4

Integration of Trigonometric Functions

§ 4·1. Integration of $\sin^m x$ and $\cos^m x$.

Case I. *When m is an odd positive integer.*

Let $m = 2n + 1$. Then

(i) $\int \sin^m x \, dx = \int \sin^{2n+1} x \, dx = \int \sin^{2n} x \cdot \sin x \, dx$

$= \int (\sin^2 x)^n \sin x \, dx = \int (1 - \cos^2 x)^n \cdot \sin x \, dx$

$= - \int (1 - t^2)^n \, dt$, putting $\cos x = t$, so that $- \sin x \, dx = dt$.

Now $(1 - t^2)^n$ can be expanded in powers of t by the binomial theorem and then term by term integration will be performed.

Thus $\int \sin^{2n+1} x \, dx$

$$= - \int \left[1 - nt^2 + \frac{n(n-1)}{2!} t^4 - \dots + (-1)^n t^{2n}\right] dt$$

$$= - \left[t - n \frac{t^3}{3} + \frac{n(n-1)}{2!} \cdot \frac{t^5}{5} - \dots + \frac{(-1)^n \cdot t^{2n+1}}{2n+1}\right]$$

$$= - \left[\cos x - \frac{n}{3} \cos^3 x + \frac{n(n-1)}{5 . 2!} \cos^5 x - \dots + \frac{(-1)^n}{(2n+1)} \cos^{2n+1} x\right].$$

(ii) Similarly $\int \cos^m x \, dx = \int \cos^{2n+1} x \, dx$, $[\because m = 2n + 1]$

$= \int \cos^{2n} x \cos x \, dx = \int (1 - \sin^2 x)^n \cos x \, dx$, **(Note)**

$= \int (1 - t^2)^n \, dt$, putting $\sin x = t$ and $\cos x \, dx = dt$.

This integral can now be easily evaluated by expanding $(1 - t^2)^n$ by the binomial theorem.

Case II. *When m is an even positive integer.*

(a) **When m is small** the integration can be done by transforming the given integrand into a sum of cosines of multiples of x by using the following trigonometrical formulae

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \ \sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

(b) **When m is large,** we apply De-Moivre's theorem as explained below :

Let $\cos x + i \sin x = y$. Then $\cos x - i \sin x = y^{-1} = 1/y$.

Therefore $\quad 2 \cos x = y + y^{-1} = y + (1/y)$

and $\quad 2i \sin x = y - y^{-1} = y - (1/y)$.

Also $y^n + (1/y^n) = (\cos x + i \sin x)^n + (\cos x - i \sin x)^n$

$= (\cos nx + i \sin nx) + (\cos nx - i \sin nx) = 2 \cos nx$,

and $\quad y^n - (1/y^n) = 2i \sin nx$.

Now $2^m \cos^m x = \{y + (1/y)\}^m$. Applying binomial theorem, we have

$$2^m \cos^m x = y^m + {}^mc_1 y^{m-1} \cdot \frac{1}{y} + {}^mc_2 y^{m-2} \cdot \frac{1}{y^2} + \dots$$
$$+ m \cdot \frac{y}{y^{m-1}} + \frac{1}{y^m}$$
$$= y^m + m\, y^{m-2} + \frac{m(m-1)}{1.2} y^{m-4} + \dots + \frac{m}{y^{m-2}} + \frac{1}{y^m}$$
$$= \left(y^m + \frac{1}{y^m}\right) + m\left(y^{m-2} + \frac{1}{y^{m-2}}\right) + \dots\dots$$
$$= 2 \cos mx + m \,.2 \cos (m-2)x + \dots$$

Thus $\int 2^m \cos^m x\, dx = 2 \int \cos mx\, dx + 2 \int m \cos (m-2)x\, dx + \dots$

$$= \frac{2}{m} \sin mx + \frac{2m}{m-2} \sin (m-2)x + \dots$$

or $$\int \cos^m x\, dx = \frac{1}{2^{m-1}} \left[\frac{1}{m} \sin mx + \frac{m}{m-2} \sin (m-2)x + \dots\right].$$

Similarly $(2i \sin x)^m = [y - (1/y)]^m$.

Expanding by binomial theorem and simplifying, we have

$$2^m i^m \sin^m x = y^m - {}^mc_1 y^{m-2} + {}^mc_2 y^{m-4} - \dots$$
$$+ {}^mc_2 \frac{1}{y^{m-4}} - \frac{m}{y^{m-2}} + \frac{1}{y^m}, \quad \text{[Note that } m \text{ is even]}$$
$$= \left(y^m + \frac{1}{y^m}\right) - m\left(y^{m-2} + \frac{1}{y^{m-2}}\right)$$
$$+ \frac{m(m+1)}{2!}\left(y^{m-4} + \frac{1}{y^{m-4}}\right) - \dots$$
$$= 2 \cos mx - m\{2 \cos (m-2)x\}$$
$$+ \frac{m(m-1)}{2!}\{2 \cos (m-4)x\} - \dots$$

Hence $\int \sin^m x\, dx$

$$= \frac{1}{2^{m-1}(-1)^{m/2}} \int [\cos mx - m \cos (m-2)x + \dots]\, dx.$$

$$[\because i^m = (i^2)^{m/2} = (-1)^{m/2}]$$

Solved Examples

Ex. 1. *Integrate* $\cos^5 x$. **(Meerut 1970)**

Sol. $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx$

$= \int (1 - t^2)^2\, dt$, [putting $\sin x = t$ so that $\cos x\, dx = dt$]

$= \int (1 - 2t^2 + t^4)\, dt = t - \frac{2}{3} t^3 + \frac{1}{5} t^5.$

$\therefore \quad \int \cos^5 x \, dx = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x.$

Similarly, $\int \sin^5 x \, dx = -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x.$

Ex. 2. *Integrate* $\sin^7 x$.

Sol. $\int \sin^7 x \, dx = \int \sin^6 x \,.\, \sin x \, dx = \int (1 - \cos^2 x)^3 . \sin x \, dx$

$= -\int (1 - t^2)^3 \, dt,$ [putting $\cos x = t$ so that $-\sin x \, dx = dt$]

$= -\int (1 - 3t^2 + 3t^4 - t^6) \, dt = -t + t^3 - \frac{3}{5} t^5 + \frac{1}{7} t^7.$

$\therefore \quad \int \sin^7 x \, dx = -\cos x + \cos^3 x - \frac{3}{5} \cos^5 x + \frac{1}{7} \cos^7 x.$

Ex. 3. *Integrate* $\cos^7 x$.

Sol. $\int \cos^7 x \, dx = \int \cos^6 x \,.\, \cos x \, dx = \int (1 - \sin^2 x)^3 \cos x \, dx$

$= \int (1 - t^2)^3 \, dt,$ [putting $\sin x = t$ so that $\cos x \, dx = dt$]

$= \int (1 - 3t^2 + 3t^4 - t^6) \, dt = t - t^3 + \frac{3}{5} t^5 - \frac{1}{7} t^7$

$= \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x.$

Ex. 4. *Integrate* $\cos^4 x$.

Sol. We have $\cos^4 x = (\cos^2 x)^2 = \{\frac{1}{2}(1 + \cos 2x)\}^2$

$= \frac{1}{4}(1 + \cos 2x)^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$= \frac{1}{4}[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{8}[3 + 4\cos 2x + \cos 4x].$

$\therefore \quad \int \cos^4 x \, dx = \frac{1}{8}[\int 3 \, dx + 4 \int \cos 2x \, dx + \int \cos 4x \, dx]$

$= \frac{1}{8}[3x + 4 \cdot \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x]$

$= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x.$

Ex. 5. *Evaluate* $\int_0^{\pi/4} \sin^4 x \, dx$.

Sol. We have $\sin^4 x = (\sin^2 x)^2 = [\frac{1}{2}(1 - \cos 2x)]^2$

$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$

$= \frac{1}{4}[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{8}[3 - 4\cos 2x + \cos 4x].$

$\therefore \quad \int_0^{\pi/4} \sin^4 x \, dx = \frac{1}{8} \int_0^{\pi/4} [3 - 4\cos 2x + \cos 4x] \, dx$

$$= \frac{1}{8}\left[3x - \frac{4 \sin 2x}{2} + \frac{\sin 4x}{4}\right]_0^{\pi/4}$$

$$= \frac{1}{8}\left[\left\{3 \cdot \frac{\pi}{4} - 2\sin\frac{\pi}{2} + \frac{1}{4}\sin \pi\right\} - 0\right] = \frac{1}{32}[3\pi - 8].$$

***Ex. 6.** *Evaluate* $\int \sin^6 x \, dx$. **(Meerut 1985, 86)**

Sol. Let $\cos x + i \sin x = y$. Then $\cos x - i \sin x = y^{-1}$.

$\therefore \quad 2i \sin x = y - y^{-1}$, or $2^6 i^6 \sin^6 x = (y - y^{-1})^6$

or $\quad -64\sin^6 x = y^6 - {}^6c_1 y^5 \cdot \frac{1}{y} + {}^6c_2 y^4 \cdot \frac{1}{y^2} - {}^6c_3 y^3 \cdot \frac{1}{y^3}$

$+ {}^6c_4 y^2 \cdot \frac{1}{y^4} - {}^6c_5 y \cdot \frac{1}{y^5} + {}^6c_6 \frac{1}{y^6}$, [by binomial theorem]

$= \left(y^6 + \frac{1}{y^6}\right) - 6\left(y^4 + \frac{1}{y^4}\right) + 25\left(y^2 + \frac{1}{y^2}\right) - 20.$

Hence $-4\sin^6 x = 2\cos 6x - 6 \,.\, 2\cos 4x + 15 \,.\, 2\cos 2x - 20,$

$[\because y^n + (1/y^n) = 2\cos nx]$

or $\quad \sin^6 x = -\frac{1}{32}[\cos 6x - 6\cos 4x + 15\cos 2x - 10].$

Now integrating, we have $\int \sin^6 x\, dx$

$= -\frac{1}{32}[\int \cos 6x\, dx - 6\int \cos 4x\, dx + 15\int \cos 2x\, dx - \int 10\, dx]$

$= -\frac{1}{32}\left[\frac{\sin 6x}{6} - 6\frac{\sin 4x}{4} + \frac{15\sin 2x}{2} - 10x\right].$

Ex. 7. *Evaluate* $\int \cos^6 x\, dx$. **(Meerut 1985 S, 87)**

Sol. Let $\cos x + i\sin x = y$. Then $\cos x - i\sin x = y^{-1}$.

$\therefore \quad 2\cos x = (y + y^{-1})$, or $2^6\cos^6 x = (y + y^{-1})^6$

or $\quad 2^6\cos^6 x = \left[y + \frac{1}{y}\right]^6 = y^6 + {}^6c_1 y^5 \cdot \frac{1}{y} + {}^6c_2 y^4 \cdot \frac{1}{y^2} + {}^6c_3 y^3 \cdot \frac{1}{y^3}$

$+ {}^6c_4 y^2 \cdot \frac{1}{y^4} + {}^6c_5 y \cdot \frac{1}{y^5} + {}^6c_6 \frac{1}{y^6}$

$= \left(y^6 + \frac{1}{y^6}\right) + {}^6c_1\left(y^4 + \frac{1}{y^4}\right) + {}^6c_2\left(y^2 + \frac{1}{y^2}\right) + 20$

$= 2\cos 6x + 6 \,.\, 2\cos 4x + 15 \,.\, 2\cos 2x + 20.$

Hence $32\cos^6 x = \cos 6x + 6\cos 4x + 15\cos 2x + 10.$

$\therefore \quad \int \cos^6 x\, dx = \frac{1}{32}\left[\frac{1}{6}\sin 6x + \frac{6\sin 4x}{4} + \frac{15\sin 2x}{2} + 10x\right].$

Ex. 8. *Evaluate* $\int \cos^8 x\, dx$.

Sol. Let $\cos x + i\sin x = y$. Then $\cos x - i\sin x = y^{-1}$

so that $\quad 2\cos x = y + y^{-1}$, or $2^8\cos^8 x = [y + y^{-1}]^8$

or $\quad 2^8\cos^8 x = y^8 + {}^8c_1 y^6 + {}^8c_2 y^4 + {}^8c_3 y^2 + {}^8c_4$

$+ {}^8c_5 \frac{1}{y^2} + {}^8c_6 \frac{1}{y^4} + {}^8c_7 \frac{1}{y^6} + \frac{1}{y^8}$

$= \left(y^8 + \frac{1}{y^8}\right) + 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) + 56\left(y^2 + \frac{1}{y^2}\right) + 70$

$= 2\cos 8x + 8.2\cos 6x + 28 \,.\, 2\cos 4x + 56 \,.\, 2\cos 2x + 70.$

$\therefore \quad \int \cos^8 x\, dx$

$= \frac{1}{128}\int (\cos 8x + 8\cos 6x + 28\cos 4x + 56\cos 2x + 35)\, dx$

$$= \frac{1}{128}\left(\frac{\sin 8x}{8} + 8\frac{\sin 6x}{6} + 2.\frac{\sin 4x}{4} + 56.\frac{\sin 2x}{2} + 35x\right).$$

§ 4· 2. Integration of $\sin^m x \cos^n x$.

Case I. *If m is odd, put* $\cos x = t$; *If n is odd, put* $\sin x = t$.

If both m and n are odd, put either $\sin x$ *or* $\cos x$ *equal to t and integrate.*

Case II. *When* $(m + n)$ *is a negative even integer.*

Convert the given integral in terms of $\tan x$ and $\sec x$ and put $\tan x = t$. Then expand by binomial theorem, if necessary, and integrate term by term.

Case III. *When both m and n are even integers.*

(i) If m and n are small even integers, then convert $\sin^m x \cos^n x$ in terms of multiples of angles by using the formulae

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x),$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x), \ \sin x \cos x = \tfrac{1}{2}\sin 2x,$$

and $$2\cos x \cos y = \cos(x + y) + \cos(x - y).$$

(ii) When m and n are large, we make use of De-Moivre's theorem as explained in § 4· 1 Case II (b), page 132; term by term integration yields the desired result.

Ex. 9 (a). *Integrate* $\sin^2 x \cos^3 x$.

Sol. Here the power of $\cos x$ being odd, we put $\sin x = t$, so that $\cos x \, dx = dt$.

Thus $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$= \int t^2 (1 - t^2) \, dt,$ [Putting $\sin x = t$ so that $\cos x \, dx = dt$]

$= \int (t^2 - t^4) \, dt = \frac{1}{3}t^3 - \frac{1}{5}t^5 = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x.$

Ex. 9 (b). *Evaluate :*

(i) $\int \sin^4 x \cos^3 x \, dx$ **(Meerut 1990 P)**

(ii) $\int \sin^3 x \cos^2 x \, dx$. **(Meerut 1991)**

Sol. (i). Let $I = \int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx.$

Put $\sin x = t$ so that $\cos x \, dx = dt$.

Then $I = \int t^4 (1 - t^2) \, dt = \int (t^4 - t^6) \, dt = \frac{t^5}{5} - \frac{t^7}{7} = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7}.$

(ii) Let $I = \int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx.$

Put $\cos x = t$ so that $-\sin x \, dx = dt$.

Then $I = \int (1 - t^2) t^2 (-dt) = -\int (t^2 - t^4) \, dt$

$$= -\frac{t^3}{3} + \frac{t^5}{5} = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5}.$$

Ex. 10. *Integrate $\sin^5 x \cos^4 x$.* **(Meerut 1981, 87 S)**

Sol. Here the power of $\sin x$ being odd, we put $\cos x = t$.

$$\therefore \quad \int \sin^5 x \cos^4 x \, dx = \int \sin^4 x \cos^4 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cos^4 x \sin x \, dx$$

$$= -\int (1 - t^2)^2 t^4 \, dt, \qquad [\because \cos x = t \text{ and } -\sin x \, dx = dt]$$

$$= -\int (1 - 2t^2 + t^4) t^4 \, dt = \int (-t^4 + 2t^6 - t^8) \, dt$$

$$= -\frac{t^5}{5} + 2\frac{t^7}{7} - \frac{t^9}{9} = -\tfrac{1}{5}\cos^5 x + \tfrac{2}{7}\cos^7 x - \tfrac{1}{9}\cos^9 x.$$

Ex. 11. *Evaluate $\int_0^{\pi/4} \sin^5 x \cos^2 x \, dx$.*

Sol. Here the power of $\sin x$ being odd, we put $\cos x = t$.

$$\therefore \quad \int_0^{\pi/4} \sin^5 x \cos^2 x \, dx = \int_0^{\pi/4} \sin^4 x \cos^2 x \, .\sin x \, dx$$

$$= \int_0^{\pi/4} (1 - \cos^2 x)^2 \cos^2 x \sin x \, dx = -\int_1^{1/\sqrt{2}} (1 - t^2)^2 t^2 \, dt,$$

[putting $\cos x = t$ so that $-\sin x \, dx = dt$. Also $t = 1$, when $x = 0$ and $t = 1/\sqrt{2}$ when $x = \pi/4$]

$$= -\int_1^{1/\sqrt{2}} (1 - 2t^2 + t^4) t^2 \, dt = -\int_1^{1/\sqrt{2}} (t^2 - 2t^4 + t^6) \, dt$$

$$= -\left[\frac{t^3}{3} - \frac{2t^5}{5} + \frac{t^7}{7}\right]_1^{1/\sqrt{2}}$$

$$= -\left[\left(\frac{1}{6\sqrt{2}} - \frac{1}{10\sqrt{2}} + \frac{1}{56\sqrt{2}}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)\right]$$

$$= -\left[\frac{71}{840\sqrt{2}} - \frac{8}{105}\right] = \frac{128 - 71\sqrt{2}}{1680}.$$

Ex. 12. *Evaluate $\int \frac{\cos^5 x}{\sin^2 x} dx$.*

Sol. Let $I = \int \frac{\cos^5 x}{\sin^2 x} dx = \int \frac{\cos^4 x}{\sin^2 x} \cos x \, dx$

$$= \int \frac{(1 - \sin^2 x)^2}{\sin^2 x} \cos x \, dx.$$

Put $\sin x = t$ so that $\cos x \, dx = dt$.

Then $I = \int \frac{(1 - t^2)^2}{t^2} dt = \int \frac{1 - 2t^2 + t^4}{t^2} dt$

$$= \int \left[\frac{1}{t^2} - 2 + t^2\right] dt = -\frac{1}{t} - 2t + \frac{t^3}{3}$$

$$= -\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3} = -\operatorname{cosec} x - 2\sin x + \tfrac{1}{3}\sin^3 x.$$

Ex. 13. *Evaluate* $\int \sin^5 x \cos^3 x \, dx$.

Sol. Here both the powers of $\cos x$ and $\sin x$ are odd; so we can put either $\sin x = t$ or $\cos x = t$. Let us put $\sin x = t$ so that $\cos x \, dx = dt$.

$$\therefore \quad \int \sin^5 x \cos^3 x \, dx = \int \sin^5 x \cos^2 x \cos x \, dx$$
$$= \int t^5 (1 - t^2) \, dt = \int (t^5 - t^7) \, dt = \tfrac{1}{6} t^6 - \tfrac{1}{8} t^8$$
$$= \tfrac{1}{6} \sin^6 x - \tfrac{1}{8} \sin^8 x.$$

***Ex. 14.** *Evaluate* $\int \sec x \tan^3 x \, dx$.

Sol. We have $\int \sec x \tan^3 x \, dx = \int \frac{\sin^3 x}{\cos^4 x} dx.$

Now the power of $\sin x$ being odd, we put $\cos x = t$ so that $-\sin x \, dx = dt$.

$$\therefore \quad \text{the given integral} = -\int \frac{(1 - t^2)\, dt}{t^4} = -\int \left(\frac{1}{t^4} - \frac{1}{t^2}\right) dt$$
$$= \left(\frac{1}{3}\frac{1}{t^3} - \frac{1}{t}\right) = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} = \frac{1}{3}\sec^3 x - \sec x.$$

***Ex. 15.** *Evaluate* $\int \sin^3 x \cos 2x \, dx$.

Sol. $I = \int \sin^3 x \cos 2x \, dx = \int \sin^2 x \, (2\cos^2 x - 1) \sin x \, dx$
$$= \int (1 - \cos^2 x)(2\cos^2 x - 1) \sin x \, dx.$$

Put $\cos x = t$, so that $-\sin x \, dx = dt$. Then
$$I = -\int (1 - t^2)(2t^2 - 1) \, dt = -\int (3t^2 - 2t^4 - 1) \, dt$$
$$= -[t^3 - \tfrac{2}{5} t^5 - t] = -\cos^3 x + \tfrac{2}{5}\cos^5 x + \cos x.$$

Ex. 16. *Integrate* $1/(\sin^3 x \cos^5 x)$.

Sol. Here the integrand is $\sin^{-3} x \cos^{-5} x$. It is of the type $\sin^m x \cos^n x$, where $m + n = -3 - 5 = -8$ *i.e.*, –ive even integer.

$$\therefore \quad I = \int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{dx}{(\sin^3 x / \cos^3 x) \cos^3 x . \cos^5 x}$$
$$= \int \frac{\sec^8 x \, dx}{\tan^3 x} = \int \frac{\sec^6 x . \sec^2 x \, dx}{\tan^3 x} \qquad \textbf{(Note)}$$
$$= \int \frac{(1 + \tan^2 x)^8 \sec^2 x \, dx}{\tan^3 x}.$$

Now put $\tan x = t$ so that $\sec^2 x \, dx = dt$.

$$\therefore \quad I = \int \frac{(1 + t^2)^3}{t^3} dt = \int \left(\frac{1}{t^3} + \frac{3}{t} + 3t + t^3\right) dt$$
$$= -\{1/(2t^2)\} + 3\log t + \tfrac{3}{2} t^2 + \tfrac{1}{4} t^4$$

$$= -\tfrac{1}{2}\cot^2 x + 3\log\tan x + \tfrac{3}{2}\tan^2 x + \tfrac{1}{4}\tan^4 x.$$

Ex. 17. *Integrate* $1/(\sin^4 x \cos^2 x)$. **(Meerut 1984 S)**

Sol. Here $m = -4, n = -2$; $m + n = -6$ *i.e.*, an even negative integer.

$$\therefore \int \frac{dx}{\sin^4 x \cos^2 x} = \int \frac{\sec^6 x}{\tan^4 x}\,dx = \int \frac{(1+\tan^2 x)^2 \sec^2 x\,dx}{\tan^4 x}$$

$$= \int \frac{(1+t^2)^2\,dt}{t^4} \text{ putting } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int \left(\frac{1}{t^4} + \frac{2}{t^2} + 1\right) dt = -\frac{1}{3t^3} - \frac{2}{t} + t$$

$$= -\tfrac{1}{3}\cot^3 x - 2\cot x + \tan x.$$

Ex. 18. *Integrate* $1/\sqrt{(\cos^3 x \sin^5 x)}$.

Sol. Here the integrand is of the type $\cos^m x \sin^n x$. We have $m = -3/2, n = -5/2$; $m + n = -4$ *i.e.*, an even negative integer.

$$\therefore \int \frac{dx}{\sqrt{(\cos^3 x \sin^5 x)}} = \int \frac{dx}{\cos^{3/2} x \sin^{5/2} x}$$

$$= \int \frac{dx}{\cos^{3/2} x\,(\sin^{5/2} x/\cos^{5/2} x)\,.\cos^{5/2} x} \quad \textbf{(Note)}$$

$$= \int \frac{dx}{\cos^4 x \tan^{5/2} x} = \int \frac{\sec^4 x}{\tan^{5/2} x}\,dx = \int \frac{\sec^2 x}{\tan^{5/2} x}\sec^2 x\,dx$$

$$= \int \frac{(1+\tan^2 x)}{\tan^{5/2} x}\sec^2 x\,dx.$$

$$= \int \frac{(1+t^2)}{t^{5/2}}\,dt, \quad \text{putting } \tan x = t \text{ and } \sec^2 x\,dx = dt$$

$$= \int (t^{-5/2} + t^{-1/2})\,dt = -\tfrac{2}{3}t^{-3/2} + 2t^{1/2}$$

$$= -\tfrac{2}{3}(\tan x)^{-3/2} + 2(\tan x)^{1/2} = 2\sqrt{(\tan x)} - \tfrac{2}{3}(\tan x)^{-3/2}.$$

Ex. 19. *Integrate* $\sqrt{(\tan x)} \sec x \operatorname{cosec} x$.

Sol. We have $\int \sqrt{(\tan x)} \sec x \operatorname{cosec} x\,dx$

$$= \int \sqrt{\left(\frac{\sin x}{\cos x}\right)} \cdot \frac{1}{\cos x \sin x}\,dx = \int \frac{dx}{\cos^{3/2} x \sin^{1/2} x}$$

$$= \int \frac{dx}{\cos^{3/2} x\,(\sin^{1/2} x/\cos^{1/2} x)\cos^{1/2} x}$$

$$= \int \frac{dx}{\cos^2 x\,.\tan^{1/2} x} = \int \frac{\sec^2 x\,dx}{\tan^{1/2} x}$$

$$= \int (\tan^{-1/2} x)\sec^2 x\,dx = \frac{\tan^{1/2} x}{(1/2)}, \text{ by power formula}$$

$$= 2\sqrt{(\tan x)}.$$

Ex. 20. *Integrate* $\sin^2 x \cos^4 x$. **(Ranchi 1974)**

Sol. We have $\sin^2 x \cos^4 x = (\sin x \ \cos x)^2 \cos^2 x$

$= (\frac{1}{2}\sin 2x)^2 \{\frac{1}{2}(1 + \cos 2x)\}$

$= \frac{1}{8}\sin^2 2x\,(1 + \cos 2x) = \frac{1}{8}\{\frac{1}{2}(1 - \cos 4x)\}(1 + \cos 2x)$

$= \frac{1}{16}(1 + \cos 2x - \cos 4x - \cos 4x \cos 2x)$

$= \frac{1}{16}[1 + \cos 2x - \cos 4x - \frac{1}{2}(\cos 6x + \cos 2x)]$

$= \frac{1}{16}[1 + \frac{1}{2}\cos 2x - \cos 4x - \frac{1}{2}\cos 6x]$

$= \frac{1}{32}(2 + \cos 2x - 2\cos 4x - \cos 6x)$.

$\therefore \quad \int \sin^2 x \cos^4 x\, dx = \frac{1}{32}\int [2 + \cos 2x - 2\cos 4x - \cos 6x]\, dx$

$= \frac{1}{32}[2x + \frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x - \frac{1}{6}\sin 6x]$.

***Ex. 21.** *Evaluate* $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$. **(Meerut 1977)**

Sol. Let $y = \cos x + i\sin x$. Then $y^{-1} = \cos x - i\sin x$.

We have $(y + y^{-1}) = 2\cos x$ and $(y - y^{-1}) = 2i\sin x$.

$\therefore \quad (2i\sin x)^4 (2\cos x)^2 = (y - y^{-1})^4 (y + y^{-1})^2$

or $\quad 2^6 i^4 \sin^4 x \cos^2 x$

$$= \left(y^4 - 4y^3\cdot\frac{1}{y} + 6y^2\cdot\frac{1}{y^2} - 4y\cdot\frac{1}{y^3} + \frac{1}{y^4}\right)\left(y + \frac{1}{y}\right)\left(y + \frac{1}{y}\right)$$

$$= \left[\left(y^4 - 4y^2 + 6 - \frac{4}{y^2} + \frac{1}{y^4}\right)\left(y + \frac{1}{y}\right)\right]\left(y + \frac{1}{y}\right)$$

$$= \left(y^5 - 3y^3 + 2y + \frac{2}{y} - \frac{3}{y^3} + \frac{1}{y^5}\right)\left(y + \frac{1}{y}\right)$$

$$= y^6 - 2y^4 - y^2 + 4 - \frac{1}{y^2} - \frac{2}{y^4} + \frac{1}{y^6}.$$

$$\therefore \quad 64\sin^4 x\,.\cos^2 x = \left(y^6 + \frac{1}{y^6}\right) - 2\left(y^4 + \frac{1}{y^4}\right) - \left(y^2 + \frac{1}{y^2}\right) + 4$$

$$= 2\cos 6x - 2\,.2\cos 4x - 2\cos 2x + 4.$$

$$\therefore \quad \int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$$

$$= \frac{1}{32}\int_0^{\pi/2} [\cos 6x - 2\cos 4x - \cos 2x + 2]\, dx$$

$$= \frac{1}{32}\left[\frac{\sin 6x}{6} - \frac{\sin 4x}{4} - \frac{\sin 2x}{2} + 2x\right]_0^{\pi/2} = \frac{1}{32}\cdot 2\cdot\frac{\pi}{2} = \frac{\pi}{32}.$$

Integration of a product of sines or cosines of multiples of angles.

****Ex. 22.** *If m and n are integers, prove that*

$$\int_0^{\pi} \cos mx \sin nx\, dx = \frac{2n}{n^2 - m^2} \text{ or } 0 \text{ according as } (n - m) \text{ is odd or even.}$$

(Meerut 1983 S, 84; Vikram 75; Kanpur 71)

Sol. The given integral $= \frac{1}{2}\int_0^{\pi} 2\cos mx \sin nx\, dx$

$$= \frac{1}{2}\int_0^{\pi} 2\sin nx \cos mx\, dx$$

$$= \frac{1}{2}\int_0^{\pi} [\sin (m+n)x + \sin (n-m)x]\, dx, \text{ by trigonometry}$$

$$= -\frac{1}{2}\left[\frac{\cos (m+n)x}{m+n} + \frac{\cos (n-m)x}{n-m}\right]_0^{\pi},$$

if $n - m \neq 0$ *i.e.*, if $n \neq m$

$$= -\frac{1}{2}\left[\left\{\frac{(-1)^{m+n}}{m+n} + \frac{(-1)^{n-m}}{n-m}\right\} - \left\{\frac{1}{m+n} + \frac{1}{n-m}\right\}\right],$$

$[\because \cos r\pi = (-1)^r]$

Case 1. $n - m$ is odd.

When $n - m$ is odd, $n + m$ is also odd because we can write

$$n + m = (n - m) + 2m.$$

$\therefore$ In this case $(-1)^{m+n} = (-1)^{n-m} = -1$.

Hence in this case the given integral

$$= -\frac{1}{2}\left[-\frac{1}{m+n} - \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{n-m}\right] = \frac{2n}{n^2 - m^2}.$$

Case 2. $n - m$ is even.

If $(n - m)$ is even, then $n + m$ is also even.

$\therefore$ $(-1)^{n-m} = (-1)^{n+m} = 1$.

Then the given integral

$$= -\frac{1}{2}\left[\frac{1}{m+n} + \frac{1}{n-m} - \frac{1}{m+n} - \frac{1}{n-m}\right] = 0.$$

Now if $n = m$, then $n - m = 0$ which is even. Also in this case the given integral

$$= \frac{1}{2}\int_0^{\pi} 2\sin mx \cos mx\, dx = \frac{1}{2}\int_0^{\pi} \sin 2mx\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{2m}\cos 2mx\right]_0^{\pi}$$

$$= -\frac{1}{4m}[\cos 2m\pi - \cos 0] = -\frac{1}{4m}(1 - 1) = 0.$$

Hence the result follows.

Ex. 23. *If m and n are integers, show that*

$$\int_0^{\pi} \sin mx \,.\, \sin nx\, dx = 0 \text{ if } m \neq n \text{ and } = \frac{\pi}{2} \text{ if } m = n.$$

Sol. The given integral

$$= \frac{1}{2}\int_0^{\pi} 2\sin mx \,.\, \sin nx\, dx$$

$$= \frac{1}{2}\int_0^{\pi} [\cos (m-n)x - \cos (m+n)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi},$$

if $m - n \neq 0$ *i.e.*, if $m \neq n$

$$= \tfrac{1}{2}[(0-0)-(0-0)] = 0, \qquad [\because \sin r\pi = 0].$$

Now when $m = n$, the given integral

$$= \int_0^{\pi} \sin^2 nx\, dx$$

$$= \tfrac{1}{2}\int_0^{\pi} (1-\cos 2nx)\, dx = \tfrac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_0^{\pi}$$

$$= \tfrac{1}{2}[(\pi-0)-(0-0)] = \tfrac{1}{2}\pi.$$

Ex. 24. *Evaluate* $\int_0^{\pi/4} \cos 3x \cos 5x\, dx$.

Sol. The given integral $= \frac{1}{2}\int_0^{\pi/4} 2\cos 3x \cos 5x\, dx$

$$= \tfrac{1}{2}\int_0^{\pi/4} (\cos 8x + \cos 2x)\, dx$$

$$= \frac{1}{2}\left[\frac{\sin 8x}{8} + \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left\{\frac{\sin 2\pi}{8} + \frac{\sin \frac{1}{2}\pi}{2}\right\} - \left\{\frac{\sin 0}{8} + \frac{\sin 0}{2}\right\}\right]$$

$$= \tfrac{1}{2}[\tfrac{1}{2}] = \tfrac{1}{4}.$$

Ex. 25. *Evaluate* $\int \cos x \cos 2x \cos 3x\, dx$.

Sol. The given integral $= \frac{1}{2}\int \cos x \,.\, (2\cos 2x \cos 3x)\, dx$

$$= \tfrac{1}{2}\int \cos x\,(\cos 5x + \cos x)\, dx$$

$$= \tfrac{1}{4}\int 2\cos x \cos 5x\, dx + \tfrac{1}{4}\int 2\cos^2 x\, dx$$

$$= \tfrac{1}{4}\int (\cos 6x + \cos 4x)\, dx + \tfrac{1}{4}\int (1+\cos 2x)\, dx$$

$$= \tfrac{1}{4}\int \cos 6x\, dx + \tfrac{1}{4}\int \cos 4x\, dx + \tfrac{1}{4}\int \cos 2x\, dx + \tfrac{1}{4}\int dx$$

$$= \frac{1}{4}\cdot\frac{\sin 6x}{6} + \frac{1}{4}\,\frac{\sin 4x}{4} + \frac{1}{4}\,\frac{\sin 2x}{2} + \frac{1}{4}\cdot x$$

$$= \tfrac{1}{4}[(1/6)\,.\,\sin 6x + \tfrac{1}{4}\sin 4x + \tfrac{1}{2}\sin 2x + x].$$

**§ 4·3. Integration of $1/(a + b\cos x)$.

(Delhi 1981; Kanpur 70; Ranchi 74; Gorakhpur 71)

We have $$I = \int \frac{dx}{a + b\cos x}$$

$$= \int \frac{dx}{a(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + b(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)} \qquad \textbf{(Note)}$$

$$= \int \frac{dx}{(a+b)\cos^2 \frac{1}{2}x + (a-b)\sin^2 \frac{1}{2}x}$$

$$= \int \frac{\sec^2 \frac{1}{2}x \, dx}{a+b+(a-b)\tan^2 \frac{1}{2}x},$$

dividing the numerator and the denominator by $\cos^2 \frac{1}{2}x$.

Now putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x \, dx = dt$, we get

$$I = 2\int \frac{dt}{(a+b)+(a-b)t^2} \qquad \text{...(1)}$$

Now two cases arise viz, $a > b$ or $a < b$.

Case I. $\mathbf{a > b}$. **(Magadh 1976)**

In this case, we have from (1)

$$I = \frac{2}{a-b}\int \frac{dt}{t^2+k^2}, \qquad \left[\text{putting } \frac{a+b}{a-b} = k^2\right]$$

$$= \frac{2}{(a-b)} \cdot \frac{1}{k}\tan^{-1}\frac{t}{k} = \frac{2}{(a-b)\sqrt{\left(\frac{a+b}{a-b}\right)}}\tan^{-1}\left\{\frac{t}{\sqrt{\left(\frac{a+b}{a-b}\right)}}\right\}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \cdot t\right\}$$

$$= \frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left[\sqrt{\left\{\frac{a-b}{a+b}\right\}}\tan\frac{x}{2}\right].$$

Case II. $\mathbf{a < b}$. In this case, from (1) we write

$$I = 2\int \frac{dt}{(a+b)-(b-a)t^2} = \frac{2}{b-a}\int \frac{dt}{\{(b+a)/(b-a)\}-t^2}$$

$$= \frac{2}{b-a} \cdot \frac{1}{2\sqrt{\{(b+a)/(b-a)\}}} \cdot \log \frac{\sqrt{\{(b+a)/(b-a)\}}+t}{\sqrt{\{(b+a)/(b-a)\}}-t}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \cdot \log \frac{\sqrt{(b+a)}+t\sqrt{(b-a)}}{\sqrt{(b+a)}-t\sqrt{(b-a)}}$$

$$= \frac{1}{\sqrt{(b^2-a^2)}} \cdot \log \frac{\sqrt{(b+a)}+\sqrt{(b-a)}\tan\frac{1}{2}x}{\sqrt{(b+a)}-\sqrt{(b-a)}\tan\frac{1}{2}x}.$$

***Ex. 26 (a).** *Integrate $1/(5+4\cos x)$.* **(Meerut 1986 P)**

Sol. We have $\displaystyle\int \frac{dx}{5+4\cos x}$

$$= \int \frac{dx}{5(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + 4(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)} \qquad \textbf{(Note)}$$

$$= \int \frac{dx}{9\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x} = \int \frac{\sec^2 \frac{1}{2}x \, dx}{9+\tan^2 \frac{1}{2}x},$$

dividing the numerator and the denominator by $\cos^2 \frac{1}{2}x$.

Now putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\, dx = dt$,

the required integral $= 2 \int \frac{dt}{9 + t^2}$

$= 2 \cdot \frac{1}{3} \tan^{-1}(t/3) = \frac{2}{3} \tan^{-1}(\frac{1}{3} \tan \frac{1}{2}x)$.

Ex. 26 (b). *Evaluate* $\int_0^{\pi/2} \frac{dx}{5 + 4\cos x}$. **(Meerut 1981, 76, 70)

Sol. Proceeding as in part (a), we get

$$\int_0^{\pi/2} \frac{dx}{5 + 4\cos x} = 2 \int_0^1 \frac{dt}{9 + t^2}$$

[Note that when $x = 0$, $t = \tan 0 = 0$ and when $x = \pi/2$, $t = \tan \frac{1}{4}\pi = 1$]

$$= 2 \times \frac{1}{3}\left[\tan^{-1} \frac{1}{3} t\right]_0^1 = \frac{2}{3}\left[\tan^{-1} \frac{1}{3} - \tan^{-1} 0\right]$$

$$= \frac{2}{3}[(\tan^{-1} \tfrac{1}{3}) - 0] = \frac{2}{3} \tan^{-1} \tfrac{1}{3}.$$

Ex. 27. *Evaluate* $\int_0^{\pi/2} \frac{dx}{4 + 5\cos x}$. (Kanpur 1974; Meerut 77)

Sol. We have $I = \int_0^{\pi/2} \frac{dx}{4 + 5\cos x}$

$$= \int_0^{\pi/2} \frac{dx}{4(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + 5(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)} \quad \textbf{(Note)}$$

$$= \int_0^{\pi/2} \frac{dx}{9\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x} = \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2}x\, dx}{9 - \tan^2 \frac{1}{2}x}.$$

Now put $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\, dx = dt$. Also $t = 0$ when $x = 0$ and $t = 1$ when $x = \pi/2$.

$$\therefore \quad I = \int_0^1 \frac{2\, dt}{9 - t^2} = 2 \cdot \frac{1}{2.3}\left[\log \frac{3 + t}{3 - t}\right]_0^1$$

$$= \frac{1}{3}\left[\log \frac{3 + 1}{3 - 1} - \log \frac{3 + 0}{3 - 0}\right] = \frac{1}{3} \log 2.$$

Ex. 28. *Integrate* $1/(3 + 2\cos x)$.

Sol. We have $\int \frac{dx}{3 + 2\cos x} = \int \frac{dx}{5\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x}$

$$= \int \frac{\sec^2 \frac{1}{2}x\, dx}{5 + \tan^2 \frac{1}{2}x} = 2 \int \frac{dt}{5 + t^2},$$

putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x \, dx = dt$

$$= 2 \cdot \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{5}} \tan \frac{1}{2}x\right).$$

Ex. 29. *Integrate* $1/(2 + \cos x)$. **(Gorakhpur 1976; Magadh 75)**

Sol. $$\int \frac{dx}{2 + \cos x} = \int \frac{dx}{3\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x} = \int \frac{\sec^2 \frac{1}{2}x \, dx}{3 + \tan^2 \frac{1}{2}x}$$

(Note)

$$= 2\int \frac{dt}{3 + t^2}, \text{ putting } \tan \tfrac{1}{2}x = t \text{ so that } \tfrac{1}{2}\sec^2 \tfrac{1}{2}x \, dx = dt$$

$$= 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{1}{2}x\right).$$

Ex. 30. *Prove that* $\displaystyle\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta} = \frac{\pi}{4}$.

(Lucknow 1981; Magadh 75; Meerut 86)

Sol. We have $\displaystyle I = \int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta}$

$$= \int_0^{\pi} \frac{d\theta}{5(\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta) + 3(\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)}$$

$$= \int_0^{\pi} \frac{d\theta}{8\cos^2 \frac{1}{2}\theta + 2\sin^2 \frac{1}{2}\theta} = \frac{1}{2}\int_0^{\pi} \frac{\sec^2 \frac{1}{2}\theta \, d\theta}{4 + \tan^2 \frac{1}{2}\theta}.$$

Now put $\tan \frac{1}{2}\theta = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}\theta \, d\theta = dt$. Also when $\theta = 0$, $t = \tan\theta = 0$, and when $\theta = \pi$, $t = \tan \frac{1}{2}\pi = \infty$.

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{4 + t^2} = \frac{1}{2}\left[\tan^{-1} \frac{1}{2}t\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\tan^{-1} \infty - \tan^{-1} 0\right] = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{4}.$$

Ex. 31 (a). *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1 + 2\cos\theta}$.

(Lucknow 1982; Kanpur 73)

Sol. We have $\displaystyle I = \int_0^{\pi/2} \frac{d\theta}{1 + 2\cos\theta}$

$$= \int_0^{\pi/2} \frac{d\theta}{(\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta) + 2(\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)}$$

$$= \int_0^{\pi/2} \frac{d\theta}{3\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta} = \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2}\theta \, d\theta}{3 - \tan^2 \frac{1}{2}\theta}.$$

Now put $\tan \frac{1}{2}\theta = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}\theta \, d\theta = dt$. Also $t = 0$ when $\theta = 0$ and $t = 1$ when $\theta = \frac{1}{2}\pi$.

$$\therefore \quad I = 2 \int_0^1 \frac{dt}{3 - t^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{3}} \left[\log \left(\frac{\sqrt{3} + t}{\sqrt{3} - t} \right) \right]_0^1 = \frac{1}{\sqrt{3}} \left[\log \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) - \log 1 \right]$$

$$= \frac{1}{\sqrt{3}} \log \left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right) = \frac{1}{\sqrt{3}} \log \left[\frac{(\sqrt{3} + 1)(\sqrt{3} + 1)}{(\sqrt{3})^2 - 1^2} \right]$$

$$= \frac{1}{\sqrt{3}} \log \left[\frac{4 + 2\sqrt{3}}{2} \right] = \frac{1}{\sqrt{3}} \log (2 + \sqrt{3}).$$

Ex. 31 (b). *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + a \cos x}$, $0 < a < 1$. **(Kanpur 1973)**

Sol. Proceed exactly as in Ex. 31 (a).

Ex. 32 (a). *Prove that* $\displaystyle\int_0^{\alpha} \frac{d\theta}{\cos \alpha + \cos \theta} = \operatorname{cosec} \alpha \log (\sec \alpha)$.

(Meerut 1983, 84)

Sol. Let $I = \displaystyle\int \frac{d\theta}{\cos \alpha + \cos \theta}$

$$= \int \frac{d\theta}{\cos \alpha \, (\cos^2 \frac{1}{2}\theta + \sin^2 \frac{1}{2}\theta) + (\cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta)}$$

$$= \int \frac{d\theta}{(1 + \cos \alpha) \cos^2 \frac{1}{2}\theta - (1 - \cos \alpha) \sin^2 \frac{1}{2}\theta}$$

$$= \frac{1}{1 - \cos \alpha} \int \frac{\sec^2 \frac{1}{2}\theta \, d\theta}{\{(1 + \cos \alpha)/(1 - \cos \alpha)\} - \tan^2 \frac{1}{2}\theta}. \quad \text{(Note)}$$

Now $\dfrac{1 + \cos \alpha}{1 - \cos \alpha} = \dfrac{2 \cos^2 \frac{1}{2}\alpha}{2 \sin^2 \frac{1}{2}\alpha} = \cot^2 \frac{1}{2}\alpha.$

Therefore $I = \dfrac{1}{2 \sin^2 \frac{1}{2}\alpha} \displaystyle\int \frac{\sec^2 \frac{1}{2}\theta \, d\theta}{\cot^2 \frac{1}{2}\alpha - \tan^2 \frac{1}{2}\theta}.$

Now putting $\tan \frac{1}{2}\theta = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}\theta \, d\theta = dt$, we get

$$I = \frac{1}{2 \sin^2 \frac{1}{2}\alpha} \int \frac{2 \, dt}{\cot^2 \frac{1}{2}\alpha - t^2} = \frac{1}{\sin^2 \frac{1}{2}\alpha} \int \frac{dt}{\cot^2 \frac{1}{2}\alpha - t^2}$$

$$= \frac{1}{\sin^2 \frac{1}{2}\alpha} \cdot \frac{1}{2 \cot \frac{1}{2}\alpha} \log \frac{\cot \frac{1}{2}\alpha + t}{\cot \frac{1}{2}\alpha - t}$$

$$= \frac{1}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}\log\left(\frac{\cot\frac{1}{2}\alpha+\tan\frac{1}{2}\theta}{\cot\frac{1}{2}\alpha-\tan\frac{1}{2}\theta}\right)$$

$$= \operatorname{cosec}\alpha\log\left(\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\theta+\sin\frac{1}{2}\alpha\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\theta-\sin\frac{1}{2}\alpha\sin\frac{1}{2}\theta}\right)$$

$$= \operatorname{cosec}\alpha\,.\log\left\{\frac{\cos\frac{1}{2}(\alpha-\theta)}{\cos\frac{1}{2}(\alpha+\theta)}\right\}.$$

Hence the given definite integral $\int_0^\alpha \frac{d\theta}{\cos\alpha+\cos\theta}$

$$= \operatorname{cosec}\alpha\left[\log\frac{\cos\frac{1}{2}(\alpha-\theta)}{\cos\frac{1}{2}(\alpha+\theta)}\right]_0^\alpha$$

$$= \operatorname{cosec}\alpha\left[\log\left(\frac{1}{\cos\alpha}\right)-\log 1\right]$$

$$= \operatorname{cosec}\alpha\log\sec\alpha.$$

Ex. 32 (b). *Evaluate* $\int \frac{\cos\alpha\cos x+1}{\cos\alpha+\cos x}\,dx.$

Sol. Given integral $= \int \frac{\cos\alpha\cos x+(\cos^2\alpha+\sin^2\alpha)}{\cos\alpha+\cos x}\,dx$ **(Note)**

$$= \int \frac{(\cos\alpha+\cos x)\cos\alpha}{(\cos\alpha+\cos x)}\,dx+\sin^2\alpha\int\frac{dx}{\cos\alpha+\cos x}$$

$$= x\cos\alpha+\sin^2\alpha\operatorname{cosec}\alpha\log\frac{\cos\frac{1}{2}(\alpha-x)}{\cos\frac{1}{2}(\alpha+x)},\text{ from Ex. 32 (a)}$$

$$= x\cos\alpha+\sin\alpha\log\frac{\cos\frac{1}{2}(\alpha-x)}{\cos\frac{1}{2}(\alpha+x)}.$$

Ex. 33. *Prove that* $\int_0^\alpha \frac{dx}{1-\cos\alpha\cos x}=\frac{\pi}{2}\operatorname{cosec}\alpha.$

(Meerut 1985 S, 87, 88P)

Sol. We have $I=\int_0^\alpha \frac{dx}{1-\cos\alpha\cos x}$

$$= \int_0^\alpha \frac{dx}{(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x)-\cos\alpha\,(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x)}$$

$$= \int_0^\alpha \frac{dx}{(1-\cos\alpha)\cos^2\frac{1}{2}x+(1+\cos\alpha)\sin^2\frac{1}{2}x}$$

$$= \frac{1}{(1+\cos\alpha)} \int_0^{\alpha} \frac{\sec^2 \frac{1}{2}x\, dx}{\{(1-\cos\alpha)/(1+\cos\alpha)\} + \tan^2 \frac{1}{2}x}$$

$$= \frac{1}{2\cos^2 \frac{1}{2}\alpha} \int_0^{\alpha} \frac{\sec^2 \frac{1}{2}x\, dx}{\tan^2 \frac{1}{2}\alpha + \tan^2 \frac{1}{2}x}.$$

Now put $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\, dx = dt$. When $x = 0$, $t = \tan 0 = 0$ and when $x = \alpha$, $t = \tan \frac{1}{2}\alpha$.

$$\therefore \quad I = \frac{1}{2\cos^2 \frac{1}{2}\alpha} \int_0^{\tan \frac{1}{2}\alpha} \frac{2\, dt}{\tan^2 \frac{1}{2}\alpha + t^2}$$

$$= \frac{1}{\cos^2 \frac{1}{2}\alpha} \int_0^{\tan \frac{1}{2}\alpha} \frac{dt}{\tan^2 \frac{1}{2}\alpha + t^2}$$

$$= \frac{1}{\cos^2 \frac{1}{2}\alpha} \frac{1}{\tan \frac{1}{2}\alpha} \left[\tan^{-1} \frac{t}{\tan \frac{1}{2}\alpha}\right]_0^{\tan \frac{1}{2}\alpha}$$

$$= \frac{1}{\cos \frac{1}{2}\alpha \sin \frac{1}{2}\alpha} [\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \frac{1}{\cos \frac{1}{2}\alpha \sin \frac{1}{2}\alpha} [\tfrac{1}{4}\pi - 0] = \frac{\pi}{2 . (2\cos \frac{1}{2}\alpha \sin \frac{1}{2}\alpha)}$$

$$= \frac{\pi}{2\sin\alpha} = \frac{\pi}{2} \operatorname{cosec} \alpha.$$

Ex. 34. *Prove that* $\displaystyle\int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2} = \frac{\pi}{1-a^2}$ *or* $\displaystyle\frac{\pi}{a^2-1}$ *according as* $a <$ *or* > 1. **(Lucknow 1979; Meerut 89; Vikram 72)**

Sol. We have $\displaystyle I = \int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}$

$$= \int_0^{\pi} \frac{dx}{(1+a^2)(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) - 2a(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$$

$$= \int_0^{\pi} \frac{dx}{(1-a)^2 \cos^2 \frac{1}{2}x + (1+a)^2 \sin^2 \frac{1}{2}x}$$

$$= \frac{1}{(1+a)^2} \int_0^{\pi} \frac{\sec^2 (\frac{1}{2}x)\, dx}{\{(1-a)/(1+a)\}^2 + \tan^2 \frac{1}{2}x}$$

$$= \frac{2}{(1+a)^2} \int_0^{\infty} \frac{dt}{\{(1-a)/(1+a)\}^2 + t^2},$$

putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\, dx = dt$

$$= \frac{2}{(1+a)^2}\left[\left(\frac{1+a}{1-a}\right)\tan^{-1}\left(t\cdot\frac{1+a}{1-a}\right)\right]_0^\infty$$

$$= \frac{2}{(1+a)(1-a)}\left[\tan^{-1}\left(t\cdot\frac{1+a}{1-a}\right)\right]_0^\infty$$

$$= \frac{2}{1-a^2}[\tan^{-1}\infty - \tan^{-1}0], \text{ if } a<1$$

[$\because$ $a<1$ means $1-a$ is $+$ ive]

$$= \frac{2}{1-a^2}[\tfrac{1}{2}\pi - 0] = \frac{\pi}{1-a^2}.$$

If $a>1$, then $I = \dfrac{2}{1-a^2}\left[\tan^{-1}\left(t\cdot\dfrac{1+a}{1-a}\right)\right]_0^\infty$

$$= \frac{2}{1-a^2}[\tan^{-1}(-\infty) - \tan^{-1}0]$$

[$\because a>1$ means $1-a$ is $-$ ive]

$$= \frac{2}{1-a^2}[-\tfrac{1}{2}\pi - 0] = \frac{\pi}{a^2-1}.$$

Ex. 35. *Evaluate* $\displaystyle\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$. **(Meerut 1985 P)**

Sol. We have

$$I = \int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x},$$

dividing the numerator and the denominator by $\cos^2 x$.

Now put $b\tan x = t$ so that $b\sec^2 x\,dx = dt$. When $x = \frac{1}{2}\pi$, $t = b\tan\frac{1}{2}\pi = \infty$ and when $x=0$, $t = b\tan 0 = 0$.

$$\therefore\quad I = \frac{1}{b}\int_0^\infty\frac{dt}{a^2+t^2} = \frac{1}{b}\cdot\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^\infty$$

$$= \frac{1}{ab}[\tan^{-1}\infty - \tan^{-1}0] = \frac{1}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2ab}.$$

***Ex. 36.** *Evaluate* $\displaystyle\int\frac{\sin 2x\,dx}{(a+b\cos x)^2}$. **(Vikram 1975)**

Sol. We have $I = \displaystyle\int\frac{\sin 2x\,dx}{(a+b\cos x)^2} = 2\int\frac{\sin x\cos x\,dx}{(a+b\cos x)^2}$.

Now put $a + b\cos x = t$ so that $-b\sin x\,dx = dt$. Also $\cos x = (t-a)/b$.

$$\therefore\quad I = -\frac{2}{b}\int\frac{(t-a)/b}{t^2}\,dt = -\frac{2}{b^2}\int\left[\frac{t}{t^2} - \frac{a}{t^2}\right]dt$$

$$= -\frac{2}{b^2}\int\left[\frac{1}{t}-\frac{a}{t^2}\right]dt = -\frac{2}{b^2}\left[\log t + \frac{a}{t}\right]$$

$$= -\frac{2}{b^2}\left[\log(a+b\cos x) + \frac{a}{a+b\cos x}\right].$$

Ex. 37. *Evaluate* $\int \frac{\cos x}{a+b\cos x}\,dx.$

Sol. We have $\int \frac{\cos x\,dx}{a+b\cos x} = \frac{1}{b}\int \frac{b\cos x\,dx}{a+b\cos x}$ **(Note)**

$$= \frac{1}{b}\int \frac{a+b\cos x - a}{(a+b\cos x)}\,dx = \frac{1}{b}\int\left(1-\frac{a}{a+b\cos x}\right)dx$$

$$= \frac{1}{b}x - \frac{a}{b}\int \frac{1}{a+b\cos x}\,dx.$$

Now proceed as in § 4·3, page 141.

Ex. 38. (a). *Evaluate* $\int \frac{dx}{1+\cos^2 x}.$ **(Lucknow 1983)**

Sol. We have $\int \frac{dx}{1+\cos^2 x} = \int \frac{\sec^2 x\,dx}{\sec^2 x + 1},$

dividing the numerator and the denominator by $\cos^2 x$

$$= \int \frac{\sec^2 x\,dx}{1+\tan^2 x + 1} = \int \frac{\sec^2 x\,dx}{2+\tan^2 x}$$

$= \int \frac{dt}{2+t^2}$, putting $\tan x = t$ so that $\sec^2 x\,dx = dt$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right).$$

Ex. 38. (b). *Evaluate* $\int \frac{dx}{1+3\sin^2 x}.$ **(Meerut 1991 P)**

Sol. We have $\int \frac{dx}{1+3\sin^2 x} = \int \frac{dx}{(\sin^2 x + \cos^2 x) + 3\sin^2 x}$ **(Note)**

$$= \int \frac{dx}{4\sin^2 x + \cos^2 x} = \int \frac{\sec^2 x\,dx}{4\tan^2 x + 1},$$

dividing the numerator and the denominator by $\cos^2 x$

$= \frac{1}{2}\int \frac{dt}{t^2+1}$, putting $2\tan x = t$ so that $2\sec^2 x\,dx = dt$

$$= \frac{1}{2}\cdot\tan^{-1}(t) = \frac{1}{2}\tan^{-1}(2\tan x).$$

Ex. 39. *Evaluate* $\int \frac{dx}{a^2 - b^2\cos^2 x}$, $a > b$.

Sol. We have $\int \frac{dx}{a^2 - b^2\cos^2 x}$

$$= \int \frac{dx}{a^2(\sin^2 x + \cos^2 x) - b^2 \cos^2 x} \qquad \textbf{(Note)}$$

$$= \int \frac{dx}{(a^2 - b^2)\cos^2 x + a^2 \sin^2 x} = \frac{1}{a^2} \int \frac{\sec^2 x \, dx}{\{(a^2 - b^2)/a^2\} + \tan^2 x}$$

$$= \frac{1}{a^2} \int \frac{dt}{\{(a^2 - b^2)/a^2\} + t^2},$$

putting $\tan x = t$ so that $\sec^2 x \, dx = dt$

$$= \frac{1}{a^2} \cdot \frac{a}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left\{ t \cdot \frac{a}{\sqrt{(a^2 - b^2)}} \right\}$$

$$= \frac{1}{a\sqrt{(a^2 - b^2)}} \tan^{-1} \left\{ \tan x \cdot \frac{a}{\sqrt{(a^2 - b^2)}} \right\}.$$

Ex. 40. *Evaluate* $\displaystyle\int \frac{dx}{a + b \cosh x}$.

Sol. Use $\cosh x = \cosh^2 \frac{1}{2}x + \sinh^2 \frac{1}{2}x$ and $\cosh^2 \frac{1}{2}x - \sinh^2 \frac{1}{2}x = 1$ and proceed exactly as in § 4·3 page 141. Discuss both the cases *i.e.*, when $a < b$ and when $a > b$.

Here we get $\displaystyle\int \frac{dx}{a + b \cosh x}$

$$= -\frac{1}{\sqrt{(a^2 - b^2)}} \log \left\{ \frac{\sqrt{(a-b)} \tanh \frac{1}{2}x - \sqrt{(a+b)}}{\sqrt{(a-b)} \tanh \frac{1}{2}x + \sqrt{(a+b)}} \right\}, \text{ when } a > b$$

or $$= \frac{2}{\sqrt{(b^2 - a^2)}} \tan^{-1} \left\{ \sqrt{\left(\frac{b-a}{b+a}\right)} \tanh \frac{1}{2}x \right\}, \text{ when } a < b.$$

§ 4·4. Integration of $1/(a + b \sin x)$.

We have $\displaystyle\int \frac{dx}{(a + b \sin x)}$

$$= \int \frac{dx}{a(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + b(2 \sin \frac{1}{2}x \cos \frac{1}{2}x)}$$

$$= \int \frac{\sec^2 \frac{1}{2}x \, dx}{a(1 + \tan^2 \frac{1}{2}x) + b(2 \tan \frac{1}{2}x)},$$

dividing Nr. & Dr. by $\cos^2 \frac{1}{2}x$

$$= 2 \int \frac{dt}{(at^2 + 2bt + a)},$$

putting $\tan \frac{1}{2}x = t$, so that $\frac{1}{2}\sec^2 \frac{1}{2}x \, dx = dt$

$$= \frac{2}{a} \int \frac{dt}{t^2 + (2b/a)t + 1} = \frac{2}{a} \int \frac{dt}{\{t + (b/a)\}^2 + 1 - (b^2/a^2)}$$

$$= \frac{2}{a} \int \frac{dt}{\{t + (b/a)\}^2 + \{(a^2 - b^2)/a^2\}} . \qquad ...(1)$$

Now two cases arise viz. $a > b$ or $a < b$.

Case I. $a > b$, i.e., $\{(a^2 - b^2)/a^2\}$ is positive.

Then $$\int \frac{dx}{a + b \sin x} = \frac{2}{a} \cdot \frac{a}{\sqrt{(a^2 - b^2)}} \cdot \tan^{-1} \left[\frac{\{t + (b/a)\}}{\sqrt{\{(a^2 - b^2)/a^2\}}} \right],$$ from (1)

$$= \frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left\{ \frac{at + b}{\sqrt{(a^2 - b^2)}} \right\}$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}} \tan^{-1} \left\{ \frac{a \tan \frac{1}{2}x + b}{\sqrt{(a^2 - b^2)}} \right\}.$$

Case II. $a < b$, i.e., $\{(a^2 - b^2)/a^2\}$ is negative.

Then $$\int \frac{dx}{a + b \sin x} = \frac{2}{a} \int \frac{dt}{\{t + (b/a)\}^2 - \{(b^2 - a^2)/a^2\}},$$ from (1)

$$= \frac{2}{a} \cdot \frac{1}{2} \sqrt{\left\{ \frac{a^2}{(b^2 - a^2)} \right\}} \log \left[\frac{\{t + (b/a)\} - \sqrt{\{(b^2 - a^2)/a^2\}}}{\{t + (b/a)\} + \sqrt{\{(b^2 - a^2)/a^2\}}} \right]$$

$$= \frac{1}{\sqrt{(b^2 - a^2)}} \log \left\{ \frac{at + b - \sqrt{(b^2 - a^2)}}{at + b + \sqrt{(b^2 - a^2)}} \right\}$$

$$= \frac{1}{\sqrt{(b^2 - a^2)}} \log \left\{ \frac{a \tan (\frac{1}{2}x) + b - \sqrt{(b^2 - a^2)}}{a \tan (\frac{1}{2}x) + b + \sqrt{(b^2 - a^2)}} \right\}.$$

§ 4·5. To evaluate $\int \frac{dx}{a \sin x + b \cos x}$.

Sol. The given integral

$$= \int \frac{dx}{a\,(2 \sin \frac{1}{2}x \cos \frac{1}{2}x) + b\,(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$$

$$= \int \frac{\sec^2 (\frac{1}{2}x)\, dx}{2a \tan \frac{1}{2}x + b\,(1 - \tan^2 \frac{1}{2}x)},$$ dividing Nr. and Dr. by $\cos^2 \frac{1}{2}x$

$$= \int \frac{2\, dt}{2at + b - bt^2},$$ putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2} \sec^2 \frac{1}{2}x\, dx = dt$

$$= \frac{2}{b} \int \frac{dt}{1 - t^2 + (2a/b)\, t} = \frac{2}{b} \int \frac{dt}{1 - \{t^2 - (2a/b)\, t\}}$$

$$= \frac{2}{b} \int \frac{dt}{\{1 + (a^2/b^2)\} - \{t - (a/b)\}^2}$$

$$= \frac{2}{b} \int \frac{dt}{\{(b^2 + a^2)/b^2\} - \{t - (a/b)\}^2}$$

$$= \frac{2}{b} \cdot \frac{1}{2} \sqrt{\left\{\frac{b^2}{(b^2+a^2)}\right\}} \log \left[\frac{\sqrt{\{(b^2+a^2)/b^2\}} + \{t-(a/b)\}}{\sqrt{\{(b^2+a^2)/b^2\}} - \{t-(a/b)\}}\right]$$

$$= \frac{1}{\sqrt{(b^2+a^2)}} \log \left[\frac{\sqrt{(b^2+a^2)} - a + bt}{\sqrt{(b^2+a^2)} + a - bt}\right]$$

$$= \frac{1}{\sqrt{(b^2+a^2)}} \log \left[\frac{\sqrt{(b^2+a^2)} - a + b \tan \frac{1}{2}x}{\sqrt{(b^2+a^2)} + a - b \tan \frac{1}{2}x}\right].$$

§ 4·6. To evaluate $\displaystyle\int \frac{dx}{a + b \cos x + c \sin x}$.

Sol. The given integral

$$I = \int \frac{dx}{a(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + b(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x) + 2c \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

(Note)

$$= \int \frac{dx}{(a-b)\sin^2 \frac{1}{2}x + 2c \sin \frac{1}{2}x \cos \frac{1}{2}x + (a+b)\cos^2 \frac{1}{2}x}.$$

Now dividing the numerator and the denominator by $\cos^2 \frac{1}{2}x$, we get

$$I = \int \frac{\sec^2 \frac{1}{2}x \, dx}{(a-b)\tan^2 \frac{1}{2}x + 2c \tan \frac{1}{2}x + (a+b)}$$

$$= \frac{1}{(a-b)} \int \frac{\sec^2 \frac{1}{2}x \, dx}{\tan^2 \frac{1}{2}x + \{2c/(a-b)\} \tan \frac{1}{2}x + \{(a+b)/(a-b)\}}$$

$$= \frac{2}{a-b} \int \frac{dt}{t^2 + \{2c/(a-b)\}t + \{(a+b)/(a-b)\}},$$

putting $\tan \frac{1}{2}x = t$, $\frac{1}{2} \sec^2 \frac{1}{2}x \, dx = dt$.

The integral can now be evaluated by the methods already discussed.

§ 4·7. Integration of $\dfrac{P \cos x + Q \sin x + R}{a \cos x + b \sin x + c}$.

To integrate such a fraction we express the numerator in the form

Numerator $= A$ (Deno.) $+ B$ (diff. coeff. of Deno.) $+ C$,

where A, B and C are constants.

Thus we write $\quad P \cos x + Q \sin x + R$

$$= A(a \cos x + b \sin x + c) + B(-a \sin x + b \cos x) + C.$$

Comparing the coefficients of $\cos x, \sin x$ and constant terms on both sides, we have

$P = Aa + Bb$ ...(1), $Q = Ab - Ba$ ...(2), $R = Ac + C$. ...(3)

Whence $A = \dfrac{Pa + Qb}{b^2 + a^2}$, $B = \dfrac{Pb - Qa}{b^2 + a^2}$ and $C = R - \dfrac{(Pa + Qb)c}{b^2 + a^2}$.

Now writing the numerator in the form mentioned above, the given integral becomes

$$I = \int \frac{A\,(\text{Dr.}) + B\,(\text{diff. coeff. of Dr.}) + C}{\text{Dr.}}\,dx$$

$$= A \int dx + B \int \frac{-a \sin x + b \cos x}{a \cos x + b \sin x + c}\,dx + C \int \frac{dx}{a \cos x + b \sin x + c}$$

$$= Ax + B \log (a \cos x + b \sin x + c) + C \int \frac{dx}{a \cos x + b \sin x + c}.$$

The last integral can now be evaluated by the methods discussed earlier.

Cor. $\displaystyle\int \frac{P \cos x + Q \sin x}{a \cos x + b \sin x}\,dx$ is a particular case of § 4·7. Here neither the numerator nor the denominator contains a constant term. So here we express the numerator in the form

Nr. $= A$ (Dr.) $+ B$ (diff. coeff. of Dr.).

***§ 4·8. Integration of $1/(a + b \tan x)$.** **(Raj. 1977)**

Let $I = \displaystyle\int \frac{dx}{a + b \tan x} = \int \frac{\cos x\,dx}{a \cos x + b \sin x}$.

Now we express the numerator $\cos x$ in the form

Nr. $= A$ (Dr.) $+ B$ (diff. coeff. of Dr.)

i.e., $\cos x = A\,(a \cos x + b \sin x) + B\,(-a \sin x + b \cos x)$.

Equating the coefficients of $\cos x$ and $\sin x$ on both sides, we get

$Aa + Bb = 1$, and $Ab - Ba = 0$.

Solving these equations, we get

$A = a/(a^2 + b^2)$ and $B = b/(a^2 + b^2)$.

Now $I = \displaystyle\int \frac{A\,(a \cos x + b \sin x) + B\,(-a \sin x + b \cos x)}{a \cos x + b \sin x}\,dx$

$$= A \int dx + B \int \frac{-a \sin x + b \cos x}{a \cos x + b \sin x}\,dx$$

$$= Ax + B \log (a \cos x + b \sin x)$$

$$= \left\{\frac{a}{a^2 + b^2}\right\} x + \frac{b}{a^2 + b^2} \log (a \cos x + b \sin x).$$

****Ex. 41.** *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{dx}{4 + 5 \sin x}$. **(Meerut 1983; Magadh 77)**

Sol. We have $I = \int_0^{\pi/2} \frac{dx}{4 + 5\sin x}$

$$= \int_0^{\pi/2} \frac{dx}{4(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + 5 \cdot 2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x + 5 \sin \frac{1}{2}x \cos \frac{1}{2}x + 2 \sin^2 \frac{1}{2}x}$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2}x \, dx}{2 + 5 \tan \frac{1}{2}x + 2 \tan^2 \frac{1}{2}x},$$

dividing Nr. and Dr. by $\cos^2 \frac{1}{2}x$.

Now put $\tan \frac{1}{2}x = t$ so that $\frac{1}{2} \sec^2 \frac{1}{2}x \, dx = dt$. When $x = 0$, $t = \tan 0 = 0$, when $x = \pi/2$, $t = \tan \frac{1}{4}\pi = 1$.

$$\therefore \quad I = \int_0^1 \frac{dt}{2t^2 + 5t + 2}$$

$$= \frac{1}{2} \int_0^1 \frac{dt}{(t^2 + \frac{5}{2}t + 1)} = \frac{1}{2} \int_0^1 \frac{dt}{(t + \frac{5}{4})^2 - \frac{9}{16}}$$

$$= \frac{1}{2} \frac{1}{2 \cdot (\frac{3}{4})} \left\{ \log \frac{(t + \frac{5}{4}) - \frac{3}{4}}{(t + \frac{5}{4}) + \frac{3}{4}} \right\}_0^1 = \frac{1}{3} \left[\log \frac{t + \frac{1}{2}}{t + \frac{8}{4}} \right]_0^1$$

$$= \frac{1}{3} \left[\log \frac{2t + 1}{2t + 4} \right]_0^1$$

$$= \frac{1}{3} (\log \tfrac{1}{2} - \log \tfrac{1}{4}) = \frac{1}{3} \log (\tfrac{1}{2} \div \tfrac{1}{4})$$

$$= \frac{1}{3} \log (\tfrac{1}{2} \times 4) = \frac{1}{3} \log 2.$$

***Ex. 42.** *Evaluate* $\int_0^{\pi/2} \frac{dx}{5 + 4 \sin x}$.

(Meerut 1981, 83 S, 87 S; Ranchi 76)

Sol. We have $\int_0^{\pi/2} \frac{dx}{5 + 4 \sin x}$

$$= \int_0^{\pi/2} \frac{dx}{5(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + 4 \cdot 2 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2}x \, dx}{5 + 8 \tan \frac{1}{2}x + 5 \tan^2 \frac{1}{2}x},$$

dividing Nr. and Dr. by $\cos^2 \frac{1}{2}x$

$$= \int_0^1 \frac{2\,dt}{5 + 8t + 5t^2},$$

[putting $\tan \frac{1}{2}x = t$ and changing the limits]

$$= \int_0^1 \frac{2\,dt}{5\,(t^2 + \frac{8}{5}t + 1)} = \int_0^1 \frac{2\,dt}{5\,[(t + \frac{4}{5})^2 + \frac{9}{25}]}$$

$$= \frac{2}{5}\left[\frac{5}{3}\tan^{-1}\left\{\frac{(t + \frac{4}{5})}{\frac{3}{5}}\right\}\right]_0^1 = \frac{2}{3}\left[\tan^{-1}\frac{5t + 4}{3}\right]_0^1$$

$$= \tfrac{2}{3}\,[\tan^{-1} 3 - \tan^{-1} \tfrac{4}{3}].$$

Ex. 43. *Evaluate* $\int \frac{dx}{3 \sin x + 4 \cos x}$.

Sol. Given integral

$$= \int \frac{dx}{3 \cdot 2 \sin \frac{1}{2}x \cos \frac{1}{2}x + 4\,(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$$

$$= \int \frac{\sec^2 \frac{1}{2}x\,dx}{4 + 6 \tan \frac{1}{2}x - 4 \tan^2 \frac{1}{2}x},$$

dividing the Nr. and the Dr. by $\cos^2 \frac{1}{2}x$

$$= \int \frac{dt}{2 + 3t - 2t^2},$$

putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\,dx = dt$

$$= \frac{1}{2}\int \frac{dt}{1 - (t^2 - \frac{3}{2}t)} = \frac{1}{2}\int \frac{dt}{1 - (t - \frac{3}{4})^2 + \frac{9}{16}}$$

$$= \frac{1}{2}\int \frac{dt}{\frac{25}{16} - (t - \frac{3}{4})^2} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{5}\log\frac{\frac{5}{4} + (t - \frac{3}{4})}{\frac{5}{4} - (t - \frac{3}{4})}$$

$$= \frac{1}{5}\log\frac{1 + 2t}{4 - 2t} = \frac{1}{5}\log\frac{1 + 2\tan \frac{1}{2}x}{4 - 2\tan \frac{1}{2}x}.$$

Ex. 44. *Evaluate* $\int \frac{dx}{5 \sin x + 12 \cos x}$.

Sol. The given integral

$$= \int \frac{dx}{5\,.2 \sin \frac{1}{2}x \cos \frac{1}{2}x + 12\,(\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$$

$$= \frac{1}{12}\int \frac{\sec^2 \frac{1}{2}x\,dx}{1 + \frac{5}{6}\tan \frac{1}{2}x - \tan^2 \frac{1}{2}x} = \frac{1}{6}\int \frac{dt}{1 + \frac{5}{6}t - t^2},$$

putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x\,dx = dt$

$$= \frac{1}{6} \int \frac{dt}{\frac{169}{144} - (t - \frac{5}{12})^2} = \frac{1}{6} \cdot \frac{1}{2} \cdot \frac{12}{13} \log \frac{\frac{13}{12} + (t - \frac{5}{12})}{\frac{13}{12} - (t - \frac{5}{12})}$$

$$= \frac{1}{13} \log \left\{ \frac{\frac{8}{12} + t}{\frac{18}{12} - t} \right\} = \frac{1}{13} \log \left(\frac{8 + 12t}{18 - 12t} \right)$$

$$= \frac{1}{13} \log \left\{ \frac{2}{3} \left(\frac{2 + 3t}{3 - 2t} \right) \right\} = \frac{1}{13} \log \left\{ \frac{2}{3} \left(\frac{2 + 3 \tan \frac{1}{2}x}{3 - 2 \tan \frac{1}{2}x} \right) \right\}.$$

****Ex. 45. (a).** *Show that* $\displaystyle\int_0^{\pi} \frac{dx}{3 + 2 \sin x + \cos x} = \frac{\pi}{4}.$

(Delhi 1979; Vikram 76; Meerut 86 S)

Sol. The given integral

$$= \int_0^{\pi} \frac{dx}{3 (\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x) + 2 (2 \sin \frac{1}{2}x \cos \frac{1}{2}x) + (\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x)}$$

$$= \int_0^{\pi} \frac{dx}{4 \cos^2 \frac{1}{2}x + 2 \sin^2 \frac{1}{2}x + 4 \sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \int_0^{\pi} \frac{\sec^2 (\frac{1}{2}x)\, dx}{4 + 2 \tan^2 \frac{1}{2}x + 4 \tan \frac{1}{2}x},$$

dividing the Nr. and the Dr. by $\cos^2 \frac{1}{2}x$

$$= \int_0^{\pi} \frac{\frac{1}{2} \sec^2 \frac{1}{2}x\, dx}{\tan^2 \frac{1}{2}x + 2 \tan \frac{1}{2}x + 2}$$

$$= \int_0^{\infty} \frac{dt}{t^2 + 2t + 2},$$

putting $\tan \frac{1}{2}x = t$ and changing the limits

$$= \int_0^{\infty} \frac{dt}{(t + 1)^2 + 1} = \left[\tan^{-1} (t + 1) \right]_0^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Ex. 45. (b). *Show that* $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + 2 \sin x + \cos x} = \frac{1}{2} \log 3.$

(Meerut 1988)

Sol. The given integral $I = \displaystyle\int_0^{\pi/2} \frac{dx}{(1 + \cos x) + 2 \sin x}$

$$= \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x + 4\sin \frac{1}{2}x \cos \frac{1}{2}x}$$

$$= \int_0^{\pi/2} \frac{\sec^2 \frac{1}{2}x\, dx}{2(1 + 2\tan \frac{1}{2}x)}, \text{ dividing the Nr. and the Dr. by } \cos^2 \frac{1}{2}x$$

$$= \int_0^1 \frac{dt}{1+2t}, \text{ putting } \tan \frac{x}{2} = t \text{ so that } \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt$$

$$= \tfrac{1}{2}\left[\log(1+2t)\right]_0^1 = \tfrac{1}{2}(\log 3 - \log 1) = \tfrac{1}{2}\log 3.$$

Ex. 46. (a). *Evaluate* $\int \frac{3\sin x + 4\cos x}{\sin x + \cos x}\, dx$. **(Meerut 1984 S)**

Sol. Here we put

Numerator ≡ A. (deno.) + B. (diff. coeff. of deno.)

i.e., $3\sin x + 4\cos x \equiv A(\sin x + \cos x) + B(\cos x - \sin x)$.

Equating the coefficients of $\sin x$ and $\cos x$ on both sides, we have

$3 = A - B$ and $4 = A + B$; whence $A = \frac{7}{2}$ and $B = \frac{1}{2}$.

∴ the given integral

$$= \int \frac{\frac{7}{2}(\sin x + \cos x) + \frac{1}{2}(\cos x - \sin x)}{\sin x + \cos x}\, dx$$

$$= \frac{7}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x}\, dx = \frac{7}{2}x + \frac{1}{2}\log(\sin x + \cos x).$$

Ex. 46. (b). *Evaluate* $\int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\, dx$.

Sol. Here we put

$(\sin x + 2\cos x) \equiv A(2\sin x + \cos x) + B(2\cos x - \sin x)$.

Equating the coefficients of $\sin x$ and $\cos x$ on both the sides, we have $1 = 2A - B$ and $2 = A + 2B$; whence $A = \frac{4}{5}$ and $B = \frac{3}{5}$.

∴ the given integral

$$= \int \frac{\frac{4}{5}(2\sin x + \cos x) + \frac{3}{5}(2\cos x - \sin x)}{2\sin x + \cos x}\, dx$$

$$= \frac{4}{5}\int dx + \frac{3}{5}\int \frac{(2\cos x - \sin x)}{2\sin x + \cos x}\, dx = \frac{4}{5}x + \frac{3}{5}\log(2\sin x + \cos x).$$

****Ex. 46 (c).** *Evaluate* $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$.

(Meerut 1981, 82, 83S, 87P; Delhi 83)

Sol. Here we put

$2\sin x + 3\cos x \equiv A(3\sin x + 4\cos x) + B(3\cos x - 4\sin x)$.

Equating the coefficients of $\sin x$ and $\cos x$ on both the sides, we have

$2 = 3A - 4B$ and $3 = 4A + 3B$; whence $A = \frac{18}{25}$ and $B = \frac{1}{25}$.

$\therefore$ the given integral

$$= \int \frac{\frac{18}{25}(3\sin x + 4\cos x) + \frac{1}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x}\,dx$$

$$= \frac{18}{25}\int dx + \frac{1}{25}\int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\,dx$$

$$= \tfrac{18}{25}x + \tfrac{1}{25}\log(3\sin x + 4\cos x).$$

Ex. 46 (d). *Evaluate* $\int \frac{17\cos x - 6\sin x}{3\sin x + 4\cos x}\,dx.$

Sol. Proceeding as above, the given integral

$$= 2x + 3\log(3\sin x + 4\cos x).$$

Ex. 47. *Evaluate* $\int \frac{\cos x\,dx}{2\sin x + 3\cos x}.$

Sol. Here we put

Numerator $\equiv A$. (deno.) $+ B$. (diff. coeff. of deno.)

i.e., $\cos x \equiv A(2\sin x + 3\cos x) + B(2\cos x - 3\sin x)$.

Equating the coefficients of $\sin x$ and $\cos x$ on both the sides, we have

$$0 = 2A - 3B \text{ and } 1 = 3A + 2B; \text{ whence } A = \tfrac{3}{13} \text{ and } B = \tfrac{2}{13}.$$

$\therefore$ the given integral

$$= \int \frac{\frac{3}{13}(2\sin x + 3\cos x) + \frac{2}{13}(2\cos x - 3\sin x)}{2\sin x + 3\cos x}\,dx$$

$$= \frac{3}{13}\int dx + \frac{2}{13}\int \frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}\,dx$$

$$= \tfrac{3}{13}x + \tfrac{2}{13}\log(2\sin x + 3\cos x).$$

Ex. 48. *Evaluate* $\int \frac{\cos x\,dx}{3\cos x + 4\sin x}.$

Sol. Proceeding as above, the given integral

$$= \tfrac{3}{25}x + \tfrac{4}{25}\log(3\cos x + 4\sin x).$$

Ex. 49. *Evaluate* $\int \frac{3 + 4\sin x + 2\cos x}{3 + 2\sin x + \cos x}\,dx.$

Sol. Here we put

Numerator $\equiv A$. (deno.) $+ B$. (diff. coeff. of deno.) $+ C$.

Thus let $3 + 4\sin x + 2\cos x$

$$\equiv A(3 + 2\sin x + \cos x) + B.(2\cos x - \sin x) + C. \qquad \text{...(1)}$$

Equating the coefficients of $\sin x$, $\cos x$ and constant terms on both sides, we have $4 = 2A - B$, $2 = A + 2B$, and $3 = C + 3A$.

Solving these equations, we have

$$A = 2,\ B = 0 \text{ and } C = -3.$$

$\therefore$ the given integral

$$= \int \frac{2(3+2\sin x+\cos x)\,dx}{3+2\sin x+\cos x} - \int \frac{3\,dx}{3+2\sin x+\cos x}$$

$$= \int 2\,dx$$

$$- \int \frac{3\,dx}{3(\cos^2\tfrac{1}{2}x+\sin^2\tfrac{1}{2}x)+4\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x+\cos^2\tfrac{1}{2}x-\sin^2\tfrac{1}{2}x}$$

$$= 2\int dx - \int \frac{3\,dx}{4\cos^2\tfrac{1}{2}x+4\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x+2\sin^2\tfrac{1}{2}x}$$

$$= 2x - \int \frac{3\sec^2\tfrac{1}{2}x\,dx}{4+4\tan\tfrac{1}{2}x+2\tan^2\tfrac{1}{2}x}$$

$$= 2x - 3\int \frac{\tfrac{1}{2}\sec^2\tfrac{1}{2}x\,dx}{2+2\tan\tfrac{1}{2}x+\tan^2\tfrac{1}{2}x}$$

$$= 2x - 3\int \frac{dt}{t^2+2t+2},$$

putting $\tan\frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2\frac{1}{2}x\,dx = dt$

$$= 2x - 3\int \frac{dt}{(t+1)^2+1} = 2x - 3\tan^{-1}(t+1)$$

$$= 2x - 3\tan^{-1}(1+\tan\tfrac{1}{2}x).$$

Ex. 50 (a). *Evaluate* $\displaystyle\int \frac{\sin x+\cos x}{3\sin x+4\cos x+1}\,dx.$

Sol. Let $\sin x+\cos x \equiv A(3\sin x+4\cos x+1)$

$$+ B(3\cos x-4\sin x)+C.$$

Equating the coefficients of $\sin x$, $\cos x$ and constant terms on both sides, we have

$$1 = 3A-4B,\; 1 = 4A+3B, \text{ and } 0 = A+C.$$

Solving, we have $A = \frac{7}{25}$, $B = -\frac{1}{25}$ and $C = -A = -\frac{7}{25}$.

$\therefore$ the given integral $\displaystyle= \frac{7}{25}\int dx - \frac{1}{25}\int \frac{3\cos x-4\sin x}{3\sin x+4\cos x+1}\,dx$

$$- \frac{7}{25}\int \frac{dx}{3\sin x+4\cos x+1}$$

$= \frac{7}{25}x - \frac{1}{25}\log(3\sin x+4\cos x+1) - \frac{7}{25}I,$ **(Note)**

where $\displaystyle I = \int \frac{dx}{3\sin x+4\cos x+1}$

$\displaystyle = \int \frac{dx}{5\cos^2\tfrac{1}{2}x+6\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x-3\sin^2\tfrac{1}{2}x}$ **(Note)**

$$= \int \frac{\sec^2\tfrac{1}{2}x\,dx}{5+6\tan\tfrac{1}{2}x-3\tan^2\tfrac{1}{2}x} = \int \frac{2\,dt}{5+6t-3t^2},$$

putting $\tan \frac{1}{2}x = t$ so that $\frac{1}{2}\sec^2 \frac{1}{2}x \, dx = dt$

$$= \frac{1}{3} \int \frac{dt}{\frac{8}{3} - (t-1)^2} = \frac{1}{3} \cdot \frac{1}{2 \cdot \sqrt{(\frac{8}{3})}} \log \frac{\sqrt{(\frac{8}{3})} + (t-1)}{\sqrt{(\frac{8}{3})} - (t-1)}$$

$$= \frac{1}{4\sqrt{6}} \log \frac{\sqrt{(\frac{8}{3})} - 1 + \tan \frac{1}{2}x}{\sqrt{(\frac{8}{3})} + 1 - \tan \frac{1}{2}x}.$$

Ex. 50 (b). *Evaluate* $\int \frac{5 \cos x + 6}{2 \cos x + \sin x + 3} dx.$ **(Meerut 1981 S)**

Sol. Let

$5 \cos x + 6 \equiv A (2 \cos x + \sin x + 3) + B (-2 \sin x + \cos x) + C.$

Equating the coefficients of $\sin x$, $\cos x$ and constant terms on both sides, we have

$$5 = 2A + B, \; 0 = A - 2B, \text{ and } 6 = 3A + C.$$

Solving these, we have

$$A = 2, B = 1 \text{ and } C = 0.$$

$\therefore$ the given integral

$$= \int \frac{2 (2 \cos x + \sin x + 3) + 1 (-2 \sin x + \cos x)}{2 \cos x + \sin x + 3} dx$$

$$= 2 \int dx + \int \frac{-2 \sin x + \cos x}{2 \cos x + \sin x + 3} dx$$

$$= 2x + \log (2 \cos x + \sin x + 3).$$

Ex. 50 (c). *Evaluate* $\int \frac{2 + 3 \cos x}{\sin x + 2 \cos x + 3} dx.$ **(Meerut 1989 S)**

Sol. Do yourself.

Ans. $\frac{6}{5}x + \frac{3}{5} \log (\sin x + 2 \cos x + 3) - \frac{8}{5} \tan^{-1} \left\{\frac{1}{3}(1 + \tan \frac{1}{2}x)\right\}$

Ex. 51. *Evaluate* $\int \cos 2x \log (1 + \tan x) \, dx.$

Sol. Integrating by parts taking $\cos 2x$ as the 2nd function, the given integral

$$= \{\log (1 + \tan x)\} \frac{\sin 2x}{2} - \int \frac{\sec^2 x}{1 + \tan x} \cdot \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2} \sin 2x \log (1 + \tan x) - \int \frac{\sin x}{\sin x + \cos x} dx.$$

Now $\int \frac{\sin x \, dx}{\sin x + \cos x}$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x) - (\cos x - \sin x)}{\sin x + \cos x} dx, \qquad \textbf{(Note)}$$

$$= \frac{1}{2} \int \left[1 - \frac{\cos x - \sin x}{\sin x + \cos x}\right] dx = \frac{1}{2} [x - \log (\sin x + \cos x)].$$

Hence the given integral

$$= \frac{1}{2} \sin 2x \log (1 + \tan x) - \frac{1}{2} [x - \log (\sin x + \cos x)].$$

Ex. 52. *Evaluate* $\int \frac{\sin x}{\sin x + \cos x} dx$. **(Meerut 1990)**

Sol. The complete solution of this problem has been given in **Ex.** 51.

Ans. $\frac{1}{2}[x - \log(\sin x + \cos x)]$.

Ex. 53. *Evaluate* $\int \frac{1}{1 + \tan x} dx$.

Sol. The given integral $I = \int \frac{1}{1 + (\sin x / \cos x)} dx$

$$= \int \frac{\cos x}{(\cos x + \sin x)} dx$$

$$= \frac{1}{2} \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{\cos x + \sin x} dx$$

$$= \frac{1}{2} \int \left[1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right] dx$$

$$= \tfrac{1}{2}[x + \log(\sin x + \cos x)].$$

Ex. 54. *Evaluate* $\int \frac{dx}{x + \sqrt{(a^2 - x^2)}}$. **(Meerut 1984)**

Sol. Let $I = \int \frac{dx}{x + \sqrt{(a^2 - x^2)}}$.

Put $x = a \sin\theta$, so that $dx = a\cos\theta \, d\theta$.

Then $I = \int \frac{a\cos\theta \, d\theta}{a\sin\theta + a\cos\theta} = \int \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta$

$= \frac{1}{2}[\theta + \log(\sin\theta + \cos\theta)]$, proceeding as in Ex. 53

$= \frac{1}{2}\theta + \frac{1}{2}\log[\sin\theta + \sqrt{(1 - \sin^2\theta)}]$

$$= \tfrac{1}{2}\sin^{-1}\frac{x}{a} + \frac{1}{2}\log\left[\frac{x}{a} + \sqrt{\left(1 - \frac{x^2}{a^2}\right)}\right]$$

$$= \tfrac{1}{2}\sin^{-1}\frac{x}{a} + \tfrac{1}{2}\log[x + \sqrt{(a^2 - x^2)}] - \tfrac{1}{2}\log a$$

$= \frac{1}{2}\sin^{-1}\frac{x}{a} + \frac{1}{2}\log[x + \sqrt{(a^2 - x^2)}]$, the constant term $-\frac{1}{2}\log a$ may be added to the constant of integration c.

Some Miscellaneous Exercises

Ex. 1. *Evaluate* $\int \frac{x^4 \, dx}{(1 + x^2)^2}$.

Sol. The given integral $I = \int \frac{(x^4 - 1) + 1}{(1 + x^2)^2} dx$

$$= \int \frac{(x^2-1)(x^2+1)}{(1+x^2)^2}\,dx + \int \frac{1}{(1+x^2)^2}\,dx$$

$$= \int \frac{x^2-1}{x^2+1}\,dx + \int \frac{1}{(1+x^2)^2}\,dx$$

$$= \int \frac{(x^2+1)-2}{x^2+1}\,dx + \int \frac{1}{(1+x^2)^2}\,dx$$

$$= \int dx - 2\int \frac{1}{1+x^2}\,dx + \int \frac{1}{(1+x^2)^2}\,dx$$

$$= x - 2\tan^{-1}x + \int \frac{1}{(1+x^2)^2}\,dx.$$

Now put $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$.

Then $I = x - 2\tan^{-1}x + \displaystyle\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$

$$= x - 2\tan^{-1}x + \int \cos^2\theta\, d\theta$$

$$= x - 2\tan^{-1}x + \tfrac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= x - 2\tan^{-1}x + \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta)$$

$$= x - 2\tan^{-1}x + \tfrac{1}{2}\theta + \tfrac{1}{4}\cdot\frac{2\tan\theta}{1+\tan^2\theta}$$

$$= x - 2\tan^{-1}x + \tfrac{1}{2}\tan^{-1}x + \tfrac{1}{2}\frac{x}{1+x^2}$$

$$= x - \frac{3}{2}\tan^{-1}x + \tfrac{1}{2}\frac{x}{1+x^2}.$$

Ex. 2. *Evaluate* $\int x^{-n}\log x\, dx$. **(Meerut 1982 S)**

Sol. Let $I = \int x^{-n}\log x\, dx$.

Integrating by parts taking x^{-n} as the second function and $\log x$ as the first function, we get

$$I = \frac{x^{-n+1}}{(-n+1)}\log x - \int \frac{x^{-n+1}}{(-n+1)}\frac{1}{x}\,dx$$

$$= \frac{1}{(1-n)x^{n-1}}\log x - \frac{1}{(1-n)}\int x^{-n}\,dx$$

$$= \frac{1}{(1-n)x^{n-1}}\log x - \frac{1}{(1-n)}\frac{x^{-n+1}}{(-n+1)}$$

$$= \frac{\log x}{(1-n)x^{n-1}} - \frac{1}{(1-n)^2x^{n-1}} = \frac{1}{(1-n)x^{n-1}}\left[\log x - \frac{1}{(1-n)}\right]$$

Ex. 3. *Evaluate* $\int x^3(\log x)^2\, dx$. **(Agra 1983)**

Sol. Let $I = \int x^3(\log x)^2\, dx$.

Integrating by parts taking x^3 as the second function and $(\log x)^2$ as the first function, we get

$$I=\frac{1}{4}x^4(\log x)^2-\int\frac{1}{4}x^4\,.\,2(\log x)\,.\,\frac{1}{x}\,dx$$
$$=\frac{1}{4}x^4(\log x)^2-\frac{1}{2}\int x^3\log x\,dx.$$

Again integrating by pratrs taking x^3 as the second function, we get

$$I=\frac{1}{4}x^4(\log x)^2-\frac{1}{2}\left[\frac{1}{4}x^4\log x-\int\frac{1}{4}\cdot x^4\cdot\frac{1}{x}\,dx\right]$$
$$=\tfrac{1}{4}x^4(\log x)^2-\tfrac{1}{8}x^4\log x+\tfrac{1}{8}\int x^3\,dx$$
$$=\tfrac{1}{4}x^4(\log x)^2-\tfrac{1}{8}x^4\log x+\tfrac{1}{8}\cdot\tfrac{1}{4}x^4$$
$$=\tfrac{1}{32}x^4\,[8(\log x)^2-4\log x+1].$$

Ex. 4. *Evaluate* $\displaystyle\int\frac{\cosh\theta\,d\theta}{\sinh\theta+\cosh\theta}$.

(Meerut 1981 P)

Sol. The given integral $I=\displaystyle\int\frac{(e^\theta+e^{-\theta})/2}{\frac{1}{2}(e^\theta-e^{-\theta})+\frac{1}{2}(e^\theta+e^{-\theta})}\,d\theta$

$$=\int\frac{(e^\theta+e^{-\theta})/2}{e^\theta}\,d\theta=\frac{1}{2}\int(1+e^{-2\theta})\,d\theta$$
$$=\frac{1}{2}\left[\theta+\frac{e^{-2\theta}}{-2}\right]=\frac{1}{2}\left[\theta-\frac{1}{2e^{2\theta}}\right]=\frac{2\theta e^{2\theta}-1}{4e^{2\theta}}.$$

Ex. 5. *Show that* $\displaystyle\int_0^\infty\frac{dx}{[x+\sqrt{(1+x^2)}]^n}=\frac{n}{n^2-1}$.

(Lucknow 1982, 79)

Sol. Let $I=\displaystyle\int_0^\infty\frac{dx}{[x+\sqrt{(1+x^2)}]^n}$.

Put $x=\sinh t$ so that $dx=\cosh t\,dt$.
Also when $x=0$, $\sinh t=0$ *i.e.*, $t=0$ and when $x=\infty$, $\sinh t=\infty$, *i.e.*, $t=\infty$.

$$\therefore\quad I=\int_0^\infty\frac{\cosh t\,dt}{(\sinh t+\cosh t)^n}=\int_0^\infty\frac{\frac{1}{2}(e^t+e^{-t})}{e^{nt}}\,dt,$$

$[\because \sinh t+\cosh t=e^t]$

$$=\frac{1}{2}\int_0^\infty[e^{-(n-1)t}+e^{-(n+1)t}]\,dt$$
$$=\frac{1}{2}\left[\frac{e^{-(n-1)t}}{-(n-1)}+\frac{e^{-(n+1)t}}{-(n+1)}\right]_0^\infty=\frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{n+1}\right]=\frac{n}{n^2-1}.$$

Ex. 6. *Evaluate* $\displaystyle\int\frac{dx}{(a^2+b^2x^2)^{3/2}}$.

Sol. Put $bx = a \tan\theta$, so that $b\,dx = a\sec^2\theta\,d\theta$. Then the given integral

$$I = \int \frac{1}{(a^2 + a^2\tan^2\theta)^{3/2}} \frac{a}{b} \sec^2\theta\,d\theta = \frac{1}{a^2 b} \int \cos\theta\,d\theta$$

$$= \frac{1}{a^2 b} \sin\theta.$$

Now $\tan\theta = bx/a$ gives $\sin\theta = bx/\sqrt{(a^2 + b^2x^2)}$.

$$\therefore \quad I = \frac{1}{a^2 b} \frac{bx}{\sqrt{(a^2 + b^2x^2)}} = \frac{x}{a^2\sqrt{(a^2 + b^2x^2)}}.$$

Ex. 7. *Evaluate* $\int (ax^2 + c)^{-3/2}\,dx$.

Sol. The given integral $I = \int \frac{dx}{(ax^2 + c)^{3/2}}$.

Put $\sqrt{a}\,x = \sqrt{c}\tan\theta$ and proceed as in Ex. 6.

The answer is $\frac{x}{c\sqrt{(ax^2 + c)}}$.

Ex. 8. *Evaluate* $\int \frac{\sec x \operatorname{cosec} x}{\log \tan x}\,dx$.

Sol. Put $\log\tan x = t$, so that $\frac{1}{\tan x}\sec^2 x\,dx = dt$

i.e., $\sec x \operatorname{cosec} x\,dx = dt$.

Then the given integral

$$I = \int \frac{dt}{t} = \log t = \log(\log\tan x).$$

Ex. 9. *Evaluate the following integrals :*

(*i*) $\int \frac{x\,dx}{1 + \sin x}$ (*ii*) $\int \frac{x}{\operatorname{cosec} x + 1}\,dx$.

Sol. (i) The given integral

$$I = \int \frac{x\,dx}{1 + \sin x}$$

$$= \int \frac{x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)}\,dx = \int \frac{x(1 - \sin x)}{1 - \sin^2 x}\,dx$$

$$= \int x \left[\frac{1 - \sin x}{\cos^2 x}\right] dx$$

$$= \int x(\sec^2 x - \sec x\tan x)\,dx.$$

Integrating by parts taking x as the first function, we have

$$I = x(\tan x - \sec x) - \int (\tan x - \sec x)\,dx$$

$$= x(\tan x - \sec x) - \int \tan x\,dx + \int \sec x\,dx$$

$$= x(\tan x - \sec x) + \log\cos x + \log(\sec x + \tan x)$$

$$= x(\tan x - \sec x) + \log\{\cos x(\sec x + \tan x)\}$$

$$= x(\tan x - \sec x) + \log(1 + \sin x).$$

(ii) The given integral

$$I=\int \frac{x\sin x}{1+\sin x}\,dx=\int \frac{x\,(1+\sin x-1)}{1+\sin x}\,dx$$

$$=\int x\left[\frac{1+\sin x}{1+\sin x}-\frac{1}{1+\sin x}\right]dx$$

$$=\int x\left[1-\frac{1}{1+\sin x}\right]dx$$

$$=\int x\,dx-\int \frac{x}{1+\sin x}\,dx.$$

Now proceeding as in part (i), we get

$$I=\tfrac{1}{2}x^2-x\,(\tan x-\sec x)-\log\,(1+\sin x).$$

Ex. 10. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{x\,dx}{1+\cos x}$ (*ii*) $\displaystyle\int \frac{x\,dx}{\sec x+1}$.

Sol. (i) The given integral

$$I=\int \frac{x\,dx}{2\cos^2\frac{1}{2}x}=\frac{1}{2}\int x\sec^2\tfrac{1}{2}x\,dx.$$

Integrating by parts taking $\sec^2\frac{1}{2}x$ as the second function, we have

$$I=\tfrac{1}{2}\left[x\,.\,2\tan\tfrac{1}{2}x-\int 1\,.\,2\tan\tfrac{1}{2}x\,dx\right]$$

$$=x\tan\tfrac{1}{2}x-\int \tan\tfrac{1}{2}x\,dx$$

$$=x\tan\tfrac{1}{2}x-2\log\sec\tfrac{1}{2}x,\qquad \left[\because \int \tan\tfrac{1}{2}x\,dx=2\log\sec\tfrac{1}{2}x\right].$$

(ii) The given integral

$$I=\int \frac{x\cos x\,dx}{1+\cos x}=\int \frac{x\,(1+\cos x-1)}{1+\cos x}\,dx$$

$$=\int x\left[1-\frac{1}{1+\cos x}\right]dx$$

$$=\int x\,dx-\int \frac{x\,dx}{1+\cos x}.$$

Now proceeding as in part (i), we get

$$I=\tfrac{1}{2}x^2-x\tan\tfrac{1}{2}x+2\log\sec\tfrac{1}{2}x.$$

Ex. 11. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \cos\left(a\log\frac{x}{b}\right)dx$ (*ii*) $\displaystyle\int \frac{\sin(\log x)}{x^3}\,dx$.

(Meerut 1984)

Sol. (i) Let $\displaystyle I=\int \cos\left(a\log\frac{x}{b}\right)dx.$

Put $\log(x/b)=t$, *i.e.*, $x/b=e^t$ *i.e.*, $x=be^t$.

Then $dx=be^t\,dt$.

$\therefore \quad I = \int b e^t \cos at \, dt = b \int e^t \cos at \, dt.$

But $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{\sqrt{(a^2+b^2)}} \cos\left(bx - \tan^{-1}\dfrac{b}{a}\right).$

$$\therefore \quad I = b \frac{e^t}{\sqrt{(1+a^2)}} \cos\left\{at - \tan^{-1}\frac{a}{1}\right\}$$

$$= b \cdot \frac{x}{b} \cdot \frac{1}{\sqrt{(1+a^2)}} \cos\left\{a \log\frac{x}{b} - \tan^{-1} a\right\}$$

$$= \frac{x}{\sqrt{(1+a^2)}} \cos\left\{a \log\frac{x}{b} - \tan^{-1} a\right\}.$$

(ii) Let $\quad I = \displaystyle\int \frac{\sin(\log x)}{x^3} dx.$

Put $\log x = t$ *i.e.,* $x = e^t$.

Then $dx = e^t \, dt$.

$$\therefore \quad I = \int \frac{\sin t}{e^{3t}} e^t \, dt = \int e^{-2t} \sin t \, dt.$$

But $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx).$

$$\therefore \quad I = \frac{e^{-2t}}{(-2)^2+(1)^2}(-2\sin t - 1 . \cos t)$$

$$= \frac{1}{5} e^{-2t}(-2\sin t - \cos t)$$

$$= \frac{1}{5} \cdot \frac{1}{x^2}(-2 \sin \log x - \cos \log x)$$

$$= -\frac{1}{5x^2}(2 \sin \log x + \cos \log x).$$

Ex. 12. *Evaluate the following integrals :*

(*i*) $\displaystyle\int \frac{x e^{\sin^{-1} x}}{\sqrt{(1-x^2)}} dx, \qquad$ (*ii*) $\int e^{\sin^{-1} x} dx.$

Sol. (i). Let $\quad I = \displaystyle\int \frac{x e^{\sin^{-1} x}}{\sqrt{(1-x^2)}} dx.$

Put $\sin^{-1} x = t$, so that $[1/\sqrt{(1-x^2)}]\, dx = dt$. Also $x = \sin t$.

$\therefore \quad I = \int \sin t . e^t \, dt.$

But $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx).$

$$\therefore \quad I = \frac{e^t}{1+1}(\sin t - \cos t) = \frac{e^t}{2}\{\sin t - \sqrt{(1-\sin^2 t)}\}$$

$$= \tfrac{1}{2} e^{\sin^{-1} x}\{x - \sqrt{(1-x^2)}\}.$$

(ii) Let $I = \int e^{\sin^{-1} x} dx$.

Put $\sin^{-1} x = t$ *i.e.*, $x = \sin t$, so that $dx = \cos t\, dt$.

Then $I = \int e^t \cos t\, dt$

$= e^t \cos t - \int e^t (-\sin t)\, dt$,

integrating by parts taking e^t as the second function

$= e^t \cos t + \int e^t \sin t\, dt$

$= e^t \cos t + e^t \sin t - \int e^t \cos t\, dt$,

again integrating by parts taking e^t as the second function

$= e^t (\cos t + \sin t) - I$.

$\therefore \quad 2I = e^t (\cos t + \sin t)$

or $\quad I = \frac{1}{2} e^t \{\sin t + \sqrt{(1 - \sin^2 t)}\}$

$= \frac{1}{2} e^{\sin^{-1} x} \{x + \sqrt{(1 - x^2)}\}$.

Ex. 13. *Evaluate the following integrals :*

(*i*) $\int \tan^{-1} \sqrt{x}\, dx$, (*ii*) $\int \frac{\cos^{-1} x}{x^3} dx$,

(*iii*) $\int \frac{\tan^{-1} x}{(1 + x)^2} dx$.

Sol. (i) Let $I = \int \tan^{-1} \sqrt{x}\, dx$.

Put $\sqrt{x} = \tan t$ *i.e.*, $x = \tan^2 t$, so that $dx = 2 \tan t \sec^2 t\, dt$.

Then $\quad I = \int (\tan^{-1} \tan t)\, 2 \tan t \sec^2 t\, dt$

$= \int t\, (2 \tan t \sec^2 t)\, dt$.

Integrating by parts taking $2 \tan t \sec^2 t$ as the second function, we have

$$I = t \tan^2 t - \int 1 . \tan^2 t\, dt = t \tan^2 t - \int (\sec^2 t - 1)\, dt$$

$$= t \tan^2 t - \int \sec^2 t\, dt + \int dt$$

$$= t \tan^2 t - \tan t + t = t (\tan^2 t + 1) - \tan t$$

$$= (x + 1) \tan^{-1} \sqrt{x} - \sqrt{x}.$$

(ii) Let $\quad I = \int \frac{\cos^{-1} x}{x^3} dx$.

Put $\cos^{-1} x = t$ *i.e.*, $x = \cos t$, so that $dx = -\sin t\, dt$.

Then $\quad I = \int \frac{t}{\cos^3 t} (-\sin t)\, dt$

$= -\int t \tan t \sec^2 t\, dt$.

Integrating by parts taking $\tan t \sec^2 t$ as the second function, we have

$$I = -t \left(\frac{1}{2} \tan^2 t\right) + \int 1 \cdot \frac{1}{2} \tan^2 t\, dt$$

$$= -\tfrac{1}{2}t\tan^2 t + \tfrac{1}{2}\int (\sec^2 t - 1)\,dt$$
$$= -\tfrac{1}{2}t\tan^2 t + \tfrac{1}{2}\tan t - \tfrac{1}{2}t$$
$$= -\tfrac{1}{2}t(1+\tan^2 t) + \tfrac{1}{2}\tan t = -\tfrac{1}{2}t\sec^2 t + \tfrac{1}{2}\tan t.$$

But if $\cos t = x$, then $\tan t = x/\sqrt{(1-x^2)}$.

$$\therefore \quad I = -\frac{1}{2}\frac{\cos^{-1}x}{x^2} + \frac{1}{2}\frac{x}{\sqrt{(1-x^2)}}.$$

(iii) Let $I = \displaystyle\int \frac{\tan^{-1}x}{(1+x)^2}\,dx.$

The integral of $1/(1+x)^2$ is $-1/(1+x)$. So integrating by parts taking $1/(1+x)^2$ as the second function, we have

$$I = (\tan^{-1}x)\left(\frac{-1}{1+x}\right) - \int \frac{1}{(1+x^2)}\cdot\frac{-1}{(1+x)}\,dx$$
$$= -\frac{\tan^{-1}x}{1+x} + \int \frac{1}{(1+x)(1+x^2)}\,dx.$$

Now let $\quad \dfrac{1}{(1+x)(1+x^2)} = \dfrac{A}{1+x} + \dfrac{Bx+C}{1+x^2}.$

Then $\quad 1 \equiv A(1+x^2) + (Bx+C)(1+x) \qquad ...(1)$

Putting $x = -1$ on both sides of (1), we have

$$1 = 2A \quad i.e., \quad A = \tfrac{1}{2}.$$

Putting $A = \tfrac{1}{2}$ in (1), we have

$$1 \equiv \tfrac{1}{2}(1+x^2) + (Bx+C)(1+x) \qquad ...(2)$$

Equating the coefficients of x^2 and constant terms on both sides of (2), we get

$$0 = \tfrac{1}{2} + B,\ 1 = \tfrac{1}{2} + C.$$

These give $B = -\tfrac{1}{2}$ and $C = \tfrac{1}{2}$.

$$\therefore \quad I = -\frac{\tan^{-1}x}{1+x} + \int \left[\frac{\frac{1}{2}}{1+x} + \frac{-\frac{1}{2}x + \frac{1}{2}}{1+x^2}\right] dx$$
$$= -\frac{\tan^{-1}x}{1+x} + \frac{1}{2}\int \frac{1}{1+x}\,dx - \frac{1}{2}\int \frac{\frac{1}{2}\cdot(2x)}{1+x^2}\,dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$
$$= -\frac{\tan^{-1}x}{1+x} + \frac{1}{2}\log(1+x) - \tfrac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1}x$$
$$= \left(\frac{1}{2} - \frac{1}{1+x}\right)\tan^{-1}x + \frac{1}{4}\log\frac{(1+x)^2}{(1+x^2)}.$$

Ex. 14. *Evaluate the following integrals :*

(i) $\displaystyle\int \frac{\sin x}{\sin(x-\alpha)}\,dx$ **(Meerut 1984)**

(ii) $\int \frac{dx}{\sin (x-\alpha) \sin (x-\beta)}$. **(Meerut 1987)**

Sol. (i) Let $I = \int \frac{\sin x}{\sin (x-\alpha)} dx$.

Put $x - \alpha = t$, so that $dx = dt$.

Then $I = \int \frac{\sin (\alpha + t)}{\sin t} dt = \int \frac{\sin \alpha \cos t + \cos \alpha \sin t}{\sin t} dt$

$= \sin \alpha \int \frac{\cos t}{\sin t} dt + \cos \alpha \int dt$

$= \sin \alpha \log \sin t + t \cos \alpha$

$= \sin \alpha \log \sin (x-\alpha) + (x-\alpha) \cos \alpha$

$= x \cos \alpha + \sin \alpha \log \sin (x-\alpha)$,

because the constant term $-\alpha \cos \alpha$ may be added to the constant of integration c.

(ii) Let $I = \int \frac{dx}{\sin (x-\alpha) \sin (x-\beta)}$

$= \int \frac{[\sin \{(x-\beta)-(x-\alpha)\}].[1/\sin (\alpha-\beta)]}{\sin (x-\alpha) \sin (x-\beta)}$ **(Note)**

$= \frac{1}{\sin (\alpha-\beta)} \int \frac{\sin (x-\beta) \cos (x-\alpha) - \cos (x-\beta) \sin (x-\alpha)}{\sin (x-\alpha) \sin (x-\beta)} dx$

$= \text{cosec} (\alpha-\beta) \left[\int \frac{\cos (x-\alpha)}{\sin (x-\alpha)} dx - \int \frac{\cos (x-\beta)}{\sin (x-\beta)} dx \right]$

$= \text{cosec} (\alpha-\beta) [\log \sin (x-\alpha) - \log \sin (x-\beta)]$

$= \text{cosec} (\alpha-\beta) \log \frac{\sin (x-\alpha)}{\sin (x-\beta)}$.

Ex. 15. *Evaluate the following integrals :*

(i) $\int \sqrt{\left\{\frac{\sin (x-\alpha)}{\sin (x+\alpha)}\right\}} dx$ (ii) $\int \frac{dx}{\sqrt{\{\sin^3 x \sin (x+\alpha)\}}}$.

Sol. (i) Let $I = \int \sqrt{\left\{\frac{\sin (x-\alpha)}{\sin (x+\alpha)}\right\}} dx$

$= \int \frac{\sin (x-\alpha)}{\sqrt{\{\sin (x+\alpha) \sin (x-\alpha)\}}}$,

multiplying the numerator and denominator by $\sqrt{\{\sin (x-\alpha)\}}$

$= \int \frac{\sin x \cos \alpha - \cos x \sin \alpha}{\sqrt{\{\sin (x+\alpha) \sin (x-\alpha)\}}} dx$

$= \int \frac{\sin x \cos \alpha}{\sqrt{\{\sin (x+\alpha) \sin (x-\alpha)\}}} dx$

$- \int \frac{\cos x \sin \alpha}{\sqrt{\{\sin (x+\alpha) \sin (x-\alpha)\}}} dx$

$$= \int \frac{\sin x \cos \alpha}{\sqrt{(\cos^2 \alpha - \cos^2 x)}} dx - \int \frac{\cos x \sin \alpha}{\sqrt{(\sin^2 x - \sin^2 \alpha)}} dx.$$

$[\because \sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A]$

In the first integral put $\cos x = t$ so that $-\sin x\, dx = dt$ and in the second integral put $\sin x = z$ so that $\cos x\, dx = dz$.

Then $I = \cos \alpha \displaystyle\int \frac{-dt}{\sqrt{(\cos^2 \alpha - t^2)}} - \sin \alpha \int \frac{dz}{\sqrt{(z^2 - \sin^2 \alpha)}}$

$= \cos \alpha \cos^{-1} \left(\dfrac{t}{\cos \alpha}\right) - \sin \alpha \cosh^{-1} \left(\dfrac{z}{\sin \alpha}\right)$

$= \cos \alpha \cos^{-1} (\cos x \sec \alpha) - \sin \alpha \cosh^{-1} (\sin x \operatorname{cosec} \alpha)$.

(ii) The given integral

$$I = \int \frac{dx}{\sqrt{\{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)\}}}$$

$$= \int \frac{dx}{\sqrt{\{\sin^4 x (\cos \alpha + \cot x \sin \alpha)\}}}$$

$$= \int \frac{dx}{\sin^2 x \sqrt{(\cos \alpha + \cot x \sin \alpha)}} = \int \frac{\operatorname{cosec}^2 x\, dx}{\sqrt{(\cos \alpha + \cot x \sin \alpha)}}$$

$$= -\frac{1}{\sin \alpha} \int (\cos \alpha + \cot x \sin \alpha)^{-1/2} (-\sin \alpha \operatorname{cosec}^2 x)\, dx,$$

adjusting suitably to apply the power formula

$= -\dfrac{1}{\sin \alpha} \dfrac{(\cos \alpha + \cot x \sin \alpha)^{1/2}}{1/2}$, by power formula

$= -2 \operatorname{cosec} \alpha \{\cos \alpha + (\cos x / \sin x) \sin \alpha\}^{1/2}$

$= -2 \operatorname{cosec} \alpha \sqrt{\{\sin (x + \alpha)/\sin x\}}$.

Ex. 16. *Evaluate the following integrals :*

(i) $\displaystyle\int \frac{(\sin \theta - \cos \theta)}{\sqrt{(\sin 2\theta)}} d\theta$

(ii) $\displaystyle\int \frac{(\sin x - \cos x)\, dx}{(\sin x + \cos x) \sqrt{(\sin x \cos x + \sin^2 x \cos^2 x)}}$.

Sol. **(i)** Let $I = \displaystyle\int \frac{(\sin \theta - \cos \theta)}{\sqrt{(\sin 2\theta)}} d\theta$

$$= \int \frac{(\sin \theta - \cos \theta)}{\sqrt{(1 + \sin 2\theta - 1)}} d\theta \qquad \textbf{(Note)}$$

$$= \int \frac{(\sin \theta - \cos \theta)\, d\theta}{\sqrt{\{(\sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta) - 1\}}}$$

$$= \int \frac{(\sin \theta - \cos \theta)\, d\theta}{\sqrt{\{(\sin \theta + \cos \theta)^2 - 1\}}}.$$

Put $\sin \theta + \cos \theta = t$, so that $(\cos \theta - \sin \theta)\, d\theta = dt$.

Then $I = -\int \frac{dt}{\sqrt{(t^2-1)}} = -\cosh^{-1} t = -\cosh^{-1}(\sin\theta + \cos\theta)$.

(ii) Let $I = \int \frac{(\sin x - \cos x)\, dx}{(\sin x + \cos x)\sqrt{(\sin x \cos x + \sin^2 x \cos^2 x)}}$

$$= \int \frac{(\sin x - \cos x)(\sin x + \cos x)\, dx}{(\sin x + \cos x)^2 \cdot \frac{1}{2}\sqrt{(4 \sin x \cos x + 4 \sin^2 x \cos^2 x)}} \quad \textbf{(Note)}$$

$$= \int \frac{-2(\cos^2 x - \sin^2 x)\, dx}{(\sin^2 x + \cos^2 x + 2 \sin x \cos x)\sqrt{\{(1 + 4 \sin x \cos x + 4 \sin^2 x \cos^2 x) - 1\}}}$$

$$= \int \frac{-2 \cos 2x\, dx}{(1 + \sin 2x)\sqrt{\{(1 + \sin 2x)^2 - 1\}}}.$$

Put $1 + \sin 2x = t$, so that $2 \cos 2x\, dx = dt$.

Then $I = -\int \frac{dt}{t\sqrt{(t^2-1)}} = \operatorname{cosec}^{-1} t = \operatorname{cosec}^{-1}(1 + \sin 2x)$.

Ex. 17. *Evaluate the integrals :*

(*i*) $\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\, d\theta$

(*ii*) $\int \sqrt{\left\{\frac{1 - \cos\theta}{\cos\theta\,(1 + \cos\theta)(2 + \cos\theta)}\right\}}\, d\theta.$

Sol. **(i)** Let

$$I = \int_0^{\pi/4} \frac{\sin 2\theta\, d\theta}{\sin^4\theta + \cos^4\theta} = \int_0^{\pi/4} \frac{2 \sin\theta \cos\theta}{\sin^4\theta + \cos^4\theta}\, d\theta$$

$$= \int_0^{\pi/4} \frac{2 \tan\theta \sec^2\theta\, d\theta}{1 + \tan^4\theta},$$

dividing the numerator and denominator by $\cos^4\theta$.

Put $\tan^2\theta = t$, so that $2 \tan\theta \sec^2\theta\, d\theta = dt$.

When $\theta = 0$, $t = \tan^2 0 = 0$ and when $\theta = \pi/4$, $t = \tan^2 \frac{1}{4}\pi = 1$.

$$\therefore \quad I = \int_0^1 \frac{dt}{1 + t^2} = \left[\tan^{-1} t\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

(ii) Multiplying the numerator and denominator by $\sqrt{(1 + \cos\theta)}$, the given integral

$$I = \int \frac{\sqrt{\{(1 - \cos\theta)(1 + \cos\theta)\}}}{(1 + \cos\theta)\sqrt{\{\cos\theta\,(2 + \cos\theta)\}}}\, d\theta$$

$$= \int \frac{\sin\theta\, d\theta}{(1+\cos\theta)\sqrt{(\cos^2\theta + 2\cos\theta)}}$$

$$= \int \frac{\sin\theta\, d\theta}{(1+\cos\theta)\sqrt{\{(1+\cos\theta)^2 - 1\}}}. \qquad \textbf{(Note)}$$

Now put $1 + \cos\theta = t$, so that $-\sin\theta\, d\theta = dt$.

Then $$I = -\int \frac{dt}{t\sqrt{(t^2-1)}}$$

$$= \operatorname{cosec}^{-1} t = \operatorname{cosec}^{-1}(1+\cos\theta) = \operatorname{cosec}^{-1}\left(2\cos^2\frac{\theta}{2}\right).$$

□