CONTENTS

1

Rings

In this chapter we shall introduce the concepts of *ring*, *integral domain*, *division ring* and *field* and give their examples and basic properties. We also discuss *subrings* and *ideals* of a ring.

1.1 Ring

Definition 1. *A* **ring** *is a non-empty set R with two binary compositions denoted by + and ·, and satisfying the following properties :*

R.1. $a + b \in R$ *for all* $a, b \in R$.

R.2. $a + b = b + a$ *for all* $a, b \in R$.

R.3. $a + (b + c) = (a + b) + c$ *for all* $a, b, c \in R$.

R.4. *There exists an element denoted by* $0 \in R$ *such that*

$$a + 0 = a \text{ for all } a \in R.$$

(0 *is called* **additive identity** *or* **zero element** *in R*)

R.5. *For each* $a \in R$, *there exists an element* $b \in R$ *such that*

$$a + b = 0.$$

(*b is called* **additive inverse** *or* **negative** *of a and is written as* $b = -a$, *so that* $a + (-a) = 0$)

R.6. $a \,.\, b \in R$ *for all* $a, b \in R$.

R.7. $a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c$ *for all* $a, b, c \in R$.

R.8. $a \,.\, (b + c) = a \,.\, b + a \,.\, c$ (*Left distributive law*)

R.9. $(a + b) \,.\, c = a \,.\, c + b \,.\, c$ (*Right distributive law*)

for all $a, b, c \in R$.

We denote a ring as $\{R, +, \cdot\}$.

Remark. Axioms R.1. – R.5. mean that $(R, +)$ is an abelian group and R.6. – R.7. mean that $(R, \cdot)$ is a semi-group.

Definition 2. *A ring is called* **finite** *or* **infinite** *according as it contains finite or infinite number of elements.*

Definition 3. *A ring* $\{R, +, \cdot\}$ *is called a* **commutative ring**, *if*

$$a \,.\, b = b \,.\, a \text{ for all } a, b \in R.$$

Definition 4. *A ring R is called* **boolean**, *if* $x^2 = x$ *for all* $x \in R$.

Definition 5. *A ring* $\{R, +, \cdot\}$ *is called a ring with* **unit element** *or* **unity** *or* **identity**, *if there exists an element* $e \in R$ *such that*

$$a \,.\, e = e \,.\, a = a \text{ for all } a \in R.$$

Definition 6. *Let R be a ring with unity e. An element $a \in R$ is called* **invertible**, *if there exists some element $b \in R$ such that*

$$a \,.\, b = b \,.\, a = e.$$

Definition 7. *If n be a positive integer and a an element of a ring R we define*

$$a^n = \underbrace{a \,.\, a \ldots a}_{n \text{ times}} \quad \text{and} \quad na = \underbrace{a + a + \ldots + a}_{n \text{ times}}.$$

1.2 Examples of Rings

Example 1.2.1. *The set* $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ *of integers is a commutative ring under the usual addition and multiplication of integers.* Here 0 is the additive identity and $-a$ is the additive inverse of $a \in \mathbf{Z}$. Note that $1 \in \mathbf{Z}$ is the unit element, since $a \,.\, 1 = 1 \,.\, a = a \;\forall\; a \in \mathbf{Z}$.

However, $\mathbf{N} = \{1, 2, 3, \ldots\}$ is **not** a ring. (Why ?)

Example 1.2.2. *The set* $\mathbf{E} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ *of all even integers is a commutative ring without unit element.*

Example 1.2.3. *The sets* $\mathbf{R}$ *(all real numbers) and* $\mathbf{Q}$ *(all rational numbers) are commutative rings with unity w.r.t. usual addition and multiplication.*

Example 1.2.4. (Ring of integers modulo $n \equiv \mathbf{Z}_n$)

For any positive integer n, $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$ is a commutative ring w.r.t. addition and multiplication modulo n, denoted as $\oplus_n$ and $\otimes_n$, respectively.

In particular, $\mathbf{Z}_2 = \{0, 1\}$, $\mathbf{Z}_3 = \{0, 1, 2\}$, $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ are rings of integers modulo 2, 3 and 6, respectively. In $\mathbf{Z}_6$, $2 \oplus_6 3 = 5$, $3 \oplus_6 4 = 1$, $4 \oplus_6 5 = 3$; $2 \otimes_6 3 = 0$, $4 \otimes_6 5 = 2$ etc.

It may be observed that $\mathbf{Z}_2$ is a boolean ring, since $0^2 = 0$ and $1^2 = 1$ in $\mathbf{Z}_2$.

Example 1.2.5. The Ring of Gaussian Integers $\equiv J[i]$ or $\mathbf{Z}[i]$:

$$\mathbf{Z}[i] = \{m + ni : m, n \text{ are integers}, i = \sqrt{-1}\}$$

is a commutative ring with unity 1 $(= 1 + 0i)$ w.r.t. the usual addition and multiplication of complex numbers.

Example 1.2.6. *The set of all 2×2 matrices :*

$$M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbf{R} \text{ (all reals)} \right\}$$

is a non-commutative ring with unity, under the addition and multiplication of 2×2 matrices. Note that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity of M_2 and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$ is the negative of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Also $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unity of M_2.

Example 1.2.7. *The set S of all 2×2 matrices over the ring $Z_2 = \{0, 1\}$ of integers modulo 2 is a finite non-commutative ring.*

Indeed, the ring S has $2^4 = 16$ elements *viz.*

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Example 1.2.8. *The set*

$$M = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a \text{ and } b \text{ are real numbers} \right\}$$

is a non-commutative ring without unity, under matrix addition and matrix multiplication.

Notice that $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$

$$\therefore \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Example 1.2.9. *The set*

$$M = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a \text{ and } b \text{ are complex numbers} \right\}$$

is a non-commutative ring with unity, under matrix addition and multiplication. The unity of M_2 is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Here $\bar{a}$ denotes the conjugate of a. Notice that

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} = \begin{pmatrix} ax - b\bar{y} & ay + b\bar{x} \\ -(\bar{a}\bar{y} + \bar{b}x) & \bar{a}\bar{x} - \bar{b}y \end{pmatrix} \in M.$$

Example 1.2.10. *Let R denote the set of all real-valued continuous functions on $[0, 1]$. Let $f, g \in R$. Define*

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)\, g(x) \ \forall\ x \in [0, 1].$$

Then R is a commutative ring with unity.

It is easy to verify that R is a ring, in which the additive identity is the zero function given by $o(x) = 0 \ \forall\ x \in [0, 1]$.

Indeed, $(f + o)(x) = f(x) + o(x) = f(x) + 0 = f(x) \ \forall\ x \in [0, 1]$.

$$\therefore \quad f + o = f \quad \text{for all } f \in R.$$

The additive inverse of $f \in R$ is $f_1 : [0, 1] \to \mathbf{R}$ such that

$f_1(x) = -f(x)$ for all $x \in [0, 1]$. We have $f + f_1 = o$.

Further $(f \cdot g)(x) = f(x)\, g(x) = g(x) f(x) = (g \cdot f)(x) \ \forall\ x \in [0, 1]$.

$\therefore f \cdot g = g \cdot f \ \forall\ f, g \in R$. Hence R *is a commutative ring with unity* $I \in R$, where $I(x) = x \ \forall\ x \in [0, 1]$.

1.3 Some Properties of Rings

Theorem 1.3.1. *If R is a ring and $a, b, c \in R$; then*

1. $a + b = a + c \Rightarrow b = c$. *(Cancellation law w.r.t. +)*
2. $-(-a) = a$.
3. *The zero element of R is unique.*
4. *The additive inverse of any element in R is unique.*

Proof. Since R is a ring, so by definition, $(R, +)$ is a group.

1. By cancellation law in a group $(G, \cdot)$, we know

$$a \,.\, b = a \,.\, c \Rightarrow b = c. \qquad \ldots(1)$$

Since $(R, +)$ is a group, so writing + for . in (1), we get

$$a + b = a + c \Rightarrow b = c. \qquad \ldots(2)$$

2. In a group $(G, \cdot)$, we know $(a^{-1})^{-1} = a$.

Since $(R, +)$ is a group in which a^{-1} is denoted by $-a$, so by (2),

$$-(-a) = a.$$

3. Let, if possible, 0 and 0′ be two zero elements in R.

Then $0 + a = a$ and $a + 0' = a \;\forall\; a \in R$. $\ldots(3)$

In particular, $0 + 0' = 0'$ and $0 + 0' = 0$.

(Take $a = 0'$ and $a = 0$ in (3), respectively).

Hence $0 = 0'$, which shows that the zero element in R is unique.

4. Let, if possible, a' and a'' be two additive inverses of a in R. Then

$$a + a' = a' + a = 0 \quad \text{and} \quad a + a'' = a'' + a = 0.$$

Now $a' = a' + 0 = a' + (a + a'') = (a' + a) + a'' = 0 + a'' = a''$.

Hence $a' = a''$, which shows that the additive inverse of any element $a \in R$ is unique.

Theorem 1.3.2. *If R is a ring, then for any $a, b, c \in R$;*

1. $a \,.\, 0 = 0 \,.\, a = 0$.
2. $a \,.\, (-b) = (-a) \,.\, b = -(a \,.\, b)$.
3. $(-a) \,.\, (-b) = a \,.\, b$.
4. $a \,.\, (b - c) = a \,.\, b - a \,.\, c$.

If, in addition, R has a unit element 1, then

5. $(-1) \,.\, a = -a$.
6. $(-1) \,.\, (-1) = 1$.

Proof. 1. We know $a + 0 = a$ for all $a \in R$.

In particular, $0 + 0 = 0$ and so $a \,.\, (0 + 0) = a \,.\, 0$

$\Rightarrow a \,.\, 0 + a \,.\, 0 = a \,.\, 0$ (left distributive law)

$\Rightarrow a \,.\, 0 + a \,.\, 0 = a \,.\, 0 + 0$.

Hence $a \,.\, 0 = 0$, by cancellation law in the group $(R, +)$.

Similarly, we can show that $0 \,.\, a = 0$.

2. We know $b + (-b) = 0$ for all $b \in R$.

$\therefore$ $a \cdot \{b + (-b)\} = a \cdot 0 = 0$, by part 1.

or $a \cdot b + a \cdot (-b) = 0$ (left distributive law).

Hence $a \cdot (-b) = -(a \cdot b)$.

Similarly, $\{a + (-a)\} \cdot b = 0 \cdot b \Rightarrow a \cdot b + (-a) \cdot b = 0$.

Hence $(-a) \cdot b = -(a \cdot b)$.

3. We have $(-a) \cdot (-b) = (-a) \cdot c$, where $c = -b$

$= -(a \cdot c)$, by part 2

$= -\{a \cdot (-b)\}$

$= -\{-(a \cdot b)\}$, by part 2

$= a \cdot b$, since $-(-x) = x$ in R.

4. $a \cdot (b - c) = a \cdot \{b + (-c)\} = a \cdot b + a \cdot (-c) = a \cdot b - a \cdot c$, by part 2.

5. Suppose that R has a unit element 1. Then $a \cdot 1 = 1 \cdot a = a$.

Now $a + (-1) \cdot a = 1 \cdot a + (-1) \cdot a = \{1 + (-1)\} \cdot a = 0 \cdot a = 0$.

Hence $(-1) \cdot a = -a$.

6. Taking $a = -1$ in part 5, we obtain

$$(-1) \cdot (-1) = -(-1) = 1.$$

Notation. *In a ring $\{R, +, \cdot\}$, we shall now write $a \cdot b$ as ab.*

EXAMPLES

Example 1.3.1. *Show that the set $R = \{a + b\sqrt{3} : a, b \in \mathbf{Q}\}$ is a ring under the usual addition and multiplication as binary compositions.*

Solution. It is easy to verify that $(R, +)$ is an abelian group with $0 = 0 + 0\sqrt{3}$ as the additive identity and $-a - b\sqrt{3}$ as the additive inverse of $a + b\sqrt{3}$.

Let $x = a + b\sqrt{3}$, $y = c + d\sqrt{3}$ and $z = e + f\sqrt{3} \in R$.

Then $xy = (a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (ad + bc)\sqrt{3} \in R$,

since $ac + 3bd \in \mathbf{Q}$ and $ad + bc \in \mathbf{Q}$.

Now $(xy)z = \{(ac + 3bd) + (ad + bc)\sqrt{3}\}(e + f\sqrt{3})$

$= (ace + 3bde + 3adf + 3bcf) + (acf + 3bdf + ade + bce)\sqrt{3}$

$= x(yz)$.

Finally,

$x(y + z) = (a + b\sqrt{3})\{(c + e) + (d + f)\sqrt{3}\}$

$= (ac + ae + 3bd + 3bf) + (ad + af + bc + be)\sqrt{3}$

$= \{(ac + 3bd) + (ad + bc)\sqrt{3}\} + \{(ae + 3bf) + (af + be)\sqrt{3}\}$

$= xy + xz$.

Similarly, $(x + y)z = xy + yz$. Hence R is a ring.

Note. R is a communicative ring, since $xy = yx \ \forall \ x, y \in R$.

Example 1.3.2. *Show that the set $\mathbf{I}$ of integers with two binary compositions $*$ and o defined by $a * b = a + b - 1$, $aob = a + b - ab$ for all integers a and b is a commutative ring with unity.*

Solution. We have $a * b = a + b - 1 \ \forall \ a, b \in \mathbf{I}$.

From (1), $a * b \in \mathbf{I} \ \forall \ a, b \in \mathbf{I}$. ...(1)

Now $a * b = a + b - 1 = b + a - 1 = b * a$. Further

$(a * b) * c = (a + b - 1) * c = (a + b - 1) + c - 1$, by (1)

$= a + (b + c - 1) - 1 = a * (b + c - 1) = a * (b * c)$, by (1)

$\therefore \ (a * b) * c = a * (b * c) \quad \forall \ a, b, c \in \mathbf{I}$.

We have $a * 1 = a + 1 - 1 = a \ \forall \ a \in \mathbf{I}$, using (1).

Thus 1 is the identity in $\mathbf{I}$.

Let $a' = 2 - a$. Then, by (1), we get

$$a * a' = a + a' - 1 = a + (2 - a) - 1 = 1.$$

Thus $2 - a$ is the inverse of $a \ \forall \ a \in \mathbf{I}$.

We now consider $aob = a + b - ab$.

From (2), $aob \in \mathbf{I} \quad \forall \quad a, b \in \mathbf{I}$. ...(2

Now $(aob)\, oc = (a + b - ab)\, oc = (a + b - ab) + c - (a + b - ab)\, c$, by (2

$= a + (b + c - bc) - a\,(b + c - bc)$

$= ao\,(b + c - bc) = ao\,(boc)$, by (2).

$\therefore \quad (aob)\, oc = ao\,(boc) \quad \forall \quad a, b, c \in \mathbf{I}$.

Finally, $ao\,(b * c) = ao\,(b + c - 1)$, by (1)

$= a + (b + c - 1) - a\,(b + c - 1)$, by (2)

$= (a + b - ab) + (a + c - ac) - 1$

$= (aob) + (aoc) - 1$, by (2).

$\therefore \quad ao\,(b * c) = (aob) * (aoc)$, by (1).

Similarly, $(a * b)\, oc = aoc * boc$.

Hence $\{\mathbf{I}, *, o\}$ is a ring.

Example 1.3.3. *If* $\{R, +, \cdot\}$ *be a ring with unit element, show that* $\{R, \oplus, \otimes\}$ *is also a ring with unit element, where*

$a \oplus b = a + b + 1$ *and* $a \otimes b = a \,.\, b + a + b \quad \forall \ a, b \in R$.

Solution. We have $a \oplus b = a + b + 1$. ...(1)

From (1) ; $a \oplus b \in R$, as R is a ring and $1 \in R$.

Now $a \oplus b = a + b + 1 = b + a + 1$, since $\{R, +, \cdot\}$ is a ring.

$\therefore \quad a \oplus b = b \oplus a$, by (1).

Using (1) and associative law w.r.t. + in the ring R,

$$(a \oplus b) \oplus c = a \oplus (b \oplus c) \quad \forall \quad a, b, c \in R.$$

Since R is a ring and $1 \in R$, so $-1 \in R$ is the identity in $\{R, \oplus, \otimes\}$.

$[\because \ a \oplus -1 = a - 1 + 1 = a + 0 = a$, R being a ring

For any $a \in R$, $a' = -a - 1 - 1 \in R$ is the additive inverse of a in $\{R, \oplus, \otimes\}$.

$[\because \ a \oplus a' = a + (-a - 1 - 1) + 1 = \{a + (-a)\} - 1 + 0 = 0 - 1 = -1$

We have $a \otimes b = a \,.\, b + a + b.$ ...(2)

From (2) ; $a \otimes b \in R$, since $\{R, +, \cdot\}$ is a ring.

Using (2) and associative law w.r.t. '$\cdot$' in the ring R, we get

$$(a \otimes b) \otimes c = a \otimes (b \otimes c) \ \forall \ a, b, c \in R.$$

Finally, $a \otimes (b \oplus c) = a \otimes (b + c + 1)$, by (1)

$= a \,.\, (b + c + 1) + a + (b + c + 1)$, by (2)

$= a \,.\, b + a \,.\, c + a \,.\, 1 + a + b + c + 1,$

by distributive law in R

$= (a \,.\, b + a + b) + (a \,.\, c + a + c) + 1$

$(\because \ a \,.\, 1 = a)$

$= a \otimes b + a \otimes c + 1$, by (2)

$\therefore \quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, by (1).

Similarly, $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c.$

By (2), $a \otimes 0 = a \,.\, 0 + a + 0 = a \quad \forall \ a \in R.$

Hence $\{R, \oplus, \otimes\}$ is a ring whose unit element is $0 \in R$.

Example 1.3.4. *If* **E** *denotes the set of all even integers, then prove that* $\{\mathbf{E}, +, *\}$ *is a commutative ring, where* $a * b = ab/2$ *and* $+$ *is the usual addition.*

Solution. It is clear that $(\mathbf{E}, +)$ is an abelian group in which 0 is the identity and $-a$ is the additive inverse of $a \in \mathbf{E}$

We have $a * b = ab/2.$ ...(1)

(*i*) (Closure law). Let $a, b \in \mathbf{E}$ so that a and b are even integers. Then $ab/2$ must be an even integer *i.e.*, $a * b \in \mathbf{E}$, by (1).

(*ii*) (Associative law). Let $a, b, c \in \mathbf{E}$. Then by (1), we have

$$a * (b * c) = a * \frac{bc}{2} = \frac{a\,(bc/2)}{2} = \frac{abc}{4}.$$

Similarly, $(a * b) * c = \dfrac{abc}{4}.$ $\quad \therefore \quad a * (b * c) = (a * b) * c.$

(*iii*) (Distributive law). We have

$$a * (b + c) = \frac{a\,(b + c)}{2}, \text{ by (1)}$$

$$= \frac{ab}{2} + \frac{ac}{2} = a * b + a * c, \text{ by (1)}.$$

(*iv*) (Commutative law). From (1), $a * b = b * a \ \forall \ a, b \in \mathbf{E}.$

Hence $\{\mathbf{E}, +, *\}$ is a commutative ring.

Example 1.3.5. *Prove that the set S of all ordered pairs* (a, b) *of real numbers is a commutative ring under the addition and multiplication compositions defined as*

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b)\,(c, d) = (ac, bd).$$

Solution. We have $(a, b) + (c, d) = (a + c, b + d)$. ...(1)

It is clear that $(S, +)$ is an abelian group in which $(0, 0)$ is the identity, since $(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)$ and $(-a, -b)$ is the additive inverse of (a, b), since $(a, b) + (-a, -b) = (0, 0)$.

Let $x = (a, b), y = (c, d)$ and $z = (e, f) \in S$.

Then $xy = (a, b)(c, d) = (ac, bd) \in S$. Also $xy = yx$.

It can be verified that $(xy)z = x(yz)$.

Now $x(y + z) = (a, b)(c + e, d + f)$, by (1)

$= (a(c + e), b(d + f)) = (ac + ae, bd + bf)$

$= (ac, bd) + (ae, bf)$, by (1).

$\therefore \quad x(y + z) = xy + xz$.

Hence R is a commutative ring.

Example 1.3.6. *Prove that a ring R is commutative if and only if*

$$(a + b)^2 = a^2 + 2ab + b^2 \text{ for all } a, b \in R.$$

Solution. Let R be a commutative ring so that $ab = ba$ for all $a, b \in R$.

Now

$$(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b),$$

$= aa + ab + ba + bb$, by distributive laws in R

$= a^2 + ab + ab + b^2$, since $ab = ba$.

Hence $(a + b)^2 = a^2 + 2ab + b^2$. ...(1)

Conversely, let (1) be true for all $a, b \in R$. We shall show that R is commutative. From (1), we have

$$(a + b)(a + b) = aa + ab + ab + bb$$

or $aa + ab + ba + bb = aa + ab + ab + bb$, by distributive laws in R.

$\therefore \quad ba = ab$, by cancellation laws in $(R, +)$.

Hence R is commutative.

Example 1.3.7. *If R is a system satisfying all the conditions for a ring with unit element with the possible exception of $a + b = b + a$, prove that the axiom $a + b = b + a$ must hold in R and that R is thus a ring.*

Solution. It is given that $1 \in R$. Consider

$(a + b) \cdot (1 + 1) = (a + b) \cdot 1 + (a + b) \cdot 1$, by left distributive law in R

$= a + b + a + b$, ...(1)

since 1 is the unit element in R.

Again

$(a + b) \cdot (1 + 1) = a \cdot (1 + 1) + b \cdot (1 + 1)$, by right distributive law

$= a \cdot 1 + a \cdot 1 + b \cdot 1 + b \cdot 1$, by left distributive law

$= a + a + b + b$, since 1 is the unit element in R. ...(2)

From (1) and (2), it follows that

$$a + a + b + b = a + b + a + b$$

So $a + b = b + a$, by cancellation law in the group $(R, +)$.

Hence R is a ring.

Example 1.3.8. *Let R be a ring such that $a^2 = a$ for all $a \in R$. Prove that R is commutative.*

Solution. It is given that $a^2 = a \ \forall \ a \in R$. ...(1)

Since $a + a \in R$, so $(a + a)^2 = a + a$, by (1)

$\Rightarrow \quad (a + a)(a + a) = a + a$

$\Rightarrow \quad aa + aa + aa + aa = a + a$, by distributive laws

$\Rightarrow \quad a^2 + a^2 + a^2 + a^2 = a + a$

$\Rightarrow \quad a + a + a + a = a + a$, by (1)

$\Rightarrow a + a = 0$, by cancellation laws in the group $(R, +)$ ...(2)

$\therefore \quad a = -a$ for all $a \in R$. ...(3)

Let $a, b \in R$. Then $a + b \in R$ and so $(a + b)^2 = a + b$.

$\Rightarrow \quad (a + b)(a + b) = a + b$

$\Rightarrow aa + ba + ab + bb = a + b$, by distributive laws

$\Rightarrow \quad a^2 + ba + ab + b^2 = a + b$

$\Rightarrow \quad a + ba + ab + b = a + b$, by (1)

$\Rightarrow ba + ab = 0$, by cancellation laws in the group $(R, +)$

$\Rightarrow ba = -ab \quad \Rightarrow \quad ba = ab \quad \forall \quad a, b \in R$, using (3).

Hence R is a commutative ring.

Example 1.3.9. *If R is a ring with unity satisfying $(xy)^2 = x^2y^2$ for all $x, y \in R$, prove that R is commutative.* [D.U., 1994]

Solution. We have $(xy)^2 = x^2y^2$ for all $x, y \in R$. ...(1)

Replacing y by $y + 1 \in R$ in (1), we obtain

$$[x(y + 1)]^2 = x^2(y + 1)^2 \quad \Rightarrow \quad (xy + x)^2 = x^2(y^2 + 2y + 1)$$

$\Rightarrow \quad (xy)^2 + (xy)x + x(xy) + x^2 = x^2y^2 + 2x^2y + x^2$. ...(2)

Using (1) and cancellation laws of $(R, +)$ in (2), we get

$$xyx + x^2y = 2x^2y \quad \text{or} \quad xyx = x^2y \quad \forall \quad x, y \in R. \qquad ...(3)$$

Replacing x by $x + 1$ in (3), $(x + 1)y(x + 1) = (x + 1)^2y$

$\Rightarrow \quad (x + 1)(yx + y) = (x + 1)(xy + y)$ ...(4)

$\Rightarrow xyx + xy + yx + y = x^2y + xy + xy + y$.

Using (3) and cancellation laws of $(R, +)$ in (4), we get

$$yx = xy \quad \forall \quad x, y \in R.$$

Hence R is a commutative ring.

Example 1.3.10. *Give an example of a non-commutative ring R without unity such that $(xy)^2 = x^2y^2 \ \forall \ x, y \in R$.*

Solution. Consider the ring R of 2×2 matrices :

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \text{ are integers} \right\}.$$

Clearly, R is non-commutative, since

$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

and so $$\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\neq\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}.$$

The possible unity $\begin{pmatrix}1&0\\0&1\end{pmatrix}\notin R$.

Let $X=\begin{pmatrix}a&b\\0&0\end{pmatrix}, Y=\begin{pmatrix}c&d\\0&0\end{pmatrix}\in R$ by arbitrary. Then

$$XY=\begin{pmatrix}ac&ad\\0&0\end{pmatrix}, X^2=\begin{pmatrix}a^2&ab\\0&0\end{pmatrix}, Y^2=\begin{pmatrix}c^2&cd\\0&0\end{pmatrix},$$

and $$(XY)^2=\begin{pmatrix}ac&ad\\0&0\end{pmatrix}\begin{pmatrix}ac&ad\\0&0\end{pmatrix}=\begin{pmatrix}a^2c^2&a^2cd\\0&0\end{pmatrix}=X^2Y^2.$$

Example 1.3.11. *Show that a ring R is commutative if and only if $a^2-b^2=(a+b)(a-b)$ for all $a, b\in R$.* [D.U., 1997

Solution. Let R be commutative. Then $ab=ba\ \forall\ a, b\in R$.

Hence $$(a+b)(a-b)=a(a-b)+b(a-b)$$
$$=a^2-ab+ba-b^2=a^2-b^2.$$

Conversely, let $a^2-b^2=(a+b)(a-b)$

or $$a^2-b^2=a^2-ab+ba-b^2$$

or $$0=-ab+ba, \text{ by cancellation laws in } (R,+)$$

$$ab=ba\quad\forall\ a, b,\in R.$$

Hence R is a commutative ring.

Example 1.3.12. *Let R be a ring such that for $x\in R$, there exists a unique $a\in R$ satisfying $xa=x$. Show that $ax=x$. Hence deduce that if R has a unique right unity e, then e is the unity of R.*

Solution. We are given $xa=x$.

We have $$x(a+ax-x)=xa+xax-xx, \text{ by left distributive law} \quad ...(1)$$
$$=x+xx-xx=x, \text{ using (1)}$$

$\therefore$ $$x(a+ax-x)=x. \quad ...(2)$$

By the uniqueness of a, (2) gives us

$$a+ax-x=a\Rightarrow ax-x=0\Rightarrow ax=x.$$

(*ii*) If e is the unique right identity of R, then

$$xe=x\quad\forall\ x\in R.$$

By part (*i*), $ex=x\ \forall\ x\in R$.

Hence $xe=ex=x\ \forall\ x\in R$ means that e is the unity of R.

Example 1.3.13. *Let R be a ring with unity $1 \in R$. Suppose for $x \neq 0 \in R$, there exists a unique $y \in R$ such that $xyx = x$. Prove that $xy = yx = 1$ i.e., x is invertible in R.*

Solution. Let $xa = 0$, where $a \in R$. Then

$$x(y + a)x = xyx + xax = x + 0x = x.$$

By the uniqueness of y in the relation $xyx = x$, it follows that

$$x(y + a)x = x \Rightarrow y + a = y \Rightarrow a = 0.$$

Hence $xa = 0 \Rightarrow a = 0$ for each $a \in R$. ...(1)

Again $xyx = x \Rightarrow xyx - x \cdot 1 = 0$ $(\because 1 \in R)$

$\Rightarrow x(yx - 1) = 0$

$\Rightarrow yx - 1 = 0$, using (1)

$\therefore \quad yx = 1.$

Similarly, we can show that

$$ax = 0 \Rightarrow x(a + y)x = x \Rightarrow a + y = y \Rightarrow a = 0, \quad ...(2)$$

and so $xyx = x \Rightarrow (xy - 1)x = 0 \Rightarrow xy - 1 = 0 \Rightarrow xy = 1$, by (2)

Hence $xy = yx = 1$.

Example 1.3.14. *Let R be a ring with unity e. If for some $x \in R$, there exists unique $y \in R$ such that $xy = e$, prove that x is invertible.*

Solution. We have

$$x(e + y - yx) = xe + xy - xyx = x + e - ex$$
$$= x + e - x = e.$$

By the uniqueness of y satisfying $xy = e$, we get

$$x(e + y - yx) = e \Rightarrow e + y - yx = y \Rightarrow yx = e.$$

Hence $xy = yx = e \Rightarrow x$ is invertible.

EXERCISES

1. If R is a ring and $a, b \in R$, prove that
$$(a + b)^2 = a^2 + ab + ba + b^2.$$
[Hint. $(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b)]$
2. Define a ring and give an example of (*i*) a non-commutative ring with unity, (*ii*) a non-commutative ring without unity, (*iii*) a commutative ring without unity, (*iv*) a commutative ring with unity.
[Hint. (*i*) See Example 1.2.6 ; (*ii*) See Example 1.2.8 ; (*iii*) The ring **E** of even integers ; (*iv*) The ring **Z** of integers.]
3. If a, b, c are any three elements in a ring R, prove that
(*i*) $a(b - c) = ab - ac$, (*ii*) $(a - b)c = ac - bc$.
4. Prove that the set $S = \{a + b\sqrt{2} : a, b \in \mathbf{Q}\}$ is a commutative ring w.r.t. usual addition and multiplication.
5. Show that the set $\{3n : n \in \mathbf{Z}\}$ is a commutative ring w.r.t. usual addition and multiplication.

6. Let S be a non-empty set and F be the collection of all subsets of S. For any $A, B \in F$, define

$$A + B = A \cup B - A \cap B \text{ and } AB = A \cap B.$$

Prove that F is a commutative ring with unity.

[**Hint.** The additive identity in F is empty set ϕ, since

$$A + \phi = A \cup \phi - A \cap \phi = A - \phi = A.$$

The additive inverse of A is A, since

$$A + A = A \cup A - A \cup A = A - A = \phi.$$

The unity in F is S, since $AS = A \cap S = A \ \forall \ A \in F$.]

7. If R is a ring such that $a^2 = a \ \forall \ a \in R$, prove that :

(i) $2a = 0 \ \forall \ a \in R$, (ii) $a + b = 0 \Rightarrow a = b$, (iii) $ab = ba \ \forall \ a, b \in R$.

[**Hint.** (i) See Example 1.3.8 for (iii). By equation (2), we have

$$a + a = 0 \Rightarrow 2a = 0 \text{ and } a = -a.$$

(ii) $a + b = 0 \Rightarrow b = -a \Rightarrow b = a$.]

8. If a, b are any two elements of a ring R and m and n are any two positive integers, then prove that

(i) $(m + n)\, a = ma + na$, (ii) $m\,(a + b) = ma + mb$,

(iii) $m\,(na) = (mn)\, a$, (iv) $(na)\,(mb) = (nm)\,(ab)$,

(v) $a^m \cdot a^n = a^{m+n}$ (vi) $(a^m)^n = a^{mn}$.

[**Hint.** (ii) $m\,(a + b) = a + b + a + b + \ldots + a + b$ (m times)

$= (a + a + \ldots + a) + (b + b + \ldots + b)$,

since $a + b = b + a$

$= ma + mb$.

(iv) $(na)\,(mb) = (a + a + \ldots + a)\,(b + b + \ldots + b)$

(n times, m times)

$= a\,(b + \ldots + b) + \ldots + a\,(b + \ldots + b)$ (n times)

(Left distributive law)

$= (ab + \ldots + ab) + \ldots + (ab + \ldots + ab)$ (n times)

(m times, m times)

$= (nm)\,(ab)$]

9. If R is a commutative ring and $a, b \in R$, then for any positive integer n, prove that

$$(a + b)^2 = a^2 + 2_{c_1} ab + b^2,$$

$$(a + b)^3 = a^3 + 3_{c_1} a^2 b + 3_{c_2} ab^2 + b^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(a + b)^n = a^n + n_{c_1} a^{n-1} b + n_{c_2} a^{n-2} b^2 + \ldots + b^n.$$

10. Let R be a ring such that for $x \in R$, there exists a unique $a \in R$ such that $ax = x$. Show that $xa = x$. Hence deduce that if R has a unique left unity e, then e is the unity of R.

[**Hint.** Consider $(a + xa - x)\, x = ax + xax - xx = x + xx - xx = x$

$\therefore \quad a + xa - x = a \Rightarrow xa = x.$]

11. Let R be a ring and $e \in R$ be such that $ex = x\ \forall\ x \in R$, then e is said to be a *left unity* of R. Show that if R has a unique left unity, then R has unity.
[**Hint.** Compare with Ex. 10 above.]

12. Let R be a ring and $e \in R$ be such that $xe = x\ \forall\ x \in R$, then e is said to be a *right unity* of R. Show that if R has a unique right unity, then R has unity.
[**Hint.** See Example 1.3.12.]

13. Let $a, b, c \in R$ be such that $ba = b$ and $a + c - ac = 0$. Show that $b = 0$.
[**Hint.** $a + c - ac = 0 \Rightarrow ba + bc - bac = 0 \Rightarrow b + bc - bc = 0 \Rightarrow b = 0$.]

14. Show that if $1 - ab$ is invertible in a ring R with unity, then so is $1 - ba$.

15. Let $\{R, +, \cdot\}$ be a ring. Show that the system $\{R, +, o\}$ is also a ring, where $xoy = y \cdot x\ \forall\ x, y \in R$.

The ring $\{R, +, o\}$ is called the **opposite ring** of R, written as R^{op}.

1.4 Integral Domain and Field

Definition 1. (Zero Divisor)

A non-zero element 'a' of a commutative ring R is called a **zero divisor,** *if there exists some non-zero element b in R such that* $ab = 0$.

Illustrations

1. In the ring M_2 of all 2×2 matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O,\ B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq O, \text{ but } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Thus A and B are zero divisors in M_2.

2. In the ring $\mathbf{Z}_6$ of integers modulo 6, 2 and 3 are zero divisors, since

$$2 \otimes_6 3 = 0 \in \mathbf{Z}_6.$$

Definition 2. (Integral Domain)

A commutative ring is called an **integral domain,** *if it has no zero divisors.* Equivalently,

A commutative ring R is called an **integral domain,** *if*

$$ab = 0 \Rightarrow \textit{either } a = 0 \textit{ or } b = 0\ ;\ a, b \in R.$$

Or

$$a \neq 0 \textit{ and } b \neq 0 \Rightarrow ab \neq 0\ ;\ a, b \in R.$$

Illustrations

1. The ring $\mathbf{Z}$ of integers is an integral domain, since for any two integers a and b : $ab = 0 \Rightarrow a = 0$ or $b = 0$.

2. The commutative ring $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ is **not** an integral domain ; since $2 \otimes_6 3 = 0$ but $2 \neq 0$ and $3 \neq 0$ in $\mathbf{Z}_6$.

3. $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$ is an integral domain, since $a \otimes_5 b \neq 0$ for all $a \neq 0$, $b \neq 0$ in $\mathbf{Z}_5$

Definition 3. (Division Ring)

A ring $\{R, +, \cdot\}$ *is called a* **division ring** *or a* **skew-field** *if its non-zero elements form a group w.r.t. the composition* '$\cdot$'.

Definition 4. (Field)

A commutative division ring is called a **field**.

Equivalently, A ring $\{R, +, \cdot\}$ *is called a* **field**, *if its non-zero elements form an abelian group w.r.t. the composition* '$\cdot$'.

The multiplicative identity of a field R is denoted by 1 and the multiplicative inverse of $a \neq 0 \in R$ is denoted by a^{-1}, so that

$$a\,a^{-1} = a^{-1}\,a = 1.$$

Note. If R is a field, then $a \neq 0 \in R \Rightarrow a^{-1} \in R$.

Remark. *The explicit properties of a field are given below :*

A **field** *is a non-empty set R with two binary compositions denoted by* $+$ *and* $\cdot$ *such that for all a, b, c in R ;*

A.1.	$a + b \in R$.
A.2.	$a + b = b + a$.
A.3.	$a + (b + c) = (a + b) + c$.
A.4.	*There exists an element* $0 \in R$ *such that* $a + 0 = a \ \forall\ a \in R$.
A.5.	*For each* $a \in R$, *there exists an element* $b \in R$ *such that* $a + b = 0$.
A.M.	$a\,.\,(b + c) = a\,.\,b + a\,.\,c$.

M.1.	$a\,.\,b \in R$.
M.2.	$a\,.\,b = b\,.\,a$.
M.3.	$a\,.\,(b\,.\,c) = (a\,.\,b)\,.\,c$.
M.4.	*There exists an element* $1 \in R$ *such that* $a\,.\,1 = a \ \forall\ a \in R$.
M.5.	*For each* $a \neq 0 \in R$, *there exists an element* $a' \in R$ *such that* $a\,.\,a' = 1$.

Example 1.4.1. $\mathbf{Q}$ *(all rationals) and* $\mathbf{R}$ *(all reals) are fields w.r.t. the usual addition and multiplication.*

Example 1.4.2. $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$ *is a field w.r.t. addition and multiplication modulo 5.*

The multiplicative inverses of $1, 2, 3, 4 \in \mathbf{Z}_5$ are $1, 3, 2, 4$; respectively.

Example 1.4.3. For any prime p, $\mathbf{Z}_p = \{0, 1, 2, \ldots, p-1\}$ *is a field w.r.t. addition and multiplication modulo p.*

For the proof, see the corollary of Theorem 1.5.3. ahead.

Example 1.4.4. $\{\mathbf{Z}_6, \oplus_6, \otimes_6\}$ *is not a field, since, for example,* $2 \neq 0 \in \mathbf{Z}_6$ *has no multiplicative inverse in* $\mathbf{Z}_6$.

Example 1.4.5. *The set* $\mathbf{C} = \{a + bi : a, b \in \mathbf{R}\}$ *of complex numbers is a field under usual addition and multiplication of complex numbers.*

It may be observed that $0 = 0 + 0i$ is the additive identity in $\mathbf{C}$. $-a - b\,i$ is the additive inverse of $a + bi$, $1 = 1 + 0i$ is the multiplicative identity in $\mathbf{C}$ and finally, if $z = a + bi \neq 0 \in \mathbf{C}$, then

$$z^{-1} = \left(\frac{a}{a^2+b^2}\right) - \left(\frac{b}{a^2+b^2}\right) i = \frac{a-bi}{a^2+b^2} \in \mathbf{C}$$

is the multiplicative inverse of z, since

$$zz^{-1} = \frac{1}{(a^2+b^2)}(a+bi)(a-bi) = \frac{a^2+b^2}{a^2+b^2} = 1.$$

Example 1.4.6. *The set*

$$S = \left\{ \begin{pmatrix} x & y \\ -\bar{x} & \bar{y} \end{pmatrix} : x, y \in \mathbf{C} \right\}$$

is a division ring, which is not a field.

We have seen in Example 1.2.9 that S is a non-commutative ring with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ under matrix addition and matrix multiplication. We now proceed to show that every non-zero element of S has its inverse under matrix multiplication. Let

$$A = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S.$$

(Here a, b, c, d are not all zero)

Then

$$A^{-1} = \begin{bmatrix} \frac{1}{k}(a-ib) & -\frac{1}{k}(c+id) \\ \frac{1}{k}(c-id) & \frac{1}{k}(a+ib) \end{bmatrix} \in S, \text{ where } k = a^2+b^2+c^2+d^2 \neq 0$$

Notice that

$$AA^{-1} = \begin{bmatrix} \frac{1}{k}(a^2+b^2+c^2+d^2) & 0 \\ 0 & \frac{1}{k}(a^2+b^2+c^2+d^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}A.$$

Hence S is a division ring, which is not a field.

Example 1.4.7. *The set*

$$Q = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \text{ are real numbers}\}$$

where $i^2 = j^2 = k^2 = ijk = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j$

is a division ring, which is not a field.

We define + in Q as follows :

$$(a_0 + a_1 i + a_2 j + a_3 k) + (b_0 + b_1 i + b_2 j + b_3 k)$$
$$= (a_0 + b_0) + (a_1 + b_1) i + (a_2 + b_2) j + (a_3 + b_3) k.$$

It is easy to verify that $(Q, +)$ is an abelian group.

The zero element in Q is $0 = 0 + 0i + 0j + 0k$, the unity in Q is $1 = 1 + 0i + 0j + 0k$, the additive inverse of $a_0 + a_1 i + a_2 j + a_3 k$ is $-a_0 - a_1 i - a_2 j - a_3 k$.

Now we proceed to show that every non-zero element in Q has its multiplicative inverse in Q.

Let $x = a_0 + a_1 i + a_2 j + a_3 k \neq 0 \in Q$, so that a_0, a_1, a_2, a_3 are not all zero. RINGS

Then $l = a_0^2 + a_1^2 + a_2^2 + a_3^2 \neq 0$.

Let $y = \frac{1}{l}(a_0 - a_1 i - a_2 j - a_3 k) \in Q$. We see that (v)

$$xy = \frac{1}{l}[(a_0 + a_1 i) + (a_2 j + a_3 k)]\,[(a_0 - a_1 i) - (a_2 j + a_3 k)]$$

(v

$$= \frac{1}{l}[(a_0^2 - a_1^2 i^2) - (a_2 j + a_3 k)(a_2 j + a_3 k) + (a_2 j + a_3 k)(a_0 - a_1 i)$$ (v

$$- (a_0 + a_1 i)(a_2 j + a_3 k)]$$ (v

$$= \frac{1}{l}[(a_0^2 + a_1^2 - (-a_2^2 - a_3^2) + a_1 a_2 k - a_1 a_3 j - a_1 a_2 k + a_1 a_3 j]$$

$$= l/l = 1.$$

Similarly, $yx = 1$. Further Q is non-commutative, as

$$(2i + 3k)(j - 4k) = 2k + 8j - 3i + 12,$$

$$(j - 4k)(2i + 3k) = -2k - 8j + 3i + 12 \neq (2i + 3k)(j - 4k).$$

Hence Q is a division ring, which is not a field.

The ring Q is called the *ring of real quaternions*.

Example 1.4.8. *The set* $R = \{a + b\sqrt{3} : a, b \in \mathbf{Q}\}$ *is a field under usual addition and multiplication.*

1.

It is easy to verify that R is a commutative ring and $1 = 1 + 0\sqrt{3} \in R$ is the multiplicative identity of R. Finally, if $x = a + b\sqrt{3} \neq 0 \in R$, then its multiplicative inverse is

d

$$x^{-1} = \left(\frac{a}{a^2 - 3b^2}\right) - \left(\frac{b}{a^2 - 3b^2}\right)\sqrt{3},$$

since

$$xx^{-1} = \frac{1}{(a^2 - 3b^2)}(a + b\sqrt{3})(a - b\sqrt{3}) = \frac{a^2 - 3b^2}{a^2 - 3b^2} = 1.$$

Example 1.4.9. *Let C be the set of all ordered pairs* (a, b) *where* a, b *are real numbers. Let the compositions of addition and multiplication in C be defined as*

$$(a, b) + (c, d) = (a + c, b + d),$$ ...(1

$$(a, b) \,.\, (c, d) = (ac - bd, bc + ad).$$ ...(2

Then C is a field.

From (1), we see that

(*i*) $(a, b) + (c, d) = (a + c, b + d) \in C$.

(*ii*) $(a, b) + (c, d) = (c, d) + (a, b)$. $[\because a + c = c + a, b + d = d +$

(*iii*) $(a, b) + (0, 0) = (a + 0, b + 0) = (a, b)\ \forall\ (a, b) \in C$.

Thus $(0, 0)$ is the additive identity in C.

(*iv*) $(a, b) + (-a, -b) = (a - a, b - b) = (0, 0)$.

Thus $(-a, -b)$ is the additive inverse of (a, b).

(v) $\{(a, b) + (c, d)\} + (e, f) = (a + c, b + d) + (e, f)$
$= ((a + c) + e, (b + d) + f) = (a + (c + e), b + (d + f))$
$= (a, b) + (c + e, d + f) = (a, b) + \{(c, d) + (e, f)\}$.

From (2), we have

(vi) $(a, b) . (c, d) \in C$.

(vii) $(a, b) . (c, d) = (c, d) . (a, b)$. $[\because ac = ca, bd = db]$

(viii) $(a, b) . (1, 0) = (a . 1 - b . 0, b . 1 + a . 0) = (a, b) \forall (a, b) \in C$.

Thus $(1, 0)$ is the multiplicative identity in C.

(ix) For $(a, b) \neq (0, 0) \in C$, the multiplicative inverse of (a, b) is

$$(a, b)^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right).$$

Using (1) and (2), it can be verified that if

$x = (a, b)$, $y = (c, d)$ and $z = (e, f) \in C$, then

(x) $x . (y . z) = (x . y) . z$, and

(xi) $x . (y + z) = x . y + x . z$.

Hence C is a field.

1.5 Basic Theorems of Integral Domains and Fields

Theorem 1.5.1. *Let R be a commutative ring. Then R is an integral domain if and only if $ab = ac \Rightarrow b = c$, where $a, b, c \in R$ and $a \neq 0$.*

Proof. *Condition is necessary*

Let R be an integral domain. Let $ab = ac$; $a, b, c \in R$.

Then $ab - ac = 0$ ...(1)

$\Rightarrow$ $a(b - c) = 0$.

Since R is an integral domain, it follows from (1),

either $a = 0$ or $b - c = 0$.

It is given that $a \neq 0$ and so $b - c = 0$. Hence $b = c$.

Condition is sufficient

Let $ab = ac \Rightarrow b = c$, $\forall$ $a, b, c \in R$ and $a \neq 0$. ...(2)

We have to show that R is an integral domain. Firstly we prove that R has no zero divisors.

Let $x, y \in R$ be such that $xy = 0$.

If $x \neq 0$, then $xy = 0 \Rightarrow xy = x0 \Rightarrow y = 0$, by (2)

Similarly, if $y \neq 0$, then $xy = 0 \Rightarrow x = 0$.

Thus R has no zero divisors. Since R is a commutative ring, it follows that R is an integral domain.

Remark. *Cancellation law may not hold in an arbitrary ring.*

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ be three elements in the ring M_2 of all 2×2 matrices over integers. Then

$$AC = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}.$$

$$BC = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}.$$

Thus $AC = BC$, but $A \neq B$.

Theorem 1.5.2. *Prove that every field is an integral domain.*

Proof. Let R be any field.

Let $ab = ac$, where $a, b, c \in R$ and $a \neq 0$.

Since $a \neq 0 \in R$, $a^{-1} \in R$ exists and $aa^{-1} = a^{-1}a = 1$.

Now $ab = ac \Rightarrow a^{-1}(ab) = a^{-1}(ac)$

$\Rightarrow (a^{-1}a)\, b = (a^{-1}a)\, c$, by associative law

$\Rightarrow \quad 1\, b = 1\, c \Rightarrow b = c$.

Thus R is a commutative ring in which

$$ab = ac \Rightarrow b = c \ (a, b, c \in R, a \neq 0).$$

Hence R is an integral domain (See Theorem 1.5.1).

Remark. *The converse of the above theorem is not true.*

For example, the set of integers $\mathbf{Z}$ is an integral domain which is not a field, since $a \neq 0 \in \mathbf{Z}$ does not have the multiplicative inverse in $\mathbf{Z}$.

Theorem 1.5.3. *Prove that a finite integral domain is a field.*

[D.U., 1999, 96]

Proof. Let $R = \{x_1, x_2, \ldots, x_n\}$ be a finite integral domain.

In order to show that R is a field, we have to prove that R has the unity and that every non-zero element of R has its multiplicative inverse in R. Let $a \neq 0 \in R$. Then

$$ax_1, ax_2, \ldots, ax_n \qquad \ldots(1)$$

are n distinct elements of R, for if $ax_i = ax_j$, $i \neq j$;

then by cancellation law in R, $x_i = x_j$, which is a contradiction.

Let r be an arbitrary element of R. Then by (1), we get

$r = ax_l$, for some l satisfying $1 \leq l \leq n$.

Since $a \in R$, $a = ax_m$, for some m satisfying $1 \leq m \leq n$, by (1).

We have $x_m r = x_m (ax_l) = (x_m a)\, x_l$

$= (ax_m)\, x_l$, since R is commutative

$= ax_l = r$.

Since R is commutative, $x_m r = r x_m = r \ \forall \ r \in R$.

It follows that $x_m \in R$ is the unity of R. We write x_m as 1.

Since $1 \in R$, $1 = ax_p$, for some p satisfying $1 \leq p \leq n$, by (1)

$= x_p a$, since R is commutative.

Thus $ab = ba = 1$, where $b = x_p \in R$.

This implies that $a^{-1} = b \in R$. Hence R is a field.

Corollary. *Show that the ring Z_p of integers modulo p is a field if and only if p is prime.* [D.U., 1998]

Proof. *Condition is necessary*

Let Z_p be a field. Let, if possible, p be not prime. Then

$$p = ab, \quad \text{where} \quad 1 < a, b < p \; ; \; a, b \in Z$$

$\Rightarrow ab \equiv 0 \pmod{p}$ $\Rightarrow$ $ab = 0$ in Z_p, where $a \neq 0, b \neq 0 \in Z_p$

$\Rightarrow Z_p$ has zero divisors $\Rightarrow$ Z_p is not an integral domain.

This is a contradiction, since Z_p is a field implies Z_p is an integral domain.

Condition is sufficient

We know Z_p is a finite commutative ring. Now we show that Z_p is an integral domain.

Let $a, b \in Z_p$ be such that $ab = 0$ in Z_p. Then p divides ab.

$\Rightarrow p \mid a$ or $p \mid b$, since p is prime

$\Rightarrow \quad a = 0$ or $b = 0$ in Z_p

So $ab = 0$ in $Z_p \Rightarrow a = 0$ or $b = 0$ in Z_p $(a, b \in Z_p)$.

Hence Z_p is a finite integral domain and so Z_p is a field.

Remark. *As an application of the above corollary, we see that*

$$Z_2 = \{0, 1\}, \; Z_3 = \{0, 1, 2\}, \; Z_5 = \{0, 1, 2, 3, 4\} \text{ etc.}$$

are all fields (finite).

Ex. *What happens if the integral domain is infinite ?* [D.U., 1996]

An infinite integral domain may not be a field. For example, Z (all integers) is an infinite integral domain, which is not a field.

Theorem 1.5.4. *Let R be a ring such that the equation $ax = b$ has a solution for all $a \neq 0 \in R$ and for all $b \in R$. Show that R is a division ring.*

Proof. Firstly, we show that R has no zero divisors, *i.e.*, to show $a \neq 0, b \neq 0 \in R \Rightarrow ab \neq 0$.

Let, if possible, $ab = 0$; where $a \neq 0, b \neq 0 \in R$. Then

$$abx = 0 \; \forall \; x \in R. \qquad \ldots(1)$$

Since $b \neq 0$, so for any $r \in R$, there exists some $x \in R$ such that

$$bx = r.$$

Using in (1), $\qquad ar = 0 \; \forall \; r \in R. \qquad \ldots(2)$

Since $a \neq 0$, $ax = a$ has a solution, say $c \in R$. Then $a = ac \Rightarrow a = 0$, by (2).

This is a contradiction.

Hence R has no zero divisors *i.e.*, $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

(3)

Let $x = e \in R$ be a solution of $ax = a$, $a \neq 0$. Then $ae = a$ and e [illegible]

We have $a(e - e^2) = ae - aee = ae - ae = 0$.

Using (3), $e - e^2 = 0$, as $a \neq 0$. Thus $e^2 = e$.

We now proceed to show that e is the unity of R. For any $x \in R$,

$(xe - x)\, e = xe^2 - xe = xe - xe = 0$, using (4).

It follows that $xe - x = 0$, as $e \neq 0$; using (3).

$\therefore \quad xe = x \ \forall\, x \in R$.

Again $e(ex - x) = e^2x - ex = ex - ex = 0$, by (4).

$\therefore \quad ex - x = 0$ or $ex = x \quad \forall\, x \in R$, using (3).

Hence $xe = ex = x \quad \forall\, x \in R \Rightarrow e$ is the unity of R.

Let $a \neq 0 \in R$. Then $ax = e$ has a solution, say $x = b \in R$.

Then $ab = e$ and $(ba - e)\, b = bab - eb = be - eb = 0$.

Using (3), $ba - e = 0$, as $b \neq 0$. Thus $ab = ba = e \Rightarrow a^{-1} = b \in R$.

Hence R is a division ring.

EXERCISES

1. Prove that the set of all real numbers of the form $a + \sqrt{2}\, b$, where a a b are rational numbers is a field under the usual addition and multiplic tion.
2. Define a ring and an integral domain. Give an example of a ring whi is not an integral domain.

   [**Hint.** $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$] is a ring, which is not an integral domai
3. Prove that every field is an integral domain, but every integral doma is not a field. Give an example of an integral domain which is also field.

   [**Hint.** The set $\mathbf{Z}$ of integers is an integral domain, which is not a fiel The ring $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$ of integers modulo 5 is both an integr domain and a field.
4. Define a division ring and give an example of it.
5. Give an example of a division ring which is not a field.

   [**Hint.** See Example 1.4.6.]
6. Show that the ring R of real-valued continuous functions on $[0, 1]$ is n an integral domain.

   [**Hint.** Refer to Example 1.2.10. Consider

   $f(x) = \begin{cases} x, & \text{if } x \leq 0 \\ 0, & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x, & \text{if } x > 0 \end{cases}$

   Then $f \neq 0 \in R$ and $g \neq 0 \in R$, but $fg = 0$.]
7. Tick the correct answer :

   (*i*) An integral domain is a field.

   (*ii*) A finite integral domain is a field.

(*iii*) A field is an integral domain.

(*iv*) A division ring is an integral domain.

(*v*) A field is a division ring.

8. Show that a non-zero finite integral domain is a field. Give an example of a finite integral domain. [D.U., 1999]

[**Hint.** See Theorem 1.5.3. $Z_5 = \{0, 1, 2, 3, 4\}$ is a finite integral domain]

9. Prove that the set of all 2×2 matrices over the finite field $Z_3 = \{0, 1, 2\}$ is a finite non-commutative ring of order $3^4 = 81$, under matrix addition and matrix multiplication.

10. Prove that the set of all 3×3 matrices over a finite field is a finite non-commutative ring under matrix addition and matrix multiplication. [D.U., 1994]

[**Hint.** If F is a field having n elements, then the required ring R has n^9 elements. Further R is non-commutative, since $AB \neq BA$, where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

11. Prove that a division ring R has no zero divisors.

[**Hint.** Let $a, b \in R$ be such that $ab = 0$. If $a \neq 0 \in R$, then $a^{-1} \in R$.

$\therefore \quad ab = 0 \Rightarrow a^{-1}(ab) = a^{-1}0 \Rightarrow (a^{-1}a)\, b = 0 \Rightarrow 1 \,.\, b = 0 \Rightarrow b = 0.$

Similarly, for $b \neq 0$, $ab = 0 \Rightarrow a = 0$.]

12. Show that a finite ring with unity and no divisors of zero is a division ring.

[**Hint.** Similar to Theorem 1.5.3]

1.6 Subring

Definition. *Let $\{R, +, \cdot\}$ be a ring. A non-empty subset S of R is called a* **subring** *of R, if $\{S, +, \cdot\}$ is a ring.*

Illustrations

1. The set **E** of even integers is a subring of the ring **Z** of integers.
2. The ring of Gaussian integers $\mathbf{Z}[i]$ is a subring of the ring **C** of complex numbers.
3. The set $S = \{0, 2\}$ is a subring of the ring $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4, under addition and multiplication modulo 4.

Theorem 1.6.1. *A non-empty subset S of a ring R is a subring of R if and only if (i) $a - b \in S$ and (ii) $ab \in S$ for all $a, b \in S$.*

Proof. *The condition is necessary*

Let S be a subring of R. Let $a, b \in S$.

By definition of a ring, $ab \in S$.

Since $(S, +)$ is a group, so $b \in S \Rightarrow -b \in S \Rightarrow a - b \in S$.

The condition is sufficient

Let the conditions (*i*) and (*ii*) be true. Consequently, for $a \in S$,

$$a - a = 0 \in S.$$

Again $0 \in S, a \in S \Rightarrow 0 - a = -a \in S$.

Further $a \in S, b \in S \Rightarrow a \in S$ and $-b \in S \Rightarrow a + b \in S$.

Let $a, b, c \in S$. Then $ab \in S$, by condition (*ii*).

Clearly $a(bc) = (ab)c$, $a(b + c) = ab + ac$, $(b + c)a = ba + ca$, are true, since $S \subseteq R$. Hence S is a ring and so S is a subring of R.

Theorem 1.6.2. *The intersection of two subrings of a ring R is a subring of R.*

Proof. Let A and B be two subrings of a ring R.

Clearly $A \cap B$ is non-empty, since $0 \in A \cap B$.

Let $a, b \in A \cap B$. Then $a, b \in A$ and $a, b \in B$.

Since A is a subring of R, $a - b \in A$ and $ab \in A$. [Theorem 1.6.1]

Similarly, $a - b \in B$ and $ab \in B$.

So $a - b \in A \cap B$ and $ab \in A \cap B$. Hence $A \cap B$ is a subring of R, by Theorem 1.6.1.

Remark 1. *The union of two subrings of R need not be a subring of R.*

Two subrings of the ring of integers $\mathbf{Z}$ are

$$A = \{..., -4, -2, 0, 2, 4, ...\}, B = \{..., -6, -3, 0, 3, 6, ...\}.$$

Then $A \cup B = \{..., -4, -3, -2, 0, 2, 3, 4, ...\}$.

We see that 3, 2 are in $A \cup B$, but $3 - 2 = 1 \notin A \cup B$.

Thus $A \cup B$ is not a subring of $\mathbf{Z}$.

Remark 2. Theorem 1.6.2 can be easily extended to an arbitrary family of subrings of R.

Theorem 1.6.3. *Show that the centre of a ring R is a subring of R.*

Proof. *The centre of a ring R, denoted by Z (R), is defined as*

$$Z(R) = \{a \in R : xa = ax \text{ for all } x \in R\}.$$

Clearly, $Z(R)$ is non-empty, since $0x = x0 \ \forall \ x \in R \Rightarrow 0 \in Z(R)$.

Let $a, b \in Z(R)$. Then $xa = ax$ and $xb = bx \ \forall \ x \in R$. ...(1)

We shall show that $a - b$ and ab are in $Z(R)$.

Consider $(a - b)x = ax - bx = xa - xb$, by (1).

Thus $(a - b)x = x(a - b) \ \forall \ x \in R \Rightarrow a - b \in Z(R)$.

Again $(ab)x = a(bx) = a(xb)$, by (1)

$= (ax)b = (xa)b$, by (1).

Thus $(ab)x = x(ab) \ \forall \ x \in R \Rightarrow ab \in Z(R)$.

Hence $Z(R)$ is a subring of R.

Theorem 1.6.4. *Show that the centre of a division ring is a field.*

[D.U., 1997]

Proof. Let R be a division ring. The *centre* of R is defined as

$$Z(R) = \{a \in R : xa = ax \ \forall\ x \in R\}. \qquad ...(1)$$

By Theorem 1.6.3, $Z(R)$ is a subring of R.

In other words, $Z(R)$ is a ring. We have to show that $Z(R)$ is a field.

Let $a, b \in Z(R)$ be arbitrary.

Using (1), $ax = xa \ \forall\ x \in R$.

In particular, $ab = ba \ \forall\ a, b \in Z(R)$

$\Rightarrow Z(R)$ is a commutative ring.

Since R is a division ring, $1 \in R$ and $1x = x1 \ \forall\ x \in R$.

Thus $1 \in Z(R)$.

Finally, we show that each non-zero element of $Z(R)$ has its multiplicative inverse in $Z(R)$.

Let $a \neq 0 \in Z(R)$ be arbitrary $\Rightarrow a \neq 0 \in R$.

$\Rightarrow \quad a^{-1} \in R$, since R is a division ring.

Let $x \neq 0 \in R$ be arbitrary, so that $x^{-1} \in R$ exists. We have

$a^{-1}x = (x^{-1}a)^{-1} = (ax^{-1})^{-1}$, since $a \in Z(R) \Rightarrow ax^{-1} = x^{-1}a$.

$\therefore \quad a^{-1}x = xa^{-1} \ \forall\ x \neq 0 \in R$.

Obviously, $a^{-1}0 = 0a^{-1}$. Thus $a^{-1}x = xa^{-1} \ \forall\ x \in R$.

It means that $a^{-1} \in Z(R) \ \forall\ a \neq 0 \in Z(R)$.

Hence $Z(R)$ is a field.

EXAMPLES

Example 1.6.1. *Show that the set*

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$$

is a subring of the ring M_2 of 2×2 matrices over integers.

Solution. Clearly, S is non-empty, since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

Let $A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in S$ and $B = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \in S$. Then

$$A - B = \begin{pmatrix} a-c & 0 \\ b-d & 0 \end{pmatrix} \in S, \quad AB = \begin{pmatrix} ac & 0 \\ bc & 0 \end{pmatrix} \in S.$$

Hence S is a subring of the ring of 2×2 matrices over integers.

Example 1.6.2. *If a is a fixed element of a ring R, show that*

$$I_a = \{x \in R : ax = 0\} \text{ is a subring of } R.$$

Solution. Since $a0 = 0$, $0 \in I_a$ and so I_a is non-empty.

Let $x, y \in I_a$ so that $ax = 0$, $ay = 0$.

Now $a(x - y) = ax - ay = 0 - 0 = 0$.

$\therefore \quad x - y \in I_a$.

Again $a(xy) = (ax)y = 0y = 0$.

$\therefore \quad xy \in I_a$. Hence I_a is subring of R.

Example 1.6.3. *(a) Give an example of a ring with unity 1 which has a subring with unity $1' \neq 1$.*

(b) Show by means of an example that a subring of a ring with unity may fail to be a ring with unity.

Solution. (a) $M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$ is a ring with unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to verify that $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbf{Z} \right\}$ is a subring of M_2 with unity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, since $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) The ring $\mathbf{Z}$ of integers is a ring with unity.

But the set $\mathbf{E}$ of even integers is a subring of $\mathbf{Z}$ without unity.

Example 1.6.4. *Prove or disprove that subring of a non-commutative ring is non-commutative.*

Solution. A subring of a non-commutative ring may be commutative.

The ring M_2 of 2×2 matrices over integers is non-commutative, since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and so $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The set $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in I \right\}$ is a subring of M_2, which is commutative, since

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 1.6.5. *Give an example of each of the following with justification :*

(i) Ring which is not commutative but has a subring which is commutative.

(ii) Ring which has no unity but has a subring which has unity.

[D.U., 2000]

Solution. (i) Refer to Example 1.6.4.

(ii) $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ is a ring which has no unity. The possible unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin R$. It can be verified that none of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a unity of R.

However, $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbf{Z} \right\}$ is a subring of R, which has $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as the unity of S.

Example 1.6.6. *Show that $S = \{0, 2, 4, 6, 8\}$ is a subring of $\mathbf{Z}_{10}$ with unity different from that of $\mathbf{Z}_{10}$, the ring of integers modulo 10.*

Solution. We know

$\mathbf{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a ring with unity 1.

It can be verified that $a \otimes_{10} b \in S$ and $a \ominus_{10} b \in S$, $\forall\ a, b \in S$. For example, $4 \otimes_{10} 6 = 4$, $8 \otimes_{10} 4 = 2$ etc. and $8 \ominus_{10} 6 = 2$, $2 \ominus_{10} 8 = 4$ etc.

Hence S is a subring of $\mathbf{Z}_{10}$, where the unity of S is 6.

$[\because\ 6 \otimes_{10} 0 = 0,\ 6 \otimes_{10} 2 = 2,\ 6 \otimes_{10} 4 = 4,\ 6 \otimes_{10} 6 = 6,\ 6 \otimes_{10} 8 = 8]$

Example 1.6.7. *What can you say about the sum of two subrings of a ring ?*

Solution. *If A and B are two subrings of a ring R, then their* **sum** *is defined as $A + B = \{a + b : a \in A, b \in B\}$.*

We show by an example that the sum of two subrings of R need not be a subring of R.

Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$, $T = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in \mathbf{Z} \right\}$.

By Example 1.6.1, S is a subring of the ring M_2 of 2×2 matrices over integers. Similarly, T is a subring of M_2. The sum of S and T is

$$S + T = \left\{ \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}.$$

It is clear that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \in S + T$, but

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \notin S + T$. Hence $S + T$ is not a subring of M_2.

Remark. We have seen that

(i) The intersection of two subrings of a ring R is a subring of R.

(ii) The union of two subrings of a ring R may not be a subring of R.

(iii) The sum of two subrings of a ring R may not be a subring of R.

Example 1.6.8. *Let e be idempotent in a ring R. Show that*

$eRe = \{eae : a \in R\}$ is a subring of R with unity e.

Solution. *An element $x \in R$ is called* **idempotent**, *if $x^2 = x$.* ...(1)

We are given that $e^2 = e$.

Clearly, eRe is non-empty, since $0 = e0e \in eRe$.

Let $x, y \in eRe$. Then $x = eae$, $y = ebe$ for some $a, b \in R$.

We have $x - y = eae - ebe = e(a-b)e \in eRe$,
since $a \in R, b \in R \Rightarrow a - b \in R$

Further $xy = eaebe = eae^2be = eaebe$, by (1)
$\Rightarrow xy = ere$, where $r = aeb \in R$
$\Rightarrow xy \in eRe$. Hence eRe is a subring of R.

For any $x \in R$, we see that

$$ex = eeae = eae = x, \quad \text{using (1)}$$

and
$$xe = eaee = eae = x, \quad \text{using (1).}$$

$$\therefore \quad ex = xe = x \quad \forall \ x \in eRe.$$

Hence e is the unity of eRe.

Example 1.6.9. *Let R be a ring such that $x^3 = x \ \forall \ x \in R$. Show that R is commutative.*

Solution. It is given that $x^3 = x \ \forall \ x \in R$.

In particular, $(x+x)^3 = x + x$ ...(1)

or
$$2x \,.\, 2x \,.\, 2x = 2x \quad \text{or} \quad 8x^3 = 2x \quad \text{or} \quad 8x = 2x.$$

$$\therefore \quad 6x = 0 \ \forall \ x \in R.$$

Again, by (1), $(x^2 - x)^3 = x^2 - x$ ...(2)

or
$$\begin{aligned} x^2 - x = (x^2 - x)(x^2 - x)^2 &= x^2 (x^2 - x)^2 - x(x^2 - x)^2 \\ &= x^2 (x^4 + x^2 - 2x^3) - x(x^4 + x^2 - 2x^3) \\ &= x^2 (x \,.\, x + x^2 - 2x) - x(x \,.\, x + x^2 - 2x), \text{ by (1)} \\ &= 2x^4 - 2x^3 - 2x^3 + 2x^2 = 2x \,.\, x - 4x + 2x^2, \text{ by (1)} \\ &= 4x^2 - 4x. \end{aligned}$$

$$\therefore \quad 3x^2 = 3x \ \forall \ x \in R.$$

Let $S = \{3x : x \in R\}$. ...(3)

Then S is a subring of R, since $3x, 3y \in S \Rightarrow 3x - 3y = 3(x-y) \in S$

and
$$3x \,.\, 3y = 9xy = 3(3xy) \in S.$$

Let $y \in S$ be arbitrary. Then $y = 3x$, for some $x \in R$.

Now $y^2 = (3x)^2 = 9x^2 = 6x^2 + 3x^2 = (6x)x + 3x^2 = 3x^2 = 3x$, by (2) and (3)

$$\therefore \quad y^2 = y \ \forall \ y \in S.$$

Hence S is a commutative subring of R.

It follows that $(3x)(3y) = (3y)(3x)$; $x, y \in R$ [See Example 1.3.8]

$$\Rightarrow \quad 6xy + 3xy = 6yx + 3yx \Rightarrow 3xy = 3yx, \text{ using (2).}$$

$$\therefore \quad 3xy = 3yx, \quad \text{for} \quad x, y \in R.$$

Using (1), $(x+y)^3 = x + y$ ...(4)

or
$$\begin{aligned} x + y = (x+y)(x+y)^2 &= (x+y)(x^2 + xy + yx + y^2) \\ &= x^3 + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y^3 \\ &= x + x^2y + xyx + xy^2 + yx^2 + yxy + y^2x + y, \text{ by (1)} \end{aligned}$$

$\therefore \quad x^2y + xyx + xy^2 + yx^2 + yxy + y^2x = 0.$ ...(5)

Again $(x-y)^3 = x - y$, by (1)

or $x - y = (x-y)(x^2 - xy - yx + y^2)$

$= x^3 - x^2y - xyx + xy^2 - yx^2 + yxy + y^2x - y^3$

$= x - x^2y - xyx + xy^2 - yx^2 + yxy + y^2x - y$, by (1)

$\therefore \quad -x^2y - xyx + xy^2 - yx^2 + yxy + y^2x = 0.$ ...(6)

Adding (5) and (6), we get

$$2xy^2 + 2yxy + 2y^2x = 0. \qquad ...(7)$$

Post-multiplying and pre-multiplying (7) by y, we get, respectively,

$2xy^3 + 2yxy^2 + 2y^2xy = 0$ and $2yxy^2 + 2y^2xy + 2y^3x = 0.$

Using (1), these equations, respectively, become

$$2xy + 2yxy^2 + 2y^2xy = 0, \qquad ...(8)$$

$$2yxy^2 + 2y^2xy + 2yx = 0. \qquad ...(9)$$

Subtracting (9) from (8), we get

$$2xy - 2yx = 0 \quad \text{or} \quad 2xy = 2yx. \qquad ...(10)$$

Subtracting (10) from (4), $xy = yx \quad \forall \ x, y \in R$.

Hence R is commutative.

EXERCISES

1. Show that $S = \{0, 2, 4\}$ and $T = \{0, 3\}$ are subrings of the ring $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ of integers modulo 6.

2. Show that the intersection of an arbitrary number of subrings of a ring R is a subring of R.

3. Show that the set of matrices $\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right\}$ is a subring of the ring of 2×2 matrices with integral elements.

4. Show that a subring of an integral domain is an integral domain.

5. Let R be the ring of 2×2 matrices over reals. Show that

$$S = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \text{ is a real number}\right\}$$

is a subring of R and has unity different from the unity of R.

[**Hint.** $\begin{pmatrix} x & x \\ x & x \end{pmatrix} - \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} x-y & x-y \\ x-y & x-y \end{pmatrix} \in S$, and

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in S.$

The unity of S is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and the unity of R is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.]

6. Let R be the ring of 3×3 matrices over reals. Show that

$$S = \left\{ \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} : x \text{ is a real number} \right\}$$

is a subring of R and has unity different from the unity of R.

[**Hint.** The unity elements of S and R are, respectively

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.]$$

7. Show that the *normalizer* $N(a)$ of an element a of a ring R :

$$N(a) = \{x \in R : xa = ax\}.$$

is a subring of R.

8. Let R be the ring of all real-valued continuous functions defined on $[0, 1]$. Show that the set

$$S = \{f \in R : f(a) = 0\}, \text{ where } a \in [0, 1]$$

is a subring of R.

[**Hint.** See Example 1.2.10. Let $f, g \in S$, so that $f(a) = g(a) = 0$.
Then $(f - g)(a) = f(a) - g(a) = 0$; $(fg)(a) = f(a)\, g(a) = 0$]

9. *A non-empty subset S of a field $\{F, +, \cdot\}$ is called a* **subfield** *of F, if $\{S, +, \cdot\}$ is a field.* Show that a subset S of a field F, containing at least two elements, is a subfield of F iff

(*i*) $a - b \in S \; \forall \; a, b \in S$, (*ii*) $a\, b^{-1} \in S \; \forall \; a \in S, b \neq 0 \in S$.

1.7 Idempotent and Nilpotent Elements

Definition 1. *An element a in a ring R is called* **idempotent**, *if* $a^2 = a$.

Definition 2. *An element a in a ring R is called* **nilpotent**, *if $a^n = 0$ for some positive integer n.*

Remark. If R is a ring with unity 1, then 0 and 1 are idempotent elements of R ($\because 0^2 = 0, 1^2 = 1$). Further 0 is always nilpotent.

Illustrations

1. In the ring M_2 of all 2×2 matrices over integers,

$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are idempotent elements, since

$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ etc. Further

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are nilpotent elements, since

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Can you find other idempotent and nilpotent elements in M_2 ?

2. In the ring $Z_4 = \{0, 1, 2, 3\}$ of integers modulo 4 ; 0 and 1 are the only idempotent elements and 0 and 2 are the only nilpotent elements. Notice that $2^2 = 0$ in Z_4.
3. In the ring $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of integers modulo 10 ; 0, 1, 5, 6 are the only idempotent elements and 0 is the only nilpotent element. Notice that in Z_{10}, $5^2 = 5, 6^2 = 6$.

EXAMPLES

Example 1.7.1. *Prove that the only idempotent elements in an integral domain R with unity are* 0 *and* 1. *What happens if R is not an integral domain ?*

Solution. Let $x \in R$ be idempotent, so that $x^2 = x$ *i.e.*, $x \,.\, x = x \,.\, 1$ as $1 \in R$.

$\therefore \quad x \,.\, (x-1) = 0 \Rightarrow x = 0$ or $x - 1 = 0$, since R is an integral domain.

Hence $x = 0$ or $x = 1$.

(*ii*) If R is not an integral domain, we may have idempotent elements other than 0 and 1. For example, $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is not an integral domain, since $2 \neq 0,\ 5 \neq 0 \in Z_{10}$, but $2 \otimes_{10} 5 = 0 \in Z_{10}$. The idempotent elements in Z_{10} are 0, 1, 5, 6. Notice that $5^2 = 5, 6^2 = 6$ in Z_{10}.

Example 1.7.2. *If R is an integral domain, then show that R does not possess any non-zero nilpotent element.* [D.U., 2000]

Solution. Let, if possible, $\exists\ a \neq 0 \in R$ such that a is nilpotent *i.e.*, $a^n = 0$, for some positive integer n

$\Rightarrow a \,.\, a \ldots a$ (n times) $= 0$

$\Rightarrow a = 0$, since R is an integral domain.

This is a contradiction. Hence the result.

Example 1.7.3. *Show that in a ring R, a non-zero idempotent element cannot be nilpotent.*

Solution. Let a be any non-zero idempotent element of R.

Then $a^2 = a \Rightarrow a^3 = a^2 = a \Rightarrow a^4 = a$ and so on.

$\therefore \quad a^n = a \neq 0$ for all positive integers n.

Hence a is not nilpotent.

Example 1.7.4. *If a and b are nilpotent elements of a commutative ring R, show that a + b is also nilpotent. Give an example to show that this may fail if R is not commutative.*

Solution. Since a and b are nilpotent, there exist positive integers m and n such that $a^m = 0,\ b^n = 0$. ...(1)

Since R is commutative, we can write

$$(a+b)^{m+n} = a^{m+n} + (m+n)c_1\, a^{m+n-1}\, b + (m+n)c_2\, a^{m+n-2}\, b^2 + \ldots + b^{m+n}$$

$$= a^m \cdot a^n + (m+n)c_1\, a^m\, a^{n-1}\, b + \ldots + b^m\, b^n$$

$$= 0, \text{ using (1).}$$

Hence $a+b$ is nilpotent.

(ii) The ring M_2 of all 2×2 matrices over the integers is non-commutative, where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ are nilpotent, since } A^2 = B^2 = O.$$

However, $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not nilpotent, since

$$(A+B)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } (A+B)^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Indeed $(A+B)^n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all positive integers n.

Example 1.7.5. *Let R be a commutative ring and $a \in R$. Show that if a is nilpotent, then ab is nilpotent for each $b \in R$.*

Solution. Since a is nilpotent, $a^n = 0$ for some $n \in \mathbf{N}$.

Since R is commutative, therefore

$$(ab)^n = a^n\, b^n \;\forall\; a, b \in R$$

$$= 0 \,.\, b^n = 0.$$

Hence ab is nilpotent for all $b \in R$.

Example 1.7.6. *Let R be a ring and $a, b \in R$. Show that ab is nilpotent implies that ba is nilpotent.*

Solution. Since ab is nilpotent, $(ab)^n = 0$ for some $n \in \mathbf{N}$.

Consider $(ba)^{n+1} = ba \,.\, ba \,.\, ba \ldots ba \,(n+1 \text{ times})$

$$= b\,(ab)\,(ab) \ldots (ab)\, a$$

$$= b\,(ab)^n\, a = b \,.\, 0 \,.\, a = 0.$$

Hence ba is nilpotent.

Example 1.7.7. *Show that $\mathbf{Z}_6$, the ring of integers mod 6, has no non-zero nilpotent element.* [D.U., 1994]

Solution. We know $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$. We have in $\mathbf{Z}_6$, $1^n = 1 \neq 0$, $3^n = 3 \neq 0$, $4^n = 4 \neq 0 \;\forall\; n \in \mathbf{N}$.

Further $2^n = 2$ or $4 \;\forall\; n \in \mathbf{N}$ and $5^n = 5$ or $1 \;\forall\; n \in \mathbf{N}$.

Hence $\mathbf{Z}_6$ has no non-zero nilpotent elements. 0 is obviously nilpotent in $\mathbf{Z}_6$.

Example 1.7.8. *If R is a ring with no non-zero nilpotent elements, then show that for any idempotent $e \in R$, $ex = xe$ for all $x \in R$.*

Solution. Since e is idempotent, $e^2 = e$. ...(1)

For any $x \in R$, we have

$$\begin{aligned}(exe - ex)^2 &= (exe - ex)(exe - ex)\\ &= exe(exe - ex) - ex(exe - ex)\\ &= exe^2xe - exe^2x - exexe + exex\\ &= exexe - exex - exexe + exex = 0, \text{ by (1).}\end{aligned}$$

It follows that $exe - ex$ is nilpotent. As given,

$$exe - ex = 0 \text{ or } exe = ex. \qquad ...(2)$$

Similarly, we can show that

$$(exe - xe)^2 = 0 \text{ and so } exe - xe = 0 \text{ or } exe = xe. \qquad ...(3)$$

From (2) and (3), $ex = xe \ \forall \ x \in R$.

In other words, $e \in Z(R)$, the centre of R.

Example 1.7.9. *In a ring R without unity, show that every idempotent is a zero divisor but is not nilpotent. Suppose now that R has no non-zero nilpotent elements. Prove that any idempotent is in the centre of R.*

[D.U., 1998]

Solution. (*i*) Let e be any idempotent in R, so that $e^2 = e$. Since R is a ring without unity, there exists some $x \in R$ such that $xe \neq x$, for otherwise,

$$xe = x \ \forall \ x \in R \Rightarrow e \text{ is the unity of } R,$$

which is contrary to the given hypothesis.

Now $$e^2 = e \Rightarrow xe^2 = xe \Rightarrow (xe - x)e = 0 \Rightarrow e = 0,$$

since $xe - x \neq 0$. Hence e is a zero divisor.

Let, if possible, e be nilpotent. Then there exists a *least* positive integer n such that $e^n = 0$. ...(1)

Now $e^n = 0 \Rightarrow e^{n-2} \cdot e^2 = 0 \Rightarrow e^{n-2} \cdot e = 0 \Rightarrow e^{n-1} = 0$,

which is a contradiction to (1).

Hence e is a zero divisor but is not nilpotent.

(*ii*) Refer to Example 1.7.8.

Example 1.7.10. *Let R be a ring such that for each $a \in R$ there exists $x \in R$ such that $a^2x = a$. Prove the following :*

(*i*) *R has no non-zero nilpotent elements.*

(*ii*) *$axa - a$ is nilpotent and so $axa = a$.*

(*iii*) *ax* and *xa are idempotents.*

Solution. (*i*) Let $a \in R$ be any nilpotent element. Then

$$a^n = 0, \text{ for some positive integer } n. \qquad ...(1)$$

As given, for $a \in R$, there exists $x \in R$ such that $a^2x = a$. ...(2)

From (2),

$$a^{n-2}(a^2x) = a^{n-2}.a \Rightarrow a^nx = a^{n-1} \Rightarrow a^{n-1} = 0, \text{ by (1).}$$

Again from (2),

$$a^{n-3}(a^2x) = a^{n-3}.a \Rightarrow a^{n-1}x = a^{n-2} \Rightarrow a^{n-2} = 0 \text{ and so on.}$$

Proceeding in this manner, $a = 0$.

Hence R has no non-zero nilpotent elements.

(*ii*) We have

$$\begin{aligned}(axa - a)^2 &= (axa - a)(axa - a)\\ &= axa(axa - a) - a(axa - a)\\ &= axa^2xa - axa^2 - a^2xa + a^2\\ &= axa^2 - axa^2 - a^2 + a^2, \text{ by (2)}\\ &= 0.\end{aligned}$$

It follows that $axa - a$ is nilpotent in R and so by part (*i*),

$$axa - a = 0. \text{ Hence } axa = a.$$

(*iii*) It is clear that

$$\begin{aligned}axa = a &\Rightarrow ax\,ax = ax \text{ and } xa\,xa = xa\\ &\Rightarrow (ax)^2 = ax \text{ and } (xa)^2 = xa\end{aligned}$$

Hence ax and xa are idempotents.

EXERCISES

1. Define nilpotent and idempotent element of a ring R. Find the idempotent and nilpotent elements in $\mathbf{Z}_6$, the ring of integers modulo 6. [D.U., 1999]

   [**Ans.** 0, 1, 3, 4 are idempotents and 0 is nilpotent]

2. Prove that the only idempotent elements in a field are 0 and 1.

3. Find the idempotent and nilpotent elements in $\mathbf{Z}_5$.

   [**Ans.** 0, 1 are idempotents and 0 is nilpotent]

4. Prove that the set S of all nilpotent elements in a commutative ring R is a subring of R.

   [**Hint.** Refer to Examples 1.7.4 and 1.7.5. Verify that $a - b \in S$, $ab \in S\ \forall\ a, b \in S$.]

5. Prove that the following statements for a ring R are equivalent :

   (*a*) R has no non-zero nilpotent elements.

   (*b*) $a^2 = 0 \Rightarrow a = 0, a \in R$.

6. Find the idempotent, nilpotent and invertible elements of $\mathbf{Z}_{20}$.

   [**Ans.** $\{0, 1, 5, 16\}$ are idempotents, $\{0, 10\}$ are nilpotents, $\{1, 3, 7, 9, 11, 13, 17, 19\}$ are invertible elements.]

7. Let a be an idempotent element in a ring R such that $a + b - ab = 0$ for some $b \in R$. Show that $a = 0$.

   [**Hint.** $a + b - ab = 0 \Rightarrow a^2 + ab - a^2b = 0 \Rightarrow a^2 = 0\ (\because\ a^2 = a)$. Hence $a = 0$]

1.8 Characteristic of a Ring

Definition 1. *A ring R is said to be of* **finite characteristic**, *if there exists a positive integer n such that $na = 0$ for all $a \in R$.*

Definition 2. *If a ring R is of finite characteristic, then the* **characteristic** *of R is defined as the smallest positive integer p such that $pa = 0$ for all $a \in R$. We write it as char $R = p$.*

Definition 3. *A ring R is said to be of* **characteristic zero**, *if $na \neq 0$ for each positive integer n and for each $a \neq 0 \in R$.*

Equivalently, if char $R = 0$, then $na = 0$ for all $a \in R \Rightarrow n = 0$, n being any positive integer.

Remark. *The above definitions can similarly be extended to any integral domain R.*

Illustrations

1. char $\mathbf{Z} = 0$, char $\mathbf{Q} = 0$, char $\mathbf{R} = 0$. Here $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ are the rings of integers, rationals and reals, respectively.
2. char $\mathbf{Z}_2 = 2$, where $\mathbf{Z}_2 = \{0, 1\}$. Notice that 2 is the smallest positive integer such that $2 \otimes_2 0 = 0$ and $2 \otimes_2 1 = 0$.
3. char $\mathbf{Z}_3 = 3$, where $\mathbf{Z}_3 = \{0, 1, 2\}$.

In general, char $\mathbf{Z}_n = n$, where $\mathbf{Z}_n = \{0, 1, 2, ..., n-1\}$ is the ring of integers modulo n.

Theorem 1.8.1. *Prove that the characteristic of any integral domain R is either zero or a prime number.* [D.U., 1996]

Proof. If char $R = 0$, then there is nothing to prove.

Let char $R = n \neq 0$, then n is the **least** positive integer such that $na = 0$ for all $a \in R$. We shall prove that n is prime. If n is not prime, then

$$n = lm, \text{ for some integers } l \text{ and } m \; ; \; 1 < l, m < n.$$

Now $na = 0 \Rightarrow (lm)\, a = 0 \Rightarrow (lm)\, ab = 0b = 0, b \in R$

$$\Rightarrow \quad \underbrace{ab + ab + ... + ab}_{(lm \text{ times})} = 0 \;\forall\; a, b \in R$$

$$\Rightarrow (\underbrace{a + a + ... + a}_{l \text{ times}})\,(\underbrace{b + b + ... + b}_{m \text{ times}}) = 0 \;\forall\; a, b \in R$$

$$\Rightarrow \quad (la)\,(mb) = 0 \;\forall\; a, b \in R. \qquad ...(1)$$

Since R is an integral domain, it follows from (1) that

$$la = 0 \;\forall\; a \in R \quad \text{or} \quad mb = 0 \;\forall\; b \in R,$$

where $1 < l < n$, $1 < m < n$.

The above two statements contradict the fact that n is the least positive integer such that $na = 0 \;\forall\; a \in R$. Hence n must be a prime number.

Corollary. *The characteristic of a field is either zero or a prime number.*

Proof. Every field is an integral domain and so the result follows.

Ex. 1. *Define the characteristic of an integral domain. Prove that the characteristic of an integral domain, if finite, must be a prime number. What is the characteristic of J_p, the ring of integers modulo a prime number p ? Justify your answer.* [D.U., 1999]

Hint. Let char $R = m$ (finite). Then there exists a smallest positive integer n such that $na = 0\ \forall\ a \in R$. Now proceed like Theorem 1.8.1.

Since p is prime, J_p is an integral domain. Hence char $J_p = p$, since p is the smallest prime number such that $pa = 0\ \forall\ a \in J_p$.

Ex. 2. *Define the characteristic of a ring. What is the characteristic of J_n, the ring of integers modulo, n being any positive integer ?* [D.U., 1996]

Hint. $J_n = \{0, 1, 2, \ldots, n-1\}$, char $J_n = n$.

Ex. 3. *Define characteristic of a ring R. What does it have to do with $(R, +)$, the additive structure of R ?* [D.U., 1998]

Hint. If char $R = n$, then n is the least positive integer such that $na = 0\ \forall\ a \in R \Rightarrow o(a) = n\ \forall\ a \in (R, +)$. Hence $o(a) =$ char R, for all $a \neq 0 \in (R, +)$.

Theorem 1.8.2. *Prove that order of a finite field F is p^n, for some prime p and some positive integer n.*

Proof. Firstly, we show that char $F \neq 0$. Let, if possible, char $F = 0$. By definition,

$$na \neq 0\ \forall\ a \neq 0 \in F \quad \text{and} \quad \forall\ n \in \mathbf{N}. \qquad \ldots(1)$$

It follows that $a, 2a, 3a, \ldots$ belong to F.

Since F is finite, we must have

$ia = ja$ for some positive integers i and j, $i > j$

$\Rightarrow$ $(i-j)\,a = 0$, where $i - j > 0$.

This contradicts (1) and so char $F \neq 0$.

We know that the characteristic of a field is either zero or a prime number. Since char $F \neq 0$, so char $F = p$, p being some prime number.

Thus p is the smallest positive integer such that $pa = 0\ \forall\ a \in F$

$\Rightarrow$ $o(a) = p$, treating $(F, +)$ as a group.

Since $(F, +)$ is a finite group and $a \in F$, by Lagrange's theorem,

$o(a)$ divides $o(F)$,

$\Rightarrow$ p divides $o(F)$, where p is prime.

Hence $o(F) = p^n$, for some positive integer n.

Corollary. *If R is a finite (non-zero) integral domain, then $o(R) = p^n$, where p is a prime number and n is a positive integer.*

Proof. The result follows, since every finite integral domain is a field.

EXAMPLES

Example 1.8.1. *Let R be a non-zero ring such that $x^2 = x$ for all $x \in R$. Prove that R is a commutative ring of characteristic 2.*

Solution. Refer to Example 1.3.8. Since $x^2 = x\ \forall\ x \in R$, R is commutative and further $2x = 0\ \forall\ x \in R$. [See equation (2) of Example 1.3.8]

Hence char $R = 2$.

Example 1.8.2. *Let R be a commutative ring of characteristic 2. Prove that* : $(a+b)^2 = a^2 + b^2 = (a-b)^2 \ \forall\ a, b \in R$.

Solution. Since R is commutative, $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Hence $(a \pm b)^2 = a^2 \pm 0 \,.\, b + b^2 = a^2 + b^2$,

since char $R = 2 \Rightarrow 2a = 0 \ \forall\ a \in R$.

Example 1.8.3. *If F is a field of characteristic p, p a prime ; then*

$$(a+b)^p = a^p + b^p \ \ \forall\ a, b \in F.$$

Solution. Since char $F = p$, $px = 0 \ \forall\ x \in F$. ...(1)

Since F is a field, we can write

$$(a+b)^p = a^p + pa^{p-1}b + \frac{1}{2!}p(p-1)a^{p-2}b^2 + \ldots + pab^{p-1} + b^p$$

$$= a^p + (pb)a^{p-1} + \frac{1}{2!}(p-1)a^{p-2}\,.\,b(pb) + \ldots + (pa)b^{p-1} + b^p$$

$$= a^p + b^p, \text{ using (1).}$$

Example 1.8.4. *Let R be a ring with characteristic n. Suppose $ma = 0$ for all $a \in R$ and for some positive integer m. Show that n divides m. Determine characteristic of $\mathbf{Z}_n$.* [D.U., 2000]

Solution. Since char $R = n$, n is the *least* positive integer such that $na = 0 \ \forall\ a \in R$. It is given that $ma = 0 \ \forall\ a \in R$ and for some positive integer m. By division algorithm, there exist integers q and r such that $m = nq + r$, where $r = 0$ or $0 < r < n$. Consider the case $0 < r < n$.

We have $0 = ma = (nq + r)\,a = q\,(na) + ra = 0 + ra = ra$.

$\therefore$ $ra = 0, \ \forall\ a \in R$; where r is a positive integer $< n$.

This is a contradiction to the fact that char $R = n$. Consequently,

$r = 0$ and so $m = nq \Rightarrow n$ divides m.

(*ii*) We know $\mathbf{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Clearly, n is the least positive integer such that $na = 0$ in $\mathbf{Z}_n$, $\forall\ a \in \mathbf{Z}_n$. Hence char $\mathbf{Z}_n = n$.

Example 1.8.5. (*i*) *If D is an integral domain and if $na = 0$ for some $a \neq 0$ in D and some integer $n \neq 0$, prove that D is of finite characteristic.* [D.U., 1998, 95]

(*ii*) *What is the relation between the characteristic of D and the number n ?* [D.U., 1995]

Solution. We are given that $na = 0$ for some $a \neq 0 \in D$

$\Rightarrow (na)\,x = 0x = 0 \ \forall\ x \in D$

$\Rightarrow (a + a + \ldots + a)\,x = 0 \quad \forall \quad x \in D$

(n times)

$\Rightarrow ax + ax + \ldots + ax = 0 \quad \forall \quad x \in D$

(n times)

$\Rightarrow a\,(x + x + \ldots + x) = 0 \quad \forall \quad x \in D$

$\Rightarrow \quad a\,(nx) = 0 \quad \forall \quad x \in D$

$\Rightarrow \quad a = 0 \quad \text{or} \quad nx = 0 \quad \forall \quad x \in D$, since D is an integral domain.

$\Rightarrow \quad nx = 0 \quad \forall \quad x \in D$, since $a \neq 0$.

Hence the characteristic of D is finite.

(*ii*) If char $D = m$, then m is the smallest positive integer such th $mx = 0 \ \forall \ x \in D$. It follows that m divides n [see Example 1.8.4]. Hen char D divides n.

Example 1.8.6. *Prove that a finite integral domain has finite chara teristic. Give an example of an integral domain which has an infin number of elements, yet is of finite characteristic.* [D.U., 19

Solution. Let D be a finite integral domain. Let char $D = 0$. Then

$$na \neq 0 \ \forall \ a \neq 0 \in D \quad \text{and} \quad \forall \ n \in \mathbf{N}.$$

It follows that $a, 2a, 3a, \ldots$ all belong to D.

Since D is finite, we must have $ia = ja$ for some positive intege i and j, $i > j$. Then $(i - j)\, a = 0$, where $i - j > 0$.

This contradicts (1) and so char $D \neq 0$.

We know that the characteristic of any integral domain is either ze or a prime number. Since char $D \neq 0$, therefore

$$\text{char } D = p \text{ (finite)}, \ p \text{ is some prime.}$$

(*ii*) Let $F = \mathbf{Z}_p = \{0, 1, 2, \ldots, p-1\}$ be the ring of integers modulo Then F is an integral domain of characteristic p *i.e.*, $pa = 0 \ \forall \ a \in \mathbf{Z}_p$.

Let $F[x]$ be the ring of polynomials over F. Since F is an integ domain, so $F[x]$ is an integral domain having infinite number of elemen with finite characteristic p.

Notice that if $f(x) = a_0 + a_1 x + \ldots + a_r x^r \in F[x]$, then by (2),

$$pf(x) = pa_0 + pa_1 x + \ldots + pa_r x^r = 0 \ \forall \ f(x) \in F[x].$$

Example 1.8.7. *Give an example of an infinite ring having fini characteristic.*

Hint. Refer to Example 1.8.6.

Example 1.8.8. *Show that each non-zero element of an integral doma D, regarded as a member of the additive group of D, is of the same ord*

Solution. Let a be any non-zero element of the additive group $(D, +)$ a let the order of a be n. Then n is the least positive integer such that $na = 0$

$$\Rightarrow \quad (na)\, x = 0x = 0 \quad \forall \quad x \in D$$

$$\Rightarrow (a + a + \ldots + a)\, x = 0 \quad \forall \quad x \in D$$

(n times)

$$\Rightarrow ax + ax + \ldots + ax = 0 \quad \forall \quad x \in D$$

(n times)

$$\Rightarrow a\,(x + x + \ldots + x) = 0 \quad \forall \quad x \in D$$

(n times)

$$\Rightarrow \quad a\,(nx) = 0 \quad \forall \quad x \in D.$$

$$\Rightarrow \quad nx = 0 \quad \forall \quad x \in D, \text{ since } a \neq 0 \in D$$

Hence $\quad o(x) = n \quad \forall \quad x \neq 0 \in D.$

Example 1.8.9. *Let R be a non-zero ring such that $x^3 = x$ for all $x \in R$. Prove that R is a commutative ring of characteristic 6.*

Solution. Refer to Example 1.6.9. R is a commutative ring such that $6x = 0 \ \forall \ x \in R$. Hence char $R = 6$.

Example 1.8.10. *Prove that if F is a finite field, its characteristic must be a prime number p and F contains p^n elements for some integer n. Further prove that if $a \in F$, then $a^{p^n} = a$.*

Solution. We know $o(F) = p^n$ [See Theorem 1.8.2].

Since non-zero elements of F (which are $p^n - 1$ in number) form a multiplicative group, by Lagrange's theorem,

$$a \in F \Rightarrow a^{p^n - 1} = e \text{ (multiplicative identity of } F)$$

Hence $a \cdot a^{p^n - 1} = a \cdot e$ or $a^{p^n} = a$, $a \in F$.

1.9 Ideals in a Ring

Definition 1. *A non-empty subset S of a ring R is called a* **left ideal** *of R, if*

(i) $(S, +)$ *is a subgroup of* $(R, +)$
i.e., $a \in S$ *and* $b \in S \Rightarrow a - b \in S$.

(ii) $a \in S$ *and* $r \in S \Rightarrow ra \in S$.

Definition 2. *A non-empty subset S of a ring R is called a* **right ideal** *of R, if*

(i) $(S, +)$ *is a subgroup of* $(R, +)$
i.e., $a \in S$ *and* $b \in S \Rightarrow a - b \in S$.

(ii) $a \in S$ *and* $r \in R \Rightarrow ar \in S$.

Definition 3. *A non-empty subset S of a ring R is called an* **ideal** *or a* **two-sided ideal** *of R, if*

(i) $(S, +)$ *is a subgroup of* $(R, +)$
i.e., $a \in S$ *and* $b \in S \Rightarrow a - b \in S$.

(ii) $a \in S$ *and* $r \in R \Rightarrow ar \in S$ *and* $ra \in S$.

In other words, a non-empty subset S of a ring R is an ideal of R, if S is both a left and right ideal of R.

Remark 1. *In a commutative ring, every left ideal or right ideal is a two-sided ideal.*

2. *Since each ideal S of a ring R is a subgroup of the additive group $(R, +)$, $0 \in S$.*

Example 1.9.1. *If $\mathbf{Z}$ be the ring of integers and n be any integer, then $(n) = \{nx : x \in \mathbf{Z}\}$ is an ideal of $\mathbf{Z}$.*

Solution. Let $a, b \in (n)$, so that $a = nx$ and $b = ny$, for some integers x and y. Then $a - b = nx = n(x - y)$, where $x - y \in \mathbf{Z}$.

Thus $a - b \in (n)$. Again for any integer r, we see that

$ra = r(nx) = (rn)\,x = (nr)\,x = n(rx)$, where rx is an integer.

$\therefore \quad ra \in (n)$. Now $ar = ra \in (n)$.

Hence (n) is an ideal of $\mathbf{Z}$.

Note. $(2) = \{..., -4, -2, 0, 2, 4, ...\}$,

$(3) = \{..., -6, -3, 0, 3, 6, ...\}$ etc. are ideals in $\mathbf{Z}$.

Theorem 1.9.1. *Every ideal of a ring R is a subring of R, but the converse need not be true.*

Proof. (*i*) Let S be an ideal of the given ring R.

Let $a, b \in S$. By definition of an ideal, $a - b \in S$.

Further $a \in S$ and $b \in S \subseteq R$ (*i.e.*, $b \in R$) $\Rightarrow$ $ab \in S$.

Hence S is a subring of R (Theorem 1.6.1).

(*ii*) The converse of part (*i*) is not true. The set $\mathbf{Z}$ of integers is a subring of the ring $\mathbf{Q}$ of rational numbers. However, $\mathbf{Z}$ is not an ideal of $\mathbf{Q}$, since $3 \in \mathbf{Z}, \frac{1}{4} \in \mathbf{Q}$, but $3 \cdot \frac{1}{4} = \frac{3}{4} \notin \mathbf{Z}$.

Theorem 1.9.2. *The intersection of two ideals of a ring R is an ideal of R.*

Proof. Let A and B be any two ideals of R.

We have to show that $A \cap B$ is an ideal of R.

We know that $0 \in A$ and $0 \in B$ and so $0 \in A \cap B$.

Thus $A \cap B$ is non-empty.

Let $x, y \in A \cap B$. Then $x, y \in A$ and $x, y \in B$.

Since A is an ideal of R, $x - y \in A$.

Similarly, $x - y \in B$ and so $x - y \in A \cap B$.

Let $r \in R$. Since A is an ideal of R, $rx \in A$ and $xr \in A$.

Similarly, $rx \in B$ and $xr \in B$.

So $rx \in A \cap B$ and $xr \in A \cap B$ $\forall$ $r \in R$ and $\forall$ $x \in A \cap B$.

Hence $A \cap B$ is an ideal of R.

Remark. *The union of two ideals of a ring R need not be an ideal of R.*

We know

$$A = (2) = \{..., -4, -2, 0, 2, 4, ...\}$$

and

$$B = (3) = \{..., -6, -3, 0, 3, 6, ...\}$$

are two ideals in the ring $\mathbf{Z}$ of integers.

Now $A \cup B = \{..., -6, -4, -3, -2, 0, 2, 3, 4, 6, ...\}$

It is easy to see that $3 \in A \cup B$ and $2 \in A \cup B$, but

$3 - 2 = 1 \notin A \cup B$. Hence $A \cup B$ is not an ideal of $\mathbf{Z}$.

Theorem 1.9.3. *The sum of two ideals of a ring R is an ideal of R*

Or

If A and B are two ideals of a ring R, then $A + B = \{a + b : a \in A, b \in B\}$ an ideal of R.

Proof. Since $0 \in A$ and $0 \in B$, $0 = 0 + 0 \in A + B$.

Thus $A + B$ is non-empty.

Let $x, y \in A + B$. Then $x = a_1 + b_1$, $y = a_2 + b_2$,

for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Now

$$x - y = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + (b_1 - b_2). \quad ...(1)$$

Since A is an ideal of R, $a_1 \in A$ and $a_2 \in A \Rightarrow a_1 - a_2 \in A$.

Similarly, $b_1 - b_2 \in B$. From (1), it follows that $x - y \in A + B$.

Let $r \in R$. Then $rx = r(a_1 + b_1) = ra_1 + rb_1$. ...(2)

Since A is an ideal of R, so $a_1 \in A$ and $r \in R \Rightarrow ra_1 \in A$.

Similarly, $rb_1 \in B$. From (2), it follows that $rx \in A + B \ \forall \ r \in R$ and $\forall \ x \in A + B$. Similarly, $xr \in A + B$.

Hence $A + B$ is an ideal of R.

Theorem 1.9.4. (Product of Two ideals)

If A and B are two ideals of a ring R, then their product AB defined as

$$AB = \left\{ a_1b_1 + a_2b_2 + ... + a_nb_n : a_i \in A, b_i \in B, 1 \le i \le n \text{ and } n \text{ being a positive integer} \right\}$$

is an ideal of R. [D.U., 1994]

Proof. Since A and B are ideals of R, $0 \in A$ and $0 \in B$ and so $0 = 0.0 \in AB$. Thus AB is non-empty.

Let x and y be any two elements of AB. Then

$$x = a_1b_1 + ... + a_nb_n \text{ and } y = \alpha_1\beta_1 + ... + \alpha_m\beta_m ;$$

where $a_i \in A, \alpha_j \in A, b_i \in B, \beta_j \in B ; 1 \le i \le n, 1 \le j \le m.$

Now
$$\begin{aligned} x - y &= (a_1b_1 + ... + a_nb_n) - (\alpha_1\beta_1 + ... + \alpha_m\beta_m) \\ &= a_1b_1 + ... + a_nb_n - \alpha_1\beta_1 - ... - \alpha_m\beta_m \\ &= a_1b_1 + ... + a_nb_n + (-\alpha_1)\beta_1 + ... + (-\alpha_m)\beta_m, \end{aligned}$$

where $-\alpha_j \in A$ for each j, since A is an ideal of R.

It follows that $x - y \in AB$. For any $r \in R$ and $x \in AB$, we have

$$\begin{aligned} rx &= r(a_1b_1 + ... + a_nb_n) = r(a_1b_1) + .. + r(a_nb_n) \\ &= (ra_1)\, b_1 + ... + (ra_n)\, b_n, \text{ where } ra_i \in A \text{ for each } i. \end{aligned}$$

(Notice that $r \in R$ and $a_i \in A \Rightarrow ra_i \in A$, as A is a **left ideal of R**)

Consequently, $rx \in AB$.

Again,
$$\begin{aligned} xr &= (a_1b_1 + ... + a_nb_n)\, r = (a_1b_1)\, r + ... + (a_nb_n)\, r \\ &= a_1\,(b_1r) + ... + a_n\,(b_nr), \quad \text{where } b_ir \in B \text{ for each } i. \end{aligned}$$

(Notice that $r \in R$ and $b_i \in B \Rightarrow b_ir \in B$, as B is a **right ideal of R**).

Consequently, $xr \in AB$. Hence AB is an ideal of R.

Remark. On carefully examining the above proof, we observe that $rx \in AB \ \forall \ r \in R$, $x \in AB$; if A is a **left ideal** of R and $xr \in AB \ \forall \ r \in R$, $x \in AB$, if B is a **right ideal** of R.

Hence Theorem 1.9.4 can be restated as :

If A is a left ideal and B is a right ideal of a ring R, then AB is a two-sided ideal of R.

Corollary 1. *If A and B are two ideals of a ring R, then*

$$AB \subseteq A \cap B.$$

Proof. We know that AB and $A \cap B$ are ideals of R.

Let x be any element of AB. Then $x = a_1b_1 + \ldots + a_nb_n$, for some $a_i \in A, b_i \in B$; $1 \le i \le n$.

Now $a_i \in A, b_i \in R \Rightarrow a_ib_i \in A$, as A is a right ideal of R

$\Rightarrow a_1b_1 + \ldots + a_nb_n \in A \Rightarrow x \in A$, as A is an ideal of R.

Again $a_i \in R, b_i \in B \Rightarrow a_ib_i \in B$, as B is a left ideal of R

$\Rightarrow a_1b_1 + \ldots + a_nb_n \in B \Rightarrow x \in B$, as B is an ideal of R.

Hence $x \in A \cap B$ and so $AB \subseteq A \cap B$.

Corollary 2. *If A and B are two ideals of a ring R, then* [D.U., 1994]

$$AB \subseteq A + B.$$

Proof. We know that $A + B$ and AB are ideals of R.

Let x be any element of AB. Then $x = a_1b_1 + a_2b_2 + \ldots + a_nb_n$, for some $a_i \in A, b_i \in B$; $1 \le i \le n$.

Now $a_1 \in A, b_1 \in R, \Rightarrow a_1b_1 \in A$, as A is a right ideal of R.

Again $a_i \in R, b_i \in B \Rightarrow a_ib_i \in B$, as B is a left ideal of R, $2 \le i \le n$

$\Rightarrow a_2b_2 + \ldots + a_nb_n \in B$, as B is an ideal of R.

$\therefore \quad a_1b_1 + (a_2b_2 + \ldots + a_nb_n) \in A + B \Rightarrow x \in A + B \quad \forall \quad x \in AB.$

Hence $AB \subseteq A + B$.

Ex. *Let U and V be two ideals of a ring R. Define $U + V$ and UV. Prove that $U + V$ as well as UV are ideals of R. Show that $UV \subseteq U + V$.* [D.U., 1994]

EXAMPLES

Example 1.9.2. *Show that the set*

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \text{ are integers} \right\}$$

is a left ideal in the ring M_2 of 2×2 matrices over integers. Further show that S is not a right ideal in M_2.

Solution. Clearly S is non-empty, since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

Let $X, Y \in S$; so that $X = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, Y = \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}; a, b, c, d \in \mathbf{Z}$.

Then $X - Y = \begin{pmatrix} a-c & 0 \\ b-d & 0 \end{pmatrix} \in S$.

Let $A \in M_2$, so that $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, $p, q, r, s \in \mathbf{Z}$.

Then $AX = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} pa+qb & 0 \\ ra+sb & 0 \end{pmatrix} \in S.$

Hence S is a left ideal of M_2.

Also S is not a right ideal of M_2, since

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in S \text{ and } T = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \in M_2,$$

but $$PT = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin S.$$

Example 1.9.3. *Show that the set*

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \text{ are integers} \right\}$$

is a right ideal of M_2, the ring of 2×2 matrices over integers, which is not a left ideal of M_2.

Solution. Let $X, Y \in S$, so that

$$X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \Rightarrow X - Y = \begin{pmatrix} a-c & b-d \\ 0 & 0 \end{pmatrix} \in S.$$

(Here a, b, c, d are some integers)

For any $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M_2$, we see that

$$XA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ 0 & 0 \end{pmatrix} \in S.$$

Hence S is a right ideal of M_2

Consider $P = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in S$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M_2$.

Then $$TP = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \notin S.$$

This shows that S is not a left ideal of M_2.

Example 1.9.4. *If S be an ideal of a ring R and $1 \in S$, prove that $S = R$.* [D.U., 1997]

Solution. We have $1 \in S$. For any $r \in R$; $1r \in S$, as S is an ideal of R. Thus $r \in S \; \forall \; r \in R$ and so $R \subseteq S$. Obviously, $S \subseteq R$. Hence $S = R$.

Example 1.9.5. *If F is a field, prove its only ideals are (0) and F itself.* [D.U., 1997]

Solution. Let U be any ideal of F.

If $U = (0)$, there is nothing to prove.

Let $U \neq (0)$. We shall show that $U = F$.

Since $U \neq (0)$, so there exists some $u \neq 0 \in U \subseteq F$.

Since F is a field and $u \neq 0 \in F$, $u^{-1} \in F$ and $uu^{-1} = u^{-1}u = 1$.

Since U is an ideal of F, $u \in U$ and $u^{-1} \in F \Rightarrow uu^{-1} = 1 \in U$.

For any $x \in F$ and $1 \in U$, $1x = x \in U$, since U is an ideal of F.

Thus $x \in U \; \forall \; x \in F \Rightarrow F \subseteq U$. Obviously, $U \subseteq F$.

Hence $U = F$.

Example 1.9.6. (*a*) *Let R be a ring with unity. Prove that no proper ideal of R can contain an invertible element of R.* [D.U., 1999, 96]

(*b*) *Deduce that a field F has no proper ideals.* [D.U., 1996]

Solution. (*a*) Let A be any proper ideal of R containing an invertible element a *i.e.*, $a \in A$, where $aa^{-1} = a^{-1}a = 1$.

Since $a^{-1} \in R$ and $a \in A$, $1 = a^{-1}a \in A$. ($\because$ A is an ideal of R)

For any $r \in R$ and $1 \in A$, $1 \,.\, r = r \in A$ ($\because$ A is an ideal of R)

$\therefore$ $r \in A \;\; \forall \;\; r \in R$ *i.e.*, $R \subseteq A$. Obviously, $A \subseteq R$.

Hence $A = R$ *i.e.*, A is not a proper ideal of R, which is a contradiction. Hence the result.

(*b*) Let A be any ideal of F. If $A = (0)$, there is nothing to prove. Let $A \neq (0)$. Then there exists some element $a \neq 0 \in A$ *i.e.*, $a \neq 0 \in F$. Since F is a field, a is invertible and $a \in A$. By part (*a*), $A = F$. Hence F has no proper ideals.

Example 1.9.7. *Let R be a ring and $a \in R$. Show that the set $S = \{r \in R : ra = 0\}$ is a left ideal of R.*

Solution. Since $0a = 0$, $0 \in S$. Thus S is non-empty.

Let $x, y \in S$, so that $xa = 0$ and $ya = 0$.

Now $(x - y)\, a = xa - ya = 0$. So $x - y \in S$.

For any $r \in R$ and $x \in S$, $(rx)\, a = r\,(xa) = r0 = 0$. So $rx \in S$.

Hence S is a left ideal of R.

Example 1.9.8. *Let R be the ring of all real valued, continuous functions on [0, 1]. Show that the set $S = \{f \in R : f(\frac{1}{2}) = 0\}$ is an ideal of R.*

Solution. Let $f, g \in S$. Then $f(\frac{1}{2}) = 0$ and $g(\frac{1}{2}) = 0$.

Consider $(f - g)(\frac{1}{2}) = f(\frac{1}{2}) - g(\frac{1}{2}) = 0 - 0 = 0$, using (1). ...(1)

$\therefore$ $(f - g)(\frac{1}{2}) = 0 \Rightarrow f - g \in S$.

Let $f \in S$ and $h \in R$. Then

$$(fh)(\tfrac{1}{2}) = f(\tfrac{1}{2})\, h(\tfrac{1}{2}) = 0 \,.\, h(\tfrac{1}{2}) = 0,$$

and

$$(hf)(\tfrac{1}{2}) = h(\tfrac{1}{2}) f(\tfrac{1}{2}) = h(\tfrac{1}{2}) \,.\, 0 = 0.$$

Thus fh, $hf \in S \; \forall \; f \in S$ and $h \in R$. Hence S is an ideal of R.

Example 1.9.9. *If U is an ideal of R, then prove that*

$$r(U) = \{x \in R : xu = 0 \ \forall \ u \in U\} \text{ is an ideal of } R.$$

Solution. Since $0u = 0 \ \forall \ u \in U$, so $0 \in r(U)$ and thus $r(U)$ is non-empty. Let $x, y \in r(U)$, so that $xu = 0$ and $yu = 0 \ \forall \ u \in U$. ...(1)

We have $(x - y)\, u = xu - yu = 0 \ \forall \ u \in U$, by (1).

Thus $x - y \in r(U)$.

Let $a \in R$ and $x \in r(U)$, so that $xu = 0 \ \forall \ u \in U$. ...(2)

Now $(ax)\, u = a(xu) = a0 = 0 \ \forall \ u \in U$, by (2).

$\therefore \quad (ax)\, u = 0 \ \forall \ u \in U \Rightarrow ax \in r(U)$.

Again $(xa)\, u = x(au) = xy$, where $y = au$.

Since U is an ideal of R, so $a \in R$ and $u \in U \Rightarrow au \in U \Rightarrow y \in U$. ...(3)

From (2) and (3), $xy = 0 \Rightarrow x(au) = 0 \Rightarrow (xa)\, u = 0$.

Thus $(xa)\, u = 0 \ \forall \ u \in U \Rightarrow xa \in r(U)$.

Hence $r(U)$ is an ideal of R.

Note. It may be observed that in (3), we have actually used the fact that U is a left ideal of R only. Thus we have another equivalent statement of Example 1.9.9 as follows :

Example 1.9.10. *If R is a ring and L is a left ideal of R. Then $\lambda(L) = \{x \in R : xa = 0 \ \forall \ a \in L\}$ is a two-sided ideal of R.* [D.U., 1995]

Example 1.9.11. *If U is an ideal of R, then prove that*

$$[R : U] = \{x \in R : rx \in U \text{ for every } r \in R\}$$

is an ideal of R and that it contains U.

Solution. Since U is an ideal of R, so $0 \in U$ *i.e.*, $r0 \in U \ \forall \ r \in R$ ($\because \ r0 = 0$). Thus $0 \in [R : U]$ and so $[R : U]$ is non-empty.

Let $x, y \in [R : U]$, so that $rx \in U$ and $ry \in U \ \forall \ r \in R$. ...(1)

It follows that $rx - ry \in U \ \forall \ r \in R$, as U is an ideal of R.

Now $r(x - y) = rx - ry \in U \ \forall \ r \in R \Rightarrow x - y \in [R : U]$.

Using (1), $(ra)\, x \in U$, since $ra \in R$.

Thus $r(ax) = (ra)\, x \in U \ \forall \ r \in R \Rightarrow ax \in [R : U]$.

Since U is an ideal of R, so $rx \in U$ and $a \in R$ implies that

$(rx)\, a \in U \Rightarrow r(xa) \in U \ \forall \ r \in R \Rightarrow xa \in [R : U]$.

Hence $[R : U]$ is an ideal of R.

Now we show that $U \subseteq [R : U]$.

Let $x \in U$. Then $rx \in U \ \forall \ r \in R$, as U is an ideal of R.

Now $rx \in U \ \forall \ r \in R$ implies that $x \in [R : U] \Rightarrow U \subseteq [R : U]$.

Hence $[R : U]$ is an ideal of R containing U.

Example 1.9.12. *Prove that $Z(R)$, the centre of a ring R, is only a subring of R and need not be an ideal of R.* [D.U., 1995]

Solution. By definition, $Z(R) = \{a \in R : xa = ax \ \forall \ x \in R\}$.

By Theorem 1.6.3, $Z(R)$ is a subring of R.

Let M_2 be the ring of all 2×2 matrices over the integers.

For any $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$ and $A = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \in M_2$, we see that

$$AX = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ap & bp \\ cp & dp \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = XA.$$

Hence $Z(M_2) = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} : p \text{ is an integer} \right\}$.

We proceed to show that $Z(M_2)$ is **not** an ideal of M_2.

For $S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in M_2$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in Z(M_2)$, we have

$$SA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \notin Z(M_2).$$

Hence $Z(M_2)$ is not an ideal of M_2.

Example 1.9.13. *Let N be the set of all nilpotent elements in a commutative ring R. Show that N is an ideal of R. Further prove that there are no non-zero nilpotent elements in the quotient ring R/N.*

[D.U., 1999, 97, 96]

Solution. (*i*) The set of all nilpotent elements of R is

$$N = \{a \in R : a^n = 0 \text{ for some positive integer } n\}.$$

Let $a, b \in N$. Then $a^n = 0$, $b^m = 0$ for some $n, m \in \mathbf{Z}^+$.

Since R is commutative, therefore ...(1)

$$(a-b)^{m+n} = a^{m+n} - (m+n)_{c_1} a^{m+n-1} \cdot b + (m+n)_{c_2} a^{m+n-2} \cdot b^2 - \ldots + (-1)^{m+n} b^{m+n}$$

$$= a^m \cdot a^n - (m+n)_{c_1} a^{m-1} \cdot a^n + \ldots + (-1)^{m+n} \cdot b^m \cdot b^n$$

$$= 0, \text{ using (1).}$$

Thus $a - b \in N$. Let $a \in N$ and $r \in R$. Then

$$(ra)^n = r^n a^n, \text{ since } R \text{ is commutative}$$

$$= r^n \cdot 0 = 0, \text{ using (1). Thus } ra \in N$$

Similarly, $(ar)^n = a^n r^n = 0 \Rightarrow ar \in N$. Hence N is an ideal of R.

(*ii*) Since N is an ideal of R, the quotient ring is

$$\frac{R}{N} = \{r + N : r \in R\}.$$ (See Chapter 2)

Let $r + N \in R/N$ be any nilpotent element in R/N.

Then $(r+N)^n = \bar{0}$, for some positive integer n

$\Rightarrow$ $(r+N)^n = N$

$(\because \bar{0} \in R/N \Rightarrow \bar{0} = N)$

$\Rightarrow r^n + N = N$, by the multiplication composition in R/N.

$\Rightarrow r^n \in N$

$\Rightarrow (r^n)^m = 0$, for some positive integer m

$\Rightarrow r^{nm} = 0 \Rightarrow r$ is nilpotent

$\Rightarrow r \in N \Rightarrow r + N = N \Rightarrow r + N = \bar{0}$.

Hence $\bar{0} = N$ is the only nilpotent element of R/N i.e., R/N has no non-zero nilpotent elements.

Ex. *Show that $0 + N$ is the only nilpotent element of R/N.* [D.U., 1996]

Hint. $\bar{0} = N = 0 + N$.

Example 1.9.14. *Let A and B be any two ideals of a ring R. Show that $A + B$ is an ideal of R generated by $A \cup B$.* [D.U., 1998]

Definition. (Ideal generated by a set) *Let S be any subset of a ring R. An ideal A of R is said to be* **generated** *by S, if*

(i) $S \subseteq A$

(ii) If I is any ideal of R such that $S \subseteq I$, then $A \subseteq I$.

We write the ideal A as $A = \langle S \rangle$. Indeed, $\langle S \rangle$ is the smallest ideal containing S.

Solution. We have to show that $A + B = \langle A \cup B \rangle$.

By Theorem 1.9.3, $A + B$ is an ideal of R.

For any $a \in A$, $a = a + 0 \in A + B$. Thus $A \subseteq A + B$.

Similarly, $B \subseteq A + B$. Consequently, $A \cup B \subseteq A + B$. ...(1)

Let I be an any ideal of R such that $A \cup B \subseteq I$. ...(2)

We shall prove that $A + B \subseteq I$. ...(3)

Let $x \in A + B$ be arbitrary. Then $x = a + b$, for $a \in A$, $b \in B$.

Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, so $a, b \in A \cup B$

$(\because A \cup B \subseteq I)$

$\Rightarrow a, b \in I$

$\Rightarrow a + b \in I$, as I is an ideal of R.

$\Rightarrow x \in I$ and so (3) is proved.

From (1), (2) and (3) ; $A + B = \langle A \cup B \rangle$.

Example 1.9.15. *If A and B are two ideals of a ring R, prove tha $A \cup B$ is an ideal of R if any only if either $A \subseteq B$ or $B \subseteq A$.*

Solution. *Condition is sufficient*

Let $A \subseteq B$. Then $A \cup B = B$, which is an ideal of R.

Let $B \subseteq A$. Then $A \cup B = A$, which is an ideal of R.

Condition is necessary

Let $A \cup B$ be an ideal of R. We have to show that either $A \subseteq B$ $B \subseteq A$. Suppose the conclusion is not true. Then $A \nsubseteq B$ and $B \nsubseteq A$. Cons quently, there exists some $x \in A$ such that $x \notin B$ and some $y \in B$ such t $y \notin A$.

Now $x \in A,\ y \in B \Rightarrow x, y \in A \cup b \Rightarrow x - y \in A \cup B$,
since $A \cup B$ is an ideal of R. We have

$$x - y \in A \cup B \Rightarrow x - y \in A \quad \text{or} \quad x - y \in B.$$

If $x - y \in A$, then $x \in A \Rightarrow x - (x - y) \in A$, as A is an ideal of R.
$\therefore \quad y \in A$, which is a contradiction.
If $x - y \in B$, then $x = (x - y) + y \in B$, which is again a contradiction.
Hence either $A \subseteq B$ or $B \subseteq A$.

Example 1.9.16. *Let A and B be two ideals of a commutative ring R with unity such that $A + B = R$. Show that $AB = A \cap B$.*

Solution. Since A and B are two ideals of R, AB is an ideal of R and

$$AB \subseteq A \cap B. \qquad \text{[See Theorem 1.9.4 and Cor. 1]}$$

Conversely, let $x \in A \cap B$ be arbitrary. Since $R = A + B$ and $1 \in R$,
so $1 \in A + B \Rightarrow 1 = a + b$, for some $a \in A, b \in B$.
$\therefore \quad x = x \cdot 1 = x(a + b) = xa + xb.$ ...(1)
Now $x \in A$ and $b \in B \Rightarrow xb \in AB$,
and $x \in B$ and $a \in A \Rightarrow ax \in AB \Rightarrow xa \in AB$,
since R is commutative.
$\therefore \quad xa + xb \in AB$, since AB is an ideal of R.
By (1), $x \in AB \ \forall \ x \in A \cap B$ and so $A \cap B \subseteq AB$.
Hence $AB = A \cap B$.

Remark. *Two ideals A and B of a ring R satisfying $A + B = R$ are called* **co-maximal ideals.**

Example 1.9.17. *If A, B and C are ideals of a ring R. Prove that*

$$A(B + C) = AB + AC.$$

Solution. By the given hypothesis, $B + C$, AB, AC, $A(B + C)$ and $AB + AC$ are ideals of R.

For any $b \in B,\ b = b + 0 \in B + C$.
$\therefore \quad B \subseteq B + C$. Similarly, $C \subseteq B + C$ $(\because\ 0 \in C)$
$\Rightarrow \quad AB \subseteq A(B + C)$ and $AC \subseteq A(B + C)$
$\Rightarrow \quad AB + AC \subseteq A(B + C)$.
Conversely, let $x \in A(B + C)$ be arbitrary. Then ...(1)

$$x = a_1 t_1 + a_2 t_2 + .. + a_n t_n, \text{ where } a_i \in A,\ t_i \in B + C.$$

Since $t_i \in B + C$, $t_i = b_i + c_i$, for some $b_i \in B$ and $c_i \in C$.

$$\therefore \quad x = a_1(b_1 + c_1) + a_2(b_2 + c_2) + \ldots + a_n(b_n + c_n)$$
$$= (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) + (a_1 c_1 + a_2 c_2 + \ldots + a_n c_n) \in AB + AC$$

$\therefore \quad A(B + C) \subseteq AB + AC.$
From (1) and (2) $A(B + C) = AB + AC.$ ...(2)

Example 1.9.18. *If A, B, C are ideals of a ring R such that $B \subseteq A$, prove that*

$$A \cap (B + C) = B + (A \cap C) = (A \cap B) + (A \cap C).$$

Solution. By the given hypothesis, $B + C$, $A \cap C$ and $A \cap (B + C)$, $B + (A \cap C)$ are also ideals of R.

Let $x \in A \cap (B + C)$ be arbitrary. Then $x \in A$ and $x \in B + C$

We have $x \in B + C \rightarrow x = b + c$, for some $b \in B$, $c \in C$

Thus $b + c \in A$ $(\because x \in A)$ and $b \in A$ $(\because B \subseteq A)$

$\rightarrow$ $b + c - b \subseteq A$ $(\because A$ is an ideal of $R)$

$\rightarrow$ $c \in A \rightarrow c \in A \cap C$

$\therefore$ $x = b + c \rightarrow x \in B + (A \cap C)$.

Consequently, $A \cap (B + C) \subseteq B + (A \cap C)$. ...(1)

Conversely, let $x \in B + (A \cap C)$ be arbitrary.

$\rightarrow x = b_1 + c_1$, for some $b_1 \in B$, $c_1 \in A \cap C$

$\rightarrow x \in B + C$, as $b_1 \in B$ and $c_1 \in C$.

Again $B \subseteq A \rightarrow b_1 \in A$. Also $c_1 \in A$.

$\therefore$ $x = b_1 + c_1 \in A$, as A is an ideal of R.

Thus $x \in A$. Also $x \in B + C$.

$\therefore$ $x \in A \cap (B + C)$.

Consequently, $B + (A \cap C) \subseteq A \cap (B + C)$. ...(2)

From (1) and (2), we obtain

$$A \cap (B + C) = B + (A \cap C).$$

Since $B \subseteq A$, so $A \cap B = B$. Hence

$$A \cap (B + C) = B + (A \cap C) = (A \cap B) + (A \cap C).$$

Example 1.9.19. *If A, B be two ideals of a ring R, then $AB \subseteq A \cap B$. Give an example to show that there exist ideals A and B such that $AB \neq A \cap B$.*

Solution. By Corollary 1 of Theorem 1.9.4, $AB \subseteq A \cap B$.

Let $A = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$, $B = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} : c, d \in \mathbf{Z} \right\}$.

Then A is a left ideal of M_2 and B is a right ideal of M_2 (M_2 being the ring of all 2×2 matrices over the integers). It follows that AB is an ideal of M_2 [See Theorem 1.9.4].

We have

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}.$$

Consequently, $AB = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : p, q, r, s \in \mathbf{Z} \right\}$.

Example 1.9.18. *If A, B, C are ideals of a ring R such that $B \subseteq A$, prove that*

$$A \cap (B + C) = B + (A \cap C) = (A \cap B) + (A \cap C).$$

Solution. By the given hypothesis, $B + C$, $A \cap C$ and $A \cap (B + C)$, $B + (A \cap C)$ are also ideals of R.

Let $x \in A \cap (B + C)$ be arbitrary. Then $x \in A$ and $x \in B + C$.

We have $x \in B + C \Rightarrow x = b + c$, for some $b \in B$, $c \in C$.

Thus $b + c \in A$ $(\because x \in A)$ and $b \in A$ $(\because B \subseteq A)$

$\Rightarrow$ $b + c - b \subseteq A$ $(\because A$ is an ideal of $R)$

$\Rightarrow$ $c \in A \Rightarrow c \in A \cap C$

$\therefore$ $x = b + c \Rightarrow x \in B + (A \cap C)$.

Consequently, $A \cap (B + C) \subseteq B + (A \cap C)$. ...(1)

Conversely, let $x \in B + (A \cap C)$ be arbitrary.

$\Rightarrow x = b_1 + c_1$, for some $b_1 \in B$, $c_1 \in A \cap C$

$\Rightarrow x \in B + C$, as $b_1 \in B$ and $c_1 \in C$.

Again $B \subseteq A \Rightarrow b_1 \in A$. Also $c_1 \in A$.

$\therefore$ $x = b_1 + c_1 \in A$, as A is an ideal of R.

Thus $x \in A$. Also $x \in B + C$.

$\therefore$ $x \in A \cap (B + C)$.

Consequently, $B + (A \cap C) \subseteq A \cap (B + C)$. ...(2)

From (1) and (2), we obtain

$$A \cap (B + C) = B + (A \cap C).$$

Since $B \subseteq A$, so $A \cap B = B$. Hence

$$A \cap (B + C) = B + (A \cap C) = (A \cap B) + (A \cap C).$$

Example 1.9.19. *If A, B be two ideals of a ring R, then $AB \subseteq A \cap B$. Give an example to show that there exist ideals A and B such that $AB \neq A \cap B$.*

Solution. By Corollary 1 of Theorem 1.9.4, $AB \subseteq A \cap B$.

Let $A = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$, $B = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} : c, d \in \mathbf{Z} \right\}$.

Then A is a left ideal of M_2 and B is a right ideal of M_2 (M_2 being the ring of all 2×2 matrices over the integers). It follows that AB is an ideal of M_2 [See Theorem 1.9.4].

We have

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}.$$

Consequently, $AB = \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} : p, q, r, s \in \mathbf{Z} \right\}$.

However, $A \cap B = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbf{Z} \right\}$.

Hence $AB \neq A \cap B$.

For example, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in AB$, but $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \subseteq A \cap B$.

Example 1.9.20. *If A is a left ideal and B is a right ideal of a ring R, then show that AB is a two-sided ideal of R. What can you say about BA?*

Solution. By Theorem 1.9.4, AB is a two-sided ideal of R.

(*ii*) Refer to the ideals A and B of M_2 as given in Example 1.9.19.

We see that

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} ac + bd & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $$BA = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbf{Z} \right\}.$$

We take $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in BA$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2$. Then

$$ST = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \notin BA \text{ and } TS = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin BA.$$

Hence BA is neither a left ideal nor a right ideal of M_2.

Example 1.9.21. *If A and B are ideals of a ring R, define*

where $$A : B = \{r \in R : rB \subseteq A\},$$
$$rB = \{rb : b \in B\}.$$

Show that $A : B$ is an ideal of R.

Solution. Let $r_1, r_2 \in A : B$. Then $r_1 B \subseteq A$ and $r_2 B \subseteq A$.

$\therefore$ $(r_1 - r_2) B = r_1 B - r_2 B \subseteq A$, since A is an ideal of R.

Thus $r_1 - r_2 \subseteq A : B$.

Let $r_1 \in A : B$ and $x \in R$. Then $r_1 B \subseteq A$ and

$(xr_1) B = x (r_1 B) \subseteq xA \subseteq A$, since A is an ideal of R.

$\therefore$ $xr_1 \in A : B$.

Again, $(r_1 x) B = r_1 (xB) \subseteq r_1 B \subseteq A$, since B is an ideal of R.

$\therefore$ $r_1 x \in A : B$. Hence $A : B$ is an ideal of R.

Example 1.9.22. *Let R be a commutative ring and let A be an ideal of R. Show that*

$$\sqrt{A} = \{x \in R : x^n \in A \text{ for some positive integer } n\}$$

is an ideal of R such that

(*i*) $A \subseteq \sqrt{A}$

(*ii*) $\sqrt{\sqrt{A}} = \sqrt{A}$

(*iii*) *If R has unity and $\sqrt{A} = R$, then $A = R$.*

Solution. Let $a, b \in \sqrt{A}$. Then $a^m \in A$ and $b^n \in A$,
for some positive integers m and n. Since R is commutative,

$$(a-b)^{m+n} = a^{m+n} - (m+n)_{c_1} a^{m+n+1} b + \ldots + (-1)^{m+n} b^{m+n}$$

$$= a^m \cdot a^n - (m+n)_{c_1} a^m \cdot a^{n-1} b + \ldots + (-1)^{m+n} b^m \cdot b^n \in A,$$

since $a^m \in A$, $b^n \in A$ and A is an ideal of R.

Thus $a - b \in \sqrt{A}$. For any $r \in A$, $a \in \sqrt{A}$; we have

$(ra)^m = r^m a^m$, since R is commutative.

Again $r^m a^m \in A$, since $a^m \in A$, $r^m \in R$ and A is an ideal of R.

$\therefore$ $ra \in \sqrt{A}$. Similarly, $ar \in \sqrt{A}$.

Hence $\sqrt{A}$ is an ideal of R.

(*i*) Obviously, $A \subseteq \sqrt{A}$ $(\because x \in A \Rightarrow x^2 \in A$, as A is an ideal of $R)$

(*ii*) We have $\sqrt{\sqrt{A}} = \sqrt{S}$, where $S = \sqrt{A}$.

By part (*i*), $S \subseteq \sqrt{S} \Rightarrow \sqrt{A} \subseteq \sqrt{\sqrt{A}}$.

Conversely, let $x \in \sqrt{\sqrt{A}} \Rightarrow x \in \sqrt{S} \Rightarrow x^n \in S$, for some $n \in \mathbf{N}$

$\Rightarrow x^n \in \sqrt{A} \Rightarrow (x^n)^m \in A$, for some $m \in \mathbf{N}$

$\Rightarrow x^{nm} \in A$, where $nm \in \mathbf{N}$

$\Rightarrow x \in \sqrt{A} \Rightarrow \sqrt{\sqrt{A}} \subseteq \sqrt{A}$.

Hence $\sqrt{\sqrt{A}} = \sqrt{A}$.

(*iii*) Let $1 \in R$ and $\sqrt{A} = R$. Then $1 \in \sqrt{A} \Rightarrow 1^n \in A$, for some positive integer $n \Rightarrow 1 \in A$ and A is an ideal of $R \Rightarrow 1 \cdot r \in A \ \forall \ r \in R \Rightarrow r \in A \ \forall \ r \in R \Rightarrow R \subseteq A$. Obviously, $A \subseteq R$. Hence $A = R$.

$\sqrt{A}$ is often called the **radical** of A. We also write $\sqrt{A}$ as $N(A)$.

Example 1.9.23. *Let R be a ring with unity. Show that*

$$\langle a \rangle = \{ \sum_{\text{finite}} xay : x, y \in R \}.$$ [D.U., 2000]

Solution. Let $S = \{ \sum_{\text{finite}} xay : x, y \in R \}$.

We have to show that S is the smallest ideal of R, which contains a.

First of all, we show that S is an ideal of R. Let $\alpha, \beta \in S$.

Then $\alpha = x_1 a y_1 + \ldots + x_n a y_n$, $\beta = s_1 a t_1 + \ldots + s_m a t_m$;

where $x_i, y_i, s_j, t_j \in R$; $1 \le i \le n$, $1 \le j \le m$.

We observe that

$$\alpha - \beta = x_1 a y_1 + \ldots x_n a y_n + (-s_1) a t_1 + \ldots + (-s_m) a t_m \in S.$$

For any $r \in R$ and $\alpha \in S$, $r\alpha = (rx_1) a y_1 + \ldots + (rx_n) a y_n \in S$

and $\alpha r = x_1 a (y_1 r) + \ldots + x_n a (y_n r) \in S$, as $r x_i \in R$ and $y_j r \in R$.

Hence S is an ideal of R containing a, since $a = 1 \cdot a \cdot 1 \in S$.

Let T be any ideal of R containing a. We shall prove that $S \subseteq T$. Since T is an ideal of R and $a \in T$, $xa \in T$ and so $xay \in T$, $\forall\ x, y \in R$.

Consequently, $\sum_{\text{finite}} xay \in T$, as T is an ideal of R.

This show that $S \subseteq T$. Hence S is the smallest ideal containing a and so $S = \langle a \rangle$.

Example 1.9.24. *Prove that any non-zero ideal in the Gaussian integers $J[i]$ must contain some positive integer.*

Solution. We know $J[i] = \{m + ni : m, n \text{ are integers}, i = \sqrt{-1}\}$ is an integral domain with unity 1. Further $J[i]$ is a principal ideal domain.

[See corollary of Theorem 3.1.1. of chapter 3]

It means every non-zero ideal A of $J[i]$ is generated by a single element of $J[i]$ *i.e.*,

$$A = \langle m + ni \rangle = \{ (m + ni)\, x : x \in J[i] \}.$$

Here $m + ni \neq 0 \in A$ *i.e.*, m and n are not both zero.

Since $m + ni \in A$ and $m - ni \in J[i]$, so

$(m + ni)(m - ni) \in A$, as A is an ideal of $J[i]$

$\Rightarrow \quad m^2 + n^2 \in A$, where $m^2 + n^2$ is a positive integer.

Hence every non-zero ideal in $J[i]$ must contain some positive integer.

EXERCISES

1. If R is a ring and $a \in R$. Show that

$$r(a) = \{x \in R : ax = 0\} \text{ is a right ideal of } R.$$

2. If A and B are ideals in a ring R such that $A \cap B = (0)$, prove that for every $a \in A$, $b \in B$, $ab = 0$.

[Hint. $a \in A$, $b \in B$ *i.e.*, $b \in R \Rightarrow ab \in A$ ($\because$ A is an ideal of R)

$a \in A$ *i.e.*, $a \in B \Rightarrow ab \in B$ ($\because$ B is an ideal of R)

Hence $ab \in A \cap B = (0) \Rightarrow ab = 0$.

3. Prove that the intersection of two left (right) ideals of a ring R is a left (right) ideal of R. What can you say about the intersection of a left ideal and a right ideal of R ?

[Hint. If A is a left ideal of R and B is a right ideal of R, then $A \cap B$ need not be even a one-sided ideal of R. Let us take

$$A = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\} \text{ and } B = \left\{ \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} : c, d \in \mathbf{Z} \right\}.$$

Then A is a left ideal and B is a right ideal of the ring M_2 of 2×2 matrices over the integers [See Examples 1.9.2 and 1.9.3].

We have $A \cap B = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbf{Z} \right\}$.

Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in A \cap B$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2$.

But $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix}\notin A\cap B$,

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\1&0\end{pmatrix}\notin A\cap B$]

4. If A and B are two left (right) ideals of a ring R, then show that $A + B$ is a left (right) ideal of R.

[**Hint.** See Theorem 1.9.3]

5. What can you about the sum of a left ideal and a right ideal of R ?

[**Hint.** Refer to the left ideal A and right ideal B of M_2 as given in Ex. 3. We have

$$A+B=\left\{\begin{pmatrix}x&z\\y&0\end{pmatrix}: x, y, z\in \mathbf{Z}\right\}.$$

Clearly, $\begin{pmatrix}1&1\\1&0\end{pmatrix}\in A+B$ and $\begin{pmatrix}1&1\\1&1\end{pmatrix}\in M_2$. But

$\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}2&2\\1&1\end{pmatrix}\notin A+B$ and

$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}=\begin{pmatrix}2&1\\2&1\end{pmatrix}\notin A+B.$

Thus $A + B$ need not be even a one-sided ideal of M_2.]

6. If A is a left ideal and B is a right ideal of a ring R, then show that AB is a two-sided ideal of R, whereas BA need not be even a one-sided ideal of R.

[**Hint.** (*i*) See the Remark of Theorem 1.9.4 for the proof of the fact that AB is a two sided ideal of R.

(*ii*) Take the left ideal A and the right ideal B of M_2 as given in Ex. 3. We see that

$$\begin{pmatrix}c&d\\0&0\end{pmatrix}\begin{pmatrix}a&0\\b&0\end{pmatrix}=\begin{pmatrix}cd+ab&0\\0&0\end{pmatrix}.$$

Thus $$BA=\left\{\begin{pmatrix}x&0\\0&0\end{pmatrix}: x\in \mathbf{Z}\right\}.$$

As shown in Ex. 3 above, BA is not even a one-sided ideal of M_2.]

7. If A, B be two ideals of a ring R, then show that $AB \subseteq A \cap B$. Give an example to show that there exist ideals A and B such that $AB \neq A \cap B$.

[**Hint.** (*i*) See Corollary 1 of Theorem 1.9.4.

(*ii*) Let A be the left ideal of M_2 and B the right ideal of M_2 as given in Ex. 3. Then

$$A\cap B=\left\{\begin{pmatrix}a&0\\0&0\end{pmatrix}: a\in \mathbf{Z}\right\} \text{ and } AB=\left\{\begin{pmatrix}x&y\\z&t\end{pmatrix}: x, y, z, t\in \mathbf{Z}\right\}=M_2,$$

since $\begin{pmatrix}a&0\\b&0\end{pmatrix}\begin{pmatrix}c&d\\0&0\end{pmatrix}=\begin{pmatrix}ac&ad\\bc&bd\end{pmatrix}$.]

8. If A, B be two ideal of a ring R, then show that $AB \subseteq A + B$. Give an example to show that there exist ideals A and B such that $AB \neq A + B$.
[**Hint.** See Cor. 2 of Theorem 1.9.4]

9. Give an example of two ideals A and B of R such that $A \subseteq B \subseteq R$, where A is an ideal of B, B is an ideal of R, but A is not an ideal of R.

[**Hint.** Let $R = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 0 & 0 & z \end{pmatrix} : x_i, y_i, z \text{ are integers} \right\}$,

$$B = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : x, y \text{ are integers} \right\},$$

$$A = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : x \text{ is an integer} \right\}.$$

Then A is an ideal of B, B is an ideal of R. But A is not an ideal of R, since $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \notin A$.

10. Let R be the ring of all real-valued, continuous functions on $[0, 1]$. Show that the set $S = \{f \in R : f(\frac{2}{3}) = 0\}$ is an ideal of R.
[**Hint.** Similar to Example 1.9.8.]

11. If R is a commutative ring and $a \in R$, then prove that
$Ra = \{ra : r \in R\}$ is an ideal of R.
[**Hint.** $x, y \in Ra \Rightarrow x = r_1 a, y = r_2 a \Rightarrow x - y = (r_1 - r_2)\, a \in Ra$, since $r_1 - r_2 \in R$. For any $r \in R$, $rx = r(r_1 a) = (rr_1)\, a \in Ra$. R is commutative $\Rightarrow xr = rx \in Ra$ *i.e.*, $xr \in Ra$.]

12. For any element a of a ring R, prove that
$Ra = \{xa : x \in R\}$ is a left ideal of R.
[**Hint.** See Ex. 11 above.]

13. Consider the ring $\mathbf{Z}$ of integers and an ideal M of $\mathbf{Z}$ consisting of all multiples of a prime p. Let N be an ideal of $\mathbf{Z}$ such that $M \subset N \subset \mathbf{Z}$. Show that $N = M$ or $N = \mathbf{Z}$. [D.U., 2000]
[**Hint.** Let $M = (p) = \{px : x \in \mathbf{Z}\}$, $N = (n) = \{nx : x \in \mathbf{Z}\}$.
$M \subset N \Rightarrow p \in (n) \Rightarrow p = nx$, for $x \in \mathbf{Z}$. Since p is prime, either $n = 1$ or $n = p \Rightarrow$ either $(n) = (1)$ or $(n) = (p) \Rightarrow$ either $N = \mathbf{Z}$ or $N = M$. Also see Example 2.6.6 of Chapter 2.

14. Let R be a ring with unity and A be any proper ideal of R. Show that no element of A can have a multiplicative inverse.
[**Hint.** See Example 1.9.6(*a*)]

15. Show that the set $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ is a subring of the ring M_2 of 2×2 matrices over the integers. Also prove that S is neither a left nor a right ideal of M_2.

16. If S is an ideal of a ring R and T is any subring of R, then show that S is an ideal of $S + T$.
[**Hint.** First show that $S + T$ is a subring of R.]

17. Show that if A is an ideal of a ring R, then $A + A = A$.
[**Hint.** For any $a \in A$, $a + 0 \in A + A$ and so $A \subseteq A + A$. Conversely, let $x \in A + A \Rightarrow x = a_1 + a_2$ for some $a_1, a_2 \in A \Rightarrow x \in A$, since A is an ideal of $R \Rightarrow A + A \subseteq A$. Hence $A + A = A$.]

18. If R is a ring and A is a left ideal of R, prove that
$$Ann\,(A) = \{x \in R : xa = 0 \ \forall\ a \in A\}$$
is a two-sided ideal of R. [D.U., 1996]
[**Hint.** Compare with Example 1.9.10]

19. Consider the ring R of all 3×3 matrices of the form :
$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, a, b, c, d, e, f \text{ are real numbers} \right\}.$$
Show that the set $I = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : a \text{ is real} \right\}$
is a left ideal of R, which is not a right ideal of R.

20. Let A be an ideal of a ring R such that $A \neq R$. Show that if R has unity, then $1 \notin A$.
[**Hint.** If $1 \in A$, then $A = R$ [See Example 1.9.4], which is a contradiction.]

1.10 Simple Ring

Definition. *A ring R is called a* **simple ring**, *if*

(i) there exist two elements a, b in R such that $ab \neq 0$.

(ii) R has no proper ideals i.e., the only ideals of R are $\{0\}$ and R.

Illustration

The ring $\mathbf{Z}_2 = \{0, 1\}$ modulo 2 is a simple ring, since $1 \otimes_2 1 = 1 \neq 0$ and $\mathbf{Z}_2$ has no proper ideals.

(Notice that $\mathbf{Z}_2$ is a field and a field has no proper ideals).

Theorem 1.10.1. *Prove that a division ring is a simple ring.*

Proof. Let R be a division ring. Since $1 \in R$, so $1 \,.\, 1 = 1 \neq 0$. Next we show that R has no proper ideals. Let A be any ideal of R. If $A = \{0\}$, there is nothing to prove. Let $A \neq \{0\}$. Then there exists some $a \neq 0 \in A \subseteq R$. So $a^{-1} \in R$ exists such that $aa^{-1} = a^{-1}a = 1$. Since A is an ideal of R, so $a \in A$ and $a^{-1} \in R \Rightarrow aa^{-1} \in A \Rightarrow 1 \in A \Rightarrow 1 \,.\, x \in A\ \forall\ x \in R$

$\Rightarrow x \in A \; \forall x \in R \Rightarrow R \subseteq A$. Obviously, $A \subseteq R$.

Hence $A = R$ and so R is a simple ring.

Theorem 1.10.2. *Let R be a commutative simple ring with unity. Prove that R is a field.* [D.U., 1998]

Or

If R be a commutative ring with unity whose only ideals are $\{0\}$ are R, then show that R is a field.

Proof. We are given that R is a commutative ring with unity (*i.e.*, $1 \in R$). Then R becomes a field if we just prove that each non-zero element of R has its multiplicative inverse. Let $a \neq 0 \in R$ be arbitrary.

Let $aR = \{ax : x \in R\}$. ...(1)

We proceed to show that aR is an ideal of R.

Since $0 = a0 \in aR$, aR is non-empty.

Let $\alpha, \beta \in aR$ be arbitrary. Then by (1), we have

$\alpha = ax, \beta = ay$, for some $x, y \in R$

$\Rightarrow \alpha - \beta = ax - ay = a(x - y) \in aR$, since $x - y \in R$.

For any $r \in R$, $\alpha \in aR$; $\alpha r = (ax) r = a(xr) \in aR$, since $xr \in R$.

Since R is commutative, $r\alpha = \alpha r \in aR$.

Thus aR is an ideal of R. Since the only ideals of R are $\{0\}$ and R, it follows that

$$aR = \{0\} \quad \text{or} \quad aR = R.$$

Since $1 \in R$, so $a \cdot 1 = a \in aR$ and $a \neq 0$. Consequently, $aR \neq \{0\}$.

Hence $aR = R$. Since $1 \in R = aR$, we have

$1 = ab$ for some $b \in R$.

$\Rightarrow \quad 1 = ab = ba$, since R is commutative

$\Rightarrow \quad a^{-1} = b \in R$. Hence R is a field.

Ex. *Prove that a commutative ring R with unity is a field if and only if it has no proper ideals.*

[**Hint.** See the above theorem and Example 1.9.5 (*i.e.*, a field has no proper ideals]

EXAMPLES

Example 1.10.1. *Show that the set of 2×2 matrices of the form*

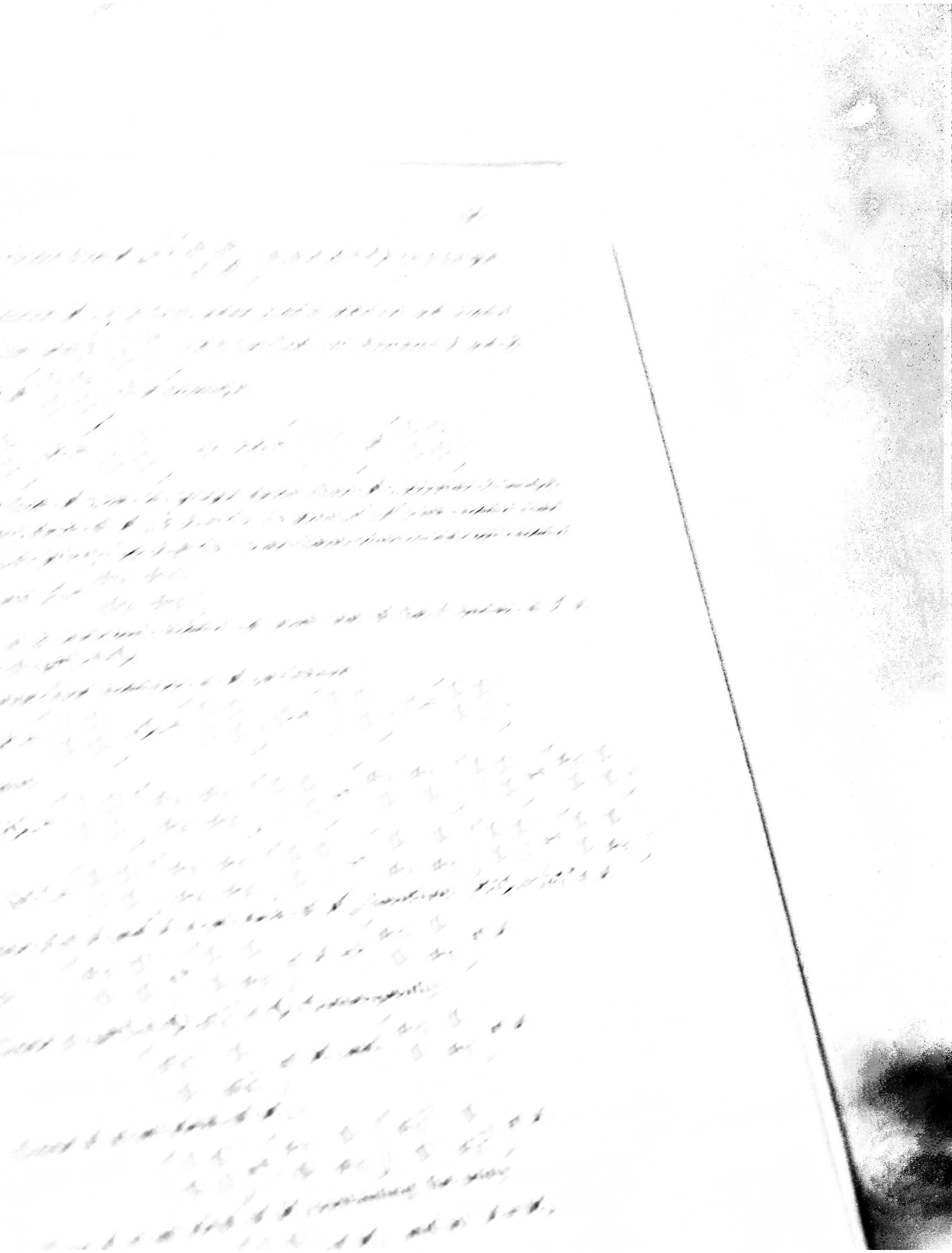
$$S = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : \begin{array}{l} a \text{ and } b \text{ are complex numbers} \\ \text{and } \bar{a}, \bar{b} \text{ their conjugates} \end{array} \right\}$$

is a simple ring.

Solution. By Example 1.4.6, S is a division ring under matrix addition and matrix multiplication. Hence by Theorem 1.10.1, S is a simple ring.

Example 1.10.2. *Define a simple ring and give an example of it.*

Please try yourself.



Example 1.10.3. *Show that* $M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Q} \right\}$ *is a simple ring.*

Solution. We know M_2 is a ring under matrix addition and matrix multiplication and has unity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can find two elements A and B in M_2 such that $AB \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. For example,

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If we show that M_2 has no proper ideals, then M_2 becomes a simple ring. Let A be any ideal of M_2. If $A = \{0\}$, 0 being a 2×2 null matrix, then there is nothing to prove. Let $A \neq \{0\}$. Then there exists a non-zero matrix $X \in A$ of the form $X = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Since X is a non-zero matrix, at least one of the 4 entries in X is non-zero. Let $a_{12} \neq 0 \in \mathbf{Q}$.

We choose four matrices in M_2 as follow :

Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

We have

$$PXQ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ 0 & 0 \end{pmatrix},$$

$$SXT = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a_{12} \end{pmatrix}.$$

Since $X \in A$ and A is an ideal of M_2, therefore $PXQ + SXT \in A$

$$\Rightarrow \quad \begin{pmatrix} a_{12} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_{12} \end{pmatrix} \in A \Rightarrow \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix} \in A.$$

Since $a_{12} \neq 0 \in \mathbf{Q}$, $a_{12}^{-1} \in \mathbf{Q}$. Consequently,

$$\begin{pmatrix} a_{12}^{-1} & 0 \\ 0 & a_{12}^{-1} \end{pmatrix} \in M_2 \text{ and } \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix} \in A.$$

Since A is an ideal of M_2,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & 0 \\ 0 & a_{12} \end{pmatrix}\begin{pmatrix} a_{12}^{-1} & 0 \\ 0 & a_{12}^{-1} \end{pmatrix} \in A.$$

Thus A is an ideal of M_2 containing the unity

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ of } M_2 \quad \text{and so} \quad A = M_2.$$

Hence M_2 is a simple ring.

Example 1.10.4. *Let R be a ring with unity. If R has no right ideals except R and $\{0\}$, then prove that R is a division ring.* [D.U., 1999]

Solution. We are given that $1 \in R$ and so R becomes a division ring if we show that each non-zero element in R has its multiplicative inverse in R. Let $a \neq 0 \in R$ be arbitrary. Let $aR = \{ax : x \in R\}$.

Since $0 = a \,.\, 0 \in aR$, aR is non-empty. ...(1)

Let $\alpha, \beta \in aR$ and $r \in R$. Then

$$\alpha = ax, \beta = ay, \text{ for some } x, y \in R, \text{ by (1)}$$

$$\Rightarrow \quad \alpha - \beta = ax - ay = a(x - y) \in aR, \text{ as } x - y \in R,$$

and $$\alpha r = (ax) r = a(xr) \in aR, \text{ as } xr \in R.$$

Thus aR is a right ideal of R and so by the given hypothesis,

$$aR = \{0\} \text{ or } aR = R.$$

Since $1 \in R$ and $a = a \,.\, 1 \in aR$ and $a \neq 0$, so $aR \neq \{0\}$.

Hence $aR = R$. Since $1 \in R$, so $1 \in aR$. Consequently,

$$1 = ab, \quad \text{for some } b \in R. \qquad ...(2)$$

From (2), it follows that each non-zero element of R has a *right* inverse. Since $b \neq 0 \in R$ (for otherwise, $1 = ab = 0$, a contradiction), there exists some $c \in R$ such that $bc = 1$.

We have $ba = ba \,.\, 1$, since 1 is the unity of R ...(3)

$$= (ba)(bc), \text{ using (3)}$$
$$= b(ab)\, c$$
$$= (b \,.\, 1)\, c, \text{ by (2)}$$
$$= bc$$
$$= 1, \text{ by (3)}$$

$$\therefore \quad ab = ba = 1$$

$$\Rightarrow \quad a^{-1} = b \in R.$$

Hence R is a division ring.

Example 1.10.5. *Let R be a ring with unity. If R has no left ideals except R and $\{0\}$, then prove that R is a division ring.*

Hint. Let $Ra = \{xa : x \in R\}$. Then Ra is a left ideal of R. Now proceed like Example 1.10.4.

Example 1.10.6. *If R be a ring having more than one element such that $aR = R \;\forall\; a \neq 0 \in R$, then R is a division ring.*

Solution. Firstly, we show that $xy = 0 \Rightarrow x = 0$ or $y = 0$; $x, y \in R$.

If $x \neq 0$ and $y \neq 0$, then by the given hypothesis, $xR = R$, $yR = R$. ...(1)

Now $0 = xy \;\Rightarrow\; 0 \,.\, R = (xy)\, R = x\,(yR) = xR = R$.

$\therefore R = \{0\}$, a contradiction to the fact that R has more than one element.

Hence (1) is proved.

Since $R \neq \{0\}$, there exists some $a \neq 0 \in R$. Further $aR = R$.

Now $a \in R \;\Rightarrow\; a \in aR \;\Rightarrow\; a = ae$ for some $e \in R$.

It may be noted that $e \neq 0$, for otherwise $a = a0 = 0$, a contradiction. Since $a \neq 0$, therefore

$$ae = a \Rightarrow ae^2 = ae \Rightarrow a(e^2 - e) = 0 \Rightarrow e^2 - e = 0, \text{ using (1)}$$

$$\therefore \quad e^2 = e. \qquad \ldots(2)$$

Let $x \in R$ be arbitrary. Then

$$(xe - x)\, e = xe^2 - xe = xe - xe = 0, \text{ using (2)}$$

$$\Rightarrow \quad xe - x = 0, \text{ since } e \neq 0 \text{ and using (1)}$$

$$\Rightarrow \quad xe = x \ \forall\ x \in R \Rightarrow e \text{ is the right unity of } R. \qquad \ldots(3)$$

Now we show that each non-zero element of R has a right inverse.

Let $x \neq 0 \in R$. By the given hypothesis, $xR = R$.

Since $e \in R$, $e \in xR \Rightarrow e = xy$, for some $y \in R$.

$\Rightarrow y$ is a right inverse of x. $\ldots(4)$

From (3) and (4), it follows that R is a division ring.

Example 1.10.7. *Let R be a ring such that the only right ideals of R are $\{0\}$ and R. Prove that either R is a division ring or that R is a ring with a prime number of elements in which $ab = 0$ for every $a, b \in R$.*

Solution. Let $A = \{a \in R : ar = 0 \ \forall\ r \in R\}$ or $A = \{a \in R : aR = 0\}$. $\ldots(1)$

We shall prove that A is a right ideal of R.

Let $a, b \in A$. Then $ar = 0$ and $br = 0 \ \forall\ r \in R$.

We have $(a - b)\, r = ar - br = 0 - 0 = 0 \ \forall\ r \in R$.

$\therefore \quad a - b \in A$.

Let $a \in A$ and $x \in R$. Then $ar = 0 \ \forall\ r \in R$. $\ldots(2)$

We have $(ax)\, r = a\,(xr) = ar_1$, where $r_1 = xr \in R$.

By (2), $\quad ar_1 = 0$ and so $(ax)\, r = 0 \ \forall\ r \in R$.

Thus $ax \in R$ and so A is a right ideal of R.

According to the given hypothesis, $A = \{0\}$ or $A = R$.

Case I. Let $A = \{0\}$. Then $aR = \{0\} \Rightarrow a = 0$.

In other words, $aR = R \ \forall\ a \neq 0 \in R$.

Hence R is a division ring. [See Example 1.10.6]

Case II. Let $A = R$. Then by (1), $ar = 0 \ \forall\ a \in R$ and $\forall\ r \in R$ $\ldots(3)$

or $\quad ab = 0 \ \ \forall\ a, b \in R$.

Let H be **any** subgroup of the additive group $(R, +)$.

Then $\quad x - y \in H \ \forall\ x \in H, y \in H$.

For $x \in H \subseteq R$, $\ r \in R \ \Rightarrow \ xr = 0 \in H$, using (3).

Thus H is a right ideal of R.

According to the given hypothesis, R has only two right ideals $\{0\}$ and R. Hence $(R, +)$ can have only two subgroups $\{0\}$ and $(R, +)$ *i.e.*, $(R, +)$ has no proper subgroups. Consequently, $(R, +)$ must be a cyclic group of prime order [by Lagrange's Theorem]. Hence R has a prime number of elements such that $ab = 0 \ \forall\ a, b \in R$, using (3).

2

Homomorphisms, Maximal & Prime Ideals & Principal Ideal Domains

In this chapter we shall discuss *Homomorphism of Rings, Quotient Rings, Imbedding of Rings, Maximal and Prime Ideals, Divisibility in Rings, Prime and Irreducible Elements and Principal Ideal Domains.*

2.1 Homomorphism of Rings

Let $\{R, +, \cdot\}$ and $\{R', \oplus, \otimes\}$ be two rings. A mapping $f: R \to R'$ is called a *homomorphism*, if

(*i*) $f(a+b) = f(a) \oplus f(b)$,

(*ii*) $f(a \cdot b) = f(a) \otimes f(b)$ $\forall$ $a, b \in R$.

The above conditions imply that f preserves the compositions of the rings R and R'.

However, if we agree to use the same compositions + and · for both R and R', then

Definition 1. *A mapping $f: R \to R'$ is called a* **homomorphism**, *if*

(*i*) $f(a+b) = f(a) + f(b)$,

(*ii*) $f(a \cdot b) = f(a) \cdot f(b)$ $\forall$ $a, b \in R$.

Remark. The compositions + and · as appearing in $f(a+b)$ and $f(a \cdot b)$ are those of R and + and · as appearing in $f(a) + f(b)$ and $f(a) \cdot f(b)$ are those of R'. It is more convenient to drop even · symbol in the condition (*ii*), which can be rewritten as :

$$f(ab) = f(a) f(b) \ \forall \ a, b \in R.$$

Definition 2. *A ring R' is called a* **homomorphic image** *of a ring R, if there exists a homomorphism f of R* **onto** *R' i.e., f is a homomorphism and for each $r' \in R'$, there exists some $r \in R$ such that $f(r) = r'$.*

Definition 3. *A mapping $f: R \to R'$ is called an* **isomorphism**, *if*

(*i*) *f is a homomorphism,*

(*ii*) *f is one-to-one i.e., $f(a) = f(b) \Rightarrow a = b$, for $a, b \in R$.*

Definition 4. *Two rings R and R' are called* **isomorphic**, *denoted as $R \approx R'$, if there exists a mapping $f: R \to R'$ such that*

(*i*) *f is a homomorphism,* (*ii*) *f is a one-to-one and* (*iii*) *f is onto.*

2.2 Examples of Homomorphisms

Example 2.2.1. *If R is a ring, then the mapping $f: R \to R$ defined as $f(x) = x$ $\forall$ $x \in R$ is a homomorphism.*

Solution. For any $x, y \in R$, we see that

$$f(x+y) = x+y = f(x)+f(y),$$

and $$f(xy) = xy = f(x)\,f(y).$$ Thus f is a homomorphism.

Example 2.2.2. *If R is a ring, the mapping $f: R \to R$ defined as $f(x) = 0 \ \forall\ x \in R$ is a homomorphism.*

Solution. For any $x, y \in R$, we see that

$$f(x+y) = 0 = 0+0 = f(x)+f(y),$$

and $$f(xy) = 0 = 0\,.\,0 = f(x)\,f(y).$$ Thus f is a homomorphism.

Example 2.2.3. *Let $\mathbf{Z}[\sqrt{2}] = \{m + n\sqrt{2} : m, n \text{ are integers}\}$. The mapping $f: \mathbf{Z}[\sqrt{2}] \to \mathbf{Z}[\sqrt{2}]$ defined as $f(m+n\sqrt{2}) = m - n\sqrt{2}$ is a homomorphism.*

Solution. Let $x = a + b\sqrt{2} \in \mathbf{Z}[\sqrt{2}]$, $y = c + d\sqrt{2} \in \mathbf{Z}[\sqrt{2}]$.

Then $x + y = (a+c) + (b+d)\sqrt{2}$, $xy = (ac+2bd) + (ad+bc)\sqrt{2}$.

We have

$$f(x+y) = (a+c) - (b+d)\sqrt{2} = (a - b\sqrt{2}) + (c - d\sqrt{2}) = f(x) + f(y),$$

$$f(xy) = (ac+2bd) - (ad+bc)\sqrt{2} = (a - b\sqrt{2})\,(c - d\sqrt{2}) = f(x)\,f(y).$$

Thus f is a homomorphism.

Example 2.2.4. *Let $R = \mathbf{Z}$ and $R' =$ set of all even integers. Then $(R', +, *)$ is a ring, where $a * b = \frac{1}{2}ab \ \forall\ a, b \in R'$. The mapping $f: R \to R'$ defined as $f(a) = 2a \ \forall\ a \in R$ is a homomorphism.*

Solution. For any $a, b \in R$; we have

$$f(a+b) = 2(a+b) = 2a + 2b = f(a) + f(b),$$

$$f(ab) = 2(ab) = \frac{(2a)\,(2b)}{2} = (2a) * (2b) = f(a) * f(b).$$

Thus f is a homomorphism.

Example 2.2.5. *Let R be the ring of all complex numbers and R' the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where a and b are real numbers. Then the mapping $f: R \to R'$ defined as*

$$f(a+ib) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ is a homomorphism.}$$

Solution. Let $x = a + ib$, $y = c + id \in R$. Then

$$x + y = (a+c) + i\,(b+d), \quad xy = (ac - bd) + i\,(ad + bc).$$

$$f(x) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, f(y) = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}, f(x+y) = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix}$$

$$\text{or} \quad f(x+y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(x) + f(y),$$

$$f(xy) = \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(x)\,f(y).$$

Hence f is a homomorphism.

Example 2.2.6. *Let R be a commutative ring such that $2x = 0\ \forall\ x \in R$. Then the mapping $f: R \to R$ defined as $f(x) = x^2$ is a homomorphism.*

Solution. Let $x, y \in R$. Then $f(x+y) = (x+y)^2 = x^2 + 2xy + y^2 = x^2 + y^2$, since R is commutative and $2x = 0\ \forall\ x \in R$.

$\therefore \quad f(x+y) = f(x) + f(y)$.

Again, $\quad f(xy) = (xy)^2 = x^2 y^2$, since R is commutative.

$\therefore \quad f(xy) = f(x) f(y)$.

Hence f is a homomorphism.

Example 2.2.7. *Show that $f: \mathbf{Z}_2 \to \mathbf{Z}_2$ defined by $f(n) = n^2 - n$ is a ring homomorphism.*

Solution. We know $\mathbf{Z}_2 = \{0, 1\}$. Let $n, m \in \mathbf{Z}_2$. Then

$$f(n+m) = (n+m)^2 - (n+m)$$
$$= (n^2 - n) + (m^2 - m) \qquad [\because\ 2nm = 0 \text{ in } \mathbf{Z}_2]$$
$$= f(n) + f(m).$$

Again
$$f(nm) = n^2 m^2 - nm$$
$$= (n^2 - n)(m^2 - m) + nm(n+m) - 2nm$$
$$= (n^2 - n)(m^2 - m) = f(n) f(m)$$

[Notice that $nm(n+m) = 0\ \forall\ n, m \in \mathbf{Z}_2$].

Hence f is a ring homomorphism.

2.3 Theorems on Homomorphisms

In the following theorems ; R and R' denote two rings.

Theorem 2.3.1. *If $R \to R'$ is a homomorphism, then*

1. $f(0) = 0'$.

2. $f(-a) = -f(a)$, $a \in R$.

Proof 1. We know $a + 0 = 0\ \forall\ a \in R$. In particular, $0 + 0 = 0$.

$\therefore \quad f(0) = f(0+0) \quad \Rightarrow \quad f(0) = f(0) + f(0)$, since f is a homo.

$\Rightarrow f(0) + 0' = f(0) + f(0)$. $\qquad [\because\ 0' \in R' \text{ and } f(0) \in R']$

Hence, by cancellation law in the additive group $(R', +)$,

$$0' = f(0).$$

2. We have $\quad a + (-a) = 0 \quad \Rightarrow \quad f[a + (-a)] = f(0)$

$\Rightarrow f(a) + f(-a) = f(0)$, since f is a homomorphism.

$\Rightarrow f(a) + f(-a) = 0'$, since $f(0) = 0'$.

Similarly, $\quad f(-a) + f(a) = 0'$.

Hence $\quad f(-a) = -f(a)$.

Remark. It is more convenient to take $f(0) = 0$, keeping in mind that 0 on the R.H.S. of the equation $[f(0) = 0]$ is the zero element of R'.

Theorem 2.3.2. *Show that :*

(*a*) *The homomorphic image of a commutative ring is a commutative ring. The converse need not be true.*

(*b*) *The homomorphic image a ring with unity is a ring with unity. The converse need not be true.*

Proof. (*a*) Let $f : R \to R'$ be an onto homomorphism, where R is a commutative ring. We have to show that R' is also a commutative ring. Let a', b' be any two arbitrary elements of R'. Since f is onto, there exist some elements $a, b \in R$ such that

$$f(a) = a' \text{ and } f(b) = b'. \qquad ...(1)$$

We have $a'\,b' = f(a)\,f(b) = f(ab)$, since f is a homomorphism

$= f(ba)$, since R is commutative

$= f(b)\,f(a)$, since f is a homomorphism

$= b'\,a'$, by (1)

$\therefore$ $a'\,b' = b'a' \;\forall\; a', b' \in R'$. Hence R' is commutative.

However, the converse of the above result may not be true i.e., if R' is a homomorphic image of a ring R, where R' is a commutative ring, then R may not be a commutative ring.

Let $$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \text{ are integers} \right\}.$$

Then R is a non-commutative ring under matrix addition and matrix multiplication. Notice that

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}.$$

We shall prove that the mapping $f : R \to \mathbf{Z}$ defined as

$$f\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\} = a \qquad ...(1)$$

is an onto homomorphism.

Let $$X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R, \quad Y = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R.$$ We have

$$X + Y = \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix} \text{ and } XY = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}.$$

Now $f(X + Y) = a + c = f(X) + f(Y)$, by (1)

and $f(XY) = ac = f(X)\,f(Y)$, by (1).

Obviously, f is onto, since for any $x \in \mathbf{Z}$

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \text{ and } f\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right\} = x. \quad (\text{Here } y \in \mathbf{Z})$$

Hence $\mathbf{Z}$ is a homomorphic image of R, where $\mathbf{Z}$ is a commutative ring and R is a non-commutative ring.

Theorem 2.3.2. *Show that :*

(*a*) *The homomorphic image of a commutative ring is a commutative ring. The converse need not be true.*

(*b*) *The homomorphic image a ring with unity is a ring with unity. The converse need not be true.*

Proof. (*a*) Let $f: R \to R'$ be an onto homomorphism, where R is a commutative ring. We have to show that R' is also a commutative ring. Let a', b' be any two arbitrary elements of R'. Since f is onto, there exist some elements $a, b \in R$ such that

$$f(a) = a' \text{ and } f(b) = b'. \qquad ...(1)$$

We have $a'b' = f(a)f(b) = f(ab)$, since f is a homomorphism

$= f(ba)$, since R is commutative

$= f(b)f(a)$, since f is a homomorphism

$= b'a'$, by (1)

$\therefore$ $a'b' = b'a' \ \forall\ a', b' \in R'$. Hence R' is commutative.

However, the converse of the above result may not be true i.e., if R' is a homomorphic image of a ring R, where R' is a commutative ring, then R may not be a commutative ring.

Let $$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \text{ are integers} \right\}.$$

Then R is a non-commutative ring under matrix addition and matrix multiplication. Notice that

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}.$$

We shall prove that the mapping $f: R \to \mathbf{Z}$ defined as

$$f\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\} = a \qquad ...(1)$$

is an onto homomorphism.

Let $$X = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R, \quad Y = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \in R.$$ We have

$$X + Y = \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix} \text{ and } XY = \begin{pmatrix} ac & ad \\ 0 & 0 \end{pmatrix}.$$

Now $f(X+Y) = a + c = f(X) + f(Y)$, by (1)

and $f(XY) = ac = f(X)f(Y)$, by (1).

Obviously, f is onto, since for any $x \in \mathbf{Z}$

$$\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \text{ and } f\left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right\} = x. \quad (\text{Here } y \in \mathbf{Z})$$

Hence $\mathbf{Z}$ is a homomorphic image of R, where $\mathbf{Z}$ is a commutative ring.

(*b*) Let $f: R \to R'$ be an onto homomorphism, where R is a ring with unity $1 \in R$. We shall prove that $f(1)$ is the unity of R'. Let $a' \in R'$ be arbitrary. Since f is onto, there exists some $a \in R$ such that $f(a) = a'$.

We have

$f(1) \cdot a' = f(1) \cdot f(a) = f(1 \cdot a) = f(a) = a'$, since f is a homomorphism.

Similarly, $a' \cdot f(1) = a'$. Hence $f(1)$ is the unity of R'.

However, the converse of the above result may not be true i.e., if R' is a homomorphic image of a ring R, where R' is a ring with unity, then R may not have unity. Refer to the above example. $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}$ is a ring without unity and $\mathbf{Z}$ is a ring with unity. The mapping $f: R \to \mathbf{Z}$ defined as $f\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\} = a$ is an onto homomorphism.

Definition. (Kernel of a Homomorphism)

If $f: R \to R'$ be a homomorphism, then the **kernel** *of f, denoted as Ker f, is defined as*

$$Ker f = \{a \in R : f(a) = 0' \in R'\}.$$

Remark. Since f is a homomorphism, $f(0) = 0'$ and so $0 \in Ker f$. This shows that $Ker f$ is always non-empty.

Theorem 2.3.3. *If $R \to R'$ is a homomorphism, then Ker f is a two-sided ideal of R.* [D.U., 1994]

Proof. By definition, $Ker f = \{a \in R : f(a) = 0' \in R'\}$.

Then $Ker f$ is non-empty, since $f(0) = 0' \Rightarrow 0 \in Ker f$.

Let $a, b \in Ker f$ be arbitrary. Then $f(a) = f(b) = 0'$.

We have $f(a - b) = f[a + (-b)] = f(a) + f(-b) = f(a) - f(b)$, since f is a homomorphism.

$\therefore \quad f(a - b) = 0' - 0' = 0' \Rightarrow a - b \in Ker f$.

For any $r \in R$ and $a \in Ker f$, we have

$f(ar) = f(a) f(r)$, since f is a homomorphism

$= 0' \cdot f(r) = 0'$, since $f(a) = 0'$

$\therefore \quad f(ar) = 0' \Rightarrow ar \in Ker f$.

Similarly, $ra \in Ker f$. Hence $Ker f$ is an ideal of R.

Theorem 2.3.4. *If $f: R \to R'$ is a homomorphism, then $Ker f = \{0\}$ if and only if f is one-to-one.*

Proof. *Condition is necessary*

Let $Ker f = \{0\}$. We shall prove that f is one-to-one.

Let $f(a) = f(b)$, where $a, b \in R$

$\Rightarrow \quad f(a) - f(b) = 0' \quad \Rightarrow f(a - b) = 0'$, since f is a homomorphism

$\Rightarrow \quad a-b \in Kerf \quad \Rightarrow \quad a-b=0 \quad \Rightarrow \quad a=b.$

Hence f is one-to-one.

Condition is sufficient

Let f be one-to-one. We shall prove $Kerf=\{0\}$.

Let $x \in Kerf$ be arbitrary. Then $f(x)=0'$

or $f(x)=f(0)$, since f is a homomorphism implies $f(0)=0'$

$\therefore \quad x=0$, since f is one-to-one.

Hence $Kerf=\{0\}$.

EXAMPLES

Example 2.3.1. *Let $f: R \to R'$ be an onto homomorphism, where R is a ring with unity 1. Show that $f(1)$ is the unity of R'.*

Solution. Let $a' \in R'$ be arbitrary. Since $f: R \to R'$ is onto, there exists some $a \in R$ such that $f(a)=a'$. We have

$$f(1) \,.\, a' = f(1) \,.\, f(a) = f(1 \,.\, a) = f(a) = a',$$

since f is a homomorphism and $1 \,.\, a = a \;\forall\; a \in R$.

Similarly, $a' \,.\, f(1) = a'$. Hence $f(1)$ is the unity of R'.

Example 2.3.2. *If R is a ring with unity 1 and f is a homomorphism of R into an integral domain R' with $Kerf \neq R$, prove that $f(1)$ is the unity of R'.* [D.U., 1999]

Solution. Let $a' \in R'$ be arbitrary. We shall prove that

$$f(1)\, a' = a'\, f(1) = a'.$$

Obviously, $\quad f(1)\, a' - f(1)\, a' = 0'$

$\Rightarrow \quad f(1 \,.\, 1)\, a' - f(1)\, a' = 0'$, since 1 is the unity of R

$\Rightarrow f(1) f(1)\, a' - f(1)\, a' = 0'$, since f is a homomorphism

$\Rightarrow \quad f(1)\,[f(1)\, a' - a'] = 0'$

$\Rightarrow \quad f(1) = 0' \;$ or $\; f(1)\, a' - a' = 0'$, since R is an integral domain

If $f(1)=0'$, then $1 \in Kerf$, where $Kerf$ is an ideal of R.

Consequently, $r = 1 \,.\, r \in Kerf \;\forall\; r \in R$ and so $Kerf = R$

This is contrary to the given hypothesis and so

$$f(1)\, a' - a' = 0' \;\text{ or }\; f(1)\, a' = a' \;\forall\; a' \in R'.$$

Similarly, by considering $a' f(1) - a' f(1) = 0'$, we can prove

$$a' f(1) = a' \;\forall\; a' \in R'.$$

Hence $f(1)$ is the unity of R'.

Example 2.3.3. *Prove that any homomorphism of a field is either an isomorphism or takes each element into 0.*

Solution. Let f be a homomorphism of a field F into a ring R $f: F \to R$ is a homomorphism. Then $Ker\, f$ is an ideal of the field F. Si the only ideals of a field F are (0) and F [See Example 1.9.5 of Chapter so

$$Ker\, f = (0) \quad \text{or} \quad Ker\, f = F.$$

Case I. Let $Ker\, f = (0)$. Then f is one-to-one (Theorem 2.3.4). Sin f is a homomorphism and one-to-one, f is an isomorphism.

Case II. Let $Ker\, f = F$. Then $x \in Ker\, f \; \forall \; x \in F$ *i.e.*, $f(x) = 0 \; \forall \; x \in$ Hence f takes each element of F into zero.

Example 2.3.4. *Prove that any homomorphism of a field is eithe one-to-one or it takes every element to zero.* [D.U., 1999

Solution. Similar to Example 2.3.3.

Example 2.3.5. *Let $f: R \to R'$ be a homomorphism.*

Let $f(R) = \{f(a) : a \in R\}$. Show that

(a) $f(R)$ is a subring of R'.

(b) If R is commutative, then $f(R)$ is commutative.

(c) If R has unity 1, then $f(1)$ is the unity of the subring $f(R)$.

Solution. (*a*) Let $x, y \in f(R)$ be arbitrary. Then

$$x = f(a), y = f(b) \quad \text{for some} \quad a, b \in R.$$

We have $x - y = f(a) - f(b) = f(a - b)$, since f is a homomorphism and $xy = f(a) f(b) = f(ab)$, since f is a homomorphism

Clearly, $a - b \in R$ and $ab \in R$ and so $f(a - b) \in f(R), f(ab) \in f(R)$.

It follows that $x - y \in f(R)$ and $xy \in f(R)$.

Hence $f(R)$ is a subring of R'.

For parts (*b*) and (*c*), refer to Theorem 2.3.2.

Remark. *$f(R)$ is called the* **homomorphic image** *of R.*

Example 2.3.6. *Let $f: R \to R'$ be a homomorphism and let A be an ideal of R. Show that $f(A)$ is an ideal of $f(R)$.*

Solution. We have $f(A) = \{f(a) : a \in A\}$.

Then $f(A)$ is non-empty, since $0 \in A \Rightarrow f(0) \in f(A)$.

Let $x, y \in f(A)$. Then $x = f(a_1), y = f(a_2)$, for $a_1, a_2 \in A$

$\Rightarrow x - y = f(a_1) - f(a_2) = f(a_1 - a_2)$, as f is a homomorphism.

Since A is an ideal of R and $a_1 a_2 \in A$, so $a_1 - a_2 \in A$

$\Rightarrow f(a_1 - a_2) \in f(A) \Rightarrow x - y \in f(A)$.

Let $\alpha \in f(R)$ and $x \in f(A)$.

Then $\alpha = f(r), x = f(a_1)$, for some $r \in R, a_1 \in A$. We have

$$\alpha x = f(r) f(a_1) = f(r a_1), \text{ as } f \text{ is a homomorphism.}$$

Since A is an ideal of R, so $r \in R$ and $a_1 \in A \Rightarrow r a_1 \in A$.

$\therefore \quad f(r a_1) \in f(A) \Rightarrow \alpha x \in f(A)$.

Similarly, $x \alpha \in f(A)$. Hence $f(A)$ is an ideal of $f(R)$.

Example 2.3.7. *Let f be an isomorphism of a ring R onto a ring R'. Show that*

(a) If R is an integral domain, then R' is also an integral domain.

(b) If R is a field, then R' is also a field.

Solution. (a) Let R be an integral domain. Since R is commutative and $f: R \to R'$ is an onto homomorphism, R' is also commutative. Now we show that R' is without zero divisors. Let $a', b' \in R'$ be such that

$$a'\,b' = 0.$$

Since f is one-to-one and onto, there exist unique $a, b \in R$ such that

$$f(a) = a' \quad \text{and} \quad f(b) = b'.$$

Now $\quad a'\,b' = 0' \;\Rightarrow\; f(a)\,f(b) = 0'$

$\Rightarrow\; f(ab) = f(0)$, since f is a homomorphism

$\Rightarrow\quad ab = 0$, since f is one-to-one

$\Rightarrow\quad a = 0$ or $b = 0$, since R is an integral domain

$\Rightarrow\; f(a) = f(0)$ or $f(b) = f(0)$

$\Rightarrow\quad a' = 0'$ or $b' = 0'$, since $f(0) = 0'$.

Thus $a'b' = 0' \;\Rightarrow\; a' = 0'$ or $b' = 0 \;\Rightarrow\; R'$ has no zero divisors.

Hence R' is an integral domain.

(b) Let R be a field. Since $f: R \to R'$ is an onto homomorphism and since R is a commutative ring with unity 1, so R' is also a commutative ring with unity $f(1)$ [See Theorem 2.3.2]. Finally, we show that every non-zero element of R' has its multiplicative inverse. Let a' be any non-zero element of R'. Since f is one-to-one and onto, there exists a unique non-zero element $a \in R$ such that $f(a) = a'$. Notice that if $a = 0$, then $f(a) = f(0) = 0'$ and so $a' = 0'$, a contradiction. Since R is a field and $a \neq 0 \in R$, $a^{-1} \in R$ exists and $aa^{-1} = a^{-1}a = 1$.

Now $\quad aa^{-1} = 1 \;\Rightarrow\; f(aa^{-1}) = f(1) \;\Rightarrow\; f(a)\,f(a^{-1}) = f(1)$

$\Rightarrow\quad a'\,f(a^{-1}) = f(1).$

Similarly, $\quad a^{-1}\,a = 1 \;\Rightarrow\; f(a^{-1})\,a' = f(1).$

$\therefore\quad (a')^{-1} = f(a^{-1}) \in R'.$

Hence R' is a field.

Example 2.3.8. *Prove that*

(a) Every isomorphic image of an integral domain is an integral domain.

(b) Every isomorphic image of a field is a field.

(c) Every isomorphic image of a division ring is a division ring.

Solution. Same as Example 2.3.7.

Example 2.3.9. *Let R be a ring with unity. Using its elements, we define a ring R' by defining*

$$a \oplus b = a + b + 1 \quad \text{and} \quad a \otimes b = ab + a + b \quad \forall\; a, b \in R.$$

Prove that R is isomorphic to R'.

Solution. Define a mapping $f: R \to R'$ as $f(a) = a - 1 \ \forall \ a \in R$.

Then $f(a+b) = a + b - 1 = (a-1) + (b-1) + 1 = f(a) + f(b) + 1$

or $f(a+b) = f(a) \oplus f(b)$.

Again $f(ab) = ab - 1$,

and $f(a) \otimes f(b) = f(a) f(b) + f(a) + f(b)$

$= (a-1)(b-1) + (a-1) + (b-1)$

$= ab - a - b + 1 + a - 1 + b - 1 = ab - 1$.

$\therefore$ $f(ab) = f(a) \otimes f(b)$ and so f is a homomorphism.

Further, f is one-to-one ; since $f(a) = f(b) \Rightarrow a - 1 = b - 1 \Rightarrow a = b$

For any $a \in R'$, $a = (a+1) - 1 \Rightarrow a = f(b)$, where $b = a + 1 \in R$

$\Rightarrow f$ is onto. Hence $R \approx R'$.

Example 2.3.10. *Prove that the only homomorphisms from* $\mathbf{Z}$ *to* $\mathbf{Z}$ *(*$\mathbf{Z}$ *being the ring of integers) are the identity and zero mappings.* [D.U., 2000]

Solution. Let $f: \mathbf{Z} \to \mathbf{Z}$ be any homomorphism.

If $f(x) = 0 \ \forall \ x \in \mathbf{Z}$, then f is the zero mapping.

Let f be a non-zero homomorphism *i.e.*, $f(x) \neq 0$ for each $x \neq 0 \in \mathbf{Z}$.

Then $\{f(1)\}^2 = f(1) . f(1) = f(1 . 1) = f(1) \neq 0$.

It follows that $f(1)$ is a non-zero idempotent element in $\mathbf{Z}$.

Since the only non-zero idempotent in $\mathbf{Z}$ is 1, so $f(1) = 1$.

For any $n \in \mathbf{Z}$ and $n > 0$, we have

$$f(n) = \underbrace{f(1 + 1 + \ldots + 1)}_{n \text{ times}} = \underbrace{f(1) + f(1) + \ldots + f(1)}_{n \text{ times}} = \underbrace{1 + 1 + \ldots + 1}_{n \text{ times}} = n.$$

If $n < 0$, we can write $n = -m$, m being a positive integer.

Then $f(n) = f(-m) = -f(m)$, since f is a homomorphism

$= -m$, since $f(m) = m > 0$, as proved above

$\therefore$ $f(n) = n$, if $n < 0$.

Obviously, $f(0) = 0$, if $n = 0$.

Hence $f(n) = n \ \forall \ n \in \mathbf{Z}$ *i.e.*, f is the identity mapping.

Example 2.3.11. *Let R and R' be two rings. A mapping $f: R \to R'$ is called an* **antihomomorphism**, *if*

$f(x+y) = f(x) + f(y)$ *and* $f(xy) = f(y) f(x) \ \forall \ x, y \in R$.

Let f, g be two antihomomorphisms of a ring R into R. Prove that $fg: R \to R$ is a homomorphism.

Solution. Let $x, y \in R$. Then

$(fg)(x+y) = f(g(x+y)) = f(g(x) + g(y)) = f(g(x)) + f(g(y))$

$= (fg)(x) + (fg)(y)$, since f and g are antihomomorphisms

Again $(fg)(xy) = f(g(xy)) = f(g(y)\, g(x)) = f(g(x)) f(g(y))$

$= (fg)(x)\,(fg)(y)$, since f and g are antihomomorphisms

Hence $fg: R \to R$ is a homomorphism.

EXERCISES

1. If $\mathbf{C}$ is the ring of complex numbers, show that the mapping $f: \mathbf{C} \to \mathbf{C}$ defined as $f(x+iy) = x - iy$, is an onto isomorphism.

2. Show that every isomorphic image of a ring without zero divisors is a ring without zero divisors.
[**Hint.** See Example 2.3.7 (*a*).]

3. Show that any non-zero homomorphism of a field F into a ring R is one-to-one.
[**Hint.** See Example 2.3.4.]

4. Let F be a field and let $f: F \to F$ be a non-zero homomorphism. Show that f need not be onto.
[**Hint.** $\mathbf{Z}_3 = \{0, 1, 2\}$, $\mathbf{Z}_5 = \{0, 1, 2, 3, 4\}$] are fields modulo 3 and modulo 5, respectively. The mapping $f: \mathbf{Z}_3 \to \mathbf{Z}_5$ defined as $f(0) = 0$, $f(1) = 1, f(2) = 2$ is a non-zero homomorphism. However, f is not onto.]

5. Show that the relation of isomorphism in the set of all rings is an equivalence relation.
[**Hint.** (*i*) $R \approx R$, since the identity mapping $I: R \to R$ is $1-1$, onto and homomorphism.
(*ii*) $R_1 \approx R_2 \Rightarrow R_2 \approx R_1$.
If $f: R_1 \to R_2$ is $1-1$, onto and homomorphism, then
$f^{-1}: R_2 \to R_1$ is $1-1$, onto and homomorphism.
(*iii*) $R_1 \approx R_2$ and $R_2 \approx R_3 \Rightarrow R_1 \approx R_3$.
Let $f: R_1 \to R_2$ be $1-1$, onto and homomorphism, and
$g: R_2 \to R_3$ be $1-1$, onto and homomorphism. Then
$gof: R_1 \to R_3$ is also $1-1$, onto and homomorphism.]

6. Give an example of a homomorphism $f: R \to R'$ such that R has unity, but R' does not have unity.
[**Hint.** $f: \mathbf{Z} \to \mathbf{E}$ (even integers) defined as $f(x) = 0 \ \forall \ x \in \mathbf{Z}$ is a homomorphism, where 1 is the unity of $\mathbf{Z}$, but $\mathbf{E}$ does not have unity.]

7. Give an example of a homomorphism $f: R \to R'$ such that 1 is the unity of R, but $f(1)$ is not the unity of R'.
[**Hint.** $f: \mathbf{Z} \to \mathbf{Z}$ defined as $f(x) = 0 \ \forall \ x \in \mathbf{Z}$ is a homomorphism, where 1 is the unity of $\mathbf{Z}$, but $f(1) = 0$ is not the unity of $\mathbf{Z}$. Notice that f is not onto.]

8. Let A be the ring of all continuous, real-valued functions defined on $[0, 1]$ and $\mathbf{R}$ the field of real numbers. Show that the mapping $\phi: A \to \mathbf{R}$ defined by $\phi(f(x)) = f(\frac{1}{2}), \forall f(x) \in A$ is an onto homomorphism. Find the kernel of ϕ. (**Ans.** $Ker\ \phi = \{f(x) \in A : f(\frac{1}{2}) = 0\}$)

9. Let $M_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbf{R} \right\}$, $\mathbf{R}$ being the field of real numbers. Show that the mapping $f : M_2 \to \mathbf{R}$ defined by $f\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right\} = a$ is an isomorphism of M_2 onto $\mathbf{R}$.

10. Let R be a commutative ring such that $px = 0 \ \forall \ x \in R$, p being a prime number. Define $f : R \to R$ such that $f(x) = x^p$. Show that f is a homomorphism.

[**Hint.** Let $x, y \in R$. Since R is commutative, $(xy)^p = x^p y^p$ and $(x+y)^p = x^p + p_{c_1} x^{p-1} y + p_{c_2} x^{p-2} y^2 + \ldots + y^p$.

$\therefore f(xy) = f(x) f(y)$ and $f(x+y) = x^p + 0 + 0 + \ldots + 0 + y^p = f(x) + f(y)$, since $px = 0 \ \forall \ x \in R$]

11. Let R, S, T be three rings such that $f : R \to S$ and $g : S \to T$ are homomorphisms. Show that $gf : R \to T$ is a homomorphism.

12. Let $f : R \to R'$ be a homomorphism. Prove that if A is a right (left) ideal of R, then $f(A)$ is a right (left) ideal of R'.

[**Hint.** Similar to Example 2.3.6]

2.4 Quotient Rings and Fundamental Theorem of Homomorphism of Rings

Theorem 2.4.1. *If U is an ideal of a ring R, then R/U is a ring and is a homomorphic image of R with kernel U.*

Proof. If U is an ideal of R, we define

$$\frac{R}{U} = \{a + U : a \in R\}.$$

We define two compositions in R/U as follow :

$$(a+U) + (b+U) = (a+b) + U,$$

$$(a+U)(b+U) = ab + U \ \forall \ a, b \in R. \qquad \ldots(1)$$

Firstly, we show that these compositions are well defined *i.e.*, we have ...(2)
to show, if

$$a + U = a' + U \text{ and } b + U = b' + U,$$

then $(a+b) + U = (a'+b') + U$ and $ab + U = a'b' + U$.

We have $a = a + 0 \in a + U = a' + U \Rightarrow a = a' + u_1$, for $u_1 \in U$.

Similarly, $b = b' + u_2$, for $u_2 \in U$.

$\therefore \quad a + b = a' + u_1 + b' + u_2 = a' + b' + u_1 + u_2$

or $(a+b) - (a'+b') = u_1 + u_2 \in U$, since U is an ideal of R.

Thus $(a+b) + U = (a'+b') + U$.

[We know $a + H = b + H \Leftrightarrow a - b \in H$, where H is a subgroup of the group $(R, +)$]

Again $ab = (a' + u_1)(b' + u_2) = a'b' + a'u_2 + u_1 b' + u_1 u_2$

or $ab - a'b' = a'u_2 + u_1 b' + u_1 u_2 \in U.$

[Notice that U is a two-sided ideal of $R \Rightarrow a' u_2 \in U, u_1 b' \in U$ and $u_1 u_2 \in U \Rightarrow a' u_2 + u_1 b' + u_1 u_2 \in U$]

$\therefore \quad ab + U = a'b' + U.$

It follows that the compositions given in (1) and (2) are well-defined. Next we show that R/U is a ring w.r.t. these compositions.

Let $a + U, b + U, c + U \in R/U$. Then

1. $(a + U) + (b + U) = (a + b) + U \in R/U$, by (1).

2. $(a + U) + (b + U) = (a + b) + U$
$= (b + a) + U = (b + U) + (a + U)$, by (1).

3. $(a + U) + \{(b + U) + (c + U)\} = (a + U) + \{(b + c) + U\}$, by (1)
$= \{a + (b + c)\} + U$, by (1)
$= \{(a + b) + c\} + U$, by asso. law in $(R, +)$
$= \{(a + b) + U\} + (c + U)$, by (1)
$= \{(a + U) + (b + U)\} + (c + U)$, by (1)

4. $(a + U) + (0 + U) = (a + 0) + U = a + U \ \forall \ a + U \in R/U.$
Thus $0 + U = U$ is the zero element of R/U.
We often write U as $\bar{0}$.

5. $(a + U) + (-a + U) = \{a + (-a)\} + U = 0 + U = U.$
Thus $-a + U \in R/U$ is the additive inverse of $a + U$.

6. $(a + U)(b + U) = ab + U \in R/U$, by (2).

7. $(a + U)\{(b + U)(c + U)\} = (a + U)(bc + U)$, by (2)
$= a(bc) + U$, by (2)
$= (ab) c + U$, by asso. law in R
$= (ab + U)(c + U)$, by (2)
$= \{(a + U)(b + U)\}(c + U)$, by (2).

8. $(a + U)\{(b + U) + (c + U)\} = (a + U)\{(b + c) + U\}$, by (1)
$= a(b + c) + U$, by (2)
$= (ab + ac) + U$, by dist. law in R
$= (ab + U) + (ac + U)$, by (1)
$= (a + U)(b + U) + (a + U)(c + U)$, by (2)

9. Similarly,
$\{(a + U) + (b + U)\}(c + U) = (a + U)(c + U) + (b + U)(c + U).$
Hence R/U is a ring.

Finally, we show that R/U is a homomorphic image of R with kernel U. We define a mapping $f: R \to R/U$ as ...(3)

$$f(a) = a + U \quad \forall \quad a \in R.$$

Clearly, f is well-defined, since $a = b \Rightarrow a + U = b + U.$

Let $a, b \in R$. From (3), we have

$$f(a+b) = (a+b)+U$$
$$= (a+U)+(b+U), \text{ by (1)}$$
$$= f(a)+f(b), \text{ by (3)}$$

and
$$f(ab) = ab+U, \text{ by (3)}$$
$$= (a+U)(b+U), \text{ by (2)}$$
$$= f(a)f(b), \text{ by (3)}.$$

Thus f is a homomorphism.

If $X \in R/U$ is arbitrary, then $X = a_1 + U$, for some $a_1 \in R$. Consequently, $X = f(a_1)$, $a_1 \in R$ and so f is onto. Hence R/U is a homomorphic image of R. We have

$$a \in Ker f \Leftrightarrow f(a) = \bar{0} \in R/U\ (\bar{0} = U)$$
$$\Leftrightarrow f(a) = U.$$
$$\Leftrightarrow a+U = U \Leftrightarrow a \in U$$

Hence $Ker f = U$.

Remarks :

1. *The ring* $R/U = \{a+U : a \in R\}$ *is called a* **quotient ring** *or* **difference ring** *of* R. *The quotient ring* R/U *is defined only when* U *is an ideal of* R.

2. *R/U is commutative, if R is commutative.*

We have

$$(a+U)(b+U) = ab+U = ba+U$$
$$= (b+U)(a+U)\ ;\ a, b \in R.$$

3. *If R has unity 1, then $1+U$ is the unity of R/U.*

We have $(a+U)(1+U) = a \cdot 1 + U = a+U \ \forall\ a+U \in R/U$.

4. *If R is a boolean ring, so is R/U.*

We have $(a+U)^2 = (a+U)(a+U) = a^2+U = a+U.$ $(\because a^2 = a)$

EXAMPLES

Example 2.4.1. *Show that the set* $A = \{5x : x \in \mathbf{Z}\}$ *is an ideal of* $\mathbf{Z}$. *Determine the quotient ring* $\frac{\mathbf{Z}}{A}$.

Solution. It is easy to verify that A is an ideal of $\mathbf{Z}$. We have $\frac{\mathbf{Z}}{A} = \{A, 1+A, 2+A, 3+A, 4+A\}$.

Notice that
$$A = \{..., -15, -10, -5, 0, 5, 10, 15, ...\},$$
$$1+A = \{..., -14, -9, -4, 1, 6, 11, 16, ...\},$$
$$2+A = \{..., -13, -8, -3, 2, 7, 12, 17, ...\},$$
$$3+A = \{..., -12, -7, -2, 3, 8, 13, 18, ...\},$$
$$4+A = \{..., -11, -6, -1, 4, 9, 14, 19, ...\}.$$

Also $5+A = A,\ 6+A = 1+A,\ 7+A = 2+A$ etc. etc.

Example 2.4.2. *In a ring R, define S as the set $\{ab - ba : a, b \in R\}$. Show that for any idea, A of R, the quotient ring R/A is commutative if and only if $S \subseteq A$.*

Solution. R/A is commutative

$\Leftrightarrow (a+A)(b+A) = (b+A)(a+A) \ \forall\ a, b \in R$

$\Leftrightarrow ab + A = ba + A \ \forall\ a, b \in R$

$\Leftrightarrow ab - ba \in A \ \forall\ a, b \in R$

$\Leftrightarrow S \subseteq A.$

Example 2.4.3. *Show that the set N of all nilpotent elements of a commutative ring R forms an ideal of R and the R/N has no non-zero nilpotent elements.*

Solution. See Example 1.9.13 of Chapter 1.

Example 2.4.4. *Let A be an ideal of a commutative ring R. Let*

$$\sqrt{A} = \{x \in R : x^n \in A \text{ for some positive integer } n\}.$$

Show that $R/\sqrt{A}$ has no non-zero nilpotent elements.

Solution. $\sqrt{A}$ is an ideal of R [See Example 1.9.22. of Chapter 1]

Let $\bar{r} = r + \sqrt{A}$ be a non-zero, nilpotent element of $R/\sqrt{A}$

$\Rightarrow \ r + \sqrt{A} \neq \sqrt{A}$ and $(r+\sqrt{A})^n = \sqrt{A}$, for some positive integer n.

$\therefore \ r^n + \sqrt{A} = \sqrt{A} \Rightarrow r^n \in \sqrt{A} \Rightarrow (r^n)^m \in A$, for some positive integer m

$\Rightarrow r^{nm} \in A$, nm being a positive integer

$\Rightarrow \ r \in \sqrt{A} \ \Rightarrow \ r + \sqrt{A} = \sqrt{A}$, which is a contradiction.

Hence $R/\sqrt{A}$ has no non-zero nilpotent elements.

Theorem 2.4.2. (Fundamental Theorem of Homomorphism)

Let $f : R \to R'$ be a homomorphism of a ring R onto a ring R'. Then

$$\frac{R}{Ker f} \approx R'. \qquad \text{[D.U., 1994]}$$

Proof. By definition, $Ker f = \{x \in R : f(x) = 0' \in R'\}$ and $Ker f$ is an ideal of R. [See Theorem 2.3.3]

Let $U = Ker f$. By Theorem 2.4.1, we have

$$\frac{R}{U} = \{a + U : a \in R\}.$$

We shall prove that $R/U \approx R'$. Define a mapping

$$\phi : \frac{R}{U} \to R' \text{ as } \phi(a+U) = f(a) \ \forall\ a \in R. \qquad \ldots(1)$$

Then ϕ is well-defined, since

$a + U = b + U \ \Rightarrow \ a - b \in U = Ker f \ \Rightarrow f(a-b) = 0'$

$\Rightarrow f(a) - f(b) = 0' \ \Rightarrow \ f(a) = f(b)$, as f is a homo.

We now proceed to show that ϕ is a homomorphism.

Let $X = a + U \in R/U, \quad Y = b + U \in R/U$. Then

$X + Y = (a+b) + U, \ XY = ab + U.$

Using (1), we have

$$\phi(X+Y) = f(a+b)$$
$$= f(a) + f(b), \text{ since } f \text{ is a homomorphism}$$
$$= \phi(a+U) + \phi(b+U), \text{ by (1)}$$
$$= \phi(X) + \phi(Y).$$

Also $\phi(XY) = f(ab)$, by (1)
$$= f(a)f(b), \text{ since } f \text{ is a homomorphism}$$
$$= \phi(a+U)\,\phi(b+U), \text{ by (1)}$$
$$= \phi(X)\,\phi(Y).$$

Thus ϕ is a homomorphism.

Next we show that ϕ is one-to-one.

Let $\phi(X) = \phi(Y) \Rightarrow f(a) = f(b)$, by (1)
$$\Rightarrow f(a) - f(b) = 0'$$
$$\Rightarrow f(a-b) = 0', \text{ since } f \text{ is a homomorphism}$$
$$\Rightarrow a - b \in Ker f = U$$
$$\Rightarrow a + U = b + U$$
$$\Rightarrow X = Y.$$

Thus ϕ is one-to-one.

Lastly, we show that ϕ is onto.

Let $r' \in R'$ be arbitrary. Since $f: R \to R'$ is onto, there exists some $r \in R$ such that

$$f(r) = r' \quad \text{or} \quad r' = \phi(r+U), \text{ by (1)}.$$

Here $r + U \in R/U$. Thus ϕ is onto.

We have shown that

$$\phi : R/U \to R' \text{ is homomorphism, } 1-1 \text{ and onto.}$$

Hence $R/U \approx R'$ or $R/Ker f \approx R'$.

Equivalently, $R' \approx R/Ker f$.

Ex. 1. *If U is any ideal of a ring R, then R/U is a homomorphic image of R. Conversely, if f is a homomorphism of a ring R onto a ring R', then R' is isomorphic to a quotient ring of R.*

[Hint. See Theorems 2.4.1 and 2.4.2.]

Ex. 2. *Show that there exists an onto homomorphism f from a ring R to a quotient ring R/I, where $I = Ker f$.*

[Hint. See Theorem 2.4.1.]

Theorem 2.4.3. *If A and B are two ideals of a ring R, then*

$$\frac{A+B}{B} \approx \frac{A}{A \cap B}.$$ [D.U., 1997]

• **Proof.** Clearly, B is an ideal of $A + B$ and $A \cap B$ is an ideal of A. Thus the quotient rings $(A+B)/B$ and $A/(A \cap B)$ are defined. We define a mapping $f: A \to \frac{A+B}{B}$ as

$$f(a) = a + B \quad \forall \quad a \in A. \qquad \ldots(1)$$

Notice that $a = a + 0 \in A + B$ and so $a + B \in (A + B)/B$.

Obviously, f is well-defined, since

$$a = a' \implies a + B = a' + B.$$

We now proceed to show that f is a homomorphism.

Let $x, y \in A$. Using (1), we have

$$f(x + y) = (x + y) + B = (x + B) + (y + B) = f(x) + f(y),$$

and

$$f(xy) = xy + B = (x + B)(y + B) = f(x) f(y).$$

Thus f is a homomorphism. Now we show that f is onto. Let $X \in (A + B)/B$ be arbitrary. Then $X = r + B$, for some $r \in A + B$. We can write $r = a_1 + b_1$, for some $a_1 \in A$, $b_1 \in B$.

Thus $X = a_1 + b_1 + B = a_1 + B.$ $(\because\ b_1 \in B \Rightarrow b_1 + B = B)$

Using (1), $X = f(a_1), a_1 \in A.$

It follows that $f: A \to (A + B)/B$ is an onto homomorphism.

By Fundamental theorem of homomorphism, we have

$$\frac{A}{Ker f} \approx \frac{A + B}{B}. \qquad \ldots(2)$$

We have $Ker f = \{x \in A : f(x) = B$, zero of $(A + B)/B\}$.

Now $x \in Ker f \Leftrightarrow x \in A$ and $f(x) = B$

$\Leftrightarrow x \in A$ and $x + B = B$, by (1)

$\Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow x \in A \cap B.$ $\qquad \ldots(3)$

$\therefore \quad Ker f = A \cap B.$

From (2) and (3), we have

$$\frac{A}{A \cap B} \approx \frac{A + B}{B} \text{ or } \frac{A + B}{B} \approx \frac{A}{A \cap B}.$$

Theorem 2.4.4. *If A and B are two ideals of a ring R, then*

$$\frac{A + B}{A} \approx \frac{B}{A \cap B}. \qquad \text{[D.U., 1995]}$$

Proof. As done in Theorem 2.4.3, the mapping $f: B \to \frac{A + B}{A}$ defined as $f(b) = b + A \ \forall\ b \in B$ is well-defined and homomorphism.

Also f is onto. Let $X \in (A + B)/A$ be arbitrary. Then $X = r + A$, for some $r \in A + B$ *i.e.*, $r = a_1 + b_1$, for some $a_1 \in A$, $b_1 \in B$.

$\therefore \quad X = a_1 + b_1 + A = b_1 + a_1 + A = b_1 + A$, since $a_1 \in A$

$\Rightarrow \quad X = f(b_1), b_1 \in B \Rightarrow f$ is onto.

By Fundamental theorem of homomorphism, we obtain

$$\frac{B}{Ker f} \approx \frac{A + B}{A} \text{ or } \frac{A + B}{A} \approx \frac{B}{Ker f}, \text{ where}$$

$$x \in Ker f \Leftrightarrow x \in B \text{ and } f(x) = A$$
$$\Leftrightarrow x \in B \text{ and } x + A = A$$
$$\Leftrightarrow x \in B \text{ and } x \in A.$$
$$\therefore \quad Ker f = A \cap B.$$

Hence $\dfrac{A+B}{A} \approx \dfrac{B}{A \cap B}$.

Theorem 2.4.5. *If A and B are two ideals of a ring R such that $B \subseteq A$, then*

$$\frac{R}{A} \approx \frac{R/B}{A/B}.$$

Proof. By the given hypothesis, the quotient rings $\dfrac{R}{A}, \dfrac{R}{B}, \dfrac{A}{B}$ are meaningful. We define a mapping

$$f: \frac{R}{B} \to \frac{R}{A} \text{ as } f(r+B) = r + A \ \forall \ r \in R. \qquad \ldots(1)$$

Then f is well-defined, since

$$r_1 + B = r_2 + B \Rightarrow r_1 - r_2 \in B \Rightarrow r_1 - r_2 \in A, \text{ as } B \subseteq A$$
$$\Rightarrow \quad r_1 + A = r_2 + A.$$

Let $X = x + B,\ Y = y + B \in R/B$. Then

$$X + Y = x + y + B,\ XY = xy + B. \text{ Using (1), we obtain}$$
$$f(X+Y) = x + y + A = (x + A) + (y + A) = f(x+B) + f(y+B)$$
$$= f(X) + f(Y),$$
$$f(XY) = xy + A = (x + A)(y + A) = f(x+B) f(x+B) = f(X) f(Y).$$

Thus f is a homomorphism.

Now we show that f is onto.

Let $S \in R/A$ be arbitrary. Then $S = r' + A$, for $r' \in R$. Using (1), we get

$$S = f(r' + B), \quad \text{where } r' + B \in R/B.$$

Thus $f: R/B \to R/A$ is an onto homomorphism. By Fundamental theorem of homomorphism, we have

$$\frac{R/B}{Ker f} \approx \frac{R}{A} \text{ or } \frac{R}{A} \approx \frac{R/B}{Ker f}.$$

We have $Ker f = \{X = x + B \in R/B : f(x+B) = A, \text{ zero of } R/A\}$ $\qquad \ldots(2)$

Thus $X \in Ker f \Leftrightarrow X = x + B \in R/B$ and $f(x+B) = A$
$\Leftrightarrow X = x + B \in R/B$ and $x + A \in A$, by (1)
$\Leftrightarrow X = x + B \in R/B$ and $x \in A$
$\Leftrightarrow X \in A/B.$

$$\therefore \quad Ker f = \frac{A}{B}. \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{R}{A} \approx \frac{R/B}{A/B}.$$

2.5 Imbedding of Rings

Definition. *A ring R is said to be* **imbedded** *in a ring R', if there exists an isomorphism of R into R' i.e., there exists a mapping $f : R \to R'$ such that (i) f is a homomorphism and (ii) f is one-to-one.*

We also say that R' is an **extension ring** *or* **over-ring** *of R.*

Remark. Since f is an isomorphism of R into R', $f(R)$ is a subring of R' and further R and $f(R)$ are isomorphic rings.

[Notice that $f : R \to f(R)$ is an onto isomorphism and so $R \simeq f(R)$.]

Thus upto isomorphism R is a subring of R' and it is in this sense that we say R is *imbedded* in R' or R' is an *extension ring* or *over-ring* of R.

Theorem 2.5.1. *Every ring can be imbedded in a ring with unity.*

Proof. Let R be any ring. We take

$$R \times \mathbf{Z} = \{(r, m) : r \in R, m \in \mathbf{Z}\},$$

$\mathbf{Z}$ denotes the ring of integers.

We define addition and multiplication in $R \times \mathbf{Z}$ as follow :

$$(r, m) + (s, n) = (r + s, m + n), \qquad ...(1)$$

$$(r, m)\,(s, n) = (rs + ms + nr, mn). \qquad ...(2)$$

It is clear that $r + s$, $rs \in R$ and

$$nr = r + r + ... + r \ (n \text{ times}) \in R, \ ms \in R.$$

Thus $r + s \in R$, $rs + ms + nr \in R$, $m + n \in \mathbf{Z}$, $mn \in \mathbf{Z}$

and so $(r, m) + (s, n) \in R \times \mathbf{Z}$, $(r, m)\,(s, n) \in R \times \mathbf{Z}$.

It is easy to verify that $(R \times \mathbf{Z}, +)$ is an abelian group in which $(0, 0) \in R \times \mathbf{Z}$ is the additive identity and $(-r, -m)$ is the additive inverse of $(r, m) \in R \times \mathbf{Z}$.

Notice that $(r, m) + (0, 0) = (r + 0, m + 0) = (r, m)$,

and $(r, m) + (-r, -m) = (r - r, m - m) = (0, 0)$.

Further, we can verify that

(*i*) $(r, m)\,\{(s, n)\,(t, l)\} = \{(r, m)\,(s, n)\}\,(t, l)$

(*ii*) $(r, m)\,\{(s, n) + (t, l)\} = (r, m)\,(s, n) + (r, m)\,(t, l)$

(*iii*) $\{(s, n) + (t, l)\}\,(r, m) = (s, n)\,(r, m) + (t, l)\,(r, m)$

(*iv*) $(r, m)\,(0, 1) = (r, m)$ for all $(r, m) \in R \times \mathbf{Z}$.

Let us verify (*ii*) and (*iv*).

L.H.S. of (*ii*) $= (r, m)\,(s + t, n + l)$, by (1)

$= (r\,(s + t) + m\,(s + t) + (n + l)\,r, m\,(n + l))$

$= (rs + rt + ms + mt + nr + lr, mn + ml)$

$= (rs + ms + nr, mn) + (rt + mt + lr, ml)$

$= (r, m)\,(s, n) + (r, m)\,(t, l) =$ R.H.S. of (*ii*).

Using (2), we see that for each $(r, m) \in R \times \mathbf{Z}$,

$$(r, m)\,(0, 1) = (r0 + m0 + 1 \,.\, r, m \,.\, 1) = (r, m).$$

Hence $R \times \mathbf{Z}$ is a ring with unity $(0, 1)$.

Finally, we show that R can be imbedded in $R \times \mathbf{Z}$. We define a mapping

$$f : R \to R \times \mathbf{Z} \text{ as } f(r) = (r, 0) \ \forall \ r \in R.$$

Then f is well-defined, for $r = s \Rightarrow (r, 0) = (s, 0)$. ...(3)

Also f is one-to-one, since $f(r) = f(s) \Rightarrow (r, 0) = (s, 0) \Rightarrow r = s$.

Next we show that f is a homomorphism.

Let $r, s \in R$. Using (3) and (1), we have

$$f(r+s) = (r+s, 0) = (r+s, 0+0) = (r, 0) + (s, 0) = f(r) + f(s).$$

Again $f(rs) = (rs, 0) = (rs + 0s + 0r, 00) = (r, 0)\ (s, 0)$

$$= f(r) f(s), \text{ using (2).}$$

Hence f is an isomorphism of R into $R \times \mathbf{Z}$ and so R is imbedded in the ring $R \times \mathbf{Z}$ with unity $(0, 1)$.

Remark. *If R is any ring, not necessarily containing unity, then its extension ring with unity is $R \times \mathbf{Z} = \{(r, m) : r \in R, m \in \mathbf{Z}\}$.*

2.5.2 Ring of Endomorphisms of an Abelian Group

Let $(G, +)$ be an additive abelian group. A homomorphism of G into G is called an **endomorphism** *of G i.e., a mapping $f : G \to G$ is an* **endomorphism,** *if*

$$f(x+y) = f(x) + f(y) \ \forall \ x, y \in G.$$

Let *Hom* (G, G) denote the set of all endomorphisms of G. *It is easy to verify that Hom (G, G) is a ring w.r.t. the compositions defined as*

$$(f+g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(g(x)) \qquad ...(1)$$

$$\forall \ x \in G \text{ and } \forall \ f, g \in Hom\ (G, G).$$

We verify some of the properties of the ring structure of *Hom* (G, G).

Let $f, g \in Hom\ (G, G)$ and $x, y \in G$. Then

$$(f+g)(x+y) = f(x+y) + g(x+y), \text{ by (1)}$$

$$= f(x) + f(y) + g(x) + g(y),$$

since f and g are homomorphisms

$$= f(x) + g(x) + f(y) + g(y),$$

since $(G, +)$ is abelian

$$= (f+g)(x) + (f+g)(y), \text{ by (1).}$$

Thus $f+g$ is a homomorphism and so $f+g \in Hom\ (G, G)$.

Similarly, $fg \in Hom\ (G, G)$.

The zero mapping $O : G \to G$ defined as $O(x) = 0 \ \forall \ x \in G$ is such that $O \in Hom\ (G, G)$ and $f + O = f \ \forall \ f \in Hom\ (G, G)$.

The additive inverse of $f \in Hom\ (G, G)$ is $-f \in Hom\ (G, G)$, where $(-f)(x) = -f(x) \ \forall \ x \in G$.

Definition. *Hom (G, G) is called the* **ring of endomorphisms** *of the additive abelian group G.*

Theorem 2.5.3. *Every ring R with unity can be imbedded in a ring of endomorphisms of some additive abelian group.*

Proof. By definition, $(R, +)$ is an additive abelian group. We write $(R, +)$ as R^+. Let $Hom\,(R^+, R^+)$ denote the set of all endomorphisms of R^+. Then $Hom\,(R^+, R^+)$ is a ring of endomorphisms of R^+ w.r.t. the compositions defined as

$$(f+g)(x) = f(x) + g(x),\ (fg)(x) = f(g(x)) \qquad \ldots(1)$$
$$\forall\ x \in R^+ \text{ and } \forall\ f, g \in Hom\,(R^+, R^+).$$

We shall prove that R can be imbedded in $Hom\,(R^+, R^+)$.

For each $a \in R$, we define

$$f_a : R^+ \to R^+ \text{ as } f_a(x) = ax \ \forall\ x \in R^+. \qquad \ldots(2)$$

Let $x, y \in R^+$. Then by (2), we obtain

$$f_a(x+y) = a(x+y) = ax + ay = f_a(x) + f_a(y).$$

Thus f_a is a homomorphism of R^+ into R^+.

It follows that $f_a \in Hom\,(R^+, R^+)$ for each $a \in R$.

We define a mapping $\theta : R \to Hom\,(R^+, R^+)$ as

$$\theta(a) = f_a \ \forall\ a \in R. \qquad \ldots(3)$$

Firstly, we show that θ is one-to-one.

Let $a, b \in R$ be such that $\theta(a) = \theta(b)$. Then by (3), we see that

$$f_a = f_b \Rightarrow f_a(x) = f_b(x) \ \forall\ x \in R^+$$
$$\Rightarrow ax = bx\ \forall\ x \in R^+, \text{ by (2)}$$
$$\Rightarrow a \cdot 1 = b \cdot 1, \text{ as } 1 \in R^+$$
$$\Rightarrow a = b$$
$$\Rightarrow \theta \text{ is one–to–one.}$$

Next we show that θ is a homomorphism.

Let $a, b \in R$. Using (3), we have $\qquad \ldots(4)$

$$\theta(a+b) = f_{a+b},\ \theta(ab) = f_{ab}.$$

For any $x \in R^+$, we have

$$f_{a+b}(x) = (a+b)x, \text{ by (2)}$$
$$= ax + bx$$
$$= f_a(x) + f_b(x), \text{ by (2)}$$
$$= (f_a + f_b)(x), \text{ by (1)}. \qquad \ldots(5)$$

$$\therefore \quad f_{a+b} = f_a + f_b.$$

Again, by (2), $f_{ab}(x) = (ab)x = a(bx) = ay,\ y = bx \in R$

$$= f_a(y), \text{ by (2)}$$
$$= f_a(bx) = f_a(f_b(x)), \text{ by (2)}$$
$$= (f_a f_b)(x), \text{ by (1)}. \qquad \ldots(6)$$

$$\therefore \quad f_{ab} = f_a f_b.$$

Using (5) and (6) in (4), we get

$$\theta(a+b) = f_a + f_b = \theta(a) + \theta(b), \text{ by (3)}$$

and $$\theta(ab) = f_a f_b = \theta(a)\,\theta(b), \text{ by (3)}.$$

Thus θ is a homomorphism. We have proved earlier that θ is one-to-one and so θ is an isomorphism of R into $Hom(R^+, R^+)$. Hence R is imbedded in $Hom(R^+, R^+)$.

Theorem 2.5.4. *Every ring R can be imbedded in a ring of endomorphisms of some additive abelian group.*

Proof. By Theorem 2.5.1, R can be imbedded in a ring R_1 with unity *i.e.*, there exists an isomorphism $f: R \to R_1$. Since R_1 is a ring with unity, by Theorem 2.5.3, R_1 can be imbedded in a ring R_2 of endomorphisms of some additive abelian group *i.e.*, there exists an isomorphism $g: R_1 \to R_2$. Hence $gof: R \to R_2$ is an isomorphism and the theorem is proved.

Theorem 2.5.5. *Show that $Hom(\mathbf{Z}^+, \mathbf{Z}^+)$ is isomorphic to $\mathbf{Z}$, where $\mathbf{Z}^+$ is the additive abelian group $(\mathbf{Z}, +)$.*

Proof. Define a mapping $\theta: \mathbf{Z} \to Hom(\mathbf{Z}^+, \mathbf{Z}^+)$ by

$$\theta(a) = f_a \ \forall\ a \in \mathbf{Z},$$

where $$f_a(x) = ax \ \forall\ x \in \mathbf{Z}^+. \qquad \ldots(1)$$

As proved in Theorem 2.5.3, θ is a ring homomorphism and one-to-one. We now proceed to show that θ is onto. Let $f \in Hom(\mathbf{Z}^+, \mathbf{Z}^+)$ be arbitrary *i.e.*, $f: \mathbf{Z}^+ \to \mathbf{Z}^+$ is a homomorphism.

Since $f(1) \in \mathbf{Z}^+$, let us take $f(1) = m \in \mathbf{Z}^+$.

We shall prove that $f = \theta(m) = f_m$.

If x is a positive integer, then

$$f(x) = \underbrace{f(1+1+\ldots+1)}_{x \text{ times}}$$

$$= \underbrace{f(1)+f(1)+\ldots f(1)}_{x \text{ times}}, \text{ since } f \text{ is a homomorphism}$$

$$= x \cdot f(1) = xm = mx. \qquad \ldots(2)$$

Using (1), $f(x) = f_m(x)$, if x is a positive integer.

If x is a negative integer, then $x = -y$, for some positive integer y. Consequently,

$$f(x) = f(-y) = -f(y), \text{ since } f \text{ is a homomorphism}$$
$$= -my, \text{ by (2)}$$
$$= m(-y) = mx = f_m(x), \text{ by (1)}.$$

$\therefore$ $f(x) = f_m(x)$, if x is a negative integer.

If $x = 0$, then $f(0) = 0$, since f is a homomorphism
$= m0 = f_m(0)$.

Thus $f(x) = f_m(x) \quad \forall\ x \in \mathbf{Z}$. ...(3)

$\Rightarrow$ $f = f_m = \theta(m)$, $m \in \mathbf{Z}$, which proves that θ is onto.

Hence $\mathbf{Z} \approx Hom(\mathbf{Z}^+, \mathbf{Z}^+)$ or $Hom(\mathbf{Z}^+, \mathbf{Z}^+) \approx \mathbf{Z}$.

Ex. *Let $f : (\mathbf{Z}, +) \to (\mathbf{Z}, +)$ be a homomorphism. Show that $f(x) = mx$ for some integer m. Also show that $f_k : (\mathbf{Z}, +) \to (\mathbf{Z}, +)$ such that $f_k(x) = kx$ is a homomorphism for each integer k. Let*

$$Hom(\mathbf{Z}, \mathbf{Z}) = \{f : (\mathbf{Z}, +) \to (\mathbf{Z}, +) \text{ is a homomorphism}\}.$$

Prove that $\theta : \mathbf{Z} \to Hom(\mathbf{Z}, \mathbf{Z})$ given by $\theta(K) = f_k$ is an isomorphism of rings. [D.U., 1998]

Hint. Refer to Theorems 2.5.3 and 2.5.5. The mapping $f_k : (\mathbf{Z}, +) \to (\mathbf{Z}, +)$ defined by $f_k(x) = kx$ is a homomorphism, since $f_x(x + y) = k(x + y) = kx + ky = f_k(x) + f_k(y)$.

The mapping $\theta : \mathbf{Z} \to Hom(\mathbf{Z}, \mathbf{Z})$ defined by $\theta(k) = f_k$ is an isomorphism. [See (3) to (6) of Theorem 2.5.3]

By Theorem 2.5.5, if f is any homomorphism of $(\mathbf{Z}, +)$ into $(\mathbf{Z}, +)$, then $f(x) = f_m(x) = mx$ for some integer m. [See (3) of Theorem 2.5.5]

Theorem 2.5.6. *Every integral domain can be imbedded in a field.* [D.U., 1996]

Proof. Let D be any integral domain. We take

$$M = \{(a, b) : a, b \in D \text{ and } b \neq 0\}.$$

We define a relation $\sim$ on M as follows :

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc. \qquad ...(1)$$

We proceed to show that $\sim$ an equivalence relation. We shall use the fact that D is commutative and cancellation laws hold in D.

(*i*) $(a, b \sim (a, b)$, since $ab = ba$ is true in D.

(*ii*) Let $(a, b) \sim (c, d)$. Then by (1), we have

$$ad = bc \Rightarrow cb = da \Rightarrow (c, d) \sim (a, b).$$

(*iii*) Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then by (1), we have

$ad = bc$ and $cf = de \Rightarrow bcf = bde \Rightarrow adf = bde \Rightarrow daf = dbe$

$\Rightarrow$ $af = be$, since $d \neq 0$ and cancellation law holds in D

$\Rightarrow$ $(a, b) \sim (e, f)$.

Thus $\sim$ is an equivalence relation on M. Let $[a, b]$ denote the equivalence class of $(a, b) \in M$ *i.e.*,

$$[a, b] = \{(x, y) \in M : (x, y) \sim (a, b)\}.$$

Clearly, $[a, b] = [c, d] \Leftrightarrow (a, b) \sim (c, d) \Leftrightarrow ad = bc.$ ...(2)

Let $F = \{[a, b] : a, b \in D \text{ and } b \neq 0\}$.

We shall prove that F is a field w.r.t. the compositions defined as

$$[a, b] + [c, d] = [ad + bc, bd], \quad \ldots(3)$$

$$[a, b]\,[c, d] = [ac, bd], \quad \ldots(4)$$

[Think of $[a, b]$ as $\frac{a}{b}$, so that

$$[a, b] + [c, d] = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad [a, b]\,[c, d] = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.]$$

Firstly, we need to show that these compositions are well-defined i.e., to show

$$[a, b] = [a', b'] \text{ and } [c, d] = [c', d'] \quad \ldots(5)$$

imply $[ad + bc, bd] = [a'd' + b'c', b'd']$ and $[ac, bd] = [a'c', b'd']$.

From the relations (5) and (2), we obtain

$$ab' = ba' \text{ and } cd' = dc' \quad \ldots(6)$$

We have $acb'd' = ab'cd'$, since D is commutative

$= ba'dc'$, using (6)

$= bda'c'$, since D is commutative.

Thus $acb'd' = bda'c' \Rightarrow [ac, bd] = [a'c', b'd']$, using (2).

Again, $(ad + bc)\,b'd' = adb'd' + bcb'd'$

$= ab'dd' + bb'cd'$, since D is commutative

$= ba'dd' + bb'dc'$, using (6)

$= bda'd' + bd\,b'c'$, since D is commutative

$= bd\,(a'd' + b'c')$.

This proves $[ad + bc, bd] = [a'd' + b'c', b'd']$, using (2).

Hence the compositions (3) and (4) are well-defined.

Let $[a, b]$, $[c, d]$, $[e, f]$ be any three elements of F.

Since $b \neq 0, d \neq 0 \in D$ and D is an integral domain, $bd \neq 0 \in D$.

It follows, by (3) and (4), that

$$[a, b] + [c, d] \in F \text{ and } [a, b]\,[c, d] \in F.$$

It is easy to verify that

(*i*) $[a, b] + [c, d] = [c, d] + [a, b]$, since D is commutative.

(*ii*) $\{[a, b] + [c, d]\} + [e, f] = [a, b] + \{[c, d] + [e, f]\}$.

(*iii*) There exists an element $[0, p] \in F$ such that

$$[a, b] + [0, p] = [a, b]\ \forall\ [a, b] \in F,$$

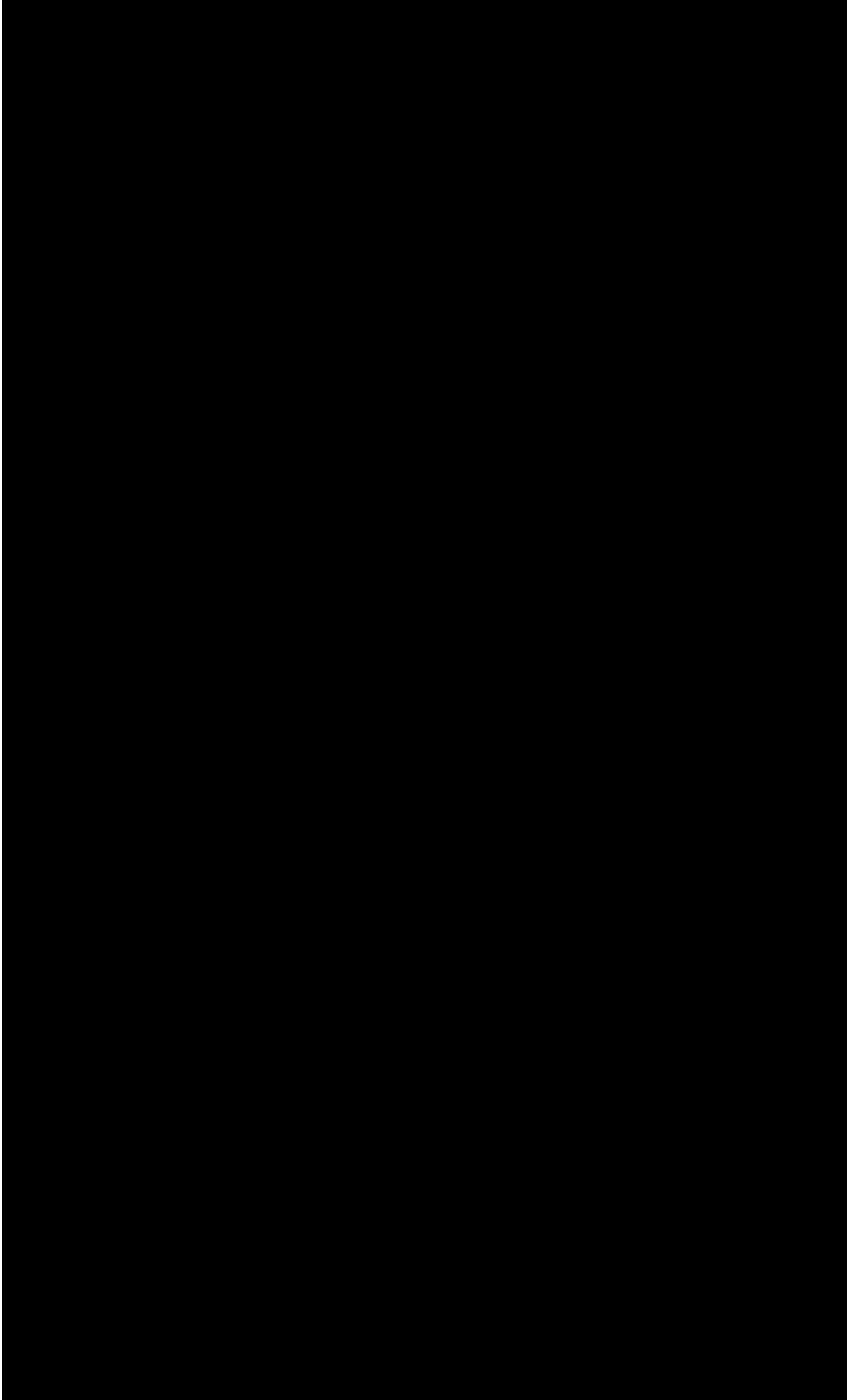
for $[a, b] + [0, p] = [ap + b.0, bp] = [ap, bp] = [a, b]$, using (3) and (2).

[Notice that $(ap)\,b = (bp)a$, since D is commutative
$\therefore \quad [ap, bp] = [a, b]$, by (2).]

(*iv*) The negative of $[a, b] \in F$ is $[-a, b] \in F$,

for $[a, b] + [-a, b] = [ab - ba, b^2] = [0, b^2] = [0, p]$, using (3) and (2).

[Notice that $0 \cdot p = b^2 \cdot 0 \Rightarrow [0, b^2] = [0, p]$, by (2)]



(v) $[a, b]\,[c, d] = [ac, bd] = [ca, db] = [c, d]\,[a, b]$, by (4).

(vi) $\{[a, b]\,[c, d]\}\,[e, f] = [ac, bd]\,[e, f]$, by (4)
$= [(ac)\,e, (bd)\,f]$, by (4)
$= [a\,(ce), b\,(df)]$, by associative law in D
$= [a, b]\,[ce, df]$, by (4)
$= [a, b]\,\{[c, d]\,[e, f]\}$.

(vii) For any $p \neq 0 \in D$, $[p, p]$ is the unity of F, since
$[a, b]\,[p, p] = [ap, bp] = [a, b]$, using (4) and (2).

[Notice that $(ap)b = (bp)\,a$ is true, as D is commutative
$\therefore \quad [ap, bp] = [a, b]$, by (2).]

(viii) Each non-zero element of F has its multiplicative inverse in F.
Let $[a, b] \neq [0, p]$. Then $a \neq 0$ and so $[b, a] \in F$. We have
$$[a, b]\,[b, a] = [ab, ba] = [p, p]\ ,\ \text{by (4) and (2).}$$

[Notice that $(ab)\,p = (ba)\,p$ is true, as D is commutative.
$\therefore \quad [ab, ba] = [p, p]$.]

Thus $[b, a]$ is the multiplicative inverse of $[a, b] \neq [0, p]$.

(ix) $[a, b]\,\{[c, d] + [e, f]\} = [a, b]\,[c, d] + [a, b]\,[e, f]$.
This property can be easily verified by using (3) and (4).
Hence F is a field. Finally, we show that D can be imbedded in F.
We define a mapping
$$f : D \to F \text{ as } f(a) = [ap, p]\ \forall\ a \in D. \qquad \ldots(7)$$

Here $p \neq 0 \in D$ is a fixed, but an arbitrary element of D.
Then f is well-defined, since
$$a = b \Rightarrow ap^2 = bp^2 \Rightarrow [ap, p] = [bp, p], \text{ by (2).}$$

Also f is one-to-one, since
$$f(a) = f(b) \Rightarrow [ap, p] = [bp, p], \text{ by (7)}$$
$\Rightarrow \quad ap \,.\, p = b \,.\, bp$, by (2)
$\Rightarrow \quad ap^2 = bp^2$, since D is commutative
$\Rightarrow \quad a = b$, since $p^2 \neq 0$ and cancellation law holds in D.

Now we show that f is a homomorphism. Let $a, b \in D$. Then
$$f(a + b) = [(a + b)\,p, p], \text{ by (7)}$$
$$= [(a + b)\,p^2, p^2], \text{ using (2)}$$
$$= [ap^2 + bp^2, p^2]$$
$$= [ap, p] + [bp, p], \text{ by (3)}$$
$$= f(a) + f(b), \text{ by (7).}$$

Again $f(ab) = [(ab)p, p]$, by (7)
$= [(ab)p^2, p^2]$, by (2)
$= [ap \cdot bp, p.p]$, since D is commutative
$= [ap, p]\,[bp, p]$, by (4)
$= f(a)\,f(b)$.

It follows that f is an isomorphism of D into F. Hence the given integral domain D is imbedded in the field F.

The field $F = \{[a, b] : a, b \in D \text{ and } b \neq 0\}$ is called **field of quotients** or **quotient field** of the integral domain D.

Definition. *The* **quotient field** *of an integral domain D may be defined as a pair (F, f), where F is a field and $f : D \to F$ is an isomorphism such that each $x \in F$ is expressible as the equivalence class $[f(a), f(b)]$ or $\frac{f(a)}{f(b)}$ for some $a, b \in D$ and $b \neq 0$.*

Remark. *The field of quotients of the integral domain* **Z** *of integers is* **Q** *(all rational numbers).*

Ex. 1. *Define the quotient field of an integral domain. Prove that every integral domain can be imbedded in its quotient field.* [D.U., 1996]

Ex. 2. *Show that the quotient field of the integral domain* **E** *of even integers is* **Q**.

Ex. 3. *Find the field of quotients of the integral domain* $3\mathbf{Z}$.

(Ans. Q)

Ex. 4. *Show that the quotient field of the integral domain* $\mathbf{Z}[i] = \{a + bi : a, b \in \mathbf{Z}\}$ *is* $K = \{x + y\,i : x, y \in \mathbf{Q}\}$.

Solution. If F be the quotient field of $\mathbf{Z}[i]$, then $\alpha \in F$ implies

$$\alpha = \frac{a+bi}{c+di}\ (a, b, c, d \in \mathbf{Z} \text{ and } c + di \neq 0) \Rightarrow \alpha = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

$$\Rightarrow \alpha = p + qi, \text{ where } p = \frac{ac+bd}{c^2+d^2} \in \mathbf{Q},\ q = \frac{bc-ad}{c^2+d^2} \in \mathbf{Q}$$

$\Rightarrow \alpha \in K \Rightarrow F \subseteq K$.

Conversely, any $\beta \in K$ is expressible as

$$\beta = \frac{a_1}{b_1} + \frac{a_2}{b_2}\,i\ ;\ a_1, a_2, b_1, b_2 \text{ are integers and } b_1 \neq 0, b_2 \neq 0$$

$$\Rightarrow \quad \beta = \frac{a_1 b_2 + b_1 a_2\, i}{b_1 b_2 + 0i} \in F \Rightarrow K \subseteq F. \text{ Hence } F = K.$$

Ex. 5. *Show that the quotient field of the integral domain* $\mathbf{Z}[\sqrt{2}] = \{m + n\sqrt{2} : m, n \in \mathbf{Z}\}$ *is* $\{x + y\sqrt{2} : x, y \in \mathbf{Q}\}$.

Hint. Proceed like Ex. 4.

Ex. 6. *Show that* **R** *(field of real numbers) can be imbedded in* **C** *(field of complex numbers).*

Hint. $f : \mathbf{R} \to \mathbf{C}$ defined as $f(x) = x + 0i\ \forall\ x \in \mathbf{R}$ is one-to-one and homomorphism.

Ex. 7. *Prove that the mapping $\phi : D \to F$ defined by $\phi(a) = [a, 1]$ is an isomorphism of D into F.*

Ex. 8. *Let D be an integral domain; $a, b \in D$. Suppose that $a^n = b^n$ and $a^m = b^m$ for two relatively prime integers m and n. Prove that $a = b$.*

[D.U., 2000, 1995]

Solution. Let $a = 0$. Then $a^n = 0$ and so $b^n = 0$ ($\because$ $a^n = b^n$).

Now $b^n = 0 \Rightarrow b.b...b$ (n times) $= 0$.

Thus $b = 0$, since D is an integral domain.

$\therefore$ $a = b$. Similarly, if $b = 0$, then $a = 0$ and so $a = b$.

We now consider the case when $a \neq 0$ and $b \neq 0$.

Since D is an integral domain, D can be imbedded in a field F.

Let $f: D \to F$ be an isomorphism. We have

$$\begin{aligned}\{f(a)\}^n &= f(a) \,.\, f(a) \ldots f(a) \ (n \text{ times}) \\ &= f(a \,.\, a \ldots a), \quad \text{since } f \text{ is a homomorphism} \\ &= f(a^n) = f(b^n), \text{ since } a^n = b^n \\ &= f(b \,.\, b \ldots b) \\ &= f(b) \,.\, f(b) \ldots f(b) \ (n \text{ times}), \text{ as } f \text{ is a homo.}\end{aligned}$$

$$\therefore \quad \{f(a)\}^n = \{f(b)\}^n. \qquad \ldots(1)$$

Similarly, $\{f(a)\}^m = \{f(b)\}^m$. $\qquad \ldots(2)$

Since n and m are relatively prime integers, there exist two integers p and q such that $np + mq = 1$. Now

$$\begin{aligned}f(a) &= \{f(a)\}^{np+mq} = [\{f(a)\}^n]^p \,.\, [\{f(a)\}^m]^q \\ &= [\{f(b)\}^n]^p \,.\, [\{f(b)\}^m]^q, \text{ by (1) and (2)} \\ &= \{f(b)\}^{np} \,.\, \{f(b)\}^{mq} = \{f(b)\}^{np+mq} = f(b).\end{aligned}$$

$$\therefore \quad f(a) = f(b) \Rightarrow a = b, \text{ since } f \text{ is one-to-one.}$$

Remark 1. In the relation $np + mq = 1$, one of the integers p and q is necessarily negative. If p is negative, then $p = -l$, for some postive integer l. Consequently, $a^p = (a^{-1})^l$, which may not exist in D. However, F being a field,

$$\{f(a)\}^p = [\{f(a)\}^{-1}]^l \in F, \quad \text{as } f(a) \neq 0 \in F.$$

Remark 2. *The conclusion of the above problem may not hold, if D is not an integral domain.*

The ring $\mathbf{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ of integers modulo 8 is not an integral domain, since $2 \neq 0$ and $4 \neq 0$, but $2 \otimes_8 4 = 0 \in \mathbf{Z}_8$.

We take $a = 2$, $b = 4 \in \mathbf{Z}_8$, $m = 3$, $n = 4$.

Then m and n are relatively prime integers satisfying

$a^m = 2^3 = 0$ in $\mathbf{Z}_8$, $b^m = 4^3 = 0$ in $\mathbf{Z}_8$;

$a^n = 2^4 = 0$ in $\mathbf{Z}_8$, $b^n = 4^4 = 0$ in $\mathbf{Z}_8$.

$\therefore$ $a^m = b^m$ and $a^n = b^m$, $(n, m) = 1$, but $a \neq b$.

Theorem 2.5.7. *Let F be the field of quotients of an integral domain D. Prove that if K is any field which contains D, then K contains a subfield isomorphic to F.*

Or

Show that the quotient field F of an integral domain D is the smallest field containing D.

Proof. We have $D \subseteq K$. The quotient field of D is

$$F = \{[a, b] : a, b \in D \text{ and } b \neq 0\}.$$

Let $[a, b] \in F$ be arbitrary. Then $a \in D, b \neq 0 \in D$ imply $a \in K$ and $b \neq 0 \in K \Rightarrow ab^{-1} \in K$, since K is a field.

Define a mapping $f: F \to K$ as

$$f\{[a, b]\} = ab^{-1} \quad \forall \quad [a, b] \in F. \qquad \ldots(1)$$

Then f is well-defined, since

$$[a, b] = [c, d] \Rightarrow ad = bc \Rightarrow b^{-1}a = cd^{-1} \Rightarrow ab^{-1} = cd^{-1}$$

[Notice that $b \neq 0, d \neq 0 \in D \Rightarrow b \neq 0, d \neq 0 \in K \Rightarrow b^{-1}, d^{-1} \in K$]

Further f is one-to-one, since

$$f\{[a, b]\} = f\{[c, d]\} \Rightarrow ab^{-1} = cd^{-1}, \text{ by (1)}$$

$$\Rightarrow \quad b^{-1}a = cd^{-1}, \text{ since } K \text{ is commutative}$$

$$\Rightarrow \quad ad = bc \Rightarrow [a, b] = [c, d].$$

Lastly, we show f is a homomorphism.

Let $[a, b], [c, d] \in F$. Then

$$[a, b] + [c, d] = [ad + bc, bd], \quad [a, b]\,[c, d] = [ac, bd].$$

Using (1), we have

$$\begin{aligned} f\{[a, b] + [c, d]\} &= (ad + bc)(bd)^{-1} \text{ in } K \\ &= (ad + bc)(d^{-1}b^{-1}) \text{ in } K \\ &= add^{-1}b^{-1} + bcd^{-1}b^{-1} \\ &= ab^{-1} + cd^{-1}, \text{ since } K \text{ is commutative} \\ &= f\{[a, b]\} + f\{[c, d]\}, \text{ by (1).} \end{aligned}$$

$$\begin{aligned} f\{[a, b]\,[c, d]\} &= f\{[ac, bd]\} = (ac)(bd)^{-1}, \text{ by (1)} \\ &= ac\,d^{-1}b^{-1} = ab^{-1}\,cd^{-1}, \text{ as } K \text{ is commutative} \\ &= f\{a, b]\}\,f\{[c, d]\}, \text{ by (1)} \end{aligned}$$

Thus $f: F \to K$ is an isomorphism. Since F is a field, $f(F)$ is a subfield of K. Consequently,

$$f: F \to f(F) \text{ is 1-1, onto and homomorphism.}$$

$\therefore$ $F \approx f(F)$ and $f(F)$ is a subfield of K.

Hence K contains a subfield $f(F)$ which is isomorphic to F.

In this sense, we say that F is the smallest field containing D.