New College

STATICS

B.A. / B.Sc. II
Semester - III

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New College
STATICS

FOR

B.A. / B.Sc. II
(Third Semester)

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CONTENTS

FRICTION

CHAPTER 6

6.1. INTRODUCTION

So far, we have been dealing with problems involving the equilibrium of smooth bodies resting in contact with other smooth bodies. We shall now discuss the problems involving the equilibrium of rough bodies resting in contact with one another. It may, however, be understood that the concept of smooth bodies is nearly an ideal one and perfectly smooth bodies do not exist in nature. All bodies are rough to some extent and therefore offer some resistance to the motion of one upon another.

Let a body A be placed against a body B at the point O and let the reaction S of the body B on the body A be along OC. If θ be the inclination of the reaction S to the normal, then it may be resolved into two components :

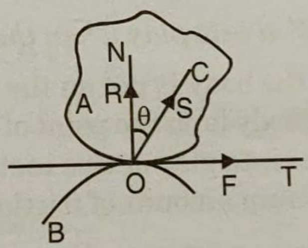

Fig. 6.1

(i) S cos θ along the common normal *i.e.*, R

(ii) S sin θ along the tangent *i.e.*, F

The first *i.e.*, R is called the *normal component* of the reaction or simply the **normal reaction**. The second *i.e.*, F is called the *tangential component* of the reaction or the **force of friction.** Evidently F is the force which resists the motion of the body A upon the body B.

Definition. *The property of roughness of bodies by virtue of which a force is exerted to resist the motion of one upon another is called* **Friction** *and the force exerted is called the* **Force of Friction.**

The force of friction is of great importance in every day life. If there were no friction between our feet and the ground, we could not walk. The motion of a motor car or a railway carriage is also due to the force of friction between the wheels and the ground.

6.2. STAGES AND KINDS OF FRICTION

Suppose a heavy body is lying on a horizontal table.

Let the body be pulled by a force P parallel to the plane and let the magnitude of P be increased gradually. It will be found that for some time the body does not move. W, the weight of the body and R the normal reaction of the table are vertical forces and therefore have no effect in the horizontal direction. So, it is the force of friction F acting in the horizontal direction which prevents the body from moving. Evidently $F = P$ and $R = W$.

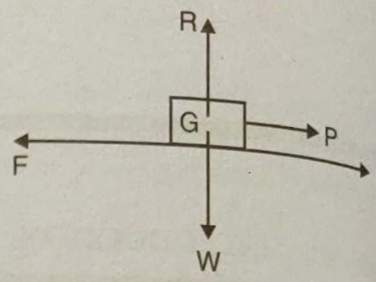

Fig. 6.2

As we increase the force P; F also increases so as to balance P. But there comes a stage, when with the increase in the value of P, the body does begin to move ultimately. It follows that the force of friction cannot go on increasing indefinitely. It can increase only up to a certain limit and if P increases beyond that limit, the body begins to move. When once the motion has started, the amount of the force of friction called into play remains constant.

We come across two stages in the above discussion; firstly when the body is in equilibrium, secondly when the body is moving.

The force of friction called into play when there is equilibrium is called **Statical Friction.**

The force of friction called into play when there is motion is called ***Dynamical*** **Friction.**

In the first case when the body is not on the point of motion, the equilibrium is said to be **non-limiting** and when the body is on the point of motion, the equilibrium is said to be **limiting** and the force of friction called into play at this instance is called **Limiting Friction**. Evidently limiting friction is the maximum amount of friction which can be called into action.

When one body moves on another, motion can take place in two ways, either by the body sliding over the other or by its rolling over the other. The forces of friction called into action in the two cases are called **sliding friction** and **rolling friction** respectively.

6.3. LAWS OF STATICAL FRICTION

(*i*) *Statical friction acts in the direction opposite to that in which the body tends to move.*

(*ii*) *The magnitude of this force is just sufficient to prevent the body from moving.*

Laws of Limiting Friction :

(*i*) *The direction of the limiting friction is opposite to that in which the body is about to move.*

(*ii*) *The magnitude of limiting friction of the point of contact between two bodies bears a constant ratio to the normal reaction at that point.*

(*iii*) *The limiting friction is independent of the shape and area of the surfaces in contact.*

Laws of Dynamical Friction :

(i) *The direction of dynamical friction is opposite to that in which the body is moving.*

(ii) *The magnitude of dynamical friction also bears constant ratio to the normal reaction and this ratio is slightly less than that in the case of limiting friction.*

(iii) *The dynamical friction is independent of the velocity of motion.*

6.4. CO-EFFICIENT OF FRICTION

[M.D.U. 2018; K.U. 2018, 2000]

Definition. *The constant ratio which the limiting friction bears to the normal reaction is called the co-efficient of friction. It is usually denoted by the letter* μ. Thus if F be the limiting friction and R be the normal reaction, then $\frac{\mathbf{F}}{\mathbf{R}} = \mu$ or $\mathbf{F} = \mu\mathbf{R}$.

6.5. RESULTANT REACTION

The limiting friction F and the normal reaction R acting at right angles to each other have a resultant force say S, which is called the **resultant reaction**

Thus $S = \sqrt{R^2 + F^2} = \sqrt{R^2 + \mu^2 R^2} = R\sqrt{1+\mu^2}$.

6.6. ANGLE OF FRICTION

[K.U. 2018, 16, 2000]

When a body is in limiting equilibrium on another, the angle which the resultant reaction S at the point of contact makes with the normal reaction is called the angle of friction and is generally denoted by the Greek letter λ.

Let F be the limiting friction, R the normal reaction and S the resultant reaction. If λ be the angle of friction which S makes with R, then

$$S \cos \lambda = R \quad \text{...(1)}$$

$$S \sin \lambda = F \quad \text{...(2)}$$

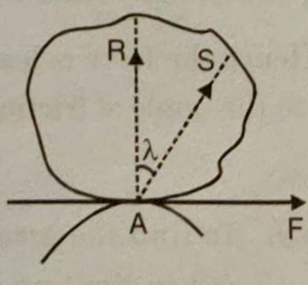

Fig. 6.3

Dividing (2) by (1), we have

$$\tan \lambda = \frac{\mathbf{F}}{\mathbf{R}} = \mu$$

Hence the tangent of the angle of friction is equal to the co-efficient of friction.

Evidently, so long as the body does not commence to move and if θ be the angle which the reaction makes with the normal in the statical stage, then $\theta < \lambda$ and $\frac{F}{R} < \mu$ *i.e.*, $F < \mu R$.

6.7. CONE OF FRICTION

The cone of friction is the cone which has the point of contact as its vertex, the normal as its axis and λ as its semi-vertical angle.

6.8. To find the least force required to drag a heavy body on a rough horizontal plane

[K.U. 2014; M.D.U. 2013]

Let W be the weight of the body and R the normal pressure. Let P be the force acting at an angle θ with the horizontal. When the body is on the point of motion, limiting friction μR or $R \tan \lambda$ will be called into play to act along AC.

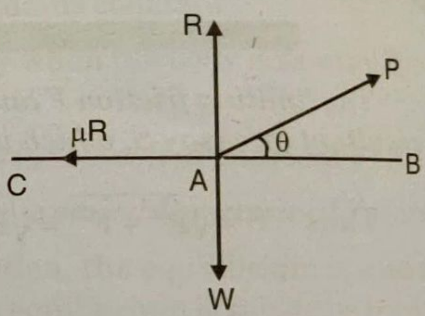

Fig. 6.4

Resolving horizontally and vertically, we have

$$P \cos \theta - \mu R = 0 \qquad \text{...(1)}$$

$$P \sin \theta + R - W = 0 \qquad \text{...(2)}$$

Eliminating R, we get

$$P (\cos \theta + \mu \sin \theta) = \mu W$$

$$\therefore \quad P = \frac{\mu W}{\cos \theta + \mu \sin \theta} = \frac{W \sin \lambda}{\cos (\theta - \lambda)} \qquad [\because \mu = \tan \lambda]$$

P is evidently least when $\cos (\theta - \lambda) = 1$ or $\theta = \lambda$.

Hence the force is least when it acts in a direction making with the horizontal an angle equal to the angle of friction and is then equal to $W \sin \lambda$, λ being the angle of friction.

6.9. To find the greatest inclination of a rough plane on which a heavy particle can rest in limiting equilibrium without the application of any other force

[M.D.U. 1990]

Let a body (treated as a particle) of weight W be placed on a rough plane and let the inclination on the plane be α to the horizon when the body is on the point of moving down the plane. Then the force of friction μR acts up the plane along the line of greatest slope through the body. Also, the weight of the body acts vertically downwards and the reaction R of the plane in a direction perpendicular to the plane.

By resolving along and perpendicular to the plane, we have

$$W \sin \alpha = \mu R \quad \text{and} \quad W \cos \alpha = R$$

$$\therefore \quad \tan \alpha = \frac{\mu R}{R} = \mu = \tan \lambda$$

$$\therefore \quad \alpha = \lambda$$

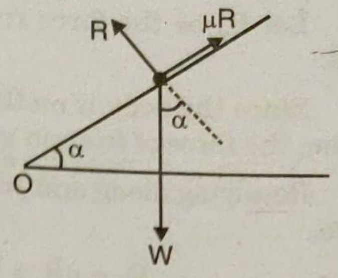

Fig. 6.5

i.e., *the greatest inclination of the plane is equal to the angle of friction.*

Cor. If $\alpha > \lambda$, the body slides down the plane and a force will be required to keep it at rest or move it up the plane. If $\alpha < \lambda$, a force will be necessary to move the body either up or down the plane.

6.10. A heavy body is placed on a rough inclined plane of inclination 'α' greater than the angle of friction, being acted upon by a force parallel to the plane and along a line of greatest slope. To find the limits between which the force must lie.

[K.U. 2013; M.D.U. 2012, 11]

Let α be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction. Let μ be the co-efficient of friction and λ the angle of friction.

Case I. *Let the body be on the point of moving up the plane.*

Let P_1 be the force acting parallel to the plane keeping the body at rest so that force of friction μR acts down the plane.

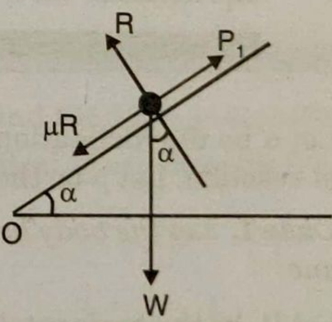

Fig. 6.6

Resolving along and perpendicular to the plane, we have

$$P_1 = \mu R + W \sin \alpha \qquad \ldots(1)$$

and

$$R = W \cos \alpha \qquad \ldots(2)$$

Eliminating R from (1) and (2),

$$P_1 = W (\sin \alpha + \mu \cos \alpha)$$

$$= W \left(\sin \alpha + \frac{\sin \lambda}{\cos \lambda} . \cos \alpha \right) \quad [\because \mu = \tan \lambda]$$

$$= W \frac{(\sin \alpha \cos \lambda + \cos \alpha \sin \lambda)}{\cos \lambda}$$

$$= W \frac{\sin (\alpha + \lambda)}{\cos \lambda} \qquad \ldots(3)$$

which gives the amount of force just sufficient to move the body up the plane.

Case II. *Let the body be on the point of moving down the plane.*

Let P_2 be the force required to keep the body at rest.

Since the body is on the point of moving down the plane, the force of friction acts up the plane.

Resolving along and perpendicular to the plane, we have

$$P_2 + \mu R = W \sin \alpha$$

and
$$R = W \cos \alpha$$

$$\therefore \quad P_2 = W \sin \alpha - \mu W \cos \alpha$$

$$= W \frac{\sin \alpha \cos \lambda - \cos \alpha \sin \lambda}{\cos \lambda}$$

$$= W \frac{\sin (\alpha - \lambda)}{\cos \lambda} \quad ...(4)$$

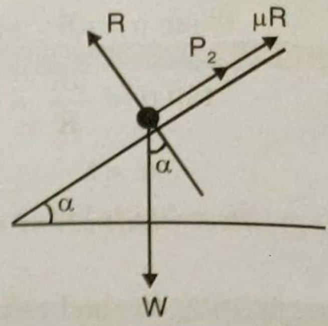

Fig. 6.7

which gives the amount of force just sufficient to support the body.

Hence P_1 and P_2 are the limiting values of the force required to keep the body at rest.

Therefore, the force must lie between P_1 and P_2.

6.11. To find the limits between which a force must lie in order to keep a body in equilibrium on a rough inclined plane, when the force acts horizontally.

[*M.D.U. 2013, 10*]

Let α be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction. Let μ be the co-efficient of friction and λ the angle of friction.

Case I. *Let the body be on the point of moving up the plane.*

Let P_1 be the horizontal force required to keep the body at rest so that force of friction μR act down the plane.

Resolving the forces acting on the body along and perpendicular to the plane, we have

$$P_1 \cos \alpha = \mu R + W \sin \alpha \quad ...(1)$$

and
$$R = W \cos \alpha + P_1 \sin \alpha \quad ...(2)$$

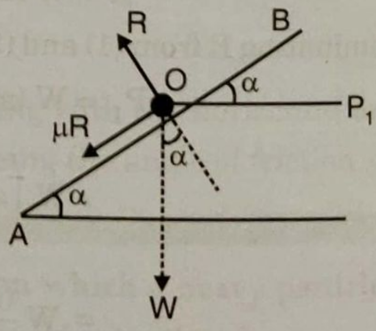

Fig. 6.8

Eliminating R from (1) and (2),

$$P_1 \cos \alpha = \mu (W \cos \alpha + P_1 \sin \alpha) + W \sin \alpha$$

or $\quad P_1 (\cos \alpha - \tan \lambda \sin \alpha) = W (\tan \lambda \cos \alpha + \sin \alpha) \quad [\because \mu = \tan \lambda]$

$$P_1 = W\frac{\sin\lambda\cos\alpha + \cos\lambda\sin\alpha}{\cos\alpha\cos\lambda - \sin\lambda\sin\alpha}$$

or

$$= W\frac{\sin(\alpha+\lambda)}{\cos(\alpha+\lambda)} = \mathbf{W\tan(\alpha+\lambda)},$$

which gives the amount of force just sufficient to move the body up the plane.

Case II. *Let the body be on the point of moving down the plane.*

Let P_2 be the force required to keep the body at rest.

Since the body is on the point of moving down the plane, the force of friction acts up the plane.

Resolving the forces along and perpendicular to the plane, we have

$$P_2\cos\alpha + \mu R = W\sin\alpha \qquad ...(3)$$

and

$$R = W\cos\alpha + P_2\sin\alpha \qquad ...(4)$$

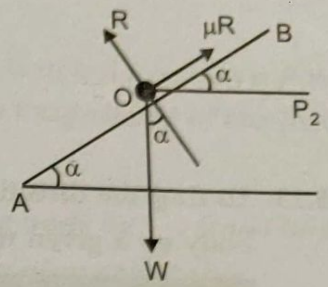

Fig. 6.9

Eliminating R from (3) and (4), we have

$$P_2\cos\alpha + \mu(W\cos\alpha + P_2\sin\alpha) = W\sin\alpha$$

or $$P_2(\cos\alpha + \tan\lambda\sin\alpha) = W(\sin\alpha - \tan\lambda\cos\alpha)$$

or $$P_2 = W\frac{\sin\alpha\cos\lambda - \sin\lambda\cos\alpha}{\cos\alpha\cos\lambda + \sin\alpha\sin\lambda} = W\frac{\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} = W\tan(\alpha-\lambda)$$

Hence P_1 and P_2 are the limiting values of the force required to keep the body at rest in which P_1 is the greatest force and P_2 is the least force.

Therefore, the force must lie between P_1 and P_2.

6.12. A heavy body is placed on a rough inclined plane. To find the force just sufficient to move the body up the plane, the force acting in a vertical plane through the line of greatest slope through the body

Let α be the inclination of the plane to the horizon, W the weight of the body and R the normal reaction. Let P be a force acting at an angle θ with the line of greatest slope through the body.

Since the body is on the point of moving up, the friction acts down the plane and is equal to μR.

Resolving along and perpendicular to the plane, we have

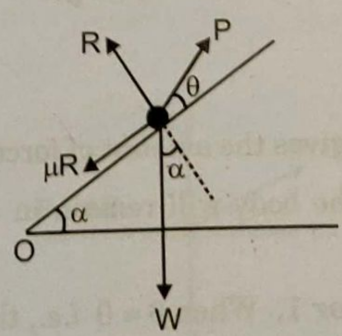

Fig. 6.10

$$P\cos\theta = \mu R + W\sin\alpha \qquad ...(1)$$

and

$$P\sin\theta + R = W\cos\alpha \qquad ...(2)$$

Eliminating R, we have

$$P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$$

i.e.,

$$P = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta} = W\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)} \quad ...(3) \quad [\because \mu = \tan\lambda]$$

Cor 1. When the force acts along the plane, then $\theta = 0$, so that

$$P = W\frac{\sin(\alpha+\lambda)}{\cos\lambda}. \quad ...(4)$$

Cor 2. When the force acts horizontally, then $\theta = -\alpha$, so that

$$P = W\frac{\sin(\alpha+\lambda)}{\cos(\alpha+\lambda)} = W\tan(\alpha+\lambda). \quad ...(5)$$

6.13. To find the direction and magnitude of the least force required to drag a heavy body up a given rough inclined plane

Equation (3) of Art. 6.12 gives the amount of force just sufficient to move the body up the plane. Clearly the force varies with θ, the angle at which the force is inclined to the plane. P will be least when $\cos(\theta-\lambda)$ is greatest *i.e.*, when $\theta = \lambda$.

Hence the required force is inclined to the plane at an angle equal to the angle of friction and is of magnitude $W\sin(\alpha+\lambda)$.

6.14. To find the force required to support a heavy body on a rough inclined plane, inclined to the horizontal at an angle greater than the angle of friction, the force acting in a vertical plane through a line of the greatest slope through the body

Let $P = P'$ when the body is on the point of motion down the plane. In this case $\alpha > \lambda$, the force of friction acts up the plane and therefore by changing the sign of μ or λ in the above equation of Art. 6.12, we get

$$P' = W\frac{\sin(\alpha-\lambda)}{\cos(\theta+\lambda)},$$

which gives the amount of force just sufficient to support the body on the plane.

The body will remain in equilibrium under the action of a force which lies between P and P′.

Cor 1. When $\theta = 0$ *i.e.*, the force acts along the plane and $\alpha > \lambda$,

$$P' = W\frac{\sin(\alpha-\lambda)}{\cos\lambda}.$$

Cor 2. When the force acts horizontally, $\theta = -\alpha$, so that

$$P' = W\frac{\sin(\alpha-\lambda)}{\cos(\alpha-\lambda)} = W\tan(\alpha-\lambda).$$

SOLVED EXAMPLES

Example 1.

A body of weight 50 kg rests on a horizontal table and is acted upon by a force of 20 kg making an angle of 30° with the horizontal. Find the magnitude of the force of friction that is called into action.

If the co-efficient of friction between the table and the body be $\frac{2}{3}$, find the least force acting in the same direction which will drag the body along the table.

[*C.D.L.U 2012*]

Solution. In the first, case, let R be the normal reaction of the table, F the force of friction which acts in the opposite direction in which the body would begin to move.

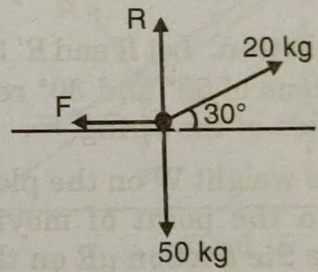

Fig. 6.11

Hence the body is in equilibrium under the action of four forces, *viz*.,

(*i*) its weight 50 kg acting vertically downwards.

(*ii*) a force of 20 kg making an angle of 30° with the horizontal.

(*iii*) the normal reaction R vertically upwards

(*iv*) the force of friction F.

Resolving vertically and horizontally, we have

$$R + 20\sin 30° = 50 \qquad ...(1)$$

and

$$F = 20\cos 30° \qquad ...(2)$$

From (2), $F = 10\sqrt{3}$ kg and from (1), R = 40.

In the second case, let P be the least force which when acting at an angle 30° to the horizon will just make the body move along the table. In this case, equilibrium is limiting and therefore,

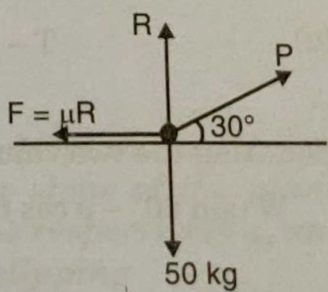

Fig. 6.12

$$F = \mu R = \frac{2}{3}R$$

Again resolving vertically and horizontally, we have

$$R + P\sin 30° = 50 \qquad ...(3)$$

and $$P \cos 30^\circ = F = \frac{2}{3}R \qquad ...(4)$$

From equation (4), $R = \frac{3}{2} P \cos 30^\circ = \frac{3\sqrt{3}}{4}P$

Substituting this value of R in (3), we get

$$\frac{3\sqrt{3}}{4}P + \frac{P}{2} = 50$$

so that $$P = \frac{200}{3\sqrt{3}+2} = \mathbf{27{\cdot}77\ kg.}$$

Example 2.

Two equal weights are attached to the ends of a string which is laid over the top of two equally rough planes having the same altitude and placed back to back; the angles of the inclination of the planes to the horizon being 30° and 60° respectively. Show that the weights will be on the point of motion if the co-efficient of friction be $2 - \sqrt{3}$.

Solution. Let R and R′ be the planes with inclinations of 60° and 30° respectively and T the tension of the string.

The weight W on the plane of inclination 60° is on the point of moving downwards, therefore the friction μR on this plane acts up the plane and the friction μR′ on the other plane acts down the plane.

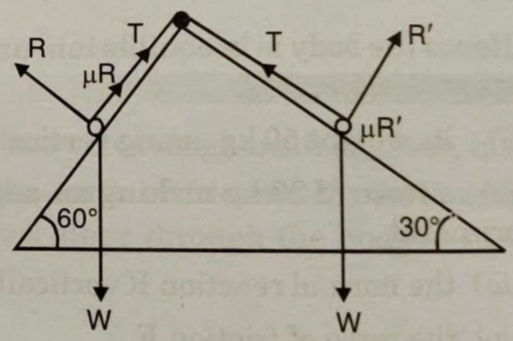

Fig. 6.13

Resolving along and perpendicular to the planes

(i) $$T + \mu R = W \sin 60^\circ; \quad R = W \cos 60^\circ$$

$\therefore$ $$T = W \sin 60^\circ - \mu R = W \sin 60^\circ - \mu W \cos 60^\circ$$
$$= W [\sin 60^\circ - \mu \cos 60^\circ]$$

(ii) $$T - \mu R' = W \sin 30^\circ, \ R' = W \cos 30^\circ$$

$\therefore$ $$T = W (\sin 30^\circ + \mu \cos 30^\circ)$$

Equating the two values of T, we get

$$W (\sin 60^\circ - \mu \cos 60^\circ) = W (\sin 30^\circ + \mu \cos 30^\circ)$$

$\therefore$ $$\mu = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{2} = 2 - \sqrt{3}.$$

Example 3.

A particle of weight W is placed on a rough plane inclined at an angle α (< λ) to the horizon. It is attached to a string which lies along a line of greatest slope and passes over a small smooth pulley at the top of the plane and carries a light scale pan hanging vertically at the end. The particle is just prevented from slipping down the plane by a weight nW in the scale pan. Show that the co-efficient of friction is given by

$$\mu = \tan\alpha - n\sec\alpha$$

Show also, that the least weight that must be added to the scale pan to drag the particle up the plane is 2W (sin α − n).

Solution. Let R be the normal reaction and T the tension in the string. For the equilibrium of nW, we have

$$T = nW$$

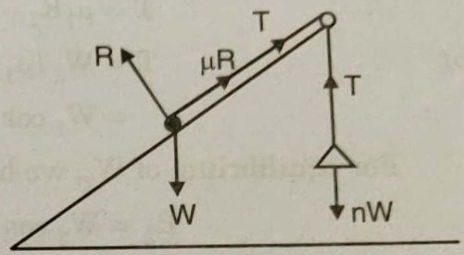

Fig. 6.14

For the equilibrium of W, resolving along the perpendicular to the plane, we get

$$T + \mu R = W\sin\alpha \quad \text{and} \quad R = W\cos\alpha$$

∴ Eliminating R, we have

$$nW + \mu W\cos\alpha = W\sin\alpha$$

Hence $\mu = \tan\alpha - n\sec\alpha.$

Let P be the weight which added to the scale pan will just drag the particle up the plane. Let T_1 be the tension in the string, then μR will act down the plane. We have for the equilibrium of the weight hanging by the string,

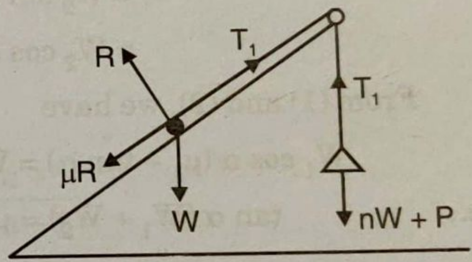

Fig. 6.15

$$T_1 = nW + P$$

Also, resolving along and perpendicular to the plane, we get

$$T_1 = \mu R + W\sin\alpha \quad \text{and} \quad R = W\cos\alpha$$

so that $nW + P = \mu W\cos\alpha + W\sin\alpha$

$$\therefore \quad P = (\mu\cos\alpha + \sin\alpha - n)\,W = (\sin\alpha - n + \sin\alpha - n)\,W$$

$$= 2W(\sin\alpha - n) \qquad [\because \mu = \tan\alpha - n\sec\alpha]$$

Example 4.

Two bodies of weights W_1 and W_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greater slope of the plane. If the coefficients of friction between the bodies and the plane be respectively μ_1 and μ_2 and $\mu_1 > \tan\alpha > \mu_2$, prove that if they are both on the point of slipping

$$\tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}. \qquad \text{[M.D.U. 2014]}$$

Solution. Let R_1 and R_2 be the normal reactions of the bodies and T the tension in the string.

Since both the bodies are on the point of slipping down the plane, the forces of friction $\mu_1 R_1$ and $\mu_2 R_2$ act up the plane.

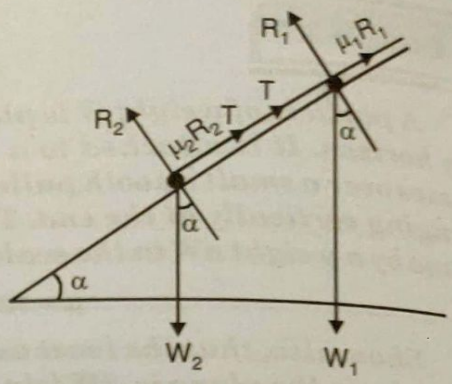

Fig. 6.16

For the equilibrium of W_1, resolving the forces along and perpendicular to the plane, we have

$$\mu_1 R_1 = W_1 \sin\alpha + T$$

and

$$R_1 = W_1 \cos\alpha$$

$\therefore$

$$T = \mu_1 R_1 - W_1 \sin\alpha$$

or

$$T = W_1 (\mu_1 \cos\alpha - \sin\alpha)$$

$$= W_1 \cos\alpha (\mu_1 - \tan\alpha) \qquad \ldots(1) \quad [\because \mu_1 > \tan\alpha, T \text{ is } +ve]$$

For equilibrium of W_2, we have

$$R_2 = W_2 \cos\alpha \quad \text{and} \quad \mu_2 R_2 + T = W_2 \sin\alpha$$

or

$$T = W_2 \sin\alpha - \mu_2 R_2$$

$$= W_2 \sin\alpha - \mu_2 W_2 \cos\alpha$$

$$= W_2 \cos\alpha (\tan\alpha - \mu_2) \qquad \ldots(2) \quad [\because \tan\alpha > \mu_2, T \text{ is } +ve]$$

From (1) and (2), we have

$$W_1 \cos\alpha (\mu_1 - \tan\alpha) = W_2 \cos\alpha (\tan\alpha - \mu_2)$$

i.e.,

$$\tan\alpha (W_1 + W_2) = \mu_1 W_1 + \mu_2 W_2$$

Hence,

$$\tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}.$$

Example 5.

The force acting parallel to a rough inclined plane of inclination α to the horizon, just sufficient to draw a weight up the plane is n times the force which will just let it be on the point of sliding down the plane. Prove that :

$$\tan\alpha = \mu \frac{n+1}{n-1}. \qquad \text{[M.D.U. 2014]}$$

Solution. Let P_1 be the force acting up the plane parallel to the plane which is just sufficient to draw a weight W up the plane.

$\therefore$

$$P_1 = \frac{W \sin(\alpha + \lambda)}{\cos\lambda} \qquad \ldots(1) \text{ [Ref. Case I of Art. 6.10]}$$

Let P_2 be the force which is just sufficient to support the body.

Then $$P_2 = \frac{W \sin(\alpha - \lambda)}{\cos \lambda} \quad ...(2) \quad \text{[Ref. Case II of Art. 6.10]}$$

Now according to the given condition,

$$P_1 = n.P_2$$

$\therefore$ From (1) and (2), we have

$$\frac{W \sin(\alpha + \lambda)}{\cos \lambda} = n \frac{W \sin(\alpha - \lambda)}{\cos \lambda}$$

i.e., $$\frac{\sin(\alpha + \lambda)}{\sin(\alpha - \lambda)} = \frac{n}{1}$$

Applying componendo and dividendo, we have

$$\frac{\sin(\alpha + \lambda) + \sin(\alpha - \lambda)}{\sin(\alpha + \lambda) - \sin(\alpha - \lambda)} = \frac{n+1}{n-1}$$

or $$\frac{2 \sin \alpha \cos \lambda}{2 \cos \alpha \sin \lambda} = \frac{n+1}{n-1}$$

or $$\tan \alpha \,.\, \cot \lambda = \frac{n+1}{n-1}$$

or $$\tan \alpha = \tan \lambda \,.\, \frac{n+1}{n-1} = \mu \,.\, \frac{n+1}{n-1}$$

Example 6.

A weight can be just supported on a rough inclined plane by a force P acting along the plane or by a force Q acting horizontally; show that the weight is $\dfrac{PQ}{\sqrt{Q^2 \sec^2 \phi - P^2}}$***, where ϕ is the angle of friction.*** [M.D.U. 2017, 11; K.U. 2014, 10, 01]

Solution. Let α be the inclination of the plane with the horizontal and W be the weight of the body.

Let μ be the coefficient of friction and ϕ is the angle of friction (given).

Case I. Let the body be just supported by the force P acting along the plane as shown in fig. 6.17.

$\therefore$ Force of friction μR is acting up the plane where R is normal reaction on the plane.

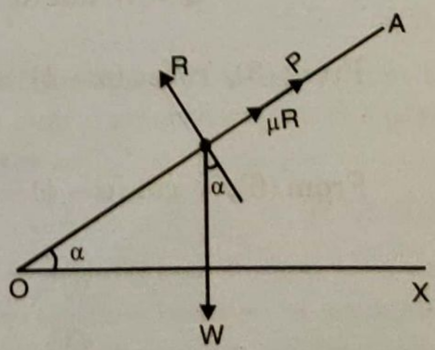

Fig. 6.17

Resolving the forces along and perpendicular to the plane, we have

$$P + \mu R = W \sin \alpha \quad ...(1)$$

and

$$R = W \cos \alpha \quad ...(2)$$

Using (2) in (1), we have

$$P = W \sin \alpha - \mu W \cos \alpha$$

$$= W(\sin \alpha - \mu \cos \alpha)$$

$$= W(\sin \alpha - \tan \phi \cos \alpha) = W\left(\frac{\sin \alpha \cos \phi - \cos \alpha \sin \phi}{\cos \phi}\right) \quad [\because \mu = \tan \phi]$$

$$\therefore \quad P = W \frac{\sin(\alpha - \phi)}{\cos \phi} \quad ...(3)$$

Case II. Let Q be horizontal force in this case which just supports the body. Let S be the normal reaction on the plane.

Resolving the forces along and perpendicular to the plane, we have

$$\mu S + Q \cos \alpha = W \sin \alpha \quad ...(4)$$

and

$$S = W \cos \alpha + Q \sin \alpha \quad ...(5)$$

Using (5) in (4), we have

$$\mu(W \cos \alpha + Q \sin \alpha) + Q \cos \alpha = W \sin \alpha$$

$$\therefore \quad Q = \frac{W \sin \alpha - \mu W \cos \alpha}{\mu \sin \alpha + \cos \alpha}$$

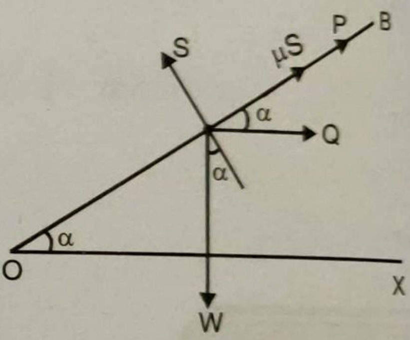

Fig. 6.18

$$= W \frac{(\sin \alpha - \tan \phi \cos \alpha)}{(\tan \phi \sin \alpha + \cos \alpha)}$$

$$= W \frac{(\sin \alpha \cos \phi - \cos \alpha \sin \phi)}{\sin \phi \sin \alpha + \cos \phi \cos \alpha} = W \frac{\sin(\alpha - \phi)}{\cos(\alpha - \phi)}$$

$$\therefore \quad Q = W \tan(\alpha - \phi) \quad ...(6)$$

From (3), $\operatorname{cosec}(\alpha - \phi) = \dfrac{W}{P} \sec \phi$ ...(7)

From (6), $\cot(\alpha - \phi) = \dfrac{W}{Q}$ ...(8)

$$\therefore \quad \frac{W^2}{Q^2} = \cot^2(\alpha - \phi)$$

or

$$\frac{W^2}{Q^2} = \operatorname{cosec}^2(\alpha - \phi) - 1 \quad [\because \operatorname{cosec}^2 \theta - \cot^2 \theta = 1]$$

or
$$\frac{W^2}{Q^2} = \frac{W^2 \sec^2 \phi}{P^2} - 1 \qquad \text{[Using (7)]}$$

or
$$W^2\left(\frac{\sec^2 \phi}{P^2} - \frac{1}{Q^2}\right) = 1 \quad \Rightarrow \quad W^2\left(\frac{Q^2 \sec^2 \phi - P^2}{P^2Q^2}\right) = 1$$

$\therefore$
$$W = \frac{\mathbf{PQ}}{\sqrt{\mathbf{Q}^2 \sec^2 \phi - \mathbf{P}^2}}.$$

EXERCISE 6.1

1. A block of weight 1000 kg rests on a fixed rough horizontal table, co-efficient of friction being 0·2. A horizontal pull of 100 kg; 200 kg and 300 kg weight is successively applied to the table and the block. What will be the corresponding frictional force between the table and the block in each case ?

2. A body of weight 80 kg rests on a rough horizontal plane while a force of 20 kg is acting on it in a direction making an angle of 60° with the horizontal. Find the force of friction that is called into play. *[C.D.L.U. 2013; M.D.U. 2011; K.U. 2000]*

3. A weight of 60 kg can just rest on a rough inclined plane of inclination 30° to the horizon. When the inclination is increased to 60°, find the least horizontal force which will support it.

4. Find how high can a particle rest inside a hollow sphere of radius 'r' if the coefficient of friction be $\frac{1}{\sqrt{3}}$.

5. A particle is at rest on the inner surface of a sphere of radius r. If the co-efficient of friction be μ, show that the greatest distance of the particle from the vertical diameter is $\frac{\mu r}{\sqrt{1+\mu^2}}$. *[K.U. 2011]*

6. Two equally rough bodies of weights W_1 and W_2 ($W_2 < W_1$) on a rough inclined plane are connected by a string which passes round a fixed smooth pulley in the plane. Find the greatest inclination of the plane consistent with the equilibrium of two bodies. *[M.D.U. 1993]*

7. Two weights resting on a rough inclined plane whose inclination α is greater than the angle of friction λ are connected by a string which passes over a smooth peg on the plane. Show that the least possible ratio of the less to the greater is
$$\sin(\alpha - \lambda) : \sin(\alpha + \lambda).$$
[K.U. 2014]

8. A body rests in contact with a rough inclined plane and is kept in equilibrium by a force acting up the plane. If the least value of the force when the plane is inclined at an angle to the horizon is equal to the greatest value of the force when the inclination of the plane to the horizon is β, prove that the angle of friction must be $\frac{\alpha - \beta}{2}$.

9. A body of weight W can just be dragged up a rough inclined plane by a force P and can just be sustained by a force Q ; P and Q both acting up the line of greatest slope through the body. Show that the co-efficient of friction is

$$\frac{P-Q}{\sqrt{4W^2-(P+Q)^2}} \text{ and angle of friction is } \sin^{-1}\frac{Q-P}{2\sqrt{W^2-PQ}}.$$

ANSWERS

1. 100 kg, 200 kg, 200 kg
2. 10 kg
3. $20\sqrt{3}$ kg
4. $\frac{r(2-\sqrt{3})}{2}$
6. $\tan^{-1}\left(\mu\frac{W_2+W_1}{W_1-W_2}\right)$

6.15. PROBLEMS ON EQUILIBRIUM OF RODS AND LADDERS

SOLVED EXAMPLES

Example 1.

A ladder whose centre of gravity divides it into two portions of lengths 'a' and 'b' rests with one end on a rough horizontal floor and the other end against a rough vertical wall. If the co-efficients of friction at the floor and wall be respectively μ and μ′, show that the inclination of the ladder to the floor, when the equilibrium is limiting, is

$$\tan^{-1}\frac{a-b\mu\mu'}{\mu(a+b)}$$

[*M.D.U. 2014; C.D.L.U. 2013*]

Solution. Let AB be the ladder and let the resultant reactions at A and B meet in C. Then the third force W must also pass through C.

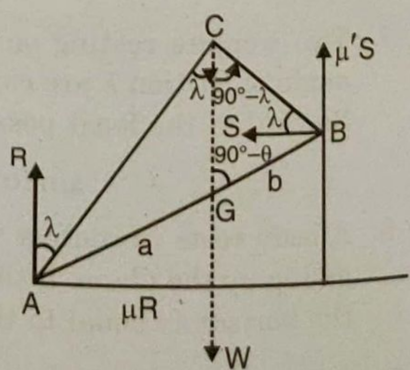

Fig. 6.19

Now, $AG = a,\ BG = b$

$\angle ACG = \lambda,\ \angle BCG = 90° - \lambda$

$\angle BGC = 90° - \theta$

where θ is the inclination of the ladder to the horizontal.

Using trigonometrical theorem for the ΔABC, we have

$$(a+b)\cot(90^\circ-\theta) = a\cot\lambda - b\cot(90^\circ-\lambda')$$

or

$$(a+b)\tan\theta = a\cot\lambda - b\tan\lambda = \frac{a}{\mu} - b\mu'$$

$\therefore$

$$\theta = \tan^{-1}\frac{a-b\mu\mu'}{(a+b)\mu}$$

which gives the inclination of the ladder to the horizontal.

Cor. If the ladder the uniform, then $b = a$. Putting $b = a$ in the result obtained above, we get

$$\theta = \tan^{-1}\frac{1-\mu\mu'}{2\mu}.$$

Example 2.

A uniform ladder of weight W, inclined to the horizon at 45°, rests with its upper extremity against a rough vertical wall and its lower extremity on the rough ground. Prove that the least horizontal force which will move the lower end towards the wall is just greater than $\frac{1}{2}W\left(\mu+\frac{1+\mu}{1-\mu'}\right)$***; where*** μ ***and*** μ' ***are the cc-efficients of friction at the lower and upper ends respectively.***

Solution. Let P be the least force which will pull the lower end towards the wall.

Resolving horizontally and vertically,

$$P = \mu R + S \quad \text{and} \quad R = W + \mu' S$$

$$\therefore \quad P = \mu(W + \mu' S) + S$$

so that

$$S = \frac{P-\mu W}{1+\mu\mu'} \qquad \text{...(1)}$$

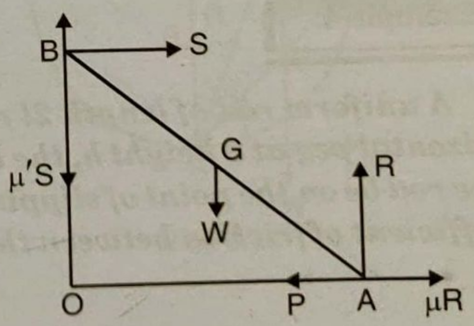

Fig. 6.20

Taking moments about A, we get

$$W.\frac{l}{\sqrt{2}} + \mu' S\frac{2l}{\sqrt{2}} = S.\frac{2l}{\sqrt{2}}, \text{ where } AB = 2l$$

$$\therefore \quad W = 2S(1-\mu') = 2\left(\frac{P-\mu W}{1+\mu\mu'}\right)(1-\mu') \qquad [Using\ (1)]$$

Hence, $P = \frac{1}{2}W\left(\mu+\frac{1+\mu}{1-\mu'}\right).$

Example 3.

One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. If ABC be a right-angle when the rod is on the point of sliding, μ the co-efficient of friction and α the angle between AB and the vertical, show that

$$\mu = \frac{\tan\alpha}{2+\tan^2\alpha}. \qquad \text{[K.U. 2016; M.D.U. 2013]}$$

Solution. Let the lines of action of the tension T in the string BC and the weight W acting through G, the mid-point of AB, meet in O. Then the line of action of the resultant reaction R′ at A also passes through O.

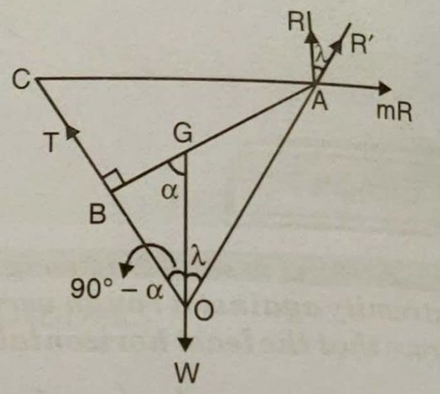

Fig. 6.21

$$\angle ABC = 90° \quad \text{and} \quad \angle BGO = \alpha$$

$\therefore$ $$\angle BOG = 90° - \alpha. \ \text{Also } \angle GOA = \lambda$$

$\therefore$ By "m - n" theorem in ΔOAB, we have

$$(a+a)\cot\alpha = a\cot\lambda - a\cot(90° - \alpha),$$

where $AB = 2a$

or $$2\cot\alpha = \cot\lambda - \tan\alpha$$

or $$\cot\lambda = 2\cot\alpha + \tan\alpha$$

$$= \frac{2}{\tan\alpha} + \tan\alpha = \frac{2\tan^2\alpha}{\tan\alpha}$$

$\therefore$ $$\mu = \tan\lambda = \frac{\tan\alpha}{2+\tan^2\alpha}.$$

Example 4.

A uniform rod of length 2l rests in a vertical plane against (and over) a smooth horizontal peg at a height h, the lower end of the rod being on level ground. Show that if the rod be on the point of slipping when its inclination to the horizontal is θ, then the co-efficient of friction between the rod and the ground is

$$\frac{l\sin\theta\sin 2\theta}{2h - l\cos\theta\sin 2\theta}.$$

Solution. Let AB be the rod resting with the end A on the level ground and a point of its length resting over the peg P. Let the normal reactions at A and P be R and S respectively and μR the force of friction. Resolving vertically and horizontally, we have

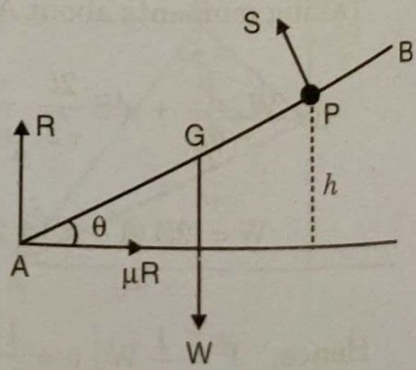

Fig. 6.22

$$R + S\cos\theta = W \quad \text{and} \quad S\sin\theta = \mu R$$

$$\therefore \qquad S = \frac{\mu W}{\sin\theta + \mu\cos\theta}$$

Taking moments about A, we have

$$W.l\cos\theta = S.h\,\text{cosec}\,\theta = \frac{\mu W}{\sin\theta + \mu\cos\theta}\,.\,h\,\text{cosec}\,\theta$$

$$\therefore \qquad l\cos\theta\sin\theta\,(\sin\theta + \mu\cos\theta) = \mu h$$

$$\text{or} \qquad \mu(h - l\sin\theta\cos^2\theta) = l\sin^2\theta\cos\theta$$

$$\text{Hence} \quad \mu = \frac{l\sin^2\theta\cos\theta}{h - l\sin\theta\cos^2\theta} = \frac{l\sin\theta\cos 2\theta}{2h - l\cos\theta\sin\theta}.$$

Example 5.

A uniform rod rests in a vertical plane within a fixed hemispherical bowl whose radius is equal to the length of the rod. If μ be the co-efficient of friction between the rod and the bowl, show that in limiting equilibrium the inclination of the rod to the horizontal is

$$\tan^{-1}\left(\frac{4\mu}{3-\mu^2}\right).$$

[M.D.U. 2012]

Solution. Let AB be the uniform rod and G, its C.G. Let O be the centre of the bowl.

Then $OA = OB = AB = r$, the radius of the bowl.

$\therefore\ \angle OAB = \angle OBA = 60°$

The rod will tend to slip in direction BA, forces of friction, therefore, at A and B will act in the opposite directions.

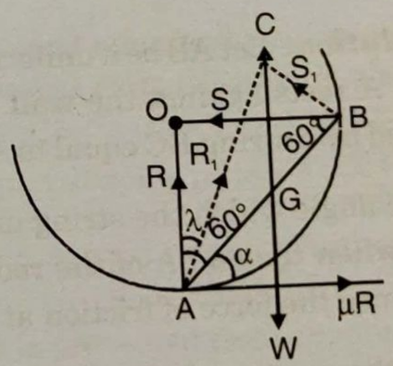

Fig. 6.23

Let R and S be the normal reactions at A and B acting along AO and BO. Let R_1 and S_1, the resultant reactions at A and B meet in C, then the vertical line of action of the weight W through G passes through C.

Let α be the inclination of the rod to the horizontal,

$$\therefore \qquad \angle CGB = 90° - \alpha$$

$$\text{Also,} \qquad \angle CAB = 60° - \lambda \text{ and } \angle CBA = 60° + \lambda$$

$\therefore$ In the ΔABC, by "$m - n$" theorem, we have

$$(r + r)\cot(90° - \alpha) = r\cot(60° - \lambda) - r\cot(60° + \lambda)$$

or
$$2\tan\alpha = \frac{1}{\tan(60^\circ - \lambda)} - \frac{1}{\tan(60^\circ + \lambda)}$$

$$= \frac{1 + \tan 60^\circ \tan\lambda}{\tan 60^\circ - \tan\lambda} - \frac{1 - \tan 60^\circ \tan\lambda}{\tan 60^\circ + \tan\lambda}$$

$$= \frac{1 + \sqrt{3}\mu}{\sqrt{3} - \mu} - \frac{1 - \sqrt{3}\mu}{\sqrt{3} + \mu}$$

$$= \frac{\sqrt{3} + 3\mu + \mu + \sqrt{3}\mu^2 - \sqrt{3} + 3\mu + \mu - \sqrt{3}\mu^2}{3 - \mu^2}$$

$\therefore$
$$2\tan\alpha = \frac{8\mu}{3 - \mu^2} \quad \text{or} \quad \tan\alpha = \frac{4\mu}{3 - \mu^2}$$

Hence,
$$\alpha = \tan^{-1}\frac{4\mu}{3 - \mu^2}.$$

Example 6.

A uniform rod rests with one extremity against a rough vertical wall, the other being supported by a string of equal length fastened to point in the wall. Prove that the least angle which the string can make with the wall is $\tan^{-1}\dfrac{3}{\mu}$.

[K.U. 2015; M.D.U. 2013]

Solution. Let AB be a uniform rod of length $2a$ whose one end A rests against the wall and the other end B is supported by a string BC equal to AB in length.

The angle which the string makes with the wall will be least when the end A of the rod is about to slip downward. Hence the force of friction at A acts up the wall.

Let the string make an angle θ with the wall. Let R be the normal reaction at A and R' be the resultant reaction at A.

Let the lines of action of tension T and resultant reaction R' meet in O. The line of action of the weight W through G, the C.G. of the rod must pass through O.

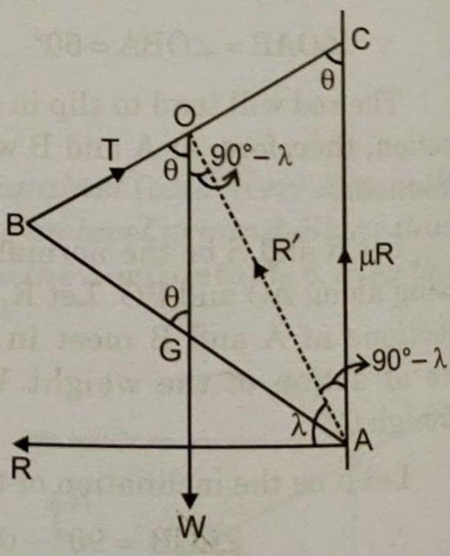

Fig. 6.24

In ΔABC, G is the mid-point of AB and OG $\parallel$ CA, therefore O is the mid point of BC.

$\therefore$
$$BO = BG \quad \text{and} \quad \angle BOG = \angle BGO = \theta$$

Also, $\angle GOA = \angle OAC = 90° - \lambda$

Applying '$m-n$' theorem in Δ ABO, we have

$$(a + a) \cot \theta = a \cot (90° - \lambda) - a \cot \theta$$

or
$$2 \cot \theta = \tan \lambda - \cot \theta$$

or
$$3 \cot \theta = \tan \lambda = \mu$$

$$\therefore \quad \tan \theta = \frac{3}{\mu} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{\mu}\right).$$

EXERCISE 6.2

1. A uniform ladder rests in limiting equilibrium with one end on a rough floor, whose co-efficient of friction is μ and with the other end against a smooth vertical wall. Show that the inclination to the vertical is $\tan^{-1}(2\mu)$. [*M.D.U. 2014*]

2. A ladder 30 m long rests with one end against a smooth vertical wall and with the other end on the ground which is rough, the co-efficient of friction being $\frac{1}{2}$. Find how high a man whose weight is 4 times that of the ladder can ascend before it begins to slip, the foot of the ladder being 6 m from the wall.

3. A beam rests with one end A on a rough horizontal plane and other end against a smooth vertical wall. If l be the length of the beam and a the distance of its CG from A, show that the inclination of the beam to the wall when on the point of slipping is $\tan^{-1}\left(\frac{l\mu}{a}\right)$, where μ is the coefficient of friction. [*K.U. 2013; C.D.L.U. 2012*]

4. A ladder rests with one end against a smooth wall and is prevented from slipping by the roughness of the ground. Show that a man weighing as much as the ladder cannot ascend to top of the ladder without causing it to slip, if its inclination to the vertical exceeds $\tan^{-1}\left(\frac{4\mu}{3}\right)$. [*M.D.U. 2000*]

   If the ascent is possible, show that as much as three times friction will be called into play when he is at the bottom.

5. A uniform ladder, of length l and weight W, rests with its foot on the rough ground and its upper end against a smooth wall, the inclination to the vertical being α. A force P is

applied horizontally to the ladder at a point distant c from the foot, so as to make the foot approach the wall. Prove that P must exceed $\frac{lW}{l-c}\left(\mu+\frac{1}{2}\tan\alpha\right)$, where μ is the co-efficient of friction at the foot. [M.D.U. 2011]

6. One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. If ABC be a right angle when the rod is on the point of sliding, μ the co-efficient of friction and α the angle between AB and the vertical, show that $\mu = \frac{\tan\alpha}{2+\tan^2\alpha}$.

7. A uniform rod rests in limiting equilibrium within a rough vertical circle. If the rod subtends an angle 2α at the centre of the circle and if λ be the angle of friction, show that the angle of inclination of the rod to the vertical is $\tan^{-1}\left(\frac{\cos 2\alpha + \cos 2\lambda}{\sin 2\lambda}\right)$.

[K.U. 2014; M.D.U. 2010]

8. A ladder inclined at 60° to the horizon rests between a rough floor and a smooth vertical wall. Show that if the ladder begins to slide down when a man has ascended so that his centre of gravity is half way up, coefficient of friction between the foot of ladder and the floor is $\frac{\sqrt{3}}{6}$.

ANSWERS

2. 88.1 m (nearly) ; to the top.

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