A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

Date 7/7/19

Keep Practising!

IMS

MAINS TEST SERIES-2019

(JUNE-2019 to SEPT.-2019)

Under the guidance of K. Venkanna

MATHEMATICS

$\frac{209}{250}$

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-5: IAS(M)/7-JULY-2019

Ti[illegible] Hours

Maximum Marks: 250

INSTRUCTIONS

1 [illegible]estion paper-cum-answer booklet has 50 pages and has [illegible] questions. Please ensure that the copy of the question [illegible]um-answer booklet you have received contains all the questions.

2 [illegible] your Name, Roll Number, Name of the Test Centre and Medium [illegible] appropriate space provided on the right side.

3 [illegible]olidated Question Paper-cum-Answer Booklet, having space [illegible]ach part/sub part of a question shall be provided to them for [illegible] the answers. Candidates shall be required to attempt answer to [illegible]sub-part of a question strictly within the pre-defined space. Any [illegible] the pre-defined space shall not be evaluated."

4. [illegible] must be written in the medium specified in the admission [illegible]ate issued to you, which must be stated clearly on the right side. [illegible]s will be given for the answers written in a medium other than [illegible]ecified in the Admission Certificate.

5. [illegible] should attempt Question Nos. 1 and 5, which are compulsory. [illegible] THREE of the remaining questions selecting at least ONE [illegible] each Section.

6. [illegible] marks carried by each question is indicated at the end of [illegible] Assume suitable data if considered necessary and indicate [illegible]e clearly.

7. [illegible]otations carry their usual meanings, unless otherwise indicated.

8. [illegible]tions carry equal marks.

9. [illegible] must be written in blue/black ink only. Sketch pen, pencil [illegible]her colour should not be used.

10. [illegible] should be done in the space provided and scored out

11. [illegible]idate should respect the instructions given by the invigilator.

12. [illegible]stion paper-cum-answer booklet must be returned in its entirety [illegible] before leaving the examination hall. Do not remove any [illegible]vigilator before leaving the examination hall.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Divyanshu Choudhary

Roll No.

Test Centre Jaipur

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Divyanshu

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMP[illegible] Whe[illegible] [illegible]ing attempted, all its parts/ sub-parts must be attempted con[illegible]. This means that before moving on to the next qu[illegible] candidates must finish attempting all parts/ sub-parts of the pr[illegible] question attempted. This is to be strictly followed Pa[illegible] [illegible]er-book are to be clearly struck out in ink. Any answers that fo[illegible] pages left blank may not be given credit.

P.T.O.

INDEX TABLE

STION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
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	(b)			06	
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	(d)			14	
	(a)				
	(b)				
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			Total Marks		

$\frac{209}{250}$

SECTION – A

1\. olve the following system of linear equations. Identify the set of solutions.

$+ 2x_2 + 3x_3 = 11$ — ①

$x_3 = -2$ — ②

$x_2 + 4x_3 = 9$ — ③

[10]

We have

let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$ where A is coefficient matrix

Reducing A to reduced row echelon form

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ which shows that this system has infinite solutions

From equation 2, $x_2 = 2 + x_3$ — ④

$\Rightarrow x_1 + 2(2 + x_3) + 3x_3 = 11$ (from ①)

$\Rightarrow x_1 + 5x_3 = 7 \Rightarrow x_1 = 7 - 5x_3$ — ⑤

So solution set is given by

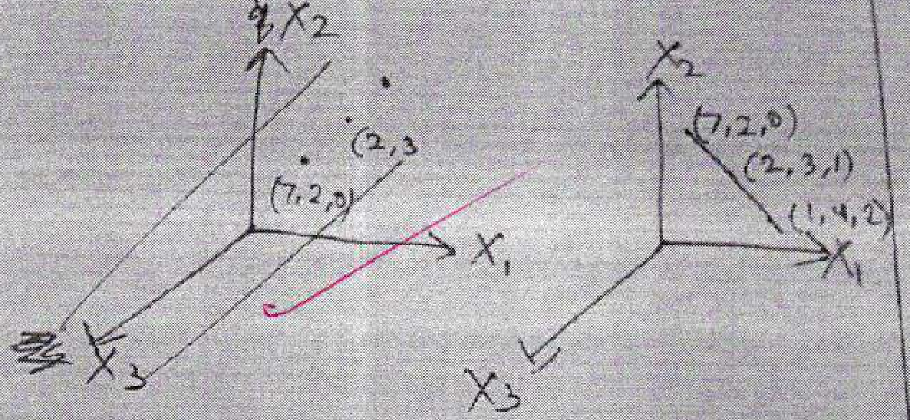
$(x_1, x_2, x_3) = (7 - 5x_3, 2 + x_3, x_3)$ where $x_3 \in R$

~~So~~ for $x_3 = 1$; $(x_1, x_2, x_3) = (2, 3, 1)$

For $x_3 = 2$; $(x_1, x_2, x_3) = (1, 4, 2)$

Now $x_1 = 7 - 5x_3$, $x_2 = 2 + x_3$

$\Rightarrow \frac{x_1 - 7}{-5} = \frac{x_2 - 2}{1} = \frac{x_3 - 0}{1}$ which represents a line



09

1. ... a, b, and c be elements of a field F, and let

$$= \begin{bmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{bmatrix}$$

rove that the characteristic polynomial for A is $x^3 - ax^2 - bx - c$ and that this is ... the minimal polynomial for A. **[10]**

For characteristic polynomial, $|A - \lambda I| = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} -\lambda & 0 & c \\ 1 & -\lambda & b \\ 0 & 1 & a-\lambda \end{vmatrix} = 0$$

$\Rightarrow -\lambda(-a\lambda + \lambda^2 - b) + c(1) = 0$

$\Rightarrow a\lambda^2 - \lambda^3 + b\lambda + c = 0$

$\Rightarrow \lambda^3 - a\lambda^2 - b\lambda - c = 0$ & by Carlyle Hamilton Theorem, this is the Characteristic eq^n whose characteristic polynomial is given by $x^3 - ax^2 - bx - c = 0$

Since a, b, c are distinct elements of a field F

??

$\Rightarrow$ The roots of given characteristic polynomial are distinct & hence it is diagonizable

Since for a diagonizable matrix, characteristic polynomial = minimal polynomial

$\therefore$ Minimal polynomial $= x^3 - ax^2 - bx - c$

07

1. [illegible] f(x + y) = f(x) + f(y) for all x, y ∈ R and f(1) = 1, then evaluate [illegible] $2^{\tan x}$ [illegible] f(sin x)

[10]

We have, $f(x+y) = f(x)+f(y)$ & $f(1)=1$

$\Rightarrow f(0) = 2f(0) \Rightarrow f(0)=0$

$\Rightarrow f(0) = f(x)+f(-x) \Rightarrow f(-x) = -f(x)$

Now

$$\lim_{x\to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)} \to 0 \text{ for } x \to 0$$

By L Hospital rule

$$\lim_{x\to 0} \frac{2^{f(\tan x)}\left[\log 2\, f'(\tan x)\sec^2 x\right] - 2^{f(\sin x)}\left[\log 2\, f'(\sin x)\cos x\right]}{2x f(\sin x) + x^2 f'(\sin x)\cos x}$$

Now we know that $f(x+y) = f(x)+f(y)$

$\Rightarrow f(x) = ax$. Since $f(1)=1 \Rightarrow f(1) = a = 1$

$\Rightarrow f(\tan x) = \tan x,\ f(\sin x) = \sin x$

$$\Rightarrow \lim_{x\to 0} \frac{2^{\tan x}\left[\log 2 \sec^2 x\right] - 2^{\sin x}\left[\log 2 \cos x\right]}{3x^2}$$

09

$\left[x^2 \sin x = x^3 \cdot \left(\frac{\sin x}{x}\right)_{x\to 0} = x^3 \text{ in denominator}\right]$

$$\Rightarrow \log 2 \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2} = \log 2 \lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x}$$

$$= \log 2 \lim_{x\to 0} \frac{\cos x + 4\sec^2 x \tan x + 2\sec^4 x}{6}$$

$$= \log 2 \left(\frac{1+2}{6}\right) = \frac{1}{2}\log 2$$

$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \;\forall\, x > 0.$ [10]

Let $f(x) = \log(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3(1+x)}$ & $f(0)=0$

$\Rightarrow f'(x) = \frac{1}{1+x} - 1 + x - \frac{1}{3}\frac{(1+x)(3x^2) - x^3}{(1+x)^2}$

$= \frac{1}{1+x} + x - 1 - \frac{1}{3}\cdot\frac{1+2x^3}{(1+x)^2}$

$= \frac{1+x+(x-1)(1+x)^2 - \frac{1}{3}(1+2x^3)}{(1+x)^2} = \frac{x^3/3 + x^2 - 1/3}{(1+x)^2}$

Let $g(x) = x^3/3 + x^2 - 1/3 \Rightarrow g'(x) = x^2 + 2x > 0$ $\forall x > 0 \Rightarrow g'(x) > 0 \;\forall\, x > 0$

$\Rightarrow g(x)$ is a monotonically increasing function

$\Rightarrow f(x)$ is a monotonically increasing function

$[\because \frac{1}{(1+x)^2} > 0 \Rightarrow f(x) > 0$ & $f(0) = 0 \Rightarrow f(x) > f(0) \;\forall x > 0]$

Let $h(x) =$ So $\log(1+x) > x - \frac{x^2}{2} + \frac{x^3}{3(1+x)}$

Let $h(x) = x - \frac{x^2}{2} + \frac{x^3}{3} = \log(1+x)$ & $h(0) = 0$

$\Rightarrow h'(x) = 1 - x + x^2\cancel{3} - \frac{1}{1+x}$

08

$= \frac{1+x-x-x^2+x^2+x^3-1}{1+x} = \frac{x^3}{1+x} > 0 \;\forall x > 0$

$\Rightarrow h(x)$ is a monotonically increasing function

$\Rightarrow h(x) > h(0) \;\forall x > 0 \Rightarrow x - \frac{x^2}{2} + \frac{x^3}{3} > \log(1+x)$

P.T.O.

A point P moves on the plane $x/a + y/b + z/c = 1$ which is fixed, and the plane through P perpendicular to OP meets the axes in A, B, C. If the planes through A, B, C parallel to the co-ordinates planes meet in a point Q, show that the locus of Q is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}$$

[10]

Let coordinates of P be (α, β, γ)

$\Rightarrow \alpha/a + \beta/b + \gamma/c = 1$ — ①

Now drs of OP are α, β, γ [$\because \alpha-0, \beta-0, \gamma-0$]

which are of the plane as well

$\Rightarrow$ Eq^n of plane can be written as

$\alpha[x-\alpha] + \beta[y-\beta] + \gamma[z-\gamma] = 0$

Now at A, y & z coordinates are 0

$\Rightarrow \alpha[x-\alpha] + \beta(-\beta) + \gamma(-\gamma) = 0$

$\Rightarrow x = \frac{\beta^2+\gamma^2}{\alpha} + \alpha = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$

So $A = \left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right)$ & similarly B & C

are $\left(0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right)$ & $\left(0, 0, \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right)$

08

Now coordinates of Q are given by (A, B, C)

Let $Q = (x_1, y_1, z_1) = \left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right)$

From ①, drs of OP are α, β, γ & also are (as OP is $\perp$ to plane) — ②

$1/a, 1/b, 1/c$

$\Rightarrow x_1\alpha + y_1\beta + z_1\gamma = \alpha^2+\beta^2+\gamma^2$

$\Rightarrow \frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z = \cancel{(1/a)}\ (1/x)^2 + (1/y)^2 + (1/z)^2$ (Substituting from ②)

Hence proved

P.T.O.

basis for the column space of the following matrix A.

$$\begin{matrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{matrix}\Bigg]$$

[08]

We have, $A^T = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$ for which we find the rowspace by converting into RRE form

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \quad (R_2 \to R_2 - R_1)$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad (R_3 \to R_3 + 2R_2)$$

So the rowspace of A^T = column space of $A = \{(1,2,-1),(0,1,-3)\}$ which is the required basis

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P.T.O.

basis for the subspace V of R^4 spanned by the vectors
(1, 2, 3, 4), (-1, -1, -4, -2), (3, 4, 11, 8).

We have

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$ which we convert

to RRE form

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \begin{Bmatrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 3R_1 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 + 2R_2)$$

So the 2 LI vectors are (1, 2, 3, 4) & ~~(-1,~~ (0, 1, -1, 2)

For finding basis of V or R^4, we need to extend this by having 2 more vectors

So (0, 1, 0, 0) & (0, 0, 1, 0) can be added as all 4 of them are LI

So required ~~Kais~~ basis is [(1, 2, 3, 4), (0, 1, -1, 2), (0, 1, 0, 0), (0, 0, 1, 0)]

06

Extended basis isn't required!!

(he maximum and minimum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, when $lx + my + nz = 0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

ret the result geometrically. [18]

By Lagrange formula, we have

let ~~$f(x)$~~ $f(x,y,z) = \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}+\lambda(lx+my+nz)+$

$$\mu\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]=0$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{2x}{a^4}+\lambda l+\frac{2x\mu}{a^2}=0 \quad \text{— ①}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{b^4}+\lambda m+\frac{2y\mu}{b^2}=0 \quad \text{— ②}$$

$$\frac{\partial f}{\partial z} = \frac{2z}{c^4}+\lambda n+\frac{2z\mu}{c^2}=0 \quad \text{— ③}$$

Multiplying ①, ② & ③ by x, y, z respectively & adding

$$\Rightarrow \left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)+\lambda(0)+2\mu=0$$

$$\Rightarrow \mu = -\left[\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right] \quad \text{— ④}$$

From ①, $2x\left(\frac{1}{a^4}+\frac{\mu}{a^2}\right) = -\lambda l$

$\Rightarrow x = \frac{-\lambda l}{2} \times \frac{a^4}{1+\mu a^2}$ & similarly

$y = \frac{-\lambda m}{2} \times \frac{b^4}{1+\mu b^2}$ & $z = \frac{-\lambda n}{2} \times \frac{c^4}{1+4c^2}$

Substituting these values in $lx+my+nz=0$,

P.T.O.

$l\left[\frac{-\lambda l}{2}\times\frac{a^4}{1+\mu a^2}\right]+m\left[\frac{-\lambda \cancel{l}m}{2}\times\frac{b^4}{1+\mu b^2}\right]+n\left[\frac{-\lambda n}{2}\times\frac{c^4}{1+\mu c^2}\right]=0$

$\Rightarrow \lambda = 0$ or $\frac{l^2a^4}{\cancel{2}(1+\mu a^2)}+\frac{m^2b^4}{(1+\mu b^2)}+\frac{n^2c^4}{(1+\mu c^2)}=0$

If $\lambda = 0$

$\Rightarrow \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4} \cancel{*} - \cancel{2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$

$\Rightarrow \left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\left[2-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right]=0$

Since ~~$\frac{x^2}{a^2} \neq$~~ 2nd term can't be 0 $\Rightarrow$

$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=0 \Rightarrow x=y=z=0$ is the ~~minimum value~~ value which is impossible by 2nd condition

$\Rightarrow \frac{l^2a^4}{(1+\mu a^2)}+\frac{m^2b^4}{(1+\mu b^2)}+\frac{n^2c^4}{(1+\mu c^2)}=0$

gives the minimum and maximum values of $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=-\mu$

Geometrically, the intersection of plane & ellipse with ~~the curve~~ another ellipse gives the minimum & maximum values.

16

(i) A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2.$$

(ii) Prove that the equation

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a cone if $u^2/a + v^2/b + w^2/c = d$. **[16]**

(i) Let equation of plane by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Since (p, q, r) lies on it $\Rightarrow \frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1$ —①

Now plane cuts x, y, z axes in a, b, C

$\Rightarrow A=(a,0,0),\ B=(0,b,0),\ C=(0,0,c)$

Now a sphere which passes through O is

$x^2+y^2+z^2+\cancel{px+qy}\,Dx+Ey+Fz=0$

Since it passes through A, B, C

$\Rightarrow a^2+aD=0,\ b^2+b\cancel{E}=0,\ c^2+\cancel{c}cF=0$

$\Rightarrow D=-a,\ E=-b,\ F=-c$

So coordinates of centre are $(a/2, b/2, c/2)$

$= (\alpha, \beta, \gamma)$ say

$\Rightarrow \cancel{\alpha}\ a=2\alpha,\ b=2\beta,\ c=2\gamma$

$\Rightarrow$ From ①, $\frac{p}{2\alpha}+\frac{\cancel{p}q}{2\beta}+\frac{\cancel{p}r}{2\gamma}=1 \Rightarrow \frac{p}{\alpha}+\cancel{p}\frac{q}{\beta}+\frac{r}{\gamma}=2$

07

So locus of centre is $\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2$

(ii) Converting into standard form, we get

$ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$

Differentiating w.r.t. x, y, z & t; we get

$2ax + 2ut = 0$; $2by + 2vt = 0$; $2cz + 2wt = 0$

$\Rightarrow x = \frac{-ut}{a}$, $y = \frac{-vt}{b}$, $z = \frac{-wt}{c}$

For $t = 1$, we have ~~x~~ $\Rightarrow \left(-u/a, -v/b, -w/c\right)$

Which satisfies $2ux + 2vy + 2wz + 2dt = 0$
(For vertex) [Keeping $t = 1$]

$\Rightarrow u\left(-u/a\right) + v\left(-v/b\right) + w\left(-w/c\right) \cancel{a} = -d$

$\Rightarrow \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$

07

(b) Determine the kernel and the range of the transformation defined by the following matrix:

$$A = \begin{bmatrix} \text{[illegible]} & 2 & 3 \\ 1 & -1 & 1 \\ \text{[illegible]} & 1 & 4 \end{bmatrix}$$ [12]

SECTION - B

nd the orthogonal trajectories of $r^n \sin n\theta = a^n$. [10]

We have $r^n \sin n\theta = a^n$

Taking log, $n \log r + \log \sin n\theta = n \log a$

$\Rightarrow \cancel{n}\left(\frac{1}{r}\right)\frac{dr}{d\theta} + \frac{1}{\sin n\theta}\cdot \cancel{n}\cos n\theta = 0$

$\Rightarrow \left(\frac{1}{r}\right)\left(\frac{dr}{d\theta}\right) = -\cot n\theta$

Now for finding orthogonal trajectories we replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$\Rightarrow \frac{1}{r}\left(r^2\frac{d\theta}{dr}\right) = \cot n\theta \Rightarrow \frac{r\,d\theta}{dr} = \cot n\theta$

$\Rightarrow \tan n\theta = \frac{dr}{r}$

$\Rightarrow \log r = \int \tan n\theta \cancel{+c}$

$\Rightarrow \log r = \frac{-1}{n}\log(\cos n\theta) \cancel{+c} + \log c$

$\Rightarrow \log\cancel{a}\, r^n = -\log(\cos n\theta) \cancel{+cn} + \log c$

$\Rightarrow \cancel{r^n}\,\cancel{=}\, \log r^n + \log \cos n\theta = \log c$

$\Rightarrow \boxed{r^n \cos n\theta = c}$

is the required trajectory

08

olve and examine for singular solution of $x^2(y - xp) = yp^2$ [10]

We have, $y - xp = \frac{yp^2}{x^2} \Rightarrow y\left(1 - \frac{p^2}{x^2}\right) = xp$

$\Rightarrow y = \frac{x^3 p}{x^2 - p^2}$

Differentiating w.r.t. x, we get

$$\Rightarrow p = \frac{(x^2-p^2)\left(3x^2p + x^3\frac{dp}{dx}\right) - (x^3p)\left(2x - 2p\frac{dp}{dx}\right)}{(x^2-p^2)^2}$$

$$\Rightarrow p(x^2-p^2)^2 = x^4p + p^2x^3\frac{dp}{dx} + x^5\frac{dp}{dx} - 3x^2p^3$$

$$\Rightarrow \cancel{x^4p} + p^5 - 2x^2p^3 = \cancel{x^4p} + p^2x^3\frac{dp}{dx} + x^5\frac{dp}{dx} - 3x^2p^3$$

$$\Rightarrow p^5 + x^2p^3 = x^2\frac{dp}{dx}(p^2 + x^2)$$

$$\Rightarrow p^2\cancel{(x^2+p^2)} = x^2\frac{dp}{dx}\cancel{(p^2+x^2)} \Rightarrow \frac{dp}{p^2} - \frac{dx}{x^2} = 0$$

$\Rightarrow \frac{1}{x} - \frac{1}{p} = \cancel{1/c}\; c$ is the required solution

For singular solⁿ, we have

$yp^2 = x^2y - x^3p \Rightarrow yp^2 + x^3p - x^2y = 0$

By taking discriminant, $x^6 + 4x^2y^2 = 0$

09

$\Rightarrow x^2(x^4 + 4y^2) = 0 \Rightarrow x = 0$ or $x^4 + 4y^2 = 0$

$x = 0$ is not a singular solⁿ by given eqⁿ which gives $0 = yp^2$ which is absurd

For $x^4 + 4y^2 = 0 \Rightarrow 4x^3 + 8yp = 0 \Rightarrow p = \frac{-2y}{x^3}$

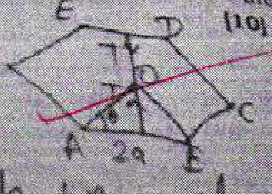
Substituting in given eqⁿ $x^2\left[y + \cancel{2y}\,\frac{2y}{x^2}\right] = y\left(\frac{-2y}{x^3}\right)^2$

$\Rightarrow 2y + x^2y = \frac{4y^3}{x^6}$ which is not true

So given eqⁿ has no singular solutions

equal rods AB, BC, CD, DE, EF and FA are ca… e freely jointed
at their extremities so as to form a hexagon, the rod A… izontal position
and the middle points of AB and DE are jointed b… tension in the
string. [10]

Let W be weight of each rod
& 2a be length of each rod



⇒ By given figure in which we apply virtual work method, we get

$$-T\,\delta(4a\cot\theta) + 6W\,\delta(2a\cot\theta) = 0$$

$$\Rightarrow \cancel{-2T}\left(-T\cdot 4 + 12W\right)\delta(a\cot\theta) = 0$$

$$\Rightarrow -4T + 12W = 0 \quad \Rightarrow T = 3W$$

08

[illegible] in a S. H. M. u, v, w be the velocities at distan[illegible] b, [illegible] fixed point on the straight line which is not the centre of force, [illegible] th[illegible] period T is given by the equation

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$ [10]

We have, by principles of SHM

$u^2 = \mu\left[A^2-(l+a)^2\right]$ where A is amplitude & l is distance from fixed point

$\Rightarrow v^2 = \mu\left[A^2-(l+b)^2\right]$ & $w^2=\mu\left[A^2-(l+c)^2\right]$

$\Rightarrow \cancel{\frac{u^2}{\mu}-a^2+2al}\ \frac{u^2}{\mu}+a^2+2al+(l^2-A^2)=0$

& similarly $\frac{v^2}{\mu}+b^2+2bl+(l^2-A^2)=0$

$\frac{w^2}{\mu}+c^2+2cl+(l^2-A^2)=0$

$$\Rightarrow \begin{vmatrix} u^2/\mu+a^2 & a & 1 \\ v^2/\mu+b^2 & b & 1 \\ w^2/\mu+c^2 & c & 1 \end{vmatrix}=0$$ [Taking 2l & l^2-A^2 as common]

$$\Rightarrow \begin{vmatrix} u^2/\mu & a & 1 \\ v^2/\mu & b & 1 \\ w^2/\mu & c & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$\Rightarrow \frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} = -\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} = \mu\cdot(a-b)(b-c)(c-a) \ \& \ \mu=\frac{4\pi^2}{T^2}$$

$$= \frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) \quad \left[\because T=\frac{2\pi}{\sqrt{\mu}}\right]$$

08

Verify Green's theorem in the plane for $\oint_C (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$. **[10]**

By Green's Theorem

Verify Green's Theorem!

$$\oint \cancel{C}\, M\,dx + N\,dy = \oint_C \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \text{Given integration} = \oint_C (2x - x - 2y)\,dx\,dy$$

$$= \oint_C (x - 2y)\,dx\,dy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dx\,dy$$

$[\because y = x \;\&\; y = x^2$ intersect at $(1,1)]$

$$\Rightarrow \int_0^1 \cancel{[xy +}\; x[y]_{x^2}^{x} - \cancel{2}[y^2]_{x^2}^{x}\,dx$$

$$= \int_0^1 x(x - x^2) - (x^2 - x^4)\, dx$$

$$= \int_0^1 (-x^3 + x^4)\, dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

04

17.

Solve $(x^3y^3 + x^2y^2 + xy + 1)\, y\, dx + (x^3y^3 - x^2y^2 - xy + 1)\, x\, dy = 0$ **[10]**

olid hemisphere rests on a plane inclined to th [illegible] e $\alpha < \sin^{-1}\frac{3}{8}$,

l the plane is rough enough to prevent an sliding [illegible] e position of

ilibrium and show that it is stable **[17]**

By given Conditions we have

$$\frac{3r/8}{\sin\alpha} = \frac{r}{\sin\theta}$$ [where r is the radius of hemisphere]

$$\Rightarrow \sin\theta\left(\frac{3}{8}\right) = \sin\alpha \Rightarrow \sin\theta = \frac{8}{3}\sin\alpha$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)$$ [Position of equilibrium]

$$\Rightarrow \sin\alpha < 3/8$$ as given

Now to determine its stability, we have by equation of stable equilibrium

$\frac{1}{h} > \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\cos\alpha$ [where ρ_1 & ρ_2 are radii of 2 contact surfaces]

Here $\rho_1 = r$ & $\rho_2 = \infty$

$\Rightarrow \frac{1}{h} > \frac{1}{r}\sec\alpha \Rightarrow h < r\cos\alpha$ — ①

[where h is distance of G (centre of gravity) from contact surface]

Now we have

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha} \text{ [by sine rule]}$$

$$\Rightarrow h = \frac{3r}{8}\frac{\sin(\theta-\alpha)}{\sin\alpha} \text{ — ②}$$

Substituting ② in ①, we get

$$\frac{3r}{8}\frac{\sin(\theta-\alpha)}{\sin\alpha} < r\cos\alpha$$

$$\Rightarrow \sin(\theta-\alpha) < \frac{8}{3}\sin\alpha\cos\alpha$$

$$\Rightarrow \sin\theta\cos\alpha - \cos\theta\sin\alpha < \frac{8}{3}\sin\alpha\cos\alpha$$

$$\Rightarrow \underset{[\because \sin\theta = 8/3 \sin\alpha]}{\frac{8}{3}\sin\alpha\cos\alpha} - \sqrt{1-\frac{64}{9}\sin^2\alpha}\ \sin\alpha < \frac{8}{3}\sin\alpha\cos\alpha$$

$\Rightarrow \sqrt{9-64\sin^2\alpha}\ \sin\alpha > 0$ which is always true as $\sin\alpha < 3/8$

Hence equilibrium is always stable

16

article moves in a straight line, its acceleration dire[illegible]ed fixed point
n the line and is always equal to $\mu(a^5/x^2)^{1/3}$ w[illegible] ce x from O.
t starts from rest at a distance a from O, then find [illegible] particle will
ive at O. [17]

We have, $\frac{d^2x}{dt^2} = -\mu \frac{a^{5/3}}{x^{2/3}}$ [as it is directed towards O]

Multiplying both sides by $2\frac{dx}{dt}$ & integrating, we get

$$(dx/dt)^2 = -6\mu a^{5/3} x^{1/3} + c$$

At $x = a$, $dx/dt = 0$ (given)

$$\Rightarrow 0 = -6\mu a^{5/3} a^{1/3} + c \Rightarrow c = 6\mu a^2$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$$

$$\Rightarrow \frac{dx}{dt} = -\sqrt{6\mu a^{5/3}}\sqrt{a^{1/3} - x^{1/3}}$$ (as it is directed towards O which is decreasing direction of x)

$$\Rightarrow dt = \frac{-1}{\sqrt{6\mu a^{5/3}}}\frac{dx}{\sqrt{a^{1/3} - x^{1/3}}}$$

Substituting $x = a\sin^6\theta$, we get

$$\int dt = \int_a^0 \frac{-1}{\sqrt{6\mu a^{5/3}}} \frac{1 \cdot 6a\sin^5\theta\cos\theta}{\sqrt{a^{1/3} - a^{1/3}\sin^2\theta}}\,d\theta$$

P.T.O.

$$\Rightarrow t = \frac{1}{\sqrt{6\mu a^{5/3}}} \int_0^{\pi/2} \frac{6a \sin^5\theta \cos\theta}{a^{1/6}\cos\theta}\, d\theta$$

$$= \frac{8}{15}\sqrt{\frac{6}{\mu}}$$ which is the required time to reach centre O.

16

[illegible] projectile aimed at a mark which is in a horiz[illegible] pl[illegible] the point of [illegible]ction, falls a metres short of it when the an[illegible] and goes b [illegible]tres beyond when the angle of projection is [illegible] projection be same in all cases, find the correct angle of p[illegible]ion. **[16]**

Let correct angle of projection be θ & velocity be v

$$\Rightarrow \frac{v^2 \sin 2\theta}{g} - \frac{v^2 \sin 2\alpha}{g} = a \quad \text{—} ①$$

$$\Rightarrow \frac{v^2 \sin 2\beta}{g} - \frac{v^2 \sin 2\theta}{g} = b \quad \text{—} ②$$

From ①, $v^2(\sin 2\theta - \sin 2\alpha) = ag$ — ③

From ②, $v^2(\sin 2\beta - \sin 2\theta) = bg$ — ④

Dividing ③ by ④, we get

$$\frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta} = \frac{a}{b}$$

$$\Rightarrow (a+b)\sin 2\theta = a\sin 2\beta + b\sin 2\alpha$$

$$\Rightarrow \sin 2\theta = \left(\frac{a}{a+b}\right)\sin 2\beta + \left(\frac{b}{a+b}\right)\sin 2\alpha$$

$$\Rightarrow 2\theta = \sin^{-1}\left[\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right]$$

$$\Rightarrow \theta = \frac{1}{2}\sin^{-1}\left[\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right]$$

is the required angle.

15

cceleration a of a particle at any time $t \geq 0$ is given … $- 6(t + 1)$ **j** +
… k. If the velocity **v** and displacement **r** are zero at t … and **r** at any
m… **[10]**

We have, $a = e^{-t} i + (-6t-6) j + 3\sin t\, k$

$\Rightarrow \frac{dv}{dt} = e^{-t} i + (-6t-6) j + 3\sin t\, k$

$\Rightarrow v(t) = -e^{-t} i + (-3t^2-6t) j - 3\cos t\, k + C_1$

Since $v(t) = 0$ at $t = 0$

$\Rightarrow 0 = -i - 3k + C_1 \Rightarrow C_1 = i + 3k$

$\Rightarrow v(t) = (1-e^{-t}) i + (-3t^2-6t) j + (3-3\cos t) k$

$\Rightarrow \frac{dx}{dt} = (1-e^{-t}) i + (-3t^2-6t) j + (3-3\cos t) k$

$\Rightarrow x(t) = (t+e^{-t}) i + (-t^3-3t^2) j + (3t-3\sin t) k + C_2$

Since $x(t) = 0$ at $t = 0$

$\Rightarrow 0 = i + C_2 \Rightarrow C_2 = -i$

$\Rightarrow x_2(t) = (t+e^{-t}-1) i + (-t^3-3t^2) j + (3t-3\sin t) k$

08

κ and τ for the space curve $x = t, y = t^2, z =$ (10)

We have

$$r = xi + yj + zk = ti + t^2 j + t^3 k$$

$$\Rightarrow \kappa = \frac{\left|dr/dt \times d^2r/dt^2\right|}{\left|dr/dt\right|^3} \quad \& \quad \tau = \frac{\left[dr/dt \;\; \frac{d^2r}{dt^2} \;\; \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

$$\Rightarrow \frac{dr}{dt} = i + 2tj + 3t^2k, \quad \frac{d^2r}{dt^2} = 2j + 6tk$$

$$\& \quad \frac{d^3r}{dt^3} = 6k$$

$$\Rightarrow \left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right| = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \boxed{6t^2 i - 6tj + 2k}$$

$$\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right| = \sqrt{36t^4 + 36t^2 + 4}$$

$$\Rightarrow \left|\frac{dr}{dt}\right|^3 = \left(\sqrt{1 + 4t^2 + 9t^4}\right)^3$$

$$2\sqrt{9t^4 + 9t^2 + 1}$$

$$\Rightarrow \kappa = \boxed{\frac{2(3t^2 i - 3tj + k)}{(1 + 4t^2 + 9t^4)^{3/2}}}$$

$$\Rightarrow \left[dr/dt \;\; d^2r/dt^2 \;\; d^3r/dt^3\right] = (6t^2 i - 6tj + 2k) \cdot (6k) = 12$$

$$\Rightarrow \tau = \frac{12}{(36t^4 + 36t^2 + 4)} = \frac{3}{9t^4 + 9t^2 + 1}$$

06

P.T.C.

$2x^2 i - 3yz\, j + xz^2 k$ and $\phi = 2z - x^3y$, find A ...

...(r) such that $\nabla\phi = \frac{r}{r^5}$ and $\phi(1) = 0$.

...the constants a and b so that the surface ax^2 ...
...onal to the surface $4x^2y + z^3 = 4$ at the point (1 ...

(i) $\nabla\phi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\phi = -3x^2y\, i - x^3 j + 2k$

Given $A = 2x^2 i - 3yz\, j + xz^2 k$

$\Rightarrow A \cdot \nabla\phi = -6x^4y + 3x^3yz + 2xz^2$

At $(x, y, z) = (1, -1, 1)$

$A \cdot \nabla\phi = -6(-1) + 3(-1) + 2 = 5$

$$A \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix} =$$

$i\left[-6yz + x^4z^2\right] - j\left[4x^2 + 3x^3yz^2\right] + k\left[-2x^5 - 9x^2y^2z\right]$

14

At $(x, y, z) = (1, -1, 1)$

$A \times \nabla\phi = i\left[6 + 1\right] - j\left[4 - 3\right] + k\left[-2 - 9\right]$

$= 7i - j - 11k$

(ii) We have

$\nabla\phi = \vec{r}/r^5$ & $\phi(1) = 0$

We know that $\nabla\phi = \phi'(r)\, \cancel{r}\, \frac{\vec{r}}{r}$

P.T.O.

$\because \phi'(r)\frac{\vec{r}}{r} = \frac{\vec{r}}{r^5} \Rightarrow \phi'(r) = \frac{1}{r^4}$

$\Rightarrow \phi(r) = \frac{-1}{3r^3} + c$ & $\phi(1) = 0$

$\Rightarrow 0 = \frac{-1}{3} + c \Rightarrow c = \frac{1}{3}$

$\Rightarrow \phi(r) = \frac{1}{3}\left(1 - \frac{1}{r^3}\right)$

(iii) Given surface is

$ax^2 - (a+2)x - byz = 0$ & it passes

through $(1, -1, 2)$

$\Rightarrow \cancel{a + (a-}\ a - (a+2) + 2b = 0 \Rightarrow 2b = 2$
$\Rightarrow \boxed{b = 1}$

Now by orthogonality

$(\text{grad} f_1) \cdot (\text{grad} f_2) = 0$; where $f_1 =$
$ax^2 - (a+2)x - byz = 0$ & $f_2 = 4x^2y + z^3 - 4$

$\Rightarrow \text{grad} f_1 = \left[2ax - (a+2)\right] i - bz\, j - by\, k$

$\text{grad} f_2 = \left[8xy\right] i + 4x^2 j + 3z^2 k$

Now at $(1, -1, 2)$

$\text{grad} f_1 = \left[\cancel{2a}\ [a-2]\right] i - [illegible]\ 2j + k$

$\text{grad} f_2 = -8i + 4j + 12k$

$\Rightarrow (a-2)(-8) - 8 + 12 = 0 \Rightarrow -8a + 16 + 4 = 0$

$\Rightarrow -8a = -20 \Rightarrow \boxed{a = 5/2}$

P.T.C.

Divergence Theorem of Gauss, evaluate the surf

$y^2+c^2z^2)^{-1/2}$ dS, [15]

is the surface of the ellipsoid

$+cz^2=1$, a, b and c being all positive constants

We have

By Gauss divergence theorem $\iint (F\cdot n)ds = \iiint(\nabla\cdot F)dv$

$\iint (a^2x^2 + \cancel{b^2y^2 + c^2z^2)}$

Now here,

$$\hat{n} = \frac{2ax\,i + 2by\,j + 2cz\,g\,k}{\sqrt{4a^2x^2+4b^2y^2+4c^2z^2}}$$

$$= \frac{ax\,i + by\,j + cz\,k}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \quad \text{—①}$$

Now $\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2}ds$ gives

$F\cdot\hat{n} = (a^2x^2+b^2y^2+c^2z^2)^{-1/2}$

Substituting value of ~~4~~ $\hat{n}$ from ①,

we get

$$F\cdot\frac{ax\,i+by\,j+cz\,k}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}}$$

$\Rightarrow F\cdot(ax\,i+by\,j+cz\,k) = 1$

Taking $F = x\,i + y\,j + z\,k$

P.T.

$\Rightarrow r\cdot(ax\,i + by\,j + cz\,k) = ax^2 + by^2 + cz^2$

$\Rightarrow F = x\,i + y\,j + z\,k$

$\Rightarrow \iint (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, ds = \iiint \nabla\cdot(x\,i + y\,j + z\,k)\, dV$

$= \iiint 3\cdot dV \left[\because \nabla\cdot(x\,i + y\,j + z\,k) = 3\right]$

$= 3\iiint dV = 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}$

(Volume of ellipsoid)

14

$= \frac{4\pi}{\sqrt{abc}}$

ROUGH SPACE

$\lambda_1 + \lambda_2 + \lambda_3 = a$

$\lambda_1\lambda_2 + \lambda_2\lambda_3 +$

$\lambda_1\lambda_3 = -b$

$\lambda_1\lambda_2\lambda_3 = c$

$a^2 = (\quad) + 2b$

$2x^2(-6yz + x^4z^2) - j(4x^2 + 3x$

$i(\cancel{12t}\; i(6t^2) - j(6t) + k(2)$

$=$