Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-I

CONTENTS

1 Linear Algebra

1. Vector Space Over R and C

1.1 Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in R^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$ is a subspace of R^4. What is the dimension of this subspace. Find one of its bases.

(2009 : 12 Marks)

Solution:

ILD for vertical reaction at A;

$$\begin{aligned} V &= \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 + x_2 + 2x_3 + x_4 \\ &= 0, 2x_1 + 3x_2 - x_3 + x_4 = 0\} \end{aligned}$$

Then clearly $(0, 0, 0, 0) \in V$ and so V is non-empty.

Again let $x = (x_1, x_2, x_3, x_4) \in V$ and $y = (y_1, y_2, y_3, y_4) \in V$, Also let $\alpha, \beta \in R$.

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4) \in R^4$$

Also, $\alpha(x_1 + x_2 + 2x_3 + x_4) + \beta(y_1 + y_2 + 2y_3 + y_4) = 0$

$\Rightarrow \alpha x + \beta y$ satisfies $x_1 + x_2 + 2x_3 + x_4 = 0$

Similarly $\alpha x + \beta y$ satisfies 2nd equation as well.

$\therefore$ $$\alpha x + \beta y \in V$$

Dimension of the Subspace :

Any element of V is a solution to equation

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So, its dimension is same as rowspace of coefficient matrix, i.e., its rank.

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

As it has two non-zero rows in row reduced form.

$$\dim(v) = \text{Rank of matrix} = 2$$

Writing the equation as matrix

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ or } Ax = 0.$$

So, $$V = \{x = (x_1, x_2, x_3, x_4) \in \text{R4} \mid Ax = 0\}$$

Clearly $$O = (0, 0, 0, 0) \in V \text{ so } V \text{ is non-empty.}$$

Let $$x = (x_1, x_2, x_3, x_4);\ y = (y_1, y_2, y_3, y_4) \in V \text{ and } \alpha, \beta \in R$$

$$A(\alpha x + by) = A(\alpha x) + A(\beta y)$$

$$\therefore \quad \begin{aligned} &= \alpha(Ax) + \beta(Ay) = \alpha \cdot 0 + \beta \cdot 0 = 0 \\ \alpha x + \beta y &\in V \end{aligned}$$

$\therefore$ V is a vector subspace.

Clearly, V is the null space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Reducing it to row reduced echelon form

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R_L \leftrightarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -5 & -1 \end{bmatrix}$$

$$\begin{aligned} \dim(\text{Row space }(A)) &= \text{Number of non-zero rows} \\ &= 2 \end{aligned}$$

By Rank-Nullity Theorem :

$$\underset{\text{(Nullity)}}{\dim(\text{Null space})} + \underset{\text{(Rank)}}{\dim(\text{Row space})} = n = 4$$

$$\dim(\text{null space}) = 2$$

$$\therefore \quad \dim(V) = 2$$

For Finding Basis :

$$\dim(\text{null space}) = 2$$

$$\therefore \quad \text{No. of free variables} = 4 - 2 = 2$$

So, we fix 2 variables.

Taking $x_3 = 1, x_4 = 0$ first.

$$\left.\begin{aligned} x_1 + x_2 &= -2 \\ 2x_1 + 3x_2 &= 1 \end{aligned}\right\} \begin{aligned} x_1 &= -7 \\ x_2 &= 5 \end{aligned} \qquad x = (-7, 5, 1, 0)$$

Taking $x_3 = 0, x_4 = 1$

$$\left.\begin{aligned} x_1 + x_2 &= -1 \\ 2x_1 + 3x_2 &= -1 \end{aligned}\right\} \begin{aligned} x_1 &= -2 \\ x_2 &= 1 \end{aligned} \qquad x = (-2, 1, 0, 1)$$

$\therefore$ $(-7, 5, 1, 0)$ and $(-2, 1, 0, 1)$ are two elements of V. And since they are linearly independent **(because of choice of 3rd and 4th element)** they form a basis.

1.2 Prove that set V of all 3×3 real symmetric matrices form a linear subspace of the space of all 3×3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V.

(2009 : 20 Marks)

Solution:

Approach : Use definition of subspaces for first part. For the 2nd impose conditions due to symmetricity on the matrix.

Let V be subset of all 3×3 symmetric matrix.

Then $I_3 \in V$ so V is not empty. Again, let $A, B \in V$.

$$\Rightarrow \quad \begin{aligned} A &= A^T \\ B &= B^T \text{ (definition f symmetric)} \end{aligned}$$

and $\alpha, \beta \in R$.

Then

$$\begin{aligned} (\alpha A + \beta B)^T &= (\alpha A)^T + (\beta B)^T \\ &= \alpha A^T + \beta B^T = \alpha A + \beta B \end{aligned}$$

$\therefore$ $\alpha A + \beta B \in V$.

So, V is a vector subspace of the space of all 3×3 real matrices over R.

Dimension : Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in V$$

$$\Rightarrow \quad A^T = A \Rightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$$

i.e.,

$$b = d, c = g, h = f$$

$$\therefore \quad A = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & i \end{bmatrix}$$

Thus, any general element has 6 variables (instead of 9 for a 3×3 real matrix). So, dimension of V is 6.

Basis : Putting each of the variables as 1 and rest 0 gives us a basis, i.e.,

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

1.3 In the n-space R^n, determine whether or not the set $\{e_1 - e_2, e_2 - e_3, \ldots\ldots, e_{n-1} - e_n, e_n - e_1\}$ is linearly independent.

(2010 : 10 Marks)

Solution:

Given the set is $\{e_1 - e_2, e_2 - e_3, \ldots\ldots, e_{n-1} - e_n, e_n - e_1\}$.

Let $a_1(e_1 - e_2) + a_2(e_2 - e_3) + \ldots\ldots + a_{n-1}(e_{n-1} - e_n) + a_n(e_n - e_1) = 0$

$\Rightarrow e_1(a_1 - a_n) + e_2(a_2 - a_1) + e_3(a_3 - a_2) + \ldots\ldots + e_n(a_n - a_{n-1}) = 0$

As $e_1, e_2, \ldots\ldots, e_n$ from basis of R^n, so they are linearly independent.

$$\therefore \quad \begin{aligned} a_1 - a_n &= 0 && \ldots(1) \\ a_2 - a_1 &= 0 && \ldots(2) \\ a_3 - a_2 &= 0 && \ldots(3) \\ a_n - a_{n-1} &= 0 && \ldots(n) \end{aligned}$$

$\therefore$ from eqn. (1) to (n) it can be deducted that

$$a_1 = a_2 = a_3 = \ldots\ldots a_{n-1} = a_n$$

and not need to be zero.

$\therefore$ The given set is linearly dependent.

1.4 Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1, 1, -1), (1, 0, 1)$ and $\{(1, 2, -3), (5, 2, 1)\}$ are identical. Also find the dimension of this subspace.

(2011 : 10 Marks)

Solution :

Let W_1 be the subspace generated by the vectors $(1, 1, -1), (1, 0, 1)$.

Consider a matrix A, whose rows are the given vectors $(1, 1, -1), (1, 0, 1)$ and reduce it to row reduced Echelon form.

$$\therefore \quad A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ by } R_2 \leftrightarrow R_2 - R_1 \qquad \text{...(i)}$$

Again W_2 be the subspace generated by the vectors (1, 2, –3), (5, 2, 1).
Consider a matrix B, whose rows are the given vectors (1, 2, –3) and (5, 2, 1) and reduce it to row reduced Echelon form.

i.e.,

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -8 & 16 \end{bmatrix} \text{ by } R_2 \to R_2 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to \frac{1}{8}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_1 \to R_1 + R_2 \qquad \text{...(ii)}$$

From (i) and (ii), we find that the non-zero rows in the row-reduced Echelon forms of matrices A and B are the same.

$\therefore$ Row space of A = Row space of B

$\therefore$ $W_1 = W_2$

Again consider the vectors (1, 1, –1) and (1, 0, 1).

Let $a_1(1, 1, -1) + a_2(1, 0, 1) = (0, 0, 0),\ a_1, a_2, a_3 \in R$

$\Rightarrow$ $(a_1, a_1, -a_1) + (a_2, 0, a_2) = (0, 0, 0)$

$\Rightarrow$ $(a_1 + a_2, a_1, -a_1 + a_2) = (0, 0, 0)$

$\Rightarrow$ $a_1 + a_2 = 0,\ a_1 = 0,\ -a_1 + a_2 = 0$

$\Rightarrow$ $a_1 = 0 = a_2$

$\Rightarrow$ The vectors (1, 1, –1) and (1, 0, 1) are linearly independent.

$\therefore$ The vectors form basis of $W_1 = W_2$.

$\therefore$ Dimension of $W_1 (= W_2) = 2$.

1.5 Prove or disprove the following statement :
If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for R^5 and V is a two-dimensional subspace of R^5, then V has a basis made of just two members of B.

(2012 : 12 Marks)

Solution :

$B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for R^5.

V is a two-dimensional subspace of R^5.

Consider the set $\{b_1, b_2\}$

Let $B' = \{b_1, b_2\}$

Since $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis of R^5.

$\Rightarrow b_1, b_2, b_3, b_4, b_5$ are linearly independent and any subset of a L.I. set of vectors is L-I = b_1, b_2 are L.I.

Also, if a basis of vector space contains n elements, then any subset of the vector space having n elements is a basis iff the subset is L.I.

Thus, V will have a basis made of just 2 members of B iff B' is a subset of V.

1.6 *Let V be the vector space of all 2 × 2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.*

(2012 : 8 Marks)

Solution:

Let $W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $|W_1| = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

and $W_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $|W_2| = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

Thus, $W_1, W_2 \in W$

Now, $W_1 + W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

and $|W_1 + W_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\therefore$ $W_1 + W_2 \not\in W$

$\Rightarrow W$ is not a subspace of V as we know that a non-empty subset W of a vector space $V(F)$ is a subspace of V iff W is closed under addition and scalar multiplication.

1.7 Let V be an n-dimensional vector space and $T : V \to U$ be an invertable linear operator. If $\beta = \{X_1, X_2, \ldots\ldots, X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, \ldots\ldots, TX_n\}$ is also a basis of U.

(2013 : 8 Marks)

Solution :

Approach : An invertible linear transformation into the same vector space is one-one onto. Anyways linearty of T allows use to prove β' is linearly independent and spans V.

An invertible linear transformation from a vector space to itself is one-one and onto.

$\therefore$ T is one-one and onto.

To prove $\beta' = \{TX_1, \ldots\ldots, TX_n\}$ is a basis, we need to show that it spans V and is linearly independent.

Spans V :

Let $x \in V$. Since T is invertible and so one-one there exists $y \in V$ such that

$$T(y) = x$$

Now $y \in V$ and β' is a basis.

So, $\exists\ C_1, C_2, \ldots\ldots, C_n \in F$ such that

$$y = C_1X_1 + \ldots\ldots + C_nX_n$$

$$T(y) = T(C_1X_1 + \ldots\ldots + C_nX_n)$$

$\Rightarrow$ $x = C_1TX_1 + \ldots\ldots + C_nTX_n$

$\therefore x \in \text{Span}\{TX_1, \ldots\ldots, TX_n\}$

β' is linearly independent.

Let it be linearly dependent, then $\exists\ C_1, C_2, \ldots\ldots C_n \in F$ not all zero such that

$$C_1TX_1 + \ldots\ldots + C_nTX_n = 0$$

$\Rightarrow$ $T(C_1X_1 + \ldots\ldots + C_nX_n) = 0$

But T is invertible $\Rightarrow$ T is one-one.

$\therefore$ Kernel $T = \{0\}$

$\Rightarrow$ $C_1X_1 + \ldots\ldots + C_nX_n = 0$

with not all of C_i's zero.

But this means $X_1, X_2, \ldots\ldots, X_n$ are not linearly independent, a contradiction.

$\therefore$ β' is linearly independent set and thus β' is a basis.

1.8 Show that the vector $X_1 = (1, 1 + i, i)$, $X_2 = (i, -i, 1 - i)$ and $X_3 = (0, 1 - 2i, 2 - i)$ in C_3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers.

(2013 : 8 Marks)

Solution :

Approach : Linear independence depends on the constants of the scalar fields over which the vector space is defined. For real constants the linear combination can not be made 0. But it can be for complex constants.
Let X_1, X_2, X_3 be three vectors over R.
Let $C_1, C_2, C_3 \in R$ such that

$$C_1X_1 + C_2X_2 + C_3X_3 = 0$$
$$\Rightarrow \quad C_1(1, 1+i, i) + C_2(i, -i, 1-i) + C_3(0, 1, -2i, 2-i) = 0$$
$$\Rightarrow \quad [C_1 + C_2 i, (C_1 + C_3) + (C_1 - C_2 - 2C_3)i, C_2 + 2C_3 + (C_1 - C_2 - C_3)i] = 0$$
$$\Rightarrow \quad C_1 + C_2 i = 0$$
$$\Rightarrow \quad (C_1 + C_3) + (C_1 - C_2 - 2C_3)i = 0 \quad \text{(using first two terms)}$$
$$\Rightarrow \quad C_1 = 0, C_2 = 0, C_3 = 0$$

So, X_1, X_2, X_3 are linearly independent over R.
Linear dependence on C.
Let $i, -1, 1 \in C$ be the constants.

$$= i(1, 1+i, i) + (-1)(i, -i, 1-i) + 1(0, 1-2i, 2-i)$$
$$= (i, i-1, -1) + (-i, i, -1+i) + (0, 1-2i, 2-i)$$
$$= (0, 0, 0)$$

$\therefore X_1, X_2, X_3$ are linearly dependent on C.

1.9 Find one vector in R^3 which generates the intersection of V and W, where V is the xy-plane and w is the space generated by the vectors (1, 2, 3) and (1, −1, 1).

(2014 : 10 Marks)

Solution:

Let $R^3 = \{(x,y,z) / x,y,z \in IR\}$ be the given vector space.
Given that V is the xy-plane.

$$\therefore \quad V = \left\{\frac{(x,y,z)\in IR^3}{z=0 : x,y \in IR}\right\} \quad \text{...(i)}$$

and given that w is the space generated by the vectors (1, 2, 3) and (1, −1, 1).
For this, we find a homogeneous system whose solution set W is generated by

$$S = \{(1, 2, 3), (1, -1, 1)\}$$
$$W = \{\alpha(1, 2, 3) + \beta(1, -1, 1); \alpha, \beta(-1)\}$$
$$= \{(\alpha + \beta, 2\alpha - \beta, 3\alpha + \beta), \alpha, \beta \in IR)\}$$
$$V = \{(x, y, 0); \lambda, y \in IR\}$$

Now for $V \cap W$

$$x = \alpha + \beta, y = 2\alpha - \beta \text{ and } 3\alpha + \beta = 0$$
$$\Rightarrow \quad \beta = -3\alpha$$
$$\Rightarrow \quad x = -2\alpha, y = 5\alpha \text{ and } z = 0$$
$$\therefore \quad V \cap W = \{-2\alpha, 5\alpha, 0; \alpha \in IR\}$$

Clearly $V \cap W$ is spanned by $(-2, 5, 0) \in IR^3$.

1.10 Let v and w be the following subspaces of R^4.

$$v = \{(a, b, c, d) : b - 2c + d = 0\}$$

and

$$w = \{(a, b, c, d) : a = d, b = 2c\}$$

Find a basis and the dimension of (i) v (ii) w (iii) $v \cap w$.

(2014 : 15 Marks)

Solution:

We observe that (a, b, c, d) G

$$\Rightarrow \quad b - 2c + d = 0$$

$$\Rightarrow \quad (a, b, c, d) = (a, 5, c, 2, cb)$$
$$= (a, 0, 0, 0) + (0, b, 0, -b) + (0, 0, c, 2c)$$
$$= a(1, 0, 0, 0) + b(0, 1, 0, -1) + c(0, 0, 1, 2)$$

This shows that every vector in v is a linear combination of the three linearly independent vectors (1, 0, 0, 0), (0, 1, 0, −1) (0, 0, 1, 2)
Thus, a basis of v is

$$A = \{(1, 0, 0, 0), (0, 1, 0, -1), (0, 0, 1, 2)\}$$

Hence, **dim $v = 3$**
Now, $(a, b, c, d) \in w \Rightarrow$

$$a = d, b = 2c$$
$$\Rightarrow \quad (a, b, c, d) = (a, 2c, c, a) = (a, 0, 0, a) + (0, 2c, c, 0)$$
$$= a(1, 0, 0, 1) + c(0, 2, 1, 0)$$

which shows that w is generated by the linearly independent set {(1,0,0,1), (0,2,1,0)}
$\therefore$ A basis for w is

$$= \{(1, 0, 0, 1), (0, 2, 1, 0)\}$$

and

$$\dim w = 2$$

$(a, b, c, d)\ v \cap w \Rightarrow$

$$(a, b, c, d) \in v \text{ and } (a, b, c, d)$$
$$\Rightarrow \quad b - 2c + d = 0, a = d, b = 2c$$
$$\Rightarrow \quad (a, b, c, d) = (0, 2c, c, 0) = c(0, 2, 1, 0)$$

Hence a basis of $v \cap w$ is {0, 2, 1, 0}
and

$$\dim(v \cap w) = 1$$

1.11 The vectors $v_1 = (1, 1, 2, 4)$, $v_2 = (2, -1, -5, 2)$, $v_3 = (1, -1, -4, 0)$ and $v_4 (2, 1, 1, 6)$ are linearly independent. Is it true? Justify your answer.

(2015 : 10 Marks)

Solution:

We form a matrix with these given vectors as rows and reduce it to row-echelon form to investigate its rank.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - \frac{2}{3}R_2 \\ R_4 \to R_4 - \frac{R_2}{3} \end{matrix}$$

Since rank (A) = 2 < Number of rows (vectors)
Hence, the vectors are not linearly independent.

1.12 Find the dimension of the subspace of R^4, spanned by the set {(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)}. Hence, find its basis.

(2015 : 12 Marks)

Solution :

We find the echelon form of the matrix formed by given vectors taking as rows.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 - R_3$$

Hence, Rank of Matrix = 3

$\therefore$ Dimension of subspace = 3

For the basis, we take the vectors from the original matrix which correspond the non-zero rows in the echelon form.

$\therefore$ Basis = {(1,0,0,0), (0,1,0,0), (1,2,0,1)}

1.13 If

$$W_1 = \{(x, y, z) : x + y - z = 0\}$$
$$W_2 = \{(x, y, z) : 3x + y - 2z = 0\}$$
$$W_3 = \{(x, y, z) : x - 7y + 3z = 0\}$$

then find $\dim(W_1 \cap W_2 \cap W_3)$ and $\dim(W_1 + W_2)$.

(2016 : 3 Marks)

Solution :

Let $(x, y, z) \in W_1 \cap W_2 \cap W_3$

$\therefore$
$$x + y - z = 0 \quad ...(i)$$
$$3x + y - 2z = 0 \quad ...(ii)$$
$$x - 7y + 3z = 0 \quad ...(iii)$$

$\Rightarrow$ $x + y - z = 0$

$\Rightarrow$ $x + y - z = 0$

$-2y + z = 0$

$\Rightarrow$ $-2y + z = 0$

$\therefore$ $y = \dfrac{z}{2}$

$x = \dfrac{z}{2}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = z\begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$\therefore$ $\dim(W_1 \cap W_2 \cap W_3) = 1$

Again $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ ...(*)

$\dim(W_1 \cap W_2) \Rightarrow$ $x + y - z = 0$

$3x + y - 2z = 0$

$\Rightarrow$ $2x - z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} z$$

$\therefore$ $\dim(W_1 \cap W_2) = 1$

(*) gives $\dim(W_1 + W_2) = 2 + 2 - 1 = 3$

1.14 Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim $V = 6$. Find the possible dimensions of subspace $U \cap W$.

(2017 : 10 Marks)

Solution:

$$\dim(U) = \dim(W) = 4$$

U and W are subspaces of V.

We know, $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

$\therefore$ $\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W)$

$= 8 - \dim(U + W)$ ...(i)

Now, $U + W$ is a subspaces of V while U and W are subspaces of $U + W$.

$\therefore$ Possible $\dim(U + W) = 4, 5, 6$

Hence, by (3)

Possible $\dim(U \cap W) = 4, 3, 2$

But $\dim(U \cap W)$ cannot be 4 otherwise $U \cap W = U = W$, but U and W are distinct.

$\therefore$ $\dim(U \cap W) = 2$ or 3

1.15 Express basic vectors $e_1 = (1, 0)$ and e_2 (0, 1) as combinations of $\alpha_1 = (2, -1)$ and $\alpha_2 = (1, 3)$.

(2018 : 10 Marks)

Solution:

Let any vector, say $(x, y) = a\alpha_1 + b\alpha_2$

$\therefore$ $(x, y) = (2a + b, -a + 3b)$

Comparing LHS and RHS, we get

$$a = \frac{3x - y}{7}, b = \frac{x + 2y}{7}$$

For $e_1 = (1, 0) : x = 1, y = 0$

$\therefore$ $a = \dfrac{3}{7}, b = \dfrac{1}{7}$

So, $e_1 = \dfrac{3}{7}\alpha_1 + \dfrac{1}{7}\alpha_2$

For $e_2 = (0, 1) : x = 0, y = 1$

$\therefore$ $a = -\dfrac{1}{7}, b = \dfrac{2}{7}$

So, $e_2 = -\dfrac{1}{7}\alpha_1 + \dfrac{2}{7}\alpha_2$

$\therefore$ $e_1 = \dfrac{3\alpha_1 + \alpha_2}{7}, e_2 = \dfrac{-\alpha_1 + 2\alpha_2}{7}$

2. Linear Transformations

2.1 Let $B = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $B' = \{(2, 1, 1), (1, 2, 1), (-1, 1, 1)\}$ be the two order bases of R^3. Then find the matrix representing the linear transformation $T: R^3 \to R^3$ which transforms B into B'. Use this matrix transformation to find $T(\bar{x})$ where $\bar{x} = (2, 3, 1)$.

(2009 : 20 Marks)

Solution:

Insight : A change of basis matrix from B to B' expresses the coordinates with respect to basis B' in terms of coordinates w.r.t. to B.

First we express elements of B' in terms of B.

Let (a, b, c) any general vector be expressed as linear combination of elements of B.

$$(a, b, c) = (1, 1, 0)x + (1, 0, 1)y + (0, 1, 1)z$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Using the augmented matrix to solve the linear system of equations

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right] R_2 \leftrightarrow R_2 - R_1 \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & c \end{array}\right]$$

$$R_2 \leftrightarrow -1R_2 \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & a-b \\ 0 & 1 & 1 & c \end{array}\right] R_3 \leftrightarrow R_3 - R_2 \left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 2 & c+b-a \end{array}\right]$$

$$\therefore \quad z = \frac{c+b-a}{2}; y = \frac{a-b+c}{2}; x = \frac{a+b-c}{2}$$

We use this to express elements of B' as coordinates of B.

$$(2, 1, 1) = 1u_1 + 1u_2 + 0u_3$$
$$(1, 2, 1) = 1u_1 + 0u_2 + 1u_3$$
$$(-1, 1, 1) = \frac{-1}{2}u_1 + \frac{-1}{2}u_2 + \frac{3}{2}u_3$$

So change of basis matrix is $\begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$

$$(2, 1, 1) = (1, 1, 0)\left(\frac{2+1-1}{2}\right) + (1, 0, 1)\left(\frac{2-1+1}{2}\right)$$

$$x, y, z = 1, 1, 0$$

Similarly, (x_2, y_2, z_2) and (x_3, y_3, z_3) matrix is

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$T(\bar{x}) = T \cdot \bar{x} = \begin{bmatrix} 1 & 1 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9/2 \\ 3/2 \\ 9/2 \end{bmatrix}$$

2.2 Let $L: R^4 \to R^3$ be a linear transformation defined by

$$L((x_1, x_2, x_3, x_4)) = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$$

Then find the rank and nullify of L. Also determine null space and range space of L.

(2009 : 20 Marks)

Solution:

Approach : We could use the standard basis vectors of R^4 or use the matrix form of the linear transformation. Let $(e_1, e_2, e_3, e_4) = B$ be the standard basis of R^4 where $e_1 = (1, 0, 0, 0)$ and similarlyo thers. $B = (e_1, e_2, e_3, e_4)$ spans R^4 so $\{L(e_1), L(e_2), L(e_3), L(e_4)\}$ will span $Im(L)$, i.e., image space of L. So a linear independent subspace will be basis of range space and its dimension rank.

$$L(e_1) = (-1, 0, -1)$$
$$L(e_2) = (-1, -1, 0)$$
$$L(e_3) = (1, 1, 0)$$
$$L(e_4) = (1, 0, 1)$$

To reduce it to a linearly independent subset we reduce the matrix with the vectors as row to row reduced echelon form

$$\begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_4]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 \leftrightarrow R_3 + R_1 \\ R_4 \leftrightarrow R_4 + R_1}]{R_2 \leftrightarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_3 - R_2]{R_2 \leftrightarrow -1R_2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $B' = \{(1, 1, 0), (0, 1, -1)\}$ is a linearly independent subset which spans $Im(L)$.

$\therefore$ $\quad$ Rank $(L) = 2$

and $\quad$ Range Space = Linear Span $\{(1, 1, 0), (0, 1, -1)\}$

Nullify and Null Space $V \in R^4$ is in null space if $L(V) = 0$

$$\Rightarrow \begin{bmatrix} x_3 + x_4 - x_1 - x_2 \\ x_3 \quad\quad -x_2 \\ x_4 - x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_3 \quad -x_2 \\ x_4 - x_1 \\ 0 \; 0 \; 0 \; 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ upon solving}$$

Thus, there are two free variables x_1 and x_2 and so nullify = dim (null space) = 2.

Giving arbitrary value to free variables

$$\begin{aligned} x_1 &= 1, x_2 = 0 && (x_1, x_2, x_3, x_4) \\ &= (1, 0, 0, 1) \\ x_2 &= 0, x_2 = 1 && (x_1, x_2, x_3, x_4) \\ &= (0, 1, 1, 0) \end{aligned}$$

$\therefore$ {(1, 0, 0, 1), (0, 1, 1, 0)} is a basis for null space of L.

2.3 **What is the null space of the differential transformation $\frac{d}{dx} : P_n \to P_n$ where P_n is the space of all polynomials of degree $\le n$ are the real numbers? What is the null space of second derivative as a transformation of P_n? What is the null space of kth derivative?**

(2010 : 12 Marks)

Solution:

Let the polynomial $\quad P_n(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$

Now, $\quad \frac{d}{dx}P_n(x) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$

If $\quad \frac{d}{dx}P_n(x) = 0 \Rightarrow a_1 + 2a_2x + ... + na_nx^{n-1} = 0$

then $\quad a_1 = a_2 = ... = a_n = 0$

$\therefore$ Null space of the transformation $\frac{d}{dx}P_n(x) \to P_n(x)$ is

$$a_1 = a_2 = ... = a_n = 0 \text{ and } a_0 \in R$$

i.e., $\quad (a_0, a_1, a_2, ..., a_n) = a_0(1, 0, 0, ..., 0)$

Now, $\quad \frac{d^2}{dx}P_n(x) = 2a_2 + 6a_3 + ... + n(n-1)a_nx^{n-2} = 0$

when $\quad a_2 = a_3 = ... = a_n = 0$

$\therefore$ Null space of $\frac{d^2}{dx}P_n(x)$ is when $a_0, a_1 \in R$ and $a_2 = a_3 = ... a_n = 0$

So, dimension of null space of k^{th} derivative is 'k'.

So, $\quad$ Null space $= a_0(1, 0, 0, ..., 0) + a_1(0, 1, 0, ..., 0)$

$$\frac{d^k}{dx}P_n = k!a_k + (k+1)!a_{k+1} + ... + n(n-1)(n-2).....(n-k+1)a_nx^{n-k}$$
$$= 0$$

when $\quad a_k = ak+1 = ... = an-1 = an = 0$

and $a_0 \in R$, $a_1 \in R$, ... $a_{k-1} \in R$

So, null space of k^{th} derivative is $a_0(1, 0, ..., 0) + a_1(0, 1, ..., 0) + ... + a_k(0, 0, ...1, 0, 0, 0)$

2.4 **Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear transformation $T : R^3 \to R^2$ so that M is the matrix of T with respect to the basis.**

$$\beta = \{v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)\}$$

of R^3 and $\beta' = [w_1 = (1, 0), w_2 = (1, 1)\}$ of R^2. Also find $T(x, y, z)$.

(2010 : 20 Marks)

Solution:

Given β is the basis of R^3 and β' be basis of R^3 for transformation T.

Now,

$$\begin{aligned} T(v_1) &= T(1, 0, 0) = 4w_1 + 0w_2 && (M \text{ is given}) \\ &= 4(1, 0) + 0.(1, 1) = (4, 0) \\ T(v_2) &= T(1, 1, 0) = 2w_1 + 1w_2 \\ &= 2(1, 0) + 1(1, 1) = (3, 1) \\ T(v_3) &= T(1, 1, 1) = 1.w_1 + 3.w_2 \\ &= 1.(1, 0) + 3(1, 1) = (4, 3) \end{aligned}$$

Now, let

$$(x, y, z) = \alpha(1, 0, 0) + \beta(1, 1, 0) + \gamma(1, 1, 1) = \alpha v_1 + \beta v_2 + \gamma v_3$$
$$(x, y, z) = (\alpha + \beta + \gamma, \beta + \gamma, \gamma)$$

Comparing LHS & RHS, we get

$$\begin{aligned} \gamma &= z && ...(1) \\ \beta + \gamma &= y \Rightarrow \beta = y - \gamma = y - z && ...(2) \text{ (from (1))} \\ \alpha + \beta + \gamma &= x \Rightarrow \alpha = x - \beta - \gamma = x - y + z - z && \text{(from (2))} \\ &= x - y && ...(3) \end{aligned}$$

$$\begin{aligned} \therefore \quad (x, y, z) &= (x - y).v_1 + (y - z).v_2 + z.v_3 \\ \therefore \quad T(x, y, z) &= T((x - y).v_1 + (y - z).v_2 + z.v_3) \\ &= (x - y)T(v_1) + (y - z)T(v_2) + z.T(v_3) \\ &= (x - y)(4, 0) + (y - z)(3, 1) + z(4, 3) \\ &= (4(x - y), 0) + (3(y - z), (y - z) + (4z, 3z) \\ &= (4x - 4y + 3y - 3z + 4z, 0 + y - z + 3z) \\ &= (4x - y + z, y + 2z) \end{aligned}$$

$\therefore$ The transformation T is $T(x, y, z) = (4x - y + z, y + 2z)$.

2.5 **Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible.**

(2010 : 10 Marks)

Solution:

Given, T be the linear transformation, such that

$$\begin{aligned} T - T^2 &= I \\ \Rightarrow \quad T(I - T) &= I \\ \Rightarrow \quad T(I - T) &= I \\ |T|\,|(I - T)| &= |I|, \text{ i.e., } |T| \times |I - T| = 1 \\ \therefore \quad |T| &\ne 0 \end{aligned}$$

$\therefore$ T is invertible.

2.6 **Find the nullity and a basis of the null space of the linear transformation A : R(4) → R(4) given by the matrix**

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

(2011 : 10 Marks)

Solution :

Let $B = \{e_1, e_2, e_3, e_4\}$ be the usual basis of R^4.

It is given that $[T; B] = A$

$\therefore$ By definition of the matrix $[T; B]$,

$$\begin{aligned} T(e_1) &= 0.e_1 + 1.e_2 + 3.e_3 + 1.e_4 \\ &= 0(1, 0, 0, 0) + 1(0, 1, 0, 0) + 3(0, 0, 1, 0) + 1(0, 0, 0, 1) \\ &= (0, 1, 3, 1) \\ T(e_2) &= (1, 0, 1, 1) \\ T(e_3) &= (-3, 1, 0, -2) \\ T(e_4) &= (-1, 1, 2, 0) \end{aligned}$$

Let (x, y, z, u) be any vector of R^4.

$$\begin{aligned} \therefore \quad T(x, y, z, u) &= T(xe_1 + ye_2 + ze_3 + ue_4) \\ &= xT(e_1) + yT(e_2) + zT(e_3) + uT(e_4) \quad [\because T \text{ is a linear mapping}] \\ &= x(0, 1, 3, 1) + y(1, 0, 1, 1) + z(-3, 1, 0, -2) + u(-1, 1, 2, 0) \\ &= (y - 3z - u, x + z + u, 3x + y + 2u, x + y - 2z) \end{aligned}$$

which is the required linear transformation.

Again let $(x, y, z, 4) \leftarrow R^4$ be any vector.

Then, $\quad T(x, y, z, u) = (0, 0, 0, 0)$

$\Rightarrow (y - 3z - 4, x + z + 4, 3x + y + 24, x + y - 2z) = (0, 0, 0, 0)$

$$\Rightarrow \quad \begin{aligned} y - 3z - u &= 0 \\ x + z + u &= 0 \\ 3x + y + 2u &= 0 \\ x + y - 2z &= 0 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x &= -z - u \\ y &= 3z + u \end{aligned}$$

Hence, u and z are independent variables.

$$\begin{aligned} \therefore \quad (x, y, z, u) &= (-z - u, 3z + u, z, u) \\ &= (-z, 3z, z, 0) + (-u, u, 0, u) \\ &= z(-1, 3, 1, 0) + u(-1, 1, 0, 1) \end{aligned}$$

Thus, basis of null space of T

$$= \{(-1, 3, 1, 0) + (-1, 1, 0, 1)\}$$

and $\quad$ nullify = dimension(null space of T) = 2

2.7 Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T : R^{(3)} \to R^{(3)}$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z, x + 2y + 5z, 2x + 4y + 6z).$$

Show that the images of above vectors under T are linearly dependent. Give the reason for the same.

(2011 : 10 Marks)

Solution :

If V is a vector space over a field F, then the vectors $V_1, V_2, \ldots\ldots, V_n \in V$ are called linearly independent (L.I.) over F if there exists scalars $\alpha_1, \alpha_2, \ldots., \alpha_n \in F$, all of them zero, such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \ldots\ldots + \alpha_n V_n = 0$$

Let $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$$\alpha_1(1, 1, 1) + \alpha_2(2, 1, 2) + \alpha_3(1, 2, 3) = (0, 0, 0)$$

$$\Rightarrow \quad \begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 0; \\ \alpha_1 + \alpha_2 + 2\alpha_3 &= 0; \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 &= 0 \end{aligned}$$

Solving these, we get, $\alpha_1 = \alpha_2 = \alpha_3 = 0$

Hence, the vectors (1, 1, 1), (2, 1, 2), (1, 2, 3) are L.I.

Again, it is given that $T : R^3 \to R^3$

defined by,

$$\begin{aligned} T(x, y, z) &= (x + 2y + 3z, x + 2y + 5z, 2x + 4y + 6z) \\ \therefore \quad T(1, 1, 1) &= (6, 8, 12) \\ T(2, 1, 2) &= (10, 14, 20) \\ T(1, 2, 3) &= (14, 20, 28) \end{aligned}$$

$\therefore$ images of the given vectors under T are (6, 8, 12), (10, 14, 20), (14, 20, 28).

Consider $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$$\alpha_1(6, 8, 12) + \alpha_2(10, 14, 20) + \alpha_3(14, 20, 28) = (0, 0, 0) \quad \ldots(i)$$

$$\Rightarrow \quad \begin{aligned} 6\alpha_1 + 10\alpha_2 + 14\alpha_3 &= 0; \\ 8\alpha_1 + 14\alpha_2 + 20\alpha_3 &= 0; \\ 12\alpha_1 + 20\alpha_2 + 28\alpha_3 &= 0. \end{aligned}$$

$$\text{or} \quad \begin{aligned} 3\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 0; \\ 4\alpha_1 + 7\alpha_2 + 10\alpha_3 &= 0; \\ 3\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 0 \end{aligned}$$

$$\Rightarrow \quad \alpha_1 = \alpha_3, \alpha_2 = -2\alpha_3$$

$\therefore$ from (i), we have

$$\alpha_3(6, 8, 12) - 2\alpha_3(10, 14, 20) + \alpha_3(14, 20, 28) = (0, 0, 0)$$

$$\Rightarrow \quad 1(6, 8, 12) - 2(10, 14, 20) + 1(14, 20, 28) = (0, 0, 0)$$

$\Rightarrow$ (6, 8, 12), (10, 14, 20), (14, 20, 28) are linearly dependent as there exists $\alpha_1, \alpha_2, \alpha_3$ not all of them zero, such that

$$\alpha_1(6, 8, 12) + \alpha_2(10, 14, 20) + \alpha_3(14, 20, 28) = (0, 0, 0).$$

2.8 Let $T : R^3 \to R^3$ be the linear transformation defined by

$$T(\alpha, \beta, \gamma) = (\alpha + 2\beta - 3\gamma, 2\alpha + 5\beta - 4\gamma, \alpha + 4\beta + \gamma)$$

Find a basis and the dimension of the image of T and the kernel of T.

(2012 : 12 Marks)

Solution :

We know that a basis of R^3 is $B = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(i) Basis for image of $T(I_m(T))$

$\because B$ is a basis of R^3,

$$\begin{aligned} \therefore \quad B_1 &= \{T(e_1), T(e_2), T(e_3)\} \text{ generates } I_m(T). \\ \text{Here,} \quad T(e_1) &= T(1, 0, 0) \\ &= (1 + 0 + 0, 2 + 0 - 0, 1 + 0 + 0) \\ &= (1, 2, 1) \\ T(e_2) &= T(0, 1, 0) = (2, 5, 4) \\ T(e_3) &= T(0, 0, 1) = (-3, -4, 1) \\ \therefore \quad B_1 &= \{(1, 2, 1), (2, 5, 4), (-3, -4, 1) \text{ generates } I_m(T). \end{aligned}$$

To find basis of $I_m(T)$, we have to find the linearly independent (L.I.) vectors from $T(e_1), T(e_2), T(e_3)$. For this, let us consider the matrix whose rows are generators of T and reduces it to echelon form.

i.e., let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 4 \\ -3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{array}{l} \text{Operate} \\ R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 + 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \leftrightarrow R_3 - 2R_2 \end{matrix}$$

$\therefore$ (1, 2, 1) and (0, 1, 2) form L.I. set of vectors which generates $I_m(T)$.

$\therefore$ Basis for the image space of T

$$= \{(1, 2, 1), (0, 1, 2)\}$$

and dimension of the image of T

$$= \text{number of elements in the above basis}$$
$$= 2$$

(ii) To find basis of kernel of T or null space of $T(N(T))$:

Let $V = (x, y, z) \in N(T)$

$\Rightarrow$ $T(V) = T(x, y, z) = 0$

$\Rightarrow (\alpha + 2\beta - 3\gamma, 2\alpha + 5\beta - 4\gamma, \alpha + 4\beta + \gamma) = (0, 0, 0)$

$$\Rightarrow \left.\begin{aligned} \alpha + 2\beta - 3\gamma &= 0 \\ 2\alpha + 5\beta - 4\gamma &= 0 \\ \alpha + 4\beta + \gamma &= 0 \end{aligned}\right] \quad ...(i)$$

For finding a basis of the kernel of T, it is equivalent to find a basis of the solution space of above equations. For this, let us consider the matrix

$$P = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \leftrightarrow R_3 - 2R_2 \end{matrix}$$

$\therefore$ The system of equations (i) are equivalent to

$$x + 2y - 3z = 0$$
$$y + 2z = 0 \text{ or } y = -2z$$

Hence, 'z' is an independent variable.

$\therefore$ Solution set is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix} z$$

Hence, $B_2 = \{(7, -2, 1)\}$ is a basis for kernel of T and dimension of $k(T) = 1$.

2.9 Consider the linear mapping $f : R^2 \to R^2$ by

$$f(x, y) = (3x + 4y, 2x - 5y)$$

Find the matrix A relative to the basis $\{(1, 0), (0, 1)\}$ and the matrix B relative to the basis $\{(1, 2), (2, 3)\}$.

(2012 : 12 Marks)

Solution :

Let $B_1 = \{(1, 0), (0, 1)\}$

Firstly, let us express any element $V = (\alpha, \beta) \in R^2$ as a linear combination of the elements of basis B_1.

Let

$$\begin{aligned} (\alpha, \beta) &= a(1, 0) + b(0, 1) \\ &= (a, 0) + (0, b) = (a, b) \end{aligned}$$

$\therefore$ $\alpha = a, \beta = b$

$\therefore$ $(\alpha, \beta) = \alpha(1, 0) + \beta(0, 1)$

Given $f : R^2 \to R^2$ defined as

$$f(x, y) = (3x + 4y, 2x - 5y)$$

and $\beta_1 = \{(1, 0), (0, 1)$ is a basis of R^2.

$$\begin{aligned} f(1, 0) &= (3.1 + 4.0, 2.1 - 5.0) = (3, 2) \\ &= 3(1, 0) + 2(0, 1) \\ f(0, 1) &= (4, -5) = 4(1, 0) + (-5)(0, 1) \end{aligned}$$

$\therefore$

$$A = [f, B_1] = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix}^t = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$$

Again, let $(\alpha, \beta) = a(1, 2) + b(2, 3)$

where $B_2 = \{(1, 2), (2, 3)\}$

$\therefore$ $\alpha = a + 2b$

$\beta = 2a + 3b$

$\Rightarrow$ $a = -3\alpha + 2\beta, b = 2\alpha - \beta$

$\therefore$ $(\alpha, \beta) = (-3\alpha + 2\beta)(1, 2) + (2\alpha - \beta)(2, 3)$

$$\begin{aligned} f(1, 2) &= (3 + 8, 2 - 10) = (11, -8) \\ &= (-33 - 16)(1, 2) + (22 + 8)(2, 3) \\ &= -49(1, 2) + 30(2, 3) \\ f(2, 3) &= (18, -11) \\ &= (-54 - 22)(1, 2) + (36 + 11)(2, 3) \\ &= -76(1, 2) + 47(2, 3) \end{aligned}$$

$\therefore$

$$B = [f, B_2] = \begin{bmatrix} -49 & 30 \\ -76 & 47 \end{bmatrix}^t = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

2.10 Let P_n denote the vector space of all real polynomials of degree at most n and $T : P_2 \to P_3$ be a linear transformation given by

$$T(P(x)) = \int_0^x p(t)dt \quad p(x) \in P_2$$

Find the matrix of T with respect to the basis $\{1, x, x^2\}$ and $\{1, x, 1 + x^2, 1 + x^3\}$ of P_2 and P_3 respectively. Also find the null space of T.

(2013 : 10 Marks)

Solution:

Let $B = \{1, x, x^2\}$ basis of P_2

and $B' = \{1, x, 1 + x^2, 1 + x^3\}$ basis of P_3

We first write linear transformation of elements of B as linear combination of B'.

$$T(1) = \int_0^x 1\,dt = x = 1\cdot x$$

$$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = \frac{1}{2}(1+x^2) - \frac{1}{2}\cdot 1$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = \frac{1}{3}(1+x^3) - \frac{1}{3}\cdot 1$$

So, the matrix of T w.r.t. B and B' are the co-ordinates written as columns.

i.e.,
$$[T]_{B,B'} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\text{Null space of } T \equiv \{p(x) \in P_2 \text{ such that } T(p) = 0\}$$
$$= \left\{p(x) \in P_2 \text{ such that } \int_0^x p_2(t)\,dt = 0\right\}$$

Now
$$\int_0^x p_2(t)\,dt = 0$$

Differentiating both sides w.r.t. x

$$p_2(x) = 0$$

$\therefore$ Null space of $T = \{0\}$

2.11 Let $V = \mathbb{R}^3$ and $T \in A(V)$, for all $a_i \in A(V)$, be defined by

$$T\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2a_1 + 5a_2 + a_3 \\ -3a_1 + a_2 - a_3 \\ -a_1 + 2a_2 + 3a_3 \end{bmatrix}$$

What is the matrix T relative to the basis $V_1 = (1, 0, 1)$, $V_2 = (-1, 2, 1)$, $V_3 = (3, -1, 1)$?

(2015 : 12 Marks)

Solution:

First we find the co-ordinates of (a, b, c) in the given basis.

$$(a, b, c) = x(1, 0, 1) + y(-1, 2, 1) + z(3, -1, 1)$$

$\therefore$
$$x - y + 3z = a$$
$$2y - z = b$$
$$x + y + z = c$$

Solving
$$x = \frac{1}{2}(-3a - 4b + 5c)$$
$$y = \frac{1}{2}(a + 2b - c)$$
$$z = (a + b - c)$$
$$T(1, 0, 1) = (3, -4, 2)$$

$$= \frac{17}{2}(1, 0, 1) - \frac{7}{2}(-1, 2, 1) - 3(3, -1, 1)$$

$$T(-1, 2, 1) = (9, 4, 8) = -\frac{3}{2}(1, 0, 1) + \frac{9}{2}(-1, 2, 1) + 5(3, -1, 1)$$

$$T(3, -1, 1) = (2, -11, -2)$$
$$= 14(1, 0, 1) + (-9)(-1, 2, 1) - 7(3, -1, 1)$$

$\therefore$ Matrix of Linear Transformation w.r.t. given basis.

$$M = \begin{bmatrix} \frac{17}{2} & -\frac{7}{2} & -3 \\ -\frac{3}{2} & \frac{9}{2} & 5 \\ 14 & -9 & -7 \end{bmatrix}^T = \begin{bmatrix} \frac{17}{2} & -\frac{3}{2} & 14 \\ -\frac{7}{2} & \frac{9}{2} & -9 \\ -3 & 5 & -7 \end{bmatrix}$$

Alternate Method : Let us denote the standard basis by β and given basis by β'.

$$\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$
$$\beta' = \{(1, 0, 1), (-1, 2, 1)$$
$$T(1, 0, 0) = (2, -3, -1)$$
$$T(0, 1, 0) = (5, 1, 2)$$
$$T(0, 0, 1) = (1, -1, 3)$$

$\therefore$
$$[T]_\beta = \begin{bmatrix} 2 & 5 & 1 \\ -3 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}$$

Transformation matrix, P so that

$$[T]_{b'} = P^{-1}[T]_\beta P$$

$$P = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

(Expressing β^* in the linear combination of vectors from β)

$$|P| = 1(2+1) + 0 + 1(1-6) = -2$$

$\therefore$
$$P^{-1} = \frac{\text{Adj.}(P)}{|P|} = -\frac{1}{2}\begin{bmatrix} 3 & -1 & -2 \\ 4 & -2 & -2 \\ -5 & 1 & 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 3 & 4 & -5 \\ -1 & -2 & 1 \\ -2 & -2 & 2 \end{bmatrix}$$

$$[T]_{\beta'} = P^{-1}[T]_\beta P$$
$$= -\frac{1}{2}\begin{bmatrix} 3 & 4 & -5 \\ -1 & -2 & 1 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 & 1 \\ -3 & 1 & -1 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17}{2} & -\frac{3}{2} & 14 \\ -\frac{7}{2} & \frac{9}{2} & -9 \\ -3 & 5 & -7 \end{bmatrix}$$

2.12 If $M_2(R)$ is space of real matrices of matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T : M_2(R) \to P_2(x)$, such that

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + c + (a - d)x + (b + c)x^2,$$

with respect to the standard bases of $M_2(R)$ and $P_2(x)$. Further find the null space of T.

(2016 : 10 Marks)

Solution:

Standard base of $M_2(R)$ is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \beta_1$$

Standard base of $P_2(x)$ is $\{1, x, x^2\} = \beta_2$

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 + 0 + (1 - 0)x + (0 + 0)x^2 = 1 + x + 0.x^2$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0 + 0 + (0 - 0)x + (1 + 0)x^2 = 0 + 0.x + x^2$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = 0 + 1 + (0 - 0)x + (0 + 1)x^2 = 1 + 0.x + x^2$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 + 0 + (0 - 1)x + (0 + 0)x^2 = 0 - x + 0.x^2$$

$\therefore$
$$M_{\beta_1}^{\beta_2} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}_{3\times 4}$$

2.13 If $T : P_2(x) \to P_3(x)$ is such that $T(f(x)) = f(x) + 5\int_0^x f(t)dt$, then choosing $\{1, 1 + x, 1 - x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T.

(2016 : 6 Marks)

Solution :

Let
$$\beta = \{1, 1 + x, 1 - x^2\}$$
$$\gamma = \{1, x, x^2, x^3\}$$

$$T(1) = 1 + 5\int_0^x 1 \cdot dt = 1 + 5x + 0.x^2 + 0.x^3$$

$$T(1 + x) = 1 + x + 5\int_0^x (1 + t)dt = 1 + x + 5\left(x + \frac{x^2}{2}\right)$$

$$= 1 + 6x + \frac{5}{2}x^2 + 0x^3$$

$$T(1 - x^2) = 1 - x^2 + 5\int_0^x (1 - t^2)dt = 1 - x^2 + 5\left(x - \frac{x^3}{3}\right)$$

$$= 1 + 5x - x^2 - \frac{5}{3}x^3$$

$\therefore$
$$M_\beta^\gamma = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 6 & \frac{5}{2} & 0 \\ 1 & 5 & -1 & -\frac{5}{3} \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 6 & 5 \\ 0 & \frac{5}{2} & -1 \\ 0 & 0 & -\frac{5}{3} \end{bmatrix}$$

2.14 If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a LT, $T : P_2(x) \to P_2(x)$ w.r.t. the bases $\{1 - x, x(1 - x), x(1 + x)\}$ and $\{1, 1 + x, 1 + x^2\}$, then find T.

(2016 : 18 Marks)

Solution:

By the definition of matrix representation of LT

$$T(1 - x) = 1.1 - 2(1 + x) + 1(1 + x^2)$$
$$T(x(1 - x)) = -1.1 + 1(1 + x) + 2(1 + x^2)$$
$$T(x(1 + xx) = 2.1 - 1(1 + x) + 3(1 + x^2)$$

Since, T is linear, hence we get

$$T(1) - T(x) = -2x + x^2 \quad \ldots(i)$$
$$T(x) - T(x^2) = x^2 + x + 2 \quad \ldots(ii)$$
$$T(x) + T(x^2) = 3x^2 - x + 4 \quad \ldots(iii)$$

(ii) + (iii) $\Rightarrow$ $2T(x) = 4x^2 + 6 \Rightarrow T(x) = 2x^2 + 3$

$\therefore$ $T(1) = 3x^2 - 2x + 3$ from (i)

and $T(x^2) = x^2 - x + 1$ from (ii)

Now, let $v = ax^2 + bx + c \in P_2(x)$ be any general element. We define T as

$$T(v) = T(ax^2 + bx + c)$$
$$= aT(x^2) + bT(x) + cT(1)$$
$$= a(x^2 - x + 1) + b(2x^2 + 3) + c(3x^2 - 2x + 3)$$
$$= (a + 2b + 3c)x^2 + (-a - 2c)x + (a + 3b + 3c)$$

2.15 Consider the matrix mapping $A : IR^4 \to IR^3$, where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and dimension of the image of A and those of the Kernel A.

(2017 : 15 Marks)

Solution:

Image of A : Col. space (A) = Row space (A^T)

$\therefore$
$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basis of image of $A = \{(1, 1, 3), (2, 3, 8)\}$

$$\text{Dim}(\text{Image}(A)) = 2$$

Dimension of Kernel of A : Let $(x, y, z, w) \in \text{Ker } A$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

We have only 2 basic variable and 2 free variable.

$\therefore$ Basis of Kernel $= \{(-7, 3, 0, 1), (1, -2, 1, 0)\}$

$\therefore$ Ker $(A) = 2$ (Solving in terms of z, w)

2.16 **Let $T: R^2 \to R^2$ be a linear map such that $T(2, 1) = (5, 7)$ and $T(1, 2) = (3, 3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2, then find Rank (A).**

(2019 : 10 Marks)

Solution:

Let $T: R^2 \to R^2$ be a linear map such that $T(2, 1) = (5, 7)$ and $T(1, 2) = (3, 3)$.

Clearly $\quad S = \{(2, 1), (1, 2)\}$ is a basis of R^2

and $\quad S_1 = \{(5, 7), (3, 3)\} \subseteq R^2$

Let $(a, b) \in R^2$, $ab \in R$

then $\quad (a, b) = x(2, 1) + y(1, 2); x, y \in R$ ...(1)

then
$$2x + y = a$$
$$x + 2y = b$$

$$\Rightarrow \quad 3y = 2b - a \Rightarrow \boxed{\begin{matrix} y = \dfrac{2b-a}{3} \\ x = \dfrac{2a-b}{3} \end{matrix}}$$

$\therefore$ from (1)

$$(a, b) = \frac{(2a-b)}{3}(2, 1) + \left(\frac{2b-a}{3}\right)(1, 2)$$

Since, T is a linear map

$$\therefore \quad T(a, b) = \left(\frac{2a-b}{3}\right)T(2, 1) + \left(\frac{2b-a}{3}\right)T(1, 2)$$

$$\Rightarrow \quad T(a, b) = \left(\frac{2a-b}{3}\right)(5, 7) + \left(\frac{2b-a}{3}\right)(3, 3)$$

$$\Rightarrow \quad T(a, b) = \left[\frac{10a-5b}{3}, \frac{14a-7b}{3}\right] + (2b-a, 2b-a)$$

$$T(a, b) = \left[\frac{10a-5b}{3} + 2b - a, \frac{14a-7b}{3} + 2b - a\right]$$

$$\therefore \quad T(a, b) = \left[\frac{7a+b}{3}, \frac{11a-b}{3}\right]$$

Let us find the matrix A corresponding to T with respect to the standard bases e_1, e_2, where

$$e_1 = (1, 0) \text{ and } e_2 = (0, 1)$$

$$T(1, 0) = \left[\frac{7}{3}, \frac{11}{3}\right] = \frac{7}{3}(1, 0) + \frac{11}{3}(0, 1)$$

$$T(0, 1) = \left[\frac{1}{3}, \frac{-1}{3}\right] = \frac{1}{3}(1, 0) - \frac{1}{3}(0, 1)$$

$$\therefore \quad A = \begin{bmatrix} \frac{7}{3} & \frac{1}{3} \\ \frac{11}{3} & \frac{-1}{3} \end{bmatrix}$$

$\because \quad |A| = -2 \neq 0$

$\therefore \quad$ Rank of $A = 2$. Hence, the solution.

3. Matrix

3.1 Find a hermitian and a skew-hermitian matrix each whose sum is the matrix

$$\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$$

(2009 : 12 Marks)

Solution:

Given any matrix A we can write it as

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$$

where A^* is the complex tranjugate of A. Also $\frac{1}{2}(A + A^*)$ is always hermitian as

$$\left[\frac{1}{2}(A + A^*)\right]^* = \frac{1}{2}(A + A^*)^* = \frac{1}{2}(A^* + A)$$
$$= \frac{1}{2}(A + A^*)$$

And $\frac{1}{2}(A + A^*)$ is skew hermitian as

$$\left[\frac{1}{2}(A - A^*)\right]^* = \frac{1}{2}(A^* - A) = -\left[\frac{1}{2}(A - A^*)\right]$$

$$\therefore \quad A = \frac{1}{2}[A + A^*] + \frac{1}{2}[A - A^*]$$
$$= B + C$$

where B is hermitian and C skew hermitian.

Taking A as given matrix.

$$B = \frac{1}{2}(A + A^*)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} + \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix} \text{ which is hermitian.}$$

and $$C = \frac{1}{2}(A - A^*)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} - \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 2i & 1 & \frac{-(i+2)}{2} \\ -1 & 3i & -1 \\ \frac{(-i+2)}{2} & 1 & 5i \end{bmatrix}$$

So, B and C are required vector where

$$A = B + C$$

and B is Hermitian and C skew Hermitian.

3.2 Find a 2×2 real matrix A which is both orthogonal and skew symmetric. Can there exist a 3×3 real matrix which is both orthogonal and skew symmetric. Justify your answer.

(2009 : 20 Marks)

Solution:

Approach : Consider the form of a skew symmetric matrix (diagonal elements zero) and impose conditions for orthogonality.

Let A be a 2×2 skew symmetric matrix and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

A is skew symmetric $\Rightarrow$ $\quad A = -A^T$

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} \Rightarrow a = d = 0 \text{ and } b = -c$$

$$\therefore \quad A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

If A is orthogonal then $AA^T = I$.

$$\Rightarrow \quad b^2 = 1 \Rightarrow b = \pm 1$$

$\therefore \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are the only matrices that are orthogonal and skew symmetric.

Again let A be a 3×3 skew symmetric matrix. Then

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \text{ as seen in previous case.}$$

Also if A is orthogonal.

$$AA^T = I$$

$$\Rightarrow \quad \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}\begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a^2+b^2 & bc & -ac \\ bc & a^2+c^2 & ab \\ -ac & ab & b^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad a^2 + b^2 = a^2 + c^2 = b^2 + c^2 = 1 \qquad \text{...(i)}$$

and $$= ab = bc = ca = 0 \qquad \text{...(ii)}$$

From (ii) two of a, b, c must be zero if $a = b = 0 \Rightarrow a^2 + b^2 = 0 \neq 1$.

Similarly in other cases it can be shown the system of equations is not compatible.

So, a 3×3 skew symmetric matrix can not be orthogonal.

3.3 If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of the matrix

$$A = \begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix}$$

1a

1

Show that $\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \leq \sqrt{1949}$

(2010 : 12 Marks)

Solution:

Given the matrix $$A = \begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix}$$

Now, finding eigen values of A

$$\begin{bmatrix} 26-\lambda & -2 & 2 \\ 2 & 21-\lambda & 4 \\ 4 & 2 & 28-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (26-\lambda)[(21-\lambda)(28-\lambda)-8] + 2[56-2\lambda-16] + 2[4-84+4\lambda) = 0$$

$$\Rightarrow \quad (26-\lambda)[588-49\lambda+\lambda^2-8] + 2[40-2\lambda] + 2[4\lambda-80] = 0$$

$$\Rightarrow \quad (26-\lambda)[580-49\lambda+\lambda^2] + 80 - 4\lambda + 8\lambda - 160 = 0$$

$$\Rightarrow \quad 15080 - 1274\lambda + 26\lambda^2 - 580\lambda + 49\lambda^2 - \lambda^3 - 80 + 4\lambda = 0$$

$$\Rightarrow \quad 15000 - 1850\lambda + 75\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow \quad \lambda^3 - 75\lambda^2 + 1850\lambda - 15000 = 0 \qquad \text{...(1)}$$

Let $\lambda_1, \lambda_2, \lambda_3$ be 3 roots of eq. (1)

$$\therefore \quad \lambda_1 + \lambda_2 + \lambda_3 = 75$$

$$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 = 1850$$

$$\lambda_1\lambda_2\lambda_3 = 15000$$

Also, $$(\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_2^1 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)$$

Putting above values in above eqn., we get

$$(75)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(1850)$$

$$\Rightarrow \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 5625 - 3700$$

$$\Rightarrow \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1925 \leq 1949$$

$$\therefore \quad \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \leq \sqrt{1949}$$

3.4 Let A and B be $n \times n$ matrices over reals. Show that I-BA is invertible if I-AB is invertible. Deduce that AB and BA have same eigen values.

(2010 : 20 Marks)

Solution:

Given A and B are two $n \times n$ matrices over reels.

Let $\quad X = (I - AB)^{-1}$ (Assuming that I–AB is invertible)

$\therefore \quad X = I + AB + (AB)^2 + (AB)^3 + ...$ (By binomial expansion)

Now, $\quad B \times A = BA + (BA)^2 + (BA)^3 + ...$

$\therefore \quad I \times B \times A = I + BA + (BA)^2 + (BA)^3 + ...$

$$= (I - BA)^{-1}$$

$\therefore (I - BA)$ is invertible if $I - AB$ is invertible.

To show that AB and BA have same eigen values. Let λ be an eigen-value of AB.

Case 1 : If $\lambda = 0$, then $\quad ABX = 0.X$ (X-eigen vector)

$\Rightarrow \quad (0.I - AB)X = 0$

$\Rightarrow \quad (0.I - AB)X = 0$

$\Rightarrow \quad 0 = |0.I - AB| = |-A|\,|B| = |B|\,|-A|$

$$= |0.I - BA|$$

$\therefore$ 0 is an eigen-value of BA also.

Case 2 : If $\lambda \neq 0$. Let λ is not an eigen-value of BA. Then

$$BAX \neq \lambda X \Rightarrow (BA - \lambda I)X \neq 0$$

$\Rightarrow \quad |BA - \lambda I| \neq 0 \Rightarrow \lambda^n \left|\frac{1}{\lambda}BA - I\right| \neq 0$

$\Rightarrow \quad \left|\frac{BA}{\lambda} - I\right| \neq 0 \Rightarrow \left|\frac{B}{\lambda} \cdot A - I\right| \neq 0$

or $\quad \left|I - \frac{B}{\lambda} \cdot A\right| \neq 0 \Rightarrow I - \frac{B}{\lambda} \cdot A$ is invertible.

$\therefore$ By above deduction, it can be concluded that $I - A \cdot \frac{B}{\lambda}$ is also invertible.

$\therefore \quad \left|I - A \cdot \frac{B}{\lambda}\right| \neq 0$

$\Rightarrow \quad \lambda^n \left|I - \frac{A - B}{\lambda}\right| \neq 0$

$\Rightarrow \quad |\lambda I - AB| \neq 0$

$\Rightarrow \lambda$ is not an eigen-value of AB which contradicts our assumption.

$\therefore \lambda$ is an eigen-value of BA also.

$\therefore$ AB and BA have same eigen-values.

3.5 **Let A be a non-singular, $n \times n$ square matrix. Show that $A(\text{adj. } A) = |A|.I_n$. Hence, show that $|\text{adj. (Adj. } A)| = |A|^{(n-1)^2}$.**

(2011 : 10 Marks)

Solution:

If A be any n-rowed square matrix, then

$$(\text{adj. } A)A = A(\text{adj. } A) = |A|I_n \quad ...(i)$$

where I_n is a unit matrix of order n and adj. A is adjoint of the matrix A.

Repalcing A by adj. A in (i), we get

$(\text{adj. } A)(\text{adj. adj. } A) = |\text{adj. } A|I_n$

$\Rightarrow \quad (\text{adj. } A)(\text{adj. adj. } A) = |A|^{n-1}I_n$ $\quad [\because |\text{adj. } A| = |A|^{n-1}]$

$\Rightarrow \quad A[(\text{adj. } A)(\text{adj. adj. } A)] = A(|A|^{n-1}I_n)$

$\Rightarrow \quad (A \text{ adj. } A)(\text{adj. adj. } A) = |A|^{n-1}(AI_n)$

$\Rightarrow \quad (|A|I_n)(\text{adj. adj. } A) = |A|^{n-1}A$ $\quad (\because AI_n = A)$

$\Rightarrow \quad |A|(I_n \text{ adj. adj. } A) = |A|^{n-1}A$

$\Rightarrow \quad \text{adj. (adj. } A) = |A|^{n-2}A$

$\Rightarrow \quad \text{adj. (adj. } A) = |A|^{n-2}A$

$\Rightarrow \quad |\text{adj. (adj. } A)| = ||A|^{n-2}A|$

$$= (|A|^{n-2})^n|A| \quad [\because |kA| = k^n|A|)$$

$$= |A|^{n(n-2)+1} = |A|^{(n-1)^2}$$

3.6 Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$. Solve the system of equations given by $AX = B$. 16

Using the above, also solve the system of equations $A^TX = B$ where A^T denotes the tranpose of matrix A.

(2011 : 10 Marks)

Solution:

We have, $\quad AX = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} = B \quad ...(i)$

The augmented matrix is

$$[A \;\; B] = \left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 3 & 4 & 5 & 6 \\ 0 & 6 & 7 & 5 \end{array}\right]$$

Reduce the matrix $[A \;\; B]$ to Echelon form by applying elementary-row transformations only.

$\therefore \quad [A \;\; B] \sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 0 & 4 & 8 & 0 \\ 0 & 6 & 7 & 5 \end{array}\right] R_2 \leftrightarrow R_2 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 6 & 7 & 5 \end{array}\right] R_2 \leftrightarrow \frac{1}{4}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -5 & 5 \end{array}\right] R_3 \leftrightarrow R_3 - 6R_2$$

$\therefore [A \;\; B]$ is in the Echelon form.

Rank of $[A \;\; B]$ = Number of non-zero rows in this Echelon form

$$= 3$$

By the same elementary transformations,

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\Rightarrow \quad \text{Rank of } A = 3$$

Since $\quad \text{Rank } A = \text{Rank}[A \;\; B]$

$\therefore$ The given equations are consistent.

Also, $\quad \text{Rank of } A = \text{Number of Unknowns}$

$\therefore$ The given equations have a unique solution.

The equation (i) is equivalent to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

$$x - z = 2$$
$$y + z = 0$$
$$-5z = 5$$

$\Rightarrow x = 1, y = 1, z = -1$ is the required solution.

2a(i)

3.7 **Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be the eigen values of a $n \times n$ square matrix A with corresponding eigen vectors $X_1, X_2, \ldots\ldots, X_n$. If B is a matrix similar to A show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A.**

(2011 : 10 Marks)

Solution:

If A and B are two square matrices of order n, then B is said to be similar to A if there exists a non-singular matrix P (i.e., $|P| \neq 0$) such that $B = P^{-1}AA$.

So, let $\quad B = P^{-1}AP$

Then,
$$\begin{aligned} B - \lambda I &= P^{-1}AP - \lambda I \\ &= P^{-1}AP - P^{-1}\lambda IP \qquad [\because P^{-1}\lambda IP = \lambda P^{-1}P = \lambda I] \\ &= P^{-1}(A - \lambda I)P \end{aligned}$$

$$\begin{aligned} \therefore \quad |B - \lambda I| &= |P^{-1}(A - \lambda I)P| \\ &= |P^{-1}|\,|A - \lambda I|\,|P| = |P^{-1}|\,|P|\,|A - \lambda I| \\ &= |P^{-1}P|\,|A - \lambda I| = |I|\,|A - \lambda I| = |A - \lambda I| \end{aligned}$$

Thus, the matrices A and B have the same characteristic determinants and hence the same characteristic equations and the same characteristic roots.

Again, if λ is an eigen value of A and X is a corresponding eigen vector, then

$$AX = \lambda X$$

$$\begin{aligned} \therefore \quad B(P^{-1}X) &= (P^{-1}AP)(P^{-1}X) = P^{-1}A(PP^{-1})X \\ &= P^{-1}AX \qquad (\because PP^{-1} = I) \\ &= P^{-1}\lambda X = \lambda(P^{-1}X) \end{aligned}$$

$\therefore$ If X is an eigen vector of A corresponding to the eigen value λ, then $P^{-1}X$ is the eigen vector of B corresponding to the eigen value λ.

2a(ii)

3.8 **Verify the Cayley-Hamilton theorem for the matrix**

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$$

Using this, show that A is non-singular and find A^{-1}.

(2011 : 10 Marks)

Solution :

Cayley-Hamilton Theorem : Every square matrix satisfies its characteristic equation.

Given, $\quad A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$

$$\begin{aligned} \therefore \quad |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -5 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)[(1-\lambda)^2 - 0] - 1(-10 - 3(1-\lambda)) \\ &= 1 - \lambda^3 - 3\lambda(1-\lambda) - 1(-13 + 3\lambda) \\ &= 1 - \lambda^3 - 3\lambda + 3\lambda^2 + 13 - 3\lambda \\ &= -\lambda^3 + 3\lambda^2 - 6\lambda + 14 \end{aligned}$$

$\therefore$ The characteristic equation of A is

$$\lambda^3 - 3\lambda^2 + 6\lambda - 14 = 0$$

We have to verify that

$$A^3 - 3A^2 + 6A - 14I = 0 \qquad \ldots\text{(i)}$$

$$A^3 = \begin{bmatrix} 2 & 15 & 0 \\ 0 & 11 & -6 \\ -30 & 0 & 2 \end{bmatrix}, \; A^2 = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix}$$

$\therefore A^3 - 3A^2 + 6A - 14I$

$$= \begin{bmatrix} 2 & 15 & 0 \\ 0 & 11 & -6 \\ -30 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -15 & 6 \\ -12 & -3 & 6 \\ 12 & 30 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 0 & -6 \\ 12 & 6 & 0 \\ 18 & -30 & 6 \end{bmatrix}$$
$$+ \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, Cayley-Hamilton theorem is verified.

Multiply (i) by A^{-1}, we have

$$A^2 - 3A + 6I - 14A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{14}(A^2 - 3A + 6I)$$
$$= \frac{1}{14}\begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

2a(ii)

3.9 Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen vaues of the matrix B^3 where $B = C^{-1}AC$.

(2011 : 10 Marks)

Solution :

We know that if A and B are two square matrices of order n, then B is said to be similar to A if there exists a non-singular matrix C such that

$$B = C^{-1}AC$$

Now,
$$B = C^{-1}AC$$
$$B^2 = (C^{-1}AC)(C^{-1}AC) = C^{-1}A(CC^{-1})AC = C^{-1}A^2C$$
$$\Rightarrow \quad B^3 = (C^{-1}A^2C)(C^{-1}AC) = C^{-1}A^3C$$

$\Rightarrow B^3$ is similar to A^3.

Also, similar matrices have the same characteristic polynomial and hence the same eigen values.

$\therefore$ Eigen values of B^3 = Eigen values of A^3

Given,
$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 0 & 0 \\ 4 & 2 & 2 \\ 4 & -2 & 6 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 8 & -8 & 8 \\ 12 & 0 & 8 \\ 12 & 8 & 0 \end{bmatrix}$$

The characteristic equation of A^3 is

$$|A^3 - \lambda I| = 0$$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -8 & 8 \\ 12 & -\lambda & 8 \\ 12 & 8 & -\lambda \end{bmatrix} = 0$$

$$\Rightarrow (8-\lambda)(\lambda^2 - 64) + (-8)(96 + 12\lambda) + 8(96 + 12\lambda) = 0$$
$$\Rightarrow 8\lambda^2 - 512 - \lambda^3 + 64\lambda - 768 - 96\lambda + 768 + 96\lambda = 0$$
$$\Rightarrow -\lambda^3 + 8\lambda^2 + 64\lambda - 512 = 0$$
$$\Rightarrow \lambda^3 - 8\lambda^2 - 64\lambda + 512 = 0$$
$$\Rightarrow \lambda^2(\lambda - 8) - 64(\lambda - 8) = 0$$
$$\Rightarrow (\lambda - 8)(\lambda^2 - 64) = 0$$
$$\Rightarrow \lambda = 8, 8, -8$$

$\therefore$ Eigen values of $A^3 = 8, 8, -8$

$\Rightarrow$ Eigen values of $B^3 = 8, 8, -8$

3.10 Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notations :

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$$

(2012 : 12 Marks)

Solution :

The given equations are :

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$$

Let A be the matrix whose rows are the cofficients of the given variable.

$$\therefore \quad A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix}$$

Reduce the matrix A to Echelon form by using row operations.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_2 \leftrightarrow R_2 - 2R_1 \\ R_3 \leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Operate} \\ R_3 \leftrightarrow R_3 - 2R_2 \end{matrix}$$

$\therefore$ The given system of equations reduces to

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_3 + 5x_4 + x_5 = 0$$

Let us take x_2, x_4 and x_5 as free variables. Put $x_2 = u$, $x_4 = s$, $x_5 = t$ and solve, we get

$$x_1 = -2u + \frac{19}{2}s - \frac{5}{2}t$$
$$x_2 = 4$$
$$x_3 = -\frac{5}{2}s - \frac{1}{2}t$$
$$x_4 = s$$
$$x_5 = t$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 4\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 19/2 \\ 0 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -s/2 \\ 0 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Bases of the space W

$$= \left\{(-2,1,0,0,0), \left(\frac{19}{2}, 0, -\frac{5}{2}, 1, 0\right), \left(-\frac{5}{2}, 0 - \frac{1}{2}, 0, 1\right)\right\}$$

Given A is an $m \times n$ matrix of rank r, the dimension of the solution space $AX = 0$ is $n - r$.

Now, Rank of A = Number of non-zero rows in the Echelon form of A $= 2$

$\therefore$ The dimension of the space $W = n - r = 5 - 2 = 3$.

3.11 Find the dimension and a basis for the space W of all solutions of the ~~following homogeneous system~~ using matrix notation :

$$\cancel{x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0}$$
$$\cancel{2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0}$$
$$\cancel{3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0}$$

~~A is a hermitian matrix~~

λ is an eigen value of $A \Rightarrow \frac{|A|}{\lambda}$ is eigen value of Adj A

(2012 : 12 Marks) 26(ii)

Solution :

Since λ is a characteristic root of a non-singular matrix A, therefore $\lambda \neq 0$, as if $\lambda = 0$, then from $AX = \lambda X$.

We have $X = A^{-1}\lambda X = 0$

$$\Rightarrow \quad X = 0, \text{ a contradiction}$$

$$\therefore \quad \lambda \neq 0$$

As λ is a characteristic root of the matrix A, there exists a non-zero vector X such that $AX = \lambda X$.

$$\Rightarrow \quad (\text{Adj. } A)(AX) = (\text{Adj. } A)(\lambda X)$$

$$\Rightarrow \quad [(\text{Adj. } A)A]X = \lambda(\text{Adj. } A)X$$

$$\Rightarrow \quad |A|IX = \lambda(\text{Adj. } A)X \qquad [\because (\text{Adj. } A)A = |A|I]$$

$$\Rightarrow \quad \frac{|A|X}{\lambda} = (\text{Adj. } A)X \qquad [\because IX = XI = X]$$

$$\Rightarrow \quad (\text{Adj. } A)X = \frac{|A|}{\lambda}X \qquad \text{...(i)}$$

Since X is a non-zero vector, from (i) we may conclude that $\frac{|A|}{\lambda}$ is a characteristic root of the matrix Adj. A.

2c

3.12 Let $H = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H\bar{P}$ is diagonal.

IFS 2013 (4b)

(2012 : 20 Marks)

Solution:

Let us consider a matrix M such that

$$M = [H, I]$$

i.e.,

$$M = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right]$$

Now apply row and column operations to reduce the matrix M to diagonal form. Same operations must also be applied to the identity matrix simultaneously.

$$\therefore \quad M \sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 2+2i & i & i & 1 & 0 \\ 0 & -i & -3 & i-2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{Operate :} \\ R_2 \leftrightarrow R_2 + iR_1 \\ R_3 \leftrightarrow R_3 + (i-2)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -2-i \\ 0 & 2+2i & i & i & 2 & 1-2i \\ 0 & -i & -3 & i-2 & 1+2i & 6 \end{array}\right] \quad \begin{array}{l} \text{Operate :} \\ C_2 \leftrightarrow C_2 - iC_1 \\ C_3 \leftrightarrow C_3 + (-2-i)C_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -2-i \\ 0 & 2+2i & i & i & 2 & 1-2i \\ 0 & 0 & \frac{i-13}{4} & \frac{5i-9}{4} & \frac{5i+3}{2} & \frac{27-i}{4} \end{array}\right] \quad \begin{array}{l} \text{Operate :} \\ R_3 \leftrightarrow R_3 + \frac{i}{2+2i}R_2 \end{array}$$

$$\left(\text{Also note } \frac{i}{2+2i} = \frac{1+i}{4}\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & \frac{-9-5i}{4} \\ 0 & 2+i & 0 & i & 2 & \frac{3-5i}{2} \\ 0 & 0 & \frac{i-13}{4} & \frac{5i-9}{4} & \frac{5i+3}{2} & \frac{31}{4} \end{array}\right] \quad \begin{array}{l} \text{Operate :} \\ C_3 \leftrightarrow C_3 + \frac{1-i}{4}C_2 \end{array}$$

$$\left(\text{Note : } \frac{-i}{2+2i} = \frac{1-i}{4}\right)$$

Thus, H has been diagonalized, and

$$P = \begin{bmatrix} 1 & -i & \frac{-9-5i}{4} \\ i & 2 & \frac{3-5i}{2} \\ \frac{5i-9}{4} & \frac{5i+3}{2} & \frac{31}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & i & \frac{5i-9}{4} \\ -i & 2 & \frac{5i+3}{2} \\ \frac{-9-5i}{4} & \frac{3-5i}{2} & \frac{31}{4} \end{bmatrix}$$

Thus, $\exists$ a non-singular matrix P such that

$$D = P^T H\bar{P} \text{ is a diagonal matrix, where}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & \frac{i-13}{4} \end{bmatrix}$$

3.13 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

1.a

by using elementary row operations. Hence solve the system of linear equations :

$$x + 3y + z = 10$$
$$2x - y + 7z = 21$$
$$3x + 2y - z = 4$$

(2013 : 10 Marks)

Solution:

Approach : Take the given matrix and an elementary matrix. Through elementary row operations reduce A to I and apply the same operations to I. When A is reduced to I, I will be reduced to A^{-1}.

$$A = I_3 A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

We reduce L.H.S. to row reduced form.

Applying $R_2 \leftrightarrow R_2 - 2R, R_3 \leftrightarrow R_3 - 3R$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & 5 \\ 0 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 \to -\frac{1}{7}R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -\frac{5}{7} \\ 0 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{7} & -\frac{1}{7} & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Apply $R_1 \leftrightarrow R_2 - 3R_1, R_3 \leftrightarrow R_3 + 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{22}{7} \\ 0 & 1 & -\frac{5}{7} \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} & 0 \\ \frac{2}{7} & -\frac{1}{7} & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow -\frac{1}{9}R_3$

$$\begin{bmatrix} 1 & 0 & \frac{22}{7} \\ 0 & 1 & -\frac{5}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} & 0 \\ \frac{2}{7} & -\frac{1}{7} & 0 \\ \frac{1}{9} & \frac{1}{9} & -\frac{1}{9} \end{bmatrix} A$$

Apply $R_1 \leftrightarrow R_1 - \frac{22}{7}R_3, R_2 \leftrightarrow R_2 + \frac{5}{7}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{13}{63} & \frac{5}{63} & \frac{22}{63} \\ \frac{23}{63} & -\frac{4}{63} & -\frac{5}{63} \\ \frac{1}{9} & \frac{1}{9} & -\frac{1}{9} \end{bmatrix}$$

Given equation

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is the solution to the linear system.

3.14 Let A be a square matrix and A^* be the adjoint, show that the eigen values of the matrices AA^* and A^*A are real. Further, show that trace (AA^*) = trace (A^*A).

(2013 : 10 Marks)

Solution:

Let λ be an eigen value of AA^*.

Then $\quad AA^*X = \lambda X$.

Taking transpose conjugate of both sides

$$(AA^*X)^* = (\lambda X)^*$$

$$\Rightarrow \quad X^*(AA^*)^* = \bar{\lambda}X^* \qquad \text{(where } \bar{\lambda} \text{ is the complex conjugate of } \lambda\text{)}$$

$$\Rightarrow \quad X^*(AA^*) = \bar{\lambda}X^*$$

$$\Rightarrow \quad X^*(AA^*)X = \bar{\lambda}X^*X$$

$$\Rightarrow \quad X^*\lambda X = \bar{\lambda}X^*X \Rightarrow (\lambda - \bar{\lambda})X^*X = 0$$

And as $X^*X = 0 \Rightarrow X = 0$ which is not true so $\lambda = \bar{\lambda}$, i.e., λ is real. So, eigen values of AA^* are real.

Similarly, considering the eigen values of A^*A in a similar manner we can prove they are real.

To show $\text{tr}(AA^*) = \text{tr}(A^*A)$.

We show the eigen values of AA^* and A^*A are same.

Let λ be any eigen values of AA^*.

$$AA^*X = \lambda X \Rightarrow A^*(AA^*X) = \lambda A^*X$$

$$\Rightarrow \quad (A^*A)A^*X = \lambda A^*X$$

$\Rightarrow \lambda$ is an eigen value of A^*A with eigen vector A^*X.

Similarly, if λ is a eigen value of A^*A it can be shown to be an eigen value of AA^* as well.

$\therefore$ Eigen values of AA^* and A^*A are the same.

$$\therefore \quad \text{tr}(AA^*) = \text{Sum of eigen values of } AA^*$$
$$= \text{Sum of eigen values of } A^*A = \text{tr}(A^*A)$$

96 8

3.15 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix}$ where $(w \neq 1)$ is a cube root of unity. If $\lambda_1, \lambda_2, \lambda_3$ denote the eigen values of A^2, show that $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$.

(2013 : 8 Marks)

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w \\ 1 & w & w^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1+w+w^2 & 1+w+w^2 \\ 1+w+w^2 & 1+w+w^2 & 1+w^3+w^3 \\ 1+w+w^2 & 1+w^3+w^3 & 1+w+w^2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \text{ as } 1 + w + w^2 = 0$$

Eigen values of A^2

$$|A^2 - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (3-\lambda)(\lambda^2 - 9) = 0 \Rightarrow \lambda = 3, 3, -3$

$\therefore \quad |\lambda_1| + |\lambda_2| + |\lambda_3| = 9 \leq 9$

$\therefore \quad |\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$

3.16 Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$$

2b

(2013 : 8 Marks)

Solution:

Reduce the matrix to row reduced echelon form using elementary row operations.
Apply $R_2 \leftrightarrow R_2 - 2R$, $R_3 \leftrightarrow R_3 - 3R_1$, $R_4 \leftrightarrow R_4 - 5R_1$, $R_5 \leftrightarrow R_5 - 8R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -2 & -3 & -3 & -2 \\ 0 & -4 & -7 & -9 & -10 \end{bmatrix} \xrightarrow[\substack{R_4 \leftrightarrow R_4 - 2R_2 \\ R_5 \leftrightarrow R_5 - 4R_2}]{R_3 \leftrightarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & -3 & -9 & -18 \end{bmatrix}$$

$$\xrightarrow[R_3 \leftrightarrow R_4]{R_5 \leftrightarrow R_5 - 3R_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 3 non-zero rows in the row reduced echelon form.

$\therefore \quad \text{Rank}(A) = 3$

2c (i)

3.17 Let A be a Hermitian matrix having all distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$. If $X_1, X_2, \ldots, X_n$ are corresponding eigen vectors then show that $n \times n$ matrix C whose k^{th} column consist of the vector X_k is non-singular.

(2013 : 8 Marks)

Solution:

We show that non singularity of C means that the columns, i.e., eigen vectors X_k are linearly independent. If C is non-singular then none of the rows in the row reduced echelon form is zero. This means no column vector is a linear combination of previous columns implying that all columns are linearly independent.

$\therefore$ We need to prove that $X_1, \ldots, X_k$ are linearly independent.

Now A is Hermitian so all eigen values are real.

Let $X_1, \ldots, X_k$ not be linearly independent. Then $\exists X_r$ such that X_r is a linear combination of previous vectors and $X_1, \ldots, X_{r-1}$ are linearly independent.

i.e., $\quad X_r = C_1X_1 + \ldots + C_{r-1}X_{r-1} \quad \ldots(i)$

where not all C_i's are zero.

Premultiplying by A

$$AX_r = C_1AX_1 + \ldots + C_{r-1}AX_{r-1}$$

$$\lambda_r X_r = C_1\lambda_1 X_1 + \ldots + C_{r-1}\lambda_{r-1}X_{r-1} \quad \ldots(ii)$$

Multiplying (i) by λ_r and subtracting from (ii)

$$0 = C_1(\lambda_1 - \lambda_r)X_1 + \ldots + C_{r-1}(\lambda_{r-1} - \lambda_r)X_{r-1}$$

But $X_1, \ldots, X_r$ are linearly independent.

$\Rightarrow \quad C_i(\lambda_i - \lambda_r) = 0 \quad \forall\, i \in 1 \text{ to } r-1$

But at least one $C_i \neq 0$

So for that i, $\quad \lambda_i = \lambda_r$ whereas $i \neq r$

i.e., λ's are not all distinct. This is a contradiction.

$\therefore X_1, \ldots, X_k$ are linearly independent.

$\therefore C$ is a singular matrix.

3.18 Using elementary row or column operations, find the rank of the matrix $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

1b

(2014 : 10 Marks)

Solution:

Given matrix is

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_4 \\ R_2 \leftrightarrow R_3 \end{matrix}$$

and we get

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 3 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

We get

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \end{bmatrix} R_4 \to 2R_4 + R_2$$

We get

$$\sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -2 & 6 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in echelon form and has 3 independent rows. So rank at matrix = 3.

3.19 Investigate the values of λ and μ so that the equations $x + 2y + \lambda z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have (i) no solution (ii) a unique selection (iii) an infinite number of solutions.

(2014 : 10 Marks)

2b

Solution:

Write the matrix equation of the given system

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] R_3 \to R_3 - R_2$$

→ If $\lambda = 3$ and $\mu \neq 0$ then $\vartheta(A|B) = 3$ and $\vartheta(A) = 2$

$\therefore$ $\vartheta(A|B) \neq \vartheta(A)$

$\therefore$ The given equation have no solutions.

→ If $\lambda \neq 3$ and μ = any value then $\vartheta(A|B) = \vartheta(A) = 3$ the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

→ If $\lambda = 3$ and $\mu = 10$ then $\vartheta(A|B) = \vartheta(A) = 2 <$ the number of unknown variables.

$\therefore$ The given equations are consistent and have infinite solutions.

10 3.20 Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Caley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5 = -4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

(2014 : 10 Marks)

Solution:

The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0 \qquad \text{...(i)}$$

$$\Rightarrow \quad (\lambda - 5)(\lambda + 1) - 8 = 0$$

$$\Rightarrow \quad \lambda = 5, -1$$

The roots of this equation are $\lambda = 5, -1$ and these are characteristic roots of A by Caley-Hamilton theorem, the matrix A must satisfy its characteristic equation (i).

So we must have

$$A^2 - 4A - 5I = 0 \qquad \text{...(ii)}$$

Verification:

We have $$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now $$A^2 - 4A - 5I = \begin{pmatrix} 9 & 16 \\ 8 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 16 \\ 8 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Now multiplying (ii) by A^{-1}, we get

$$A^2A^{-1}, 4AA^{-1} - 5IA^{-1} = 0A^{-1}$$

$$\Rightarrow \quad A - 4I - 5A - 1 = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I)$$

Now $$A - 4I = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$$

The characteristic equation of A is $\lambda^2 - 4\lambda - 5 = 0$. Dividing the polynomial

$\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10 = 0$ by the polynomial $\lambda^2 - 4\lambda - 5$, we get

$$\lambda^5 - 4A^4 - 7A^3 + 11A^2 - \lambda - 10 = (A^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5$$

$$\therefore A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = (A^2 - 4A - 5I)(A^3 - 2A + 3I) + A + 5I$$

But $$A^2 - 4A - 5I = 0$$

Therefore we get

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$$

which is a linear polynomial in A.

3.21 Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors. 4(c)(i)

(2014 : 8 Marks)

Solution:

First, we find Eigen values :

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -(\lambda + 2)[\lambda(\lambda - 1) - 12] + 4(\lambda + 3) + 3(\lambda + 3) = 0$$

$$\Rightarrow \quad -(\lambda + 2)[\lambda^2 - \lambda - 12) + 7(\lambda + 3) = 0$$

$$\Rightarrow \quad -(\lambda + 2)(\lambda - 4)(\lambda + 3) + 7(\lambda + 3) = 0$$

Solving we get

$$\lambda = -3, -3, 5$$

So, eigen values are $-3, -3, 5$.

Finding eigen vectors

For $\lambda = -3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + 2y - 3z = 0 \Rightarrow z = \frac{x + z_0}{3}$$

So, two eigen vectors corresponding to $\lambda = -3$ are

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$$

For $\lambda = 5$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving we get $x = -7$, $y = -27$

So, $$V_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

So, eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2/3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

4c (ii)

3.22 Prove that the eigen values of a unitary matrix have absolute value 1.

(2014 : 7 Marks)

Solution:

Let A be an unitary matrix

$$\therefore \quad A^{\theta} A = I \quad \text{...(i)}$$

Let λ be a characteristic not of A.

$\therefore$ $\exists$ a non-zero column matrix.

i.e., characteristic vector x such that

$$Ax = \lambda x \quad \text{...(ii)}$$

Taking conjugate transpose on the sides, we get

$$(Ax)^{\theta} = (\lambda x)^{\theta}$$

$$\Rightarrow \quad x^{\theta} A^{\theta} = \bar{\lambda} x^{\theta} \quad \text{...(iii)}$$

$$\Rightarrow \quad x^{\theta} A^{\theta} = (\bar{\lambda} x^{\theta})(\lambda x)$$

$$\Rightarrow \quad x^{\theta}(1)x = |\lambda|^2 (x^{\theta} x) \quad \text{(by (i))} \qquad (\because \bar{\lambda}\lambda = |\lambda|^2)$$

$$\Rightarrow \quad x^{\theta} x = |\lambda|^2 (x^{\theta} x)$$

$$\Rightarrow \quad (1 - |\lambda|^2)(x^{\theta} x) = 0$$

$$\Rightarrow \quad 1 - |\lambda|^2 = 0 \qquad (\because x \neq 0 \Rightarrow x^{\theta} x \neq 0)$$

$$\Rightarrow \quad |\lambda|^2 = 1$$

$$\Rightarrow \quad |\lambda| = 1$$

3.23 Reduce the following matrix to row echelon form and hence find its rank :

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$$

(2015 : 10 Marks)

Solution:

Let us denote given matrix by A, then reducing A to echelon form

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + R_1 \\ R_4 \to R_4 + 5R_2 \end{matrix}$$

This is the row-echelon form of A.

$$\text{Rank}(A) = 2$$

3.24 If matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then find A^{30}.

2a 13

(2015 : 12 Marks)

Solution:

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$A^{16} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}$$

Note that, $$A^{30} = A^{16+8+4+2} = A^{16}A^8A^4A^2$$

$$A^{30} = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 12 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 15 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$$

3.25 Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. 2c

(2015 : 12 Marks)

Solution:

Character polynomial; $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$(\lambda+2)(\lambda-3)(\lambda-6) = 0$$

$$\lambda = -2, 3, 6$$

For Eigen-Value $\lambda = -2$

$$(A - 2I)V = 0$$

$\therefore$ $$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $$3x + y + 3z = 0$$

$$x + 7y + z = 0$$

$\Rightarrow$ $$y = 0$$

$$x + z = 0,\ z = -x$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = -x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \text{Eigen-vector for } \lambda = -2.$$

For $\lambda = 3$:

$$(A - 3I)V = 0$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $$-2x + y + 3z = 0$$

$$x + 2y + z = 0$$

$$3x + y - 2z = 0$$

$\therefore$ $$5y + 5z = 0 \Rightarrow y = -z \text{ and } x = -y$$

$\therefore$ $$V^T = (1, -1, 1) \text{ for } \lambda = 3.$$

For $\lambda = 6$:

$$(A - 6I)V = 0$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $$-5x + y + 3z = 0$$

$$x - y + z = 0$$

$$3x + y - 5z = 0$$

$\therefore$ $$x = z,\ y = 2x$$

$$V^T = (1, 2, 1)$$

3.26 Using elementary row operations, find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. 1a

(2016 : 6 Marks)

Solution:

Consider $$A = IA$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 \to R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$R_3 \to R_{\frac{3}{2}} + R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} A$$

$$\begin{matrix} R_2 \to R_2 - R_3 \\ R_1 \to R_1 - R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} A$$

$$R_1 \to R_1 - 2R_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{3}{2} & 1 & \frac{1}{2} \end{bmatrix} A$$

$\Rightarrow$ $$I = A^{-1}A$$

15

3.27 If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then find $A^{14} + 3A - 2I$.

(2016 : 4 Marks)

Solution:

Characteristic polynomial of A

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$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[-6+3\lambda-2\lambda+\lambda^2+6]-1(-15-5\lambda+12)+3(-5+4-2\lambda) = 0$$

$$(1-\lambda)(l+\lambda^2)+3+5\lambda-3-6\lambda = 0$$

$$\lambda+\lambda^2-\lambda^2-\lambda^3-\lambda = 0$$

$$\text{i.e., } \lambda^3 = 0$$

Using Cayley Hamilton theorem, i.e., every square matrix satisfies its character polynomial.

$$A^3 = 0$$

$$A^{14} = A^{12}.A^2 = 0 \cdot A^2 = 0$$

$\therefore$

$$A^{14}+3A-2I = 3A-2I$$

$$= \begin{bmatrix} 3 & 3 & 9 \\ 15 & 6 & 18 \\ -6 & -3 & -9 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 9 \\ 15 & 4 & 18 \\ -6 & -3 & -11 \end{bmatrix}$$

3.28 Using elementary row operations; find the condition that the linear equations 16

$$x-2y+z = a$$
$$2x+7y-3z = b$$
$$3x+5y-2z = c$$

have a solution.

(2016 : 7 Marks)

Solution:

The given system, $AX = B$, where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 7 & -3 \\ 3 & 5 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

This system has a solution if

$$\text{Rank}[A \mid B] = \text{Rank}[A]$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 7 & -3 \\ 3 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 11 & -5 \\ 0 & 11 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 11 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $\text{Rank}(A) = 2$

System has solution in $\text{Rank}(A \mid B) = 2$.

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & a \\ 2 & 7 & -3 & b \\ 3 & 5 & -2 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & a \\ 0 & 11 & -5 & b-2a \\ 0 & 11 & -5 & c-3a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & a \\ 0 & 11 & -5 & b-2a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

$R_3 \to R_3 \to R_2$

$$P[A \mid B] = 2 \Rightarrow c-b-a = 0, \text{ i.e., } c = a+b$$

16

3.29 If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then find the eigen values and eigen vectors of A.

(2016 : 8 Marks)

Solution:

To find eigen values : $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(1-\lambda)^2-1] = 0 \quad \text{(Expanding along 3rd row)}$$

$$(1-\lambda)(\lambda^2-2\lambda) = 0$$

$$\Rightarrow \quad \lambda(\lambda-1)(\lambda-2) = 0$$

$$\Rightarrow \quad \lambda = 0, 1, 2$$

Now, eigen vectors :

For $\lambda = 0$, $\quad Ax = \lambda x$, i.e., $[A-\lambda I]x = 0$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow x+y = 0 \Rightarrow x = -y; z = 0$$

$\therefore$ Eigen space $= (-y, y, 0) = y(-1, 1, 0)$

For $\lambda = 1$,

$$(A-\lambda I)x = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow y = 0; x = 0$$

$\therefore$ Eigen space $= (0, 0, z) = z(0, 0, 1)$

For $\lambda = 2$

$$(A-2I)x = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow \begin{matrix} -x+y = 0 \\ x-y = 0 \\ -z = 0 \end{matrix} \Rightarrow x = y$$

Eigen Space $= (x, y, z) = (x, x, 0) = x(1, 1, 0)$

3.30 Prove that eigen values of a Hermitian matrix are all real. 17

(2016 : 8 Marks)

Solution:

Let A be Hermitian matrix $\Rightarrow A^\theta = A$

Let λ be an eigen value of A.

$\therefore$ $\quad AX = \lambda X$ for some non-zero vector X.

$(AX)^\theta = (\lambda X)^\theta$, taking conjugate transpose

$\Rightarrow \quad X^\theta A^\theta = X^\theta \bar{\lambda}$

$\Rightarrow \quad X^\theta A = \bar{\lambda}(X^\theta)$

$\Rightarrow \quad X^\theta AX = X^\theta \bar{\lambda} X$ (Post multiplying by X)

$\Rightarrow \quad X^\theta \lambda X = X^\theta \bar{\lambda} X$

i.e., $\quad \lambda(X^\theta X) = \bar{\lambda}(X^\theta X)$ (λ is constant)

i.e., $\quad \lambda = \bar{\lambda}$

$\Rightarrow \lambda$ is purely real.

18

3.31 Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

(2017 : 10 Marks)

Solution:

Matrix P is formed by using eigen-vector of A as column vectors.

To find eigen-vecotrs :

$$\det(A-\lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 - 5\lambda + 6 - 2 = 0$

i.e., $(\lambda-4)(\lambda-1) = 0$

$$\lambda = 1, 4$$

Eigen-vector for $\lambda = 1$

$$(A-I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x + 2y = 0$

$$\text{Eigen-vector} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda = 4$, $(A-4I)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x - y = 0$$

$$\text{Eigen-vector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $P = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ such that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

16

3.32 Show that similar matrices have the same characteristic polynomial.

(2017 : 10 Marks)

Solution:

Let matrices A and B are similar. So, there exist a matrix P such that

$$B = P^{-1}AP$$

Character Polynomial of B

$$= |B - \lambda I| = |P^{-1}AP - \lambda P^{-1}P|$$

$$= |P-1(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P|$$

$$= |A - \lambda I| \qquad \left(\because |P^{-1}| = \frac{1}{|P|}\right)$$

Hence, Proved.

Method II : Let A has eigen-value λ and corresponding eigen vector v. So,

$$Av = \lambda v \qquad ...(i)$$

We show that $B = P^{-1}AP$ has same eigen-value corresponding to eigen vector

$$u = P^{-1}v$$

$$Bu = (P^{-1}AP)(P^{-1}v)$$

$$= P^{-1}A(PP^{-1})v = P^{-1}Av$$

$$= P^{-1}(\lambda v) = \lambda(P^{-1}v)$$

$$= \lambda v$$

Hence, A and B have same eigen values. Therefore, have same characteristic polynomial.

3.33 Prove that distinct non-zero eigen vectors of a matrix are linearly independent.

(2017 : 10 Marks)

Solution:

Let $x_1, x_2, ..., x_m$ be distinct non-zero eigen vectors corresponding to distinct eigen-values, $\lambda_1, \lambda_2, ...\lambda_m$ of a matrix A.

We have $\quad Ax_i = \lambda_i x_i \quad \forall \quad i = 1, 2, ... x_n \qquad ...(i)$

$\therefore \quad (A - \lambda_i I)x_i = 0 \qquad ...(ii)$

Now, let if possible x_1, x_2, x_m be L.D.

We can choose 'r' so that $1 \le r < m$ and $x_1, x_2, ..., x_r$ are L.I. but $x_1, x_2, ... x_r, x_{r+1}$ are L.D. Hence, we can choose scalars $a_1, a_2, ... a_{r+1}$, not all zero such that

$$a_1x_1 + ... + a_{r+1} \cdot x_{r+1} = 0 \qquad ...(iii)$$

$\Rightarrow \quad A(a_1x_1 + ... + a_{r+1}x_{r+1}) = 0$

i.e., $\quad a_1(\lambda_1x_1) + ... + a_{r+1}(\lambda_{r+1}x_{r+1}) = 0 \qquad ...(iv)$

Multiplying (iii) by scalar λ_{r+1} and subtracting from (iv).

$$a(\lambda_1 - \lambda_{r+1})x_1 + ... + a_r(\lambda_r - \lambda_{r+1})x_r = 0 ...(v)$$

But $x_1, ... x_r$ are L.I. and $\lambda_i \neq j$ for $i \neq j$.

$\therefore \quad a_1 = 0, a_2 = 0, ..., a_r = 0$

Putting these values in (iii), we get

$$a_{r+1}x_{r+1} = 0 \Rightarrow a_{r+1} = 0 \text{ as } x_{r+1} \neq 0$$

Now, this contradicts our assumptions that scalars $a_1, a_2, ..., a_{r+1}$ are not all zero. Hence, our initial assumption was wrong and $x_1, ... x_m$ are L.I.

3.34 Consider the following system of equations in x, y, z :

4b

$$x + 2y + 2z = 1$$

$$x + ay + 3z = 3$$

$$x + 11y + az = b$$

(i) For which values of 'a' does the system have a unique solution?

(ii) For which pair of values (a, b) does the system have more than one solution?

(2017 : 15 Marks)

Solution:

For ease of solving, rearranging the equations :

$$x + 2y + 2z = 1$$

$$x + 11y + az = b$$

$$x + ay + 3z = 3$$

Augmented matrix is

$$M = \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 1 & 11 & a & b \\ 1 & a & 3 & 3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 9 & a-2 & b-1 \\ 0 & a-2 & 1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 9 & a-2 & b-1 \\ 0 & 0 & \frac{9-(a-2)^2}{9} & 2-\frac{(b-1)(a-2)}{9} \end{array}\right] R_3 \to R_3 - \frac{(a-2)}{9}R_2$$

$$M = \left[\begin{array}{ccc:c} 1 & 2 & 2 & 1 \\ 0 & 9 & a-2 & b-1 \\ 0 & 0 & \frac{9-(a-2)^2}{9} & 2-\frac{(b-1)(a-2)}{9} \end{array}\right]$$

(i) For system to have unique solution,

$$\text{Rank}(A) = 3$$

$\therefore$ $\frac{9-(a-2)^2}{9} \neq 0$, i.e., $(a-2)^2 \neq 9$

i.e., $(a-2) \neq \pm 3$, i.e., $a \neq -1$ or 5

$\therefore$ System has unique solution for $a \in R \sim \{-1, 5\}$

(ii) For system to have multiple solutions Rank $(A) < 3$ and Rank $(M) < 3$.

$\therefore$ $a = -1$ or 5

For $a = -1$, $2-\frac{(b-1)(a-2)}{9} = 0 \Rightarrow b = -5$

For $a = 5$, $2-\frac{(b-1)(a-2)}{9} = 0 \Rightarrow b = 7$

$\therefore$ Pair $(a, b) = (-1, -5), (5, 7)$ gives multiple solutions.

1a

3.35 Let A be a matrix 3×2 matrix and B a 2×3 matrix. Show that $C = A \cdot B$ is a singular matrix.

(2018 : 10 Marks)

Solution:

Given, A is a 3×2 matrix and B is 2×3 matrix.

$\therefore \rho(A) \le 2$ and $\rho(B) \le 3$ and $\rho(AB) \le \text{Min.}\{\rho(A), \rho(B)\}$

Let $\rho(A) = r_A$, $\rho(B) = r_B$ and $\rho(AB) = r_{AB}$

$\therefore \exists$ a non-singular matrix D such that

$$DA = \begin{bmatrix} P \\ O \end{bmatrix},$$

where P is of other $r_A \times 2$ and O is a zero matrix of order $(3 - r_A) \times 2$.

By post-multiplying both sides by B, we get

$$DAB = \begin{bmatrix} P \\ O \end{bmatrix} B$$

So, $\rho(DAB) = \rho(AB) = r_{AB}$

$\therefore$ rank of $\begin{bmatrix} P \\ O \end{bmatrix} B = r_{AB}$

$\because$ P has only non-zero rows

$\therefore$ $r_{AB} \le r_A$

i.e., $\rho(AB) \le \rho(A)$ ...(i)

Also, $\rho(AB) = \rho(AB)^T = \rho(B^T A^T)$

$\le r(B^T)$

$\le \rho(B)$

$= r_B$

$\therefore$ $\rho(AB) \le \rho(B)$ ...(ii)

From eqn. (i) and (ii), we have

Now, $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$

$\therefore$ $C = A \cdot B$ $\therefore$ C is 3×3 matrix

$\rho(C) = \rho(AB) \le 2$

$\therefore AB$ is a singular matrix.

3.36 Show that if A and B are similar $n \times n$ matrices, then they have same eigen-values.

2a

Solution: **(2017 : 12 Marks)**

Given A & B are similar $n \times n$ matrices.

$\therefore \exists$ non-singular matrix C, such that

$$B = C^{-1}AC$$

Now, let $AX = \lambda X$

$\Rightarrow$ $|A - \lambda I| = 0$ is the characteristic equation of A. (i)

Let $BX = \mu X$

$\Rightarrow$ $|B - \mu I| = 0$

$\Rightarrow$ $|C^{-1}|\,|A - \mu I|\,|C| = 0$

$\Rightarrow$ $|C|^{-1}\,|A - \mu I| = 0$

$\Rightarrow$ $|A - \mu I| = 0$ ...(ii)

From (i) and (ii), it can be concluded that μ and λ are same. Therefore, A and B have same eigen-values.

3.37 For the system of linear equation

$$x + 3y - 2z = -1$$
$$5y + 3z = -8$$
$$x - 2y - 5z = 7$$

determine which of the following statements are true and which are false.

3a

(i) The system has no solution.

(ii) The system has a unique solution.

(iii) The system has infinitely many solutions.

(2018 : 13 Marks)

Solution:

The given system of equations can be written as

$$\underbrace{\begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & 3 \\ 1 & -2 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}}_{B}$$

$$AX = B$$

The augmented matrix $[A|B]$ can be written as :

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 1 & -2 & -5 & 7 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & -5 & -3 & 8 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, $\rho(A) = \rho(A|B) = 2 < 3$

$\therefore$ System has infinitely many solutions.

3.38 If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ then show that $AB = 6I_3$. Use this result to solve the following system of equations :

1D

$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

(2019 : 10 Marks)

Solution:

Given:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I_3. \text{ Hence Proved.}$$

$$A.B = 6I_3$$
$$A.B.B^{-1} = 6I_3B^{-1}$$
$$A = 6B^{-1}$$
$$B^{-1} = \frac{1}{6}A$$
$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}; C = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and from the linear equation

$$BX = C \Rightarrow B^{-1}BX = B^{-1}C$$
$$X = B^{-1}C$$
$$X = \frac{1}{6}AC$$
$$X = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix} = \frac{1}{6} \times 6 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$x = 1, y = 1, z = 2$ is the required result.

3.39 Let A and B be two orthogonal matrices of same order and $\det A + \det B = 0$. Show that $A + B$ is a singular matrix.

(2019 : 15 Marks)

Solution:

Given that A and B are two orthogonal matrices of some order and $\det A + \det B = 0$. We know that all orthogonal matrices are unvertible, so we can write

$$A + B = A(I + A^{-1}B)$$

Now it is enough to show that $(I + A^{-1}B)$ is singular. ...(1)

[$\because$ the product of matrices is singular as long as at least one of the matrices is singular]

Also,
$$\det(A^{-1} * B) = \det(A^{-1}) * \det B$$
$$= \det(A^{-1}) * (-\det(A)) \qquad [\because \det(A) + \det(B) = 0]$$

Now, since the determinant of an orthogonal matrix is either 1 or –1,

$$\det(A)^{-1} * (-\det(A)) = -1$$

[$\because$ This product is the product of –1 and 1 (in some order)] ...(2)

Using (2), we have

$$\det(I + A^{-1}B) = \det I + \det(A^{-1} * B)$$
$$= \det I + (-1)$$
$$= 1 - 1$$
$$= 0$$

$\Rightarrow A + B$ is singular matrix [$\because$ A matrix is singular. Its determinant is 0]

Alternatively, since A and B are orthogonal matrices

i.e.,
$$\left.\begin{aligned} A \cdot A^T = I \Rightarrow |A|^2 = 1 \\ B \cdot B^T = I \Rightarrow |B|^2 = 1 \end{aligned}\right\} \qquad ...(1)$$

Also, given that $|A| + |B| = 0 \Rightarrow |A| = -|B|$ ...(2)

Consider,
$$|A + B|^2 = |A|^2 + |B|^2 + 2|A|\,|B|$$
$$= 2 + 2(-|B|^2)$$

Using (1) and (2)

$$= 2 - 2(1)$$
$$= 0$$
$$\Rightarrow \quad |A + B|^2 = 0$$
$$\Rightarrow \quad |A + B| = 0$$

$\Rightarrow A + B$ is singular matrix. Hence, proved.

3.40 Let

3c

$$A = \begin{pmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{pmatrix}$$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in R^4 \,\middle|\, A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$

(2019 : 15+5 = 20 Marks)

Solution:

The given matrix is

$$A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$$

Now, we use row transformations to obtain an echelon matrix.

$$\sim \begin{bmatrix} 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \\ 5 & 7 & 2 & 1 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2; \\ R_2 \leftrightarrow R_3; \\ R_3 \leftrightarrow R_4 \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -8 & 1 \\ 0 & 1 & 21 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_2 \\ R_3 \to R_3 - (R_1 + R_2) \\ R_4 \to R_4 - (R_2 + R_3) \\ \end{matrix} \qquad ...(I)$$

which is clearly in the echelon form and the number of non-zero is 2.

$\therefore$ $\qquad \rho(A) = 2 \qquad ...(1)$

Let $\qquad R^4 = \left\{ \dfrac{(x_1, x_2, x_3, x_4)}{x_1, x_2, x_3, x_4 \in R} \right\}$ by given vector space.

Let $\qquad V = \left\{ \dfrac{(x_1, x_2, x_3, x_4) \in R^4}{A \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix}} = 0 \right\} \subseteq R^4$

We have $\qquad Ax = 0$

$$\Rightarrow \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -8 & 1 \\ 0 & 1 & 21 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{[From (I)]}$$

$\Rightarrow \qquad x_1 + x_2 - 8x_3 + x_4 = 0 \qquad ...(1)$

$\Rightarrow \qquad x_2 + 21x_3 - 2x_4 = 0 \qquad ...(2)$

From (2),

$$x_2 = -21x_3 + 2x_4$$

$\therefore$ From (1)

$$x_2 - 21x_3 + 2x_4 - 8x_3 + x_4 = 0$$

$\Rightarrow \qquad x_2 - 29x_3 + 3x_4 = 0$

$\Rightarrow \qquad x_1 = 29x_3 - 3x_4$

$\therefore$ There are two for variables x_3 and x_4 (say)

$\therefore \qquad \dim V = 2$ and

$$V = \left\{ \frac{(29x_3 - 3x_4, -21x_3 + 2x_4, x_3, x_4}{x_3, x_4 \in R} \right\} \qquad ...(3)$$

Hence, the result.

3.41 State the Cayley-Hamilton theorem. Use this theorem to find A^{100}, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

4a. 22

(2019 : 15 Marks)

Solution:

Statement : "Cayley-Hamilton Theorem"

Every square matrix A is a zero of its characteristic polynomial.

OR

Every square matrix satisfies its non-characteristic equation.

The given matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial for the given matrix

$$A = |A - \lambda I|$$
$$= (1 - \lambda)(\lambda^2 - 1)$$
$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

By Cayley-Hamilton theorem,we have

$$-A^3 + A^2 + A - I = 0$$

$\Rightarrow \qquad A^3 = A^2 + A - I \qquad ...(1)$

Now, $\qquad A^4 = A^3 + A^2 - A$

$\qquad = (A^2 + A - I) + A^2 - A \qquad$ (using (1))

$\qquad A^4 = 2A^2 - I$

Similarly, $\qquad A^5 = 2A^2 + A - 2I$

$\qquad A^6 = 3A^2 - 2I \qquad ...(2)$

$\qquad \vdots$

$\qquad A^8 = 4A^2 - 3I \qquad ...(3)$

$\qquad \vdots$

$\qquad A^{10} = 5A^2 - 4I \qquad ...(4)$

$\qquad \vdots$

From (2), (3), (4) and inductive sequence, we have

$$A^{2n} = nA^2 - (n-1)I$$

$\therefore \qquad A^{100} = 50A^2 - 49I$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{100} = \begin{bmatrix} 50 & 0 & 0 \\ 50 & 50 & 0 \\ 50 & 0 & 50 \end{bmatrix} - \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$\therefore \quad A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 50 & 1 & 0 \\ 50 & 0 & 1 \end{bmatrix}$$

which is the required matrix.

■■■■

2 Calculus

1. Function of a Real Variable

1.1 Suppose that f'' is continuous on [1, 2] and that f has three zeros in the interval (1, 2). Show that f'' has at least one zero in the interval (1, 2).

(2009 : 12 Marks)

Solution:

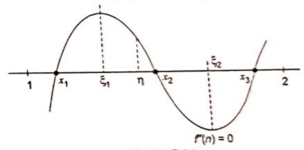
Insight : This question uses the fact that **continuity of any derivative of a function ensures continuity and differentiability of lower order derivatives** and the Rolle's theorem.

f'' is continuous on [1, 2]

$\Rightarrow f'$ is continuous and differentiable on [1, 2]

$\Rightarrow f$ is continuous and differentiable on [1, 2]

f has three zeros in (1, 2). Let them be x_1, x_2, x_3 with $x_1 < x_2 < x_3$.



Inflection Point

In the interval $[x_1, x_2]$, applying Rolle's theorem.

f is continuous on $[x_1, x_2]$.

f is differentiable on (x_1, x_2).

$$f(x_1) = f(x_2) = 0$$

$\Rightarrow \exists\, \xi_1 \in (x_1, x_2)$ such that $f'(\xi) = 0$ by Rolle's theorem.

Similarly, applying Rolle's theorem in interval $[x_2, x_3]$, $\exists\, \xi_2 \in (x_2, x_3)$ such that $f'(\xi_2) = 0$.

As $\xi_1 < x_2$ and $\xi_2 > x_2 \Rightarrow \xi_1 < \xi_2$.

Applying Rolle's theorem on f' in (ξ_1, ξ_2).

f' is continuous on $[\xi_1, \xi_2]$.

f' is differentiable on (ξ_1, ξ_2) as f'' is continuous on that interval.

$$f'(\xi_1) = f'(\xi_2) = 0$$

$\Rightarrow \exists\, \eta \in (\xi_1, \xi_2)$ so that $f''(\eta) = 0$ by Rolle's theorem.

Also, $(x_1, x_2) \subset (1, 2) \Rightarrow \eta \in (1, 2)$

1.2 If f is the derivative of some function defined on $[a, b]$ prove that there exists a number $\eta \in [a, b]$ such that

$$\int_a^b f(t)\,dt = f(\eta)(b-a)$$

(2009 : 12 Marks)