Date : ....

NO.1 INSITITUTE FOR IAS/IFoS EXAMINATIONS

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MATHEMATICS CLASSROOM TEST 2022-23

Under the guidance *of* K. Venkanna

MATHEMATICS

REAL & CALCULUS CLASS TEST

Date: 30 Jan.-2022

Time: 03:00 Hours Maximum Marks: 250

INSTRUCTIONS

1. Write your details in the appropriate space provided on the right side.
2. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
3. Candidates should attempt All Question.
4. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/notations carry their usual meanings, unless otherwise indicated.
6. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
7. All rough work should be done in the space provided and scored out finally.
8. The candidate should respect the instructions given by the invigilator.
9. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Mobile No.

Email.: (In Block Letter)

Test Centre

Medium

I have read all the instructions and shall abide by them

_ _

Signature of the Candidate

I have verified the information filled by the candidate above

_ _

Signature of the invigilator

P.T.O.

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Total Marks

1. (i) Prove that the following sets are bounded $\{n^{1/n} : n \in N\}, \left\{\left(1+\frac{1}{n}\right)^n : n \in N\right\}$, $\{a^{1/n}$: $a > 0$ and $n \in N\}$.

Give supremum and infimum of each of these sets.

(ii) Applying Lagrange's mean value theorem to the function defined by $f(x) = \log(1 + x)$ for all $x \geq 0$, show that $0 < [\log(1 + x)]^{-1} - x^{-1} < 1$ whenever $x > 0$.

[15]

2. Show that the series $\sum \frac{1}{n^3 + n^4 x^2}$ is uniformly convergent for all real x. If s(x) be the sum function verify that s′(x) is obtained by term-by-term differentiation.

[14]

3. If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}, x \neq y$ show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u.$

[10]

4. Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx.$ **[10]**

5. f (x) is defined as follows:

$$f(x)=\begin{cases} \frac{1}{2}(b^2-a^2) & \text{for} \quad 0<x<a \\ \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x} & \text{for} \quad a<x<b \\ \frac{1}{3}\frac{b^3-a^3}{x} & \text{for} \quad x>b \end{cases}$$

Prove that f (x) and f′(x) are continuous but f″(x) is discontinuous. **[14]**

6. (i) Find the maximum and the minimum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$ on the interval [2, 3].

(ii) If $u = \sin^{-1}\sqrt{\frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}}$ then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-1/6$.

Hence show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$ **[20]**

7. Consider the function $f(x)=\int_0^x (t^2-5t+4)(t^2-5t+6)\,dt$

(i) Find the critical points of the function f(x).

(ii) Find the points at which local minimum occurs.

(iii) Find the points at which local maximum occurs.

(iv) Find the number of zeros of the function f(x) in [0, 5] **[15]**

8. Prove that the sequence (a_n) satisfying the condition $|a_{n+1} - a_n| \leq \alpha\ |a_n - a_{n-1}|$, $0 < \alpha < 1$ for all natural numbers $n \geq 2$, is a Cauchy sequence. **[10]**

9. Show that the sequence $\{f_n\}$, where

$$f_n(x)=\begin{cases} n^2x, & 0\le x\le 1/n \\ -n^2x+2n, & 1/n\le x\le 2/n \\ 0, & 2/n\le x\le 1 \end{cases}$$

is not uniformly convergent on $[0,1]$. **[13]**

10. (i) Show that the function f(x) = 1/x, x > 0 is continuous in (0, 1) but not uniformly continuous.

(ii) Determine whether $f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$

is Riemann-integrable on [0, 1] and justify your answer **[15]**

11. Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all x > 0. Deduce that

$\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n} < \log 2,$ n being a positive integer. **[15]**

12. (i) Prove that $f(x) = \sin\frac{1}{x}, x \neq 0$

$= 0, \quad x = 0$

is not uniformly continuous on $[0, \infty[$.

(ii) Define an open set. Prove that the union of a arbitrary family of open sets is open. show also that the intersection of a finite family of open sets is open. Does it hold for an arbitrary family of open sets ? Explain the reason for your answer by example. **[16]**

13. Given :

$$\Delta(x)=\begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$$

where f is a real valued differentiable function and α is a constant.

Find $\lim\limits_{x\to 0}\dfrac{\Delta(x)}{x}$. **[10]**

14. (i) If $u = x^2 + y^2$, $v = x^2 - y^2$, where $x = r\cos\theta$, $y = r\sin\theta$, then find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

(ii) If $\int_0^x f(t)\,dt = x + \int_x^1 t f(t)\,dt$, then find the value of f(1).

(iii) Express $\int_a^b (x-a)^m (b-x)^n\,dx$ in terms of Beta function. **[7+5+8=20]**

15. Given that $f(x,y)=\left|x^2-y^2\right|$. Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.

Hence show that $f_{xy}(0, 0) = f_{yx}(0, 0)$. **[15]**

16. Test the uniform convergence of the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} + \frac{x^4}{\left(1+x^4\right)^3} + \ldots \text{ on } [0,\ 1]$$ **[10]**

17. Find the maximum and minimum values of $f(x) = x^3 - 9x^2 + 26x - 24$ for $0 \le x \le 1$.

[13]

18. Find the stationary values of $x^2 + y^2 + z^2$ subject to the conditions $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$. Interpret the result geometrically. **[15]**

ROUGH SPACE