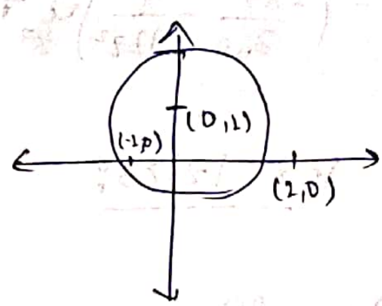
Q.1 Using Cauchy Integral formula, evaluate $\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz$ where C is circle $|z-i|=2$

Soln... Acc. to Cauchy Integral formula:

$$f^n(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}$$



$$\therefore I(z) = \int \frac{z+2}{(z+1)^2(z-2)}\,dz = \int \frac{\frac{(z+2)}{(z-2)}}{(z+1)^2}\,dz$$

where $f(z) = \frac{z+2}{z-2}$ & $a=-1$

$$\therefore f'(-1) = \frac{1}{2\pi i}\int \frac{f(z)}{(z+1)^2}$$

$$\Rightarrow \left[\frac{(z-2)-(z+2)}{(z-2)^2}\right]_{z=1} = \frac{1}{2\pi i}\int \frac{f(z)}{(z+1)^2}\,dz$$

$$\Rightarrow -\frac{4}{9} = \frac{1}{2\pi i}\int \frac{f(z)}{(z+1)^2}\,dz$$

$$\Rightarrow \int_C \frac{z+2}{(z+1)^2(z-2)} = -\frac{8\pi i}{9} \quad \text{Ans.}$$

Q. Expand Laurent's series for $f(z) = \frac{1}{z^2(z^2+2z-3)}$ about $z=0$ for regions (i) $1<|z|<3$ (ii) $|z|>3$.

Soln... Resolving into partial fractions: $f(z) = \frac{-20}{9z} + \left(\frac{-1}{3z^2}\right) + \frac{1}{4(z-1)} - \frac{1}{36(z+3)}$

(i) for $1<|z|<3$:

$$f(z) = \frac{-20}{9z} - \frac{1}{3z^2} - \frac{1}{4z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{36}\left[\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{-20}{9z} - \frac{1}{3z^2} - \frac{1}{4z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{108}\left[1-\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

$$= \left(-\frac{89}{36z} - \frac{7}{12z^2}\right) + \frac{1}{4z}\left(\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{108}\left[1-\frac{z}{3}+\frac{z^2}{9}+\cdots\right]$$

(ii) for $|z|>3$:

$$f(z) = \frac{-20}{9z} - \frac{1}{3z^2} - \frac{1}{4z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{36z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{-20}{9z} - \frac{1}{3z^2} - \frac{1}{4z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{36z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\cdots\right)$$

$$= -\frac{5}{2z} - \frac{1}{2z^2} - \frac{1}{4z}\left(\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{36z}\left(\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right)$$