LAPLACE TRANSFORMS

CHAPTER 5

5.1. INTRODUCTION

In mathematics, a transformation is a device which converts one function into another *e.g.*, operation of differentiation on differentiable function is a transformation because it converts a function into another function known as $f'(x)$. Likewise operation of integration is also a transformation. The Laplace transformation, which we are going to study in this chapter, reduces a given initial value problem to an algebraic equation. Here the solution of initial value problem is directly found without finding the general solution. Laplace transformation is a method for solving differential equations in Physics and Engineering and its knowledge has become an essential part of background required for Scientists and Engineers.

5.2. LAPLACE TRANSFORMATION

[*M.D.U. 2006*]

Let $f(t)$ be a function defined for all $t \geq 0$. Then the Laplace transform of $f(t)$ is denoted by $L\{f(t)\}$ and it is defined to be

$\int_0^\infty e^{-st} f(t)\, dt$ provided the integral exists. Thus

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad \ldots(1)$$

where s is known as a parameter, which may be real.

Clearly integral in (1) is a function of s and we will denote it by $F(s)$

$\therefore \qquad L\{f(t)\} = F(s)$

We can write it as $\qquad f(t) = L^{-1}[F(s)]$

$f(t)$ is called inverse Laplace transform of $F(s)$.

Note :

1. The symbol 'L' is called the **Laplace transform operator** and when operated upon $f(t)$, it transforms into $F(s)$.
2. The Laplace transform of $f(t)$ is said to exist if the integral in (1) converges for some value of s, otherwise it does not exist.

5.3. LAPLACE TRANSFORM OF SOME ELEMENTARY FUNCTIONS

To show that

(*i*) $L(1) = \frac{1}{s}, s > 0$

(*ii*) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ **if n is any real number > -1 and $s > 0$**

$L(t^n) = \frac{n!}{s^{n+1}}, n = 0, 1, 2, \ldots$

(*iii*) $L(e^{at}) = \frac{1}{s-a}, s > a$

(*iv*) $L(\sin at) = \frac{a}{s^2 + a^2}, s > 0$

(*v*) $L(\cos at) = \frac{s}{s^2 + a^2}, s > 0$

(*vi*) $L(\sinh at) = \frac{a}{s^2 - a^2}, s > |a|$

(*vii*) $L(\cosh at) = \frac{s}{s^2 - a^2}, s > |a|$

Proof. (*i*) By definition,

$$L(1) = \int_0^\infty 1 e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}, s > 0$$

[*$s > 0$ because integral is convergent for $s > 0$*]

(*ii*) $$L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt$$

Put $st = x$ so that $dt = \frac{1}{s} dx$

$\therefore \quad L(t^n) = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n \, dx$

or $\quad L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$, provided $s > 0$ and $n + 1 > 0$ *i.e.*, $n > -1$

[Using definition of Gamma function]

$= \frac{n!}{s^{n+1}}$ $\quad$ [$\because \Gamma(n+1) = n!$ *if n is a +ve integer or zero*]

Otherwise $\quad L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ if $n > -1$ and $s > 0$.

***Deductions*:** Above formula is true for $n > -1$.

Putting $n = 0, 1$ in it, we have

$$L(1) = \frac{\Gamma(1)}{s} = \frac{1}{s} \quad \text{and} \quad L(t) = \frac{1!}{s^2} = \frac{1}{s^2}$$

(iii) $\quad L(e^{at}) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}, \; s > a$$

[s > a because integral is convergent for s > a]

(iv) $\quad L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$

$$= \left[\frac{e^{-st}}{s^2 + a^2}(-s \sin at - a \cos at)\right]_0^\infty$$

$$\left[\because \int e^{ax} \sin(bx + c)\, dx = \frac{e^{ax}}{a^2 + b^2}[a \sin(bx + c) - b \cos(bx + c)]\right]$$

For $s > 0$, integral is convergent.

$\therefore \quad L(\sin at) = \frac{a}{s^2 + a^2}$ for $s > 0$.

(v) $\quad L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt = \left[\frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at)\right]_0^\infty$

$$\left[\because \int e^{ax} \cos(bx + c)\, dx = \frac{e^{ax}}{(a^2 + b^2)}[a \cos(bx + c) + b \sin(bx + c)]\right]$$

$$\therefore \quad L(\cos at) = \frac{s}{s^2 + a^2}, \; s > 0$$

(vi) $$L(\sinh at) = \int_0^\infty e^{-st} \sinh at \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2} \right) dt = \frac{1}{2} \int_0^\infty \left(e^{-(s-a)t} - e^{-(s+a)t} \right) dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty = \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right], \text{ for } s > a, s > -a$$

$$\therefore \quad L(\sinh at) = \frac{a}{s^2 - a^2}, \; s > |a|.$$

(vii) $$L(\cosh at) = \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2} \right) dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{-(s-a)t} + e^{-(s+a)t} \right) dt$$

$$= \frac{1}{2} \left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \text{ for } s > a, s > -a$$

$$\therefore \quad L(\cosh at) = \frac{s}{s^2 - a^2}, \text{ for } s > |a|.$$

5.4. LINEAR PROPERTY OF LAPLACE TRANSFORM

To show that $L[a f(t) + b g(t) + c h(t)] = a L\{f(t)\} + b L\{g(t)\} + c L\{h(t)\}$, where a, b, c are constants and f, g, h are functions of t.

[M.D.U. 2006]

Proof. $$L[a f(t) + b g(t) + c h(t)] = \int_0^\infty e^{-st} [a f(t) + b g(t) + c h(t)] \, dt$$

$$= a \int_0^{\infty} e^{-st} f(t)\, dt + b \int_0^{\infty} e^{-st} g(t)\, dt + c \int_0^{\infty} e^{-st} h(t)\, dt$$

$$= a\, L\{f(t)\} + b\, L\{g(t)\} + c\, L\{h(t)\}$$

Note :

1. Above result can be generalised
2. Due to this property, L is called **Linear operator.**

SOLVED EXAMPLES

Example 1. *Find the Laplace transform of $e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t$.*

Solution. Here $f(t) = e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t$

$$\therefore \quad L(f(t)) = L(e^{2t} + 4t^3 - 2 \sin 3t + 3 \cos 3t)$$

$$= L(e^{2t}) + 4\, L(t^3) - 2\, L(\sin 3t) + 3\, L(\cos 3t) \quad [\textit{By Linear property}]$$

$$= \frac{1}{s-2} + 4 \cdot \frac{3!}{s^4} - 2\, \frac{3}{s^2+9} + 3 \cdot \frac{s}{s^2+9} \quad [\textit{Refer Art. 5.3}]$$

$$= \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^2+9}.$$

Example 2. *Find the Laplace transform of $(\sin t - \cos t)^2$.*

Solution.
$$L[(\sin t - \cos t)^2] = L[\sin^2 t + \cos^2 t - 2 \sin t \cos t]$$

$$= L[1 - \sin 2t] = L(1) - L(\sin 2t)$$

$$= \frac{1}{s} - \frac{2}{s^2 + 2^2} = \frac{s^2 + 4 - 2s}{s(s^2+4)}.$$

Example 3. *Find the Laplace transform of $\sin 2t \cos 3t$.* [K.U. 2017]

Solution.
$$L(\sin 2t \cos 3t) = L\left(\frac{1}{2} \cdot 2 \cos 3t \sin 2t\right)$$

$$= L\left[\frac{1}{2}(\sin 5t - \sin t)\right]$$

$$[\because 2 \cos A \sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2}[L(\sin 5t) - L(\sin t)] \quad [\textit{Linear property}]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right] = \frac{1}{2}\left[\frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{4s^2-20}{2(s^2+25)(s^2+1)} = \frac{2(s^2-5)}{(s^2+25)(s^2+1)}.$$

Example 4. *Find the Laplace transform of* $\cos^3 2t$.

Solution. $$L(\cos^3 2t) = L\left[\frac{1}{4}(\cos 6t + 3\cos 2t)\right] \qquad [\because \cos 3A = 4\cos^3 A - 3\cos A]$$

$$= \frac{1}{4}[L(\cos 6t) + 3L(\cos 2t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{3s}{s^2+4}\right]$$

$$= \frac{s[s^2+4+3s^2+108]}{4(s^2+36)(s^2+4)} = \frac{s(s^2+28)}{(s^2+36)(s^2+4)}.$$

Example 5. *Find the Laplace transform of sin at sin bt.* [M.D.U. 2017]

Solution. $$L(\sin at \sin bt) = L\left[\frac{1}{2}\{\cos(a-b)t - \cos(a+b)t\}\right]$$

$$= \frac{1}{2}[L\cos(a-b)t - L\cos(a+b)t]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right]$$

$$= \frac{1}{2}s\left[\frac{s^2+(a+b)^2-s^2-(a-b)^2}{[s^2+(a-b)^2][s^2+(a+b)^2]}\right]$$

$$= \frac{2abs}{[s^2+(a-b)^2][s^2+(a+b)^2]}.$$

Example 6. *Find the Laplace transform of the function* $\sin(wt+\delta)$.

Solution. $$L(\sin(wt+\delta)) = L(\sin wt\cos\delta + \cos wt\sin\delta)$$

$$= \cos\delta\, L(\sin wt) + \sin\delta\, L(\cos wt) \qquad \text{[By Linear property]}$$

$$= \cos\delta \cdot \frac{w}{s^2+w^2} + \sin\delta \cdot \frac{s}{s^2+w^2}, \; s > 0$$

$$= \frac{1}{s^2+w^2}[w\cos\delta + s\sin\delta].$$

Example 7. *Find the Laplace transform of the function $\sinh^3 2t$.* [K.U. 2013]

Solution. $L(\sinh^3 2t) = L\left[\dfrac{e^{2t} - e^{-2t}}{2}\right]^3$

$$= L\left[\frac{1}{8}(e^{6t} - 3e^{2t} + 3e^{-2t} - e^{-6t})\right]$$

$$= \frac{1}{8}[L(e^{6t}) - 3L(e^{2t}) + 3L(e^{-2t}) - L(e^{-6t})]$$

$$= \frac{1}{8}\left[\frac{1}{s-6} - \frac{3}{s-2} + \frac{3}{s+2} - \frac{1}{s+6}\right], \; s > a \qquad \left[\because L(e^{at}) = \frac{1}{s-a}, \; s > a\right]$$

$$= \frac{1}{8}\left[\left(\frac{1}{s-6} - \frac{1}{s+6}\right) + 3\left(\frac{1}{s+2} - \frac{1}{s-2}\right)\right]$$

$$= \frac{1}{8}\left[\frac{s+6-s+6}{s^2-36} + 3.\frac{s-2-s-2}{s^2-4}\right]$$

$$= \frac{1}{8}\left[\frac{12}{s^2-36} - \frac{12}{s^2-4}\right]$$

$$= \frac{12}{8}\left[\frac{s^2-4-s^2+36}{(s^2-4)(s^2-36)}\right] = \frac{48}{(s^2-4)(s^2-36)}.$$

Example 8. *Find the Laplace transform of the function* $f(t) = 1 + 2\sqrt{t} + \dfrac{3}{\sqrt{t}}$.

Solution. $f(t) = 1 + 2\sqrt{t} + \dfrac{3}{\sqrt{t}}$

$\therefore$ $Lf(t) = L(1) + 2L(\sqrt{t}) + 3L(t^{-1/2})$

$$= \frac{1}{s} + 2\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} + 3\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{1}{s} + 2.\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} + 3.\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$$

$$= \frac{1}{s} + 2.\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3.\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}}$$

$$= \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + 3.\frac{\sqrt{\pi}}{s^{1/2}}. \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

EXERCISE 5.1

1. Using standard formula, write the Laplace transform of the following functions :

(i) $t^{5/3}$ (ii) e^{-10t} (iii) $\cosh 7t$

(iv) $\sin 4t$ (v) t

Find the Laplace transform of the following functions :

2. $t^{10} + t^4 + 2$
3. $e^{-5t} + t^2 + 6t$
4. $e^{4t} + e^{2t} + t^3 + \sin^2 t$
5. $\sin 2t \sin 3t$
6. $\sin 6t \sin 4t$
7. $\cos^2 2t$
8. $\sin^3 2t$
9. $\cos^3 t$
10. $\cos^3 4t + t$
11. $\sin 3t \cos 2t$
12. $\cos 4t \cos t$
13. $\cos (at + b)$
14. $\sin 2t \cos 3t + 4$
15. $t - \sin 2t$
16. $t - \sinh 2t$
17. $\cosh at - \cos at$

ANSWERS

1. (i) $\dfrac{10\,\Gamma\left(\frac{2}{3}\right)}{9s^{8/3}}$ (ii) $\dfrac{1}{s+10}, s > -10$ (iii) $\dfrac{s}{s^2 - 49}, s > 7$

(iv) $\dfrac{4}{s^2 + 16}, s > 0$ (v) $\dfrac{1}{s^2}, s > 0$

2. $\dfrac{10!}{s^{11}} + \dfrac{4!}{s^5} + \dfrac{2}{s}, \; s > 0$

3. $\dfrac{1}{s+5} + \dfrac{2!}{s^3} + \dfrac{6}{s^2}, \; s > 0$

4. $\dfrac{1}{s-4} + \dfrac{1}{s-2} + \dfrac{6}{s^4} + \dfrac{1}{2s} - \dfrac{s}{2(s^2+4)}, \; s > 4$

5. $\dfrac{12s}{(s^2+1)(s^2+25)}$

6. $\dfrac{48s}{(s^2+4)(s^2+100)}$

7. $\dfrac{s^2+8}{s(s^2+16)}$

8. $\dfrac{48}{(s^2+4)(s^2+36)}$

9. $\dfrac{s(s^2+7)}{(s^2+9)(s^2+1)}$

10. $\dfrac{s\,(s^2+112)}{(s^2+144)\,(s^2+16)} + \dfrac{1}{s^2},\ s>0$

11. $\dfrac{3s^2+15}{(s^2+25)(s^2+1)},\ s>0$

12. $\dfrac{s\,(s^2+17)}{(s^2+25)\,(s^2+9)},\ s>0$

13. $\dfrac{s\cos b - a\sin b}{s^2+a^2}$

14. $\dfrac{2\,(s^2-5)}{(s^2+25)\,(s^2+1)} + \dfrac{4}{s}$

15. $\dfrac{4-s^2}{s^2(s^2+4)}$

16. $\dfrac{4+s^2}{s^2\,(4-s^2)}$

17. $\dfrac{2a^2 s}{s^4-a^4},\ s>|a|$

5.5. FIRST SHIFTING PROPERTY

If $L\,[f(t)] = F(s)$, then $L\,[e^{at} f(t)] = F(s-a)$

[If $L[f(t)] = F(s)$, then Laplace transform of function obtained by multiplying $f(t)$ by e^{at} is obtained by $F(s-a)$ *i.e.*, changing s to $s-a$ in $F(s)$].

Proof.

$$L[e^{at} f(t)] = \int_0^\infty e^{-st}\, e^{at}\, f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, f(t)\, dt = \int_0^\infty e^{-rt} f(t)\, dt \qquad [\text{say } s-a=r]$$

$$= F(r) = F(s-a).$$

5.6. SOME STANDARD RESULTS OBTAINED BY APPLYING SHIFTING PROPERTY

In Art. 5.3., we have discussed the Laplace transform of some elementary functions. Multiplying the function, whose Laplace transform is required, by e^{at} and changing s to $s-a$ in $F(s)$ we can obtain some standard results which are given below :

1. $L\,(e^{at}\,.\,1) = \dfrac{1}{s-a},\ s>a$

2. $L\,(e^{at}\,t^n) = \dfrac{n!}{(s-a)^{n+1}},\ n>-1$ and $s>a$

3. $L\,(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

4. $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

5. $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$

6. $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$

In each case $s > a$.

5.7. CHANGE OF SCALE PROPERTY

To show that :

(*i*) $\mathbf{L}\,f(at) = \dfrac{1}{a}\,\mathrm{F}\left(\dfrac{s}{a}\right)$

(*ii*) $\mathbf{L}\,f\left(\dfrac{t}{a}\right) = a\,\mathrm{F}(as)$, **where** $\mathbf{L}\,[f(t)] = \mathrm{F}(s)$.

Proof. (*i*) $$\mathrm{L}\,f(at) = \int_0^{\infty} e^{-st} f(at)\,dt$$

Put $at = u$ so that $a\,dt = du$

$$\therefore \quad \mathrm{L}\,f(at) = \int_0^{\infty} e^{\left(-\frac{s}{a}\right)u} f(u)\,\frac{du}{a} = \frac{1}{a}\int_0^{\infty} e^{\left(-\frac{s}{a}\right)u} f(u)\,du$$

$$= \frac{1}{a}\,\mathrm{F}\left(\frac{s}{a}\right)$$ [*Here* $\frac{s}{a}$ *is in place of* s]

(*ii*) $$\mathrm{L}\,f\left(\frac{t}{a}\right) = \int_0^{\infty} e^{-st} f\left(\frac{t}{a}\right) dt$$

Put $\dfrac{t}{a} = u$ *i.e.*, $t = au$ so that $dt = a\,du$

$$\therefore \quad \mathrm{L}\,f\left(\frac{t}{a}\right) = \int_0^{\infty} e^{-asu} f(u)\,a\,du = a\int_0^{\infty} e^{(-as)u} f(u)\,du$$

$$= a\,\mathrm{F}(as).$$ [*Here 'as' is in place of s*]

SOLVED EXAMPLES

Example 1. *Find the Laplace transform of $e^{-at} \sinh bt$.* [M.D.U. 2015]

Solution. We know that $L(\sinh bt) = \dfrac{b}{s^2 - b^2}$

$$\therefore \quad L(e^{-at} \sinh bt) = \frac{b}{[s-(-a)]^2 - b^2} \quad \text{[By shifting property, changing } s \text{ to } s-(-a)]$$

$$= \frac{b}{(s+a)^2 - b^2}.$$

Example 2. *If the Laplace transform of the function $f(t)$ for $t \geq 0$ is $F(s)$, then show that*

$$L[(\cosh at) f(t)] = \frac{1}{2}[F(s-a) + F(s+a)].$$ [M.D.U. 2004]

Solution. $L[e^{at} f(t)] = F(s-a)$

$\therefore \quad L[e^{-at} f(t)] = F(s+a)$ [By shifting property]

Now,
$$L[(\cosh at) f(t)] = L\left(\frac{e^{at} + e^{-at}}{2}\right) f(t)$$

$$= \frac{1}{2}[L(e^{at} f(t)) + L(e^{-at} f(t))]$$

$$= \frac{1}{2}[F(s-a) + F(s+a)].$$

Example 3. *Find the Laplace transform of $\sinh 3t \cos^2 t$.* [K.U. 2014, 07, 04; M.D.U. 2009]

Solution.
$$L(\sinh 3t \cos^2 t) = L\left[\sinh 3t \left(\frac{1+\cos 2t}{2}\right)\right]$$

$$= \frac{1}{2} L[\sinh 3t + \sinh 3t \cos 2t]$$

$$= \frac{1}{2}[L(\sinh 3t) + L(\sinh 3t \cos 2t)]$$

$$= \frac{1}{2}\left[\frac{3}{s^2 - 3^2} + L\left(\frac{e^{3t} - e^{-3t}}{2}\right)\cos 2t\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2 - 9} + \frac{1}{2}\{L(e^{3t}\cos 2t) - L(e^{-3t}\cos 2t)\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-9} + \frac{1}{2}\left\{\frac{s-3}{(s-3)^2+2^2} - \frac{s+3}{(s+3)^2+2^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-9} + \frac{1}{2}\left\{\frac{s-3}{(s^2-6s+13)} - \frac{s+3}{(s^2+6s+13)}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-9} + \frac{1}{2}\left\{\frac{(s-3)(s^2+6s+13)-(s+3)(s^2-6s+13)}{(s^2+13)^2-36s^2}\right\}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2-9} + \frac{1}{2}\left\{\frac{6s^2-78}{s^4-10s^2+169}\right\}\right]$$

$$= \frac{3}{2}\left[\frac{1}{s^2-9} + \frac{s^2-13}{s^4-10s^2+169}\right].$$

Example 4. *Find the Laplace transform of cosh at sin at.*

Solution. $L(\cosh at \sin at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at$

$$= \frac{1}{2}[L(e^{at}\sin at) + L(e^{-at}\sin at)]$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]$$

$$= \frac{a}{2}\left[\frac{s^2+2as+2a^2+s^2-2as+2a^2}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{a}{2}\left[\frac{2s^2+4a^2}{(s^2+2a^2)^2-4a^2s^2}\right] = \frac{a(s^2+2a^2)}{(s^4+4a^4)}.$$

Example 5. *Find the Laplace transform of the function $te^{-4t}\sin 3t$.*

Solution. Here $\quad L(t) = \frac{1}{s^2}$

$$L(t\,e^{i3t}) = \frac{1}{(s-3i)^2} \qquad \text{[Note the step } e^{at} = e^{i\,3t},\ a = 3i]$$

$$L(t\cos 3t + i\,t\sin 3t) = \frac{(s+3i)^2}{(s^2+9)^2} \qquad [e^{i\,3t} = \cos 3t + i\sin 3t]$$

$$L(t\cos 3t + i\,t\sin 3t) = \frac{(s^2-9)+6si}{(s^2+9)^2}$$

$$L(t\cos 3t) + i\,L(t\sin 3t) = \frac{s^2-9}{(s^2+9)^2} + \frac{6si}{(s^2+9)^2}$$

Equating imaginary parts on both sides, we have

$$L(t\sin 3t) = \frac{6s}{(s^2+9)^2}$$

By using shifting property, we get

$$L(te^{-4t}\sin 3t) = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}.$$

Example 6. *Find the Laplace transform of the function $e^{-2t}\sin t\cos 3t$.*

[K.U. 2015; M.D.U. 2006]

Solution.

$$L(e^{-2t}\sin t\cos 3t) = \frac{1}{2}L(e^{-2t}\,2\cos 3t\sin t)$$

$$= \frac{1}{2}L\{e^{-2t}(\sin 4t - \sin 2t)\}$$

$$= \frac{1}{2}[L(e^{-2t}\sin 4t) - L(e^{-2t}\sin 2t)]$$

$$= \frac{1}{2}\left[\frac{4}{(s+2)^2+16} - \frac{2}{(s+2)^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2+16s+32-2s^2-8s-40}{(s^2+4s+20)(s^2+4s+8)}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2+8s-8}{(s^2+4s+20)(s^2+4s+8)}\right]$$

$$= \frac{s^2+4s-4}{(s^2+4s+20)(s^2+4s+8)}.$$

Example 7. *Find the Laplace transform of the function $e^{-3t}(2\cos 5t - 3\sin 5t)$, $t \geq 0$.*

Solution. $L[e^{-3t}(2\cos 5t - 3\sin 5t)]$

$$= 2L(e^{-3t}\cos 5t) - 3L(e^{-3t}\sin 5t),\quad t \geq 0$$

$$= 2\left(\frac{s+3}{(s+3)^2+5^2}\right) - 3\left(\frac{5}{(s+3)^2+5^2}\right)$$

[*By shifting property, changing s to s + 3*]

$$= \frac{2s+6-15}{(s+3)^2+25}, s > -3$$

$$= \frac{2s-9}{(s+3)^2+25}, s > -3.$$

Example 8. *Find the Laplace transform of the following functions of t for $t \geq 0$:*

(*i*) $e^{-t} \sin^2 t$ (*ii*) $e^{-5t} \cosh 3t$

Solution. (*i*) $L(e^{-t} \sin^2 t) = \frac{1}{2} L[e^{-t}(1-\cos 2t)]$

$$= \frac{1}{2}[L(e^{-t}) - L(e^{-t}\cos 2t)]$$

$$= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+2^2}\right] = \frac{1}{2}\left[\frac{s^2+2s+5-s^2-2s-1}{(s+1)(s^2+2s+5)}\right]$$

$$= \frac{1}{2}\left[\frac{4}{(s+1)(s^2+2s+5)}\right] = \frac{2}{(s+1)(s^2+2s+5)}.$$

(*ii*) $L(e^{-5t}\cosh 3t) = \frac{s+5}{(s+5)^2-9}$ [*By shifting property*]

$$= \frac{s+5}{s^2+10s+16}.$$

Method II. $L(e^{-5t}\cosh 3t) = L\left[e^{-5t}\frac{(e^{3t}+e^{-3t})}{2}\right]$

$$= \frac{1}{2} L[e^{-2t} + e^{-8t}] = \frac{1}{2}[L(e^{-2t}) + L(e^{-8t})]$$

$$= \frac{1}{2}\left[\frac{1}{s+2} + \frac{1}{s+8}\right] = \frac{1}{2}\left[\frac{s+8+s+2}{(s+2)(s+8)}\right]$$

$$= \frac{s+5}{s^2+10s+16}.$$

Example 9. *Show that* $L\left(\sinh\frac{1}{2}t \cdot \sin\frac{1}{2}\sqrt{3}\,t\right) = \frac{\sqrt{3}\,s}{2(s^4+s^2+1)}$.

[*K.U. 2015; M.D.U. 2014, 12, 11*]

Solution. $L\left(\sinh\frac{1}{2}t \cdot \sin\frac{\sqrt{3}}{2}t\right) = L\left(\frac{e^{t/2}-e^{-t/2}}{2}\sin\frac{\sqrt{3}}{2}t\right)$

$$= \frac{1}{2}\left[L\left(e^{t/2}\sin\frac{\sqrt{3}}{2}t\right) - L\left(e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right)\right]$$

$$= \frac{1}{2}\left[\frac{\frac{\sqrt{3}}{2}}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

[$\because$ *If* $L\,[f(t)] = F(s)$, *then* $L\, e^{at} f(t) = F(s-a)$]

$$= \frac{\sqrt{3}}{4}\left[\frac{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}-\left(s-\frac{1}{2}\right)^2-\frac{3}{4}}{\left[\left(s-\frac{1}{2}\right)^2+\frac{3}{4}\right]\left[\left(s+\frac{1}{2}\right)^2+\frac{3}{4}\right]}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{2s}{(s^2-s+1)(s^2+s+1)}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{(s^2+1)^2-s^2}\right] = \frac{\sqrt{3}\,s}{2(s^4+s^2+1)}.$$

Example 10. (*i*) *If* $L[f(t)] = \dfrac{20-4s}{s^2-4s+20}$, *find* $L[f(3t)]$.

(*ii*) *If* $L\,[f(t)] = \dfrac{s^2-s+1}{(2s+1)^2\,(s-1)}$, *find* $L[f(2t)]$.

Solution. (*i*) We have $L[f(t)] = \dfrac{20-4s}{s^2-4s+20}$ [*Given*]

By change of scale property, we know that $L\,f(at) = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$

$$\therefore \quad L[f(3t)] = \frac{1}{3} F\left[\frac{20-4\left(\frac{s}{3}\right)}{\left(\frac{s}{3}\right)^2-4\left(\frac{s}{3}\right)+20}\right]$$

$$= \frac{1}{3}\left[\frac{\frac{60-4s}{3}}{\frac{s^2-12s+180}{9}}\right] = \frac{4(15-s)}{s^2-12s+180}.$$

(*ii*) We have $\quad L\,[f(t)] = \dfrac{s^2-s+1}{(2s+1)^2\,(s-1)}$

$$\therefore \quad L[f(2t)] = \frac{1}{2} F\left[\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2} + 1}{\left(2 \cdot \frac{s}{2} + 1\right)^2 \left(\frac{s}{2} - 1\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\frac{s^2 - 2s + 4}{4}}{(s+1)^2 \left(\frac{s-2}{2}\right)}\right] = \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}.$$

EXERCISE 5.2

Find the Laplace transform of the following functions :

1. $t^3 e^{-3t}$
2. $(t+2)^2 . e^t$
3. $e^{-2t} \sin 4t$
4. $t^2 e^t \sin 4t$
5. $e^{3t} \sin^2 t$
6. $e^{4t} \sin 2t \cos t$
7. $e^{-t} \cos t \cos 2t$
8. $e^{-t} (\sin 2t - 2t \cos 2t$
9. $\sin t \cos^3 t \, e^{-2t}$ [M.D.U. 2008]
10. $\sinh at \cos at$
11. $\sinh at \sin at$
12. If $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\frac{1}{s}$, show that $L\left[\frac{\sin 3t}{t}\right] = \tan^{-1}\left(\frac{3}{s}\right)$.

ANSWERS

1. $\frac{6}{(s+3)^4}$
2. $\frac{2[2s^2 - 2s + 1]}{(s-1)^3}$
3. $\frac{4}{s^2 + 4s + 20}$
4. $\frac{8[3s^2 - 6s - 1\ldots}{(s^2 - 2s + 17)^{\ldots}}$
5. $\frac{2}{(s-3)(s^2 - 6s + 13)}$
6. $\frac{1}{2}\left[\frac{3}{s^2 - 8s + 25} + \frac{1}{s^2 - 8s + 17}\right], s > 4$
7. $\frac{(s+1)(s^2 + 2s + 6)}{(s^2 + 2s + 10)(s^2 + 2s + 2)}$
8. $16(s^2 + 2s + 5)^{-2}$
9. $\frac{s^2 + 4s + 14}{(s^2 + 4s + 20)(s^2 + 4s + 8)}$
10. $\frac{a(s^2 - 2a^2)}{(s^2 - 2a^2)^2 + 4a^2 s^2}$
11. $\frac{2a^2 s}{(s^2 - 2a^2)^2 + 4a^2 s^2}$

5.8. PIECE-WISE CONTINUITY OF A FUNCTION IN AN INTERVAL

Let an interval $[a, b]$ in which the function is defined be divided into a finite number of sub-intervals. Suppose the function is continuous in each of the sub-intervals except at the end points of these intervals where it is discontinuous and at these points of discontinuties, the function has finite jumps at the end points of these intervals. Then the function $f(t)$ is called *sectionally continuous* or *piece wise continuous* in interval $[a, b]$. In piece-wise continuity, left hand and right hand limit exists in every sub-interval though at the end points of the interval it has jumps.

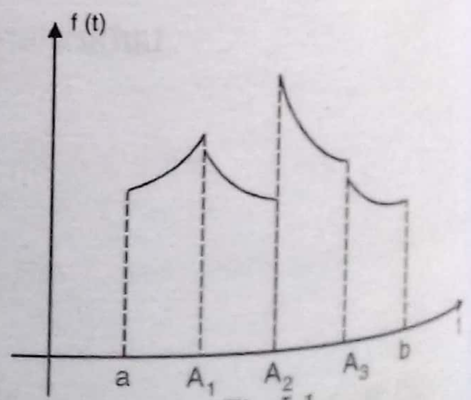


Fig. 5.1

Graph of a piece-wise continuous function in an interval is shown in fig 5.1.

5.9. FUNCTION OF EXPONENTIAL ORDER

A function $f(t)$ is said to be of *exponential order* $a > 0$ if $\lim_{t \to \infty} e^{-at} f(t)$ exists and is a finite quantity.

In other words, using the definition of a limit of a function as $t \to \infty$, we can say there exists a real number $M > 0$ such that

$$|e^{-at} f(t)| < M \text{ for all } t > T$$

or

$$|f(t)| < M e^{at} \text{ for all } t > T$$

Illustration. (*i*) We know that $|t| < e^t$ for all $t \geq 0$

Compare it with $|f(t)| < M e^{at}$ for $t > T$

Here $f(t) = t$, $M = 1$, $a = 1$ and $T = 0$

$\therefore$ $f(t) = t$ is of exponential order 1.

(*ii*) We know that $|\cos t| \leq e^t$ for $t \geq 0$

$\therefore$ $f(t) = \cos t$ is of exponential order 1.

(*iii*) Consider $\lim_{t \to \infty} e^{-at} t^n, a > 0$

Now, $\lim_{t \to \infty} \frac{t^n}{e^{at}}$ is of the form $\frac{\infty}{\infty}$

$$\therefore \quad \lim_{t \to \infty} \frac{t^n}{e^{at}} = \lim_{t \to \infty} \frac{n!}{a^n e^{at}} \qquad \text{[By applying L' Hospital rule successively } n \text{ times]}$$

$$= 0 = \text{a finite number}$$

$\therefore$ $f(t) = t^n$ is of exponential order $a > 0$.

5.10. THEOREM (SUFFICIENT CONDITION FOR THE EXISTENCE OF LAPLACE TRANSFORM)

Laplace transform of $f(t)$ exists for all $s > a$ iff $f(t)$ is piece-wise continuous in every finite interval in its domain $t \geq 0$ and is of exponential order a.

Proof. Let $f(t)$ be piece-wise continuous in $[0, T]$

Then $e^{-st} f(t)$ is also piece-wise continuous in $[0, T]$

$$\therefore \quad \int_0^T e^{-st} f(t)\, dt \text{ exists.}$$

Laplace transform exists if the integral converges *i.e.*, has a finite limit as $T \to \infty$.

Now, $$|L[f(t)]| = \left| \int_0^\infty e^{-st} f(t)\, dt \right|$$

$$\leq \int_0^\infty e^{-st} |f(t)|\, dt \qquad \text{[Property of integral]}$$

$$\leq \int_0^\infty e^{-st} Me^{at}\, dt \qquad [\because f(t) \text{ is of exponential order } a]$$

$$\leq M \int_0^\infty e^{-(s-a)t}\, dt$$

$$\leq \frac{M}{s-a} \text{ provided } s > a$$

$\therefore$ $\int_0^\infty e^{-st} f(t)\, dt$ converges. Hence Laplace transform of $f(t)$ exists.

Remark :

Above condition is only sufficient for the existence of Laplace transform. Converse is not true. There may be a function having Laplace transform but may not satisfy the existence condition.

Note that the function $f(t) = t^{-1/2}$ is not continuous on any interval $[0, T]$ as $t \to 0+$, $f(t) \to \infty$. But $\int_0^T e^{-st} e^{-1/2}\, dt$ can be determined for all $T > 0$.

Laplace transform of $t^{-1/2} = \int_0^\infty e^{-st} t^{-1/2}\, dt$.

Put $st = z$ so that $dt = \frac{dz}{s}$

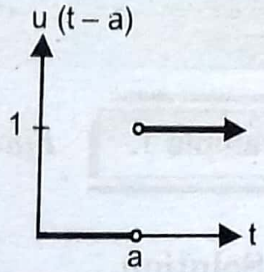
$$\therefore \quad L(t^{-1/2}) = \int_0^\infty e^{-z} \left(\frac{z}{s}\right)^{-1/2} \frac{dz}{s}$$

$$= \int_0^\infty \frac{e^{-z} z^{-1/2}}{\sqrt{s}}\, dz = \frac{\Gamma\left(1-\frac{1}{2}\right)}{\sqrt{s}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}.$$

5.10.1 Unit Step Function

If $a \geq 0$, then function of t defined as below and denoted by $u(t-a)$ is called unit step function

$$u(t-a) = \begin{cases} 0 \text{ if } 0 \leq t < a \\ 1 \text{ if } t > a \end{cases}$$



Note :

Unit step function is also known as **Heaviside's unit function.**

5.10.2 Second Shifting Theorem

If $f(t)$ is a function of t for $t \geq 0$ whose Laplace transform F (s) exists, then for any constant $a \geq 0$, the function

$$g(t) = f(t-a)\, u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the Laplace transform e^{-as} F (s).

[M.D.U. 2009]

Proof. We know that $L f(t) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$ ...(1)

[By definition of Laplace transform]

Now, $$L[g(t)] = \int_0^\infty e^{-st} g(t)\, dt = \int_0^a e^{-st} g(t)\, dt + \int_a^\infty e^{-st} g(t)\, dt$$

$$= \int_0^a 0\, dt + \int_a^\infty e^{-st} g(t)\, dt \qquad [\because g(t) = 0 \text{ if } t < a]$$

$$= \int_a^\infty e^{-st} f(t-a)\, dt \qquad [\because g(t) = f(t-a) \text{ if } t > a]$$

Put $t - a = u$ so that $dt = du$

$$\therefore \quad L[g(t)] = \int_0^\infty e^{-s(a+u)} f(u)\, du$$

$$= e^{-as} \int_0^\infty e^{-su} f(u)\, du = e^{-as}[F(s)]$$

Note :

Students should remember the result **$L f(t-a)\, u(t-a) = e^{-as} F(s)$.**

SOLVED EXAMPLES

Example 1. *Find the Laplace transform of* $f(t) = \begin{cases} e^t & \text{if} \quad 0 < t < 1 \\ 0 & \text{if} \quad t > 1 \end{cases}$

Solution.

$$L f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^\infty e^{-st} f(t)\, dt \qquad \text{[Property of definite integral]}$$

$$= \int_0^1 e^{-st} e^t\, dt + \int_1^\infty e^{-st} \times 0\, dt$$

$$= \int_0^1 e^{-(s-1)t}\, dt + 0 = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1$$

$$= -\frac{1}{s-1} e^{-(s-1)} + \frac{1}{s-1} = \frac{e^{1-s}}{1-s} - \frac{1}{1-s} = \frac{e^{1-s} - 1}{1-s}.$$

Example 2. *Find the Laplace transform of* $f(t) = \begin{cases} \sin t & \text{if} \quad 0 < t < \pi \\ 0 & \text{if} \quad t > \pi \end{cases}$

Solution.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} f(t)\, dt + \int_\pi^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \sin t\, dt + 0 = \left[\frac{e^{-st}}{s^2+1}(-s \sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-\pi s}}{s^2+1}(0+1) - \frac{1}{s^2+1}(0-1) = \frac{1}{1+s^2}(1 + e^{-\pi s}).$$

Example 3. *Find the Laplace transform of* $f(t) = |t-1| + |t+1|, t \geq 0.$ [M.D.U. 2007]

Solution. Here $f(t) = |t-1| + |t+1|, t \geq 0$

i.e.,

$$f(t) = \begin{cases} -(t-1) + t + 1 = 2 & \text{when} \quad 0 \leq t < 1 \\ t - 1 + t + 1 = 2t & \text{when} \quad t \geq 1 \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, 2dt + \int_1^\infty e^{-st}\, 2t\, dt$$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^1 + 2\left[t\left(\frac{e^{-st}}{-s}\right) - 1.\frac{e^{-st}}{s^2}\right]_1^\infty$$

$$= 2\left[\frac{e^{-s}}{-s} + \frac{1}{s}\right] + 2\left[0 - 0 + \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}\right]$$

$$= 2\frac{e^{-s}}{-s} + \frac{2}{s} + \left[\frac{2e^{-s}}{s} + \frac{2e^{-s}}{s^2}\right]$$

$$= \frac{2}{s}\left[-e^{-s} + 1 + e^{-s} + \frac{e^{-s}}{s}\right] = \frac{2}{s}\left[1 + \frac{e^{-s}}{s}\right].$$

Example 4. *Find the Laplace transform of* $g(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right), & t > \dfrac{2\pi}{3} \\ 0, & 0 < t < \dfrac{2\pi}{3} \end{cases}$

(i) by definition (ii) by using second shifting theorem.

[K.U. 2012; M.D.U. 2007]

Solution. (*i*) $\mathrm{L}[g(t)] = \int_0^\infty e^{-st}\, g(t)\, dt$

$$= \int_0^{\frac{2\pi}{3}} e^{-st} \times 0\, dt + \int_{\frac{2\pi}{3}}^\infty e^{-st} \cos\left(t - \frac{2\pi}{3}\right) dt$$

$$= 0 + \left[\frac{e^{-st}}{s^2+1}\left\{-s\cos\left(t - \frac{2\pi}{3}\right) + 1\sin\left(t - \frac{2\pi}{3}\right)\right\}\right]_{\frac{2\pi}{3}}^\infty$$

$$= 0 + 0 - \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}[-s] = \frac{e^{-\frac{2\pi s}{3}}.s}{s^2+1}.$$

(*ii*) Here $g(t) = f\left(t - \dfrac{2\pi}{3}\right) u\left(t - \dfrac{2\pi}{3}\right)$, where $f(t) = \cos t$

[Ref. def. of u (t − a)]

$$\mathrm{L}f(t) = \frac{s}{s^2+1}$$

$$\therefore \quad \mathrm{L}[g(t)] = \frac{e^{-\frac{2\pi s}{3}}.s}{s^2+1}$$

Example 5. *Find the Laplace transform of* $f(x) = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$. [K.U. 2014]

Solution.
$$\mathrm{L}f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} f(t)\, dt + \int_1^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

[*Property of definite integral*]

$$= \int_0^1 0\, e^{-st}\, dt + \int_1^2 e^{-st}\, t\, dt + \int_2^\infty e^{-st} \times 0\, dt$$

$$= \int_1^2 e^{-st}\, t\, dt = \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_1^2$$

$$= -\frac{2}{s} e^{-2s} - \frac{e^{-2s}}{s^2} + \frac{1}{s} e^{-s} + \frac{e^{-s}}{s^2}$$

$$= e^{-s}\left(\frac{1}{s} + \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right).$$

Example 6. *Find the Laplace transform of the function* $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 3, & t > 2 \end{cases}$. [K.U. 2013]

Solution.
$$\mathrm{L}f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} f(t)\, dt + \int_2^\infty e^{-st} f(t)\, dt$$

$$= \int_0^2 e^{-st} . t\, dt + \int_2^\infty e^{-st} . 3\, dt$$

$$= t . \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=2} - \int_0^2 1 . \frac{e^{-st}}{-s}\, dt + 3 \lim_{T \to \infty} \int_2^T e^{-st}\, dt$$

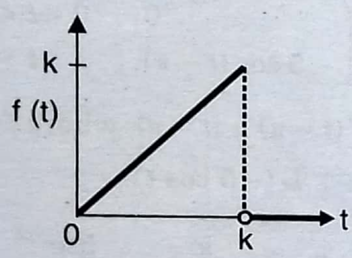
$$= -\frac{2e^{-2s}}{s} + 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=2} + 3 \lim_{T \to \infty} \left[\frac{e^{-st}}{-s}\right]_{t=2}^{t=T}$$

$$= -\frac{2e^{-2s}}{s} - \frac{1}{s^2}[e^{-2s} - 1] - \frac{3}{s}\lim_{T\to\infty}[e^{-sT} - e^{-2s}]$$

$$= -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{3e^{-2s}}{s}, \; s > 0 \qquad \left[\because \lim_{T\to\infty} e^{-sT} = 0\right]$$

$$= \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}, \; s > 0.$$

Example 7. *Find the Laplace transform of the function whose graph is given as follows :*



Solution. We have the following function from the given graph :

$$f(t) = \begin{cases} t, & 0 \le t \le k \\ 0, & t > k \end{cases}$$

$$\therefore \quad \mathrm{L}f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^k e^{-st} f(t)\, dt + \int_k^\infty e^{-st} f(t)\, dt$$

$$= \int_0^k e^{-st}.t\, dt + \int_k^\infty e^{-st}.0.\, dt$$

$$= \int_0^k t.e^{-st}\, dt + 0$$

$$= t.\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=k} - \int_0^k 1.\frac{e^{-st}}{-s}\, dt$$

$$= -\frac{ke^{-sk}}{s} - 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=k}$$

$$= -\frac{ke^{-sk}}{s} - \frac{1}{s^2}[e^{-sk} - 1]$$

$$= \frac{1}{s^2}(1 - e^{-sk}) - \frac{k}{s}e^{-sk}, \quad s > 0.$$

Example 8. *Find the Laplace transform of* $5 \cos t . u(t - \pi)$.

Solution. Let $g(t) = 5 \cos t . u(t - \pi)$

$$= \begin{cases} 0, & 0 < t < \pi \\ 5 \cos t, & t > \pi \end{cases}$$

$$= \begin{cases} 0, & 0 < t < \pi \\ -5 \cos(t - \pi), & t > \pi \end{cases}$$

$= f(t - \pi)\, u(t - \pi)$, where $f(t) = -5 \cos t$

$\therefore \quad L(g(t)) = e^{-\pi s} L(-5 \cos t)$

$$= -5e^{-\pi s}\frac{s}{s^2+1} = -\frac{5se^{-\pi s}}{s^2+1}, \quad s > 0.$$

EXERCISE 5.3

Find the Laplace transform of the following functions :

1. $f(t) = \begin{cases} \frac{t}{T}, & 0 < t < T \\ 1, & t > T \end{cases}$

2. $f(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ t - 1, & \frac{1}{2} \le t \le 1 \\ 0, & t > 1 \end{cases}$

[K.U. 2014]

3. $f(t) = \begin{cases} 4, & 0 \le t < 3 \\ 2, & t > 3 \end{cases}$

4. $f(t) = \begin{cases} \frac{1}{\varepsilon} & \text{if } 0 \le t \le \varepsilon \\ 0 & \text{if } t > \varepsilon \end{cases}$ [K.U. 2013]

[This function is known as Dirac delta function]

5. $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

6. $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

7. $g(t) = \begin{cases} \sin\left(t - \frac{\pi}{6}\right), & t > \frac{\pi}{6} \\ 0, & 0 < t < \frac{\pi}{6} \end{cases}$

8. $g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2} \end{cases}$

[M.D.U. 2008]

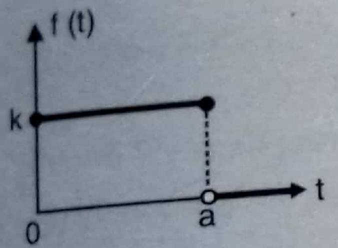
9. $g(t) = \begin{cases} 0, & 0 < t < \frac{\pi}{2} \\ \sin t, & t > \frac{\pi}{2} \end{cases}$

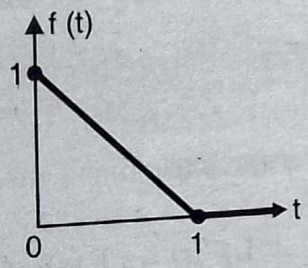
10. Find the Laplace transform of the function $g(t) = \begin{cases} 0, & 0 < t < 5 \\ t, & t > 5 \end{cases}$

(i) using definition,

(ii) using second shifting theorem. Verify that the results are same.

Find the Laplace transforms of the functions whose graphs are given below :

11. 

12. 

[*M.D.U. 2014*]

ANSWERS

1. $\frac{1}{s^2 T}[1 - e^{-sT}]$

2. $\frac{1}{s^2}[1 - se^{-\frac{1}{2}s} - e^{-s}]$

3. $\frac{2}{s}(2 - e^{-3s}), s > 0$

4. $\frac{1}{\varepsilon s}[1 - e^{-\varepsilon s}]$

5. $\frac{2}{s^2+4}(1 - e^{-\pi s})$

6. $\frac{s}{s^2+1}(1 - e^{-2\pi s})$

7. $\frac{1}{s^2+1}(e^{-\frac{\pi}{6}s})$

8. $-\frac{e^{-\frac{\pi}{2}s}}{s^2+1}, s > 0$

9. $e^{-\frac{\pi}{2}s}\frac{s}{s^2+1}, s > 0$

10. $\left(\frac{5s+1}{s^2}\right)e^{-5s}, s > 0$

11. $\frac{k}{s}[1 - e^{-as}], s > 0$

12. $\frac{1}{s} + \frac{1}{s^2}\left(\frac{1}{e^s} - 1\right), s \neq 0.$

5.11. LAPLACE TRANSFORMS OF DERIVATIVES

Theorem. ***For $t \geq 0$, let $f(t)$ be a continuous function of exponential order a on $[0, \infty)$ $\left(\text{so that } \lim_{t \to \infty} e^{-st} f(t) = 0\right)$. Let $f'(t)$ be also of exponential order and is continuous or piece-wise continuous on $[0, \infty)$, then***

$$L[f'(t)] = s F(s) - f(0) \text{ for } s > a, \text{ where } L f(t) = F(s).$$

Proof. Let $f(t)$ be continuous on $[0, \infty)$, then

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty + s \int_0^\infty e^{-st} f(t)\, dt \qquad \text{[Integrating by parts]}$$

$$= \lim_{t \to \infty} e^{-st} f(t) - f(0) + s\, F(s) \qquad \ldots(1)$$

Now as $f(t)$ is continuous exponential of order a

$\therefore$ $|f(t)| \le M e^{at}$ for some constant M.

Thus we have $|e^{-st} f(t)| = e^{-st} |f(t)| \le e^{-st} M e^{at}$

$\le M e^{-(s-a)t} \to 0$ as $t \to \infty$ if $s > a$

$\therefore$ $\lim_{t \to \infty} e^{-st} f(t) \to 0$

From (1), $L f'(t) = s\, F(s) - f(0)$

We can also show that, if $f'(t)$ is piece-wise continuous and has a finite jump at a point T, even then $L f'(t) = s\, F(s) - f(0)$, as the integral can be broken up in the sum of integrals in different ranges from $0 \to \infty$.

Hence $\mathbf{L f'(t) = s F(s) - f(0)}$.

5.12. LAPLACE TRANSFORM OF *n*th ORDER DERIVATIVE OF *f(t)*

Let $f(t)$, $f'(t)$, $f''(t)$, ... $f^{n-1}(t)$ be continuous on $[0, \infty)$ and are of exponential order, then

$$\mathbf{L[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots - f^{n-1}(0).}$$

Proof. $L[f^n(t)] = \int_0^\infty e^{-st} f^n(t)\, dt$

Integrating by parts using the general rule of integration, by parts*

$$L[f^n(t)] = [e^{-st} f^{n-1}(t) - (-se^{-st}) f^{n-2}(t) + ((-s)^2 e^{-st}) f^{n-3}(t) + \ldots$$

$$+ (-1)^{n-1} (-s)^{n-1} e^{-st} f(t)]_0^\infty + (-1)^n (-s)^n \int_0^\infty e^{-st} f(t)\, dt$$

* **Note.** General rule of integration by parts is as under :

$$\int uv\, dx = uv_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots\ldots + (-1)^{n-1} u^{(n-1)} v_n + \ldots\ldots$$

where dashes stands for derivatives and subscripts stands for integration.

Assuming that $\lim\limits_{t \to \infty} e^{-st} f^{m}(t) = 0,\ m = 0, 1, 2, \ldots\ldots (n-1)$, we have

$$L[f^{n}(t)] = -f^{n-1}(0) - sf^{n-2}(0) - s^{2}f^{n-3}(0) \ldots\ldots - s^{n-1}f(0) + s^{n}\int_{0}^{\infty} e^{-st} f(t)\,dt$$

$$= s^{n}\,F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots\ldots - f^{n-1}(0).$$

5.13. EFFECT OF MULTIPLICATION OF $f(t)$ BY t^n IN FINDING LAPLACE TRANSFORM

If $L[f(t)] = F(s)$, then show that $L[t^{n} f(t)] = (-1)^{n} \dfrac{d^{n}}{ds^{n}}[F(s)]$, where $n = 1, 2, 3, \ldots\ldots$

Proof. We shall prove the result by using the principle of mathematical induction.

Verification of the result for $n = 1$.

We know that $\displaystyle\int_{0}^{\infty} e^{-st} f(t)\,dt = F(s)$

Differentiating both sides w.r.t. s using the Leibnitz rule for differentiation under integral sign, we have

$$\int_{0}^{\infty} \frac{\partial}{\partial s}(e^{-st})\, f(t)\,dt = \frac{d}{ds}F(s)$$

i.e., $$\int_{0}^{\infty} -te^{-st}\, f(t)\,dt = \frac{d}{ds}F(s)$$

i.e., $$\int_{0}^{\infty} e^{-st}\,[t f(t)] = (-1)^{1}\frac{d}{ds}F(s)$$

i.e., $$L[t f(t)] = (-1)\frac{d}{ds}F(s)$$

Thus the result is true for $n = 1$.

Let us assume that the result is true for $n = m$

i.e., $$L\,[t^{m} f(t)] = (-1)^{m}\frac{d^{m}}{ds^{m}}F(s)$$

i.e., $$\int_{0}^{\infty} e^{-st}\, t^{m}\,[f(t)]\,dt = (-1)^{m}\frac{d^{m}}{ds^{m}}F(s) \qquad \ldots(1)$$

Now we shall prove that the result is also true for $n = m + 1$.

Differentiating (1) w.r.t. s using Leibnitz rule of differentiation under integral sign, we have

$$\frac{d}{ds}\int_0^\infty e^{-st}\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$$

or $$\int_0^\infty \frac{\partial}{\partial s} e^{-st}\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$$

or $$\int_0^\infty -te^{-st}\, t^m f(t)\, dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}} F(s)$$

or $$\int_0^\infty e^{-st}\, t^{m+1} f(t)\, dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s)$$

or $$L[t^{m+1} f(t)] = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} F(s)$$

Thus the result is true for $m + 1$ when true for m. But it is true for 1 also.

$\therefore$ By principle of mathematical induction it is true for every positive integer n.

5.14. EFFECT OF DIVISION OF $f(t)$ BY t IN FINDING LAPLACE TRANSFORM

If $f(t)$ is piece-wise continuous on $[0, \infty)$ and is of exponential order n and $L[f(t)] = F(s)$, then

$$L\left(\frac{1}{t} f(t)\right) = \int_s^\infty F(s)\, ds.$$

Proof. $F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides w.r.t. s from s to ∞, we have

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds$$

As the limits are independent, changing the order of integration, we have

$$\int_s^\infty F(s)\, ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt$$

$$= \int_0^\infty \left(\frac{e^{-st}}{-t} f(t) \right)_s^\infty dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = \mathrm{L}\left(\frac{1}{t} f(t)\right)$$

Hence, $$\mathbf{L}\left(\frac{1}{t} f(t)\right) = \int_s^\infty \mathbf{F}(s)\, ds.$$

SOLVED EXAMPLES

Example 1. *Evaluate* : $L(e^{at})$

Solution. Let $f(t) = e^{at}$, then $f'(t) = ae^{at}$

Also, $f(0) = 1$

Now, $\mathrm{L} f'(t) = s\,\mathrm{F}(s) - f(0)$ [*Ref. Art.* 5.11]

$\therefore$ $\mathrm{L}(a\, e^{at}) = s\,\mathrm{L}(e^{at}) - 1$

i.e., $a(\mathrm{L} e^{at}) = s\,\mathrm{L}(e^{at}) - 1$

or $1 = (s-a)\,\mathrm{L}(e^{at}) \Rightarrow \mathrm{L}(e^{at}) = \dfrac{1}{s-a}.$

Example 2. *Given* $L(\sin\sqrt{t}) = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$; *show that* $L\left(\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\dfrac{\pi}{s}}\, e^{-\frac{1}{4s}}$.

Solution. Given $f(t) = \sin\sqrt{t}$

$\therefore$ $f'(t) = \dfrac{1}{2\sqrt{t}} \cos\sqrt{t}$

$f(0) = 0$

Now, $\mathrm{L}[f'(t)] = s\,\mathrm{F}(s) - f(0)$ ...(1) $[\because f(0) = 0]$

$\therefore$ $\mathrm{L}[f'(t)] = s\,\mathrm{F}(s)$

i.e., $$\mathrm{L}\left(\frac{1}{2\sqrt{t}} \cos\sqrt{t}\right) = s\,\mathrm{L}(\sin\sqrt{t}) = s\,\frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

$\therefore$ $$\mathrm{L}\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-\frac{1}{4s}}.$$

Example 3. *Find the Laplace transform of t sin 3t cos 2t.*

Solution. $L(\sin 3t \cos 2t) = \frac{1}{2} L[2 \sin 3t \cos 2t]$

$$= \frac{1}{2} L[\sin 5t + \sin t] = \frac{1}{2}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right]$$

$$\therefore \quad L(t \sin 3t \cos 2t) = -\frac{1}{2}\frac{d}{ds}\left[\frac{5}{s^2+25} + \frac{1}{s^2+1}\right] \qquad \text{[Using Art. 5.13]}$$

$$= -\frac{1}{2}\left[\frac{(-5)\,2s}{(s^2+25)^2} + \frac{(-1)\,2s}{(s^2+1)^2}\right] = \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}.$$

Example 4. *Given* $L[t^2] = \frac{2}{s^3}$*, find the Laplace transform of* t^3 *and* t^4.

Solution. We have $\quad L[t^2] = \frac{2}{s^3}$

$$\therefore \quad L[t^3] = L[t^1 . t^2] = (-1)^1 \frac{d}{ds} L[t^2] \qquad \text{[Using Art. 5.13]}$$

$$= (-1)\frac{d}{ds}\left(\frac{2}{s^3}\right) = (-1)\left(\frac{-6}{s^4}\right)$$

$$= \frac{6}{s^4} \qquad \text{...(1)}$$

Similarly, $\quad L[t^4] = L[t^1 . t^3] = (-1)^1 \frac{d}{ds} L[t^3]$

$$= (-1)\frac{d}{ds}\left(\frac{6}{s^4}\right) \qquad \left[\because \ L[t^3] = \frac{6}{s^4}, \text{ from (1)}\right]$$

$$= (-1)\left(\frac{-24}{s^5}\right)$$

$$= \frac{24}{s^5}.$$

Example 5. *Find the Laplace transform of sin at – at cos at.*

[M.D.U. 2008, 06; K.U. 2008, 04]

Solution. $L(\sin at - at \cos at) = L(\sin at) - L(at \cos at)$

$$= \frac{a}{s^2+a^2} - a(-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) \qquad \text{[Using Art. 5.13]}$$

$$= \frac{a}{s^2 + a^2} + a\,\frac{(s^2 + a^2).1 - s.2s}{(s^2 + a^2)^2}$$

$$= \frac{a}{(s^2 + a^2)} + \frac{a\,(a^2 - s^2)}{(s^2 + a^2)^2} = \frac{a\,(s^2 + a^2) + a^3 - as^2}{(s^2 + a^2)^2}$$

$$= \frac{2a^3}{(s^2 + a^2)^2}.$$

Method II. Let $f(t) = \sin at - at\cos at$

$\therefore$ $f'(t) = a\cos at - a\,[-at\sin at + \cos at] = a^2 t\sin at$

$f(0) = 0$

Now, $L f'(t) = s\,F(s) - f(0)$

$\therefore$ $L\,(a^2 t\sin at) = s\,F(s)$ $[\because f(0) = 0]$

i.e., $$sF(s) = a^2\,L(t\sin at) = a^2\,(-1)\,\frac{d}{ds}\left(\frac{a}{s^2 + a^2}\right)$$

$$= a^2(-1)\left(\frac{-a}{(s^2 + a^2)^2}\,2s\right) = \frac{2a^3 s}{(s^2 + a^2)^2}$$

$\therefore$ $$F(s) = \frac{2a^3}{(s^2 + a^2)^2}.$$

Example 6. *Find the Laplace transform of $t^2\cos at$.* [K.U. 2007, 06]

Solution. $L\,(\cos at) = \dfrac{s}{s^2 + a^2}$

$\therefore$ $$L\,(t^2\cos at) = (-1)^2\,\frac{d^2}{ds^2}\left[\frac{s}{s^2 + a^2}\right] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{s^2 + a^2 - s.\,2s}{(s^2 + a^2)^2}\right] = \frac{d}{ds}\left[\frac{a^2 - s^2}{(s^2 + a^2)^2}\right]$$

$$= \frac{(s^2 + a^2)^2\,(-2s) - (a^2 - s^2)\,2\,(s^2 + a^2)\,2s}{(s^2 + a^2)^4}$$

$$= -\frac{2s}{(s^2 + a^2)^4}\,[(s^2 + a^2)^2 + 2\,(a^4 - s^4)]$$

$$= \frac{-2s}{(s^2 + a^2)^4}\,[s^4 + a^4 + 2a^2 s^2 + 2a^4 - 2s^4]$$

$$= \frac{-2s}{(s^2+a^2)^4}[3a^4 - s^4 + 2a^2s^2]$$

$$= \frac{-2s}{(s^2+a^2)^4}[(3a^2 - s^2)(s^2+a^2)]$$

$$= -\frac{2s(3a^2-s^2)}{(s^2+a^2)^3} = \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}.$$

Example 7. *Evaluate* $L(t e^{-t} \sin 3t)$. *[K.U. 2005]*

Solution. $L(e^{-t}\sin 3t) = \frac{3}{(s+1)^2+9} = \frac{3}{s^2+2s+10}$ *[By shifting property]*

$$\therefore \quad L(t e^{-t}\sin 3t) = (-1)\frac{d}{ds}\left(\frac{3}{s^2+2s+10}\right) \quad \text{[By Art. 5.13]}$$

$$= (-1)(3)(-1)(s^2+2s+10)^{-2}(2s+2)$$

$$= \frac{6(s+1)}{(s^2+2s+10)^2}.$$

Example 8. *Evaluate* $L\left(\frac{e^{-t}\sin t}{t}\right)$. *[K.U. 2016; M.D.U. 2013]*

Solution. $L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1^2}$

$$\therefore \quad L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty \frac{ds}{(s+1)^2+1^2} \quad \text{[By Art. 5.14]}$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1).$$

Example 9. *Evaluate* $L\left(\frac{1-\cos 2t}{t}\right)$. *[K.U. 2016]*

Solution. $L(1-\cos 2t) = L(1) - L(\cos 2t) = \frac{1}{s} - \frac{s}{s^2+4}$

$$\therefore \quad L\left(\frac{1-\cos 2t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds \quad \text{[By Art. 5.14]}$$

$$= \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty = \left(\log \frac{s}{\sqrt{s^2+4}}\right)_s^\infty$$

$$= \lim_{s\to\infty} \log \frac{s}{\sqrt{s^2+4}} - \log \frac{s}{\sqrt{s^2+4}}$$

$$= \lim_{s\to\infty} \log \frac{1}{\sqrt{1+\frac{4}{s^2}}} - \log \frac{s}{\sqrt{s^2+4}}$$

$$= 0 - \log \frac{s}{\sqrt{s^2+4}} = \log \sqrt{\frac{s^2+4}{s^2}}$$

$$= \frac{1}{2}\log \frac{s^2+4}{s^2}.$$

Example 10. *Find the Laplace transform of* $\frac{\sin at}{t}$. *Does the transform of* $\frac{\cos at}{t}$ *exist.*

Solution. $$\mathrm{L}(e^{iat}) = \int_0^\infty e^{-st} e^{iat}\, dt = \int_0^\infty e^{-(s-ia)t}\, dt$$

$$= \left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty = \frac{1}{s-ia}$$

$$\mathrm{L}(\cos at + i \sin at) = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating the real and imaginary parts, we have

$$\mathrm{L}(\cos at) = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathrm{L}(\sin at) = \frac{a}{s^2+a^2}$$

Now, $$\mathrm{L}\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\, ds = \frac{a}{a}\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}$$

and $$\mathrm{L}\left(\frac{\cos at}{t}\right) = \int_s^\infty \frac{s}{s^2+a^2}\, ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\lim_{s \to \infty} \log (s^2 + a^2) - \log (s^2 + a^2)\right]$$

which does not exist as $\lim_{s \to \infty} \log (s^2 + a^2) =$ infinite.

Thus the transform of $\frac{\cos at}{t}$ does not exist.

5.15. LAPLACE TRANSFORM OF A PERIODIC FUNCTION

Let a function $f(t)$ be periodic with period w so that $f(t + nw) = f(t)$, $n = 1, 2, 3, \ldots\ldots$, then

$$L[f(t)] = \int_0^w \frac{e^{-st} f(t)}{1 - e^{-sw}}\, dt.$$ [M.D.U. 2014]

Proof $$L f(t) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^w e^{-st} f(t)\, dt + \int_w^{2w} e^{-st} f(t)\, dt + \int_{2w}^{3w} e^{-st} f(t)\, dt + \ldots..$$

[*Property of definite integral*]

$\Rightarrow$ $$L f(t) = \sum_{n=0}^{\infty} \int_{nw}^{(n+1)w} e^{-st} f(t)\, dt$$

Put $t = x + nw$ so that $dt = dx$

When $t = nw$, $x = 0$ and when $t = (n + 1)\, w$, $x = w$

$\therefore$ $$L f(t) = \sum_{n=0}^{\infty} \int_0^w e^{-s(x+nw)} f(x + nw)\, dx$$

$$= \sum_{n=0}^{\infty} \int_0^w e^{-sx} e^{-nws} f(x)\, dx \qquad [\because f(x + nw) = f(x)]$$

$$= \sum_{n=0}^{\infty} e^{-nws} \int_0^w e^{-sx} f(x)\, dx$$

$$= (1 + e^{-ws} + e^{-2ws} + \ldots..) \int_0^w e^{-sx} f(x)\, dx$$

$$= \frac{1}{1 - e^{-sw}} \int_0^w e^{-sx} f(x)\, dx = \int_0^w \frac{e^{-st} f(t)}{1 - e^{-sw}}\, dt.$$

Example 11. *Find the Laplace transform of periodic function*

$$f(t)=\frac{kt}{T},\ 0<t<T\ ;\ f(t+T)=f(t).$$ [M.D.U. 2015]

Solution. As the function is periodic with period T

$$L[f(t)]=\int_0^T \frac{e^{-st}f(t)}{1-e^{-sT}}\,dt$$ [Ref. Art. 5.15]

$$=\left(\frac{1}{1-e^{-sT}}\right)\int_0^T e^{-st}\frac{kt}{T}\,dt \qquad \left[\because\ f(t)=\frac{kt}{T},\ \text{(Given)}\right]$$

$$=\left(\frac{1}{1-e^{-sT}}\right)\frac{k}{T}\int_0^T e^{-st}\,t\,dt$$

$$=\left(\frac{1}{1-e^{-sT}}\right)\frac{k}{T}\left[t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right]_0^T \qquad \text{[Integrating by parts]}$$

$$=\frac{k}{T}\left(\frac{1}{1-e^{-sT}}\right)\left[-\frac{Te^{-sT}}{s}-\frac{e^{-sT}}{s^2}+\frac{1}{s^2}\right]$$

$$=\frac{k}{T}\left(\frac{1}{1-e^{-sT}}\right)\left[-\frac{Te^{-sT}}{s}+\frac{1}{s^2}(1-e^{-sT})\right]$$

$$=-\frac{ke^{-sT}}{s(1-e^{-sT})}+\frac{k}{s^2T}.$$

Example 12. *Find* $L\,f(t)$, *where* $f(t)=\begin{cases}\sin t & \text{if } 0<t<\pi\\ 0 & \text{if } \pi<t<2\pi\end{cases}$ *given that* $f(t)$ *has period* 2π.

Solution. If $f(t)$ is a periodic function with period 2π, then

[Ref. Art. 5.15]

$$L[f(t)]=\int_0^{2\pi}\frac{e^{-st}f(t)}{1-e^{-2\pi s}}\,dt$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi}e^{-st}f(t)\,dt+\int_{\pi}^{2\pi}e^{-st}f(t)\,dt\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi}e^{-st}\sin t\,dt\right] \qquad [\text{As } f(t)=0 \text{ for } \pi<t<2\pi]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-st}}{s^2+1}(-s \sin t - \cos t)\right]_0^{\pi}$$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{-s\pi}}{s^2+1}(-0+1) - \frac{1}{s^2+1}(0-1)\right]$$

$$= \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}.$$

EXERCISE 5.4

Find the Laplace transform of the following functions :

1. $t \sinh at$
2. $t \sin^2 t$
3. $t^2 e^{2t}$
4. $t e^{-2t} \sin 2t$
5. $t e^{-t} \cosh t$
6. $(t^2 - 3t + 2) \sin 3t$ [M.D.U. 2009]
7. $t^n e^{at}$
8. $(1 + t e^{-t})^3$
9. $t^2 \sin at$ [M.D.U. 2005]
10. $t \cos at$
11. $\dfrac{\cos 2t - \cos 3t}{t}$
12. $\dfrac{e^{at} - \cos bt}{t}$
13. $t^2 e^{-3t} \sin 2t$
14. $\dfrac{e^{-at} - e^{-bt}}{t}$
15. Given $L\left[2\sqrt{\dfrac{t}{\pi}}\right] = \dfrac{1}{s^{3/2}}$; show that $L\left(\dfrac{1}{\sqrt{\pi t}}\right) = \dfrac{1}{s^{1/2}}$. [K.U. 2015]
16. Use Laplace transformation to prove that $\displaystyle\int_0^{\infty} \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

[**Hint.** $L\left(\dfrac{\sin at}{t}\right) = \cot^{-1}\dfrac{s}{a}$ (See example)

Using definition, $\displaystyle\int_0^{\infty} e^{-st}\frac{\sin t}{t}\, dt = \cot^{-1}\frac{s}{a}$

Take $s = 0$ and $a = 1$ $\therefore$ $\displaystyle\int_0^{\infty} \frac{\sin t}{t}\, dt = \cot^{-1}(0) = \frac{\pi}{2}$]

ANSWERS

1. $\dfrac{2as}{(s^2-a^2)^2}$

2. $\dfrac{2(3s^2+4)}{s^2(s^2+4)^2}$

3. $\dfrac{2}{(s-2)^3}$

4. $\dfrac{4s+8}{(s^2+4s+8)^2}$

5. $\dfrac{s^2+2s+2}{(s^2+2s)^2}$

6. $\dfrac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}$

7. $\dfrac{n!}{(s-a)^{n+1}}, s > a$

8. $\dfrac{1}{s}+\dfrac{3}{(s+1)^2}+\dfrac{6}{(s+2)^3}+\dfrac{6}{(s+3)^4}$

9. $\dfrac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

10. $\dfrac{s^2-a^2}{(s^2+a^2)^2}$

11. $\dfrac{1}{2}\log\left|\dfrac{s^2+9}{s^2+4}\right|$

12. $\dfrac{1}{2}\log\dfrac{s^2+b^2}{(s-a)^2}$

13. $-4\left[\dfrac{-3s^2-18s-23}{(s^2+6s+13)^3}\right]$

14. $\log\left(\dfrac{s+b}{s+a}\right)$

5.16. LAPLACE TRANSFORM OF INTEGRALS

If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(u)\,du\right] = \dfrac{1}{s}F(s)$. [K.U. 2011]

Proof. Let $L[f(t)] = F(s)$ ...(1)

and suppose $\phi(t) = \int_0^t f(u)\,du$ ...(2)

From (2), taking $t = 0$, we have $\phi(0) = 0$ ...(3)

Differentiating (2) w.r.t. t using Leibnitz rule of differentiation under integral sign, we have

$$\phi'(t) = \int_0^t \frac{\partial}{\partial t} f(u)\,du + \frac{dt}{dt}f(t) - 0$$

$$= 0 + f(t) - 0$$

i.e., $\phi'(t) = f(t)$ ...(4)

Now, $L[\phi'(t)] = s\bar{\phi}(s) - \phi(0)$ *[Denoting* $L\,\phi(t) = \bar{\phi}(s)$*] [Using Art. 5.11]*

$= s\,\bar{\phi}\,(s)$ *[By (3)]*

$\therefore$ $s\,\bar{\phi}\,(s) = L\,[\phi'(t)]$ *[Reversing the sides]*

or $\bar{\phi}\,(s) = \dfrac{1}{s} L\,[f(t)]$ $[\because \phi'(t) = f(t)$ *by (4)*$]$

or $L\,[\phi\,(t)] = \dfrac{1}{s} L\,[f(t)]$

or $$L\left(\int_0^t f(u)\,du\right) = \frac{1}{s} F(s)$$

Evaluation of integral :

If $L\,[f(t)] = F\,(s)$ *i.e.,* $\displaystyle\int_0^\infty e^{-st} f(t)\,dt = F(s)$

Taking the limits as $s \to 0$

$$\int_0^\infty f(t)\,dt = F\,(0)$$

assuming the integral to be convergent.

SOLVED EXAMPLES

Example 1. *Evaluate* $\displaystyle\int_0^\infty t e^{-2t} \cos t\,dt.$ *[M.D.U. 2015]*

Solution. $\displaystyle\int_0^\infty e^{-2t}\, t \cos t\,dt = \int_0^\infty e^{-st}\, t\cos t\,dt$ *[Taking s = 2]*

$= L\,(t\cos t)$ *[By def. of Laplace transformation]*

$$= -\frac{d}{ds}\cdot\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{(s^2+1)1 - s\,.\,2s}{(s^2+1)^2} = -\frac{1-s^2}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2} = \frac{2^2-1}{(2^2+1)^2} = \frac{3}{25}.$$ $[\because s = 2]$

Example 2. *Evaluate* $\int_0^\infty \frac{\sin mt}{t}\, dt$.

Solution. We know that $L(\sin mt) = \frac{m}{s^2 + m^2}$

$$\therefore \quad L\left(\frac{\sin mt}{t}\right) = \int_s^\infty \frac{m}{s^2 + m^2}\, ds$$

$$= \frac{1}{m}\left(m \tan^{-1} \frac{s}{m}\right)_s^\infty = \left(\tan^{-1} \frac{s}{m}\right)_s^\infty$$

$$= \begin{cases} \frac{\pi}{2} - \tan^{-1} \frac{s}{m}, & \text{if } m > 0 \\ -\frac{\pi}{2} - \tan^{-1} \frac{s}{m}, & \text{if } m < 0 \end{cases}$$

$$\therefore \quad \int_0^\infty e^{-st} \frac{\sin mt}{t}\, dt = \begin{cases} \frac{\pi}{2} - \tan^{-1} \frac{s}{m}, & \text{if } m > 0 \\ -\frac{\pi}{2} - \tan^{-1} \frac{s}{m}, & \text{if } m < 0 \end{cases}$$

Taking limits as $s \to 0$, we have

$$\int_0^\infty \frac{\sin mt}{t}\, dt = \frac{\pi}{2} \text{ if } m > 0 \quad \text{and} \quad -\frac{\pi}{2} \quad \text{if } m < 0.$$

Example 3. *Evaluate* $\int_0^\infty t e^{-t} \sin^4 t\, dt$. [M.D.U. 2018, 14; K.U. 2015]

Solution. $L(t \sin^4 t) = L\left[t\left(\frac{1 - \cos 2t}{2}\right)^2\right]$

$$= L\left[t\left(\frac{1 - 2\cos 2t + \cos^2 2t}{4}\right)\right]$$

$$= L\left[\frac{1}{4}t\left(1 - 2\cos 2t + \frac{1 + \cos 4t}{2}\right)\right]$$

$$= \frac{1}{4} L\left[t\left(\frac{3 - 4\cos 2t + \cos 4t}{2}\right)\right]$$

$$= \frac{1}{8}(-1)\frac{d}{ds}[L(3 - 4\cos 2t + \cos 4t)]$$

$$= -\frac{1}{8}\frac{d}{ds}\left[\frac{3}{s} - \frac{4s}{s^2+4} + \frac{s}{s^2+16}\right]$$

$$= -\frac{1}{8}\left[-\frac{3}{s^2} - \frac{(s^2+4)4 - 4s.2s}{(s^2+4)^2} + \frac{(s^2+16).1 - s.2s}{(s^2+16)^2}\right]$$

$$\therefore \quad \int_0^\infty e^{-st}\, t \sin^4 t\, dt = -\frac{1}{8}\left[-\frac{3}{s^2} - \frac{16-4s^2}{(s^2+4)^2} + \frac{16-s^2}{(s^2+16)^2}\right]$$

Taking $s = 1$, we have

$$\int_0^\infty e^{-t}\, t \sin^4 t\, dt = -\frac{1}{8}\left[-\frac{3}{1} - \frac{16-4}{(1+4)^2} + \frac{16-1}{(1+16)^2}\right]$$

$$= -\frac{1}{8}\left[-3 - \frac{12}{25} + \frac{15}{289}\right] = \frac{3}{8} + \frac{3}{50} - \frac{15}{8 \times 289}$$

$$= \left(\frac{3}{8} - \frac{15}{8 \times 289}\right) + \frac{3}{50} = \left(\frac{867 - 15}{2312}\right) + \frac{3}{50}$$

$$= \frac{852}{2312} + \frac{3}{50} = \frac{42600 + 6936}{2312 \times 50} = \frac{49536}{2312 \times 50} = \frac{3096}{7225}.$$

Example 4. *Evaluate* $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt$.

Solution. $\displaystyle L(e^{-at} - e^{-bt}) = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore \quad L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds = \left(\log\frac{s+a}{s+b}\right)_s^\infty$$

$$= \lim_{s \to \infty} \log\frac{s+a}{s+b} - \log\frac{s+a}{s+b}$$

$$= \lim_{s \to \infty} \log\frac{1 + \frac{a}{s}}{1 + \frac{b}{s}} - \log\frac{s+a}{s+b} = 0 - \log\frac{s+a}{s+b}$$

$$\therefore \quad \int_0^\infty e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right) dt = \log\frac{s+b}{s+a}$$

Taking the limit as $s \to 0$,

$$\int_0^{\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}.$$

Example 5. *Show that* $\int_0^{\infty} t e^{-3t} \sin t \, dt = \frac{3}{50}$. [K.U. 2018]

Solution.

$$L(t \sin t) = (-1) \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = (-1) \cdot \frac{(-1)\, 2s}{(s^2+1)^2}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$\therefore \quad \int_0^{\infty} e^{-st}\, t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

Taking the limits as $s \to 3$, we have

$$\int_0^{\infty} e^{-3t}\, t \sin t \, dt = \frac{6}{100} = \frac{3}{50}.$$

Example 6. *Evaluate* $L\left(\int_0^t \frac{e^t \sin t}{t}\, dt\right)$.

Solution. We know that

$$L \int_0^t f(u)\, du = \frac{1}{s} F(s), \text{ where } L[f(t)] = F(s)$$ [Ref. Art. 5.16]

Here $$f(t) = \frac{e^t \sin t}{t}$$

Now, $$L(e^t \sin t) = \frac{1}{(s-1)^2+1}$$

$$\therefore \quad L\left(\frac{e^t \sin t}{t}\right) = \int_s^{\infty} \frac{1}{(s-1)^2+1}\, ds = \left[\tan^{-1}(s-1)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$$

$$\therefore \quad L \int_0^t \frac{e^t \sin t}{t}\, dt = \frac{1}{s} F(s) = \frac{\cot^{-1}(s-1)}{s}.$$

Example 7. *Evaluate* $L \int_0^t e^{-t} \cos t \, dt$.

Solution. Here $f(t) = e^{-t} \cos t$

$$F(s) = L\,(e^{-t} \cos t) = \frac{s+1}{(s+1)^2+1}$$

Now, $$L \int_0^t e^{-t} \cos t \, dt = \frac{1}{s}.F(s)$$ [*Ref. Art.* 5.16]

$$= \frac{1}{s}.\frac{(s+1)}{(s+1)^2+1} = \frac{1}{s}.\frac{s+1}{s^2+2s+2}.$$

Note :

Before solving example 8, students should note the following :

$$\sqrt{-i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \,; \; \sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \,.$$

Example 8. *Show that* $\int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. [*K.U.* 2017; *M.D.U.* 2008]

Solution. Put $x^2 = t$ so that $dx = \frac{1}{2}\frac{dt}{x} = \frac{1}{2\sqrt{t}}\, dt$

The limits of integration remain the same.

$$\therefore \quad \int_0^\infty \sin x^2 \, dx = \int_0^\infty \sin t \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty \frac{e^{it} - e^{-it}}{2i} . \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{4i}\left[\int_0^\infty \frac{e^{it}}{\sqrt{t}} dt - \int_0^\infty \frac{e^{-it}}{\sqrt{t}} dt\right] \quad ...(1)$$

$$\text{1st integral in (1)} = \int_0^\infty \frac{e^{-(-i)t}}{\sqrt{t}} dt = L\left(\frac{1}{\sqrt{t}}\right) \quad [s = -i]$$

$$= L\,(t^{-1/2}) = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{-i}} \qquad \text{[Taking } s = -i\text{]}$$

$$= \frac{\sqrt{\pi}}{\frac{1-i}{\sqrt{2}}} \qquad \text{[See note before this example]}$$

$$= \frac{\sqrt{2}\sqrt{\pi}}{1-i} \qquad \ldots(2)$$

$$\text{2nd integral in (1)} = \int_0^\infty \frac{e^{-it}}{\sqrt{t}}\, dt$$

$$= L\left(\frac{1}{\sqrt{t}}\right) \qquad [s = i]$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{i}} \qquad [s = i]$$

$$= \frac{\sqrt{\pi}}{\frac{1+i}{\sqrt{2}}} \qquad \text{[See note before this example]}$$

$$= \frac{\sqrt{2}\sqrt{\pi}}{1+i} \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$\int_0^\infty \sin x^2\, dx = \frac{1}{4i}\left[\frac{\sqrt{\pi}\sqrt{2}}{1-i} - \frac{\sqrt{\pi}\sqrt{2}}{1+i}\right]$$

$$= \frac{\sqrt{\pi}\sqrt{2}}{4i} \cdot \frac{1+i-1+i}{(1-i)(1+i)}$$

$$= \frac{\sqrt{\pi}\sqrt{2}}{4i} \cdot \frac{2i}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

EXERCISE 5.5

1. Evaluate $\int_0^\infty te^{-2t} \sin t\, dt$ *M.D.U. 2007]*

2. Evaluate $\int_0^\infty t^3 e^{-t} \sin t\, dt$

3. Prove that $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}\,dt = \log 3$

4. Show that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

5. Show that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\left(\frac{2}{3}\right)$

6. Evaluate $L\int_0^t \frac{\cos at - \cos bt}{t}\,dt$

7. Evaluate $L\int_0^t \frac{\sin t}{t}\,dt$

8. Prove that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

9. Show that $\int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ [M.D.U. 2005]

10. Show that : $L\int_0^t \left(\frac{1-e^{-2x}}{x}\right)dx = \frac{1}{s}\log\left(1+\frac{2}{s}\right)$. B.A 2018

11. Using Laplace transform of integrals, solve the following :

(i) Using $L(1) = \frac{1}{s}$, find the Laplace transform of $\frac{1}{7}\sin \pi t$.

(ii) Using $L(1) = \frac{1}{s}$ and $L(\sinh at) = \frac{a}{s^2-a^2}$, find $L(4\cosh 6t)$

(iii) Using $L(t) = \frac{1}{s^2}$, find the value of $L(t^2)$ and $L(t^3)$.

(iv) Using $L(\cosh at) = \frac{s}{s^2-a^2}$, find $L(2\sinh pt)$.

ANSWERS

1. $\frac{4}{25}$

2. 0

6. $\frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}$

7. $\frac{1}{s}\cot^{-1}s$

11. (i) $\frac{1}{7}\cdot\frac{\pi}{(s^2+\pi^2)}$

(ii) $\frac{4s}{s^2-36}$

(iii) $\frac{6}{s^4}$

(iv) $\frac{2p}{s^2-p^2}$

5.17. LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

5.17.1 To find L $J_0(t)$. [K.U. 2016]

Solution. We know that

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4(2n+2)(2n+4)} - \frac{x^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \ldots\right]$$

Taking $n = 0$,

$$J_0(x) = \left[1 - \frac{x^2}{2.2} + \frac{x^4}{2.4.2.4} - \frac{x^6}{2.4.6.2.4.6} + \ldots\right]$$

Changing x to t, we have

$$J_0(t) = \left[1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \ldots\right]$$

$$\therefore \quad L[J_0(t)] = \left[\frac{1}{s} - \frac{2!}{2^2}\cdot\frac{1}{s^3} + \frac{4!}{2^2.4^2}\cdot\frac{1}{s^5} - \frac{6!}{2^2.4^2.6^2}\cdot\frac{1}{s^7}\ldots\right] \qquad \left[\because \ L(t^n) = \frac{n!}{s^{n+1}}\right]$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\cdot\frac{1}{s^2} + \frac{4!}{2^2.4^2}\cdot\frac{1}{s^4} - \frac{6!}{2^2.4^2.6^2}\cdot\frac{1}{s^6} + \ldots\right]$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{1}{s^2}\right)^2 - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{1}{s^2}\right)^3 + \ldots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2} = \frac{1}{s}\left[\frac{1+s^2}{s^2}\right]^{-1/2} = \frac{1}{\sqrt{1+s^2}}.$$

5.17.2. To find L $J_0(at)$. [K.U. 2012]

From previous article, we have L $J_0(t) = \dfrac{1}{\sqrt{1+s^2}}$

Let L $f(t) = F(s)$ (say), where $f(t) = J_0(t)$ and $F(s) = \dfrac{1}{\sqrt{1+s^2}}$

By change of scale property,

$$\mathrm{L}f(at) = \frac{1}{a}\mathrm{F}\left(\frac{s}{a}\right)$$

i.e.,
$$\mathrm{L}\,\mathrm{J}_0(at) = \frac{1}{a}\cdot\frac{1}{\sqrt{1+\frac{s^2}{a^2}}} = \frac{1}{\sqrt{a^2+s^2}}.$$

5.17.3. To find L [$t\,\mathrm{J}_0\,(at)$].

From above, we have $\mathrm{L}\,[\mathrm{J}_0(at)] = \dfrac{1}{\sqrt{a^2+s^2}}$

$$\therefore \quad \mathrm{L}\,[t\,\mathrm{J}_0(at)] = -\frac{d}{ds}\left(\frac{1}{\sqrt{a^2+s^2}}\right) \qquad \text{[Ref. Art. 5.13]}$$

$$= (-1)\left(-\frac{1}{2}\right)\frac{1}{(a^2+s^2)^{3/2}}\,2s = \frac{s}{(a^2+s^2)^{3/2}}$$

5.17.4. To prove that $\int_0^\infty \mathrm{J}_0(t)\,dt = 1$. [K.U. 2012]

We know that
$$\mathrm{L}\,[\mathrm{J}_0\,(t)] = \frac{1}{\sqrt{1+s^2}} \qquad \text{[Ref. Art. 5.17.1]}$$

$$\therefore \quad \int_0^\infty e^{-st}\,\mathrm{J}_0(t)\,dt = \frac{1}{\sqrt{1+s^2}}$$

Taking $s = 0$,
$$\int_0^\infty \mathrm{J}_0(t)\,dt = 1.$$

5.17.5. To find the Laplace transform of error function $erf\,(\sqrt{t})$, where $\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$ denotes $erf\,(t)$.

Taking $\sqrt{t}$ in place of t, we have

$$erf(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \dots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \ldots\right]$$

$$L[erf(\sqrt{t})] = L\left[\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{(5).2!} - \ldots\right)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}} - \frac{\Gamma\left(\frac{5}{2}\right)}{3s^{5/2}} + \frac{\Gamma\left(\frac{7}{2}\right)}{(5)\,2!\,s^{7/2}} - \ldots\right]$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{s^{3/2}}\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{1} - \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{3s} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{(5)\,2!\,s^2} - \ldots\right]$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{s^{3/2}}\left[\frac{1}{2} - \frac{\frac{1}{2}\cdot\frac{3}{2}}{3s} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{(5)\,2!\,s^2} - \ldots\right]$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}\,\frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\cdot\frac{1}{s^2} - \ldots\right] \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{1}{s^{3/2}}\left[1 + \frac{1}{s}\right]^{-1/2}$$

$$= \frac{1}{s^{3/2}}\cdot\frac{s^{1/2}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}.$$

5.17.6. Laplace transform of exponential integral function $\int_t^\infty \frac{e^{-x}}{x}\,dx$, which is denoted by E($t$). [K.U. 2018; M.D.U. 2012, 11]

Given $$E(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx \qquad \ldots(1)$$

The integral on R.H.S. is a function of t say $f(t)$

Let $$f(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx = -\int_\infty^t \frac{e^{-x}}{x}\,dx \qquad \ldots(2)$$

Differentiating w.r.t. t, we have

$$f'(t) = -\left[0 + \frac{dt}{dt}\frac{e^{-t}}{t} - 0\right] \quad \text{[Using Leibnitz rule]}$$

or $$f'(t) = -\frac{e^{-t}}{t}$$

or $$t\,f'(t) = -e^{-t}$$

or $$\mathrm{L}\,[t\,f'(t)] = -\mathrm{L}\,(e^{-t})$$

$\therefore$ $$-\frac{d}{ds}[s\mathrm{F}(s) - f(0)] = -\frac{1}{s+1}$$

or $$\frac{d}{ds}s\mathrm{F}(s) = \frac{1}{s+1} \quad [\because f(0) \text{ is constant i.e., independent of } s]$$

Integrating both sides, $\quad s\,\mathrm{F}(s) = \log(s+1) + c \quad$ ...(3)

Let $s \to 0$. Proceeding to limits

$$\lim_{s \to 0} s\mathrm{F}(s) = \lim_{s \to 0} \log(s+1) + c$$

i.e., $$\lim_{s \to 0} s\mathrm{F}(s) = c$$

i.e., $${}^{*}\lim_{t \to \infty} f(t) = c \quad \text{[See box after the proof]}$$

or $$0 = c \quad \left[\text{By (2), } \lim_{t \to \infty} f(t) = 0\right]$$

$\therefore$ Eq. (3) becomes $\quad s\,\mathrm{F}(s) = \log(s+1)$

or $$\mathrm{F}(s) = \frac{1}{s}\log(s+1)$$

or $$\mathrm{L\,E}\,(t) = \frac{1}{s}\log(s+1).$$

---

* We have used the result $\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,\mathrm{F}(s)$ where $\mathrm{L}\,f(t) = \mathrm{F}(s)$.

Its proof is as follows :

We know that $\quad \mathrm{L}\,[f'(t)] = s\,\mathrm{F}(s) - f(0)$

i.e., $$\int_0^\infty e^{-st} f'(t)\,dt = s\,\mathrm{F}(s) - f(0)$$

Taking $s \to 0$, we have

$$\lim_{s \to 0}[s\mathrm{F}(s) - f(0)] = \lim_{s \to 0}\int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty \lim_{s\to 0} e^{(-st)} f'(t)\,dt = \int_0^\infty f'(t)\,dt = [f(t)]_0^\infty$$

$$= \lim_{t\to\infty} f(t) - \lim_{t\to 0} f(t) = \lim_{t\to\infty} f(t) - f(0)$$

$$\lim_{s\to 0} s\,F(s) = \lim_{t\to\infty} f(t)$$

This theorem is known as **final value theorem.**

5.17.7. To find the Laplace transform of sine integral *i.e.*, $\int_0^t \frac{\sin u}{u}\,du$. [K.U. 2006]

$$\int_0^t \frac{\sin u}{u}\,du = \int_0^t \frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \ldots\right] du$$

$$= \int_0^t \left[1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \ldots\right] du$$

$$= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \ldots$$

$$L\int_0^t \frac{\sin u}{u}\,du = L(t) - \frac{1}{3.3!}L(t^3) + \frac{1}{5.5!}L(t^5) - \frac{1}{7.7!}L(t^7) + \ldots$$

$$= \frac{1}{s^2} - \frac{1}{3.3!}\frac{3!}{s^4} + \frac{1}{5.5!}\frac{5!}{s^6} - \frac{7!}{7.7!}\frac{1}{s^8} + \ldots$$

$$= \frac{1}{s}\left[\frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \ldots\right]$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}. \qquad \left[\text{By expansion of } \tan^{-1} x. \text{ Here } x = \frac{1}{s}\right]$$

5.17.8. To find the Laplace transform of cosine integral *i.e.*, $\int_t^\infty \frac{\cos x}{x}\,dx$.

Let $$f(t) = \int_t^\infty \frac{\cos x}{x}\,dx$$

$$= -\int_\infty^t \frac{\cos x}{x}\,dx$$

$$\therefore \quad f'(t) = -\frac{\cos t}{t}$$

[*Using Leibnitz theorem of differentiation under integral sign*]

or $$t f'(t) = -\cos t$$

$$\therefore \quad L[t f'(t)] = -\frac{s}{s^2+1}$$

$$\therefore \quad -\frac{d}{ds}[s\,F(s) - f(0)] = -\frac{s}{s^2+1}$$

or $$-\frac{d}{ds} s\,F(s) = -\frac{s}{s^2+1} \qquad [\because f(0) \text{ is constant}]$$

Integrating, $$s\,F(s) = \frac{1}{2}\log(s^2+1) + c \qquad ...(1)$$

Let $s \to 0$

$$\therefore \quad \lim_{s \to 0} s\,F(s) = \lim_{s \to 0} \frac{1}{2}\log(s^2+1) + c = c \qquad ...(2)$$

$$\lim_{t \to \infty} f(t) = c$$ [*See result proved in box after Art. 5.17.6*]

or $$0 = c \qquad \left[\because f(t) = \int_t^\infty \frac{\cos x}{x}\,dx\right]$$

$\therefore$ From (1), $$s\,F(s) = \frac{1}{2}\log(s^2+1)$$

or $$F(s) = \frac{1}{2s}\log(s^2+1)$$

or $$L\left(\int_t^\infty \frac{\cos x}{x}\,dx\right) = \frac{1}{2s}\log(s^2+1).$$

Example 1. *Prove that* $\displaystyle\int_0^\infty \frac{J_0(t) - \cos t}{t}\,dt = \log_e 2$.

Solution. We have $L[J_0(t) - \cos t] = \dfrac{1}{\sqrt{1+s^2}} - \dfrac{s}{s^2+1}$ $\qquad \left[\because L\,J_0(t) = \dfrac{1}{\sqrt{1+s^2}}\right]$

$$\therefore \quad L\left(\frac{J_0(t) - \cos t}{t}\right) = \int_s^\infty \left(\frac{1}{\sqrt{1+s^2}} - \frac{s}{s^2+1}\right) ds \qquad [\textit{Ref. Art. 5.14}]$$

$$= \left[\log\left(s+\sqrt{s^2+1}\right) - \frac{1}{2}\log(s^2+1) \right]_s^\infty$$

$$= \left[\log\left(s+\sqrt{s^2+1}\right) - \log\sqrt{s^2+1} \right]_s^\infty$$

$$= \left[\log\left(\frac{s+\sqrt{s^2+1}}{\sqrt{s^2+1}}\right) \right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left[\frac{s\left(1+\sqrt{1+\frac{1}{s^2}}\right)}{s\sqrt{1+\frac{1}{s^2}}}\right] - \log\left(\frac{s+\sqrt{s^2+1}}{\sqrt{s^2+1}}\right)$$

$$= \log 2 + \log\left(\frac{\sqrt{s^2+1}}{s+\sqrt{s^2+1}}\right)$$

$$\therefore \int_0^\infty e^{-st}\left(\frac{J_0(t)-\cos t}{t}\right)dt = \log 2 + \log\left(\frac{\sqrt{s^2+1}}{s+\sqrt{s^2+1}}\right) \quad \text{...(1)}$$

Putting $s = 0$ in (1), we get

$$\int_0^\infty \frac{J_0(t)-\cos t}{t}\,dt = \log 2 + \log 1 = \log_e 2.$$

EXERCISE 5.6

Evaluate :

1. $L[J_1(t)]$
2. $L[t\,(J_1(t))]$
3. $L\,e^{-at}\,[J_0(at)]$

ANSWERS

1. $1-\frac{s}{\sqrt{1+s^2}}$
2. $\frac{1}{(1+s^2)^{3/2}}$
3. $\frac{1}{\sqrt{s^2+2as+2a^2}}$

Summary

Table of Results of Laplace Transforms

No.	Operation	$f(t)$	$L\{f(t)\} = F(s)$
1.	Linear Property	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\}$
2.	First Translation or Shifting Theorem	$e^{at} f(t)$	$F(s-a)$
3.	Second Translation or Shifting Theorem	$G(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$	$e^{-as} F(s)$
4.	Change of Scale Property	$f(at)$ $f\left(\frac{t}{a}\right)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$ $aF(as)$
5.	Differentiation Theorem	$f'(t)$ $f^n(t)$	$sF(s) - f(0)$ $s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots - f^{n-1}(0)$
6.	Multiplication Theorem	$tf(t)$ $t^n f(t)$	$-F'(s)$ $(-1)^n \frac{d^n}{ds^n}[F(s)]$
7.	Division Theorem	$\frac{1}{t} f(t)$	$\int_s^\infty F(s)\, ds$
8.	Integral Theorem	$\int_0^t f(u)\, du$	$\frac{1}{s} F(s)$
9.	Fundamental Theorem for Periodic functions	$L[f(t)] = \int_0^p \frac{e^{-st} f(t)}{1 - e^{-sp}}\, dt,$ where $f(t)$ is a periodic function of period p.	
10.	Initial Value Theorem	$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s\, L\{f(t)\}$	
11.	Final Value Theorem	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, L\{f(t)\}$	

INVERSE LAPLACE TRANSFORMS

C H A P T E R 6

6.1. INVERSE LAPLACE TRANSFORM

Suppose $L[f(t)] = F(s)$. Then $f(t)$ is called ***Inverse Laplace transform*** of $F(s)$ and we write

$$L^{-1}[F(s)] = f(t).$$

Illustrations :

(i) $L^{-1}\left(\frac{1}{s^2}\right)$ is t as $L(t) = \frac{1}{s^2}$

(ii) $L^{-1}\left(\frac{1}{s+2}\right)$ is e^{-2t} as $L(e^{-2t}) = \frac{1}{s+2}$

(iii) $L^{-1}\left(\frac{3}{s^2+9}\right)$ is $\sin 3t$ as $L(\sin 3t) = \frac{3}{s^2+9}$.

Students should note that inverse transform is unique as given by **Lerchs theorem**, which is stated below (without proof).

If $f(t)$ is piece-wise continuous in every finite interval $0 \le t \le a$ and of exponential order for $t > a$, then the inverse Laplace transform $L^{-1}[F(s)] = f(t)$ is unique.

Following results of inverse Laplace are direct consequence of Laplace transformation of some standard functions given earlier which the students should memorize.

1. $L^{-1}\left(\frac{1}{s}\right) = 1, s > 0$, as $L(1) = \frac{1}{s}$ if $s > 0$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{\Gamma(n+1)}$ if $n > 1$, as $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}, s > 0, n > -1$

3. $L^{-1}\left(\dfrac{1}{s^2+a^2}\right)=\dfrac{1}{a}\sin at, s>0$, as $L\left(\dfrac{1}{a}\sin at\right)=\dfrac{1}{a}\cdot\dfrac{a}{s^2+a^2}=\dfrac{1}{s^2+a^2}$

4. $L^{-1}\left(\dfrac{s}{s^2+a^2}\right)=\cos at, s>0$ as $L(\cos at)=\dfrac{s}{s^2+a^2}$

5. $L^{-1}\left(\dfrac{1}{s^2-a^2}\right)=\dfrac{1}{a}\sinh at, s>|a|$, as $L\left(\dfrac{1}{a}\sinh at\right)=\dfrac{1}{a}\cdot\dfrac{a}{s^2-a^2}=\dfrac{1}{s^2-a^2}$

6. $L^{-1}\left(\dfrac{s}{s^2-a^2}\right)=\cosh at, s>|a|$ as $L(\cosh at)=\dfrac{s}{s^2-a^2}$

7. $L^{-1}\left(\dfrac{1}{s-a}\right)=e^{at}, s>a$ as $L(e^{at})=\dfrac{1}{s-a}, s>a$

8. $L^{-1}\left(\dfrac{1}{(s-a)^2+b^2}\right)=\dfrac{1}{b}e^{at}\sin bt$, as $L\left(\dfrac{1}{b}e^{at}\sin bt\right)=\dfrac{1}{b}\dfrac{b}{(s-a)^2+b^2}=\dfrac{1}{(s-a)^2+b^2}$.

[*By shifting property*]

9. $L^{-1}\left(\dfrac{s-a}{(s-a)^2+b^2}\right)=e^{at}\cos bt$, as $L(e^{at}\cos bt)=\dfrac{s-a}{(s-a)^2+b^2}$

10. $L^{-1}\left(\dfrac{1}{(s-a)^2-b^2}\right)=\dfrac{1}{b}e^{at}\sinh bt$, as $L\left(\dfrac{1}{b}e^{at}\sinh bt\right)=\dfrac{1}{b}\cdot\dfrac{b}{(s-a)^2-b^2}=\dfrac{1}{(s-a)^2-b^2}$

11. $L^{-1}\left(\dfrac{s-a}{(s-a)^2-b^2}\right)=e^{at}\cosh bt$, as $L(e^{at}\cosh bt)=\dfrac{s-a}{(s-a)^2-b^2}$.

6.1.1. To show that

(*i*) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]=\dfrac{1}{2a}t\sin at$

(*ii*) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]=\dfrac{1}{2a^3}(\sin at-at\cos at)$.

Solution. (*i*) We have $L(t)=\dfrac{1}{s^2}$

$$L(te^{iat}) = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$$

$$L[t(\cos at + i\sin at)] = \frac{(s^2-a^2)+2ias}{(s^2+a^2)^2}$$

Comparing the real and imaginary parts, we have

$$L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2} \quad ...(1)$$

$$L(t\sin at) = \frac{2as}{(s^2+a^2)^2} \quad ...(2)$$

It follows from (2) that $t\sin at = L^{-1}\left(\dfrac{2as}{(s^2+a^2)^2}\right)$

Thus $\quad L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right) = \dfrac{1}{2a}t\sin at.$

(*ii*) From (1), $\quad L(t\cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$

$$\therefore \quad t\cos at = L^{-1}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{s^2+a^2-2a^2}{(s^2+a^2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s^2+a^2}\right) - 2a^2L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$$

$$= \frac{1}{a}\sin at - 2a^2\, L^{-1}\frac{1}{(s^2+a^2)^2}$$

By transposition, $2a^2\, L^{-1}\left(\dfrac{1}{(s^2+a^2)^2}\right) = \dfrac{1}{a}\sin at - t\cos at$

or

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = \frac{1}{2a^3}(\sin at - at\cos at).$$

SOLVED EXAMPLES

Example 1. *Find the inverse Laplace transform of* $\frac{1}{s^{7/2}}$. [M.D.U. 2013]

Solution.
$$L^{-1}\left(\frac{1}{s^{7/2}}\right) = \frac{t^{\frac{7}{2}-1}}{\Gamma\left(\frac{7}{2}\right)} \qquad \left[\because L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\Gamma(n)}\right]$$

$$= \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{8t^{5/2}}{15\sqrt{\pi}} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{8t^2}{15}\sqrt{\frac{t}{\pi}}.$$

Example 2. *Find* $L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right)$. [M.D.U. 2015]

Solution.
$$L^{-1}\left(\frac{1}{s}\sin\frac{1}{s}\right) = L^{-1}\left[\frac{1}{s}\left(\frac{1}{s} - \frac{1}{3!}\cdot\frac{1}{s^3} + \frac{1}{5!}\cdot\frac{1}{s^5} - \ldots\right)\right]$$

$$= L^{-1}\left[\frac{1}{s^2} - \frac{1}{3!}\cdot\frac{1}{s^4} + \frac{1}{5!}\cdot\frac{1}{s^6} - \ldots\right]$$

$$= \frac{t}{1!} - \frac{1}{3!}\cdot\frac{t^3}{3!} + \frac{1}{5!}\cdot\frac{t^5}{5!} - \ldots$$

$$= t - \frac{t^3}{(3!)^2} + \frac{t^5}{(5!)^2} - \ldots$$

Example 3. *Find the inverse Laplace transform of* $\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}$.

Solution.
$$L^{-1}\left[\frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}\right] = \frac{1}{4}L^{-1}\left(\frac{2s-5}{s^2+\frac{25}{4}}\right) - L^{-1}\left(\frac{4s-18}{s^2-9}\right)$$

$$= \frac{1}{4}\left[2L^{-1}\left(\frac{s}{s^2+\frac{25}{4}}\right) - 5L^{-1}\left(\frac{1}{s^2+\frac{25}{4}}\right)\right] - \left[4L^{-1}\left(\frac{s}{s^2-9}\right) - 18L^{-1}\left(\frac{1}{s^2-9}\right)\right]$$

$$= \frac{1}{4}\left[2\cos\frac{5}{2}t - 5\,\frac{1}{5/2}\sin\frac{5}{2}t\right] - 4\cosh 3t + 18\,.\,\frac{1}{3}\sinh 3t$$

[*Using formulas 3, 4, 5, 6; Art. 6.1*]

$$= \frac{1}{2}\left[\cos\frac{5}{2}t - \sin\frac{5}{2}t\right] - 4\cosh 3t + 6\sinh 3t\,.$$

Example 4. *Find the inverse Laplace transform of* $\dfrac{1+2s}{(s+2)^2\,(s-1)^2}$.

Solution. Let $$\frac{1+2s}{(s+2)^2\,(s-1)^2} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s+2)^2} + \frac{D}{(s-1)^2} \qquad ...(1)$$

Multiplying both sides by $(s+2)^2\,(s-1)^2$, we have

$$1 + 2s = A\,(s+2)\,(s-1)^2 + B\,(s-1)\,(s+2)^2 + C\,(s-1)^2 + D\,(s+2)^2 \qquad ...(2)$$

Putting $s = 1$ in (2) : $\quad 3 = 9D \;\Rightarrow\; D = \dfrac{1}{3}$

Putting $s = -2$ in (2) : $\;-3 = 9C \;\Rightarrow\; C = -\dfrac{1}{3}$

Comparing the coefficient of s^3 on both sides of (2), we have

$$0 = A + B \qquad ...(3)$$

Comparing the constant term on both sides of (2), we have

$$1 = 2A - 4B + C + 4D$$

or $$1 = 2A - 4B - \frac{1}{3} + \frac{4}{3}$$

$\therefore$ $$2A - 4B = 0$$

i.e., $$A - 2B = 0 \qquad ...(4)$$

Solving (3), and (4), we have $A = 0,\; B = 0$

$\therefore$ $$\frac{1+2s}{(s+2)^2\,(s-1)^2} = -\frac{1}{3}\,\frac{1}{(s+2)^2} + \frac{1}{3}\,\frac{1}{(s-1)^2}$$

$\therefore$ $$L^{-1}\left(\frac{1+2s}{(s+2)^2\,(s-1)^2}\right) = -\frac{1}{3}\,e^{-2t}\,t + \frac{1}{3}\,e^{t}\,t = \frac{1}{3}\,t\,(e^{t} - e^{-2t})\,.$$

Example 5. *Find* $L^{-1}\left(\dfrac{s}{(s+1)^2\,(s^2+1)}\right)$. [*M.D.U. 2015*]

Solution. Let $$\frac{s}{(s+1)^2\,(s^2+1)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1} \qquad ...(1)$$

Multiplying both sides by $(s+1)^2(s^2+1)$, we get

$$s = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2 \qquad ...(2)$$

Putting $s=-1$ in (2): $-1 = 2B \Rightarrow B = -\frac{1}{2}$

Comparing the coefficients of s^3, s^2 and constant term in (2), we have

$$0 = A + C \qquad ...(3)$$

$$0 = A + B + 2C + D \qquad ...(4)$$

$$0 = A + B + D \qquad ...(5)$$

Putting the value of $B = -\frac{1}{2}$ in (4) and (5), we have

$$A + 2C + D = \frac{1}{2} \qquad ...(6)$$

$$A + D = \frac{1}{2} \qquad ...(7)$$

Solving (6) and (7), we have $C = 0$

$\therefore$ From (3), $\quad A = 0$

Thus from (7), $\quad D = \frac{1}{2}$

$$\therefore \quad \frac{s}{(s+1)^2(s^2+1)} = \frac{-\frac{1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1} \qquad [From (1)]$$

$$\therefore \quad L^{-1}\left(\frac{s}{(s+1)^2(s^2+1)}\right) = -\frac{1}{2}L^{-1}\left(\frac{1}{(s+1)^2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{1}{2}[e^{-t}\,t - \sin t].$$

Example 6. *Find* $L^{-1}\left(\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}\right)$.

Solution. Let $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+2s+2} \qquad ...(1)$

Multiplying both sides by $(s^2+1)(s^2+2s+2)$, we have

$$s^2 + s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1) \qquad ...(2)$$

Comparing the coefficients of s^3, s^2, s and constant terms in (2), we get

$$0 = A + C \qquad ...(3)$$

$$1 = 2A + B + D \quad \text{....(4)}$$

$$1 = 2A + 2B + C \quad \text{...(5)}$$

$$0 = 2B + D \quad \text{...(6)}$$

Putting $D = -2B$ from (6) and $C = -A$ from (3) in (4) and (5), we have

$$1 = 2A + B - 2B \quad i.e., \quad 1 = 2A - B \quad \text{..(7)}$$

and

$$1 = 2A + 2B - A \quad i.e., \quad 1 = A + 2B \quad \text{...(8)}$$

Solving (7) and (8), we have

$$A = \frac{3}{5}, \quad B = \frac{1}{5}$$

From (3), $\quad C = -A = -\frac{3}{5}$

From (6), $\quad D = -2B = -\frac{2}{5}$

Putting the values of A, B, C, D in (1), we get

$$\frac{s^2+s}{(s^2+1)(s^2+2s+2)} = \frac{\frac{3}{5}s + \frac{1}{5}}{s^2+1} + \frac{-\frac{3}{5}s - \frac{2}{5}}{s^2+2s+2}$$

$$\therefore \; L^{-1}\left(\frac{s^2+s}{(s^2+1)(s^2+2s+2)}\right) = \frac{1}{5}L^{-1}\left(\frac{3s+1}{s^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{3s+2}{(s+1)^2+1}\right)$$

$$= \frac{1}{5}\left[3L^{-1}\left(\frac{s}{s^2+1}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)\right] - \frac{1}{5}L^{-1}\left(\frac{3s+3-1}{(s+1)^2+1}\right)$$

$$= \frac{1}{5}[3\cos t + \sin t] - \frac{1}{5}\left[3L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{5}[3\cos t + \sin t - 3e^{-t}\cos t + e^{-t}\sin t]$$

$$= \frac{1}{5}[(1-e^{-t})\,3\cos t + (1+e^{-t})\sin t].$$

Example 7. *Find the inverse Laplace transform of* $\frac{s}{s^4+s^2+1}$. [K.U. 2016; M.D.U. 2014]

Solution.

$$\frac{s}{s^4+s^2+1} = \frac{s}{s^4+2s^2+1-s^2} = \frac{s}{(s^2+1)^2-s^2}$$

$$= \frac{s}{(s^2+1+s)(s^2+1-s)} \quad \text{...(1)}$$

Let $\dfrac{s}{(s^2+s+1)(s^2-s+1)} = \dfrac{As+B}{s^2+s+1} + \dfrac{Cs+D}{s^2-s+1}$ ...(2)

Multiplying both sides by $(s^2 + s + 1)(s^2 - s + 1)$, we have

$$s = (As + B)(s^2 - s + 1) + (Cs + D)(s^2 + s + 1) \quad ...(3)$$

Comparing the coefficients of s^3, s^2, s and constant terms in (3), we get

$$0 = A + C \quad ...(4)$$

$$0 = -A + B + C + D \quad ...(5)$$

$$1 = A - B + C + D \quad ...(6)$$

$$0 = B + D \quad ...(7)$$

From (4) and (6), $\quad 1 = -B + D$ ...(8)

Solving (7) and (8), we have

$$D = \frac{1}{2}, B = -\frac{1}{2}$$

From (5), $\quad 0 = -A + C$ ...(9) $[\because B + D = 0]$

Solving (4) and (9), $\quad A = 0, C = 0$

Substituting the values of A, B, C and D in (2), we get

$$\frac{s}{s^4+s^2+1} = \frac{-1}{2(s^2+s+1)} + \frac{1}{2(s^2-s+1)}$$

$$\therefore \quad L^{-1}\left(\frac{s}{s^4+s^2+1}\right) = \left(\frac{-1}{2}\right)L^{-1}\left(\frac{1}{s^2+s+1}\right) + \frac{1}{2}L^{-1}\frac{1}{(s^2-s+1)}$$

$$= -\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$= -\frac{1}{2}e^{-\frac{1}{2}t}\frac{1}{\frac{\sqrt{3}}{2}}\sin\frac{\sqrt{3}}{2}t + \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

$$= \frac{1}{\sqrt{3}}(e^{-t/2} - e^{-t/2})\sin\frac{\sqrt{3}}{2}t = \frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t.$$

Example 8. *Find the inverse Laplace transform of*

$$\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}. \quad \text{[K.U. 2004]}$$

Solution. We have $L^{-1}\left(\dfrac{1}{s^2+4s+13}\right) = L^{-1}\left(\dfrac{1}{(s+2)^2+9}\right)$

$$= \frac{1}{3}e^{-2t}\sin 3t$$

$$L^{-1}\left(\frac{s+4}{s^2+8s+97}\right) = L^{-1}\left(\frac{s+4}{(s+4)^2+81}\right)$$

$$= e^{-4t}\cos 9t$$

$$L^{-1}\left(\frac{s+2}{s^2-4s+29}\right) = L^{-1}\left(\frac{s-2+4}{(s-2)^2+25}\right)$$

$$= L^{-1}\left(\frac{s-2}{(s-2)^2+25}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2+25}\right)$$

$$= e^{2t}\cos 5t + 4.\frac{1}{5}e^{2t}\sin 5t$$

$$\therefore\ L^{-1}\left(\frac{1}{s^2+4s+13} - \frac{s+4}{s^2+8s+97} + \frac{s+2}{s^2-4s+29}\right)$$

$$= \frac{1}{3}e^{-2t}\sin 3t + e^{-4t}\cos 9t + e^{2t}\cos 5t + \frac{4}{5}e^{2t}\sin 5t.$$

EXERCISE 6.1

Find the Inverse Laplace transform of the following functions :

1. $\dfrac{1}{s}\cos\dfrac{1}{s}$

2. $\dfrac{3s-8}{4s^2+25}$

3. $\dfrac{3(s^2-2)^2}{2s^5}$

4. $\dfrac{5}{s^2} + \left(\dfrac{\sqrt{s}-1}{s}\right)^2 - \dfrac{7}{3s+2}$

5. $\dfrac{3s}{s^2+2s-8}$

6. $\dfrac{s}{(s+3)^2+4}$

7. $\dfrac{1}{s^2-6s+10}$

8. $\dfrac{s^2+s-2}{s(s+3)(s-2)}$ [M.D.U. 2014; K.U. 2007]

9. $\dfrac{s^3}{s^4-a^4}$

10. $\dfrac{s}{(s^2-1)^2}$

11. $\dfrac{1}{s^3 - a^3}$

12. $\dfrac{s^2 - 10s + 13}{(s-7)(s^2 - 5s + 6)}$

13. $\dfrac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}$

14. $\dfrac{a(s^2 - 2a^2)}{s^4 + 4a^4}$

15. $\dfrac{s}{s^4 + 4a^4}$

16. $\dfrac{s+3}{(s^2 + 6s + 13)^2}$

17. $\dfrac{s+2}{(s^2 + 4s + 5)^2}$ BoA 2018

18. $\dfrac{s^2 + 6}{(s^2+1)(s^2+4)}$

19. $\dfrac{2s - 3}{s^2 + 4s + 13}$

20. $\dfrac{3s + 7}{s^2 - 2s - 3}$

21. $\dfrac{1}{s^2 (s-a)^2}$

22. $\dfrac{1}{s^2(s^2+1)(s^2+9)}$.

ANSWERS

1. $\left[1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \ldots..\right]$

2. $\dfrac{3}{4}\cos\dfrac{5}{2}t - \dfrac{4}{5}\sin\dfrac{5}{2}t$

3. $\dfrac{3}{2} - 3t^2 + \dfrac{1}{4}t^4$

4. $1 + 6t - 4\sqrt{\dfrac{t}{\pi}} - \dfrac{7}{3}e^{-\frac{2}{3}t}$

5. $e^{2t} + 2e^{-4t}$

6. $e^{-3t}\left(\cos 2t - \dfrac{3}{2}\sin 2t\right)$

7. $e^{3t}\sin t$

8. $\dfrac{1}{3} + \dfrac{4}{15}e^{-3t} + \dfrac{2}{5}e^{2t}$

9. $\dfrac{1}{2}[\cosh at + \cos at]$

10. $\dfrac{1}{2}t\sinh t$

11. $\dfrac{1}{3a^2}\left[e^{at} - e^{-\frac{at}{2}}\cos\dfrac{\sqrt{3}}{2}at - \sqrt{3}e^{-\frac{a}{2}t}\sin\dfrac{\sqrt{3}}{2}at\right]$

12. $-\dfrac{2}{5}e^{7t} - \dfrac{3}{5}e^{2t} + 2e^{3t}$

13. $\dfrac{1}{2}e^t - e^{2t} + \dfrac{5}{2}e^{3t}$

14. $\cos at \sinh at$

15. $\dfrac{1}{2a^2}\sin at \sinh at$

16. $e^{-3t}\dfrac{1}{4}t\sin 2t$

17. $e^{-2t}\frac{1}{2}t\sin t$ **18.** $\frac{1}{3}(5\sin t - \sin 2t)$

19. $\frac{1}{3}e^{-2t}(6\cos 3t - 7\sin 3t)$ **20.** $4e^{3t} - e^{-t}$

21. $\frac{1}{a^3}[(2+at)-(2-at)e^{at}]$ **22.** $\frac{1}{9}t - \frac{1}{8}\sin t + \frac{1}{216}\sin 3t$.

6.2. OTHER METHODS FOR FINDING INVERSE TRANSFORMS

6.2.1. If $L^{-1}[F(s)] = f(t)$, then $L^{-1}[F(s-a)] = e^{at} L^{-1} F(s)$. [*First Shifting Property*]

Proof. Here $L^{-1} F(s) = f(t)$ ...(1)

$\therefore$ $F(s) = L[f(t)]$

i.e., $F(s-a) = L[e^{at} f(t)]$

or $L^{-1} F(s-a) = e^{at} f(t)$

or $L^{-1} F(s-a) = e^{at} L^{-1} F(s)$ [*Using (1)*]

(This property is known as shifting property of inverse Laplace transformation)

Remember. If $L^{-1}(F(s)) = f(t)$, then $L^{-1}(e^{-as}F(s)) = f(t-a)\,u(t-a)$ for any $a > 0$ where $u(t-a)$ is a step function.

6.2.2. If $L^{-1} F(s) = f(t)$ and $f(0) = 0$, then $L^{-1}[s F(s)] = \frac{d}{dt} f(t)$.

[*Ref. Art. 5.11*]

Proof. We know that $L f'(t) = s F(s) - f(0)$

$= s F(s)$ [*Given $f(0) = 0$*]

$\therefore$ $f'(t) = L^{-1} s F(s)$

Reversing the sides, $L^{-1}[s F(s)] = \frac{d}{dt} f(t)$.

Generalisation : In the same way as above, from Laplace transform of nth order derivative, we have

$$L\left(\frac{d^n}{dt^n} f(t)\right) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots f^{n-1}(0)$$

If we assume $f(0) = f'(0) = f''(0) = \ldots\ldots f^{n-1}(0) = 0$, then

$$L\left(\frac{d^n}{dt^n} f(t)\right) = s^n F(s)$$

or
$$\frac{d^n}{dt^n} f(t) = L^{-1}[s^n F(s)]$$

Hence
$$L^{-1}[s^n F(s)] = \frac{d^n}{dt^n} f(t).$$

6.2.3. If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$

Proof. As
$$L^{-1} F(s) = f(t)$$

$\therefore$
$$F(s) = L f(t)$$

From the transform of integrals, we have

$$L \int_0^t f(u)\,du = \frac{F(s)}{s} \qquad \text{[Ref. Art. 5.16]}$$

or
$$\int_0^t f(u)\,du = L^{-1}\left(\frac{F(s)}{s}\right)$$

Reversing the sides and changing the variable u to t, we have

$$L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(t)\,dt$$

Note :

By the repeated application of above theorem, we have

$$L^{-1}\left(\frac{F(s)}{s^2}\right) = \int_0^t \int_0^t f(t)\,dt$$

$$L^{-1}\left(\frac{F(s)}{s^3}\right) = \int_0^t \int_0^t \int_0^t f(t)\,dt \quad \text{and so on.}$$

6.2.4. If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[-\frac{d}{ds} F(s)\right] = t\, f(t)$.

Proof. As
$$L^{-1}[F(s)] = f(t)$$

$\therefore$
$$F(s) = L[f(t)]$$

Reversing the sides, we have

$$L[f(t)] = F(s)$$

$$\therefore \quad L[t f(t)] = -\frac{d}{ds} F(s) \qquad \text{[Ref. Art. 5.13]}$$

or

$$t f(t) = L^{-1}\left[-\frac{d}{ds}[F(s)]\right]$$

Reversing the sides again, we have

$$L^{-1}\left[-\frac{d}{ds}[F(s)]\right] = t f(t).$$

6.2.5. If $L^{-1}[F(s)] = f(t)$, then $L^{-1}\left[\int_s^{\infty} F(s)\, ds\right] = \frac{f(t)}{t}$, provided the inverse transform of $\int_s^{\infty} f(s)\, ds$ can be calculated.

Proof. As $\quad L^{-1}[F(s)] = f(t)$

$$\therefore \quad L[f(t)] = F(s)$$

We know that if $\quad L f(t) = F(s)$, then

$$L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s)\, ds \qquad \text{[Ref. Art. 5.14]}$$

$$\therefore \quad \frac{f(t)}{t} = L^{-1}\left[\int_s^{\infty} F(s)\, ds\right]$$

SOLVED EXAMPLES

Example 1. *Find the inverse Laplace transform of* $\frac{1}{s(s+2)^3}$. [K.U. 2006]

Solution. $L^{-1}\frac{1}{s(s+2)^3} = e^{-2t} L^{-1}\left(\frac{1}{(s-2)s^3}\right)^*$ [By Art. 6.2.1]

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* How to use shifting property :

It reads as $L^{-1} F(s-a) = e^{at} L^{-1} F(s)$

Change s to $s + a$ on L.H.S. and obtain F(s) and multiply by e^{at} to get R.H.S.

$$= e^{-2t}\, L^{-1}\left[\frac{\left(\frac{1}{s-2}\right)}{s^3}\right]$$

$$= e^{-2t}\int_0^t\int_0^t\int_0^t e^{2t}\,dt \qquad \text{[By Art. 6.2.3]}$$

$$= e^{-2t}\int_0^t\int_0^t\left(\frac{e^{2t}}{2}\right)_0^t dt = \frac{e^{-2t}}{2}\int_0^t\int_0^t (e^{2t}-1)\,dt$$

$$= \frac{1}{2}e^{-2t}\int_0^t\left(\frac{e^{2t}}{2}-t\right)_0^t dt$$

$$= \frac{1}{2}e^{-2t}\int_0^t\left(\frac{e^{2t}}{2}-t-\frac{1}{2}\right)dt = \frac{1}{4}e^{-2t}\int_0^t (e^{2t}-2t-1)\,dt$$

$$= \frac{1}{4}e^{-2t}\left[\frac{e^{2t}}{2}-t^2-t\right]_0^t = \frac{1}{4}e^{-2t}\left[\left(\frac{e^{2t}}{2}-t^2-t\right)-\frac{1}{2}\right]$$

$$= \frac{1}{2}e^{-2t}\left[\frac{e^{2t}}{2}-t^2-t-\frac{1}{2}\right].$$

Example 2. *Find the inverse Laplace transform of* $\dfrac{1}{s^3(s^2+1)}$. *[M.D.U. 2018; K.U. 2016]*

Solution. $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

$$\therefore \quad L^{-1}\left(\frac{1}{s^3(s^2+1)}\right) = \int_0^t\int_0^t\int_0^t \sin t\,dt \qquad \text{[By Art. 6.2.3]}$$

$$= \int_0^t\int_0^t [-\cos t]_0^t\,dt = \int_0^t\int_0^t (1-\cos t)\,dt$$

$$= \int_0^t [t-\sin t]_0^t\,dt = \int_0^t (t-\sin t)\,dt$$

$$= \left[\frac{t^2}{2}+\cos t\right]_0^t = \frac{t^2}{2}+\cos t - 1.$$

Example 3. *Find the inverse Laplace transform of* $\log \frac{1+s}{s}$. [K.U. 2015]

Solution. Let $L^{-1} F(s) = f(t)$, where $F(s) = \log \frac{1+s}{s}$ ...(1)

$\therefore$ $$L^{-1}\left[-\frac{d}{ds} F(s)\right] = t f(t)$$ [By Art. 6.2.4.]

i.e., $$L^{-1}\left[-\frac{d}{ds} \log \frac{1+s}{s}\right] = t f(t)$$

or $$L^{-1}\left[-\frac{d}{ds} (\log (1+s) - \log s)\right] = t f(t)$$

or $$L^{-1}\left[-\frac{1}{1+s} + \frac{1}{s}\right] = t f(t)$$

or $$L^{-1}\left[\frac{1}{s} - \frac{1}{1+s}\right] = t f(t)$$

or $$(1 - e^{-t}) = t f(t)$$

$\therefore$ $$f(t) = \frac{1}{t}(1 - e^{-t})$$

i.e., $$L^{-1}\left(\log \frac{1+s}{s}\right) = \frac{1}{t}(1 - e^{-t}).$$ [By (1)]

Example 4. *Find the inverse Laplace transform of* $\log \frac{s^2+1}{(s-1)^2}$.

[K.U. 2013; M.D.U. 2005]

Solution. Let $L^{-1}[F(s)] = f(t)$, where $F(s) = \log \frac{s^2+1}{(s-1)^2}$

$\therefore$ $$L^{-1}\left[-\frac{d}{ds}\{F(s)\}\right] = t f(t)$$ [By Art. 6.2.4]

$\therefore$ $$-L^{-1}\frac{d}{ds}\left(\log \frac{s^2+1}{(s-1)^2}\right) = t f(t)$$

or $$L^{-1}\left[-\frac{d}{ds}\log (s^2+1) + \frac{d}{ds}\log (s-1)^2\right] = t f(t)$$

or $$L^{-1}\left[\frac{-2s}{s^2+1} + \frac{2}{s-1}\right] = t f(t)$$

or $$-2L^{-1}\frac{s}{s^2+1}+2L^{-1}\frac{1}{s-1}=t\,f(t)$$

or $$-2\cos t+2e^t=t\,f(t)$$

or $$f(t)=\frac{1}{t}[2e^t-2\cos t]=\frac{2}{t}[e^t-\cos t].$$

Example 5. *Find the inverse Laplace transform of* $\frac{s^2}{(s+a)^3}$.

Solution. Here $L^{-1}\frac{s^2}{(s+a)^3}$ *i.e.*, $L^{-1}F(s)$ is required, where

$$F(s)=\frac{s^2}{(s+a)^3}$$

$$\therefore \quad F(s-a)=\frac{(s-a)^2}{s^3}$$

Now, $$e^{at}L^{-1}F(s)=L^{-1}F(s-a)$$ [By Art. 6.2.1]

$$=L^{-1}\left(\frac{(s-a)^2}{s^3}\right)=L^{-1}\left(\frac{1}{s}-\frac{2a}{s^2}+\frac{a^2}{s^3}\right)$$

$$=1-2at+\frac{a^2t^2}{2!}$$

$$\therefore \quad L^{-1}[F(s)]=e^{-at}\left(1-2at+\frac{a^2t^2}{2!}\right).$$

Example 6. *Find the inverse Laplace transform of* $\cot^{-1}\frac{s}{\pi}$.

Solution. Let $$L^{-1}\left(\cot^{-1}\frac{s}{\pi}\right)=f(t)$$

$$\therefore \quad L^{-1}\left[-\frac{d}{ds}\cot^{-1}\frac{s}{\pi}\right]=t\,f(t)$$ [By Art. 6.2.4]

or $$L^{-1}\left[\frac{1}{1+\frac{s^2}{\pi^2}}\left(\frac{1}{\pi}\right)\right]=t\,f(t)$$

or $$L^{-1}\left[\frac{\pi}{s^2+\pi^2}\right]=t\,f(t)$$

or $$\pi L^{-1}\left[\frac{1}{s^2+\pi^2}\right]=t\,f(t)$$

$$\pi \cdot \frac{1}{\pi} \sin \pi t = t f(t)$$

or

$$f(t) = \frac{1}{t} (\sin \pi t)$$

∴

i.e.,

$$L^{-1}\left(\cot^{-1} \frac{s}{\pi}\right) = \frac{\sin \pi t}{t}.$$

Example 7. *Find the inverse Laplace transform of* $\tan^{-1} \frac{2}{s^2}$. [K.U. 2017, 14]

Solution. Let $$L^{-1}\left(\tan^{-1} \frac{2}{s^2}\right) = f(t)$$

∴ $$L^{-1}\left[-\frac{d}{ds} \tan^{-1} \frac{2}{s^2}\right] = t f(t) \quad \text{[By Art. 6.2.4]}$$

or $$L^{-1}\left[(-1) \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right)\right] = t f(t)$$

or $$L^{-1}\left[\frac{4s}{s^4+4}\right] = t f(t)$$

or $$L^{-1}\left[\frac{4s}{(s^2+2)^2 - 4s^2}\right] = t f(t)$$

or $$L^{-1}\left[\frac{4s}{(s^2+2s+2)(s^2-2s+2)}\right] = t f(t) \quad \text{...(1)}$$

Let $$\frac{4s}{(s^2+2s+2)(s^2-2s+2)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2-2s+2}$$

∴ $$4s = (As+B)(s^2-2s+2) + (Cs+D)(s^2+2s+2) \quad \text{...(2)}$$

Comparing the coefficient of s^3, s^2, s and constants in (2), we have

$$0 = A + C \quad \text{...(3)}$$

$$0 = -2A + B + 2C + D \quad \text{...(4)}$$

$$4 = 2A - 2B + 2C + 2D \Rightarrow 2 = A - B + C + D \quad \text{...(5)}$$

$$0 = 2B + 2D \Rightarrow 0 = B + D \quad \text{...(6)}$$

Putting the value of B + D (= 0) in (4),

$$0 = -2A + 2C \Rightarrow 0 = -A + C \quad \text{...(7)}$$

Solving (3) and (7), we have

$$A = 0, C = 0$$

From (5), $2 = -B + D$ ...(8)

From (6), $0 = B + D$ ...(9)

Solving (8) and (9), we have $D = 1$, $B = -1$

$$\therefore \quad \frac{4s}{(s^2+2s+2)(s^2-2s+2)} = \frac{-1}{s^2+2s+2} + \frac{1}{s^2-2s+2}$$

$$= \frac{-1}{(s+1)^2+1} + \frac{1}{(s-1)^2+1}$$

$$\therefore \quad L^{-1}\left(\frac{4s}{(s^2+2s+2)(s^2-2s+2)}\right) = L^{-1}\left(\frac{-1}{(s+1)^2+1}\right) + L^{-1}\left(\frac{1}{(s-1)^2+1}\right)$$

$$= -e^{-t}\sin t + e^{t}\sin t$$

Substituting this value in (1), we have

$$(-e^{-t}+e^{t})\sin t = t\,f(t)$$

or

$$2\sin t\sinh t = t\,[f(t)]$$

or

$$f(t) = \frac{1}{t}(2\sin t\sinh t).$$

Example 8. *Find the inverse Laplace transform of* $-\dfrac{e^{-\frac{\pi}{2}s}}{s^2+1}$.

Solution. Let $\quad F(s) = -\dfrac{1}{s^2+1}$

$$\therefore \quad L^{-1}(F(s)) = L^{-1}\left(-\frac{1}{s^2+1}\right) = -\sin t$$

$$\therefore \quad L^{-1}\left(-\frac{e^{-\frac{\pi}{2}s}}{s^2+1}\right) = L^{-1}\left(e^{-\frac{\pi}{2}s}F(s)\right) = f\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)$$

$$= -\sin\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right) = \cos t \,.\, u\left(t-\frac{\pi}{2}\right)$$

$$= \begin{cases} 0, & \text{if } 0 \le t < \pi/2 \\ \cos t, & \text{if } \quad t > \pi/2 \end{cases}$$

Example 9. *Find* $L^{-1}\left(\dfrac{e^{-5s}}{(s-2)^4}\right)$.

Solution. Let $\quad F(s) = \dfrac{1}{(s-2)^4}$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s-2)^4}\right) = e^{2t}\, L^{-1}\left(\frac{1}{s^4}\right)$$

$$= e^{2t}\,\frac{t^3}{3!} = \frac{1}{6}\,t^3 e^{2t}$$

$$L^{-1}\left(\frac{e^{-5s}}{(s-2)^4}\right) = L^{-1}(e^{-5}\,F(s))$$

$$= f(t-5)\,u\,(t-5)$$

$$= \frac{1}{6}(t-5)^3\, e^{2(t-5)}\,.\,u\,(t-5)$$

Example 10. *Find the inverse Laplace transform of* $\frac{1}{s(s-6)^4}$. [M.D.U. 2008]

Solution. We know that $L^{-1}\,F(s-a) = e^{at}\,L^{-1}\,F(s)$

Using this formula, we have

$$L^{-1}\left(\frac{1}{s\,(s-6)^4}\right) = e^{6t}\,L^{-1}\left(\frac{1}{(s+6)\,s^4}\right)$$

$$= e^{6t}\,L^{-1}\left(\frac{\frac{1}{s+6}}{s^4}\right) = e^{6t}\int_0^t\int_0^t\int_0^t\int_0^t e^{-6t}\,dt \qquad ...(1)$$

Now,
$$\int_0^t e^{-6t}\,dt = \left[\frac{e^{-6t}}{-6}\right]_0^t = \frac{1}{6}(1-e^{-6t})$$

$$\int_0^t\int_0^t e^{-6t}\,dt = \int_0^t \frac{1}{6}(1-e^{-6t})\,dt$$

$$= \frac{1}{6}\left[t - \frac{e^{-6t}}{-6}\right]_0^t = \frac{1}{6}\left[t + \frac{1}{6}e^{-6t} - \frac{1}{6}\right]$$

$$\int_0^t\int_0^t\int_0^t e^{-6t}\,dt = \frac{1}{36}\int_0^t (6t + e^{-6t} - 1)\,dt = \frac{1}{36}\left[3t^2 - \frac{e^{-6t}}{6} - t\right]_0^t$$

$$= \frac{1}{36}\left[3t^2 - \frac{e^{-6t}}{6} - t + \frac{1}{6}\right] = \frac{1}{216}\,[18t^2 - e^{-6t} - 6t + 1]$$

$$\int_0^t\int_0^t\int_0^t\int_0^t e^{-6t}\,dt = \frac{1}{216}\int_0^t (18t^2 - e^{-6t} - 6t + 1)\,dt$$

$$= \frac{1}{216}\left[6t^3 + \frac{e^{-6t}}{6} - 3t^2 + t\right]_0^t$$

$$= \frac{1}{216}\left[6t^3 + \frac{e^{-6t}}{6} - 3t^2 + t - \frac{1}{6}\right]$$

$$= \frac{1}{1296}(36t^3 + e^{-6t} - 18t^2 + 6t - 1)$$

$\therefore$ From (1), $L^{-1}\left(\frac{1}{s(s-6)^4}\right) = \frac{e^{6t}}{1296}(36t^3 + e^{-6t} - 18t^2 + 6t - 1).$

Example 11. *For $a > 0$, prove that $L^{-1}\{F(s)\} = f(t)$ implies*

$$L^{-1}\{F(as+b)\} = \frac{1}{a}e^{-\frac{bt}{a}} f\left(\frac{t}{a}\right).$$

Solution. We have $L^{-1}\{F(s)\} = f(t)$

$\Rightarrow$ $$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$\Rightarrow$ $$F(as+b) = \int_0^\infty e^{-(as+b)t} f(t)\,dt$$

$$= \int_0^\infty e^{-ast} e^{-bt} f(t)\,dt \quad \text{...(1)}$$

Put $at = x$, so that $dt = \frac{1}{a}dx$

$\therefore$ From (1) $$F(as+b) = \int_0^\infty e^{-sx} e^{-b\frac{x}{a}} \frac{1}{a} f\left(\frac{x}{a}\right) dx$$

$$= \int_0^\infty e^{-st} e^{-\frac{bt}{a}} \frac{1}{a} f\left(\frac{t}{a}\right) dt$$

$$= \int_0^\infty e^{-st}\left(\frac{1}{a}e^{-\frac{bt}{a}} f\left(\frac{t}{a}\right)\right) dt$$

$$= L\left(\frac{1}{a}e^{-\frac{bt}{a}} f\left(\frac{t}{a}\right)\right)$$

Hence, $L^{-1}\{F(as+b)\} = \frac{1}{a}e^{-\frac{bt}{a}} f\left(\frac{t}{a}\right).$

EXERCISE 6.2

Find the Inverse Laplace transform of the following functions :

1. $\frac{1}{(s-a)^3}$
2. $\frac{1}{s^3(s+1)}$
3. $\frac{1}{s(s+1)^3}$
4. $\frac{1}{s^2(s+2)}$
5. $\frac{1}{s^2(s^2+a^2)}$ [K.U. 2008]
6. $\frac{s}{(s+a)^2}$
7. $\frac{s}{a^2s^2+b^2}$
8. $\frac{1}{s^4(s^2+1)}$
9. $\log\frac{s+2}{s+1}$
10. $\log\frac{(s+1)^2}{(s+2)(s+3)}$ [K.U. 2008, 05]
11. $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$
12. $s\log\frac{s-1}{s+1}$
13. $\log\left(1-\frac{a^2}{s^2}\right)$
14. $\log\frac{s^2+1}{s(s+1)}$
15. $\tan^{-1}\frac{2}{s}$
16. $\frac{s+2}{s^2(s+1)(s-2)}$
17. $\cot^{-1}(s+1)$
18. $\frac{s}{(s^2+a^2)^2}$ [M.D.U. 2006]
19. $\frac{s^2}{(s^2+a^2)^2}$ [M.D.U. 2013]
20. $\frac{1}{(s^2+a^2)^2}$
21. $\frac{s+1}{(s^2+2s+2)^2}$

ANSWERS

1. $e^{at}\frac{t^2}{2!}$
2. $\frac{1}{2}t^2 - e^{-t} - t + 1$
3. $1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)$

4. $\frac{1}{4}(2t + e^{-2t} - 1)$
5. $\frac{1}{a^3}[at - \sin at]$
6. $e^{-at}[1 - at]$
7. $\frac{1}{a^2}\cos\frac{b}{a}t$
8. $\frac{1}{6}t^3 + \sin t - t$
9. $\frac{1}{t}[e^{-t} - e^{-2t}]$
10. $\frac{1}{t}[e^{-2t} + e^{-3t} - 2e^{-t}]$
11. $\frac{1}{t}[\cos at - \cos bt]$
12. $\frac{2}{t^2}[\sinh t - t\cosh t]$
13. $\frac{2}{t}[1 - \cosh at]$
14. $\frac{1}{t}[1 + e^{-t} - 2\cos t]$
15. $\frac{1}{t}\sin 2t$
16. $\frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} - t$
17. $\frac{1}{t}e^{-t}\sin t$
18. $\frac{1}{2a}t\sin at$
19. $\frac{1}{2a}[\sin at + at\cos at]$
20. $\frac{1}{2a^3}[\sin at - at\cos at]$
21. $\frac{1}{2}e^{-t}\, t\sin t$

5.3. CONVOLUTION THEOREM

Statement. ***If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, then***

$$L^{-1}[F(s)\,.\,G(s)] = \int_0^t f(u)\, g(t-u)\, du$$

[M.D.U. 2012, 11, 07; K.U. 2011]

Proof. Let $\phi(t) = \int_0^t f(u)\, g(t-u)\, du$

$\therefore$
$$L\,\phi(t) = L\left\{\int_0^t f(u)\, g(t-u)\, du\right\}$$

$$= \int_0^\infty e^{-st}\int_0^t f(u)\, g(t-u)\, du\, dt$$

$$= \int_0^\infty\int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

Integration is over the area, shown dotted in the fig. 6.1.

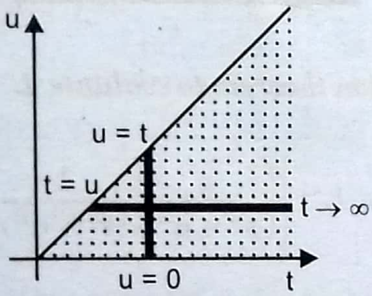

Fig. 6.1

We change the order of integration by suppossing the area to be made of horizontal strips as shown.

$$L\,\phi(t) = \int_0^\infty \left(\int_u^\alpha e^{-st} f(u)\, g(t-u)\, dt \right) du$$

[*For horizontal strip, u varies from 0 to* ∞]

or

$$L\,\phi(t) = \int_0^\infty \left(\int_u^\infty e^{-s(t-u)} e^{-su} f(u)\, g(t-u) \right) dt\, du$$

$$= \int_0^\infty e^{-su} f(u) \left(\int_u^\infty e^{-s(t-u)} g(t-u)\, dt \right) du$$

$$= \int_0^\infty e^{-su} f(u) \left(\int_0^\infty e^{-sv} g(v)\, dv \right) du \quad [\textit{Putting } t-u=v \textit{ for inner integral}]$$

$$= \int_0^\infty e^{-su} f(u)\, G(s)\, du = \int_u^\infty e^{-su} f(u)\, du \,.\, G(s)$$

$$= F(s) \,.\, G(s)$$

$$\therefore \quad \int_0^t f(u)\, g(t-u)\, du = L^{-1}[F(s) \,.\, G(s)]$$

Note :

1. $\bar{f} * \bar{g}$ *implies* $\int_0^t f(u)\, g(t-u)\, du$.

2. $L^{-1}[F(s) \,.\, G(s)]$ *can be written as* $\int_0^t g(u)\, f(t-u)\, du$.

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SOLVED EXAMPLES

Example 1. *Apply convolution theorem to evaluate* $L^{-1} \dfrac{s}{(s^2+a^2)^2}$.

Solution. $$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\frac{1}{(s^2+a^2)}\right] \quad \text{...(1)}$$

Now, $$L^{-1}[F(s)] = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at \; [= f(t) \text{ say}]$$

$$L^{-1}[G(s)] = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at \; [= g(t) \text{ say}]$$

$\therefore$ From (1), by convolution theorem

$$L^{-1}\frac{s}{(s^2+a^2)^2} = L^{-1}\left(\frac{s}{s^2+a^2}\right)\left(\frac{1}{(s^2+a^2)}\right)$$

$$= \frac{1}{a}\int_0^t (\cos au \sin a(t-u)\, du$$

$$= \frac{1}{2a}\int_0^t 2\cos au \sin(at-au)\, du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au - at)]\, du$$

$$= \frac{1}{2a}\left[u \sin at + \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - 0 - \frac{\cos at}{2a}\right] = \frac{1}{2a} t \sin at.$$

Example 2. *Find the inverse Laplace transform of the function* $\dfrac{s}{(s^2+a^2)^3}$.

[K.U. 2015; M.D.U. 2009, 07, 04]

Solution. $$L^{-1}\frac{s}{(s^2+a^2)^3} = L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)\left(\frac{1}{s^2+a^2}\right) \quad \text{...(1)}$$

$$= L^{-1}[F(s) \,.\, G(s)] \text{ (say)}$$

Now,
$$L^{-1}(F(s)) = L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{1}{2a} t \sin at = f(t) \qquad \text{[By Example 1]}$$

$$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at = g(t)$$

∴ From (1), by convolution theorem,

$$L^{-1}\left(\frac{s}{(s^2+a^2)^3}\right) = \int_0^t \frac{1}{2a} u \sin au \, \frac{1}{a} \sin a(t-u)\, du$$

$$= \int_0^t \frac{1}{2a^2} u \sin au \sin (at - au)\, du$$

$$= \frac{1}{4a^2}\int_0^t u\,[2 \sin au \sin (at - au)]\, du$$

$$= \frac{1}{4a^2}\int_0^t u\,[\cos (2au - at) - \cos at]\, du$$

$$= \frac{1}{4a^2}\int_0^t u \cos (2au - at)\, du - \frac{1}{4a^2}\int_0^t u \cos at\, du$$

$$= \frac{1}{4a^2}\left[u\,\frac{\sin (2au - at)}{2a} - 1\left[-\frac{\cos (2au-at)}{4a^2}\right]_0^t - \frac{1}{4a^2}\left[\frac{u^2}{2}\cos at\right]_0^t\right]$$

$$= \frac{1}{4a^2}\left[\frac{t \sin at}{2a} + \frac{\cos at}{4a^2} - \frac{\cos at}{4a^2}\right] - \frac{1}{8a^2} t^2 \cos at$$

$$= \frac{1}{8a^3}[t \sin at - at^2 \cos at]$$

$$= \frac{t}{8a^3}[\sin at - at \cos at].$$

Example 3. *Use convolution theorem to evaluate* $L^{-1}\left(\frac{1}{(s+1)(s+9)^2}\right)$.

[K.U. 2018; M.D.U. 2018]

Solution. $L^{-1}\left(\frac{1}{(s+1)(s+9)^2}\right) = L^{-1}\left(\frac{1}{(s+1)}\cdot\frac{1}{(s+9)^2}\right)$

$$= \int_0^t e^{-9u} . u . e^{-(t-u)}\, du \qquad \left[L^{-1}\left(\frac{1}{s+1}\right) = e^{-t},\ L^{-1}\left(\frac{1}{(s+9)^2}\right) = te^{-9t}\right]$$

$$= \int_0^t ue^{-8u}\, e^{-t}\, du$$

$$= e^{-t}\left[\frac{ue^{-8u}}{-8} - (1)\frac{e^{-8u}}{(-8)^2}\right]_0^t = e^{-t}\left[-\frac{te^{-8t}}{8} - \frac{e^{-8t}}{64} + \frac{1}{64}\right]$$

$$= \frac{e^{-t}}{64}[-8\,te^{-8t} - e^{-8t} + 1] = \frac{e^{-t}}{64}[1 - e^{-8t}(1+8t)].$$

EXERCISE 6.3

Using convolution theorem, evaluate :

1. $L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right)$ [M.D.U. 2009]

2. $L^{-1}\left(\frac{s}{(s^2+1)(s^2+4)}\right)$

3. $L^{-1}\left(\frac{1}{(s-1)(s+3)}\right)$

4. $L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right)$

5. $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$

6. $L^{-1}\left(\frac{1}{s(s^2+4)}\right)$

7. $L^{-1}\left(\frac{1}{(s+2)^2(s-2)}\right)$

8. $L^{-1}\left(\frac{1}{s(s+1)^3}\right)$

9. $L^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right)$ [K.U. 2016; M.D.U. 2005]

10. $L^{-1}\left(\frac{s^2}{(s^2+4)^2}\right)$ [K.U. 2016; M.D.U. 2016]

11. $L^{-1}\left(\frac{1}{s^3(s^2+1)}\right)$

12. $L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right)$ [M.D.U. 2014]

ANSWERS

1. $\frac{a\sin at - b\sin bt}{a^2-b^2}$

2. $\frac{1}{3}(\cos t - \cos 2t)$

3. $\frac{1}{4}(e^t - e^{-3t})$

4. $\frac{1}{2}(\sin t - \cos t + e^{-t})$

5. $\frac{1}{2a^3}(\sin at - at\cos at)$

6. $\frac{1}{4}(1-\cos 2t)$

7. $\frac{1}{16}(e^{2t} - e^{-2t} - 4te^{-2t})$

8. $1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)$

9. $\frac{1}{a^3}[at - \sin at]$

10. $\frac{1}{2}\left(t\cos 2t + \frac{1}{2}\sin 2t\right)$

11. $\frac{1}{2}t^2 - 1 + \cos t$

12. $\frac{1}{2}e^{-2t}\,t\sin t$.

SOLUTION OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMATION

CHAPTER 8

8.1. INTRODUCTION

In physics and Engineering sometimes it is required to solve differential equations subject to some initial conditions. General solution of such equations contains many arbitrary constants which can be determined using initial conditions. But Laplace transform method solves the differential equations giving us the particular solution even without finding the general solution and eliminates the labour of determining the constants. Laplace transform method is used to solve many types of ordinary differential, integral and partial differential equations. However in the following articles, we will solve the linear differential equations with constants or with variable coefficients and simultaneous linear equations with constant coefficients.

8.2. SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS BY TRANSFORM METHOD

Step 1. Take Laplace transform of both sides of differential equations using the method of transform of derivatives, using the initial conditions.

Step 2. Step 1 gives an algebraic equation called *subsidiary equation*.

Step 3. Divide by the coefficients of $\overline{y}$ which is used in place of F (s).

Step 4. Take the inverse transform of both sides.

Note :

In transform method we are to use the Laplace transform of derivatives.

If $f(t)$ is a function and their derivatives are denoted by $f'(t)$, $f''(t)$ ..... etc, we have seen

$$Lf'(t) = sF(s) - f(0)$$
$$Lf''(t) = s^2F(s) - sf(0) - f'(0)$$
$$\dots\dots\dots\dots\dots\dots$$
$$Lf^n(t) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) \dots\dots f^{n-1}(0)$$

where $Lf(t) = F(s)$.

In further study if we use the function as $y = f(t)$, then we shall use $\bar{y}$ in place of $F(s)$

i.e., $L\,f(t) = \bar{y}$ and $y(0), y'(0), y''(0)$ ..... will be used in place of $f(0), f'(0), f''(0)$ .....

In light of these notations, we write

$$L\left(\frac{dy}{dt}\right) = s\bar{y} - y(0)$$

$$L\left(\frac{d^2y}{dt^2}\right) = s^2\bar{y} - sy(0) - y'(0)$$

$$\dots\dots\dots\dots\dots\dots$$

and so on.

SOLVED EXAMPLES

Example 1. *Solve the following equation by transform method*

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}, \text{ where } y(0) = y'(0) = 1.$$ [K.U. 2015. 14]

Solution. The given equation is $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-t}$ ...(1)

Taking Laplace transform of both sides of (1), we have

$$[s^2\bar{y} - sy(0) - y'(0)] + 4[s\bar{y} - y(0)] + 3\bar{y} = \frac{1}{s+1}$$

or $$[s^2\bar{y} - s - 1] + 4[s\bar{y} - 1] + 3\bar{y} = \frac{1}{s+1}$$ [Given $y(0) = y'(0) = 1$]

or $$(s^2 + 4s + 3)\bar{y} = \frac{1}{s+1} + s + 1 + 4$$

or $$(s^2 + 4s + 3)\bar{y} = \frac{1}{s+1} + s + 5 = \frac{s^2 + 6s + 5 + 1}{s+1}$$

$$\bar{y} = \frac{s^2+6s+6}{(s+1)(s^2+4s+3)} = \frac{s^2+6s+6}{(s+1)(s+3)(s+1)}$$

$$= \frac{s^2+6s+6}{(s+3)(s+1)^2}$$

Taking inverse Laplace transform, we have

$$y = L^{-1}\left(\frac{s^2+6s+6}{(s+3)(s+1)^2}\right) \qquad \text{...(2)}$$

Let $$\frac{s^2+6s+6}{(s+3)(s+1)^2} = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \qquad \text{...(3)}$$

$\therefore$ $$s^2 + 6s + 6 = A(s+1)^2 + B(s+1)(s+3) + C(s+3) \qquad \text{...(4)}$$

Putting $s = -3$ in (4),

$$9 - 18 + 6 = 4A \quad \Rightarrow \quad A = -\frac{3}{4}$$

Putting $s = -1$ in (4),

$$1 - 6 + 6 = (-1+3)C \quad \Rightarrow \quad C = \frac{1}{2}$$

Comparing the constant term on both sides of (4), we have

$$6 = A + 3B + 3C$$

$\therefore$ $$6 = -\frac{3}{4} + 3B + \frac{3}{2}$$

or $$6 + \frac{3}{4} - \frac{3}{2} = 3B \quad \Rightarrow \quad B = \frac{7}{4}$$

Putting the values of A, B and C in (3), we have

$$\frac{s^2+6s+6}{(s+3)(s+1)^2} = \frac{-3}{4(s+3)} + \frac{7}{4}\cdot\frac{1}{s+1} + \frac{1}{2(s+1)^2}$$

$\therefore$ $$L^{-1}\left(\frac{s^2+6s+6}{(s+3)(s+1)^2}\right) = -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\,t$$

Substituting its value in (2), we get

$$y = -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\,t.$$

Example 2. *Solve* $\dfrac{d^4y}{dt^4} - k^4y = 0$ *by transform method, where* [K.U. 2017]

$$y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0.$$

Solution. The given equation is $\dfrac{d^4y}{dt^4} - k^4y = 0$

Taking Laplace transform of both sides of the equation, we have

$$[s^4\bar{y} - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0)] - k^4\bar{y} = 0$$

or $$(s^4\bar{y} - s^3 - 0 - 0 - 0) - k^4\bar{y} = 0$$

$$[\because y(0) = 1 \text{ and } y'(0) = y''(0) = y'''(0) = 0]$$

or $$(s^4 - k^4)\bar{y} = s^3$$

or $$\bar{y} = \frac{s^3}{s^4 - k^4}$$

$\therefore$ $$y = L^{-1}\left(\frac{s^3}{s^4 - k^4}\right) \quad \text{...(1)}$$

Let $$\frac{s^3}{s^4 - k^4} = \frac{s^3}{(s-k)(s+k)(s^2+k^2)}$$

$$= \frac{A}{s-k} + \frac{B}{s+k} + \frac{Cs + D}{s^2 + k^2} \quad \text{...(2)}$$

Multiplying both sides of (2) by $(s-k)(s+k)(s^2+k^2)$, we have

$$s^3 = A(s+k)(s^2+k^2) + B(s-k)(s^2+k^2) + (Cs + D)(s^2 - k^2) \quad \text{...(3)}$$

Putting $s = k$ in (3) : $\quad k^3 = A(2k)(2k^2) \quad \Rightarrow \quad A = \dfrac{1}{4}$

Putting $s = -k$ in (3) : $\quad -k^3 = B(-2k)(2k^2) \quad \Rightarrow \quad B = \dfrac{1}{4}$

Comparing the coefficients of s^3 and constant terms in (3), we have

$$1 = A + B + C \quad \text{...(4)}$$

and $$0 = Ak^3 - Bk^3 - Dk^2 \quad \text{...(5)}$$

From (4), $$C = 1 - A - B = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

From (5), $$Dk^2 = k^3(A - B) = k^3\left(\frac{1}{4} - \frac{1}{4}\right) \quad \Rightarrow \quad D = 0$$

Substituting these values in (2), we have

$$\frac{s^3}{s^4-k^4} = \frac{1}{4}\cdot\frac{1}{s-k} + \frac{1}{4}\cdot\frac{1}{s+k} + \frac{\frac{1}{2}s}{s^2+k^2}$$

Taking inverse Laplace transform,

$$L^{-1}\left(\frac{s^3}{s^4-k^4}\right) = \frac{1}{4}e^{kt} + \frac{1}{4}e^{-kt} + \frac{1}{2}\cos kt$$

$$= \frac{1}{2}\left(\frac{e^{kt}+e^{-kt}}{2}\right) + \frac{1}{2}\cos kt = \frac{1}{2}[\cosh kt + \cos kt].$$

Example 3. *Solve* $\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y = \sin t$ *by transform method, where*

$y(0) = y'(0) = y''(0) = y'''(0) = 0.$ [M.D.U. 2015, 09]

Solution. The given equation is $\frac{d^4y}{dt^4} + 2\frac{d^2y}{dt^2} + y = \sin t$

Applying Laplace transform on both sides, we have

$$[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = \frac{1}{s^2+1}$$

or $$(s^4\bar{y} - 0 - 0 - 0 - 0) + 2(s^2\bar{y} - 0 - 0) + \bar{y} = \frac{1}{s^2+1} \quad [\because y(0) = y'(0) = y''(0) = y'''(0) = 0]$$

$\therefore$ $$(s^4 + 2s^2 + 1)\bar{y} = \frac{1}{s^2+1}$$

or $$\bar{y} = \frac{1}{(s^2+1)(s^4+2s^2+1)} = \frac{1}{(s^2+1)^3} \quad ...(1)$$

$\therefore$ $$y = L^{-1}\left(\frac{1}{(s^2+1)^3}\right)$$

Now, $$L^{-1}\left(\frac{1}{(s^2+1)^3}\right) = L^{-1}\left(\frac{1}{(s^2+1)^2}\cdot\frac{1}{(s^2+1)}\right)$$

We shall use convolution theorem to find the inverse transform.

$$\left[\textit{Here } L^{-1}\left(\frac{1}{(s^2+1)^2}\right) = \frac{1}{2}(\sin t - t\cos t)\ (= f(t)\ say)\ \text{(By Art. 6.1.1.)}\right.$$

$$\left.\textit{and } L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t\ (= g(t)\ say)\right]$$

$$\therefore L^{-1}\left(\frac{1}{(s^2+1)^3}\right) = \int_0^t \frac{1}{2}(\sin u - u\cos u)\sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\sin u \sin(t-u) - u\cos u \sin(t-u)]\,du$$

$$= \frac{1}{4}\int_0^t [\{\cos(2u-t) - \cos t\} - u\{\sin t - \sin(2u-t)\}]\,du$$

$$= \frac{1}{4}\left[\left(\frac{\sin(2u-t)}{2} - u\cos t - \frac{u^2}{2}\sin t\right)_0^t + \int_0^t u\sin(2u-t)\,du\right]$$

$$= \frac{1}{4}\left[\frac{\sin t}{2} - t\cos t - \frac{t^2}{2}\sin t + \frac{\sin t}{2}\right] + \frac{1}{4}\left[u\left(-\frac{\cos(2u-t)}{2}\right) - (1)\left(-\frac{\sin(2u-t)}{4}\right)\right]_0^t$$

[*Integrating by parts*]

$$= \frac{1}{4}\left[\sin t - t\cos t - \frac{t^2}{2}\sin t\right] + \frac{1}{4}\left[-\frac{t\cos t}{2} + \frac{\sin t}{4} - 0 + \frac{\sin t}{4}\right]$$

$$= \frac{1}{4}\left[\sin t - t\cos t - \frac{t^2}{2}\sin t - \frac{t\cos t}{2} + \frac{\sin t}{4} + \frac{\sin t}{4}\right]$$

$$= \frac{1}{4}\left[\frac{3}{2}\sin t - \frac{3}{2}t\cos t - \frac{t^2}{2}\sin t\right]$$

$$= \frac{1}{8}[3\sin t - 3t\cos t - t^2\sin t]$$

$$= \frac{1}{8}[(3-t^2)\sin t - 3t\cos t].$$

Example 4. *Solve* $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$ *by transform method, where* $y(0) = 0$, $y'(0) = 1$. [*K.U. 2013*]

Solution. The given equation is $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$

Applying Laplace transform on both sides, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

or $$(s^2\bar{y} - 0 - 1) + 2(s\bar{y} - 0) + 5\bar{y} = \frac{1}{s^2+2s+2} \qquad [\because y(0) = 0, y'(0) = 1]$$

or $$(s^2+2s+5)\bar{y} = \frac{1}{s^2+2s+2} + 1 = \frac{s^2+2s+3}{s^2+2s+2}$$

$\therefore$ $$\bar{y} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

or $$y = L^{-1}\left(\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right) \qquad ...(1)$$

Let $$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5} \qquad ...(2)$$

$\therefore$ $$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2) \qquad ...(3)$$

Comparing the coefficients of s^3, s^2, s and constant terms in (3), we get

$$0 = A + C \qquad ...(4)$$

$$1 = 2A + B + 2C + D \qquad ...(5)$$

$$2 = 5A + 2B + 2C + 2D \qquad ...(6)$$

$$3 = 5B + 2D \qquad ...(7)$$

Solving the above equations, we find

$$A = C = 0 \quad \text{and} \quad B = \frac{1}{3}, \quad D = \frac{2}{3}$$

$\therefore$ From (2), $$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{\frac{1}{3}}{(s+1)^2+1} + \frac{\frac{2}{3}}{(s+1)^2+4}$$

$\therefore$ $$L^{-1}\left(\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right) = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\frac{1}{2}\sin 2t$$

Putting this value in (1), we have

$$y = \frac{1}{3}e^{-t}(\sin t + \sin 2t).$$

EXERCISE 8.1

Solve the following equations by transform method :

1. $\frac{d^2y}{dt^2} + y = 6\cos 2t$, where $y'(0) = 1,\ y(0) = 3$. [M.D.U. 2013, 09; K.U. 2009, 05]

2. $\frac{d^2y}{dt^2} + y = t$, where $y(0) = 1,\ y'(0) = -2$.

3. $(D^2 + D)x = 2$ when $x(0) = 3,\ x'(0) = 1$.

4. $\frac{dx}{dt} + x = \sin wt$ when $x(0) = 2$.

5. $(D^2 - 1)x = a\cosh t$ when $x(0) = x'(0) = 0$.

6. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t,\ y = \frac{dy}{dt} = 0$ when $t = 0$. B.A 2018 [M.D.U. 2008, 05]

7. $(D^2 + m^2)y = a\cos nt,\ t > 0$ and $y = 0 = \frac{dy}{dt}$ when $t = 0$

8. $y'' - 3y' + 2y = 4t + e^{3t}$ when $y(0) = 1, y'(0) = -1$.

9. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2,\ t > 0,\ y = 1,\ \frac{dy}{dt} = 0$ when $t = 0$.

10. $(D^2 + 2D + 1)y = 3te^{-t},\ t > 0$ when $y = 4,\ \frac{dy}{dt} = 2$ when $t = 0$. [M.D.U. 2007]

11. $(D + 2)^2 y = 4e^{-2t},\ y(0) = -1$ and $y'(0) = 4$.

12. $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ when $y = 1,\ \frac{dy}{dt} = 2,\ \frac{d^2y}{dt^2} = 2$ at $t = 0$.

13. $\frac{d^2x}{dt^2} + x = t\cos 2t,\ x(0) = 0;\ \frac{dx}{dt} = 0$ at $x = 0$.

14. $\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y\right) = t^2 e^t$ given $y(0) = 1,\ y'(0) = 0, y''(0) = -2$.

15. $(D^2 + n^2)x = a\sin(nt + \alpha)$ where $x(0) = 0, x'(0) = 0$.

16. $(D^4 + 2D^2 + 1)y = 0$ where $y(0) = 0, y'(0) = 1, y''(0) = 2, y'''(0) = -3$.

17. $(D^3 - 2D^2 + 5D)y = 0$ if $y(0) = 0, y'(0) = 1,\ y\left(\frac{\pi}{8}\right) = 1$.

18. $\frac{d^2y}{dt^2} + y = \sin t\sin 2t,\ t > 0,\ y(0) = 1,\ y'(0) = 0$.

19. $\frac{d^2x}{dt^2}+x=f(t)$ if $x(0)=x'(0)=0.$

20. $(D^2+D)y=t^2+2t,\ y(0)=4,\ y'(0)=-2.$ [M.D.U. 2007]

ANSWERS

1. $y=5\cos t+\sin t-2\cos 2t$
2. $y=t-3\sin t+\cos t$
3. $x=2t+e^{-t}+2$
4. $x=\left(\frac{w}{1+w^2}+2\right)e^{-t}+\frac{\sin wt-w\cos wt}{1+w^2}$
5. $x=\frac{a}{2}t\sinh t$
6. $y=-\frac{1}{40}e^{-3t}+\frac{1}{8}e^{t}-\frac{1}{10}(\cos t+2\sin t)$
7. $y=\frac{a}{m^2-n^2}[\cos nt-\cos mt]$
8. $y=(2t+3)+\frac{1}{2}[e^{3t}-e^{t}]-2e^{2t}$
9. $y=-\frac{1}{3}+\frac{8}{15}e^{3t}+\frac{4}{5}e^{-2t}$
10. $y=\frac{1}{2}e^{-t}t^3+4e^{-t}+6te^{-t}$
11. $y=e^{-2t}(2t^2+2t-1)$
12. $y=\frac{1}{3}(e^{-2t}+5e^{t})-e^{-t}$
13. $x=-\frac{5}{9}\sin t+\frac{4}{9}\sin 2t-\frac{1}{3}t\cos 2t$
14. $y=e^{t}\left[\frac{1}{60}t^5+1-t-\frac{1}{2}t^2\right]$
15. $x=\frac{a}{2n^2}[\cos\alpha\sin nt-nt\cos(nt+\alpha)]$
16. $y=t(\sin t+\cos t)$
17. $y=1+e^{t}(\sin 2t-\cos 2t)$
18. $y=\frac{1}{4}t\sin t+\frac{15}{16}\cos t+\frac{1}{16}\cos 3t$
19. $x=\int_0^t \sin(t-u)f(u)\,du$
20. $y=\frac{1}{3}t^3+2e^{-t}+2$

8.3. SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENT BY TRANSFORM METHOD

Some of the ordinary differential equations with variable co-efficients can be solved by Laplace transform. This method is useful in solving the equations having the terms of the form $t^m y^n(t)$ whose Laplace transform is $(-1)^m\frac{d^m}{ds^m}L[y^n(t)]$.

This method is illustrated by the solved examples given below :

SOLVED EXAMPLES

Example 1. *Solve* $t\dfrac{d^2y}{dt^2}+(1-2t)\dfrac{dy}{dt}-2y=0,\ y(0)=1,\ y'(0)=2.$

[K.U. 2018, 16; M.D.U. 2017]

Solution. The given equation is $t\dfrac{d^2y}{dt^2}+(1-2t)\dfrac{dy}{dt}-2y=0$

Taking Laplace transform of both sides, we have

$$L\left(t\frac{d^2y}{dt^2}\right)+L\left(\frac{dy}{dt}\right)-2L\left(t\frac{dy}{dt}\right)-2L(y)=0$$

or $$-\frac{d}{ds}[s^2\bar{y}-s\,y(0)-y'(0)]+s\bar{y}-y(0)+2\frac{d}{ds}[s\bar{y}-y(0)]-2\bar{y}=0$$

or $$-\frac{d}{ds}[(s^2\bar{y}-s+2)+(s\bar{y}-1)+2\frac{d}{ds}(s\bar{y}-1)-2\bar{y}=0\ [\because y(0)=1,y'(0)=2]$$

or $$-\left[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-1\right]+(s\bar{y}-1)+2\left[s\frac{d\bar{y}}{ds}+\bar{y}\right]-2\bar{y}=0$$

or $$s^2\frac{d\bar{y}}{ds}+2s\bar{y}-1-s\bar{y}+1-2s\frac{d\bar{y}}{ds}-2\bar{y}+2\bar{y}=0$$

or $$(s^2-2s)\frac{d\bar{y}}{ds}+s\bar{y}=0$$

or $$(s^2-2s)\frac{d\bar{y}}{ds}=-s\bar{y}$$

or $$\frac{d\bar{y}}{\bar{y}}=\frac{-s}{s^2-2s}ds$$

Integrating, $$\log\bar{y}=-\log(s-2)+\log c_1$$

or $$\log\bar{y}=\log\frac{c_1}{s-2}$$

or $$\bar{y}=\frac{c_1}{s-2}$$

Taking inverse transform, $$y=c_1e^{2t}$$

Taking $t=0$, $$c_1=1$$

Hence $y=e^{2t}$ is the required solution.

Example 2. *Solve* $t\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+ty=\sin t$, *when* $y(0)=1$. *[K.U. 2017, 12, 11]*

Solution. The given equation is $t\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+ty=\sin t$

Taking Laplace transform on both sides, we have

$$L\left(t\frac{d^2y}{dt^2}\right)+2L\left(\frac{dy}{dt}\right)+L(ty)=L(\sin t)$$

or $$-\frac{d}{ds}[s^2\bar{y}-s\,y(0)-y'(0)]+2[s\bar{y}-y(0)]+\left(-\frac{d}{ds}\right)(\bar{y})=\frac{1}{1+s^2}$$

or $$-\left[\left(s^2\frac{d\bar{y}}{ds}+2s\bar{y}\right)-y(0)-0\right]+2s\bar{y}-2y(0)-\frac{d\bar{y}}{ds}=\frac{1}{1+s^2}$$

or $$s^2\frac{d\bar{y}}{ds}+2s\bar{y}-1-2s\bar{y}+2+\frac{d\bar{y}}{ds}=\frac{-1}{1+s^2} \qquad [\because y(0)=1]$$

or $$(s^2+1)\frac{d\bar{y}}{ds}=-\frac{1}{1+s^2}-1=-\left[\frac{2+s^2}{1+s^2}\right]$$

or $$\frac{d\bar{y}}{ds}=-\frac{(2+s^2)}{(1+s^2)^2}=-\left[\frac{(1+s^2)+1}{(1+s^2)^2}\right]$$

$$=-\left[\frac{1}{1+s^2}+\frac{1}{(1+s^2)^2}\right]$$

Taking inverse transform on both sides, we have

$$L^{-1}\frac{d\bar{y}}{ds}=-\left[L^{-1}\left(\frac{1}{1+s^2}\right)+L^{-1}\left(\frac{1}{1+s^2}\right)\left(\frac{1}{1+s^2}\right)\right]$$

Remember that $L^{-1}\left(\dfrac{d\bar{y}}{ds}\right)=-ty$*

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* $$L^{-1}\left[-\frac{d}{ds}F(s)\right]=t\,f(t)$$

$\therefore$ $$L^{-1}\frac{d\bar{y}}{ds}=-t\,f(t)=-ty.$$

$$\therefore \quad -ty = -\left[\sin t + \int_0^t \sin u \sin (t-u)\, du\right] \qquad \text{[Convolution theorem]}$$

or
$$ty = \left[\sin t + \frac{1}{2}\int_0^t [\cos (2u - t) - \cos t]\, du\right]$$

or
$$ty = \left[\sin t + \frac{1}{2}\left(\frac{\sin (2u-t)}{2} - u \cos t\right)_0^t\right]$$

or
$$ty = \left[\sin t + \frac{1}{2}\left\{\frac{\sin t}{2} - t \cos t + \frac{\sin t}{2}\right\}\right]$$

or
$$ty = \left[\frac{3}{2}\sin t - \frac{1}{2} t \cos t\right]$$

$$\therefore \quad y = \frac{1}{2}\left[\frac{3 \sin t}{t} - \cos t\right].$$

EXERCISE 8.2

Solve the following differential equations using transform method :

1. $t\dfrac{d^2y}{dt^2} + (t-1)\dfrac{dy}{dt} - y = 0$, $y(0) = 5$, $y(\infty) = 0$. [M.D.U. 2014]

2. $t\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + 4t\, y = 0$, if $y(0) = 3$, $y'(0) = 0$.

3. $\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} - y = 0$, if $y(0) = 0$, $y'(0) = 1$.

4. $t\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + ty = 0$ under the conditions $y(0) = 1$ and $y'(0) = 0$.

5. $ty'' + 2y' + ty = 0$, $y(0) = 1$, $y(\pi) = 0$.

ANSWERS

1. $y = 5e^{-t}$
2. $y = 3\, J_0(2t)$
3. $y = t$
4. $y = J_0(t)$
5. $y = \dfrac{\sin t}{t}$

8.4. SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS BY TRANSFORM METHOD

The following solved examples will illustrate the method.

Example 1. *Solve the following simultaneous equations*

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$$

when $x(0) = 3,\ y(0) = 0.$ [K.U. 2016; M.D.U. 2014, 12, 11, 04]

Solution. The given equations are

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1 \qquad \text{...(1)}$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0 \qquad \text{...(2)}$$

Taking Laplace transform of (1) and (2), we have

$$3\,[s\,\bar{x} - x(0)] + [s\,\bar{y} - y(0)] + 2\,\bar{x} = \frac{1}{s}$$

and

$$s\,\bar{x} - x(0) + 4\,[s\,\bar{y} - y(0)] + 3\,\bar{y} = 0$$

As $x(0) = 3$ and $y(0) = 0$

$$\therefore \quad 3s\,\bar{x} - 9 + s\,\bar{y} - 0 + 2\bar{x} = \frac{1}{s}$$

and

$$s\,\bar{x} - 3 + 4s\,\bar{y} + 3\,\bar{y} = 0$$

i.e.,

$$(3s+2)\,\bar{x} + s\,\bar{y} = 9 + \frac{1}{s} \qquad \text{...(3)}$$

and

$$s\,\bar{x} + (4s+3)\,\bar{y} = 3 \qquad \text{...(4)}$$

Multiplying (3) by $(4s + 3)$ and (4) by s, we have

$$(3s+2)(4s+3)\,\bar{x} + s\,(4s+3)\,\bar{y} = 9\,(4s+3) + \frac{4s+3}{s} \qquad \text{...(5)}$$

and

$$s^2\,\bar{x} + s\,(4s+3)\,\bar{y} = 3s \qquad \text{...(6)}$$

Subtracting (6) from (5), we have

$$[(3s+2)(4s+3)-s^2]\bar{x} = 9.(4s+3)+\frac{4s+3}{s}-3s$$

or

$$(11s^2+17s+6)\bar{x} = 9.(4s+3)+\frac{4s+3}{s}-3s$$

or

$$(s+1)(11s+6)\bar{x} = 9.(4s+3)+\frac{4s+3}{s}-3s$$

$$\therefore \quad \bar{x} = \frac{9.(4s+3)}{(s+1)(11s+6)}+\frac{4s+3}{s(s+1)(11s+6)}-\frac{3s}{(s+1)(11s+6)}$$

$$= \left(\frac{A}{s+1}+\frac{B}{11s+6}\right)+\left(\frac{C}{s}+\frac{D}{s+1}+\frac{E}{11s+6}\right)-\left(\frac{F}{s+1}+\frac{G}{11s+6}\right)$$

$$= \left(\frac{9}{5}.\frac{1}{s+1}+\frac{81}{5}.\frac{1}{11s+6}\right)+\left(\frac{1}{2}.\frac{1}{s}-\frac{1}{5}.\frac{1}{s+1}-\frac{33}{10}.\frac{1}{11s+6}\right)-\left(\frac{3}{5}.\frac{1}{s+1}-\frac{18}{5}.\frac{1}{11s+6}\right)$$

$$\therefore \quad \bar{x} = \frac{1}{2s}+\frac{1}{s+1}+\frac{33}{2}.\frac{1}{11s+6}$$

Taking inverse transform, we have

$$x = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s+1}\right)+\frac{3}{2}L^{-1}\left(\frac{1}{s+\frac{6}{11}}\right)$$

$$= \frac{1}{2}+e^{-t}+\frac{3}{2}e^{-\frac{6}{11}t}$$

Again multiplying (3) by s and (4) by $(3s+2)$, we have

$$s(3s+2)\bar{x}+s^2\bar{y} = 9s+1 \qquad \text{...(7)}$$

$$s(3s+2)\bar{x}+(4s+3)(3s+2)\bar{y} = 3(3s+2) \qquad \text{...(8)}$$

Subtracting (7) from (8), we have

$$[(4s+3)(3s+2)-s^2]\bar{y} = 9s+6-9s-1$$

or

$$(s+1)(11s+6)\bar{y} = 5$$

$$\therefore \quad \bar{y} = \frac{5}{(s+1)(11s+6)} = -\frac{1}{s+1}+\frac{11}{11s+6}$$

Taking inverse transform, we have

$$y = L^{-1}\left(-\frac{1}{s+1}\right) + L^{-1}\left(\frac{1}{s+\frac{6}{11}}\right)$$

$$= -e^{-t} + e^{-\frac{6}{11}t}$$

Hence the required solution is $x = \frac{1}{2} + e^{-t} + \frac{3}{2}e^{-\frac{6}{11}t}$

$$y = -e^{-t} + e^{-\frac{6}{11}t}.$$

Example 2. *The co-ordinate (x, y) of a particle moving along a plane curve at any time t is given by*

$$\frac{dy}{dt} + 2x = \sin 2t$$

$$\frac{dx}{dt} - 2y = \cos 2t.$$

If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms that particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$. [M.D.U. 2016]

Solution. The given parametric equations are

$$\frac{dy}{dt} + 2x = \sin 2t$$

$$\frac{dx}{dt} - 2y = \cos 2t$$

Taking Laplace transforms, we have

$$[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2+4} \qquad ...(1)$$

and $$[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2+4} \qquad ...(2)$$

Given $x(0) = 1$, $y(0) = 0$

$\therefore$ From (1) and (2), we have

$$s\bar{y} + 2\bar{x} = \frac{2}{s^2+4} \qquad ...(3)$$

and $$s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \qquad ...(4)$$

Now we shall solve (3) and (4) for $\bar{x}$ and $\bar{y}$

Multiplying (3) by 2 and (4) by s, we get

$$2s\bar{y} + 4\bar{x} = \frac{4}{s^2+4} \quad \text{...(5)}$$

and

$$s^2\bar{x} - 2s\bar{y} = \frac{s^2}{s^2+4} + s \quad \text{...(6)}$$

Adding (5) and (6), we have

$$(s^2+4)\bar{x} = \frac{4+s^2}{s^2+4} + s$$

$$\therefore \quad \bar{x} = \frac{1}{s^2+4} + \frac{s}{s^2+4}$$

Taking inverse transforms, we have

$$x = \frac{1}{2}\sin 2t + \cos 2t \quad \text{...(7)}$$

Multiplying (3) by s and (4) by 2, we have

$$s^2\bar{y} + 2s\bar{x} = \frac{2s}{s^2+4} \quad \text{...(8)}$$

and

$$2s\bar{x} - 4\bar{y} = \frac{2s}{s^2+4} + 2 \quad \text{...(9)}$$

Subtracting (9) from (8), we get $(s^2+4)\bar{y} = -2$

$$\therefore \quad \bar{y} = -\frac{2}{s^2+4}$$

Taking inverse transforms, $y = -\frac{2}{2}\sin 2t = -\sin 2t$ ...(10)

Equation (7) and (10) are parametric equations of curve.

We now eliminate t to get the equation of curve along which the particle moves.

Writing (7) and (10) together $2x = \sin 2t + 2\cos 2t$

$$y = -\sin 2t$$

Adding these, $2x + y = 2\cos 2t$ ...(11)

Also, $2y = -2\sin 2t$ ...(12)

Squaring and adding (11) and (12), we get

$$4x^2 + 4xy + 5y^2 = 4.$$

Example 3. *Solve the simultaneous equations using Laplace transform method :*

$$(D^2+2)\,x - Dy = 1$$

$$Dx + (D^2+2)\,y = 0, \text{ when } x(0)=0,\ x'(0)=0,\ y(0)=0 \text{ and } y'(0)=0.$$

[K.U. 2017, 04; M.D.U. 2017]

Solution. The given system of equation is

$$D^2x + 2x - Dy = 1 \quad ...(1)$$

$$Dx + D^2y + 2y = 0 \quad ...(2)$$

Taking Laplace transform of both sides, we have

$$L(x'') + 2L(x) - L(y') = L(1) \quad ...(3) \quad \left[D = \frac{d}{dt} \text{ and } D^2 = \frac{d^2}{dt}\right]$$

and

$$L(x') + L(y'') + 2\,L(y) = L(0) \quad ...(4)$$

From (3), $[s^2\bar{x} - sx(0) - x'(0) + 2\bar{x} - (s\bar{y}) - y(0)] = \frac{1}{s}$ ...(5)

From (4), $[s\bar{x} - x(0)] + [s^2y - sy(0) - y'(0)] + 2\bar{y} = 0$ ...(6)

From (5), $(s^2+2)\bar{x} - s\bar{y} - \frac{1}{s} = 0$ ...(7) $[\because x(0)=0, x'(0)=0 \text{ and } y(0)=0, y'(0)=0]$

From (6), $s\bar{x} + (s^2+2)\bar{y} = 0$ ...(8)

Solving (7) and (8), we have

$$\frac{\bar{x}}{0+\frac{s^2+2}{s}} = \frac{\bar{y}}{-1-0} = \frac{1}{(s^2+2)^2+s^2}$$

$\therefore$

$$\bar{x} = \frac{\frac{s^2+2}{s}}{(s^2+2)^2+s^2}$$

i.e.,

$$\bar{x} = \left(\frac{s^2+2}{s}\right)\left[\frac{1}{(s^2+2)^2+s^2}\right]$$

i.e.,

$$\bar{x} = \frac{(s^2+2)}{s(s^2+1)(s^2+4)}$$

Breaking up into partial fractions,

$$\bar{x} = \frac{1}{2s} - \frac{s}{3(s^2+1)} - \frac{s}{6(s^2+4)}$$

and

$$\bar{y} = \frac{-1}{(s^2+2)^2+s^2}$$

i.e.,

$$\bar{y} = \frac{-1}{(s^2+1)(s^2+4)}$$

Breaking up into partial fractions,

$$\bar{y} = \frac{-1}{3(s^2+1)} + \frac{1}{3(s^2+4)}$$

Taking inverse transform,

$$x = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{3}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{6}L^{-1}\frac{s}{(s^2+4)}$$

$$= \frac{1}{2}.1 - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t$$

Taking inverse transform,

$$y = -\frac{1}{3}L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s^2+4}\right)$$

$$y = -\frac{1}{3}\sin t + \frac{1}{3}.\frac{1}{2}\sin 2t$$

Hence the solution is $x = \frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t$, $y = -\frac{1}{3}\sin t + \frac{1}{6}\sin 2t$.

EXERCISE 8.3

Solve the following simultaneous equations using Laplace transform method :

1. $\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x = 15e^{-t}$; $\frac{d^2y}{dt^2} - 4\frac{dx}{dt} + 3y = 15\sin 2t$
   when $x = (0) = 35$, $x'(0) = -48$; $y(0) = 27$, $y'(0) = -55$
2. $\frac{dx}{dt} - y = e^t$; $\frac{dy}{dt} + x = \sin t$, when $x(0) = 1$, $y(0) = 0$. B.A 2018
3. $(D^2+2)x - Dy = 2t+5$; $(D-1)x + (D+1)y = -1-2t$ when $x = 3$, $Dx = 0$, $y = -3$, $Dy = 5$, when $t = 0$
4. $(D-2)x + 3y = 0$; $2x + (D-1)y = 0$, if $x(0) = 8$ and $y(0) = 3$. [M.D.U. 2015]
5. $Dx + Dy = t$; $D^2x - y = e^{-t}$ when $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$. [M.D.U. 2006]
6. $\frac{dx}{dt} + 5x - 2y = t$; $\frac{dy}{dt} + 2x + y = 0$ when $x(0) = 0$, $y(0) = 0$.
7. $(D-2)x - (D+1)y = 6e^{3t}$; $(2D-3)x + (D-3)y = 6e^{3t}$, when $x(0) = 3$, $y(0) = 0$.
8. $\frac{dx}{dt} = 5x + y$; $\frac{dy}{dt} = x + 5y$ when $x(0) = -3$, $y(0) = 7$. [M.D.U. 2013]

ANSWERS

1. $x = 30\cos t - 45\sin 3t + 3e^{-t} + 2\cos 2t$
   $y = 30\cos 3t - 60\sin t - 3e^{-t} + \sin 2t$
2. $x = \frac{1}{2}[2\sin t - t\cos t + \cos t + e^t]$
   $y = \frac{1}{2}[\cos t + t\sin t - \sin t - e^t]$
3. $x = 2 + e^{-2t} + t + \sin t$
   $y = 1 - t - 3e^{-2t} - \cos t$
4. $x = 5e^{-t} + 3e^{4t}$
   $y = -2e^{4t} + 5e^{-t}$
5. $x = 2 + \frac{1}{2}[t^2 + \cos t - 3\sin t + e^{-t}]$
   $y = 1 + \frac{1}{2}[3\sin t - e^{-t} - \cos t]$
6. $x = \frac{1}{27} - \frac{1}{27}e^{-3t} + \frac{1}{9}t - \frac{2}{9}te^{-3t}$
   $y = \frac{4}{27} - \frac{4}{27}e^{-3t} - \frac{2}{9}t - \frac{2}{9}e^{-3t}t$
7. $x = (1+2t)e^t + 2e^{3t}$; $y = (1-t)e^t - e^{3t}$
8. $x = -5e^{4t} + 2e^{6t}$; $y = 5e^{4t} + 2e^{6t}$.