LINEAR ALGEBRA

: IFoS - 2014 :

Q. Show that $u_1 = (1,-1,0)$, $u_2 = (1,1,0)$ and $u_3 = (0,1,1)$ form a basis of $\mathbb{R}^3$. Express $(5,3,4)$ in terms of u_1, u_2, u_3.

→ Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Reducing it to echelon form using elementary row transformations;

$R_2 \to R_2 - R_1 \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim^{R_3 \to 2R_3 - R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ → Echelon form.

We observe that there are ~~on~~ three non-zero rows in echelon form of the matrix A which consists of the vectors $u_1, u_2, u_3 \in \mathbb{R}^3$.

$\Rightarrow S = \{u_1, u_2, u_3\}$ is a L.I. subset of $\mathbb{R}^3$.

WKT $\dim \mathbb{R}^3 = 3$. So, every L.I. subset of $\mathbb{R}^3$ consisting of three elements is a basis of $\mathbb{R}^3$.

$\therefore S = \{u_1, u_2, u_3\}$ constitute a basis of $\mathbb{R}^3$.

Let $(x,y,z) \in \mathbb{R}^3$, $a,b,c \in \mathbb{R}$ such that

$(x,y,z) = au_1 + bu_2 + cu_3 = a(1,-1,0) + b(1,1,0) + c(0,1,1)$

$(x,y,z) = (a+b, -a+b+c, c)$

On comparison, $c = z$.

$a + b = x$ —— ①

$-a + b + c = y$

$-a + b = y - c = y - z$ —— ②

① + ②:

$2b = x + y - z \Rightarrow b = \frac{1}{2}(x+y-z)$.

① − ②: $2a = x - y + z \Rightarrow a = \frac{1}{2}(x-y+z)$

$\therefore (x,y,z) = \frac{1}{2}(x-y+z)u_1 + \frac{1}{2}(x+y-z)u_2 + zu_3$.

$\therefore (5,3,4) = \frac{1}{2}(5-3+4)(1,-1,0) + \frac{1}{2}(5+3-4)(1,1,0) + 4(0,1,1)$

$= 3u_1 + 2u_2 + 4u_3$

(2) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. find all eigen values & corresponding eigen vectors of B viewed as a matrix over

(i) The real field $\mathbb{R}$ (ii) The complex field $\mathbb{C}$.

$\rightarrow$ Char. equation of B is given by $|B-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = 0$

$-(1-\lambda)(1+\lambda) + 2 = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

To find eigen vectors corresponding to the eigen values

(a) $\underline{\lambda = i}$: $(A - iI)X = 0$

$\Rightarrow \begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to (1-i)R_2 - 2R_1$,

$\begin{bmatrix} 1-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\Rightarrow (1-i)x - y = 0 \Rightarrow (1-i)x = y$.

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1-i)x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$.

(b) $\underline{\lambda = -i}$: $(A + iI)X = 0$

$\Rightarrow \begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$R_2 \to (1+i)R_2 - 2R_1$

$\begin{bmatrix} 1+i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$(1+i)x - y = 0$

$y = (1+i)x$.

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (1+i)x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$.

$\therefore X_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

(i) <u>If matrix is taken over the real field $\mathbb{R}$</u>:

Eigen value $a+ib$ is written as $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ & eigen vector $\begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

$\therefore$ The eigen value $\lambda_1 = i$ & corresponding eigen vector $X_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$ can be expressed as $\lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $X_1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

The eigen value $\lambda_2 = -i$ corresponding eigen vector $X_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ can be expressed as $\lambda_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

(ii) <u>If matrix is taken over the complex field $\mathbb{C}$</u>

Eigen value $\lambda_1 = i \longrightarrow$ Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

Eigen value $\lambda_2 = -i \longrightarrow$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

(3) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, Prove that $A^n = A^{n-2} + A^2 - I$, $n \geq 3$

$\longrightarrow$ The char. equation of A is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(\lambda^2 - 1) = 0$$

$$\Rightarrow \cancel{(\lambda-1)^2(\lambda+1) = 0}$$

$$\Rightarrow \lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$\Rightarrow \lambda^3 = \lambda^2 + \lambda - 1 \quad \text{—— (1)}$$

By Cayley-Hamilton's Theorem, A satisfies the Char. eqn (1).

$\therefore A^3 = A^2 + A - I$ —— (2)

Let $S(n) \equiv A^n = A^{n-2} + A^2 - I$. Then $S(3) \equiv A^3 = A + A^2 - I$ is true by (2).

Let S be true for $n = k$ for some $k \in \mathbb{Z}^+$, $k \geq 3$. Then

$S(k) \equiv A^k = A^{k-2} + A^2 - I$ is true.

Then, for $n = k+1$,

$S(k+1) \equiv A^{k+1} = A A^k = A[A^{k-2} + A^2 - I] = A^{k-1} + A^3 - A$.

$A^{k+1} = A^{(k+1)-2} + A + A^2 - I - A$ $[A^3 = A + A^2 - I$ from (2)$]$.

$A^{k+1} = A^{(k+1)-2} + A^2 - I$

Hence $S(k+1)$ is also true.

Therefore, by the principal of mathematical induction

$A^n = A^{n-2} + A^2 - I \quad \forall\, n \geq 3$.

(4) Show that the mapping $T: V_2(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ defined as $T(a,b) = (a+b, a-b, b)$ is a linear transformation. Find range, rank and nullity of T.

$\longrightarrow$ Let $\alpha_1 = (a_1, b_1)$, $\beta_1 = (a_2, b_2) \in \mathbb{R}^2$. Let $a, b \in \mathbb{R}$. Then

$a\alpha_1 + b\beta_1 = a(a_1, b_1) + b(a_2, b_2) = (aa_1 + ba_2, ab_1 + bb_2) \in \mathbb{R}^2$

$T(a\alpha + b\beta_1) = (aa_1 + ba_2 + ab_1 + bb_2, aa_1 + ba_2 - ab_1 - bb_2, ab_1 + bb_2)$

$= (a(a_1+b_1) + b(a_2+b_2), a(a_1-b_1) + b(a_2-b_2); ab_1 + bb_2)$

$= a(a_1+b_1, a_1-b_1, b_1) + b(a_2+b_2, a_2-b_2, b_2)$

$\Rightarrow T(a\alpha_1 + b\beta_1) = aT(\alpha_1) + bT(\beta_1)$

Hence, T is a linear transformation.

Range of $T = \{(a,b,c) \in \mathbb{R}^3 \,/\, T(x,y) = (a,b,c) \text{ for } (x,y) \in \mathbb{R}^2\}$

Let $S = \{(1,0),(0,1)\}$ be the basis of $\mathbb{R}^2$. Then

$T(1,0) = (1,1,0)$ & $T(0,1) = (1,-1,0)$.

Then Range of T = Subspace generated by $(1,1,0),(1,-1,0)$.

$= L\{(1,1,0),(1,-1,0)\}$.

Nullity of T: Let nullspace of T be defined as

$N_A(T) = \{\alpha \in \mathbb{R}^2 / T(\alpha) = (0,0,0)\}$.

Let $(x,y) \in \mathbb{R}^2$ such that $T(x,y) = (0,0,0) \Rightarrow (x,y) \in N_A(T)$.

$T(x,y) = (x+y,\, x-y,\, y) = (0,0,0)$

on comparison, $x+y=0,\ x-y=0,\ y=0$

$\Rightarrow x=0, y=0$.

$\therefore N_A(T) = \{(0,0)\}$.

$\therefore$ Nullity $(T) = 0$.

Rank $(T) = 2$ since the basis of Range of T contains two vectors.

(5) Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain matrix P such that $P^{-1}AP$ is a diagonal matrix.

$\rightarrow$ Char. eq^n of A is given by $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow (-2-\lambda)[(1-\lambda)(-\lambda) - 12] + 2[6+2\lambda] - 3[-4+(1-\lambda)] = 0$

$\rightarrow (-2-\lambda)[\lambda^2 - \lambda - 12] + 12 + 4\lambda + 9 + 3\lambda = 0$

$\rightarrow -2\lambda^2 - \lambda^3 + 2\lambda + \lambda^2 + 24 + 12\lambda + 12 + 4\lambda + 9 + 3\lambda = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0 \Rightarrow \lambda = 5, -3, -3$.

Hence, the eigen values of A are 5, -3, -3.

Eigen vector of A corresponding to the eigen values.

(i) $\lambda = 5$: $(A-5I)X = 0$

$$\Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 7R_1$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, it is in echelon form

$-8y - 16z = 0 \Rightarrow y = -2z$

$-x - 2y - 5z = 0 \Rightarrow x = -2y - 5z$

$x = 4\cancel{y}z - 5z$

$\Rightarrow x = -z$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -2z \\ z \end{bmatrix} = z\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

(ii) $\lambda = -3$: $(A+3I)X = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, it is in echelon form

$\therefore x + 2y - 3z = 0$

$x = -2y + 3z$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y+3z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Hence the eigen vectors of A corresponding to eigen value $\lambda = -3$ are $X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and corresponding to eigen value $\lambda = 5$ are $X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Since there are three L.I. eigen vectors of the 3x3 matrix A, then A is diagonalizable.

Let $P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

Then, $\underline{\underline{P^{-1}AP = D}}$

(6) Consider the linear mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x,y) = (3x+4y, 2x-5y)$, with usual basis. Find the matrix associated with linear transformation relative to basis $S = \{u_1, u_2\}$ where $u_1 = (1,2)$, $u_2 = (2,3)$.

→ Let $a, b \in \mathbb{R}$ and $(x,y) \in \mathbb{R}^2$ such that

$(x,y) = au_1 + bu_2 = a(1,2) + b(2,3)$

$\Rightarrow (x,y) = (a+2b, 2a+3b)$. Comparing both sides, we have

$\underbrace{a+2b = x}_{①}$, $\underbrace{2a+3b = y}_{②}$

$\Rightarrow$ ~~$(x,y) = (2y-x)(1,2) + (x-y)(2,3)$~~

$(x,y) = (2y-3x)(1,2) + (2x-y)(2,3)$

$② - 2 \times ① \equiv \quad -b = y - 2x$

$b = 2x - y$

$① \equiv a + 2b = x$

$a = x - 2b = x - 4x + 2y$

$= 2y - 3x$.

Now:

$T(u_1) = T(1,2) = (11, -8) = -49(1,2) + 30(2,3)$

$T(u_1) = -49u_1 + 30u_2$.

$T(u_2) = T(2,3) = (18, -11) = -76(1,2) + 47(2,3)$

$T(u_2) = -76u_1 + 47u_2$.

Then, the required matrix of T is given by

$$A = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$