MAINS TEST SERIES-2021

TEST-6 (BATCH-II) &
TEST-16 (BATCH-I)

FULL SYLLABUS (PAPER-I & PAPER-II)

Answer Key

IMS (INSTITUTE OF MATHEMATICAL SCIENCES) TM

1.(a) If $G/Z(G)$ is cyclic, show that G is abelian. Where $Z(G)$ is centre of G.

Solⁿ: Let $G/Z(G) = \langle xZ(G) \rangle$

Let $a, b \in G$. Then there exist an integer such that $aZ(G) = x^i Z(G)$ as $G/Z(G)$ is cyclic.

This implies $a = x^i z$ for some $z \in Z(G)$.

Similarly, there exist an integer j such that $b = x^j w$ for some $w \in Z(G)$.

Now consider $ab = x^i z x^j w = x^i x^j z w$ as $z \in Z(G)$, then $ab = x^i x^j z w = x^{i+j} z w = x^j x^i z w$

$= x^j w x^i z$ as $w \in Z(G)$,

which is ba.

Hence for all $a, b \in G$

$$ab = ba$$

i.e. G is abelian.

1.(b) → Prove that a finite integral domain has finite characteristic. Give an example of an integral domain which has an infinite number of elements, yet is of finite characteristic.

Sol^n: Let D be a finite integral domain.
Let char $D=0$. then

$$na \neq 0 \ \forall\ a \neq 0 \in D \text{ and } \forall\ n \in N. \quad \text{——— (1)}$$

It follows that $a, 2a, 3a, \ldots$ all belong to D.
Since D is finite, we must have $ia = ja$ for some positive integers i and j, $i > j$. Then

$$(i-j)a = 0, \text{ where } i-j > 0.$$

This contradicts (1) and so char $D \neq 0$.
We know that the characteristic of any integral domain is either zero or a prime number.

$\because$ char $D \neq 0$, therefore

char $D = p$ (finite), p is some prime.

Let $F = Z_p = \{0, 1, 2, \ldots, p-1\}$ be the ring of integers modulo p. Then F is an integral domain of characteristic p i.e., $pa = 0 \ \forall\ a \in Z_p$ ——— (2)

Let $F[x]$ be the ring of polynomials over F. Since F is an integral domain, so $F[x]$ is an integral domain having infinite number of elements with finite characteristic p.

Notice that if $f(x) = a_0 + a_1 x + \cdots + a_r x^r \in F[x]$, then by (2),

$$pf(x) = pa_0 + pa_1 x + \cdots + pa_r x^r = 0$$
$$\forall\ f(x) \in F[x].$$

1.(C) → A twice differentiable function f is such that $f(a)=f(b)=0$ and $f(c)>0$, for $a<c<b$. Prove that there is atleast one value ξ between a and b for which $f''(\xi)<0$.

Solⁿ: Let us consider the function f on $[a,b]$. Since f'' exists, f' and f both exist and are continuous on $[a,b]$. Since c is a point between a and b, applying Lagrange's Mean value Theorem to f on the intervals $[a,c]$ and $[c,b]$ respectively, we get

$$\frac{f(c)-f(a)}{c-a}=f'(\xi_1),\ a<\xi_1<c$$

and $$\frac{f(b)-f(c)}{b-c}=f'(\xi_2),\ c<\xi_2<b$$

But $f(a)=f(b)=0$

$$\therefore\ f'(\xi_1)=\frac{f(c)}{c-a} \text{ and } f'(\xi_2)=-\frac{f(c)}{b-c}$$

where $a<\xi_1<c<\xi_2<b$.

Again $f'(x)$ is continuous and derivable on $[\xi_1,\xi_2]$. Therefore, by Mean value Theorem,

$$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1}=f''(\xi),\text{ where } \xi_1<\xi<\xi_2$$

Substituting the values of $f'(\xi_2)$ and $f'(\xi_1)$, we get

$$f''(\xi) = \frac{-f(c)}{\xi_2 - \xi_1}\left(\frac{1}{b-c} + \frac{1}{c-a}\right)$$

$$= \frac{-(b-a)f(c)}{(\xi_2-\xi_1)(b-c)(c-a)} < 0.$$

1.(d) → Evaluate the following integrals by using cauchy's integral formula:

(i) $\int_C \frac{(\sin z)^6}{\left(z-\frac{\pi}{6}\right)^3} dz$, where 'c' is circle $|z|=1$.

(ii) $\int_C \frac{e^{3z}dz}{z+i}$ if 'c' is circle $|z+1+i|=2$.

Sol^n: (i) Let $I = \int_C \frac{(\sin z)^6 dz}{\left(z-\frac{\pi}{6}\right)^3}$, where c is $|z|=1$.

$$z = \frac{\pi}{6} = \frac{3.14}{6} = 0.52 \text{ lies inside } C.$$

Take $f(z) = (\sin z)^6$ Then

$$I = \int_C \frac{f(z)dz}{\left(z-\frac{\pi}{6}\right)^3} = \frac{2\pi i}{2!} f''\left(\frac{\pi}{6}\right) = \pi i f''\left(\frac{\pi}{6}\right), \text{ by}(R_3)$$

$$f(z) = (\sin z)^6 \Rightarrow f'(z) = 6(\sin z)^5 \cos(z).$$

$$\Rightarrow f''(z) = 6\left[5(\sin z)^4(\cos z)^2 - (\sin z)^6\right]$$

$$\Rightarrow f''\left(\frac{\pi}{4}\right) = 6\left[5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right] = \frac{21}{16}$$

$$I = \pi i f''\left(\frac{\pi}{6}\right) = \pi i\left(\frac{21}{16}\right)$$

(ii) Let $I = \int_C \frac{e^{3z}dz}{z+i}$, where 'c' is $|z+1+i|=2$

Take $f(z) = e^{3z}$ and $z+i=0 \Rightarrow z=-i$

if $z=-i$, then $|z+1+i| = |-i+1+i| = 1 < 2 = R$.

$\therefore$ $z=-i$ lies inside C.

By (R_2), $I = 2\pi i f(-i) = 2\pi i e^{-3i}$

$\boxed{\text{Note}}$: Here we use two results:

→ (R_1) if $f(z)$ is analytic within and on a closed contour C, then $\int_C f(z)\,dz = 0$.

→ (R_2) if $z=a$ is a point inside a closed contour C and $f(z)$ is analytic within and on C, then

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} = f(a)$$

(R_3) $$f^{(n)}(a) \; \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} \text{ for } n=1,2,3,\ldots$$

where $z=a$ is inside C.

(1)(e) A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Rs 20 per chair and Rs 30/- per table. Both products are processed on three machines M_1, M_2, M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows

Machines	Chairs	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution? Solve geographically.

Sol^n Let firm makes production of x chairs and y tables.

Contribution of chairs and tables

$P = 20x + 30y$ (Maximize)

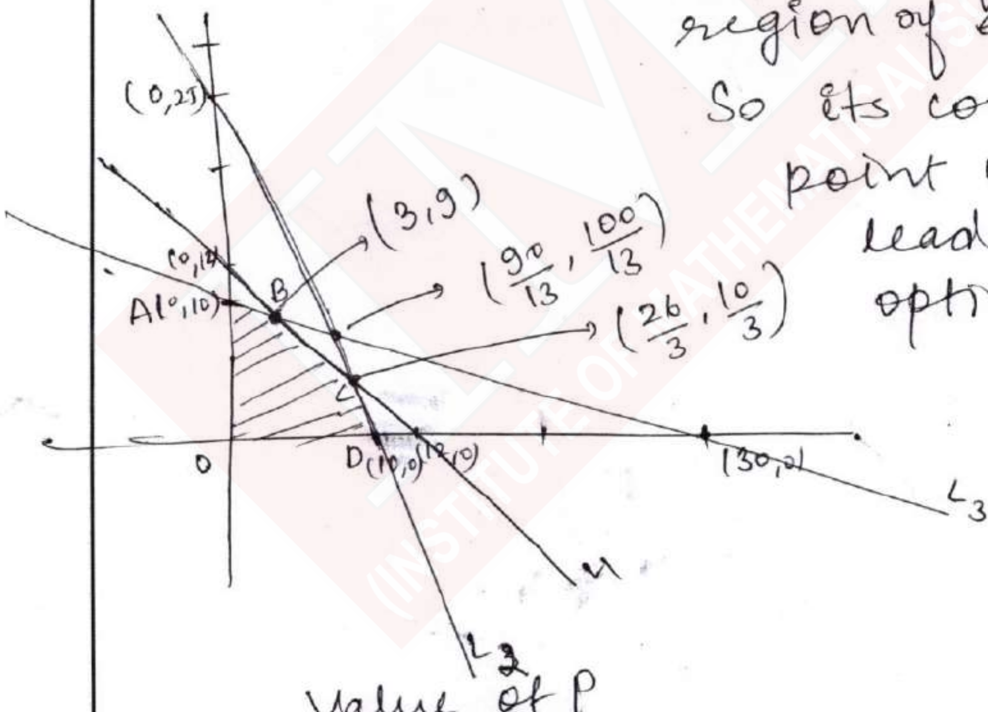
Time requirement on machines M_1, M_2, M_3 by chairs and tables –

$3x + 3y \le 36$ —— L_1

$5x + 2y \le 50$ —— L_2

$2x + 6y \le 60$ —— L_3

Geographically – shadded region OABCD is feasible region of equations. So its corner point will lead to optimality



Value of P

$O(0,0) \Rightarrow 0$

$A(0,10) \Rightarrow 20\times0 + 30\times10 = 300$

$B(3,9) \Rightarrow 20\times3 + 30\times9 = 330$ ⟶ Max

$C\left(\frac{26}{3}, \frac{10}{3}\right) \Rightarrow 20\times\frac{26}{3} + 30\times\frac{10}{3} = 273.3$

$D(10,0) \Rightarrow 20\times10 + 30\times0 = 200$

For maximum contribution – production of 3 chairs and 9 tables per week.

2.a(i) Let x belong to a group. If $x^2 \neq e$ while $x^6 = e$, Prove that $x^4 \neq e$ and $x^5 \neq e$. What can we say about the order of x?

Solⁿ: We have $x^2 \neq e$ and $x^6 = e$.

So $x^4 = x^{-2}$. since $x^2 \neq e$

we obtain $x^{-2} \neq e$. Then we get

$$x^4 = x^{-2} \neq e.$$

Now multiplying by x^{-1} both sides of the equation $x^6 = e$, we have

$$x^5 = x^{-1} \neq e \quad \text{since } x^2 = e$$

then we have

$$x^{-1} \neq x \neq e.$$

The order of x is either 3 or 6.

2.a(ii) If $|a| = n$, show that $|a^t| = \frac{n}{\gcd(n,t)}$.

Solⁿ: Let $|a| = n$. Then $(a^t)^{\frac{n}{\gcd(n,t)}} = (a^n)^{\frac{t}{\gcd(n,t)}} = e$

as we know that $\frac{t}{\gcd(n,t)}$ is an integer.

If $|a^t| = s$, then by above $s \mid \frac{n}{\gcd(n,t)}$.

The equality $|a^t| = s$, implies that $a^{ts} = e$.

This implies that $n \mid ts$ and $n \leq ts$.

Let $K = \gcd(n,t)$.

Then $\gcd\left(\frac{t}{K}, \frac{n}{K}\right) = 1$.

Let $ts = nm$ for some $m \in Z$. Then $\frac{t}{K}s = \frac{n}{K}m$.

So $\frac{t}{K} \mid m$ as $\frac{t}{K}$ and $\frac{n}{K}$ are relatively prime.

Hence by cancelling the $\frac{t}{K}$ in the above equation

we have $s = \frac{n}{K}m_1 = \frac{n}{\gcd(n,t)}m_1$ for some

$m_1 \in Z$. Then $\frac{n}{\gcd(n,t)} \mid s$

So m_1 must be 1 and we obtain

$s = \frac{n}{\gcd(n,t)} = |a^t|$.

2-(b) → Every bounded infinite subset of $\mathbb{R}$ has at least one limit point (in $\mathbb{R}$).

Sol^n: Let S be a bounded subset of $\mathbb{R}$ containing infinite number of elements. since S is a non-empty bounded subset of $\mathbb{R}$, $\sup S$ and $\inf S$ exist.

Let $a_1 = \inf S$, $b_1 = \sup S$.

Then $x \in S \Rightarrow a_1 \leq x \leq b_1$

i.e., $x \in [a_1, b_1]$.

Thus S is contained in the closed and bounded interval $I_1 = [a_1, b_1]$.

Let $c_1 = \frac{a_1+b_1}{2}$. Then at least one of the closed intervals $[a_1, c_1]$, $[c_1, b_1]$ must contain infinitely many elements of S. Because, otherwise, S would be a finite set. we take one such subinterval containing infinitely many elements of S and call it $I_2 = [a_2, b_2]$.

$I_2 \subset I_1$ and $|I_2| = \frac{b_1-a_1}{2}$.

Let $c_2 = \frac{a_2+b_2}{2}$. Then at least one of the closed intervals $[a_2, c_2]$, $[c_2, b_2]$ must contain infinitely many elements of S. we take one such subinterval containing infinitely many elements of S and call it $I_3 = [a_3, b_3]$.

$I_3 \subset I_2 \subset I_1$ and $|I_3| = \frac{b_1-a_1}{2}$.

Let $c_3 = \frac{a_3+b_3}{2}$. continuing in a similar manner we obtain a family of closed and bounded intervals $\{I_n\}$ such that

(i) $I_1 \supset I_2 \supset I_3 \supset \cdots\cdots$

(ii) $|I_n| = \frac{1}{2^{n-1}}(b_1 - a_1)$, for each $n \in N$

(iii) I_n contains infinitely many elements of S, for each $n \in N$.

So $\{I_n : n \in N\}$ is a family of nested closed and bounded intervals and $\inf\{(b_n - a_n) : n \in N\} = 0$.

By the nested intervals theorem, there exists precisely one point x such that $\{x\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$.

We now prove that x is a limit point of S.

Let $\epsilon > 0$. since $\inf\{(b_n - a_n) : n \in N\} = 0$, there exists a natural number m such that $0 \leq b_m - a_m < \epsilon$.

$\because x \in I_m$ and $b_m - a_m < \epsilon$, $I_m \subset N(x, \epsilon)$.

Since I_m contains infinitely many elements of S, $N(x,\epsilon)$ contains infinitely many elements of S and this happens for each $\epsilon > 0$.

$\therefore$ x is a limit point of S.

Thus S has a limit point.

2.(c) By using contour integration prove that

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p}, \ 0<p<1$$

Solⁿ: Let c denote unit circle $|z|=1$.

and $I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2}$ s.t. $0<p<1$

Then $I = \frac{1}{2}\int_0^{2\pi} \frac{(1+\cos 6\theta)\,d\theta}{1-2p\cos 2\theta + p^2}$

$$= R.P.\ \frac{1}{2}\int_0^{2\pi} \frac{(1+e^{i6\theta})\,d\theta}{1-p(e^{i2\theta}+e^{-2\theta})+p^2}$$

Putting $z=e^{i\theta}$ so that $dz = ie^{i\theta}d\theta$, $\frac{dz}{iz} = d\theta$,

we get $I = R.P.\ \frac{1}{2}\int_C \frac{(1+z^6)}{1-p(z^2+z^{-2})+p^2}\,\frac{dz}{iz}$

$$= R.P.\left(\frac{-1}{2ip}\right)\int_C \frac{z(1+z^6)\,dz}{z^4-\left(\frac{1+p^2}{p}\right)z^2+1}$$

or $I = R.P.\ \frac{-1}{2ip}\int_C f(z)\,dz$ ——— (1)

$$f(z) = \frac{z(1+z^6)}{z^4-\left(\frac{1+p^2}{p}\right)z^2+1}$$

Poles of $f(z)$ are given by $z^4-\left(\frac{1+p^2}{p}\right)z^2+1=0$

or $pz^4-(1+p^2)z^2+p=0$

or $z^2 = \dfrac{(1+p^2) \pm [(1+p^2)^2 - 4p^2]^{1/2}}{2p}$

or $z^2 = \dfrac{(1+p^2) \pm (1-p^2)}{2p} = \dfrac{1}{p}, p$ so that

$$z = \pm \frac{1}{\sqrt{p}}, \pm\sqrt{p}$$

The poles lying within the unit circle C are $\pm\sqrt{p}$ as $0 < p < 1$

$$\text{Res}(z=\sqrt{p}) + \text{Res}(z=-\sqrt{p})$$

$$= \lim_{z\to\sqrt{p}} (z-\sqrt{p}) f(z) + \lim_{z\to-\sqrt{p}} (z+\sqrt{p}) f(z)$$

$$= \lim_{z\to\sqrt{p}} \frac{(z-\sqrt{p}) z(1+z^6)}{(z^2-p)(z^2-1/p)} + \lim_{z\to-\sqrt{p}} \frac{(z+\sqrt{p}) z(1+z^6)}{(z^2-p)(z^2-1/p)}$$

$$= \frac{(1+p^3)\sqrt{p}}{(\sqrt{p}+\sqrt{p})(p-1/p)} + \frac{(-\sqrt{p})(1+p^3)}{(-\sqrt{p}-\sqrt{p})(p-1/p)}$$

as $z^2 - p = (z-\sqrt{p})(z+\sqrt{p})$

$$= \frac{p(1+p^3)}{p^2-1}$$

$\int_C f(z)dz = 2\pi i(\text{Sum of residues within } C) = \dfrac{2\pi i p(1+p^3)}{p^2-1}$

Now by (1), $I = R.P.\left(-\dfrac{1}{2ip}\right)\dfrac{2\pi i p(1+p^3)}{p^2-1}$

$$= R.P.\left(\frac{1+p^2-p}{1-p}\right)\pi$$

or $I = \left(\dfrac{1+p^2-p}{1-p}\right)\pi$.

3.(a) → Let f be an isomorphism of a ring R onto a ring R'. Show that

(i) If R is an integral domain, then R' is also an integral domain.

(ii) If R is a field, then R' is also a field.

Solⁿ: (i) Let R be an integral domain. Since R is commutative and $f: R \to R'$ is an onto homomorphism, R' is also commutative. Now we show that R' is without zero divisors.

Let $a', b' \in R'$ be such that $a'b' = 0$.

$\because$ f is one to one and onto, there exist unique $a, b \in R$ such that

$$f(a) = a' \text{ and } f(b) = b'.$$

Now $a'b' = 0' \Rightarrow f(a)f(b) = 0'$

$\Rightarrow f(ab) = f(0)$, since f is a homomorphism

$\Rightarrow ab = 0$, since f is one to one

$\Rightarrow a = 0$ or $b = 0$, since R is an integral domain

$\Rightarrow f(a) = f(0)$ or $f(b) = f(0)$

$\Rightarrow a' = 0'$ or $b' = 0'$, since $f(0) = 0'$.

Thus $a'b' = 0' \Rightarrow a' = 0'$ or $b' = 0'$

$\Rightarrow R'$ has no zero divisors.

Hence R' is an integral domain.

(ii) Let R be a field. Since $f: R \to R'$ is an onto homomorphism and since R is a commutative ring with unity 1, so R' is also a commutative ring with unity $f(1)$. Finally, we show that every non-zero element of R' has its multiplicative inverse.

Let a' be any non-zero element of R'. Since f is one to one and onto, there exists a unique non-zero element $a \in R$ such that $f(a) = a'$. Notice that if $a = 0$, then $f(a) = f(0) = 0'$ and so $a' = 0'$, a contradiction. Since R is a field and $a \neq 0 \in R$, $a^{-1} \in R$ exists and $aa^{-1} = a^{-1}a = 1$.

Now $aa^{-1} = 1$

$\Rightarrow f(aa^{-1}) = f(1)$

$\Rightarrow f(a) f(a^{-1}) = f(1)$

$\Rightarrow a' f(a^{-1}) = f(1)$.

Similarly, $a^{-1}a = 1$

$\Rightarrow f(a^{-1}) a' = f(1)$.

$\therefore (a')^{-1} = f(a^{-1}) \in R'$.

Hence R' is a field.

3.(b) A function f is defined on $[0,1]$ by

$$f(x) = x, \text{ if } x \text{ is rational}$$
$$= 1-x, \text{ if } x \text{ is irrational.}$$

Show that f is not integrable on $[0,1]$.

Solⁿ: f is bounded on $[0,1]$. Let P_n be the partition of $[0,1]$ defined by

$P_n = (x_0, x_1, x_2, \ldots, x_{2n})$ where

$$x_r = \frac{r}{2n}, \ 0 \leq r \leq 2n.$$

Let us choose α_r in $[x_{r-1}, x_r]$ by $\alpha_r = x_r$, for $r = 1, 2, \ldots n$

and $\alpha_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = n+1, \ldots, 2n$.

Then $S(P_n, f, \alpha) = \frac{1}{2n}\Big[(x_1 + x_2 + \cdots + x_n) + \left(1 - x_{n+1} + \frac{1}{\sqrt{5}n}\right) + \cdots + \left(1 - x_{2n} + \frac{1}{\sqrt{5}n}\right)\Big]$

$$= \frac{1}{2n}\left[\frac{1+2+\cdots+n}{2n} + n + \frac{1}{\sqrt{5}} - \frac{(n+1)+\cdots+2n}{2n}\right]$$

$$= \frac{1}{2n}\left[\frac{2n}{4} + \frac{1}{\sqrt{5}}\right].$$

Let us choose β_r in $[x_{r-1}, x_r]$ by $\beta_r = x_r - \frac{1}{\sqrt{5}n}$, for $r = 1, 2, \ldots n$.

and $\beta_r = x_r$, for $r = n+1, \ldots, 2n$.

Then $S(P_n, f, \beta) = \frac{1}{2n}\Big[\left(1 - x_1 + \frac{1}{\sqrt{5}n}\right) + \cdots + \left(1 - x_n + \frac{1}{\sqrt{5}n}\right) + (x_{n+1} + \cdots + x_{2n})\Big]$

$$= \frac{1}{2n}\left[n - \frac{1+2+\cdots+n}{2n} + \frac{1}{\sqrt{5}} + \frac{(n+1)+\cdots+2n}{2n}\right]$$

$$= \frac{1}{2n}\left[\frac{6n}{4} + \frac{1}{\sqrt{5}}\right].$$

Let us consider the sequence of partitions $\{P_n\}$.

$\|P_n\| = \frac{1}{2n}$. $\lim \|P_n\| = 0$.

$\lim\limits_{n\to\infty} S(P_n, f, \alpha) = \frac{1}{4}$, $\lim\limits_{n\to\infty} S(P_n, f, \beta) = \frac{3}{4}$.

Since for two different choices of intermediate points ξ_r, the Riemann sums $S(P_n, f, \xi)$ converge to two different limits, f is not integrable on $[0,1]$.

3.(C) Solve the following LPP by using simplex method.

Maximize $Z = 8x_2$, subject to the constraints:

$$x_1 - x_2 \geq 0,\ 2x_1 + 3x_2 \leq -6 \text{ and}$$

x_1, x_2 are unrestricted.

Sol: In this problem, the variables x_1 and x_2 are unrestricted in sign, i.e. x_1 and x_2 may be +ve, −ve or zero. But, the simplex method can be used only when the variables are non-negative (≥ 0). This difficulty can be immediately removed by using the transformation:

$$x_1 = x_1' - x_1'' \text{ and } x_2 = x_2' - x_2'' \text{ such that}$$

$x_1' \geq 0,\ x_1'' \geq 0,\ x_2' \geq 0,\ x_2'' \geq 0.$

Therefore, the given problem becomes:

maximize $z = 8x_2' - 8x_2''$, subject to the constraints:

$$(x_1' - x_1'') - (x_2' - x_2'') \geq 0$$

$$-2(x_1' - x_1'') - 3(x_2' - x_2'') \geq 6$$

$$x_1', x_1'', x_2', x_2'' \geq 0.$$

Now introducing the surplus variables $x_3 \geq 0, x_4 \geq 0$. and artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the given problem becomes:

$$\text{Max } z = 0x_1' + 0x_1'' + 8x_2' - 8x_2'' + 0x_3 + 0x_4 - Ma_1 - Ma_2,$$

Subject to $x_1' - x_1'' - x_2' + x_2'' - x_3 + a_1 = 0$

$$-2x_1' + 2x_1'' - 3x_2' + 3x_2'' - x_4 + a_2 = 6$$

$$x_1', x_1'', x_2', x_2'', x_3, x_4, a_1, a_2 \geq 0.$$

Basic Var.	C_B	X_B	x_1'	x_1''	x_2'	x_2''	x_3	x_4	A_1	A_2	Min. Ratio X_B/x_K
$c_j \rightarrow$			0	0	8	-8	0	0	-M	-M	
$\leftarrow a_1$	-M	0	1	-1	-1	$\boxed{1}$ ↑	-1	0	1	0	0 ←
a_2	-M	6	-2	2	-3	3	0	-1	0	1	6/3
	Z = -6M		M	-M	(4M-8)	(-4M+8) ↑	M	M	0 ↓	1	$\leftarrow \Delta_j$
$\rightarrow x_2'$	-8	0	1	-1	-1	1	-1	0	x	0	
$\leftarrow a_2$	-M	6	-5	$\boxed{5}$	0	0	3	-1	x	1	6/5 ←
	Z = -6M		(5M-8)	(-5M+8) ↑	0	0	(-3M+8)	M	x	0 ↓	$\leftarrow \Delta_j$
x_2''	-8	6/5	0	0	-1	1	-2/5	1/5	x	x	
$\rightarrow x_1''$	0	6/5	-1	1	0	0	3/5	-1/5	x	x	
	Z = -48/5		0	0	0	0	16/5	8/5	x	x	$\Delta_j \geq 0$

Remember that the coefficients of slack or surplus variables in the objective function are always zero and the coefficient of artificial variables is taken a largest negative quantity $-M$ where $M>0$.

Since all $\Delta_j \geq 0$, an optimum solution is obtained as: $x'=0$, $x_1''=6/5$, $x_2'=0$, $x_2''=6/5$.

Since $x_1 = x_1' - x_1''$ and $x_2 = x_2' - x_2''$, transforming the solution to original variables, we get

$$x_1 = 0 - 6/5 = -6/5$$

$$x_2 = 0 - 6/5 = -6/5$$

$$\max. z = -48/5.$$

4.(a) → Find the g.c.d. of $11+7i$ and $18-i$ in $Z[i]$.

Sol^n: We have $\frac{18-i}{11+7i} = \frac{(18-i)(11-7i)}{(11+7i)(11-7i)} = \frac{191-137i}{170}$

$$= (1-i) + \left(\frac{21}{170} + \frac{33}{170}i\right).$$

$$\therefore (18-i) = (1-i)(11+7i) + \left(\frac{21}{170} + \frac{33i}{170}\right)(11+7i)$$

or $(18-i) = (1-i)(11+7i) + 3i$, where $d(3i) < d(11+7i)$.

Now we consider

$$\frac{11+7i}{3i} = \frac{7}{3} + \frac{11}{3i} = \frac{7}{3} - \frac{11}{3}i = (2-3i) + \left(\frac{1}{3} - \frac{2}{3}i\right)$$

or $11+7i = (2-3i)3i + \left(\frac{1}{3} - \frac{2}{3}i\right)(3i)$

or $11+7i = (2-3i)3i + (2+i)$, where $d(2+i) < d(3i)$.

Again $\frac{3i}{2+i} = \frac{3i(2-i)}{(2+i)(2-i)} = \frac{3+6i}{5} = (1+2i) + \left(-\frac{2}{5} - \frac{4}{5}i\right)$

or $3i = (1+2i)(2+i) + \left(-\frac{2}{5} - \frac{4}{5}i\right)(2+i)$

or $3i = (1+2i)(2+i) - 2i$, where $d(-2i) < d(2+i)$.

Further $\frac{2+i}{-2i} = -\frac{1}{i} - \frac{1}{2} = -\frac{1}{2} + i$

or $2+i = i(-2i) + i$, where $d(i) < d(-2i)$.

Finally, $-\frac{2i}{i} = -2$ or $-2i = (-2)i$

Thus i is the g.c.d. of $11+7i$ and $18-i$, where i is a unit in $Z[i]$.

Hence $11+7i$ and $18-i$ are co-prime elements in $Z[i]$.

4.(b) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and f is continuous at a point of $\mathbb{R}$; Prove that f is uniformly continuous on $\mathbb{R}$.

Solⁿ: Let f be continuous at a point $c \in \mathbb{R}$.
Let us choose $\epsilon > 0$. There exists a positive δ such that $|f(c+h) - f(c)| < \epsilon$ for all h satisfying $|h| < \delta$.
But $|f(c+h) - f(c)| = |f(c) + f(h) - f(c)| = |f(h)|$.
Continuity of f at c implies $|f(h)| < \epsilon$ for all h satisfying $|h| < \delta$.

Let x_1, x_2 be any two points in $\mathbb{R}$ such that $|x_1 - x_2| < \delta$. Then $|f(x_1 - x_2)| < \epsilon$.

$f(x+y) = f(x) + f(y)$ gives $f(0+0) = f(0) + f(0)$
or $f(0) = 2f(0)$ or $f(0) = 0$.

Also $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$.
Therefore $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

$$|f(x_1 - x_2)| = |f(x_1) + f(-x_2)|$$
$$= |f(x_1) - f(x_2)|$$

Thus $|f(x_1) - f(x_2)| < \epsilon$ for any two points x_1, x_2 in $\mathbb{R}$ satisfying $|x_1 - x_2| < \delta$.

δ depends on ϵ only and not on the points x_1, x_2 in $\mathbb{R}$.
This proves that f is uniformly continuous on $\mathbb{R}$.

4(c) If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is an analytic function of $z=x+iy$, find $f(z)$ interms of z

Sol^n: Now $f(z)=u+iv$ so that $if(z)=iu-v$

$F(z)=(1+i)f(z)=(u-v)+i(u+v)=U+iV$, say

Here $U=u-v=(x-y)(x^2+4xy+y^2)$

$$\frac{\partial U}{\partial x}=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=x^2+4xy+y^2+(x-y)(2x+4y)$$

$$=3x^2+6xy-3y^2$$

$$\frac{\partial U}{\partial y}=\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=-(x^2+4xy+y^2)+(x-y)(4x+2y)$$

$$=3x^2-6xy-3y^2.$$

Let $\frac{\partial U}{\partial x}=\phi_1(x,y)$ and $\frac{\partial U}{\partial y}=\phi_2(x,y)$.

By Milne's method we have

$$F'(z)=\phi_1(z,0)-i\phi_2(z,0).$$

$$F(z)=\int[\phi_1(z,0)-i\phi_2(z,0)]\,dz+C.$$

$$=\int(3z^2-3iz^2)\,dz+C$$

$$=3\int(1-i)z^2\,dz+C$$

$$\Rightarrow F(z)=(1-i)z^3+C$$

$$\Rightarrow(1+i)f(z)=\frac{1-i}{\cancel{1+i}}z^3+\frac{C}{\cancel{1+i}}$$

$$\Rightarrow f(z)=\frac{1-i}{1+i}z^3+\frac{C}{1+i}$$

$$\Rightarrow f(z)=-iz^3+C_1 \quad \text{where } C_1=\frac{C}{1+i}.$$

4.C.(ii) The function $f(z)$ has a double pole at $z=0$ with residue 2, a simple pole at $z=1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If $f(2)=5$, $f(-1)=2$, find $f(z)$.

Soln: $\text{Res}(z=1)=2$, order 1 and $\text{Res}(z=0)=2$, order 2.

Hence $f(z)$ will be expressible as

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \frac{2}{z-1} + \frac{2}{z} + \frac{b}{z^2} \quad \text{———} (1)$$

Hence $|f(z)| \leq M$ where $M>0$. Since $f(z)$ is bounded as $|z| \to \infty$. It follows that $f(z)$ has no singularity at $z=\infty$, showing thereby $f(t)$ has no singularity at $t=0$ where $t=1/z$. As a result of which principle part of $f(t)$ contains no terms so that $a_n = 0 \ \forall n$. For $\sum_{1}^{\infty} a_n t^{-n}$ is the principle part of $f(t)$, by (1).

Now (1) becomes $f(z) = a_0 + \frac{2}{z-1} + \frac{2}{z} + \frac{b}{z^2}$ ——— (2)

This $\Rightarrow f(2) = 5 = a_0 + 2 + 1 + \frac{b}{4}$

and $f(-1) = 2 = a_0 - 1 - 2 + b$

$\Rightarrow a_0 = 1, \ b = 4.$

$$\therefore f(z) = 1 + \frac{2}{z-1} + \frac{2}{z} + \frac{4}{z^2}.$$

4.(d) → Five salesmen are to be assigned to five territories. Based on the past performance, the following table shows the annual sales (in rupees lakhs) that can be generated by each salesman in each territory. Find the optimum assignment.

Salesman	Territory T_1	T_2	T_3	T_4	T_5
S_1	26	14	10	12	9
S_2	31	27	30	14	16
S_3	15	18	16	25	30
S_4	17	12	21	30	25
S_5	20	19	25	16	10

Soln: Step 1. Since the matrix represents the sales which can be generated by each territory, the objective function of the assignment problem is, therefore, to maximize the total sales generated. But the algorithm for assignment problem is for minization of the objective function, we therefore, convert the given problem to minimization problem, by subtracting all the elements of the given matrix from the maximum element 31 to obtain the adjoining matrix.

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	5	17	21	19	22
S_2	0	4	1	17	15
S_3	16	13	15	6	1
S_4	14	19	10	1	6
S_5	11	12	6	15	21

Step 2.(i) Row Substraction

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	0	12	16	14	17
S_2	0	4	1	17	15
S_3	15	12	14	5	0
S_4	13	18	9	0	5
S_5	5	6	0	9	15

Step 2.(ii) column subtraction

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	0	8	16	14	17
S_2	0	0	1	17	15
S_3	15	8	14	5	0
S_4	13	14	9	0	5
S_5	5	2	0	9	15

Step 3. Minimum straight lines to cover zeros.

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5	
S_1	0	8	16	14	17	
S_2	0	0	1	17	15	$\to L_2$
S_3	15	8	14	5	0	
S_4	13	14	9	0	5	
S_5	5	2	0	9	15	
	L_1		L_3	L_4	L_5	

Step 4. Since number of lines is 5, the optimality criteria is satisfied.

Salesman \ Territory	T_1	T_2	T_3	T_4	T_5
S_1	$\boxed{0}$	8	16	14	17
S_2	$\cancel{0}$	$\boxed{0}$	1	17	15
S_3	15	8	14	5	$\boxed{0}$
S_4	13	14	9	$\boxed{0}$	5
S_5	5	2	$\boxed{0}$	9	15

The optimum assignment is:

$$S_1 \to T_1, \quad S_2 \to T_2, \quad S_3 \to T_5,$$

$$S_4 \to T_4, \quad S_5 \to T_3$$

and the maximum sales generated are:

$$26 + 27 + 30 + 30 + 25 = 138.$$

Q.1. (b) → Solve $(D^2-4D'^2)z = (4x/y^2)-(y/x^2)$

Solution:

Here A.E. is $m^2-4=0$

$\Rightarrow m = 2,-2.$

$\therefore$ C.F. $= \phi_1(y+2x)+\phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I. = \frac{1}{(D+2D')(D-2D')}\left(\frac{4x}{y^2}-\frac{y}{x^2}\right)$$

$$= \frac{1}{D+2D'}\int\left\{\frac{4x}{(c-2x)^2}-\frac{c-2x}{x^2}\right\}dx,$$

where $c = y + 2x$.

$$= \frac{1}{D+2D'} \int \left\{ -\frac{2}{c-2x} + \frac{2c}{(c-2x)^2} - \frac{c}{x^2} + \frac{2}{x} \right\} dx$$

$$= \frac{1}{D+2D'} \left\{ \log(c-2x) + \frac{c}{c-2x} + \frac{c}{x} + 2\log x \right\}$$

$$= \frac{1}{D+2D'} \left[\log y + \frac{y+2x}{y} + \frac{y+2x}{x} + 2\log x \right]$$

$$= \int \left\{ \log(c'+2x) + 1 + 2\cdot\frac{x}{c'+2x} + \frac{c'+2x}{x} + 2 + 2\log x \right\} dx$$

[Taking $c' = y - 2x$]

$$= x\log(c'+2x) + 5x + c\log x + 2x\log x - 2x$$

$$= x\log y + y\log x + 3x, \quad \text{as } c' = y - 2x$$

$$\boxed{\therefore\ z = \phi_1(y+2x) + \phi_2(y-2x) + x\log y + y\log x + 3x.}$$

Hence, the result.

6(b) Reduce $y^2\left(\frac{\partial^2 z}{\partial x^2}\right) - x^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$ to canonical form

Sol^n: Rewriting the given equation $x^2 r - y^2 t = 0$ —— ①

Comparing ① with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence ① is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x, -y/x$ and hence corresponding characteristic equations become $(dy/dx) + y/x = 0$ and $(dy/dx) - y/x = 0$

Integrating, we get

$$\frac{x^2+y^2}{2} = c_1, \quad \frac{y^2-x^2}{2} = c_2$$

To reduce ① to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u = \frac{x^2+y^2}{2} \text{ and } v = \frac{y^2-x^2}{2}$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v} = x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v} = y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x\left\{\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]\right\}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x^2\frac{\partial^2 z}{\partial u^2} - x^2\frac{\partial^2 z}{\partial u\partial v} - x^2\frac{\partial^2 z}{\partial v\partial u} + x^2\frac{\partial^2 z}{\partial v^2}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x^2\frac{\partial^2 z}{\partial u^2} + x^2\frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u\partial v}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + y\left[\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)y + \left(\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}\right)y\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Substituting the values of r and t in ①, we get

$$y^2\left\{\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + x^2\frac{\partial^2 z}{\partial u^2} + x^2\frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u \partial v}\right\}$$

$$-x^2\left\{\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + y^2\frac{\partial^2 z}{\partial u^2} + 2y^2\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2}\right\} = 0$$

$$\Rightarrow (y^2 - x^2)\frac{\partial z}{\partial u} - (y^2 + x^2)\frac{\partial z}{\partial v} - 4x^2y^2\frac{\partial^2 z}{\partial u \partial v} = 0$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2y^2}\left[2v\frac{\partial z}{\partial u} - 2u\frac{\partial z}{\partial v}\right]$$

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = \frac{1}{(x^2+y^2)^2 - (y^2-x^2)^2}\left[2v\frac{\partial z}{\partial u} - 2u\frac{\partial z}{\partial v}\right]$$

$$= \frac{2}{4u^2 - 4v^2}\left[v\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial v}\right]$$

$$= \frac{1}{2(u^2 - v^2)}\left[v\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial v}\right]$$

which is the required canonical form.

5(c) By using Newton Raphson method, show that the equation $2e^{-x} = \frac{1}{x+2} + \frac{1}{x+1}$ has two roots greater than -1. Calculate these roots correct to five decimal places.

Sol'n: Let $f(x) = 2e^{-x} - \frac{1}{x+2} - \frac{1}{x+1}$

and $f'(x) = -2e^{-x} + \frac{1}{(x+2)^2} + \frac{1}{(x+1)^2}$

$f(-0.9) = -5.99$; $f(0) = 0.5$; $f(1) = -0.0976$
$f(-0.9) < 0$; $f(0) = > 0$; $f(1) < 0$

$\therefore$ The equation $f(x)=0$ has two roots which are greater than -1 lie in the intervals $(-0.9, 0)$ and $(0, 1)$.
Let the initial approximation be $x_0 = -0.9$ in the interval $(-0.9, 0)$

By Newton - Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——} (1)$$

we get

$x_1 = -0.83754$, $x_2 = -0.76713$, $x_3 = -0.71318$

$x_4 = -0.69213$, $x_5 = -0.68978$, $x_6 = -0.68975$

$x_7 = -0.68975$

For second root put $x_0 = 1$, by (1) we get

$x_1 = 0.73947$, $x_2 = 0.76963$, $x_3 = 0.77009$, $x_4 = 0.77009$

$\therefore$ The two roots are given by -0.689752 and 0.770091 which are greater than -1.

Ques:-5(d) Convert the following:

(i) $(41.6875)_{10}$ to binary number

Solve:-

2	41	
2	20	1
2	10	0
2	5	0
2	2	1
	1	0

$0.6875 \times 2 = \underline{1}.3750 \rightarrow 1$
$0.3750 \times 2 = \underline{0}.7500 \rightarrow 0$
$0.7500 \times 2 = \underline{1}.5000 \rightarrow 1$
$0.5000 \times 2 = \underline{1}.0000 \rightarrow 1$
0.0000

$$\boxed{\therefore (41.6875)_{10} \longleftrightarrow (101001.1011)_2}$$

(ii) $(101101)_2$ to decimal number.

Solve:- $1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0$

$= 32+0+8+4+0+1$

$= 45.$

$$\boxed{\therefore (101101)_2 \longleftrightarrow (45)_{10}}$$

(iii) $(AF63)_{16}$ to decimal number.

Solve :- $A\times 16^3 + F\times 16^2 + 6\times 16^1 + 3\times 16^0$

$= 10\times 16^3 + 15\times 256 + 6\times 16 + 3$

$= 40960 + 3840 + 96 + 3 = 44899.$

$$\boxed{\therefore (AF63)_{16} \longleftrightarrow (44899)_{10}}$$

(iv) $(101111011111)_2$ to Hexadecimal number

Solve:- $(\underbrace{1011}_{B}\ \underbrace{1101}_{D}\ \underbrace{1111}_{F})_2 \longleftrightarrow (BDF)_{16}$

$$\boxed{\therefore (101111011111)_2 \longleftrightarrow (BDF)_{16}}$$

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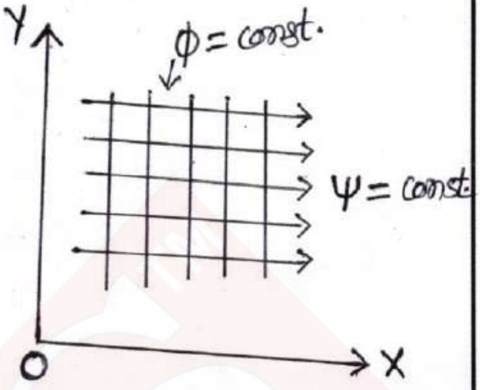
5.(e) A velocity field is given by $q = -xi + (y+t)j$. Find the stream function and the stream lines for this fied at $t=2$.



Sol^n: Here $q = ui + vj$

$$= -xi + (y+t)j$$

$$\Rightarrow u = -x, \; v = y+t$$

We know that

$$-\frac{\partial \psi}{\partial y} = u = -x \quad \text{—— (1)}$$

and $\frac{\partial \psi}{\partial x} = v = y+t$ —— (2)

By Integrating (1) with regard to y, we have

$$\psi = xy + f(x,t) \quad \text{—— (3)}$$

where $f(x,t)$ is an integration constant.

or $$\frac{\partial \psi}{\partial x} = y + \frac{\partial f}{\partial x} \quad \text{—— (4)}$$

From (2) and (4), we have

$$y + \frac{\partial f}{\partial x} = y + t \Rightarrow \frac{\partial f}{\partial x} = t$$

$$\Rightarrow f(x,t) = xt + g(t) \quad \text{—— (5)}$$

From (3) and (5), we have

$$\psi = xy + xt + g(t)$$

At $t=2$, $\psi = x(y+2) + g(2)$

The stream lines are given by $\psi = $ const.

Therefore $x(y+2) = $ const.,

which represent rectangular hyperbolas.

2 (a)i → Form partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2-y) + g(x^2+y)$.

Sol^n

Given $z = f(x^2-y) + g(x^2+y)$ —— ①

differentiating ① partially w.r.to x and y, we get

$$\frac{\partial z}{\partial x} = 2x f'(x^2-y) + 2x g'(x^2+y).$$

$$= 2x\{f'(x^2-y) + g'(x^2+y)\} \quad \text{—— ②}$$

and $\dfrac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y)$ —— ③

Differentiating ② and ③ w.r.to x and y respectively, we get -

$$\frac{\partial^2 z}{\partial x^2} = 2\{f'(x^2-y) + g'(x^2+y)\} + 4x^2\{f''(x^2-y) + g''(x^2+y)\} \quad \text{—— ④}$$

and $\dfrac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y)$ —— ⑤

Again ② $\Rightarrow f'(x^2-y) + g'(x^2+y)$

$$= \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right) \quad \text{—— ⑥}$$

Substituting the values of $f''(x^2-y) + g''(x^2+y)$ and $f'(x^2-y) + g'(x^2+y)$ from ⑤ and ⑥ in ④ we have.

$$\frac{\partial^2 z}{\partial x^2} = 2\left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2}$$

$$x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2}$$

Which is the required partial differential eq^n.

6.(b) Find a complete integral of $16p^2z^2+9q^2z^2+4z^2-4=0$.

Soln: Given equation is

$$f(x,y,z,p,q)=16p^2z^2+9q^2z^2+4z^2-4=0 \quad \text{——— (1)}$$

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{-p(32p^2z+18q^2z+8z)}=\frac{dq}{q(32p^2z+18q^2z+8z)}$$

$$=\frac{dz}{-p(32pz^2)-q(18qz^2)}=\frac{dx}{-32pz^2}=\frac{dy}{-18qz^2}.$$

Taking the first and second fractions,

$$(1/p)dp=(1/q)dq.$$

Integrating, $\log p=\log q+\log a$ so that $p=aq$. ——— (2)

Solving (1) and (2) for p and q, we have

$$p=\frac{2a\sqrt{(1-z^2)}}{z\sqrt{(16a^2+9)}} \text{ and } q=\frac{2\sqrt{(1-z^2)}}{z\sqrt{(16a^2+9)}} \quad \text{——— (3)}$$

$$\therefore \; dz=pdx+qdy$$

$$=\frac{2\sqrt{(1-z^2)}}{z\sqrt{(16a^2+9)}}(adx+dy), \text{ using (3)}$$

or $$(1/2)\sqrt{(16a^2+9)}\,(1-z^2)^{-1/2}(-2zdz)=-2(adx+dy) \quad \text{——— (4)}$$

Putting $1-z^2=t$ so that $-2zdz=dt$, (4) becomes

or $(1/2)\sqrt{(16a^2+9)}\,t^{-1/2}dt=-2(adx+dy)$.

Integrating,

$$\sqrt{(16a^2+9)}\, t^{1/2} = -2(ax+y)+b$$

or $\sqrt{(16a^2+9)}\sqrt{(1-z^2)} + 2(ax+y) = b,$

as $t = 1-z^2$.

Ques : 6(c)> Find the characteristics of the equation

$$pq = xy$$

and determine the integral surface which passes through the curve $z = x, y = 0$.

Solution:-

given equation is — $pq = xy$ —— ①

Integral surface passes through $z = x$ and $y = 0$ —— ②

Consider, parametric form

$\equiv x = \lambda, y = 0, z = \lambda$ —— ③

where λ is a parameter.

Let, the initial values of p, q, x, y, z be p_0, q_0, x_0, y_0, z_0 respectively —— ④

$x_0 = \lambda, y_0 = 0, z_0 = \lambda$ and $p_0 q_0 = x_0 y_0 = 0$ —— ⑤

we have $z_0' = x_0' p_0 + y_0' q_0 \Rightarrow 1 = 1(p_0) + 0$

$\Rightarrow \boxed{p_0 = 1}$

$\therefore q_0 = \frac{0}{p_0} = \frac{0}{1} = 0 \Rightarrow \boxed{q_0 = 0}$

$\therefore x_0 = \lambda, y_0 = 0, z_0 = \lambda, p_0 = 1, q_0 = 0$ —— ⑥

Consider $f(x, y, z, p, q) = pq - xy = 0$ —— ⑦

Hence, the characteristic equations are :-

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = q \quad \text{—— ⑧}$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = p \quad \text{—— ⑨}$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = pq + qp = 2pq \quad \text{—— (10)}$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -(-y) - p(0) = y \quad \text{—— (11)}$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -(-x) - q(0) = x \quad \text{—— (12)}$$

from (8) and (12),

$$\frac{dx/dt}{dq/dt} = \frac{q}{x} \Rightarrow \frac{dx}{dq} = \frac{q}{x}$$

$$\Rightarrow x\,dx - q\,dq = 0 \Rightarrow x^2 - q^2 = C_1^2 \quad \text{—— (13)}$$

from (9) & (11), we have

$$\frac{dy/dt}{dp/dt} = \frac{p}{y} \Rightarrow \frac{dy}{dp} = \frac{p}{y}$$

$$\Rightarrow y\,dy - p\,dp = 0 \Rightarrow y^2 - p^2 = C_2^2 \quad \text{—— (14)}$$

from (6) & (13)

$$\lambda^2 - 0^2 = C_1^2 \Rightarrow C_1^2 = \lambda^2 \Rightarrow x^2 = \lambda^2 + q^2 \quad \text{—— (15)}$$

from (6) and (14)

$$0 - 1^2 = C_2^2 \Rightarrow C_2^2 = 1 \Rightarrow y^2 = 1 + p^2 \quad \text{—— (16)}$$

Also from (8);

$$q = \frac{dx}{dt} \Rightarrow \frac{dq}{dt} = \frac{d^2x}{dt^2} = x \quad \left[\because \frac{dq}{dt} = x\right].$$

$$\therefore \frac{d^2x}{dt^2} = x \Rightarrow \frac{d^2x}{dt^2} - x = 0$$

$$\boxed{x = Ae^{t} + Be^{-t}} \quad \text{—— (17)}$$

from (9) & (11),

$$\frac{d^2y}{dt^2} = \frac{dp}{dt} = y$$

$$\therefore \frac{d^2y}{dt^2} - y = 0 \Rightarrow \boxed{y = Ce^{t} + De^{-t}} \quad \text{——} (18)$$

from (12) & (17),

$$\frac{dq}{dt} = Ae^{t} + Be^{-t} \Rightarrow \boxed{q = Ae^{t} - Be^{-t}} \quad \text{——} (19),$$

from (11) and (18)

$$\frac{dp}{dt} = y \Rightarrow \frac{dp}{dt} = Ce^{t} + De^{-t}$$

$$\boxed{p = Ce^{t} - De^{-t}} \quad \text{——} (20)$$

$$\therefore z = 2pq = 2\left[Ae^{t} - Be^{-t}\right]\left[Ce^{t} - De^{-t}\right]$$

$$z = 2ACe^{t} + 2BDe^{-2t} - 2(AD + BC) \quad \text{——} (21)$$

Initial Condition :- At $t = 0$, $q_0 = 0$ & $q_0 = 1$

$\therefore 0 = A - B$, $C - D = 1$; $\lambda = A + B$. ; $0 = C + D$

$pq - xy = 0$

$\therefore C(C - D)(A - B) = (A + B)(C + D)$

$\therefore C = \frac{1}{2}$, $D = -\frac{1}{2}$, $A = B = \lambda/2$.

$\therefore$ Characteristics :-

$x = Ae^{t} + Be^{-t}$ $\qquad$ $y = Ce^{t} + De^{-t}$

$p = Ce^{t} - De^{-t}$ $\qquad$ $q = Ce^{t} - De^{-t}$

$z = 2ACe^{2t} + 2BDe^{-2t} - 2(AD + BC)$

6.(d) → A thin rectangular homogeneous thermally conducting plate lies in the xy-plane defined by $0 \le x \le a$, $0 \le y \le b$. The edge $y=0$ is held at the temperature $Tx(x-a)$, where T is a constant, while the remaining edges are held at $0°$. The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate.

Solⁿ: Since no heat sources and sinks are present in the plate, the steady state temperature u must satisfy $\nabla^2 u = 0$. Hence the problem is to solve

$$\text{PDE: } \nabla^2 u = 0$$

$$\text{BCs: } u(0,y)=0,\quad u(a,y)=0,\quad u(x,b)=0,$$
$$u(x,0) = Tx(x-a)$$

This is a typical Dirichlet's problem. The general solution satisfying the first three B.Cs is given by

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\}$$

$$\text{where } A_n = \frac{2}{a}\frac{1}{\sinh(-n\pi b/a)}\int_0^a f(x)\sin(n\pi x/a)\,dx.$$

$$\text{(or) } A_n \sinh(-n\pi b/a) = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx.$$

Using the last BC: $u(x,0) = Tx(x-a) = f(x)$, we get

$$A_n \sinh\frac{-n\pi b}{a} = \frac{2}{a}\int_0^a Tx(x-a)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{2T}{a}\int_0^a x(x-a)\sin\left(\frac{n\pi}{a}x\right)dx.$$

$$= -\frac{a}{n\pi}\cdot\frac{2T}{a}\left[\int_0^a x(x-a)\, d\left\{\cos\left(\frac{n\pi}{a}x\right)\right\}\right]$$

$$= -\frac{2T}{n\pi}\left[(x-a)\cos\left(\frac{n\pi}{a}x\right)\right]_0^a - \frac{a}{n\pi}\int_0^a (2x-a)\, d\left[\sin\left(\frac{n\pi}{a}x\right)\right]$$

$$= \frac{2aT}{n^2\pi^2}\left[(2x-a)\sin\left(\frac{n\pi}{a}x\right)\right]_0^a - \int_0^a 2\sin\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{2aT}{n^2\pi^2}\left\{a\sin n\pi + \frac{2a}{n\pi}\left[\cos\left(\frac{n\pi}{a}x\right)\right]_0^a\right\}$$

$$= \frac{2aT}{n^2\pi^2}\,\frac{2a}{n\pi}(\cos n\pi - 1)$$

$$= \frac{4a^2T}{n^3\pi^3}\left[(-1)^n - 1\right]$$

Thus the required temperature distribution is given by

$$u(x,y) = \sum_{n=1}^{\infty} \operatorname{cosech}\left(-\frac{n\pi}{a}b\right)\frac{4Ta^2}{n^3\pi^3}\left[(-1)^n - 1\right]\sin\left(\frac{n\pi}{a}x\right)$$
$$\sinh\left[\frac{n\pi}{a}(y-b)\right].$$

7(a) Find the solution of the following system of equations

$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$, $-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$

$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$, $-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$

using Gauss-Seidel method and perform the first five iterations.

Sol'n: The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \longleftarrow (1)$$

By Gauss-Seidel method, system (1) can be written as

$$x_1^{K+1} = 0.5 + 0.25x_2^{(K)} + 0.25x_3^{(K)}$$

$$x_2^{K+1} = 0.5 + 0.25x_1^{K+1} + 0.25x_4^{(K)}$$

$$x_3^{K+1} = 0.25 + 0.25x_1^{K+1} + 0.25x_4^{(K)}$$

$$x_4^{K+1} = 0.25 + 0.25x_2^{K+1} + 0.25x_3^{K+1}$$

where $K = 0, 1, 2, 3 \ldots$

Now taking $x^{(0)} = 0$ (i.e. $x_2^{(0)} = 0$, $x_3^{(0)} = 0$, $x_4^{(0)} = 0$) which is initial solution.

K=0

$$x_1^{(1)} = 0.5 + 0.25x_2^{(0)} + 0.25x_3^{(0)} = 0.5 + 0 + 0 = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 0.5 + (0.25)(0.5) + 0 = 0.625$$

$$x_3^{(1)} = 0.25 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 0.25 + (0.25)(0.5) + (0.25)(0) = 0.375$$

$$x_4^{(1)} = 0.25 + 0.25x_1^{(1)} + 0.25x_3^{(1)} = 0.25 + (0.25)(0.625) + (0.25)(0.375) = 0.5$$

K=1

$$x_1^{(2)} = 0.5 + 0.25x_2^{(1)} + 0.25x_3^{(1)} = 0.5 + (0.25)(0.625) + (0.25)(0.375) = 0.75$$

$$x_2^{(2)} = 0.5 + 0.25x_1^{(2)} + 0.25x_4^{(1)} = 0.5 + (0.25)(0.75) + (0.25)(0.5) = 0.8125$$

$$x_3^{(2)} = 0.25 + 0.25x_1^{(2)} + 0.25x_4^{(2)} = 0.25[1 + 0.75 + 0.5] = 0.5625$$

$$x_4^{(2)} = 0.25 + 0.25x_2^{(2)} + 0.25x_3^{(2)} = (0.25)[1 + 0.8125 + 0.5625]$$

$$= 0.59375$$

K=2

$x_1^{(3)} = 0.5 + 0.25x_2^{(2)} + 0.25x_3^{(2)} = 0.5 + (0.25)(0.8125) + (0.25)(0.5625) = 0.84375$

$x_2^{(3)} = 0.5 + 0.25x_1^{(3)} + 0.25x_4^{(2)} = 0.5 + (0.25)(0.84375) + (0.25)(0.59375) = 0.85938$

$x_3^{(3)} = 0.25 + 0.25x_1^{(3)} + 0.25x_4^{(2)} = 0.25[1 + 0.84375 + 0.59375] = 0.60938$

$x_4^{(3)} = 0.25 + 0.25x_2^{(3)} + 0.25x_3^{(3)} = 0.25[1 + 0.85938 + 0.60938] = 0.61719$

K=3

$$x_1^{(4)} = 0.5 + 0.25x_2^{(3)} + 0.25x_3^{(3)} = 0.5 + (0.25)x_2^{(3)} + 0.25x_3^{(3)}$$
$$= 0.5 + (0.25)(0.85938) + (0.25)(0.60938)$$
$$= 0.86719$$

$$x_2^{(4)} = 0.5 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.5 + 0.25(0.86719) + 0.25(0.61719)$$
$$= 0.87110$$

$x_3^{(4)} = 0.25 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.25[1 + 0.46719 + 0.61719] = 0.62110$

$x_4^{(4)} = 0.25 + 0.25x_2^{(4)} + 0.25x_{(3)}^{(4)} = 0.25[1 + 0.8711 + 0.6211] = 0.62305$

K=4

$$x_1^{(5)} = 0.5 + 0.25x_2^{(4)} + 0.25x_3^{(4)} = 0.5 + (0.25)(0.8715) + (0.25)(0.6211)$$
$$= 0.87305$$

$$x_2^{(5)} = 0.5 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.5 + 0.25(0.87305) + 0.25(0.62305)$$
$$= 0.87402$$

$$x_3^{(5)} = 0.25 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.25[1 + 0.87305 + 0.62305]$$
$$= 0.62402$$

$$x_4^{(5)} = 0.25 + 0.25x_2^{(5)} + 0.25x_3^{(5)} = 0.25[1 + 0.87402 + 0.62402]$$
$$= 0.62451$$

The solution is given by

$x_1 = 0.87305$, $x_2 = 0.87402$, $x_3 = 0.62402$

$x_4 = 0.62451$.

5(C) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below.

Time in minutes	0	2	4	6	8	10	12
velocity in km/hr	0	22	30	27	18	7	0

Apply Simpson's rule to find the distance covered by the car.

sol^n: we know velocity $v = \frac{ds}{dt}$ —① where s = distance, t = time.

① ⇒ $ds = v\,dt$

∴ so distance covered by car in 12 min. is

$$S = \int_0^S ds = \int_0^{12} v\,dt \quad \text{——} ②$$

Here given

Time	$t_0=0$	$t_1=2$	$t_2=4$	$t_3=6$	$t_4=8$	$t_5=10$	$t_6=12$
velocity	$v_0=0$	$v_1=22$	$v_2=30$	$v_3=27$	$v_4=18$	$v_5=7$	$v_6=0$

② ⇒ $S = \int_0^{12} v\,dt$

Using Simpson's $\frac{1}{3}$rd rule

we get $S = \frac{h}{3}\left[(v_0+v_6)+4(v_1+v_3+v_5)+2(v_2+v_4)\right]$

$= \frac{h}{3}\left[(0+0)+4(22+27+7)+2(30+18)\right]$ —③

Since $h = 2\text{min} = \frac{2}{60} = \frac{1}{30}$ hour.

③ ⇒ $S = \frac{1/30}{3}\left[4(56)+2(48)\right] = 3.556$ km.

Hence distance covered by car is 3.556 km.

3(b) Solve the initial value problem $u' = -2tu^2$, $u(0)=1$ with $h=0.2$ on the interval $[0, 0.4]$. Use the fourth order classical Runge-Kutta method. Compare with the exact solution.

Sol: Given that $\frac{du}{dt} = -2tu^2 = f(t,u)$

$h = 0.2$

$t_0 = 0, \; u_0 = 1$

Now $k_1 = hf(t_0, u_0) = -2(0.2)(1)^2 = 0$

$k_2 = hf(t_0 + \frac{h}{2}, u_0 + \frac{k_1}{2}) = -2(0.2)(\frac{0.2}{2})(1)^2 = -0.04$

$k_3 = hf(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}k_2) = -2(0.2)(\frac{0.2}{2})(0.98)^2$

$= -0.038416$

$k_4 = hf(t_0 + h, u_0 + k_3) = -2(0.2)(0.2)(0.961584)^2$

$= -0.0739715$

By Runge Kutta fourth order method

$u_1 = u(0.2) = u_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1 + \frac{1}{6}[0 - 0.08 - 0.076832 - 0.0739715]$

$= 0.9615328$

$\therefore u(0.2) = 0.9615328$

For second step, we have

$t_1 = 0.2, \; u_1 = 0.9615328$

$k_1 = hf(t_1, u_1) = -2(0.2)(0.2)(0.9615328)^2$

$= -0.0739636$

$k_2 = hf(t_1 + \frac{h}{2}, u_1 + \frac{k_1}{2}) = -2(0.2)(0.3)(0.924551)^2$

$= -0.1025753$

$k_3 = hf(t_1 + \frac{h}{2}, u_1 + \frac{k_2}{2}) = -2(0.2)(0.3)(0.910245)^2$

$= -0.0994255$

$k_4 = hf(t_1+h,\ u_1+k_3) = -2(0.2)(0.4)(0.8621073)^2$
$= -0.1189166$

∴ By Runge kutta fourth order method

$u(0.4) = u_2 = u_1 + \frac{1}{6}[-0.0739636 - 0.2051506 - 0.1988510 - 0.1189166]$

$u(0.4) = 0.8620525.$

The exact solution of $u' = -2tu^2$

$\frac{du}{dt} = -2tu^2$

$\Rightarrow \frac{du}{u^2} = -2t\,dt$

$\Rightarrow -\frac{1}{u} = -t^2+c$ — ①

$\Rightarrow -1 = -0+c \quad (\because u=1,\ t=0)$

$\Rightarrow c = -1$

∴ from ① $-\frac{1}{u} = -t^2-1$

$\Rightarrow \frac{1}{u} = 1+t^2$

$\Rightarrow \boxed{u = \frac{1}{1+t^2}}$

The exact solution is

$u(0.2) = 0.961538$

$u(0.4) = 0.862069$

∴ The absolute errors in the numerical solutions

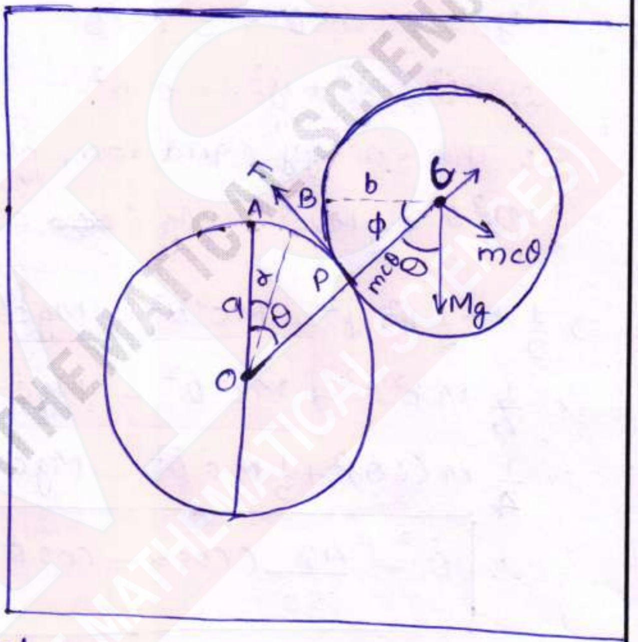
are $\epsilon(0.2) = |0.961539 - 0.961533| = 0.000006$

$\epsilon(0.4) = |0.862069 - 0.862053| = 0.000016$

Q.7(C)) A rough solid Cylinder rolls down a second rough cylinder which is fixed with its axis horizontal. If the plane through their axis makes an angle 'α' with the vertical when first cylinder is at rest, show that the cylinders will separate when this angle of inclination is $\cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$.

Solution:-

Let, O be the centre 'a' be the radius of fixed cylinder. Let, 'C' be the centre of and 'b' be the radius of the cylinder resting on the fixed cylinder. with its point 'B' in contact of 'A' of the fixed cylinder such that OA makes an angle 'α' to the vertical.



upper cylinder rolls & at time 't', Let 'P' be the point of contact of the two cylinder such that the line OC joining the centres of makes an angle 'θ' to the Vertical.

Let, CB make an angle ϕ to the vertical at time 't'.

Since, there is pure rolling.

$\therefore$ Arc AP = Arc BP or $a(\theta-\alpha) = b(\phi-\theta)$

i.e $b\phi = (a+b)\theta - a\alpha$ i.e $\boxed{b\phi = c\theta}$ — (1)

where, $\alpha \to$ is very small

$a+b = c$.

Let, R be the normal reaction,
F the friction acting on the upper sphere.
Therefore, the equation of motion of cylinder along CO is $\Rightarrow \boxed{Mc\dot{\theta}^2 = Mg\cos\theta - R}$ ——— (2)

The coordinates (x_c, y_c) of the centre 'C' referred to the horizontal and vertical lines through 'O' as axes are given by –

$$x_c = OC\sin\theta = c\sin\theta$$
$$y_c = OC\cos\theta = c\cos\theta$$
$$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$$

$\therefore$ The energy equation, gives

$$\frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}Mv_c^2 = Mg(c\cos\alpha - c\cos\theta)$$
$$\Rightarrow \frac{1}{2}M\cdot\frac{1}{2}b^2\cdot\dot{\phi}^2 + Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta)$$
$$\Rightarrow \frac{1}{4}Mb^2\dot{\phi}^2 + Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta)$$
$$\Rightarrow \frac{1}{4}M(c\dot{\theta})^2 + \frac{1}{2}Mc^2\dot{\theta}^2 = Mgc(\cos\alpha - \cos\theta) \text{ – from (1)}$$
$$\therefore \boxed{\dot{\theta}^2 = \frac{4g}{3c}(\cos\alpha - \cos\theta)}$$

Substituting in (2), we get

$$R = Mg\cos\theta - Mc\dot{\theta}^2 = Mg\cos\theta - M\cancel{c}\cdot\frac{4g}{3\cancel{c}}(\cos\alpha - \cos\theta)$$
$$\boxed{R = \frac{1}{3}Mg(7\cos\theta - 4\cos\alpha)}$$

The cylinders will separate when $R = 0$

i.e $\frac{1}{3}Mg(7\cos\theta - 4\cos\alpha) = 0$ $M \neq 0$, $g \neq 0$

$$\therefore 7\cos\theta - 4\cos\alpha = 0$$
$$\cos\theta = \frac{4\cos\alpha}{7} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{4}{7}\cos\alpha\right)}$$

Which is required result.

Ques:- 8(b)> Use Hamilton's equations to find the cartesian equations of motion of a particle moving in three dimensions in a force field of potential V.

Solution:- Let (x, y, z) be the co-ordinates of a particles moving in three dimensions, at time t.

$$\boxed{\therefore \text{K.E} \Rightarrow T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)}$$

Since; V is the potential energy,

$$\therefore L = T - V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V$$

Here x, y, z are the generalised co-ordinates

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad ; \; p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \; ; \; p_z = \frac{\partial L}{\partial \dot{z}} m\dot{z} \quad \text{— (1)}$$

Since, L does not contain 't' explicitely, therefore.

$$H = T + V = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + V$$

$$\text{Or} \Rightarrow \; H = \frac{1}{2m}(p_x^2+p_y^2+p_z^2) + V \quad \text{— (using (1))}$$

Hence, the six Hamilton's equations are, (note that V is a function of x, y, z and t)

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \quad \text{— } (H_1)$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{— } (H_2)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y} \quad \text{— } (H_3)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad \text{— } (H_4)$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z} \quad \text{--- } (H_5)$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \text{--- } (H_6)$$

from (H_1) and (H_2), we have

$$\boxed{m\ddot{x} = \dot{p}_x = -\frac{\partial V}{\partial x}} \quad \text{--- } (2)$$

Similarly from (H_3), (H_4) and (H_5), (H_6), we get

$$\boxed{m\ddot{y} = \dot{p}_y = -\frac{\partial V}{\partial y}} \quad \text{--- } (3)$$

$$\boxed{m\ddot{z} = \dot{p}_z = -\frac{\partial V}{\partial z}} \quad \text{--- } (4)$$

If X, Y, Z are the external forces parallel to the axes, at (x, y, z), then

$$-\frac{\partial V}{\partial x} = X, \quad -\frac{\partial V}{\partial y} = Y, \quad -\frac{\partial V}{\partial z} = Z$$

Hence, from (2), (3), (4) the equations of motion of a particle moving in three dimensions are.

$$\boxed{m\ddot{x} = X; \quad m\ddot{y} = Y; \quad m\ddot{z} = Z}$$

Which are required solution

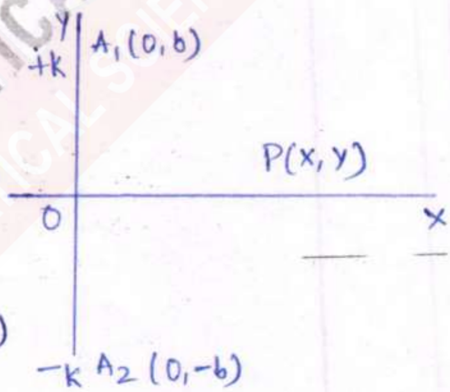
8(c) When an infinite liquid contains two parallel equal and opposite vortices at a distance 2b, Prove that the stream lines relative to the vortices are given by the equation

$$\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]+\frac{y}{b}=c$$

the origin being the middle point of the join which is taken for the axis of y.

Sol'n: Suppose there are two vortices of strengths $k, -k$ at A_1, A_2 respectively such that origin O is the middle point of $A_1A_2=2b$ and A_1, A_2 lie along y-axis. Both vortices will move along a line parallel to x-axis with velocity.

$$q=\frac{k}{2\pi(A_1A_2)}=\frac{k}{2\pi\cdot 2b}=\frac{k}{4\pi b}$$



The complex potential W at P due to these two vortices is given by

$$W=\frac{ki}{2\pi}\log(z-ib)-\frac{ki}{2\pi}\log(z+ib)$$

$$=\frac{ki}{2\pi}\log[x+i(y-b)]-\frac{ki}{2\pi}\log[x+i(y+b)]$$

Equating imaginary parts from bothsides,

$$\psi=\frac{k}{4\pi}\log[x^2+(y-b)^2]-\frac{k}{4\pi}\log[x^2+(y+b)^2]$$

$$\Rightarrow \psi=\frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]$$

To reduce the vortex system to rest, we superimpose a velocity $\frac{k}{4\pi b}$ along x-axis to the System. Let ψ' be the stream function due to this addition, then

$$-\frac{\partial\psi'}{\partial y}=-\frac{\partial\phi'}{\partial x}=-\frac{k}{4\pi b} \quad \therefore \psi'=\frac{ky}{4\pi b}$$

Hence the streamlines relative to vortices are given by

$$\varphi = \frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{ky}{4\pi b} = \text{Const.}$$

$$\Rightarrow \log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{y}{b} = C$$

If we take $PA_1 = r_1 = [x^2+(y-b)^2]^{1/2}$

and $PA_2 = r_2 = [x^2+(y+b)^2]^{1/2}$, then the last gives

$$\log\frac{r_1^2}{r_2^2} + \frac{y}{b} = \text{Const} \Rightarrow \log\frac{r_1}{r_2} + \frac{y}{2b} = C.$$