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Maxima and Minima

(Of Functions of a Single Independent Variable)

§ 1. Definitions.

A function $f(x)$ is said to be **maximum** at $x = a$, if there exists a positive number δ such that

$$f(a+h) < f(a)$$

for all values of h, other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be **minimum** at $x = a$, if there exists a positive number δ such that

$$f(a+h) > f(a)$$

for all values of h, other than zero, in the interval $(-\delta, \delta)$.

Maximum and minimum values of a function are also called its **extreme values** or **turning values** and the points at which they are attained are called **points of maxima and minima.**

The points at which a function has extreme values are called **turning points.**

§ 2. Properties of maxima and minima.

(i) At least one maximum or one minimum must lie between two equal values of a function.

(ii) Maximum and Minimum values must occur **alternately.**

(iii) There may be several maximum or minimum values of the same function.

(iv) A function $y = f(x)$ is maximum at $x = a$, if dy/dx changes sign from +ive to –ive as x passes through a.

(v) A function $y = f(x)$ is minimum at $x = a$, if dy/dx changes sign from –ive to +ive as x passes through a.

(vi) If the sign of dy/dx does not change while x passes through a, then y is neither maximum nor minimum at $x = a$.

**§ 3. Conditions for maximum or minimum values.

(Kanpur 1980, 79; Agra 79; Lucknow 76)

Necessary Conditions : *A necessary condition for $f(x)$ to be a maximum or a minimum at $x = a$ is that $f'(a) = 0$.*

Let $f(x)$ be a given function of x and suppose $f(x)$ can be expanded in the neighbourhood of $x = a$ by Taylor's theorem.

$$\text{Then } f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^n(a+\theta h)$$

$$\therefore \quad f(a+h) - f(a) = hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^n}{n!}f^n(a+\theta h). \quad \text{...(1)}$$

By definition, $f(x)$ is maximum at $x = a$ if $f(a+h) - f(a) < 0$ for all $h \neq 0$ in $(-\delta, \delta)$, where δ is a sufficiently small +ive number. Similarly $f(x)$ is minimum at $x = a$ if $f(a+h) - f(a) > 0$ for all $h \neq 0$ in $(-\delta, \delta)$, where δ is a sufficiently small +ive number. Thus for $f(x)$ to be a maximum or a minimum at $x = a$, the expression $f(a+h) - f(a)$ must be of invariable sign (negative in the case of maximum and positive in the case of minimum) for all sufficiently small non-zero values of h, positive or negative.

Now when h is made sufficiently small, the sign of the right hand side of (1) and therefore the sign of $f(a+h) - f(a)$ is ultimately governed by the sign of $hf'(a)$ as it is the term of lowest degree in h in (1). But if $f'(a) \neq 0$, the sign of $hf'(a)$ will change with a change of sign of h. *Therefore the necessary condition that $f(x)$ should have a maximum or a minimum at $x = a$ is that $f'(a) = 0$.*

Stationary values. Definition. *A function $f(x)$ is said to be stationary at $x = a$ if $f'(a) = 0$.*

Thus for a function $f(x)$ to be a maximum or a minimum at $x = a$ it must be stationary at $x = a$.

Sufficient conditions for maximum or minimum values.

Now, if $f'(a) = 0$, then (1) becomes

$$f(a+h) - f(a) = \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots + \frac{h^n}{n!}f^n(a+\theta h). \quad \text{...(2)}$$

Now when h is made sufficiently small, the sign of the right hand side of (2) and therefore the sign of $f(a+h) - f(a)$ is ultimately governed by the sign of $\frac{1}{2}h^2 f''(a)$ as it is the term of lowest degree in h in (2). But h^2 is always +ive. Therefore $\frac{1}{2}h^2 f''(a)$ and $f''(a)$ are of the same sign. Hence if $f'(a) \neq 0$, then for all sufficiently small non zero values of h the expression $f(a+h) - f(a)$ is of invariable sign and its sign is the same as that of $f''(a)$.

Therefore, there is a **maximum** *of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$* **is negative.**

Similarly, there is a **minimum** *of $f(x)$ at $x = a$ if $f'(a) = 0$ and $f''(a)$* **is positive.**

Note. If $f''(a)$ is also equal to zero, then we can show that for a maximum or a minimum of $f(x)$ at $x = a$, we must have $f'''(a) = 0$. Again, if $f^{iv}(a)$ is negative, there will be a maximum at $x = a$ and if $f^{iv}(a)$ is positive, there will be a minimum at $x = a$.

In general if

$f'(a) = f''(a) = f'''(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$,

then n must be an even integer for maximum or minimum. Also for a maximum $f^{(n)}(a)$ must be negative and for a minimum $f^{n}(a)$ must be positive.

§ 4. Working rule for Maxima and Minima of $f(x)$.

1. Find $f'(x)$ and equate it to zero.

2. Solve the resulting equation for x. Let its roots be $a_1, a_2, \ldots\ldots$ Then $f(x)$ is stationary at $x = a_1, a_2, \ldots\ldots$ Thus $x = a_1, a_2, \ldots\ldots$ are the only points at which $f(x)$ can be a maximum or a minimum.

3. Find $f''(x)$ and substitute in it by turns $x = a_1, a_2, \ldots\ldots$

4. If $f''(a_1)$ is –ive we have a maximum at $x = a_1$. If $f''(a_1)$ is +ive, we have a minimum at $x = a_1$.

5. If $f''(a_1) = 0$, find $f'''(x)$ and put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. if $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is –ive, we have a maximum at $x = a_1$; if it is positive, there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on.

Repeat the above process for each root of the equation $f'(x) = 0$.

Solved Examples

Ex.1. *Prove that $x^5 - 5x^4 + 5x^3 - 10$ has a maximum for $x = 1$, a minimum for $x = 3$, and for $x = 0$, it has neither a maximum nor a minimum.* **(Vikram 1977; Kashmir 71; Meerut 85)**

Sol. Let $f(x) = x^5 - 5x^4 + 5x^3 - 10$.

For a maximum or minimum of $f(x)$ it is necessary that

$f'(x) = 0$ *i.e.*, $5x^4 - 20x^3 + 15x^2 = 0$ *i.e.*,

$5x^2(x^2 - 4x + 3) = 0$ *i.e.*, $5x^2(x-3)(x-1) = 0$ *i.e.*, $x = 0, 1$ or 3.

Thus $f(x)$ is stationary at $x = 0, 1$ and 3.

Now $f''(x) = 20x^3 - 60x^2 + 30x$.

We have $f''(1) = 20 - 60 + 30 = -10$ *i.e.*, –ive.

Hence $f(x)$ has a maximum value at $x = 1$.

Again $f''(3) = 540 - 540 + 90 = 90$ *i.e.*, +ive.

Hence $f(x)$ has a minimum value at $x = 3$.

Further we have $f''(0) = 0$. So we find $f'''(x)$. We have

$$f'''(x) = 60x^2 - 12x + 30 \text{ which is } \neq 0 \text{ when } x = 0.$$

Hence at $x = 0, f(x)$ has neither a maximum nor a minimum.

Ex. 2 (a). *Find the maximum value of* $(x-1)(x-2)(x-3)$.

(Agra 1982; Meerut 71; Raj. 76)

Sol. Let $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$.

Then $f'(x) = 3x^2 - 12x + 11$.

For a maximum or minimum value of $f(x), f'(x) = 0$

i.e., $$3x^2 - 12x + 11 = 0$$

i.e., $$x = \frac{12 \pm \sqrt{(144 - 4 \times 3 \times 11)}}{6} = \frac{6 \pm \sqrt{(36 - 33)}}{3} = 2 \pm \frac{1}{\sqrt{3}}.$$

Also $f''(x) = 6x - 12$.

Now $f''[2 + (1/\sqrt{3})] = +$ive. Therefore $f(x)$ has minimum at $x = 2 + (1/\sqrt{3})$.

Again $f''[2 - (1/\sqrt{3})] = -$ive. Therefore $f(x)$ has a minimum at $x = 2 - (1/\sqrt{3})$.

The maximum value of $f(x)$ is $= f[2 - (1/\sqrt{3})]$

$$= [1 - (1/\sqrt{3})](-1/\sqrt{3})[-1 - (1/\sqrt{3})]$$

$$= (1 - \tfrac{1}{3})(1/\sqrt{3}) = 2/(3\sqrt{3}).$$

Ex. 2 (b). *Find the maximum and minimum values of y where*

$$y = (x-1)(x-2)^2.$$ **(Bihar 1976)**

Sol. Let $f(x) = y = (x-1)(x-2)^2$.

Then

$$f'(x) = \frac{dy}{dx} = (x-2)^2 + 2(x-1)(x-2) = (x-2)(3x-4).$$

For a maximum or minimum of y, $dy/dx = f'(x) = 0$

i.e., $(x-2)(3x-4) = 0$, whence $x = 2, 4/3$.

Also $f''(x) = d^2y/dx^2 = 6x - 10$.

Now $f''(2) = 2$, *i.e.,* +ive $\Rightarrow f(x)$ has a minimum at $x = 2$.

Again $f''(4/3) = -2$, *i.e.,* –ive $\Rightarrow f(x)$ has a maximum at $x = 4/3$.

***Ex. 3 (a).** *Show that* $\dfrac{x}{1 + x \tan x}$ *is maximum when* $x = \cos x$.

(Magadh 1971; Ranchi 74; Meerut 84, 90 P)

Sol. Let $f(x) = \dfrac{x}{1 + x \tan x}$.

Then $$f'(x) = \frac{(1 + x\tan x) - x(x\sec^2 x + \tan x)}{(1 + x\tan x)^2} = \frac{1 - x^2 \sec^2 x}{(1 + x\tan x)^2}.$$

For a maximum or minimum value of $f(x), f'(x) = 0$

i.e., $1 - x^2 \sec^2 x = 0$ *i.e.,* $x = \pm \cos x$.

$$\text{Now } f''(x) = (1 - x^2 \sec^2 x) \frac{d}{dx} \frac{1}{(1 + x \tan x)^2}$$
$$+ \frac{1}{(1 + x \tan x)^2} (-2x \sec^2 x - 2x^2 \sec^2 x \tan x)$$
$$= (1 - x^2 \sec^2 x) \frac{d}{dx} \frac{1}{(1 + x \tan x)^2} - \frac{2x \sec^2 x}{(1 + x \tan x)^2} \cdot (1 + x \tan x)$$
$$= (1 - x^2 \sec^2 x) \frac{d}{dx} \frac{1}{(1 + x \tan x)^2} - \frac{2x \sec^2 x}{1 + x \tan x}.$$

When $x = \cos x$, $f''(x) = 0 - \dfrac{2}{\cos x (1 + \sin x)}$ *i.e.*, $-$ive.

Therefore $f(x)$ has a maximum value when $x = \cos x$.

***Ex. 3 (b).** *Show that* $\sin x (1 + \cos x)$ *is a maximum at* $x = \pi/3$. **(Agra 1984; Kanpur 1979; Meerut 74, 85 P; Raj. 76; Vikram 70)**

Sol. Let $f(x) = \sin x (1 + \cos x) = \sin x + \frac{1}{2} \sin 2x$.

Then $f'(x) = \cos x + \cos 2x$.

For a maximum or a minimum value of $f(x)$, $f'(x) = 0$

i.e., $\cos x + \cos 2x = 0$ *i.e.*, $2 \cos^2 x + \cos x - 1 = 0$

i.e., $(2 \cos x - 1)(\cos x + 1) = 0$.

$\therefore \quad \cos x = \frac{1}{2}, -1$ *i.e.*, $x = \pi/3, \pi$.

Now $f''(x) = -\sin x - 2 \sin 2x$.

$\therefore \quad f''(\pi/3) = -\sin(\pi/3) - 2 \sin(2\pi/3) =$ –ive.

Hence $f(x)$ is maximum at $x = \pi/3$.

Ex. 4 (a). *Find the maximum value of* $(\log x)/x$ *in* $0 < x < \infty$.

(Meerut 1981, 8?, 87; Kanpur 78; Garhwal 83)

Sol. Let $f(x) = (\log x)/x$. ...(1)

Then $f'(x) = [x \cdot (1/x) - \log x]/x^2 = (1 - \log x)/x^2$.

For a maximum or a minimum value of $f(x)$, $f'(x) = 0$

i.e., $1 - \log x = 0$ *i.e.*, $x = e$.

Now $f''(x) = (1 - \log x)(-2/x^3) - (1/x^3)$.

$\therefore \quad f''(e) = -1/e^3 =$ –ive. Therefore $f(x)$ is a maximum at $x = e$.

Putting $x = e$ in (1), the maximum value of $f(x) = (\log e)/e = 1/e$.

Ex. 4 (b). *Prove that the maximum value of* $(1/x)^x$ *is* $e^{1/e}$.

(Meerut 1983 P, 87; Kanpur 88)

Sol. Let $y = (1/x)^x$.

Then $\log y = x (\log 1 - \log x) = -x \log x$.

$\therefore \quad (1/y)(dy/dx) = -1 . \log x - x (1/x) = -(1 + \log x)$

or $\quad dy/dx = -y (1 + \log x) = -(1/x)^x (1 + \log x)$. ...(1)

For a maximum or a minimum of y it is necessary that

$$dy/dx = 0 \text{ i.e., } -(1/x)^x (1 + \log x) = 0$$

i.e., $1 + \log x = 0$ *i.e.,* $\log x = -1$ *i.e.,* $x = e^{-1} = 1/e$.

Now differentiating (1) again, we have

$$d^2y/dx^2 = -(dy/dx)(1 + \log x) - y(1/x)$$
$$= -(dy/dx)(1 + \log x) - (1/x)^x .(1/x). \quad ...(2)$$

When $x = 1/e$, we have $dy/dx = 0$. Therefore when $x = 1/e$, we have from (2)

$$d^2y/dx^2 = 0 - (e)^{1/e} . e, \text{ which is negative.}$$

Hence y is maximum when $x = 1/e$ and putting $x = 1/e$ in the value of y, the maximum value of y is $e^{1/e}$.

Ex. 4 (c). *Prove that* $x^2 \log(1/x)$ *has a maximum value when* $x = e^{-1/2}$. **(Lucknow 1981)**

Sol. Let $y = x^2 \log(1/x) = x^2(\log 1 - \log x) = -x^2 \log x$.

Then $\quad dy/dx = -2x \log x - x^2 (1/x)$
$$= -x(2 \log x + 1) \quad ...(1)$$

For a maximum or a minimum of y, we must have

$$dy/dx = 0 \text{ i.e., } -x(2 \log x + 1) = 0$$

i.e., $x = 0$, or $2 \log x + 1 = 0$ giving $\log x = -\frac{1}{2}$ or $x = e^{-1/2}$.

Now differentiating (1) again, we have

$$d^2y/dx^2 = -(2 \log x + 1) - x.(2/x) = -(2 \log x + 1) - 2.$$

When $x = e^{-1/2}$, we have $2 \log x + 1 = 0$. Therefore when $x = e^{-1/2}$, we have $d^2y/dx^2 = -2$, which is negative.

Hence y is maximum when $x = e^{-1/2}$.

Note that when $x = 0$, the value of y does not exist.

Ex. 5 (a). *If* $dy/dx = (x-a)^{2n}(x-b)^{2p+1}$, *where n and p are positive integers, show that* $x = a$ *gives neither a maximum nor a minimum value of y, but* $x = b$ *gives a minimum.* **(Lucknow 1977)**

Sol. For a maximum or a minimum of y, we must have

$$dy/dx = 0 \text{ i.e., } (x-a)^{2n}(x-b)^{2p+1} = 0$$

i.e., $\quad x = 0 \quad$ or $\quad x = b$.

To test nature of y at $x = b$. When x is slightly less than b, dy/dx is negative as $2p + 1$ is odd. Also when x is slightly greater than b, dy/dx is positive. Thus dy/dx changes sign from negative to positive while x passes through the value b. Hence y is minimum at $x = b$.

To test the nature of y at $x = a$. Here we see that the sign of dy/dx when x is slightly less than a is the same as when x is slightly greater than a since $2n$ is even. Thus dy/dx does not change sign while x passes through the value a. Therefore y is neither a maximum nor a minimum at $x = a$.

Ex. 5 (b). *Discuss the maxima and minima of y if*

$$dy/dx = (x-a)^{10}(x-b)^{13}.$$ **(Meerut 1990)**

Sol. Proceed as in Ex. 5 (a). Here y is minimum at $x = b$ and is neither a maximum nor a minimum at $x = a$.

Ex. 5 (c). *Find the maximum and minimum value of y if* $dy/dx = (x-a)^4(x-b)^3$. **(Lucknow 1983)**

Sol. Proceed as in Ex. 5 (a). Here y is minimum at $x = b$ and is neither a maximum nor a minimum at $x = a$.

Ex. 5 (d). *If* $dy/dx = x(x-1)^2(x-3)^3$, *find the maximum and minimum value of* y. **(Lucknow 1979)**

Sol. For a maximum or a minimum of y, we must have

$$dy/dx = 0 \text{ i.e., } x(x-1)^2(x-3)^3 = 0$$

i.e., $$x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = 3.$$

Nature of y at $x = 0$. When x is slightly less than $0, dy/dx$ is positive. Also when x is slightly greater than $0, dy/dx$ is negative. Thus dy/dx changes sign from positive to negative while x passes through the value 0. Hence y is maximum at $x = 0$

Nature of y at $x = 1$. When x is slightly less than $1, dy/dx$ is negative and when x is slightly greater than $1, dy/dx$ is again negative. Thus dy/dx does not change sign while x passes through the value 1. Therefore y is neither a maximum nor a minimum at $x = 1$.

Nature of y at $x = 3$. When x is slightly less than $3, dy/dx$ is negative and when x is slightly greater than $3, dy/dx$ is positive. Thus dy/dx changes sign from negative to positive while x passes through the value 3. Hence y is minimum at $x = 3$.

Ex. 6 (a). *Show that* $\sin^p\theta\cos^q\theta$ *attains a maximum when* $\theta = \tan^{-1}\sqrt{(p/q)}$. **(Meerut 1987)**

Sol. Let $y = \sin^p\theta\cos^q\theta$. For a maximum or minimum of y, we have $dy/d\theta = 0$

i.e., $$p\sin^{p-1}\theta\cos^{q+1}\theta - q\sin^{p+1}\theta\cos^{q-1}\theta = 0$$

i.e., $$\sin^{p-1}\theta\cos^{q-1}\theta\,(p\cos^2\theta - q\sin^2\theta) = 0$$

i.e., $$\sin\theta = 0 \text{ or } \cos\theta = 0 \text{ or } \tan^2\theta = p/q$$

i.e., $$\theta = 0 \text{ or } \theta = \pi/2 \text{ or } \theta = \tan^{-1}\sqrt{(p/q)}.$$

Now $y = 0$ at $\theta = 0$ and also at $\theta = \pi/2$. When $0 < \theta < \pi/2, y$ is +ive. Also $\tan^{-1}\sqrt{(p/q)}$ is the only value of θ lying between 0 and $\pi/2$ at which $dy/d\theta = 0$. Hence y is maximum when $\theta = \tan^{-1}\sqrt{(p/q)}$. This is clear from the graph of y.

Ex. 6 (b). *Find the maximum and minimum values of*

$$\sin^2\theta + \sin^2\phi, \text{ where } \theta + \phi = \alpha.$$

Sol. Let $u = \sin^2\theta + \sin^2\phi$, where $\theta + \phi = \alpha$.

Then $\quad u = \sin^2 \theta + \sin^2 (\alpha - \theta)$.

$\therefore \quad du/d\theta = 2 \sin \theta \cos \theta - 2 \sin (\alpha - \theta) \cos (\alpha - \theta)$
$= \sin 2\theta - \sin (2\alpha - 2\theta)$.

For a maximum or a minimum of u, we have $du/d\theta = 0$

i.e., $\quad \sin 2\theta - \sin (2\alpha - 2\theta) = 0$, *i.e.,* $\sin 2\theta = \sin (2\alpha - 2\theta)$

i.e., $\quad 2\theta = n\pi + (-1)^n (2\alpha - 2\theta)$.

If n is odd, we have $2\theta = n\pi - 2\alpha + 2\theta$ which does not give any value of θ. If n is even, say $2m$, we have $2\theta = 2m\pi + 2\alpha - 2\theta$ which gives $\theta = (m\pi + \alpha)/2$. Thus u is stationary when $\theta = (m\pi + \alpha)/2$. Also then $\phi = \alpha - \theta = \alpha - (m\pi + \alpha)/2 = (\alpha - m\pi)/2$.

Now $d^2u/d\theta^2 = 2 \cos 2\theta + 2 \cos (2\alpha - 2\theta) = 4 \cos \alpha \cos (2\theta - \alpha)$. When $\theta = (m\pi + \alpha)/2$, we have $d^2u/d\theta^2 = 4 \cos \alpha \cos m\pi$.

If m is odd, say $2r + 1$, then $d^2u/d\theta^2 = -4 \cos \alpha$ which is $-$ ive. Therefore u is maximum when $\theta = \{(2r + 1)\pi + \alpha\}/2$.

If m is even, say $2r$, then $d^2u/d\theta^2 = 4 \cos \alpha$ which is $+$ive. Therefore u is minimum when $\theta = (2r\pi + \alpha)/2$.

The maximum value of $u = \sin^2 \frac{1}{2} (m\pi + \alpha) + \sin^2 \frac{1}{2} (\alpha - m\pi)$, (where m is odd) $= \cos^2 \frac{1}{2}\alpha + \cos^2 \frac{1}{2}\alpha = 2 \cos^2 \frac{1}{2}\alpha = 1 + \cos \alpha$.

The minimum value of $u = \sin^2 (m\pi + \alpha)/2 + \sin^2 (\alpha - m\pi)/2$, (where m is even) $= \sin^2 \alpha/2 + \sin^2 \alpha/2 = 2 \sin^2 \alpha/2 = 1 - \cos \alpha$.

Ex. 7. *Prove that the maximum and minimum values of the function* $y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C)$ *are those values of* y *for which* $(ax^2 + 2bx + c) - y(Ax^2 + 2Bx + C)$ *is a perfect square.*

Sol. We have $y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C)$.

$\therefore \quad y(Ax^2 + 2Bx + C) = ax^2 + 2bx + c$. ...(1)

Differentiating (1) w.r.t. x, we get

$(dy/dx)(Ax^2 + 2Bx + C) + y(2Ax + 2B) = 2ax + 2b$. ...(2)

For a maximum or a minimum of y we have $dy/dx = 0$. Putting $dy/dx = 0$ in (2), we get

$y(Ax + B) = ax + b$. ...(3)

Thus the maximum and the minimum values of y are obtained by eliminating x between (1) and (3). From (3), we get

$x = -(By - b)/(Ay - a)$. Also (1) can be written as

$x^2 (Ay - b) + 2x(By - b) + Cy - c = 0$. ...(4)

Putting $x = -(By - b)/(Ay - a)$ in (4), we get

$$\frac{(By - b)^2}{Ay - a} - \frac{2(By - b)^2}{Ay - a} + Cy - c = 0$$

or $\quad (By - b)^2 - (Cy - c)(Ay - a) = 0$, ...(5)

giving the maximum and minimum values of y.

Now the condition that the expression

$$(ax^2 + 2bx + c) - y(Ax^2 + 2Bx + C)$$

or $$x^2(a - Ay) + 2x(b - By) + c - Cy$$

be a perfect square is that

$$4(b - By)^2 - 4(a - Ay)(c - Cy) = 0$$

or $$(By - b)^2 - (Ay - a)(Cy - c) = 0. \qquad \text{...(6)}$$

Since the equations (5) and (6) are identical, therefore maximum and the minimum values of y are those for which

$$(ax^2 + 2bx + c) - y(Ax^2 + 2Bx + C)$$

is a perfect square.

Ex. 8. *Find the largest and the smallest values of the function* $x^3 - 18x^2 + 96x$ *in the interval* $[0, 9]$. **(Meerut 1981)**

Sol. Let $f(x) = x^3 - 18x^2 + 96x$. ...(1)

Then $f'(x) = 3x^2 - 36x + 96 = 3(x^2 - 12x + 32)$
$= 3(x - 4)(x - 8)$.

For a maximum or minimum value of $f(x)$, $f'(x) = 0$ *i.e.*, $(x - 4)(x - 8) = 0$ *i.e.*, $x = 4, 8$. Both these values lie in the interval $[0, 9]$.

Now $f''(x) = 6x - 36 = 6(x - 6)$.

At $x = 4$, $f''(x) =$ –ive. Therefore $f(x)$ is a maximum at $x = 4$.

At $x = 8$, $f''(x) =$ +ive. Therefore $f(x)$ is a minimum at $x = 8$.

Thus in the interval $[0, 9]$, the function $f(x)$ is maximum at $x = 4$ and is minimum at $x = 8$. We have $f(4) = 64 - 288 + 384 = 160 =$ the maximum value of $f(x)$ in $[0, 9]$. Also the minimum value of $f(x)$ in $[0, 9] = f(8) = 512 - 1152 + 768 = 128$.

Now at the ends of the interval $[0, 9]$ we have $f(0) = 0$, and $f(9) = 135$. The value 0 is less than the minimum value 128. Therefore 0 is the smallest value of $f(x)$ in $[0, 9]$. The value $f(9)$ *i.e.*, 135 is less than the maximum value 160. Therefore 160 is the largest value of $f(x)$ in $[0, 9]$.

§ 5. Application of Maxima and Minima to Geometrical and other problems.

Sometimes the principle of maxima and minima is applied to geometrical and other problems. In such cases we first construct from the given conditions of the problem the function whose maximum and minimum values are to be determined. If this function contains more than one variable, it is just possible to express it in terms of a single variable by eliminating the other variables with the help of the conditions of the problem. Now we apply the rules for finding maxima or minima to this resulting function of a single variable.

Sometimes it is easy to find by inspection that out of the stationary values of the function which value gives a maximum and which a minimum. Thus it is not always necessary to find the second derivative.

Remember : 1. For a sphere of radius r, volume $= \frac{4}{3}\pi r^3$, surface $= 4\pi r^2$.

2. For a right circular cylinder of height h and radius of the base r, volume $= \pi r^2 h$, curved surface $= 2\pi rh$, total surface $= 2\pi rh + 2\pi r^2$.

3. For a right circular cone of height h and radius of the base r, volume $= \frac{1}{3}\pi r^2 h$, curved surface $= \pi rl$, where l is the slant height such that $l = \sqrt{(r^2 + h^2)}$.

Solved Examples

Ex. 9. *The sum of one number and three times a second number is 60. Among the possible numbers satisfying the condition, find the pair whose product is maximum.* **(Meerut 1976)**

Sol. Let the two numbers be x and y. Then as given,

$$x + 3y = 60. \qquad ...(1)$$

Now let $z = xy =$ the product of the two numbers.

Then $z = x \cdot \frac{1}{3}(60 - x)$, from (1)

$= 20x - \frac{1}{3}x^2$.

We have $dz/dx = 20 - \frac{2}{3}x$.

For a maximum or a minimum of $z, dz/dx = 0$ *i.e.*,

$20 - \frac{2}{3}x = 0$ *i.e.*, $x = 30$.

Putting $x = 30$ in (1), we get $3y = 60 - 30$ *i.e.*, $y = 10$.

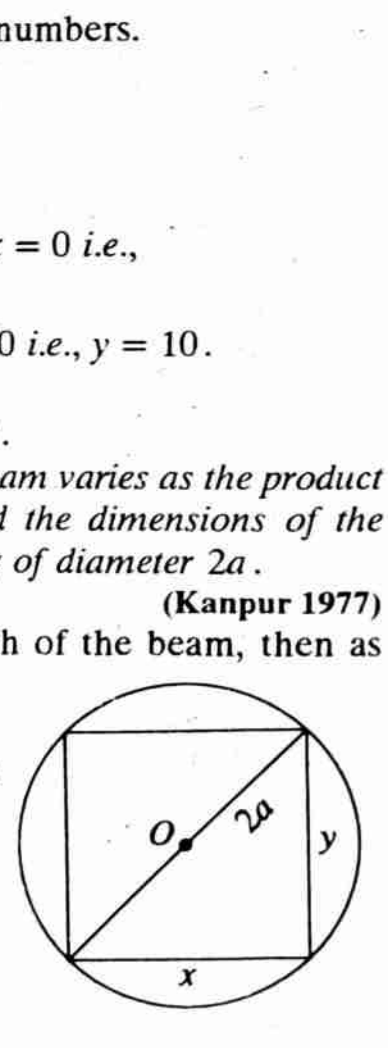
Now $d^2z/dx^2 = -2/3$ which is $-$ ive.

Hence z is maximum when $x = 30, y = 10$.

Ex. 10 (a). *The strength of a rectangular beam varies as the product of its breadth and the square of its depth. Find the dimensions of the strongest beam that can be cut from a round log of diameter $2a$.* **(Kanpur 1977)**

Sol. Let x be the breadth and y the depth of the beam, then as given

$$x^2 + y^2 = 4a^2. \qquad ...(1)$$



If S denotes the strength of the beam, then $S \propto xy^2$, or $S = kxy^2$, where k is some constant.

Now $S = kx(4a^2 - x^2)$, from (1).

$$\therefore \quad \frac{dS}{dx} = k(4a^2 - 3x^2).$$

For a maximum or a minimum of S, $\dfrac{dS}{dx} = 0$

i.e., $4a^2 - 3x^2 = 0$ *i.e.*, $x = 2a/\sqrt{3}$ because x cannot be –ive.

Now $d^2S/dx^2 = -6kx =$ negative, when $x = 2a/\sqrt{3}$. Therefore S is maximum when $x = 2a/\sqrt{3}$. When $x = 2a/\sqrt{3}$; we have from (1), $y = \sqrt{(4a^2 - \frac{1}{3} \cdot 4a^2)} = 2a\sqrt{(2/3)}$.

Hence for the strongest beam, depth $= \sqrt{(2/3)}$. diameter of the log and breadth $= (1/\sqrt{3})$. diameter of the log.

Ex. 10 (b). *Assuming that the strength of a beam of rectangular cross section varies as the product of its breadth and the cube of its depth, find the breadth of the strongest beam which can be cut from a circular log of diameter a.*

Sol. Let x be the breadth and y be the depth of the beam.

Then $x^2 + y^2 = a^2$. ...(1)

Let $S =$ the strength of the beam $= kxy^3$, where k is some constant.

Now $S = ky^3(a^2 - y^2)^{1/2}$, [from (1)].

Since S is positive, therefore S is maximum or minimum according as S^2 is maximum or minimum.

Let $u = S^2 = k^2y^6(a^2 - y^2) = k^2a^2y^6 - k^2y^8$.

We have $du/dy = 6k^2a^2y^5 - 8k^2y^7 = k^2y^5(6a^2 - 8y^2)$.

For a maximum or a minimum of u, $du/dy = 0$

i.e., $k^2y^5(6a^2 - 8y^2) = 0$ *i.e.*, $6a^2 - 8y^2 = 0$ since y cannot be zero, *i.e.*, $y^2 = 6a^2/8 = 3a^2/4$.

Now $d^2u/dy^2 = 30k^2a^2y^4 - 56k^2y^6 = 2k^2y^4(15a^2 - 28y^2) = -$ ive, when $y^2 = 3a^2/4$. Therefore u is maximum, and hence S is maximum, when $y^2 = 3a^2/4$. When $y^2 = 3a^2/4$, we have from (1), $x^2 = a^2/4$ or $x = a/2$. Hence for the strongest beam, the breadth $= a/2$.

Ex. 11. *The sum of the sides of a rectangle is constant. If the area is to be a maximum show that the rectangle must be a square.*

Sol. Let x and y be the sides of the rectangle.

Then $x + y = c$, a constant. ...(1)

Now area $A = xy - x(c - x) = cx - x^2$.

We have $dA/dx = c - 2x$.

For a maximum or minimum of A, $dA/dx = 0$ *i.e.*, $c - 2x = 0$ *i.e.*, $x = c/2$.

Now $d^2A/dx^2 = -2$, which is negative therefore A is maximum when $x = c/2$. When $x = c/2$, we have from (1), $y = x - \frac{1}{2}c = c/2$.

Thus for the area of the rectangle to be maximum, we have

$x = y = c/2$ *i.e.*, the rectangle must be a square.

**Ex. 12. *Show that the maximum rectangle inscribed in a circle is a square.* (Kanpur 1979; Vikram 74; Ranchi 76; Meerut 77; Agra 83; Gorakhpur 79)

Sol. Let x and y be the sides of the rectangle inscribed in a circle of diameter a.

Then $\quad x^2 + y^2 = a^2.$ ...(1)

Let A be the area of the rectangle.

Then $\quad A = xy = x(a^2 - x^2)^{1/2}.$

Since A is positive, therefore A is maximum or minimum according as A^2 is maximum or minimum.

Let $u = A^2 = x^2(a^2 - x^2) = x^2a^2 - x^4$. Then $du/dx = 2xa^2 - 4x^3$.

For a maximum or a minimum of u, $du/dx = 0$

i.e., $2xa^2 - 4x^3 = 0$ *i.e.*, $2x(a^2 - 2x^2) = 0$ *i.e.*, $x = a/\sqrt{2}$, since $x \neq 0$ and x cannot be $-$ ive.

Now $\quad d^2u/dx^2 = 2a^2 - 12x^2.$

When $\quad x = a/\sqrt{2}$, $d^2u/dx^2 = 2a^2 - \frac{1}{2} \times 12a^2 = -4a^2 = -$ ive.

Hence u is maximum when $x = a/\sqrt{2}$.

When $x = a/\sqrt{2}$, we have from (1), $y = a/\sqrt{2}$ showing that the maximum rectangle inscribed in a circle is a square.

Ex. 13 (a). *Find the coordinates of the point on the parabola* $y = x^2$ *which is nearest to the point* $(3, 0)$.

Sol. Let (x, y) be any point on the parabola $y = x^2$. ...(1)

Let s be the distance of (x, y) from $(3, 0)$.

Then $s^2 = (x-3)^2 + (y-0)^2 = (x-3)^2 + y^2 = (x-3)^2 + x^4$, [from (1)].

Now s is maximum or minimum according as s^2 is maximum or minimum. Let $u = s^2 = (x-3)^2 + x^4$. Then $du/dx = 2(x-3) + 4x^3$. For a maximum or a minimum of u, $du/dx = 0$ *i.e.*,

$$2(x-3) + 4x^3 = 0 \text{ i.e., } 2x^3 + x - 3 = 0 \text{ giving } x = 1.$$

Now $d^2u/dx^2 = 2 + 12x^2 =$ +ive, when $x = 1$. Hence u is minimum when $x = 1$. When $x = 1$, we have from (1), $y = 1$. Thus the required point is $(1, 1)$.

Ex. 13 (b). *What point on the curve* $xy^2 = 1$ *is nearest to the origin ?* (Lucknow 1983)

Sol. Proceed as in Ex. 13 (a). Here $s^2 = x^2 + y^2 = x^2 + (1/x)$. The required point is $(1/2^{1/3}, 2^{1/6})$.

Ex. 14. *A piece of wire of length l is cut into two parts, one of which is bent in the shape of a cricle and the other into the shape of a square. How should the wire be cut so that the sum of the areas of the circle and the square is minimum ?* (Meerut 1973)

Sol. Let the lengths of the two parts of the wire be x and y.

Then $\quad x + y = l$. ...(1)

Let the part of length x be bent in the shape of a circle, say of radius r. Then $x =$ the circumference of the circle $= 2\pi r$, so that $r = x/(2\pi)$.

The area of the circle $= \pi r^2 = \pi (x^2/4\pi^2) = x^2/4\pi$.

Further suppose that the part of length y is bent in the shape of a square. The length of each side of the square is then $y/4$ and area of the square $= (y/4)^2 = y^2/16$.

Let A be the sum of the areas of the circle and the square.

Then $A = \dfrac{x^2}{4\pi} + \dfrac{y^2}{16} = \dfrac{x^2}{4\pi} + \dfrac{(l-x)^2}{16}$, [from (1)].

We have $\dfrac{dA}{dx} = \dfrac{2x}{4\pi} - \dfrac{2(l-x)}{16} = \dfrac{x}{2\pi} - \dfrac{l}{8} + \dfrac{x}{8}$.

For a maximum or minimum of A, $dA/dx = 0$ *i.e.*,

$$\frac{x}{8}\left(1 + \frac{4}{\pi}\right) - \frac{l}{8} = 0 \text{ i.e., } x = \frac{\pi l}{\pi + 4}.$$

Now $\dfrac{d^2A}{dx^2} = \dfrac{1}{2\pi} + \dfrac{1}{8} =$ +ive for $x = \dfrac{\pi l}{\pi + 4}$.

Hence A is minimum when $x = \pi l/(\pi + 4)$.

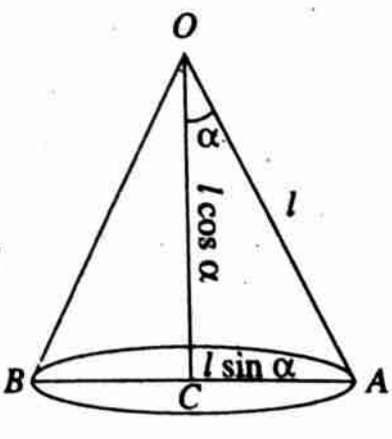
Putting $x = \dfrac{\pi l}{\pi + 4}$ in (1), we get $y = l - \dfrac{\pi l}{\pi + 4} = \dfrac{4l}{\pi + 4}$.

Hence the wire should be cut in length $\pi l/(\pi + 4)$ to be bent in the shape of a circle and the length $4l/(\pi + 4)$ to be bent in the shape of a square.

Ex. 15. *Show that the semi-vertical angle of the cone of maximum volume and of given slant height is* $\tan^{-1}\sqrt{2}$.

(Meerut 1977, 83, 88; Agra 86; Allahabad 87)

Sol. Let α be the semi-vertical angle of the cone and $OA = l$ be its slant height.



Then $V =$ volume of the cone

$$= \tfrac{1}{3}\pi (l \sin\alpha)^2 (l\cos\alpha)$$
$$= \tfrac{1}{3}\pi l^3 \sin^2\alpha \cos\alpha.$$

Since l is given, therefore V is a function of α. We have

$$\frac{dV}{d\alpha} = \tfrac{1}{3}\pi l^3 (2\sin\alpha\cos^2\alpha - \sin^3\alpha)$$
$$= \tfrac{1}{3}\pi l^3 \sin\alpha\,(2\cos^2\alpha - \sin^2\alpha).$$

For a maximum or minimum of V,

$$dV/d\alpha = 0$$

i.e., $\sin\alpha\,(2\cos^2\alpha - \sin^2\alpha) = 0$ *i.e.*, $\sin\alpha = 0$ or $\tan^2\alpha = 2$

i.e., $\alpha = \tan^{-1}\sqrt{2}$, because the values $\alpha = 0$ and $\tan\alpha = -\sqrt{2}$ are inadmissible.

Now $d^2V/d\alpha^2 = \frac{1}{3}\pi l^3 \cos\alpha\,(2\cos^2\alpha - \sin^2\alpha)$

$$+ \tfrac{1}{3}\pi l^3 \sin\alpha\,(-4\cos\alpha\sin\alpha - 2\sin\alpha\cos\alpha)$$

< 0, when $\tan\alpha = \sqrt{2}$ *i.e.*, when $2\cos^2\alpha = \sin^2\alpha$.

Hence V is maximum when $\alpha = \tan^{-1}\sqrt{2}$.

Ex. 16 (a). *Show that the cone of greatest volume which can be inscribed in a given sphere is such that three times its altitude is twice the diameter of the sphere.* **(Agra 1988; Gorakhpur 76; Lucknow 75)**

Sol. Let x be the radius of the base and y be the height of a cone inscribed in a given sphere of radius a.

Let V be the volume of the cone.

Then $V = \frac{1}{3}\pi x^2 y$. ...(1)

From figure, we have

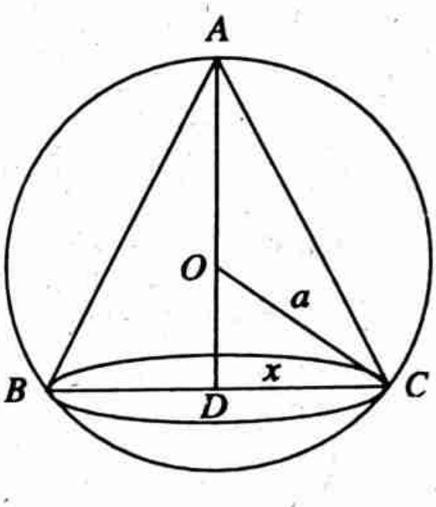
$OD = AD - AO = y - a$.

$\therefore \quad a^2 = (y-a)^2 + x^2$

or $\quad x^2 = a^2 - (y-a)^2$

$= a^2 - y^2 + 2ay - a^2$

$= y\,(2a - y)$.



Putting the value of x^2 in (1), we get

$$V = \tfrac{1}{3}\pi y^2 (2a - y).$$

We have

$$\frac{dV}{dy} = \tfrac{1}{3}\pi\,[2y\,(2a-y) - y^2]$$

$$= \tfrac{1}{3}\pi y\,(4a - 3y).$$

For a maximum or minimum of V, $\dfrac{dV}{dy} = 0$

i.e., $y\,(4a - 3y) = 0$ *i.e.*, $y = (2/3)\,(2a)$, since $y = 0$ is inadmissible.

Also $d^2V/dy^2 = \frac{1}{3}\pi\,(4a - 3y) + \frac{1}{3}\pi y\,(-3)$

$= -$ ive, when $y = 4a/3$.

Therefore V is maximum when $y = \frac{2}{3}(2a)$,

i.e., when $3y = 2\,(2a)$.

Ex. 16 (b). *What are the dimensions of the largest cone (cone of maximum volume) that can be inscribed in a sphere of radius* 10 *cm.* ? **(Meerut 1989)**

Ans. Radius $\dfrac{20\sqrt{2}}{3}$ cm., altitude $\dfrac{40}{3}$ cm.

Ex. 17. *Show that the semi-vertical angle of the right circular cone of given total surface (including area of the base) and maximum volume is* $\sin^{-1}\frac{1}{3}$.

(Allahabad 1973; Bihar 75; Jodhpur 76; Kashmir 75; Gorakhpur 1986; Kanpur 85)

Sol. Let x be the radius of the base, h be the height and y be the slant height of the cone. Then the total surface of the cone $=$ constant $\Rightarrow \pi x^2 + \pi xy =$ constant. ...(1)

Now $V =$ volume of the cone $= \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2 (y^2 - x^2)^{1/2}$, since $h = \sqrt{(y^2 - x^2)}$. Therefore $V^2 = \frac{1}{9}\pi^2 x^4 (y^2 - x^2)$.

Now V is maximum or minimum according as V^2 or $9V^2/\pi^2$ is maximum or minimum.

Let $z = 9V^2/\pi^2 = x^4 (y^2 - x^2)$.

Then z can be regarded as a function of x because y is connected with x by (1).

We have $dz/dx = 4x^3 (y^2 - x^2) + x^4 \{2y\,(dy/dx) - 2x\}$. ...(2)

Differentiating (1) w.r.t. x, we get

$$\pi\left(2x + y + x\frac{dy}{dx}\right) = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{2x + y}{x}.$$

Substituting this value of dy/dx in (2), we get

$$dz/dx = 4x^3y^2 - 4x^5 + x^4\left[-2y\frac{(2x + y)}{x} - 2x\right]$$

$$= 2x^3y^2 - 6x^5 - 4x^4y.$$

For a maximum or a minimum of z, we must have $dz/dx = 0$.

Now $\quad dz/dx = 0 \quad \Rightarrow \quad 2x^3 (y^2 - 2xy - 3x^2) = 0$

$\Rightarrow \quad 2x^3 (y - 3x)(y + x) = 0$

i.e., $y = 3x$, since $x \neq 0$ and $y \neq -x$.

Again $\quad \dfrac{d^2z}{dx^2} = 6x^2y^2 + 4x^3y\dfrac{dy}{dx} - 30x^4 - 16x^3y - 4x^4\dfrac{dy}{dx}.$

When $y = 3x$, $\quad \dfrac{dy}{dx} = -5. \qquad$ So when $y = 3x$, we have

$$\frac{d^2z}{dx^2} = 54x^4 - 30x^4 - 48x^4 + 20x^4 < 0.$$

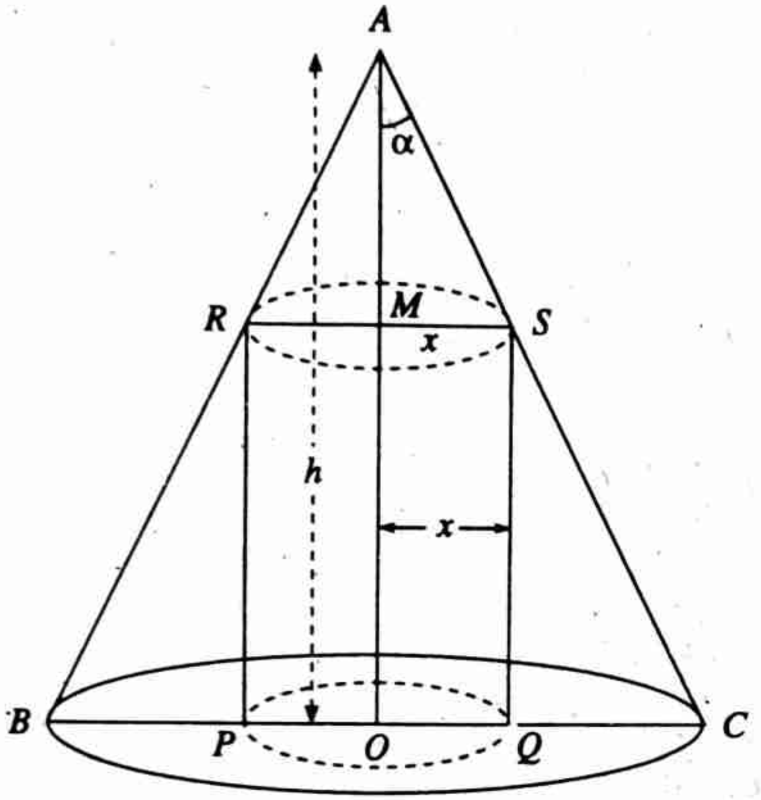
$\therefore \quad z$ is maximum when $y = 3x$.

Hence V is maximum when $y = 3x$ or $x/y = 1/3$, or $\sin\alpha = 1/3$ where α is the semi-vertical angle of the cone.

****Ex. 18 (a).** *Show that the volume of the greatest cylinder which can be inscribed in a cone of height* h *and semi-vertical angle* α *is* $(4/27)\,\pi h^3 \tan^2\alpha$.

(Meerut 1982, 87; Lucknow 81, 79; Agra 76; Allahabad 74; Gorakhpur 74, 72; Andhra 71; Magadh 78; Garhwal 77)

Sol. Let $PQRS$ be a cylinder of radius x inscribed in the cone ABC of height h and semi-vertical angle α.



From the figure, $AM = x \cot \alpha$.

$\therefore$ height of the cylinder $= h - x \cot \alpha$, and $V =$ volume of the cylinder $= \pi x^2 (h - x \cot \alpha)$.

For a maximum or a minimum value of V, $\dfrac{dV}{dx} = 0$.

Now $\dfrac{dV}{dx} = \pi \{x^2 (-\cot \alpha) + (h - x \cot \alpha) 2x\} = 0$

gives $x(-3x \cot \alpha + 2h) = 0$

i.e., $x = 0$ or $(2h/3) \tan \alpha$.

Also $\dfrac{d^2V}{dx^2} = \pi (2h - 6x \cot \alpha)$,

which is –ive at $x = \dfrac{2h}{3} \tan \alpha$.

Hence V is maximum when $x = (2h/3) \tan \alpha$.

Also from (1), the maximum value of V

$$= \pi \cdot \frac{4h^2}{9} \cdot (\tan^2 \alpha) \left(h - \frac{2h}{3}\right) = \frac{4\pi h^3}{27} \tan^2 \alpha.$$

Ex. 18 (b). *Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.*

2

Maxima and Minima

(Of Functions of two Independent Variables)

§ 1. Definition.

Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively. Then $f(a, b)$ is said to be a *maximum or a minimum* value of $f(x, y)$ according as $f(a+h, b+k)$ is *less* or *greater* than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

§ 2. Necessary conditions for the existence of a Maximum or a Minimum of f (x, y) at x = a, y = b.

(Meerut 1981, 83, 89, 89S, 90, 91P)

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y)$ at $x = a, y = b$ if the expression $f(a+h, b+k) - f(a, b)$ is of invariable sign for all sufficiently small independent values of h and k provided both of them are not equal to zero. If the sign of $f(a+h, b+k) - f(a, b)$ is negative, we shall have a maximum of $f(x, y)$ at $x = a, y = b$. If it is positive, $f(x, y)$ has a minimum at $x = a, y = b$.

By Taylor's theorem for two variables, we have

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{\substack{x=a,\\ y=b}} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a,\\ y=b}} + \cdots$$

$$\therefore \quad f(a+h, b+k) - f(a, b) = h\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}} + k\left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}}$$

+ terms of the second and higher orders in h and k. ...(1)

By taking h and k, sufficiently small, the first degree terms in h and k can be made to govern the sign of the right hand side and therefore of the left hand side of (1).

Thus the sign of $[f(a+h, b+k) - f(a, b)]$

$$= \text{the sign of} \left[h \left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} + k \left(\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}} \right] \qquad ...(2)$$

Taking $k = 0$, we find that if $\left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} \neq 0$, the right hand side of (2) changes sign when h changes sign. Therefore $f(x, y)$ cannot have a maximum or minimum at $x = a$, $y = b$ if $\left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} \neq 0$. Similarly taking $h = 0$, we can see that $f(x, y)$ cannot have a maximum or a minimum at $x = a, y = b$ if $\left(\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}} \neq 0$.

Thus *a set of necessary conditions that $f(x, y)$ should have a maximum or a minimum at $x = a$, $y = b$ is that*

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)_{\substack{\mathbf{x}=\mathbf{a} \\ \mathbf{y}=\mathbf{b}}} = \mathbf{0} \textit{ and } \left(\frac{\partial \mathbf{f}}{\partial \mathbf{y}} \right)_{\substack{\mathbf{x}=\mathbf{a} \\ \mathbf{y}=\mathbf{b}}} = \mathbf{0}.$$

The above conditions are necessary but not sufficient for the existence of maxima or minima.

§ 3. Stationary and Extreme Points.

A point (a, b) is called *a stationary point*, if both the first order partial derivatives of the function $f(x, y)$ vanish at that point. A stationary point which is either a maximum or a minimum is called an *extreme point* and the value of the function at the point is called an *extreme value*. A stationary point is not necessarily an extreme point. Thus a stationary point may be a maximum or a minimum or neither of these two. To decide whether a point is really an exreme point a further investigation is necessary.

§ 4. Sufficient condition for Maxima or Minima.

(Meerut 1981, 83, 89, 89P, 89S, 90, 91P)

Let $r = \left(\frac{\partial^2 f}{\partial x^2} \right)_{\substack{x=a \\ y=b}}$, $s = \left(\frac{\partial^2 f}{\partial x \, \partial y} \right)_{\substack{x=a \\ y=b}}$ and $t = \left(\frac{\partial^2 f}{\partial y^2} \right)_{\substack{x=a \\ y=b.}}$

If $\left(\frac{\partial f}{\partial x} \right)_{\substack{x=a \\ y=b}} = 0$ and $\left(\frac{\partial f}{\partial y} \right)_{\substack{x=a \\ y=b}} = 0$, *i.e.* if the necessary conditions for the existence of maxima or minima are satisfied, we have

$$f(a+h, b+k) - f(a, b) = \frac{1}{2!}(rh^2 + 2shk + tk^2) + R_3,$$

where R_3 consists of terms of third and higher orders in h and k.

For sufficiently small values of h and k, the sign of

$$\tfrac{1}{2}(rh^2 + 2shk + tk^2) + R_3$$

is the same as that of

$$rh^2 + 2shk + tk^2.$$

Now the following three different case arise :

Case I. $rt - s^2 > 0$. In this case obviously neither r nor t can be zero. Therefore we write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[r^2h^2 + 2srhk + rtk^2]$$

$$= \frac{1}{r}[(rh + sk)^2 + (rt - s^2)k^2].$$

Since $rt - s^2$ is positive, therefore

$$(rh + sk)^2 + (rt - s^2)k^2$$

is positive for all values of h and k except when

$rh + sk = 0, k = 0$ *i.e.* when $h = 0, k = 0$ which is obviously not possible.

Thus in this case the expression $rh^2 + 2shk + tk^2$ will have the *same* sign for all values of h and k. This sign is determined by the sign of r.

Thus $f(x, y)$ will have a maximum or a minimum at $x = a$, $y = b$ if $rt > s^2$. Further $f(x, y)$ is a maximum or a minimum according as r is negative or positive.

Case II. $rt - s^2 < 0$. In this case if $r \neq 0$, we can write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[(rh + sk)^2 + (rt - s^2)k^2].$$

If $k = 0, h \neq 0$, the sign of this expression will be the same as that of r. But if $k \neq 0$, $rh + sk = 0$, the sign of this expression will be opposite to that of r since $rt - s^2$ is negative. Thus in this case the expression $rh^2 + 2shk + tk^2$ is not of invariable sign.

A similar argument can be given if $t \neq 0$.

In case $r = 0$ as well as $t = 0$, we have

$$rh^2 + 2shk + tk^2 = 2shk,$$

which obviously does not keep the same sign for all values of h and k.

Thus $f(x, y)$ will have neither a maximum nor a minimum at $x = a$, $y = b$, if $rt < s^2$.

Case III. $\mathbf{rt - s^2 = 0.}$ If $r \neq 0$, we can write

$$rh^2 + 2shk + tk^2 = \frac{1}{r}[(rh+sk)^2 + (rt - s^2)k^2]$$

$$= \frac{1}{r}(rh+sk)^2, \qquad [\because rt - s^2 = 0].$$

This expression becomes zero when $rh + sk = 0$. Therefore the nature of the sign of

$$f(a+h, b+k) - f(a,b)$$

depends upon the consideration of R_3. The case is, therefore, doubtful.

In case $r = 0$, we must have $s = 0$, because of the condition $rt - s^2 = 0$.

$$\therefore \qquad rh^2 + 2shk + tk^2 = tk^2,$$

which is zero when $k = 0$ whatever h may be. The case is again doubtful.

Thus, if $rt - s^2 = 0$, the case is doubtful and further investigation is needed to determine whether $f(x,y)$ is a maximum or a minimum at $x = a$, $y = b$, or not.

§ 5. Working Rule for Maxima and Minima.

Suppose $f(x,y)$ is a given function of x and y. Find $\partial f/\partial x$ and $\partial f/\partial y$ and solve the simultaneous equations $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. In order to solve these equations we may either eliminate one of the variables, to factorise the equations. In the latter case each factor of the first equation must be solved in conjuction with each factor of the second equation. Suppose solving these equations we get the pairs of values of x and y as (a_1, b_1), (a_2, b_2) etc. Then all these pairs of roots will give stationary values of $f(x,y)$.

To discuss the maximum or minimum at $x = a_1, y = b_1$, we should find

$$r = \left(\frac{\partial^2 u}{\partial x^2}\right)_{\substack{x=a_1\\ y=b_1}}, \quad s = \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)_{\substack{x=a_1\\ y=b_1}}, \quad t = \left(\frac{\partial^2 u}{\partial y^2}\right)_{\substack{x=a_1\\ y=b_1}}.$$

Then calculate $\quad rt - s^2.$

If $rt - s^2 > 0$ and r is negative, $f(x,y)$ is maximum at $x = a_1$, $y = b_1$. If $rt - s^2 > 0$ and r is positive, $f(x,y)$ is minimum at $x = a_1$, $y = b_1$. If $rt - s^2 < 0$, $f(x,y)$ is neither maximum nor minimum at $x = a_1$, $y = b_1$.

If $rt - s^2 = 0$, the case is doubtful and further investigation will be required to decide it. We shall leave this case.

Solved Examples

Ex. 1. *Discuss the maximum or minimum values of u where*

$$u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3.$$

(Kanpur 1975; Meerut 84S)

Sol. We have $\partial u/\partial x = 2a^2y - 6axy + 3x^2y + y^3$,

and $\partial u/\partial y = 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2$.

Also $r = \partial^2u/\partial x^2 = -6ay + 6xy$,

$s = \partial^2u/\partial x\,\partial y = 2a^2 - 6ax + 3x^2 + 3y^2$,

and $t = \partial^2u/\partial y^2 = -6ay + 6xy$.

For a maximum or minimum of u, we have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0.$$

Thus, we have

$$\left.\begin{array}{r} y(2a^2 - 6ax + 3x^2 + y^2) = 0 \\ \text{and} \quad 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 = 0 \end{array}\right\}$$

Therefore we have to consider the pairs of equations, *viz.*,

$$\left.\begin{array}{r} y = 0 \\ 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 = 0 \end{array}\right\} \quad ...(1)$$

and

$$\left.\begin{array}{r} 2a^2 - 6ax + 3x^2 + y^2 = 0 \\ 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 = 0 \end{array}\right\} \quad ...(2)$$

Putting $y = 0$ in the second equation of the pair (1), we get

$$2a^2x - 3ax^2 + x^3 = 0$$

i.e., $x(x^2 - 3ax + 2a^2) = 0$

i.e., $x(x - a)(x - 2a) = 0$

i.e., $x = 0, x = a, x = 2a.$

Thus the pair (1) gives the following values of x and y:

$$x = 0, y = 0;\ x = a, y = 0;\ x = 2a, y = 0.$$

Multiplying the first equation of the pair (2) by x and subtracting it from the second equation of the pair, we get

$$3ax^2 - 3ay^2 - 2x^3 + 2xy^2 = 0$$

or $(x^2 - y^2)(3a - 2x) = 0.$

$\therefore \quad x = \frac{3}{2}a \quad$ and $\quad x = \pm y.$

When $x = \frac{3}{2}a$, the first equation of the pair (2) gives

$$y = \pm \tfrac{1}{2}a.$$

When $x = y$, we have $2a^2 - 6ay + 4y^2 = 0$

i.e., $y = a, \frac{1}{2}a.$

Also when $x = -y$, we have $2a^2 + 6ay + 4y^2 = 0$

i.e., $y = -a, -\frac{1}{2}a.$

Thus in all we get the following pairs of values of x and y which make the function u stationary :

$$(0,0), (a,0), (2a,0), (\tfrac{3}{2}a, \tfrac{1}{2}a), (\tfrac{3}{2}a, -\tfrac{1}{2}a)$$
$$(a,a), (\tfrac{1}{2}a, \tfrac{1}{2}a), (a,-a), (\tfrac{1}{2}a, -\tfrac{1}{2}a).$$

For (0, 0).

$r = 0, s = 2a^2, t = 0$ so that $rt - s^2$ is negative.

Therefore we have neither a maximum nor a minimum of u at $(0, 0)$.

Similarly, we can show that u has neither a maximum nor a minimum at $(a, 0)$, $(2a, 0)$, (a, a), $(a, -a)$.

For $\left(\frac{3a}{2}, \frac{a}{2}\right)$,

$r = \frac{3}{2}a^2, s = \frac{1}{2}a^2, t = \frac{3}{2}a^2$ so that $rt - s^2$ is positive. Since r is positive, therefore u has minimum at this point.

Similarly, we can show that u has a maximum at $(\frac{1}{2}a, -\frac{1}{2}a)$.

For $\left(\frac{3a}{2}, -\frac{a}{2}\right)$,

$r = -\frac{3}{2}a^2, s = -\frac{1}{2}a^2, t = -\frac{3}{2}a^2$ so that $rt - s^2$ is positive. Since r is negative, therefore u has a maximum at this point.

Similarly, we can show that u has a maximum at $(a/2, a/2)$.

Ex. 2. (a). *Find the extreme values of $xy(a - x - y)$.*

(Meerut 1980, 83, 91S)

Sol. Let $u = xy(a - x - y)$.

Then $$\frac{\partial u}{\partial x} = ay - 2xy - y^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = ax - x^2 - 2xy.$$

Also $$r = \frac{\partial^2 u}{\partial x^2} = -2y,\ s = \frac{\partial^2 u}{\partial y\, \partial x} = a - 2x - 2y,$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -2x.$$

For a maximum or minimum of u, we have
$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0.$$

Thus, we have
$$\left.\begin{aligned} ay - 2xy - y^2 &= 0 \\ ax - x^2 - 2xy &= 0 \end{aligned}\right\}.$$

These equations can be written as
$$y(a - 2x - y) = 0, x(a - x - 2y) = 0,$$
so that we have to consider the four pairs of equations, *viz.*,
$$y = 0, x = 0; a - 2x - y = 0, x = 0;$$
$$y = 0, a - x - 2y = 0;$$
$$a - 2x - y = 0, a - x - 2y = 0.$$

Solving these, we get the following pairs of values of x and y which make the function stationary :

$$(0, 0), (0, a), (a, 0), (\tfrac{1}{3}a, \tfrac{1}{3}a).$$

For (0, 0),

$r = 0, s = a, t = 0$ so that $rt - s^2$ is negative.

Therefore we have neither a maximum nor a minimum of u at $(0, 0)$.

For (0, a),

$r = -2a, s = -a, t = 0$ so that $rt - s^2$ is negative.

Therefore u has not an extreme value at $(0, a)$.

Similarly, we may show that u has not an extreme value at $(a, 0)$.

For $(\frac{1}{3}\mathbf{a}, \frac{1}{3}\mathbf{a})$,

$r = -\frac{2}{3}a, s = -\frac{1}{3}a, t = -\frac{2}{3}a$ so that $rt - s^2$ is positive.

Therefore u has an extreme value at $(\frac{1}{3}a, \frac{1}{3}a)$ and it will be a maximum or minimum according as, r, is negative or positive *i.e.*, according as, a, is positive or negative.

$\therefore$ the extreme value of u is $(1/27)\, a^3$.

Ex. 2. (b). *Find the points (x, y) where the function $xy\,(1 - x - y)$ is maximum or minimum. What is the maximum value of the function ?*

(Kanpur 1981; Meerut 80, 83S, 89)

Sol. Proceed as in Ex. 2 (a). Here $a = 1$.

Ans. The given function is maximum at the point $(\frac{1}{3}, \frac{1}{3})$.

Maximum value $= 1/27$.

Ex. 3. *Discuss the maxima and minima of the function*

$$u = x^2 + y^2 + \frac{2}{x} + \frac{2}{y}.$$

(Meerut 1986)

Sol. We have $u = x^2 + y^2 + \dfrac{2}{x} + \dfrac{2}{y}$.

$$\therefore \quad \frac{\partial u}{\partial x} = 2x - \frac{2}{x^2} \text{ and } \frac{\partial u}{\partial y} = 2y - \frac{2}{y^2}.$$

Also $$r = \frac{\partial^2 u}{\partial x^2} = 2 + \frac{4}{x^3},\ s = \frac{\partial^2 u}{\partial x\, \partial y} = 0$$

and $$t = \frac{\partial^2 u}{\partial y^2} = 2 + \frac{4}{y^3}.$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow 2x - \frac{2}{x^2} = 0 \Rightarrow 2\,(x^3 - 1) = 0 \Rightarrow x^3 = 1 \Rightarrow x = 1$$

and $\dfrac{\partial u}{\partial y}=0 \Rightarrow 2y-\dfrac{2}{y^2}=0 \Rightarrow 2(y^3-1)=0 \Rightarrow y^3=1 \Rightarrow y=1$.

$\therefore$ (1, 1) is the only point at which u is stationary *i.e.*, at which u may have a maximum or a minimum.

Now at the point(1, 1),

$r=2+4=6$, $s=0$ and $t=2+4=6$.

$\therefore$ $rt-s^2=36-0=36$ which is >0.

$\therefore$ the stationary value of u at (1, 1) is an extreme value.

Since $r>0$, therefore u has a minimum at this point.

Hence u is minimum at the point $x=1$, $y=1$.

Ex. 4. *Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.*

(Meerut 1990 S)

Sol. Let (x_r, y_r), $r=1, 2, 3$ be the vertices of the triangle and (x, y) be any point inside the triangle.

Let $u=\sum_{r=1}^{3}[(x-x_r)^2+(y-y_r)^2]$.

Then $$\frac{\partial u}{\partial x}=\Sigma\, 2(x-x_r)=2[(x-x_1)+(x-x_2)+(x-x_3)]$$

and $$\frac{\partial u}{\partial y}=\Sigma\, 2(y-y_r)=2[(y-y_1)+(y-y_2)+(y-y_3)].$$

Also $$r=\frac{\partial^2 u}{\partial x^2}=6,\ s=\frac{\partial^2 u}{\partial x\,\partial y}=0 \text{ and } t=\frac{\partial^2 u}{\partial y^2}=6.$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x}=0 \Rightarrow 2[(x-x_1)+(x-x_2)+(x-x_3)]=0 \Rightarrow x=\frac{x_1+x_2+x_3}{3}$$

and

$$\frac{\partial u}{\partial y}=0 \Rightarrow 2[(y-y_1)+(y-y_2)+(y-y_3)]=0 \Rightarrow y=\frac{y_1+y_2+y_3}{3}.$$

$\therefore$ $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$ is the only point at which u may have a maximum or a minimum.

At this point, $r=6$, $s=0$, $t=6$, so that $rt-s^2=36$ which is >0.

$\therefore$ the stationary value of u at the point

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

is an extreme value.

Since $r>0$, therefore this extreme value of u is a minimum.

Hence the required point at which u is minimum is the point

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right),$$

which is the centroid of the given triangle.

Ex. 5. *Find the values x and y for which the expression*

$$(a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 + \dots + (a_nx + b_ny + c_n)^2$$

becomes a minimum.

Sol. Let $u = (a_1x + b_1y + c_1)^2 + \dots + (a_nx + b_ny + c_n)^2$

$$= \Sigma\,(a_1x + b_1y + c_1)^2.$$

Then $$\frac{\partial u}{\partial x} = 2\,\Sigma\, a_1\,(a_1x + b_1y + c_1)$$

and $$\frac{\partial u}{\partial y} = 2\,\Sigma\, b_1\,(a_1x + b_1y + c_1).$$

Also $r = \dfrac{\partial^2 u}{\partial x^2} = 2\,\Sigma\, a_1^2$, $s = \dfrac{\partial^2 u}{\partial x\, \partial y} = 2\,\Sigma\, a_1b_1$ and $t = \dfrac{\partial^2 u}{\partial y^2} = 2\,\Sigma\, b_1^2$.

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \Rightarrow (\Sigma\, a_1^2)\,x + (\Sigma\, a_1b_1)\,y + \Sigma\, a_1c_1 = 0 \qquad \dots(1)$$

and $$\frac{\partial u}{\partial y} = 0 \Rightarrow (\Sigma\, a_1b_1)\,x + (\Sigma\, b_1^2)\,y + \Sigma\, b_1c_1 = 0. \qquad \dots(2)$$

$\therefore$ the only point (x, y) at which u is stationary is obtained by solving the linear equations (1) and (2) in x and y.

At this point $r = 2\,\Sigma\, a_1^2$, $s = 2\,\Sigma\, a_1b_1$, $t = 2\,\Sigma\, b_1^2$ so that $rt - s^2 = 4\,[\Sigma\, a_1^2\, \Sigma\, b_1^2 - (\Sigma\, a_1b_1)^2] = 4\,\Sigma\,(a_1b_2 - a_2b_1)^2$ which is > 0.

$\therefore$ the stationary value of u at the point given by the equations (1) and (2) is an extreme value.

Since $r = 2\,\Sigma\, a_1^2 > 0$, therefore this extreme value of u is a minimum.

Hence the values of x and y for which u is minimum satisfy the linear equations (1) and (2) in x and y.

Ex. 6. *Examine the following surface for high and low points :*

$$z = x^2 + xy + 3x + 2y + 5. \qquad \textbf{(Meerut 68)}$$

Sol. We have $\partial z/\partial x = 2x + y + 3$ and $\partial z/\partial y = x + 2$.

Also $r = \partial^2 z/\partial x^2 = 2$, $s = \partial^2 z/\partial x\, \partial y = 1$, $t = \partial^2 z/\partial y^2 = 0$.

For a maximum or a minimum of z, we must have

$$\frac{\partial z}{\partial x} = 0 \Rightarrow 2x + y + 3 = 0 \qquad \dots(1)$$

and $$\frac{\partial z}{\partial y} = 0 \Rightarrow x + 2 = 0. \qquad ...(2)$$

Solving the equations (1) and (2), we get $x = -2, y = 1$.

$\therefore$ z may have a maximum or a minimum only at the point $x = -2, y = 1$.

At this point $rt - s^2 = 2 \,.\, 0 - 1 = -1$ which is < 0 and so the stationary value of z at the point $x = -2, y = 1$ is neither a maximum nor a minimum.

Hence the given surface has no high and low points.

Ex. 7. *Show that the minimum value of* $u = xy + (a^3/x) + (a^3/y)$ *is* $3a^2$. **(Meerut 1977, 88P, 92, 95; Agra 73)**

Sol. For a maximum or a minimum of u, we have

$$\frac{\partial u}{\partial x} \equiv y - \frac{a^3}{x^2} = 0, \quad \text{and} \quad \frac{\partial u}{\partial y} \equiv x - \frac{a^3}{y^2} = 0.$$

Solving these equations, we get $x = y = a$. Thus u is stationary at the point (a, a).

Now $$r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3}, \; s = \frac{\partial^2 u}{\partial x \, \partial y} = 1,$$

and $$t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}.$$

At $x = a$ and $y = a$, we have $r = 2a^3/a^3 = 2$, $s = 1$ and $t = 2$. These give $rt - s^2 = 4 - 1 = 3$ which is > 0.

Thus at (a, a) we have $rt - s^2 > 0$ and r is +ive. Therefore u is minimum at $x = a$ and $y = a$.

The minimum value of $u = a \,.a + (a^3/a) + (a^3/a) = 3a^2$.

Ex. 8. *Examine for maximum and minimum values the function*

$$u = x^4 + 2x^2y - x^2 + 3y^2.$$ **(Meerut 1990 P)**

Sol. For a maximum or minimum of u, we must have

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &\equiv 4x^3 + 4xy - 2x = 0, \\ \text{and} \quad \frac{\partial u}{\partial y} &\equiv 2x^2 + 6y = 0 \end{aligned}\right\}; \quad \begin{aligned} &\text{giving } x = \pm \tfrac{1}{2}\sqrt{3} \\ &\text{and} \quad y = -\tfrac{1}{4}. \end{aligned}$$

Now $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 + 4y - 2$, $s = \dfrac{\partial^2 u}{\partial x \, \partial y} = 4x$ and $t = \dfrac{\partial^2 u}{\partial y^2} = 6$.

When $x = \frac{1}{2}\sqrt{3}, y = -\frac{1}{4}$, we have

$r = 12\,(3/4) + 4\,(-\frac{1}{4}) - 2 = 6$, $s = 4\,(\frac{1}{2}\sqrt{3}) = 2\sqrt{3}$ and $t = 6$.

$\therefore$ $rt - s^2 = 6 \times 6 - (2\sqrt{3})^2$ which is > 0. Also $r > 0$. Hence u has a minimum when $x = \frac{1}{2}\sqrt{3}, y = -\frac{1}{4}$.

Again when $x = -\frac{1}{2}\sqrt{3}, y = -\frac{1}{4}$, we have

$r = 6,\ s = -2/\sqrt{3}$ and $t = 6$. Therefore $rt - s^2 = +$ive. Also $r > 0$. Hence there is again a minimum when $x = -\frac{1}{2}\sqrt{3}, y = -\frac{1}{4}$.

Ex. 9. *Examine for maximum and minimum values of the function*

$$z = x^2 - 3xy + y^2 + 2x. \qquad \textbf{(Meerut 1978, 84 P, 97)}$$

Sol. For a maximum or minimum of z, we must have

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &\equiv 2x - 3y + 2 = 0 \\ \text{and} \quad \frac{\partial z}{\partial y} &\equiv -3x + 2y = 0 \end{aligned}\right\} ; \text{ giving } x = 4/5, y = 6/5.$$

Thus, z is stationary at the point $(4/5, 6/5)$.

Also $r = \dfrac{\partial^2 z}{\partial x^2} = 2;\ s = \dfrac{\partial^2 z}{\partial x\, \partial y} = -3;\ t = \dfrac{\partial^2 z}{\partial y^2} = 2.$

$\therefore \quad rt - s^2 = 2 \times 2 - (-3)^2 = -5$ which is < 0.

Hence z is neither maximum nor minimum at $x = 4/5, y = 6/5$.

Ex. 10. *Discuss the maximum or minimum values of*

$$u = x^3y^2(1 - x - y). \qquad \textbf{(Meerut 1984)}$$

Sol. We have $\partial u/\partial x = 3x^2y^2(1 - x - y) + x^3y^2(-1)$

$$= 3x^2y^2 - 4x^3y^2 - 3x^2y^3,$$

and $\quad \partial u/\partial y = 2x^3y(1 - x - y) + x^3y^2(-1) = 2x^3y - 2x^4y - 3x^3y^2.$

For a maximum or minimum of u, we must have $\partial u/\partial x = 0$ and $\partial u/\partial y = 0$ *i.e.*,

$$\left.\begin{aligned} x^2y^2(3 - 4x - 3y) &= 0 \\ \text{and} \quad x^3y(2 - 2x - 3y) &= 0 \end{aligned}\right\} ; \text{ giving } x = \tfrac{1}{2}, y = \tfrac{1}{3};$$

$$x = 0, -\infty < y < \infty;$$

$$\text{and } y = 0, -\infty < x < \infty.$$

Now $r = \dfrac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3 = 6xy^2(1 - 2x - y),$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = 6x^2y - 8x^3y - 9x^2y^2 = x^2y(6 - 8x - 9y),$$

and $\quad t = \dfrac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y = 2x^3(1 - x - 3y).$

When $x = \frac{1}{2}, y = \frac{1}{3}$, we have

$$r = 6 \cdot \tfrac{1}{2} \cdot \tfrac{1}{9}(1 - 2 \cdot \tfrac{1}{2} - \tfrac{1}{3}) = -\tfrac{1}{9},$$

$$s = \tfrac{1}{4} \cdot \tfrac{1}{3}(6 - 8 \cdot \tfrac{1}{2} - 9 \cdot \tfrac{1}{3}) = -\tfrac{1}{12}$$

and $t = 2 \cdot \frac{1}{8}(1 - \frac{1}{2} - 3 \cdot \frac{1}{3}) = -\frac{1}{8}$.

$\therefore\ rt - s^2 = (-\frac{1}{9})(-\frac{1}{8}) - (\frac{1}{12})^2 = \frac{1}{72} - \frac{1}{144} = +$ive. Also r is $-$ive.

Hence u is maximum at $x = \frac{1}{2}, y = \frac{1}{3}$.

When $x = 0$ and $-\infty < y < \infty$, we have $rt - s^2 = 0$ Also when $y = 0$ and $-\infty < x < \infty$, we have $rt - s^2 = 0$. So these cases are doubtful and need further investigation.

Ex. 11. *Determine the points where a function $x^3 + y^3 - 3axy$ has a maximum or minimum.* **(Meerut 1984, 86 P, 88, 95BP; Agra 80)**

Sol. Let $u = x^3 + y^3 - 3axy$.

We have $\partial u/\partial x = 3x^2 - 3ay$, $\partial u/\partial y = 3y^2 - 3ax$,
$r = \partial^2 u/\partial x^2 = 6x$, $s = \partial^2 u/\partial x\, \partial y = -3a$, $t = \partial^2 u/\partial y^2 = 6y$.

For a maximum or minimum of u, we must have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0.$$

From $\partial u/\partial x = 0$, we get $x^2 - ay = 0$, ...(1)

and from $\partial u/\partial y = 0$, we get $y^2 - ax = 0$. ...(2)

Solving the equations (1) and (2), we get

$(y^2/a)^2 - ay = 0$, or $y^4 - a^3 y = 0$, or $y(y^3 - a^3) = 0$, or $y = 0, a$.

From (1), when $y = 0$, we get $x = 0$

and when $y = a$, we get $x = \pm a$.

But $x = -a$ and $y = a$ do not satisfy the equation (2). Therefore we reject these values.

Hence the only solutions are $x = 0, y = 0$; $x = a, y = a$. Thus u is stationary at the points $(0, 0)$ and (a, a).

At x = 0, y = 0, we get

$$r = 0,\ s = -3a \text{ and } t = 0.$$

$\therefore$ $rt - s^2 = 0 - (-3a)^2 = -9a^2$ *i.e.*, negative and so there is neither maximum nor minimum of u at $x = 0, y = 0$.

At x = a, y = a, we get

$$r = 6a,\ s = -3a \text{ and } t = 6a.$$

$\therefore$ $rt - s^2 = (6a).(6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2$ *i.e.*, > 0.

Also $r = 6a$ which is > 0 if $a > 0$ and is < 0 if $a < 0$. Thus at the point (a, a), there is maximum or minimum of u according as $a < 0$ or $a > 0$.

Hence there is a maximum at the point (a, a) if $a < 0$ and a minimum if $a > 0$.

Ex. 12. *Examine the function $z = x^2 y - y^2 x - x + y$ for maxima and minima.* **(Agra 1974)**

Sol. For max. or min. of z, we have

$$\partial z/\partial x = 2xy - y^2 - 1 = 0,$$

and
$$\partial z/\partial y = x^2 - 2xy + 1 = 0.$$

Solving these equations, we get $x = 1, y = 1$; $x = -1, y = -1$.

Now $r = \partial^2 z/\partial x^2 = 2y$, $s = \partial^2 z/\partial x\, \partial y = 2x - 2y$,

$$t = \partial^2 z/\partial y^2 = -2x.$$

For x = 1, y = 1, we have $r = 2, s = 0, t = -2$ so that

$rt - s^2 = -4$ *i.e.*, –ive.

Hence z has neither a max. nor a min. at $(1, 1)$.

Thus $(1, 1)$ is a saddle point.

For x = − 1, y = − 1, we have $r = -2, s = 0, t = 2$ so that $rt - s^2 = -4$ *i.e.*, –ive.

Hence z has neither a max. nor min. at $(-1, -1)$.

Thus $(-1, -1)$ is also a saddle point.

Ex. 13. *Find all maximum or minimum of the function*

$$f(x, y) = y^2 + x^2y + x^4.$$ **(Meerut 1985, 81 S)**

Sol. Let $u = y^2 + x^2y + x^4$.

Then $\partial u/\partial x = 2xy + 4x^3$, $\partial u/\partial y = 2y + x^2$,

$$r = \frac{\partial^2 u}{\partial x^2} = 2y + 12x^2,\ s = \frac{\partial^2 u}{\partial x\, \partial y} = 2x,\ t = \frac{\partial^2 u}{\partial y^2} = 2.$$

For a maximum or minimum of u, we must have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0$$

i.e., $2xy + 4x^3 = 0$ and $2y + x^2 = 0$

i.e., $2x\,(y + 2x^2) = 0$ and $2y + x^2 = 0$.

To slove these equations we have to consider two pairs of equations, viz.,

$$x = 0,\ 2y + x^2 = 0;\ y + 2x^2 = 0,\ 2y + x^2 = 0.$$

The first pair gives $x = 0, y = 0$ and the second pair also gives

$$x = 0, y = 0.$$

Therefore the maximum or minimum of u may exist on the point $x = 0, y = 0$.

Again, at $x = 0, y = 0$, we get

$$r = 0,\ s = 0 \text{ and } t = 2.$$

$$\therefore \quad rt - s^2 = 0 \times 2 - 0 = 0$$

and hence the case is **doubtful** and further investigation is needed.

We may tackle the doubtful case at the point $(0, 0)$ by the following consideration :

We can write $f(x, y) = (\frac{1}{2}x^2 + y)^2 + \frac{3}{4}x^4$.

We have $f(0, 0) = 0$ *i.e.*, the value of $f(x, y)$ at $(0, 0)$ is zero.

Also for all (x, y) in the neighbourhood of $(0, 0)$, we have $f(x, y) > 0$ *i.e.*, $f(x, y) > f(0, 0)$.

Thus at all points (x, y) in the neighbourhood of $(0, 0)$, the value of $f(x, y)$ is greater than its value at $(0, 0)$. Hence $f(x, y)$ is minimum at $(0, 0)$.

Ex. 14. *Let* $f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$. *Show that* $f(x, y)$ *has neither a maximum nor a minimum at* $(0, 0)$.

(Meerut 1981, 83P, 97)

Sol. We have $\partial f/\partial x = 2x - 2y + 3x^2 + 5x^4$

and $\partial f/\partial y = -2x + 2y - 3y^2$.

Also $r = \dfrac{\partial^2 f}{\partial x^2} = 2 + 6x + 20x^3$, $s = \dfrac{\partial^2 f}{\partial x\,\partial y} = -2$, $t = \dfrac{\partial^2 f}{\partial y^2} = 2 - 6y$.

For a maximum or a minimum of $f(x, y)$, we must have

$$\partial f/\partial x = 0 \quad \text{and} \quad \partial f/\partial y = 0.$$

Obviously $x = 0$, $y = 0$ is a solution of the equations

$$\partial f/\partial x = 0 \quad \text{and} \quad \partial f/\partial y = 0.$$

$\therefore$ the value of $f(x, y)$ is stationary at the point $(0, 0)$.

At the point $(0, 0)$, $r = 2$, $s = -2$, $t = 2$,

so that $rt - s^2 = 4 - 4 = 0$.

$\therefore$ the stationary value of $f(x, y)$ at the point $(0, 0)$ gives rise to **doubtful case** and needs further investigation.

We can write $f(x, y) = (x - y)^2 + (x - y)(x^2 + xy + y^2) + x^5$.

We have $f(0, 0) = 0$.

In the neighbourhood of the point $(0, 0)$, where $x = y$, we have $f(x, y) = x^5$ which is positive when $x > 0$ and is negative when $x < 0$.

Thus in the neighbourhood of the point $(0, 0)$, there are points at which $f(x, y)$ takes values less than its value at the point $(0, 0)$ and there are points at which $f(x, y)$ takes values greater than its value at $(0, 0)$. Hence $f(x, y)$ cannot have a maximum or a minimum at the point $(0, 0)$.

Ex. 15. *Discuss the maxima and minima of the function* $u = \sin x \sin y \sin(x + y)$. **(Meerut 1974, 85 S, 90 S, 91,93 P, 94, 98)**

Sol. We have $u = \sin x \sin y \sin(x + y)$.

Since the function u is periodic with period π both for x and y, therefore it is sufficient to consider the values of x and y between 0 and π.

Now $\partial u/\partial x = \sin y\,[\sin x \cos(x + y) + \cos x \sin(x + y)]$
$= \sin y \sin(x + x + y) = \sin y \sin(2x + y)$

and $\partial u/\partial y = \sin x\,[\sin y \cos(x + y) + \cos y \sin(x + y)]$
$= \sin x \sin(2y + x)$.

Also $r = \partial^2 u/\partial x^2 = 2 \sin y \cos(2x + y)$,

$s = \partial^2 u/\partial x\,\partial y = \cos y \sin(2x + y) + \sin y \cos(2x + y)$
$= \sin(2x + y + y) = \sin 2(x + y)$

and $t = \partial^2 u/\partial y^2 = 2 \sin x \cos(2y + x)$.

For a maximum or a minimum of u, we must have

$$\partial u/\partial x = 0 \quad \text{and} \quad \partial u/\partial y = 0$$

i.e., $\sin y \sin(2x + y) = 0$ and $\sin x \sin(2y + x) = 0$.

To find the values of x and y satisfying these equations and lying between 0 and π, we have to consider the following pairs of equations :

$x = 0, y = 0$; $2x + y = \pi$, $2y + x = \pi$; $2x + y = 2\pi$, $2y + x = 2\pi$.

Solving these, we get the following pairs of values of x and y between 0 and π, which make the function u stationary :

$x = 0, y = 0$; $x = \pi/3, y = \pi/3$; $x = 2\pi/3, y = 2\pi/3$.

For (0, 0),

$$r = 0, s = 0, t = 0 \text{ so that } rt - s^2 = 0.$$

Therefore at the point $(0, 0)$, the case is doubtful and further investigation is needed.

For $(\pi/3, \pi/3)$,

$$r = 2\sin\tfrac{1}{3}\pi . \cos\pi = -\sqrt{3},$$

$$s = \sin(4\pi/3) = \sin(\pi + \tfrac{1}{3}\pi) = -\sin\tfrac{1}{3}\pi = -\sqrt{3}/2$$

and $\quad t = 2\sin\tfrac{1}{3}\pi . \cos\pi = -\sqrt{3}$.

$$\therefore \quad rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - (-\sqrt{3}/2)^2 = 3 - \tfrac{3}{4}$$

$$= \tfrac{9}{4} = + \text{ive}.$$

Hence u has an extreme value at $(\pi/3, \pi/3)$.

Also as $r = -\sqrt{3}$ *i.e.*, –ive, therefore u is maximum at $x = \pi/3$, $y = \pi/3$.

For x = $2\pi/3$, y = $2\pi/3$, we have

$$r = 2\sin\tfrac{2}{3}\pi . \cos 2\pi = 2(\sqrt{3}/2).1 = \sqrt{3},$$

$$s = \sin(8\pi/3) = \sin(2\pi + \tfrac{2}{3}\pi) = \sin\tfrac{2}{3}\pi = \sqrt{3}/2,$$

and $\quad t = 2\sin\tfrac{2}{3}\pi \cos 2\pi = \sqrt{3}$.

$$\therefore \quad rt - s^2 = (\sqrt{3})(\sqrt{3}) - (\sqrt{3}/2)^2 = 3 - \tfrac{3}{4} = \tfrac{9}{4} = + \text{ive}.$$

Hence u has an exrtreme value at $(2\pi/3, 2\pi/3)$.

Also as $r = \sqrt{3}$ *i.e.*, +ive, therefore u is minimum at

$$x = 2\pi/3, y = 2\pi/3.$$

Thus the given function is maximum at $x = y = \pi/3$ and minimum at $x = y = 2\pi/3$.

Ex. 16. *Discuss the maximum and minimum values of*

$$u = 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y) + \cos(x+y). \qquad \textbf{(Meerut 1990)}$$

Sol. We have $u = \sin x + \sin y + \cos(x + y)$. For a maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} \equiv \cos x - \sin(x+y) = 0, \qquad ...(1)$$

and $$\frac{\partial u}{\partial y} \equiv \cos y - \sin(x+y) = 0. \qquad ...(2)$$

From (1) and (2), we get $\cos x = \cos y$ or $x = y$. Putting $x = y$ in (1), we get

$$\cos x - \sin 2x = 0 \quad \text{or} \quad \cos x - 2\sin x \cos x = 0$$

or $$\cos x\,(1 - 2\sin x) = 0.$$

$$\therefore \quad \cos x = 0 \quad \text{or} \quad \sin x = \tfrac{1}{2}.$$

Now $\cos x = 0$ gives $x = 2n\pi \pm \frac{1}{2}\pi$ and $\sin x = \frac{1}{2}$ gives $x = n\pi + (-1)^n \frac{1}{6}\pi$.

Thus u is stationary when $x = y = 2n\pi \pm \frac{1}{2}\pi$ and when

$$x = y = n\pi + (-1)^n \tfrac{1}{6}\pi.$$

Now $r = \dfrac{\partial^2 u}{\partial x^2} = -\sin x - \cos(x + y)$, $s = \dfrac{\partial^2 u}{\partial x\,\partial y} = -\cos(x + y)$ and $t = \dfrac{\partial^2 u}{\partial y^2} = -\sin y - \cos(x + y)$.

When $x = y$, we have $r = -\sin x - \cos 2x$, $s = -\cos 2x$ and $t = -\sin x - \cos 2x$.

Now when $x = y = n\pi + (-1)^n \frac{1}{6}\pi$, we have $\sin x = \frac{1}{2}$ and $\cos 2x = 1 - 2\sin^2 x = 1 - 2\cdot\frac{1}{4} = \frac{1}{2}$. Therefore $r = -\frac{1}{2} - \frac{1}{2} = -1$, $s = -\frac{1}{2}$ and $t = -1$. These give $rt - s^2 = 1 - \frac{1}{4} = +$ive. Since $rt - s^2$ is $+$ive and r is $-$ ive, therefore u is maximum when

$$x = y = n\pi + (-1)^n \tfrac{1}{6}\pi.$$

Again when $x = y = 2n\pi - \frac{1}{2}\pi$, we have $\sin x = -1$, $\cos x = 0$ and $\cos 2x = 2\cos^2 x - 1 = 0 - 1 = -1$.

Therefore $r = 1 + 1 = 2$, $s = 1$ and $t = 2$.

These give $rt - s^2 = 4 - 1 = +$ive. Since $rt - s^2$ is $+$ive and r is $+$ive, therefore u is minimum when $x = y = 2n\pi - \frac{1}{2}\pi$.

Further when $x = y = 2n\pi + \frac{1}{2}\pi$, we have $\sin x = 1$, $\cos x = 0$, and $\cos 2x = -1$. Therefore $r = 0$, $s = 1$ and $t = 0$. These give $rt - s^2 = -1 = -$ ive. Therefore u has neither a maximum nor a minimum at $x = y = 2n\pi + \frac{1}{2}\pi$.

Ex. 17. *Show that the distance l of any point (x, y, z) on the plane $2x + 3y - z = 12$ from the origin is given by*

$$l = \sqrt{[x^2 + y^2 + (2x + 3y - 12)^2]}.$$

Hence find the point on the plane that is nearest to the origin.

(Meerut 1975)

Sol. If l is the distance from $(0, 0, 0)$ of any point (x, y, z), then $l = \sqrt{(x^2 + y^2 + z^2)}$. If the point (x, y, z) lies on the plane $2x + 3y - z = 12$, then $l = \sqrt{[x^2 + y^2 + (2x + 3y - 12)^2]}$,

Sol. Draw figure as in Ex. 18 (a). Here let r be the radius and α the semi-vertical angle of the given cone. Then the height AO of the cone $= r \cot \alpha$.

Let $PQRS$ be a cylinder of radius x inscribed in the cone ABC.

From the figure, $AM = x \cot \alpha$.

$\therefore$ the height MO of the cylinder $PQRS = r \cot \alpha - x \cot \alpha$.

If S be the curved surface of the cylinder $PQRS$, then

$$S = 2\pi x (r \cot \alpha - x \cot \alpha) = 2\pi \cot \alpha (xr - x^2).$$

We have $dS/dx = 2\pi \cot \alpha (r - 2x)$.

For a maximum or a minimum value of S, we must have

$$dS/dx = 0 \quad i.e., \ 2\pi \cot \alpha (r - 2x) = 0$$

i.e., $\quad r - 2x = 0 \quad i.e., \ x = r/2$.

Also $d^2 S/dx^2 = 2\pi \cot \alpha . (-2)$, which is negative.

$\therefore$ S is maximum when $x = r/2$ *i.e.,* when the radius of the cylinder is half that of the cone.

Ex. 19. *A given quantity of metal is to be cast into a half cylinder i.e., with a rectangular base and semi-circular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of the ends is $\pi/(\pi + 2)$.* **(Agra 1980)**

Sol. Let l be the length and r be the radius of each end of the half cylinder.

If V is the volume of the half cylinder, then $V = \frac{1}{2}\pi r^2 l =$ constant, so that $l = 2V/(\pi r^2)$.

Now let S be the total surface of the half cylinder. Then $S =$ area of the rectangular base + curved surface + area of semi-circular ends $= 2rl + \pi rl + 2(\frac{1}{2}\pi r^2) = (2 + \pi) rl + \pi r^2$

i.e., $S = (2 + \pi)(2V/\pi r) + \pi r^2$, from (1).

For a maximum or a minimum of S, $dS/dr = 0$.

Now $\dfrac{dS}{dr} \equiv \dfrac{-2(2+\pi)V}{\pi r^2} + 2\pi r = 0$ gives

$$-\frac{2(2+\pi)\frac{1}{2}\pi r^2 l}{\pi r^2} + 2\pi r = 0, \qquad [\because V = \tfrac{1}{2}\pi r^2 l]$$

or $\quad (l/2r) = \pi/(\pi + 2)$.

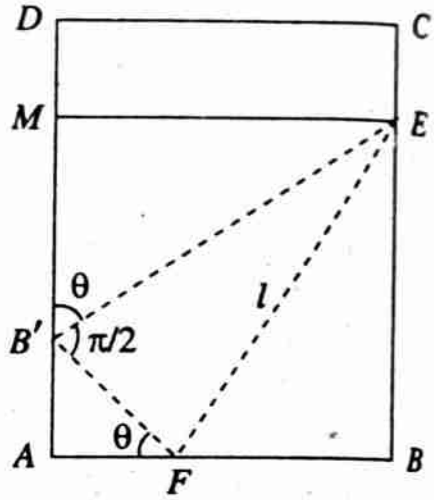
Also $$\frac{d^2S}{dr^2} = \frac{\pi + 2}{\pi} \cdot \frac{4V}{r^3} + 2\pi = +\text{ive}.$$

Hence the total surface S is minimum when

$$(l/2r) = \pi/(\pi + 2)$$

i.e., when the ratio of the length of the cylinder to the diameter of the ends is $\pi : \pi + 2$.

Ex. 20. *One corner of a long rectangular sheet of paper of width 1 foot is folded so as to reach the opposite edge of the sheet. Find the minimum length of the crease.*



Sol. Let the corner B of the rectangular sheet $ABCD$ of width 1 foot be folded along EF so as to reach the opposite edge AD at B'. Let l feet be the length of the crease EF.

Also let $BF = x$. Then $B'F = BF = x$; $AF = AB - BF = 1 - x$.

Let $\angle AFB' = \theta$; then $\angle EB'M = \theta$, where EM is perpendicular to AD.

Now $\cos\theta = \dfrac{AF}{B'F} = \dfrac{1-x}{x}$. ...(1)

From $\Delta\, B'ME$, $B'E = ME \operatorname{cosec}\theta$

$= \operatorname{cosec}\theta = 1/\sqrt{(1-\cos^2\theta)} = x/\sqrt{(2x-1)}$, from (1).

From the right-angled tirangle $EB'F$, we have

$$EF^2 = EB'^2 + B'F^2 = \frac{x^2}{(2x-1)} + x^2 = \frac{2x^3}{2x-1}$$

or $$l^2 = \frac{2x^3}{2x-1} \quad ...(2)$$

Now l is maximum or minimum when l^2 is maximum or minimum.

Let $l^2 = u$. Then $u = \dfrac{2x^3}{2x-1}$. Therefore

$$\frac{du}{dx} = \frac{6x^2(2x-1) - 4x^3}{(2x-1)^2} = \frac{2x^2(4x-3)}{(2x-1)^2} = \frac{8x^2}{(2x-1)^2}\left(x - \frac{3}{4}\right).$$

For a maximum or a minimum of u,

$$\frac{du}{dx} = 0 \text{ i.e., } \frac{8x^2}{(2x-1)^2}\left(x - \frac{3}{4}\right) = 0,$$

i.e., $x = 3/4$, since $x = 0$ is not possible if there is to be a folding.

Now $$\frac{d^2u}{dx^2} = \left(x - \frac{3}{4}\right)\frac{d}{dx}\left\{\frac{8x^2}{(2x-1)^2}\right\} + \frac{8x^2}{(2x-1)^2}\cdot 1$$

$=$ +ive, when $x = 3/4$. Therefore u is minimum, and hence l is minimum, when $x = 3/4$. Putting $x = 3/4$ in (2), we get

$$l^2 = \frac{2.(27/64)}{1/2} = \frac{27}{16}.$$

Therefore $l = (3\sqrt{3})/4$ feet is the minimum length of the crease.

Ex. 21. *A rectangular sheet of metal has four equal square portions removed at the corners and the sides are then turned up so as to form an* *open rectangular box. Show that when the volume contained in the box is a maximum, the depth will be*

$$\frac{1}{6}\left\{(a+b)-\sqrt{(a^2-ab+b^2)}\right\},$$

where a, b are sides of the original rectangle.

Sol. Let x be the length of the side of each of the square portions removed. Then the length of the box $= a - 2x$, the breadth $= b - 2x$, and the depth $= x$. If V is the volume of the box, then

$$V = x(a-2x)(b-2x),\ (a > b)$$

or $$V = 4x^3 - 2x^2(a+b) + abx.$$

For a maximum or a minimum of V, we have

$$\frac{dV}{dx} = 12x^2 - 4(a+b)x + ab = 0$$

or $$x = [4(a+b) \pm \sqrt{\{16(a+b)^2 - 48ab\}}]/24$$
$$= \tfrac{1}{6}\left\{(a+b) \pm \sqrt{(a^2-ab+b^2)}\right\}.$$

Now the +ive sign before the radical is not admissible, because then the value of x becomes $> \frac{1}{2}b$. Obviously the other value of x obtained by taking the –ive sign before the radical is less than $b/2$.

Now $d^2V/dx^2 = 24x - 4(a+b)$
$$= 4\{(a+b) - \sqrt{(a^2-ab+b^2)}\} - 4(a+b),$$
when $x = \frac{1}{6}\{(a+b) - \sqrt{(a^2-ab+b^2)}\}$
$$= -4\sqrt{(a^2-ab+b^2)}, \text{ which is –ive.}$$

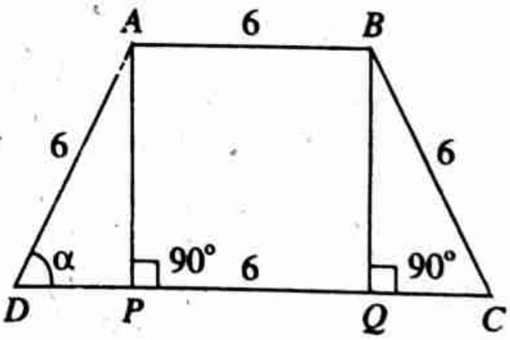
Hence V is maximum for this value of x.

Ex. 22. *The three sides of a trapezium are equal, each being 6 centimetres long. Find the area of the trapezium when it is maximum.*

(Agra 1971)

Sol. From figure,

$$AP = BQ = 6\sin\alpha \text{ and}$$
$$DP = QC = 6\cos\alpha.$$
$\therefore$ $$CD = 6 + 12\cos\alpha.$$



If S is the area of the trapezium in square centimetres, then

$$S = \tfrac{1}{2}(AB + CD)\,BQ$$
$$= \tfrac{1}{2}(6 + 6 + 12\cos\alpha)\,6\sin\alpha$$
$$= 36(\sin\alpha + \sin\alpha\cos\alpha)$$
$$= 36(\sin\alpha + \tfrac{1}{2}\sin 2\alpha).$$

We have

$$(dS/d\alpha) = 36(\cos\alpha + \cos 2\alpha) = 72\cos(3\alpha/2)\cos(\alpha/2).$$

For a maximum or a minimum of S,

$dS/d\alpha = 0$ *i.e.*, $\cos(3\alpha/2)\cos(\alpha/2) = 0$

i.e., $3\alpha/2 = \pi/2$ or $\alpha/2 = \pi/2$ *i.e.*, $\alpha = \pi/3$ or $\alpha = \pi$.

But $\alpha = \pi$ is inadmissible.

Now $d^2S/d\alpha^2 = 36(-\sin\alpha - 2\sin 2\alpha) =$ –ive when $\alpha = \pi/3$.

$\therefore$ S is maximum when $\alpha = \pi/3$. Also the maximum value of S is $36\left(\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right) = 27\sqrt{3}$ square centimetres.

Ex. 23. *A thin closed rectangular box is to have one rectangular edge n times the length of another edge and the volume of the box is given to be S. Prove that the least surface S is given by*

$$nS^3 = 54(n+1)^2 V^2.$$

Sol. Let the lengths of the edges be x, nx and y. Then $V =$ the volume of the rectangular box $= x \cdot nx \cdot y = nx^2y =$ const. ...(1)

Differentiating (1) w.r.t. x, we get

$$0 = nx^2(dy/dx) + 2nxy \text{ or } (dy/dx) = -2y/x. \qquad ...(2)$$

Now $S =$ the surface of the box $= 2nx^2 + 2(1+n)xy$. ...(3)

Here S is a function of x and y. But y is connected with x by (1). So we can regard S as a function of x only. For a maximum or a minimum of S, we have $dS/dx = 0$.

Now $\dfrac{dS}{dx} = 4nx + 2(1+n)\left(x\dfrac{dy}{dx} + y\right) = 4nx - 2(1+n)y$, from (2).

$\therefore$ $dS/dx = 0$ gives $2nx = (1+n)y$.

Also $\dfrac{d^2S}{dx^2} = 4n - 2(1+n)\dfrac{dy}{dx} = 4n + \{4(1+n)y\}/x$, from (2)

$= +$ ive.

Therefore S is least when $2nx = (1+n)y$. For this least S, on putting $y = 2nx/(1+n)$ in (1) and (3), we get

$$V = \frac{2n^2x^3}{1+n} \text{ and } S = 2nx^2 + 2(1+n)x \cdot \frac{2nx}{1+n} = 6nx^2.$$

$\therefore$ $S^3 = 6^3n^3x^6 = 6^3n^3[(1+n)^2V^2/(4n^4)]$.

Hence $nS^3 = 54(n+1)^2V^2$, when S is least.

Ex. 24. *Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is* $6r\sqrt{3}$.

Sol. Let θ be the semi-vertical angle of an isosceles triangle in which a circle of radius r can be inscribed.

Then $AQ = r\cot\theta$

and $OA = r\operatorname{cosec}\theta$.

$\therefore$ $AP = r + r\operatorname{cosec}\theta = r(1 + \operatorname{cosec}\theta)$, and $BP = AP\tan\theta$

$= r(1+\operatorname{cosec}\theta)\tan\theta = r(\tan\theta + \sec\theta)$.

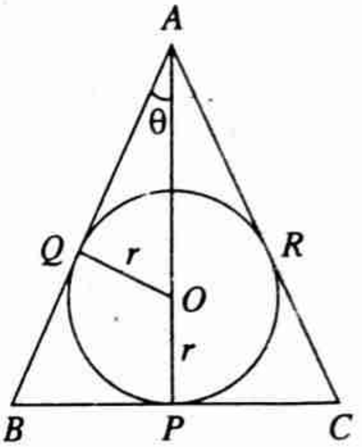
Now if s is the perimeter of the triangle ABC, then

$$s = 2AQ + 4PB \text{ [Note that } AQ = AR, BQ = BP = CP = CR]$$
$$= 2r\cot\theta + 4r(\tan\theta + \sec\theta) = 2r(\cot\theta + 2\tan\theta + 2\sec\theta).$$
$$\therefore\ ds/d\theta = 2r(-\operatorname{cosec}^2\theta + 2\sec^2\theta + 2\sec\theta\tan\theta).$$

For s to be a maximum or a minimum, we have $ds/d\theta = 0$

i.e., $\quad -\operatorname{cosec}^2\theta + 2\sec^2\theta + 2\sec\theta\tan\theta = 0$

i.e., $\quad -\dfrac{1}{\sin^2\theta} + \dfrac{2}{\cos^2\theta} + \dfrac{2\sin\theta}{\cos^2\theta} = 0$



or $\quad 2\sin^3\theta + 2\sin^2\theta - \cos^2\theta = 0$

or $\quad 2\sin^3\theta + 3\sin^2\theta - 1 = 0.$

Solving this equation, we get $\sin\theta = -1$ or $\sin\theta = \frac{1}{2}$. But $\sin\theta = -1$ is inadmissible. Therefore $\sin\theta = \frac{1}{2}$ or $\theta = 30°$.

Also

$$d^2s/d\theta^2$$
$$= 2r[2\operatorname{cosec}^2\theta\cot\theta + 4\sec^2\theta\tan\theta + 2\sec^3\theta + 2\sec\theta\tan^2\theta]$$
$$= +\text{ ive for } \theta = 30°.$$

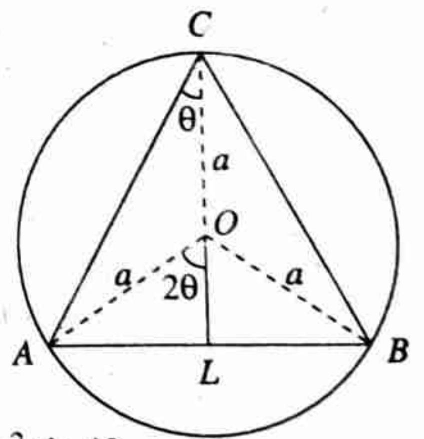
Hence $\theta = 30°$ gives least perimeter, which is

$$= 2r\left[\sqrt{3} + \frac{2}{\sqrt{3}} + 2\cdot\frac{2}{\sqrt{3}}\right] = \frac{18r}{\sqrt{3}} = 6r\sqrt{3}.$$

Ex. 25. *Show that the triangle of maximum area that can be inscribed in a circle of radius a is an equilateral triangle.*

(Meerut 1982 S)

Sol. First we observe that among all the inscribed triangles having AB as base, the area of that triangle is greatest for which the altitude of C w.r.t. AB is greatest. Evidently such a triangle is an isosceles triangle. Let θ be the semi-vertical angle of such a triangle ABC inscribed in a given circle of radius a.



From fig., $S =$ area of the triangle $ABC = 2\,\Delta AOC + \Delta AOB$

$$= 2\cdot\tfrac{1}{2}a^2\sin(\pi - 2\theta) + \tfrac{1}{2}a^2\sin 4\theta$$
$$= a^2\sin 2\theta + \tfrac{1}{2}a^2\sin 4\theta.$$
$$\therefore\quad \frac{dS}{d\theta} = 2a^2\cos 2\theta + 2a^2\cos 4\theta.$$

For a maximum or minimum of S, we have $dS/d\theta = 0$

i.e., $\quad \cos 2\theta + \cos 4\theta = 0,\quad$ *i.e.,* $2\cos 3\theta\cos\theta = 0$

i.e., $\quad 3\theta = \pi/2$ or $\theta = \pi/2,\quad$ *i.e.,* $\theta = \pi/6$ or $\theta = \pi/2.$

But $\theta = \pi/2$ is not admissible. Hence $\theta = \pi/6$.

Also, $d^2S/d\theta^2 = -4a^2 \sin 2\theta - 8a^2 \sin 4\theta = -\text{ive}$ for $\theta = \pi/6$.

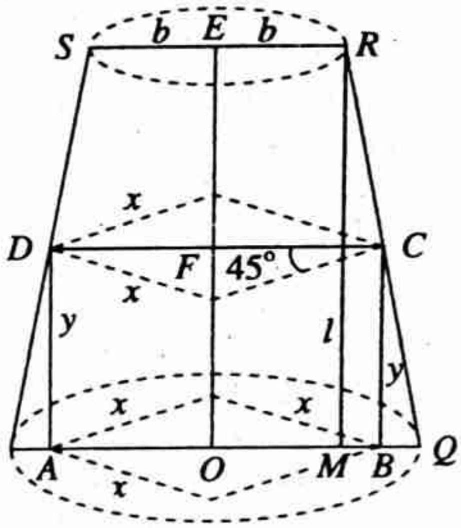
Hence area is maximum when $\theta = \pi/6$ or $2\theta = \pi/3$, *i.e.*, when the triangle is equilateral.

Ex. 26. *A tree trunk l feet long is in the shape of frustum of a cone, the radii of its ends being a and b feet $(a > b)$. It is required to cut from it a beam of uniform square section. Prove that the beam of the greatest volume that can be cut is $al/\{3(a-b)\}$ feet long.*

Sol. Let x be a side of the square base and y be the length BC of the beam to be cut. Then

$$OB = \tfrac{1}{2} \text{ (diagonal of the base)} = x/\sqrt{2}.$$

In the figure AB and CD are the parallel diagonals of the ends of the beam to be cut. Now from the figure we have



$$\frac{MQ}{MR} = \frac{BQ}{BC}$$

or $$\frac{OQ - OM}{l} = \frac{OQ - OB}{y}$$

or $$\frac{a-b}{l} = \frac{a - (x/\sqrt{2})}{y}.$$

$$\therefore \quad y = \frac{l}{a-b}\left(a - \frac{x}{\sqrt{2}}\right). \quad \ldots(1)$$

Now, the volume V of the beam is given by

$$V = x^2 y = x^2 \cdot \frac{l}{a-b}\left(a - \frac{x}{\sqrt{2}}\right), \text{ from (1).}$$

For a maximum or a minimum of V, we have

$$\frac{dV}{dx} = \frac{l}{a-b}\left[x^2\left(-\frac{1}{\sqrt{2}}\right) + \left(a - \frac{x}{\sqrt{2}}\right)2x\right] = 0$$

or $\quad x(2a - \sqrt{2}x - x/\sqrt{2}) = 0$. Therefore $x = 0$ or $x = 2\sqrt{2}\,a/3$.

Obviously $x = 0$ is not admissible.

Also $$\frac{d^2V}{dx^2} = \frac{l}{a-b}[2a - 2\sqrt{2}x - \sqrt{2}x] = \frac{l}{a-b}(2a - 3\sqrt{2}x)$$

$$= -\text{ ive at } x = \frac{2\sqrt{2}}{3}a.$$

$\therefore$ V is maximum when $x = (2\sqrt{2}\,a)/3$. Then from (1),

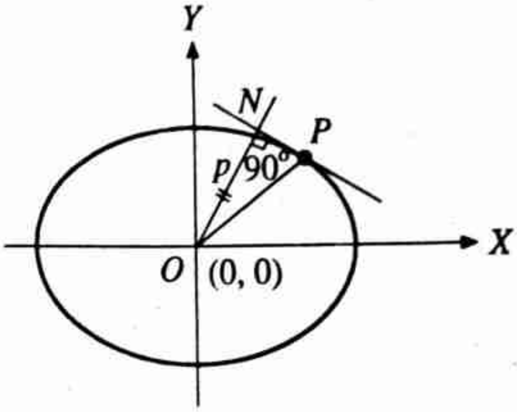
$$y = \frac{l}{a-b}\left[a - \frac{2a}{3}\right] = \frac{al}{3(a-b)} \text{ feet.}$$

Ex. 27. *N is the foot of the perpendicular drawn from the centre O onto the tangent at a variable point P on the ellipse*

$$x^2/a^2 + y^2/b^2 = 1, (a > b).$$

Find the maximum length of PN and also the maximum area of the triangle OPN.

Sol. Let P be the point $(a \cos t, b \sin t)$. Then the equation of the tangent at P to the given ellipse is



$$\frac{x}{a} \cos t + \frac{y}{b} \sin t = 1$$

or $\qquad bx \cos t + ay \sin t = ab.$

Let $ON = p$.

Then

$$p = \frac{ab}{\sqrt{(b^2 \cos^2 t + a^2 \sin^2 t)}}.$$

$$\therefore \quad \frac{a^2b^2}{p^2} = b^2 \cos^2 t + a^2 \sin^2 t = b^2 (1 - \sin^2 t) + a^2 (1 - \cos^2 t)$$

$$= a^2 + b^2 - (a^2 \cos^2 t + b^2 \sin^2 t) = a^2 + b^2 - OP^2,$$

because $OP^2 = a^2 \cos^2 t + b^2 \sin^2 t$.

$\therefore \quad OP^2 = a^2 + b^2 - (a^2b^2/p^2)$.

We have

$$PN^2 = OP^2 - ON^2 = a^2 + b^2 - (a^2b^2/p^2) - p^2 = z, \text{ say...(1)}$$

Now PN is maximum or minimum according as PN^2 is maximum or minimum. But PN^2 *i.e.*, z is a function of p. For a maximum or a minimum of z, we have

$$\frac{dz}{dp} \equiv \frac{2a^2b^2}{p^3} - 2p = 0 \text{ i.e., } p^4 = a^2b^2 \text{ i.e., } p^2 = ab.$$

Also $\dfrac{d^2z}{dp^2} = -\dfrac{6a^2b^2}{p^4} - 2 =$ –ive, when $p^2 = ab$. Therefore z is maximum when $p^2 = ab$. Putting $p^2 = ab$ in (1), we get

$$PN^2 = a^2 + b^2 - \frac{a^2b^2}{ab} - ab = (a - b)^2; \quad \therefore \quad PN = (a - b).$$

Hence the maximum length of $PN = a - b$.

Now let S be the area of the triangle OPN.

Then $S = \frac{1}{2} ON . PN = \frac{1}{2} p . PN$. Therefore

$$S^2 = \tfrac{1}{4} p^2 . PN^2 = \tfrac{1}{4} p^2 \left(a^2 + b^2 - \frac{a^2b^2}{p^2} - p^2 \right), \qquad \text{from (1).}$$

Let $u = S^2 = \frac{1}{4} \{ p^2 (a^2 + b^2) - a^2b^2 - p^4 \}$.

Now S is maximum or minimum according as S^2, *i.e.*, u is maximum or minimum. For a maximum or minimum of u, we have

$$\frac{du}{dp} \equiv \frac{1}{4}\{2p(a^2+b^2)-4p^3\}=0, \text{ i.e., } p^2=\frac{a^2+b^2}{2}, \text{ since } p \neq 0.$$

Also $\dfrac{d^2u}{dp^2}=\dfrac{1}{4}\{2(a^2+b^2)-12p^2\}=$ –ive, when $p^2=\dfrac{a^2+b^2}{2}$.

Hence u is maximum when $p^2=(a^2+b^2)/2$. Putting $p^2=(a^2+b^2)/2$ in the value of S^2, we get

$$S^2=\tfrac{1}{4}\{\tfrac{1}{2}(a^2+b^2)^2-a^2b^2-\tfrac{1}{4}(a^2+b^2)^2\}$$
$$=\tfrac{1}{16}\{(a^2+b^2)^2-4a^2b^2\}=\tfrac{1}{16}(a^2-b^2)^2.$$

Hence the maximum area of the triangle $OPN=\frac{1}{4}(a^2-b^2)$.

Ex. 28. *An open rectangular tank, with a square base and vertical sides, is to be constructed of sheet metal to hold a given quantity of water. Show that the cost of the material will be least when the depth is half the width.* **(Meerut 1986 P)**

Sol. Let x be the width of the tank and y be its depth. If V be the given capacity of the tank, then

$$V=x^2y \quad \text{or} \quad y=V/x^2 \qquad \text{...(1)}$$

Now if S be the area of the sheet required to construct the tank, then $S=$ area of the base + area of the four vertical walls

$$=x^2+4xy=x^2+4x(V/x^2), \qquad \text{from (1)}$$
$$=x^2+4V/x.$$

For a maximum or minimum of S, we have

$$\frac{dS}{dx}\equiv 2x-\frac{4V}{x^2}=0 \text{ i.e., } 2V=x^3.$$

Also $\dfrac{d^2S}{dx^2}=2+\dfrac{8V}{x^3}=$ +ive for $x^3=2V$. Hence S is least when

$2V=x^3$ or when $2x^2y=x^3$, $\qquad [\because V=x^2y$, from (1)]

or when $y=x^3/(2x^2)=x/2$ or when depth $=\frac{1}{2}$ width.

Ex. 29. *Investigate the maxima and minima of $ax+by$ when $xy=c^2$.* **(Meerut 1980; Kanpur 85)**

Sol. Let $\qquad z=ax+by \qquad$ ...(1)

Given $xy=c^2$ or $y=c^2/x$. Substituting this value of y in (1), we get $z=ax+bc^2/x$.

Now for Max. or Min. values of z, we must have $dz/dx=0$.

$\therefore \quad dz/dx=a-bc^2/x^2=0$

or $\quad a=bc^2/x^2$ or $x^2=bc^2/a$ or $x=\pm c\sqrt{(b/a)}$.

Again, $d^2z/dx^2=2bc^2/x^3=$ +ive for $x=c\sqrt{(b/a)}$

and $\quad d^2z/dx^2=$ –ive for $x=-c\sqrt{(b/a)}$.

Hence z is maximum when $x=-c\sqrt{(b/a)}$ and minimum when $x=c\sqrt{(b/a)}$.

$\therefore$ maximum value of $z = -ac\sqrt{(b/a)} - bc^2/\{c\sqrt{(b/a)}\}$
$= -c\sqrt{(ab)} - c\sqrt{(ab)} = -2c\sqrt{(ab)}$.

And minimum value of $z = ac\sqrt{(b/a)} + bc^2/\{c\sqrt{(b/a)}\}$
$= c\sqrt{(ab)} + c\sqrt{(ab)} = 2c\sqrt{(ab)}$.

Ex. 30. *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is* $2a/\sqrt{3}$.

(Meerut 1980, 85, 88P; Kanpur 80)

Sol. Let a be the radius of the sphere and O be its centre. Then $OA = a$ where A is the one of the lower point of contact of the cylinder and the sphere. Also let MN and AN represent the height and radius of the cylinder (draw the figure).

Let $MN = x$ then $ON = x/2$;

$\therefore$ $AN^2 = OA^2 - ON^2 = a^2 - x^2/4$.

Thus $V =$ volume of the cylinder
$= \pi AN^2 . MN = \pi (a^2 - x^2/4) x = \pi (a^2 x - x^3/4)$.

Differentiating, we get $dV/dx = \pi (a^2 - 3x^2/4)$.

For max. or min. of V, $dV/dx = 0$ or $(a^2 - 3x^2/4) = 0$

or $3x^2 = 4a^2$ or $x = 2a/\sqrt{3}$.

Now $d^2V/dx^2 = -3\pi x/2 =$ –ive when $x = 2a/\sqrt{3}$. Hence volume of the cylinder is max. when height of the cylinder is $2a/\sqrt{3}$.

Ex. 31. *Assuming that the petrol burnt in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c kilometres per hour is* $(3/2)\,c$ *kilometres per hour.* **(Allahabad 1979; Meerut S-80; Agra 79)**

Sol. Let the speed of the boat be v kilometres per hour and the distance travelled in 't' hours be a kilometres. When the boat is going against the current of c kilometres per hour, its resultant speed is $(v - c)$ kilometres per hour.

$\therefore$ $t =$ distance/speed $= a/(v - c)$ hours.

Now the petrol burnt per hour is kv^3, where k is a constant. If x is the petrol burnt in t hours, then

$$x = \{a/(v - c)\}\,kv^3 = a\,kv^3/(v - c).$$

For the most economical speed, x should be minimum.

Now $dx/dv = ak\,\{3v^2(v - c) - v^3\}/(v - c)^2$
$= ak\,(2v^3 - 3v^2 c)/(v - c)^2$
$= ak\,.v^2(2v - 3c) \cdot \dfrac{1}{(v - c)^2}$.

For max. or min. value of x, $dx/dv = 0$

or $v^2(2v - 3c) = 0$

or $2v - 3c = 0$, $[\because v \neq 0]$

or $v = 3c/2$.

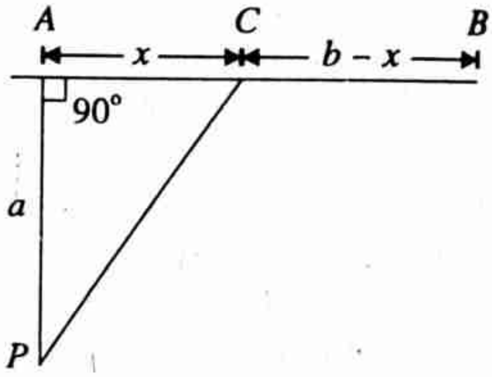
Also $\dfrac{d^2x}{dv^2} = ak\left[-\dfrac{2}{(v-c)^3}\cdot v^2(2v-3c) + \dfrac{1}{(v-c)^2}\cdot 6v(v-c)\right]\cdot$

When $v = \dfrac{3c}{2}$, $\dfrac{d^2x}{dv^2} = ak\cdot\dfrac{b\,.(3c/2)}{(3c/2)-c} = 18ak$ *i.e.*, +ive.

Hence x is minimum when $v = 3c/2$ *i.e.*, the most economical speed is $c/2$ kilometres per hour.

Ex. 32. *A person being in a boat a kilometres from the nearest point of the beach, wishes to reach as quickly as possible a point b kms. from that point along the shore. The ratio of his rate of walking to his rate of rowing is* $\sec\alpha$. *Prove that he should land at a distance* $b - a\cot\alpha$ *from the place to be reached.* **(Allahabad 1980)**

Sol. Let the initial position of the boat be P and the point of the beach nearest to P be A. Then $PA = a$. Suppose the point to be reached is B. Then $AB = b$. Let the rate of rowing be u kms. per hour. Then the rate of walking $= u\sec\alpha$ kms. per hour.



Suppose the person lands at C, where $AC = x$, and then walks to B. Thus he covers the distance PC by boat and the distance CB by walking. If the total time taken to reach B is t hours, then

$$t = \frac{PC}{u} + \frac{CB}{u\sec\alpha} = \frac{\sqrt{(a^2+x^2)}}{u} + \frac{(b-x)}{u\sec\alpha}.$$

We have $\dfrac{dt}{dx} = \dfrac{x}{u\sqrt{(a^2+x^2)}} - \dfrac{1}{u\sec\alpha}$.

For a maximum or minimum of t, we have $dt/dx = 0$

i.e., $\dfrac{x}{\sqrt{(a^2+x^2)}} - \dfrac{1}{\sec\alpha} = 0$ *i.e.*, $\dfrac{x^2}{a^2+x^2} = \cos^2\alpha$

i.e., $x^2 = a^2\cos^2\alpha + x^2\cos^2\alpha$ *i.e.*, $x^2\sin^2\alpha = a^2\cos^2\alpha$

i.e., $x = a\cot\alpha$.

Now $\dfrac{d^2t}{dx^2} = \dfrac{1}{u\sqrt{(a^2+x^2)}} - \dfrac{x^2}{u(a^2+x^2)^{3/2}}$.

When $x = a\cot\alpha$,

$$\frac{d^2t}{dx^2} = \frac{1}{ua\operatorname{cosec}\alpha} - \frac{a^2\cot^2\alpha}{ua^3\operatorname{cosec}^3\alpha}$$

$$= \frac{1}{ua}[\sin\alpha - \cos^2\alpha\sin\alpha] = \frac{1}{ua}\sin^3\alpha = \text{+ive}.$$

Therefore t is minimum when $x = a\cot\alpha$.

Thus to reach B in the shortest possible time, the person should land at C where $AC = a \cot \alpha$ or $BC = b - a \cot \alpha$.

Ex. 33. *The velocity of waves of wave-length λ on deep water is proportional to* $\sqrt{\left(\dfrac{\lambda}{a} + \dfrac{a}{\lambda}\right)}$, *where a is a certain linear magnitude, prove that the velocity is a minimum when $\lambda = a$.*

Sol. Let $v = \mu \sqrt{(\lambda/a + a/\lambda)}$, where μ is some constant.

Then $v^2 = \mu^2 (\lambda/a + a/\lambda) = z$, say.

Now v is maximum or minimum according as v^2 *i.e.*, z is maximum or minimum.

Now $dz/d\lambda = \mu^2 (1/a - a/\lambda^2)$. For a maximum or a minimum of z, we have $dz/d\lambda = 0$ *i.e.*, $1/a - a/\lambda^2 = 0$ *i.e.*, $\lambda^2 = a^2$ *i.e.*, $\lambda = a$, because $\lambda = -a$ is inadmissible.

Now $d^2z/d\lambda^2 = \mu^2 (2a/\lambda^3) =$ +ive when $\lambda = a$. Hence z or v is minimum when $\lambda = a$.

Ex. 34. *Prove that the minimum radius vector of the curve $a^2/x^2 + b^2/y^2 = 1$ is of length $a + b$.* **(Lucknow 1982, 80; Meerut 82)**

Sol. Let r be the radius vector of the curve. Then changing the equation of the curve to polar coordinates by putting $x = r \cos \theta$ and $y = r \sin \theta$, we get

$$a^2/(r^2 \cos^2 \theta) + b^2/(r^2 \sin^2 \theta) = 1$$

or $$r^2 = a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta = z, \text{ say.}$$

Now $|r|$ is max. or min. according as r^2 *i.e.*, z is maximum or minimum. We have $dz/d\theta = 2a^2 \sec^2 \theta \tan \theta - 2b^2 \operatorname{cosec}^2 \theta \cot \theta$.

For a maximum or a minimum of z, we have $dz/d\theta = 0$

i.e., $$2a^2 \sec^2 \theta \tan \theta - 2b^2 \operatorname{cosec}^2 \theta \cot \theta = 0$$

i.e., $$(\sec^2 \theta \tan \theta)/(\operatorname{cosec}^2 \theta \cot \theta) = b^2/a^2$$

i.e., $$\tan^4 \theta = b^2/a^2$$

i.e., $\tan^2 \theta = b/a$ because $\tan^2 \theta$ cannot be –ive.

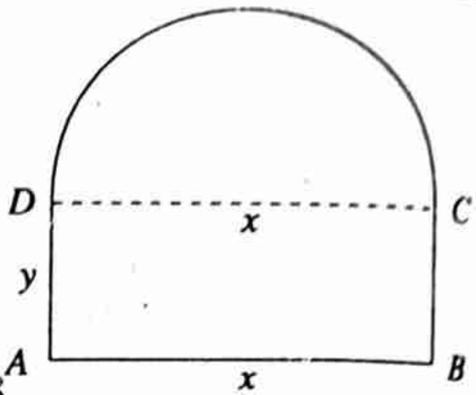
Now $d^2z/d\theta^2 = 4a^2 \sec^2 \theta \tan^2 \theta + 2a^2 \sec^4 \theta + 4b^2 \operatorname{cosec}^2 \theta \cot^2 \theta$
$+ 2b^2 \operatorname{cosec}^4 \theta =$ +ive when $\tan^2 \theta = b/a$.

Therefore z or r is minimum when $\tan^2 \theta = b/a$.

Now $r = \sqrt{(a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta)}$
$= \sqrt{\{a^2 (1 + \tan^2 \theta) + b^2 (1 + \cot^2 \theta)\}}$.

Therefore the minimum value of $r = \sqrt{\{a^2 (1 + b/a) + b^2 (1 + a/b)\}}$
$= \sqrt{(a^2 + 2ab + b^2)} = \sqrt{(a + b)^2} = a + b$.

Ex. 35. *A figure consists of a semi-circle with a rectangle on its diameter. Given that the perimeter of the figure is 20 feet, find its dimensions in order that its area may be maximum.*



Sol. Let x be the breadth and y be the height of the rectangle. Then the diameter of the semi-circle is x. Therefore the perimeter of the figure

$= x + 2y + \pi x/2 = 20$. ...(1)

Let A be the area of the figure.

Then $A = xy + \frac{1}{2}\pi (x/2)^2$

$= x(10 - x/2 - \pi x/4) + \pi x^2/8$,

[$\because$ from (1), $y = 10 - x/2 - \pi x/4$]

$= 10x - x^2/2 - \pi x^2/8$.

For a maximum or a minimum of A, we have

$$dA/dx = 10 - x - \pi x/4 = 0$$

i.e., $$x(1 + \pi/4) = 10$$

i.e., $$x = 40/(\pi + 4).$$

Now $d^2A/dx^2 = -1 - \frac{1}{4}\pi$, which is negative when $x = 40/(\pi + 4)$.

Hence A is maximum when $x = 40/(\pi + 4)$.

When $x = 40/(\pi + 4)$, we get from (1),

$$y = 10 - 20/(\pi + 4) - 10\pi/(\pi + 4) = 20/(\pi + 4).$$

Hence the area of the figure is maximum when the radius of the semi-circle = the height of the rectangle = $20/(\pi + 4)$ feet.

****Ex. 36.** *Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.* **(Meerut 1983, 89P; Jhansi 88; Agra 82; Bundelkhand 78; Gorakhpur 77, 75)**

Sol. Let r be the radius and h be the height of the conical tent. Since the capacity of the tent is given, therefore its volume

$$V = \tfrac{1}{3}\pi r^2 h = \text{constant}. \quad ...(1)$$

The amount of canvas required for the tent = the curved surface of the cone = S, say.

We have, $S = \pi r \sqrt{(r^2 + h^2)}$. Therefore $S^2 = \pi^2 r^2 (r^2 + h^2)$

or $$S^2 = \pi^2 r^2 \{r^2 + 9V^2/(\pi^2 r^4)\}, \qquad [\because \text{from (1)}, h = 3V/(\pi r^2)]$$

$$= \pi^2 r^4 + 9V^2/r^2 = z, \text{ say}.$$

Now S is maximum or minimum according as S^2 *i.e.,* z is maximum or minimum. We have $dz/dr = 4\pi^2 r^3 - 18V^2/r^3$. For a maximum or minimum of z, we have $dz/dr = 0$,

i.e., $4\pi^2 r^3 - 18V^2/r^3 = 0$ *i.e.,* $r^6 = 9V^2/2\pi^2$.

Now $d^2z/dr^2 = 12\pi^2 r^2 + 54V^2/r^4$ which is +ive when $r^6 = 9V^2/2\pi^2$.

Hence z or S is minimum when $r^6 = 9V^2/2\pi^2$ or $r^6 = \pi^2 r^4 h^2/2\pi^2$,

$[\because$ from (1), $3V = \pi r^2 h]$

or $\qquad h^2 = 2r^2 \quad$ or $\quad h = r\sqrt{2}$.

Ex. 37 (a). *A gas-holder is a cylindrical vessel closed at the top and open at the bottom (which dips into water). What should be the ratio of the height to the diameter in order that for a given volume its construction may require the least amount of material ?*

Sol. Let h be the height and $2r$ be the diameter of the cylindrical vessel.

Since the capacity of the vessel is given, therefore its volume

$$V = \pi r^2 h = \text{constant}. \qquad ...(1)$$

The amount of material required for the construction of the vessel is least when its whole surface, say S, is least. We have

$S =$ area of the top + area of the curved surface

$= \pi r^2 + 2\pi rh = \pi r^2 + 2\pi r . V/(\pi r^2)$, from (1)

$= \pi r^2 + 2V/r$.

For a maximum or minimum of S, we must have

$$dS/dr = 2\pi r - 2V/r^2 = 0 \text{ i.e., } \pi r^3 = V.$$

Now $\qquad d^2S/dr^2 = 2\pi + 4V/r^3 =$ +ive when $\pi r^3 = V$.

Hence S is minimum when $\pi r^3 = V$

or $\qquad \pi r^2 h = \pi r^3, \qquad [\because$ from (1), $V = \pi r^2 h]$

or $\qquad h = r \quad$ or $\quad h/2r = \frac{1}{2}$.

Ex. 37 (b). *A can in the form of a closed right circular cylinder is to be made of sheet metal to have capacity V. Find the height of the can and the diameter of its base so that the metal used may be minimum.*

(Meerut 1988 S)

Sol. Let h be the height and $2r$ be the diameter of the cylindrical can. Since the capacity of the can is given, therefore its volume

$$V = \pi r^2 h = \text{constant}. \qquad ...(1)$$

The amount of metal used for the construction of the can is least when the whole surface, say S, is least. We have

$S =$ area of the two circular faces + area of the curved surface

$= 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r .(V/\pi r^2)$, from (1)

$= 2\pi r^2 + (2V/r)$.

For a maximum or minimum of S, we must have

$$dS/dr = (-2V/r^2) + 4\pi r = 0 \text{ i.e., } 2\pi r^3 = V$$

i.e., $\qquad r = (V/2\pi)^{1/3}$.

Now $d^2S/dr^2 = (4V/r^3) + 4\pi$ which is > 0 when $2\pi r^3 = V$.

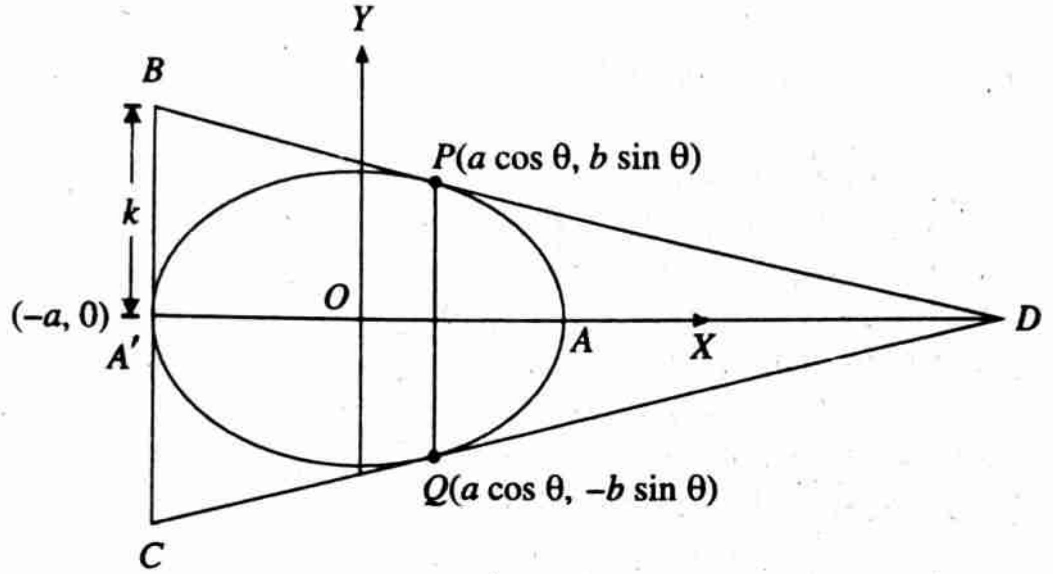
Hence S is minimum when $2\pi r^3 = V$.

When $2\pi r^3 = V$, we have from (1) $2\pi r^3 = \pi r^2 h$ or $h = 2r$.

$\therefore$ the required height of the can $= 2\,(V/2\pi)^{1/3} =$ the diameter of the base of the can.

Ex. 38. *An ellipse is inscribed in an isosceles triangle of height h and base $2k$, and having one axis lying along the perpendicular from the vertex of the triangle to the base. Show that the maximum area of the ellipse is $\sqrt{3}\,.\pi hk/9$.*

Sol. Let BCD be the given isosceles triangle whose base is BC, vertex D and height DA'. Then $BC = 2k$ and $DA' = h$.



An ellipse is inscribed in this triangle so that one axis of the ellipse is along $A'A$. Let $A'A = 2a$ and let the length of the other axis of the ellipse be $2b$. Let the equation of the ellipse be

$$x^2/a^2 + y^2/b^2 = 1,$$

where the x-axis is along OA and the y-axis is perpendicular to OA.

Here O is the middle point of $A'A$.

The sides BD, CD and BC of the triangle touch the ellipse at the points P, Q and A' respectively.

Here PQ is a double ordinate.

Let P be the point $(a \cos\theta, b \sin\theta)$.

Then the equation of the tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$ is

$$(x/a)\cos\theta + (y/b)\sin\theta = 1. \qquad \text{...(1)}$$

The coordinates of the point B are $(-a, k)$.

Since the point B lies on (1),

therefore $-\dfrac{a}{a}\cos\theta + \dfrac{k}{b}\sin\theta = 1$ or $\dfrac{k}{b}\sin\theta = 1 + \cos\theta$

or $$b = k\sin\theta/(1 + \cos\theta). \qquad \text{...(2)}$$

The coordinates of the point D are $(h - a, 0)$.

[Note that $DA' = h$ and $A'O = a$]

Since the point D also lies on (1),

therefore $\{(h-a)/a\} \cos\theta = 1$ or $h\cos\theta - a\cos\theta = a$

or $$a = h\cos\theta/(1+\cos\theta) \qquad ...(3)$$

Let S be the area of the ellipse.

Then $S = \pi ab = \dfrac{\pi hk \sin\theta\cos\theta}{(1+\cos\theta)^2}$, from (2) and (3).

We have $dS/d\theta$

$$= \pi hk \frac{(\cos^2\theta - \sin^2\theta)(1+\cos\theta)^2 + 2\sin^2\theta\cos\theta(1+\cos\theta)}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{(2\cos^2\theta - 1)(1+\cos\theta)^2 + 2(1-\cos^2\theta)\cos\theta(1+\cos\theta)}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{(1+\cos\theta)^2\{(2\cos^2\theta - 1) + 2\cos\theta(1-\cos\theta)\}}{(1+\cos\theta)^4}$$

$$= \pi hk \frac{2\cos\theta - 1}{(1+\cos\theta)^2}, \text{ since } \cos\theta \neq -1.$$

For a maximum or minimum of S, we have

$$dS/d\theta = 0 \text{ i.e., } 2\cos\theta - 1 = 0 \text{ i.e., } \cos\theta = \tfrac{1}{2} \text{ i.e., } \theta = \pi/3.$$

Now $$\frac{d^2S}{d\theta^2} = \pi hk(2\cos\theta - 1)\frac{d}{d\theta}\left\{\frac{1}{(1+\cos\theta)^2}\right\} + \frac{\pi hk}{(1+\cos\theta)^2}(-2\sin\theta),$$

which is negative when $\cos\theta = \frac{1}{2}$ or $\theta = \frac{1}{3}\pi$.

Hence S is maximum when $\theta = \frac{1}{3}\pi$.

The maximum area of the ellipse

$$= \pi hk \sin(\pi/3)\cos(\pi/3)/\{1+\cos(\pi/3)\}^2$$
$$= \pi hk(\sqrt{3}/4)/(1+\tfrac{1}{2})^2 = \sqrt{3}\,.\,\pi hk/9.$$

Ex. 39. *In a submarine telegraph cable the speed of signalling varies as $x^2 \log(1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1 : \sqrt{e}$.*

Sol. Let S be the speed of signalling.

Then $S = \mu x^2 \log(1/x) = -\mu x^2 \log x$, where μ is a constant.

We have $dS/dx = -\mu\{2x\log x + x^2(1/x)\} = -\mu x(2\log x + 1)$.

For a max. or a min. of S, we have $dS/dx = 0$

i.e., $\quad x(2\log x + 1) = 0$ *i.e.,* $x = 0$ or $\log x = -\frac{1}{2}$.

But $x = 0$ is inadmissible.

Therefore $\log x = -\frac{1}{2}$ or $x = e^{-1/2} = 1/\sqrt{e}$.

Now $d^2S/dx^2 = -\mu(2\log x + 1) - \mu x.(2/x)$
$= -\mu(2\log x + 1) = -2\mu.$

When $x = 1/\sqrt{e}$, we have $2\log x + 1 = 0$.

$\therefore$ when $x = 1/\sqrt{e}$, we have $d^2S/dx^2 = -2\mu$ which is –ive.

Hence S is maximum when $x = 1/\sqrt{e}$.

Ex. 40. *Show that if $x = a$ is an approximate position for a maximum or minimum of $f(x)$, then*

$$f(a) - \{f'(a)\}^2/f''(a)$$

is, in general, a better approximation than $f(a)$ to the maximum or minimum value.

Sol. We have $f(x) = f[a + (x - a)]$

$$= f(a) + (x-a)f'(a) + \{(x-a)^2/2!\}f''(a) + \dots \quad \text{...(1)}$$

[By Taylor's Theorem]

Differentiating (1) w.r.t. x, we get

$$f'(x) = f'(a) + (x-a)f''(a) + \dots \quad \text{...(2)}$$

For a maximum or a minimum of $f(x)$, we have $f'(x) = 0$.

Retaining only the first two terms on the right side of (2), we see that $f'(x) = 0$ is the same equation approximately as

$$f'(a) + (x-a)f''(a) = 0.$$

This gives $x - a = -f'(a)/f''(a)$.

Substituting this value of $x - a$ in (1), we see that

$f(a) - \{f'(a)\}^2/f''(a)$ is a better approximation than $f(a)$ for the maximum or minimum value of $f(x)$.

□

$[\because z = 2x + 3y - 12$, from the equation of the plane$]$.

$\therefore \quad l^2 = x^2 + y^2 + (2x + 3y - 12)^2$
$= 5x^2 + 10y^2 + 12xy - 48x - 72y + 144$
$= u$, say.

Now l is maximum or minimum according as l^2 *i.e.*, u is maximum or minimum.

For a maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} \equiv 10x + 12y - 48 = 0.$$

and
$$\frac{\partial u}{\partial y} \equiv 20y + 12x - 72 = 0.$$

Solving these equations, we get

$x = 12/7$, and $y = 18/7$.

Also $r = \dfrac{\partial^2 u}{\partial x^2} = 10$, $s = \dfrac{\partial^2 u}{\partial x\, \partial y} = 12$, and $t = \dfrac{\partial^2 u}{\partial y^2} = 20$.

$\therefore \quad rt - s^2 = 10 \times 20 - 12^2 = +$ive. Since $rt - s^2 > 0$ and $r > 0$, therefore u is minimum and hence l is minimum when $x = 12/7$, and $y = 18/7$. Putting these values of x and y in the equation of the plane, we get $z = 2\,.(12/7) + 3\,.\,(18/7) - 12 = -6/7$.

Therefore the required point is $(12/7, 18/7, -6/7)$.

Ex. 18. *Locate the stationary points of* $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ *and determine their nature.* **(Gorakhpur 1981; Meerut 84R, 89P, 96)**

Sol. Let $\quad u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Then $\dfrac{\partial u}{\partial x} = 4x^3 - 4x + 4y$ and $\dfrac{\partial u}{\partial y} = 4y^3 + 4x - 4y$.

The stationary points are given by

$$\frac{\partial u}{\partial x} = 0 \text{ i.e., } 4x^3 - 4x + 4y = 0, \qquad ...(1)$$

and
$$\frac{\partial u}{\partial y} = 0 \text{ i.e., } 4y^3 + 4x - 4y = 0. \qquad ...(2)$$

Now we shall find the points (x, y) satisfying the simultaneous equations (1) and (2).

Adding (1) and (2), we get

$4x^3 + 4y^3 = 0$ *i.e.*, $x^3 + y^3 = 0$ *i.e.*, $(x + y)(x^2 - xy + y^2) = 0$.

$\therefore \quad$ either $x + y = 0$, $\qquad ...(3)$

or $\quad x^2 - xy + y^2 = 0$. $\qquad ...(4)$

First we solve the simultaneous equations (1) and (3).

From (3), we have $y = -x$.

Putting $y = -x$ in (1), we get

$4x^3 - 8x = 0$ *i.e.*, $x^3 - 2x = 0$ *i.e.*, $x(x^2 - 2) = 0$

i.e., $x = 0, \sqrt{2}$ or $-\sqrt{2}$.

The corresponding values of y are $y = 0, -\sqrt{2}, \sqrt{2}$.

Thus the points $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ satisfy (1) and (2).

If we solve the equations (1) and (4), we get $(0, 0)$ as the only real solution.

Hence the function u is stationary at the points

$$(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2}).$$

We have $r = \dfrac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $s = \dfrac{\partial^2 u}{\partial x \, \partial y} = 4$,

$$t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4.$$

At (0, 0), $r = -4$, $s = 4$, $t = -4$, so that

$$rt - s^2 = 16 - 16 = 0.$$

Thus at the point $(0, 0)$, the case is doubtful and further investigation is needed.

At $(\sqrt{2}, -\sqrt{2})$, $r = 20, s = 4, t = 20$, so that

$$rt - s^2 = 400 - 16 = + \text{ive}.$$

Therefore u has an extreme value at this point.

Since r is positive, therefore u has a minimum at this point.

At $(-\sqrt{2}, \sqrt{2})$, $r = 20$, $s = 4$, $t = 20$, so that $rt - s^2$ is positive. Since r is positive, therefore u has a minimum at this point also.

Important Note. We may tackle the doubtful case at the point $(0, 0)$ by the following consideration :

We have $u = x^4 + y^4 - 2(x - y)^2$.

At the point $(0, 0)$, we have $u = 0$.

At the points in the neighbourhood of the point $(0, 0)$ where $x \neq y$, the value of u is approximately given by

$$u = -2(x - y)^2,$$

[Neglecting the terms $x^4 + y^4$ because the numerical values of x and y are small].

Thus at such points u is –ive.

Again at the points in the neighbourhood of the point $(0, 0)$, where $x = y$, we have $u = x^4 + y^4$ which is positive.

Thus in the neighbourhood of the point $(0, 0)$, there are points at which u takes values less than its value at the point $(0, 0)$ and there are points at which u takes values greater than its value at the point $(0, 0)$. Hence u cannot have a maximum or a minimum value at the point $(0, 0)$.

Ex. 19. *Investigate the maxima and minima of*

$$2(x-y)^2 - x^4 - y^4,$$

leaving aside any doubtful case that may arise. **(Meerut 69; Roorkee 61)**

Sol. Proceed exactly as in Ex. 18. In fact the function $f(x,y)$ given in this problem is the function given in Ex. 18 with sign changed.

Ans. $f(x,y)$ is maximum at $x=\sqrt{2}, y=-\sqrt{2}$ and is minimum at $x=-\sqrt{2}, y=\sqrt{2}$.

Ex. 20. *Find the minimum value of* $x^2+y^2+z^2$ *when*

$$ax+by+cz=p.$$

(Gorakhpur 1978, 80; Meerut 83, 89 S, 90 P, 91 P, 98)

Sol. Let $u=x^2+y^2+z^2$.

Here u is a function of three variables x, y and z. But we can eliminate one variable with the help of the given relation, viz.,

$$ax+by+cz=p.$$

From this relation, we have $z=\dfrac{p-ax-by}{c}$.

Putting this value of z in the value of u, we get

$$u=x^2+y^2+\frac{(p-ax-by)^2}{c^2},$$

where u has been expressed as a function of two independent variables x and y.

We have $\dfrac{\partial u}{\partial x}=2x-\dfrac{2a}{c^2}(p-ax-by)$,

and $\dfrac{\partial u}{\partial y}=2y-\dfrac{2b}{c^2}(p-ax-by)$.

Solving $\dfrac{\partial u}{\partial x}=0$ and $\dfrac{\partial u}{\partial y}=0$, we get

$$x=\frac{ap}{a^2+b^2+c^2}, \text{ and } y=\frac{bp}{a^2+b^2+c^2}.$$

Again, we get $r=\dfrac{\partial^2 u}{\partial x^2}=2+\dfrac{2a^2}{c^2}$, $s=\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{2ab}{c^2}$,

and $t=\dfrac{\partial^2 u}{\partial y^2}=2+\dfrac{2b^2}{c^2}$.

$$\therefore \quad rt-s^2=4\left(1+\frac{a^2}{c^2}\right)\left(1+\frac{b^2}{c^2}\right)-\frac{4a^2b^2}{c^4}=4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right).$$

Since $rt-s^2$ is positive and r, is also positive, therefore u, is minimum for the values of x and y found above.

The minimum value of u, therefore, is $\dfrac{p^2}{(a^2+b^2+c^2)}$.

Ex. 21. *Discuss the maxima and minima of the function*

$$u = x^2y^2 - 5x^2 - 8xy - 5y^2.$$ **(Meerut 1993)**

Sol. The given function is $u = x^2y^2 - 5x^2 - 8xy - 5y^2$.

We have $\partial u/\partial x = 2xy^2 - 10x - 8y$

and $\partial u/\partial y = 2x^2y - 8x - 10y$.

Also $r = \dfrac{\partial^2 u}{\partial x^2} = 2y^2 - 10$, $s = \dfrac{\partial^2 u}{\partial x \partial y} = 4xy - 8$, $t = \dfrac{\partial^2 u}{\partial y^2} = 2x^2 - 10$.

For a maximum or a minimum of u, we must have

$\partial u/\partial x = 0 \Rightarrow \quad 2xy^2 - 10x - 8y = 0$ ...(1)

and $\partial u/\partial y = 0 \Rightarrow \quad 2x^2y - 8x - 10y = 0$ ...(2)

From (2), we have $y(x^2 - 5) = 4x$.

Putting $y = \dfrac{4x}{x^2 - 5}$ in (1), we get

$$x \cdot \frac{16x^2}{(x^2-5)^2} - 5x - \frac{16x}{x^2-5} = 0$$

or $\quad 16x^3 - 5x(x^2-5)^2 - 16x(x^2-5) = 0$

or $\quad x[16x^2 - 5x^4 + 50x^2 - 125 - 16x^2 + 80] = 0$

or $\quad x[-5x^4 + 50x^2 - 45] = 0$

or $\quad x[x^4 - 10x^2 + 9] = 0$

or $\quad x(x^2 - 1)(x^2 - 9) = 0$.

$\therefore \quad x = 0, \pm 1, \pm 3$.

From (2), when $x = 0$, we have $y = 0$,

when $x = 1, y = -1$, when $x = -1, y = 1$, when $x = 3$, $y = 3$ and when $x = -3, y = -3$.

$\therefore$ the function u is stationary at the points $(0, 0), (1, -1), (-1, 1), (3, 3)$ and $(-3, -3)$.

At $(0, 0)$, $r = -10$, $s = -8$, $t = -10$,

so that $\quad rt - s^2 = 100 - 64 = 36 > 0$.

$\therefore$ the stationary value of u at $(0, 0)$ is an extreme value.

Since $r < 0$, so u is maximum at $(0, 0)$.

At $(1, -1)$, $r = -8$, $s = -12$, $t = -8$ so that $rt - s^2 = 64 - 144 < 0$. Therefore the stationary value of u at $(1, -1)$ is neither maximum nor minimum.

Similarly we can show that the stationary value of u at each of the points $(-1, 1)$, $(3, 3)$ and $(-3, -3)$ is neither maximum nor minimum.

Hence the only point at which u has an extreme value is $(0, 0)$ and u is maximum at this point.

Ex. 22. *Find points on $z^2 = xy + 1$ nearest to the origin.*

(Meerut 1994, 95)

Sol. Let l be the distance from the origin $(0, 0, 0)$ of any point (x, y, z) on the surface

$$z^2 = xy + 1. \qquad ...(1)$$

Then $\quad l = \sqrt{(x^2 + y^2 + z^2)}$

$$= \sqrt{(x^2 + y^2 + xy + 1)}. \qquad [\because z^2 = xy + 1, \text{from}(1)]$$

$\therefore \quad l^2 = x^2 + y^2 + xy + 1 = u$, say.

Since l is always > 0, therefore l is maximum or minimum according as l^2 *i.e.*, u is maximum or minimum.

For a maximum or minimum of u, we must have

$$\frac{\partial u}{\partial x} \equiv 2x + y = 0 \qquad ...(2)$$

and
$$\frac{\partial u}{\partial y} \equiv 2y + x = 0. \qquad ...(3)$$

Solving the equations (2) and (3), we get $x = 0, y = 0$.

Also $r = \dfrac{\partial^2 u}{\partial x^2} = 2,\ s = \dfrac{\partial^2 u}{\partial x \partial y} = 1,\ t = \dfrac{\partial^2 u}{\partial y^2} = 2.$

$\therefore \quad rt - s^2 = 2.2 - 1 = 3$ which is > 0.

Since at $x = 0, y = 0, rt - s^2 > 0$ and $r > 0$, therefore u is minimum at $x = 0, y = 0$. Hence l is minimum when $x = 0, y = 0$. Putting $x = 0$, $y = 0$ in the equation (1), we get $z^2 = 1$ *i.e.*, $z = \pm 1$.

$\therefore$ The required points are $(0, 0, 1)$ and $(0, 0, -1)$.

□

3

Maxima and Minima

(Of Functions of Several Independent Variables)

§ 1. Definition.

Let $f(x, y, z, ...)$ be any function of several independent variables $x, y, z, ...$ supposed to be continuous for all values of these variables in the neighbourhood of their values $a, b, c, ...$ respectively. Then $f(a, b, c, ...)$ is said to be a **maximum** or a **minimum** value of $f(x, y, z, ...)$ according as $f(a+h, b+k, c+l, ...)$ is less or greater than $f(a, b, c, ...)$ for all sufficiently small independent values of $h, k, l, ...$, positive or negative, provided they are not all zero.

§ 2. Necessary Conditions for the Existence of Maxima or Minima.

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y, z, ...)$ for those values of $x, y, z, ...$ for which the expression $f(x+h, y+k, z+l, ...) - f(x, y, z, ...)$ is of **invariable sign** for all sufficiently small independent values of $h, k, l, ...$ provided they are not all equal to zero. There will be a maximum or a minimum according as this sign is negative or positive.

Expanding by Taylor's theorem for several variables, we have

$$f(x+h, y+k, z+l, ...)$$
$$= \left[1 + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + \ldots\right) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + \ldots\right)^2 + \ldots\right] f(x, y, z, ...).$$

$$\therefore \quad f(x+h, y+k, z+l, ...) - f(x, y, z)$$
$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots\right) + \text{terms of the second and higher orders in } h, k, l, \ldots . \quad \ldots(1)$$

Now by taking $h, k, l, ...$ sufficiently small, the first degree terms in $h, k, l, ...$ can be made to govern the sign of the right hand side and therefore of the left hand side of (1). But if $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + \ldots$ is not equal to zero, the sign of this expression will change by changing the sign of each of $h, k, l, \ldots$. Hence as a necessary condition for the occurrence of a maximum or a minimum or $f(x, y, z, \ldots)$, we must have

$$h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+l\frac{\partial f}{\partial z}+\ldots=0. \qquad \ldots(2)$$

Since (2) is true whatever be the values of $h, k, l,\ldots$ independent of each other, we must have as a necessary consequence

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}=\mathbf{0},\ \frac{\partial \mathbf{f}}{\partial \mathbf{y}}=\mathbf{0},\ \frac{\partial \mathbf{f}}{\partial \mathbf{z}}=\mathbf{0},\ldots.$$

If there are n independent vairables, we have then obtained n simultaneous equations to give us the values $a, b, c,\ldots$ of the n variables $x, y, z,\ldots$ for which $f(x, y, z,\ldots)$ may have a maximum or minimum value.

The conditions $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0,\ldots$ *are necessary but not sufficient for the existence of maxima and minima.*

§ 3. Stationary and Extreme points.

A point $(a_1, a_2, \ldots, a_n)$ is called a **stationary point**, if all the first order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ vanish at that point. Also then the value of the function $f(x_1, x_2, \ldots, x_n)$ is said to be stationary at that point. A stationary point which is either a maximum or a minimum is called an **extreme point** and the value of the function at that point is called an **extreme value.** A stationary point is not necessarily an extreme point. Thus a stationary value may be a maximum or a minimum or neither of these two. To decide whether a stationary point is really an extreme point, a further investigation is required.

§ 4. Lagrange's necessary and sufficient condition for the maxima or minima of a function of three independent variables.

Necessary Conditions. Let $f(x, y, z)$ be a function of three independent variables x, y and z. Then as derived in § 2, for $f(x, y, z)$ to be a maximum or a minimum at any point (a, b, c), it is necessary that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}=\mathbf{0},\ \frac{\partial \mathbf{f}}{\partial \mathbf{y}}=\mathbf{0} \text{ and } \frac{\partial \mathbf{f}}{\partial \mathbf{z}}=\mathbf{0}$$

at that point.

Hence the points where the value of the function $f(x, y, z)$ is stationary (*i.e.*, may be a maximum or a minimum) are obtained by solving the simultaneous equations

$$\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0,\ \frac{\partial f}{\partial z}=0.$$

Sufficient Conditions. Before deriving the sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables, we obtain the following two algebraic lemmas regarding the signs of quadratic expressions.

Lemma 1. Let $I_2 = ax^2 + 2hxy + by^2$ be a quadratic expression in two variables x and y. We can write

$$I_2 = \frac{1}{a}[a^2x^2 + 2ahxy + aby^2], \text{ if } a \neq 0$$

$$= \frac{1}{a}[(ax + hy)^2 + (ab - h^2)y^2].$$

The expression within the square brackets will be positive if $ab - h^2$ is positive and in that case the sign of the expression I_2 will be the same as that of a.

In case $ab - h^2$ is not positive, we can say nothing about the sign of the expression within the square brackets and hence nothing about the sign of the given quadratic expression I_2.

Lemma 2. In three variables x, y and z,

$$I_3 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$= \frac{1}{a}[a^2x^2 + aby^2 + acz^2 + 2fayz + 2gazx + 2haxy], \text{ if } a \neq 0$$

$$= \frac{1}{a}[a^2x^2 + 2ax(gz + hy) + aby^2 + acz^2 + 2fayz]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + aby^2 + acz^2 + 2fayz - (gz + hy)^2]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + (ab - h^2)y^2 + 2yz(fa - gh) + (ac - g^2)z^2].$$

Now I_3 will be of the same sign as a provided the expression within the square brackets is positive which will of course be so if

$$ab - h^2 \text{ and } \{(ab - h^2)(ac - g^2) - (fa - gh)^2\}$$

are both positive *i.e.*, if

$$ab - h^2 \text{ and } a(abc + 2fgh - af^2 - bg^2 - ch^2)$$

are both positive.

Thus I_3 will be positive if

$$a, \quad \begin{vmatrix} a & h \\ h & b \end{vmatrix}, \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

be all positive and will be negative if these three expressions are alternately negative and positive.

Now we are in a position to derive Lagrange's sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables at a stationary point.

Let a set of the values of x, y, z obtained by solving the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

be a, b, c.

Let the values of the six second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y\, \partial z}, \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } \frac{\partial^2 f}{\partial x\, \partial y}$$

at the point (a, b, c) be denoted by A, B, C, F, G and H respectively.

Then, we have

$$f(a+h, b+k, c+l) - f(a, b, c)$$

$$= \frac{1}{2!}(Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk) + R_3, \quad ...(1)$$

where R_3 consists of terms of third and higher orders of small quantities h, k and l. By taking h, k and l sufficiently small, the second degree terms in h, k and l can be made to govern the sign of the right hand side and therefore of the left hand side of (1). If this group of terms forms an expression of invariable sign for all such values of h, k and l, we shall have a maximum or a minimum value of $f(x, y, z)$ at (a, b, c) according as that sign is negative or positive.

Hence by our lemma 2, if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c).

§ 5. Working Rule for finding the maxima and minima of a function of three independent variables.

Suppose $f(x, y, z)$ is a given function of three independent variables x, y and z. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ and solve the simultaneous equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $\partial f/\partial z = 0$. All the triads (a, b, c) of the values of x, y and z obtained on solving these equations will give the stationary values of $f(x, y, z)$ *i.e.*, will give the points at which the function $f(x, y, z)$ may be a maximum or a minimum.

To discuss the maximum or minimum of $f(x, y, z)$ at any point (a, b, c) obtained on solving the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $\partial f/\partial z = 0$, we find the values at this point of the six partial derivatives of second order of $f(x, y, z)$ symbolically denoted as follows :

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}, F = \frac{\partial^2 f}{\partial y\, \partial z}, G = \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } H = \frac{\partial^2 f}{\partial x\, \partial y}.$$

If the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c).

Solved Examples

Ex. 1. *Discuss the maximum or minimum values of u where*

$$u = x^2 + y^2 + z^2 + x - 2z - xy.$$

Sol. For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0,$$

$$\frac{\partial u}{\partial y} = -x + 2y = 0,$$

and $$\frac{\partial u}{\partial z} = 2z - 2 = 0.$$

These equations give $x = -2/3, y = -1/3, z = 1$.

$\therefore$ $(-2/3, -1/3, 1)$ is the only point at which u is stationary *i.e.*, at which u may have a maximum or a minimum.

Now $\dfrac{\partial^2 u}{\partial x^2} = 2, \dfrac{\partial^2 u}{\partial y^2} = 2, \dfrac{\partial^2 u}{\partial z^2} = 2,$

$$\frac{\partial^2 u}{\partial y\, \partial z} = 0, \frac{\partial^2 u}{\partial z\, \partial x} = 0 \text{ and } \frac{\partial^2 u}{\partial x\, \partial y} = -1.$$

If A, B, C, F, G and H denote the respective values of these six partial derivatives of second order at the point $(-2/3, -1/3, 1)$, then

$$A = 2, B = 2, C = 2, F = 0, G = 0, H = -1.$$

Now we have

$$A = 2, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6.$$

Since these three expressions are all positive, we have a minimum of u when $x = -2/3, y = -1/3, z = 1$.

Ex. 2. *Show that the point such that the sum of the squares of its distances from n given points shall be minimum, is the centre of the mean position of the given points.*

Sol. Let the n given points be (a_1, b_1, c_1), (a_2, b_2, c_2), ..., (a_n, b_n, c_n) and let (x, y, z) be the coordinates of the required point.

If u denotes the sum of the squares of the distances of (x, y, z) from the n given points, then

$$u = \Sigma\left[(x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2\right]$$
$$= \Sigma (x - a_1)^2 + \Sigma (y - b_1)^2 + \Sigma (z - c_1)^2.$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2\,\Sigma (x - a_1) = 2nx - 2\,\Sigma a_1 = 0,$$

$$\frac{\partial u}{\partial y} = 2\,\Sigma (y - b_1) = 2ny - 2\,\Sigma b_1 = 0,$$

and
$$\frac{\partial u}{\partial z} = 2\,\Sigma (z - c_1) = 2nz - 2\,\Sigma c_1 = 0.$$

Solving these equations, we get

$$x = \frac{\Sigma a_1}{n}, y = \frac{\Sigma b_1}{n}, z = \frac{\Sigma c_1}{n}.$$

Now
$$A = \frac{\partial^2 u}{\partial x^2} = 2n, B = \frac{\partial^2 u}{\partial y^2} = 2n, C = \frac{\partial^2 u}{\partial z^2} = 2n,$$

$$F = \frac{\partial^2 u}{\partial y\, \partial z} = 0, G = \frac{\partial^2 u}{\partial z\, \partial x} = 0, H = \frac{\partial^2 u}{\partial x\, \partial y} = 0.$$

We have $A = 2n$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2n & 0 \\ 0 & 2n \end{vmatrix} = 4n^2,$

and
$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2n & 0 & 0 \\ 0 & 2n & 0 \\ 0 & 0 & 2n \end{vmatrix} = 8n^3.$$

Since these three expressions are all positive, u is minimum when

$$x = \frac{\Sigma a_1}{n}, y = \frac{\Sigma b_1}{n} \text{ and } z = \frac{\Sigma c_1}{n}$$

i.e., u is minimum when the point (x, y, z) is the centre of the mean position of the n given points.

Ex. 3. *Find the maximum value of u where*

$$u = \frac{xyz}{(a + x)(x + y)(y + z)(z + b)}.$$ **(Meerut 1998)**

Sol. We have

$$\log u = \log x + \log y + \log z - \log (a + x) - \log (x + y) - \log (y + z) - \log (z + b).$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a + x} - \frac{1}{x + y} = \frac{ay - x^2}{x(a + x)(x + y)}$$

or
$$\frac{\partial u}{\partial x} = \frac{(ay - x^2)\, u}{x(a + x)(x + y)}.$$

Similarly $\dfrac{\partial u}{\partial y} = \dfrac{(xz - y^2)\, u}{y\,(x + y)\,(y + z)}$

and $\dfrac{\partial u}{\partial z} = \dfrac{(by - z^2)\, u}{z\,(y + z)\,(z + b)}$.

Now for a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \text{ i.e., } \quad ay - x^2 = 0$$

$$\frac{\partial u}{\partial y} = 0 \text{ i.e., } \quad xz - y^2 = 0$$

and $$\frac{\partial u}{\partial z} = 0 \text{ i.e., } \quad by - z^2 = 0.$$

From the above equations, it follows that a, x, y, z and b are in geometrical progression. Let r be the common ratio of this geometrical progression. Then

$$ar^4 = b \quad \text{or} \quad r = (b/a)^{1/4}.$$

Also $x = ar, y = ar^2, z = ar^3$.

Substituting these values, we get

$$u = \frac{ar \,.ar^2 \,.ar^3}{a\,(1 + r)\,.ar\,(1 + r)\,.ar^2\,(1 + r)\,.ar^3\,(1 + r)}$$

$$= \frac{1}{a\,(1 + r)^4} = \frac{1}{a\,[1 + (b/a)^{1/4}]^4} = \frac{1}{(a^{1/4} + b^{1/4})^4}.$$

This gives a stationary value of u. To decide whether this value of u is a maximum or a minimum we proceed to find the second order partial derivatives of u.

We have

$$\frac{\partial^2 u}{\partial x^2} = \frac{-\,2xu}{x\,(a + x)\,(x + y)} + (ay - x^2)\,\frac{\partial}{\partial x}\left[\frac{u}{x\,(a + x)\,(x + y)}\right].$$

$\therefore$ when $x = ar, y = ar^2, z = ar^3$, we have

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{-\,2.\,ar.\,u}{ar\,.a\,(1 + r)\,.ar\,(1 + r)} = \frac{-\,2u}{a^2 r\,(1 + r)^2},$$

which is negative.

Hence the above stationary value of u is a maximum.

Ans. Maximum value of $u = \dfrac{1}{(a^{1/4} + b^{1/4})^4}$.

Note. In the complicated problems in order to find whether a stationary value of u is a maximum or a minimum, it is sufficient to find the value of a second partial differential coefficient of u with respect to any of the independent variables. The value of u will be maximum or minimum according as the value of this second partial derivative at the stationary point under consideration is –ive or +ive.

Ex. 4. *Show that* $u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ *has minimum at* $(1, 1, 1)$ *and maximum at* $(-1, -1, -1)$.

Sol. For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 3(x + y + z)^2 - 3 - 24yz = 0, \qquad ...(1)$$

$$\frac{\partial u}{\partial y} = 3(x + y + z)^2 - 3 - 24zx = 0 \qquad ...(2)$$

and $$\frac{\partial u}{\partial z} = 3(x + y + z)^2 - 3 - 24xy = 0. \qquad ...(3)$$

Subtracting (2) from (1), we get

$$24z(x - y) = 0$$

which has $x = y$ for one of its solutions.

Similarly subtracting (3) from (1), we get

$$24y(x - z) = 0$$

which has $x = z$ for one of its solutions.

Thus the equations (1), (2) and (3) are satisfied when

$$x = y = z.$$

Putting $y = x$ and $z = x$ (1), we get

$$27x^2 - 3 - 24x^2 = 0 \text{ or } 3x^2 = 3 \text{ or } x^2 = 1 \text{ or } x = \pm 1.$$

$\therefore$ $x = y = z = 1$ and $x = y = z = -1$ are solutions of the equations (1), (2) and (3).

Hence u is stationary at the points $(1, 1, 1)$

and $(-1, -1, -1)$.

Now $A = \dfrac{\partial^2 u}{\partial x^2} = 6(x + y + z)$, $B = \dfrac{\partial^2 u}{\partial y^2} = 6(x + y + z)$,

$C = \dfrac{\partial^2 u}{\partial z^2} = 6(x + y + z)$, $F = \dfrac{\partial^2 u}{\partial y\, \partial z} = 6(x + y + z) - 24x$,

$G = \dfrac{\partial^2 u}{\partial z\, \partial x} = 6(x + y + z) - 24y$, $H = \dfrac{\partial^2 u}{\partial x\, \partial y} = 6(x + y + z) - 24z$.

Nature of u at (1, 1, 1). At the stationary point $(1, 1, 1)$, we have $A = 18, B = 18, C = 18, F = -6, G = -6, H = -6$.

$\therefore$ at the point $(1, 1, 1)$, we have

$$A = 18, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix}$$

$$= 6^3 . \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

$$= 6^3. \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}, \text{ by } R_1 + R_2 + R_3$$

$$= 6^3. \begin{vmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{vmatrix}, \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= 6^3. 16.$$

Since these three expressions are all positive, we have a minimum of u at the point $(1, 1, 1)$.

Nature of u at the stationary point $(-1, -1, -1)$.

At the stationary point $(-1, -1, -1)$, we have

$$A = -18 = B = C, F = 6 = G = H.$$

$\therefore$ at the point $(-1, -1, -1)$, we have

$$A = -18, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288$$

and
$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -6^3. 16.$$

Since these three expressions are alternately negative and positive, we have a maximum of u at the point $(-1, -1, -1)$.

Ex. 5. *Find the maximum or minimum values of u where*

$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4.$$

Sol. We have $\partial u/\partial x = ay^2z^3 - 2xy^2z^3 - y^3z^3 - y^2z^4$

$$= y^2z^3 (a - 2x - y - z),$$

$$\partial u/\partial y = 2a\,xyz^3 - 2x^2yz^3 - 3xy^2z^3 - 2xyz^4$$

$$= xyz^3 (2a - 2x - 3y - 2z)$$

and $\partial u/\partial z = 3axy^2z^2 - 3x^2y^2z^2 - 3xy^3z^2 - 4xy^2z^3$

$$= xy^2z^2 (3a - 3x - 3y - 4z).$$

For a maximum or a minimum of u, we must have

$$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$$

Now one solution of the equations $\partial u/\partial x = 0$, $\partial u/\partial y = 0$, $\partial u/\partial z = 0$ is given by the equations

$$2x + y + z = a, 2x + 3y + 2z = 2a, 3x + 3y + 4z = 3a.$$

Solving these equations, we get $x = a/7, y = 2a/7, z = 3a/7$.

$\therefore$ u is stationary at the point $(a/7, 2a/7, 3a/7)$.

Now $A = \dfrac{\partial^2 u}{\partial x^2} = y^2z^3. (-2)$.

At the stationary point $(a/7, 2a/7, 3a/7)$, the value of A is – ive.

$\therefore$ u has a maximum value at the point

$$(a/7, 2a/7, 3a/7).$$

Putting $x = a/7, y = 2a/7, z = 3a/7$ in the value of u, the maximum value of u at the point $(a/7, 2a/7, 3a/7) = 108a^7/7$.

Ex. 6. *Find the maximum value of*

$$(ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}.$$

Sol. Let $u = (ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}$.

Then $\log u = \log(ax + by + cz) - (\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax + by + cz} - 2\alpha^2x, \frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax + by + cz} - 2\beta^2y,$$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax + by + cz} - 2\gamma^2z.$$

For a maximum or a minimum of u, we must have

$$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$$

These give $x\,(ax + by + cz) = \dfrac{a}{2\alpha^2}$ ...(1)

$$y\,(ax + by + cz) = \frac{b}{2\beta^2} \qquad ...(2)$$

and $$z\,(ax + by + cz) = \frac{c}{2\gamma^2}. \qquad ...(3)$$

Multiplying (1), (2), (3) by a, b, c respectively and adding, we get

$$(ax + by + cz)^2 = \tfrac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right),$$

so that $$(ax + by + cz) = \sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\right\}} = R, \text{say}.$$

Then $x = \dfrac{a}{2\alpha^2 R}, y = \dfrac{b}{2\beta^2 R}, z = \dfrac{c}{2\gamma^2 R}$.

$\therefore \quad u$ is stationary when $x = \dfrac{a}{2\alpha^2 R}, y = \dfrac{b}{2\beta^2 R}, z = \dfrac{c}{2\gamma^2 R}$.

Again $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{a^2}{(ax + by + cz)^2} - 2\alpha^2.$$

Now at a stationary point, we have $\partial u/\partial x = 0$.

$\therefore \quad$ at the stationary point found above, we have

$\dfrac{\partial^2 u}{\partial x^2} = -u\left[\dfrac{a^2}{(ax + by + cz)^2} + 2\alpha^2\right]$, which is –ive because u is positive for the values of x, y, z found above.

$\therefore \quad u$ is maximum at the stationary point found above.

Also putting the values of x, y, z found above in the value of u, the maximum value of u

$$= R \, . \, e^{-\frac{1}{4R^2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)} = R \, . e^{-\frac{1}{4R^2} . 2R^2}$$

$$= R \, . e^{-1/2} = \frac{R}{\sqrt{e}} = \frac{1}{\sqrt{e}} \cdot \sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}$$

$$= \sqrt{\left\{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}} .$$

§ 6. Lagrange's method of undetermined multipliers.

Let $u = f(x_1, x_2, ..., x_n)$

be a function of n variables $x_1, x_2, ..., x_n$. Let these variables be connected by m equations

$$\phi_1 (x_1, x_2, ..., x_n) = 0 , \ \phi_2 (x_1, x_2, ..., x_n) = 0 , ...,$$
$$\phi_m (x_1, x_2, ..., x_n) = 0$$

so that only $n - m$ of the n variables are independent.

For a maximum or a minimum of u, we have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + ... + \frac{\partial u}{\partial x_n} dx_n = 0 .$$

Also differentiating the m given equations connecting the variables, we get

$$d\phi_1 = \frac{\partial \phi_1}{\partial x_1} dx_1 + \frac{\partial \phi_1}{\partial x_2} dx_2 + \frac{\partial \phi_1}{\partial x_3} dx_3 + ... + \frac{\partial \phi_1}{\partial x_n} dx_n = 0 ,$$

$$d\phi_2 = \frac{\partial \phi_2}{\partial x_1} dx_1 + \frac{\partial \phi_2}{\partial x_2} dx_2 + \frac{\partial \phi_2}{\partial x_3} dx_3 + ... + \frac{\partial \phi_2}{\partial x_n} dx_n = 0 ,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$d\phi_m = \frac{\partial \phi_m}{\partial x_1} dx_1 + \frac{\partial \phi_m}{\partial x_2} dx_2 + \frac{\partial \phi_m}{\partial x_3} dx_3 + ... + \frac{\partial \phi_m}{\partial x_n} dx_n = 0 .$$

Multiplying the above $m + 1$ equations obtained on differentiation by $1, \lambda_1, \lambda_2, ..., \lambda_m$ respectively and adding, we get an equation which may be written as

$$P_1 \, dx_1 + P_2 \, dx_2 + P_3 \, dx_3 + ... + P_n \, dx_n = 0 , \qquad ...(1)$$

where $$P_r = \frac{\partial u}{\partial x_r} + \lambda_1 \frac{\partial \phi_1}{\partial x_r} + \lambda_2 \frac{\partial \phi_2}{\partial x_r} + ... + \lambda_m \frac{\partial \phi_m}{\partial x_r} .$$

Now the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$ are at our choice. We choose them such that they satisfy the m linear equations

$$P_1 = 0 , P_2 = 0 , ..., P_m = 0 .$$

Then the equation (1) reduces to

$$P_{m+1} \, dx_{m+1} + P_{m+2} \, dx_{m+2} + ... + P_n \, dx_n = 0 . \qquad ...(2)$$

It is immaterial which of the $n - m$ of the n variables $x_1, x_2, ..., x_n$ are regarded as independent. Let us regard the $n - m$ variables $x_{m+1}, x_{m+2}, ..., x_n$ as independent. Then since the $n - m$ quantities $dx_{m+1}, dx_{m+2}, ..., dx_n$ are all independent of one another, their coefficients must be separately zero in the relation (2). Hence we must have

$$P_{m+1} = 0, P_{m+2} = 0, ..., P_n = 0.$$

Thus we get $m + n$ equations

$$P_1 = 0, P_2 = 0, ..., P_n = 0$$

and $$\phi_1 = 0, \phi_2 = 0, ..., \phi_m = 0,$$

which together with the relation $u = f(x_1, x_2, ..., x_n)$ determine the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$, the values of $x_1, x_2, ..., x_n$ and u at the stationary point. This method is known as **Lagrange's method of undetermined multipliers.** It is very convenient to apply and it often gives us the maximum or minimum values of u without actually determining the values of the multipliers $\lambda_1, ..., \lambda_m$. The only drawback of this method is that it does not determine the nature of the stationary point.

Solved Examples

Ex. 1(a). *If* $u = x^2 + y^2 + z^2$, *where* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$, *find the maximum or minimum values of* u. **(Meerut 1991 S)**

Sol. We have $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y, z are connected by the relation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \quad ...(2)$$

For a maximum or a minimum of u, we have

$$du = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0. \quad ...(3)$$

Also differentiating the given relation (2), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz + 2fy\,dz + 2fz\,dy + 2gz\,dx + 2gx\,dz + 2hx\,dy + 2hy\,dx = 0$$

or $$(ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0. \quad ...(4)$$

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy, dz to zero, we have

$$x + \lambda(ax + hy + gz) = 0, \quad ...(5)$$

$$y + \lambda(hx + by + fz) = 0, \quad ...(6)$$

and $$z + \lambda(gx + fy + cz) = 0. \quad ...(7)$$

Multiplying (5) by x, (6) by y, (7) by z and adding, we get

$$x^2 + y^2 + z^2 + \lambda (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

or $\quad u + \lambda . 1 = 0,$ using (1) and (2).

$\therefore \quad \lambda = -u.$

Hence from (5), we have

$$x - u (ax + hy + gz) = 0$$

or $\quad x (1 - au) - huy - guz = 0$

or $$\left(a - \frac{1}{u}\right) x + hy + gz = 0. \qquad ...(8)$$

Similarly from (6) and (7), we have

$$hx + (b - 1/u) y + fz = 0, \qquad ...(9)$$

and $$gx + fy + (c - 1/u) z = 0. \qquad ...(10)$$

Eliminating x, y, z from (8), (9), (10), we get

$$\begin{vmatrix} a - (1/u) & h & g \\ h & b - (1/u) & f \\ g & f & c - (1/u) \end{vmatrix} = 0. \qquad ...(11)$$

Hence the required maximum or minimum values of u are the roots of the equation (11).

Ex. 1(b). *Find the maxima and minima of* $x^2 + y^2$ *subject to the condition*

$$ax^2 + 2hxy + by^2 = 1.$$ **(Meerut 1996)**

Sol. Let $\quad u = x^2 + y^2, \qquad ...(1)$

where the variables x and y are connected by the relation

$$ax^2 + 2hxy + by^2 = 1. \qquad ...(2)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow \quad 2xdx + 2ydy = 0 \quad \Rightarrow \quad xdx + ydy = 0. \qquad ...(3)$

Also differentiating the given relation (2), get

$$2axdx + 2hxdy + 2hydx + 2bydy = 0$$

or $$(ax + hy) dx + (hx + by) dy = 0 \qquad ...(4)$$

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy to zero, we have

$$x + \lambda (ax + hy) = 0 \qquad ...(5)$$

and $$y + \lambda (hx + by) = 0 \qquad ...(6)$$

Multiplying (5) by x, (6) by y and adding, we get

$$x^2 + y^2 + \lambda (ax^2 + 2hxy + by^2) = 0$$

or $\quad u + \lambda . 1 = 0,$ using (1) and (2).

$\therefore \quad \lambda = -u.$

Hence from (5), we have

$$x - u (ax + hy) = 0 \quad \text{or} \quad x (1 - au) - huy = 0$$

or $$\left(a - \frac{1}{u}\right) x + hy = 0 \qquad ...(7)$$

Similarly from (6), we have

$$hx + \left(b - \frac{1}{u}\right) y = 0 \qquad \ldots(8)$$

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} a - \frac{1}{u} & h \\ h & b - \frac{1}{u} \end{vmatrix} = 0 \quad \text{or} \quad \left(a - \frac{1}{u}\right)\left(b - \frac{1}{u}\right) = h^2 \qquad \ldots(9)$$

Hence the required maximum or minimum values of $u = x^2 + y^2$ are the roots of the equation (9).

Ex. 2. *Find the stationary values of $x^2 + y^2 + z^2$ subject to the conditions*

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0.$$

Interpret the result geometrically. **(Meerut 1991)**

Sol. Let $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y and z are connected by the relations

$$ax^2 + by^2 + cz^2 = 1, \qquad \ldots(2)$$

and $$lx + my + nz = 0. \qquad \ldots(3)$$

For a stationary value of u, we have

$$du = 0$$

$$2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$x\,dx + y\,dy + z\,dz = 0. \qquad \ldots(4)$$

Also differentiating the given relations (2) and (3), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0$$

i.e., $$ax\,dx + by\,dy + cz\,dz = 0 \qquad \ldots(5)$$

and $$l\,dx + m\,dy + n\,dz = 0. \qquad \ldots(6)$$

Multilpying (4) by 1, (5) by λ and (6) by μ and adding, and then equating the coefficients of dx, dy, dz to zero, we get

$$x + \lambda ax + \mu l = 0, \qquad \ldots(7)$$

$$y + \lambda by + \mu m = 0, \qquad \ldots(8)$$

and $$z + \lambda cz + \mu n = 0. \qquad \ldots(9)$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$x^2 + y^2 + z^2 + \lambda\,(ax^2 + by^2 + cz^2) + \mu\,(lx + my + nz) = 0,$$

or $$u + \lambda\,.1 + \mu\,.0 = 0, \text{ using (1), (2) and (3)}$$

or $$\lambda = -u.$$

Substituting for λ in the equations (7), (8) and (9), we get

$$x = \frac{\mu l}{au - 1}, y = \frac{\mu m}{bu - 1}, z = \frac{\mu n}{cu - 1}.$$

Substituting these values of x, y, z in (3), we get

$$\frac{\mu l^2}{au - 1} + \frac{\mu m^2}{bu - 1} + \frac{\mu n^2}{cu - 1} = 0$$

or $$\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0. \qquad ...(10)$$

Hence the stationary (*i.e.*, maximum or minimum) values of u are given by the equation (10). The equation (10) is a quadratic in u and so it gives two stationary values of u.

Geometrical interpretation. The surface $ax^2+by^2+cz^2=1$ represents an ellipsoid (or a hyperboloid) whose centre is origin, and $lx+my+nz=0$ is a plane passing through the origin. Therefore the point (x, y, z) satisfying both the conditions (2) and (3) lies on the conic in which (2) and (3) intersect. Also $x^2+y^2+z^2$ gives the square of the distance of (x, y, z) from the origin which is also the centre of the conic of intersection. The maximum and minimum values of this distance are the major and minor semi-axes of the conic. So the equation (10) gives the squares of the lengths of the semi-axes of the conic of intersection.

Ex. 3. *Find the maximum and minimum values of*
$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4},$$
when $lx+my+nz=0$ and $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Interpret the result geometrically.

Sol. Let $u=x^2/a^4+y^2/b^4+z^2/c^4$. Then for a maximum or a minimum of u, we have
$$du=0$$
$$\Rightarrow \quad \frac{x}{a^4}dx+\frac{y}{b^4}dy+\frac{z}{c^4}dz=0 \qquad ...(1)$$

Also differentiating the two given equations connecting the variables x, y and z, we get
$$l\,dx+m\,dy+n\,dz=0, \qquad ...(2)$$
and $$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \qquad ...(3)$$

Multiplying (1), (2) and (3) by $1, \lambda$ and μ respectively and adding, and then equating to zero the coefficients of dx, dy and dz, we get
$$\frac{x}{a^4}+\lambda l+\mu\frac{x}{a^2}=0, \qquad ...(4)$$
$$\frac{y}{b^4}+\lambda m+\mu\frac{y}{b^2}=0, \qquad ...(5)$$
and $$\frac{z}{c^4}+\lambda n+\mu\frac{z}{c^2}=0. \qquad ...(6)$$

Multiplying the equations (4), (5) and (6) by x, y and z respectively and adding, we get
$$u+\lambda.0+\mu.1=0 \quad \text{or} \quad \mu=-u.$$

Putting $\mu=-u$ in (4), we get

$$\frac{x}{a^4} + \lambda l - \frac{xu}{a^2} = 0, \text{ or } \frac{x}{a^2}\left\{u - \frac{1}{a^2}\right\} = \lambda l, \text{ or } x = \frac{\lambda l a^4}{a^2 u - 1}.$$

Similarly from (5) and (6), we get

$$y = \frac{\lambda m b^4}{b^2 u - 1} \quad \text{and} \quad z = \frac{\lambda n c^4}{c^2 u - 1}.$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{l^2 a^4}{a^2 u - 1} + \frac{m^2 b^4}{b^2 u - 1} + \frac{n^2 c^4}{c^2 u - 1} = 0. \quad ...(7)$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical interpretation. The equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at any point (x, y, z) on it is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2} = 1. \quad ...(8)$$

If p be the length of the perpendicular from origin which is also the centre of the ellipsoid to the tangent plane (1), then

$$p^2 = \frac{1}{x^2/a^4 + y^2/b^4 + z^2/c^4}.$$

If the point (x, y, z) on the ellipsoid also lies on the given plane $lx + my + nz = 0$, the problem consists of finding out the maximum or minimum values of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx + my + nz = 0$ and the ellipsoid.

Ex. 4. *Find the maximum and minimum values of*

$$u = a^2x^2 + b^2y^2 + c^2z^2,$$

where $x^2 + y^2 + z^2 = 1$ *and* $lx + my + nz = 0$.

Sol. We have $u = a^2x^2 + b^2y^2 + c^2z^2$, ...(1)

where the variables x, y, z are connected by relations

$$x^2 + y^2 + z^2 = 1 \quad ...(2)$$

and

$$lx + my + nz = 0 \quad ...(3)$$

For a maximum or a minimum of u, we have

$$du = 0$$

$$\Rightarrow \quad 2a^2x\,dx + 2b^2y\,dy + 2c^2z\,dz = 0$$

$$\Rightarrow \quad a^2x\,dx + b^2y\,dy + c^2z\,dz = 0. \quad ...(4)$$

Also differentiating the two given equations (2) and (3) connecting the variables x, y and z, we get

$$2x\,dx + 2y\,dy + 2z\,dz = 0$$

i.e., $x\,dx + y\,dy + z\,dz = 0$ ...(5)

and $l\,dx + m\,dy + n\,dz = 0$ ...(6)

Multiplying (4), (5) and (6) by $1, \lambda$ and μ respectively and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$a^2x + \lambda x + \mu l = 0, \qquad ...(7)$$

$$b^2y + \lambda y + \mu m = 0, \qquad ...(8)$$

and $$c^2z + \lambda z + \mu n = 0. \qquad ...(9)$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda .1 + \mu .0 = 0 \text{ or } \lambda = -u.$$

Putting $\lambda = -u$ in (7), we get

$$a^2x - ux + \mu l = 0$$

or $$x = \frac{\mu l}{u - a^2}.$$

Similarly from (8) and (9), we get

$$y = \frac{\mu m}{u - b^2} \quad \text{and} \quad z = \frac{\mu n}{u - c^2}.$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{\mu l^2}{u - a^2} + \frac{\mu m^2}{u - b^2} + \frac{\mu n^2}{u - c^2} = 0$$

or $$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0. \qquad ...(10)$$

The equation (10) gives the required maximum or minimum values of u.

Ex. 5. *Find the maximum and minimum values of u^2 when*

$$u^2 = a^2x^2 + b^2y^2 + c^2z^2$$

while $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$.

Sol. Proceed exactly as in solved example 4 taking the function as u^2 in place of u. The required maximum or minimum values of u^2 are the roots of the equation

$$\frac{l^2}{u^2 - a^2} + \frac{m^2}{u^2 - b^2} + \frac{n^2}{u^2 - c^2} = 0.$$

Ex. 6. *Show that the maximum and minimum values of*

$$u = x^2 + y^2 + z^2$$

subject to the conditions

$$px + qy + rz = 0 \text{ and } x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

are given by the equation

$$\frac{a^2p^2}{u - a^2} + \frac{b^2q^2}{u - b^2} + \frac{c^2r^2}{u - c^2} = 0.$$

Sol. Do yourself. Proceed exactly as in solved examples 2 and 4.

Ex. 7. *Find the maximum value of $x^m y^n z^p$ subject to the condition*

$$x + y + z = a. \qquad \textbf{(Meerut 1994P, 95BP)}$$

Sol. Let $u = x^m y^n z^p$, ...(1)

where the variables x, y, z are connected by the relation

$$x + y + z = a. \qquad ...(2)$$

From (1), $\log u = m \log x + n \log y + p \log z$.

$$\therefore \quad \frac{1}{u} du = \frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz.$$

For a maximum or a minimum of u, we have

$$du = 0$$

$$\Rightarrow \quad \frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz = 0. \qquad ...(3)$$

Differentiating the given equation (2) connecting the variables x, y and z, we get

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{m}{x} + \lambda = 0, \frac{n}{y} + \lambda = 0, \frac{p}{z} + \lambda = 0.$$

From these, we get $x = -m/\lambda, y = -n/\lambda, z = -p/\lambda$.

Putting these values of x, y, z in $x + y + z = a$, we get

$$-\left(\frac{m}{\lambda} + \frac{n}{\lambda} + \frac{p}{\lambda}\right) = a \quad \text{or} \quad -\frac{1}{\lambda}(m + n + p) = a$$

or $$-\frac{1}{\lambda} = \frac{a}{m + n + p}.$$

$\therefore$ u is stationary when

$$x = \frac{am}{m + n + p}, y = \frac{an}{m + n + p}, z = \frac{ap}{m + n + p}.$$

Let us now find the nature of this stationary value of u.

Since the variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = m \log x + n \log y + p \log z.$$

$$\therefore \quad \frac{1}{u} \frac{\partial u}{\partial x} = \frac{m}{x} + \frac{p}{z} \frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$1 + (\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -1.$$

$$\therefore \quad \frac{1}{u} \frac{\partial u}{\partial x} = \frac{m}{x} - \frac{p}{z},$$

so that $$\frac{1}{u} \frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2} \left(\frac{\partial u}{\partial x}\right)^2 = -\frac{m}{x^2} + \frac{p}{z^2} \frac{\partial z}{\partial x} = -\frac{m}{x^2} - \frac{p}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{m}{x^2} + \frac{p}{z^2}\right] = -x^m y^n z^p \left[\frac{m}{x^2} + \frac{p}{z^2}\right],$$

which is –ive for the values of x, y, z found above.

Hence at the stationary point found above the value of u is maximum and this maximum value

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p = \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

Ex. 8. *Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition*

$$ax + by + cz = p + q + r.$$

Sol. For complete solution of this question proceed as in solved example 7.

Let $$u = x^p y^q z^r, \qquad ...(1)$$

where the variables x, y, z are connected by the relation

$$ax + by + cz = p + q + r. \qquad ...(2)$$

From (1), $\log u = p \log x + q \log y + r \log z$.

$$\therefore \quad \frac{1}{u} du = \frac{p}{x} dx + \frac{q}{y} dy + \frac{r}{z} dz.$$

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow \quad \frac{p}{x} dx + \frac{q}{y} dy + \frac{r}{z} dz = 0. \qquad ...(3)$$

Also differentiating the given equation (2), we get

$$a\,dx + b\,dy + c\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{p}{x} + \lambda a = 0, \frac{q}{y} + \lambda b = 0, \frac{r}{z} + \lambda c = 0.$$

From these, we get $x = -p/\lambda a, y = -q/\lambda b, z = -r/\lambda c$.

Putting these values of x, y, z in (2), we get

$$-\left(\frac{p}{\lambda} + \frac{q}{\lambda} + \frac{r}{\lambda}\right) = p + q + r \quad \text{or} \quad -\frac{1}{\lambda}(p + q + r) = p + q + r$$

or $$\lambda = -1.$$

$\therefore$ u is stationary when $x = p/a, y = q/b, z = r/c$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = p \log x + q \log y + r \log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} + \frac{r}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$a + c\,(\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -\,a/c.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} - \frac{r}{z}\cdot\frac{a}{c},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{p}{x^2} + \frac{a}{c}\cdot\frac{r}{z^2}\frac{\partial z}{\partial x} = -\frac{p}{x^2} - \frac{a^2}{c^2}\cdot\frac{r}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -\,u\left[\frac{p}{x^2} + \frac{a^2}{c^2}\cdot\frac{r}{z^2}\right] = -\,x^p y^q z^r\left[\frac{p}{x^2} + \frac{a^2}{c^2}\cdot\frac{r}{z^2}\right],$$

which is –ive for the values of x, y, z found above.

Hence u is maximum at the stationary point found above and this maximum value of $u = (p/a)^p.\,(q/b)^q.\,(r/c)^r$.

Ex. 9. *Find the minimum value of $x + y + z$, subject to the condition*

$$(a/x) + (b/y) + (c/z) = 1.$$

Sol. Let $u = x + y + z$, ...(1)

where the variables x, y, z are connected by the relation

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1. \qquad ...(2)$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow dx + dy + dz = 0. \qquad ...(3)$$

Also differentiating the given equation (2), we get

$$-\frac{a}{x^2}\,dx - \frac{b}{y^2}\,dy - \frac{c}{z^2}\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$1 - \frac{\lambda a}{x^2} = 0,\ 1 - \frac{\lambda b}{y^2} = 0,\ 1 - \frac{\lambda c}{z^2} = 0.$$

From these, we get $x = \sqrt{(\lambda a)}, y = \sqrt{(\lambda b)}, z = \sqrt{(\lambda c)}$.

Putting these values of x, y, z in (2), we get

$$\frac{a}{\sqrt{(\lambda a)}} + \frac{b}{\sqrt{(\lambda b)}} + \frac{c}{\sqrt{(\lambda c)}} = 1 \text{ or } \frac{1}{\sqrt{\lambda}}(\sqrt{a} + \sqrt{b} + \sqrt{c}) = 1$$

or $$\sqrt{\lambda} = \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

$\therefore$ u is stationary when

$$x = \sqrt{a}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}),\ y = \sqrt{b}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}),$$
$$z = \sqrt{c}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$-\frac{a}{x^2} - \frac{c}{z^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{az^2}{cx^2}.$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1 - \frac{az^2}{cx^2},$$

so that $$\frac{\partial^2 u}{\partial x^2} = \frac{2az^2}{cx^3} - \frac{2az}{cx^2}\frac{\partial z}{\partial x} = \frac{2az^2}{cx^3} + \frac{2az}{cx^2}\cdot\frac{az^2}{cx^2},$$

which is positive for the values of x, y, z found above.

Hence u is minimum at the stationary point found above and the minimum value of u

$$= \sqrt{a}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{b}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{c}\,(\sqrt{a} + \sqrt{b} + \sqrt{c})$$
$$= (\sqrt{a} + \sqrt{b} + \sqrt{c})^2.$$

Ex. 10. *Find the minimum value of* $x^2 + y^2 + z^2$, *given that*

$$ax + by + cz = p.$$ **(Meerut 1991 P)**

Sol. Do yourself. Proceed as in solved example 9. The required minimum value of u is $p^2/(a^2 + b^2 + c^2)$.

Ex. 11. *Find the maximum or minimum value of* $x^p y^q z^r$ *subject to the condition*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Sol. Let $u = x^p y^q z^r$, ...(1)

where the variables x, y, z are connected by the relation

$$(a/x) + (b/y) + (c/z) = 1. \qquad \text{...(2)}$$

From (1), $\log u = p \log x + q \log y + r \log z$.

$$\therefore \quad \frac{1}{u} du = \frac{p}{x} dx + \frac{q}{y} dy + \frac{r}{z} dz.$$

For a maximum or a minimum of u, we have $du = 0$.

$$\therefore \quad \frac{p}{x} dx + \frac{q}{y} dy + \frac{r}{z} dz = 0. \qquad \text{...(3)}$$

Also differentiating the given equation (2), we get

$$-\frac{a}{x^2} dx - \frac{b}{y^2} dy - \frac{c}{z^2} dz = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{p}{x}-\frac{\lambda a}{x^2}=0,\ \frac{q}{y}-\frac{\lambda b}{y^2}=0,\ \frac{r}{z}-\frac{\lambda c}{z^2}=0.$$

From these, we get $x=a\lambda/p,\ y=b\lambda/q,\ z=c\lambda/r$.

Putting these values of x,y,z in (2), we get

$$\frac{p}{\lambda}+\frac{q}{\lambda}+\frac{r}{\lambda}=1 \text{ or } \frac{1}{\lambda}(p+q+r)=1 \text{ or } \lambda=p+q+r.$$

$\therefore$ u is stationary when

$$\frac{px}{a}=\frac{qy}{b}=\frac{rz}{c}=p+q+r.$$

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u=p\log x+q\log y+r\log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{p}{x}+\frac{r}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$-\frac{a}{x^2}-\frac{c}{z^2}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial z}{\partial x}=-\frac{az^2}{cx^2}.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{p}{x}-\frac{r}{z}\cdot\frac{az^2}{cx^2}=\frac{p}{x}-\frac{arz}{cx^2},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2}-\frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2=-\frac{p}{x^2}+\frac{2arz}{cx^3}-\frac{ar}{cx^2}\frac{\partial z}{\partial x}$$

$$=-\frac{p}{x^2}+\frac{2arz}{cx^3}+\frac{ar}{cx^2}\cdot\frac{az^2}{cx^2}.$$

But at the stationary point, we have $\partial y/\partial x=0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2}=u\left[-\frac{p}{x^2}+\frac{2arz}{cx^3}+\frac{a^2rz^2}{c^2x^4}\right]$$

$$=x^py^qz^r\left[-p\cdot\frac{p^2}{a^2(p+q+r)^2}+\frac{2ar}{c}\cdot\frac{c(p+q+r)}{r}\cdot\frac{p^3}{a^2(p+q+r)^3}+\frac{a^2r}{c^2}\cdot\frac{c^2(p+q+r)^2}{r^2}\cdot\frac{p^4}{a^4(p+q+r)^4}\right]$$

$$=x^py^qz^r\left[-\frac{p^3}{a^2(p+q+r)^2}+\frac{2p^2}{a^2(p+q+r)^2}+\frac{p^4}{ra^2(p+q+r)^2}\right]$$

$$=x^py^qz^r\left[\frac{p^3}{a^2(p+q+r)}+\frac{p^4}{ra^2(p+q+r)^2}\right],$$

which is +ive for the values of x,y,z found above.

Hence u is minimum at the stationary point given by

$$\frac{px}{a}=\frac{qy}{b}=\frac{rz}{c}=p+q+r.$$

Also the minimum value of u

$$=\left[\frac{a\,(p+q+r)}{p}\right]^p\left[\frac{b\,(p+q+r)}{q}\right]^q\left[\frac{c\,(p+q+r)}{r}\right]^r$$

$$=\frac{a^pb^qc^r}{p^pq^qr^r}(p+q+r)^{p+q+r}.$$

Ex. 12. *Find the minimum value of* $x^4+y^4+z^4$, *where* $xyz=c^3$.

Sol. Let $\quad u=x^4+y^4+z^4,$ ...(1)

where the variables x,y,z are connected by the relation

$$xyz=c^3. \qquad ...(2)$$

For a maximum or a minimum of u, we have

$$du=0 \Rightarrow 4x^3\,dx+4y^3\,dy+4z^3\,dz=0$$

$$\Rightarrow x^3\,dx+y^3\,dy+z^3\,dz=0. \qquad ...(3)$$

Also from the given relation (2), we have

$$\log x+\log y+\log z=\log c^3.$$

Differentiating this, we get

$$(1/x)\,dx+(1/y)\,dy+(1/z)\,dz=0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$x^3+\frac{\lambda}{x}=0,\ y^3+\frac{\lambda}{y}=0,\ z^3+\frac{\lambda}{z}=0.$$

From these, we get $x^4=y^4=z^4=-\lambda$.

Now from (2), $x^4y^4z^4=c^{12}$.

$\therefore \quad -\lambda^3=c^{12} \quad$ or $\quad \lambda=-c^4$.

$\therefore \quad u$ is sationary when $x^4=y^4=z^4=c^4$ *i.e.*, when $x=y=z=c$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have $\dfrac{\partial u}{\partial x}=4x^3+4z^3\dfrac{\partial z}{\partial x}$.

Now from (2), we have $\log x+\log y+\log z=\log c^3$.

Differentiating this partially w.r.t. x taking y as constant, we get

$$\frac{1}{x}+\frac{1}{z}\frac{\partial z}{\partial x}=0 \quad \text{or} \quad \frac{\partial z}{\partial x}=-\frac{z}{x}.$$

$$\therefore \quad \frac{\partial u}{\partial x}=4x^3-4z^3\cdot\frac{z}{x}=4x^3-4\frac{z^4}{x},$$

so that $$\frac{\partial^2 u}{\partial x^2}=12x^2+\frac{4z^4}{x^2}-\frac{16}{x}z^3\frac{\partial z}{\partial x}$$

$$= 12x^2 + \frac{4z^4}{x^2} - \frac{16z^3}{x}\left(-\frac{z}{x}\right) = 12x^2 + \frac{4z^4}{x^2} + \frac{16z^4}{x^2}.$$

At the stationary point (c, c, c) found above, we have

$$\frac{\partial^2 u}{\partial x^2} = 12c^2 + 4c^2 + 16c^2 = 32c^2, \text{ which is +ive.}$$

$\therefore$ u is minimum at the point $x = y = z = c$ and the minimum value of $u = c^4 + c^4 + c^4 = 3c^4$.

Ex. 13. *Find the maximum value of* u, *when*

$u = x^2y^3z^4$ *and* $2x + 3y + 4z = a$.

Sol. Let $u = x^2y^3z^4$, ...(1)

where the variables x, y, z are connected by the relation

$$2x + 3y + 4z = a. \qquad ...(2)$$

From (1), $\log u = 2 \log x + 3 \log y + 4 \log z$.

$$\therefore \quad \frac{1}{u} du = \frac{2}{x} dx + \frac{3}{y} dy + \frac{4}{z} dz.$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow (2/x)\, dx + (3/y)\, dy + (4/z)\, dz = 0. \qquad ...(3)$$

Differentiating the given equation (2), we have

$$2\, dx + 3\, dy + 4\, dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{2}{x} + 2\lambda = 0, \frac{3}{y} + 3\lambda = 0, \frac{4}{z} + 4\lambda = 0.$$

From these, get $x = -1/\lambda, y = -1/\lambda, z = -1/\lambda$.

Putting these values of x, y, z in (2), we get

$$-\frac{2}{\lambda} - \frac{3}{\lambda} - \frac{4}{\lambda} = a \text{ or } -\frac{9}{\lambda} = a \text{ or } \lambda = -\frac{9}{a}.$$

$\therefore$ u is stationary when $x = y = z = a/9$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = 2 \log x + 3 \log y + 4 \log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} + \frac{4}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$2 + 4\,(\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -1/2.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} + \frac{4}{z}\cdot\left(-\frac{1}{2}\right) = \frac{2}{x} - \frac{2}{z},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{2}{x^2} + \frac{2}{z^2}\frac{\partial z}{\partial x} = -\frac{2}{x^2} - \frac{1}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left(\frac{2}{x^2}+\frac{1}{z^2}\right) = -x^2y^3z^4\left(\frac{2}{x^2}+\frac{1}{z^2}\right),$$

which is –ive for $x = y = z = a/9$.

Hence at the stationary point $x = y = z = a/9$, u is maximum and the maximum value of u

$$= (a/9)^2\,(a/9)^3\,(a/9)^4 = (a/9)^9.$$

Ex. 14. *Given $u = 5xyz/(x + 2y + 4z)$. Find the values of x, y, z for which u is maximum subject to the condition $xyz = 8$.* **(Meerut 1994)**

Sol. We have $u = 5xyz/(x + 2y + 4z)$, ...(1)

where the variables x, y, z are connected by the relation

$$xyz = 8. \qquad ...(2)$$

From (1) and (2), we have $u = 40/(x + 2y + 4z)$.

$$\therefore \quad du = \frac{-40}{(x+2y+4z)^2}(dx + 2\,dy + 4\,dz).$$

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow \quad dx + 2\,dy + 4\,dz = 0. \qquad ...(3)$$

From (2), $\log x + \log y + \log z = \log 8$.

Differentiating this, we get

$$(1/x)\,dx + (1/y)\,dy + (1/z)\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$1 + (\lambda/x) = 0,\ 2 + (\lambda/y) = 0,\ 4 + (\lambda/z) = 0.$$

From these, we get $x = -\lambda, y = -\lambda/2, z = -\lambda/4$.

Putting these values of x, y, z in (2), we get

$$-\lambda^3/8 = 8 \text{ or } \lambda^3 = -64 \text{ or } \lambda = -4.$$

$\therefore$ u is stationary at the point given by $x = 4, y = 2, z = 1$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

We have $u = 40/(x + 2y + 4z)$.

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{40}{(x+2y+4z)^2}\left[1 + 4\frac{\partial z}{\partial x}\right].$$

From (2), $\log x + \log y + \log z = \log 8$.

$$\therefore \quad (1/x) + (1/z)\,(\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -z/x.$$

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{40}{(x+2y+4z)^2}\left[1 - 4\frac{z}{x}\right],$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \frac{80}{(x+2y+4z)^3}\left[1 + 4\frac{\partial z}{\partial x}\right]\left[1 - 4\frac{z}{x}\right] - \frac{40}{(x+2y+4z)^2}\left[\frac{4z}{x^2} - \frac{4}{x}\frac{\partial z}{\partial x}\right]$$

$$= \frac{80}{(x+2y+4z)^3}\left[1-\frac{4z}{x}\right]^2 - \frac{40}{(x+2y+4z)^2}\left[\frac{4z}{x^2}+\frac{4z}{x^2}\right].$$

$\therefore$ at the stationary point $(4, 2, 1)$ found above, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{80}{12^3}[1-1]^2 - \frac{40}{144}\left[\frac{1}{4}+\frac{1}{4}\right], \text{ which is –ive.}$$

$\therefore$ u is maximum at the point given by $x = 4, y = 2, z = 1$.

Ex. 15. *Divide a number a into three parts such that their product will be maximum.*

Sol. Let $u = xyz$, ...(1)

where the variables x, y and z are connected by the relation

$$x + y + z = a. \qquad ...(2)$$

From (1), $\log u = \log x + \log y + \log z$.

$\therefore$ $(1/u)\, du = (1/x)\, dx + (1/y)\, dy + (1/z)\, dz$.

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow$ $(1/x)\, dx + (1/y)\, dy + (1/z)\, dz = 0$. ...(3)

Also differentiating the equation (2), we have

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{1}{x}+\lambda = 0, \frac{1}{y}+\lambda = 0, \frac{1}{z}+\lambda = 0.$$

From these, we get $x = y = z = -1/\lambda$.

Putting these values of x, y, z in (2), we get

$$-3/\lambda = a \quad \text{or} \quad \lambda = -3/a.$$

$\therefore$ u is stationary at the point given by

$$x = y = z = a/3.$$

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\log u = \log x + \log y + \log z$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x}.$$

But from (2), $1 + (\partial z/\partial x) = 0$ or $\partial z/\partial x = -1$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{z},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{1}{x^2} + \frac{1}{z^2}\frac{\partial z}{\partial x} = -\frac{1}{x^2} - \frac{1}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point $(a/3, a/3, a/3)$, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{1}{x^2}+\frac{1}{z^2}\right] = -xyz\left[\frac{1}{x^2}+\frac{1}{z^2}\right]$$

which is negative for $x = y = z = a/3$.

Hence u is maximum when $x = y = z = a/3$ and the maximum value of $u = (a/3)^3$.

Ans. The required three parts of a are $a/3, a/3, a/3$ and the maximum value of the product is $(a/3)^3$.

Ex. 16. *Show that the maximum and minimum of the radii vectors of the sections of the surface*

$$(x^2 + y^2 + z^2)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

by the plane $\lambda x + \mu y + \nu z = 0$
are given by the equation

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0.$$

Sol. We have to find the maximum and minimum values of r, where

$$r^2 = x^2 + y^2 + z^2. \qquad ...(1)$$

Also the variables x, y, z are connected by the relations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (x^2 + y^2 + z^2)^2 = r^4 \qquad ...(2)$$

and $$\lambda x + \mu y + \nu z = 0. \qquad ...(3)$$

From (1), $2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz$.

For a maximum or a minimum of r, we have

$$dr = 0 \Rightarrow x\,dx + y\,dy + z\,dz = 0. \qquad ...(4)$$

Differentiating (2), we get

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 4r^3\,dr.$$

But for a maximum or a minimum of r, we have $dr = 0$.

$$\therefore \quad \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0. \qquad ...(5)$$

Also differentiating (3), we get

$$\lambda dx + \mu dy + \nu dz = 0. \qquad ...(6)$$

Multiplying (4) by 1, (5) by λ_1 and (6) by λ_2, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$x + \frac{x}{a^2}\lambda_1 + \lambda\lambda_2 = 0 \qquad ...(7)$$

$$y + \frac{y}{b^2}\lambda_1 + \mu\lambda_2 = 0 \qquad ...(8)$$

$$z + \frac{z}{c^2}\lambda_1 + \nu\lambda_2 = 0. \qquad ...(9)$$

Multiplying (7), (8), (9) by x, y, z respectively and adding, we get

$$r^2 + r^4.\lambda_1 + 0.\lambda_2 = 0 \quad \text{or} \quad \lambda_1 = -1/r^2.$$

$\therefore$ from (7), we have $x - \dfrac{x}{a^2} \cdot \dfrac{1}{r^2} + \lambda\lambda_2 = 0$

or $$x = \frac{a^2 r^2 \lambda \lambda_2}{1 - a^2 r^2}.$$

Similarly from (8) and (9), we have

$$y = \frac{b^2 r^2 \mu \lambda_2}{1 - b^2 r^2} \quad \text{and} \quad z = \frac{c^2 r^2 \nu \lambda_2}{1 - c^2 r^2}.$$

Substituting these values of x, y, z in $\lambda x + \mu y + \nu z = 0$, we get

$$\frac{a^2 r^2 \lambda^2 \lambda_2}{1 - a^2 r^2} + \frac{b^2 r^2 \mu^2 \lambda_2}{1 - b^2 r^2} + \frac{c^2 r^2 \nu^2 \lambda_2}{1 - c^2 r^2} = 0$$

or $$\frac{a^2 \lambda^2}{1 - a^2 r^2} + \frac{b^2 \mu^2}{1 - b^2 r^2} + \frac{c^2 \nu^2}{1 - c^2 r^2} = 0. \quad ...(10)$$

The equation (10) gives the maximum and minimum values of r.

Ex. 17. *Find the points where*

$$u = ax^p + by^q + cz^r$$

has extreme values subject to the condition

$$x^l + y^m + z^n = k.$$

Sol. We have $u = ax^p + by^q + cz^r$, ...(1)

where the variables x, y, z are connected by the relation

$$x^l + y^m + z^n = k. \quad ...(2)$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow apx^{p-1}\,dx + bqy^{q-1}\,dy + crz^{r-1}\,dz = 0. \quad ...(3)$$

Also differentiating (2), we get

$$lx^{l-1}\,dx + my^{m-1}\,dy + nz^{n-1}\,dz = 0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$ap\,x^{p-1} + \lambda l\,x^{l-1} = 0, \quad ...(5)$$

$$bq\,y^{q-1} + \lambda m\,y^{m-1} = 0, \quad ...(6)$$

and $$cr\,z^{r-1} + \lambda n\,z^{n-1} = 0. \quad ...(7)$$

From (5), we have

$$ap\,x^{p-1} = -\lambda l\,x^{l-1} \quad \text{or} \quad ap\,x^{p-l} = -\lambda l$$

or $$\frac{x^{p-l}}{l/pa} = -\lambda.$$

Similarly from (6) and (7), we have

$$\frac{y^{q-m}}{m/qb} = -\lambda \quad \text{and} \quad \frac{z^{r-n}}{n/rc} = -\lambda.$$

Hence the values of x, y, z for which u has extreme values are given by $\dfrac{x^{p-l}}{l/pa} = \dfrac{y^{q-m}}{m/qb} = \dfrac{z^{r-n}}{n/rc}$.

Ex. 18. *If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $u = x^2 + y^2 + xy$ will be the roots of the equation*

$$4(u-a)(u-b) = ab.$$

Sol. We have $\quad u = x^2 + y^2 + xy, \quad$ ...(1)

where the variables x and y are connected by the relation

$$ax^2 + by^2 = ab. \quad \text{...(2)}$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow 2x\,dx + 2y\,dy + y\,dx + x\,dy = 0$$

$$\Rightarrow \quad (2x+y)\,dx + (2y+x)\,dy = 0. \quad \text{...(3)}$$

Also differentiating (2), we have

$$2ax\,dx + 2by\,dy = 0 \text{ or } ax\,dx + by\,dy = 0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx and dy, we get

$$(2x+y) + \lambda ax = 0 \quad \text{...(5)}$$

and

$$(2y+x) + \lambda by = 0. \quad \text{...(6)}$$

Multiplying (5) by x and (6) by y and adding, we get

$$2(x^2+y^2+xy) + \lambda(ax^2+by^2) = 0$$

or

$$2u + \lambda ab = 0 \text{ or } \lambda = -2u/ab.$$

Putting $\lambda = -2u/ab$ in (5), we get

$$(2x+y) - \frac{2u}{b}x = 0$$

or

$$2(b-u)x + by = 0. \quad \text{...(7)}$$

Similarly putting $\lambda = -2u/ab$ in (6), we get

$$(2y+x) - \frac{2u}{a}y = 0$$

or

$$ax + 2(a-u)y = 0. \quad \text{...(8)}$$

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} 2(b-u) & b \\ a & 2(a-u) \end{vmatrix} = 0$$

or $\quad 4(b-u)(a-u) - ab = 0$

or $\quad 4(u-a)(u-b) = ab$, which gives the maximum and minimum values of u.

Ex. 19. *Show that the maximum and minimum values of $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ subject to the conditions $lx + my + nz = 0$ and $x^2 + y^2 + z^2 = 1$ are given by the equation*

$$\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Sol. We have $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \quad$ ...(1)

where the variables x, y and z are connected by the relations

$$lx + my + nz = 0 \qquad ...(2)$$

and $$x^2 + y^2 + z^2 = 1. \qquad ...(3)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow \quad 2(ax + gz + hy)\,dx + 2(by + fz + hx)\,dy + 2(cz + fy + gx)\,dz = 0$

$\Rightarrow \quad (ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0. \quad ...(4)$

Also differentiating (2) and (3), we get

$$l\,dx + m\,dy + n\,dz = 0 \qquad ...(5)$$

and $$x\,dx + y\,dy + z\,dz = 0. \qquad ...(6)$$

Multiplying (4) by 1, (5) by λ and (6) by μ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(ax + hy + gz) + \lambda l + \mu x = 0 \qquad ...(7)$$

$$(hx + by + fz) + \lambda m + \mu y = 0 \qquad ...(8)$$

and $$(gx + fy + cz) + \lambda n + \mu z = 0. \qquad ...(9)$$

Multiplying (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda.0 + \mu.1 = 0 \text{ or } \mu = -u.$$

Putting $\mu = -u$ in (7), (8) and (9), we get

$$(a - u)x + hy + gz + \lambda l = 0, \qquad ...(10)$$

$$hx + (b - u)y + fz + \lambda m = 0, \qquad ...(11)$$

and $$gx + fy + (c - u)z + \lambda n = 0. \qquad ...(12)$$

Also the relation (2) can be written as

$$lx + my + nz + \lambda.0 = 0. \qquad ...(13)$$

Eliminating x, y, z and λ from (10), (11), (12) and (13), we get

$$\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

which gives the maximum and minimum values of u.

Ex. 20. *Prove that if $x + y + z = 1$, $ayz + bzx + cxy$ has an extreme value equal to*

$$\frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Prove also that if a, b, c are all positive and c lies between $a + b - 2\sqrt{(ab)}$ and $a + b + 2\sqrt{(ab)}$ this value is true maximum and that if a, b, c are all negative and c lies between $a + b \pm 2\sqrt{(ab)}$, it is true minimum.

Sol. Let $u = ayz + bzx + cxy, \qquad ...(1)$

where the variables x, y and z are connected by the relation

$$x + y + z = 1. \qquad ...(2)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow \quad (bz + cy)\,dx + (cx + az)\,dy + (ay + bx)\,dz = 0. \qquad ...(3)$

Also differentiating (2), we get

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$bz + cy + \lambda = 0,\; cx + az + \lambda = 0,\; ay + bx + \lambda = 0.$$

$$\therefore \quad -\lambda = bz + cy = cx + az = ay + bx.$$

From these, we have

$$z = \frac{ay + bx - cx}{a} = \frac{ay + bx - cy}{b}.$$

$$\therefore \quad bx\,(a + c - b) = ay\,(b + c - a)$$

or
$$\frac{x}{a\,(b + c - a)} = \frac{y}{b\,(a + c - b)} = \frac{z}{c\,(a + b - c)}, \qquad \text{(by symmetry)}$$

$$= \frac{x + y + z}{2\Sigma bc - \Sigma a^2} = \frac{1}{2\Sigma bc - \Sigma a^2}. \qquad ...(5)$$

Hence u is stationary for the values of x, y and z given by (5).

Also the stationary value of u

$$= ayz + bzx + cxy$$

$$= \frac{abc\,(2bc + 2ca + 2ab - a^2 - b^2 - c^2)}{(2bc + 2ca + 2ab - a^2 - b^2 - c^2)^2}$$

$$= \frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Now let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\dfrac{\partial u}{\partial x} = (bz + cy) + (ay + bx)\dfrac{\partial z}{\partial x}$,

regarding y as a constant.

Also from (2), $1 + (\partial z/\partial x) = 0$ or $\partial z/\partial x = -1$.

$$\therefore \quad \frac{\partial u}{\partial x} = bz + cy - ay - bx,$$

so that $\quad r = \dfrac{\partial^2 u}{\partial x^2} = b\dfrac{\partial z}{\partial x} - b = -b - b = -2b,$

$s = \dfrac{\partial^2 u}{\partial x\,\partial y} = b\dfrac{\partial z}{\partial y} + c - a = c - a - b.$ [$\because$ from (2), $\partial z/\partial y = -1$]

Similarly $t = \dfrac{\partial^2 u}{\partial y^2} = -2a.$

$$\therefore \quad rt - s^2 = 4ab - (c - a - b)^2$$
$$= \{2\sqrt{(ab)} + c - a - b\}\{2\sqrt{(ab)} - c + a + b\}$$
$$= [c - \{a + b - 2\sqrt{(ab)}\}][\{a + b + 2\sqrt{(ab)}\} - c].$$

Hence $rt - s^2$ will be positive when $c > a + b - 2\sqrt{(ab)}$ and $< a + b + 2\sqrt{(ab)}$ whether a, b, c are all +ive or –ive.

But when a, b, c are all +ive, r is −ive and so the stationary value is a true maximum in this case. Also when a, b, c are all −ive, r is +ive and so the stationary value is a true minimum in this case.

Ex. 21. *Find the maximum or minimum value of $x^2 + y^2 + z^2$, subject to the conditions*

$$lx + my + nz = 1, \; l'x + m'y + n'z = 1.$$

Sol. Let $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y and z are connected by the relations

$$lx + my + nz = 1, \quad \text{...(2)}$$

and $$l'x + m'y + n'z = 1. \quad \text{...(3)}$$

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow \quad 2xdx + 2ydy + 2zdz = 0$$

$$\Rightarrow \quad xdx + ydy + zdz = 0. \quad \text{...(4)}$$

Also differentiating (2) and (3), we get

$$ldx + mdy + ndz = 0 \quad \text{...(5)}$$

and $$l'dx + m'dy + n'dz = 0. \quad \text{...(6)}$$

Multiplying (4) by 1, (5) by λ and (6) μ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$x + l\lambda + l'\mu = 0, \quad \text{...(7)}$$

$$y + m\lambda + m'\mu = 0, \quad \text{...(8)}$$

and $$z + n\lambda + n'\mu = 0. \quad \text{...(9)}$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda .1 + \mu .1 = 0. \quad \text{...(10)}$$

Again multiplying the equations (7), (8) and (9) by l, m and n respectively and adding, we get

$$1 + \lambda \Sigma l^2 + \mu \Sigma ll' = 0. \quad \text{...(11)}$$

Next multiplying the equations (7), (8) and (9) by l', m' and n respectively and adding, we get

$$1 + \lambda\, \Sigma ll' + \mu\, \Sigma l'^2 = 0. \quad \text{...(12)}$$

Now eliminating λ and μ from (10), (11) and (12), we get

$$\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma l^2 & \Sigma ll' \\ 1 & \Sigma ll' & \Sigma l'^2 \end{vmatrix} = 0.$$

The above equation gives the maximum or minimum value of u.

Note. If we wish to find an explicit expression for the extreme value of u and also wish to say whether it is maximum or minimum we proceed as follows :

Solving the equations (11) and (12) for λ and μ, we get

$$\frac{\lambda}{\Sigma ll' - \Sigma l'^2} = \frac{\mu}{\Sigma ll' - \Sigma l^2} = \frac{\ }{\Sigma l^2 \Sigma l'^2 - (\Sigma ll')^2}$$

or $$\lambda = \frac{\Sigma ll' - \Sigma l'^2}{\Sigma\,(mn' - m'n)^2} \quad \text{and} \quad \mu = \frac{\Sigma ll' - \Sigma l^2}{\Sigma\,(mn' - m'n)^2}$$

[$\because \Sigma l^2\, \Sigma l'^2 - (\Sigma ll')^2 = \Sigma\,(mn' - m'n)^2$, by Lagrange's identity]

Putting these values of λ and μ in (10), the maximum or minimum value of u is given by

$$u = -\lambda - \mu = \frac{\Sigma l^2 + \Sigma l'^2 - 2\Sigma ll'}{\Sigma\,(mn' - m'n)^2} = \frac{\Sigma\,(l - l')^2}{\Sigma\,(mn' - m'n)^2}.$$

To find the nature of this stationary value of u.

Since there are two relations amongst the variables x, y and z, therefore only one variable will be independent. Let it be x. Then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx}$$

$$= 2x + 2y \cdot \frac{dy}{dx} + 2z \cdot \frac{dz}{dx}.$$

Now differentiating (2) and (3) w.r.t. x, we get

$$l + m\frac{dy}{dx} + n\frac{dz}{dx} = 0$$

and $$l' + m'\frac{dy}{dx} + n'\frac{dz}{dx} = 0.$$

Solving these, we get

$$\frac{dy/dx}{nl' - n'l} = \frac{dz/dx}{m'l - l'm} = \frac{1}{mn' - m'n}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{nl' - n'l}{mn' - m'n} \text{ and } \frac{dz}{dx} = \frac{m'l - l'm}{mn' - m'n}.$$

$$\therefore \quad \frac{du}{dx} = 2x + 2y \cdot \frac{nl' - n'l}{mn' - m'n} + 2z \cdot \frac{m'l - l'm}{mn' - m'n}$$

and so $$\frac{d^2u}{dx^2} = 2 + 2\frac{dy}{dx} \cdot \frac{nl' - n'l}{mn' - m'n} + 2\frac{dz}{dx} \cdot \frac{m'l - l'm}{mn' - m'n}$$

$$= 2 + 2\left(\frac{nl' - n'l}{mn' - m'n}\right)^2 + 2\left(\frac{m'l - l'm}{mn' - m'n}\right)^2,$$

which is +ive.

[illegible]e the stationary value of u found above is the minimum value.

— Ex. 22. *Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.*

Sol. Let x, y, z be the dimensions of the rectangular parallelopiped, V be its volume and S be its surface. Then

$$S = 2xy + 2yz + 2zx \qquad \text{...(1)}$$

and $$xyz = V = \text{some constant}. \qquad \text{...(2)}$$

For a maximum or minimum of S, we have

$$dS = 2\,(y + z)\,dx + 2\,(z + x)\,dy + 2\,(x + y)\,dz = 0$$

i.e., $$(y+z)\,dx+(z+x)\,dy+(x+y)\,dz=0. \qquad ...(3)$$

Also differentiating (2) and observing that V is constant, we have

$$yz\,dx+zx\,dy+xy\,dz=0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y+z)+\lambda yz=0, \qquad ...(5)$$

$$(z+x)+\lambda zx=0, \qquad ...(6)$$

and $$(x+y)+\lambda xy=0. \qquad ..(7)$$

These give $-\lambda=\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{z}+\dfrac{1}{x}=\dfrac{1}{x}+\dfrac{1}{y}\cdot$

$\therefore \quad \dfrac{1}{y}-\dfrac{1}{x}=0 \quad$ or $\quad x=y.$

Similarly $y=z$.

Hence for a stationary value of S, we have

$$x=y=z=V^{1/3}, \text{ from (2).}$$

Thus S is stationary when the rectangular parallelopiped is a cube.

Let us now find the nature of this stationary value of S.

Here S is a function of three variables x, y, z which are connected by the relation (2) so that only two variables are independent. Let us regard x and y as independent variables and z to be dependent on them.

Then from (1), $\dfrac{\partial S}{\partial x}=2y+2y\dfrac{\partial z}{\partial x}+2z+2x\dfrac{\partial z}{\partial x}\cdot$

Also from (2), $yz+xy\dfrac{\partial z}{\partial x}=0$ *i.e.*, $\dfrac{\partial z}{\partial x}=-\dfrac{z}{x}\cdot$

$$\therefore \quad \frac{\partial S}{\partial x}=2y-\frac{2yz}{x}+2z-2z=2y-\frac{2yz}{x}$$

and $$\frac{\partial^2 S}{\partial x^2}=\frac{2yz}{x^2}-\frac{2y}{x}\cdot\frac{\partial z}{\partial x}=\frac{2yz}{x^2}+\frac{2yz}{x^2}=\frac{4yz}{x^2}=4 \text{ at } x=y=z.$$

Similarly by symmetry $\dfrac{\partial^2 S}{\partial y^2}=4$ at $x=y=z$.

Also $\dfrac{\partial^2 S}{\partial x\,\partial y}=2-\dfrac{2z}{x}-\dfrac{2y}{x}\dfrac{\partial z}{\partial y}\cdot$

But differentiating (2) partially w.r.t. y taking x as constant, we get

$$xz+xy\frac{\partial z}{\partial y}=0 \quad \text{or} \quad \frac{\partial z}{\partial y}=-\frac{z}{y}\cdot$$

$$\therefore \quad \frac{\partial^2 S}{\partial x\,\partial y}=2-\frac{2z}{x}-\frac{2y}{x}\left(-\frac{z}{y}\right)=2-\frac{2z}{x}+\frac{2z}{x}=2.$$

Thus at the stationary point $x=y=z=V^{1/3}$, we have

$$r=\frac{\partial^2 S}{\partial x^2}=4,\ s=\frac{\partial^2 S}{\partial x\,\partial y}=2 \text{ and } t=\frac{\partial^2 S}{\partial y^2}=4.$$

$\therefore \quad rt - s^2 = 4 \times 4 - 2^2 = 12$ which is > 0.

Also $r = 4$ which is > 0.

Hence the stationary value of S given by $x = y = z = V^{1/3}$ is a minimum.

Thus of all rectangular paralleopipeds of the same volume, the cube has the least surface.

Ex. 23. *Discuss the maxima and minima of the function*

$$u = \sin x \sin y \sin z,$$

where x, y, z are the angles of a triangle.

Sol. We have $u = \sin x \sin y \sin z$, ...(1)

where $\qquad x + y + z = \pi$. ...(2)

For a maximum or a minimum of u, we must have

$$du = \cos x \sin y \sin z \, dx + \sin x \cos y \sin z \, dy + \sin x \sin y \cos z \, dz = 0 \quad ....(3)$$

Also form (2), we have

$$dx + dy + dz = 0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\cos x \sin y \sin z + \lambda = 0,$$

$$\sin x \cos y \sin z + \lambda = 0,$$

and $\qquad \sin x \sin y \cos z + \lambda = 0.$

From these, we get

$$-\lambda = \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$$

or $\quad \cot x = \cot y = \cot z \qquad$ (dividing by $\sin x \sin y \sin z$)

i.e., $\quad x = y = z = \pi/3, \qquad$ from (2).

Thus u is stationary when $x = y = z = \pi/3$.

Let us now find the nature of this stationary value of u.

Since variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent and z to be dependent on them by the relation (2).

Then from (1),

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \frac{\partial z}{\partial x}.$$

Also form (2),

$$1 + \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1.$$

$$\therefore \quad \frac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z$$

and $$\frac{\partial^2 u}{\partial x^2} = -\sin y \sin z \sin x + \sin y \cos x \cos z \frac{\partial z}{\partial x}$$

$$- \cos x \sin y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial x}$$

$$= -2 \sin x \sin y \sin z - 2 \sin y \cos x \cos z.$$

Also $\dfrac{\partial^2 u}{\partial x \, \partial y} = \cos y \sin z \cos x + \sin y \cos x \cos z \dfrac{\partial z}{\partial y}$

$$- \sin x \cos y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial y}$$

$$= \cos y \sin z \cos x - \sin y \cos x \cos z$$

$$- \sin x \cos y \cos z - \sin x \sin y \sin z.$$

Hence putting $x = y = z = \pi/3$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= -\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = -\sqrt{3},$$

$$s = \frac{\partial^2 u}{\partial x \, \partial y} = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} = -\frac{\sqrt{3}}{2}.$$

Also by symmetry, $t = \dfrac{\partial^2 u}{\partial y^2} = -\sqrt{3}$.

$\therefore$ at the stationary point $x = y = z = \pi/3$, we have

$rt - s^2 = 3 - (3/4) = 9/4$ which is > 0

and $r = -\sqrt{3}$ which is < 0.

$\therefore$ at the stationary point $x = y = z = \pi/3$, u is maximum.

Hence u is maximum when $x = y = z = \pi/3$ and the maximum value of $u = \left(\dfrac{\sqrt{3}}{2}\right)^3 = \dfrac{3\sqrt{3}}{8}$.

Ex. 24. *In a plane triangle ABC, find the maximum value of $u = \cos A \cos B \cos C$.* **(Meerut 1994P, 95BP)**

Sol. We have $\quad u = \cos A \cos B \cos C$, ...(1)

where the variables A, B and C are connected by the relation

$$A + B + C = \pi. \quad \text{...(2)}$$

From (1), $\log u = \log \cos A + \log \cos B + \log \cos C$.

$$\therefore \quad \frac{1}{u} du = -\tan A \, dA - \tan B \, dB - \tan C \, dC.$$

For a maximum or a minimum of u, we must have $du = 0$

$$\Rightarrow \quad \tan A \, dA + \tan B \, dB + \tan C \, dC = 0. \quad \text{...(3)}$$

Also differentiating (2), we get

$$dA + dB + dC = 0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dA, dB and dC, we get

$$\tan A + \lambda = 0, \ \tan B + \lambda = 0, \ \tan C + \lambda = 0$$

or $\quad -\lambda = \tan A = \tan B = \tan C$

or $\quad A = B = C = \pi/3$, from (2).

Thus u is stationary when $A = B = C = \pi/3$ *i.e.*, the triangle is equilateral.

Now to show that the stationary value of u given by

$$A = B = C = \pi/3 \text{ is maximum.}$$

Let us regard A and B as independent variables and C as a function of A and B given by (2).

From (1), $\log u = \log\cos A + \log\cos B + \log\cos C$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = -\tan A - \tan C \cdot \frac{\partial C}{\partial A}.$$

Also differentiating (2) partially w.r.t. A taking B as constant, we get

$$1 + (\partial C/\partial A) = 0 \text{ or } \partial C/\partial A = -1.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = -\tan A + \tan C,$$

so that $\dfrac{1}{u}\dfrac{\partial^2 u}{\partial^2 A} - \dfrac{1}{u^2}\left(\dfrac{\partial u}{\partial A}\right)^2 = -\sec^2 A + \sec^2 C \cdot \dfrac{\partial C}{\partial A}$

$$= -(\sec^2 A + \sec^2 C).$$

But at the stationary point $\partial u/\partial A = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial A^2} = -u(\sec^2 A + \sec^2 C)$$

$$= -\cos A \cos B \cos C (\sec^2 A + \sec^2 A),$$

which is –ive for $A = B = C = \pi/3$.

Hence u is maximum when $A = B = C = \pi/3$ and the maximum value of $u = \left(\cos\dfrac{\pi}{3}\right)^3 = \left(\dfrac{1}{2}\right)^3 = 1/8$.

Ex. 25. *Find a plane triangle ABC such that*

$$u = \sin^m A \sin^n B \sin^p C$$

has maximum value.

Sol. We have $u = \sin^m A \sin^n B \sin^p C$, ...(1)

where the variables A, B and C are connected by the relation

$$A + B + C = \pi. \qquad \text{...(2)}$$

From (1), $\log u = m\log\sin A + n\log\sin B + p\log\sin C$.

$$\therefore \quad \frac{1}{u}\,du = m\cot A\,dA + n\cot B\,dB + p\cot C\,dC.$$

For a maximum or a minimum of u, we must have $du = 0$

$$\Rightarrow \quad m\cot A\,dA + n\cot B\,dB + p\cot C\,dC = 0. \qquad \text{...(3)}$$

Also differentiating (2), we get

$$dA + dB + dC = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dA, dB and dC, we get

$$m \cot A + \lambda = 0, n \cot B + \lambda = 0, p \cot C + \lambda = 0.$$

$$\therefore \quad -\lambda = m \cot A = n \cot B = p \cot C.$$

Hence u is stationary when A, B, C are given by

$$m \cot A = n \cot B = p \cot C$$

or
$$\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}.$$

Now to show that the above stationary value of u is maximum.

Let us regard A and B as independent variables and C as a function of A and B given by (2).

From (1), $\log u = m \log \sin A + n \log \sin B + p \log \sin C$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = m \cot A + p \cot C \cdot \frac{\partial C}{\partial A}.$$

But from (2), $1 + (\partial C/\partial A) = 0$ or $\partial C/\partial A = -1$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = m \cot A - p \cot C,$$

so that
$$\frac{1}{u}\frac{\partial^2 u}{\partial A^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial A}\right)^2 = -m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C \cdot \frac{\partial C}{\partial A}$$

$$= -(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C).$$

But at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial A^2} = -u\,(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C)$$

$$= -\sin^m A \sin^n B \sin^p C\,(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C),$$

which is obviously –ive if A, B, C are the angles of a triangle.

Hence u is maximum when A, B, C are given by

$$\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}.$$

Ex. 26. *Show that if the perimeter of a triangle is constant, its area is a maximum when it is equilateral.*

Sol. Let a, b, c denote the sides of a triangle, $2s$ its constant perimeter and u its area.

Then
$$u^2 = s(s-a)(s-b)(s-c), \qquad \text{...(1)}$$

where the variables a, b, c are connected by the relation

$$a + b + c = 2s. \qquad \text{...(2)}$$

From (1), $2 \log u = \log s + \log(s-a) + \log(s-b) + \log(s-c)$.

$$\therefore \quad \frac{2}{u}\,du = -\frac{1}{s-a}\,da - \frac{1}{s-b}\,db - \frac{1}{s-c}\,dc.$$

For a maximum or a minimum of u, we must have $du = 0$

$$\Rightarrow \quad \frac{da}{s-a} + \frac{db}{s-b} + \frac{dc}{s-c} = 0. \qquad \text{...(3)}$$

Also differentiating (2), we have

$$da + db + dc = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of da, db and dc, we get

$$\frac{1}{s-a}+\lambda=0, \frac{1}{s-b}+\lambda=0, \frac{1}{s-c}+\lambda=0.$$

$$\therefore \quad -\lambda=\frac{1}{s-a}=\frac{1}{s-b}=\frac{1}{s-c}$$

or $\qquad s-a=s-b=s-c$ or $a=b=c$.

Hence u is stationary when $a=b=c$ *i.e.*, the triangle is equilateral.

Now to show that the stationary value of u given by $a=b=c$ is maximum.

Let us regard a and b as independent variables and c as a function of a and b given by (2).

From (1), differentiating logarithmically, we have

$$\frac{2}{u}\frac{\partial u}{\partial a}=-\frac{1}{s-a}-\frac{1}{s-c}\frac{\partial c}{\partial a}.$$

But from (2), $1+(\partial c/\partial a)=0$ or $\partial c/\partial a=-1$.

$$\therefore \quad \frac{2}{u}\frac{\partial u}{\partial a}=-\frac{1}{s-a}+\frac{1}{s-c},$$

so that $$\frac{2}{u}\frac{\partial^2 u}{\partial a^2}-\frac{2}{u^2}\left(\frac{\partial u}{\partial a}\right)^2=-\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\cdot\frac{\partial c}{\partial a}$$

$$=-\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right].$$

But at the stationary point, we have $\partial u/\partial a=0$.

$\therefore$ at the stationary point found above, we have

$$\frac{2}{u}\frac{\partial^2 u}{\partial a^2}=-\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right]$$

or $$\frac{\partial^2 u}{\partial a^2}=-\frac{u}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right],$$ which is –ive.

Hence u is maximum when $a=b=c$ *i.e.*, the area of the triangle is maximum when it is equilateral.

Ex. 27. *Find the triangle of maximum area inscribed in a circle.*

Sol. Let x, y, z denote the angles of a triangle inscribed in a given circle of radius k. If u be the area of the triangle, then

$$u=\tfrac{1}{2}k^2(\sin 2x+\sin 2y+\sin 2z), \qquad \ldots(1)$$

where the variables x, y, z are connected by the relation

$$x+y+z=\pi. \qquad \ldots(2)$$

From (1), $du=k^2(\cos 2x\, dx+\cos 2y\, dy+\cos 2z\, dz)$.

For a maximum or a minimum of u, we must have $du=0$

$$\therefore \quad \cos 2x\, dx+\cos 2y\, dy+\cos 2z\, dz=0. \qquad \ldots(3)$$

Also differentiating (2), we have

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$\cos 2x + \lambda = 0, \cos 2y + \lambda = 0, \cos 2z + \lambda = 0$$

or $\quad -\lambda = \cos 2x = \cos 2y = \cos 2z$

or $\quad 1 - 2\sin^2 x = 1 - 2\sin^2 y = 1 - 2\sin^2 z$

or $\quad \sin^2 x = \sin^2 y = \sin^2 z$

or $\quad \sin x = \sin y = \sin z$

or $\quad x = y = z = \pi/3$, from (2).

Thus u is stationary when $x = y = z = \pi/3$ *i.e.*, the triangle is equilateral.

Now to show that the stationary value of u given by $x = y = z = \pi/3$ is maximum.

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\dfrac{\partial u}{\partial x} = k^2 \left(\cos 2x + \cos 2z \dfrac{\partial z}{\partial x}\right)$ ·

Also differentiating (2) partially w.r.t. x taking y as a constant, we get

$$1 + (\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -1.$$

$$\therefore \quad \frac{\partial u}{\partial x} = k^2 (\cos 2x - \cos 2z),$$

so that $\quad \dfrac{\partial^2 u}{\partial x^2} = k^2 \left[-2\sin 2x + 2\sin 2z \cdot \dfrac{\partial z}{\partial x}\right]$

$$= -2k^2 (\sin 2x + \sin 2z).$$

Also $\dfrac{\partial^2 u}{\partial x\, \partial y} = k^2 \left(2\sin 2z \cdot \dfrac{\partial z}{\partial y}\right) = -2k^2 \sin 2z.$

Hence putting $x = y = z = \pi/3$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2k^2 \left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = -2k^2\sqrt{3},$$

$$s = \frac{\partial^2 u}{\partial x\, \partial y} = -2k^2 \cdot \frac{\sqrt{3}}{2} = -k^2\sqrt{3}.$$

Also by symmetry, $t = \dfrac{\partial^2 u}{\partial y^2} = -2k^2\sqrt{3}$.

$\therefore \quad$ at the stationary point $x = y = z = \pi/3$, we have

$\quad rt - s^2 = 12k^4 - 3k^4 = 9k^4$ which is > 0

and $\quad r = -2k^2\sqrt{3}$ which is –ive.

$\therefore \quad$ at the stationary point $x = y = z = \pi/3$, u is maximum.

Hence the triangle of maximum area inscribed in a circle is equilateral.

Ex. 28. *Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $8abc/3\sqrt{3}$. **(Meerut 1995)**

or

Find the maximum value of u :

$$u=8xyz, \text{ given } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(Meerut 1995)

Sol. Let (x, y, z) denote the coordinates of the vertex of the rectangular parallelopiped which lies in the positive octant and let V denote its volume. Then, we have to find the maximum value of

$$V=8xyz \qquad ...(1)$$

subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad ...(2)$$

For a maximum or a minimum of V, we have

$$dV=8yz\,dx+8zx\,dy+8xy\,dz=0$$

i.e.,

$$yz\,dx+zx\,dy+xy\,dz=0. \qquad ...(3)$$

Also differentiating (2), we get

$$\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$$

i.e.,

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$yz+\frac{\lambda x}{a^2}=0, zx+\frac{\lambda y}{b^2}=0 \text{ and } xy+\frac{\lambda z}{c^2}=0.$$

From these, we get

$$\frac{x}{a^2}=-\frac{yz}{\lambda}, \frac{y}{b^2}=-\frac{zx}{\lambda}, \frac{z}{c^2}=-\frac{xy}{\lambda}$$

or

$$\frac{x^2}{a^2}=-\frac{xyz}{\lambda}, \frac{y^2}{b^2}=-\frac{xyz}{\lambda}, \frac{z^2}{c^2}=-\frac{xyz}{\lambda}$$

or

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{x^2/a^2+y^2/b^2+z^2/c^2}{3}=\frac{1}{3}, \text{ using (2)}$$

or

$$x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}.$$

Thus V is stationary when $x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial V}{\partial x} = 8yz + 8xy \dfrac{\partial z}{\partial x}$.

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2 z}.$$

$$\therefore \quad \frac{\partial V}{\partial x} = 8yz + 8xy \cdot \left(-\frac{c^2 x}{a^2 z}\right) = 8yz - \frac{8c^2x^2y}{a^2 z}$$

and so $$\frac{\partial^2 V}{\partial x^2} = 8y\frac{\partial z}{\partial x} - \frac{16c^2xy}{a^2 z} + \frac{8c^2x^2y}{a^2z^2}\cdot\frac{\partial z}{\partial x}$$

$$= 8y \cdot \left(\frac{-c^2x}{a^2z}\right) - \frac{16c^2xy}{a^2z} - \frac{8c^2x^2y}{a^2z}\cdot\frac{c^2x}{a^2z},$$

which is –ive when $x = a/\sqrt{3}, y = b/\sqrt{3}, z = c/\sqrt{3}$.

Hence V is maximum when $x = a/\sqrt{3}, y = b/\sqrt{3}, z = c/\sqrt{3}$ and the maximum value of $V = \dfrac{8abc}{3\sqrt{3}}$.

Note. In complicated problems to show that whether the stationary value of a function is maximum or minimum, it will be sufficient to see whether the second partial differential coefficient of the function w.r.t. any of the independent variables is negative or positive.

Ex. 29. *Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.*

Sol. Referred to the centre as origin, let the equation of the sphere be $x^2 + y^2 + z^2 = a^2$.

Let (x, y, z) denote the coordinates of that vertex of the rectangular parallelopiped inscribed in the sphere which lies in the positive octant and let V denote the volume of the rectangular parallelopiped. Then, we have to find the maximum value of

$$V = 8xyz \qquad \text{...(1)}$$

subject to the condition

$$x^2 + y^2 + z^2 = a^2. \qquad \text{...(2)}$$

From (1), $\log V = \log 8 + \log x + \log y + \log z$.

$$\therefore \quad \frac{1}{V}dV = \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz.$$

For a maximum or a minimum of V, we must have $dV = 0$

$$\Rightarrow \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0. \qquad \text{...(3)}$$

Also differentiating (2), we have

$$x\,dx + y\,dy + z\,dz = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\frac{1}{x} + \lambda x = 0, \frac{1}{y} + \lambda y = 0, \frac{1}{z} + \lambda z = 0$$

or $\quad - 1/\lambda = x^2 = y^2 = z^2 \quad$ or $\quad x = y = z.$

Thus V is stationary when $x = y = z = a/\sqrt{3}$, from (2).

The lengths of the edges of the rectangular parallelopiped are $2x, 2y, 2z$. So V is stationary when the rectangular parallelopiped is a cube.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\log V = \log 8 + \log x + \log y + \log z$.

$$\therefore \quad \frac{1}{V}\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{z} \cdot \frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$2x + 2z\,(\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -x/z.$$

$$\therefore \quad \frac{1}{V}\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{z} \cdot \frac{-x}{z} = \frac{1}{x} - \frac{x}{z^2},$$

so that $$\frac{1}{V}\frac{\partial^2 V}{\partial x^2} - \frac{1}{V^2}\left(\frac{\partial V}{\partial x}\right)^2 = -\frac{1}{x^2} - \frac{1}{z^2} + \frac{2x}{z^3}\frac{\partial z}{\partial x} = -\frac{1}{x^2} - \frac{1}{z^2} - \frac{2x^2}{z^4}.$$

But at the stationary point, we have $\partial V/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 V}{\partial x^2} = -V\left[\frac{1}{x^2} + \frac{1}{z^2} + \frac{2x^2}{z^4}\right] = -8xyz\left[\frac{1}{x^2} + \frac{1}{z^2} + \frac{2x^2}{z^4}\right],$$

which is –ive when $x = y = z = a/\sqrt{3}$.

Thus V is maximum when $x = y = z = a/\sqrt{3}$.

Hence the rectangular solid of maximum volume inscribed in a sphere is a cube.

Ex. 30. *A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Sol. Let the given capacity of the box be V, its three edges be x, y, z and its surface be S. Then

$$S = xy + 2yz + 2zx \qquad \text{...(1)}$$

and $$xyz = V. \qquad \text{...(2)}$$

For a maximum or a minimum of S, we have

$$dS = (y + 2z)\,dx + (x + 2z)\,dy + 2\,(x + y)\,dz = 0. \qquad \text{...(3)}$$

Also from (2), since V is constant, we have

$$yz\,dx + zx\,dy + xy\,dz = 0. \qquad \text{..(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$(y + 2z) + \lambda yz = 0, \qquad \text{...(5)}$$

$$(x + 2z) + \lambda zx = 0, \qquad \text{...(6)}$$

and $$2x + 2y + \lambda xy = 0. \qquad ...(7)$$

Multiplying (5) by x, (6) by y and subtracting, we get

$$2zx - 2zy = 0 \text{ or } 2z(x-y) = 0, \text{ or } x = y.$$

[The root $z = 0$ is inadmissible becuase the depth of the box cannot be zero.]

Similarly, from the equations (6) and (7), we get $y = 2z$.

Hence the dimensions of the box for a stationary value of S are

$$x = y = 2z = (2V)^{1/3}, \text{ from (2).}$$

Let us now find the nature of this stationary value of S.

Regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial S}{\partial x} = y + 2y\dfrac{\partial z}{\partial x} + 2z + 2x\dfrac{\partial z}{\partial x}\cdot$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$yz + xy\frac{\partial z}{\partial x} = 0 \text{ i.e., } \frac{\partial z}{\partial x} = -\frac{z}{x}\cdot$$

$$\therefore \quad \frac{\partial S}{\partial x} = y - \frac{2yz}{x} + 2z - 2z = y - \frac{2yz}{x}$$

and so $$\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 2 \text{ at } x = y = 2z.$$

Thus at the stationary point $x = y = 2z = (2V)^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 2, \text{ which is positive.}$$

Similarly we can find $s = \dfrac{\partial^2 S}{\partial x\,\partial y}$ and $t = \dfrac{\partial^2 S}{\partial y^2}$ at the stationary point $x = y = 2z = (2V)^{1/3}$ and can show that $rt - s^2$ is positive.

Since at the stationary point $x = y = 2z = (2V)^{1/3}$, $rt - s^2 > 0$ and $r > 0$, therefore the stationary value of S at this point is a minimum.

Hence the dimensions of the box requiring least meterial for its construction are given by $x = y = 2z = (2V)^{1/3}$.

□

4

Envelopes and Evolutes

§ 1. One Parameter family of curves.

An equation of the form

$$F(x, y, \alpha) = 0 \qquad ...(1)$$

in which α is a constant, represents a curve. If α is a parameter *i.e.*, if α can take all real values, then (1) is the equation of a *one parameter family of curves with parameter* α. If we give different values to α we get different members of this family. On any particular curve belonging to this family the value of α is constant but it changes from one curve to another.

An equation of the type $F(x, y, \alpha, \beta) = 0$ also defines a family of curves but in this case we have two parameters α and β.

§ 2. Envelope of a one parameter family of curves.

Definition. Let $F(x, y, \alpha) = 0$ be a family of curves, the parameter being α. Suppose P is a point of intersection of two members $F(x, y, \alpha) = 0$ and $F(x, y, \alpha + \delta\alpha) = 0$ of this family corresponding to the parameter values α and $\alpha + \delta\alpha$. As $\delta\alpha \to 0$, let P tend to a definite point Q on the member α. The locus of Q (for varying values of α) is called the envelope of the family.

Thus the envelope of a one parameter family of curves is the locus of the limiting positions of the points of intersection of any two members of the family when one of them tends to coincide with the other which is kept fixed. **(Allahabad 1977; Meerut 84 R)**

§ 3. Method of finding the Envelope.

(Kanpur 1980; Allahabad 77; Meerut 84 R)

Suppose $\quad F(x, y, \alpha) = 0 \qquad ...(1)$

is the equation of a family of curves with parameter α.

Consider the two members

$$F(x, y, \alpha) = 0 \text{ and } F(x, y, \alpha + \delta\alpha) = 0 \qquad ...(2)$$

of this family corresponding to the parameter values α and $\alpha + \delta\alpha$.

The co-ordinates of the points of intersection of the curves (2) satisfy the equations

$$F(x, y, \alpha) = 0, \; F(x, y, \alpha + \delta\alpha) - F(x, y, \alpha) = 0$$

and therefore the equations

$$F(x,y,\alpha)=0,\ \frac{F(x,y,\alpha+\delta\alpha)-F(x,y,\alpha)}{\delta\alpha}=0.$$

Taking limits as $\delta\alpha\to 0$, we see the co-ordinates of the limiting positions of the points of intersection of the curves (2) satisfy the equations

$$F(x,y,\alpha)=0 \text{ and } \frac{\partial F(x,y,\alpha)}{\partial\alpha}=0. \qquad ...(3)$$

Thus, for all values of α, the co-ordinates of the points on the envelope satisfy the equations (3). Therefore eliminating α between the equations (3), we shall get the envelope of the family of curves (1).

Working Rule. *The equation of the envelope of the family of curves $F(x,y,\alpha)=0$ where α is the parameter, is obtained by eliminating α between the equations*

$$\mathbf{F(x,y,\alpha)=0}$$

and
$$\frac{\partial \mathbf{F(x,y,\alpha)}}{\partial\alpha}=\mathbf{0}.$$ **(Meerut 1984 R)**

Here $\dfrac{\partial F(x,y,\alpha)}{\partial\alpha}$ is the partial derivative of $F(x,y,\alpha)$ with respect to the parameter α while x and y have been regarded as constants.

Note. The equations $x=\phi(\alpha),\ y=\psi(\alpha)$ obtained on solving $F(x,y,\alpha)=0$ and $\dfrac{\partial F(x,y,\alpha)}{\partial\alpha}=0$ are the parametric equations of the envelope, α being the parameter.

§ 4. Envelope in case the equation of the family of curves is a quadratic in the parameter. **(Agra 1973)**

Let the equation of the family of curves be $F(x,y,\alpha)=0$, the parameter being α. Suppose this equation can be arranged as a quadratic in α. Let this quadratic be

$$A\alpha^2+B\alpha+C=0, \qquad ...(1)$$

where A, B and C are some functions of x and y.

Differentiating (1) partially with respect to α, we get

$$2A\alpha+B=0. \qquad ...(2)$$

Eliminating α between (1) and (2), we get the envelope.

From (2), we have $\alpha=-B/2A$.

Substituting this value of α in (1), we get

$$A(-B/2A)^2+B(-B/2A)+C=0$$

or
$$B^2-4AC=0,$$

which is the required equation of the envelope.

Remember. *The envelope of the family of curves*

$$A\alpha^2+B\alpha+C=0,$$

where A, B, C are functions of x and y, is

$$\mathbf{B^2-4AC=0}.$$

§ 5. Geometrical significance of the envelope.

In general the envelope of a family of curves touches each member of the family. **(Allahabad 1977)**

Let the equation of the family of curves be

$$F(x, y, \alpha) = 0, \ \alpha \text{ being the parameter.} \quad \ldots(1)$$

The envelope of (1) is obtained by eliminating α between (1) and

$$\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0. \quad \ldots(2)$$

Obviously we can take (1) as the equation of the envelope provided we regard α as a function of x and y given by (2).

Let (x, y) be a point common to the member 'α' of the family and the envelope. If at the point (x, y) we do not have $\partial F/\partial x = 0 = \partial F/\partial y$, then at this point the slope of the tangent to the member 'α' of the family is

$$-\frac{\partial F/\partial x}{\partial F/\partial y}. \quad \ldots(3)$$

[Refer the chapter on partial differentiation § 4]

Also the slope of the tangent to the envelope at the point (x, y) is

$$-\left\{\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial x}\right\} \Big/ \left\{\frac{\partial F}{\partial y} + \frac{\partial F}{\partial \alpha}\frac{\partial \alpha}{\partial y}\right\}. \quad \ldots(4)$$

Note that $F(x, y, \alpha) = 0$ is also the equation of the envelope provided α is not a constant but is a function of x and y given by $\partial F/\partial \alpha = 0$.

Since at every point of the envelope $\partial F/\partial \alpha = 0$, therefore the two slopes given by (3) and (4) are the same.

Hence the slopes of the tangents to the member of the family and the envelope at the common points are equal. This means that the curves of the family and the envelope have the same tangent at the points in common *i.e.*, they touch each other at these points.

Each point on the envelope is a point on some curve of the family and each curve of the family has some point which is on the envelope. At these common points both touch each other. Hence, *in general, the envelope of a family of curves touches each curve of the family and at each point is touched by some member of the family.*

Note. If $\partial F/\partial x$ and $\partial F/\partial y$ are both zero for any point on the curve, the slopes of the tangents cannot be found from (3) and (4). In this case the above argument breaks down and the envelope may not touch a curve at the points where $\partial F/\partial x = 0 = \partial F/\partial y$ *i.e.*, at the singular points.

If the given family of curves is a family of straight lines or a family of conics we have no singular points. Hence the envelope of a family of straight lines or of conics touches each member of the family at all their common points without exception.

Solved Examples

Ex. 1. *Find the envelope of the family of straight lines $y = mx + a/m$, the parameter being m.*

Sol. The equation of the given family of straight lines is

$$y = mx + (a/m), \text{ the parameter being } m. \quad ...(1)$$

Differentiating (1) partially with respect to m, we get

$$0 = x - (a/m^2) \text{ or } m = (a/x)^{1/2}. \quad ...(2)$$

Eliminating m between (1) and (2), we get the required envelope.

Putting $m = (a/x)^{1/2}$ in (1), we get

$$y = x \cdot \frac{a^{1/2}}{x^{1/2}} + a \cdot \frac{x^{1/2}}{a^{1/2}} = 2a^{1/2} . x^{1/2}.$$

Squaring, we get $y^2 = 4ax$, which is the required envelope.

Ex. 2. *Find the envelope of the straight lines*

$$(x/a) \cos\theta + (y/b) \sin\theta = 1,$$

the parameter being θ and interpret the result geometrically.

Sol. The equation of the given family of straight lines is

$$(x/a) \cos\theta + (y/b) \sin\theta = 1, \quad ...(1)$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-(x/a) \sin\theta + (y/b) \cos\theta = 0. \quad ...(2)$$

Eliminating θ between (1) and (2), we get the envelope of the family of straight lines (1). So squaring and adding (1) and (2), we get

$$(x^2/a^2)(\cos^2\theta + \sin^2\theta) + (y^2/b^2)(\sin^2\theta + \cos^2\theta) = 1$$

or

$$x^2/a^2 + y^2/b^2 = 1, \quad ...(3)$$

which is the required envelope of the family of straight lines (1).

Geometrical interpretation. The equation (3) represents an ellipse whose centre is origin. Whatever may be the value of θ, the straight line (1) always touches the ellipse (3) and the ellipse (3) is also touched at each point by some straight line belonging to the family (1).

Ex. 3 (a). *Find the envelope of the family of curves $(a^2/x) \cos\theta - (b^2/y) \sin\theta = c^2/a$, θ being the parameter.*

Sol. The equation of the given family of curves is

$$(a^2/x) \cos\theta - (b^2/y) \sin\theta = c^2/a, \quad ...(1)$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-(a^2/x) \sin\theta - (b^2/y) \cos\theta = 0. \quad ...(2)$$

Squaring and adding (1) and (2), we get

$$\frac{a^4}{x^2} + \frac{b^4}{y^2} = \frac{c^4}{a^2},$$

which is the required envelope of the family of curves (1).

Ex. 3(b). *Find the envelope of $x^2 \sin \alpha + y^2 \cos \alpha = a^2$, where α is the parameter.* **(Meerut 1993P, 98)**

Sol. The equation of the given family of curves is

$$x^2 \sin \alpha + y^2 \cos \alpha = a^2, \qquad \text{...(1)}$$

where α is the parameter.

Differentiating (1) partially with respect to 'α', we get

$$x^2 \cos \alpha - y^2 \sin \alpha = 0. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$x^4 + y^4 = a^4$$

which is the required envelope of the family of curves (1).

Ex. 3(c). *Find the envelope of the family of circles*

$(x - c)^2 + y^2 = R^2$, where c is the parameter.

(Meerut 1994P, 95 BP)

Sol. The given family of circles is

$$(x - c)^2 + y^2 = R^2, \qquad \text{...(1)}$$

where c is the parameter.

Differentiating (1) partially with respect 'c', we get

$$-2(x - c) = 0 \quad \text{or} \quad x - c = 0. \qquad \text{...(2)}$$

Eliminating c between (1) and (2), we get the envelope of the family (1).

Putting $x - c = 0$ in (1), we get $y^2 = R^2$ as the required envelope of the family (1).

Hence the envelope of the family (1) consists of the straight lines $y = \pm R$.

Ex. 3 (d). *Find the envelope of the family of curves*

$$(y - c)^2 - \tfrac{2}{3}(x - c)^3 = 0,$$

where c is the parameter. **(Meerut 1995)**

Sol. The equation of the given family of curves is

$$(y - c)^2 - \frac{2}{3}(x - c)^3 = 0, \qquad \text{...(1)}$$

where c is the parameter.

Differentiating (1) partially with respect to 'c', we get

$$2(y - c)(-1) - \frac{2}{3} \cdot 3(x - c)^2(-1) = 0$$

or $\quad (y - c) - (x - c)^2 = 0$

or $\quad y - c = (x - c)^2. \qquad \text{...(2)}$

Eliminating c between (1) and (2), we get the envelope of the family (1).

Putting $y - c = (x - c)^2$ in (1), we get

$$(x - c)^4 - \frac{2}{3}(x - c)^3 = 0.$$

or $$(x - c)^3 \left[x - c - \frac{2}{3}\right] = 0.$$

$\therefore$ $$x - c = 0 \quad \text{or} \quad x - c = \frac{2}{3}.$$

Putting $x - c = 0$ in (2), we get

$$y - c = 0.$$

Eliminating c between $x - c = 0$ and $y - c = 0$, we get $x = y$.

Again putting $x - c = \frac{2}{3}$ in (2), we get $y - c = \frac{4}{9}$.

Eliminating c between $x - c = \frac{2}{3}$ and $y - c = \frac{4}{9}$, we get

$$x - y = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}.$$

Hence the required envelope of the family of curves (1) consists of the straight lines

$$x - y = 0 \quad \text{and} \quad x - y = \frac{2}{9}.$$

Ex. 4. *Find the envelope of the family of straight lines $x \operatorname{cosec} \theta - y \cot \theta = c$, the parameter being θ.*

Sol. The equation of the given family of straight lines is

$$x \operatorname{cosec} \theta - y \cot \theta = c$$

or $$\frac{x}{\sin \theta} - \frac{y \cos \theta}{\sin \theta} = c$$

or $$x - y \cos \theta = c \sin \theta$$

or $$y \cos \theta + c \sin \theta = x, \qquad \text{...(1)}$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$- y \sin \theta + c \cos \theta = 0. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$y^2 + c^2 = x^2 \quad \text{or} \quad x^2 - y^2 = c^2,$$

which is the required envelope of the given family of straight lines.

Ex. 5 (a). *Find the envelope of the family of straight lines $x \cos^n \theta + y \sin^n \theta = a$ for different values of θ.*

Sol. The equation of the given family of straight lines is

$$x \cos^n \theta + y \sin^n \theta = a, \qquad \text{...(1)}$$

the parameter being θ.

Differentiating (1) partially with repsect to θ, we get

$$- nx \cos^{n-1} \theta \sin \theta + ny \sin^{n-1} \theta \cos \theta = 0$$

or $$\frac{\cos^{n-1} \theta \sin \theta}{\sin^{n-1} \theta \cos \theta} = \frac{y}{x} \quad \text{or} \quad \cot^{n-2} \theta = \frac{y}{x}$$

or $$\frac{1}{\cot^{n-2}\theta}=\frac{x}{y} \text{ or } \cot^{2-n}\theta=\frac{x}{y}$$

or $$\cot\theta=\frac{x^{1/(2-n)}}{y^{1/(2-n)}}.$$

$\therefore$ $$\cos\theta=\frac{x^{1/(2-n)}}{[x^{2/(2-n)}+y^{2/(2-n)}]^{1/2}}$$

and $$\sin\theta=\frac{y^{1/(2-n)}}{[x^{2/(2-n)}+y^{2/(2-n)}]^{1/2}}.$$

Putting the values of $\cos\theta$ and $\sin\theta$ in (1), the required envelope of the family of straight lines (1) is

$$x\cdot\frac{x^{n/(2-n)}}{[x^{2/(2-n)}+y^{2/(2-n)}]^{n/2}}+y\cdot\frac{y^{n/(2-n)}}{[x^{2/(2-n)}+y^{2/(2-n)}]^{n/2}}=a$$

or $$\frac{x^{2/(2-n)}+y^{2/(2-n)}}{[x^{2/(2-n)}+y^{2/(2-n)}]^{n/2}}=a$$

or $$[x^{2/(2-n)}+y^{2/(2-n)}]^{1-(n/2)}=a$$

or $$[x^{2/(2-n)}+y^{2/(2-n)}]^{(2-n)/2}=a$$

or $$x^{2/(2-n)}+y^{2/(2-n)}=a^{2/(2-n)},$$

raising both sides to the power $2/(2-n)$.

Ex. 5 (b). *Find the envelope of the family of straight lines*

$$x\cos^3\alpha+y\sin^3\alpha=a,$$

the parameter being α. **(Rohilkhand 1980; Meerut 98)**

Sol. Proceed as in Ex. 5 (a).

The required envelope is $a^2(x^2+y^2)=x^2y^2$

or $$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}.$$

Ex. 6. *If* $x^{2/3}+y^{2/3}=k^{2/3}$ *is the envelope of the lines*

$$x/a+y/b=1,$$

then find the necessary relation between a, b *and* k. **(Lucknow 1977)**

Sol. The equation of the given family of straight lines is

$$\frac{x}{a}+\frac{y}{b}=1, \qquad \text{...(1)}$$

where a and b are parameters.

If the envelope of the family of straight lines (1) is the curve

$$x^{2/3}+y^{2/3}=k^{2/3}, \qquad \text{...(2)}$$

then each member of the family of straight lines (1) touches the curve (2).

So to find the necessary relation between a, b and k we have to find the condition that the straight line (1) touches the curve (2).

The coordinates (x, y) of any point on the curve (2) may be taken

$x = k\cos^3\theta$, $y = k\sin^3\theta$, where θ is a parameter.

We have $dx/d\theta = -3k\cos^2\theta\sin\theta$

and $\qquad dy/d\theta = 3k\sin^2\theta\cos\theta$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{3k\sin^2\theta\cos\theta}{-3k\cos^2\theta\sin\theta} = -\dfrac{\sin\theta}{\cos\theta}$ = slope of the tangent to the curve (2) at the point $(k\cos^3\theta, k\sin^3\theta)$.

$\therefore$ the equation of the tangent to the curve (2) at the point $(k\cos^3\theta, k\sin^3\theta)$ is

$$y - k\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - k\cos^3\theta)$$

or $\qquad y\cos\theta - k\sin^3\theta\cos\theta = -x\sin\theta + k\sin\theta\cos^3\theta$

or $\qquad x\sin\theta + y\cos\theta = k\sin\theta\cos\theta$

or $$\frac{x}{k\cos\theta} + \frac{y}{k\sin\theta} = 1. \qquad \text{...(3)}$$

Now suppose the straight line (1) touches the curve (2) at the point $(k\cos^3\theta, k\sin^3\theta)$. Then (1) and (3) are the equations of the same straight line. So comparing the coefficients of like terms in (1) and (3), we get

$$a = k\cos\theta,\ b = k\sin\theta.$$

Squaring and adding, we get $a^2 + b^2 = k^2$,
which is the required relation between a, b and k so that the line (1) touches the curve (2).

Ex. 7. *Find the envelope of the family of straight lines*

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2,$$

where the parameter is θ. **(Garhwal 1983; Meerut 84 S, 89)**

Sol. The equation of the given family of straight lines is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2,\ \theta \text{ being the parameter.} \qquad \text{...(1)}$$

Differentiating (1) partially with respect to θ, we get

$$\frac{ax\sin\theta}{\cos^2\theta} + \frac{by\cos\theta}{\sin^2\theta} = 0. \qquad \text{...(2)}$$

Eliminating θ between (1) and (2), we get the required envelope.

From (2), we get $\tan^3\theta = -(by/ax)$.

$\therefore \quad \tan\theta = -(by)^{1/3}/(ax)^{1/3}$.

$$\therefore \quad \sin\theta = \frac{(by)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}},\ \cos\theta = -\frac{(ax)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}$$

or $$\sin\theta = -\frac{(by)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}},\ \cos\theta = \frac{(ax)^{1/3}}{\sqrt{[(ax)^{2/3} + (by)^{2/3}]}}.$$

Substituting these values in (1), we get

$$\pm [(ax)^{2/3} + (by)^{2/3}] [(ax)^{2/3} + (by)^{2/3}]^{1/2} = a^2 - b^2$$

or $$\pm [(ax)^{2/3} + (by)^{2/3}]^{3/2} = (a^2 - b^2)$$

i.e., $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3},$$

which is the equation of the required envelope.

Ex. 8. *Find the envelope of the family of straight lines* $y = mx + \sqrt{(a^2m^2 + b^2)}$, *the parameter being* m.

Sol. The equation of the given family of straight lines is

$$y = mx + \sqrt{(a^2m^2 + b^2)}$$

or $$y - mx = \sqrt{(a^2m^2 + b^2)}$$

or $$(y - mx)^2 = a^2m^2 + b^2$$

or $$y^2 - 2mxy + m^2x^2 - a^2m^2 - b^2 = 0$$

or $$m^2 (x^2 - a^2) - 2xym + y^2 - b^2 = 0. \qquad ...(1)$$

The equation (1) is a quadratic in the parameter m. So the required envelope is obtained by equating to zero the discriminant of (1). Hence the required envelope is

$$(-2xy)^2 - 4 (x^2 - a^2) (y^2 - b^2) = 0$$

or $$x^2y^2 - (x^2 - a^2) (y^2 - b^2) = 0$$

or $$x^2y^2 - x^2y^2 + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

or $$x^2b^2 + a^2y^2 = a^2b^2$$

or $$x^2/a^2 + y^2/b^2 = 1,$$ which is an ellipse.

Ex. 9. *Find the envelope of the family of straight lines*

$$x/a + y/b = 1,$$

where the two parameters, a, b are connected by the relation $ab = c^2$, c being a constant. **(Rohilkhand 1982)**

Sol. The equation of the given family of straight lines is

$$x/a + y/b = 1, \qquad ...(1)$$

where the parameters a, b are connected by the relation

$$ab = c^2. \qquad ...(2)$$

We shall eliminate one parameter, say b.

From (2), we have $b = c^2/a$. Putting the value of b in (1), we get

$$\frac{x}{a} + \frac{y}{c^2/a} = 1$$

or $$\frac{x}{a} + \frac{ay}{c^2} = 1, \qquad ...(3)$$

which is the equation of the given family and it contains only one parameter a. We can arrange (3) as a quadratic in a. Thus (3) can be written as

$$c^2x + a^2y = ac^2 \quad \text{or} \quad a^2y - ac^2 + c^2x = 0. \qquad ...(4)$$

The equation (4) is a quadratic in the parameter a. So the required envelope is

$$(-c^2)^2 - 4yc^2x = 0 \quad \text{or} \quad c^4 - 4xyc^2 = 0 \text{ or } xy = c^2/4.$$

Ex. 10. *Find the envelope of the straight lines $y = m^2x + 1/m^2$, where m is the parameter.*

Sol. The equation of the given family of straight lines can be written as

$$m^2y = m^4x + 1, \quad \text{or} \quad m^4x - m^2y + 1 = 0. \qquad ...(1)$$

The equation (1) is a quadratic in m^2. So the required envelope is obtained by equating to zero the discriminant of (1). Hence the required envelope is

$$(-y)^2 - 4x.1 = 0 \text{ or } y^2 = 4x.$$

Ex. 11. *Find the envelope of the straight lines $y = mx + am^3$, the parameter being m.*

Sol. We have $\quad y = mx + am^3. \qquad ...(1)$

Differentiating (1) partially w.r.t. 'm', we have

$$0 = x + 3am^2 \quad \text{or} \quad m^2 = -x/(3a). \qquad ...(2)$$

From (1), $y^2 = m^2(x + am^2)^2$

$= (-x/3a)(x - x/3)^2$, substituting for m^2 from (2)

$= -4x^3/27a$.

Hence $27ay^2 + 4x^3 = 0$, is the required envelope.

Ex. 12. *Find the envelope of the family of straight lines $y = mx + am^p$, the parameter being m.*

Sol. The equation of the given family of straight lines is

$$y = mx + am^p, \qquad ...(1)$$

the parameter being m.

Differentiating (1) partially with respect to m, we get

$$0 = x + pam^{p-1}. \qquad ...(2)$$

Eliminating m between (1) and (2), we get the envelope of the family of straight lines (1).

From (2), $\quad m^{p-1} = -\dfrac{x}{pa}. \qquad ...(3)$

Now the equation (1) can be written as

$$y = m(x + am^{p-1}).$$

Raising both sides to the power $p - 1$, we get

$y^{p-1} = m^{p-1}(x + am^{p-1})^{p-1}$

or $\quad y^{p-1} = -\dfrac{x}{pa}\left[x - a \cdot \dfrac{x}{pa}\right]^{p-1}$, substituting for m^{p-1} from (3)

or $\quad y^{p-1} = -\dfrac{x}{pa} \cdot \dfrac{x^{p-1}(p-1)^{p-1}}{p^{p-1}}$

or $\quad (p-1)^{p-1}x^p + p^p ay^{p-1} = 0$, which is the required envelope.

Ex. 13. *Find the envelope of the family of the straight lines* $x \cos m\theta + y \sin m\theta = a (\cos n\theta)^{m/n}$, *where* θ *is the parameter.*

Sol. We have $x \cos m\theta + y \sin m\theta = a (\cos n\theta)^{m/n}$. ...(1)

Differentiating (1), partially w.r.t. θ, we have

$$-mx \sin m\theta + my \cos m\theta = a (m/n) (\cos n\theta)^{(m/n)-1} (-n \sin n\theta)$$

or $\quad -x \sin m\theta + y \cos m\theta = -a (\cos n\theta)^{(m/n)-1} (\sin n\theta)$. ...(2)

Squaring and adding (1) and (2), we have

$$(\cos^2 m\theta + \sin^2 m\theta)(x^2+y^2) = a^2 (\cos n\theta)^{2(m/n-1)} (\cos^2 n\theta + \sin^2 n\theta)$$

or $\quad x^2 + y^2 = a^2 (\cos n\theta)^{2(m-n)/n}$. ...(3)

Multiplying (1) by $\sin n\theta$ and (2) by $\cos n\theta$ and adding, we get

$$-x(\sin m\theta \cos n\theta - \cos m\theta \sin n\theta) + y(\cos m\theta \cos n\theta + \sin m\theta \sin n\theta) = 0$$

or $\quad -x \sin (m-n)\theta + y \cos (m-n)\theta = 0$

or $\quad \tan (m-n)\theta = y/x$ ...(4)

Now let (r, ϕ) denote the polar coordinates of the point (x, y) so that $x = r\cos\phi$ and $y = r \sin\phi$. Then from (4), we have $\tan (m-n)\theta = \tan\phi$ or $(m-n)\theta = \phi$ or $\theta = \phi/(m-n)$. Substituting the value of θ in (3), we get

$$(r^2\cos^2\phi + r^2\sin^2\phi) = a^2 [\cos \{n\phi/(m-n)\}]^{2(m-n)/n}$$

or $\quad r^2 = a^2 [\cos\{n\phi/(m-n)\}]^{2(m-n)/n}$

or $\quad r^{n/(m-n)} = a^{n/(m-n)} \cos \{n\phi/(m-n)\}$,

which is the required envelope where (r, ϕ) are the polar coordinates of the point (x, y).

Ex. 14. *Find the envelope of the following systems of circles :*

(a) $(x-\alpha)^2 + y^2 = 4\alpha$,

(b) $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$, *where* α *is the parameter.*

Sol. **(a).** We have $(x-\alpha)^2 + y^2 = 4\alpha$ or $x^2 - 2\alpha x + \alpha^2 + y^2 = 4\alpha$

or $\quad \alpha^2 - 2\alpha(x+2) + (x^2+y^2) = 0$.

This equation is a quadratic in α. Hence the required envelope is

$$4(x+2)^2 - 4.(x^2+y^2) = 0$$

or $\quad x^2 + 4x + 4 - x^2 - y^2 = 0$ or $y^2 - 4x - 4 = 0$.

(b) We have $(x-\alpha)^2 + (y-\alpha)^2 = 2\alpha$

or $\quad x^2 - 2\alpha x + \alpha^2 + y^2 - 2\alpha y + \alpha^2 = 2\alpha$

or $\quad 2\alpha^2 - 2(x+y+1)\alpha + (x^2+y^2) = 0$.

This equation is a quadratic in α. Hence the required envelope is

$$4(x+y+1)^2 - 4.2(x^2+y^2) = 0$$

or $\quad (x+y+1)^2 = 2(x^2+y^2)$.

Ex. 15. *Find the envelope of the following families of curves :*

(a) $tx^2 + t^2y = a$,

(b) $y^2 = t^2 (x - t)$, *the parameter being t in each case.*

Sol. (a). We have $tx^2 + t^2 y = a$ or $t^2 y + tx^2 - a = 0$.

This equation is a quadratic in t. Hence the required envelope is $(x^2)^2 - 4y.(-a) = 0$ or $x^4 + 4ay = 0$.

(b) Here $y^2 = t^2 (x - t) = t^2 x - t^3$. ...(1)

Differentiating (1) partially w.r.t. the parameter 't', we get

$$0 = 2tx - 3t^2 \text{ or } t = 2x/3.$$

Substituting this value of t in (1), we get

$$y^2 = (2x/3)^2 (x - 2x/3) = 4x^3/27$$

or $4x^3 = 27y^2$, which is the required envelope.

Ex. 16. *Find the envelope of the straight lines*

$$x \cos \alpha + y \sin \alpha = l \sin \alpha \cos \alpha,$$

where the parameter is the angle α. Give the geometrical interpretation.

(Agra 1976; Gorakhpur 80; Meerut 84, 90P, 96)

Sol. Here $x \cos \alpha + y \sin \alpha = l \sin \alpha \cos \alpha$,

or $x \operatorname{cosec} \alpha + y \sec \alpha = l$. **(Note)** ...(1)

Differentiating (1) partially w.r.t. the parameter α, we have $x(-\operatorname{cosec} \alpha \cot \alpha) + y \sec \alpha \tan \alpha = 0$

or $\tan \alpha = x^{1/3}/y^{1/3}$.

$\therefore$ $\operatorname{cosec} \alpha = \sqrt{(1 + \cot^2 \alpha)} = \sqrt{\{1 + (y^{2/3}/x^{2/3})\}}$

$= \sqrt{(x^{2/3} + y^{2/3})}/x^{1/3}$, ...(2)

and $\sec \alpha = \sqrt{(1 + \tan^2 \alpha)} = \sqrt{(x^{2/3} + y^{2/3})}/y^{1/3}$. ...(3)

Eliminating α between (1), (2) and (3), we have

$$\frac{x (x^{2/3} + y^{2/3})^{1/2}}{x^{1/3}} + y . \frac{(x^{2/3} + y^{2/3})^{1/2}}{y^{1/3}} = l$$

or $(x^{2/3} + y^{2/3})^{(3/2)} = l$, or $x^{2/3} + y^{2/3} = l^{2/3}$,

which is the required envelope.

Geometrical Interpretation. The equation (1) may be written as $x/(l \sin \alpha) + y/(l \cos \alpha) = 1$, which shows that the intercepts on the axes made by the line are $l \sin \alpha$ and $l \cos \alpha$. Hence the length of the line between the axes is $\sqrt{(l^2 \sin^2 \alpha + l^2 \cos^2 \alpha)}$ *i.e.*, l which is constant. Hence the result may be interpreted geometrically as follows :

If a straight line of constant length l slides between the axes, the envelope of the straight line is the astroid

$$x^{2/3} + y^{2/3} = l^{2/3}.$$

Ex. 17. *From any point of the ellipse $x^2/a^2 + y^2/b^2 = 1$, perpendiculars are drawn to the axes and the feet of these perpendiculars are joined. Show that the straight line thus formed always touches the curve*

$$(x/a)^{2/3} + (y/b)^{2/3} = 1.$$ **(Rohilkhand 1978)**

Sol. Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$. Also let PL and PM be the perpendiculars from P on the x-axis and y-axis respectively.

Then L is the point $(a\cos\theta, 0)$ and M is the point $(0, b\sin\theta)$.

$\therefore$ equation of the line LM is

$$\frac{x}{a\cos\theta} + \frac{y}{b\sin\theta} = 1. \quad ...(1) \qquad \text{(Intercepts form)}$$

We have to find the envelope of (1), where θ is the parameter.

Differentiating (1) partially w.r.t. θ, we have

$$\frac{x\sin\theta}{a\cos^2\theta} - \frac{y\cos\theta}{b\sin^2\theta} = 0, \text{ or } \frac{x}{a}\sin^3\theta - \frac{y}{b}\cos^3\theta = 0$$

or $(x/a)\sin^3\theta = (y/b)\cos^3\theta$ or $\tan^3\theta = (y/b)/(x/a)$

or $\tan\theta = (y/b)^{1/3}/(x/a)^{1/3}$.

$$\therefore \quad \sin\theta = \frac{(y/b)^{1/3}}{\{(x/a)^{2/3} + (y/b)^{2/3}\}^{1/2}},$$

$$\cos\theta = \frac{(x/a)^{1/3}}{\{(x/a)^{2/3} + (y/b)^{2/3}\}^{1/2}}.$$

Substituting these values of $\sin\theta$ and $\cos\theta$ in (1), we get

$$\frac{(x/a)\{(x/a)^{2/3} + (y/b)^{2/3}\}^{1/2}}{(x/a)^{1/3}} + \frac{(y/b)\{(x/a)^{2/3} + (y/b)^{2/3}\}^{1/2}}{(y/b)^{1/3}} = 1$$

or $\{(x/a)^{2/3} + (y/b)^{2/3}\}^{1/2}.\{(x/a)^{2/3} + (y/b)^{2/3}\} = 1$

or $\{(x/a)^{2/3} + (y/b)^{2/3}\}^{3/2} = 1$ or $(x/a)^{2/3} + (y/b)^{2/3} = 1$,

which is the envelope of the line LM.

Hence the line LM always touches the curve

$$(x/a)^{2/3} + (y/b)^{2/3} = 1.$$

Ex. 18. *Projectiles are fired from a gun with a constant initial velocity V_0. Supposing the gun can be given any elevation and is kept always in the same vertical plane, what is the envelope of the possible trajectories, assuming their equation to be*

$$y = x\tan\alpha - \frac{gx^2}{2V_0^2\cos^2\alpha}?$$

Sol. The given equation is

$$y = x\tan\alpha - gx^2/(2V_0^2\cos^2\alpha),$$

where α is parameter and V_0 is constant

or $\qquad y = x\tan\alpha - (\frac{1}{2}gx^2/V_0^2)\sec^2\alpha$

or $\qquad y = x\tan\alpha - (\frac{1}{2}gx^2/V_0^2)(1 + \tan^2\alpha)$

or $\quad (\frac{1}{2}gx^2/V_0^2)\tan^2\alpha - x\tan\alpha + (y + \frac{1}{2}gx^2/V_0^2) = 0,$

which is a quadratic equation in $\tan\alpha$.

$\therefore$ the envelope is given by

$$x^2 - 4\,(\tfrac{1}{2}gx^2/V_0^2)\,(y + \tfrac{1}{2}gx^2/V_0^2) = 0$$

or $\quad 1 - (2g/V_0^2)\,(y + \frac{1}{2}gx^2/V_0^2) = 0$

or $\quad V_0^2/2g = y + \frac{1}{2}gx^2/V_0^2$, multiplying by $V_0^2/2g$

or $\quad V_0^4 = g^2x^2 + 2V_0^2\,gy$, which is the required envelope.

Ex. 19. *Obtain the envelope of the family of curves given by*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2 - \alpha^2} = 1,$$

where α is the parameter. **(Lucknow 1979; Meerut 91P, 92, 94)**

Sol. The given equation is

$$(x^2/\alpha^2) + y^2/(k^2 - \alpha^2) = 1$$

or $\quad x^2k^2 - \alpha^2x^2 + y^2\alpha^2 = k^2\alpha^2 - \alpha^4,$

or $\quad \alpha^4 + \alpha^2\,(y^2 - x^2 - k^2) + x^2k^2 = 0,$

which is a quadratic equation in the parameter α^2.

$\therefore$ the envelope is given by $(y^2 - x^2 - k^2)^2 = 4k^2x^2$

or $\quad y^2 - x^2 - k^2 = \pm\, 2kx$

or $\quad y^2 = x^2 + k^2 \pm 2kx$

or $\quad y^2 = (x \pm k)^2$

or $\quad y = \pm\,(x \pm k).$

Hence the required envelope consists of the four straight lines $x \pm y = \pm\, k$.

Ex. 20. *Find the envelope of the ellipse*

$$x = a\sin(\theta - \alpha),\ y = b\cos\theta,$$

where α is the parameter. **(Agra 1979; Lucknow 83)**

Sol. The given ellipse is

$$x = a\sin(\theta - \alpha),\ y = b\cos\theta, \text{ where } \alpha \text{ is the parameter}$$

or $\quad x = a\,(\sin\theta\cos\alpha - \cos\theta\sin\alpha)\ \ y = b\cos\theta$

or $\quad x = a\,\{\sqrt{(1 - y^2/b^2)}\cos\alpha - (y/b)\sin\alpha\}$, eliminating θ. ...(1)

Differentiating (1) partially w.r.t. the parameter α, we have

$$0 = a\,\{-\sqrt{(1 - y^2/b^2)}\sin\alpha - (y/b)\cos\alpha\}. \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$x^2 = a^2\,\{1 - y^2/b^2 + y^2/b^2\},$$

or $\quad x^2 = a^2$ or $x = \pm\, a$ as the required envelope.

Ex. 21. *Show that the envelope of the polars of points on the ellipse $x^2/h^2 + y^2/k^2 = 1$ with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\dfrac{h^2x^2}{a^4} + \dfrac{k^2y^2}{b^4} = 1$.* **(Allahabad 1981, 78)**

Sol. Any point on the ellipse $x^2/h^2 + y^2/k^2 = 1$ is

$$(h \cos \theta, k \sin \theta),$$

where θ is the parameter. And polar of this point with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{xh \cos \theta}{a^2} + \frac{yk \sin \theta}{b^2} = 1. \quad \text{...(1)}$$

We have to find the envelope of the family (1) where θ is the parameter.

Differentiating (1) partially w.r.t. θ, we have

$$-(xh/a^2) \sin \theta + (yk/b^2) \cos \theta = 0. \quad \text{...(2)}$$

Squaring and adding (1) and (2), we have

$$\left(\frac{xh}{a^2}\right)^2 + \left(\frac{yk}{b^2}\right)^2 = 1 \quad \text{or} \quad \frac{x^2h^2}{a^4} + \frac{y^2k^2}{b^4} = 1,$$

which is the required envelope.

Ex. 22. *Find the envelope of the family of straight lines $x \cos \alpha + y \sin \alpha = a$, the parameter being α, and interpret the result geometrically.* **(Meerut 1994)**

Sol. The equation of the given family of straight lines is

$$x \cos \alpha + y \sin \alpha = a, \quad \text{...(1)}$$

the parameter being α.

Differentiating (1) partially with respect to α, we get

$$-x \sin \alpha + y \cos \alpha = 0. \quad \text{...(2)}$$

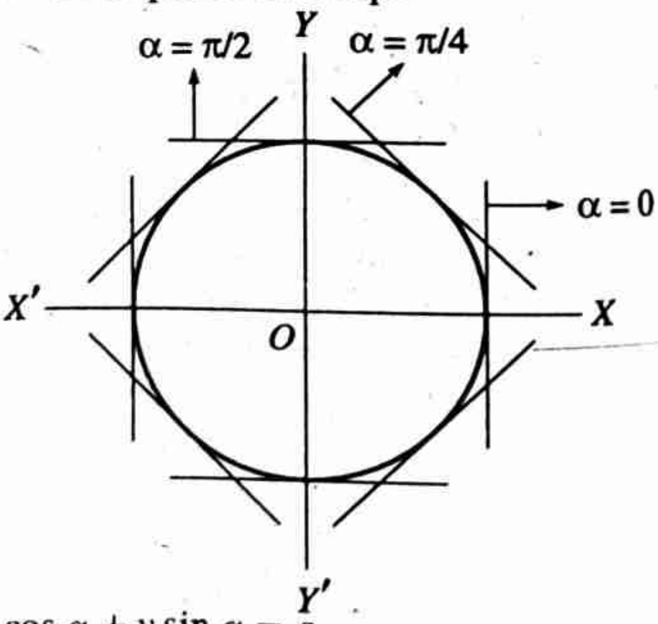
Eliminating α between (1) and (2), we get the envelope. So squaring and adding (1) and (2), we get

$$(x \cos \alpha + y \sin \alpha)^2 + (-x \sin \alpha + y \cos \alpha)^2 = a^2$$

or $\quad x^2 (\cos^2 \alpha + \sin^2 \alpha) + y^2 (\sin^2 \alpha + \cos^2 \alpha) = a^2$

or $\quad x^2 + y^2 = a^2$, which is the required envelope.

Geometrical interpretation : $x^2 + y^2 = a^2$ is the equation of a circle whose centre is origin and radius is a. This circle is the envelope of the family of straight lines $x \cos \alpha + y \sin \alpha = a$. So for each value of α, the straight line $x \cos \alpha + y \sin \alpha = a$ touches the circle $x^2 + y^2 = a^2$. Also the circle $x^2 + y^2 = a^2$ is touched at each point by some straight line belonging to the family $x \cos \alpha + y \sin \alpha = a$.



Ex. 23. *Find the envelope of the family of circles*

$$x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = c^2,$$

where α is the parameter, and interpret the result.

(Lucknow 1983; Agra 82; Meerut 90S, 92, 97)

Sol. The equation of the given family of circles can be written as

$$2ax \cos \alpha + 2ay \sin \alpha = x^2 + y^2 - c^2. \quad ...(1)$$

[Note that we have brought the terms containing $\cos \alpha$ and $\sin \alpha$ to one side and the rest of the terms to the other side].

Differentiating (1) partially with respect to α, we get

$$- 2ax \sin \alpha + 2ay \cos \alpha = 0. \quad ...(2)$$

Squaring and adding (1) and (2), we get

$$4a^2x^2 + 4a^2y^2 = (x^2 + y^2 - c^2)^2$$

or $$(x^2 + y^2 - c^2)^2 = 4a^2 (x^2 + y^2), \quad ...(3)$$

which is the required envelope.

Interpretation. The equation (3) can be written as

$$(x^2 + y^2)^2 - 2 (2a^2 + c^2) (x^2 + y^2) + c^4 = 0.$$

Solving it as a quadratic in $(x^2 + y^2)$, we get

$$x^2 + y^2 = \frac{2 (2a^2 + c^2) \pm \sqrt{\{4 (2a^2 + c^2)^2 - 4c^4\}}}{2}$$

$$= 2a^2 + c^2 \pm 2a \sqrt{(c^2 + a^2)}$$

$$= (a^2 + c^2) \pm 2a \sqrt{(a^2 + c^2)} + a^2$$

$$= [\sqrt{(a^2 + c^2)} \pm a]^2.$$

Therefore the required envelope consists of the two circles $x^2 + y^2 = [\sqrt{(a^2 + c^2)} + a]^2$ and $x^2 + y^2 = [\sqrt{(a^2 + c^2)} - a]^2$. These are the circles with centre at origin and radii $\sqrt{(a^2 + c^2)} \pm a$.

Ex. 24. *Find the envelope of the circles which pass through the origin and whose centres lie on the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

(Gorakhpur 1982; Kanpur 80)

Sol. Any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$(a \cos \theta, b \sin \theta).$$

Its distance from the origin is $\sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}$.

Therefore the equation of the given family of circles is

$$(x - a \cos \theta)^2 + (y - b \sin \theta)^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$$

or $$x^2 + y^2 - 2ax \cos \theta - 2by \sin \theta = 0. \quad ...(1)$$

We are to find the envelope of the family of circles (1), where θ is the parameter. The equation (1) may be written as

$$2ax \cos \theta + 2by \sin \theta = x^2 + y^2. \quad ...(2)$$

Differentiating (2) partially with respect to θ, we get

$$- 2ax \sin \theta + 2by \cos \theta = 0. \quad ...(3)$$

Squaring and adding (2) and (3), we get

$$4a^2x^2 + 4b^2y^2 = (x^2 + y^2)^2$$

or $$(x^2+y^2)^2 = 4\,(a^2x^2+b^2y^2),$$
which is the required envelope.

Ex. 25. *Find the envelope of the circles drawn upon the radii vectors of the ellipse $x^2/a^2+y^2/b^2=1$ as diameter.*

(Meerut 1983S, 86S, 89P, 96; Kanpur 79)

Sol. Any point on the given ellipse is $(a\cos\theta, b\sin\theta)$.

So the equation of the circle on the radius vector to this point as diameter is

$$(x-0)\,(x-a\cos\theta)+(y-0)\,(y-b\sin\theta)=0$$

or $$x^2+y^2-ax\cos\theta-by\sin\theta=0$$

or $$ax\cos\theta+by\sin\theta=x^2+y^2. \quad ...(1)$$

We have to find the envelope of (1) where θ is the parameter.

Differentiating (1) partially w.r.t. θ, we get

$$-ax\sin\theta+by\cos\theta=0. \quad ...(2)$$

Squaring and adding (1) and (2), we get

$a^2x^2+b^2y^2=(x^2+y^2)^2$ as the required envelope.

Ex. 26. *Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse.*

(Lucknow 1981; Meerut 84 R)

Sol. Let the equation of the given ellipse be $x^2/a^2+y^2/b^2=1$.

Let the extremity of a diamter be $P\,(a\cos\theta, b\sin\theta)$. Then the extremity of the conjugate diameter is

$$Q\,\{a\cos(\tfrac{1}{2}\pi+\theta),\, b\sin(\tfrac{1}{2}\pi+\theta)\}\ \textit{i.e.,}\ (-a\sin\theta, b\cos\theta).$$

$\therefore$ equation of the straight line PQ joning the extremities P and Q is

$$y-b\sin\theta=\frac{b\,(\sin\theta-\cos\theta)}{a\,(\cos\theta+\sin\theta)}\,(x-a\cos\theta)$$

or $$ay\,(\cos\theta+\sin\theta)-ab\,(\sin\theta\cos\theta+\sin^2\theta)$$
$$=bx\,(\sin\theta-\cos\theta)-ab\,(\sin\theta\cos\theta-\cos^2\theta)$$

or $$(ay+bx)\cos\theta+(ay-bx)\sin\theta=ab\,(\cos^2\theta+\sin^2\theta)=ab. \quad ...(1)$$

Differentiating (1) partially w.r.t. θ, we have

$$-(ay+bx)\sin\theta+(ay-bx)\cos\theta=0. \quad ...(2)$$

Squaring and adding (1) and (2), we get

$$(ay+bx)^2+(ay-bx)^2=a^2b^2$$

or $$\frac{x^2}{a^2/2}+\frac{y^2}{b^2/2}=1,$$

which is the required envelope and is a similar ellipse.

Ex. 27. *Show that the envelope of the circles whose centres lie on the rectangular hyperbola $x^2-y^2=a^2$ and which pass through the origin is the lemniscate $r^2=4a^2\cos 2\theta$.*

Sol. Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ is $(a \cosh \theta, a \sinh \theta)$. Its distance from the origin is

$$\sqrt{(a^2 \cosh^2 \theta + a^2 \sinh^2 \theta)}.$$

Therefore the equation of the circle whose centre is the point $(a \cosh \theta, a \sinh \theta)$ and which passes through the origin is

$$(x - a \cosh \theta)^2 + (y - a \sinh \theta)^2 = a^2 \cosh^2 \theta + a^2 \sinh^2 \theta$$

or $\quad x^2 + y^2 - 2ax \cosh \theta - 2ay \sinh \theta = 0$

or $\quad 2ax \cosh \theta + 2ay \sinh \theta = x^2 + y^2. \quad$ ...(1)

We are to find the envelope of the family of circles (1), where θ is the parameter.

Differentiating (1) partially with respect to θ, we get

$$2ax \sinh \theta + 2ay \cosh \theta = 0. \quad ...(2)$$

Squaring and subtracting (1) and (2), we get

$$4a^2x^2 (\cosh^2 \theta - \sinh^2 \theta) + 4a^2y^2 (\sinh^2 \theta - \cosh^2 \theta) = (x^2 + y^2)^2$$

or $\quad 4a^2x^2 - 4a^2y^2 = (x^2 + y^2)^2 \quad [\because \cosh^2 \theta - \sinh^2 \theta = 1]$

or $\quad 4a^2 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) = (r^2)^2$, changing to polars

or $\quad 4a^2r^2 \cos 2\theta = r^4$

or $\quad r^2 = 4a^2 \cos 2\theta$, which is the required envelope.

Ex. 28. *Show that the envelope of circles described on the central radii of a rectangular hyperbola is a lemniscate $r^2 = a^2 \cos 2\theta$.*

Sol. Referred to the centre as origin, let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$.

Any point on it is $(a \cosh \phi, a \sinh \phi)$.

Then the equation of the circle having $(0, 0)$ and $(a \cosh \phi, a \sinh \phi)$ as the ends of the diameter is

$$(x - 0)(x - a \cosh \phi) + (y - 0)(y - a \sinh \phi) = 0$$

or $\quad ax \cosh \phi + ay \sinh \phi = x^2 + y^2. \quad$ ...(1)

Differentiating (1) partially w.r.t. ϕ, we have

$$ax \sinh \phi + ay \cosh \phi = 0. \quad ...(2)$$

Squaring (2) and subtracting from the square of (1), we get

$$a^2x^2 (\cosh^2 \phi - \sinh^2 \phi) + a^2y^2 (\sinh^2 \phi - \cosh^2 \phi) = (x^2 + y^2)^2$$

or $\quad a^2 (x^2 - y^2) = (x^2 + y^2)^2, \quad [\because \cosh^2 \phi - \sinh^2 \phi = 1]$

or $\quad a^2 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) = (r^2)^2 \quad$ [changing to polars]

or $\quad a^2r^2 \cos 2\theta = r^4 \quad$ or $\quad r^2 = a^2 \cos 2\theta$,

which is the required envelope.

Ex. 29. *Find the envelope of the circles drawn on the radii vectors of the parabola $y^2 = 4ax$ as diameter.* **(Kanpur 1977)**

Sol. Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$. Equation of the circle drawn on the line joining the origin $(0, 0)$ to the point $(at^2, 2at)$ as diameter is

$$(x-0)(x-at^2)+(y-0)(y-2at)=0$$

or $$x^2+y^2-axt^2-2aty=0. \qquad ...(1)$$

We are to find the envelope of the family of circles (1), where t is the parameter.

Differentiating (1) partially with respect to t, we get

$$0-2axt-2ay=0. \qquad ...(2)$$

Eliminating t between (1) and (2), we get the required envelope. From (2), we get $t=-y/x$.

Putting this value of t in (1), we get

$$x^2+y^2-ax.(y^2/x^2)+2ay.(y/x)=0$$

or $$x^2+y^2-(ay^2/x)+(2ay^2/x)=0$$

or $$x^2+y^2+(ay^2/x)=0$$

or $$ay^2+x(x^2+y^2)=0,$$

which is the required envelope.

Ex. 30. *A circle moves with its centre on the parabola $y^2=4ax$ and always passes through the vertex of the parabola. Show that envelope of the circle is the curve $x^3+y^2(x+2a)=0$.* **(Rohilkhand 1979; Meerut 90)**

Sol. Let any point P on the parabola $y^2=4ax$ be $(at^2, 2at)$.

$\therefore$ the centre of the circle is $(at^2, 2at)$ and it passes through origin $(0, 0)$.

Equation of any circle with centre $(-g, -f)$ and passing through the origin is $x^2+y^2+2gx+2fy=0$.

$\therefore$ equation of the circle whose centre is $(at^2, 2at)$, and which passes through the origin is

$$x^2+y^2-2ax\,t^2-4ay\,t=0$$

or $$2ax\,t^2+4ay\,t-(x^2+y^2)=0. \qquad ...(1)$$

This equation is quadratic in the parameter t. Hence the required envelope is

$$(4ay)^2+4.2ax.(x^2+y^2)=0$$

or $$2ay^2+x^3+xy^2=0 \quad \text{or} \quad x^3+y^2(x+2a)=0.$$

Ex. 31. *Show that the envelope of the family of circles whose diameters are double ordinates of the parabola $y^2=4ax$ is the parabola $y^2=4a(x+a)$.* **(Allahabad 1980; Kanpur 78)**

Sol. Equation of the parabola is $y^2=4ax$.

We know that coordinates of the extremities of a double ordinate of the parabola may be written as

$$(at^2, 2at) \text{ and } (at^2, -2at).$$

Hence the equation of the circle described on the double ordinate as diameter is

$$(x-at^2)(x-at^2)+(y-2at)(y+2at)=0$$

or $\qquad x^2 - 2ax\,t^2 + a^2 t^4 + y^2 - 4a^2t^2 = 0$

or $\qquad a^2t^4 - 2at^2 (x + 2a) + x^2 + y^2 = 0. \qquad$ ...(1)

The equation (1) is a quadratic in t^2. Hence the required envelope is

$$4a^2 (x + 2a)^2 - 4a^2 (x^2 + y^2) = 0$$

or $\qquad x^2 + 4ax + 4a^2 - x^2 - y^2 = 0$

or $\qquad y^2 = 4ax + 4a^2 \quad$ or $\quad y^2 = 4a (x + a)$.

Ex. 32. *Find the envelope of a family of parabolas, of given latus rectum and parallel axes, when the locus of their foci is a given straight line $y = px + q$.*

Sol. Let $4a$ be the given latus rectum of a family of parabolas having their axes parallel to the axis of x. Then the equation of any parabola of the family having its vertex at the point (h, k) is

$$(y - k)^2 = 4a (x - h). \qquad \text{...(1)}$$

Hence the coordinates of the focus are $(a + h, k)$.

Now the focus lies on the line $y = px + q$, (given).

$\therefore \quad k = p (a + h) + q, \quad$ or $\quad \{(k - q)/p\} - a = h$.

Substituting this value of h in (1), we have

$$(y - k)^2 = 4a [x - \{(k - q)/p\} + a]$$

or $\qquad y^2 - 2ky + k^2 = 4ax - 4ak/p + 4aq/p + 4a^2$

or $\qquad k^2 + 2 (2a/p - y) k + (y^2 - 4ax - 4aq/p - 4a^2) = 0$,

which is a quadratic equation in the parameter k.

$\therefore \quad$ the envelope is given by

$$4 (2a/p - y)^2 - 4 (y^2 - 4ax - 4aq/p - 4a^2) = 0$$

or $\qquad 4a^2/p^2 - 4ay/p + y^2 - y^2 + 4ax + 4aq/p + 4a^2 = 0$

or $\qquad p^2x - py + (pq + ap^2 + a) = 0$.

Ex. 33. *Show that the envelope of the straight lines*

$$y \cos \theta - x \sin \theta = a - a \sin \theta \log \tan (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi),$$

where θ is the parameter, is the catenary

$$y = a \cosh (x/a). \qquad \textbf{(Agra 1981)}$$

Sol. The equation of the given family of straight lines may be written as

$$y \cot \theta - x = a \operatorname{cosec} \theta - a \log \tan (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi). \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. θ, we have

$$-y \operatorname{cosec}^2 \theta = -a \operatorname{cosec} \theta \cot \theta - \frac{a \cdot \frac{1}{2} \sec^2 (\frac{1}{2}\theta + \frac{1}{4}\pi)}{\tan (\frac{1}{2}\theta + \frac{1}{4}\pi)}$$

or $\quad y \operatorname{cosec}^2 \theta = a \operatorname{cosec} \theta \cot \theta + a/\{2 \sin (\frac{1}{2}\theta + \frac{1}{4}\pi) \cos (\frac{1}{2}\theta + \frac{1}{4}\pi)\}$

or $\quad \dfrac{y}{\sin^2 \theta} = \dfrac{a \cos \theta}{\sin^2 \theta} + \dfrac{a}{\sin (\frac{1}{2}\pi + \theta)} = \dfrac{a (\cos^2 \theta + \sin^2 \theta)}{\sin^2 \theta . \cos\theta}$

or $\qquad y = a/\cos \theta. \qquad$ ...(2)

Substituting this value of y in (1), we get

$$\frac{a}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}-x=\frac{a}{\sin\theta}-a\log\tan\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi\right)$$

or $\quad -x=-a\log\tan\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi\right)$ or $x/a=\log\tan\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi\right)$.

$\therefore \quad e^{x/a}=\tan\left(\tfrac{1}{2}\theta+\tfrac{1}{4}\pi\right)=\left(1+\tan\tfrac{1}{2}\theta\right)/\left(1-\tan\tfrac{1}{2}\theta\right)$,

and $\quad e^{-x/a}=\left(1-\tan\tfrac{1}{2}\theta\right)/\left(1+\tan\tfrac{1}{2}\theta\right)$.

$$\therefore \quad e^{x/a}+e^{-x/a}=\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}+\frac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta}$$

$$=\frac{2\left(1+\tan^2\frac{1}{2}\theta\right)}{\left(1-\tan^2\frac{1}{2}\theta\right)}=\frac{2}{\cos\theta}.$$

But $2/\cos\theta=2y/a$, from (2).

$\therefore \quad 2y/a=\left(e^{x/a}+e^{-x/a}\right)$

or $\quad y=a\cdot\frac{1}{2}\left(e^{x/a}+e^{-x/a}\right)=a\cosh(x/a)$,

which is the required envelope.

Ex. 34. *Show that the radius of curvature of the envelope of the family of lines $x\cos\alpha+y\sin\alpha=f(\alpha)$ is $f(\alpha)+f''(\alpha)$.*

Sol. The given equation of the family of lines is

$$x\cos\alpha+y\sin\alpha=f(\alpha), \qquad ...(1)$$

where α is the parameter.

Differentiating (1) partially w.r.t. α, we have

$$-x\sin\alpha+y\cos\alpha=f'(\alpha). \qquad ...(2)$$

Squaring (1) and (2) and adding, we get

$$x^2+y^2=\{f(\alpha)\}^2+\{f'(\alpha)\}^2,$$

or $\quad r^2=\{f(\alpha)\}^2+\{f'(\alpha)\}^2$, changing to polars. $\qquad$ ...(3)

Now the envelope of the family of lines (1) touches each member of the family (1). Therefore (1) is a tangent to the envelope of (1).

Hence if p be the length of the perpendicular from the pole (*i.e.*, origin) upon the tangent (1) to the envelope of (1), then

$$p=f(\alpha). \qquad ...(4)$$

Therefore (3) may be regarded as the pedal equation of the envelope where α is given by (4).

Differentiating (4) and (3) w.r.t. 'p', we have

$$1=f'(\alpha)(d\alpha/dp) \qquad ...(5)$$

and $\quad 2r(dr/dp)=2f(\alpha)f'(\alpha).(d\alpha/dp)+2f'(\alpha)f''(\alpha).(d\alpha/dp)$.

$\therefore \quad \rho$ (*i.e.*, the radius of curvature of the envelope)

$$=r\frac{dr}{dp}=f(\alpha)f'(\alpha)\left(\frac{d\alpha}{dp}\right)+f'(\alpha)f''(\alpha)\left(\frac{d\alpha}{dp}\right)$$

$$=\{f(\alpha)+f''(\alpha)\}\{f'(\alpha)(d\alpha/dp)\}$$

$$=f(\alpha)+f''(\alpha), \text{ from (5).}$$

Alternative method. To find the radius of curvature of the envelope of the given family of lines we can also proceed as follows :

The equations (1) and (2) are parametric equations of the envelope of the family of lines (1), the parameter being α.

First we shall solve (1) and (2) for x and y to express x and y as functions of the parameter α.

Multiplying (1) by $\cos\alpha$, (2) by $\sin\alpha$ and subtracting, we get

$$x = f(\alpha)\cos\alpha - f'(\alpha)\sin\alpha. \quad \text{...(6)}$$

Again multiplying (1) by $\sin\alpha$, (2) by $\cos\alpha$ and adding, we get

$$y = f(\alpha)\sin\alpha + f'(\alpha)\cos\alpha. \quad \text{..(7)}$$

Now from (6), $\dfrac{dx}{d\alpha} = f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha - f''(\alpha)\sin\alpha$

$$- f'(\alpha)\cos\alpha$$

$$= -[f(\alpha) + f''(\alpha)]\sin\alpha \quad \text{...(8)}$$

and from (7), $\dfrac{dy}{d\alpha} = f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha + f''(\alpha)\cos\alpha$

$$- f'(\alpha)\sin\alpha$$

$$= [f(\alpha) + f''(\alpha)]\cos\alpha. \quad \text{...(9)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\alpha}{dx/d\alpha} = -\cot\alpha.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\alpha) = \frac{d}{d\alpha}(-\cot\alpha)\cdot\frac{d\alpha}{dx}$$

$$= \operatorname{cosec}^2\alpha \cdot \frac{-1}{[f(\alpha) + f''(\alpha)]\sin\alpha} = \frac{-\operatorname{cosec}^3\alpha}{f(\alpha) + f''(\alpha)}.$$

Now the required radius of curvature ρ

$$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = [1 + \cot^2\alpha]^{3/2} \cdot \frac{f(\alpha) + f''(\alpha)}{-\operatorname{cosec}^3\alpha}$$

$$= -[f(\alpha) + f''(\alpha)].$$

Neglecting the negative sign, $\rho = f(\alpha) + f''(\alpha)$.

Ex. 35 (a). *Find the envelope of the straight lin*

$$x/a + y/b = 1, \quad \text{...(1)}$$

where the parameters a and b are related by the equation

$$a^n + b^n = c^n, \text{ c being a constant.} \quad \text{....(2)}$$

(Kaapur 1983; Agra 80, 78)

Sol. Let us regard a and b as functions of some other parameter t. Differentiating (1) and (2) w.r.t. t, taking x and y as constants, we have

$$\frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0 \quad \text{and} \quad a^{n-1}\frac{da}{dt} + b^{n-1}\frac{db}{dt} = 0.$$

Equating the values of $\dfrac{da/dt}{db/dt}$ obtained from these equations, we have

$$\frac{(x/a^2)}{a^{n-1}} = \frac{(y/b^2)}{b^{n-1}} \qquad ...(3)$$

or $$\frac{(x/a)}{a^n} = \frac{(y/b)}{b^n} = \frac{(x/a)+(y/b)}{a^n+b^n} = \frac{1}{c^n}, \text{ from (1) and (2).}$$

$$\left[\textbf{Note that } \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right].$$

Thus $a^{n+1} = c^n x$ and $b^{n+1} = c^n y$.

Substituting in (2), we have

$$(c^n x)^{n/(n+1)} + (c^n y)^{n/(n+1)} = c^n,$$

or $$x^{n/(n+1)} + y^{n/(n+1)} = c^{n/(n+1)},$$

which is the required equation of the envelope.

Ex. 35 (b). *Find the envelope of the family of straight lines*

$$x/a + y/b = 1,$$

where a, b *are connected by the relation (i)* $a + b = c$, *(ii)* $a^2 + b^2 = c^2$, *c is a constant.* **(Agra 1982)**

Sol. Proceed as in Ex. 35 (a).

Ans. (i) $x^{1/2} + y^{1/2} = c^{1/2}$. **(ii)** $x^{2/3} + y^{2/3} = c^{2/3}$.

Ex. 36. *Find the envelope of the straight lines* $x/a + y/b = 1$ *when* $a^m b^n = c^{m+n}$, *where c is a constant.*

(Lucknow 1981; Allahabad 77; Meerut 91)

Sol. The equation of the given family of straight lines is

$$x/a + y/b = 1, \qquad ...(1)$$

where the parameters a and b are connected by the relation

$$a^m b^n = c^{m+n}. \qquad ...(2)$$

Taking log of (2), we get

$$m \log a + n \log b = (m+n) \log c. \qquad ...(3)$$

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (3) w.r.t. t taking x and y as constants, we have

$$-\frac{x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0 \quad \text{or} \quad \frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0 \qquad ...(4)$$

and $$\frac{m}{a}\cdot\frac{da}{dt} + \frac{n}{b}\frac{db}{dt} = 0. \qquad ...(5)$$

From (4) and (5) comparing the coefficients of (da/dt) and (db/dt), we get

$$\frac{(x/a^2)}{(m/a)} = \frac{(y/b^2)}{(n/b)} \quad \text{or} \quad \frac{x/a}{m} = \frac{y/b}{n} = \frac{x/a + y/b}{m+n} = \frac{1}{m+n}.$$

(Note)

$\therefore \quad a = x(m+n)/m,\ b = y(m+n)/n.$

Substituting these values of a and b in (2), we get

$$\{x^m (m+n)^m/m^m\} \times \{y^n (m+n)^n/n^n\} = c^{m+n}$$

or $$(m+n)^{m+n} x^m y^n = m^m n^n c^{m+n},$$

which is the required envelope.

***Ex. 37.** *Prove that the envelope of ellipses having the axes of coordinates as principal axes and the sum of their semi-axes constant and equal to c, is the astroid* $x^{2/3} + y^{2/3} = c^{2/3}$.

Sol. Let the equation of the family of ellipses be

$$x^2/a^2 + y^2/b^2 = 1, \qquad ...(1)$$

where a and b are parameters.

Given that the sum of their semi-axes is equal to c, we have

$$a + b = c. \qquad ..(2)$$

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (2) w.r.t. t, we have

$$-\frac{2x^2}{a^3} \cdot \frac{da}{dt} - \frac{2y^2}{b^3} \cdot \frac{db}{dt} = 0 \quad \text{and} \quad \frac{da}{dt} + \frac{db}{dt} = 0.$$

Comparing the coefficients of (da/dt) and (db/dt), we get

$$\frac{x^2/a^3}{1} = \frac{y^2/b^3}{1} \text{ i.e., } \frac{x^2/a^2}{a} = \frac{y^2/b^2}{b} = \frac{x^2/a^2 + y^2/b^2}{a+b} = \frac{1}{c}.$$

$\therefore \quad a^3 = cx^2$ and $b^3 = cy^2$.

Substituting these values of a and b in (2), we have

$$c^{1/3} x^{2/3} + c^{1/3} y^{2/3} = c \quad \text{or} \quad x^{2/3} + y^{2/3} = c^{2/3}.$$

Hence the requied envelope is the astroid $x^{2/3} + y^{2/3} = c^{2/3}$.

Ex. 38. *Find the envelope of a system of concentric and co-axial eilipses of constant area.* **(Allahabad 1979)**

Sol. Let the equation of a system of concentric and coaxial ellipses be $x^2/a^2 + y^2/b^2 = 1$, where a and b are parameters. ..(1)

Also given that the area of each ellipse is constant, we have

$\pi ab = \pi c^2$ or $ab = c^2$, where c is a constant. ...(2)

Let us regard a and b as functions of t.

Differentiating (1) and (2) partially w.r.t. t, we get

$$-\frac{2x^2}{a^3} \cdot \frac{da}{dt} - \frac{2y^2}{b^3} \cdot \frac{db}{dt} = 0 \text{ and } b\frac{da}{dt} + a\frac{db}{dt} = 0.$$

Comparing the coefficients of (da/dt) and (db/dt), we get

$$\frac{x^2/a^3}{b} = \frac{y^2/b^3}{a} \text{ i.e., } \frac{x^2/a^2}{1} = \frac{y^2/b^2}{1} = \frac{x^2/a^2 + y^2/b^2}{1+1} = \frac{1}{2}.$$

$$\therefore \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2}$$

or $$a^2 = 2x^2 \quad \text{and} \quad b^2 = 2y^2.$$

$\therefore$ from (2), $2x^2 . 2y^2 = c^4$ or $4x^2y^2 = c^4$,

which is the required envelope.

Ex. 39. *Show that the envelope of the family of parabolas*

$$(x/a)^{1/2} + (y/b)^{1/2} = 1,$$

under the condition $ab = c^2$, is a hyperbola whose asymptotes coincide with the axes.

Sol. The given equation of the family of parabolas is

$$(x/a)^{1/2} + (y/b)^{1/2} = 1, \qquad ...(1)$$

where $$ab = c^2. \qquad ...(2)$$

Let us regard the parameters a and b as functions of t.

Differentiating (1) and (2) w.r.t. t taking x and y as constants, we have

$$\{-x^{1/2}/(2a^{3/2})\}\,(da/dt) + \{-y^{1/2}/(2b^{3/2})\}\,(db/dt) = 0$$

i.e., $$(x^{1/2}/a^{3/2})\,(da/dt) + (y^{1/2}/b^{3/2})\,(db/dt) = 0 \qquad ...(3)$$

and $$b\,(da/dt) + a\,(db/dt) = 0. \qquad ...(4)$$

Comparing the coefficients of (da/dt) and (db/dt) from (3) and (4), we have

$$\frac{x^{1/2}/a^{3/2}}{b} = \frac{y^{1/2}/b^{3/2}}{a} \quad \text{or} \quad (x/a)^{1/2} = (y/b)^{1/2}. \qquad \textbf{(Note)}$$

$\therefore$ from (1), $(x/a)^{1/2} = (y/b)^{1/2} = \frac{1}{2}$

or $x/a = y/b = \frac{1}{4}$ or $a = 4x,\ b = 4y$.

Substituting these values of a and b in (2), we have

$$4x . 4y = c^2 \quad \text{or} \quad 16xy = c^2.$$

Hence the required envelope is a hyperbola $16xy = c^2$, whose asymptotes are $x = 0,\ y = 0$ *i.e.*, the coordinate axes.

***Ex. 40.** *A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.* **(Meerut 1991 S)**

Sol. Take the two fixed straight lines at right angles as the coordinate axes OX and OY. Let the equation of the sliding line AB be $x/a + y/b = 1$ whose end A remains on the x-axis and B on the y-axis. Thus a and b are the intercepts OA and OB of the line AB on the axes of x and y respectively.

We have $a^2 + b^2 = OA^2 + OB^2 = AB^2 = l^2$, ...(1)

where l is the given length of the line AB.

The coordinates of the ends A and B are $(a, 0)$ and $(0, b)$ respectively. Hence the equation of the circle described on the line AB as diameter is

$$(x - a)\,(x - 0) + (y - 0)\,(y - b) = 0$$

or $$ax + by = x^2 + y^2. \qquad ...(2)$$

We have to find the envelope of (2) where the parameters a and b are connected by (1).

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (2) w.r.t. t taking x and y as constants, we have

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0 \text{ and } x\frac{da}{dt}+y\frac{db}{dt}=0.$$

Comparing the coefficients of da/dt and db/dt from these, we have

$$\frac{x}{a}=\frac{y}{b}$$

or $$\frac{ax}{a^2}=\frac{by}{b^2}=\frac{ax+by}{a^2+b^2}=\frac{x^2+y^2}{l^2}. \quad \textbf{(Note)}$$

$\therefore \quad a=l^2x/(x^2+y^2)$, and $b=l^2y/(x^2+y^2)$.

Substituting these values of a and b in (1), we have

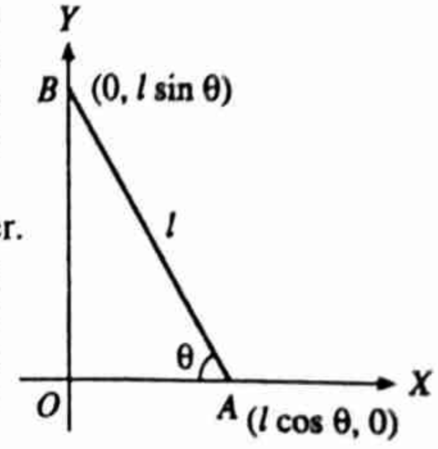
$$l^4x^2/(x^2+y^2)^2+l^4y^2/(x^2+y^2)^2=l^2$$

or $$l^4(x^2+y^2)=l^2(x^2+y^2)^2$$

or $$(x^2+y^2)=l^2,$$

which is the required envelope and is a circle.

Alternative solution. Take the two fixed straight lines at right angles as the coordinate axes. Let l be the length of the sliding line. Take any position AB of the sliding line.



Let $\angle BAO=\theta$, then θ is parameter.

The coordinates of A are $(l\cos\theta, 0)$ and those of B are $(0, l\sin\theta)$. Hence the equation of the circle described on the line AB as diameter is

$$(x-l\cos\theta)(x-0)+(y-0)(y-l\sin\theta)=0$$

or $$x^2+y^2-xl\cos\theta-yl\sin\theta=0. \quad ...(1)$$

We are to find the envelope of the family of circles (1), where θ is the parameter. The equation (1) may be written as

$$xl\cos\theta+yl\sin\theta=x^2+y^2. \quad ...(2)$$

Differentiating (2) partially with respect to θ, we get

$$-xl\sin\theta+yl\cos\theta=0. \quad ...(3)$$

Squaring and adding (2) and (3), we get

$$x^2l^2+y^2l^2=(x^2+y^2)^2$$

or $$l^2(x^2+y^2)=(x^2+y^2)^2$$

or $$x^2+y^2=l^2,$$

which is the required envelope and is a circle.

Ex. 41. *Find the envelope of the family of curves*

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1,$$

where the parameters a and b are connected by the relation

$$a^p+b^p=c^p.$$

(Allahabad 1976; Lucknow 82; Meerut 83, 89S, 97)

Sol. Here we shall give another method for solving a problem of two parameters connected by a relation.

The equation of the given family of curves is

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1, \qquad \text{...(1)}$$

where the parameters a and b are connected by the relation

$$a^p+b^p=c^p. \qquad \text{...(2)}$$

Since there is a relation between a and b, therefore we shall regard b as a function of a. Now we shall differentiate (1) and (2) with respect to a regarding x and y as constants and b as a function of a.

From (1), we get

$$-\frac{mx^m}{a^{m+1}}-\frac{my^m}{b^{m+1}}\frac{db}{da}=0$$

i.e,
$$\frac{db}{da}=-\frac{x^m/a^{m+1}}{y^m/b^{m+1}}. \qquad \text{...(3)}$$

Again from (2), we get

$$pa^{p-1}+pb^{p-1}(db/da)=0$$

i.e.,
$$db/da=-a^{p-1}/b^{p-1}. \qquad \text{...(4)}$$

Equating the two values of (db/da), we get

$$\frac{x^m/a^{m+1}}{y^m/b^{m+1}}=\frac{a^{p-1}}{b^{p-1}} \quad \text{or} \quad \frac{x^m/a^m}{y^m/b^m}=\frac{a^p}{b^p}. \qquad \text{...(5)}$$

Eliminating a and b between (1), (2) and (5), we get the required envelope. From (5), we have

$$\frac{x^m/a^m}{a^p}=\frac{y^m/b^m}{b^p}=\frac{x^m/a^m+y^m/b^m}{a^p+b^p}=\frac{1}{c^p}. \qquad \textbf{[Note]}$$

$\therefore \quad x^m/a^{p+m}=1/c^p \quad$ or $\quad a^{p+m}=x^m c^p$

or $\quad a=(x^m c^p)^{1/(p+m)}$

or $\quad a^p=(x^m c^p)^{p/(p+m)}=x^{mp/(p+m)}\,c^{p^2/(p+m)}.$

Similarly $b^p=y^{mp/(p+m)}\,c^{p^2/(p+m)}$.

Substituting these values of a^p and b^p in (2), we get

$$c^{p^2/(p+m)}\{x^{mp/(p+m)}+y^{mp/(p+m)}\}=c^p$$

or
$$x^{mp/(p+m)}+y^{mp/(p+m)}=c^{p-p^2/(p+m)}$$

or
$$x^{mp/(p+m)}+y^{mp/(p+m)}=c^{mp/(p+m)},$$

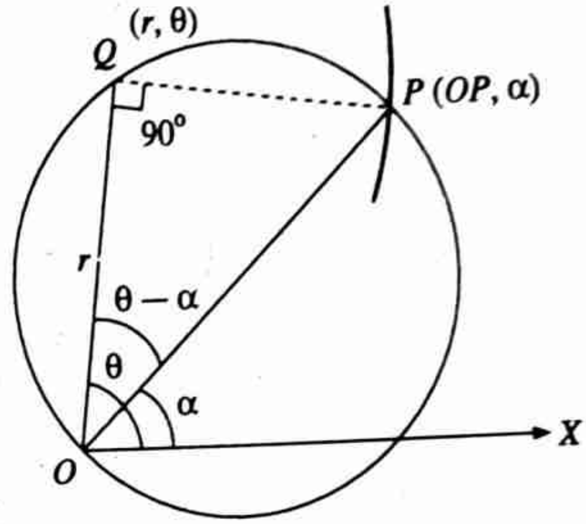
which is the required envelope.

Ex. 42. *Find the envelope of the circles drawn on the radii-vectors of the curve $r^n = a^n \cos n\theta$ as diameter.* **(Agra 1977; Meerut 84, 86)**

Sol. Let P be any point on the curve $r^n = a^n \cos n\theta$. If α is the vectorial angle of P, then the radius vector OP is given by

$$(OP)^n = a^n \cos n\alpha.$$

$$\therefore \quad OP = a (\cos n\alpha)^{1/n}.$$



Let Q be any point (r, θ) on the circle drawn on OP as diameter. From the right angled triangle OQP, we have

$$OQ = OP \cos \angle POQ.$$

$$\therefore \quad r = a (\cos n\alpha)^{1/n} \cos (\theta - \alpha), \quad \text{...(1)}$$

is the equation of the circle drawn on OP as diameter.

We are to find the envelope of the family of circles (1), where α is the parameter.

Taking logarithm of both sides of (1), we get

$$\log r = \log a + (1/n) \log \cos n\alpha + \log \cos (\theta - \alpha). \quad \text{...(2)}$$

Differentiating (2) partially with respect to α, we get

$$0 = 0 + \frac{1}{n} \frac{n}{\cos n\alpha} (-\sin n\alpha) + \frac{\sin (\theta - \alpha)}{\cos (\theta - \alpha)}$$

or $\quad \tan n\alpha = \tan (\theta - \alpha)$.

$\therefore \quad n\alpha = \theta - \alpha$ (taking principal values only)

or $\quad \alpha = \theta/(n + 1)$.

Substituting this value of α in (1), we get the required envelope.

Thus from (1), $r = a (\cos n\alpha)^{1/n} \cos n\alpha \qquad [\because \theta - \alpha = n\alpha]$

or $\quad r = a (\cos n\alpha)^{1 + 1/n}$ or $r = a (\cos n\alpha)^{(n + 1)/n}$

or $\quad r^{n/(n + 1)} = a^{n/(n + 1)} \cos \{n\theta/(n + 1)\}$,

which is the required envelope.

Ex. 43 (a). *Find the envelope of the straight lines drawn at right angles to the radii vectors of the cardioid $r = a(1 + \cos \theta)$ through their extremities.*

Sol. Let P be any point on the cardioid $r = a (1 + \cos \theta)$. If α is the vectorial angle of P, then the radius vector OP is given by

$$OP = a (1 + \cos \alpha).$$

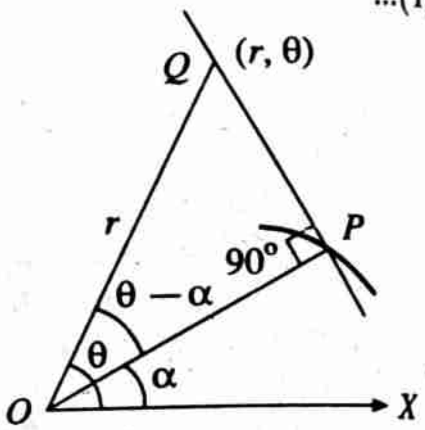
Let Q be any point on the straight line drawn through P and at right angles to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ.$$

$\therefore$ $$a(1 + \cos\alpha) = r\cos(\theta - \alpha)$$

or $$r\cos(\theta - \alpha) = 2a\cos^2 \tfrac{1}{2}\alpha, \quad ...(1)$$

is the equation of the straight line drawn through P and at right angles to OP.



We are to find the envelope of the family of straight lines (1), where α is the parameter.

Taking logarithm of both sides of (1), we get

$$\log r + \log\cos(\theta - \alpha) = \log 2a + 2\log\cos(\alpha/2). \quad ...(2)$$

Differentiating (2) partially with respect to α, we get

$$0 + \frac{1}{\cos(\theta - \alpha)}\sin(\theta - \alpha) = 0 - \frac{2}{\cos\frac{1}{2}\alpha}(\sin\tfrac{1}{2}\alpha)\cdot\tfrac{1}{2}$$

or $$\tan(\theta - \alpha) = -\tan\tfrac{1}{2}\alpha = \tan(-\tfrac{1}{2}\alpha).$$

$\therefore$ $\theta - \alpha = n\pi - \frac{1}{2}\alpha$, where n is any integer.

$\therefore$ $\alpha = 2\theta - 2n\pi.$

Substituting this value of α in (1), we get

$$r\cos\{\theta - (2\theta - 2n\pi)\} = 2a\cos^2(\theta - n\pi)$$

or $$r\cos(2n\pi - \theta) = 2a\cos^2\theta$$

or $$r\cos\theta = 2a\cos^2\theta \quad \text{or} \quad r = 2a\cos\theta.$$

Therefore the required envelope is $r = 2a\cos\theta$. It is a circle passing through the pole.

Ex. 43 (b). *Find the envelopes of the straight lines drawn at right angles to the radii vectors of the following curves through their extremities.*

(i) $r = ae^{\theta\cot\alpha}$, **(Meerut 1984 R)**

(ii) $r^n = a^n \cos n\theta$, **(iii)** $r = a + b\cos\theta$.

Sol. (i). The equation of the given curve is $r = ae^{\theta\cot\alpha}$. ...(1)

Let (r_1, θ_1) be any point P on (1). [See fig. of Ex. 43 (a)]

Then $$r_1 = ae^{\theta_1\cot\alpha}, \quad ...(2)$$

r_1 and θ_1 being parameters.

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP = OQ\cos\angle POQ$$

or $$r_1 = r\cos(\theta - \theta_1)$$

or $$r\cos(\theta - \theta_1) = ae^{\theta_1\cot\alpha}, \quad ...(3)$$

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

Taking logarithm of both sides of (3), we get

$$\log r + \log \cos (\theta - \theta_1) = \log a + \theta_1 \cot \alpha. \quad \text{...(4)}$$

Differentiating (4) partially with respect to θ_1, we get

$$0 + \frac{1}{\cos (\theta - \theta_1)} \cdot [-\sin (\theta - \theta_1)] . (-1) = 0 + \cot \alpha$$

or $\tan (\theta - \theta_1) = \cot \alpha = \tan (\frac{1}{2}\pi - \alpha)$.

$\therefore \quad \theta - \theta_1 = \frac{1}{2}\pi - \alpha \quad$ or $\quad \theta_1 = \theta + \alpha - \frac{1}{2}\pi$.

Substituting this value of θ_1 in (3), the required envelope of the family of straight lines (3) is

$$r \cos (\tfrac{1}{2}\pi - \alpha) = a\, e^{(\theta + \alpha - \frac{1}{2}\pi) \cot \alpha}$$

or $\quad r \sin \alpha = a\, e^{(\alpha - \frac{1}{2}\pi) \cot \alpha}\, e^{\theta \cot \alpha}$.

(ii) The equation of the given curve is $r^n = a^n \cos n\theta$. ...(1)

Let (r_1, θ_1) be any point P on (1). Then r_1 and θ_1 are parameters connected by

$$r_1^n = a^n \cos n\theta_1. \quad \text{...(2)}$$

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ$$

or $\quad r_1 = r \cos (\theta - \theta_1)$

or $\quad r \cos (\theta - \theta_1) = a (\cos n\theta_1)^{1/n}$, ...(3)

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

Taking logarithm of both sides of (3), we get

$$\log r + \log \cos (\theta - \theta_1) = \log a + (1/n) \log \cos n\theta_1. \quad \text{...(4)}$$

Differentiating (4) partially with respect to θ_1, we get

$$0 + \frac{1}{\cos (\theta - \theta_1)} \sin (\theta - \theta_1) = 0 + \frac{1}{n} \cdot \frac{-n \sin n\theta_1}{\cos n\theta_1}.$$

or $\quad \tan (\theta - \theta_1) = -\tan n\theta_1 = \tan (-n\theta_1)$.

$\therefore \quad \theta - \theta_1 = -n\theta_1 \quad$ or $\quad \theta_1 (1 - n) = \theta \quad$ or $\quad \theta_1 = \theta/(1 - n)$.

Substituting this value of θ_1 in (3), the required envelope of the family of straight lines (3) is

$$r\cos\left[\theta-\frac{\theta}{1-n}\right]=a\left[\cos\left(\frac{n\theta}{1-n}\right)\right]^{1/n}$$

or $$r\cos\left(\frac{\theta-n\theta-n}{1-n}\right)=a\left[\cos\left(\frac{n\theta}{1-n}\right)\right]^{1/n}$$

or $$r\cos\left(\frac{n\theta}{1-n}\right)=a\left[\cos\left(\frac{n\theta}{1-n}\right)\right]^{1/n}$$

or $$r=a\left[\cos\left(\frac{n\theta}{1-n}\right)\right]^{(1/n)-1}=a\left[\cos\left(\frac{n\theta}{1-n}\right)\right]^{(1-n)/n}$$

or $$r^{n/(1-n)}=a^{n/(1-n)}\cos\{n\theta/(1-n)\},$$

raising both sides to the power $n/(1-n)$.

(iii) The equation of the given curve is $r=a+b\cos\theta$. ...(1)

Let (r_1, θ_1) be any point P on (1). Then r_1 and θ_1 are parameters connected by

$$r_1=a+b\cos\theta_1. \qquad \text{...(2)}$$

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP=OQ\cos\angle POQ$$

or $$r_1=r\cos(\theta-\theta_1)$$

or $$r\cos(\theta-\theta_1)=a+b\cos\theta_1, \qquad \text{...(3)}$$

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

The equation (3) may be written as

$$r\cos\theta\cos\theta_1+r\sin\theta\sin\theta_1=a+b\cos\theta_1$$

or $$(r\cos\theta-b)\cos\theta_1+r\sin\theta\sin\theta_1=a. \qquad \text{...(4)}$$

Differentiating (4) partially with respect to θ_1, we get

$$-(r\cos\theta-b)\sin\theta_1+r\sin\theta\cos\theta_1=0. \qquad \text{...(5)}$$

To eliminate θ_1, squaring and adding (4) and (5), we get

$$(r\cos\theta-b)^2+r^2\sin^2\theta=a^2$$

or $$r^2-2br\cos\theta+b^2-a^2=0,$$

which is the required envelope of the family of straight lines (3).

Ex. 44. *Find the envelopes of circles described on the radii vectors of the following curves as diameters :*

(a) $l/r=1+e\cos\theta$ **(b)** $r^3=a^3\cos 3\theta$

(c) $r\cos^n(\theta/n)=a$. **(Meerut 1984 S)**

Sol. (a). For figure see Ex. 42. Let $P(OP, \alpha)$ be any point on the curve $l/r=1+e\cos\theta$. Then

$$l/OP = 1 + e \cos \alpha. \qquad ...(1)$$

Let (r, θ) be any point Q on the circle described on OP as diameter. Now from the figure we have the equation of the circle as

$$r = OP \cos (\theta - \alpha)$$

or $\quad r = \{l/(1 + e \cos \alpha)\} \cos (\theta - \alpha),$ [from (1)]

or $\quad r \{1 + e \cos \alpha\} = l (\cos \theta \cos \alpha + \sin \theta \sin \alpha)$

or $\quad r (l \cos \theta - re) \cos \alpha + l \sin \theta \sin \alpha = r.$ ...(3)

We have to find the envelope of the family of circles (3), where α is the parameter.

Partially differentiating (3) w.r.t. α, we have

$$-(l \cos \theta - re) \sin \alpha + l \sin \theta \cos \alpha = 0. \qquad ...(4)$$

Squaring (3) and (4) and adding, we have

$$(l \cos \theta - re)^2 + l^2 \sin^2 \theta = r^2$$

or $\quad l^2 \cos^2 \theta - 2\, ler \cos \theta + r^2 e^2 + l^2 \sin^2 \theta = r^2$

or $\quad l^2 - 2ler \cos \theta + r^2 e^2 = r^2$, which is the required envelope.

(b) The equation of the given curve is $r^3 = a^3 \cos 3\theta$. ...(1)

Let (r_1, θ_1) be any point P on (1).

Then $\quad r_1^3 = a^3 \cos 3\theta_1.$ ...(2)

Equation of the circle drawn on OP as diameter is

$$r = r_1 \cos (\theta - \theta_1).$$ [Proceed as in part (a)]

Substituting for r_1 from (2), this equation becomes

$$r = a (\cos 3\theta_1)^{1/3} \cos (\theta - \theta_1), \qquad ...(3)$$

where θ_1 is the parameter.

Taking log, $\log r = \log a + \frac{1}{3} \log \cos 3\theta_1 + \log \cos (\theta - \theta_1)$.

Differentiating partially w.r.t. θ_1, we have

$$0 = 0 + \tfrac{1}{3} (-3 \sin 3\theta_1 / \cos 3\theta_1) + \sin (\theta - \theta_1)/\cos (\theta - \theta_1)$$

or $\quad \tan 3\theta_1 = \tan (\theta - \theta_1)$ *i.e.*, $3\theta_1 = \theta - \theta_1$ or $\theta_1 = \theta/4$.

Substituting this value of θ_1 in (3), we have

$$r = a \{\cos (3\theta/4)\}^{1/3} \cos (\theta - \theta/4) = a \{\cos (3\theta/4)\}^{4/3}$$

or $\quad r^3 = a^3 \cos^4 (3\theta/4)$, which is the required envelope.

(c) The given equation is $r \cos^n (\theta/n) = a$. ...(1)

Let (r_1, θ_1) be any point P on (1) ; then

$$r_1 \cos^n (\theta_1/n) = a. \qquad ...(2)$$

Equation of the circle drawn on OP as diameter is

$$r = r_1 \cos (\theta - \theta_1).$$

Substituting for r_1 from (2), this equation becomes

$$r = a \cos (\theta - \theta_1)/\cos^n (\theta_1/n), \qquad ...(3)$$

where θ_1 is the parameter.

Taking log, $\log r = \log a + \log\cos(\theta - \theta_1) - n\log\cos(\theta_1/n)$.

Differentiating partially w.r.t. θ_1, we have

$$0 = 0 + \frac{\sin(\theta-\theta_1)}{\cos(\theta-\theta_1)} - n\left\{-(1/n)\frac{\sin(\theta_1/n)}{\cos(\theta_1/n)}\right\}$$

or $\quad \tan(\theta-\theta_1) + \tan(\theta_1/n) = 0$

or $\quad \tan(\theta-\theta_1) = -\tan(\theta_1/n) = \tan(-\theta_1/n)$

or $\quad \theta - \theta_1 = -\theta_1/n \quad$ or $\quad \theta = \theta_1(1 - 1/n) = \theta_1(n-1)/n$

or $\quad \theta_1 = n\theta/(n-1)$.

Substituting this value of θ_1 in (3), we have

$$r = \frac{n\cos\{\theta - n\theta/(n-1)\}}{\cos^n\{\theta/(n-1)\}} = \frac{a\cos\{-\theta/(n-1)\}}{\cos^n\{\theta/(n-1)\}}$$

$$= \frac{a}{\cos^{n-1}\{\theta/(n-1)\}}$$

or $\quad r\cos^{n-1}\{\theta/(n-1)\} = a$, which is the required envelope.

Ex. 45. *Find the envelopes of the straight lines, drawn through the extremities of, and at right angles to the radii vectors of the following curves :*

(a) $r\cos(\theta+\alpha) = p$, $\qquad$ **(b)** $r^n\cos n\theta = a^n$.

Sol. (a). Draw figure taking help from the figure of Ex. 43 (a). The given equation is $r\cos(\theta+\alpha) = p$. ...(1)

Let (r_1, θ_1) be any point P on the curve (1), then

$$r_1\cos(\theta_1+\alpha) = p. \qquad ...(2)$$

Now the equation of the line through (r_1, θ_1) and perpendicular to OP is $r = r_1\sec(\theta-\theta_1)$.

Substituting for r_1 from (2), this equation becomes

$$r = p\frac{\sec(\theta-\theta_1)}{\cos(\theta_1+\alpha)} = p\sec(\theta-\theta_1)\sec(\theta_1+\alpha). \qquad ...(3)$$

We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

Taking logarithm of both sides of (3), we get

$$\log r = \log p + \log\sec(\theta-\theta_1) + \log\sec(\theta_1+\alpha).$$

Differentiating partially w.r.t θ_1, we get

$$0 = 0 + \frac{-\sec(\theta-\theta_1)\tan(\theta-\theta_1)}{\sec(\theta-\theta_1)} + \frac{\sec(\theta_1+\alpha)\tan(\theta_1+\alpha)}{\sec(\theta_1+\alpha)}$$

or $\quad \tan(\theta-\theta_1) = \tan(\theta_1+\alpha)$

or $\quad \theta - \theta_1 = \theta_1 + \alpha \quad$ or $\quad \theta_1 = (\theta - \alpha)/2.$

Substituting this value of θ_1 in (3), we have

$$r = p \sec \left(\theta - \frac{\theta - \alpha}{2}\right) \sec \left(\frac{\theta - \alpha}{2} + \alpha\right)$$

or $\quad r = p \sec \frac{1}{2}(\theta + \alpha) \sec \frac{1}{2}(\theta + \alpha)$

or $\quad r \cos^2 \frac{1}{2}(\theta + \alpha) = p$, which is the required envelope.

(b) The given equation is $r^n \cos n\theta = a^n$. ...(1)

Let (r_1, θ_1) be any point P on the curve (1); then

$$r_1^n = a^n \sec n\theta_1. \qquad ...(2)$$

The equation of the line through (r_1, θ_1) and perpendicular to OP is

$$r = r_1 \sec(\theta - \theta_1).$$

Substituting for r_1 from (2), this equation becomes

$$r = a (\sec n\theta_1)^{1/n} \sec(\theta - \theta_1), \qquad ...(3)$$

where θ_1 is the parameter.

Taking logarithm of both sides of (3), we get

$$\log r = \log a + (1/n) \log \sec n\theta_1 + \log \sec(\theta - \theta_1).$$

Differentiating partially w.r.t. θ_1, we have

$$0 = 0 + \frac{1}{n} \cdot \frac{n \sec n\theta_1 \tan n\theta_1}{\sec \theta_1} - \frac{\sec(\theta - \theta_1)\tan(\theta - \theta_1)}{\sec(\theta - \theta_1)}$$

or $\quad \tan(\theta - \theta_1) = \tan n\theta_1$

or $\quad \theta - \theta_1 = n\theta_1 \quad$ or $\quad \theta_1 = \theta/(n+1).$

Substituting this value of θ_1 in (3), we have

$$r = a [\sec \{n\theta/(n+1)\}]^{1/n} \sec \{\theta - \theta/(n+1)\}$$

or $\quad r = a [\sec \{n\theta/(n+1)\}]^{(n+1)/n}$

or $\quad r^{n/(n+1)} \cos \{n\theta/(n+1)\} = a^{n/(n+1)}$,

which is the required envelope.

§ 6. Evolute of a curve.

The evolute of a curve is the envelope of the normals of that curve.

(Agra 1980; Meerut 84 S)

We define the evolute of a curve as the locus of the centre of curvature for that curve. But the centre of curvature of a curve for a given point P on it is the limiting position of the intersection of the normal at P with the normal at any other consecutive point Q as $Q \to P$. Therefore by the definition of envelope, *the envelope of the normals to a curve is the evolute of that curve.*

**Ex. 46(a). *Find the evolute of the parabola $y^2 = 4ax$.*
(Delhi 1983; Agra 83; Meerut 83S, 94P, 98)

Sol. We know that the evolute of a curve is the envelope of the normals to that curve.

Equation of any normal to the parabola $y^2 = 4ax$ is
$$y = mx - 2am - am^3, \quad \text{...(1)}$$
where m is the parameter.

So the envelope of (1) is the evolute of $y^2 = 4ax$.

Differentiating (1) partially with respect to m, we get
$$0 = x - 2a - 3am^2 \text{ i.e., } m = \{(x - 2a)/3a\}^{1/2}.$$
Substituting this value of m in (1), we get
$$y = \left(\frac{x - 2a}{3a}\right)^{1/2}\left[x - 2a - a \cdot \frac{x - 2a}{3a}\right]$$
$$= \left(\frac{x - 2a}{3a}\right)^{1/2}(x - 2a) \cdot \frac{2}{3} = \frac{2\,(x - 2a)^{3/2}}{3\sqrt{(3a)}}.$$
Squaring, we get $27ay^2 = 4\,(x - 2a)^3$,
which is the required evolute.

Ex. 46(b). *Find the equation of the evolute of the parabola*
$$y^2 = 2px.$$ **(Meerut 1994P, 95)**

Sol. Equation of any normal to the parabola $y^2 = 2px$ is
$$y = mx - pm - \tfrac{1}{2}pm^3,$$
where m is the parameter.

Now proceed as in Ex. 46 (a). The required evolute is
$$27py^2 = 8\,(x - p)^3.$$

Ex. 46(c). *Prove that the normals to a given curve are always tangent lines to its evolute.* **(Meerut 1993P)**

Sol. We know that the evolute of a given curve is the envelope of the family of normals to that curve.

Again we know that the envelope of a family of curves touches each member of the family.

Hence the normals to a given curve are always tangent lines to its evolute.

Ex. 47 (a). *Find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ onsidered as the envelope of its normals.*
(Rohilkhand 1983; Agra 81; Meerut 94, 95BP)

Sol. The given ellipse is $x^2/a^2 + y^2/b^2 = 1$. ...(1)

The evolute of the ellipse (1) is the envelope of the family of normals to the ellipse (1). The coordinates (x, y) of any point P on the ellipse (1) may be taken as
$$x = a\cos\theta,\ y = b\sin\theta, \text{ where } \theta \text{ is parameter.}$$

We have $dx/d\theta = -a \sin\theta$, $dy/d\theta = b \cos\theta$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b \cos\theta}{-a \sin\theta}$.

$\therefore$ slope of the normal to the ellipse (1) at the point

$$(a \cos\theta, b \sin\theta) = -\frac{dx}{dy} = \frac{a \sin\theta}{b \cos\theta}.$$

$\therefore$ Equation of the normal to the ellipse (1) at the point $(a \cos\theta, b \sin\theta)$ is

$$y - b \sin\theta = \frac{a \sin\theta}{b \cos\theta}(x - a \cos\theta)$$

or $\quad by \cos\theta - b^2 \sin\theta \cos\theta = ax \sin\theta - a^2 \sin\theta \cos\theta$

or $\quad ax \sin\theta - by \cos\theta = (a^2 - b^2) \sin\theta \cos\theta$

or $\quad \dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2.$ ...(2)

Now the evolute of the ellipse (1) is the envelope of the family (2) of normals to the ellipse (1), where θ is the parameter.

To find the envelope of the family of straight lines (2), proceed as in Ex. 7.

Thus the envelope of the family of straight lines (2) is the curve $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$,
which is the required evolute of the ellipse (1).

Ex. 47 (b). *Find the evolute of the hyperbola $x^2/a^2 - y^2/b^2 = 1$.*

Sol. The given hyperbola is $x^2/a^2 - y^2/b^2 = 1$. ...(1)

The evolute of the hyperbola (1) is the envelope of the family of normals to the hyperbola (1). The coordinates (x, y) of any point P on the hyperbola (1) may be taken as

$x = a \sec\theta, y = b \tan\theta$, where θ is parameter.

We have $dx/d\theta = a \sec\theta \tan\theta$, $dy/d\theta = b \sec^2\theta$.

$\therefore$ slope of the normal to the hyperbola (1) at the point

$$(a \sec\theta, b \tan\theta) = -\frac{dx}{dy} = -\frac{dx/d\theta}{dy/d\theta} = -\frac{a \sec\theta \tan\theta}{b \sec^2\theta}$$

$$= -\frac{a \tan\theta}{b \sec\theta}.$$

$\therefore$ Equation of the normal to the hyperbola (1) at the point $(a \sec\theta, b \tan\theta)$ is

$$y - b \tan\theta = -\frac{a \tan\theta}{b \sec\theta}(x - a \sec\theta)$$

or $\quad ax \tan\theta + by \sec\theta = (a^2 + b^2) \sec\theta \tan\theta$

or $\quad ax \cos\theta + by \cot\theta = a^2 + b^2.$ ...(2)

Now the evolute of the hyperbola (1) is the envelope of the family of straight lines (2), where θ is the parameter.

Differentiating (2) partially with respect to θ, we get

$$-ax\sin\theta - by\operatorname{cosec}^2\theta = 0$$

or $$ax\sin\theta = -by\operatorname{cosec}^2\theta$$

or $$\sin^3\theta = -\frac{by}{ax} \quad \text{or} \quad \sin\theta = -\frac{(by)^{1/3}}{(ax)^{1/3}}.$$

$$\therefore \quad \cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{\left[1 - \frac{(by)^{2/3}}{(ax)^{2/3}}\right]}$$

$$= \frac{[(ax)^{2/3} - (by)^{2/3}]^{1/2}}{(ax)^{1/3}}$$

and $$\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{[(ax)^{2/3} - (by)^{2/3}]^{1/2}}{(by)^{1/3}}.$$

Substituting the values of $\cos\theta$ and $\cot\theta$ in (2), the envelope of the family of straight lines (2) is

$$ax\cdot\frac{[(ax)^{2/3} - (by)^{2/3}]^{1/2}}{(ax)^{1/3}} - by\cdot\frac{[(ax)^{2/3} - (by)^{2/3}]^{1/2}}{(by)^{1/3}} = a^2 + b^2$$

or $$(ax)^{2/3}[(ax)^{2/3} - (by)^{2/3}]^{1/2} - (by)^{2/3}[(ax)^{2/3} - (by)^{2/3}]^{1/2} = a^2 + b^2$$

or $$[(ax)^{2/3} - (by)^{2/3}][(ax)^{2/3} - (by)^{2/3}]^{1/2} = a^2 + b^2$$

or $$[(ax)^{2/3} - (by)^{2/3}]^{3/2} = (a^2 + b^2)$$

or $$(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3},$$

which is the required evolute of the hyperbola (1).

Ex. 47 (c). *Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.*

(Meerut 1990P, 92)

Sol. The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The evolute of the curve (1) is the envelope of the family of the normals to the curve (1). The coordinates (x, y) of any point P on the curve (1) may be taken as

$$x = a\cos^3\theta,\ y = a\sin^3\theta \text{ where } \theta \text{ is parameter.}$$

We have $dx/d\theta = -3a\cos^2\theta\sin\theta$, $dy/d\theta = 3a\sin^2\theta\cos\theta$.

$\therefore$ slope of the normal to the curve (1) at the point

$$(a\cos^3\theta, a\sin^3\theta) = -\frac{dx}{dy} = -\frac{dx/d\theta}{dy/d\theta} = -\frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta}.$$

$\therefore$ Equation of the normal to the curve (1) at the point $(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = \frac{\cos\theta}{\sin\theta}(x - a\cos^3\theta)$$

or $$x\cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta)$$

$$= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$

or $\qquad x\cos\theta - y\sin\theta = a(\cos^2\theta - \sin^2\theta)$. ...(2)

Now the evolute of the curve (1) is the envelope of the family of straight lines (2), where θ is parameter.

Differentiating (2) partially with respect to θ, we get

$$-x\sin\theta - y\cos\theta = a(-2\cos\theta\sin\theta - 2\sin\theta\cos\theta)$$

or $\qquad x\sin\theta + y\cos\theta = 4a\sin\theta\cos\theta$. ...(3)

Now to find the envelope of the family of straight lines (2), we have to eliminate θ between (2) and (3).

Solving (2) and (3) for x and y, we get

$$x = a(\cos^3\theta + 3\cos\theta\sin^2\theta)$$

and $\qquad y = a(3\sin\theta\cos^2\theta + \sin^3\theta)$.

$$\therefore\quad x + y = a(\cos^3\theta + 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + \sin^3\theta)$$
$$= a(\cos\theta + \sin\theta)^3$$

and $x - y = a(\cos^3\theta - 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta - \sin^3\theta)$

$$= a(\cos\theta - \sin\theta)^3.$$

$$\therefore\quad (x+y)^{2/3} + (x-y)^{2/3}$$
$$= a^{2/3}[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2]$$

or $\qquad (x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$,

which is the required evolute of the curve (1).

Ex. 48. *Prove that the evolute (i.e., the locus of the centre of curvature) of the hyperbola $2xy = a^2$ is*

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}.$$

(Delhi 1982; Agra 75; Meerut 91 P, 98)

Sol. The given curve is $xy = a^2/2 = c^2$, (say). ...(1)

The evolute of (1) is the envelope of the normals of (1).

Let $P(ct, c/t)$ be any point on the hyperbola (1).

Also from (1), differentiating we get $dy/dx = -c^2/x^2$.

$\therefore$ the slope of the normal to (1) at $P = -dx/dy = x^2/c^2 = t^2$.

Hence the equation of the normal at P is

$$y - c/t = t^2(x - ct) \qquad ...(2)$$

Differentiating (2) partially w.r.t. t, we get

$$c/t^2 = 2tx - 3ct^2 \quad \text{or} \quad 2x = 3ct + c/t^3. \qquad ...(3)$$

Substituting this value of x in (2), we get

$$2y = 3c/t + ct^3. \qquad ...(4)$$

Adding (3) and (4), we have $2(x+y) = c(t + 1/t)^3$ ...(5)

Subtracting (4) from (3), we have $2(x-y) = c\{(1/t) - t\}^3$ ...(6)

Eliminating t from (5) and (6), we have

$$2^{2/3}[(x+y)^{2/3} - (x-y)^{2/3}] = 4c^{2/3},$$

[Raising (5) and (6) to the power 2/3 and subtracting]

or $\qquad (x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3} = (4a/\sqrt{2})^{2/3}$

$$= (2\sqrt{2}\,a)^{2/3} = 2a^{2/3},$$

which is the required evolute.

Ex. 49. *Prove that evolute of the ellipses $b^2x^2 + a^2y^2 = a^2b^2$ is the envelope of the family of ellipses given by*

$$a^2x^2 \sec^4\alpha + b^2y^2 \operatorname{cosec}^4\alpha = (a^2 - b^2)^2,$$

α being the variable parameter. **(Meerut 1990)**

Sol. The given ellipse is $b^2x^2 + a^2y^2 = a^2b^2$, or $x^2/a^2 + y^2/b^2 = 1$. The evolute of this ellipse is

$$(\mathbf{ax})^{2/3} - (\mathbf{by})^{2/3} = (\mathbf{a}^2 - \mathbf{b}^2)^{2/3}. \qquad \text{[See Ex. 47 (a)]}$$

The given family of ellipses is

$$a^2x^2 \sec^4\alpha + b^2y^2 \operatorname{cosec}^4\alpha = (a^2 - b^2)^2, \qquad ...(1)$$

where α is the parameter.

Differentiating (1) partially w.r.t. α, we have

$$4a^2x^2 \sec^4\alpha \tan\alpha - 4b^2y^2 \operatorname{cosec}^4\alpha \cot\alpha = 0$$

or $$\frac{4a^2x^2 \sin\alpha}{\cos^5\alpha} - \frac{4b^2y^2 \cos\alpha}{\sin^5\alpha} = 0$$

or $$a^2x^2 \sin^6\alpha - b^2y^2 \cos^6\alpha = 0 \quad \text{or} \quad \tan^6\alpha = (b^2y^2/a^2x^2)$$

or $$\tan\alpha = (by)^{1/3}/(ax)^{1/3}.$$

$$\therefore \quad \sec\alpha = \frac{\{(by)^{2/3} + (ax)^{2/3}\}^{1/2}}{(ax)^{1/3}},$$

$$\operatorname{cosec}\alpha = \frac{\{(by)^{2/3} + (ax)^{2/3}\}^{1/2}}{(by)^{1/3}}.$$

Substituting these values in (1), we have

$$\frac{(ax)^2\,\{(by)^{2/3} + (ax)^{2/3}\}^2}{(ax)^{4/3}} + \frac{(by)^2\,\{(by)^{2/3} + (ax)^{2/3}\}^2}{(by)^{4/3}} = (a^2 - b^2)^2$$

or $$\{(ax)^{2/3} + (by)^{2/3}\}^2\,\{(ax)^{2/3} + (by)^{2/3}\} = (a^2 - b^2)^2$$

or $$\{(ax)^{2/3} + (by)^{2/3}\}^3 = (a^2 - b^2)^2$$

or $$(\mathbf{ax})^{2/3} + (\mathbf{by})^{2/3} = (\mathbf{a}^2 - \mathbf{b}^2)^{2/3},$$

which is the envelope of the family of ellipses (1). We see that this envelope is the same as the evolute of the given ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Ex. 50. *Prove that the evolute of the tractrix*

$$x = a\,(\cos t + \log\tan \tfrac{1}{2}t),\ y = a \sin t$$

is the catenary $y = a\cosh(x/a)$.

(Rohilkhand 1977; Kanpur 76; Meerut 84 S, 86 S, 89 P, 90 S, 91 S)

Sol. The given tractrix is $x = a\,(\cos t + \log\tan\frac{1}{2}t)$, $y = a\sin t$.

Differentiating these equations w.r.t. t, we get

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{\frac{1}{2}\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}\right\} = a\left\{-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right\}$$

$$= a\,\{-\sin t + (1/\sin t)\} = a\,(1-\sin^2 t)/\sin t = a\cos^2 t/\sin t$$

and $\dfrac{dy}{dt} = a\cos t$. $\quad\therefore\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a\cos^2 t/\sin t}{a\cos t} = \tan t$.

$\therefore$ the slope of the normal at the point 't'

$$= -(dx/dy) = -\cot t.$$

Hence the equation of the normal at the point 't' of the tractrix is

$$y - a\sin t = -\cot t\,(x - a\cos t - a\log\tan\tfrac{1}{2}t)$$

or $\quad \tan t\,(y - a\sin t) + x - a\cos t - a\log\tan\frac{1}{2}t = 0$

or $\quad x + y\tan t - a\sin^2 t/\cos t - a\cos t - a\log\tan\frac{1}{2}t = 0$

or $\quad x + y\tan t - a\,(\sin^2 t + \cos^2 t)/\cos t - a\log\tan\frac{1}{2}t = 0$

or $\quad x + y\tan t - a\sec t - a\log\tan\frac{1}{2}t = 0. \qquad$ ...(1)

Now (1) is the equation of the family of normals of the given tractrix, the parameter being t. The envelope of the family (1) is the evolute of the given tractrix.

Differentiating (1) partially w.r.t. 't', we have

$$y\sec^2 t - a\sec t\tan t - \tfrac{1}{2}a\,(\sec\tfrac{1}{2}t/\tan\tfrac{1}{2}t) = 0$$

or $\quad y\sec^2 t - a\sec t\tan t - a\,\mathrm{cosec}\,t = 0$

or $\quad \dfrac{y}{\cos^2 t} - \dfrac{a\sin t}{\cos^2 t} - \dfrac{a}{\sin t} = 0$

or $\quad y\sin t - a\,(\sin^2 t + \cos^2 t) = 0$

or $\quad y\sin t = a \quad$ or $\quad y = a/\sin t. \qquad$ ...(2)

Substituting this value of y in (1), we have

$$x + a\tan t/\sin t - a\sec t - a\log\tan\tfrac{1}{2}t = 0$$

or $\quad x + a\sec t - a\sec t - a\log\tan\frac{1}{2}t = 0$

or $\quad x = a\log\tan\frac{1}{2}t \quad$ or $\quad x/a = \log\tan\frac{1}{2}t$

or $\quad \tan\frac{1}{2}t = e^{x/a}. \qquad$ ...(3)

Now from (2), $y = \dfrac{a}{\sin t} = \dfrac{a\,(1+\tan^2\frac{1}{2}t)}{2\tan\frac{1}{2}t}$

$$= \tfrac{1}{2}a\,(\cot\tfrac{1}{2}t + \tan\tfrac{1}{2}t) = \tfrac{1}{2}a\,(e^{x/a} + e^{-x/a}), \qquad \text{from (3)}$$

$$= a\cosh(x/a).$$

Hence the envelope of the family of normals (1) *i.e.*, the evolute of the given tractrix is $y = a\cosh(x/a)$.

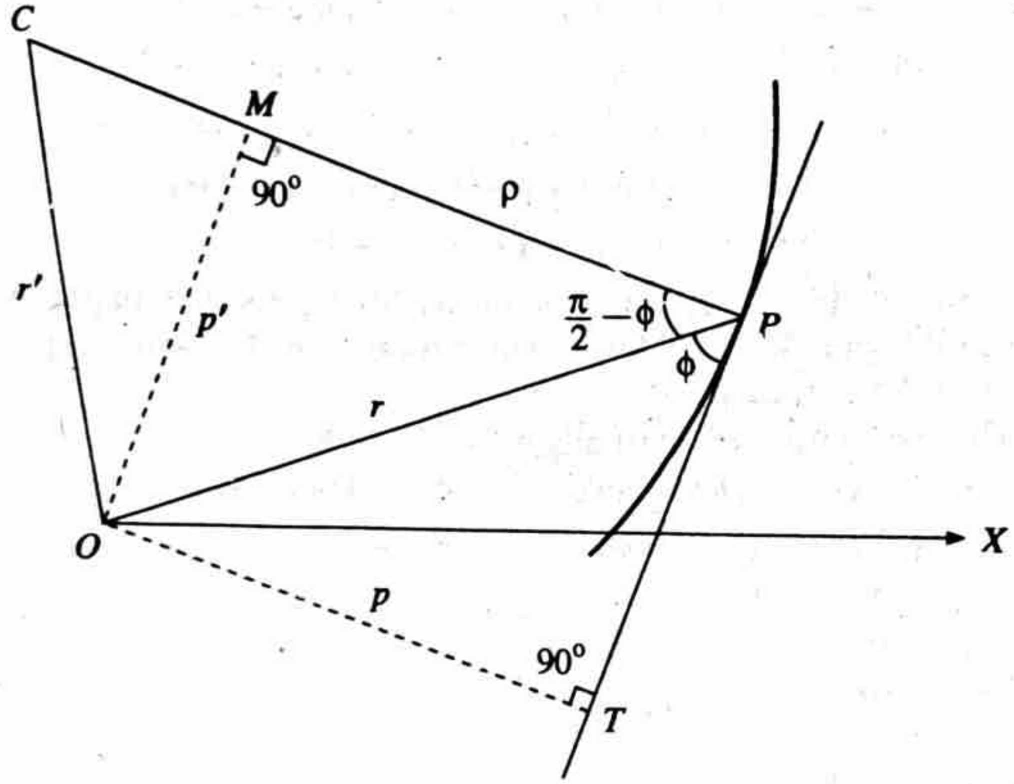
§ 7. Evolute of Polar Curves.

In the case of polar curves there is no standard method of finding the evolute of a curve. However, if a curve is given in pedal form, we can easily find the pedal equation of its evolute by the method given below.

Let the pedal equation of the given curve be

$$p = f(r). \qquad ...(1)$$

Let C be the centre of curvature corresponding to the point P on the given curve.



Then $PC = \rho$ and PC is normal to the given curve at the point P. Corresponding to the point P on the given curve the point on the evolute is C. Since the evolute of a curve is the envelope of the normals of that curve, therefore the normal PC of the given curve is tangent to the evolute at the point C.

If OT is the perpendicular from P to the tangent to the given curve at the point P, then $OT = p$ and $OP = r$.

Draw OM perpendicular from the pole O to the tangent PC to the evolute at the point C. If $OC = r'$ and $OM = p'$, then the relation between p' and r' will be the pedal equation of the evolute.

From the $\Delta\, OPC$, we have

$$OC^2 = OP^2 + PC^2 - 2OP \, . \, PC \cos \angle \, OPC$$

i.e., $$r'^2 = r^2 + \rho^2 - 2r\rho \sin \phi$$

i.e., $$r'^2 = r^2 + \rho^2 - 2\rho p, \qquad [\because p = r \sin \phi]. \qquad ...(2)$$

Now $OTPM$ is a rectangle and so $MP = OT = p$. From the right angled triangle OMP, we have $OM^2 = OP^2 - MP^2$

i.e., $$p'^2 = r^2 - p^2. \quad ...(3)$$

Also $$\rho = r\frac{dr}{dp}. \quad ...(4)$$

Eliminating r, p and ρ between the equations (1), (2), (3) and (4), we get the pedal equation of the evolute.

Ex. 51. *Show that the evolute of an equiangular spiral is an equiangular spiral.* **(Agra 1980; Meerut 84)**

Sol. Draw figure as in § 7.

Let the pedal equation of the given equiangular spiral be

$$p = r\sin\alpha. \quad ...(1)$$

We have $dp/dr = \sin\alpha$.

$$\therefore \quad \rho = r\frac{dr}{dp} = r\cdot\frac{1}{\sin\alpha} = r\operatorname{cosec}\alpha. \quad ...(2)$$

Corresponding to the point (p, r) on the given curve (1), let the point on the evolute be (p', r'), the co-ordinates in each case being expressed in pedal form.

Then $\quad r'^2 = r^2 + \rho^2 - 2\rho p$

$= r^2 + r^2\operatorname{cosec}^2\alpha - 2r\operatorname{cosec}\alpha . r\sin\alpha,$ [from (1) and (2)]

$= r^2\operatorname{cosec}^2\alpha - r^2 = r^2\cot^2\alpha. \quad ...(3)$

Also $$p'^2 = r^2 - p^2 = r^2 - r^2\sin^2\alpha = r^2\cos^2\alpha \quad ...(4)$$

Dividing (4) by (3), we get

$$\frac{p'^2}{r'^2} = \frac{r^2\cos^2\alpha}{r^2\cot^2\alpha} = \sin^2\alpha.$$

$$\therefore \quad p'^2 = r'^2\sin^2\alpha \quad \text{or} \quad p' = r'\sin\alpha.$$

Hence the locus of the point (p', r') is $p = r\sin\alpha$. This is the pedal equation of the evolute and is an equiangular spiral.

Ex. 52. *Show that the evolute of the curve whose pedal equation is* $r^2 - a^2 = mp^2$ *is the curve whose pedal equation is*

$$r^2 - (1-m)\,a^2 = mp^2.$$ **(Meerut 1985; Lucknow 79)**

Sol. Draw figure as in § 7. The pedal equation of the given curve is

$$r^2 - a^2 = mp^2. \quad ...(1)$$

Differentiating both sides of (1) with respect to p, we have

$$2r\frac{dr}{dp} = 2mp.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = mp. \quad ...(2)$$

Corresponding to the point (p, r) on the given curve (1), let the point on the evolute by (p', r'), the coordinates in each case being expressed in pedal form.

Then $\quad r'^2 = r^2 + \rho^2 - 2\rho p$

$= r^2 + m^2p^2 - 2mpp$, substituting for ρ from (2)

$= a^2 + mp^2 + m^2p^2 - 2mp^2$, substituting for r^2 from (1).

$\therefore \quad r'^2 - a^2 = m^2p^2 - mp^2 = mp^2(m-1)$. ...(3)

Also $\quad p'^2 = r^2 - p^2 = a^2 + mp^2 - p^2$.

$\therefore \quad p'^2 - a^2 = (m-1)p^2$. ...(4)

To eliminate p, dividing (3) by (4), we get

$$\frac{r'^2 - a^2}{p'^2 - a^2} = m$$

or
$$r'^2 - a^2 = mp'^2 - ma^2$$

or
$$r'^2 - (1-m)a^2 = mp'^2.$$

Hence the locus of the point (p', r') is

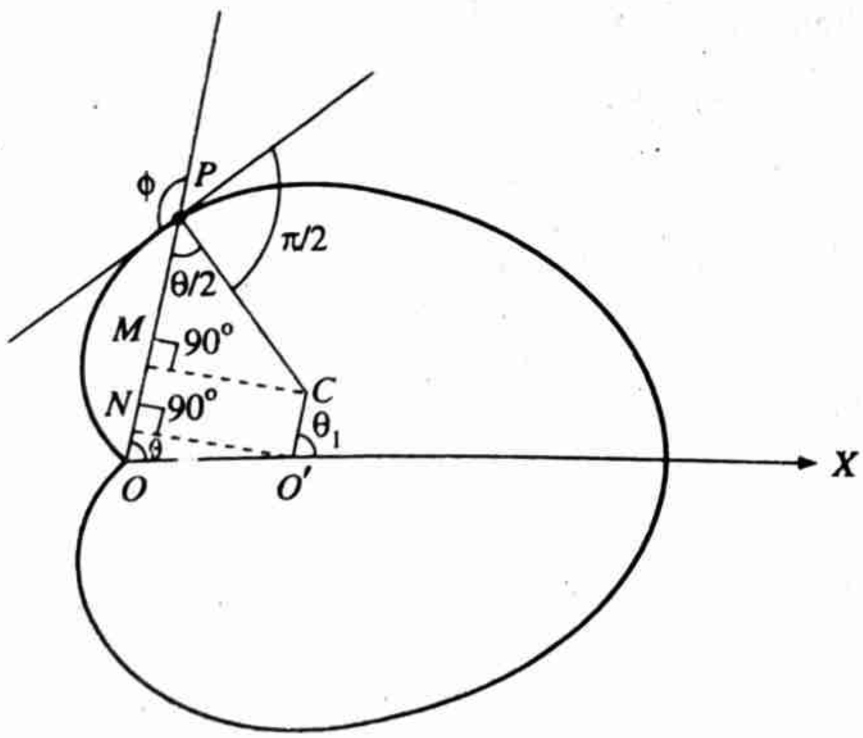
$$r^2 - (1-m)a^2 = mp^2.$$

This is the pedal equation of the evolute of the curve (1).

Ex. 53. *Prove that the evolute of the cardioid $r = a(1 + \cos\theta)$ is the cardioid $r = \frac{1}{3}a(1 - \cos\theta)$, the pole of the latter equation being at the point $(\frac{2}{3}a, 0)$.* **(Meerut 1983, 84 R, 86, 89, 89 S, 91)**

Sol. The given cardioid is $r = a(1 + \cos\theta)$. ...(1)

Let O be the pole and OX the initial line. The cardioid (1) has been drawn in the figure.



Take any point $P(r, \theta)$ on the given cardioid (1). Also let O' be the given point $(\frac{2}{3}a, 0)$ [this will be on the initial line because $\theta = 0$].

Let C be the centre of curvature of the curve (1) corresponding to the point P. Then $CP = \rho =$ radius of carvature of (1) at the point P.

We have to find the locus of the point C w.r.t. O' as pole. Let $O'C = r_1$ and $\angle CO'X = \theta_1$. Draw CM and $O'N$ perpendiculars from C and O' respectively to OP.

Let us first find the value of ρ. Taking logarithm of both sides of (1), we get $\log r = \log a + \log(1 + \cos\theta)$.

Differentiating w.r.t. 'θ', we get

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= -\tan\tfrac{1}{2}\theta = \cot(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta).$$

$\therefore \quad \phi = \frac{1}{2}\pi + \frac{1}{2}\theta.$

Now $\quad p = r\sin\phi = r\sin(\frac{1}{2}\pi + \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta.$

From (1), $\quad r = 2a\cos^2\frac{1}{2}\theta = 2a(p/r)^2.$

$\therefore$ the pedal equation of the curve (1) is

$$r^3 = 2ap^2. \qquad ...(2)$$

Differentiating (2) w.r.t. 'p', we get

$$3r^2(dr/dp) = 4ap.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\left(\frac{r^3}{2a}\right)^{1/2}, \qquad \left[\because \text{ from (2)}, p^2 = \frac{r^3}{2a}\right]$$

$$= \frac{4a}{3}\left(\frac{r}{2a}\right)^{1/2} = \frac{4a}{3}\left\{\frac{a(1+\cos\theta)}{2a}\right\}^{1/2} = \frac{4a}{3}\left(\frac{2a\cos^2\frac{1}{2}\theta}{2a}\right)^{1/2}$$

$$= \frac{4}{3}a\cos\tfrac{1}{2}\theta.$$

Since $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta, \quad \therefore \quad \angle CPM = \frac{1}{2}\theta.$

$\therefore \quad CM = PC\sin\frac{1}{2}\theta = \rho\sin\frac{1}{2}\theta = \frac{4}{3}a\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta = \frac{2}{3}a\sin\theta.$

$\therefore \quad$ Also $O'N = OO'\sin\theta = \frac{2}{3}a\sin\theta.$

Thus $CM = O'N$ and consequently $O'NMC$ is a rectangle. Therefore $O'C$ is parallel to OP,

i.e., $\qquad \theta_1 = \theta. \qquad ...(4)$

Now $r_1 = O'C = NM = OP - ON - PM$

$= OP - OO'\cos\theta - PC\cos\frac{1}{2}\theta$

$= r - \frac{2}{3}a\cos\theta - \frac{4}{3}a\cos\frac{1}{2}\theta\cos\frac{1}{2}\theta$

$= a\,(1 + \cos\theta) - \frac{2}{3} a \cos\theta - \frac{2}{3} a\,(1 + \cos\theta)$

$= \frac{1}{3} a\,(3 + 3\cos\theta - 2\cos\theta - 2 - 2\cos\theta) = \frac{1}{3} a\,(1 - \cos\theta)$

$= \frac{1}{3} a\,(1 - \cos\theta_1).$ $\qquad [\because \theta_1 = \theta].$

Evolute is the locus of the centre of curvature. Hence generalising $r_1 = \frac{1}{3} a\,(1 - \cos\theta_1)$, the required equation of the evolute is $r = \frac{1}{3} a\,(1 - \cos\theta)$ referred to O' as pole.

§ 8. Length of arc of an evolute.

Remember. *The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two given points.*

Ex. 54. *Show that the whole length of the evolute of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *is* $4\,(a^2/b - b^2/a)$. **(Meerut 1993, 94P)**

Sol. The given equation of the ellipse is

$$x^2/a^2 + y^2/b^2 = 1. \qquad ...(1)$$

Now ρ at the point $(a\cos t, b\sin t)$ of (1)

$$= (a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}/ab.$$

Now at the ends of major and minor axes t is equal to 0 and $\frac{1}{2}\pi$ respectively.

$\therefore \quad \rho_1 = \rho$ at the end of major axis

$$= (b^2)^{3/2}/ab = b^2/a$$

and $\quad \rho_2 = \rho$ at the end of minor axis $= (a^2)^{3/2}/ab = a^2/b$.

As the ellipse and its evolute are symmetrical about both the axes [see Ex. 47 (a)], therefore the whole length of the evolute of the ellipse

$$= 4\,(\rho_2 - \rho_1) = 4\,(a^2/b - b^2/a).$$

□

5

Jacobians

§ 1. Definition.

If $u_1, u_2, \ldots, u_n$ are functions of n independent variables $x_1, x_2, \ldots,$ x_n then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots\cdots\cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots\cdots\cdots & \dfrac{\partial u_2}{\partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots\cdots\cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ and is denoted either by $\dfrac{\partial\,(u_1, u_2, \ldots, u_n)}{\partial\,(x_1, x_2, \ldots, x_n)}$ or by $J\,(u_1, u_2, \ldots, u_n)$. The second notation is used when there is no doubt as regards the independent variables.

Thus if u and v are functions of two independent variables x and y, we have

$$\frac{\partial\,(u, v)}{\partial\,(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = J\,(u, v).$$

Similarly if u, v and w are functions of three independent variables x, y and z, we have

$$\frac{\partial\,(u, v, w)}{\partial\,(x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = J\,(u, v, w).$$

Note. If the functions $u_1, u_2, \ldots, u_n$ of n independent variables $x_1, x_2, \ldots, x_n$ are of the following forms,

$u_1 = f_1(x_1), u_2 = f_2(x_1, x_2), \ldots, u_n = f_n(x_1, x_2, \ldots, x_n)$, then

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & \ldots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \ldots \ldots \frac{\partial u_n}{\partial x_n},$$

i.e., in such cases the Jacobian reduces to the principal diagonal term of the determinant.

Solved Examples

Ex. 1. *If $x = c \cos u \cosh v, y = c \sin u \sinh v$, prove that*

$$\frac{\partial(x, y)}{\partial(u, v)} = \tfrac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

(Agra 1982)

Sol. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -c \sin u \cosh v & c \cos u \sinh v \\ c \cos u \sinh v & c \sin u \cosh v \end{vmatrix}$$

$$= -c^2 \sin^2 u \cosh^2 v - c^2 \cos^2 u \sinh^2 v$$

$$= -\tfrac{1}{2} c^2 [(1 - \cos 2u) \cosh^2 v + (1 + \cos 2u) \sinh^2 v]$$

$$= -\tfrac{1}{2} c^2 [\cosh^2 v + \sinh^2 v - \cos 2u (\cosh^2 v - \sinh^2 v)]$$

$$= -\tfrac{1}{2} c^2 (\cosh 2v - \cos 2u) = \tfrac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

Ex. 2. *If $x = u(1 + v), y = v(1 + u)$, find the Jacobian of x, y with respect to u, v.*

Sol. We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 + v & u \\ v & 1 + u \end{vmatrix}$$

$$= (1 + v)(1 + u) - uv = 1 + u + v + uv - uv = 1 + u + v.$$

Ex. 3. *If $x = r\cos\theta, y = r\sin\theta$, show that*

(*i*) $\dfrac{\partial(x, y)}{\partial(r, \theta)} = r$, (*ii*) $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}$.

(Meerut 1992, 95BP, 97; Rohilkhand 77; Agra 79)

Sol. **(i)** We have

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

(ii) From the given relations, we have

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = y/x.$$

Now differentiating $r^2 = x^2 + y^2$ partially w.r.t. x and y, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

and $$2r\frac{\partial r}{\partial y} = 2y \quad \text{or} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

Again differentiating $\tan\theta = y/x$ partially w.r.t. x and y, we get

$$\sec^2\theta \cdot \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

or $$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2\sec^2\theta} = -\frac{y}{r^2\cos^2\theta\sec^2\theta} = -\frac{y}{r^2}$$

and $$\sec^2\theta \cdot \frac{\partial \theta}{\partial y} = \frac{1}{x} \quad \text{or} \quad \frac{\partial \theta}{\partial y} = \frac{1}{x\sec^2\theta} = \frac{\cos^2\theta}{x} = \frac{x^2}{r^2}\cdot\frac{1}{x} = \frac{x}{r^2}.$$

$$\therefore \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -\dfrac{y}{r^2} & \dfrac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

Ex. 4. *If $u = \dfrac{y^2}{2x}, v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.*

Sol. We have $u = \dfrac{y^2}{2x}$ and $v = \dfrac{x}{2} + \dfrac{y^2}{2x}$.

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{y^2}{2x^2} & \dfrac{y}{x} \\ \dfrac{1}{2} - \dfrac{y^2}{2x^2} & \dfrac{y}{x} \end{vmatrix}$$

$$= -\frac{y^2}{2x^3} - \frac{y}{x}\left(\frac{1}{2} - \frac{y^2}{2x^2}\right) = -\frac{y^3}{2x^3} - \frac{y}{2x} + \frac{y^3}{2x^3} = -\frac{y}{2x}.$$

Ex. 5. *If* $u_1 = x_2x_3/x_1$, $u_2 = x_3x_1/x_2$, $u_3 = x_1x_2/x_3$, *prove that* $J(u_1, u_2, u_3) = 4$.

(Meerut 1991; Agra 84; Rohilkhand 90)

Sol. We have $J(u_1, u_2, u_3)$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \begin{vmatrix} 0 & 0 & 2x_1x_2 \\ 2x_2x_3 & 0 & 0 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}, \quad \text{adding } R_2 \text{ to } R_1 \text{ and then } R_3 \text{ to } R_2$$

$$= \frac{1}{x_1^2x_2^2x_3^2} \cdot (2x_1x_2 \cdot 2x_2x_3^2x_1) = 4.$$

Ex. 6. *If* $x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r\cos\theta$, *show that*

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta.$$

(Meerut 1991S, 94, 98; Agra 86; Rohilkhand 80)

Sol. We have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi)$$
$$+ r\sin\theta\,(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi), \text{ expanding the determinant along the third row}$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta.$$

Ex. 7. *Find the Jacobian* $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ *being given*

$x = r\cos\theta\cos\phi, y = r\sin\theta\sqrt{(1 - m^2\sin^2\phi)}$,
$z = r\sin\phi\sqrt{(1 - n^2\sin^2\theta)}$, *where* $m^2 + n^2 = 1$. **(Agra 1985)**

Sol. Here $x^2 + y^2 + z^2$

$$= r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta - r^2m^2\sin^2\theta\sin^2\phi + r^2\sin^2\phi - r^2n^2\sin^2\phi\sin^2\theta$$

$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\phi - \sin^2\theta\sin^2\phi) \quad [\because m^2 + n^2 = 1]$$

$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\phi\cos^2\theta)$$

$$= r^2(\sin^2\theta + \cos^2\theta) = r^2.$$

$$\left.\begin{aligned} \therefore \quad & x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r} + z\frac{\partial z}{\partial r} = r\,;\, x\frac{\partial x}{\partial\theta} + y\frac{\partial y}{\partial\theta} + z\frac{\partial z}{\partial\theta} = 0 \\ \text{and} \quad & x\frac{\partial x}{\partial\phi} + y\frac{\partial y}{\partial\phi} + z\frac{\partial z}{\partial\phi} = 0. \end{aligned}\right\} \quad ...(1)$$

Now
$$J(x, y, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \frac{1}{x}\begin{vmatrix} x\frac{\partial x}{\partial r} & x\frac{\partial x}{\partial\theta} & x\frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \frac{1}{x}\begin{vmatrix} r & 0 & 0 \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$, by adding $yR_2 + zR_3$ to R_1 and using the relations (1)

$$= \frac{r}{x}\begin{vmatrix} \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \frac{r}{x}\left\{\frac{\partial y}{\partial\theta}\frac{\partial z}{\partial\phi} - \frac{\partial z}{\partial\theta}\frac{\partial y}{\partial\phi}\right\}$$

$$= \frac{r}{x}\left\{r\cos\theta\sqrt{(1 - m^2\sin^2\phi)}\,.\,r\cos\phi\sqrt{(1 - n^2\sin^2\theta)} - \frac{r\sin\phi\,.\,n^2\sin\theta\cos\theta}{\sqrt{(1 - n^2\sin^2\theta)}}\cdot\frac{r\sin\theta\,.\,m^2\sin\phi\cos\phi}{\sqrt{(1 - m^2\sin^2\phi)}}\right\}$$

$$= \frac{r^3 \cos\theta \cos\phi}{x} \left[\frac{(1 - m^2 \sin^2 \phi)(1 - n^2 \sin^2 \theta) - n^2 m^2 \sin^2 \theta \sin^2 \phi}{\sqrt{[(1 - n^2 \sin^2 \theta)(1 - m^2 \sin^2 \phi)]}} \right]$$

$$= \frac{r^3 \cos\theta \cos\phi}{r \cos\theta \cos\phi} \left[\frac{1 - m^2 \sin^2 \phi - n^2 \sin^2 \theta + m^2 n^2 \sin^2 \phi \sin^2 \theta - m^2 n^2 \sin^2 \theta \sin^2 \phi}{\sqrt{[(1 - n^2 \sin^2 \theta)(1 - m^2 \sin^2 \phi)]}} \right]$$

$$= \frac{r^2 (m^2 \cos^2 \phi + n^2 \cos^2 \theta)}{\sqrt{[(1 - n^2 \sin^2 \theta)(1 - m^2 \cos^2 \phi)]}}. \qquad [\because m^2 + n^2 = 1]$$

Ex. 8. *If* $y_1 = r \sin\theta_1 \sin\theta_2, y_2 = r \sin\theta_1 \cos\theta_2$,

$y_3 = r \cos\theta_1 \sin\theta_3,\ y_4 = r \cos\theta_1 \cos\theta_3$, *find the value of the Jacobian*

$$\frac{\partial (y_1, y_2, y_3, y_4)}{\partial (r, \theta_1, \theta_2, \theta_3)}.$$

(Meerut 1991P, 93, 98)

Sol. Squaring and adding the given relations, we have

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2.$$

$$\left.\begin{aligned} \therefore \quad & y_1 \frac{\partial y_1}{\partial r} + y_2 \frac{\partial y_2}{\partial r} + y_3 \frac{\partial y_3}{\partial r} + y_4 \frac{\partial y_4}{\partial r} = r \\ \text{and} \quad & y_1 \frac{\partial y_1}{\partial \theta_r} + y_2 \frac{\partial y_2}{\partial \theta_r} + y_3 \frac{\partial y_3}{\partial \theta_r} + y_4 \frac{\partial y_4}{\partial \theta_r} = 0, \ r = 1, 2, 3. \end{aligned}\right\} \quad ...(1)$$

Also $y_3^2 + y_4^2 = r^2 \cos^2 \theta_1$, so that

$$\left.\begin{aligned} & y_3 \frac{\partial y_3}{\partial \theta_1} + y_4 \frac{\partial y_4}{\partial \theta_1} = - r^2 \cos\theta_1 \sin\theta_1 ; \\ & y_3 \frac{\partial y_3}{\partial \theta_r} + y_4 \frac{\partial y_4}{\partial \theta_r} = 0, \quad r = 2, 3. \end{aligned}\right\} \quad ...(2)$$

Now the required Jacobian

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \frac{\partial y_1}{\partial \theta_3} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix}.$$

Operating $y_1 R_1 + (y_2 R_2 + y_3 R_3 + y_4 R_4)$, and using the results (1), we get

$$J = \frac{1}{y_1} \begin{vmatrix} r & 0 & 0 & 0 \\ \dfrac{\partial y_2}{\partial r} & \dfrac{\partial y_2}{\partial \theta_1} & \dfrac{\partial y_2}{\partial \theta_2} & \dfrac{\partial y_2}{\partial \theta_3} \\ \dfrac{\partial y_3}{\partial r} & \dfrac{\partial y_3}{\partial \theta_1} & \dfrac{\partial y_3}{\partial \theta_2} & \dfrac{\partial y_3}{\partial \theta_3} \\ \dfrac{\partial y_4}{\partial r} & \dfrac{\partial y_4}{\partial \theta_1} & \dfrac{\partial y_4}{\partial \theta_2} & \dfrac{\partial y_4}{\partial \theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1} \begin{vmatrix} \dfrac{\partial y_2}{\partial \theta_1} & \dfrac{\partial y_2}{\partial \theta_2} & \dfrac{\partial y_2}{\partial \theta_3} \\ \dfrac{\partial y_3}{\partial \theta_1} & \dfrac{\partial y_3}{\partial \theta_2} & \dfrac{\partial y_3}{\partial \theta_3} \\ \dfrac{\partial y_4}{\partial \theta_1} & \dfrac{\partial y_4}{\partial \theta_2} & \dfrac{\partial y_4}{\partial \theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1 y_3} \begin{vmatrix} \dfrac{\partial y_2}{\partial \theta_1} & \dfrac{\partial y_2}{\partial \theta_2} & \dfrac{\partial y_2}{\partial \theta_3} \\ -r^2 \cos\theta_1 \sin\theta_1 & 0 & 0 \\ \dfrac{\partial y_4}{\partial \theta_1} & \dfrac{\partial y_4}{\partial \theta_2} & \dfrac{\partial y_4}{\partial \theta_3} \end{vmatrix},$$

adding $y_4 R_3$ to $y_3 R_2$ and using the results (2)

$$= \frac{r}{y_1 y_3} \cdot r^2 \cos\theta_1 \sin\theta_1 \left[\frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_4}{\partial \theta_3} - \frac{\partial y_4}{\partial \theta_2} \cdot \frac{\partial y_2}{\partial \theta_3} \right]$$

$$= \frac{r^3 \cos\theta_1 \sin\theta_1}{y_1 y_3} [(-r \sin\theta_1 \sin\theta_2)(-r\cos\theta_1 \sin\theta_3) - 0]$$

$$= \frac{r^5 \sin^2\theta_1 \cos^2\theta_1 \sin\theta_2 \sin\theta_3}{r^2 \sin\theta_1 \cos\theta_1 \sin\theta_2 \sin\theta_3} = r^3 \sin\theta_1 \cos\theta_1 .$$

Ex. 9. *If $x = \sin\theta\sqrt{(1 - c^2 \sin^2\phi)}$, $y = \cos\theta\cos\phi$, then show that*

$$\frac{\partial(x, y)}{\partial(\theta, \phi)} = -\sin\phi \, \frac{[(1 - c^2)\cos^2\theta + c^2\cos^2\phi]}{\sqrt{(1 - c^2\sin^2\phi)}} .$$

Sol. We have $\dfrac{\partial(x, y)}{\partial(\theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} = \dfrac{\partial x}{\partial \theta} \cdot \dfrac{\partial y}{\partial \phi} - \dfrac{\partial x}{\partial \phi} \cdot \dfrac{\partial y}{\partial \theta}$

$$= \cos\theta\sqrt{(1 - c^2\sin^2\phi)} \, . \, (-\cos\theta\sin\phi)$$
$$- \sin\theta \cdot \tfrac{1}{2}(1 - c^2\sin^2\phi)^{-1/2} . (-2c^2 \sin\phi\cos\phi) . (-\sin\theta\cos\phi)$$

$$= -\frac{\sin\phi}{\sqrt{(1-c^2\sin^2\phi)}}[\cos^2\theta(1-c^2\sin^2\phi)+c^2\sin^2\theta\cos^2\phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1-c^2\sin^2\phi)}}[\cos^2\theta - c^2\cos^2\theta\sin^2\phi + c^2(1-\cos^2\theta)\cos^2\phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1-c^2\sin^2\phi)}}[\cos^2\theta - c^2\cos^2\theta(\sin^2\phi+\cos^2\phi)+c^2\cos^2\phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1-c^2\sin^2\phi)}}[\cos^2\theta - c^2\cos^2\theta + c^2\cos^2\phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1-c^2\sin^2\phi)}}[(1-c^2)\cos^2\theta + c^2\cos^2\phi].$$

Ex. 10. *If* $y_1 = 1 - x_1,\ y_2 = x_1(1-x_2),\ y_3 = x_1x_2(1-x_3), \ldots,$ $y_n = x_1x_2\ldots x_{n-1}(1-x_n)$, *prove that*

$$J(y_1, y_2, \ldots, y_n) = (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}. \qquad \textbf{(Agra 1978)}$$

Sol. Here y_1 is a function of x_1 only, y_2 is a function of x_1, x_2 only, y_3 is a function of x_1, x_2, x_3 only, ..., and y_n is a function of $x_1, x_2, \ldots, x_n$.

$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \ldots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial y_2}{\partial x_2}\cdot\frac{\partial y_3}{\partial x_3}\cdot\ldots\cdot\frac{\partial y_n}{\partial x_n}$$

$$= (-1).(-x_1)(-x_1x_2)\ldots(-x_1x_2x_3\ldots x_{n-1})$$

$$= (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

Ex. 11. *If* $y_1 = \cos x_1, y_2 = \sin x_1 \cos x_2, y_3 = \sin x_1 \sin x_2 \cos x_3,$ $\ldots, y_n = \sin x_1 \sin x_2 \sin x_3 \ldots \sin x_{n-1} \cos x_n$, *find the Jacobian of* $y_1,$ $y_2, \ldots, y_n$ *with respect to* $x_1, x_2, \ldots, x_n$.

Sol. Here y_1 is a function of x_1 only, y_2 is a function of x_1, x_2 only, y_3 is a function of x_1, x_2, x_3 only, ... and y_n is a function of $x_1,$ $x_2, \ldots, x_n$.

$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \text{the principal diagonal term of the determinant}$$

$$= \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3} \cdot \cdots \cdot \frac{\partial y_n}{\partial x_n}$$

$$= (-\sin x_1) \cdot (-\sin x_1 \sin x_2) \cdot (-\sin x_1 \sin x_2 \sin x_3) \ldots$$
$$\cdot (-\sin x_1 \sin x_2 \sin x_3 \ldots \sin x_n)$$

$$= (-1)^n \sin^n x_1 \sin^{n-1} x_2 \sin^{n-2} x_3 \ldots \sin x_n .$$

Ex. 12. *If* $u_1 = x_2^2 + 2a_1 x_2 x_3 + x_3^2,\ u_2 = x_1^2 + 2a_2 x_3 x_1 + x_3^2,$
$u_3 = x_1^2 + 2a_3 x_1 x_2 + x_2^2,$
find the Jacobian of u_1, u_2, u_3 *w.r.t.* x_1, x_2, x_3.

Sol. We have $J(u_1, u_2, u_3)$

$$= \begin{vmatrix} 0 & 2(x_2 + a_1 x_3) & 2(a_1 x_2 + x_3) \\ 2(x_1 + a_2 x_3) & 0 & 2(a_2 x_1 + x_3) \\ 2(x_1 + a_3 x_2) & 2(a_3 x_1 + x_2) & 0 \end{vmatrix}$$

$$= 8\,[-(x_1 + a_2 x_3)\{-(a_1 x_2 + x_3)(a_3 x_1 + x_2)\}$$
$$+ (x_1 + a_3 x_2)(x_2 + a_1 x_3)(a_2 x_1 + x_3)],$$

expanding the determinant along the first column

$$= 8\,[(a_2 x_3 + x_1)(a_1 x_2 + x_3)(a_3 x_1 + x_2)$$
$$+ (x_1 + a_3 x_2)(x_2 + a_1 x_3)(x_3 + a_2 x_1)].$$

Ex. 13. *Find the Jacobian of* $y_1, y_2, y_3, \ldots, y_n$, *being given*
$y_1 = x_1(1 - x_2),\ y_2 = x_1 x_2 (1 - x_3), \ldots, y_{n-1} = x_1 x_2 \ldots x_{n-1}(1 - x_n),$
$y_n = x_1 x_2 x_3 \ldots x_n .$

Sol. Adding all the given relations, we get

$$y_1 + y_2 + \ldots + y_n = x_1 .$$

$$\left.\begin{aligned} \therefore \quad & \frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial y_n}{\partial x_1} = 1 \\ \text{and} \quad & \frac{\partial y_1}{\partial x_r} + \frac{\partial y_2}{\partial x_r} + \ldots + \frac{\partial y_n}{\partial x_r} = 0, \ r = 2, 3, \ldots, n. \end{aligned}\right\} \quad \ldots(1)$$

Now
$$\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_1} & \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \\ 1 & 0 & 0 & \cdots & 0 \end{vmatrix},$$

adding $R_1, R_2, ..., R_{n-1}$ to R_n and using the relations (1)

$$= (-1)^{n-1} \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \frac{\partial y_1}{\partial x_4} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_2}{\partial x_4} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \frac{\partial y_{n-1}}{\partial x_4} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \end{vmatrix},$$

expanding the determinant along the nth row

$$= (-1)^{n-1} \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & 0 & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \frac{\partial y_{n-1}}{\partial x_4} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \end{vmatrix}$$

$$= (-1)^{n-1} \cdot \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial y_2}{\partial x_3} \cdot \frac{\partial y_3}{\partial x_4} \cdot \cdots \cdot \frac{\partial y_{n-1}}{\partial x_n}$$

$$= (-1)^{n-1} \cdot (-x_1) \cdot (-x_1 x_2) \cdot (-x_1 x_2 x_3) \cdot \cdots \cdot (-x_1 x_2 \ldots x_{n-1})$$

$$= (-1)^{n-1} \cdot (-1)^{n-1} x_1^{n-1} x_2^{n-2} x_3^{n-3} \ldots x_{n-1}$$

$$= (-1)^{2n-2} x_1^{n-1} x_2^{n-2} x_3^{n-3} \ldots x_{n-1}$$

$$= x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

Ex. 14. *If* $y_1 = r\cos\theta_1$, $y_2 = r\sin\theta_1 \cos\theta_2$, $y_3 = r\sin\theta_1 \sin\theta_2 \cos\theta_3, \dots, y_{n-1} = r\sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-2}\cos\theta_{n-1}$ *and* $y_n = r\sin\theta_1 \sin\theta_2 \dots \sin\theta_{n-1}$, *prove that*

$$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(r, \theta_1, \dots, \theta_{n-1})} = r^{n-1}\sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \dots \sin\theta_{n-2}.$$

Sol. If we square and add, we get $y_1^2 + y_2^2 + \dots + y_n^2 = r^2$.

$$\left.\begin{aligned} \therefore \quad & y_1\frac{\partial y_1}{\partial r} + y_2\frac{\partial y_2}{\partial r} + \dots + y_n\frac{\partial y_n}{\partial r} = r \\ \text{and} \quad & y_1\frac{\partial y_1}{\partial \theta_k} + y_2\frac{\partial y_2}{\partial \theta_k} + \dots + y_n\frac{\partial y_n}{\partial \theta_k} = 0,\ k = 1, 2, \dots, n-1. \end{aligned}\right\} \quad \dots(1)$$

Now $\dfrac{\partial(y_1, y_2, \dots, y_n)}{\partial(r, \theta_1, \dots, \theta_{n-1})}$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \dots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \dots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial y_n}{\partial r} & \frac{\partial y_n}{\partial \theta_1} & \frac{\partial y_n}{\partial \theta_2} & \dots & \frac{\partial y_n}{\partial \theta_{n-1}} \end{vmatrix}$$

$$= \frac{1}{y_n}\begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \dots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \dots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial y_{n-1}}{\partial r} & \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \dots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \\ r & 0 & 0 & \dots & 0 \end{vmatrix},$$

operating $y_nR_n + (y_1R_1 + y_2R_2 + \dots + y_{n-1}R_{n-1})$ and using the results (1)

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \cdots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \cdots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \cdots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix},$$

expanding the determinant along the nth row

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \frac{\partial y_{n-1}}{\partial \theta_3} & \cdots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot \frac{\partial y_1}{\partial \theta_1} \cdot \frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_3}{\partial \theta_3} \cdot \ldots \frac{\partial y_{n-1}}{\partial \theta_{n-1}}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot (-r \sin \theta_1)(-r \sin \theta_1 \sin \theta_2). \ldots$$
$$. (-r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1})$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} (-1)^{n-1} r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 . \ldots$$
$$. \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= \frac{r}{r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1}} \cdot r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2$$
$$\ldots \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2}.$$

§ 2. Case of Functions of Functions.

We shall establish the formula for two variables and the result can be easily extended to any number of variables.

Theorem. *If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2, then*

$$\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} = \frac{\partial (u_1, u_2)}{\partial (y_1, y_2)} \cdot \frac{\partial (y_1, y_2)}{\partial (x_1, x_2)}.$$ **(Meerut 1991)**

Proof. We have

$$\left.\begin{aligned}\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1},\ \frac{\partial u_1}{\partial x_2}=\frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2}+\frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2},\\ \frac{\partial u_2}{\partial x_1}=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1}+\frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1},\ \frac{\partial u_2}{\partial x_2}=\frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2}+\frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2}.\end{aligned}\right\}\quad ...(1)$$

Now $\dfrac{\partial (u_1, u_2)}{\partial (y_1, y_2)} \cdot \dfrac{\partial (y_1, y_2)}{\partial (x_1, x_2)}$

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial y_2}\\ \dfrac{\partial u_2}{\partial y_1} & \dfrac{\partial u_2}{\partial y_2}\end{vmatrix}\times\begin{vmatrix}\dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2}\\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2}\end{vmatrix}$$

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_1}+\dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_2}+\dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_2}\\ \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_1}+\dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_2}+\dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_2}\end{vmatrix},$$

applying row-by-column multiplication rule

$$=\begin{vmatrix}\dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2}\\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2}\end{vmatrix},\ \text{using the relations (1)}$$

$$=\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)}.$$

Note. The above formula resembles very much with the formula

$$\frac{df}{dx}=\frac{df}{dt}\cdot\frac{dt}{dx},$$

for the derivative of the function of a function.

Generalization of the above formula. *If $u_1, u_2, ..., u_n$ are functions of $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are functions of $x_1, x_2, ..., x_n$, then*

$$\frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, ..., x_n)}=\frac{\partial (u_1, u_2, ..., u_n)}{\partial (y_1, y_2, ..., y_n)}\cdot\frac{\partial (y_1, y_2, ..., y_n)}{\partial (x_1, x_2, ..., x_n)}.$$

The proof may be easily extended as in the case of two variables and has been left as an exercise for the students.

§ 3. Jacobian of Implicit Functions.

Theorem. *Suppose $u_1, u_2, ..., u_n$ instead of being given explicitly in terms of $x_1, x_2, ..., x_n$ are connected with them by equations such as*

$$F_1(u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0,$$
$$F_2(u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F_n(u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0.$$

Then, we have

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(x_1, x_2, ..., x_n)}}{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(u_1, u_2, ..., u_n)}}.$$

Proof. Here also we shall establish the result for two variables and the proof can be extended easily for n variables. The students should themselves write the proof for n variables on the basis of the proof given below for two variables.

In the case of two variables, the connecting relations are

$$\left.\begin{aligned} F_1(u_1, u_2, x_1, x_2) = 0, \\ F_2(u_1, u_2, x_1, x_2) = 0. \end{aligned}\right\} \qquad \ldots(1)$$

From relations (1), we have by differentiation

$$\left.\begin{aligned}
\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} = 0, \\
\frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_2} = 0, \\
\frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_1} = 0, \\
\frac{\partial F_2}{\partial x_2} + \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_2} = 0.
\end{aligned}\right\} \qquad \ldots(2)$$

Now $\dfrac{\partial(F_1, F_2)}{\partial(u_1, u_2)} \cdot \dfrac{\partial(u_1, u_2)}{\partial(x_1, x_2)}$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \end{vmatrix},$$

applying row-by-column multiplication rule

$$= \begin{vmatrix} \dfrac{-\partial F_1}{\partial x_1} & \dfrac{-\partial F_1}{\partial x_2} \\ \dfrac{-\partial F_2}{\partial x_1} & \dfrac{-\partial F_2}{\partial x_2} \end{vmatrix}, \text{ using the relations (2)}$$

$$= (-1)^2 \frac{\partial (F_1, F_2)}{\partial (x_1, x_2)}.$$

Accordingly, we have

$$\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} = (-1)^2 \frac{\dfrac{\partial (F_1, F_2)}{\partial (x_1, x_2)}}{\dfrac{\partial (F_1, F_2)}{\partial (u_1, u_2)}}.$$

Solved Examples

Ex. 1. *Prove that*

$$\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)} = 1.$$

(Meerut 1991P, 93P, 95; Rohilkhand 81; Agra 82)

Sol. Let $u = f_1 (x, y), v = f_2 (x, y)$. ...(1)

Obviously x and y can also be expressed as functions of u and v. Differentiating relations (1) partially with respect to u and v, we get

$$\left. \begin{aligned} 1 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \quad 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \\ 0 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \quad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}. \end{aligned} \right\} \quad ...(2)$$

Now $$\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix},$$

applying row-by-column multiplication

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \text{ using the relations (2)}$$

$= 1.$

Ex. 2. *If* $u^3 + v + w = x + y^2 + z^2,$

$u + v^3 + w = x^2 + y + z^2,$

$$u + v + w^3 = x^2 + y^2 + z,$$

prove that
$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

(Meerut 1991 S)

Sol. The given relations can be written as

$$F_1 \equiv u^3 + v + w - x - y^2 - z^2 = 0,$$
$$F_2 \equiv u + v^3 + w - x^2 - y - z^2 = 0,$$
$$F_3 \equiv u + v + w^3 - x^2 - y^2 - z = 0.$$

Now $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \dfrac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \dfrac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)}$. ...(1)

Here $\dfrac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$

$= -1(1 - 4yz) + 2x(2y - 4yz) - 2x(4yz - 2z)$
$= -1 + 4(yz + zx + xy) - 16xyz.$

And $\dfrac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$

$= 3u^2(9v^2w^2 - 1) - 1(3w^2 - 1) + 1 . (1 - 3v^2)$
$= 2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2.$

$\therefore$ From (1), $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = \dfrac{1 - 4(yz + zx + xy) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$.

Ex. 3. *If* $u^3 + v^3 = x + y, u^2 + v^2 = x^3 + y^3$, *show that*

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u - v)}.$$

(Rohilkhand 1976)

Sol. The given relations can be written as

$$F_1 \equiv u^3 + v^3 - x - y = 0$$

and $$F_2 \equiv u^2 + v^2 - x^3 - y^3 = 0.$$

Now $\dfrac{\partial (u, v)}{\partial (x, y)} = (-1)^2 \dfrac{\partial (F_1, F_2)}{\partial (x, y)} \Big/ \dfrac{\partial (F_1, F_2)}{\partial (u, v)}$. ...(1)

We have $\dfrac{\partial (F_1, F_2)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$

$$= 3y^2 - 3x^2 = 3(y^2 - x^2)$$

and $\dfrac{\partial (F_1, F_2)}{\partial (u, v)} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6u^2v - 6uv^2 = 6uv(u - v).$

$\therefore$ From (1), $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{3 (y^2 - x^2)}{6uv (u - v)} = \dfrac{1}{2} \dfrac{y^2 - x^2}{uv (u - v)}$.

Ex. 4. *If $x + y + z = u, y + z = uv, z = uvw$, show that*

$$\frac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v.$$

(Agra 1977)

Sol. The given relations can be written as

$$F_1 \equiv x + y + z - u = 0$$

$$F_2 \equiv y + z - uv = 0$$

and $$F_3 \equiv z - uvw = 0.$$

Now $$\frac{\partial (x, y, z)}{\partial (u, v, w)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)}. \quad ...(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$

$$= (-1) \,.\, (-u) \,.\, (-uv) = -u^2 v$$

and $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

$\therefore$ From (1), $\dfrac{\partial (x, y, z)}{\partial (u, v, w)} = - \dfrac{-u^2 v}{1} = u^2 v$.

Ex. 5. *If $u^3 = xyz, \dfrac{1}{v} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}, w^2 = x^2 + y^2 + z^2$, prove that*

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{-v (y - z)(z - x)(x - y)(x + y + z)}{3u^2 w (yz + zx + xy)}.$$

Sol. The given relations can be written as

$$F_1 \equiv u^3 - xyz = 0$$

$$F_2 \equiv \frac{1}{v} - \frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0$$

and $$F_3 \equiv w^2 - x^2 - y^2 - z^2 = 0.$$

Now $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \quad ...(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -yz & -zx & -xy \\ 1/x^2 & 1/y^2 & 1/z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= \frac{2}{x^2y^2z^2} \begin{vmatrix} x^2yz & y^2zx & z^2xy \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \frac{2}{xyz} \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} = -\frac{2}{xyz} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= -\frac{2}{xyz} \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}, \text{ by } C_2 - C_1 \text{ and } C_3 - C_1$$

$$= -\frac{2}{xyz}(y-x)(z-x) \begin{vmatrix} 1 & 1 \\ y^2+xy+x^2 & z^2+zx+x^2 \end{vmatrix}$$

$$= -\frac{2}{xyz}(y-x)(z-x) \begin{vmatrix} 1 & 0 \\ y^2+xy+x^2 & (z-y)(x+y+z) \end{vmatrix}, \text{ by } C_2 - C_1$$

$$= -\frac{2}{xyz}(y-x)(z-x)(z-y)(x+y+z)$$

$$= -\frac{2}{xyz}(x-y)(y-z)(z-x)(x+y+z).$$

Also $\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & -1/v^2 & 0 \\ 0 & 0 & 2w \end{vmatrix} = -6u^2w/v^2.$

Hence from (1), $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= -\frac{-2(x-y)(y-z)(z-x)(x+y+z)}{xyz} \cdot \frac{-v^2}{6u^2w}$$

$$= -\frac{(x-y)(y-z)(z-x)(x+y+z) \, . \, v}{3u^2wxyz} \cdot \frac{xyz}{yz+zx+xy}$$

$$\left[\because \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \text{ so that } v = \frac{xyz}{yz+zx+xy}\right]$$

$$= -\frac{v(y-z)(z-x)(x-y)(x+y+z)}{3u^2w(yz+zx+xy)}.$$

Ex. 6. *If* $u^3 + v^3 + w^3 = x + y + z,$
$u^2 + v^2 + w^2 = x^3 + y^3 + z^3,$
$u + v + w = x^2 + y^2 + z^2,$

then prove that

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$ **(Agra 1983; Meerut 98)**

Sol. The given relations can be written as

$$F_1 \equiv u^3 + v^3 + w^3 - x - y - z = 0,$$

$$F_2 \equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0,$$

and
$$F_3 \equiv u + v + w - x^2 - y^2 - z^2 = 0.$$

Now $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)}. \quad ...(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= 6 (y-x)(z-x) \begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix}$$

$$= 6 (y-x)(z-x)(z-y) = 6 (x-y)(y-z)(z-x).$$

Also $$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ u & v & w \\ u^2 & v^2 & w^2 \end{vmatrix}$$

$$= -6 (u-v)(v-w)(w-u).$$

Hence from (1), $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -\frac{6 (x-y)(y-z)(z-x)}{-6 (u-v)(v-w)(w-u)}$$

$$= \frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$

Ex. 7. *Compute the Jacobian* $\dfrac{\partial (u, v)}{\partial (r, \theta)}$ *where*

$u = 2xy, v = x^2 - y^2, x = r \cos \theta, y = r \sin \theta.$ **(Indore 1979)**

Sol. Here we have case of functions of functions. Using the formula for finding the Jacobian in the case of functions of functions, we have

$$\frac{\partial (u, v)}{\partial (r, \theta)} = \frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, \theta)}. \quad ...(1)$$

Now $\dfrac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4(x^2 + y^2).$

Also $\dfrac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r(\cos^2\theta + \sin^2\theta) = r.$

Hence from (1), $\dfrac{\partial (u, v)}{\partial (r, \theta)} = -4(x^2 + y^2)\, r = -4r^2 . r = -4r^3.$

[$\because$ from $x = r\cos\theta, y = r\sin\theta$, we have $x^2 + y^2 = r^2$]

Ex. 8. *If* $u_1 = x_1 + x_2 + x_3 + x_4, u_1u_2 = x_2 + x_3 + x_4,$ $u_1u_2u_3 = x_3 + x_4, u_1u_2u_3u_4 = x_4$, *show that*

$$\frac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_1, u_2, u_3, u_4)} = u_1^3 u_2^2 u_3.$$

Sol. The given relations can be written as

$F_1 \equiv u_1 - x_1 - x_2 - x_3 - x_4 = 0, F_2 \equiv u_1u_2 - x_2 - x_3 - x_4 = 0,$

$F_3 \equiv u_1u_2u_3 - x_3 - x_4 = 0$ and $F_4 \equiv u_1u_2u_3u_4 - x_4 = 0.$

Now

$$\frac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_1, u_2, u_3, u_4)} = (-1)^4 \frac{\partial (F_1, F_2, F_3, F_4)}{\partial (u_1, u_2, u_3, u_4)} \bigg/ \frac{\partial (F_1, F_2, F_3, F_4)}{\partial (x_1, x_2, x_3, x_4)} \quad ...(1)$$

We have $\dfrac{\partial (F_1, F_2, F_3, F_4)}{\partial (u_1, u_2, u_3, u_4)}$ = the principal diagonal term of the Jacobian determinant

$$= \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial F_3}{\partial u_3} \cdot \frac{\partial F_4}{\partial u_4} = 1 . u_1 . u_1u_2 . u_1u_2u_3 = u_1^3 u_2^2 u_3.$$

Also $\dfrac{\partial (F_1, F_2, F_3, F_4)}{\partial (x_1, x_2, x_3, x_4)} = \begin{vmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1.$

Hence from (1), $\dfrac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_1, u_2, u_3, u_4)} = \dfrac{u_1^3 u_2^2 u_3}{1} = u_1^3 u_2^2 u_3.$

Ex. 9. *Given* $y_1 (x_1 - x_2) = 0, y_2 (x_1^2 + x_1x_2 + x_2^2) = 0$, *show that*

$$\frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = 3y_1y_2 \frac{x_1 + x_2}{x_1^3 - x_2^3}.$$

Sol. The given relations can be written as

$$F_1 \equiv y_1 (x_1 - x_2) = 0$$

and $$F_2 \equiv y_2 (x_1^2 + x_1 x_2 + x_2^2) = 0.$$

Now $$\frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = (-1)^2 \frac{\partial (F_1, F_2)}{\partial (x_1, x_2)} \Big/ \frac{\partial (F_1, F_2)}{\partial (y_1, y_2)}. \qquad \text{...(1)}$$

We have $$\frac{\partial (F_1, F_2)}{\partial (x_1, x_2)} = \begin{vmatrix} y_1 & -y_2 \\ y_2 (2x_1 + x_2) & y_2 (x_1 + 2x_2) \end{vmatrix}$$

$$= y_1 y_2 (x_1 + 2x_2 + 2x_1 + x_2)$$

$$= 3y_1 y_2 (x_1 + x_2).$$

Also $$\frac{\partial (F_1, F_2)}{\partial (y_1, y_2)} = \begin{vmatrix} x_1 - x_2 & 0 \\ 0 & x_1^2 + x_1 x_2 + x_2^2 \end{vmatrix} = x_1^3 - x_2^3.$$

Hence from (1), $$\frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = \frac{3y_1 y_2 (x_1 + x_2)}{x_1^3 - x_2^3}.$$

Ex. 10. *If* $u = x (1 - r^2)^{-1/2}$, $v = y (1 - r^2)^{-1/2}$,

$w = z (1 - r^2)^{-1/2}$, *where* $r^2 = x^2 + y^2 + z^2$,

show that $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (1 - r^2)^{-5/2}.$$

Sol. From the given relations, we have

$x^2 = u^2 (1 - r^2) = u^2 (1 - x^2 - y^2 - z^2)$,

$y^2 = v^2 (1 - x^2 - y^2 - z^2)$ and $z^2 = w^2 (1 - x^2 - y^2 - z^2)$.

The above relations can be written as

$$F_1 \equiv x^2 - u^2 (1 - x^2 - y^2 - z^2) = 0,$$

$$F_2 \equiv y^2 - v^2 (1 - x^2 - y^2 - z^2) = 0,$$

and $$F_3 \equiv z^2 - w^2 (1 - x^2 - y^2 - z^2) = 0.$$

Now

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)}. \qquad \text{...(1)}$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} 2x (1 + u^2) & 2yu^2 & 2zu^2 \\ 2xv & 2y (1 + v^2) & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z (1 + w^2) \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1 + u^2 & u^2 & u^2 \\ v^2 & 1 + v^2 & v^2 \\ w^2 & w^2 & 1 + w^2 \end{vmatrix}$$

$$= 8xyz (1 + u^2 + v^2 + w^2) \begin{vmatrix} 1 & 1 & 1 \\ v^2 & 1 + v^2 & v^2 \\ w^2 & w^2 & 1 + w^2 \end{vmatrix}$$

$$= 8xyz\,(1 + u^2 + v^2 + w^2)\begin{vmatrix} 1 & 0 & 0 \\ v^2 & 1 & 0 \\ w^2 & 0 & 1 \end{vmatrix}$$

$$= 8xyz\,(1 + u^2 + v^2 + w^2)$$

$$= 8xyz\left[1 + \frac{x^2}{1 - r^2} + \frac{y^2}{1 - r^2} + \frac{z^2}{1 - r^2}\right] = 8xyz\left[1 + \frac{x^2 + y^2 + z^2}{1 - r^2}\right]$$

$$= 8xyz\left[1 + \frac{r^2}{1 - r^2}\right] = \frac{8xyz}{1 - r^2}.$$

Also $\dfrac{\partial\,(F_1, F_2, F_3)}{\partial\,(u, v, w)}$

$$= \begin{vmatrix} -2u\,(1 - r^2) & 0 & 0 \\ 0 & -2v\,(1 - r^2) & 0 \\ 0 & 0 & -2w\,(1 - r^2) \end{vmatrix}$$

$$= -8uvw\,(1 - r^2)^3.$$

Hence from (1), $\dfrac{\partial\,(u, v, w)}{\partial\,(x, y, z)} = -\dfrac{8xyz}{1 - r^2} \cdot \dfrac{1}{-8uvw\,(1 - r^2)^3}$

$$= \frac{xyz}{uvw\,(1 - r^2)^4} = \frac{xyz}{xyz\,(1 - r^2)^{-3/2}\,(1 - r^2)^4}$$

$$= (1 - r^2)^{-5/2}.$$

Ex. 11. *If λ, μ, ν are the roots of the equation in k,*

$$\frac{x}{a + k} + \frac{y}{b + k} + \frac{z}{c + k} = 1,$$

prove that

$$\frac{\partial\,(x, y, z)}{\partial\,(\lambda, \mu, \nu)} = -\frac{(\mu - \nu)\,(\nu - \lambda)\,(\lambda - \mu)}{(b - c)\,(c - a)\,(a - b)}.$$

Sol. The given equation in k can be written as

$$(a + k)\,(b + k)\,(c + k) - x\,(b + k)\,(c + k) - y\,(c + k)\,(a + k) - z\,(a + k)\,(b + k) = 0$$

or $$k^3 + k^2\,(a + b + c - x - y - z) + k\,\{ab + bc + ca - x\,(b + c) - y\,(c + a) - z\,(a + b)\} + abc - xbc - yca - zab = 0.$$

Since λ, μ, ν are the roots of this equation, therefore from theory of equations, we have

$$\lambda + \mu + \nu = x + y + z - a - b - c$$

$$\lambda\mu + \mu\nu + \nu\lambda = ab + bc + ca - x\,(b + c) - y\,(c + a) - z\,(a + b)$$

and $\lambda\mu\nu = xbc + yca + zab - abc.$

The above relations can be written as

$$F_1 \equiv \lambda + \mu + \nu - x - y - z + a + b + c = 0,$$

$$F_2 \equiv \lambda\mu + \mu\nu + \nu\lambda + x\,(b + c) + y\,(c + a) + z\,(a + b) - ab - bc - ca = 0$$

and $F_3 \equiv \lambda\mu\nu - xbc - yca - zab + abc = 0.$

Now $$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(\lambda,\mu,\nu)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}. \qquad ...(1)$$

We have $$\frac{\partial(F_1,F_2,F_3)}{\partial(\lambda,\mu,\nu)} = \begin{vmatrix} 1 & 1 & 1 \\ \mu+\nu & \lambda+\nu & \mu+\lambda \\ \mu\nu & \nu\lambda & \lambda\mu \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \mu+\nu & \lambda-\mu & \lambda-\nu \\ \mu\nu & \nu(\lambda-\mu) & \mu(\lambda-\nu) \end{vmatrix}$$

$$= (\lambda-\mu)(\lambda-\nu)\begin{vmatrix} 1 & 1 \\ \nu & \mu \end{vmatrix} = (\lambda-\mu)(\lambda-\nu)(\mu-\nu)$$

$$= -(\lambda-\mu)(\mu-\nu)(\nu-\lambda).$$

Also $$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ b+c & c+a & a+b \\ -bc & -ca & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix} = -(b-c)(c-a)(a-b).$$

Hence from (1), $$\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = -\frac{-(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{-(b-c)(c-a)(a-b)}$$

$$= -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

Ex. 12. *The roots of the equation in* λ,

$$(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$$

are u, v, w. *Prove that*

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$$

Sol. The given equation in λ can be written as

$$3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0.$$

Since u, v, w are the roots of this equation, therefore fro... theory of equations, we have

$$u+v+w = x+y+z,$$

$$uv+vw+wu = x^2+y^2+z^2,$$

and $$uvw = \tfrac{1}{3}(x^3+y^3+z^3).$$

The above relations can be written as

$$F_1 \equiv u+v+w-x-y-z = 0,$$

$$F_2 \equiv uv+vw+wu-x^2-y^2-z^2 = 0$$

and $$F_3 \equiv uvw - \tfrac{1}{3}(x^3+y^3+z^3) = 0.$$

Now $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \dfrac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \dfrac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)}$ ...(1)

We have $\dfrac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$

$= -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -2 (y-z)(z-x)(x-y).$

Also $\dfrac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$

$= -(v-w)(w-u)(u-v).$ [Refer solved example 11]

Hence from (1), $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = -\dfrac{-2(y-z)(z-x)(x-y)}{-(v-w)(w-u)(u-v)}$

$= -2 \dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}.$

Ex. 13. *Prove that*

$$\frac{\partial (y_1, y_2, ..., y_n)}{\partial (x_1, x_2, ..., x_n)} \cdot \frac{\partial (x_1, x_2, ..., x_n)}{\partial (y_1, y_2, ..., y_n)} = 1.$$

(Meerut 1983)

Sol. Let $\left.\begin{array}{l} y_1 = f_1 (x_1, x_2, ..., x_n), y_2 = f_2 (x_1, x_2, ..., x_n), \\ ..., y_n = f_n (x_1, x_2, ..., x_n) \end{array}\right\}$ ...(1)

These relations may be put in the form

$x_1 = F_1 (y_1, y_2, ..., y_n), x_2 = F_2 (y_1, y_2, ..., y_n), ...,$

$x_n = F_n (y_1, y_2, ..., y_n).$

Differentiating the relations (1) partially w.r.t. $y_1, y_2, ..., y_n$, we have

$$\text{(A)} \quad \left\{\begin{array}{l} 1 = \dfrac{\partial y_1}{\partial x_1} \cdot \dfrac{\partial x_1}{\partial y_1} + \dfrac{\partial y_1}{\partial x_2} \cdot \dfrac{\partial x_2}{\partial y_1} + \cdots + \dfrac{\partial y_1}{\partial x_n} \cdot \dfrac{\partial x_n}{\partial y_1} = \Sigma \dfrac{\partial y_1}{\partial x_r} \cdot \dfrac{\partial x_r}{\partial y_1} \\ 0 = \dfrac{\partial y_1}{\partial x_1} \cdot \dfrac{\partial x_1}{\partial y_2} + \dfrac{\partial y_1}{\partial x_2} \cdot \dfrac{\partial x_2}{\partial y_2} + \cdots + \dfrac{\partial y_1}{\partial x_n} \cdot \dfrac{\partial x_n}{\partial y_2} = \Sigma \dfrac{\partial y_1}{\partial x_r} \cdot \dfrac{\partial x_r}{\partial y_2} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 = \dfrac{\partial y_1}{\partial x_1} \cdot \dfrac{\partial x_1}{\partial y_n} + \dfrac{\partial y_1}{\partial x_2} \cdot \dfrac{\partial x_2}{\partial y_n} + \cdots + \dfrac{\partial y_1}{\partial x_n} \cdot \dfrac{\partial x_n}{\partial y_n} = \Sigma \dfrac{\partial y_1}{\partial x_r} \cdot \dfrac{\partial x_r}{\partial y_n} \end{array}\right.$$

and similar other relations.

Now $\dfrac{\partial (y_1, y_2, ..., y_n)}{\partial (x_1, x_2, ..., x_n)} \cdot \dfrac{\partial (x_1, x_2, ..., x_n)}{\partial (y_1, y_2, ..., y_n)}$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots, & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \Sigma \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \cdots & \Sigma \frac{\partial y_1}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_n} \\ \Sigma \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \Sigma \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \cdots & \Sigma \frac{\partial y_2}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma \frac{\partial y_n}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_1} & \Sigma \frac{\partial y_n}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_2} & \cdots & \Sigma \frac{\partial y_n}{\partial x_r} \cdot \frac{\partial x_r}{\partial y_n} \end{vmatrix},$$

applying row-by-column multiplication

$$= \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{vmatrix}, \text{ using the relations (A)}$$

$= 1.$

Ex. 14. *If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that*

$$\frac{\partial (y, z)}{\partial (x, z)} = \left(\frac{\partial y}{\partial x}\right)_{z = const.}$$

Sol. We have $f(x, y, z) = 0 \Rightarrow y$ is a function of x and z.

Also from the equation, $z = z$, z may be regarded as a function of x and z.

$$\therefore \quad \frac{\partial (y, z)}{\partial (x, z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}} & \left(\frac{\partial y}{\partial z}\right)_{x = \text{const.}} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}} & \left(\frac{\partial y}{\partial z}\right)_{x = \text{const.}} \\ 0 & 1 \end{vmatrix} \quad \left[\because \frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial z} = 1\right]$$

$$= \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}}$$

§ 4. Necessary and sufficient condition for a Jacobian to vanish.

Theorem. *Let $u_1, u_2, ..., u_n$ be functions of n independent variables $x_1, x_2, ..., x_n$. In order that these n functions may not be independent, i.e., there may exist between these n functions a relation*

$$F(u_1, u_2, ..., u_n) = 0, \qquad ...(1)$$

it is necessary and sufficient that the Jacobian $\dfrac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}$ should vanish identically.

Proof. The condition is necessary *i.e.*, if there exists between $u_1, u_2, ..., u_n$ a relation

$$F(u_1, u_2, ..., u_n) = 0 \qquad ...(1)$$

their Jacobian is necessarily zero.

Differentiating (1) partially with respect to $x_1, x_2, ..., x_n$, we get

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_1} = 0,$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_2} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_n} + \cdots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_n} = 0.$$

Eliminating $\dfrac{\partial F}{\partial u_1}, \dfrac{\partial F}{\partial u_2}, \ldots, \dfrac{\partial F}{\partial u_n}$ from these equations, we get

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \cdots & \dfrac{\partial u_n}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_2}{\partial x_n} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix} = 0$$

or $$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = 0.$$

The condition is sufficient, *i.e.*, if the Jacobian $J(u_1, u_2, ..., u_n)$ is zero, then there must exist a relation between $u_1, u_2, ..., u_n$.

The equations connecting the functions $u_1, u_2, ..., u_n$ and the variables $x_1, x_2, ..., x_n$ are always capable of being put into the following form :

$$\phi_1(x_1, x_2, ..., x_n, u_1) = 0$$
$$\phi_2(x_2, x_3, ..., x_n, u_1, u_2) = 0$$
$$\text{......................................}$$
$$\phi_r(x_r, x_{r+1}, ..., x_n, u_1, u_2, ..., u_r) = 0$$
$$\text{......................................}$$
$$\phi_n(x_n, u_1, u_2, ..., u_n) = 0.$$

Then, we have

$$J = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\dfrac{\partial(\phi_1, \phi_2, ..., \phi_n)}{\partial(x_1, x_2, ..., x_n)}}{\dfrac{\partial(\phi_1, \phi_2, ..., \phi_n)}{\partial(u_1, u_2, ..., u_n)}}$$

$$= (-1)^n \frac{\dfrac{\partial\phi_1}{\partial x_1} \dfrac{\partial\phi_2}{\partial x_2} \cdots \dfrac{\partial\phi_n}{\partial x_n}}{\dfrac{\partial\phi_1}{\partial u_1} \dfrac{\partial\phi_2}{\partial u_2} \cdots \dfrac{\partial\phi_n}{\partial u_n}}. \qquad \text{[See note after § 1]}$$

Now, if $J = 0$, we have

$$\frac{\partial\phi_1}{\partial x_1} \cdot \frac{\partial\phi_2}{\partial x_2} \cdots \frac{\partial\phi_r}{\partial x_r} \cdots \frac{\partial\phi_n}{\partial x_n} = 0$$

i.e., $\dfrac{\partial\phi_r}{\partial x_r} = 0$ for *some value* of r between 1 and n.

Hence, for that particular value of r the function ϕ_r must not contain x_r; and accordingly the corresponding equation is of the form

$$\phi_r(x_{r+1}, ..., x_n, u_1, u_2, ..., u_r) = 0.$$

Consequently between this and the remaining equations $\phi_{r+1} = 0, \phi_{r+2} = 0, ..., \phi_n = 0$, the variables $x_{r+1}, x_{r+2}, ..., x_n$ can be eliminated so as to give a final equation between $u_1, u_2, ..., u_n$ alone.

Hence the theorem is established.

Solved Examples

Ex. 1. *Show that the functions*

$$u = x + y - z, v = x - y + z, w = x^2 + y^2 + z^2 - 2yz$$

are not independent of one another. Also find the relation between them.

Sol. Here

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix}, \text{ adding } C_2 \text{ to } C_3$$

$$= 0.$$

Since the Jacobian is zero, the functions are not independent.
Now $u + v = 2x$ and $u - v = 2(y - z)$.
Therefore $(u + v)^2 + (u - v)^2 = 4(x^2 + y^2 + z^2 - 2yz) = 4w$.
This is the required relation between u, v, w.

Ex. 2. *Show that* $ax^2 + 2hxy + by^2$ *and* $Ax^2 + 2Hxy + By^2$ *are independent unless* $\dfrac{a}{A} = \dfrac{h}{H} = \dfrac{b}{B}$.

Sol. Let $u = ax^2 + 2hxy + by^2, v = Ax^2 + 2Hxy + By^2$. If the functions u, v are not independent, then

$$\frac{\partial (u, v)}{\partial (x, y)} = 0$$

or
$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

or
$$\begin{vmatrix} 2(ax + hy) & 2(hx + by) \\ 2(Ax + Hy) & 2(Hx + By) \end{vmatrix} = 0$$

or
$$(ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$$

or
$$(aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2 = 0.$$

Since the variables x, y are independent, the coefficients of x^2 and y^2 in the above equation must be separately zero. Hence, we have

$$aH - Ah = 0 \quad \text{and} \quad Bh - bH = 0$$

whence
$$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$

Ex. 3. *If* $u = x^2 + y^2 + z^2, v = x + y + z, w = xy + yz + zx$, *show that the Jacobian* $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$ *vanishes identically. Also find the relation between u, v and w.*

Sol. We have $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}, \text{ by } R_1 + R_3$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0, \text{ the first two rows being identical.}$$

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and there must exist a relation between them.

We have $v^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$
$= u + 2w.$

Thus $v^2 = u + 2w$ is the required relation between u, v and w.

Ex. 4. *If* $u = (x+y)/(1-xy)$ *and* $v = \tan^{-1} x + \tan^{-1} y$, *find* $\dfrac{\partial(u,v)}{\partial(x,y)}$. *Are u and v functionally related ? If so, find the relationship.*

Sol. We have

$$\frac{\partial u}{\partial x} = \frac{1.(1-xy) - (-y)(x+y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2},$$

$$\frac{\partial u}{\partial y} = \frac{1.(1-xy) - (-x)(x+y)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2},$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \frac{\partial v}{\partial y} = \frac{1}{1+y^2}.$$

Now $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{1+y^2}{(1-xy)^2} \cdot \frac{1}{1+y^2} - \frac{1+x^2}{(1-xy)^2} \cdot \frac{1}{1+x^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

Since the Jacobian of the functions u, v is zero, therefore these functions are not independent and so they must be functionally related.

We have

$$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} = \tan^{-1} u.$$

Thus $v = \tan^{-1} u$ or $\tan v = u$ is the required relation between u and v.

Ex. 5. *Show that the functions* $u = 3x + 2y - z, v = x - 2y + z$ *and* $w = x(x + 2y - z)$ *are not independent and find the relation between them.*

Sol. We have

$$u = 3x + 2y - z, v = x - 2y + z, w = x^2 + 2xy - xz.$$

$$\therefore \quad \frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 1 \\ 2x + 2y & 2x & -x \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & -1 & -1 \\ 1 & 1 & 1 \\ 2x + 2y & -x & -x \end{vmatrix}$$

$= 0$, the last two columns being identical.

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $u^2 - v^2 = (3x + 2y - z)^2 - (x - 2y + z)^2$

$$= (3x + 2y - z + x - 2y + z)(3x + 2y - z - x + 2y - z)$$

$$= 4x (2x + 4y - 2z) = 8x (x + 2y - z) = 8w.$$

Thus $u^2 - v^2 = 8w$ is the required relation between u, v and w.

Ex. 6. *Show that the functions*

$$u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$$

are not independent. Find the relation between them. **(Meerut 1990)**

Sol. We have $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y + z & z + x & x + y \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & 1 \\ y + z & z + x & x + y \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & 0 \\ y + z & x - y & x - z \\ x^2 - yz & (y - x)(x + y + z) & (z - x)(x + y + z) \end{vmatrix},$$

by $C_2 - C_1$ and $C_3 - C_1$

$$= 3 (x - y)(x - z) \begin{vmatrix} 1 & 0 & 0 \\ y + z & 1 & 1 \\ x^2 - yz & -(x + y + z) & -(x + y + z) \end{vmatrix} = 0,$$

the last two columns being identical.

Since the Jacobian of the functions u, v and w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $\quad w = x^3 + y^3 + z^3 - 3xyz$

$$= (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)$$

$$= (x + y + z)\,[(x + y + z)^2 - 3\,(yz + zx + xy)]$$
$$= u\,(u^2 - 3v) = u^3 - 3uv.$$

$\therefore\quad u^3 = 3uv + w$ is the required relation between u, v and w.

Ex. 7. *If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w.* **(Meerut 1981)**

Sol. We have $\dfrac{\partial\,(u, v, w)}{\partial\,(x, y, z)}$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x - 4y + 6z & -x + 2y - 3z \end{vmatrix},$$

by $C_2 - 2C_1$ and $C_3 - C_1$

$$= -2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 2y - z & -x + 2y - 3z & -x + 2y - 3z \end{vmatrix} = 0,$$

the last two columns being identical.

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $u^2 - v^2 = (x + 2y + z)^2 - (x - 2y + 3z)^2$

$$= (x + 2y + z + x - 2y + 3z)\,(x + 2y + z - x + 2y - 3z)$$
$$= (2x + 4z)\,(4y - 2z) = 4\,(x + 2z)\,(2y - z)$$
$$= 4\,(2xy - xz + 4yz - 2z^2) = 4w.$$

Therefore $u^2 - v^2 = 4w$ is the required relation between u, v and w.

Ex. 8. *If* $u = x + y + z + t$, $v = x + y - z - t$, $w = xy - zt$, $r = x^2 + y^2 - z^2 - t^2$, *show that*

$$\frac{\partial\,(u, v, w, r)}{\partial\,(x, y, z, t)} = 0$$

and hence find a relation between u, v, w and r.

Sol. We have $\dfrac{\partial\,(u, v, w, r)}{\partial\,(x, y, z, t)}$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ y & x & -t & -z \\ 2x & 2y & -2z & -2t \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ y & x & -t & -z \\ 2x & 2y & -2z & -2t \end{vmatrix}, \text{ by } R_1 + R_2$$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ y & x-y & -t & -z \\ 2x & 2(y-x) & -2z & -2t \end{vmatrix}, \text{ by } C_2 - C_1$$

$$= 2 \begin{vmatrix} 0 & -1 & -1 \\ x-y & -t & -z \\ -2(x-y) & -2z & -2t \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -1 \\ x-y & z-t & -z \\ -2(x-y) & 2(t-z) & -2t \end{vmatrix}, \text{ by } C_2 - C_3$$

$$= 2(x-y)(z-t) \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & -z \\ -2 & -2 & -2t \end{vmatrix}$$

$= 0$, the first two columns being identical.

Since the Jacobian of the functions u, v, w, r is zero, therefore these functions are not independent and so there must exist a relation between them.

Now let us find a relation between u, v, w, r. We have

$$uv = (x+y+z+t)(x+y-z-t) = (x+y)^2 - (z+t)^2$$
$$= (x^2+y^2-z^2-t^2) + 2(xy-zt) = r + 2w.$$

Hence $uv = r + 2w$ is the required relation between u, v, w and r.

Ex. 9. *If $f(0) = 0$ and $f'(x) = \dfrac{1}{1+x^2}$, prove without using the method of integration, that*

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

Sol. Let $u = f(x) + f(y), v = \dfrac{x+y}{1-xy}$.

We have $\dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} f'(x) & f'(y) \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix}$

$$= \begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

Hence there must exist a functional relation between u and v.

Let $u = \phi(v)$

i.e., $$f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right).$$

Putting $y = 0$, we obtain $f(x) = \phi(x)$. $[\because f(0) = 0]$

Thus the functions f and ϕ are equal.

Hence $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$.

Ex. 10. *If the functions u, v, w of three independent variables x, y, z are not independent, prove that the Jacobian of u, v, w with respect to x, y, z vanishes.*

Sol. Since the functions u, v and w are not independent, therefore there exists a relation between them. Let it be

$$F(u, v, w) = 0. \qquad ...(1)$$

Now we are to prove that the Jacobian of u, v, w with respect to x, y, z must be equal to zero.

Differentiating (1) partially w.r.t. x, y, z, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = 0, \qquad ...(2)$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} = 0, \qquad ...(3)$$

and $$\frac{\partial F}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} = 0. \qquad ...(4)$$

Eliminating $\partial F/\partial u, \partial F/\partial v, \partial F/\partial w$ from (2), (3), (4), we get

$$\begin{vmatrix} \partial u/\partial x & \partial v/\partial x & \partial w/\partial x \\ \partial u/\partial y & \partial v/\partial y & \partial w/\partial y \\ \partial u/\partial z & \partial v/\partial z & \partial w/\partial z \end{vmatrix} = 0,$$

or $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$, which proves the required result.

□

6

Singular Points

§ 1. Singular Points : Definition.

A point on the curve at which the curve behaves in an extraordinary manner is called a **singular point.**

There are two types of singular points :

(i) points of inflexion (ii) multiple points.

§ 2. Concavity and convexity.

Let P be a given point on a curve and AB a given straight line which does not pass through P. Then the curve is said to be *concave* or *convex* at P with respect to AB, according as a sufficiently small arc

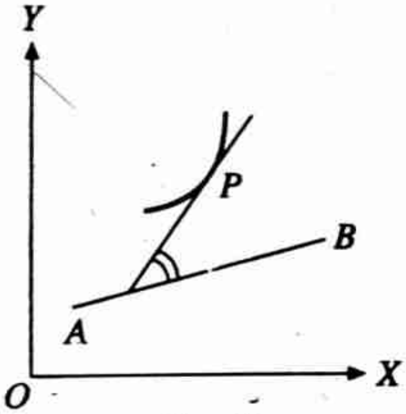


Fig. 1

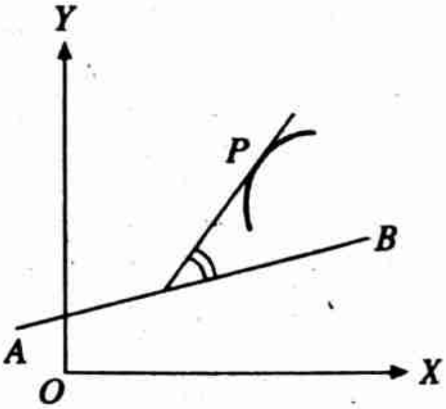
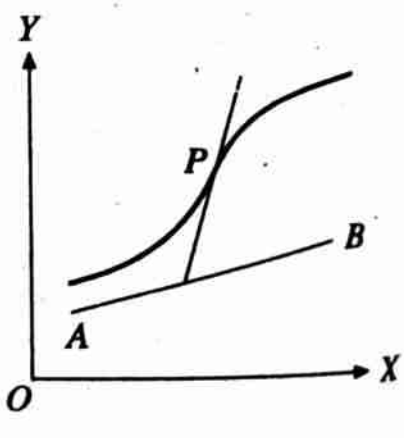


Fig. 2

of the curve containing P lies entirely *within* or *without* the acute angle formed by the tangent at P to the curve with the line AB. Thus in the Fig. 1 the curve at P is convex to AB, and in Fig. 2 it is concave to AB.

§ 3. Point of inflexion.

A point P on a curve is said to be a point of inflexion, if the curve is concave on one side and convex on the other side of P with respect to any line AB. Thus at a point of inflexion the curve changes its direction of bending from concavity to convexity or vice versa. The two portions of the curve on the two sides of P lie on *different* sides

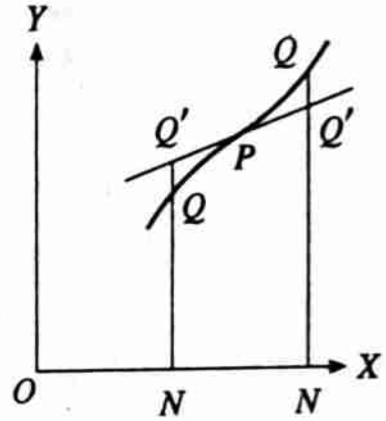


of the tangent at P, *i.e.*, the curve *crosses* the tangent at P.

Thus *a point where the curve crosses the tangent is a point of inflexion.*

Note. The position of a point of inflexion of a curve will in no way depend on the choice of coordinate axes. In particular, the positions of x and y axes may be interchanged without affecting the position of the points of inflexion on the curve.

§ 4. Test for point of inflexion.



Let the equation of the curve be $y = f(x)$ and let P be the point (x, y) on this curve. Suppose the tangent at P is not parallel to y-axis so that at P the value $f'(x)$ is finite. Let Q be the point $(x + h, y + k)$ on the curve in the neighbourhood of P. The point Q may be taken on either side of P. Suppose the ordinate QN of Q meets the tangent to the curve at P in Q'.

The equation of the tangent at P is

$$Y - y = f'(x)(X - x), \quad \text{...(1)}$$

where (X, Y) are the current coordinates.

Putting $X = x + h$ in (1), we get

$$NQ' - y = f'(x)\{x + h - x\}$$

or

$$NQ' = f(x) + hf'(x), \quad [\because y = f(x)] \quad \text{...(2)}$$

Also from the equation of the curve, we get

$$NQ = f(x + h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots$$

$$+ \frac{1}{(n-1)!}h^{n-1}f^{(n-1)}(x) + \frac{1}{n!}h^n f^{(n)}(x + \theta h), \quad \text{...(3)}$$

on expanding by Taylor's theorem, if $0 < \theta < 1$.

From (2) and (3), by subtraction, we get

$$NQ - NQ' = \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^n}{n!}f^{(n)}(x + \theta h). \quad \text{...(4)}$$

If $f''(x) \neq 0$, then by taking h sufficiently small, the second degree terms in h on the R.H.S. of (4) can be made to govern its sign. Therefore $(NQ - NQ')$ will be of the same sign as $(h^2/2!)f''(x)$. But $(h^2/2!)f''(x)$ will be of invariable sign whether h is positive or negative *i.e.*, whether Q lies to the right or the left of P. Therefore on both sides of P the curve wil be either concave or convex. Hence *the necessary condition for the existence of a point of inflexion at P is that*

$$\mathbf{f''(x) = 0}.$$

Now if $f''(x) = 0$, we have from (4)

$$NQ - NQ' = \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{iv}(x) + \dots + \frac{h^n}{n!} f^{(n)}(x + \theta h). \qquad ...(5)$$

If $f'''(x) \neq 0$, then for sufficiently small values of h the sign of the right hand side of (5) is the same as that of $(h^3/3!) f'''(x)$, which changes sign when h changes sign. Thus with respect to x-axis, the curve will be concave on one side of P and convex on the other side of P. So there will be a point of inflexion at P.

Hence **there will be a point of inflexion at P, if**

$$\frac{d^2y}{dx^2} = 0 \text{ but } \frac{d^3y}{dx^3} \neq 0.$$

Generalisation. If $f''(x) = f'''(x) = f^{iv}(x) = \dots = f^{(n-1)}(x) = 0$, and $f^{(n)}(x) \neq 0$, it is easy to see from the value of $NQ - NQ'$, that there will be a point of inflexion if n is odd. If n is even, there will not be a point of inflexion at P.

Cor. The position of a point of inflexion is independent of the choice of coordinate axes. Therefore on interchanging x and y in the above results, we can say that, *there will be a point of inflexion at P, if*

$$\frac{d^2x}{dy^2} = 0, \text{ but } \frac{d^3x}{dy^3} \neq 0.$$

It will become necessary for us to use this criterion if the tangent at P is parallel to y-axis *i.e.*, if dy/dx is infinite at P. It will also be useful where the equation of the curve is of the form

$$x = f(y).$$

Remember :

(i) A curve is **concave** upwards if d^2y/dx^2 is positive.

(ii) A curve is **convex** upwards if d^2y/dx^2 is negative.

(iii) At the point of inflexion, $d^2y/dx^2 = 0$ and $d^3y/dx^3 \neq 0$.

(iv) If dy/dx becomes infinite then we must find the points of inflexion by considering d^2x/dy^2.

Solved Examples

Ex. 1. *Find the points of inflexion of the curve*

$$y = 3x^4 - 4x^3 + 1.$$ **(Allahabad 1977)**

Sol. Differentiating the equation of the curve with respect to x, we get $dy/dx = 12x^3 - 12x^2$ and $d^2y/dx^2 = 36x^2 - 24x$.

For the points of inflexion, we must have

$$d^2y/dx^2 = 0$$

i.e., $36x^2 - 24x = 0$, *i.e.*, $12x(3x-2) = 0$,

i.e., $x = 0$ or $\frac{2}{3}$ for the points of inflexion.

Now $d^3y/dx^3 = 72x - 24$.

When $x = 0$, $d^3y/dx^3 \neq 0$, therefore $x = 0$ gives a point of inflexion.

Similarly, when $x = \frac{2}{3}$, $d^3y/dx^3 \neq 0$; therefore $x = \frac{2}{3}$ also gives a point of inflexion.

From the equation of the curve, we have

$y = 1$, when $x = 0$ and $y = \frac{11}{27}$, when $x = \frac{2}{3}$.

Hence $(0, 1)$ and $(\frac{2}{3}, \frac{11}{27})$ are the required points of inflexion.

Important. Instead of finding d^3y/dx^3, we can use another criterion for points of inflexion. If $d^2y/dx^2 = 0$ at $x = a$ and the sign of d^2y/dx^2 changes while x passes through a, then there will be a point of inflexion at $x = a$.

***Ex. 2.** *Find the points of inflexion of the curve*

$$y(a^2 + x^2) = x^3.$$

(Meerut 1983 S, 89; G.N.U. 73)

Sol. Differentiating the equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(a^2+x^2).3x^2 - x^3.2x}{(a^2+x^2)^2} = \frac{3a^2x^2 + x^4}{(a^2+x^2)^2}$$

and
$$\frac{d^2y}{dx^2} = \frac{(a^2+x^2)^2.(6a^2x + 4x^3) - (3a^2x^2 + x^4).2(a^2+x^2).2x}{(a^2+x^2)^4}$$

$$= \frac{x[6a^4 + 10a^2x^2 + 4x^4 - 12a^2x^2 - 4x^4]}{(a^2+x^2)^3} = \frac{2a^2x(3a^2 - x^2)}{(a^2+x^2)^3}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$.

$$\therefore \quad \frac{2a^2x(3a^2 - x^2)}{(a^2+x^2)^3} = 0 \quad \text{or} \quad 2a^2x(3a^2 - x^2) = 0$$

i.e., $x = 0$ or $x = \pm\sqrt{3}a$, (for the points of inflexion).

Now

$$\frac{d^3y}{dx^3} = 2a^2 . \frac{[(a^2+x^2)^3(3a^2 - 3x^2) - (3a^2x - x^3).3(a^2+x^2)^2.2x]}{(a^2+x^2)^6}$$

$$= \frac{6a^2[(a^2+x^2)(a^2-x^2) - 2x(3a^2x - x^3)]}{(a^2+x^2)^4}$$

$$= \frac{6a^2(a^4 - x^4 - 6a^2x^2 + 2x^4)}{(a^2+x^2)^4} = \frac{6a^2(a^4 - 6a^2x^2 + x^4)}{(a^2+x^2)^4}.$$

When $x = 0$, $d^3y/dx^3 = 6/a^2 \neq 0$,

when $x = \sqrt{3}a$, $d^3y/dx^3 = -3/4a^2 \neq 0$ and

when $x = -\sqrt{3}a$, $d^3y/dx^3 = -3/4a^2 \neq 0$.

Therefore $x = 0, \pm\sqrt{3}a$ give us points of inflexion.

From the equation of the curve, we have
when $x = 0, y = 0$; when $x = \sqrt{3}a, y = (3\sqrt{3}/4)\,a$
and when $x = -\sqrt{3}a, y = -(3\sqrt{3}/4)\,a$.

Hence the points of inflexion are

$$(0, 0),\ (\sqrt{3}a, 3\sqrt{3}a/4),\ (-\sqrt{3}a, -3\sqrt{3}a/4).$$

Ex. 3. *Find the points of inflexion of the curve*

$$x^2 y = a^2(x - y) \text{ or } (a^2 + x^2)\,y = a^2 x.$$

(Agra 1973; Vikram 70)

Sol. The curve is $y = a^2x/(x^2 + a^2)$. Differentiating the equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = a^2 \cdot \frac{(x^2 + a^2).1 - x.2x}{(x^2 + a^2)^2} = \frac{a^2(a^2 - x^2)}{(x^2 + a^2)^2}$$

and
$$\frac{d^2y}{dx^2} = a^2 \cdot \frac{(x^2 + a^2)^2(-2x) - (a^2 - x^2).2(x^2 + a^2).2x}{(x^2 + a^2)^4}$$

$$= a^2 \cdot \frac{-2x(x^2 + a^2) - 4x(a^2 - x^2)}{(x^2 + a^2)^3}$$

$$= a^2 \cdot \frac{2(x^3 - a^2x)}{(x^2 + a^2)^3} = \frac{2a^2x(x^2 - 3a^2)}{(x^2 + a^2)^3}.$$

For the points of inflexion, putting $d^2y/dx^2 = 0$, we have

$$2a^2x(x^2 - 3a^2) = 0. \quad \therefore \quad x = 0 \quad \text{or} \quad \pm\sqrt{3}a.$$

Now

$$\frac{d^3y}{dx^3} = \frac{2a^2[(x^2 + a^2)^3.(3x^2 - 3a^2) - (x^3 - 3a^2x)\{3.(a^2 + x^2)^2 2x\}]}{(x^2 + a^2)^6}$$

$$= 6a^2 \cdot \frac{(x^2 + a^2)(x^2 - a^2) - 2x(x^3 - 3a^2x)}{(x^2 + a^2)^4}$$

$$= \frac{6a^2[x^4 - a^4 - 2x^4 + 3a^2x^2]}{(x^2 + a^2)^4} = \frac{6a^2[3a^2x^2 - x^4 - a^4]}{(x^2 + a^2)^4}.$$

When $x = 0$, $d^3y/dx^3 = -6/a^2 \neq 0$,
when $x = \sqrt{3}a$, $d^3y/dx^3 = 3/4a^2 \neq 0$,
and when $x = -\sqrt{3}a$, $d^3y/dx^3 = 3/4a^2 \neq 0$.

Hence $x = 0, \pm\sqrt{3}a$ give us points of inflexion.

From the equation of the curve, we have
when $x = 0, y = 0$; when $x = \sqrt{3}a, y = (\sqrt{3}/4)\,a$
and when $x = -\sqrt{3}a, y = -(\sqrt{3}/4)\,a$.

Hence the points of inflexion are

$$(0, 0),\ \left(\sqrt{3}a, \frac{\sqrt{3}}{4}a\right) \text{ and } \left(-\sqrt{3}a, -\frac{\sqrt{3}}{4}a\right).$$

Ex. 4. *Find the points of inflexion of the curve*

$$x = \log(y/x).$$

(Meerut 1991)

Sol. The given curve is $x = \log(y/x)$ or $y/x = e^x$

or $\qquad y = x\,e^x.$ ...(1)

Differentiating (1), we get

$$dy/dx = x\,e^x + e^x = (x+1)\,e^x$$

and $\qquad d^2y/dx^2 = e^x + (x+1)\,e^x = e^x\,(x+2).$

For points of inflexion $d^2y/dx^2 = 0$.

$\therefore \quad e^x\,(x+2) = 0$ *i.e.*, $x = -2$, $\qquad (\because e^x \neq 0)$.

Now $d^3y/dx^3 = e^x + (x+2)\,e^x = e^x\,(x+3) \neq 0$ at $x = -2$.

$\therefore$ there is a point of inflexion at $x = -2$.

From (1), when $x = -2$, $y = -2e^{-2} = -2/e^2$.

Hence the point of inflexion is $(-2, -2/e^2)$.

Ex. 5. *Find the points of inflexion of the curve*

$$xy = a^2 \log (y/a).$$

Sol. The given curve is $x = (a^2/y) \log (y/a)$. ...(1)

Differentiating (1) taking y as independent variable and x dependent variable, we have

$$\frac{dx}{dy} = -\frac{a^2}{y^2}\log\left(\frac{y}{a}\right) + \frac{a^2}{y}\cdot\frac{1}{y/a}\cdot\frac{1}{a}$$

$$= \frac{a^2}{y^2}\left(1 - \log\frac{y}{a}\right),$$

and

$$\frac{d^2x}{dy^2} = \frac{a^2}{y^2}\left(-\frac{1}{y/a}\cdot\frac{1}{a}\right) + \left(1 - \log\frac{y}{a}\right)\left(-2\cdot\frac{a^2}{y^3}\right)$$

$$= -\frac{a^2}{y^3}\left(3 - 2\log\frac{y}{a}\right).$$

For the points of inflexion, putting $d^2x/dy^2 = 0$, we have

$$3 - 2\log (y/a) = 0 \text{ or } y = ae^{3/2}.$$

Now

$$\frac{d^3x}{dy^3} = -\frac{a^2}{y^3}\cdot\left(-2\cdot\frac{1}{y/a}\cdot\frac{1}{a}\right) + \left(3 - 2\log\frac{y}{a}\right)\cdot\frac{3a^2}{y^4}$$

$$= \frac{a^2}{y^4}\left(5 - 2\log\frac{y}{a}\right).$$

When $y = a\,e^{3/2}$ or $\log (y/a) = 3/2$, we have

$$\frac{d^3x}{dy^3} = \frac{a^2}{a^4e^6}\left(5 - 2\cdot\frac{3}{2}\right) = \frac{2}{a^2e^6} \neq 0.$$

Hence $y = a\,.e^{3/2}$ gives a point of inflexion.

From (1), putting $y = a\,e^{3/2}$, we have

$$x = \frac{a^2}{a\,e^{3/2}}\cdot\frac{3}{2} = \frac{3}{2}\,a\,e^{-3/2}.$$

Hence the point of inflexion is $\left(\frac{3}{2}\,a\,e^{-3/2}, a\,e^{3/2}\right)$.

Ex. 6. *Find the points of inflexion of the curve*

$$x = (\log y)^3.$$

Sol. Differentiating the given equation w.r.t. y, we get

$$\frac{dx}{dy} = 3(\log y)^2 \times \frac{1}{y} = \frac{3(\log y)^2}{y}$$

and $$\frac{d^2x}{dy^2} = 3\,\frac{y\,[2(\log y).1/y] - (\log y)^2}{y^2}$$

$$= \frac{3\log y}{y^2}[2 - \log y] = 3 \cdot \frac{2\log y - (\log y)^2}{y^2}$$

$$= 3y^{-2}[2\log y - (\log y)^2].$$

For the points of inflexion, putting $d^2x/dy^2 = 0$, we have

$$2\log y - (\log y)^2 = 0 \quad \text{or} \quad \log y\,[2 - \log y] = 0.$$

$\therefore$ $\log y = 0$ *i.e.*, $y = 1$ or $\log y = 2$ *i.e.*, $y = e^2$.

Now $$\frac{d^3x}{dy^3} = 3y^{-2}\left[\frac{2}{y} - (2\log y)\cdot\frac{1}{y}\right] + [2\log y - (\log y)^2](-6y^{-3})$$

$$= 6y^{-3}[1 - \log y - 2\log y + (\log y)^2]$$

$$= 6y^{-3}[(\log y)^2 - 3\log y + 1].$$

When $y = 1$, $d^3x/dy^3 = 6 \neq 0$,

and when $y = e^2$, $\dfrac{d^3x}{dy^3} = \dfrac{-6}{e^6} \neq 0$.

Hence $y = 1, e^2$ give us the points of inflexion.

From the curve when $y = 1$, $x = 0$ and when $y = e^2$, $x = 8$.

Hence the points of inflexion are $(0, 1)$ and $(8, e^2)$.

***Ex. 7.** *Show that every point in which the sine curve*

$$y = c \sin(x/a)$$

meets the axis of x is a point inflexion. **(Meerut 1984 S)**

Sol. The given curve meets x-axis where $y = 0$

i.e., where $\sin(x/a) = 0 = \sin n\pi$

or $x/a = n\pi$ or $x = an\pi$, where n is any integer.

Differentiating the given equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = \frac{c}{a}\cos\frac{x}{a}, \quad \frac{d^2y}{dx^2} = -\frac{c}{a^2}\sin\frac{x}{a}.$$

For points of inflexion, putting $d^2y/dx^2 = 0$, we have

$$\sin(x/a) = 0 \text{ i.e., } x = an\pi.$$

Now $$\frac{d^3y}{dx^3} = -\frac{c}{a^3}\cos\frac{x}{a}.$$

When $x = an\pi$, $\dfrac{d^3y}{dx^3} = -\dfrac{c}{a^3}\cos n\pi = -(-1)^n \cdot \dfrac{c}{a^3} \neq 0$.

Hence the points of inflexion are given by $x = an\pi$. These are the points where the curve cuts the x-axis.

***Ex. 8.** *Show that points of inflexion of the curve*

$$y^2 = (x-a)^2 (x-b)$$

lie on the line $3x + a = 4b$. **(Meerut 1990)**

Sol. The given curve is $y^2 = (x-a)^2(x-b)$

or $$y = \pm (x-a)\sqrt{(x-b)}. \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \pm \left[(x-a)\cdot\frac{1}{2\sqrt{(x-b)}} + \sqrt{(x-b)}\right]$$

$$= \pm \frac{x-a+2x-2b}{2\sqrt{(x-b)}} = \pm \frac{3x-2b-a}{2\sqrt{(x-b)}}$$

$$= \pm \tfrac{1}{2}(3x-2b-a).(x-b)^{-1/2}$$

and $$\frac{d^2y}{dx^2} = \pm \tfrac{1}{2}[(3x-2b-a).(-\tfrac{1}{2})(x-b)^{-3/2} + (x-b)^{-1/2}.3]$$

$$= \pm \tfrac{1}{2}\left[\frac{a+2b-3x}{2(x-b)^{3/2}} + \frac{3}{(x-b)^{1/2}}\right]$$

$$= \pm \frac{1}{2\sqrt{(x-b)}}\left[\frac{a+2b-3x}{2(x-b)} + 3\right]$$

$$= \pm \frac{1}{2\sqrt{(x-b)}}\cdot\frac{a-4b+3x}{2(x-b)}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$

i.e., $\quad a - 4b + 3x = 0 \quad$ or $\quad 3x + a = 4b$.

Hence the points of inflexion lie on the straight line

$$3x + a = 4b.$$

Ex. 9. *Show that origin is a point of inflexion of the curve* $a^{m-1}.y = x^m$ *if m is odd and greater than 2.*

Sol. The given curve is $a^{m-1}.y = x^m$

or $$y = x^m/a^{m-1}. \qquad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{m}{a^{m-1}}\cdot x^{m-1};\ \frac{d^2y}{dx^2} = \frac{m(m-1)}{a^{m-1}}\cdot x^{m-2}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$

or $\dfrac{m(m-1)}{a^{m-1}}\cdot x^{m-2} = 0 \quad$ or $\quad x^{m-2} = 0 \quad$ or $\quad x = 0$, if $m > 2$.

Also $\dfrac{d^3y}{dx^3} = \dfrac{m(m-1)(m-2)}{a^{m-1}}\cdot x^{m-3};......;\dfrac{d^my}{dx^m} = \dfrac{m!}{a^{m-1}}$.

At $x = 0$, $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = ... = \dfrac{d^{m-1}y}{dx^{m-1}} = 0$ and $\dfrac{d^my}{dx^m} \neq 0$.

Hence there is a point of inflexion at $x = 0$ (*i.e.*, at origin) if m is odd and greater than 2 and no point of inflexion if m is even.

Ex. 10. *Show that the abscissae of the points of inflexion on the curve $y^2 = f(x)$ satisfy the equation $[f'(x)]^2 = 2f(x) f''(x)$.*

Sol. The curve is $y^2 = f(x)$, or $y = [f(x)]^{1/2}$. ...(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \tfrac{1}{2} \cdot [f(x)]^{-1/2} . f'(x),$$

and $$\frac{d^2y}{dx^2} = \tfrac{1}{2} [\{f(x)\}^{-1/2} . f''(x) + f'(x) . (-1/2) \{f(x)\}^{-3/2} . f'(x)]$$

$$= \tfrac{1}{2} [\{f(x)\}^{-1/2} f''(x) - \tfrac{1}{2} \{f(x)\}^{-3/2} \{f'(x)\}^2]$$

$$= \tfrac{1}{2} \{f(x)\}^{-3/2} [f(x) . f''(x) - \tfrac{1}{2} \{f(x)\}^2].$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$.

$\therefore \quad 2f(x) . f''(x) - [f'(x)]^2 = 0$

or $\quad [f'(x)]^2 = 2f(x) . f''(x)$.

***Ex. 11.** *Show that the line joining the points of inflexion of the curve $y^2 (x - a) = x^2 (x + a)$ subtends an angle of $\pi/3$ at the origin.*

(Meerut 1990 P)

Sol. The given curve is

$$y^2 = \frac{x^2 (x + a)}{(x - a)} \quad \text{or} \quad y = \pm \frac{x \sqrt{(x + a)}}{\sqrt{(x - a)}}. \qquad ...(1)$$

Differentiating (1) w.r t. x, we get

$$\frac{dy}{dx} = \pm \frac{\sqrt{(x - a)} . \{x \cdot \tfrac{1}{2} (x + a)^{-1/2} + (x + a)^{1/2}\} - x \sqrt{(x + a)} \cdot \tfrac{1}{2} (x - a)^{-1/2}}{(x - a)}$$

$$= \pm \frac{[\sqrt{(x - a)} (3x + 2a)/\{2 \sqrt{(x + a)}\}] - [x \sqrt{(x + a)}/\{2 \sqrt{(x - a)}\}]}{(x - a)}$$

$$= \pm \frac{(x - a) (3x + 2a) - x (x + a)}{2 (x - a)^{3/2} (x + a)^{1/2}} = \pm \frac{x^2 - ax - a^2}{(x - a)^{3/2} (x + a)^{1/2}},$$

and $$\frac{d^2y}{dx^2} = \pm \frac{(x - a)^{3/2} (x + a)^{1/2} . [2x - a] - [x^2 - ax - a^2] \times [\tfrac{3}{2} (x - a)^{1/2} . (x + a)^{1/2} + (x - a)^{3/2} . \tfrac{1}{2} (x + a)^{-1/2}]}{(x - a)^3 . (x + a)}$$

$$= \pm \frac{2 (x^2 - a^2) (2x - a) - (x^2 - ax - a^2) (4x + 2a)}{2 (x - a)^{5/2} . (x + a)^{3/2}},$$

(on simplification)

$$= \pm \frac{a^2 (x + 2a)}{(x - a)^{5/2} . (x + a)^{3/2}}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$.

$\therefore \quad x = -2a$, and hence from (1) $y = \pm 2a/\sqrt{3}$.

Hence the points of inflexion are

$$P(-2a, 2a/\sqrt{3}) \quad \text{and} \quad Q(-2a, -2a/\sqrt{3}).$$

Let the line joining the points of inflexion (*i.e.*, the line PQ) subtend an angle θ at the origin. We have

$$m_1 = \text{slope of } OP = \frac{-2a/\sqrt{3}}{2a} = -1/\sqrt{3}$$

and $$m_2 = \text{slope of } OQ = \frac{2a/\sqrt{3}}{2a} = \frac{1}{\sqrt{3}}.$$

$$\therefore \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{(1/\sqrt{3}) + (1/\sqrt{3})}{1 - (1/3)} = \frac{2/\sqrt{3}}{2/3} = \sqrt{3}.$$

$\therefore \quad \theta = \pi/3$ *i.e.*, the required angle is $\pi/3$.

Ex. 12. *Show that the points of inflexion on the curve* $y = b\,e^{-(x/a)^2}$ *are given by* $x = \pm\, a/\sqrt{2}$.

Sol. The given curve is $y = b\,e^{-x^2/a^2}$. ...(1)

$$\therefore \quad \frac{dy}{dx} = b\,e^{-x^2/a^2} \cdot \left(\frac{-2x}{a^2}\right) = y \cdot \left(\frac{-2x}{a^2}\right),$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} \cdot \left(\frac{-2x}{a^2}\right) + y \cdot \frac{-2}{a^2} = y \cdot \left(\frac{-2x}{a^2}\right) \cdot \left(\frac{-2x}{a^2}\right) - \frac{2y}{a^2}$$

$$= y\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right].$$

and $$\frac{d^3y}{dx^3} = \frac{dy}{dx}\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] + y \cdot \frac{8x}{a^4}$$

$$= y \cdot \left(\frac{-2x}{a^2}\right)\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] + y \cdot \frac{8x}{a^4}.$$

For the points of inflexion of (1), we must have $d^2y/dx^2 = 0$.

$$\therefore \quad y\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] = 0$$

or $$\frac{4x^2}{a^4} - \frac{2}{a^2} = 0 \quad [\because y = b\,e^{-x^2/a^2} \text{ cannot be zero for any real } x]$$

or $$x^2 = \frac{a^2}{2} \quad \text{or} \quad x = \pm\frac{a}{\sqrt{2}}.$$

The curve (1) will have points of inflexion where $x = \pm\, a/\sqrt{2}$, provided d^3y/dx^3 is not zero at these points.

Now for $x = \pm\dfrac{a}{\sqrt{2}}$, we have $\dfrac{d^3y}{dx^3} = 0 + y \cdot \dfrac{8}{a^4} \cdot \left(\pm\dfrac{a}{\sqrt{2}}\right) \neq 0.$

Hence the points of inflexion of (1) are given by $x = \pm\, a/\sqrt{2}$.

Ex. 13. *Find the points of inflexion on the curve* $y^2 = x(x+1)^2$ *and also obtain the equations of the inflexional tangents.*

(Meerut 1983, 89 P, 91 S, 98)

Sol. The given curve is $y^2 = x(x+1)^2$ (symmetry about x-axis)

or $\quad y = \pm (x+1) x^{1/2}$.

Let us take $y = (x+1) x^{1/2}$.

Then $\dfrac{dy}{dx} = 1 \,.\, x^{1/2} + (x+1) \,.\, \tfrac{1}{2} x^{-1/2} = x^{1/2} + \dfrac{x+1}{2x^{1/2}}$

$$= \frac{2x + x + 1}{2x^{1/2}} = \frac{3x+1}{2x^{1/2}}$$

and $\dfrac{d^2y}{dx^2} = \tfrac{1}{2} \cdot \dfrac{3 \,.\, x^{1/2} - (3x+1) \,.\, \tfrac{1}{2} x^{-1/2}}{x} = \dfrac{6x - 3x - 1}{4x \,.\, x^{1/2}} = \dfrac{3x-1}{4x^{3/2}}$.

For the points of inflexion of the given curve, we must have

$d^2y/dx^2 = 0$.

$\therefore \quad \dfrac{3x-1}{4x^{3/2}} = 0 \quad$ or $\quad 3x - 1 = 0 \quad$ or $\quad x = \tfrac{1}{3}$.

Now there will be points of inflexion where $x = \tfrac{1}{3}$ if we have

$d^3y/dx^3 \neq 0$ at $x = \tfrac{1}{3}$.

We have $\dfrac{d^3y}{dx^3} = \tfrac{1}{4} \cdot \dfrac{3 \,.\, x^{3/2} - (3x-1) \,.\, (3/2)\, x^{1/2}}{x^3}$

$$= \tfrac{1}{4} \cdot \frac{3x^{1/2}\,[2x - 3x + 1]}{2 \,.\, x^3} = \frac{3}{8x^{5/2}}(1 - x) \neq 0, \text{ at } x = \tfrac{1}{3}.$$

$\therefore$ the given curve has points of inflexion where $x = \tfrac{1}{3}$. Putting $x = \tfrac{1}{3}$ in the equation of the given curve, we get

$$y = \pm \frac{4}{3} \cdot \left(\frac{1}{3}\right)^{1/2} = \pm \frac{4}{3\sqrt{3}}.$$

Hence the points of inflexion on the given curve are

$$\left(\frac{1}{3}, \pm \frac{4}{3\sqrt{3}}\right).$$

Now tangents at the points of inflexion are called inflexional tangents.

The equation of the given curve can be written as

$$y^2 = x(x^2 + 2x + 1) = x^3 + 2x^2 + x.$$

Differentiating with respect to x, we get

$$2y \frac{dy}{dx} = 3x^2 + 4x + 1 \text{ or } \frac{dy}{dx} = \frac{3x^2 + 4x + 1}{2y}.$$

$\therefore \quad$ at the point $\left(\dfrac{1}{3}, \dfrac{4}{3\sqrt{3}}\right)$, $\dfrac{dy}{dx} = \dfrac{(1/3) + (4/3) + 1}{2 \,.\, (4/3\sqrt{3})} = \sqrt{3}$

and at the point $\left(\dfrac{1}{3}, \dfrac{-4}{3\sqrt{3}}\right)$, $\dfrac{dy}{dx} = -\sqrt{3}$.

$\therefore$ inflexional tangent at the point $\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right)$ is

$$y - \frac{4}{3\sqrt{3}} = \sqrt{3}\left(x - \frac{1}{3}\right) \quad \text{or} \quad \sqrt{3}x - y - \frac{\sqrt{3}}{3} + \frac{4}{3\sqrt{3}} = 0$$

or $$\sqrt{3}x - y + \frac{1}{3\sqrt{3}} = 0 \quad \text{or} \quad 9x - 3\sqrt{3}\,y + 1 = 0$$

and inflexional tangent at the point $\left(\frac{1}{3}, \frac{-4}{3\sqrt{3}}\right)$ is

$$y + \frac{4}{3\sqrt{3}} = -\sqrt{3}\left(x - \frac{1}{3}\right) \quad \text{or} \quad \sqrt{3}x + y + \frac{4}{3\sqrt{3}} - \frac{\sqrt{3}}{3} = 0$$

or $$\sqrt{3}x + y + \frac{1}{3\sqrt{3}} = 0 \quad \text{or} \quad 9x + 3\sqrt{3}y + 1 = 0.$$

Hence the inflexional tangents are $9x \pm 3\sqrt{3}y + 1 = 0$.

Ex. 14. *Prove that the curve*

$$y = \frac{1 - x}{1 + x^2}$$

has three points of inflexion which lie in a straight line. **(Meerut 1989 S)**

Sol. The given curve is $y = \dfrac{1 - x}{1 + x^2}$. ...(1)

$$\therefore \quad \frac{dy}{dx} = \frac{-1.(1 + x^2) - 2x(1 - x)}{(1 + x^2)^2} = \frac{-1 - 2x + x^2}{(1 + x^2)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{(-2 + 2x)(1 + x^2)^2 - (-1 - 2x + x^2).2(1 + x^2).2x}{(1 + x^2)^4}$$

$$= \frac{2[(1 + x^2)(x - 1) - 2x(-1 - 2x + x^2)]}{(1 + x^2)^3}$$

$$= \frac{2[x - 1 + x^3 - x^2 + 2x + 4x^2 - 2x^3]}{(1 + x^2)^3} = \frac{2[-x^3 + 3x^2 + 3x - 1]}{(1 + x^2)^3}$$

$$= \frac{-2(x^3 - 3x^2 - 3x + 1)}{(1 + x^2)^3} = \frac{-2[(x^3 + 1) - 3x(x + 1)]}{(x^2 + 1)^3}$$

$$= \frac{-2[(x + 1)(x^2 - x + 1) - 3x(x + 1)]}{(x^2 + 1)^3}$$

$$= \frac{-2(x + 1)(x^2 - 4x + 1)}{(x^2 + 1)^3}.$$

For the points of inflexion of the given curve, we must have

$$d^2y/dx^2 = 0.$$

$\therefore \quad (x + 1)(x^2 - 4x + 1) = 0$, which gives

$$x = -1, \frac{4 \pm \sqrt{(16 - 4)}}{2}$$

i.e., $$x = -1, 2 \pm \sqrt{3}.$$

Now we observe that each of the factors $x + 1$, $x - (2 + \sqrt{3})$ and $x - (2 - \sqrt{3})$ occurs in first degree in d^2y/dx^2. Therefore the sign of d^2y/dx^2 changes when x passes through each of the values $-1, 2 + \sqrt{3}$ and $2 - \sqrt{3}$; or we also conclude that $d^3y/dx^3 \neq 0$ at any of the points $x = -1, 2 \pm \sqrt{3}$.

Hence there are points of inflexion where $x = -1, 2 \pm \sqrt{3}$.

From the equation of the curve, we have

when $x = -1, y = 1$

when $x = 2 + \sqrt{3}$, $y = \dfrac{1 - 2 - \sqrt{3}}{1 + 4 + 3 + 4\sqrt{3}} = \dfrac{-1 - \sqrt{3}}{8 + 4\sqrt{3}} = -\dfrac{1}{4} \cdot \dfrac{1 + \sqrt{3}}{2 + \sqrt{3}}$

$$= -\frac{1}{4} \cdot \frac{(1 + \sqrt{3})(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = -\frac{-1 + \sqrt{3}}{4}$$

$$= -\frac{1 - \sqrt{3}}{4}$$

and when $x = 2 - \sqrt{3}$, $y = \dfrac{1 - 2 + \sqrt{3}}{1 + 4 + 3 - 2\sqrt{3}} = \dfrac{-1 + \sqrt{3}}{4(2 - \sqrt{3})} = \dfrac{1 + \sqrt{3}}{4}$.

Hence the given curve has three points of inflexion

$$(-1, 1), \left(2 + \sqrt{3}, \frac{1 - \sqrt{3}}{4}\right) \text{ and } \left(2 - \sqrt{3}, \frac{1 + \sqrt{3}}{4}\right).$$

Let us name these points as A, B and C respectively.

The slope of the line $AB = \dfrac{\frac{1}{4}(1 - \sqrt{3}) - 1}{2 + \sqrt{3} + 1} = \dfrac{1}{4} \cdot \dfrac{1 - \sqrt{3} - 4}{3 + \sqrt{3}}$

$$= \frac{1}{4} \cdot \frac{-3 - \sqrt{3}}{3 + \sqrt{3}} = -\frac{1}{4}$$

and the slope of the line $AC = \dfrac{\frac{1}{4}(1 + \sqrt{3}) - 1}{2 - \sqrt{3} + 1} = \dfrac{1}{4} \cdot \dfrac{1 + \sqrt{3} - 4}{3 - \sqrt{3}}$

$$= \frac{1}{4} \cdot \frac{-3 + \sqrt{3}}{3 - \sqrt{3}} = -\frac{1}{4}.$$

Since the slope of the line $AB =$ the slope of the line AC, therefore the points A, B and C lie in a straight line.

Ex. 15. *Investigate the points of inflexion of the curve*

$$y = (x - 1)^4 (x - 2)^3.$$ **(Meerut 1984)**

Sol. The given curve is $y = (x - 1)^4 (x - 2)^3$. ...(1)

$$\therefore \quad \frac{dy}{dx} = 4(x - 1)^3 (x - 2)^3 + 3(x - 1)^4 (x - 2)^2$$

$$= (x - 1)^3 (x - 2)^2 [4(x - 2) + 3(x - 1)]$$

$$= (x - 1)^3 (x - 2)^2 (7x - 11)$$

and $\dfrac{d^2y}{dx^2} = 3(x - 1)^2 (x - 2)^2 (7x - 11)$

$$+ 2(x - 1)^3 (x - 2)(7x - 11) + 7(x - 1)^3 (x - 2)^2$$

$$= (x-1)^2 (x-2) [3 (x-2)(7x-11) + 2(x-1)(7x-11) + 7(x-1)(x-2)]$$

$$= (x-1)^2 (x-2) [21x^2 - 75x + 66 + 14x^2 - 36x + 22 + 7x^2 - 21x + 14]$$

$$= (x-1)^2 (x-2)(42x^2 - 132x + 102)$$

$$= 6 (x-1)^2 (x-2)(7x^2 - 22x + 17).$$

For the points of inflexion of the given curve, we must have $d^2y/dx^2 = 0$

i.e., $(x-1)^2 (x-2)(7x^2 - 22x + 17) = 0$, which gives

$$x = 1, 2, \frac{22 \pm \sqrt{[(22)^2 - 4.7.17]}}{14}$$

i.e., $x = 1, 2, \dfrac{11 \pm \sqrt{2}}{7}$.

Thus the given curve may have points of inflexion where

$$x = 1, 2, (11 \pm \sqrt{2})/7.$$

Since $(x-1)$ occurs as a factor of second degree in d^2y/dx^2, therefore the sign of d^2y/dx^2 does not change as x passes through 1. Consequently $x = 1$ does not give a point of inflexion. Thus there is a point of undulation at $x = 1$.

Again each of the factors $x - 2$, $[x - \frac{1}{7}(11 + \sqrt{2})]$ and $[x - \frac{1}{7}(11 - \sqrt{2})]$ occurs in first degree in d^2y/dx^2 and so the sign of d^2y/dx^2 changes as x passes through each of the values 2, $(11 + \sqrt{2})/7$ and $(11 - \sqrt{2})/7$. Hence the given curve has points of inflexion at $x = 2, (11 \pm \sqrt{2})/7$.

Ex. 16. *Investigate the points of inflexion of the curve*

$$y = (x-2)^6 (x-3)^5.$$ **(Meerut 1990 S)**

Sol. Proceed exactly as in Ex. 15.

Here $$\frac{dy}{dx} = (x-3)^4 (x-2)^5 (11x - 28)$$

and $$\frac{d^2y}{dx^2} = 10 (x-3)^3 (x-2)^4 (11x^2 - 56x + 71).$$

For the points of inflexion of the given curve, we must have

$$d^2y/dx^2 = 0$$

i.e., $$10 (x-3)^3 (x-2)^4 (11x^2 - 56x + 71) = 0$$ which gives

$$x = 2, 3, (28 \pm \sqrt{3})/11.$$

Thus the given curve may have points of inflexion where

$$x = 2, 3, (28 \pm \sqrt{3})/11.$$

Since $(x-2)$ occurs as a factor of even degree in d^2y/dx^2, therefore the sign of d^2y/dx^2 does not change as x passes through 2. Consequently

$x = 2$ does not give a point of inflexion. Thus there is a point of undulation at $x = 2$.

Again each of the factors $x - 3$, $[x - \frac{1}{11}(28 + \sqrt{3})]$ and $[x - \frac{1}{11}(28 - \sqrt{3})]$ occurs in odd degree in d^2y/dx^2 and so the sign of d^2y/dx^2 changes as x passes through each of the values 3, $(28 + \sqrt{3})/11$ and $(28 - \sqrt{3})/11$. Hence the given curve has points of inflexion at $x = 3, (28 \pm \sqrt{3})/11$.

Ex.17. *Find the points of inflexion on the curve*
$$x = a(2\theta - \sin\theta),\ y = a(2 - \cos\theta).$$
(G.N.U. 1973; Meerut 91)

Sol. Differentiating w.r.t. θ, we have
$$dx/d\theta = a(2 - \cos\theta) \text{ and } dy/d\theta = a\sin\theta.$$
$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta}{2 - \cos\theta}.$$
And
$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{2 - \cos\theta}\right) \cdot \frac{d\theta}{dx}$$
$$= \frac{(2 - \cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(2 - \cos\theta)^2} \cdot \frac{1}{a(2 - \cos\theta)}$$
$$= \frac{2\cos\theta - (\cos^2\theta + \sin^2\theta)}{a(2 - \cos\theta)^3} = \frac{2\cos\theta - 1}{a(2 - \cos\theta)^3}.$$
Now for the points of inflexion, we must have $d^2y/dx^2 = 0$

i.e. $\quad 2\cos\theta - 1 = 0 \quad$ or $\quad \cos\theta = \frac{1}{2} = \cos\frac{1}{3}\pi$.

$\therefore \quad \theta = 2n\pi \pm \frac{1}{3}\pi$, where n is any integer.

Substituting the value of θ in the given equation of the curve we get the points of inflexion as
$$\left[a\left(4n\pi \pm \frac{2\pi}{3} \mp \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right].$$
$$\left[\because \sin\left(2n\pi \pm \frac{\pi}{3}\right) = (-1)^{2n}\sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}\right].$$

Ex. 18. *Show that the points of inflexion on the curve $r = b\theta^n$ are given by $r = b\{-n(n+1)\}^{n/2}$.*

Sol. Differentiating the given equation of the curve w.r.t. θ, we get
$$dr/d\theta = nb\,\theta^{n-1} \text{ and } d^2r/d\theta^2 = n(n-1)\,b\theta^{n-2}.$$
We know that at the point of inflexion, the radius of curvature is infinite. Hence at the point of inflexion, we have
$$\mathbf{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2) = 0} \qquad \textbf{[Note]}$$
or
$$(b\theta^n)^2 + 2(nb\theta^{n-1})^2 - (b\theta^n)\{n(n-1)b\theta^{n-2}\} = 0,$$
substituting the values of r, $dr/d\theta$, $d^2r/d\theta^2$

or $$b^2\theta^{2n}\left[1+\frac{2n^2}{\theta^2}-\frac{n(n-1)}{\theta^2}\right]=0$$

or $$b^2\,\theta^{2n-2}\,[\theta^2+n^2+n]=0 \text{ giving } \theta^2=-\,n\,(n+1).$$

Now from the equation of the curve, we have

$$r=b\theta^n=b\,(\theta^2)^{n/2}. \qquad ...(1)$$

Putting $\theta^2=-\,n\,(n+1)$ in (1), we see that the points of inflexion are given by

$$r=b\,\{-\,n\,(n+1)\}^{n/2}.$$

Ex. 19. *Find the points of inflexion on the curve*

$$r\,(\theta^2-1)=a\theta^2. \qquad \textbf{(G.N.U. 1972; P.U. 70 S)}$$

Sol. We have $r=c\theta^2/(\theta^2-1)$.

$$\therefore \quad \frac{dr}{d\theta}=a\,[(\theta^2-1).2\theta-\theta^2.2\theta]/(\theta^2-1)^2$$

$$=-\,2a\theta/(\theta^2-1)^2,$$

and $$\frac{d^2r}{d\theta^2}=-\,2a\,[(\theta^2-1)^2.1-\theta.2\,(\theta^2-1).2\theta]/(\theta^2-1)^4$$

$$=2a\,(3\theta^2+1)/(\theta^2-1)^3.$$

At the point of inflexion, we have

$$r^2+2\,(dr/d\theta)^2-r\,(d^2r/d\theta^2)=0$$

or $$\frac{a^2\theta^4}{(\theta^2-1)^2}+\frac{8a^2\theta^2}{(\theta^2-1)^4}-\frac{2a^2\theta^2(3\theta^2+1)}{(\theta^2-1)^4}=0$$

or $$\frac{a^2\theta^2(\theta^2-3)(\theta^2+2)}{(\theta^2-1)^3}=0$$

or $$\theta^2(\theta^2-3)(\theta^2+2)=0$$

$$\therefore \quad \theta^2=0,\,3,\,-\,2.$$

Rejecting the values $\theta^2=-\,2$ and 0 we see that the points of inflexion are given by $\theta^2=3$ *i.e.*, $\theta=\pm\sqrt{3}$.

§ 5. Multiple points. (Agra 1970; Kanpur 71; Indore 70)

A point through which more than one branches of a curve pass is called a multiple point on the curve. A point on the curve is called a **double point** if two branches of the curve pass through it, a **triple point** if three branches pass through it. In general if r branches pass through a point, it is called a multiple point of the r^{th} order.

§ 6. Classification of Double Points.

(P.U. 1970; Pbi. U. 73; G.N.U. 74 S)

(i) Node. If the two branches through a double point on a curve are real and have *different* tangents there, the double point is called a *node*. (Fig. 1)

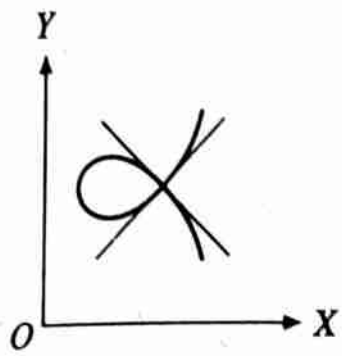


Fig. 1

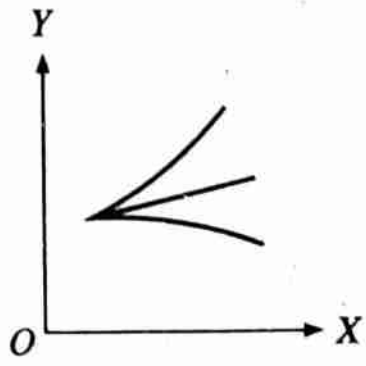


Fig. 2

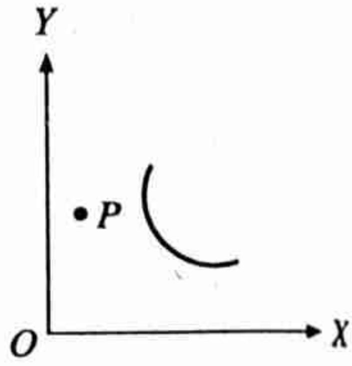


Fig. 3

(ii) Cusp. If the two branches through a double point on a curve are real and have *coincident* tangents there, then the double point is called a *cusp*. (Fig. 2)

(iii) Conjugate point. If there are no real points on the curve in the neighbourhood of a point P on the curve, then P is called a *conjugate point* (or an *isolated point*). The process of finding the tangents usually gives imaginary tangents at such a point. (Fig. 3)

Note : Since through a double point two branches of the curve pass, therefore in the process of finding tangents at a double point we must get *two* tangents there, one for each branch. If the two tangents are real and distinct, the double point will be a node. If the tangents are imaginary, the double point will be a conjugate point. If the two tangents are real and coincident, the double point may be a cusp or a conjugate point. The possibility of the double point being a conjugate point in this case arises on account of the fact that sometimes imaginary expressions $A \pm iB$ become real by chance when $B = 0$. In such cases the double point will be a cusp if there are other real points of the curve in its neighbourhood, otherwise it will be a conjugate point.

§ 7. Species of Cusps.

We know that two branches of a curve have a common tangent at a cusp. A cusp is said to be *single* or *double* according as the curve lies entirely on *one* side of the common normal or on *both* sides. Also it is of the *first* or *second species* according as the two branches lie on *opposite* sides or on the *same* side of the common tangent. We have the following *five* different types of cusps :

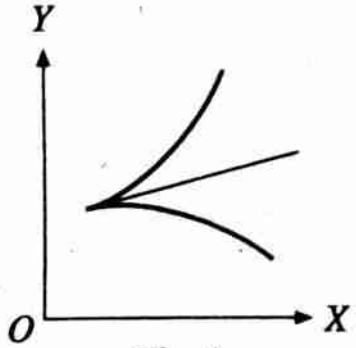


Fig. 1

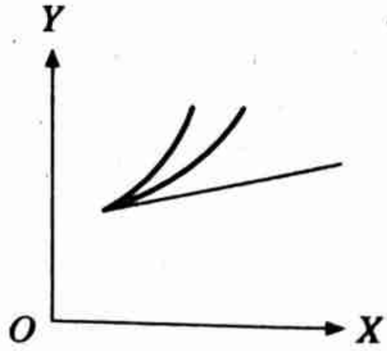


Fig. 2

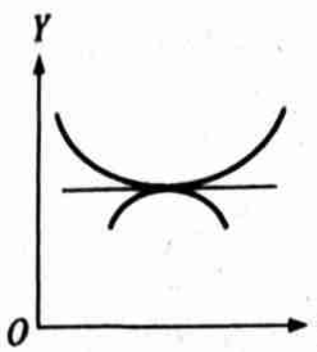


Fig. 3

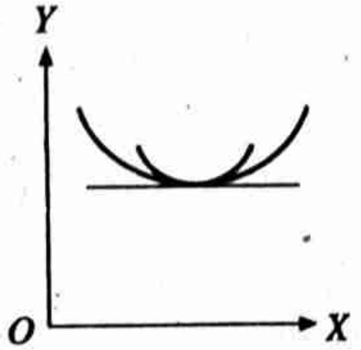


Fig. 4

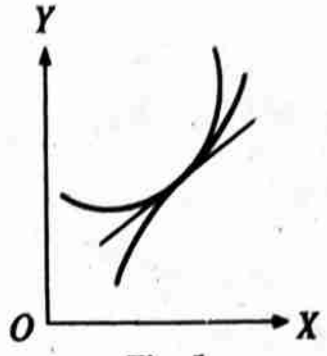


Fig. 5

Single cusp of the first species as shown in Fig. 1.

Single cusp of the second species as shown in Fig. 2.

Double cusp of the first species as shown in Fig. 3.

Double cusp of the second species as shown in Fig. 4.

Double cusp with change of species as shown in Fig. 5. Here the two branches lie on *both the sides* of the common normal but on one side they lie on the same and on the other on opposite sides of the common tangent. Such a point is called a point of **osculinflexion.**

§ 8. Tangents at Origin.

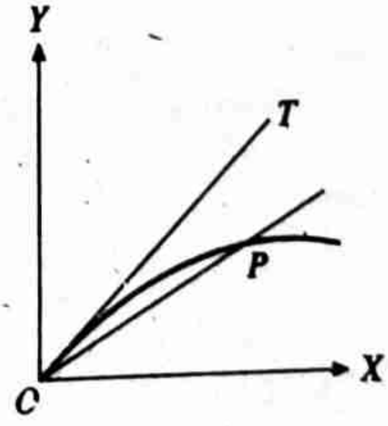
In order to know the nature of a double point it is necessary to find the tangent or tangents there. Now we shall find a simple rule for writing down the *tangent* or *tangents* at the origin to rational algebraic curves.

If a curve passes through the origin and is given by a rational, integral, algebraic equation, the equation to the tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

Let the equation of the curve when arranged according to ascending powers of x and y be

$$(a_1x + a_2y) + (b_1x^2 + b_2xy + b_3y^2) + (c_1x^3 + c_2x^2y + \dots) + \dots = 0, \quad \dots(1)$$

where the constant term is absent since the curve passes through the origin.



Let $P(x, y)$ be any point on the curve. The slope of the chord OP is y/x. Therefore the equation to OP is

$$Y = (y/x)X,$$

where (X, Y) are current coordinates.

As $P \to O$ *i.e.,* as $x \to 0$ and $y \to 0$, chord OP tends to the tangent at O.

Excluding for the present the case when the tangent is the y-axis *i.e.*, when $\lim_{x \to 0} \left(\frac{y}{x}\right) = \pm \infty$, we have the equation of the tangent at O as

$$Y = \left\{ \lim_{x \to 0} \left(\frac{y}{x}\right) \right\} = X. \qquad ...(2)$$

Case I. Let $a_2 \neq 0$. Dividing (1) by x and taking limit as $x \to 0$, we get

$$a_1 + a_2 \left\{ \lim_{x \to 0} \left(\frac{y}{x}\right) \right\} = 0. \qquad ...(3)$$

Eliminating $\lim_{x \to 0} \left(\frac{y}{x}\right)$ between (2) and (3), we get

$$a_1 X + a_2 Y = 0,$$

as the equation of tangent at the origin to the curve (1).

Replacing the current coordinates X, Y by x, y this equation becomes

$$a_1 x + a_2 y = 0, \qquad ...(4)$$

which is obviously the equation obtained by equating to zero the lowest degree terms in (1).

If $a_2 = 0$, then a_1 is also zero from (3), and we get the next case.

Case II. Let $a_1 = 0, a_2 = 0$, but b_2 and b_3 are not both zero. Dividing (1) by x^2 and taking limit as $x \to 0$, we get

$$b_1 + b_2 \lim_{x \to 0} \left(\frac{y}{x}\right) + b_3 \lim_{x \to 0} \left(\frac{y}{x}\right)^2 = 0,$$

or $$b_1 + b_2 m + b_3 m^2 = 0, \qquad ...(5)$$

where $$\lim_{x \to 0} \left(\frac{y}{x}\right) = m.$$

Equation (5) is a quadratic in m, showing that there are two tangents at the origin in this case. Eliminating m between (2) and (5), we get

$$b_1 x^2 + b_2 xy + b_3 y^2 = 0, \qquad ...(6)$$

as the equation of the tangents at the origin to (1) in this case. In equation (6), we have taken x, y as current coordinates. Obviously the equation (6) is obtained by equating to zero the lowest degree terms in the equation of the curve (1), where

$$a_1 = a_2 = 0.$$

If $b_2 = b_3 = 0$, then by (5), $b_1 = 0$.

Case III. If $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, we can show by the same process that the rule still holds; and so on.

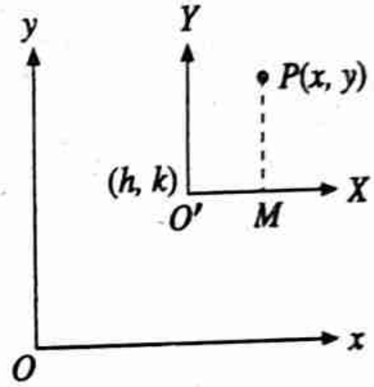
If tangent at the origin is the y-axis, we can easily show by supposing the axes of x and y to be interchanged for a moment, that the rule is still true.

Hence the equation of the tangent or tangents at the origin is obtained by equating to zero the lowest degree terms in the equation of the curve.

Cor. If the origin is double point on a curve, then the curve has two tangents at the origin. Therefore the equation of the curve should not contain the constant and the first degree terms and the second degree terms should be the lowest degree terms in the equation of the curve.

§ 9. Change of origin.

Let (x, y) be the coordinates of a point P with reference to Ox and Oy as coordinate axes. Referred to Ox and Oy as coordinate axes, let (h, k) be the coordinates of a point O'. Draw a line $O'X$ parallel to Ox and a line $O'Y$ parallel to Oy. Let (X, Y) be the coordinates of P with reference to $O'X$ and $O'Y$ as coordinate axes.



Obviously, we have

$$x = X + h \quad \text{and} \quad y = Y + k.$$

Thus to obtain the equation of the curve referred to the point (h, k) as origin, the coordinate axes remaining parallel to their original directions, we should put $X + h$ in place of x and $Y + k$ in place of y in the equation of the curve, where X, Y are the current coordinates in the new equation.

If in the new equation also we take x, y as the current coordinates, then in order to shift the origin to the point (h, k), we should replace x by $x + h$ and y by $y + k$ in the given equation of the curve.

§ 10. Tangents at the point (h, k) to a curve.

If we are to find the tangents at the point (h, k) to a curve, we should first shift the origin to the point (h, k) in the equation of the curve. Then the equation of the tangents at the new origin will be obtained by equating to zero the lowest degree terms in the new equation of the curve.

****§ 11. Position and Character of Double points.** **(Gorakhpur 1982)**

Let $f(x, y) = 0$ be any curve and P be any point (x, y) on it. The slope of the tangent at P is equal to dy/dx and it is given by the equation

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

or
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad ...(1)$$

At a multiple point of a curve, the curve has at least two tangents and accordingly dy/dx must have at least two values at a multiple point. The equation (1) is of first degree in dy/dx. It can be satisfied for more than one value of dy/dx, if and only if,

$$\partial f/\partial x = 0,\ \partial f/\partial y = 0.$$

Therefore the necessary and sufficient conditions for any point (x, y) of the curve $f(x, y) = 0$ to be a multiple point are that

$$\boldsymbol{\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0.}$$

Rule. To find the multiple points of the curve $f(x, y) = 0$, we should simultaneously solve the equations,

$$\boldsymbol{\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0,\ f(x, y) = 0.}$$

Classification of double points.

Differentiating (1) with respect to x again, we get

$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) + \frac{d}{dx}\left\{\frac{\partial f}{\partial y}\frac{dy}{dx}\right\} = 0$$

or
$$\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) + \frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) \cdot \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2y}{dx^2} = 0$$

or
$$\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \left\{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right\}\frac{dy}{dx}\right]$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \left\{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right\} \cdot \frac{dy}{dx}\right] \cdot \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2y}{dx^2} = 0$$

or
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x\,\partial y} \cdot \frac{dy}{dx} + \frac{\partial^2 f}{\partial x\,\partial y} \cdot \frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y} \cdot \frac{d^2y}{dx^2} = 0$$

or $\dfrac{\partial^2 f}{\partial x^2} + 2 \cdot \dfrac{\partial^2 f}{\partial x\,\partial y} \cdot \dfrac{dy}{dx} + \dfrac{\partial^2 f}{\partial y^2} \cdot \left(\dfrac{dy}{dx}\right)^2 = 0$, since at a multiple point $\partial f/\partial y = 0$.

Therefore at the multiple point, the values of dy/dx are given by the equation

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + 2\frac{\partial^2 f}{\partial x\,\partial y}\left(\frac{dy}{dx}\right) + \frac{\partial^2 f}{\partial x^2} = 0. \qquad ...(2)$$

If $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x\,\partial y}, \dfrac{\partial^2 f}{\partial y^2}$ are not all zero, the equation (2) will be a quadratic in dy/dx and the multiple point will be a double point.

The two tangents will be real and distinct, coincident, or imaginary according as

$$4\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 - 4\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} >, = \text{ or } < 0$$

i.e., in general the double point will be a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \mathbf{y}}\right)^2 >, = \textbf{ or } < \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{y}^2}\right).$$

Note. If $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y^2} = 0$, then the point (x, y) will be a multiple point of order higher than the second.

§ 12. Nature of cusp at the origin.

Suppose the origin is a cusp. Then the curve will have two coincident tangents at the origin. Therefore the equation of the curve must be of the form

$$(ax + by)^2 + \text{ terms of third and higher degrees} = 0. \qquad \text{...(1)}$$

The common tangent at the origin to the two branches of the curve is

$$ax + by = 0. \qquad \text{...(2)}$$

Let P be the perpendicular to (2) from any point (x, y) on (1) in the neighbourhood of the origin. Then

$$P = \frac{ax + by}{\sqrt{(a^2 + b^2)}},$$

which is proportional to $ax + by$. Let us put

$$p = ax + by. \qquad \text{...(3)}$$

Eliminate x or y (whichever is convenient) between (1) and (3). Suppose we eliminate y. Then we shall get a relation between p and x. Since p is small and also there are only two branches of the curve (2) through the origin, therefore terms involving powers of p above the second will be neglected. Thus we shall get a quadratic in p of the form

$$Ap^2 + Bp + C = 0, \qquad \text{...(4)}$$

where A, B and C are some functions of x. Solving (4), we get

$$p = \{-B \pm \sqrt{(B^2 - 4AC)}\}/2A. \qquad \text{...(5)}$$

Also if p_1, p_2 are the roots of (4), we get

$$p_1 p_2 = C/A. \qquad \text{...(6)}$$

The following different cases arise :

(*i*) If for all values of x, positive or negative, provided they are numerically small, the values of p given by (5) are imaginary, the origin will be a conjugate point.

(*ii*) If for all numerically small values of x, positive or negative, the values of p given by (5) are real, there will be a double cusp at the origin.

(*iii*) If the reality of the values of p given by (5) depends on the sign of x, there will be a single cusp at the origin.

(*iv*) If for numerically samll values of x for which p is real, the sign of $p_1 p_2$ is positive, then p_1 and p_2 will be of the same sign. Therefore the two perpendiculars lie on the same side of the common tangent and there will be a cusp of the second species. If on the other hand, the sign of $p_1 p_2$ is negative, then p_1 and p_2 are of opposite signs. Therefore the two perpendiculars lie on opposite sides of the common tangent and there will be a cusp of the first species.

Note. While investigating the sign of an expression for sufficiently small values of x, we should keep in mind only those terms which involve the lowest power of x.

§ 13. Nature of a cusp at any point.

If there is a cusp at the point (h, k), we should first shift the origin to (h, k) and then apply the methods given in § 12.

Solved Examples

***Ex. 1.** *Examine the nature of the origin on the curve*

$$x^4 - ax^2y + axy^2 + a^2y^2 = 0.$$

(Meerut 1983 S; Gorakhpur 76)

Sol. Equating to zero, the lowest degree terms in the given curve, the tangents at the origin are given by

$$a^2y^2 = 0 \text{ or } y^2 = 0 \text{ } i.e., y = 0, y = 0.$$

Thus there are two real and coincident tangents at the origin.

$\therefore$ origin is either a **Cusp or a Conjugate point.**

Now the equation of the given curve is

$$ay^2(x + a) - ax^2y + x^4 = 0.$$

Solving it for y, we have

$$y = \frac{ax^2 \pm \sqrt{\{a^2x^4 - 4ax^4(x + a)\}}}{2a(x + a)}$$

$$= \frac{ax^2 \pm x^2\sqrt{(-4ax - 3a^2)}}{2a(x + a)}.$$

Now for small values of $x \neq 0$, $(-4ax - 3a^2)$ is – ive.

$\therefore$ y is imaginary in the neighbourhood of origin.

Hence origin is a conjugate point.

Ex. 2. *Find the nature of the origin on the curve*

$$y^2 = 2x^2y + x^4y - 2x^4.$$

Sol. Equating to zero, the lowest degree terms in the given curve, the tangents at the origin are given by $y^2 = 0$ *i.e.*, $y = 0, y = 0$.

$\therefore$ origin is either a *cusp or a conjugate point.*

Now the equation of the given curve can be written as

$$y^2 - x^2y(2 + x^2) + 2x^4 = 0.$$

Solving it for y, we have

$$y = \frac{x^2(2 + x^2) \pm \sqrt{\{x^4(2 + x^2)^2 - 8x^4\}}}{2}$$

$$= \frac{x^2(x^2 + 2) + x^2\sqrt{(x^4 + 4x^2 - 4)}}{2}.$$

Now for small values of $x \neq 0$, $x^4 + 4x^2 - 4$ is negative.

$\therefore y$ is imaginary in the neighbourhood of origin.

Hence origin is a *conjugate point.*

Ex. 3. *Find the nature of the origin for the curve*

$$y^2(a^2 + x^2) = x^2(a^2 - x^2).$$

Sol. Equating to zero, the lowest degree terms in the equation of the curve, the tangents at the origin are

$$a^2y^2 - a^2x^2 = 0 \text{ or } y = \pm x,$$

which are **real and distinct.** Therefore the origin is a **node.**

Ex. 4. *Examine the nature of the origin on the curve*

$$a^2/x^2 - b^2/y^2 = 1.$$

Sol. The curve can be written as $a^2y^2 - b^2x^2 = x^2y^2$. ...(1)

Equating to zero the lowest degree terms in (1), the tangents at the origin are $a^2y^2 - b^2x^2 = 0$ or $y = \pm(b/a)x$, which are **real and distinct**. Therefore the origin is a **node.**

Ex. 5. *Show that the curve $x^3 + x^2y = ay^2$ has the cusp at the origin.*

(Meerut 1992)

Sol. The equation of the given curve may be written as

$$ay^2 - x^2y - x^3 = 0. \qquad ...(1)$$

Equating to zero, the lowest degree terms in (1), the tangents at the origin are $y^2 = 0$ *i.e.*, $y = 0$ and $y = 0$.

$\therefore$ origin is either a *cusp or a conjugate point.*

Now from (1), $y = [x^2 \pm \sqrt{(x^4 + 4ax^3)}]/(2a)$.

For small values of $x \neq 0$, $x^4 + 4ax^3$ has the same sign as $4ax^3$, which is +ive when x is +ive and − ive when x is −ive.

$\therefore$ when x is +ive, y has two real values one +ive and the other −ive, whereas y is imaginary when x is negative.

Hence there is a single cusp of the first kind at the origin on the right side of the y-axis.

Ex. 6 (a). *Determine the existence and nature of the double points on the curve* $y^2 = (x-2)^2 (x-1)$.

(Rohilkhand 1982; Garhwal 83; Meerut 91S, 97)

Sol. The equation of the given curve is

$$f(x, y) \equiv y^2 - (x-2)^2 (x-1) = 0. \qquad \text{...(1)}$$

We have $\partial f / \partial x = -2(x-2)(x-1) - (x-2)^2$

$$= -(x-2)\{2(x-1) + (x-2)\}$$

$$= -(x-2)(3x-4)$$

and $\partial f / \partial y = 2y$.

For double points, $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $f(x, y) = 0$.

Here $\partial f / \partial x = 0$ gives $(x-2)(3x-4) = 0$ *i.e.*, $x = 2, 4/3$,
and $\partial f / \partial y = 0$ gives $y = 0$.

$\therefore$ the possible double points are $(2, 0)$, $(4/3, 0)$.

Out of these only $(2, 0)$ satisfies the equation of the curve. Therefore $(2, 0)$ is the only double point on the given curve.

Nature of the double point at (2, 0). Shifting the origin to the point $(2, 0)$, the equation of the curve becomes

$$y^2 = (x + 2 - 2)^2 (x + 2 - 1)$$

i.e.,
$$y^2 = x^2 (x+1) \qquad \text{...(2)}$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are

$$y^2 - x^2 = 0 \text{ i.e., } y^2 = x^2 \text{ i.e., } y = \pm x.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(2, 0)$ on the given curve.

Ex. 6 (b). *Find the position and nature of the double points on the following curve :*

$$f(x, y) \equiv x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0. \quad \textbf{(Meerut 1984 R)}$$

Sol. At a double point, we have

$$(\partial f / \partial x) = 4x^3 - 4x = 0 \text{ and } (\partial f / \partial y) = -6y^2 - 6y = 0.$$

These give $x(x^2 - 1) = 0$ and $y(y+1) = 0$

i.e., $x = 0, \pm 1; y = 0, -1$.

Out of these points only $(0, -1)$, $(1, 0)$ and $(-1, 0)$ satisfy the equation of the curve, which are double points.

Also $\dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4$, $\dfrac{\partial^2 f}{\partial x \, \partial y} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = -12y - 6$.

At the point $(0, -1)$,

$$\frac{\partial^2 f}{\partial x^2} = -4, \ \frac{\partial^2 f}{\partial x \, \partial y} = 0, \ \frac{\partial^2 f}{\partial y^2} = 6.$$

$$\therefore \quad \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2 > \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$

The above relation also holds at the points $(1, 0)$ and $(-1, 0)$. Hence there are nodes at the points

$$(0, -1), (0, 1) \text{ and } (-1, 0).$$

Ex. 6 (c). *Examine the nature of the double points of the curve* $2(x^3 + y^3) - 3(3x^2 + y^2) + 12x = 4.$ **(Lucknow 1978; Meerut 89, 96)**

Sol. The equation of the given curve is

$$f(x, y) \equiv 2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0. \qquad ...(1)$$

We have $\dfrac{\partial f}{\partial x} = 6x^2 - 18x + 12$

and $\dfrac{\partial f}{\partial y} = 6y^2 - 6y.$

For the double points,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \text{ and } f(x, y) = 0.$$

Here $\dfrac{\partial f}{\partial x} = 0$ gives $6x^2 - 18x + 12 = 0$

i.e., $x^2 - 3x + 2 = 0$ *i.e.,* $(x-1)(x-2) = 0$ *i.e.,* $x = 1, 2$

and $\dfrac{\partial f}{\partial y} = 0$ gives $6y^2 - 6y = 0$

i.e., $y(y-1) = 0$ *i.e.,* $y = 0, 1.$

$\therefore$ the possible double points are $(1, 0)$, $(1, 1)$, $(2, 0)$ and $(2, 1)$.

Out of these only $(1, 1)$ and $(2, 0)$ satisfy the equation of the curve. Therefore $(1, 1)$ and $(2, 0)$ are the only double points on the given curve.

Now $\dfrac{\partial^2 f}{\partial x^2} = 12x - 18$, $\dfrac{\partial^2 f}{\partial x\, \partial y} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = 12y - 6.$

At the point $(1, 1)$, $\dfrac{\partial^2 f}{\partial x^2} = -6$, $\dfrac{\partial^2 f}{\partial x\, \partial y} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = 6.$

$\therefore$ at the point $(1, 1)$, $\left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 = 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = -36.$

Thus at the point $(1, 1)$, $\left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}.$

Therefore there is a node at the point $(1, 1)$.

At the point $(2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = 6$, $\dfrac{\partial^2 f}{\partial x\, \partial y} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = -6.$

$\therefore$ at the point $(2, 0)$, $\left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}.$

Thus there is a node at the point $(2, 0)$.

Ex. 6 (d). *Find the nature of the origin on the curve*
$$a^4y^2 = x^4(x^2 - a^2).$$ **(Gorakhpur 1980)**

Sol. The given curve is $a^4y^2 = x^4(x^2 - a^2)$. ...(1)

Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $a^4y^2 = 0$ *i.e.*, $y = 0, y = 0$ are two real and coincident tangents at the origin.

Thus the origin may be a cusp or a conjugate point.

From (1), $y = \pm (x^2/a^2)\sqrt{(x^2 - a^2)}$.

For small values of $x \neq 0$, +ive or –ive, $(x^2 - a^2)$ is –ive *i.e.*, y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin. Hence origin is a conjugate point and not a cusp.

Ex. 6 (e). *Determine the position and character of the double points on the curve*
$$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$$ **(Meerut 1991, 93)**

Sol. The equation of the given curve is
$$f(x, y) \equiv x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0. \quad ...(1)$$

We have $\partial f/\partial x = 3x^2 - 14x + 15$

and $\partial f/\partial y = -2y + 4$.

For the double points,
$$\partial f/\partial x = 0, \; \partial f/\partial y = 0 \quad \text{and} \quad f(x, y) = 0.$$

Here $\partial f/\partial x = 0$ gives $3x^2 - 14x + 15 = 0$

i.e., $3x^2 - 9x - 5x + 15 = 0$ *i.e.*, $(x - 3)(3x - 5) = 0$

i.e., $x = 3, 5/3$

and $\partial f/\partial y = 0$ gives $-2y + 4 = 0$ *i.e.*, $y = 2$.

$\therefore$ the possible double points are $(3, 2)$ and $(5/3, 2)$. Out of these only $(3, 2)$ satisfies the equation of the curve. Hence $(3, 2)$ is the only double point of the given curve.

Nature of the double point (3, 2).

Now $\partial^2 f/\partial x^2 = 6x - 14$, $\partial^2 f/\partial x\,\partial y = 0$, $\partial^2 f/\partial^2 y = -2$.

At the point $(3, 2)$, $\dfrac{\partial^2 f}{\partial x^2} = 4$, $\dfrac{\partial^2 f}{\partial x\,\partial y} = 0$, $\dfrac{\partial^2 f}{\partial y^2} = -2$.

$\therefore$ at the point $(3, 2)$, $\left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2 = 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = -8$.

Thus at the point $(3, 2)$, $\left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}$.

Therefore there is a node at the point $(3, 2)$.

Alternative method for finding the nature of the double point (3, 2).

Shifting the origin to the point $(3, 2)$, the equation of the curve becomes

$$(x+3)^3-(y+2)^2-7(x+3)^2+4(y+2)+15(x+3)-13=0$$

or $x^3+9x^2+27x+27-y^2-4y-4-7x^2-42x-63+4y+8+15x$
$+45-13=0$

or $x^3+2x^2-y^2=0$. ...(2)

Equating to zero the lowest degree terms in (2), the tangents at the new origin are

$$y^2-2x^2=0 \text{ i.e., } y^2=2x^2 \text{ i.e., } y=\pm x\sqrt{2}.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point (3, 2) on the given curve.

Ex. 7. *Show that origin is a node, a cusp or conjugate point on the curve $y^2=ax^2+bx^3$ according as a is +ive, zero or –ive.*

Sol. Here $f(x,y)\equiv y^2-ax^2-bx^3=0$. ...(1)

At a double point, we must have

$$(\partial f/\partial x)=-2ax-3bx^2=0 \text{ and } (\partial f/\partial y)=2y=0.$$

These give $2ax+3bx^2=0$ and $y=0$

i.e., $x=0,\ -2a/3b$ and $y=0$

i.e., $x=0,y=0$ and $x=-2a/3b,\ y=0$.

Out of these (0, 0) satisfies the equation of the curve. Hence (0, 0) is the only double point on the curve.

Now equating to zero, the lowest degree terms in (1), the tangents at the origin are $y^2-ax^2=0$. ...(2)

When **a is positive** (*i.e.,* $a>0$), (2) gives two real and distinct tangents, showing that origin is a **node.**

When $\boldsymbol{a=0}$, (2) becomes $y^2=0$ *i.e.,* $y=0, y=0$ (giving two real and coincident tangents of the curve). Also the given equation reduces to $y^2=bx^3$ *i.e.,* the branches of the curve near the origin are real.

Hence **origin is a cusp when a = 0.**

When **a is negative** (*i.e.,* $a<0$), (2) reduces to $y^2+kx^2=0$, where $a=-k$ (say), k being a +ive quantity. Thus the two tangents are imaginary and hence origin is a **conjugate point when a < 0.**

***Ex. 8.** *Show that the curve $y^2=bx\tan(x/a)$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.*

Sol. Here $f(x,y)\equiv y^2-bx\tan(x/a)=0$. ...(1)

At a double point, we must have

$$\frac{\partial f}{\partial x}=-b\tan\frac{x}{a}-\frac{b}{a}x\sec^2\left(\frac{x}{a}\right)=0 \text{ and } \frac{\partial f}{\partial y}=2y=0.$$

These give $x=0$, $y=0$. Also (0, 0) satisfies the given curve. Thus (0, 0) is a double point.

Also $\qquad \dfrac{\partial^2 f}{\partial x\, \partial y} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = 2$, at $(0, 0)$

and $\qquad \dfrac{\partial^2 f}{\partial x^2} = -\dfrac{b}{2}\sec^2\dfrac{x}{a} - \dfrac{b}{a}\sec^2\dfrac{x}{a} - \dfrac{2bx}{a^2}\sec\dfrac{x}{a}\cdot\tan\dfrac{x}{a}$

$$= -2\,(b/a),\ \text{at } (0, 0).$$

Also we know that (see § 11) the double point will be a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 >, = \text{or} < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right). \qquad \textbf{(Remember)}$$

Here $\left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2 = 0$ and $\left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right) = -4\dfrac{b}{a}$.

Now $0 > -4b/a$ if a and b have like signs

and $\qquad 0 < -4b/a$ if a and b have unlike signs.

$\therefore$ origin is a node or conjugate point according as a and b have like or unlike signs.

***Ex. 9.** *Show that each of the curves*

$$(x\cos\alpha - y\sin\alpha - b)^3 = c\,(x\sin\alpha + y\cos\alpha)^2$$

for all values of α has a cusp; show that all the cusps lie on a circle.

Sol. Here $f(x, y) \equiv (x\cos\alpha - y\sin\alpha - b)^3$

$$- c\,(x\sin\alpha + y\cos\alpha)^2 = 0.$$

$\therefore \qquad \partial f/\partial x = 3(x\cos\alpha - y\sin\alpha - b)^2\cos\alpha$

$$- 2c\,(x\sin\alpha + y\cos\alpha)\sin\alpha,$$

and $\qquad \partial f/\partial y = -3\,(x\cos\alpha - y\sin\alpha - b)^2\sin\alpha$

$$- 2c\,(x\sin\alpha + y\cos\alpha)\cos\alpha.$$

Also $(\partial^2 f/\partial x^2) = 6\,(x\cos\alpha - y\sin\alpha - b)\cos^2\alpha - 2c\sin^2\alpha$,

$(\partial^2 f/\partial y^2) = 6\,(x\cos\alpha - y\sin\alpha - b)\sin^2\alpha - 2c\cos^2\alpha$,

and $\qquad (\partial^2 f/\partial x\, \partial y) = -6\,(x\cos\alpha - y\sin\alpha - b)\cos\alpha\sin\alpha$

$$- 2c\sin\alpha\cos\alpha.$$

Now at a double point $f = 0$, $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. Thus

$$3\,(x\cos\alpha - y\sin\alpha - b)^2\cos\alpha = 2c\,(x\sin\alpha + y\cos\alpha)\sin\alpha \qquad ...(1)$$

$$-3\,(x\cos\alpha - y\sin\alpha - b)^2\sin\alpha = 2c\,(x\sin\alpha + y\cos\alpha)\cos\alpha \qquad ...(2)$$

Multiplying (1) by $\sin\alpha$ and (2) by $\cos\alpha$ and adding, we get

$$x\sin\alpha + y\cos\alpha = 0. \qquad ...(3)$$

Again multiplying (1) by $\cos\alpha$ and (2) by $\sin\alpha$ and subtracting, we get

$$(x\cos\alpha - y\sin\alpha - b)^2 = 0 \quad \text{or} \quad x\cos\alpha - y\sin\alpha = b. \qquad ...(4)$$

Solving (3) and (4), we get

$$x = b\cos\alpha,\ y = -b\sin\alpha. \qquad ...(5)$$

The values of x and y found in (5) satisfy

$$f = 0,\ (\partial f/\partial x) = 0 \text{ and } (\partial f/\partial y) = 0.$$

Hence $(b \cos \alpha, -b \sin \alpha)$ is a double point of each curve obtained by giving any real value to α in the given equation.

Eliminating α from (5), we find the locus of these double points as $x^2 + y^2 = b^2$ (*i.e*, a circle).

Also $(\partial^2 f/\partial x^2) = -2c \sin^2 \alpha$, at $(b \cos \alpha, -b \sin \alpha)$

$(\partial^2 f/\partial y^2) = -2c \cos^2 \alpha$, at $(b \cos \alpha, -b \sin \alpha)$

and $(\partial^2 f/\partial x\, \partial y) = -2c \sin \alpha \cos \alpha$, at $(b \cos \alpha, -b \sin \alpha)$.

From these, we have $\left(\dfrac{\partial^2 f}{\partial x^2}\right)\left(\dfrac{\partial^2 f}{\partial y^2}\right) = \left(\dfrac{\partial^2 f}{\partial x\, \partial y}\right)^2$.

Hence the double point $(b \cos \alpha, -b \sin \alpha)$ is a cusp.

Ex. 10. *Find the double points on* $x^3 + y^3 = 3axy$.

(Rohilkhand 1982 P; Meerut 98)

Sol. Here $f(x, y) \equiv x^3 + y^3 - 3axy = 0$. ...(1)

$\therefore$ $(\partial f/\partial x) = 3x^2 - 3ay$, $(\partial f/\partial y) = 3y^2 - 3ax$. ...(2)

Now at a double point $f = 0$, $(\partial f/\partial x) = 0$ and $(\partial f/\partial y) = 0$.

$\therefore$ from (2), by equating each of $\partial f/\partial x$ and $\partial f/\partial y$ to zero, we get

$$x^2 = ay \text{ and } y^2 = ax.$$

Solving, we get $x(x^3 - a^3) = 0$ or $x = 0, a$.

When $x = 0, y = 0$ and when $x = a, y = a$.

Out of these points only $(0, 0)$ satisfies the euqation $f(x, y) = 0$ of the given curve. Therefore $(0, 0)$ is the only double point.

Now equating the lowest degree terms in (1) to zero, we get $xy = 0$ or $x = 0$, $y = 0$ as the tangents at the origin. These being real and distinct implies that origin is a node.

Ex. 11. *Find the position and nature of the double points of the curve* $a^4 y^2 = x^4 (2x^2 - 3a^2)$. **(Gorakhpur 1980)**

Sol. Here $f(x, y) \equiv 2x^6 - 3a^2 x^4 - a^4 y^2 = 0$. ...(1)

$\therefore$ $(\partial f/\partial x) = 12x^5 - 12a^2 x^3$, $(\partial f/\partial y) = -2a^4 y$. ...(2)

Now at a double point $f = 0$, $(\partial f/\partial x) = 0$ and $(\partial f/\partial y) = 0$.

Here $(\partial f/\partial x) = 0$ gives $12x^3(x^2 - a^2) = 0$ or $x = 0, a, -a$

and $(\partial f/\partial y) = 0$ gives $-2a^4 y = 0$ or $y = 0$.

Thus $(0, 0)$, $(a, 0)$, $(-a, 0)$ are the *possible double points*.

But out of these only $(0, 0)$ satisfies the given equation (1). Hence $(0, 0)$ is the only double point.

Now equating to zero the lowest degree terms in the equation of the curve we get the tangents at the origin as $a^2 y^2 = 0$ *i.e.*, $y = 0$, $y = 0$ are two coincident tangents at the origin.

Thus the origin may be a cusp or a conjugate point.

Now from (1), $y = \pm \dfrac{x^2}{a^2} \sqrt{(2x^2 - 3a^2)}$.

For small values of $x \neq 0$, +ive or –ive, $(2x^2 - 3a^2)$ is – ive *i.e.*, y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin.

Hence origin is a *conjugate point and not a cusp*.

Ex. 12. *Find the position and nature of the double points on the curve*

$$x^3 + x^2 + y^2 - x - 4y + 3 = 0. \qquad \textbf{(G.N.U. 1971)}$$

Sol. Here $f(x, y) \equiv x^3 + x^2 + y^2 - x - 4y + 3 = 0$. ...(1)

$\therefore \quad (\partial f/\partial x) = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$,

and $\quad (\partial f/\partial y) = 2y - 4 = 2(y - 2)$.

But for double points $(\partial f/\partial x) = 0$, $(\partial f/\partial y) = 0$ and $f = 0$.

Here $(\partial f/\partial x) = 0$ gives $(3x - 1)(x + 1) = 0$ or $x = \frac{1}{3}, -1$

and $\quad (\partial f/\partial y) = 0$ gives $2(y - 2) = 0$ *i.e.*, $y = 2$.

Thus $(\frac{1}{3}, 2)$ and $(-1, 2)$ are the possible double points.

Out of these only $(-1, 2)$ satisfies the given equation (1). Hence $(-1, 2)$ is the only double point on the curve.

Shifting the origin to $(-1, 2)$ by putting $x = X - 1$ and $y = Y + 2$ in (1), the equation of the curve becomes

$$(X - 1)^3 + (X - 1)^2 + (Y + 2)^2 - (X - 1) - 4(Y + 2) + 3 = 0$$

or $\quad X^3 - 2X^2 + Y^2 = 0$. ...(2)

Now equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $-2X^2 + Y^2 = 0$ or $Y = \pm \sqrt{2}X$. The two tangents being real and distinct, the new origin is a node. Hence the point $(-1, 2)$ is a node.

***Ex. 13 (a).** *Determine the position and character of the double points on the curve* $y^3 = (x - a)^2 (2x - a)$. **(Gorakhpur 1981; Meerut 89 S)**

Sol. Here $f(x, y) \equiv y^3 - (x - a)^2 (2x - a) = 0$. ...(1)

$$\therefore \quad (\partial f/\partial x) = -2(x - a)^2 - 2(x - a)(2x - a)$$
$$= -2(x - a)(3x - 2a) = -6x^2 + 10ax - 4a^2,$$

and $\quad (\partial f/\partial y) = 3y^2$.

Now for double points $(\partial f/\partial x) = 0$, $(\partial f/\partial y) = 0$ and $f = 0$.

Here $(\partial f/\partial x) = 0$ gives $(x - a)(3x - 2a) = 0$ or $x = a, 2a/3$ and $(\partial f/\partial y) = 0$ gives $y = 0$.

Thus $(a, 0)$ and $(2a/3, 0)$ are the possible double points.

Out of these only $(a, 0)$ satisfies the given curve. Hence $(a, 0)$ is the only double point.

Shifting the origin to $(a, 0)$ by putting $x = X + a$ and $y = Y + 0$ in (1), the equation of the curve becomes

$$Y^3 = (X + a - a)^2 (2X + 2a - a) = X^2 (2X + a). \qquad ...(2)$$

Now equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $aX^2 = 0$ *i.e.*, $X = 0, X = 0$ are two real coincident tangents at the new origin. Therefore the new origin may be a cusp or a conjugate point.

Now for the curve (2), the tangent at the origin is $X = 0$. Let $X = p$. Putting $X = p$ in (2) and neglecting powers of p greater than 2, we get

$$ap^2 = Y^3. \qquad ...(3)$$

From (3), we see that for sufficiently small positive values of Y, p is real and the two values of p are of opposite signs. Also for numerically small negative values of Y, p is imaginary. Hence for the curve (2) there is a single cusp of the first species at the origin.

Thus for the given curve the point $(a, 0)$ is a single cusp of the first kind.

Ex. 13 (b). *Find the double points on the curve*

$$y^3 = x^3 + ax^2.$$ **(Rohilkhand 1983; Gorakhpur 78)**

Sol. Proceed as in Ex. 13 (a). Here for the given curve $(0, 0)$ is the only double point and is a single cusp of the first kind.

***Ex. 14.** *Determine the existence and nature of the double points on the curve* $(x - 2)^2 = y (y - 1)^2$.

(Meerut 1983, 91P; Lucknow 77; Rohilkhand 76; Kanpur 71)

Sol. Here $f(x, y) \equiv (x - 2)^2 - y (y - 1)^2 = 0. \qquad ...(1)$

$\therefore \quad (\partial f/\partial x) = 2 (x - 2)$ and $(\partial f/\partial y) = -(y - 1)^2 - 2y (y - 1)$.

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x, y) = 0$.

Here $(\partial f/\partial x) = 0$ gives $x = 2$ and $(\partial f/\partial y) = 0$ gives $y = 1, \frac{1}{3}$.

Thus $(2, 1)$ and $(2, \frac{1}{3})$ are the possible double points.

Out of these only $(2, 1)$ satisfies the equation of the curve. Hence $(2, 1)$ is the only double point.

Shifting the origin to $(2, 1)$ (by putting $x = X + 2, y = Y + 1$), the equation of the curve changes to $X^2 = (Y + 1) Y^2. \qquad ...(2)$

Now equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $Y^2 = X^2$ or $Y = \pm X$. The two tangents being real and distinct, the new origin is a node. Hence the double point $(2, 1)$ is a node.

Ex. 15. *Find the position and nature of double points of the curve* $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$ **(Agra 1982)**

Sol. Here $f(x, y) \equiv x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0. \qquad ...(1)$

$\therefore \quad (\partial f/\partial x) = 4x^3 - 4a^2x$ and $(\partial f/\partial y) = -6ay^2 - 6a^2y$.

For the double points, $(\partial f/\partial x) = 0$, $(\partial f/\partial y) = 0$ and $f = 0$.

Here $(\partial f/\partial x) = 0$ gives $4x(x^2 - a^2) = 0$ *i.e.*, $x = 0, a, -a$

and $\quad (\partial f/\partial y) = 0$ gives $-6ay(y + a) = 0$ *i.e.*, $y = 0, -a$.

Thus $(0,0), (0,-a), (a,0), (a,-a), (-a,0), (-a,-a)$ are the possible double points.

Out of these only $(0,-a), (a,0), (-a,0)$ satisfy the equation of the curve, hence these are the only double points.

Nature of $(0, -a)$. Shifting origin to $(0,-a)$ (by putting $x = X + 0$ and $y = Y - a$) the equation (1) changes to

$$X^4 - 2a(Y-a)^3 - 3a^2(Y-a)^2 - 2a^2X^2 + a^4 = 0$$

or $$X^4 - 2a(Y^3 - 3aY^2 + 3a^2Y - a^3) - 3a^2(Y^2 - 2aY + a^2) - 2a^2X^2 + a^4 = 0$$

or $$X^4 - 2aY^3 + 3a^2Y^2 - 2a^2X^2 = 0. \quad \text{...(2)}$$

Equating to zero, the lowest degree terms in (2), the tangents at the new origin $(0,-a)$ are given by $3a^2Y^2 - 2a^2X^2 = 0$

or $\quad 3Y^2 = 2X^2$ or $\sqrt{3}Y = \pm\sqrt{2}X$.

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node. Hence there is a node at the point $(0,-a)$ on the given curve.

Nature of the double point $(a, 0)$: Shifting origin to $(a,0)$ [by putting $x = X + a$ and $y = Y + 0$] equation (1) changes to

$$(X+a)^4 - 2aY^3 - 3a^2Y^2 - 2a^2(X+a)^2 + a^4 = 0$$

or $$X^4 + 4aX^3 - 2aY^3 + 4a^2X^2 - 3a^2Y^2 = 0. \quad \text{...(3)}$$

Now equating to zero, the lowest degree terms in (3), the tangents at the new origin $(a,0)$ are given by

$$4a^2X^2 - 3a^2Y^2 = 0 \quad \text{or} \quad Y = \pm(2/\sqrt{3})X.$$

These tangents being real and distinct, the new origin is a node.

Hence the point $(a,0)$ is a node.

Nature of the double point $(-a, 0)$: The given curve (1) is symmetrical about y-axis becuase in the equation of the curve the powers of x are all even. Since the double point $(a,0)$ is a node, therefore by symmetry the double point $(-a,0)$ is also a node.

Thus $(0,-a), (a,0)$ and $(-a,0)$ are the three nodes of the given curve.

Ex. 16. *Examine the nature of the origin on the curve*

$$(2x + y)^2 - 6xy(2x + y) - 7x^3 = 0.$$

Sol. The tangents at the origin are $(2x + y)^2 = 0$. Thus there are two coincident tangents at the origin. Therefore the origin may be a cusp or a conjugate point.

Let $\quad p = 2x + y.$

Putting $y = p - 2x$ in the equation of the curve, we get

$$p^2 - 6xp\,(p - 2x) - 7x^3 = 0$$

or $\quad p^2\,(1 - 6x) + 12x^2 p - 7x^3 = 0.$ ...(1)

Let p_1, p_2 be the roots of (1). Then

$$p = \frac{-\,12x^2 \pm \sqrt{\{144x^4 + 28x^3\,(1 - 6x)\}}}{2\,(1 - 6x)}$$

i.e., $\quad p = \dfrac{-\,6x^2 \pm \sqrt{(7x^3 - 6x^4)}}{(1 - 6x)},$ ...(2)

and $\quad p_1 p_2 = -\,\dfrac{7x^3}{1 - 6x}.$ ...(3)

From (2), we see that for small positive values of x, p is real and for numerically small negative values of x, p is imaginary. Therefore, there is a single cusp at the origin.

Also when x is +ive and very small, then from (3) we notice that $p_1 p_2$ is –ive. Therefore p_1 and p_2 are of opposite signs.

Hence there is a single cusp of the first species at the origin.

***Ex. 17.** *Examine the curve $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ for singular points and show that it has a cusp of first kind.*

(Delhi 1983; K.U. 76)

Sol. Here $f(x, y) \equiv x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0.$ ...(1)

$\therefore \quad (\partial f/\partial x) = 3x^2 + 4x + 2y + 5;\ (\partial f/\partial y) = 2x - 2y - 2.$

For double points we must have $\dfrac{\partial f}{\partial x} = 0,\ \dfrac{\partial f}{\partial y} = 0$ and $f = 0.$

Here $\partial f/\partial x = 0$ gives $3x^2 + 4x + 2y + 5 = 0$

and $\partial f/\partial y = 0$ gives $2x - 2y - 2 = 0$ or $2y = 2x - 2.$

Solving these equations, we get $x = -1, -1$ and $y = -2$. Thus $(-1, -2)$ is the only possible double point.

Since $(-1, -2)$ satisfies the equation of the given curve, therefore it is a double point.

Shifting the origin to $(-1, -2)$ (by putting $x = X - 1$ and $y = Y - 2$), the equation of the curve changes to

$$(X - 1)^3 + 2\,(X - 1)^2 + 2\,(X - 1)\,(Y - 2) - (Y - 2)^2 + 5\,(X - 1) - 2\,(Y - 2) = 0$$

or $\quad X^3 - X^2 + 2XY - Y^2 = 0 \quad$ or $\quad (Y - X)^2 = X^3$ ...(2)

Equating to zero the lowest degree terms in (2), the tangents at the new origin are $(Y - X)^2 = 0$ *i.e.*, $Y - X = 0, Y - X = 0$ are two coincident tangents at the new origin. Therefore the new origin may be a cusp or a conjugate point.

Now the tangent to the curve (2) at the origin is $Y - X = 0.$

Let $Y - X = p$.

Putting $Y - X = p$ in (2), we get

$$p^2 = X^3. \quad ...(3)$$

From (3), we see that for sufficiently small positive values of X, p is real and the two values of p are of opposite signs. Also for negative values of X, p is imaginary. Hence for the curve (2) there is a single cusp of the first kind at the origin.

Hence the point $(-1, -2)$ on the given curve is a single cusp of first species.

***Ex. 18.** *Examine the nature of double points on the curve*

$$(x+y)^3 = \sqrt{2}\,(y - x + 2)^2.$$

(K.U. 1977; G.N.U. 73; Meerut 1990)

Sol. Here $f(x, y) \equiv (x+y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0. \quad ...(1)$

$\therefore \quad (\partial f/\partial x) = 3\,(x+y)^2 + 2\sqrt{2}\,(y - x + 2),$

and $\quad (\partial f/\partial y) = 3\,(x+y)^2 - 2\sqrt{2}\,(y - x + 2).$

For double points we must have $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$ and $f = 0$.

Now $\partial f/\partial x = 0$ gives $3\,(x+y)^2 + 2\sqrt{2}\,(y - x + 2) = 0 \quad ...(2)$

and $\partial f/\partial y = 0$ gives $3\,(x+y)^2 - 2\sqrt{2}\,(y - x + 2) = 0. \quad ...(3)$

Adding (2) and (3), we get $x + y = 0$.

Subtracting (3) from (2), we get $y - x + 2 = 0$.

Solving these equations, we get $x = 1, y = -1$.

Also $(1, -1)$ satisfies the equation of the given curve.

Hence $(1, -1)$ is a double point.

Shifting the origin to $(1, -1)$, the given equation of the curve reduces to $(X + 1 + Y - 1)^3 = \sqrt{2}\,(Y - 1 - X - 1 + 2)^2$

or $\quad (X + Y)^3 = \sqrt{2}\,(Y - X)^2. \quad ...(4)$

Equating to zero, the lowest degree terms in (4), the tangents at the new origin $(1, -1)$ are given by $(Y - X)^2 = 0$, *i.e.*, $Y - X = 0$, $Y - X = 0$ are two real and coincident tangents at the new origin.

$\therefore$ the new origin $(1, -1)$ is either a cusp or a conjugate point. In equation (4), putting $Y - X = p$ *i.e.*, $Y = p + X$, we get

$$(2X + p)^3 = \sqrt{2}p^2$$

or $\quad p^3 + p^2\,(6X - \sqrt{2}) + 12X^2 p + 8X^3 = 0$

or $\quad (6X - \sqrt{2})\,p^2 + 12X^2 p + 8X^3 = 0, \quad$ (neglecting p^3 as $p \to 0$).

$$\therefore \quad p = \frac{-12X^2 + \sqrt{\{144X^4 - 32X^3\,(6X - \sqrt{2})\}}}{2\,(6X - \sqrt{2})}$$

$$= \frac{-6X^2 \pm \sqrt{(8\sqrt{2}X^3 - 12X^4)}}{(6X - \sqrt{2})} = \frac{-6X^2 \pm \sqrt{(8\sqrt{2}X^3)}}{(6X - \sqrt{2})},$$

neglecting $-12X^4$.

When X is +ive and small p is real and the two values of p are of opposite signs.

When X is –ive and small p is imaginary.

Thus there is a single cusp of the first kind at the new origin.

Hence on the given curve the double point $(1, -1)$ is a single cusp of the first species.

Ex. 19. *Find the position and nature of double points on the curve*

$$x^4 - 4y^3 - 12y^2 - 8x^2 + 16 = 0.$$

(K.U. 1976 S; G.N.U. 74 S; Delhi 77, 74)

Sol. For complete solution of this problem see Ex. 15. If we put $a = 2$ there, we get the curve given here.

***Ex. 20.** *Find the position and nature of double points on the curve*

$$x^2y^2 = (a + y)^2 (b^2 - y^2).$$

Distinguish between the cases $b >, = or < a$. **(Meerut 84 S, 89 P; 90 S)**

Sol. Here $f(x, y) \equiv x^2y^2 - (a + y)^2 (b^2 - y^2) = 0.$ ...(1)

$\therefore$ $(\partial f/\partial x) = 2xy^2,$

$(\partial f/\partial y) = 2x^2y - 2(a + y)(b^2 - y^2) + 2y(a + y)^2.$

For the double points, $(\partial f/\partial x) = 0, (\partial f/\partial y) = 0$ and $f = 0$.

Now $(\partial f/\partial x) = 0$ gives $2xy^2 = 0$ or $x = 0, y = 0$

and $(\partial f/\partial y) = 0$ gives $2x^2y - 2(a + y)(b^2 - y^2) + 2y(a + y)^2 = 0$

or $(a + y)[y(a + y) - (b^2 - y^2)] = 0$, when $x = 0$

or $(y + a)(2y^2 + ay - b^2) = 0.$

$\therefore$ when $x = 0, y = -a$ or $y = [-a \pm \sqrt{(a^2 + 8b^2)}]/4.$

For $y = 0$, $\partial f/\partial y = 0$ gives no value of x.

$\therefore$ the possible double points are

$(0, -a), (0, \{-a \pm \sqrt{(a^2 + 8b^2)}\}/4).$

Out of these only $(0, -a)$ satisfies the given equation of the curve.

$\therefore$ $(0, -a)$ is the only double point on the given curve.

Nature of (0, – a) : Shifting the origin to $(0, -a)$ by putting $x = X + 0$ and $y = Y - a$ in (1), the given equation becomes

$$X^2 (Y - a)^2 = \{a + Y - a\}^2 \{b^2 - (Y - a)^2\}$$

or $$(Y^2 - 2aY + a^2) X^2 = Y^2 (b^2 - a^2 + 2aY - Y^2) \qquad ...(3)$$

Equating to zero, the lowest degree terms in (3), the tangents at the new origin are given by

$$(a^2 - b^2) Y^2 + a^2X^2 = 0 \text{ or } \sqrt{(b^2 - a^2)}\, Y = \pm aX. \qquad ...(4)$$

When b > a, the two tangents given by (4) are real and distinct.

$\therefore$ the new origin is a node. Hence when $b > a$, the point $(0, -a)$ is a node on the given curve.

When b = a, the two tangents given by (4) become $X = 0, X = 0$ which are real and coincident.

$\therefore$ the new origin is a cusp or a conjugate point.

Putting $b = a$, the equation (3) becomes

$$(Y^2 - 2aY + a^2) X^2 = Y^2 (2aY - Y^2). \qquad \text{...(5)}$$

Now for the curve (5) the tangent at the origin is $X = 0$. Let $X = p$. Putting $X = p$ in (2), we get

$$p^2 = \frac{2aY^3 - Y^4}{Y^2 - 2aY + a^2} = \frac{2aY^3}{a^2},$$ neglecting $-Y^4$ in the numerator and $Y^2 - 2aY$ in the denominator.

Now for small +ive values of Y, p is real and the two values of p are of opposite signs. Also for numerically small negative values of Y, p is imaginary. Thus, in the neighbourhood of origin the curve lies only on one side of the line $Y = 0$ and the two branches of the curve lie on opposite sides of the common tangent $X = 0$. Hence for the curve (5) there is a single cusp of the first kind at the origin. Hence when $b = a$, the point $(0, -a)$ is a single cusp of the first kind on the given curve.

When $b < a$, the two tangents given by (4) become imaginary. Hence the new origin *i.e.*, the point $(0, -a)$ on the given curve is a conjugate point.

Ex. 21. *Prove that the curve $y^2 = (x - a)^2 (x - b)$ has at $x = a$, a conjugate point if $a < b$, a node if $a > b$ and a cusp if $a = b$.*

(Gorakhpur 1979; Delhi 75; Meerut 86 S)

Sol. Here $f(x, y) \equiv (x - a)^2 (x - b) - y^2 = 0$. ...(1)

At $x = a, y = 0$; $\therefore$ the point is $(a, 0)$.

Shifting the origin to $(a, 0)$ by putting $x = X + a$ and $y = Y + 0$ in (1), the given equation of the curve changes to

$$Y^2 = X^2 (X + a - b). \qquad \text{...(2)}$$

Equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by

$$Y^2 = (a - b) X^2 \quad \text{or} \quad Y = \pm \sqrt{(a - b)}\, X. \qquad \text{...(3)}$$

When b > a, the two tangents given by (3) become imaginary. Hence the new origin *i.e.*, the point $(a, 0)$ on the given curve is a conjugate point.

When b = a, the two tangents given by (3) are $Y = 0, Y = 0$ *i.e.*, they are real and coincident. Therefore the new origin is a cusp or a conjugate point.

In this case from (2), we get

$$Y^2 = X^3 \quad \text{or} \quad Y = \pm X\sqrt{X}, \qquad (\because a = b).$$

When X is small and +ive, the two values of Y are real.

$\therefore$ the curve has real branches at the new origin.

$\therefore$ the point $(a, 0)$ on the given curve is a cusp.

When b < a, the two tangents given by (3) are real and distinct. Hence the new origin *i.e.*, the point $(a, 0)$ on the given curve is a node.

Ex. 22. *Determine the positions and character of the double points on* $y(y-6) = x^2(x-2)^3 - 9$. **(Gorakhpur 1981, Meerut 90 P)**

Sol. Here $f(x, y) \equiv x^2(x-2)^3 - y(y-6) - 9 = 0$. ...(1)

$\therefore \quad (\partial f/\partial x) = 2x(x-2)^3 + 3(x-2)^2 . x^2$

$= x(x-2)^2 [2(x-2) + 3x] = x(x-2)^2(5x-4)$

and $\quad (\partial f/\partial y) = -2y + 6$.

For double points, $(\partial f/\partial x) = 0$, $(\partial f/\partial y) = 0$ and $f = 0$.

Here $(\partial f/\partial x) = 0$ gives $x = 0, 2, 4/5$

and $\quad (\partial f/\partial y) = 0$ gives $y = 3$.

$\therefore \quad (0, 3), (2, 3)$ and $(4/5, 3)$ are the possible double points.

Out of these only $(0, 3)$ and $(2, 3)$ satisfy (1) *i.e.*, these are the only two double points on the curve.

Nature of the double point (0, 3) : Shifting the origin to $(0, 3)$ by putting $x = X + 0$ and $y = Y + 3$, the equation (1) changes to

$$(Y+3)(Y+3-6) = X^2(X-2)^3 - 9$$

or $\quad Y^2 - 9 = X^2(X-2)^3 - 9$ or $Y^2 = X^2(X-2)^3$. ...(2)

Equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $Y^2 = -8X^2$, which are imaginary.

$\therefore$ the new origin *i.e.*, the point $(0, 3)$ on the given curve is a conjugate point.

Nature of the double point (2, 3) : Shifting the origin to $(2, 3)$ by putting $x = X + 2$ and $y = Y + 3$, the equation (1) changes to

$$(Y+3)(Y+3-6) = (X+2)^2 X^3 - 9$$

or $\quad Y^2 = X^3(X+2)^2$. ...(3)

Equating to zero the lowest degree terms in (3), the tangents at the new origin are given by $Y^2 = 0$ or $Y = 0, Y = 0$ whih are real and coincident.

$\therefore$ the new origin *i.e.*, the point $(2, 3)$ on the given curve is either a cusp or a conjugate point.

Now from (3), $\quad Y = \pm X(X+2)\sqrt{X}$.

When X is small and +ive, Y is real and the values of Y are of opposite signs. When X is small and –ive, Y is imaginary. Thus the curve has real branches at the new origin only on one side of the line $X = 0$ and the two branches of the curve lie on opposite sides of their common tangent $Y = 0$. Hence the new origin *i.e.*, the point $(2, 3)$ on the curve (1) is a single cusp of the first kind.

Ex. 23. *Determine the position and character of the double points on the curve*

$$y^2 - x(x-a)^2 = 0, (a > 0).$$ Meerut [illegible]

Sol. The equation of the given curve is

$$f(x, y) \equiv y^2 - x(x-a)^2 = 0. \qquad ...(1)$$

We have $\partial f/\partial x = -(x-a)^2 - 2x(x-a) = -(x-a)(3x-a)$
and $\quad \partial f/\partial y = 2y$.

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x, y) = 0$.

Here $\quad \partial f/\partial x = 0$ gives $\quad (x-a)(3x-a) = 0$ *i.e.* $x = a, a/3$
and $\quad \partial f/\partial y = 0$ gives $\quad y = 0$.

$\therefore$ the possible double points are $(a, 0)$, $(a/3, 0)$.

Out of these only $(a, 0)$ satisfies the equation of the curve. Therefore $(a, 0)$ is the only double point on the given curve.

Nature of the double point at $(a, 0)$. Shifting the origin to the point $(a, 0)$, the equation of the curve becomes

$$y^2 - (x+a)x^2 = 0. \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are

$$y^2 - ax^2 = 0 \quad \textit{i.e.,} \quad y^2 = ax^2 \quad \textit{i.e.,} \quad y = \pm\sqrt{a}x.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(a, 0)$ on the given curve.

Ex. 24. *Determine the position and character of the double points on the curve*

$$y^2 - x^3 = 0. \qquad \textbf{(Meerut 1994P)}$$

Sol. The equation of the given curve is

$$f(x, y) \equiv y^2 - x^3 = 0. \qquad ...(1)$$

We have $\partial f/\partial x = -3x^2$ and $\partial f/\partial y = 2y$.

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x, y) = 0$.

$\therefore$ the only double point of (1) is (0, 0).

Equating to zero the lowest degree terms in (1), the tangents to (1) at the origin are

$$y^2 = 0 \qquad \textit{i.e.,} \qquad y = 0 \text{ and } y = 0.$$

$\therefore$ origin is either a cusp or a conjugate point.

Now from (1), $\qquad y = \pm\sqrt{(x^3)}$.

For small values of $x \neq 0$, the values of y are real when x is +ive and are imaginary when x is −ive.

Also when x is +ive, the two values of y are of opposite signs.

Hence there is a single cusp of the first kind at the origin (0, 0) on the right side of the y-axis.

Ex. 25. *Determine the position and character of the double points on the curve*

$$a^4y^2 = x^4 (a^2 - x^2).$$ **(Meerut 1995)**

Sol. The equation of the given curve is

$$f(x,y) \equiv a^4y^2 - x^4 (a^2 - x^2) = 0. \qquad ...(1)$$

We have $\partial f/\partial x = -4x^3 (a^2 - x^2) + 2x^5 = 2x^3 (3x^2 - 2a^2)$

and $\quad \partial f/\partial y = 2a^4y.$

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x,y) = 0$.

Here $\quad \partial f/\partial x = 0$ gives $\quad x = 0, \pm\sqrt{(2/3)}\, a$

and $\quad \partial f/\partial y = 0$ gives $\quad y = 0.$

$\therefore$ the possible double points are $(0, 0)$, $(\sqrt{(2/3)}\, a, 0)$, $(-\sqrt{(2/3)}\, a, 0)$.

Out of these only (0, 0) satisfies the equation of the curve. Therefore (0, 0) is the only double point on the given curve.

Nature of the double point at (0, 0). Equating to zero the lowest degree terms in (1), the tangents to (1) at (0, 0) are given by

$$a^4y^2 = 0 \qquad \text{i.e.,} \qquad y = 0 \text{ and } y = 0.$$

$\therefore$ origin is either a cusp or a conjugate point.

Now from (1), $\quad y = \pm\dfrac{1}{a^2}\sqrt{(a^2x^4 - x^6)} = \pm\dfrac{1}{a^2}\sqrt{(a^2x^4)},$

keeping only the lowest degree terms in x under the radical sign.

For small values of $x \neq 0$, the values of y are real both when x is +ive and x is −ive.

Also the two values of y are of opposite signs both when x is +ive and x is −ive.

Hence there is a double cusp of the first kind at (0, 0).

Ex. 26. *Determine the position and character of the double points on the curve*

$$y^2 = x^2 (9 - x^2).$$ **(Meerut 1995 BP)**

Sol. The equation of the given curve is

$$f(x,y) \equiv y^2 - x^2 (9 - x^2) = 0. \qquad ...(1)$$

We have $\partial f/\partial x = -2x (9 - x^2) + 2x^3 = 2x (2x^2 - 9)$

and $\quad \partial f/\partial y = 2y.$

For double points $\quad \partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x,y) = 0$.

Here $\quad \partial f/\partial x = 0$ gives $\quad x = 0, \pm 3/\sqrt{2}$
and $\quad \partial f/\partial y = 0$ gives $\quad y = 0.$
$\therefore$ the possible double points are $(0, 0)$, $(3/\sqrt{2}, 0)$, $(-3/\sqrt{2}, 0)$.

Out of these only $(0, 0)$ satisfies the equation of the curve. Therefore $(0, 0)$ is the only double point on the given curve.

Equating to zero the lowest degree terms in (1), the tangents to (1) at $(0, 0)$ are

$$y^2 - 9x^2 = 0 \quad i.e., \quad y^2 = 9x^2 \quad i.e., \quad y = \pm 3x.$$

Thus there are two real and distinct tangents at $(0, 0)$. Hence origin is a node on the given curve.

Ex. 27. *Show that the curve*

$$(xy + 1)^2 + (x - 1)^3 (x - 2) = 0$$

has a single cusp of the first species at the point $(1, -1)$.

(Meerut 1993 P)

Sol. The given curve is

$$(xy + 1)^2 + (x - 1)^3 (x - 2) = 0. \qquad ...(1)$$

The point $(1, -1)$ satisfies the equation (1) and so it lies on (1). Shifting the origin to $(1, -1)$, the equation (1) becomes

$$[(x + 1)(y - 1) + 1]^2 + x^3 (x - 1) = 0$$

or $$[xy - (x - y)]^2 + x^3 (x - 1) = 0$$

or $$x^2y^2 - 2xy(x - y) + (x - y)^2 + x^3 (x - 1) = 0 \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangents to (2) at the new origin are

$$(x - y)^2 = 0 \qquad i.e., \qquad x - y = 0, x - y = 0.$$

$\therefore$ the new origin is either a cusp or a conjugate point.

In equation (2), putting $x - y = p$ *i.e.*, $y = x - p$, we get

$$x^2 (x - p)^2 - 2x (x - p) p + p^2 + x^3 (x - 1) = 0$$

or $$x^4 - 2x^3p + x^2p^2 - 2x^2p + 2xp^2 + p^2 + x^3 (x - 1) = 0$$

or $$p^2 (x^2 + 2x + 1) - (2x^3 + 2x^2) p + 2x^4 - x^3 = 0$$

or $$p^2 (x + 1)^2 - 2x^2 (x + 1) p + 2x^4 - x^3 = 0.$$

$\therefore$ $$p = \frac{2x^2 (x + 1) \pm \sqrt{[4x^4 (x + 1)^2 - 4 (x + 1)^2 (2x^4 - x^3)]}}{2 (x + 1)^2}$$

$$= \frac{2x^2 (x + 1) \pm 2(x + 1) \sqrt{(x^4 - 2x^4 + x^3)}}{2(x + 1)^2}$$

$$= \frac{x^2 (x + 1) \pm (x + 1) \sqrt{(x^3)}}{(x + 1)^2},$$

keeping only the lowest degree terms in x under the radical sign.

Also $p_1 p_2 = \dfrac{2x^4 - x^3}{(x+1)^2} = \dfrac{-x^3}{(x+1)^2}$, keeping only the lowest degree terms in x in the numerator.

For small values of $x \neq 0$, the values of p are real when x is + ive and are imaginary when x is − ive.

Also when x is positive, $p_1 p_2$ is −ive *i.e.*, the two values of p are of opposite signs.

Hence the new origin on the curve (2) is a single cusp of the first kind.

Hence the given curve (1) has a single cusp of the first species at the point $(1, -1)$.

□

7

Curve Tracing

§ 1. Curve Tracing. Cartesian Equations.

To find the approximate shape of a curve whose cartesian equation is given, we should adopt the following procedure :

1. **Symmetry.** First we should find if the curve is symmetrical about any line. In this connection the following rules are helpful :

(*i*) If in the equation of a curve the powers of y are all even, the curve is symmetrical about the axis of x *i.e.*, the shape of the curve above and below the axis of x is symmetrical. The obvious reason is that the equation of the curve in this case remains unchanged if we replace y by $-y$. Thus the parabola $y^2 = 4ax$ is symmetrical about the axis of x.

(*ii*) If in the equation of a curve the powers of x are all even, the curve is symmetrical about the axis of y.

Note. The curve $x^2 + y^2 = 4ax$ is symmetrical about x-axis, but the curve $a^2x^2 = y^3(2a - y)$ is not symmetrical about x-axis, for besides containing a term in y^4, the equation also contains a term in y^3 so that even powers as well as odd powers of y occur in the equation.

(*iii*) If the equation of a curve remains unchanged when x is replaced by $-x$ and y is replaced by $-y$, then the curve is symmetrical in opposite quadrants. For example, the curve $xy = c^2$ is symmetrical in opposite quadrants.

(*iv*) If the equation of a curve remains unchanged when x and y are interchanged the curve is symmetrical about the line $y = x$, (*i.e.*, the straight line passing through the orgin and making an angle $45°$ with the positive direction of the axis of x). For example, the curve $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$.

2. **Nature of the Origin on the Curve.** We should see whether the curve passes through the origin or not. If the point $(0, 0)$ satisfies the equation of the curve, it passes through the origin. In order to know the shape of a curve at any point, we should draw the tangent or tangents to the curve at that point. Therefore if the curve passes through the origin, we should find the equation to the tangents at origin by equating to zero the lowest degree terms in the equation of the

curve. If there are two tangents at the origin, then the origin will be a double point on the curve. We should also observe the nature of the double point.

3. Points of intersection of the curve with the co-ordinate axes. We should find the points where the curve cuts the co-ordinate axes. To find the points where the curve cuts the x-axis we should put $y = 0$ in the equation of the curve and solve the resulting equation for x. Similarly the points of intersection with the y-axis are obtained by putting $x = 0$ and solving the resulting equation for y. **We should also obtain the tangents to the curve at the points where it meets the co-ordinate axes.** In order to find the tangent at the point (h, k), we should shift the origin to (h, k) and then the tangent or tangents at this new origin will be obtained by equating to zero the lowest degree terms. The value of dy/dx at the point (h, k) can also be used to find the slope of the tangent at that point.

4. We should solve the equation of the curve for y or x whichever is convenient. Suppose we solve for y. Starting from $x = 0$, we should see the nature of y as x increases and then tends to $+\infty$. Similarly we should see the nature of y as x decreases and then tends to $-\infty$. **We should pay special attention to those values of x for which $y = 0$ or $\rightarrow$ infinity.**

If we solve the equation of the curve for y and the curve is symmetrical about y-axis or there is symmetry in opposite quadrants, then we should consider only positive values of x. The curve for negative values of x can be drawn from symmetry and there is no necessity of considering them afresh.

However, if we solve the equation for y and there is symmetry only about x-axis, then we are to consider both positive as well as negative values of x. If the curve is symmetrical in opposite qudrants, or if there is symmetry about the x-axis, then only positive values of y need be considered.

If $y \rightarrow$ infinity as $x \rightarrow a$, then the line $x = a$ will be an asymptote of the curve. Similarly if $x \rightarrow$ infinity as $y \rightarrow b$, then the line $y = b$ will be an asymptote of the curve.

5. Regions where the curve does not exist. We should find out if there is any region of the plane such that no part of the curve lies in it. Such a region is easily obtained on solving the equation for one variable in terms of the other. The curve will not exist for those values of one variable which make the other imaginary. For example, in the curve

$$a^2y^2 = x^2 (x - a) (2a - x),$$

we find that for $0 < x < a$, y^2 is negative, *i.e.*, y is imaginary. Therefore the curve does not exist in the region bounded by the lines $x = 0$ and $x = a$. For $a < x < 2a$, y^2 is positive *i.e.*, y is real. Therefore the curve exists in the region bounded by the lines $x = a$ and $x = 2a$. Thus if y is imaginary when x lies between a and b, the curve does not exist in the region bounded by the lines $x = a$ and $x = b$.

6. Asymptotes. We should find all the asymptotes of the curve. If an infinte branch of the curve has an asymptote, then ultimately it must be drawn parallel to the asymptote. The asymptotes parallel to the x-axis can be obtained by equating to zero the coefficient of the highest power of x in the equation of the curve. Similarly the asymptotes parallel to the y-axis can be obtained by equating to zero the coefficient of the highest power of y in the equation of the curve.

7. The sign of dy/dx. We should calculate the value of dy/dx from the equation of the curve. Then we shall find the points at which dy/dx **vanishes or becomes infinite**. These will give us the points where the tangent is parallel or perpendicular to the x-axis.

If in any region $a < x < b$, dy/dx remains throughout positive, then in this region y increases continuously as x increases. If in any region $a < x < b$, dy/dx remains throughout negative, then in this region y decreases continuously as x increases.

8. Special Points. If necessary, we should find the co-ordinates of a few points on the curve.

9. Points of inflexion. While drawing the curve if it appears that the curve possesses some points of inflexion, then their positions can be accurately located by putting d^2y/dx^2 or d^2x/dy^2 equal to zero and solving the resulting equation.

Taking all the above isolated facts into consideration, we can draw the approximate shape of the curve.

Solved Examples

Ex. 1. *Trace the curve* $ay^2 = x^3$. **(semicubical parabola).**

Sol. We note the following facts about this curve :

(i) Since in the equation of the curve the powers of y are all even, therefore the curve is *symmetrical about the axis of* x.

(ii) The curve passes through the origin.

(iii) Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin. Therefore the tangents at origin are $ay^2 = 0$ *i.e.*, $y = 0, y = 0$. Thus the origin is a double point and it may be a cusp since there are two coincident tangents at the origin.

(iv) The curve does not intersect the coordinate axes anywhere except the origin.

(v) Solving the equation of the curve for y, we get

$$y^2 = x^3/a.$$

When $x = 0, y^2 = 0$.

When $x > 0, y^2$ is positive *i.e.*, y is real. Therefore the curve exists in the region $x > 0$.

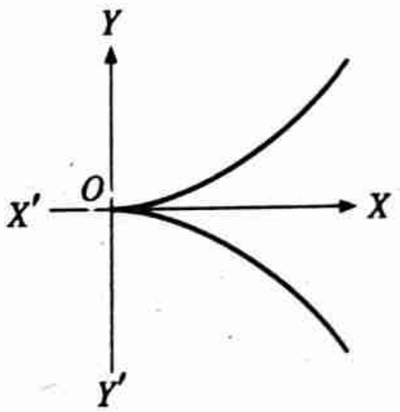
As x increases, y^2 also increases and when $x \to \infty, y^2 \to \infty$.

When $x < 0, y^2$ is negative *i.e.*, y is imaginary.

Therefore the curve does not exist in the region $x < 0$.

(vi) Obviously the curve has no asymptotes.

(vii) The curve exists in the neighbourhood of origin where $x > 0$. Also x-axis is a common tangent to the two branches of the curve passing through origin. Hence origin is a cusp.



Taking all these facts into consideration, the shape of the cruve is as shown in the adjoining figure.

Ex. 2. *Trace the curve* $y = x^3$. **(Cubical parabola)**

Sol. (i). If we change the signs of x and y both, the equation of the curve does not change. Therefore the curve is symmetrical in opposite quadrants.

(ii) The curve passes through the origin and tangent at the origin is $y = 0$ *i.e.*, x-axis.

(iii) When $y = 0, x = 0$; when $x = 0, y = 0$. Thus the origin is the only point of the curve on the coordinate axes. Hence the curve does not intersect the coordinate axes anywhere except at the origin.

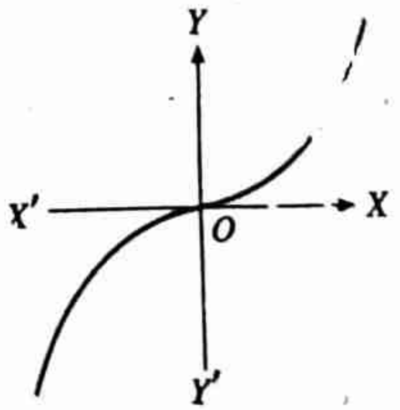
(iv) No asymptotes.

(v) When x is +ive, y is +ive; when x is –ive, y is –ive.

Hence the curve lies in the first and the third quadrants.

(vi) Special points :

x	0	1	2	3	$\to$	∞
y	0	1	8	27	$\to$	∞



Now we first trace the curve in the first qaudrant and then by symmetry in the third quadrant. Shape of the curve is as shown in the adjoining figure.

Ex. 3. *Trace the curve* $x^3y = x + 1$.

Sol. The given curve may be written as

$$y = (x+1)/x^3 = (1/x^2) + (1/x^3).$$

(i) No symmetry.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x = -1$; when $x = 0$, we do not get any value of y. Hence the curve cuts the coordinates axes only at the point $(-1, 0)$.

(iv) Asymptotes parallel to x-axis : $y = 0$ *i.e.*, x-axis. Asymptotes parallel to y-axis : $x^3 = 0$ or $x = 0$ *i.e.*, y-axis.

(v) When x is positive, y is positive ; when $x \to \infty, y \to 0$; when $x \to 0$ from the right, $y \to \infty$; when $x \to 0$ from the left, $y \to -\infty$. At $x = -1, y = 0$.

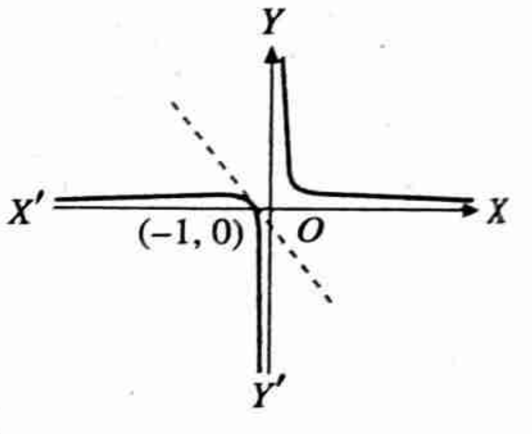
When $x < -1, y$ is positive ; when $x \to -\infty, y \to 0$.

Hence when $x > 0$, the curve is in the first quadrant, when $-1 < x < 0$, the curve is in the third quadrant and when $x < -1$, the curve is in the second quadrant.

(vi) Special points :

x	$-\infty$	-2	-1	$-\frac{1}{2}$	-0
y	0	$\frac{1}{8}$	0	-4	$-\infty$

x	$+0$	1	2	∞
y	∞	2	$\frac{3}{8}$	0



Hence the curve is as shown in the figure.

Ex. 4. *Trace the curve* $y = x(x^2 - 1)$. **(Meerut 1993)**

Sol. **(i).** There is symmetry in opposite quadrants.

[$\because$ by putting $-x$ for x and $-y$ for y, the equation of the curve remains unchanged]

(ii) The curve passes through $(0, 0)$. Tangent at the origin is $y = -x$.

(iii) When $y = 0, x = 0, \pm 1$; when $x = 0, y = 0$.

Hence the curve intersects the coordinate axes at the points $(0, 0), (\pm 1, 0)$.

From the equation of the curve, $dy/dx = 3x^2 - 1$. At $(\pm 1, 0)$, $dy/dx = 2$ *i.e.*, tangents at the points $(\pm 1, 0)$ make an angle lying between 60° and 90° with the positive direction of x-axis.

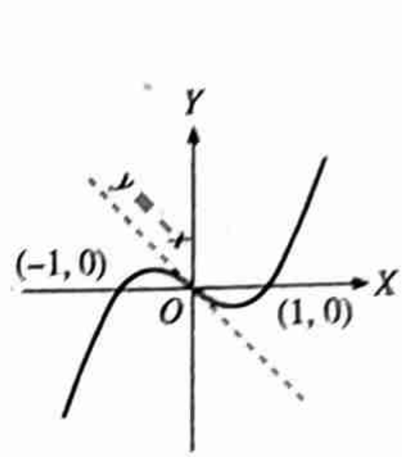
(iv) No asymptotes.

(v) When x lies between 0 and 1, y is negative.

When $x > 1, y$ is positive and when $x \to \infty, y \to \infty$.

(iv) Special points :

x	0	$\frac{1}{2}$	1	2	3	∞
y	0	$-\frac{3}{8}$	0	6	24	∞



Trace the curve first on the right hand side of the x-axis (*i.e.*, in I and IV quadrants) and then by symmetry on the left hand side of the x-axis (*i.e.*, in II and III quadrants). The curve is as shown in the figure.

Ex. 5. *Trace the curve* $x = (y-1)(y-2)(y-3)$.

(Agra 1978; Meerut 85 S)

Sol. **(i).** The curve is not symmetrical about the coordinate axes or about the line $x = y$ or in opposite quadrants.

(ii) The curve does not pass through the origin.

(iii) Taking y as the independent variable, we have

when $y = 0, x = -6$; when $y = 1, x = 0$.

When $0 < y < 1, x$ is negative, as then all the three factors are negative.

When $1 < y < 2, x$ is positive as one factor is +ive and two are –ive. Also x becomes zero at $y = 2$.

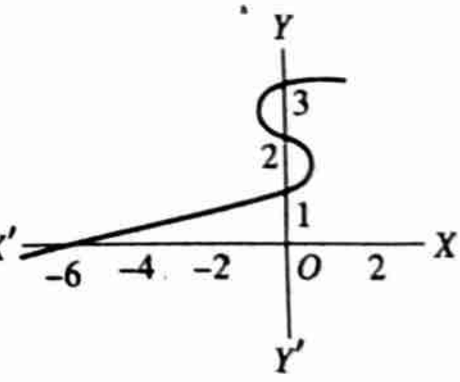
When $2 < y < 3, x$ is negative and x becomes zero at $y = 3$.

When $y > 3, x$ is positive. As $y \to \infty, x \to \infty$.

When $y < 0, x$ is negative and when $y \to -\infty, x \to -\infty$.

(iv) When $y \to \pm\infty, x \to \pm\infty$. For very large values of y, x is approximately equal to y^3.

Hence there are no linear asymptotes.



(v) When $y = \frac{3}{2}, x = \frac{3}{8}$;

when $y = \frac{5}{2}, x = -\frac{3}{8}$.

Hence the shape of the curve is as shown in the figure.

Ex. 6. *Trace the curve* $y(x^2 + 4a^2) = 8a^3$.

Sol. We have $y = 8a^3/(x^2 + 4a^2)$. ...(1)

(i) The curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0$, we get no value of x; when $x = 0, y = 2a$.

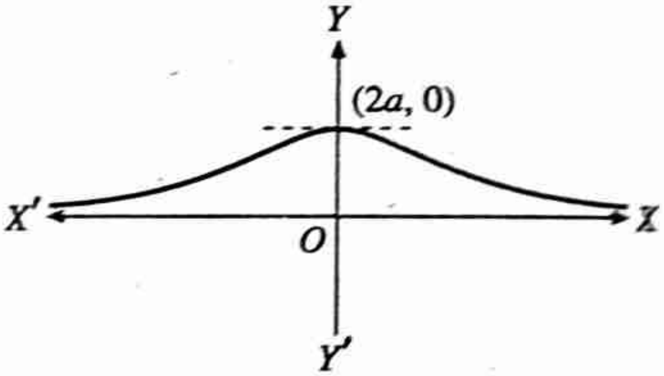
Hence the curve cuts the coordinate axes only at the point $(0, 2a)$.

Shifting the origin to the point $(0, 2a)$ the equation of the curve becomes $(y + 2a)(x^2 + 4a^2) = 8a^3$ or $x^2y + 4a^2y + 2ax^2 = 0$. Equating to zero the lowest degree terms in it, we get the tangent at the new origin as $4a^2y = 0$ *i.e.*, $y = 0$. Thus the new x-axis is tangent at the new origin.

(iv) Asymptote parallel to x-axis is $y = 0$.

Asymptotes parallel to y-axis are given by $x^2 + 4a^2 = 0$ or $x^2 = -4a^2$ *i.e.*, imaginary asymptotes.

(v) For all values of x, y is positive *i.e.*, the curve lies in I and II quadrants. Also y is greatest when x^2 is 0. Thus the greatest value of y is $2a$.



(iv) Special points :

x	0	a	$2a$	$\to \infty$
y	$2a$	$8a/5$	a	$\to 0$.

First trace the curve on the right hand side of the y-axis and then by symmetry on its left hand side. The curve is as shown in the figure.

Ex. 7. *Trace the curve* $y(x^2 - 1) = (x^2 + 1)$.

(Agra 1980, 77; Meerut 93P)

Sol. We have $y = (x^2 + 1)/(x^2 - 1)$ or $y = 1 + 2/(x^2 - 1)$

or $$y = -1 + 2x^2/(x^2 - 1).$$

(i) The curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x$ is imaginary; when $x = 0, y = -1$.

Hence the curve cuts the coordinate axes only at the point $(0, -1)$.

From the equation of the curve, $dy/dx = -4x/(x^2 - 1)^2$.

$\therefore$ at the point $(0, -1)$, $dy/dx = 0$ *i.e.*, the tangent to the curve at this point is parallel to the x-axis.

(iv) The curve is $y(x^2 - 1) - x^2 - 1 = 0$.

$\therefore$ asymptote parallel to x-axis is $y - 1 = 0$ or $y = 1$.

Solved Examples

**Ex. 27. *Find the radius of curvature at the point (r, θ) on each of the following curves :*

(i) $r = a(1 - \cos\theta)$

(Meerut 1980, 83, 84, 86, 92, 94P, 96 BP, 98; Kanpur 87)

(ii) $r = a(1 + \cos\theta)$ **(Rohilkhand 1991; Meerut 94P)**

(iii) $r = 3(1 + \cos\theta)$ **(Agra 1975)**

(iv) $r^2 \cos 2\theta = a^2$ **(Rohilkhand 1987; Jhansi 88; Kanpur 70)**

(v) $r^n = a^n \cos n\theta$ **(Kanpur 1989; Meerut 94, 96P)**

(vi) $r^n = a^n \sin n\theta$ **(Rohilkhand 1983; Meerut 88)**

(vii) $r^2 = a^2 \cos 2\theta$

(viii) $l/r = 1 + e\cos\theta$. **(Meerut 1990S)**

Sol. **(i)** The given curve is $r = a(1 - \cos\theta)$. ...(1)

$\therefore$ $dr/d\theta = a\sin\theta$ and $d^2r/d\theta^2 = a\cos\theta$.

Hence $\rho = \dfrac{[r^2 + (dr/d\theta)^2]^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}$

$$= \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)}$$

$$= \frac{a^2[1 + \cos^2\theta - 2\cos\theta + \sin^2\theta]^{3/2}}{a^2[1 + \cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$$

$$= \frac{a[2(1-\cos\theta)]^{3/2}}{3(1-\cos\theta)} = \frac{2\sqrt{2}}{3}a(1-\cos\theta)^{1/2}$$

$$= \frac{2\sqrt{2}}{3}a\sqrt{\left(\frac{r}{a}\right)}, \text{ from (1)}$$

$$= \tfrac{2}{3}\sqrt{(2ar)}.$$

Note. We could have solved this problem more easily by changing the equation of the curve to pedal form. The method is given in the part (ii) of this example that just follows :

(ii) The given curve is $r = a(1 + \cos\theta)$ ...(1)

Taking log of (1), we get $\log r = \log a + \log(1 + \cos\theta)$.

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}.$$

$\therefore$ $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan\frac{1}{2}\theta = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$;

so that $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$.

Now $p = r\sin\phi = r\sin(\frac{1}{2}\pi + \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta$.

From (1), $r = 2a\cos^2\frac{1}{2}\theta = 2a(p/r)^2$. $[\because p/r = \cos\frac{1}{2}\theta]$

$\therefore$ the pedal equation of the curve (1) is

$$r^3 = 2ap^2. \quad ...(2)$$

Differentiating (2) w.r.t. p, we get $3r^2 \dfrac{dr}{dp} = 4ap$.

$$\therefore \quad \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\left(\frac{r^3}{2a}\right)^{1/2} \qquad \left[\because \text{ from (2)}, p^2 = \frac{r^3}{2a}\right]$$
$$= \tfrac{2}{3}\sqrt{(2ar)}.$$

If we want the value of ρ in terms of θ, then putting $r = a(1 + \cos\theta)$ in the value of ρ, we get

$$\rho = \tfrac{2}{3}\sqrt{\{2a^2(1+\cos\theta)\}} = \tfrac{2}{3}\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)} = \frac{4a}{3}\cos\tfrac{1}{2}\theta.$$

(iii) Proceed as in part (ii). The value of ρ is $4\cos\frac{1}{2}\theta$.

(iv) The given curve is $r^2\cos 2\theta = a^2$. ...(1)

Taking log of (1), we get $2\log r + \log\cos 2\theta = \log a^2$.

Differentiating w.r.t. θ, we get $\dfrac{2}{r}\dfrac{dr}{d\theta} + \dfrac{-2\sin 2\theta}{\cos 2\theta} = 0$.

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta = \cot(\tfrac{1}{2}\pi - 2\theta); \text{ so that } \phi = \tfrac{1}{2}\pi - 2\theta.$$

Now $p = r\sin\phi = r\sin(\frac{1}{2}\pi - 2\theta) = r\cos 2\theta = r(a^2/r^2)$, from (1).

$\therefore$ the pedal equation of the curve (1) is $p = a^2/r$.

$$\therefore \qquad dp/dr = -a^2/r^2.$$

Hence $\quad \rho = r\dfrac{dr}{dp} = r\cdot\left(\dfrac{-r^2}{a^2}\right) = -\dfrac{r^3}{a^2}$.

Neglecting the negative sign, we have $\rho = r^3/a^2$.

(v) The given curve is $r^n = a^n\cos n\theta$. ...(1)

Taking logarithm and differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n\sin n\theta}{\cos n\theta} = -n\tan n\theta\,; \quad \therefore \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta.$$

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta = \cot(\tfrac{1}{2}\pi + n\theta);$$

so that $\qquad \phi = \frac{1}{2}\pi + n\theta$.

Now $p = r\sin\phi = r\sin(\frac{1}{2}\pi + n\theta) = r\cos n\theta = r(r^n/a^n)$, from (1).

$\therefore$ the pedal equation of (1) is $p = r^{n+1}/a^n$.

$$\therefore \quad \frac{dp}{dr} = \frac{1}{a^n}(n+1)r^n.$$

Hence $\rho = r\dfrac{dr}{dp} = r\cdot\dfrac{a^n}{(n+1)r^n} = \dfrac{a^n}{(n+1)r^{n-1}}$.

(vi) Proceed exactly as in part (v). **Ans.** $\rho = a^n/[(n+1)r^{n-1}]$.

(vii) The given curve is $r^2 = a^2\cos 2\theta$. ...(1)

Taking logarithm, $2\log r = \log\cos 2\theta + 2\log a$.

Differentiating w.r.t. θ, we get $\dfrac{2}{r}\dfrac{dr}{d\theta} = -\dfrac{2\sin 2\theta}{\cos 2\theta}$.

$$\therefore \quad \cot \phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan 2\theta = \cot\left(\tfrac{1}{2}\pi + 2\theta\right);$$

so that $$\phi = \tfrac{1}{2}\pi + 2\theta.$$

Now $p = r \sin \phi = r \sin\left(\tfrac{1}{2}\pi + 2\theta\right) = r \cos 2\theta = r\,(r^2/a^2)$, from (1).

Hence the pedal equation of the curve (1) is $a^2 p = r^3$. ...(2)

Differentiating (2) w.r.t. r, we get

$$a^2\,(dp/dr) = 3r^2 \quad \text{or} \quad (dr/dp) = a^2/3r^2.$$

Hence $\rho = r\,(dr/dp) = r\,.\,(a^2/3r^2) = a^2/3r.$

(viii) The given curve is $l/r = 1 + e\cos\theta$.

Let $1/r = u$. Then the equation of the curve is

$$u = (1/l)\,(1 + e\cos\theta).$$

Differentiating, we get

$$\frac{du}{d\theta} = \frac{1}{l}(-e\sin\theta) \quad \text{and} \quad \frac{d^2u}{d\theta^2} = \frac{1}{l}(-e\cos\theta).$$

$$\therefore \quad \rho = \frac{[u^2 + (du/d\theta)^2]^{3/2}}{u^3\,(u + d^2u/d\theta^2)},$$

[See the formula for ρ on page 224]

$$= \frac{(1/l^3)\,[(1 + e\cos\theta)^2 + e^2\sin^2\theta]^{3/2}}{(1/l)^4\,(1 + e\cos\theta)^3\,[(1 + e\cos\theta) + (-e\cos\theta)]}$$

$$= \frac{l\,[1 + e^2 + 2e\cos\theta]^{3/2}}{(1 + e\cos\theta)^3}.$$

Ex. 27 (a). *Prove that for the cardioid $r = a\,(1 + \cos\theta)$, ρ^2/r is constant.* **(Meerut 1988 P)**

Sol. Proceeding as in Ex. 27 part (ii), we get

$$\rho = \tfrac{2}{3}\sqrt{(2ar)}.$$

$\therefore \quad \rho^2/r = 8a/9$ which is constant.

Ex. 28. *Show that the curvatures of the curves $r = a\theta$ and $r\theta = a$ at their common point are in the ratio* 3 : 1. **(Bhopal 1973; Meerut 87)**

Sol. The given curves are

$$r = a\theta \quad \text{...(1)} \qquad \text{and} \qquad r\theta = a \quad \text{...(2)}$$

Eliminating r between (1) and (2), we get $a\theta^2 = a$ or $\theta^2 = 1$.

Thus at the common point of (1) and (2), we have $\theta^2 = 1$.

For the first curve, $dr/d\theta = a$ and $d^2r/d\theta^2 = 0$.

$$\therefore \quad \rho = \frac{[r^2 + (dr/d\theta)^2]^{3/2}}{r^2 + 2\,(dr/d\theta)^2 - r\,(d^2r/d\theta^2)}$$

$$= \frac{[a^2\theta^2 + a^2]^{3/2}}{a^2\theta^2 + 2a^2} = \frac{a\,(\theta^2 + 1)^{3/2}}{\theta^2 + 2}.$$

$$\therefore \quad \rho\,(\text{at } \theta^2 = 1) = \frac{a\,.\,(1 + 1)^{3/2}}{1 + 2} = \frac{2a\sqrt{2}}{3}.$$

But curvature $= 1/\rho$. Therefore the curvature of the first curve at $\theta^2 = 1$ is $3/(2a\sqrt{2})$.

For the second curve, $r = a/\theta$; $dr/d\theta = -a/\theta^2$; $d^2r/d\theta^2 = 2a/\theta^3$.

$$\therefore \quad \rho = \frac{[(a^2/\theta^2) + (a^2/\theta^4)]^{3/2}}{(a^2/\theta^2) + (2a^2/\theta^4) - (2a^2/\theta^4)} = \frac{a\,(\theta^2+1)^{3/2}}{\theta^4}.$$

$$\therefore \quad \rho\,(\text{at } \theta^2 = 1) = \frac{a\,(1+1)^{3/2}}{1} = 2a\sqrt{2}.$$

$\therefore$ curvature of the second curve at $\theta^2 = 1$ is $1/(2a\sqrt{2})$.

Hence at their common point

$$\frac{\text{curvature of first curve}}{\text{curvature of second curve}} = \frac{3/(2a\sqrt{2})}{1/(2a\sqrt{2})} = \frac{3}{1}.$$

Ex. 29 (a). *Find the radius of curvature of the curve*

$$r = a \sin n\theta.$$

(b) *Find the radius of curvature at (r, θ) on the curve*

$$r = 6\,(1 - \sin^2 \tfrac{1}{2}\theta).$$

Sol. (a). Here $r = a \sin n\theta$.

Therefore $dr/d\theta = an \cos n\theta$; $d^2r/d\theta^2 = -an^2 \sin n\theta = -n^2 r$.

$$\therefore \quad \rho = [r^2 + (dr/d\theta)^2]^{3/2}/[r^2 + 2\,(dr/d\theta)^2 - r\,(d^2r/d\theta^2)]$$
$$= [r^2 + a^2n^2 \cos^2 n\theta]^{3/2}/[r^2 + 2a^2n^2 \cos^2 n\theta + n^2r^2].$$

Now $a^2 \cos^2 n\theta = a^2 (1 - \sin^2 n\theta) = a^2 - a^2 \sin^2 n\theta = a^2 - r^2$.

So putting $a^2 \cos^2 n\theta = a^2 - r^2$ in the value of ρ, we get

$$\rho = [r^2 + n^2 (a^2 - r^2)]^{3/2}/[r^2 + 2n^2 (a^2 - r^2) + n^2r^2]$$
$$= [r^2 + n^2a^2 - n^2r^2]^{3/2}/[r^2 + 2a^2n^2 - n^2r^2].$$

$$\therefore \quad \text{Curvature} = \frac{1}{\rho} = [r^2 + 2a^2n^2 - n^2r^2]/[r^2 + n^2a^2 - n^2r^2]^{3/2}.$$

(b) Proceed as in part (a) of this Example. Here $\rho = 4 \cos \frac{1}{2}\theta$.

Ex. 30. *Find the radius of curvature at the point (r, θ) on the curve*

$$\theta = a^{-1} (r^2 - a^2)^{1/2} - \cos^{-1} (a/r). \qquad \textbf{(Agra 1982)}$$

Sol. We have

$$\frac{d\theta}{dr} = \frac{1}{a} \cdot \frac{1}{2} \frac{2r}{\sqrt{(r^2 - a^2)}} + \frac{1}{\sqrt{\{1 - (a/r)^2\}}} \left(-\frac{a}{r^2}\right)$$
$$= \frac{r}{a\sqrt{(r^2 - a^2)}} - \frac{a}{r\sqrt{(r^2 - a^2)}} = \frac{r^2 - a^2}{ar\sqrt{(r^2 - a^2)}} = \frac{\sqrt{(r^2 - a^2)}}{ar}.$$

$$\therefore \quad \frac{dr}{d\theta} = \frac{ar}{\sqrt{(r^2 - a^2)}}.$$

Now

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4} \cdot \frac{a^2r^2}{r^2 - a^2} = \frac{1}{r^2} + \frac{a^2}{r^2 (r^2 - a^2)}$$
$$= \frac{r^2 - a^2 + a^2}{r^2 (r^2 - a^2)} = \frac{1}{r^2 - a^2}.$$

$\therefore$ the pedal equation of the given curve is $\dfrac{1}{p^2} = \dfrac{1}{r^2 - a^2}$

or

$$p^2 = r^2 - a^2. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. p, we get $2p = 2r\,(dr/dp)$.

$\therefore \quad \rho = r\,(dr/dp) = p = (r^2 - a^2)^{1/2}$, from (1).

Ex. 31. *Find the radius of curvature of the curve $2a/r = 1 + \cos\theta$; hence show that the square of the radius of curvature varies as the cube of the focal distance.* **(Meerut 1996)**

Sol. The given curve is $2a/r = 1 + \cos\theta$, ...(1)

which is a parabola with focus as pole.

Proceeding as in the chapter on tangents and normals, we get pedal equation of the curve as $p^2 = ar$. ...(2)

Differentiating (2) w.r.t. p, we get

$2p = a\,(dr/dp); \quad \therefore \quad dr/dp = 2p/a.$

Now $\rho = r\,(dr/dp) = r\,(2p/a) = (2r/a)\,p = (2r/a)\,.\,\sqrt{(ar)}$, from (2)

$= (2/\sqrt{a})\,r^{3/2}$.

$\therefore \rho^2 = (4/a)\,r^3$. Thus $\rho^2 \propto r^3$, where r is the distance of the point from the pole (*i.e.* focus). Hence ρ^2 varies as the cube of the focal distance.

Ex. 32. (a). *Prove that for any curve $\dfrac{r}{\rho} = \sin\phi\left(1 + \dfrac{d\phi}{d\theta}\right)$, where ρ is the radius of curvature and $\tan\phi = r\dfrac{d\theta}{dr}$.*

(Meerut 1983S; Rohilkhand 88, 90; Gorakhpur 88; Kanpur 86; Indore 72; Allahabad 70)

Sol. We know that $\psi = \theta + \phi$. ...(1)

Differentiating (1) w.r.t. s, we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right).$$

$$\therefore \quad \frac{1}{\rho} = \frac{\sin\phi}{r}\left(1 + \frac{d\phi}{d\theta}\right), \qquad \left[\because \rho = \frac{ds}{d\psi} \text{ and } \sin\phi = r\frac{d\theta}{ds}\right]$$

or

$$\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right).$$

Ex. 32 (b). *Prove that for any curve $\dfrac{d^2r}{ds^2} = \dfrac{\sin^2\phi}{r} - \dfrac{\sin\phi}{\rho}$.*

Sol. We have $dr/ds = \cos\phi$.

Differentiating w.r.t. 's', we get

$$\frac{d^2r}{ds^2} = -\sin\phi\cdot\frac{d\phi}{ds} = -\sin\phi\cdot\frac{d}{ds}(\psi - \theta), \qquad [\because \phi = \psi - \theta]$$

$$= -\sin\phi\left(\frac{d\psi}{ds} - \frac{d\theta}{ds}\right) = -\sin\phi\left[\frac{1}{\rho} - \frac{1}{r}\sin\phi\right].$$

$$\left[\because \rho = \frac{ds}{d\psi} \text{ and } \sin\phi = r\frac{d\theta}{ds}\right]$$

$$= -\frac{\sin\phi}{\rho} + \frac{1}{r}\sin^2\phi = \frac{\sin^2\phi}{r} - \frac{\sin\phi}{\rho}.$$

Ex. 33. *Show that at any point on the equiangular spiral $r = ae^{\theta \cot \alpha}$, $\rho = r \operatorname{cosec} \alpha$, and that it subtends a right angle at the pole.*
(Meerut 1977, 84, 94; Agra 87; Kanpur 80, 78, 71)

Sol. The equation of the given curve is $r = ae^{\theta \cot \alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we have

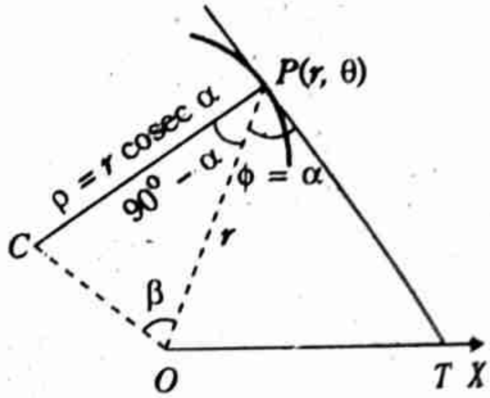
$$dr/d\theta = ae^{\theta \cot \alpha} \cdot \cot \alpha = r \cot \alpha.$$

$\therefore$ $(1/r)\, dr/d\theta = \cot\alpha$ or $\cot\phi = \cot\alpha$ or $\phi = \alpha$.

Now $p = r\sin\phi = r\sin\alpha$. Thus the pedal equation of (1) is $p = r\sin\alpha$. Therefore $dp/dr = \sin\alpha$.

Now $\rho = r\dfrac{dr}{dp} = \dfrac{r}{\sin\alpha} = r\operatorname{cosec}\alpha$. **(First part proved)**

Second part. Let $P(r,\theta)$ be any point on the given curve; PT is the tangent and PC is the normal to the curve at P. Let C be the centre of curvature of the point P of the curve. Then PC = the radius of curvature of the curve at $P = r\operatorname{cosec}\alpha$.



Join OP and OC, where O is the pole. Let $\angle POC = \beta$. Then to prove that $\beta = 90°$.

We have $\angle OPT = \phi = \alpha$,

[$\because$ for this curve $\phi = \alpha$, as already proved]

$\therefore$ $\angle OPC = 90° - \alpha$, since PC is normal at P *i.e.*, perpendicular to the tangent PT.

Now in $\Delta\, OPC$, we have

$$\angle OCP = 180° - \{(90° - \alpha) + \beta\} = (90° + \alpha - \beta).$$

Hence applying the sine theorem for $\Delta\, OPC$, we get

$$\frac{OP}{\sin \angle OCP} = \frac{PC}{\sin\beta}; \quad \text{or} \quad \frac{r}{\sin(90° + \alpha - \beta)} = \frac{\rho}{\sin\beta}$$

or $$\frac{r}{\cos(\alpha - \beta)} = \frac{r\operatorname{cosec}\alpha}{\sin\beta}, \qquad [\because \rho = r\operatorname{cosec}\alpha]$$

or $\sin\alpha\sin\beta = \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, $[\because r \neq 0]$

or $\cos\alpha\cos\beta = 0$ or $\cos\beta = 0$, $[\because \cos\alpha \neq 0]$

$\therefore$ $\beta = 90°$.

Ex. 34. *If ρ_1, ρ_2 be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then show that*

$$\rho_1^2 + \rho_2^2 = 16a^2/9.$$ **(Meerut 1980, 88)**

Sol. The given curve is $r = a(1 + \cos\theta)$. ...(1)

Let PQ be any chord of the curve (1) passing through the pole, and let P be the point (r_1, θ_1) and Q be the point (r_2, θ_2). Then $\theta_2 = \pi + \theta_1$.

[**Note :** To understand this point draw the figure of a chord passing through the pole.]

Since both the points (r_1, θ_1) and (r_2, θ_2) lie on the given cardioid (1), therefore

$$r_1 = a(1 + \cos\theta_1) \quad \text{and} \quad r_2 = a(1 + \cos\theta_2) \qquad ...(2)$$

Now let ρ be the radius of curvature of (1) at the point (r, θ). Then proceeding as in Ex. 27, part (ii) on page 225, we get

$$\rho = \tfrac{2}{3}\sqrt{(2ar)} \quad \text{or} \quad \rho^2 = \tfrac{8}{9}ar.$$

If ρ_1 and ρ_2 be the radii of curvature at the points P and Q, we have

$$\rho_1^2 = \tfrac{8}{9}ar_1 \quad \text{and} \quad \rho_2^2 = \tfrac{8}{9}ar_2.$$

$$\therefore \quad \rho_1^2 + \rho_2^2 = \frac{8}{9}a(r_1 + r_2) = \frac{8}{9}a[a(1 + \cos\theta_1) + a(1 + \cos\theta_2)], \quad \text{[from (2)]}$$

$$= (8a^2/9)[1 + \cos\theta_1 + 1 + \cos(\pi + \theta_1)], \qquad [\because \theta_2 = \pi + \theta_1]$$

$$= (8a^2/9)[1 + \cos\theta_1 + 1 - \cos\theta_1] = (16a^2/9).$$

Ex. 35. *If the equation to a curve be given in polar co-ordinates and if* $u = \dfrac{1}{r}$, *prove that the curvature is given by* $\left(\dfrac{d^2u}{d\theta^2} + u\right)\sin^3\phi$, *where* $\tan\phi = r(d\theta/dr)$.

Sol. We know that $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad ...(1)$$

Let $u = \dfrac{1}{r}$. Then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ so that $\left(\dfrac{du}{d\theta}\right)^2 = \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$.

Therefore (1) becomes $1/p^2 = u^2 + (du/d\theta)^2$. ...(2)

Differentiating (2) w.r.t. r, we get

$$-\frac{2}{p^3}\frac{dp}{dr} = 2u\frac{du}{dr} + \left(2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right)\frac{d\theta}{dr},$$

$$\left[\because \frac{d}{dr}\left(\frac{du}{d\theta}\right)^2 = \left\{\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2\right\}\frac{d\theta}{dr}\right]$$

$$= 2\left[u\frac{du}{dr} + \frac{d^2u}{d\theta^2}\frac{du}{dr}\right], \qquad \left[\because \frac{du}{d\theta}\frac{d\theta}{dr} = \frac{du}{dr}\right]$$

$$= 2\left(u + \frac{d^2u}{d\theta^2}\right)\frac{du}{dr}.$$

$$\therefore \quad -\frac{1}{r^3 \sin^3 \phi} \frac{dp}{dr} = \left(u + \frac{d^2u}{d\theta^2}\right)\left(-\frac{1}{r^2}\right),$$

$$\left[\because p = r \sin \phi. \text{ Also } u = \frac{1}{r} \Rightarrow \frac{du}{dr} = -\frac{1}{r^2}\right]$$

or $$\frac{1}{r}\frac{dp}{dr} = \sin^3 \phi \left(u + \frac{d^2u}{d\theta^2}\right).$$

Hence curvature, *i.e.*, $\dfrac{1}{\rho} = \sin^3 \phi \left(u + \dfrac{d^2u}{d\theta^2}\right)$. $\left[\because \rho = r\dfrac{dr}{dp}\right]$.

§ 7. Tangential Polar formula for Radius of Curvature.

(Agra 1983; Lucknow 79; Gorakhpur 80, 77; Allahabad 74)

A relaltion between p and ψ, holding for every point of a curve, is called the tangential polar equation of the curve. Thus the tangential polar equation of the curve is of the form $p = f(\psi)$.

To find the radius of curvature of a curve for which the relation between p and ψ is given.

We know that $\quad p = r \sin \phi$. ...(1)

Also $\dfrac{dp}{d\psi} = \dfrac{dp}{dr} \cdot \dfrac{dr}{ds} \cdot \dfrac{ds}{d\psi}$

$= (dp/dr) \cdot \cos \phi \cdot \rho$, $\quad [\because ds/d\psi = \rho$ and $dr/ds = \cos \phi]$

$= (dp/dr) \cos \phi \cdot r\,(dr/dp)$. $\quad [\because \rho = r\,(dr/dp)]$

$\therefore \quad dp/d\psi = r \cos \phi$. ...(2)

Squaring and adding (1) and (2), we have

$$r^2 = p^2 + (dp/d\psi)^2 \qquad ...(3)$$

Differentiating both sides of (3) w.r.t. 'p', we get

$$2r\frac{dr}{dp} = 2p + 2\frac{dp}{d\psi} \cdot \frac{d^2p}{d\psi^2} \cdot \frac{d\psi}{dp}$$

or $$r\frac{dr}{dp} = p + \frac{d^2p}{d\psi^2}. \quad \text{Hence} \quad \rho = \mathbf{p} + \frac{\mathbf{d^2p}}{\mathbf{d\psi^2}}.$$

Ex. 36. *Find the radius of curvature of the curve, $p = a \sin b\psi$.*

Sol. We have $\quad p = a \sin b\psi$.

$\therefore \quad dp/d\psi = ab \cos b\psi$ and $d^2p/d\psi^2 = -ab^2 \sin b\psi$.

$\therefore \quad \rho = p + (d^2p/d\psi^2) = a \sin b\psi - ab^2 \sin b\psi$

$= a\,(p/a) - ab^2 \cdot (p/a)$, from (1)

$= p - b^2 p = (1 - b^2)\,p$, *i.e.*, ρ varies as p.

Ex. 37. *Find the radius of curvature at any point of the ellipse*

$$p^2 = a^2 \cos^2 \psi + b^2 \sin^2 \psi.$$

Sol. We have $\quad p^2 = a^2 \cos^2 \psi + b^2 \sin^2 \psi$.

Proceeding as in Ex. 11 page 34 we have

$$p + \frac{d^2p}{d\psi^2} = \frac{a^2b^2}{p^3}. \qquad \text{[Give complete proof here]}$$

$$\therefore \qquad \rho = p + \frac{d^2p}{d\psi^2} = \frac{a^2b^2}{p^3}.$$

§ 8. Miscellaneous formulae for radius of curvature, when x and y are given as functions of arc length.

(a) We have $\cos\psi = dx/ds$.

Differentiating w.r.t., 's', we get $(-\sin\psi)(d\psi/ds) = d^2x/ds^2$

or $$-\sin\psi \cdot \frac{1}{\rho} = \frac{d^2x}{ds^2}. \qquad \left[\because \frac{1}{\rho} = \frac{d\psi}{ds}\right] \qquad ...(1)$$

$$\therefore \qquad \rho = -\frac{\mathbf{dy/ds}}{\mathbf{d^2x/ds^2}}. \qquad \left[\because \sin\psi = \frac{dy}{ds}\right].$$

(b) We have $\sin\psi = dy/ds$.

Differentiating w.r.t. 's', we get

$$\cos\psi \cdot \frac{1}{\rho} = \frac{d^2y}{ds^2}. \qquad ...(2)$$

$$\therefore \qquad \rho = \frac{\mathbf{dx/ds}}{\mathbf{d^2y/ds^2}}.$$

(c) Squaring and adding (1) and (2), we get

$$\frac{1}{\rho^2} = \left(\frac{\mathbf{d^2x}}{\mathbf{ds^2}}\right)^2 + \left(\frac{\mathbf{d^2y}}{\mathbf{ds^2}}\right)^2.$$

(d) We know that $\cos\psi = \dfrac{dx}{ds} = \dfrac{dx}{d\psi}\cdot\dfrac{d\psi}{ds} = \dfrac{1}{\rho}\cdot\dfrac{dx}{d\psi}$

and $$\sin\psi = \frac{dy}{ds} = \frac{dy}{d\psi}\cdot\frac{d\psi}{ds} = \frac{1}{\rho}\cdot\frac{dy}{d\psi}.$$

Squaring and adding these, we obtain

$$\cos^2\psi + \sin^2\psi = \frac{1}{\rho^2}\left[\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2\right]$$

or $$\rho^2 = (dx/d\psi)^2 + (dy/d\psi)^2.$$

§ 9. Radius of curvature at the origin.

(a) Newton's method of finding the radius of curvature at the origin.

(i) When the curve passes through the origin, and the axis of x is the tangent at the origin, we have

$$y(0) = 0 \text{ and } \left(\frac{dy}{dx}\right)_{x=0} = 0, \text{ i.e., } y_1(0) = 0. \qquad ...(1)$$

Now, by Maclaurin's theorem, y can be expanded as

$$y = y(0) + xy_1(0) + (x^2/2!)\,y_2(0) + (x^3/3!)\,y_3(0) + \ldots$$

or $$y = 0 + 0 + (x^2/2!)\,y_2(0) + (x^3/3!)\,y_3(0) + \ldots, \qquad \text{[by (1)]}.$$

Dividing both sides by x^2, we get

$$\frac{y}{x^2} = \frac{1}{2!}y_2(0) + \frac{x}{3!}y_3(0) + \ldots \qquad ...(2)$$

Since the curve passes through the origin, therefore $x \to 0 \Rightarrow y \to 0$. Hence taking limit of both sides of (2) when $x \to 0, y \to 0$, we get

$$\lim_{x \to 0} \frac{y}{x^2} = \frac{1}{2!} y_2(0), \qquad \text{[Other terms vanish].}$$

Therefore ρ **(at the origin)** $= \dfrac{[1 + \{y_1(0)\}^2]^{3/2}}{y_2(0)} = \dfrac{1}{y_2(0)}$,

$$[\because y_1(0) = 0]$$

$$= \lim_{\mathbf{x} \to 0, \mathbf{y} \to 0} \frac{\mathbf{x}^2}{\mathbf{2y}}.$$

(ii) *Similarly we can prove that if a curve passes through the origin, and the axis of y is the tangent there, then*

$$\rho \textbf{ (at the origin)} = \lim_{\mathbf{x} \to 0, \mathbf{y} \to 0} \frac{\mathbf{y}^2}{\mathbf{2x}}.$$

The above two formulae are known as **Newton's formulae.**

Note. *If a curve passes through the origin and is given by a rational integral, algebraic equation, then the tangents at the origin are easily obtained by equating to zero the lowest degree terms in the equation of the curve.*

(b) Expansion method for finding the radius of curvature at the origin. When a curve passes through the origin, but neither of the co-ordinate axes is a tangent at the origin, we cannot apply Newton's formula to find the radius of curvature at the origin. In such cases we can apply the following method known as the **expansion method.**

Since the curve passes through the origin, therefore $y(0) = 0$, *i.e.*, the value of y at $x = 0$ is 0.

Let $\left(\dfrac{dy}{dx}\right)_{(0,0)} = y_1(0) = p$ and $\left(\dfrac{d^2y}{dx^2}\right)_{(0,0)} = y_2(0) = q.$

Then $\rho \text{ (at origin)} = \dfrac{(1 + p^2)^{3/2}}{q}$. ...(1)

Now by Maclaurin's theorem, we have

$$y = y(0) + xy_1(0) + (1/2!)x^2y_2(0) + \dots$$
$$= px + \tfrac{1}{2}qx^2 + \dots,$$

because the curve passes through the origin.

Thus to get the values of p and q, we should obtain from the equation of the curve an expansion for y in ascending powers of x by algebraic or trigonom tric methods. The coefficient of x in this expansion will be equal to p and the coefficient of x^2 will be equal to $\frac{1}{2}q$, as is obvious from the Maclaurin's expansion for y. Putting these values of p and q in (1), we shall get the ρ at the origin.

(c) Radius of curvature at the pole. Suppose the curve passes through the pole and the initial line is the tangent at the pole. The relations between the cartesian and polar co-ordinates are $x = r\cos\theta$, $y = r\sin\theta$. By Newton's formula, we have

$$\rho \text{ (at the pole)} = \lim_{x \to 0, y \to 0} \frac{x^2}{2y} = \lim_{\theta \to 0} \frac{r^2 \cos^2\theta}{2r\sin\theta}$$

$$= \lim_{\theta \to 0} \left(\frac{r}{2\theta} \cdot \frac{\theta}{\sin\theta} \cdot \cos^2\theta\right) \qquad \textbf{(Note)}$$

$$= \lim_{\theta \to 0} \left(\frac{r}{2\theta}\right), \quad \left[\because \text{ as } \theta \to 0, \frac{\theta}{\sin\theta} \to 1 \text{ and } \cos\theta \to 1\right].$$

Solved Examples

Ex. 38. *Find the radius of curvature for the curve*

$$x = c\log\{s + \sqrt{(s^2 + c^2)}\},\ y = \sqrt{(s^2 + c^2)}.$$

Sol. We have $\dfrac{dx}{ds} = c\,\dfrac{1 + \frac{1}{2}(s^2 + c^2)^{-1/2}(2s)}{s + \sqrt{(s^2 + c^2)}} = \dfrac{c}{\sqrt{(s^2 + c^2)}}$,

and $\dfrac{dy}{ds} = \dfrac{s}{(s^2 + c^2)^{1/2}}$.

$$\therefore \quad \frac{d^2y}{ds^2} = \frac{1 \cdot \sqrt{(s^2 + c^2)} - s \cdot \frac{1}{2}(s^2 + c^2)^{-1/2}(2s)}{s^2 + c^2} = \frac{c^2}{(s^2 + c^2)^{3/2}}.$$

$$\therefore \quad \rho = \frac{dx/ds}{d^2y/ds^2} = \frac{c}{\sqrt{(s^2 + c^2)}} \cdot \frac{(s^2 + c^2)^{3/2}}{c^2} = \frac{1}{c}(s^2 + c^2) = \frac{y^2}{c}.$$

Ex. 39 (a). *Show that for the curve* $s^2 = 8ay$,

$\rho = 4a\sqrt{[1 - (y/2a)]}$. **(Lucknow 1981; Agra 71; Indore 71)**

Sol. We have $s^2 = 8ay$. Differentiating it w.r.t. s, we get

$$2s = 8a\frac{dy}{ds} = 8a\sin\psi. \qquad \left[\because \sin\psi = \frac{dy}{ds}\right]$$

$$\therefore \quad s = 4a\sin\psi.$$

Now $\rho = \dfrac{ds}{d\psi} = 4a\cos\psi = 4a\sqrt{(1 - \sin^2\psi)} = 4a\sqrt{\left[1 - \dfrac{s^2}{16a^2}\right]}$,

$$= 4a\sqrt{\left[1 - \frac{8ay}{16a^2}\right]}, \qquad [\because s^2 = 8ay]$$

$$= 4a\sqrt{[1 - (y/2a)]}.$$

Ex. 39 (b). *Show that for the curve* $s = ae^{x/a}$, $a\rho = s(s^2 - a^2)^{1/2}$.

(Rohilkhand 1989; Kanpur 85; Agra 86)

Sol. We have $s = ae^{x/a}$.

Therefore $ds/dx = ae^{x/a} \cdot 1/a = e^{x/a} = s/a$.

$\therefore \quad s = a(ds/dx) = a\sec\psi$. $[\because \cos\psi = dx/ds \Rightarrow \sec\psi = ds/dx]$

Now $\rho = \dfrac{ds}{d\psi} = a\sec\psi\tan\psi = s\sqrt{(\sec^2\psi - 1)}$

$$= s\sqrt{\left(\frac{s^2}{a^2}-1\right)} = \frac{s}{a}(s^2-a^2)^{1/2}.$$

Hence $\quad a\rho = s\,(s^2-a^2)^{1/2}.$

Ex. 40. *Prove that for any curve*

$$\frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{ds}\right)$$

(Meerut 1989 S; Rohilkhand 85; Kanpur 87)

Sol. We have $\quad \frac{d}{dx}\left(\frac{dy}{ds}\right) = \frac{d}{dx}(\sin\psi)$

$$= \cos\psi\cdot\frac{d\psi}{dx} = \cos\psi\cdot\frac{d\psi}{ds}\cdot\frac{ds}{dx} = \cos\psi\cdot\frac{1}{\rho}\cdot\sec\psi,$$

$$\left[\because \sec\psi = \frac{ds}{dx}\right]$$

$$= \frac{1}{\rho}.$$

Ex. 41. *Find the radius of curvature at the origin of the curve*

$$y = x^4 - 4x^3 - 18x^2.$$

Sol. Here $\quad dy/dx = 4x^3 - 12x^2 - 36x,\ d^2y/dx^2 = 12x^2 - 24x - 36.$

$\therefore$ At $(0,0)$, $dy/dx = 0$ and $d^2y/dx^2 = -36.$

$$\therefore\quad \rho \text{ at } (0,0) = \frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2} \text{ at } (0,0) = \frac{(1+0)^{3/2}}{-36} = \frac{1}{36}$$

(numerically).

Ex. 42. *Find the radius of curvature at the origin for the following curves :*

(*a*) $a(y^2 - x^2) = x^3,$

(*b*) $5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0,$

(Kanpur 1989)

(*c*) $x^3 - y^3 - 2x^2 + 6y = 0,$

(*d*) $a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + \ldots = 0.$

Sol. (a) The given curve passes through the origin. The tangents at origin are $y^2 - x^2 = 0$, *i.e*, $y = \pm x$. Thus neither of the co-ordinate axes is a tangent at the origin. So we cannot apply Newton's formula for finding ρ at origin. From the equation of the curve, we have

$$y^2 = x^2 + \frac{x^3}{a} = x^2\left(1+\frac{x}{a}\right).$$

$$\therefore\quad y = \pm x\left(1+\frac{x}{a}\right)^{1/2} = \pm x\left(1+\tfrac{1}{2}\cdot\frac{x}{a}+\ldots\right), \quad \text{...(1)}$$

(by binomial theorem).

Let $\left(\frac{dy}{dx}\right)_{(0,0)} = p$ and $\left(\frac{d^2y}{dx^2}\right)_{(0,0)} = q$. Then by Maclaurin's expansion, we get for this curve

$$y = px + \tfrac{1}{2}qx^2 + \ldots \qquad \ldots(2)$$

Comparing (1) and (2), we get

$$p = 1, q = 1/a; \quad \text{or} \quad p = -1, q = -1/a.$$

Now ρ (at origin) $= \dfrac{(1 + p^2)^{3/2}}{q}$.

$\therefore$ When $p = 1, q = 1/a$, we have

$$\rho \text{ (at origin)} = \frac{(1+1)^{3/2}}{1/a} = (2\sqrt{2})\, a,$$

and when $p = 1, q = -1/a$, we have

$$\rho \text{ (at origin)} = \frac{(1+1)^{3/2}}{-1/a} = -(2\sqrt{2})\, a.$$

(b) The given curve passes through the origin (0, 0). Equating to zero the lowest degree terms in the equation of the curve, we get the tangent at origin as $4x = 0$, *i.e*, $x = 0$, *i.e.*, y-axis.

$\therefore$ by Newton's formula

$$\rho \text{ (at the origin)} = \lim_{x \to 0, y \to 0} \left(\frac{y^2}{2x}\right).$$

Now dividing each term in the equation of the curve by $2x$, we get

$$\tfrac{3}{2}x^2 + 7y\left(\frac{y^2}{2x}\right) + 2xy + \tfrac{1}{2}y^2 + x + \tfrac{3}{2}y + \frac{y^2}{2x} + 2 = 0. \qquad \ldots(1)$$

Taking limits of both sides of (1) when $x \to 0$, $y \to 0$ and remembering that $\lim_{x \to 0} (y^2/2x) = \rho$ (at origin), we get

$$0 + 0 \cdot \rho + 0 + 0 + 0 + 0 + \rho + 2 = 0$$

i.e., ρ (at origin) $= -2$ or 2 (numerically).

(c) The given curve passes through the origin (0, 0). The tangent at origin obtained by equating to zero the lowest degree terms is $y = 0$, *i.e.*, x-axis. Therefore by Newton's formula

$$\rho \text{ (at origin)} = \lim_{x \to 0, y \to 0} \left(\frac{x^2}{2y}\right).$$

Now dividing each term in the equation of the curve by $2y$, we get

$$x\,(x^2/2y) - \tfrac{1}{2}y^2 - 2\,(x^2/2y) + 3 = 0. \qquad \ldots(1)$$

Taking limits of both sides of (1) when $x \to 0, y \to 0$, we get

$$0 \cdot \rho - \tfrac{1}{2} \cdot 0 - 2 \cdot \rho + 3 = 0; \quad \text{or} \quad \rho \text{ (at the origin)} = 3/2.$$

(d) The given curve passes through the origin. Since neither of the co-ordinate axes is a tangent at the origin, therefore Newton's method cannot be applied to find ρ at the origin. Also we cannot put the equation of the curve in the form

$$y = px + \tfrac{1}{2}qx^2 + \ldots$$

So substituting $px + \frac{1}{2}qx^2 + \ldots$ for y in the equation of the curve, we get the identity

$$a_1x + a_2(px + \tfrac{1}{2}qx^2 + \ldots) + b_1x^2 + b_2x(px + \tfrac{1}{2}qx^2 + \ldots) + \ldots = 0$$

or $\quad x(a_1 + a_2p) + x^2(\frac{1}{2}a_2q + b_1 + b_2p + b_3p^2) + \ldots = 0.$

Equating to zero the coefficients of x and x^2, we get

$$a_1 + a_2p = 0 \text{ and } \tfrac{1}{2}a_2q + b_1 + b_2p + b_3p^2 = 0.$$

Solving these, we get

$$p = -a_1/a_2, \text{ and } q = (2/a_2^3)(a_1b_2a_2 - b_1a_2^2 - a_1^2b_3).$$

$$\therefore \quad \rho \text{ (at the origin)} = \frac{(1 + p^2)^{3/2}}{q} = \frac{[1 + (a_1^2/a_2^2)]^{3/2}}{(2/a_2^3)(a_1a_2b_2 - a_2^2b_1 - a_1^2b_3)}$$

$$= \frac{(a_1^2 + a_2^2)^{3/2}}{2(a_1a_2b_2 - a_2^2b_1 - a_1^2b_3)}.$$

Ex. 43. *Find the radius of curvature at the origin for the curve*

$$3x^2 + 4x^3 - 12y = 0.$$

Sol. The given curve is $3x^2 + 4x^3 - 12y = 0.$ ...(1)

The curve (1) passes through the origin.

To find the tangents at the origin, equating to zero the lowest degree terms in (1), we get

$y = 0$ *i.e.*, x-axis as the tangent at origin.

$\therefore$ by Newton's formula ρ (at the origin) $= \lim_{x \to 0} \dfrac{x^2}{2y}.$ ...(2)

Dividing both sides of (1) by $2y$, we get

$$3 \cdot (x^2/2y) + 4x(x^2/2y) - 6 = 0. \qquad \text{...(3)}$$

Taking limits of both sides of (3) as $x \to 0$ and $y \to 0$, we have

$$3 \lim_{x \to 0} \frac{x^2}{2y} + 4 \cdot \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{x^2}{2y} - 6 = 0$$

or $\quad 3\rho + 4 \cdot 0 \cdot \rho - 6 = 0,$ from (2)

$\therefore \quad \rho = 2.$

Ex. 44. *Show that the radii of curvature at the origin on the curve $x^3 + y^3 = 3axy$ is each equal to $3a/2$.*

Sol. The curve is $\quad x^3 + y^3 = 3axy,$ ...(1)

which obviously passes through the origin.

Equating to zero the lowest degree terms in (1), the tangents at the origin are given by $3axy = 0$, *i.e.*, are $x = 0$ and $y = 0$.

$$\left.\begin{aligned} \therefore \rho \text{ (at the origin)} &= \lim_{x \to 0} \left(\frac{y^2}{2x}\right) = \rho_1 \text{ (say)} \\ \text{and} \qquad &= \lim_{x \to 0} \left(\frac{x^2}{2y}\right) = \rho_2 \text{ (say)} \end{aligned}\right\} \qquad \text{...(2)}$$

Now dividing both sides of (1) by $2xy$, we get

$$(x^2/2y) + (y^2/2x) = 3a/2. \quad ...(3)$$

Taking limits of (3) as $x \to 0$ and $y \to 0$, we get

$$\lim_{x \to 0} \frac{x^2}{2y} + \lim_{x \to 0} \left(\frac{y^2}{2x}\right) = \frac{3a}{2}$$

or $$\lim_{x \to 0} \frac{x^2}{2y} + \lim_{x \to 0} \frac{1}{4} xy \frac{2y}{x^2} = \frac{3a}{2}$$

or $$\rho_2 + 0 \,.\, (1/\rho_2) = 3a/2 \quad \text{or} \quad \rho_2 = 3a/2.$$

Similarly $\rho_1 = 3a/2$.

Ex. 45. *Find the radius of curvature at the origin of the curve*

$$y = x^3 + 5x^2 + 6x.$$

Sol. The given curve is $y = 6x + 5x^2 + x^3$, ...(1)
which obviously passes through the origin.

Let $\left(\dfrac{dy}{dx}\right)_{(0,0)} = p$ and $\left(\dfrac{d^2y}{dx^2}\right)_{(0,0)} = q.$

Then by Maclaurin's expansion, we get for this curve

$$y = px + \tfrac{1}{2}qx^2 + \ldots \quad ...(2)$$

Comparing (1) and (2), we get $p = 6, q/2 = 5$, *i.e*, $q = 10$.

Hence ρ at the origin $= \dfrac{(1+p^2)^{3/2}}{q} = \dfrac{(1+36)^{3/2}}{10} = \dfrac{1}{10} 37 \sqrt{37}.$

Ex. 46. *Show that the radii of curvature of the curve*

$$y^2 = x^2 (a + x)/(a - x)$$

at the origin are $\pm a \sqrt{2}$. **(Magadh 1972)**

Sol. Obviously the given curve passes through the origin. From the equation of the given curve, we have

$$y^2 = \frac{x^2 (a + x)}{a - x}$$

or $$y = \pm \frac{x (x + a)^{1/2}}{(a - x)^{1/2}} = \pm x \left(1 + \frac{x}{a}\right)^{1/2} \left(1 - \frac{x}{a}\right)^{-1/2}$$

$$= \pm x \left\{1 + \frac{1}{2}\frac{x}{a} + \ldots\right\} \left\{1 + \frac{1}{2}\frac{x}{a} + \ldots\right\},$$

expanding by binomial theorem

$$= \pm x \left(1 + \frac{x}{a} + \ldots\right) \quad ...(1)$$

Let $\left(\dfrac{dy}{dx}\right)_{(0,0)} = p$ and $\left(\dfrac{d^2y}{dx^2}\right)_{(0,0)} = q.$

Then by Maclaurin's expansion, we get for this curve

$$y = px + \tfrac{1}{2}qx^2 + \ldots \quad ...(2)$$

Comparing (1) and (2), we get

$$p = 1, q = 2/a; \quad \text{or} \quad p = -1, q = -2/a.$$

But ρ (at origin) $= \dfrac{(1+p^2)^{3/2}}{q}$.

$\therefore$ when $p = 1, q = 2/a$, we have ρ (at origin)

$= \dfrac{(1+1)^{3/2}}{2/a} = a\sqrt{2}$ and when $p = -1, q = -2/a$, we have

$$\rho \text{ (at origin)} = \frac{(1+1)^{3/2}}{-2/a} = -a\sqrt{2}.$$

Ex. 47. *Find the radius of curvature of the curve $r = a \sin n\theta$ at the origin (pole).*

Sol. As $\theta = 0 \Rightarrow r = 0$, so the pole lies on the given curve. Also $\theta = 0$ is a tangent at pole.

$$\therefore \quad \rho \text{ at the pole} = \lim_{\theta \to 0}\left(\frac{r}{2\theta}\right) = \lim_{\theta \to 0} \frac{a \sin n\theta}{2\theta}$$

$$= \lim_{\theta \to 0}\left(\tfrac{1}{2}na \cdot \frac{\sin n\theta}{n\theta}\right) = \tfrac{1}{2}na \cdot 1 = \tfrac{1}{2}na \cdot$$

Ex. 48. *Apply Newton's method to prove that the radius of curvature at the lowest point of the catenary $y = c \cosh (x/c)$ is equal to c.*

Sol. Shifting the origin to the lowest point of the catenary *i.e.*, the point $(0, c)$, we have $y + c = c \cosh (x/c)$

or $$y + c = c\left[1 + \frac{x^2}{2!\,c^2} + \frac{x^4}{4!\,c^4} + \dots\right]$$

or $$y = \frac{x^2}{2c} + \frac{x^4}{24c^3} + \dots \qquad \dots(1)$$

Now the tangent to (1) at the new origin is $y = 0$, (*i.e.*, the new x-axis).

Dividing both sides of (1) by $2y$, we get

$$\frac{1}{2} = \frac{1}{2c} \cdot \frac{x^2}{2y} + \frac{x^2}{24c^3} \cdot \frac{x^2}{2y} + \dots \qquad \dots(2)$$

Taking the limits of both sides of (2) when $x \to 0, y \to 0$, we have

$$\frac{1}{2} = \frac{1}{2c} \cdot \rho, \qquad \text{(Other terms vanish)}$$

or $$\rho = c.$$

Ex. 49. *Apply Newton's formula to find the radius of curvature at the origin of the cycloid*

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta).$$

Sol. Differentiating, $\dfrac{dx}{d\theta} = a(1 + \cos\theta)$ and $\dfrac{dy}{d\theta} = a \sin\theta$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a \sin\theta}{a(1 + \cos\theta)} = \tan \tfrac{1}{2}\theta.$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at origin}} = \tan 0 = 0, \qquad [\because \theta = 0 \text{ at } x = 0, y = 0].$$

Thus the axis of x is tangent at the origin.

$$\therefore \quad \rho \text{ (at the origin)} = \lim_{x \to 0} \left(\frac{x^2}{2y}\right)$$

$$= \lim_{\theta \to 0} \left[\frac{a^2 (\theta + \sin \theta)^2}{2a (1 - \cos \theta)}\right], \qquad \text{[form 0/0]}$$

$$= \lim_{\theta \to 0} \left[\frac{a}{2} \frac{2 (\theta + \sin \theta) (1 + \cos \theta)}{\sin \theta}\right],$$

by L'Hospital's rule

$$= \lim_{\theta \to 0} \left[a \frac{(\theta + \sin \theta) (- \sin \theta) + (1 + \cos \theta)^2}{\cos \theta}\right] = 4a.$$

Ex. 50. *Find the radius of curvature at the origin for the curve* $x = t - \frac{1}{3} t^3, y = t^2$.

Sol. We have $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{1 - t^2}$.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at origin}} = 0, \qquad [\because \text{ at origin } x = 0, y = 0 \Rightarrow t = 0].$$

Thus the tangent at origin is x-axis.

$$\therefore \quad \rho \text{ (at the origin)} = \lim_{x \to 0} \left(\frac{x^2}{2y}\right)$$

$$= \lim_{t \to 0} \left\{\frac{(t - \frac{1}{3} t^3)^2}{2t^2}\right\} = \lim_{t \to 0} \tfrac{1}{2} (1 - \tfrac{1}{3} t^2)^2 = \tfrac{1}{2}.$$

§ 10. Co-ordinates of centre of curvature.

To find the centre of curvature for any point (x, y) of the curve

$$y = f(x) \quad \text{or} \quad f(x, y) = 0.$$

Let the equation of the curve be $y = f(x)$. Let P be the given point (x, y) on the curve and Q a point $(x + \delta x, y + \delta y)$ in the neighbourhood of P. (See the figure of § 1 on page 206). Let N be the point of intersection of the normals at P and Q. As $Q \to P$, suppose $N \to C$. Then C is the centre of curvature for the point P. Let the co-ordinates of C be (α, β).

From the equation of the curve, we have

$$dy/dx = f'(x) = \phi(x), \text{ (say)}.$$

The equation of normal at $P(x, y)$ is

$$(Y - y) \phi(x) + (X - x) = 0. \qquad ...(1)$$

The equation of the normal at $Q(x + \delta x, y + \delta y)$ is

$$\{Y - (y + \delta y)\} \phi(x + \delta x) + \{X - (x + \delta x)\} = 0. \qquad ...(2)$$

Subtracting (1) from (2), we get

$$(Y - y) \{\phi(x + \delta x) - \phi(x)\} - \delta y . \phi(x + \delta x) - \delta x = 0.$$

Dividing by δx, we get

$$(Y-y)\left\{\frac{\phi(x+\delta x)-\phi(x)}{\delta x}\right\}-\phi(x+\delta x)\frac{\delta y}{\delta x}-1=0. \quad ...(3)$$

The value of Y obtained from this equation gives us the y-co-ordinate of the point of intersection of (1) and (2).

Now as $Q \to P, \delta x \to 0$ and Y obained from (3) $\to \beta$.

Therefore taking limit of (3) as $\delta x \to 0$, we get

$$(\beta-y)\lim_{\delta x \to 0}\left\{\frac{\phi(x+\delta x)-\phi(x)}{\delta x}\right\}$$

$$-\lim_{\delta x \to 0}\phi(x+\delta x).\lim_{\delta x \to 0}\frac{\delta y}{\delta x}-1=0$$

or $$(\beta-y)\frac{d}{dx}\phi(x)-\phi(x)\cdot\frac{dy}{dx}-1=0$$

or $$(\beta-y)\frac{d}{dx}\left(\frac{dy}{dx}\right)-\frac{dy}{dx}\cdot\frac{dy}{dx}-1=0, \qquad \left[\because \ \phi(x)=\frac{dy}{dx}\right]$$

or $$(\beta-y)\frac{d^2y}{dx^2}-\left\{\left(\frac{dy}{dx}\right)^2+1\right\}=0.$$

$$\therefore \quad \beta=\mathbf{y}+\frac{\mathbf{1+(dy/dx)^2}}{\mathbf{d^2y/dx^2}}. \quad ...(4)$$

Also (α, β) lines on (1). Therefore, we get

$$(\beta-y)(dy/dx)+(\alpha-x)=0$$

i.e., $$(\alpha-x)=-(\beta-y)\frac{dy}{dx}=-\frac{dy}{dx}\cdot\frac{1+(dy/dx)^2}{d^2y/dx^2}, \quad \text{[from (4)]}$$

$$\therefore \quad \alpha=\mathbf{x}-\frac{\mathbf{(dy/dx)\ \{1+(dy/dx)^2\}}}{\mathbf{d^2y/dx^2}}. \quad ...(5)$$

Hence the co-ordinates (α, β) of the centre of curvature are given by (4) and (5).

Evolute of a curve. Definicion. *The locus of the centres of curvature of all points of a given plane curve is called the evolute of the curve.*

Cor. Equation of the circle of Curvature at P (x, y).

If (α, β) be the co-ordinates of the centre of curvature and ρ be the radius of curvature at any point (x, y) on a curve then the equation of the circle of curvature at that point is

$$(X-\alpha)^2+(Y-\beta)^2=\rho^2.$$

Solved Examples

Ex. 51. *Find the co-ordinates of the centre of curvature for the point (x, y) on the parabola $y^2 = 4ax$.* **(Kanpur 1998; Kurukshetra 84)**

Also find the equation of the evolute of the parabola.

(Kanpur 1974, 75; R.U. 79)

Sol. We have $y^2 = 4ax$. ...(1)

Differentiating, $\dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{2a}{\sqrt{(4ax)}} = a^{1/2}x^{-1/2} = \sqrt{\left(\dfrac{a}{x}\right)}$,

and $$d^2y/dx^2 = -\tfrac{1}{2}a^{1/2}x^{-3/2}.$$

If (α, β) be the centre of curvature for the point (x, y), then

$$\alpha = x - \frac{(dy/dx)\{1 + (dy/dx)^2\}}{d^2y/dx^2} = x - \frac{[1 + (a/x)] \cdot \sqrt{(a/x)}}{-\frac{1}{2}a^{1/2}x^{-3/2}}$$

$$= x + 2(x + a) = 3x + 2a. \quad ...(2)$$

Also $$\beta = y + \frac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y + \frac{[1 + (a/x)]}{-\frac{1}{2}a^{1/2}x^{-3/2}}$$

$$= 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{1/2}(x + a) = 2a^{-1/2}x^{1/2}[a - (x + a)]$$
$$= -2a^{-1/2}.x^{3/2}. \quad ...(3)$$

Hence the required centre of curvature is the point

$$(3x + 2a, -2a^{-1/2}x^{3/2}).$$

Eliminating x from (2) and (3), we get

$$\left(\frac{\alpha - 2a}{3}\right)^3 = \frac{a\beta^2}{4} \quad \text{or} \quad 27a\beta^2 = 4(\alpha - 2a)^3.$$

Therefore the locus of (α, β) is

$$27ay^2 = 4(x - 2a)^3,$$

which is the required evolute of the parabola.

Ex. 52. (a). *Find the centre of curvature at the point 't' on the ellipse*
$$x = a\cos t, y = b\sin t.$$

Sol. Differentiating, $\dfrac{dx}{dt} = -a\sin t$, $\dfrac{dy}{dt} = b\cos t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t,$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\right\}.\frac{dt}{dx}$$

$$= \frac{b}{a}\text{coses}^2 t\left(\frac{1}{-a\sin t}\right) = -\frac{b}{a^2}\text{cosec}^3 t.$$

If (α, β) be the centre of curvature for the point 't', then

$$\alpha = x - \frac{[1 + (dy/dx)^2](dy/dx)}{d^2y/dx^2}$$

$$= x - \frac{\{1 + (b^2/a^2)\cot^2 t\}\{-(b/a)\cot t\}}{-(b/a^2)\text{cosec}^3 t}$$

$$= a\cos t - (1/a)(a^2\sin^2 t + b^2\cos^2 t)\cos t$$

$$= \frac{a^2(\cos t - \sin^2 t\cos t) - b^2\cos^3 t}{a} = \frac{a^2 - b^2}{a}\cos^3 t.$$

Also $$\beta = y + \frac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y + \frac{\{1 + (b^2/a^2)\cot^2 t\}}{-(b/a^2)\text{cosec}^3 t}$$

$$= b \sin t - \frac{\sin t}{b} \cdot (a^2 \sin t + b^2 \cos^2 t)$$

$$= \frac{1}{b} \sin t \, [b^2 - a^2 \sin^2 t - b^2 \cos^2 t]$$

$$= \frac{b^2 - a^2}{b} \sin^3 t = - \frac{a^2 - b^2}{b} \sin^3 t .$$

$\therefore$ the required centre of curvature is the point

$$\left(\frac{a^2 - b^2}{a} \cos^3 t , - \frac{a^2 - b^2}{b} \sin^3 t \right) .$$

Ex. 52. (b). *Prove that the centre of curvature* (α, β) *for the curve*

$$x = 3t, \; y = t^2 - 6$$

is $\alpha = - \frac{4}{3} t^3 , \beta = 3t^2 - \frac{3}{2} .$ **(Meerut 1989S)**

Ex. 53. *Find the co-ordinates of the centre of curvature of the rectangular hyperbola* $xy = a^2$.

Sol. We have $y = a^2/x$; $\therefore$ $dy/dx = - a^2 x^{-2}$,

and $d^2y/dx^2 = 2a^2 x^{-3}$.

Let (α, β) be the co-ordinates of the centre of curvature at the point (x, y) . Then

$$\alpha = x - \frac{(dy/dx) \, [1 + (dy/dx)^2]}{d^2y/dx^2} = x - \frac{(- a^2/x^2) \, (1 + a^4/x^4)}{2a^3/x^3}$$

$$= x + \frac{(x^4 + a^4)}{2x^3} = \frac{3x^4 + x^2y^2}{2x^3} = \frac{3x}{2} + \frac{y^2}{2x}, \qquad [\because a^4 = x^2y^2]$$

Also $$\beta = y + \frac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y + \frac{[1 + (a^4/x^4)]}{2a^2/x^3}$$

$$= y + \frac{x^4 + a^4}{2a^2x} = y + \frac{x^4 + x^2y^2}{2x^2y}, \qquad [\because a^2 = xy]$$

$$= y + \frac{x^2 + y^2}{2y} = \frac{3y^2 + x^2}{2y} = \frac{3}{2} y + \frac{x^2}{2y} .$$

Hence the required centre of curvature is the point

$$\left(\frac{3x}{2} + \frac{y^2}{2x}, \frac{3y}{2} + \frac{x^2}{2y} \right) .$$

Ex. 54. *Find the co-ordinates of the centre of curvature of the curve* $a^2y = x^3$.

Sol. Here $a^2y = x^3$; $\therefore$ $\dfrac{dy}{dx} = \dfrac{3x^2}{a^2}$ and $\dfrac{d^2y}{dx^2} = \dfrac{6x}{a^2}$.

$$\therefore \quad \alpha = x - \frac{[1 + (dy/dx)^2] \, (dy/dx)}{d^2y/dx^2} = x - \tfrac{1}{2} x \left(1 + \frac{9x^4}{a^4} \right)$$

$$= \frac{x}{2} \cdot \left(1 - \frac{9x^4}{a^4} \right) .$$

Also $\beta = y + \dfrac{[1 + (dy/dx)^2]}{d^2y/dx^2} = \dfrac{x^3}{a^2} + \dfrac{[1 + (9x^4/a^4)]}{6x/a^2} = \dfrac{5x^3}{2a^2} + \dfrac{a^2}{6x}$.

$\therefore$ the required centre of curvature is

$$\left[\frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right), \left(\frac{5x^3}{2a^2} + \frac{a^2}{6x}\right)\right].$$

Ex. 55. *Find the co-ordinates of the centre of curvature of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *at a given point* (x, y). **(Gorakhpur 1973)**

Sol. Here $dy/dx = -b^2x/(a^2y)$

and $d^2y/dx^2 = -b^4/(a^2y^3)$. [See Ex. 10, on page 213]

$\therefore \quad \alpha = x - \dfrac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2} = x - \dfrac{(b^4x^2 + a^4y^2)}{a^4b^2}$,

and $\beta = y + \dfrac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y - \dfrac{(b^4x^2 + a^4y^2)}{a^2b^4}$.

Ex. 56. *Find the centre of curvature of the curve* $y = x^3 - 6x^2 + 3x + 1$ *at the point* $(1, -1)$. **(Kanpur 1985)**

Sol. We have $dy/dx = 3x^2 - 12x + 3$; $d^2y/dx^2 = 6x - 12$.

$\therefore \quad (dy/dx)_{(1,-1)} = -6$; $(d^2y/dx^2)_{(1,-1)} = -6$.

$\therefore \quad \alpha = 1 - \dfrac{[1 + (-6)^2](-6)}{-6} = 1 - 37 = -36,$

and $\beta = -1 + \dfrac{[1 + (-6)^2]}{-6} = -1 - \dfrac{37}{6} = -\dfrac{43}{6}$.

$\therefore$ the required centre of curvature is $(-36, -\frac{43}{6})$.

Ex. 57. *Prove that the centres of curvature at points of the cycloid* $x = a(t - \sin t)$, $y = a(1 - \cos t)$ *lie on an equal cycloid.*

Sol. We have $\dfrac{dx}{dt} = a(1 - \cos t)$, $\dfrac{dy}{dt} = a \sin t$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a(1 - \cos t)} = \cot \tfrac{1}{2}t,$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\cot \tfrac{1}{2}t) = \left\{\dfrac{d}{dt}(\cot \tfrac{1}{2}t)\right\} \cdot \dfrac{dt}{dx}$

$= -\tfrac{1}{2}\operatorname{cosec}^2 \tfrac{1}{2}t \cdot \dfrac{1}{a(1 - \cos t)} = \dfrac{-1}{4a \sin^4 \frac{1}{2}t}$.

$\therefore \quad \alpha = x - \dfrac{(dy/dx)[1 + (dy/dx)^2]}{d^2y/dx^2}$

$= a(t - \sin t) - \dfrac{\cot \frac{1}{2}t\,[1 + \cot^2 \frac{1}{2}t]}{-(1/4a)\operatorname{cosec}^2 \frac{1}{2}t}$

$= a(t - \sin t) + 4a \cos \tfrac{1}{2}t \,.\, \sin \tfrac{1}{2}t$

$= at - a \sin t + 2a \sin t = a(t + \sin t)$ ...(1)

Also $\quad \beta = y + \dfrac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y + \dfrac{1 + \cot^2 \frac{1}{2} t}{- (1/4a) \operatorname{cosec}^4 \frac{1}{2} t}$

$= a (1 - \cos t) - 4a \sin^2 \frac{1}{2} t$

$= a - a \cos t - 2a + 2a \cos t = - a (1 - \cos t).$ ...(2)

$\therefore$ the centre of curvature at the point 't' of the given cycloid is the point

$$[a (t + \sin t), \ - a (1 - \cos t)].$$

Now the locus of the centre of curvature is obtained by generalising (α, β) obtained from (1) and (2). The required locus is the curve

$$x = a (t + \sin t), y = - a (1 - \cos t),$$

which is an equal cycloid.

Ex. 58. *Prove that the co-ordinates* (α, β) *of the centre of curvature at any point* (x, y) *can be expressed in the form*

$$\alpha = x - (dy/d\psi) \text{ and } \beta = y + (dx/d\psi). \quad \textbf{(Kanpur 1986)}$$

Sol. Let (α, β) be the co-ordinates of the centre of curvature at any point (x, y) of a given curve.

We have $\rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{(1 + \tan^2 \psi)^{3/2}}{d^2y/dx^2}, \left[\because \dfrac{dy}{dx} = \tan \psi\right]$

$\therefore \quad d^2y/dx^2 = (1/\rho) \sec^3 \psi.$

Now, $\alpha = x - \dfrac{(dy/dx) [1 + (dy/dx)^2]}{d^2y/dx^2} = x - \dfrac{\tan \psi (1 + \tan^2 \psi)}{(1/\rho) \sec^3 \psi}$

$= x - \rho \sin \psi$

$= x - \dfrac{ds}{d\psi} \cdot \dfrac{dy}{ds}, \qquad \left[\because \rho = \dfrac{ds}{d\psi} \text{ and } \sin \psi = \dfrac{dy}{ds}\right]$

$\therefore \quad \alpha = x - (dy/d\psi).$

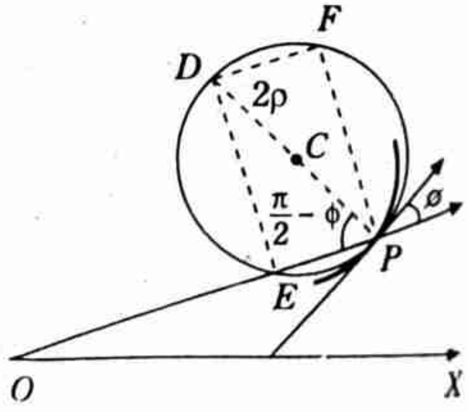
Similarly, $\quad \beta = y + \dfrac{(1 + \tan^2 \psi)}{(1/\rho) \sec^3 \psi} = y + \rho \cos \psi$

$= y + \dfrac{ds}{d\psi} \cdot \dfrac{dx}{ds}, \qquad \left[\because \rho = \dfrac{ds}{d\psi} \text{ and } \cos \psi = \dfrac{dx}{ds}\right]$

$\therefore \quad \beta = y + (dx/d\psi).$

§ 11.(i) Chord of curvature through the origin (pole).

Let C be the centre of curvature at the point P on any given curve. Then $CP = \rho =$ radius of curvature at P. The circle whose centre is C and radius CP, is the circle of curvature at P. Any chord of this circle through P is a chord of curvature.



Let O be the pole. Join OP to meet the circle of curvature in E. Then

PE is the chord of curvature through the origin.

PD is the diameter of the circle of curvature. We have $PD = 2\rho$ and $\angle PED = 90^\circ$, being an angle in a semi-circle.

Also PD is normal to the curve at P; therefore $\angle EPD = \frac{1}{2}\pi - \phi$.

Now from the right-angled triangle PED, we have

$$PE = PD \cos\left(\tfrac{1}{2}\pi - \phi\right) = 2\rho \sin \phi.$$

$\therefore$ **chord of curvature through the pole $= 2\rho \sin \phi$.**

Deduction. *To find the chord of curvature through the pole for the curve $p = f(r)$.* **(Lucknow 1982)**

The required chord of curvature

$$= 2\rho \sin \phi = 2r \frac{dr}{dp} \cdot \frac{p}{r}, \qquad \left[\because \rho = r\frac{dr}{dp} \text{ and } p = r \sin \phi\right]$$

$$= 2p \frac{dr}{dp} = \frac{2f(r)}{f'(r)}, \qquad \left[\because p = f(r) \text{ and } \frac{dp}{dr} = f'(r)\right]$$

(ii) Chord of curvature perpendicular to the radius vector.

In the figure of this article suppose a line through P, perpendicular to the radius vector OP, meets the circle of curvature in F. Then PF is the chord of curvature perpendicular to the radius vector.

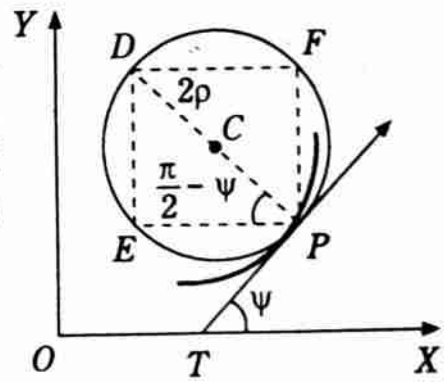
We have $\quad PF = ED = 2\rho \sin\left(\frac{1}{2}\pi - \phi\right) = 2\rho \cos \phi.$

$\therefore$ **chord of curvature perpendicular to the radius vector**

$$= 2\rho \cos \phi.$$

§ 12. Chords of curvature parallel to the co-ordinate axes.

In the adjoining figure, C is the centre of curvature at any point P on a given curve. We have $CP = \rho$ and $PD = 2\rho$. PE and PF are the chords of curvature parallel to the x-axis and y-axis respectively.



Also $\angle PED = \frac{1}{2}\pi$,

$\angle EPT = \angle PTX = \psi$ and

$\angle EPD = \frac{1}{2}\pi - \psi$.

We have $\quad PE = PD \cos\left(\frac{1}{2}\pi - \psi\right) = 2\rho \sin \psi.$

$\therefore$ **chord of curvature parallel to the axis of x $= 2\rho \sin \psi$.**

Again $\quad PF = ED = PD \sin\left(\frac{1}{2}\pi - \psi\right) = 2\rho \cos \psi.$

$\therefore$ **chord of curvature parallel to the axis of y $= 2\rho \cos \psi$.**

Solved Examples

Ex. 59. *Show that the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$ is $2r/(n+1)$.*

(Meerut 1983; Lucknow 77; Gorakhpur 89)

Sol. We have $r^n = a^n \cos n\theta$.

Taking logarithm, $n \log r = n \log a + \log \cos n\theta$.

Differentiating w.r.t. θ, we have $\dfrac{n}{r}\dfrac{dr}{d\theta} = -n \tan n\theta$.

$\therefore \quad \cot \phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta = \cot\left(\tfrac{1}{2}\pi + n\theta\right)$; so that $\phi = \tfrac{1}{2}\pi + n\theta$.

Now $\quad \sin \phi = \sin\left(\tfrac{1}{2}\pi + n\theta\right) = \cos n\theta = (r/a)^n$. ...(1)

Also $\quad p = r \sin \phi = r\,(r/a)^n = r^{n+1}/a^n$.

Thus $p = r^{n+1}/a^n$ is the pedal equation of the curve. Differentiating w.r.t. r, we have

$$\frac{dp}{dr} = \frac{(n+1)\,r^n}{a^n}.$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r \cdot \frac{a^n}{(n+1)\,r^n} = \frac{a^n}{(n+1)\,r^{n-1}}. \qquad \text{...(2)}$$

Hence the chord of curvature through the pole

$$= 2\rho \sin \phi = 2\frac{a^n}{(n+1)\,r^{n-1}}\left(\frac{r^n}{a^n}\right), \qquad \text{[from (1) and (2)]}$$

$= 2r/(n+1)$.

Ex. 60. *Find the chord of curvature through the pole of the cardioid, $r = a(1 + \cos\theta)$.* **(Gorakhpur 1978; Jiwaji 72; Meerut 84)**

Sol. We have $r = a(1 + \cos\theta)$.

Differentiating, $dr/d\theta = -a \sin\theta$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\tfrac{1}{2}\theta = \tan\left(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta\right);$$

so that $\qquad \phi = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta$

Now $\quad p = r\sin\phi = r\sin\left(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta\right) = r\cos\tfrac{1}{2}\theta$.

$\therefore \quad 2p^2 = r^2\left(2\cos^2\tfrac{1}{2}\theta\right) = r^2(1+\cos\theta) = r^2 \cdot (r/a) = r^3/a$.

Thus $2p^2 a = r^3$ is the pedal equation of the curve.

Differentiating w.r.t. r, we have $4ap\dfrac{dp}{dr} = 3r^2$.

$$\therefore \quad \rho = r\frac{dr}{dp} = r \cdot \frac{4ap}{3r^2} = \frac{4ap}{3r}.$$

Hence the chord of curvature through the pole

$$= 2\rho\sin\phi = 2 \cdot \frac{4ap}{3r} \cdot \frac{p}{r}, \qquad [\because\ p = r\sin\phi]$$

$$= \frac{8ap^2}{3r^2} = \frac{8}{3r^2} \cdot \frac{r^3}{2}, \qquad [\because 2ap^2 = r^3]$$
$$= 4r/3.$$

Ex. 61. *Show that the chord of curvature through the pole of the curve $r = a\,e^{m\theta}$ is $2r$.*

Sol. We have $r = a\,e^{m\theta}$. Therefore $dr/d\theta = a\,e^{m\theta} \,.\, m = mr$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = r \cdot \frac{1}{mr} = \frac{1}{m} = \tan\alpha, \text{ say.} \qquad \textbf{(Note)}$$

$\therefore \quad \phi = \alpha$.

Now $p = r\sin\phi = r\sin\alpha$. Therefore $dp/dr = \sin\alpha$.

$$\therefore \quad \rho = r \cdot \frac{dr}{dp} = r \cdot \frac{1}{\sin\alpha} = r\,\text{cosec}\,\alpha.$$

Hence the chord of curvature through the pole $= 2\rho\sin\phi$
$$= 2r\,\text{cosec}\,\alpha\sin\alpha = 2r.$$

Ex. 62. *Show that the length of the chord of curvature, parallel to y-axis at the origin, in the parabola $y = mx + (x^2/a)$ is $(1 + m^2)\,a$.*

(Allahabad 1971)

Sol. We have $\quad y = mx + (x^2/a)$.

$\therefore \quad dy/dx = m + (2x/a)$, $d^2y/dx^2 = 2/a$.

$\therefore \quad$ at the origin (0, 0), we have $dy/dx = m$ and $d^2y/dx^2 = 2/a$.

$$\therefore \quad \rho \text{ (at origin)} = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \text{ at } (0, 0) = \frac{(1 + m^2)^{3/2}}{2/a}$$
$$= \frac{a\,(1 + m^2)^{3/2}}{2}.$$

Also at the origin, we have $\tan\psi = dy/dx$ at $(0, 0) = m$;

$\therefore \quad \cos\psi = 1/\sqrt{(1 + m^2)}$.

Now the chord of curvature parallel to the y-axis $= 2\rho\cos\psi$.

$\therefore$ at the origin, the chord of curvature parallel to the y-axis
$$= 2 \cdot \frac{a\,(1 + m^2)^{3/2}}{2} \cdot \frac{1}{\sqrt{(1 + m^2)}} = a\,(1 + m^2).$$

Ex. 63. *Show that in the curve $y = a\cosh(x/a)$ the chord of curvature parallel to the axis of x is of length $a\sinh(2x/a)$.*

(Jiwaji 1972; Gorakhpur 66)

Sol. We have $\quad y = a\cosh(x/a)$.

Therefore $\quad dy/dx = a\,\{\sinh(x/a)\}\,.\,(1/a) = \sinh(x/a)$;
$$d^2y/dx^2 = (1/a)\cosh(x/a).$$

Now the chord of curvature parallel to the axis of $x = 2\rho\sin\psi$

$$= 2\rho \cdot \frac{1}{\text{cosec}\,\psi} = 2\rho\frac{1}{\sqrt{(1 + \cot^2\psi)}} = 2\rho\frac{\tan\psi}{\sqrt{(1 + \tan^2\psi)}}$$
$$= 2 \cdot \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \cdot \frac{dy/dx}{\sqrt{[1 + (dy/dx)^2]}} = \frac{2\,(dy/dx)\,[1 + (dy/dx)^2]}{d^2y/dx^2}$$

$$= \frac{2 \sinh (x/a) [1 + \sinh^2 (x/a)]}{(1/a) \cosh (x/a)} = \frac{2a \sinh (x/a) \cosh^2 (x/a)}{\cosh (x/a)}$$

$= 2a \sinh (x/a) \cosh (x/a) = a \sinh \ (2x/a)$.

Ex. 64. *In the curve $y = a \log \sec (x/a)$ prove that the chord of curvature parallel to the axis of y is of constant length.*

(Lucknow 1980; Kanpur 86)

Sol. We have $\quad y = a \log \sec (x/a)$. ...(1)

Differentiating,

$$\frac{dy}{dx} = a \cdot \frac{1}{\sec (x/a)} \cdot \sec (x/a) \,.\, \tan (x/a) \,.\, \frac{1}{a} = \tan \frac{x}{a},$$

and $$\frac{d^2y}{dx^2} = \frac{1}{a} \sec^2 \left(\frac{x}{a}\right)$$

Now chord of curvature parallel to y-axis

$$= 2\rho \cos \psi = \frac{2\rho}{\sec \psi} = \frac{2\rho}{\sqrt{\{1 + (dy/dx)^2\}}}$$

$$= 2 \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} \cdot \frac{1}{\{1 + (dy/dx)^2\}^{1/2}}$$

$$= 2 \frac{1 + (dy/dx)^2}{d^2y/dx^2} = 2 \frac{1 + \tan^2 (x/a)}{(1/a) \sec^2 (x/a)} = 2a = \text{ constant.}$$

Ex. 65. *If C_x, C_y be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$, prove that*

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2a\, C_x} \cdot$$

(Lucknow 1981; Meerut 89)

Sol. We have, $C_x =$ the chord of curvature parallel to the x-axis

$$= 2\rho \sin \psi = 2\rho \frac{1}{\operatorname{cosec} \psi} = 2\rho \frac{1}{\sqrt{(1 + \cot^2 \psi)}} = 2\rho \frac{\tan \psi}{\sqrt{(1 + \tan^2 \psi)}}$$

$$= 2 \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{(1 + y_1^2)}} = \frac{2y_1}{y_2} (1 + y_1^2). \qquad ...(1)$$

Here $y = ae^{x/a}$; $y_1 = dy/dx = e^{x/a}$; and $y_2 = d^2y/dx^2 = (1/a)\, e^{x/a}$.

$\therefore$ from (1), we get

$$C_x = \frac{2e^{x/a}}{(1/a)\, e^{x/a}} [1 + e^{2x/a}] = 2a\, (1 + e^{2x/a}).$$

Again $C_y =$ the chord of curvature parallel to y-axis

$$= 2\rho \cos \psi = \frac{2\rho}{\sec \psi} = \frac{2\rho}{\sqrt{(1 + \tan^2 \psi)}} = 2 \frac{(1 + y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{(1 + y_1^2)}}$$

$$= \frac{2\, (1 + y_1^2)}{y_2} = \frac{2}{(1/a)\, e^{x/a}} [1 + e^{2x/a}] = \frac{2a}{e^{x/a}} (1 + e^{2x/a}).$$

$$\therefore \quad \frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{4a^2(1+e^{2x/a})^2}[1+e^{2x/a}]$$

$$= \frac{1}{4a^2(1+e^{2x/a})} = \frac{1}{2a\,C_x}.$$

Ex. 66. *Show that the chord of curvature through the focus of a parabola is four times the focal distance of the point and the chord of curvature parallel to the axis has the same length.* **(Lucknow 1979)**

Sol. Let the equation of the parabola referred to focus as pole and axis as initial line, (*i.e.*, x-axis) be

$$2a/r = 1 + \cos\theta. \qquad ...(1)$$

Taking log of both sides of (1), we get

$$\log 2a - \log r = \log(1+\cos\theta).$$

Differentiating w.r.t. θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = -\tan\tfrac{1}{2}\theta.$$

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \tan\tfrac{1}{2}\theta = \cot(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta);$$

so that $\qquad \phi = \frac{1}{2}\pi - \frac{1}{2}\theta.$

Now $\qquad p = r\sin\phi = r\sin(\frac{1}{2}\pi - \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta.$

$$\therefore \quad p^2 = r^2\cos^2\tfrac{1}{2}\theta = \tfrac{1}{2}r^2(2\cos^2\tfrac{1}{2}\theta) = \tfrac{1}{2}r^2(1+\cos\theta)$$

$$= \tfrac{1}{2}r^2(2a/r), \qquad \text{from (1).}$$

$\therefore$ the pedal equation of the parabola (1) is $p^2 = ar$.

Differentiating it with respect to p, we get

$$2p = a\frac{dr}{dp}; \quad \therefore \quad \frac{dr}{dp} = \frac{2p}{a}.$$

Now the chord of curvature through the focus

= the chord of curvature through the pole

$$= 2\rho\sin\phi = 2r\frac{dr}{dp}\cdot\frac{p}{r}, \qquad \left[\because\ p = r\sin\phi \text{ and } \rho = r\frac{dr}{dp}\right]$$

$$= 2p\frac{dr}{dp} = 2p\left(\frac{2p}{a}\right) = \frac{4p^2}{a} = \frac{4ar}{a} = 4r$$

$= 4 \times$ the distance of the point from the pole

$= 4 \times$ the focal distance of the point. **This proves the first result.**

Again $\qquad \psi = \theta + \phi.$

But for this curve $\phi = \frac{1}{2}\pi - \frac{1}{2}\theta$ or $\theta = \pi - 2\phi$.

$$\therefore \qquad \psi = \pi - 2\phi + \phi = \pi - \phi.$$

Now the chord of curvature parallel to the axis of the parabola = the chord of curvature parallel to x-axis

$$= 2\rho \sin \psi = 2\rho \sin (\pi - \phi) = 2\rho \sin \phi = 2r \frac{dr}{dp} \frac{p}{r} = 2p \frac{dr}{dp}$$

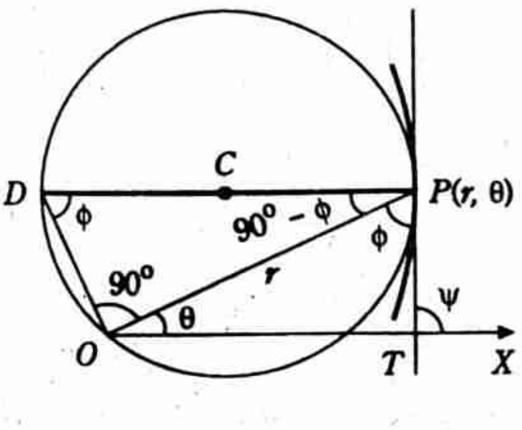
$$= 2p \left(\frac{2p}{a}\right) = \frac{4p^2}{a} = \frac{4ar}{a} = 4r$$

= four times the focal distance of the point.

Ex. 67. *Prove that the points on the curve $r = f(\theta)$, the circle of curvature at which passes through the origin are given by the equation*

$$f(\theta) + f''(\theta) = 0.$$

Sol. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ and let C be the centre of the circle of curvature at P. Also let the circle of curvature at P pass through the pole O.



Let PC produced meet the circle of curvature in D. Tangent at P, (*i.e.*, PT) makes an angle ϕ with the radius vector OP. Thus in the figure, $OP = r$, $PD = 2\rho$, $\angle OPT = \phi$.

From ΔPOD, we get

$$OP = PD \sin \phi \quad \text{or} \quad r = 2\rho \sin \phi. \qquad ...(1)$$

Now $\quad \tan \phi = r \dfrac{d\theta}{dr} = \dfrac{r}{r_1}, \qquad \left(\text{where } r_1 = \dfrac{dr}{d\theta}\right).$

$\therefore \quad \sin \phi = r/\sqrt{(r^2 + r_1^2)}. \qquad ...(2)$

Also $\quad \rho = \dfrac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}, \qquad \left(\text{where } r_2 = \dfrac{d^2r}{d\theta^2}\right) \qquad ...(3)$

Now putting the values of $\sin \phi$ and ρ from (2) and (3) in (1), we get

$$r = \frac{2(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \cdot \frac{r}{\sqrt{(r^2 + r_1^2)}} = \frac{2r(r^2 + r_1^2)}{r^2 + 2r_1^2 - rr_2}$$

or $\quad r^2 + 2r_1^2 - rr_2 = 2r^2 + 2r_1^2, \qquad [\because r \neq 0]$

or $\quad r^2 + rr_2 = 0 \quad \text{or} \quad r + r_2 = 0, \qquad [\because r \neq 0]$

or $\quad f(\theta) + f''(\theta) = 0, \qquad [\because r = f(\theta), r_2 = d^2r/d\theta^2 = f''(\theta)]$

Ex. 68. *If C_r and C_p be the chords of curvature of the cardioid, through the pole and perpendicular to the radius vector, then*

$$3(C_r^2 + C_p^2) = 8a \cdot C_r.$$

Sol. The equation of the cardioid is $r = a(1 + \cos \theta). \qquad ...(1)$
Taking log of both sides of (1), we get

$$\log r = \log a + \log (1 + \cos \theta).$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = -\tan\tfrac{1}{2}\theta.$$

$\therefore$ $\cot\phi = -\tan\frac{1}{2}\theta = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$; so that $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$.

Now $p = r\sin\phi = r\sin(\frac{1}{2}\pi + \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta$.

$\therefore$ $p^2 = r^2\cos^2\frac{1}{2}\theta = \frac{1}{2}r^2(2\cos^2\frac{1}{2}\theta) = \frac{1}{2}r^2(1+\cos\theta) = \frac{1}{2}r^2(r/a)$, from (1).

Thus the pedal equation of (1) is $p^2 = \dfrac{r^3}{2a}\cdot$

Differentiating it w.r.t. p, we get

$$2p = \frac{3r^2}{2a}\frac{dr}{dp}; \quad \therefore \quad \frac{dr}{dp} = \frac{4ap}{3r^2}.$$

Now $\quad \rho = r\dfrac{dr}{dp} = r\left(\dfrac{4ap}{3r^2}\right) = \dfrac{4ap}{3r}.$

$\therefore \quad C_r = 2\rho\sin\phi = \dfrac{8ap}{3r}\dfrac{p}{r} = \dfrac{8ap^2}{3r^2},$

and $\quad C_p = 2\rho\cos\phi.$

Now $C_r^2 + C_p^2 = (2\rho\sin\phi)^2 + (2\rho\cos\phi)^2 = 4\rho^2$

$$= 4\cdot\frac{16a^2p^2}{9r^2} = \frac{8a}{3}\cdot\frac{8ap^2}{3r^2} = \frac{8a}{3}C_r.$$

Hence $\quad 3(C_r^2 + C_p^2) = 8aC_r.$

Ex. 69. *At the point upon the Archimedean spiral $r = a\theta$ at which tangent makes half a right angle with the radius vector, prove that*

$$C_r = C_\theta = 4a/3,$$

where C_r and C_θ denote the chords of curvature through the pole and perpendicular to the radius vector.

Sol. The curve is $\quad r = a\theta.$ ...(1)

$\therefore \quad dr/d\theta = a, \quad d^2r/d\theta^2 = 0.$

Now $\quad \tan\phi = r\dfrac{d\theta}{dr} = r\cdot\dfrac{1}{a} = (a\theta)\cdot\dfrac{1}{a} = \theta.$

$\therefore$ at the point where $\phi = 45°$, we have $\tan\phi = 1$ and so $\theta = 1$.

Now $\quad \rho = \dfrac{[r^2 + (dr/d\theta)^2]^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}.$

Therefore at the point $\theta = 1$, we have

$\rho = \dfrac{(a^2+a^2)^{3/2}}{a^2+2a^2}, \qquad [\because$ on the curve $r = a\theta$, $r = a$ when $\theta = 1]$

$= (2\sqrt{2}a)/3.$

$$\therefore \quad C_r = 2\rho \sin\phi = 2 \cdot \frac{2\sqrt{2}a}{3} \sin 45^\circ = 2 \cdot \frac{2\sqrt{2}a}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4a}{3},$$

and
$$C_\theta = 2\rho \cos\phi = 2 \cdot \frac{2\sqrt{2}a}{3} \cos 45^\circ = 2 \cdot \frac{2\sqrt{2}a}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4a}{3}.$$

Hence $C_r = C_\theta = 4a/3.$

Ex. 70. *Show that in any curve the chord of curvature perpendicular to the radius vector is $2\rho\sqrt{(r^2 - p^2)}/r$.* **(Kanpur 1985)**

Sol. The chord of curvature perpendicular to the radius vector

$$= 2\rho \cos\phi = 2\rho\sqrt{(1 - \sin^2\phi)}$$

$$= 2\rho\sqrt{[1 - (p/r)^2]}, \qquad [\because p = r\sin\phi]$$

$$= [2\rho\sqrt{(r^2 - p^2)}]/r.$$

□

8

Asymptotes

§ 1. Definition. (Lucknow 1977; Nagpur 77)

I. *An asymptote is a straight line which cuts a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity.*

II. *An asymptote is a straight line, at a finite distance from the origin, to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity.*

§ 2. Determination of Asymptotes.

If $y = mx + c$ is an asymptote to the curve $y = f(x)$, to show that

$$\mathbf{m} = \lim_{\mathbf{x} \to \infty} (\mathbf{y}/\mathbf{x}).$$

Let $y = mx + c$ be an asymptote of the curve $y = f(x)$ or $f(x, y) = 0$ so that m and c are both finite.

The equation of the tangent at $P(x, y)$ to the curve $y = f(x)$ is

$$Y - y = (dy/dx)(X - x)$$

or
$$Y = (dy/dx) X + [y - x(dy/dx)]. \quad \text{...(1)}$$

From (1), as $x \to \infty$, $\dfrac{dy}{dx}$ and $y - x\dfrac{dy}{dx}$ must both tend to finite limits, say m and c, in order that an asymptote might exist. Thus the tangent (1) tends to the asymptote $y = mx + c$ if

$$\lim_{x \to \infty} \frac{dy}{dx} = m \text{ and } \lim_{x \to \infty} \left(y - x\frac{dy}{dx}\right) = c.$$

Then we have $\displaystyle\lim_{x \to \infty} \frac{y - x(dy/dx)}{x} = 0,$ [$\because$ c is finite]

or
$$\lim_{x \to \infty} \left(\frac{y}{x} - \frac{dy}{dx}\right) = 0 \text{ or } \lim_{x \to \infty} \frac{y}{x} = \lim_{x \to \infty} \frac{dy}{dx} = m.$$

Also
$$c = \lim_{x \to \infty} \left(y - x\frac{dy}{dx}\right) = \lim_{x \to \infty} (y - mx),$$

$$\left[\because \lim_{x \to \infty} \frac{dy}{dx} = m\right]$$

Therefore, if $\mathbf{y = mx + c}$ is an asymptote to the curve $\mathbf{y = f(x)}$, then

$$\mathbf{m} = \lim_{\mathbf{x} \to \infty} \frac{\mathbf{y}}{\mathbf{x}} \textbf{ and } \mathbf{c} = \lim_{\mathbf{x} \to \infty} (\mathbf{y} - \mathbf{mx}).$$

§ 3. The asymptotes of the general rational algebraic curve.

Let $f(x, y) = 0$ be the equation of any rational algebraic curve of the n^{th} degree.

Let the equation of the curve on being arranged in groups of homogeneous terms, be

$$a_0 y^n + a_1 y^{n-1} x + a_2 y^{n-2} x^2 + \dots + a_{n-1} y x^{n-1} + a_n x^n$$
$$+ b_1 y^{n-1} + b_2 y^{n-2} x + \dots + b_{n-1} y x^{n-2} + b_n x^{n-1}$$
$$+ c_2 y^{n-2} + \dots + \dots = 0. \qquad \text{...(1)}$$

This equation can also be written as

$$x^n \phi_n (y/x) + x^{n-1} \phi_{n-1} (y/x) + \dots = 0, \qquad \text{...(2)}$$

where $\phi_r (y/x)$ is a polynomial in y/x of degree r.

Let $y = mx + c$ be a line which is not parallel to the y-axis. It meets (2) in points whose abscissae are given by

$$x^n \phi_n \left(\frac{mx + c}{x}\right) + x^{n-1} \phi_{n-1} \left(\frac{mx + c}{x}\right) + \dots = 0$$

or $$x^n \phi_n \left(m + \frac{c}{x}\right) + x^{n-1} \phi_{n-1} \left(m + \frac{c}{x}\right) + \dots = 0.$$

Expanding each term of the type $\phi_r (m + c/x)$, by Taylor's theorem, we get

$$x^n [\phi_n (m) + (c/x) \phi_n' (m) + (c^2/2x^2) \phi_n'' (m) + \dots]$$
$$+ x^{n-1} [\phi_{n-1} (m) + (c/x) \phi_{n-1}' (m) + \dots]$$
$$+ x^{n-2} [\phi_{n-2} (m) + (c/x) \phi_{n-2}' (m) + \dots] + \dots = 0.$$

Arranging the terms according to the descending powers of x, we have

$$x^n \phi_n (m) + x^{n-1} [\phi_{n-1} (m) + c\phi_n' (m)]$$
$$+ x^{n-2} [\phi_{n-2} (m) + c\phi_{n-1}' (m) + \tfrac{1}{2} c^2 \phi_n'' (m)] + \dots = 0. \qquad \text{...(3)}$$

If $y = mx + c$ is an asymptote, this equation must have two infinite roots and consequently

$$\phi_n (m) = 0, \qquad \text{...(4)}$$

and $$\phi_{n-1} (m) + c\phi_n' (m) = 0, \qquad \text{...(5)}$$

as can be seen on dividing (3) by x^n.

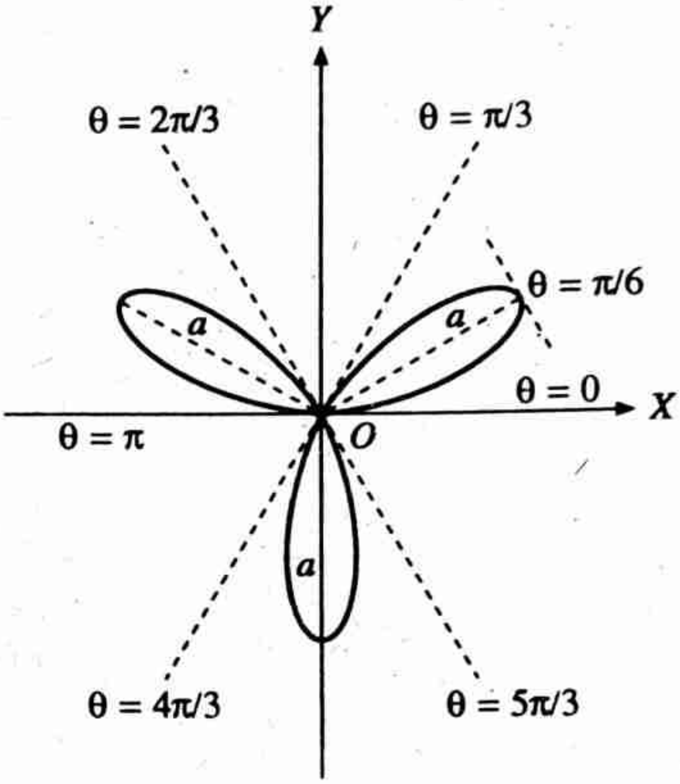
Solving (4) and (5) simultaneously, we get the values for m and c, and hence the asymptotes are determined by substituting the corresponding values of m and c in the equation $y = mx + c$.

Note. All the imaginary values of m will be rejected.

****Working rule for finding the asymptotes.**

(i) Substitute $mx + c$ for y in the equation of the curve and arrange it in descending powers of x.

Here one loop of the curve lies in the region $0 < \theta < \frac{\pi}{3}$, one loop in the region $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ and one loop lies in the region $\frac{2\pi}{3} < \theta < \pi$. If θ increases beyond π to 2π, the same branches of the curve are repeated and we do not get any new branch. Hence the complete curve is as shown in the adjoining figure.



Important Note. The above curve is a particular case of the curves of the type $r = a \sin n\theta$ which have n loops when n is odd and $2n$ loops when n is even.

Ex. 9. *Trace the curve $r = a + b \cos \theta$, when $a > b$.* **(Limacon)**

(Meerut 1991 P)

Sol. **(i)** The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $a + b \cos \theta = 0$ *i.e.*, $\theta = \cos^{-1}(-a/b)$.

Since $a > b$, therefore $(a/b) > 1$ and so $\cos^{-1}(-a/b)$ gives no real values of θ. Thus in the given curve, r cannot be equal to zero.

r is maximum when $\cos \theta = 1$ *i.e.*, $\theta = 0$. Then $r = a + b$.

Also r is minimum when $\cos \theta = -1$ *i.e.*, $\theta = \pi$. Then $r = a - b$, which is positive because $a > b$.

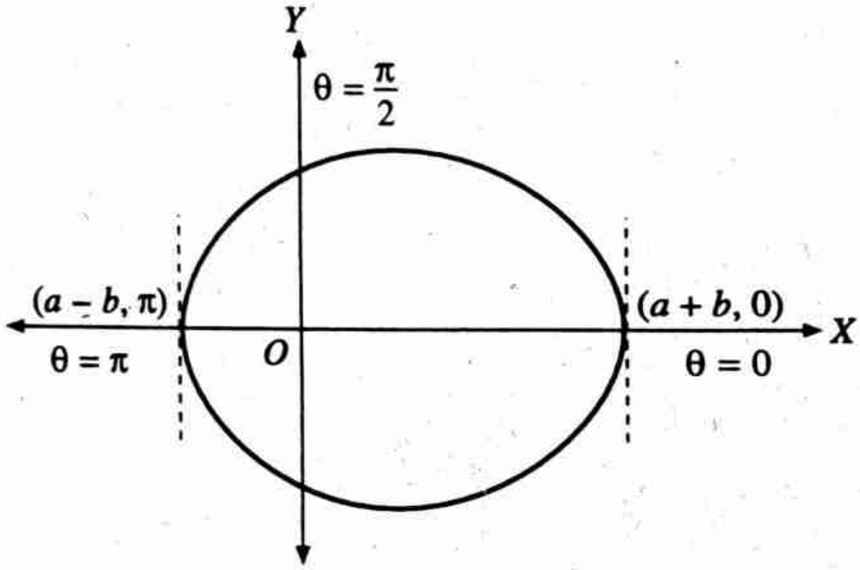
(iii) $\dfrac{dr}{d\theta} = -b \sin \theta$. When $0 < \theta < \pi, dr/d\theta$ is throughout negative. Therefore r decreases continuously as θ increases from 0 to π.

Also $\quad \tan \phi = r \dfrac{d\theta}{dr} = -\dfrac{a + b \cos \theta}{\sin \theta}$.

We have $\phi = 90^\circ$ when $\theta = 0$ and π. Therefore at the points $\theta = 0$ and $\theta = \pi$, the tangent to the curve is perpendicular to the radius vector.

(iv) The following table gives the corresponding values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$a+b$	$a+\frac{1}{2}b$	a	$a-\frac{1}{2}b$	$a-b$



The variation of θ from π to 2π need not be considered because of the symmetry about the initial line. Hence the curve is as shown in the figure.

***Ex. 10.** *Trace the curve $r = a + b\cos\theta$, when $a < b$.* **(Limacon)**

(Meerut 1989)

Sol. **(i)** The curve is symmetrical about the initial line.

(ii) $r = 0$ when $a + b\cos\theta = 0$ *i.e.*, $\theta = \cos^{-1}\left(-\frac{a}{b}\right)$.

Since $\frac{a}{b} < 1$, therefore $\cos^{-1}\left(-\frac{a}{b}\right)$ is real.

Therefore the radius vector $\theta = \cos^{-1}\left(-\frac{a}{b}\right)$ is tangent to the curve at the pole.

r is maximum when $\cos\theta = 1$, *i.e.*, $\theta = 0$. Then $r = a + b$.

Also r is minimum when $\cos\theta = -1$, *i.e.*, $\theta = \pi$.

Then $r = a - b$, which is negative, $(\because a < b)$.

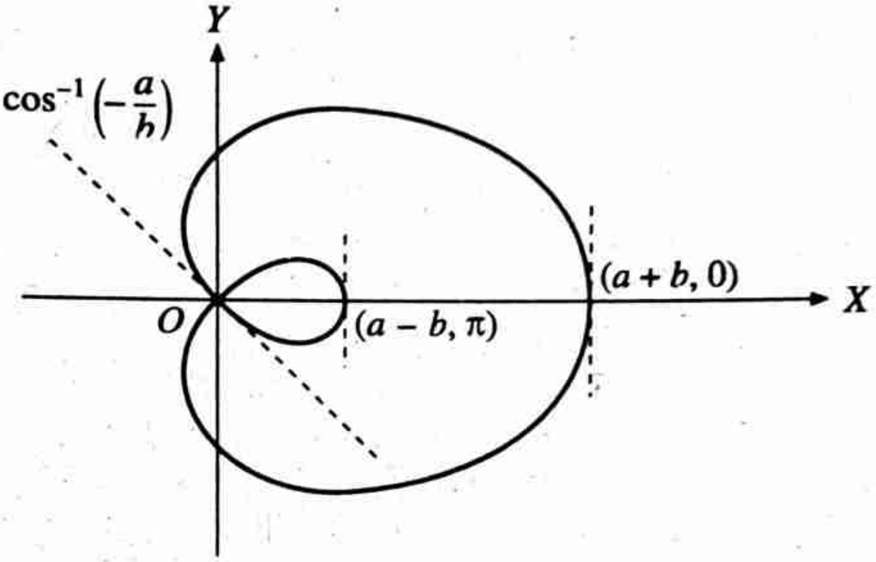
(iii) $\frac{dr}{d\theta} = -b\sin\theta$.

$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = -\frac{(a + b\cos\theta)}{b\sin\theta}$.

$\phi = 90^\circ$ when $\theta = 0$ and π. Therefore at the points $\theta = 0$ and $\theta = \pi$, the tangent is perpendicular to the radius vector.

(iv) The following table gives the corresponding values of r and θ.

θ	0	$\pi/2$	$\cos^{-1}\left(-\frac{a}{b}\right)$	$\cos^{-1}\left(-\frac{a}{b}\right) < \theta < \pi$	π
r	$a+b$	a	0	r is negative	$a-b$



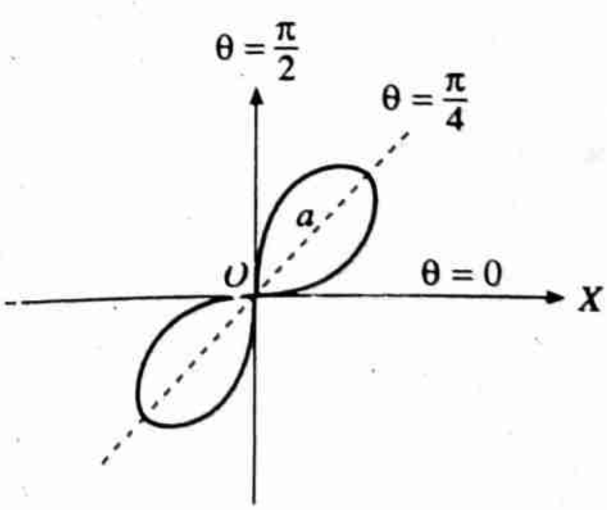
The variation of θ from π to 2π need not be considered because of the symmetry about the initial line. Hence the curve is as shown in the adjoining figure.

Ex. 11. *Trace the curve* $r^2 = a^2 \sin 2\theta$.

Sol. **(i)** The given curve is not symmetrical about the initial line, but it is symmetrical about the pole.

(ii) We have $r = 0$ when $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \pi/2$.

Thus two consecutive values of θ for which r is zero are 0 and $\pi/2$. Therefore one loop of the curve lies between the lines $\theta = 0$ and $\theta = \pi/2$ and both these lines are tangents to the curve at the pole.



(iii) r^2 is maximum when $\sin 2\theta = 1$ *i.e.*, $2\theta = \pi/2$ *i.e.*, $\theta = \pi/4$. When $\theta = \pi/4$, $r^2 = a^2$ or $r = \pm a$. Thus when $0 < \theta < \pi/2$, the greatest value of the radius vector of this curve is a and it occurs at $\theta = \pi/4$.

(iv) We have $2r \dfrac{dr}{d\theta} = 2a^2 \cos 2\theta$ so that $\dfrac{dr}{d\theta} = \dfrac{a^2 \cos 2\theta}{r}$.

$$\therefore \quad \cot \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r^2} = \frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = \cot 2\theta .$$

When $\theta = \pi/4, \cot \phi = \cot \frac{1}{2}\pi = 0$ so that $\phi = 90^\circ$.

Thus at the point $\theta = \pi/4$ the tangent to the curve is perpendicular to the radius vector.

(v) When $\pi < 2\theta < 2\pi$ *i.e.*, $\frac{1}{2}\pi < \theta < \pi$, $\sin 2\theta$ is < 0.

Thus when $\frac{1}{2}\pi < \theta < \pi$, r^2 is –ive *i.e.*, r is imaginary and so the given curve does not exist in the region $\frac{1}{2}\pi < \theta < \pi$.

Taking into consideration all the above facts the shape of the curve is as shown in the figure.

Ex. 12. *Trace the curve $r = a \sin 2\theta$.*

Sol. **(i)** The given curve is neither symmetrical about the initial line nor it is symmetrical about the pole.

(ii) We have $r = 0$ when $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ etc. *i.e.*, $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$ etc.

Thus two consecutive values of θ for which r is zero are 0 and $\pi/2$. Therefore one loop of the curve lies between the lines $\theta = 0$ and $\theta = \pi/2$ and both these lines are tangents to the curve at the pole.

(iii) r is maximum when $\sin 2\theta = 1$. Therefore when $0 < \theta < \pi/2, r$ is maximum when $2\theta = \pi/2$ *i.e.*, $\theta = \pi/4$. Thus when $0 < \theta < \pi/2$, the greatest value of the radius vector of this curve is a and it occurs at $\theta = \pi/4$.

(iv) We have $\dfrac{dr}{d\theta} = 2a \cos 2\theta$.

$$\therefore \quad \cot \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{2a \cos 2\theta}{a \sin 2\theta} = 2 \cot 2\theta .$$

When $\theta = \pi/4$, $\cot \phi = 2 \cot \frac{1}{2}\pi = 0$ so that $\phi = 90^\circ$.

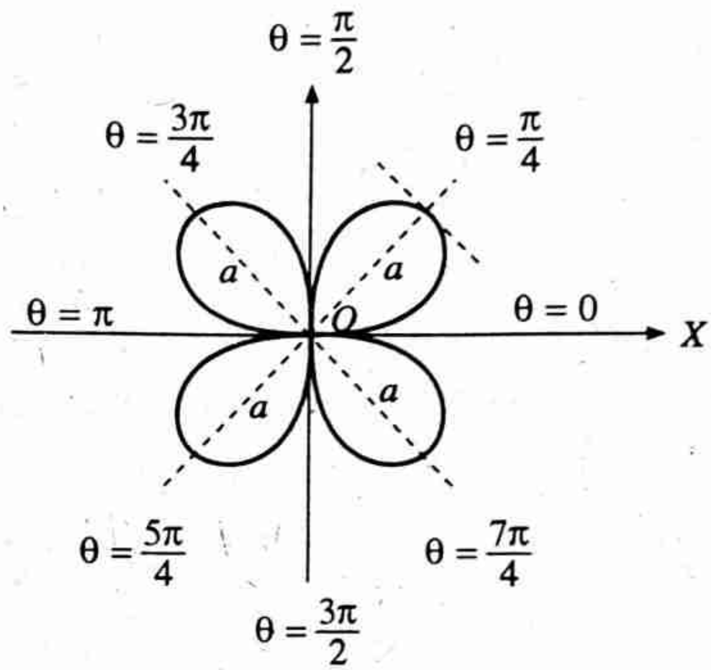
Thus at the point $\theta = \pi/4$ the tangent to the curve is perpendicular to the radius vector.

(v) The following table gives the corresponding values of $2\theta, \theta$ and r.

2θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0

We observe that the given curve consists of four similar loops which lie in the regions $0 < \theta < \pi/2, \pi/2 < \theta < \pi, \pi < \theta < 3\pi/2$ and $3\pi/2 < \theta < 2\pi$. All these four loops lie within a circle of radius a and centre at the pole.

When $\pi/2 < \theta < \pi$, we have $\pi < 2\theta < 2\pi$ so that $\sin 2\theta$ is negative. Thus when $\pi/2 < \theta < \pi$, r is –ive and so for these values of θ the points of the curve lie on the opposite side of the pole. Similarly when θ takes values between $3\pi/2$ and 2π, r is again negative and consequently for these values of θ also the curve lies on the opposite side of the pole.



Taking into consideration all the above facts the shape of the curve is as shown in the figure.

Ex. 13. *Trace the curve* $r = ae^{m\theta}$. **(Equiangular Spiral)**

Sol. (i) The curve is not symmetrical about the initial line.

(ii) As $\theta \to \infty, r \to \infty$ and as $\theta \to -\infty, r \to 0$. Also r is always positive. When $\theta = 0, r = a$.

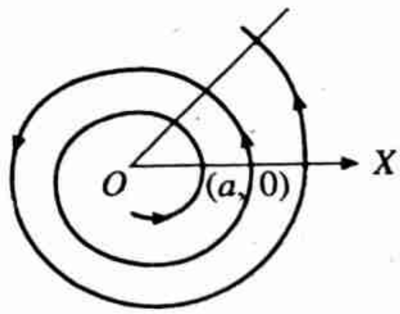
(iii) $\dfrac{dr}{d\theta} = ame^{m\theta}$.

When $-\infty < \theta < \infty, \dfrac{dr}{d\theta}$ is throughout positive. Therefore r increases continuously as θ increases from $-\infty$ to ∞.

(iv) $\tan \phi = r\dfrac{d\theta}{dr} = \dfrac{a\,e^{m\theta}}{ame^{m\theta}} = \dfrac{1}{m}$.

$\therefore \quad \phi = \tan^{-1}\left(\dfrac{1}{m}\right) = \text{constant}.$

Thus in this curve the angle between the radius vector and the tangent always remains constant. Hence the name of the curve.



Hence the shape of the curve is as shown in the adjoining diagram.

Ex. 14. *Trace the curve* $r^2 \cos 2\theta = a^2$. **(Meerut 1989 S)**

Sol. The given curve is $r^2 \cos 2\theta = a^2$ or $r^2 (\cos^2\theta - \sin^2\theta) = a^2$. Changing to cartesian coordinates by putting $r\cos\theta = x$ and $r\sin\theta = y$, the equation of the curve becomes

$$x^2 - y^2 = a^2, \qquad ...(1)$$

which is the equation of a rectangular hyperbola.

(i) The curve (1) is symmetrical about both the axes.

(ii) The curve does not pass through the origin.

(iii) The curve (1) cuts the x-axis where $y = 0$. Putting $y = 0$ in the equation (1), we get $x^2 = a^2$ or $x = \pm a$. Therefore the curve (1) cuts x-axis at the points $(a, 0)$ and $(-a, 0)$.

The curve (1) cuts the y-axis where $x = 0$. Putting $x = 0$ in the equation (1), we get $y^2 = -a^2$ or $y = \pm ia$ which are not real. Thus the curve (1) does not cut y-axis.

(iv) Tangent at (a, 0). Shifting the origin to $(a, 0)$, the equation of the curve becomes

$$(x + a)^2 - y^2 = a^2 \qquad \text{or} \qquad x^2 + 2ax - y^2 = 0.$$

$\therefore$ the tangent at the new origin is the line $2ax = 0$ *i.e.*, the line $x = 0$ *i.e.*, the new y-axis.

(v) Solving the equation of the curve for y, we have

$$y^2 = x^2 - a^2.$$

When $0 < x < a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x > a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region.

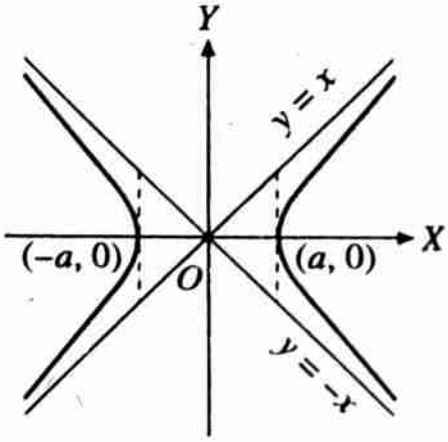
When $x \to \infty$, $y^2 \to \infty$.

(vi) Asymptotes. The curve has no asymptotes parallel to coordinate axes.

The equation of the given curve can be written as

$$y^2 - x^2 + a^2 = 0.$$

Putting $y = m$ and $x = 1$ in the highest *i.e.*, second degree terms, we get $\phi_2(m) = m^2 - 1$.



So the slopes of the asymptotes are given by the equation $\phi_2(m) = 0$ *i.e.*, $m^2 - 1 = 0$ which gives $m = \pm 1$.

Now c is given by the equation $c\,\phi'_2(m) + \phi_1(m) = 0$

i.e., $c \cdot 2m = 0$. [$\because \phi_1(m) = 0$, there being no first degree terms in the equation of the curve]

Putting $m = \pm 1$, we get $c = 0$.

$\therefore$ the asymptotes are the lines $y = x$ and $y = -x$.

The shape of the curve is as shown in the figure.

Ex. 15. *Trace the curve* $r = \frac{1}{2} + \cos 2\theta$. **(Meerut 1993)**

Sol. The given curve is symmetrical about the initial line.

In the given equation of the curve $r = \frac{1}{2} + \cos 2\theta$ putting $r = 0$, we get $\cos 2\theta = -\frac{1}{2}$ *i.e.*, $2\theta = \pm 2\pi/3$ or $\pm 4\pi/3$

i.e., $\theta = \pm \pi/3$ or $\pm 2\pi/3$.

The following table gives the corresponding values of r and θ.

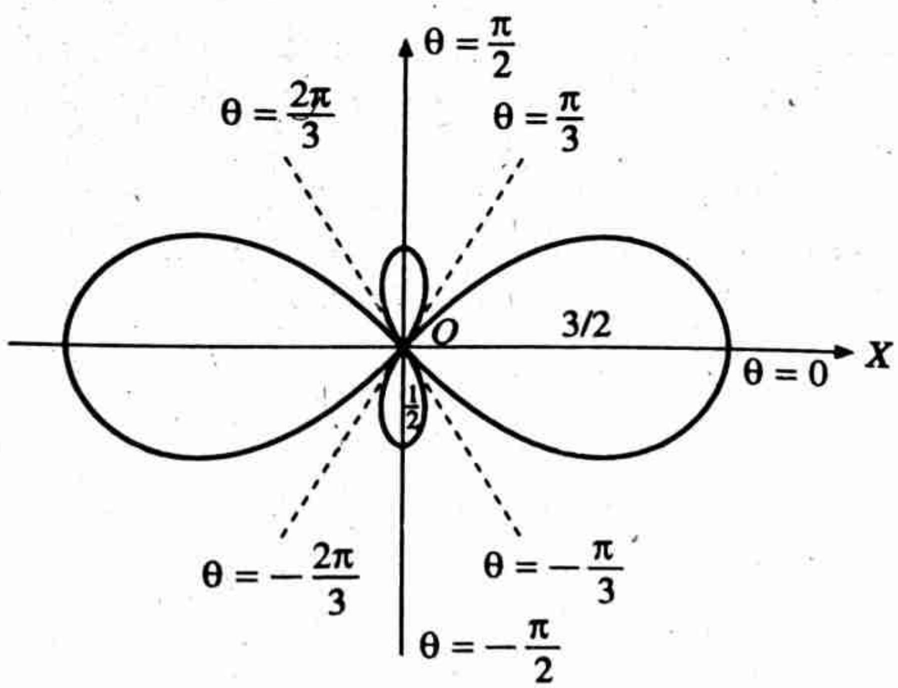
θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$

The greatest radius vector of the loop lying between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$ is given by $\theta = 0$ and it is equal to $3/2$. The greatest radius vector of the loop lying between $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$ is given by $\theta = \frac{1}{2}\pi$ and its absolute value is $\frac{1}{2}$.

Thus we observe that the larger loop lies between $\theta = -\pi/3$ and $\theta = \pi/3$ and is symmetrical about the initial line $\theta = 0$.

Also the smaller loop lies between $\theta = \pi/3$ and $\theta = 2\pi/3$.

We first trace the curve from $\theta = 0$ to $\theta = \pi$. The variation of θ from π to 2π need not be considered because of the symmetry about the initial line. The curve has four loops and is as shown in the figure.



§ 3. Parametric Equations.

If the equation to a curve is given in a parametric form, $x = f(t), y = \phi(t)$, then in some cases the curve can be easily traced by eliminating the parameter. But if it is not convenient to eliminate t, a series of values are given to t and the corresponding values of x, y and (dy/dx) are found. Then we plot the different points and observe the slopes of the tangents at these points given by the values of (dy/dx).

****Ex. 1.** *Trace the curve*

$$x = a(t + \sin t), y = a(1 - \cos t), \qquad \textbf{(Cycloid)}$$

when $-\pi \le t \le \pi$.

Sol. Here $\dfrac{dx}{dt} = a(1 + \cos t)$ and $\dfrac{dy}{dt} = a \sin t$.

Therefore $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a(1 + \cos t)} = \tan \dfrac{t}{2}$.

(i) $y = 0$, when $\cos t = 1$ *i.e.*, $t = 0$.

When $t = 0, x = 0, (dy/dx) = \tan 0 = 0$.

Therefore the curve passes through the origin and the axis of x is tangent at the origin.

(ii) y is maximum when $\cos t = -1$, *i.e.*, $t = \pi$ and $-\pi$. When $t = \pi, x = a\pi, y = 2a$ and $(dy/dx) = \infty$.

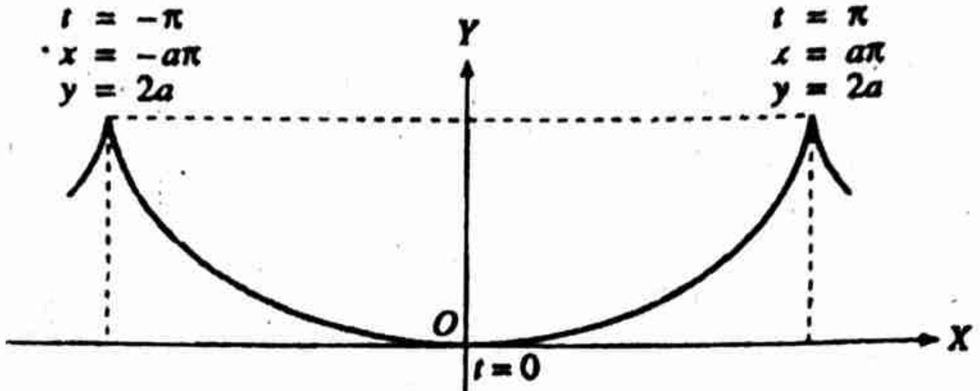
Therefore at the point $t = \pi$, whose cartesian coordinates are $(a\pi, 2a)$, the tangent is perpendicular to the x-axis. When $t = -\pi, x = -a\pi, y = 2a, (dy/dx) = -\infty$.

(iii) In this curve y cannot be negative. Therefore the curve lies entirely above the axis of x. Also no portion of the curve lies in the region $y > 2a$.

(iv) Corresponding values of x, y and (dy/dx) for different values of t are given in the following table :

t	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$-a\pi$	$-a(\frac{1}{2}\pi + 1)$	0	$a(\frac{1}{2}\pi + 1)$	$a\pi$
y	$2a$	a	0	a	$2a$
dy/dx	$-\infty$	-1	0	1	∞

If we put $-t$ in place of t in the equation of the curve, we get $x = -a(t + \sin t)$, and $y = a(1 - \cos t)$. Thus for every value of y, there are two equal and opposite values of x. Therefore the curve is symmetrical about the y-axis. Hence the shape of the curve is as shown



in the diagram. The portion of the cycloid included between two successive cusps is called an arch of the cycloid.

Ex. 2. *Trace the curve $x = a(t + \sin t), y = a(1 + \cos t)$.*

Sol. (i) Differentiating, we get

$$(dx/dt) = a(1 + \cos t)$$

and $$(dy/dt) = -a \sin t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 + \cos t)} = \frac{-2a \sin \frac{1}{2}t \cos \frac{1}{2}t}{2a \cos^2 \frac{1}{2}t} = -\tan \tfrac{1}{2}t.$$

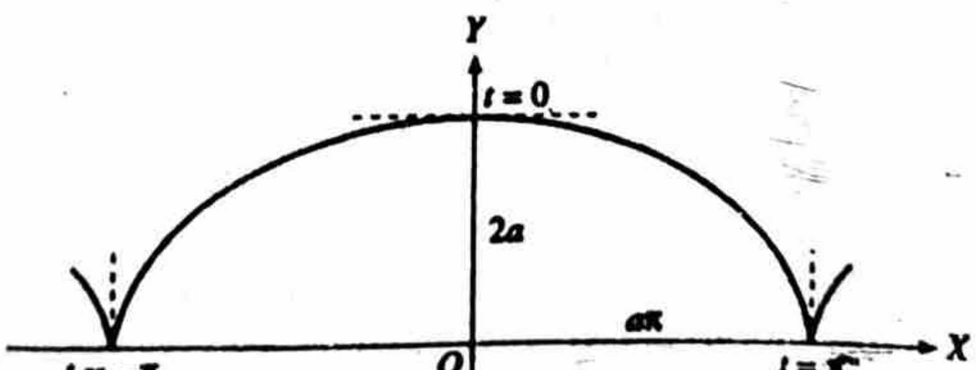
(ii) We have $y = 0$, when $\cos t = -1$ *i.e.*, $t = -\pi, \pi$. When $t = \pi, x = a\pi, dy/dx = -\infty$. Thus at the point $(a\pi, 0)$ the tangent to the curve is perpendicular to the x-axis. Again when $t = -\pi, x = -a\pi, dy/dx = \infty$.

(iii) y is maximum when $\cos t = 1$ *i.e.*, $t = 0$. When $t = 0, x = 0, y = 2a$ and $dy/dx = 0$. Thus at the point $(0, 2a)$ the tangent to the curve is parallel to the x-axis.

(iv) In this curve y cannot be negative. Also no portion of the curve lies in the region $y > 2a$.

(v) Corresponding values of x, y and (dy/dx) for different values of t are given in the following table :

t	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$-a\pi$	$-a(\frac{1}{2}\pi + 1)$	0	$a(\frac{1}{2}\pi + 1)$	$a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	$-\infty$



From above, $(-a\pi, 0)$ is a point on the curve with tangent inclined to x-axis at the angle $\psi = \pi/2$, ($\because \tan\psi = dy/dx = \infty$). Arguing as in Ex. 1, the curve is symmetrical about the y-axis.

Hence the shape of the curve is as shown in the figure.

Ex. 3. *Trace the curve* $x = a(t - \sin t),\ y = a(1 - \cos t)$.

(Meerut 1995)

Sol. The parametric equations of the given cycloid are

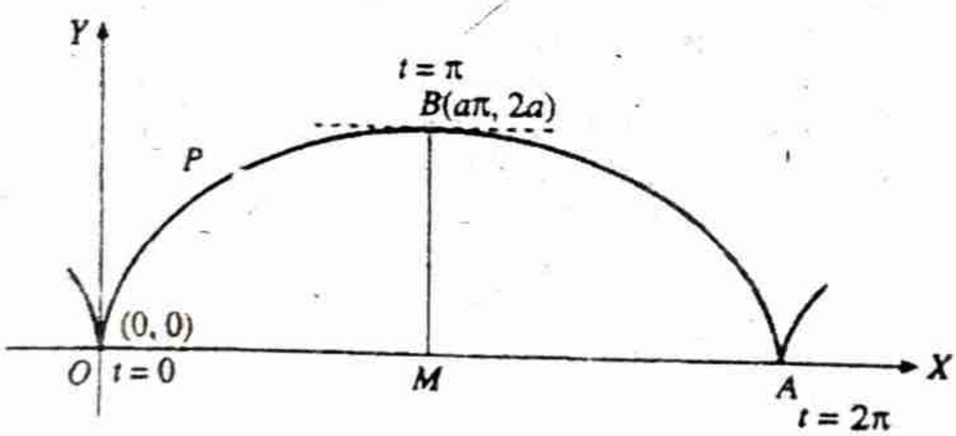
$$x = a(t - \sin t), y = a(1 - \cos t).$$

We have $dx/dt = a(1 - \cos t),\ dy/dt = a\sin t$.

$$\therefore\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\sin^2\frac{1}{2}t} = \cot\tfrac{1}{2}t.$$

In this curve $y = 0$ when $\cos t = 1$ *i.e.*, $t = 0, 2\pi$. When $t = 0$, $x = a(0 - \sin 0) = 0, y = 0$ and $dy/dx = \cot 0 = \infty$. Thus the curve passes through the point $(0, 0)$ and the axis of y is tangent to the curve at this point.

In this curve y is maximum when $\cos t = -1$ *i.e.*, $t = \pi$. When $t = \pi, x = a(\pi - \sin\pi) = a\pi, y = 2a, dy/dx = \cot\frac{1}{2}\pi = 0$. Thus at the point $t = \pi$, whose cartesian coordinates are $(a\pi, 2a)$, the tangent to the curve is parallel to x-axis. This curve does not exist in the region $y > 2a$.



In this curve y cannot be –ive because $\cos t$ cannot be greater than 1. Thus one complete arch of the given cycloid lying between $0 \le t \le 2\pi$ is as shown in the figure.

***Ex. 4.** *Trace the curve*

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,\ y = a\sin t. \qquad \textbf{(Tractrix)}$$

Sol. (i) If we put $-t$ in place of t in the equation of the curve, we get $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{1}{2}t$, and $y = -a\sin t$. Thus for every value of x, there are two equal and opposite values of y. Therefore the curve is symmetrical about the x-axis.

Again if we put $\pi - t$ in place of t in the equation of the curve, we get

$$x = -a\cos t + \tfrac{1}{2}a\log\cot^2\tfrac{1}{2}t = -a\cos t - \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,$$

and $\quad y = a\sin t.$

Thus for every value of y, there are two equal and opposite values of x. Therefore the curve is symmetrical about the y-axis.

(ii) Differentiating the equations of the curve w.r.t. 't', we get

$$\frac{dx}{dt} = -a\sin t + \tfrac{1}{2}a\frac{1}{\tan^2\frac{1}{2}t}\cdot\left(2\tan\tfrac{1}{2}t\sec^2\tfrac{1}{2}t\right)\cdot\tfrac{1}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{1}{2}t\cos\frac{1}{2}t} = -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t},$$

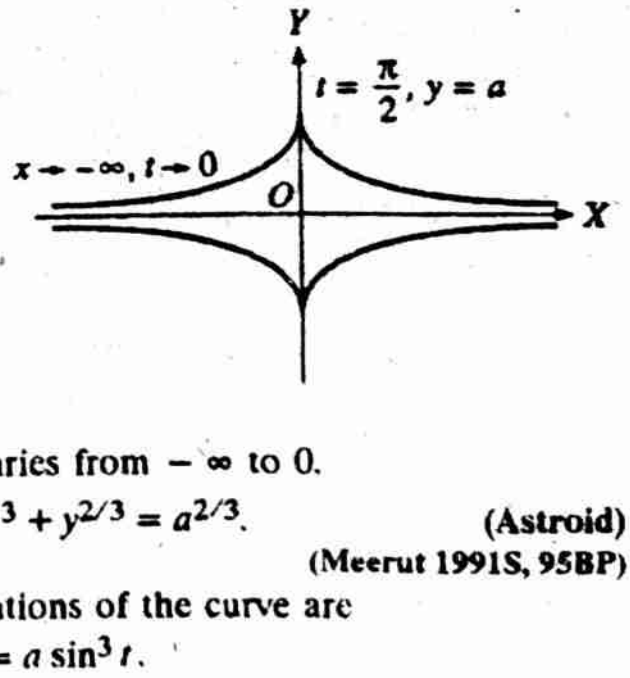
and $\quad dy/dt = a\cos t.$

Therefore $dy/dx = \tan t$.

(iii) We have $y = 0$ when $\sin t = 0$ *i.e.*, $t = 0$. When $t \to 0, x \to -\infty$. Thus $x \to -\infty$ when $y \to 0$, showing that the line $y = 0$ is an asymptote of the curve.

(iv) y is maximum when $\sin t = 1$ *i.e.*, $t = \frac{1}{2}\pi$. When $t = \frac{1}{2}\pi$, $x = 0, y = a$ and $dy/dx = \tan\frac{1}{2}\pi = \infty$. Thus the curve passes through the point $(0, a)$ and the tangent at this point is the y-axis.

(v) In this curve the numerical value of y cannot be greater than a. Thus the curve does not exist in the regions $y > a$ and $y < -a$. The shape of the curve is as shown in the figure. The curve has four infinite branches and for the branch in the second quadrant t varies from 0 to $\frac{1}{2}\pi$ while x varies from $-\infty$ to 0.



Ex. 5. *Trace the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. **(Astroid)**

(Meerut 1991S, 95BP)

Sol. The parametric equations of the curve are

$$x = a\cos^3 t, y = a\sin^3 t.$$

We have $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\dfrac{3a\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\tan t.$

Also the equation of the curve can be written as

$$\left(\frac{x^2}{a^2}\right)^{1/3} + \left(\frac{y^2}{a^2}\right)^{1/3} = 1.$$

We observe the following facts about the curve.

(i) The curve is symmetrical about both the axes. It is also symmetrical about the line $y = x$.

(ii) The curve does not pass through the origin.

(iii) The curve cuts the x-axis, where $y = 0$

i.e., $\left(\frac{x^2}{a^2}\right)^{1/3} = 1$ *i.e.*, $\frac{x^2}{a^2} = 1$ *i.e.*, $x = \pm a$.

Thus the curve cuts the x-axis at $(a, 0)$ and $(-a, 0)$. Similarly the curve crosses the y-axis at $(0, a)$ and $(0, -a)$.

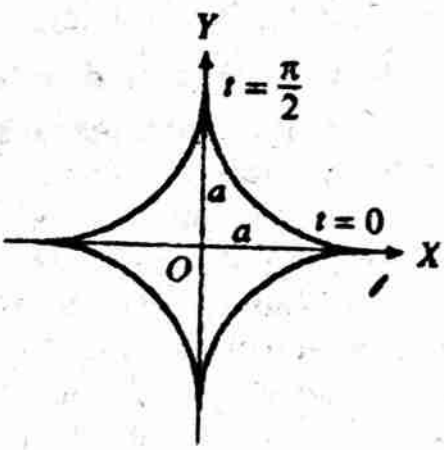
(iv) At the point $(a, 0)$, we have $x = a$. Therefore $\cos^3 t = 1$ and thus $t = 0$. When $t = 0$, $\frac{dy}{dx} = 0$.

Hence at the point $(a, 0)$, the x-axis is tangent to the curve.

Again at the point $(0, a)$, we have $y = a$. Therefore $\sin^3 t = 1$ and thus $t = \frac{\pi}{2}$. When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = -\infty$.

Hence at the point $(0, a)$, the y-axis is tangent to the curve.

(v) The values of $\sin t$ and $\cos t$ cannot numerically exceed 1. Therefore in this curve the values of x and y cannot numerically exceed a. Therefore the entire curve lies in the region bounded by the lines $x = a, x = -a, y = a$ and $y = -a$.



Hence the shape of the curve is as shown in the figure. □

1

Functions of a Real Variable, Limits, Continuity and Differentiability

§ 1. Definitions.

Differential Calculus is the study of the rates of change of continuously varying co-related quantities and with the help of these rates, it studies the problems and properties connected with such quantities.

A constant is a quantity which, during any set of mathematical operations, retains the same value.

A variable is a quantity, or a symbol representing a number, which is capable of assuming different values.

A continuous variable is a variable that can take all the numerical values between two given numbers.

An independent variable is one which may take up any arbitrary value that may be assigned to it.

A dependent variable is a symbol which can assume its value as a result of some other variable taking some assigned value.

Functions. If one quantity depends upon another, so that it assumes a definite value when a system of definite values is given to other *i.e.*, if a symbol y has one definite value for every value of the variable x, **then y is called a function of x** and is written as $y = f(x)$, or $y = F(x)$, or $y = \phi(x)$ etc.

The function itself is a *dependent variable*, and the variables to which values are given are *independent variables*. Thus $u = f(x, y, z)$ represents that u is a function of several variables x, y, z.

Domain of a variable x consists of the totality of values that it can take. If a variable x takes all real values between a and b, its domain is represented by the interval (a, b). If a and b belong to the domain (a, b), it is said to be a **closed domain** or a **closed interval** and if a and b do not belong to the domain (a, b), then it is an **open domain** or an **open interval.**

Explicit and Implicit functions. A function is said to be *explicit* when it is expressed directly in terms of the independent variable or variables. Thus a function is explicit if it is written as $y = f(x)$. But a function y which instead of being written in the form $y = f(x)$ is written in the form $f(x, y) = 0$ is called an *implicit* function of x.

Even and odd functions. A function $f(x)$ is said to be an *even function of* x if $f(-x) = f(x)$ and an *odd function of* x if

$$f(-x) = -f(x).$$

Algebraic functions. Functions which consist of a finite number of terms involving powers and root of the independent variable, are called algebraic functions *e.g.*, $(x^2 + \sqrt{x})$, $2x^2 + 9x + 7$, $x^{7/2}$, etc.

Transcendental functions. All non-algebraic functions are called transcendental functions. For example, $\sin x$, $\sqrt{\tan x}$, $x^2 + \cos x$, $\sin^{-1} x$, e^x, 3^x, $\log x$, $\log(1 + x)$ etc. are transcendental functions.

§ 2. Limit. Definition. **(Delhi 1983, 81; Kashmir 83; Allahabad 72; Meerut 80)**

A function $f(x)$ is said to tend to the limit A as x tends to a, if, corresponding to any arbitrary chosen positive number ε, however small (but not zero), there exists a positive number δ, such that

$$| f(x) - A | < \varepsilon,$$

for all values of x for which $0 < | x - a | < \delta$.

We write it as $\lim_{x \to a} f(x) = A$.

Limit on the right : A function $f(x)$ is said to tend to the limit A as $x \to a$ **from the right**, if, corresponding to any arbitrary chosen +ive number ε, however small (but not zero), there exists a +ive number δ, such that

$$| f(x) - A | < \varepsilon,$$

for all values of x for which $0 < x - a < \delta$.

The limit of $f(x)$ as $x \to a$ from the right is called the *right hand limit* of $f(x)$ and is denoted by $\lim_{x \to a + 0} f(x)$ or by $f(a + 0)$.

We calculate $f(a + 0)$ by evaluating $\lim_{h \to 0} f(a + h)$, where h is +ive and sufficiently small.

Limit on the left. A function $f(x)$ is said to tend to the limit A as $x \to a$ from the left, if corresponding to any arbitrary chosen +ive number ε, however small (but not zero), there exists a +ive number δ, such that

$$| f(x) - A | < \varepsilon,$$

for all values of x for which $0 < a - x < \delta$.

The limit of $f(x)$ as $x \to a$ from the left is called the *left hand limit* of $f(x)$ and is denoted by $\lim_{x \to a - 0} f(x)$ or by $f(a - 0)$. We calculate $f(a - 0)$ by evaluating $\lim_{h \to 0} f(a - h)$, where h is +ive and sufficiently small.

Note 1. $\lim_{x \to a} f(x)$ exists only if, limit on the left = limit on the right.

Note 2. If $f_1(x) \to A$ and $f_2(x) \to B$, as $x \to a$, where A and B are both finite, we have

(i) $\lim_{x \to a} \{f_1(x) \pm f_2(x)\} = A \pm B$,

(ii) $\lim_{x \to a} \{f_1(x) \cdot f_2(x)\} = A \cdot B$,

(iii) $\lim_{x \to a} \left\{\dfrac{f_1(x)}{f_2(x)}\right\} = \dfrac{A}{B}$, provided $B \neq 0$.

Note 3. The limit of $f(x)$ as $x \to a$ is not necessarily the same as the value of the function at $x = a$. In fact the limit of $f(x)$ as $x \to a$ may exist even if the function $f(x)$ is not defined at $x = a$.

Examples on Limits

Ex. 1. *The limit of a sum is equal to the sum of the limits.*

Sol. Let $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$.

Let ε be any arbitrary chosen positive number.

We can choose δ_1, δ_2 such that

$$|f_1(x) - A| < \tfrac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta_1,$$

and

$$|f_2(x) - B| < \tfrac{1}{2}\varepsilon, \text{ when } 0 < |x - a| < \delta_2.$$

Let δ be any positive number smaller than both δ_1 and δ_2. Then

$$|f_1(x) - A| < \tfrac{1}{2}\varepsilon \text{ and } |f_2(x) - B| < \tfrac{1}{2}\varepsilon, \quad \text{for } 0 < |x - a| < \delta.$$

Now from algebra, we have

$$|\{f_1(x) + f_2(x)\} - (A + B)| = |\{f_1(x) - A\} + \{f_2(x) - B\}|$$
$$\leq |f_1(x) - A| + |f_2(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ i.e., } < \varepsilon.$$

$\therefore \quad \lim_{x \to a} \{f_1(x) + f_2(x)\} = A + B.$

Note. Similarly we can prove that

$$\lim_{x \to a} [f_1(x) - f_2(x)] = A - B.$$

Ex. 2. *The limit of a product is equal to the product of the limits.* **(Meerut 1975)**

Sol. Let $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$. We shall prove the result when $A \neq 0$ and $B \neq 0$. A similar proof can be given when $A = 0$ or $B = 0$ or $A = 0$ and $B = 0$.

We can write $f_1(x) f_2(x) - AB$

$$= [f_1(x) - A][f_2(x) - B] + A[f_2(x) - B] + B[f_1(x) - A].$$

$\therefore$ by algebra, we have $|f_1(x) f_2(x) - AB|$

$$\leq |f_1(x) - A| \,.\, |f_2(x) - B| + |A| \,.\, |f_2(x) - B| + |B| \,.\, |f_1(x) - A|$$

Now $\lim_{x \to a} f_1(x) = A$ and $\lim_{x \to a} f_2(x) = B$. Therefore, for a given ε, we can take δ such that

$$|f_1(x) - A| < \frac{\varepsilon}{3|B|} \text{ and } |f_2(x) - B| < \frac{\varepsilon}{3|A|},$$

when $0 < |x - a| < \delta$. Then

$$|f_1(x) f_2(x) - AB| < \frac{\varepsilon}{3} \cdot \frac{\varepsilon}{3|A||B|} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.$$

Since only small values of ε need by considered, we can take ε such that $\frac{\varepsilon}{3|A||B|} < 1$. Then $|f_1(x) f_2(x) - AB| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ *i.e.*, $< \varepsilon$, when $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} f_1(x) \cdot f_2(x) = AB$.

Ex. 3. *Show that* $\lim_{x \to 2} \frac{|x-2|}{x-2}$ *does not exist.*

Sol. Let $f(x) = |x - 2| / (x - 2)$.

Here the right hand limit *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{|2+h-2|}{(2+h-2)} = 1,$$

and left hand limit *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{|2-h-2|}{(2-h-2)}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1.$$

Since $f(2+0) \neq f(2-0)$, therefore $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

Ex. 4. *Evaluate* $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$. **(Delhi 1981)**

Sol. We have

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)}$$

$$= \lim_{x \to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \frac{(1 + x)^{1/3} - (1 - x)^{1/3}}{x}$. **(Delhi 1980)**

Sol. We have

$$\lim_{x \to 0} \frac{(1 + x)^{1/3} - (1 - x)^{1/3}}{x} = \lim_{h \to 0} \frac{(1 + h)^{1/3} - (1 - h)^{1/3}}{h}$$

$$= \lim_{h \to 0} \frac{1 + \frac{1}{3} h + \dots - (1 - \frac{1}{3} h + \dots)}{h} = \lim_{h \to 0} \frac{(2/3)\, h + \dots}{h} = \frac{2}{3}.$$

Ex. 6. *Evaluate the following limits if they exist :*

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. **(Delhi 1982)**

Sol. Let $f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2}$.

We have, Right Hand Limit *i.e.*, $f(2 + 0) = \lim_{h \to 0} f(2 + h)$

$$= \lim_{h \to 0} \frac{(2 + h - 2)(2 + h + 2)}{2 + h - 2} = \lim_{h \to 0} \frac{h(h + 4)}{h}$$

$$= \lim_{h \to 0} (h + 4) = 4.$$

Also, Left Hand Limit *i.e.*, $f(2 - 0) = \lim_{h \to 0} f(2 - h)$

$$= \lim_{h \to 0} \frac{(2 - h - 2)(2 - h + 2)}{2 - h - 2} = \lim_{h \to 0} \frac{-h(-h + 4)}{-h}$$

$$= \lim_{h \to 0} (-h + 4) = 4.$$

Since $f(2 + 0) = f(2 - 0)$, therefore $\lim_{x \to 2} f(x)$ exists and is equal to 4.

(b) $\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$. Proceed as in part (a).

Here the limit exists and is equal to 1.

(c) $\lim_{x \to 2} \frac{x^2 + 3x + 2}{x - 2}$.

Sol. Let $f(x) = \frac{x^2 + 3x + 2}{x - 2}$.

We have, right hand limit *i.e.*, $f(2 + 0) = \lim_{h \to 0} f(2 + h)$

$$= \lim_{h \to 0} \frac{(2 + h)^2 + 3(2 + h) + 2}{2 + h - 2} = \lim_{h \to 0} \frac{12 + 7h + h^2}{h}$$

$$= \lim_{h \to 0} \left(\frac{12}{h} + 7 + h\right) = \infty.$$

Also, left hand limit *i.e.*, $f(2-0) = \lim_{h \to 0} f(2-h)$

$$= \lim_{h \to 0} \frac{(2-h)^2 + 3(2-h) + 2}{2-h-2} = \lim_{h \to 0} \frac{12 - 7h + h^2}{-h}$$

$$= \lim_{h \to 0} \left(-\frac{12}{h} + 7 - h\right) = -\infty.$$

Since $f(2+0) \neq f(2-0)$, therefore $\lim_{x \to 2} f(x)$ does not exist.

(d) $\lim_{x \to 0} \dfrac{1+3x^2}{x}$. Let $f(x) = \dfrac{1+3x^2}{x}$.

We have R.H.L. *i.e.*, $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{1+3h^2}{h} = \lim_{h \to 0} \left(\frac{1}{h} + 3h\right) = \infty.$$

Again L.H.L. *i.e.*, $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{1+3h^2}{-h} = \lim_{h \to 0} \left(-\frac{1}{h} - 3h\right) = -\infty.$$

Since $f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} f(x)$ does not exist.

(e) *Prove that* $\lim_{x \to 0} (1+x)^{1/x} = e$.

Sol. Let $f(x) = (1+x)^{1/x}$. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1+h)^{1/h}$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{h} \cdot h + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)}{1.2} h^2 + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)\left(\frac{1}{h} - 2\right)}{1.2.3} h^3 + \ldots\right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{1!} + \frac{1.(1-h)}{2!} + \frac{1.(1-h)(1-2h)}{3!} + \ldots\right]$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (1-h)^{-1/h} = e.$$

Since both $f(0+0)$ and $f(0-0)$ exist and are equal to e, therefore $\lim_{x \to 0} (1+x)^{1/x} = e$.

(f) $\lim_{x \to 0} \frac{\sin x}{x}$ **(Lucknow 1982, 79)**

Sol. Let $f(x) = \frac{\sin x}{x}$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin h}{h}$

$$= \lim_{h \to 0} \frac{h - \frac{h^3}{3!} + \frac{h^5}{5!} - \ldots\ldots}{h} = \lim_{h \to 0} \left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \ldots\ldots\right) = 1.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Since $f(0+0) = f(0-0) = 1$, therefore $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

(g) $\lim_{x \to \infty} \frac{\sin x}{x}$.

Sol. Put $x = 1/y$. Then $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{y \to 0} y \sin(1/y)$.

Now let $f(y) = y \sin(1/y)$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin(1/h) = 0,$$

since $\sin(1/h)$ lies between -1 and 1.

Also $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (-h) \sin(-1/h) = \lim_{h \to 0} h \sin(1/h) = 0.$$

Since both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero, therefore $\lim_{y \to 0} y \sin(1/y) = 0$. Hence $\lim_{x \to \infty} \frac{\sin x}{x} = 0$.

(h) $\lim_{x \to 0} \sin \frac{1}{x}$. **(Delhi 1975)**

Sol. Let $f(x) = \sin(1/x)$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \sin \frac{1}{h}.$$

As $h \to 0$, the value of $\sin(1/h)$ oscillates between $+1$ and -1, passing through zero and intermediate values an infinite number of times. Thus there is no definite number A to which $\sin(1/h)$ tends as h tends to zero. Therefore the right hand limit $f(0+0)$ does not exist.

Similarly the left hand limit $f(0-0)$ also does not exist. Hence $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

(i) $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ · Proceed as in part (h). This limit also does not exist.

(j) $\lim_{x \to 0} \frac{e^x - 1}{x}$.

Sol. Let

$$f(x) = \frac{e^x - 1}{x} = \frac{1 + x + \frac{x^2}{2!} + \ldots - 1}{x} = \frac{x\left(1 + \frac{x}{2!} + \ldots\right)}{x}.$$

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{h\,[1 + (h/2!) + \ldots]}{h} = \lim_{h \to 0} \left(1 + \frac{h}{2!} + \ldots\right) = 1.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{-h\,[1 - (h/2!) + \ldots]}{-h} = 1.$$

Thus $f(0+0) = f(0-0) = 1$. Therefore $\lim_{x \to 0} (e^x - 1)/x = 1$.

(k) $\lim_{x \to 0} \frac{a^x - 1}{x}$ · Let $f(x) = \frac{a^x - 1}{x}$

$$= \frac{1 + x \log a + (x^2/2!)(\log a)^2 + \ldots - 1}{x}$$

$$= \frac{x\left[\log a + \frac{1}{2}x(\log a)^2 + \ldots\right]}{x}.$$

Now

$$f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h\left[\log a + \frac{h}{2}(\log a)^2 + \ldots\right]}{h} = \log a.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \log a$.

Thus $f(0+0) = f(0-0) = \log a$. Therefore

$$\lim_{x \to 0} \frac{(a^x - 1)}{x} = \log a.$$

(l) $\lim_{x \to 0} \frac{1}{x} . e^{1/x}$.

Sol. Let $f(x) = \frac{1}{x} e^{1/x}$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{h} e^{1/h} = \infty.$$

[Since both $(1/h)$ and $e^{1/h}$ tend to ∞ as $h \to 0$].

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -\frac{1}{h} e^{-1/h}$$

$$= \lim_{h \to 0} \frac{-1}{h\, e^{1/h}} = \lim_{h \to 0} \frac{-1}{h\left(1 + \frac{1}{h} + \frac{1}{2!}\frac{1}{h^2} + \dots\right)}$$

$$= \lim_{h \to 0} \frac{-1}{h + 1 + (1/2h) + \dots} = 0.$$

Since both the limits $f(0+0)$ and $f(0-0)$ are unequal, therefore $\lim_{x \to 0} \frac{1}{x} e^{1/x}$ does not exist.

(m) $\lim_{x \to 0} e^{-1/x}$. Let $f(x) = e^{-1/x}$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} e^{-1/h}$

$$= \lim_{h \to 0} \frac{1}{e^{1/h}} = 0.$$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} e^{1/h} = \infty$.

Since $f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} e^{-1/x}$ does not exist.

(n) $\lim_{x \to 0} \frac{1}{1 - e^{1/x}}$. Let $f(x) = \frac{1}{1 - e^{1/x}}$.

We have $f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{1}{1 - e^{1/h}} = 0.$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$

$$= \lim_{h \to 0} \frac{1}{1 - (1/e^{1/h})} = 1.$$

$\because \quad f(0+0 \neq f(0-0), \quad \therefore \quad \lim_{x \to 0} \frac{1}{1 - e^{1/x}}$ does not exist.

(o) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$. Let $f(x) = \frac{(1+x)^n - 1}{x}$.

We have $f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{(1+h)^n - 1}{h}$

$$= \lim_{h \to 0} \frac{1 + nh + \frac{n(n-1)}{2!} h^2 + \dots - 1}{h}$$

$$= \lim_{h \to 0} \frac{h\left[n + \frac{n(n-1)}{2!} h + \dots\right]}{h}$$

$$= \lim_{h \to 0} \left[n + \frac{n(n-1)}{2!} h + \dots\right] = n.$$

Again $f(0-0) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{(1-h)^n - 1}{-h}$

$$= \lim_{h \to 0} \frac{1 + n(-h) + \frac{n(n-1)}{2!}(-h)^2 + \dots - 1}{-h} = n.$$

Thus $f(0+0) = f(0-0) = n$. Therefore $\lim_{x \to 0} f(x) = n$.

(p) $\lim_{x \to a} \frac{x^m - a^m}{x - a}$. Let $f(x) = \frac{x^m - a^m}{x - a}$.

We have $f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \frac{(a+h)^m - a^m}{a + h - a}$

$$= \lim_{h \to 0} \frac{a^m\left[\left(1 + \frac{h}{a}\right)^m - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{a^m}{h}\left[1 + m \cdot \frac{h}{a} + \frac{m(m-1)}{2!}\frac{h^2}{a^2} + \dots - 1\right]$$

$$= \lim_{h \to 0} a^m \left[\frac{h}{a} + \frac{m(m-1)}{2}\frac{h}{a^2} + \dots\right] = a^m \cdot \frac{m}{a} = ma^{m-1}.$$

Again $f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{(a-h)^m - a^m}{a - h - a} = ma^{m-1}$.

Thus $f(a+0) = f(a-0) = ma^{m-1}$.

Therefore $\lim_{x \to a} f(x) = ma^{m-1}$.

Ex. 7. *Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case :*

(*i*) $\lim_{x \to 2} \frac{2x^2 - 8}{x - 2}$; (*ii*)* $\lim_{x \to 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$;

(Delhi 1983, 79)

(*iii*) $\lim_{x \to 0} f(x)$, *where* $f(x)$ *is defined as*

$f(x) = x$, *when* $x > 0$

$f(x) = 0,$ when $x = 0$
$f(x) = -x,$ when $x < 0.$

Sol. **(i)** Let $f(x) = \dfrac{2x^2 - 8}{x - 2}.$

Here the right hand limit, *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{2(2+h)^2 - 8}{2 + h - 2}$$

$$= \lim_{h \to 0} \frac{2(4 + 4h + h^2) - 8}{h} = \lim_{h \to 0} \frac{8h + 2h^2}{h} = \lim_{h \to 0} \frac{h(8 + 2h)}{h}$$

$$= \lim_{h \to 0} (8 + 2h) = 8.$$

Again the left hand limit, *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{2(2-h)^2 - 8}{2 - h - 2}$$

$$= \lim_{h \to 0} \frac{2(4 - 4h + h^2) - 8}{-h} = \lim_{h \to 0} \frac{-8h + 2h^2}{-h}$$

$$= \lim_{h \to 0} \frac{-h(8 - 2h)}{-h} = \lim_{h \to 0} (8 - 2h) = 8.$$

Since $f(2+0) = f(2-0) = 8$, therefore $\lim_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$ exists and is equal to 8.

(ii) Let $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}.$

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h \to 0} \frac{e^{1/h}[1 - (1/e^{1/h})]}{e^{1/h}[1 + (1/e^{1/h})]} = \lim_{h \to 0} \frac{1 - (1/e^{1/h})}{1 + (1/e^{1/h})} = 1.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h \to 0} \frac{(1/e^{1/h}) - 1}{(1/e^{1/h}) + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Since $f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

(iii) Here the right hand limit *i.e.*, $f(0+0) = \lim_{h \to 0} f(0+h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h, \qquad [\because h > 0 \text{ and } f(x) = x \text{ when } x > 0]$

$= 0.$

Again the left hand limit *i.e.*, $f(0 - 0) = \lim_{h \to 0} f(0 - h),$

where h is +ive and sufficiently small

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h),$

$[\because -h < 0 \text{ and } f(x) = -x \text{ when } x < 0]$

$= \lim_{h \to 0} h = 0.$

Since both the limits $f(0 + 0)$ and $f(0 - 0)$ exist and are equal to zero, therefore $\lim_{x \to 0} f(x)$ exists and is equal to zero.

Ex. 8. *Define the limit of a function. Discuss the existence of the limit of the function* $f(x) = 1$, *if* $x < 1$; $f(x) = 2 - x$, *if* $1 < x < 2$; $f(x) = 2$, *if* $x \geq 2$ *at* $x = 1$ *and* $x = 2$. **(Delhi 1982)**

Sol. For definition of limit see § 2 on page 2.

At x = 1, we have, right hand limit

i.e., $\quad f(1 + 0) = \lim_{h \to 0} f(1 + h),$

where h is +ive and sufficiently small

$= \lim_{h \to 0} [2 - (1 + h)] = \lim_{h \to 0} (1 - h) = 1;$

and left hand limit *i.e.*, $f(1 - 0) = \lim_{h \to 0} f(1 - h) = \lim_{h \to 0} (1) = 1.$

Since R.H.L. = L.H.L. = 1, therefore $\lim_{x \to 1} f(x)$ exists and is equal to 1.

At x = 2, we have, right hand limit

i.e., $\quad f(2 + 0) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} (2) = 2;$

and left hand limit

i.e., $f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} [2 - (2 - h)]$

$= \lim_{h \to 0} h = 0.$

$\because$ left hand limit $\neq$ right hand limit,

$\therefore$ $\lim_{x \to 2} f(x)$ does not exist.

§ 3. **Continuity. Definition.** **(Meerut 1983, 82, 81; Delhi 83, 81, 80)**

A function $f(x)$ defined for $x = a$ is said to be continuous at $x = a$ if

(i) *$f(a)$ i.e., the value of $f(x)$ at $x = a$ is a definite number.*

(ii) *the limit of the function $f(x)$ as $x \to a$ exists and is equal to the value of $f(x)$ at $x = a$.*

Otherwise the function is **discontinuous** at $x = a$.

A function is said to be continuous in an interval (a, b) if it is continuous at every point of that interval.

Arithmetical Definition of Continuity. *A function $f(x)$ is said to be continuous at $x = a$, if for any arbitrarily chosen positive number ε, however small (but not zero), we can find a corresponding number δ such that*

$$|f(x) - f(a)| < \varepsilon$$

for all values of x for which $|x - a| < \delta$.

This definition, as given by Cauchy, is known as Cauchy's definition of continuity.

Note. On comparing the definitions of limit and continuity, we find that a function $f(x)$ is continuous at $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

Thus $f(x)$ is continuous at $x = a$ if we have $f(a + 0) = f(a - 0) = f(a)$; otherwise it is discontinuous at $x = a$.

§ 4. **Differentiability. Definition.** **(Lucknow 1970; Mysore 71; Delhi 82, 80)**

A function $f(x)$ is said to be differentiable at $x = a$ if both

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},\ h > 0 \text{ and } \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h},\ h > 0,$$

exist and have a common value (finite or infinite). This common value is called the **derivative** of $f(x)$ at the point $x = a$ and is denoted by $f'(a)$. Another name for derivative is *differential coefficient.*

Progressive and regressive derivatives.

Progressive derivative or *the right hand differential coefficient* of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},\ h > 0.$$

It is denoted by $Rf'(a)$.

Regressive derivative or *the left hand differential coefficient* of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}, h > 0.$$

It is denoted by $Lf'(a)$.

A function $f(x)$ is said to be differentiable at $x = a$ if both $Rf'(a)$ and $Lf'(a)$ exist and are equal; otherwise it is said to be non-differentiable.

A necessary condition for the existence of a finite derivative.

Theorem. *Continuity is a necessary but not a sufficient condition for the existerce of a finite derivative.* **(Delhi 1982, 77, Meerut 86 S)**

Proof. Let $f(x)$ have a finite derivative $f'(a)$ at $x = a$. Then

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a). \quad \text{...(1)}$$

Now $f(a+h) - f(a) = \dfrac{f(a+h) - f(a)}{h} \times h, h \neq 0.$

Taking limit of both sides when $h \to 0$, we get

$$\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times \lim_{h \to 0} h$$

$$= f'(a) \times 0, \quad \text{[from (1)]}$$

$$= 0, \text{ since } f'(a) \text{ is finite.}$$

Now $\quad \lim_{h \to 0} [f(a+h) - f(a)] = 0$

implies $\lim_{h \to 0} f(a+h) = f(a)$. ...(2)

From the condition (2) we see that $f(x)$ is necessarily continuous at $x = a$. Thus continuity is a necessary condition for the existence of a finite derivative.

In order to show that continuity is not a sufficient condition for differentiability, we should give an example of a function which is continuous at a point but is not differentiable at that point. The students can select one such example out of the many that are given ahead. The function given in Ex. 1 (a) is a very good illustration of a continuous function which is not differentiable.

Examples on Continuity and Differentiability

****Ex. 1 (a).** *Show that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$, where $|x|$ means the numerical value of x.* **(Kashmir 1983; Meerut 1981, 75; Delhi 81, 74, 73)**

Sol. To test the continuity of f (x) at x = 0.

We have $f(0) = |0| = 0$;

$$f(0+h) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h|$$

$$= \lim_{h \to 0} h = 0;$$

and $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0} |-h|$

$$=\lim_{h\to 0} h=0.$$

Since $f(0+0)=f(0-0)=f(0)$, therefore $f(x)$ is continuous at $x=0$.

To test the differentiability of f (x) at x = 0.

We have, $Rf'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$, where h is +ive and sufficiently small

$$=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{|h|-|0|}{h}=\lim_{h\to 0}\frac{h}{h}$$

$$=\lim_{h\to 0} 1=1;$$

and $Lf'(0)=\lim_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}$, h being +ive and sufficiently small

$$=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}=\lim_{h\to 0}\frac{|-h|-|0|}{-h}=\lim_{h\to 0}\frac{h-0}{-h}$$

$$=\lim_{h\to 0}(-1)$$

$$=-1.$$

Since $Rf'(0)\neq Lf'(0)$, therefore the function $f(x)$ is not differentiable at $x=0$.

Ex. 1 (b). *Show that a function which is derivable at $x=a$ is necessarily continuous at $x=a$ but the converse is not true.* **(Delhi 1977)**

Sol. For the Ist part of the question see theorem on § 4 and for the second part see Ex. 1 (a).

Ex. 2. *Show that $f(x)=(x^2-1)/(x-1)$ is continuous for all values of x except $x=1$.* **(Meerut 1977)**

How may this function be defined to make it continuous at $x=1$?

Sol. Let us first test the continuity of $f(x)$ at $x=a$ where $a\neq 1$. We have

$f(a)=\dfrac{a^2-1}{a-1}$ = a definite real number because $a-1\neq 0$;

$$f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\frac{(a+h)^2-1}{a+h-1}=\frac{a^2-1}{a-1};$$

and $f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\dfrac{(a-h)^2-1}{a-h-1}=\dfrac{a^2-1}{a-1}.$

Thus $f(a+0) = f(a-0) = f(a)$. Therefore $f(x)$ is continuous at $x = a$ where $a \neq 1$.

From the formula defining $f(x)$, we have $f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$ which is meaningless. Thus $f(x)$ has no definite value at $x = 1$. Therefore $f(x)$ is not continuous at $x = 1$. To make $f(x)$ continuous at $x = 1$, we shall find $\lim_{x \to 1} f(x)$. We have

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \frac{(1+h)^2 - 1}{1+h-1} = \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2+h)}{h} = \lim_{h \to 0} (2+h) = 2;$$

and $$f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{(1-h)^2 - 1}{1-h-1}$$
$$= \lim_{h \to 0} \frac{-2h + h^2}{-h}$$
$$= \lim_{h \to 0} (2-h) = 2.$$

Thus $\lim_{x \to 1} f(x) = 2$. So if we take $f(x) = \dfrac{(x^2-1)}{(x-1)}$ when $x \neq 1$ and $f(x) = 2$ when $x = 1$, it will become continuous at $x = 1$.

Note. In a similar manner we can show that $f(x) = \dfrac{(x^2 - a^2)}{(x-a)}$ is continuous for all values of x except for $x = a$. **(Meerut 1969)**

Ex. 3. *Test the following functions for continuity* :

(a) $f(x) = x^3$ *at* $x = 2$.

Sol. We have $f(2) = 2^3 = 8$;

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^3 = 2^3 = 8;$$

and $$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} (2-h)^3 = 2^3 = 8.$$

Since $f(2) = f(2+0) = f(2-0)$, therefore $f(x)$ is continuous at $x = 2$.

(b) $f(x) = x \sin(1/x)$, $x \neq 0$ *and* $f(0) = 0$ *at* $x = 0$.

(Delhi 1983, 80; Meerut 74; Allahabad 74)

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin\frac{1}{h} = 0.$$

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (-h) \sin\left(-\frac{1}{h}\right) = 0.$$

Also $f(0) = 0$. Thus $f(0+0) = f(0-0) = f(0)$. Therefore $f(x)$ is continuous at $x = 0$.

Note. If we are given that $f(0) = 2$, the above function becomes discontinuous at $x = 0$.

(c) $f(x) = \sin(1/x), x \neq 0$ *and* $f(0) = 0$ *at* $x = 0$. **(Meerut 1973)**

Sol. As shown in Ex. 3 part (h) on page 7, neither the left hand limit nor the right hand limit of the function $\sin(1/x)$ exists at $x = 0$. Hence the function $\sin(1/x)$ is discontinuous at $x = 0$.

(J) $f(x) = 2^{1/x}$, *when* $x \neq 0$; $f(0) = 0$ *at* $x = 0$.

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 2^{1/h} = \infty.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} 2^{-1/h} = 2^{-\infty} = 0.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is doscontinuous at $x = 0$.

Note. A function $f(x)$ is said to have an **infinite discontinuity** at a point $x = a$, if one or both of the limits $f(a+0)$ and $f(a-0)$ are infinite.

(e) $f(x) = \dfrac{1}{1 - e^{-1/x}}, x \neq 0; f(0) = 0$ *at* $x = 0$.

Sol. We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$$

$$= \frac{1}{1 - e^{-\infty}} = \frac{1}{1 - 0} = 1.$$

Again

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{1}{1 - e^{1/h}}$$

$$= \frac{1}{1 - e^{\infty}} = \frac{1}{1 - \infty} = 0.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is discontinuous at $x = 0$.

(f) $f(x) = x \cos(1/x)$ *for* $x \neq 0$; $f(0) = 0$ *at* $x = 0$.

Sol. We have

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)=\lim_{h\to 0} h\cos\frac{1}{h}=0.$$

Again $\quad f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0} -h\cos\left(\frac{-1}{h}\right)=0.$$

Also $f(0)=0$. Thus $f(0+0)=f(0-0)=f(0)$. Therefore $f(x)$ is continuous at $x=0$.

Ex. 4. *Define continuity of a function at a point and in an interval. Discuss the continuity of*
$f(x)=2, x\le 0; f(x)=3x+2, 0<x\le 1; f(x)=x/(x-1), 1<x$ *at* $x=0$ *and at* $x=1$. **(Delhi 1983, 81)**

Sol. For definition of continuity see § 3 on page 13.

(i) To test f (x) for continuity at x = 0. We have

$f(0)=2$; Also right hand limit $=f(0+0)=\lim_{h\to 0} f(0+h)$

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0}(3h+2)=2$$

and left hand limit $=f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0}(2)=2.$$

Since L.H.L. = R.H.L. $=f(0)=2$, the function is continuous at $x=0$.

(ii) To test f (x) for continuity at x = 1. We have
$f(1)=3.1+2=5$.

Also right hand limit $=f(1+0)=\lim_{h\to 0} f(1+h)$

$$=\lim_{h\to 0}\frac{1+h}{(1+h)-1}=\lim_{h\to 0}\frac{1+h}{h}=\infty,$$

and left hand limit $=f(1-0)=\lim_{h\to 0} f(1-h)$

$$=\lim_{h\to 0}[3(1-h)+2]=\lim_{h\to 0}[5-3h]=5.$$

Since right hand limit $\ne$ left hand limit, the function is not continuous at $x=1$.

Ex. 5. *Discuss the continuity of the function* $f(x)$ *at* $x=\frac{1}{2}$ *where*
$f(x)=x, 0\le x<\frac{1}{2}; f(x)=1, x=\frac{1}{2}; f(x)=1-x, \frac{1}{2}<x\le 1$. **(Delhi 1979)**

Sol. We have $f(x)=1$, when $x=\frac{1}{2}$.

No· R.H.L. $= \lim_{h \to 0} f(\frac{1}{2}+h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [1-(\frac{1}{2}+h)]$, $[\because f(x) = 1-x$ when $\frac{1}{2} < x \le 1]$

$= \lim_{h \to 0} (\frac{1}{2}-h) = \frac{1}{2}$,

and L.H.L. $= \lim_{h \to 0} f(\frac{1}{2}-h)$

$= \lim_{h \to 0} (\frac{1}{2}-h) = \frac{1}{2}$, $[\because f(x) = x$ when $0 \le x < \frac{1}{2}]$

Since R.H.L. = L.H.L. $\neq$ the value of $f(x)$ at $x = \frac{1}{2}$, therefore the function is discontinuous at $x = \frac{1}{2}$.

****Ex. 6.** *A function $\phi(x)$ is defined as follows :*

$$\begin{array}{lll} \phi(x) = 0 & \textit{for} & x = 0 \\ \phi(x) = \frac{1}{2} - x & \textit{for} & 0 < x < \frac{1}{2} \\ \phi(x) = \frac{1}{2} & \textit{for} & x = \frac{1}{2} \\ \phi(x) = \frac{3}{2} - x & \textit{for} & \frac{1}{2} < x < 1 \\ \phi(x) = 1 & \textit{for} & x = 1. \end{array}$$

Find the points of discontinuity. Also draw the graph of the function.

(Meerut 1990; Lucknow 79; Delhi 77)

Sol. We shall test the function for continuity at $x = 0, \frac{1}{2}$ and 1.

(i) For $x = 0$, we have $\phi(0) = 0$, $\phi(0+0) = \lim_{h \to 0} \phi(0+h)$

$= \lim_{h \to 0} \phi(h) = \lim_{h \to 0} (\frac{1}{2}-h) = \frac{1}{2}$. Since $\phi(0) \neq \phi(0+0)$, the function $\phi(x)$ is discontinuous at $x = 0$.

(ii) For $x = \frac{1}{2}$, we have $\phi(\frac{1}{2}) = \frac{1}{2}$, $\phi(\frac{1}{2}-0) = \lim_{h \to 0} \phi(\frac{1}{2}-h)$

$= \lim_{h \to 0} [\frac{1}{2}-(\frac{1}{2}-h)]$, (Note that $0 < \frac{1}{2} - h < \frac{1}{2}$)

$= \lim_{h \to 0} h = 0$.

Since $\phi(\frac{1}{2}-0) \neq \phi(\frac{1}{2})$, the function $\phi(x)$ is discontinuous at $x = \frac{1}{2}$.

(iii) For $x = 1$, we have $\phi(1) = 1$, $\phi(1-0) = \lim_{h \to 0} \phi(1-h)$

$= \lim_{h \to 0} [3/2-(1-h)]$, [Note that $\frac{1}{2} < 1 - h < 1$]

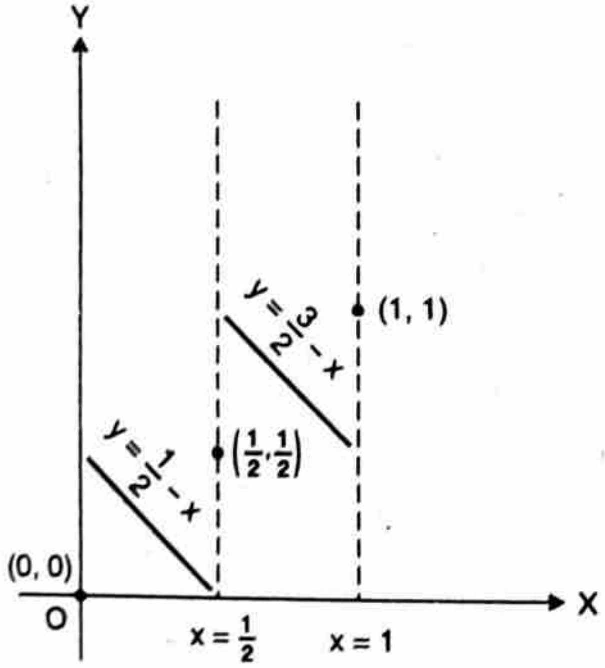
$= \lim_{h \to 0} (\frac{1}{2}+h) = \frac{1}{2}$.

Since $\phi(1) \neq \phi(1-0)$, the function $\phi(x)$ is discontinuous at $x = 1$.

Hence the function $\phi(x)$ has three points of discontinuity at $x = 0, \frac{1}{2}$ and 1.

The graph of the function consists of the point $(0, 0)$; the segment of the line $y = \frac{1}{2} - x,\ 0 < x < \frac{1}{2}$;
the point $(\frac{1}{2}, \frac{1}{2})$; the segment of the line

$$y = (3/2) - x,\ \tfrac{1}{2} < x < 1;\ \text{and the point } (1, 1).$$



Thus the graph is as shown in the figure. From the graph we observe that the function is discontinuous at $x = 0, \frac{1}{2}$ and 1.

Ex. 7. *Define continuity of a function $f(x)$ at $x = a$. Discuss the continuity of the function $f(x)$ defined as :*

$$f(0) = 0; f(x) = 1 - x,\ 0 < x < 1; f(1) = 1;$$
$$f(x) = 2 - x,\ 1 < x < 2; f(2) = 0$$

at the points $x = 0, 1, 2$. **(Delhi 1980)**

Sol. For definition of continuity refer § 3 page 13.

(i) To test f(x) for continuity at x = 0. We have $f(0) = 0$.

Now $f(0+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1-h)$

$[\because f(x) = 1 - x \text{ for } 0 < x < 1]$

$= 1$.

As $f(0+0) \neq f(0)$, therefore $f(x)$ is discontinuous at $x = 0$.

(ii) To test f(x) for continuity at x = 1. We have $f(1) = 1$.

Now $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [1-(1-h)]$

$[\because f(x) = 1 - x \text{ for } 0 < x < 1]$

$= 0,$

and $f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [2-(1+h)],$

$[\because f(x) = 2 - x, \text{ for } 1 < x < 2]$

$= 1.$

$\because$ $f(1-0) \neq f(1+0)$, therefore $f(x)$ is discontinuous at $x = 1$.

(iii) To test f (x) for continuity at x = 2. We have $f(2) = 0$.

Now $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2-(2-h)],$

$[\because f(x) = 2 - x \text{ for } 1 < x < 2]$

$= 0.$

Since $f(2) = f(2-0)$, $f(x)$ is continuous at $x = 2$.

Note that here $f(x)$ is not defined for $x < 2$. So the question of finding $f(2+0)$ does not arise. Here $f(x)$ will be continuous at $x = 2$ if $f(2) = f(2-0)$.

Ex. 8. *Discuss the points of discontinuity of the function given by :*

$$f(x) = \begin{cases} -x, & \text{for } x \leq 0 \\ x, & \text{for } 0 < x \leq 1 \\ 2-x, & \text{for } 1 < x \leq 2 \\ 1, & \text{for } x > 2. \end{cases}$$ **(Meerut 1973)**

Sol. Here we shall test $f(x)$ for continuity at the points $x = 0$, 1 and 2. Obviously at all other points the function is continuous.

(i) Continuity at x = 0. We have $f(0) = 0$, because $f(x) = -x$ for $x = 0$.

Now left hand limit of $f(x)$ as $x \to 0$ *i.e.,*

$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} [-(-h)],$

$[\because f(x) = -x \text{ or } x < 0]$

$= \lim_{h \to 0} (h) = 0;$

and right hand limit of $f(x)$ as $x \to 0$ *i.e.,*

$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h,$

$[\because f(x) = x \text{ for } 0 < x \leq 1]$

$= 0.$

Thus $f(0-0)=f(0+0)=f(0)$ and so $f(x)$ is continuous at $x=0$.

(ii) Continuity at x = 1. We have $f(1)=1$, because $f(x)=x$ for $0<x\le 1$.

Now left hand limit of $f(x)$ as $x\to 1$ *i.e.*,

$$f(1-0)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}(1-h),$$

$$[\because f(x)=x \text{ for } 0<x\le 1]$$

$$=1;$$

and $$f(1+0)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}[2-(1+h)],$$

$$[\because f(x)=2-x \text{ for } 0<x\le 2]$$

$$=\lim_{h\to 0}(1-h)=1.$$

Since $f(1-0)=f(1+0)=f(1)$, therefore $f(x)$ is continuous at $x=1$.

(iii) Continuity at x = 2. We have $f(2)=2-2=0$, because $f(x)=2-x$, for $1<x\le 2$.

Now left hand limit of $f(x)$ as $x\to 2$ *i.e.*,

$$f(2-0)=\lim_{h\to 0} f(2-h)=\lim_{h\to 0}[2-(2-h)],$$

$$[\because f(x)=2-x, \text{ for } 1<x\le 2]$$

$$=\lim_{h\to 0}(h)=0;$$

and $$f(2+0)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}(1),$$

$$[\because f(x)=1 \text{ for } x>2]$$

$$=1.$$

Since $f(2+0)\ne f(2-0)$, the function $f(x)$ is discontinuous at $x=2$.

Ex. 9. *Determine the continuity of the function $f(x)$ defined as follows :*

$$f(0)=0;\ f(x)=\tfrac{1}{2}-x,\ 0<x<\tfrac{1}{2};\ f(\tfrac{1}{2})=0;$$
$$f(x)=4x^2-1,\ \tfrac{1}{2}<x<3/4;\ f(x)=1-x^2,\ 3/4\le x\le 1.$$

(Meerut 1982)

Sol. Proceed exactly as in Ex. 8. The function is discontinuous at $x=0,\ 3/4$.

Ex. 10 (a). *A function $f(x)$ is defined as follows :*

$$f(x)=\begin{cases}(x^2/a)-a, & \text{when } x<a\\ 0, & \text{when } x=a\\ a-(a^2/x), & \text{when } x>a.\end{cases}$$

Prove that the function $f(x)$ is continuous at $x=a$. **(Delhi 1983, 78)**

Sol. We have $f(a)=0$.

Also the right hand limit of $f(x)$ at $x=a$ *i.e.*,

$$f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\left[a-\frac{a^2}{(a+h)}\right],$$

$$[\because f(x)=a-(a^2/x) \text{ for } x>a]$$

$$=a-(a^2/a)=a-a=0;$$

and the left hand limit of $f(x)$ at $x=a$ *i.e.*,

$$f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\left[\frac{(a-h)^2}{a}-a\right],$$

$$[\because f(x)=(x^2/a)-a \text{ for } x<a]$$

$$=(a^2/a)-a=a-a=0.$$

$\because$ $f(a+0)=f(a-0)=f(a)$, therefore $f(x)$ is continuous at $x=a$.

Ex. 10 (b). *Discuss the continuity at $x=a$ of the function $f(x)$ given by :—*

$$\begin{aligned} f(x)&=(x^2/a)-a, && \text{for } 0<x<a\\ &=0, && \text{for } x=a\\ &=a-(a^3/x^2), && \text{for } x>a.\end{aligned}$$

Sol. Proceed exactly as in Ex. 10 (a).

Here $f(a+0)=f(a-0)=f(a)=0$, therefore the given function is continuous at $x=a$.

Ex. 11. *Examine for continuity at $x=0$ of the following functions*

(*i*) $f(x)=\cos(1/x)$, *when $x\neq 0$ and $f(0)=0$.* **(Meerut 1981)**

(*ii*) $f(x)=\sin x\cos(1/x)$, *when $x\neq 0$ and $f(0)=0$.*

Sol. (i) We have $f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$$=\lim_{h\to 0}\cos(1/h).$$

As $h\to 0$ the value of $\cos(1/h)$ oscillates between $+1$ and -1. Obviously there is no definite number A to which $\cos(1/h)$ tends as $h\to 0$. Thus $f(0+0)$ does not exist. Therefore $f(x)$ is discontinuous at $x=0$.

(ii) We have $f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$=\lim_{h\to 0}\sin h\cos(1/h)=0\times$ a number which oscillates between -1 and $+1=0$.

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \sin(-h)\cos(-1/h)$$

$$= -\lim_{h \to 0} \sin h \cos(1/h) = 0.$$

Also $f(0) = 0$. Thus $f(0+0) = f(0-0) = f(0)$. Hence the function $f(x)$ is continuous at $x = 0$.

Ex. 12. *Examine the function*

$f(x) = (x-a)\sin\{1/(x-a)\}$, when $x \neq a$ and $f(a) = 0$ for continuity at $x = a$.

Sol. We have $f(a+0) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} (a+h-a)\sin\{1/(a+h-a)\} = \lim_{h \to 0} h\sin(1/h) = 0.$$

Again

$$f(a-0) = \lim_{h \to 0} f(a-h)$$

$$= \lim_{h \to 0} (a-h-a)\sin\{1/(a-h-a)\}$$

$$= \lim_{h \to 0} -h\sin(-1/h) = 0.$$

Also $f(a) = 0$. Thus $f(a+0) = f(a-0) = f(a)$. Therefore $f(x)$ is continuous at $x = a$.

Ex. 13. *Investigate the points of continuity and discontinuity of the following function :*

$$f(x) = \frac{1}{x-a}\operatorname{cosec}\left(\frac{1}{x-a}\right), \; x \neq a;$$

$$f(x) = 0, \qquad x = a.$$

(Lucknow 1980)

Sol. We have $f(a+0) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \frac{1}{a+h-a}\operatorname{cosec}\left(\frac{1}{a+h-a}\right) = \lim_{h \to 0} \frac{1}{h\sin(1/h)}$$

$= +\infty$, since $h\sin(1/h) \to 0$ as $h \to 0$.

Again

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{1}{a-h-a}\operatorname{cosec}\left(\frac{1}{a-h-a}\right)$$

$$= \lim_{h \to 0} -\frac{1}{h}\cdot\frac{1}{\sin\{-(1/h)\}} = \lim_{h \to 0} \frac{1}{h\sin(1/h)}$$

$= +\infty$, since $h\sin(1/h) \to 0$ as $h \to 0$.

Also as given $f(a) = 0$.

Since $f(a+0) = f(a-0) \neq f(a)$, the function $f(x)$ is discontinuous at $x = a$, having an infinite discontinuity of second kind.

Ex. 14. *Discuss the continuity at $x = 0, 1, 2$ of function $f(x)$ defined as follows :*

$f(x) = -x^2$ for $x \leq 0$, $f(x) = 5x - 4$ for $0 < x \leq 1$, $f(x) = 4x^2 - 3x$ for $1 < x < 2$ and $f(x) = 3x + 4$ for $x \geq 2$.

(Delhi 1982; Meerut 81)

Sol. (i) Conitinuity at x = 0. We have $f(0) = -0^2 = 0$;

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h)^2$$

$$= \lim_{h \to 0} -h^2 = 0;$$

and $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (5h - 4)$, $[\because 0 < h < 1]$

$$= -4.$$

Since $f(0+0) \neq f(0-0)$, the function $f(x)$ is discontinuous at $x = 0$.

(ii) Continuity at x = 1. We have $f(1) = 5 \times 1 - 4 = 1$;

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [4(1+h)^2 - 3(1+h)], \quad [\because 1 < 1+h < 2]$$

$$= \lim_{h \to 0} [4h^2 + 5h + 1] = 1;$$

and $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} [5(1-h) - 4]$, $[\because 0 \leq 1-h < 1]$

$$= \lim_{h \to 0} (1 - 5h) = 1.$$

Thus $f(1+0) = f(1-0) = f(1)$. Therefore $f(x)$ is continuous at $x = 1$.

(iii) Continuity at x = 2. We have $f(2) = 3 \times 2 + 4 = 10$;

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [4(2-h)^2 - 3(2-h)]; \quad [\because 1 < 2-h < 2]$$

$$= \lim_{h \to 0} [4h^2 - 13h + 10] = 10;$$

and $f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} [3(2+h) + 4]$

$$= \lim_{h \to 0} (3h + 10) = 10.$$

Thus $f(2) = f(2-0) = f(2+0)$. Therefore $f(x)$ is continuous at $x = 2$.

Ex. 15. *Consider the function $f(x) = x - [x]$, where x is a positive variable, and $[x]$ denotes the integral part of x and show that it is discontinuous for all integral values of x, and continuous for all others. Draw the graph.* **(Meerut 1983, 90)**

Sol. From the definition of the function $f(x)$, we have

$$f(x) = x - (n-1) \quad \text{for } n-1 < x < n,$$
$$f(x) = 0 \quad \text{for } x = n,$$
$$f(x) = x - n \quad \text{for } n < x < n+1$$

and so on, where n is an integer.

We shall test the function $f(x)$ for continuity at $x = n$. We have $f(n) = 0$;

$$f(n+0) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} [(n+h) - n] = \lim_{h \to 0} h = 0;$$

$$[\because n < n+h < n+1]$$

and $f(n-0) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} [(n-h) - (n-1)]$,

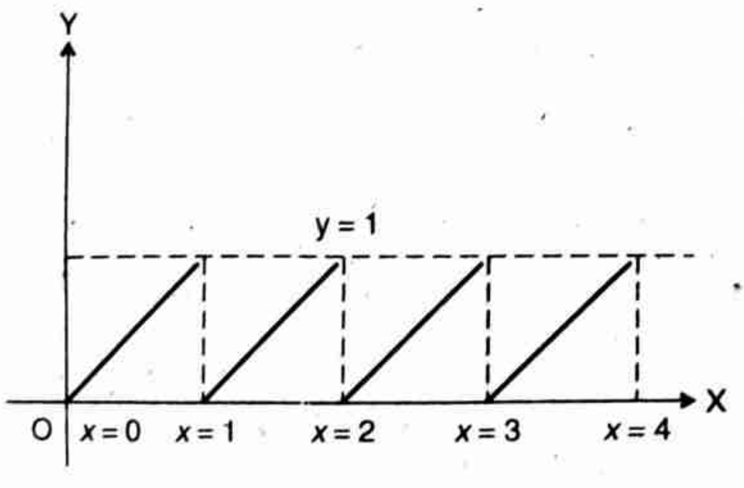
$$[\because n-1 < n-h < n]$$

$$= \lim_{h \to 0} (1-h) = 1.$$

Since $f(n+0) \neq f(n-0)$, the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. It is obviously continuous for all other values of x.

Since x is a positive variable, putting $n = 1, 2, 3, 4, 5, \ldots,$ we observe that the graph of $f(x)$ consists of the following :

$y = x$ when $0 < x < 1$, $\quad y = 0$ when $x = 1$
$y = x - 1$ when $1 < x < 2$, $\quad y = 0$ when $x = 2$
$y = x - 2$ when $2 < x < 3$, $\quad y = 0$ when $x = 3$
$y = x - 3$ when $3 < x < 4$, $\quad y = 0$ when $x = 4$



and so on.

The graph of the function thus obtained is shown by thick lines from $x = 0$ to $x = 4$.

Ex. 16. *Let $y = [x]$ or $E(x)$, where $E(x)$ denotes the integral part of x. Prove that this function is dis-continuous where x has an integral value. Also draw the graph.* **(Allahabad 1973)**

Sol. From the definition of $E(x)$, we have

$$E(x) = n - 1 \text{ for } n - 1 \le x < n,$$
$$E(x) = n \text{ for } n \le x < n + 1,$$
$$E(x) = n + 1 \text{ for } n + 1 \le x < n + 2,$$

and so on where n is an integer.

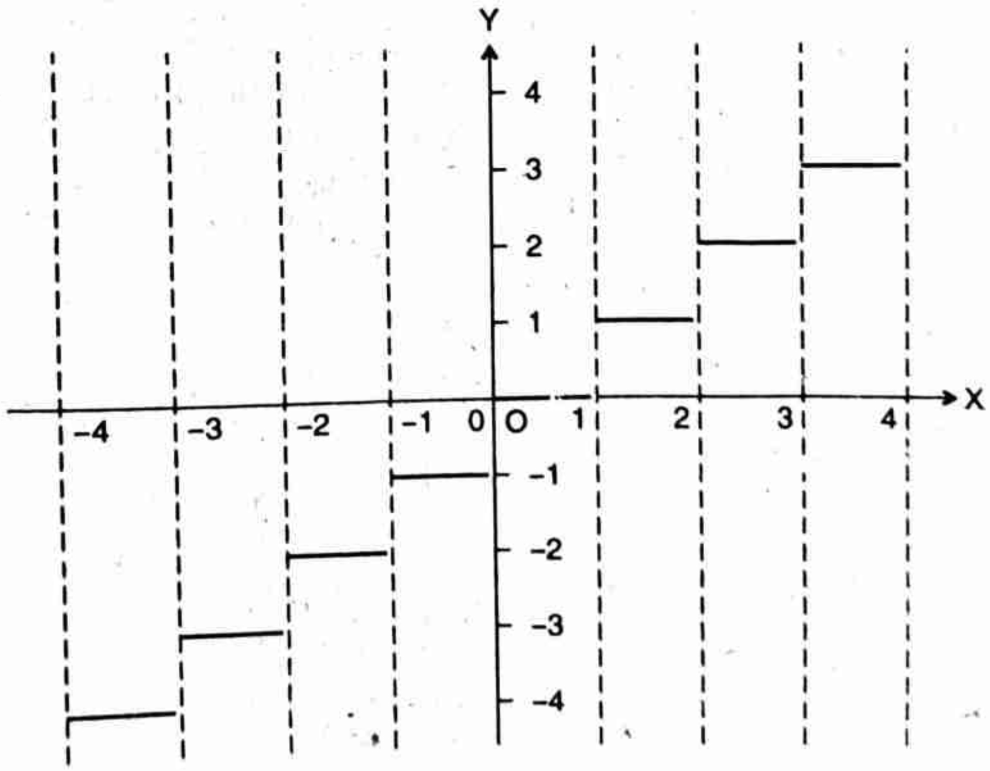
We consider $x = n$. Then $E(n) = n$, $E(n - 0) = n - 1$ and $E(n + 0) = n$.

Since $E(n + 0) \ne E(n - 0)$, the function $E(x)$ is discontinuous at $x = n$ *i.e.*, when x has an integral value. Evidently it is continuous for all other values of x.

To draw the graph, we put

$n = \dots\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\dots$, so that

$$y = -4, \text{ when } -4 \le x < -3$$
$$y = -3, \text{ when } -3 \le x < -2$$
$$y = -2, \text{ when } -2 \le x < -1$$
$$y = -1, \text{ when } -1 \le x < 0$$
$$y = 0, \text{ when } 0 \le x < 1$$
$$y = 1, \text{ when } 1 \le x < 2$$



$y = 2,$ when $2 \le x < 3$

$y = 3,$ when $3 \le x < 4$

$y = 4,$ when $4 \le x < 5$ and so on.

The graph is shown by thick lines.

Ex. 17. *A function $f(x)$ is defined as follows :*

$$f(x) = 1 + x \text{ if } x \le 2 \text{ and } f(x) = 5 - x \text{ if } x \ge 2.$$

Is the function continuous at $x = 2$? **(Lucknow 1982; Meerut 86, 88)**

Sol. We have $f(2) = 1 + 2$ or $5 - 2 = 3$;

$f(2+0) = \lim_{h \to 0} f(2+h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [5 - (2+h)]$, $\quad [\because 2 + h > 2 \text{ and } f(x) = 5 - x \text{ if } x > 2]$

$= \lim_{h \to 0} (3 - h) = 3;$

and $f(2-0) = \lim_{h \to 0} f(2-h)$, where h is +ive and sufficiently small

$= \lim_{h \to 0} [1 + (2-h)]$, $\quad [\because 2 - h < 2 \text{ and } f(x) = 1 + x \text{ if } x < 2]$

$= \lim_{h \to 0} (3 - h) = 3.$

Since $f(2+0) = f(2-0) = f(2)$, the function $f(x)$ is continuous at $x = 2$.

Ex. 18. *Discuss the continuity of the function $f(x)$ defined as follows :*

$f(x) = x^2$ for $x < -2$, $f(x) = 4$ for $-2 \le x \le 2$, $f(x) = x^2$ for $x > 2$.

Sol. We shall test $f(x)$ for continuity only at the points $x = -2$ and 2. It is obviously continuous at all other points.

(i) **Continuity of f(x) at x = − 2.** We have $f(-2) = 4$;

$f(-2+0) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} 4 = 4;$

$f(-2-0) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} (-2-h)^2,$

$[\because -2 - h < -2]$

$= 4.$

Thus $f(-2+0) = f(-2-0) = f(-2)$. Therefore $f(x)$ is continuous at $x = -2$.

(ii) **Continuity of f(x) at x = 2.** We have $f(2) = 4$;

$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4.$

and $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4.$

Thus $f(x)$ is also continuous at $x = 2$.

Ex. 19. *Let*

$$f(x) = \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} + \dots \infty.$$

Is $f(x)$ continuous at the origin ? Give reasons for your answer.

Sol. Here the function $f(x)$ forms an infinite G.P. whose first terms a is $\dfrac{x^2}{1+x^2}$ and common ratio r is $\dfrac{1}{1+x^2}$ which is less than one when $x \neq 0$.

$$\therefore f(x) = \frac{a}{1-r} = \frac{x^2/(1+x^2)}{1-1/(1+x^2)} = \frac{x^2/(1+x^2)}{x^2/(1+x^2)} = 1,$$

provided $x \neq 0$.

Thus $f(x) = 1$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$.

We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1.$$

Thus $f(0+0) \neq f(0)$ and so $f(x)$ is not continuous at $x = 0$.

Ex. 20. *Discuss the continuity and differentiability of the function $f(x)$ defined as follows :*

$$f(x) = 1 \text{ for } -\infty < x < 0, f(x) = 1 + \sin x \text{ for } 0 \leq x < \pi/2,$$

$$f(x) = 2 + \left(x - \frac{\pi}{2}\right)^2 \text{ for } \frac{\pi}{2} \leq x < \infty.$$

Sol. We shall test $f(x)$ for continuity and differentiability at $x = 0$ and $\pi/2$. It is obviously continuous as well as differentiable at all other points.

(i) Continuity and differentiability of f (x) at x = 0.

We have $f(0) = 1 + \sin 0 = 1$; $f(0+0) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} (1 + \sin h) = 1; \text{ and}$$

$$f(0-0) = \lim_{h \to 0} f(0-h)$$

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} 1 = 1$. Thus $f(0) = f(0+0) = f(0-0)$ and therefore $f(x)$ is continuous at $x = 0$.

Now $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{(1 + \sin h) - (1 + \sin 0)}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1;$$

and $Lf'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \dfrac{f(-h) - f(0)}{-h}$

$= \lim_{h \to 0} \dfrac{1 - (1 + \sin 0)}{-h} = \lim_{h \to 0} \dfrac{0}{-h} = \lim_{h \to 0} 0 = 0.$

Since $Rf'(0) \neq Lf'(0)$, $f(x)$ is not differentiable at $x = 0$.

(ii) Continuity and differentiability of f (x) at x = $\pi/2$.

We have $f\left(\dfrac{\pi}{2}\right) = 2 + \left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\right)^2 = 2;$

$f\left(\dfrac{\pi}{2} + 0\right) = \lim_{h \to 0} f\left(\dfrac{\pi}{2} + h\right) = \lim_{h \to 0} \left[2 + \left\{\left(\dfrac{\pi}{2} + h\right) - \dfrac{\pi}{2}\right\}^2\right]$

$= \lim_{h \to 0} (2 + h^2) = 2;$

and $f\left(\dfrac{\pi}{2} - 0\right) = \lim_{h \to 0} f\left(\dfrac{\pi}{2} - h\right) = \lim_{h \to 0} \left[1 + \sin\left(\dfrac{\pi}{2} - h\right)\right]$

$= \lim_{h \to 0} (1 + \cos h) = 1 + 1 = 2.$

Thus $f\left(\dfrac{\pi}{2} + 0\right) = f\left(\dfrac{\pi}{2} - 0\right) = f\left(\dfrac{\pi}{2}\right)$

and therefore $f(x)$ is continuous at $x = \pi/2$.

Now $Rf'\left(\dfrac{\pi}{2}\right) = \lim_{h \to 0} \dfrac{f(\pi/2 + h) - f(\pi/2)}{h}$

$= \lim_{h \to 0} \dfrac{[2 + \{\pi/2 + h - \pi/2\}] - [2 + (\pi/2 - \pi/2)^2]}{h}$

$= \lim_{h \to 0} \dfrac{2 + h^2 - 2}{h} = \lim_{h \to 0} h = 0;$

and $Lf'\left(\dfrac{\pi}{2}\right) = \lim_{h \to 0} \dfrac{f(\pi/2 - h) - f(\pi/2)}{-n}$

$= \lim_{h \to 0} \dfrac{1 + \sin(\pi/2 - h) - 2}{-h}$

$= \lim_{h \to 0} \dfrac{-1 + \cos h}{-h} = \lim_{h \to 0} \dfrac{1 - \cos h}{h} = \lim_{h \to 0} \dfrac{2 \sin^2 (h/2)}{h}$

$= \lim_{h \to 0} \left[\dfrac{\sin h/2}{h/2} \sin h/2\right] = 1 \times 0 = 0.$

Thus $Rf'(0) = Lf'(0)$ and so $f(x)$ is differentiable at $x = \pi/2$.

Ex. 21. *Examine the following curve for continuity and differentiability :*

$y = x^2$ *for* $x \leq 0$, $y = 1$ *for* $0 < x < 1$, $y = 1/x$ *for* $x > 1$.

(Meerut 1986 P)

Sol. Let $y = f(x)$. We need to test $f(x)$ for continuity and differentiability at the points $x = 0$ and 1. It is obviously continuous and differentiable at all other points.

At $x = 0$. We have $f(0) = 0^2 = 0$; $f(0+0) = \lim_{h \to 0} f(0+h)$ $= \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1$. Since $f(0+0) \neq f(0)$, the function $f(x)$ is not continuous at $x = 0$. Consequently it is also not differentiable at $x = 0$.

At $x = 1$. The function $f(x)$ is not defined at $x = 1$. Therefore it is discontinuous as well as non-differentiable at $x = 1$.

Ex. 22. *Show that $f(x) = |x - 1|$, $0 \leq x \leq 2$ is not derivable at $x = 1$. Is it continuous in $[0, 2]$?* **(Delhi 1979, 77)**

Sol. (i) To test for continuity at $x = 1$. We have $f(1) = 0$.

Also

$$\text{R.H.L.} = f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} |1 + h - 1| = 0;$$

$$\text{and L.H.L.} = f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} |1 - h - 1| = 0.$$

Since $f(1+0) = f(1-0) = f(1)$, therefore $f(x)$ is continuous at $x = 1$.

In fact $f(x)$ is continuous at every point in the interval $[0, 2]$.

(ii) To test $f(x)$ for differentiability at $x = 1$.

We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{|1 + h - 1| - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1;$$

$$\text{and } Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{|1 - h - 1| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1.$$

Since $Rf'(1) \neq Lf'(1)$, the function $f(x)$ is not differentiable at $x = 1$.

Ex. 23. *Show that the function $f(x)$, where*

$$f(x) = 2 + x \text{ if } x \geq 0;\ f(x) = 2 - x \text{ if } x < 0$$

is not derivable at the point $x = 0$. **(Delhi 1981, 78)**

Sol. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{2 + h - 2}{h} = 1,$$

$$\text{and } Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\{2 - (-h)\} - 2}{-h} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

Ex. 24. *Examine the following curve for continuity and differentiability at the points $x = \pm 1$:*

$$y = x - 1 \text{ for } x \geq 1,\ y = \tfrac{3}{2} + x \text{ for } x \leq -1,$$

$$y = 1 + x + x^2 + x^3 + \dots \infty \text{ for } -1 < x < 1.$$

Sol. Obviously $y = \dfrac{1}{1-x}$ for $-1 < x < 1$. Let $y = f(x)$.

(i) **At $x = 1$.** We have $f(1) = 1 - 1 = 0$;

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1 + h - 1) = 0;$$

and $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \dfrac{1}{1-(1-h)} = \lim_{h \to 0} \dfrac{1}{h} = \infty$.

Since $f(1+0) \neq f(1-0)$, the function $f(x)$ is not continuous at $x = 1$. Consequently $f(x)$ is also not differentiable at $x = 1$.

(ii) **At $x = -1$.** We have $f(-1) = \frac{3}{2} - 1 = \frac{1}{2}$;

$$f(-1+0) = \lim_{h \to 0} f(-1+h) = \lim_{h \to 0} \frac{1}{1-(-1+h)}$$

$$= \lim_{h \to 0} \frac{1}{2-h} = \tfrac{1}{2};$$

and $f(-1-0) = \lim_{h \to 0} f(-1-h) = \lim_{h \to 0} \left[\tfrac{3}{2} + (-1-h)\right]$

$$= \lim_{h \to 0} \left(\tfrac{1}{2} - h\right) = \tfrac{1}{2}.$$

Since $f(-1+0) = f(-1-0) = f(-1)$, the function $f(x)$ is continuous at $x = -1$.

Now $$Rf'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1}{1-(-1+h)} - \left(\tfrac{3}{2} - 1\right)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{2-h} - \tfrac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{2 - 2 + h}{2h(2-h)} = \lim_{h \to 0} \frac{1}{2(2-h)} = \tfrac{1}{4}.$$

Again $$Lf'(-1) = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{\left[\tfrac{3}{2} + (-1-h)\right] - \left(\tfrac{3}{2} - 1\right)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{2} - h - \frac{1}{2}}{-h} = \lim_{h \to 0} 1 = 1.$$

Since $Rf'(-1) \neq Lf'(-1)$, the function $f(x)$ is not differentiable at $x = -1$.

Ex. 25. *Examine the function defined below for continuity at* $x = 0$:

$$f(x) = \frac{\sin^2 ax}{x^2} \text{ for } x \neq 0, f(x) = 1 \text{ for } x = 0.$$

Sol. We have $f(0) = 1$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah}\right)^2 \cdot a^2 = 1 . a^2 = a^2;$$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin^2(-ah)}{(-h)^2}$

$$= \lim_{h \to 0} \frac{\sin^2 ah}{h^2} = a^2.$$

Now $f(x)$ is continuous at $x = 0$ if and only if

$f(0) = f(0+0) = f(0-0)$.

Therefore $f(x)$ is discontinuous at $x = 0$ unless $a = \pm 1$.

Ex. 26. *If $f(x) = -x$ for $x \leq 0$ and $f(x) = +x$ for $x \geq 0$, prove that $f(x)$ is continuous but not differentiable at $x = 0$.*

Sol. (i) To test f (x) for continuity at x = 0. We have $f(0) = 0$; $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h = 0$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h)$

$$= \lim_{h \to 0} h = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function $f(x)$ is continuous at $x = 0$.

(ii) To test f (x) for differentiability at x = 0.

We have $Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h - 0}{h} = \lim_{h \to 0} 1 = 1.$$

Again $Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{-(-h) - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

Ex. 27. *Give an example to show that a continuous function need not be differentiable.* **(Delhi 1980; Meerut 72)**

Sol. The students can give the example given in Ex. 1 (a) page 14. Here we shall give one more such example. Consider the function $f(x)$ defined as follows :

$$f(x) = x \sin(1/x) \text{ for } x \neq 0, f(0) = 0.$$

As already proved in part (b) of Ex. 3 page 16, this function $f(x)$ is continuous at $x = 0$. Now we shall show that the function $f(x)$ is not differentiable at $x = 0$. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h}, \text{ which does not exist.}$$

Similarly $Lf'(0)$ does not exist. Hence $f(x)$ is not differentiable at $x = 0$.

****Ex. 28.** *If $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is continuous and differentiable at $x = 0$.* **(Delhi 1982, 80; Meerut 77)**

Sol. (i) To test f(x) for continuity at x = 0.

We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h)^2 \sin(-1/h)$$

$$= -\lim_{h \to 0} h^2 \sin \frac{1}{h} = 0.$$

Now $f(0+0) = f(0-0) = f(0)$ implies that $f(x)$ is continuous at $x = 0$.

(ii) To test f(x) for differentiability at x = 0. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Again $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h) - 0}{-h} = \lim_{h \to 0} \frac{-h^2 \sin(1/h)}{-h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Now $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$. The derivative of $f(x)$ at $x = 0$ has the value zero.

Ex. 29. *If* $f(x) = e^{-1/x^2} \sin \frac{1}{x}$ *for* $x \neq 0$ *and* $f(0) = 0$, *test the differentiability of* $f(x)$ *at* $x = 0$.

Sol. We have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h) - 0}{h} = \lim_{h \to 0} \frac{\sin(1/h)}{he^{1/h^2}}$$

$$= \lim_{h \to 0} \frac{\sin(1/h)}{h\left[1 + \frac{1}{h^2} + \frac{1}{h^4 2!} + \ldots\right]} = \lim_{h \to 0} \frac{\sin(1/h)}{h + \frac{1}{h} + \frac{1}{2h^3} + \ldots}$$

$$= \frac{\text{a finite quantity}}{0 + \infty} = 0,$$ [$\because$ $\sin(1/h)$ oscillates between -1 and 1 and hence finite]

Also $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{e^{-1/h^2} \sin(-1/h) - 0}{-h} = \lim_{h \to 0} \frac{e^{-1/h^2} \sin(1/h)}{h} = 0$$

as before.

Now $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$ and its derivative is zero there.

Ex. 30. *If* $f(x) = \frac{xe^{1/x}}{1 + e^{1/x}}$ *for* $x \neq 0$ *and* $f(0) = 0$, *show that* $f(x)$ *is continuous at* $x = 0$, *but* $f'(0)$ *does not exist.*

(Delhi 1982; Meerut 85 S, 87)

Sol. Here $f(0) = 0$. Also right hand limit *i.e.,*

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{he^{1/h}}{1 + e^{1/h}} = \lim_{h \to 0} \frac{h}{e^{-1/h} + 1} \text{ (Note) } = \frac{0}{0+1} = 0;$$

and the left hand limit *i.e.,*

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \left[\frac{-he^{-1/h}}{1+e^{-1/h}}\right] = \frac{0}{1+0} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

We now proceed to find the derivative of $f(x)$ at $x = 0$.

We have,

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{he^{1/h}}{1+e^{1/h}}\right) - 0}{h} = \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{h \to 0} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{0+1} = 0$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\dfrac{-he^{-1/h}}{1+e^{-1/h}} - 0}{-h} = \lim_{h \to 0} \frac{e^{-1/h}}{1+e^{-1/h}} = \frac{0}{1+0} = 0.$$

Since $Rf'(0) \neq Lf'(0)$, the derivative of $f(x)$ at $x = 0$ does not exist.

Ex. 31. *Let* $f(x) = x\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$, $x \neq 0$; $f(0) = 0$.

Show that $f(x)$ *is continuous but not derivable at* $x = 0$.

Sol. We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} h\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} = \lim_{h \to 0} h\frac{1 - e^{-2/h}}{1 + e^{-2/h}},$$

dividing Nr. and Dr. by $e^{1/h}$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} -h \cdot \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \lim_{h \to 0} -h\frac{e^{-2/h} - 1}{e^{-2/h} + 1} = 0 \times \frac{0-1}{0+1} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, the function is continuous at $x = 0$.

Now to test f (x) for differentiability at x = 0, we have

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \left[h \, \frac{e^{1/h} - e^{-1/h}}{e^{1/h} - e^{-1/h}} - 0 \right] \Big/ h = \lim_{h \to 0} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1-0}{1+0} = 1,$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \left[(-h) \cdot \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}} - 0 \right] \Big/ (-h)$$

$$= \lim_{h \to 0} \frac{e^{-2/h} - 1}{e^{-2/h} + 1} = \frac{0-1}{0+1} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function is not differentiable at $x = 0$.

□

2

Rolle's Theorem, Mean value Theorems, Taylor's and Maclaurin's Theorems

§ 1. Rolle's Theorem.

(Meerut 1985, 91; Agra 82, 80, 77, 73; Indore 70; Gorakhpur 78)

If a function $f(x)$ is such that

(i) $f(x)$ is continuous in the closed interval $a \le x \le b$,

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$,

(iii) $f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.

Proof. Since $f(a) = f(b)$, unless the function $f(x)$ is a constant in which case the theorem is at once established, $f(x)$ should either increase or decrease when x takes values greater than a. Suppose it increases; then since it again takes a value $f(b) = f(a)$, it must cease to increase and begin to dercrease at some point c, such that $a < c < b$.

At this point c the function $f(x)$ has a maximum value and so
$f(c+h) - f(c)$ and $f(c-h) - f(c)$
are both negative, h being small and positive.

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0 \quad \text{and} \quad \frac{f(c-h) - f(c)}{-h} > 0.$$

Obviously as $h \to 0$, the above expressions tend to being $-$ive and $+$ive respectively unless each of them has the limit zero.

If they have different limits, then $Rf'(c) \ne Lf'(c)$ and therefore $f'(c)$ does not exist, contradicting the hypothesis.

Hence each of the above limits must be zero,

i.e., $\quad f'(c) = 0$ where $a < c < b$.

Note 1. There may be more than one point like c at which $f'(x)$ vanishes.

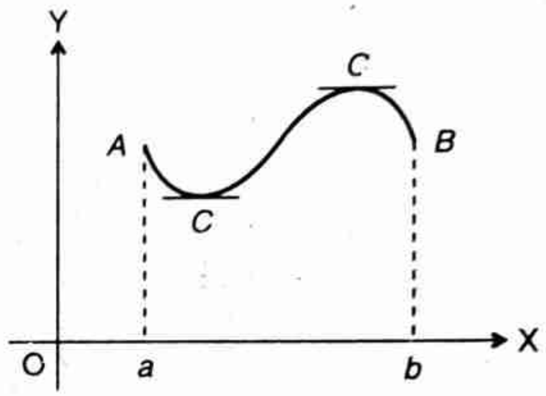
Note 2. Rolle's therorem will not hold good

(i) if $f(x)$ is discontinuous at some point in the interval
$a \le x \le b$,

or **(ii)** if $f'(x)$ does not exist at some point in the interval
$a < x < b$,

or **(iii)** if $f(a) \ne f(b)$.

Geometrical interpretation of Rolle's Theorem. Suppose the function $f(x)$ satisfies the conditions of Rolle's theorem in the interval



$[a, b]$. Then its geometrical interpretation is that on the curve $y = f(x)$ there is at least one point lying in the open interval (a, b) the tangent at which is parallel to the axis of x.

Solved Examples

Ex. 1 (a). *Discuss the applicability of Rolle's theorem for* $f(x) = 2 + (x-1)^{2/3}$ *in the interval* $[0, 2]$. **(G.N.U. 1977)**

Sol. Given $f(x) = 2 + (x-1)^{2/3}$. Obviously $f(0) = 3 = f(2)$, showing that the third condition of Rolle's therorem is satisfied.

The function $f(x)$, being an algebraic function of x, is continuous in the closed interval $[0, 2]$. Thus the first condition of Rolle's theorem is satisfied.

Now $f'(x) = \frac{2}{3} . [1/(x-1)^{1/3}]$. We observe that at $x = 1, f'(x)$ is not finite while $x = 1$ is a point of the open interval $0 < x < 2$. Thus the second condition for Rolle's theorem is not satisfied.

Hence the Rolle's theorem is not applicable for the function $2 + (x-1)^{2/3}$ in the given interval $[0, 2]$.

Ex. 1 (b). *Discuss the applicability of Rolle's theorem in the interval* $[-1, 1]$ *to the function* $f(x) = |x|$. **(Meerut 1974)**

Sol. Here $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$, so that $f(-1) = f(1)$.

Also the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but it is not differentiable at $x = 0$ which is a point of the open interval $(-1, 1)$. Therefore the second condition for Rolle's theorem is not satisfied, *i.e.*, the Rolle's theorem is not applicable here.

Ex. 2. *Are the conditions of Rolle's theorem satisfied in the case of the following functions ?*

(i) $f(x) = x^2$ *in* $2 \leq x \leq 3$,

(ii) $f(x) = \cos(1/x)$ *in* $-1 \leq x \leq 1$,

(iii) $f(x) = \tan x$ in $0 \leq x \leq \pi$.

Sol. (i) The function $f(x) = x^2$ is continuous and differentiable in the interval $[2, 3]$. Thus the first two conditions of Rolle's theorem are satisfied.

Also $f(2) = 4$ and $f(3) = 9$, so that $f(2) \neq f(3)$. Hence the third condition is not satisfied.

(ii) Here $f(-1) = \cos(-1) = \cos 1$ and $f(1) = \cos 1$. Thus $f(-1) = f(1)$ *i.e.*, the third condition is satisfied.

But the first two conditions of Rolle's theorem are not satisfied as the function is discontinuous at $x = 0$ and consequently is not differentiable there.

(iii) Here $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$. Thus $f(0) = f(\pi)$ *i.e.*, the third condition is satisfied.

But the first two conditions of Rolle's theorem are not satisfied here as the function is not continuous at $x = \pi/2$ and consequently is non-differentiable there.

Ex. 3. *Discuss the applicability of Rolle's theorem to* $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$, *in the interval* $[a, b]$. **(Meerut 1990)**

Sol. We have $f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)a}\right] = \log 1 = 0,$

and $\qquad f(b) = \log\left[\dfrac{b^2 + ab}{(a+b)b}\right] = \log 1 = 0.$

Hence $f(a) = f(b) = 0$.

Also $\qquad Rf'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log \frac{(x^2 + 2xh + h^2 + ab)(a+b)x}{(a+b)(x+h)(x^2+ab)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2 + 2xh + h^2 + ab)}{(x^2+ab)} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{1 + \frac{xh + h^2}{x^2 + ab}\right\} - \log\left\{1 + \frac{h}{x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2hx + h^2}{x^2 + ab} - \frac{h}{x} + \ldots\ldots\right] \qquad \ldots(1)$$

$$\left[\because \log(1+y) = y - \tfrac{1}{2}y^2 + \ldots\ldots\right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $Lf'(x) = \lim_{h \to 0} \left[\frac{f(x-h) - f(x)}{-h}\right]$

$$= \lim_{h \to 0} \frac{1}{(-h)} \left[\frac{-2hx + h^2}{x^2 + ab} - \frac{(-h)}{x} + \dots\right],$$

[replacing h by $-h$ in (1)]

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Thus $Rf'(x) = Lf'(x)$, showing that $f(x)$ is differentiable for all values of x in $[a, b]$. Consequently $f(x)$ is also continuous for all values of x in $[a, b]$. Hence all the three conditioins of Rolle's theorem are staisfied.

$\therefore$ $f'(x) = 0$ for at least one value of x in the open interval $a < x < b$.

Now $f'(x) = 0$ where $\dfrac{2x}{x^2 + ab} - \dfrac{1}{x} = 0$ or $2x^2 - (x^2 + ab) = 0$ or $x^2 = ab$ or $x = \sqrt{(ab)}$, which being the geometric mean of a and b lies in the open interval (a, b). Hence Rolle's theorem is verified.

Ex. 4. *Verify Rolle's theorem in the case of the functions*

(i) $f(x) = 2x^3 + x^2 - 4x - 2$, **(Agra 1982, 80)**

(ii) $f(x) = \sin x$ *in* $[0, \pi]$,

(iii) $f(x) = (x - a)^m (x - b)^n$, *where m and n are +ive integers, and x lies in the interval* $[a, b]$. **(Agra 1981)**

Sol. (i) Here $f(x)$ is a rational integral function of x. So it is continuous and differentiable for all real values of x. Thus the first two conditions of Rolle's theorem are satisfied in any interval.

Now let $f(x) = 0$. Then $2x^3 + x^2 - 4x - 2 = 0$

or $\quad (x^2 - 2)(2x + 1) = 0$ *i.e,* $x = \pm \sqrt{2}, -\frac{1}{2}$.

Thus $f(\sqrt{2}) = f(-\sqrt{2}) = f(-\frac{1}{2}) = 0$.

Let us consider the interval $[-\sqrt{2}, \sqrt{2}]$. In this interval all the conditions of Rolle's theorem are satisfied. Therefore there is at least one value of x in the open interval $(-\sqrt{2}, \sqrt{2})$ where $f'(x) = 0$.

Now $f'(x) = 0$ where $6x^2 + 2x - 4 = 0$

or $\quad 3x^2 + x - 2 = 0$ or $(3x - 2)(x + 1) = 0$ or $x = -1, 2/3$.

Thus $\quad f'(-1) = f'(2/3) = 0$.

Since both the points $x = -1$ and $x = 2/3$ lie in the open interval $(-\sqrt{2}, \sqrt{2})$, Rolle's theorem is verified.

(ii) Here $f(0) = \sin 0 = 0$ and $f(\pi) = \sin \pi = 0$. Thus

$$f(0) = f(\pi) = 0.$$

Further $\sin x$ is continuous and differentiable in $[0, \pi]$. Hence all the three conditions of Rolle's theorem are satisfied. Therefore $f'(x) = 0$ for at least one value of x in the open interval $(0, \pi)$.

Now $f'(x) = 0$ gives $\cos x = 0$ or $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \cdots$ Since $x = \pi/2$ lies in the open interval $(0, \pi)$, the Rolle's theorem is verified.

(iii) Here $f(x) = (x - a)^m (x - b)^n$.

As m and n are positive integers, $(x - a)^m$ and $(x - b)^n$ are polynomials in x on being expanded by binomial theorem. Hence $f(x)$ is also a polynomial in x. Consequently $f(x)$ is continuous and differentiable in the closed interval $[a, b]$. Also $f(a) = f(b) = 0$. Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x) = 0$ for at least one value of x lying in the open interval (a, b).

Now $f'(x) = (x - a)^m . n (x - b)^{n-1} + m (x - a)^{m-1} (x - b)^n$.

The equation $f'(x) = 0$, on being solved, gives

$$x = a, b, \frac{na + mb}{m + n}.$$

Out of these values the value $\frac{na + mb}{m + n}$ is a point lying in the open interval (a, b) as it divides the interval (a, b) internally in the ratio $m : n$. Thus the Rolle's theorem is verified.

Ex. 5. *Verify Rolle's theorem for*

(i) $f(x) = x^3 - 4x$ *in* $[-2, 2]$, **(G.N.U. 1975)**

(ii) $f(x) = x(x + 3) e^{-x/2}$ *in* $[-3, 0]$, **(Gorakhpur 1970)**

(iii) $f(x) = e^x (\sin x - \cos x)$ *in* $[\pi/4, 5\pi/4]$.

Sol. **(i)** Here $f(x) = x^3 - 4x$. Since $f(x)$ is a polynomial in x, therefore it is continuous and differentiable for every real value of x; Also $f(-2) = 0 = f(2)$.

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore$ there must exist at least one number, say c, in the open interval $(-2, 2)$ for which $f'(c) = 0$.

Now $f'(x) = 0$ gives $3x^2 - 4 = 0$ or

$$x = \pm \frac{2}{\sqrt{3}} = \pm 1.155 \text{ (approx)}.$$

Both these values lie in the open interval $(-2, 2)$. Thus the theorem is verified.

(ii) Here $f(x) = x(x + 3) e^{-x/2} = (x^2 + 3x) e^{-x/2}$.

We have $f'(x) = (2x + 3) e^{-x/2} + (x^2 + 3x) . e^{-x/2} . (-\frac{1}{2})$

$$= e^{-x/2} [2x + 3 - \tfrac{1}{2}(x^2 + 3x)] = -\tfrac{1}{2}(x^2 - x - 6) e^{-x/2},$$

which exists for every value of x in the interval $[-3, 0]$. Therefore $f(x)$ is differentiable and also continuous in the interval $[-3, 0]$.

Also $f(-3) = 0 = f(0)$. Therefore all the three conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist at least one number, say c, in the open interval $(-3, 0)$ for which $f'(c) = 0$ *i.e.*, $-\frac{1}{2}(c^2 - c - 6)\, e^{-c/2} = 0$ or $c^2 - c - 6 = 0$ or $(c-3)(c+2) = 0$ or $c = 3, -2$.

The value $c = -2$ lies in the open interval $(-3, 0)$. Hence the theorem is verified.

(iii) Here $f(x) = e^x (\sin x - \cos x)$.

We have $f(\pi/4) = e^{\pi/4} \{\sin(\pi/4) - \cos(\pi/4)\}$

$= e^{\pi/4} [(1/\sqrt{2}) - (1/\sqrt{2})] = 0$

and $$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left[\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right] = e^{5\pi/4}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 0.$$

$\therefore \quad f(\pi/4) = f(5\pi/4) = 0.$

Further the function $f(x)$ is continuous and differentiable in $[\pi/4, 5\pi/4]$. Therefore all the three conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist at least one number, say c, in the open interval $(\pi/4, 5\pi/4)$ for which $f'(c) = 0$.

Now $f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x) = 2e^x \sin x$.

From $f'(x) = 0$ we get $2e^x \sin x = 0$

or $\quad \sin x = 0, \qquad [\because e^x \neq 0]$

or $\quad x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Out of these values $x = \pi$ lies in the open interval $(\pi/4, 5\pi/4)$. Thus the Rolle's theorem is verified.

Ex. 6. *If $f(x)$, $\phi(x)$, $\psi(x)$ have derivatives when $a \le x \le b$, show that there is a value c of x lying between a and b such that*

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

(Agra 1973)

Sol. Consider the following function

$$F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$$

On expanding the determinant, we observe that the function $F(x)$ is of the form $A f(x) + B\phi(x) + C\psi(x)$, where A, B, C are some real numbers.

Since the functions $f(x)$, $\phi(x)$ and $\psi(x)$ have derivatives when $a \le x \le b$, therefore the function $F(x)$ also possesses derivatives when

$a \leq x \leq b$. Consequently $F(x)$ is also continuous when $a \leq x \leq b$. Further $F(a) = F(b) = 0$ because then the two rows of the determinant become identical. Thus $F(x)$ satisfies all the three conditions of Rolle's theorem. Hence $F'(x) = 0$ for at least one value of x, say $x = c$, lying between a and b. Thus there is a value c of x lying between a and b such that

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

****§ 2. Lagrange's mean value theorem or First mean value theorem.** **(Lucknow 1983, 81; Gorakhpur 77; Meerut 81, 84P, 86, 91; Delhi 76, Agra 78; Alld. 81)**

If a function $f(x)$ is

(i) continuous in the closed interval $a \leq x \leq b$,

and (ii) differentiable in the open interval (a, b) i.e., $a < x < b$,

then there exists at least one value 'c' of x lying in the open interval $a < x < b$ such that

$$\frac{\mathbf{f(b) - f(a)}}{\mathbf{b - a}} = \mathbf{f'(c)}.$$

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + Ax, \quad \text{...(1)}$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$ *i.e.*,

$$f(a) + Aa = f(b) + Ab$$

or
$$-A = \frac{f(b) - f(a)}{b - a} \quad \text{...(2)}$$

Now $f(x)$ is given to be continuous in $a \leq x \leq b$ and differentiable in $a < x < b$.

Again, A being a constant, Ax is also continuous in $a \leq x \leq b$ and differentiable in $a < x < b$.

$\therefore$ $\phi(x) = f(x) + Ax$ is continuous in $a \leq x \leq b$ and differentiable in $a < x < b$. Also by our choice of A, we have $\phi(a) = \phi(b)$. Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem in the interval $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $a < x < b$, such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$, from (1).

$\therefore$ $\phi'(c) = 0$ gives $f'(c) + A = 0$

or
$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}, \text{ from (2).}$$

This proves the theorem.

Another form of Lagrange's mean value theorem.

If a function $f(x)$ is

(*i*) *continuous in the closed interval* $[a, a+h]$,
and (*ii*) *differentiable in the open interval* $(a, a+h)$, *then there exists at least one number* θ *lying between* 0 *and* 1 *such that*

$$\mathbf{f(a+h) = f(a) + hf'(a+\theta h)}. \qquad \textbf{(K.U. 1974)}$$

Proof. Let $a + h = b$. Then $b - a = h =$ the length of the interval.

Now give the complete proof of Lagrange's mean value theorem.

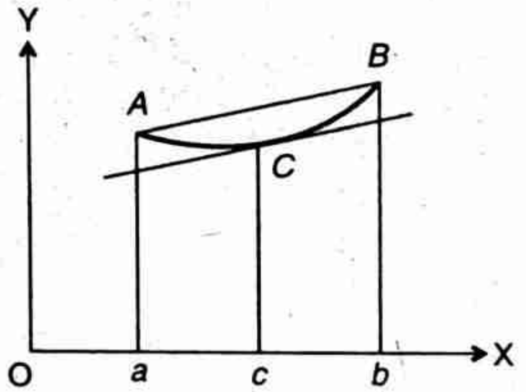
Since c lies between a and $a + h$, therefore it is greater than a by a fraction of h and may be written as $c = a + \theta h$, where $0 < \theta < 1$.

Hence the result of Lagrange's mean value theorem can be written as

$$f(a+h) - f(a) = hf'(a+\theta h), \qquad [0 < \theta < 1].$$

Geometrical interpretation of the mean value theorem.
(Meerut 1977, 78, 85 P; Lucknow 80)

In the figure let ACB be the graph of $f(x)$ in (a, b) and let the chord AB make an angle α with the x-axis so that



$$\tan\alpha = \frac{f(b) - f(a)}{b - a}$$
$$= f'(c), \text{ by the Mean Value Theorem}$$

where $a < c < b$.

Thus there is some point c within (a, b) such that the tangent to the curve at the point $[c, f(c)]$ is parallel to the chord AB.

§ 3. Some important deductions from mean value theorem.

Theorem 1. *If a function* $f(x)$ *be such that* $f'(x)$ *is zero throughout the interval* (a, b), *then* $f(x)$ *must be constant throughout the interval.*

Proof. Let x_1, x_2 be any two points in the interval (a, b) such that $x_2 > x_1$. Since $f'(x)$ exists through out the interval (a, b), therefore $f(x)$ satisfies the conditions of Lagrange's mean value theorem in the interval $[x_1, x_2]$. So applying this theorem for $f(x)$ in the interval $[x_1, x_2]$, we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c), \text{ where } x_1 < c < x_2.$$

But by hypothesis $f'(x)=0$ throughout the interval (a,b).

$\therefore \quad f'(c)=0$ or $f(x_2)-f(x_1)=0$ or $f(x_2)=f(x_1)$.

Thus the values of $f(x)$ at every two points of (a,b) are equal. Hence $f(x)$ must be constant throughout (a,b).

Theorem 2. *If $f(x)$ and $\phi(x)$ be two functions such that $f'(x)=\phi'(x)$ throughout the interval (a,b), then $f(x)$ and $\phi(x)$ differ only by a constant.*

Proof. Consider the function $F(x)=f(x)-\phi(x)$.

Throughout the interval (a,b), we have

$$F'(x)=f'(x)-\phi'(x)=0, \qquad [\because f'(x)=\phi'(x)].$$

Therefore, from theorem 1, we have

$$F(x)=\text{const}, \text{ or } f(x)-\phi(x)=\text{const}.$$

Theorem 3. *If $f(x)$ is*

(i) *continuous in the closed interval $[a,b]$,*

(ii) *differentiable in the open interval (a,b)*

and **(iii)** *$f'(x)$ is –ive in $a<x<b$, then*

$f(x)$ is a monotonically decreasing function in the closed interval $[a,b]$.

Proof. Let x_1, x_2 be any two points belonging to the closed interval $[a,b]$ such that $x_2 > x_1$.

Applying Lagrange's mean value theorem to $f(x)$ in the interval $[x_1,x_2]$, we have

$$f(x_2)-f(x_1)=(x_2-x_1)f'(c), \text{ where } x_1<c<x_2. \qquad ...(1)$$

Now $x_2-x_1>0$. Since by hypothesis $f'(x)$ is negative for every x in (a,b), therefore $f'(c)<0$. Hence from (1), we have

$$f(x_2)-f(x_1)<0 \text{ i.e., } f(x_2)<f(x_1).$$

Thus $f(x)$ is a decreasing function of x in $[a,b]$.

Similarly we can prove that **a function having a positive derivative for every value of x in an interval is a monotonically increasing function in that interval.** **(Mysore 1971)**

Corollary. *The function $f(x)$ is strictly decreasing or increasing in $[a,b]$ if $f'(x)<0$ or $(f'(x)>0)$ for every x in (a,b) except for a finite number of points where the derivative is zero.*

§ 4. Cauchy's mean value theorem or second mean value theorem.

(Meerut 1991; Gorakhpur 82; Allahabad 82; Agra 79; Luck. 82)

If two functions $f(x)$ and $g(x)$ are

(i) *continuous in the closed interval $[a,b]$,*

(ii) *differentiable in the open interval* (a, b),

and **(iii)** $g'(x) \neq 0$ *for any point of the open interval* (a, b), *then there exists at least one value c of x in the open interval* (a, b), *such that*

$$\frac{\mathbf{f(b) - f(a)}}{\mathbf{g(b) - g(a)}} = \frac{\mathbf{f'(c)}}{\mathbf{g'(c)}}, \ \mathbf{a < c < b}.$$

Proof. First we note that $g(b) - g(a) \neq 0$. For if $g(b) - g(a) = 0$ *i.e.*, $g(b) = g(a)$, then the function $g(x)$ satisfies the conditions of Rolle's theorem and so its derivative $g'(x)$ should vanish for at least one value of x lying in the open interval (a, b). But this is contrary to our hypothesis.

Now consider the function $F(x)$ defined by

$$F(x) = f(x) + Ag(x), \qquad ...(1)$$

where A is a constant to be determined such that $F(a) = F(b)$ *i.e.*,

$$f(a) + Ag(a) = f(b) + Ag(b)$$

or
$$-A = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad ...(2)$$

Since $g(b) - g(a) \neq 0$, therefore A is a definite real number.

Now the function $F(x)$ obviously satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Therefore there exists, at least one value, say c, of x in the open interval (a, b) such that $F'(c) = 0$.

But $F'(x) = f'(x) + Ag'(x)$, from (1).

$\therefore \quad F'(c) = 0$ gives $f'(c) + Ag'(c) = 0$

or
$$-A = \frac{f'(c)}{g'(c)}. \qquad ...(3)$$

From (2) and (3), we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Another form. Let $b = a + h$. Then $a + \theta h = a$ when $\theta = 0$ and $a + \theta h = b$ when $\theta = 1$. Therefore $a + \theta h$, where $0 < \theta < 1$, means some value between a and b. So putting $b = a + h$ and $c = a + \theta h$, the result of the above theorem can be written as

$$\frac{\mathbf{f(a+h) - f(a)}}{\mathbf{g(a+h) - g(a)}} = \frac{\mathbf{f'(a+\theta h)}}{\mathbf{g'(a+\theta h)}}, \ \mathbf{0 < \theta < 1}.$$

Note. Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem.

Let us set $g(x) = x$ in Cauchy's mean value theorem which is justified because $g(x) = x$ satisfies all the conditions of Cauchy's mean value theorem. But $g(x) = x$ means $g(b) = b$, $g(a) = a$, $g'(x) = 1$ and so $g'(c) = 1$. Putting these values in Cauchy's mean value theorem, we get

$$\frac{f(b) - f(a)}{b - a} = f'(c), \ (a < c < b)$$

which is nothing but the result of Lagrange's mean value theorem.

Solved Examples

Ex. 7 (a). *If $f(x) = (x-1)(x-2)(x-3)$ and $a = 0$, $b = 4$, find 'c' using Lagrange's mean value theorem.*

Sol. We have

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

$$\therefore \quad f(a) = f(0) = -6, \text{ and } f(b) = f(4) = 6.$$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{6 - (-6)}{4 - 0} = \frac{12}{4} = 3.$$

Also $f'(x) = 3x^2 - 12x + 11$, so that $f'(c) = 3c^2 - 12c + 11$.

Substituting these values in Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c), \ (a < c < b), \text{ we have}$$

$$3 = 3c^2 - 12c + 11 \text{ or } 3c^2 - 12c + 8 = 0$$

or $$c = \frac{12 \pm \sqrt{(144 - 96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

Both of these values of c lie in the open interval $(0, 4)$. Hence both of these are the required values of c.

Ex. 7 (b). *Find 'c' of the mean value theorem, if*

$$f(x) = x(x-1)(x-2);\ a = 0,\ b = \tfrac{1}{2}.$$

Sol. Here $f(a) = f(0) = 0$ and

$$f(b) = f(\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{2} - 1)(\tfrac{1}{2} - 2) = \tfrac{3}{8}.$$

$$\therefore \quad \frac{f(b) - f(a)}{b - a} = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0} = \frac{3}{4}.$$

Now $f(x) = x^3 - 3x^2 + 2x$.

$\therefore \quad f'(x) = 3x^2 - 6x + 2$, so that $f'(c) = 3c^2 - 6c + 2$.

Substituting these values in Lagrange's mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c), \ (a < c < b), \text{ we have}$$

$$\tfrac{3}{4} = 3c^2 - 6c + 2 \text{ or } 12c^2 - 24c + 5 = 0.$$

$\therefore$

$$c = \frac{24 \pm \sqrt{(24 \times 24 - 4 \times 12 \times 5)}}{24} = \frac{24 \pm 4\sqrt{(36 - 15)}}{24} = 1 \pm \frac{\sqrt{21}}{6}.$$

Out of these two values of c only $1 - \dfrac{\sqrt{21}}{6}$ lies in the open interval $(0, \tfrac{1}{2})$ which is therefore the required value of c.

Ex. 8. *Find 'c' so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ in the following cases :}$$

(i) $f(x) = x^2 - 3x - 1;\ a = -11/7,\ b = 13/7.$

(ii) $f(x) = e^x;\ a = 0,\ b = 1.$

Sol. (i) Here $f(a) = f\left(-\frac{11}{7}\right) = \frac{121}{49} + \frac{33}{7} - 1 = \frac{303}{49}$

and $$f(b) = f\left(\frac{13}{7}\right) = \frac{169}{49} - \frac{39}{7} - 1 = -\frac{153}{49}.$$

$\therefore$ $$\frac{f(b) - f(a)}{b - a} = \frac{-456/49}{24/7} = -\frac{19}{7}.$$

Now $f'(x) = 2x - 3$; $\therefore$ $f'(c) = 2c - 3$.

From Lagrange's mean value theorem, we have
$$2c - 3 = -19/7 \quad \text{or} \quad c = 1/7.$$

(ii) Here $f(a) = f(0) = e^0 = 1$, and $f(b) = f(1) = e^1 = e$. Also $f'(x) = e^x$, so that $f'(c) = e^c$

$\therefore$ using Lagrange's mean value theorem, we have
$$\frac{e-1}{1-0} = e^c \text{ or } e^c = e - 1 \text{ or } c = \log_e (e-1).$$

Ex. 9. *Compute the value of θ in the first mean value theorem*
$$f(x+h) = f(x) + hf'(x + \theta h),$$
if $f(x) = ax^2 + bx + c$.

Sol. We have $f(x) = ax^2 + bx + c$.

$\therefore$ $f(x+h) = a(x+h)^2 + b(x+h) + c$,

$f'(x) = 2ax + b$, $f'(x + \theta h) = 2a(x + \theta h) + b$.

Putting all these values in the Lagrange's mean value theorem, we have
$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x + \theta h) + b] \quad ...(1)$$

The relation (1) is identically true for all values of x. So when $x \to 0$, we get
$$ah^2 + bh + c = c + h[2a\theta h + b]$$
or $$ah^2 = 2a\theta h^2 \quad \text{or} \quad \theta = 1/2.$$

Ex. 10. *A function $f(x)$ is continuous in the closed interval $0 \le x \le 1$ and differentiable in the open interval $0 < x < 1$, prove that $f'(x_1) = f(1) - f(0)$, where $0 < x_1 < 1$.*

Sol. Here $a = 0$, $b = 1$. Therefore
$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0).$$

If we take $c = x_1$ and substitute these values in the result of Lagrange's mean value theorem, we get
$$f(1) - f(0) = f'(x_1) \text{ where } 0 < x_1 < 1.$$

Ex. 11. *Separate the intervals in which the polynomial $2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.*

Sol. Let $f(x) = 2x^3 - 15x^2 + 36x + 1$.

Then $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$.

Now $f'(x) > 0$ for $x < 2$; $f'(x) < 0$ for $2 < x < 3$; $f'(x) > 0$ for $x > 3$; $f'(x) = 0$ for $x = 2$ and 3.

Thus $f(x)$ is positive in the intervals $(-\infty, 2)$ and $(3, \infty)$ and negative in the interval $(2, 3)$.

Hence $f(x)$ is monotonically increasing in the intervals $(-\infty, 2]$, $[3, \infty)$ and monotonically decreasing in the interval $[2, 3]$.

Ex. 12. *Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.*

Sol. Let $f(x) = x^3 - 3x^2 + 3x + 2$.

Then $f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$.

We see that $f'(x) > 0$ for every real value of x except 1 where its value is zero. Hence $f(x)$ is monotonically increasing in every interval.

Ex. 13 (a). *Show that*

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.$$

(Delhi 1973)

Sol. Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$.

Then

$$f'(x) = \frac{1}{1+x} - \frac{1.(1+x) - x.1}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

We see that $f'(x) > 0$ for $x > 0$. Therefore $f(x)$ is monotonically increasing in the interval $[0, \infty)$. But $f(0) = 0$. Therefore $f(x) > f(0) = 0$ for $x > 0$ *i.e.,* $\left[\log(1+x) - \dfrac{x}{1+x}\right] > 0$ for $x > 0$.

Hence $\log(1+x) > \dfrac{x}{1+x}$ for $x > 0$. ...(1)

Again let $\phi(x) = x - \log(1+x)$; then

$$\phi'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}.$$

We see that $\phi'(x) > 0$ for $x > 0$. Therefore $\phi(x)$ is monotonically increasing in the interval $[0, \infty)$. But $\phi(0) = 0$. Therefore $\phi(x) > \phi(0) = 0$ for $x > 0$ *i.e.,* $[x - \log(1+x)] > 0$ for $x > 0$.

Hence $x > \log(1+x)$ for $x > 0$. ...(2)

From (1) and (2), we have

$$\frac{x}{1+x} < \log(1+x) < x \text{ when } x > 0.$$

Ex. 13 (b). *Prove that for every $x > 0$, $\dfrac{x}{1+x^2} < \tan^{-1} x < x$.*

(Lucknow 1982)

Sol. Proceed exactly in the same way as in Ex. 13 (a). First take

$$f(x) = \tan^{-1} x - \frac{x}{1+x^2}.$$

Then $$f'(x) = \frac{2x^2}{(1+x^2)^2}.$$

Again take $\phi(x) = x - \tan^{-1} x$.

Then $\phi'(x) = \dfrac{x^2}{1+x^2}$.

Ex. 14. *State the conditions for the validity for the formula*

$$f(x+h) = f(x) + hf'(x+\theta h)$$

and investigate how far these conditions are satisfied and whether the result is true, when $f(x) = x \sin(1/x)$ *(being defined to be zero at* $x = 0$*) and* $x < 0 < x + h$.

Sol. The conditions for the validity of the given formula are :

(i) The function $f(x)$ must be continuous in the closed interval $[x, x+h]$.

(ii) The function $f(x)$ must be differentiable in the open interval $(x, x+h)$.

(iii) θ is a real number such that $0 < \theta < 1$.

Now consider the function $f(x)$ defined as :

$$f(x) = x \sin(1/x) \text{ for } x \neq 0, f(0) = 0.$$

The first condition is satisfied because $f(x)$ is continuous in the closed interval $[x, x+h]$ for $x < 0 < x + h$. [The students should show here that $f(x)$ is continuous at $x = 0$].

But the second condition is not satisfied because $f(x)$ is not differentiable at $x = 0$ which is a point lying in the open interval $(x, x+h)$ for $x < 0 < x + h$. [Show here that $f(x)$ is not differentiable at $x = 0$].

Hence the result of the given formula is not true for this function $f(x)$.

Ex. 15. *Verify Cauchy's mean value theorem for the functions* x^2 *and* x^3 *in the interval* $[1, 2]$.

Sol. Let $f(x) = x^2$ and $g(x) = x^3$. Both $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$ and differentiable in the open interval $(1, 2)$. Also $g'(x) = 3x^2 \neq 0$ for any point in the open interval $(1, 2)$. Therefore by Cauchy's mean value theorem there exists at least one real number c in the open interval $(1, 2)$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)} \qquad \text{...(1)}$$

Now $\dfrac{f(2) - f(1)}{g(2) - g(1)} = \dfrac{4-1}{8-1} = \dfrac{3}{7}$. Also $f'(x) = 2x$, $g'(x) = 3x^2$.

Therefore $\dfrac{f'(c)}{g'(c)} = \dfrac{2c}{3c^2} = \dfrac{2}{3c}$.

Substituting these values in (1), we get $\dfrac{3}{7} = \dfrac{2}{3c}$ or $c = \dfrac{14}{9}$ which lies in the open interval $(1, 2)$. This verifies the theorem.

Ex. 16. *If in the Cauchy's mean value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

Sol. Here $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = -e^a e^b = -e^{a+b}$.

Also $\dfrac{f'(x)}{g'(x)} = \dfrac{e^x}{-e^{-x}}$, so that $\dfrac{f'(c)}{g'(c)} = \dfrac{e^c}{-e^{-c}} = e^{2c}$.

Substituting these values in Cauchy's mean value theorem, we get

$$-e^{a+b} = -e^{2c} \text{ or } 2c = a + b \text{ or } c = \tfrac{1}{2}(a + b).$$

Hence c is the arithmetic mean between a and b.

Ex. 17. *If, in the Cauchy's mean value theorem, we write*

(i) *$f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$, then c is the geometric mean between a and b, and if*

(ii) *$f(x) = 1/x^2$ and $g(x) = 1/x$, then c is the harmonic mean between a and b.*

Sol. (i) Here $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{\sqrt{b} - \sqrt{a}}{(1/\sqrt{b}) - (1/\sqrt{a})} = -\sqrt{(ab)}$.

Also $\dfrac{f'(x)}{g'(x)} = \dfrac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}}$, so that $\dfrac{f'(c)}{g'(c)} = -\dfrac{c^{-1/2}}{c^{-3/2}} = -c$.

Substituting these values in Cauchy's mean value theorem, we get $-\sqrt{(ab)} = -c$ or $c = \sqrt{(ab)}$ *i.e.*, c is the geometric mean between a and b.

(ii) From the Cauchy's mean value theorem, we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Putting $f(x) = 1/x^2$ and $g(x) = 1/x$, we get

$$\frac{(1/b^2) - (1/a^2)}{(1/b) - (1/a)} = \frac{-2c^{-3}}{-c^{-2}} \text{ or } \frac{a+b}{ab} = \frac{2}{c} \text{ or } c = \frac{2ab}{a+b}$$

i.e., c is the harmonic mean between a and b.

§ 5. Taylor's theorem with Lagrange's form of remainder after n terms.

(Delhi 1971; K.U. 73; Meerut 90)

If $f(x)$ is a single valued function of x such that

(i) *all the derivatives of $f(x)$ upto $(n-1)^{th}$ are continuous in $a \le x \le a + h$,*

and **(ii)** *$f^{(n)}(x)$ exists in $a < x < a + h$, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h), \text{ where } 0 < \theta < 1.$$

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \dots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{A}{n!}(a+h-x)^n,$$

where A is a constant to be determined such that

$$\phi(a) = \phi(a+h).$$

Now

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{A}{n!}h^n,$$

and $\phi(a+h) = f(a+h)$.

Therefore A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots\dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A. \quad \dots(1)$$

Now, by hypothesis, all the functions

$$f(x), f'(x), f''(x), \dots, f^{(n-1)}(x)$$

are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $(a, a+h)$.

Also $(a+h-x)$, $(a+h-x)^2/2!, \dots, (a+h-x)^n/n!$, all being polynomials , are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $(a, a+h)$. Further A is a constant.

$\therefore$ $\phi(x)$ is continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $(a, a+h)$. Also by our choice of A, $\phi(a) = \phi(a+h)$. Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ $\phi'(a+\theta h) = 0$, where $0 < \theta < 1$.

Now

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x)$$
$$+ \dots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{A}{(n-1)!}(a+h-x)^{n-1}$$
$$= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A], \text{ since other terms cancel in pairs.}$$

$\therefore \quad \phi'(a+\theta h)=0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h)-A]=0$$

or
$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h)-A]=0.$$

Now $h \neq 0$. Also $(1-\theta)\neq 0$ because $0<\theta<1$.

$\therefore \quad f^{(n)}(a+\theta h)-A=0$ or $A=f^{(n)}(a+\theta h)$.

Substituting this value of A in (1), we get

$$\mathbf{f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots}$$
$$\mathbf{+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h).}$$

This is **Taylor's development** of $f(a+h)$ in ascending integral powers of h. The $(n+1)^{th}$ term $\mathbf{\frac{h^n}{n!}f^{(n)}(a+\theta h)}$ is called **Lagrange's form of remainder** after n terms in Taylor's expansion of $f(a+h)$.

Note. If we take $n=1$, we observe that Lagrange's mean value theorem is a particular case of Taylor's theorem.

Corollary. (Maclaurin's development). Instead of considering the interval $[a, a+h]$, let us take the interval $[0, x]$. Then changing a to 0 and h to x in Taylor's theorem, we get

$$\mathbf{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots\ldots}$$
$$\mathbf{+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\theta x),}$$

which is known as **Maclaurin's theorem** or **Maclaurin's development** of $f(x)$ in the interval $[0,x]$ with **Lagrange's form of remainder** $\mathbf{\frac{x^n}{n!}f^{(n)}(\theta x)}$.

§ 6. Taylor's theorem with Cauchy's form of remainder.

(G.N.U. 1975; Meerut 91)

If $f(x)$ is a single valued function of x such that

(i) *all the derivatives of $f(x)$ upto $(n-1)^{th}$ are continuous in $a \le x \le a+h$,*

and **(ii)** *$f^{(n)}(x)$ exists in $a<x<a+h$, then*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots$$
$$+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h),\ \textit{where}\ 0<\theta<1.$$

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots\ldots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x) A,$$

where A is a constant to be determined such that

$$\phi(a) = \phi(a+h).$$

Now $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA,$$

and $\phi(a+h) = f(a+h)$.

Therefore A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots$$
$$+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA. \quad \ldots(1)$$

As shown in § 5, we can easily show that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore \quad \phi'(a+\theta h) = 0$, where $0 < \theta < 1$.

Now $\phi'(x) = \dfrac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - A$, since other terms cancel in pairs. $\therefore \quad \phi'(a+\theta h) = 0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - A = 0$$

or $$A = \frac{h^{n-1}}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a+\theta h).$$

Substituting this value of A in (1), we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)}$$
$$\mathbf{+ \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a+\theta h).}$$

The $(n+1)^{th}$ term $\dfrac{\mathbf{h^n}}{\mathbf{(n-1)!}} \mathbf{(1-\theta)^{n-1} f^{(n)}(a+\theta h)}$ is called **Cauchy's form of remainder** after n terms in Taylor's expansion of $f(a+h)$ in ascending integral powers of h.

Corollary. (Maclaurin's development with Cauchy's form of remainder). Changing a to 0 and h to x in the above result, we get

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)}$$
$$\mathbf{+ \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta x),}$$

which is known as **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Cauchy's form** of remainder after n terms.

Solved Examples

Ex. 18. *Expand the following by Maclaurin's theorem with Lagrange's form of remainder after n terms :*

(i) a^x, **(ii)** e^x.

Sol. **(i)** Here $f(x) = a^x$. ...(1)

$\therefore$ $f^{(n)}(x) = a^x (\log a)^n$. ...(2)

Putting $x = 0$ in (1) and (2), we get

$$f(0) = a^0 = 1, f^{(n)}(0) = a^0 (\log a)^n = (\log a)^n.$$

$\therefore$ $f'(0) = \log a, f''(0) = (\log a)^2, \ldots, f^{(n-1)}(0) = (\log a)^{n-1}$.

Also changing x to θx in (2), we get

$$f^{(n)}(\theta x) = a^{\theta x} (\log a)^n.$$

Now by Maclaurin's theorem with Lagrange's form of remainder after n terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x), \text{ where } 0 < \theta < 1. \quad \ldots(3)$$

Substituting the values found above in (3), we get

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \ldots + \frac{x^{n-1}}{(n-1)!} (\log a)^{n-1} + \frac{x^n}{n!} a^{\theta x} (\log a)^n.$$

Here Lagrange's form of remainder after n terms

$$= \frac{x^n}{n!} a^{\theta x} (\log a)^n, \text{ where } 0 < \theta < 1.$$

(ii) Here $f(x) = e^x$. Therefore $f^{(n)}(x) = e^x$.

Putting $x = 0$, in these, we get

$$f(0) = e^0 = 1, f^{(n)}(0) = e^0 = 1. \text{ Also } f^{(n)}(\theta x) = e^{\theta x}.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after n terms, we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x}.$$

Ex. 19. *Show that*

(i) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!}\sin \theta x,$$

for every real value of x. **(K.U. 1973)**

(ii) $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n}, \text{ for } x > -1.$$

(Punjab University 1975)

Sol. **(i)** Here $f(x) = \sin x$. ...(1)

We know that $\sin x$ possesses derivatives of every order for every real number x and

$$f^{(n)}(x) = \sin(x + \tfrac{1}{2}n\pi). \qquad \text{...(2)}$$

Putting $x = 0$ in (1) and (2), we get

$$f(0) = \sin 0 = 0, f^{(n)}(0) = \sin(\tfrac{1}{2}n\pi).$$

$$\therefore \quad f'(0) = \sin(\tfrac{1}{2}\pi) = 1, f''(0) = \sin\pi = 0,$$

$$f'''(0) = \sin(3\pi/2) = -1,$$

$$f^{iv}(0) = \sin 2\pi = 0,$$

$$f^{v}(0) = \sin(5\pi/2) = \sin(2\pi + \tfrac{1}{2}\pi) = \sin\tfrac{1}{2}\pi = 1, \ldots,$$

$$f^{(2n-2)}(0) = \sin(n-1)\pi = 0,$$

$$f^{(2n-1)}(0) = \sin\{\tfrac{1}{2}(2n-1)\pi\} = \sin(n\pi - \tfrac{1}{2}\pi)$$

$$= (-1)^n \sin(-\pi/2),$$

$$[\because \sin(n\pi + \theta) = (-1)^n \sin\theta]$$

$$= (-1)^n(-1) = (-1)^{n+1} = (-1)^{n-1}.$$

Also changing n to $2n$ and x to θx in (2), we get

$$f^{(2n)}(\theta x) = \sin(\theta x + n\pi) = (-1)^n \sin\theta x.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after $2n$ terms *i.e.*,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{2n-1}}{(2n-1)!}f^{(2n-1)}(0) + \frac{x^{2n}}{(2n)!}f^{(2n)}(\theta x),$$

we $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!}\sin\theta x.$$

(ii) Here $f(x) = \log(1+x)$. ...(1)

We know that $\log(1+x)$ possesses derivatives of every order when $(1+x) > 0$ *i.e.*, $x > -1$.

Also, $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$ ...(2)

Putting $x = 0$ in (1) and (2), we get

$$f(0) = \log 1 = 0,\ f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Also changing x to θx in (2), we get

$$f^{(n)}(\theta x) = (-1)^{n-1}(n-1)!\,(1+\theta x)^{-n}.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after n terms *i.e.*,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x),$$

we get

$$\log(1+x) = 0 + \frac{x}{1!}\cdot 1 + \frac{x^2}{2!}\cdot(-1)\cdot 1! + \frac{x^3}{3!}(-1)^2\cdot 2! + \dots + \frac{x^{n-1}}{(n-1)!}(-1)^{n-2}(n-2)! + \frac{x^n}{n!}(-1)^{n-1}(n-1)!\,(1+\theta x)^{-n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n}.$$

Ex. 20. *Find θ, if*

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x+\theta h),\ 0 < \theta < 1,\ \textit{and}$$

(*i*) $f(x) = ax^3 + bx^2 + cx + d$, **(Lucknow 1981)**

(*ii*) $f(x) = x^3$. **(Lucknow 1983, 80)**

Sol. (i) Here $f(x) = ax^3 + bx^2 + cx + d$.

$\therefore\ f(x+h) = a(x+h)^3 + b(x+h)^2 + c(x+h) + d$,

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b$,

and so $f''(x+\theta h) = 6a(x+\theta h) + 2b$.

Putting these values in the given relation

$f(x+h) = f(x) + hf'(x) + (h^2/2!) f''(x+\theta h)$, we have

$$a(x+h)^3 + b(x+h)^2 + c(x+h) + d$$
$$= ax^3 + bx^2 + cx + d + h(3ax^2 + 2bx + c)$$
$$+ (h^2/2!) \{6a(x+\theta h) + 2b\} \quad \ldots(1)$$

The relation (1) is an identity in x. Letting $x \to 0$ on both sides of (1), we have

$$ah^3 + bh^2 + ch + d = d + ch + (h^2/2)(6a\theta h + 2b)$$

or $$ah^3 + bh^2 + ch + d = d + ch + 3a\theta h^3 + bh^2$$

or $$ah^3 = 3a\theta h^3 \quad \text{or} \quad \theta = 1/3. \qquad [\because ah^3 \neq 0]$$

(ii) Proceed as in part (i) of this question. The required value of θ is 1/3.

□

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