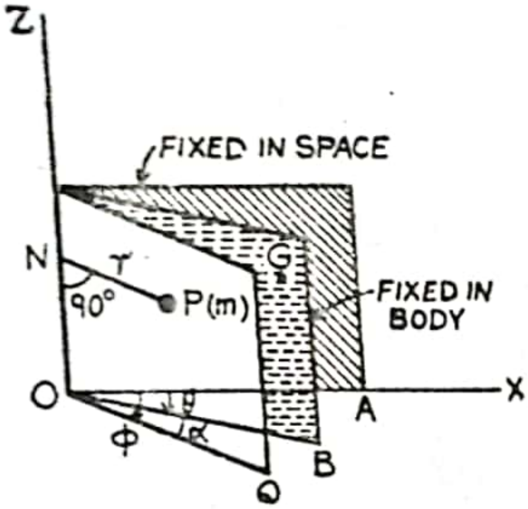
4

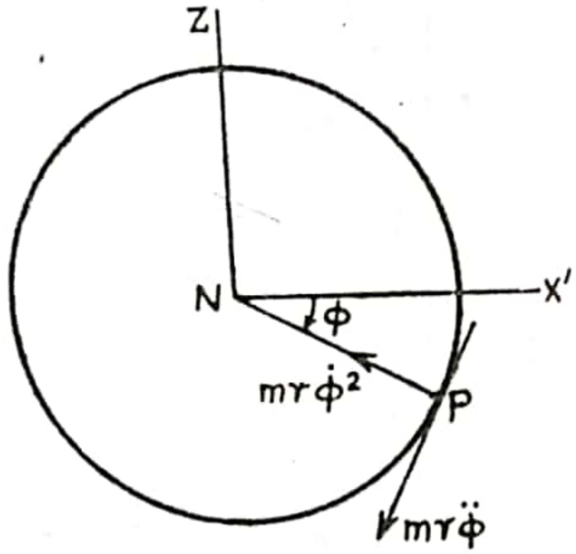
Motion About a Fixed Axis

§ 4·1. Moment of the Effective Forces about the Axis of Rotation.

[Meerut 83, 89, 90]

Let a rigid body rotate about the fixed axis OZ, perpendicular to the plane of the paper. Let a plane ZOA through OZ and fixed in space, cutting the plane of the paper along OA, be taken as the plane of reference. Let a plane ZOB through the axis OZ and the centre of gravity G of the body be fixed in the body.





Consider an element of mass m at the point P of the body and let the plane ZOQ through OZ and P cut the plane of the paper along OQ. Let the plane ZOQ make an angle α with the plane ZOB. Obviously when the body rotates about OZ, α remains constant.

At time t, let the planes ZOB and ZOQ make angles θ and ϕ respectively with the plane ZOA, *i.e.* $\angle AOB=\theta$ and $\angle AOQ=\phi$.

$\therefore\quad \theta+\alpha=\phi$, so that $\dot{\theta}=\dot{\phi}$ and $\ddot{\theta}=\ddot{\phi}$.

Let PN be the perpendicular from P to OZ and $PN=r$. Then as the body rotates about OZ, the point P describes a circle of radius r about N as centre. The plane of this circle is perpendicular to OZ.

The components of the acceleration of mass m at P are $r\dot{\phi}^2$ and $r\ddot{\phi}$ along and perpendicular to PN respectively.

Therefore the effective forces on the particle of mass m at P are $mr\dot{\phi}^2$ and $mr\ddot{\phi}$ along and perpendicular to PN.

But $\dot{\phi}=\dot{\theta}$ and $\ddot{\phi}=\ddot{\theta}$. Therefore the effective forces on the particle m are $mr\dot{\theta}^2$ and $mr\ddot{\theta}$ along and perpendicular to PN.

The effective force $mr\dot{\theta}^2$ cuts the axis OZ at N, so its moment about OZ will be zero.

$\therefore$ Moment of the effective forces on m about OZ is

$$NP.mr\ddot{\theta}=r.mr\ddot{\theta}=mr^2\ddot{\theta}.$$

Hence the moment of the effective forces of the whole body about OZ is

$$\Sigma\, mr^2\ddot{\theta}=\ddot{\theta}\Sigma\, mr^2=Mk^2\ddot{\theta},$$

where k is the radius of gyration of the body about OZ.

§ 4·2. Equation of Motion of the Body about the Axis of Rotation.

[Meerut 79, 84, 85, 86; Raj. 81]

When a body rotates about a fixed axis say OZ, the impressed forces include, besides the external forces, the reaction on the axis of rotation OZ. The moment of the reaction on the axis OZ about OZ will be zero. Since by D' Alembert's principle the reversed effective forces and the impressed forces are in equilibrium, therefore the algebraic sum of their moments about OZ will be zero

i.e., Moment of the effective forces about OZ

$=$Moment of all the external forces about OZ

or $$Mk^2\ddot{\theta}=L,$$

where L is the moment of all the external forces about the axis of rotation OZ.

This equation is called the equation of motion of the body about the axis of rotation.

§ 4·3. Moment of Momentum about the Axis of Rotation.

[Meerut 82; Raj. 81]

See the figures in § 4·1. The velocity of the particle m at P is $r\dot{\phi}$ perpendicular to NP.

$\therefore$ The moment of momentum of the particle m at P about $OZ=NP.(mr\dot{\phi})=r.(mr\dot{\phi})=mr^2\dot{\theta}$. $\quad (\because\ \dot{\phi}=\dot{\theta})$

$\therefore$ Moment of momentum of the whole body about OZ is

$$\Sigma\, mr^2\dot{\theta}=\dot{\theta}\Sigma\, mr^2=Mk^2\dot{\theta},$$

where k is the radius of gyration of the body about OZ.

§ 4·4. Kinetic Energy.

[Meerut 83; Raj. 81]

Refer the figure in § 4·1. The velocity of the particle m at P is $r\dot{\phi}$ perpendicular to NP.

$\therefore$ The kinetic energy of the particle m at P

$$=\tfrac{1}{2}m\,(r\dot{\phi})^2=\tfrac{1}{2}mr^2\dot{\theta}^2 \qquad (\because\ \dot{\phi}=\dot{\theta})$$

$\therefore$ Kinetic energy of the whole body is

$$\Sigma\,\tfrac{1}{2}mr^2\dot{\theta}^2=\tfrac{1}{2}\dot{\theta}^2\,\Sigma\,mr^2=\tfrac{1}{2}Mk^2\dot{\theta}^2,$$

where k is the radius of gyration of the body about OZ.

Solved Examples

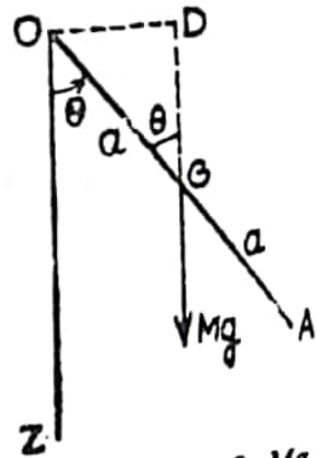
Ex. 1. *A uniform rod, of mass M and length $2a$, can turn freely about one end which is fixed. It is started with angular velocity ω from the position in which it hangs vertically. Then*

(i) *find its angular velocity at any instant,* [Meerut 80, 81, 82]

(ii) *find the least angular velocity with which it must begin to move so that it may perform complete revolution in a vertical plane.*

and (iii) *find the time of describing an angle θ from its vertical position in case (i).*

Sol. (i) Let the uniform rod OA of length $2a$ and mass M turn freely about one end O which is fixed. At time t, let the rod be inclined at an angle θ to the vertical OZ.



Here the axis of rotation is the horizontal line through O perpendicular to the plane in which the rod rotates, OZ is the line fixed in space and OA is the line fixed in the body.

Taking moments about O, the equation of motion is

$$Mk^2\ddot{\theta}=-Mg.OD$$

(Negative sign is taken on R. H. S. as the moment of Mg is in the direction of θ decreasing)

or $\quad \tfrac{4}{3}Ma^2\ddot{\theta}=-Mg.a\sin\theta \quad [\because\ k^2=\tfrac{4}{3}a^2]$

or $\quad \ddot{\theta}=-(3g/4a)\sin\theta.$

Multiplying both sides by $2\dot{\theta}$ and integrating, we have

$$\dot{\theta}^2=(3g/2a)\cos\theta+C,$$

where C is the constant of integration.

But initially, when $\theta=0$, $\dot{\theta}=\omega$.

$\therefore\quad \omega^2=(3g/2a)+C \quad$ or $\quad C=\omega^2-(3g/2a).$

$$\therefore\quad \dot{\theta}^2=\omega^2-\frac{3g}{2a}(1-\cos\theta), \qquad \dots(1)$$

which gives the angular velocity of the rod at any instant.

(ii) The least angular velocity ω with which the rod must begin to move to perform a complete revolution in the vertical plane is that for which $\dot{\theta}=0$ when $\theta=\pi$.

Therefore, from (1), we have $0=\omega^2-(3g/2a)(1+1)$

or $$\omega=\sqrt{(3g/a)},$$

which is the least angular velocity with which the rod must begin to move to perform a complete revolution in the vertical plane.

(iii) If $\omega=\sqrt{\left(\frac{3g}{a}\right)}$, then from (1), we have

$$\dot{\theta}^2=\frac{3g}{a}-\frac{3g}{2a}.2\sin^2\tfrac{1}{2}\theta=\frac{3g}{a}(1-\sin^2\tfrac{1}{2}\theta)=\frac{3g}{a}\cos^2\tfrac{1}{2}\theta$$

or $$\dot{\theta}=\frac{d\theta}{dt}=\sqrt{\left(\frac{3g}{a}\right)}\cos\tfrac{1}{2}\theta$$

(+ve sign is taken on R.H.S. as θ increases with increase in t)

or $$dt=\sqrt{\left(\frac{a}{3g}\right)}\sec\tfrac{1}{2}\theta\, d\theta.$$

If t is the time of describing an angle θ, then integrating, we have

$$t=\sqrt{\left(\frac{a}{3g}\right)}\int_0^\theta \sec\tfrac{1}{2}\theta\, d\theta$$
$$=\sqrt{\left(\frac{a}{3g}\right)}\left[2\log\tan\left(\frac{\theta}{4}+\frac{\pi}{4}\right)\right]_0^\theta$$
$$=2\sqrt{\left(\frac{a}{3g}\right)}\log\tan\left(\frac{\theta}{4}+\frac{\pi}{4}\right).$$

Ex. 2. *A fine string has two masses M and M' tied to its ends and passes over a rough pulley of mass m, whose centre is fixed. If the string does not slip over the pulley, show that M will descend with acceleration $\dfrac{M-M'}{M+M'+(mk^2/a^2)}g$, where a is the radius and k the radius of gyration of the pulley.*

[Meerut 85, 87, 89; Kanpur 81; Rohilkhand 83]

If the pulley be not sufficiently rough to prevent sliding and M be the descending mass, show that its acceleration is $\dfrac{M-M'e^{\mu\pi}}{M+M'e^{\mu\pi}}g$ and that the pulley will now spin with an angular acceleration equal to $\dfrac{2MM'ga(e^{\mu\pi}-1)}{mk^2(M+M'e^{\mu\pi})}$.

Sol. Let in time t, the mass M move through a distance x

downwards and the mass M' move through a distance x upwards. If θ be the angle turned by the pulley during this time, then

$x=a\theta$, so that $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$. ...(1)

If T_1 and T_2 are the tensions in the two parts of the string, then the equations of motion of M and M' are

$$M\ddot{x}=Mg-T_1 \quad ...(2)$$

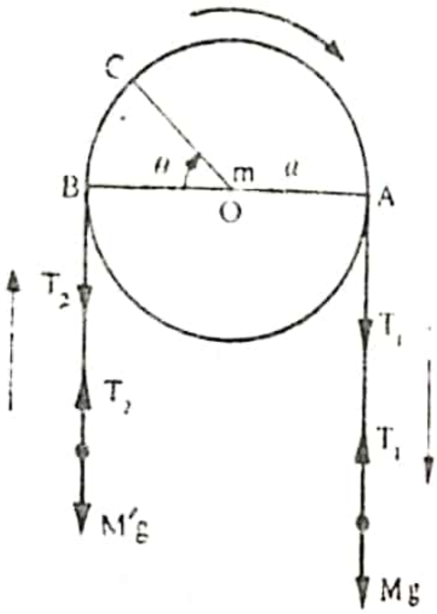
and
$$M'\ddot{x}=T_2-M'g. \quad ...(3)$$

Taking moments about the fixed horizontal axis through O, the equation of motion of the pulley is

$$mk^2\ddot{\theta}=T_1.a-T_2.a \quad ...(4)$$

or
$$\frac{mk^2}{a^2}\ddot{x}=T_1-T_2 \quad ...(5)$$

$[\because \ \ddot{\theta}=\ddot{x}/a$, from (1)]



Adding (2), (3) and (5), we have

$$[M+M'+(mk^2/a^2)]\ \ddot{x}=(M-M')g$$

or
$$\ddot{x}=\frac{M-M'}{M+M'+(mk^2/a^2)}\ g,$$

which is the required acceleration.

Second part. When the pulley is not rough enough to prevent sliding then the equation (1) does not hold.

If μ is the coefficient of friction, then in this case we have

$$T_1=T_2\ e^{\mu\pi} \quad ...(6)$$

Solving (2), (3) and (6), we have

$$T_1=\frac{2MM'\ g\ e^{\mu\pi}}{M+M'\ e^{\mu\pi}},\ T_2=\frac{2MM'\ g}{M+M'\ e^{\mu\pi}}$$

and
$$\ddot{x}=\frac{M-M'\ e^{\mu\pi}}{M+M'\ e^{\mu\pi}}\ g,$$

which gives the acceleration of the descending mass M.

Substituting the values of T_1 and T_2 in (4), we have

$$\ddot{\theta}=\frac{2MM'\ ga\ (e^{\mu\pi}-1)}{mk^2\ (M+M'\ e^{\mu\pi})},$$

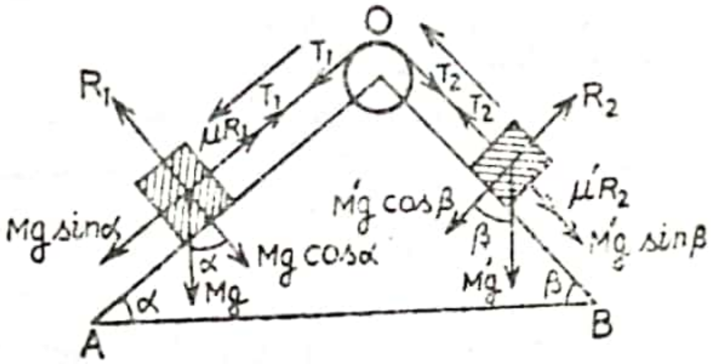
which gives the angular acceleration of the pulley.

Ex. 3. *Two unequal masses, M and M', rest on two rough planes inclined at angles α and β to the horizontal; they are connected by a fine string passing over a small pulley, of mass m and radius* *a, which is placed at the common vertex of the two planes; show that the acceleration of either mass is*

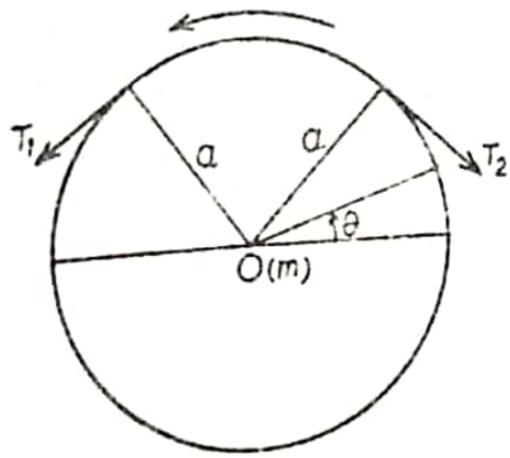
$$\frac{g\,[M(\sin\alpha-\mu\cos\alpha)-M'(\sin\beta+\mu'\cos\beta)]}{M+M'+(mk^2/a^2)}$$

where μ and μ' are the coefficients of friction, k the radius of gyration of the pulley about its axis and M the mass which moves downwards.

Sol.



Let the mass M move a distance x downwards and M' the same distance x upwards in time t. If the pulley turns through an angle θ in this time t, then $x=a\theta$, so that $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$.



Now by Newton's second law of motion the equations of motion of the masses M and M' are

$$M\ddot{x}=Mg\sin\alpha-\mu R_1-T_1, \quad \text{...(1)}$$

and

$$M'\ddot{x}=T_2-M'g\sin\beta-\mu'R_2. \quad \text{...(2)}$$

Also taking moments about the axis of rotation through the centre O of the pulley, the equation of motion of the pulley is

$$mk\ddot{\theta}^2=T_1a-T_2a$$

or

$$\frac{mk^2}{a^2}\ddot{x}=T_1-T_2. \quad \text{...(3)}$$

$[\because \ \ddot{\theta}=\ddot{x}/a]$

Adding (1), (2) and (3), we have

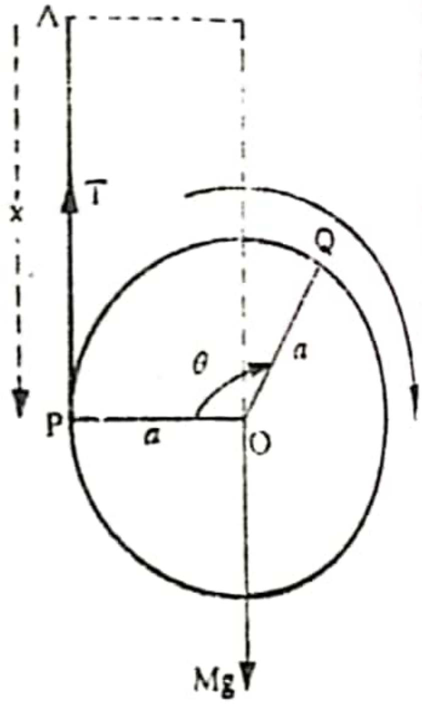
$$\left(M+M'+\frac{mk^2}{a^2}\right)\ddot{x}=Mg\sin\alpha-\mu Mg\cos\alpha-M'g\sin\beta-\mu'M'g\cos\beta$$

$(\because \ R_1=Mg\cos\alpha$ and $R_2=M'g\cos\beta)$

or $\quad \ddot{x}=\dfrac{g\,[M\,(\sin\alpha-\mu\cos\alpha)-M'\,(\sin\beta+\mu'\cos\beta)]}{M+M'+(mk^2/a^2)}$,

which is the required acceleration of either of the masses M or M'.

Ex. 4. *A vertical thread unwounds itself from a reel, the upper end of the thread being fixed. Prove that the downwards acceleration of the reel is* $\dfrac{a^2}{a^2+k^2}g$ *and the tension of the thread is* $\dfrac{k^2}{a^2+k^2}Mg$, *where* M *is the mass of the reel,* a *its radius, and* k *its radius of gyration about its axis.*



Sol. Let the upper end of the thread be fixed to a point A. Initially the centre O of the reel will be in a horizontal line through A. Since there is no horizontal force, the centre of gravity O of the reel will move vertically downwards. In time t, let x be the distance moved by the C.G. O vertically downwards and θ the tension in the string then the equation of motion of the C.G. of the reel is

$$M\ddot{x}=Mg-T. \qquad \ldots(1)$$

Taking moments about the horizontal axis through O, the equation of motion of the reel is

$$Mk^2\,\ddot{\theta}=T.a \qquad \ldots(2)$$

Also $x=a\theta$, so that $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$.

$\therefore$ from (2), we have

$$T=\frac{Mk^2\ddot{x}}{a^2}. \qquad \left[\because\ \ddot{\theta}=\frac{\ddot{x}}{a}\right]$$

Substituting in (1), we have

$$M\ddot{x}=Mg-\frac{Mk^2}{a^2}\ddot{x}.$$

$$\therefore\quad \ddot{x}=\frac{a^2}{a^2+k^2}g,$$

which is the required acceleration of the reel.

Also $T=\dfrac{Mk^2}{a^2}\ddot{x}=\dfrac{Mk^2}{a^2}\cdot\dfrac{a^2}{a^2+k^2}g=\dfrac{k^2}{a^2+k^2}Mg$,

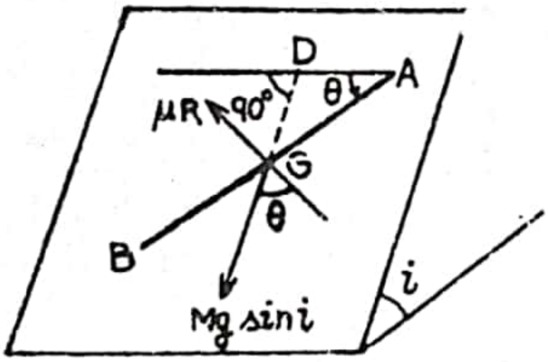
which is the required tension of the thread.

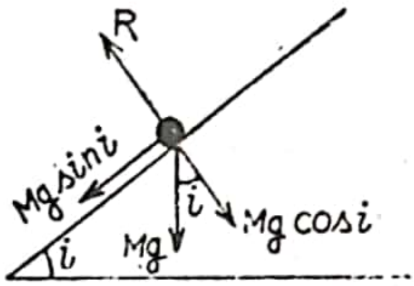
Ex. 5. *A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizontal is i and whose coefficient friction is* μ, *about a smooth pin fixed through the end A ; the bar is held in the horizontal position in the plane and allowed to fall from this position. If* θ *be the angle through which it falls from rest, show that*

$$\frac{\sin\theta}{\theta}=\mu\cot i.$$

[Meerut 83, 86, 88, 90; Kanpur 82; Raj. 82]

Sol.





Let AB be a rod movable on a rough inclined plane about the end A. Let the rod fall through an angle θ from its orginal horizontal position, in time t. If R is the reaction of the plane, then $\quad R=Mg\cos i.$

The external forces acting on the rod in the inclined plane are (i) the resolved part $Mg\sin i$ of its weight Mg acting downwards along the line of greatest slope, (ii) the frictional force μR, acting perpendicular to AB at its middle point G and (iii) the reaction at A

To avoid the reaction at A, taking moments about A, we have

$$Mk^2\ddot{\theta}=Mg\sin i.AD-\mu R.AG$$
$$=Mg\sin i.a\cos\theta-\mu Mg\cos i.a$$

or $$k^2\ddot{\theta}=ag(\sin i\cos\theta-\mu\cos i).$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we have

$$k^2\dot{\theta}^2=2ag(\sin i.\sin\theta-\mu\theta\cos i)+c.$$

But initially when $\theta=0$, $\dot{\theta}=0$; $\therefore\ c=0.$

$\therefore\quad k^2\dot{\theta}^2=2ag(\sin i\sin\theta-\mu\theta\cos i).$

The rod will come to rest when $\dot{\theta}=0$.

$\therefore\quad 0=2ag(\sin i\sin\theta-\mu\theta\cos i)$

or $\quad(\sin\theta)/\theta=\mu\cot i.$

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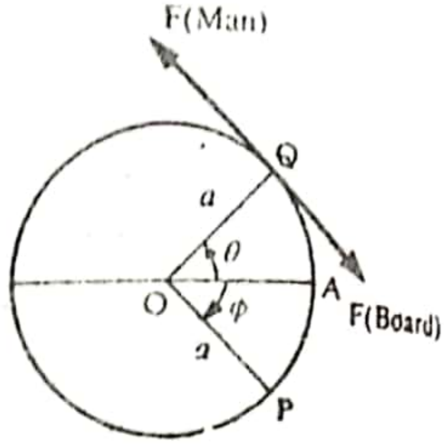
Ex. 6. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through the centre. A man whose weight is equal to that of the board walks on and around it at the edge. When he has completed the circuit, what will be his position in space ?*



Sol. As the man moves in the edge of the board, obviously the board will turn in the opposite direction.

In time t, let θ be the angle described by the man and ϕ be the angle described by the board in the opposite direction.

Let F be the frictional force between the feet of the man and the board acting along the tangent to the edge of the board in the horizontal plane. Let M be the mass of the board or that of the man.

$\therefore$ The equation of motion of the man is

$$Ma^2\ddot{\theta}=Fa \quad \text{or} \quad Ma\ddot{\theta}=F. \qquad ...(1)$$

Also the equation of motion of the board is

$$Mk^2\ddot{\phi}=Fa \quad \text{or} \quad M(\tfrac{1}{2}a^2)\,\ddot{\phi}=Fa$$

[$\because$ $k^2=a^2/2$ for the circular disc]

or $$M(\tfrac{1}{2}a)\,\ddot{\phi}=F. \qquad ...(2)$$

From (1) and (2), we have

$$Ma\ddot{\theta}=M(\tfrac{1}{2}a)\,\ddot{\phi} \quad \text{or} \quad 2\ddot{\theta}-\ddot{\phi}=0.$$

Integrating, we have $2\dot{\theta}-\dot{\phi}=c_1$.

But when $\dot{\theta}=0$, we have $\dot{\phi}=0$, so that $c_1=0$.

$\therefore$ $2\dot{\theta}-\dot{\phi}=0$.

Integrating again, we have $2\theta-\phi=c_2$.

But when $\theta=0$, we have $\phi=0$, so that $c_2=0$.

$\therefore$ $2\theta-\phi=0. \qquad ...(3)$

When the man completes the circuit at the edge of the board, we have

$$\theta+\phi=2\pi. \qquad ...(4)$$

Solving (3) and (4), we get $\theta=2\pi/3$,

which is the angle described by the man in space.

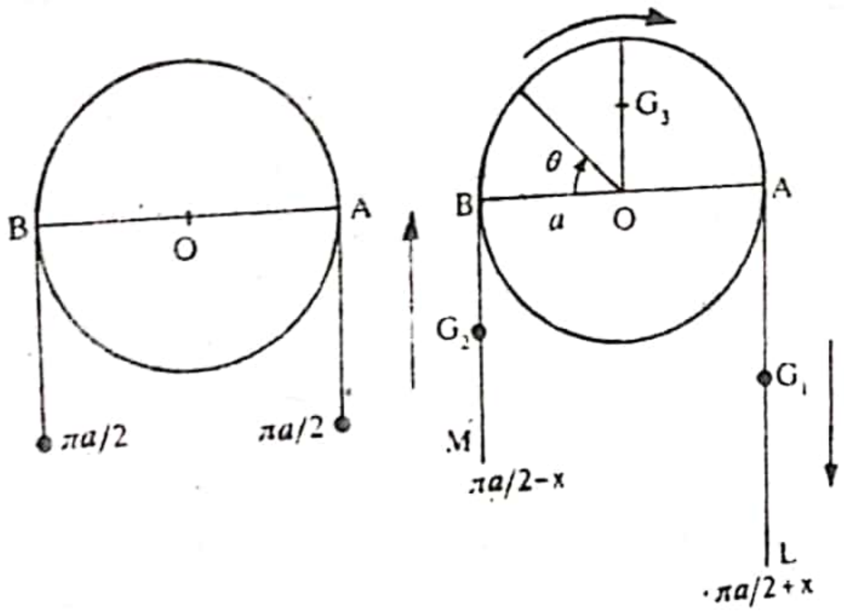
Remark. Solving (3) and (4), we have $\phi=4\pi/3$ *i.e.* when the man completes the circuit at the edge of the board, the board turns through an angle $4\pi/3$ in the opposite direction.

Ex. 7 *A uniform vertical circular plate of radius a, is capable of revolving about a smooth horizontal axis through its centre; a rough perfectly flexible chain, whose mass is equal to that of the plate and whose length is equal to its circumference hangs over its rim in equilibrium, if one end be slightly displaced, show that the velocity of the chain when the other end reaches the plate is*

$$\sqrt{(\pi a g/6)}.$$ **[Meerut 84; Raj. 83]**

Sol. Let x be the distance described by the chain and θ the angle turned by the plate in time t. Then if v is the velocity of the chain, we have

$$v=\dot{x}=a\dot{\theta}. \qquad ...(1)$$



Let M be the mass of the plate or that of the string.

The total K.E. of the system at time t

$=$K.E. of the chain$+$K.E. of the plate

$=\frac{1}{2}Mv^2+\frac{1}{2}Mk^2\dot{\theta}^2=\frac{1}{2}Mv^2+\frac{1}{2}M\,(a^2/2)\,(v/a)^2$ $\quad [\because \quad k^2=a^2/2]$

$=\frac{3}{4}Mv^2.$

Let w be the weight per unit length of the chain.

Then the weight of $AL=(\frac{1}{2}\pi a+x)\,w$, and the depth of its C.G. 'G_1' below $AB=\frac{1}{2}\,(\frac{1}{2}\pi a+x)$, the weight of $BM=(\frac{1}{2}\pi a-x)\,w$ and the depth of its C.G. 'G_2' below $AB=\frac{1}{2}\,(\frac{1}{2}\pi a-x)$.

and the weight of the semi-circular chain $AB = \pi a w$ and the height of its C.G. 'G_3' above $AB = 2a/\pi$.

$\therefore$ the depth x_1 of the C. G. of the chain below AB at time t is given by

$$x_1 = \frac{(\frac{1}{2}\pi a + x)w.\frac{1}{2}(\frac{1}{2}\pi a + x) + (\frac{1}{2}\pi a - x)w.\frac{1}{2}(\frac{1}{2}\pi a - x) + \pi a w.(-2a/\pi)}{(\frac{1}{2}\pi a + x)\,w + (\frac{1}{2}\pi a - x)\,w + \pi a w}$$

$$= \frac{a^2\pi^2 + 4x^2 - 8a^2}{8\pi a}.$$

Putting $x = 0$, the depth x_2 of the C.G. of the chain below AB at time $t = 0$ is given by

$$x_2 = \frac{a^2\pi^2 - 8a^2}{8\pi a}.$$

$\therefore$ the displacement of the C.G. of the chain in the vertical downward direction in time t

$$= x_1 - x_2 = \frac{4x^2}{8\pi a} = \frac{x^2}{2\pi a}.$$

$\therefore$ Work done by the system in time t

$=$ work done by the chain $+$ work done by the plate

$$= Mg\left(\frac{x^2}{2\pi a}\right) + 0.$$

Hence the energy equation gives,

the change in the K.E. $=$ the work done by the forces.

$$\therefore \quad \tfrac{3}{4} M v^2 = \frac{Mgx^2}{2\pi a} \quad \text{or} \quad v^2 = \frac{2gx^2}{3\pi a}, \qquad \ldots(2)$$

which gives the velocity of the chain in terms of x.

When the end M reaches the pulley, we have $x = \frac{1}{2}\pi a$.

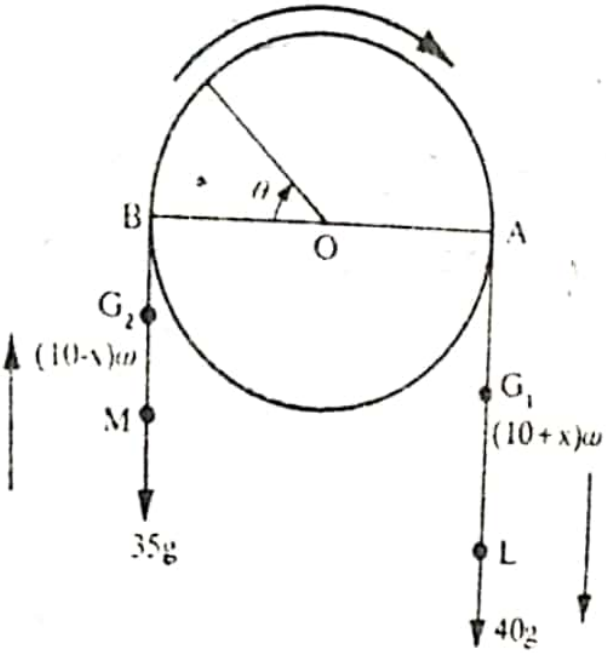
Putting $x = \frac{1}{2}\pi a$ in (2), the velocity v of the chain when one end of the chain reaches the pulley is given by

$$v^2 = \frac{2g}{3\pi a}(\tfrac{1}{2}\pi a)^2 = \frac{\pi a g}{6} \quad \text{or} \quad v = \sqrt{\left(\frac{\pi a g}{6}\right)}$$

Ex. 8. *A uniform string of length 20 feet and mass 40 lbs. hangs in equal length over circular pulley of mass 10 lbs. and small radius, the axis of the pulley being horizontal. Masses of 40 and 35 lbs. are attached to the ends of the string and motion takes place. Show that the time taken by the smaller mass to reach the pulley is*

$$\tfrac{1}{4}\sqrt{(15)}\,\log\,(9 + 4\sqrt{5})\ \textit{seconds.}$$

Sol. If w is the weight per unit length of the string, then proceeding as in Ex. 7, the depth x_1 of the C.G. of the string below AB at time t is given by



$$x_1=\frac{(10+x)w.\frac{1}{2}(10+x)+(10-x)w.\frac{1}{2}(10-x)}{(10+x)w+(10-x)w}$$

$$=\tfrac{1}{20}(100+x^2).$$

Putting $x=0$, the depth x_2 of the C.G. of the string below AB at time $t=0$ is given by

$$x_2=\tfrac{1}{20}(100).$$

$\therefore$ Vertical downward displacement of the C.G. of the string in time t

$$=x_1-x_2=\tfrac{1}{20}x^2.$$

$\therefore$ Work done by the system in time t

$=$ work done by the weights $+$ work done by the string

$=40gx-35gx+40g.(\tfrac{1}{20}x^2)=gx(5+2x).$

Also we have $x=a\theta$.

$\therefore$ Velocity v of the string $=\dot{x}=a\dot{\theta}$.

Now the K.E. of the system at time t

$$=\tfrac{1}{2}.40v^2+\tfrac{1}{2}.35v^2+\tfrac{1}{2}.40v^2+\tfrac{1}{2}.10k^2\dot{\theta}^2$$

$$=20v^2+\frac{35}{2}v^2+20v^2+5.\frac{a^2}{2}\left(\frac{v}{a}\right)^2=60v^2.$$

$\therefore$ The energy equation gives,

the change in the K.E. $=$ the work done by the forces.

$\therefore$ $60v^2=gx(5+2x)$

or
$$v=\frac{dx}{dt}=\frac{4}{\sqrt{(15)}}\sqrt{(x^2+\tfrac{5}{2}x)} \qquad [\because\ g=32]$$

or
$$dt=\frac{\sqrt{(15)}}{4}\frac{dx}{\sqrt{(x^2+\frac{5}{2}x)}}.$$

When the end M reaches the plate, we have $x=10$ feet.

$\therefore$ Integrating between the limits $x=0$ to $x=10$, the required time t is given by

$$t=\frac{\sqrt{(15)}}{4}\int_0^{10}\frac{dx}{\sqrt{(x^2+\frac{5}{2}x)}}=\frac{\sqrt{(15)}}{4}\int_0^{10}\frac{dx}{\sqrt{\{(x+\frac{5}{4})^2-(\frac{5}{4})^2\}}}$$

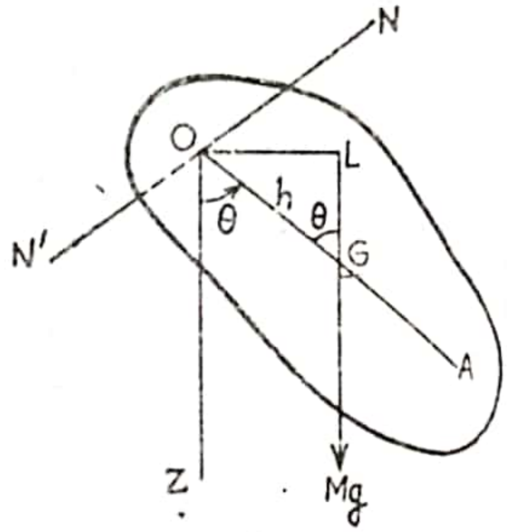
$$=\frac{\sqrt{(15)}}{4}\left[\log\left(x+\frac{5}{4}\right)+\sqrt{\left\{\left(x+\frac{5}{4}\right)^2-\left(\frac{5}{4}\right)^2\right\}}\right]_0^{10}$$

$$=\tfrac{1}{4}\sqrt{(15)}\log(9+4\sqrt{5}) \text{ seconds.}$$

COMPOUND PENDULUM

§ 4·5. Compound pendulum. (Def.). *A* **compound pendulum** *is a rigid body of any form and constitution, which is free to turn about a fixed horizontal axis.*

§ 4·6. Time of a complete small oscillation of a compound pendulum. **[Meerut 80, 82, 87, 89, 90]**



Let a body be free to turn about a fixed horizontal axis NON'. Let the vertical plane through the centre of gravity 'G' of the body meet the axis of rotation NON' at the point O. This plane is the plane of rotation of the line OG and let this plane be taken as the plane of the paper.

Let $OG=h$, so that h is the distance of the centre of gravity G of the body from the axis of rotation. Let OZ be the vertical line through O. At any time t, let $\angle GOZ=\theta$. Then θ is the inclination at time t of the plane through G and the fixed axis (*i.e.*, the plane fixed in the body) to the vertical plane through the fixed axis (*i.e.*, the plane fixed in space).

The impressed forces on the body are

(i) its weight Mg acting vertically downwards at G, and

(ii) the reaction of the fixed axis.

Let k be the radius of gyration of the body about the fixed axis.

To avoid the reaction of the fixed axis, taking moments about the fixed axis NON', we have

$$Mk^2\ddot{\theta}=-Mg.OL=-Mgh\sin\theta$$

($-$ve sign is taken as the moment of Mg is in the direction of θ decreasing)

or $\quad \ddot{\theta}=-(gh/k^2).\theta,\quad$ [taking $\sin\theta=\theta$, as θ is very small]

which shows that the motion of the body about the fixed axis is simple harmonic motion.

Hence the time of a complete oscillation of the compound pendulum is given by

$$T=\frac{2\pi}{\sqrt{(gh/k^2)}} \quad \text{or} \quad T=2\pi\sqrt{\left(\frac{k^2}{gh}\right)}.$$

§ 4·7. Simple Equivalent pendulum.

A simple pendulum having the same periodic time as the compound pendulum is called a simple equivalent pendulum.

If l is the length of a simple pendulum, then its time period

$$=2\pi\sqrt{(l/g)}.$$

The period of a compound pendulum$=2\pi\sqrt{(k^2/gh)}$.

If the simple pendulum of length l is the simple equivalent pendulum then

$$2\pi\sqrt{\left(\frac{l}{g}\right)}=2\pi\sqrt{\left(\frac{k^2}{gh}\right)}. \quad \therefore \quad l=\frac{k^2}{h}.$$

Thus k^2/h is the length of simple equivalent pendulum of a compound pendulum.

§ 4·8. Minimum time of oscillation of a compound pendulum.

[Meerut 80, 82, 90]

Let K be the radius of gyration of the body about a line through the centre of gravity G, parallel to the axis of rotation. Then M I. of the body about the axis of rotation

$$=Mk^2=MK^2+Mh^2, \text{ so that } k^2=K^2+h^2.$$

(Recall that k is the radius of gyration of the boby about the axis of rotation).

$\therefore$ The length of the simple equivalent pendlum is

$$l=\frac{k^2}{h}=\frac{K^2+h^2}{h}=\frac{K^2}{h}+h. \qquad \ldots(1)$$

Now the time of oscillation of a compound pendulum will be least when the length l of the simple equivalent pendulum is minimum

i.e., when $\dfrac{dl}{dh}=\dfrac{d}{dh}\left(\dfrac{K^2}{h}+h\right)=-\dfrac{K^2}{h^2}+1=0$ or $h=K$.

Also $\dfrac{d^2l}{dh^2}=\dfrac{2K^2}{h^3}$, which is positive when $h=K$.

$\therefore$ l is minimum when $h=K$.

Hence the time of oscillation of a compound pendulum is minimum when $h=K$.

Putting $h=K$ in (1), the minimum value of $l=2K$.

Thus the minimum time of oscillation of a compound pendulum is given by

$$T=2\pi\sqrt{\left(\frac{l}{g}\right)}=2\pi\sqrt{\left(\frac{2K}{g}\right)}.$$

Note. In case $h=0$ or ∞, then from (1), $l=\infty$, *i.e.*, if the axis of rotation (the axis of suspension) either passes through the centre of gravity or be at infinity, the corresponding simple equivalent pendulum is of infinite length and hence the time of oscillation of such a compound pendulum is also infinite.

§ 4·9. Definition.

(i) Centre of Suspension. [Meerut 80, 81, 82, 84P, 88]

The point O, where the perpendicular from the centre of gravity 'G' of the body, cuts the axis of rotation, is called the **centre of suspension** *of the body. Thus the centre of suspension O of a body is the point where the vertical plane through the centre of gravity of the body meets the axis of rotation.*

(ii) Centre of Oscillation. [Meerut 80, 81, 82, 84P, 88]

If O is the centre of suspension and G the centre of gravity of the body then the point O′ on OG produced such that $OO'=k^2/h$ *(i.e., the length of simple equivalent pendulum) is called the* **centre of oscillation** *of the body.*

Remark. From the definition of centre of oscillation, we conclude that if the whole mass of the body were collected at the centre of oscillation and hanged from the centre of suspension by a string of length $l=(k^2/h)$, the time of oscillation of this simple pendulum will be the same as that of the body swinging about the centre of suspension.

The time of oscillation $T=2\pi\sqrt{(l/g)}$ of a simple pendulum depends upon the length l of the string and not upon the mass attached at the other end of the string.

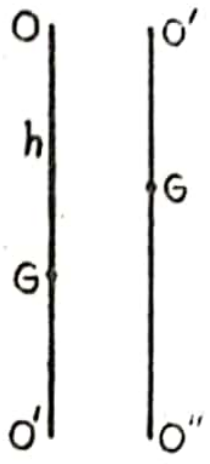
Hence we conclude that if an additional mass is attached at the centre of oscillation of a compound pendulum, its period of oscillation remains unaltered.

§ 4·10. The centre of suspension and the centre of oscillation of a compound pendulum are convertible.

[Meerut 80, 81, 82, 83, 84, 84P, 86, 88; Raj. 79]

Let O' be the centre of oscillation of a compound pendulum when O is the centre of suspension.

Then $$OO'=k^2/h,$$
where k is the radius of gyration of the body about the axis of rotation through O and $h=OG$, where G is the centre of gravity of the body.



Now let the body be suspended from O' instead of O. Then we shall prove that the body will swing about O' with O as the centre of oscillation.

Let K be the radius of gyration of the body about an axis through the centre of gravity G and parallel to the axis of rotation through O.

The M.I. about the axis of rotation through O

$$=Mk^2=MK^2+M.OG^2 \quad \text{or} \quad k^2=K^2+OG^2.$$

$$\therefore \quad OO'=\frac{k^2}{h}=\frac{K^2+OG^2}{OG} \qquad (\because \quad OG=h)$$

or $$K^2=OG.OO'-OG^2=OG.(OO'-OG)$$

or $$K^2=OG.O'G. \qquad \ldots(1)$$

If O'' is the centre of oscillation of the body when the centre of suspension is O', then proceeding as above, we have

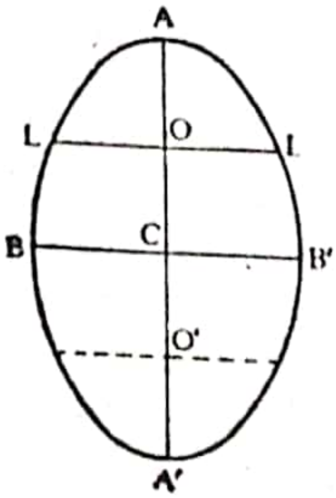
$$K^2=O'G.O''G. \qquad \ldots(2)$$

From (1) and (2), we conclude that O'' and O are the same points *i.e.*, if the point O' is the centre of suspension then O is the centre of oscillation. Hence the centre of suspension and centre of oscillation are convertible or reversible.

Solved Examples

Ex. 1. *Find the length of the simple equivalent pendulum for an elliptic lamina when the axis is a latus rectum.*

Sol. Let C be the centre, O, O' the two focii and LL' the latus rectum of an elliptic lamina.



Here the latus rectum LL' is taken as the axis of rotation. If h is the depth of the C.G. of the lamina below LL' and k the radius of gyration about it, then

$$h=OC=ae$$

and $$Mk^2=\text{M.I. about } LL'$$
$$=M(\tfrac{1}{4}a^2+OC^2)=M(\tfrac{1}{4}a^2+a^2e^2)$$

or $$k^2=a^2(\tfrac{1}{4}+e^2).$$

$\therefore$ Length of the simple equivalent pendulum $=k^2/h$

$$=\frac{a^2\left(\frac{1}{4}+e^2\right)}{ae}=\left(\frac{1}{4e}+e\right)a.$$

Ex. 2. *An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, prove that the eccentricity is $\frac{1}{2}$.*

Sol. (Refer figure of Ex. 1).

Let C be the centre, O, O' the two focii and LL' the latus rectum of an elliptic lamina.

Here the latus rectum LOL' is the horizontal axis of rotation. If the other latus rectum passes through the centre of oscillation O', then the length l of the simple equivalent pendulum is given by

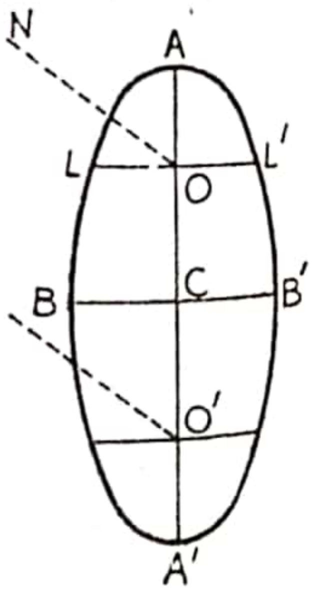
$$l=OO'=2ae. \quad \ldots(1)$$

But $\quad l=a\left(\frac{1}{4e}+e\right)$ [See Ex. 1]

$\therefore \quad a\left(\frac{1}{4e}+e\right)=2ae$

or $\quad \frac{1}{4}+e^2=2e^2$ or $e^2=\frac{1}{4}$ or $e=\frac{1}{2}$.

Ex. 3. *A uniform elliptic board swings about a horizontal axis at right angles to the plane of the board and passing through one focus. If the centre of oscillation be the other focus prove that its eccentricity is $\sqrt{(2/5)}$.*



Sol. Let C be the centre and O, O' the two focii of the elliptic board. Here the axis of rotation is a horizontal line ON through one focus O and at right angles to the plane of the board.

Let k be the radius of gyration of the board about the axis of rotation ON. Then

$Mk^2=$ M.I. of the board about the axis of rotation ON

$=$ M.I. of the board about the axis through the centre of gravity *i.e.*, the centre C of the board and $\parallel$ to ON + M.I. of total mass M at C about ON

$=\frac{1}{4}M(a^2+b^2)+M.OC^2$

$=\frac{1}{4}M(a^2+b^2)+M.(ae)^2.$

$\therefore \quad k^2=\frac{1}{4}[a^2+a^2(1-e^2)+4a^2e^2]=\frac{1}{4}a^2(2+3e^2).$

Also $h=$ Depth of C.G. of the lamina below the axis of rotation ON

$=OC=ae.$

$\therefore \quad l=$length of simple equivalent pendulum

$=k^2/h=a\ (2+3e^2)/4e.$

But it is given that the other focus O' is the centre of oscillation.

$$\therefore \quad l=OO'=2ae.$$

Hence equating the two values of l, we have

$$a(2+3e^2)/4e=2ae \quad \text{or} \quad 2+3e^2=8e^2. \qquad \therefore \quad e=\sqrt{(2/5)}.$$

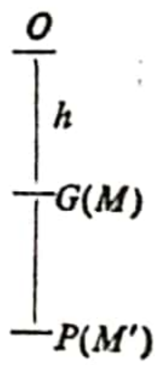
Ex. 4. *A pendulum is supported at O and P is the centre of oscillation. Show that if an additional weight is rigidly attached at P, the period of oscillation is unaltered.*

Sol. Let M be the mass and h the depth of the C G. below the centre of suspension O of the body forming the compound pendulum. Since P is the centre of oscillation, therefore

$OP=k^2/h$, where k is the radius of gyration of the body about the axis of rotation through O.

And the period of oscillation $T=2\pi\sqrt{(k^2/gh)}$.

Let an additional mass M' be attached at the centre of oscillation P. If h' is the depth of the C G. of the combined body below O and k' is the radius of gyration of the combined body about the axis of rotation through O, then we have



$$h'=\frac{M.OG+M'.OP}{M+M'}=\frac{Mh+M'.(k^2/h)}{M+M'}$$

or $\quad (M+M')\ h'=h\ [M+M'.(k^2/h^2)]. \quad ...(1)$

Also M.I. of the combined body about the axis of rotation

=(M.I of the original body+M.I. of the mass M' attached at P), about the axis of rotation.

$\therefore\ (M+M')\ k'^2=Mk^2+M'.OP^2=Mk^2+M'(k^4/h^2)$

or $(M+M')\ k'^2=k^2\ [M+M'.(k^2/h^2)]. \quad ...(2)$

Dividing (2) by (1), we get $k'^2/h'=k^2/h$.

$\therefore$ The period of oscillation T' when an additional weight is rigidly attached at P is given by

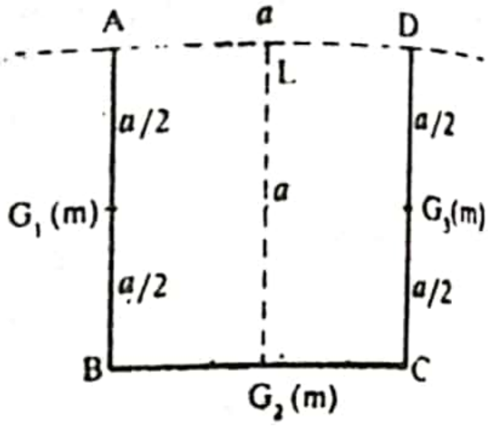
$$T'=2\pi\sqrt{(k'^2/gh')}=2\pi\sqrt{(k^2/gh)}=T.$$

Hence the period of oscillation is unaltered.

Note. Also see remark on § 4·9.

Ex. 5. *Three uniform rods AB, BC, CD each of length a, are freely joined at B and C and suspended from the points A and D which are in the same horizontal line and at a distance a apart.*

Prove that when the rods move in a vertical plane, the length of equivalent pendulum is $5a/6$. [Meerut 80]



Sol. Here the horizontal line AD is the axis of rotation of the compound pendulum formed by the rods AB, BC and CD each of length a. Let m be the mass of each rod.

Let h be the depth of the C.G. of the system below AD and k be its radius of gyration about AD.

Then $h=\dfrac{m.AG_1+m.LG_2+m.DG_3}{m+m+m}$

$=\frac{1}{3}(a/2+a+a/2)=\frac{2}{3}a.$

Also M.I. of the system about AD

$=$Sum of the moments of inertia of the three rods about AD.

$\therefore$ $(m+m+m)\,k^2=\frac{4}{3}m\,(a/2)^2+ma^2+\frac{4}{3}M\,(a/2)^2$

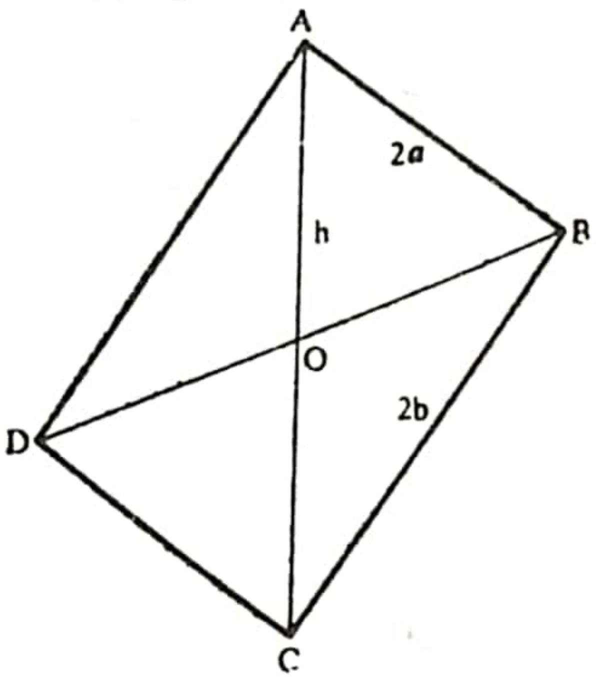
or $\qquad k^2=\frac{5}{9}a^2.$

$\therefore$ Length l of the simple equivalent pendulum is given by

$$l=k^2/h=\tfrac{5}{9}a^2/(\tfrac{2}{3}a)=5a/6.$$

Ex. 6. *A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of the diagonal.* [Meerut 80, 81]

Sol. Let the rectangular plate $ABCD$ of sides of lengths



$AB=2a$ and $BC=2b$ swing in its vertical plane about a horizontal line through A and perpendicular to its plane as the axis of rotation. If O is the centre and M the mass of the plate, then

$$h = \text{Depth of the C.G. of the plate below } A$$
$$= AO = \tfrac{1}{2}AC = \tfrac{1}{2}\sqrt{(AB^2+BC^2)}$$
$$= \tfrac{1}{2}\sqrt{(4a^2+4b^2)}$$
$$= \sqrt{(a^2+b^2)}. \qquad \ldots(1)$$

Also if k is the radius of gyration of the plate about the axis of rotation through A, then

$$Mk^2 = \text{M.I. of the plate about the axis of rotation through } A$$
$$= \frac{M(a^2+b^2)}{3} + M.AO^2$$
$$= \frac{M}{3}h^2 + Mh^2 \qquad \text{[from (1)]}$$
$$= \tfrac{4}{3}Mh^2.$$
$$\therefore \quad k^2 = \tfrac{4}{3}h^2.$$

$\therefore$ Period of oscillation $T = 2\pi\sqrt{\left(\dfrac{k^2}{gh}\right)} = 2\pi\sqrt{\left(\dfrac{4h}{3g}\right)}$.

But it is given that $T=1$ second.

$$\therefore \quad 2\pi\sqrt{\left(\frac{4h}{3g}\right)} = 1 \quad \text{or} \quad 16\pi^2 h = 3g \quad \text{or} \quad h = \frac{3g}{16\pi^2}.$$

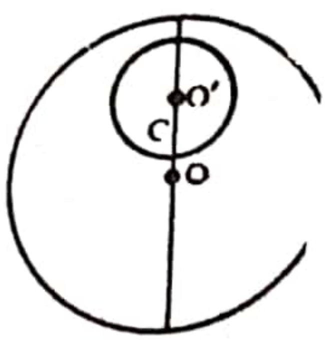
Hence the length of the diagonal of the plate

$$= AC = 2AO = 2h = \frac{3g}{8\pi^2}.$$

Ex. 7. *A flat circular disc of radius a has a hole in it of radius b whose centre is at a distance c from the centre of the disc ($c < a - b$). The disc is free to oscillate in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Show that the length of the equivalent pendulum is* $c + \frac{1}{2}.\dfrac{a^4-b^4}{a^2c}$.

Sol. Let O be the centre of the disc and O' that of the hole of radius b.

$$\therefore \quad OO' = c \text{ (given).}$$



The disc is free to oscillate about a smooth horizontal circular rod of radius b passing through the hole, therefore O' is the centre of suspension and the horizontal line through O' and perpendicular to the given disc is the axis of rotation.

If ρ is the density of the disc, then

m_1 = mass of the whole disc without hole $= \pi a^2 \rho$

and m_2 = mass of the hole of radius $b = \pi b^2 \rho$.

$\therefore$ h = Depth of the C.G. of the disc with hole below O'

$$= \frac{m_1 . O'O - m_2 . 0}{m_1 - m_2} = \frac{\pi a^2 \rho . c - 0}{\pi a^2 \rho - \pi b^2 \rho} = \frac{a^2 c}{a^2 - b^2}.$$

If k is the radius of gyration of the disc with hole about the axis of rotation, then $(m_1 - m_2)\, k^2$ = M.I. of the disc with hole about the axis of rotation

=(M.I. of the disc without hole−M.I. of the hole), about the axis of rotation

$$=(m_1 . \tfrac{1}{2}a^2 + m_1 O'O^2) - m_2 . \tfrac{1}{2}b^2 = m_1 (\tfrac{1}{2}a^2 + c^2) - m_2 . \tfrac{1}{2}b^2.$$

$$\therefore \quad (\pi a^2 \rho - \pi b^2 \rho)\, k^2 = \tfrac{1}{2}\, \pi a^2 \rho\, (a^2 + 2c^2) - \tfrac{1}{2} \pi b^2 \rho . b^2$$

or $$k^2 = \frac{a^4 + 2a^2 c^2 - b^4}{2\,(a^2 - b^2)}.$$

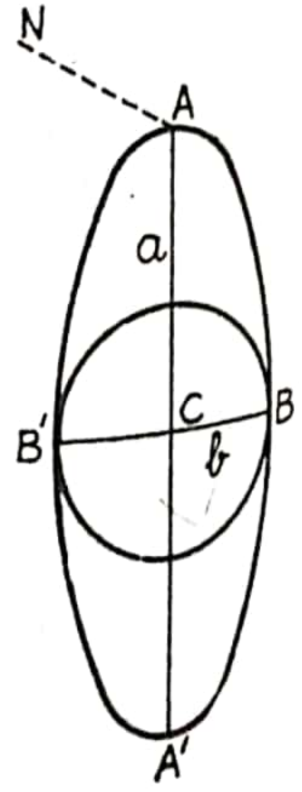
$\therefore$ Length of the simple equivalent pendulum $= k^2/h$

$$= \frac{a^4 + 2a^2 c^2 - b^4}{2a^2 c} = c + \tfrac{1}{2} . \frac{a^4 - b^4}{a^2 c}.$$

Ex. 8. *An ellipse of axes 2a, 2b and a circle of radius b are cut from the same sheet of thin uniform metal and are superposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of the major axis; show that the length of the equivalent simple pendulum is*

$$(5a^2 - ab + 2b^2)/4a. \qquad \text{[Meerut 89]}$$

Sol. Let ρ be the mass per unit area of the sheet.



$\therefore$ m_1 = mass of the ellipse $= \pi ab \rho$

and m_2 = mass of the circle $= \pi b^2 \rho$.

$\therefore$ Total mass of the figure $= m_1 + m_2$.

Here the axis of rotation is the horizontal line AN through the end A of the major axis and perpendicular to the plane of the figure. If k is the radius of gyration of the body about the axis of rotation AN, then

$(m_1 + m_2)\, k^2$ = M I. of the figure about the axis of rotation AN

= sum of the moments of inertia of the ellipse and the circle about the axis of rotation AN

$$=\{\tfrac{1}{4}m_1(a^2+b^2)+m_1a^2\}+\{\tfrac{1}{2}m_2b^2+m_2a^2\}.$$

$$\therefore\quad (\pi ab\rho+\pi b^2\rho)\,k^2=\tfrac{1}{4}\,\pi ab\rho\,(5a^2+b^2)+\tfrac{1}{2}\,\pi b^2\rho\,(b^2+2a^2)$$

or $$(a+b)\,\pi b\rho k^2=\tfrac{1}{4}\,\pi b\rho\,[5a^3+ab^2+2b^3+4a^2b]$$

or $$(a+b)\,k^2=\tfrac{1}{4}\,[5a^2(a+b)-ab(a+b)+2b^2(a+b)]$$

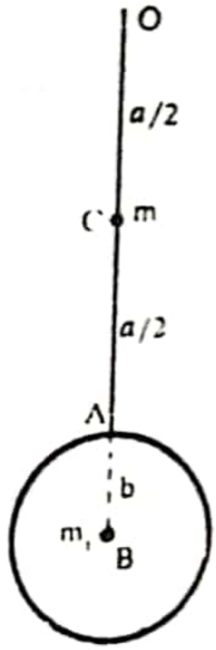
or $$k^2=\tfrac{1}{4}\,[5a^2-ab+2b^2].$$

Also h=depth of the centre of gravity 'C' of the figure below $AN=AC=a$.

$\therefore$ Length of the simple equivalent pendulum

$$=k^2/h=(5a^2-ab+2b^2)/4a.$$

Ex. 9. *Find the time of oscillation of a compound pendulum, consisting of a rod of mass m and length a, carrying at one end a sphere of mass m_1 and diameter $2b$, the other end of the rod being fixed.* **[Meerut 83; Raj. 80]**



Sol. Let OA be the rod of mass m and length a attached at A to a sphere of radius b and mass m_1. If h is the depth of the C G. of the system below O and k its radius of gyration about the axis through O, then

$$(m+m_1)\,k^2=\tfrac{4}{3}m\,(a/2)^2+[\tfrac{2}{5}m_1b^2+m_1.OB^2]$$

or $$k^2=\frac{\tfrac{1}{3}ma^2+m_1\,[\tfrac{2}{5}b^2+(a+b)^2]}{(m+m_1)}$$

and $$h=\frac{m.OG+m_1\,OB}{m+m_1}=\frac{m.\tfrac{1}{2}a+m_1.(a+b)}{m+m_1}.$$

$\therefore$ The time of oscillation of the compound pendulum

$$=2\pi\sqrt{\left(\frac{k^2}{gh}\right)}=\frac{2\pi}{\sqrt{g}}\sqrt{\left\{\frac{\tfrac{1}{3}ma^2+m_1\,[\tfrac{2}{5}b^2+(a+b)^2]}{\tfrac{1}{2}ma+m_1\,(a+b)}\right\}}.$$

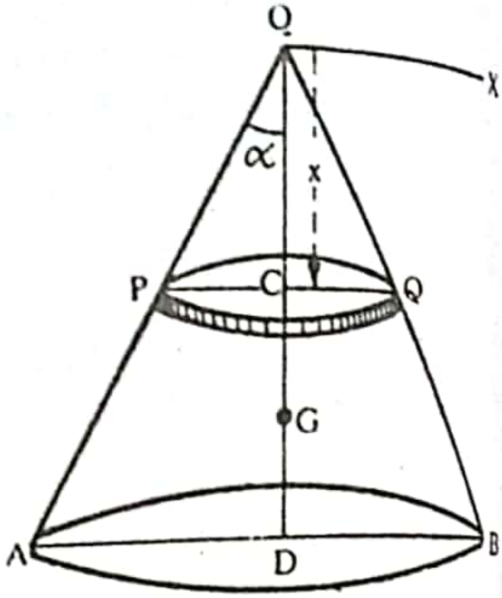
Ex. 10. *A solid homogeneous cone of height h and vertical angle 2α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is $\tfrac{1}{5}h\,(4+\tan^2\alpha)$.*

[Meerut 80, 81, 82, 87, 89; Raj. 79; Rohilkhand 83]

Sol. Let the cone OAB of height h and vertical angle 2α oscillate about the horizontal axis OX through the vertex O.

$\therefore$ M=mass of the cone$=\tfrac{1}{3}\pi h^3\tan^2\alpha.\rho$, where ρ is the mass per unit volume.

To find the M. I. of the cone about OX, consider an elementary disc PQ of thickness δx, parallel to OX at a distance x from this axis.



Mass δm of this elementary disc$=\rho\,\pi x^2 \tan^2\alpha . \delta x$.

$\therefore$ M.I. of the disc about $OX=\frac{1}{4}\delta m . CP^2+\delta m . OC^2$

$=(\frac{1}{4}x^2\tan^2\alpha+x^2)\rho\pi x^2\tan^2\alpha\,\delta x$

$=\rho\pi\tan^2\alpha.(\frac{1}{4}\tan^2\alpha+1)\,x^4\delta x.$

Therefore M.I. of the cone about OX

$$=\int_0^h \rho\pi\tan^2\alpha\,(\tfrac{1}{4}\tan^2\alpha+1)\,x^4 dx$$

$$=\tfrac{1}{5}\rho\pi h^5\tan^2\alpha\,(\tfrac{1}{4}\tan^2\alpha+1)=\tfrac{3}{20}\,M\,(\tan^2\alpha+4)\,h^2.$$

If k be the radius of gyration of the cone about OX, then

$$Mk^2=\text{M.I. of the cone about } OX=\tfrac{3}{20}\,M\,(\tan^2\alpha+4)\,h^2.$$

$$\therefore \quad k^2=\tfrac{3}{20}\,(\tan^2\alpha+4)\,h^2.$$

Also depth of the C.G. of the cone below $OX=OG=3h/4$.

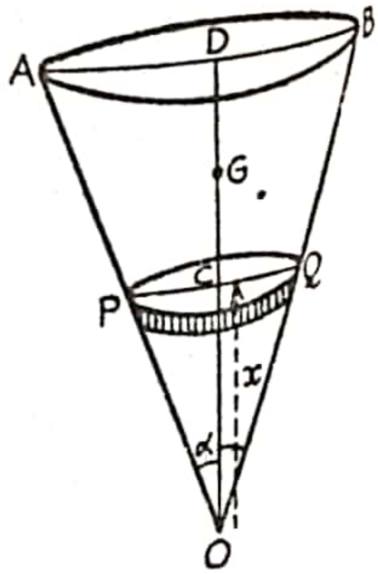
$\therefore$ Length of the simple equivalent pendulum$=k^2/OG$

$$=\tfrac{1}{5}h\,(4+\tan^2\alpha).$$

Ex. 11. *A solid homogeneous cone of height h and semi-vertical angle α, oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\frac{1}{5}h\,(2+3\tan^2\alpha)$.*

[Meerut 81, 84, 90]

Sol. Let O be the vertex of a cone of height h and semi-vertical angle α, which oscillates about a diameter AB of its base.



Let ρ be the density and M be the mass of the cone. Then

$$M=\tfrac{1}{3}\,\pi h^3\tan^2\alpha\rho.$$

To find the M.I. of the cone about AB, consider an elementary disc PQ of thickness δx, parallel to AB and at a distance x from the vertex O.

$\therefore$ Mass of the disc $=\delta m=\rho\pi x^2 \tan^2 \alpha\delta x$.

$\therefore$ M.I. of the disc about $AB=\frac{1}{4}\delta m.CP^2+\delta m.DC^2$

$$=\rho\pi x^2 \tan^2 \alpha \left[\tfrac{1}{4}x^2 \tan^2 \alpha+(h-x)^2\right] \delta x.$$

$\therefore$ M.I. of the cone about AB

$$=\int_0^h \rho\pi \tan^2 \alpha \left[\tfrac{1}{4}x^4 \tan^2 \alpha+(h-x)^2 x^2\right] \delta x$$

$$=\tfrac{1}{4}\rho\pi \tan^2 \alpha \int_0^h (x^4 \tan^2 \alpha+4h^2x^2-8hx^3+4x^4)\, dx$$

$$=\tfrac{1}{4}\rho\pi \tan^2 \alpha \left[\tfrac{1}{5}h^5 \tan^2 \alpha+\tfrac{4}{3}h^5-2h^5+\tfrac{4}{5}h^5\right]$$

$$=\tfrac{1}{60}\rho\pi \tan^2 \alpha.h^5 (3 \tan^2 \alpha+2)=\tfrac{1}{20}Mh^2 (3 \tan^2 \alpha+2).$$

$\therefore$ If k is the radius of gyration of the cone about AB, then

$$Mk^2=\tfrac{1}{20} Mh^2 (3 \tan^2 \alpha+2)$$

or $$k^2=\tfrac{1}{20}h^2 (3 \tan^2 \alpha+2).$$

And $h'=$ depth of the C.G. of the cone below AB

$=DG=h/4$.

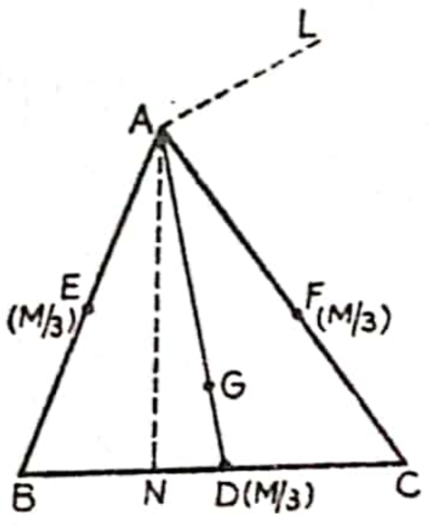
$\therefore$ Length of simple equivalent pendulum $=k^2/h'$

$$=\tfrac{1}{5}h\,(2+3 \tan^2 \alpha).$$

Ex. 12. *A uniform triangular lamina can oscillate in its own plane about the angle A. Prove that the length of the simple equivalent pendulum is* $\dfrac{3\,(b^2+c^2)-a^2}{4\sqrt{[2\,(b^2+c^2)-a^2]}}$, *the axis through A being horizontal.* **[Meerut 86, 89]**

Sol. Let the triangular lamina ABC of mass M oscillate in its own plane about an axis AL through one vertex A and perpendicular to its plane.

M.I. of the ΔABC about AL is equal to the sum of moments of inertias of three particles each of mass $M/3$ placed at the middle points of the sides.



$\therefore$ If k is the radius of gyration of the triangle about AL, then

$$Mk^2=(M/3).AD^2+(M/3).AE^2+(M/3)\,AF^2. \quad \ldots(1)$$

Now $AE=\frac{1}{2}AB=c/2$, $AF=\frac{1}{2}AC=b/2$.

Also $AB^2+AC^2=2\,(AD^2+DC^2)$.

$$\therefore\ AD^2=\tfrac{1}{2}\,(AB^2+AC^2)-DC^2$$
$$=\tfrac{1}{2}\,(b^2+c^2)-\tfrac{1}{4}a^2$$
$$=\tfrac{1}{4}\,(2b^2+2c^2-a^2).$$

$\therefore$ from (1), we have

$$k^2=\tfrac{1}{3}.\tfrac{1}{4}(2b^2+2c^2-a^2)+\tfrac{1}{3}(\tfrac{1}{2}c)^2+\tfrac{1}{3}(\tfrac{1}{2}b)^2$$
$$=\frac{1}{12}[3(b^2+c^2)-a^2].$$

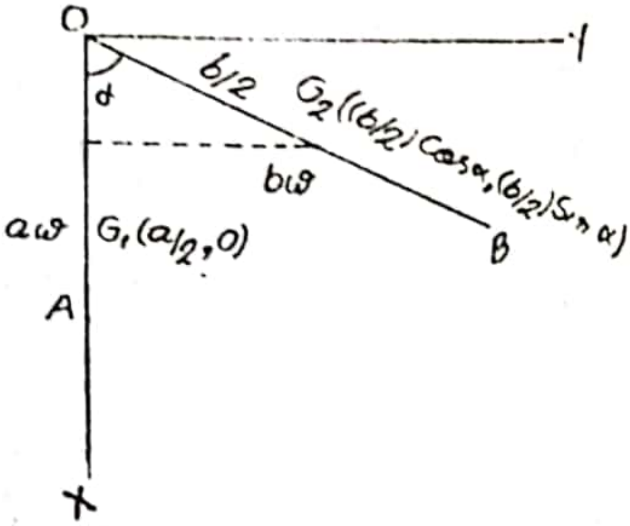
And h=Distance of C.G. 'G' of the $\triangle$ below AL
$$=AG=\tfrac{2}{3}AD=\tfrac{1}{3}\sqrt{(2b^2+2c^2-a^2)}.$$

$\therefore$ Length of the simple equivalent pendulum
$$=\frac{k^2}{h}=\frac{3(b^2+c^2)-a^2}{4\sqrt{[2(b^2+c^2)-a^2]}}.$$

Ex. 13. *A bent lever, whose arms are of lengths a and b, the angle between them being α, makes small oscillations in its own plane about the fulcrum, show that the length of the corresponding simple pendulum is* $\frac{2}{3}\frac{a^3+b^3}{\sqrt{(a^4+2a^2b^2\cos\alpha+b^4)}}$.

[Meerut 80, 81, 83, 88]

Sol. Let the bent lever with arms $OA=a$ and $OB=b$, and $\angle AOB=\alpha$, oscillate in its own plane about the fulcrum O. If w is the weight per unit length of the arm, then the weights aw and bw of the arms will act at their middle points G_1 and G_2.



Taking the co-ordinate axes along and perpendicular to OA in the plane of the lever, the coordinates of G_1 and G_2 are

$$\left(\frac{a}{2},0\right) \text{ and } \left(\frac{b}{2}\cos\alpha,\frac{b}{2}\sin\alpha\right) \text{ respectively.}$$

If $(\bar{x},\bar{y})$ are the coordinates of the C.G. 'G' of the lever, then

$$\bar{x}=\frac{aw.(a/2)+bw.(b/2)\cos\alpha}{aw+bw}=\frac{a^2+b^2\cos\alpha}{2(a+b)}$$

and
$$\bar{y}=\frac{aw.0+bw.(b/2)\sin\alpha}{aw+bw}=\frac{b^2\sin\alpha}{2(a+b)}.$$

$\therefore$ h=perpendicular distance of the C.G. 'G' of the lever from the axis of rotation

$$=OG=\sqrt{(\bar{x}^2+\bar{y}^2)}=\frac{1}{2(a+b)}\sqrt{[(a^2+b^2\cos\alpha)^2+(b^2\sin\alpha)^2]}$$

$$=\frac{1}{2(a+b)}\sqrt{[a^4+2a^2b^2\cos\alpha+b^4(\cos^2\alpha+\sin^2\alpha)]}$$

$$=\frac{\sqrt{[a^4+2a^2b^2\cos\alpha+b^4]}}{2(a+b)}.$$

If k is the radius of gyration about the axis of rotation through O, then

$$(aw+bw)\,k^2=\frac{4}{3}\,aw.\left(\frac{a}{2}\right)^2+\frac{4}{3}\,bw.\left(\frac{b}{2}\right)^2$$

or
$$k^2=\frac{a^3+b^3}{3(a+b)}.$$

Hence the length of the corresponding simple pendulum is

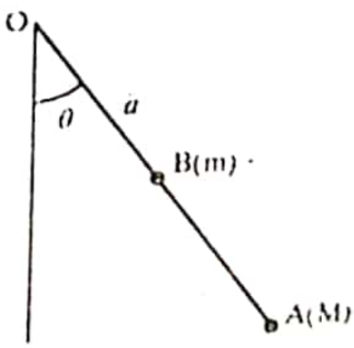
$$=\frac{k^2}{h}=\frac{2}{3}\cdot\frac{a^3+b^3}{\sqrt{(a^4+2a^2b^2\cos\alpha+b^4)}}.$$

Ex. 14. *A simple circular pendulum is formed of a mass M suspended from a fixed point by a weightless wire of length l. If a mass m, very small compared with M, be knotted on to the wire at a distance a from the point of suspension, show that the time of a small vibration of the pendulum is approximately diminished by*

$$\frac{m}{2M}\cdot\frac{a}{l}\left(1-\frac{a}{l}\right) \text{ of itself.}$$ [Meerut 84, 85]

Sol. If T is the period of the simple pendulum before knotting the mass m, then

$$T=2\pi\sqrt{(l/g)}. \qquad \text{...(1)}$$



Let T' be the period of the pendulum when a small mass m is knotted at a point B of the wire such that $OB=a$, then

$$T'=2\pi\sqrt{(k^2/gh)}, \qquad \text{...(2)}$$

where k is the radius of gyration of the system about the axis of rotation through O and h the distance of the C.G. of the system from the same axis.

$\therefore\quad (m+M)\,k^2=M.OA^2+m.OB^2=Ml^2+ma^2$

or $\quad k^2=\dfrac{Ml^2+ma^2}{m+M}$ and $h=\dfrac{M.OA+m.OB}{M+m}=\dfrac{Ml+ma}{M+m}.$

$\therefore\quad$ From (2), we have

$$T'=2\pi\sqrt{\left[\frac{Ml^2+ma^2}{g(Ml+ma)}\right]}$$

$$=2\pi\sqrt{\left(\frac{Ml^2}{gMl}\right)}\cdot\left[1+\frac{ma^2}{Ml^2}\right]^{1/2}\cdot\left[1+\frac{ma}{Ml}\right]^{-1/2}$$

$$=2\pi\sqrt{\left(\frac{l}{g}\right)}\cdot\left(1+\frac{ma^2}{2Ml^2}\right)\left(1-\frac{ma}{2Ml}\right)$$

[expanding by binomial theorem and neglecting higher powers of m/M which is very small]

$$=T\left(1-\frac{ma}{2Ml}+\frac{ma^2}{2Ml^2}\right) \qquad \text{[again neglecting } m^2/M^2\text{]}$$

$$=T-\left(\frac{ma}{2Ml}-\frac{ma^2}{2Ml^2}\right)T$$

or $$T-T'=\frac{m}{2M}\cdot\frac{a}{l}\left(1-\frac{a}{l}\right).T.$$

$\therefore$ The period T is diminished by

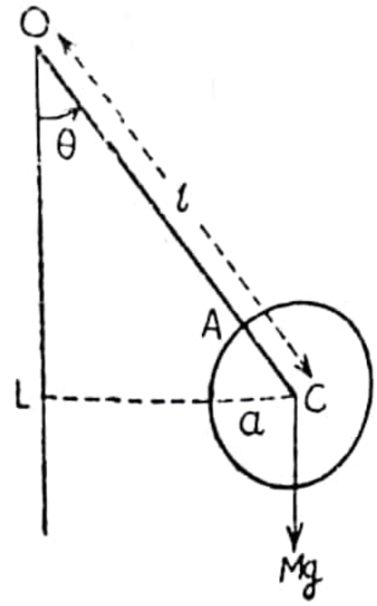
$\frac{m}{2M}\cdot\frac{a}{l}\left(1-\frac{a}{l}\right)$ of itself.

Ex. 15. *A sphere of radius a, is suspended by a fine wire from a fixed point at a distance l from its centre, show that the time of a small oscillation is given by*

$$2\pi\sqrt{\left(\frac{5l^2+2a^2}{5lg}\right)}\left[1+\tfrac{1}{4}\sin^2\left(\tfrac{1}{2}\alpha\right)\right]$$

where α represents the amplitude of the vibration.

[Meerut 81, 84; Kanpur 83; Raj. 81]



Sol. Let C be the centre of the sphere and O the fixed point such that $OC=l$.

M.I. of the sphere about the axis of rotation through O

$=\frac{2}{5}Ma^2+Ml^2=M\left(\frac{2}{5}a^2+l^2\right)$.

At time t, let θ be the inclination of OC to the vertical.

$\therefore$ Taking moments about the fixed axis through O, the equation of motion of the sphere is

$$M\left(\tfrac{2}{5}a^2+l^2\right)\ddot{\theta}=-Mg.CL$$

(Negative sign is taken as the moment of Mg is in the direction of θ decreasing)

or $$M\left(\tfrac{2}{5}a^2+l^2\right)\ddot{\theta}=-Mgl\sin\theta$$

or $$\ddot{\theta}=-\frac{5gl}{2a^2+5l^2}\sin\theta.$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we have

$$\dot{\theta}^2=\frac{10gl}{2a^2+5l^2}\cos\theta+C.$$

But when $\theta=\alpha$, $\dot{\theta}=0$; $\therefore$ $C=-\frac{10gl}{2a^2+5l^2}\cos\alpha.$

$$\therefore\quad \dot{\theta}^2=\frac{10gl}{2a^2+5l^2}(\cos\theta-\cos\alpha)$$

$$=\frac{10gl}{2a^2+5l^2}[(1-2\sin^2\tfrac{1}{2}\theta)-(1-2\sin^2\tfrac{1}{2}\alpha)]$$

$$=\frac{20gl}{2a^2+5l^2}(\sin^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\theta).$$

$$\therefore\quad \dot{\theta}=\frac{d\theta}{dt}=\sqrt{\left(\frac{20gl}{2a^2+5l^2}\right)}\sqrt{(\sin^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\theta)}$$

or $$dt=\sqrt{\left(\frac{2a^2+5l^2}{20gl}\right)}\cdot\frac{d\theta}{\sqrt{(\sin^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\theta)}}.$$

Integrating between the limits $\theta=0$ to $\theta=\alpha$, the time t_1 from the vertical position to the extreme position on one side is given by

$$t_1=\frac{1}{2}\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_0^\alpha\frac{d\theta}{\sqrt{(\sin^2\tfrac{1}{2}\alpha-\sin^2\tfrac{1}{2}\theta)}}.$$

Put $\sin\tfrac{1}{2}\theta=\sin\tfrac{1}{2}\alpha\sin\phi$

so that $\tfrac{1}{2}\cos\tfrac{1}{2}\theta\, d\theta=\sin\tfrac{1}{2}\alpha\cos\phi\, d\phi$

or $$d\theta=\frac{2\sin\tfrac{1}{2}\alpha\cos\phi\, d\phi}{\cos\tfrac{1}{2}\theta}=\frac{2\sin\tfrac{1}{2}\alpha\cos\phi}{\sqrt{(1-\sin^2\tfrac{1}{2}\theta)}}\, d\phi$$

$$=\frac{2\sin\tfrac{1}{2}\alpha\cos\phi}{\sqrt{(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)}}\, d\phi.$$

When $\theta=0$, $\phi=0$ and when $\theta=\alpha$, $\phi=\tfrac{1}{2}\pi$.

$$\therefore\quad t_1=\frac{1}{2}\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_0^{\pi/2}\frac{2\sin\tfrac{1}{2}\alpha\cos\phi}{\sin\tfrac{1}{2}\alpha\cos\phi.\sqrt{(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi)}}\, d\phi$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_0^{\pi/2}\left(1-\sin^2\tfrac{1}{2}\alpha\sin^2\phi\right)^{-1/2} d\phi$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_0^{\pi/2}\left(1+\frac{1}{2}\sin^2\frac{1}{2}\alpha\sin^2\phi+\dots\right) d\phi,$$

$$[\because\ (1-x)^{-1/2}=1+\tfrac{1}{2}x+\dots]$$

$$=\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\left[\left(\phi\right)_0^{\pi/2}+\tfrac{1}{2}\sin^2\tfrac{1}{2}\alpha.\ \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)}+\dots\right]$$

$=\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\left[\frac{\pi}{2}+\left(\frac{1}{2}\sin^2\frac{1}{2}\alpha\right).\frac{\pi}{4}\right]$, neglecting higher powers of $\sin\frac{1}{2}\alpha$ which is small as α is very small

$=\frac{\pi}{2}\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}.\left[1+\frac{1}{4}\sin^2\frac{1}{2}\alpha\right].$

$\therefore$ Time for one small complete oscillation

$$=4t_1=2\pi\sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\left[1+\frac{1}{4}\sin^2\frac{1}{2}\alpha\right].$$

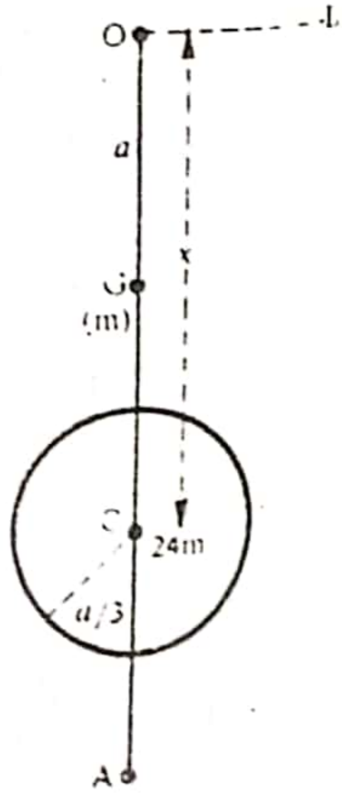
Ex. 16. *A uniform rod of mass m and length 2a can oscillate about a horizontal axis through one end. A circular disc of mass 24m and radius $\frac{1}{3}a$ can have its centre clamped to any point of the rod and its plane contains the axis of rotation. Show that for oscillations under gravity the length of the simple equivalent pendulum lies between a/2 and 2a.*

Sol. Let OA be the uniform rod of mass m and length $2a$ oscillating about a horizontal axis OL through O. Let a circular disc of mass $24m$ and radius $(a/3)$ have its centre clamped to a point C of the rod such that its plane contains the axis of rotation and $OC=x$.

If k is the radius of gyration of the system about OL, then

$$(m+24m)\,k^2=\frac{4}{3}\,ma^2+\left[24m.\frac{1}{4}\left(\frac{a}{3}\right)^2+24m.OC^2\right]$$

or $\quad 25mk^2=\frac{4}{3}\,ma^2+\frac{2}{3}\,ma^2+24mx^2\quad$ or $\quad k^2=\frac{2a^2+24x^2}{25}.$



Also h=depth of the C.G. of the system below OL

$$=\frac{m.OG+24m.OC}{m+24m}=\frac{1}{25}\,(a+24x).$$

$\therefore$ the length l of the simple equivalent pendulum is given by

$$l=\frac{k^2}{h}=\frac{2a^2+24x^2}{a+24x}.$$

Now for a max. or a mini. of l,

$$\frac{dl}{dx}=\frac{48x\,(a+24x)-(2a^2+24x^2).24}{(a+24x)^2}$$

$$=\frac{48\,(12x^2+ax-a^2)}{(a+24x)^2}=0$$

or $\quad 12x^2+ax-a^2=0$

or $\quad (4x-a)\,(3x+a)=0.$

$\therefore \quad x=\frac{a}{4}\;$ or $\;-\frac{a}{3}.$

But $x=-a/3$ gives an inadmissible value ; $\therefore \quad x=a/4$.

When $\quad x=a/4,\ l=\dfrac{2a^2+(24a^2/16)}{a+6a}=\dfrac{a}{2}$.

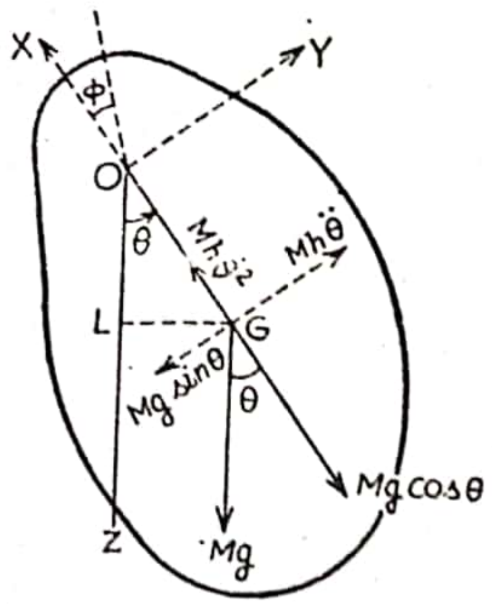
Also putting $x=0$ or $2a$, the other extreme value of l is $2a$. Hence the length of the simple equivalent pendulum lies between $a/2$ and $2a$.

§ 4·11. Reactions of the Axis of Rotation.

A body moves about a fixed axis under the action of forces and both the forces and the body are symmetrical with respect to the plane through the C.G. and perpendicular to the axis, find the reactions of the axis of rotation. [Meerut 83, 84]

Let the plane perpendicular to the axis of rotation through the C G. 'G' meet the axis at O, *i.e.* O is the centre of suspension of the body By the symmetry of the forces and the body, the actions on the axis of the body reduce to a single force acting at O in the plane of rotation.

Let X, Y be the components of the single force of reaction along and perpendicular to GO.



Since the centre of gravity 'G' of the body describes a circle around O of radius $OG=h$, therefore the components of its acceleration along and perpendicular to GO will be $h\dot{\theta}^2$ and $h\ddot{\theta}$ respectively.

$\therefore$ Equations of motion of the C.G. 'G' are

$$Mh\dot{\theta}^2=X-Mg\cos\theta, \quad \ldots(1)$$

and $$Mh\ddot{\theta}=Y-Mg\sin\theta. \quad \ldots(2)$$

Also for the equation of motion of the body about the axis of rotation, taking moments about the axis of rotation through O, we have

$$Mk^2\ddot{\theta}=-Mg.LG$$

or $$Mk^2\ddot{\theta}=-Mgh\sin\theta, \quad \ldots(3)$$

where k is the radius of gyration about the axis of rotation.

From (1), (2) and (3), we can find X and Y as follows :

(i) Eliminating $\ddot{\theta}$ between (2) and (3), we get Y.

(ii) Multiplying both sides of (3) by $2\dot{\theta}$ and integrating and determining the constant of integration by using the initial conditions, we get $\dot{\theta}^2$.

Substituting the value of $\dot{\theta}^2$ in (1), we can get the value of X.

$\therefore$ Resultant reaction at $O=\sqrt{(X^2+Y^2)}$.

If the reaction is inclined at an angle ϕ to X, *i.e.*, to GO, then

$$\phi=\tan^{-1}(Y/X).$$

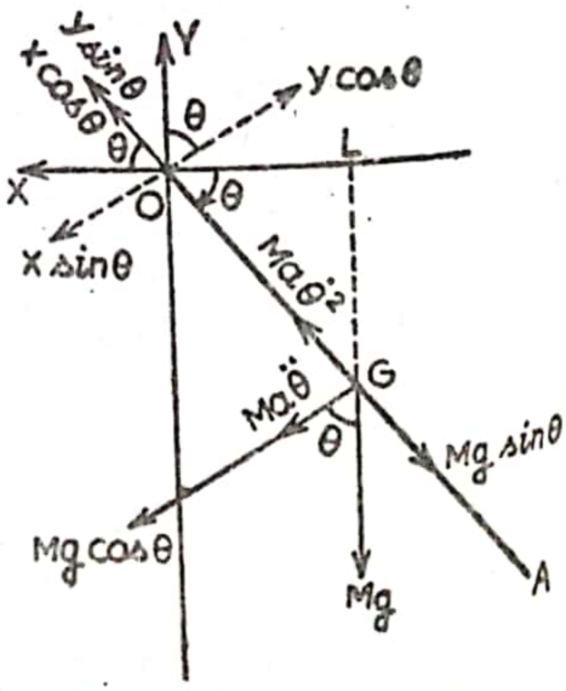
Solved Examples

Ex. 1. *A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from the horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.* **[Meerut 80, 82, 84, 85, 87]**

Sol. Let the rod OA of length $2a$, attached to a smooth hinge at O and held initially in a horizontal position, turn through an angle θ to the horizontal in time t. If X and Y are the horizontal and vertical components of the strain (*i.e.*, the reaction) on the hinge, then the equations of motion of G along and perpendicular to GO are

$$Ma\dot{\theta}^2=X\cos\theta+Y\sin\theta-Mg\sin\theta \quad \ldots(1)$$

and $$Ma\ddot{\theta}=X\sin\theta-Y\cos\theta+Mg\cos\theta. \quad \ldots(2)$$



M.I. of the rod about the fixed axis through O and perpendicular to the rod $=\frac{4}{3}Ma^2$.

$\therefore$ Taking moments about O, the equation of motion of the rod is

$$M\tfrac{4}{3}a^2\ddot{\theta}=Mg\,OL=Mga\cos\theta.$$

$$\ddot{\theta}=\frac{3g}{4a}\cos\theta. \quad \ldots(3)$$

Substituting the value of $\ddot{\theta}$ from (3) in (2), we have

$$X\sin\theta-Y\cos\theta+Mg\cos\theta=\frac{3g}{4}M\cos\theta$$

or $\quad X \sin \theta - Y \cos \theta = -\frac{1}{4} Mg \cos \theta$. $\quad$ ...(4)

Multiplying both sides of (3) by $2\dot{\theta}$ and integrating, we have

$$\dot{\theta}^2 = \frac{3g}{2a} \sin \theta + C.$$

But initially in the horizontal position of the rod when $\theta = 0$, $\dot{\theta} = 0$.

$\therefore \quad C = 0$.

$\therefore \quad \dot{\theta}^2 = \frac{3g}{2a} \sin \theta$.

Putting the value of $\dot{\theta}^2$ in (1), we have

$$X \cos \theta + Y \sin \theta - Mg \sin \theta = \frac{3Mg}{2} \sin \theta$$

or $\quad X \cos \theta + Y \sin \theta = \frac{5}{2} Mg \sin \theta$. $\quad$ ...(5)

Multiplying (4) by $\sin \theta$, (5) by $\cos \theta$ and adding, we have

$$X = (\tfrac{5}{2} - \tfrac{1}{4}) Mg \sin \theta \cos \theta = \tfrac{9}{8} Mg \sin 2\theta.$$

Again multiplying (5) by $\sin \theta$ and (4) by $\cos \theta$ and subtracting, we have

$$Y = (\tfrac{5}{2} \sin^2 \theta + \tfrac{1}{4} \cos^2 \theta) Mg.$$

Obviously horizontal strain X is maximum, when $\sin 2\theta = 1$, *i.e.* when $2\theta = 90°$ or $\theta = 45°$.

And then the vertical strain

$$Y = (\tfrac{5}{2} \sin^2 45° + \tfrac{1}{4} \cos^2 45°) Mg = \tfrac{11}{8} Mg$$

$$= \tfrac{11}{8} \text{ times the weight of the rod.}$$

Ex. 2. *A circular area can turn freely about a horizontal axis which passes through a point O of its circumference and is perpendicular to its plane. If motion commences when the diameter through O is vertically above O, show that when the diameter has turned through an angle θ, the components of the strain at O along and perpendicular to this diameter are respectively $\frac{1}{3}W$ (7 cos $\theta - 4$) and $\frac{1}{3}W \sin \theta$.* [Meerut 83]

Sol. Let a be the radius of the circular area. Initially the diameter OA is vertically above O. At time t, let the diameter make an angle θ to the vertical. If X and Y are the components of the strain at O along and perpendicular to the diameter OA, then the equations of motion of G are

$$Ma\dot{\theta}^2 = Mg \cos \theta - X, \quad \text{...(1)}$$

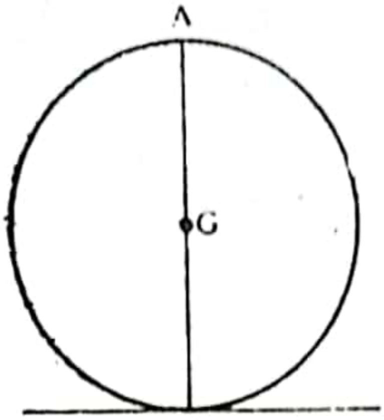
and

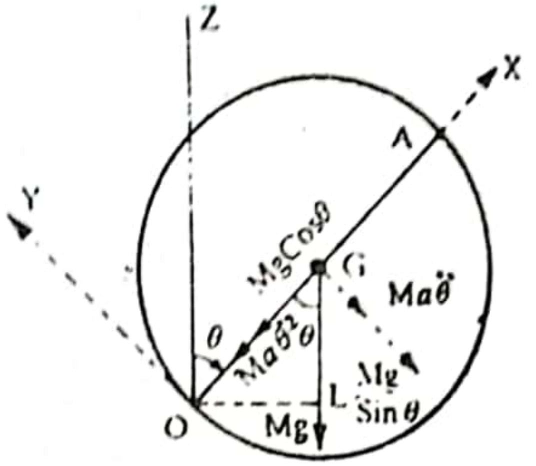
$$Ma\ddot{\theta} = Mg \sin \theta - Y. \quad \text{...(2)}$$

M. I. of the circular area about the axis of rotation through

O and perpendicular to it $= M\frac{1}{2}a^2 + Ma^2 = \frac{3}{2}Ma^2$.

$\therefore$ Taking moments about the axis of rotation, we get
$\frac{3}{2}Ma^2\ddot{\theta} = Mg\,OL = Mg.a \sin\theta$ or $\ddot{\theta} = (2g/3a)\sin\theta$. ...(3)





Multiplying both sides of (3) by $2\dot{\theta}$ and integrating, we have $\dot{\theta}^2 = -(4g/3a)\cos\theta + C$. But initially when $\theta = 0$, we have $\dot{\theta} = 0$, so that $C = 0$.

$$\therefore \quad \dot{\theta}^2 = (4g/3a)(1 - \cos\theta).$$

Substituting the value of $\dot{\theta}^2$ in the equation (1), we have
$X = Mg\cos\theta - Ma.(4g/3a)(1 - \cos\theta) = \frac{1}{3}W(7\cos\theta - 4)$.

[$\because$ $W = Mg$]

Again substituting the value of $\ddot{\theta}$ from (3) in (2), we have

$$Y = Mg\sin\theta - Ma.(2g/3a)\sin\theta = \tfrac{1}{3}W\sin\theta.$$

Ex. 3. *A circular disc of weight W can turn freely about a horizontal axis perpendicular to its plane which passes through a point O on its circumference. If it starts from rest with the diameter vertically above O, show that the resultant pressure on the axis when that diameter is horizontal and vertically below O are respectively $\frac{1}{3}\sqrt{(17)}W$ and $\frac{11}{3}W$. Further prove that the axis must be able to bear at least $\frac{11}{3}$ times the weight of the disc.*

Sol. From Ex. 2, the components of the reaction on the axis of rotation, along and perpendicular to the diameter OA are

$$X = (W/3)(7\cos\theta - 4) \text{ and } Y = \tfrac{1}{3}W\sin\theta.$$

Case I. When the diameter OA is horizontal, then in this position $\theta = \pi/2$. $\therefore$ $X = -\frac{4}{3}W$ and $Y = \frac{1}{3}W$.

$\therefore$ The resultant pressure on the axis in this case

$$= \sqrt{(X^2 + Y^2)} = \sqrt{[(\tfrac{16}{9} + \tfrac{1}{9})W^2]} = \tfrac{1}{3}\sqrt{(17)}W.$$

Case II. When the diameter OA is vertically below O, then in this position $\theta = \pi$. $\therefore$ $X = -\frac{11}{3}W$ and $Y = 0$.

$\therefore$ The resultant pressure on the axis in this case

$$=\sqrt{(X^2+Y^2)}=\sqrt{(\tfrac{121}{9}W^2+0)}=\tfrac{11}{3}W.$$

Finally the resultant pressure P on the axis of rotation at any time t is given by

$$P=\sqrt{(X^2+Y^2)}=\sqrt{[(W^2/9)\{(7\cos\theta-4)^2+\sin^2\theta)\}]}$$
$$=(W/3)\sqrt{(49\cos^2\theta-56\cos\theta+16+\sin^2\theta)}$$
$$=(W/3)\sqrt{(48\cos^2\theta-56\cos\theta+17)},$$

which is greatest when $\cos\theta=-1$ *i.e.* when $\theta=\pi$.

$\therefore$ Maximum pressure on the axis

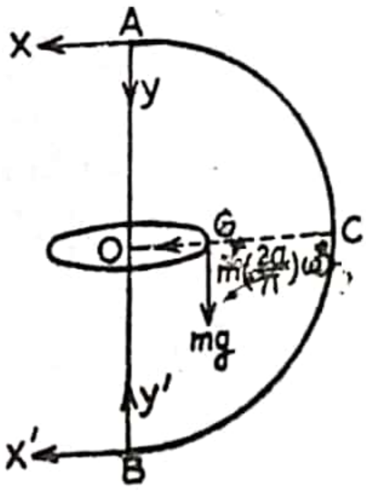
$$=(W/3)\sqrt{(48\cos^2\pi-56\cos\pi+17)}=\tfrac{11}{3}W$$

Hence the axis must be able to bear at least $\frac{11}{3}$ times the weight of the disc.

Ex. 4. *A uniform semi-circular arc, of mass m and radius a, is fixed at its ends to two points in the same vertical line and is rotating with constant angular velocity ω. Show that the horizontal thrust on the upper end is* $m.\dfrac{g+\omega^2 a}{\pi}$.

[Meerut 80, 81, 82, 83, 84, 86, 88]

Sol. Let a uniform semi-circular arc, of mass m, centre O and radius a rotate with constant angular velocity ω about the vertical line AB where A and B are the ends of the arc.



The centre of gravity G of the arc is on the central radius OC such that $OG=2a/\pi$.

As the arc rotates about AB, the C.G. 'G' describes a circle of radius $OG=2a/\pi$ with O as its centre. Since the arc rotates with uniform angular velocity ω, therefore the only effective force on G will be $m(2a/\pi)\omega^2$ along GO.

Let the horizontal and vertical components of reaction at the upper end A be X and Y and those at the lower end B be X' and Y' respectively.

Taking moments about B, we have

moment of the effective force = sum of the moments of the external forces i.e., $m \frac{2a}{\pi} \omega^2.BO = -mg.OG + X.BA$

or $m \frac{2a}{\pi} \omega^2 a = mg.\frac{2a}{\pi} + X.2a$

or $X = \frac{mg}{\pi} + \frac{ma\omega^2}{\pi} = m.\frac{g+\omega^2 a}{\pi}$.

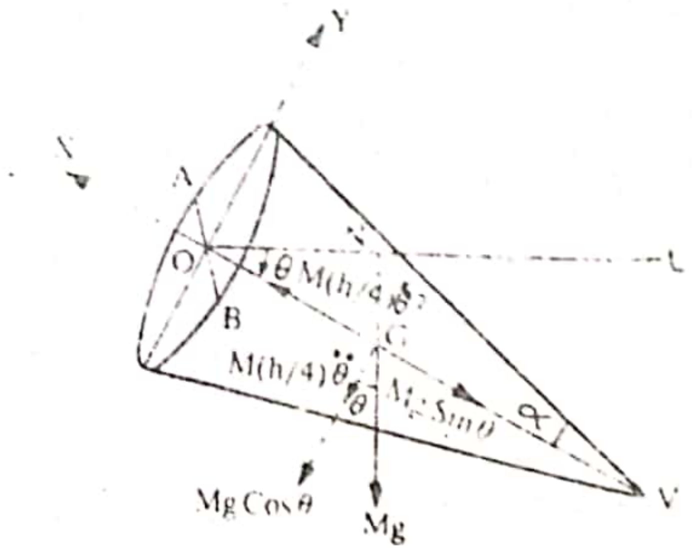
Ex. 5. *A right cone of angle 2α can turn freely about an axis passing through the centre of its base and perpendicular to the axis; if the cone starts from rest with its axis horizontal, show that when, the axis is vertical, the thrust on the fixed axis to the weight of the cone is* $(1+\frac{1}{8}\cos^2\alpha):(1-\frac{1}{8}\cos^2\alpha)$. [Meerut 81; Raj. 80]

Sol. Let a cone of angle 2α and mass M turn freely round a horizontal diameter AOB of the base. Initially the axis OV of the cone is along the horizontal line OL.

At time t, let the axis OV of the cone turn through an angle θ to the horizontal.

If h is the height of the cone and G its C.G., then $OG = h/4$.

$\therefore$ Accelerations of G along and perpendicular to GO are $M(h/4)\dot{\theta}^2$ and $M(h/4)\ddot{\theta}$ respectively.



If X and Y are the components of the reaction (*i.e.* the thrust) on the fixed axis AOB, along and perpendicular to GO, then the equations of motion of G are

$$M(h/4)\dot{\theta}^2 = X - Mg\sin\theta \qquad ...(1)$$

and
$$M(h/4)\ddot{\theta} = Mg\cos\theta - Y. \qquad ...(2)$$

Now the moment of inertia Mk^2 of the cone about the line AOB is given by

$$Mk^2 = \frac{1}{20}Mh^2(2+3\tan^2\alpha).$$ [See Ex. 11 on page 58]

So taking moments about O, we have

$Mk^2\ddot{\theta}=Mg\ ON$ or $\frac{3}{20}Mh^2(2+3\tan^2\alpha)\ddot{\theta}=Mg(h/4)\cos\theta$

or $$h\ddot{\theta}=\frac{5g}{2+3\tan^2\alpha}\cos\theta. \quad ...(3)$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we have

$$h\dot{\theta}^2=\frac{10g}{2+3\tan^2\alpha}\sin\theta+C.$$

But initially when OV is horizontal, $\theta=0$ and $\dot{\theta}=0$, so that $C=0$.

$$\therefore\quad h\dot{\theta}^2=\frac{10g}{2+3\tan^2\alpha}\sin\theta. \quad ...(4)$$

Now from (1) and (4), we have

$$X=\tfrac{1}{4}M.\frac{10g}{2+3\tan^2\alpha}\sin\theta+Mg\sin\theta=\frac{9+6\tan^2\alpha}{2(2+3\tan^2\alpha)}.Mg\sin\theta$$

and from (2) and (3), we have

$$Y=Mg\cos\theta-\tfrac{1}{4}M.\frac{5g}{2+3\tan^2\alpha}\cos\theta=\frac{3+12\tan^2\alpha}{4(2+3\tan^2\alpha)}.Mg\cos\theta.$$

When the axis of the cone is vertical *i.e.* when $\theta=\pi/2$, then

$$X=\frac{9+6\tan^2\alpha}{2(2+3\tan^2\alpha)}.Mg,\ Y=0.$$

$\therefore$ the resultant thrust on the fixed axis in this position

$=P=\sqrt{(X^2+Y^2)}=X \qquad [\because\ Y=0]$

$$=\frac{9+6\tan^2\alpha}{2(2+3\tan^2\alpha)}.Mg=\frac{9\cos^2\alpha+6\sin^2\alpha}{2(2\cos^2\alpha+3\sin^2\alpha)}.Mg$$

or $$\frac{P}{Mg}=\frac{6(\cos^2\alpha+\sin^2\alpha)+3\cos^2\alpha}{2[3(\cos^2\alpha+\sin^2\alpha)-\cos^2\alpha]}=\frac{6+3\cos^2\alpha}{2(3-\cos^2\alpha)}=\frac{1+\frac{1}{2}\cos^2\alpha}{1-\frac{1}{3}\cos^2\alpha}.$$

§ 4·12. Centre of Percussion (Definition).

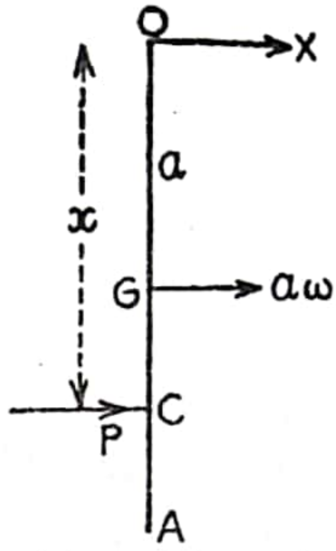
If a body, rotating about a fixed axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of action of the striking force is called a **centre of percussion.** *The line of action of the force is called the* **line of percussion.**

If the line of action of the blow is known, the axis about which the body begins to turn is called the axis of spontaneous rotation. Obviously this coincides with the fixed axis in the first case. [Meerut 80, 80S, 82, 82P, 83P, 84, 84P, 86]

§ 4·13. Centre of Percussion of a Rod.

Let a rod OA of mass M and length $2a$ be suspended freely from the end O. Let a horizontal blow P be given to the rod at the point C such that $OC=x$.

If ω is the angular velocity of the rod due to the impulse, then the centre of gravity G of the rod will move with velocity $a\omega$. Let X be the impulsive action at O.



Since the change in the moment of momentum or the angular momentum about the axis is equal to the moment of the impulsive forces about this axis, therefore taking moments about the axis through O, we have

$$Mk^2(\omega-0)=Px. \quad ...(1)$$

Also consider the motion of the centre of gravity 'G' of the rod. The velocity of the centre of gravity 'G' of the rod after the blow is $a\omega$ perpendicular to the rod *i.e.*, in the horizontal direction. Since the change in the momentum in the horizontal direction is equal to the total external impulse in that direction, therefore we have

$$M(a\omega-0)=P+X. \quad ...(2)$$

If the blow is given through the centre of percussion, then $X=0$.

$\therefore$ From (2), we have $Ma\omega=P$. ...(3)

Dividing (1) by (3), we have

$x=k^2/a=$Length of the equivalent simple pendulum, *i.e.* the centre of percussion of a rod is at a depth $l=(k^2/a)$ from the centre of suspension.

Hence the centre of percussion of a rod coincides with its centre of oscillation.

§ 4·14. Centre of Percussion (In general case) [Meerut 86]

To find the centre of percussion of a body for a given fixed axis proceed as follows.

(i) Find the depth $\bar{x}$ of the centre of gravity of the body below the axis of rotation.

(ii) Find the point where the fixed axis is principal axis of the body. Take a point vertically below this point at a depth $k^2/\bar{x}$ and draw an axis perpendicular to the plane containing the fixed axis and the centre of gravity.

(iii) Any point on this line is a centre of percussion of the body for the given fixed axis.

Solved Examples

Ex. 1. *Find the position of the centre of percussion of a uniform rod with one end fixed.*

Sol. Let $2a$ be the length of the rod and M be its mass. Then h=depth of its C.G. from the fixed end$=a$

and Mk^2=M.I. about the axis of rotation through one end

$$=M.\tfrac{1}{3}a^2+Ma^2=\tfrac{4}{3}Ma^2.$$

$\therefore\quad k^2=4a^2/3.$

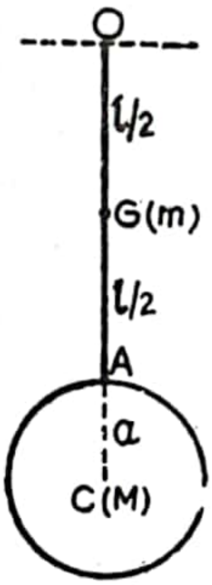
$\therefore$ Distance of the centre of percussion from the fixed end

$$=\frac{k^2}{h}=\frac{4a^2/3}{a}=\frac{4}{3}a.$$

Ex. 2. *A pendulum is constructed of a solid sphere, of mass M and radius a, which is attached to the end of a rod, of mass m and length b. Show that there will be no strain on the axis if the pendulum be struck at a distance*

$$[M\{\tfrac{2}{5}a^2+(a+b)^2\}+\tfrac{1}{3}mb^2]\div[M(a+b)+\tfrac{1}{2}mb]$$

from the axis. [Meerut 89]



Sol. There will be no strain on the axis of the pendulum if it be struck at its centre of percussion which is at a depth k^2/h below O.

Now M.I. of the pendulum about the axis of rotation through O

$$(M+m)\,k^2=M[\tfrac{2}{5}a^2+(a+b)^2]+\tfrac{4}{3}m\,(b/2)^2.$$

$$\therefore\quad k^2=\frac{[M\{\tfrac{2}{5}a^2+(a+b)^2\}+\tfrac{1}{3}mb^2]}{(M+m)}.$$

Also h=Depth of C.G. of the system below O

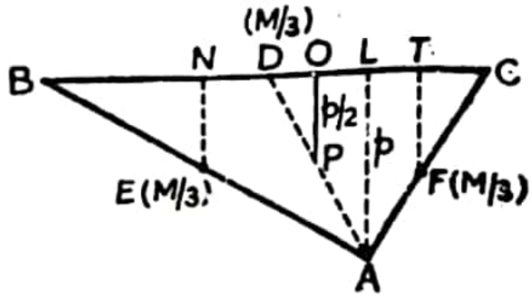
$$=\frac{M.OC+m.OG}{M+m}=\frac{M.(a+b)+m.(b/2)}{M+m}.$$

$\therefore$ Depth of the centre of percussion below O

$$=\frac{k^2}{h}=\frac{[M\{\tfrac{2}{5}a^2+(a+b)^2\}+\tfrac{1}{3}mb^2]}{[M(a+b)+\tfrac{1}{2}mb]}.$$

Ex. 3. *Find the centre of percussion of a triangle ABC which is free to move about its side BC.* [Meerut 80, 82, 83, 84, 87]

Sol. Let ABC be the triangle which is free to move about its side BC.



(i) We proceed to find a point on BC where it is a principal axis of the triangle.

Draw the perpendicular AL from A on BC. Let D be the middle point of BC. Then BC is the principal axis of the triangle at the middle point O of DL. (See Ex. 9 on page 33)

(ii) To find k^2/h.

Let M be the mass of the triangle ABC. Then M.I. of $\triangle ABC$ about BC is equal to the sum of M.I. of three particles each of mass $M/3$ placed at the middle points D, E and F of the sides of the triangle.

$$\therefore \quad Mk^2=\frac{M}{3}.\,0+\frac{M}{3}\,EN^2+\frac{M}{3}\,FT^2=0+\frac{M}{3}\left(\frac{p}{2}\right)^2+\frac{M}{3}\left(\frac{p}{2}\right)^2$$

where $AL=p$

or $\qquad k^2=\frac{1}{6}p^2$.

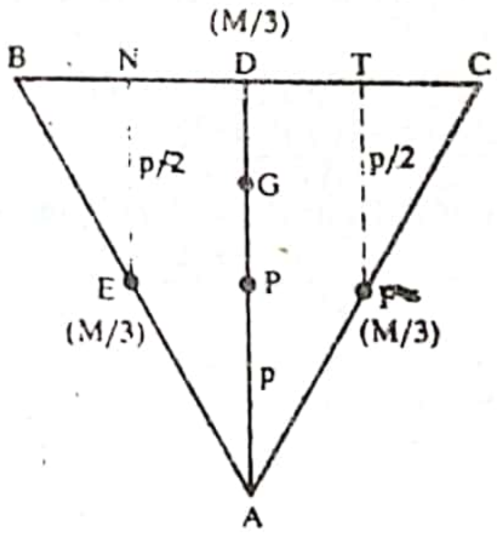
Also $\qquad h=$ Depth of C. G. of $\triangle ABC$ below $BC=p/3$.

$$\therefore \quad \frac{k^2}{h}=\frac{\frac{1}{6}p^2}{(\frac{1}{3}p)}=\frac{p}{2}.$$

(iii) Hence a point vertically below O at a depth $p/2$ below BC is the centre of percussion. Thus the middle point P of the median AD is the centre of percussion.

Ex. 4. *Find how an equilateral triangular lamina must be struck so that it may commence to rotate about a side.*

[Meerut 80, 82, 86, 88]



Sol. Let the equilateral triangle ABC rotate about the side BC. The blow must be given at the centre of percussion of the triangle so that it may commence to rotate about BC.

The perpendicular AD from A on BC will be the median of the triangle. Let $AD=p$.

Here BC will be the principal axis at the point D.

Therefore the centre of percussion will be a point on DA at a depth (k^2/h) below D.

If M is the mass of the $\triangle ABC$ and E, F the middle points of the sides AB and AC, then

$M.k^2$=M. I. of $\triangle ABC$ about BC

$$=\frac{M}{3}.0+\frac{M}{3}.\ EN^2+\frac{M}{3}.\ FT^2=0+\frac{M}{3}\left(\frac{p}{2}\right)^2+\frac{M}{3}\left(\frac{p}{2}\right)^2.$$

$\therefore \quad k^2=p^2/6.$

Also h=Depth of the C. G. of the $\triangle ABC$ below BC

$=\frac{1}{3}DA=\frac{1}{3}p.$

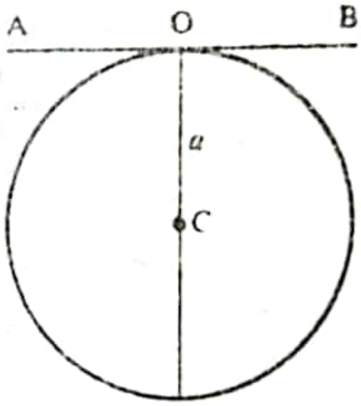
$$\therefore \quad \frac{k^2}{h}=\frac{p^2/6}{p/3}=\frac{p}{2}=\frac{AD}{2}.$$

Therefore the centre of percussion of the equilateral $\triangle ABC$ is at the middle point of the median AD.

Hence the equilateral triangle must be struck at the middle point of the median bisecting the side about which the lamina commences to rotate.

Ex. 5. *Find the position of the centre of percussion of a uniform circular plate with axis a horizontal tangent.*

Sol. Let C be the centre, M the mass and AOB the horizontal tangent of a uniform circular plate.



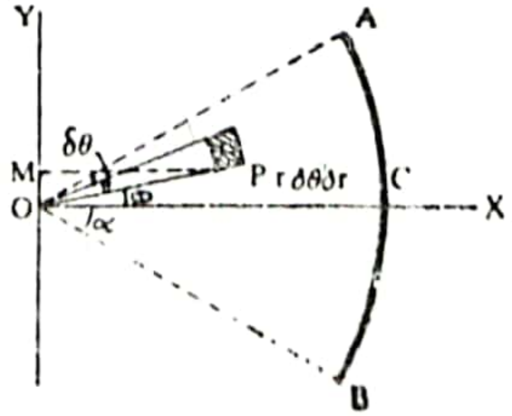
Then the centre of percussion of the plate is k^2/h below O.

Now Mk^2=M. I. of the plate about $AB=\frac{1}{4}\ Ma^2+Ma^2$.

$\therefore \quad k^2=\frac{5}{4}a^2$, where a is the radius and $h=OC=a$.

$\therefore$ Distance of the centre of percussion below the highest point O $=k^2/h=\frac{5}{4}a.$

Ex. 6. *Find the position of the centre of percussion of a sector of a circle, axis in the plane of the sector, perpendicular to its symmetrical radius, and passing through the centre of the circle.*



Sol. Let OAB be the sector of the circle of radius a such that $\angle AOB = 2\alpha$.

Take the x-axis along its symmetrical radius OC and y-axis perpendicular to OX and in the plane of the sector.

Consider an element $r\delta\theta\delta r$ at $P(r, \theta)$. Its mass $\delta m = \rho . r\delta\theta\delta r$.

$\therefore \quad Mk^2 =$ M. I. of the sector about OY

$$= \int_{\theta=-\alpha}^{\alpha} \int_{r=0}^{a} (r \cos\theta)^2 . \rho r d\theta dr \qquad [\because \quad PM = r\cos\theta]$$

$$= \rho \int_{-\alpha}^{\alpha} \cos^2\theta . \left[\tfrac{1}{4} r^4\right]_0^a d\theta = \frac{a^4}{8}\rho \int_{-\alpha}^{\alpha} (1+\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{8} a^4 \rho \left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\alpha}^{\alpha} = \tfrac{1}{8} a^4 \rho . 2\,[\alpha + \tfrac{1}{2}\sin 2\alpha].$$

$\therefore \quad k^2 = \dfrac{a^2}{4\alpha}(\alpha + \sin\alpha\cos\alpha)$, since $M = a^2\alpha\,\rho$.

Also $h =$ Distance of the C. G. of the sector from OY

$$= \frac{2}{3}\,\frac{a\sin\alpha}{\alpha}.$$

Hence the distance of the centre of percussion on OX from O

$$= \frac{k^2}{h} = \frac{3a}{8} . \frac{\alpha + \sin\alpha\cos\alpha}{\sin\alpha} .$$
