* Complex Integration And Cauchy's theorem *

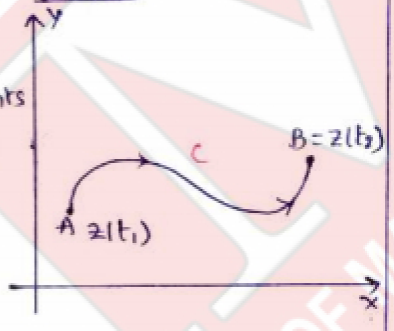
*** Continuous Curve or arc :-**

Let $\phi(t)$ and $\psi(t)$ be two real valued continuous functions of a real variable 't' in $t_1 \le t \le t_2$ then $z = x+iy = \phi(t) + i\psi(t) = z(t)$, $t_1 \le t \le t_2$ defines a continuous curve in the z-plane joining the points $z(t_1)$ and $z(t_2)$. Then the point $A: z(t_1)$ is called the initial point of the curve and $B: z(t_2)$ is called the terminal point of the curve.

— If B does not coincide with A, it is also called an arc.

→ **Simple Open Curve :-**

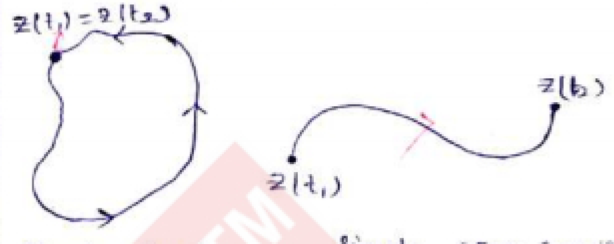
If the initial and terminal points of curve doesnot coincide & does not intersect itself. the curve is said to be simple open curve.

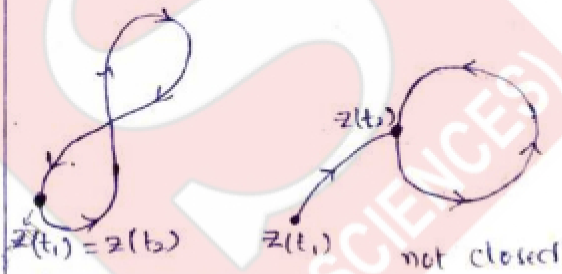


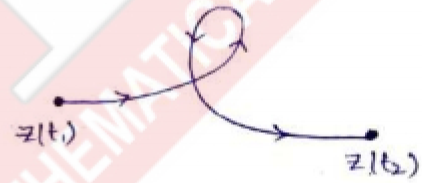
→ **closed curve :-** If the initial and terminal points of a curve 'c' coincide (i.e. $z(t_1) = z(t_2)$) then the given curve is called a closed curve

→ **Simple closed curve. :-**

A closed curve which doesnot intersect itself is called simple closed curve (Jordan curve).







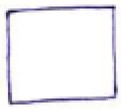
*** Smooth Curve :**

If $\phi(t)$ and $\psi(t)$ (i.e. $z(t)$) have continuous derivatives in $t_1 \le t \le t_2$, the curve is called a smooth curve (or) arc.

*** Piece-wise (or) Sectionally Smooth Curve (or) Contour :-**

A curve which is composed of a finite number of smooth arcs is called piecewise smooth (or) sectionally smooth curve (or) contour.

Ex:- Boundary of a square.



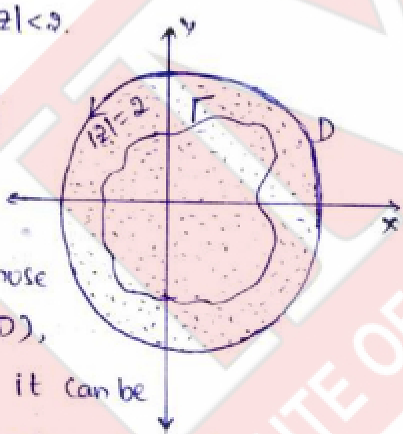
* Simply - Connected domain (Region):-

A domain D is said to be simply-connected if every simple closed curve contained in D contains only points of R inside. (or)

A domain D is called simply-connected if any simple closed curve which lies in D can be shrunk to a point without leaving D.

For example: Suppose D is the domain defined by $|z|<2$.

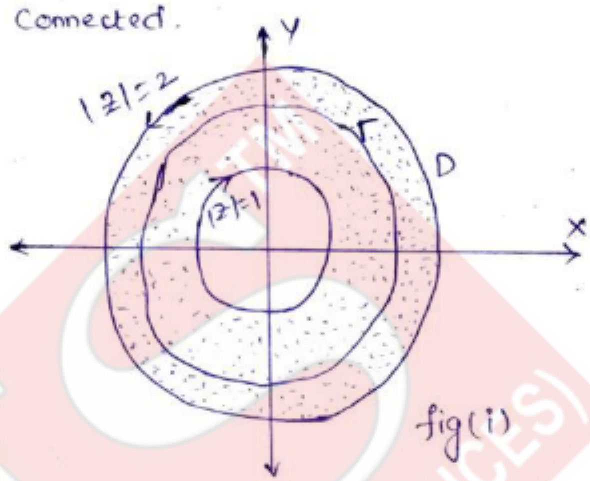
If Γ is any simple closed curve lying in D (i.e. whose points are in D), we see that it can be shrunk to a point which lies in D, and does not leave D, so that D is simply-connected.



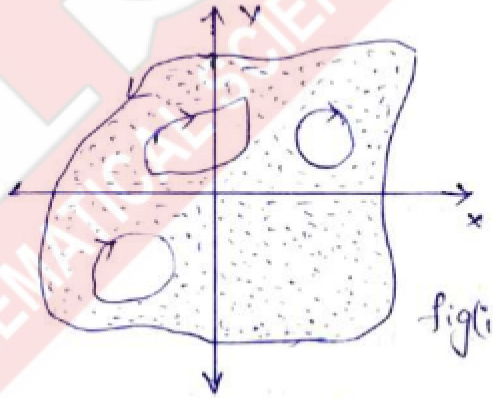
* Multiply - Connected :-

A domain which is not simply connected is called multiply-connected.

For example: If D is the domain defined by $1<|z|<2$ (shown shaded in figure (i)), then there is a simple closed curve Γ lying in D which can not possibly be shrunk to a point without leaving D, so that D is multiply-connected.



fig(i)



fig(ii)

Geometrically, a simply connected domain is one which does not have any 'holes' in it, while a multiply connected domain is one which does. thus the multiply-connected domains of figures (i) & (ii) have respectively one and three holes in them.

* Positive orientation :-

The boundary C of a domain is said to have positive orientation or to be traversed in the +ve sense, if a person walking on C always has the domain to his left.

Ex:- Let us observe the fig(i),

In the multiply-connected domain the outer boundary has +ve orientation, if traversed in counter clockwise, where as the inner boundary has +ve orientation if traversed clockwise as indicated in the figure (i).

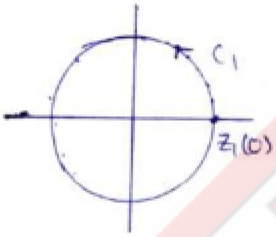
fig(i)

Ex ②:

$C_1 : z_1(t) = e^{it}$

$= \cos t + i \sin t;$

$0 \le t \le 2\pi$

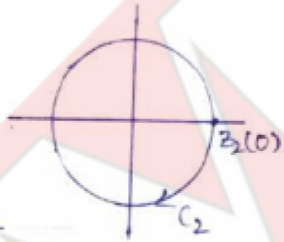


$C_2 : z_2(t) = e^{-it}$

$= \cos t - i \sin t;$

$0 \le t \le 2\pi$

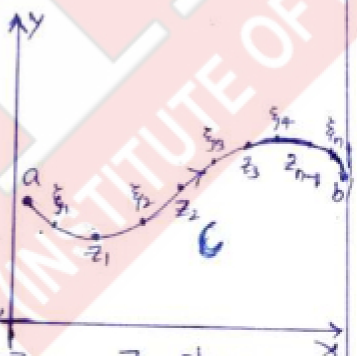
$z_1(t) = \phi(t) + i\psi(t)$, $y = \psi(t)$, $x = \phi(t)$, $t=0$: $x=1, y=0$



* Complex Line Integrals:

Let $f(z)$ be continuous at all points of a curve 'c' which we shall assume has a finite length.

Subdivide 'C' into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$.



On each arc joining z_{k-1} to z_k (where $k : 1 \to n$) choose a point ξ_k.

Form the sum

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1}).$$

On writing $z_k - z_{k-1} = \Delta z_k$,

this becomes $S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$

$$= \sum_{k=1}^{n} f(\xi_k)\, \Delta z_k.$$

Let the number of subdivisions 'n' increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum 'S_n' approaches a limit which does not depend on the mode of subdivision and we denote this limit by $\int_a^b f(z)\,dz$ (or) $\int_C f(z)\,dz$

This is called the complex line integral or line integral of $f(z)$ along a curve 'c'.

In such a case $f(z)$ is said to be integrable along 'c'.

*Note (1): If $f(z)$ is analytic at all points of a region R and if 'c' is a curve lying in R then $f(z)$ is certainly integrable along 'c'.

Note(2): Suppose that the smooth curve 'c' is parametrised by $z(t) = x(t) + iy(t)$, $a \le t \le b$.

then we can write

$$\int_C f(z)\,dz = \int_a^b f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

Note(3): In the case of a real function $f(x)$, the integral of $f(x)$, i.e. $\int_a^b f(x)\,dx$ depends upon the nature of the function and end points of the interval $[a,b]$. But in the case of complex functions, the value of $\int_a^b f(z)\,dz$ depends upon the nature of the function $f(z)$ and on the path 'C', joining the two points, not just the end points of 'C'.

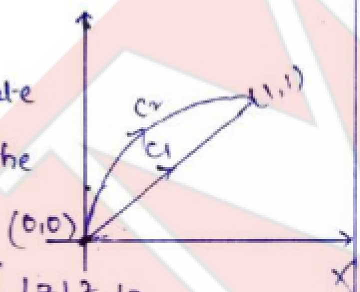
This can be observed from the following examples.

Example: Find $\int_C |z|^2 dz$ along the curves (a) $C = C_1 : z_1(t) = t+it$ $(0 \le t \le 1)$

(b) $C = C_2 : z_2(t) = t^2+it$ $(0 \le t \le 1)$

Sol^n: For these two curves the initial point $(0,0)$ and the terminal point is $(1,1)$.

Now let us evaluate the integral along the curves



(a) $\int_C f(z)\,dz = \int_C |z|^2 dz$

$= \int_{C_1} |z|^2 dz = \int_0^1 |t+it|^2 (1+i)\,dt$

$z = t+it$, $\phi(t) = x+iy$, $x = t$, $y = t$ $\psi(t)$

$\boxed{y = x}$

$\boxed{\begin{array}{l}\frac{dz}{dt} = 1+i \\ dz = (1+i)dt\end{array}}$

$= (1+i)\int_0^1 (t^2+t^2)\,dt$

$= (1+i)\left(\frac{2t^3}{3}\right)_0^1$

$= \frac{2}{3} + \frac{2}{3} i$

$\therefore$ In this case the path 'C_1' is the straight line joining $(0,0)$ and $(1,1)$.

(b) $\int_C f(z)\,dz = \int_{C_2} |z|^2 dz = \int_0^1 |t^2+it|^2 (2t+i)\,dt$

$= \int_0^1 2t\,(t^4+t^2)\,dt + i\int_0^1 (t^4+t^2)\,dt$

$= \frac{5}{6} + \frac{8}{15} i$

$\therefore$ In this case, the path C_2 is a parabola joining $(0,0)$ & $(1,1)$.

Now comparing these two, we get

$$\int_{C_1} |z|^2 dz \neq \int_{C_2} |z|^2 dz.$$

But this need not be the case in all cases.

Example (H-10)

verify that $\int_C z^2 dz$ along both the paths

(a) $C_1 : z(t) = t+it$, $(0 \le t \le 1)$

(b) $C_2 : z(t) = t^2+it$, $(0 \le t \le 1)$

is equal.

* Parametrization :-

In general the equations of curves in a plane will be in terms of variables x and y. then x, y are expressed in terms of a parameter.

Example (1)

Find the Parametrized curve tracing of the straight line segment from $z = i$ to $z = 1-i$.

Sol^n: In general the equation of the straight line interms of a parameter is given by $z(t) = (a+bt)+i(c+dt)$,

where a, b, c, d are determined by depending upon the initial and terminal points of the line segment and the interval of 't' which accounts the points on the given curve.

The first point $t = t_1$ is given by $z = i$.

$\therefore i = (a+bt_1)+i(c+dt_1)$

comparing the real and imaginary parts, we get

$a+bt_1 = 0$, $c+dt_1 = 1$

Similarly the second point $t = t_2$ is given as $z = 1-i$.

$\therefore 1-i = (a+bt_2)+i(c+dt_2)$

From this we get $a+bt_2 = 1$; $c+dt_2 = -1$.

$\therefore$ There are four equations in six unknowns a, b, c, d, t_1, t_2.

Now choose t_1, t_2 are arbitrarily

$\therefore$ let $t_1 = 0$, $t_2 = 1$.

Then solving the above equations we get $a = 0, c = 1, b = 1, d = -2$

$\therefore$ The parametric equation of the given line segment is

$$z(t) = t+i(1-2t); \quad 0 \le t \le 1.$$

Example (2): Find the parametric equation of the parabola $y = 2x^2-3$ that initial point $z = -1-i$ and the terminal point $z = 2+5i$

Sol^n: In this case the x is of second degree and y is of first degree. So we take the general parametric equation as

$$z(t) = (a+bt)+i(c+dt^2) \quad \text{——} \ (1)$$

Let $t \in [0,1]$. Then determine the values of a, b, c & d. In this case $(-1,-1)$ and $(2,5)$ are the initial & terminal points.

when $t = 0$, $a = -1$, $c = -1$

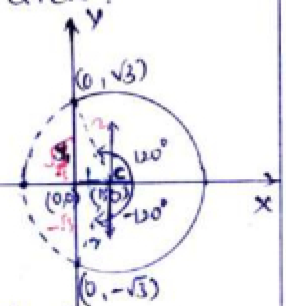
$t = 1$, $b = 3$, $d = 6$

$\therefore$ the Parametric equation of the given Parabola is

$$z(t) = (3t-1)+i(6t^2-1); \quad 0 \le t \le 1.$$

Example (3): The part of the circle $|z-1| = 2$ in the right half of the z-plane.

Sol^n:- The general parametric equation of a circle is $z(t) = z_0 + Re^{it}$, where z_0 is the centre of the circle, R the radius of the circle.



In the present case; Centre of the circle is at $(1,0)$ and $R=2$

$\therefore$ The parametric equation of the circle is

$$z(t) = 1 + 2e^{it}$$

$$= (1+2\cos t) + i2\sin t \quad \text{——— ①}$$

But in this case we require the circle of the right half of the z-plane.

For the first point $(0,-\sqrt{3})$ let us find.

on using ①, we get

$$0 + (-\sqrt{3})i = (1+2\cos t) + i2\sin t$$

$$\cos t = -\tfrac{1}{2}, \quad \sin t = -\tfrac{\sqrt{3}}{2}$$

$\therefore$ t satisfying the above two is $t = \frac{-2\pi}{3}$

Similarly let us consider the end point $(0,\sqrt{3})$.

then $\cos t = -\tfrac{1}{2}$, $\sin t = \sqrt{3}/2$

$\therefore$ Solving these two we get

$$t = 2\pi/3$$

$\therefore$ The required parametric equation is

$$z(t) = (1+2\cos t) + 2i\sin t,$$

$$-\frac{2\pi}{3} \le t \le \frac{2\pi}{3}.$$

Example (4):- Evaluate $\int_C x\,dz$ along the line segment 'C' from origin to $(1+i)$.

Sol^n: The parametric equation of the curve in this example is $z(t) = t + it$. $0 \le t \le 1$.

$\therefore$ $x(t) = t$, $y(t) = t$ & $dz = (1+i)dt$

$$\therefore \int_C x\,dz = \int_0^1 t(1+i)\,dt = (1+i)\left(\frac{t^2}{2}\right)_0^1$$

$$= \frac{1+i}{2}$$

$\rightarrow$ Evaluate $\int_C y\,dz$, $\int_C \bar{z}\,dz$ along the line segment 'C' from

(i) origin to $1+i$

(ii) origin to $1-i$.

Note:- Suppose $C: z = z(t)$ is a smooth curve defined on the interval $[a,b]$. Breaking this interval into two subintervals $[a,c]$ and $[c,b]$.

we obtain two curves C_1 & C_2 from $z(t)$ by restricting the parameter 't' to the intervals $[a,c]$ and $[c,b]$ respectively. For any function $f(z)$ continuous on 'C'

$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt.$$

$$= \int_a^c f(z(t))\,z'(t)\,dt + \int_c^b f(z(t))\,z'(t)\,dt$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

By the sum of two curves, we

mean the curve formed by joining the initial point of one curve to the terminal point of the other.

Similarly, the curve 'c' can be expressed as the "sum" of n curves with

$$\int_C f(z)\,dz = \int_{C_1+C_2+\cdots+C_n} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

Examples:

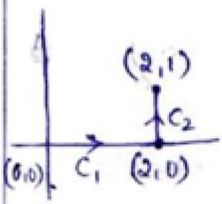
→ find $\int_C z\,dz$ along the contour.

$$C: z(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2+i(t-1), & 1 \le t \le 2 \end{cases}$$

Sol^n: Defining curves C_1 & C_2 by restricting the parameter 't' of 'c' to the intervals $[0,1]$ & $[1,2]$ respectively.

we have $\int_C z\,dz = \int_{C_1} z\,dz + \int_{C_2} z\,dz$.

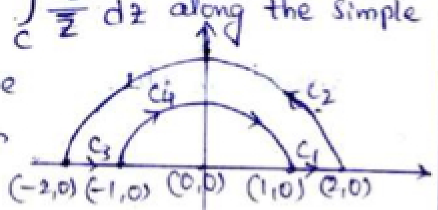
$$= \int_0^1 (2t)\,2dt + \int_1^2 [2+i(t-1)](i\,dt)$$



$$= \int_0^1 4t\,dt - \int_1^2 (t-1)\,dt + \int_1^2 2i\,dt$$

$$= 2 - \tfrac{1}{2} + 2i = \tfrac{3}{2} + 2i$$

→ Evaluate $\int_C \frac{z}{\bar{z}}\,dz$ along the simple closed curve 'c' as shown below.



Sol^n: The given contour 'C' is the sum of fours arcs C_1, C_2, C_3, C_4 as shown in figure.

The parametric equations of these curves are given by

$C_1: z: z(t) = t \;;\; 1 \le t \le 2$

$C_2: z: z(t) = 2(\cos t + i\sin t);\; 0 \le t \le \pi$

$C_3: z: z(t) = t \;;\; -2 \le t \le -1$

$C_4: z: z(t) = \cos t + i\sin t \;;\; \pi \le t \le 0.$

$$\therefore \int_C \frac{z}{\bar{z}}\,dz = \int_{C_1} \frac{z}{\bar{z}}\,dz + \int_{C_2} \frac{z}{\bar{z}}\,dz + \int_{C_3} \frac{z}{\bar{z}}\,dz + \int_{C_4} \frac{z}{\bar{z}}\,dz$$

$$= \int_1^2 \frac{t}{t}\,dt + \int_0^\pi \frac{2e^{it}}{2e^{-it}}\,2ie^{it}\,dt + \int_{-2}^{-1} \frac{t}{t}\,dt + \int_\pi^0 \frac{e^{it}}{e^{-it}}\,ie^{it}\,dt$$

$$= 2 + \tfrac{2}{3}(-1-1) + \tfrac{1}{3}(1+1)$$

$$= 4/3$$

Arc Length:-

The arc length L of a curve in the plane, defined parametrically by the equation $x = \phi(t)$, $y = \psi(t)$, $a \le t \le b$. is given by

$$L = \int_a^b \sqrt{(\phi'(t))^2 + (\psi'(t))^2}\,dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\int_a^b |z'(t)|\,dt = \int_a^b \left| \frac{dx}{dt} + i\frac{dy}{dt} \right| dt$$

Example: Find the length of the contour $z(t) = 3e^{2it}+2$; $-\pi \le t \le \pi$

Sol'n: Since $z(t) = 3e^{2it}+2$, $-\pi \le t \le \pi$

$$= [3\cos(2t)+2] + i\,[3\sin(2t)]$$

$\therefore \phi(t) = 3\cos(2t)+2$, $\psi(t) = 3\sin(2t)$

$\phi'(t) = -6\sin(2t)$ $\quad \psi'(t) = 6\cos(2t)$

Arc length

$$L = \int_{-\pi}^{\pi} \sqrt{36\sin^2(2t) + 36\cos^2(2t)}\,dt$$

$$= 6\int_{-\pi}^{\pi} dt = 12\pi$$

Note: If $f(z)$ is integrable along a curve 'c' having finite length 'L' and if there exists a +ve number M such that $|f(z)| \le M$ on 'c' then

$$\left|\int_c f(z)\,dz\right| < ML.$$

i.e. $\left|\int_c f(z)dz\right| \le \int_c |f(z)|\,|dz| \le M\int_c |dz| = ML$

Example: Find an upper bound for $\left|\int_c \frac{dz}{z^2+10}\right|$ where 'c' is the circle

$C: z(t) = 2e^{it}$, $(-\pi \le t \le \pi)$

Sol'n: $\left|\int_c \frac{dz}{z^2+10}\right| \le \int_c \frac{|dz|}{|z^2+10|} \le \int_c \frac{|dz|}{|10-|z|^2|}$

$\because z(t) = 2e^{it}$
$dz = 2ie^{it}dt$
$|dz| = 2dt$
$\int_c |dz| = \int_{-\pi}^{\pi} 2dt = 4\pi$

$$= \frac{1}{6}\int_c |dz|$$

$$= \frac{1}{6}(4\pi)$$

(or) Circle length is $2\pi r$
$\therefore$ Here $r=2$
$\therefore 4\pi$

$$= 2\pi/3$$

→ Evaluate $\int_c |z|\,dz$ along the straight line 'c' joining the origin to the point $1+i$.

Sol'n: we parametrize the line by

$C_1: z(t) = t+it$; $(0 \le t \le 1)$

$z'(t) = 1+i$

i.e. $dz = (1+i)dt$

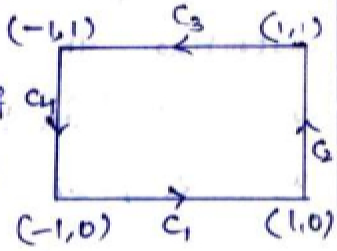
$$\therefore \int_c |z|\,dz = \int_0^1 |t+it|\,(1+i)\,dt$$

$$= \sqrt{2}\,(1+i)\int_0^1 t\,dt$$

$$= \frac{1+i}{\sqrt{2}}$$

→ Evaluate $\int_c |z|\,dz$ along the rectangle 'c' having corners $-1, 1, 1+i, -1+i$

Sol'n: Given contour is the sum of four smooth curves (straight lines)



we have $\int_c |z|\,dz = \int_{C_1} |z|\,dz + \int_{C_2} |z|\,dz + \int_{C_3} |z|\,dz + \int_{C_4} |z|\,dz$

where $C_1: z_1(t) = t$; $(-1 \le t \le 1)$

$C_2: z_2(t) = 1+it$; $(0 \le t \le 1)$

$C_3: z_3(t) = -1+i$; $(-1 \le t \le 1)$

$C_4: z_4(t) = -1-it$; $(-1 \le t \le 0)$

solving we obtain,

$$\int_C |z|dz = \int_{-1}^{1} |t|dt + i\int_{0}^{1}\sqrt{1+t^2}\,dt - \int_{-1}^{1}\sqrt{t^2+1}\,dt - i\int_{-1}^{0}\sqrt{1+t^2}\,dt$$

$$= \int_{-1}^{1}\left[|t| - \sqrt{t^2+1}\right]dt = 1-\sqrt{2}-\log(\sqrt{2}+1)$$

→ Evaluate $\int_C |z|dz$ along the circle having centre at the origin and radius 'r'.

Sol'n: We parametrize by

$C: z(t) = re^{it}$ $(0 \le t \le 2\pi)$,

then $\int_C |z|dz = \int_0^{2\pi} |re^{it}|\, ire^{it}dt$

$$= ir^2\int_0^{2\pi} e^{it}dt$$

$$= ir^2\int_0^{2\pi}(\cos t + i\sin t)dt$$

$$= 0.$$

Note: Since integrating around a circle is such a common occurence, we introduce the notation $\int_{|z-z_0|=r} f(z)dz$, which will be interpreted as the integral of $f(z)$ around the contour consisting of the circle $|z-z_0|=r$ oriented in the positive sense.

Ex: Evaluate $\int_{|z|=r} z^n dz$ for each integer n.

Sol'n: $\int_{|z|=r} z^n dz = \int_0^{2\pi}(re^{it})^n(ire^{it})dt$

$$= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}dt$$

$$= ir^{n+1}\int_0^{2\pi}[\cos(n+1)t + i\sin(n+1)t]dt$$

$$= \begin{cases} 0\ ; & \text{if } n \neq -1 \\ 2\pi i\ ; & \text{if } n = -1 \end{cases}$$

→ Evaluate $\int_C \frac{dz}{z-z_0}$ where C represents a circle $|z-z_0| = r$.

Sol'n: The parametric equation of C is

$$z = z_0 + re^{it}\ ;\ 0 \le t \le 2\pi$$

$$\therefore \int_C \frac{1}{z-z_0}dz = \int_0^{2\pi}(r^{-1}e^{-it})(rie^{it})dt \qquad \left(\because \frac{dz}{dt} = rie^{it}\right)$$

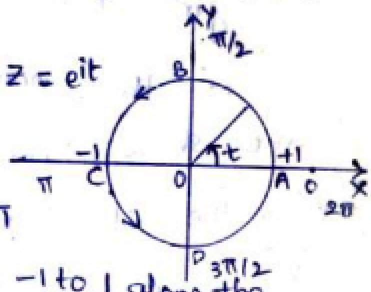
$$= \int_0^{2\pi} i\,dt = 2\pi i$$

(Q) → Prove that the value of the integral of $\frac{1}{z}$ along a semi-circular arc $|z| = 1$ from -1 to $+1$ is $-\pi i$ (or) πi according as the arc lies above (or) below the real axis.

Sol'n: The parametric equation of the circle $|z| = 1$ is $z = e^{it}$

$\Rightarrow \frac{dz}{dt} = ie^{it}$

where $0 \le t \le 2\pi$



As z moves from -1 to 1 along the upper semi-circular arc,

t varies from π to 0.

∴ In this case, we have

$$I_1 = \int_{CBA}\frac{dz}{z} = \int_{\pi}^{0}\frac{ie^{it}}{e^{it}}dt = i\int_{\pi}^{0}dt = -\pi i \quad \text{--- (1)}$$

Also if z moves along the lower semi circle from -1 to 1.

t varies from π to 2π,

we have

$$I_2 = \int_{CDA} \frac{1}{z}\,dz = \int_{\pi}^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_{\pi}^{2\pi} dt = \pi i$$

$$① \equiv \int_{CBA} \frac{1}{z}\,dz = -\pi i \Rightarrow \int_{ABC} \frac{1}{z}\,dz = \pi i \quad ——③$$

$\therefore$ from ② & ③, we have,

$$\int_{ABCDA} \frac{1}{z}\,dz = \int_{ABC} \frac{1}{z}\,dz + \int_{CDA} \frac{1}{z}\,dz$$

$$= \pi i + \pi i = 2\pi i$$

$\therefore$ The integral of $\frac{1}{z}$ round the entire circle $|z|=1$ is $2\pi i$.

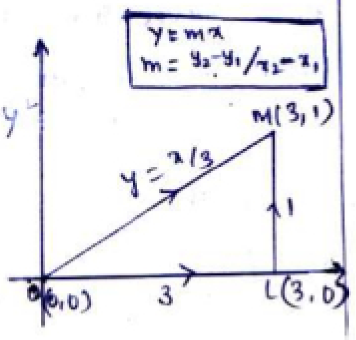
(II) Integrate z^2 along the straight line OM and along the path OLM consisting of two straight line segments OL and LM where O is the origin, L is the point $z=3$ and M the point $z=3+i$. Hence show that the integral of z^2 along the closed path OLMO is zero.

Sol^n: We have

$$z^2 = (x+iy)^2 = x^2-y^2+2ixy$$

on the line OM

$x = 3y \Rightarrow dx = 3dy$



Also y varies from 0 to 1 as z moves on OM from O to M.

$$\therefore I_1 = \int_{OM} z^2\,dz$$

$$= \int_{OM} (x^2-y^2+2ixy)(dx+idy) \quad (\because dz = dx+idy)$$

$$= \int_0^1 (9y^2-y^2+2i(3y)y)(3dy+idy)$$

$$= \int_0^1 (8y^2+6y^2 i)(3+i)dy$$

$$= \int_0^1 (8+6i)(3+i)y^2 dy$$

$$= 6 + \frac{26}{3}i \quad ——①$$

Now we consider the integral of z^2 along the path OLM.

Now we observe that on OL,

$y=0 \Rightarrow dy=0$

and along LM, $x=3 \Rightarrow dx=0$.

Also as z moves along OL from O to L.

x varies from 0 to 3

and z moves along LM from L to M,

y varies from 0 to 1.

$$\therefore I_2 = \int_{OLM} z^2 dz = \int_{OL} z^2 dz + \int_{LM} z^2 dz$$

$$= \int_{OL} (x^2-y^2+2ixy)(dx+idy) + \int_{LM} (x^2-y^2+2ixy)(dx+idy)$$

$$= \int_0^3 x^2 dx - \int_0^1 (9-y^2+6iy)\,i\,dy$$

$$= 6 + \frac{26}{3}i \quad ——②$$

$$\therefore I_1 = I_2$$

Since $\int_{OM} z^2 dz = 6 + \frac{26}{3}i$,

$$\Rightarrow \int_{MO} z^2 dz = -\left(6 + \frac{26i}{3}\right) \quad \text{--- (3)}$$

$$\therefore \int_{OLMO} z^2 dz = \int_{OLM} z^2 dz + \int_{MO} z^2 dz = 0$$

(by (2) & (3))

Note:

we have seen in the above two problems (I) & (II) that the integral of $\frac{1}{z}$ round the circle $|z| = 1$ is $2\pi i \neq 0$. whereas the integral of z^2 round the closed contour OLMO is zero. As a matter of fact the function $\frac{1}{z}$ is not analytic at $z=0$ which is an interior point (the centre) of the circle $|z|=1$ where as the function z^2 is analytic throughout the interior and the boundary of the triangle OLM. we shall prove a general result known as Cauchy's fundamental theorem which states

If $f(z)$ is analytic at all points within and on the closed contour C, then

$$\int_C f(z) dz = 0.$$

* Line Integrals :-

Now we adopt another method to evaluate a complex integral, in which we express the integral as a sum of two real line integrals, then they are evaluated. Eventhough this method is not a better method, but helps us in proving a theorem called Green's theorem, which in turn helps us in proving Cauchy's - weak theorem. This weak theorem tells us that $\int_C f(z)\,dz = 0$ when 'c' is closed curve and $f(z)$ is analytic with continuous derivative in a domain D.

As a consequence of this theorem it is proved that the value of the $\int_C f(z)\,dz$ depends upon the initial and terminal points of the curve 'c' when it is a simple open curve.

→ **Line Integral :-**

we know that

$f(z) = u(x,y) + iv(x,y)$.

Now let us consider the integral of this function $f(z)$ along a contour 'C', parametrised by $C: z = z(t) = x(t)+iy(t)$; $a \le t \le b$.

$$\therefore \int_C f(z)\,dz = \int_a^b f[z(t)]\, z'(t)\,dt$$

$$= \int_a^b \{u[z(t)] + iv[z(t)]\}\{x'(t)+iy'(t)\}dt$$

or simply

$$\int_C f(z)\,dz = \int_a^b (ux'-vy')\,dt + i\int_a^b (uy'+vx')\,dt$$

$$= \int_a^b (u\,dx - v\,dy) + i\int_a^b (u\,dy + v\,dx)$$

Thus the complex integral is expressed interms of two real line integrals.

Thus a complex integral can be computed by using two methods i.e. one using parametric form and the other line integral form.

*** Example**: Find $\int_C z^2 dz$, where 'c' is a contour parametrized by

$C: z(t) = t^2 + it$, $0 \le t \le 1$.

Sol^n: **method (1)**: $\int_C z^2 dz = \int_0^1 (t^2+it)^2(2t+i)\,dt$

$$= \frac{-2}{3} + i\,2/3$$

Method (2): $\int_C z^2 dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)$

Since $z(t) = t^2 + it$, we get

$x(t) = t^2$, $y(t) = t$

$dx = 2t\,dt$, $dy = dt$.

Since $z = x+iy$

$\Rightarrow z^2 = (x^2-y^2) + i2xy$

$\therefore u(x,y) = x^2-y^2$; $v(x,y) = 2xy$

$u(z(t)) = t^4 - t^2$; $v(z(t)) = 2t^3$

$$\therefore \int_C z^2 dz = \int_C \left[(t^4-t^2)2tdt - 2t^3 dt\right] + i\int_C \left[(t^4-t^2)dt + 2t^3(2tdt)\right]$$

$$= \int_0^1 (t^4-t^2)2tdt - 2\int_0^1 t^3 dt + i\int_0^1 (t^4-t^2)dt + 4i\int t^4 dt$$

$$= -\tfrac{2}{3} + \tfrac{2}{3}i$$

* Fundamental Theorem Of Calculus

Let $f(x)$ be continuous in a closed interval $[a,b]$. Then there exists a function $F(x)$ such that $F'(x) = f(x)$ on $[a,b]$ with $F(b)-F(a) = \int_a^b f(x)dx$.

* Green's Theorem :-

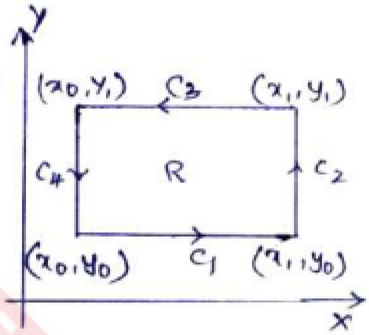
Let $P(x,y)$ and $Q(x,y)$ be continuous with continuous partials in a simply connected closed region R whose boundary is a rectangle (contour) 'C'. Then

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C is traversed in the positive sense.

Proof:

Now let us consider a rectangle whose sides are parallel to the coordinate axes as shown in figure.



Now the contour 'C' can be divided into four straight lines C_1, C_2, C_3, C_4 as shown in the figure.

$\therefore C : C_1 + C_2 + C_3 + C_4$

$$\therefore \int_C Pdx + Qdy = \int_{C_1+C_2+C_3+C_4} Pdx + Qdy$$

$$= \int_{C_1} Pdx+Qdy + \int_{C_2} Pdx+Qdy + \int_{C_3} Pdx+Qdy + \int_{C_4} Pdx+Qdy.$$

But let us note that $dy=0$ on C_1 and C_3 and $dx=0$ on C_2 and C_4.

$$\therefore \int_C Pdx+Qdy = \int_{x_0}^{x_1} P(x,y_0)dx + \int_{y_0}^{y_1} Q(x_1,y)dy + \int_{x_1}^{x_0} P(x,y_1)dx + \int_{y_1}^{y_0} Q(x_0,y)dy$$

$$= \int_{x_0}^{x_1} \left[P(x,y_0) - P(x,y_1)\right]dx + \int_{y_0}^{y_1} \left[Q(x,y) - Q(x_0,y)\right]dy$$

By using the fundamental theorem of calculus, we get

$$= \int_{x_0}^{x_1}\left[\int_{y_1}^{y_0} \frac{\partial P(x,y)}{\partial y} dy\right]dx + \int_{y_0}^{y_1}\left[\int_{x_0}^{x_1} \frac{\partial Q(x,y)}{\partial x} dx\right]dy$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1} -\frac{\partial P}{\partial y} dy\,dx + \int_{x_0}^{x_1}\int_{y_0}^{y} \frac{\partial Q}{\partial x} dx\,dy$$

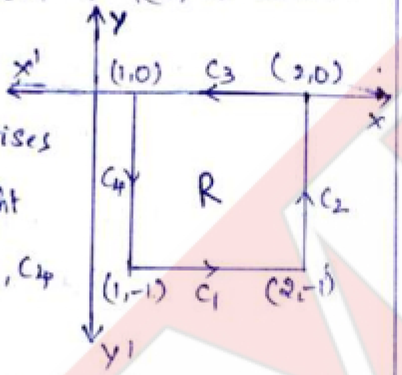
(Order of integration can be changed)

$$\therefore \int_C P\,dx + Q\,dy = \int_{y_0}^{y_1}\int_{x_0}^{x_1}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

Hence the result.

Example: Evaluate $\int_C x^2 y\,dx + (2x+1)y^2 dy$ where C is a square whose vertices are (1,0), (1,−1), (2,−1), (2,0).

sol^n: Method(1):



Here 'C' comprises of four straight lines C_1, C_2, C_3, C_4 respectively.

Therefore, we get

$$\int_C (x^2 y\,dx + (2x+1)y^2 dy = \int_{C_1+C_2+C_3+C_4} x^2 y\,dx + (2x+1)y^2 dy$$

Let us ~~obseidve~~ that on C_1 & C_3 $dy = 0$ and C_2 & C_4 $dx = 0$

$$\therefore \int_C x^2 y\,dx + (2x+1)y^2 dy = \int_{C_1} x^2 y\,dx + \int_{C_2}(2x+1)y^2 dy$$

$$+ \int_{C_3} x^2 y\,dx + \int_{C_4}(2x+1)y^2 dy$$

$$= \int_1^2 x^2(-1)\,dx + \int_{-1}^0 (4+1)y^2 dy$$

$$+ \int_2^1 x^2(0)\,dx + \int_0^{-1}(2(1)+1)y^2 dy$$

$$= -8/3 + 1/3 + 5/3 - 1 = -5/3$$

Method (2): Using the green's theorem comparing

$\int_C x^2 y\,dx + (2x+1)y^2 dy$ with $\int_C P\,dx + Q\,dy$ we get

$$P = x^2 y, \quad Q = (2x+1)y^2$$

$$\therefore \frac{\partial P}{\partial y} = x^2, \quad \frac{\partial Q}{\partial x} = 2y^2$$

Here x varies from 1 to 2 and y varies from −1 to 0

$$\therefore \int_C x^2 y\,dx + (2x+1)y^2 dy = \int_{x=1}^{2}\int_{y=-1}^{0}(2y^2 - x^2)\,dy\,dx$$

$$= \int_{x=1}^{2}\left[\frac{2y^3}{3} - x^2 y\right]_{y=-1}^{0} dx$$

$$= -5/3$$

$\therefore$ Green's theorem is verified.

* Cauchy's theorem :-

Let $f(z)$ be analytic in a region R and on its boundary 'C' then $\int_C f(z)\,dz = 0$

This fundamental theorem, often called Cauchy's integral theorem (or) Cauchy's theorem, is valid for both simply and multiply connected regions. It was first proved by use of Green's theorem with added restriction that $f'(z)$ be continuous on R. However, Goursat gave a proof which removed this restriction. For this reason the theorem is sometimes called the Cauchy-Goursat theorem when one desires to emphasize the removal of this restriction.

* Cauchy's weak theorem:-

If $f(z)$ is analytic (with a continuous derivative) in a simply connected domain D and C is a closed contour in D then $\int_C f(z)dz = 0$.

Proof: Let $f(z) = u+iv$

$$= u(x,y) + iv(x,y)$$

Since $f(z)$ is analytic in D,

we get $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ (By using Cauchy-Riemann Conditions)

$$\therefore \int_C f(z)dz = \int_C (u+iv)(dx+idy)$$

$$(\because z = x+iy \Rightarrow dz = dx+idy)$$

$$= \int_C (udx - vdy) + i\int_C vdx + udy \quad \text{—①}$$

Since $f'(z)$ is continuous,

$\therefore$ the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in D.

So on applying Green's theorem to the equation ①, we get

$$\int_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy$$

$$+ i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

But by Cauchy-Riemann Conditions, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

on using these we get

$$\int_C f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dxdy$$

$$+ i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)dxdy$$

$$= 0$$

***Corollary:-** Let $f(z)$ be analytic with continuous partial derivatives in a simply connected domain D. Let C_1 and C_2 be any contours in the domain with the same initial and terminal points. Then $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

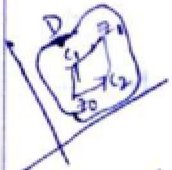
Proof: Suppose that both the contours C_1 and C_2 have z_0 and z_1 as the initial and terminal points.

Let $C = C_1 - C_2$

Then C is a closed contour in D

$\therefore$ by Cauchy's weak theorem,

$$\int_C f(z)dz = \int_{C_1 - C_2} f(z)dz = 0$$



$$= \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\therefore \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Note(1): This corollary says that the integral is independent of the path in the domain. That means the value of the integral just depends upon the initial and terminal points, provided only the contour stays inside the domain, where the function is analytic and is having continuous differentials.

Note(2): On using the above result, we can get the following convenience in evaluating integrals.

If C and C_1 are two curves whose initial & terminal points are the same in a domain where $f(z)$ is analytic, such that $\int_C f(z)\,dz$ is difficult to be evaluated, where as $\int_{C_1} f(z)\,dz$ can be evaluated with easier, then note that $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$.

*Example: If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,1)$ and $(2,3)$. Find the value of $\int_C (12z^2 - 4iz)\,dz$.

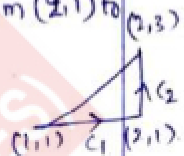
Sol^n: The shape of the given curve $C: y = x^3 - 3x^2 + 4x + 1$ is not of a standard curve for which we know the parametric equation.

But the given function $f(z) = 12z^2 - 4iz$ is analytic everywhere and is having continuous partials.

So its integral is independent of the path joining $(1,1)$ and $(2,3)$.

Hence any path can be chosen.

In particular, let us choosen the straight line paths from $(1,1)$ to $(2,1)$ denoted by C_1 and then from $(2,1)$ to $(2,3)$ denoted by C_2.



$$\therefore \int_C (12z^2 - 4iz)\,dz = \int_{C_1+C_2} (12z^2 - 4iz)\,dz$$

$$= \int_{C_1} (12z^2 - 4iz)\,dz + \int_{C_2} (12z^2 - 4iz)\,dz$$

Case(i): Along the path from $(1,1)$ to $(2,1)$ then $y = 1 \Rightarrow dy = 0$ and $z = x + iy$

$\Rightarrow z = x + i$

$dz = dx$ and

x : 1 to 2.

$$\therefore \int_{C_1} (12z^2 - 4iz)\,dz = \int_{x=1}^{2} \left[12(x+i)^2 - 4i(x+i)\right] dx$$

$$= \left[4(x+i)^3 - 2i(x+i)^2\right]_{x=1}^{2}$$

$$= 20 + 30i$$

Case(ii): Along the path from $(2,1)$ to $(2,3)$.

Then $x = 2$, $\therefore dx = 0$,

$z = x + iy = 2 + iy \Rightarrow dz = i\,dy$,

and y : 1 to 3.

$$\therefore \int_{C_2} (12z^2-4iz)\,dz = \int_{y=1}^{3} \left[12(2+iy)^2-4i(2+iy)\right] i\,dy$$

$$= \left[4(2+iy)^3 - 2i(2+iy)^2\right]_1^3$$

$$= -176+8i$$

$$\therefore \int_C (12z^2-4iz)\,dz = 20+30i-176+8i$$

$$= -156+38i$$

Method(2):- Since the value of the integral is independent of path, its value depends upon the initial and terminal points of the path only.

So we can directly integrate

$$\therefore \int_{1+i}^{2+3i} (12z^2-4iz)\,dz = \left[(4z^3-2iz^2)\right]_{1+i}^{2+3i}$$

$$= -156+38i$$

* Already we have shown that $\int_C f(z)dz = 0$ when $f(z)$ is analytic having continuous derivative in a domain D and 'C' is closed contour in D, (OR) the value of $\int_{z_1}^{z_2} f(z)dz$ is independent of path.

Now we prove the same eventhough $f(z)$ is not having continuous derivative in D.

In that attempt we first prove that every continuous function having anti derivative in a domain D also has the property that its integral over a path in D joining two points is independent of path.

$F'(z) = f(z)$ i.e. $f(z)$ has antiderivative

Then we show that a function analytic in a simply connected domain has an antiderivative. Then we prove the Cauchy's theorem in most general form.

we also try to evaluate $\int_C f(z)dz$ by using Cauchy's theorem. There we observe that if the given function $f(z)$ is analytic in and on closed contour 'C' then we say $\int_C f(z)dz = 0$ without actually evaluating it. But when $f(z)$ is not analytic at a point z_0 inside 'C', then we evaluate $\int_C f(z)dz$ by using the parametric equation of the curve 'C'. If 'C' is an odd shaped curve for which the parametric equation is not known, then we construct a circle C_1 around the point z_0 and use the Cauchy's theorem for multiple connected region to obtain

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz.$$

Now the second integral can be easily evaluated.

Now we will prove several forms of Cauchy's theorem (also called the Cauchy's - Goursat theorem), each involving different geometric and topological considerations. Goursat showed that the Cauchy weak theorem can be proved without assuming the continuity of $f'(z)$. In its simplest form, the theorem is proved for a rectangle.

Theorem (I) → Cauchy's – Goursat theorem: Let $f(z)$ be analytic in a domain containing a rectangle and its interior then $\int_C f(z)dz = 0$.

Theorem (1) → Let $f(z)$ be continuous in a domain D and suppose there is a differentiable function $F(z)$ such that $F'(z) = f(z)$ in D. Then for any contour 'c' in D parametrized by $z(t)$, $a \le t \le b$, we have

$$\int_C f(z)dz = F(z(b)) - F(z(a)).$$

Proof: Since $F(z)$ has continuous derivative in D, we get

$$\int_C f(z)dz = \int_C F'(z)dz = \int_a^b F'(z)\,z'(t)\,dt$$

$$= \int_a^b \frac{d}{dt}\left[F(z(t))\right]dt = \left[F(z(t))\right]_{t=a}^{b}$$

$$= F(z(b)) - F(z(a))$$

Note (1): If we denote $z(a) = z_0$ and $z(b) = z_1$, we get

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0).$$

Example: the function $f(z) = z^2$ is continuous everywhere and has antiderivative $F(z) = \frac{z^3}{3}$.

Hence for any contour 'c' in the plane

$$\int_C f(z)dz = \int_{z_0}^{z_1} z^2 dz = \frac{z_1^3}{3} - \frac{z_0^3}{3}.$$

Note (2): The theorem (1) looks deceptively (misleading) similar to fundamental theorem of calculus.

There is an important difference. The fundamental theorem says that a continuous function $f(x)$ defined on $[a,b]$ has an antiderivative $F(x)$ satisfying $\int_a^b f(x)dx = F(b) - F(a)$.

But theorem (1) merely asserts that if the continuous function $f(z)$ has antiderivative then the conclusion follows. That continuity is not a sufficient condition for the existence of an antiderivative.

Example: Evaluate $\int_{-i}^{i} |z|\,dz$ along different contours. Does $|z|$ have an antiderivative ?

Sol'n: $\int_{-i}^{i} |z|\,dz = \begin{cases} 0 & \text{along a straight line } z(t) = i(2t-1);\ 0 \le t \le 1 \\ 2i & \text{along a circle } z = e^{it};\ -\pi/2 \le t \le \pi/2 \end{cases}$

$F'(z) = f(z)$
$F(z) = \int_{-i} f(z)$

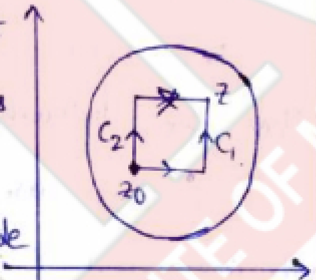
The integral (solution) depends on the path of integration. Hence the integrand is not analytic in a domain containing the path.

So antiderivative of $|z|$ doesnot exist.

Theorem-(II)

* Now we prove that Cauchy-Goursat theorem in a circular disc :-

→ Let $f(z)$ be analytic in a domain containing the closed circle $|z-z_0| \le r$. Then $\int_{|z-z_0|=r} f(z)\,dz = 0$.



Proof: we know that when $f(z)$ is continuous in a domain D and there is a differentiable function $F(z)$ such that $F'(z) = f(z)$ in D, then $\int_C f(z)\,dz = 0$.

where 'C' is a simple closed contour in D.

In this theorem we want to use of the above result and prove the theorem, because the given contour $|z-z_0| = r$ is closed and $f(z)$ is analytic in D. implies $f(z)$ is continuous in D. Therefore now we have to establish that the antiderivative $F(z)$ to $f(z)$ exists in D.

For this consider any point z in $|z-z_0| \le r$. Let us denote the curve consisting of horizontal line segment from $z_0 = x_0 + iy_0$ to $x + iy_0$ follow by vertical line segment from $x + iy_0$ to $z = x + iy$ by C_1.

Similarly let us denote the curve consisting of the vertical line segment from $z_0 = x_0 + iy_0$ to $x_0 + iy$ and the horizontal line segment from $x_0 + iy$ to $x + iy$ by C_2

Now let us define,

$$F(z) = \int_{C_1} f(z)\,dz$$

$$= \int_{x_0}^{x} f(t + iy_0)\,dt + \int_{y_0}^{y} f(x + it)\,i\,dt \quad \text{—(1)}$$

because, the first line is $z(t) = t + iy_0$; $x_0 \le t \le x$

the second line is $z(t) = x + it$; $y_0 \le t \le y$.

Since $C_1 - C_2$ is a rectangle in D, by Cauchy's - Goursat theorem, we get,

$$\int_{C_1 - C_2} f(z)\,dz = 0.$$

$$\Rightarrow \int_{C_1-C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)dz$$

from ①, $F(z) = \int_{C_2} f(z)dz$

$$\Rightarrow F(z) = \int_{y_0}^{y} f(x_0+it)dt + \int_{x_0}^{x} f(t+iy)\,dt \quad \text{——— ②}$$

because: the first line is

$z(t) = x_0+it;\ y_0 \le t \le y$

the second line is $z(t) = t+iy;\ x_0 \le t \le x$.

There is a result in differential calculus, which states that

$$\frac{\partial}{\partial \alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}dx + f(b(\alpha),\alpha)\frac{db}{d\alpha} - f(a(\alpha),\alpha)\frac{da}{d\alpha}$$

Keeping this in view differentiating ①, with respect to y and observe the following:

In the present case two variables are y and t. But the other term 'x' present in ① can be treated a parameter and one variable y is absent in $f(x+it)$. Treating that way we get,

$$\frac{\partial}{\partial y}\int_{y_0}^{y} f(x+it)i\,dt = i\left[\int_{y_0}^{y} \frac{\partial f(x+it)}{\partial y}i\,dt + f(x+iy)\frac{d(y)}{dy} - f(x+iy_0)\frac{d(y_0)}{dy}\right]$$

$$= i[0 + f(x+iy) - 0]$$

$$= if(x+iy).$$

$$= if(z)$$

$\therefore$ from ①,

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int_{x_0}^{x} f(t+iy_0)dt + \int_{y_0}^{y} f(x+it)\,i\,dt\right]$$

$$= \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)dt + \frac{\partial}{\partial y_0}\int_{y_0}^{y} f(x+it)dt$$

$$= 0 + if(z) \quad \left(\because \frac{\partial}{\partial y}\int_{x_0}^{x} f(t+iy_0)dt = \int_{x_0}^{x}\frac{\partial f(t+iy_0)}{\partial y}dt + f(x+iy_0)\frac{dx}{dy} - f(x_0+iy_0)\frac{dx_0}{dy} = 0\right)$$

$$\left.\begin{array}{l}\boxed{\dfrac{\partial F}{\partial y} = if(z)} \\ \Rightarrow \boxed{-i\dfrac{\partial F}{\partial y} = f(z)}\end{array}\right\} \longrightarrow ③$$

Similarly let us take the partial derivative of ② with respect to 'x'.

But before that let us observe the following.

In this case 'x' and 't' are two variables and y is treated as a parameter. Then

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} f(t+iy)\,dt$$

$$= \int_{x_0}^{x} \frac{\partial f(t+iy)}{\partial x}dt + f(x+iy)\frac{d(x)}{dx} - f(x_0+iy)\frac{d(x_0)}{dx}$$

$= 0 + f(x+iy).1 - 0$

$= f(x+iy)$

$= f(z)$ ——— (4)

Comparing (3) & (4), we get,

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

Since $f(z)$ is analytic in the domain $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are continuous in the domain.

$\therefore$ Now $F(z)$ is such that its real and imaginary parts have continuous partial derivatives which satisfy Cauchy-Riemann conditions

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}.$$

So we conclude that $F(z)$ is analytic in the domain as z is arbitrary, so we get

$$F'(z) = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

$\Rightarrow F'(z) = f(z).$

So antiderivative of $f(z)$ exists in D.

Hence we get $\int_C f(z)\,dz = 0.$

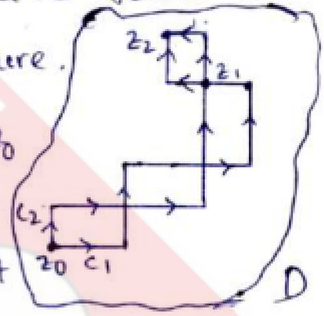
* **Theorem (III): (Cauchy – Goursat):**

If $f(z)$ is analytic in a simply connected domain D and 'C' is closed contour lying in D then

$$\int_C f(z)\,dz = 0.$$

Proof: For proving the theorem first we try to establish the existence of an antiderivative $F(z)$ to the given function $f(z)$.

For this let us follow the following procedure.



Let us fix point z_0 in D and choose any arbitrary point z in D.

Then let us construct two curves C_1 & C_2 joining the points z_0 & z as shown in figure.

Now define $F(z) = \int_{C_1} f(z)\,dz$

Since $C_1 - C_2$ is a closed curve and integral around each rectangle is zero, we get

$$\int_{C_1-C_2} f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\therefore \int_C f(z)\,dz = \int_{C_2} f(z)\,dz \quad \text{——— (1)}$$

Suppose that $z_1 = x_1 + iy_1$ is the last point of intersection of C_1 & C_2 b/w z_0 and $z = x+iy$.

Suppose that in this rectangle, C_1 consists of horizontal followed by verticle line where as C_2 consists of verticle line followed by horizontal as shown in figure.

the value of the integral $f(z)$ from z_0 to z_1 is the same along both contours C_1 & C_2 and we denote by a constant 'K'. since z_1 is a constant then consider

$$F(z) = \int_{C_1} f(z)\,dz = K + \int_{x_1}^{x} f(t+iy_1)\,dt + \int_{y_1}^{y} f(x+it)\,i\,dt \quad \text{---} (2)$$

from (1),

$$F(z) = \int_{C_2} f(z)\,dz = \int_{y_1}^{y} f(x+it)\,i\,dt + \int_{x_1}^{x} f(t+iy)\,dt \quad \text{---} (3)$$

Now Differentiating (2) with respect to 'y' we get,

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int_{y_1}^{y} f(x+it)\,i\,dt$$

$$= i f(x+iy) = i f(z)$$

similarly differentiating (3) with respect to 'x' we get

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int_{x_1}^{x} f(t+iy)\,dt\right)$$

$$= f(x+iy)$$

$$= f(z)$$

Comparing the above two results, we get,

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

$\therefore$ $F(z)$ satisfies the Cauchy's - Riemann equations.

since $f(z)$ continuous in D, the partials $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ of $F(z)$ is such that its real and imaginary parts satisfy Cauchy's - Riemann equations and have continuous partials.

So $F(z)$ is analytic in D.

$$\therefore F'(z) = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} = f(z)$$

$\therefore$ antiderivative $f(z)$ exists in D.

Hence $\int_C f(z)\,dz = 0$.

Problem

→ what can be said about $\int_C \frac{1}{z}dz$ if the closed contour C pass-through the origin.

Sol'n: At $z=0$, the integrand $\frac{1}{z}$ is undefined.

so it is not analytic on C

$$\therefore \int_C \frac{1}{z}\,dz \neq 0.$$

→ Suppose $f(z)$ is analytic on a closed contour 'C'. Does $\int_C f(z)\,dz = 0$?

Sol'n: Need not be.

Because, it is given that $f(z)$ is analytic only on 'C', but the nature

$f(z)$ inside is not known

$\therefore$ we cannot definitely say that

$$\int_C f(z)\,dz = 0$$

For example $\int_{|z|=1} \frac{1}{z}\,dz \neq 0$

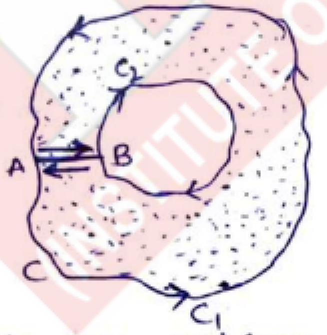
*** We Prove the Cauchy-Goursat theorem for multiply-Connected Region :-**

Statement: Let $f(z)$ be analytic in a multiply connected domain and its boundary be 'c'. Then $\int_C f(z)\,dz = 0$.

Proof: The Proof of this theorem lies in converting the multiply-connected domain into a simply connected domain.

For this we follow the following procedure.

Let us consider a multiply connected domain with boundary C.



Let C_1 and C_2 be its outer and inner boundaries respectively, which are both +vely oriented. Then let us introduce a cut AB by constructing a line AB, which joins one point on each of the outer boundary C_1 and with the inner boundary C_2.

Then the domain bounded by the contour C_1 & the line segment AB, the contour C_2 and the line segment BA is simply connected domain.

$\therefore$ by Cauchy's theorem for simply connected domain,

we have

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz = 0.$$

Since $\int_{BA} f(z)\,dz = -\int_{AB} f(z)\,dz$; we get

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

Hence the theorem.

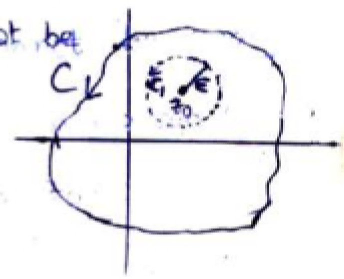
Example: Evaluate $\int_C \frac{1}{z-z_0}\,dz$ along simple closed contour 'C' having the

(i) z_0 as an interior point.

(ii) z_0 as an exterior point.

Sol'n: (a) the given function $\frac{1}{z-z_0} = f(z)$ is not defined at $z = z_0$.

$\therefore$ the function cannot be analytic in a simply connected domain which includes the point z_0.



To avoid this we make the following construction.

Construct a circle C_1 with radius ϵ and z_0 as centre.

Then $C+C_1$ is the boundary of multiply connected domain.

$\therefore$ by Cauchy's theorem for multiply connected domain, we get

$$\int_{C+C_1} \frac{1}{z-z_0}\,dz = 0.$$

$$\Rightarrow \int_C \frac{1}{z-z_0}\,dz + \int_{C_1} \frac{1}{z-z_0}\,dz = 0$$

$$\Rightarrow \int_C \frac{1}{z-z_0}\,dz = \int_{-C_1} \frac{1}{z-z_0}\,dz \quad \text{—(1)}$$

Let $z-z_0 = \epsilon e^{it}$, $0 \le t \le 2\pi$

$(\because z = z_0 + \epsilon e^{it})$

$dz = i\epsilon e^{it}dt$

$$\therefore \int_C \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{i\epsilon e^{it}}{\epsilon e^{it}}\,dt$$

$$= i\int_0^{2\pi} dt = i[2\pi - 0]$$

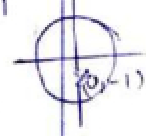
$$= 2\pi i$$

(b) If z_0 is outside C then $f(z) = \frac{1}{z-z_0}$ is analytic everywhere inside and on C

$\therefore$ By Cauchy's theorem $\int_C \frac{1}{z-z_0}\,dz = 0$.

(i) If C is the circle $|z-2|=5$, determine whether $\int_C \frac{dz}{z-3}$ is zero.

H.W (ii) find $\int_C \frac{dz}{1+z^2}$ where C is a circle given by (a) $|z+i|=1$ (b) $|z-i|=1$



Sol^n: (i) Putting $z-2 = 5e^{it}$; $dz = 5ie^{it}$, we get

$$\therefore \int_C \frac{1}{z-3}\,dz = \int_0^{2\pi} \frac{5ie^{it}}{5e^{it}-1}\,dt$$

$$= i\int_0^{2\pi}\left[1 - \tfrac{1}{5}e^{-it}\right]^{-1}dt$$

$$= i\int_0^{2\pi}\left[1 + \tfrac{1}{5}e^{-it} + \tfrac{1}{5^2}e^{-2it} + \cdots\right]dt \quad \text{—(1)}$$

Now $\int_0^{2\pi} e^{-mit}\,dt = -\left[e^{-mit}\right]_0^{2\pi}$

$$= \frac{-1}{mi}\left[e^{-2\pi mi} - e^0\right]$$

$$= \frac{-1}{mi}[1-1]$$

$= 0$ when $m \neq 0$.

$$\therefore \int_C \frac{1}{z-3}\,dz = i\int_0^{2\pi} dt = 2\pi i \neq 0.$$

The reason, the integral is not zero is that $\frac{1}{z-3}$ is not analytic at $z=3$ which is an interior point of the circle $|z-2|=5$.