Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
Yogesh Kumbhejkar
AIR 8 - CSE 2015

LINEAR PROGRAMING PROBLEM

① Convex Sets

A set S is convex set if
$\forall x_1, x_2 \in S$; if $x_3 = \lambda x_1 + (1-\lambda)x_2$
then $x_3 \in S$ for $0 < \lambda < 1$

Extreme corner point or vertex of convex set
A vertex pt. is a pt. which does not lie on a segment joining any 2 other points of the set.
So, whole boundary of circle has vertices.

② Hyperplane

In a space E^n, a hyperplane is a set satisfying condition
$$S = \{x \in E^n \mid c_1x_1 + c_2x_2 + \cdots + c_nx_n = d\}$$
So a line in 2D, plane in 3D & so on.

A hyperplane divides E^n in 2 half spaces

Open half spaces $S_1 = \{x \in E^n \mid c_1x_1 + \cdots + c_nx_n < d\}$
$S_2 = \{x \in E^n \mid c_1x_1 + \cdots + c_nx_n > d\}$

Closed half spaces $S_1 = \{x \in E^n \mid c_1x_1 + \cdots + c_nx_n \leq d\}$
$S_2 = \{x \in E^n \mid c_1x_1 + \cdots + c_nx_n \geq d\}$

③ Convex Combination

$x_1, x_2 \ldots, x_m \in E^n$ then $X = \sum_{i=1}^{m} \mu_i x_i$ is a convex combination where $\sum_{i=1}^{m} \mu_i = 1$

④ Convex Hull

Let A be a non-convex set. Then smallest convex set containing A is its convex hull. Obv. it is intersection of all convex sets containing A.

Page

(5) Convex Polyhedron

Convex hull of finite no. of points is convex polyhedron given by $S = \{ x = \sum \alpha_i x_i \,/\, \sum \alpha_i = 1 \}$

(6) Structure of LPP

We have decision variables $x_1 \ldots x_n$ whose optimal value has to be determined.

Then we have constraints who could be inequalities or equalities.

We keep all deision variables as non-negative.

& We have objective function whose value has to be either minimized or maximized.

(7) Some Terminology

(a) Sol^n :- It satisfies all constraints of ~~in~~ LPP but may not satisfy $x_1, x_2 \ldots x_n \geq 0$

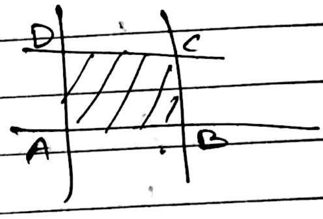
(b) Feasible Sol^n :- Satisfies all constraints including non-negativity

(c) Optimal feasible Sol^n :- It optimizes objective function

(d) Feasible Region :- Common region determined by all constraints including non-negativity

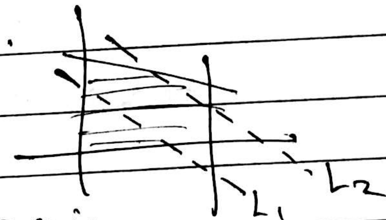
(8) GRAPHICAL METHOD

(a) Corner Point Method



(1) Draw the lines & find intersection Points. Shade the feasible region.
Find objective function value at all corner points A, B, C & D.

(b) (2) Iso-profit or Iso-cost method



Draw lines & Shade feasible region.
If objective function is
$a_1x + a_2y$, draw 2 lines
$a_1x + a_2y = c_1$ & $a_1x + a_2y = c_2$ in region.
Depending on where to minimize or maximize, move one of the lines up or down.
The point where it last touches feasible region, is your optimal point.

(9) UNBOUNDED PROBLEM

We have to maximize objective function & if feasible region is unbounded, we say this problem is unbounded & $Z \to \infty$. So optimal sol^n can't be found.

(10) UNFEASIBLE PROBLEM

If the constraints are such that $\nexists$ no feasible region, we say this problem is infeasible.

Scanned by CamScanner

Date
Page

(11) ALTERNATIVE OPTIMUM SOLUTION

An LPP having more than 1 optimal solution is possible when objective function line is parallel to binding constraint line.

(12) Redundant Constraint

When removing a constraint doesn't change feasible region, that is redundant constraint.

SIMPLEX METHOD

① Insight into SIMPLEX:

ⓐ We start with one of the vertices of feasible region & next vertex is adjacent to this vertex.

ⓑ But this next vertex is chosen only after considering that it will give better value of objective function.

ⓒ & Since there are finite vertices & we are going only to next better vertex, Simplex gives solution in finite number of steps.

② Slack & Surplus Variable

A constraint $a_1x_1 + a_2x_2 \leq b$ $\quad x_1, x_2 \geq 0$

is same as $a_1x_1 + a_2x_2 + \underbrace{S_1}_{\text{Slack Variable}} = b$ $\quad x_1, x_2, S_1 \geq 0$

A constraint $a_1x_1 + a_2x_2 \geq b$ $\quad x_1, x_2 \geq 0$

is same as $a_1x_1 + a_2x_1 - \underbrace{S_1}_{\text{Surplus variable}} = b$ $\quad x_1, x_2, S_1 \geq 0$

Now for each constraint containing surplus variable, we have to add an artificial variable so that we can have a 'feasible' basic solⁿ at the start.

$\therefore \quad a_1x_1 + a_2x_2 \geq b \Rightarrow a_1x_1 + a_2x_2 - S_1 + A_1 = b$

$x_1, x_2, S_1, A_1 \geq 0$

This way initial basic feasible solⁿ can have

$x_1 = 0, x_2 = 0, S_1 = 0$ & $A_1 = b$ where $b \geq 0$

Otherwise it would have been $S_1 = -b$ & $S_1 \geq 0$ unsatisfied.

Artificial variable is introduced even in an equality for initial basic feasible solution

$a_1x_1 + a_2x_2 = b \Rightarrow a_1x_1 + a_2x_2 + A_1 = b$

$x_1, x_2, A_1 \geq 0$

& initial basic feasible solⁿ has $x_1 = 0, x_2 = 0, A_1 = b$

classmate

Date
Page

③ Canonical Form

$$\text{Max } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t. } a_{i_1}x_1 + a_{i_2}x_2 + \dots + a_{in}x_n \leq b_i$$

$$i = 1, 2, \dots m$$

$$x_1, x_2 \dots x_n \geq 0$$

is known as canonical form.
No restriction on sign of b_i.

Standard Form

$$\text{Max. or minimize } Z = c_1x_1 + \dots + c_nx_n$$

$$\text{s.t. } a_{i_1}x_1 + a_{i_2}x_2 + \dots + a_{in}x_n = b_i$$

$$i = 1, 2 \dots m$$

$$x_1, x_2 \dots, x_n \geq 0$$

$$b_i \geq 0$$

So in cannonical form no constraint on b_i but maximize compulsory.
In standard form b_i is non-negative but obj. fn. can be max. or minimize.

Unrestricted Variable

An unrestricted variable x is shown as $x = x' - x''$ where $x', x'' \geq 0$ & thus fit into standard form.

(4) CONVERSION TO STD. FORM & INITIAL BASIC FEASIBLE SOL.

e.g. $\max \quad Z = 80x_1 + 60x_2$

s.t. $x_1 + 3x_2 \leq 25$

& $x_1 + x_2 = 10$

→ std. form

$\max \quad Z = 80x_1 + 60x_2 + 0S_1 - MA_1$

s.t. $x_1 + 3x_2 + S_1 + 0A_1 = 25$

$x_1 + x_2 + 0S_1 + A_1 = 10$

Ini. Basic Feasible Sol^n: $x_1 = 0, x_2 = 0$, $S_1 = 25, A_1 = 10$

$-MA_1$ is a large penalty here.

If the problem were minimization, it would be M.

(5) Matrix Form of LPP would be

Max. or Min $\quad Z = C^T X$

s.t. $\quad AX = B \quad \& \; X \geq 0$

(6) Initial Basic Feasible Solution.

Here we put all original decision variables to 0. Also surplus variables are 0. Slack & Artificial variables are equal to corresponding b_i which gives ini. basic feasible solution.

Whenever a decision variable appears only once, we can use that as Artificial Variable by adjusting its coefficient to 1

e.g. $x_1 + 3x_2 \leq 25$

$2x_1 + 2x_2 + 4x_3 \geq 40$

↓ $x_1 + 3x_2 + 0x_3 + S_1 + 0S_2 = 25$

$\frac{x_1}{2} + \frac{x_2}{2} + x_3 + 0S_1 - S_2 = 10$

So, x_3 is the artificial variable here & I.B.F.S. is

$x_1 = 0, x_2 = 0, x_3 = 10, S_1 = 25, S_2 = 0$

Date
Page

(7) BASIC SOLUTION

If we have a system of n variables & m equations $(n > m)$; then we can put $n-m$ variables equal to 0, & solve system with remaining m variables.

These m variables are called basis variables & in the matrix form; the columns of E^n are 'basic' corresponding to these variables

Obv. these vectors have to be independent for it to be a basic sol^n.

e.g. is $[1,0,1,0,0,0]^T$ a basic feasible sol^n of

$$x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$$
$$2x_1 + 3x_2 + 4x_3 + x_6 = 6$$

?

$\rightarrow$ corresponding vectors are $\begin{bmatrix}1\\2\end{bmatrix}$ & $\begin{bmatrix}2\\4\end{bmatrix}$ which are not linearly independent. $\therefore$ It is not a basic sol^n although it is feasible.

Degenerate :- If one or more basic variables are 0 in sol^n; it is degenerate.

O/w it is non-degenerate.

Total no. of basic $sol^n = {}^nC_m$

Some Question Types

① In examples of type, find all basic sol^n & check whether they are feasible or optimal, ~~kind of~~ A table is better option.

e.g. $2x_1 + x_2 + 4x_3 = 11$ Max. $x_1 + x_2 + 2x_3$
$3x_1 + x_2 + 5x_3 = 14$

→ 3 variable & 2 eq. ⇒ 2 basic variables

No.	Basic Variable	Non Basic Variable	Value of Basic	Is it Feasible	Is it Degenerate	Is it Optimal
1	x_1, x_2	x_3	$x_1 = 3, x_2 = 5$	Yes	No.	$Z = 8$

& so on. looks good :)

② 2011 IAS → For LPP max $Z = x_1 + 2x_2 + 3x_3$ s.t.

$$x_1 - x_2 + 3x_3 = 4$$
$$2x_1 + x_2 + x_3 = 6 \qquad x_1, x_2, x_3 \geq 0$$

$x_1 = 2, x_2 = 1, x_3 = 1$ is a feasible solution.
Reduce this sol^n to a basic feasible solution.

→ Problem is simple but just have to ensure $x_1, x_2, x_3 \geq 0$.

so, $2\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} + 1\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix}$ is given. —(1)

We will find lin. dependence bet^n 3 vectors $\lambda_1\begin{bmatrix}1\\2\end{bmatrix} + \lambda_2\begin{bmatrix}-1\\1\end{bmatrix} + \lambda_3\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

One set of sol^n would be $\lambda_1 = 4, \lambda_2 = -5, \lambda_3 = -3$

∴ $4\begin{bmatrix}1\\2\end{bmatrix} = 5\begin{bmatrix}-1\\1\end{bmatrix} + 3\begin{bmatrix}3\\1\end{bmatrix}$ putting this in ① gives

$$\frac{7}{2}\begin{bmatrix}-1\\1\end{bmatrix} + \frac{5}{2}\begin{bmatrix}3\\1\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4\\8\end{bmatrix}$$

∴ We have a basic sol^n which satisfies non-negativity.

SIMPLEX METHOD

(1) First convert given problem to std. problem by introducing slack/surplus/artificial variables as necessary.
It should be maximization problem & all b_is should be positive

(2) Find initial basic feasible solution. n variable & m equations means $n-m$ can be put to 0.
Put it in std. format table

	C_j	3	4	0	0	-M	b	θ
C_B	Basic variable	x_1	x_2	S_1	S_2	A_1		$\theta = b/a_j$ ← min +ve no. focussed on
	S_1							
	S_2							
	A_1	etc.						

$Z_j = \Sigma a_{ij} C_B$

$C_j = c_j - Z_j$

maximum +ve C_j is incoming variable

if all are 0, optimality attained

(3) Essentially C_j indicates how much improvement we can have if we bring in that particular variable. ∴ We choose maximum +ve no.
θ on other hand shows how much movement freedom we have given the constraints we have. −ve θ shows nothing to worry. +ve θ shows limit of movement. ∴ We choose min +ve θ to ensure all variables stay within limit.

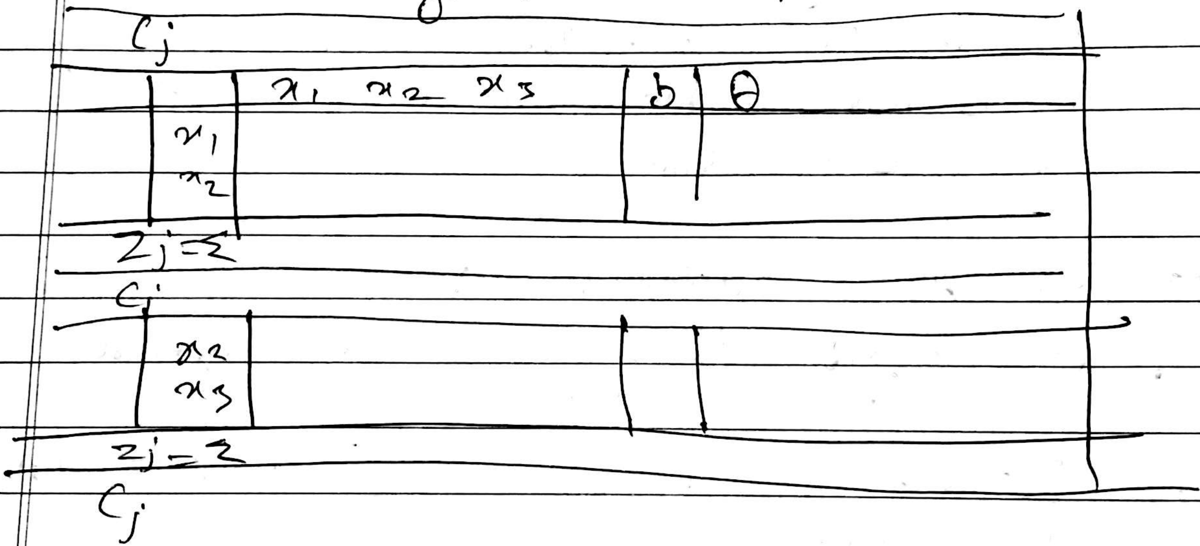
If some C_j is +ve & all θ are −ve ⇒ unbounded solution

(4)

(5)

(4) This also explains that when we see a non-basic variable with $C_j = 0$, it means if we make that variable as basic, objective function value won't change. So if optimal table has a non-basic variable C_j as 0; alternate optimal sol^n exists.
Then we make it incoming variable & find alternate optimal sol^n.

(5) If you don't have much time in exam, club 2-3 simplex tables in single table as follows :



Big-M-Method

(1) Artificial variable is introduced when constraint is $(\geq)$ or $(=)$ type. Then we change objective function by adding huge penalty $-MA_1$ to Z. Then following will happen in Solve simplex as usual.

case (i) Either A_i disappears or is present in basic variable with 0 value in b. Then we have optimal solution in both cases. Obv. latter case is degenerate sol^n.

case (ii) Optimality condition is satisfied & A_i is present with +ve value. Then feasible sol^n to original problem is not possible & what we have is pseudo-optimal ~~opt~~ solution.

Note when A_i is ousted from basic variable, we drop its column (from next simplex table) as it is not needed anymore.

So, $Z = 3x+2y$ becomes $Z^* = 3x+2y-MA_1$
& we solve table.

II PHASE METHOD

(1) In phase I, we try to force all Artificial Variables to 0. For that we max $Z^* = -A_1 - A_2 \cdots$

After reaching optimality in phase I, check below

case (i) If atleast one A_i is +ve, there exists no feasible solution to our problem.

(ii) Z^* is 0 but one Artificial variable exists Basic variable at 0. Here no need of phase 2! This is our optimal solⁿ.

(iii) Z^* is 0 & no artificial var. is basic, then d all A_i from table & move to phase II using same table. This time C_j; C_B will change as per original objective function Z.

DUALITY

(1) For dual finding. following std. structure is needed.
For dual finding. following std. structure is needed.

(a) Minimization should have all constraints ($\geq$).
no restriction of sign of b_i

(b) Maximization should have all constraints ($\leq$)
again no restriction on b_i sign.

Then say Max $Z = x_1 + 2x_2$ s.t. $2x_1 - 3x_2 \leq 4$ & $13x_1 + 4x_2 = 5$
$(x_1, x_2 \geq 0)$

becomes $\Rightarrow$ ($\because 13x_1 + 4x_2 = 5 \Rightarrow 13x_1 + 4x_2 \leq 5$ & $-13x_1 - 4x_2 \leq -5$)

$\therefore$ Min $Z' = 4y_1 + 5y_2 - 5y_3$ s.t. $2y_1 + 13y_2 - 13y_3 \geq 1$
& $-3y_1 + 4y_2 - 4y_3 \geq 2$

(obv. $y_1, y_2, y_3 \geq 0$)

If we have unrestricted variables, put $x = x' - x''$, $x' \geq 0$, $x'' \geq 0$
& again same transpose.

(2) The objective value function value is same for both duals. If primal has unbounded sol^n then dual has no feasible sol^n & vice versa.

(3) Dual promblem answer from primal simplex table.
As you convert a problem to dual, you can form a corrspondance in surplus / slack variable & original primal variable.

i.e. Max $Z = x_1 + 2x_2$ s.t. $3x_1 + 4x_2 \leq 5$ & $6x_1 + x_2 \leq 8$

Dual is Min $Z = 5y_1 + 8y_2$ s.t. $3y_1 + 6y_2 \geq 1$ & $4y_1 + y_2 \geq 2$

std. form is Max $Z' = -5y_1 - 8y_2$ s.t. $3y_1 + 6y_2 - S_1 + A_1 = 1$
& $4y_1 + y_2 - S_2 + A_2 = 2$

We can see x_1 corresponds to A_1 & x_2 to A_2 from coefficient.

So after we have optimality in this dual, feasible sol^n of primal will be Z_j corresponding to A_1, A_2. Take $\pm$ sign as needed.

Dual Simplex

1. Convert to maximization form & all constraints of ($\leq$) type. Don't care of b_i sign. So we will have only slack variables here & no surplus or artificial variable.

Essentially we change what we do for b_i (θ) & c_j here & also +ve/-ve signs.

(a) Optimality reached if → all $c_j \leq 0$ & all $b_i \geq 0$
(b) If any $c_j > 0$ → the method fails

(c) We first find outgoing variable here.
If all $c_j \leq 0$; & $b_i < 0$ the most negative b_i goes out

(d) How to find incoming? Obv θ calculation.
Take ratio of c_j with the above ~~cat~~ row. The ~~most~~ ~~posit~~ least positive ratio comes in.

(e) Keep on iterating till we get $c_j \leq 0$ & $b_i \geq 0$.
If ~~a selected be~~ all θ ratios are -ve, ∄ no feasible solution.

TRANSPORTATION PROBLEM

1. If we have m sources & n destinations, basic feasible sol^n will have m+n-1 variables +ve ~~at the most~~. (balanced prob.)
A (m,n) TP has m+n constraints & mn variables.

2. Initial Basic Feasible Solution.
(1) North-West Corner Method
Start with North-West Corner & assign as much value as possible then move right or below where you can assign max. value. With each move you will cross a row or column. Thus reach the end.

② Matrix Minima or Least Cost Method
Find cell with cheapest route, allot max value, then find next cell with cheapest value where some allocation can be done & so on.

③ Vogel's Approximation Method
ⓐ Find difference betⁿ min. & 2^{nd} min. costs in each row & column.
Where the diff. is highest allocate max. possible at min. cost cell.
If ∃ tie, choose least cost cell.
Recalculate diff. again for reduced table & iterate.

Vogel's gives best table generally with less cost.

④ Closed Chain / Open Chain
A given feasible solution is basic if it doesn't form a closed chain with non-zero cells.
Given ex. is not basic as it has a closed chain ⇒ cells are not linearly independent.

5	10	15
15	x	x
x	25	20

Above 3 methods always give a basic i.e. lin. independent solⁿ.

⑤ Degenerate B.F.S.
If IBFS has less than m+n−1 non-zero cells, then we find addition cell with zero allocation to have complete solⁿ.
Obv. Such cell is determined by ensuring it forms no closed chain with other non-zero guys.

⑥ UV method for transportation problem optimization.
① First find a non-degenerate IBFS.
Then assign dual variable u, v value to each row/col.
The row (or column) that contains most basic cells gets $u_i = 0$ (v_j) value; rest value determined by condition $\Delta_{ij} = u_i + v_j - c_{ij} = 0$

if we have degenerate soln i.e. occupied cells < m+n-1; we add zero allocation to a cell s.t. no closed chain is formed and we can also determine u, v properly.

Imp. to remember this.

for all basic cells.

(2) Then find Δ_{ij} value for all the cells. (net evaluations)
The cell with most +ve Δ_{ij} value is entering cell. (say r_n)

(3) Now with this new cell r_n form a closed chain. Assign $r_n = \theta$ & alternatively ~~+θ~~ subtract & add θ around chain. The cell who has $-\theta$ & smallest allocation goes.

(4) Again find the u & v values & iterate.
If $\Delta_{ij} \leq 0$ for all cells, optimality reached.

(5) Don't forget to calculate the cost after optimality.

Obviously this is all for Balanced problems. For nonbalanced problems we add dummy row or column & ~~put~~ cost = 0 for every new cell & carry on.

ASSIGNMENT PROBLEM

(i) Reduce given table by subtracting min. no. from each row & then each column. (This is HUNGARIAN METHOD)

(ii) Cover all zeroes with min. lines. If lines (r) = n i.e. order of matrix then we have optimal table

(iii) If r < n; pick min. no. from all uncovered cells & subtract it from all uncovered cells. Add it to intersection of covered lines.
This should give r = n eventually.

(iv) Now, examine rows till a row with only one zero is found. Circle that zero & cross all other zeroes in ~~that~~ its column. Similarly after rows do this for column.
If some zeroes still remain, do randomly for rest, its okay!

(v) Remember above is for minimization of cost. If you are manufacturer & want to maximize profit, take -ve of all costs in the table & carry on for minimization.

10	12	9	11
5	10	7	8
12	14	13	11
8	15	11	9

→

1	0	0	2
0	2	2	3
1	0	2	0
0	4	3	1

→

2	0	0	2
0	1	1	2
2	0	2	0
0	3	2	0

→

2	~~0~~	0	2
0	1	1	2
2	0	2	~~0~~
~~0~~	3	2	0