TEST SERIES – 2018

ANSWER KEY – TEST-1 (PAPER-I)

Linear Algebra, Calculus and 3D

1.(a) (i) Let V be a vector space over K. Show that, if for $\alpha \in K$ and $x \in V$, $\alpha x = x$ then $\alpha = 1$ or $x = 0$.

(ii) Let V be a vector space over $\mathbb{C}$. Define another scalar multiplication $*$ on V: $\alpha * x = Re(\alpha)x$, $\alpha \in \mathbb{C}$, $x \in V$, where $Re(\alpha)$ is the real part of α. Is V a vector space w.r.t original addition and scalar multiplication $*$?.

Sol (i) we have $\alpha x = x$; $\alpha \in K$; $x \in X$

$\Rightarrow \alpha x = 1.x$; $1 \in K$

$\Rightarrow (\alpha - 1)x = 0$ —— (1)

if $x \neq 0$ then to prove that $\alpha = 1$.

If possible let $\alpha \neq 1$, i.e $\alpha - 1 \neq 0$.

$\therefore (\alpha - 1)^{-1}$ exists in K.

we have $(\alpha-1)x = 0 \Rightarrow (\alpha-1)^{-1}(\alpha-1)x = (\alpha-1)^{-1}0$

$\Rightarrow [(\alpha-1)^{-1}.(\alpha-1)]x = 0$

$\Rightarrow x = 0$.

$\therefore$ which is a contradiction to the fact that $x \neq 0$.

$\therefore \alpha - 1 \neq 0$ is wrong

$\therefore \alpha - 1 = 0$

$\Rightarrow \alpha = 1$ and $x \neq 0$.

If $(\alpha-1)x = 0$ and $\alpha - 1 \neq 0$

then $(\alpha-1)^{-1}$ exists in K.

$\therefore (\alpha-1)x = 0 \Rightarrow (\alpha-1)^{-1}((\alpha-1)x) = (\alpha-1)^{-1}0$

$\Rightarrow (\alpha-1)^{-1}.(\alpha-1)x = 0$.

$\Rightarrow 1.x = 0$.

$\Rightarrow x = 0$.

(ii) let $V(\mathbb{C})$ be a given vector space.

Define another scalar multiplication $*$ on V $\alpha * x = Re(\alpha)x$; $\alpha \in \mathbb{C}$; $x \in V$. —(1)

clearly $V(C)$ is not a vector space w.r.t original addition and scalar multiplication $*$ defined in (1).

because. $\alpha = a+ib$, $\beta = c+id \in C$. and $x \in V$.

we have $(\alpha\beta)x = [(ac-bd)+i(ad+bc)]x$

$= (ac-bd)x$ (by (1))

and $\alpha(\beta x) = (a+ib)[(c+id)x]$

$= (a+ib)[cx]$ (by (1))

$= (ac)x$ (by (1))

$\therefore (\alpha\beta)x \neq \alpha(\beta x)$

1(b) Suppose U and W are two-dimensional subspace of R^3. Show that $U \cap W \neq \{0\}$. Inparticular, find the possible dimensions of $U \cap W$.

Solⁿ: Suppose $U = W$.

Then $U \cap W = U = W$.

and hence $\dim(U \cap W) = 2$

$(\because \dim U = \dim W = 2)$

Suppose $U \neq W$.

Then $U + W$ properly contains U (and W).

Hence $\dim(U+W) > \dim U = 2$.

But $U + W \subseteq R^3$, which has dimension 3.

$\therefore \dim(U+W) = 3$.

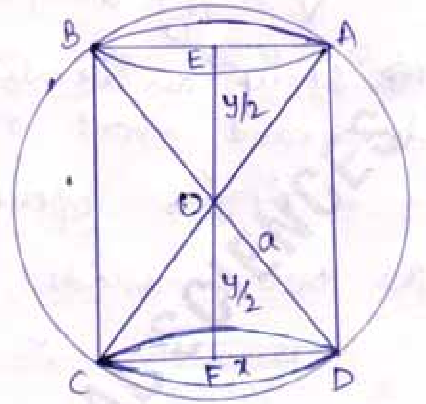
$\therefore$ By using the theorem

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$$
$$= 2+2-3$$
$$= 1.$$

i.e, $U \cap W$ is a line through the origin

1(c) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$.

Sol^n: Let ABCD be the given sphere of radius a so that $OD = a$.



Let ABCD be a right circular cylinder of radius $FD(=x)$ and height $EF(=y)$ which can be inscribed in the given sphere.

From ΔOFD,

$$x^2 + y^2/4 = a^2 \Rightarrow x^2 = a^2 - y^2/4$$

Note that if O is the Centre of the sphere then it will also be the middle point of the height of the cylinder. Let V be the volume of the cylinder. then, we have

$$V = \pi x^2 y$$
$$= \pi(a^2 - y^2/4) y$$
$$= \pi(a^2 y - y^3/4) \quad \text{--- (1)}$$
$$\frac{dv}{dy} = \pi\left(a^2 - \frac{3y^2}{4}\right) \quad \text{--- (2)}$$

For maximum or minimum value of V, $dv/dy = 0$

So that $\pi\left(a^2 - \frac{3y^2}{4}\right) = 0 \Rightarrow y = 2a/\sqrt{3}$, y being +ve only.

Again, from (2) $d^2y/dx^2 = -(3\pi y/2)$, which is −ve when $y = 2a/\sqrt{3}$

$\therefore$ $y = 2a/\sqrt{3}$ gives the cylinder of maximum volume inscribed in a given sphere.

1(d). Find the volume of the region in $\mathbb{R}^3$ bounded by the paraboloids with equations $z = 10 - x^2 - y^2$ and $z = x^2 + y^2 - 8$

sol^n:

We will find volume V by evaluating

$$V = \iiint_D dx\,dy\,dz .$$

Note that the paraboloid $z = 10 - x^2 - y^2$ opens downward about z-axis and the paraboloid $z = x^2 + y^2 - 8$ opens upward about the z-axis.

The two paraboloids intersect when

$$10 - x^2 - y^2 = x^2 + y^2 - 8$$

i.e, when $x^2 + y^2 = 9$.

Now we may describe the region in the xy-plane described by $x^2+y^2 \leq 9$ as the set of points (x, y) for which $-3 \leq x \leq 3$ and, for every such fixed x

$$-\sqrt{3-x^2} \leq y \leq \sqrt{3-x^2}.$$

Moreover, once we have fixed x & y so that (x, y) is inside the circle $x^2+y^2=9$, then (x, y, z) is in D provided $x^2+y^2-8 \leq z \leq 10 - x^2 - y^2$.

Hence, we have

$$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2-8}^{10-x^2-y^2} dz\,dy\,dx.$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left[z\right]_{x^2+y^2-8}^{10-x^2-y^2} dy\,dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (18 - 2x^2 - 2y^2)\,dy\,dx$$

$$= \int_{-3}^{3}\left(18y-2x^2y-\frac{2y^3}{3}\right)_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3}\left[36\sqrt{9-x^2}-4x^2\sqrt{9-x^2}-\frac{4}{3}(9-x^2)^{3/2}\right] dx$$

$$= \int_{-3}^{3}\sqrt{9-x^2}\left[36-4x^2-\frac{4}{3}(9-x^2)\right] dx = \frac{8}{3}\int_{-3}^{3}(9-x^2)^{3/2} dx$$

Using the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta d\theta$

$$\therefore V = \frac{8}{3}\int_{-3}^{3}(9-x^2)^{3/2} dx$$

$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2}(9-9\sin^2\theta)^{3/2}\, 3\cos\theta\, d\theta$$

$$= 216\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta$$

$$= 216\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 216\int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\, d\theta$$

$$= 216\left[(\theta)_{-\pi/2}^{\pi/2} + (\sin 2\theta)_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{2} d\theta\right.$$

$$= 54\pi + 27(\theta)_{-\pi/2}^{\pi/2} + \frac{27}{4}(\sin 4\theta)_{-\pi/2}^{\pi/2}$$

$$= 81\pi.$$

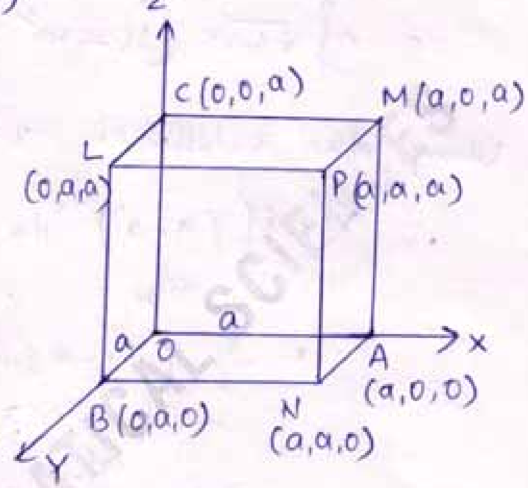
1(e) → Find the angle between the diagonals of a cube.

Sol^n: Take O, a corner of the cube as origin and OA, OB, OC the three edges through it, as the axes.

Let $OA = OB = OC = a$. Then the coordinates of the various points are

$O(0,0,0)$, $A(a,0,0)$, $B(0,a,0)$, $C(0,0,a)$, $L(0,a,a)$,

$M(a,0,a)$, $N(a,a,0)$, $P(a,a,a)$.



The four diagonals are AL, BM, CN and OP.

The direction ratio's of the diagonal AL are

$0-a,\ a-0,\ a-0$

| using $x_2-x_1, y_2-y_1, z_2-z_1$

i.e. $-a, a, a$

$\Rightarrow -1, 1, 1$ (Cancelling a)

Similarly the direction ratio's of the diagonal BM are

$a-0,\ 0-a,\ a-0$

$\Rightarrow a, -a, a$

$\Rightarrow 1, -1, 1$ (Cancelling a)

If θ be the angle between the diagonals AL and BM, then

$$\cos\theta = \frac{1(-1)+(-1)(1)+1(1)}{\sqrt{1+1+1}\sqrt{1+1+1}}$$

$$\left|\because \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right.$$

$$= \frac{-1-1+1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3} = \frac{1}{3} \text{ (numerically)}$$

$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$

Similarly the angle between other two diagonals is also the same.

2(a)(i) Let $A=\begin{pmatrix}2&2\\1&3\end{pmatrix}$. Is A diagonalizable? If yes, find P such that $P^{-1}AP$ is diagonal.

(b) If interchanging the eigen vectors of P, does P still diagonalize A?

Sol^n: we have $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}2-\lambda & 2\\ 1 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (2-\lambda)(3-\lambda)-2=0$$

$$\Rightarrow \lambda^2-5\lambda+4=0$$

$$\Rightarrow \lambda=1,4.$$

which are the eigenvalues of A.

Let us find the eigenvector corresponding to $\lambda=1$

i.e, $(A-\lambda I)X=0$

$$\Rightarrow \begin{pmatrix}1&2\\1&2\end{pmatrix}X=0$$

$$\Rightarrow x+2y=0$$

$\therefore X_1=\begin{pmatrix}2\\-1\end{pmatrix}$ is a non-zero solution of the system and so is an eigenvector of A corresponding to $\lambda=1$.

Let us find the eigen vector corresponding to $\lambda=4$

i.e $(A-I\lambda)X=0$

$$\Rightarrow \begin{bmatrix}-2&2\\1&-1\end{bmatrix}X=0$$

$\Rightarrow x-y=0$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a non-zero solution and so is an eigenvector of A corresponding to $\lambda=4$.

Since A has two independent eigen vectors.

A is diagonalizable.

Let $P = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) If interchanging the eigenvectors of P.

i.e, $P = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$, then P still diagonalize A.

However, now $P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

In other words, the order of the eigen values in $P^{-1}AP$ corresponds to the eigenvectors in P.

2(a)(ii) Show that no skew-symmetric matrix can be of rank 1.

Sol^n: Let

$$A = \begin{bmatrix} 0 & h & g & l \\ -h & 0 & f & m \\ -g & -f & 0 & n \\ -l & -m & -n & 0 \end{bmatrix}$$

be an 4×4 skew-symmetric matrix

If h, g, l, m, n are all equal to zero, the matrix A will be of rank zero. If at least one of these elements, say, g is not equal to zero, then at least one 2-rowed minor of the matrix A, i.e. the minor $\begin{vmatrix} 0 & g \\ -g & 0 \end{vmatrix}$ is not equal to zero as its value is g^2 which is not equal to zero.

$\therefore$ the rank of the matrix A is ≥ 2.

Thus in either case the rank of the matrix A is not equal to one.

$\boxed{\text{Note}}$: The method of proof can be given in the case of a skew symmetric matrix of any order.

2(b) Prove that the function $f(x,y) = \sqrt{|xy|}$ is not differentiable at the point $(0,0)$, but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

Sol^n: Now at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

If the function is differentiable at $(0,0)$ then by definition

$$f(h,k) - f(0,0) = 0h + 0k + h\phi + k\varphi$$

where ϕ and φ are functions of h and k, and tend to zero as $(h,k)\to(0,0)$

putting $h=\rho\cos\theta$, $k=\rho\sin\theta$ and dividing by ρ, we get

$$|\cos\theta\sin\theta|^{1/2} = \phi\cos\theta + \varphi\sin\theta$$

Now for arbitrary θ, $\rho\to 0$ implies that $(h,k)\to(0,0)$

Taking the limits as $\rho\to 0$, we get

$$|\cos\theta\sin\theta| = 0$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at $(0,0)$ and Consequently the partial derivatives f_x, f_y cannot be continuous at $(0,0)$, for otherwise the function would be differentiable there at,

Let us now see that it is actually so

For $(x,y)\neq(0,0)$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{|x+h|\,|y|}-\sqrt{|x||y|}}{h}$$

$$= \lim_{h\to 0}\sqrt{|y|}\,\frac{|x+h|-|x|}{h\left[\sqrt{|x+h|}-\sqrt{|x|}\right]}$$

Now as $h\to 0$, we can take $x+h>0$, i.e. $|x+h| = x+h$, when $x>0$ and $x+h<0$ (or) $|x+h| = -(x+h)$, when $x<0$

$$\therefore \quad f_x(x,y) = \begin{cases} \frac{1}{2}\sqrt{\frac{|y|}{|x|}}, & \text{when } x>0 \\ -\frac{1}{2}\sqrt{\frac{|y|}{|x|}}, & \text{when } x<0 \end{cases}$$

Similarly

$$f_y(x,y) = \begin{cases} \frac{1}{2}\sqrt{\frac{|x|}{|y|}}, & \text{when } y>0 \\ -\frac{1}{2}\sqrt{\frac{|x|}{|y|}}, & \text{when } y<0 \end{cases}$$

which are, obviously, not continuous at the origin.

2(c) → A cone has as base the circle $x^2+y^2+2ax+2by=0$, $z=0$ and passes through the fixed point $(0,0,c)$. If the section of the cone by zx-plane is a rectangular hyperbola, Prove that the vertex lies on a fixed circle.

Sol^n: Let (α,β,γ) be the vertex of the cone. Any line through (α,β,γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ — ①

It meets the plane $z=0$ in $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$ and if this point lies on the given conic, we have

$$\left(\alpha-\frac{l\gamma}{n}\right)^2+\left(\beta-\frac{m\gamma}{n}\right)^2+2a\left(\alpha-\frac{l\gamma}{n}\right)+2b\left(\beta-\frac{m\gamma}{n}\right)=0 \quad \text{— ②}$$

Eliminating l,m,n between ① & ②, the equation of the cone is

$$\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2+2a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$
$$+2b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]=0$$

$(\alpha z - x\gamma)^2 + (\beta z - \gamma y)^2 + 2a(\alpha z - xy)(z-\gamma) + 2b(\beta z - y\gamma)(z-\gamma) = 0$

If this cone passes through $(0,0,c)$, then

$(\alpha c)^2 + (\beta c)^2 + 2a(\alpha c)(c-\gamma) + 2b(\beta c)(c-\gamma) = 0$ — (3)

Again the section of the cone by zx-plane

i.e. $y=0$ is

$(\alpha z - \gamma x)^2 + (\beta z)^2 + 2a(\alpha z - x\gamma)(z-\gamma) + 2b(\beta z)(z-\gamma) = 0$

and if this section is a rectangular hyperbola in the zx-plane, then the sum of the coefficients of x^2 and z^2 should be zero

i.e. $\gamma^2 + (\alpha^2 + \beta^2 + 2a\alpha + 2b\beta) = 0$ — (4)

$\therefore$ the locus of (α, β, γ) from (3) and (4) is

$c(x^2+y^2) + 2ax(c-z) + 2by(c-z) = 0$ — (5)

and $x^2 + y^2 + z^2 + 2ax + 2by = 0$ — (6)

Multiplying (6) by c and subtracting (5) from the result so obtained, we get

$cz^2 + 2azx + 2byz = 0$ (or) $2ax + 2by + cz = 0$ — (7)

which is the equation of a plane.

$\therefore$ the required locus of the vertex is given by (6) and (7) which taken together represent a circle.

3(a) In C^3, let $\alpha_1 = (1, 0, -i)$, $\alpha_2 = (1+i, 1-i, 1)$, $\alpha_3 = (i, i, i)$. Prove that these vectors form a basis for C^3. What are the co-ordinates of the vector (a, b, c) in this basis?

Sol: let $C^3 = C\times C\times C = \{(a+ib, c+id, e+if) \mid a,b,c,d,e,f \in \mathbb{R}\}$ be a given vector space.

clearly $\dim(C^3) = 3$ over the field 'C'

let us construct a Matrix A whose rows are given vectors α_1, α_2 and α_3.

$$A = \begin{bmatrix} 1 & 0 & -i \\ 1+i & 1-i & 1 \\ i & i & i \end{bmatrix} \Rightarrow |A| = 1(\cancel{i}+1-\cancel{i}) - 0(i+1) - i(i-\cancel{1}+i+\cancel{1})$$

$$\Rightarrow |A| = 1 - i(2i)$$
$$= 1+2 = 3 \neq 0$$

$\therefore$ $\alpha_1, \alpha_2, \alpha_3$ are Linearly independent.

and $\dim(C^3) = 3 =$ The number of linearly independent vectors.

$\therefore$ α_1, α_2 and α_3 form a basis for C^3.

let $(a, b, c) = (x_1+iy_1)(1, 0, -i) + (x_2+iy_2)(1+i, 1-i, 1) + (x_3+iy_3)(i, i, i)$. ——(1)

where $x_1+iy_1, x_2+iy_2, x_3+iy_3 \in C$.

Here a, b, c also required to mention in complex form please.

try your self.

3.(b) Let $U = \text{span}(u_1, u_2, u_3)$ and $W = \text{span}(v_1, v_2)$ be subspaces of $\mathbb{R}^4$ where $u_1 = (1,2,-1,3)$, $u_2 = (2,4,1,-2)$, $u_3 = (3,6,+3,-7)$, $v_1 = (1,2,-4,11)$, $v_2 = (2,4,-5,14)$. S.T $U = W$.

Sol: Form the matrix A whose rows are u_i's $(i=1,2,3)$ and reduce it to row reduced echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{bmatrix} \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2,\ R_2 \to \frac{R_2}{3}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

Now form the matrix whose rows are v_i's $(i=1,2,3)$ and reduce it to row reduced echelon form.

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{bmatrix} R_2 \to \tfrac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \tfrac{1}{3} \\ 0 & 0 & 1 & -8/3 \end{bmatrix} R_1 \to R_1 + 4R_2$$

Since the non-zero rows of the row reduced matrices are same.

$\therefore$ The row spaces of A & B are equal.

$\therefore$ $U = W$.

3(c). The ellipsoid with equation $x^2 + 2y^2 + z^2 = 4$ is heated so that its temperature at (x, y, z) is given by $T(x, y, z) = 70 + 10(x+z)$. Find the hottest and coldest points on the ellipsoid.

Sol'n: Given that the ellipsoid $x^2 + 2y^2 + z^2 = 4$ — ① is heated so that its temperature at a point (x, y, z) is given by

$T(x, y, z) = 70 + 10(x+z)$ — ②

we have to find the hottest and coldest points on the ellipsoid.

Now, by using Lagrange's multiplier method.

10

Consider the function

$$f = 70 + 10(x+z) + \lambda\left[x^2+2y^2+z^2-4\right]$$

$$df = \left[10+2x\lambda\right]dx + 4y\lambda\, dy + (10+2z\lambda)\,dz.$$

for stationary values $f_x = f_y = f_z = 0$

$\Rightarrow$ $10+2x\lambda = 0$ —③

$4y\lambda = 0$

$10+2z\lambda = 0$ —④

Now $10+2x\lambda = 0 \Rightarrow \lambda \neq 0.$

and so $0 = 4y\lambda \Rightarrow y = 0.$

from ③ & ④, we have

$-2x\lambda = -2z\lambda$

$\Rightarrow x = z.$

Substituting $x=z$ & $y=0$ in equation ①, we have

$x^2+0+x^2=4$

$2x^2=4$

$x = \pm\sqrt{2}$

$\therefore$ The points are $(\sqrt{2},0,\sqrt{2})$ and $(-\sqrt{2},0,-\sqrt{2})$

At $(\sqrt{2},0,\sqrt{2})$;

$T(x,y,z) = 70 + 10(2\sqrt{2}) = 70+20\sqrt{2}$
$= 70+28.28 = 98.28 \simeq 98$

At $(-\sqrt{2},0,-\sqrt{2}) = 70 - 20\sqrt{2} = 70-28.28$
$= 41.72$
$\simeq 42.$

Thus the hottest point on the ellipsoid is $(\sqrt{2},0,\sqrt{2})$
and the coldest point on the ellipsoid is $(-\sqrt{2},0,-\sqrt{2})$

21

3(d) (i) A variable plane, which remains at a constant distance p from the origin, cuts the coordinate axes at A, B, and C. Show that the locus of the centroid of $\triangle ABC$ is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

(ii) Find the S.D between lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and } \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

Find also its equations and the points in which it meets the given lines.

Sol'n (i) Let the equation of the variable plane be

$$x/a+y/b+z/c=1 \quad \text{—— (1)}$$

The plane (1) meets the axes in A, B and C whose coordinates are $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

Also the distance of the plane (1) from $(0,0,0)$ is given as p, so we have,

$$p=\frac{1}{\sqrt{(1/a)^2+(1/b)^2+(1/c)^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad \text{—— (2)}$$

The planes through A, B and C parallel to the coordinates planes are given by $x=a$, $y=b$ and $z=c$ respectively.

the required locus is obtained by eliminating a, b, c from the equations of these planes and the relation (2)

Substituting the values of a, b, c from (3) in (2), we have the required locus, as $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

(ii) Given lines are $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ — (1)

and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ — (2)

Any point on the line ① is $(3-3r, 8-r, 3+r)$, say point P and any point on the line ② is $(-3-3r', -7+2r', 6+4r')$ say point Q.

Then the direction ratios of the line PQ are

$$(3+3r)-(-3-3r'), (8-r)-(-7+2r'), (3+r)-(6+4r')$$

$$3r+3r'+6, -r-2r'+15, r-4r'-3 \quad \text{—} \ ③$$

Now if PQ is the S.D between the given lines then PQ is ⊥ar to both ① and ②, the conditions for the same are

$$3(3r+3r'+6)-1(-r-2r'+15)+1(r-4r'-3)=0$$

and $-3(3r+3r'+6)+2(-r-2r'+15)+4(r-4r'-3)=0$

$\Rightarrow \ 11r+7r'=0$ and $7r+29r'=0$

solving these we find $r=0$ and $r'=0$

Substituting these values of r and r', we find that the Coordinates of P and Q are $(3,8,3)$ and $(-3,-7,6)$ respectively.

And the dir's of the line PQ from ③ are $6,15,-3$ (or) $2,5,-1$

Now required S.D

$$= PQ = \sqrt{[(3-(-3))^2+(8-(-7))^2+(3-6)^2]}$$

$$= \sqrt{36+225+9} = 3\sqrt{30}.$$

Also PQ is a line through $P(3,8,3)$ and of direction ratios $2,5,-1$ so its equations are

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{1}$$

4(a) Let V be a subspace of $\mathbb{R}^5$ generated by

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2 \\ 3 \end{bmatrix},\ v_2 = \begin{bmatrix} 2 \\ 7 \\ -5 \\ 6 \\ 5 \end{bmatrix},\ v_3 = \begin{bmatrix} 3 \\ 6 \\ -3 \\ 0 \\ 13 \end{bmatrix}.$$

and let W be a subspace generated by

$$w_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \\ 1 \end{bmatrix},\ w_2 = \begin{bmatrix} 5 \\ 16 \\ -3 \\ 12 \\ 6 \end{bmatrix},\ w_3 = \begin{bmatrix} 3 \\ 8 \\ 3 \\ 4 \\ 2 \end{bmatrix}$$

find a basis for $V+W$ and $V \cap W$.

Sol^n: Since $V+W$ is the space spanned by all six vectors

Hence from the matrix whose rows are the given six vectors and then reduce to echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 2 & 7 & -5 & 6 & 5 \\ 3 & 6 & -3 & 0 & 13 \\ 1 & 3 & 0 & 2 & 1 \\ 5 & 16 & -3 & 12 & 6 \\ 3 & 8 & 3 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 4 \\ 0 & 0 & +2 & 0 & -2 \\ 0 & 1 & 7 & 2 & -9 \\ 0 & -1 & 9 & -2 & -7 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 5R_1 \\ R_6 \to R_6 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 8 & 0 & -8 \\ 0 & 0 & 8 & 0 & -8 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 3R_1 \\ R_5 \to R_5 - R_2 \\ R_6 \to R_6 + R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_5 \to R_5 - 4R_4 \\ R_6 \to R_6 - 4R_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \leftrightarrow R_3$$

Clearly it is an echelon form.

The non-zero rows of the echelon matrix,

$(1,3,-2,2,3)$, $(0,1,-1,2,-1)$, $(0,0,2,0,-2)$, $(0,0,0,0,1)$ form a basis of $V+W$.

2nd part $\therefore \underline{\dim(V+W) = 4.}$

let us construct the matrices A & B whose rows are the span vectors of V and span vectors of W respectively.

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 2 & 7 & -5 & 6 & 5 \\ 3 & 6 & -3 & 0 & 13 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 5 & 16 & -3 & 12 & 6 \\ 3 & 8 & 3 & 4 & 2 \end{bmatrix}$$

13

Let us convert A & B into the row-reduced echelon forms:

$$A \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3 + 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 6 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - 3R_2$$

clearly it is in row-reduced echelon form ——(1)

$$B \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 9 & -4 & -2 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - 3R_2$$

Clearly it is in row-reduced echelon form. ——(2)

from ① & ②, we have

$$V = \{x(1,0,1,-4,6) + y(0,1,-1,2,-1) + z(0,0,0,0,1) / x,y,z \in \mathbb{R}\}$$

$$= \{(x, y, x-y, -4x+2y, 6x-y+z) / x,y,z \in \mathbb{R}\} \quad \text{— (3)}$$

and

$$W = \{a(1,0,9,-4,-2) + b(0,1,-3,2,1) / a,b \in \mathbb{R}\}$$

$$= \{(a, b, 9a-3b, -4a+2b, -2a+b) / a,b \in \mathbb{R}\} \quad \text{— (4)}$$

To find $V \cap W$:

from (3) & (4), we have

$x = a$ —(i), $y = b$ —(ii), $x - y = 9a - 3b$ —(iii)

$-4x+2y = -4a+2b$ —(iv), $6x - y + z = -2a + b$ —(v)

$(v) \equiv 6a - b + z = -2a + b$

$\Rightarrow z = -2a + b - 6a + b$

$\Rightarrow \boxed{z = -8a + 2b}$

$(iii) \equiv a - b = 9a - 3b \Rightarrow -8a + 2b = 0$

$\Rightarrow \boxed{a = \frac{b}{4}}$

$$\therefore V \cap W = \left\{\left(\frac{b}{4}, b, \frac{-7}{4}b, b, \frac{b}{2}\right) / b \in \mathbb{R}\right\}.$$

clearly the basis of $V \cap W$ is

$$\left\{\left(\frac{1}{4}, 1, -\frac{7}{4}, 1, \frac{1}{2}\right)\right\}.$$

and $\dim(V \cap W) = 1$.

4(b) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$

Sol'n: It is $(\infty,-\infty)$ form and

therefore $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) = \lim_{x\to 0}\frac{\sin^2 x - x^2}{x^2\sin^2 x}\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{\sin 2x - 2x}{2x\sin^2 x + x^2\sin 2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{\cos 2x - 1}{\sin^2 x + 2x\sin 2x + x^2\cos 2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-2\sin 2x}{3\sin 2x + 6x\cos 2x - 2x^2\sin 2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{-\cos 2x}{3\cos 2x - 4x\sin 4x - x^2\cos 2x}$$

$$= -\frac{1}{3}$$

4(c) Evaluate $\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}\,dx$

Sol'n: $I = \int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}\,dx$

$$I = \int_0^{\pi/2}\frac{\sin^2(\pi/2 - x)}{\sin(\pi/2 - x)+\cos(\pi/2 - x)}\,dx$$

$$= \int_0^{\pi/2}\frac{\cos^2 x}{\sin x+\cos x}\,dx$$

$$\therefore 2I = \int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\pi/2}\frac{dx}{\sin x+\cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin(x + \pi/4)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi} \operatorname{cosec}(x + \pi/4)\, dx$$

$$= \frac{1}{\sqrt{2}} \left[\log \tan\left(\tfrac{1}{2}x + \tfrac{1}{8}\pi\right) \right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} \left[\log \tan \tfrac{3}{8}\pi - \log \tan \tfrac{1}{8}\pi \right]$$

$$= \frac{1}{\sqrt{2}} \left[\log \frac{\tan \frac{3}{8}\pi}{\tan \frac{1}{8}\pi} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\log \frac{\tan\left(\frac{\pi}{2} - \frac{1}{8}\pi\right)}{\tan \frac{1}{8}\pi} \right]$$

$$= \frac{1}{\sqrt{2}} \log \frac{\cot \pi/8}{\tan \frac{1}{8}\pi}$$

$$= \frac{1}{2} \log \frac{\cos^2 \frac{1}{8}\pi}{\sin^2 \frac{1}{8}\pi}$$

$$= \frac{1}{\sqrt{2}} \log \frac{1 + \cos \pi/4}{1 - \cos \pi/4}$$

$$= \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

$$= \sqrt{2} \log(\sqrt{2}+1)$$

$$\therefore\ I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

16

4(d) Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose generators are parallel to the line $\frac{x}{0}=\frac{y}{\pm\sqrt{(a^2-b^2)}}=\frac{z}{c}$ meet the plane $z=0$ in circles.

Sol^n: Let $P(\alpha,\beta,\gamma)$ be any point on the enveloping cylinder. Then the equations of the generator through $P(\alpha,\beta,\gamma)$ are.

$$\frac{x-\alpha}{0}=\frac{y-\beta}{\pm\sqrt{(a^2-b^2)}}=\frac{z-\gamma}{c}=r\,(\text{say})$$

Any point on this generator is $(\alpha, \beta\pm r\sqrt{(a^2-b^2)}, \gamma+cr)$

If this point lies on the given ellipsoid, then we get

$$\frac{\alpha^2}{a^2}+\frac{\{\beta\pm r\sqrt{(a^2-b^2)}\}^2}{b^2}+\frac{(\gamma+cr)^2}{c^2}=1$$

$$\Rightarrow r^2\left[\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right]+2r\left[\frac{c\gamma}{c^2}\pm\frac{\beta\sqrt{(a^2-b^2)}}{b^2}\right]+\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]=0 \quad \text{--- (1)}$$

Since this generator is a tangent to the given ellipsoid, so the two values of r obtained from (1) must be equal and the condition for the same is

"$b^2=4ac$"

i.e. $$\left[\frac{\gamma}{c}\pm\frac{\beta\sqrt{(a^2-b^2)}}{b^2}\right]^2=\left[\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]$$

$$\Rightarrow \frac{\gamma^2}{c^2}+\frac{\beta^2(a^2-b^2)}{b^4}\pm\frac{2\beta\gamma\sqrt{(a^2-b^2)}}{b^2c}=\left(\frac{a^2}{b^2}\right)\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ or the equation of the cylinder is

$$\frac{z^2}{c^2}+\frac{y^2(a^2-b^2)}{b^4}\pm\frac{2yz\sqrt{(a^2-b^2)}}{b^2c}=\frac{a^2}{b^2}\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]$$

this meets the plane $z=0$ in the curve

$$\frac{y^2(a^2-b^2)}{b^4} = \frac{a^2}{b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right),\ z=0$$

i.e. $\frac{x^2}{b^2}+\frac{y^2}{b^2}=\frac{a^2}{b^2}$, $z=0$ i.e. $x^2+y^2=a^2$, $z=0$.

which is a circle of radius a on the plane $z=0$.

5(a) → Let $a=\begin{bmatrix}0\\1\\-1\end{bmatrix}$, $b=\begin{bmatrix}0\\1\\1\end{bmatrix}$, $c=\begin{bmatrix}-1\\1\\0\end{bmatrix}$

and $d=\begin{bmatrix}1\\0\\1\end{bmatrix}$ be vectors in $\mathbb{R}^3$.

Let $W_1=\langle a,b\rangle$, $W_2=\langle b,c\rangle$ and $W_3=\langle c,d\rangle$

S.T $W_2=W_3$ and $W_1\cap W_2=\langle b\rangle$

Also identify the subspace W_1+W_2.

Is $W_1\cup W_2$ a subspace of $\mathbb{R}^3$?

Sol. let $A=\begin{bmatrix}0&1&-1\\0&1&1\end{bmatrix}$, $B=\begin{bmatrix}0&1&1\\-1&1&0\end{bmatrix}$

$C=\begin{bmatrix}-1&1&0\\1&0&1\end{bmatrix}$.

let us convert A, B and C into row-reduced echelon forms:.

$$A\sim\begin{bmatrix}0&1&-1\\0&0&2\end{bmatrix}^{R_2\to R_2-R_1}\sim\begin{bmatrix}0&1&0\\0&0&2\end{bmatrix}^{R_1\to R_1+\frac{1}{2}R_2}$$

$$\sim\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}^{R_2\to\frac{1}{2}R_2}$$

— 1

17

$$B \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_1 \to (-1) R_1 \quad \text{--- (2)}$$

$$\text{and } C \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_1 \to (-1) R_1 \quad \text{--- (3)}$$

since the row-reduced echelon forms of B & C have the same non-zero rows. $\therefore$ $W_2 = W_3$.

from (1), $W_1 = \{x(0,1,0) + y(0,0,1) / x, y \in \mathbb{R}\}$

$= \{(0, x, y) / x, y \in \mathbb{R}\}$ --- (4)

and $W_2 = \{a(1,0,1) + b(0,1,1) / a, b \in \mathbb{R}\}$

$= \{(a, b, a+b) / a, b \in \mathbb{R}\}$ --- (5)

To find $W_1 \cap W_2$;

from ④ and ⑤, we have

$a = 0,\ b = x;\ a + b = y$

$\therefore \boxed{b = y} \Rightarrow \boxed{x = y.}$

$\therefore W_1 \cap W_2 = \{(0, x, x) / x \in \mathbb{R}\}$

$= \langle (0,1,1) \rangle = \langle b \rangle$.

The subspace $W_1 + W_2$ of $\mathbb{R}^3$ spanned a, b, c vectors.

we have

$$D = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} R_3 \to R_3 - R_2$$

Clearly it is in echelon form.

Since and no row is zero row.

$\therefore$ a, b, c are linearly independent vectors.

$\therefore \dim(W_1 + W_2) = 3 = \dim(\mathbb{R}^3)$

$\therefore W_1 + W_2 = \mathbb{R}^3$

clearly $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^3$.

because $W_1 = \{(0, x, y) / x, y \in \mathbb{R}\}$

and $W_2 = \{(a, b, a+b) / a, b \in \mathbb{R}\}$
$= \{a(1,0,1) + b(0,1,1) / a, b \in \mathbb{R}\}$

let $\alpha = (0, 1, -1) \in W_1$

$\beta = (-1, 1, 0) \in W_2$.

then $\alpha, \beta \in W_1 \cup W_2$

$\because \alpha + \beta = (-1, 2, -1) \notin W_1 \cup W_2$

(or)

$\because W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

$\therefore W_1 \cup W_2$ is not a subspace

5(b) What condition must be placed on a, b and c so that the following system in unknowns x, y, and z has a solution?

$x+2y-3z=a$
$2x+6y-11z=b$
$x-2y+7z=c$.

Sol: The matrix form of the given system of equations is

$$\begin{bmatrix}1 & 2 & -3\\ 2 & 6 & -11\\ 1 & -2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

$R_2 \to R_2-2R_1$
$R_3 \to R_3-R_1$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & -4 & +10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c-a\end{bmatrix}$$

$R_3 \to R_3+2R_2$

$$\begin{bmatrix}1 & 2 & -3\\ 0 & 2 & -5\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-2a\\ c+2b-5a\end{bmatrix}$$

The system will have no solution if $c+2b-5a \neq 0$.

Thus the system will have atleast one solution if $c+2b-5a=0$ i.e, $5a=2b+c$.
which is the required condition.

Note: In this case the system will have infinitely many solutions. In otherwords, the system cannot have a unique solution.

What conditions must be placed on a, b and c so that the following system in unknowns x, y and z has a solution?

$x+2y-3z=a$

$2x+6y-11z=b$

$x-2y+7z=c$

5(C). Test the convergence of the integral $\int_1^2 \frac{dx}{\sqrt{x^4-1}}$.

Sol^n: In the given integral the integrand $f(x) = \frac{1}{\sqrt{x^4-1}}$ is unbounded at the lower limit of integration $x=1$.

Take $\phi(x) = \frac{1}{\sqrt{x^2-1}}$

then $\lim_{x\to1} \frac{f(x)}{\phi(x)} = \lim_{x\to1}\left[\frac{1}{\sqrt{x^4-1}}\sqrt{x^2-1}\right]$

$$= \lim_{x\to1} \frac{1}{\sqrt{x^2+1}} = \frac{1}{\sqrt{2}}$$

which is finite and non-zero.

$\therefore$ by Comparison test,

$\int_1^2 f(x)\,dx$ and $\int_1^2 \phi(x)\,dx$

are either both convergent or both divergent.

But $\int_1^2 \phi(x)\,dx = \int_1^2 \frac{dx}{\sqrt{x^2-1}} = \lim_{\varepsilon\to0}\int_{1+\varepsilon}^2 \frac{dx}{\sqrt{x^2-1}}$

$$= \lim_{\varepsilon\to0}\left[\log\{x+\sqrt{x^2-1}\}\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon\to0}\left[\log(2+\sqrt{3}) - \log\{1+\varepsilon+\sqrt{\varepsilon^2+\varepsilon}\}\right]$$

$$= \log(2+\sqrt{3})$$

which is a definite real number

$\therefore \int_1^2 \phi(x)\,dx$ is convergent.

Hence $\int_1^2 \frac{1}{\sqrt{x^4-1}}\,dx$ is also convergent.

5(d) show that $\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$ represents a pair of planes. Find angle between them also.

Sol'n: The given equation can be written as.

$a(z-x)(x-y)+b(y-z)(x-y)+c(y-z)(z-x)=0.$

$\Rightarrow ax^2+by^2+cz^2-(b+c-a)yz-(c+a-b)zx-(a+b-c)xy=0$ —①

If it represents a pair of planes we should have

$$\begin{vmatrix} a & h & f \\ h & b & f \\ g & f & c \end{vmatrix}=0 \text{ i.e } \begin{vmatrix} a & -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} -2a & a+b-c & c+a-b \\ a+b-c & -2b & b+c-a \\ c+a-b & b+c-a & -2c \end{vmatrix}=0$$

Adding all the columns to first we find the det. on the left vanishes and hence the given equation represents a pair of planes.

Also comparing ① with the equation.

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

we find $a=a,\ b=b,\ c=c$

$f=\frac{1}{2}(a-b-c),\ g=\frac{1}{2}(b-c-a),\ h=\frac{1}{2}(c-a-b)$

$\therefore$ Required angle $=\tan^{-1}\left[\frac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c}\right]$ —②

Now $f^2+g^2+h^2-bc-ca-ab$

$=\frac{1}{4}(a-b-c)^2+\frac{1}{4}(b-c-a)^2+\frac{1}{4}(c-a-b)^2-bc-ca-ab$

21

$$=\frac{1}{4}\left[(a-b-c)^2+(b-c-a)^2+(c-a-b)^2-4bc-4ca-4ab\right]$$

$$=\frac{1}{4}\left[3(a^2+b^2+c^2)-6(ab+bc+ca)\right]$$

$$=\frac{3}{4}\left[a^2+b^2+c^2-2ab-2bc-2ca\right]$$

Substituting this value in ②, we have the required

$$\text{angle} = \tan^{-1}\left[\frac{\sqrt{\{3(a^2+b^2+c^2-2ab-2bc-2ca)\}}}{a+b+c}\right]$$

5(e) Find the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0$, $2x+3y+4z=8$ is a great circle.

Sol^n: The equation of any sphere through the given circle is

$$(x^2+y^2+z^2+7y-2z+2)+\lambda(2x+3y+4z-8)=0 \quad \text{——} ①$$

$$\Rightarrow x^2+y^2+z^2+2\lambda x+(7+3\lambda)y+(4\lambda-2)z+(2-8\lambda)=0$$

Its centre is $\left[-\lambda, -\frac{1}{2}(7+3\lambda), 1-2\lambda\right]$ —— ②

Now if the given circle is a great circle of the sphere ①, then the centre of the sphere ① must lie on the plane of the circle i.e. on the plane $2x+3y+4z=8$

$\therefore$ from ② we get $2(-\lambda)+3\left[-\frac{1}{2}(7+3\lambda)\right]+4(1-2\lambda)=8$

$$\Rightarrow -2\lambda-\frac{21}{2}-\frac{9}{2}\lambda+4-8\lambda=8$$

$$\Rightarrow \lambda=-1$$

Substituting the value of λ in ①, we get the required equation as

$$(x^2+y^2+z^2-7y-2z+2)-(2x+3y+4z-8)=0$$

$$x^2+y^2+z^2-2x+4y-6z+10=0$$

6(a) Let $\alpha_1 = (1,1,-2,1)$, $\alpha_2 = (3,0,4,-1)$,
$\alpha_3 = (-1,2,5,2)$

let $\alpha = (4,-5,9,-7)$, $\beta = (3,1,-4,4)$
$\gamma = (-1,1,0,1)$.

(i) which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by the α_i ?

(ii) which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by the α_i?

(iii) Does this suggest a theorem ?

Sol (i) let $\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3$. $x, y, z \in \mathbb{R}$.

$$\Rightarrow \left.\begin{aligned} x+3y-z &= 4 \\ x+0y+2z &= -5 \\ -2x+4y+5z &= 9 \\ x-y+2z &= -7. \end{aligned}\right\} \text{—(1)}$$

$\Rightarrow AX = B$ —(2)

where $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 0 & 2 \\ -2 & 4 & 5 \\ 1 & -1 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ -5 \\ 9 \\ -7 \end{bmatrix}$

we have

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 1 & 0 & 2 & -5 \\ -2 & 4 & 5 & 9 \\ 1 & -1 & 2 & -7 \end{array}\right]$$

22

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -3 & 3 & -9 \\ 0 & 10 & 3 & 17 \\ 0 & -4 & 3 & -11 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 10 & 3 & 17 \\ 0 & -4 & 3 & -11 \end{array}\right] R_2 \to -\tfrac{1}{3} R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 13 & -13 \\ 0 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 10R_2 \\ R_4 \to R_4 + 4R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right] R_3 \to \tfrac{1}{13} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] R_4 \to R_4 + R_3.$$

clearly it is in echelon form

$\therefore \rho(A) = \rho(A|B) = 3 =$ The number of variables x, y, z.

$\therefore \alpha$ is a L.C. of $\alpha_1, \alpha_2, \alpha_3$.

and α is in the subspace of $\mathbb{R}^4$ and spanned by α_i; $i = 1, 2, 3$.

please check your self for β and γ.

(ii) since α, β, γ are real vectors in C^3 $\therefore$ α is also belong to the subspace of $\mathbb{C}^4$ spanned by the α_i, $i = 1, 2, 3$

[please check your self for β and γ.]

(iii) since α is a linear combination of α_1, α_2 and α_3.

$\therefore$ $\alpha, \alpha_1, \alpha_2$ and α_3 are Linearly dependent.

[Rough work] ~~Check for β.~~

6(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by
$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$
If $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$, $\beta' = \{(1,0), (0,1)\}$ be ordered basis of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, then find the matrix of T relative to β, β'. Also find rank and nullity.

Sol: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1) \quad \text{--- (1)}$$

$$T(1,0,-1) = (1,-3)$$
$$T(1,1,1) = (2,1)$$
$$T(1,0,0) = (1,-1)$$

further $T(1,0,-1) = (1,-3) = 1(1,0) + -3(0,1)$

$$T(1,1,1) = (2,1) = 2(1,0) + 1(0,1)$$
$$T(1,0,0) = (1,-1) = 1(1,0) + (-1)(0,1)$$

Hence the matrix of T relative to β, β' is

$$[T]_{\beta,\beta'} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Let $(x_1, x_2, x_3) \in \ker T$ be arbitrary. Then

$$T(x_1, x_2, x_3) = (0,0)$$
$$\Rightarrow (x_1 + x_2, 2x_3 - x_1) = 0$$
$$\Rightarrow x_1 + x_2 = 0 \text{ and } 2x_3 - x_1 = 0$$
$$\Rightarrow x_1 = 2x_3, \; x_2 = -x_1 = -2x_3$$
$$\therefore (x_1, x_2, x_3) = (2x_3, -2x_3, x_3)$$

$= x_3(2,-2,1)$

Hence $\ker T = \{x_3(2,-2,1) / x_3 \in \mathbb{R}\}$

This shows that $\ker T$ is spanned by $(2,-2,1) \neq (0,0,0)$ and so $\{(2,-2,1)\}$ is a basis of $\ker T$.

$\therefore \dim \ker T = 1$

i.e, $\text{nullity}(T) = 1$

W.k.T $\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^3 = 3$

Hence $\text{rank}\, T = 3 - 1 = 2$.

6(c) (i) Show that the real field $\mathbb{R}$ is a vector space of infinite dimension over the rational field $\mathbb{Q}$.

(ii) Let V be the vector space of ordered pairs of complex numbers over the real field $\mathbb{R}$. Show that V is of dimension 4.

Sol^n: we claim that, for any n, $\{1, \pi, \pi^2, \ldots \pi^n\}$ is linearly independent over $\mathbb{Q}$.

for suppose
$a_0 + a_1\pi + a_2\pi^2 + \ldots + a_n\pi^n = 0$,
where $a_i \in \mathbb{Q}$, and not all the a_i are zero.

Then π is a root of the non-zero polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ over $\mathbb{Q}$.

This is impossible, since π is a transcendental number

i.e, π is not a root of any non-zero polynomial over Q.

Accordingly, the $(n+1)$ real numbers $1, \pi, \pi^2, --- \pi^n$

are linearly independent over Q..

Thus for any finite n, R cannot be of dimension n over Q.

i.e., R is of infinite dimension over Q.

(ii) we claim that $B = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis of V.

Suppose $v \in V$.

Then $v = (z, w)$, where z, w are complex numbers

and so $v = (a+bi, c+di)$ where a, b, c, d are real numbers.

Then $v = a(1,0) + b(i,0) + c(0,1) + d(0,i)$.

Thus B generates V.

The proof is complete if we show that B is independent.

Suppose $x_1(1,0) + x_2(i,0) + x_3(0,1) + x_4(0,i) = 0$ where $x_1, x_2, x_3, x_4 \in R$.

Then
$(x_1+x_2 i, x_3+x_4 i) = (0,0)$

and so $x_1+x_2 i = 0$
$x_3+x_4 i = 0$

$\Rightarrow x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$

and so B is independent

6(d) If H is any Hermitian matrix, then
$A = (H+iI)^{-1}(H-iI) = (H-iI)(H+iI)^{-1}$
is unitary and every unitary matrix can be thus expressed provided, -1, is not a characteristic root of A.

Sol: Since H is a Hermitian matrix.

$\therefore H^{\theta} = H$

We know that the characteristic roots of H are real.

$\therefore$ Neither i nor $-i$ is a root of the equation $|H-\lambda I| = 0$

$\Rightarrow |H-iI| \neq 0$ and $|H+iI| \neq 0$

$\Rightarrow (H-iI)$ is non-singular and $(H+iI)$ is non-singular.

Given that $A = (H+iI)^{-1}(H-iI)$

consider $A^{\theta} = \left[(H+iI)^{-1}(H-iI)\right]^{\theta}$

$= (H-iI)^{\theta}\left[(H+iI)^{-1}\right]^{\theta}$

$$= \left[H^\theta + (iI)^\theta\right]\left\{\left[H + iI\right]^\theta\right\}^{-1}$$

$$= \left[H + \bar{i} I^\theta\right]\left[H^\theta + \bar{i} I^\theta\right]^{-1}$$

$$= (H + iI)(H - iI)^{-1}$$

$$\therefore A^\theta A = (H+iI)^{-1}(H+iI)^{-1}(H+iI)(H+iI)$$

$$= (H+iI)(H+iI)^{-1}(H-iI)^{-1}(H-iI)$$

$$(\because (H-iI)^{-1}(H+iI)^{-1} = (H+iI)^{-1}(H-iI)^{-1})$$

$$A^\theta A = I$$

$\Rightarrow A$ is unitary.

Now to show that every unitary matrix can be expressed in the form provided -1 is not an eigenvalue of A for some suitable Hermitian

$\to 1$

Let A be a unitary matrix and given matrix H

i.e to show that $H^\theta = H$

then $A = (H+iI)^{-1}(H-iI)$ — ①

pre multiply by $(H+iI)$

$\Rightarrow (H+iI)A = H - iI$

$\Rightarrow HA + iA = H - iI$

$\Rightarrow i(A+I) = H(I-A)$

$\Rightarrow -i(I+A) = H(A-I)$ — ②

Since 1 is not the characteristic root of A

$\Rightarrow |A-I| \neq 0 \Rightarrow A-I$ is non-singular.

Hence from ② $H = -i(A+I)(A-I)^{-1}$

Now to show that H is Hermitian matrix.

$H^\theta = \left[-i(A+I)(A-I)^{-1}\right]^\theta$

$= -\bar{i}\left[(A-I)^{-1}\right]^\theta [A+I]^\theta$

$= i\left[(A-I)^\theta\right]^{-1}(A+I)^\theta$

$H^\theta = i(A^\theta - I)^{-1}(A^\theta + I)$ ——— (3)

Since $(A^\theta+I)(A^\theta-I) = (A^\theta-I)(A^\theta+I)$

On pre & post multiplying by $(A^\theta-I)^{-1}$ throughout the above equⁿ

$\Rightarrow (A^\theta-I)^{-1}(A^\theta+I)(A^\theta-I)(A^\theta-I)^{-1} = (A^\theta-I)^{-1}(A^\theta-I)(A^\theta+I)(A^\theta-I)^{-1}$

$\Rightarrow (A^\theta-I)^{-1}(A^\theta+I) = (A^\theta+I)(A^\theta-I)^{-1}$

Now from (3)

$H^\theta = i(A^\theta+I)(A^\theta-I)^{-1}$ (since A is unitary $\Rightarrow A^\theta A = I$)

$= i(A^\theta + A^\theta A)(A^\theta - A^\theta A)^{-1}$

$= i(A^\theta + AA^\theta)(A^\theta - AA^\theta)^{-1}$ $(\because A^\theta A = AA^\theta)$

$\Rightarrow i(I+A)A^\theta\left[(I-A)A^\theta\right]^{-1}$

$= i(I+A)A^\theta (A^\theta)^{-1}(I-A)^{-1}$

$= i(I+A)(I-A)^{-1}$ $(\because A^\theta (A^\theta)^{-1} = I)$

$\Rightarrow H^\theta = -i(A+I)(A-I)^{-1}$

$\therefore H^\theta = H$

$\Rightarrow H$ is Hermitian.

Hence every unitary matrix can be written in the form $(H+iI)^{-1}(H-iI)$

7(a) By using the transformation $x+y=u$, $y=uv$, Prove that $\iint \{xy(1-x-y)\}^{1/2}\, dx\, dy$ taken over the area of the triangle bounded by lines $x=0$, $y=0$, $x+y=1$ is $\frac{2\pi}{105}$

Sol^n: Given $x+y=u$, $y=uv$

$$\Rightarrow x = u - uv$$
$$x = u(1-v)$$

Now $$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$
$$= u - vu + uv = u$$

But $$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\, du\,dv$$

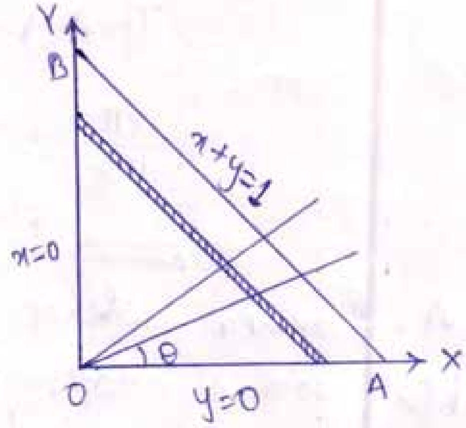
$$\therefore dx\,dy = u\,du\,dv$$

and $$\sqrt{xy(1-x-y)} = \sqrt{u(1-v)uv(1-u)}$$
$$= uv^{1/2}\sqrt{(1-u)(1-v)}$$

Clearly region of integration is OAB.

The integration formulae are

$x+y=u$, $y=uv$
$$= (x+y)v$$
$$\Rightarrow y = \frac{v}{1-v}x$$



i.e. Clearly the area for new variables is to be divided by the lines parallel to $x+y=1$ and by lines

$$y = \frac{v}{1-v}x$$

i.e. $y = x\tan\theta$

where $\tan\theta = \frac{v}{1-v}$

where θ varies from 0 to $\pi/2$ and so v varies from 0 to 1. and $u = x+y$, varies from 0 to 1.

i.e. limits of u are 0 to 1

Hence the given integral

$$= \int_0^1\int_0^1 uv^{1/2}\sqrt{(1-u)(1-v)}\; u\,du\,dv$$

$$= \int_0^1 u^2(1-u)^{1/2}du \int_0^1 v^{1/2}(1-v)^{1/2}dv$$

$$= \int_0^1 u^{3-1}(1-u)^{3/2-1}du \int_0^1 v^{3/2-1}(1-v)^{3/2-1}dv$$

$$= B\left(3,\tfrac{3}{2}\right)B\left(\tfrac{3}{2},\tfrac{3}{2}\right)$$

$$= \frac{\Gamma 3\,\Gamma 3/2}{\Gamma(3+3/2)}\cdot\frac{\Gamma 3/2\,\Gamma 3/2}{\Gamma 3}$$

$$= \frac{2\pi}{105}$$

7(b) A farmer wishes to build a rectangular bin, with a top, to hold a volume of 1000 cubic meters. find the dimensions of the bin that will minimize the amount of material needed in its construction.

Solⁿ: Let x and y be the dimensions of the base of bin and z be the height, all measured in meters, then the farmer wishes to minimize the surface area of the bin, given by

$$S = 2xy + 2yz + 2zx. \quad \text{——} (A)$$

Subject to the constraint on the volume. namely

$$xyz = 1000 \quad \text{——} (B)$$

Let us consider the function F of independent variables x, y, z

where $F = (2xy + 2yz + 2zx) + \lambda(xyz - 1000)$.

$$\therefore dF = [2y + 2z + \lambda(yz)]\,dx + (2x + 2z + \lambda xz)\,dy + (2y + 2x + \lambda xy)\,dz.$$

At stationary points, $dF = 0$

$$\therefore F_x = 0 \Rightarrow 2y + 2z + \lambda yz = 0 \quad \text{——} (1)$$

$$F_y = 0 \Rightarrow 2x + 2z + \lambda xz = 0 \quad \text{——} (2)$$

$$F_z = 0 \Rightarrow 2y + 2x + \lambda xy = 0 \quad \text{——} (3)$$

Multiplying (1) by x & (2) by y and subtracting we get

$$2zx - 2yz = 0$$

$$\Rightarrow 2z(x - y) = 0$$

$$\Rightarrow z = 0 \text{ or } x = y.$$

Here $z = 0$ is neglected, as it will not satisfy eqn (A)

$$\therefore \boxed{x = y} \quad \text{——} (4)$$

Again multiplying (2) by y and (3) by z and subtracting, we get

$2xy - 2yz = 0$

$\Rightarrow y(x-z) = 0$

$\Rightarrow y = 0$ or $x = z$

Here y is neglected

$\therefore \boxed{x = z}$ — (5)

$\therefore$ from (4) & (5) we have,

$x = y = z$

From (3), $xyz = 1000$

$x^3 = 1000$

$\Rightarrow x = 10$

$\therefore x = y = z = 10$ m.

$\therefore$ The dimensions of the bin that will minimize the amount of material needed in its construction are

$x = y = z = 10$ m.

7(c) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, $x \neq y$ show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u.$$

Sol^n: Here $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ is not a homogeneous function.

However, we write

$$\tan u = \frac{x^3+y^3}{x-y} \ (=z) \text{ say}$$

$$\Rightarrow z = x^2\left[\frac{1+(y/x)^3}{1-(y/x)}\right]$$

$\therefore$ z is a homogeneous function of x, y of degree 2.

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \quad \text{---} \ (2)$$

But from (1)

$$\frac{\partial z}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \ \& \ \frac{\partial z}{\partial y} = \sec^2 u\frac{\partial u}{\partial y} \quad \text{---} \ (3)$$

(ii) from (3)

$$\frac{\partial^2 z}{\partial x^2} = \sec^2 u\frac{\partial^2 u}{\partial x^2} + 2\sec^2 u\tan u\left(\frac{\partial u}{\partial x}\right)^2$$

$$\frac{\partial^2 z}{\partial y^2} = \sec^2 u\frac{\partial^2 u}{\partial y^2} + 2\sec^2 u\tan u\left(\frac{\partial u}{\partial y}\right)^2$$

and $$\frac{\partial^2 z}{\partial x \partial y} = \sec^2 u\frac{\partial^2 u}{\partial x \partial y} + 2\sec^2 u\tan u\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

By corollary of Euler's theorem,

we have

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 2(2-1)z$$

$$\Rightarrow \sec^2 u\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)$$
$$+2\sec^2 u\tan u\left[x^2\left(\frac{\partial u}{\partial x}\right)^2+2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+y^2\left(\frac{\partial u}{\partial y}\right)^2\right]=2\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2\tan u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)^2$$
$$=2\sin u\cos u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u-2\tan u\sin^2 2u$$
(by (4))

$$=(1-2\tan u\sin 2u)\sin 2u$$
$$=(1-4\sin^2 u)\sin 2u$$

7(d) If $f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Show that both the partial derivatives exist at $(0,0)$ but the function is not continuous there at.

Sol'n: $f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}$

$$=\lim_{h\to 0}\frac{0-0}{h}=0$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}$$

$$=\lim_{k\to 0}\frac{0-0}{k}=0$$

$\therefore f$ possesses both the partial derivatives at $(0,0)$

Now let $(x,y)\to(0,0)$ along the straight line $y=mx$.

$$\lim_{x\to 0}f(x,mx)=\lim_{x\to 0}\frac{mx^2}{x^2(1+m^2)}$$
$$=\frac{m}{1+m^2}$$

which depends upon m.

$\therefore \lim_{(x,y)\to(0,0)}f(x,y)$ does not exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$

8.(a) If the feet of the three normals from P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ prove that the feet of the other three lie on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$ and P lies on the line $a(b^2-c^2)x = b(c^2-a^2)y = c(a^2-b^2)z$.

Sol^n:- Let (x_1, y_1, z_1) be the given point, then feet of the six normals from it to the given ellipsoid are given by

$$\alpha = \frac{a^2x_1}{a^2+\lambda}, \quad \beta = \frac{b^2y_1}{b^2+\lambda}, \quad \gamma = \frac{c^2z_1}{c^2+\lambda} \quad \text{--- (1)}$$

and the six values of λ are given by the equation

$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}-1=0 \quad \text{--- (2)}$$

Now the equation of the plane PQR is given as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the three of six feet of the normals lie on it, so we have

$$\frac{\frac{1}{a}a^2x_1}{(a^2+\lambda)^2}+\frac{\frac{1}{b}b^2y_1}{(b^2+\lambda)^2}+\frac{\frac{1}{c}c^2z_1}{(c^2+\lambda)^2}-1=0$$

i.e. $$\frac{ax_1}{(a^2+\lambda)^2}+\frac{by_1}{(b^2+\lambda)^2}+\frac{cz_1}{(c^2+\lambda)^2}-1=0 \quad \text{--- (3)}$$

Substituting the values from (1) in the equation of the plane PQR.

Let the equation of the plane P'Q'R' be

$l'x + m'y + n'z = p'$ ——— (4)

Since the remaining three feet of the normals lie on it so we have

$$\frac{l'a^2x_1}{a^2+\lambda} + \frac{m'b^2y_1}{b^2+\lambda} + \frac{n'c^2z_1}{c^2+\lambda} - p' = 0 \text{ ——— } (5)$$

Hence we find that equation (2) is the product of (3) and (5). so comparing them we have

$$\frac{l'a^2x_1^2}{a'(a^2+\lambda)^2} = \frac{a^2x_1^2}{(a^2+\lambda)^2} \Rightarrow l'a^2x_1^2 = a^2x_1^2$$

$$\Rightarrow l' = \frac{1}{a}.$$

Similarly, $m' = \frac{1}{b}$, $n' = \frac{1}{c}$ and $p' = -1$

The six feet of the normals, therefore lie on

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1\right) = 0$$

(or) $\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2 - 1 = 0$

$$\Rightarrow \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) + 2\left(\frac{yz}{bc}+\frac{zx}{ac}+\frac{xy}{ab}\right) = 1$$

$$\Rightarrow \frac{yz}{bc} + \frac{zx}{ac} + \frac{xy}{ab} = 0$$

($\because$ the feet also lie on $\sum \frac{x^2}{a^2} = 1$)

Hence the feet of the normals lie on

$$\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=0 \quad \text{—— (6)}$$

we know that if the normals be drawn from the point $P(x_1, y_1, z_1)$ then the feet of the normals lie on the cone.

$$\frac{a^2x_1(b^2-c^2)}{x}+\frac{b^2y_1(c^2-a^2)}{y}+\frac{c^2z_1(a^2-b^2)}{z}=0 \quad \text{—— (7)}$$

$$a^2x_1(b^2-c^2)yz+b^2y_1(c^2-a^2)xz+c^2z_1(a^2-b^2)xy=0 \quad \text{—— (8)}$$

Comparing (6) and (8), we get

$$\frac{a^2(b^2-c^2)x_1}{1/bc}=\frac{b^2(c^2-a^2)y_1}{1/ca}=\frac{c^2(a^2-b^2)z_1}{1/ab}$$

$$\Rightarrow a(b^2-c^2)x_1=b(c^2-a^2)y_1=c(a^2-b^2)z_1$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is the line

$$a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z$$

Hence proved.

8(b) If A and A' are the extremities of the major axis of the principal elliptic section and any generator meets two generators of the same system through A and A' in P and P' respectively, then prove that

$$AP.\ A'P'=b^2+c^2$$

Solⁿ: we know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ are given by}$$

$$x = \frac{a(1+\lambda\mu)}{\lambda+\mu}, \quad y = \frac{b(\lambda-\mu)}{\lambda+\mu}, \quad z = \frac{c(1-\lambda\mu)}{\lambda+\mu} \quad \text{--- (1)}$$

The extremities of the major axis of the principal elliptic section are $A(a,0,0)$ and $A'(-a,0,0)$.

At A and A' from (1) we have

$$\lambda-\mu=0, \quad 1-\lambda\mu=0$$

$$\Rightarrow \lambda=\mu \text{ and } 1-\lambda^2=0$$

$$\Rightarrow \lambda=\pm 1$$

Now consider the generator through $A(a,0,0)$ corresponding to $\lambda=1$ and then it's points of intersection P with a generator of μ-system is obtained from (1) by putting $\lambda=1$ and is $\left[a, \frac{b(1-\mu)}{1+\mu}, \frac{c(1-\mu)}{1+\mu}\right]$

(or) (a, bt, ct) where $t = \frac{1-\mu}{1+\mu}$

$$\therefore AP^2 = (a-a)^2 + (bt-0)^2 + (ct-0)^2 = (b^2+c^2)t^2 \quad \text{--- (2)}$$

Again the generator through $A'(-a,0,0)$ corresponding to $\lambda=-1$ meets the generator of μ-system at P', whose coordinates are obtained from (1) by putting $\lambda=-1$ and is

$$\left[-a, \frac{b(1+\mu)}{1-\mu}, \frac{c(1+\mu)}{-(1-\mu)}\right]$$

(or) $\left(-a, \frac{b}{t}, -\frac{c}{t}\right)$ where $t = \frac{1-\mu}{1+\mu}$

$$\therefore (A'P')^2 = (-a-a)^2 + \left(\frac{b}{t}-0\right)^2 + \left(-\frac{c}{t}-0\right)^2$$

$$= \frac{b^2+c^2}{t^2} \quad \text{--- (3)}$$

$\therefore$ from (2) and (3) we get

$$AP^2 \cdot A'P'^2 = (b^2+c^2)^2$$

$$\Rightarrow AP \cdot A'P' = b^2+c^2$$

8(C). Prove that $z(ax+by+cz)+\alpha x+\beta y=0$ represents paraboloid and the equations to the axis are $ax+by+2cz=0$, $(a^2+b^2)z+a\alpha+b\beta=0$.

Sol^n: Given equation is $cz^2+byz+azx+\alpha x+\beta y=0$.

$\therefore$ Here $a=0,\ b=0,\ c=c,\ f=b/2,\ g=a/2,\ h=0,\ u=\alpha/2$

$v=\beta/2,\ w=0$ and $d=0$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{(or)}$$

$$\Rightarrow -\lambda\left[-\lambda(c-\lambda)-\frac{b^2}{4}\right]+\left(\frac{a}{2}\right)\left(\frac{a\lambda}{2}\right)]=0$$

$$\Rightarrow 4\lambda^3-4a\lambda^2-(a^2+b^2)\lambda=0$$

$$\Rightarrow \lambda\left[4\lambda^2-4a\lambda-a^2-b^2\right]=0$$

$$\Rightarrow \lambda=0 \text{ and } \lambda=\frac{4a\pm\sqrt{(16a^2+16a^2+16b^2)}}{8}=\frac{a\pm\sqrt{(2a^2+b^2)}}{2}$$

Let $\lambda_1=\frac{1}{2}\left[a+\sqrt{(2a^2+b^2)}\right]$, $\lambda_2=\frac{1}{2}\left[a-\sqrt{(2a^2+b^2)}\right]$, $\lambda_3=0$.

Now putting $\lambda=0$ in the determinant given by (I) & associating each row with l_3, m_3, n_3, we have

$0.l_3+0.m_3+(a/2)n_3=0$, $0.l_3+0.m_3+(b/2)n_3=0$

and $(a/2)l_3+(b/2)m_3+cn_3=0$

These gives $n_3=0$ and $al_3+bm_3=0$.

$$\text{i.e. } \frac{l_3}{b}=\frac{m_3}{-a}=\frac{n_3}{0}=\frac{\sqrt{(l_3^2+m_3^2+n_3^2)}}{\sqrt{b^2+a^2+0}}=\frac{1}{\sqrt{(a^2+b^2)}}$$

$$\therefore l_3=\frac{b}{\sqrt{(a^2+b^2)}},\quad m_3=-a/\sqrt{(a^2+b^2)},\quad n_3=0$$

Now $k=ul_3+vm_3+wn_3$

$$= \frac{\alpha}{2}\frac{b}{\sqrt{(a^2+b^2)}} + \frac{\beta}{2}\left[\frac{-a}{\sqrt{a^2+b^2}}\right] + 0 = \frac{b\alpha - a\beta}{2\sqrt{a^2+b^2}} \neq 0$$

$\therefore$ Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$.

$$\Rightarrow \frac{1}{2}\left[a + \sqrt{(2a^2+b^2)}\right]x^2 + \frac{1}{2}\left[a - \sqrt{(2a^2+b^2)}\right]y^2 + \frac{(b\alpha - a\beta)}{\sqrt{(a^2+b^2)}}z = 0 \quad \text{—— (2)}$$

Now as $a + \sqrt{(2a^2+b^2)} > 0$ and $a - \sqrt{(2a^2+b^2)} < 0$, so (2) represents a hyperbolic paraboloid.

Also if $F(x,y,z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations.

$$\frac{\partial F/\partial x}{l_3} = \frac{\partial F/\partial y}{m_3} = \frac{\partial F/\partial z}{n_3} = 2k$$

and $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

i.e. any two of the equations

$$\frac{\alpha + az}{b/\sqrt{(a^2+b^2)}} = \frac{\beta + bz}{-a/\sqrt{a^2+b^2}} = \frac{ax+by+2cz}{0} = \frac{2(b\alpha - a\beta)}{2\sqrt{(a^2+b^2)}}$$

and $k\left[\frac{bx}{\sqrt{(a^2+b^2)}} - \frac{ay}{\sqrt{(a^2+b^2)}} + 0\right] + \frac{\alpha}{2}x + \frac{\beta}{2}y = 0$, on

Substituting the values.

Thus we have $\frac{\alpha + az}{b} = \frac{\beta + bz}{-a}$, $ax + by + 2cz = 0$

and $\frac{b\alpha - a\beta}{2\sqrt{(a^2+b^2)}}\left[\frac{bx - ay}{\sqrt{(a^2+b^2)}}\right] + \frac{1}{2}(\alpha x + \beta y) = 0$

i.e. $(a^2+b^2)z + a\alpha + b\beta = 0$, $ax + by + 2cz = 0$

and $(b\alpha - a\beta)(bx - ay) + (\alpha x + \beta y)(a^2+b^2) = 0$

Now if (x_1, y_1, z_1) be the vertex of the paraboloid then x, y, z satisfies above three equations.

i.e. $(a^2+b^2)z_1 + a\alpha + b\beta = 0$ —— (3)

$$ax_1 + by_1 + 2cz_1 = 0 \quad \text{--- (4)}$$

and $(b\alpha - a\beta)(bx - ay) + (\alpha x + \beta y)(a^2+b^2) = 0$ --- (5).

and the equations of the axis are

$$\frac{x-x_1}{l_3} = \frac{y-y_1}{m_3} = \frac{z-z_1}{n_3}$$

$$\frac{x-x_1}{b/\sqrt{(a^2+b^2)}} = \frac{y-y_1}{-a/\sqrt{(a^2+b^2)}} = \frac{z-z_1}{0} \quad \text{--- (6)}$$

Substituting values of l_3, m_3, n_3

These give $z - z_1 = 0 \Rightarrow z = z_1 = -\dfrac{a\alpha + b\beta}{(a^2+b^2)}$ from (3)

$$(a^2+b^2)z + a\alpha + b\beta = 0 \quad \text{--- (7)}$$

Again from first two fractions of (6), we get

$$a(x-x_1) + b(y-y_1) = 0$$

$$ax + by = ax_1 + by_1 = -2cz_1 \text{ from (4)}$$

$$= -2c\left[-\frac{a\alpha + b\beta}{a^2+b^2}\right], \text{ from (3)}$$

$$= -2cz, \text{ from (7)}$$

$$\Rightarrow ax + by + 2cz = 0 \quad \text{--- (8)}$$

Hence from (7) and (8) the equations of the axis of the paraboloid are

$$(a^2+b^2)z + a\alpha + b\beta = 0, \quad ax + by + 2cz = 0$$