IfoS 2014 Hydrostatic, S&D

8/7/20

5c) A particle whose mass is m, is acted upon by a force $m\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

(8M)

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5. (c) Given $\frac{d^2x}{dt^2} = -\mu\left[x + \frac{a^4}{x^3}\right]$, ...(i)

the –ve sign being taken because the force is attractive.

Integrating it after multiplying throughout by 2 (dx/dt), we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + C.$$

When $x = a$, $dx/dt = 0$, so that $C = 0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

or $$\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4 - x^4)}}{x}, \quad \text{...(ii)}$$

the –ve sign is taken because the particle is moving in the direction of x decreasing. If t_1 be the time taken to reach the origin, then integrating (ii), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4 - x^4)}}dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4 - x^4)}}.$$

Put $x^2 = a^2 \sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\therefore \quad t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2} \frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}$$

$$= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2} d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{\mu}} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4\sqrt{\mu}}.$$

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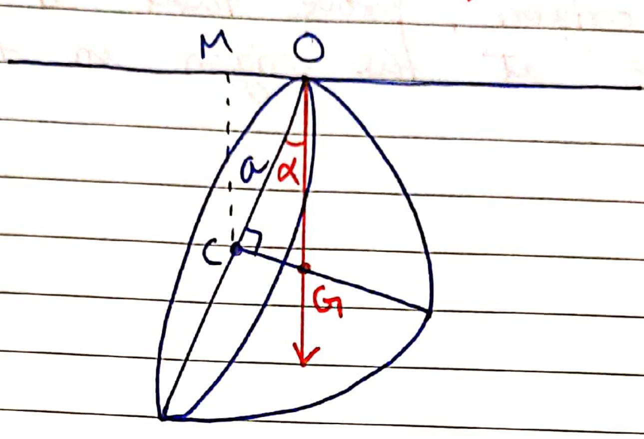
5d) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12:\sqrt{73}$. (8m)



Let 'a' be the radius of the hemisphere and O the point of rim from which it is suspended.
If G be the C.G. of the hemisphere, then

$$CG = \frac{3}{8}a$$

and OG must be vertical.
If α be the inclination of the base to the vertical, then

$$\tan\alpha = \frac{3}{8} \quad \text{--- (1)}$$

The whole pressure (thrust) on the base

$$= w \cdot \pi a^2 \cdot (a\cos\alpha)$$

(Here w = weight per unit volume of liquid.

Depth of the C.G. of the base below the surface of liquid $= CM = a\cos\alpha$.)

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Weight of the liquid contained

$$= w.\left(\frac{2}{3}\pi a^3\right)$$

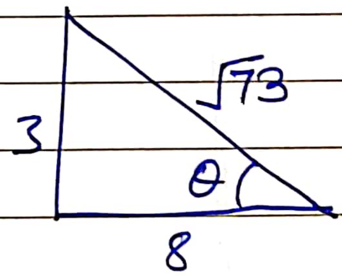
$\therefore$ Required ratio is

$$= \frac{\cancel{w}.\cancel{\pi}a^{\cancel{2}}(\cancel{a}\cos\alpha)}{\cancel{w}.\frac{2}{3}\cancel{\pi}\cancel{a^3}}$$

$$= \frac{3}{\cancel{2}}\cdot\frac{\cancel{8}^{4}}{\sqrt{73}} = \frac{12}{\sqrt{73}}$$

[from ① $\tan\alpha = \frac{3}{8}$

$\Rightarrow \cos\alpha = \frac{8}{\sqrt{73}}$]



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6b) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2 - \frac{1}{2}\right)$.

(10m)

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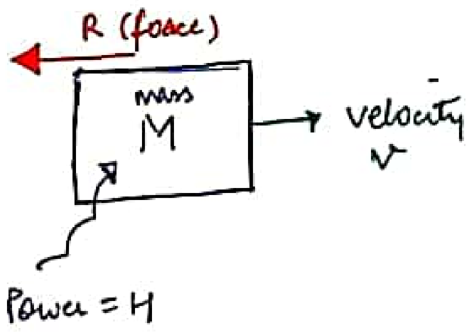

Energy equation for time dt

energy supplied $= H\,dt$

Energy lost due to ~~do~~ Resistance

$= $ Force $\times$ distance

$= R\,v\,dt$

Assuming no change of PE, $\Delta PE = 0$

$\sum$ energy supplied $-$ $\sum$ energy lost $= \Delta(K.E.)$.

$$H\,dt - Rv\,dt = d\cdot\left(\tfrac{1}{2}mv^2\right)$$

$$H\,dt - Rv\,dt = mv\,dv$$

$$H - Rv = mv\frac{dv}{dt}$$

for max. velocity; $\frac{dv}{dt} = 0$; $\Rightarrow$ acceleration $= 0$

$$H - RV = 0$$

$$\boxed{V_{max} = H/R}$$

now integrating $\quad H - Rv = mv\frac{dv}{dt}$

$$dt = m\left(\frac{v\,dv}{\cancel{H}-Rv}\right) = \frac{m}{R}\left(\frac{Rv}{H-Rv}\right)dv$$

$$dt = \frac{m}{R}\left(\frac{Rv - H + H}{H - Rv}\right)dv = \frac{m}{R}\left(\frac{H}{H-Rv} - 1\right)dv$$

$$\int_0^t dt = \int_0^{\frac{V_{max}}{2}} \frac{M}{R}\cdot\left(\left(\frac{H}{H-Rv}\right)dv - dv\right)$$

$V_{max} = \frac{H}{2R}$.

$$t = \int_0^{\frac{H}{2R}} \frac{M}{R}\left(\frac{\ln(H-RV)H}{-R} - V\right) = \boxed{\frac{MH}{R^2}\left(-\ln\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\right)} \#$$

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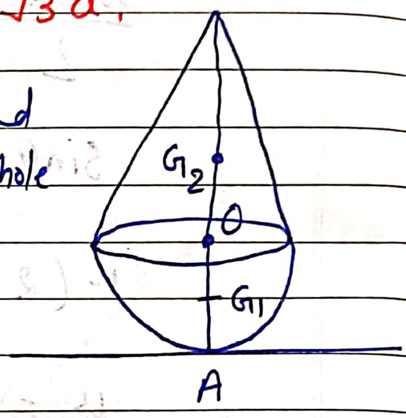
2. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius 'a'. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.



Let us first try to find out the C.G. of the whole body:

As we know C.G. of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its flat base, where 'a' is radius of sphere.

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

w_1 = weight of hemisphere

$$= \frac{2}{3}\pi a^3 \rho g$$

x_2 = distance of centre of gravity of cone from table

$$= AO + OG_2 = a + \frac{H}{4}$$

w_2 = weight of cone

$$= \frac{1}{3}\pi a^2 H \rho g, \quad H - \text{Height of cone}$$

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h = distance of C.G. of combined body from horizontal plane.

$$= \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$$

$$= \frac{\frac{2}{3}\pi a^3 \rho g \cdot \frac{5a}{8} + \frac{1}{3}\pi a^2 H \rho g \left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H \rho g}$$

$$= \frac{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)}{2a + H}$$

$$h = \frac{5a^2 + H(4a + H)}{4(2a + H)}$$

Let R = radius of lower surface $= \infty$

r = radius of upper surface $= a$

For stable equilibrium, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$\frac{4(2a+H)}{5a^2 + H(4a+H)} > \frac{1}{a} + \frac{1}{\infty}$$

$$a(8a + \cancel{4H}) > 5a^2 + \cancel{4a}H + H^2$$

$$3a^2 > H^2$$

or $\boxed{H < \sqrt{3}\, a}$

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8a) A semi-circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coeff of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$$\sin^{-1}\left(\frac{3\pi}{4}\cdot\frac{\mu+\mu^2}{1+\mu^2}\right).$$

(15m)

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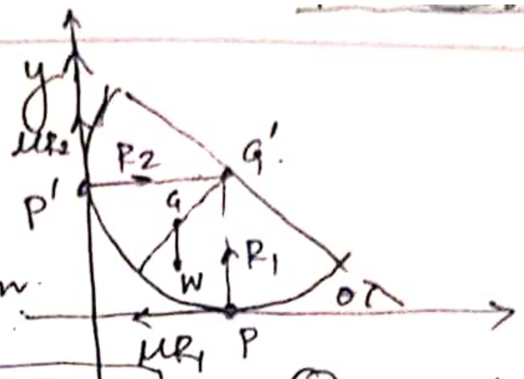

At equilibrium before motion

$\sum F_x = 0 \Rightarrow \boxed{R_2 - \mu R_1 = 0}$ — ①

$GG' = \frac{4r}{3\pi}$

$\sum F_y = 0 \Rightarrow \boxed{\mu R_2 + R_1 - W = 0}$ — ②

~~taking moment at~~

$\sum M_{G'} = \mu R_2 r - W \frac{4r}{3\pi} \sin\theta + \mu R_1 r = 0$ — ⑤

$\Rightarrow \mu \cancel{r}(R_1 + R_2) = W \frac{4\cancel{r}}{3\pi} S_\theta$

from ① $R_2 = \mu R_1$.

eq^n ② $= \mu^2 R_1 + R_1 = W$

$R_1 = \frac{W}{1+\mu^2}$ — ③

$\therefore R_2 = \frac{\mu W}{1+\mu^2}$ — ④

$\mu\left(\frac{1+\mu}{1+\mu^2}\right)\cancel{W} = \cancel{W}\frac{4}{3\pi} S_\theta$

$S_\theta = \frac{3\pi}{4}\left(\frac{\mu+\mu^2}{1+\mu^2}\right)$

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are each one-third of the sphere. Prove that $27\sigma = 122\rho$.

Sol. Proceed as in Ex. 11.

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Ex. 14. *A body floating in water has volumes* V_1, V_2, V_3 *above the surface, when the densities of the surrounding air are respectively* ρ_1, ρ_2, ρ_3. *Prove that*

$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0. \qquad \textbf{(Rohilkhand 1991, 93)}$$

Sol. Let V be the volume and W the weight of the body. Then the volumes immersed in water in the three cases are

$$(V - V_1), (V - V_2) \text{ and } (V - V_3).$$

Let ρ be the density of water.

For equilibrium, wt. of the body = wt. of water displaced + wt. of air displaced

$$\therefore \quad W = (V - V_1)\rho g + V_1 \rho_1 g \quad \text{or} \quad W - V\rho g = V_1 g(\rho_1 - \rho)$$

or
$$\frac{W - V\rho g}{V_1} = g(\rho_1 - \rho) \qquad ...(1)$$

Similarly
$$\frac{W - V\rho g}{V_2} = g(\rho_2 - \rho) \qquad ...(2)$$

and
$$\frac{W - V\rho g}{V_3} = g(\rho_3 - \rho) \qquad ...(3)$$

Multiplying (1) by $(\rho_2 - \rho_3)$, (2) by $(\rho_3 - \rho_1)$ and (3) by $(\rho_1 - \rho_2)$ and adding, we get

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$$(W - V\rho g)\left[\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3}\right] = 0$$

or
$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0.$$

Note. The above result can be put in the form

$$V_2 V_3 (\rho_2 - \rho_3) + V_3 V_1 (\rho_3 - \rho_1) + V_1 V_2 (\rho_1 - \rho_2) = 0$$

or
$$\rho_1 V_1 (V_2 - V_3) + \rho_2 V_2 (V_3 - V_1) + \rho_3 V_3 (V_1 - V_2) = 0.$$

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