SuccessClap: Book 5 Vector Calculus

Vector Calculus

9.0 INTRODUCTION

In Science and Engineering we often deal with the analysis of forces and velocities and other quantities which are vectors. These vectors are not constants but vary with position and time. Hence, they are functions of one or more variables.

Vector Calculus extends the concepts of differential calculus and integral calculus of real functions in an interval to vector functions and thus enabling us to analyse problems over curves and surfaces in three dimension. Vector Calculus finds applications in a wide variety of fields such as fluid flow, heat flow, solid mechanics, electrostatics etc.

In Vector Calculus we deal mainly with two kinds of functions, **scalar point functions** and **vector point functions and their fields**.

9.1 SCALAR AND VECTOR POINT FUNCTIONS

Definition 9.1 If to each point $P(\vec{r})$ (the point P with position vector $\vec{r}$) of a region R in space there is a unique scalar or real number denoted by $\boldsymbol{\phi}(\vec{r})$, then **$\boldsymbol{\phi}$ is called a scalar point function in $\boldsymbol{R}$**. The region R is called a **scalar field**.

Definition 9.2 If to each point $P(\vec{r})$ of a region R in space there is a unique vector denoted by $\vec{F}(\vec{r})$, then **$\vec{\mathbf{F}}$ is called a vector point function in $\boldsymbol{R}$**. The region R is called a **vector field**.

Note

1. In applications, the domain of definition of point functions may be points in a region of space, points on a surface or points on a curve.
2. If we introduce cartesian coordinate system, then $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ or

   $\vec{r} = (x, y, z)$ and instead of $\vec{F}(\vec{r})$ and $\boldsymbol{\phi}(\vec{r})$ we can write

   $\vec{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$ or

   $\vec{F}(x, y, z) = F_1(x, y, z)\vec{i} + F_2(x, y, z)\vec{j} + F_3(x, y, z)\vec{k}$

   and $\boldsymbol{\phi}(\vec{r})$ as $\boldsymbol{\phi}(x, y, z)$
3. A vector or scalar field that has a geometrical or physical meaning should depend only on the points P where it is defined but not on the particular choice of the cartesian coordinates. In otherwords, the scalar and vector fields have the property of invariance under a transformation of space coordinates.

Examples of scalar field

1. Temperature T within a body is scalar field, namely temperature field.
2. When an iron bar is heated at one end, the temperature at various points will attain a steady state and the temperature will depend only on the position.

3. The pressure of air in earth's atmosphere is a scalar field called pressure field.
4. $\phi(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ defines a scalar field.

Examples of vector field

1. The velocity of a moving fluid at any instant is a vector point function and defines a vector field.
2. Earth's magnetic field is a vector field.
3. Gravitational force on a particle in space defines a vector field.
4. $\vec{F}(x, y, z) = x^2\vec{i} - y^2\vec{j} + z\vec{k}$ defines a vector field.

Note Vector and scalar functions may also depend on time or on other parameters.

Definition 9.3 Derivative of a Vector Function

A vector function $\vec{f}(t)$ is said to be differentiable at a point t, if $\lim\limits_{\Delta t \to 0} \dfrac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$ exists.

Then it is denoted by $\dfrac{d\vec{f}}{dt}$ or $\vec{f}\,'$ and is called the derivative of the vector function $\vec{f}$ at t.

Note

1. If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ then $\vec{f}(t)$ is differentiable at t if and only if its components $f_1(t)$, $f_2(t), f_3(t)$ are differentiable at t and $\dfrac{d\vec{f}(t)}{dt} = f_1'(t)\vec{i} + f_2'(t)\vec{j} + f_3'(t)\vec{k}$
2. If the derivative of $\dfrac{d\vec{f}}{dt}$ w.r.to t exists, it is denoted by $\dfrac{d^2\vec{f}}{dt^2}$. Similarly, we denote higher derivatives.
3. If $\vec{c}$ is a constant vector, then $\dfrac{d\vec{c}}{dt} = \vec{0}$.

   For $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$ and $\dfrac{d\vec{c}}{dt} = 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$.

9.1.1 Geometrical Meaning of Derivative

Let $\vec{r}(t)$ be the position vector of a point P with respect to the origin O.

As t varies continuously over a time interval P traces the curve C. Thus, the vector function $\vec{r}(t)$ represents a curve C in space.

Let $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ be the position vectors of neighbouring points P and Q on the curve C.

Then
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{r} + \Delta\vec{r} - \vec{r} = \Delta\vec{r}$$

$\therefore$ $\dfrac{\Delta\vec{r}}{\Delta t}$ is along the chord PQ.

If $\lim\limits_{\Delta t \to 0} \dfrac{\Delta\vec{r}}{\Delta t}$ exists, it is denoted by $\dfrac{d\vec{r}}{dt}$ and $\dfrac{d\vec{r}}{dt}$ is in the directing of the tangent at P to the curve.

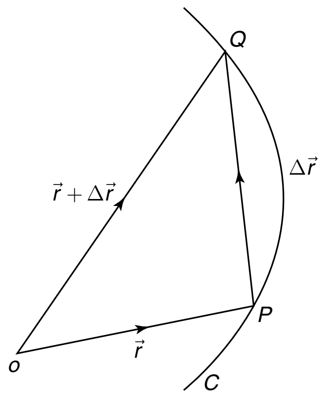

Fig. 9.1

If $\dfrac{d\vec{r}}{dt} \neq 0$, then $\dfrac{d\vec{r}}{dt}$ or $\vec{r}'(t)$ is called a tangent vector to the curve C at P.

The unit tangent vector at P is $= \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} = \hat{u}(t)$.

Both $\vec{r}'(t)$ and $\hat{u}(t)$ are in the direction of increasing t. Hence, their sense depends on the orientation of the curve C.

9.2 DIFFERENTIATION FORMULAE

If $\vec{f}$ and $\vec{g}$ are differentiable vector functions of t and ϕ is a scalar function of t then

1. $\dfrac{d}{dt}(\vec{f} \pm \vec{g}) = \dfrac{d\vec{f}}{dt} \pm \dfrac{d\vec{g}}{dt}$
2. $\dfrac{d}{dt}(\phi\vec{f}) = \phi\dfrac{d\vec{f}}{dt} + \dfrac{d\phi}{dt}\vec{f}$
3. $\dfrac{d}{dt}(\vec{f} \cdot \vec{g}) = \vec{f} \cdot \dfrac{d\vec{g}}{dt} + \dfrac{d\vec{f}}{dt} \cdot \vec{g}$
4. $\dfrac{d}{dt}(\vec{f} \times \vec{g}) = \vec{f} \times \dfrac{d\vec{g}}{dt} + \dfrac{d\vec{f}}{dt} \times \vec{g}$
5. $\dfrac{d}{dt}(\vec{f} \cdot \vec{g} \times \vec{h}) = \dfrac{d\vec{f}}{dt} \cdot \vec{g} \times \vec{h} + \vec{f} \cdot \dfrac{d\vec{g}}{dt} \times \vec{h} + \vec{f} \cdot \vec{g} \times \dfrac{d\vec{h}}{dt}$.

Note If $\vec{f}$ is a continuous function of a scalar s and s is a continuous function of t, then $\dfrac{d\vec{f}}{dt} = \dfrac{d\vec{f}}{ds}\dfrac{ds}{dt}$.

6. Let $\vec{f}(t)$ be a vector function. $\vec{f}(t)$ changes if its magnitude is changed or its direction is changed or both magnitude and direction are changed. We shall find conditions under which a vector function will remain constant in magnitude or in direction.

   (i) Let $\vec{f}(t)$ be a vector of constant length k.

   Then $$\vec{f} \cdot \vec{f} = |\vec{f}|^2 = k^2$$

   Differentiating w.r.to t, we get

   $$\frac{d\vec{f}}{dt} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad 2\vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \quad \Rightarrow \quad \vec{f} \cdot \frac{d\vec{f}}{dt} = 0$$

   $$\therefore \quad \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} = \text{is} \perp \text{to } \vec{f} \cdot$$

   (ii) Let $\vec{f}(t)$ be a vector function with constant direction and let $\vec{a}$ be the unit vector in that direction

   Then $\vec{f}(t) = \phi\vec{a}$, where $\phi = |\vec{f}|$

   $$\therefore \quad \frac{d\vec{f}}{dt} = \frac{d\phi}{dt}\vec{a} + \phi\frac{d\vec{a}}{dt}.$$

   But $\vec{a}$ is a constant vector, since its direction is fixed and magnitude is 1. $\therefore \dfrac{d\vec{a}}{dt} = \vec{0}$

   $$\therefore \quad \frac{d\vec{f}}{dt} = \frac{d\phi}{dt}\vec{a}$$

   Now $$\vec{f} \times \frac{d\vec{f}}{dt} = \phi\vec{a} \times \frac{d\phi}{dt}\vec{a} = \phi\frac{d\phi}{dt}\vec{a} \times \vec{a} = \vec{0} \qquad (\because \ \vec{a} \times \vec{a} = \vec{0})$$

   $$\therefore \quad \frac{d\vec{f}}{dt} = \vec{0} \quad \text{or} \quad \frac{d\vec{f}}{dt} \text{ is parallel to } \vec{f}.$$

9.3 LEVEL SURFACES

Let ϕ be a continuous scalar point function defined in a region R in space. Then the set of all points satisfying the equation $\phi(x, y, z) = C$, where C is a constant, determines a surface which is called a **level surface** of ϕ. At every point on a level surface the function ϕ takes the same value C. If C is an arbitrary constant, the for different values of C, we get different level surfaces of ϕ.
No two level surfaces intersect. For, if $\phi = C_1$ and $\phi = C_2$ be two level surfaces of ϕ intersecting at a point P. Then $\phi(P) = C_1$ and $\phi(P) = C_2$ and so ϕ has two values at P which contradicts the uniqueness of value of the function ϕ. So, $\phi = C_1$ and $\phi = C_2$ do not intersect.

Thus, **only one level surface of ϕ passes through a given point**

For example, if $\phi(x, y, z)$ represents the temperature of (x, y, z) in a region R of space, then the level surfaces of equal temperature are called **isothermal surfaces**.

9.4 GRADIENT OF A SCALAR POINT FUNCTION OR GRADIENT OF A SCALAR FIELD

9.4.1 Vector Differential Operator

The symbolic vector $\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}$ is called **Hamiltonian operator** or **vector differential operator** and is denoted by ∇ (read as del or nabla).

$$\therefore \qquad \nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

It is also known as del operator. This operator can be applied on a scalar point function $\phi(x, y, z)$ or a vector point function $\vec{F}(x, y, z)$ which are differentiable functions. This gives rise to three field quantities namely gradient of a scalar, divergence of a vector and curl of a vector function.

Definition 9.4 Gradient

If $\phi(x, y, z)$ is a scalar point function continuously differentiable in a given region R of space, then the gradient of ϕ is defined by $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$.

It is abbreviated as grad ϕ. Thus, grad $\phi = \nabla\phi$.

Note Since $\nabla\phi$ is a vector, the gradient of a scalar point function is always a vector point function. Thus, $\nabla\phi$ defines a vector field.

Gradient is of great practical importance because some of the vector fields in applications can be obtained from scalar fields and scalar fields are easy to handle.

9.4.2 Geometrical Meaning of $\nabla\phi$

Let $\phi(x, y, z)$ be a scalar point function. Let $\phi(x, y, z) = C$ be a level surface of ϕ. Let P be a point on this surface with position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Then the differential $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ is tangent to the surface at P.

$$\text{Now} \qquad \nabla\phi \cdot d\vec{r} = \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = d\phi = 0 \qquad [\because \phi = C]$$

$\therefore$ $\nabla \phi$ is normal to the surface $\phi(x, y, z) = C$ at P.

So, a unit normal to the surface at P is $\vec{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

There is another unit normal in the opposite direction $= -\dfrac{\nabla \phi}{|\nabla \phi|}$.

9.4.3 Directional Derivative

The directional derivative of a scalar point function ϕ in a given direction $\vec{a}$ is the rate of change of ϕ in that direction. It is given by the component of $\nabla \phi$ in the direction of $\vec{a}$

$\therefore$ the directional derivative $= \nabla \phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$.

Since $\nabla \phi \cdot \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{|\nabla \phi||\vec{a}|}{|\vec{a}|} \cos \theta$, where θ is the angle between $\nabla \phi$ and $\vec{a}$.

$$= |\nabla \phi| \cos \theta$$

So, the directional derivative at a given point is maximum if $\cos \theta$ is maximum.
i.e., $\cos \theta = 1 \Rightarrow \theta = 0$.

$\therefore$ the maximum directional derivative at a point is in the direction of $\nabla \phi$ and the maximum directional derivative is $|\nabla \phi|$.

Note

1. The directional derivative is minimum when $\cos \theta = -1 \Rightarrow \theta = \pi$
   $\therefore$ the minimum directional derivative is $-|\nabla \phi|$
2. In fact, the vector $\nabla \phi$ is in the direction in which ϕ increases rapidly.
   i.e., outward normal and $-\nabla \phi$ points in the direction in which ϕ decreases rapidly.

9.4.4 Equation of Tangent Plane and Normal to the Surface

(i) Equation of tangent plane

Let A be a given point on the surface $\phi(x, y, z) = C$.
Let $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ be the position vector of A.
Let P be any point on the tangent plane to the surface at the point A and let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of P.
Then $\nabla \phi$ at A is normal to the surface and $\vec{r} - \vec{r}_0$ lies on the tangent plane at A.
$\therefore$ the equation of the tangent plane at the point A is $(\vec{r} - \vec{r}_0) . \nabla \phi = 0$

Note The cartesian equation of the plane at the point $A(x_0, y_0, z_0)$ is

$$(x - x_0)\frac{\partial \phi}{\partial x} + (y - y_0)\frac{\partial \phi}{\partial y} + (z - z_0)\frac{\partial \phi}{\partial z} = 0$$

where the partial derivatives are evaluated at the point (x_0, y_0, z_0).

(ii) Equation of the normal at the point A

Let A be a given point on the surface $\phi(x, y, z) = C$ and let $\vec{r}_0 = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ be the position vector of A.

Let $\vec{r}$ be the position vector of any point P on the normal at the point A. Then $\vec{r} - \vec{r}_0$ is parallel to the normal at the point A.

$\therefore$ the equation of the normal at the point A is $(\vec{r} - \vec{r}_0) \times \nabla\phi = 0$.

The cartesian equation of the normal at the point A is

$$\frac{x - x_0}{\dfrac{\partial\phi}{\partial x}} = \frac{y - y_0}{\dfrac{\partial\phi}{\partial y}} = \frac{z - z_0}{\dfrac{\partial\phi}{\partial z}},$$

where the partial derivatives are evaluated at (x_0, y_0, z_0).

9.4.5 Angle between Two Surfaces at a Common Point

We know that the angle between two planes is the angle between their normals.

We define angle between two surfaces at a point of intersection P is the angle between their tangent planes at P and hence, the angle between their normals at P.

The angle between two surfaces $f(x, y, z) = C_1$ and $g(x, y, z) = C_2$ at a common point P is the angle between their normals at the point P.

The normal at P to the surface $f(x, y, z) = C_1$ is ∇f.

The normal at P to the surface $g(x, y, z) = C_2$ is ∇g.

If θ is the angle between the normals at the point P, then $\cos\theta = \dfrac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|}$

(i) If $\theta = \dfrac{\pi}{2}$, then the normals are perpendicular and $\cos\theta = 0 \Rightarrow \dfrac{\nabla f \cdot \nabla g}{|\nabla f||\nabla g|} = 0 \Rightarrow \nabla f \cdot \nabla g = 0$

$\therefore$ if two surfaces are orthogonal at the point P then $\nabla f \cdot \nabla g = 0$

Conversely, if $\nabla f \cdot \nabla g = 0$, then $\theta = \dfrac{\pi}{2}$ That is they are orthogonal.

(ii) If $\theta = 0$, the normals at the common point coincide.

$\therefore$ the two tangent planes coincide and the surfaces touch at the common point.

9.4.6 Properties of Gradients

If f and g are scalar point functions which are differentiable, then

1. $\nabla C = 0$, where C is constant.
2. $\nabla(Cf) = C\nabla f$, where C is a constant.
3. $\nabla(f \pm g) = \nabla f \pm \nabla g$
4. $\nabla(fg) = f\nabla g + g\nabla f$
5. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ if $g \neq 0$

1. $\nabla C = 0$, C is constant.

Proof We know $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$ (1)

$$= \sum \vec{i}\,\frac{\partial \phi}{\partial x}$$

$\therefore$ $$\nabla C = \sum \vec{i}\,\frac{\partial C}{\partial x} = 0 \qquad \left[\because \text{C is a constant } \frac{\partial C}{\partial x} = 0, \frac{\partial C}{\partial y} = 0, \frac{\partial C}{\partial z} = 0\right]$$ ■

2. $\nabla C\phi = C\nabla\phi$

Proof We have $$\nabla C\phi = \sum \vec{i}\,\frac{\partial}{\partial x}(C\phi) = C\sum \vec{i}\,\frac{\partial \phi}{\partial x} = C\nabla\phi \qquad \text{[using (1)]}$$ ■

3. $\nabla(f \pm g) = \nabla f \pm \nabla g$

Proof We have $$\nabla(f \pm g) = \sum \vec{i}\,\frac{\partial}{\partial x}(f \pm g) \qquad \text{[using (1)]}$$

$$= \sum \left[\vec{i}\,\frac{\partial f}{\partial x} \pm \vec{i}\,\frac{\partial g}{\partial x}\right] = \sum \vec{i}\,\frac{\partial f}{\partial x} \pm \sum \vec{i}\,\frac{\partial g}{\partial x} = \nabla f \pm \nabla g$$

$\therefore$ $$\nabla(f \pm g) = \nabla f \pm \nabla g$$ ■

4. $\nabla(fg) = f\nabla g + g\nabla f$

Proof We have $$\nabla(fg) = \sum \vec{i}\,\frac{\partial}{\partial x}(fg)$$

$$= \sum \vec{i}\left[f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right]$$

$$= \sum \vec{i}\left(f\frac{\partial g}{\partial x}\right) + \sum \vec{i}\left(g\frac{\partial f}{\partial x}\right)$$

$$= f\sum \vec{i}\,\frac{\partial g}{\partial x} + g\sum \vec{i}\,\frac{\partial f}{\partial x} = f\nabla g + g\nabla f$$

$\therefore$ $$\nabla(fg) = f\nabla g + g\nabla f$$ ■

5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Proof We have $$\nabla\left(\frac{f}{g}\right) = \sum \vec{i}\,\frac{\partial}{\partial x}\left(\frac{f}{g}\right)$$

$$= \sum \vec{i}\left[\frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}\right]$$

$$= \frac{1}{g^2}\left[g\sum \vec{i}\,\frac{\partial f}{\partial x} - f\sum \vec{i}\,\frac{\partial g}{\partial x}\right] = \frac{g\nabla f - f\nabla g}{g^2}$$

$\therefore$ $$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$$ ■

WORKED EXAMPLES

EXAMPLE 1

Find grad ϕ for the following functions.

(i) $\phi(x, y, z) = 3x^2y - y^3z^2$ at the point $(1, -2, 1)$

(ii) $\phi(x, y, z) = \log(x^2 + y^2 + z^2)$ at the point $(1, 2, 1)$.

Solution.

(i) Given $\quad \phi(x, y, z) = 3x^2y - y^3z^2$

We know $\quad \text{grad } \phi = \nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r. to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 6xy, \quad \frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2, \quad \frac{\partial\phi}{\partial z} = -2y^3z$$

At the point (1, −2, 1), $\quad \dfrac{\partial\phi}{\partial x} = 6\cdot 1(-2) = -12$

$$\frac{\partial\phi}{\partial y} = 3\cdot 1^2 - 3\cdot(-2)^2 1^2 = 3 - 12 = -9$$

$$\frac{\partial\phi}{\partial z} = -2\cdot(-2)^3\cdot 1 = 16$$

$\therefore$ **at the point (1, −2, 1),** $\quad \nabla\phi = -12\vec{i} - 9\vec{j} + 16\vec{k}.$

(ii) Given $\quad \phi(x, y, z) = \log(x^2 + y^2 + z^2)$

We know $\quad \text{grad } \phi = \nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get,

$$\frac{\partial\phi}{\partial x} = \frac{1}{x^2 + y^2 + z^2}\cdot 2x, \quad \frac{\partial\phi}{\partial y} = \frac{1}{x^2 + y^2 + z^2}\cdot 2y, \quad \frac{\partial\phi}{\partial z} = \frac{1}{x^2 + y^2 + z^2}\cdot 2z$$

At the point (1, 2, 1), $\quad \dfrac{\partial\phi}{\partial x} = \dfrac{2\cdot 1}{1^2 + 2^2 + 1^2} = \dfrac{2}{6} = \dfrac{1}{3}$

$$\frac{\partial\phi}{\partial y} = \frac{2\cdot 2}{1^2 + 2^2 + 1^2} = \frac{4}{6} = \frac{2}{3}$$

$$\frac{\partial\phi}{\partial z} = \frac{2\cdot 1}{1^2 + 2^2 + 1^2} = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ **at the point (1, 2, 1),** $\quad \text{grad } \phi = \dfrac{1}{3}\vec{i} + \dfrac{2}{3}\vec{j} + \dfrac{1}{3}\vec{k} = \dfrac{1}{3}[\vec{i} + 2\vec{j} + \vec{k}].$

EXAMPLE 2

Find the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point (1, −2, −1) in the direction of the vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution.

Given $$\phi(x, y, z) = x^2yz + 4xz^2$$

We know $$\text{grad } \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz$$

At the point (1, −2, −1), $$\frac{\partial\phi}{\partial x} = 2\cdot 1(-2)(-1) + 4(-1)^2 = 8$$

$$\frac{\partial\phi}{\partial y} = 1^2\cdot(-1) = -1$$

$$\frac{\partial\phi}{\partial z} = 1^2(-2) + 8\cdot 1(-1) = -2 - 8 = -10$$

$\therefore$ **at the point (1, −2, −1),** $\nabla\phi = 8\vec{i} - \vec{j} - 10\vec{k}$

Given direction is $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$\therefore$ the directional derivative of ϕ at the point (1, −2, −1) in the direction of $\vec{a}$ is

$$\nabla\phi\cdot\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k})\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{\sqrt{4+1+4}} = \frac{16+1+20}{\sqrt{9}} = \frac{37}{3}$$

EXAMPLE 3

If $\vec{r} + x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ prove that (i) $\nabla r = \frac{\vec{r}}{r}$, (ii) $\nabla r^n = nr^{n-2}\vec{r}$, (iii) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ (iv) $\nabla(\log r) = \frac{\vec{r}}{r^2}$.

Solution.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2$ (1)

(i) $\nabla r = \frac{\vec{r}}{r}$

We know $$\nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

Differentiating (1) partially w.r.to x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \quad \nabla r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} = \frac{1}{r}[x\vec{i} + y\vec{j} + z\vec{k}] = \frac{\vec{r}}{r}$$

(ii) $\nabla r^n = nr^{n-2}\vec{r}$

We know
$$\nabla r^n = \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$
$$= \vec{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \vec{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \vec{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right)$$
$$= n\,r^{n-1}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = \frac{n\,r^{n-1}}{r}[x\vec{i} + y\vec{j} + z\vec{k}] = nr^{n-2}\vec{r}$$

(iii) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

We know,
$$\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$= \vec{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$
$$= -\frac{1}{r^2}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{\vec{r}}{r^3}$$

(iv) $\nabla(\log r) = \frac{\vec{r}}{r^2}$

We know, $\nabla(\log r) = \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$
$$= \vec{i}\left(\frac{1}{r}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right) = \frac{1}{r}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = \frac{\vec{r}}{r^2}$$

EXAMPLE 4

Find the directional derivative of the function $2yz + z^2$ in the direction of the vector $\vec{i} + 2\vec{j} + 2\vec{k}$ at the point (1, –1, 3).

Solution.

Given $\phi = 2yz + z^2$

We know $\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 2z, \quad \frac{\partial\phi}{\partial z} = 2y + 2z$$

At the point (1, −1, 3), $\dfrac{\partial \phi}{\partial x} = 0, \quad \dfrac{\partial \phi}{\partial y} = 2(3) = 6, \quad \dfrac{\partial \phi}{\partial z} = 2(-1) + 2 \cdot 3 = 4$

∴ **at the point (1, −1, 3),** $\nabla \phi = 6\vec{j} + 4\vec{k}$

Given direction is $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

∴ the directional derivative of ϕ at the point (1, –1, 3) in the direction of $\vec{a}$ is

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = (6\vec{j} + 4\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{1+4+4}} = \frac{12+8}{\sqrt{9}} = \frac{20}{3}$$

EXAMPLE 5

Find the directional derivative of $x^3 + y^3 + z^3$ at the point (1, −1, 2) in the direction of $\vec{i} + 2\vec{j} + \vec{k}$.

Solution.

Given $\phi(x, y, z) = x^3 + y^3 + z^3$

We know $\nabla \phi = \vec{i}\dfrac{\partial \phi}{\partial x} + \vec{j}\dfrac{\partial \phi}{\partial y} + \vec{k}\dfrac{\partial \phi}{\partial z}$

Now differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial \phi}{\partial x} = 3x^2, \quad \frac{\partial \phi}{\partial y} = 3y^2, \quad \frac{\partial \phi}{\partial z} = 3z^2$$

At the point (1, –1, 2), $\dfrac{\partial \phi}{\partial x} = 3 \cdot 1^2 = 3, \quad \dfrac{\partial \phi}{\partial y} = 3(-1)^2 = 3, \quad \dfrac{\partial \phi}{\partial z} = 3 \cdot 2^2 = 12$

∴ **at the point (1, −1, 2),** $\nabla \phi = 3\vec{i} + 3\vec{j} + 12\vec{k}$

Given direction is $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$

∴ the directional derivative of ϕ at the point (1, –1, 2) in the direction of $\vec{a}$ is

$$\nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = (3\vec{i} + 3\vec{j} + 12\vec{k}) \cdot \frac{(\vec{i} + 2\vec{j} + \vec{k})}{\sqrt{1+4+1}} = \frac{3+6+12}{\sqrt{6}} = \frac{21}{\sqrt{6}} = 21\frac{\sqrt{6}}{6} = \frac{7\sqrt{6}}{2}$$

EXAMPLE 6

Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, –1).

Solution.

The given surface is $x^3 + y^3 + 3xyz = 3$, which is taken as $\phi = C$

∴ $\phi = x^3 + y^3 + 3xyz$

We know that $\nabla \phi$ is normal to the surface.

So, unit normal to the surface is $\vec{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

Now $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively,

we get, $$\frac{\partial\phi}{\partial x} = 3x^2 + 3yz, \quad \frac{\partial\phi}{\partial y} = 3y^2 + 3xz, \quad \frac{\partial\phi}{\partial z} = 3xy$$

At the point (1, 2, −1), $$\frac{\partial\phi}{\partial x} = 3\cdot 1^2 + 3\cdot 2(-1) = -3$$

$$\frac{\partial\phi}{\partial y} = 3\cdot 2^2 + 3\cdot 1(-1) = 9 \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 3\cdot 1\cdot 2 = 6$$

$\therefore$ **at the point (1, 2, −1),** $\nabla\phi = -3\vec{i} + 9\vec{j} + 6\vec{k}$

$\therefore$ unit normal to the given surface at the point (1, 2, −1) is

$$\vec{n} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{9+81+36}} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{126}}$$

Note If the surface equation is written as $x^3 + y^3 + 3xyz - 3 = 0$, then we take $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$. Here $C = 0$.

EXAMPLE 7

Find a unit normal to the surface $x^2y + 2xz^2 = 8$ at the point (1, 0, 2).

Solution.

Given $$\phi(x, y, z) = x^2y + 2xz^2$$

We know, $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2, \quad \frac{\partial\phi}{\partial y} = x^2, \quad \frac{\partial\phi}{\partial z} = 4xz$$

At the point (1, 0, 2), $$\frac{\partial\phi}{\partial x} = 2\cdot 1\cdot 0 + 2\cdot 2^2 = 8, \quad \frac{\partial\phi}{\partial y} = 1^2 = 1, \quad \frac{\partial\phi}{\partial z} = 4\cdot 1\cdot 2 = 8$$

$\therefore$ **at the point (1, 0, 2),** $\nabla\phi = 8\vec{i} + \vec{j} + 8\vec{k}$

$\therefore$ unit normal vector to the given surface at the point (1, 0, 2) is

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{64+1+64}} = \frac{8\vec{i} + \vec{j} + 8\vec{k}}{\sqrt{129}}$$

EXAMPLE 8

Find the maximum value of the directional derivative of $\phi = x^3yz$ at the point (1, 4, 1).

Solution.

Given $$\phi = x^3yz$$

We know, $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

The directional derivative is maximum in the direction of $\nabla\phi$ and the maximum value $= |\nabla\phi|$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 3x^2yz, \quad \frac{\partial\phi}{\partial y} = x^3z, \quad \frac{\partial\phi}{\partial z} = x^3y$$

At the point (1, 4, 1), $\quad \dfrac{\partial\phi}{\partial x} = 3\cdot1\cdot4\cdot1 = 12, \ \dfrac{\partial\phi}{\partial y} = 1^3\cdot1 = 1 \ \text{ and } \ \dfrac{\partial\phi}{\partial z} = 1^3\cdot4 = 4$

$\therefore$ **at the point (1, 4, 1),** $\quad \nabla\phi = 12\vec{i} + \vec{j} + 4\vec{k}$

Maximum value of the directional derivative $= |\nabla\phi| = |12\vec{i} + \vec{j} + 4\vec{k}| = \sqrt{144+1+16} = \sqrt{161}$

EXAMPLE 9

In what direction from the point (1, 1, −2), is the directional derivative of $\phi = x^2 - 2y^2 + 4z^2$ maximum? Also find the maximum directional derivative.

Solution.

Given $\quad \phi = x^2 - 2y^2 + 4z^2$

We know that the directional derivative is maximum in the direction of $\nabla\phi$. The maximum value $= |\nabla\phi|$

We have $\quad \nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z}$

Differentiating ϕ partially w.r.to x, y, z respectively, we get

$$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -4y, \quad \frac{\partial\phi}{\partial z} = 8z$$

At the point (1, 1, −2), $\quad \dfrac{\partial\phi}{\partial x} = 2\cdot1 = 2, \quad \dfrac{\partial\phi}{\partial y} = -4\cdot1 = -4, \quad \dfrac{\partial\phi}{\partial z} = 8(-2) = -16$

$\therefore$ **at the point (1, 1, −2),** $\quad \nabla\phi = 2\vec{i} - 4\vec{j} - 16\vec{k} = 2[\vec{i} - 2\vec{j} - 8\vec{k}]$

$\therefore$ the directional derivative is maximum in the direction of $2(\vec{i} - 2\vec{j} - 8\vec{k})$

$$\text{Maximum value} = |\nabla\phi| = |2(\vec{i} - 2\vec{j} - 8\vec{k})| = 2\sqrt{1+4+64} = 2\sqrt{69}$$

EXAMPLE 10

Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point (2, −1, 2).

Solution.

The given surfaces are

$$x^2 + y^2 + z^2 = 9 \quad (1) \qquad \text{and} \qquad x^2 + y^2 - z = 3 \quad (2)$$

$P(2, -1, 2)$ is a common point of (1) and (2)

Let $\quad f = x^2 + y^2 + z^2$ and $g = x^2 + y^2 - z$

Now, $$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

Differentiating f partially w.r.to x, y, z respectively we get,

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y, \qquad \frac{\partial f}{\partial z} = 2z$$

At the point (2, −1, 2), $$\frac{\partial f}{\partial x} = 2\cdot 2 = 4, \quad \frac{\partial f}{\partial y} = 2(-1) = -2, \quad \frac{\partial f}{\partial z} = 2(+2) = +4$$

$\therefore$ **at the point (2, −1, 2),** $$\nabla f = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

Now $$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

Differentiating g partially w.r.to x, y, z respectively, we get

$$\frac{\partial g}{\partial x} = 2x, \qquad \frac{\partial g}{\partial y} = 2y, \qquad \frac{\partial g}{\partial z} = -1$$

at the point (2, −1, 2), $$\frac{\partial g}{\partial x} = 2\cdot 2 = 4, \quad \frac{\partial g}{\partial y} = 2(-1) = -2, \quad \frac{\partial g}{\partial z} = -1$$

$\therefore$ **at the point (2, −1, 2),** $$\nabla g = 4\vec{i} - 2\vec{j} - \vec{k}$$

If $\boldsymbol{\theta}$ is the angle between the surfaces (1) and (2) at (2, –1, 2), then

$$\cos\boldsymbol{\theta} = \frac{\nabla f\cdot\nabla g}{|\nabla f||\nabla g|} = \frac{(4\vec{i} - 2\vec{j} + 4\vec{k})}{\sqrt{16+4+16}}\cdot\frac{(4\vec{i} - 2\vec{j} - \vec{k})}{\sqrt{16+4+1}} = \frac{16+4-4}{\sqrt{36}\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$\therefore$ $$\boldsymbol{\theta} = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

EXAMPLE 11

Show that the surfaces $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$ are orthogonal at the point (1, −1, 2).

Solution.

The given surfaces are

$$5x^2 - 2yz - 9x = 0 \quad (1) \quad \text{and} \quad 4x^2y + z^3 - 4 = 0 \quad (2)$$

Let $f = 5x^2 - 2yz - 9x$ and $g = 4x^2y + z^3 - 4$

To prove (1) and (2) cut orthogonally at the point (1, –1, 2),

i.e., to prove $\nabla f\cdot\nabla g = 0$

Now $$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 10x - 9, \quad \frac{\partial f}{\partial y} = -2z \quad \text{and} \quad \frac{\partial f}{\partial z} = -2y$$

$\therefore$ $$\nabla f = (10x - 9)\vec{i} - 2z\vec{j} - 2y\vec{k}$$

and $$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$ $$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

At the point (1, −1, 2), $$\nabla f = (10-9)\vec{i} - 2\cdot 2\vec{j} - 2(-1)\vec{k} = \vec{i} - 4\vec{j} + 2\vec{k}$$

and $$\nabla g = 8\cdot 1\cdot(-1)\vec{i} + 4\cdot 1^2\vec{j} + 3\cdot 2^2\vec{k} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$ $$\nabla f\cdot\nabla g = (\vec{i} - 4\vec{j} + 2\vec{k})\cdot(-8\vec{i} + 4\vec{j} + 12\vec{k}) = -8 - 16 + 24 = 0$$

Hence, the two surfaces cut orthogonally at the point (1, −1, 2).

EXAMPLE 12

Find a and b if the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at the point (1, −1, 2).

Solution.

The given surfaces are

$$ax^2 - byz - (a+2)x = 0 \quad (1) \qquad \text{and} \quad 4x^2y + z^3 - 4 = 0 \quad (2)$$

Let $f = ax^2 - byz - (a+2)x$ and $g = 4x^2y + z^3 - 4$

Given the surfaces (1) and (2) cut orthogonally at the point (1, −1, 2).

$\therefore$ $$\nabla f\cdot\nabla g = 0 \quad (3)$$

Now $$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial x} = 2ax - a - 2, \quad \frac{\partial f}{\partial y} = -bz \quad \text{and} \quad \frac{\partial f}{\partial z} = -by$$

$\therefore$ $$\nabla f = (2ax - a - 2)\vec{i} - bz\vec{j} - by\vec{k}$$

and $$\nabla g = \vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}$$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2 \quad \text{and} \quad \frac{\partial g}{\partial z} = 3z^2$$

$\therefore$ $$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

At the point (1, −1, 2), $$\nabla f = (2a - a - 2)\vec{i} - b\cdot 2\vec{j} - b(-1)\vec{k}$$

$\Rightarrow$ $$\nabla f = (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and $$\nabla g = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$\therefore$ $$\nabla f\cdot\nabla g = ((a-2)\vec{i} - 2b\vec{j} + b\vec{k})\cdot(-8\vec{i} + 4\vec{j} + 12\vec{k})$$
$$= -8(a-2) - 8b + 12b = -8a + 4b + 16$$

From (3), $$\nabla f\cdot\nabla g = 0 \Rightarrow -8a + 4b + 16 = 0 \Rightarrow 2a - b = 4 \quad (4)$$

Since (1, −1, 2) is a point on the surface $f = 0$, we get

$$a + 2b - (a+2) = 0 \Rightarrow 2b = 2 \Rightarrow b = 1$$

$\therefore$ (4) $\Rightarrow$ $\quad 2a = 4 + b = 4 + 1 = 5 \quad \Rightarrow \quad a = \frac{5}{2}$

$\therefore$ $$a = \frac{5}{2},\ b = 1$$

EXAMPLE 13

Find the angle between the normals to the surface $xy = z^2$ at the points (1, 4, 2) and (–3, –3, 3).

Solution.

The given surface is $\quad xy - z^2 = 0$

$\therefore$ $\quad \phi = xy - z^2$

We know $\nabla\phi$ is normal to the surface at the point (x, y, z)

Let $\vec{n}_1, \vec{n}_2$, be the normals to the surface at the points (1, 4, 2) and (–3, –3, 3) respectively.

$\therefore$ $\quad \vec{n}_1 = \nabla\phi$ at the point (1, 4, 2)

and $\quad \vec{n}_2 = \nabla\phi$ at the point (–3, – 3, 3)

Now $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y, \quad \frac{\partial\phi}{\partial y} = x \quad \text{and} \quad \frac{\partial\phi}{\partial z} = -2z$$

$\therefore$ $$\nabla\phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

At the point (1, 4, 2), $\quad \nabla\phi = 4\vec{i} + \vec{j} - 4\vec{k} \quad \therefore \quad \vec{n}_1 = 4\vec{i} + \vec{j} - 4\vec{k}$

At the point (−3, −3, 3), $\quad \nabla\phi = -3\vec{i} - 3\vec{j} - 6\vec{k} \quad \therefore \quad \vec{n}_2 = -3\vec{i} - 3\vec{j} - 6\vec{k}$

If $\boldsymbol{\theta}$ is the angle between the normals, then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(4\vec{i} + \vec{j} - 4\vec{k})\cdot(-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{16+1+16}\ \sqrt{9+9+36}}$$

$$= \frac{-12 - 3 + 24}{\sqrt{33}\ \sqrt{54}} = \frac{9}{\sqrt{33}\ \sqrt{54}} = \frac{1}{\sqrt{22}}$$

$\therefore$ $$\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

EXAMPLE 14

Find the directional derivative of the function $\phi = xy^2 + yz^3$ at the point (2, −1, 1) in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at the point (−1, 2, 1).

Solution.

Given $\quad \phi = xy^2 + yz^3$

$\therefore$ $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

At the point (2, −1, 1), $\quad \nabla\phi = (-1)^2\vec{i} + (-4+1)\vec{j} + 3(-1)1^2\vec{k} = \vec{i} - 3\vec{j} - 3\vec{k}$

The directional derivative of ϕ in the direction of the normal to the surface $x\log z - y^2 + 4 = 0$ at the point (–1, 2, 1) is required.

Let $$f = x\log z - y^2 + 4$$

$\therefore$ $$\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z} = \log z\vec{i} - 2y\vec{j} + \frac{x}{z}\vec{k}$$

At the point (−1, 2, 1), $$\nabla f = \log 1\vec{i} - 4\vec{j} + \left(\frac{-1}{1}\right)\vec{k} = 0\vec{i} - 4\vec{j} - \vec{k} = -4\vec{j} - \vec{k}$$

$\therefore$ $$\vec{a} = -4\vec{j} - \vec{k}$$

Required directional derivative is $= \nabla\phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \frac{(-4\vec{j} - \vec{k})}{\sqrt{16+1}} = \frac{12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

EXAMPLE 15

If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$, then find ϕ if $\phi(1, -2, 2) = 4$.

Solution.

Given $$\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k} \quad (1)$$

But $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \quad (2)$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$, from (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \quad (3) \qquad \frac{\partial\phi}{\partial y} = x^2z^3 \quad (4) \qquad \frac{\partial\phi}{\partial z} = 3x^2yz^2 \quad (5)$$

Integrating (3) partially w.r.to x, we get

$$\phi = x^2yz^3 + f_1(y, z) \quad (6)$$

Integrating (4) partially w.r.to y, we get,

$$\phi = x^2z^3y + f_2(x, z) \quad (7)$$

Integrating (5) partially w.r.to z, we get,

$$\phi = x^2yz^3 + f_3(x, y) \quad (8)$$

From (6), (7), (8), ϕ is obtained by adding all the terms and an arbitrary constant C, but omitting $f_1(y, z), f_2(x, z), f_3(x, y)$ and choosing only one of the repeated terms.

Thus, $$\phi = x^2yz^3 + C$$

Given $$\phi(1, -2, 2) = 4$$

$\therefore$ $$1 \times (-2) \times 8 + C = 4 \quad \Rightarrow \quad C = 4 + 16 = 20$$

$\therefore$ $$\phi = x^2yz^3 + 20$$

EXAMPLE 16

Find the equation of the tangent plane and the equation of the normal to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at the point (3, 2, 1).

Solution.

The given surface is $\quad x^2 - 4y^2 + 3z^2 + 4 = 0$

Let $$\phi = x^2 - 4y^2 + 3z^2 + 4$$

$\therefore$ $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} - 8y\vec{j} + 6z\vec{k}$$

At the point (3, 2, 1), $$\nabla\phi = 6\vec{i} - 16\vec{j} + 6\vec{k}$$

We know that the equation of the tangent plane at the point (x_0, y_0, z_0) is

$$(x - x_0)\frac{\partial\phi}{\partial x} + (y - y_0)\frac{\partial\phi}{\partial y} + (z - z_0)\frac{\partial\phi}{\partial z} = 0$$

Now $$\frac{\partial\phi}{\partial x} = 2x, \quad \frac{\partial\phi}{\partial y} = -8y \quad \text{and} \quad \frac{\partial\phi}{\partial z} = 6z$$

Here $(x_0, y_0, z_0) = (3, 2, 1)$ $\therefore$ $\dfrac{\partial\phi}{\partial x} = 6, \quad \dfrac{\partial\phi}{\partial y} = -16 \quad \text{and} \quad \dfrac{\partial\phi}{\partial z} = 6$

$\therefore$ the equation of the tangent plane at the point (3, 2, 1) is

$$(x-3)6 + (y-2)(-16) + (z-1)6 = 0$$

$\Rightarrow$ $$3(x-3) - 8(y-2) + 3(z-1) = 0$$ [dividing by 2]

$\Rightarrow$ $$3x - 8y + 3z - 9 + 16 - 3 = 0$$

$\Rightarrow$ $$3x - 8y + 3z + 4 = 0$$

The equation of the normal at the point (x_0, y_0, z_0) is

$$\frac{x - x_0}{\frac{\partial\phi}{\partial x}} = \frac{y - y_0}{\frac{\partial\phi}{\partial y}} = \frac{z - z_0}{\frac{\partial\phi}{\partial z}}$$

The equation of the normal at the point (3, 2, 1) is

$$\frac{x-3}{6} = \frac{y-2}{-16} = \frac{z-1}{6} \quad \Rightarrow \quad \frac{x-3}{3} = \frac{y-2}{-8} = \frac{z-1}{3}.$$

EXAMPLE 17

If the directional derivative of $\phi(x, y, z) = a(x + y) + b(y + z) + c(z + x)$ has maximum value 12 at the point (1, 2, 1) in the direction parallel to the line $\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-1}{3}$, find the value of a, b, c.

Solution.

Given $$\phi = a(x + y) + b(y + z) + c(z + x)$$

$\therefore$ $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$\Rightarrow$ $$\nabla\phi = (a + c)\vec{i} + (a + b)\vec{j} + (b + c)\vec{k}$$

We know that the directional derivative is maximum in the direction of $\nabla\phi$.

But given it is maximum in the direction parallel to the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-1}{3}$.

$$\therefore \qquad \frac{a+c}{1}=\frac{a+b}{2}=\frac{b+c}{3}=K$$

$$\Rightarrow \qquad a+c=K \quad (1) \qquad a+b=2K \quad (2) \qquad b+c=3K \quad (3)$$

Adding we get,

$$a+c+a+b+b+c=K+2K+3K$$

$$\Rightarrow \qquad 2(a+b+c)=6K \quad \Rightarrow \quad a+b+c=3K \qquad (4)$$

Using (3), (4) $\Rightarrow$ $a+3K=3K \quad \Rightarrow \quad a=0$

From (1), $0+c=K \quad \Rightarrow \quad c=K$

From (2), $0+b=2K \quad \Rightarrow \quad b=2K$

Given the maximum value of directional derivative = 12

$$\Rightarrow \qquad |\nabla\phi|=12$$

$$\Rightarrow \qquad \sqrt{(a+c)^2+(a+b)^2+(b+c)^2}=12$$

$$\Rightarrow \qquad (a+c)^2+(a+b)^2+(b+c)^2=144$$

$$\Rightarrow \qquad K^2+4K^2+9K^2=144$$

$$\Rightarrow \qquad 14K^2=144 \quad \Rightarrow \quad K^2=\frac{144}{14} \quad \Rightarrow \quad K=\pm\frac{12}{\sqrt{14}}$$

$$\therefore \qquad a=0,\ b=\pm\frac{24}{\sqrt{14}},\ c=\pm\frac{12}{\sqrt{14}}$$

EXAMPLE 18

If $\vec{u}=x+y+z$, $\vec{v}=x^2+y^2+z^2$, $\vec{w}=xy+yz+zx$, then show that the vectors ∇u, ∇v, ∇w are coplanar.

Solution.

Given $\qquad u=x+y+z, \quad v=x^2+y^2+z^2, \quad w=xy+yz+zx$

Now,

$$\nabla u=\vec{i}\,\frac{\partial u}{\partial x}+\vec{j}\,\frac{\partial u}{\partial y}+\vec{k}\,\frac{\partial u}{\partial z}=\vec{i}+\vec{j}+\vec{k}$$

$$\nabla v=\vec{i}\,\frac{\partial v}{\partial x}+\vec{j}\,\frac{\partial v}{\partial y}+\vec{k}\,\frac{\partial v}{\partial z}=2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla w=\vec{i}\,\frac{\partial w}{\partial x}+\vec{j}\,\frac{\partial w}{\partial y}+\vec{k}\,\frac{\partial w}{\partial z}=(y+z)\vec{i}+(z+x)\vec{j}+(x+y)\vec{k}$$

We know that three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if their scalar triple product $\vec{a} \cdot \vec{b} \times \vec{c} = 0$.

$\therefore$ $\nabla u, \nabla v, \nabla w$ are coplanar, if $\nabla u \cdot \nabla v \times \nabla w = 0$

Now $$\nabla u \cdot \nabla v \times \nabla w = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix} \quad R_2 \to R_2 + R_3$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0 \qquad [\text{since } R_1 = R_2]$$

$\therefore$ the vectors $\nabla u, \nabla v, \nabla w$ are coplanar.

EXERCISE 9.1

1. If $\phi(x, y, z) = 3xz^2y - y^3z^2$, find $\nabla\phi$ at the point $(1, -2, -1)$.
2. If $\phi = 2xz - y^2$ find grad ϕ at the point $(1, 3, 2)$.
3. Find the directional derivative of $\phi = 3x^2 + 2y - 3z$ at the point $(1, 1, 1)$ in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$.
4. Find the directional derivative of $xyz - xy^2z^2$ at the point $(1, 2, -1)$ in the direction of the vector $\vec{i} - \vec{j} - 3\vec{k}$.
5. Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point P $(1, 2, 3)$ in the direction of the line PQ where $Q = (5, 0, 4)$.
6. Find the unit normal vector to the surface
   (i) $x^2 + 2y^2 + z^2 = 7$ at the point $(1, -1, 2)$.
   (ii) $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.
   (iii) $x^2 + y^2 - z = 1$ at the point $(1, 1, 1)$.
   (iv) $x^2 + y^2 = z$ at the point $(1, 2, 5)$.
7. Find the angle between the surfaces $x^2 + y + z = 2$ and $x\log z = y^2 - 1$ at the point $(1, 1, 1)$.
8. Find the angle between the surfaces $2yz + z^2 = 3$ and $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$.
9. Find the angle between the surfaces $xyz = 4$ and $x^2 + y^2 + z^2 = 9$ at the point $\vec{i} + 2\vec{j} + 2\vec{k}$.
10. Find the equation of the tangent plane and normal line to the surface $xz^2 + x^2y - z + 1 = 0$ at the point $(1, -3, 2)$.
11. Find the equation of the tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$.
12. Find the equation of the tangent plane and normal line to the surface $2z - x^2 = 0$ at the point $P(2, 0, 2)$.
13. Find ϕ if
    (i) $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (8z^3 - 3x^2yz^2)\vec{k}$

(ii) $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ if $\phi(1, -2, 2) = 4$

(iii) $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

(iv) $\nabla\phi = (2xyz + x)\vec{i} + x^2z\vec{j} + x^2y\vec{k}$

(v) $\nabla\phi = (y + \sin z)\vec{i} + x\vec{j} + x\cos z\vec{k}$.

14. Find the angle between the normals to the intersecting surfaces $xy - z^2 - 1 = 0$ and $y^2 - 3z - 1 = 0$ at the point (1, 1, 0).
15. Find the angle between the normals to the surface $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).
16. Find the values of a and b so that the surfaces $ax^3 - by^2z = (a + 3)x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at the point (2, –1, –3).
17. The temperature at any point in space is given by $T = xy + yz + zx$. Find the direction in which the temperature changes most rapidly from the point (1, 1, 1) and determine the maximum rate of change.
18. In what direction is the directional derivative of the function $\phi = x^2 - 2y^2 + 4z^2$ from the point (1, 1, –1) is maximum and what is its value?
19. Find the maximum value of the directional derivative of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point (1, 1, –4).
20. Find $\nabla\phi$ at the point (1, 1, 1) if $\phi(x, y, z) = x^2y + y^2x + z^2$.
21. Find the directional derivative of $\phi(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point (1, 1, –1) in the direction $2\vec{i} - \vec{j} - \vec{k}$.
22. Find the directional derivative of the function $\phi = xy + yz + zx$ in the direction of the vector $2\vec{i} + 3\vec{j} + 6\vec{k}$ at the point (3, 1, 2).
23. Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at (1, 2, 3) in the direction of $2\vec{i} + \vec{j} - \vec{k}$.
24. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $P(1, -2, -1)$ in the direction of PQ, where Q is (3, –3, –3).
25. Find a unit normal to the surface $xy^3z^2 = 4$ at the point (–1, –1, 2).
26. In what direction from (3, 1, –2) is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.
27. What is the greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3)?
28. Find the angle between the spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at the point (4, –3, 2).
29. Find ϕ if $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$.

ANSWERS TO EXERCISE 9.1

1. $-6\vec{i} - 9\vec{j} - 4\vec{k}$ 2. $4\vec{i} - 6\vec{j} + 2\vec{k}$ 3. $\dfrac{19}{3}$ 4. $\dfrac{29}{\sqrt{11}}$ 5. $\dfrac{28}{\sqrt{21}}$

6. (i) $\dfrac{\vec{i} - 2\vec{j} + 2\vec{k}}{3}$ (ii) $\dfrac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ (iii) $\dfrac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$ (iv) $\dfrac{2\vec{i} + 4\vec{j} - 5\vec{k}}{3\sqrt{5}}$

7. $\cos^{-1}\left(\dfrac{1}{\sqrt{30}}\right)$ 8. $\cos^{-1}\sqrt{\dfrac{3}{5}}$ 9. $\cos^{-1}\sqrt{\dfrac{2}{3}}$

10. $2x - y - 3z + 1 = 0, \dfrac{x-1}{-2} = \dfrac{y+3}{1} = \dfrac{z-2}{3}$

11. $7x - 3y + 8z - 26 = 0, \dfrac{x-1}{7} = \dfrac{y+1}{-3} = \dfrac{z-2}{3}$

12. $2x - z = 2; \dfrac{x-2}{-2} = \dfrac{y}{0} = \dfrac{z-2}{1}$

13. (i) $\boldsymbol{\phi} = xy^2 - x^2yz^3 + 3y + 2z^4 + c$ (ii) $\boldsymbol{\phi} = x^2yz^3 + 20$

(iii) $\boldsymbol{\phi} = 3x^2y + xz^3 - yz + c$ (iv) $\boldsymbol{\phi} = x^2yz + \dfrac{x^2}{2} + c$ (v) $\boldsymbol{\phi} = xy + x\sin z + c$

14. $\cos^{-1}\left(\dfrac{2}{\sqrt{26}}\right)$

15. $\cos^{-1}\dfrac{13}{3\sqrt{22}}$

16. $a = -\dfrac{7}{3}, b = \dfrac{64}{9}$

17. $\vec{i} + \vec{j} + \vec{k}, 2\sqrt{3}$

18. $2\vec{i} - 4\vec{j} - 8\vec{k}, 2\sqrt{21}$

19. 1652

20. $\nabla\boldsymbol{\phi} = 3\vec{i} + 3\vec{j} + 2\vec{k}$

21. $\dfrac{16}{\sqrt{6}}$

22. $\dfrac{45}{7}$

23. $\dfrac{86}{\sqrt{6}}$

24. $\dfrac{37}{3}$

25. $-\dfrac{(\vec{i} + 3\vec{j} - \vec{k})}{\sqrt{11}}$

26. $96\sqrt{19}$

27. 9

28. $\boldsymbol{\theta} = \cos^{-1}\left(\sqrt{\dfrac{19}{29}}\right)$

29. $\boldsymbol{\phi} = 3x^2y + xz^3 - yz + c$

9.5 DIVERGENCE OF A VECTOR POINT FUNCTION OR DIVERGENCE OF A VECTOR FIELD

Definition 9.5 If $\vec{F}(x, y, z)$ be a vector point function continuously differentiable in a region R of space, then **the divergence of** $\vec{F}$ is defined by

$$\nabla \cdot \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y} + \vec{k} \cdot \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as div $\vec{F}$ and thus, div $\vec{F} = \nabla \cdot \vec{F}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then $\qquad \nabla \cdot \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

If $\vec{F}$ is a constant vector, then $\nabla \cdot \vec{F} = 0$ and conversely if $\nabla \cdot \vec{F} = 0$, then $\vec{F}$ is a constant vector.

Note (i) From the definition it is clear that div $\vec{F}$ is a scalar point function. So, the divergence of a vector field is a scalar point function. The notation $\nabla \cdot \vec{F}$ is not a scalar product in the usual sense, since $\nabla \cdot \vec{F} \neq \vec{F} \cdot \nabla$. In fact $\vec{F} \cdot \nabla = F_1 \dfrac{\partial}{\partial x} + F_2 \dfrac{\partial}{\partial y} + F_3 \dfrac{\partial}{\partial z}$ is a scalar operator.

9.5.1 Physical Interpretation of Divergence

Physical interpretation of divergence applied to a vector field is that it gives approximately the 'loss' of the physical quantity at a given point per unit volume per unit time.

(i) If $\vec{v}(x, y, z)$ is the moving fluid at a point (x, y, z), then the 'loss' of the fluid per unit volume per unit time at the point is given by div $\vec{v}$. Thus, divergence gives a measure of the outward flux per unit volume of the flow at (x, y, z).
If there is no 'loss' of fluid anywhere, then div $\vec{v} = 0$ and the fluid is said to be incompressible.

(ii) If $\vec{v}$ represents an electric flux, div $\vec{v}$ is the amount of electric flux which diverges per unit volume in unit time.

(iii) If $\vec{v}$ represents the heat flux, div $\vec{v}$ is the rate at which heat is issuing from a point per unit volume.

Definition 9.6 Solenoidal Vector

If div $\vec{F} = 0$ everywhere in a region R, **then** $\vec{F}$ **is called a solenoidal vector point function** and ***R*** **is called a solenoidal field.**

9.6 CURL OF A VECTOR POINT FUNCTION OR CURL OF A VECTOR FIELD

Definition 9.7 If $\vec{F}(x, y, z)$ be a vector point function continuously differentiable in a region R, then **the curl of** $\vec{F}$ is defined by

$$\nabla \times \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

It is abbreviated as curl $\vec{F}$

Thus,
$$\textbf{curl } \vec{F} = \nabla \times \vec{F}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\begin{aligned}\text{curl } \vec{F} &= \nabla \times \vec{F}\\ &= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\\ &= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] + \vec{j}\left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]\end{aligned}$$

This is symbolically written as
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $\vec{F}$ is a constant vector, then curl $\vec{F} = \vec{0}$

9.6.1 Physical Meaning of Curl $\vec{F}$

If $\vec{F}$ represents the linear velocity of the point P of a rigid body that rotates about a fixed axis (e.g., top) with constant angular velocity $\vec{\boldsymbol{\omega}}$, then curl $\vec{F}$ at P is equal to $2\vec{\boldsymbol{\omega}}$.

If the body is not rotating, then $\vec{\boldsymbol{\omega}} = \vec{0}$ $\quad\therefore\quad$ Curl $\vec{F} = \vec{0}$

Definition 9.8 Irrotational Vector Field

Let $\vec{F}(x, y, z)$ be a vector point function. If curl $\vec{F} = \vec{0}$ at all points in a region R, then $\vec{F}$ is said to be an **irrotational vector in *R***. The vector field R is called an **irrotational vector field**.

Definition 9.9 Conservative Vector Field

A vector field $\vec{F}$ is said to be **conservative** if there exists a scalar function $\boldsymbol{\phi}$ such that $\vec{F} = \nabla\boldsymbol{\phi}$

Note

1. In a conservative vector field $\vec{F} = \nabla\boldsymbol{\phi}$
$$\therefore \quad \nabla \times \vec{F} = \nabla \times \nabla\boldsymbol{\phi} = \vec{0} \quad \Rightarrow \quad \vec{F} \text{ is irrotational.}$$
2. This scalar function $\boldsymbol{\phi}$ is called the **scalar potential of $\vec{F}$.**
Only irrotational vectors will have scalar potential $\boldsymbol{\phi}$.

WORKED EXAMPLES

EXAMPLE 1

Prove that $\nabla \times \nabla\phi = 0$, where ϕ is a scalar point function.

Solution.

We have $$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}, \quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$\therefore$ $$\nabla \times \nabla\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \vec{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \vec{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$$= 0 \qquad \left[\text{Assuming } \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}, \frac{\partial^2\phi}{\partial z\partial x} = \frac{\partial^2\phi}{\partial x\partial z}, \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}\right]$$

$\therefore$ $\nabla\phi$ is always an irrotational vector.

EXAMPLE 2

Find the divergence and curl of the vector $\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$ at the point (2, –1, 1).

Solution.

Given $$\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}$$

$\therefore$ $$\text{div } \vec{v} = \nabla\cdot\vec{v} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + 2xz - y^2$$

At the point (2, −1, 1), $\nabla\cdot\vec{v} = (-1)\cdot 1 + 3\cdot 4 + 2\cdot 2\cdot 1 - (-1)^2 = -1 + 12 + 4 - 1 = 14$

and $$\text{Curl } \vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \vec{i}[0 - 2yz - 0] - \vec{j}[z^2 - 0 - xy] + \vec{k}[6xy - xz]$$

$$= -2yz\vec{i} - (z^2 - xy)\vec{j} + (6xy - xz)\vec{k}$$

At the point (2, −1, 1),

$$\nabla \times \vec{v} = -2(-1)\cdot 1\vec{i} - (1^2 - 2(-1))\vec{j} + [6\cdot 2(-1) - 1\cdot 2]\vec{k} = 2\vec{i} - 3\vec{j} - 14\vec{k}$$

EXAMPLE 3

Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution.

Given $$\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$$

$\vec{F}$ is irrotational if curl $\vec{F} = \vec{0}$

Now curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right]$$

$$= \vec{i}[-1+1] - \vec{j}[3z^2 - 3z^2] + \vec{k}[6x - 6x] = \vec{0}.$$

∴ $\vec{F}$ is irrotational vector.

EXAMPLE 4

Prove that (i) div $\vec{r} = 3$, (ii) curl $\vec{r} = \vec{0}$ where $\vec{r}$ is the position vector of a point (x, y, z) in space.

Solution.

Given $\vec{r}$ is the position vector of a point (x, y, z) in space.

∴ $$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

(i) div $\vec{r} = \nabla \cdot \vec{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(ii) Curl $\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] - \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \vec{i}[0 - 0] - \vec{j}[0 - 0] + \vec{k}[0 - 0] = \vec{0}$$

∴ $\vec{r}$ is an irrotational vector.

EXAMPLE 5

Find the value of a if the vector
$\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$ is solenoidal.

Solution.
Given $$\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} + (axyz - 2x^2y^2)\vec{k}$$
is solenoidal.

$$\therefore \quad \nabla \cdot \vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(2x^2y + yz) + \frac{\partial}{\partial y}(xy^2 - xz^2) + \frac{\partial}{\partial z}(axyz - 2x^2y^2) = 0$$

$$\Rightarrow \quad 4xy + 2xy + axy = 0$$

$$\Rightarrow \quad 6xy + axy = 0$$

$$\Rightarrow \quad xy(6+a) = 0 \Rightarrow (6+a) = 0 \Rightarrow a = -6 \qquad [\because x \neq 0, y \neq 0]$$

EXAMPLE 6

Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$ is irrotational and solenoidal.

Solution.

Given $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$.
We have to prove $\vec{F}$ is irrotational and solenoidal.
i.e., to prove $\nabla \times \vec{F} = \vec{0}$ and $\nabla \cdot \vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \vec{i}(3x - 3x) - \vec{j}[3y - 2z - (-2z + 3y)] + \vec{k}[3z + 2y - (2y + 3z)] = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) + \frac{\partial}{\partial y}(3xz + 2xy) + \frac{\partial}{\partial z}(3xy - 2xz + 2z)$$

$$= -2 + 2x + (-2x + 2) = 0$$

$\therefore$ $\vec{F}$ is solenoidal.

EXAMPLE 7

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$, prove that $r^n\vec{r}$ is solenoidal if $n = -3$ and irrotational for all values of n.

Solution.

Given $$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore \quad r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2} \Rightarrow r^2 = x^2 + y^2 + z^2 \qquad (1)$$

$$r^n\vec{r} = r^n(x\vec{i} + y\vec{j} + z\vec{k}) = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\therefore \quad \text{div}\,(r^n\vec{r}) = \nabla \cdot (r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z) \qquad (2)$$

But $\dfrac{\partial}{\partial x}(r^n x) = r^n + x \cdot nr^{n-1}\dfrac{\partial r}{\partial x}$, $\dfrac{\partial}{\partial y}(r^n y) = r^n + y \cdot nr^{n-1}\dfrac{\partial r}{\partial y}$

and $\dfrac{\partial}{\partial z}(r^n z) = r^n + z \cdot nr^{n-1}\dfrac{\partial r}{\partial z}$

We have, $r^2 = x^2 + y^2 + z^2$, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \frac{\partial}{\partial x}(r^n x) = r^n + nxr^{n-1} \cdot \frac{x}{r} = r^n + nx^2 r^{n-2}$$

$$\frac{\partial}{\partial y}(r^n y) = r^n + nyr^{n-1} \cdot \frac{y}{r} = r^n + ny^2 r^{n-2}$$

and $$\frac{\partial}{\partial z}(r^n z) = r^n + nzr^{n-1} \cdot \frac{z}{r} = r^n + nz^2 r^{n-2}$$

Substitute in (2).

$$\therefore \quad \text{div}\,(r^n \vec{r}) = r^n + nx^2 r^{n-2} + r^n + ny^2 r^{n-2} + r^n + nz^2 r^{n-2}$$
$$= 3r^n + nr^{n-2}(x^2 + y^2 + z^2) = 3r^n + nr^{n-2} \cdot r^2 = 3r^n + nr^n = (n+3)r^n$$

If $n = -3$, then div $(r^n \vec{r}) = 0$ $\quad \therefore$ $r^n \vec{r}$ is solenoidal if $n = -3$

$$\text{Now } \nabla \times r^n \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(r^n z) - \frac{\partial}{\partial z}(r^n x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right]$$

$$= \vec{i}\left(nzr^{n-1}\frac{\partial r}{\partial y} - nyr^{n-1}\frac{\partial r}{\partial z}\right) - \vec{j}\left(nzr^{n-1}\frac{\partial r}{\partial x} - nxr^{n-1}\frac{\partial r}{\partial z}\right) + \vec{k}\left(nyr^{n-1}\frac{\partial r}{\partial x} - nxr^{n-1}\frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left(nzr^{n-1}\frac{y}{r} - nyr^{n-1}\frac{z}{r}\right) - \vec{j}\left(nzr^{n-1} \cdot \frac{x}{r} - nxr^{n-1}\frac{z}{r}\right) + \vec{k}\left(nyr^{n-1} \cdot \frac{x}{r} - nxr^{n-1} \cdot \frac{y}{r}\right)$$

$$= \vec{i}(nr^{n-2}yz - nr^{n-2}yz) - \vec{j}(nr^{n-2}xz - nr^{n-2}xz) + \vec{k}(nr^{n-2}xy - nr^{n-2}xy) = \vec{0}$$

$\therefore$ $\nabla \times (r^n \vec{r}) = \vec{0}$ for all values of n.

Hence, $r^n \vec{r}$ is irrotational for all values of n.

EXAMPLE 8

Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2 \vec{k}$ is irrotational and find its scalar potential.

Solution.

Given $$\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2 \vec{k}$$

Now $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - 4 & 3xz^2 \end{vmatrix}$$
$$= \vec{i}(0-0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2y\cos x - 2y \cos x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

Hence, there exist a scalar function ϕ such that $\vec{F} = \nabla\phi$

$$\Rightarrow \quad (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k} = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}$$

$$\therefore \quad \frac{\partial \phi}{\partial x} = y^2 \cos x + z^3 \quad (1) \qquad \frac{\partial \phi}{\partial y} = 2y \sin x - 4 \quad (2) \quad \text{and} \quad \frac{\partial \phi}{\partial z} = 3xz^2 \quad (3)$$

Integrating (1) w.r.to x, $\quad \phi = y^2 \sin x + z^3 x + f_1(y, z) \quad (4)$

Integrating (2) w.r.to y, $\quad \phi = y^2 \sin x - 4y + f_2(x, z) \quad (5)$

Integrating (3) w.r.to z, $\quad \phi = xz^3 + f_3(x, y) \quad (6)$

From (4), (5), (6), $\phi = y^2 \sin x + xz^3 - 4y + c$ is the scalar potential, where c is an arbitrary constant.

EXAMPLE 9

(i) Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.
(ii) Find a, b, c if $(x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$ is irrotational.

Solution.

(i) Let $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$

Given $\vec{F}$ is solenoidal.

$$\therefore \quad \nabla \cdot \vec{F} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$$\Rightarrow \quad 3 + a + 2 = 0 \Rightarrow a = -5$$

(ii) Let $\vec{F} = (x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$

Given $\vec{F}$ is irrotational.

$$\therefore \quad \nabla \times \vec{F} = \vec{0} \quad \Rightarrow \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}\left[\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right] = 0$$

$$\Rightarrow \quad \vec{i}(c+1) - \vec{j}(-1-a) + \vec{k}(b-1) = \vec{0}$$
$$\Rightarrow \quad (c+1)\vec{i} + (1+a)\vec{j} + (b-1)\vec{k} = \vec{0}$$
$$\therefore \quad c+1=0,\ 1+a=0,\ b-1=0$$
$$\therefore \quad a=-1, \quad b=1 \text{ and } c=-1$$

EXAMPLE 10

Determine $f(r)$ so that the vector $f(r)\,\vec{r}$ is both solenoidal and irrotational.

Solution.

If $\vec{r}$ is not specified, it will always represent the position vector of any point (x, y, z).

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \text{and} \quad r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \quad \therefore \quad r^2 = x^2+y^2+z^2 \tag{1}$$

$$\therefore \quad f(r)\vec{r} = f(r)(x\vec{i} + y\vec{j} + z\vec{k}) = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}$$

Given $f(r)\,\vec{r}$ is solenoidal.

$$\therefore \quad \nabla\cdot(f(r)\vec{r}) = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) = 0 \tag{2}$$

But $$\frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\frac{\partial r}{\partial x}$$

$$\frac{\partial}{\partial y}(f(r)y) = f(r) + yf'(r)\frac{\partial r}{\partial y}$$

and $$\frac{\partial}{\partial z}(f(r)z) = f(r) + zf'(r)\frac{\partial r}{\partial z}$$

Differentiating (1) we get, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}, \quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}, \quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \quad \frac{\partial}{\partial x}(f(r)x) = f(r) + xf'(r)\cdot\frac{x}{r} = f(r) + \frac{x^2}{r}f'(r)$$

Similarly, $$\frac{\partial}{\partial y}(f(r)y) = f(r) + \frac{y^2}{r}f'(r)$$

and $$\frac{\partial}{\partial z}(f(r)z) = f(r) + \frac{z^2}{r}f'(r)$$

$$\therefore \ (2) \quad \Rightarrow \quad f(r) + \frac{x^2}{r}f'(r) + f(r) + \frac{y^2}{r}f'(r) + f(r) + \frac{z^2}{r}f'(r) = 0$$

$$\Rightarrow \quad 3f(r) + \frac{f'(r)}{r}(x^2+y^2+z^2) = 0$$

$$\Rightarrow \quad 3f(r) + \frac{f'(r)}{r}\cdot r^2 = 0$$

$$\Rightarrow \quad 3f(r) + rf'(r) = 0 \quad \Rightarrow \quad \frac{f'(r)}{f(r)} = -\frac{3}{r}$$

[here r is real variable.]

Integrating w.r.to 'r', we get $\int \frac{f'(r)}{f(r)}\,dr = -3\int \frac{1}{r}\,dr$

$\Rightarrow$ $$\log_e f(r) = -3\log_e r + \log c$$

$\Rightarrow$ $$\log_e f(r) = -\log_e r^3 + \log_e c = \log_e \frac{c}{r^3} \quad \Rightarrow \quad f(r) = \frac{c}{r^3}$$

where c is the constant of integration.

$$\text{Now } \nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}(f(r)y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(f(r)z) - \frac{\partial}{\partial z}(f(r)x)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x)\right]$$

$$= \sum \vec{i}\left[zf'(r)\cdot\frac{\partial r}{\partial y} - y\cdot f'(r)\cdot\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\cdot\frac{y}{r} - y\cdot f'(r)\cdot\frac{z}{r}\right] = \sum \vec{i} f'(r)\left[\frac{yz}{r} - \frac{yz}{r}\right] = \vec{0}$$

$\therefore$ $f(r)\vec{r}$ is irrotational for all $f(r)$ and it is solenoidal for $f(r) = \frac{c}{r^3}$, where c is arbitrary constant.

Hence, the required function is $f(r) = \frac{c}{r^3}$, for which $f(r)\vec{r}$ is both solenoidal and irrotational.

EXERCISE 9.2

1. If $\vec{F} = xy^2 + 2x^2yz\vec{j} - 3yz^2\vec{k}$, then find div $\vec{F}$ and curl $\vec{F}$ at (1, 1, –1).
2. If $F = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ then find curl curl $\vec{F}$.
3. Find div $\vec{F}$ and curl $\vec{F}$ at (1, 1, 1) if $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$.
4. Show that the following vectors are solenoidal.
   (i) $\vec{F} = (2 + 3y)\vec{i} + (x - 2z)\vec{j} + x\vec{k}$
   (ii) $\vec{F} = (y^2 - z^2 + 3yz - 2x)\vec{i} + (3xz + 2xy)\vec{j} + (3xy - 2xz + 2z)\vec{k}$
   (iii) $\vec{F} = 3x^2y\vec{i} - 4xy^2\vec{j} + 2xyz\vec{k}$
5. Find the value of a if $\vec{F} = ay^4z^2\vec{i} + 4x^3z^2\vec{j} + 5x^2y^2\vec{k}$ is solenoidal.
6. If the vector $3x\vec{i} + (x + y)\vec{j} - az\vec{k}$ is solenoidal, then find a.
7. Show that the following vectors are irrotational.
   (i) $\vec{F} = (y^2 + 2xz^2)\vec{i} + (2xy - z)\vec{j} + (2x^2z - y + 2z)\vec{k}$
   (ii) $\vec{F} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x - y)\vec{k}$
   (iii) $\vec{F} = (4xy - z^2)\vec{i} + 2x^2\vec{j} - 3xz^2\vec{k}$

8. Find the value of a if $\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ is irrotational.
9. If $\vec{F} = (ax^2 + 2y^2 + 1)\vec{i} + (4xy + by^2z - 3)\vec{j} + (c - y^3)\vec{k}$ is irrotational, then find the values of a, b, c.
10. Show that $F = (2x + 3y + z^2)\vec{i} + (3x + 2y + z)\vec{j} + (y + 2zx)\vec{k}$ is irrotational and hence, find its scalar potential.
11. Prove that $\vec{F} = (y^2 \cos x + z^3)\vec{i} + (2y \sin x - 4)\vec{j} + 3xz^2\vec{k}$ is irrotational and find its scalar potential.
12. Show that $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational, find its scalar potential.
13. Find the div $\vec{\mathrm{F}}$ and curl $\vec{\mathrm{F}}$, where $\vec{\mathrm{F}} = \mathrm{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.
14. If $\vec{v} = \vec{w} \times \vec{r}$, prove that $\vec{w} = \frac{1}{2}\mathrm{curl}\, \vec{v}$, where $\vec{w}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z).
15. If $\vec{r}$ is the position vector of a point (x, y, z) in space and $\vec{A}$ is a constant vector, prove that $\vec{A} \times \vec{r}$ is solenoidal.
16. Prove that the vector $\vec{\mathrm{F}} = (x + 3y)\vec{i} + (y - 3z)\vec{j} + (x - 2z)\vec{k}$ is solenoidal.
17. Show that $\vec{v} = xyz^2\vec{u}$ is solenoidal, where

    $\vec{u} = (2x^2 + 8xy^2z)\vec{i} + (3x^3y - 3xy)\vec{j} - (4y^2z^2 + 2x^3z)\vec{k}$.

ANSWERS TO EXERCISE 9.2

1. $5; -5\vec{i} - 6\vec{k}$
2. $2[z\vec{i} + x\vec{j} + y\vec{k}]$
3. $6; -2\vec{i} + 2\vec{k}$
5. a can be any real number
6. 4
8. 2
9. $a = 3, b = -3, c = 2$
10. $\phi = x^2 + y^2 + 3xy + yz + z^2x + c$
11. $\phi = y^2 \sin x + xz^3 - 4y + c$
12. $\phi = 3x^2y + xz^3 - yz + c$
13. div $\vec{\mathrm{F}} = b(x + y + z)$ Curl $\vec{\mathrm{F}} = \vec{\mathrm{O}}$

9.7 VECTOR IDENTITIES

We shall list the vector identities into two categories.

(i) ∇ operator applied once to point functions.
(ii) ∇ operator applied twice to point functions.

TYPE 1.

If f and g are scalar point functions we have already proved the following results.

1. $\nabla c = 0$, where c is a constant.
2. $\nabla(c\phi) = c\nabla\phi$, where c is constant.
3. $\nabla(f \pm g) = \nabla f \pm \nabla g$
4. $\nabla(fg) = f\nabla g + g\nabla f$
5. $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{f^2}$

6. If $\vec{F}$ and $\vec{G}$ are vector point functions, then $\nabla \cdot (\vec{F} + \vec{G}) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$.

Proof

$$\begin{aligned}\nabla \cdot (\vec{F}+\vec{G}) &= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(\vec{F}+\vec{G})\\ &= \vec{i}\cdot\left(\frac{\partial \vec{F}}{\partial x}+\frac{\partial \vec{G}}{\partial x}\right)+\vec{j}\cdot\left(\frac{\partial \vec{F}}{\partial y}+\frac{\partial \vec{G}}{\partial y}\right)+\vec{k}\cdot\left(\frac{\partial \vec{F}}{\partial z}+\frac{\partial \vec{G}}{\partial z}\right)\\ &= \left(\vec{i}\cdot\frac{\partial \vec{F}}{\partial x}+\vec{j}\cdot\frac{\partial \vec{F}}{\partial y}+\vec{k}\cdot\frac{\partial \vec{F}}{\partial z}\right)+\left(\vec{i}\cdot\frac{\partial \vec{G}}{\partial x}+\vec{j}\cdot\frac{\partial \vec{G}}{\partial y}+\vec{k}\cdot\frac{\partial \vec{G}}{\partial z}\right)\\ &= \nabla\cdot\vec{F}+\nabla\cdot\vec{G}\end{aligned}$$

Similarly, $\nabla\cdot(\vec{F}-\vec{G}) = \nabla\cdot\vec{F}-\nabla\cdot\vec{G}$ ■

7. If f is a scalar point function and $\vec{G}$ is a vector point function, then $\nabla \cdot (f\vec{G}) = \nabla f \cdot \vec{G} + f(\nabla\cdot\vec{G})$

Proof Let $\vec{G} = G_1\vec{i}+G_2\vec{j}+G_3\vec{k}$, then $f\vec{G} = fG_1\vec{i}+fG_2\vec{j}+fG_3\vec{k}$

$$\therefore \quad \nabla\cdot(f\vec{G}) = \frac{\partial}{\partial x}(fG_1)+\frac{\partial}{\partial y}(fG_2)+\frac{\partial}{\partial z}(fG_3)$$

$$= f\frac{\partial G_1}{\partial x}+\frac{\partial f}{\partial x}G_1+f\frac{\partial G_2}{\partial y}+\frac{\partial f}{\partial y}G_2+f\frac{\partial G_3}{\partial z}+\frac{\partial f}{\partial z}G_3$$

$$= \frac{\partial f}{\partial x}G_1+\frac{\partial f}{\partial y}G_2+\frac{\partial f}{\partial z}G_3+f\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}+\frac{\partial G_3}{\partial z}\right)$$

$$\therefore \quad \nabla\cdot(f\vec{G}) = \nabla f\cdot\vec{G}+f(\nabla\cdot\vec{G})$$ ■

8. If f is a scalar point function and $\vec{G}$ is a vector point function, then $\nabla \times (f\vec{G}) = \nabla f \times \vec{G} + f(\nabla \times G)$

Proof Let $\vec{G} = G_1\vec{i}+G_2\vec{j}+G_3\vec{k}$ $\quad\therefore\quad$ $f\vec{G} = fG_1\vec{i}+fG_2\vec{j}+fG_3\vec{k}$

$$\text{Now } \nabla\times(f\vec{G}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ fG_1 & fG_2 & fG_3\end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(fG_3)-\frac{\partial}{\partial z}(fG_2)\right]-\vec{j}\left[\frac{\partial}{\partial x}(fG_3)-\frac{\partial}{\partial z}(fG_1)\right]+\vec{k}\left[\frac{\partial}{\partial x}(fG_2)-\frac{\partial}{\partial y}(fG_1)\right]$$

$$= \vec{i}\left[f\frac{\partial G_3}{\partial y}+G_3\frac{\partial f}{\partial y}-f\frac{\partial G_2}{\partial z}-G_2\frac{\partial f}{\partial z}\right]-\vec{j}\left[f\frac{\partial G_3}{\partial x}+G_3\frac{\partial f}{\partial x}-f\frac{\partial G_1}{\partial z}-G_1\frac{\partial f}{\partial z}\right]$$

$$+\vec{k}\left[f\frac{\partial G_2}{\partial x}+G_2\frac{\partial f}{\partial x}-f\frac{\partial G_1}{\partial y}-G_1\frac{\partial f}{\partial y}\right]$$

$$= f\left[\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z}\right)\vec{i}-\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)\vec{j}+\left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)\vec{k}\right]$$

$$+\left(\frac{\partial f}{\partial y}G_3-\frac{\partial f}{\partial z}G_2\right)\vec{i}-\left(\frac{\partial f}{\partial x}G_3-\frac{\partial f}{\partial z}G_1\right)\vec{j}+\left(\frac{\partial f}{\partial x}G_2-\frac{\partial f}{\partial y}G_1\right)\vec{k}$$

$$= f\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix}$$

$\therefore \quad \nabla\times(f\vec{G}) = f(\nabla\times\vec{G})+(\nabla f)\times\vec{G}.$ ■

9. If $\vec{F}$ and $\vec{G}$ are vector point functions, then

$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + (\vec{G}\cdot\nabla)\vec{F} + \vec{F}\times(\nabla\times\vec{G}) + \vec{G}\times(\nabla\times\vec{F})$$

Proof We know that $\nabla f = \vec{i}\frac{\partial f}{\partial x}+\vec{j}\frac{\partial f}{\partial y}+\vec{k}\frac{\partial f}{\partial z} = \sum \vec{i}\frac{\partial f}{\partial x}$

$$\therefore \quad \nabla(\vec{F}\cdot\vec{G}) = \sum \vec{i}\frac{\partial}{\partial x}(\vec{F}\cdot\vec{G})$$

$$= \sum \vec{i}\left[\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}+\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right]$$

$$= \sum\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i}+\sum\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{i} \tag{1}$$

We know that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

$$\therefore \quad (\vec{a}\cdot\vec{b})\vec{c} = (\vec{a}\cdot\vec{c})\vec{b}-\vec{a}\times(\vec{b}\times\vec{c})$$

$$\therefore \quad \left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i} = (\vec{F}\cdot\vec{i})\left(\frac{\partial \vec{G}}{\partial x}\right)-\vec{F}\times\left(\frac{\partial \vec{G}}{\partial x}\times\vec{i}\right)$$

$$= (\vec{F}\cdot\vec{i})\left(\frac{\partial \vec{G}}{\partial x}\right)+\vec{F}\times\left(\vec{i}\times\frac{\partial \vec{G}}{\partial x}\right)$$

$$\therefore \quad \sum\left(\vec{F}\cdot\frac{\partial \vec{G}}{\partial x}\right)\vec{i} = \left(\vec{F}\cdot\sum \vec{i}\frac{\partial}{\partial x}\right)\vec{G}+\vec{F}\times\sum\left(\vec{i}\times\frac{\partial \vec{G}}{\partial x}\right)$$

$$= (\vec{F}\cdot\nabla)\vec{G}+\vec{F}\times(\nabla\times\vec{G}) \tag{2}$$

Interchanging $\vec{F}$ and $\vec{G}$, we get

$$\sum\left(\vec{G}\cdot\frac{\partial \vec{F}}{\partial x}\right)\vec{i} = (\vec{G}\cdot\nabla)\vec{F}+\vec{G}\times(\nabla\times\vec{F}) \tag{3}$$

Substituting (2) and (3) in (1) we get

$$\nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G}+\vec{F}\times(\nabla\times\vec{G})+(G\cdot\nabla)\vec{F}+\vec{G}\times(\nabla\times\vec{F})$$

$$\therefore \quad \nabla(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$$ ■

10. If $\vec{F}$ and $\vec{G}$ are vector point functions then

$$\nabla\cdot(\vec{F}\times\vec{G}) = \vec{G}\cdot(\nabla\times\vec{F}) - \vec{F}\cdot(\nabla\times\vec{G})$$

$$\text{i.e., div } \vec{F}\times\vec{G} = \vec{G}\cdot\text{Curl } \vec{F} - \vec{F}\cdot\text{ Curl } \vec{G}.$$

Proof

$$\nabla\cdot(\vec{F}\times\vec{G}) = \sum \vec{i}\cdot\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum \vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$$= \sum \vec{i}\cdot\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum \vec{i}\cdot\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

In a scalar triple product $\cdot$ and $\times$ can be interchanged.

$\therefore$ we get $$\nabla\cdot(\vec{F}\times\vec{G}) = \sum\left(\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right)\cdot\vec{G} - \sum\left(\vec{i}\times\frac{\partial\vec{G}}{\partial x}\right)\cdot\vec{F}$$

$\Rightarrow$ $$\nabla\cdot(\vec{F}\times\vec{G}) = (\nabla\times\vec{F})\cdot\vec{G} - (\nabla\times\vec{G})\cdot\vec{F}$$ ■

11. If $\vec{F}$ and $\vec{G}$ are vector product functions, then

$$\nabla\times(\vec{F}\times\vec{G}) = \vec{F}(\nabla\cdot\vec{G}) - \vec{G}(\nabla\cdot\vec{F}) + (\vec{G}\cdot\nabla)\vec{F} - (\vec{F}\cdot\nabla)\vec{G}$$

Proof

$$\nabla\times(\vec{F}\times\vec{G}) = \sum \vec{i}\times\frac{\partial}{\partial x}(\vec{F}\times\vec{G})$$

$$= \sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G} + \vec{F}\times\frac{\partial\vec{G}}{\partial x}\right)$$

$\Rightarrow$ $$\nabla\times(\vec{F}\times\vec{G}) = \sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) + \sum \vec{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) \quad (1)$$

We know $$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$\therefore$ $$\sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) = \sum\left[(\vec{i}\cdot\vec{G})\frac{\partial\vec{F}}{\partial x} - \left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}\right]$$

$$= \sum(\vec{G}\cdot\vec{i})\frac{\partial\vec{F}}{\partial x} - \sum\left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}$$ ■

$$= \vec{G}\cdot\left(\sum\vec{i}\frac{\partial}{\partial x}\right)\vec{F} - \left(\sum\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)\vec{G}$$

$\Rightarrow$ $$\sum \vec{i}\times\left(\frac{\partial\vec{F}}{\partial x}\times\vec{G}\right) = (\vec{G}\cdot\nabla)\vec{F} - (\nabla\cdot\vec{F})\vec{G} \quad (2)$$

Similarly, $$\sum \vec{i}\times\left(\vec{F}\times\frac{\partial\vec{G}}{\partial x}\right) = \sum\left[\left(\vec{i}\cdot\frac{\partial\vec{G}}{\partial x}\right)\vec{F} - \sum(\vec{i}\cdot\vec{F})\frac{\partial\vec{G}}{\partial x}\right]$$

$$= \sum \left(\vec{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - \sum (\vec{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x}$$

$$= \left(\sum \vec{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - \sum \left(\vec{F} \cdot \vec{i} \frac{\partial}{\partial x} \right) \vec{G}$$

$$= (\nabla \cdot \vec{G}) \vec{F} - \vec{F} \cdot \left(\sum \vec{i} \frac{\partial}{\partial x} \right) \vec{G}$$

$$\Rightarrow \qquad \sum \vec{i} \times \left(\vec{F} \times \frac{\partial \vec{G}}{\partial x} \right) = (\nabla \cdot \vec{G}) \vec{F} - (\vec{F} \cdot \nabla) \vec{G} \qquad (3)$$

Substituting (2) and (3) in (1), we get

$$\nabla \times (\vec{F} \times \vec{G}) = (\vec{G} \cdot \nabla) \vec{F} - (\nabla \cdot \vec{F}) \vec{G} + (\nabla \cdot \vec{G}) \vec{F} - (\vec{F} \cdot \nabla) \vec{G}$$

$$\therefore \qquad \text{Curl } \vec{F} \times \vec{G} = \vec{F} (\text{div } \vec{G}) - \vec{G} (\text{div } \vec{F}) + (\vec{G} \cdot \nabla) \vec{F} - (\vec{F} \cdot \nabla) \vec{G}$$ ■

TYPE II – Identities – ∇ Applied Twice

1. If f is scalar point function, then div grad $f = \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

Proof We know, grad $f = \nabla f = \vec{i} \dfrac{\partial f}{\partial x} + \vec{j} \dfrac{\partial f}{\partial y} + \vec{k} \dfrac{\partial f}{\partial z}$

$$\text{div (grad } f) = \nabla \cdot \nabla f$$

$$= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left(\vec{i} \frac{\partial f}{\partial x} + \vec{j} \frac{\partial f}{\partial y} + \vec{k} \frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\therefore \qquad \text{div (grad } f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ ■

Note $\nabla \cdot \nabla = \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is a scalar operator called the Laplacian operator.

2. If $\vec{F}$ is a vector point function, then div curl $\vec{F} = 0$.

Proof Let $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$, where F_1, F_2, F_3 are scalar functions of x, y, z.

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i} \left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right] - \vec{j} \left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right] + \vec{k} \left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right]$$

$$\therefore \quad \text{div Curl } \vec{F} = \nabla\cdot\nabla\times\vec{F} = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial^2 F_2}{\partial x\partial z}+\frac{\partial^2 F_1}{\partial y\partial z}-\frac{\partial^2 F_3}{\partial y\partial x}+\frac{\partial^2 F_2}{\partial z\partial x}-\frac{\partial^2 F_1}{\partial z\partial y}$$

$$\Rightarrow \quad \text{div Curl } \vec{F} = 0 \qquad \left[\text{Since } \frac{\partial^2 F_3}{\partial x\partial y}=\frac{\partial^2 F_3}{\partial y\partial x}, \frac{\partial^2 F_2}{\partial x\partial z}=\frac{\partial^2 F_2}{\partial z\partial x}, \frac{\partial^2 F_1}{\partial y\partial z}=\frac{\partial^2 F_1}{\partial z\partial y}\right] \blacksquare$$

3. If $\vec{F}$ is a vector point function, then

curl (Curl $\vec{F}$) = $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$.

Proof Let $\vec{F} = F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$, where F_1, F_2, F_3 are scalar functions.

Then
$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-\vec{j}\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+\vec{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

$$\therefore \quad \text{Curl curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} \end{vmatrix}$$

$$= \sum\vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]$$

$$= \sum\vec{i}\left[\frac{\partial^2 F_2}{\partial y\partial x}-\frac{\partial^2 F_1}{\partial y^2}-\frac{\partial^2 F_1}{\partial z^2}+\frac{\partial^2 F_3}{\partial z\partial x}\right]$$

$$= \sum\vec{i}\left\{\frac{\partial^2 F_2}{\partial y\partial x}+\frac{\partial^2 F_3}{\partial z\partial x}-\left(\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum\vec{i}\left\{\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_2}{\partial x\partial y}+\frac{\partial^2 F_3}{\partial x\partial z}-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum\vec{i}\left\{\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right)-\left(\frac{\partial^2 F_1}{\partial x^2}+\frac{\partial^2 F_1}{\partial y^2}+\frac{\partial^2 F_1}{\partial z^2}\right)\right\}$$

$$= \sum\vec{i}\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{F})-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)F_1\right\}$$

$$= \sum \vec{i}\left\{\frac{\partial}{\partial x}(\nabla\cdot\vec{F}) - \nabla^2 F_1\right\}$$

$$= \left(\sum \vec{i}\frac{\partial}{\partial x}\right)(\nabla\cdot\vec{F}) - \nabla^2\left(\sum \vec{i}F_1\right)$$

$$\therefore \qquad \nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$$

■

WORKED EXAMPLES

EXAMPLE 1

Prove that $\nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$.

Solution.
We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \qquad \nabla\left(\frac{1}{r^n}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r^n}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r^n}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r^n}\right)$$

$$= \vec{i}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial x}\right) + \vec{j}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial y}\right) + \vec{k}\left(\frac{-n}{r^{n+1}}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{n}{r^{n+1}}\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right] = -\frac{n}{r^{n+2}}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{n}{r^{n+2}}\vec{r}$$

$$\therefore \qquad \nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$$

Note We have $\quad \nabla\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$

If $n = 1, 2, 3, 4, \ldots$

Then $\quad \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}\vec{r}, \quad \nabla\left(\frac{1}{r^2}\right) = -\frac{2}{r^4}\vec{r}, \quad \nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\vec{r}, \quad \nabla\left(\frac{1}{r^4}\right) = -\frac{4}{r^6}\vec{r}$ and so on.

EXAMPLE 2

Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

Solution.

We have $\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r^2 = x^2 + y^2 + z^2$

$$\therefore \qquad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla(r^n) = \sum \vec{i}\frac{\partial}{\partial x}(r^n) = \sum \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\, nr^{n-1}\frac{x}{r} = nr^{n-2}\sum x\vec{i} = nr^{n-2}(x\vec{i}+y\vec{j}+z\vec{k}) = nr^{n-2}\vec{r} \quad \text{if } n \geq 3 \qquad (1)$$

$$\nabla^2(r^n) = \nabla\cdot(\nabla r^n) = \nabla\cdot(nr^{n-2}\vec{r})$$

$$= n[\nabla r^{n-2}\cdot\vec{r} + r^{n-2}(\nabla\cdot\vec{r})]$$

$$= n[(n-2)r^{n-4}\vec{r}\cdot\vec{r} + r^{n-2}3] \qquad \text{[using (1)]}$$

$$= n[(n-2)r^{n-4}r^2 + 3r^{n-2}] = nr^{n-2}[n-2+3] = n(n+1)r^{n-2}$$

Note We have $\nabla(r^n) = nr^{n-2}\vec{r}$
If $n = 1, 2, 3, 4, \ldots$

$$\nabla(r) = \frac{1}{r}\vec{r}, \quad \nabla(r^2) = 2\cdot r^{2-2}\vec{r} = 2\vec{r}, \quad \nabla(r^3) = 3r\vec{r}, \quad \nabla(r^4) = 4r^2\vec{r} \;\ldots\nabla(r^{n-2}) = (n-2)r^{n-4}\vec{r}, \text{ etc.}$$

EXAMPLE 3

Prove that $\nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4}$**.**

Solution.

We have $\nabla\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}\vec{r}, \quad \nabla\left(\frac{1}{r^4}\right) = \frac{-4}{r^6}\vec{r}$ and $\nabla\cdot\vec{r} = 3$

$$\therefore \quad \nabla\cdot\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \nabla\cdot\left(r\frac{-3}{r^5}\vec{r}\right) = \nabla\cdot\left(\frac{-3}{r^4}\vec{r}\right)$$

$$= -3\left[\nabla\left(\frac{1}{r^4}\right)\cdot\vec{r} + \frac{1}{r^4}\nabla\cdot\vec{r}\right]$$

$$= -3\left[-\frac{4}{r^6}(\vec{r}\cdot\vec{r}) + \frac{3}{r^4}\right] = -3\left[-\frac{4}{r^6}r^2 + \frac{3}{r^4}\right] = -3\left[\frac{-4}{r^4} + \frac{3}{r^4}\right] = \frac{3}{r^4}$$

EXAMPLE 4

If ϕ and ψ satisfy Laplace equation, prove that the vector $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal.

Solution.
Given ϕ and ψ satisfy Laplace equation.

$$\therefore \quad \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (1) \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \qquad (2)$$

To prove $(\phi\nabla\psi - \psi\nabla\phi)$ is solenoidal, we have to prove $\operatorname{div}(\phi\nabla\psi - \psi\nabla\phi) = 0$
Now $\operatorname{div}(\phi\nabla\psi - \psi\nabla\phi) = \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)$

$$= \nabla\cdot(\phi\nabla\psi) - \nabla\cdot(\psi\nabla\phi)$$

$$= \nabla\phi\cdot\nabla\psi + \phi(\nabla\cdot\nabla\psi) - [\nabla\psi\cdot\nabla\phi + \psi(\nabla\cdot\nabla\phi)]$$

$$= \phi\nabla^2\psi - \psi\nabla^2\phi \qquad [\because\ \nabla\phi\cdot\nabla\psi = \nabla\psi\cdot\nabla\phi]$$

$$= 0 \qquad \text{[from (1) and (2)]}$$

$\therefore\quad (\phi\,\nabla\,\psi - \psi\,\nabla\,\phi)$ is solenoidal.

EXAMPLE 5

Show that $\nabla^2 f(r) = \dfrac{d^2 f}{dr^2} + \dfrac{2}{r}\dfrac{df}{dr}$.

Solution.

We have, $\quad \nabla f(r) = f'(r)\dfrac{\vec{r}}{r}$

$$\therefore\ \nabla^2 f(r) = \nabla\cdot\nabla f(r) = \nabla\cdot f'(r)\frac{\vec{r}}{r} = \left(\nabla\frac{f'(r)}{r}\right)\cdot\vec{r} + \frac{f'(r)}{r}(\nabla\cdot\vec{r})$$

$$= \left(\nabla\frac{f'(r)}{r}\right)\cdot\vec{r} + \frac{3f'(r)}{r} \qquad [\because\ \nabla\cdot\vec{r} = 3]$$

$$= \left(\frac{r\nabla f'(r) - f'(r)\nabla r}{r^2}\right)\cdot\vec{r} + \frac{3f'(r)}{r} \qquad \left[\because\ \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}\right]$$

$$= \frac{\left(rf''(r)\frac{\vec{r}}{r} - f'(r)\frac{\vec{r}}{r}\right)\cdot\vec{r}}{r^2} + \frac{3f'(r)}{r} \qquad \left[\because\ \nabla f'(r) = f''(r)\frac{\vec{r}}{r} \text{ and } \nabla r = \frac{\vec{r}}{r}\right]$$

$$= \frac{[rf''(r) - f'(r)]\vec{r}\cdot\vec{r}}{r^3} + \frac{3f'(r)}{r}$$

$$= \frac{[rf''(r) - f'(r)]}{r^3}r^2 + \frac{3f'(r)}{r} \qquad [\because\ \vec{r}\cdot\vec{r} = r^2]$$

$$= \frac{rf''(r) - f'(r)}{r} + \frac{3f'(r)}{r} = f''(r) + \frac{2f'(r)}{r} = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

9.8 INTEGRATION OF VECTOR FUNCTIONS

Let $\vec{f}(t)$ and $\vec{F}(t)$ be two vector functions of a scalar variable t such that $\dfrac{d}{dt}\vec{F}(t) = \vec{f}(t)$. Then $\vec{F}(t)$ is called an indefinite integral of $\vec{f}(t)$ with respect to t and is written as $\int \vec{f}(t)dt = \vec{F}(t) + \vec{c}$, where $\vec{c}$ is an arbitrary constant vector independent of t and is called the constant of integration.

The definite integral of $\vec{f}(t)$ between the limits $t = t_1$ and $t = t_2$ is given by

$$\int_{t_1}^{t_2} \vec{f}(t)\,dt = \left[\vec{F}(t)\right]_{t_1}^{t_2} = \vec{F}(t_2) - \vec{F}(t_1).$$

As in the case of differentiation of vectors, in order to integrate a vector function, we integrate its components.

If $\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$, then

$$\int \vec{f}(t)\,dt = \vec{i}\int f_1(t)\,dt + \vec{j}\int f_2(t)\,dt + \vec{k}\int f_3(t)\,dt$$

9.8.1 Line Integral

An integral evaluated over a curve C is called a line integral. We call C as the path of integration. We assume every path of integration of a line integral to be piecewise smooth consisting of finitely many smooth curves.

Definition 9.10 A line integral of a vector point function $\vec{F}(\vec{r})$ over a curve C, where $\vec{r}$ is the position vector of any point on C, is defined by $\int_C \vec{F}\cdot d\vec{r}$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$ and $\int_C \vec{F}\cdot d\vec{r} = \int_C F_1\,dx + F_2\,dy + F_3\,dz$

Here F_1, F_2, F_3 are functions of x, y, z, where x, y, z depend on a parameter $t \in [a, b]$, since $\vec{r}(t)$ is the equation of the curve C.

Then we can write $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt.$

If the path of integration C is a closed curve, we write $\oint_C$ instead of $\int_C$.

Note

1. Since $\frac{d\vec{r}}{dt}$ is a tangent vector to the curve C the line integral $\int_C \vec{F}\cdot d\vec{r}$ is also called the tangential line integral of $\vec{F}$ over C and **line integral is a scalar**.
2. Two other types of line integrals are also considered. $\int_C \vec{F}\times d\vec{r}$ and $\int_C \boldsymbol{\phi}\,d\vec{r}$ are vectors.

WORKED EXAMPLES

EXAMPLE 1

If $\vec{F} = 3xy\,\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the arc of the parabola $y = 2x^2$ from (0, 0) to (1, 2).

Solution.

Given $\vec{F} = 3xy\vec{i} - y^2\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$, where $\vec{r}$ is the position vector of any point (x, y) on $y = 2x^2$.

$\therefore \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

and $\quad \vec{F}\cdot d\vec{r} = (3xy\vec{i} - y^2\vec{j})\cdot(dx\vec{i} + dy\vec{j}) = 3xydx - y^2dy$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C (3xy\,dx - y^2 dy)$$

Equation of C is $y = 2x^2 \quad \therefore \quad dy = 4x\,dx$.

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3x \cdot 2x^2 dx - 4x^4 \cdot 4x\,dx)$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1 = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = -\frac{7}{6}$$

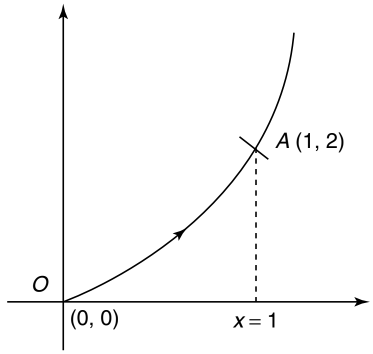

Fig. 9.2

EXAMPLE 2

If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from (0, 0, 0) to (1, 1, 1) along the curve C given by $x = t, y = t^2, z = t^3$.

Solution.

Given $\quad \vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$

and $\quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore \quad d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

and $\quad \vec{F} \cdot d\vec{r} = \left[(3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\vec{k}\right] \cdot \left[dx\vec{i} + dy\,\vec{j} + dz\vec{k}\right]$

$$= (3x^2 + 6y)dx - 14yzdy + 20xz^2dz$$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_C (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

Given $\quad x = t, \quad y = t^2, \quad z = t^3$ is the curve.

$\therefore \quad dx = dt, \quad dy = 2t\,dt, \quad dz = 3t^2\,dt$

When $\quad x = 0, t = 0$ and $x = 1, t = 1. \quad$ Limits for t are $t = 0, t = 1$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3 \cdot t^2 + 6 \cdot t^2)dt - 14 \cdot t^5 \cdot 2t\,dt + 20t^7 \cdot 3t^2\,dt$$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10}\right]_0^1 = 3 - 4 + 6 = 5.$$

EXAMPLE 3

Evaluate the line integral $\int_C (y^2 dx - x^2 dy)$ around the triangle whose vertices are (1, 0), (0, 1), (–1, 0) in the positive sense.

Solution.

Given the path C consists of the sides of the $\Delta\, ABC$, where $A(-1, 0)$, $B(1, 0)$ and $C(0, 1)$.

Equation of AB is $y = 0$

Equation of BC is $\dfrac{y - 0}{0 - 1} = \dfrac{x - 1}{1 - 0} \quad \Rightarrow \quad y = -x + 1$

Equation of CA is $\dfrac{y-1}{1-0} = \dfrac{x-0}{0+1} \Rightarrow y = x+1$

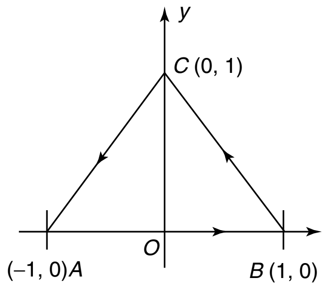

Fig. 9.3

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy) + \int_{CA} (y^2dx - x^2dy)$$

On AB, $y = 0$, $\therefore$ $dy = 0$ and x varies from -1 to 1

$$\therefore \quad \int_{AB} (y^2dx - x^2dy) = \int_{-1}^{1} 0\ dx = 0$$

On BC, $y = -x + 1$ $\therefore$ $dy = -dx$ and From B to C, x varies from 1 to 0.

$$\therefore \quad \int_{BC} (y^2dx - x^2dy) = \int_1^0 (-x+1)^2 dx - x^2(-dx) = \int_1^0 (x^2 - 2x + 1 + x^2)dx$$

$$= \int_1^0 (2x^2 - 2x + 1)dx$$

$$= \left[2\frac{x^3}{3} - 2\frac{x^2}{2} + x\right]_1^0 = 0 - \left(\frac{2}{3} - 1 + 1\right) = -\frac{2}{3}$$

On CA, $y = x + 1$ $\therefore$ $dy = dx$ and From C to A, x varies from 0 to -1

$$\therefore \quad \int_{CA} (y^2dx - x^2dy) = \int_0^{-1} (x+1)^2 dx - x^2 dx = \int_0^{-1} (x^2 + 2x + 1 - x^2)dx$$

$$= \int_0^{-1} (2x+1)dx = [x^2 + x]_0^{-1} = 1 - 1 - 0 = 0$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = 0 + \left(-\frac{2}{3}\right) + 0 = -\frac{2}{3}$$

EXAMPLE 4

If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the straight line joining (0, 0, 0) to (1, 1, 1).

Solution.

Given $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \therefore\ d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$$\therefore \quad \vec{F}\cdot d\vec{r} = \left[(3x^2+6y)\vec{i} - 14yz\,\vec{j} + 20xz^2\,\vec{k}\right]\cdot\left[dx\vec{i} + dy\,\vec{j} + dz\vec{k}\right]$$
$$= (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$$

Equation of the line joining (0, 0, 0) to (1, 1, 1) is

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} \quad \Rightarrow \quad x = y = z = t, \text{ say}$$

$$\therefore \quad dx = dt, \quad dy = dt, \quad dz = dt$$

At the point (0, 0, 0), $t = 0$ and at the point (1, 1, 1), $t = 1$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2+6t)dt - 14t^2 dt + 20t^3 dt$$
$$= \int_0^1 (3t^2 + 6t - 14t^2 + 20t^3)dt$$
$$= \int_0^1 (20t^3 - 11t^2 + 6t)dt = \left[20\frac{t^4}{4} - 11\frac{t^3}{3} + 6\frac{t^2}{2}\right]_0^1 = 5 - \frac{11}{3} + 3 = \frac{13}{3}$$

Definition 9.11 Work Done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which is moved along arc AB then $\int_A^B \vec{F}\cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in displacing the particle from A to B.

Conservative force field

A line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in domain D if and only if $\vec{F} = \nabla\phi$ for some scalar function ϕ defined in D. Such a force field is called a conservative field.

In the conservative field the total work done by $\vec{F}$ from A to B is

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi\cdot d\vec{r}$$
$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$
$$= \int_C d\phi = \int_A^B d\phi$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = [\phi]_A^B = \phi(B) - \phi(A)$$

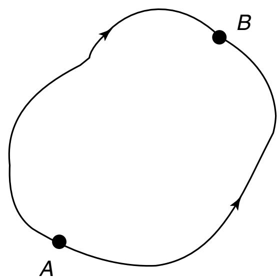

Fig. 9.4

So, in a conservative field the work done depends on the value of ϕ at the end points A and B of the path, but not on the path.

Note

1. ϕ is scalar potential.
2. If $\vec{F}$ is conservative, then $\vec{F} = \nabla\phi \Rightarrow \nabla\times\vec{F} = \nabla\times\nabla\phi = \vec{0}$
   $\therefore$ $\vec{F}$ is irrotational.
3. If C is a simple closed curve and $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r} = 0$.

WORKED EXAMPLES

EXAMPLE 5

Show that $\vec{F} = (e^x z - 2xy)\vec{i} - (x^2 - 1)\vec{j} + (e^x + z)\vec{k}$ is a conservative field. Hence, evaluate $\int_C \vec{F}\cdot d\vec{r}$ where the end points of C are (0, 1, –1) and (2, 3, 0).

Solution.

To prove that $\vec{F}$ is conservative, we have to prove $\nabla\times\vec{F} = \vec{0}$

Now
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x z - 2xy & 1 - x^2 & e^x + z \end{vmatrix}$$
$$= \vec{i}[0] - \vec{j}(e^x - e^x) + \vec{k}(-2x + 2x) = \vec{0}$$

Hence, $\vec{F}$ is conservative. $\quad\therefore\quad \vec{F} = \nabla\phi$

$$\Rightarrow \quad (e^x z - 2xy)\vec{i} + (1 - x^2)\vec{j} + (e^x + z)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = e^x z - 2xy \quad (1) \qquad \frac{\partial\phi}{\partial y} = 1 - x^2 \quad (2) \qquad \frac{\partial\phi}{\partial z} = e^x + z \quad (3)$$

Integrating (1) w. r. to x, $\quad \phi = ze^x - x^2y + f_1(y, z)$

Integrating (2) w. r. to y, $\quad \phi = (1 - x^2)y + f_2(x, z)$

Integrating (3) w. r. to z, $\quad \phi = e^x z + \dfrac{z^2}{2} + f_3(x, y)$

$$\therefore \quad \phi = ze^x - x^2y + y + \frac{z^2}{2} + C$$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = [\phi]_{(0,1,-1)}^{(2,3,0)}$$

$$= [ze^x - x^2y + y + \frac{z^2}{2} + c]_{(0,1,-1)}^{(2,3,0)}$$

$$= \left[0 - 2^2\cdot 3 + 3 + C - \left(-1 - 0 + 1 + \frac{1}{2} + C\right)\right] = -12 + 3 - \frac{1}{2} = -\frac{19}{2}.$$

EXAMPLE 6

If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$, then check whether the integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path *C*.

Solution.

Given $$\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$$

Now $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy-3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left\{\frac{\partial}{\partial y}(-2x^3z) - \frac{\partial}{\partial z}(2x^2)\right\} - \vec{j}\left\{\frac{\partial}{\partial x}(-2x^3z) - \frac{\partial}{\partial z}(4xy-3x^2z^2)\right\}$$

$$+\vec{k}\left\{\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(4xy-3x^2z^2)\right\}$$

$$= \vec{i}\{0-0\} - \vec{j}\{-6x^2z+6x^2z\} + \vec{k}\{4x-4x\} = \vec{0}$$

$\therefore$ $\vec{F}$ is conservative.

Hence, $\int_C \vec{F}\cdot d\vec{r}$ is independent of the path C.

EXAMPLE 7

Show that $\vec{F} = (2xy + z^3)i + x^2\vec{j} + 3xz^2\vec{k}$ is a conservative field. Find the scalar potential and work done in moving an object in this field from (1, –2, 1) to (3, 1, 4).

Solution.

Given $$\vec{F} = (2xy + z^3)i + x^2\vec{j} + 3xz^2\vec{k}$$

Now $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy+z^3)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy+z^3)\right]$$

$$= \vec{i}[0-0] - \vec{j}[3z^2-3z^2] + \vec{k}[2x-2x] = \vec{0}$$

$\therefore$ $\vec{F}$ is conservative.

So, there exists a scalar function ϕ such that $\vec{F} = \nabla\phi$.

$$\Rightarrow \qquad (2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \qquad \frac{\partial\phi}{\partial x}=2xy+z^3 \quad (1) \qquad \frac{\partial\phi}{\partial y}=x^2 \quad (2) \qquad \frac{\partial\phi}{\partial z}=3xz^2 \quad (3)$$

Integrating (1) partially w.r.to x, $\phi = x^2y + z^3x + f_1(y, z)$
Integrating (2) partially w.r.to y, $\phi = x^2y + f_2(x, z)$
Integrating (3) partially w.r.to z, $\phi = xz^3 + f_3(x, y)$
$\therefore$ $\phi = x^2y + xz^3 + C$

Since $\vec{F}$ is conservative, work done by the force $\vec{F}$ from (1, –2, 1) to (3, 1, 4) is equal to

$$[\phi]_{(1,-2,1)}^{(3,1,4)} = \left[x^2y+xz^3+C\right]_{(1,-2,1)}^{(3,1,4)}$$
$$= 3^2\cdot 1+3\cdot 4^3+C-\left[(1^2(-2)+1\cdot 1^3)+C\right]=9+192+1=202 \text{ units.}$$

EXERCISE 9.3

1. Prove that if $\vec{F}=\phi\nabla\psi$, then $\vec{F}\cdot(\nabla\times\vec{F})=0$.
2. Prove that Curl (ϕ grad ϕ) $=\vec{0}$.
3. Show that $\nabla\cdot(\phi\nabla\psi-\psi\nabla\phi)=\phi\nabla^2\psi-\psi\nabla^2\phi$.
4. Prove that $\nabla\times(\phi\nabla\psi)=\nabla\phi\times\nabla\psi$.
5. Prove that $\nabla\times[f(r)\vec{r}]=\vec{0}$.
6. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$, along the straight line joining the points (1, –2, 1) and (3, 2, 4).
7. Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(2y+3)\vec{i}+xz\vec{j}+(yz-x)\vec{k}$ along the line joining the points (0, 0, 0) to (2, 1, 1).
8. Find the work done in moving a particle in the force field $\vec{F}=3x^2\vec{i}+(2xz-y)\vec{j}-z\vec{k}$ from $t=0$ to $t=1$ along the curve $x=2t^2$, $y=t$, $z=4t^3$.
9. Show that $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$ is conservative. Find its scalar potential and find the work done in moving a particle from (1, –2, 1) to (3, 1, 2).
10. Find the work done by the force $\vec{F}=-xy\vec{i}+y^2\vec{j}+z\vec{k}$ in moving a particle over a circular path $x^2+y^2=4$, $z=0$ from (2, 0, 0) to (0, 2, 0).
11. Find the work done when a force $\vec{F}=(x^2-y^2+x)\vec{i}-(2xy+y)\vec{j}$ moves a particle in the xy plane from (0, 0) to (1, 1) along the curve $y^2=x$. If the path is $y=x$, whether the work done is different or same. If it is same, state the reason.
12. Find the total work done in moving a particle in a force field given by $\vec{F}=3xy\vec{i}-5z\vec{j}+10x\vec{k}$ along the curve $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$.

13. For the vector function $\vec{F} = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$ determine $\int_C \vec{F}\cdot d\vec{r}$ around the unit circle with centre at the origin in the xy plane.
14. Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (x - 3y)\vec{i} + (x - 2y)\vec{j}$ and C is the closed curve in the xy plane. $x = 2\cos t$, $y = 2\sin t$ and $t = 0$ to $t = 2\pi$.
15. Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.
16. Prove that $\nabla \times (\nabla r^n) = \vec{0}$.
17. If $\vec{F} = 5xy\vec{i} + 2y\vec{j}$, then evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the part of the curve $y = x^2$ between $x = 1$ and $x = 2$.
18. Show that the vector field $\vec{F}$, where $\vec{F} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2xz)\vec{k}$, is conservative and find its scalar potential.

ANSWERS TO EXERCISE 9.3

6. 211 [*Hint*: $\vec{F}$ is conservative]
7. 5
8. $\frac{13}{6}$
9. 34
10. $\frac{16}{3}$
11. $\frac{-2}{3}, \frac{-2}{3}$, $\vec{F}$ is conservative.
12. 303
13. 0
14. 24π
17. $\frac{135}{4}$
18. $\phi = xy + xy^2 + yz + xz^2 + c$.

9.9 GREEN'S THEOREM IN A PLANE

Green's theorem gives a relation between a double integral over a region R in the xy plane and the line integral over a closed curve C enclosing the region R. It helps to evaluate line integral easily.

Statement of Green's theorem

If $P(x, y)$ and $Q(x, y)$ are continuous functions with continuous partial derivatives in a region R in the xy plane and on its boundary C which is a simple closed curve then

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where C is described in the anticlockwise sense (which is the positive sense).

Green's theorem in a plane

Proof Let R be the region in the xy-plane bounded by the simple closed curve C traced in the anticlockwise sense, which is the positive sense. We assume any line parallel to the axes meet the curve in not more than two points. The curve C consists of two arcs APB and BQA as in figure.

Let $y = f_1(x)$ and $y = f_2(x)$ be the equations of these arcs.

Clearly, $f_1(x) \le f_2(x)$ in $[a, b]$

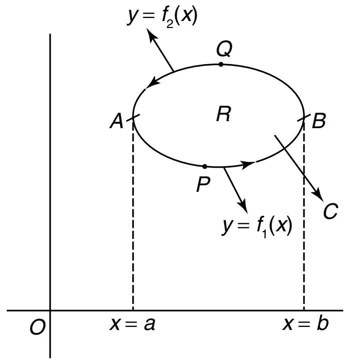

Fig. 9.5

Now, $$\iint_R \frac{\partial P}{\partial y} dx\, dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy \right] dx$$

$$= \int_a^b [P(x, y)]_{f_1(x)}^{f_2(x)} dx$$

$$= \int_a^b \left[P((x, f_2(x)) - P((x, f_1(x)) \right] dx$$

$$= \int_a^b P((x, f_2(x)) dx - \int_a^b P((x, f_1(x)) dx$$

However, $\int_a^b P(x, f_2(x))\, dx$ is numerically equal to the line integral $\int_{AQB} P(x, y)\, dx$ taken along the curve AQB.

But the positive sense is BQA (anticlockwise)

$\therefore$ $$\int_a^b P(x, f_2(x)) dx = -\int_{BQA} P(x, y)\, dx$$

Similarly, $$\int_a^b P(x, f_1(x)) dx = \int_{APQ} P(x, y)\, dx$$

$\therefore$ $$\iint_R \frac{\partial P}{\partial y} dy = -\int_{APB} P(x, y) dx - \int_{BQA} P(x, y) dx$$

$$= -\left\{ \int_{APB} P(x, y) dx + \int_{BQA} P(x, y) dx \right\} = -\oint_C P(x, y)\, dx$$

$\Rightarrow$ $$\int_C P(x, y) dx = -\iint_R \frac{\partial P}{\partial y} dx\, dy \qquad (1)$$

Now, we regard the curve C as constituted of the arcs QAP and PBQ.

Let their equations be $x = \phi_1(y)$ and $x = \phi_2(y)$

Then $\phi_1(y) \le \phi_2(y)$ in $[c, d]$

$$\iint_R \frac{\partial Q}{\partial x} dx\, dy = \int_{y=c}^{y=d} \left[\int_{x=\phi_1(y)}^{x=\phi_2(y)} \frac{\partial Q}{\partial x} dx \right] dy$$

$$= \int_c^d \left[Q(x, y) \right]_{x=\phi_1(y)}^{x=\phi_2(y)} \, dy$$

$$= \int_c^d \left[Q(\phi_2(y), y) - \phi(\phi_1(y), y) \right] dy$$

$$= \int_c^d Q(\phi_2(y), y) \, dy - \int_c^d Q(\phi_2(y), y) \, dy$$

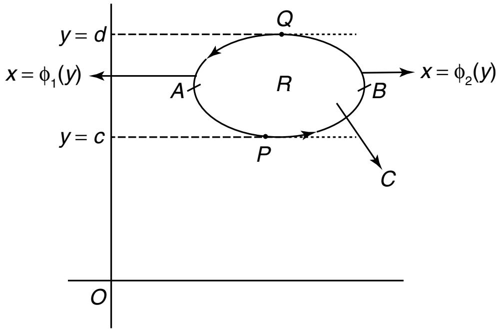

Fig. 9.6

But, $\int_c^d Q(\phi_2(y), y)\, dy$ is the line integral $\int_{PBQ} Q(x, y)\, dy$

and $\int_c^d Q(\phi_2(y), y)\, dy$ is the line integral $\int_{PAQ} Q(x, y)\, dy$

However, the positive sense of arc is QAP.

$$\therefore \qquad \int_c^d Q(\phi_2(y), y)\, dy = -\int_{QAP} Q(x, y)\, dy$$

$$\therefore \qquad \iint_R \frac{\partial Q}{\partial x} dx\, dy = \int_{PBQ} Q(x, y) dy + \int_{QAP} Q(x, y) dy = \int_C Q(x, y) dy$$

$$\therefore \qquad \int_C Q(x, y) dy = \iint_R \frac{\partial Q}{\partial x} dx\, dy \tag{2}$$

Adding the equations (1) and (2), we get

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = -\iint_R \frac{\partial P}{\partial y} dx\, dy + \iint_R \frac{\partial Q}{\partial x} dx\, dy$$

$$\Rightarrow \qquad \oint_C P\, dx + Q\, dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\, dy$$

Note We have proved the theorem by taking a simple closed region. The theorem is also valid in a region which can be divided into regions enclosed by simple closed curves.

Corollary Area of the region R bounded by C is $= \iint_R dxdy = \frac{1}{2} \oint_C (xdy - ydx)$

Proof In Green's theorem, take $P = -y$ and $Q = x$. $\therefore \frac{\partial P}{\partial y} = -1$ and $\frac{\partial Q}{\partial x} = 1$

Then
$$\oint_C (-ydx + xdy) = \iint_R (1+1) dxdy = 2 \iint_R dxdy$$

$$\therefore \qquad \frac{1}{2} \oint_C (xdy - ydx) = \iint_R dxdy \qquad ■$$

9.9.1 Vector Form of Green's Theorem

Let $\quad \vec{F} = P\vec{i} + Q\vec{j} \quad$ and $\quad \vec{r} = x\vec{i} + y\vec{j}$

$\therefore \quad d\vec{r} = dx\vec{i} + dy\vec{j} \quad$ and $\quad \vec{F}.d\vec{r} = P\,dx + Q\,dx$

Now, $$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \vec{i}\left[0 - \frac{\partial Q}{\partial z}\right] - \vec{j}\left[0 - \frac{\partial P}{\partial z}\right] + \vec{k}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \qquad \left[\because \frac{\partial Q}{\partial z} = 0; \frac{\partial P}{\partial z} = 0\right]$$

$$\therefore \quad \nabla \times \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\therefore$ Green's theorem becomes $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} \cdot \vec{k}\, d\,R,$ where $d\,R = dx\,dy$

WORKED EXAMPLES

EXAMPLE 1

Using Green's theorem evaluate $\displaystyle\int_C [(x^2 - y^2)dx + 2xy\,dy]$, where C is the closed curve of the region bounded by $y^2 = x$ and $x^2 = y$.

Solution.

Green's theorem is $\displaystyle\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

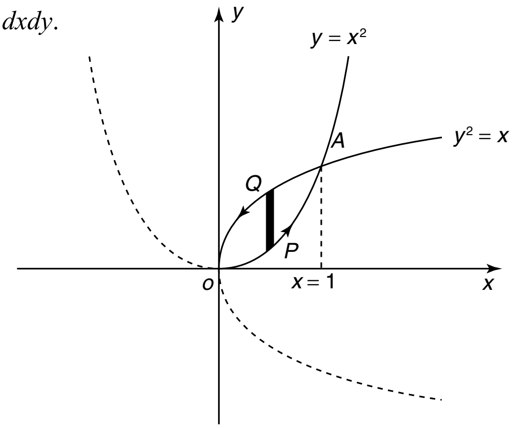

Fig. 9.7

The given line integral is $\displaystyle\int_C [(x^2 - y^2)dx + 2xydy]$

Here $\quad P = x^2 - y^2 \quad$ and $\quad Q = 2xy$

$$\therefore \quad \frac{\partial P}{\partial y} = -2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2y$$

$$\therefore \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y + 2y = 4y$$

$$\therefore \int_C (x^2 - y^2)\,dx + 2xydy = \iint_R 4ydxdy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 4ydydx$$

$$= 4\int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx = 2\int_0^1 (x - x^4)dx = 2\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{5}\right] = \frac{3}{5}$$

EXAMPLE 2

Evaluate $\int_C [(\sin x - y)dx - \cos x dy]$, where C is the triangle with vertices $(0, 0), \left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$

Given line integral is $\int_C [(\sin x - y)dx - \cos x dy]$

Here $\qquad P = \sin x - y \quad$ and $\quad Q = -\cos x$

$\therefore \qquad \frac{\partial P}{\partial y} = -1 \quad$ and $\quad \frac{\partial Q}{\partial x} = \sin x$

$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin x + 1$

$$\int_C [(\sin x - y)dx - \cos x dy] = \iint_R (\sin x + 1)dxdy$$

Equation of OB is $\frac{y - 0}{0 - 1} = \frac{x - 0}{0 - \frac{\pi}{2}} \quad \Rightarrow \quad y = \frac{2x}{\pi}$

Equation of AB is $\quad x = \frac{\pi}{2}$

In this region R, x varies from $\frac{\pi y}{2}$ to $\frac{\pi}{2}$ and y varies from 0 to 1.

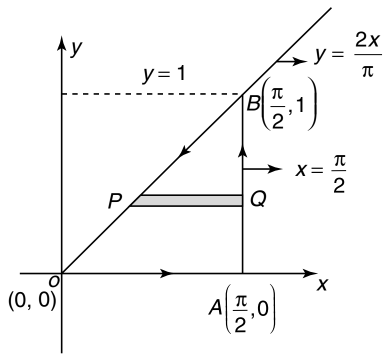

Fig. 9.8

$$\therefore \int_C [(\sin x - y)dx - \cos x dy] = \int_0^1 \left[\int_{\pi y/2}^{\pi/2} (\sin x + 1)dx\right] dy$$

$$= \int_0^1 [-\cos x + x]_{\pi y/2}^{\pi/2}\, dy$$

$$= \int_0^1 \left[\left(-\cos\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(-\cos\frac{\pi y}{2} + \frac{\pi y}{2}\right)\right] dy$$

$$= \int_0^1 \left(\frac{\pi}{2} + \cos\frac{\pi y}{2} - \frac{\pi y}{2}\right) dy$$

$$= \left[\frac{\pi}{2} y + \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} - \frac{\pi}{2}\frac{y^2}{2}\right]_0^1 = \frac{\pi}{2} + \frac{2}{\pi}\sin\frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2} + \frac{2}{\pi} - \frac{\pi}{4} = \frac{2}{\pi} + \frac{\pi}{4}$$

EXAMPLE 3

Evaluate by Green's theorem $\int_C e^{-x}(\sin y\,dx + \cos y\,dy)$, C being the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

The given line integral is $\int_C e^{-x}(\sin y dx + \cos y dy)$

Here $\qquad P = e^{-x}\sin y \quad$ and $\quad Q = e^{-x}\cos y$

$$\therefore \qquad \frac{\partial P}{\partial y} = e^{-x}\cos y \quad \text{and} \quad \frac{\partial Q}{\partial x} = -e^{-x}\cos y$$

$$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x}\cos y - e^{-x}\cos y = -2e^{-x}\cos y$$

$$\therefore \oint_C e^{-x}(\sin y\,dx + \cos y\,dy) = \iint_R -2e^{-x}\cos y\,dxdy$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{\pi} e^{-x}\cos y\,dxdy$$

$$= -2\left[\int_0^{\frac{\pi}{2}}\cos y\,dy\right]\left[\int_0^{\pi} e^{-x}dx\right]$$

$$= -2[\sin y]_0^{\pi/2}\left[\frac{e^{-x}}{-1}\right]_0^{\pi} = 2\left(\sin\frac{\pi}{2}\right)(e^{-\pi} - e^0) = 2(e^{-\pi} - 1)$$

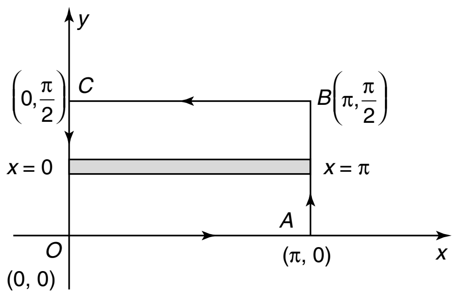

Fig. 9.9

EXAMPLE 4

Find the area bounded between the curves $y^2 = 4x$ and $x^2 = 4y$ using Green's theorem.

Solution.

We know, by Green's theorem the area bounded by a simple closed curve C is

$$\frac{1}{2}\oint_C (x\,dy - y\,dx)$$

Here C consists of the curves C_1 and C_2.

$$\therefore \qquad \text{area} = \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx\right] = \frac{1}{2}[I_1 + I_2]$$

On C_1: $\quad x^2 = 4y$

$$\therefore \qquad 2x\,dx = 4\,dy \quad \Rightarrow \quad dy = \frac{1}{2}x\,dx$$

and x varies from 0 to 4.

$$\therefore \qquad I_1 = \int_{C_1} x\,dy - y\,dx$$

$$= \int_0^4 x\cdot\frac{1}{2}x\,dx - \frac{x^2}{4}\,dx$$

$$= \int_0^4\left(\frac{x^2}{2} - \frac{x^2}{4}\right)dx = \int_0^4 \frac{x^2}{4}\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$

$$= \frac{64}{4\cdot 3} = \frac{16}{3}$$

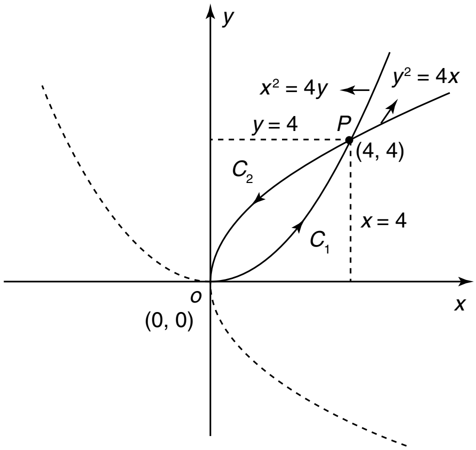

Fig. 9.10

On C_2: $y^2 = 4x \quad \therefore \quad 2y\,dy = 4\,dx \quad \Rightarrow \quad dx = \dfrac{1}{2}y\,dy$
and y varies from 4 to 0.

$$\therefore \qquad I_2 = \int_{C_2} x\,dy - y\,dx$$

$$= \int_4^0 \frac{y^2}{4}\,dy - y\cdot\frac{1}{2}y\,dy$$

$$= \int_4^0\left(\frac{y^2}{4} - \frac{y^2}{2}\right)dy = \int_4^0 -\frac{y^2}{4}\,dy = \frac{1}{4}\int_0^4 y^2\,dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3}$$

$$\therefore \qquad \text{area} = \frac{1}{2}\left[\frac{16}{3} + \frac{16}{3}\right] = \frac{16}{3}$$

EXAMPLE 5

Verify Green's theorem in the plane for $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$.

Solution.

Green's theorem is $\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

The given integral is $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

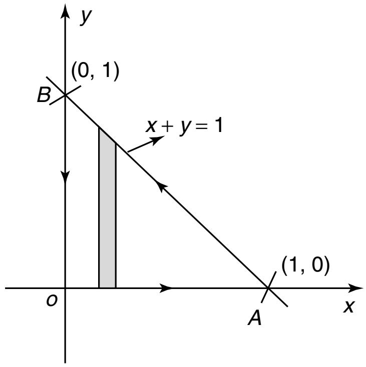

Fig. 9.11

Here $\qquad P = 3x^2 - 8y^2 \quad$ and $\quad Q = 4y - 6xy$

$$\therefore \qquad \frac{\partial P}{\partial y} = -16y \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = -6y$$

$$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -6y + 16y = 10y$$

$$\therefore \quad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} 10y \, dy \, dx$$

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx = 5\int_0^1 (1-x)^2 \, dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{-5}{3}[0-1] = \frac{5}{3}$$

$$\Rightarrow \quad \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \frac{5}{3} \qquad (1)$$

We shall now compute the line integral $\int_C Pdx + Qdy$

Now $\qquad \int_C Pdx + Qdy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$$= \int_{OA} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \int_{AB} (3x^2 - 8y^2)dx + (4y - 6xy)dy$$

$$+ \int_{BO} (3x^2 - 8y^2)dx + (4y - 6xy)dy = I_1 + I_2 + I_3$$

On *OA*: $y = 0 \quad \therefore \; dy = 0$ and x varies from 0 to 1.

$$\therefore \qquad I_1 = \int_0^1 3x^2 dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1$$

On *AB*: $x + y = 1 \quad \Rightarrow \quad y = 1 - x \qquad \therefore \; dy = -dx$ and x varies 1 to 0.

$$\therefore \qquad I_2 = \int_1^0 (3x^2 - 8(1-x)^2)dx + [4(1-x) - 6x(1-x)](-dx)$$

$$= \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6(x - x^2)]dx$$

$$= \left[x^3 - \frac{8(1-x)^3}{-3} - 4\frac{(1-x)^2}{-2} + 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right) \right]_1^0$$

$$= \left[0 + \frac{8}{3} + 2 + 0 - \left\{1 + 6\left(\frac{1}{2} - \frac{1}{3}\right)\right\} \right] = \frac{8}{3} + 2 - 1 - 1 = \frac{8}{3}$$

On *BO*: $x = 0 \quad \therefore \quad dx = 0$ and y varies from 1 to 0

$$\therefore \qquad I_3 = \int_1^0 4y\,dy = 2\left[y^2\right]_1^0 = -2$$

$$\therefore \qquad \int Pdx + Qdy = \int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \qquad (2)$$

(1) and (2) give the same value. Hence, Green's theorem is verified.

EXAMPLE 6

Verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2 dy$, where C is the boundary, of the area between $y = x^2$ and $y = x$.

Solution.

Green's theorem is

$$\int_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

The given line integral is

$$\int_C (xy + y^2)\,dx + x^2 dy$$

Here $\qquad P = xy + y^2 \quad$ and $\quad Q = x^2$

$$\therefore \qquad \frac{\partial P}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2x$$

$$\therefore \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - x - 2y = x - 2y$$

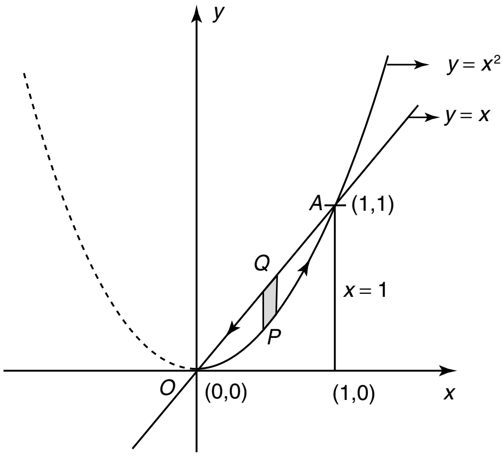

Fig. 9.12

$$\therefore \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_0^1 \int_{x^2}^{x} (x - 2y)\,dy\,dx = \int_0^1 \left[xy - \frac{2y^2}{2} \right]_{x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - (x^3 - x^4)]\,dx = \int_0^1 [(x^4 - x^3)]\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = -\frac{1}{20} \qquad (1)$$

We shall now compute the line integral $\int_C Pdx + Qdy$

Now $$\int_C Pdx + Qdy = \int_C (xy + y^2)\,dx + x^2 dy$$
$$= \int_{C_1} (xy + y^2)\,dx + x^2 dy + \int_{C_2} (xy + y^2)\,dx + x^2 dy = I_1 + I_2$$

On C_1: $y = x^2$, $\therefore$ $dy = 2x\,dx$ and x varies from 0 to 1.

$$I_1 = \int_0^1 (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx$$
$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$
$$= \int_0^1 (3x^3 + x^4)\,dx = \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

On C_2: $y = x$, $\therefore$ $dy = dx$ and x varies from 1 to 0.

$$\therefore \qquad I_2 = \int_1^0 (x \cdot x + x^2)\,dx + x^2\,dx = \int_0^1 3x^2\,dx = 3\left[\frac{x^3}{3}\right]_1^0 = -1$$

$$\therefore \qquad \int_C Pdx + Qdy = \frac{19}{20} - 1 = -\frac{1}{20} \qquad (2)$$

(1) and (2) give the same value.
Hence, Green's theorem is verified.

9.10 SURFACE INTEGRALS

Suppose a surface is bounded by a simple closed curve C, then we can regard the surface as having two sides separated by C. One of which is arbitrarily chosen as the positive side and the other is the negative side. If the surface is a closed surface, then the outerside is taken as the positive side and the inner side is the negative side. A unit normal at any point of the positive side of the surface is denoted by $\vec{n}$ and is called the outward drawn normal and its direction is considered positive.

Any integral which is evaluated over a surface is called a surface integral.

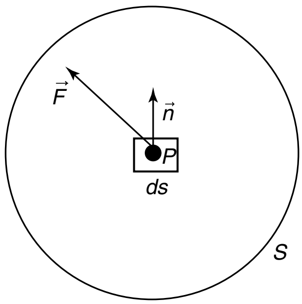

Fig. 9.13

Definition 9.12 Surface Integral

Let S be a surface of finite area which is smooth or piecewise smooth (e.g. a sphere is a smooth surface and a cube is a piecewise smooth surface). Let $\vec{F}(x, y, z)$ be a vector point function defined at each point of S. Let P be any point on the surface and let $\vec{n}$ be the outward unit normal at P. Then the surface integral of $\vec{F}$ over S is defined as $\iint_S \vec{F} \cdot \vec{n}\,dS$

If we associate a vector $d\vec{S}$ (called vector area) with the differential of surface area dS such that $|d\vec{S}| = dS$ and direction of $d\vec{S}$ is $\vec{n}$, then

$$d\vec{S} = \vec{n}\, dS$$

$\therefore$ $\iint_S \vec{F}\cdot\vec{n}\, dS$ can also be written as $\iint_S \vec{F}\cdot\overrightarrow{dS}$

Note

1. In physical application the integral $\iint_S \vec{F}\cdot\overrightarrow{dS}$ is called the normal flux of $\vec{F}$ through the surface S, because this integral is a measure of the volume emerging from S per unit time.

9.10.1 Evaluation of Surface Integral

To evaluate a surface integral over a surface it is usually expressed as a double integral over the orthogonal projection of S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface S in not more than one point.

Let R be the orthogonal projection of S on the xy plane.

Then the element surface dS is projected to an element area $dx\, dy$ in the xy plane as in fig.

$\therefore$ $dx\, dy = dS \cos\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is the angle between the planes of dS and xy–plane.

Let $\vec{n}$ be the unit normal to dS and $\vec{k}$ is the unit normal to the xy–plane.

Since angle between the planes is equal to the angle between the normals,

$\boldsymbol{\theta}$ is the angle between the normals $\vec{n}$ and $\vec{k}$.

$$\therefore \qquad \cos\boldsymbol{\theta} = \frac{\vec{n}\cdot\vec{k}}{|\vec{n}||\vec{k}|}$$

$$= \vec{n}\cdot\vec{k} \ \left[\text{Since } |\vec{n}| = 1, |\vec{k}| = 1\right]$$

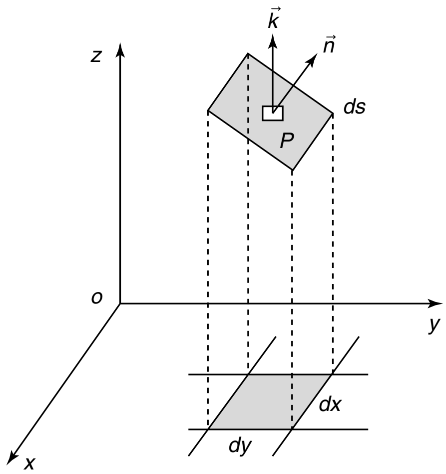

Fig. 9.14

We take the acute angle between the normals and

So, we take $|\vec{n}\cdot\vec{k}|$

$$\therefore \qquad dx\, dy = dS|\vec{n}\cdot\vec{k}| \quad \Rightarrow \quad dS = \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$$

Hence, $\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$

Similarly, taking the projection on the yz and zx planes, we get

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dy\, dz}{|\vec{n}\cdot\vec{i}|}$$

and
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\frac{dz\, dx}{|\vec{n}\cdot\vec{j}|}$$

Corollary

The surface area
$$\iint_S dS = \iint_R \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|} = \iint_{R_1} \frac{dy\, dz}{|\vec{n}\cdot\vec{i}|} = \iint_{R_2} \frac{dz\, dx}{|\vec{n}\cdot\vec{j}|}$$

9.11 VOLUME INTEGRAL

Any integral which is evaluated over a volume bounded by a surface is called a volume integral.

If V is the volume bounded by a surface S, then

$\iiint_V \phi(x, y, z)dV$ and $\iiint_V \vec{F}\, dV$ are called volume integrals.

If we divide V into rectangular blocks by drawing planes parallel to the coordinate planes, then

$$dV = dx\, dy\, dz.$$

$$\therefore \qquad \iiint_V \phi dV = \iiint_V \phi(x, y, z)\, dxdydz$$

If
$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

then
$$\iiint_V \vec{F}dV = \vec{i}\iiint_V F_1 dxdydz + \vec{j}\iiint_V F_2 dxdydz + \vec{k}\iiint_V F_3 dxdydz$$

WORKED EXAMPLES

EXAMPLE 1

Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$ if $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and S is the surface of the plane $3x + 2y + 6z = 6$ contained in the first octant.

Solution.

Given $\vec{F} = 4y\vec{i} + 18z\vec{j} - x\vec{k}$ and the surface $3x + 2y + 6z = 6$.

Let
$$\phi = 3x + 2y + 6z$$

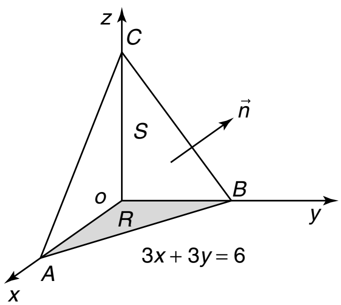

Fig. 9.15

Let R be the projection of S in the xy plane.

$\therefore$ R is the $\Delta\, AOB$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_R \vec{F}\cdot\vec{n}\, \frac{dx\, dy}{|\vec{n}\cdot\vec{k}|}$$

where $\vec{n}$ is unit normal to S and $\vec{k}$ is the unit normal to xy–plane.

Normal to the surface is $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

$$\therefore \text{ unit normal is } \quad \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9+4+36}} = \frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$$

$$\therefore \qquad \vec{F}\cdot\vec{n} = (4y\vec{i} + 18z\vec{j} - x\vec{k})\cdot\frac{1}{7}(3\vec{i} + 2\vec{j} + 6\vec{k})$$

$$= \frac{1}{7}(12y + 36z - 6x) = \frac{6}{7}(2y + 6z - x)$$

$$\vec{n}\cdot\vec{k} = \frac{1}{7}(3\vec{i}+2\vec{j}+6\vec{k})\cdot\vec{k} = \frac{6}{7}$$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \frac{6}{7}(2y+6z-x)\frac{dx\,dy}{\frac{6}{7}} = \iint_R (2y+6z-x)dx\,dy$$

We have $3x + 2y + 6z = 6$

$\Rightarrow \qquad 6z = 6 - 3x - 2y$

$\therefore \qquad 2y + 6z - x = 2y + 6 - 3x - 2y - x = 6 - 4x$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R (6-4x)\,dx\,dy$$

The plane $3x + 2y + 6z = 6$ meets the xy–plane $z = 0$ in line AB.

$\therefore$ the equation of AB is $3x + 2y = 6$

$\therefore$ the point A is (2, 0) and the point B is (0, 3)

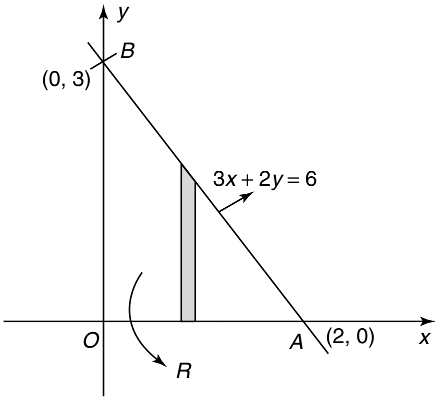

Fig. 9.16

Now $\qquad 3x + 2y = 6 \quad \Rightarrow \quad y = \dfrac{6-3x}{2}$

$\therefore$ In R, x varies from 0 to 2 and y varies from 0 to $\dfrac{6-3x}{2}$

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^2\int_0^{\frac{6-3x}{2}} (6-4x)\,dy\,dx = 2\int_0^2\int_0^{\frac{6-3x}{2}} (3-2x)\,dy\,dx$$

$$= 2\int_0^2 \Big[(3-2x)\,y\Big]_0^{\frac{6-3x}{2}} dx$$

$$= 2\int_0^2 (3-2x)\frac{(6-3x)}{2}\,dx$$

$$= 3\int_0^2 (3-2x)(2-x)\,dx$$

$$= 3\int_0^2 (6-7x+2x^2)\,dx$$

$$= 3\left[6x - \frac{7x^2}{2} + 2\frac{x^3}{3}\right]_0^2$$

$$= 3\left[6\times 2 - 7\times\frac{4}{2} + 2\times\frac{8}{3}\right] = 3\left[12 - 14 + \frac{16}{3}\right] = -6 + 16 = 10$$

EXAMPLE 2

Evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$ if $\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$ and S is part of the surface $x^2+y^2+z^2=1$, which lies in the first octant.

Solution.

Given $\vec{F}=yz\vec{i}+zx\vec{j}+xy\vec{k}$ and the surface is $x^2+y^2+z^2=1$

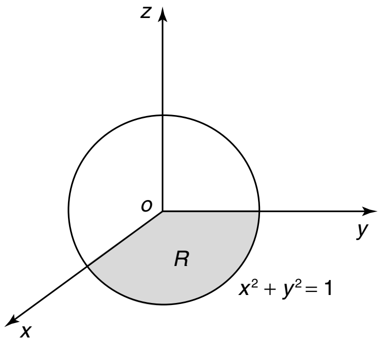

Fig. 9.17

Let $$\phi = x^2+y^2+z^2$$

The normal to the surface is $$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$\therefore$ unit normal is $$\vec{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$=\frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{2\sqrt{x^2+y^2+z^2}}=x\vec{i}+y\vec{j}+z\vec{k} \qquad [\because x^2+y^2+z^2=1]$$

$\therefore$ $$\vec{F}\cdot\vec{n}=(yz\vec{i}+zx\vec{j}+xy\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$
$$= xyz+xyz+xyz=3xyz$$

The projection of the surface of the sphere in the first octant into the xy plane is R, which is the quadrant of the circle $x^2+y^2=1$, $z=0$, $x\ge 0$, $y\ge 0$ and $\vec{k}$ is the unit normal to R.

$\therefore$ $$\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_R \vec{F}\cdot\vec{n}\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

$$=\iint_R 3xyz\frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

But $$\vec{n}\cdot\vec{k}=(x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k}=z$$

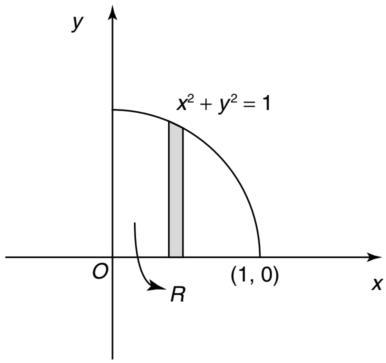

Fig. 9.18

$\therefore$ $$\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_R 3xyz\,\frac{1}{z}dx\,dy$$

$$=\iint_R 3xy\;dx\;dy$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}} 3xy\;dx\;dy$$

$$=3\int_0^1\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} x\,dx=\frac{3}{2}\int_0^1 x(1-x^2)dx$$

$$=\frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{3}{2}\cdot\frac{1}{4}=\frac{3}{8}$$

EXAMPLE 3

Evaluate $\iint \vec{F}\cdot\vec{n}\,dS$, where $\vec{F}=4xz\,\vec{i}-y^2\vec{j}+yz\vec{k}$ and S is the surface of the cube bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$.

Solution.

Given $\vec{F}=4xz\,\vec{i}-y^2\vec{j}+yz\vec{k}$

S is the surface of the cube, which is piecewise smooth surface consisting of six smooth surfaces.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{\text{ABEF}} \vec{F}\cdot\vec{n}\,dS + \iint_{\text{OCDG}} \vec{F}\cdot\vec{n}\,dS + \iint_{\text{BCDE}} \vec{F}\cdot\vec{n}\,dS + \iint_{\text{OAFG}} \vec{F}\cdot\vec{n}\,dS + \iint_{\text{OABC}} \vec{F}\cdot\vec{n}\,dS + \iint_{\text{DEFG}} \vec{F}\cdot\vec{n}\,dS$$

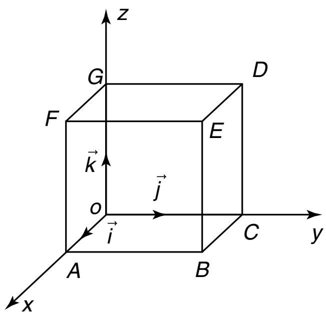

Fig. 9.19

On the face *ABEF*: $x=1, \quad \vec{n}=\vec{i}$

$\therefore$ $$\vec{F}\cdot\vec{n}=(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot\vec{i}=4xz=4z$$

and $$dS=\frac{dy\,dz}{|\vec{n}\cdot\vec{i}|}=\frac{dy\,dz}{|\vec{i}\cdot\vec{i}|}=dy\,dz$$

$$\therefore \iint_{\text{ABEF}} \vec{F}\cdot\vec{n}\,dS=\int_0^1\int_0^1 4z\,dz\,dy=4\cdot[y]_0^1\left[\frac{z^2}{2}\right]_0^1=4\cdot1\cdot\frac{1}{2}=2$$

On the face *OCDG*: $x=0, \quad \vec{n}=-\vec{i}$

$$\vec{F}\cdot\vec{n}=(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(-\vec{i})=-4xz=0$$

$$\therefore \iint_{\text{OCDG}} \vec{F}\cdot\vec{n}\,dS=0$$

On the face *BCDE*: $y=1, \quad \vec{n}=\vec{j}$

$\therefore$ $$dS=\frac{dx\,dz}{|\vec{n}\cdot\vec{j}|}=\frac{dx\,dz}{|\vec{j}\cdot\vec{j}|}=dx\,dz$$

and $$\vec{F}\cdot\vec{n}=(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot\vec{j}=-y^2=-1$$

$\therefore$ $$\iint_{\text{BCDE}} \vec{F}\cdot\vec{n}\,dS=\int_0^1\int_0^1(-1)dx\,dz=-[x]_0^1[z]_0^1=-1$$

On the face *OAFG*: $y=0, \quad \vec{n}=-\vec{j}$

$\therefore$ $$\vec{F}\cdot\vec{n}=(4xz\vec{i}-y^2\vec{j}+yz\vec{k})\cdot(-\vec{j})=y^2=0$$

$\therefore$ $$\iint_{\text{OAFG}} \vec{F}\cdot\vec{n}\,dS=0$$

On the face *DEFG*: $z=1, \quad \vec{n}=\vec{k}$

$\therefore$ $$\vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{k} = yz = y$$

and $$dS = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|} = \frac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = dx\,dy$$

$\therefore$ $$\iint_{\text{DEFG}} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = 1\times\frac{1}{2} = \frac{1}{2}$$

On the face *OABC*: $z = 0,\quad \vec{n} = -\vec{k}$

$$\vec{F}\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{k}) = -yz = 0$$

$\therefore$ $$\iint_{\text{OABC}} \vec{F}\cdot\vec{n}\,dS = 0$$

$\therefore$ $$\iint_S \vec{F}\cdot\vec{n}\,dS = 2 + (-1) + \frac{1}{2} = \frac{3}{2}\cdot$$

9.12 GAUSS DIVERGENCE THEOREM

The divergence theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss divergence theorem

Let V be the volume bounded by a closed surface S. If a vector function $\vec{F}$ is continuous and has continuous partial derivatives inside and on S, then the surface integral of $\vec{F}$ over S is equal to the volume integral of divergence of $\vec{F}$ taken throughout V.

i.e., $$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{F}\,dV$$

If $\vec{n}$ is the outward normal to the surface $d\vec{S} = \vec{n}\,dS$

$$\therefore\ \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

Proof Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$\therefore$ $$\vec{F}\cdot\vec{n} = F_1(\vec{i}\cdot\vec{n}) + F_2(\vec{j}\cdot\vec{n}) + F_3(\vec{k}\cdot\vec{n})$$

and $$\vec{F}\cdot\vec{n}\,dS = F_1(\vec{i}\cdot\vec{n})dS + F_2(\vec{j}\cdot\vec{n})dS + F_3(\vec{k}\cdot\vec{n})dS$$
$$= F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy$$

But $$\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

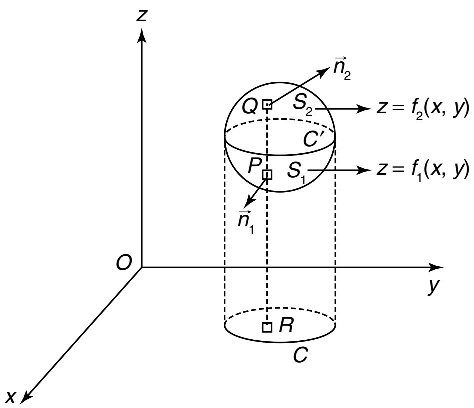

Fig. 9.20

Hence, Gauss theorem in Cartesian form is

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) \equiv \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz$$

We shall assume that S is a closed surface such that any line drawn parallel to coordinate axes cuts S in almost two points. The lines drawn parallel to Z-axis touching the surface S determine the curve C' on it and intersect the xy-plane along the curve C. Now, the curve C' divides the surface S into two parts S_1 and S_2.

S_1 and S_2 are called the lower and upper surfaces.

Let $z = f_1(x, y)$ and $z = f_2(x, y)$ be the equations of S_1 and S_2, respectively.

The projection of S on the xy-plane is the region R bounded by C.

Now consider the triple integral $\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz$ over the volume V enclosed by S.

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial F_3}{\partial z}\right] dx\,dy$$

$$= \iint_R \left[F_3(x,y,z)\right]_{z=f_1(x,y)}^{z=f_2(x,y)} dx\,dy$$

$$= \iint_R \left[F_3\left(x,y,f_2(x,y)\right) - F_3\left(x,y,f_1(x,y)\right)\right] dx\,dy$$

$$\Rightarrow \quad \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R F_3\left(x,y,f_2(x,y)\right) dx\,dy - \iint_R F_3\left(x,y,f_1(x,y)\right) dx\,dy \qquad (1)$$

Let a line parallel to the z-axis meet S_1 at the point P and S_2 at the point Q. Let dS_1 and dS_2 be element surface at P and Q, respectively and their projections in the xy-plane be $dx\,dy$.

Let $\vec{n}_1$ be the outward unit normal at P to S_1 and $\vec{n}_2$ be the outward unit normal at Q to S_2.

Let the angle between $\vec{n}_2$ and $\vec{k}$ be $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_2$ is acute, since $\vec{k}$ is unit vector in the direction of the positive z-axis.

Then $$dx\,dy = \cos\gamma_2\,dS_2 = \vec{k}\cdot\vec{n}_2\,d\,S_2$$

Let the angle between $\vec{n}_1$ and $\vec{k}$ be $\boldsymbol{\gamma}_1$ and it is obtuse. [$\because$ $\vec{k}$ is upward and $\vec{n}_1$ is downward]

$\therefore$ $$dx\,dy = -\cos\gamma_1\,dS_1 = -\vec{k}\cdot\vec{n}_1\,d\,S_1$$

Hence, $$\iint_R F_3\left(x,y\,f_2(x,y)\right) dx\,dy = \iint_{S_2} F_3\vec{k}\cdot\vec{n}_2\;dS_2$$

and $$\iint_R F_3\left(x,y\,f_1(x,y)\right) dx\,dy = -\iint_{S_1} F_3\vec{k}\cdot\vec{n}_1\;dS_1$$

Substituting in (1), we get

$$\iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_{S_2} F_3\vec{k}\cdot\vec{n}_2\;dS_2 + \iint_{S_1} F_3\vec{k}\cdot\vec{n}_1\;dS_1$$

$$\Rightarrow \qquad \iiint_V \frac{\partial F_3}{\partial z} dx\, dy\, dz = \iint_S F_3 \vec{k} \cdot \vec{n}\ dS \tag{2}$$

Similarly, projecting S on the yz- and zx-planes, we get

$$\iiint_V \frac{\partial F_2}{\partial y} dx\, dy\, dz = \iint_S F_2 \vec{j} \cdot \vec{n}\ dS \tag{3}$$

and

$$\iiint_V \frac{\partial F_1}{\partial x} dx\, dy\, dz = \iint_S F_1 \vec{i} \cdot \vec{n}\ dS \tag{4}$$

Adding equations (2), (3) and (4), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \cdot \vec{n}\ dS$$

$$\Rightarrow \qquad \iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS$$

9.12.1 Results Derived from Gauss Divergence Theorem

The following results are immediate consequence of Gauss divergence theorem:

(1) $\iint_S \phi\, \vec{n}\, dS = \iiint_V \nabla\ \phi\, dV$ (2) $\iint_S \vec{F} \times \vec{n}\, dS = -\iiint_V \nabla \times \vec{F}\, dV$

where ϕ is the scalar point function defined in the region V enclosed by the closed surface S.

Solution.

(1) $\iint_S \phi\, \vec{n}\, dS = \iiint_V \nabla\ \phi\, dV$.

Gauss divergence theorem is

$$\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \vec{n}\, dS \tag{1}$$

Let $\vec{F} = \phi \vec{a}$, where $\vec{a}$ is an arbitrary constant vector.

$\therefore$ (1) becomes $\iiint_V \left(\nabla \cdot \phi\ \vec{a} \right) dS = \iint_S \phi\ \vec{a} \cdot \vec{n}\, dS$ (2)

Now, $\nabla \cdot \phi \vec{a} = \nabla \phi \cdot \vec{a} + \phi (\nabla \cdot \vec{a}) = \nabla \phi \cdot \vec{a}$ $[\because\ \nabla \cdot \vec{a} = 0]$

$$\therefore \qquad \iiint_V \left(\nabla \cdot \phi \vec{a} \right) dV = \iiint_V \left(\nabla \phi \cdot \vec{a} \right) dV \tag{3}$$

and

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \phi \vec{a} \cdot \vec{n}\, dS = \iint_S \phi\ \vec{n}\, dS \cdot \vec{a} \tag{4}$$

$\therefore$ Using (3) and (4) in (2), we get

$$\iiint_V \nabla \phi \cdot \vec{a}\, dV = \iint_S \phi \vec{n}\, dS \cdot \vec{a}$$

$$\Rightarrow \qquad \vec{a}\cdot\iiint_V \nabla\phi\, dV = \vec{a}\cdot\iint_S \phi\,\vec{n}\, dS$$

$$\Rightarrow \qquad \iiint_V \nabla\phi\, dV = \iint_s \phi\ \vec{n}\, dS \qquad [\because \vec{a} \text{ is arbitrary}]$$

2. $\iint_S \vec{F}\times\vec{n}\, dS = -\iiint_V \nabla\times\vec{F}\, dV$

Gauss divergence theorem is $\iiint_V \nabla\cdot\vec{F}\, dV = \iint_S \vec{F}\cdot\vec{n}\, dS$ (1)

Let $\vec{F} = \vec{a}\times\vec{F}$, where a is an arbitrary constant vector.

$$\therefore \qquad \nabla\cdot\vec{F} = \nabla\cdot(\vec{a}\times\vec{F}) = \vec{F}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{F}) = -\vec{a}\cdot(\nabla\times\vec{F}) \qquad [\because \nabla\times\vec{a} = 0]$$

and
$$\vec{F}\cdot\vec{n} = \vec{a}\times\vec{F}\cdot\vec{n} = \vec{a}\cdot(\vec{F}\times\vec{n})$$

$\therefore$ (1) becomes $-\iiint_V \vec{a}\cdot(\nabla\times\vec{F})\, dV = \iint_S (\vec{a}\cdot\vec{F}\times\vec{n})\, dS$

$$\Rightarrow \qquad -\vec{a}\cdot\iiint_V \nabla\times\vec{F}\, dV = \vec{a}\cdot\iint_S \vec{F}\times\vec{n}\, dS$$

$$\Rightarrow \qquad -\iiint_V \nabla\times\vec{F}\, dV = \iint_S \vec{F}\times\vec{n}\, dS$$

$$\Rightarrow \qquad \iint_S \vec{F}\times\vec{n}\, dS = -\iiint_V \nabla\times\vec{F}\, dV\cdot$$

If S is closed surface, then prove that Eq.

(1) $\iint_S d\vec{S} = \iiint_V \nabla\cdot\vec{n}\, dV$ (2) $\iint_S d\vec{S} = 0$ (3) $\iint_S \vec{r}\times\vec{n}\, dS = 0$

(4) $\iiint_V (\nabla\times\vec{n})\, dV = 0$ (5) $\iint_S \frac{\vec{r}}{r^3}\cdot\vec{n}\, dS = 0$ (6) $\iint_S r^4\vec{n}\ dS = 4\iiint_V r^4\vec{r}\ dV$

(7) $\iint_S f(r)\vec{r}\times\vec{n}\, dS = 0$ (8) $\iint_S (\nabla r^2\cdot\vec{n})\, dS = 6V$ (9) $\iint_S (\nabla\times\vec{r})\cdot\vec{n}\, dS = 0$

Solution.

(i) To prove $\iint_S dS = \iiint_V \nabla\cdot\vec{n}\, dV$.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV$ (1)

Let $\vec{F} = \vec{n}$ $\quad\therefore\quad$ $\nabla\cdot\vec{F} = \nabla\cdot\vec{n}$ and $\vec{F}\cdot\vec{n} = \vec{n}\cdot\vec{n} = 1$

$\therefore$ (1) becomes
$$\iint_S dS = \iiint_V \nabla\cdot\vec{n}\, dV.$$

(2) To prove $\iint_S dS = 0$.

We have $\iiint_V \nabla\phi\, dV = \iint_S \phi\ \vec{n}\, dS$ (1)

Let $\qquad \boldsymbol{\phi} = 1.$

$\therefore \qquad \nabla\boldsymbol{\phi} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(1) = 0 \quad \therefore \quad \iiint_V \nabla\boldsymbol{\phi}\, dV = 0$

$\therefore \qquad \iint_S \vec{n}\, dS = 0 \quad \Rightarrow \quad \iint_S dS = 0$ [using (1)]

(3) To prove $\boldsymbol{\iint_S \vec{r} \times \vec{n}\, dS = 0.}$

We have $\iint_S \vec{F} \times \vec{n}\, dS \equiv -\iiint_V \nabla \times \vec{F}\, dV$

Let $\qquad \vec{F} = \vec{r}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \qquad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = 0$$

$\therefore \qquad \iiint_V \nabla \times \vec{F}\, dv = 0 \quad \therefore \iint_S \vec{F} \times \vec{n}\, dS = 0 \Rightarrow \iint_S \vec{r} \times \vec{n}\, dS = 0$ [using (1)]

(4) To prove $\boldsymbol{\iint_V \nabla \times \vec{n}\, dV = 0.}$

We have $\iint_S \vec{F} \times \vec{n}\, dS = -\iiint_V \nabla \times \vec{F}\, dV$ (1)

Let $\qquad \vec{F} = \vec{n} \qquad \therefore \quad \vec{F} \times \vec{n} = \vec{n} \times \vec{n} = 0$

$\therefore \qquad \iint_S \vec{F} \times \vec{n}\, dS = 0 \quad \therefore \quad \iiint_V \nabla \times \vec{F}\, dV = 0 \quad \Rightarrow \quad \iiint_V \nabla \times \vec{n}\, dV = 0$ [using (1)]

(5) To prove $\boldsymbol{\iint_S \frac{\vec{r}}{r^3} \cdot \vec{n}\, dS = 0.}$

Gauss divergence theorem is $\iint_S \vec{F}.\vec{n}\, dS = \iiint_V \Delta \cdot \vec{F}\, dS$ (1)

Let $\vec{F} = \dfrac{\vec{r}}{r^3} \quad \therefore \quad \nabla \cdot \vec{F} = \nabla \cdot \left(\dfrac{\vec{r}}{r^3}\right) = \dfrac{1}{r^3}(\nabla \cdot \vec{r}) + \nabla\left(\dfrac{1}{r^3}\right) \cdot \vec{r}$

$\therefore \qquad \nabla \cdot \vec{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

Now $\qquad r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \quad 2r\dfrac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

Similarly, $\qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\therefore \qquad \nabla\left(\frac{1}{r^3}\right) = \nabla(r^{-3}) = \vec{i}\,\frac{\partial}{\partial x}(\overline{r}^3) + \vec{j}\,\frac{\partial}{\partial y}(\overline{r}^3) + \vec{k}\,\frac{\partial}{\partial z}(\overline{r}^3)$$

$$= \vec{i}\,(-3)\overline{r}^4\,\frac{\partial r}{\partial x} + \vec{j}\,(-3)\overline{r}^4\,\frac{\partial r}{\partial y} + \vec{k}\,(-3)\overline{r}^4\,\frac{\partial r}{\partial z}$$

$$= -\frac{3}{r^4}\left[\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right] = -\frac{3}{r^5}[x\vec{i} + y\vec{j} + z\vec{k}] = -\frac{3}{r^4}\vec{r}$$

$$\therefore \qquad \nabla\cdot\vec{F} = \nabla\cdot\left(\frac{\vec{r}}{r^3}\right) = \frac{3}{r^3} - \frac{3}{r^5}(\vec{r}\cdot\vec{r}) = \frac{3}{r^3} - \frac{3}{r^3} = 0$$

$$\therefore \qquad \iiint_V \nabla\cdot\vec{F}\,dV = 0 \quad\Rightarrow\quad \iint_S \vec{F}\cdot\vec{n}\,dS = 0 \qquad \text{[using (1)]}$$

$$\therefore \qquad \iint_S \frac{r}{r^3}\cdot\vec{n}\,dS = \iiint_V\left(\nabla\cdot\frac{\vec{r}}{r^3}\right)dS = 0.$$

(6) To prove $\iint_S r^4\vec{n}\,dS = 4\iiint_V r^2\vec{r}\,dV$.

We have $\quad \iiint_V \nabla\phi\,dV = \iint_S \phi\,\vec{n}\,dS$

Let $\phi = r^4 \quad \therefore \quad \nabla\phi = \vec{i}\,\frac{\partial}{\partial x}(r^4) + \vec{j}\,\frac{\partial}{\partial y}(r^4) + \vec{k}\,\frac{\partial}{\partial z}(r^4)$

$$= 4r^3\cdot\frac{x}{r}\vec{i} + 4r^3\cdot\frac{y}{r}\vec{j} + 4r^3\cdot\frac{z}{r}\vec{k} = 4r^2[x\vec{i} + y\vec{j} + z\vec{k}] = 4r^2\vec{r}$$

$\therefore$ (1) becomes, $\iiint_V 4r^2\vec{r}\,dV = \iint_S r^4\,\vec{n}\,dS \quad\Rightarrow\quad 4\iiint_V r^2\vec{r}\,dV = \iint_S r^4\,\vec{n}\,dS$

(7) To prove: $\iint_S f(r)\vec{r}\times\vec{n}\,dS = 0$.

We have $\quad \iint_S \vec{F}\times\vec{n}\,dS = -\iiint_V \nabla\times\vec{F}\,dV \qquad (1)$

Let $\quad \vec{F} = f(r)\vec{r}, \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \qquad \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(x)x & f(x)y & f(x)z \end{vmatrix} = \sum\vec{i}\left[\frac{\partial}{\partial y}f(r)z - \frac{\partial}{\partial z}f(r)y\right]$$

Now $\left[\dfrac{\partial}{\partial y}f(r)z - \dfrac{\partial}{\partial z}f(r)y\right] = \left[f(r)\dfrac{\partial z}{\partial y} + zf'(r)\dfrac{\partial r}{\partial y}\right] - \left[f(r)\dfrac{\partial y}{\partial z} + y\,f'(r)\dfrac{\partial r}{\partial z}\right]$

$$= 0 + z\,f'(r)\frac{y}{r} - 0 - yf'(r)\cdot\frac{z}{r} = \frac{f'(r)}{r}[yz - yz] = 0$$

$\therefore \qquad \vec{i}\left[\dfrac{\partial}{\partial y}(f(r)z)-\dfrac{\partial}{\partial z}(f(r)y)\right]=0$

Similarly, $\vec{j}\left[\dfrac{\partial}{\partial x}(f(r)z)-\dfrac{\partial}{\partial z}(f(r)y)\right]=0$ and $\vec{k}\left[\dfrac{\partial}{\partial x}(f(r)y)-\dfrac{\partial}{\partial z}(f(r)x)\right]=0$

$\therefore \qquad \nabla\times\vec{F}=0 \quad \therefore \quad \iiint_V \nabla\times\vec{F}\,dV=0$

$\therefore$ (1) becomes $\qquad \iint_S \vec{F}\times\vec{n}\,dS=0 \quad \Rightarrow \quad \iint_S f(r)\vec{r}\times\vec{n}\,dS=0$

(8) To prove $\iint_S (\nabla r^2\cdot\vec{n})\,dS=6V$.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS=\iiint_V \nabla\cdot\vec{F}\,dV$ (1)

Let $\vec{F}=\nabla r^2$ and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k} \quad \Rightarrow \quad r^2=x^2+y^2+z^2$

$\therefore \qquad \nabla\cdot\vec{F}=\nabla\cdot\nabla r^2$

Now, $\qquad \nabla r^2=\vec{i}\dfrac{\partial}{\partial x}(x^2+y^2+z^2)+\vec{j}\dfrac{\partial}{\partial y}(x^2+y^2+z^2)+\vec{k}\dfrac{\partial}{\partial z}(x^2+y^2+z^2)$

$$=2x\vec{i}+2y\vec{j}+2z\vec{k}=2[x\vec{i}+y\vec{j}+z\vec{k}]=2\vec{r}$$

$\therefore \qquad \nabla\cdot\nabla r^2=\nabla\cdot 2\vec{r}=2\nabla\cdot\vec{r}$

But $\qquad \nabla\cdot\vec{r}=\left(\vec{i}\dfrac{\partial}{\partial x}+\vec{j}\dfrac{\partial}{\partial y}+\vec{k}\dfrac{\partial}{\partial z}\right)\cdot(x\vec{i}+y\vec{j}+z\vec{k})=1+1+1=3$

$\therefore \qquad \nabla\cdot\nabla r^2=2\cdot 3=6 \quad \therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV=\iiint_V 6\,dV=6V$

$\therefore$ (1) becomes $\iint_S \vec{F}\cdot\hat{n}\,dS=6V \quad \Rightarrow \quad \iint_S \nabla r^2\cdot\vec{n}\,dS=6V$

WORKED EXAMPLES

EXAMPLE 1

Let V be the region bounded by a closed surface S. Let f and g be scalar point functions that together with their derivatives in any directions are uniformly continuous within the region V. Then

$$\iiint_V (f\nabla^2 g-g\nabla^2 f)\,dV=\iint_S (f\nabla g-g\nabla f)\cdot\vec{n}\,dS.$$

Solution.

Gauss divergence theorem is

$$\iiint_V \nabla\cdot\vec{F}\,dV=\iint_S \vec{F}\cdot\vec{n}\,dS$$

Put $\vec{F} = f\nabla g$ $\therefore$ $\nabla\cdot\vec{F} = \nabla\cdot(f\nabla g) = f(\nabla\cdot\nabla g) + \nabla f\cdot\nabla g = f\nabla^2 g + \nabla f\cdot\nabla g$

and $\vec{F}\cdot\vec{n} = (f\nabla g)\cdot\vec{n}$

$\therefore$ by divergence theorem becomes

$$\iiint_V (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV = \iint_S (f\nabla g\cdot\vec{n})\,dS \tag{1}$$

Interchanging f and g, we get

$$\iiint_V (g\nabla^2 f + \nabla g\cdot\nabla f)\,dV = \iint_S (g\nabla f\cdot\vec{n})\,dS \tag{2}$$

(1) − (2) ⇒
$$\iiint_V (f\nabla^2 g - g\nabla^2 f)\,dV = \iint_S (f\nabla g - g\nabla f)\cdot\vec{n}\,dS \tag{3}$$

Note This result is known as **Green's theorem**.
Equation (1) is called Green's first identity and equation (3) is called Green's second identity.

EXAMPLE 2

Prove that $\iiint_V \frac{1}{r^2}\,dV = \iint_S \frac{\vec{r}}{r^2}\cdot\vec{n}\,dS$.

Solution.

Gauss divergence theorem is

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\vec{n}\,dS \tag{1}$$

Put $\vec{F} = \frac{\vec{r}}{r^2} = r^{-2}\vec{r}$. Then $\nabla\cdot\vec{F} = \nabla\cdot(r^{-2}\vec{r}) = (\nabla\cdot\vec{r})r^{-2} + \nabla r^{-2}\cdot\vec{r}$

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

and $r^2 = x^2 + y^2 + z^2$

$\therefore$ $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla r^{-2} = \vec{i}\frac{\partial}{\partial x}(r^{-2}) + \vec{j}\frac{\partial}{\partial y}(r^{-2}) + \vec{k}\frac{\partial}{\partial z}(r^{-2})$$

$$= \vec{i}(-2)r^{-3}\frac{\partial r}{\partial x} + \vec{j}(-2)r^{-3}\frac{\partial r}{\partial y} + \vec{k}(-2)r^{-3}\frac{\partial r}{\partial z}$$

$$= -2r^{-3}\frac{x}{r}\vec{i} - 2r^{-3}\frac{y}{r}\vec{j} - 2r^{-3}\frac{z}{r}\vec{k} = \frac{-2}{r^4}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{2\vec{r}}{r^4}$$

$\therefore$
$$\nabla\cdot\vec{F} = 3r^{-2} + \left(\frac{-2}{r^4}\vec{r}\cdot\vec{r}\right) = \frac{3}{r^2} - \frac{2}{r^4}\times r^2 = \frac{3}{r^2} - \frac{2}{r^2} = \frac{1}{r^2}$$

$\therefore$ (1) becomes $\iiint_V \frac{1}{r^2}\,dV = \iint_S \frac{\vec{r}}{r^2}\cdot\vec{n}\,dS$

EXAMPLE 3

Using divergence theorem, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $\qquad \vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\therefore \qquad \nabla\cdot\vec{F} = \dfrac{\partial}{\partial x}(4xz) + \dfrac{\partial}{\partial y}(-y^2) + \dfrac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$

$\therefore$

$$\iint \vec{F}\cdot\vec{n}\,dS = \int_0^2\int_0^2\int_0^2 (4z - y)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 (4z - y)[x]_0^2\,dy\,dz$$

$$= \int_0^2\int_0^2 (4z - y)2\,dy\,dz$$

$$= 2\int_0^2 \left[4zy - \frac{y^2}{2}\right]_0^2 dz$$

$$= 2\int_0^2 \left(4z\cdot 2 - \frac{4}{2}\right)dz = 2\cdot\int_0^2 (8z - 2)\,dz$$

$$= 2\left[\frac{8z^2}{2} - 2z\right]_0^2 = 2\left[8\cdot\frac{4}{2} - 2\cdot 2\right] = 2[16 - 4] = 2\times 12 = 24.$$

EXAMPLE 4

Using Gauss divergence theorem, evaluate $\iint_S \vec{F}\cdot\vec{n}\,dS$ where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and S is the sphere $x^2 + y^2 + z^2 = a^2$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$

Given $\qquad \vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$

$\therefore \qquad \nabla\cdot\vec{F} = \dfrac{\partial}{\partial x}(x^3) + \dfrac{\partial}{\partial y}(y^3) + \dfrac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2)$

$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V 3(x^2 + y^2 + z^2)\,dx\,dy\,dz$

We shall evaluate this triple integral by using spherical polar coordinates.
$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$

then $$dx\,dy\,dz = \left|\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}\right| dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

and $x^2 + y^2 + z^2 = r^2$
Here r varies from 0 to a, θ varies from 0 to π and ϕ varies from 0 to 2π.

$$\therefore \quad \iint_S \vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3r^4\sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^{a} r^4\,dr$$

$$= 3\left[\phi\right]_0^{2\pi}\left[-\cos\theta\right]_0^{\pi}\left[\frac{r^5}{5}\right]_0^{a}$$

$$= 3\cdot 2\pi(-\cos\pi + \cos 0)\cdot\frac{a^5}{5}$$

$$= 6\pi\cdot 2\cdot\frac{a^5}{5} = \frac{12\pi}{5}a^5$$

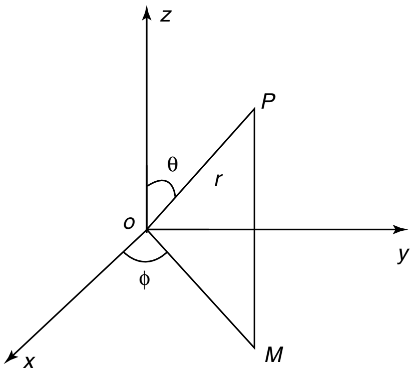

Fig. 9.21

Note We have $\operatorname{div}\vec{F} = 3(x^2 + y^2 + z^2)$. Since the equation of the surface is $x^2 + y^2 + z^2 = a^2$, we cannot take $\operatorname{div}\vec{F} = 3a^2$ because $\vec{F}$ is defined in the volume inside and on S.
But $x^2 + y^2 + z^2 = a^2$ is true only for points on S.

EXAMPLE 5

Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0$, $x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution.
Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Given $$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz \qquad [\because\ dV = dx\,dy\,dz]$$

$$= \int_0^1\int_0^1 (4z - y)\left[x\right]_0^1 dy\,dz = \int_0^1\int_0^1 [4z - y]\,dy\,dz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right] dz = \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow \qquad \iiint_V \nabla \cdot \vec{F}\, dv = \frac{3}{2} \tag{1}$$

We shall now evaluate $\iint_S \vec{F} \cdot \vec{n}\, dS$

Here the surface S consists of the six faces of the cube.

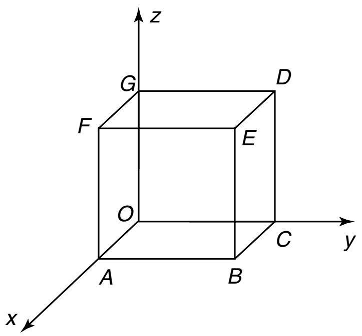

Fig. 9.22

$$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}\,dS + \iint_{S_3} \vec{F}\cdot\vec{n}\,dS + \iint_{S_4} \vec{F}\cdot\vec{n}\,dS + \iint_{S_5} \vec{F}\cdot\vec{n}\,dS + \iint_{S_6} \vec{F}\cdot\vec{n}\,dS$$

We shall simplify the computation and put it in the form of a table.

Face	Equation	Outward normal $\vec{n}$	$\vec{F}\cdot\vec{n}$	dS
S_1 = ABEF	$x = 1$	$\vec{i}$	$4xz = 4z$	$dy\,dz$
S_2 = OCDG	$x = 0$	$-\vec{i}$	$-4xz = 0$	$dy\,dz$
S_3 = BCDE	$y = 1$	$\vec{j}$	$-y^2 = -1$	$dx\,dz$
S_4 = OAFG	$y = 0$	$-\vec{j}$	$y^2 = 0$	$dx\,dz$
S_5 = DEFG	$z = 1$	$\vec{k}$	$yz = y$	$dx\,dy$
S_6 = OABC	$z = 0$	$-\vec{k}$	$-yz = 0$	$dx\,dy$

$$\therefore \qquad \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 4z\,dy\,dz = 4[y]_0^1 \left[\frac{z^2}{2}\right]_0^1 = 4\cdot 1\cdot\frac{1}{2} = 2$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} 0\,dy\,dz = 0$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 -1\,dx\,dz = -[x]_0^1[z]_0^1 = -1$$

$$\iint_{S_4} \vec{F}\cdot\vec{n}\,dS = \iint_{S_4} 0\,dx\,dz = 0$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,dS = \int_0^1\int_0^1 y\,dx\,dy = [x]_0^1\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$$

and

$$\iint_{S_6} \vec{F}\cdot\vec{n}\,dS = \iint_{S_6} 0\,dx\,dy = 0$$

$\therefore \qquad \iint_S \vec{F}\cdot\vec{n}\,dS = 2+0+(-1)+0+\frac{1}{2}+0=\frac{3}{2}$ (2)

From (1) and (2), $\qquad \iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 6

Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by the planes $x = \pm 1$, $y = \pm 1, z = \pm 1$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$

Given $\qquad \vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\therefore \qquad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial y}(z)+\frac{\partial}{\partial z}(yz) = 2x+0+y = 2x+y$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x+y)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[x^2+yx\right]_{-1}^{1} dy\,dz = \int_{-1}^{1}\int_{-1}^{1}\left[1+y-(1-y)\right]dy\,dz = \int_{-1}^{1}\int_{-1}^{1} 2y\,dy\,dz = 0$$

$$\left[\int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd function, Here } y \text{ is odd function}\right]$$

$$\Rightarrow \quad \iiint_V \nabla\cdot\vec{F}dV = 0 \qquad (1)$$

We shall now compute $\iint_S \vec{F}\cdot\vec{n}\,dS$

S is the surface consisting of the six faces of the cube.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \vec{F}\cdot\vec{n}\,dS$$
$$+ \iint_{S_3} \vec{F}\cdot\vec{n}\,dS + \iint_{S_4} \vec{F}\cdot\vec{n}\,dS$$
$$+ \iint_{S_5} \vec{F}\cdot\vec{n}\,dS + \iint_{S_6} \vec{F}\cdot\vec{n}\,dS$$

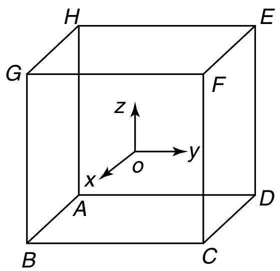

Fig. 9.23

We shall simplify the computations and put it in the form of a table.

Faces	Equation	Outward normal $\vec{n}$	$\vec{F}\cdot\vec{n}$	dS
$S_1 = $ BCFG	$x = 1$	$\vec{i}$	$x^2 = 1$	$dy\,dz$
$S_2 = $ ADEH	$x = -1$	$-\vec{i}$	$-x^2 = -1$	$dy\,dz$
$S_3 = $ CDEF	$y = 1$	$\vec{j}$	z	$dz\,dx$
$S_4 = $ ABGH	$y = -1$	$-\vec{j}$	$-z$	$dz\,dx$
$S_5 = $ EFGH	$z = 1$	$\vec{k}$	$yz = y$	$dx\,dy$
$S_6 = $ ABCD	$z = -1$	$-\vec{k}$	$-yz = y$	$dx\,dy$

$$\therefore \quad \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = [y]_{-1}^{1}[z]_{-1}^{1} = (1+1)(1+1) = 4$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} -1dy\,dz = -[y]_{-1}^{1}[z]_{-1}^{1} = -[1+1][1+1] = -4$$

$$\iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} zdz\,dx = 0 \qquad [\because z \text{ is odd function}]$$

$$\iint_{S_4} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} -zdz\,dx = -\int_{-1}^{1}\int_{-1}^{1} zdz\,dx = 0$$

$$\iint_{S_5} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy = 0 \qquad [\because y \text{ is odd function}]$$

$$\iint_{S_6} \vec{F}\cdot\vec{n}\,dS = \int_{-1}^{1}\int_{-1}^{1} y\,dx\,dy = 0$$

$$\therefore \quad \iint_{S} \vec{F}\cdot\vec{n}\,dS = 4-4+0+0+0+0 = 0 \qquad (2)$$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 7

Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface of the region, bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0, z = 3$.

Solution.

Gauss divergence theorem is $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \text{div } \vec{F}dV$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k} \quad \therefore \quad \nabla\cdot\vec{F} = \dfrac{\partial}{\partial x}(4x) + \dfrac{\partial}{\partial y}(-2y^2) + \dfrac{\partial}{\partial z}(z^2)$

$$= 4 - 4y + 2z$$

and z varies from 0 to 3,

Also given $x^2 + y^2 = 4$

$$\Rightarrow \qquad y^2 = 4 - x^2 \quad \Rightarrow \quad y = \pm\sqrt{4-x^2}$$

$$\text{and} \qquad y = 0 \quad \Rightarrow x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dz\,dy\,dx$$

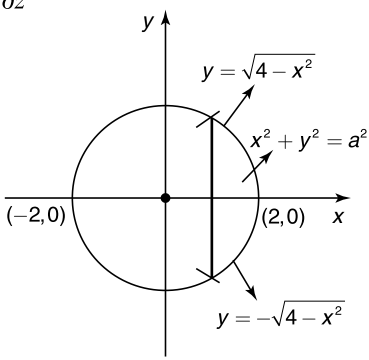

Fig. 9.24

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[(4-4y)z + 2\frac{z^2}{2} \right]_0^3 dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4-4y)\cdot 3 + 9]\,dy\,dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [21-12y]\,dy\,dx$$

$$= \int_{-2}^{2} \left[21y - 12\frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} [21\left(\sqrt{4-x^2} + \sqrt{4-x^2}\right) - 6(4-x^2-(4-x^2))]dx$$

$$= \int_{-2}^{2} 42\sqrt{4-x^2}\,dx$$

$$= 84\int_{0}^{2} \sqrt{4-x^2}\,dx \qquad \left[\because \sqrt{4-x^2} \text{ is even function}\right]$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2 = 84\,[0 + 2\sin^{-1}1 - 0] = 84\cdot 2\,\frac{\pi}{2} = 84\pi$$

$$\iiint_V \nabla\cdot\vec{F}dV = 84\,\pi \tag{1}$$

We shall now compute the surface integral $\iint_S \vec{F}\cdot\,\vec{n}\,dS$.

S consists of the bottom surface S_1, top surface S_2 and the curved surface S_3 of the cylinder.

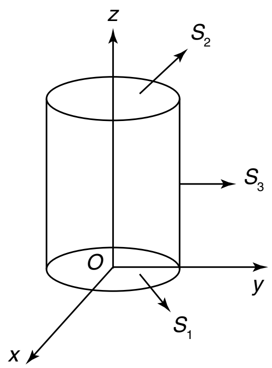

Fig. 9.25

On S_1: Equation is $z = 0$, $\vec{n} = -\vec{k}$

$$\therefore \qquad \vec{F}\cdot\vec{n} = -z^2 = 0 \quad \Rightarrow \quad \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = 0$$

On S_2: Equation is $z = 3$, $\vec{n} = \vec{k}$

$$\therefore \qquad \vec{F}\cdot\vec{n} = z^2 = 9, \quad dS = \frac{dx\,dy}{\left|\vec{n}\cdot\vec{k}\right|} = \frac{dx\,dy}{\left|\vec{k}\cdot\vec{k}\right|} = dx\,dy$$

$$\therefore \qquad \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S_2} 9\,dx\,dy = 9\iint_{S_2} dx\,dy$$

$$= 9\text{ (area of the circle } S_2) = 9\,\pi\,2^2 = 36\pi.$$

On S_3: Equation of the cylinder is $x^2 + y^2 = 4$

Let $\phi = x^2 + y^2$

$$\therefore \quad \nabla\phi = \vec{i}\frac{\partial}{\partial x}\phi + \vec{j}\frac{\partial}{\partial y}\phi + \vec{k}\frac{\partial\phi}{\partial z} = \vec{i}\,2x + 2y\vec{j} + 0k = 2(x\vec{i} + y\vec{j})$$

$$\therefore \text{ the normal } \quad \vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{x^2 + y^2}} = \frac{2(x\vec{i} + y\vec{j})}{2\sqrt{4}} = \frac{1}{2}(x\vec{i} + y\vec{j})$$

$$\therefore \quad \vec{F}\cdot\vec{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\frac{1}{2}(x\vec{i} + y\vec{j}) = 2x^2 - y^3$$

Since S_3 is the surface of a cylinder $x^2 + y^2 = 4$, we use cylindrical polar coordinates to evaluate $\iint_{S_3} \vec{F}\cdot\vec{n}\,dS$

$\therefore \; x = 2\cos\theta, \quad y = 2\sin\theta, \quad z = z \quad \therefore \quad dS = 2\,d\theta\,dz$

θ varies from 0 to 2π and z varies from 0 to 3

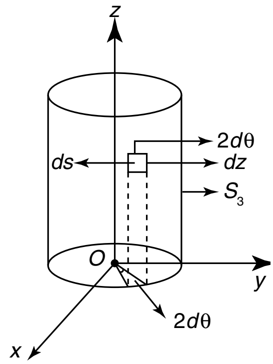

Fig. 9.26

$$\therefore \quad \iint_{S_3} \vec{F}\cdot\vec{n}\,dS = \int_0^3\int_0^{2\pi} (2\cdot 4\cos^2\theta - 8\sin^3\theta)\,2d\theta\,dz$$

$$= 16\int_0^3\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta\,dz$$

$$= 16\int_0^3\int_0^{2\pi}\left[\frac{1+\cos 2\theta}{2} - \frac{1}{4}(3\sin\theta - \sin 3\theta)\right]d\theta\,dz$$

$$= 16\int_0^3\left[\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right)\right]_0^{2\pi} dz$$

$$= 16\int_0^3\left\{\frac{1}{2}\left[2\pi + \frac{\sin 4\pi}{2} - 0\right] - \frac{1}{4}\left[-3\cos 2\pi + \frac{\cos 6\pi}{3} - \left(-3\cos 0 + \frac{\cos 0}{3}\right)\right]\right\}dz$$

$$= 16\int_0^3\left(\pi + \frac{3}{4} - \frac{1}{12} - \frac{3}{4} + \frac{1}{12}\right)dz$$

$$= 16\pi\int_0^3 dz = 16\pi[z]_0^3 = 16\pi\times 3 = 48\pi$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = 36\pi + 48\pi = 84\pi \qquad (2)$$

From (1) and (2), $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 8

Verify Gauss divergence theorem for $\vec{F} = a(x + y)\vec{i} + a(y - x)\vec{j} + z^2\vec{k}$ over the region bounded by the upper hemisphere $x^2 + y^2 + z^2 = a^2$ and the plane $z = 0$.

Solution.

Gauss divergence theorem is

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

Given $\quad \vec{F} = a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}$

$$\therefore \quad \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(a(y-x)) + \frac{\partial}{\partial z}(z^2) = a + a + 2z = 2(a+z)$$

$$\therefore \quad \iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V 2(a+z)\,dV$$

$$= 2a\iiint_V dV + 2\iiint_V z\,dV$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} z\,dz\,dy\,dx$$

$$= 2aV + 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 2a\frac{2\pi}{3}a^3 + \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a^2-x^2-y^2)\,dy\,dx \qquad \left[\because V = \frac{2}{3}\pi a^3\right]$$

$$= \frac{4\pi a^4}{3} + \int_{-a}^{a} 2\int_0^{\sqrt{a^2-x^2}}(a^2-x^2-y^2)\,dy\,dx \qquad [\because a^2-x^2-y^2 \text{ is even in } y]$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)y - \frac{y^3}{3}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)\sqrt{a^2-x^2} - \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

$$= \frac{4\pi a^4}{3} + 2\int_{-a}^{a}\left[(a^2-x^2)^{3/2} - \frac{(a^2-x^2)^{3/2}}{3}\right] dx$$

$$= \frac{4\pi a^4}{3} + 2\cdot\frac{2}{3}\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{4\pi a^4}{3} + \frac{4}{3}\times 2\int_0^a (a^2-x^2)^{3/2}\,dx = \frac{4\pi a^4}{3} + \frac{8}{3}I \qquad [\because (a^2-x^2)^{3/2} \text{ is even}]$$

where $\quad I = \int_0^a (a^2-x^2)^{3/2}\,dx$

Put $\quad x = a\sin\theta \quad \therefore \quad dx = a\cos\theta\,d\theta$

When $x = 0$, $\quad \sin\theta = 0 \Rightarrow \theta = 0$ and when $x = a$, $\quad \sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{3/2} a \cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} a^3 \cos^3 \theta \cdot a \cos\theta d\theta$$

$$= a^4 \int_0^{\pi/2} \cos^4 \theta \, d\theta = a^4 \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2} = a^4 \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi a^4}{16}$$

$$\iiint_V \nabla \cdot \vec{F} = \frac{4\pi a^4}{3} + \frac{8}{3} \cdot \frac{3\pi a^4}{16} = \frac{(8+3)}{6} \pi a^4 = \frac{11}{6} \pi a^4 \qquad (1)$$

Now we shall compute the double integral $\iint_S \vec{F} \cdot \vec{n} \, dS$

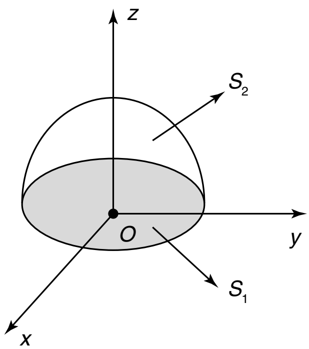

Fig. 9.27

S consists of S_1 and S_2

$$\therefore \qquad \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_{S_1} \vec{F} \cdot \vec{n} \, dS_1 + \iint_{S_2} \vec{F} \cdot \vec{n} \, dS_2$$

On S_1: $z = 0$, $\vec{n} = -\vec{k}$

$$\therefore \quad \vec{F} \cdot \vec{n} = (a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}) \cdot (-\vec{k}) = -z^2 = 0$$

$$\therefore \quad \iint_{S_1} \vec{F} \cdot \vec{n} \, dS = 0$$

On S_2: $x^2 + y^2 + z^2 = a^2$

Let $\qquad \phi = x^2 + y^2 + z^2$

$$\therefore \qquad \nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$= 2(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\therefore \qquad \vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a} \text{ and } \vec{n} \cdot \vec{k} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a} \cdot \vec{k} = \frac{z}{a}$$

$$\vec{F} \cdot \vec{n} = [a(x+y)\vec{i} + a(y-x)\vec{j} + z^2\vec{k}] \cdot \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$= (x+y)x + (y-x)y + \frac{z^3}{a} = x^2 + y^2 + \frac{z^3}{a}$$

$$\iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iint_R \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n} \cdot \vec{k}|},$$ where R is the projection of S_2 on the xy–plane.

$$\therefore \quad \iint_{S_2} \vec{F} \cdot \vec{n} \, dS = \iint_R \left(x^2 + y^2 + \frac{z^3}{a} \right) \frac{dx\,dy}{\frac{z}{a}}$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + z^2 \right) dx\,dy$$

$$= \iint_R \left(\frac{a(x^2 + y^2)}{z} + [a^2 - x^2 - y^2] \right) dx\,dy$$

Changing to polar coordinate, we have

$$x = r\cos\boldsymbol{\theta},\quad y = r\sin\boldsymbol{\theta},\quad r^2 = x^2 + y^2 \quad \text{and} \quad dx\,dy = r\,dr\,d\boldsymbol{\theta}$$

$$\therefore \quad \iint\limits_{S_2} \vec{F}\cdot\vec{n}\,dS = \int_0^a\int_0^{2\pi}\left\{\frac{ar^2}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\boldsymbol{\theta}$$

$$= \int_0^a\int_0^{2\pi}\left\{\frac{-a(a^2-r^2)+a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\boldsymbol{\theta}$$

$$= \int_0^a\int_0^{2\pi}\left\{-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr\,d\boldsymbol{\theta}$$

$$= \int_0^{2\pi} d\boldsymbol{\theta}\int_0^a\left\{-a\sqrt{a^2-r^2} + \frac{a^3}{\sqrt{a^2-r^2}} + (a^2-r^2)\right\} r\,dr$$

$$= [\boldsymbol{\theta}]_0^{2\pi}\int_0^a (-a\sqrt{a^2-r^2})r + a^3(a^2-r^2)^{-1/2}r + (a^2-r^2)\,r]\,dr$$

$$= 2\boldsymbol{\pi}\left\{\int_0^a + \frac{a}{2}(a^2-r^2)(-2r)dr - \frac{a^3}{2}\int_0^a (a^2-r^2)^{-1/2}(-2r)dr + \int_0^a (a^2r - r^3)dr\right\}$$

$$= 2\boldsymbol{\pi}\left\{\frac{a}{2}\left[\frac{(a^2-r^2)^{3/2}}{\frac{3}{2}}\right]_0^a - \frac{a^3}{2}\left[\frac{(a^2-r^2)^{1/2}}{\frac{1}{2}}\right]_0^a + \left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^a\right\}$$

$$= 2\boldsymbol{\pi}\left[\frac{a}{3}(0-a^3) - a^3(0-a) + \frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= 2\boldsymbol{\pi}\left[-\frac{a^4}{3} + a^4 + \frac{a^4}{4}\right] = 2\boldsymbol{\pi}\times\frac{11a^4}{12} = \frac{11\boldsymbol{\pi}a^4}{6}$$

$$\therefore \quad \iint\limits_S \vec{F}\cdot\vec{n}dS = 0 + \frac{11\boldsymbol{\pi}a^4}{6} = \frac{11\boldsymbol{\pi}a^4}{6} \qquad (2)$$

From (1) and (2), $\iint\limits_S \vec{F}\cdot\vec{n}\,dS = \iiint\limits_V \nabla\cdot\vec{F}dV$

Hence, Gauss's divergence theorem is verified.

EXAMPLE 9

Evaluate $\iint\limits_S x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy$ over the surface $z = 0, z = h, x^2 + y^2 = a^2$.

Solution.

We know Gauss divergence theorem in cartesian form is

$$\iint\limits_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint\limits_V\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz$$

Given surface integral is $\iint_S x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy$

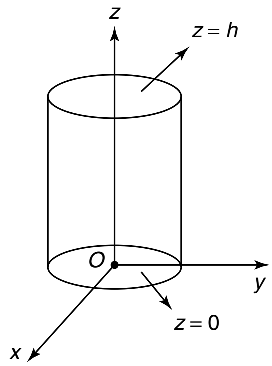

Fig. 9.28

Here $F_1 = x^3$, $\quad F_2 = x^2 y$, $\quad F_3 = x^2 z$

$$\therefore \quad \frac{\partial F_1}{\partial x} = 3x^2, \quad \frac{\partial F_2}{\partial y} = x^2, \quad \frac{\partial F_3}{\partial z} = x^2$$

$$\therefore \quad \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2 = 5x^2$$

$$\iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \iiint_V 5x^2\,dx\,dy\,dz$$

$$= 5\int_{z=0}^{h}\int_{y=-a}^{a}\int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} x^2\,dx\,dy\,dz$$

$$= 5\int_0^h\int_{y=-a}^{a}\left[2\int_0^{\sqrt{a^2-y^2}} x^2\,dx\right]dy\,dz \qquad [\because x^2 \text{ is even}]$$

$$= 10\int_{z=0}^{h}\int_{y=-a}^{a}\left[\frac{x^3}{3}\right]_0^{\sqrt{a^2-y^2}} dydz$$

$$= \frac{10}{3}\int_{z=0}^{h}\int_{y=-a}^{a}(a^2-y^2)^{3/2}\,dydz$$

$$= \frac{10}{3}\int_0^h dz\left[2\int_0^a (a^2-y^2)^{3/2}\,dy\right]$$

$$= \frac{20}{3}[z]_0^h\int_0^a (a^2-y^2)^{3/2}\,dy = \frac{20}{3}h\int_0^a (a^2-y^2)^{3/2}\,dy = \frac{20h}{3}\times I$$

where $$I = \int_0^a (a^2-y^2)^{3/2}\,dy$$

Put $\quad y = a\sin\theta \quad \therefore dy = a\cos\theta\,d\theta$

When $\quad y = 0, \quad \sin\theta = 0 \quad \Rightarrow \quad \theta = 0$ and when $y = a, \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \dfrac{\pi}{2}$

$$\therefore\ I = \int_0^{\pi/2}(a^2 - a^2\sin^2\theta)^{3/2}a\cos\theta d\theta = a^4\int_0^{\pi/2}\cos^3\theta\cos\theta d\theta$$

$$= a^4\int_0^{\pi/2}\cos^4\theta d\theta = a^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^4}{16}$$

$$\therefore\ \iint_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \frac{20}{3}h\times\frac{3\pi a^4}{16} = \frac{5}{4}\pi a^4 h$$

9.13 STOKE'S THEOREM

Stoke's theorem gives a relation between **line integral** and **surface integral**.

Theorem 9.1 If S is an open surface bounded by a simple closed curve C and if $\vec{F}$ is continuous having continuous partial derivatives in S and on C, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F}\cdot \vec{n}\, dS$,

where C is traversed in the positive direction.

Proof Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ and $\vec{r}$ be the position vector of any point P on S.

$$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \quad F\cdot d\vec{r} = \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\cdot(dx\vec{i} + dy\vec{j} + dz\vec{k}) = F_1dx + F_2dy + F_3dz$$

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \oint_C (F_1dx + F_2dy + F_3dz)$$

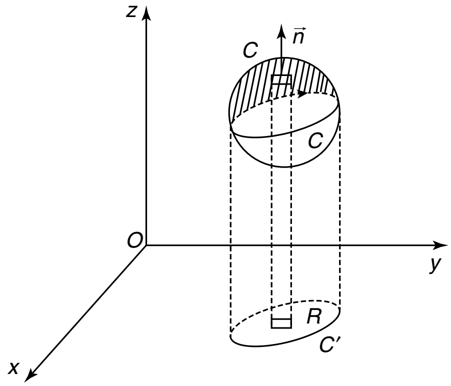

Fig. 9.29

Let $z = f(x, y)$ be the equation of the surface S enclosed by the curve C.

Any line parallel to Z-axis intersects the surface in at most one point. The positive direction of the normal $\vec{n}$ is that it makes an acute angle with the positive Z-axis (or $\vec{k}$).

The projection of S on the xy-plane is a region R enclosed by C'.

Now,
$$\oint_C F_1dx = \oint_C F_1(x, y, z)dx$$
$$= \oint_{C'} F_1\left((x, y, f(x, y)\right)dx = \oint_{C'} P(x, y)dx$$

where $\quad P(x, y) = F_1\left(x, y\, f(x, y)\right)$

By Green's theorem,

$$\oint_{C'} P(x, y)dx = \iint_R -\frac{\partial P}{\partial y}\,dx\,dy \qquad [\because\ Q = 0 \text{ here}]$$

But $\quad P(x, y) = F_1\left(x, y\, f(x, y)\right)$

$$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y} \qquad [\because P(x, y) = F_1(x, y, z) \text{ and } z = f(x, y)] \qquad (1)$$

$$\therefore \quad \oint_{C'} P(x, y)dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot\frac{\partial f}{\partial y}\right)dx\,dy \qquad (2)$$

Now $\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS = \iint_S \nabla\times(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot\vec{n}\,dS$

Consider $\iint_S (\nabla \times F_1\vec{i})\cdot \vec{n}\,dS$

But $$\nabla \times F_1\vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}\left(0 - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(0 - \frac{\partial F_1}{\partial y}\right) = \frac{\partial F_1}{\partial z}\vec{j} - \frac{\partial F_1}{\partial y}\vec{k}$$

$\therefore$ $$(\nabla \times F_1\vec{i})\cdot \vec{n} = \frac{\partial F_1}{\partial z}\vec{j}\cdot\vec{n} - \frac{\partial F_1}{\partial y}\vec{k}\cdot\vec{n} \tag{3}$$

We have $$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$$ [since $z = f(x, y)$]

$\therefore$ $$\frac{\partial \vec{r}}{\partial y} = \vec{j} + \frac{\partial f}{\partial y}\vec{k}$$

But $\frac{\partial \vec{r}}{\partial y}$ is a tangent vector to S at P, and hence, $\frac{\partial \vec{r}}{\partial y}$ is $\perp$ to $\vec{n}$. $\therefore$ $\frac{\partial \vec{r}}{\partial y}\cdot\vec{n} = 0$

Substituting in (4), we get $\vec{j}\cdot\vec{n} + \frac{\partial f}{\partial y}\vec{k}\cdot\vec{n} = 0 \Rightarrow \vec{j}\cdot\vec{n} = -\frac{\partial f}{\partial y}\vec{k}\cdot\vec{n}$

$\therefore$ (3) $\Rightarrow$ $$\nabla \times F_1\vec{i}\cdot\vec{n} = \frac{\partial F_1}{\partial z}\left(-\frac{\partial f}{\partial y}\vec{k}\cdot\vec{n}\right) - \frac{\partial F_1}{\partial y}\vec{k}\cdot\vec{n} = -\left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)\vec{k}\cdot\vec{n}$$

$\therefore$ $$\iint_S (\nabla \times F_1\vec{i})\cdot\vec{n}\,dS = -\iint_S \left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)(\vec{k}\cdot\vec{n})\,dS$$

$\Rightarrow$ $$\iint_S (\nabla \times F_1\vec{i})\cdot\vec{n}\,dS = -\iint_R \left(\frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial F_1}{\partial y}\right)dx\,dy \tag{5}$$

From (2) and (5), we get

$$\oint_{C'} F_1\,dx = \iint_S \nabla \times F_1\vec{i}\cdot\vec{n}\,dS$$

Similarly, $$\oint_{C'} F_2\,dy = \iint_S (\nabla \times F_2\vec{j})\cdot\vec{n}\,dS \tag{6}$$

and $$\oint_{C'} F_3\,dz = \iint_S (\nabla \times F_3\vec{k})\cdot\vec{n}\,dS \tag{7}$$

Adding (5), (6), and (7), we get

$$\oint_{C'} F_1dx + F_2dy + F_3dz = \iint_S \nabla \times \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)\cdot\vec{n}\,dS$$

$\Rightarrow$ $$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla \times \vec{F}\cdot\vec{n}\,dS$$

Note

If S is the region R in the xy–plane, bounded by the simple closed curve C, then $\vec{n} = \vec{k}$ is the outward unit normal.

$\therefore$ Stoke's theorem in the plane is $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{Curl } \vec{F} \cdot \vec{k}\, dR$,

which is Green's theorem.

Cartesian form of Stoke's theorem

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

and $\quad \vec{F} \cdot d\vec{r} = F_1\,dx + F_2\,dy + F_3\,dz$

$\therefore$ the cartesian form of Stoke's theorem is $\oint_C (F_1\,dx + F_2\,dy + F_3\,dz)$

$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dydz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dzdx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy\right]$$

Note

If $\vec{F} = P\vec{i} + Q\vec{j}$ and $\vec{r} = x\vec{i} + y\vec{j}$, then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$ and $\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\therefore \qquad \text{Curl } \vec{F} \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\therefore$ Stokes theorem in the plane is $\oint_C (P\,dx + Q\,dy) = \iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)dxdy$

which is Green's thorem.

WORKED EXAMPLES

EXAMPLE 1

Prove that $\oint_C \vec{r} \cdot d\vec{r} = 0$, where C is the simple closed curve.

Solution.

Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on C. $\quad \therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Stokes theorem is $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, dS$

Here $\vec{F} = \vec{r}$.

$$\therefore \qquad \text{Curl } \vec{F} = \text{Curl } \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) + (0-0) = \vec{0}$$

$$\therefore \qquad \oint_C \vec{r}\cdot d\vec{r} = 0$$

EXAMPLE 2

If A is solenoidal, then prove that $\iint_S \nabla^2 \vec{A}\cdot\vec{n} dS = -\oint_C \text{Curl } \vec{A}\cdot d\vec{r}$.

Solution.

Given $\vec{A}$ is solenoidal. $\quad\therefore\quad \nabla\cdot\vec{A} = 0$

We know $\quad \nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = -\nabla^2\vec{A}$

Stoke's theorem is $\quad \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$

Putting $\vec{F} = \nabla\times\vec{A}$, we get $\nabla\times\vec{F} = -\nabla^2\vec{A}$

$$\therefore \qquad \iint_S -\nabla^2\vec{A}\cdot\vec{n}\, dS = \oint_C \nabla\times\vec{A}\cdot d\vec{r}$$

$$\Rightarrow \qquad \iint_S \nabla^2\vec{A}\cdot\vec{n}\, dS = -\oint_C \text{Curl } \vec{A}\cdot d\vec{r}$$

EXAMPLE 3

Prove that $\oint_C \phi\, d\vec{r} = -\iint_S \nabla\phi\times\vec{n}\, dS$.

Solution.

Stoke's theorem is $\quad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S \nabla\times\vec{F}\cdot\vec{n}\, dS$

Put $\vec{F} = \phi\,\vec{a}$, where $\vec{a}$ an arbitrary constant vector.

$$\therefore \qquad \oint_C (\phi\vec{a})\cdot d\vec{r} = \iint_S \nabla\times\phi\vec{a}\cdot\vec{n}\, dS$$

We know curl $\quad \phi\vec{a} = \nabla\times\phi\vec{a} = \nabla\phi\times\vec{a} + \phi\nabla\times\vec{a} = \nabla\phi\times\vec{a} \qquad [\because\ \nabla\times\vec{a} = \vec{0}]$

$$\therefore \qquad \oint_C (\phi\vec{a})\cdot d\vec{r} = \iint_S (\nabla\phi\times\vec{a})\cdot\vec{n}\, dS$$

$$\Rightarrow \quad \oint_C \phi \vec{a}\cdot d\vec{r} = -\iint_S (\vec{a}\times\nabla\phi)\cdot\vec{n}\,dS$$

$$\Rightarrow \quad \vec{a}\cdot\left(\oint_C \phi\, d\vec{r}\right) = -\iint_S \vec{a}\cdot\left(\nabla\phi\times\vec{n}\right)dS \qquad \text{[Interchanging dot and cross]}$$

$$\Rightarrow \quad \vec{a}\cdot\left(\oint_C \phi\, d\vec{r}\right) = -\vec{a}\cdot\iint_S \nabla\phi\times\vec{n}\,dS = \vec{a}\cdot\left(-\iint_S \nabla\phi\times\vec{n}\,dS\right)$$

$$\therefore \quad \oint_C \phi\, d\vec{r} = -\iint_S \nabla\phi\times\vec{n}\,dS \qquad [\because\ \vec{a} \text{ is arbitrary}]$$

EXAMPLE 4

If S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$, then show that $\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = 0$.

Solution.

Suppose the sphere is cut by a plane into two parts S_1 and S_2 and let C be the curve binding these two parts.

Then $\quad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \text{Curl } \vec{F}\cdot\vec{n}\,dS + \iint_{S_2} \text{Curl } \vec{F}\cdot\vec{n}\,dS$

By Stoke's theorem, $\iint_{S_1} \text{Curl } \vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

and $\quad \iint_{S_2} \text{Curl } \vec{F}\cdot\vec{n}\,dS = -\oint_C \vec{F}\cdot d\vec{r}$, because for S_2

the positive sense of the curve C is the opposite direction of C in S_1

$$\therefore \quad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r} - \oint_C \vec{F}\cdot d\vec{r} = 0$$

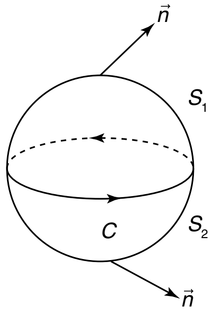

Fig. 9.30

EXAMPLE 5

Evaluate $\int_C (xy\,dx + xy^2 dy)$ by Stoke's theorem, where C is the square in the xy–plane with vertices (1, 0), (−1, 0), (0, 1), (0, −1).

Solution.

Stoke's theorem is $\quad \oint_C \vec{F}\cdot dr = \iint_S \text{curl } \vec{F}\cdot\vec{n}\,dS$

Given $\quad \int_C (xy\,dx + xy^2 dy)$ and $\vec{r} = x\vec{i} + y\vec{j} \quad \therefore \quad d\vec{r} = dx\vec{i} + dy\vec{j}$.

Here $\quad \vec{F}\cdot d\vec{r} = xy\,dx + xy^2 dy \quad \therefore \quad \vec{F} = xy\,\vec{i} + xy^2\,\vec{j}$

$$\therefore \qquad \text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(y^2 - x)$$

$$\Rightarrow \qquad \text{Curl } \vec{F} = (y^2 - x)\vec{k}$$

Also given C is the square in the xy plane with vertices (1, 0), (–1, 0), (0, 1), (0, –1).

$$\therefore \qquad \vec{n} = \vec{k} \text{ and } dS = dx\, dy$$

$$\therefore \qquad \text{Curl } \vec{F} \cdot \vec{n} = (y^2 - x)\vec{k} \cdot \vec{k} = y^2 - x$$

$$\therefore \qquad \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_R (y^2 - x)\, dx\, dy$$

where R is the region inside the square.

That is $\displaystyle\int_C xy\, dx + xy^2 dy = \iint_R (y^2 - x)\, dx\, dy$

We shall now evaluate this double integral.

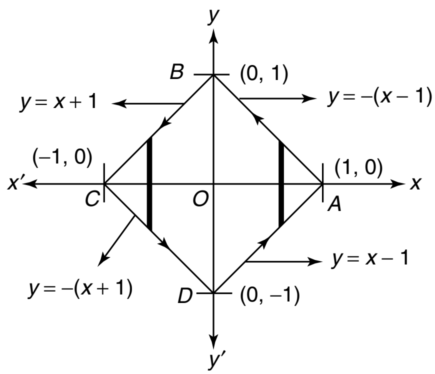

Fig. 9.31

Equation of AB in intercept form is

$$\frac{x}{1} + \frac{y}{1} = 1 \quad \Rightarrow \quad x + y = 1 \quad \Rightarrow \quad y = -x + 1 \quad \Rightarrow \quad y = -(x - 1)$$

Equation of BC is $\dfrac{x}{-1} + \dfrac{y}{1} = 1 \quad \Rightarrow \quad y - x = 1 \quad \Rightarrow \quad y = x + 1$

Equation of CD is $\dfrac{x}{-1} + \dfrac{y}{-1} = 1 \quad \Rightarrow \quad x + y = -1 \quad \Rightarrow \quad y = -(x + 1)$

Equation of AD is $\dfrac{x}{1} + \dfrac{y}{-1} = 1 \quad \Rightarrow \quad y - x = -1 \quad \Rightarrow \quad y = x - 1$

$$\therefore \int_C (xy\, dx + xy^2 dy) = \int_{-1}^{0} \int_{-(x+1)}^{x+1} (y^2 - x)\, dy dx + \int_{0}^{1} \int_{x-1}^{-(x-1)} (y^2 - x)\, dy dx$$

$$= \int_{-1}^{0} \left[\frac{y^3}{3} - xy\right]_{-(x+1)}^{x+1} dx + \int_{0}^{1} \left[\frac{y^3}{3} - xy\right]_{x-1}^{-(x-1)} dx$$

$$= \int_{-1}^{0} \frac{1}{3}\left\{\left[(x+1)^3 - (-(x+1))^3\right] - x\left[x + 1 - (-(x+1))\right]\right\} dx$$

$$+ \int_{0}^{1} \left\{\frac{1}{3}\left[(-(x-1))^3 - (x-1)^3\right] - x[-(x-1) - (x-1)]\right\} dx$$

$$= \int_{-1}^{0} \left\{ \frac{1}{3}[(x+1)^3 + (x+1)^3] - x[(x+1)+(x+1)] \right\} dx$$

$$+ \int_{0}^{1} \left\{ -\frac{1}{3}[(x-1)^3 + (x-1)^3] + x[x-1+x-1] \right\} dx$$

$$= \int_{-1}^{0} \left[\frac{2}{3}(x+1)^3 - 2x(x+1) \right] dx + \int_{0}^{1} \left[-\frac{2}{3}(x-1)^3 + 2x(x-1) \right] dx$$

$$= \left[\frac{2}{3}\frac{(x+1)^4}{4} - 2\left(\frac{x^3}{3} + \frac{x^2}{2} \right) \right]_{-1}^{0} + \left[-\frac{2}{3}\frac{(x-1)^4}{4} + 2\left(\frac{x^3}{3} - \frac{x^2}{2} \right) \right]_{0}^{1}$$

$$= \frac{2}{3}\left(\frac{1}{4}\right) - 2\left\{ 0 - \left[\frac{1}{3}(-1)^3 + \frac{(-1)^2}{2} \right] \right\} - \frac{2}{3}\left[0 - \left(\frac{1}{4}\right) \right] + 2\left[\frac{1}{3} - \frac{1}{2} \right]$$

$$= \frac{1}{6} - \frac{2}{3} + 1 + \frac{1}{6} + \frac{2}{3} - 1 = \frac{2}{6} = \frac{1}{3}$$

EXAMPLE 6

Evaluate $\int_C [(x+y)dx + (2x-z)dy + (y+z)dz]$ where C is the boundary of the triangle with the vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6), using Stoke's theorem.

Solution.

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS,$$

where S is the surface of the triangle ABC bounded by the curve C, consisting of the sides of the triangle in the figure.

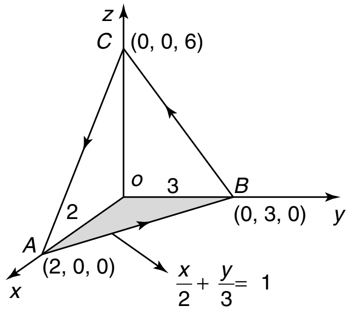

Fig. 9.32

Given $\vec{F} \cdot d\vec{r} = (x+y)dx + (2x-z)dy + (y+z)dz$

Here $\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}$

$$\therefore \quad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z) \right] - \vec{j}\left[\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y) \right] + \vec{k}\left[\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y) \right]$$

$$= \vec{i}[1-(-1)] - \vec{j}[0-0] + \vec{k}(2-1)] = 2\vec{i} + \vec{k}$$

Equation of the plane ABC is $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$ [intercept form]

$\therefore$
$$\phi = \frac{x}{2} + \frac{y}{3} + \frac{z}{6} \quad , \quad \frac{\partial \phi}{\partial x} = \frac{1}{2} \quad , \quad \frac{\partial \phi}{\partial y} = \frac{1}{3} \quad , \quad \frac{\partial \phi}{\partial z} = \frac{1}{6}$$

$\therefore$
$$\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} = \frac{1}{2}\vec{i} + \frac{1}{3}\vec{j} + \frac{1}{6}\vec{k} = \frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})$$

$\therefore$
$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\frac{1}{6}(3\vec{i} + 2\vec{j} + \vec{k})}{\frac{1}{6}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k})$$

$\therefore$
$$\text{Curl } \vec{F} \cdot \vec{n} = (2\vec{i} + \vec{k}) \cdot \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k}) = \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}}$$

$\therefore$
$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_S \frac{7}{\sqrt{14}}\, dS = \frac{7}{\sqrt{14}} \iint_R \frac{dx\, dy}{|\vec{n} \cdot \vec{k}|}$$

where R is the orthogonal projection of S on the xy–plane.

But
$$\vec{n} \cdot \vec{k} = \frac{1}{\sqrt{14}}(3\vec{i} + 2\vec{j} + \vec{k}) \cdot \vec{k} = \frac{1}{\sqrt{14}}$$

$\therefore$
$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \frac{7}{\sqrt{14}} \iint_R \frac{dx dy}{\frac{1}{\sqrt{14}}}$$

$$= 7 \iint_R dx\, dy = 7 \times \text{Area of } \Delta OAB = 7 \cdot \frac{1}{2} \cdot 2 \cdot 3 = 21$$

$\therefore$ $\oint_C [(x+y)dx + (2x - z)dy + (y+z)dz] = 21.$

EXAMPLE 7

Using Stoke's theorem, evaluate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0), (1, 1, 0).

Solution.

Given
$$\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$$

Stoke's theorem is

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n}\, dS$$

Now
$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -x-z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x-z) - \frac{\partial}{\partial z}(x^2)\right]\vec{i} - \left[\frac{\partial}{\partial x}(-x-z) - \frac{\partial}{\partial z}(y^2)\right]\vec{j}$$

$$+\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right]\vec{k}$$

$$= (0)\,\vec{i} - [-1]\,\vec{j} + [2x-2y]\,\vec{k} = \vec{j} + 2\,(x-y)\,\vec{k}.$$

Given C is the boundary of the triangle formed by the points (0, 0, 0), (1, 0, 0) and (1, 1, 0) which lie in the xy–plane. $\therefore\quad \vec{n} = \vec{k}$

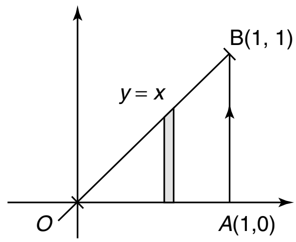

Fig. 9.33

$$\therefore \qquad \text{curl}\,\vec{F}\cdot\vec{n} = 2\,(x-y)$$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S 2\,(x-y)\,dx\,dy$$

Equation of OB is $y = x$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = 2\int_0^1\int_0^x (x-y)\,dy\,dx$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2} - 0\right] dx = 2\int_0^1 \frac{x^2}{2}\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

EXAMPLE 8

Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$, where S is the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0$ and $z = 2$ above the xy–plane.

Solution.

Given $\qquad \vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}.$

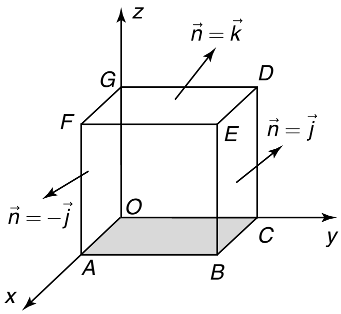

Fig. 9.34

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$$

Now $\qquad \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-xz) - \frac{\partial}{\partial z}(yz+4)\right] - \vec{j}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial z}(y-z+2)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(yz+4) - \frac{\partial}{\partial y}(y-z+2)\right]$$

$$= \vec{i}[(0-y)] - \vec{j}[-z-(-1)] + \vec{k}(0-1) = -y\vec{i} + (z-1)\vec{j} - \vec{k}$$

We shall compute $\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS$.

Given S is the open surface consisting of 5 faces of the cube except the face $OABC$.

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

$$+ \iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n}\, dS + \iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

Face	Equation	Outward normal $\vec{n}$	$\vec{F} \cdot \vec{n}$	dS
S_1 = ABEF	$x = 2$	$\vec{i}$	$-y$	$dy\, dz$
S_2 = OCDG	$x = 0$	$-\vec{i}$	y	$dy\, dz$
S_3 = BCDE	$y = 2$	$\vec{j}$	$z - 1$	$dx\, dz$
S_4 = OAFG	$y = 0$	$-\vec{j}$	$-(z - 1)$	$dx\, dz$
S_5 = DEFG	$z = 2$	$\vec{k}$	-1	$dx\, dy$

$\therefore$

$$\iint_{S_1} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -y\, dy\, dz = \int_0^2 dz \cdot \int_0^2 (-y)\, dy = [z]_0^2 \left[\frac{-y^2}{2}\right]_0^2 = 2(-2) = -4$$

$$\iint_{S_2} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 y\, dy\, dz = \int_0^2 dz \int_0^2 y\, dy = [z]_0^2 \left[\frac{y^2}{2}\right]_0^2 = 2 \cdot 2 = 4$$

$$\iint_{S_3} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 (z-1)\, dz\, dx = \int_0^2 dx \cdot \int_0^2 (z-1)\, dz = [x]_0^2 \cdot \left[\frac{(z-1)}{2}\right]_0^2$$

$$= 2 \cdot \frac{1}{2}\left\{(2-1)^2 - (-1)^2\right\} = 1 - 1 = 0$$

$$\iint_{S_4} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -(z-1)\, dz\, dx = 0 \qquad \text{[as above]}$$

and

$$\iint_{S_5} \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^2\int_0^2 -1\, dx\, dy = -[x]_0^2 [y]_0^2 = -4$$

$\therefore$

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = -4 + 4 + 0 + 0 - 4 = -4 \qquad (1)$$

We shall now compute the line integral over the simple closed curve C bounding the surface consisting of the edges OA, AB, BC and CO in $z = 0$ plane

$\therefore$ $$\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

Now $$\vec{F} \cdot d\vec{r} = \left[(y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}\right] \cdot \left[dx\vec{i} + dy\vec{j} + dz\vec{k}\right]$$
$$= (y - z + 2)dx + (yz + 4)dy - xzdz$$

$\Rightarrow$ $$\vec{F} \cdot d\vec{r} = (y + 2)dx + 4dy \qquad [\because z = 0]$$

On OA: $y = 0$ $\quad \therefore\ dy = 0$ and $\vec{F} \cdot d\vec{r} = 2dx$ and x varies from 0 to 2

$\therefore$ $$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^2 2dx = 2[x]_0^2 = 4$$

On AB: $x = 2$ $\quad \therefore\ dx = 0$ and $\vec{F} \cdot d\vec{r} = 4dy$ and y varies from 0 to 2

$\therefore$ $$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^2 4dy = 4[y]_0^2 = 8$$

On BC: $y = 2$ $\quad \therefore\ dy = 0$ and $\vec{F} \cdot d\vec{r} = 4dx$ and x varies from 2 to 0

$\therefore$ $$\int_{BC} \vec{F} \cdot d\vec{r} = \int_2^0 4dx = 4[x]_2^0 = 4(-2) = -8$$

On CO: $x = 0$ $\quad \therefore\ dx = 0$, $\vec{F} \cdot d\vec{r} = 4dy$ and y varies from 2 to 0

$\therefore$ $$\int_{CO} \vec{F} \cdot d\vec{r} = \int_2^0 4dy = 4[y]_2^0 = -8$$

$\therefore$ $$\int_C \vec{F} \cdot d\vec{r} = 4 + 8 - 8 - 8 = -4 \qquad (2)$$

From (1) and (2), $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

Hence, Stoke's theorem is verified.

EXAMPLE 9

Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\,\vec{j}$ in the rectangular region in the xy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution.

Given $$\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$$

Stoke's theorem is

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(2y+2y) = 4y\vec{k}$$

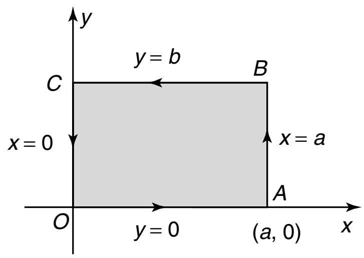

Fig. 9.35

Since the surface is a rectangle in the xy–plane, normal $\vec{n} = \vec{k}$

$\therefore$ $$\text{Curl } \vec{F} \cdot \vec{n} = 4y\vec{k} \cdot \vec{k} = 4y$$

$$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^a \int_0^b 4y\, dx\, dy$$

$\Rightarrow$ $$\iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS = \int_0^a dx \int_0^b 4y\, dy = [x]_0^a\, 4\left[\frac{y^2}{2}\right]_0^b = 2ab^2 \tag{1}$$

We shall now compute the line integral.

$\therefore$ $$\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

Now $$\vec{F} \cdot d\vec{r} = (x^2 - y^2)dx + 2xydy$$

On OA: $y = 0$ $\quad \therefore$ $dy = 0$ and $\vec{F} \cdot d\vec{r} = x^2 dx$ and x varies from 0 to a

$\therefore$ $$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a$ $\quad \therefore$ $dx = 0$ and $\vec{F} \cdot d\vec{r} = 2aydy$ and y varies from 0 to b

$\therefore$ $$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2aydy = 2a\left[\frac{y^2}{2}\right]_0^b = ab^2$$

On BC: $y = b$ $\quad \therefore$ $dy = 0$ and $\vec{F} \cdot d\vec{r} = (x^2 - b^2)dx$ and x varies from a to 0

$\therefore$ $$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 (x^2 - b^2)dx = \left[\frac{x^3}{3} - b^2 x\right]_a^0 = 0 - \left(\frac{a^3}{3} - b^2 a\right) = ab^2 - \frac{a^3}{3}$$

On CO: $x = 0$ $\quad \therefore$ $dx = 0$ and $\vec{F} \cdot d\vec{r} = 0$

$\therefore$ $$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

$\therefore$ $$\int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2 \tag{2}$$

From (1) and (2), $$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{Curl } \vec{F} \cdot \vec{n}\, dS$$

Hence, Stoke's theorem is verified.

Note Stoke's theorem in the plane is Green's theorem. This is indeed Green's theorem verification.

EXAMPLE 10

Verify Stoke's theorem for the vector field $\vec{F} = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ over the upper half surface $x^2 + y^2 + z^2 = 1$, bounded by its projections on the xy–plane.

Solution.

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot\vec{n}\, dS$$

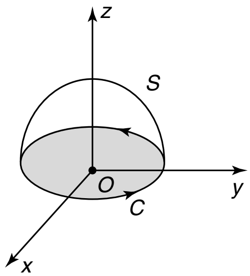

Fig. 9.36

Given $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

$$\therefore \quad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right]$$

$$= \vec{i}[-2yz + 2yz] - \vec{j}[0-0] + \vec{k}[0-(-1)] = \vec{k}$$

$$\therefore \quad \vec{F}\cdot\vec{n} = \vec{k}\cdot\vec{n}$$

The surface is the upper hemisphere $x^2 + y^2 + z^2 = 1$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S \vec{k}\cdot\vec{n}\, dS = \iint_R \vec{k}\cdot\vec{n}\, \frac{dxdy}{|\vec{k}\cdot\vec{n}|},$$

where R is the projection of S on the xy–plane.

$\therefore$ R is the circle $x^2 + y^2 = 1$ in the xy–plane.

$$\therefore \quad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_R dx\, dy$$

$$\Rightarrow \quad \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \text{area of the circle} = \pi \cdot 1^2 = \pi \qquad (1)$$

Now C is the circle $x^2 + y^2 = 1$ in the $z = 0$ plane.

Parametric equations are $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$

$$\therefore \quad \int_C \vec{F}\cdot d\vec{r} = \int_C [(2x-y)dx - yz^2dy - y^2zdz] = \oint_C (2x-y)dx \qquad [\because\ z = 0]$$

Now $x = \cos\theta \quad \Rightarrow \quad dx = -\sin\theta d\theta$

$$\therefore \quad \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-2\sin\theta\cos\theta + \sin^2\theta)\,d\theta$$

$$= \int_0^{2\pi} \left[-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$\Rightarrow \quad \oint_C \vec{F}\cdot d\vec{r} = \frac{1}{2}\left[(\cos 4\pi - \cos 0) + 2\pi - \frac{\sin 4\pi}{2} - 0\right] = \frac{1}{2}[1 - 1 + 2\pi] = \pi \qquad (2)$$

From (1) and (2), $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS$

Hence, Stoke's theorem is verified.

EXAMPLE 11

Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$.

Solution.

Stoke's theorem is

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS$$

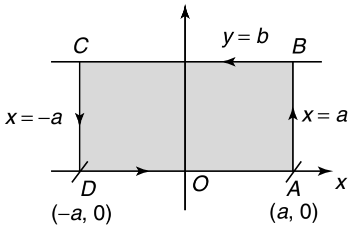

Fig. 9.37

Given $\quad \vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$

$$\therefore \quad \text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0 - 0] - \vec{j}(0 - 0) + \vec{k}(-2y - 2y) = -4y\vec{k}$$

Since S is the rectangular surface, $\vec{n} = \vec{k}$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = \iint_S -4y\vec{k}\cdot\vec{k}\, dx\, dy$$

$$= -4\int_0^b \int_{-a}^{a} y\, dx\, dy = -4\left[\frac{y^2}{2}\right]_0^b [x]_{-a}^{a} = -2b^2 \cdot 2a = -4ab^2$$

$$\iint_S \text{Curl } \vec{F}\cdot\vec{n}\, dS = -4ab^2 \qquad (1)$$

We shall now compute the line integral $\oint_C \vec{F}\cdot d\vec{r}$.

Now $\quad \vec{F}\cdot d\vec{r} = [(x^2 + y^2)\vec{i} - 2xy\vec{j}]\cdot[dx\vec{i} + dy\vec{j}] = (x^2 + y^2)dx - 2xy\, dy$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

On AB: $x = a \quad \therefore\ dx = 0$ and $\vec{F}\cdot d\vec{r} = -2ay\,dy$ and y varies from 0 to b

$$\therefore \qquad \int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b (-2a)y\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$

On BC: $y = b \quad \therefore\ dy = 0$ and $\vec{F}\cdot d\vec{r} = (x^2+b^2)dx$ and x varies from a to $-a$

$$\therefore \qquad \int_{BC} F\cdot d\vec{r} = \int_a^{-a} (x^2+b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_a^{-a}$$

$$= \frac{1}{3}(-a^3 - a^3) + b^2(-a-a) = \frac{-2}{3}a^3 - 2ab^2$$

On CD: $x = -a \quad \therefore\ dx = 0$ and $\vec{F}\cdot d\vec{r} = 2ay\,dy$ and y varies from b to 0

$$\therefore \qquad \int_{CD} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy = 2a\left[\frac{y^2}{2}\right]_b^0 = a(0-b^2) = -ab^2$$

On DA: $y = 0 \quad \therefore\ dy = 0$ and $\vec{F}\cdot d\vec{r} = x^2 dx$ and x varies from $-a$ to a

$$\therefore \qquad \int_{DA} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx = 2\int_0^a x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^a = \frac{2}{3}a^3$$

$$\therefore \qquad \oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \frac{2}{3}a^3 - 2ab^2 - ab^2 + \frac{2}{3}a^3 = -4ab^2$$

From (1) and (2), $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\ \vec{F}\cdot\vec{n}\ dS$

Hence, Stoke's theorem is verified.

EXAMPLE 12

Verify stokes theorem for $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$, where S is the open surface of the cube formed by the planes $x = -a, x = a, y = -a, y = a, z = -a,\ z = a$ in which $z = -a$ is cut open.

Solution.

Stoke's theorem is $\oint \vec{F}.\,d\vec{r} = \iint_S \text{curl}\ \vec{F}.\vec{n}\ ds$

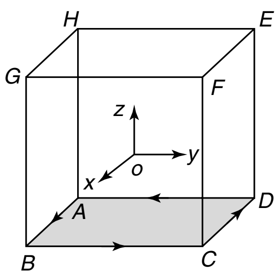

Fig. 9.38

Given $\qquad \vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$

$$\therefore \qquad \text{Curl F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2 y) - \frac{\partial}{\partial z}(z^2 x)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2 y) - \frac{\partial}{\partial z}(y^2 z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(z^2 x) - \frac{\partial}{\partial y}(y^2 z)\right]$$

$$= (x^2 - 2zx)\vec{i} + (y^2 - 2xy)\vec{j} + (z^2 - 2yz)\vec{k}$$

We shall now compute $\iint_S$ *curl* $\vec{F}.\vec{n}\, dS$

Given S is the open surface consisting of the five faces of the cube except face $ABCD$

$$\therefore \iint_S \text{curl } \vec{F}.\vec{n}dS = \iint_{S_1} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_2} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_3} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_4} \text{curl } \vec{F}.\vec{n}dS + \iint_{S_5} \text{curl } \vec{F}.\vec{n}dS$$

Face	Equation	Normal $\vec{n}$	Curl $\vec{F}.\vec{n}$	dS
$S_1 = BCFG$	$x = a$	$\vec{i}$	$a^2 - 2az$	$dy\, dz$
$S_2 = ADEH$	$x = -a$	$-\vec{i}$	$-(a^2 + 2az)$	$dy\, dz$
$S_3 = CDEF$	$y = a$	$\vec{j}$	$a^2 - 2ax$	$dz\, dx$
$S_4 = ABGH$	$y = -a$	$-\vec{j}$	$-(a^2 + 2ax)$	$dz\, dx$
$S_5 = EFGH$	$z = a$	$\vec{k}$	$a^2 - 2ay$	$dx\, dy$

$$\iint_{S_1} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2az)dydz$$

$$= \left[\int_{-a}^{a} dy\right]\left[\int_{-a}^{a} a^2 - 2az\, dydz\right] = [y]_{-a}^{a}\left[a^2 z - 2a\frac{z^2}{2}\right]_{-a}^{a} = [a+a]\left[a^2(a+a) - a(a^2 - a^2)\right] = 4a^4$$

$$\iint_{S_2} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az)dydz \quad = -\left[\int_{-a}^{a} dy\right]\left[\int_{-a}^{a} a^2 + 2az\right]dz$$

$$= -[y]_{-a}^{a}\left[a^2 z + 2a\frac{z^2}{2}\right]_{-a}^{a} = -[a+a]\left[a^2(a+a) + a(a^2 - a^2)\right] = -4a^4$$

Similarly, $\displaystyle\iint_{S_3} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ax)dzdx = 4a^4$

$$\iint_{S_4} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2ax)dzdx = -4a^4$$

and $\displaystyle\iint_{S_5} \text{curl } \vec{F}.\vec{n}dS = \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ay)\, dxdy = 4a^4$

$$\therefore \quad \iint_S \text{curl } \vec{F}.\vec{n}dS = 4a^4 - 4a^4 + 4a^4 - 4a^4 + 4a^4 = 4a^4 \qquad (1)$$

We shall now compute the line integral over the simple closed curve C consisting of the edges AB, BC, CD, DA. Here $z = -a$, $dz = 0$

$$\therefore \quad \vec{F}.d\vec{r} = y^2 zdx + z^2 xdy + x^2 ydz = -ay^2 dx + a^2 xdy$$

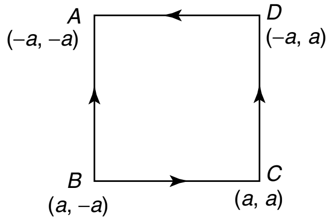

Fig. 9.39

On *AB*: $y = -a \therefore dy = 0$

$\vec{F}.d\vec{r} = -a^3 dx$ and x varies from $-a$ to a.

$$\therefore \quad \int_{AB} \vec{F}.d\vec{r} = \int_{-a}^{a} -a^3 dx = -a^3 [x]_{-a}^{a} = -a^3.2a = -2a^4$$

On *BC*: $x = a \therefore dx = 0$, $\vec{F}.d\vec{r} = a^3 dy$ and y varies from $-a$ to a.

$$\therefore \quad \int_{BC} \vec{F}.d\vec{r} = \int_{-a}^{a} a^3\, dy = a^3 [r]_{-a}^{a} = a^3.2\,a = 2\,a^4$$

On *CD*: $y = a \therefore dy = 0, \vec{F}.d\vec{r} = -a^3 dx$ and x varies from a to $-a$

$$\therefore \quad \int_{CD} \vec{F}.d\vec{r} = \int_{a}^{-a} -a^3 dx = -a^3 [x]_{a}^{-a} = -a^3(-2a) = 2a^4$$

On *DA*: $x = -a \therefore dx = 0, \vec{F}.d\vec{r} = -a^3 dy$ and y varies from a to $-a$.

$$\therefore \quad \int_{DA} \vec{F}.d\vec{r} = \int_{a}^{-a} -a^3 dy = -a^3 [y]_{a}^{-a} = -a^2(-2a) = 2a^4 \tag{2}$$

$$\therefore \quad \oint_{c} \vec{F}\cdot d\vec{r} = -2\,a^4 + 2\,a^4 + 2\,a^4 + 2\,a^4 = 4\,a^4 \tag{3}$$

From (1) and (2), we get $\iint_S \text{curl } \vec{F}.\vec{n}ds = \oint_C \vec{F}.d\vec{r}$

Hence, Stoke's theorem is verified.

EXERCISE 9.4

1. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = 12x^2 y\vec{i} - 3yz\vec{j} + 2z\vec{k}$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.
2. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = (2x^2 - 3z)\vec{i} + 2y\vec{j} - 4xz\vec{k}$, where S is the surface of the solid bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.
3. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = z\vec{i} + x\vec{j} - y^2 z\vec{k}$ and S is the curved surface of the cylinder $x^2 + y^2 = 1$ included in the first octant between the planes $z = 0$ and $z = 2$.
4. If $\vec{F} = xy^2\vec{i} - yz^2\vec{j} + zx^2\vec{k}$, find $\iint_S \vec{F}\cdot\vec{n}\, dS$ over the sphere $x^2 + y^2 + z^2 = 1$.
5. Evaluate $\iint_S \vec{F}\cdot\vec{n}\, dS$, where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

6. Evaluate $\oint_C (x^2 + xy)dx + (x^2 + y^2)dy$, where C is the square formed by the lines $y = \pm 1$, $x = \pm 1$, by Green's theorem.
7. Using Green's theorem evaluate $\oint_C (x^2 + y)dx - xy^2\, dy$ taken around the square whose vertices are (0, 0), (1, 0), (1, 1) (0, 1)
8. Using Green's theorem find the value of $\int_C (xy - x^2)dx + x^2 y\, dy$ along the closed curve C formed by $y = 0$, $x = 1$ and $y = x$.
9. Verify Green theorem for $\int_C (15x^2 - 4y^2)dx + (2y - 3x)\, dy$, where C is the curve enclosing the area bounded by $y = x^2$, $x = y^2$
10. Verify Green theorem in the plane for $\int_C (3x^2 - 8y^3)dx + (4y - 6xy)\, dy$, where C is the boundary of the region defined by $x = 0$, $y = 0$, $x + y = 1$.
11. Using Green's theorem find the area of $x^{2/3} + y^{2/3} = a^{2/3}$.

    [**Hint:** Area $= \frac{1}{2}\int_C (xdy - ydx)$, C is the boundary of the curve]
12. Using Green's theorem in xy plane find the area of the region in the xy plane bounded by $y^3 = x^2$ and $y = x$.
13. Using Green's theorem evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)\, dy$, where C is the boundary of the area in the xy plane bounded by x–axis and the semi circle $x^2 + y^2 = 1$ in the upper half of the plane.
14. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{i} + yz\vec{k}$ taken over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.
15. Verify Gauss divergence theorem for $\vec{F} = (x^3 - yz)\vec{i} - 2x^2 y\vec{j} + 2\vec{k}$ over the parallelopiped bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $y = 2$, $z = 0$, $z = 3$.
16. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over a unit cube.
17. Verify Gauss divergence theorem for $\vec{F} = (x^3 - yz)\vec{i} - zx^2 y\vec{j} + 2\vec{k}$ over the cube $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$.
18. Verify the divergence theorem for $\vec{F} = 2xy\vec{i} + yz^2\vec{j} + xz\vec{k}$, where S is the rectangular parallelopiped bounded by $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 1$, $z = 3$.
19. Using divergence theorem show that

    $\iint_S x^2 dy + y^2 dzdy + 2z(xy - x - y)dxdy = \frac{1}{2}$, where S is the surface of the cube $x = y = z = 0$, $y = z = 1$.
20. Use divergence theorem to evaluate $\iint_S (2xy\vec{i} + yz^2\vec{j} + xz\vec{k}).d\vec{S}$, where S is the surface of the region bounded by $x = y = z = 0$, $y = 3$, $x + 2z = 6$.
21. Prove that $\iint_S [x(y - z)\vec{i} + y(z - x)\vec{j} + z(x - y)\vec{k}]\cdot d\vec{S} = 0$, where S is any closed surface.

22. Verify Stoke's theorem for $\vec{F} = 2z\vec{i} + x\vec{j} + y^2\vec{k}$, where S is the surface of the paraboloid $z = 4 - x^2 - y^2$ and C is the simple closed curve in the xy plane.
23. Verify Stoke's theorem for $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C its boundary.
24. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j} + xyz\vec{k}$ over the surface of the box bounded by the planes $x = 0, y = 0, x = a, y = b, z = c$ above the xy plane.
25. Verify Stoke's theorem for $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the xy plane bounded by $x = 0, x = a, y = 0, y = b$.
26. Verify Stoke's theorem for $\vec{F} = -y^3\vec{i} + x^3\vec{j}$ and the closed curve C is the boundary of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
27. If ϕ is scalar point function, use Stoke's theorem to prove curl (grad ϕ) = 0.
28. Evaluate $\iint\limits_S \nabla \times \vec{F} \cdot \vec{n}\, dS$, where S is the surface $x^2 + y^2 + z^2 = a^2$ above the xy–plane and $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}\cdot$
29. Evaluate $\int\limits_C yzdx + zx\,dy + xy\,dz$, where C is the curve $x^2 + y^2 = 1, z = y^2$.
30. Evaluate $\iint \nabla \times \vec{F} \cdot \vec{n}\, dS$ for $\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$ over the surface of the cylinder $x^2 + y^2 = 4$, bounded by the plane $z = 9$ and open at the end $z = 0$.
31. Find the area of a circle of radius a using Green's theorem.
32. Using Green's theorem evaluate $\oint\limits_C [(2xy - x^2)dx + (x^2 + y^2)dy]$ where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$
33. Verify Green's theorem in a plane for the integral $\int\limits_C (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 4$.
34. Verify Green's theorem in the plane for $\oint\limits_C [(x^2 - xy^3)dx + (y^2 - 2xy)dy]$ where C is the square with vertices (0, 0) (2, 0), (2, 2), (2, 0).
35. Evaluate $\iiint\limits_V \nabla \cdot \vec{F}\, dV$ if $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the volume of the region enclosed by the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.
36. If S is any closed surface enclosing volume V and $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$ prove that $\iint\limits_S \vec{F} \cdot \vec{n}\, dS = (a + b + c)V$
37. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped bounded by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.
38. Verify Stoke's theorem for $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\vec{k}$ where S is the open surface of the cube formed by the planes $x = -a, x = a, y = -a, y = a, z = -a, z = a$ in which $z = -a$ is cut open.

39. Evaluate $\iint_S \text{Curl}\, \vec{F} \cdot \vec{n}\, dS$, where $\vec{F} = (y - z)\vec{i} + yz\,\vec{j} - xz\,\vec{k}$ and S is the open surface bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ above the xy plane.

ANSWERS TO EXERCISE 9.4

1. $\frac{49}{120}$	2. $\frac{16}{3}$	3. 3	4. $\frac{4}{3}\pi$	5. $\frac{3}{2}$	6. 0	7. $-\frac{4}{3}$
8. $-\frac{1}{12}$	11. $\frac{3}{8}\pi a^3$	12. $\frac{1}{10}$	13. $\frac{4}{3}$	20. $\frac{351}{2}$	28. 0	29. 0
30. 8π	31. πa^2	32. 0	35. 3	36. $(a + b + c)\, V$	39. -1	

SHORT ANSWER QUESTIONS

1. If $\vec{r} = x\vec{i} + y\,\vec{j} + z\,\vec{k}$ and $|\vec{r}| = r$, then find ∇r.

2. Find grad ϕ at the point (1, –2, –1), where $\phi = 3x^2y - y^3z^2$.

3. What is the greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3)?

4. Find the unit normal vector to the surface $x^2 + xy + z^2 = 4$ at the point (1, –1, 2).

5. Find the directional derivative of $\phi = xyz$ at (1, 1, 1) in the direction of $\vec{i} + \vec{j} + \vec{k}$

6. The temperature at a point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at the point (4, 4, 2) desires to fly in such a direction that it gets cooled faster. Find the direction in which it should fly.

7. Find the normal derivative of $\phi = x^3 - y^3 + z$ at the point (1, 1, 1).

8. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point (2, –1, 2).

9. Find the equation of the tangent plane to the surface $x^2 + y^2 - z = 0$ at the point (2, –1, 5).

10. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, find div (curl $\vec{F}$).

11. Prove that $\vec{F} = (2x^2y + yz)\vec{i} + (xy^2 - xz^2)\vec{j} - (6xy + 2x^2y^2)\vec{k}$ is solenoidal.

12. Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoial.

13. If ϕ is a scalar point function, prove that $\nabla\phi$ is solenoidal and irrotational if ϕ is a solution of Laplace equation.

14. Find the values of a, b, c if $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

15. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

16. Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{i} - (2x + y)\vec{j}$ moves a particle from the origin to the point (1, 1) along $y^2 = x$.

17. Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

18. Evaluate $\int_C (x^2 - xy)dx + (x^2 + y^2)dy$, where C is the square formed by the lines $y = \pm 1, x = \pm 1$ using Green's theorem.

19. Using Stoke's theorem prove that curl (grad ϕ) = 0.

20. If S any closed surface show that $\iint_S$ curl $\vec{F} \cdot \vec{n}\, dS = 0$.

OBJECTIVE TYPE QUESTIONS.

A. Fill up the blanks

1. $\nabla\left(\frac{1}{r}\right) =$ ________

2. If $\phi\,(x, y, z) = x^2y + xy^2 + z^2$, then $\nabla\,\phi$ at (1, 1, 1) is =________.

3. The directional derivative of $\phi = x^3 + y^3 + z^3$ at $(1, -1, 2)$ in the direction of $\vec{i} + 2\vec{j} + \vec{k}$ is = ________.

4. The unit normal to the surface $xy^2z^3 = 1$ at the point (1, 1, 1) is = ________.

5. The greatest rate of increase of $\phi = xyz^2$ at the point (1, 0, 3) is = ________.

6. Equation of the normal to the surface $x^2 + y^2 + z^2 = 25$ at the point (1, 0, 3) is = ________.

7. If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$, then $curl\ \vec{F}$ = ________.

8. If $\vec{F} = (3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal, then value of a is = ________.

9. If $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ and the curve c is the line joining the points (1, –2, 1) and (3, 2, 4), then $\int_C \vec{F} \cdot d\vec{r}$ = ________.

10. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the region bounded by the cube $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$, then $\iiint_v \nabla \cdot \vec{F} dv$ = ________.

B. Choose the correct answer

1. If $\phi = x^2 + y^2 + z^2 - 8$, then grad ϕ at (2, 0, 2) is

(a) $\vec{i} + 4\vec{k}$ (b) $\vec{i} + \vec{j} + \vec{k}$ (c) $4\vec{i} + \vec{k}$ (d) $4\vec{i} + 4\vec{j} + 4\vec{k}$

2. $div\left(\frac{\vec{r}}{r}\right)$ is equal to

(a) $\frac{1}{r}$ (b) $\frac{2}{r}$ (c) $\frac{3}{r}$ (d) $\frac{4}{r}$

3. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then $curl\ \vec{r}$ is equal to

(a) $\vec{o}$ (b) $\vec{i}$ (c) $\vec{j}$ (d) $\vec{k}$

4. If $\phi = x^2 - y^2$, then $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ is equal to

(a) 0 (b) 2 (c) −2 (d) 1

5. If $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xy^2 - y)\vec{k}$, then ϕ is equal to

(a) $xz - yz + c$ (b) $3x^2y + xz^3$ (c) $xz^3 - yz + c$ (d) $3x^2y - yz + c$

6. The unit normal at (1, 2, 5) on $x^2 + y^2 = z$ is

(a) $\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$ (b) $\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$ (c) $\frac{\vec{i} + 4\vec{j} - 5\vec{k}}{\sqrt{42}}$ (d) $\frac{2\vec{i} + 4\vec{j} - 5\vec{k}}{3\sqrt{5}}$

7. The equation of the tangent plane to the surface at (2, 0, 2) is

(a) $x - y - z = 0$ (b) $2x - z = 2$ (c) $3x + y - 2z = 2$ (d) None of these

8. If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, then $\int_c \vec{F} \cdot \vec{dr}$, where c is the segment on $y = x$ from (0, 0) to (1, 1) is

(a) $-\frac{7}{6}$ (b) $\frac{7}{12}$ (c) $\frac{7}{6}$ (d) $-\frac{7}{12}$

9. Find the work done when the force $\vec{F} = 5xy\vec{i} + 2y\vec{j}$ displaces a particle from the points corresponding to $x = 1$ to $x = 2$ along $y = x^3$

(a) 24 (b) 64 (c) −84 (d) 94

10. Using Green's theorem in the plane, evaluate $\int_c (2x - y)dx + (x + y)dy$, where c is the circle $x^{2+} + y^2 = 4$ in the plane

(a) 2π (b) 4π (c) -4π (d) 8π

ANSWERS

A. Fill up the blanks

1. $-\frac{\vec{r}}{r^3}$ 2. $3\vec{i} + 3\vec{j} + 2\vec{k}$ 3. $\frac{7\sqrt{6}}{2}$ 4. $\frac{\vec{i} + 2\vec{j} + 3\vec{k}}{\sqrt{14}}$ 5. 9

6. $\frac{x-4}{4} = \frac{y}{0} = \frac{z-3}{3}$ 7. $\vec{o}$ 8. −5 9. 21 10. 3

B. Choose the correct answer

1. (c) 2. (c) 3. (a) 4. (a) 5. (b) 6. (d) 7. (b) 8. (b) 9. (d) 10. (d)