Set-III Linear Transformation

* Defn.

Vectorspace Homomorphism:

Let U and V be two vectorspaces over the same field F. Then the mapping $f: U \to V$ is called a homomorphism (or linear transformation) from U into V. if (i) $f(\alpha+\beta) = f(\alpha) + f(\beta) \ \forall\ \alpha, \beta \in U$

(ii) $f(a\alpha) = a f(\alpha) \ \forall\ a \in F, \alpha \in U$.

— If f is onto function then V is called the homomorphic image of U.

— If 'f' is one-one onto function then 'f' is called isomorphism. Then it is said that 'U' isomorphic to V denoted by $U \cong V$.

→ Let U(F) and V(F) be two vectorspaces. Then the function $T: U \to V$ is a linear transformation of U into V iff $T(a\alpha+b\beta) = aT(\alpha) + bT(\beta) \ \forall\ a, b \in F$; $\alpha, \beta \in U$.

proof: Now suppose that the function $T: U \to V$ is a linear transformation.

$\therefore \forall\ a, b \in F, \alpha, \beta \in U$;

$$T(a\alpha+b\beta) = T(a\alpha) + T(b\beta) \quad \text{(by defn condition (i))}$$

$$= aT(\alpha) + bT(\beta) \quad \text{(by defn condition (ii))}$$

Conversely suppose that

$$T(a\alpha+b\beta) = aT(\alpha) + bT(\beta) \ \forall\ a, b \in F; \ \alpha, \beta \in U.$$

Taking $a=1, b=1$ in F we get

$$T(\alpha+\beta) = T(\alpha) + T(\beta)$$

Taking $b=0$ in F we get

$$T(a\alpha) = aT(\alpha)$$

$\therefore$ T is a linear Transformation.

Note(1):- The condition $T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$ completely characterises linear transformation

Note(2): Suppose $T: U \to V$ is linear transformation. Then for any $a_i \in F$ and any $\alpha_i \in U$,

$$T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n).$$

Note(3):- If $T: U \to U$ (i.e. T transforms U onto itself) then T is called a linear operator on U.

Note(4): If $T: U \to F$ (i.e. T transforms U onto the field F) then T is called a linear function on U.

* Zero Transformation:

Theorem → Let U(F) and V(F) be two vector spaces. Let the mapping $T: U \to V$ be defined by

$$T(\alpha) = \hat{0} \quad \forall\ \alpha \in U.$$

where $\hat{0}$ is the zero vector of V. Then T is a linear transformation.

Proof. $\forall\ a, b \in F$ and $\alpha, \beta \in U$

$\Rightarrow a\alpha + b\beta \in U$ ($\because$ U is a vector space).

$\therefore$ By definition, we have

$$T(a\alpha+b\beta) = \hat{0}$$
$$= a\hat{0} + b\hat{0}$$
$$= aT(\alpha) + bT(\beta).$$

$\therefore$ T is a linear transformation.

Such a L.T. is called the zero transformation and is denoted by '0'.

* Identity operator :-

Theorem ⟶ Let $V(F)$ be a vectorspace and the mapping $I: V \to V$ be defined by $I(\alpha) = \alpha \ \forall \alpha \in V$. Then, I is a linear operator from V into itself. (61)

proof $\forall a, b \in F$; $\alpha, \beta \in V$

$\Rightarrow a\alpha + b\beta \in V$ ($\because V$ is a vectorspace).

By defn, we we have

$I(a\alpha + b\beta) = a\alpha + b\beta$

$= a I(\alpha) + b I(\beta)$.

$\therefore I$ is a linear transformation from V into ~~itself~~ itself and is called the identity operator.

* Negative of Transformation :-

Theorem ⟹ Let $U(F)$ and $V(F)$ be two vectorspaces and $T: U \to V$ be a linear transformation. Then the mapping $(-T)$ defined by $(-T)(\alpha) = -T(\alpha)$ $\forall \alpha \in U$ is a linear Transformation.

proof $\forall a, b \in F$ and $\alpha, \beta \in U$

$\Rightarrow a\alpha + b\beta \in U$ ($\because U$ is a vectorspace)

Now by definition,

$(-T)(a\alpha + b\beta) = -T(a\alpha + b\beta)$

$= -[aT(\alpha) + bT(\beta)]$ ($\because T$ is LT)

$= -aT(\alpha) - bT(\beta)$

$= a[-T(\alpha)] + b[-T(\beta)]$

$= a[(-T)(\alpha)] + b[(-T)(\beta)]$

$\Rightarrow -T$ is L.T.

* properties of Linear transformation:

Let $T: U \to V$ be a linear transformation from the vectorspace $U(F)$ to the vector space $V(F)$. Then

(i) $T(\bar{0}) = \hat{0}$ where $\bar{0} \in U$ and $\hat{0} \in V$

(ii) $T(-\alpha) = -T(\alpha)$ $\forall$ $\alpha \in U$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta)$ $\forall$ $\alpha, \beta \in U$.

Sol (i) $\alpha, \bar{0} \in U \Rightarrow T(\alpha), T(\bar{0}) \in V$

Now $T(\alpha) + T(\bar{0}) = T(\alpha + \bar{0})$ $(\because T$ is LT$)$

$= T(\alpha)$

$= T(\alpha) + \hat{0}$ $(\because \hat{0} \in V)$

$\therefore T(\alpha) + T(\bar{0}) = T(\alpha) + \hat{0}$

By L.C.L, $T(\bar{0}) = \hat{0}$.

(ii) $T(-\alpha) = T(-1 \cdot \alpha)$

$= (-1)\, T(\alpha)$

$= -T(\alpha)$.

(iii) $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$= T(\alpha) + T(-\beta)$ $(\because T$ is LT$)$

$= T(\alpha) - T(\beta)$ (by (ii))

* Determination of Linear Transformation:-

Let $U(F)$ and $V(F)$ be two vector spaces and $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of U. Let $\{\delta_1, \delta_2, \ldots \delta_n\}$ be a set of n vectors in V. Then there exists a unique linear transformation $T: U \to V$ s.t

$T(\alpha_i) = \delta_i$ for $i = 1, 2, \ldots n$.

Proof

Let $U(F)$ and $V(F)$ be two vector spaces.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $U(F)$

$\therefore$ S is LI and 'S' spans $U(F)$.
i.e $L(S) = U$

Let $\alpha \in U$, $\exists\, a_1, a_2, \ldots, a_n \in F$ s.t

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ $(\in U)$ ——— (1).

(i) Existence of T:

Let $S_2 = \{\delta_1, \delta_2, \ldots, \delta_n\} \subset V$

then $\delta_1, \delta_2, \ldots, \delta_n \in V \Rightarrow (a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n) \in V$.

we define $T: U \to V$ s.t

$T(\alpha) = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$ ——— (2) where $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

$\therefore$ T is a fn from U into V.

Now $\alpha_i = 0\cdot\alpha_1 + 0\alpha_2 + \ldots + 1\cdot\alpha_i + 0\alpha_{i+1} + \ldots + 0\alpha_n$.

By defn of T fn

$T(\alpha_i) = T(0\alpha_1 + 0\alpha_2 + \ldots + 1\alpha_i + 0\alpha_{i+1} + \ldots + 0\alpha_n)$

$= 0\delta_1 + 0\delta_2 + \ldots + 1\delta_i + 0\delta_{i+1} + \ldots + 0\delta_n$ (by (2)).

$= \delta_i \quad \forall\, i = 1, 2, 3, \ldots n.$ ——— (3)

(ii) To show that T is LT:

Let $a, b \in F$ and $\alpha, \beta \in U$

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$.

$\therefore T(\alpha) = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$

& $T(\beta) = b_1\delta_1 + b_2\delta_2 + \ldots + b_n\delta_n$.

$\therefore a\alpha + b\beta = a(a_1\alpha_1 + \dots + a_n\alpha_n) +$
$b(b_1\alpha_1 + b_2\alpha_2 + \dots + b_n\alpha_n)$

$= (aa_1 + bb_1)\alpha_1 + \dots + (aa_n + bb_n)\alpha_n$

$T(a\alpha + b\beta) = T[(aa_1+bb_1)\alpha_1 + \dots + (aa_n+bb_n)\alpha_n]$

$= (aa_1+bb_1)\delta_1 + \dots + (aa_n+bb_n)\delta_n$ (by ②)

$= a(a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n) +$
$b(b_1\delta_1 + b_2\delta_2 + \dots + b_n\delta_n)$

$= aT(\alpha) + bT(\beta)$.

$\therefore$ T is L.T.

(iii) To show that T is unique:

Let $T': U \to V$ be another L.T. s.t

$T'(\alpha_i) = \delta_i$; for $i = 1, 2, 3, \dots n$.

If $\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$

then $T'(\alpha) = T'(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$

$= a_1T'(\alpha_1) + a_2T'(\alpha_2) + \dots + a_nT'(\alpha_n)$ ($\because T'$ is L.T).

$= a_1\delta_1 + a_2\delta_2 + \dots + a_n\delta_n$.

$= T(\alpha)$.

$\therefore T' = T$ and hence T is unique.

NOTE:- In determining the L.T the assumption that $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ is basis of V is essential.

→ Let $S = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ and $S' = \{\beta_1, \beta_2, \cdots \beta_n\}$ be two ordered bases of 'n' dimensional vectorspace $V(F)$. (62)

Let $\{a_1, a_2, \cdots a_n\}$ be an ordered set of 'n' scalars such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$
$$\beta = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n.$$ Then

$T(\alpha) = \beta$ where T is the linear operator on V defined by $T(\alpha_i) = \beta_i$, $i = 1, 2, 3, \cdots n$.

Sol $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n)$ ($\because T$ is LT)

$= a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$

$= \beta$.

Problems

→ The mapping $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ is defined by $T(x, y, z) = (x-y, x-z)$. Show that T is a linear transformation.

Sol. Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be two vectors of $V_3(\mathbb{R})$.

For $a, b \in \mathbb{R}$,

$T(a\alpha + b\beta) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$

$= T[(ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)]$

$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$= ((ax_1 + bx_2) - (ay_1 + by_2), (ax_1 + bx_2) - (az_1 + bz_2))$

$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$

$= \left(a(x_1-y_1), a(x_1-z_1)\right) + \left(b(x_2-y_2), b(x_2-z_2)\right)$

$= a\left(x_1-y_1, x_1-z_1\right) + b\left(x_2-y_2, x_2-z_2\right)$

$= a\,T(x_1, y_1, z_1) + b\,T(x_2, y_2, z_2)$ (by defn).

$= a\,T(\alpha) + b\,T(\beta)$.

$\therefore T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)\ \forall\ a, b \in F;\ \alpha, \beta \in V_3(\mathbb{R})$.

$\therefore$ T is a linear transformation from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$.

$\rightarrow$ The mapping $T: V_3(\mathbb{R}) \to V_1(\mathbb{R})$ is defined by $T(a,b,c) = a^2+b^2+c^2$.

Can T be a linear transformation?

Sol. Let $\alpha = (a,b,c)$ and $\beta = (x,y,z)$ be two vectors of $V_3(\mathbb{R})$.

for $p, q \in \mathbb{R}$.

$T(p\alpha + q\beta) = T\left[p(a,b,c) + q(x,y,z)\right]$

$= T\left[(pa, pb, pc) + (qx, qy, qz)\right]$

$= T\left[pa+qx, pb+qy, pc+qz\right]$

$= (pa+qx)^2 + (pb+qy)^2 + (pc+qz)^2$ (by hyp)

Now $p\,T(\alpha) + q\,T(\beta) =$

$= p\,T(a,b,c) + q\,T(x,y,z)$

$= p(a^2+b^2+c^2) + q(x^2+y^2+z^2)$ (by hyp).

$\therefore T(p\alpha + q\beta) \neq p\,T(\alpha) + q\,T(\beta)$.

$\therefore$ T is not a L.T from $V_3(\mathbb{R})$ to $V_1(\mathbb{R})$.

→ Let V be the vectorspace of polynomials in the variable x over ℝ. (63)

Let $f(x) \in V(\mathbb{R})$; show that

(i) $D : V \to V$ defined by $D f(x) = \frac{d}{dx} f(x)$

(ii) $I : V \to V$ defined by $I f(x) = \int_0^x f(x)\, dx$

are linear transformations.

Sol Let $f(x), g(x) \in V(\mathbb{R})$ and $a, b \in \mathbb{R}$

(i) $D[a f(x) + b g(x)] = \frac{d}{dx}[a f(x) + b g(x)]$ (by Defn)

$= \frac{d}{dx}[a f(x)] + \frac{d}{dx}[b g(x)]$

$= a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x)$

$= a D f(x) + b D g(x)$

∴ D is a linear transformation and D is called differential operator.

(ii) $I[a f(x) + b g(x)] = \int_0^x (a f(x) + b g(x))\, dx$

$= \int_0^x (a f(x))\, dx + \int_0^x (b g(x))\, dx$

$= a \int_0^x f(x)\, dx + b \int_0^x g(x)\, dx.$

$= a I f(x) + b I g(x).$

∴ I is L.T. and I is called integral operator.

→ Let $P_n(\mathbb{R})$ be the vectorspace of all polynomials of degree 'n' over a field ℝ. If a linear operator T on $P_n(\mathbb{R})$ is such that

$T f(x) = f(x+1)$, $f(x) \in P_n(\mathbb{R})$.

Show that $T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots + \frac{D^n}{n!}$.

Sol

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ $\forall x \in \mathbb{R}$.

Now $\left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right] f(x)$

$= \left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right] (a_0 + a_1 x + \cdots + a_n x^n)$

$= (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + \frac{1}{1!}(0 + a_1 + 2a_2 x + \cdots + n a_n x^{n-1})$

$+ \frac{1}{2!}(0 + 0 + 2a_2 + 6a_3 + \cdots + n(n-1) a_n x^{n-2})$

$+ \cdots \cdots$

$+ \frac{1}{n!}(0 + 0 + \cdots + a_n n!)$.

$= a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_n(x+1)^n$

$= f(x+1)$.

$= T f(x)$. (by hyp).

$\therefore T = \left(1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right)$.

→ Is the mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, 0)$ a linear transformation?

Sol We have $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, 0).$$

Let $\alpha, \beta \in \mathbb{R}^3$, where $\alpha = (x_1, y_1, z_1)$ &

$\beta = (x_2, y_2, z_2)$

For $a, b \in \mathbb{R}$,

$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$.

$\therefore T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$= (|ax_1 + bx_2|, 0)$.

And $aT(\alpha) + bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$

$= a(|x_1|, 0) + b(|x_2|, 0)$

$= (a|x_1| + b|x_2|, 0)$.

Clearly $T(a\alpha + b\beta) \neq aT(\alpha) + bT(\beta)$. (64)

Hence T is not a linear transformation.

→ Let T be a linear transformation on a vectorspace U into V (i.e $T: U \to V$ is LT) prove that the vectors $x_1, x_2, \ldots, x_n \in U$ are LI if $T(x_1), T(x_2), \ldots, T(x_n)$ are LI.

Sol. Given $T: U(F) \to V(F)$ is LT.

and $x_1, x_2, \ldots, x_n \in U$.

Let there exist $a_1, a_2, \ldots, a_n \in F$ s.t

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = \bar{0}$ —— (1).

($\because \bar{0} \in U$).

$\Rightarrow T(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = T(\bar{0})$

$\Rightarrow a_1T(x_1) + a_2T(x_2) + \ldots + a_nT(x_n) = \hat{0}$

($\because T$ is LT).

$\hat{0} \in V(F)$).

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$

($\because T(x_1), T(x_2), \ldots, T(x_n)$ are LI).

From (1), $x_1, x_2, \ldots, x_n$ are LI.

→ Let V be a vectorspace of $n\times n$ matrices over the field F. M is a fixed matrix in V. The mapping $T: V \to V$ is defined by $T(A) = AM + MA$ where $A \in V$. Show that T is linear.

Sol Let $a, b \in F$ and $A, B \in V$. Then

$T(A) = AM + MA$ & $T(B) = BM + MB$.

$\therefore T(aA + bB) = (aA + bB)M + M(aA + bB)$

$= a(AM + MA) + b(BM + MB)$

$= aT(A) + bT(B)$.

$\therefore T$ is a linear transformation.

→ Describe explicitly the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ s.t $T(2,3)=(4,5)$ and $T(1,0)=(0,0)$.

Sol. first of all we have to show that the set $S=\{(2,3),(1,0)\}$ is a basis of $\mathbb{R}^2$.

for this we have to show S is LI and $L(S)=\mathbb{R}^2$.

Let $a(2,3)+b(1,0)=\bar{0}$; $a,b\in\mathbb{R}$.

$\Rightarrow (2a+b,\ 3a+0)=(0,0)$

$\Rightarrow 2a+b=0,\ 3a=0.$

$\Rightarrow a=0, b=0.$

$\therefore$ S is LI.

W.K.T $L(S)\subseteq\mathbb{R}^2$ —— (1).

Let $(x,y)\in\mathbb{R}^2$ then $(x,y)=a(2,3)+b(1,0)$

$\Rightarrow (x,y)=(2a+b,\ 3a+0)$

$\Rightarrow 2a+b=x;\ \ 3a=y$

$\Rightarrow 2\left(\frac{y}{3}\right)+b=x$; $\boxed{a=\frac{y}{3}}$

$\Rightarrow \boxed{b=x-\frac{2y}{3}}$

$\therefore (x,y)=\frac{y}{3}(2,3)+\left(x-\frac{2y}{3}\right)(1,0)$

$\in L(S).$

$\therefore (x,y)\in L(S).$

$\therefore \mathbb{R}^2\subseteq L(S)$ —— (2).

from (1) & (2), we have $L(S)=\mathbb{R}^2$.

$\therefore S$ is a basis of $\mathbb{R}^2$.

$\therefore S=\{(2,3),(1,0)\}$ is a basis of $\mathbb{R}^2$ and

$S'=\{(4,5),(0,0)\}$ is a set of two vectors in $\mathbb{R}^2$.

$\therefore\ \exists$ a unique linear transformation 65

$T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t

$$T(x,y) = T\left[\frac{y}{3}(2,3) + \left(x - \frac{2y}{3}\right)(1,0)\right]$$

$$= \frac{y}{3}T(2,3) + \left(x - \frac{2y}{3}\right)T(1,0)$$

$$= \frac{y}{3}(4,5) + \left(x - \frac{2y}{3}\right)(0,0)$$

$$= \left(\frac{4y}{3}, \frac{5y}{3}\right)$$ which is the reqd transformation.

→ Show that $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a L.T such that $T(1,0,0) = (1,0,0)$, $T(1,1,0) = (1,1,1)$ and $T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$.

HW → Find $T(x,y,z)$ where $T: \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$.

HW → Find a linear transformation

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$

(ii) $T: V_2 \to V_2$ s.t $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$

(iii) $T: V_3 \to V_3$ s.t $T(0,1,2) = (3,1,2)$ and $T(1,1,1) = (2,2,2)$

(iv) $T: V_2(\mathbb{R}) \to V_3(\mathbb{R})$ s.t $T(1,1) = (3,-1,5)$ and $T(0,1) = (2,1,-1)$

* Sum of Linear Transformations *

Defn → Let T_1 and T_2 be two linear transformations from $U(F)$ into $V(F)$. Then their sum T_1+T_2 is defined by $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$.

Theorem → Let $U(F)$ and $V(F)$ be two vector spaces. Let T_1 and T_2 be two linear transformations from U into V. Then the mapping T_1+T_2 defined by $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall\, \alpha \in U$ is a linear transformation.

proof: Given the $T_1 : U \to V$ and $T_2 : U \to V$ are linear transformation.

and $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)$ $\forall \alpha \in U$,

$T_1(\alpha), T_2(\alpha) \in V$

since $T_1(\alpha), T_2(\alpha) \in V$

$\Longrightarrow T_1(\alpha) + T_2(\alpha) \in V$.

Hence $(T_1+T_2) : U \to V$

Let $a, b \in F$ and $\alpha, \beta \in U$. Then

$(T_1+T_2)(a\alpha + b\beta) = T_1(a\alpha+b\beta) + T_2(a\alpha+b\beta)$ (by hyp).

$= \left(aT_1(\alpha) + bT_1(\beta)\right) + \left(aT_2(\alpha) + bT_2(\beta)\right)$ ($\because$ T_1 & T_2 are L.T)

$= a\left(T_1(\alpha) + T_2(\alpha)\right) + b\left(T_1(\beta) + T_2(\beta)\right)$.

$= a(T_1+T_2)(\alpha) + b(T_1+T_2)(\beta)$. (by hyp).

$\therefore$ T_1+T_2 is a L.T from U into V.

* **Scalar multiplication of a L.T:-**

→ Let $T : U(F) \to V(F)$ be a linear transformation and $a \in F$. Then the function (aT) defined by $(aT)(\alpha) = aT(\alpha)$ $\forall \alpha \in U$. is a linear transformation.

proof Given the $T : U(F) \to V(F)$

and $(aT)(\alpha) = a\,T(\alpha)$ $\forall a \in F$, $\alpha \in U$

Now $T(\alpha) \in V \Rightarrow a.T(\alpha) \in V$

$\therefore (aT) : U \to V$.

for $c, d \in F$ and $\alpha, \beta \in U$

$\Longrightarrow (aT)(c\alpha + d\beta) = aT(c\alpha + d\beta)$ (by hyp).

$= a[cT(\alpha) + dT(\beta)]$ $(\because T$ is L.T.$)$ (66)

$= acT(\alpha) + adT(\beta)$.

$= c(aT)(\alpha) + d(aT)(\beta)$.

Hence (aT) is a L.T. from U into V.

problems

→ Let $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ and $H: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ be the two linear transformations defined by $T(x,y,z) = (x-y, y+z)$ and $H(x,y,z) = (2x, y-z)$

Find (i) $H+T$ (ii) aH.

Sol (i) $(H+T)(x,y,z) = H(x,y,z) + T(x,y,z)$

$= (2x, y-z) + (x-y, y+z)$

$= (3x-y, 2y)$

(ii) $(aH)(x,y,z) = aH(x,y,z)$

$= a(2x, y-z)$

$= (2ax, ay-az)$

→ Let $G: V_3 \to V_3$ and $H: V_3 \to V_3$ be two linear operators defined by $G(e_1) = e_1 + e_2$, $G(e_2) = e_3$, $G(e_3) = e_2 - e_3$.

and $H(e_1) = e_3$, $H(e_2) = 2e_2 - e_3$, $H(e_3) = 0$.

where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$.

Find (i) $G+H$ (ii) $2G$.

Sol Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $V_3(\mathbb{R})$.

Sol^n:
$e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

Now $G(e_1) = e_1 + e_2$

$\Rightarrow G(1,0,0) = (1,0,0) + (0,1,0)$

$= (1,1,0)$

$\therefore \boxed{G(1,0,0) = (1,1,0)}$

$G(e_2) = e_3$

$\Rightarrow \boxed{G(0,1,0) = (0,0,1)}$

$G(e_3) = e_2 - e_3$

$\Rightarrow G(0,0,1) = (0,1,0) - (0,0,1)$

$\boxed{G(0,0,1) = (0,1,-1)}$

Again $H(e_1) = e_3$

$\Rightarrow \boxed{H(1,0,0) = (0,0,1)}$

$H(e_2) = 2e_2 - e_3$

$\Rightarrow \boxed{H(0,1,0) = (0,2,-1)}$

$\Rightarrow H(e_3) = \bar{0}$

$\Rightarrow \boxed{H(0,0,1) = (0,0,0)}$

(i) $(G+H)(e_1) = G(e_1) + H(e_1)$

$\Rightarrow (G+H)(1,0,0) = (1,1,0) + (0,0,1)$

$= (1,1,1)$

$(G+H)(e_2) = G(e_2) + H(e_2) \Rightarrow (G+H)(0,1,0) = (0,2,0)$

$(G+H)(e_3) = G(e_3) + H(e_3) \Rightarrow (G+H)(0,0,1) = (0,1,-1)$

(ii) $(2G)(e_1) = 2G(e_1) = 2e_1 + 2e_2$

$(2G)(e_2) = 2G(e_2) = 2e_3$

$(2G)(e_3) = 2G(e_3) = 2e_2 - 2e_3$ etc.

* Product of Linear Transformation :- (67)

→ Let U(F), V(F) and W(F) are three vectorspaces. and $T: V \to W$ and $H: U \to V$ are two linear transformations. Then the composite function TH (called the product of linear transformations) defined by

$$(TH)\alpha = T[H(\alpha)] \quad \forall \alpha \in U.$$

is a linear transformation from U into W.

Note: The range of H is the domain of T.

→ Let H, H' be two linear transformations from U(F) to V(F). Let T, T' be the linear transformations from V(F) to W(F) and $a \in F$. Then

(i) $T(H+H') = TH + TH'$

(ii) $(T+T')H = TH + T'H$

(iii) $a(TH) = (aT)H = T(aH)$

* → Algebra of Linear operators:

→ Let A, B, C be linear operators on a vector space V(F). Also let O be the zero-operator and I the identity operator on V. Then

(i) $AO = OA = O$

(ii) $AI = IA = A$

(iii) $A(B+C) = AB + AC$

(iv) $(A+B)C = AC + BC$

(v) $A(BC) = (AB)C$

→ Let $T: R^3 \to R^2$ and $H: R^3 \to R^2$ be defined by $T(x, y, z) = (3x, y+z)$ and $H(x, y, z) = (2x - z, y)$

Compute (i) $T+H$ (ii) $4T - 5H$ (iii) TH (iv) HT.

Sol Since T and H map $\mathbb{R}^3 \to \mathbb{R}^2$, the linear transformations $T+H$ and $4T-5H$ are defined.

(i) $(T+H)(x,y,z) = T(x,y,z) + H(x,y,z)$
$= (3x, y+z) + (2x-z, y)$
$= (5x-z, 2y+z)$.

(ii) $(4T-5H)(x,y,z) = 4T(x,y,z) - 5H(x,y,z)$
$= 4(3x, y+z) - 5(2x-z, y)$.
$= (2x+5z, -y+4z)$.

(iii) and (iv) both TH and HT are not defined because the range of T is not equal to the domain of H and vice versa.

→ Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are two linear transformations defined by
$T_1(x,y,z) = (3x, 4y-z)$
$T_2(x,y) = (-x, y)$. Compute T_1T_2 and T_2T_1.

Sol. (i) Since the range of T_2 i.e $\mathbb{R}^2$ is not equal to the domain of T_1 (i.e $\mathbb{R}^3$)

$\therefore T_1T_2$ is not defined.

(ii) But the range T_1 i.e $\mathbb{R}^3$ is equal to the domain of T_2

$\therefore T_2T_1$ is defined.

$\therefore (T_2T_1)(x,y,z) = T_2[T_1(x,y,z)]$
$= T_2[3x, 3y-z)$
$= (-3x, 3y-z)$

→ Let $P(R)$ be the vectorspace of all polynomials in x & D, T be two linear operators on P defined by $D[f(x)] = \frac{df}{dx}$ and (68)

$$T[f(x)] = xf(x) \quad \forall f(x) \in P(R)$$

Show that (i) $TD \neq DT$ (ii) $(TD)^2 = T^2D^2 + TD$

Sol (i) $(TD) f(x) = T[D f(x)]$

$= T\left[\frac{df}{dx}\right]$ (by hyp)

$= x f'(x)$.

and $(DT) f(x) = D[T f(x)]$

$= D[x f(x)]$ (by hyp)

$= \frac{d}{dx}(x f(x))$

$= x f'(x) + f(x)$

Clearly $TD \neq DT$

Also $(DT) f(x) - (TD) f(x) = f(x)$

$\Rightarrow (DT - TD) f(x) = I f(x)$

$\Rightarrow (DT - TD) = I$.

(ii) $(TD)^2 f(x) = [(TD)(TD)] f(x)$

$= (TD)[(TD) f(x)]$

$= (TD)[x f'(x)]$

$= T[D(x f'(x))]$

$= T\left[\frac{d}{dx}(x f'(x))\right]$

$= T[(x f''(x) + f'(x))]$

$= T[x f''(x) + f'(x)]$

$= x[x f''(x) + f'(x)]$

$= x^2 f''(x) + x f'(x)$.

Now $(T^2D^2)f(x)$

$$= T^2 D[Df(x)]$$

$$= T^2 D\left[\frac{df}{dx}\right]$$

$$= T^2\left[\frac{d^2f}{dx^2}\right]$$

$$= T\left(T\left[\frac{d^2f}{dx^2}\right]\right)$$

$$= T\left(x\frac{d^2f}{dx^2}\right)$$

$$= x\left(x\frac{d^2f}{dx^2}\right)$$

$$= x^2\frac{d^2f}{dx^2}.$$

$$\therefore (T^2D^2+TD)(f(x)) = (T^2D^2)(f(x)) + (TD)f(x)$$

$$= x^2\frac{d^2f}{dx^2} + x\frac{df}{dx}$$

$$\therefore (TD)^2(f(x)) = (T^2D^2+TD)f(x)$$

$\forall\ f(x) \in P$

Hence $(TD)^2 = T^2D^2 + TD.$

HW → Let P be the polynomial space in one indeterminate x with real co-efficients. Let $D: P \to P$ and $S: P \to P$ be two linear operators defined by

$$Df(x) = \frac{df}{dx} \text{ and } Sf(x) = \int_0^x f(x)\,dx$$

$\forall\ f(x) \in P.$

Show that $DS = I$ and $SD \neq I$ where I is the identity transformation.

HW → Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $H: \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformation defined by

$$T(x, y, z) = (x-3y-2z, y-4z).$$

and $H(x, y) = (2x, 4x-y, 2x+3y)$

Find HT and TH. Is product commutative

HW → Define on $\mathbb{R}^2$ linear operators H and T as follows $H(x,y) = (0, x)$ and $T(x,y) = (x, 0)$ and show that

$TH = 0$, $HT \neq TH$ and $T^2 = T$.

* Transformations as vectors *

→ Let $L(U,V)$ be the set of all linear transformations from a vector space $U(F)$ into a vectorspace $V(F)$. Then $L(U,V)$ be a vector space relative to the operations of vector addition and scalar multiplication defined as

(i) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$

(ii) $(aT)(\alpha) = aT(\alpha)$ $\forall$ $\alpha \in U$, $a \in F$ and

$T, H \in L(U,V)$.

The set $L(U,V)$ is also denoted by $HOM(U,V)$.

proof Let $L(U,V) = \{T: U \to V / T \text{ is a L.T.}\}$

Given that $T: U \to V$ and

$H: U \to V$ are L.T. in $L(U,V)$

and $(T+H)(\alpha) = T(\alpha) + H(\alpha)$ $\forall$ $\alpha \in U$.

$T(\alpha), H(\alpha) \in V$.

Since $T(\alpha)$, $H(\alpha) \in V$

$\Longrightarrow T(\alpha) + H(\alpha) \in V$.

$\therefore (T+H): U \to V$.

Let $a, b \in F$ and $\alpha, \beta \in U$ then

$(T+H)(a\alpha + b\beta) = T(a\alpha + b\beta) + H(a\alpha + b\beta)$

$= (a\,T(\alpha) + b\,T(\beta)) + (a\,H(\alpha) + b\,H(\beta))$ ($\because$ T & H are L.T.)

$= a(T+H)(\alpha) + b(T+H)(\beta)$.

$\therefore$ T+H is a L.T. from U into V.

$\therefore$ $T+H \in L(U,V)$.

$\therefore$ Internal composition is satisfied by $L(U,V)$

Given that $T: U \to V$ is L.T. in $L(U,V)$.

and $(aT)(\alpha) = a\,T(\alpha)$ $\forall$ $a \in F$, $\alpha \in U$

Now $T(\alpha) \in V \Rightarrow a.T(\alpha) \in V$

$\therefore (aT): U \to V$

for $c, d \in F$ and $\alpha, \beta \in U$

$\Rightarrow (aT)(c\alpha + d\beta) = a\,T(c\alpha + d\beta)$ (by hyp)

$= a[c\,T(\alpha) + d\,T(\beta)]$

$= ac\,T(\alpha) + ad\,T(\beta)$

$= c(aT)(\alpha) + d(aT)(\beta)$

$\therefore$ (aT) is a L.T. from U into V.

$\therefore$ $aT \in L(U,V)$.

$\therefore$ External composition is satisfied in $L(U,V)$ over the field F.

(I) (i) $\forall$ $T, H \in L(U,V) \Rightarrow T+H \in L(U,V)$

$\therefore$ closure prop is satisfied.

(ii) $\forall$ $T, H, G \in L(U,V)$

$[(T+H)+G](\alpha) = (T+H)(\alpha) + G(\alpha)$
$= [T(\alpha) + H(\alpha)] + G(\alpha)$

$= T(\alpha) + [H(\alpha) + G(\alpha)]$ ($\because$ + in V is associative)

$= T(\alpha) + [H + G](\alpha)$

$= [T + (H + G)](\alpha)$

$\therefore (T+H) + H = T + (H + G)$

$\therefore$ ~~A~~sso. prop is satisfied in $L(U,V)$.

(iii) Let 'O' be the zero transformation from U into V

i.e $O(\alpha) = \hat{0} \quad \forall \alpha \in U, \hat{0} \in V$

Now $(O + T)(\alpha) = O(\alpha) + T(\alpha)$

$= \hat{0} + T(\alpha)$

$= T(\alpha)$ ($\because \hat{0}$ is additive identity in V)

$\therefore O + T = T$

sly $T + O = T$

$\therefore \forall T \in L(U,V) \; \exists O \in L(U,V)$ s.t

$O + T = T = T + O$.

~~Here~~ 'O' is the additive identity in $L(U,V)$.

(iv) for $T \in L(U,V)$,

let us define $(-T)$ as $(-T)(\alpha) = -T(\alpha)$ $\forall \alpha \in U$.

Then $(-T) \in L(U,V)$.

Now $(-T + T)(\alpha) = (-T)(\alpha) + T(\alpha)$

$= -T(\alpha) + T(\alpha)$

$= \hat{0}$ ($\because \hat{0} \in V$).

$= O(\alpha)$

$\therefore (-T) + T = O \quad \forall T \in L(U,V)$.

sly $T + (-T) = O \quad \forall T \in L(U,V)$.

$\therefore \forall T \in L(U,V), \exists -T \in L(U,V)$ s.t

$(-T) + T = O = T + (-T)$

Here $-T$ is additive inverse of T in $L(U,V)$.

(iv) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$

$= H(\alpha) + T(\alpha)$ ($\because$ addition in V is commutative)

$= (H+T)(\alpha)$

$\therefore T+H = H+T$.

$\therefore$ commutative prop is satisfied in $L(U,V)$.

$\therefore (L(U,V), +)$ is an abelian grp.

(II) $\forall\, a,b \in F,\ T,H \in L(U,V)$;

$\Longrightarrow$ (i) $[a(T+H)](\alpha) = a\,(T+H)(\alpha)$ (by hyp(iii))

$= a\,[T(\alpha) + H(\alpha)]$ (by hyp(i))

$= a\,T(\alpha) + a\,H(\alpha)$ (by hyp(ii))

$= (aT)(\alpha) + (aH)(\alpha)$ (by hyp (i))

$= (aT + aH)(\alpha)$

$\therefore a(T+H) = aT + aH$.

(ii) $[(a+b)T](\alpha) = (a+b)\,T(\alpha)$ (by hyp (ii))

$= a\,T(\alpha) + b\,T(\alpha)$

$= (aT)(\alpha) + (bT)(\alpha)$ (by hyp (ii))

$= (aT + bT)(\alpha)$.

$\therefore (a+b)T = aT + bT$.

(iii) $[(ab)T](\alpha) = ab(T(\alpha))$ (by hyp)

$= a(bT(\alpha))$

$= a[(bT)(\alpha)] = [a(bT)](\alpha)$

$\therefore (ab)T = a(bT)$

(iv) $(1 \cdot T)(\alpha) = 1 \cdot T(\alpha)$

$= T(\alpha)$; (multiplication for V is identity). $1 \in F$

$\therefore 1 \cdot T = T$

$\therefore L(U,V)$ is a vectorspace over the field F.

→ $L(U,V)$ be the vectorspace of all linear transformations from $U(F)$ to $V(F)$. So that $\dim U = n$ and $\dim V = m$.

Then $\dim L(U,V) = mn$.

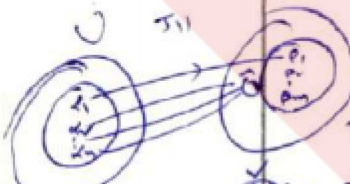
Proof: Given that $L(U,V)$ is the vectorspace of all linear transformations from $U(F)$ to $V(F)$

i.e. $L(U,V) = \{T : U \to V / T$ is a linear transformation$\}$.

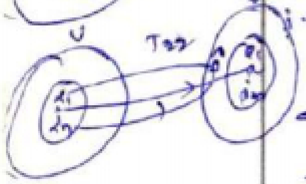
Since $\dim U = n$ and $\dim V = m$

Let $B_1 = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B_2 = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

$\therefore$ there exists uniquely a linear transformation T_{ij} from U to V such that



$T_{11}(\alpha_1) = \beta_1$, $T_{11}(\alpha_2) = \hat{0}, \ldots \ldots T_{11}(\alpha_n) = \hat{0}$ where $\beta_1, \hat{0} \in V$



i.e. $T_{ij}(\alpha_i) = \beta_j$, $i = 1, 2, \ldots n$, $j = 1, 2, \ldots m$.

and $T_{ij}(\alpha_k) = \hat{0}$ $k \neq i$

Thus there are "mn" T_{ij}'s $\in L(U,V)$.

we shall show that $S = \{T_{ij}\}$ of mn elts is a basis for $L(U,V)$.

(i) To prove S is LI:

Let a_{ij}'s $\in F$, let us suppose that $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij} = \hat{0}$ $(\because \hat{0} \in L(U,V))$

for $\alpha_k \in V$, $k=1,2,3,\dots n$ we get

$$\left[\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} T_{ij}\right](\alpha_k) = O(\alpha_k)$$

$$\Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k) = \hat{0} \quad (\because \hat{0}\in V)$$

$$\Rightarrow \sum_{j=1}^{m} a_{kj} T_{kj}(\alpha_k) = \hat{0}.$$
where $1\le k\le n$.

$\left(\because \sum_{j=1}^{m}\left(a_{1j} T_{1j}(\alpha_k) + a_{2j} T_{2j}(\alpha_k) + \dots + a_{nj} T_{nj}(\alpha_k)\right) = \hat{0}\right.$

$\Rightarrow \sum_{j=1}^{m}\left(a_{1j} T_{1j}(\alpha_1) + a_{2j} T_{2j}(\alpha_1) + \dots + a_{nj} T_{nj}(\alpha_1)\right) = \hat{0}$ if $k=1$

$\Rightarrow \sum_{j=1}^{m}(a_{1j}\beta_j) = 0$

$\left. \text{Sly } \sum_{j=1}^{m}(a_{2j}\beta_j) = \hat{0} \text{ if } k=2 \text{ and so on}\right)$

$$\Rightarrow a_{k1} T_{k1}(\alpha_k) + a_{k2} T_{k2}(\alpha_k) + \dots + a_{km} T_{km}(\alpha_k) = \hat{0}$$

$$\Rightarrow a_{k1}\beta_1 + a_{k2}\beta_2 + \dots + a_{km}\beta_m = \hat{0}$$

$$\Rightarrow a_{k1} = a_{k2} = \dots = a_{km} = 0 \quad (\because B_2 \text{ is a basis of } V)$$

$\therefore S = \{T_{ij}\}$ is L.I. set.

(ii) To show that $L(S) = L(U,V)$.

Let $T\in L(U,V)$ then the vector $T(\alpha_1)\in V$

It can be expressed as L.C. of elts of B_2.

i.e. $T(\alpha_1) = b_{11}\beta_1 + b_{21}\beta_2 + \dots + b_{m1}\beta_m$.

In general for $i=1,2,\dots n$

$$T(\alpha_i) = b_{1i}\beta_1 + b_{2i}\beta_2 + \dots + b_{mi}\beta_m. \quad \text{——— (1)}$$

consider the linear transformation,

$$H = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} T_{ij}$$

clearly H is a linear combination of $S=\{T_{ij}\}$

$\therefore H\in L(U,V)$

Let $\alpha_k\in U$ for $k=1,\dots n$.

Since $T_{ij}(\alpha_k) = \hat{0}$ for $k\ne i$ & $T_{kj}(\alpha_k) = \beta_j$

we have $H(\alpha_k) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij} T_{ij}(\alpha_k)$

$$= \sum_{j=1}^{m} b_{kj} T_{kj}(\alpha_k)$$

$$= \sum_{j=1}^{m} b_{kj}\, \beta_j$$

i.e $H(\alpha_k) = b_{k1}\beta_1 + b_{k2}\beta_2 + \cdots + b_{km}\beta_m$.

$= T(\alpha_k)$ (by ①)

$\therefore H(\alpha_k) = T(\alpha_k)$ for each k

$\Rightarrow H = T$

Thus T is a linear combination of elts of S

i.e $L(S) = L(U,V)$

$\therefore$ S is a basis set of $L(U,V)$.

$\therefore \dim L(U,V) = mn$.

problems

→ find the dimension of $L(\mathbb{R}^3, \mathbb{R}^2)$

since $\dim \mathbb{R}^3 = 3$ and $\dim \mathbb{R}^2 = 2$

$\therefore \dim(L(\mathbb{R}^3, \mathbb{R}^2)) = 6$.

→ find the dimension of $L(\mathbb{C}^3, \mathbb{R}^2)$.

Sol since $\mathbb{C}^3$ is a vectorspace over 'C' and $\mathbb{R}^2$ is a vectorspace over $\mathbb{R}$

Hence $\dim(L(\mathbb{C}^3, \mathbb{R}^2))$ does not exist.

→ Let $V = \mathbb{C}^3$ be a vector space over $\mathbb{R}$. find the dimension of $L(V, \mathbb{R}^2)$ (i.e dim of $Hom(V, \mathbb{R}^2)$).

Sol since $V = \mathbb{C}^3$ is a vectorspace over $\mathbb{R}$

i.e let $V = \{(a_1+ib_1, a_2+ib_2, a_3+ib_3) / a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}\}$

$\therefore \dim V = 6$

and obviously $\dim \mathbb{R}^2 = 2$

$\therefore \dim(Hom(U,V)) = 6 \times 2 = 12$.

* Range and Nullspace of a Linear Transformation:-

Def Let $U(F)$ and $V(F)$ be two vectorspaces and let $T: U \to V$ be a linear transformation. The range of T is defined to be the set

$$\text{Range}(T) = R(T)$$
$$= \{T(\alpha) / \alpha \in U\}.$$

Obviously the range of T is a subset of V. i.e $R(T) \subseteq V$.

→ Let $U(F)$ and $V(F)$ be two vectorspaces. Let $T: U(F) \to V(F)$ be a linear transformation. Then the range set $R(T)$ is a subspace of $V(F)$.

proof Let $R(T) = \{T(\alpha) / \alpha \in U\}$

For $\bar{0} \in U \Rightarrow T(\bar{0}) = \hat{0} \in R(T)$

$\therefore R(T)$ is non-empty set and $R(T) \subseteq V$.

Let $\alpha_1, \alpha_2 \in U$ and $\beta_1, \beta_2 \in R(T)$ be s.t

$T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$.

For $a, b \in F$, $a\alpha_1 + b\alpha_2 \in U$ ($\because U$ is v.s).

$\Rightarrow T(a\alpha_1 + b\alpha_2) \in R(T)$.

But $T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$

$= a\beta_1 + b\beta_2$ ($\because T$ is LT)

$\in R(T)$.

$\therefore a, b \in F$ and $\beta_1, \beta_2 \in R(T)$

$\Rightarrow a\beta_1 + b\beta_2 \in R(T)$.

$\therefore R(T)$ is subspace of $V(F)$.

$R(T)$ is called the range space.

* **Nullspace or Kernal:**

Let $U(F)$ and $V(F)$ be two vectorspaces and $T: U \to V$ be a linear transformation. The nullspace denoted by $N(T)$ is the set of all vectors $\alpha \in U$ s.t $T(\alpha) = \hat{0}$ (zero vector of V).

The nullspace of $N(T)$ is also called the kernel of T.

i.e $N(T) = \{\alpha \in U / T(\alpha) = \hat{0} \in V\}$

Obviously the nullspace $N(T) \subseteq U$.

→ Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ is a linear transformation. Then nullspace $N(T)$ is a subspace of $U(F)$.

proof Let $N(T) = \{\alpha \in U / T(\alpha) = \hat{0} \in V\}$

$\because T(\bar{0}) = \hat{0} \Rightarrow \bar{0} \in N(T)$. $(\because \bar{0} \in U, \hat{0} \in V)$.

$\therefore N(T)$ is a non-empty subset of U.

Now $\alpha, \beta \in N(T) \Rightarrow T(\alpha) = \hat{0}, T(\beta) = \hat{0}$.

For $a, b \in F$, $T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ $(\because T$ is LT$)$.

$= a.\hat{0} + b.\hat{0}$

$= \hat{0}$.

$\therefore T(a\alpha + b\beta) = \hat{0}$.

By definition $a\alpha + b\beta \in N(T)$.

$\therefore \forall a, b \in F$ and $\alpha, \beta \in N(T) \Rightarrow a\alpha + b\beta \in N(T)$.

$\therefore$ nullspace $N(T)$ is a subspace of $U(F)$.

→ Let $T: U(F) \to V(F)$ be a linear transformation. If U is finite dimensional then the range space $R(T)$ is a finite dimensional subspace of $V(F)$.

proof: Given that U is finite dimensional.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the basis set of $U(F)$.

Let $\beta \in R(T)$

Then $\exists\ \alpha \in U$ such that $T(\alpha) = \beta$.

$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n$ for a's $\in F$.

$\Rightarrow T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$\Rightarrow \beta = a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n)$

But $S' = \{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\} \subset R(T) \Rightarrow L(S') \subseteq R(T)$ — (1).

Now $\beta \in R(T)$ and $\beta =$ l.c. of ell's of S'

$\Rightarrow \beta \in L(S') \Rightarrow R(T) \subseteq L(S')$ — (2).

from (1) & (2), we have $R(T) = L(S')$

Thus $R(T)$ is spanned by a finite set S'.

$\therefore R(T)$ is finite dimensional subspace of $V(F)$.

* **Dimension of Range and Kernel:**

→ Let $T: U(F) \to V(F)$ be a linear transformation where U is finite dimensional vector space.

Rank: Then the rank of T denoted by $\rho(T)$ is the dimension of range space $R(T)$.

i.e., $\rho(T) = \dim R(T)$.

Nullity: The nullity of T denoted by $\nu(T)$ is the dimension of null space $N(T)$.

$\nu(T) = \dim N(T)$.

Theorem

Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a linear transformation. Let U be finite dimensional then $\rho(T) + \nu(T) = \dim U$.

i.e, rank(T) + nullity$(T) = \dim U$.

Proof: The null space $N(T)$ is a subspace of finite dimensional space $U(F)$.

$\Rightarrow N(T)$ is finite dimensional.

Let $S=\{\alpha_1,\alpha_2,\ldots\ldots\alpha_k\}$ be a basis of $N(T)$.

$\therefore \dim N(T) = \nu(T) = k$.

$\therefore T(\alpha_1)=\hat{0},\ T(\alpha_2)=\hat{0}\ \ldots\ldots\ T(\alpha_k)=\hat{0}.$ —— (1)

As S is L.I. it can be extended to form a basis of $U(F)$.

Let $S_1=\{\alpha_1,\alpha_2\ldots\alpha_k,\theta_1,\theta_2,\ldots\theta_m\}$ be the extended basis of $U(F)$.

$\therefore \dim U = k+m$.

Now we show that the set of images of additional vectors $S_2=\{T(\theta_1),T(\theta_2),\ldots T(\theta_m)\}$ is a basis of $R(T)$.

Clearly $S_2\subseteq R(T)$.

(i) To prove S_2 is LI.

Let $a_1,a_2,\ldots a_m\in F$ such that

$a_1T(\theta_1)+a_2T(\theta_2)+\ldots\ldots+a_mT(\theta_m)=\hat{0}.$

$\Rightarrow T(a_1\theta_1+a_2\theta_2+\ldots\ldots+a_m\theta_m)=\hat{0}.$ ($\because T$ is LT)

$\Rightarrow a_1\theta_1+a_2\theta_2+\ldots\ldots+a_m\theta_m\in N(T).$

But each vector in $N(T)$ is a l.c. of elts of basis 'S'.

$\therefore$ for some $b_1,b_2,\ldots\ldots b_k\in F$,

Let $a_1\theta_1+a_2\theta_2+\ldots+a_m\theta_m=b_1\alpha_1+b_2\alpha_2+\ldots+b_k\alpha_k$

$\Rightarrow a_1\theta_1+a_2\theta_2+\ldots+a_m\theta_m-b_1\alpha_1-b_2\alpha_2-\ldots\ldots-b_k\alpha_k=0$

$\Rightarrow a_1=0,\ a_2=0,\ \ldots a_m=0,\ b_1=0,b_2=0\ldots b_k=0$

($\because S_1$ is LI)

$\Rightarrow S_2$ is L.I. set.

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space $R(T)$, then $\exists\, \alpha \in U$ s.t

$$T(\alpha) = \beta.$$

Now $\alpha \in U \Rightarrow$ there exist c's, d's $\in F$ such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \cdots + d_m\theta_m$$

$$\Rightarrow T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \cdots + d_m\theta_m)$$

$$= c_1T(\alpha_1) + c_2T(\alpha_2) + \cdots + c_kT(\alpha_k) + d_1T(\theta_1) + d_2T(\theta_2) + \cdots + d_mT(\theta_m)$$

$$\Rightarrow \beta = d_1T(\theta_1) + d_2T(\theta_2) + \cdots + d_mT(\theta_m) \quad (\because \text{by } (1))$$

$\Rightarrow \beta \in L(S_2)$.

$\therefore S_2$ is a basis of $R(T)$.

and $\dim R(T) = m$.

$\therefore \dim R(T) + \dim N(T) = m + k = \dim U$.

i.e., $\rho(T) + \nu(T) = \dim U$.

problems

→ If $T: V_4(\mathbb{R}) \to V_3(\mathbb{R})$ is a linear transformation defined by $T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$, for $a,b,c,d \in \mathbb{R}$ then verify $\rho(T) + \nu(T) = \dim V_4(\mathbb{R})$.

Sol. Let $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ be the standard basis set of $V_4(\mathbb{R})$

$\therefore$ the transformation T on S will be

$T(1,0,0,0) = (1,1,1)$, $T(0,1,0,0) = (-1,0,1)$

$T(0,0,1,0) = (1,2,3)$, $T(0,0,0,1) = (1,-1,-3)$

Let $S_1 = \{(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)\}$

$\therefore S_1 \subseteq R(T)$.

Now we verify whether S_1 is L.I or not.
If not, we find least L.I set by forming the matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad R_4 \to R_4 + 2R_2$$

Clearly which is in echelon form.

$\therefore$ The non-zero rows of vectors $\{(1,1,1), (0,1,2)\}$ constitute the L.I set forming the basis of $R(T)$.

$\Longrightarrow \boxed{\dim R(T) = 2.}$

<u>Basis for nullspace of T:</u>

$N(T) = \{\alpha \in V_4 \,/\, T(\alpha) = \hat{0}\}$

Let $\alpha \in N(T) \Longrightarrow T(\alpha) = \hat{0}$

$\therefore T(a,b,c,d) = \hat{0}$ where $\hat{0} = (0,0,0) \in V_3(\mathbb{R})$

$\Longrightarrow (a-b+c+d,\ a+2c-d,\ a+b+3c-3d) = (0,0,0)$

$\Longrightarrow a-b+c+d = 0$

$a+2c-d = 0$

$a+b+3c-3d = 0$

we have to solve them for a,b,c,d.

coefficient matrix $= \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$.

$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

Clearly which is in echelon form.

$\therefore$ The equivalent system of equations are

$a - b + c + d = 0 \implies \boxed{b = 2d - c}$

$b + c - 2d = 0 \implies \boxed{a = d - 2c}$

$\therefore$ The number of free variables is 2 namely c, d and the values of a & b depend on ~~these~~ and hence $\boxed{\text{nullity}(T) = \dim(N(T)) = 2}$

choosing $c = 1, d = 0$, we get $a = -2, b = -1$

$\therefore (a, b, c, d) = (-2, -1, 1, 0)$

choosing $c = 0, d = 1$, we get $a = 1, b = 2$

$\therefore (a, b, c, d) = (1, 2, 0, 1)$

$\therefore \{(-2, -1, 1, 0), (1, 2, 0, 1)\}$ constitute a basis of $N(T)$.

$\therefore \dim(R(T)) + \dim(N(T)) = 2 + 2 = 4 = \dim V_4(\mathbb{R})$.

$\therefore N(T) = \{(d - 2c, 2d - c, c, d) / c, d \in \mathbb{R}\}$.

→ verify the Rank-Nullity theorem for the linear map $T: V_4 \to V_3$ defined by $T(e_1) = f_1 + f_2 + f_3$, $T(e_2) = f_1 - f_2 + f_3$, $T(e_3) = f_1$, $T(e_4) = f_1 + f_3$ when $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are standard basis V_4 and V_3 respectively.

Sol^n: Let $e_1 = (1,0,0,0)$; $e_2 = (0,1,0,0)$; $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$

and $f_1 = (1,0,0)$ $f_2 = (0,1,0)$, $f_3 = (0,0,1)$

$\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3\}$ are the standard basis of V_4 and V_3 respectively.

we have $T(e_1) = f_1 + f_2 + f_3$

$$\Rightarrow T(1,0,0,0) = (1,0,0) + (0,1,0) + (0,0,1)$$
$$= (1,1,1)$$

$$T(e_2) = f_1 - f_2 + f_3$$

$$\Rightarrow T(0,1,0,0) = (1,0,0) - (0,1,0) + (0,0,1)$$
$$= (1,-1,1)$$

$$T(e_3) = f_1$$

$$\Rightarrow T(0,0,1,0) = (1,0,0)$$

$$T(e_4) = f_1 + f_3$$

$$\Rightarrow T(0,0,0,1) = (1,0,0) + (0,0,1)$$
$$= (1,0,1)$$

Let $\alpha \in V_4$

Then α can be written as $\alpha = ae_1 + be_2 + ce_3 + de_4$

Then $T(\alpha) = T(ae_1 + be_2 + ce_3 + de_4)$

$$= aT(e_1) + bT(e_2) + cT(e_3) + dT(e_4)$$
$$= a(1,1,1) + b(1,-1,1) + c(1,0,0) + d(1,0,1)$$
$$= (a+b+c+d,\ a-b,\ a+b+d)$$

consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$R_4 \to R_4 - R_1$
$R_3 \to R_3 - R_1$
$R_2 \to R_2 - R_1$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

$R_3 \to R_3 - \frac{1}{2}R_2$, $R_4 \to R_4 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly which is in echelon form.

$\therefore$ The non-zero rows of vectors

$\{(1,1,1), (0,-2,0), (0,0,-1)\}$ constitute the L.I set forming the basis of $R(T)$.

$$\Rightarrow \boxed{\dim R(T) = 3}$$

Basis for null space of T:

$N(T) = \{\alpha \in V_4 \,/\, T(\alpha) = \hat{0}\}$

Let $\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\Rightarrow (a+b+c+d, a-b, a+b+d) = (0,0,0)$

$\Rightarrow$ $a+b+c+d = 0$ —— (1)

$a-b = 0$ —— (2)

$a+b+d = 0$ —— (3).

We have to then for a, b, c, d.

from (1) & (3), we get $\boxed{c=0}$

from (2) & (3) we get $\boxed{d=-2a}$

from (1), we get $\boxed{b=a}$.

The number of free variables is 1 namely 'a' and the values d & b depend on 'a' and hence nullity(T) = dim N(T) = 1.

choosing $a=1$, we get $b=1$, $d=-2$

$(a,b,c,d) = (1,1,0,-2)$.

$\therefore$ $\{(1,1,0,-2)\}$ constitute a basis of $N(T)$.

$\therefore$ $\dim R(T) + \dim N(T) = 3+1$
$= 4$
$= \dim(V_4)$.

H.W $\rightarrow$ Let $T: V_4 \rightarrow V_3$ be a linear transformation defined by $T(\alpha_1) = (1,1,1)$; $T(\alpha_2) = (1,-1,1)$;
$T(\alpha_3) = (1,0,0)$; $T(\alpha_4) = (1,0,1)$.

Then verify the $\rho(T) + \nu(T) = \dim V_4$.

$\rightarrow$ find a linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$ whose range is spanned by $(1,2,0,-4),(2,0,-1,-3)$.

Sol Given the $R(T)$ spanned by
$\{(1,2,0,-4),(2,0,-1,-3)\}$.

Let us include a vector $(0,0,0,0)$ in this set which will not effect the spanning property.

so the $S = \{(1,2,0,-4),(2,0,-1,-3),(0,0,0,0)\}$

Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of $\mathbb{R}^3$.

W.K.T there exists a transformation

T s.t $T(\alpha_1) = (1,2,0,-4)$
$T(\alpha_2) = (2,0,-1,-3)$
$T(\alpha_3) = (0,0,0,0)$

Now if $\alpha \in \mathbb{R}^3 \Rightarrow \alpha = (a,b,c) = a\alpha_1 + b\alpha_2 + c\alpha_3$

77

$\Rightarrow T(\alpha) = T(a,b,c) = T(a\alpha_1 + b\alpha_2 + c\alpha_3)$

$\Rightarrow T(a,b,c) = aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3)$

$= a(1,2,0,-4) + b(2,0,-1,-3)$

$+ c(0,0,0,0)$

$\therefore T(a,b,c) = (a+2b, 2a, -b, -4a-3b)$

is the reqd transformation.

→ ① Find $F: \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation whose range is spanned by $(1,-1,2,3)$ and $(2,3,-1,0)$

Sol: consider the standard basis for $\mathbb{R}^3$ is $\{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

Then $F(e_1) = (1,-1,2,3)$

$F(e_2) = (2,3,-1,0)$ and

$F(e_3) = (0,0,0,0)$.

W.K.T. $(x,y,z) = xe_1 + ye_2 + ze_3$

$F(x,y,z) = F(xe_1 + ye_2 + ze_3)$

$= xF(e_1) + yF(e_2) + zF(e_3)$ ($\because F$ is L.T.)

$= (x, -x, 2x, 3x) + (2y, 3y, -y, 0)$

$+ (0,0,0,0)$

$= (x+2y, -x+3y, 2x-y, 3x)$

2nd → Let V be a vectorspace of all 2×2 matrices over reals. Let P be a fixed matrix of V; $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ and $T: V \to V$ be a linear operator defined by $T(A) = PA, A \in V$

find the nullity T.

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

The nullspace $N(T)$ is the set of all 2×2 matrices whose T-image is $\hat{0}$.

$\Rightarrow T(A) = PA = \hat{0}$

$\Rightarrow T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a-c=0,\ b-d=0$

$\Rightarrow a=c,\ b=d$

The free variables are c, & d

Hence $\dim N(T) = 2$.

H.W. → Describe explicitly the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose range space is spanned by $\{(1,0,-1),(1,2,2)\}$.

→ Find the null space, range, rank and nullity of the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+y, x-y, y)$

Sol (i) Let $\alpha = (x,y) \in \mathbb{R}^2$

Then $N(T) = \{\alpha \in \mathbb{R}^2 \,/\, T(\alpha) = \hat{0}\}$.

$\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\Rightarrow T(x,y) = \hat{0}$ where $\hat{0} = (0,0,0) \in \mathbb{R}^3$.

$\Rightarrow (x+y, x-y, y) = (0,0,0)$

$$\left.\begin{array}{l} x+y=0 \\ x-y=0 \\ y=0 \end{array}\right\} \Longrightarrow x=0,\ y=0.$$

$\therefore \alpha=(x,y)=(0,0)\in\mathbb{R}^2.$

$\therefore$ the nullspace of T consists of only zero vector of $\mathbb{R}^2$

$\therefore$ nullity $T=\dim N(T)=0.$

(ii) Range space of $T=\{\beta\in\mathbb{R}^3 / T(\alpha)=\beta \text{ for } \alpha\in\mathbb{R}^2\}.$

$\therefore$ The range space consists of all vectors of the type $(x+y,\ x-y,\ y)$ for all $(x,y)\in\mathbb{R}^2$

(iii) $\dim R(T)+\dim N(T)=\dim\mathbb{R}^2$

$\Rightarrow \dim R(T)+0=2$

$\Rightarrow \dim R(T)=2$

$\Rightarrow \boxed{\text{rank of } T=2}$

$\longrightarrow$ Show that $T:\mathbb{R}^4\to\mathbb{R}^4$ given by $T(x,y,z,t)=(2x,3y,0,0)$ is a linear transformation. Find its rank and nullity.

Sol Let $\alpha=(x_1,y_1,z_1,t_1)$ and $\beta=(x_2,y_2,z_2,t_2)$ be two vectors of $\mathbb{R}^4$.

For $a,b\in\mathbb{R}$

$$T(a\alpha+b\beta)=T\left[a(x_1,y_1,z_1,t_1)+b(x_2,y_2,z_2,t_2)\right]$$

$$=T\left[ax_1+bx_2,\ ay_1+by_2,\ az_1+bz_2,\ at_1+bt_2\right]$$

$$=\left(2(ax_1+bx_2),\ 3(ay_1+by_2),\ 0,\ 0\right) \text{ (by hyp)}$$

$= a(2x_1, 3y_1, 0, 0) + b(2x_2, 3y_2, 0, 0)$

$= a\,T(\alpha_1) + b\,T(\beta)$.

$\therefore$ T is a linear transformation.

Now we have $N(T) = \{(x, y, z, t) \in \mathbb{R}^4 / T(x, y, z, t) = (0, 0, 0, 0)\}$

$\therefore (x, y, z, t) \in N(T)$

$\iff T(x, y, z, t) = (0, 0, 0, 0)$

$\iff (2x, 3y, 0, 0) = (0, 0, 0, 0)$

$\iff x = 0,\ y = 0.$

$\therefore N(T) = \{(0, 0, z, t) / z, t \in \mathbb{R}\}$.

since $(0, 0, z, t) = z(0, 0, 1, 0) + t(0, 0, 0, 1)$

$\therefore N(T)$ is spanned by the set

$S = \{e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$.

Clearly which is L.I.

$\therefore S$ is basis of $N(T)$.

$\therefore \dim N(T) = 2$.

$\boxed{\text{Nullity of } T = 2}$ i.e $\nu(T) = 2$.

W.K.T

$\dim R(T) + \dim N(T) = \dim \mathbb{R}^4$

$\Rightarrow \dim R(T) + 2 = 4$

$\Rightarrow \boxed{\dim R(T) = 2}$.

HW $\Rightarrow$ Show that $T: \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(x, y, z, t) = (x+y, x-y, 0, 0)$ is a linear transformation. find rank and nullity

Clearly which is LI.

$\therefore$ S is basis of ~~R(~~ker T.

$\therefore$ dim ker T = 1

i.e Nullity T = 1.

→ Find the range, rank, Kernel and nullity of the linear transformation $T: R^2 \to R^3$ defined by

$$T(x, y, z) = (x+2y-z, y+z, x+y-2z)$$

→ Let $T: R^3 \to R^3$ defined by

$$T(x, y, z) = (x-y+2z, 2x+y-z, -x-2y)$$

find the null space of T.

→ Let T be the linear transformation from R^3 to R^4 defined by

$$T(x_1, x_2, x_3) = (2x_1+x_2+x_3, x_1+x_2, x_1+x_3, 3x_1+x_2+2x_3).$$

for each $(x_1, x_2, x_3) \in R^3$.

Determine a basis for the null space of T. What is the dimension of the Range space of T?

→ Let $T: \mathbb{R}^5 \to \mathbb{R}^5$ be a linear mapping given by

$$T(a, b, c, d, e) = (b-d, d+e, b, 2d+e, b+e).$$

Obtain bases for its nullspace and range space.

→ Show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation, where $f(x, y, z) = 3x+y-z$. What is the dimension of the kernel? find a basis for the kernel.

→ Let $T: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation defined by $T(x, y, z) = (x+y, y+z)$. find a basis, dimension of each of the range and null space of T.

→ ~~Let~~ $T: V_3(R) \to V_2(\mathbb{R})$ is defined by

$T(a, b, c) = (a, b) \;\; \forall (a, b, c) \in R^3$.

P.T. T is a linear transformation. find the kernel of T.

→ Let $V(F)$ be a vectorspace and T be a linear operator on V. Prove that the following statements are true.

(i) The intersection of the range of T and null space of T is the zero subspace

i.e $R(T) \cap N(T) = \{\bar{0}\}$.

(ii) If $T[T(\alpha)] = \bar{0}$, then $T(\alpha) = \bar{0}$.

Sol (i) $\Rightarrow$ (ii)

Let $R(T) \cap N(T) = \{\bar{0}\}$.

Let $T(\alpha) = \beta \quad \therefore \beta \in R(T)$ ——— (1)

Now $T[T(\alpha)] = \bar{0} \Rightarrow T(\beta) = \bar{0} \Rightarrow \beta \in N(T)$. ——— (2)

from (1) & (2) $\beta \in R(T) \cap N(T)$

But $R(T) \cap N(T) = \{\bar{0}\} \Rightarrow \beta = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$.

$\therefore T[T(\alpha)] = \bar{0} \Rightarrow T(\alpha) = \bar{0}$.

(ii) $\Rightarrow$ (i) :

Given ~~that~~ $T[T(\alpha)] = \bar{0} \Rightarrow T(\alpha) = \bar{0}$.

Let $\beta \in R(T) \cap N(T)$

$\Rightarrow \beta \in R(T)$ and $\beta \in N(T)$

Now $\beta \in R(T) \Rightarrow T(\alpha) = \beta$ for some $\alpha \in V$

and $\beta \in N(T) \Rightarrow T(\beta) = \bar{0}$

$\Rightarrow T[T(\alpha)] = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$

$\Rightarrow \beta = \bar{0} \quad (\because T(\alpha) = \beta)$

$\therefore R(T) \cap N(T) = \{\bar{0}\}$.

Note:- If $T: U \to V$ is a linear transformation, then $\rho(T) \le \min(\dim U, \dim V)$.

→ Is there a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^2$ for which rank $T = 3$ and nullity $T = 2$?

Sol If $T: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation, then

$$\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^4$$

$$\text{i.e. } 3 + 2 = 4$$

This is impossible.

Hence T is not a linear transformation.

→ Let T be a linear transformation from $\mathbb{R}^7$ onto a 3-dimensional subspace of $\mathbb{R}^5$. Find dim ker T.

Sol. Let W be a 3-dimensional subspace of $\mathbb{R}^5$ such that $T: \mathbb{R}^7 \to W$ is an onto L.T.

We have

$$T(\mathbb{R}^7) = W \Rightarrow \dim T(\mathbb{R}^7) = \dim W = 3.$$

$$\therefore \text{Rank}(T) = \dim(T(\mathbb{R}^7)) = 3.$$

$$\therefore \text{Rank}(T) + \text{Nullity}(T) = \dim \mathbb{R}^7$$

$$\Rightarrow 3 + \text{Nullity}(T) = 7.$$

$$\Rightarrow \text{Nullity}(T) = 7 - 3$$

$$\Rightarrow \text{Nullity}(T) = 4.$$

$$\therefore \dim \ker T = 4.$$

→ Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ having a 3-dimensional kernel. Find dim Range T.

Sol Given that $T:\mathbb{R}^5 \to \mathbb{R}^3$ is L.T and having a 3-dimensional kernel.

$\therefore$ dim Ker $T = 3 \Rightarrow$ nullity$(T) = 3$.

$\therefore$ rank(T) + nullity(T) = dim $\mathbb{R}^5$

$\Rightarrow$ rank$(T) + 3 = 5$

$\Rightarrow$ rank$(T) = 2$

$\Rightarrow \boxed{\text{dim Range } T = 2}$

* Singular and Non-singular Transformation *.

Singular transformation:

A linear transformation $T: U(F) \to V(F)$ is said to be singular if the nullspace of T consists of atleast one non-zero vector.

i.e If there exists a vector $\alpha \in U$ s.t $T(\alpha) = \hat{0}$ for $\alpha \neq \bar{0}$ then T is singular.

*Non-Singular Transformation:

A linear transformation $T: U(F) \to V(F)$ is said to be non-singular if the null space consists of one zero vector alone.

i.e $\alpha \in U$ and $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

$\Rightarrow N(T) = \{\bar{0}\}$.

Theorem ⇒ Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a linear transformation. Then T is non-singular iff the set of images of a linearly independent set is linearly independent.

proof (i) Let T be non-singular and let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a L.I. subset of U. Then its T-images set be $S' = \{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$.

Now to prove S' is L.I.

For some $a_1, a_2, \ldots a_n \in F$,

Let $a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n) = \hat{0}$ $\quad (\because \hat{0} \in V)$.

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n] = \hat{0}$ $\quad (\because T \text{ is L.T})$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = \bar{0}$ $\quad (\because T \text{ is non-singular})$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$ $\quad (\because S \text{ is L.I.})$.

$\therefore S'$ is L.I.

(ii) Let the T-images of any L.I. set be L.I. then to prove T is non-singular.

Let $\alpha \in U$ and $\alpha \neq \bar{0}$. Then the set $B = \{\alpha\}$ is L.I. set and image set $B' = \{T(\alpha)\}$ is given to be L.I.

$\Rightarrow T(\alpha) \neq \hat{0}$

$\therefore \alpha \neq \bar{0} \Rightarrow T(\alpha) \neq \hat{0}$

$\therefore T$ is non-singular.

problems → A linear mapping $T: R^3 \to R^3$ is defined by $T(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)$ show that T is non-singular.

Sol^n Let $T(x,y,z) = \hat{0}$

$\Rightarrow (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta, z) = (0,0,0)$

$\Rightarrow x\cos\theta - y\sin\theta = 0$ — (i)

$x\sin\theta + y\cos\theta = 0$ — (ii)

$z = 0$

Squaring and adding eqns (i) & (ii)

$x^2 + y^2 = 0$

$\Rightarrow x = 0, y = 0$

$\therefore x=0,\ y=0,\ z=0$

$\therefore$ we have $T(x,y,z) = \hat{0}$

$\Rightarrow (x,y,z) = (0,0,0)$

$\therefore$ T is non-singular

→ Show that a linear transformation $T: U \to V$ over the field F is non-singular iff T is one-one

Sol^n (i) Let T be non-singular

i.e, $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now for $\alpha_1, \alpha_2 \in U$,

$T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{0}$ $(\because \hat{0} \in V)$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{0}$ $(\because$ T is L.T$)$

$\Rightarrow \alpha_1 - \alpha_2 = \bar{0}$ $(\because$ T is non-singular$)$

$\Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one.

(ii) Let T be one-one.

$\therefore$ zero elt $\hat{0}$ of V is the T-image of only one element $\in U$.

$\Rightarrow$ null space of T consists of only one elt.

Since null space $N(T) \subset U$ (subspace of U), it must consist of $\bar{0}$.

$\Rightarrow$ null space $N(T)$ consists of only one $\bar{0}$ element

$\Rightarrow N(T) = \{\bar{0}\}$.

$\Rightarrow T$ is non-singular.

→ Let $T: U \to V$ be a linear transformation of $U(F)$ into $V(F)$ where $U(F)$ is finite dimensional. Prove that U and the range space of T have the same dimension iff T is non-singular.

Sol: (i) Let $\dim U = \dim R(T)$

W.K.T
$\dim U = \dim R(T) + \dim N(T)$

$\Rightarrow \dim R(T) = \dim R(T) + \dim N(T)$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow$ The null space of T is the zero space $\{\bar{0}\}$.

$\therefore T$ is non-singular.

(ii) Let T be non-singular. Then $N(T) = \{\bar{0}\}$ and nullity $T = 0$ i.e $\dim(N(T)) = 0$.

As $\dim U = \dim R(T) + \dim N(T)$
$= \dim R(T) + 0$

$\Rightarrow \dim U = \dim R(T)$.

→ If U and V are finite dimensional vector spaces of the same dimension, then a linear mapping $T: U \to V$ is one-one iff it is onto.

Sol: T is one-one $\Leftrightarrow N(T) = \{\bar{0}\}$
$\Leftrightarrow \dim N(T) = 0$
$\Leftrightarrow \dim R(T) + \dim N(T) = \dim U = \dim V$.
$\Leftrightarrow R(T) = V$
$\Leftrightarrow T$ is onto.

* **Inverse function :**

Let $T: U \to V$ be a one-one onto mapping. Then the mapping $T^{-1}: V \to U$ defined by

$T^{-1}(\beta) = \alpha \Leftrightarrow T(\alpha) = \beta$, $\alpha \in U$, $\beta \in V$, is called the inverse mapping of T.

Note:- If $T: U \to V$ is one-one onto mapping then the mapping $T^{-1}: V \to U$ is also one-one onto.

→ Let $U(F)$ and $V(F)$ be two vectorspaces and $T: U \to V$ be a one-one onto linear transformation. Then T^{-1} is a linear transformation and the T is said to be invertible.

Sol. Let $\beta_1, \beta_2 \in V$ and $a, b \in F$

Since T is one-one onto function,

$\exists$ unique vectors $\alpha_1, \alpha_2 \in U$ s.t

$$T(\alpha_1) = \beta_1 \text{ and } T(\alpha_2) = \beta_2$$

Hence by the definition of T^{-1}

$$\alpha_1 = T^{-1}(\beta_1) \text{ and } \alpha_2 = T^{-1}(\beta_2)$$

Also $\alpha_1, \alpha_2 \in U$ and $a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in U$

$$\therefore T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2) \quad (\because T \text{ is LT})$$

$$= a\beta_1 + b\beta_2$$

$\therefore$ by the defn of inverse

$$T^{-1}(a\beta_1 + b\beta_2) = a\alpha_1 + b\alpha_2$$

$$= aT^{-1}(\beta_1) + bT^{-1}(\beta_2)$$

$\therefore T^{-1}$ is a linear transformation from V into U.

→ A linear transformation T on a finite dimensional vectorspace is invertible iff T is non-singular.

Sol Let $U(F)$ and $V(F)$ be two vectorspaces and have the same dimension.
Let $T: U \to V$ be a linear transformation.

(i) Let T be non-singular.
i.e for $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now to prove T is invertible,
it is enough to show that T is one-one onto.
Since T is non-singular,
i.e $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$; $N(T) = \{\bar{0}\}$
$\Rightarrow \dim N(T) = 0$.

for $\alpha_1, \alpha_2 \in U$, $T(\alpha_1) = T(\alpha_2)$
$\Rightarrow T(\alpha_1) - T(\alpha_2) = 0$
$\Rightarrow T(\alpha_1 - \alpha_2) = \bar{0}$ ($\because T$ is LT)
$\Rightarrow \alpha_1 - \alpha_2 = \bar{0}$ ($\because T$ is non-singular)
$\Rightarrow \alpha_1 = \alpha_2$
$\therefore T$ is one-one.

W.K.T $\dim U = \dim R(T) + \dim N(T)$
$= \dim R(T)$ ($\because \dim N(T) = 0$).
Also $T: U \to V$ is one-one
$\Rightarrow V = R(T)$
$\Rightarrow T$ is onto.

(ii) Let T be invertible so that T is one-one onto.
Now to prove T is non-singular
for $\alpha \in U$, $T(\alpha) = \hat{0} = T(\bar{0})$ ($\because T$ is LT)
$\Rightarrow T(\alpha) = T(\bar{0}) \Rightarrow \alpha = \bar{0}$ ($\because T$ is one-one)

T is non-singular.

Note: Let $U(F)$ and $V(F)$ be two finite dimensional vectorspace s.t $\dim U = \dim V$. If $T: U \to V$ is a linear transformation then the following are equivalent.

1. T is invertible
2. T is non-singular
3. The range of T is V
4. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is any basis of U, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V
5. There is some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of U s.t $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V.

Here we shall have a series of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$

problems.

→ If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is invertible operator defined by $T(x,y,z) = (2x, 4x-y, 2x+3y-z)$. Find T^{-1}.

Sol Since T is invertible

$T(\alpha) = \beta \Rightarrow T^{-1}(\beta) = \alpha$. ; $\alpha \in \mathbb{R}^3$, $\beta \in \mathbb{R}^3$.

Now $T(x,y,z) = (a,b,c) \Rightarrow T^{-1}(a,b,c) = (x,y,z)$

Now $(2x, 4x-y, 2x+3y-z) = (a,b,c)$

$\Rightarrow 2x = a,\ 4x-y = b,\ 2x+3y-z = c$

solving $x = a/2,\ y = 2a-b,\ z = 7a-3b-c$.

Hence $T^{-1}(a,b,c) = (a/2, 2a-b, 7a-3b-c)$.

→ The set $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$. $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$ is a linear operator defined by $T(e_1) = e_1+e_2$, $T(e_2) = e_2+e_3$, $T(e_3) = e_1+e_2+e_3$.

show that T is non-singular and find its invertible.

Sol Let $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$

Now $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by

$T(e_1)=e_1+e_2 \Rightarrow T(1,0,0)=(1,1,0)$

$T(e_2)=e_2+e_3 \Rightarrow T(0,1,0)=(0,1,1)$

$T(e_3)=e_1+e_2+e_3 \Rightarrow T(0,0,1)=(1,1,1)$

Let $\alpha=(x,y,z)\in V_3(\mathbb{R})$

$\therefore \alpha = x(1,0,0)+y(0,1,0)+z(0,0,1)$

$\Rightarrow T(\alpha) = x\,T(1,0,0)+y\,T(0,1,0)+z\,T(0,0,1)$

$= x(1,1,0)+y(0,1,1)+z(1,1,1)$

$\therefore$ The transformation is given by

$T(x,y,z) = (x+z,\ x+y+z,\ y+z)$.

Now if $T(x,y,z)=\hat{0}$ then

$(x+z,\ x+y+z,\ y+z)=(0,0,0)$

$\Rightarrow x+z=0,\ x+y+z=0,\ y+z=0.$

$\Rightarrow x=y=z=0$

$\therefore T(\alpha)=\hat{0} \Rightarrow \alpha=\bar{0}$.

Hence T is non-singular and therefore T^{-1} exists.

Let $T(x,y,z)=(a,b,c)$ then $T^{-1}(a,b,c)=(x,y,z)$

now $(x+z,\ x+y+z,\ y+z)=(a,b,c)$

$\Rightarrow x+z=a,\ x+y+z=b,\ y+z=c$

$\Rightarrow \boxed{x=b-c}$ $\boxed{y=b-a}$ $\boxed{z=a-b+c}$

$\therefore\ T^{-1}(a,b,c) = (x,y,z)$

$= (b-c,\ b-a,\ a-b+c)$.

HW → Show that each of the following linear operators T on $\mathbb{R}^3$ is invertible and find T^{-1}.

(a) $T(x,y,z) = (2x,\ 4x-y,\ 2x+3y-z)$

(b) $T(a,b,c) = (a-3b-2c,\ b-4c,\ c)$

(c) $T(a,b,c) = (3a,\ a-b,\ 2a+b+c)$

(d) $T(x,y,z) = (x+y+z,\ y+z,\ z)$

(e) $T(a,b,c) = (a-2b-c,\ b-c,\ a)$.

HW → The set $\{e_1,e_2,e_3\}$ is the standard basis set of $V_3(\mathbb{R})$. The linear operator $T:\mathbb{R}^3\to\mathbb{R}^3$ is defined below. Show that T is invertible and find T^{-1}.

(i) $T(e_1) = e_1+e_2,\ T(e_2) = e_1-e_2+e_3,\ T(e_3) = 3e_1+4e_3$.

(ii) $T(e_1) = e_1-e_2,\ T(e_2) = e_2,\ T(e_3) = e_1+e_2-7e_3$.

(iii) $T(e_1) = e_1-e_2+e_3,\ T(e_2) = 3e_1-5e_3,\ T(e_3) = 3e_1-2e_3$.

2002 → Show that the mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ where $T(a,b,c) = (a-b,\ b-c,\ a+c)$ is linear and non-singular.

* Matrix of Linear Transformation *

→ Let $U(F)$ and $V(F)$ be two finite dimensional vectorspaces such that $\dim U = n$ and $\dim V = m$. Let $T : U \to V$ be a linear transformation.

Let $B_1 = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be the ordered basis of U and $B_2 = \{e_1, e_2, \ldots e_m\}$ be the ordered basis V.

— For every $\alpha \in U \Rightarrow T(\alpha) \in V$ and $T(\alpha)$ can be expressed as a linear combination of elements of of the basis B_2.

If there exists a's $\in F$ s.t

$$\left.\begin{aligned}
T(\alpha_1) &= a_{11}e_1 + a_{21}e_2 + \cdots\cdots + a_{m1}e_m \\
T(\alpha_2) &= a_{12}e_1 + a_{22}e_2 + \cdots\cdots + a_{m2}e_m \\
&\cdots\cdots\cdots \\
T(\alpha_j) &= a_{1j}e_1 + a_{2j}e_2 + \cdots\cdots + a_{mj}e_m \\
&\cdots\cdots\cdots \\
T(\alpha_n) &= a_{1n}e_1 + a_{2n}e_2 + \cdots\cdots + a_{mn}e_m
\end{aligned}\right\} \to (A)$$

Writing the co-ordinates $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)$ successively as columns of a matrix we get,

$$\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
- & - & - & - & - & - \\
- & - & - & - & - & - \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
- & - & \cdots & \cdots & \cdots & - \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}_{m \times n}.$$

This matrix represented as $[a_{ij}]_{m\times n}$ is called the matrix of the linear transformation T w.r.t to the bases B_1 and B_2.

symbolically $[T : B_1; B_2]$ or $[T] = [a_{ij}]_{m\times n}$.

Thus the matrix $[a_{ij}]_{m\times n}$ completely determines the linear transformation through the relations given in (A).

Hence the matrix $[a_{ij}]_{m\times n}$ represents the transformation T.

imp Note:- Let $T : V \to V$ be a linear operator so that $\dim V = n$.

If $B_1 = B_2 = B$ (say) then the above said matrix is called the matrix of T relative to the ordered basis B.

It is denoted by $[T ; B] = [T]_B = [a_{ij}]_{n\times n}$.

problems

→ Let $T : V_2 \to V_3$ be defined by

$T(x, y) = (x+y, 2x-y, 7y)$

find $[T : B_1, B_2]$ where B_1 and B_2 are the standard bases of V_2 and V_3.

Sol B1 is standard bases of V_2 and B_2 is standard basis of V_3.

$\therefore B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Now $T(1,0) = (1,2,0)$
$= 1(1,0,0) + 2(0,1,0) + 0(0,0,1)$

$T(0,1) = (1,-1,7)$
$= 1(1,0,0) - 1(0,1,0) + 7(0,0,1)$.

$\therefore$ The matrix of T relative to the bases B_1 and B_2 is

$$[T; B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}.$$

→ Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x, y, z) = (3x + 2y - 4z,\ x - 5y + 3z).$$

Find the matrix of T relative to the bases $B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,3), (2,5)\}$.

<u>Sol</u> Let $(a,b) \in \mathbb{R}^2$ and

let $(a,b) = p(1,3) + q(2,5)$

$= (p + 2q,\ 3p + 5q)$

$\Rightarrow p + 2q = a,\ 3p + 5q = b$

solving $p = -5a + 2b,\ q = 3a - b$

$\therefore (a,b) = (-5a + 2b)(1,3) + (3a - b)(2,5)$. —— (1)

Now $T(1,1,1) = (1, -1)$

$= -7(1,3) + 4(2,5)$ (from (1)).

$T(1,1,0) = (5, -4)$

$= -33(1,3) + 19(2,5)$ (from (1))

$T(1,0,0) = (3,1)$

$= -13(1,3) + 8(2,5)$ (from (1))

$\therefore$ The matrix of L.T relative to B_1 and B_2 is

$$[T : B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

→ If the matrix of a linear operator T on $V_3(\mathbb{R})$ w.r.t the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$ Describe explicitly $T : V_3(\mathbb{R}) \to V_3(\mathbb{R})$.

what is the matrix of T w.r.t the basis $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$.

Sol (i) Let the standard basis of $V_3(\mathbb{R})$ be

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$

$$\therefore \text{ Given } [T]_B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$\therefore T(\alpha_1) = 0\,\alpha_1 + 1\,\alpha_2 + (-1)\alpha_3$$
$$= 0(1,0,0) + 1(0,1,0) + (-1)(0,0,1)$$
$$= (0,1,-1)$$

$$T(\alpha_2) = 1\cdot\alpha_1 + 0\cdot\alpha_2 + (-1)\alpha_3 = (1,0,-1)$$

$$T(\alpha_3) = 1\cdot\alpha_1 + (-1)\alpha_2 + 0\cdot\alpha_3 = (1,-1,0).$$

Let $(a,b,c) \in V_3(\mathbb{R})$ then

$$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= a\alpha_1 + b\alpha_2 + c\alpha_3.$$

$$\therefore T(a,b,c) = aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3)$$
$$= a(0,1,-1) + b(1,0,-1) + c(1,-1,0)$$
$$= (b+c,\ a-c,\ -a-b)$$

which is the reqd L.T.

(ii) Let $B_2 = \{e_1, e_2, e_3\}$ where

$e_1 = (0,1,-1)$, $e_2 = (1,-1,1)$, $e_3 = (-1,1,0)$

Using the transformation

$T(a,b,c) = (b+c, a-c, -a-b)$,

we have

$T(e_1) = T(0,1,-1) = (0,1,-1)$

$T(e_2) = T(1,-1,1) = (0,0,0)$

$T(e_3) = T(-1,1,0) = (1,-1,0)$.

Now let $(a,b,c) = x e_1 + y e_2 + z e_3$

$= x(0,1,-1) + y(1,-1,1) + z(-1,1,0)$

$= (y-z, x-y+z, -x+y)$

$\Rightarrow y-z = a$
$x-y+z = b$
$-x+y = c$
$\Rightarrow$ $\boxed{x = a+b}$, $\boxed{y = a+b+c}$, $\boxed{z = b+c}$

$\therefore (a,b,c) = (a+b)e_1 + (a+b+c)e_2 + (b+c)e_3$ —— (1)

$\therefore T(e_1) = (0,1,-1)$
$= 1\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$ (from (1))

$T(e_2) = (0,0,0) = 0\cdot e_1 + 0\cdot e_2 + 0\cdot e_3$

$T(e_3) = (1,-1,0) = 0\cdot e_1 + 0\cdot e_2 - 1\cdot e_3$.

$\therefore [T; B_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

→ Let $D: P_3 \to P_2$ be the polynomial differential transformation $D(p) = \frac{dp}{dx}$. Find the matrix of D relative to the standard bases.

$B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$.

Sol $D(1) = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$D(x^3) = 3x^2 = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$.

∴ The matrix of D relative to B_1 and B_2 is $[T : B_1, B_2]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

2005

→ Let T be a linear transformation on $\mathbb{R}^3$, whose matrix relative to the standard basis of $\mathbb{R}^3$ is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}.$$

Find the matrix of T relative to the basis $B = \{(1,1,1), (1,1,0), (0,1,1)\}$.

2007

→ If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$T(x, y) = (2x - 3y, x + y)$.

Compute the matrix of T relative to the basis $B = \{(1,2), (2,3)\}$.

→ Let $R_3[x] = \{a_0 + a_1 x + a_2 x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$.
Define $T: \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ by $T(f(x)) = \frac{d}{dx} f(x)$ — (1)
for all $f(x) \in \mathbb{R}_3[x]$. Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

Sol Let $f(x), g(x) \in \mathbb{R}_3[x]$ and $a, b \in \mathbb{R}$

By (1), we have

$$T(af(x) + bg(x)) = \frac{d}{dx}(af(x) + bg(x))$$
$$= a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$$
$$= aT(f(x)) + bT(g(x))$$

$\therefore$ T is a linear transformation.

Now
$$T(1) = \frac{d}{dx}(1) = 0$$
$$T(x) = \frac{d}{dx}(x) = 1$$
$$T(x^2) = \frac{d}{dx}(x^2) = 2x$$

Again
$$T(1) = 0 = 0\cdot1 + 0x + 0x^2$$
$$T(x) = 1 = 1\cdot1 + 0\cdot x + 0\cdot x^2$$
$$T(x^2) = 2x = 0\cdot1 + 2\cdot x + 0\cdot x^2.$$

Hence the matrix representation of T w.r.t. the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now $T(1) = \frac{d}{dx}(1) = 0$

$T(1+x) = \frac{d}{dx}(1+x) = 1$.

$T(1+x+x^2) = \frac{d}{dx}(1+x+x^2) = 1+2x$.

Again $T(1) = 0 = 0\cdot 1 + 0(1+x) + 0(1+x+x^2)$.

$T(1+x) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0(1+x+x^2)$

$T(1+x+x^2) = 1+2x = (-1)\cdot 1 + 2(1+x) + 0(1+x+x^2)$

Hence the matrix representation of T w.r.t the basis $\{1, 1+x, 1+x+x^2\}$

is $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

HW → Let $R_4[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 / a_i \in \mathbb{R}\}$.

Define $T: R_4[x] \longrightarrow R_4[x]$ as

$T(f(x)) = \frac{d}{dx}(f(x))$ for all $f(x) \in R_4[x]$.

Let $\beta = \{1, x, x^2, x^3\}$ be an ordered basis $R_4[x]$. Find $[T]_\beta$.

→ Let V be the vector space polynomials of degree ≤ 3 over F. Let T be a linear transformation defined on V by $T(a_0 + a_1x + a_2x^2 + a_3x^3) =$

$a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$. ——(1)

Compute the matrix of T relative to the bases (a) $\{1, x, x^2, x^3\}$ (b) $\{1, 1+x, 1+x^2, 1+x^3\}$.

89(i)

sol

(a) $T(1) = 1 = 1 + 0x + 0x^2 + 0x^3$

$T(x) = x+1 = 1 + 1\cdot x + 0x^2 + 0\cdot x^3$ by (1)

$T(x^2) = (x+1)^2 = 1 + 2x + 1\cdot x^2 + 0\cdot x^3$

$T(x^3) = (x+1)^3 = 1 + 3x + 3x^2 + 1\cdot x^3.$

Hence the matrix representation of T w.r.t the basis $\{1, x, x^2, x^3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) $T(1) = 1$

$T(1+x) = 1 + (1+x)$

$T(1+x^2) = 1 + (x+1)^2 = 1 + (1 + x^2 + 2x)$

$T(1+x^3) = 1 + (x+1)^3$

$= 1 + (x^3 + 1 + 3x^2 + 3x)$

Again $T(1) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+x^2) + 0\cdot(1+x^3).$

$T(1+x) = 1 + (1+x) = 1\cdot 1 + 1\cdot(1+x) + 0(1+x^2) + 0(1+x^3).$

$T(1+x^2) = -1\cdot 1 + 2(1+x) + 1\cdot(1+x^2) + 0\cdot(1+x^3).$

$T(1+x^3) = 1 + (1+x^3) + (3x + 3x^2)$

$= -5(1) + 3(1+x) + 3(1+x^2) + 1\cdot(1+x^3)$

Hence the matrix representation of T w.r.t the basis $\{1, 1+x, 1+x^2, 1+x^3\}$ is

$$\begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Consider the vector space

$X := \{ p(x) \mid p(x)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$ over the real field $\mathbb{R}$. Define the map $D : X \to X$ by $D(p(x)) := p_1 + 2p_2 x + 3p_3 x^2$.

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$.

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the ordered basis $\{1, x, x^2, x^3\}$ for X.

Sol. Let $p(x), q(x) \in X$, $a, b \in \mathbb{R}$.

Given map $D : X \to X$ defined by

$$D(p(x)) = p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

i.e $D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2$ —— (1)

Now $D[ap(x) + bq(x)] = D[a(p_0 + p_1 x + p_2 x^2 + p_3 x^3) + b(q_0 + q_1 x + q_2 x^2 + q_3 x^3)]$

$= D[(ap_0 + bq_0) + (ap_1 + bq_1)x + (ap_2 + bq_2)x^2 + (ap_3 + bq_3)x^3]$

$= (ap_1 + bq_1) + 2(ap_2 + bq_2)x + 3(ap_3 + bq_3)x^2$

$$= a\left(p_1 + 2p_2 x + 3p_3 x^2\right) + b\left(q_1 + 2q_2 x + 3q_3 x^2\right)$$

$$= a\,D(p(x)) + b\,D(q(x))$$

$\therefore$ $D : X \to X$ is a linear transformation.

Now

From ①

$D(1) = 0 = 0 + 0x + 0x^2 + 0x^3$

$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0 + 2\cdot x + 0x^2 + 0x^3$

$D(x^3) = 3x^2 = 0 + 0x + 0x^2 + 3\cdot x^2$

Hence the matrix representation of D w.r.t the ordered basis $\{1, x, x^2, x^3\}$

is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$