Main Test Series - 2018

Test-18 - paper-II - Answer key

1(a) Show that A_4 has no subgroup of order 6.

Sol^n. Let $S = \{1,2,3,4\}$ then

$S_4 = \{f / f : S \to S$ is a permutation on $S\}$

is a permutation group of order $4!$.

Let $A_4 = \{f / f : S \to S$ is even permutation on $S\}$

is an even permutation group of order $\frac{4!}{2}$

If possible suppose that A_4 has a subgroup H of order 6.

i.e, Let $H < A_4$ such that $o(H) = 6$

$\therefore$ By Lagrange's theorem

$$\frac{o(A_4)}{o(H)} = \frac{12}{6} = 2$$

$\therefore (G:H) = 2$

$\therefore H \trianglelefteq A_4$ (i.e, H is a normal subgroup of A_4)

$\therefore \frac{A_4}{H} = \{Hf / f \in A_4\}$ is a quotient group

$= \{He, Hf\}$ where f is not identity permutation.

let $\phi : A_4 \to \frac{A_4}{H}$ be a homomorphism and onto such that $\phi(x) = Hx \;\forall\; x \in A_4$. — (1)

let $x \in A_4$ be a 3-cycle permution (i.e, even permutation) such that $O(x) = 3$ then $\frac{O(x)}{O(\phi(x))} = \frac{3}{O(\phi(x))}$ — (2)

Since $O\left(\frac{A_4}{H}\right) = 2$

$\therefore O(\phi(x)) = 1$

$\Rightarrow \phi(x) = He \Rightarrow \phi(x) = H$ (by (1))

$\Rightarrow Hx = H$

$\Rightarrow x \in H$

$\therefore$ H contains all 3-cycle elements.

but A_4 has 8 3-cycles

and $8 > 6$ which is a contradiction.

$\therefore A_4$ has no subgroup of order 6.

1(b) Prove that every field is an integral domain, but every integral domain is not a field. Give an example of an integral domain which is also a field.

Sol: Let F be a field then by definition F is a commutative ring with unity and every non-zero element is invertible w.r.t $\times$.

In order to prove that a field is an I.D. we have to prove that a field F has no zero divisors.

Let $a, b \in F$ and $a \neq 0$.

Since F is a field.

For $a \neq 0 \in F \Rightarrow a^{-1}$ exists in F.

$\therefore a a^{-1} = a^{-1} a = 1$

Now we have

$ab = 0$

$\Rightarrow a^{-1}(ab) = a^{-1}0$

$\Rightarrow (a^{-1}a)b = 0$

$\Rightarrow 1b = 0$

$\Rightarrow b = 0$

Similarly we can prove that $a, b \in F$
$b \neq 0$ and $ab = 0 \Rightarrow a = 0$

$\therefore a, b \in F$ and $ab = 0$
$\Rightarrow$ either $a = 0$ or $b = 0$

$\therefore$ F has no zero divisors.

$\therefore$ A field is an ID

The converse of the above need not be true
i.e., every ID need not be a field.
For example, the set of integers $\mathbb{Z}$ is an integral domain which is not a field.
Since $a \neq 0 \in \mathbb{Z}$ doesnot have the multiplicative inverse in $\mathbb{Z}$.

The ring $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ of integers modulo 5 is both an integral domain and a field.

1(c) prove that b/w any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

Sol Let $x=a$ & $x=b$ be the roots of $e^x \sin x + 1 = 0$ then

$e^a \sin a + 1 = 0$ & $e^b \sin b + 1 = 0$ —— (1)

Let $f(x) = e^x \sin x + 1 \;\forall x \in [a,b]$

since e^x & $\sin x$ are continuous and differentiable for all reals.

$\therefore f(x) = e^x \sin x + 1$ is continuous and differentiable in $[a,b]$

and $f(a) = f(b) = 0$ (by (1))

$\therefore$ f has been satisfied the conditions of Rolle's theorem.

$\therefore \exists$ atleast one $x \in (a,b)$ s.t $f'(x) = 0$ —— (2)

$\therefore f'(x) = e^x \sin x + e^x \cos x$.

$\Rightarrow f'(x) = e^x(\sin x + \cos x)$

$\therefore$ (1) $\equiv e^x(\sin x + \cos x) = 0 \;\forall x \in (a,b)$

$\Rightarrow \sin x + \cos x = 0$ ($\because e^x \neq 0 \;\forall x$)

$\Rightarrow \tan x + 1 = 0$; $x \in (a,b)$

1(d) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$

Sol: Given that $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$.

comparing the given integral with

$\int \frac{f(z)}{z-z_0} dz$ when $C: |z|=1$

Since $f(z) = \frac{\cos z}{z-4}$ and $z_0 = 0$ is a point inside $|z|=1$.

∴ we can apply Cauchy's integral formula

$$\int_{|z|=1} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

$$= 2\pi i f(0)$$

$$= 2\pi i \left[\frac{\cos(0)}{0-4}\right]$$

$$= \frac{2\pi i}{-4}$$

$$= \frac{-\pi i}{2}$$

1(d) (ii) $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1| = 1$.

Soln: The integrand has singularities, where $z^2-1=0$ i.e at $z=1$ and $z=-1$.

The circle $|z-1|=1$ has center at $z=1$, $f(z)=3z^2+z$, is an analytic function

Also, $\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$

$$\int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz$$

By cauchy's integral formula $\int_C \frac{3z^2+z}{z-1}\,dz = 2\pi i f(1) = 8\pi i$. where $f(z) = 3z^2+z$

By cauchy's theorem $\int_C \frac{3z^2+z}{z+1}\,dz = 0$

$\therefore$ from (1), we have $\int_C \frac{3z^2+z}{z^2-1}\,dz = 4\pi i$

1(e) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table:

Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Step(1):

(i) After subtracting the minimum of each row from all elements of that row the reduced matrix is given by

3	1	0	6
5	7	0	4
2	0	1	3
0	2	2	3

(ii) Subtracting the minimum elements of column from elements of that column, we get

3	1	0	3
5	7	0	1
2	0	1	0
0	2	2	0

Step(2): Cover all the zeros by minimum no. of horizontal and vertical lines. A symmetric approach for this is to look for a row or column containing maximum no of zeros

We can cover all the zeros by 3 lines only
so, $r = 3 < 4 = n$
so, go to step (3)

3	1	0	3
5	7	0	1
2	0	1	0
0	2	2	0

Step(3):

1 is the least uncovered element. Subtract 1 from all the uncovered elements. Add 1 to elements at intersection of the covering lines namely 1 at position (3,3) and 2 at (4,3). Leave other uncovered elements unchanged and the reduced matrix so obtained is

2	0	0	2
4	6	0	0
2	0	2	0
0	2	3	0

Again, cover the zeros by minimum no. of horizontal and vertical lines. we required exactly 4 lines to cover all the zeros. As $r = 4 = n$; optimal assignment can be made at this stage.

2	0	~~0~~	2
4	6	0	~~0~~
2	~~0~~	2	0
0	2	3	~~0~~

It may be noted that an assignment problem can have more than one optimum solution.

Optimum solution - I

2	~~0~~	0	2
4	6	~~0~~	0
2	0	2	~~0~~
0	2	3	~~0~~

Optimum solution-I

Man	Job	Man hours
1	B	3
2	C	2
3	D	7
4	A	5

Optimum solution II

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

3(a)(i) If $\beta \in S_7$ and $\beta^4 = (2\,1\,4\,3\,5\,6\,7)$ then find β.

(ii) let $GL(2,R)$ be the group of all non singular 2x2 matricy over R. show that each of the following set is a sub group of $GL(2,R)$.

$$H = \left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in GL(2,R) \mid ad \neq 0 \right\}$$

Sol (ii): $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$, Hence $H \neq 0$. let $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \in H$

and $B = \begin{bmatrix} a_1 & 0 \\ c_1 & d_1 \end{bmatrix} \in H$. Since $B \in H$, $a_1 d_1 \neq 0$.

Hence.

$$B^{-1} = \begin{bmatrix} \frac{d_1}{a_1 d_1} & 0 \\ -\frac{c_1}{a_1 d_1} & \frac{a_1}{a_1 d_1} \end{bmatrix} = \begin{bmatrix} 1/a_1 & 0 \\ -\frac{c_1}{a_1 d_1} & \frac{1}{d_1} \end{bmatrix}$$

Then.

$$AB^{-1} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{a_1} & 0 \\ \frac{c_1}{a_1 d_1} & 1/d_1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{a_1} & 0 \\ \frac{c}{a_1} - \frac{dc_1}{a_1 d_1} & \frac{d}{d_1} \end{bmatrix}$$

Since $ad \neq 0$, $a_1 d_1 \neq 0$ we find that $\frac{a}{a_1} \frac{d}{d_1} = \frac{ad}{a_1 d_1} \neq 0$

Hence $AB^{-1} \in H$ consequently, H is a sub group of G.

(i) Since $\beta^{28} = (\beta^4)^7 = I$ (identity permutation)

we know that $o(\beta)/28$

so that $o(\beta) \neq 1, 2$ (or) 4.

If $o(\beta) = 14$ then β is written in disjoint cycle form would need at least one 7-cycle and one 2-cycle

But that requires atleast 9 symbols and we have only 7.

Likewise $o(\beta) = 28$ requires at least one 7-cycle and one 4-cycle

so $o(\beta) = 7$.

Ans.

2(b) → Prove that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

Solⁿ: Let $f_n(x) = \frac{x^n}{n^2}$

$$|f_n(x)| = \left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2} = M_n \text{ for } 0 \leq x \leq 1$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

∴ By Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent for $0 \leq x \leq 1$.

∴ The series can be integrated term by term

$$\Rightarrow \int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n^2}\, dx$$

$$= \sum_{n=1}^{\infty} \frac{x^{n+1}}{n^2(n+1)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

$$\therefore \int_0^1 \left(\sum \frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}$$

2(c). A function f is defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{2^n}, & \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; \ (n=0,1,2\ldots) \\ 0; & x=0. \end{cases}$$

prove that (i) f is integral on $[0,1]$ (ii) $\int_0^1 f = 2/3$

Sol Let $f(x) = \begin{cases} \frac{1}{2^n} & \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; \ n=0,1,2\ldots \\ 0; & \text{when } x=0. \end{cases}$

then $f(x) = \begin{cases} \frac{1}{2^0}=1 & \text{when } \frac{1}{2^1} < x \le \frac{1}{2^0}=1 \\ \frac{1}{2} & \text{when } \frac{1}{2^2} < x \le \frac{1}{2^1} \\ \frac{1}{2^2}; & \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ \vdots \\ \frac{1}{2^{n-1}} & \text{when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ \vdots \\ 0 & \text{when } x=0. \end{cases}$

Since f is bounded and continuous on $[0,1]$ except at the points $1/2, 1/2^2, 1/2^3, \ldots$

$\therefore$ The set of points of discontinuity of f on $[0,1]$ is $\{1/2, 1/2^2, 1/2^3, \ldots\}$

which has only one limit point 0.
Since the set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.
$\therefore$ f is integrable on $[0,1]$.

Now $\int_{1/2^n}^{1} f(x)\,dx = \int_{1/2}^{1} f(x) + \int_{1/2^2}^{1/2} f(x)\,dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$

$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2}-\frac{1}{2^3}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$

$= \left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$

$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots + \left(\frac{1}{2^2}\right)^{n-1}\right]$

$\int_{1/2^n}^{1} f(x)\,dx = \frac{1}{2}\left[\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}\right] = \frac{2}{3}\left(1-\frac{1}{4^n}\right)$

$\Rightarrow \lim_{n\to\infty} \int_{1/2^n}^{1} f(x)\,dx = \lim_{n\to\infty} \frac{2}{3}\left(1-\frac{1}{4^n}\right) \Rightarrow \int_0^1 f(x)\,dx = \frac{2}{3}.$

2(d)) Use the method of contour integration to prove that $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta} d\theta = 2\pi a \left\{1 - \frac{a}{\sqrt{(a^2-1)}}\right\}$, where $a>1$. 10

Sol'n: $I = \int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta} d\theta = \int_0^{2\pi} \frac{2a\cos\theta}{2a+2\cos\theta}$

$I = \text{real part of} \int_0^{2\pi} \frac{2ae^{i\theta}}{2a+(e^{i\theta}+e^{-i\theta})} d\theta$

$= \text{real Part of} \int_C \frac{2az}{2a+z+\frac{1}{z}} \frac{dz}{iz}$ writing $e^{i\theta}=z$, $d\theta = \frac{dz}{iz}$

where C is the unit circle $|z|=1$

$= \text{real part of} \int_C \frac{-2iaz}{z^2+2az+1} dz$

$= \text{real Part of} \int_C \frac{-2iaz}{(z-\alpha)(z-\beta)}$

where $\alpha = -a+\sqrt{a^2-1}$, $\beta = -\sqrt{(a^2-1)}$

$= \text{real part of} \int_C f(z)dz$ where $f(z) = \frac{-2iaz}{(z-\alpha)(z-\beta)}$

Poles of $f(z)$ are given by $(z-\alpha)(z-\beta)=0$

i.e. $z=\alpha$ and $z=\beta$ are the two simple poles

The value of β is obviously greater than unity while that of α is less Thus only the pole α lies within the contour C.

Residue of $f(z)$ at the simple pole $z=\infty$ is

$= \lim_{z\to\infty} (z-a)f(z)$

$= \lim_{z\to\infty} (z-a) \frac{-2iaz}{(z-\alpha)(z-\beta)} = \frac{-2aiz}{\alpha-\beta}$

$= \frac{-2ia\{-a+\sqrt{(a^2-1)}\}}{2\sqrt{(a^2-1)}} = ai\left[\frac{a}{\sqrt{(a^2-1)}} - 1\right]$

Hence by Cauchy's residue theorem we have

$$= 2\pi i \cdot ai\left\{\frac{a}{\sqrt{(a^2-1)}} - 1\right\}$$

$$= 2\pi a\left\{1 - \frac{a}{\sqrt{(a^2-1)}}\right\}$$ which is purely real.

Hence I = real part of $\int_C f(z)\,dz$

$$= 2\pi a\left\{1 - \frac{a}{\sqrt{(a^2-1)}}\right\}$$

3(a) (i) Let G be a group. Show that if $G/Z(G)$ is cyclic, then G is abelian.

(ii) Show that the ring Z_p of integers modulo p is a field if and only if p is prime.

Sol^n:

(i) Let n be an integer

Define $f: \mathbb{Z} \to \mathbb{Z}$ by $f(t) = nt \ \forall t \in \mathbb{Z}$

Let $r, s \in \mathbb{Z}$. Then $f(r+s) = n(r+s)$
$= nr + ns$

Hence f is a homomorphism.

We denote this homomorphism by f_n.

We show that any homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ is one of these $f_n, n \in \mathbb{Z}$.

To show this, let us consider an integer $m \in \mathbb{Z}$.

Now if f is a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$,

then $f(m) = f(m1) = m f(1)$.

So we find that f is completely determined if we know $f(1)$.

If $f(1) = n$, then $f(m) = nm = f_n(m)$.

Hence $f = f_n$ and all the homomorphism of $\mathbb{Z}$ into $\mathbb{Z}$ are given by $f_n, n = 0, \pm 1, \pm 2, \ldots$

(ii) Let us write $Z(G) = N$. Then $\frac{G}{N}$ is cyclic.

Suppose it is generated by Ng.

Let $a, b \in G$ be any two elements

then $Na, Nb \in \frac{G}{N}$

$\Rightarrow Na = (Ng)^n; \ Nb = (Ng)^m$ for some n, m

$\Rightarrow Na = Ng \cdot Ng \cdots\cdots Ng = Ng^n$
and $Nb = Ng^m$.

$\Rightarrow ag^{-n} \in N,\ bg^{-m} \in N$

$\Rightarrow ag^{-n} = x,\ bg^{-m} = y$ for some $x, y \in N$.

$\Rightarrow a = xg^n,\ b = yg^m$

$\Rightarrow ab = (xg^n)(yg^m) = x(g^n y)g^m$

$= x(yg^n)g^m$ as $g \in N = Z(G)$

$= xyg^n g^m$

$= xyg^{n+m}$

Similarly, $ba = (yg^m)(xg^n) = y(g^m x)g^n$

$= y(xg^m)g^n$

$= yx\,g^{m+n}$

$\Rightarrow ab = ba$ as $xy = yx$ as $x, y \in Z(G)$

$\therefore$ G is abelian

3(b) Show that the sequence $\{f_n\}$ where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2 x + 2n & 1/n \le x \le 2/n \\ 0 & \frac{2}{n} \le x \le 1 \end{cases}$$

is not uniformly convergent on $[0,1]$.

Soln The sequence converges to f, where $f(x) = 0$ for all $x \in [0,1]$. Each function f_n and f are continuous on $[0,1]$.

Also $$\int_0^1 f_n\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (-n^2 x + 2n)\,dx + \int_{2/n}^{1} 0\,dx = 1$$

But $$\int_0^1 f\,dx = 0$$

$$\lim_{n\to\infty} \int_0^1 f_n\,dx \neq \int_0^1 f\,dx$$

So by the theorem

(i) If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable, then f is integrable on $[a,b]$ and the sequence $\left\{ \int_a^x f_n\,dt \right\}$ converges.

uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt \quad \forall x \in [a,b]$$

(B) If a series Σf_n converges uniformly to f on $[a,b]$ and each term $f_n(x)$ is integrable then 'f' is integrable on $[a,b]$ and the series $\Sigma\left(\int_a^x f_n\,dt\right)$ converges uniformly to $\int_a^x f\,dt$ on $[a,b]$ i.e

$$\int_a^x f\,dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right);\ \forall x \in [a,b]$$

$\therefore$ from the above theorem the sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

4(a) In the ring $Z[i]$, show that $I = \{a+bi \in Z[i] / a, b$ are both even$\}$ is an ideal of $Z[i]$, but not a maximal ideal of $Z[i]$.

Sol^n: Let $x = a+bi$, $y = c+di \in I$ and $u = r+si \in Z[i]$. then exist $r_1, r_2, r_3, r_4 \in Z$ such that

$$a = 2r_1, \; b = 2r_2, \; c = 2r_3, \; d = 2r_4$$

Then $x-y = (a+ib)-(c+id) = (a-c)+(b-d)i$

$$= 2(r_1-r_3) + 2(r_2-r_4)i \in I$$

and $ux = (r+si)(a+bi) = (ra-sb)+(rb+sa)i$

$$= 2(rr_1-sr_2) + 2(rr_2+sr_1)i \in I$$

Since $Z[i]$ is a Commutative ring, it follows that I is an ideal of $Z[i]$.

Let $J = \{a+bi \in Z[i] / 2$ divides $a^2+b^2\}$.

Since $0+0i \in J$, $J \neq \phi$.

Let $x = a+ib$, $y = c+id \in J$ and $u = r+si \in Z[i]$.

Then $2 / a^2+b^2$ and $2 / c^2+d^2$

Now $x-y = (a-c)+(b-d)i$

Since $(a-c)^2+(b-d)^2 = a^2+b^2+c^2+d^2-2(ac+bd)$,

it follows that $2 / (a-c)^2+(b-d)^2$

Again $ux = (r+si)(a+bi) = (ra-sb)+(rb+sa)i$

and $(ra-sb)^2+(rb+sa)^2 = r^2(a^2+b^2)+s^2(a^2+b^2)$

Hence $2 / (ra-sb)^2+(rb+sa)^2$ implies that

$$ux = xu \in J.$$

Consequently J is an ideal.

Now $1+2i \in Z[i]$, but $1+2i \notin J$

Hence $J \neq Z[i]$.

Again $1+i \in J$ but $1+i \notin I$.

Hence we find that $I \subset J \subset Z[i]$.

Consequently, I is not a maximal ideal [illegible] [illegible] [illegible] the [illegible] of [illegible]

If

[illegible]

is continuous but not

uniformly continuous on $\mathbb{R}^2$

4(b) Prove that the function f defined by $f(x) = \sin \frac{1}{x} \ \forall x > 0$ is continuous but not uniformly continuous on R^+.

Sol^n: let 'a' be any arbitary +ve real number

$$\therefore f(a) = \sin \frac{1}{a} \ \forall a \in \mathbb{R}^+ \quad \text{—— (1)}$$

Now $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \sin \frac{1}{x}$

put $x = a - h,\ h > 0$

$$\lim_{x \to a^-} f(x) = \lim_{h \to a^+} \sin \frac{1}{a-h}$$

$$= \sin \frac{1}{a} \quad \text{—— (2)}$$

Similarly

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sin \frac{1}{x}$$

Put $x = a+h,\ h > 0$

$$\therefore \lim_{x \to a^+} f(x) = \lim_{h \to 0^+} \sin \frac{1}{a+h}$$

$$= \sin \frac{1}{a} \quad \text{—— (3)}$$

from (1), (2) and (3)

$$\lim_{x \to a^+} f(x) = f(a) = \lim_{x \to a^-} f(x)$$

$\therefore$ f is continuous at 'a'

Since 'a' is arbitary & $a \in \mathbb{R}^+$, f is continuous on $\mathbb{R}^+$

Now, let $\varepsilon > 0$, we shall show that for each $\delta > 0$

$\exists x_1, x_2 \in \mathbb{R}^+$ such that

$|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)| \geq \epsilon$

Consider for any $\delta>0$ $\exists$ $m\in N$ such that

$$\frac{1}{2n(n\pi+\pi/2)}<\delta \quad \forall\, n\geq m$$

Take $x_1=\dfrac{1}{m\pi+\pi/2}$ and $x_2=\dfrac{1}{m\pi}$

$\therefore$ $x_1, x_2 \in \mathbb{R}^+$ and

$$|x_1-x_2| = \left|\frac{1}{m\pi+\pi/2}-\frac{1}{m\pi}\right| = \left|\frac{-\pi/2}{m\pi(m\pi+\pi/2)}\right|$$

$$= \frac{1}{2m(m\pi+\pi/2)} < \delta$$

But $|f(x_1)-f(x_2)| = |\sin(m\pi+\pi/2)-\sin m\pi|$

$$= |\cos m\pi - 0|$$

$$= |(-1)^m| = 1$$

which is not less than each $\epsilon>0$

Hence, f is not uniformly continuous on $\mathbb{R}^+$

4(d) Obtain the dual of the LP problem.

$$\min z = x_1+x_2+x_3$$

Subject to the constraints: $x_1-3x_2+4x_3=5$,

$$x_1-2x_2\leq 3$$

$$2x_2-x_3\geq 4$$

$x_1, x_2\geq 0$, and x_3 is unrestricted.

Sol^n. Since the problem is of minimization type all constraints should be of ($\geq$) type. Multiply the second constraint throughout by -1, so that $-x_1+2x_2\geq -3$. and we write the first equality constraint in the form of two inequalities of $\geq$ type.

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$\therefore$ The given problem can be written as

Minimize $Z = x_1 + x_2 + x_3$

subject to

$$\left.\begin{array}{l} x_1 - 3x_2 + 4x_3 \geq 5 \\ -x_1 + 3x_2 - 4x_3 \geq -5 \\ -x_1 + 2x_2 \quad \geq -3 \\ 2x_2 - x_3 \geq 4 \end{array}\right\} \longrightarrow (1)$$

$x_1, x_2 \geq 0$, x_3 is unrestricted.

Since x_3 is unrestricted

put $x_3 = x_3' - x_3''$

The equation (1) can be written as

$\text{Min} Z = x_1 + x_2 + x_3' - x_3''$

subject to

$$x_1 - 3x_2 + 4x_3' - 4x_3'' \geq 5$$
$$-x_1 + 3x_2 - 4x_3' + 4x_3'' \geq -5$$
$$-x_1 + 2x_2 \geq -3$$
$$2x_2 - x_3' + x_3'' \geq 4.$$

$x_1, x_2, x_3', x_3'' \geq 0.$

Let y_1, y_2, y_3 and y_4 be the dual variables associated with the above 4 constraints.

Then the dual is given by

Maximize $W = 5y_1 - 5y_2 - 3y_3 + 4y_4$

subject to

$$y_1 - y_2 - y_3 + 0y_4 \leq 1$$
$$-3y_1 + 3y_2 + 2y_3 + 2y_4 \leq 1$$
$$4y_1 - 4y_2 \quad - y_4 \leq 1.$$
$$-4y_1 + 4y_2 \quad + y_4 \leq -1$$

$y_i \geq 0.$

This dual can be written in more compact form as:

$$\text{Max } w = 5y' - 3y_3 + 4y_4$$

Subject to

$$y' - y_3 \leq 1$$
$$-3y' + 2y_3 + 2y_4 \leq 1$$
$$4y' - \quad - y_4 \leq 1$$
$$-4y' \quad + y_4 \leq -1$$

$y', y_3, y_4 \geq 0$ and $y' (= y_1 - y_2)$ unrestricted

(or)

$$\text{Max } w = 5y' - 3y_3 + 4y_4$$

subject to.

$$y' - y_3 \leq 1$$
$$-3y' + 2y_3 + 2y_4 \leq 1$$
$$4y' - y_4 = 1$$

$y_3, y_4 \geq 0$ and y' is unrestricted.

4(c) If the function $f(z)$ is analytic and one valued in $|z-a|<R$, Prove that for $0<r<R$

$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta$, where $P(\theta)$ real part of $(a+re^{i\theta})$

Solⁿ: Since $f(z)$ is regular in $|z-a|<R$, therefore it must be regular in $|z-a|=r$, $(r<R)$

Hence $f(z)$ can be expanded in a Taylor's Series within the Circle $|z-a|=r$

i.e $f(z) = \sum_0^\infty a_m(z-a)^m$

$= \sum_0^\infty a_m r^m (e^{i\theta})^m$ Since $z-a = re^{i\theta}$

So that $\overline{f(z)} = \sum_0^\infty \bar{a}_m r^m (e^{i\theta})^m$

Now, Consider the integral $\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}}$

$$\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi} \sum_0^\infty \bar{a}_m r^m e^{-im\theta} \frac{re^{i\theta} \cdot i d\theta}{r^{n+1}e^{i(n+1)\theta}}$$

$$= \sum_0^\infty \bar{a}_m \cdot r^{m-n} \cdot i \int_0^{2\pi} e^{-i(m+n)\theta} = 0 \text{ for all values of } n$$

Particularly $\int_C \overline{f(z)} \frac{dz}{(z-a)^2} = 0$ ——— (1)

we know that $f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}dz \text{ from (1)}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})+\overline{f(a+re^{i\theta})}}{r^2e^{2i\theta}} re^{i\theta} \cdot i d\theta$$ Since $z-a = re^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\operatorname{Re} f(a+re^{i\theta})}{r^2e^{2i\theta}} re^{i\theta} \cdot i d\theta$$ Since $z+\bar{z} = 2\operatorname{Re}(z)$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta.$$ [Since $P(\theta) = \operatorname{Re} f(a+re^{i\theta})$]

This proves the result.

5(a) → Find the integral surface of the linear partial differential equation $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0,\ z=1$.

Sol Given $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$

Lagranges equation $Pp+Qq=R \Rightarrow \frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$\equiv \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2+z-(x^2+z)+x^2-y^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

integrating $\ln(xyz)=c$

$$\underline{xyz=c_1} \quad \text{— (1)}$$

$$\frac{x\,dx+y\,dy}{x^2(y^2+z)-y^2(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

$$x\,dx+y\,dy=dz$$

$$\Rightarrow x^2+y^2=2z+c_2 \quad \text{— (2)}$$

Given $x+y=0$; $z=1$ — (3)

from (1) & (3) $-x^2=c_1$

from (2) & (3) $2x^2=2+c_2 \Rightarrow \underline{2c_1+c_2+2=0}$

$$\boxed{2xyz+x^2+y^2-2z+2=0}$$ Integral Surface

5(b) Find the complete integral $z(p^2-q^2)=x-y$

Sol

Rewriting $\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2-\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2=x-y$

Let $\sqrt{z}\,dz=dZ \Rightarrow \frac{2}{3}\cdot z^{3/2}=Z$ —(1)

$P=\frac{\partial Z}{\partial x} \quad Q=\frac{\partial Z}{\partial y}$

$\therefore P^2-Q^2=x-y$

$P^2-x=Q^2-y$

Let $P^2-x=a \;;\; Q^2-y=a$

$P=(x+a)^{1/2} \quad Q=(y+a)^{1/2}$

$\therefore dZ=P\,dx+Q\,dy$

$dZ=(x+a)^{1/2}dx+(y+a)^{1/2}$

$Z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y+a)^{3/2}+b$

from(1) $\frac{2}{3}z^{3/2}=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y+a)^{3/2}+b$

$\therefore z^{3/2}=(x+a)^{3/2}+(y+a)^{3/2}+b$ a, b are arbitrary Constants

5(c) Compute the integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-x^2/2}\, dx .$$

using Simpson's $\frac{1}{3}$ rule, taking $h = 0.125$.

Sol^n Given that $f(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2}$

$a = 0,\ b = 1$

x	0	0.125	0.250	0.375	0.5	0.625	0.750	0.875	1.0
$y = f(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2}$	0.79779 y_0	0.7917 y	0.7733 y_2	0.7437 y_3	0.7041 y_4	0.6563 y_5	0.6023 y_6	0.5441 y_7	0.4835 y_4

By using Simpson's $\frac{1}{3}$ rule, we have

$$\sqrt{2/\pi} \int_0^1 e^{-x^2/2}\, dx = \frac{h}{3}\Big[y_0 + y_8 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) \Big]$$

$$= \frac{0.125}{3}\Big[0.7979 + 0.4839 + 4(0.7917 + 0.7477 + 0.6563 + 0.5441) + 2(0.7733 + 0.7041 + 0.6023) \Big]$$

$$= \frac{0.125}{3}(1.2818 + 10.9432 + 4.1594)$$

$$= 0.6827$$

5(d) Give a Boolean expression for the following statements.

(i) Y is a 1 only if A is a 1 and B is a 1 or A is a 0 and B is a 0.

(ii) Y is a 1 only if A, B and C are all 1s or if only one of the variables is a 0.

Sol^n: (i) Truth table for given conditions.

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

$$\boxed{Y = A'B' + AB}$$

(ii)

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$Y = A'BC + AB'C + ABC' + ABC$

Simplify the expression

$= A'BC + ABC + AB'C + ABC + ABC' + ABC$

$= BC(A'+A) + AC(B'+B) + A(C'+C) = BC + AC + AB$

$$\boxed{\therefore Y = BC + AC + AB}$$

5(c) → If the velocity potential of a fluid is $\phi = (z/r^3)\tan^{-1}(y/x)$ where $r^2 = x^2+y^2+z^2$, then show that the streamlines lies on the surface $x^2+y^2+z^2 = c(x^2+y^2)^{2/3}$, c being an arbitrary constant.

Sol^n → The velocity potential ϕ is given by

$$\phi(x,y,z) = (x^2+y^2+z^2)^{-3/2}\, z\tan^{-1}(y/x)$$

$$= r^{-3} z \tan^{-1}(y/x) \quad \text{—— (1)}$$

where $\quad r^2 = x^2+y^2+z^2$ —— (2)

so that $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial z}{\partial r} = \dfrac{z}{r}$ —— (3)

$$u = -\frac{\partial \phi}{\partial x} = 3zxr^{-5}\tan^{-1}\frac{y}{x} + \frac{zyr^{-3}}{x^2+y^2}$$

$$v = -\frac{\partial \phi}{\partial y} = 3zyr^{-5}\tan^{-1}\frac{y}{x} - \frac{zxr^{-3}}{x^2+y^2}$$

$$w = -\frac{\partial \phi}{\partial z} = 3z^2r^{-5}\tan^{-1}\frac{y}{x} - r^{-3}\tan^{-1}\frac{y}{x}$$

The equation of line flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{3zxr^{-5}\tan^{-1}\frac{y}{x} + \frac{zyr^{-3}}{x^2+y^2}} = \frac{dy}{3zyr^{-5}\tan^{-1}\frac{y}{x} - \frac{zxr^{-3}}{x^2+y^2}}$$

$$= \frac{dz}{(3z^2r^{-5} - r^{-3})\tan^{-1}\frac{y}{x}}$$

—— (4)

each member of equation (4) is

$$= \frac{x\,dx + y\,dy + z\,dz}{(3x^2+3y^2+3z^2)r^{-2} - 1} = \frac{x\,dx + y\,dy}{(3x^2+3y^2)/r^2}$$

or, $$\frac{x\,dx + y\,dy + z\,dz}{2} = \frac{r^2(x\,dx + y\,dy)}{3(x^2+y^2)}$$

or, $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \frac{2}{3}\,\frac{2x\,dx + 2y\,dy}{x^2+y^2} \quad \text{——— (5)}$$

Integrating (5)

$$\log(x^2+y^2+z^2) = (2/3)\log(x^2+y^2) + \log c$$

or, $x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}$, c being an arbitrary constant ——— (6)

(6) gives the required series of the surface.

6(a) Find a Partial differential equation by eliminating a, b, c from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

Sol'n: Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— ①

Differentiating ① w.r.t x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow c^2x + a^2z\frac{\partial z}{\partial x} = 0 \text{ —— ②}$$

and $\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \Rightarrow c^2y + b^2z\frac{\partial z}{\partial y} = 0$ —— ③

Differentiating ② w.r.t x and ③ w.r.t y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \text{ —— ④}$$

& $c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0$ —— ⑤

from ②, $c^2 = \frac{-a^2z}{x}\left(\frac{\partial z}{\partial x}\right)$

Putting this value of c^2 in ④ and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} \text{ —— ⑥}$$

Similarly, from ③ & ⑤

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \text{ —— ⑦}$$

Differentiating ② partially w.r.t y, we get

$$a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\frac{\partial^2 z}{\partial x\partial y}\right\} = 0$$

i.e. $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial x\partial y} = 0$ —— ⑧

∴ ⑥, ⑦ and ⑧ are three possible forms of the required partial differential equation.

6(b) → $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$

Sol: Auxiliary equation $m^3 - 7m - 6 = 0$

$(m+2)(m-3)(m+1) = 0$

$m = -1, -2, 3$

$\therefore C.F = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

Particular Integral $= \dfrac{\sin(x+2y) + e^{3x+y}}{(D_x+2D_y)(D_x-3D_y)(D_x+D_y)}$

$\dfrac{\sin(x+2y)}{D_x^3 - 7D_xD_y^2 - 6D_y^3} = \dfrac{1}{1^3 - 7(1)(2)^2 - 6(2)^3}\iiint \sin v\, dv\,dv\,dv \qquad v = x+2y$

$= -\frac{1}{75}\cos(x+2y)$

$\dfrac{e^{3x+y}}{(D_x+2D_y)(D_x-3D_y)(D_x+D_y)} = \dfrac{1}{(D_x-3D_y)}\left[\dfrac{1}{(3+2)(3+1)}\iint e^v\, dv\,dv\right] \qquad v = 3x+y$

$= \dfrac{1}{20}\,\dfrac{e^{(3x+y)}}{D_x-3D_y}$

$= \dfrac{1}{20}\,\dfrac{x\cdot e^{(3x+y)}}{1\cdot 1}$

$= \frac{x}{20}\cdot e^{3x+y}$

$\therefore y = CF + PI = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20}e^{3x+y}$

6(c) Reduce $\frac{\partial^2 z}{\partial x^2} = (1+y)^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form. —(1)

Sol. Comparing with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

here $R = 1$, $S = 0$ and $T = -(1+y)^2$

$S^2 - 4RT = (1+y)^2 > 0$ ∴ hyperbolic

Quadratic reduces to $\lambda^2 - (1+y)^2 = 0$

$$\lambda = 1+y,\ -(1+y)$$

Corresponding characteristic equations are

$$\frac{dy}{dx} + 1 + y = 0 \qquad \frac{dy}{dx} - (1+y) = 0$$

$$\ln(1+y) + x = c_1 \qquad \ln(1+y) - x = c_2$$

$$u = \log(1+y) + x \qquad v = \log(1+y) - x$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad —(2)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right)$$

$$= \frac{-1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{(1+y)}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{-1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

from ①, ② and ③

$$\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) - \frac{(1+y)^2}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0$$

$$\boxed{4\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}}$$ Required Canonical form.

6(d) → The ends A and B of a rod 20 cm along have the temperatures at 30° and 80° until steady state prevails. The temperatures of the ends are changed to 40° and 60° respectively. Find the temperature distribution in the rod at time t.

Sol'n: Let the equation for conduction of heat is $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ —— ①

Prior to temperature change at the end B when $t=0$, the heat flow was independent of time (steady state condition, for which $\partial u/\partial t = 0$) when u depends only on x, ① reduces to

$$\frac{d^2u}{dx^2} = 0 \Rightarrow u = c_1 x + c_2 \text{ —— ②}$$

Given that $u = 30$ for $x = 0$ and $u = 80$ for $x = 20\,\text{cm}$ } —— ③.

by using ③.

② becomes $u = \frac{5}{2}x + 30$

∴ The initial condition is given by

$$u(x,0) = \frac{5}{2}x + 30 \text{ —— ④}$$

Given that the boundary condition are

$u(0,t) = 40 \quad \forall t$ ——— (5)

$u(20,t) = 60 \quad \forall t$ ——— (6)

Now the boundary values are non-zero, so we modify the procedure as follows.

we split the temperature function $u(x,t)$ into two parts as

$u(x,t) = u_1(x) + u_2(x,t)$ ——— (7)

where $u_1(x)$ is a solution of (1) involving x only and satisfying the boundary conditions (5) and (6) and $u_2(x,t)$ is then a function defined by (7).

Hence $u_1(x)$ is a steady state solution of the form (2) and $u_2(x,t)$ may be treated as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 40$ for $x=0$ & $u_1(x) = 60$ for $x=20$

using (2), we get $\boxed{u_1(x) = x+40}$ ——— (8)

$\left[\because u = c_1x+c_2,\ 40 = c_1(0)+c_2 \Rightarrow c_2 = 40 \ \& \ 60 = c_1(20)+40 \Rightarrow c_1 = 1,\ u_1(x) = 1(x)+40 = x+40\right]$

putting $x=0$ in (7) and using (5),

we get

$\underline{\underline{u_2(0,t)}} = u(0,t) - u_1(0) = 40-40 = 0$ ——— (9)

putting $x=20$ in (7) and using (6), we get

$\underline{\underline{u_2(20,t)}} = u(20,t) - u_1(20) = 60-60 = 0$ ——— (10)

Also, $u_2(x,0) = u_1(x,0) - u_1(x)$

$= \frac{5}{2}x + 30 - (x+40)$

$\underline{\underline{u_2(x,0)}} = \frac{3}{2}x - 10$ ——— (11)

Hence the boundary conditions and initial condition to the transient solution $u_2(x,t)$ are given by (9), (10) and (11).

So we now solve $\frac{\partial u_2}{\partial t} = k \frac{\partial^2 u_2}{\partial x^2}$ ——— (12)

subject to boundary conditions (9) & (10) and initial condition (11).

Now taking $u_2(x,t) = X(x)\,T(t)$

$\therefore$ From (12), we have $XT' = k X'' T$

$\Rightarrow \frac{X''}{X} = \frac{T'}{kT} = \mu$ (say)

$\Rightarrow X'' - \mu X = 0$ ——— (13) & $T' = \mu k T$ ——— (14).

Using (9) & (10), (13) gives

$X(0)T(t) = 0$ & $X(20)T(t) = 0$

$\Rightarrow X(0) = 0$ & $X(20) = 0$ ——— (15)

($\because T(t) \neq 0$ otherwise leading to $u=0$)

we now solve (13) under B.C (15).

Three cases arise:

Case(1) Let $\mu = 0$, the solution of (13) is $X(x) = Ax + B$

Using B.C. (15), we get $A = B = 0$

$\therefore X(x) = 0$ so that $u = 0$ which doesnot satisfy (11)

so we reject $\mu = 0$.

Case (2): Let $\mu = \lambda^2$, $\lambda \neq 0$. Then $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$.

by using B.C (15), we get $A = B = 0$ so that $X(x) = 0$ and hence $u = 0$

which doesnot satisfy (11).

so we reject $\mu = \lambda^2$.

Case(3): Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then $X(x) = A\cos\lambda x + B\sin\lambda x$.

Using B.C (15), we get

$A = 0$ & $A\cos 20\lambda + B\sin 20\lambda = 0$

$\Rightarrow B\sin 20\lambda = 0$

$\Rightarrow \sin 20\lambda = 0$ $(\because B \neq 0)$

$\Rightarrow 20\lambda = n\pi$, $n = 1, 2, \ldots$

$\Rightarrow \lambda = \frac{n\pi}{20}$, $n = 1, 2, \cdots$ —— (16)

Hence non-zero solutions $X_n(x)$ of (13) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{20}\right)$.

Using (16), (14) gives $\frac{dT}{T} = -\frac{n^2\pi^2}{400}k\,dt$

$$\Rightarrow T_n(t) = C_n e^{-\frac{n^2\pi^2 k t}{400}} \quad \left(\because \mu = -\lambda^2 = -\frac{n^2\pi^2}{(20)^2}\right)$$

$$\therefore u_2(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{20}\right) e^{-\frac{n^2\pi^2 k}{400}t}$$

where $D_n = B_n C_n$.

where $D_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin\frac{n\pi x}{20}\,dx$

$$= \frac{1}{10}\left[\left(\frac{3x}{2} - 10\right)\left(-\cos\frac{n\pi x}{20}\right)\frac{20}{n\pi} + \frac{3}{2}\sin\frac{n\pi x}{20}\cdot\left(\frac{20}{n\pi}\right)^2\right]$$

$$= \frac{1}{10}\left[\frac{20}{n\pi}\left[20(-\cos n\pi) - 10\right] + 0\right]$$

$$= -\frac{20}{n\pi}\left[2\cos n\pi + 1\right]$$

$$= -\frac{20}{n\pi}\left[2(-1)^n + 1\right]$$

$$\therefore u_2(x,t) = -\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{\left[2(-1)^n + 1\right]}{n}\sin\left(\frac{n\pi x}{20}\right)e^{-\frac{n^2\pi^2 k t}{400}} \quad \text{—— (17)}$$

$\therefore$ From (8) & (17), (7) gives

$$u(x,t) = x + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{\left[2(-1)^n + 1\right]}{n}\sin\frac{n\pi x}{20}e^{-\frac{n^2\pi^2 k t}{400}}$$

which is the required solution

4(a)(i) A NOR gate has three inputs A, B, C. Which combination of inputs will give High output?

(ii) Implement the expression $Y = AB + CD$ using only NAND gates.

Soln: (i) Given that NOR gate has three inputs A, B, C. The high output (Y) be 1

i.e $Y = 1 = \overline{A+B+C}$

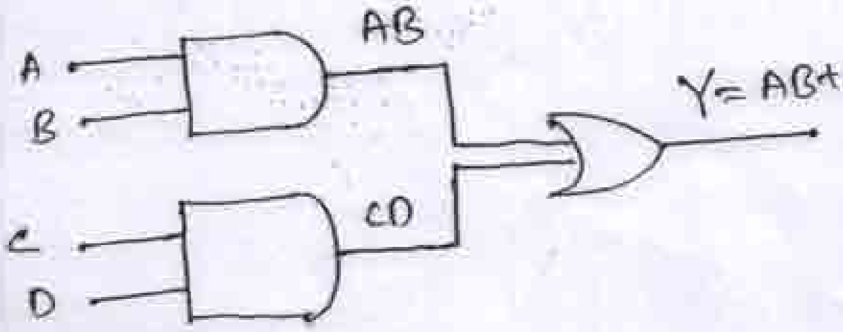
Since $\overline{A+B+C} = 1$

$\Rightarrow A+B+C = 0$

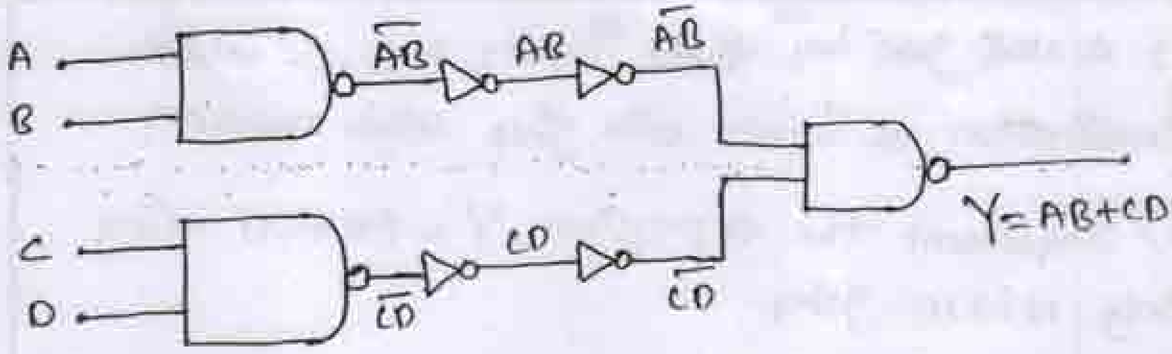
This is possible only if $A = B = C = 0$.

(ii) The straight forward implementation uses two AND gates and one OR gate as shown in Fig. 4.51(a). Each AND gate can be replaced by a NAND gate and NOT gate in series. The OR gate can be also replaced by NAND gates. This is shown in Fig. 4.51(b). It is seen that NOT gates 1 and 2 are in series and can be eliminated (because $\overline{\overline{A}} = A$). Similarly NOT gates 3 and 4 are in series and can be eliminated. Thus we get the logic circuit shown in Fig 4.51(c)

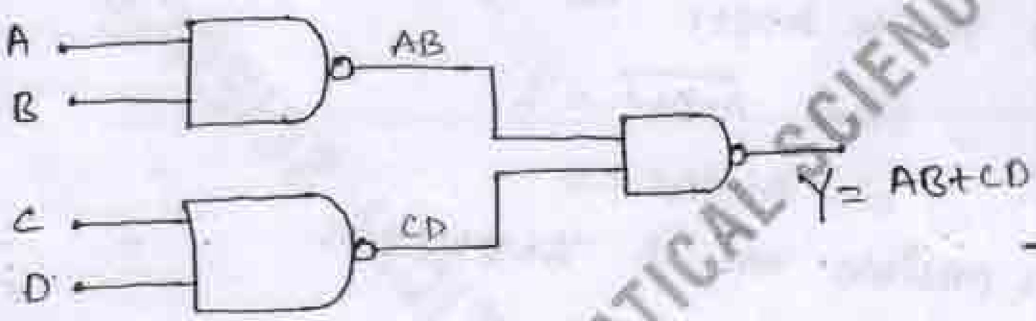
(a)



(b)


A
B
$\overline{AB}$
AB
$\overline{AB}$
C
D
$\overline{CD}$
CD
$\overline{CD}$
Y = AB+CD

(c)


A
B
AB
C
D
CD
Y = AB+CD

7(b) Find the solution of the following system of equations

$$x_1 - \frac{x_2}{4} - \frac{x_3}{4} = 1/2$$

$$-\frac{x_1}{4} + x_2 - \frac{x_4}{4} = 1/2$$

$$-\frac{x_1}{4} + x_3 - \frac{x_4}{4} = 1/4$$

$$-\frac{x_2}{4} - \frac{x_3}{4} + x_4 = 1/4.$$

using Gauss-Seidel method and perform the first five iterations.

Soln: The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \longrightarrow (1)$$

By Gauss-Seidel method, system (1) can be written as

$$x_1^{k+1} = 0.5 + 0.25x_2^{(k)} + 0.25x_3^{(k)}$$

$$x_2^{k+1} = 0.5 + 0.25x_1^{k+1} + 0.25x_4^{(k)}$$

$$x_3^{(k+1)} = 0.25 + 0.25x_1^{(k+1)} + 0.25(x_4)^{(k)}$$

$$x_4^{(k+1)} = 0.25 + 0.25x_2^{(k+1)} + 0.25x_3^{(k+1)}$$

where $k = 0, 1, 2, 3, \ldots$

Now taking $x^{(0)} = 0$ (i.e $x_2^{(0)} = 0$, $x_3^{(0)} = 0$, $x_4^{(0)} = 0$)

(which is the initial solution)

$K=0$

$x_1^{(1)} = 0.5 + 0.25\, x_2^{(0)} + 0.25\, x_3^{(0)} = 0.5 + 0 + 0 = 0.5$

$x_2^{(1)} = 0.5 + 0.25\, x_1^{(1)} + 0.25\, x_4^{(0)} = 0.5 + (0.25)(0.5) + 0 = 0.625$

$x_3^{(1)} = 0.25 + 0.25\, x_1^{(1)} + 0.25\, x_4^{(0)} = 0.25 + (0.25)(0.5) + (0.25)(0)$
$= 0.375$

$x_4^{(1)} = 0.25 + 0.25\, x_2^{(1)} + 0.25\, x_3^{(1)} = 0.25 + (0.25)(0.625) + (0.25)(0.375)$
$= 0.5$

$K=2$

$x_1^{(2)} = 0.5 + 0.25\, x_2^{(1)} + 0.25\, x_3^{(1)} = 0.5 + (0.25)(0.625) + (0.25)(0.375)$
$= 0.75$

$x_2^{(2)} = 0.5 + 0.25\, x_1^{(2)} + 0.25\, x_4^{(1)} = 0.5 + (0.25)(0.75) + (0.25)(0.5)$
$= 0.8125$

$x_3^{(2)} = 0.25 + 0.25\, x_1^{(2)} + 0.25\, x_4^{(1)} = 0.25\,[1 + 0.75 + 0.5]$
$= 0.5625$

$x_4^{(2)} = 0.25 + 0.25\, x_2^{(2)} + 0.25\, x_3^{(2)} = (0.25)\,[1 + 0.8125 + 0.5625]$
$= 0.59375$

$K=2$

$x_1^{(3)} = 0.5 + 0.25\, x_2^{(2)} + 0.25\, x_3^{(2)} = 0.5 + (0.25)(0.8125) + (0.25)(0.5625)$
$= 0.84375$

$x_2^{(3)} = 0.5 + 0.25\, x_1^{(3)} + 0.25\, x_4^{(2)} = 0.5 + (0.25)(0.84375) + (0.25)(0.59375)$
$= 0.85938$

$x_3^{(3)} = 0.25 + 0.25\, x_1^{(3)} + 0.25\, x_4^{(2)} = 0.25\,[1 + 0.84375 + 0.59375]$
$= 0.60938$

$x_4^{(3)} = 0.25 + 0.25\, x_2^{(3)} + 0.25\, x_3^{(3)} = 0.25\,[1 + 0.85938 + 0.60938]$
$= 0.\underline{61719}$

K = 3

$$x_1^{(4)} = 0.5 + 0.25x_2^{(3)} + 0.25x_3^{(3)} = 0.5 + (0.25)(0.85938) + (0.25)(0.60938) = 0.86719.$$

$$x_2^{(4)} = 0.5 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.5 + 0.25(0.86719) + (0.25)(0.61719) = 0.87110$$

$$x_3^{(4)} = 0.25 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.25\left[1 + 0.46719 + 0.61719\right] = 0.62110$$

$$x_4^{(4)} = 0.25 + 0.25x_2^{(4)} + 0.25x_3^{(4)} = 0.25\left[1 + 0.8711 + 0.6211\right] = 0.62305.$$

K = 4.

$$x_1^{(5)} = 0.5 + 0.25x_2^{(4)} + 0.25x_3^{(4)} = 0.5 + (0.25)(0.8711) + (0.25)(0.6211) = 0.87305.$$

$$x_2^{(5)} = 0.5 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.5 + 0.25 \times 0.87305 + 0.25 \times 0.62305 = 0.87402.$$

$$x_3^{(5)} = 0.25 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.25\left[1 + 0.87305 + 0.62305\right] = 0.62402$$

$$x_4^{(5)} = 0.25 + 0.25x_2^{(5)} + 0.25x_3^{(5)} = 0.25\left[1 + 0.87402 + 0.62402\right] = 0.62451$$

The solution is given by

$x_1 = 0.87305$; $x_2 = 0.87402$

$x_3 = 0.62402$; $x_4 = 0.62451$

7(c) Using Runge-Kutta method of order 4, find y for $x = 0.1, 0.2, 0.3$ given that $dy/dx = xy + y^2$, $y(0) = 1$.

Sol'n: Given that $f(x, y) = \frac{dy}{dx} = xy + y^2$; $y(0) = 1$

To find $y(0.1)$

Here $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$K_1 = hf(x_0, y_0) = (0.1) f(0.1) = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0.1) f(0.05, 1.05) = 0.1155$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0.1) f(0.05, 1.0577) = 0.1172$

$K_4 = hf(x_0 + h, y_0 + K_3) = (0.1) f(0.1, 1.1172) = 0.13598$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}(0.1 + 0.231 + 0.2344 + 0.13598)$

$= 0.11687$

$\therefore y(0.1) = y_1 = y_0 + K = 1.1169$

To find $y(0.2)$: Here $x_1 = 0.1$, $y_1 = 1.1169$, $h = 0.1$

$K_1 = hf(x_1, y_1) = (0.1) f(0.1, 1.1169) = 0.1359$

$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0.1) f(0.15, 1.1848) = 0.1581$

$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = (0.1) f(0.15, 1.1959) = 0.1609$

$K_4 = hf(x_1 + h, y_1 + K_3) = (0.1) f(0.2, 1.2778) = 0.1888$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.1605$

$y(0.2) = y_2 = y_1 + K = 1.2773$.

To find $y(0.3)$: Here $x_2 = 0.2$, $y_2 = 1.2773$, $h = 0.1$

$K_1 = hf(x_2, y_2) = (0.1) f(0.2, 1.2773) = 0.1887$

$K_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_1}{2}\right) = (0.1) f(0.25, 1.3716) = 0.2224$

$K_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{K_2}{2}\right) = (0.1) f(0.25, 1.3885) = 0.2275$

$K_4 = hf(x_2 + h, y_2 + K_3) = (0.1) f(0.3, 1.5048) = 0.2267$

$K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.2267$

$\therefore y(0.3) = y_3 = y_2 + K = 1.504$

7(d) (i) Simplify the expression $A = XY + \overline{X}\overline{Z} + X\overline{Y}Z(XY+Z)$

(ii) Simplify the Boolean expression $Y = \overline{A\cdot B} + \overline{\overline{A}+B}$. Prepare truth table to show that the simplified expression is correct.

Sol^n: (i) $A = XY + \overline{X}\overline{Z} + X\overline{Y}Z(XY+Z)$

$= XY + \overline{X}\overline{Z} + XXY\overline{Y}Z + X\overline{Y}ZZ$

$= XY + \overline{XZ} + X\overline{Y}Z \qquad (\because ZZ = Z \text{ and } Y\overline{Y} = 0)$

$= XY + \overline{X} + \overline{Z} + X\overline{Y}Z$

$= XY + \overline{Z} + \overline{X} + X\overline{Y}Z \qquad$ (by commutative law)

$= XY + \overline{Z} + \overline{X} + \overline{Y}Z$

$= \overline{X} + Y + \overline{Z} + \overline{Y}$

$= 1$

(ii) $Y = \overline{A\cdot B} + \overline{\overline{A}+B}$

$= \overline{A} + \overline{B} + \overline{\overline{A}}\cdot\overline{B}$

$= \overline{A} + \overline{B}$

$= \overline{A\cdot B}$

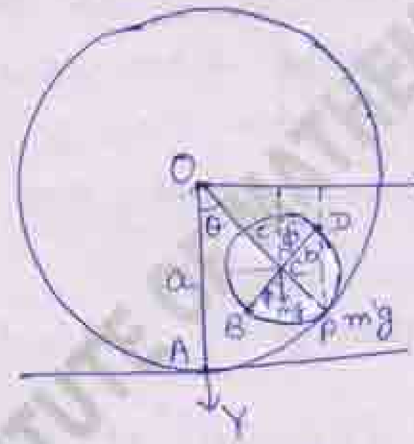
(Using De Morgan's theorem)

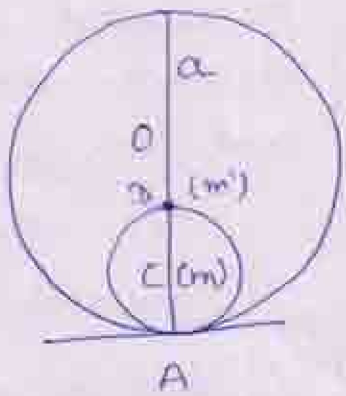
A	B	$A\cdot B$	$\overline{A\cdot B}$	$\overline{A}$	$\overline{A}+B$	$\overline{\overline{A}+B}$	$\overline{A\cdot B}+\overline{\overline{A}+B}$
0	0	0	1	1	1	0	1
0	1	0	1	1	1	0	1
0	0	0	1	0	0	1	1
0	1	1	0	0	1	0	0

8(a) A perfectly rough sphere of mass m and radius b rests at the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m' and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length

$$\frac{(a-b)\cdot 4m'+7m/5}{m+m'(2-a/b)}$$

Sol^n





Let O be the centre of the fixed spherical cavity and C the centre of the sphere of mass m and radius b resting at the lowest point A of the cavity. A particle of mass m' is attached at the highest point D_0 of the sphere. In time t, let the line OC joining centres and the diameter B_0D_0

turn through angles θ and ϕ respectively from the verticals, i.e at time t B & D correspond to the point B_0 and D_0 at time $t=0$.

Since there is no slipping b/w sphere and cavity, therfore if P is their point of contact at time t, then,

Arc AP = Arc PB i.e $a\theta = b(\theta+\phi)$

or $b\phi = (a-b)\theta = c\theta$, where $a-b=c$ (say)

$\therefore b\phi = c\theta$ ——(1)

Referred to the centre O as origin, horizontal and vertical lines OX and OY as axes, the coordinates (x_c, y_c) of C and (x_D, y_D) of D respectively are

$x_c = c\sin\theta$, $y_c = c\cos\theta$

$x_D = c\sin\theta + b\sin\phi$, $y_D = c\cos\theta - b\cos\phi$

$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = (c\cos\theta\dot{\theta})^2 + (-c\sin\theta\dot{\theta})^2 = c^2\dot{\theta}^2$

and $v_D^2 = \dot{x}_D^2 + \dot{y}_D^2 = (c\cos\theta\dot{\theta} + b\cos\phi\dot{\phi})^2$
$+ (-c\sin\theta\dot{\theta} + b\sin\phi\dot{\phi})^2$

$= c^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2bc\dot{\theta}\dot{\phi}\cos(\theta+\phi) = c^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2bc\dot{\theta}\dot{\phi}$

($\because \theta$ and ϕ are small)

If T be the kinetic energy and W the work function of the system, then we have.

$T =$ K.E. of the sphere + K.E. of the particle

$= \left[\frac{1}{2} m \cdot \frac{2}{5} b^2 \dot{\phi}^2 + \frac{1}{2} m v_C^2\right] + \left[\frac{1}{2} m' v_D^2\right]$

$= \frac{1}{2} m \left(\frac{2}{5} b^2 \dot{\phi}^2 + c^2 \dot{\theta}^2\right) + \frac{1}{2} m' \left(c^2 \dot{\theta}^2 + b^2 \dot{\phi}^2 + 2bc\dot{\theta}\dot{\phi}\right)$

$= \frac{1}{2} m \left(\frac{2}{5} b^2 \dot{\phi}^2 + b^2 \dot{\phi}^2\right) + \frac{1}{2} m' \left(b^2 \dot{\phi}^2 + b^2 \dot{\phi}^2 + 2b \cdot b\, \dot{\phi} \cdot \dot{\phi}\right)$ (using ①)

$= \frac{1}{10} b^2 (7m + 20m') \dot{\phi}^2$

and $W = -mg(OC' - y_C) + m'g(y_D - OD_0)$

$= -mg(c - c\cos\theta) + m'g\{c\cos\theta - b\cos\phi - (a - 2b)\}$

$= (m + m') cg \cos\theta - m' bg \cos\phi + C$

$= (m + m') cg \cos(b\phi/c) - m' bg \cos\phi + C$ $(\because c\theta = b\phi)$

$\therefore$ Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{5} b^2 (7m + 20m') \dot{\phi}\right] - 0 = -(m + m') cg \cdot \frac{b}{c} \sin\left(\frac{b}{c}\phi\right) + m' bg \sin\phi$

or $\frac{1}{5} b^2 (7m + 20m') \ddot{\phi} = -(m + m') bg \cdot \frac{b}{c} \phi + m' bg \phi$ $(\because \phi$ is small$)$

or $b\left(\frac{7}{5} m + 4m'\right) \ddot{\phi} = -\frac{g}{c}\left[(m + m') - \frac{c}{b} m'\right] \phi$

or $b\left(4m'+\frac{7}{5}m\right)\ddot{\phi} = \frac{-g}{a-b}\left[(m+m')-\frac{a-b}{b}m'\right]\phi$

$[\because c = a-b]$

$$\ddot{\phi} = \frac{-g}{a-b}\cdot\frac{m+(2-a/b)m'}{4m'+\frac{7}{5}m}\phi = -\mu\phi \text{ (say)}$$

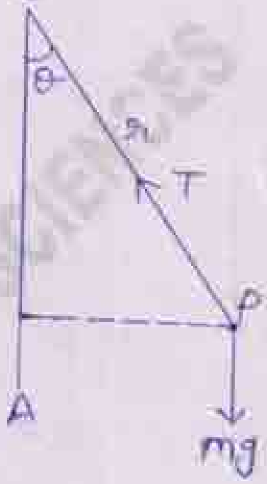
which represent S.H.M.

$\therefore$ The length of simple equivalent pendulum is

$$\frac{g}{\mu} = (a-b)\frac{4m'+7m/5}{m+m'(2-a/b)}$$

8(b) Use Hamilton's equations to write down the equations of motion of a pendulum bob suspended from a coil spring and allowed to swing in a vertical plane.

Soln: At time t, let r be the stretched of length the spring of natural length r_0. If θ is the inclination of the spring to the vertical at this time t, then the velocity of m at P $\dot{r}$ and $r\dot{\theta}$ along and perpendicular to OP respectively.



$\therefore$ Velocity of m at $P = \dot{r}^2 + (r\dot{\theta})^2$

Thus if T and V are the kinetic and the potential energies of the system, at time t, then $T = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)$

and V = Workdone against the forces

= Workdone against the force Mg + workdone against the tension in the spring

$$= -Mg \cdot r\cos\theta \int_0^r \lambda\left(\frac{r-r_0}{r_0}\right) dr$$

$\because$ Tension $= \lambda\left(\frac{r-r_0}{r_0}\right)$

λ is the modulus of elasticity of the spring.

$= -Mgr\cos\theta + (k/2)(r-r_0)^2$, where $k = (\lambda/r_0)$

$\therefore$ $L = T - V = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + Mgr\cos\theta - (k/2)(r-r_0)^2$

Here r and θ are the generalised coordinates.

$\therefore$ $p_r = \frac{\partial L}{\partial \dot{r}} = M\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Mr^2\dot{\theta}$ —— ①

Since L does not contain t explicitly.

$\therefore$ $H = T + V = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mgr\cos\theta + (k/2)(r-r_0)^2$

Substituting $\dot{r} = p_r/M$ and $\dot{\theta} = p_\theta/(Mr^2)$ from ①

$$H = \left(\frac{1}{2M}\right)\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + mgr\cos\theta + \left(\frac{k}{2}\right)(r-r_0)^2 \quad \text{—— ②}$$

Hence the four Hamilton's equations are

$\dot{p}_r = -\frac{\partial H}{\partial r}$ i.e. $\dot{p}_r = \frac{p_\theta^2}{Mr^3} + mg\cos\theta - k(r-r_0)$ —— (H_1)

$\dot{r} = \frac{\partial H}{\partial p_r}$ i.e. $\dot{r} = \frac{p_r}{M}$ —— (H_2)

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$ i.e. $\dot{p}_\theta = -Mgr\sin\theta$ —— (H_3)

and $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{Mr^2}$ —— (H_4)

Differentiating (H_2), we have

$\dot{r} = \frac{1}{m} p_r$ or $M\ddot{r} = \dot{p}_r = \frac{p_\theta^2}{Mr^3} + Mg\cos\theta - k(r - r_0)$

[from (H_1)]

$\therefore M\ddot{r} = \frac{(Mr^2\dot{\theta})^2}{Mr^3} + Mg\cos\theta - k(r - r_0)$

(substituting from (H_4))

or $M\ddot{r} - M H_2 r\dot{\theta}^2 - Mg\cos\theta + k(r - r_0) = 0$ —— (3)

Also from (H_2), we have $Mr^2\dot{\theta} = \dot{p}_\theta$

Differentiating $M(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \dot{p}_\theta$

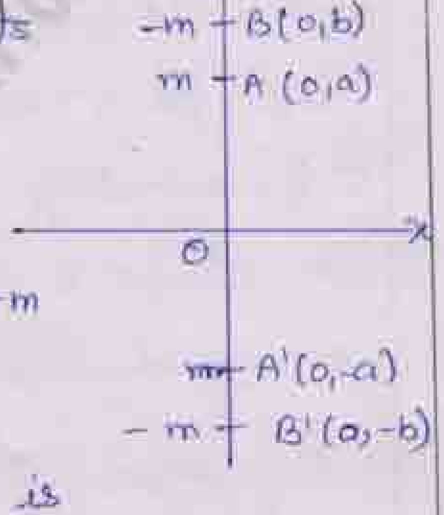
or $M(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = -Mgr\sin\theta$ [from (H_3)]

or $r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0$ —— (4)

Equations (3) & (4) are the required equations of motion.

Q(c) If the fluid fills the region of space on the positive side of x-axis is a rigid boundary, and if there be a source $+m$ at the point $(0,a)$ and an equal sink at $(0,b)$ and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundry is $\pi \rho m^2 (a-b)^2 / ab(a+b)$, where ρ is the density of the fluid.

Sol^n The object system consists of source $+m$ at $A(0,a)$, i.e. at $z = ia$ and sink $-m$ at $z = ib$. The image system consist of source $+m$ at A' $(z = -ia)$ and sink $-m$ at B' $(z = -ib)$ w.r.t. the positive line OX which is rigid boundry. The complex potential due to object system with rigid boundry is equivalent to the object system and its image system with no rigid boundry.



$$\therefore \quad w = -m\log(z-ia) + m\log(z-ib) - m\log(z+ia) + m\log(z+ib)$$

or $\quad w = -m\log(z^2+a^2) + m\log(z^2+b^2)$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right] = \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)|z|}{|z^2+a^2||z^2+b^2|}$$

for any point on x-axis, we have $z = x$ so that

$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis. Let P_0 be the pressure at $x = \infty$. By Bernoulli's equation for steady motion.

$$\therefore \quad \frac{P}{\rho} + \frac{1}{2}q^2 = c$$

In view of $P = P_0$, $q = 0$ when $x = \infty$, we get $c = P_0/\rho$.

$$\frac{P_0 - P}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundry is given by,

$$P = \int_{-\infty}^{\infty}(P_0 - P)\,dx = \int_{-\infty}^{\infty}\frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx.$$

$$= 4\rho m^2(a^2-b^2)^2\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

$$= 4m^2\rho\int_0^\infty\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right\}-\frac{a^2}{(x^2+a^2)^2}-\frac{b^2}{(x^2+b^2)^2}\right]dx.$$

$$= 4m^2\rho\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b}-\frac{\pi}{2a}\right\}-\frac{\pi}{4a}-\frac{\pi}{4b}\right]$$

$$= \frac{\pi\rho m^2(a-b)^2}{ab(a+b)} \quad (\text{Ans.})$$

for $\int_0^\infty \frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty = \frac{\pi}{2a}$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^\infty \cos^2\theta\,d\theta, \quad x = a\tan\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{\pi}{2}\cdot\frac{1}{2a^3}$$

$$= \frac{\pi}{4a^3}$$