VECTOR SPACES   Set-I

Field: Let F be a non-empty set and $+^n$ and $\times^n$ be binary operations on F. Then algebraic structure $(F,+,\cdot)$ is said to be field if the following properties are satisfied.

(I) $(F,+)$ is an abelian group.

i) Closure property : $\forall a,b \in F \Rightarrow a+b \in F$

ii) Asso. prop. :- $\forall a,b,c \in F \Rightarrow (a+b)+c = a+(b+c)$.

(iii) Existence of left identity : $\forall a \in F \; \exists 0 \in F$ s.t $0+a=a$

Here '0' is the identity elt.

(iv) Existence of left inverse :

for each $a \in F$, $\exists -a \in F$ s.t $(-a)+a=0$ (left identity)

Here $-a$ is the inverse of a in F.

(v) comm. prop. : $\forall a,b \in F$, $a+b=b+a$

(II) $(F,\cdot)$ is an abelian group

i) closure prop : $\forall a,b \in F \Rightarrow a\cdot b \in F$

ii) Asso. prop : $\forall a,b,c \in F \Rightarrow (a\cdot b)\cdot c = a\cdot(b\cdot c)$

iii) Existence of left identity:

$\forall a \in F \; \exists 1 \in F$ s.t $1\cdot a = a$.

Here 1 is the identity in F.

(iv) Existence of left inverse:

For each $0 \neq a \in F$ $\exists \frac{1}{a} \in F$ s.t $\frac{1}{a}\cdot a = 1$

$\therefore \frac{1}{a}$ is the inverse of a in F.

(v) comm. prop : $\forall a,b \in F$; $ab=ba$

(III) $\times^n$ is distributive w.r.t $+^n$.

i.e, $\forall a,b,c \in F \Rightarrow a\cdot(b+c) = ab+ac$.

Ex: $(I,+,\cdot)$ is not a field.

$(Q,+,\cdot)$, $(R,+,\cdot)$, $(C,+,\cdot)$ are fields.

$(Q^*,+,\cdot)$, $(R^*,+,\cdot)$, $(C^*,+,\cdot)$ are not fields.

Defn Subfield: Let F be a field and $K \subseteq F$.
If K is a field w.r.t same binary operation in F then K is called subfield of F.

Ex I is not a subfield of Q
Q is a subfield of $\mathbb{R}$.
$\mathbb{R}$ is " " $\mathbb{C}$

Defns
→ Internal composition:

Let A be any set. If $a * b \in A$ $\forall a, b \in A$ then $*$ is said to be internal composition on A.

→ External Composition:
Let V and F be any two sets if $a \circ \alpha \in V$ $\forall a \in F, \alpha \in V$ then '$\circ$' is said to be an external composition in V over F.

→ vector Space or Linear Space:
Let $(F, +, \cdot)$ be a field. The elts of F are called scalars.
Let V be a non-empty set whose elts are called vectors.
The following compositions are defined.
i) An internal composition in V called vector addition
(ii) An external composition in V over the field F called scalar multiplication.
If these compositions satisfy the following axioms then V is called vector space over the field F.

$\boxed{I}$. $(V, +)$ is an abelian group.
(i) closure prop: $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$
(ii) Asso. prop: $\forall \alpha, \beta, \gamma \in V \Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$.

(iii) Existence of identity:

$\forall \alpha \in V, \exists\, 0 \in V$ s.t $\alpha + 0 = 0 + \alpha = \alpha$

Here the identity elt $0 \in V$ is called zero vector.

(iv) Existence of Inverse:

for each $\alpha \in V$, $\exists -\alpha \in V$ s.t $\alpha + (-\alpha) = -\alpha + \alpha = 0$

(v) Comm. prop:

$\forall \alpha, \beta \in V \Rightarrow \alpha + \beta = \beta + \alpha$

$\boxed{\text{II}}$. The two compositions i.e, Scalar x^n and vector $+^n$.

$\forall\ a, b \in F;\ \alpha, \beta \in V \Rightarrow$

(i) $a\cdot(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

iv) $1\alpha = \alpha$; 1 is the unity elt of the field F.

Note: $\boxed{1}$. When V is a vector space over field F then we shall denote it by V(F) and we say that V(F) is a vector space.

$\boxed{2}$. If F is the field $\mathbb{R}$ of real no's then V is called real vector space. Similarly $V(\mathbb{Q}), V(\mathbb{C})$ are called rational, complex vector spaces respectively.

Problems: (1) $V = I$, $F = \mathbb{Q}$

Is V(F) a vector space?

Sol^n Internal Composition.

$\forall \alpha, \beta \in I \Rightarrow \alpha + \beta \in I$

$\therefore$ vector $+^n$ is an internal composition on I.

External Composition:

$\forall a \in \mathbb{Q}, \alpha \in I \Rightarrow a\alpha$ need not be an integer.

Ex $a = \frac{1}{2} \in \mathbb{Q}$, $\alpha = 3 \in I \Rightarrow \frac{1}{2}\cdot 3 = \frac{3}{2} \notin I$.

$\therefore$ scalar x^n is not an external composition. $\therefore$ I over $\mathbb{Q}$

$\therefore I(Q)$ is not a vector space.

Note: If $V \subseteq F$ then $V(F)$ is not a vector space.
(except $V=\{0\} \subseteq F$)

(2) $V = \mathbb{R}$; $F = \mathbb{Q}$

Solⁿ $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta \in \mathbb{R}$.

and $\forall a \in \mathbb{Q} \subseteq \mathbb{R}$, $\alpha \in \mathbb{R} \Rightarrow a\alpha \in \mathbb{R}$.

$\therefore$ Internal and external compositions are satisfied.

$\boxed{I}$ i) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta \in \mathbb{R}$

$\therefore$ Closure prop. is satisfied.

ii) $\forall \alpha, \beta, \gamma \in \mathbb{R}$
$\Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \alpha \in \mathbb{R}$ $\exists\, 0 \in \mathbb{R}$ s.t $\alpha+0 = 0+\alpha = \alpha$

$\therefore$ Identity prop. is satisfied.

$\therefore 0$ is identity elt.

(iv) for each $\alpha \in \mathbb{R}$ $\exists -\alpha \in \mathbb{R}$ s.t $\alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (identity elt in $\mathbb{R}$)

$\therefore$ Inverse of α is $-\alpha$.

$\therefore$ Inverse prop. is satisfied.

(v) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta = \beta+\alpha$

$\therefore$ comm. prop. is satisfied

$\therefore (\mathbb{R}, +)$ is an abelian group.

$\boxed{II}$ $\forall a, b \in \mathbb{Q} \subseteq \mathbb{R}$; $\alpha, \beta \in \mathbb{R}$

(i) $a(\alpha+\beta) = a\alpha + a\beta$ (LDL in $\mathbb{R}$)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in $\mathbb{R}$)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop. in $\mathbb{R}$)

(iv) $1\cdot\alpha = \alpha$ $\forall \alpha \in \mathbb{R}$. (1 is identity w.r.t x^n in $\mathbb{R}$)

$\therefore \mathbb{R}(\mathbb{Q})$ is vector space.

Note: If $F \subseteq V$ then $V(F)$ is a vector space.
Similarly $\mathbb{C}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ are also vector spaces

→ A field K can be regarded as a vector space over any subfield F of K. (3)

Sol^n Given that K is a field and F is a subfield of K.

∴ F is also field w.r.t some b-o's defined in K.

Let us consider the elts of K as vectors.

$\forall\, \alpha, \beta \in K \Rightarrow \alpha+\beta \in K$.

and let us consider the elts of the subfield F as scalars.

Now $a \in F \subseteq K,\ \alpha \in K \Rightarrow a\alpha \in K$.

∴ Internal and external compositions are satisfied.

I. Since K is a field.

∴ $(K,+)$ is an abelian group

II. $\forall\, a, b \in F \subseteq K$; $\alpha, \beta \in K$

(i) $a(\alpha+\beta) = a\alpha + a\beta$ (LDL in K)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in K)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop in K)

(iv) $1\alpha = \alpha \quad \forall\, \alpha \in K$. and 1 is the identity elt of the subfield F.

($\because$ 1 is also identity elt of the field K).

∴ $1\alpha = \alpha \quad \forall\, \alpha \in K$.

∴ $K(F)$ is a vector space.

Note: If F is any field, then F itself is a vector space over the field F.

i.e, $F(F)$ is a vector space.

→ V = Set of all vectors and F is a field of real no's.

Sol^n $\forall\, \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha}+\bar{\beta} \in V$ and

$a \in F, \bar{\alpha} \in V \Rightarrow a\bar{\alpha} \in V$

$\therefore$ Internal and external compositions are satisfied.

$\boxed{\text{I}}$. (i) $\forall\ \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha}+\bar{\beta} \in V$

$\therefore$ Closure prop. is satisfied.

(ii) $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V \Rightarrow (\bar{\alpha}+\bar{\beta})+\bar{\gamma} = \bar{\alpha}+(\bar{\beta}+\bar{\gamma})$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall\ \bar{\alpha} \in V\ \exists\ \bar{0} \in V$ s.t $\bar{\alpha}+\bar{0} = \bar{0}+\bar{\alpha} = \bar{\alpha}$

$\therefore \bar{0}$ is the identity vector in V.

(iv) for each $\bar{\alpha} \in V\ \exists\ -\bar{\alpha} \in V$ s.t $\bar{\alpha}+(-\bar{\alpha}) = (-\bar{\alpha})+\bar{\alpha} = \bar{0}$ (zero vector)

$\therefore$ inverse of $\bar{\alpha}$ is $-\bar{\alpha}$

(v) $\forall\ \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha}+\bar{\beta} = \bar{\beta}+\bar{\alpha}$

$\therefore$ comm. prop. is satisfied.

$\boxed{\text{II}}$. $\forall\ a, b \in \mathbb{R}$; $\bar{\alpha}, \bar{\beta} \in V$

(i) $a(\bar{\alpha}+\bar{\beta}) = a\bar{\alpha}+a\bar{\beta}$

(ii) $(a+b)\bar{\alpha} = a\bar{\alpha}+b\bar{\alpha}$

(iii) $(ab)\bar{\alpha} = a(b\bar{\alpha})$

(iv) $1\bar{\alpha} = \bar{\alpha}\ \ \forall\ \bar{\alpha} \in V$

$\therefore$ V($\mathbb{R}$) is a vector space.

$\rightarrow$ V = Set of all $m \times n$ matrices with their elts as real numbers and $F = \mathbb{R}$.

Note : If V = the set of all $m \times n$ matrices with their elts as rational numbers and $F = \mathbb{R}$. then V(F) is not a vector space.

Because there is no external composition

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \in V$; $\sqrt{7} \in \mathbb{R}$. then $\sqrt{7}A = \begin{bmatrix} \sqrt{7} & 2\sqrt{7} & 3\sqrt{7} \\ 0 & \sqrt{7} & 2\sqrt{7} \end{bmatrix} \notin V$.

($\because$ the elts of resulting matrix are not rational numbers)

→ Similarly, If V = the set of all $m \times n$ matrices with their elts as real numbers. (4)

and $F = \mathbb{C}$ (complex numbers)

then $V(F)$ is not vector space.

→ If V = the set of all $m \times n$ matrices with their elts as integers. and $F = \mathbb{Q}$. (rational numbers) then $V(F)$ is not a vector space.

→ V = the set of all ordered n-tuples and F is any field.

Sol^n Let $V = \{(a_1, a_2, \dots a_n) / a_1, a_2, \dots a_n \in F\}$

Let $\alpha, \beta \in V$

Choose $\alpha = (a_1, a_2, \dots a_n)$

$\beta = (b_1, b_2, \dots b_n)$

where $a_1, a_2, \dots a_n \in F$

$b_1, b_2, \dots b_n \in F$

$\Rightarrow \alpha + \beta = (a_1, a_2 \dots a_n) + (b_1, b_2, \dots b_n)$

$= (a_1 + b_1, a_2 + b_2, \dots a_n + b_n) \in V$

$\because (a_1 + b_1, a_2 + b_2 \dots a_n + b_n \in F)$

$\therefore$ Internal composition is satisfied.

and $a \in F, \alpha \in V$

$\Rightarrow a\alpha = a(a_1, a_2 \dots a_n)$

$= (aa_1, aa_2 \dots aa_n) \in V$

$\because aa_1, aa_2 \dots aa_n \in F$

$\therefore$ External composition is satisfied.

$\boxed{\text{I}}$. (i) $\forall \alpha, \beta \in V$

$\Rightarrow \alpha + \beta = (a_1 + b_1, a_2 + b_2, \dots a_n + b_n) \in V$

$\because a_1 + b_1, a_2 + b_2, \dots a_n + b_n \in V$

(ii) $\forall \alpha, \beta, \gamma \in V$

$(\alpha + \beta) + \gamma = [(a_1, a_2 \dots a_n) + (b_1, b_2, \dots b_n)] + (c_1, c_2 \dots c_n)$

$= (a_1+b_1, a_2+b_2, \ldots a_n+b_n) + (c_1, c_2 \ldots c_n)$

$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \ldots (a_n+b_n)+c_n)$

$= (a_1+(b_1+c_1), a_2+(b_2+c_2), \ldots, a_n+(b_n+c_n))$

(by asso. prop. w.r.t $+^n$ in F)

$= (a_1, a_2 \ldots a_n) + (b_1+c_1, b_2+c_2, \ldots b_n+c_n)$

$= (a_1, a_2, \ldots a_n) + [(b_1, b_2 \ldots b_n) + (c_1, c_2 \ldots c_n)]$

$= \alpha + (\beta + \gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) we have $0 = (0, 0, \ldots 0) \in V$ where $0 \in F$

if $\alpha = (a_1, a_2, \ldots a_n) \in V$ where $a_1, a_2, \ldots a_n \in F$

then $0+\alpha = (0, 0, 0, \ldots 0) + (a_1, a_2, \ldots a_n)$

$= (0+a_1, 0+a_2, \ldots 0+a_n)$

$= (a_1, a_2 \ldots a_n)$

$= \alpha$

Similarly $\alpha + 0 = \alpha$.

$\therefore 0+\alpha = 0+\alpha = \alpha$

$\therefore 0 = (0, 0, 0 \ldots 0)$ is the identity elt in V.

(iv) If $\alpha = (a_1, a_2, \ldots a_n) \in V$ where $a_1, a_2 \ldots a_n \in F$

then $-\alpha = -(a_1, a_2, \ldots a_n)$

$= (-a_1, -a_2, \ldots -a_n) \in V$

where $-a_1, -a_2, \ldots -a_n \in F$

Now $(-\alpha) + \alpha = ((-a_1)+a_1, (-a_2)+a_2, \ldots (-a_n)+a_n)$

$= (0, 0, \ldots 0)$

Similarly $\alpha + (-\alpha) = 0$

$\therefore (-\alpha) + \alpha = \alpha + (-\alpha) = 0$ $\therefore -\alpha$ is the inverse of α in V.

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(v). $\forall \alpha, \beta \in V \Rightarrow \alpha+\beta = (a_1, a_2, \cdots a_n) + (b_1, b_2 \cdots b_n)$

$= (a_1+b_1, a_2+b_2, \cdots\cdots a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \cdots\cdots b_n+a_n)$

$= (b_1\ b_2 \cdots b_n) + (a_1\ a_2 \cdots a_n)$

$= \beta+\alpha.$

$\therefore (V,+)$ is an abelian group.

$\boxed{\text{II}}$. for $\alpha, \beta \in V$; $a, b \in F$

(i) $a(\alpha+\beta) = a[(a_1, a_2, \cdots\cdots a_n) + (b_1, b_2, \cdots b_n)]$

$= a[a_1+b_1, a_2+b_2, \cdots\cdots a_n+b_n]$

$= (a(a_1+b_1), a(a_2+b_2) \cdots a(a_n+b_n))$

$= (aa_1+ab_1, aa_2+ab_2, \cdots aa_n+ab_n)$ (By LDL in F)

$= (aa_1, aa_2 \cdots aa_n) + (ab_1, ab_2, \cdots ab_n)$

$= a(a_1, a_2, \cdots a_n) + a(b_1, b_2, \cdots b_n)$

$= a\alpha + a\beta.$

(ii) $(a+b)\alpha = (a+b)(a_1, a_2, \cdots a_n)$

$= ((a+b)a_1, (a+b)a_2, \cdots (a+b)a_n)$

$= (aa_1+ba_1, aa_2+ba_2, \cdots aa_n+ba_n)$ (By RDL in F)

$= (aa_1, aa_2 \cdots aa_n) + (ba_1, ba_2 \cdots ba_n)$

$= a(a_1\ a_2 \cdots a_n) + b(a_1\ a_2 \cdots a_n)$

$= a\alpha + b\alpha.$

(iii) $(ab)\alpha = (ab)(a_1\ a_2 \cdots a_n)$

$= ((ab)a_1, (ab)a_2, \cdots (ab)a_n)$

$= (a(ba_1), a(ba_2), \cdots a(ba_n))$ (By asso. prop in F)

$= a(ba_1, ba_2 \cdots ba_n)$

$= a[b(a_1, a_2, \cdots a_n)]$

$= a(b\alpha)$

(iv) $1\alpha = 1(a_1, a_2, \cdots a_n)$
$= (1a_1, 1a_2 \cdots 1a_n)$
$= (a_1, a_2, \cdots a_n)$ $\left(\because 1 \in F, a_1 \in F \Rightarrow 1a_1 \in F \therefore 1a_1, 1a_2 \cdots 1a_n \in F. \right)$
$= \alpha \quad \forall \alpha \in V.$

$\therefore V(F)$ is a vector space.

Note

[1] → The vector space of all ordered n-tuples over F is denoted by $V_n(F)$.

→ Some times denote it by $F^{(n)}$ or F^n.

$\therefore V_n(F)$ or $F^{(n)} = \{(a_1, a_2, \cdots a_n) / a_1, a_2 \cdots a_n \in F\}$

[2]. $V_2(F) = \{(a_1, a_2) / a_1, a_2 \in F\}$ is a vector space of all ordered pairs over F.

Similarly $V_3(F) = \{(a_1, a_2, a_3) / a_1, a_2, a_3 \in F\}$ is the vector space of all ordered triples or triads over F.

→ $F[x]$ = The set of all polynomials and F is any field.

Sol^n $F[x] = \{ f(x) / f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots \cdots ; a_0, a_1, \cdots \cdots a_n \cdots \in F \}$

Now $f(x), g(x) \in F[x]$

Choose $f(x) = \sum_{i=0}^{\infty} a_i x^i$

$g(x) = \sum_{i=0}^{\infty} b_i x^i$; where $a_i, b_i \in F$, $i = 0, 1, 2, \cdots$

$\Rightarrow f(x) + g(x) = (a_0 x^0 + a_1 x + a_2 x^2 + \cdots) + (b_0 + b_1 x + b_2 x^2 + \cdots)$
$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots \cdots$
$= \sum (a_i + b_i) x^i \in F[x].$
$\because a_i + b_i \in F, i = 0, 1, 2, \cdots$

$\therefore$ External Composition is satisfied.

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Now $f(x) \in F[x]$; $a \in F$

$$a f(x) = a(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$= aa_0 + (aa_1)x + (aa_2)x^2 + \cdots$$
$$= \Sigma (aa_i) x^i \in F[x]$$

$\left(\because a, a_i \in F, i = 0,1,2\ldots \Rightarrow aa_i \in F.\right)$

$\therefore$ External composition is satisfied.

[I] (i) $\forall f(x), g(x) \in F[x]$, where $f(x) = \Sigma a_i x^i$, $g(x) = \Sigma b_i x^i$, $a_i, b_i \in F$, $i = 0,1,2,\ldots$

$$\Rightarrow f(x) + g(x) = \Sigma a_i x^i + \Sigma b_i x^i$$
$$= \Sigma (a_i + b_i) x^i \in F[x] \quad [\because a_i + b_i \in F]$$

(ii) $\forall f(x), g(x), h(x) \in F[x]$

$$\Rightarrow [f(x) + g(x)] + h(x) = \left[\Sigma a_i x^i + \Sigma b_i x^i\right] + \Sigma c_i x^i$$
$$= \Sigma (a_i + b_i) x^i + \Sigma c_i x^i$$
$$= \Sigma [(a_i + b_i) + c_i] x^i$$
$$= \Sigma [a_i + (b_i + c_i)] x^i \quad (\because \text{By asso. prop in } F)$$
$$= \Sigma a_i x^i + \Sigma (b_i + c_i) x^i$$
$$= \Sigma a_i x^i + \left[\Sigma b_i x^i + \Sigma c_i x^i\right]$$
$$= f(x) + [g(x) + h(x)]$$

$\therefore$ Asso. prop. is satisfied.

(iii) we have $0 = 0 + 0x + 0x^2 + \cdots$
(zero polynomial)

$$= \Sigma 0 x^i \; ; \; 0 \in F$$
$$\in F[x]$$

if $f(x) = \Sigma a_i x^i \in F[x]$; $a_i \in F$, $i = 0,1,2,\ldots$

then
$$0 + f(x) = \Sigma 0 x^i + \Sigma a_i x^i$$
$$= \Sigma (0 + a_i) x^i$$
$$= \Sigma a_i x^i$$
$$= f(x)$$

Similarly $f(x)+0 = f(x)$.

$\therefore 0+f(x) = f(x)+0 = f(x) \quad \forall f(x) \in F[x]$.

$\therefore$ Identity elt is the zero polynomial.

(iv) if $f(x) \in F[x]$ then $-f(x) \in F[x]$.

i.e, $f(x) = a_0 + a_1 x + \cdots \in F[x]$; $a_0, a_1, a_2 \cdots \in F$

then $-f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \cdots + \cdots \in F[x]$;

$-a_0, -a_1, -a_2 \cdots \in F$.

we have

$$(-f(x)) + f(x) = (-a_0+a_0) + (-a_1+a_1)x + (-a_2+a_2)x^2 + \cdots$$

$$= 0 + 0x + 0x^2 + \cdots$$

$$= 0 \text{ (zero polynomial)}$$

Similarly $f(x) + (-f(x)) = 0$

$\therefore (-f(x)) + f(x) = f(x) + (-f(x)) = 0$

$\therefore -f(x)$ is the inverse polynomial of $f(x)$ in $F[x]$

(v) $\forall f(x), g(x) \in F[x]$

$$\Rightarrow f(x) + g(x) = (a_0+b_0)x^0 + (a_1+b_1)x + \cdots$$

$$= (b_0+a_0)x^0 + (b_1+a_1)x + \cdots$$

(By asso. prop in F)

$$= (b_0 + b_1 x + \cdots) + (a_0 + a_1 x + \cdots)$$

$$= g(x) + f(x)$$

$\therefore$ Commutative property is satisfied.

$\therefore (F[x], +)$ is an abelian group.

$\boxed{\text{II}}$. $\forall f(x), g(x) \in F[x]$; $a, b \in F$

(i) $a(f(x)+g(x)) = a\left[(a_0 + a_1 x + a_2 x^2 + \cdots) + (b_0 + b_1 x + b_2 x^2 + \cdots)\right]$

$$= a\left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots\right]$$

$$= a(a_0+b_0) + a(a_1+b_1)x + a(a_2+b_2)x^2 + \cdots$$

$$= (aa_0 + ab_0) + (aa_1 + ab_1)x + (aa_2 + bb_2)x^2 + \cdots$$

(By asso. prop. in F.)

$$= (aa_0 + (aa_1)x + (aa_2)x^2 + \cdots) + (ab_0 + (ab_1)x + (ab_2)x^2 + \cdots)$$

$$= a(a_0 + a_1x + a_2x^2 + \cdots) + a(b_0 + b_1x + b_2x^2 + \cdots)$$
$$= a\,f(x) + a\,g(x)$$

(ii) $(a+b)f(x) = (a+b)(a_0 + a_1x + a_2x^2 + \cdots)$
$$= [(a+b)a_0] + [(a+b)a_1]x + [(a+b)a_2]x^2 + \cdots$$
$$= (aa_0 + ba_0) + (aa_1 + ba_1)x + \cdots$$
$$= (aa_0 + (aa_1)x + \cdots) + (ba_0 + (ba_1)x + \cdots)$$
$$= a(a_0 + a_1x + \cdots) + b(a_0 + a_1x + \cdots)$$
$$= a\,f(x) + b\,f(x).$$

Similarly

(iii) $(ab)f(x) = a\,b f(x)$

iv) $1\,f(x) = f(x).\ \forall f(x) \in F[x]$.

→ Let F be the field and let P_n be the set of all polynomials (of degree atmost n) over the field F.
S.T P_n is vector space over the field F.

Sol^n Let $P_n = \{ f(x) \mid f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_0, a_1, \ldots a_n \in F \}$.

$\forall f(x), g(x) \in P_n$

Choose $f(x) = a_0 + a_1x + \cdots + a_nx^n$
$g(x) = b_0 + b_1x + \cdots + b_nx^n$
$a_0, a_1, \ldots a_n;\ b_1 b_2 \ldots b_n \in F$.

$\Rightarrow f(x) + g(x) = (a_0+b_0) + (a_1+b_1)x + \cdots + (a_n+b_n)x^n$
$\in P_n$. $(\because a_0+b_0, a_1+b_1, \ldots a_n+b_n \in F$ and polynomial of degree atmost $n)$

$\forall f(x) \in P_n;\ c \in F$

$\Rightarrow c f(x) = ca_0 + (ca_1)x + (ca_2)x^2 + \cdots + (ca_n)x^n.$
$\in P_n$ $(\because ca_0, ca_1, \ldots ca_n \in F$ and polynomial of degree atmost $n)$

$\therefore$ Internal and external compositions are satisfied.

[I] (i) $\forall f(x), g(x) \in P_n \Rightarrow f(x)+g(x) \in P_n$

$\therefore$ Closure prop. is satisfied.

(ii) $\forall f(x), g(x), h(x) \in P_n$

$\Rightarrow (f(x)+g(x))+h(x)$

$= f(x)+(g(x)+h(x))$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall f(x) \in P_n \; \exists \; I(x) = 0 + 0x + 0x^2 + \cdots + 0x^n \in P_n$ or F.

s.t $f(x)+I(x) = f(x)$.

Similarly $I(x)+f(x) = f(x)$

$\therefore f(x)+I(x) = I(x)+f(x) = f(x)$

$\forall f(x) \in P_n$

$\therefore I(x)$ is the identity polynomial in P_n.

iv) Inverse prop.

(v) Commutative prop:

$\therefore (P_n, +)$ is an abelian group.

[II]. $\forall f(x), g(x) \in P_n$; $a, b \in F$

we have (i) $a(f(x)+g(x)) = a[(a_0+a_1x+a_2x^2+\cdots a_nx^n)$
$+(b_0+b_1x+\cdots b_nx^n)]$

$= a[(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n]$

$= a(a_0+b_0)+a(a_1+b_1)x+\cdots+a(a_n+b_n)x^n$

$=(aa_0+ab_0)+(aa_1+ab_1)x+\cdots(aa_n+ab_n)x^n$

(By LDL in F)

$=(aa_0+aa_1x+\cdots+aa_nx^n)+(ab_0+(ab_1)x+\cdots(ab_n)x^n$

$= a(a_0+a_1x+\cdots+a_nx^n)+a(b_0+b_1x+\cdots+b_nx^n)$

$= af(x)+ag(x)$.

(ii) Sly $(a+b)f(x) = af(x)+bf(x)$

(iii) Sly $(ab)f(x) = a(bf(x))$

(v) Sly $1f(x) = f(x) \; \forall f(x) \in P_n$. $\therefore P_n(F)$ is a vector space

→ Let F be any field and S be any non-empty set. Let V be the set of all functions from S to F. (8)

i.e, $V = \{f / f: S \to F\}$

Let us define sum of two vectors f and g in V as follows.

$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$

and the product of the scalar 'c' in F and the function f in V as follows:

$(cf)(x) = c\,f(x) \quad \forall x \in S.$

then $V(F)$ is vectorspace.

Sol^n $\forall f, g \in V \Rightarrow (f+g)(x) = f(x) + g(x) \quad \forall x \in S$ (By defn)

Since $f(x), g(x) \in F$ and F is a field.

$\Rightarrow f(x) + g(x) \in F$

$\therefore (f+g)(x) = f(x) + g(x) \in F$

$\therefore (f+g): S \to F$

$\therefore f+g \in V.$

Internal composition is satisfied.

$\forall f \in V, c \in F \Rightarrow (cf)(x) = c\,f(x) \quad \forall x \in S$ (By defn)

Since $f(x) \in F$, $c \in F$ and F is a field.

$\therefore c\,f(x) \in F$

$\therefore cf: S \to F.$

$\Rightarrow cf \in V; \forall c \in F, f \in V$

External composition is satisfied.

$\boxed{I}$ (i) $\forall f, g \in V \Rightarrow f+g \in V$. Closure prop. is satisfied

(ii) $\forall f, g, h \in V$

$\Rightarrow [(f+g)+h](x) = (f+g)(x) + h(x) = (f(x)+g(x)) + h(x)$

$= f(x) + [g(x) + h(x)]$

$= [f+(g+h)](x)$

By asso. prop in F
i.e, $f(x), g(x), h(x) \in F$
$\Rightarrow [f(x)+g(x)]+h(x)$
$= f(x)+[g(x)+h(x)]$

$\therefore (f+g)+h = f+(g+h)$

(iii) If $\forall x \in S$, $I(x) = 0 \in F$ then $I \in V$. (i.e $I: S \to F$)

now $(I+f)(x) = I(x)+f(x)$
$= 0+f(x)$
$= f(x) \quad \forall f(x) \in F$

$\therefore I+f = f \quad \forall f \in V$

Sly $f+I = f \quad \forall f \in V$

$\therefore I+f = f+I = f \quad \forall f \in V$

$\therefore$ Identity elt $= I \in V$.

(iv) if $f \in V$, then $-f = (-1)f \in V$.

now $[f+(-f)](x) = f(x)+(-f)(x)$
$= f(x)+[-1 f(x)]$
$= f(x)-f(x)$
$= 0 = I(x)$.

$\therefore f+(-f) = 0 = I$

Sly $(-f)+f = 0 = I$.

$\therefore f+(-f) = -f+f = 0 = I$.

$\therefore$ Inverse of f is $-f$ in V.

(v) $\forall f, g \in V \Rightarrow (f+g)(x) = f(x)+g(x)$ (By defn)
$= g(x)+f(x)$ By ~~comm~~ in F
$= (g+f)(x)$ i.e, $f(x), g(x) \in F$
$\Rightarrow f(x)+g(x)$
$= g(x)+f(x)$

$\therefore f+g = g+f$

$\therefore$ Commutative prop. is satisfied.

$\boxed{II}$ $\forall a, b \in F$, $f, g \in V$

(i) $[a(f+g)](x) = a(f+g)(x)$ (By defn)
$= a(f(x)+g(x))$ (By defn)
$= af(x)+ag(x)$ (By LDL in F)
$= (af)(x)+(ag)(x) = (af+ag)(x)$ $\therefore a(f+g) = af+ag$

(ii) $[(a+b)f](x) = (a+b)f(x)$ (By defn)

$= af(x) + b.f(x)$

$= (af)(x) + (bf)(x)$

$= (af+bf)(x)$

$\therefore (a+b)f = af+bf.$

(iii) $[(ab)f](x) = (ab)f(x)$ (by defn)

$= a(bf(x))$

$= a(bf)(x)$

$\therefore (ab)f = a(bf)$

(iv) $(1f)(x) = 1\cdot f(x)$ (By defn)

$= f(x)$ By identity in F $\forall f(x) \in F$

$\therefore 1f = f \quad \forall f \in V$

$\therefore V(F)$ is a vector space.

→ Let V be the set of all pairs (x, y) of real numbers and let F be the field of real numbers. Examine in each of the following cases whether V is a vector space over the field of real numbers or not?

(1) $(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

$c(x,y) = (cx, y)$

not II(iv) fail (2) $(x,y) + (x_1, y_1) = (x+x_1, 0)$

$c(x,y) = (cx, 0)$

not II(ii) fail (3) $(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

$c(x,y) = (|c|x, |c|y)$

not II(ii) fail (4) $(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

$c(x,y) = (c^2x, c^2y)$

not (5) If (v) fail

$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$

$c(x, y) = (0, cy)$.

Sol^n → ① $\boxed{I}$ Let $\alpha = (x, y)$, $\beta = (x_1, y_1) \in V$ where $x, y, x_1, y_1 \in \mathbb{R}$.

(i) $\alpha + \beta = (x, y) + (x_1, y_1)$

$= (x + x_1, y + y_1) \in V$ $(\because x + x_1, y + y_1 \in \mathbb{R})$

$\therefore$ closure prop. is satisfied.

(ii) $(\alpha + \beta) + \gamma = [(x, y) + (x_1, y_1)] + (x_2, y_2)$

$= (x + x_1, y + y_1) + (x_2, y_2)$

$= ((x + x_1) + x_2, (y + y_1) + y_2)$ (By defn)

$= (x + (x_1 + x_2), y + (y_1 + y_2))$; (By asso. prop in $\mathbb{R}$)

$= (x, y) + (x_1 + x_2, y_1 + y_2)$

$= (x, y) + [(x_1, y_1) + (x_2, y_2)]$

$= \alpha + (\beta + \gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \alpha = (x, y) \in V$ $\exists\, 0 = (0, 0) \in V$, $0 \in \mathbb{R}$

s.t $\alpha + 0 = (x, y) + (0, 0)$

$= (x + 0, y + 0)$

$= (x, y)$

$= \alpha$

S^{ly} $0 + \alpha = \alpha$.

$\therefore 0 + \alpha = \alpha + 0 = \alpha$

$\therefore (0, 0)$ is the identity in V.

(iv) For each $\alpha \in V$ $= (x, y)$ $\exists\, -\alpha = (-x, -y) \in V$; $-x, -y \in \mathbb{R}$.

s.t $\alpha + (-\alpha) = (x, y) + (-x, -y)$

$= (x-x, y-y)$ (By defn)

$= (0, 0)$

$\text{Sly } (-\alpha)+\alpha = (0,0)$

$\therefore \alpha+(-\alpha) = (-\alpha)+\alpha = (0,0)$

$\therefore -\alpha$ is the inverse of α.

(v) $\forall\ \alpha, \beta \in V \Rightarrow \alpha+\beta = \beta+\alpha$ (by defn)

$\therefore$ comm. prop. is satisfied.

$\therefore (V,+)$ is an abelian group.

$\boxed{\text{II}}$ $\forall\ \alpha, \beta \in V$; $a, b \in \mathbb{R}$

(i) $a(\alpha+\beta) = a\left[(x,y)+(x_1,y_1)\right]$

$= a(x+x_1, y+y_1)$ (By defn)

$= (a(x+x_1), y+y_1)$ (by defn) — (1)

and $a\alpha + a\beta = a(x,y) + a(x_1,y_1)$

$= (ax, y) + (ax_1, y_1)$

$= (a(x+x_1), y+y_1)$ — (2)

$\therefore$ from (1) & (2)

$a(\alpha+\beta) = a\alpha + a\beta$.

(ii) $(a+b)\alpha = (a+b)(x,y)$

$= ((a+b)x, y)$ — (1)

and $a\alpha + b\alpha = a(x,y) + b(x,y)$

$= (ax, y) + (bx, y)$

$= ((a+b)x, 2y)$ — (2)

$\therefore$ from (1) & (2) we have

$(a+b)\alpha \neq a\alpha + b\alpha$.

$\therefore V(\mathbb{R})$ is not a vector space.

Sol^n (2), $\boxed{\text{II}}$ ii

Let $\alpha = (x,y) \in V$; $x, y \in \mathbb{R}$

then $1\alpha = 1(x,y) = (1x, 0)$ (By defn)

$= (x, 0)$

But $(x,0) \neq (x,y)$ (if $y \neq 0$)

$\therefore 1\alpha \neq \alpha\ \forall\ \alpha \in V$

$\therefore V(\mathbb{R})$ is not a vector space.

→ Let $V(F)$ be a vector space and $\vec{0}$ be the zero vector of V. Then

(i) $a\vec{0} = \vec{0}$ $\forall a \in F$

(ii) $0\alpha = \vec{0}$ $\forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha)$ $\forall a \in F$, $\forall \alpha \in V$

(iv) $(-a)\alpha = -(a\alpha)$ $\forall a \in F$, $\forall \alpha \in V$

(v) $a(\alpha - \beta) = a\alpha - a\beta$ $\forall a \in F$, and $\forall \alpha, \beta \in V$.

(vi) $a\alpha = 0 \Rightarrow a = 0$ (or) $\alpha = 0$

proof:

(i) we have $a0 = a(0+0)$ $\quad (\because 0 = 0+0)$

$= a0 + a0$ $\quad (\because a(\alpha+\beta) = a\alpha + a\beta$ $a \in F, \alpha, \beta \in V)$

$\Rightarrow a0 + 0 = a0 + a0$

$(\because a0 \in V$ and $0 + a0 = a0)$

Given that $V(F)$ be a vector space

$\therefore$ V is an abelian group w.r.t addition

Therefore by ~~right~~ left cancellation law in V,

we get $0 = a0$

$\Rightarrow \boxed{a0 = 0}$ $\forall a \in F$.

(ii) we have $0\alpha = (0+0)\alpha$ $\quad (\because 0 = 0+0)$

$= 0\alpha + 0\alpha$

$\Rightarrow 0 + 0\alpha = 0\alpha + 0\alpha$

$(\because 0\alpha \in V$ and $0 + 0\alpha = 0\alpha)$

Since V is an abelian group w.r.t addition.

Therefore by right cancellation law in V, we get $0 = 0\alpha$

$\therefore 0\alpha = 0 \quad \forall \alpha \in V.$

(iii) we have $a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$

$\Rightarrow a0 = a\alpha + a(-\alpha)$

$\Rightarrow 0 = a\alpha + a(-\alpha)$

$\Rightarrow a(-\alpha)$ is the additive inverse of α

$\Rightarrow a(-\alpha) = -a\alpha$

$\therefore a(-\alpha) = -a\alpha \quad \forall a \in F, \ \forall \alpha \in V$

(iv) we have $[a + (-a)]\alpha = a\alpha + (-a)\alpha$

$\Rightarrow 0\alpha = a\alpha + (-a)\alpha$

$\Rightarrow 0 = a\alpha + (-a)\alpha$

$\Rightarrow (-a)\alpha$ is the additive inverse of $a\alpha$.

$\Rightarrow (-a)\alpha = -a\alpha$

$\therefore (-a)\alpha = -a\alpha \quad \forall a \in F, \ \forall \alpha \in V.$

(v) we have $a(\alpha - \beta) = a(\alpha + (-\beta))$

$= a\alpha + a(-\beta)$

$= a\alpha + [-(a\beta)] \quad (\because a(-\beta) = -a\beta)$

$= a\alpha - a\beta$

$\therefore a(\alpha - \beta) = a\alpha - a\beta \quad \forall a \in F, \ \forall \alpha, \beta \in V.$

(vi) Let $a\alpha = 0$ and $a \neq 0$.

Then a^{-1} exists because 'a' is a non-element of the field F.

$\therefore a\alpha = 0 \Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$

$\Rightarrow 1\alpha = 0$
$\Rightarrow \alpha = 0$

Again let $a\alpha = 0$ and $\alpha \neq 0$.

Then to prove that $a = 0$.

If possible suppose that $a \neq 0$.

Then a^{-1} exists.

$\therefore a\alpha = 0 \Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$

$\Rightarrow 1\alpha = 0$

$\Rightarrow \alpha = 0$

Then we get a contradiction that α must be a zero vector.

Therefore a must be equal to 0.

Hence $\alpha \neq 0$ and $a\alpha = 0$

$\Rightarrow a = 0$

$\therefore a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

→ Let $V(F)$ be a vector space. Then

(i) If $a, b \in F$ and α is a non-zero vector of V, we have $a\alpha = b\alpha \Rightarrow a = b$

(ii) If $\alpha, \beta \in V$ and a is a non-zero element of F. we have

$a\alpha = a\beta \Rightarrow \alpha = \beta$.

proof: (i) we have $a\alpha = b\alpha$

$\Rightarrow a\alpha - b\alpha = 0$

$\Rightarrow (a-b)\alpha = 0$

$\Rightarrow a-b = 0$

$\Rightarrow a = b$

(ii) we have $a\alpha = a\beta$

$\Rightarrow a\alpha - a\beta = 0$

$\Rightarrow a(\alpha-\beta) = 0$

$\Rightarrow \alpha-\beta = 0$, Since $a \neq 0$

$\Rightarrow \alpha = \beta$

→ On R^n, define two operations

$$\alpha \oplus \beta = \alpha - \beta$$
$$c \cdot \alpha = -c\alpha$$

The operations on the right are the usual ones. which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

→ Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = cf(t)$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real-valued.

10(iii)

→ Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$u+v = u \cdot v$ for all $u, v \in R^+$

$\alpha u = u^\alpha$ for all $u \in R^+$ and real scalar α.

prove that R^+ is a real vector space.

→ which of the following subsets of V_4 are vector spaces for coordinatewise addition and scalar multiplication?

The set of all vectors $(x_1, x_2, x_3, x_4) \in V_4$ such that

(a) $x_4 = 0$ (b) $x_1 = 0$ (c) $x_2 > 0$ (d) $x_3^2 \geq 0$ (e) $x_1^2 < 0$

(f) $2x_1 + 3x_2 = 0$ (g) $x_1 + \frac{2}{3}x_2 - 3x_3 + x_4 = 1$. (h) $x_1 = 1$

Ans: (a), (b), (f) and (h) are vector spaces.

→ which of the following subsets of P are vector spaces?

The set of all polynomials p such that

(a) degree of $p \leq n$ (b) degree of $p = 3$

(c) degree of $p \geq 4$ (d) $p(1) = 0$

(e) $p(2) = 1$ (f) $p'(1) = 0$

(g) p has integral coefficients.

Ans: (a), (d) and (f) are vector spaces.

Notations:

$\mathcal{C}[a,b]$ = the set of all real-valued functions defined and continuous on the closed interval $[a,b]$.

$\mathcal{C}^{(1)}[a,b]$ = the set of all real-valued functions defined on $[a,b]$ and whose first derivatives are continuous on $[a,b]$.

$\mathcal{C}^{(n)}[a,b]$ = the set of all real-valued functions defined on $[a,b]$, differentiable n-times and whose n^{th} derivatives are continuous on $[a,b]$. These functions are called n-times continuously differentiable functions.

→ Which of the following subsets $\mathcal{C}[0,1]$ are vector spaces.?

The set of all functions $f \in \mathcal{C}[a,b]$ such that

yes (a) $f(1/2) = 0$ (b) $f(3/4) = 0$ yes (c) $f'(x) = x f(x)$

yes (d) $f(0) = f(1)$ (e) $f(x) = 0$ at a finite number of points in $[0,1]$

yes (f) f has a local minima at $x = 1/2$

(g) f has a local extrema at $x = 1/2$.

Ans: (a), (c), (d) & (f) are vector spaces.

10 (iv)

→ Is Z_7 a vector space over Z_5 ?

Sol^n: NO.

$Z_5 = \{0, 1, 2, 3, 4\}$ is not subfield of

$Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

because:

$2+3=0$ in Z_5 but $2+3 \neq 0$ in Z_7.

Hence Z_5 is not a vector space over Z_5.

→ Let $K = Z_3$, the integers modulo 3. How many elements are there in the vector space $V = K^4$?

Sol^n: There are three choices 0, 1 or 2, for each of the four components of a vector in V.

Hence V has $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ elements.

→ Can $\mathbb{C}^2$ (pairs of complex numbers) be defined as a vector space: (a) over $\mathbb{R}$? (b) over $\mathbb{Q}$? (c) over $\mathbb{C}$? (d) over $\mathbb{Z}$?

Sol^n: (a), (b), (c) are vector spaces.

where as (d) is not a vector space.

because $\mathbb{Z}$ is not a field.

→ Can $\mathbb{R}^n$ be defined as a vector space:

(a) over $\mathbb{Q}$ (b) over $\mathbb{R}$ (c) over $\mathbb{C}$?

Sol^n: (a), (b) are vector spaces.

where as (c) is not a vector space

because $\mathbb{C}$ is not a subfield of $\mathbb{R}$.

→ Let $V = \{\langle a_n \rangle : a_n \in \mathbb{R} \ \forall n \in \mathbb{N}\}$ i.e., V is the set of all real sequences. prove that V is a vector space over $\mathbb{R}$, where addition and scalar multiplication are defined component wise.

Miscellaneous results and notations

→ Let $\mathcal{F}(I)$ be the set of all real-valued functions defined on the interval I. with pointwise addition and scalar multiplication $\mathcal{F}(I)$ becomes a real vector space.

The zero of this space is the function 0 given by $0(x) = 0$ for all $x \in I$.

→ Note: If, instead of the real valued function, we use complex valued functions defined on I and pointwise addition and scalar multiplication, then we get a complex vector space (using complex scalars).

we denote this complex vector space by $\mathcal{F}_c(I)$.

→ Let $\mathcal{P}(I)$ denote the set of all polynomials p with real coefficients defined on the interval I.

where p is a function whose value at x is $p(x) = \alpha_0 + \alpha_1 x + \cdots\cdots + \alpha_n x^n$ for all $x \in I$;

where α_i's are real numbers and n is a non-negative integer.

Using pointwise addition and scalar multiplication as for functions, we find that $\mathcal{P}(I)$ is a real vector space.

If we take complex coefficients for the

polynomials and use complex scalars, then we get the complex vector space $\mathcal{P}_C(I)$.

In both cases the vector space '0' of the space is the zero polynomial given by

$0(x) = 0$ for all $x \in I$

→ $\mathcal{C}[a,b]$, $\mathcal{C}^{(1)}[a,b]$, $\mathcal{C}^{(n)}[a,b]$ are real vector spaces under pointwise addition and scalar multiplication.

we have sum of two continuous (differentiable) functions is continuous (differentiable) and any scalar multiple of a continuous (differentiable) function is continuous (differentiable).

By changing the domain of definitions of continuity and differentiability to the open interval (a, b), we get, similarly, the real vector space $\mathcal{C}(a,b)$ and $\mathcal{C}^{(n)}(a,b)$ for each positive integer n.

<u>Note</u>: By changing real-valued function to complex-valued functions and using complex scalars, we get the complex vector spaces $\mathcal{C}_C[a,b]$ and $\mathcal{C}_C(a,b)$

→ Let $\mathcal{C}^\infty[a,b]$ stand for the set of all functions defined on $[a,b]$ and having derivatives of all orders on $[a,b]$. This is a real vector space for the usual operations. It is called the space of infinitely differentiable functions on $[a, b]$.

SUBSPACE:

Let $V(F)$ be a vector space and $W \subseteq V$ if W is a vector space w.r.t the internal and external compositions in V then W is called a subspace of V.

Theorems

$\longrightarrow$ $V(F)$ is a vector space, W is a subset of V ($W \subseteq V$); W is a subspace of $V(F)$ iff the internal and external compositions are satisfied in W.

i.e, (i) $\forall \alpha, \beta \in W \Rightarrow \alpha+\beta \in W$

(ii) $\forall a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

proof Necessary part:

Let W be a subspace of $V(F)$.

$\therefore$ By defn W is a vector space w.r.t the internal and external compositions in V.

$\therefore$ Internal and external compositions are satisfied in W.

i.e, (i) $\forall \alpha, \beta \in W \Rightarrow \alpha+\beta \in W$ and

(ii) $\forall \alpha \in W, a \in F \Rightarrow a\alpha \in W$.

Sufficient condition:

Let $W \subseteq V$ and internal and external compositions be satisfied in W.

i.e, (i) $\forall \alpha, \beta \in W \Rightarrow \alpha+\beta \in W$.

(ii) $\forall \alpha \in W, a \in F \Rightarrow a\alpha \in W$.

[I]. (i) $\forall \alpha, \beta \in W \subseteq V \Rightarrow \alpha+\beta \in W$. (by hypothesis.)

$\therefore$ Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ By Asso. prop. in V.

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \alpha, \beta \in W \subseteq V$.

$\Rightarrow \alpha+\beta = \beta+\alpha$. (By comm. prop in V)

$\therefore$ comm. prop. is satisfied in W.

(iv) Take $a = 0 \in F, \alpha \in W \Rightarrow a\alpha = 0\alpha \in W$ (by hyp.)

$\Rightarrow 0 \in W$

$\therefore 0+\alpha = \alpha+0 = \alpha \quad \forall \alpha \in W \subseteq V$ (By identity prop in V)

$\therefore$ Identity satisfied in W.

(iv) $1 \in F \Rightarrow -1 \in F$

Take $a = -1 \in F$; $\alpha \in V$

$\Rightarrow a\alpha = (-1)\alpha \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore \alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (By inverse prop. in V)

$\therefore$ inverse of α is $-\alpha$.

$\therefore (W,+)$ is an abelian group.

$\boxed{\text{II}}$ $\forall \alpha, \beta \in W \subseteq V$, $a, b \in F$

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

iv) $1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$

By axioms w.r.t external compositions in V.

$\therefore W(F)$ is a vector space.

$\therefore W(F)$ is a subspace of $V(F)$.

$\longrightarrow$ $V(F)$ is a vector space, W is a subset of $V(F)$ (i.e, $W \subseteq V$); W is a subspace of $V(F)$ iff $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.

Proof: N.C:

Let W be a subspace of $V(F)$.

$\therefore$ By defn W is a vector space w.r.t internal and external compositions in V.

$\therefore a, b \in F$, $\alpha, \beta \in W$ ~~$\Rightarrow a\alpha + b\beta \in W$~~

$\Rightarrow a\alpha, b\beta \in W$. (By external composition in W)

$\Rightarrow a\alpha + b\beta \in W$ ... (by internal comp in W) (12)

$\boxed{I}$ S.C (i) Take $a = b = 1 \in F$

$1 \in F,\ \alpha, \beta \in W \subseteq V \Rightarrow 1\alpha + 1\beta \in W$ (By hyp)

$\Rightarrow \alpha + \beta \in W$

Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in W \subset V$

$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (By asso. prop. in V)

(iii) $\forall \alpha, \beta \in W \subseteq V$

$\Rightarrow \alpha + \beta = \beta + \alpha$ (By comm. prop in V)

$\therefore$ Asso. prop and comm. prop are satisfied in W.

(iv) Take $a = b = 0 \in F$

$0 \in F,\ \alpha, \beta \in W \subseteq V$

$\Rightarrow 0\alpha + 0\beta \in W$ (by hyp)

$\Rightarrow 0 \in W$

$\therefore \forall \alpha \in W \subseteq V\ \exists\ 0 \in W$ s.t

$\alpha + 0 = \alpha = 0 + \alpha$ (By identity in V)

$\therefore 0$ is identity elt in W.

(v) $1 \in F \Rightarrow -1 \in F$

Take $a = -1 \in F$; $b = 0 \in F$

$\alpha, \beta \in W \subseteq V$

$\Rightarrow (-1)\alpha + 0\beta \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore$ If $\alpha \in W \subseteq V$ then $-\alpha \in W \subseteq V$

$\therefore \alpha + (-\alpha) = \alpha + (-\alpha) = 0$ (By inverse axiom in V)

$\therefore$ inverse of α is $-\alpha$

$\therefore (W, +)$ is an abelian group.

$\boxed{II}$. $\forall \alpha, \beta \in W \subseteq V;\ a, b \in F$.

(i) $a(\alpha + \beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$; (iv) $1\alpha = \alpha\ \ \forall \alpha \in W \subseteq V$

By axioms w.r.t external compositions in V.

$\therefore$ $W(F)$ is a vector space

$\therefore$ $W(F)$ is a subspace of $V(F)$.

$\rightarrow$ $V(F)$ is a vectorspace ; $W \subseteq V$; W is a subspace of $V(F)$ iff (i) $\forall\, \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

proof: N.C:- Let W be a subspace of V.

$\therefore$ By defn W is a vector space w.r.t the internal and external compositions in V.

By internal composition:

$\forall\, \alpha, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$ (By inverse axiom in W)

$\Rightarrow \alpha + (-\beta) \in W$ (By closure prop in W)

$\Rightarrow \alpha - \beta \in W$

By external composition

$a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

S.C:- Let $W \subseteq V$; (i) $\forall\, \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $\forall\, a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$\boxed{I}$. (i) Take $a = 0 \in F$

$0 \in F, \alpha \in W \subseteq V$

$\Rightarrow 0\alpha \in W$ (by hyp)

$\Rightarrow 0 \in W$.

$\therefore 0 + \alpha = \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V$

(by identity axiom of V)

$\therefore$ ~~Ident~~ Identity prop. is satisfied in W.

and '0' is the identity in W.

(ii) Take $\alpha = 0 \in W$, $\beta = \alpha \in W$

$\Rightarrow 0 - \alpha \in W$ (by hyp)

$\Rightarrow -\alpha \in W$.

$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (by inverse axiom of V)

$\therefore$ Inverse prop. is satisfied in W. and inverse of α is $-\alpha$

(iii) $\alpha, \beta \in W \subseteq V \Rightarrow \alpha, -\beta \in W$ $(\because \alpha \in W \Rightarrow -\alpha \in W)$

$\Rightarrow \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$

$\therefore$ Closure prop. is satisfied in W.

(iv) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$\therefore$ Asso. prop. is satisfied.

(v) $\forall \alpha, \beta \in W \subseteq V \Rightarrow \alpha + \beta = \beta + \alpha$

$\therefore$ comm. prop is satisfied.

$\therefore$ $(W, +)$ is an abelian group.

$\boxed{\text{II}}$. $\forall \alpha, \beta \in W \subseteq V$, $a, b \in F$

(i) $a(\alpha + \beta) = a\alpha + a\beta$
(ii) $(a+b)\alpha = a\alpha + b\alpha$
(iii) $(ab)\alpha = a(b\alpha)$

By axioms w.r.t external composition in V

(iv) $1\alpha = \alpha$ $\forall \alpha \in W \subseteq V$

$\therefore$ $W(F)$ is a vector space.

$\therefore$ $W(F)$ is a subspace of $V(F)$.

$\rightarrow$ $V(F)$ is a vector space and $W \subseteq V$; W is a subspace of $V(F)$ iff $a \in F$, $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$.

$(\Longleftrightarrow)$

Proof N.C:-

Let W be a subspace of $V(F)$.

$\therefore$ By defn W is a vector space w.r.t the internal & external compositions in V.

By external composition in W

$\forall a \in F, \alpha \in W \Rightarrow a\alpha \in W$

By internal composition

$\forall a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$.

S.C:- Let $W \subseteq V$ &

$\forall \alpha, \beta \in W, a \in F \Rightarrow a\alpha + \beta \in W$

$\boxed{\text{I}}$ (i) Take $a = 1 \in F$

$1 \in F,\ \alpha, \beta \in W \Rightarrow 1\cdot\alpha + \beta \in W$ (by hyp)

$\Rightarrow \alpha + \beta \in W$

$\therefore$ Closure prop. is satisfied in W

(ii) $\forall\ \alpha, \beta, \gamma \in W \subseteq V \Rightarrow (\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$

(iii) $\alpha + \beta = \beta + \alpha$

$\therefore$ Asso. & comm. prop. is satisfied in W.

(iv) $1 \in F, \Rightarrow -1 \in F$

Take $a = -1 \in F$, $\beta = \alpha \in W$

$-1 \in F,\ \alpha, \alpha \in W \Rightarrow (-1)\alpha + \alpha \in W$ (by hyp)

$\Rightarrow -\alpha + \alpha = 0 \in W$

$\therefore 0 + \alpha = \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V$

(by identity of V)

$\therefore$ Identity prop is satisfied in W.

0 is the identity in W.

(v) $-1 \in F;\ \alpha, 0 \in W \Rightarrow -1\cdot\alpha + 0 \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (By inverse axiom of V)

$\therefore$ Inverse prop is satisfied in W

$\therefore -\alpha$ is inverse of α

$\therefore (W, +)$ is an abelian group.

$\boxed{\text{II}}$ $\forall\ \alpha, \beta \in W \subseteq V$, $a, b \in F$

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

(iv) $1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$

$\therefore$ W(F) is a vector space

$\therefore$ W(F) is a subspace of V(F)

Algebra of subspaces

→ The intersection of any two subspaces of a vector space $V(F)$ is also a subspace of $V(F)$.

proof: Let W_1 & W_2 be any two subspaces of $V(F)$.

Let $W = W_1 \cap W_2$

$a, b \in F$; $\alpha, \beta \in W$

$\Rightarrow a, b \in F$; $\alpha, \beta \in W_1 \cap W_2$

$\Rightarrow a, b \in F$; ($\alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$)

$\Rightarrow a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2$

($\because W_1$ & W_2 are two subspaces)

$\therefore W_1 \cap W_2$ is also subspace of $V(F)$

$\therefore$ The intersection of two subspaces is also a subspace.

→ The arbitrary intersection of subspaces i.e., the intersection of any family of subspaces of a vector space is also a subspace.

proof: Let $W_1, W_2, \ldots$ be the given family of subspaces of the vector space $V(F)$.

Let $W = W_1 \cap W_2 \cap \ldots$

$= \bigcap_{i \in N} W_i \quad (i = 1, 2, \ldots)$

$a, b \in F$, $\alpha, \beta \in W \Rightarrow a, b \in F$; $\alpha, \beta \in \bigcap_{i \in N} W_i$

$\Rightarrow a, b \in F$, $\alpha, \beta \in W_i \ \forall i \in N$

$\Rightarrow a\alpha + b\beta \in W_i \ \forall i \in N$ ($\because W_i$ is a subspace for all $i \in N$)

$\Rightarrow a\alpha + b\beta \in \bigcap_{i \in N} W_i = W$

$\therefore W = \bigcap_{i \in N} W_i$ is a subspace of $V(F)$

$\therefore$ The intersection of any family of subspaces of a vector space is also a subspace.

→ The union of two subspaces of a vectorspace need not be a subspace

Solⁿ $V_3(F) = \{(a_1, a_2, a_3) / a_1, a_2, a_3 \in F\}$ is a vector space

Let $W_1 = \{(0, a, b) / a, b \in F\} \subseteq V_3$

and $W_2 = \{(x, 0, y) / x, y \in F\} \subseteq V_3$.

$a_1, a_2 \in F$;

$\alpha = (0, a, b)$,

$\beta = (0, c, d) \in W_1$

$a, b, c, d \in F$

$\Rightarrow a_1\alpha + a_2\beta = a_1(0, a, b) + a_2(0, c, d)$

$= (0, a_1a, a_1b) + (0, a_2c, a_2d)$

$= (0, a_1a + a_2c, a_1b + a_2d) \in W_1$.

$\therefore W_1$ is a subspace. $(\because 0, a_1a + a_2c, a_1b + a_2d \in W_1)$

Now $a_1, a_2 \in F$; $\alpha = (x_1, 0, y_1)$, $\beta = (x_2, 0, y_2) \in W_2$

$x_1, y_1, x_2, y_2 \in F$

$\Rightarrow a_1\alpha + a_2\beta = a_1(x_1, 0, y_1) + a_2(x_2, 0, y_2)$

$= (a_1x_1, 0, a_1y_1) + (a_2x_2, 0, a_2y_2)$

$= (a_1x_1 + a_2x_2, 0, a_1y_1 + a_2y_2) \in W_2$

$\therefore W_2$ is a subspace of $V(F)$ $(\because a_1x_1 + a_2x_2, 0, a_1y_1 + a_2y_2 \in W_2)$

If $F = Q$ then we have

$(0, \frac{1}{2}, 3) \in W_1$, $(1, 0, 3) \in W_2$

$W_1 \cup W_2 = \{(0, \frac{1}{2}, 3), (1, 0, 3)\}$

$\Rightarrow (0, \frac{1}{2}, 3), (1, 0, 3) \in W_1 \cup W_2$.

$\Rightarrow (0, \frac{1}{2}, 3) + (1, 0, 3) = (1, \frac{1}{2}, 6) \notin W_1 \cup W_2$.

$(\because$ neither $(1, \frac{1}{2}, 6) \in W_1$ nor $(1, \frac{1}{2}, 6) \in W_2)$

$\therefore W_1 \cup W_2$ is not closed under vector addition.

$\therefore W_1 \cup W_2$ is not a subspace of $V_3(F)$.

→ The union of two subspaces is a subspace iff one is contained in the other. (15)

proof: Let W_1 and W_2 be two subspaces of the vector space $V(F)$.

N.C Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

$W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ (subspace of $V(F)$)

$W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$ (subspace of $V(F)$)

$\therefore W_1 \cup W_2$ is a subspace of $V(F)$.

S.C: Let $W_1 \cup W_2$ be a subspace of $V(F)$

then we prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If possible suppose that $W_1 \not\subseteq W_2$ or $W_2 \not\subseteq W_1$

if $W_1 \not\subseteq W_2$

let $\alpha \in W_1$ then $\alpha \notin W_2$

if $W_2 \subseteq W_1$

let $\beta \in W_2$ then $\beta \notin W_1$

Now $\alpha \in W_1$, $\beta \in W_2$

$\Rightarrow \alpha, \beta \in W_1 \cup W_2$

$\Rightarrow \alpha + \beta \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow \alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$

Now $\alpha + \beta \in W_1$, $\alpha \in W_1$

$\Rightarrow (\alpha + \beta) - \alpha \in W_1$ ($\because W_1$ is a subspace)

$\Rightarrow \beta \in W_1$

which is contradiction to $\beta \notin W_1$

and $\alpha + \beta \in W_2$, $\beta \in W_2$

$\Rightarrow (\alpha + \beta) - \beta \in W_2$ ($\because W_2$ is a subspace)

$\Rightarrow \alpha \in W_2$

which contradiction to $\alpha \notin W_2$

$\therefore$ our assumption that $W_1 \not\subseteq W_2$ or $W_2 \not\subseteq W_1$ is wrong

$\therefore W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Note: [1] Let $V(F)$ be any vector space.

Then V itself and the subset of V consisting of the zero vector alone are always subspaces V.

These two subspaces are called improper subspaces.

If V has any other subspace then it is called a proper subspace.

[2]. The subspace of V consisting of the zero vector only is called the zero subspace of V.

Problems

→ Let $W = \{(a_1, a_2, 0) / a_1, a_2 \in F\} \subseteq V_3(F)$.

then S.T W is a subspace of $V_3(F)$.

Soln Let $a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (a_1, a_2, 0)$

$\beta = (b_1, b_2, 0)$

where $a_1, a_2, b_1, b_2 \in F$

$\Rightarrow a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$

$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$

$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$

∴ W is a subspace of $V_3(F)$. $aa_1 + bb_1, aa_2 + bb_2 \in F$.

→ Let $W = \{(x_1, x_2, x_3) / a_1x_1 + a_2x_2 + a_3x_3 = 0$

a_1, a_2, a_3 are fixed elts in F

$x_1, x_2, x_3 \in F\} \subseteq V_3(F)$.

S.T W is a subspace of $V_3(F)$.

Soln $\forall a, b \in F$, $\alpha, \beta \in W$

Choose $\alpha = (x_1, x_2, x_3)$: $a_1x_1 + a_2x_2 + a_3x_3 = 0$

$\beta = (y_1, y_2, y_3)$: $a_1y_1 + a_2y_2 + a_3y_3 = 0$

$\Rightarrow a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$

$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$

$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

and $a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$

$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$

$= a(0) + b(0)$

$= 0$

$\therefore a\alpha + b\beta \in W$

∴ W is a subspace of $V_3(F)$.

→ P.T the set of all solutions (a, b, c) of the equation $a+b+2c=0$ is a subspace of the vectorspace $V_3(\mathbb{R})$.

Sol^n Let $W = \{(a, b, c) / a+b+2c=0; a, b, c \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$.

let $a, b \in \mathbb{R}$, $\alpha, \beta \in W$

Choose $\alpha = (a_1, b_1, c_1)$; $a_1+b_1+2c_1=0$

$\beta = (a_2, b_2, c_2)$; $a_2+b_2+2c_2=0$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.

→ S.T the set W of the elts of the vector space $V_3(\mathbb{R})$ of the form $(x+2y, y, -x+3y)$ where $x, y \in \mathbb{R}$ is a subspace of $V_3(\mathbb{R})$.

Sol^n Let $W = \{(x+2y, y, -x+3y) / x, y, z \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$

let $a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (x_1+2y_1, y_1, -x_1+3y_1)$

$\beta = (x_2+2y_2, y_2, -x_2+3y_2)$

$\Rightarrow a\alpha+b\beta = a(x_1+2y_1; y_1, -x_1+3y_1) + b(x_2+2y_2, y_2, -x_2+3y_2)$

$= (ax_1+2ay_1, ay_1, -ax_1+3ay_1) + (bx_2+2by_2, by_2, -bx_2+3by_2)$

$= \left(ax_1+bx_2+2(ay_1+by_2), ay_1+by_2, -[ax_1+bx_2]+3(ay_1+by_2)\right)$

$\in W$.

$\therefore a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $V_3(\mathbb{R})$.

→ which of the following sets of vectors $\alpha = (a_1, a_2, \ldots a_n)$ in $\mathbb{R}^n$ are subspace of $\mathbb{R}^n$?

(i) all α s.t $a_1 \leq 0$

(ii) all α s.t a_3 is an integer

(iii) all α s.t $a_2 + 4a_3 = 0$

(iv) all α s.t $a_1 + a_2 + \cdots + a_n = k$ (constant)

Sol^n (i) Let $W = \{\alpha \mid \alpha = (a_1, a_2, \ldots a_n)$ and $a_1 \leq 0\} \subseteq \mathbb{R}^n$ i.e., $V_n(\mathbb{R})$

If $a_1 = -3$ then $a_1 < 0$

Let $\alpha = (-3, a_2, a_3, \ldots a_n) \in W$

and if $a = -2 \in \mathbb{R}$

then $a\alpha = -2(-3, a_2 \ldots a_n)$

$= (6, -2a_2, \ldots -2a_n) \notin W$ $(\because a_1 = 6 > 0.)$

$\therefore \forall \alpha \in W, a \in \mathbb{R} \Rightarrow a\alpha \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^n$.

(ii) Let $W = \{\alpha \mid \alpha = (a_1 a_2 \ldots a_n)$ and a_3 is an integer$\} \subseteq \mathbb{R}^n$.

if $a_3 = -3$ is an integer.

Let $\alpha = (a_1, a_2, -3, \ldots a_n) \in W$

and $a = \frac{1}{2} \in \mathbb{R}$.

then $a\alpha = \left(\frac{a_1}{2}, \frac{a_2}{2}, -\frac{3}{2}, \frac{a_4}{2}, \ldots a_n\right) \notin W$

$(\because -3/2$ is not an integer$)$.

$\therefore \forall \alpha \in W, a \in \mathbb{R} \Rightarrow a\alpha \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^n$.

(iii) Let $W = \{\alpha \mid \alpha = (a_1, a_2, \ldots a_n)$ and $a_2 + 4a_3 = 0\} \subseteq \mathbb{R}^n$.

Now $a, b \in \mathbb{R}$, $\alpha, \beta \in W$

Choose $\alpha = (a_1, a_2, \ldots a_n)$ and $a_2 + 4a_3 = 0$

$\beta = (b_1, b_2, \ldots b_n)$ and $b_2 + 4b_3 = 0$

$\Rightarrow a\alpha+b\beta = a(a_1,a_2,a_3,\ldots a_n) + b(b_1,b_2,b_3 \ldots b_n)$

$= (aa_1+bb_1, aa_2+bb_2, aa_3+bb_3, \ldots aa_n+bb_n)$ — (1)

Now we have

$(aa_2+bb_2)+4(aa_3+bb_3) = a(a_2+4a_3)+b(b_2+4b_3)$

$= a(0)+b(0)$

$= 0$

$\therefore$ (1) $\equiv a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $\mathbb{R}^n$.

(iv) Let $W = \{\alpha / \alpha = (a_1,a_2,\ldots a_n)$ and $a_1+a_2+\ldots+a_n = k$ (const)$\} \subseteq \mathbb{R}^n$.

Let $a,b \in F$, $\alpha,\beta \in W$

Choose $\alpha = (a_1,a_2,\ldots a_n)$ and $a_1+a_2+\ldots+a_n = k$

$\beta = (b_1,b_2,\ldots b_n)$ and $b_1+b_2+\ldots+b_n = k$.

$\Rightarrow a\alpha+b\beta = a(a_1,a_2,\ldots a_n) + b(b_1,b_2,\ldots b_n)$

$= (aa_1+bb_1, aa_2+bb_2, \ldots aa_n+bb_n)$ — (1)

Now we have

$(aa_1+bb_1)+(aa_2+bb_2)+\ldots+(aa_n+bb_n)$

$= a(a_1+a_2+\ldots+a_n)+b(b_1+b_2+\ldots+b_n)$

$= ak+bk$

$= (a+b)k$.

If $k=0$ then (1) $\equiv a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $\mathbb{R}^n$

If $k \neq 0$ then $a\alpha+b\beta \notin W$.

$\therefore W$ is not a subspace of $\mathbb{R}^n$.

$\rightarrow$ S.T W is not a subspace of $\mathbb{R}^3 = V$, where $W = \{(a,b,c) / a^2+b^2+c^2 \leq 1\} \subseteq V$.

Sol^n Let $\alpha = (0,1,0)$, $\beta = (1,0,0) \in W$

then $\alpha+\beta = (1,1,0) \notin W$ $(\because 1^2+1^2+0 = 2 > 1)$

$\therefore W$ is not a subspace of $V = \mathbb{R}^3$.

→ S.T W is not subspace of $V = \mathbb{R}^3$.

where $W = \{(a, b, c) / a, b, c \in \mathbb{Q}\} \subseteq \mathbb{R}^3$.

Soln Let $a_1 = \sqrt{2} \in \mathbb{R}$, $\alpha = (1, 2, 3) \in W$

$$\Rightarrow a_1\alpha = \sqrt{2}(1, 2, 3)$$
$$= (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$$

∴ W is not a subspace of V. $(\because \sqrt{2}, 2\sqrt{2}, 3\sqrt{2} \notin \mathbb{Q})$

→ S.T W is not a subspace of $V = \mathbb{R}^n$.

where $W = \{(a_1, a_2, \dots a_n) / a_1 \geq 0\}$.

Soln If $a_1 = 3$ then $a_1 > 0$

$$\alpha = (3, a_2, a_3, \dots a_n)$$

If $a = -2 \in \mathbb{R}$

then $a\alpha = (-6, -2a_2, -2a_3, \dots, -2a_n) \notin W$

$(\because a_1 = -6 < 0)$

∴ W is not a subspace of $\mathbb{R}^n$.

→ S.T W is not a subspace of $\mathbb{R}^n$.

where $W = \{(a_1, a_2 \dots a_n) / a_2 = a_1^2\} \subseteq \mathbb{R}^n$

Let $a \in \mathbb{R}$, $\alpha = (a_1, a_2, \dots a_n) \in W$ and $a_2 = a_1^2$.

$\Rightarrow a\alpha$ need not be an elt of W.

for example

Let $a = \frac{1}{2} \in \mathbb{R}$, $\alpha = (2, 4, a_3, \dots a_n) \in W$

$$\Rightarrow a\alpha = (1, 2, \tfrac{a_3}{2}, \dots \tfrac{a_n}{2}) \notin W$$

$(\because 2 \neq 1^2$ i.e, $a_2 \neq a_1^2)$

→ Let V be the real vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$.

which of the following sets of functions are subspaces of V.

(i) $W = \{f / f(3) = 0\}$

(ii) $W = \{f / f(7) = f(1)\}$

(iii) $W = \{f / f(-x) = -f(x)\}$

(iv) $W = \{f / f(7) = 2 + f(1)\}$

(v) $W = \{f \mid f(x^2) = [f(x)]^2\}$

(vi) W consists of the continuous functions

(vii) W consists of the differentiable functions.

Sol^n: (i) Let $a, b \in \mathbb{R}$; $f, g \in W$ s.t $f(3)=0$ & $g(3)=0$

$$\Rightarrow (af+bg)(3) = (af)(3)+(bg)(3)$$
$$= a f(3) + b g(3)$$
$$= a(0) + b(0)$$
$$= 0$$

$\therefore af+bg \in W$.

$\therefore$ W is a subspace of V

(ii) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(7) = f(1)$ and $g(7) = g(1)$

$$\Rightarrow (af+bg)(7) = (af)(7)+(bg)(7)$$
$$= a f(7) + b g(7)$$
$$= a f(1) + b g(1)$$
$$= (af)(1) + (bg)(1)$$
$$= (af+bg)(1)$$

$\therefore af+bg \in W$

$\therefore$ W is a subspace of V.

(iii) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(-x) = -f(x)$ & $g(-x) = -g(x)$

$$\Rightarrow (af+bg)(-x) = (af)(-x)+(bg)(-x)$$
$$= a f(-x) + b g(-x)$$
$$= a[-f(x)] + b[-g(x)]$$
$$= -[a f(x) + b g(x)]$$
$$= -[(af)(x) + (bg)(x)]$$
$$= -(af+bg)(x).$$

$\therefore af+bg \in W$.

$\therefore$ W is a subspace of V.

(iv) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(7) = 2 + f(1)$ & $g(7) = 2 + g(1)$

$$\Rightarrow (af+bg)(7) = (af)(7) + (bg)(7)$$
$$= a f(7) + b g(7)$$
$$= a[2+f(1)] + b(2+g(1))$$
$$= 2a + a f(1) + 2b + b g(1)$$
$$= (2a+2b) + (af+bg)(1) \quad \text{—— (1)}$$

Let $a=1$, $b=1$ then

$$(f+g)(7) = 4 + (f+g)(1)$$
$$\neq 2 + (f+g)(1)$$

$\therefore f+g \notin W$

$\therefore W$ is not a subspace.

(v) Let $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(x^2) = [f(x)]^2$ & $g(x^2) = [g(x)]^2$

$$\Rightarrow (af+bg)(x^2) = a f(x^2) + b g(x^2)$$
$$= a[f(x)]^2 + b[g(x)]^2 \quad \text{—— (1)}$$

Now $(af+bg)(x^2) = [(af+bg)(x)]^2$
$$= [a f(x) + b g(x)]^2$$
$$= a^2[f(x)]^2 + b^2[g(x)]^2 + 2ab\, f(x)\, g(x) \quad \text{—— (2)}$$

$\therefore$ from (1) & (2)

$$a[f(x)]^2 + b[g(x)]^2 \neq a^2[f(x)]^2 + b^2[g(x)]^2 + 2ab\, f(x) g(x)$$

$\therefore af+bg \notin W$.

$\therefore W$ is not a subspace

(v) If f and g are continuous functions and $a, b \in \mathbb{R}$ then $af+bg$ is also continuous function.

$\therefore af+bg \in W$

$\therefore W$ is a subspace of V.

(vi) If f and g are differentiable functions and $a, b \in \mathbb{R}$ then $af+bg$ is also differentiable.

$\therefore W$ is a subspace of V.

→ Let $W = \{(x_1, x_2, \dots, x_n) \in V_n / x_1 = 0\}$. prove that W is a subspace of V_n.

→ prove that $W = \{(x_1, x_2, \dots, x_n) \in V_n^C / \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0,\ \alpha_i\text{'s are given constants}\}$

where V_n^C – the set of all ordered n-tuples of complex numbers.

is a subspace of V_n^C.

→ which of the following sets are subspaces of V_3?

(a) $\{(x_1, x_2, x_3) / x_1 x_2 = 0\}$ (b) $\{(x_1, x_2, x_3) / \frac{x_2}{x_1} = \sqrt{2}\}$

(c) $\{(x_1, x_2, x_3) / \sqrt{2} x_1 = \sqrt{3} x_2\}$ (d) $\{(x_1, x_2, x_3) / x_3$ is an integer$\}$

(e) $\{(x_1, x_2, x_3) / x_1^2 + x_2^2 + x_3^2 \leq 1\}$ (f) $\{(x_1, x_2, x_3) / x_1 + x_2 + x_3 \geq 0\}$

(g) $\{(x_1, x_2, x_3) / x_1 = \sqrt{2} x_2$ and $x_3 = 3x_2\}$

(h) $\{(x_1, x_2, x_3) / x_1 - 2x_2 = x_3 - \frac{3x_2}{2}\}$.

(i) $\{(x_1, x_2, x_3) / x_1 = 2x_2$ or $x_3 = 3x_2\}$.

Ans: (c), (g) & (h) are subspaces of V_3.

→ which of the following sets are subspaces of $\mathcal{P}$?

(a) $\{p \in \mathcal{P} /$ degree of $p = 4\}$ (b) $\{p \in \mathcal{P} /$ degree of $p \leq 3\}$

(c) $\{p \in \mathcal{P} /$ degree of $p \geq 5\}$ (d) $\{p \in \mathcal{P} /$ degree of $p \leq 4$ and $p'(0) = 0\}$

(e) $\{p \in \mathcal{P} / p(1) = 0\}$.

Ans: (b), (d) & (e) are subspaces of $\mathcal{P}$.

→ which of the following sets are subspaces of $\mathcal{C}(a,b)$?

(a) $\{f \in \mathcal{C}(a,b) / f(x_0) = 0,\ x_0 \in (a,b)\}$

(b) $\{f \in \mathcal{C}(a,b) / f'(x) = 0$ for all $x \in (a,b)\}$

(c) $\{f \in \mathcal{C}(a,b) / f\left(\frac{a+b}{2}\right) = 1\}$

(d) $\{f \in \mathcal{C}(a,b) / f(x) = x^2 f(x)\}$

(e) $\{f \in \mathcal{C}(a,b) / 2f'''(x) + 3x f''(x) - f'(x) + x^2 f(x) = 0\}$

(f) $\{f \in \mathcal{C}(a,b) / \int_a^b f(x)\,dx = 0\}$

Ans: (a), (b), (d), (e) and (f) are subspaces of $\mathcal{C}(a,b)$.

→ $\mathcal{C}[a,b]$ is a subspace of $\mathcal{F}[a,b]$.

because the sum of two continuous functions is continuous and any scalar multiple of a continuous function is again continuous, we find that addition and scalar multiplication are closed in $\mathcal{C}[a,b]$.

This observation not only proves that $\mathcal{C}[a,b]$ is a vector space, but also that it is a subspace of $\mathcal{F}[a,b]$.

Note: The spaces $\mathcal{C}[a,b]$, $\mathcal{C}^{(1)}[a,b]$, $\mathcal{C}^{(n)}[a,b]$ and $\mathcal{P}[a,b]$ are subspaces of $\mathcal{F}[a,b]$.

Further, note that

(a) $\mathcal{P}[a,b]$ is a subspace of $\mathcal{C}[a,b]$

(b) $\mathcal{C}^{(1)}[a,b]$ is a subspace of $\mathcal{C}[a,b]$.

(c) $\mathcal{C}^{(n)}[a,b]$ is a subspace of $\mathcal{C}[a,b]$ for every positive integer n.

(d) $\mathcal{C}^{(n)}[a,b]$ is a subspace of $\mathcal{C}^{(m)}[a,b]$ for every $m<n$.

(e) $\mathcal{P}[a,b]$ is a subspace of $\mathcal{C}^{(n)}[a,b]$ for every positive integer n.

(f) Similar results are true for functions defined on (a,b).

→ Let V be the vector space of all real sequences $\langle \alpha_n \rangle$.

(i) prove that $W = \{\langle a_n \rangle \in V : \lim_{n\to\infty} a_n = 0\}$ is a subspace of V.

(ii) prove that $U = \{\langle a_n \rangle \in V : \sum_{n=1}^{\infty} a_n^2$ is finite$\}$ is a subspace of V and is contained in W.

Sol^n: Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in W$, $\langle b_n \rangle \in W$.

$\therefore \lim_{n\to\infty} a_n = 0$ & $\lim_{n\to\infty} b_n = 0$

Now $\alpha\langle a_n \rangle + \beta\langle b_n \rangle = \langle \alpha a_n + \beta b_n \rangle$.

where $\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$

$= \alpha \cdot 0 + \beta \cdot 0$

$= 0$.

$\therefore \alpha\langle a_n \rangle + \beta\langle b_n \rangle \in W$.

$\therefore$ W is a subspace of V.

(ii) Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in U$, $\langle b_n \rangle \in U$.

$\therefore \sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are finite.

i.e, each one of them is a convergent series.

It follows that $\alpha\sum_{n=1}^{\infty} a_n^2 + \beta\sum_{n=1}^{\infty} b_n^2$ is finite

i.e, $\alpha\langle a_n \rangle + \beta\langle b_n \rangle \in U$.

Hence U is a subspace of V.

Let $\langle a_n \rangle \in U$ be arbitrary.

Then $\sum_{n=1}^{\infty} a_n^2$ is a convergent series and

So $\underset{n\to\infty}{Lt}\, a_n^2 = 0$

$\Rightarrow \underset{n\to\infty}{Lt}\, a_n \; \underset{n\to\infty}{Lt}\, a_n = 0$

$\Rightarrow \underset{n\to\infty}{Lt}\, a_n = 0$

$\Rightarrow \langle a_n \rangle \in W$

Hence $U \subseteq W$.

→ Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$ of real numbers.

Let (i) $W_1 = \{A \in V / A^2 = A\}$

(ii) $W_2 = \{A \in V / \det A = 0\}$

Show that W_1 and W_2 are not subspaces of V.

→ Show that W is a subspace of V where W consists of all matrices which commute with a given matrix T; that is, $W = \{A \in V / AT = TA\}$.

Sol^n: Given that W consists of all matrices which commute with a given matrix

i.e, $W = \{A \in V / AT = TA\}$.

Since $OT = O = TO$

$\therefore O \in W$.

$\therefore W$ is non-empty.

Now suppose $A, B \in W$

i.e, $AT = TA$ and $BT = TB$.

for any scalars $a, b \in F$,

$$(aA + bB)T = (aA)T + (bB)T$$
$$= a(AT) + b(BT)$$

$$= a(TA) + b(TB)$$
$$= T(aA) + T(bB)$$
$$= T(aA + bB).$$

Thus $aA+bB$ commutes with T.

$\Rightarrow aA+bB \in W$.

Hence W is a subspace of V.

→ Show that W is a subspace of V, where W consists of the bounded functions.

[A function $f \in V$ is bounded if there exists $M > 0$ $(M \in \mathbb{R})$ such that $|f(x)| \le M$ for every $x \in \mathbb{R}$]

Sol^n: Since $0(x) = 0$ for every $x \in \mathbb{R}$.

Clearly 0 is bounded.

$\therefore 0 \in W$

i.e, W is non empty.

Now let $f, g \in W$ with M_f and M_g bounds for f and g respectively. i.e, $|f(x)| \le M_f$ & $|g(x)| \le M_g$ $\forall x \in \mathbb{R}$

Then for any scalars $a, b \in F$ and

$$|(af+bg)(x)| = |af(x) + bg(x)|$$
$$\le |af(x)| + |bg(x)|$$
$$= |a|\,|f(x)| + |b|\,|g(x)|$$
$$\le |a| M_f + |b| M_g.$$

$\Rightarrow |a|M_f + |b|M_g$ is a bound for the function $af+bg$.

Thus W is a subspace of V.

scan 055.jpg

→ which of the following sets of vectors $\alpha = (a_1, a_2 \ldots a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$.

(b) all α such that $a_1 + 3a_2 = a_3$.

(c) all α such that $a_2 = a_1^2$

(d) all α such that $a_1 a_2 = 0$

(e) all α such that a_2 is rational.

Linear Combination:

Defn: Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ then any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where $a_1, a_2 \ldots a_n \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$.

Linear Span: Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$. Then the collection of all linear combinations of a finite number of elements of 'S' is called linear span of S and is denoted by $L(S)$.

i.e, $L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \,/\, \alpha_1, \alpha_2, \ldots \alpha_n \in S, \; a_1, a_2, \ldots a_n \in F\}$.

Smallest subspace containing any subset of V(F): 23

$S \subseteq V$,
Let $U \subseteq V$ or $U \subseteq X \subseteq V$

Defn: Let $V(F)$ be a vector space and S be any subset of V (i.e., $S \subseteq V$). If U is a subspace of V containing S and U is contained in every subspace of V containing 'S' then U is called the smallest subspace of V containing S.

→ The smallest subspace of V containing S is also called the subspace of V generated or spanned by S and is denoted by $\{S\}$. i.e, $\{S\} = U$

→ If $\{S\} = V$ then we say that V is spanned by S.

Theorem:

→ If $V(F)$ is a vector space, $S \subseteq V$, then the linear span of S is the smallest subspace of $V(F)$ containing S.

(i.e, $L(S)$ is a subspace of $V(F)$ generated by S i.e, $L(S) = \{S\}$.)

proof: Given that $V(F)$ is a vector space and $S \subseteq V$.

Let $S = \{\alpha_1, \alpha_2, \ldots \ldots \alpha_n\} \subseteq V$

and $L(S) = \left\{ a_1\alpha_1 + a_2\alpha_2 + \ldots \ldots + a_n\alpha_n \,\middle|\, \begin{array}{l}\alpha_1, \alpha_2, \ldots \alpha_n \in S \\ a_1, a_2, \ldots a_n \in F\end{array} \right\}$

$\subseteq V$

Now $\forall\, a, b \in F$; $\alpha, \beta \in L(S)$

Choose $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

where a's, b's $\in F$ and α's $\in S$

$\Rightarrow a\alpha+b\beta = a(a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n)$
$+b(b_1\alpha_1+b_2\alpha_2+\cdots+b_n\alpha_n)$

$= (aa_1+bb_1)\alpha_1+(aa_2+bb_2)\alpha_2+\cdots\cdots+(aa_n+bb_n)\alpha_n$

$\in L(S)$. $(\because aa_1+bb_1, aa_2+bb_2, \ldots\ldots aa_n+bb_n \in F)$

$\therefore L(S)$ is a subspace of $V(F)$.

Let $\alpha_i \in S$; $i=1,2,\ldots n$

then $\alpha_i = 1\alpha_i$
= linear combination of α_i
$\in L(S)$

$\therefore \alpha_i \in L(S)$

$\therefore S \subseteq L(S)$

Now let W be any subspace of $V(F)$ containing S.
$\therefore S \subseteq W$.

if $\alpha \in L(S)$ then α = the linear combination of a finite no. of elts of S.

$\in W$ $(\because S \subseteq W)$

$\therefore$ If $\alpha \in L(S)$ then $\alpha \in W$

$\therefore L(S) \subseteq W$.

$\therefore S \subseteq L(S) \subseteq W \subseteq V$.

$\therefore L(S)$ is the smallest subspace of V containing S.

i.e, $L(S) = \{S\}$.

Note: If in any case, we are to prove that $L(S) = V$ then we are enough to prove that $V \subseteq L(S)$ because w.k.t $L(S) \subseteq V$ $(\because L(S)$ is a subspace of $V)$

In order to prove that $V \subseteq L(S)$

for this each elt of V can be expressed as linear combination of a finite no. of elts of S.

$\therefore$ Each elt of V will also the elt of $L(S)$.

i.e, let $\alpha \in V \Rightarrow \alpha =$ the l.c. of finite no. of elts of S.

$$\in L(S)$$

$$\therefore \alpha \in L(S)$$

$$\therefore V \subseteq L(S).$$

$$\therefore V \subseteq L(S) \text{ and } L(S) \subseteq V$$

$$\Rightarrow L(S) = V.$$

Ex The subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $V_3(F)$. (i.e, $S \subseteq V_3(F)$) generates or spans the entire vector space $V_3(F)$ i.e, $L(S) = V_3$

Sol^n W.K.T $L(S) \subseteq V_3$ —— (1)

Let $\alpha = (a,b,c) \in V_3$ then

$$\alpha = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$\in L(S)$$

$$\therefore \alpha \in L(S)$$

$$\therefore V_3 \subseteq L(S). \text{ —— (2)}$$

$\therefore$ from (1) & (2) we have $L(S) = V_3$.

Def^n Linear Sum of two Subspaces

Let W_1 & W_2 be any two subspaces of $V(F)$ then the set $\{\alpha_i + \alpha_j / \alpha_i \in W_1 \text{ \& } \alpha_j \in W_2\} \subseteq V$ is called linear sum of W_1 & W_2 and is denoted by $W_1 + W_2$.

i.e, $W_1 + W_2 = \{\alpha_i + \alpha_j / \alpha_i \in W_1, \alpha_j \in W_2\} \subseteq V$.

Theorem: Let W_1 and W_2 be two subspaces of $V(F)$, then the linear sum $W_1 + W_2$ is a subspace of $V(F)$ and $W_1 + W_2 = L(W_1 \cup W_2)$.
i.e, $W_1 + W_2 = \{W_1 \cup W_2\}$.

proof: Given that

$V(F)$ is a vectorspace.

W_1 & W_2 are two subspaces of $V(F)$.

$W_1+W_2 = \{\alpha_i+\alpha_j / \alpha_i \in W_1, \alpha_j \in W_2\} \subseteq V$.

Let $a, b \in F$; $\alpha, \beta \in W_1+W_2$

Choose $\alpha = \alpha_i+\alpha_j$; $\alpha_i \in W_1$, $\alpha_j \in W_2$
$\beta = \alpha_k+\alpha_l$; $\alpha_k \in W_1$, $\alpha_l \in W_2$

$\Rightarrow a\alpha + b\beta = a(\alpha_i+\alpha_j)+b(\alpha_k+\alpha_l)$

$= (a\alpha_i+b\alpha_k)+(a\alpha_j+b\alpha_l)$

$\in W_1+W_2$.

Since W_1 is a subspace
$\therefore a\alpha_i+b\alpha_k \in W_1$
and W_2 is a subspace
$\therefore a\alpha_j+b\alpha_l \in W_2$.

$\therefore W_1+W_2$ is a subspace of $V(F)$.

Now $0 \in W_1$, $x \in W_2 \Rightarrow 0+x \in W_1+W_2$
$\Rightarrow x \in W_1+W_2$

$\therefore W_2 \subseteq W_1+W_2$ —— (1)

$y \in W_1$, $0 \in W_2 \Rightarrow y+0 \in W_1+W_2$
$\Rightarrow y \in W_1+W_2$

$\therefore W_1 \subseteq W_1+W_2$ —— (2)

$\therefore$ from (1) & (2) we have

$W_1 \cup W_2 \subseteq W_1+W_2 \subseteq V$

W.K.T Linear Span of $W_1 \cup W_2$ (i.e, $L(W_1 \cup W_2)$)
is the smallest subspace of $V(F)$ containing $W_1 \cup W_2$

$\therefore L(W_1 \cup W_2) \subseteq W_1+W_2$ —— (3)

Let $\alpha \in W_1+W_2 \Rightarrow \alpha = \alpha_i+\alpha_j \in W_1+W_2$

Now $\alpha_i \in W_1$, $\alpha_j \in W_2 \Rightarrow \alpha_i, \alpha_j \in W_1 \cup W_2$

Now $\alpha \in W_1+W_2 \Rightarrow \alpha = \alpha_i+\alpha_j$
$= 1\alpha_i + 1\alpha_j$

$= $ l.c. of finite no. of elts of $W_1 \cup W_2$.
$\in L(W_1 \cup W_2)$

$\therefore W_1 + W_2 \subseteq L(W_1 \cup W_2)$ —— (4)

$\therefore$ from (3) & (4)
we have $L(W_1 \cup W_2) = W_1 + W_2$.

→ If S, T are subsets of $V(F)$ then

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$.

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$.

proof Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ then

any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$

Since $S \subseteq T$

$\Rightarrow S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq T$.

$\therefore \alpha \in L(T)$.

$\therefore$ If $\alpha \in L(S)$ then $\alpha \in L(T)$.

$\therefore L(S) \subseteq L(T)$.

(ii) Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and $T = \{\beta_1, \beta_2, \ldots \beta_p\} \subseteq V$

then $S \cup T = \{\alpha_1, \alpha_2 \ldots \alpha_n, \beta_1, \beta_2, \ldots \beta_p\} \subseteq V$.

Let $\alpha \in L(S \cup T)$ then

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p$

Since $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$

and $b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p \in L(T)$.

$\therefore \alpha \in L(S) + L(T)$

$\therefore L(S \cup T) \subseteq L(S) + L(T)$ —— (1)

Let $\gamma \in L(S) + L(T)$ then $\gamma = \beta + \delta$.

where $\beta \in L(S)$ & $\delta \in L(T)$.

NOW β = L.C. of finite no. of elts of S and

δ = L.C of finite no. of elts of T.

$\therefore \beta+\delta$ = L.C. of finite no. of elts of $S \cup T$.

$\therefore \gamma = \beta+\delta \in L(S \cup T)$

$\therefore$ If $\gamma \in L(S)+L(T)$ then

$\gamma \in L(S \cup T)$

$\therefore L(S)+L(T) \subseteq L(S \cup T)$ — (2)

$\therefore$ from (1) & (2) we have

$L(S \cup T) = L(S)+L(T)$.

(iii) $L(L(S))$ is the smallest subspace of V containing $L(S)$.

But $L(S)$ is a subspace of V.

$\therefore$ the smallest subspace of V containing $L(S)$ is $L(S)$ itself.

i.e, $L(S) \subseteq L(L(S)) \subseteq L(S) \subseteq V$

$\therefore L(L(S)) = L(S)$

$\left(L(S) \subseteq L(L(S)) \subseteq \underset{L(S)}{W} \subseteq V\right)$

Defn: Linear dependence of vectors:

$V(F)$ is a vector space and $S=\{\alpha_1,\alpha_2,\ldots\alpha_n\}\subseteq V$

If $\exists$ atleast one non-zero scalar $a_1,a_2,\ldots a_n\in F$

such that $a_1\alpha_1+a_2\alpha_2+\ldots\ldots+a_n\alpha_n=0$

Then S is called linear dependent.

Linear Independence of vectors:

$V(F)$ is a vector space and $S=\{\alpha_1,\alpha_2,\ldots\ldots\alpha_n\}\subseteq V$

If $a_1\alpha_1+a_2\alpha_2+\ldots\ldots+a_n\alpha_n=0$; $a_i\in F$, $1\le i\le n$

$\Rightarrow a_1=a_2=\ldots\ldots=a_n=0$

i.e, $a_i=0$ for each $1\le i\le n$

Ex $V_n(F)=\{(a_1,a_2,\ldots a_n)/ a_1,a_2,\ldots a_n\in F\}$

is a vector space.

$S=\{(1,0,0,\ldots 0),(0,1,0,\ldots 0),(0,0,1,0,\ldots 0)$

$\ldots\ldots(0,0,\ldots 0,1)\}\subseteq V_n(F)$.

Now $a_1\alpha_1+a_2\alpha_2+\ldots\ldots+a_n\alpha_n=0$

$\Rightarrow a_1(1,0,0,\ldots 0)+a_2(0,1,0,\ldots 0)$

$+\ldots\ldots+a_n(0,0,\ldots 1)=(0,0,\ldots 0)$

$\Rightarrow(a_1,0,\ldots 0)+(0,a_2,\ldots 0)+\ldots\ldots+$

$(0,0,\ldots a_n)=(0,0,\ldots 0)$

$\Rightarrow(a_1,a_2,\ldots a_n)=(0,0,\ldots 0)$

$\Rightarrow a_1=a_2=\ldots=a_n=0$

$\therefore S$ is L.I.

Ex S.T $S = \{(1,2,1), (3,1,5), (3,-4,7)\} \subseteq \mathbb{R}^3$ is L.D.

Sol^n $\forall\ a, b, c \in \mathbb{R}$

$$a(1,2,1) + b(3,1,5) + c(3,-4,7) = (0,0,0)$$

$$\Rightarrow (a, 2a, a) + (3b, b, 5b) + (3c, -4c, 7c) = (0,0,0)$$

$$\Rightarrow (a+3b+3c,\ 2a+b-4c,\ a+5b+7c) = (0,0,0)$$

$$\Rightarrow a+3b+3c = 0 \quad \text{---(1)}$$

$$2a+b-4c = 0 \quad \text{---(2)}$$

$$a+5b+7c = 0 \quad \text{---(3)}$$

Solving these equations, we get

$$(1)-(3) \equiv -2b-4c = 0$$

$$\Rightarrow b = -2c \quad \text{---(4)}$$

$$\therefore (1) \equiv a-6c+3c = 0$$

$$\Rightarrow a-3c = 0$$

$$\Rightarrow a = 3c \quad \text{---(5)}$$

Substituting (4) & (5) in (2) we get

$$6c-2c-4c = 0$$

$$\Rightarrow 0 = 0$$

$\therefore \exists$ non-zero values for a, b, c to satisfy the equations (1) (2) & (3)

$\therefore$ The given set is L.D.

Theorem

→ If two vectors are linear dependent then one of them is a scalar multiple of the other.

Solⁿ Let α, β be two linear dependent vectors of the vector space $V(F)$.

$\therefore \exists$ atleast one of the scalar $a, b \in F$ is non-zero s.t $a\alpha + b\beta = 0$

if $a \neq 0$ then $a\alpha = -b\beta$.

$\Rightarrow \alpha = \left(\frac{-b}{a}\right)\beta$.

$\therefore \alpha$ is scalar multiple of β.

If $b \neq 0$ then $b\beta = -a\alpha$

$\Rightarrow \beta = \left(-\frac{a}{b}\right)\alpha$

$\therefore \beta$ is the scalar multiple of α.

$\therefore$ One of the vectors α and β is scalar multiple of the other.

Theorem A set consisting of single non-zero vector is always L.I.

proof Let $V(F)$ be a vector space.

$S = \{\alpha\} \subseteq V$; $\alpha \neq 0$

if $a \in F$ then $a\alpha = 0$

$\Rightarrow a = 0 \quad (\because \alpha \neq 0)$

$\therefore S$ is L.I.

Theorem: If the set $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ of vectors of $V(F)$ is L.I then none of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ can be zero vector.

proof: Given that

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ is L.I

$\therefore a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$, $a_1, a_2, \ldots a_n \in F$

$\Rightarrow a_1 = a_2 = a_3 = \ldots = a_n = 0$

If possible let $\alpha_k = 0$; $1 \le k \le n$.

then $0\alpha_1 + 0\alpha_2 + \cdots + a_k\alpha_k + 0\alpha_{k+1} + \cdots + 0\alpha_n = 0$

for any $a_k \neq 0$ in F.

Since $a_k \neq 0$

$\therefore$ S is LD.

which is contradiction to the hypothesis that S is L.I.

Our assumption that $\alpha_k = 0$; $1 \le k \le n$ is wrong.

$\therefore$ None of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ can be zero vector.

Theorem $\rightarrow$ A set of vectors which containing the zero vector is LD.

proof Let V(F) be the vector space.

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and $\alpha_k = 0$; $1 \le k \le n$.

Consider linear combination

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k + a_{k+1}\alpha_{k+1} + \cdots + a_n\alpha_n = 0$.

Taking $a_1 = a_2 = \cdots = a_{k+1} = \cdots = a_n = 0$

and $a_k \neq 0$

$\therefore 0\alpha_1 + 0\alpha_2 + \cdots + a_k\alpha_k + 0\alpha_{k+1} + \cdots + 0\alpha_n = 0$

$\Rightarrow a_k\alpha_k = 0$

$\Rightarrow a_k \neq 0 \quad (\because \alpha_k = 0)$

$\therefore$ S is L.D.

Theorem A subset of a LI set is L.I.

proof V(F) is a vector space.

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ is LI

Now let $S' = \{\alpha_1, \alpha_2, \ldots \alpha_k\} \subseteq S \quad (1 \le k \le n)$

then $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = 0$; $a_1, a_2, \ldots a_k \in F$.

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_k\alpha_k + 0\alpha_{k+1} + 0\alpha_{k+2} + \cdots\cdots + \cdots + 0\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \cdots\cdots = a_k = 0$ ($\because S$ is LI)

$\therefore S'$ is LI.

Theorem A superset of a linear dependent set of vectors is LD

proof Let $V(F)$ be a vector space.

and $S = \{\alpha_1, \alpha_2, \cdots\cdots \alpha_n\} \subseteq V$ is LD.

Now let $S' = \{\alpha_1, \alpha_2, \cdots \alpha_n, \beta_1, \beta_2, \cdots \beta_k\} \supseteq S$

Since S is LD

$\therefore \exists$ atleast one of the scalar $a_1, a_2, \cdots\cdots a_n \in F$ is not zero s.t

$a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n = 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \cdots\cdots + 0\beta_k = 0$

Since in the above relation the scalar coefficients not all zero.

$\therefore S'$ is LD.

Theorem Let $V(F)$ be vector space. and $S = \{\alpha_1, \alpha_2 \cdots \alpha_n\} \subseteq V$ (contains non-zero vectors) if S is LD then one of the vectors of S say α_i $(1 < i \leq n)$ is a linear combination of its preceding vectors.

proof $V(F)$ is a vectorspace

$S = \{\alpha_1 \alpha_2 \cdots \alpha_n\} \subseteq V$

and S contains non-zero vectors.

Since S is LD.

$\therefore \exists$ atleast one scalar $a_1, a_2, \cdots\cdots a_n \in F$ is non-zero s.t $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k + \cdots + a_n\alpha_n = 0$ — (1)

Suppose that the maximum value of k for which $a_k \neq 0$ is 'i'.

i.e, $a_i \neq 0$ and $a_{i+1} = a_{i+2} = \cdots\cdots = a_n = 0$

if this maximum value is one then $a_1 \neq 0$

and $a_2 = a_3 = \cdots\cdots = a_n = 0$

$(1) \equiv a_1\alpha_1 + 0\alpha_2 + \cdots\cdots + 0\alpha_n = 0$

$\Rightarrow a_1\alpha_1 = 0$

$\Rightarrow \alpha_1 = 0 \quad (\because a_1 \neq 0)$

which is contradiction to the hypothesis that S contains non-zero vectors.

$\therefore i \neq 1.$

$\therefore 1 < i \leq n$

$(1) \equiv a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots\cdots + a_{i-1}\alpha_{i-1} + a_i\alpha_i + 0\alpha_{i+1} + 0\alpha_{i+2} + \cdots\cdots + 0\alpha_n = 0$

$\Rightarrow a_i\alpha_i = -a_1\alpha_1 - a_2\alpha_2 - \cdots\cdots - a_{i-1}\alpha_{i-1}$

$\Rightarrow \alpha_i = \left(-\frac{a_1}{a_i}\right)\alpha_1 + \left(-\frac{a_2}{a_i}\right)\alpha_2 + \cdots\cdots + \left(-\frac{a_{i-1}}{a_i}\right)\alpha_{i-1}$

$\therefore \alpha_i \, (1 < i \leq n)$ is a linear combination of its preceding vectors.

→ Let V(F) be the vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ (contains non-zero vectors)

If one of the vectors of 'S' say $\alpha_i \, (1 < i \leq n)$ is a linear combination of its preceding vectors then S is LD.

proof Given that V(F) is a vectorspace.

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and one of the vectors of S say α_i $(1<i\le n)$ is a linear combination of its preceding vectors.

$$\therefore \alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1}$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + (-1)\alpha_i = 0$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + (-1)\alpha_i + 0\alpha_{i+1} + 0\alpha_{i+2} + \cdots + 0\alpha_n = 0$$

$\therefore$ Co-efficient of $\alpha_i = -1 \neq 0$

$\therefore$ S is LD.

Theorem Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2 \cdots \alpha_n\} \subseteq V$ if one of the vectors of S is a linear combination of all the remaining vectors then S is LD.

proof: $V(F)$ is a vector space

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and one of the vectors of S is a linear combination of all the remaining vectors.

$$\therefore \alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + a_{i+1}\alpha_{i+1} + \cdots + a_n\alpha_n$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + (-1)\alpha_i + a_{i+1}\alpha_{i+1} + a_{i+2}\alpha_{i+2} + \cdots + a_n\alpha_n = 0$$

$\therefore$ The coefficient of $\alpha_i \neq 0$.

$\therefore$ S is LD.

Theorem If in a vector space $V(F)$, a vector β is a linear combination of the set of vectors $\alpha_1, \alpha_2 \cdots \alpha_n$ then the set of vectors $\beta, \alpha_1, \alpha_2 \ldots \alpha_n$ is L.D.

Sol^n Since β is a linear combination of $\alpha_1, \alpha_2 \ldots \alpha_n$

$\therefore \exists$ scalars $a_1, a_2, \dots a_n \in F$ s.t

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \dots\dots + a_n\alpha_n$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots\dots + a_n\alpha_n + (-1)\beta = 0$$

$\therefore$ In the above relation the coefficient of $\beta = -1 \neq 0$

$\therefore$ In the above relation not all the scalar coefficients are zero.

$\therefore$ The set of vectors $\alpha_1, \alpha_2, \dots \alpha_n, \beta$ is LD.

→ write the vector $\alpha = (1,-2,5)$ as a linear combination of the elements of the set $\{(1,1,1), (1,2,3), (2,-1,1)\} \subseteq \mathbb{R}^3$.

Sol'n: $\alpha = (1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1)$

$$= (a+b+2c,\ a+2b-c,\ a+3b+c)$$

$$\Rightarrow a+b+2c = 1 \quad \text{—— (1)}$$

$$a+2b-c = -2 \quad \text{—— (2)}$$

$$a+3b+c = 5 \quad \text{—— (3)}$$

$$(1)-(2) \equiv -b+3c = 3 \quad \text{—— (4)}$$

$$(2)-(3) \equiv -b-2c = -7 \quad \text{—— (5)}$$

$$(4)-(5) \equiv 5c = 10$$

$$\Rightarrow \boxed{c=2}$$

$$(4) \equiv -b = -3$$

$$\Rightarrow \boxed{b=3}$$

$$(1) \equiv a+3+4 = 1$$

$$\Rightarrow \boxed{a=-6}$$

$$\therefore (1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$$

H.W.

→ Express $\alpha = (2,-5,3)$ in $\mathbb{R}^3$ as a linear combination of the vectors $e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$ and $e_3 = (1,-5,7)$.

→ Express the polynomial $\alpha = t^2+4t-3$ as a linear combination of the polynomials $e_1 = t^2-2t+5$, $e_2 = 2t^2-3t$ and $e_3 = t+3$.

Sol^n $\alpha = ae_1 + be_2 + ce_3$; where a, b, c are unknown scalars. ——(I)

$\Rightarrow t^2+4t-3 = a(t^2-2t+5) + b(2t^2-3t) + c(t+3)$

$= (a+2b)t^2 + (-2a-3b+c)t + (5a+3c)$.

$\Rightarrow a+2b = 1$ ——(1)

$-2a-3b+c = 4$ ——(2)

$5a+3c = -3$ ——(3)

$2\times(2) + 3\times(1) \equiv -a+2c = 11$ ——(4)

$(3) + 5\times(4) \equiv 13c = 52$ $\Rightarrow \boxed{c=4}$; $(4) \equiv -a = 3 \Rightarrow \boxed{a=-3}$

$(1) \equiv -3+2b = 1$

$\Rightarrow \boxed{b=2}$

$\therefore$ (I) $\equiv$ $\alpha = -3e_1 + 2e_2 + 4e_3$.

→ Write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$

Sol^n $E = xA + yB + zC$ where x, y, z are unknown scalars. ——(I)

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & y \end{pmatrix} + \begin{pmatrix} 0 & 2z \\ 0 & -z \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} x & x+2z \\ x+y & y-z \end{pmatrix}$$

$\therefore \boxed{x=3}$

$x+2z = 1$; $x+y = 1$

$\Rightarrow \boxed{z=-1}$ $\Rightarrow \boxed{y=-2}$

$\therefore$ (I) $\equiv$ $E = 3A - 2B + (-1)C$

→ Determine whether α & β are L.D.
where (a) $\alpha = (3,4)$, $\beta = (1,-3)$
(b) $\alpha = (2,-3)$, $\beta = (6,-9)$

Sol^n (a) Since no vector is a scalar multiple of the other.

$\therefore$ α & β are not L.D.

b) Since β is a scalar multiple of α.

i.e, $(6,-9) = 3(2,-3)$
i.e, $\beta = 3\alpha$.
$\therefore$ α & β are L.D. vectors

→ Determine whether α & β are LD
where (a) $\alpha = (4,3,-2)$, $\beta = (2,-6,7)$
(b) $\alpha = (-4,6,-2)$, $\beta = (2,-3,1)$

Sol^n a) Neither is a scalar multiple of the other.
$\therefore$ α and β are not LD.

b) $\alpha = (-2)\beta$.
$\therefore$ α and β are LD.

→ S.T $S = \{(1,2,4), (1,0,0), (0,1,0), (0,0,1)\}$ is a LD subset of $V_3(\mathbb{R})$.

Sol^n Since one of the vector of S is a linear combination of all the remaining vectors.
i.e, $(1,2,4) = 1(1,0,0) + 2(0,1,0) + 4(0,0,1)$
$\therefore$ S is LD.

Echelon form of a matrix:

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A matrix 'A' is said to be in echelon form if the number of zeroes preceding the non-zero elt of a row increases row by row and the elts of last row or rows may be all zeroes.

Ex: $\begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are all echelon matrices.

Note: [1] The rank of matrix in echelon form is equal to the no. of non-zero rows of the matrix.

ex: $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the matrix A in echelon form

∴ The no. of non-zero rows in echelon form $= 2$

∴ $\rho(A) = 2$

Note [2]. Let
$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \text{—— (I)}$$

given system of 3 non-homogeneous linear equations in 3 unknowns x, y, z.

Now write the single matrix equation –

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times1}$

and the matrix $[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$ is called the augmented matrix of the given system of equations.

* **working rule for finding the solutions of the equation $AX=B$:-**

→ Now the augmented matrix $[A|B]$ reduce to an echelon form by appling only elementary row operations.

→ This echelon form will enable us to know the ranks of the augmented matrix $[A|B]$ and the coefficient matrix A.

Then the following cases arise:

(i) If $\rho(A) = \rho(A|B)$ = the no. of unknowns.
then the given system (I) is consistent and has unique solution.

(ii) If $\rho(A) = \rho(A|B) <$ no. of unknowns.
then the given system (I) is consistent and has infinite solutions.

(iii) If $\rho(A) \neq \rho(A|B)$ then the given system is inconsistent and has no solution.

Note [3]. Let

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 0 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}\right\} \text{—(II)}$$

be the given system of 3 homogeneous linear equations in 3 unknowns x, y, z.

Now write the single matrix equation

$$AX = 0$$

where coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} ; \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}.$$

working rule for finding the solutions of the equation $AX=0$:-

→ Reduce the coefficient matrix A to echelon form by applying elementary row operations only.

This echelon form will help us to know the rank of the matrix A.

→ If $\rho(A)$ = no. of unknowns.
then the system (II) possesses a zero solution (trivial solution)
i.e, $x=0, y=0, z=0$

→ If $\rho(A) <$ no. of unknowns
then there will be a non-zero solution (non-trivial solution)

Problem → Determine whether or not $\alpha = (3, 9, -4, -2)$ in $\mathbb{R}^4$ is a linear combination of $\alpha_1 = (1, -2, 0, 3)$, $\alpha_2 = (2, 3, 0, -1)$ and $\alpha_3 = (2, -1, 2, 1)$

Sol^n:- Let $x, y, z \in \mathbb{R}$.

$$\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3.$$

$$\Rightarrow (3, 9, -4, -2) = x(1, -2, 0, 3) + y(2, 3, 0, -1) + z(2, -1, 2, 1)$$

$$\Rightarrow x + 2y + 2z = 3$$
$$-2x + 3y - z = 9$$
$$2z = -4$$
$$3x - y + z = -2$$

Now write the single matrix equation $AX=B$

i.e., $$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 3 & -1 \\ 0 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -4 \\ -2 \end{bmatrix}$$

Augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -2 & 3 & -1 & 9 \\ 0 & 0 & 2 & -4 \\ 3 & -1 & 1 & -2 \end{array}\right]$

$R_2 \to R_2 + 2R_1$
$R_4 \to R_3 - 3R_1$
$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & -5 & -11 \end{array}\right]$$

$R_4 \to R_4 + R_2$
$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$R_4 \to R_4 + R_3$
$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = \rho(A/B) = 3 =$ no. of unknowns x, y, z.

$\therefore$ The given system is consistent and has unique solⁿ.

for solving the unknowns x, y, z..
we write the echelon matrix equation

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ -4 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y + 2z = 3$

$7y + 3z = 15$

$2z = -4 \Rightarrow \boxed{z = -2}$; $\boxed{x = 1}$ and $\boxed{y = 3}$.

$\therefore \alpha = 1\alpha_1 + 3\alpha_2 - 2\alpha_3$.

$\therefore \alpha$ is a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Note 1. If the system of linear equations are consistent. then it has a solⁿ and the vector α is a linear combination of α_i $(1 \le i \le n)$.

2. If the given system of linear equations are not consistent then it has no solution and the vector α is not a linear combination of α_i $(1 \le i \le n)$

→ P.T the set $\{(-1,2,1), (3,0,-1), (-5,4,3)\} \subseteq \mathbb{R}^3$ is LD.

Sol^n Let $a, b, c \in \mathbb{R}$ then

$a(-1,2,1) + b(3,0,-1) + c(-5,4,3) = (0,0,0)$.

$\Rightarrow -a + 3b - 5c = 0$ —(1)

$2a + 4c = 0$ —(2)

$a - b + 3c = 0$ —(3)

Solving the above equations, we get

(1)+(3) ≡ $2b - 2c = 0 \Rightarrow \boxed{b = c}$

(2) ≡ $\boxed{a = -2c}$

(3) ≡ $-2c - c + 3c = 0$

$\Rightarrow 0 = 0$

∴ ∃ non-zero values for a, b, c to satisfy the equations (1), (2), (3).

∴ The given set is LD.

→ Determine whether or not the vectors $(1,-2,1), (2,1,-1)$ $(7,-4,1)$ are LD.

Sol^n If $a, b, c \in F$, then

$a(1,-2,1) + b(2,1,-1) + c(7,-4,1) = (0,0,0)$.

$$\Rightarrow \left.\begin{array}{l} a + 2b + 7c = 0 \\ -2a + b - 4c = 0 \\ a - b + c = 0 \end{array}\right\} \text{—(1)}$$

Coefficient matrix $A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}$

Now $|A| = 1(-3) - 2(2) + 7(1)$

$= -3 - 4 - 7 = 0$

∴ $\rho(A) <$ no. of unknowns a, b, c.

∴ The system of equations possess a non-zero solution.

∴ The given vectors are LD.

Note: [1]. Consider the system of three linear equations in three unknown variables.

$$\left.\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{array}\right\} \text{—(1)}$$

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

→ If $|A| \neq 0$,
then the system ① possesses a trivial solution (zero soln)
i.e, $x=0, y=0, z=0$.

→ If $|A| = 0$, the system ① possesses a non trivial solution (non-zero soln).

→ Determine whether $(2,-3,7)$, $(0,0,0)$, $(3,-1,-4)$ are LD.

Method ①

Soln: Let $a, b, c \in \mathbb{R}$. then
$$a(2,-3,7) + b(0,0,0) + c(3,-1,-4) = (0,0,0).$$

$$\Rightarrow \left.\begin{aligned} 2a + 0b + 3c &= 0 \\ -3a + 0b - c &= 0 \\ 7a + 0b - 4c &= 0 \end{aligned}\right\} \text{— ①}$$

The coefficient matrix $A = \begin{bmatrix} 2 & 0 & 3 \\ -3 & 0 & -1 \\ 7 & 0 & -4 \end{bmatrix}$

and $|A| = 0$

∴ The System of equations posses a non-zero solution.

∴ The given vectors are LD.

Method ②

form the matrix 'A' whose rows are the given vectors

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$\Rightarrow |A| = 0$

∴ The given vectors are LD.

Method ③

form the matrix 'A' whose rows are the given vectors and reduce to echelon form

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 2 & -3 & 7 \\ 3 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \to 2R_2 - 3R_1$ $\begin{bmatrix} 2 & -3 & 7 \\ 0 & 7 & -29 \\ 0 & 0 & 0 \end{bmatrix}$

which is an echelon form.

Since the echelon form has a zero row.

$\therefore$ The given vectors are LD.

→ In $V_3(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers, examine each of the following sets of vectors for linear dependent.

(i) $\{(2,1,2), (8,4,8)\}$ (ii) $\{(1,2,0), (0,3,1), (-1,0,1)\}$

(iii) $\{(2,3,5), (4,9,25)\}$.

→ P.T the set $\{(1,2,1), (3,1,5), (2,-4,7)\} \subseteq \mathbb{R}^3$ is LI.

→ Examine the vectors $(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)$ are LI in $\mathbb{R}^4$

Soln Now form the matrix 'A' whose rows are given vectors and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$
$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_2 \to -\frac{1}{3}R_2$
$R_3 \to \frac{1}{2}R_3$
$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_3 \to R_3 + R_2$
$R_4 \to R_4 + R_2$
$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly which is in echelon form.

Since this echelon form has two zero rows.

$\therefore$ The given vectors are LD

→ Determine whether $(1,2,-3), (1,-3,2), (2,-1,5)$ are L.D.

Sol^n: Now form the matrix A whose rows are given vectors and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{bmatrix}$$

$$\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & -5 & 11 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Clearly which is in echelon form.

Since the echelon form has no zero rows.

∴ The given vectors are L.I.

→ Let V be the vector space of functions from $\mathbb{R} \to \mathbb{R}$.

Show that $f, g, h \in V$ are L.I.

where $f(t) = e^{2t}$, $g(t) = t^2$; $h(t) = t$.

Sol^n: Let $a, b, c \in \mathbb{R}$ then $af + bg + ch = 0$

Now for every value of t,

we have $af(t) + bg(t) + ch(t) = 0$

$\Rightarrow ae^{2t} + bt^2 + ct = 0$.

if $t = 0$, then $ae^0 + b(0) + c(0) = 0$

$\Rightarrow \boxed{a = 0}$ —— (1)

if $t = 1$ then $ae^{2(1)} + b(1)^2 + c(1) = 0$

$\Rightarrow ae^2 + b + c = 0$ —— (2)

if $t = 2$ then $ae^{2(2)} + b(2)^2 + c(2) = 0$

$\Rightarrow ae^4 + 4b + 2c = 0$ —— (3)

(2) ≡ $b + c = 0$ —— (4)

(3) ≡ $4b + 2c = 0$ —— (5) $(\because a = 0)$

$2 \times (4) - (5) \equiv -2b = 0$

$\Rightarrow \boxed{b = 0}$

(4) ≡ $\boxed{c = 0}$

∴ f, g, h are L.I.

→ S.T the functions $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ are LI.

Sol^n Let $a, b, c \in \mathbb{R}$

then $af + bg + ch = 0$

for every value of t

we have $af(t) + bg(t) + ch(t) = 0$

$\Rightarrow a\sin t + b\cos t + ct = 0$ ——(I)

if $t = 0$ then $a(0) + b(1) + c(0) = 0$

$\Rightarrow \boxed{b = 0}$ ——(1)

if $t = \pi/2$ then $a(1) + b(0) + c(\pi/2) = 0$

$\Rightarrow a + c(\pi/2) = 0$ ——(2)

if $t = \pi$ then $a(0) + b(-1) + c\pi = 0$

$\Rightarrow \boxed{-b + c\pi = 0}$ ——(3)

from (1) & (3) $0 + c\pi = 0$

$\Rightarrow \boxed{c = 0}$ ——(4)

from (2) & (4) $a + 0 = 0$

$\Rightarrow \boxed{a = 0}$

$\therefore f(t), g(t), h(t)$ are LI.

Second method:

diff (I) three times w.r.t t

——(1)

——(2)

——(3)

→ Find the values of K for which the vectors $(1,1,1,1)$, $(1,3,-2,k)$, $(2, 2k-2, -k-2, 3k-1)$ and $(3, k+2, -3, 2k+1)$ are LI in $\mathbb{R}^4$.

Sol^n form the matrix A whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{bmatrix}$$

Since the given vectors are LI.

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{vmatrix} \neq 0$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 3R_1$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & k-1 \\ 0 & 2k-4 & -k-4 & 3k-3 \\ 0 & k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

$$\Rightarrow 1 \begin{vmatrix} 2 & -3 & k-1 \\ 2k-4 & -k-4 & 3k-3 \\ k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

proceeding in this way.

→ If α_1 & α_2 are vectors of $V(F)$ and $a, b \in F$.
S.T the set $\{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\}$ is LD.

Sol^n Let $S = \{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\} \subseteq V(F)$.
Since one of the vector of S is a l.c. of the remaining vectors.
i.e., $\underline{a\alpha_1 + b\alpha_2} = a\alpha_1 + b\alpha_2$
$\therefore S$ is LD.

→ Let $\alpha_1, \alpha_2, \alpha_3$ be vectors of $V(F)$, $a, b \in F$.
S.T the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD if the set $\{\alpha_1 + a\alpha_2 + b\alpha_3, \alpha_2, \alpha_3\}$ is LD.

Sol^n Since the set $\{\alpha_1 + a\alpha_2 + b\alpha_3, \alpha_2, \alpha_3\} \subseteq V$ is LD.
$\therefore \exists$ atleast one non-zero scalar $x, y, z \in F$ s.t
$x(\alpha_1 + a\alpha_2 + b\alpha_3) + y(\alpha_2) + z(\alpha_3) = 0$
$\Rightarrow x\alpha_1 + (ax + y)\alpha_2 + (bx + z)\alpha_3 = 0$
If $x \neq 0$ then the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD.
If $x = 0$ then atleast one of y & z is not zero
$\therefore$ Atleast one of $ax + y$ & $bx + z$ is not zero.
$\therefore$ the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD.

→ If α, β, γ are LI vectors of $V(F)$. Where F is field of Complex numbers then $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$ are also LI.

Sol^n Let $a, b, c \in F$ then
$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$
$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0$
Since α, β, γ are LI
$\therefore a + c = 0$ — (1)
$a + b = 0$ — (2)
$b + c = 0$ — (3)
(1) − (3) ≡ $a - b = 0$ — (4)
(2) + (4) ≡ $2a = 0 \Rightarrow \boxed{a = 0}$
(4) ≡ $\boxed{b = 0}$ and (3) ≡ $\boxed{c = 0}$
$\therefore \alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$ are LI.

→ Let $\mathbb{C}(\mathbb{C})$ be a vector space. then show that $\{1, i\} \subseteq \mathbb{C}(\mathbb{C})$ is LD.

Let $S = \{1, i\} \subseteq \mathbb{C}(\mathbb{C})$

Soln: Since one of the vector of S is scalar multiple of other.

i.e, $i = i(1)$ (34(ii))

$\therefore$ S is LD.

→ Let $\mathbb{C}(\mathbb{R})$ be a vector space then show that $\{1, i\} \subseteq \mathbb{C}(\mathbb{R})$ is LI.

Soln Let $a, b \in \mathbb{R}$ then $a(1) + b(i) = 0 + 0(i)$

$\Rightarrow a = 0,\ b = 0$

$\therefore$ $\{1, i\}$ is LI.

→ S.T the set $\{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{C})$ is LD. over the field of complex numbers.

Soln: Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{C})$

Since one of the vector of S is a scalar multiple of other.

i.e, $(1+i, 2i) = (1+i)(1, 1+i)$

$\therefore$ S is LD.

→ S.T $\{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{R})$ is LI. over the field of real numbers.

Soln Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq \mathbb{C}^2(\mathbb{R})$

Let $a, b \in \mathbb{R}$ then

$a(1+i, 2i) + b(1, 1+i) = (0, 0)$

$\Rightarrow (a + ia, 2ia) + (b, b + ib) = (0, 0)$

$\Rightarrow (a + ia + b,\ 2ia + b + ib) = (0, 0)$

$\Rightarrow$ $a(1+i) + b = 0$ —(1)

$b(1+i) + 2ia = 0$ —(2)

(1) $\equiv$ $(a+b) + ia = 0 + i0$

$\Rightarrow a + b = 0$ & $\boxed{a = 0}$

$\Rightarrow \boxed{b = 0}$ $\therefore$ S is LI.

→ In the vector space $F[x]$ of all polynomials over the field F, the infinite set $S = \{1, x, x^2, \ldots\}$ is LI.

Soln Let $S' = \{x^{m_1}, x^{m_2}, \ldots\ldots x^{m_n}\}$ be any finite subset of S having n vectors.

Hence $m_1, m_2, \ldots m_n$ are non-negative integers

Let $a_1, a_2, \ldots a_n \in F$ s.t

$$a_1 x^{m_1} + a_2 x^{m_2} + \ldots + a_n x^{m_n} = 0x^{m_1} + 0x^{m_2} + \ldots + 0x^{m_n}$$

$$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$$

$\therefore$ Every finite subset of S is LI.

$\therefore$ S is LI.

→ Let $S = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(3,5) \in L(S)$.

Soln $(3,5) = 3(1,0) + 5(0,1)$

$\Rightarrow (3,5) \in L(S)$.

→ Let $S = \{(1,0,0), (0,1,0)\} \subseteq \mathbb{R}^3(\mathbb{R})$. find $L(S)$.

Do $(3,2,0)$, and $(2,5,1)$ belong to $L(S)$.

Soln $L(S) = \{\alpha(1,0,0) + \beta(0,1,0) / \alpha, \beta \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

$= \{(\alpha, \beta, 0) / \alpha, \beta \in \mathbb{R}\}$

$\therefore (3,2,0) \in L(S)$.

but $(2,5,1) \notin L(S)$. $(\because 1 \neq 0)$

→ Let $S = \{(2,3), (1,4)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(4,1) \in L(S)$.

Soln $(4,1) = \alpha(2,3) + \beta(1,4)$; $\alpha, \beta \in \mathbb{R}$.

$\Rightarrow 2\alpha + \beta = 4$

$3\alpha + 4\beta = 1$

$\Rightarrow \alpha = 3, \beta = -2$.

$\therefore (4,1) = 3(2,3) - 2(1,4)$

$\therefore (4,1) \in L(S)$

→ Is the vector $(2,-5,3)$ in the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$? (94)(iii)

Sol^n Let $\alpha = (2,-5,3)$, $\alpha_1 = (1,-3,2)$, $\alpha_2 = (2,-4,-1)$
$\alpha_3 = (1,-5,7)$

Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{R}^3(\mathbb{R})$

Let $\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$; $a, b, c \in \mathbb{R}$

then $(2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$

$\Rightarrow a + 2b + c = 2$ — (1)
$-3a - 4b - 5c = -5$ — (2)
$2a - b + 7c = 3$ — (3)

$3\times(1) + (2) \equiv 2b - 2c = 1 \Rightarrow b - c = 1/2$ — (4)

$2\times(1) - (3) \equiv 5b - 5c = 1 \Rightarrow b - c = 1/5$ — (5)

$\therefore$ The equations (4) & (5) are inconsistent.

$\therefore \alpha$ cannot be expressed as a l.c. of S.

$\therefore \alpha$ is not in the subspace of $\mathbb{R}^3$ spanned by S.

→ In the vector space $\mathbb{R}^3$ let $\alpha = (1,2,1)$, $\beta = (3,1,5)$, $\gamma = (3,-4,7)$.
S.T the subspaces spanned by $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \gamma\}$ are the same.

Sol^n: Let $S = \{\alpha, \beta\} \subseteq V_3(\mathbb{R})$
$T = \{\alpha, \beta, \gamma\} \subseteq V_3(\mathbb{R})$
and $L(S)$ & $L(T)$ be two subspaces spanned by S & T.

we have to S.T $L(S) = L(T)$.

Since $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ — (1)

Let $x \in L(T)$ then
$x = a\alpha + b\beta + c\gamma$; $a, b, c \in \mathbb{R}$.

Let $\gamma = a_1\alpha + a_2\beta$; $a_1, a_2 \in \mathbb{R}$.

$\Rightarrow (3,-4,7) = a_1(1,2,1) + a_2(3,1,5)$

$\Rightarrow a_1 + 3a_2 = 5$ — (i)

$2a_1 + a_2 = -4$ — (ii)

$a_1 + 5a_2 = 7$ — (iii)

(i) − (iii) $\equiv -2a_2 = -4$

$\boxed{a_2 = 2}$

and $\boxed{a_1 = -3}$

$\therefore$ (3) $\equiv \gamma = -3\alpha + 2\beta$.

$\therefore$ (2) $\equiv x = a\alpha + b\beta + c(-3\alpha + 2\beta)$

$= (a-3c)\alpha + (b+2c)\beta$

$=$ L.C. of α & β.

$\therefore x \in L(S)$.

$\therefore L(T) \subseteq L(S)$. — (4)

$\therefore$ from (1) & (4)

we have $L(S) = L(T)$

→ Is the vector $(3,-4,6)$ in the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,2,-1)$, $(2,2,1)$ and $(1,-2,3)$?

→ Let $\alpha_1 = (1,1,-2,1)$, $\alpha_2 = (3,0,4,-1)$, $\alpha_3 = (-1,2,5,2)$.

Show that $(4,-5,9,-7)$ is spanned by $\alpha_1, \alpha_2, \alpha_3$.

T(H) → Is the vector $(3,-1,0,-1)$ in the subspace of $\mathbb{R}^4$ spanned by the vectors $\alpha_1 = (2,-1,3,2)$, $\alpha_2 = (-1,1,1,-3)$ and $\alpha_3 = (1,1,9,-5)$?

→ Let $V = \mathbb{R}^3(\mathbb{R})$ and $S = \{\alpha_1 = (1,1,0), \alpha_2 = (0,-1,1), \alpha_3 = (1,0,1)\}$

prove that $(a,b,c) \in L(S)$ iff $a = b + c$.

Sol^n: By definition of $L(S)$, $(a,b,c$

$(a,b,c) \in L(S)$

$\Leftrightarrow (a,b,c) = \alpha(1,1,0) + \beta(0,-1,1) + \gamma(1,0,1)$; $\alpha,\beta,\gamma \in \mathbb{R}$

$\Leftrightarrow (a,b,c) = (\alpha+\gamma, \alpha-\beta, \beta+\gamma)$

$\Leftrightarrow a = \alpha+\gamma,\ b = \alpha-\beta,\ c = \beta+\gamma$

$\Leftrightarrow a = b + c$

If v_1, v_2, v_3 are three vectors in a vector space $V(F)$ such that $v_1+v_2+v_3=0$, then show that $\{v_1, v_2\}$ spans the same subspace as $\{v_2, v_3\}$. (3 F(iv))

Sol^n: Let $S=\{v_1, v_2\}$ and $T=\{v_2, v_3\}$.

We shall prove that $L(S)=L(T)$

Let $x \in L(S)$.

Then $x = \alpha v_1 + \beta v_2$; $\alpha, \beta \in F$

$\Rightarrow x = \alpha(-v_2-v_3)+\beta v_2$

$\Rightarrow x = (\beta-\alpha)v_2 - \alpha v_3 \in L(T)$

$\therefore L(S) \subseteq L(T)$.

Conversely, let $y \in L(T)$

Then $y = a v_2 + b v_3$; $a, b \in F$

$\Rightarrow y = a v_2 + b(-v_1-v_2)$

$\Rightarrow y = -b v_1 + (a-b) v_2 \in L(S)$

$\therefore L(T) \subseteq L(S)$.

Hence $L(S)=L(T)$

→ Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which spans W.