Set-III

1

Solution of system of linear equations

Introduction:

System of linear equations arise in a large number of areas, both directly in modelling physical situations and indirectly in the numerical solution of other mathematical models.
These applications occur in all areas of the physical, biological and engineering sciences.
For instance, in physics, the problem of steady state temperature in a plate is reduced to solving linear equations.
Linear algebraic systems are also involved in the optimization theory, least squares fitting of data, numerical solution of boundary value problems for ordinary and partial differential eqns, statistical inference etc. Hence the numerical solution of systems of linear algebraic eqns play a very important role.

Numerical methods for solving linear algebraic systems may be divided into two types, direct and iterative.

Direct methods are those which, in the absence of round-off or other errors, yield the exact solution in a finite number of elementary arithmetic operations.

Iterative methods start with an initial approximation and by applying a suitably chosen algorithm, lead to successively better approximations.

The general form of a system of 'm' linear eqns in 'n' unknowns $x_1, x_2, \ldots x_n$ can be represented in matrix form as under:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad \text{—— (1)}$$

$$\Rightarrow \quad AX = B \quad \text{—— (2)}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

- The solution of the system of eqns (2) gives 'n' unknown values $x_1, x_2, \ldots x_n$ which satisfy the system simultaneously.
- A system of eqns (2) is said to be consistent if it has atleast one solution. If no solution exists, then the system is said to be inconsistent.
- The system of eqns (2) is said to be homogeneous if $b=0$, that is all the elements $b_1, b_2, \ldots b_m$ are zero, otherwise the system is called non-homogeneous.
- In this lesson, we consider only non-homogeneous, and we restrict $m=n$ (i.e, the number of eqns = the no. of unknowns)
- A non-homogeneous system of n linear eqns in 'n' unknowns has a unique solution iff the coefficient matrix A is non-singular. (ie, $|A| \neq 0$)

- If A is non-singular, then A^{-1} exists and the solution of system (2) can be expressed as $x = A^{-1}B$.
- In case the matrix A is singular, then the system (2) has no solution if $b \neq 0$ or has an infinite number of solutions if $b=0$.

Here we assume that A is non-singular matrix.

The methods of solution of the system (2) may be classified into two types:

(i) Direct Methods: which in the absence of round-off errors give the exact solution in a finite number of steps.

(ii) Iterative Methods: Starting with an approximate solution vector $x^{(0)}$, these methods generate a sequence of approximate solution vectors $\{x^{(k)}\}$ which converge to the exact solution vector x as the number of iterations $k \to \infty$.

Thus iterative methods are infinite processes. Since we perform only a finite number of iterations, these methods can only find some approximation to the solution vector x.

Direct Methods for Special Matrices:

we now discuss three special forms of matrix A in eqn (2) for which the solution vector x can be obtained directly.

case (i): $A = D$, where D is a diagonal matrix.

In this case the system of eqns (2) are of the form

$$\left.\begin{array}{l} a_{11}x_1 \cdots\cdots = b_1 \\ \quad a_{22}x_2 \cdots = b_2 \\ \vdots \qquad \ddots \qquad \vdots \\ \qquad\qquad a_{nn}x_n = b_n \end{array}\right\} \longrightarrow (3)$$

and $|A| = \det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdots\cdots a_{nn}$

Since the matrix A is non-singular, $a_{ii} \neq 0$ for $i = 1, 2, 3, \cdots n$ and we obtain the solution as $x_i = \dfrac{b_i}{a_{ii}}$, $i = 1, 2, 3, \cdots n$.

Case(ii): $A = L$, where L is a lower triangular matrix ($a_{ij} = 0$, $j > i$). The system of eqns (2) is now of the form

$$\left.\begin{array}{l} a_{11}x_1 \qquad\qquad\qquad = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \longrightarrow (4)$$

and $|A| = a_{11}a_{22} \cdots\cdots a_{nn}$

Since the coefficient matrix A is non-singular, $a_{ii} \neq 0$, $i = 1, 2, \cdots n$.

Solving the first eqn and then successively solving second, third and so on, we obtain.

$$x_1 = \frac{b_1}{a_{11}}$$

$$x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}}$$

$$x_3 = \frac{(b_3 - a_{31}x_1 - a_{32}x_2)}{a_{33}}$$

$$\vdots$$

$$x_n = \left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right)\frac{1}{a_{nn}}.$$

(3)

In general, we have for any i, $x_i = \dfrac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}}$
$i = 1, 2, \ldots n.$

Since the unknowns in this method are solved by forward substitution, this method is called the forward substitution method.

Case(iii): $A = U$, where U is an upper triangular matrix ($a_{ij} = 0, j < i$). The system (2) is now of the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \cdots\cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{(n-1)(n-1)}x_{n-1} + a_{(n-1)n}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned}\right\} \text{—(5)}$$

and $|A| = a_{11}a_{22} \cdots\cdots a_{nn}$

Since the coefficient matrix A is non singular.
$a_{ii} \neq 0$, $i = 1, 2, \ldots n$.

Solving unknowns in the order $x_n, x_{n-1}, \ldots x_2, x_1$, we get

$$x_n = \frac{b_n}{a_{nn}}$$

$$x_{n-1} = \frac{(b_{n-1} - a_{(n-1)n}x_n)}{a_{(n-1)(n-1)}}$$

$$\vdots$$

$$x_1 = \left(b_1 - \sum_{j=2}^{n} a_{1j}x_j\right) \Big/ a_{11}$$

In general we have for any i, $x_i = \dfrac{b_i - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$
$i = 1, 2, \ldots n.$

The unknowns are solved by back substitution and this method is called the back substitution method.
Thus, the eqns (2) are exactly solvable, if the matrix A in (2) can be transformed into any one of the forms D, L or U.

Direct Methods :-

Gaussian Elimination Method :

In the Gaussian elimination method, the solution to the system of eqns (2) is obtained in two stages.

In the first stage, the given system of eqns is reduced to an equivalent upper triangular form using elementary transformations. In the second stage, the upper triangular system is solved using back substitution procedure by which we obtain the solution in the order $x_n, x_{n-1}, x_{n-2}, \ldots x_2, x_1$.

This method is explained by considering a system of 'n' eqns in 'n' unknowns in the form as follows.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n \end{array}\right\} \text{—} (6)$$

Stage I: we divide the first eqn by a_{11} and then subtract this eqn multiplied by $a_{21}, a_{31}, \ldots a_{n1}$ from the 2nd, 3rd ....nth eqn. Then the system (6) reduces to the following form:

$$\left.\begin{array}{l} x_1 + a'_{12}x_2 + \cdots\cdots + a'_{1n}x_n = b'_1 \\ \quad a'_{22}x_2 + \cdots\cdots + a'_{2n}x_n = b'_2 \\ \quad \vdots \\ \quad a'_{n2}x_2 + \cdots\cdots + a'_{nn}x_n = b'_n \end{array}\right\} \text{—} (7)$$

Here, we can observe that the last $(n-1)$ eqns are independent of x_1, i.e, x_1 is eliminated from the last $(n-1)$ eqns.

This procedure is repeated with the second eqn of (7) i.e, we divide the second eqn by a'_{22} and then x_2 is eliminated from $3^{rd}, 4^{th}, \ldots n^{th}$ eqns of (7). The same procedure is repeated again and again till the given system assumes the following upper triangular form:

$$\left.\begin{array}{r} c_{11}x_1 + c_{12}x_2 + \cdots\cdots + c_{1n}x_n = d_1 \\ c_{22}x_2 + \cdots\cdots + c_{2n}x_n = d_2 \\ \vdots \qquad\qquad \\ c_{nn}x_n = d_n \end{array}\right\} \text{--- (8)}$$

<u>Stage II</u>:

Now, the values of the unknowns are determined by back substitution procedure, in which we obtain x_n from the last eqn of (8). and then substituting this value of x_n in the preceding eqn, we get the value of x_{n-1}. Continuing this way, we can find values of all other unknowns in the order $x_n, x_{n-1}, \ldots x_2, x_1$.

In this method, we observe that the determinant of the coefficient matrix is obtained as a by-product, i.e,

$$|A| = c_{11} \cdot c_{22} \cdots\cdots c_{nn}.$$

Example: Solve the following system of eqns using Gaussian elimination method.

$$\left.\begin{aligned} 2x+3y-z&=5\\ 4x+4y-3z&=3\\ -2x+3y-z&=1 \end{aligned}\right\} \longrightarrow \text{(i)}$$

Soln: The given system of eqns (i) is solved in two stages.

Stage I (Reduction to upper-triangular form):- we divide the first eqn by '2' and then subtract the resulting eqn (multiplied by 4 and −2) from the second eqn and third eqn respectively. Thus, we eliminate x from the 2nd and 3rd eqns.

The resulting new system is given by

$$\left.\begin{aligned} x+\frac{3}{2}y-\frac{z}{2}&=\frac{5}{2}\\ -2y-z&=-7\\ 6y-2z&=6 \end{aligned}\right\} \longrightarrow \text{(ii)}$$

Now, we divide the second eqn of (ii) by −2 and eliminate y from the last eqn and the modified system is given by

$$\left.\begin{aligned} x+\frac{3}{2}y-\frac{z}{2}&=\frac{5}{2}\\ y+\frac{z}{2}&=7/2\\ -5z&=-15 \end{aligned}\right\} \longrightarrow \text{(iii)}$$

Stage II: (Back substitution):-

from the last eqn of (iii)

we get $\boxed{z=3}$

using this value of z, the second eqn of (iii) gives,

$$y = \frac{7}{2} - \frac{3}{2} = 2$$

$$\Rightarrow \boxed{y = 2}$$

Using these values of y and z in the first eqn of (iii), we get

$$\boxed{x = 1}.$$

Thus, the solution of the given system is $x = 1,\ y = 2,\ z = 3$.

Note: we can write the above procedure more conveniently in matrix form. Since the arithmetic operations we have performed here affect only the elements of the matrix A and the matrix B, we consider the agumented matrix i.e, $[A|B]$ and perform the elementary row operations on the augmented matrix.

$$[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & a'_{32} & a'_{33} & b'_3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1 \\ R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a^2_{33} & b^2_3 \end{array}\right] \quad R_3 \to R_3 - \frac{a'_{32}}{a'_{22}} R_2$$

i.e, $[A|B] \xrightarrow[\text{elimination}]{\text{Gaussian}} [U|C]$

which is in desired form.

where a'_{22}, a'_{23}, a'_{32}, a'_{33}, b'_2, b'_3, a^2_{33}, b^2_3 are given by eqns 7 & 8.

→ The diagonal elements a_{11}, a_{22}^{1} and a_{33}^{2} which have been assumed to be non-zero are called pivot elements.

→ we observe that for a linear system of order 3, the elimination was performed in $3-1=2$ stages.
In general for a system of 'n' eqns given by eqns (2) the elimination is performed in $(n-1)$ stages.

At the i^{th} stage of elimination, we eliminate x_i starting from $(i+1)^{th}$ row upto the n^{th} row.
Some times, it may happen that the elimination process stops in less than $(n-1)$ stages.

<u>But this is possible only</u> when no eqns containing the unknowns are left or when the coefficients of all the unknowns in the remaining eqns become zero. Thus if the process stops at the r^{th} stage of elimination then we get a derived system of the form.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n &= b_1 \\
a_{22}^{1}x_2 + \cdots\cdots + a_{2n}^{1}x_n &= b_2^{1} \\
\vdots \qquad\qquad & \\
a_{rr}^{(r-1)}x_2 + \cdots\cdots + a_{rn}^{(r-1)}x_n &= b_r^{(r-1)} \\
0 &= b_{r+1}^{(r-1)} \\
\vdots \quad & \\
0 &= b_n^{(r-1)}
\end{aligned}
$$

where $r \le n$ and $a_{11} \neq 0$, $a_{22}^{1} \neq 0, \cdots a_{rr}^{(r-1)} \neq 0$.

In the solution of system of linear eqns we can expect two different situations.
(i) $r=n$ (ii) $r<n$.

Ex (1) Solve the following system of eqns by using Gaussian elimination method.

$$\left.\begin{aligned} 4x_1+x_2+x_3 &= 4 \\ x_1+4x_2-2x_3 &= 4 \\ -x_1+2x_2-4x_3 &= 2 \end{aligned}\right\} \text{---}$$

Soln: we have

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 1 & 4 & -2 & 4 \\ -1 & 2 & -4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & 15/4 & 9/4 & 3 \\ 0 & 9/4 & 15/4 & 3 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{1}{4}R_1 \\ R_3 \to R_3 + \frac{R_1}{4} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 1 & 1 & 4 \\ 0 & \frac{15}{4} & -\frac{9}{4} & 3 \\ 0 & 0 & \frac{-12}{5} & \frac{6}{5} \end{array}\right] \quad R_3 \to R_3 - \frac{3}{5}R_2$$

Using back substitution method, we get

$$x_3 = -\tfrac{1}{2},\ x_2 = \tfrac{1}{2},\ x_1 = 1$$

Also $|A| = -36$.

Thus in this case we observe that $r=n=3$ and the given system of eqn has a unique solution. Also the coefficient matrix is non-singular.

Ex (2): Solve the system of eqns

$$\begin{aligned} 3x_1+2x_2+x_3 &= 3 \\ 2x_1+x_2+x_3 &= 0 \\ 6x_1+2x_2+4x_3 &= 6 \end{aligned}$$

Using Gauss elimination method. Does the solution exist?

Soln: we have

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{2}{3}R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & -1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \begin{array}{l} \to b_1 \\ \to b_2' \\ \to b_3' \end{array} \quad R_3 \to R_3 - 6R_2$$

In this case $r<n$ and elements b_1, b_2' and b_3' are all non-zero. Since we cannot determine x_3 from the last eqn, the system has no solution.

In such situation we say that the eqns are inconsistent. Also $|A|=0$. i.e., the coefficient matrix is singular.

Ex(3): Solve the system of eqns

$$16x_1 + 22x_2 + 4x_3 = -2$$
$$4x_1 - 3x_2 + 2x_3 = 9$$
$$12x_1 + 25x_2 + 2x_3 = -11$$

using Gauss elimination method.

Sol^n: we have

$$[A|B] = \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 4 & -3 & 2 & 9 \\ 12 & 25 & 2 & -11 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & 19/2 \\ 0 & \frac{17}{2} & -1 & -19/2 \end{array}\right] \quad R_2 \to R_2 - \frac{1}{4}R_1$$

$$\sim \left[\begin{array}{ccc|c} 16 & 22 & 4 & -2 \\ 0 & -\frac{17}{2} & 1 & \frac{19}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now in this case $r<n$ and elements b_1, b_1' are non-zero, but $b_3^{(2)}$ is zero.

Also the last eqn is satisfied for any value of x_3.

Thus we get $x_3 =$ any value

$$x_2 = -\frac{2}{17}\left(\frac{19}{2} - x_3\right)$$

$$x_1 = \frac{1}{16}\left(-2 - 22x_2 - 4x_3\right).$$

Hence the system of eqns has infinitely many solutions.

Also $|A| = 0$.

→ We now summarise these conclusions as follows:

(i) If $r = n$ then the system of eqns (2) has unique solution which can be obtained by using the back substitution method. Moreover the coefficient matrix A in this case is non-singular.

(ii) If $r < n$ and all the elements $b_{r+1}^{(r-1)}, b_{r+2}^{(r-1)}, \ldots\ldots b_n^{(r-1)}$ are not zero then the system has no solution.

In this case we say that the system of eqns is inconsistent.

(iii) If $r < n$ and all the elements $b_{r+1}^{(r-1)}, b_{r+2}^{(r-1)}, \ldots\ldots b_n^{(r-1)}$, if present, are zero, then the system has infinite number of solutions.

In this case the system has only r linearly independent rows.

In both the cases (ii) and (iii), the matrix A is singular.

→ Use the Gaussian elimination method solve the following system of eqns.

(1)
$$x_1 + 2x_2 + x_3 = 3$$
$$3x_1 - 2x_2 - 4x_3 = -2$$
$$2x_1 + 3x_2 - x_3 = -6$$

$\boxed{\text{Ans: } x_3 = 5,\ x_2 = -3,\ x_1 = 4}$

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(2) $3x_1+18x_2+9x_3=18$
$2x_1+3x_2+3x_3=117$
$4x_1+x_2+2x_3=283$

$\boxed{\text{Ans: } x_3=4,\ x_2=-13,\ x_1=72}$

(3) $\begin{bmatrix}1 & 2 & -3 & 1\\ 0 & 1 & 3 & 1\\ 2 & 3 & 1 & 1\\ 1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-5\\ 6\\ 4\\ 1\end{bmatrix}$

$\boxed{\text{Ans: } x_4=-1,\ x_3=2,\ x_2=1,\ x_1=0}$

(4) $\begin{bmatrix}3 & 2 & -1 & -4\\ 1 & -1 & 3 & -1\\ 2 & 1 & -3 & 0\\ 0 & -1 & 8 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}10\\ -4\\ 16\\ 3\end{bmatrix}$

[Ans: inconsistent. we cannot determine x_4 from the last eqn.]

(5) $\begin{bmatrix}2 & -1 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\\ 0\\ 1\end{bmatrix}$

$\boxed{\text{Ans: } x_5=x_4=x_3=x_2=x_1=1}$

→ we can apply Gaussian elimination method to a system of eqns of any order. However, what happens if any one of the diagonal elements i.e, the pivotes in the triangularization process vanishes. Then the method will fail. In such situations we modify the Gaussian elimination method and this procedure is called pivoting.

→ In the elimination process, if any one of the pivot elements $a_{11}^1, a_{22}^2, \ldots a_{nn}^n$ vanishes or becomes very small compared to other elements in that row then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This strategy is called pivoting.

The pivoting is of the following two types:

(1) Partial pivoting: In the first stage of elimination, the first column is searched for the largest

element in magnitude and this largest element is then brought at the position of the first pivote by interchanging the first row with the row having the largest element in magnitude in the first column. In the second stage of elimination, the second column is searched for the largest element in magnitude among the $(n-1)$ elements leaving the first element and then this largest element in magnitude is brought at the position of the second pivot by interchanging the second row with row having the largest element in the second column. This searching and interchanging of rows is repeated in all the $(n-1)$ stages of the elimination.

Complete pivoting:

We search the matrix A for the largest element in magnitude and bring it as the first pivot. This requires not only an interchanging of eqns but also an interchange of the position of the variables.

- Complete pivoting is much more complicated and is not often used.

→ Solve the system of eqns

$$x_1 + x_2 + x_3 = 6$$
$$3x_1 + 3x_2 + 4x_3 = 20$$
$$2x_1 + x_2 + 3x_3 = 13$$

using Gauss elimination method with partial pivoting.

Soln: Now let us try first to solve the system without pivoting.

we have $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

In the above matrix the second pivot has the value zero and the elimination procedure cannot be continued further unless, pivoting is used.

Now let us use the partial pivoting.

In the first column 3 is the largest element interchanging the rows 1st & 2nd,

we get $[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right]$

$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -1/3 & -2/3 \\ 0 & -1 & 1/3 & -1/3 \end{array}\right]$ $R_2 \to R_2 - \frac{1}{3}R_1$, $R_3 \to R_3 - \frac{2}{3}R_1$

In the second column, 1 is the largest element in magnitude leaving the first element.

Interchange the second and third rows,

we have $[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & -1 & 1/3 & -1/3 \\ 0 & 0 & -1/3 & -2/3 \end{array}\right]$

Clearly the resultant matrix is in triangular form and no further elimination is required.

Using the back substitution method, we obtain the solution $x_3 = 2,\ x_2 = 1,\ x_1 = 3$.

→ solve the system of eqns

$0.0003x_1 + 1.566x_2 = 1.569$

$0.3454x_1 - 0.436x_2 = 3.018$

Using Gauss elimination method with and without pivoting.

Soln: <u>without pivoting.</u>

we have

$$[A|B] = \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -0.436 & 3.018 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1802.0 & -1803.0 \end{array}\right]$$

Clearly which is in triangular form and no further elimination is required.

Using back substitution method

we obtain the solution $x_2 = 1.001$
$x_1 = 3.333$

which is highly inaccurate compared to the exact solution.

<u>with pivoting:</u>

we interchange the 1^{st} and 2^{nd} rows

we get

$$[A|B] = \left[\begin{array}{cc|c} 0.3454 & -0.436 & 3.018 \\ 0.0003 & 1.566 & 1.569 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 0.345 & -0.436 & 3.018 \\ 0 & 1.566 & 1.566 \end{array}\right]$$

Clearly which is in triangular form and no further elimination is required.

Using back substitution method we obtain the solution $x_2 = 1$ & $x_1 = 10$

which is the <u>exact solution</u>.

H.W → Solve the system of eqns

$$x+y+z=7$$
$$3x+3y+4z=24$$
$$2x+y+3z=16$$ by Gaussian elimination method with partial pivoting

$\boxed{\text{Ans: } z=3,\ y=1,\ x=3}$

→ Solve the Gaussian elimination method with partial pivoting, the following system of eqns

$$0x_1+4x_2+2x_3+8x_4=24$$
$$4x_1+10x_2+5x_3+4x_4=32$$
$$4x_1+5x_2+6.5x_3+2x_4=26$$
$$9x_1+4x_2+4x_3+0x_4=21$$

$\boxed{\text{Ans: } x_1=1,\ x_2=1,\ x_3=2,\ x_4=2}$

Gauss-Jordan Elimination method:

This method is a variation of the Gauss elimination method.

In the Gauss elimination method, using elementary row operations, we transform the matrix A to an upper triangular matrix U and obtain the solution by back substitution method.

In Gauss-Jordan elimination method not only the elements below the diagonal but also the elements above the diagonal of A are made zero at the same time.

In otherwords, we transform the matrix A to a diagonal matrix D. This diagonal matrix may then be reduced to an identity matrix by dividing each row by its pivot element.

Alternatively, the diagonal elements can also be made unity at the same time when the reduction is performed.

This transforms the coefficient matrix into an identity matrix, on completion of the Gauss-Jordan method, we have

$$[A|B] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|d]$$

The solution is given by

$$x_i = d_i, \ i = 1, 2, \ldots n.$$

- pivoting can be used to make the pivot non-zero or to make it the largest element in magnitude in that column as discussed earlier.
- Generally the Gauss-Jordan elimination method requires more number of operations compared to the Gaussian elimination method. We therefore, do not use this method for solving system of eqns but is very commonly used for finding the inverse matrix.

This is done by augmenting the matrix 'A' by the identity matrix I of the order same as that of A. Using elementary row operations on the augmented matrix $[A|I]$ we reduce the matrix A to the form 'I' and in the process the matrix I is transformed to A^{-1}.

i.e,

$$[A|I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|A^{-1}].$$

→ Solve the system of eqns

$x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_2 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$ by using the Gauss-Jordan method with pivoting.

Sol^n: we have

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 5 & 3 & 4 \end{array}\right] \quad \text{(Interchanging first \& second row)}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 0 & \frac{1}{4} & \frac{5}{4} & -\frac{1}{2} \\ 0 & \frac{11}{4} & \frac{15}{4} & -\frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{1}{4}R_1 \\ R_3 \to R_3 - \frac{3}{4}R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 3 & -1 & 6 \\ 0 & \frac{11}{4} & \frac{15}{4} & -\frac{1}{2} \\ 0 & \frac{1}{4} & \frac{5}{4} & -\frac{1}{2} \end{array}\right] \quad \text{(Interchanging 2nd \& 3rd row)}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & -\frac{56}{11} & \frac{72}{11} \\ 0 & \frac{11}{4} & \frac{15}{4} & -\frac{1}{2} \\ 0 & 0 & \frac{10}{11} & -\frac{5}{11} \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - \frac{1}{11}R_2 \\ R_1 \to R_1 - \frac{12}{11}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & \frac{11}{4} & 0 & \frac{11}{8} \\ 0 & 0 & \frac{10}{11} & \frac{5}{11} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + \frac{56}{10}R_3 \\ R_2 \to R_2 - \frac{33}{8}R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{R_1}{4} \\ R_2 \to \frac{4}{11}R_2 \\ R_3 \to \frac{11}{10}R_3 \end{array}$$

which is the desired form.

$\therefore x_1 = 1;\ x_2 = 1/2;\ \underline{x_3 = -1/2}.$

→ Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$ using Gauss - Jordan method.

Sol:

Using the augmented matrix $[A|I]$,

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 2 & -3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to \frac{1}{3}R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{11}{3} & -\frac{7}{3} & -\frac{2}{3} & 1 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & \frac{7}{11} & 2/11 & -3/11 & 0 \\ 0 & -\frac{7}{3} & \frac{1}{3} & -\frac{1}{3} & 0 & 1 \end{array}\right] \quad R_2 \to -\frac{3}{11} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5/11 & 3/11 & 1/11 & 0 \\ 0 & 1 & 7/11 & 2/11 & -3/11 & 0 \\ 0 & 0 & 20/11 & 1/11 & -7/11 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{1}{3}R_2 \\ R_3 \to R_3 + \frac{7}{3}R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5/11 & 3/11 & 1/11 & 0 \\ 0 & 1 & 7/11 & 2/11 & -3/11 & 0 \\ 0 & 0 & 1 & 1/20 & -7/20 & 11/20 \end{array}\right] . \quad R_3 \to \frac{11}{20} R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/4 & -1/4 \\ 0 & 1 & 0 & 3/20 & -1/20 & -7/20 \\ 0 & 0 & 1 & 1/20 & -7/20 & 11/20 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{5}{11}R_3 \\ R_2 \to R_2 - \frac{7}{11}R_3 \end{array}$$

$$= [I | A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 1/4 & 1/4 & -1/4 \\ 3/20 & -1/20 & -7/20 \\ 1/20 & -7/20 & 11/20 \end{bmatrix}$$

→ Find the inverse of the coefficient matrix of the system
$x_1+x_2+x_3=1$
$4x_1+3x_2-x_3=6$
$3x_1+5x_2+3x_3=4$ by the Gauss Jordan method with partial pivoting and hence solve the system.

Sol^n: we have
$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\6\\4\end{bmatrix}$$
Using the augmented matrix $[A|I]$, we obtain
$$\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\4&3&-1&0&1&0\\3&5&3&0&0&1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}4&3&-1&0&1&0\\1&1&1&1&0&0\\3&5&3&0&0&1\end{array}\right]\quad R_1\leftrightarrow R_2$$
$$\sim\left[\begin{array}{ccc|ccc}1&3/4&-1/4&0&1/4&0\\1&1&1&1&0&0\\3&5&3&0&0&1\end{array}\right]\quad R_1\to\tfrac{1}{4}R_1$$
$$\sim\left[\begin{array}{ccc|ccc}1&3/4&-1/4&0&1/4&0\\0&1/4&5/4&1&-1/4&0\\1&11/4&15/4&0&-3/4&1\end{array}\right]\quad \begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-3R_1\end{array}$$
continuing in this way, we get
$$\sim\left[\begin{array}{ccc|ccc}1&0&0&7/5&1/5&-2/5\\0&1&0&-3/2&0&1/2\\0&0&1&11/10&-1/5&-1/10\end{array}\right]=[I|A^{-1}]$$
∴ Solution of the system is
$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=A^{-1}B=\begin{bmatrix}7/5&1/5&-2/5\\-3/2&0&1/2\\11/10&-1/5&-1/10\end{bmatrix}\begin{bmatrix}1\\6\\4\end{bmatrix}=\begin{bmatrix}1\\1/2\\-1/2\end{bmatrix}$$

→ Find the inverse of the following matrices by using Gauss-Jordan method.

(1) $A=\begin{bmatrix}2&0&0&0\\1&1/2&0&0\\2&0&-3&0\\1&-7/2&-17&\frac{55}{3}\end{bmatrix}$ (2) $A=\begin{bmatrix}1&3/2&2&1/2\\0&1&-4&1\\0&0&1&2/3\\0&0&0&1\end{bmatrix}$ (3) $A=\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}$ (4) $A=\begin{bmatrix}1&1&2\\1&2&4\\2&4&7\end{bmatrix}$

→ Using Gauss ~~elimination~~ Jordan method, find the inverse of the matrix $A=\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix}$

3.8.1 Gaussian Elimination Method

In this method, if A is a given matrix, for which we have to find the inverse; at first, we place an identity matrix, whose order is same as that of A, adjacent to A which we call an *augmented* matrix. Then the inverse of A is computed in two stages. In the first stage, A is converted into an upper triangular form, using Gaussian elimination method as discussed in Section 3.2. In the second stage, the above upper triangular matrix is reduced to an identity matrix by row transformations. All these operations are also performed on the adjacently placed identity matrix. Finally, when A is transformed into an identity matrix, the adjacent matrix gives the inverse of A. In order to increase the accuracy of the result, it is essential to employ partial pivoting. To understand the sequence of the steps involved, we consider an example.

Example 3.9 Use the Gaussian elimination method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

Solution At first, we place an identity matrix of the same order adjacent to the given matrix. Thus, the augmented matrix can be written as

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

Stage I (Reduction to upper triangular form): Let R_1, R_2 and R_3 denote the first, second and third rows of a matrix. In the first column of Eq. (1), 4 is the largest element, thus interchanging R_1 and R_2 to bring the pivot element 4 to the place of a_{11}, we have the augmented matrix in the form

$$\left[\begin{array}{ccc:ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Divide R_1 by 4 to get

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{3}$$

Perform $R_2 - R_1 \longrightarrow R_2$, which gives

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

Perform $R_3 - 3R_1 \longrightarrow R_3$ in Eq. (4), which yields

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \end{array}\right] \tag{5}$$

Now, looking at the second column for the pivot, the max (1/4, 11/4) is 11/4. Therefore, we interchange R_2 and R_3 in Eq. (5) and get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{11}{4} & \frac{15}{4} & 0 & -\frac{3}{4} & 1 \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \tag{6}$$

Now, divide R_2 by the pivot $a_{22} = 11/4$, and obtain

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & \frac{1}{4} & \frac{5}{4} & 1 & -\frac{1}{4} & 0 \end{array}\right] \tag{7}$$

Performing $R_3 - (1/4)R_2 \longrightarrow R_3$ in (7) yields

$$\left[\begin{array}{ccc|ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & \frac{10}{11} & 1 & -\frac{2}{11} & -\frac{1}{11} \end{array}\right] \tag{8}$$

Finally, we divide R_3 by (10/11), thus getting an upper triangular form

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{15}{11} & 0 & -\frac{3}{11} & \frac{4}{11} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \tag{9}$$

Stage II (Reduction to an identity matrix): Multiply R_3 by –1/4 and 15/11 respectively and subtract it from R_1 and R_2 of Eq. (9), we get

$$\left[\begin{array}{ccc:ccc} 1 & \frac{3}{4} & 0 & \frac{11}{40} & \frac{1}{5} & -\frac{1}{40} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \tag{10}$$

Finally, performing $R_1 - (3/4)\ R_2 \longrightarrow R_1$ in Eq. (10), we obtain

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right]$$

Thus, we have

$$A^{-1} = \begin{bmatrix} \frac{7}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{bmatrix} \tag{11}$$

We can easily cheque $[A]\ [A^{-1}] = [I]$.

3.8.2 Gauss–Jordan Method

This method is similar to Gaussian elimination method, with the essential difference that the stage *I* of reducing the given matrix to an upper triangular form is not needed. However, the given matrix can be directly reduced to an identity matrix using elementary row transformations. This technique is illustrated in the following example.

Example 3.10 Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

by Gauss–Jordan method.

Solution Let R_1, R_2 and R_3 denote the first, second and third rows of a matrix. We place an identity matrix adjacent to the given matrix as a first step and the resulting augmented matrix is given by

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{1}$$

Performing $R_2 - 4R_1 \longrightarrow R_2$, we get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \tag{2}$$

Now, performing $R_3 - 3R_1 \longrightarrow R_3$, we obtain

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \tag{3}$$

Carrying out further operations $R_2 + R_1 \longrightarrow R_1$ and $R_3 + 2R_2 \longrightarrow R_3$, we arrive at

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \tag{4}$$

Now, dividing the third row by -10, we get

proceeding in this way we get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{10} & -\frac{1}{5} & -\frac{1}{10} \end{array}\right] \tag{5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right] = [I \mid A^{-1}].$$

Indirect methods:-

Iteration Method:

Direct methods provide the exact solution in a finite number of steps provided exact arithmetic is used and there is no round-off error. Also, direct methods are generally used when the matrix A is having few zero elements, and the order of the matrix is not very large say $n<50$.

Iterative methods, on the other hand, start with an initial approximation and by applying a suitably choosen algorithm, lead to successively better approximations. Even if the process converges, it gives only an approximate solution. These methods are generally used when the matrix A is sparse (many elements are zero) and the order of the matrix 'A' is very large say $n>50$. Sparse matrices have very few non-zero elements. In most cases these non-zero elements lie on or near the main diagonal giving rise to trigonal, or five diagonal matrix systems.

— It may be noted that there are no fixed rules to decide when to use direct methods and when to use iterative methods.

However, when the coefficient matrix is sparse or large, the use of iterative methods is ideally suited to find the solution which take advantage of the sparse nature of the matrix involved.

The General iteration method

We start with some initial approximate solution vector $x^{(0)}$ and generate a sequence of approximations $\{x^{(k)}\}$ which converge to the exact solution vector x as $k \to \infty$. If the method is convergent, each iteration produces a better approximation to the exact solution. We repeat the iterations till the required accuracy is obtained.

Therefore, in an iterative method the amount of computation depends on the desired accuracy whereas in direct methods the amount of computation is fixed. The number of iterations needed to obtain the desired accuracy also depends on the initial approximation, closer the initial approximation to the exact solution, faster will be the convergence.

→ Now consider the system of eqns

$$Ax = B \quad \text{——} \quad (1)$$

where A is $n\times n$ non-singular matrix.

$$\Rightarrow \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{nn}x_n = b_n \end{array}\right\} \text{——} (2)$$

we assume the diagonal coefficients $a_{ii} \neq 0$ $(i = 1, 2, \cdots n)$

If some $a_{ii} = 0$ then we arrange the eqns, so that this condition holds

Now we rewrite the system ② as

$$\left.\begin{aligned} x_1 &= -\frac{1}{a_{11}}(a_{12}x_2 + a_{13}x_3 + \cdots\cdots + a_{1n}x_n) + \frac{b_1}{a_{11}} \\ x_2 &= -\frac{1}{a_{22}}(a_{21}x_1 + a_{23}x_3 + \cdots\cdots + a_{2n}x_n) + \frac{b_2}{a_{22}} \\ & - - - - - - - - - - - - \\ & - - - - - - - - - - - - \\ x_n &= -\frac{1}{a_{nn}}(a_{n1}x_1 + a_{n2}x_2 + \cdots\cdots + a_{n\,n-1}x_{n-1}) + \frac{b_n}{a_{nn}} \end{aligned}\right\} \quad ③$$

In matrix form, system ③ can be written as

$$\underline{X = HX + C}.$$

$$\text{where } H = \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & -\frac{a_{13}}{a_{11}} & \cdots\cdots & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}} & 0 & -\frac{a_{23}}{a_{22}} & \cdots\cdots & -\frac{a_{2n}}{a_{22}} \\ - & - & - & - & - \\ - & - & - & - & - \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots\cdots & -\frac{a_{n\,n-1}}{a_{nn}} & 0 \end{bmatrix} \quad ④$$

and the elements of C are $c_i = \frac{b_i}{a_{ii}}$ $(i = 1, 2, \ldots n)$

To solve ③, we make an initial guess $x^{(0)}$ of the solution vector and substitute into the RHS of eqn ③. The solution of equation ③ will then yield a vector $x^{(1)}$, which hopefully is a better approximation to the solution than $x^{(0)}$. We then substitute $x^{(1)}$ into the RHS of eqn ③ and get another approximation $x^{(2)}$. We continue in this manner until the successive iterations $x^{(k)}$ have converged to the required number of significant figures.

In general we can write the iteration method for solving the linear system

of eqns (1) in the form $x^{(k+1)} = Hx^{(k)} + C$, ——— (5)
$k = 0, 1, 2 \ldots$

where $x^{(k)}$ and $x^{(k+1)}$ are the approximations for x at the k^{th} and $(k+1)^{th}$ iterations respectively.

H is called the iteration matrix and depends on A and C is a column vector and depends on both A and B.

The matrix H is generally a constant matrix.

when the method (5) is cgt, then

$$\underset{k\to\infty}{Lt}\, x^{(k)} = \underset{k\to\infty}{Lt}\, x^{(k+1)} = x.$$

and we obtain from eqn (5),

$$\boxed{X = HX + C} \quad \text{——— (6)}$$

If we define the error vector at the k^{th} iteration as $\epsilon^{(k)} = x^{(k)} - x$ ——— (7)

then subtracting eqn (6) from eqn (5) (ie (5) − (6) ⇒)

we obtain

$$x^{(k+1)} - x = H\left[x^{(k)} - x\right]$$

$$\Rightarrow x^{(k+1)} - x = H\epsilon^{(k)}$$

$$\Rightarrow \epsilon^{(k+1)} = H\epsilon^{(k)} \quad \left(\because \epsilon^{(k)} = x^{(k)} - x\right) \text{——— (8)}$$

$$(8) \Rightarrow \epsilon^{(k)} = H\epsilon^{(k-1)}$$
$$= H\left(H\epsilon^{(k-2)}\right)$$
$$= H^2\epsilon^{(k-2)}$$
$$= H^2\left(H\epsilon^{(k-3)}\right)$$
$$= H^3\epsilon^{(k-3)} = \ldots\ldots = H^k\epsilon^{(0)}.$$

where $\epsilon^{(0)}$ is the error in the initial approximate vector. Thus, for the convergence of the iterative method, we must have $\underset{k\to\infty}{Lt}\, \epsilon^{(k)} = 0$ independent of $\epsilon^{(0)}$

Gauss-Seidel Iteration method:

Consider the system of eqns (2) written in form (3)

For this system of eqns, we define the Gauss-Seidel method as:

$$\left.\begin{aligned}
x_1^{(k+1)} &= -\frac{1}{a_{11}}\left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1\right)\\
x_2^{(k+1)} &= -\frac{1}{a_{22}}\left(a_{22}x_1^{(k+1)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2\right)\\
&\vdots\\
x_n^{(k+1)} &= -\frac{1}{a_{nn}}\left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \cdots + a_{n,n-1}x_{n-1}^{(k+1)} - b_n\right)
\end{aligned}\right\} \rightarrow \text{(I)}$$

(or)

$$x_i^{(k+1)} = -\frac{1}{a_{ii}}\left[\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} - b_i\right]$$

$i = 1, 2, 3, \ldots n$

Note that, in the first eqn of system (I), we substitute the initial approximation $\left(x_2^{(0)}, x_3^{(0)}, \ldots x_n^{(0)}\right)$ on RHS.

In the second eqn, we substitute $\left(x_1^{(1)}, x_3^{(0)}, \ldots x_n^{(0)}\right)$ on RHS.

In third eqn, we substitute $\left(x_1^{(1)}, x_2^{(1)}, x_4^{(0)}, \ldots x_n^{(0)}\right)$.

We continue in this manner until all the components have been improved. At the end of this first iteration, we will have an improved vector $\left(x_1^{(1)}, x_2^{(1)}, \ldots x_n^{(1)}\right)$

The entire process is then repeated. In other words the method uses an improved component as soon as it becomes available. It is for this reason the method is also called the method of successive displacements.

→ perform four iterations (rounded to four decimal places) using the Gauss-seidel method for solving the system of eqns

$$\begin{bmatrix} -8 & 1 & 1 \\ 1 & -5 & 1 \\ 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \\ 7 \end{bmatrix} \quad \text{with } X^{(0)} = 0$$

The exact solution is $X = (-1\ -4\ -3)^T$

Soln: Given system is

$$\left.\begin{aligned} -8x_1 + x_2 + x_3 &= 1 \\ x_1 - 5x_2 + x_3 &= 16 \\ x_1 + x_2 - 4x_3 &= 7 \end{aligned}\right\} \text{—— (i)}$$

$$\Rightarrow \left.\begin{aligned} x_1 &= -\frac{1}{8}(1 - x_2 - x_3) \\ x_2 &= -\frac{1}{5}(16 - x_1 - x_3) \\ x_3 &= -\frac{1}{4}(7 - x_1 - x_2) \end{aligned}\right\} \text{—— (ii)}$$

By the Gauss-seidel method, System (ii) can be written as

$$x_1^{(k+1)} = -\frac{1}{8}\left(1 - x_2^{(k)} - x_3^{(k)}\right)$$

$$x_2^{(k+1)} = -\frac{1}{5}\left(16 - x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = -\frac{1}{4}\left(7 - x_1^{(k+1)} - x_2^{(k+1)}\right)$$

where $k = 0, 1, 2, \ldots$

Now taking $X^{(0)} = 0$, we obtain the following iterations.

$k = 0$

$$x_1^{(1)} = -\frac{1}{8}\left(1 - x_2^{(0)} - x_3^{(0)}\right) = -\frac{1}{8}(1 - 0 - 0)$$

$$= -\frac{1}{8} = -0.125$$

$$x_2^{(1)} = -\frac{1}{5}\left[16 - x_1^{(1)} - x_3^{(0)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.125 - 0\right]$$
$$= -3.225$$
$$x_3^{(1)} = -\frac{1}{4}\left[7 - x_1^{(1)} - x_2^{(1)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.125 + 3.225\right]$$
$$= -2.5875$$

K=1:

$$x_1^{(2)} = -\frac{1}{8}\left[1 - x_2^{(1)} - x_3^{(1)}\right]$$
$$= -\frac{1}{8}\left[1 + 3.225 + 2.5875\right]$$
$$= -0.8516$$
$$x_2^{(2)} = -\frac{1}{5}\left[16 - x_1^{(2)} - x_3^{(1)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.8516 + 2.5875\right]$$
$$= -3.8878$$
$$x_3^{(2)} = -\frac{1}{4}\left[7 - x_1^{(2)} - x_2^{(2)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.8516 + 3.8878\right]$$
$$= -2.9349$$

K=2:

$$x_1^{(3)} = -\frac{1}{8}\left[1 - x_2^{(2)} - x_3^{(2)}\right]$$
$$= -\frac{1}{8}\left[1 + 3.8878 + 2.9349\right]$$
$$= -0.9778$$
$$x_2^{(3)} = -\frac{1}{5}\left[16 - x_1^{(3)} - x_3^{(2)}\right]$$
$$= -\frac{1}{5}\left[16 + 0.9778 + 2.9349\right]$$
$$= -3.9825$$
$$x_3^{(3)} = -\frac{1}{4}\left[7 - x_1^{(3)} - x_2^{(3)}\right]$$
$$= -\frac{1}{4}\left[7 + 0.9778 + 3.9825\right]$$
$$= -2.9901$$

K=3:

$$x_1^{(4)} = -\frac{1}{8}[1+3.9825+2.9901]$$
$$= -0.9966$$

$$x_2^{(4)} = -\frac{1}{5}[16+0.9966+2.9901]$$
$$= -3.9973$$

$$x_3^{(4)} = -\frac{1}{4}[7+0.9966+3.9973]$$
$$= -2.9985$$

which is a good approximation to the exact solution $x = (-1\ -4\ -3)^T$ with maximum error 0.0034

→ Solve the following eqns

IAS-2014

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1$$

using Gauss-Seidel method of iteration and perform three iterations.

Sol: The given system of eqns can be written as

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(7+x_2) \\ x_2 &= \frac{1}{2}(1+x_1+x_3) \\ x_3 &= \frac{1}{2}(1+x_2) \end{aligned}\right] \text{—— (1)}$$

By the Gauss-Seidel method, system (1) can be written as

$$x_1^{(k+1)} = \frac{1}{2}\left(7+x_2^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{2}\left(1+x_1^{(k+1)}+x_3^{(k)}\right)$$
$$x_3^{k+1} = \frac{1}{2}\left(1+x_2^{(k+1)}\right)$$

where $k=0,1,2,\ldots$

Now taking $x^{(0)}=0$, we obtain the following iterations (18)

$\underline{k=0}$: $x_1^{(1)} = \frac{1}{2}(7+0) = \frac{7}{2} = 3.5$

$$x_2^{(1)} = \frac{1}{2}\left(1+x_1^{(1)}+x_3^{(0)}\right)$$
$$= \frac{1}{2}(1+3.5+0)$$
$$= \frac{4.5}{2} = 2.25$$

$$x_3^{(1)} = \frac{1}{2}\left(1+x_2^{(1)}\right)$$
$$= \frac{1}{2}(1+2.25) = \frac{1}{2}(3.25)$$
$$= 1.625$$

$\underline{k=1}$:

$$x_1^{(2)} = \frac{1}{2}\left(7+x_2^{(1)}\right)$$
$$= \frac{1}{2}(7+2.25) = \frac{9.25}{2} = 4.625$$

$$x_2^{(2)} = \frac{1}{2}\left(1+x_1^{(2)}+x_3^{(1)}\right)$$
$$= \frac{1}{2}(1+4.625+1.625)$$
$$= \frac{1}{2}(7.250) = 3.625$$

$$x_3^{(2)} = \frac{1}{2}\left(1+x_2^{(2)}\right)$$
$$= \frac{1}{2}(1+3.625) = \frac{4.625}{2} = 2.3125$$

$\underline{k=2}$:

$$x_1^{(3)} = \frac{1}{2}\left(7+x_2^{(2)}\right)$$
$$= \frac{1}{2}(7+3.625) = \frac{10.625}{2} = 5.3125$$

$$x_2^{(3)} = \frac{1}{2}\left(1+x_1^{(3)}+x_3^{(2)}\right)$$
$$= \frac{1}{2}(1+5.3125+2.3125)$$
$$= \frac{8.625}{2} = 4.3125$$

$$x_3^{(3)} = \frac{1}{2}\left(1+x_2^{(3)}\right)$$
$$= \frac{1}{2}(1+4.3125) = 2.6563.$$

→ Use the Gauss-seidel method for solving the following system of eqns

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

with $x^{(0)} = [0.5 \quad 0.5 \quad 0.5 \quad 0.5]^T$.

Compare the results with the exact solution is $x = [1 \ 1 \ 1 \ 1]^T$

IFoS-2011 (★)

2001, 30M → Using the Gauss-seidel method and starting solution $x_1 = x_2 = x_3 = 0$, determine the solution the solution in two iterations of the following system of eqns

$$10x_1 - x_2 - x_3 = 8$$
$$x_1 + 10x_2 + x_3 = 12$$
$$x_1 - x_2 + 10x_3 = 10$$

Compare the approximate solution with the exact solution

IFoS-12

2004, 15M. → Using Gauss-Seidel iterative method, find the solution of the following system:

$$4x - y + 8z = 26$$
$$5x + 2y - z = 6$$
$$x - 10y + 2z = -13$$ up to three iterations

→ Find the solution of the following system of eqns

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$
$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$
$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$
$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$ using Gauss seidel and perform the first five iterations

IAS-2012 → Solve the system eqns

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 2z = 25$$ by Gauss-seidel iterative method and perform the first three iterations