Indian Forest Service Examination-2012

D-GT-M-NUA

MATHEMATICS

Paper I

Time Allowed : Three Hours	Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

Important Note : All parts/sub-parts of a question must be attempted contiguously. That is, candidates must finish attempting all the parts/sub-parts of each question they are answering in the answer-book before moving on to the next question.

Pages left blank must be clearly struck out. Answers that follow any pages left blank may not be given credit.

(Contd.)

Section – A

1. (a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. 8

(b) Show that the set of all functions which satisfy the differential equation

$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space. 8

(c) If the three thermodynamic variables P, V, T are connected by a relation, $f(P, V, T) = 0$ show that, $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1.$ 8

(d) If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ then show that

$g = \sqrt{\left(\frac{n}{2\mu}\right)}.$ 8

(e) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone

$$4x^2 - y^2 + 3z^2 = 0.$$ 8

2. (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a+b, 0)$.

Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0)$, $e_2' = (0, 1, 1)$, $e_3' = (1, 1, 1)$.

Also, show that there exists an invertible matrix P such that

$$B = \overset{-1}{P} A P$$ 10

(b) Verify Cayley–Hamilton theorem for the matrix

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express

$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. 10

(c) Find the equations of the tangent plane to the ellipsoid $2x^2 + 6y^2 + 3z^2 = 27$ which passes through the line $x - y - z = 0 = x - y + 2z - 9$. 10

(d) Show that there are three real values of λ for which the equations :

$(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ 10

3\. (a) Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ corresponding to the basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 10

(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. 10

(c) If $2C$ is the shortest distance between the lines

$$\frac{x}{l} - \frac{z}{n} = 1, \; y = 0$$

and $\frac{y}{m} + \frac{z}{n} = 1, \; x = 0$

then show that

$$\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}.$$ 10

(d) Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin. 10

4\. (a) Find by triple Integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$. 10

(b) Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2+y^2+2fx+2gy+2hz=0$$

if $af+bg+ch=0$. 10

(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its equation. 10

(d) Evaluate the following in terms of Gamma function :

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}\,dx.$$ 10

Section – B

5. (a) Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y.$ 8

(b) Solve and find the singular solution of $x^3p^2+x^2py+a^3=0.$ 8

(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$ 8

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3 + 3ab^2 : a^3 + 3a^2b$ where a and b are the sides BC & AC respectively. 8

(e) If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that grad u, grad v and grad w are coplanar. 8

6. (a) Solve :

$$x^2 y \frac{d^2 y}{dx^2} + \left(x \frac{dy}{dx} - y \right)^2 = 0. \qquad 10$$

(b) Find the value of $\iint_S \left(\vec{\nabla} \times \vec{F} \right) \cdot \vec{ds}$

taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when

$$\vec{F} = \left(y^2 + z^2 - x\right)\vec{i} + \left(z^2 + x^2 - y^2\right)\vec{j} + \left(x^2 + y^2 - z^2\right)\vec{k}.$$

10

(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is

$$\frac{2u}{g\sqrt{\left(1+3\sin^2\beta\right)}},$$

range on the plane is $\frac{2u^2}{g} \cdot \frac{\sin\beta}{1+3\sin^2\beta}$

and the vertical height of the point struck is

$\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$ above the point of projection.

10

(d) Solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2\cos x$. 10

7. (a) A particle is moving with central acceleration $\mu\left[r^5 - c^4 r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4 + y^4 = c^4$. 13

(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. 14

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}.$$ 13

8. (a) Solve $x = y\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$. 10

(b) Find the value of the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$, where the vector field,

$$\vec{F} = (\sin y)\,\vec{i} + x(1+\cos y)\,\vec{j}.$$ 10

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$ 10

(d) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1-x)^{-2}$. 10
