1=1$ & $a_n=\sqrt{2+a_{n-1}}$ $\forall n\geq 2$

$\therefore a_2=\sqrt{2+a_1}$

$=\sqrt{2+1}$

$=\sqrt{3}>1=a_1$

$\therefore a_2>a_1$.

Similarly $a_3>a_2$

Now suppose, $a_n > a_{n-1}$ for some n, $n>1$.

$\Rightarrow 2+a_n > 2+a_{n-1}$

$\Rightarrow \sqrt{2+a_n} > \sqrt{2+a_{n-1}}$

$\Rightarrow a_{n+1} > a_n$

$\therefore$ By mathematical induction

$a_n < a_{n+1}$ $\forall n$

$\therefore (a_n)$ is an increasing sequence.

$\therefore (a_n)$ is monotonic sequence

→ Show that the sequence (x_n) defined by $x_n = \left(1+\frac{1}{n}\right)^n$ is cgt.

Sol^n: Given that $x_n = \left(1+\frac{1}{n}\right)^n$ $\forall n \in N$

$\Rightarrow x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

and $x_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$

$\therefore x_n \leq x_{n+1}$ $\forall n$

$\therefore (x_n)$ is an increasing sequence.

$\therefore (x_n)$ is monotonic sequence

Since $x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$

$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}$

$= 1+1+\frac{1}{2\cdot1}+\frac{1}{3\cdot2\cdot1}+\cdots+\frac{1}{n(n-1)\cdots3\cdot2\cdot1}$

$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}$

$= 1+\dfrac{\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$ (By G.P. i.e. $\frac{a(1-r^n)}{1-r}$)

$= 1+2\left(1-\frac{1}{2^n}\right)$

$= 3-\frac{1}{2^{n-1}}$

< 3 $\forall n$

$\therefore x_n < 3$ $\forall n$

$\therefore (x_n)$ is bdd above.

Since (x_n) is monotonically increasing & bdd above.

$\therefore (x_n)$ is cgt.

Note: Clearly $2 \leq x_n$ $\forall n$

$\therefore 2 \leq x_n < 3$ $\forall n$

$\Rightarrow 2 \leq \lim_{n\to\infty} x_n < 3$

$\Rightarrow 2 \leq \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n < 3$

$\Rightarrow \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

→ Discuss the convergence of the sequence (x_n) where

(i) $x_n = \frac{n+1}{n}$ (ii) $x_n = \frac{n}{n^2+1}$

~~(iii) $x_n = 1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^n}$.~~

→ P.T the sequence $\left\{\frac{2n-7}{3n+2}\right\}$,

(i) is monotonically increasing

(ii) is bounded

(iii) tends to the limit $\frac{2}{3}$.

Sol^n: Let $x_n = \frac{2n-7}{3n+2}$ $\forall n$.

$x_{n+1} = \frac{2n-5}{3n+5}$

$\therefore x_{n+1} - x_n = \frac{25}{(3n+5)(3n+2)}$

$> 0 \ \forall n$

$\therefore x_{n+1} \geq x_n \ \forall n$

$\therefore (x_n)$ is monotonically increasing

(ii) The given sequence is

$\{-1, -\frac{3}{8}, -\frac{1}{11}, \frac{1}{14}, \frac{3}{17}, \cdots\}$ approach to 1

Clearly $x_n \geq -1 \ \forall n$

and also $1 - x_n = 1 - \frac{2n-7}{3n+2}$

$= \frac{n+9}{3n+2} > 0 \ \forall n$

$\therefore 1 - x_n > 0 \ \forall n$

$\Rightarrow 1 > x_n \ \forall n$

$\Rightarrow x_n < 1 \ \forall n$

$\Rightarrow -1 \leq x_n < 1 \ \forall n$

$\therefore (x_n)$ is bounded.

(iii) Since (x_n) is M↑ and bdd above.

$\therefore$ It cgs.

Now $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n-7}{3n+2}$

$= \lim_{n\to\infty} \frac{2-\frac{7}{n}}{3+\frac{2}{n}}$

$= \frac{2}{3}$.

$\therefore (x_n)$ cgs to $2/3$.

H.W. → P.T the sequence whose n^{th} term is $\frac{3n+4}{2n+1}$

(i) is monotonically decreasing

(ii) is bdd and

(iii) cgs to $3/2$.

1999 P-II ~~Show that the sequence~~ (x_n) defined by $x_{n+1} = \sqrt{3x_n}$, $x_1 = 1$ cgs to 3.

Sol^n: Given that $x_1 = 1$,

$x_{n+1} = \sqrt{3x_n} \ \forall n$

Now $x_2 = \sqrt{3x_1}$

$= \sqrt{3(1)}$

$= \sqrt{3} > 1 = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$

Now suppose $x_{n+1} > x_n$

$\Rightarrow 3x_{n+1} > 3x_n$

$\Rightarrow \sqrt{3x_{n+1}} > \sqrt{3x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$

$\therefore$ By mathematical induction

$x_{n+1} > x_n \ \forall n$

$\therefore (x_n)$ is monotonically increasing.

Now $x_1 = 1 < 3$

$x_2 = \sqrt{3x_1}$

$= \sqrt{3} < 3$

$x_3 = \sqrt{3x_2}$

$= \sqrt{3 \cdot \sqrt{3}} < 3$

Suppose $x_n < 3$

then $x_{n+1} = \sqrt{3x_n}$

$< \sqrt{3 \cdot 3} = 3$

$\therefore x_{n+1} < 3$.

$\therefore$ By mathematical induction

$x_n < 3 \ \forall n$.

$\therefore (x_n)$ is bdd above by 3.

Since (x_n) M. $\uparrow$ and bdd above.

$\therefore$ It is cgt.

Now let $\lim\limits_{n\to\infty} x_n = l$

then $\lim\limits_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{3x_n} \ \forall n$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = \sqrt{3 \cdot \lim\limits_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{3l}$

$\Rightarrow l^2 - 3l = 0$

$\Rightarrow l(l-3) = 0$

$\Rightarrow l = 0, \ l = 3$.

But $l \neq 0$, since $x_n \geq 1$

$\therefore \lim\limits_{n\to\infty} x_n = 3$

H.W. $\rightarrow$ Show that the seque (x_n), where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}} \ \forall n \geq 2$ is cgt and cgs to 2.

$\rightarrow$ P.T the sequence $\{x$ defined by $x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{7}$ cgs to the +ve root of the equation $x^2 - x - 7 =$

Solⁿ: Given $x_1 = \sqrt{7}$, $x_{n+1} = \sqrt{7+x_n}$

$x_2 = \sqrt{7+x_1}$

$= \sqrt{7+\sqrt{7}} > \sqrt{7} = x_1$

$\therefore x_2 > x_1$

Similarly $x_3 > x_2$.

Suppose $x_{n+1} > x_n$ for some 'n'.

$\Rightarrow 7+x_{n+1} > 7+x_n$

$\Rightarrow \sqrt{7+x_{n+1}} > \sqrt{7+x_n}$

$\Rightarrow x_{n+2} > x_{n+1}$

$\therefore$ By mathematical induction

$x_{n+1} > x_n \ \forall n$.

$\therefore (x_n)$ is M ↑

Now $x_1 = \sqrt{7} < 7$

$x_2 = \sqrt{7+\sqrt{7}} \le 7$

Similarly $x_3 < 7$.

Suppose $x_n < 7$

$\Rightarrow 7+x_n < 14$

$\Rightarrow \sqrt{7+x_n} < \sqrt{14}$

$\Rightarrow x_{n+1} < \sqrt{14} < \sqrt{49} = 7$

$\Rightarrow x_{n+1} < 7$

$\therefore$ By mathematical induction

$x_n < 7 \ \forall n$.

$\therefore (x_n)$ is bdd above.

Since (x_n) is M ↑ & bdd above.

$\therefore$ It is cgt.

Let $\lim_{n\to\infty} x_n = l$ &

$\lim_{n\to\infty} x_{n+1} = l$

Now $x_{n+1} = \sqrt{7+x_n}$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \sqrt{7+\lim_{n\to\infty} x_n}$

$\Rightarrow l = \sqrt{7+l}$

$\Rightarrow l^2 - l - 7 = 0$

$\Rightarrow l = \dfrac{1 \pm \sqrt{29}}{2}$

But $l = \dfrac{1-\sqrt{29}}{2} < 0$, where as $x_n > 0 \ \forall n$.

$\therefore l \ne \dfrac{1-\sqrt{29}}{2}$.

$\therefore x_n$ cgs to $\dfrac{1+\sqrt{29}}{2}$,

which is +ve root of the equation $x^2 - x - 7 = 0$

H.W. P.T. the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2+x_n}$ cgs to the +ve root of the equation

$x^2 - x - 2 = 0$

$\to$ Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ $\forall n$

Show that (x_n) is bdd and monotone, find the limit.

Sol^n: Given that

$x_1 = 8$, $x_{n+1} = \frac{1}{2}x_n + 2$

$x_2 = \frac{1}{2}x_1 + 2$

$= \frac{1}{2}(8) + 2 = 6 < 8 = x_1$

$\therefore x_2 < x_1$

Similarly $x_3 < x_2$.

Now Suppose $x_{n+1} < x_n$ for some 'n'.

$\Rightarrow \frac{1}{2}x_{n+1} < \frac{1}{2}x_n$

$\Rightarrow \frac{1}{2}x_{n+1} + 2 < \frac{1}{2}x_n + 2$

$\Rightarrow x_{n+2} < x_{n+1}$

$\therefore$ By mathematical induction

$x_{n+1} < x_n$ $\forall n$.

$\therefore (x_n)$ is Monotonically decreasing.

But w.k.T every decreasing sequence is always bdd below.

Since $x_n > x_{n+1}$ $\forall n \in N$.

$\Rightarrow x_n > \frac{1}{2}x_n + 2$ $\forall n$

$\Rightarrow 2x_n > x_n + 4$ $\forall n$

$\Rightarrow x_n > 4$ $\forall n$

$\therefore (x_n)$ is bdd below

$\therefore (x_n)$ is bdd.

Since (x_n) is M $\downarrow$ and bdd below

$\therefore$ It is cgt.

Let $\lim_{n\to\infty} x_n = l$ & $\lim_{n\to\infty} x_{n+1} = l$

Since $x_{n+1} = \frac{x_n}{2} + 2$ $\forall n$

$\Rightarrow \lim_{n\to\infty} x_{n+1} = \frac{1}{2}\lim_{n\to\infty} x_n + 2$

$\Rightarrow l = \frac{1}{2}l + 2$

$\Rightarrow l = 4$

$\therefore \lim_{n\to\infty} x_n = 4$.

H.W. $\to$ Let $Y = (y_n)$ be defined inductively by $y_1 = 1$;

$y_{n+1} = \frac{1}{4}(2y_n + 3)$ $\forall n \geq 1$.

Show that $\lim Y = 3/2$

H.W. $\to$ Let $Z = (z_n)$ be the seque of real numbers defined by $z_1 = 1$, $z_{n+1} = \sqrt{2z_n}$ for $n \in N$ then show that

$\lim_{n\to\infty}(z_n) = 2$

→ Let $y_1 = \sqrt{p}$, $p>0$; $y_{n+1} = \sqrt{p+y_n}$ for $n \in N$

Show that (y_n) cgs and find limit.

Solⁿ: $y_1 = \sqrt{p}$; $p > 0$ &

$$y_{n+1} = \sqrt{p+y_n} \quad \forall n$$

Now $y_2 = \sqrt{p+y_1}$

$= \sqrt{p+\sqrt{p}} > \sqrt{p} = y_1$

$\therefore y_2 > y_1$

Similarly $y_3 > y_2$

Now Suppose, $y_{n+1} > y_n$

$\Rightarrow p + y_{n+1} > p + y_n$; $p>0$

$\Rightarrow \sqrt{p+y_{n+1}} > \sqrt{p+y_n}$

$\Rightarrow y_{n+2} > y_{n+1}$

$\therefore$ By mathematical induction

$y_{n+1} > y_n \quad \forall n$

$\therefore (y_n)$ is M ↑.

Since $y_n < y_{n+1} \quad \forall n$

$\Rightarrow y_n < \sqrt{p+y_n} \quad \forall n$

$\Rightarrow y_n^2 < p + y_n \quad \forall n$

$\Rightarrow y_n^2 - y_n - p < 0 \quad \forall n$

$$\Rightarrow \left[y_n - \frac{1+\sqrt{1+4p}}{2}\right]\left[y_n - \frac{1-\sqrt{1+4p}}{2}\right] < 0$$

$$\Rightarrow \left[y_n - \frac{1+\sqrt{1+4p}}{2}\right] < 0$$

$\left(\because y_n - \frac{1-\sqrt{1+4p}}{2} > 0\right)$

$\Rightarrow y_n < \frac{1+\sqrt{1+4p}}{2} \quad \forall n$

$< \frac{1+1+\sqrt{1+4p}}{2}$

$< 1 + \frac{\sqrt{1+4p}}{2}$

$< 1 + \frac{\sqrt{4p}}{2}$

$< 1 + \sqrt{p}$

$\therefore (y_n)$ is bdd.

Since (y_n) is M ↑ & bdd above.

$\therefore$ It is cgt.

To find limit of (y_n):

Let $\lim_{n\to\infty} y_n = l$ & $\lim_{n\to\infty} y_{n+1} = l$

Now $y_{n+1} = \sqrt{p+y_n}$; $p>0$

$\Rightarrow \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \sqrt{p+y_n}$

$\Rightarrow l = \sqrt{p+l}$

$\Rightarrow l^2 - l - p = 0$

$\Rightarrow l = \frac{1 \pm \sqrt{1+4p}}{2}$

$\therefore l = \frac{1+\sqrt{1+4p}}{2}$ $\left(\because l = \frac{1-\sqrt{1+4p}}{2} < 0 \text{ but } y_n > 0\right)$

$\therefore \lim_{n\to\infty} y_n = \frac{1}{2}\left(1+\sqrt{1+4p}\right)$.

→ Let $y_n = \sqrt{n+1} - \sqrt{n}$ $\forall n \in N$

Show that (y_n) and $(\sqrt{n}\, y_n)$ converge. Find their limits.

Sol: (i) $y_n = \sqrt{n+1} - \sqrt{n}$

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\leq \frac{1}{\sqrt{n} + \sqrt{n}} \quad (\because \sqrt{n+1} \geq \sqrt{n})$$

$\therefore y_n \leq \frac{1}{2\sqrt{n}}$ —— (A)

Since $0 < y_n$ $\forall n$ —— (B)

from (A) & (B), we have

$$0 < y_n \leq \frac{1}{2\sqrt{n}}$$

which is in the form of $x_n < y_n \leq z_n$ $\forall n$.

with $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} z_n$

$\therefore$ By squeeze theorem $\lim y_n = 0$

$\therefore (y_n)$ cgs to zero.

(ii) $\sqrt{n}\, y_n = \sqrt{n}\left(\frac{1}{\sqrt{n+1} + \sqrt{n}}\right)$

$$\leq \sqrt{n}\left(\frac{1}{2\sqrt{n}}\right) \quad (\text{By (A)})$$

$$= \frac{1}{2}$$

$\therefore \sqrt{n}\, y_n \leq \frac{1}{2}$ —— (C)

Now since

$$\sqrt{n+1} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$$

$$\Rightarrow 2\sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$$

$$\Rightarrow \frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

$$\Rightarrow \frac{\sqrt{n}}{2\sqrt{n+1}} < \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n+1}}$$

$$\Rightarrow \frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}\, y_n \quad \text{—— (D)}$$

from (C) & (D), we have

$$\frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}\, y_n \leq \frac{1}{2}$$

of the form

$$x_n < v_n \leq z_n.$$

with $\lim x_n = \lim z_n = \frac{1}{2}$

$\therefore$ By squeeze theorem

$$\lim_{n\to\infty} (\sqrt{n}\, y_n) = \frac{1}{2}$$

→ Establish the convergence of the sequence (y_n), where $y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$; $n \in N$.

Sol: $y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}$

Now $y_{n+2} - y_{n+1} = \left[\frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}\right] - \left[\cancel{\frac{1}{n+1}} + \frac{1}{\cdots} + \cdots + \cancel{\frac{1}{2n}}\right]$

$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$= \frac{1}{2(n+1)(2n+1)} > 0 \ \forall n$

$\therefore y_{n+1} > y_n \ \forall n$

$\therefore (y_n)$ is M $\uparrow$

Now $y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$

$< \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1}$

$< \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}$

$= \frac{n}{n}$

$= 1$

$\therefore y_n < 1 \ \forall n \in N.$

$\therefore (y_n)$ is bdd above.

$\therefore (y_n)$ is cgt

→ If (b_n) is bdd sequence and $\lim_{n\to\infty} (a_n) = 0$ then show that $\lim_{n\to\infty} (a_n b_n) = 0$.

Sol: Since (b_n) is bdd sequence.

$\exists\ M > 0$ such that

$|b_n| \le M \ \forall n$ ——— (1)

and since $\lim_{n\to\infty} a_n = 0$

i.e., $a_n \to 0$ as $n \to \infty$

$\therefore$ Given $\epsilon > 0$, $\exists\ K \in Z^+$ such that $|a_n - 0| < \frac{\epsilon}{M}$, $(M > 0)$ $\forall n \ge K$ ——— (2)

Now we have

$|a_n b_n - 0| = |a_n b_n|$

$= |a_n|\,|b_n|$

$< \frac{\epsilon}{M} \cdot M \quad \forall n \ge K$

$= \epsilon$

$\therefore |a_n b_n - 0| < \epsilon \ \forall n \ge K$

$\therefore \lim_{n\to\infty} a_n b_n = 0$

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Set – III

Infinite Series

→ If $\{x_n\}$ is a sequence of real numbers, then the expression $x_1+x_2+\cdots+x_n+\cdots$ (i.e., the sum of the terms of the sequence, which are infinite in number) is called an infinite series.

The infinite series $x_1+x_2+\cdots+x_n+\cdots$ is usually denoted by $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$

The numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ are called the first, second, third, ... n^{th} term (or general term) ...... of the series.

Series of positive terms:– If all the terms of the series $\sum x_n = x_1+x_2+x_3+\cdots+x_n+\cdots$ are positive i.e, if $x_n>0$ $\forall n$, then the series $\sum x_n$ is called a series of positive terms.

Alternating Series:– A series in which the terms are alternatively +ve and −ve is called an alternating series.

i.e, $\sum(-1)^{n-1}x_n = x_1-x_2+x_3-x_4+\cdots+(-1)^{n-1}x_n+\cdots$ where $x_n>0$ $\forall n$ is an alternating series.

Partial Sums:– If $\sum x_n = x_1+x_2+\cdots+x_n+\cdots$ is an infinite series where the terms may be +ve or −ve then $S_n = x_1+x_2+\cdots+x_n$ is called the n^{th} partial sum of $\sum x_n$.

∴ The n^{th} partial sum of an infinite series is the sum of its first 'n' terms.

$s_1, s_2, s_3, \ldots$ are the first, second, third, ... partial sums of the series.

Since $n \in N$, $\{S_n\}$ is a sequence ~~of~~ called the sequence of partial sums of the infinite series $\sum x_n$.

$\therefore$ To every infinite series $\sum x_n$, there corresponds a sequence of $\{S_n\}$ of its partial sums.

Nature of infinite Series:-

(i) The series $\sum x_n$ is said to be cgs if the sequence of its partial sums cgs.

i.e, $\sum x_n$ is cgt if $\lim\limits_{n\to\infty} S_n = l$ (finite)

(ii) If $\lim\limits_{n\to\infty} S_n = +\infty$ (or) $-\infty$ then the series $\sum u_n$ is called dgs.

(iii) The series $\sum x_n$ is neither cgt nor dgt, the series $\sum x_n$ is called oscillatory series.

→ The series $\sum x_n$ oscillates ~~oscillatory~~ finitely if the sequence $\{S_n\}$ of its partial sums oscillates finitely.

i.e, $\sum x_n$ oscillates finitely if $\{S_n\}$ is bounded and neither cgt nor dgt.

→ The Series $\sum x_n$ oscillates infinitely if the sequence $\{S_n\}$ of its partial sums oscillates infinitely.

i.e, $\sum x_n$ oscillates infinitely if $\{S_n\}$ is unbounded and neither cgs nor dgs.

Discuss the cgs and dgs.

(1) $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} n$

$= 1+2+3+\cdots+n+\cdots\cdots$

Let $S_n = 1+2+\cdots\cdots+n$

$= \frac{n(n+1)}{2}$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt to $+\infty$

$\therefore \sum x_n$ is dgt to $+\infty$.

(2) $\sum x_n = \sum n^2$

$= 1^2+2^2+\cdots+n^2+\cdots\cdots$

Let $S_n = 1^2+2^2+\cdots\cdots+n^2$

$= \frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty} S_n = +\infty$

$\therefore \{S_n\}$ is dgt.

$\therefore \sum x_n$ is dgt.

(3) $\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \cdots\cdots + \frac{1}{3^{n-1}} + \frac{1}{3^n} + \cdots\cdots$

$S_n = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$

$= 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\cdots + \frac{1}{3^{n-1}}$

$= \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}}$

$= \frac{3}{2}\left(1-\frac{1}{3^{n-1}}\right)$

$\lim_{n\to\infty} S_n = 3/2$

$\therefore \{S_n\}$ is cgt.

$\therefore \sum x_n$ is cgt.

→ $\sum_{n=1}^{\infty} \frac{1}{2^n}$

→ $\sum_{n=1}^{\infty} K$ (Constant) $= K+K+\cdots+K+\cdots$

$S_n = K+K+\cdots\cdots+K.$ (n terms)

$= nK.$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt.

$\therefore$ The series $\sum x_n$ is dgt.

Note:– Every constant sequence is cgt but the constant series is dgt.

→ $\sum \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} + \cdots$

Let $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1 - 0$

$= 1$

$\therefore \{S_n\}$ is cgt to 1

$\therefore \sum x_n$ is cgt to 1.

Arithmetic Series:–

$\sum x_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d) + \cdots$

Let $S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-1)d)$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\lim S_n = \infty$

$\therefore \{S_n\}$ is dgt.

$\therefore \sum x_n$ is dgt.

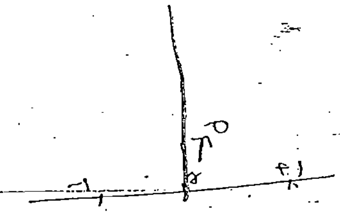
* Geometric series

$\sum_{n=0}^{\infty} r^n = 1+r+r^2+\cdots+r^{n-1}+\cdots$

(i) cgs if $-1<r<1$ i.e $|r|<1$
(ii) dgs if $r\geq 1$
(iii) oscillates finitely if $r=-1$
(iv.) oscillates infinitely if $r<-1$.

Sol Let $S_n = 1+r+r^2+\cdots+r^{n-1}$

then $S_n = \frac{1-r^n}{1-r}$



(i) if $-1<r<1$ i.e $|r|<1$

$\therefore r^n \to 0$ as $n\to\infty$

$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt} \frac{(1-r^n)}{1-r} = \frac{1}{1-r}$ (finite number)

$\therefore \{S_n\}$ is cgt.

$\therefore \sum r^n$ is cgt.

(ii) If $r\geq 1$

subcase (i): If $r=1$

then $S_n = 1+1+\cdots+1$ (n times)

$= n(1)$

$= n$

now $\underset{n\to\infty}{Lt} S_n = \infty$

$\therefore \{S_n\}$ is dgt $\Rightarrow \sum r^n$ is dgt.

subcase (ii): If $r>1$

i.e $1<r\to\infty$

$\therefore r^n \to\infty$ as $n\to\infty$

$\underset{n\to\infty}{Lt} S_n = \underset{n\to\infty}{Lt}\frac{(1-r^n)}{1-r}$ or $\underset{n\to\infty}{Lt}\frac{r^n-1}{r-1}$

$= +\infty$ or $+\infty$

$\therefore \{S_n\}$ is dgt

$\sum r^n$ is dgt.

(iii) If $r=-1$

then $S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$\therefore Lt\ S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$\therefore \{S_n\}$ is oscillatory seq.

$\therefore \sum r^n$ is oscillatory series.

This oscillatory series is finite oscillatory series.

(IV) If $r<-1$ then $-r>1$

$\Rightarrow x>1$ where $x=-r$.

$\Rightarrow 1 < x^n \to \infty$ as $n \to \infty$

i.e. $(-r)^n \to \infty$ as $n \to \infty$.

$\therefore S_n = \dfrac{(1-(-r)^n)}{1-(-r)} = \dfrac{1-(-r)^n}{1+r}$

$= \begin{cases} \dfrac{1+r^n}{1+r} & \text{if } n \text{ is odd} \\ \dfrac{1-r^n}{1+r} & \text{if } n \text{ is even} \end{cases}$

Now $Lt\ S_n = \begin{cases} +\infty & \text{if } n \text{ is odd} \\ -\infty & \text{if } n \text{ is even.} \end{cases}$

$\therefore$ The series $\sum r^n$ is oscillatory series.

This oscillates infinitely.

Note: ① The g.s cgs only when common ratio is numerically less than 1.

② In an infinite series, if the terms are changed, a finite number of terms are added (or) omitted and each term of the series is multiplied and divided by the fixed non-zero number 'k' then the nature of the series does not change.

Problems

* The geometric series

$\sum_{n=0}^{\infty} r^n = 1+r+r^2+ \cdots + r^{n-1} + \cdots$

(i) Converges if $-1 < r < 1$ i.e. $|r| < 1$

(ii) diverges if $r \geq 1$

(iii) oscillates finitely if $r = -1$

(iv) oscillates infinitely if $r < -1$.

* P-Test (or) P-Series:

The series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots$

(i) Converges if $p > 1$

(ii) diverges if $p \leq 1$.

* **The n^{th} term Test:**

If the series $\sum x_n$ Converges then $\lim_{n\to\infty} (x_n) = 0$.

Note:— (1) $\sum x_n$ Converges $\Rightarrow \lim x_n = 0$

(2) $\lim x_n = 0 \Rightarrow \sum x_n$ may (or) may not Convergent

(3) $\lim x_n \neq 0 \Rightarrow \sum x_n$ is not Convergent.

(4) the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is Called the harmonic series.

* A positive term series either Converges (or) diverges to $+\infty$.

* If $x_n > 0 \; \forall n$ and $\lim_{n\to\infty} x_n \neq 0$ then $\sum x_n$ diverges to $+\infty$.

→ **Comparison Test:** Let $X = (x_n)$ and $Y = (y_n)$ be non-negative sequences of real numbers and suppose that for some $K \in N$, we have $0 \leq x_n \leq y_n$ for $n \geq K$.

then

(a) the convergence of $\sum y_n \Rightarrow$ the Convergence of $\sum x_n$

(b) the divergence of $\sum x_n \Rightarrow$ the divergence of $\sum y_n$.

→ **Limit Comparison Test:**

Suppose that $X = (x_n)$ and $Y = (y_n)$ are strictly +ve sequences and suppose that the following limit exists in $\mathbb{R}$:

$$r = \lim \left(\frac{x_n}{y_n}\right)$$

(a) If $r \neq 0$ (finite) then $\sum x_n$ is Convergent (or divergent) iff $\sum y_n$ is convergent (or divergent).

(b) If $r = 0$ and if $\sum y_n$ is convergent then $\sum x_n$ is convergent.

(c) If $r = \infty$ and if $\sum y_n$ diverges then $\sum x_n$ diverges.

Problems

→ The series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ Converges.

Sol^n:- Clearly the inequality

$$0 < \frac{1}{n^2+n} < \frac{1}{n^2} \quad \forall n$$

$\therefore$ which is in the form of $0 < x_n < y_n \; \forall n$

where $x_n = \frac{1}{n^2+n}$ & $y_n = \frac{1}{n^2}$.

Now $\sum y_n = \sum \frac{1}{n^2}$ is of the form $\sum \frac{1}{n^p}$ where $p = 2 > 1$

$\therefore$ By P-Test

$\sum y_n$ is convergent.

$\therefore$ By Comparison Test

$\sum x_n$ is convergent.

(Or)

Let $x_n = \frac{1}{n^2+n} = \frac{1}{n^2(1+1/n)}$

Let $y_n = \frac{1}{n^2}$

then $\frac{x_n}{y_n} = \frac{1}{1+1/n}$

$\therefore \lim_{n\to\infty} \frac{x_n}{y_n} = 1 \neq 0.$

Since $\sum y_n = \sum \frac{1}{n^2}$ is Convergent by P-Test.

$\therefore$ By Comparison test

$\therefore \sum x_n$ is convergent.

H.W. The series $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$ is Convergent.

$\longrightarrow$ The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent

* By using partial fractions, show that

ⓐ $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$

ⓑ $\sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0$ if $\alpha > 0$.

ⓒ $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$

sol'n: (a) Let $x_n = \frac{1}{(n+1)(n+2)}$

$= \frac{1}{n+1} - \frac{1}{n+2}$

$\therefore$ The n^{th} partial sum

$S_n = x_0 + x_1 + \cdots + x_{n-1}$

$= \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right) \cdots\cdots \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$

$= 1 - \frac{1}{n+1}$

$\therefore \lim S_n = 1$

$\therefore \sum x_n = 1$

ⓑ Let $x_n = \frac{1}{(\alpha+n)(\alpha+n+1)}$

$= \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}$

$\therefore S_n = x_0 + x_1 + \cdots + x_{n-1}$

$= \left(\frac{1}{\alpha} - \cancel{\frac{1}{\alpha+1}}\right) + \left(\cancel{\frac{1}{\alpha+1}} - \cancel{\frac{1}{\alpha+2}}\right) + \cdots + \left(\cancel{\frac{1}{\alpha+n-1}} - \frac{1}{\alpha+n}\right)$

$= \frac{1}{\alpha} - \frac{1}{\alpha+n}$

$\therefore \lim S_n = \frac{1}{\alpha} > 0$ if $\alpha > 0$.

ⓒ Let $x_n = \frac{1}{n(n+1)(n+2)}$

$= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

$\therefore S_n = x_1 + x_2 + \cdots + x_{n-1} + x_n$

→ Discuss the Convergence or divergence of the following Series.

ⓐ $\sqrt{1/4}+\sqrt{2/6}+\sqrt{3/8}+\cdots\cdots+\sqrt{\frac{n}{2(n+1)}}+\cdots$

ⓑ $\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\frac{1}{\sqrt{3.4}}+\cdots$

Sol'n: Let $\sum x_n=\sum\sqrt{\frac{n}{2(n+1)}}$

$$=\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\cdots+\sqrt{\frac{n}{2(n+1)}}+\cdots$$

Here $x_n=\sqrt{\frac{n}{2(n+1)}}$

$$=\sqrt{\frac{1}{2(1+1/n)}}$$

$\therefore$ $\lim x_n=\lim\frac{1}{\sqrt{2(1+1/n)}}$

$$=\frac{1}{\sqrt{2}}\neq 0$$

$\therefore$ $\lim x_n\neq 0$.

since $x_n>0$ $\forall n$. and $\lim x_n\neq 0$.

$\therefore$ $\sum x_n$ diverges to $+\infty$.

ⓑ Given that

$\frac{1}{\sqrt{1.2}}+\frac{1}{\sqrt{2.3}}+\cdots+\frac{1}{\sqrt{n(n+1)}}$

$$=\sum\frac{1}{\sqrt{n(n+1)}}$$

Let $x_n=\frac{1}{\sqrt{n(n+1)}}$

$$=\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}\right)$$

Let $y_n=1/n$

then, $x_n/y_n=\frac{1}{\sqrt{1+1/n}}$

$\therefore$ $\lim\frac{x_n}{y_n}=1\neq 0$

Since $\sum y_n=\sum\frac{1}{n}$ is divergent. (by P-test)

$\therefore$ By Comparison test the series $\sum x_n$ is divergent.

ⓒ $\frac{2}{1^P}+\frac{3}{2^P}+\frac{4}{3^P}+\frac{5}{4^P}+\cdots$

Sol'n :- Let $x_n=\frac{n+1}{n^P}$

$$=\frac{1}{n^{P-1}}(1+1/n)$$

Let $y_n=\frac{1}{n^{P-1}}$

then $\frac{x_n}{y_n}=1+1/n$.

$\therefore$ $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)=1\neq 0$.

Since $\sum y_n=\sum\frac{1}{n^{P-1}}$ is Convergent.

if $P-1>1$

i.e. $P>2$

$\therefore$ By Comparison test $\sum x_n$ is convergent.

also $\sum y_n=\sum\frac{1}{n^{P-1}}$ is divergent.

if $P-1\leq 1$

i.e. $P\leq 2$

$\therefore$ By comparison test $\sum x_n$ is divergent.

H.W.

→ $\sum(3n-1)^{-1}$

→ $\sum(n+1/2)^{-2}$

→ $\sum\left(\frac{n^2-n+1}{n^4+1}\right)$

→ $\sum\frac{1}{n^P(n+1)^P}$

→ $\Sigma \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

Sol'n: Let $x_n = \sqrt{n^3+1} - \sqrt{n^3}$

$$= \left(\sqrt{n^3+1} - \sqrt{n^3}\right) \times \frac{\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{n^3+1-n^3}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{n^{3/2}\left(1+\sqrt{1+\frac{1}{n^3}}\right)}$$

Let $y_n = \frac{1}{n^{3/2}}$

then $\frac{x_n}{y_n} = \frac{1}{1+\sqrt{1+\frac{1}{n^3}}}$

$\therefore$ lt $\frac{x_n}{y_n} = \frac{1}{2} \neq 0.$

Since $\Sigma y_n = \Sigma \frac{1}{n^{3/2}}$ is convergent by P-Test. Here $P = 3/2 > 1$

$\therefore$ By Comparison test Σx_n is convergent.

→ $\Sigma\left(\sqrt{n^2+1} - n\right)$

→ $\Sigma\left(\sqrt{n^4+1} - n^2\right)$

→ $\Sigma\left(\sqrt[3]{n+1} - \sqrt[3]{n}\right)$

Sol'n: Let $x_n = \sqrt[3]{n+1} - \sqrt[3]{n}$

$$= (n+1)^{1/3} - n^{1/3}$$

$$= n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3} - 1\right]$$

$$= n^{1/3}\left[\left(1+\frac{1}{3}\cdot\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^2}+\cdots\right)-1\right]$$

$$= n^{1/3}\left[\left(1+\frac{1}{3n}-\frac{1}{9n^2}+\cdots\right)-1\right]$$

$$= n^{1/3}\left[\frac{1}{3n}-\frac{1}{9n^2}+\cdots\right]$$

$$= \frac{1}{n^{2/3}}\left[\frac{1}{3}-\frac{1}{9n}+\cdots\right]$$

Let $y_n = \frac{1}{n^{2/3}}$

then $\frac{x_n}{y_n} = \frac{1}{3} - \frac{1}{9n} + \cdots$

$\therefore$ lt $\frac{x_n}{y_n} = \frac{1}{3} \neq 0.$

Since $\Sigma y_n = \Sigma \frac{1}{n^{2/3}}$ is divergent by P-Test where $P = 2/3 < 1$

$\therefore$ By comparison test Σx_n is divergent.

→ $\Sigma\left(\sqrt[3]{n^3+1} - n\right)$

Note: Rationalisation is effective only square roots are involved where as this method of Binomial expansion is general.

→ Test the convergence of the series.

(a) $\Sigma \sin \frac{1}{n}$ (b) $\Sigma \frac{1}{n}\sin\frac{1}{n}$

(c) $\Sigma \frac{1}{\sqrt{n}} \sin\frac{1}{n}$ (d) $\Sigma \cos\frac{1}{n}$

(e) $\Sigma \tan^{-1}\frac{1}{n}$ (f) $\Sigma \cot^{-1} n^2$

(g) $\Sigma \frac{1}{\sqrt{n}} \tan\frac{1}{n}$

Sol'n: (a) $\Sigma \sin\frac{1}{n}$

Let $a_n = \sin\frac{1}{n}$

and let $y_n = \frac{1}{n}$

then $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n}$

$= \underset{1/n \to 0}{Lt} \dfrac{\sin 1/n}{1/n}$ $(\because n\to\infty \Rightarrow \frac{1}{n}\to 0)$

$= 1$

$\neq 0$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent.

by P-Test where $P=1$.

$\therefore$ By Comparison Test

$\sum x_n$ is divergent.

ⓑ. Let $x_n = \frac{1}{n}\sin\frac{1}{n}$

Let $y_n = \frac{1}{n^2}$

then $\underset{n\to\infty}{Lt} \dfrac{x_n}{y_n} = \underset{n\to\infty}{Lt} \dfrac{\sin 1/n}{1/n}$

$= 1 \neq 0.$

Since $\sum y_n = \sum \frac{1}{n^2}$ is Convergent

by P-Test where $P=2>1$.

$\therefore$ By Comparison test $\sum x_n$ is Convergent.

ⓒ. Let $x_n = \frac{1}{\sqrt{n}}\sin\frac{1}{n}$

Let $y_n = \frac{1}{n^{3/2}}$

ⓓ Let $x_n = \cos\frac{1}{n}$

$= 1 - \dfrac{(1/n)^2}{2!} + \dfrac{(1/n)^4}{4!} - \dfrac{(1/n)^6}{6!} + \cdots$

$= 1 - \dfrac{1}{n^2 \cdot 2!} + \dfrac{1}{n^4 \cdot 4!} - \cdots$

Let $y_n = 1/n$

then $\dfrac{x_n}{y_n} = n - \dfrac{1}{n.2!} + \dfrac{1}{n^3.4!} \cdots$

$\therefore Lt \dfrac{x_n}{y_n} = \infty.$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent

(by P-Test)

$\therefore$ By comparison test $\sum x_n$ is divergent.

(or)

$\underset{n\to\infty}{Lt}\, x_n = \underset{n\to\infty}{Lt} \cos 1/n$

$= 1 \neq 0.$

Since $\sum x_n$ is a series of +ve terms.

i.e. $x_n > 0\ \forall n$.

and $Lt\, x_n \neq 0$.

$\therefore \sum x_n$ is divergent.

(e). Let $x_n = \tan^{-1} 1/n$

$= \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \frac{1}{7n^7} + \cdots$

(f) Let $x_n = \cot^{-1} n^2$

$= \tan^{-1}(1/n^2)$

$\because \cot^{-1} x = \theta$
$\Rightarrow x = \cot\theta$
$\Rightarrow x = \frac{1}{\tan\theta}$
$\Rightarrow \tan\theta = 1/n$
$\Rightarrow \theta = \tan^{-1}(1/n)$

(g) Let $x_n = \frac{1}{\sqrt{n}} \tan^{-1}(1/n)$

Let $y_n = \frac{1}{n^{3/2}}$

$\rightarrow \sum \dfrac{1}{n^{1+1/n}}$.

Sol^n:- Let $x_n = \dfrac{1}{n.n^{1/n}}$

let $y_n = 1/n$ then $\dfrac{x_n}{y_n} = \dfrac{1}{n^{1/n}}$

$\therefore \underset{n\to\infty}{Lt} \dfrac{x_n}{y_n} = \underset{n\to\infty}{Lt} \dfrac{1}{n^{1/n}} = \dfrac{1}{1}$

$= 1$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent by P-Test

$\therefore$ By Comparison test $\sum x_n$ is divergent.

D'Alembert's Ratio Test :—

If Σu_n is a series of +ve terms such that (a) $Lt \frac{u_n}{u_{n+1}} = l$ then

(i) Σu_n converges if $l>1$

(ii) Σu_n diverges if $l<1$

(iii) Σu_n may converge (or) diverge if $l=1$

(i.e. the test fails if $l=1$)

Note: In general the ratio test is applied when factorials & combination of powers involved.

Discuss the convergence of the following series:

→ $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$

Sol^n:— Let $u_n = \frac{n!}{n^n}$ then

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{n(1+1/n)^{n+1}}{(n+1)}$$

$$= (1+1/n)^n$$

$$\therefore Lt \frac{u_n}{u_{n+1}} = Lt (1+1/n)^n$$

$$= e > 1$$

∴ By D'Alemberts Ratio test.

∴ Σu_n is convergent.

→ $\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \cdots$

Sol^n:— Let $u_n = \frac{n^2 (n+1)^2}{n!}$

then $u_{n+1} = \frac{(n+1)^2 (n+2)^2}{(n+1)!}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^2 (n+1)^2}{n!} \times \frac{(n+1)!}{(n+1)^2 (n+2)^2}$$

$$= \frac{(n+1)}{(1+\frac{2}{n})^2}$$

$$= n \cdot \frac{(1+1/n)}{(1+2/n)^2}$$

$$\therefore Lt\left(\frac{u_n}{u_{n+1}}\right) = \infty > 1$$

∴ By D' Alembert's Ratio test

Σu_n is Convergent.

→ $\frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \cdots$

Sol^n:— Let $u_n = \frac{2^n}{n^2+1}$

then $u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{2^n}{n^2+1} \times \frac{(n+1)^2+1}{2^{n+1}}$$

$$= \frac{1}{2} \frac{(1+1/n)^2 + 1/n^2}{(1+1/n^2)}$$

$$\therefore Lt \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

∴ By D' Alemberts ratio test

Σu_n is divergent.

→ $\Sigma \frac{x^n}{3^n \cdot n^2}, \; x>0.$

Sol^n:— Let $u_n = \frac{x^n}{3^n \cdot n^2}$

then $u_{n+1} = \frac{x^{n+1}}{3^{n+1} (n+1)^2}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{x^n}{3^n \cdot n^2} \times \frac{3^{n+1} \cdot (n+1)^2}{x^{n+1}}$$

$$= \frac{3}{x}\left(1+\frac{1}{n}\right)^2$$

Now $\lim \frac{u_n}{u_{n+1}} = \frac{3}{x}$

$\therefore$ By D'Alembert's test

$\sum u_n$ Converges if $3/x > 1$

i.e. $x < 3$

and diverges if $3/x < 1$ i.e. $x > 3$.

if $x = 3$ then $\lim \frac{u_n}{u_{n+1}} = 1$

$\therefore$ Ratio Test fails.

Now if $x = 3$; $u_n = \frac{3^n}{3^n \cdot n^2} = \frac{1}{n^2}$

$\therefore \sum u_n = \sum \frac{1}{n^2}$ is Convergent by P-Test.

$\therefore \sum u_n$ is Convergent if $x \leq 3$

and divergent if $x > 3$.

$\rightarrow \sum \frac{x^n}{a+\sqrt{n}}$; $x > 0$

$\rightarrow \sum \sqrt{\frac{n+1}{n^3+1}} \cdot x^n$; $x > 0$

$\rightarrow 1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots$; $x > 0$.

Sol^n :- Let $u_n = \frac{x^n}{2n}$ (leaving the first term)

then $u_{n+1} = \frac{x^{n+1}}{2(n+1)}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x^2}\left(1+\frac{1}{n}\right)$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$ By D'Alembert's Ratio test

$\sum u_n$ is convergent

if $\frac{1}{x^2} > 1$

i.e. $x^2 < 1$

i.e. if $x < 1$

and the series divergent

if $\frac{1}{x^2} < 1$

i.e. $x^2 > 1$

i.e. $x > 1$

If $x^2 = 1$ i.e. $x = 1$, then the ratio test fails.

if $x = 1$ then $u_n = \frac{1}{2n}$

Let $v_n = \frac{1}{n}$ then

$$\frac{u_n}{v_n} = \frac{1}{2}$$

$$\therefore \lim \frac{u_n}{v_n} = \frac{1}{2} \neq 0.$$

Since $\sum v_n = \sum \frac{1}{n}$ is divergent by P-Test.

$\therefore$ By comparison test $\sum v_n$ is divergent

$\therefore \sum u_n$ is divergent

if $x \geq 1$

and $\sum u_n$ is Convergent

if $x < 1$.

* Cauchy's Root Test :-

$\rightarrow$ If $\sum x_n$ is a series of +ve terms such that

(a) $\lim (u_n)^{1/n} = l$ then

(i) $\sum u_n$ Converges if $l < 1$

(ii) $\sum u_n$ diverges if $l > 1$

(iii) $\sum u_n$ may Converge or diverge if $l = 1$.

(i.e. the test fails if $l = 1$).

(b) $\lim\limits_{n\to\infty} (u_n)^{1/n} = \infty$ then $\sum u_n$ is divergent.

Note: The root test is used when powers are involved.

Problems:

* Test the convergence of the following series:

→ (a) $\sum \left(\frac{n}{n+1}\right)^{n^2}$ (b) $\sum \frac{x^n}{n^n}$

(c) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

(d) ~~$\sum n^2 x^n, x>0$~~ (e) $\sum \left(\frac{n+1}{3n}\right)^n$

Sol'n :- (a) Let $x_n = \left(\frac{n}{n+1}\right)^{n^2}$

then $(x_n)^{1/n} = \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}$

$$= \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n+1}{n}\right)^{-n}$$

$$= \left[\left(1+\frac{1}{n}\right)^{+n}\right]^{-1}$$

Now $\underset{n\to\infty}{Lt} (x_n)^{1/n} = e^{-1}$

$$= \frac{1}{e} < 1$$

∴ By Cauchy's root test

$\sum x_n$ converges.

(c). Let $x_n = \frac{1}{(\log n)^n}$

then $x_n^{1/n} = \frac{1}{\log n}$

$\therefore \underset{n\to\infty}{Lt} x_n^{1/n} = \underset{n\to\infty}{Lt} \frac{1}{\log n} = 0 < 1.$

∴ By root test.

$\sum x_n$ is convergent.

→ $\sum 5^{-n-(-1)^n}$

Sol'n :- Let $x_n = 5^{-n-(-1)^n}$

then $(x_n)^{1/n} = 5^{-1-\frac{(-1)^n}{n}}$

$$= 5^{-1-\frac{1}{n}} \text{ if } n \text{ is even}$$

$$= 5^{-1+\frac{1}{n}} \text{ if } n \text{ is odd.}$$

$\therefore \underset{n\to\infty}{Lt} x_n^{1/n} = 5^{-1}$

$$= \frac{1}{5} < 1$$

∴ By root test

$\sum x_n$ is convergent.

→ $\sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$

→ $\sum_{n=1}^{\infty} 3^{-2n-5(-1)^n}$

Note :- Cauchy's root test is more general than D'Alembert's ratio test.

because:

(i) $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n}$ exists ⇒ $\underset{n\to\infty}{Lt} u_n^{1/n}$ exists.

and $\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} u_n^{1/n}$

(Cauchy's second theorem on limits)

∴ whenever ratio test is applicable, so is the root test.

(ii) If $\underset{n\to\infty}{Lt} u_n^{1/n}$ exists then

$\underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n}$ may not exist.

∴ when the ratio test fails the root test succeeds.

$\therefore$ The root test is more general than the ratio test.

→ Show that Cauchy's root test establishes the convergence of the series $\sum 3^{-n-(-1)^n}$ while D'Alembert's ratio test fails to do so.

Sol'n: Let $u_n = 3^{-n-(-1)^n}$

then $\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} 3^{-1-\frac{(-1)^n}{n}}$

$= 3^{-1}$

$= 1/3 < 1$

$\therefore$ By root test $\sum u_n$ is Convergent

Now if n is odd (so that $n+1$ is even)

$u_n = 3^{-n+1}$, $u_{n+1} = 3^{-(n+1)-1}$

$= 3^{-n-2}$

$\therefore \lim \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{3^{-n+1}}{3^{-n-2}}$

$= \lim_{n\to\infty} \left(\frac{3^1}{3^{-2}}\right)$

$= 3^3$

$= 27 > 1$

when n is even (so that $n+1$ is odd)

$\therefore u_n = 3^{-n-1}$, $u_{n+1} = 3^{-n}$

$\therefore \lim \frac{u_n}{u_{n+1}} = 3^{-1} = 1/3 < 1.$

$\therefore \lim \frac{u_n}{u_{n+1}}$ does not exist.

$\therefore$ D'Alembert's ratio test fails.

* Raabe's Test :-

If $\sum u_n$ is a series of +ve terms such that

$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then

(i) $\sum u_n$ Converges if $l > 1$

(ii) $\sum u_n$ diverges if $l < 1$.

(iii) The test fails if $l = 1$.

Note! – Raabe's test is stronger than D'Alembert's ratio test and may succeed where the ratio test fails.

For example :-

$\sum \frac{1}{n^2}$

Let $u_n = \frac{1}{n^2}$

$u_{n+1} = \frac{1}{(n+1)^2}$

$\therefore \lim \frac{u_n}{u_{n+1}} = \lim \frac{(n+1)^2}{n^2}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2$

$= 1$

Here $l = 1$,

$\therefore$ the ratio test fails.

But $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\left(1+1/n\right)^2 - 1\right]$

$= \lim_{n\to\infty} n\left[\frac{(n+1)^2}{n^2} - 1\right]$

$$= \lim_{n\to\infty} n\left[\frac{n^2+1+2n-n^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{n}+2\right) = 2 > 1$$

$\therefore$ By Raabe's test

$\sum u_n$ is Convergent.

Note :-

① If $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \infty$ then

$\sum u_n$ is Convergent.

② $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = -\infty$ then

$\sum u_n$ is divergent.

③ In general Rabbe's test is used when D'Alemberts ratio test fails and the ratio $\frac{u_n}{u_{n+1}}$ doesnot involve the number 'e'

— when $\frac{u_n}{u_{n+1}}$ involves 'e' we apply logarithmic test after the ratio test and not Rabbe's test.

* **Logarithmic test :**

~~If $\sum u_n$ is a series of +ve terms such that $\lim_{n\to\infty} n \log\frac{u_n}{u_{n+1}} = l$~~

~~then (i) $\sum u_n$ Converges if $l > 1$~~

~~(ii) $\sum u_n$ diverges if $l < 1$~~

~~(iii) The test fails if $l = 1$~~

~~$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots$~~

Sol'n : Let $u_n = \frac{n^n \cdot x^n}{n!}$

then $u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n \cdot x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} \cdot x^{n+1}}$$

$$= \frac{(n+1) \cdot n^n}{(n+1)^{n+1} \cdot x}$$

$$= \frac{1}{(1+1/n)^n} \cdot \frac{1}{x}$$

Now $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{ex}$

$\therefore$ By D'Alembert's test, the series $\sum u_n$ Converges if $\frac{1}{ex} > 1$

i.e. $ex < 1$

i.e. $x < 1/e$

and diverges if $1/ex < 1$

i.e. $ex > 1$

i.e. $x > 1/e$.

If $x = 1/e$ then the ratio test fails.

Now if $x = 1/e$ then $\frac{u_n}{u_{n+1}} = \frac{1}{(1+1/n)^n} e$

Since $\frac{u_n}{u_{n+1}}$ involves the number e.

$\therefore$ we apply the logarithmic test.

Now $\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left[\frac{1}{(1+1/n)^n} e\right]$

$$= \log e - n\log(1+1/n)$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \cdots$$

$$\therefore \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}}$$

$$= \lim_{n\to\infty} n\left[\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right]$$

$$= \frac{1}{2} < 1$$

$\therefore$ By logarithmic test the series $\sum u_n$ is divergent.

$\therefore \sum u_n$ is divergent if $x \geq 1/e$ and converges if $x < 1/e$.

→ Discuss the convergence of the series.

$$1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots \quad (x > 0)$$

Sol'n :- Neglecting the first term,

let $u_n = \dfrac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots(2n)} x^n$

$$u_{n+1} = \frac{1.3.5\cdots(2n-1)(2n+1)}{2.4.6\cdots(2n)(2n+2)} x^{n+1}$$

Now $\dfrac{u_n}{u_{n+1}} = \dfrac{2n+2}{2n+1} \cdot \dfrac{1}{x} = \dfrac{1+1/n}{1+\frac{1}{2n}} \cdot \dfrac{1}{x}$

$\therefore \lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$

$\therefore$ By Ratio test $\sum u_n$ converges if $\frac{1}{x} > 1$ i.e. $x < 1$ & diverges if $\frac{1}{x} < 1$ i.e. $x > 1$

If $x = 1$ then the ratio fails.

but when $x = 1$, $\dfrac{u_n}{u_{n+1}} = \dfrac{2n+2}{2n+1}$

Clearly which is not involving in e. So we apply the Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1} = \frac{1}{2+1/n}$$ 9

Now $\lim_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2} < 1.$

$\therefore$ By Raabe's Test, $\sum u_n$ is divergent.

Hence $\sum u_n$ is convergent if $x \leq 1$ and divergent if $x \geq 1$.

~~2008 → Discuss the convergence of the series $\frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \cdots (x>0)$~~

Gauss Test :-

If $\sum u_n$ is a series of positive terms such that $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{\lambda}{n} + \dfrac{\alpha_n}{n^{1+\delta}}$

where $\delta > 0$ and (α_n) is a bounded sequence, then

(i) $\sum u_n$ converges if $\lambda > 1$

(ii) $\sum u_n$ diverges if $\lambda \leq 1$.

Note :-

(1) The test never fails as we know that the series diverges for $\lambda = 1$. Moreover, the test is applied after the failure of Ratio test and when it is possible to expand $\frac{u_n}{u_{n+1}}$ in powers of $1/n$ by Binomial Theorem (or) by any other method.

(2) Raabe's Test (or) Gauss Test if $\dfrac{u_n}{u_{n+1}}$ does not involve the number e.

③ If $\frac{u_n}{u_{n+1}}$ involves the number e, apply logarithmic test.

④ For application of Gauss Test, expand $\frac{u_n}{u_{n+1}}$ in powers of $1/n$ as

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right) \text{ where}$$

$O\left(\frac{1}{n^2}\right)$ stands for terms of order $1/n^2$ and higher powers of $1/n$.

* De Morgan's and Bertrand's Test

If $\sum u_n$ is a series of positive terms such that

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right] = l$$

then (1) $\sum u_n$ Converges if $l>1$.

(2) $\sum u_n$ diverges if $l<1$.

Note:- This test is to be applied when both D'Alembert's ratio test and Raabe's test fails.

* An alternative to Bertrand's Test:—

If $\sum u_n$ is a series of positive terms such that

$$\lim_{n\to\infty}\left[\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right] = l$$

then ① $\sum u_n$ Converges if $l>1$.

② $\sum u_n$ diverges if $l<1$.

Note:- This test is to be applied when the logarithmic test fails.

Problems:-

→ $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + \cdots\cdots$

Let $u_n = \frac{1.3.5 \cdots (2n-1)}{2.4.6 \cdots 2n}$

then $u_{n+1} = \frac{1.3.5 \cdots (2n-1)(2n+1)}{2.4.6 \cdots (2n)(2n+2)}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots 2n} \times \frac{2.4.6\cdots(2n)(2n+2)}{1.3.5\cdots(2n-1)(2n+1)}$$

$$= \frac{2n+2}{2n+1} = \frac{1+1/n}{1+\frac{1}{2n}}$$

Now $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1$.

$\therefore$ D'Alembert's Ratio test fails.

Now we apply the Raabe's test

$$\therefore n\left[\frac{u_n}{u_{n+1}}-1\right] = n\left[\frac{2n+2}{2n+1}-1\right]$$

$$= \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = 1/2 < 1$$

$\therefore$ By Raabe's test, $\sum u_n$ diverges.

→ $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \cdots$

Sol'n:- Let $u_n = \frac{1^2.3^2.5^2\cdots(2n-1)^2}{2^2.4^2.6^2\cdots(2n)^2}$

then $u_{n+1} = \frac{1^2.3^2.5^2\cdots(2n-1)^2(2n+1)^2}{2^2.4^2.6^2\cdots(2n)^2(2n+2)^2}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{(1+1/n)^2}{(1+1/2n)^2}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = 1$$

$\therefore$ D'Alembert's Ratio test fails

Now apply Raabe's test

$$\therefore n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{(2n+2)^2}{(2n+1)^2} - 1\right]$$

$$= n\left[\frac{4n+3}{(2n+1)^2}\right]$$

$$= \frac{4n^2+3n}{(2n+1)^2}$$

$$= \frac{1+\frac{3}{4n}}{\left(1+\frac{1}{2n}\right)^2}$$

$$\therefore \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = 1$$

$\therefore$ Raabe's test fails.

Now we apply Gauss test

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}$$

$$= \left(1+\frac{1}{n}\right)^2 \left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2} - \cdots\right)$$

$$= \left(1-\frac{2}{2n}+\frac{3}{4n^2} - \cdots\right) + \left(\frac{2}{n} - \frac{4}{2n^2} + \frac{6}{4n^3} - \cdots\right) + \left(\frac{1}{n^2} - \frac{2}{2n^3} + \frac{3}{4n^4} - \cdots\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{4n^2} + \cdots$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Now comparing with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

we have $\lambda = 1$

$\therefore$ By Gauss Test :-

Σu_n is divergent.

~~Note:- when D' Almeberts test fails then we may directly apply Gauss test.~~

$\rightarrow$ $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \cdots$

Sol'n :- omitting the first term, we have

$$u_n = \frac{2^2.4^2.6^2 \cdots (2n)^2}{3^2.5^2.7^2 \cdots (2n+1)^2}$$

$\rightarrow$ $1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \cdots$

Sol'n :- Leaving the first term, we have

$$u_n = \frac{3.6.9 \cdots (3n)}{7.10.13 \cdots (3n+4)} \cdot x^n$$

$$\Rightarrow u_{n+1} = \frac{3.6.9 \cdots (3n)(3n+3)}{7.10.13 \cdots (3n+4)(3n+7)} x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$= \frac{1+7/3n}{1+1/n} \cdot \frac{1}{x}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio test

$\sum u_n$ Converges if $\frac{1}{x} > 1$
i.e. $x < 1$

and $\sum u_n$ diverges if $\frac{1}{x} < 1$
i.e. $x > 1$

If $x = 1$ then, the ratio test fails.

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

Now we apply Gauss Test.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

$$= \left(1+\frac{7}{3n}\right)\left(1+\frac{1}{n}\right)^{-1}$$

$$= \left(1+\frac{7}{3n}\right)\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\cdots\right)$$

$$= \left(1-\frac{1}{n}+\frac{1}{n^2}-\cdots\right)+\left(\frac{7}{3n}-\frac{7}{3n^2}+\cdots\right)$$

$$= 1+\frac{4}{3n}-\frac{4}{3n^2}+\cdots$$

$$= 1+\left(\frac{4}{3}\right)\frac{1}{n}+o\left(\frac{1}{n^2}\right)$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+o\left(\frac{1}{n^2}\right)$$

where $\lambda = 4/3 > 1$

$\therefore$ By Gauss Test,

$\therefore \sum u_n$ is Convergent.

$\therefore$ The given series Converges if $x \leq 1$ and diverges if $x > 1$.

$$\rightarrow \frac{x}{1} + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\cdot\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7} + \cdots$$

Sol^n:– Neglecting the first term, we have

$$u_n = \frac{1.3.5.7\cdots(2n-1)}{2.4.6\cdots(2n)}\,\frac{x^{2n+1}}{(2n+1)}$$

* Cauchy's Condensation Test:

Let $\sum_{n=1}^{\infty} a(n)$ be such that $(a(n))$ is a decreasing sequence of strictly positive numbers.

$\sum_{n=1}^{\infty} a(n)$ Converges (or diverges) iff $\sum_{n=1}^{\infty} 2^n a(2^n)$ Converges (or diverges).

Problems:–

(i) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(iii) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$

(iv) $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$

Sol^n:–(i) Here given that

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

Put $\sum_{n=2}^{\infty} a(n) = \sum_{n=2}^{\infty} \frac{1}{\log n}$

Here $a(n) = \frac{1}{\log n}$.

Since $(\log n)$ is an increasing sequence.

$\therefore (a(n)) = \left(\frac{1}{\log n}\right)$ is a decreasing sequence.

$\therefore \displaystyle\sum_{n=2}^{\infty} 2^n a(2^n) = \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{\log(2^n)}$

$= \displaystyle\sum_{n=2}^{\infty} 2^n \cdot \frac{1}{n \cdot \log 2}$

$= \displaystyle\frac{1}{\log 2} \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{n}$ —— (1)

Let $u_n = \frac{2^n}{n}$

then $u_n^{1/n} = \frac{2}{n^{1/n}}$

$\therefore \displaystyle\lim_{n\to\infty} u_n^{1/n} = \frac{2}{1}$

$= 2 > 1$

$\therefore$ By Cauchy's root test,

$\sum u_n$ is divergent.

$\therefore \displaystyle\sum_{n=2}^{\infty} 2^n a(2^n)$ is divergent.

$\therefore$ By Cauchy's condensation test

$\sum a(n) = \sum \frac{1}{\ln n}$

is divergent.

(ii) Given that

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \sum \frac{1}{n \log n}$

put $\sum a(n) = \sum \frac{1}{n \log n}$

Here $a(n) = \frac{1}{n \log n}$

Since $(n \log n)$ is an increasing sequence.

$\therefore (a(n)) = \frac{1}{(n \log n)}$ is a decreasing sequence.

$\therefore \displaystyle\sum_{n=2}^{\infty} 2^n a(2^n) = \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \log(2^n)}$

$= \displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log 2}$

$= \displaystyle\frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n}$

is divergent by P-Test where $P=1$.

$\therefore$ By Cauchy's condensation test

$\sum a(n) = \sum \frac{1}{n \ln n}$ is divergent.

(iii) Given $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)} = \sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)}$

Put $\sum a(n) = \sum \frac{1}{n(\log n)(\log \log n)}$

Here $a(n) = \frac{1}{n(\log n)(\log \log n)}$

Since $(n(\log n)\cdot(\log \log n))$ is an increasing sequence.

$\therefore (a(n))$ is a decreasing sequence

$\therefore \sum 2^n a(2^n) = \sum 2^n \frac{1}{2^n (\log 2^n)(\log \log 2^n)}$

$= \sum \frac{1}{(n \log 2) \log(n \log 2)}$

$= \sum x_n$ (say) —— (A)

Since $\log 2 < 1$

$\Rightarrow n\log 2 < n$

$\Rightarrow \log(n\log 2) < \log n$

$\Rightarrow \dfrac{1}{\log(n\log 2)} > \dfrac{1}{\log n}$

$\Rightarrow \dfrac{1}{n\log 2} \cdot \dfrac{1}{\log(n\log 2)} > \dfrac{1}{n\log 2} \cdot \dfrac{1}{\log n}$

$\Rightarrow x_n > \dfrac{1}{\log 2} \cdot \dfrac{1}{n\log n}$

$= y_n$ (say) ——— (B)

$\therefore x_n > y_n \ \forall n$

i.e. $y_n < x_n \ \forall n$ ——— (C)

Since $\sum y_n = \sum \dfrac{1}{\log 2} \cdot \dfrac{1}{n\log n}$

$= \dfrac{1}{\log 2} \sum \dfrac{1}{n\log n}$

diverges (by (ii))

$\therefore$ By comparison test,

$\sum x_n$ diverges.

$\therefore$ By Cauchy's Condensation test,

$\sum a(n)$ diverges.

(iv) Given that

$$\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$$

$$= \sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

Put $\sum_{n=4}^{\infty} a(n) =$

$$\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

Here $a(n) = \dfrac{1}{n(\log n)(\log \log n)(\log \log \log n)}$

Since $(n(\log n)(\log \log n)(\log \log \log n))$ is an increasing sequence.

$\therefore (a(n))$ is a decreasing sequence.

$\therefore \sum 2^n a(2^n) =$

$$\sum 2^n \frac{1}{2^n(\log 2^n)(\log \log 2^n)(\log \log \log 2^n)}$$

$$= \sum \frac{1}{(n\log 2)(\log(n\log 2))(\log \log(n\log 2))}$$

$$= \frac{1}{\log 2} \sum \frac{1}{n(\log n\log 2)(\log \log(n\log 2))}$$

$= \sum x_n$ (say)

Since $\log 2 < 1$

$\Rightarrow \log n \log 2 < \log n$ ——— (A)

$\Rightarrow \log \log n \log 2 < \log \log n$

$\Rightarrow \dfrac{1}{\log \log n \log 2} > \dfrac{1}{\log \log n}$ ——— (B)

(A) $\equiv$

$\dfrac{1}{\log n \log 2} > \dfrac{1}{\log n}$ ——— (C)

But $\dfrac{1}{n} = \dfrac{1}{n}$ ——— (D) from ⓓ, ⓒ, ⓑ give,

$$\frac{1}{n(\log n\log 2)\cdot(\log\log n \log 2)} > \frac{1}{n(\log n)(\log\log n)}$$

$= y_n$ say

$\therefore x_n > y_n \ \forall n$

i.e. $y_n < x_n \ \forall n$

But by ⓑ,

$$\sum y_n = \sum \frac{1}{n(\log n)(\log\log n)}$$

diverges (by(iii))

$\therefore$ By Comparison test

$\sum x_n$ also diverges.

$\therefore$ By Cauchy's Condensation test,

$\sum a(n)$ diverges.

→ If $c>1$ then show that the following series are convergent.

ⓐ $\sum \frac{1}{n(\log n)^c}$

ⓑ $\sum \frac{1}{n(\log n)(\log\log n)^c}$

Sol'n: (a) $\sum a(n) = \sum \frac{1}{n(\log n)^c}$

is decreasing for $c>1$.

$$\therefore \sum 2^n a(2^n) = \sum 2^n \cdot \frac{1}{2^n(\log 2^n)^c}$$

$$= \sum \frac{1}{(n\log 2)^c}$$

$$= \frac{1}{(\log 2)^c}\sum \frac{1}{n^c}$$

Since $\sum \frac{1}{n^c}$ is convergent for $c>1$.

$\therefore \sum 2^n a(2^n)$ is convergent.

$\therefore$ By Cauchy's Condensation test,

$\sum a(n)$ is convergent.

Comparison test :-

Note: Let $\sum u_n$ and $\sum v_n$ be two series of +ve terms and let h & k be +ve real numbers such that

~~$hv_n < u_n < kv_n \ \forall n$~~

Then the series -

$\sum u_n$ & $\sum v_n$ converge (or) diverge together.

→ Examine the following series for convergence.

(i) $\sum \frac{1}{(\log n)^{\log n}}$ (ii) $\sum \frac{1}{(\log\log n)^{\log n}}$

(iii) $\sum r^{\log n}$

Sol^n :- (i) Since $\lim_{n\to\infty} \log(\log n) = \infty$

$\therefore$ we can find n

so large that $\log(\log n) > 2$

$\log n[\log(\log n)] > 2\log n$

$\log(\log n)^{\log n} > \log n^2$

$\Rightarrow (\log n)^{\log n} > n^2$

$\Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$ ——— (1)

Since $\sum \frac{1}{n^2}$ is convergent (by P-Test)

$\therefore \sum \frac{1}{(\log n)^{\log n}}$ is convergent.

(ii) Since $\lim_{n\to\infty} \log(\log\log n) = \infty$

~~$\therefore$ we can find a n~~

So large that

$\log(\log\log n) > 2$

$\Rightarrow \log n . \log(\log\log n) > 2\log n$

$\Rightarrow \log(\log\log n)^{\log n} > \log n^2$

$\Rightarrow (\log\log n)^{\log n} > n^2$

$\Rightarrow \frac{1}{(\log\log n)^{\log n}} < \frac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ By comparison test.

$\sum \frac{1}{(\log\log n)^{\log n}}$ is convergent

(iii) Since the multiplication of numbers is commutative.

$\therefore \log n \log r = \log r \log n$

$\Rightarrow \log(r^{\log n}) = \log(n^{\log r})$

$\Rightarrow r^{\log n} = n^{\log r}$

$\therefore \sum r^{\log n} = \sum n^{\log r}$

$= \sum \frac{1}{n^{-\log r}}$

By P-Test it is convergent

if $-\log r > 1$

i.e. if $\log r < -1$

i.e if $\log r < -\log e$

i.e. if $\log r < \log e^{-1}$

i.e. if $r < \frac{1}{e}$

$\therefore \sum r^{\log n}$ converges iff $r < 1/e$.

* Alternating Series :—

A series with terms alternatively +ve and −ve is called an alternative series.

i.e. $u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1} u_n + \cdots$

where $u_n > 0 \;\forall n$ is alternating series and is shortly written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n.$$

* Leibnitz's Test on Alternating Series :—

The alternating series

$\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots,$

$u_n > 0 \;\forall n$

Converges if (i) $u_n \geq u_{n+1} \;\forall n$ and

(ii) $\lim_{n\to\infty} u_n = 0$

Note :— The alternating series will not be convergent if any one of the two conditions is not satisfied.

* Absolute and Conditional Convergence :—

→ A series $\sum_{n=1}^{\infty} u_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

→ If $\sum_{n=1}^{\infty} u_n$ converges but not absolutely.

i.e. $\sum_{n=1}^{\infty} |u_n|$ diverges then the series $\sum_{n=1}^{\infty} u_n$ is called conditional convergent (or) semi-convergent (or) non-absolutely convergent.

Note :— Every absolutely convergent series is convergent but convergent series need not be absolute convergent.

Ex :—

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Sol'n :— Let $u_n = 1/n$

then $u_n > 0 \;\forall n$

Since $\frac{1}{n} > \frac{1}{n+1} \;\forall n.$

$\Rightarrow u_n > u_{n+1} \;\forall n$

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

$\therefore$ By Leibnitz's test,

$\sum \frac{(-1)^{n-1}}{n}$ convergent.

But the series $\sum \left|\frac{(-1)^{n-1}}{n}\right|$

$= \sum \frac{1}{n}$ is divergent (by P-Test)

Problems :—

Test the convergence and absolute convergence of the series.

(i) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sol^n:- The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

Here $v_n = \frac{1}{2n-1} > 0 \ \forall n$

$v_{n+1} = \frac{1}{2n+1} \ \forall n$

Since $2n-1 < 2n+1 \ \forall n$

$\frac{1}{2n-1} > \frac{1}{2n+1} \ \forall n$

$\Rightarrow v_n > v_{n+1} \ \forall n$

and $\lim_{n\to\infty} v_n = 0$.

$\therefore$ By Leibnitz's test, the series is Convergent.

Now $|u_n| = \frac{1}{2n-1}$

Since $\frac{1}{2n-1} > \frac{1}{2n} \ \forall n$.

$\therefore \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ is divergent (by P-Test).

$\therefore$ By Comparison test,

$\sum \frac{1}{2n-1}$ is divergent.

$\therefore \sum |u_n|$ is divergent.

$\therefore$ The series

$\sum \frac{(-1)^{n-1}}{n-1}$ is Conditional Convergent.

$\rightarrow \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$

$\rightarrow \frac{1}{1\cdot 3} - \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} - \frac{1}{4\cdot 6} + \cdots$

Sol'n:- The given series is

$$\sum v_n = \sum \frac{(-1)^{n-1}}{n.(n+2)} = \sum (-1)^{n-1} v_n.$$

It is an alternating series.

$\rightarrow \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \cdots$

Sol'n:- The given series is

$$\sum u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\log(n+1)}.$$

$$= \sum (-1)^{n-1} v_n.$$

It is alternating series.

Here $v_n = \frac{1}{\log(n+1)}$

and $v_{n+1} = \frac{1}{\log(n+2)}$

Since $(n+1) < (n+2) \ \forall n$

$\log(n+1) < \log(n+2) \ \forall n$

$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{\log(n+2)} \ \forall n$

$\Rightarrow v_n > v_{n+1} \ \forall n$.

and $\lim v_n = \lim_{n\to\infty} \frac{1}{\log(n+1)}$

$= 0$

$\therefore$ By Leibnitz's test, the series is Convergent.

Now $|u_n| = \frac{1}{\log(n+1)}$

Since $\log(n+1) < (n+1) \ \forall n$

$$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{n+1} \ \forall n \quad \text{——} \ (A)$$

Since $\sum \frac{1}{n+1} = \sum x_n$ (say)

~~let~~ $x_n = \frac{1}{n+1} = \frac{1}{n(1+1/n)}$

and $y_n = 1/n$

then $\frac{x_n}{y_n} = \frac{1}{(1+1/n)}$

$\therefore \lt \frac{x_n}{y_n} = 1 \neq 0.$

Since $\sum y_n = \sum \frac{1}{n}$ is divergent (by P-Test)

$\therefore$ By Comparison test

$\sum x_n = \sum \frac{1}{n+1}$ is divergent.

again by Comparison test

$\sum |u_n| = \sum \frac{1}{\log(n+1)}$ is divergent.

$\therefore \sum u_n = \sum \frac{(-1)^{n-1}}{n(n+1)}$

is conditional Convergent.

→ Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$ is absolutely Convergent.

Sol'n :- $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$

$|u_n| = \frac{2^n}{n!}$, $|u_{n+1}| = \frac{2^{n+1}}{(n+1)!}$

$\therefore \lt \left|\frac{u_n}{u_{n+1}}\right| = \lt \left(\frac{n+1}{2}\right) = \infty > 1$

$\therefore$ By D'Alembert's ratio test.

$\sum |u_n|$ is Convergent.

$\therefore$ The given alternating series is absolutely Convergent.

Abel's Test :-

If $\sum_{n=1}^{\infty} a_n$ is Convergent and the sequence $\{b_n\}$ is Monotonic and bounded, then $\sum_{n=1}^{\infty} a_n b_n$ is Convergent

Problem ① : Test the Convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(1+1/n\right)^n$$

Sol'n :- Let $a_n = \frac{(-1)^{n-1}}{n}$ and

$$b_n = \left(1+\frac{1}{n}\right)^n \ \forall n$$

clearly $\sum a_n$ is Convergent (by Leibnitz's Test) and the Sequence $\{b_n\}$ is monotone (increasing) and bounded.

Hence by Abel's Test,

the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Problem ② Test the Convergence of

$$1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \cdots$$

Sol'n :- $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} + \frac{1}{7.4^2} + \cdots$

$$= \sum_n \frac{(-1)^{n-1}}{(2n-1)n^2}$$

Let $a_n = \frac{(-1)^{n-1}}{n^2}$ and $b_n = \frac{1}{2n-1}$

Problem:- Show that the series

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3}-n}{\log n} \text{ is convergent.}$$

Sol'n:- Let $a_n = (n^3+1)^{1/3} - n$, $b_n = \frac{1}{\log n}$

then the series can be written as

$$\sum_{n=1}^{\infty} a_n b_n .$$

Now $a_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\cdot\frac{1}{n^6} + \cdots +\right]$$

$$= \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^2} + \cdots\right]$$

Take $c_n = \frac{1}{n^2}$ then

$$\frac{a_n}{c_n} = \frac{1}{3} - \frac{1}{9n^2} + \cdots$$

$\lim_{n\to\infty} \frac{a_n}{c_n} = \frac{1}{3}$ which is finite and non-zero.

∴ By Comparison test, Σa_n and Σc_n converge (or) diverge together.

But $\Sigma c_n = \Sigma \frac{1}{n^2}$ is convergent.

∴ Σa_n is convergent.

Also $\{b_n\}$ is a monotonically decreasing sequence of +ve terms and bounded below.

∴ By Abel's test the series $\sum_{n=2}^{\infty} a_n b_n$ is convergent.

* Dirichlet's Test

If $\sum_{n=1}^{\infty} a_n$ is a series whose n^{th} partial sum $\{S_n\}$ is bounded and $\{b_n\}$ is a monotonic sequence converging to zero then $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

→ Note: Leibnitz's Test as a particular case of Dirichlet's Test:

The series $\sum_{n=1}^{\infty} (-1)^{n-1}$ has bounded partial sums,

since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd.} \\ 0 & \text{if } n \text{ is even} \end{cases}$ $\left[\because S_n = 1 + (-1) + 1 + (-1) + \cdots + (-1)^{n-1}\right]$

If $\{a_n\}$ is a monotonically decreasing sequence of +ve numbers convergent to '0' i.e. If (i) $a_n > 0$ then

(ii) $a_n \geq a_{n+1}$ $\forall n$.

(iii) $a_n \to 0$ as $n \to \infty$

then by Dirichlet's test,

the series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

i.e. the alternating series.

$a_1 - a_2 + a_3 - a_4 + \cdots$ is convergent.

Problem:- Discuss the convergence of

$$\Sigma \frac{(-1)^{n-1}}{n^p}, \ (p>0)$$

Sol'n:- Let $a_n = (-1)^{n-1}$ and $b_n = \frac{1}{n^p}$, $(p>0)$

then the series $\Sigma a_n = \Sigma (-1)^{n-1}$ has bounded partial sums,

Since $S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

and the sequence $\{b_n\} = \left\{\frac{1}{n^p}\right\} (p>0)$ is monotonically decreasing sequence of +ve numbers convergent to '0'.

i.e. (i) $b_n > 0 \; \forall n$

(ii) $b_n \geq b_{n+1} \; \forall n$

(iii) $b_n \to 0$ as $n \to \infty$

Hence by Dirichlet's Test $\sum_{n=1}^{\infty} a_n b_n$ is Convergent.

* Rearrangement of Terms :—

A series $\sum_{n=1}^{\infty} b_n$ is said to arise from a series $\sum_{n=1}^{\infty} a_n$ by a rearrangement of terms if there exists a one-to-one Correspondence between the terms of the two series so that every a_n is some b_n and conversely.

for example, the series.

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \text{-----}$

is a rearrangement of series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{------}$

i.e. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{-----}$

on rearranging the terms so that each positive term is followed by two negative terms, —

the series

$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \text{---}$

— If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum arranged. But this is not so when infinite series are involved. An arrangement (or equally well derangements) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

Dirichlet's Theorem (1) :—

(i) If $\sum_{n=1}^{\infty} a_n$ is a +ve term series converging to 's', then any derangement $\sum_{n=1} a_n$ also converges to 's'.

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent positive term series then so also is $\sum_{n=1}^{\infty} b_n$.

Dirichlet's Theorem (2) :—

If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series then every derangement $\sum_{n=1}^{\infty} b_n$ also converges absolutely to the same sum as the original series.

* Riemann's Theorem :—

A conditionally convergent series can be made by derangement of terms. (i) to converge to any real number.

(ii) to diverge to any $+\infty$ or $-\infty$.

(iii) to oscillate finitely or infinitely.

Problem: (1) Discuss the convergence of the series $1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}-\frac{1}{4^2}+\frac{1}{6^2}$ ----

Sol'n:- The given series is a rearrangement of the series

$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+$ ----

which is absolutely convergent.

Hence by the Dirichlet's theorem, the given series is convergent.

Note:- Riemann's method is of theoretical importance only. For pratical applications, the method given by Pringsheim's is useful. (Imp)

Pringsheim's Method:-

Let $f(n)$ be a +ve fn decreasing to zero as $n\to\infty$. Then by Leibnitz's test, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ is convergent.

Let the terms of the series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ be rearranged by taking alternatively α positive terms and β negative terms.

If $g=m f(m)$ and $K=\alpha/\beta$. then the alternation in the sum due to this rearrangement is $\frac{1}{2}g\log K$.

In particular, if $f(n)=1/n$.

then $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+$ ------

then we know that the series is conditionally convergent and its sum is $\log 2$.

Also $g=m f(m)$

$=m\cdot\frac{1}{m}=1$

$\therefore$ If the terms are rearranged by taking alternately α +ve terms & β -ve terms, then the sum of new series is

$\log 2+\frac{1}{2}g\log K=\log 2+\frac{1}{2}\log K$

Problems: (1) find the sum of the series

$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}$ ----

Sol'n: The given series

$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}$ -----

is rearrangement of the series

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ ---- $=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

and is conditionally convergent and whose sum is $\log 2$.

Here the rearranged given series is formed by taking alternatively one +ve and two -ve terms.

Let α be the +ve terms & β be the -ve terms.

then $K=\alpha/\beta=1/2$

and $g=m f(m)=m\cdot\frac{1}{m}=1$

$\therefore$ the sum of the rearranged given series is $\log 2 + \frac{1}{2} g \log k$.

$$= \log 2 + \tfrac{1}{2}\log \tfrac{1}{2}$$
$$= \log 2 - \tfrac{1}{2}\log 2$$
$$= \tfrac{1}{2}\log 2.$$

→ Find the sum of the series.

(i) $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} - \cdots$

(ii) $1 + 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots$

(iii) $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} - \frac{1}{8} + \cdots$

→ Investigate what derangement of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ will reduce its sum to zero.

Sol^n:- The given series is

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = \sum (-1)^{n-1}\frac{1}{n}.$$

It is conditionally convergent with sum $\log 2$.

Let it be deranged by taking alternately α +ve & β -ve terms so that $k = \alpha/\beta$.

and $g = m f(m)$
$$= m \cdot \frac{1}{m} = 1$$

$\therefore$ The sum of the deranged

$$= \log 2 + \tfrac{1}{2} g \log k.$$
$$= \log 2 + \tfrac{1}{2}\log k.$$

But the sum is given to be zero.

$$\therefore \log 2 + \tfrac{1}{2}\log k = 0$$
$$\Rightarrow \tfrac{1}{2}\log k = -\log 2$$
$$\Rightarrow \log k = -2\log 2$$
$$\Rightarrow \log k = \log \tfrac{1}{4}$$
$$\Rightarrow k = \tfrac{1}{4}$$
$$\Rightarrow \alpha/\beta = \tfrac{1}{4}$$
$$\Rightarrow \alpha = 1 \text{ (one +ve term)}$$
$$\& \beta = 4 \text{ (four -ve terms)}.$$

$\therefore$ To get the sum zero, one +ve term should be followed by four -ve terms.

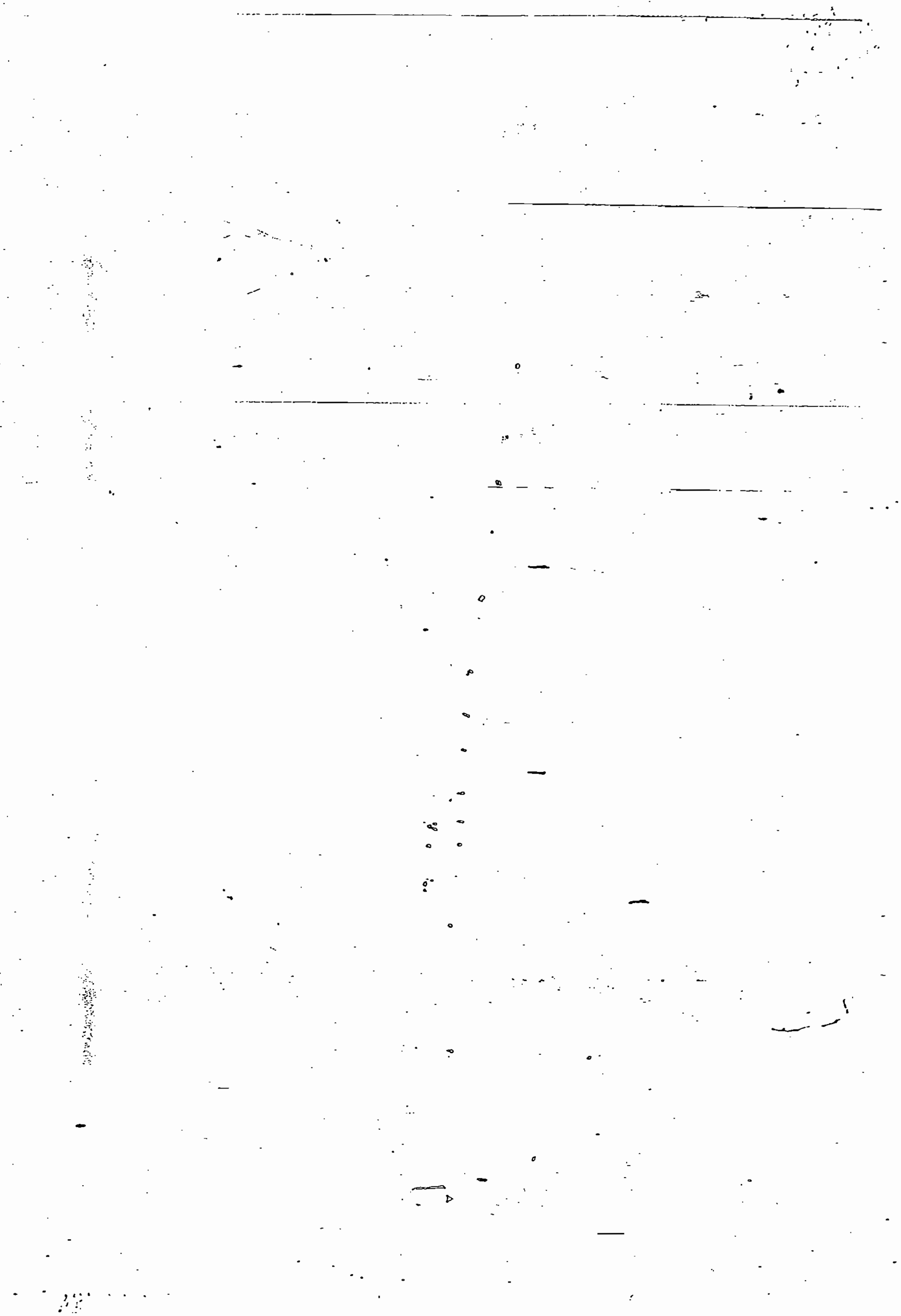
The deranged series.

$$1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{3} - \tfrac{1}{10} - \tfrac{1}{12} - \tfrac{1}{14} - \tfrac{1}{16} + \tfrac{1}{5} + \cdots$$

H.W. → what derangement of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ will reduce its sum to $\frac{1}{2}\log 2$.

$\boxed{2007}$ H.W. → Rearrange the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ to converge to 1.

i.e. what derangement of the series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ will reduce its sum to 1.



Cauchy product of Two Infinite series :-

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two infinite series then their product, called the cauchy product, is defined as $\sum_{n=1}^{\infty} c_n$

where $c_n = a_1 b_n + a_2 b_{n-1} + \cancel{a_3 b_{n-2} + \cdots} \cdots + a_n b_1$

$$= \sum_{r=1}^{n} a_r b_{n-r+1} \text{ for each } n \in N$$

Thus $\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$

$$= (a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots)$$

$$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \cdots$$

$$= c_1 + c_2 + c_3 + \cdots$$

The terms in the product are so arranged that all the terms which have the same sum of suffices are bracketed together.

Note:- (1) The cauchy product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is defined as $\sum_{n=0}^{\infty} c_n$

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$

$$= \sum_{r=0}^{n} a_r b_{n-r} \text{ for each } n \in N.$$

(2) $c_n = \sum_{r=1}^{n} a_r b_{n-r+1} = \sum_{r=1}^{n} a_{n-r+1} b_r$

and $c_n = \sum_{r=0}^{n} a_r b_{n-r} = \sum_{r=0}^{n} a_{n-r} b_r$

(3) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then it is not necessary that $\sum_{n=1}^{\infty} c_n = \left(\sum a_n\right)\left(\sum_{n=1} b_n\right)$ must converge

→ $\sum_{n=1}^{\infty} c_n$ converges if

(i) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of non-negative terms (or)

(i) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent
(or)
(ii) $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and one of them is absolutely convergent.

i.e

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series of non-negative terms converging to A and B respectively then their cauchy product $\sum_{n=1}^{\infty} c_n$ converges to AB.

(ii) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two absolutely convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$ Then their cauchy product $\sum_{n=1}^{\infty} c_n$ is also absolutely convergent and $\sum_{n=1}^{\infty} c_n = AB$.

(iii) Merten's Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series and let $\sum_{n=1}^{\infty} a_n$ converge absolutely. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$; then their product $\sum_{n=1}^{\infty} c_n$ converges to AB.

→ Cesaro's Theorem:

If two sequences (a_n) and (b_n) converge to 'a' and 'b' respectively, then the sequence (x_n) where $x_n = \dfrac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n}$ converge to ab.

→ Abel's Test:- Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series such that $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$. If their cauchy product $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = AB$.

Problems

(1) S.T the Cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ with itself is not convergent.

Sol Given that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Let $a_n = b_n = \frac{(-1)^{n-1}}{n}$, $\forall n$

$\therefore$ By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The Cauchy's product of the two series is $\sum_{n=1}^{\infty} c_n$

where $c_n = a_1 b_n + a_2 b_{n-1} + \cdots\cdots a_n b_1$

$$= \frac{(-1)^0}{1} \cdot \frac{(-1)^{n-1}}{n} + \frac{(-1)^1}{2} \cdot \frac{(-1)^{n-2}}{n-1} + \cdots \cdots + \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^0}{1}$$

$$= (-1)^{n-1} \left[\frac{1}{1 \cdot n} + \frac{1}{2 \cdot (n-1)} + \frac{1}{3 \cdot (n-2)} + \cdots + \frac{1}{n \cdot 1} \right]$$

$$\geq (-1)^{n-1} \left(\frac{1}{n \cdot n} + \frac{1}{n \cdot n} + \cdots + \frac{1}{n \cdot n} \right)$$

$\left(\because r \leq n \Rightarrow \frac{1}{r} \geq \frac{1}{n} \right)$ here $r = 1$ to n

$$= (-1)^{n-1} \left[\frac{1}{n \cdot n} (n \text{ times}) \right]$$

$$= (-1)^{n-1} \left[\frac{n}{n \cdot n} \right]$$

$$= (-1)^{n-1} \left[\frac{1}{n} \right]$$

$c_n \geq \frac{(-1)^{n-1}}{n}$

$\frac{(-1)^{n-1}}{n} \leq c_n$

$\sum \frac{(-1)^{n-1}}{n} \leq \sum c_n$

If dgt then dgt

$\therefore c_n \geq (-1)^{n-1} \left(\frac{1}{n} \right) \quad \forall n$

$\Rightarrow |c_n| \geq \left| (-1)^{n-1} \frac{1}{n} \right| \quad \forall n$

$\Rightarrow \frac{1}{n} \leq |c_n| \quad \forall n$

Since $\sum \frac{1}{n}$ is dgt. $\Rightarrow \sum |c_n|$ is dgt (By comparison test)

$\Rightarrow \lim_{n \to \infty} c_n \neq 0$

Hence $\sum_{n=1}^{\infty} c_n$ can not converge.

Note:- The above example illustrates that the cauchy product of two conditionally convergent series need not be necessarily convergent.

(2) → Show that the cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ with itself is not convergent.

Sol. Let $a_n = b_n = \frac{(-1)^{n-1}}{\sqrt{n}}$, $n \in N$

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The cauchy product of the two series is $\sum_{n=1}^{\infty} c_n$; where

$$c_n = a_1 b_n + a_2 b_{n-1} + \cdots\cdots + a_n b_1$$

$$= \frac{(-1)^0}{\sqrt{1}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^{n-2}}{\sqrt{n-1}} + \cdots$$

$$\cdots\cdots + \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{(-1)^0}{\sqrt{1}}$$

$$= (-1)^{n-1}\left[\frac{1}{\sqrt{1 \cdot n}} + \frac{1}{\sqrt{2(n-1)}} + \cdots + \frac{1}{\sqrt{n \cdot 1}}\right]$$

$$\geq (-1)^{n-1}\left[\frac{1}{\sqrt{n \cdot n}} + \frac{1}{\sqrt{n \cdot n}} + \cdots\cdots + \frac{1}{\sqrt{n \cdot n}}\right]$$

$$= (-1)^{n-1}\left[\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots\cdots + \frac{1}{n}\right]$$

$$= (-1)^{n-1}\left[\frac{n}{n}\right]$$

$$= (-1)^{n-1}$$

$$\Longrightarrow |c_n| \geq 1 \ \forall\, n \in N$$

$$\Longrightarrow \lim_{n\to\infty} c_n \neq 0.$$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge.

③ → Show that the cauchy product of the convergent series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ with itself is not convergent.

Sol Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}, \forall n \in N$.

By Leibnitz's test, the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent (but not absolutely).

The cauchy product of the two series is ~~$\sum_{n=1}^{\infty} c_n$, where~~

$$c_n = a_1 b_n + a_2 b_{n-1} + \cdots\cdots + a_n b_1$$

$$= \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \cdots\cdots\cdots + \frac{(-1)^n}{\sqrt{n+1}} \cdot \frac{(-1)^1}{\sqrt{2}}.$$

$$= (-1)^{n+1}\left[\frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{3n}} + \cdots + \frac{1}{\sqrt{(n+1)\cdot 2}}\right]$$

$$\geq (-1)^{n+1}\left[\frac{1}{\sqrt{(n+1)(n+1)}} + \frac{1}{\sqrt{(n+1)(n+1)}} + \cdots\cdots + \frac{1}{\sqrt{(n+1)(n+1)}}\right]$$

$$= (-1)^{n+1}\left(\frac{n}{n+1}\right).$$

$$\therefore c_n \geq (-1)^{n+1}\left(\frac{n}{n+1}\right) \; \forall n \in N.$$

$$|c_n| \geq \frac{n}{n+1}, \; \cancel{\forall n \in N}.$$

Since $\sum_{n=1}^{\infty} \frac{n}{n+1}$ is divergent (By comparison test)

$\Rightarrow \sum |c_n|$ is divergent

$\Rightarrow \lim_{n\to\infty} c_n \neq 0.$

Hence $\sum_{n=1}^{\infty} c_n$ cannot converge

④ → Show that the cauchy product of two divergent series $\sum_{n=1}^{\infty} a_n = 2 + 2^1 + 2^2 + 2^3 + \cdots$ and $\sum_{n=1}^{\infty} b_n = -1 + 1 + 1 + 1 + 1 + \cdots\cdots$ is convergent.

Sol for $n \geq 2$,

$\sum a_n$ and $\sum b_n$ are geometric series with common ratios 2 and 1 respectively.

Since the geometric series $\sum r^n$ is divergent for $r \geq 1$

$\therefore$ the series $\sum a_n$ and $\sum b_n$ are both divergent

The cauchy product of the two given series is $\sum_{n=1}^{\infty} c_n$, where

$$c_n = a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \cdots + a_{n-1} b_2 + a_n b_1.$$

$$= 2\cdot 1 + 2\cdot 1 + 2^2\cdot 1 + 2^3\cdot 1 + \cdots \cdots + 2^{n-2}\cdot 1 + 2^{n-1}\cdot(-1)$$

$$= 2 + (2 + 2^2 + 2^3 + \cdots + 2^{n-2}) - 2^{n-1}$$

$$= 2 + \frac{2(2^{n-2}-1)}{2-1} - 2^{n-1} \qquad \left(\because r \geq 1 \Rightarrow \frac{a(r^n-1)}{r-1}\right)$$

$$= 2 + 2^{n-1} - 2 - 2^{n-1}$$

$$= 0 \quad \forall n \geq 2$$

and $c_1 = a_1 b_1$

$= 2(-1)$

$= -2$

$\boxed{\lim_{n} S_n = -2}$

Thus $\sum_{n=1}^{\infty} c_n = -2 + 0 + 0 + 0 + \cdots$

Clearly which cgs to -2.

(5) $\rightarrow$ Show that the cauchy product of two divergent series $\sum_{n=0}^{\infty} a_n = 1 - \frac{3}{2} - (3/2)^2 - (3/2)^3 - \cdots$

and $\sum_{n=0}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \cdots$

is convergent.

Sol for $n \geq 1$, $\sum a_n$ is a geometric series with common ratio $3/2$ (>1). (20)

$\Rightarrow \sum a_n$ is divergent.

Also $\sum b_n$ is a series of positive terms and $b_n \geq 1$ $\forall n$ [?]

Since $\lim_{n\to\infty} b_n \neq 0$

$\therefore \sum b_n$ is divergent.

The cauchy product of the two given series is $\sum_{n=0}^{\infty} c_n$, where

$$c_0 = a_0 b_0 = 1 \times 1 = 1$$

and for $n \geq 1$,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0$$

$$= 1\cdot\left(\tfrac{3}{2}\right)^{n-1}\left(2^n + \tfrac{1}{2^{n+1}}\right) - \left(\tfrac{3}{2}\right)\cdot\left(\tfrac{3}{2}\right)^{n-2}\left(2^{n-1} + \tfrac{1}{2^n}\right)$$

$$- \left(\tfrac{3}{2}\right)^2\cdot\left(\tfrac{3}{2}\right)^{n-3}\left(2^{n-2} + \tfrac{1}{2^{n-1}}\right) - \cdots$$

$$\cdots - \left(\tfrac{3}{2}\right)^{n-1}\left(2 + \tfrac{1}{2^2}\right) - \left(\tfrac{3}{2}\right)^n,$$

$\frac{2^{n-1}}{2^{n-2}} = 2$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\left(2^n + \tfrac{1}{2^{n+1}}\right) - \left(2^{n-1} + 2^{n-2} + \cdots + 2\right) - \left(\tfrac{1}{2^n} + \tfrac{1}{2^{n-1}} + \cdots + \tfrac{1}{2^2}\right)\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[2^n + \tfrac{1}{2^{n+1}} - \frac{2(2^{n-1}-1)}{2-1} - \frac{\frac{1}{2^2}\left(1 - \frac{1}{2^{n-1}}\right)}{1 - 1/2}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[2^n + \tfrac{1}{2^{n+1}} - 2^n + 2 - 1/2 + \tfrac{1}{2^n}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\tfrac{3}{2} + \tfrac{1}{2^{n+1}} + \tfrac{1}{2^n}\right] - \left(\tfrac{3}{2}\right)^n$$

$$= \left(\tfrac{3}{2}\right)^{n-1}\left[\tfrac{3}{2} + \tfrac{3}{2^{n+1}} - \tfrac{3}{2}\right] = \frac{3^n}{2^{2n}} = \left(\tfrac{3}{4}\right)^n$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$$

clearly which is a geometric series of positive terms with common ratio $\frac{3}{4}$ (<1) is absolutely convergent.

HW ⑥ → S.T the 'cauchy' product of two divergent series $\sum_{n=1}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \cdots$

and $\sum_{n=1}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2}\left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right)^2\left(2^3 + \frac{1}{2^4}\right) + \cdots$ is convergent

(Note: In example ⑤ a_n is the $(n+1)^{th}$ term of $\sum_{n=0}^{\infty} a_n$ where as in example ⑥, a_n is the n^{th} term of $\sum_{n=1}^{\infty} a_n$).

→ prove that the cauchy product of the two series $3 + \sum_{n=1}^{\infty} 3^n$ and $-2 + \sum_{n=1}^{\infty} 2^n$ is absolutely convergent, although both ~~the~~ series are divergent.

Sol^n: Let $\sum_{n=0}^{\infty} a_n = 3 + 3 + 3^2 + 3^3 + \cdots = 3 + \sum_{n=1}^{\infty} 3^n$

and $\sum_{n=0}^{\infty} b_n = -2 + 2 + 2^2 + 2^3 + \cdots = -2 + \sum_{n=1}^{\infty} 2^n$

→ Show that

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{n\cdot 1}\right]$$

Sol^n: Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1} a_n$,

then $\sum_{n=1}^{\infty} a_n$ converges (conditionally).

By Abel's test, if the cauchy product $\sum_{n=1}^{\infty} c_n$ of $\sum_{n=1}^{\infty} a_n$ (21)
with itself converges,

then $\left(\sum_{n=1}^{\infty} a_n\right)^2 = \sum_{n=1}^{\infty} c_n$ ——— (1)

Now,

$$c_n = 1\cdot\frac{(-1)^{n-1}}{n} - \frac{1}{2}\cdot\frac{(-1)^{n-2}}{n-1} + \cdots + \frac{(-1)}{n-1}\cdot\left(-\frac{1}{2}\right) + \frac{(-1)}{n}\cdot 1$$

$$= (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots + \frac{1}{(n-1)\cdot 2} + \frac{1}{n\cdot 1}\right] \quad \text{——— (2)}$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\frac{n+1}{1\cdot n} + \frac{n+1}{2(n-1)} + \cdots + \frac{n+1}{(n-1)2} + \frac{n+1}{n\cdot 1}\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\left[\left(1+\frac{1}{n}\right) + \left(\frac{1}{2}+\frac{1}{n-1}\right) + \cdots + \left(\frac{1}{n-1}+\frac{1}{2}\right) + \left(\frac{1}{n}+1\right)\right]$$

$$= \frac{(-1)^{n-1}}{n+1}\left[2 + \frac{2}{2} + \cdots + \frac{2}{n-1} + \frac{2}{n}\right]$$

$$= (-1)^{n-1}\,\frac{2}{n+1}\left[1+\frac{1}{2}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right]$$

$$\therefore |c_n| = \frac{2}{n+1}\left[1+\frac{1}{2}+\cdots+\frac{1}{n}\right]$$

$$= \frac{2n}{n+1}\left[\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n}\right]$$

$$= \frac{2}{1+\frac{1}{n}}\left[\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n}\right] \to 0 \text{ as } n\to\infty$$

(by cauchy's first theorem on limits)

Also
$$|c_{n+1}| - |c_n| = \frac{2}{n+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right) - \frac{2}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$= \left(\frac{2}{n+2}-\frac{2}{n+1}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) + \frac{2}{(n+2)(n+1)}$$

$$= \frac{-2}{(n+2)(n+1)}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-1\right)$$

$$= \frac{-2}{(n+2)(n+1)}\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) < 0$$

$\Rightarrow \quad |C_n| > |C_{n+1}|$

$\therefore$ By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{1\cdot n} + \frac{1}{2(n-1)} + \cdots\cdots + \frac{1}{n\cdot 1}\right] \text{ (by (3))}$$

converges.

Hence, from (1), we have

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots\cdots\right)^2 = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{1\cdot n}+\frac{1}{2(n-1)}+\cdots+\frac{1}{n\cdot 1}\right]$$

H.W. ~~Show that~~

$$\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)^2 = 2\left[\frac{1}{2}-\frac{1}{3}\left(1+\frac{1}{2}\right)+\frac{1}{4}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{5}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\cdots\right]$$

H.W. Show that

$$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)^2 = \frac{1}{2}-\frac{1}{4}\left(1+\frac{1}{3}\right)+\frac{1}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)-\cdots$$

* Infinite products :-

(92)

If (a_n) is a sequence, then the product $a_1 a_2 a_3 \cdots\cdots a_n \cdots\cdots$ is called an infinite product and is denoted by $\prod_{n=1}^{\infty} a_n$ or simply by $\prod a_n$

i.e $\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \cdots\cdots a_n \cdots\cdots$

'a_n' is called the n^{th} factor of the product.

The product of first 'n' terms of the sequence (a_n) is called the n^{th} partial product and is denoted by P_n.

Thus $P_n = a_1 a_2 a_3 \cdots\cdots a_n$

$= \prod_{r=1}^{n} a_r$

— The sequence (P_n) is called the sequence of partial products of the sequence (a_n)

* Convergence of Infinite products :

Let $P_n = \prod_{r=1}^{n} a_r$ be the n^{th} partial product of the infinite product $\prod_{n=1}^{\infty} a_n$.

(i) If no factor a_n is 'zero', then the product $\prod_{n=1}^{\infty} a_n$ converges if the sequence (P_n) converges to a non-zero finite number P (say),

i.e if $\lim_{n\to\infty} P_n = P$ then P is called the value of the product and we write $\prod_{n=1}^{\infty} a_n = P$.

— If $\lim_{n\to\infty} P_n = \infty$ then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to ∞

— If $\lim_{n\to\infty} P_n = 0$, then the product $\prod_{n=1}^{\infty} a_n$ is said to diverge to '0'.

(ii) If infinitely many factors 'a_n' are zero, ~~the the product~~ $\prod_{n=1}^{\infty} a_n$ is said to diverge to '0'.

(iii) If finitely many factors 'a_n' are zero, then the product $\prod_{n=1}^{\infty} a_n$ said to converge ~~if it~~ converges when the zero factors are removed.

(iv) If a finite number of factors are negative, then there exists a positive integer 'm' such that $a_n > 0 \; \forall \; n > m$ and the product $\prod_{n=1}^{\infty} a_n$ is said to converge if the product $\prod_{n=m+1}^{\infty} a_n$ converges. Since $\prod_{n=1}^{\infty} a_n = a_1 a_2 \cdots a_m \prod_{n=m+1}^{\infty} a_n$.

(v) If the sequence (P_n) oscillates, then the product $\prod_{n=1}^{\infty} a_n$ is said to oscillate.

Note: ① It is usually convenient to write the factors of the infinite product as $1+a_n$ instead of a_n.

Thus an infinite product is usually written as $\prod_{n=1}^{\infty} (1+a_n)$ and $P_n = \prod_{r=1}^{n} (1+a_r)$.

② We shall assume throughout our discussion that $a_n > -1$ i.e. $1+a_n > 0 \; \forall \; n$ so that $\log(1+a_n)$ is defined for all n.

③ For $a_n > -1$, ~~let P_n denote~~ the n^{th} partial product of $\prod_{n=1}^{\infty} (1+a_n)$, then

$$P_n = (1+a_1)(1+a_2)\cdots\cdots(1+a_n)$$

$$\Rightarrow \log P_n = \log(1+a_1) + \log(1+a_2) + \cdots + \log(1+a_n)$$

$= S_n$

where $S_n = \sum_{r=1}^{n} \log(1+a_r)$ is the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \log(1+a_n)$

$$\Rightarrow P_n = e^{S_n}$$

If $\underset{n\to\infty}{Lt}\, S_n = S$ then $\underset{n\to\infty}{Lt}\, P_n = e^{S}$.

Thus, to say that the product $\prod_{n=1}^{\infty}(1+a_n)$ diverges to 'zero',

i.e $\underset{n\to\infty}{Lt}\, P_n = 0$ is equivalent to saying that the series $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to $-\infty$

i.e $\underset{n\to\infty}{Lt}\, S_n = -\infty$. $\left(\text{i.e } \underset{n\to\infty}{Lt}\, P_n = e^{-\infty} = 0.\right)$

Problems

→ Show that the infinite product

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots$$

converges to $\frac{1}{2}$.

Sol The given infinite product is

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots = \prod_{n=1}^{\infty}\left(1-\frac{1}{(n+1)^2}\right)$$

$$= \prod_{n=1}^{\infty} \frac{(n+1)^2-1}{(n+1)^2}$$

$$= \prod_{n=1}^{\infty} \frac{n(n+2)}{(n+1)^2}$$

$$= \prod_{n=1}^{\infty} \left(\frac{n}{n+1}\cdot\frac{n+2}{n+1}\right)$$

$$\therefore P_n = \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\cdots\cdots\left(\frac{n}{n+1}\cdot\frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{2}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdots\cdots\frac{n}{n+1}\right)\left(\frac{\cancel{3}}{2}\cdot\frac{\cancel{4}}{\cancel{3}}\cdots\cdots\frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{n+1}\right)\left(\frac{n+2}{2}\right) = \frac{1}{2}\left(1+\frac{1}{n+1}\right)$$

$$\Rightarrow \underset{n\to\infty}{Lt}\, P_n = \frac{1}{2}$$

Hence the given infinite product converges to $\frac{1}{2}$

i.e $\prod_{n=1}^{\infty}\left(1-\frac{1}{(n+1)^2}\right)=\frac{1}{2}$.

H.W → show that the infinite product

$\left(1-\frac{2}{2\cdot 3}\right)\left(1-\frac{2}{3\cdot 4}\right)\left(1-\frac{2}{4\cdot 5}\right)\cdots$

converges to $\frac{1}{3}$.

→ show that the infinite products (i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ and (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ are both divergent

Sol (i) Given ~~infinite~~ product is

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)=\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)$

$=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdots\frac{(n+1)}{n}\cdots$

Let $P_n=\prod_{k=1}^{n}\left(\frac{k+1}{k}\right)$

$=\frac{\cancel{2}}{1}\cdot\frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{5}}{\cancel{3}}\cdot\frac{7}{\cancel{5}}\cdots\frac{(n+1)}{\cancel{n}}$

$=n+1$

Now $\lim_{n\to\infty}P_n=\infty$.

∴ The given infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$ is dgt and dgs to ∞.

i.e $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)=\infty$.

(ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)=\prod_{n=2}^{\infty}\left(\frac{n-1}{n}\right)$

∴ $P_n=\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{4}\cdots\frac{\cancel{n}}{n+1}=\frac{1}{n+1}$

$\Rightarrow\lim_{n\to\infty}P_n=0$

∴ $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ dgs to '0'

→ show that the infinite product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots$ is convergent.

Sol already we have done by the following method

(24)

Let $P = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots$

Then $\log P = \log\left(1-\frac{1}{2^2}\right) + \log\left(1-\frac{1}{3^2}\right) + \cdots\cdots$

$\sum_{n=2}^{\infty} \log\left(1+\frac{(-1)^n}{n}\right) = \log\left(1-\frac{1}{4}\right)$, $\log\frac{3}{4}$ (by $\sum$)

$$= \sum_{n=2}^{\infty} \log\left(1-\frac{1}{n^2}\right)$$

$$= \sum_{n=2}^{\infty} a_n \text{ (say)} \quad \text{--- (1)}$$

Now $a_n = \log\left(1-\frac{1}{n^2}\right)$

$$= -\left[\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \cdots\right]$$

$$\left(\because \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\right)$$

$$= -\frac{1}{n^2}\left[1 + \frac{1}{2n^2} + \frac{1}{3n^4} + \cdots\right].$$

Let $b_n = \frac{1}{n^2}$ $\forall n$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} -\left[1 + \frac{1}{2n^2} + \cdots\right]$

$$= -1$$
$$\neq 0.$$

Here $\sum b_n = \sum \frac{1}{n^2}$ is cgt (by p-test)

$\therefore$ By comparison test,

$\therefore \sum a_n$ is cgt.

and is convergent to finite number 'S' (say)

$\therefore$ from (1), $\log P =$ a finite number 'S' when $n \to \infty$

i.e. $P =$ a finite number e^S when $n \to \infty$

$\therefore$ The given product is cgt.

~~Hw: S.T. the infinite product~~ is convergent

~~$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots$~~

(1) Show that the infinite product

$$\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots\frac{(2n-1)}{2n}\cdot\frac{(2n+1)}{2n}\cdots$$

tends to a finite limit as $n\to\infty$.

(2) ~~S.T the~~ infinite product

$$\left(1+\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1-\tfrac{1}{5}\right)\cdots\cdots$$

converges to 1.

Sol: Given infinite product is

$$\left(1+\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1-\tfrac{1}{5}\right)\cdots$$

Let $P=\left(1+\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1-\tfrac{1}{5}\right)\cdots\left(1+\tfrac{1}{2n}\right)\left(1-\tfrac{1}{2n+1}\right)\cdots$

then $\log P=\log\left[\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{4}\right)\right]+\log\left[\left(1+\tfrac{1}{4}\right)\left(1-\tfrac{1}{5}\right)\right]$

$$+\cdots+\log\left[\left(1+\tfrac{1}{2n}\right)\left(1-\tfrac{1}{2n+1}\right)\right]+\cdots$$

$$=\sum_{n=1}^{\infty}\log\left[\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n+1}\right)\right]$$

$$=\sum_{n=1}^{\infty}\log\left[1+\frac{1}{2n}-\left(1+\frac{1}{2n}\right)\left(\frac{1}{2n+1}\right)\right]$$

$$=\sum_{n=1}^{\infty}\log\left[1+\left(\frac{1}{2n}-\frac{1}{2n+1}\right)-\frac{1}{2n(2n+1)}\right]$$

$$=\sum_{n=1}^{\infty}\log\left[1+\frac{1}{2n(2n+1)}-\frac{1}{2n(2n+1)}\right]$$

$$=\sum_{n=1}^{\infty}\log[1]=0.$$

$\therefore \log P=0\Rightarrow P=e^{0}=1.$

$\therefore$ The given infinite product is convergent and is convergent to 1.

A necessary condition for convergence:

If the product $\prod_{n=1}^{\infty}(1+a_n)$ is convergent, then

$\lim_{n\to\infty} a_n=0$.

proof Given that $\prod_{n=1}^{\infty}(1+a_n)$ is cgt

and it cgs to P (say) (25)

$\therefore P \neq 0$; $\lim_{n\to\infty} P_n = P$ and $\lim_{n\to\infty} P_{n-1} = P$.

now $\dfrac{P_n}{P_{n-1}} = \dfrac{\cancel{(1+a_1)(1+a_2)\cdots(1+a_{n-1})}(1+a_n)}{\cancel{(1+a_1)(1+a_2)\cdots(1+a_{n-1})}}$

$\dfrac{P_n}{P_{n-1}} = (1+a_n)$

$\therefore \lim_{n\to\infty}(1+a_n) = \lim_{n\to\infty} \dfrac{P_n}{P_{n-1}} = \dfrac{\cancel{P}}{\cancel{P}} = 1$

$\lim_{n\to\infty}(1+a_n) = 1 \Rightarrow 1 + \lim_{n\to\infty} a_n = 1$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

Note(i): If $\lim_{n\to\infty} a_n \neq 0$ then $\prod_{n=1}^{\infty}(1+a_n)$ must not cgt

Note(ii):- The converse of the above need not be true. i.e. if $\lim_{n\to\infty} a_n = 0$. i.e $a_n \to 0$ as $n\to\infty$ then $\prod_{n=1}^{\infty}(1+a_n)$ need not be cgt.

For examples

The infinite product $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{\infty}(1+a_n)$

Here $a_n = \frac{1}{n} \to 0$ as $n\to\infty$

but the product is divergent. (already we have done)

→ **General principle of convergence of an infinite product:**

Necessary and sufficient condition

A necessary and sufficient condition for the convergence of the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is that for every $\epsilon > 0$, there exists a positive integer 'm' s.t $\left|\dfrac{P_{n+p}}{P_n} - 1\right| < \epsilon \;\forall\, n \geq m, p \geq 1$.

Note: In order to establish the convergence (or divergence) of infinite product, we now give the following statements:

→ If $a_n \geq 0$ then the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

→ If $-1 < a_n \leq 0$, then the series $\sum_{n=1}^{\infty} a_n$ and the product $\prod_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

→ If $0 \leq b_n < 1$ then $\prod_{n=1}^{\infty}(1-b_n)$ converges to non-zero finite limit, if $\sum_{n=1}^{\infty} b_n$ converges and diverges to zero if $\sum_{n=1}^{\infty} b_n$ diverges.

→ If the series $\sum_{n=1}^{\infty} a_n^2$ is convergent, then the product $\prod_{n=1}^{\infty}(1+a_n)$ and series $\sum_{n=1}^{\infty} a_n$ converge or diverge together.

→ If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then we have $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \log(1+a_n)$ converge or diverge together.

Also $\sum_{n=1}^{\infty} \log(1+a_n)$ and $\sum_{n=1}^{\infty}(1+a_n)$ converge or diverge together.

∴ we have

→ if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \log(1+a_n)$ converges and therefore $\prod_{n=1}^{\infty}(1+a_n)$ converges.

→ if $\sum_{n=1}^{\infty} a_n$ diverges to ∞, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to ∞ and therefore $\prod_{n=1}^{\infty}(1+a_n)$ diverges ~~to ∞~~.

→ if $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$, then $\sum_{n=1}^{\infty} \log(1+a_n)$ diverges to $-\infty$ and therefore $\prod_{n=1}^{\infty}(1+a_n)$ diverges to zero.

Also if $\sum_{n=1}^{\infty} a_n^2$ diverges and $\sum_{n=1}^{\infty} a_n$ converges or oscillates finitely, then $\prod_{n=1}^{\infty}(1+a_n)$ diverges to zero.

→ ~~Absolute convergence of Infinite products~~:

Def The product $\prod_{n=1}^{\infty}(1+a_n)$ is said to be absolutely convergent if the product $\prod_{n=1}^{\infty}(1+|a_n|)$ is convergent.

→ The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff ~~the series~~ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

→ The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff the series $\sum_{n=1}^{\infty} \log(1+a_n)$ is absolutely convergent.

→ Every absolutely convergent infinite product is convergent.

i.e If $\prod_{n=1}^{\infty}(1+a_n)$ is an absolutely convergent (i.e $\prod_{n=1}^{\infty}(1+|a_n|)$ is cgt)

~~then~~ $\prod_{n=1}^{\infty}(1+a_n)$ is cgt.

Note:- The factors of an absolutely convergent infinite product may be rearranged in any order without affecting its convergence.

problems

→ Discuss the convergence of the infinite products:

(i) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{3/2}}\right)$ (iii) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right), \alpha>1$

(iv) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^{\alpha}}\right), 0<\alpha\leq 1$ (v) $\prod_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)$ (vi) $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$

Sol (i) The given product is

$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right) = \prod_{n=1}^{\infty}(1+a_n)$, where $a_n=\frac{1}{n^2}>0$ $\forall n$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt (by p-test) here $p=2>1$

$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)$

is cgt.

→ Discuss the convergence of the infinite products:

(i) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (ii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)$ (iii) $\prod_{n=2}^{\infty}\left(1-\frac{1}{\sqrt{n}}\right)$

(iv) $\frac{3}{4}\cdot\frac{6}{7}\cdot\frac{9}{10}\cdots\frac{3n}{3n+1} = \prod_{n=1}^{\infty}\left(\frac{3n}{3n+1}\right)$

$= \prod_{n=1}^{\infty}\left(1-\frac{1}{3n+1}\right)$

Sol (i) The given product is

$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \prod_{n=2}^{\infty}(1-b_n)$, where $b_n = \frac{1}{n^2}$ and $n \geq 2$

so that $0 < b_n < 1$.

$\therefore$ The product $\prod_{n=2}^{\infty}(1-b_n)$ and the series $\sum_{n=2}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty}\frac{1}{n^2}$ is cgt (by p-test).

$\therefore$ The given product is cgt.

(iv) The product is $\prod_{n=1}^{\infty}\left(1-\frac{1}{3n+1}\right) = \prod_{n=1}^{\infty}(1-b_n)$

where $b_n = \frac{1}{3n+1}$ and $n \geq 1$

so that $0 < b_n < 1$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{3n+1}$ dgs.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ dgs to zero.

→ Discuss the convergence of the infinite products:

(i) $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)$ (ii) $\prod_{n=1}^{\infty}\left(1+n\sin\frac{\theta}{n^2}\right)$ (27)

(iii) $\prod_{n=1}^{\infty}\left(1+\frac{x}{n^p}\right)$; where x is the real number.

Sol The given product is $\prod_{n=1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)=\prod_{n=1}^{\infty}(1+a_n)$

where $a_n=\sin^2\frac{\theta}{n}\geq 0$ $\forall n$.

∴ The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum a_n$ converge or diverge together.

Now $a_n=\sin^2\frac{\theta}{n}=\left(\sin\frac{\theta}{n}\right)^2$

$$=\left(\frac{\theta}{n}-\frac{1}{3!}\cdot\frac{\theta^3}{n^3}+\frac{1}{5!}\cdot\frac{\theta^5}{n^5}-\cdots\right)^2$$

$\frac{\theta^3}{n}\to 0$ $=0$

$$=\frac{\theta^2}{n^2}-2\left(\frac{1}{3!}\cdot\frac{\theta^4}{n^4}\right)+\cdots\cdots$$

$$=\frac{1}{n^2}\left[\theta^2-2\left(\frac{1}{3!}\cdot\frac{\theta^4}{n^2}\right)+\cdots\cdots\right]$$

Take $b_n=\frac{1}{n^2}$

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n}=\theta^2$$

Since $\sum_{n=1}^{\infty} b_n=\sum\frac{1}{n^2}$ is cgt

∴ By comparison test $\sum_{n=1}^{\infty} a_n$ is cgt.

Hence the given product is cgt.

(iii) $\prod_{n=1}^{\infty}\left(1+\frac{x}{n^p}\right)=\prod_{n=1}^{\infty}(1+a_n)$

where, $a_n=\frac{x}{n^p}\geq 0$ $\forall n$.

So The product $\prod_{n=1}^{\infty}(1+a_n)$ and the series $\sum a_n$ converge or diverge together.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a}{n^p} = a \sum_{n=1}^{\infty} \frac{1}{n^p}$

which is cgt if $p>1$

and dgt if $p \le 1$.

Hence the given product is also convergent if $p>1$ and dgt if $p \le 1$.

→ S.T $(1+a)\left(1+\frac{a}{2}\right)\left(1+\frac{a}{3}\right)\cdots\cdots$

dgs to $+\infty$ or to '0' according as $a>0$ or $a<0$.

Sol The given product is

$$\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right) = \prod_{n=1}^{\infty}(1+a_n)$$

where $a_n = \frac{a}{n}$.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a}{n} = a \sum_{n=1}^{\infty} \frac{1}{n}$

which dgs to $+\infty$ if $a>0$

dgs to $-\infty$ if $a<0$.

Hence the given product dgs to ∞ if $a>0$

dgs to $-\infty$ if $a<0$.

→ Discuss the convergence of the product:

(i) $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right)$

(ii) $\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\cdots\cdots$

(iii) $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\cdots$

(iv) $\left(1+\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{\sqrt{3}}\right)\left(1+\frac{1}{\sqrt{4}}\right)\left(1\mp\frac{1}{\sqrt{5}}\right)\cdots\cdots$

(v) $\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{\sqrt{n}}\right)$.

(28)

Sol (1) The given product is

$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n}\right) = \prod_{n=2}^{\infty}(1+a_n) \text{ where } a_n = \frac{(-1)^n}{n}$$

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$ is cgt by Leibnitz's test

and $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \frac{1}{n^2}$ is also cgt

$\therefore$ The given product is cgt.

→ Discuss the convergence of the infinite product $(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{2})(1+\frac{1}{2})\cdots$

Sol The given product is $\prod_{n=1}^{\infty}\left(1+(-1)^n\cdot\frac{1}{2}\right)$

$$= \prod_{n=1}^{\infty}(1+a_n)$$

where $a_n = (-1)^n \cdot \frac{1}{2}$.

Now $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{2}$

$$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n$$

$$= \frac{1}{2}(-1+1-1+1-1\cdots)$$

$\therefore$ which oscillates b/w $-\frac{1}{2}$ and 0.

$\therefore \sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty}\frac{1}{4} = \frac{1}{4}+\frac{1}{4}+\cdots$

which is dgt (to $+\infty$)

Hence the given product dgs to zero.

HW Discuss the convergence of $\prod_{n=1}^{\infty}(1+(-1)^n)$.

→ Show that the infinite product $\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^\alpha}\right)$ is convergent if $\alpha > \frac{1}{2}$.

Sol The given product is
$$\prod_{n=2}^{\infty}\left(1+\frac{(-1)^n}{n^\alpha}\right)=\prod_{n=2}^{\infty}(1+a_n)$$
where $a_n=\frac{(-1)^n}{n^\alpha}$.

Now $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty}\frac{(-1)^n}{n^\alpha}$ cgs if $\alpha>0$ (by Leibnitz's test)

Also $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty}\frac{1}{n^{2\alpha}}$ cgs if $2\alpha>1$

i.e if $\alpha>\frac{1}{2}$

Hence the given product cgs if $\alpha>\frac{1}{2}$.

Discuss the convergence of the product
$$\prod_{n=1}^{\infty}\left[1+\left(\frac{nx}{n+1}\right)^n\right]$$

Sol Here $a_n=\left(\frac{nx}{n+1}\right)^n$

$\therefore a_n^{1/n}=\frac{nx}{n+1}=\frac{x}{1+\frac{1}{n}}$

$\therefore \lim_{n\to\infty} a_n^{1/n}=x$.

$\therefore$ By Cauchy's root test, the series $\sum_{n=1}^{\infty} a_n$ is cgt if $x>1$ and dgt if $x<1$.

Hence the given product is cgt if $x<1$ and dgt if $x>1$.

If $x=1$ then $a_n=\left(\frac{n}{n+1}\right)^n=\frac{1}{(1+\frac{1}{n})^n}$

$\therefore \lim_{n\to\infty} a_n=\frac{1}{e}\neq 0$ and $a_n>0$ $\forall n$.

$\therefore \sum a_n$ is dgt.

Hence $\prod_{n=1}^{\infty}(1+a_n)$ is dgt.

Thus the given product is cgt if $x<1$ and dgt if $x\geq 1$.

(25)

→ Discuss absolute convergence of the following infinite products:

(i) $\prod_{n=1}^{\infty} \cos\frac{\theta}{n}$ (ii) $\prod_{n=1}^{\infty}\left[\frac{\sin x/n}{x/n}\right]$.

Sol (i) Here $1+a_n = \cos\frac{\theta}{n}$.

$$1-\frac{1}{2!}\cdot\frac{\theta^2}{n^2}+\frac{1}{4!}\cdot\frac{\theta^4}{n^4}-\cdots$$

$$\Rightarrow a_n = -\frac{1}{2!}\frac{\theta^2}{n^2}+\frac{1}{4!}\frac{\theta^4}{n^4}-\cdots$$

$$= \frac{1}{n^2}\left(-\frac{\theta^2}{2!}+\frac{\theta^4}{4!\,n^2}-\cdots\right)$$

Now $|a_n| = \frac{1}{n^2}\left|-\frac{\theta^2}{2!}+\frac{\theta^4}{4!\,n^2}-\cdots\right|$.

Let $b_n = \frac{1}{n^2}$ then $\lim_{n\to\infty}\frac{|a_n|}{b_n} = \frac{\theta^2}{2}$, (a finite quantity)

But $\sum b_n = \sum\frac{1}{n^2}$ is cgt (by p-Test).

$\therefore \sum|a_n|$ is cgt (by comparison test).

$\therefore \sum a_n$ is absolutely convergent.

$\therefore$ the product $\prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\cos\frac{\theta}{n}$ is absolutely cgt.

(ii) Here $1+a_n = \frac{\sin\frac{x}{n}}{\frac{x}{n}}$

$$= \frac{1}{\left(\frac{x}{n}\right)}\left[\frac{x}{n}-\frac{1}{3!}\cdot\frac{x^3}{n^3}+\frac{1}{5!}\cdot\frac{x^5}{n^5}-\cdots\right]$$

$$= 1-\frac{1}{3!}\cdot\frac{x^2}{n^2}+\frac{1}{5!}\cdot\frac{x^4}{n^4}-\cdots$$

$$\Rightarrow a_n = -\frac{1}{3!}\frac{x^2}{n^2}+\frac{1}{5!}\frac{x^4}{n^4}-\cdots$$

$$= \frac{1}{n^2}\left(-\frac{x^2}{3!} + \frac{x^4}{5!\,n^2} - \cdots\right)$$ proceed in this way.

→ Prove that $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is absolutely convergent for any real x.

Sol Here $1+a_n = \left(1+\frac{x}{n}\right)e^{-x/n}$

$$= \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{x^2}{2!\,n^2}-\frac{x^3}{3!\,n^3}\cdots\right)$$

$$= 1-\frac{x^2}{n^2}+\frac{x^2}{2n^2}+\frac{x^3}{2n^3}-\frac{x^3}{6n^3}\cdots$$

$$\Rightarrow a_n = \frac{-x^2}{2n^2}+\frac{x^3}{3n^3}\cdots\cdots$$

$$= \frac{1}{n^2}\left(-\frac{x^2}{2}+\frac{x^3}{3n}\cdots\cdots\right)$$

proceed in this way.

HW → P.T $\prod_{n=1}^{\infty}\left(1+\frac{x}{n\pi}\right)e^{-\frac{x}{n\pi}}$ is absolutely cgt for all values of x.

HW → P.T $\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}$ is absolutely cgt.

→ Test the absolutely convergence of the infinite product $\prod_{n=1}^{\infty}\left(\frac{x+x^{2n}}{1+x^{2n}}\right)$.

Sol Here $1+a_n = \frac{x+x^{2n}}{1+x^{2n}} = \frac{(1+x^{2n})+(x-1)}{1+x^{2n}}$

$$\Rightarrow 1+a_n = 1+\frac{x-1}{1+x^{2n}}$$

$$\Rightarrow \boxed{a_n = \frac{x-1}{1+x^{2n}}}$$

Now, when $|x| > 1$, we have

$$|a_n| = \left|\frac{x-1}{1+x^{2n}}\right| = \frac{|x-1|}{|1+x^{2n}|} = \frac{|x-1|}{1+x^{2n}} < \frac{|x-1|}{x^{2n}} \quad \text{—— (1)}$$

Now $\sum_{n=1}^{\infty} \frac{1}{x^{2n}} = \sum u_n$ (say).

Here $u_n = \frac{1}{x^{2n}}$, then

$$u_n^{1/n} = \frac{1}{x^2}$$

$\therefore \lim_{n\to\infty} u_n^{1/n} = \frac{1}{x^2} \; (<1) \quad (\because |x| > 1)$

$\therefore$ By Cauchy's n^{th} root test $\sum \frac{1}{x^{2n}}$ is cgt.

$\therefore$ By comparison test,

$\sum |a_n|$ is cgt

$\therefore \sum a_n$ is absolutely cgt.

Hence $\prod_{n=1}^{\infty} (1+a_n)$ is absolutely convergent.

Now, when $|x| < 1$ (i.e $-1 < x < 1$),

we have

$$1 + a_n = \frac{x + x^{2n}}{1+x^{2n}} = \frac{x(1+x^{2n-1})}{1+x^{2n}} \to x \quad (\because -1 < x < 1)$$

$\therefore 1 + a_n \to x$ as $n \to \infty$.

$\Rightarrow a_n \to x - 1$ as $n \to \infty$.

$\Rightarrow a_n$ doesnot tend to 0 $(\because -1 < x < 1)$

i.e $\lim_{n\to\infty} a_n \neq 0$

$\therefore$ The product $\prod_{n=1}^{\infty} (1+a_n)$ is divergent.

Now, when $x = 1$,

every factor is unity

i.e $\prod_{n=1}^{\infty}\left(\frac{x+x^{2n}}{1+x^{2n}}\right) = \left(\frac{x+x^2}{1+x^2}\right)\left(\frac{x+x^4}{1+x^4}\right)\cdots\cdots$

$= \frac{2}{2}\cdot\frac{2}{2}\cdots\cdots$

$= 1\cdot 1\cdots\cdots$

Hence the product is convergent.

Now, when $x = -1$,

every factor is zero.

Hence the product is divergent.

→ Discuss the convergence of the infinite product $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$.

Sol Here $1+a_n = 1+\frac{x^n}{x^{2n}+1}$ so that $a_n = \frac{x^n}{x^{2n}+1}$

Now $a_{n+1} = \frac{x^{n+1}}{x^{2n+2}+1}$

$$\therefore \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{x^n}{x^{2n}+1}\cdot\frac{x^{2n+2}+1}{x^{n+1}}\right|$$

$$= \left|\frac{x^{2n+2}+1}{x(x^{2n}+1)}\right| = \frac{|x^{2n+2}+1|}{|x|\,|x^{2n}+1|}$$

$\therefore$ If $|x|<1$ (i.e. $-1<x<1$),

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{|x|} > 1$$

$\therefore$ By ratio test, $\Sigma|a_n|$ cgs and hence $\prod(1+a_n)$ cgs absolutely.

If $|x|>1$,

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}+1}{x^{2n+1}+x}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x+\frac{1}{x^{2n+1}}}{1+\frac{1}{x^{2n}}}\right| = |x| > 1$$

$\therefore$ By ratio test, $\Sigma|a_n|$ cgs and hence $\prod(1+a_n)$ cgs absolutely.

If $x=1$, $a_n = \frac{1}{2}$ $\forall n$

$$\underset{n\to\infty}{Lt}\, a_n \neq 0.$$

~~$\therefore$ The product $\prod_{n=1}^{\infty}(1+a_n)$~~ is dgt.

(Or)

If $x=1$, $a_n = \frac{1}{2}$ $\forall n$.

$$\therefore \sum a_n = \sum_{n=1}^{\infty}\frac{1}{2} = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots\cdots$$

Let $S_n = \left(\frac{1}{2}+\frac{1}{2}+\cdots\cdots+\frac{1}{2}\right)$ (n times)

$$= \frac{n}{2}$$

$$\underset{n\to\infty}{Lt}\, S_n = \infty.$$

$\therefore \sum a_n$ is dgt.

hence the product $\prod_{n=1}^{\infty}(1+a_n)$ is dgt

If $x=-1$, the product $\prod_{n=1}^{\infty}\left(1+\frac{x^n}{x^{2n}+1}\right)$ becomes

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\cdots\cdots$$

which dgs to '0' (already we have done.)

→ Show that $\prod_{n=2}^{\infty}\left[1-\left(1-\frac{1}{n}\right)^{-n}x^{-n}\right]$ cgs absolutely for $|x|>1$.

Sol: Here $1+a_n = 1-\left(1-\frac{1}{n}\right)^{-n}x^{-n}$ so that

$$a_n = -\left(1-\frac{1}{n}\right)^{-n}x^{-n}.$$

Now

$$a_{n+1} = -\left(1-\frac{1}{n+1}\right)^{-n-1}x^{-n-1}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{-\left(1-\frac{1}{n}\right)^{-n}x^{-n}}{-\left(1-\frac{1}{n+1}\right)^{-n-1}x^{-n-1}} = \frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{\left(1-\frac{1}{n}\right)^{n}}\,x$$

$\therefore \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{e}{e}|x|$

$= |x| > 1.$

$\therefore$ By ratio test, $\sum |a_n|$ is cgt.

Hence the infinite product cgs absolutely.

→ Show that $\prod_{n=0}^{\infty}(1+x^{2^n})$ cgs $\frac{1}{1-x}$ if $|x|<1$.

Sol Given infinite product is

$$\prod_{n=0}^{\infty}(1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots\cdots(1+x^{2^{n-1}})(1+x^{2^n})\cdots\cdots$$

Let

$$P_n=\prod_{n=0}^{n-1}(1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots\cdots(1+x^{2^{n-1}})$$

$$=\left(\frac{1}{1-x}\right)\left[(1-x)(1+x)(1+x^2)(1+x^{2^2})(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{2^1})(1+x^{2^1})(1+x^{2^2})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-(x^2)^2)(1+x^{2^2})(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{4(=2^2)})(1+x^4)(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-(x^4)^2)(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{8(=2^3)})(1+x^8)\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-(x^8)^2)(1+x^{2^4})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{16})(1+x^{2^4})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{2^4})(1+x^{2^4})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$\cdots\cdots\cdots\cdots$

$$=\frac{1}{1-x}(1-x^{2^n}).$$

Now if $|x| < 1$ (i.e. $-1 < x < 1$)

then $x^{2^n} \to 0$ as $n \to \infty$

$\therefore P_n \to \frac{1}{1-x}$ as $n \to \infty$

$\therefore$ The infinite product $\prod_{n=0}^{\infty}(1+x^{2^n})$ cgs to $\frac{1}{1-x}$.

HW S.T $\prod_{n=0}^{\infty}\left[1+\left(\frac{1}{2}\right)^{2^n}\right]$ cgs to 2

(Hint: put $x = 1/2$).

Set - IV

Limits and Continuity

Real valued functions:

Constant function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = K$, $(K \in \mathbb{R})$ is called a constant function.

Range of $f = \{K\}$ a singleton set.

Identity function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is called the Identity function.

Range of $f = \mathbb{R}$ = Domain of f.

Polynomial function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, a_2, \ldots a_n \in \mathbb{R}$. $n \in \mathbb{N}$ and $a_n \neq 0$ is called a polynomial function of n^{th} degree.

If $a_0 = a_1 = \cdots = a_n = 0$ then $f(x) = 0 \;\forall x \in \mathbb{R}$.

In this case we say that f is a zero polynomial function.

Rational function: If f, g are two polynomial functions and $A = \{x / x \in \mathbb{R},\ g(x) \neq 0\}$. then the function $h: A \to \mathbb{R}$ defined by $h(x) = \frac{f(x)}{g(x)}$ is called a rational function.

Ex: $h(x) = \frac{1}{x}$ is a rational function with domain $\mathbb{R} - \{0\}$.

~~Power function: A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^n$ where $n \in \mathbb{R}$ is called power function.~~

~~... is the square root function defined ...~~

→ Absolute value function (or) Mod function:

The function $f: \mathbb{R} \to \mathbb{R}$ defined by

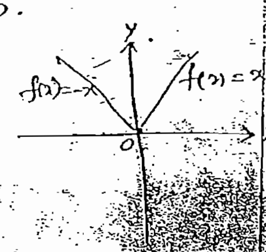
$f(x) = x$ if $x \geq 0$

$= -x$ if $x < 0$

is called mod function.

It is denoted by $f(x) = |x|$.

Range of $f = [0, \infty)$.



→ Signature function:

The function $f: \mathbb{R} \to \mathbb{R}$ defined by

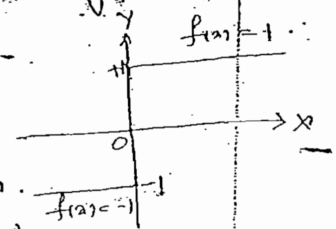
$f(x) = 1$; $x > 0$

$= 0$; $x = 0$

$= -1$; $x < 0$

is called signature function.

It is denoted by $f(x) = sgn(x)$.



i.e, $sgn(x) = 1$ if $x > 0$

$= 0$ if $x = 0$

$= -1$ if $x < 0$

i.e, $sgn(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Range of $sgn(x) = \{-1, 0, 1\}$

→ Integral part function or Step function or greatest integer function:

The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$ = integral part of x, is called step function.

i.e, $f(x) = [x]$ is a greatest integer $\leq x$, is called the greatest integer function.

i.e, for every $x \in \mathbb{R}$, $\exists$ unique $n \in \mathbb{Z}$ such that $n \le x < n+1$ and $[x] = n$.

The range of the step function $= \mathbb{Z}$.

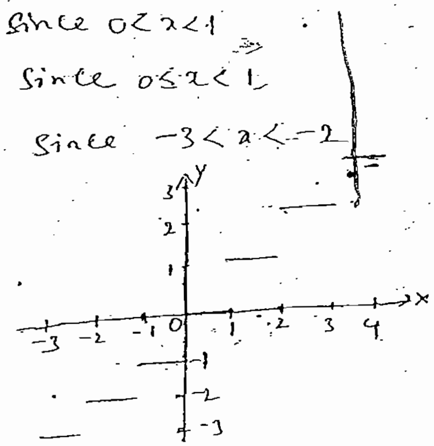
Ex: $x = 2.5$; $[x] = 2$ since $2 < x < 3$.
i.e, $2 < x < 2+1$

$x = 0.1$; $[x] = 0$ since $0 < x < 1$

$x = 0$; $[x] = 0$ since $0 \le x < 1$

$x = -2.5$; $[x] = -3$ since $-3 < x < -2$

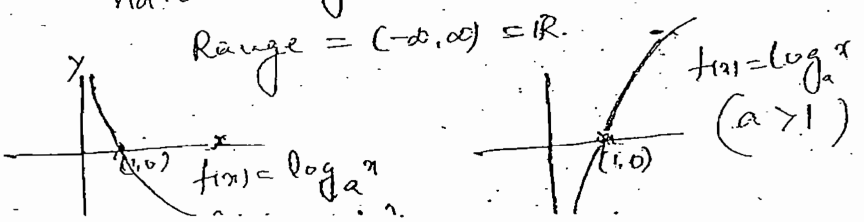
$x = 1.5$; $[x] = 1$.



→ Exponential function:
The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$ is called exponential function.
The range of exponential function $= \mathbb{R}^+$.
The range of exponential function is also
→ If $a \in \mathbb{R}^+ - \{1\}$ then $f(x) = a^x$ from $\mathbb{R} \to \mathbb{R}^+$ called exponential function.

→ Logarithmic function:
The exponential function $f: \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^x$ is both 1-1 and onto. The inverse function of this exponential function is called Logarithmic function.
$f: \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \log x$ is the natural logarithmic function.

Range $= (-\infty, \infty) = \mathbb{R}$.



→ Trigonometric functions:

— The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is called sine function.

~~Range $f = [-1, 1]$~~

— The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x$ is called cosine function.

Range $f = [-1, 1]$

— If $A = \{x \in \mathbb{R} / x = n\pi + \frac{\pi}{2} ; n \in \mathbb{Z}\}$ (or) $(2n+1)\frac{\pi}{2}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sin x}{\cos x} = \tan x$$

is called tangent function.

Domain $f = \mathbb{R} - A$ (odd values)

Range $f = \mathbb{R}$.

— If $A = \{x \in \mathbb{R} / x = n\pi : n \in \mathbb{Z}\}$ then the function $f: \mathbb{R} - A \to \mathbb{R}$ defined by $f(x) = \frac{\cos x}{\sin x} = \cot x$, is called cotangent function.

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R}$.

— If $A = \{x \in \mathbb{R} / x = (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$ defined by $f(x) = \frac{1}{\cos x} = \sec x$

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R} - (-1, 1)$.

— If $A = \{x \in \mathbb{R} / x = n\pi : n \in \mathbb{Z}\}$ then the function $f: (\mathbb{R} - A) \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sin x} = \operatorname{cosec} x$ is called cosecant function.

Domain $f = \mathbb{R} - A$

Range $f = \mathbb{R} - (-1, 1)$.

Boundedness of a function:

A function f is said to be bounded if its range is bounded. Otherwise it is unbounded.

i.e, A ~~function~~ f is said to be bounded on a domain D if there exist two real numbers h, k such that $h \le f(x) \le k$ $\forall x \in D$.

where h is called a lower bound of f,
k is called an upperbound of f.

(or)

A function f is said to be bounded on a domain D if there exist a ~~+ve~~ real number M such that $|f(x)| \le M$ $\forall x \in D$.
(i.e, $M > 0$)

Ex: $f(x) = \sin x$, $f(x) = \cos x$ are bounded functions on $\mathbb{R}$.

But $f(x) = \tan x$ is not bounded on $\mathbb{R}$.

Cluster point of a set or Limit point of a set:

Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if for every $\delta > 0$ there exists atleast one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

i.e, $0 < |x - c| < \delta$.

(or)

Let $A \subseteq \mathbb{R}$, A point $c \in \mathbb{R}$ is a cluster point of A if every δ-nbd of c contains atleast one point of A other than c.

i.e, $\delta > 0$, $(c - \delta, c + \delta)$ contains atleast one point of the set A other than c.

(or)

A point $c \in \mathbb{R}$ is a cluster point of A if every

nbd of c contains infinitely many points of A.
i.e, $\delta>0$, $(c-\delta, c+\delta)$ contains infinitely many points of A.

Ex: (1) For the open interval $A_1 = (0,1)$, every point of the closed interval $[0,1]$ is a cluster point of A_1.
— The points 0 & 1 are cluster points of A_1 but donot belong to A_1.
— All the points of A_1 are cluster points of A_1.

(2) A finite set has no cluster points.

(3) The infinite set N has no cluster points.

(4) The set $A_4 = \{\frac{1}{n} / n \in N\}$ has only the point '0' as a cluster point.
None of the points in A_4 is a cluster point of A_4.

Note: A cluster point of the set A may (or) may not belong to the set A.

(A real no l is said to be $\lim$ of a fn f at at pt a if given $\varepsilon>0$ (however small), $\exists$ a $\delta>0$)

Limit of a function:

Let $A \subseteq R$ and let c be a cluster point of A. For a function $f: A \to R$, [a real number L is said to be a limit of f at c, if given any $\varepsilon>0$, there exists a $\delta>0$ (depending on ε, i.e, $\delta(\varepsilon)$) such that if $x \in A$ and $0<|x-c|<\delta$ then

$|f(x)-L|<\varepsilon$.

i.e, $|f(x)-L|<\varepsilon$ whenever $0<|x-c|<\delta$

i.e, $f(x) \in (L-\varepsilon, L+\varepsilon)$ $\forall x \in (c-\delta, c+\delta)$; $x \neq c$

'ef Notes
& brief Ans

Note: (1) If L is a limit of f at c then we ... L ... to L at c

we write $\underset{x \to c}{Lt} f(x) = L$ (or) $\underset{x \to c}{Lt} f = L$

we also say that $f(x)$ approaches L as x approaches c.

i.e, $f(x) \to L$ as $x \to c$.

(2) If the limit of 'f' at 'c' doesnot exist, we say that f diverges at c.

(3) If 'c' is not a cluster point of A then the limit of a function 'f' doesnot discuss at 'c'.

(4) The function f may (or) may not be defined at the limit points.

Ex: If $A = (0,1)$ and if $f: A \to \mathbb{R}$ then 1 is a cluster point of A

but f is not defined at 1.

similarly at '0'

(5) In order to prove that $\underset{x \to c}{Lt} f(x) \neq L$, we have to show that for any $\epsilon > 0$, and any $\delta > 0$ there is $x \in A$, $0 < |x - c| < \delta \Rightarrow |f(x) - L| \geq \epsilon$.

(6) If $f: A \to \mathbb{R}$ and if c is a cluster point of A then f can have only one limit at 'c'.

Sequential Criterion:

Let $f: A \to \mathbb{R}$ and let 'c' be a cluster point of A ~~then~~ the following are equivalent.

(i) ~~$\underset{x \to c}{Lt} f(x) = L$~~

(ii) for every (x_n) in A cgs to c such that

$x_n \neq c \ \forall n \in N$, the sequence $(f(x_n))$ cgs to L.

→ Use the $\epsilon-\delta$ definition of limit, to show that

$\underset{x \to c}{Lt} \frac{1}{x} = \frac{1}{c}$, $c > 0$.

Sol: Let $f(x) = \frac{1}{x}$; $x > 0$

and let $c > 0$.

To show that $\underset{x \to c}{Lt} f(x) = \frac{1}{c}$.

For this we are enough to show that for any $\epsilon > 0$, $\exists$ a $\delta > 0$ (depends on ϵ) such that

$\left|f(x) - \frac{1}{c}\right| < \epsilon$ whenever $0 < |x-c| < \delta$.

Now we have $\left|f(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right|$

$$= \left|\frac{c-x}{xc}\right|$$

$$= \frac{|x-c|}{|cx|} \quad \text{——} (1)$$

for $x \to c$,

By taking x sufficiently close to c

we have $0 < |x-c| < \frac{1}{2}c$. $(\because \delta \leq \frac{1}{2}c)$

$\Rightarrow |x-c| > 0$ and $|x-c| < \frac{1}{2}c$.

$x \neq c$ and $-\frac{1}{2}c < x-c < \frac{1}{2}c$.

$x \neq c$ and $\frac{c}{2} < x < \frac{3}{2}c$.

$x \neq c$ and $\frac{c}{2} < x$.

$x \neq c$ and $xc > \frac{c^2}{2}$

and $|xc| > \frac{c^2}{2}$

and $\frac{1}{|xc|} < \frac{2}{c^2}$

$\therefore$ (1) $\equiv$ $\left|f(x) - \frac{1}{c}\right| < \frac{2}{c^2}|x-c|$

$< \epsilon$ whenever $|x-c| < \frac{c^2}{2}\epsilon$.

Choosing $\delta = \min\{\frac{1}{2}c, \frac{c^2}{2}\epsilon\}$

$\therefore |f(x) - \frac{1}{c}| < \epsilon$ whenever $0 < |x-c| < \delta$.

$\therefore f(x) \to \frac{1}{c}$ as $x \to c$.

i.e., $\lim_{x \to c} f(x) = \frac{1}{c}$; $c > 0$.

→ Use either the ϵ-δ definition of limit (or) the sequential criterion for limits to establish the following limits.

(1) $\lim_{x \to 2} \frac{1}{1-x} = -1$ (2) $\lim_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$

(3) $\lim_{x \to 0} \frac{x^2}{|x|} = 0$ (4) $\lim_{x \to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

Sol^n:

(i) ϵ-δ method

Let $f(x) = \frac{1}{1-x}$; then we prove that

$\lim_{x \to 2} f(x) = -1$.

for this we are enough to prove that for each $\epsilon > 0$ $\exists$ a $\delta > 0$ such that $|f(x) - (-1)| < \epsilon$ whenever $0 < |x-2| < \delta$.

we have

$$|f(x) - (-1)| = \left| \frac{1}{1-x} - (-1) \right|$$

$$= \left| \frac{1}{1-x} + 1 \right|$$

$$= \left| \frac{2-x}{1-x} \right|$$

$$= \left| \frac{x-2}{x-1} \right|$$

$$\therefore |f(x) - (-1)| = \left| \frac{x-2}{x-1} \right| \quad \text{——— (1)}$$

for $x \to 2$,

By taking x sufficiently close to 2

we have $0 < |x-2| < 1 \quad (\because 0 < \delta \leq 1)$

$\Rightarrow |x-2| > 0$ and $|x-2| < 1$

$\Rightarrow x \neq 2$ and $-1 < x-2 < 1$

$\Rightarrow x \neq 2$ and $2-1 < x < 2+1$

i.e, $1 < x < 3$

Since $x > 1$

$\Rightarrow x-1 > 0 \Rightarrow \frac{1}{x-1} > 0$

$\Rightarrow \left|\frac{1}{x-1}\right| > 0$

$\Rightarrow 0 < \left|\frac{1}{x-1}\right| \leq 1$

$\Rightarrow \frac{1}{|x-1|} \leq 1$

$\therefore$ (1) $\equiv |f(x) - (-1)| \leq 1 \cdot |x-2|$

$< \epsilon$ whenever $|x-2| < \frac{\epsilon}{1}$

choosing $\delta = \min\left\{1, \frac{\epsilon}{1}\right\}$

$\therefore |f(x) - (-1)| < \epsilon$ whenever $0 < |x-2| < \delta$

$\therefore f(x) \to -1$ as $x \to 2$

$\therefore \lim_{x \to 2} f(x) = -1$

Sequential Method:

Let $f(x) = \frac{1}{1-x}$; $c = 2$

Take $x_n = \frac{2n}{n+1} \quad \forall n \in N$

$\therefore \lim_{n \to \infty} x_n = 2$

$\therefore f(x_n) = \frac{1}{1-x_n}$

$= \frac{1}{1 - \frac{2n}{n+1}} = \frac{n+1}{1-n}$

$$\therefore \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\left(\frac{1+n}{1-n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n}+1}{\frac{1}{n}-1}\right)$$
$$= \frac{0+1}{0-1} = -1$$

$\therefore$ The sequence $(f(x_n))$ cgs to $-1 = L$

$$\therefore \lim_{x\to 2} f(x) = L$$
$$\Rightarrow \lim_{x\to 2} f(x) = -1$$

(3) By $\epsilon-\delta$ method:

$$\left|\frac{x^2}{|x|} - 0\right| = \frac{|x^2|}{|x|}$$
$$= \frac{|x|^2}{|x|}$$
$$= |x| < \epsilon \text{ (say)}$$

$\therefore \left|\frac{x^2}{|x|} - 0\right| < \epsilon$ whenever $|x| < \delta = \epsilon$.

$\forall \epsilon > 0$, $\exists\ \delta = \epsilon > 0$ such that

$\left|\frac{x^2}{|x|} - 0\right| < \epsilon$ whenever $|x - 0| < \delta$.

$$\therefore \lim_{x\to 0} \frac{x^2}{|x|} = 0$$

Divergence Criteria:

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

(a) If $L \in \mathbb{R}$ then f doesnot have limit L at c iff there exists a sequence (x_n) in A with $x_n \neq c$ such that the sequence (x_n) cgs to c but the sequence $(f(x_n))$ doesnot converge to L.

(b) The function f doesnot have a limit at 'c' iff there exists a sequence (x_n) in A with $x_n \neq c$ $\forall n \in N$ such that the sequence (x_n) cgs to 'c' but the sequence $(f(x_n))$ doesnot converge in $\mathbb{R}$.

Ex: $\lim_{x \to 0} \frac{1}{x}$ doesnot exist in $\mathbb{R}$.

Sol^n: Let $f(x) = \frac{1}{x}$; $c=0$

Let $x_n = \frac{1}{n}$ $\forall n$

then $\lim_{n \to \infty} x_n = 0 = c$

$\therefore (x_n)$ cgs to '0'.

Now $f(x_n) = \frac{1}{x_n} = \frac{1}{1/n} = n$ $\forall n$

$\therefore \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} n = +\infty$

$\therefore (f(x_n))$ is not cgt in $\mathbb{R}$.

$\therefore \lim_{x \to 0} f(x)$ doesnot exist in $\mathbb{R}$.

→ Show that the following limits do not exist.

(a) $\lim_{x \to 0} \frac{1}{x^2}$ $(x>0)$ (b) $\lim_{x \to a} \frac{1}{\sqrt{x}}$ $(x>0)$

(c) $\lim_{x \to 0} sgn(x)$ (d) $\lim_{x \to 0} (x + sgn(x))$ (e) $\lim_{x \to 0} \sin(\frac{1}{x})$

(f) $\lim_{x \to 0} \sin(\frac{1}{x^2})$

Sol^n: (c) $\lim_{x \to 0} sgn(x)$

Let $f(x) = sgn(x) = \begin{cases} +1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$

Now $sgn(x) = \frac{|x|}{x}$ if $x \neq 0$.

Now we have to show that $sgn(x)$ doesnot have a limit at $x=0$.

Let $x_n = \frac{(-1)^n}{n}$ $\forall n$ then $\underset{n\to\infty}{Lt}\, x_n = 0$

$\therefore (x_n)$ cgs to '0'.

Now $sgn(x_n) = \frac{(-1)^n/n}{|(-1)^n/n|} = (-1)^n$ $\forall n$

$\therefore \underset{n\to\infty}{Lt}\, sgn(x_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

$\therefore$ $sgn(x_n)$ doesnot convergent

$\therefore$ $\underset{x\to 0}{Lt}\, sgn(x)$ doesnot exist

(e) $\underset{x\to 0}{Lt}\, \sin\left(\frac{1}{x}\right)$

Let $f(x) = \sin(1/x)$; $c=0$

By introducing two sequences (x_n) & (y_n).

Let $x_n = \frac{1}{n\pi}$ $\forall n$

then $\underset{n\to\infty}{Lt}\, x_n = 0$

Now $f(x_n) = \sin(n\pi)$
$= 0$ $\forall n$

$\therefore \underset{n\to\infty}{Lt}\, f(x_n) = 0$

and let $y_n = \frac{1}{\frac{1}{2}\pi + 2n\pi}$

then $\underset{n\to\infty}{Lt}\, y_n = 0$

Now $f(y_n) = \sin\left(\frac{1}{\frac{1}{\frac{1}{2}\pi+2n\pi}}\right)$

$= \sin\left(\frac{1}{2}\pi + 2n\pi\right)$

$= 1$ $\forall n$

$\therefore \underset{n\to\infty}{Lt}\, f(y_n) = 1$

$\therefore \underset{n\to\infty}{Lt}\, f(x_n) \neq \underset{n\to\infty}{Lt}\, f(y_n)$

$\therefore \underset{n\to\infty}{Lt}\, f(x)$ doesnot exist.

(f) $\underset{x\to 0}{Lt}\, \sin\left(\frac{1}{x^2}\right)$

Let $f(x) = \sin\left(\frac{1}{x^2}\right)$; $c=0$

Let $x_n = \frac{1}{\ }$

Algebra of limits:

~~Let~~ $A \subseteq \mathbb{R}$. Let f & g be two functions on A to $\mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Further let $b \in \mathbb{R}$ if $\lim\limits_{x \to c} f = L$ and $\lim\limits_{x \to c} g = M$

then (i) $\lim\limits_{x \to c} (f \pm g) = L \pm M$

(ii) $\lim\limits_{x \to c} (fg) = LM$

(iii) $\lim\limits_{x \to c} (bf) = bL$

(iv) $\lim\limits_{x \to c} \left(\frac{f}{g}\right) = \frac{L}{M}$ provided $M \neq 0$.

v) $\lim\limits_{x \to c} |f|x = \lim\limits_{x \to c} |f(x)| = |L|$

Theorem Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.

if $a \leq f(x) \leq b \quad \forall x \in A \; ; x \neq c$

and $\lim\limits_{x \to c} f(x)$ exists.

then $a \leq \lim\limits_{x \to c} f(x) \leq b$.

Squeeze theorem:

Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. if $f(x) \leq g(x) \leq h(x) \quad \forall x \in A; \; x \neq c$.

and if $\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} h(x)$

then $\lim\limits_{x \to c} g(x) = L$

Note: if $x \in \mathbb{R}, x \geq 0$ then

we have (i) $-x \leq S(x) \leq x$

(ii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$

(iii) $x - \frac{1}{6}x^3 \leq S(x) \leq x$

(iv) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Here $\sin x = S(x)$ & $\cos x = C(x)$

Sol: Let $-1 \le C(t) \le 1 \ \forall t \in \mathbb{R}$

if $x \ge 0$ then

$$-\int_{t=0}^{x} dt \le \int_0^x C(t)\,dt \le \int_0^x dt$$

$$\Rightarrow \boxed{-x \le S(x) \le x}$$

Integrating, we get

$$-\frac{x^2}{2} \le -\cos x + 1 \le \frac{x^2}{2}$$

$$\Rightarrow \frac{-x^2}{-2} \le \cos x - 1 \le \frac{x^2}{2}$$

$$\Rightarrow 1 - \frac{x^2}{2} \le \cos x \le 1 + \frac{x^2}{2}$$

$$\Rightarrow \boxed{1 - \frac{x^2}{2} \le \cos x \le 1} \quad (\because \text{range of } \cos x \le 1)$$

and so on.

Problems:

→ $\underset{x\to 0}{Lt}\ x^{3/2} = 0$; $(x > 0)$

Sol: Let $g(x) = x^{3/2}$; $x > 0$

We have $x < x^{1/2} \le 1$ for $0 < x \le 1$

$\Rightarrow x^2 < x^{3/2} \le x$ for $0 < x \le 1$

is of the form $f(x) \le g(x) \le h(x)$.

where $f(x) = x^2$, $g(x) = x^{3/2}$; $h(x) = x$.

$\therefore \underset{x\to 0}{Lt}\ f(x) = 0 = \underset{x\to 0}{Lt}\ h(x)$

$\therefore$ By squeeze theorem

$$\underset{x\to 0}{Lt}\ g(x) = 0$$

→ $\underset{x\to 0}{Lt}\ \sin x = 0$

Since $-x \le \sin x \le x \ \forall x \ge 0$.

is of the form $f(x) = -x$; $g(x) = \sin x$; $h(x) = x$.

$\therefore \underset{x\to 0}{Lt}\ f(x) = 0 = \underset{x\to 0}{Lt}\ h(x)$

$\therefore$ By squeeze theorem $\underset{x\to 0}{Lt}\, g(x) = 0$.

$\rightarrow \underset{x\to 0}{Lt} \cos x = 1$

Solⁿ: Since $1-\frac{x^2}{2} \le \cos x \le 1$

$\rightarrow \underset{x\to 0}{Lt}\left(\frac{\cos x - 1}{x}\right) = 0$

Solⁿ: Since $1-\frac{x^2}{2} \le \cos x \le 1 \quad \forall x \ge 0$

$\Rightarrow -\frac{x^2}{2} \le \cos x - 1 \le 0$

$\Rightarrow -\frac{x}{2} \le \frac{\cos x - 1}{x} \le 0.$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = -\frac{x}{2}$; $g(x) = \frac{\cos x - 1}{x}$

and $h(x) = 0$.

$\therefore \underset{x\to 0}{Lt} f(x) = 0 = \underset{x\to 0}{Lt} h(x)$

$\therefore$ By squeeze theorem

$\underset{x\to 0}{Lt}\, g(x) = 0$

$\rightarrow \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1$

Sol: Since $x - \frac{x^3}{6} \le \sin x \le x \quad \forall x > 0$

$\Rightarrow 1-\frac{x^2}{6} \le \frac{\sin x}{x} \le 1 \quad \forall x > 0$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = 1-\frac{x^2}{6}$; $g(x) = \frac{\sin x}{x}$

and $h(x) = 1$

$\therefore \underset{x\to 0}{Lt} f(x) = 1 = \underset{x\to 0}{Lt} h(x)$

$\therefore$ By squeeze theorem

$\underset{x\to 0}{Lt}\, g(x) = 1$

$\rightarrow \underset{x\to 0}{Lt}\left(x \sin \frac{1}{x}\right) = ?$

Let $f(x) = x \sin \frac{1}{x}$

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Since $-1 \le \sin\frac{1}{x} \le 1$; $x \neq 0$

$\Rightarrow -x \le x\sin\frac{1}{x} \le x$; $x \neq 0$

is of the form $f(x) \le g(x) \le h(x)$

where $f(x) = -x$; $g(x) = x\sin\frac{1}{x}$ and $h(x) = x$

$\therefore \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0$

$\therefore$ By squeeze theorem,

$\lim\limits_{x\to 0} g(x) = 0$

HW $\rightarrow$ $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = ?$; $x \neq 0$

$\rightarrow$ $\lim\limits_{x\to 0} \operatorname{sgn}\left(\sin\frac{1}{x}\right) = ?$

Sol^n: Let $f(x) = \operatorname{sgn}\left(\sin\frac{1}{x}\right)$; $x \neq 0$

$$= \frac{\sin\frac{1}{x}}{\left|\sin\frac{1}{x}\right|} ; x \neq 0$$

Since $\lim\limits_{x\to 0} \sin\frac{1}{x}$ doesnot exist.

$\therefore \lim\limits_{x\to 0} \operatorname{sgn}\left(\sin\frac{1}{x}\right)$ doesnot exist.

One-Sided Limits:

$\rightarrow$ Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$

if $c \in \mathbb{R}$ is a cluster point of the set

$A \cap (c, \infty) = \{x \in A / x > c\}$

then we say that $L \in \mathbb{R}$ is a right-hand limit of f at c if given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \epsilon$

i.e, $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$.

— The right-hand limit (RHL) is denoted by

$\lim\limits_{x\to c^+} f(x)$ (or) $\lim\limits_{x\to c^+} f$

(ii) If $c \in \mathbb{R}$ is a cluster point of the set

$A \cap (-\infty, c) = \{x \in A / x < c\}$,

then we say that $L \in \mathbb{R}$ is a left-hand limit of 'f' at 'c'.

~~if given any $\epsilon > 0$, $\exists$ a $\delta > 0$ such that~~

for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \epsilon$

i.e., $|f(x) - L| < \epsilon$ whenever $0 < c - x < \delta$.

– The left-hand limit (LHL) is denoted by

$\underset{x \to c-}{Lt} f(x)$ (or) $\underset{x \to c-}{Lt} f$.

Existence of a limit:

$$\underset{x \to c}{Lt} f(x) = L \iff \underset{x \to c+}{Lt} f(x) = L = \underset{x \to c-}{Lt} f(x)$$

→ Sequential Criteria:

Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$ then the following statements are equivalent.

(i) $\underset{x \to c+}{Lt} f(x) = L$

(ii) for every sequence (x_n) that cgs to 'c' such that $x_n \in A$ and $x_n > c$ $\forall n \in \mathbb{N}$, the sequence $(f(x_n))$ cgs to L.

$\boxed{\text{In this way for left-hand limit.}}$

Examples:

→ $\underset{x \to 0}{Lt}\, Sgn(x) = ?$

Sol: Let $f(x) = Sgn(x)$; $x \neq 0$

$= \frac{x}{|x|}$, $x \neq 0$

$= \{ 1$ if $x > 0$

Now $\lim_{x\to 0+} f(x) = 1$ & $\lim_{x\to 0-} f(x) = -1$

$\therefore \lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$

$\therefore \lim_{x\to 0} f(x)$ doesnot exist.

→ $\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = ?$

Let $f(x) = \sin\left(\frac{1}{x}\right)$; $x \neq 0$ (i.e, $x<0$ or $x>0$)

LHL: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sin\frac{1}{x}$

Now $x\to 0+$ (i.e $x>0$), $\sin\frac{1}{x}$ is finite and oscillates between -1 & 1.

$\therefore$ It doesnot tend to any unique number.

$\therefore \lim_{x\to 0+} \sin\frac{1}{x}$ doesnot exist.

Similarly, $\lim_{x\to 0-} \sin\frac{1}{x}$ doesnot exist.

$\therefore \lim_{x\to 0} \sin\frac{1}{x}$ doesnot exist.

→ $\lim_{x\to 0} x\sin\frac{1}{x} = ?$

Sol^n: Let $f(x) = x\sin\frac{1}{x}$; $x \neq 0$

LHL: $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x\sin\frac{1}{x}$

$= 0 \times$ [finite number between -1 & $+1$]

$= 0$

RHL: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x\sin\frac{1}{x}$

$= 0 \times$ (finite number b/w -1 & $+1$)

$= 0$

$\therefore \lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0$

$\therefore \lim_{x\to 0} f(x) = 0$

Limits at infinity and Infinite limits :

(i) $\underset{x\to\infty}{Lt} f(x) = L$.

A function $f(x)$ is said to tend to 'L' as $x\to\infty$, if given any $\epsilon>0$ (however small) $\exists$ a +ve number ~~K (depends on ϵ)~~ such that $x \geq K \Rightarrow |f(x)-L| < \epsilon$.

(ii) $\underset{x\to-\infty}{Lt} f(x) = L$.

A function $f(x)$ is said to tend to L as $x\to-\infty$, if given any $\epsilon>0$, (however small) $\exists$ a +ve number K (depends on ϵ) such that $x \leq -K \Rightarrow |f(x)-L| < \epsilon$

(iii) $\underset{x\to c}{Lt} f(x) = +\infty$

A function $f(x)$ is said to tend to ∞ as $x\to c$ if given any $k>0$ (however large) $\exists$ a +ve number 'δ' such that $0 < |x-c| < \delta \Rightarrow f(x) > K$.

(iv) $\underset{x\to c}{Lt} f(x) = -\infty$

A function $f(x)$ is said to tend to $-\infty$ as $x\to c$, if given $K>0$ (however large), $\exists$ a $\delta>0$ such that $0<|x-c|<\delta \Rightarrow f(x) < -K$

(v) $\underset{x\to\infty}{Lt} f(x) = +\infty$

A function $f(x)$ is said to tend to $+\infty$ as $x\to\infty$ if any $K>0$ (however large), $\exists$ a number $K'>0$ such that $x \geq K' \Rightarrow f(x) > K$.

(vi) $\underset{x\to\infty}{Lt} f(x) = -\infty$.

A function $f(x)$ is said to tend to $-\infty$ as $x\to\infty$, if given $k>0$ (however large), $\exists$ a number $k'>0$ such that $x \geq k' \Rightarrow f(x) < -k$.

(vii) $\underset{x\to-\infty}{Lt} f(x) = \infty$.

A function $f(x)$ is said to tend to ∞ as $x\to-\infty$ if given $K>0$ (however large), $\exists$ a number $K'>0$ (depends on K) such that $x < -K' \Rightarrow f(x) > K$.

(viii) $\underset{x\to-\infty}{Lt} f(x) = -\infty$.

A function $f(x)$ is said to tend to $-\infty$ as $x\to-\infty$ if given any $K>0$ (however large), $\exists$ $K'>0$ (depends on k) $x < -K' \Rightarrow f(x) < -K$

Continuous functions:

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in A$ be a cluster point of A then we say that 'f' is continuous at 'c' if $\underset{x \to c}{Lt} f(x) = f(c)$

$$(\text{or}) \quad \underset{x \to c-}{Lt} f(x) = \underset{x \to c+}{Lt} f(x) = f(c)$$

(or)

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in A$ be a cluster point of A then we say that f is continuous at 'c' if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depending on ϵ) such that if $x \in A$ satisfying $|x - c| < \delta$

then $|f(x) - f(c)| < \epsilon$

i.e, $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.

i.e, $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$

$\forall x \in (c - \delta, c + \delta)$

Continuous from the left at a point:

A function f is continuous from the left (or left continuous) at the point $x = c$ if $\underset{x \to c-}{Lt} f(x) = f(c)$.

(or)

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, $c \in A$ is a cluster point of $A \cap (-\infty, c]$ then we say that f is left continuous at 'c', if given any $\epsilon > 0$ (however small), $\exists$ a $\delta > 0$ ~~(depends on ϵ) such that~~

$$c - \delta < x \leq c \Rightarrow |f(x) - f(c)| < \epsilon$$

<u>Continuity from the right at a point</u>:

A function f is continuous from the right (or) right continuous at the point $x = c$ if $\underset{x \to c+}{Lt} f(x) = f(c)$

(or)

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, $c \in A$ is a cluster point of $A \cap [c, \infty) = \{x \in A / x \geq c\}$ then we say that f is right continuous at 'c', if given any $\epsilon > 0$ (however small) $\exists$ a $\delta > 0$ (depends on ϵ) such that

$$c \leq x < c + \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

<u>Discontinuity</u>:

if f is not continuous at 'c' then f is said to be discontinuous at c

i.e, $\underset{x \to c}{Lt} f(x) \neq f(c)$

(or)

$A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$; $c \in A$ is a cluster point of A then f is not continuous at 'c', if $\epsilon > 0$, $\exists\ \delta > 0$ s.t x is any point of A satisfying $|x - c| < \delta \Rightarrow |f(x) - f(c)| \geq \epsilon$.

Note:- ① If c is a cluster point of A then the following three conditions must hold for f to be continuous at 'c'.

(i) f should be defined at 'c' (i.e $f(c)$ exists).

(ii) $\underset{x \to c}{Lt} f(x)$ exists and

(iii) $f(c)$ & $\underset{x \to c}{Lt} f(x)$ are equal.

② f is discontinuous at $x = c$ because of any one of the following reason:

(i) f is not defined at 'c'

(ii) $\underset{x \to c}{Lt} f(x)$ does not exist

i.e $\underset{x \to c-}{Lt} f(x) \neq \underset{x \to c+}{Lt} f(x)$

(or)

one of the limit does not exist or

both of the limits do not exist.

and

(iii) $\underset{x \to c}{Lt} f(x)$ & $f(c)$ exist but are not equal. (12)

$\Rightarrow$ Sequential Criterion for continuity:-

A function $f: A \to \mathbb{R}$ is continuous at the point $c \in A$ iff for every sequence (x_n) in A that cgs to 'c', the sequence $(f(x_n))$ cgs to $f(c)$.

$\Rightarrow$ Discontinuity Criterion:

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at 'c' iff there exists a sequence (x_n) in A that cgs to 'c', but the sequence $(f(x_n))$ does not converge to $f(c)$.

$\rightarrow$ Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ if $B \subseteq A$, we say that f is continuous on the set B if f is continuous at every point of B.

* Continuity in an open interval:-

A function f is said to be continuous in an open interval (a,b), if it is continuous at every point of (a,b).

i.e. $\lim_{x\to c} f(x) = f(c)$; $x = c \in (a,b)$

* Continuity in a closed interval:-

A function f is said to be conti in a closed interval $[a,b]$ if it is

(i) right conti at 'a' i.e. $\lim_{x\to a+} f(x) = f(a)$

(ii) left conti at 'b' i.e. $\lim_{x\to b-} f(x) = f(b)$

(iii) conti on (a,b) i.e. $\lim_{x\to c} f(x) = f(c)$; $c \in (a,b)$.

→ A function which is not continuous even at a single point of an interval is said to be discontinuous in that interval.

* Types of Discontinuity:

(I) Removable discontinuity:-

If $\lim_{x\to c} f(x)$ exists but is not equal to ... then f ... said to be removable discontinuity at 'c'

i.e. $\left(\lim_{x\to c+} f(x) = \lim_{x\to c-} f(x)\right) \neq f(c)$

Ex:- 1

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$$

Sol

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{\sin x}{x}$

$= 1.$

at $x=0$,

$f(0) = 2.$

$\therefore \lim_{x\to 0} f(x) \neq f(0).$

Ex:- 2

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x > 2 \\ 4 - x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \end{cases}$$

Sol LHL

$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (4-x)$

$= 4 - 2$

$= 2$

RHL

$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} x^2 - 2$

$= 4 - 2$

$= 2$

at $x = 2$

$f(2) = 1$

$\therefore \left(\lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x)\right) \neq f(2)$

(2) Discontinuity of first kind (or) jump discontinuity (or) Ordinary discontinuity :-

If $\lim_{x \to c-} f(x)$ and $\lim_{x \to c+} f(x)$ both exist but are not equal and $f(c)$ exists, it is equal to the either (or) neither of $\lim_{x \to c-} f(x)$ (or) $\lim_{x \to c+} f(x)$ then f is called discontinuity of first kind.

→ f is said to be discontinuity of first kind from the left at 'c' if $\lim_{x \to c-} f(x)$ exists but it is not equal to $f(c)$.

→ f is said to be discontinuity of the first kind from right at 'c' if $\lim_{x \to c+} f(x)$ exist but is not equal to $f(c)$.

Ex:- $f(x) = \begin{cases} x^2 - 2 & \text{if } x > 2 \\ 3 - x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \end{cases}$

(3) Disconti. of second kind :-

→ If $\lim_{x \to c-} f(x)$ & $\lim_{x \to c+} f(x)$ both donot exist they f is called disconti. of 2^{nd} kind at c.

→ f is said to be a disconti. of the second kind from the left at 'c' if $\lim_{x \to c-} f(x)$ does not exist.

→ f is said to be a disconti of the second kind from the right at 'c' if $\lim_{x \to c+} f(x)$ does not exist.

Ex:- $f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Sol LHL

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \sin(1/x)$

$= l \quad (\because -1 \le l \le 1)$

$\therefore l$ is finite number but it is not fixed because l rotates with -1 to 1.

$\therefore \lim_{x \to 0-} f(x)$ does not exist.

Hly RHL does not exist

(4) Mixed discontinuity:-

If a function f has discontinuity of the second kind on one side of 'c' and other side a discontinuity of first kind, f (or) may be continuous then f is called a mixed discontinuity at 'c'

(or)

If one of the limits $\lim_{x \to c-} f(x)$ & $\lim_{x \to c+} f(x)$ exist but not the other then f is called mixed discontinuous at c.

i.e $\lim_{x \to c-} f(x)$ does not exist and $\lim_{x \to c+} f(x)$ exists and may (or) may not equal to $f(c)$

(or)

$\lim_{x \to c+} f(x)$ does not exist and $\lim_{x \to c-} f(x)$ exists and may (or) may not equal to $f(c)$.

Ex:- $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x>0 \\ 2-x & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases}$

Sol

(5) Infinite disconti:-

If one (or) both limits $\lim_{x \to c-} f(x)$ & $\lim_{x \to c+} f(x)$ ~~are $-\infty$ (or) $+\infty$ then~~ f is called infinite discontinuity at 'c'.

Ex:- $f(x) = \begin{cases} \frac{1}{x-2} & ; x \neq 2 \\ 0 & ; x = 2 \end{cases}$

Sol

Algebra of continuous functions:-

If $f(x)$ & $g(x)$ are continuous functions at $x=c$ ~~then~~ $\lim_{x \to c} f(x) = f(c)$ & $\lim_{x \to c} g(x) = g(c)$.

(i) $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} (f(x) \pm g(x))$
$= \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
$= f(c) \pm g(c)$.
$= (f \pm g)(c)$

(ii) $\lim_{x \to c} (f \cdot g)(x) = \lim_{x \to c} (f(x) \cdot g(x))$
$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
$= f(c) \cdot g(c)$
$= (f g)(c)$

(iii) $\lim_{x \to c} (a f)(x) = \lim_{x \to c} [a \cdot f(x)]$
$= a \lim_{x \to c} f(x)$

(iv) $\lim_{x\to c}\left(\frac{f}{g}\right)(x) = \lim_{x\to c}\frac{f(x)}{g(x)}$

$= \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

$= \dfrac{f(c)}{g(c)}$

$= \left(\frac{f}{g}\right)(c)$

provided $g \neq 0$.

problems

Using $\epsilon-\delta$ definition, prove that

(i) $f(x) = 3x+1$ is continuous at $x=2$.

(ii) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x\neq 2 \\ 4 & \text{if } x=2 \end{cases}$

Sol (i) $f(x) = 3x+1$;

at $x=2$

$f(2) = 3(2)+1$
$= 7.$

Let $\epsilon>0$ be given,

we have

$|f(x)-f(2)| = |3x+1-7|$

$= |3x-6|$

$= 3|x-2| \leq \epsilon$

whenever $|x-2| < \frac{\epsilon}{3}$

If we choose $\delta = \frac{\epsilon}{3}$, then

$|f(x)-f(2)| < \epsilon$ whenever $|x-2|<\delta$

$\therefore f(x)$ is conti. at $x=2$.

(ii) $f(x) = \frac{x^2-4}{x-2}$; $x\neq 2$

at $x=2$; $f(2) = 4$.

Let $\epsilon > 0$ be given,

Now we have

$|f(x)-f(2)| = \left|\frac{x^2-4}{x-2} - 4\right|$

$= \left|\frac{x^2-4-4x+8}{x-2}\right|$

$= \left|\frac{x^2-4x+4}{x-2}\right|$

$= \left|\frac{(x-2)^2}{x-2}\right|$

$= |x-2| < \epsilon$

whenever $|x-2| < \frac{\epsilon}{1}$

choosing $\delta = \frac{\epsilon}{1}$,

$\therefore |f(x)-f(2)| < \epsilon$ whenever $|x-2|<\delta$

$\therefore f(x)$ is conti. at $x=2$

→ The constant function $f(x) = b$ is conti on $\mathbb{R}$

→ $g(x) = x$ is conti. on $\mathbb{R}$.

→ $h(x) = x^2$ is conti on $\mathbb{R}$

→ $\phi(x) = \frac{1}{x}$ is conti on $A = \{x\in\mathbb{R} / x\neq 0\}$

Sol^n Let $x = c \in \mathbb{R}$ $(c\neq 0)$

then $\phi(c) = \frac{1}{c}$

and $\lim_{x\to c}\phi(x) = \frac{1}{c}$.

$\therefore \lim_{x\to c}\phi(x) = \phi(c)$.

$\therefore \phi(x)$ is conti at $x = c \in \mathbb{R}$ $(c\neq 0)$

→ $\phi(c) = \frac{1}{3}$ is not conti at $x = 0$

because

ϕ is not defined at $x=0$ and

$\underset{x \to 0}{Lt}\ \phi$ does not exist

→ The signum function sgn is not conti. at $x=0$

Sol Let $f(x) = sgn(x)$

$$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0. \end{cases}$$

$\underset{x \to 0-}{Lt} f(x) = -1$ & $\underset{x \to 0+}{Lt} f(x) = 1$

$\therefore \underset{x \to 0-}{Lt} f(x) \neq \underset{x \to 0+}{Lt} f(x)$

$\therefore \underset{x \to 0}{Lt} f(x)$ does not exist.

$\therefore f(x)$ is not cont at $x=0$.

8.→ Let $A = \mathbb{R}$ and

let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

which is known as

Dirichlet's function.

→ S.T that the Dirichlet's function is not continuous any point of $\mathbb{R}$.

Sol Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Let $x = c \in \mathbb{R}$ then

c is either rational or irrational number.

If c is a rational number:

$\therefore f(c) = 1.$

Let (x_n) be sequence of irrational numbers that converges to 'c'

Since $f(x_n) = 0\ \forall n$

($\because x_n$ is irrational)

$\therefore \underset{n \to \infty}{Lt} f(x_n) = 0 \neq f(c)$

$\therefore (f(x_n))$ does not converge to $f(c)$.

$\therefore f(x)$ is not continuous at the rational number 'c'.

If c is an irrational number:

$f(c) = 0$

Let (x_n) be a sequence of rational numbers that converges to 'c'

Since $f(x_n) = 1\ \forall n$.

$\therefore \underset{n \to \infty}{Lt} f(x_n) = 1 \neq f(c)$

$\therefore (f(x_n))$ does not converge to $f(c)$.

$\therefore f(x)$ is not conti. at the irrational number 'c'.

2006

→ Prove that the function f defined by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

is nowhere continuous.

(1) → prove that the function f defined by

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is rational} \\ \frac{1}{3} & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere.

→ Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 2x, & \text{for } x \text{ rational} \\ x+3, & \text{for } x \text{ irrational.} \end{cases}$$

Find all points at which g is continuous.

Sol Given that

$$g(x) = \begin{cases} 2x, & \text{for } x \text{ rational} \\ x+3 & \text{for } x \text{ irrational} \end{cases}$$

Let x be any real number. For each $n \in \mathbb{N}$, $\exists$ a rational number a_n and an irrational number b_n such that

$x - 1/n < a_n < x + 1/n$ and

$x - 1/n < b_n < x + 1/n$ then

$\Rightarrow |a_n - x| < \frac{1}{n}$ and $|b_n - x| < \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\Rightarrow \lim_{n\to\infty} a_n = x$ & $\lim_{n\to\infty} b_n = x$.

$\Rightarrow \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$ ——— (1)

If g is continuous at 'x' then we must have

$$\lim_{n\to\infty} g(a_n) = g(x) = \lim_{n\to\infty} g(b_n)$$

But $g(a_n) = 2a_n$ &

$g(b_n) = b_n + 3$.

$\therefore \lim_{n\to\infty} 2a_n = g(x) = \lim_{n\to\infty}(b_n+3)$

$\Rightarrow 2\lim_{n\to\infty} a_n = g(x) = \lim_{n\to\infty} b_n + 3$

$\Rightarrow 2x = g(x) = x+3$ (by (1))

$\Rightarrow 2x = x+3$

$\Rightarrow \boxed{x = 3}$

$\therefore$ 3 is the only possible point of continuous and discontinuous at every other point.

Now we show that g is continuous at $x=3$.

at $x=3$; $g(3) = 6$.

Let $\epsilon > 0$ be given,

for a rational number 'x', we have

$|g(x) - g(3)| = |2x - 6|$
$= 2|x-3|$ → (2)

for an irrational number 'x' we have

$|g(x) - g(3)| = |x+3-6| = |x-3|$ → (3)

from ②,

$|g(x)-g(3)| = 2|x-3| < \epsilon$ whenever $|x-3| < \frac{\epsilon}{2}$

choosing $\delta = \frac{\epsilon}{2}$

$\therefore |g(x)-g(3)| < \epsilon$ whenever $|x-3| < \delta$

from ③,

$|g(x)-g(3)| = 1 \cdot |x-3| < \epsilon$ whenever $|x-3| < \frac{\epsilon}{1}$

choosing $\delta = \frac{\epsilon}{1}$

$\therefore |g(x)-g(3)| < \epsilon$ whenever $|x-3| < \delta$

$\therefore g(x)$ is continuous at $x=3$.

P-3
2001 → Let f be defined on $\mathbb{R}$ by setting $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$

show that f is continuous at $x = \frac{1}{2}$ but discontinuous at every other point.

HW → show that the function f defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at $x=0$.

~~2001 → S.T the function f(x) defined by:~~

~~$f(x) = \begin{cases} x & \text{when } x \text{ is rational} \\ -x & \text{when } x \text{ is irrational} \end{cases}$~~

* Examine the continuity of the following functions at the indicated point.

(i) $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$

(ii) $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x=0$

(iii) $f(x) = \begin{cases} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x=0$.

(iv) $f(x) = \begin{cases} (x-a)\dfrac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$ at $x=a$.

(v) $f(x) = \begin{cases} \dfrac{x\, e^{1/x}}{1+e^{1/x}}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x=0$

Sol

(i) Since $x \to 0^- \Rightarrow \frac{1}{x} \to -\infty$

and $x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty$.

LHL

$$\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= \frac{e^{-\infty}-1}{e^{-\infty}+1} = \frac{0-1}{0+1} = -1.$$

RHL

$$\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= \lim_{x\to 0^+}\left[\frac{1-e^{-1/x}}{1+e^{-1/x}}\right]$$

$$= \frac{1-e^{-\infty}}{1+e^{-\infty}}$$

$$= \frac{1-0}{1+0} = 1.$$

$\therefore \lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist

(iv) Since $x\to c^- \Rightarrow (x-c)\to 0^-$
$\Rightarrow \frac{1}{x-c} \to -\infty$

and $x\to c^+ \Rightarrow (x-c)\to 0^+$
$\Rightarrow \frac{1}{x-c} \to +\infty$.

LHL

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^-} (x-c)\left[\frac{e^{\frac{1}{x-c}}-1}{e^{\frac{1}{x-c}}+1}\right]$$

$$= 0\times\left[\frac{e^{-\infty}-1}{e^{-\infty}+1}\right]$$

$$= 0\left[\frac{0-1}{0+1}\right] = 0\times(-1) = 0$$

RHL

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^+}(x-c)\left[\frac{e^{\frac{1}{x-c}}-1}{e^{\frac{1}{x-c}}+1}\right]$$

$$= \lim_{x\to c^+}(x-c)\left[\frac{1-e^{-\frac{1}{x-c}}}{1+e^{-\frac{1}{x-c}}}\right]$$

$$= 0\times\left[\frac{1-e^{-\infty}}{1+e^{-\infty}}\right]$$

$$= 0\times\left[\frac{1-0}{1+0}\right]$$

$$= 0\times 1 = 0$$

(16)

at $x = c$,

$f(c) = 0$.

$$\therefore \left(\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)\right) = f(c)$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c).$$

$\therefore f(x)$ is conti. at $x=c$.

→ Discuss the continuity of the following functions at $x=0$

(i) $f(x) = \begin{cases} 2^{1/x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(ii) $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(iii) $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$

(iv) $f(x) = \begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(v) $f(x) = \begin{cases} \cos\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$

(vi) $f(x) = \begin{cases} x\cos\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$

(vii) $f(x) = \begin{cases} x^2\cos\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0. \end{cases}$

(viii) $f(x) = \begin{cases} \cos x & \text{if } x\geq 0 \\ -\cos x & \text{if } x<0 \end{cases}$

Sol (i) Since $x \to 0+$
$\Rightarrow \frac{1}{x} \to +\infty$

$x \to 0-$
$\Rightarrow \frac{1}{x} \to -\infty.$

LHL

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} 2^{\frac{1}{x}}$

$= 2^{-\infty}$

$= \frac{1}{2^{\infty}} = \frac{1}{\infty}$
$= 0.$

RHL

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2^{\frac{1}{x}}$

$= 2^{\infty} = \infty.$

$\therefore \lim_{x \to 0+} f(x)$ does not exist

$\therefore f$ is dis conti at $x=0$

(ii) Since $x \to 0- \Rightarrow \frac{1}{x} \to -\infty$
$x \to 0+ \Rightarrow \frac{1}{x} \to +\infty.$

LHL $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \sin \frac{1}{x}$

$= l \quad (\because -1 \le l \le 1)$

Since l is finite number but it is not fixed number because l rotates with -1 to 1

$\therefore \lim_{x \to 0-} f(x)$ does not exist.

Sly RHL does not exist.

$\therefore f$ is disconti at $x=0$.

(iii) Since $x \to 0+ \Rightarrow \frac{1}{x} \to \infty$
$x \to 0- \Rightarrow \frac{1}{x} \to -\infty.$

LHL

$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} x \sin \frac{1}{x}$

$= 0 \times l \quad (\because -1 \le l \le 1)$

$= 0.$

RHL $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x \sin \frac{1}{x}$

$= 0 \times l \quad (\because -1 \le l \le 1).$

$= 0.$

at $x=0$
$f(0) = 0.$

$\therefore \left[\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x)\right] = f(0)$

$\Rightarrow \lim_{x \to 0} f(x) = f(0).$

$\therefore f$ is conti at $x=0.$

(v) Since $x \to 0- \Rightarrow \frac{1}{x} \to -\infty$
$x \to 0+ \Rightarrow \frac{1}{x} \to +\infty.$

LHL $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \cos \frac{1}{x}$

$= l$

Since l is finite number but is not fixed because l rotates with -1 to $+1$.

$\therefore \lim_{x \to 0-} f(x)$ does not exist.

Sly RHL does not exist

$\therefore f$ is not continuous at $x=0.$

→ Discuss the continuity of the following functions at $x=a$.

(i) $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x\neq a \\ 0 & \text{if } x=a. \end{cases}$

(ii) $f(x) = \begin{cases} (x-a)\cos\left(\frac{1}{x-a}\right) & \text{if } x\neq a \\ 0 & \text{if } x=a. \end{cases}$

(iii) $f(x) = \begin{cases} \frac{1}{x-a}\operatorname{cosec}(x-a) & \text{if } x\neq a \\ 0 & \text{if } x=a. \end{cases}$

→ Examine the discontinuity of the following functions at the indicated point.

(i) $f(x) = \begin{cases} \frac{|x|}{x}, & \text{when } x\neq 0 \\ 1 & \text{when } x=0 \end{cases}$ at $x=0$

(ii) $f(x) = |x| + |x-1|$ at $x=0$ and $x=1$.

(iii) $f(x) = \begin{cases} \frac{x-|x|}{x}, & \text{if } x\neq 0 \\ 1 & \text{if } x=0; \end{cases}$

(iv) $f(x) = \begin{cases} \frac{|x-2|}{x-2} & \text{if } x\neq 2 \\ -1 & \text{if } x=2 \end{cases}$; at $x=2$

(v) $f(x) = \begin{cases} \frac{x}{|x|+x^2} & \text{if } x\neq 0 \\ 0 & \text{if } x=0; \end{cases}$ at $x=0$.

(vi) $f(x) = \begin{cases} \frac{|x|}{x^2+2x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0, \end{cases}$ at $x=0$

(vii) $f(x) = \begin{cases} \frac{2|x|+x^2}{x}, & \text{if } x\neq 0, \\ & \text{if } x=0. \end{cases}$ at $x=0$

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Sol. (i)

Since $|x| = x$ if $x\geq 0$

$= -x$ if $x<0$

LHL –

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-}\frac{(-x)}{x}$

$= \lim_{x\to 0^-} (-1)$

$= -1$.

RHL

$\lim_{x\to 0^+} f(x) = +1$.

at $x=0$, $f(0)=1$.

$\therefore \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

$\therefore f(x)$ is not conti at $x=0$

(ii) $f(x) = |x| + |x-1|$.

$x<0$, $0\leq x\leq 1$, $1<x$.

if $x<0$ then $|x| = -x$ &

$|x-1| = -(x-1)$

$= 1-x$

$\therefore f(x) = |x| + |x-1|$

$= -x + 1 - x$

$= 1-2x$

If $0 \le x \le 1$ then $|x| = x$

and $|x-1| = -(x-1)$

$= 1-x$.

$\therefore f(x) = |x| + |x-1|$

$= x + 1 - x$

$= 1$.

If $x > 1$ then $|x| = x$ and

$|x-1| = x-1$.

$\therefore f(x) = x + x - 1$

$= 2x - 1$.

$$\therefore f(x) = \begin{cases} 1-2x & \text{if } x<0 \\ 1 & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } x>1 \end{cases}$$

Continuity at $x=0$:-

at $x=0$, $f(0) = 1$.

LHL $\lim_{x\to 0^-} f(x) = \lim_{x \to 0^-} (1-2x)$

$= 1 - 2(0) = 1$.

RHL $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1)$

$= 1$.

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$.

$\therefore f$ is conti at $x=0$.

Continuity at $x=1$:

At $x=1$, $f(1) = 1$.

LHL $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1) = 1$

RHL $\lim f(x) = \lim (2x-1)$

$= 2(1) - 1$

$= 1$.

$\therefore \left(\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)\right) = f(1)$

$\Rightarrow \lim_{x\to 1} f(x) = f(1)$.

$\therefore f$ is conti. at $x=1$.

(iv)

at $x=2$ $f(2) = -1$.

LHL $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{|x-2|}{x-2}$

$= \lim_{x\to 2^-} \frac{-(x-2)}{x-2}$

$\left(\because x \to 2^- \text{ i.e. } x<2 \text{ i.e. } (x-2)<0\right)$

$= \lim_{x\to 2^-} (-1)$

$= -1$

RHL $\lim_{x\to 2^+} f(x) = +1$.

$\therefore \lim_{x\to 2^-} f(x) \ne \lim_{x\to 2^+} f(x)$.

$\therefore \lim_{x\to 2} f(x)$ does not exist

$\therefore f$ is not conti. at $x=2$

Imp

(IFoS 2008) Let $f: \mathbb{R} \to \mathbb{R}$ be

~~such that~~

$$f(x) = \begin{cases} \cancel{\dfrac{\sin((a+1)x) + \sin x}{x}}, & \cancel{x<0} \\ \cancel{c}, & \cancel{x=0} \\ \cancel{\dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}}, & \cancel{x>0} \end{cases}$$

Determine the values of a, b, c

RHL $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (x) = 0.$

$\therefore$ LHL = RHL = $f(0)$

$\therefore$ f is continuous at '0'.

LHD : $Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0-} \frac{-x-0}{x}$

$= -1$

RHD :- $Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0+} \frac{x-0}{x}$

$= +1$

$\therefore$ LHD $\neq$ RHD

$\therefore$ f is not differentiable at '0'.

Note(2): If f is not continuous at any point, it can not be derivable at that point.

Note(3): $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at 'c'. Then

(a) If $\alpha \in \mathbb{R}$ then the function αf is differentiable at 'c' and $(\alpha f)'(c) = \alpha f'(c)$

(b) The function $f+g$ is differentiable at 'c' and $(f+g)'(c) = f'(c) + g'(c)$

(c) the function fg is differentiable at 'c' and

$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$

(d) If $g(c) \neq 0$ then the function f/g is differentiable at 'c' and $(f/g)'(c) = \frac{f'(c)g(c) - g'(c)f(c)}{(g(c))^2}$

Problems :

→ use the definition to find the derivative of each of the following functions.

(a). $f(x) = x^3 \ \forall \ x \in \mathbb{R}$

(b). $g(x) = \frac{1}{x} \ \forall \ x \in \mathbb{R}, \ x \neq 0.$

(c). $h(x) = \sqrt{x} \ \forall \ x > 0.$

(d). $k(x) = \frac{1}{\sqrt{x}}$ for $x > 0.$

Sol'n (c) let $x = c > 0$

then $h(c) = \sqrt{c}$

Now $h'(c) = \lim_{x\to c} \frac{h(x)-h(c)}{x-c}$

$= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c}$

$= \lim_{x\to c} \left(\frac{\sqrt{x}-\sqrt{c}}{x-c}\right) \times \left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right)$

$= \lim_{x\to c} \frac{\cancel{(x-c)}}{\cancel{(x-c)}(\sqrt{x}+\sqrt{c})}$

$= \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}}$

$= \frac{1}{\sqrt{c}+\sqrt{c}} = \frac{1}{2\sqrt{c}}$ exists.

$\therefore$ f' is defined for all +ve values of $\mathbb{R}$ and $f'(x) = \frac{1}{2\sqrt{x}}$.

→ Show that $f(x) = x^{1/3}$; $x \in \mathbb{R}$ is not differentiable at $x=0$.

Sol'n: at $x=0$; $f(0)=0$

Now $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0} \frac{x^{1/3}-0}{x}$

$= \lim_{x\to 0} \left(\frac{1}{x}\right)^{2/3}$ does not exist.

$\therefore f$ is not differentiable at $x=0$.

→ Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational.} \end{cases}$$

Show that f is differentiable at $x=0$ and find $f'(0)$.

Sol'n: Let $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$= L$ — ①

(i) when x is rational number

then $f'(0) = \lim_{x\to 0} \frac{x^2-0}{x-0}$

$= \lim_{x\to 0} \frac{x^2}{x}$

$= \lim_{x\to 0} x = 0$

When x is irrational number

then $f'(0) = \lim_{x\to 0} \frac{0-0}{x-0}$

$= \lim (0) = 0$.

$\therefore f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0 = L$

Now we have

$\left|\frac{f(x)-f(0)}{x-0} - L\right| = \left|\frac{f(x)-f(0)}{x-0} - 0\right|$

$= \left|\frac{x^2-0}{x} - 0\right|$ ($\because x$ is rational)

$= |x| < \epsilon$ whenever $|x| < \epsilon/1$.

Choosing $\delta = \epsilon/1$

$\therefore \left|\frac{f(x)-f(0)}{x-0} - L\right| < \epsilon$ whenever $0 < |x-0| < \delta$ — ⓐ

Now we have

$\left|\frac{f(x)-f(0)}{x-0} - L\right| = \left|\frac{f(x)-f(0)}{x-0} - 0\right|$

$= \left|\frac{0-0}{x-0} - 0\right|$ ($\because x$ is irrational)

$= 0 < \epsilon$ whenever $0 < |x-0| < \delta$

$\therefore \left|\frac{f(x)-f(0)}{x-0} - L\right| < \epsilon$ whenever $0 < |x-0| < \delta$

$\therefore f$ is differentiable at $x=0$.

and $f'(0) = 0$.

P-I 2006 Find a & b so that $f'(2)$ exists where

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a+bx^2 & \text{if } |x| \le 2. \end{cases}$$

... for which the function is continuous at $x=0$.

Sol^n: at $x=0$

$f(0)=c$

LHL

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\sin(a+1)x+\sin x}{x}$$

$$= \lim_{x\to 0-}\left[\frac{\sin(a+1)x}{x} + \frac{\sin x}{x}\right]$$

$$= \lim_{x\to 0-}\left[(a+1)\frac{\sin(a+1)x}{(a+1)x}\right] + \lim_{x\to 0-}\frac{\sin x}{x}$$

$$= (a+1)\lim_{(a+1)x\to 0^-}\frac{\sin(a+1)x}{(a+1)x} + \lim_{x\to 0^-}\frac{\sin x}{x}$$

$\left(\because x\to 0^- \Rightarrow (a+1)x\to 0^-\right)$

$$= (a+1)(1)+(1) \quad \left(\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\right)$$

$$= a+2$$

RHL

$$\lim_{x\to 0+} f(x) = \lim \frac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0+} \frac{x^{1/2}(1+bx)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0+} \frac{(1+bx)^{1/2}-1}{bx}$$

$$= \lim_{x\to 0+} \frac{(\sqrt{1+bx}-1)}{bx} \times \frac{(\sqrt{1+bx}+1)}{(\sqrt{1+bx}+1)}$$

$$= \lim_{x\to 0+} \frac{1+bx-1}{bx(\sqrt{1+bx}+1)}$$

$$= \lim_{x\to 0+} \frac{1}{(\sqrt{1+bx}+1)}$$

$$= \frac{1}{\sqrt{1+b(0)}+1} \quad (\because b\neq 0)$$

$= \frac{1}{2}$ This is the ...

So that b can have any non-zero real value.

Since f is conti at $x=0$, we have,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$\Rightarrow a+2 = \frac{1}{2} = c$

$\Rightarrow a+2 = \frac{1}{2}$ & $c=\frac{1}{2}$

$\Rightarrow a = -\frac{3}{2}$ & $c=\frac{1}{2}$

$\therefore a = -\frac{3}{2},\ b\neq 0$ & $c=\frac{1}{2}$

→ Discuss the continuity of the function $f(x)=[x]$ at the points $\frac{1}{2}$ & 1, where $[x]$ denotes the greatest integer $\leq x$.

Sol: $f(x)=[x]$.

conti at $x=\frac{1}{2}$:-

at $x=\frac{1}{2}$, $f(\frac{1}{2}) = [\frac{1}{2}] = 0$

LHL

$$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to\frac{1}{2}^-}[x]$$

$$= 0 \quad \left(\because x\to \tfrac{1}{2}^- \Rightarrow x < \tfrac{1}{2} = .5,\ x = .1, .2, .3, .4\right)$$

RHL $\lim f(x) = \lim [x] = 0$

$\therefore \left(\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)\right) = f(\tfrac{1}{2})$

$\therefore f(x)$ is conti at $x = \tfrac{1}{2}$

conti at $x=1$:

~~$f(1) = [1] = 1.$~~

LHL $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x]$

$= 0 \quad \left(\because x\to 1^- \Rightarrow x<1,\ x = \cdot 6, \cdot 7, \cdot 8, \cdot 9 \ldots\right)$

RHL $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [x]$

$= 1 \quad \left(\because x\to 1^+ \Rightarrow x>1,\ x = 1\cdot 1,\ 1\cdot 2 \ldots\right)$

$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore f(x)$ is not conti at $x=1$

→ Discuss the continuity of f at $x=1$, where

$f(x) = [1-x] + [x-1]$.

Sol $f(1) = [1-1] + [1-1]$

$= [0] + [0]$

$= 0.$

LHL $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [1-x] + [x-1]$

$= \lim_{x\to 1^-} [1-x] + \lim_{x\to 1^+} [x-1]$

$= 0 + (-1) = -1$

$\left(\because x\to 1^- \Rightarrow x<1 \Rightarrow x = \cdot 5, \cdot 6, \cdot 7, \cdot 8, \cdot 9\right)$

RHL

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left([1-x] + [x-1]\right)$

$= \lim_{x\to 1^+} [1-x] + \lim_{x\to 1^+} [x-1]$

$= -1 + 0$

$= -1.$

$\therefore \left(\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)\right) \neq f(1)$

$\therefore f$ is not conti at $x=1$.

→ Show that the function f defined by

$$f(x) = \begin{cases} [x-1] + |x-1| & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$$

is discontinuity at $x=1$.

→ Examine the continuity of f at $x=2$, where

$$f(x) = \begin{cases} x-[x], & \text{if } x<2 \\ 1 & \text{if } x=2 \\ 3x-5 & \text{if } x>2 \end{cases}$$

Determine the points of continuity of the following functions:

(i) $f(x) = [x]$, (ii) $f(x) = x[x]$

(iii) $h(x) = x - [x]$ (iv) $k(x) = \left[\frac{1}{x}\right]$, $x \neq 0$

Sol (i) $f(x) = [x]$; $x \in \mathbb{R}$

(a) Let $x = c \in \mathbb{Z}$ (i.e. integral value)

then $f(c) = [c] = c$

LHL

$\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c-} [x]$

putting $x = c - h$ ($\because h > 0$, $h \to 0$)

$\therefore \lim\limits_{x \to c-} f(x) = \lim\limits_{h \to 0} [c-h]$

$= \lim\limits_{h \to 0} (c-1)$ ($\because c-1 < c-h < c$)

$= c-1$

$[2-h] = 1 = 2-1$

RHL

$\lim\limits_{x \to c+} f(x) = \lim\limits_{x \to c+} [x]$

putting $x = c+h$ ($\because h > 0$)

$\therefore \lim\limits_{x \to c+} f(x) = \lim\limits_{h \to 0} [c+h]$

$= \lim\limits_{h \to 0} (c)$ ($\because c < c+h < c+1$)

$= c$

$\therefore \lim\limits_{x \to c-} f(x) \neq \lim\limits_{x \to c+} f(x)$

$\therefore f(x)$ is not continuous at $x = c \in \mathbb{Z}$.

(b) Let $x = c \in \mathbb{R} - \mathbb{Z}$

i.e. x = non-integral value

If n is the greatest integer less than 'c' then $[c] = n$.

where $n < c < n+1$

Now $f(c) = [c] = n$.

LHL

$\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c-} [x]$

$= \lim\limits_{h \to 0} [c-h]$ (put $x = c-h$, $h > 0$)

$= \lim\limits_{h \to 0} (n)$ ($\because n < (c-h) < n+1$)

$= n$

$x \to c-$
$x \to c = 2.5$
$x = 2.1$ Lt $\to 2$
$f(x) = 2 = n$

RHL

$\lim\limits_{x \to c+} f(x) = \lim\limits_{x \to c+} [x]$

$= \lim\limits_{h \to 0} [c+h]$ (put $x = c+h$, $h > 0$)

$= \lim\limits_{h \to 0} (n)$ ($\because n < (c+h) < n+1$)

$= n$.

$\therefore \left(\lim\limits_{x \to c-} f(x) = \lim\limits_{x \to c+} f(x)\right) = f(c)$.

$\therefore f$ is conti. at $x = c \in \mathbb{R} - \mathbb{Z}$

i.e. f is conti at the non-integral values.

(iv) $k(x) = \left[\frac{1}{x}\right]$; $(x \neq 0)$, $x \in \mathbb{R}$

(a) Let $x = c$ be an integer (except 0 & ± 1).

If n is the greatest integer less than $\frac{1}{c}$ then $[\frac{1}{c}] = n$.

where $n < \frac{1}{c} < n+1$

NOW

LHL

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^-} [1/x]$$

$$= \lim_{h \to 0} \left[\frac{1}{c-h}\right] \quad \left(\text{put } x = c-h;\ h>0 \to 0\right)$$

$$= \lim_{h \to 0} (n) \quad \left(\because n < \frac{1}{c-h} < n+1\right)$$

$$= n.$$

RHL

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} \left[\frac{1}{x}\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{c+h}\right] \quad \left(\text{put } x = c+h;\ h>0\right)$$

$$= \lim_{h \to 0} (n) \quad \left(\because n < \frac{1}{c+h} < n+1\right)$$

$$= n.$$

$$\therefore \left(\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)\right) = f(c).$$

$\therefore f(x)$ is conti at $x = c \in Z$.
except 0 & ±1

(b) Let $x = c \in R - Z$.

then (1) $\frac{1}{x} = 1/c$ is an integer if $x = c = \pm 1/2, \pm 1/3 \ldots$

(2) $\frac{1}{x} = 1/c$ is not an integer if $x = c \neq \pm 1/2, \pm 1/3, \ldots$

(1) If $\frac{1}{x} = 1/c$ is an integer then $[\frac{1}{c}] = 1/c$.

LHL

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^-} \left[\frac{1}{x}\right]$$

$$= \lim_{h \to 0} \left[\frac{1}{c-h}\right] \quad \left(\text{put } x = c-h;\ h>0\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{c} - 1\right) \quad \left(\because \frac{1}{c} - 1 < \frac{1}{c-h} < \frac{1}{c}\right)$$

$$= \frac{1}{c} - 1$$

$x \to c^-$
$x \to c = \frac{1}{2}^-$
$\frac{1}{x} \to \frac{1}{c} = 2^-$
$\frac{1}{x} = 1.6, 1.7, 1.8, 1.9$
$[\frac{1}{x}] = 1 = 2-1$
$\frac{1}{c} - 1$

RHL

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} \frac{1}{x}$$

$$= \lim_{h \to 0} \left[\frac{1}{c+h}\right] \quad \left(\text{put } x = c+h;\ h>0\right)$$

$$= \lim_{h \to 0} \left(\frac{1}{c}\right) \quad \left(\frac{1}{c} < \frac{1}{c+h} < \frac{1}{c}+1\right)$$

$$= \frac{1}{c}.$$

$$\therefore \lim_{x \to c^+} f(x) \neq \lim_{x \to c^-} f(x)$$

$\therefore f$ is not conti at $x = \pm 1/2, \pm \frac{1}{3}, \ldots$

(2) If $\frac{1}{x}=\frac{1}{c}$ is not an integer if $x \neq \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

Let n be the greatest integer less than $\frac{1}{c}$. then $\left[\frac{1}{c}\right]=n$. where $n<\frac{1}{c}<n+1$.

LHL $\underset{x\to c-}{Lt} f(x) = \underset{x\to c-}{Lt}\left[\frac{1}{x}\right]$

$= \underset{h\to 0}{Lt}\left[\frac{1}{c-h}\right]$ (Put $x=c-h$, $h>0$)

$= \underset{h\to 0}{Lt}(n)$ $\left(\because n<\frac{1}{c-h}<n+1\right)$

$= n$

RHL $\underset{x\to c+}{Lt} f(x) = \underset{x\to c+}{Lt}\left[\frac{1}{x}\right]$

$= \underset{h\to 0}{Lt}\left[\frac{1}{c+h}\right]$ put $x=c+h$, $h>0$

$= \underset{h\to 0}{Lt}(n)$

$= n$

$\therefore \underset{x\to c-}{Lt} f(x) = \underset{x\to c+}{Lt} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x \neq \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

Exp → Show that the absolute function $f(x)=|x|$ is continuous at every point $c\in\mathbb{R}$

Sol Given that $f(x)=|x|$ $\forall x\in\mathbb{R}$

Let $x=c\in\mathbb{R}$ then $f(c)=|c|$

Now we will show that $\underset{x\to c}{Lt} f(x)=f(c)$

i.e $f(x)\to f(c)$ as $x\to c$.

Let $\epsilon>0$ be given.

Now we have

$|f(x)-f(c)| = \left||x|-|c|\right|$

$\leq |x-c|$

$<\epsilon$ whenever $|x-c|<\frac{\epsilon}{1}$.

Choosing $\delta=\frac{\epsilon}{1}$.

$\therefore |f(x)-f(c)|<\epsilon$ whenever $|x-c|<\delta$.

i.e $f(x)\to f(c)$ as $x\to c$.

$\underset{x\to c}{Lt} f(x) = f(c)$.

$\therefore f(x)$ is continuous at $x=c$

(20)

(1) Exp Let $K>0$ and let $f:\mathbb{R}\to\mathbb{R}$ satisfy the condition $|f(x)-f(y)|\leq K|x-y|$ $\forall x,y\in\mathbb{R}$. Show that f is continuous at every point $c\in\mathbb{R}$.

Sol $K>0$, $f:\mathbb{R}\to\mathbb{R}$ satisfies the condition

$|f(x)-f(y)|\leq K|x-y|$ $\forall x,y\in\mathbb{R}$ —(1)

Now we shall show that $\underset{x\to c}{Lt} f(x)=f(c)$.

i.e $f(x)\to f(c)$ as $x\to c$.

For this we are enough to show that, given any $\epsilon>0$ (however small), $\exists$ a $\delta>0$ (depends on ϵ) s.t $|f(x)-f(c)|<\epsilon$ whenever $|x-c|<\delta$

Now from (1),

we have

$|f(x)-f(y)| \le K|x-y|$

$\forall x, y \in (c-\delta, c+\delta)$

Taking $x=x$, $y=c$, we get

$|f(x)-f(c)| \le K|x-c|$

$< \epsilon$ whenever

$|x-c| < \frac{\epsilon}{K}$

Choosing $\delta = \frac{\epsilon}{K}$

$\therefore |f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta$

i.e $f(x) \to f(c)$ as $x \to c$

i.e $\lim_{x \to c} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x = c \in \mathbb{R}$.

Now $\lim_{x \to 0^-} (g \circ f)(x) = \lim_{x \to 0^-} (2)$

$= 2$

and

$\lim_{x \to 0^+} (g \circ f)(x) = \lim_{x \to 0^+} (2)$

$= 2$

$\therefore \left(\lim_{x \to 0^-} (g \circ f)(x) = \lim_{x \to 0^+} (g \circ f)(x) \right) = 2$

$\Rightarrow \lim_{x \to 0} (g \circ f)(x) = 2$ —— (1)

But at $x = 0$,

$(g \circ f)(0) = 0$.

—— (2)

from (1) & (2)

$\lim_{x \to 0} (g \circ f)(x) \neq (g \circ f)(0)$.

→ Let g be defined on $\mathbb{R}$

by $g(x) = \begin{cases} 0 & \text{for } x = 1 \\ 2 & \text{for } x \neq 1 \end{cases}$

and let $f(x) = x+1$ $\forall x \in \mathbb{R}$.

Show that $\lim_{x \to 0} (g \circ f)(x) \neq (g \circ f)(0)$

Sol $g(x) = \begin{cases} 2 & \text{for } x \neq 1 \\ 0 & \text{for } x = 1. \end{cases}$

and $f(x) = x+1$ $\forall x \in \mathbb{R}$.

Now $f(x) = x+1$ $\forall x \in \mathbb{R}$.

Now $(g \circ f)(x) = g(f(x))$

$= g(x+1)$

$= \begin{cases} 2 & \text{for } x+1 \neq 1 \\ 0 & \text{for } x+1 = 1 \end{cases}$

$= \begin{cases} 2 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$

Def: A function $f: \mathbb{R} \to \mathbb{R}$ is said to be additive if $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$

Prove that if f is continuous at some x_0, then it is continuous at every point of $\mathbb{R}$.

Sol

Continuity at the point x_0:

$\lim_{x \to x_0^-} f(x) = \lim_{h \to 0} f(x_0 - h)$ | putting $x = x_0 - h$, $h > 0$

$= \lim_{h \to 0} [f(x_0) + f(-h)]$ | $\because f(x+y) = f(x) + f(y)$

$= \lim_{h \to 0} f(x_0) + \lim_{h \to 0} f(-h)$

$= f(x_0) + \lim_{h \to 0} f(-h)$

$\lim f(-h)$

Sly $\lim_{x\to x_0^+} f(x) = \lim_{h\to 0} f(x_0+h)$ | putting $x = x_0+h$, $h>0$

$= \lim_{h\to 0}[f(x_0) + f(h)]$ | $\because f(x+y) = f(x)+f(y)$

$= \lim_{h\to 0} f(x_0) + \lim_{h\to 0} f(h)$

$= f(x_0) + f(h)$.

$\therefore \boxed{\lim_{x\to x_0^+} f(x) = f(x_0) + f(h)}$

Since f is continuous at $x=x_0$

$\therefore \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0)$

$\Rightarrow f(x_0) + \lim_{h\to 0} f(-h) = f(x_0) + \lim_{h\to 0} f(h) = f(x_0)$.

$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 0 \longrightarrow (1)$

Let c be any real number

then $\lim_{x\to c^-} f(x) = \lim_{h\to 0} f(c-h)$ | putting $x = c-h$; $h>0$

$= \lim_{h\to 0}(f(c) + f(-h))$ | $\because f(x+y) = f(x)+f(y)$

$= \lim_{h\to 0} f(c) + \lim_{h\to 0} f(-h)$

$= f(c) + 0$ (from (1))

$\therefore \boxed{\lim_{x\to c^-} f(x) = f(c)}$.

Also

$\lim_{x\to c^+} f(x) = \lim_{h\to 0} f(c+h)$ | putting $x = c+h$; $h>0$

$= \lim_{h\to 0}[f(c) + f(h)]$

$= \lim_{h\to 0} f(c) + \lim_{h\to 0} f(h)$

$= f(c) + 0$ (from (1))

$\boxed{\lim_{x\to c^+} f(x) = f(c)}$.

$\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

$\therefore f(x)$ is continuous at $x = c \in \mathbb{R}$

Theorem → If a function f is continuous at c then $|f|$ is also continuous at c.

proof Since f is continuous at $x = c$.

$\therefore$ Given $\epsilon > 0$, $\exists$ a $\delta > 0$ s.t $|f(x) - f(c)| < \epsilon$ whenever $|x-c|<\delta \longrightarrow (1)$

Now we have $\big||f(x)| - |f(c)|\big| \le |f(x) - f(c)| < \epsilon$ whenever $|x-c|<\delta$ (from (1))

$\therefore \big||f(x)| - |f(c)|\big| < \epsilon$ whenever $|x-c|<\delta$.

$\therefore |f|$ is continuous at c.

Note:- The converse of the above theorem need not be true

i.e. If $|f|$ is continuous at c then f need not

For example

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

sol

$|f|(x) = |f(x)| = 1 \quad \forall x \in \mathbb{R}$.

$\lim_{x\to 0} |f|(x) = \lim_{x \to 0} (1)$

$= 1$

at $x = 0$,

$|f|(0) = |f(0)|$

$= 1$

$\therefore \lim_{x\to 0} |f|(x) = |f|(0)$.

$\therefore |f|$ is conti at $x=0$

Now

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} (-1) = -1$ &

$\lim_{x\to 0+} f(x) = \lim_{x \to 0+} (+1) = +1$.

$\therefore \lim_{x\to 0+} f(x) \neq \lim_{x\to 0-} f(x)$

$\therefore f(x)$ is not at $x=0$.

Theorem (1) Let $f, g : \mathbb{R} \to \mathbb{R}$ continuous at 'c' and let $h(x) = \max\{f(x), g(x)\}$ i.e, $\sup\{f(x), g(x)\}$ for $x \in \mathbb{R}$

show that $h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)| \quad \forall x \in \mathbb{R}$.

Use this to show that h is continuous at 'c'.

proof: Since $f : \mathbb{R} \to \mathbb{R}$ & $g : \mathbb{R} \to \mathbb{R}$ are two continuous functions at 'c'

$\therefore \lim_{x\to c} f(x) = f(c)$ &

$\lim_{x \to c} g(x) = g(c)$ —— (1)

$\Rightarrow$ Since

$h(x) = \sup\{f(x), g(x)\}$; $x \in \mathbb{R}$

ie $h(x) = \max\{f(x), g(x)\}$; $x \in \mathbb{R}$

(i) $h(x) = \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) \leq g(x) \end{cases}$ —— (A)

Now since, $\forall x \in \mathbb{R}$,

$\frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$

$= \begin{cases} \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}[f(x) - g(x)], & \text{if } f(x) \geq g(x) \\ \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}[-(f(x) - g(x))] & \text{if } f(x) \leq g(x) \end{cases}$

$= \begin{cases} f(x) & \text{if } f(x) \geq g(x) \\ g(x) & \text{if } f(x) \leq g(x). \end{cases}$ —— (B)

$= h(x)$ (by (A)).

(ii) To show h is conti at 'c'

since f & g are continuous at 'c'

$\therefore f+g$ is also conti at 'c'

$\Rightarrow \frac{1}{2}(f+g)$ is also conti at 'c' —— (C)

also $(f-g)$ is conti at 'c'

$\Rightarrow |f-g|$ is also continuous at 'c'.

$\Rightarrow \frac{1}{2}|f-g|$ is also conti at 'c' —— (D) from (C) & (D)

$\frac{1}{2}(f+g)+\frac{1}{2}|f-g|$ is also continuous at 'c'.

$\therefore h(x) = \frac{1}{2}\left(f(x)+g(x)\right)+\frac{1}{2}|f(x)-g(x)|$ is

continuous at $x=c$

(or)

$$\lim_{x\to c} h(x) = \lim_{x\to c} \frac{1}{2}\left[\left[f(x)+g(x)\right]+\frac{1}{2}|f(x)-g(x)|\right]$$

$$= \frac{1}{2}\left[\lim_{x\to c} f(x)+\lim_{x\to c} g(x)\right]+\frac{1}{2}\left[\left|\lim_{x\to c} f(x)+\lim_{x\to c} g(x)\right|\right]$$

$$= \frac{1}{2}\left[f(c)+g(c)\right]+\frac{1}{2}\left[|f(c)-g(c)|\right]$$

by (1)

$$= h(c)$$

$\therefore \lim_{x\to c} h(x) = h(c)$.

$\therefore h(x)$ is continuous at 'c'.

H.W → Let $f, g: \mathbb{R}\to\mathbb{R}$ be continuous at 'c' and let

$h(x) = \inf\{f(x), g(x)\}$

i.e, $h(x) = \min\{f(x), g(x)\}$ for $x\in\mathbb{R}$

Show that $h(x) = \frac{1}{2}\left(f(x)+g(x)\right)-\frac{1}{2}|f(x)-g(x)|$ ~~$\forall x\in\mathbb{R}$~~

Use this to show that h is continuous at 'c'.

Theorem → A function f is continuous at 'c' iff for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x_1)-f(x_2)|<\epsilon$ whenever $x_1, x_2 \in (c-\delta, c+\delta)$.

proof: (i) Let f be a continuous function at 'c'. then for each $\epsilon>0$, $\exists$ a $\delta>0$ such that

$|f(x)-f(c)| < \frac{\epsilon}{2}$ whenever $|x-c|<\delta$.

$\Rightarrow |f(x)-f(c)| < \epsilon/2$ whenever $-\delta<x-c<\delta$.

$\Rightarrow |f(x)-f(c)| < \epsilon/2$ whenever $c-\delta<x<c+\delta$.

$\Rightarrow |f(x)-f(c)| < \epsilon/2$ whenever $x\in(c-\delta, c+\delta)$

Now for $x_1, x_2 \in (c-\delta, c+\delta)$

$|f(x_1)-f(c)|<\epsilon/2$ & $|f(x_2)-f(c)|<\epsilon/2$ —— (1)

Now we have

$$|f(x_1)-f(x_2)| = |f(x_1)-f(c)+f(c)-f(x_2)|$$
$$\leq |f(x_1)-f(c)|+|f(x_2)-f(c)|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \quad (\text{by (1)})$$

$\therefore |f(x_1)-f(x_2)| < \epsilon$ whenever $x_1, x_2 \in (c-\delta, c+\delta)$

(ii) Conversely suppose that for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x_1)-f(x_2)| < \epsilon$ whenever $x_1, x_2 \in (c-\delta, c+\delta)$.

Taking $x_1 = x$ & $x_2 = c$

we have $|f(x)-f(c)| < \epsilon$ whenever $x \in (c-\delta, c+\delta)$

$\therefore f$ is continuous at $x = c$

Theorem → If a function f is continuous at 'c' then it is bounded in some nbd of 'c'.

proof: Since f is continuous at 'c'.

Given $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x)-f(c)| < \epsilon$ whenever $|x-c| < \delta; x \in D_f$.

$\Rightarrow f(c)-\epsilon < f(x) < f(c)+\epsilon$ whenever $c-\delta < x < c+\delta; x \in D_f$

Let $M = \max\{|f(c)-\epsilon|, |f(c)+\epsilon|\}$

then $-M \leq f(x) \leq M$ whenever $x \in (c-\delta, c+\delta) \cap D_f$

$\Rightarrow |f(x)| \leq M$ whenever $x \in (c-\delta, c+\delta) \cap D_f$

$\therefore f$ is bounded in some nbd of 'c'.

Ex: $f(x) = \sin x$ is continuous for all $x \in \mathbb{R}$ and the range of $\sin x$ is $[-1, 1]$

$\therefore -1 \leq \sin x \leq 1 \quad \forall x \in \mathbb{R}$.

$\inf = -1$ & $\sup = 1$

$\therefore f$ is bdd. (for each nbd of x)

2-2-08 / 12/4

If f is a continuous function of x satisfying the functional equation

$$f(x+y) = f(x) + f(y)$$

show that $f(x) = ax$, where 'a' is a constant. $\forall x \in \mathbb{R}$.

[22.1]

Sol^n: Given that 'f' is continuous and $f(x+y) = f(x) + f(y)$ —— (1)

Taking $x = 0 = y$ in (1)

(1) $\Rightarrow$ $f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow f(0) + 0 = f(0) + f(0)$

$\Rightarrow f(0) = 0$

Taking $y = -x$.

(1) $\Rightarrow$ $f(x+(-x)) = f(x) + f(-x)$

$\Rightarrow f(0) = f(x) + f(-x)$

$\Rightarrow 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x)$

If x be a +ve integer,

we have

$$f(x) = f(1+1+1+\cdots+1)$$
$$= f(1) + f(1) + f(1) + \cdots + f(1)$$
$$= x f(1)$$
$$= ax, \text{ say}$$

where $f(1) = a$.

~~Let~~ Now. let x be a −ve integer.

we write $x = -y$ so that y is +ve integer.

we have

$f(x) = f(-y)$

$= -f(y) \quad [\because f(-y) = -f(y)]$

$= -ay$

$= a(-y)$

$= ax.$

Again let $x = \frac{p}{q}$ be a rational number; q being +ve.

we have

$f(p) = f(\frac{p}{q} \cdot q)$

$= f(\frac{p}{q} + \frac{p}{q} + \cdots \; q \text{ times})$

$= f(p/q) + f(p/q) + \cdots \; q \text{ times}$

$= q f(p/q)$

$\Rightarrow f(p) = q f(p/q)$

$\Rightarrow ap = q f(p/q) \quad (\because f(p) = ap)$

$\Rightarrow f(p/q) = a \cdot \frac{p}{q}$

$\Rightarrow f(x) = ax. \quad (\because \frac{p}{q} = x)$

Now, suppose that x is any real number. Let $\{x_n\}$ be a sequence of rational numbers such that $\lim x_n = x$.

we have, x_n being rational

$f(x_n) = ax_n$ ——— (2)

Let $n \to \infty$.

As f is a continuous function we obtain from (2)

$f(x) = ax \; \forall x.$

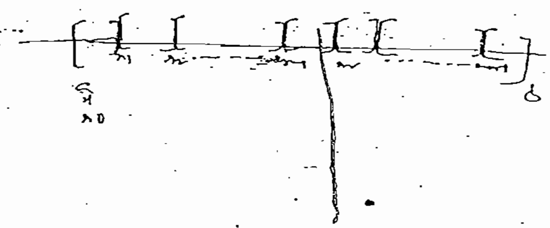
Hence the result

Partition of a closed interval :—

Let $[a,b]$ be a closed interval.

If $a = x_0 < x_1 < x_2 < \dots x_{r-1} < x_r < x_{r+1} < \dots < x_n = b$

where $r = 1, 2, 3, \dots$



— then the finite set

$P = \{x_0, x_1, x_2, \dots x_{r-1}, x_r, \dots x_n\}$ is called a partition of $[a,b]$

— The $(n+1)$ points $x_0, x_1, \dots x_n$ are called partition points of the set 'P'.

— The closed intervals $[x_0, x_1]$, $[x_1, x_2]$, ---- $[x_{r-1}, x_r]$, ---- $[x_{n-1}, x_n]$ are called the n subinterval of the closed interval $[a,b]$.

— The r^{th} subinterval $[x_{r-1}, x_r]$ is denoted by I_r and its length $x_r - x_{r-1}$ is denoted by δ_r.

i.e. $\delta_r = x_r - x_{r-1}$.

⟹ If f is cont on $[a,b]$ then given $\epsilon > 0$ (however small), the closed interval $[a,b]$ can be divided into a finite number of subintervals in each of which the oscillation of f is less than ϵ

i.e. $|f(x_1) - f(x_2)| < \epsilon$ for any two points x_1 & x_2 in the same subinterval.

Theorem → If f is continuous on $[a,b]$ then f is bdd on that interval.

proof Since f is continuous on $[a,b]$

∴ Given $\epsilon > 0$ (however small), $[a,b]$ can be divided into finite number of subintervals in each of which the oscillation of f is less than ϵ

i.e. Let $[a=c_0, c_1]$, $[c_1, c_2]$, ---- $[c_{n-1}, c_n]$

s.t $|f(x_1) - f(x_2)| < \epsilon$ ——— (1)

for any two points x_1, x_2 belonging two the same subinterval.

Let x be any point of the first subinterval $[c_0, c_1]$ then by (1),

$|f(x) - f(c_0)| < \epsilon$

$\therefore |f(x)| = |f(x) - f(c_0) + f(c_0)|$

$\le |f(x) - f(c_0)| + |f(c_0)|$

$< \epsilon + |f(c_0)|.$

In particular $x = c_1$

$|f(c_1)| < \epsilon + |f(c_0)|.$ ——— (2)

Let $x \in [c_1, c_2]$ then by (1)

$|f(x) - f(c_1)| < \epsilon$

$\therefore |f(x)| = |f(x) - f(c_1) + f(c_1)|$

$\le |f(x) - f(c_1)| + |f(c_1)|$

$< \epsilon + |f(c_1)|$

$= 2\epsilon + |f(c_0)|$

$\therefore |f(x)| < 2\epsilon + |f(c_0)|.$

In particular $x = c_2$,

$|f(c_2)| < 2\epsilon + |f(c_0)|.$ ——— (3)

proceeding,

Similarly, we have

$|f(x)| < n\epsilon + |f(c_0)|$

$\forall x \in [c_{n-1}, c_n].$

∴ This inequality is satisfied over the whole interval $[a,b]$ ——— (4)

∴ f is bdd on $[a,b]$.

Note :- The converse of the above theorem need not be true

i.e If f is bdd on $[a,b]$ the f need not continuous on $[a,b]$

Ex :-

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \forall x \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right].$$

Sol $f(x) = \sin \frac{1}{x}$

$\Rightarrow f(-2/\pi) = \sin\left(\frac{-\pi}{2}\right)$

$= -1$

$\& \ f\left(\frac{2}{\pi}\right) = \sin(\pi/2)$

$= 1.$

$\therefore -1 \le f(x) \le 1 \quad \forall x \in \left[\frac{-2}{\pi}, \frac{2}{\pi}\right];$

∴ f is bdd on $\left[\frac{-2}{\pi}, \frac{2}{\pi}\right]$.

but is not continuous at $x=0$.

Because:

at $x=0$; $f(0)=0$.

Now since $-1 \le \sin \frac{1}{x} \le 1$ $\forall x \in [-\frac{1}{\pi}, \frac{1}{\pi}]$, $x \ne 0$.

is of the form

$g(x) \le f(x) \le h(x)$

where

$g(x) = -1$, $f(x) = \sin \frac{1}{x}$ &

$h(x) = -1$

with $\lim\limits_{x\to 0} g(x) \ne \lim\limits_{x\to 0} h(x)$

∴ By Squeeze theorem

$\lim\limits_{x\to 0} f(x)$ does not exist.

∴ $f(x)$ is not conti. at $x=0$

∴ $f(x)$ is not conti on $[-\frac{1}{\pi}, \frac{1}{\pi}]$

Note:- → If f is conti. on (a,b) then f need not be bdd in that interval.

Ex:- $f(x) = \frac{1}{x}$ $\forall x \in (0,1)$.

Since f is conti. on $(0,1)$ but is not bdd on $(0,1)$.

Because: $\because x > 0$

$\Rightarrow \frac{1}{x} > 0$

$\Rightarrow 0 < \frac{1}{x} < \infty$ $\forall x \in (0,1)$

$\Rightarrow 0 < \frac{1}{x}$ $\forall x \in (0,1)$

$\Rightarrow 0 < f(x)$ $\forall x \in (0,1)$

∴ f is not bdd.

Theorem: → If f is continuous on $[a,b]$ then f attains its bounds.

(or)

If f is conti on $[a,b]$ then f attains its supremum & infimum atleast once in $[a,b]$

proof: Let f be conti. on $[a,b]$ then f is bdd on $[a,b]$.

∴ sup f & inf f on $[a,b]$ exist.

Let M = sup of f (l.u.b) &

m = inf of f on $[a,b]$ (g.l.b)

∴ $f(x) \le M$ & $f(x) \ge m$ $\forall x \in [a,b]$

now we have to show that f attains its sup & inf atleast once in $[a,b]$.

i.e $\exists\, \alpha, \beta \in [a,b]$ such that $f(\alpha) = M$ & $f(\beta) = m$.

now if possible suppose that f doesn't attain M on $[a,b]$.

∴ $f(x) \ne M$ $\forall x \in [a,b]$.

$M - f(x) \ne 0$ $\forall x \in [a,b]$.

Since M is constant it is continuous for all x.

and f is continuous on $[a,b]$.

∴ $M - f(x)$ is continuous on $[a,b]$

$\Rightarrow \frac{1}{M - f(x)}$ is also conti on $[a,b]$ ($\because M - f(x) \ne 0$)

$\Rightarrow \frac{1}{\quad}$ is bdd on $[a,b]$.

$\therefore \exists$ +ve real number k (i.e $k>0$)

s.t $\frac{1}{M-f(x)} \le K \ \forall x \in [a,b]$

$\Rightarrow M-f(x) \ge \frac{1}{K} \ \forall x \in [a,b]$

$\Rightarrow M-\frac{1}{K} \ge f(x) \ \forall x \in [a,b]$

$\Rightarrow f(x) \le M-\frac{1}{K} \ \forall x \in [a,b]$

$< M \ \forall x \in [a,b]$.

$\Rightarrow M-\frac{1}{K}$ is an upper bound of f on $[a,b]$

and this upper bound less than sup of f on $[a,b]$.

$\therefore$ which is contradiction to the hypothesis that M is sup (lub) of f on $[a,b]$

$\therefore \exists \alpha \in [a,b]$ s.t $f(\alpha)=M$

$\therefore f$ attains its sup atleast once on $[a,b]$

sly f attains its inf atleast once on $[a,b]$

Note1:- The above theorem is not true if the interval is not closed

Ex:- $f(x)=x \ \forall x \in [0,1)$

f is conti. on $(0,1]$ and is bdd on $(0,1]$

because $f(0)=0$ $f(1)=1$
$\downarrow$ inf $\downarrow$ sup

$0 \le f(x) < 1$

clearly f attains sup but not attains inf on $(0,1]$.

Sly f on $[0,1)$ attains the infimum but not the sup

Sly f on $(0,1)$ doesnot attain inf & sup.

Note(2) The converse of above theorem need not be true

Ex:- $f(x)=\begin{cases}\sin(\frac{1}{x}); & x\ne 0\\ 0; & x=0\end{cases}$

$\forall x \in [-\frac{2}{\pi}, \frac{2}{\pi})$

Theorem

If f is continuous on $[a,b]$ then f is bdd and attains its bounds atleast once on $[a,b]$

proof: Above two theorems proofs combined.

Sign preservation theorem:

If f is continuous on $[a,b]$ and $a<c<b$ such that $f(c)\neq 0$ then $\exists$ a $\delta>0$ such that $f(x)$ ~~has the same sign~~ as $f(c)$ $\forall x\in(c-\delta, c+\delta)$.

Theorem: If a function f is continuous on $[a,b]$ and $f(a)$ & $f(b)$ are of opposite signs then $\exists$ atleast one point $c\in(a,b)$ such that $f(c)=0$

Intermediate value theorem:

If f is continuous on $[a,b]$ and $f(a)\neq f(b)$ then f assumes every value between $f(a)$ & $f(b)$ atleast once.

Uniform Continuity:-

w.k.T a function f is continuous at a point x_0 of an interval I, if given $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x)-f(x_0)|<\epsilon$ whenever $|x-x_0|<\delta$.

Here δ depends, in general, not only on ϵ but also on the point x_0 at which the continuity of 'f' is considered.

i.e; $\delta=\delta(\epsilon, x_0)$.

For example:

$f(x)=x^2 \quad \forall x\in\mathbb{R}$.

Let $\epsilon=1/4$ & $x_0=0$

then $|f(x)-f(x_0)| = |x^2-0|$

$= |x^2|$

$= |x|^2<\epsilon$

whenever $|x|<\frac{\sqrt{\epsilon}}{1}$.

Since $\epsilon = \frac{1}{4}$

$\therefore |f(x) - f(x_0)| < \frac{1}{4}$ whenever $|x| < \frac{1}{2}$

Taking $\delta = \frac{1}{2}$

$\therefore |f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

$\therefore \delta = \frac{1}{2}$ works at $x_0 = 0$ corresponding to $\epsilon = \frac{1}{4}$

Now let $\epsilon = \frac{1}{4}$ and $x_0 = 1$, then $\delta = \frac{1}{2}$ doesnot work.

<u>because:</u> Let $x = 1.4$ then

$|x - x_0| = |1.4 - 1| = 0.4 < \frac{1}{2}$

But $|f(x) - f(x_0)| = |1.96 - 1|$

$= 0.96 \not< \frac{1}{4} (= \epsilon)$

$\therefore \epsilon > 0$, the same value of δ doesnot work for different points of the interval.

$\therefore$ If a continuous function f is such that given $\epsilon > 0$, we can find a uniform $\delta > 0$ which depends ~~only~~ on ϵ and not on the point x_0 at which the continuity is considered, then we say that f is uniformly continious

<u>Defn:</u> A ~~function~~ function defined on an interval I is said to be uniformly continuous on I, if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depends on ϵ only) such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$ where $x_1, x_2 \in I$.

<u>Note:</u> [1] Uniform Continuity of a function is a global property we talk of

[2]. Continuity on the otherhand is a local property.

[3]. A function f is not uniformly continuous on I if $\exists$ some $\epsilon > 0$ for which no $\delta > 0$ works.
i.e, for any $\delta > 0$, $\exists$ $x_1, x_2 \in I$ such that
$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \epsilon.$$

Problems

→ Every constant function is uniformly continuous on $\mathbb{R}$.

Sol^n: Let $f(x) = c \; (\in \mathbb{R})$ constant function.

Given $\epsilon > 0$,
Now choosing $\delta > 0$ such that
$|x_1 - x_2| < \delta$; $x_1, x_2 \in \mathbb{R}$
$\Rightarrow |f(x_1) - f(x_2)| = |c - c|$
$= 0 < \epsilon$

$\therefore f(x) = c \; (\in \mathbb{R})$ is uniformly continuous on $\mathbb{R}$.

→ The identity function $f(x) = x \;\; \forall x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

Sol^n: Given $f(x) = x \;\; \forall x \in \mathbb{R}$

Let $\epsilon > 0$ be given.
Let $x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$.

Now we have
$|f(x_1) - f(x_2)| = |x_1 - x_2|$
$< \epsilon$ whenever $|x_1 - x_2| < \frac{\epsilon}{1}$

Choosing $\delta = \frac{\epsilon}{1}$
$\therefore |f(x_1) - f(x_2)| < \epsilon$.
$\therefore f(x) = x$ is uniformly continuous on $\mathbb{R}$.

→ S.T $f(x) = x^2$ is uniformly continuous on $[-1,1]$

Sol^n: Let $\epsilon > 0$ be given and let $x_1, x_2 \in [-1,1]$

$\Rightarrow x_1 \in [-1,1]$ & $x_2 \in [-1,1]$.

~~$\Rightarrow -1 \le x_1 \le 1$ & $-1 \le x_2 \le 1$~~

$\Rightarrow |x_1| \le 1$ & $|x_2| \le 1$.

Now we have

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| \\
&= |(x_1 - x_2)(x_1 + x_2)| \\
&= |x_1 - x_2|\,|x_1 + x_2| \\
&\le (|x_1| + |x_2|)(|x_1 - x_2|) \\
&\le (1+1)|x_1 - x_2| \\
&= 2|x_1 - x_2| \\
&< \epsilon \text{ whenever } |x_1 - x_2| < \frac{\epsilon}{2}
\end{aligned}
$$

Choosing $\delta = \epsilon/2$

$\therefore |f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$.

$\therefore f$ is uniformly continuous on $[-1,1]$.

→ S.T $f(x) = \dfrac{x}{x+1}$ is uniformly continuous on $[0,2]$

Sol^n Let $\epsilon > 0$ be given,

Let $x_1, x_2 \in [0,2]$

$\Rightarrow 0 \le x_1 \le 2$ & $0 \le x_2 \le 2$ — (1)

we have

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= \left| \frac{x_1}{x_1+1} - \frac{x_2}{x_2+1} \right| \\
&= \left| \frac{x_1 - x_2}{(x_1+1)(x_2+1)} \right| \\
&= \frac{|x_1 - x_2|}{|x_1+1|\,|x_2+1|} \quad \text{— (2)}
\end{aligned}
$$

① $\equiv$ $1 < x_1+1 \le 3$ & $1 \le x_2+1 \le 3$.

$\Rightarrow |x_1+1| \ge 1$ & $|x_2+1| \ge 1$

$\Rightarrow \frac{1}{|x_1+1|} \le 1$ & $\frac{1}{|x_2+1|} \le 1$

$\therefore$ ~~② $\equiv |f(x_1)-f(x_2)| \le (1)(1)|x_1-x_2|$~~

$< \epsilon$ whenever $|x_1-x_2| < \frac{\epsilon}{1}$

Choosing $\delta = \frac{\epsilon}{1}$.

$\therefore |f(x_1)-f(x_2)| < \epsilon$ whenever $|x_1-x_2| < \delta$

$\therefore f$ is uniformly continuous on $[0,2]$

$\rightarrow$ S.T $f(x) = \frac{2x}{2x-1}$ is uniformly continuous on $[1,\infty)$

Sol^n: Let $\epsilon > 0$, be given.

Let $x_1, x_2 \in [1,\infty)$

then $x_1 \ge 1$ & $x_2 \ge 1$ ——— ①

we have

$$|f(x_1)-f(x_2)| = \left|\frac{2x_1}{2x_1-1} - \frac{2x_2}{2x_2-1}\right|$$

$$= \frac{2|x_1-x_2|}{|2x_1-1||2x_2-1|} \text{ ——— ②}$$

① $\equiv$ $2x_1-1 \ge 1$ & $2x_2-1 \ge 1$

$\Rightarrow |2x_1-1| \ge 1$ & $|2x_2-1| \ge 1$

$\Rightarrow \frac{1}{|2x_1-1|} \le 1$ & $\frac{1}{|2x_2-1|} \le 1$.

$\therefore$ ② $\equiv$

we have

$|f(x_1)-f(x_2)| \le (1)(1)(2)|x_1-x_2|$

$< \epsilon$ whenever $|x_1-x_2| < \epsilon/2$

Choosing $\delta = \epsilon/2$.

$\therefore |f(x_1)-f(x_2)| < \epsilon$ whenever $|x_1-x_2| < \delta$.

$\therefore f$ is uniformly continuous on $[1,\infty)$

Note: Every uniformly continuous is always continuous but not converse.
i.e, every continuous function need not be uniformly continuous.

for example:
$f(x) = x^2$ is continuous on $\mathbb{R}$, but not uniformly continuous on $\mathbb{R}$.

because:
Let $\epsilon > 0$ be given.
Now we shall show that for each $\delta > 0$,
$\exists\, x_1, x_2 \in \mathbb{R}$ such that $|x_1 - x_2| < \delta$
$\Rightarrow |f(x_1) - f(x_2)| \geq \epsilon$.

Taking $x_2 = x_1 + \frac{\delta}{2}$

$$\therefore |x_1 - x_2| = |x_1 - x_1 - \tfrac{\delta}{2}|$$
$$= \frac{\delta}{2} < \delta.$$

$$\text{Now } |f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$$
$$= |x_1 - x_2|\,|x_1 + x_2|$$
$$= \frac{\delta}{2}\,|x_1 + x_1 + \tfrac{\delta}{2}|$$
$$= \left(\frac{\delta}{2}\right)|2x_1 + \tfrac{\delta}{2}|$$
$$= x_1\delta + \frac{\delta^2}{4} \quad (\because x_1 > 0)$$
$$> \epsilon.$$

Since $\frac{\delta^2}{4} > 0$ and $x_1\delta < \epsilon\ \forall\, x_1 \in \mathbb{R}$; $x_1 > 0$
It is impossible, δ depends on ϵ & x_1
$\therefore$ The given function is not uniformly continuous on $\mathbb{R}$.

→ If a function f is continuous on $[a, b]$ then it is uniformly continuous on $[a, b]$.

Ex: $f(x) = x^2 + 5x + 3 \quad \forall x \in [0, 4]$

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Differentiability

Geometrical Meaning of Derivative at a point :

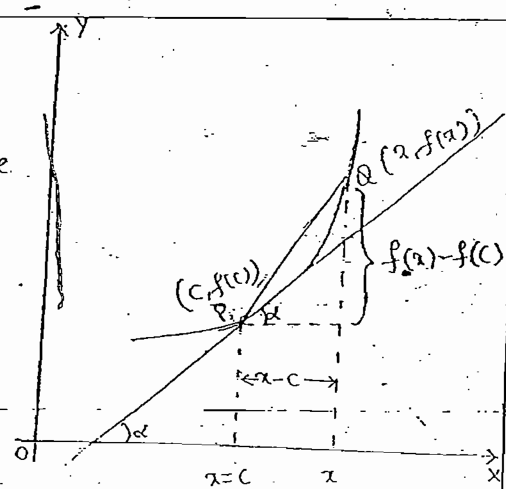
Consider the curve $y=f(x)$ defined in an open interval (a,b).

Let $x = c \in (a,b)$.

Let $y=f(x)$ be differentiable at $x=c$.

Let $P(c, f(c))$ be a point on the curve $y=f(x)$.

and let $Q(x, f(x))$ be a neighbouring point on the curve.



Now the slope of the chord $PQ = \dfrac{f(x)-f(c)}{x-c}$ $\left[\text{i.e. } \dfrac{y_2-y_1}{x_2-x_1}\right]$

Taking limit as $Q \to P$

i.e. $x \to c$, we get

$$\lim_{Q\to P} (\text{slope of the chord } PQ) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{——— (1)}$$

As $Q \to P$, chord PQ becomes tangent at P.

$\therefore$ from (1), we have

slope of the tangent at P.

$$= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \left[\frac{d}{dx}(f(x))\right]_{x=c} \text{(or) } \{f'(x)\}_{x=c}$$

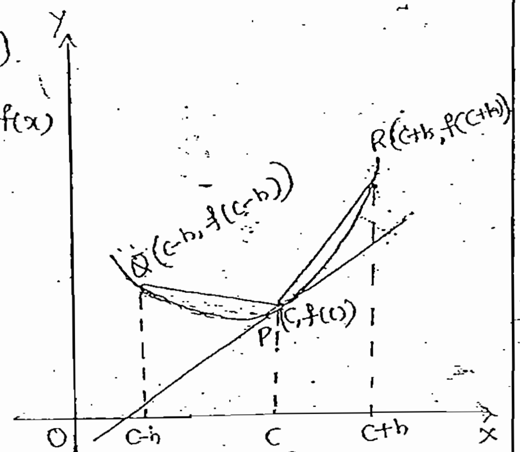
i.e. the derivative of a function at a point $x=c$ is the slope of the tangent to the curve $y=f(x)$ at the point $(c, f(c))$.

→ If a function is not differentiable at $x=c$ only if the point $(c, f(c))$ is a corner point of the curve $y=f(x)$ i.e., the curve suddenly changes its direction at a point $(c, f(c))$.

→ Consider the function $f(x)$ defined on (a,b).

~~Let~~ $P(c, f(c))$ be a point on the curve $y = f(x)$.

Let $Q(c-h, f(c-h))$ & $R(c+h, f(c+h))$ be two neighbouring points on the left hand side (LHS) and RHS respectively of the point P.



Now slope ~~of~~ the chord $PQ = \dfrac{f(c-h) - f(c)}{(c-h) - c}$

$$= \frac{f(c-h) - f(c)}{-h}$$

and slope of chord $PR = \dfrac{f(c+h) - f(c)}{c+h-c}$

$$= \frac{f(c+h) - f(c)}{h}$$

Now taking limit as $Q \to P$ i.e. $h \to 0$

$$\therefore \lim_{Q \to P} (\text{slope of Chord } PQ) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \quad \text{--- (1)}$$

$$\text{Similarly } \lim_{R \to P} (\text{slope of Chord } PR) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{--- (2)}$$

As $Q \to P$ & $R \to P$, the chords PQ & PR become tangent at P.

$\therefore$ from (1) & (2), we have the slope of the tangent at P.

$$\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\therefore f(x)$ is differentiable at $x = c$

$$\Leftrightarrow \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

* **Derivative of a function at a point:**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$, then f is said to be derivable (or differentiable) at c, if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ (or) } \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists. The limit is called the derivative (or) the differential coefficient of the function 'f' at $x=c$ and is denoted by $f'(c)$.

i.e. $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

(or)

Let $f:[a,b]\to\mathbb{R}$ and $c\in(a,b)$. Then we say that a real number L is the derivative of 'f' at 'c' if given any $\epsilon>0$, $\exists\, \delta(\epsilon)>0$ such that if $x\in I$ satisfies $0<|x-c|<\delta$

then $\left|\frac{f(x)-f(c)}{x-c}-L\right|<\epsilon$

In this case, we say that f is differentiable at 'c' and we write $f'(c)$ for L.

→ **Left-hand Derivative:-**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$ if

$$\lim_{x\to c-}\frac{f(x)-f(c)}{x-c} \text{ (or) } \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

exists.

then this limit is called the left-hand derivative of 'f' at 'c' and is denoted by $f'(c-0)$ (or) $f'(c-)$ (or) $Lf'(c)$.

→ **Right-hand Derivative:-**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$.

If $\lim_{x\to c+}\frac{f(x)-f(c)}{x-c}$ (or) $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$

exists. Then this limit is called the right hand derivative of 'f' at 'c' and denoted by $f'(c+)$ (or) $f'(c+0)$ (or) $Rf'(c)$.

Note:- The derivative $f'(c)$ exists $\iff Lf'(c)=Rf'(c)$.

→ **Derivability in an interval:**

* A function $f:[a,b]\to\mathbb{R}$ is said to be derivable in the open interval (a,b) if $f'(c)$ exists for each $c\in(a,b)$.

* A function $f:[a,b]\to\mathbb{R}$ is said to be derivable in $[a,b]$ if

(i) $f'(c)$ exists at $c\in(a,b)$

(ii) $Rf'(a)$ exists

(iii) $Lf'(b)$ exists.

* A function $f: I \to \mathbb{R}$ is said to be derivable on I if f is derivable at every point of I.

Ex :- ① $f(x) = x^2 \ \forall x \in \mathbb{R}$

let $x = c \in \mathbb{R}$

then $f(c) = c^2$

Now $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

$= \lim_{x \to c} \frac{x^2 - c^2}{x - c}$

$= \lim_{x \to c} (x + c)$

$= 2c$ (exists)

$\therefore f(x)$ is derivable function at $x = c \in \mathbb{R}$

$\therefore f'(x)$ is defined on $\mathbb{R}$ and $f'(x) = 2x \ \forall x \in \mathbb{R}$.

Ex :- ② $f(x) = |x| \ \forall x \in \mathbb{R}$

Sol'n : $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

at $x = 0$, $f(0) = 0$.

LHD $Lf'(0) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0-} \frac{-x - 0}{x}$

$= \lim_{x \to 0} (-1) = -1$

RHD :- $Rf'(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{x \to 0+} \frac{x - 0}{x}$

$= \lim_{x \to 0+} (1) = 1$

$\therefore Lf'(0) \neq Rf'(0)$

Theorem :- If $f: I \to \mathbb{R}$ has a derivative at $c \in I$ then f is continuous at 'c'.

Proof : Since f has a derivative at $c \in I$.

$\therefore f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \ \forall c \in I.$ —— ①

Now for $x \in I$; $x \neq c$,

we have

$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) * (x - c)$

Now applying limit on bothsides at $x \to c$, we get

$\lim_{x \to c} (f(x) - f(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) (x - c)$

$= f'(c) \lim_{x \to c} (x - c)$ (by ①)

$= f'(c) \times (0)$

$= 0$

$\therefore \lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0.$

$\Rightarrow \lim_{x \to c} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x = c$.

Note: (1) The converse of the above theorem need not be true.

Ex:- $f(x) = |x| \ \forall x \in \mathbb{R}$

$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

At $x = 0$, $f(0) = 0$

LHL $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-x)$

$= 0$

Sol'n: Since f is derivable at $|x|=2$

i.e. at $x=2$

$\therefore f$ is continuous at $|x|=2$

Now LHL

$$\lim_{|x|\to 2-} f(x) = \lim_{|x|\to 2-} (a+bx^2)$$

$$= \lim_{|x|\to 2-} (a+b|x|^2)$$

$$= (a+4b)$$

Now RHL

$$\lim_{|x|\to 2+} f(x) = \lim_{|x|\to 2+} \left(\frac{1}{|x|}\right)$$

$$= \frac{1}{2}$$

and at $|x|=2$ i.e. $x=2$

$$f(2) = a+4b$$

Since f is continuous at $|x|=2$

$$\therefore \lim_{|x|\to 2-} f(x) = \lim_{|x|\to 2+} f(x) = f(2)$$

$$\Rightarrow a+4b = \frac{1}{2} = a+4b$$

$$\Rightarrow \boxed{a+4b = \frac{1}{2}} \quad \text{——— (I)}$$

Now LHD:

$$Lf'(2) = \lim_{|x|\to 2-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{|x|\to 2-} \frac{(a+bx^2)-(a+4b)}{x-2}$$

$$= \lim_{|x|\to 2-} \left(\frac{bx^2-4b}{x-2}\right)$$

$$= \lim_{|x|\to 2-} \frac{b(x^2-4)}{x-2}$$

$$= \lim_{x\to 2} b(x+2)$$

$$= b(2+2) = \underline{4b}.$$

Now RHD:

$$Rf'(2) = \lim_{|x|\to 2+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{|x|\to 2+} \left[\frac{\frac{1}{|x|}-(a+4b)}{x-2}\right]$$

$$= \lim_{|x|\to 2+} \left[\frac{\frac{1}{|x|}-\frac{1}{2}}{x-2}\right] \text{(using (I))}$$

$$= \lim_{|x|\to 2+} \left[\frac{2-|x|}{2|x|(x-2)}\right]$$

$$= -\lim_{|x|\to 2+} \left[\frac{|x|-2}{2|x|(x-2)}\right]$$

$$= -\lim_{|x|\to 2+} \left[\frac{\cancel{(x-2)}}{2|x|\cancel{(x-2)}}\right]$$

$$= -\lim_{|x|\to 2+} \left[\frac{1}{2|x|}\right]$$

$$= -\frac{1}{2(2)} = -\frac{1}{4}$$

Since f is derivable at $|x|=2$.

$$\therefore Lf'(2) = Rf'(2)$$

$$\Rightarrow 4b = -\frac{1}{4}$$

$$\Rightarrow \boxed{b = -\frac{1}{16}}$$

$$\text{(I)} \equiv a + 4\left(-\frac{1}{8}\right) = \frac{1}{2}$$

$$\Rightarrow a - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \boxed{a = \frac{1}{4}}$$

$\therefore a = \frac{1}{4}$ & $b = -\frac{1}{16}$.

03. For all real numbers x, $f(x)$ is given as

$$f(x) = \begin{cases} e^x + a\sin x & ; x < 0 \\ b(x-1)^2 + x - 2 & ; x \geq 0 \end{cases}$$

Find the values of a & b for which f is differentiable at $x=0$.

The function f defined by

$$f(x) = \begin{cases} x^2 + 3x + a & \text{if } x \leq 1 \\ bx + 2 & \text{if } x > 1 \end{cases}$$

is given to be derivable for every x. Find the values of a and b at $x=1$.

Sol:- Since f is derivable for every x, f must be derivable at $x=1$ and hence f must be continuous at $x=1$.

For what choice of a & b, if y will the function

$$f(x) = \begin{cases} ax - 6 & \text{if } x > 1 \\ bx^2 & \text{if } x \leq 1 \end{cases}$$

become differentiable at $x = 1$?

(i) Determine if $f(x)$ has derivative at $x=0$ when

$$f(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Examine the function

$$f(x) = \begin{cases} x^2 \cos 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

for the existence of derivative at $x=0$

Discuss the continuity and differentiability of the following functions at $x=a$.

(i) $$f(x) = \begin{cases} (x-a)\cdot \sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x = a \end{cases}$$

(ii) $$f(x) = \begin{cases} (x-a)^2 \sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x = a \end{cases}$$

Sol^n (i): since $x \to a- \Rightarrow (x-a) \to 0-$

$\Rightarrow \frac{1}{x-a} \to -\infty$

$x \to a+ \Rightarrow (x-a) \to 0+$

$\Rightarrow \frac{1}{(x-a)} \to +\infty$

Continuous at $x=a$:-

at $x=a$

$f(a) = 0$

LHL $\lim_{x\to a-} f(x) = \lim_{x\to a-} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= 0 \times l \quad (\because -1 \leq l \leq 1)$

$= 0$

RHL $\lim_{x\to a+} f(x) = \lim_{x\to a+} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= 0 \times l \quad (\because -1 \leq l \leq 1)$

$= 0$

$\therefore$ LHL $=$ RHL $= f(0)$

$\therefore$ f is continuous at $x=a$.

Differentiable at $x=a$:

LHD: $\lim_{x\to a-} \left(\frac{f(x) - f(a)}{x-a}\right)$

$= \lim_{x\to a-} \left(\frac{(x-a)\sin\frac{1}{x-a} - 0}{x-a}\right)$

$= \lim_{x\to a-} \sin\left(\frac{1}{x-a}\right) = l$

$(\because -1 < l < 1)$

$\therefore f$ is continuous and differentiable over each subinterval. The only doubtful points are the breaking points $x=1$, and $x=2$.

At $x=1$;

$$f(1)=1$$

LHL $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1-}(3-2x)$

$= 3-2(1)$

$= 1$

RHL $\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1}(1) = 1$

$\therefore$ LHL = RHL = $f(1)$

$\therefore f$ is continuous at $x=1$.

Now LHD:

$$Lf'(1) = \lim_{x\to 1-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1-} \frac{3-2x-1}{x-1}$$

$$= \lim_{x\to 1-}\left(\frac{-2x+2}{x-1}\right)$$

$$= -2 \lim_{x\to 1-} \frac{(x-1)}{x-1}$$

$$= -2(1)$$

$$= -2$$

RHD: $Rf'(1) = \lim\limits_{x\to 1+} \frac{f(x)-f(1)}{x-1}$

$$= \lim_{x\to 1+} \frac{1-1}{x-1} = 0$$

$\therefore Lf'(1) \neq Rf'(1)$.

$\therefore f$ is not differentiable at $x=1$.

Similarly we can easily show that f is continuous at $x=2$ but not differentiable at $x=2$.

$\therefore f$ is continuous on $[0,3]$ Also differentiable on $[0,3]$ except at $x=1$ and $x=2$.

H.W. Discuss the continuity and differentiability of the function $f(x) = |x-2| + 2|x-3|$ in $[1,4]$.

→ Determine where each of the following functions from $\mathbb{R}\to\mathbb{R}$ is differentiable and find derivative.

(a) $f(x) = |x| + |x+1|$

(b) $g(x) = 2x + |x|$

(c) $h(x) = x|x|$.

Sol'n: (a) $f(x) = |x| + |x+1|$ the value of f depends on

$x<0$, $x>0$;

$x+1>0$, $x+1<0$.

$$\underset{x}{\longleftarrow} \; -1 \; \underset{x}{\longleftrightarrow} \; 0 \; \underset{x}{\longrightarrow}$$

(or) $x+1<0$, $x+1>0$, $x<0$, $x>0$.

i.e. $x<-1$, $x>-1$, $x<0$, $x>0$.

i.e. $x\le -1$, $-1<x<0$; $x>0$.

if $x<-1$; $|x| = -x$ & $|x+1| = -(x+1)$

$\therefore \boxed{f(x) = -2x-1}$

if $-1<x<0$; $|x| = -x$ & $|x+1| = x+1$

$\therefore \boxed{f(x) = 1}$

if $x>0$; $|x| = x$ & $|x+1| = x+1$

$\therefore \boxed{f(x) = 2x+1}$

$$f(x) = \begin{cases} -2x-1 & , x<-1 \\ 1 & , -1<x<0 \\ 2x+1 & , x\geq 0 \end{cases}$$

$$f'(x)\Big/_{x<-1} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-2(x+h)+1-(-2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{h}$$

$$= \lim_{h\to 0} (-2)$$

$$= -2$$

$$f'(x)\Big/_{-1<x<0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{1-1}{h}\right) = 0$$

$$f'(x)\Big/_{x>0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)+1-(2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h}$$

$$= \lim_{h\to 0} (2)$$

$$= 2$$

$$\therefore f'(x) = \begin{cases} -2 & \text{if } x \leq -1 \\ 0 & \text{if } -1 \leq x<0 \\ 2 & \text{if } x>0. \end{cases}$$

(c) $f(x) = x|x|$

The value of f depends on $x<0$ and $x>0$.

if $x<0$ then $|x| = -x$

$\therefore \boxed{f(x) = -x^2}$

if $x>0$ then $|x| = x$

$\therefore f(x) = x^2$.

$$\therefore f(x) = \begin{cases} -x^2 & \text{if } x<0 \\ x^2 & \text{if } x>0 \end{cases}$$

$$f'(x)\Big/_{x<0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h)^2-x^2}{h}$$

$$= -2x$$

and $f'(x)\Big/_{x>0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$$

$$= 2x$$

$$\therefore f'(x) = \begin{cases} -2x & ; x<0 \\ 2x & ; x>0 \end{cases}$$

$$= 2|x|.$$

→ Prove that if $f: \mathbb{R} \to \mathbb{R}$ is an even function (i.e. $f(-x) = f(x)$ $\forall x \in \mathbb{R}$) and has a derivative at every point then the derivative f is an odd function.

(i.e. $f'(-x) = -f'(x)$ $\forall x \in \mathbb{R}$). Also

Examine its continuity and derivability at $x = \pi/2$.

→ Show that the function defined by $f(x) = |x| + |x-1|$ is continuous but not derivable at $x = 0$ and $x = 1$.

Sol'n:-

$\xleftarrow{\;x\;}$ 0 $\xleftarrow{\;x\;}$ 1 $\xrightarrow{\;x\;}$

$x < 0$; $0 \le x \le 1$; $1 < x$

if $x < 0$;

$|x| = -x$

$|x-1| = 1-x$

$\therefore \boxed{f(x) = 1-2x}$

if $0 \le x \le 1$; $|x| = x$,

$|x-1| = 1-x$

$\therefore \boxed{f(x) = 1}$

if $x > 1$;

$|x| = x$

$|x-1| = x-1$

$\therefore \boxed{f(x) = 2x-1}$

$$\therefore f(x) = \begin{cases} 1-2x & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

Continuity at $x = 0$:

Derivability at $x = 0$; not

Continuity at $x = 1$;

Derivability at $x = 1$; not

→ Show that the function $f(x)$ defined by $f(x) = |x-1| + 2|x-2|$ is continuous but not derivable at 1 and 2.

→ Discuss the continuity and differentiability of the function $f(x) = |x-1| + |x-2|$ in the interval $[0, 3]$.

Sol'n:-

0 1 2 3

$0 \le x \le 1$; $1 \le x \le 2$; $2 \le x \le 3$.

if $0 \le x \le 1$

$|x-1| = |1-x|$ &

$|x-2| = 2-x$

$\therefore \boxed{f(x) = 3-2x}$

$\therefore$ if $1 \le x \le 2$;

$|x-1| = x-1$ &

$|x-2| = 2-x$

$\therefore \boxed{f(x) = 1}$

if $2 \le x \le 3$;

$|x-1| = x-1$ &

$|x-2| = x-2$

$\therefore \boxed{f(x) = 2x-3}$

$$\therefore f(x) = \begin{cases} 3-2x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 \le x \le 2 \\ 2x-3 & \text{if } 2 \le x \le 3 \end{cases}$$

Since f is a linear (polynomial) function or constant function over the various subintervals.

Here l is finite but not fixed because it rotates with -1 to $+1$.

$\therefore$ LHD does not exist.

Similarly RHD does not exist.

$\therefore f$ is not differentiable at $x=a$

P-I 2002 → Let $f(x) = \begin{cases} x^p \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Obtain condition p ~~show~~ so that

(i) f is continuous at $x=0$ and

(ii) f is differentiable at $x=0$.

Sol'n: (i) at $x=0$

$$f(0) = 0$$

LHL $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^p(\sin 1/x)$ —— (1)

RHL $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x^p(\sin 1/x)$ —— (2)

f is continuous at $x=0$ if the limits (1) & (2) both must be zero.

This is possible only when $p>0$.

$\therefore$ the required condition for continuity of f at $x=0$ is $p>0$.

(ii) LHD $Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0-} \frac{x^p \sin 1/x - 0}{x}$$

$$= \lim_{x\to 0-} x^{(p-1)} \sin 1/x \quad \text{—— (3)}$$

RHD $Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0+} \frac{x^p \sin 1/x - 0}{x}$$

$$= \lim_{x\to 0+} x^{(p-1)} \sin 1/x \quad \text{—— (4)}$$

f is differentiable at $x=0$ if the limits (3) & (4) both must be zero.

This is possible only when $(p-1)>0$.

$\therefore$ The required condition for differentiability of f at $x=0$ is $p>1$.

H.W. Let $f(x) = \begin{cases} x^m \sin 1/x ; & x \neq 0 \\ 0 ; & x=0 \end{cases}$

What conditions should be imposed on 'm' so that

(i) f may be continuous at $x=0$.

(ii) f may be differentiable at $x=0$

H.W. Show that the following function is continuous at $x=1$, for all values of p.

$$f(x) = \begin{cases} px+1 & \text{if } x \geq 1 \\ x^2+p & \text{if } x<1 \end{cases}$$

Find the left-hand & right-hand derivatives of $f(x)$ at $x=1$.

Hence find the condition for the existence of the derivative at that point.

H.W. Let $f(x) = \begin{cases} x\left[\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right] ; & x \neq 0 \\ 0 ; & x=0 \end{cases}$

Show that f is continuous but not differentiable at $x=0$.

H.W. A function $f(x)$ is defined as follows.

$$f(x) = \begin{cases} 1+\sin x & \text{for } 0<x<\pi/2 \\ 2+(x-\pi/2)^2 & \text{for } x \geq \pi/2 \end{cases}$$

Prove if $g: \mathbb{R} \to \mathbb{R}$ is a differentiable odd function then g' is an even function.

Sol'n :- (I) Since f is even function

$\therefore f(-x) = f(x) \quad \forall x \in \mathbb{R}$

Let $x = c \in \mathbb{R}$ then

$$f(-c) = f(c)$$

Now $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ —— (1)

Now $f'(-c) = \lim_{x \to c} \frac{f(-x) - f(-c)}{-x - (-c)}$

$= \lim_{x \to c} \frac{f(x) - f(c)}{-(x - c)}$ ($\because f$ is even)

$= -\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

$= -f'(c)$ (from (1))

i.e $f'(-x) = -f'(x)$

$\therefore f'$ is an odd function.

(II) Since g is odd function

$\therefore g(-x) = -g(x) \quad \forall x \in \mathbb{R}$.

Let $x = c \in \mathbb{R}$. then $g(-c) = -g(c)$

Now $g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$ —— (2)

Now $g'(-c) = \lim_{x \to c} \frac{g(-x) - g(-c)}{-x - (-c)}$

$= \lim_{x \to c} \frac{-g(x) + g(c)}{-(x - c)}$

$= \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$

$= g'(c)$

$\therefore g'(-c) = g'(c)$

i.e. $g'(-x) = g'(x)$

$\therefore g'$ is an even function.

P-I 2007 → Let $f(x)$ ($x \in (-\pi, \pi)$) be defined by $f(x) = \sin|x|$. Is f is continuous on $(-\pi, \pi)$? If it is continuous, then is it differentiable on $(-\pi, \pi)$?

Sol'n :- $f(x) = \sin|x| = \begin{cases} \sin x & x > 0 \\ \sin(-x) & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$

$= \begin{cases} \sin x & x \geq 0 \\ -\sin x & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$

→ Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that g is differentiable for all $x \in \mathbb{R}$. Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

Sol'n :- $g'(x) = 2x \sin\left(\frac{1}{x^2}\right) + x^2\left(\frac{-2}{x^3}\right)\cos\frac{1}{x^2}$

$= 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right)$ (A)

$\therefore g'(x)$ is well defined for $x \neq 0$.

Now at $x = 0$:

$g(0) = 0$

$$\therefore g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x^2}\right) - 0}{x}$$

$$= \lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) \quad \text{—— (B)}$$

Now we have

$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1 \;\; \forall x \in \mathbb{R};\, x \neq 0.$

$\Rightarrow -x \le x \sin\left(\frac{1}{x^2}\right) \le x \;\; \forall x > 0$. is of the form

$$f(x) \le t(x) \le h(x)$$

Here $f(x) = -x$; $t(x) = x \sin\left(\frac{1}{x^2}\right)$

$h(x) = x$

$\therefore \lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$

$\therefore$ By Squeeze theorem

$$\lim_{x\to 0} x \sin\left(\frac{1}{x^2}\right) = 0.$$

$\therefore g'(0) = 0$

$\therefore g$ is differentiable at $x = 0$.

(A) $\equiv g'(x)$ is not bounded on $[-1, 1]$ as $0 \in [-1, 1]$.

If $\gamma > 0$ is a rational number let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^{\gamma} \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

Determine these values of γ for which $f'(0)$ exists.

Sol'n:— At $x = 0$; $f(0) = 0$

Now $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x\to 0} \frac{x^{\gamma} \sin 1/x}{x}$$

$$= \lim_{x\to 0} x^{\gamma - 1} \sin(1/x)$$

$$= \lim_{x\to 0} x^{\gamma - 1} \lim_{x\to 0} \sin(1/x)$$

$$= 0 \times \lim_{x\to 0} \sin(1/x) \quad (\because \gamma > 0 \Rightarrow (\gamma - 1) > 0)$$

$$= 0.$$

$\therefore f'(0)$ exists for $\gamma > 1$.

Extreme Value (Definition):

→ A real number 'x' is called an interior point of a set A if A is neighbourhood of x.

i.e. $\exists\ \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset A$.

Ex:- (1) Every point of (a,b) is its interior point.

(2). Every point $[a,b]$ is its interior point except a & b.

→ The function $f: I \to \mathbb{R}$ is said to have a relative maximum (or) maximum value (or) maxima at $c \in I$ if $f(c)$ is the greatest value of the function f in a small neighbourhood of $V = V_\delta(c)$ of c.

i.e. for all $x \in (c-\delta, c+\delta)$; $\delta > 0$ such that $f(x) \le f(c)\ \forall\ x \in V \cap I$.

→ The function $f: I \to \mathbb{R}$ is said to have a relative minimum (or) minimum value (or) minima at $c \in I$ if $f(c)$ is the least value of the function in a small neighbourhood $V = V_\delta(c)$ [i.e. $(c-\delta, c+\delta)$] of c.

i.e. for all $x \in (c-\delta, c+\delta)$; $\delta > 0$ such that $f(x) \ge f(c)\ \forall\ x \in V \cap I$.

→ The function $f: I \to \mathbb{R}$ is said to have relative extremum (or) extreme value at $c \in I$, if f has either relative maximum (or) relative minimum at 'c'.

→ **Interior Extremum Theorem:**

Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum at 'c'. If the derivative of 'f' at 'c' exists then $f'(c) = 0$.

Proof :- Since f has a relative extremum at 'c'.

Suppose that f has a relative maximum at 'c'.

$$f(x) \le f(c)\ \forall\ x \in I \cap V_\delta(c). \quad \text{--- (1)}$$

If possible let $f'(c) \ne 0$.

then $f'(c) > 0$ or $f'(c) < 0$.

Case (i) : If $f'(c) > 0$

i.e. $\displaystyle Lt_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$.

$\therefore \dfrac{f(x) - f(c)}{x - c} > 0\ \forall\ x \in I \cap V_\delta(c);\ x \ne c$

Now if $x \in V_\delta(c)$ and $x > c$

then $f(x) - f(c) = \left(\dfrac{f(x) - f(c)}{x - c}\right)(x-c) > 0$

$\Rightarrow f(x) - f(c) > 0$

$\Rightarrow f(x) > f(c)$ --- (2)

But (1) & (2) are contradiction.

$\therefore f'(c) \not> 0$ --- (A)

Case (ii): If $f'(c) < 0$ then

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} < 0$$

$$\therefore \frac{f(x)-f(c)}{x-c} < 0 \ \forall\, x \in I \cap V_\delta(c);\ x \neq c$$

If $x \in V_\delta(c)$ and $x < c$ then

$$f(x) - f(c) = \left[\frac{f(x)-f(c)}{(x-c)}\right] \times (x-c) > 0$$

$\therefore f(x) - f(c) > 0$

$\Longrightarrow f(x) > f(c)$ —— (3)

But (1) & (3) are contradiction.

$\therefore f'(c) \not< 0$. —— (B)

from (A) & (B)

$$f'(c) = 0$$

Note: (1) If f has relative extremum at 'c' then $f'(c)$ may not exist. if it exists then $f'(c) = 0$.

Ex:- $f(x) = |x| \ \forall\, x \in [-1,1]$

Sol'n Let $x = c = 0 \in [-1,1]$

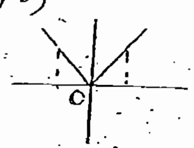
for $x = V_\delta(0)$

$\Longrightarrow x \in (-\delta, \delta)$

(i) $x \in (-\delta, 0)$

$\Rightarrow f(x) > f(0) = 0$

$\therefore f(x)$ has minimum at $x = 0$



(ii) $x \in (0, \delta)$

$\Rightarrow f(x) > f(0) = 0$

$\therefore f(x)$ has minimum at $x=0$

$\therefore f$ has relative extremum at $x=0$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0} \frac{|x|-0}{x-0}$$

~~$= \lim_{x\to 0} \frac{|x|}{x}$~~

Now $\lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-(x)}{x}$

$$= \lim_{x\to 0-} (-1) = -1$$

Now $\lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = \lim_{x\to 0+} (1) = 1$

$\therefore f'(0)$ does not exist.

Note (2): (a) The converse of above theorem need not be true.

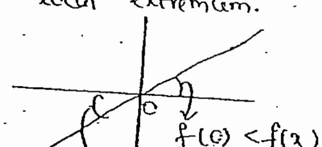
If $f'(c) = 0$ then $f(c)$ may not be an extreme value.

Ex:- $f(x) = x^3 \ \forall\, x \in \mathbb{R}$

$f'(x) = 3x^2$

At $x=0$; $f'(0) = 0$.

But f is strictly increasing in $\mathbb{R}$ and has no local extremum.



Definition:- The point c is said to be stationary point and $f(c)$ the stationary value of the function f if $f'(c) = 0$.

*Rolle's Theorem:- [Only problems]

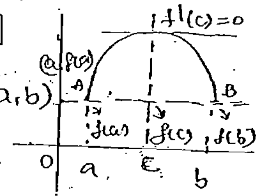
Suppose that f is continuous on $I = [a,b]$ that the derivative f'

exists at every point of (a,b) and $f(a)=f(b)=0$. Then there exists at least one point $c\in(a,b)$ such that $f'(c)=0$.

Proof: Case(i)

If $f(x)=0$ on $I=[a,b]$ then $f'(x)=0\ \forall x\in[a,b]$

$\therefore f'(c)=0\ \forall x=c\in(a,b)$



Case(ii):

If $f(x)\neq 0\ \forall x\in[a,b]$ then $f(x)>0$ or $f(x)<0$.

Suppose that $f(x)>0\ \forall x\in[a,b]$

i.e. f assumes the +ve values in $I=[a,b]$

Since f is continuous on $I=[a,b]$

$\therefore f$ attains its supremum (lub) at least once in $[a,b]$.

i.e. let f attains its supremum at some point $c\in[a,b]$.

i.e. $f(c)=\sup\{f(x)\,/\,x\in I=[a,b]\}>0$ —— (A)

at $x=c\in I$

$\Rightarrow x\in(c-\delta, c+\delta)$

Since f takes some +ve values.

$\therefore f(x)\leq f(c)$
$\forall x\in I\cap(c-\delta, c+\delta)$

$\therefore f$ has relative maximum at c.

from (A)

$f(c)>0$.

Since $f(a)=f(b)=0$

and then $c\neq a$, $c\neq b$

$\Rightarrow c\in(a,b)$

Since f' exists at every point of (a,b).

$\therefore f'(c)$ exists.

$\therefore f$ has relative maximum at c and f has derivative at c.

$\therefore$ By interior extremum theorem $f'(c)=0$ for at least one point $c\in(a,b)$.

Hence the theorem.

* Failure of Rolle's Theorem:-

Rolle's theorem fails to hold good for a function which does not satisfy all the three conditions of the theorem.

The theorem is not applicable if their (i) f is not continuous in $[a,b]$ (or) (ii) f is not derivable in (a,b) (or) (iii) $f(a)\neq f(b)$.

Note: The converse of Rolle's theorem is not true i.e. $f'(x)=0$ at $x=c\in(a,b)$ without $f(x)$ satisfying all the three conditions of Rolle's theorem.

Ex:- $f(x)=\begin{cases}0 & \text{when } 0\leq x\leq 1\\ x+1 & \text{when } 1<x\leq 2\end{cases}$ $\forall x\in[0,2]$.

Clearly f is not continuous and not

derivable at $x=1$.

$\therefore f$ is not continuous in $[0,2]$ and f is not derivable in $(0,2)$.

Also $f(0)\neq f(2)$.

But $f(x)=0 \ \forall x\in(0,1)\subset(0,2)$.

i.e $f'(x)=0$ for at least one point $x\in(0,2)$.

Note(2): Another form of Rolle's theorem

If f is continuous on $[a, a+h]$ derivable on $(a, a+h)$ and ~~$f(a)=f(a+h)=0$~~ then ~~$\exists$ at least~~ one real number $\theta\in(0,1)$ such that $f'(a+\theta h)=0$

Here. $b=a+h$; $h>0$ and $c=a+\theta h$

Since $c\in(a,b)$

$\Rightarrow a<c<b$

$\Rightarrow a<a+\theta h<a+h$

$\Rightarrow 0<\theta h<h$

$\Rightarrow 0<\theta<1 \quad (\because h>0)$

$\Rightarrow \theta\in(0,1)$.

Problems:

Verify Rolle's theorem in the following cases:

(i) $f(x)=(x-a)^m(x-b)^n$ where m & n are +ve integers, in the interval $[a,b]$.

Sol'n :- we have

$$f(x)=(x-a)^m(x-b)^n$$

(i) Since m & n are +ve integers.

$\therefore f(x)$ is polynomial in x (on expansion by binomial theorem).

Since every polynomial function is continuous function of x for all values of x.

$\therefore f(x)$ is continuous function for all values of x.

$\therefore$ It is continuous on $[a,b]$.

(ii) $f'(x)=m(x-a)^{m-1}(x-b)^n+n(x-a)^m(x-b)^{n-1}$

$$=(x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$$

$$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$$

exists in (a,b)

$\therefore f(x)$ is derivable in (a,b).

(iii) $f(a)=f(b)=0$.

$\therefore f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore \exists$ at least one value $x=c\in(a,b)$ such that $f'(c)=0$

$$f'(c)=(c-a)^{m-1}(c-b)^{n-1}[c(m+n)-(mb+na)]$$

$$=0$$

$\Rightarrow c(m+n)-(mb+na)=0 \quad (\because c\neq a,\ c\neq b)$

$\Rightarrow c(m+n)=mb+na$

$\Rightarrow c=\dfrac{mb+na}{m+n}\in(a,b)$

$\therefore$ Rolle's theorem is verified.

H.W. $f(x) = (x-a)^3(x-b)^4 \ \forall x \in [a,b]$

$\rightarrow f(x) = 2+(x-1)^{2/3} \ \forall x \in [0,2]$

Sol'n :- Since $f'(x) = \frac{2}{3}(x-1)^{-1/3}$

$= \frac{2}{3(x-1)^{1/3}}$

which does not exist in $x = 1 \in (0,2)$

$\therefore f'(x)$ does not exist in $(0,2)$

$\therefore f$ is not derivable in $(0,2)$.

$\therefore$ Rolle's theorem is not applicable to $f(x)$ in $[0,2]$.

$\rightarrow f(x) = e^x \sin x \ \forall x \in [0,\pi]$

Sol'n :- (i) Since e^x & $\sin x$ are both Continuous functions for values of x.

$\therefore e^x \sin x$ is also Continuous for all values of x.

$\therefore f(x)$ is continuous in $[0,\pi]$.

(ii) $f'(x) = e^x \cos x + e^x \sin x$ which exists in $(0,\pi)$.

$\therefore f(x)$ is derivable in $(0,\pi)$.

(iii) $f(0) = e^0 \sin(0)$

$= 0$

$f(\pi) = e^\pi \sin(\pi)$

$= 0$

$\therefore f(0) = f(\pi) = 0$

$\therefore$ The Conditions of Rolle's theorem are satisfied.

$\therefore \exists$ atleast one value $c \in (0,\pi)$ such that $f'(c) = 0$

$f'(c) = e^c(\cos c + \sin c) = 0$

$\Rightarrow \cos c + \sin c = 0 \quad (\because e^c \neq 0)$

$\Rightarrow \cos c = -\sin c$

$\Rightarrow 1 = -\tan c$

$\Rightarrow \tan c = -1$

$\Rightarrow \tan c = -\tan(\pi/4)$

$= \tan(\pi - \pi/4)$

$\Rightarrow c = \pi - \pi/4$

$\Rightarrow c = 3\pi/4 \in (0,\pi)$

$\therefore$ Rolle's theorem is verified.

H.W. $f(x) = x(x+3)e^{-x/2} \ \forall x \in [-3,0]$

$\rightarrow f(x) = |x| \ \forall x \in [-1,1]$.

Sol'n :- (i) Since $f(x) = |x|$ is continuous for all values of x.

$\therefore$ It is continuous in $[-1,1]$.

(ii) Since $f(x)$ is not derivable at $x = 0 \in (-1,1)$

$\therefore f$ is not derivable in $(-1,1)$

$\therefore$ The Rolle's is not applicable to $f(x) = |x|$ in $[-1,1]$

$\rightarrow f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right] \ \forall x \in [a,b]$

$0 \notin [a,b]$.

Sol'n : (i) $f(x) = \log(x^2+ab) - \log[x(a+b)]$

$= \log(x^2+ab) - \log x - \log(a+b)$

It is continuous in $[a,b]$ $0 \notin [a,b]$

(ii) $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$= \frac{x^2-ab}{x(x^2+ab)}$ exists in (a,b).

$\therefore f(x)$ is derivable in (a,b).

(iii) $f(a) = \log\left[\dfrac{(a^2+ab)}{a(a+b)}\right]$

$= \log\left(\dfrac{a^2+ab}{a^2+ab}\right)$

$= \log(1) = 0$

$f(b) = \log\left[\dfrac{b^2+ab}{b(a+b)}\right]$

$= \log(1) = 0$

$\therefore f(a) = f(b) = 0$

$\therefore$ The Conditions of Rolle's theorem are satisfied.

$\therefore \exists$ atleast one point $c \in (a,b)$ such that $f'(c) = 0$.

$f'(c) = \dfrac{c^2-ab}{c(\cancel{a^2}+ab)} = 0$

$\Rightarrow c^2 - ab = 0$

$\Rightarrow c^2 = ab$

$\Rightarrow c = \pm\sqrt{ab}$

$\Rightarrow c = +\sqrt{ab} \in (a,b)$

$\therefore$ Rolle's is verified. (neglecting $-\sqrt{ab}$)

H.w.

$\rightarrow f(x) = \log\left(\dfrac{x^2+3}{4x}\right) \ \forall x \in [1,3]$

$\rightarrow f(x) = x^2 - 6x + 8 \ \forall x \in [2,4]$

$\rightarrow f(x) = 8x - x^2 \ \forall x \in [2,6]$

$\rightarrow f(x) = \begin{cases} x^2+1 & \text{for } 0 \le x \le 1 \\ 3-x & \text{for } 1 \le x \le 2 \end{cases}$

Sol^n:- Here $f(x)$ is defined in $[0,2]$

Since $f(x) = x^2+1$ for $0 \le x \le 1$ i.e. $x \in [0,1]$ is a polynomial.

$\therefore$ It is continuous & derivable in $[0,1]$.

Since $f(x) = 3-x$ for $1 \le x \le 2$ is a polynomial.

$\therefore$ It is continuous & derivable in $[1,2]$

Since the domain of function $f(x)$ is $[0,2]$ which is partioned at $x=1$ we are not sure about the continuity and derivability of $f(x)$ at $x=1$

Now LHL $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x^2+1)$

$= 2$

RHL $\lim\limits_{x\to 1^+} f(x) = 2$

at $x=1$

$f(1) = 2$

$\therefore f$ is Continuous at $x=1$.

LHD $Lf'(1) = \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1}$

$= \lim\limits_{x\to 1^-} \dfrac{x^2 \ldots}{x-\ldots}$

$= \lim\limits_{x\to 1^-}\left(\dfrac{x^2-1}{x-1}\right)$

$= \lim_{x\to 1-} (x+1)$

$= 2.$

RHD $\quad Rf'(1) = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1+} \frac{3-x-2}{x-1}$

$= \lim_{x\to 1+} \frac{1-x}{x-1}$

$= -1$

$\therefore$ LHD $\neq$ RHD

$\therefore f$ is not derivable at $x=1$

$\therefore f$ is not derivable in $(0,2)$

$\therefore$ Rolle's theorem is not applicable to $f(x)$ in $[0,2]$.

Imp → Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$

Show that the function $a_0x^n + a_1x^{n-1} + \cdots + a_n$ vanishes atleast once in $(0,1)$.

Sol'n:- Let $f(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + a_2\frac{x^{n-1}}{n-1} + \cdots + \frac{a_{n-1}}{2}x^2 + a_nx \quad \forall x \in [0,1].$

Since $f(x)$ is a polynomial which is continuous & derivable for all x.

$\therefore f$ is continuous in $[0,1]$ & derivable in $(0,1)$.

Also $f(0) = 0$.

and $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a$

$= 0$ (given)

$\therefore$ The conditions of Rolle's theorem is satisfied.

$\therefore \exists$ atleast one point $x \in (0,1)$ such that $f'(x) = 0$.

$\Rightarrow f'(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$

H.W. By considering the function $(x-4)\log x$, show that the equation $x\log x = 4-x$ is satisfied by at least one value of $x \in (1,4)$.

Sol'n:- Let $f(x) = (x-4)\log x$

→ Show that between any two roots of $e^x\cos x = 1$, $\exists$ atleast one root of $e^x\sin x - 1 = 0$.

i.e. $\sin x - e^{-x} = 0$

Sol'n: Let $x=a$ & $x=b$ be two distinct roots of the given equation $e^x\cos x = 1$

$\therefore e^a\cos a = 1$ & $e^b\cos b = 1$

$\Rightarrow \cos a = e^{-a}$ & $\cos b = e^{-b}$ —— (1)

Let $f(x) = -\cos x + e^{-x} \; \forall x \in [a,b]$

(i) Since $\cos x$ & e^{-x} are continuous in $[a,b]$.

$\therefore f(x)$ is continuous in $[a,b]$

ii) $f'(x) = \sin x - e^{-x}$

which exists for all $x \in (a,b)$.

$\therefore f$ is derivable in (a,b).

iii) $f(a) = -\cos a + e^{-a}$

$= 0$ (by ①)

& $f(b) = -\cos b + e^{-b}$

$= 0$ (by ①)

$\therefore f(a) = f(b) = 0$

$\therefore$ the conditions of Rolle's theorem are satisfied.

$\therefore \exists$ at least one point $c \in (a,b)$ such that $f'(c) = 0$.

$\Rightarrow f'(c) = \sin c - e^{-c} = 0$

$\Rightarrow \sin c = e^{-c}$

$\Rightarrow e^{c}\sin c - 1 = 0$

$\Rightarrow x = c \in (a,b)$ is a root of the equation $e^{x}\sin x - 1 = 0$

$\therefore e^{\sin x} - 1$ has at least one root b/w any two roots of the equation.

$e^{x}\cos x = 1$.

H.W. Prove that b/w any two roots of $e^{x}\sin x = 1$, $\exists$ atleast one real root of

$e^{x}\cos x + 1 = 0$.

* Lagrange's Mean Value theorem :—

(First Mean Value theorem of Differential Calculus) —

Statement: Suppose that f is continuous on $I = [a,b]$ and f has a derivative in (a,b). Then there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

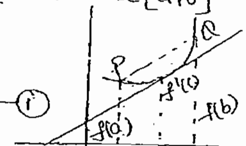
i.e. $f(b)-f(a) = f'(c)(b-a)$

Proof:- Consider the function

$\phi(x) = f(x) - f(a) - k(x-a)\ \forall x \in [a,b]$

where

$$k = \frac{f(b)-f(a)}{b-a} \quad \text{—(1)}$$



Since $f(x)$ is continuous on $I = [a,b]$

since $(x-a)$ is polynomial it continuous on I and $f(a)$ & k are constants.

$\therefore \phi(x)$ is continuous on $[a,b]$.

Now $\phi'(x) = f'(x) - k$ exists in (a,b) —(2)

$[\because f'(x)$ exists in $(a,b)]$

Now $\phi(a) = 0$

and $\phi(b) = f(b) - f(a) - k(b-a)$

$= (f(b)-f(a)) - \left(\frac{f(b)-f(a)}{b-a}\right)(b-a)$

$= 0$

$\because \phi(a) = \phi(b) = 0$

$\therefore \phi(x)$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ at least one $c \in (a,b)$ such that

$\phi'(c) = 0$.

$\phi'(c) = f'(c) - k = 0$

$\Rightarrow f'(c) = k$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Another statement:—

If a function f defined on $[a,b]$ is

(i) Continuous on $[a, a+h]$

(ii) derivable on $(a, a+h)$ then $\exists$ atleast one real number $\theta \in (0,1)$ such that $f(a+h) = f(a) + hf'(a+\theta h)$

Here $b = a+h$

& $c = a+\theta h$

* Deductions from Lagrange's Mean Value theorem :—

→ If a function f is continuous on closed interval $I = [a,b]$ and derivable on (a,b) and $f'(x) = 0\ \forall x \in (a,b)$ then f is constant on $I = [a,b]$

Sol^n:— Let x_1, x_2 (with $x_1 < x_2$) be any two distinct points of $[a,b]$ so that $[x_1, x_2] \subset [a,b]$.

Then f satisfies both conditions of Lagrange's Mean value theorem on $[x_1, x_2]$.

$\therefore \exists\ c \in (x_1, x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \quad \text{--- (1)}$$

But $f'(x) = 0\ \forall x \in (a,b)$ and $x_1 < c < x_2$

$\therefore f'(c) = 0$

from (1), $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = 0$

$$\Rightarrow f(x_2) - f(x_1) = 0$$

$$\Rightarrow f(x_1) = f(x_2)$$

Since x_1 & x_2 are any two distinct points of $[a,b]$.

it follows that f keeps the same value for every $x \in [a,b]$.

$\therefore f(x)$ is constant on $[a,b]$.

→ If two functions f & g are continuous on $[a,b]$, differentiable on (a,b) and $f'(x) = g'(x)\ \forall x \in [a,b]$ then $f-g$ is a constant on $[a,b]$.

Sol'n: Let us consider $\phi(x) = f(x) - g(x)\ \forall x \in [a,b]$

Since f & g continuous on $[a,b]$ and differentiable on (a,b).

$\therefore \phi$ is continuous on $[a,b]$ and differentiable on (a,b).

$\therefore \phi'(x) = f'(x) - g'(x)$ exists on (a,b).

Since $f'(x) = g'(x)\ \forall x \in [a,b]$

$\therefore \phi(x) = 0\ \forall x \in [a,b]$.

Since ϕ is continuous on $[a,b]$, differentiable on (a,b) and

$$\phi'(x) = 0 \quad \forall x \in (a,b)$$

$\therefore \phi$ is a constant function on $[a,b]$.

i.e. $f-g$ is constant on $[a,b]$.

* Increasing and Decreasing Functions :

If in a part of the domain of the function $f(x)$,

→ $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ then $f(x)$ is called monotonically increasing function in that part.

→ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then $f(x)$ is called strictly monotonically increasing function in that part.

→ $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Monotonically decreasing.

→ $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Strictly Monotonically decreasing

Theorem:

Let $f: I \to \mathbb{R}$ be differentiable on I then

(a) f is increasing on I iff $f'(x) \geq 0 \quad \forall x \in I$.

(b) f is decreasing on I iff $f'(x) \leq 0 \quad \forall x \in I$.

Proof:- (a) Suppose that $f'(x) \geq 0 \quad \forall x \in I$.

Let $x_1, x_2 \in I$ with $x_1 < x_2$, so that $[x_1, x_2] \subset I$.

Since f is differentiable on I

$\therefore$ It is differentiable on $[x_1, x_2]$ and therefore it is continuous on $[x_1, x_2]$.

$\therefore f$ satisfies both the conditions of Lagranges mean value theorem on $[x_1, x_2]$.

$\therefore \exists \; c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{--- (1)}$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$,

$f'(x) \geq 0 \; \forall \; x \in I$ and $x_1 < c < x_2$

$$\Rightarrow f'(c) \geq 0$$

$\therefore$ from (1), $f(x_2) - f(x_1) \geq 0$

$$\Rightarrow f(x_1) \leq f(x_2).$$

Since $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

$\therefore f$ is an increasing on I.

Conversely - Suppose that f is differentiable on I and f is an increasing on I.

Now for $x \neq c \in I$ then $x > c$ or $x < c$

Case (i) if $x > c$ (i.e. $(x - c) > 0$)

then $f(x) \geq f(c)$. ($\because f$ is increasing on I).

$$\Rightarrow f(x) - f(c) \geq 0$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c > 0)$$

$$\Rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \text{--- (2)}$$

Case (ii) If $x < c$ (i.e. $(x - c) < 0$)

then $f(x) \leq f(c)$ ($\because f$ is increasing on I)

$$\Rightarrow f(x) - f(c) \leq 0$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c < 0)$$

$$\Rightarrow \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \text{--- (3)}$$

Since f is differentiable on I.

Let f be differentiable at $c \in I$.

$$\therefore f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$\therefore$ from (2) & (3) we have

$$f'(c) \geq 0.$$

(b) The Proof of part (b) is similar.

Problems:

* Verify Lagrange's mean value theorem for the following functions in the specified intervals:

→ $f(x) = x(x-1)(x-2) \forall x \in [0, \frac{1}{2}]$

Sol'n: $f(x) = x^3 - 3x^2 + 2x$ is a polynomial in x.

which is continuous in $[0, \frac{1}{2}]$.

$f'(x) = 3x^2 - 6x + 2$ exists in $(0, \frac{1}{2})$

∴ f is differentiable in $(0, \frac{1}{2})$.

∴ f satisfies the conditions of Lagrange's Mean value Theorem.

∴ $\exists\ c \in (0, \frac{1}{2})$ such that

$$f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8 - 0}{\frac{1}{2}}$$

$$\Rightarrow 3c^2 - 6c + 2 = 3/4$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$\Rightarrow c = \frac{24 \pm \sqrt{336}}{24}$$

$$\Rightarrow c = \frac{24 \pm 4\sqrt{21}}{24}$$

$$\Rightarrow c = \frac{6 + \sqrt{21}}{6}$$

Now the two values of c are

$1 + \frac{1}{6}\sqrt{21}$, $1 - \frac{1}{6}\sqrt{21}$

In these two values of c the second value $1 - \frac{1}{6}\sqrt{21} \in (0, \frac{1}{2})$.

∴ ∃ atleast one value of

$c = 1 - \frac{1}{6}\sqrt{21} \in (0, \frac{1}{2})$ such that

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c)$$

∴ The Lagranges Mean value theorem is verified.

H.W.

→ $f(x) = x^2 - 3x + 2 \ \forall x \in [-2, 3]$

→ $f(x) = x^3 + x^2 - 6x \ \forall x \in [-1, 4]$

→ $f(x) = e^x$ on $[0, 1]$

→ $f(x) = \log x \ \forall x \in [1, e]$ where $e = 2.71828$.

Sol'n: since $f(x) = \log x$ is continuous for all +ve values of x.

∴ It is continuous on $[1, e]$ and

$f'(x) = \frac{1}{x}$ exists in $(1, e)$.

∴ f is derivable in $(1, e)$

∴ f satisfies the conditions of Lagranges Mean Value theorem.

∴ ∃ at least one $c \in (1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log_e e - \log(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$\Rightarrow e-1 = c$

$\Rightarrow c = e-1 \in (1, e)$

$\therefore$ The Lagrange's Mean Value theorem is Satisfied.

11.4. $f(x) = \sqrt{x^2-4} \quad \forall\, x \in [2,4]$

$\rightarrow f(x) = \begin{cases} 2 & \text{if } x=1 \\ x^2 & \text{if } 1<x<2 \\ 1 & \text{if } x=2 \end{cases}$

Sol'n: Since $f(x) = x^2$ is a polynomial function in $1<x<2$ and every polynomial function is Continuous for all Values of x.

$\therefore$ It is Continuous on $(1,2)$

Now at $x=1$;

$f(1) = 2$

Now $\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1+} x^2$

$= 1$

$\therefore \lim\limits_{x\to 1+} f(x) \neq f(1)$.

At $x=2$;

$f(2) = 1$

Now $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} x^2$

$= 4$

$\therefore \lim\limits_{x\to 2-} f(x) \neq f(2)$

$\therefore f(x)$ is not at $x=1$ & 2

$\therefore f(x)$ is continuous in $(1,2)$ but not in $[1,2]$

$\therefore f(x)$ does not Satisfy the Conditions of Lagrange's Mean Value theorem.

$\therefore$ Lagrange's Mean value theorem is not applicable to $f(x)$.

$\rightarrow f(x) = |x| \quad \forall\, x \in [-1,2]$

Sol'n: It is Continuous on $[-1,2]$ and it is differentiable at each point in $(-1,2)$ except at $x=0$.

$\therefore f(x)$ is not differentiable in $(-1,2)$

$\therefore f(x)$ does not Satisfy the Conditions of Lagrange's Mean Value theorem.

$\therefore$ Lagrange's Mean value theorem is not applicable to $f(x)$.

$\rightarrow$ If $f(x) = (x-1)(x-2)(x-3)$; $a=0, b=4$ find c of Lagrange Mean value theorem.

Sol'n: $f(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

$f(a) = f(0)$

$= -6$

$f(b) = f(4)$

$= (3)(2)(1) = 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$3c^2-12c+11 = \frac{6-(-6)}{4-0}$$

$$\Rightarrow 3c^2-12c+11 = 12/4$$

$$\Rightarrow 3c^2-12c+11 = 3$$

$$\Rightarrow 3c^2-12c+8=0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144-96}}{2\times 3}$$

$$\Rightarrow c = \frac{12\pm\sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12\pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2}{\sqrt{3}} \in (0,4).$$

→ $f(x) = 1/x \ \forall\ x \in [-1,1]$

Sol'n: $f(0)$ is not finite while $0 \in [-1,1]$.

$\therefore$ LHL

$$\lim_{x\to 0-} f(x) = -\infty \ \&$$

RHL

$$\lim_{x\to 0+} f(x) = \infty$$

$\therefore f(x)$ is not continuous at $x=0$.

$\therefore f(x)$ is not continuous on $[-1,1]$

$\because$ Lagranges Mean Value theorem is not applicable to $f(x)$.

eg. → $f(x) = x^{1/3}$ in $[-1,1]$.

Sol'n: $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

does not exist at $x=0 \in (-1,1)$

Lagranges Mean value theorem is not applicable to $f(x)$.

However $\frac{f(1)-f(-1)}{1-(-1)} = f'(c)$

$$\Rightarrow \frac{1-(-1)}{-2} = \frac{1}{3c^{2/3}}$$

$$\Rightarrow 3c^{2/3} = 1$$

$$\Rightarrow c^{2/3} = 1/3$$

$$\Rightarrow c^{1/3} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow c = \frac{1}{3\sqrt{3}} \in (-1,1)$$

$\therefore$ The hypothesis of Lagranges Mean Value theorem is not valid.

i.e., the two conditions of Lagranges Mean Value theorem are sufficient but not necessary.

→ Show that if $x>0$, $\log(1+x) > \frac{x}{1+x}$ and hence prove that $x^{-1}\log(1+x)$ decreases monotonically as x increases from 0 to ∞.

Sol'n:– Let $f(x) = \log(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \left[\frac{(1+x)\cdot 1 - x}{(1+x)^2}\right]$$

$$= \cancel{\frac{1}{1+x}} - \cancel{\frac{1}{1+x}} + \frac{x}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2} > 0 \quad (\because x>0)$$

$\therefore f'(x) > 0$ when $x>0$

i.e. $f(x)$ is an increasing when $x>0$.

$\therefore f(x) > f(0)$.

Now $f(0) = \log(1+0) - \frac{0}{1+0}$

$= \log 1 - 0$

$= 0$

$\therefore f(x) > 0$

$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$

$\Rightarrow \log(1+x) > \frac{x}{1+x}$

Let $F(x) = x^{-1}\log(1+x)$

$= \frac{\log(1+x)}{x}$

$F'(x) = \frac{x\frac{1}{1+x} - \log(1+x).1}{x^2}$

$= \frac{-\left[\log(1+x) - \frac{x}{1+x}\right]}{x^2}$

$= \frac{-f(x)}{x^2} < 0$ for $x>0$

$(\because f(x)>0)$

$\therefore F'(x) < 0$ for $x>0$

$\therefore F(x)$ is an decreasing function in $(0,\infty)$.

$\boxed{2004\ P\text{-}I}$ Show that

$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$; $x>0$

sol'n: Let $f(x) = x - \frac{x^2}{2} - \log(1+x)$

$f'(x) = 1 - x - \frac{1}{1+x}$

$= \frac{1-x^2-1}{1+x}$

$= \frac{-x^2}{1+x} < 0$ for $x>0$.

$\therefore f'(x) < 0$ for $x>0$.

$\therefore f(x)$ is a decreasing function for $x>0$.

$\therefore f(0) > f(x)$.

Now $f(0) = 0 - 0 - \log 1$

$= 0$

$\therefore f(x) < 0$

$\Rightarrow x - \frac{x^2}{2} - \log(1+x) < 0$

$\Rightarrow x - \frac{x^2}{2} < \log(1+x)$ —— (1)

Now let $g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$

$\Rightarrow g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\left[\frac{(1+x)2x - x^2(1)}{(1+x)^2}\right]$

$= \frac{1}{1+x} - 1 + \frac{1}{2}\left[\frac{2x+x^2}{(1+x)^2}\right]$

$= \frac{1}{1+x} - 1 + \frac{1}{2}\frac{2x+x^2}{(1+x)^2}$

$= \frac{2(1+x) - 2(1+x)^2 + 2x + x^2}{2(1+x)^2}$

$= \frac{-x^2}{2(1+x)^2} < 0$ for $x>0$.

$\therefore g'(x) < 0$ for $x \geq 0$.

$\therefore g(x)$ is a decreasing function for $x>0$.

$\therefore g(0) > g(x)$.

But $g(0) = 0$

$\therefore g(x) < 0$

$$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$$

$$\Rightarrow \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \text{—— (2)}$$

Combining (1) & (2),

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$$

$\boxed{2002 \text{ P-I}}$ → show that

$$\frac{b-a}{\sqrt{1-a^2}} \le \sin^{-1}b - \sin^{-1}a \le \frac{b-a}{\sqrt{1-b^2}}$$

for $0 < a < b < 1$.

Sol'n: Let $f(x) = \sin^{-1}x \ \forall x \in [a,b]$

where $a > 0$; $b < 1$

i.e. $0 < a < b < 1$.

$f(x)$ is continuous & derivable in $[a,b]$ and $f'(x) = \frac{1}{\sqrt{1-x^2}} \ \forall x \in (a,b)$

$\therefore$ By Lagrange's Mean value theorem; $\exists\ c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a} \quad \text{—— (1)}$$

Since $c \in (a,b)$

$\Rightarrow a < c < b$

$\Rightarrow a^2 < c^2 < b^2$

$\Rightarrow -a^2 > -c^2 > -b^2$

$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}} \quad \text{—— (I)}$$

→ Prove that $\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}$

Sol'n: putting $b = 3/5$, $a = 1/2$ then from (I),

$$\frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-1/4}} < \sin^{-1}3/5 - \sin^{-1}1/2 < \frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-9/25}}$$

$$\Rightarrow \frac{1}{10} \times \frac{2}{\sqrt{3}} < \sin^{-1}(0.6) - \pi/6 < \frac{1}{10} \times \frac{5}{4}$$

$$\Rightarrow \frac{1}{5\sqrt{3}} < \sin^{-1}(0.6) - \frac{\pi}{6} < \frac{1}{8}$$

$$\Rightarrow \frac{\sqrt{3}}{15} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{\pi}{6} + 1/8$$

$\boxed{2008}$ (6m) → If $x > 0$, show that

$$\frac{x}{1+x} < \log(1+x) < x.$$

Sol'n: Let $f(t) = \log(1+t) \ \forall t \in [0,x]$

where $x > 0$.

It is continuous & differentiable in $[0,x]$.

and $f'(t) = \frac{1}{1+t} \ \forall t \in (0,x)$

By Lagrange's Mean value theorem, $\exists\ c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-0}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since $c \in (0,x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$ $(\because x>0)$

$\Rightarrow \frac{x}{1+x} < \log(1+x) < x.$

2000 P-I → Use the Mean Value theorem to Prove that $\frac{2}{7} < \log(1.4) < \frac{2}{5}$.

Sol'n : Let $f(t) = \log(1+t)$ $\forall\, t \in [0,x]$ where $x>0$.

$f(t)$ is continuous & differentiable on $[0,x]$.

and $f'(t) = \frac{1}{1+t}$ $\forall\, t \in (0,x)$

By Lagrange's Mean Value theorem, $\exists\, c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since $c \in (0,x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$

$\Rightarrow \frac{x}{1+x} < \log(1+x) < x$ $(\because x>0)$

Putting $x = 2/5$, we get.

$$\frac{2/5}{1+2/5} < \log(1+2/5) < 2/5$$

$$\Rightarrow \frac{2}{5}\times\frac{5}{7} < \log(7/5) < 2/5$$

$$\Rightarrow \frac{2}{7} < \log(1.4) < 2/5.$$

H.W. → show that

$\frac{x^2}{2} > x - \log(1+x) > \frac{x^2}{2(1+x)}$ for $x>1$

H.W. → Prove that

$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x>0$.

→ Apply Lagrange's mean value theorem to the function $\log(1+x)$ to show that

$0 < [\log(1+x)]^{-1} - x^{-1} < 1$ $\forall\, x>0$.

Sol'n :- Let $f(t) = \log(1+t)$ $\forall\, t \in [0,x]$ where $x>0$.

which is continuous & differentiable on $[0,x]$.

and $f'(t) = \frac{1}{1+t} \ \forall \ t \in (0,x)$.

By Lagranges mean value theorem

$\exists \ c \in (0,x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$

$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log(1)}{x-0}$

$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x}$ —— ①

Since $c \in (0,x)$; $x > 0$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$.

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$ (by ①)

$\Rightarrow (1+x) > \frac{x}{\log(1+x)} > 1$

$\Rightarrow \frac{1}{x} + 1 > \frac{1}{\log(1+x)} > \frac{1}{x}$

$\Rightarrow 1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 0$

$\Rightarrow 0 < [\log(1+x)]^{-1} - x^{-1} < 0$ for $x > 0$.

Q. Use Lagranges mean value theorem to prove that $1+x < e^x < 1+xe^x$ $\forall x > 0$.

Let $f(t) = e^t \ \forall \ t \in [0,x]$ where $x > 0$.

Q. show that

$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$ if $0 < u < v$ and deduce that

$\pi/4 + 3/25 < \tan^{-1} 4/3 < \pi/4 + 1/6$.

Sol^n: Let $f(x) = \tan^{-1}x \ \forall x \in [u,v]$ where $0 < u < v$.

$\longrightarrow$ use the Mean Value theorem to prove that $|\sin x - \sin y| \leq |x-y| \ \forall x,y \in \mathbb{R}$

Sol^n: If $x = y$ then there is nothing to prove.

If $x > y$ then consider the function $f(t) = \sin t \ \forall \ t \in [y,x]$.

Clearly f is continuous on $[y,x]$ and $f'(t) = \cos t$ exists on $[y,x]$

$\therefore$ By mean value theorem

$\exists \ c \in (y,x)$ such that

$f'(c) = \frac{f(x)-f(y)}{x-y}$

$\Rightarrow \cos c = \frac{\sin x - \sin y}{x-y}$

$\Rightarrow \left| \frac{\sin x - \sin y}{x-y} \right| = |\cos c|$

$\Rightarrow \frac{|\sin x - \sin y|}{|x-y|} = |\cos c| \leq 1$

$\Rightarrow |\sin x - \sin y| \leq |x-y|$

$\therefore \ \forall \ x,y \in \mathbb{R}$

$|\sin x - \sin y| \leq |x-y|$.

H.W. Use the Mean value Theorem to prove that $\frac{x-1}{x} < \ln x < (x-1)$ for $x > 1$

Let $f(t) = \ln t \ \forall \ t \in [1,x]$ where $x > 1$.

$\Rightarrow f(t) = \log t$.

2007 P-II

Using Lagrange's Mean value theorem, show that $|\cos b - \cos a| \le |b-a|$

→ If a function f is such that its derivative f' is continuous on $[a,b]$ and derivable on (a,b), then show that

$$f(b) = f(a) + (b-a)f'(a) + \tfrac{1}{2}(b-a)^2 f''(c).$$

Sol^n:- Let $\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 K,\ \forall x \in [a,b]$

where $K = \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$

(or) Let the function ϕ on $[a,b]$ defined by

$\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 K$

where K is a constant to be determined such that $\phi(a) = \phi(b)$

$f(a) + (b-a)f'(a) + (b-a)^2 K = f(b)$

$K = \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$

Since f' is continuous on $[a,b]$

$\Rightarrow f'$ exists on $[a,b]$

$\Rightarrow f$ is derivable on $[a,b]$

$\Rightarrow f$ is continuous on $[a,b]$

$\therefore$ The functions f and f' are continuous function on $[a,b]$ and derivable on (a,b)

$(b-x), (b-x)^2$ and K are continuous on $[a,b]$ and derivable on (a,b).

$\therefore \phi(x)$ is continuous on $[a,b]$ and derivable on (a,b).

Now

$\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 K$.

$\Rightarrow \phi(a) = \cancel{f(a)} + \cancel{(b-a)f'(a)} + \cancel{(b-a)^2}\left[\dfrac{f(b) - \cancel{f(a)} - \cancel{(b-a)f'(a)}}{\cancel{(b-a)^2}}\right]$

$= f(b)$

and $\phi(b) = f(b)$

$\therefore \phi(a) = \phi(b)$

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists\ c \in (a,b)$ such that $\phi'(c) = 0$ ——(1)

but

$\phi'(x) = f'(x) + (-1)f'(x) + (b-x)f''(x) + 2(b-x)(-1)K$

$\rightarrow \phi'(c) = (b-c)f''(c) + 2(b-c)(-1)K$

$\Rightarrow 0 = (b-c)[f''(c) - 2K]$ (by (1))

$\Rightarrow f''(c) - 2K = 0$ ($\because b-c \neq 0$ i.e. $c \in (a,b) \Rightarrow a < c < b$)

$\Rightarrow f''(c) = 2K$

$\Rightarrow K = \frac{1}{2} f''(c)$

$$\Rightarrow \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2} = \frac{1}{2} f''(c)$$

$\Rightarrow f(b) - f(a) - (b-a)f'(a) = \frac{1}{2}(b-a)^2 f''(c)$

$\Rightarrow f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$

$\rightarrow$ If a function f is twice differentiable on $[a, a+h]$ then show that $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$ for some real number θ where $\theta \in (0,1)$.

Sol'n :— Since f is twice differentiable on $[a, a+h]$

$\Rightarrow f', f''$ exist on $[a, a+h]$

$\Rightarrow f, f'$ are differentiable on $[a, a+h]$

$\Rightarrow f, f'$ are continuous on $[a, a+h]$

Let $\phi(x) =$

$$f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} K$$

where K is a constant to be determined such that $\phi(a) = \phi(a+h)$

$\Rightarrow f(a) + hf'(a) + \frac{h^2}{2!} K = f(a+h)$

$$\Rightarrow K = \frac{f(a+h) - f(a) - hf'(a)}{\left(\frac{h^2}{2!}\right)} \text{ ——(1)}$$

Since f & f' are continuous on $[a, a+h]$, $[a+h-x]$ and $\frac{(a+h-x)^2}{2!} K$ are continuous functions on $[a, a+h]$

$\Rightarrow \phi$ is continuous on $[a, a+h]$.

Since f & f' are derivable on $(a, a+h)$ and $(a+h-x)$, $\frac{(a+h-x)^2}{2!} \cdot K$ are derivable on $(a, a+h)$.

$\Rightarrow \phi$ is derivable on $(a, a+h)$.

Also $\phi(a) = \phi(b)$

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ a real numbers $\theta \in (0,1)$ such that

$\phi'(a+\theta h) = 0$ ——(2)

But $\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)K$

$= (a+h-x)[f''(x) - K]$

$\Rightarrow \phi'(a+\theta h) = (a+h-a-\theta h)\left[f''(a+\theta h) - K\right]$

$\Rightarrow 0 = (h-\theta h)\left[f''(a+\theta h) - K\right]$ (by ②)

$\Rightarrow f''(a+\theta h) - K = 0 \quad (\because h-\theta h \neq 0)$

$\Rightarrow f''(a+\theta h) = K$

$$\Rightarrow f''(a+\theta h) = \frac{f(a+h)-f(a)-hf'(a)}{\left(\frac{h^2}{2!}\right)}$$

$\Rightarrow \frac{h^2}{2!} f''(a+\theta h) = f(a+h) - f(a) - hf'(a)$

$\Rightarrow f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$

P-II 2006, 2011, P-I 2010, 12M

A twice differentiable function f on $[a,b]$ is such that $f(a)=f(b)=0$ and $f(c)>0$ for $a<c<b$. Prove that there is atleast one value ξ, $a<\xi<b$ for which $f''(\xi)<0$.

Sol: f is twice differentiable on $[a,b]$

$\Rightarrow f', f''$ exist on $[a,b]$

$\Rightarrow f, f'$ are differentiable on $[a,b]$

$\therefore f, f'$ are continuous functions on $[a,b]$.

Since $a<c<b$, applying Lagrange's Mean value theorem to on the intervals $[a,c]$ and $[c,b]$ we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1)$$

where $a<\xi_1<c$ and

$\frac{f(b)-f(c)}{b-c} = f'(\xi_2)$ where $c<\xi_2<b$

But $f(a)=f(b)=0$

$\therefore f'(\xi_1) = \frac{f(c)}{c-a}$ and

$f'(\xi_2) = \frac{-f(c)}{b-c}$ where

$a<\xi_1<c<\xi_2<b$.

Again f' is continuous and derivable on $[\xi_1, \xi_2]$

$\therefore$ By Lagrange's Mean value theorem we have

$$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} = f''(\xi)$$

where $\xi_1<\xi<\xi_2$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$, we get

$$f''(\xi) = \frac{\frac{-f(c)}{b-c} - \frac{f(c)}{c-a}}{\xi_2-\xi_1}$$

$$= \frac{-f(c)}{\xi_2-\xi_1}\left[\frac{1}{b-c} - \frac{1}{c-a}\right]$$

$$= \frac{-f(c)}{\xi_2-\xi_1}\left[\frac{b-a}{b-c}\right]$$

Since $a<\xi_1<c<\xi_2<b$ and $f(c)>0$

$\therefore f''(\xi) < 0$ where $a < \xi < b$.

Cauchy's Mean Value theorem (Second Mean Value theorem)

Statement: Let f and g be continuous on $[a,b]$ and differentiable on (a,b) and assume that $g'(x) \neq 0$ $\forall x \in (a,b)$ then $\exists$ $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof:- Let $\phi(x) = f(x) - f(a) - K[g(x) - g(a)]$ $\forall x \in [a,b]$

where $K = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

If possible, let $g(a) = g(b)$.

$\because$ Since $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b)

$\therefore$ g satisfies the conditions of Rolle's theorem.

$\therefore \exists$ $c \in (a,b)$ such that $g'(c) = 0$.

Which is contradiction to $g'(x) \neq 0$ $\forall x \in (a,b)$.

$\therefore g(a) \neq g(b)$

$\therefore \phi(x)$ is well defined.

Since $f(x)$ & $g(x)$ are continuous functions on $[a,b]$.

and $f(a), g(a)$ and K are constants. These are continuous for all x.

$\therefore \phi(x)$ is continuous on $[a,b]$.

and $\phi'(x) = f'(x) - Kg'(x)$ exists on (a,b).

because f & g are differentiable functions on (a,b).

$\therefore \phi$ is differentiable function on (a,b)

Now $\phi(a) = 0$

and $\phi(b) = f(b) - f(a) - K[g(b) - g(a)]$

$$= [f(b)-f(a)] - \frac{f(b)-f(a)}{g(b)-g(a)}[g(b)-g(a)] = 0$$

$\therefore \phi(a) = \phi(b)$.

$\therefore \phi(x)$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ $c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x) - Kg'(x)$ $\forall x \in (a,b)$.

$\Rightarrow \phi'(c) = f'(c) - Kg'(c)$

$\Rightarrow 0 = f'(c) - Kg'(c)$ $(\because \phi'(c) = 0)$

$\Rightarrow f'(c) = Kg'(c)$

$\Rightarrow K = \dfrac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

* **Another form of the statement:**

If two functions f and g defined on $[a, a+h]$ are

(i) Continuous on $[a, a+h]$

(ii) differentiable on $(a, a+h)$

(iii) $g'(x) \neq 0$ for any $x \in (a, a+h)$

then $\exists$ atleast one real numbers $\theta \in (0,1)$ such that

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)}$$

→ If f', g' are continuous and differentiable on $[a,b]$ then show that for $a<c<b$.

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$

Sol'n :- Let us consider

$$\phi(x) = f(x) + (b-x)f'(x) + k\{g(x)+(b-x)g'(x)\} \quad \forall x \in [a,b]$$

where k is a constant to be determined such that $\phi(a) = \phi(b)$.

$$f(a)+(b-a)f'(a)+k[g(a)+(b-a)g'(a)] = f(b) + kg(b).$$

$$\Rightarrow k = \frac{f(b)-f(a)-(b-a)f'(a)}{g(a)+(b-a)g'(a)-g(b)} \quad \text{(1)}$$

Since f', g' are continuous and differentiable functions on $[a,b]$.

$\therefore \phi(x)$ is continuous and differentiable on $[a,b]$

$\therefore \phi(x)$ satisfies the conditions Rolle's theorem on an interval $[a,b]$

$\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$.

But $\phi'(x) = \cancel{f'(x)} + (b-x)f''(x) - \cancel{f'(x)}$
$+ k[\cancel{g'(x)} + (b-x)g''(x) - \cancel{g'(x)}]$

$= (b-x)f''(x) + k(b-a)g''(x)$

$\Rightarrow \phi'(c) = (b-c)f''(c) + k(b-c)g''(c)$

$\Rightarrow 0 = (b-c)f''(c) + k(b-c)g''(c) \quad (\because \phi'(c) = 0)$

$$\Rightarrow k = -\frac{f''(c)}{g''(c)} \quad (\because b-c \neq 0).$$

$$\Rightarrow \frac{f(b)-f(a)-(b-a)f'(a)}{g(a)+(b-a)g'(a)-g(b)} = \frac{-f''(c)}{g''(c)}$$

$$\Rightarrow \frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$

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$\Rightarrow$ If $f'(x)$ and $g'(x)$ exist for all $x \in [a,b]$ and if $g'(x)$ doesnot vanish any where on (a,b) then prove that for some c between a and b.

$$\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

Sol^n: Let us consider

$$\phi(x) = f(x)g(x) - f(a)g(x) - g(b)f(x) \; \forall x \in [a,b]$$

Since f' and g' exists in $[a,b]$.

$\therefore$ f and g are derivable functions on $[a,b]$.

$\therefore$ f and g are Continuous functions on $[a,b]$.

$\therefore$ $\phi(x)$ is Continuous and derivable on $[a,b]$

and $\phi(a) = -f(a)g(b)$

$\phi(b) = -f(a)g(b)$

$\therefore \phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies the Conditions of Rolle's theorem on $[a,b]$

$\therefore \exists$ at least one point $c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x)g(x) + f(x)g'(x) - f(a)g'(x) - g(b)f'(x)$.

$\Rightarrow \phi'(c) = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$

$\Rightarrow 0 = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$

$(\because \phi'(c)=0)$

$g'(c)[f(c)-f(a)] + f'(c)[g(c)-g(b)] = 0$

$\Rightarrow g'(c)[f(c)-f(a)] = -f'(c)[g(b)-g(c)]$

$\Rightarrow \dfrac{f(c)-f(a)}{g(b)-g(c)} = \dfrac{f'(c)}{g'(c)}$

$[\because g'(x) \neq 0 \; \forall x \in (a,b)]$

Generalised Mean Value Theorem:

If three functions f, g and h defined on $[a,b]$ are

(i) Continuous on $[a,b]$

(ii) Differentiable on (a,b)

there exists a real number $c \in (a,b)$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof:- Consider the function ϕ on $[a,b]$ defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$= f(x)\begin{vmatrix} g(a) & h(a) \\ g(b) & h(b) \end{vmatrix} - g(x)\begin{vmatrix} f(a) & h(a) \\ f(b) & h(b) \end{vmatrix} + h(x)\begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}$$

$$= A\,f(x) + B\,g(x) + C\,h(x)$$

where A, B, C are Constants.

Since f, g, h are continuous functions on $[a,b]$

$\therefore \phi(x)$ is continuous on $[a,b]$ and f, g, h are differentiable on (a,b)

$\therefore \phi(x)$ is differentiable on (a,b).

and $\phi(a) = \phi(b) = 0$.

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$. — (i)

But $\phi'(x) = Af'(x) + Bg'(x) + Ch'(x)$ in (a,b)

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow \phi'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow 0 = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

where $c \in (a,b)$.

$\rightarrow$ when h is a constant function the above theorem reduces to Cauchy's mean value theorem.

Let $h(x) = K$ (constant) then $h(a) = h(b) = K$ and $h'(c) = 0$.

Substituting generalised mean value theorem, we get,

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & K \\ f(b) & g(b) & K \end{vmatrix} = 0$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

which is the Cauchy's Mean value theorem.

$\longrightarrow$ when $g(x)=x$ and $h(x)=K$ (Constant) the above theorem (generalised Mean value) reduces to Lagrange's mean value theorem.

$g(x)=x$ and $h(x)=K$

$\Rightarrow g'(x)=1$ and $h'(x)=0$

$\Rightarrow g'(c)=1$, $h'(c)=0$ and

$g(a)=a$; $g(b)=b$; $h(a)=h(b)=K$

$\therefore$ From generalised mean value theorem,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & K \\ f(b) & b & K \end{vmatrix} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

which is the Lagrange's Mean Value theorem.

Problems :-

$\rightarrow$ Verify Cauchy's Mean Value theorem for the following pairs of functions in the specified intervals.

$f(x)=x^2$ & $g(x)=x^3$ $\forall x \in [1,2]$

Sol'n: Since f & g are Continuous on $[1,2]$ and differentiable on $(1,2)$

Also $g'(x)=3x^2 \neq 0$ for any $x \in (1,2)$.

$\therefore f$ & g Satisfy the conditions of Cauchy's Mean Value theorem.

$\therefore \exists\, c \in (1,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)} \quad \text{—— (1)}$$

But $g'(x)=3x^2$ & $f'(x)=2x$

$\therefore f'(c)=2c$ & $g'(c)=3c^2$

$$(1) \equiv \frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow 9c = 14$$

$$\Rightarrow c = \frac{14}{9} \in (1,2)$$

$\therefore$ Cauchy's Mean Value theorem is verified.

→ Find 'c' of Cauchy's Mean Value Theorem for the following pairs of functions.

(i) $f(x)=e^{x}$, $g(x)=e^{-x}$ $\forall x\in[a,b]$

Sol'n: $f(a)=e^{a}$; $f(b)=e^{b}$

$g(a)=e^{-a}$; $g(b)=e^{-b}$

$f'(x)=e^{x} \Rightarrow f'(c)=e^{c}$

$g'(x)=-e^{-x} \Rightarrow g'(c)=-e^{-c}$

$$\therefore \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{e^{c}}{-e^{-c}}=\frac{e^{b}-e^{a}}{e^{-b}-e^{-a}}$$

$$\Rightarrow -e^{2c}=\frac{e^{b}-e^{a}}{\frac{1}{e^{b}}-\frac{1}{e^{a}}}$$

$$=\frac{-[-e^{b}+e^{a}]}{\frac{e^{a}-e^{b}}{e^{a}\cdot e^{b}}}$$

$$=-e^{a}\cdot e^{b}$$

$$=-e^{(a+b)}$$

$\Rightarrow 2c=a+b$

$\Rightarrow c=\frac{a+b}{2}\in(a,b)$

H.W.

(ii) $f(x)=x^{2}$, $g(x)=x$ $\forall x\in[a,b]$

(iii) $f(x)=\sin x$, $g(x)=\cos x$ $\forall x\in[-\pi/2,0]$

→ Show that $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}=\cot\theta$

where $0<\alpha<\theta<\beta<\pi/2$.

Sol'n: Let $f(x)=\sin x$

$g(x)=\cos x$ $\forall x\in[\alpha,\beta]$

Since f and g are both continuous on $[\alpha,\beta]$ and differentiable on (α,β).

$g'(x)=-\sin x\neq 0$ for any $x\in(\alpha,\beta)$

$\therefore$ By Cauchy's Mean value theorem

$\exists\,\theta\in(\alpha,\beta)$ such that

$$\frac{f'(\theta)}{g'(\theta)}=\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} \quad \text{--- (1)}$$

But $f'(x)=\cos x$; $g'(x)=-\sin x$

$\Rightarrow f'(\theta)=\cos\theta$; $g'(\theta)=-\sin\theta$

$$\therefore (1)\equiv \frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha}=\frac{\cos\theta}{-\sin\theta} ;\ \theta\in(\alpha,\beta)$$

$$\Rightarrow \frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}=\cot\theta,\ \theta\in(\alpha,\beta).$$

* Miscellaneous Problems :-

→ Assuming f'' to be continuous on $[a,b]$, show that

$$f(c)-f(a)\left(\frac{b-c}{b-a}\right)-\left(\frac{c-a}{b-a}\right)f(b) = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in $[a,b]$ i.e. $c, \xi \in [a,b]$.

Sol^n: We have to show that

$$(b-a)f(c)-(b-c)f(a)-(c-a)f(b) = \frac{1}{2}(b-a)(c-a)(c-b)f''(\xi)$$

Let us consider the function for $x \in [a,b]$ defined by

$$\phi(x) = (b-a)f(x)-(b-x)f(a)-(x-a)f(b) - (b-a)(x-a)(x-b)K$$

where K is a constant to be determined such that $\phi(c)=0$.

$$0 = (b-a)f(c)-(b-c)f(a)-(c-a)f(b) - (b-a)(c-a)(c-b)K$$

$$\therefore K = \frac{(b-a)f(c)-(b-c)f(a)-(c-a)f(b)}{(b-a)(c-a)(c-b)}$$

Clearly $\phi(a)=\phi(b)=0$ and $\phi(x)$ is differentiable in $[a,b]$

$\therefore$ The function ϕ satisfies all the conditions of Rolle's theorem on each intervals $[a,c]$ and $[c,b]$.

$\therefore \exists$ two numbers ξ_1, ξ_2 in (a,c) and (c,b) such that $\phi'(\xi_1)=0$ and $\phi'(\xi_2)=0$

But $\phi'(x) = (b-a)f'(x)+f(a)-f(b) - (b-a)\{2x-(a+b)\}K$.

which is continuous on $[a,b]$ and derivable on (a,b).

$\therefore$ Continuous and derivable on $[\xi_1, \xi_2]$.

Also $\phi'(\xi_1)=\phi'(\xi_2)=0$

$\therefore$ By Rolle's theorem,

$\exists\, \xi \in (\xi_1, \xi_2)$ such that $\phi''(\xi)=0$

But $\phi''(x) = (b-a)f''(x)-2(b-a)K$.

$\therefore f''(\xi)-2K=0$ $\quad (\because b-a \neq 0$ & $\phi''(\xi)=0)$

$\Rightarrow K = \frac{1}{2}f''(\xi)$ where $a<\xi_1<\xi<\xi_2<b$ — (2)

from (1) & (2), we have

$$\frac{(b-a)f(c)-(b-c)f(a)-(c-a)f(b)}{(b-a)(c-a)(c-b)} = \frac{1}{2}f''(\xi)$$

$$\Rightarrow f(c)-\left(\frac{b-c}{b-a}\right)f(a)-\left(\frac{c-a}{b-a}\right)f(b) = \frac{1}{2}(c-a)(c-b)f''(\xi).$$

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→ Let $\mathbb{R}$ be the set of real numbers and $f: \mathbb{R} \to \mathbb{R}$ such that for all x and y in $\mathbb{R}$, $|f(x)-f(y)| \le |x-y|^3$. Prove that $f(x)$ is a constant function.

Sol^n:- Given $|f(x)-f(y)| \le |x-y|^3$

$$\forall\, x, y \in \mathbb{R} \quad \text{--- (1)}$$

Let $y \in \mathbb{R}$ and x be choosen arbitrarily close to y but not equal to y.

$$\therefore (1) \equiv \left|\frac{f(x)-f(y)}{x-y}\right| \le |x-y|^2$$

Taking limit when $x \to y$ we get

$$\lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{x\to y}|x-y|^2$$

$$\Rightarrow \left|\lim_{x\to y}\frac{f(x)-f(y)}{x-y}\right| \le \left|\lim_{x\to y}(x-y)\right|^2$$

$$\Rightarrow |f'(x)| = 0 \left[\because \lim_{x\to y}\frac{f(x)-f(y)}{x-y} = f'(x) \text{ and } |f'(x)| \ge 0.\right]$$

$\Rightarrow f'(x) = 0$

$\therefore f(x)$ is constant.

[2004] → Prove that an equation of the form $x^n = \alpha$ where $n \in \mathbb{N}$ and $\alpha > 0$ is a real number, has a positive root.

(or)

show that $x^n - \alpha = 0$ has at most one real +ve root if n is a +ve integer.

Sol^n:- Let $f(x) = x^n - \alpha$

then $f'(x) = nx^{n-1}$

Since $f'(x) > 0$ for $x > 0$.

hence $f(x)$ is increasing on $(0,\infty)$

Let $x_1, x_2 \in (0,\infty)$ and $0 < x_1 < r < x_2$ such that $f(r) = 0$.

then $f(x_1) < f(r) < f(x_2)$ (or)

$$f(x_1) < 0 < f(x_2)$$

$\therefore$ This shows that if $x \ne r$, $f(x) \ne 0$ on $(0,\infty)$

i.e. $x^n - \alpha = 0$ has at most one real +ve root.

2008 → Prove that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ whenever $0 < x < \pi/2$.

Sol^n:- $\dfrac{\tan x}{x} - \dfrac{x}{\sin x} = \dfrac{\tan x \sin x - x^2}{x \sin x}$.

Since $x \sin x > 0 \ \forall\, x \in (0, \pi/2)$

$\therefore$ we are enough to show that

$\tan x . \sin x - x^2 > 0 \ \forall\, x \in (0, \pi/2)$

Let $f(x) = \tan x . \sin x - x^2 \ \forall\, x \in (0, \pi/2)$

$\Rightarrow f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$.

$= \sin x (\sec^2 x + 1) - 2x$.

We cannot decide about the sign of $f'(x)$ (because of the presence of $2x$ term)

Let $g(x) = f'(x) \ \forall x \in (0, \pi/2)$.

$\Rightarrow g'(x) = \cos x(\sec^2 x + 1) + \sin x(2\sec^2 x \tan x) - 2$

$= \sec x + \cos x - 2 + 2\sin^2 x \sec^3 x$

$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x \sec^3 x$.

Since $g'(x) > 0 \ \forall x \in (0, \pi/2)$.

$\Rightarrow g(x)$ is an increasing function $(0, \pi/2)$.

$\Rightarrow g(0) < g(x)$ in $0 < x < \pi/2$.

Since $g(0) = 0$

$\therefore g(x) > 0$.

$\Rightarrow f'(x) > 0$ whenever $0 < x < \pi/2$

$\therefore f$ is an increasing function in $0 < x < \pi/2$.

$\Rightarrow f(0) < f(x)$

$\Rightarrow 0 < f(x)$

$\Rightarrow \tan x \sin x - x^2 > 0$ in $(0, \pi/2)$.

$\Rightarrow \dfrac{\tan x \sin x - x^2}{x \sin x} > 0$

$\Rightarrow \dfrac{\tan x}{x} - \dfrac{x}{\sin x} > 0$

$\Rightarrow \dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ whenever $0 < x < \frac{\pi}{2}$

→ Prove that if f be defined for all real x such that $|f(x) - f(y)| < (x-y)^2$ then f is constant.

Sol'n:- Here we have to show that $f'(x) = 0 \ \forall x \in \mathbb{R}$.

Let $x = c \in \mathbb{R}$.

Now we have

$$\left|\frac{f(x) - f(c)}{x - c} - 0\right| \text{ for } x \neq c.$$

$$= \left|\frac{f(x) - f(c)}{x - c}\right|$$

$$= \frac{|f(x) - f(c)|}{|x - c|} < \frac{(x-c)^2}{|x-c|} = \frac{|x-c|^2}{x-c}$$

(by hyp).

$= |x - c| < \epsilon$ whenever $|x - c| < \frac{\epsilon}{1}$

$\therefore \left|\dfrac{f(x) - f(c)}{x - c} - 0\right| < \epsilon$ whenever $|x - c| < \delta$ by choosing $\delta = \frac{\epsilon}{1}$

$\therefore \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} = 0$

i.e. $f'(c) = 0 \ \forall c \in \mathbb{R}$.

$\Rightarrow f$ is constant function.

→ Find the interval in which the function $f(x) = \sin(\log_e x) - \cos(\log_e x)$ is strictly increases.

Sol'n: Given that

$f(x) = \sin(\log_e x) - \cos(\log_e x)$

Here domain is $x > 0$ as $\log_e x$

exists when $x>0$,

$$f'(x) = \frac{\cos(\log_e x) + \sin(\log_e x)}{x}$$

$$= \frac{\sqrt{2}\{\sin\frac{\pi}{4}\cos(\log_e x) + \cos\frac{\pi}{4}\sin(\log_e x)\}}{x}$$

$$= \frac{\sqrt{2}\sin(\pi/4 + \log_e x)}{x}$$

Since $f(x)$ is strictly increases when $f'(x) \geq 0$.

i.e. $\sin(\pi/4 + \log_e x) \geq 0$

$\Rightarrow 2n\pi \leq \frac{\pi}{4} + \log_e x \leq (2n+1)\pi \ \forall n \in Z$

$\Rightarrow 2n\pi - \pi/4 \leq \log_e x \leq 2n\pi + \pi - \pi/4$

$\Rightarrow e^{2n\pi - \pi/4} \leq x \leq e^{2n\pi + 3\pi/4}$

$\therefore f(x)$ is strictly increasing when

$x \in \left[e^{2n\pi - \pi/4}, e^{2n\pi + 3\pi/4}\right]$

→ Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0 \ \forall \ x \in (0,1)$. Find the intervals of increase and decrease of $g(x)$.

Sol'n :— We have

$$g(x) = f(x) + f(1-x) \quad \text{—— (1)}$$

then $g'(x) = f'(x) - f'(1-x)$ —— (2)

Since $f''(x) > 0 \ \forall \ x \in (0,1)$

$\therefore f'(x)$ is increasing on $(0,1)$.

Hence two cases:

Case(i): $x > 1-x$ and $f'(x)$ is increasing for $\forall x > \frac{1}{2}$ in $(0,1)$

$\Rightarrow f'(1-x) < f'(x) \ \forall x > \frac{1}{2}$

$\Rightarrow f'(x) - f'(1-x) \geq 0 \ \forall x > \frac{1}{2}$

$x \in (0,1)$, $1-x \in (0,1)$; if f is $\uparrow$ & $x_1 < x_2$ then $f(x_1) < f(x_2)$

$\therefore g'(x) > 0 \ \forall x > \frac{1}{2}$ in $(0,1)$

i.e. $g'(x) > 0 \ \forall \ x \in (\frac{1}{2}, 1)$

$\Rightarrow g(x)$ is increasing in $(\frac{1}{2}, 1)$.

Case(ii)

$x < 1-x$ and $f'(x)$ increasing for $0 < x < \frac{1}{2}$ in $(0,1)$.

$\Rightarrow f'(x) < f'(1-x)$ for $0 < x < \frac{1}{2}$

$\Rightarrow f'(x) - f'(1-x) < 0$ for $0 < x < \frac{1}{2}$

$\Rightarrow g'(x) < 0$; $x \in (0, \frac{1}{2})$

$\Rightarrow g(x)$ is decreasing function in $(0, \frac{1}{2})$.

→ show that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad 0 < x < \pi/2$$

Sol'n :- Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad \forall x \in [0, \pi/2]$$

then f is continuous in $[0, \pi/2]$

and derivable in $(0, \pi/2)$

and $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$;

$x \in (0, \pi/2)$ —— (i)

Let $F(x) = x\cos x - \sin x$; $x \in (0, \pi/2)$

$F'(x) = \cos x - x\sin x - \cos x$

$= -x\sin x$

< 0 ; $x \in (0, \pi/2)$

$\therefore F$ is decreasing in $(0, \pi/2)$

$\therefore F(x) < F(0)$ for $x > 0$ in $(0, \pi/2)$

$\Rightarrow F(x) < 0$ for $x \in (0, \pi/2)$

$(\because F(0) = 0)$

$\Rightarrow f'(x) < 0$; $x \in (0, \pi/2)$

$\therefore f(x)$ is decreasing in $(0, \pi/2)$

$\Rightarrow f(0) > f(x) > f(\pi/2)$ for

$0 < x < \pi/2$

$\Rightarrow 1 > \dfrac{\sin x}{x} > \dfrac{2}{\pi}$

$\Rightarrow \dfrac{2}{\pi} < \dfrac{\sin x}{x} < 1 \;\; \forall x \in (0, \pi/2)$

Taylor's Theorem:

Statement:

If a function f defined on $[a,b]$, is such that

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a,b]$

(ii) the n^{th} derivative $f^{(n)}$ exists on (a,b)

then there exist atleast one real number $c \in (a,b)$ such that
$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p\,(n-1)!} f^{n}(c)$$

where p is a given +ve integer.

proof: Let $G(x) = F(x) - \left(\frac{b-x}{b-a}\right)^p F(a)$

where
$$F(x) = f(b) - f(x) - (b-x) f'(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)$$

Since the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a,b]$.

$\therefore f, f', f'', \ldots\ldots f^{(n-1)}$ are continuous on $[a,b]$.

and $(b-x)^r$, $r = 1, 2, 3, \ldots\ldots n-1$

is continuous for all x.

$\therefore F(x)$ is continuous on $[a,b]$.

$\therefore G(x)$ is continuous on $[a,b]$.

Since the n^{th} derivative $f^{(n)}$ exists on (a,b).

$\therefore f, f', f'', \ldots\ldots f^{(n-1)}$ are differentiable on (a,b)

and $(b-x)^r$; $r = 1, 2, 3, \ldots\ldots (n-1)$.

is differentiable for all x.

$\therefore F(x)$ is differentiable on (a,b).

$\therefore G(x)$ is differentiable on (a,b).

Now $G(a) = F(a) - \left(\frac{b-a}{b-a}\right)^p F(a)$

$= 0$

and $G(b) = F(b) - 0$

$\cancel{= f(b) - f(b) - (b-b) f'(b) - \cdots - \frac{(b-b)^{n-1}}{(n-1)!} f^{(n-1)}(b)}$

$= 0$

$\therefore G(a) = G(b) = 0$

$\therefore G(x)$ satisfies the conditions of Rolle's theorem,

$\exists$ atleast one real number $c \in (a,b)$

such that $G'(c) = 0$.

But $G'(x) = F'(x) + p \frac{(b-x)^{p-1}}{(b-a)^p} F(a)$

Now $F'(x) = 0 - f'(x) + f'(x) - (b-x) f''(x) + (b-x) f''(x)$

$- \frac{(b-x)^2}{2!} f'''(x) + \cdots + \frac{(n-1)(b-x)^{n-2}}{(n-1)!} f^{(n-1)}(x)$

$- \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$

$\therefore F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$ —— (2).

(1) $\equiv$ $G'(c) = F'(c) + p \frac{(b-c)^{p-1}}{(b-a)^p} F(a)$

$\Rightarrow 0 = F'(c) + p \frac{(b-c)^{p-1}}{(b-a)^p} F(a) \quad (\because G'(c) = 0)$

$\Rightarrow F'(c) = -\frac{p(b-c)^{p-1}}{(b-a)^p} F(a)$

$\Rightarrow -\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) = -\frac{p(b-c)^{p-1}}{(b-a)^p} F(a)$ (from (2))

$\Rightarrow F(a) = \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$.

$\Rightarrow f(b) - f(a) - \frac{(b-a)}{1!} f'(a) - \frac{(b-a)^2}{2!} f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a)$

$= \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$

$$\Rightarrow f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$
$$\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$$
$$= P_n(x) + R_n(x)$$

Note: [1]. After n terms $R_n(x) = T_{n+1}$
$$= \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$$

for some point $c \in (a,b)$.

This formula for R_n is referred to as the Roche's form (or derivative form) of the remainder.

[2].

(i) For $p=1$,
$$R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c)$$ Called Cauchy's form of remainder.

(ii) for $p=n$,
$$R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$$ Called Lagrange's form of remainder.

[3]. Another form of Taylor's Theorem:

If a function f defined on $[a, a+h]$ is such that

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$

(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$

then $\exists$ $\theta \in (0,1)$ such that
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^{n}(a+\theta h)$$

where p is +ve integer.

4. Maclaurin's theorem:

putting $a=0$, $h=x$ in Taylor's theorem.

i.e, If a function f defined on $[0,x]$ is such that (i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[0,x]$

(ii) the n^{th} derivative $f^{(n)}$ exists on $(0,x)$

then $\exists\ \theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-p}}{p\,(n-1)!} f^{n}(\theta x).$$

Taylor's and Maclaurin's series:

→ Let a function f be continuous derivatives of every order in $[a, a+h]$ then for all $n \in \mathbb{N}$ we have by Taylor's theorem

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

where $\theta \in (0,1)$

Let $P_n = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$

$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$ (which is Taylor's remainder after n terms)

Then $f(a+h) = P_n + R_n$

if $R_n \to 0$ as $n \to \infty$,

we have $\lim_{n\to\infty} P_n = f(a+h)$

$\Rightarrow$ The infinite series

$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \cdots$

converges to $f(a+h)$

$\therefore f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots\cdots$ is called Taylor's series which is cgs to $f(a+h)$

if $\underset{n\to\infty}{Lt} R_n = 0$

Hence if $f : [a, a+h] \to R$ possesses continuous derivatives of every order in $[a, a+h]$ and Taylor's remainder $R_n \to 0$ as $n \to \infty$

then $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$
$\cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \cdots\cdots$

$\rightarrow$ If we put $a = 0$, $h = x$; we get

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots\cdots$

This is called Maclaurin's series.

Note: This series is useful in finding the expansion of functions.

Problems:

If $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x+\theta h)$, find the value of θ as $x \to a$ if $f(x) = (x-a)^{5/2}$

Sol: Given $f(x) = (x-a)^{5/2}$

$\Rightarrow f(x+h) = (x+h-a)^{5/2}$

and $f'(x) = \frac{5}{2}(x-a)^{3/2}$

$\Rightarrow f''(x) = \frac{15}{4}(x-a)^{1/2}$

$\Rightarrow f''(x+\theta h) = \frac{15}{4}(x+\theta h-a)^{1/2}$

$\therefore f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h)$.

$\Rightarrow (x+h-a)^{5/2} = (x-a)^{5/2} + h\left(\frac{5}{2}\right)(x-a)^{3/2} + \frac{h^2}{2}\left(\frac{15}{4}\right)(x+\theta h-a)^{1/2}$

when $x \to a$, we get

$$h^{5/2} = \frac{h^2}{2}\left(\frac{15}{4}\right)(\theta h)^{1/2}$$

$$\Rightarrow h^{5/2} = \frac{h^{5/2}}{2}\left(\frac{15}{4}\right)\theta^{1/2}$$

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$$\Rightarrow \frac{8}{15} = \theta^{1/2}$$

$$\Rightarrow \theta = \frac{64}{225}$$

H.W. If $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$; find the value of θ as $x \to 1$ if $f(x) = (1-x)^{5/2}$.

$\rightarrow$ Using Taylor's theorem, Show that

(i) $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$.

(ii) $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x, \quad x > 0$

(iii) $x - \frac{x^3}{3!} < \sin x < x, \quad x > 0$

(iv) $x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}; \quad x \geq 0$

Sol^n (i) $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$.

Case(1) Let $x = 0$.

then $\cos x = 1$; $1 - \frac{x^2}{2} = 1$

$\therefore \cos x = 1 - \frac{x^2}{2}$.

Case(2): Let $x > 0$ and $f(x) = \cos x$

$\Rightarrow f'(x) = -\sin x$, $f''(x) = -\cos x$

Since $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$

where $0 < \theta < 1$

$\therefore \cos x = 1 - \frac{x^2}{2} \cos \theta x$.

But $\cos \theta x < 1$; $\forall \theta x$; $x > 0$.

$\therefore 1 - \frac{x^2}{2} \cos \theta x > 1 - \frac{x^2}{2}$

$\Rightarrow \cos x > 1 - \frac{x^2}{2}$.

Case(3):

Let $x<0 \Rightarrow -x>0$

put $y=-x$; $y>0$

By Case(2), $\cos y > 1-\frac{y^2}{2}$

$\Rightarrow \cos(-x) > 1-\frac{(-x)^2}{2}$

$\Rightarrow \cos x > 1-\frac{x^2}{2}$

∴ combining all cases,

$\cos x \geq 1-\frac{x^2}{2} \quad \forall x \in \mathbb{R}$

(ii) $1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$; $x>0$.

Solⁿ: Let $f(x)=e^x$; $x>0$

then $f'(x)=e^x=f''(x)$

Since $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x)$

where $0<\theta<1$.

$= 1+x+\frac{x^2}{2}e^{\theta x}$ —— (A)

Now $0<\theta<1$

$\Rightarrow 0<\theta x<x$; $x>0$

$\Rightarrow e^0<e^{\theta x}<e^x$

$\Rightarrow 1<e^{\theta x}<e^x$

$\Rightarrow \frac{x^2}{2}<\frac{x^2}{2}e^{\theta x}<\frac{x^2}{2}e^x$

$\Rightarrow 1+x+\frac{x^2}{2}<1+x+\frac{x^2}{2}e^{\theta x}<1+x+\frac{x^2}{2}e^x$

$\Rightarrow 1+x+\frac{x^2}{2}<e^x<1+x+\frac{x^2}{2}e^x$ (by (A))

→ Expand e^x as an infinite series.

Solⁿ: Let $f(x)=e^x$

$f(x)=e^x \Rightarrow f(0)=1$

$f'(x)=e^x \Rightarrow f'(0)=1$

⋮

$f^n(x)=e^x \Rightarrow f^{(n)}(\theta x)=e^{\theta x}$

Clearly f and its derivatives exist and are continuous for every value of x.

$$R_n(x) = \frac{x^n}{n!} e^{\theta x} \quad (LFR)$$

$$\underset{n\to\infty}{Lt} R_n = e^{\theta x} \underset{n\to\infty}{Lt} \frac{x^n}{n!} \quad \text{——} ①$$

Now let $a_n = \frac{x^n}{n!}, \forall n \in N$

$$\Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!} \quad \forall n \in N$$

$$\therefore \underset{n\to\infty}{Lt}\left(\frac{a_{n+1}}{a_n}\right) = \underset{n\to\infty}{Lt} \frac{x}{n+1}$$

$$= 0 < 1$$

$$\therefore \underset{n\to\infty}{Lt}\, a_n = 0 \quad \left(\because \text{if } \underset{n\to\infty}{Lt}\left(\frac{a_{n+1}}{a_n}\right) = l < 1 \text{ then } \underset{n\to\infty}{Lt}\, a_n = 0\right)$$

$$\therefore \underset{n\to\infty}{Lt} R_n = 0$$

$\therefore$ The conditions of Maclaurin's series are satisfied.

$\forall x \in \mathbb{R}$,

$$e^x = f(x)$$
$$= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

→ Expand $\sin x$ as infinite series

Sol^n: Let $f(x) = \sin x$.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

Here $R_n(x)$ must be tend to '0'.

Now $f(x) = \sin x$; $f'(x) = \cos x$; $f''(x) = -\sin x$; $f'''(x) = -\cos x$

$\Rightarrow f(0) = 0$ $\quad \therefore f'(0) = 1$ $\quad \therefore f''(0) = 0$ $\quad f'''(0) = -1$

$f^{iv}(x) = \sin x$; $\quad f^{(v)}(x) = +\cos x$

$\therefore f^{iv}(0) = 0$ $\quad \therefore f^{(v)}(0) = 1$.

Generally $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$

$\therefore f$ and all its derivatives exist and continuous for every real value of x.

$$R_n(x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

Now $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} \times \lim_{n\to\infty} \sin\left(\theta x + \frac{n\pi}{2}\right)$

$= 0 \times l \quad (-1 \le l \le 1)$

$= 0$

$\therefore \lim_{n\to\infty} R_n = 0$

$\therefore$ The conditions of Maclaurin's series is satisfied $\forall x \in \mathbb{R}$.

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

Similarly $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$

$\rightarrow f(x) = e^x$

$\rightarrow f(x) = a^x$

$\rightarrow f(x) = \log(1+x)$ etc.

* Extreme Values of a function: Maxima and Minima:—

[Some definitions discussed in pg No. (8)].

* Theorem (First derivative Test):

→ Let f be continuous on $I = [a,b]$ and let 'c' be an interior point on I. Assume that f is differentiable on (a,c) and (c,b). Then –

(i) If there is a neighbourhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c+\delta$ then 'f' has maximum at 'c'.

(ii) If there is neighbourhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c+\delta$ then f has a minimum at 'c'.

→ **Theorem** :—

Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$ let $c \in I$ and assume that f has a derivative at 'c' then.

(i) If $f'(c) > 0$ then $\exists$ a $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$. Such that $c < x < c+\delta$

(ii) If $f'(c) < 0$ then $\exists$ a $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$. Such that $c-\delta < x < c$.

* Darboux's Theorem:—

If f is differentiable on $I = [a,b]$ and if k is a number between $f'(a)$ & $f'(b)$ then $\exists$ $c \in (a,b)$ such that $f'(c) = k$.

Proof:- Since k is number between $f'(a)$ & $f'(b)$.

Suppose that $f'(a) < k < f'(b)$.

Now we define

$g(x) = kx - f(x) \quad \forall x \in [a,b]$ — ①

Since f is differentiable on I.

$\therefore f$ is continuous on I and kx is a polynomial which is continuous on I

$\therefore g(x)$ is continuous on I.

$\therefore g(x)$ attains its supremum (infimum) atleast once on $[a,b] = I$.

Since $g'(x) = k - f'(x) \; \forall x \in [a,b]$

$\Rightarrow g'(a) = (k - f'(a)) > 0$

$(\because f'(a) < k < f'(b))$

$\Rightarrow g'(a) > 0$.

We know that g has derivative at 'a' and $g'(a) > 0$. then $\exists$ a $\delta > 0$ such that $g(x) > g(a)$ $\forall x \in I$. Such that $a < x < a+\delta$.

$\therefore g$ does not have the maximum at $x = a$.

Similarly g does not have the minimum at $x = b$.

$\therefore g$ has maximum at $c \in (a,b)$

∴ Interior extremum theorem

$g'(c) = 0 \ \forall \ c \in (a,b)$

$f'(c) = k \ \forall \ c \in (a,b)$.

* Generalised Test :—

Let I be an interval, let $x_0 \in I$ and let $n \geq 2$.

Suppose that the derivatives $f', f'', \ldots, f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and that

$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$

but $f^n(x_0) \neq 0$.

i) If n is even and $f^n(x_0) > 0$ then f has minimum at x_0.

, If n is even and $f^n(x_0) < 0$ then f has maximum at x_0.

i) If n is odd, then f has neither a minimum nor maximum at x_0.

First Method

Working Rule for Finding Maxima and Minima :—

) Denote the given function by $f(x)$

) Find $f'(x)$ and equate it to zero.

Let its roots be $x_1, x_2, \ldots$

) Find $f''(x)$, Put $x = x_1$. If $f''(x_1) < 0$, $f(x)$ has a maximum at $x = x_1$.

If $f''(x_1) > 0$, $f(x)$ has a minimum at $x = x_1$.

(4) If $f''(x_1) = 0$, find $f'''(x_1)$.

If $f'''(x_1) \neq 0$, there is neither maximum nor minimum at $x = x_1$.

If $f'''(x_1) = 0$, find $f^{iv}(x_1)$

If $f^{iv}(x_1) < 0$, $f(x)$ has a maxima at $x = x_1$

If $f^{iv}(x_1) > 0$, $f(x)$ has a minimum at $x = x_1$, so on,

* Working Rule for Finding Maxima and Minima :—

(Second Method by First Derivative Test)

(1). Denote the given function by $f(x)$.

(2). Find $f'(x)$ and equate it to zero.

Let its roots be $x_1, x_2, x_3 \ldots$

(3) Test these values in succession.

Consider $x = x_1$ (say).

If there is a neighbourhood $x \in (x_1 - \delta, x_1 + \delta)$ such that

- $f'(x) \geq 0$ for $x_1 - \delta < x < x_1$ and $f'(x) \leq 0$ for $x_1 < x < x_1 + \delta$ then f has maximum at x_1.
- $f'(x) \leq 0$ for $x_1 - \delta < x < x_1$ and $f'(x) \geq 0$ for $x_1 < x < x_1 + \delta$ then f has minimum at x.
- $f'(x) \leq 0$ (≥ 0 only) for $x_1 - \delta < x < x_1$ and $x_1 < x < x_1 + \delta$ then f is neither maximum nor minimum at x_1.

(ii) Similarly test all these values of x obtained in (2).

Problems:-

→ Examine the following function for extreme values: $(x-3)^5 (x+1)^4$.

Sol^n:- Let $f(x) = (x-3)^5 (x+1)^4$

$f'(x) = (x-3)^5 \, 4(x+1)^3 + (x+1)^4 \, 5(x-3)^4$

$= (x-3)^4 (x+1)^3 [(x-3)4 + (x+1)5]$

$= (x-3)^4 (x+1)^3 [4x-12+5x+5]$

$= (x-3)^4 (x+1)^3 [9x-7]$.

For maximum or minimum $f'(x)=0$

$\Rightarrow (x-3)^4 (x+1)^3 (9x-7)=0$

$\Rightarrow \boxed{x = 3, -1, 7/9}$

Second Method:-

Take $x=3 \in (3-\delta, 3+\delta)$; $\delta>0$ for $3-\delta < x < 3 \Rightarrow f'(x)>0$ and for $3<x<3+\delta \Rightarrow f'(x)>0$.

$\therefore f'(x)>0$ for $3-\delta<x<3$ and $3<x<3+\delta$

$\therefore f(x)$ is neither minimum nor maximum at $x=3$.

Take $x=-1 \in (-1-\delta, -1+\delta)$, $\delta>0$

for $-1-\delta<x<-1 \Rightarrow f'(x)>0$ and for $-1<x<-1+\delta \Rightarrow f'(x)<0$

$\therefore$ By first derivative test for extrema,

$f(x)$ has maximum at $x=-1$.

$\therefore f_{max} = f(-1) = 0$.

Take $x = 7/9 \in (7/9-\delta, 7/9+\delta)$

for $7/9-\delta<x<7/9$;

$\Rightarrow f'(x)<0$

for $7/9<x<7/9+\delta \Rightarrow f'(x)>0$.

$\therefore f(x)$ has minimum at $x = 7/9$.

(By first derivative test for extreme)-

$\therefore f_{min} = f(7/9) = \dfrac{-4^{13}.5^5}{3^{18}}$.

H.W → Examine for maxima and minima of the function defined by

$f(x) = x^2 (1-x)^3$

→ Show that $\sin x(1+\cos x)$ is maximum when $x = \pi/3$.

First Method:-

Sol^n: Let $f(x) = \sin x\,(1+\cos x)$

then $f'(x) = \cos x\,(1+\cos x) + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x + \cos x$

$= \cos 2x + \cos x$

and $\boxed{f''(x) = -2\sin 2x - \sin x}$

For maxima or minima $f'(x)=0$

$\Rightarrow \cos 2x + \cos x = 0$

$\Rightarrow 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ either $\frac{3x}{2} = \frac{\pi}{2}$ (or) $\frac{x}{2} = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{3}$ (or) $x = \pi$.

Here we consider only the point $x = \pi/3$.

$$f''(\pi/3) = -2\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)$$
$$= -2\sin(120^\circ) - \sin 60^\circ$$
$$= -2\sin(180^\circ - 60^\circ) - \sin 60^\circ$$
$$= -2\sin 60^\circ - \sin 60^\circ$$
$$= -3\sin 60^\circ$$
$$= -3\sqrt{3}/2 < 0.$$

$\therefore f(x)$ has a maximum at $x = \pi/3$.

$$\therefore f_{max} = f(\pi/3) = \sin\pi/3\,(1 + \cos\pi/3)$$
$$= \frac{\sqrt{3}}{2}(1 + 1/2)$$
$$= \frac{\sqrt{3}}{2}\left(3/2\right) = \frac{3\sqrt{3}}{4}.$$

→ Find the maximum and minimum values if any of the function $(1-x)^2 e^x$.

Sol^n:- Let $f(x) = (1-x)^2 e^x$

then $f'(x) = (1-x)^2 e^x - 2(1-x)e^x$
$$= [1 + x^2 - \cancel{2x} - 2 + \cancel{2x}]e^x$$
$$= (x^2 - 1)e^x$$

For maximum or minimum.

$$f'(x) = 0$$
$$\Rightarrow e^x(x^2 - 1) = 0$$
$$\Rightarrow x^2 - 1 = 0 \quad (e^x \neq 0)$$
$$\Rightarrow x = \pm 1$$

when $x = 1$: $f''(x) = e^x(x^2-1) + e^x(2x)$
$$= e^x[x^2 + 2x - 1]$$
$$\therefore f''(1) = e^1(1+2-1)$$
$$= 2e > 0$$

$\therefore f$ is minimum at $x = 1$.

$\therefore f_{min} = f(1) = 0$.

when $x = -1$: $f''(x) = e^x(x^2 + 2x - 1)$
$$\therefore f''(-1) = e^{-1}(1-2-1)$$
$$= \frac{-2}{e} < 0$$

$\therefore f$ is maximum at $x = -1$.

$\therefore f_{max} = f(-1) = \frac{4}{e}$.

→ Find the maximum value of $\frac{\log x}{x}$, $0 < x < \infty$.

Sol^n: Let $f(x) = \frac{\log x}{x}$

then $f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$
$$= \frac{1 - \log x}{x^2}$$

and $f''(x) = \frac{x^2(-1/x) - (1 - \log x)2x}{x^4}$
$$= \frac{-x - 2x + 2x\log x}{x^4}$$

For maximum or minimum.

$$f'(x) = 0$$
$$\Rightarrow 1 - \log x = 0$$
$$\Rightarrow \log x = 1$$
$$\Rightarrow \boxed{x = e}$$

when $x = e$:

$$f''(e) = \frac{-e - 2e + 2e\log e}{e^4}$$
$$= \frac{-3e + 2e}{e^4}$$
$$= \frac{-e}{e^4}$$
$$= \frac{-1}{e^3} < 0.$$

$\therefore f$ is maximum at $x=e$.

$\therefore f_{max} = f(e) = \frac{\log e}{e} = \frac{1}{e}$.

→ Prove that the function $\left(\frac{1}{x}\right)^x$, $x>0$ has a maximum at $x = 1/e$.

Sol^n Let $f(x) = \left(\frac{1}{x}\right)^x$; $x>0$

$\Rightarrow \log f(x) = x \log 1/x$

$\Rightarrow \log f(x) = x[-\log x]$

$\Rightarrow \log f(x) = -x \log x$

$\Rightarrow \frac{1}{f(x)} f'(x) = -\left[x\left(\frac{1}{x}\right) + \log x\right]$

$\Rightarrow \boxed{f'(x) = -f(x)[1+\log x]}$

And $f''(x) = -f(x)\frac{1}{x} - f'(x)[1+\log x]$

For maximum (or) minimum $f'(x)=0$

$\Rightarrow -f(x)[\log x + 1] = 0$

$\Rightarrow 1+\log x = 0 \quad (\because f(x) \neq 0)$

$\Rightarrow \log x = -1$

$\Rightarrow \boxed{x = e^{-1}}$

Now when $x = e^{-1}$:

$f''(e^{-1}) = -f(e^{-1})\frac{1}{e^{-1}} - f'(e^{-1})[1+\log(e^{-1})]$

$= -(e)^{e^{-1}}\frac{1}{e^{-1}} - 0[1+\log(e^{-1})]$

$= -e^{e^{-1}} \cdot e^{1}$

$= -(e)^{1/e} \cdot e$

< 0

$\therefore f$ is maximum at $x = 1/e$

$\therefore f_{max} = f(1/e) = \left(\frac{1}{1/e}\right)^{1/e} = (e)^{1/e}$

→ H.W. Prove that the function x^x, $x>0$ has a minimum at $x = 1/e$.

→ find the maximum and minimum values of the following functions:

① $2x^3 - 9x^2 - 24x - 20$.

② $(x-1)(x-2)(x-3)$

→ For each of the following functions on $\mathbb{R} \to \mathbb{R}$. Find points of extrema, the intervals on which the function is increasing, and those on it is decreasing.

(i) $f(x) = x^2 - 3x + 5$

(ii) $g(x) = 3x - 4x^2$

(iii) $h(x) = x^3 - 3x - 4$

Sol^n (i) $f(x) = x^2 - 3x + 5$

$f'(x) = 2x - 3$

for maximum (or) minimum $f'(x) = 0$

$x = 3/2$

Now $x = 3/2 \in (3/2 - \delta, 3/2 + \delta)$

For $3/2 - \delta < x < 3/2 \Rightarrow f'(x) < 0$

and $3/2 < x < 3/2 + \delta \Rightarrow f'(x) > 0$

$\therefore$ By first derivative test for extrema $f(x)$ has minimum at $x = 3/2$.

$f_{min} = f(3/2) = (3/2)^2 - 3(3/2) + 5$

$= 9/4 - 9/2 + 5$

$= \frac{9 - 18 + 20}{4}$

$= 11/4$

Now $f'(x) = 2x-3$

if $x < 3/2 \Rightarrow f'(x) < 0$.

$\xrightarrow{\quad}\ 3/2\ \xleftarrow{\quad}$

$\therefore f(x)$ is an decreasing in $(-\infty, 3/2)$

if $x > 3/2$

$\Rightarrow f'(x) > 0$

$\therefore f(x)$ is an increasing in $(3/2, \infty)$.

$h(x) = x^3 - 3x - 4$

Sol'n:- $h'(x) = 3x^2 - 3$

For maximum or minimum $h'(x) = 0$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

At $x=1$: $h(x)$ has minimum.

At $x=-1$: $h(x)$ has maximum.

Now $h'(x) = 3x^2 - 3$

$= 3(x-1)(x+1)$ — ①

$x \quad -1 \quad +1 \quad x$

if $x < -1$

$\Rightarrow (x-1) < 0$; $(x+1) < 0$

$\therefore$ ① $\equiv h'(x) > 0$

$\therefore h(x)$ is increasing in $(-\infty, -1)$.

if $-1 < x < 1 \Rightarrow (x-1) < 0$, $(x+1) > 0$

$\therefore$ ① $\equiv h'(x) < 0$

$\therefore h(x)$ is decreasing in $(-1, 1)$.

if $x > 1 \Rightarrow (x-1) > 0$; $(x+1) > 0$.

$\therefore$ ① $\equiv h'(x) > 0$

$\therefore h(x)$ is increasing in $(1, \infty)$.

$\therefore$ In $(-\infty, -1) \cup (1, \infty)$,

$h(x)$ is increasing.

and in $(-1, 1)$, $h(x)$ is decreasing.

* L' Hospital's Rules *

Indeterminate Forms :-

If $A = \underset{x\to c}{Lt} f(x)$ and $B = \underset{x\to c}{Lt} g(x)$.

and if $B \neq 0$ then $\underset{x\to c}{Lt} \frac{f(x)}{g(x)} = \frac{A}{B}$

However, if $B=0$ then it has no conclusion.

- If $B=0$ and $A\neq 0$ then the limit is infinite. (when it exists).
- If $A=0$ & $B=0$ then $\underset{x\to c}{Lt} \frac{f(x)}{g(x)}$ is said to be indeterminate form.

Ex:- (1) If α is any real number and if we define $f(x) = \alpha x$ and $g(x) = x$ then

$$\underset{x\to 0}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 0}{Lt} \frac{\alpha x}{x} \quad \Big| \frac{0}{0} \text{ form}$$

$$= \underset{x\to 0}{Lt} (\alpha)$$

$$= \alpha$$

Ex:-(2): If $f(x) = x^2-1$ and $g(x) = x-1$ with $a=1$

then we have

$$\underset{x\to a}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 1}{Lt} \frac{x^2-1}{x-1} \quad \Big| \frac{0}{0} \text{ form}$$

$$= \underset{x\to 1}{Lt} (x+1)$$

$$= 2.$$

→ other indeterminate form are represented by the symbols $\frac{\infty}{\infty}, 0\cdot\infty, 0^0, 1^\infty, \infty^0$ and $\infty-\infty$.

→ our attention will be focused on the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

The other indeterminate cases are usually reduced to the form $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$ by taking logarithms, exponentials, or algebraic manipulations.

* we first establish an elementary result that is based simply on the definition of the derivative.

Theorem :-

Let f and g be defined on $[a,b]$ let $f(a) = g(a) = 0$ and $g(x) \neq 0$ for $x \in (a,b)$ (i.e. $a<x<b$). If f and g are differentiable at a and if $g'(a) \neq 0$. then the limit of $\frac{f}{g}$ at a exists and is equal to $\frac{f'(a)}{g'(a)}$

i.e $\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

* Working Rule for finding the Value of $\underset{x\to a}{Lt} \frac{f(x)}{g(x)}$:

where $f(a) = 0 = g(a)$.

(1) Differentiate the numerator and denominator separately.

(2) Put $x=a$ and remove the word limit.

3) If the indeterminate form $\frac{0}{0}$ still persists, repeat the above process.

Problems :-

* Evaluate the following limits:

— $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$.

Sol^n: $\lim_{x\to 0} \frac{(1+x)^n-1}{x}$ $\Big|\frac{0}{0}$ form

$= \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}$ (By differentiate numerator & denominator separately)

$= n$

$\xrightarrow{H.W} \lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

$\xrightarrow{H.W} \lim_{x\to 0} \frac{x-\sin x}{x^3}$

$\xrightarrow{H.W} \lim_{x\to 0} \frac{\log(1-x^2)}{\log\cos x}$

$\longrightarrow \lim_{x\to 1} \left(\frac{x^x - x}{1-x+\log x}\right)$

Sol^n: Let $u = x^x$

then $\log u = x\log x$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x$

$\Rightarrow \frac{du}{dx} = u(1+\log x)$

$= x^x(1+\log x)$

Now $\lim_{x\to 1}\left(\frac{x^x - x}{1-x+\log x}\right)$ $\Big|$ form $\frac{0}{0}$

$= \lim_{x\to 1} \frac{x^x(1+\log x) - 1}{-1+\frac{1}{x}}$ $\Big|$ form $\frac{0}{0}$

$= \lim_{x\to 1} \frac{x^x\left(\frac{1}{x}\right) + x^x(1+\log x)(1+\log x)}{\frac{-1}{x^2}}$

$= \frac{1^1\left(\frac{1}{1}\right) + 1^1(1+\log 1)^2}{\frac{-1}{1^2}}$

$= \frac{1+1(1+0)^2}{-1}$

$= \frac{1+1}{-1} = -2$

* L' Hospital's Rule - I :-

Let $-\infty \le a < b \le \infty$ and let f, g be differentiable on (a,b) such that $g'(x) \neq 0$ $\forall x \in (a,b)$

Suppose that

$\lim_{x\to a+} f(x) = 0 = \lim_{x\to a+} g(x)$

(a) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ then $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$

(b) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ then

$\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

Problems :

Evaluate the following limits.

→ (a) $\lim_{x\to 0} \frac{e^x - 2\cos x + e^{-x}}{x\sin x}$

(b) $\lim_{x\to 0} \frac{\tan x - x}{x^2 \tan x}$

(c) $\lim_{x\to a} \frac{x^a - a^x}{x^x - a^a}$

(d) $\lim_{x\to 0} \dfrac{\cosh x - \cos x}{x \sin x}$

(e) $\lim_{x\to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$

(f) $\lim_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

Sol^n (b): $\lim_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$ $\Big|\ \frac{0}{0}$ form

$= \lim_{x\to 0} \dfrac{\tan x - x}{x^3} \cdot \dfrac{x}{\tan x}$

$= \lim_{x\to 0} \dfrac{\tan x - x}{x^3} \lim_{x\to 0} \dfrac{x}{\tan x}$

$= \lim_{x\to 0} \dfrac{\tan x - x}{x^3} (1)$

$= \lim_{x\to 0} \dfrac{\tan x - x}{x^3}$ $\Big|\ \frac{0}{0}$ form

$= \lim_{x\to 0} \dfrac{\sec^2 x - 1}{3x^2}$ $\Big|\ \frac{0}{0}$ form

$= \lim_{x\to 0} \dfrac{2\sec x \sec x \tan x}{6x}$

$= \lim_{x\to 0} \dfrac{2\sec^2 x}{6} \cdot \lim_{x\to 0} \dfrac{\tan x}{x}$

$= \dfrac{2(1)}{6} \cdot (1)$

$= 1/3$

→ (c) $\lim_{x\to a} \dfrac{x^a - a^x}{x^x - a^a}$

Let $u = x^x$ then $\log u = x\log x$

$\Rightarrow \dfrac{1}{u}\dfrac{du}{dx} = x\dfrac{1}{x} + \log x$

$\Rightarrow \dfrac{du}{dx} = x^x(1+\log x)$

$\lim_{x\to a} \left(\dfrac{x^a - a^x}{x^x - a^a}\right)$

$= \lim_{x\to a} \dfrac{a x^{a-1} - a^x \log a}{x^x(1+\log a) - 0}$

$= \dfrac{a\cdot a^{a-1} - a^a \log a}{a^a(1+\log a)}$

$= \dfrac{a\cdot a^{-1} - \log a}{1+\log a}$

$= \dfrac{1-\log a}{1+\log a}$

→ what is wrong with the following application of L'Hospital rule:

$\lim_{x\to 1} \dfrac{x^3+3x-4}{2x^2+x-3} = \lim_{x\to 1} \dfrac{3x^2+3}{4x+1} = \lim_{x\to 1} \dfrac{6x}{4}$

$= 3/2$

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Sol^n: $\lim_{x\to 1} \dfrac{x^3+3x-4}{2x^2+x-3}$ $\Big|\ \frac{0}{0}$ form

$= \lim_{x\to 1} \dfrac{3x^2+3}{4x+1}$

Now the expression $\dfrac{3x^2+3}{4x+1}$ is not of the form $\frac{0}{0}$ as $x\to 1$

∴ It is not correct to apply L'Hospital's Rule to evaluate $\lim_{x\to 1} \dfrac{3x^2+3}{4x+1}$

∴ $\lim_{x\to 1} \dfrac{3x^2+3}{4x+1} = \dfrac{3(1)+3}{4(1)+1} = 6/5.$

→ what is wrong with the following use of L'Hospital's Rule:

$\lim_{x\to 1} \dfrac{x^4-4x^3+3}{3x^2-x-2} = \lim_{x\to 1} \dfrac{4x^3-12x^2}{6x-1}$

$$= \lim_{x\to 1} \frac{12x^2-24x}{6}$$

$$= -2$$

→ For what value of 'a' does $\frac{\sin 2x + a\sin x}{x^2}$ tend to a finite limit l as $x\to 0$? when 'a' has this value, what is the value of l?

Sol'n: $\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ | form $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} \quad \text{— (i)}$$

The denominator (i) → 0 as $x\to 0$

but (i) → a a finite limit l.

∴ The numerator $(2\cos 2x + a\cos x)$ must be tend to zero as $x\to 0$.

∴ $2\cos(0) + a\cos(0) = 0$

⟹ $2+a=0$

⟹ $\boxed{a=-2}$

with this value of a

$$\text{(i)} \equiv \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \Big|\ \tfrac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \Big|\ \tfrac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8(1)+2(1)}{6} = -6/6 = -1$$

$$\therefore l = -1$$

→ Find the values of 'a' and 'b' in order that $\lim_{x\to 0} \frac{x(1-a\cos x)+b\sin x}{x^3}$ may be equal to $1/3$.

Sol'n: $\lim_{x\to 0} \frac{x(1-a\cos x)+b\sin x}{x^3}$ | form $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{x(a\sin x) + (1-a\cos x) + b\cos x}{3x^2} \quad \text{— (1)}$$

The denominator of (1) → 0 as $x\to 0$

but (1) → $1/3$ as $x\to 0$

∴ The numerator of (1).

$x(a\sin x) + (1-a\cos x) + b\cos x$ tends to zero as $x\to 0$.

⟹ $0(0) + (1-a(1)) + b(1) = 0$

⟹ $1-a+b=0$ —— (2)

If the relation (2) holds then from (1)

$$\lim_{x\to 0} \frac{ax\sin x + (1-a\cos x) + b\cos x}{3x^2}$$

(is of the form $\frac{0}{0}$)

$$= \lim_{x\to 0} \frac{a\sin x + ax\cos x + a\sin x - b\sin x}{6x} \quad \Big|\ \text{form } \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{a\cos x + a\cos x - ax\sin x + a\cos x - b\cos x}{6}$$

$$= \frac{a(1)+a(1)-a(0)+a(1)-b(1)}{6}$$

$$= \frac{3a-b}{6}$$

but the limit of (1) equal to $1/3$ (given)

$$\therefore \frac{3a-b}{6} = \frac{1}{3}$$

$\Rightarrow 3a-b=2$ —— (3)

From (2) & (3), we get

$a=\frac{1}{2}, \; b=-\frac{1}{2}$

H.W. Find the values p and q for which $\lim_{x\to 0}\frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

2006 P-I Find the values of a and b such that

$$\lim_{x\to 0}\frac{a\sin^2 x+b\log\cos x}{x^4}=\frac{1}{2}$$

Sol^n: $\lim_{x\to 0}\frac{a\sin^2x+b\log\cos x}{x^4}$ | form $\frac{0}{0}$

$$=\lim_{x\to 0}\frac{a(2\sin x\cos x)+\frac{b}{\cos x}(-\sin x)}{4x^3}$$

$$=\lim_{x\to 0}\frac{a\sin 2x-b\tan x}{4x^3} \quad \Big|\frac{0}{0}\text{ form}$$

$$=\lim_{x\to 0}\frac{2a\cos 2x-b\sec^2 x}{12x^2} \quad \text{—— (1)}$$

The denominator of (1) $\to 0$ as $x\to 0$.

but (1) $\to$ a finite limit value $\frac{1}{2}$

$\therefore$ The numerator of (1) must be zero as $x\to 0$.

$\therefore$ (1) $\equiv 2a\cos(0)-b\sec^2(0)=0$

$\Rightarrow \boxed{2a-b=0}$ —— (2)

with this form (1).

$$\lim_{x\to 0}\frac{2a\cos 2x-b\sec^2x}{12x^2} \quad \Big|\frac{0}{0}\text{ form.}$$

$$=\lim_{x\to 0}\frac{-4a\sin 2x-b[2\sec^2x\tan x]}{24x} \quad \Big|\frac{0}{0}\text{ form}$$

$$=\lim_{x\to 0}\left[\frac{-4a\sin 2x}{24x}-\frac{2b\sec^2x\tan x}{24x}\right]$$

$$=\frac{-a}{3}\lim_{2x\to 0}\frac{\sin 2x}{2x}-\frac{b}{12}\lim_{x\to 0}\sec^2x\lim_{x\to 0}\frac{\tan x}{x}$$

$$=\frac{-a}{3}(1)-\frac{b}{12}(1)(1)$$

$$=\frac{-a}{3}-\frac{b}{12}=\frac{-4a-b}{12}$$

but limit of (1) is equal to $\frac{1}{2}$

$\therefore \frac{-4a-b}{12}=\frac{1}{2}$

$\Rightarrow -4a-b=6$ —— (3)

(2) $-$ (3) $\equiv 6a=6$

$\Rightarrow \boxed{a=1}$

(2) $\equiv 2(1)-b=0$

$\Rightarrow 2=b$

$\Rightarrow \boxed{b=2}$

$\therefore a=1;\; b=2$

*L' Hospital's Rule - 2:

Let $-\infty\le a<b\le\infty$ and let f, g be differentiable on (a,b) such that $g'(x)\neq 0\;\forall x\in(a,b)$

Suppose that $\lim_{x\to a+}g(x)=\pm\infty$.

ⓐ If $\underset{x\to a+}{Lt} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$ then ..

$$\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L$$

ⓑ If $\underset{x\to a+}{Lt} \dfrac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then

$$\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L.$$

Note: In most of the Problems of the form $\frac{\infty}{\infty}$, it is necessary to change it into the form $\frac{0}{0}$ at the proper stage, otherwise the process will never end.

Problems:

Evaluate the following limits:

ⓐ $\underset{x\to 0}{Lt} \dfrac{\log x^2}{\cot x^2}$

Sol^n:

$= \underset{x\to 0}{Lt} \dfrac{2\log x}{\cot x^2}$ | form $\frac{\infty}{\infty}$ $\because \log 0 = -\infty$ & $\cot 0 = \infty$

$= \underset{x\to 0}{Lt} \dfrac{2\frac{1}{x}}{(-\operatorname{cosec}^2 x^2)(2x)}$

$= \underset{x\to 0}{Lt} \dfrac{-\cancel{2}}{\cancel{2}x^2 \operatorname{cosec}^2 x^2}$

$= \underset{x\to 0}{Lt} \left(\dfrac{-\sin^2(x^2)}{x^2}\right)$ | $\frac{0}{0}$ form

$= \underset{x\to 0}{Lt} \dfrac{-2\sin^2 x^2 \cdot \cos x^2 \cdot (\cancel{2x})}{\cancel{2x}}$

$= \underset{x\to 0}{Lt} -\sin(2x^2)$

$= 0$

→ $\underset{\theta\to \pi/2}{Lt} \dfrac{\log(\theta - \pi/2)}{\tan\theta}$

→ $\underset{x\to 0}{Lt} \dfrac{\operatorname{cosec} x}{\log x}$

→ $\underset{x\to \pi/2}{Lt} \dfrac{\tan 5x}{\tan x}$

→ $\underset{x\to 0+}{Lt} \dfrac{\log(\tan x)}{\log x}$

Sol^n: $\underset{x\to 0+}{Lt} \dfrac{\log(\tan x)}{\log x}$ | $\frac{\infty}{\infty}$ form.

$= \underset{x\to 0+}{Lt} \left[\dfrac{\frac{1}{\tan x}\sec^2 x}{1/x}\right]$

$= \underset{x\to 0+}{Lt} \dfrac{x}{\sin x \cos x}$

$= \underset{x\to 0+}{Lt} \dfrac{2x}{\sin 2x}$ | $\frac{0}{0}$ form

$= \underset{x\to 0+}{Lt} \dfrac{2}{2\cos 2x}$

$= \dfrac{2}{2(1)} = 1$

* Other Indeterminate forms:

The indeterminate forms $\infty-\infty$, $0\times\infty$, 1^{∞}, 0^0, ∞^0 can be reduced to any one of the two indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic manipulations and the exponential functions.

This is illustrated by the following examples.

Form $\infty-\infty$

$\rightarrow \lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

Sol'n: $\lim_{x\to 0+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ | $\infty-\infty$ form

$= \lim_{x\to 0+}\frac{\sin x - x}{x\sin x}$ | form $\frac{0}{0}$

$= \lim_{x\to 0+}\frac{\cos x - 1}{x\cos x+\sin x}$ | form $\frac{0}{0}$

$= \lim_{x\to 1+}\frac{\sin x}{-x\sin x+2\cos x}$

$= \frac{0}{2} = 0$

H.W. $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\tan x}\right)$

H.W. $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

H.W. $\lim_{x\to \pi/2}(\sec x-\tan x)$

H.W. $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]$

$\rightarrow \lim_{x\to 0}\left[\frac{\pi}{4x}-\frac{\pi}{2x(e^{\pi x}+1)}\right]$

$\rightarrow \lim_{x\to 1}\left[\frac{2}{x^2-1}-\frac{1}{x-1}\right]$

$\rightarrow \lim_{x\to 4}\left[\frac{1}{\log(x-3)}-\frac{1}{x-4}\right]$

$\rightarrow \lim_{x\to 0}\left[\frac{1}{e^x-1}-\frac{1}{x}\right]$

$\rightarrow \lim_{x\to 0}\left[\cot^2 x-\frac{1}{x^2}\right]$

Sol'n:– $\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$ | $\infty-\infty$ form

$= \lim_{x\to 0}\left(\frac{\cos^2 x}{\sin^2 x}-\frac{1}{x^2}\right)$

$= \lim_{x\to 0}\left(\frac{x^2\cos^2 x-\sin^2 x}{x^2\sin^2 x}\right)$

$= \lim_{x\to 0}\frac{x^2\cos^2 x-\sin^2 x}{x^4}\cdot\frac{x^2}{\sin^2 x}$

$= \lim_{x\to 0}\frac{x^2\cos^2 x-\sin^2 x}{x^4}\left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2$

$= \lim_{x\to 0}\frac{x^2\cos^2 x-\sin^2 x}{x^4}\cdot(1)^2$

$= \lim_{x\to 0}\frac{x^2\left(\frac{1+\cos 2x}{2}\right)-\left(\frac{1-\cos 2x}{2}\right)}{x^4}$

$= \lim_{x\to 0}\frac{x^2(1+\cos 2x)-(1-\cos 2x)}{2x^4}$

$= \lim_{x\to 0}\frac{(x^2-1)+(x^2+1)\cos 2x}{2x^4}$ | $\frac{0}{0}$ form

$$= \lim_{x\to 0} \frac{2x - (x^2+1)\left(\frac{\sin 2x}{2}\right) + 2x\cos 2x}{8x^3}$$

$$= \lim_{x\to 0} \left[\frac{2 + 2\cos 2x - 4x\sin 2x - 4x\sin 2x - 4(x^2+1)\cos 2x}{24x^2}\right]$$

$\left|\frac{0}{0}\text{ form}\right.$

$$= \lim_{x\to 0} \left[\frac{-8\sin 2x - 16x\cos 2x - 8x\cos 2x + 2(4x^2+2)\sin 2x}{48x}\right]$$

$$= \lim_{x\to 0} \frac{-24x\cos 2x + (8x^2-4)\sin 2x}{48x} \quad \left|\text{form } \frac{0}{0}\right.$$

$$= \lim_{x\to 0} \frac{-24\cos 2x + 48x\sin 2x + 16x\sin 2x + 2(8x^2-4)\cos 2x}{48}$$

$$= \frac{-24+0+0-8}{48} = \frac{-32}{48} = -2/3.$$

Form $0\times\infty$:

→ Evaluate the following limits:

→ $\lim_{x\to 0+} x\ln x$

Sol'n $\lim_{x\to 0+} x\ln x \quad |\ 0\times\infty\text{-form}$

$= \lim_{x\to 0+} \frac{\log x}{1/x} \quad \left|\ \frac{\infty}{\infty}\text{-form}\right.$

$= \lim_{x\to 0+} \frac{1/x}{-1/x^2}$

$= \lim_{x\to 0+} \frac{-x^2}{x}$

$= \lim_{x\to 0+} (-x)$

$= 0$

→ $\lim_{x\to 0} x^3 \ln x$

→ $\lim_{x\to 0} \frac{1}{x(\log x)^2}$

Sol'n: $\lim_{x\to 0} \frac{1}{x(\log x)^2} \quad |\ 0\times\infty$ form

$= \lim_{x\to 0} \frac{1/x}{(\log x)^2} \quad \left|\ \frac{\infty}{\infty}\text{ form}\right.$

$= \lim_{x\to 0} \left[\frac{-1/x^2}{2(\log x)\,1/x}\right]$

$= \lim_{x\to 0} \frac{-1/x}{2\log x} \quad \left|\ \frac{\infty}{\infty}\text{ form}\right.$

$= \lim_{x\to 0} \frac{1/x^2}{2(1/x)}$

$= \lim_{x\to 0} \frac{1}{2x}$

$= \infty$

→ $\lim_{x\to\infty} \frac{x^3}{e^x} = l.$

Forms : $0^0,\ 1^\infty,\ \infty^\infty$

→ $\lim_{x\to 0+} x^x = ?$

Sol'n: Let $\lim_{x\to 0} x^x = l \quad |\ 0^0$ form

$\Rightarrow \log\left[\lim_{x\to 0} x^x\right] = \log l$

$\Rightarrow \lim_{x\to 0+}\left[\log(x^x)\right] = \log l$

$\Rightarrow \log l = \lim_{x\to 0+}\left[x\log x\right] \quad |\ 0\times\infty$ form

$= \lim_{x\to 0+} \frac{\log x}{1/x} \quad \left|\ \frac{\infty}{\infty}\text{ form}\right.$

$$= \lim_{x\to 0+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0+} \left(\frac{-x^2}{x}\right)$$

$$= \lim_{x\to 0+} (-x) = 0$$

$\therefore \log l = 0$

$\Rightarrow l = e^0$

$\Rightarrow \boxed{l=1}$

$\rightarrow \lim_{x\to\infty} (1+1/x)^x = ?$

Sol'n: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ | ∞^∞ form

Let $l = \lim_{x\to\infty} (1+1/x)^x$

$$\log l = \lim_{x\to\infty} [x\log(1+1/x)]$$

| $0\times\infty$ form

$$= \lim_{x\to\infty} \frac{\log(1+1/x)}{1/x}$$ | $\frac{0}{0}$ form

$$= \lim_{x\to\infty} \left[\frac{\frac{1}{1+1/x}\left(-\cancel{1/x^2}\right)}{-\cancel{1/x^2}}\right]$$

$$= \lim_{x\to\infty} \frac{x}{1+x}$$ | $\frac{\infty}{\infty}$ form

$$= \lim_{x\to\infty} \frac{1}{1} = 1$$

$\therefore \log l = 1$

$\Rightarrow \boxed{l = e^1}$

$\rightarrow \lim_{x\to 0+} \left(1+\frac{1}{x}\right)^x = ?$

Sol'n: $\lim_{x\to 0+} (1+1/x)^x$ | ∞^0 form

Let $l = \lim_{x\to 0+} (1+1/x)^x$

$$\Rightarrow \log l = \lim_{x\to 0+} \left[\log(1+1/x)^x\right]$$

$$= \lim_{x\to 0+} \left[x\log(1+1/x)\right]$$

| $0\times\infty$ form

$$= \lim_{x\to 0+} \left[\frac{\log(1+1/x)}{1/x}\right]$$

$$= \lim_{x\to 0+} \left[\frac{\frac{1}{1+1/x}\cdot\frac{-1}{\cancel{x^2}}}{-1/\cancel{x^2}}\right]$$

$$= \left(\frac{1}{1+\frac{1}{0}}\right)$$

$$= 0$$

$\rightarrow$ (a) $\lim_{x\to\infty} \frac{\ln x}{x^2}$ $(0,\infty)$

(b) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ $(0,\infty)$

(c) $\lim_{x\to 0} x\ln\sin x$ $(0,\pi)$

(d) $\lim_{x\to\infty} \frac{x+\ln x}{x\ln x}$ $(0,\infty)$

$\rightarrow$ (a) $\lim_{x\to 0+} x^{2x}$ $(0,\infty)$

(b) $\lim_{x\to 0} (1+3/x)^x$; $(0,\infty)$

(c) $\lim_{x\to\infty} (1+3/x)^x$; $(0,\infty)$

(a) $\underset{x\to\infty}{Lt}\ x^{1/x}$; $(0,\infty)$

(b) $\underset{x\to 0+}{Lt}\ (\sin x)^{x}$; $(0,\pi)$

(c) $\underset{x\to 0+}{Lt}\ x^{\sin x}$; $(0,\infty)$

(d) $\underset{x\to \pi/2-}{Lt}\ (\sec x-\tan x)$; $(0,\pi/2)$