Set II

The plane

Definition:-

Plane is defined as the surface which is such that the line joining any two points on it lies wholly in the surface.

General Equation of First Degree:-

An equation of first degree in x, y, z is of the form

$ax+by+cz+d=0$ where a, b, c are given real numbers and a, b, c are not all zero. (ie $a^2+b^2+c^2 \neq 0$)

[It will be shown that this locus is a plane if it is such that if P & Q are any two points on the locus then every point of the line PQ is also a point on the locus].

Theorem:-

Prove that the general equation of first degree in x, y, z represents a plane.

Proof:-

Let the first degree equation in x, y, z be

$$ax+by+cz+d=0 \quad \text{——} (1)$$

where $a^2+b^2+c^2 \neq 0$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the locus given by (1) and let R be any point on the line PQ.

Dividing it in the ratio $m_1 : m_2$

Now we have $ax_1+by_1+cz_1+d=0$ —— (2)

and $ax_2+by_2+cz_2+d=0$ —— (3)

Now multiplying (2) by m_2 and (3) by m_1 and adding we get

$a(m_2x_1+m_1x_2)+b(m_2y_1+m_1y_2)+c(m_2z_1+m_1z_2)+d(m_1+m_2)=0$

$$\Rightarrow a\left(\frac{m_2x_1+m_1x_2}{m_1+m_2}\right)+b\left(\frac{m_2y_1+m_1y_2}{m_1+m_2}\right)+c\left(\frac{m_2z_1+m_1z_2}{m_1+m_2}\right)+d=0$$

Clearly this equation has a point

$$R\left(\frac{m_2x_1+m_1x_2}{m_1+m_2}, \frac{m_2y_1+m_1y_2}{m_1+m_2}, \frac{m_2z_1+m_1z_2}{m_1+m_2}\right)$$

it lies on (1) for all values of m_1, m_2

If P, Q lie on (1) then the point R of the line joining P & Q lies on (1). i.e., the line PQ lies wholly on (1).

$\therefore$ The surface represented by (2) is a plane surface. the equation of first degree in x, y, z always represents a plane.

* One point form :-

To show that the equation of any plane through (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Sol^n :- Let the equation of plane be

$ax+by+cz+d=0$ ——— ①.

$a^2+b^2+c^2 \neq 0$

Since the plane ① is passing through the point (x_1, y_1, z_1)

$\therefore$ $ax_1+by_1+cz_1+d=0$ ——②

$\therefore$ ② − ① $\equiv a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

which is the required equation.

Note: 1. The general equation of the plane is $ax+by+cz+d=0$

(or) $\frac{a}{d}x+\frac{b}{d}y+\frac{c}{d}z+1=0$

(or) $a'x+b'y+c'z+1=0$.

2. The equation of a plane through the origin is given by $ax+by+cz=0$.

* Intercepts :-

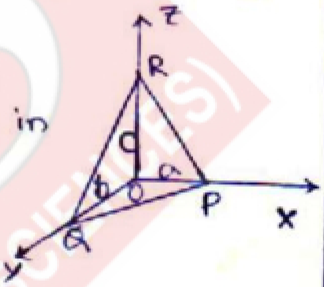
If a plane cuts (or) intercepts x-axis at $P(a,0,0)$, Y-axis at $Q(0,b,0)$ and z-axis at $R(0,0,c)$ then 'a' is called x-intercept, 'b' is called y-intercept and 'c' is called z-intercept of the plane.

* Intercept form :-

To find the equation of a plane interms of the intercepts a, b, c which it makes on the axis i.e. the equation of the plane intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol^n :- Let the equation of plane be $Ax+BY+Cz+D=0$ —①.

Let the plane meet the axes in P, Q, R.



$\therefore$ $OP=a$; $OQ=b$; $OR=c$.

Now the coordinates of P, Q, R are $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$

Since the point $P(a,0,0)$ lies on ① we get

$Aa+D=0 \Rightarrow A=\frac{-D}{a}$

Similarly $B=\frac{-D}{b}$ and $C=\frac{-D}{c}$

$\therefore$ ① $\equiv \frac{-D}{a}x-\frac{D}{b}y-\frac{D}{c}z+D=0$

$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1=0$.

which is the required equation i.e., equation of the plane in intercept form is

$$\frac{x}{\text{intercept on }x\text{-axis}}+\frac{y}{\text{intercept on }Y\text{-axis}}+\frac{z}{\text{intercept on }z\text{-axis}}=1.$$

→ Reduction of the general equation of the plane to the intercept form.

Sol'n :- Let the plane be

$$ax+by+cz+d=0 \quad \text{——} ①$$

$$\Rightarrow ax+by+cz=-d$$

$$\Rightarrow \frac{ax}{-d}+\frac{by}{-d}+\frac{cz}{-d}=1$$

$$\Rightarrow \frac{x}{(-d/a)}+\frac{y}{(-d/b)}+\frac{z}{(-d/c)}=1$$

$$\Rightarrow \frac{x}{A}+\frac{y}{B}+\frac{z}{C}=1$$

where $A=-d/a$, $B=-d/b$, $C=-d/c$

which is the required equation of the plane in the intercept form.

Note: To find the intercept on x-axis

Sol'n :- Put $y=0, z=0$ in ①

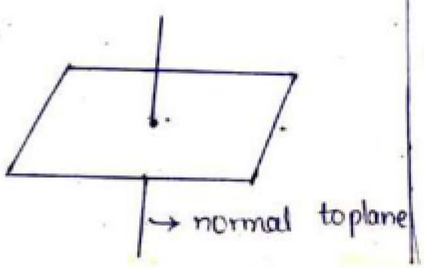
$$ax+d=0 \Rightarrow x=-d/a$$

Similarly on Y-axis is $-d/b$

Z-axis is $-d/c$

*Normal to a plane :-

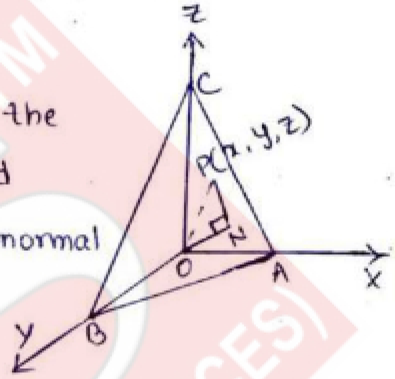
A line which is perpendicular to a plane is called a normal to the plane.



*Normal Form :-

To find the equation of a plane in terms of p, the length of a perpendicular from the origin on the plane, and l, m, n the direction cosines of the perpendicular.

Sol'n :-



Let ABC be the given plane and let ON be the normal (i.e. ⊥) from 'O' to the plane.

$\therefore$ $ON = p$ and the d.c's of ON are l, m, n.

Let $P(x,y,z)$ be any point on the plane.

Now join OP and PN then $ON \perp PN$

$\therefore$ ON = Projection of 'OP' on 'ON'

$$p = l(x-0)+m(y-0)+n(z-0)$$

$$(\because l(x_2-x_1)+m(y_2-y_1)+n(z-z_1))$$

$$\Rightarrow p = lx+my+nz$$

which is the required equation of the plane.

Note:— 1. The equation $lx+my+nz=p$ is called the normal form of the equation of a plane.

2. The equation of a plane in the form $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ where $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$ and p is always positive.

3. A equation $lx+my+nz=P$ is in the normal form if

(i) $(\text{Coefficient of } x)^2 + (\text{coefficient of } y)^2 + (\text{coefficient of } z)^2 = 1$ and

(ii) Constant term on the R.H.S is +ve.

* Reduction of General form to Normal form :-

To reduce the equation $Ax+By+Cz+D=0$ to the normal form $lx+my+nz=P$.

Sol^n :- the given equations

$$Ax+By+Cz+D=0 \quad \text{— ①}$$

$$lx+my+nz-P=0 \quad \text{— ②}$$

[Now we are required to find l,m,n and P in terms of A,B,C,D)

Now comparing the coefficients in ① & ②, we have

$$\frac{l}{A} = \frac{m}{B} = \frac{n}{C} = \frac{-P}{D}$$

$$\Rightarrow l = \frac{-PA}{D},\ m = \frac{-BP}{D},\ n = \frac{-CP}{D}$$

$\because$ l,m,n are direction cosines of normal to the plane.

$$\therefore l^2+m^2+n^2 = 1$$

$$\Rightarrow \frac{(A^2+B^2+C^2)}{D^2}P^2 = 1$$

$$\Rightarrow P = \pm\frac{D}{\sqrt{A^2+B^2+C^2}}$$

Case(i): If D is +ve then as P is always +ve,

$$\therefore P = \frac{D}{\sqrt{A^2+B^2+C^2}}$$

$$\therefore l = \frac{-A}{\sqrt{\Sigma A^2}},\ m = \frac{-B}{\sqrt{\Sigma A^2}},\ n = \frac{-C}{\sqrt{\Sigma A^2}}$$

$\therefore$ The normal form of ① is (Putting these values in ②)

$$\frac{-A}{\sqrt{\Sigma A^2}}x + \frac{-B}{\sqrt{\Sigma A^2}}y + \frac{-C}{\sqrt{\Sigma A^2}}z = \frac{D}{\sqrt{\Sigma A^2}}$$

Case(ii) If D is −ve then since P is always +ve

$$\therefore P = \frac{-D}{\sqrt{\Sigma A^2}},\ l = \frac{A}{\sqrt{\Sigma A^2}},\ m = \frac{B}{\sqrt{\Sigma A^2}},\ n = \frac{C}{\sqrt{\Sigma A^2}}$$

$\therefore$ The normal form of ① is (Putting these values in ②)

$$\frac{A}{\sqrt{\Sigma A^2}}x + \frac{B}{\sqrt{\Sigma A^2}}y + \frac{C}{\sqrt{\Sigma A^2}}z = \frac{-D}{\sqrt{\Sigma A^2}}$$

* Direction Cosines of the normal to a plane :-

From the above, the d.c's of normal to the plane $Ax+By+Cz+D=0$ are

$$\pm\frac{A}{\sqrt{\Sigma A^2}},\ \pm\frac{B}{\sqrt{\Sigma A^2}},\ \pm\frac{C}{\sqrt{\Sigma A^2}}$$

(+ sign if D is −ve
− sign if D is +ve).

Note: 1. The d.c's of the normal to a plane are proportional to coefficient of x, coefficient of y, coefficient of z in the equation of plane.

i.e. these give d.r's of normal to the plane and not that of the plane (d.rs are related to a line only.)

***Angle between two planes:-**

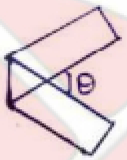
Let the two planes be given by

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{---(1)}$$

$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{---(2)}$$

Now the angle between two planes is equal to the angle between their normals.

Now the d.r's of the normal to plane (1) are (a_1, b_1, c_1).



Similarly the d.r's of the normal to plane (2) are (a_2, b_2, c_2)

$\therefore$ the d.c's of their normals are $\left(\frac{a_1}{\sqrt{\Sigma a_1^2}}, \frac{b_1}{\sqrt{\Sigma a_1^2}}, \frac{c_1}{\sqrt{\Sigma a_1^2}}\right)$ & $\left(\frac{a_2}{\sqrt{\Sigma a_2^2}}, \frac{b_2}{\sqrt{\Sigma a_2^2}}, \frac{c_2}{\sqrt{\Sigma a_2^2}}\right)$.

$\therefore$ If θ is the angle between two planes is angle between their normals.

$$\therefore \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\Sigma a_1^2}\ \sqrt{\Sigma a_2^2}}$$

*** Expression For $\sin\theta$ and $\tan\theta$**

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \frac{a_1a_2 + b_1b_2 + c_1c_2}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$$

$$\sin^2\theta = \frac{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2) - (a_1a_2+b_1b_2+c_1c_2)}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$$

$$\sin\theta = \pm\frac{\sqrt{(a_1b_2-b_1a_2)^2+(b_1c_2-c_1b_2)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

$$\sin\theta = \pm\frac{\sqrt{\Sigma(a_1b_2-b_1a_2)^2}}{\sqrt{\Sigma a_1^2}\cdot\sqrt{\Sigma a_2^2}}$$

Now $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$$= \frac{\sqrt{\Sigma(a_1b_2-b_1a_2)^2}}{\Sigma a_1a_2}$$

***Parallelism and Perpendicularity of two planes :-**

$\rightarrow$ If the two planes are $\|$lel then $\theta = 0°$

$\sin\theta = \sin 0 = 0$.

$$\sqrt{\Sigma (a_1b_2 - b_1a_2)^2} = 0$$

$a_1b_2 - b_1a_2 = 0$, $b_1c_2 - c_1b_2 = 0$

and $c_1a_2 - a_1c_2 = 0$

$\Rightarrow a_1b_2 = b_1a_2$; $b_1c_2 = c_1b_2$ & $c_1a_2 = a_1c_2$

$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$; $\frac{b_1}{b_2} = \frac{c_1}{c_2}$ & $\frac{c_1}{c_2} = \frac{a_1}{a_2}$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

→ If the two planes are ⊥lar then $\theta = 90°$.

$\cos\theta = \cos 90° = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$

* Plane through three Points :-

Prove that the equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Sol'n :- Let the plane equation be

$$ax + by + cz + d = 0 \quad \text{--- (1)}$$

Since the equation (1) passing through the points (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3)

$\therefore$ The equation (1) becomes

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{--- (2)}$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{--- (3)}$$

$$ax_3 + by_3 + cz_3 + d = 0 \quad \text{--- (4)}$$

Eliminating a, b, c, d from (1), (2), (3) and (4), we get.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

which is the required equation of the plane.

* Condition for Four Points to be Coplanar :-

The condition for four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) to be coplanar is that

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Sol'n :- The equation of plane through the three points (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3).

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Since it passes through the point (x_4, y_4, z_4) then the equation ① becomes

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Problems :—

1. Find the intercepts of the plane $2x-3y+z=12$ on the coordinate axes.

Sol'n :- The given plane equation is $2x-3y+z=12$

$$\Rightarrow \frac{2}{12}x - \frac{3}{12}y + \frac{1}{12}z = 1$$

$$\Rightarrow \frac{1}{6}x - \frac{1}{4}y + \frac{1}{12}z = 1$$

$\therefore$ The required intercepts are $6, -4, 12$.

2. Find the intercepts made on the coordinate axes by the plane $x+2y-2z=9$. Find also the d.c's of the normal to the plane.

Sol'n :- The given plane equation is $x+2y-2z=9$.

$$\Rightarrow \frac{x}{9} + \frac{y}{9/2} + \frac{z}{(-9/2)} = 1$$

$\therefore$ The required intercepts are $9, 9/2, -9/2$.

Now the direction ratios of the normal to the given plane are $1, 2, -2$.

$\therefore$ The direction cosines of the normal to the plane are

$$\frac{1}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}, \frac{-2}{\sqrt{1+4+4}}$$

i.e. $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$.

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3. Find the d.c's of the normals to the planes (i) $2x-3y+6z=7$ (ii) $x+2y+2z-1=0$

4. Show that the normal to the planes $x-y+z=1$, $3x+2y-z+2=0$ are $\perp$lar to eachother.
(Hint : $l_1l_2+m_1m_2+n_1n_2=0$).

5. Show that the plane $x+2y-3z+4=0$ is perpendicular to each of the planes $2x+5y+4z+1=0$; $4x+7y+6z+2=0$.

→6. find the angle between the following Pairs of planes.

(i) $2x-y+2z=3;\ 3x+6y+2z=4;$

[Ans: $\theta=\cos^{-1}(4/21)$]

(ii) $2x-y+z=6;\ x+y+2z=7$

(Ans: $\theta=\pi/3$)

(iii) $3x-4y+5z=0;\ 2x-y-2z=5$

(Ans: $\theta=\pi/2$)

→7. find the equation of the plane through the points $P(2,2,-1)$, $Q(3,4,2)$, $R(7,0,6)$.

→8. Show that the four points $(-6,3,2)$, $(3,-2,4)$, $(5,7,3)$ and $(-13,17,-1)$ are coplanar.

→9. find the equations of the two planes through the points $(0,4,-3)$, $(6,-4,3)$ which cutoff from the axes intercepts whose sum is zero.

Soln:- Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ — ①

and given that $a+b+c=0$ — ②

equation ① passes through the points $(0,4,-3)$ and $(6,-4,3)$.

$$\therefore \frac{0}{a}+\frac{4}{b}+\frac{-3}{c}=1$$

$$\Rightarrow \frac{4}{b}-\frac{3}{c}=1 \quad \text{— ③}$$

and $\frac{6}{a}+\frac{-4}{b}+\frac{3}{c}=1$

$$\Rightarrow \frac{6}{a}-\frac{4}{b}+\frac{3}{c}=1 \quad \text{— ④}$$

Adding ③ & ④, we get

$$\frac{6}{a}=2 \Rightarrow \boxed{a=3}$$

from ② we have $b+c=-3$ — ⑤

$$\boxed{b=-3-c} \quad \text{— ⑥}$$

from ③ we have

$$\frac{-4}{3+c}-\frac{3}{c}=1$$

$$\Rightarrow -4c-9-3c=3c+c^2$$

$$\Rightarrow -9-10c=c^2$$

$$\Rightarrow c^2+10c+9=0$$

$$\Rightarrow c^2+9c+c+9=0$$

$$\Rightarrow c(c+9)+1(c+9)=0$$

$$\Rightarrow (c+1)(c+9)=0$$

$$\Rightarrow \boxed{c=-1,\ c=-9}$$

from ⑥, we have

$$b=-3+1,\ -3+9$$

$$\boxed{b=-2,6}$$

∴ from ① we have

$$\frac{x}{3}+\frac{y}{-2}+\frac{z}{-1}=1$$

$$\frac{x}{3}+\frac{y}{6}+\frac{z}{6}=1$$

* **Equation of any plane Parallel to a given plane:**

→ The equation of a plane Parallel to $ax+by+cz+d=0$

is $ax+by+cz+K=0$.

Proof :- Let the given plane be

$$ax+by+cz+d=0 \quad \text{—— (1)}$$

and let any plane ‖lel to (1) be $a_1x+b_1y+c_1z+d=0$

—— (2)

Since the planes are ‖lel

$\therefore \dfrac{a_1}{a}=\dfrac{b_1}{b}=\dfrac{c_1}{c}=K$ (say)

$a_1=aK$; $b_1=bK$; $c_1=cK$

from (2) we have

$$aKx+bKy+cKz+d=0$$

$$\Rightarrow ax+by+cz+d/K=0$$

$$\Rightarrow ax+by+cz+\lambda=0$$

where $\lambda=d/K$.

$\therefore$ The equation of any plane ‖lel to the plane $ax+by+cz+d=0$ is $ax+by+cz+\lambda=0$.

i.e. the equation of two ‖lel planes differ by a constant.

Problems :-

1. find the equation of the plane through the point (x_1,y_1,z_1) and ‖lel to the plane $ax+by+cz+d=0$.

Sol^n :- Any plane ‖lel to given plane is

$$ax+by+cz+K=0 \quad \text{—— (1)}$$

Since it passes through (x_1,y_1,z_1)

$$\therefore ax_1+by_1+cz_1+K=0 \quad \text{—— (2)}$$

$$\therefore (1)-(2) \equiv a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

which is the required equation of the plane.

2. find equation of the plane through the point $(1,2,-1)$ and ‖lel to the plane $2x+3y+4z+5=0$

Sol^n :- Given equation of a plane is

$$2x+3y+4z+5=0 \quad \text{—— (1)}$$

Equation of any plane ‖lel to (1) is

$$2x+3y+4z+d=0 \quad \text{—— (2)}$$

since it is passing through the point $(1,2,-1)$

$$\therefore 2(1)+3(2)+4(-1)+\cancel{d}=0$$

$$\Rightarrow 2+6-4+K=0$$

$$\Rightarrow \boxed{K=-\cancel{2}}$$

$\therefore$ from (2), we have

$$2x+3y+4z-\cancel{2}=0$$

3. find the equation of the plane through the points $(2,2,1)$ and $(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.

Sol^n :- Any plane passes through the point $(2,2,1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0 \quad \text{—— (1)}$$

and it passes through the point $(9,3,6)$ is

$a(9-2)+b(3-2)+c(6-1)=0$

$\Rightarrow 7a+b+5c=0$ —— ②

Since the plane is ⊥lar to the given plane $2x+6y+6z=9$.

$\therefore 2a+6b+6c=9$ —— ③

[Two planes ⊥lar $\therefore a_1a_2+b_1b_2+c_1c_2=0$].

solving ② & ③

$7a+b+5c=0$

$2a+6b+6c=0$, we get

$$\frac{a}{6-30}=\frac{b}{10-42}=\frac{c}{42-2}$$

$$\frac{a}{-24}=\frac{b}{-32}=\frac{c}{40}$$

$$\Rightarrow \frac{a}{-3}=\frac{b}{-4}=\frac{c}{5}=k \text{ (say)}$$

$\Rightarrow a=-3k,\ b=-4k,\ c=5k$

① $\equiv -3k(x-2)-4k(y-2)+5k(z-1)=0$

$\Rightarrow 3x-6+4y-8-5z+5=0$

$\Rightarrow 3x+4y-5z-9=0$

which is the required equation.

Second Method :-

Let the equation of plane be

$Ax+By+Cz+D=0$ —— ①

it passes through the points $(2,2,1)$ & $(9,3,6)$

$2A+2B+C+D=0$ —— ②

$9A+3B+6C+D=0$ —— ③

Since ① is ⊥lar to $2x+6y+6z=9$

$\therefore 2A+6B+6C=0$ —— ④

Now ③−② $\equiv 7A+B+5C=0$ —— ⑤

solving ④ & ⑤

$$\frac{A}{30-6}=\frac{B}{42-10}=\frac{C}{2-42}$$

$$\Rightarrow \frac{A}{24}=\frac{B}{32}=\frac{C}{-40}$$

$$\Rightarrow \frac{A}{3}=\frac{B}{4}=\frac{C}{-5}$$

$\therefore A=3,\ B=4,\ C=-5$

$\therefore$ ② $\equiv 6+8-5=-D \Rightarrow D=-9$

$\therefore$ ① $\equiv 3x+4y-5z-9=0$

H.W.

4. find the equation of the plane which passes through $A(1,1,1)$ and $B(1,-1,1)$ and is ⊥lar to the plane $x+2y+2z=5$

5. obtain the equation of the plane which passes through the Point $(-1,3,2)$ and is ⊥lar to each of the two planes $x+2y+2z=5$, $3x+2y+2z=8$

(Ans: ~~2x+4y+3z+8=0~~ $y-z=1$).

*Equation of some particular planes :—

(a) Equation to the coordinate planes :—

(i) yz-plane: Any point on yz-plane will have its x-coordinate as '0', i.e. the equation of the yz-plane is $x=0$.

(ii) zx-plane: Its equation is $y=0$

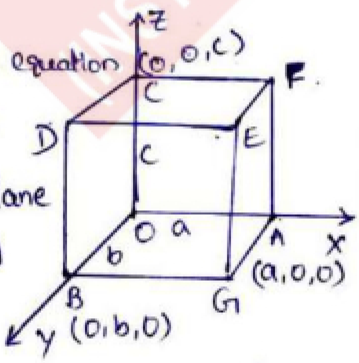
(iii) xy-plane: Its equation is $z=0$

(b) Equations to the plane ||lel to coordinate planes :—

Any point on a plane ||lel to yz-plane at a distance 'a' from it will have its x-coordinate as 'a'.

∴ The equation $\boxed{x=a}$ represents a plane parallel to yz-plane and at a distance 'a' from it.

Similarly the equation to the plane ||lel to xz-plane is $\boxed{y=b}$ and the equation to the plane ||lel to xy-plane is $\boxed{z=c}$.



(C) Equations to the plane ⊥lar to the coordinate planes:—

The equation of the yz-plane is $x=0$ i.e. $1.x+0.y+0.z=0$ —— ①

Let the equation to the plane be

$Ax+By+Cz+D=0$ —— ②

⊥lar to yz-plane ($x=0$)

$\therefore A(1)+B(0)+C(0)=0$

$\Rightarrow A=0.$

② ≡ $By+Cz+D=0$ which is required plane ⊥lar to yz-plane.

Similarly the equations to the planes ⊥ to zx and xy-planes are $Ax+Cz+D=0$ and $Ax+By+D=0$ respectively.

(d) Equations of the planes perpendicular to coordinate axes:—

Any plane perpendicular to x-axis is evidently parallel to yz-plane and its equation is $x=a$.

Similarly the equations to the planes ⊥lar to y & z axes are $y=b$ & $z=c$ respectively.

Note:-1. The equations $ax+by+r=0$; $by+cz+p=0$, $cz+ax+q=0$

represents planes respectively ⊥ to the xy, yz, zx planes.

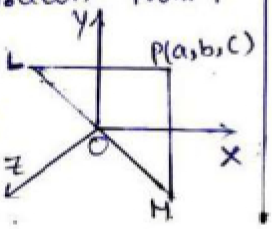
→ 2. $ax+by+cz+d=0$ represents plane ⊥ respectively to yz, zx, xy planes, if a, b, c separately vanish.

H.W: → find the equation of the plane through the points (1, 1, 1), (1, −1, 1), (−7, −3, −5) and show that it is ⊥ to the xz-plane.
(Ans: $3x-4z+1=0$)

→ Obtain the equation of the plane passing through the point (−2, −2, 2) and containing the line joining the points (1, 1, 1) and (1, −1, 2)   (Ans: $x-3y-6z+8=0$)

→ If from the point P(a, b, c), ⊥ lars PL, MP be drawn to yz & zx planes, then find the equation of the plane OLM.

Solⁿ :- Given that P(a, b, c) be any point and the ⊥ lars PL & MP be drawn from P on yz & zx planes



Since PL is ⊥ to yz − plane (x=0).

∴ The point L(0, b, c)
and MP is ⊥ lar to zx − plane (i.e. y = 0)
∴ the point M(a, 0, c)
∴ we have O(0, 0, 0), L(0, b, c) & M(a, 0, c).

Now the equation of the plane through three points OLM is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 0 & 1 \\ 0 & b & c & 1 \\ a & 0 & c & 1 \end{vmatrix} = 0.$$

→ show that the equations of the three planes passing through the points (1, −2, 4), (3, −4, 5) and ⊥ to the xy, yz, zx planes are $\cancel{x-y-3=0}$, $y+2z-6=0$ and $x-2z+7=0$ respectively.
$x-2y+1=0$

Solⁿ :- Let the equation of the plane passing through the point (1, −2, 4) is

$$A(x-1)+B(y+2)+C(z-4)=0 \quad \text{—— (1)}$$

Since it is passing through (3, −4, 5) is

$$2A-2B+C=0 \quad \text{—— (2)}$$

(1) is ⊥ to xy − plane

$$A(0)+B(0)+C=0 \quad \text{—— (3)}$$

solving (2) & (3)

$$\frac{A}{-2} = \frac{B}{2} = \frac{C}{0}$$

$\Rightarrow A=-1, B=1, C=0$

$\therefore$ ① $\equiv -1+1+y+2=0$

$\Rightarrow x-y-3=0$

Continue in this way.

→ Show that $(-1,4,-3)$ is the circum centre of the triangle formed by the points $(3,2,-5)$, $(-3,8,-5)$, $(-3,2,1)$.

Sol'n :- Let $P(-1,4,-3)$ be the circum centre of the triangle formed by the points $A(3,2,-5)$, $B(-3,8,-5)$ and $C(-3,2,1)$.

if (i) $PA = PB = PC$ and

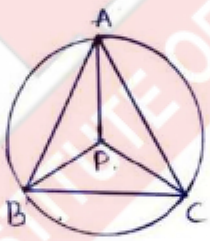
(ii) P, A, B, C are coplanar.

Now PA = distance

PB = distance

PC = distance

and coplanar.



→ Show that the four points $(6,-4,4)$, $(0,0,-4)$ intersects the join of $(-1,-2,-3)$, $(1,2,-5)$.

Sol'n :- The four points are $P(6,-4,4)$, $Q(0,0,-4)$, $R(-1,-2,-3)$, $S(1,2,-5)$

Now we are enough to P.T the line joining P, Q intersects the line joining R, S.

if the points P, Q, R, S lie on the same plane.

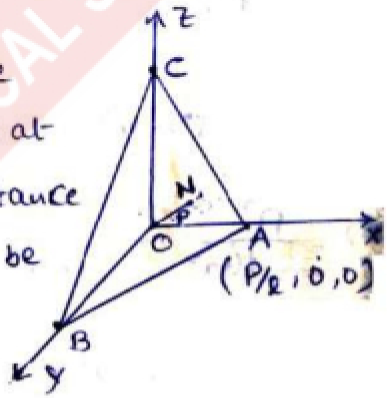
$\therefore$ i.e. we are enough to Prove that all the four points are coplanar.

1994 A variable plane is at a constant distance P from the origin and meet the coordinate axes in A, B, C. Show that the equation of the locus of the centroid of tetrahedron OABC is.

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}$$

Sol'n :- Let the variable plane at a constant distance 'p' from origin be

$lx+my+nz=p$ —— ①



where l, m, n are the d.c's of normal to the plane.

Since the plane ① meets x-axis where putting $y=0, z=0$

$\therefore lx=p \Rightarrow \boxed{x=P/l}$

$\therefore$ The coordinates of A are $(P/l, 0, 0)$

Similarly plane ① meets y-axis, & z-axis.

$\therefore y=P/m \; ; \; z=P/n$

The coordinates of B, C are $(0, P/m, 0)$ & $(0, 0, P/n)$

Now let (x, y, z) be centroid of the tetrahedron of OABC.

$$(x, y, z) = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

$$= \left(\frac{0+P/l+0+0}{4}, \frac{0+0+P/m+0}{4}, \frac{0+0+0+P/n}{4}\right)$$

$\Rightarrow x = \frac{P}{4l}, \quad y = \frac{P}{4m}, \quad z = \frac{P}{4n}$

$\Rightarrow l = \frac{P}{4x} \quad m = \frac{P}{4y}, \quad n = \frac{P}{4z}$

Since $l^2+m^2+n^2 = 1$

$\Rightarrow \frac{P^2}{16x^2} + \frac{P^2}{16y^2} + \frac{P^2}{16z^2} = 1$

$\Rightarrow x^{-2} + y^{-2} + z^{-2} = 16P^{-2}$

which is the required locus of centroid of tetrahedron.

1996 → A variable plane is at a constant distance 3p from the origin and meets the coordinate axes in A, B and C. Show that the locus of the centroid of the triangle ABC is $x^{-2}+y^{-2}+z^{-2} = p^{-2}$.

Sol'n :- Let the variable plane at a constant distance 3p from the origin be $lx+my+nz = 3p$ —①

where l, m, n are the d.c's of the normals to the plane.

Since the plane ① meets coordinate axes in A, B and C.

Now let the plane ① meets the x-axis where $y=0, z=0$.

$\therefore lx + m(0) + n(0) = 3p$

$lx = 3p$

$\Rightarrow \boxed{x = 3P/l}$

$\therefore$ The coordinates of A are $\left(\frac{3P}{l}, 0, 0\right)$

Now the plane ① meets the y-axis where $x=0, z=0$.

$\therefore my = 3p \Rightarrow \boxed{y = \frac{3P}{m}}$.

$\therefore$ The Coordinates of B are $\left(0, \frac{3P}{m}, 0\right)$

and the plane ① meets the z-axis where $x=0, y=0$.

$\therefore nz = 3p \Rightarrow \boxed{z = \frac{3P}{n}}$.

$\therefore$ The coordinate of C are $(0, 0, 3P/n)$.

Let (x, y, z) be the Centroid of ΔABC is (x, y, z)

$$= \left(\frac{3P/l+0+0}{3}, \frac{0+3P/m+0}{3}, \frac{0+0+3P/n}{3}\right)$$

$$= \left(P/l, P/m, P/n\right)$$

$\Rightarrow x = p/l, \; y = p/m, \; z = p/n.$

$\Rightarrow l = p/x, \; m = p/y, \; n = p/z$

Since $l^2+m^2+n^2=1$

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 1$$

$\Rightarrow x^{-2} + y^{-2} + z^{-2} = p^{-2}$

which is the locus of P.

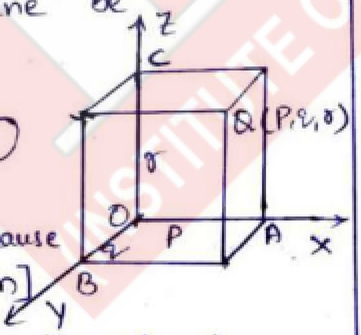
→ A variable plane passes through a fixed point (a,b,c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the coordinate planes is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

Sol^n:- Let the equation of variable plane be

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1 \quad \text{—(1)}$$

[Here we have taken p,q,r because a,b,c are given]



Since it passes through the point (a,b,c)

$$\therefore \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 1 \quad \text{—(2)}$$

Since the plane (1) meets coordinate axes in A, B, C.

$\therefore OA = p, \; OB = q, \; OC = r$

Now the eq^n of the plane through A$(p,0,0)$ and $\parallel$ to yz-plane is

$x = p$ —(3)

Sly the equations of planes through B$(0,q,0)$ & C$(0,0,r)$ and $\parallel$ lel to zx & xy planes are $y=q$, $z=r$.
—(4) —(5)

$\therefore$ These planes (3) (4) & (5) meet in the point Q where $x=p, \; y=q, \; z=r$ —(6)

The required locus is obtained by putting values of p,q,r from (6) in (2), we get

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

$\therefore$ which is required equation of the locus.

✱ **Equation to the plane through the line of intersection of two given planes:**

Let the two given planes be

$P_1 = A_1x + B_1y + C_1z + D_1 = 0$ —(1)

& $P_2 = A_2x + B_2y + C_2z + D_2 = 0$ —(2)

Then $P_1 + \lambda P_2 = 0$

i.e. $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$ —(3)

For all values of λ (3) represents a plane through the line of intersection of the plane (1) & (2)

Since if any point (x_1, y_1, z_1) satisfies both (1) & (2) then for all

Values of λ it satisfies ③ also.

Hence the equation ③ which is a first degree equation in x, y, z represents a plane through the line of intersection of ① & ②.

Problems :

→ find the equation of a plane passing through the intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and the point $(1,1,1)$.

Sol'n :- Given that $x+y+z-6=0$ —— ①

$$2x+3y+4z+5=0 \text{ —— ②}$$

Now the equation of any plane through the line of intersection of the given planes is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \text{ —— ④}$$

Since it passes through the point $(1,1,1)$.

we get $-3+14\lambda=0$

$$\Rightarrow \lambda = 3/14$$

∴ putting $K = 3/14$ in ① we obtain

$$20x+23y+26z-69=0$$

which is the required equation of the plane.

→ Obtain the equation of the plane through the intersection of the planes $x+2y+3z+4=0$ and $4x+3y+2z+1=0$ and the point origin.

→ find the equation of the plane passing through the line of intersection of the planes $2x-y=0$ and $3z-y=0$ are perpendicular to the plane $4x+5y-3z=8$.

→ find the equation of the plane which is ⊥ to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$, $2x+y-z+5=0$

(Ans: $51x+15y-50z+173=0$).

→ find the equation of the plane through the line of intersection of the planes.

$ax+by+cz+d=0$; $a_1x+b_1y+c_1z+d_1=0$ and ⊥ to the xy-plane.

(Ans: $x(ac_1-a_1c)+y(bc_1-b_1c)+(dc_1-d_1c)=0$.)

→ find the equation of the plane through the point $(2,3,4)$ and ∥ to the plane $5x-6y+7z=3$.

Sol'n :- Let the plane $5x-6y+7z+k=0$ —— ①

be ∥ to $5x-6y+7z-3=0$

Since ① passes through the point $(2,3,4)$

continue in this way—

→ find the equation of the plane through $(2,3,-4)$ and $(1,-1,3)$ $\parallel$ to the x-axis.

sol^n:- the equation of a plane is passing through the point $(2,3,-4)$ then

$$A(x-2)+B(y-3)+C(z+4)=0 \quad \text{——} (1)$$

and it passes through the point $(1,-1,3)$

$$A(1-2)+B(-1-3)+C(3+4)=0 \quad \text{——} (2)$$

and since (1) is $\parallel$ to x-axis i.e $\perp$ to yz-plane.

$\therefore x=0$.

$A=0$ i.e. $A+0B+0C=0$. ——(3)

solving (2) & (3) we get A,B,C in (1)

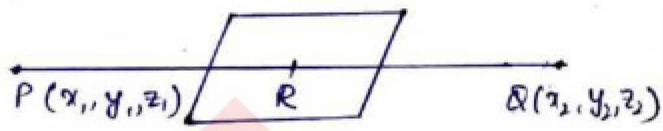
proceeding in this way.

* Two sides of a Plane:-

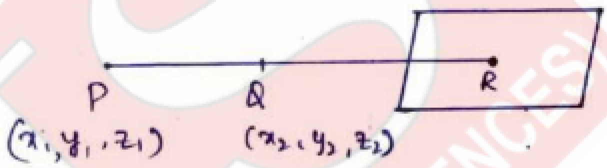
Show that the two points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ are the same (or) opposite sides of the plane $ax+by+cz+d=0$ according as the expressions $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the same (or) of different signs.

sol^n:- Equation of the plane is

$$ax+by+cz+d=0 \quad \text{——} (1).$$



Let the line PQ be divided by the given plane (1) at R in the ratio $k:1$.



$\therefore$ the coordinates of R are

$$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$$

since this point lies in the plane (1)

$$a\left(\frac{kx_2+x_1}{k+1}\right)+b\left(\frac{ky_2+y_1}{k+1}\right)+c\left(\frac{kz_2+z_1}{k+1}\right)+d=0$$

$$\Rightarrow \boxed{k=\frac{-(ax_1+by_1+cz_1+d)}{ax_2+by_2+cz_2+d}} \quad \text{——} (2).$$

Case(i):- If the expressions $ax_1+by_1+cz_1+d$ & $ax_2+by_2+cz_2+d$ are of opposite signs, then from (2) k is +ve.

$\therefore$ the plane (1) devides PQ internally in the ratio $k:1$

$\therefore$ P & Q lie on the opposite sides of the plane.

Case (ii) :- If the expressions $(ax_1+by_1+cz_1+d)$ and $(ax_2+by_2+cz_2+d)$ are of same signs then from ② k is $-$ve.

$\therefore$ The plane ① devides PQ externally in the ratio $k:1$

$\therefore$ The points P & Q lie on the same side of the plane.

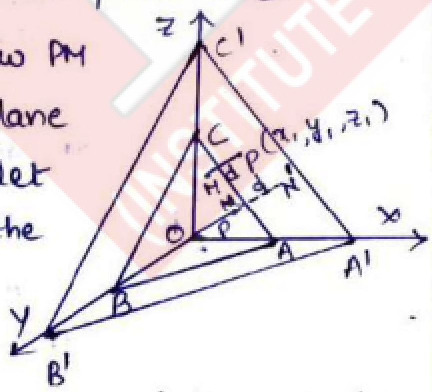
* **Length of the perpendicular from a point to a plane :—**

→ The perpendicular distance of the point (x_1, y_1, z_1) from the plane $lx+my+nz=p$ is

$$d = |lx_1+my_1+nz_1-p|.$$

Sol'n :- Let $P(x_1, y_1, z_1)$ be the given point and let $lx+my+nz=p$ be the given plane ABC.

From P draw PM ⊥ to the plane ABC, and let PM = d be the required ⊥ distance.



from 'O' draw ON ⊥ to ABC plane. ON = P and the d.c's of ON are l, m, n through $P(x_1, y_1, z_1)$ draw a plane A'B'C' ||lel to the plane ABC to meet ON in N'.

$\therefore ON' = ON + NN'$

$= p+d$

Since the direction cosines of ON' are same as those of ON. i.e; l, m, n.

$\therefore$ Equation of the plane A'B'C' is

$$lx+my+nz = p+d$$

since it passes through the point $P(x_1, y_1, z_1)$.

$\therefore lx_1+my_1+nz_1 = p+d$

$$\Rightarrow \boxed{d = (lx_1+my_1+nz_1-p)} \quad \text{①}$$

If P & O are on the same side of the plane.

then $d = -(lx_1+my_1+nz_1-p)$ ②

Combining ① & ②, the complete ⊥lar distance formula is

$$\boxed{\begin{array}{l} d = \pm(lx_1+my_1+nz_1-p) \\ \text{(or)}\ \ d = |lx_1+my_1+nz_1-p| \end{array}}$$

1984 → To find the perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$.

Sol'n :- Equation of the given plane is $ax+by+cz+d=0$ ①

To reduce ① to the normal form

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

which is the equation of the plane in normal form

(taking d to be −ve then R.H.S is +ve)

(or) $$\frac{ax+by+cz+d}{\sqrt{a^2+b^2+c^2}} = 0$$

$\therefore$ $\perp$lar distance of (x_1, y_1, z_1) from the plane is

$$\delta = \pm \frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$$

(or) $$\delta = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

Problems :-

→ Show that the points (1, 1, 1) and (−3, 0, 1) are on opposite sides and equidistant from the plane $3x+4y-12z+13=0$.

Sol^n :- the given plane is

$3x+4y-12z+13=0$ —— ①

For (1, 1, 1), put $x=1, y=1, z=1$ in L.H.S. expression of ①,

$\therefore ax_1+by_1+cz_1+d = 3(1)+4(1)-12(1)+13$
$= 20-12 = 8$ (+ve)

For point (−3, 0, 1), put $x=-3$, $y=0$, $z=1$ in L.H.S expression of ①

$\therefore ax_2+by_2+cz_2+d = 3(-3)+0-12(1)+13$
$= -9-12+13 = -8$ (−ve)

$\therefore$ The expressions $ax_1+by_1+cz_1+d$ & $ax_2+by_2+cz_2+d$ have opposite signs.

$\therefore$ the given two points (1, 1, 1) & (−3, 0, 1) are on opposite sides of plane ①

Now the given plane ①; distance from (1, 1, 1) is $\frac{|8|}{\sqrt{9+16+144}} = \frac{|8|}{\sqrt{169}} = \frac{8}{13}$

and also from (−3, 0, 1) is $\frac{|-8|}{13} = \frac{8}{13}$

$\therefore$ The given points are equidistant from the given plane.

→ Show that the origin and the point (2, −4, 3) lie on different sides of the plane $x+3y-5z+7=0$.

→ Find the distances of the points (2, 3, 4) and (1, 1, 4) from the plane

$3x-6y+2z-11=0.$

Distance between parallel planes :-

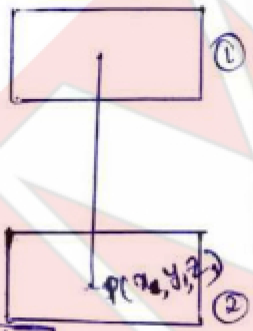
→ The distance between the parallel planes $ax+by+cz+d_1=0$; $ax+by+cz+d_2=0$ is $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

Sol'n :- Let the given planes be

$ax+by+cz+d_1=0$ —— (1)

and $ax+by+cz+d_2=0$ —— (2).

Let $P(x_1,y_1,z_1)$ be a point in the plane $ax+by+cz+d_2=0$ is



$ax_1+by_1+cz_1=-d_2$

$\Rightarrow \boxed{ax_1+by_1+cz_1=-d_2}$

The distance between two ||lel planes = ⊥lar distance from P to the plane (1)

$$= \frac{|ax_1+by_1+cz_1+d_1|}{\sqrt{a^2+b^2+c^2}} = \frac{|-d_2+d_1|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Problems :-

→ Find the distance between the ||lel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Sol'n :- The given planes are

$x+2y-2z+1=0$ —— (1)

and $x+2y-2z+5/2=0$ —— (2)

∴ distance between two ||lel planes is $= \dfrac{|1-5/2|}{\sqrt{1+4+4}} = \dfrac{(-3)}{3} = 1$

→ Show that the distance between the parallel planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$ is $1/6$.

→ Find the locus of the point whose distance from $(1,0,0)$ is twice its distance from the plane $3x+4y-z+2=0$.

Sol'n :- Let $P(x_1,y_1,z_1)$ be any point in the locus.

The distance of $P(x_1,y_1,z_1)$ from the point $(1,0,0) = \sqrt{(x_1-1)^2+y_1^2+z_1^2}$

Now the distance of (x_1,y_1,z_1) from the given plane

$$= \frac{|3x_1+4y_1-z_1+2|}{\sqrt{9+16+1}} = \frac{|3x_1+4y_1-z_1+2|}{\sqrt{26}}$$

Now given that

$$\sqrt{(x_1-1)^2+y_1^2+z_1^2} = 2\left(\frac{3x_1+4y_1-z_1+2}{\sqrt{26}}\right)$$

Squaring on both sides, we get

$$(x_1-1)^2+y_1^2+z_1^2 = 4\,\frac{(3x_1+4y_1-z_1+2)^2}{26}$$

$13(x_1^2+1-2x_1+y_1^2+z_1^2) = 2(9x_1^2+16y_1^2+z_1^2+4 \cdots)$

$\therefore$ Required locus of (x_1, y_1, z_1) is

$5x^2+19y^2-11z^2+48xy-12xz-16yz-50x+32y-8z-5=0$

→ find the locus of the point whose distance from origin is 3 times its distance from the plane $2x+2z=3$.

→ find the locus of a point, the sum of the squares of whose distances from the planes $x+y+z=0$, $x-z=0$, $x-2y+z=0$ is 9.

Sol'n:- Let $P(x_1, y_1, z_1)$ be any point in the locus and given planes are

$x+y+z=0$ — ①

$x-z=0$ — ② ; $x-2y+z=0$ — ③

Since the sum of the squares of distances of P from the given planes $=9$

$$\therefore \left(\frac{x_1+y_1+z_1}{\sqrt{1+1+1}}\right)^2+\left(\frac{x_1-z_1}{\sqrt{1+1}}\right)^2+\left(\frac{x_1-2y_1+z_1}{\sqrt{1+4+1}}\right)^2=9$$

$$\frac{(x_1+y_1+z_1)^2}{3}+\frac{(x_1-z_1)^2}{2}+\frac{(x_1-2y_1+z_1)^2}{6}=9$$

$\Rightarrow 2(x_1^2+y_1^2+z_1^2+2x_1y_1+2y_1z_1+2x_1z_1)+3(x_1^2+z_1^2-2x_1z_1)$

$+(x_1^2+4y_1^2+z_1^2-4x_1y_1-4y_1z_1+2x_1z_1) = 54$

$\Rightarrow 6x_1^2+6y_1^2+6z_1^2=54$

$\Rightarrow x_1^2+y_1^2+z_1^2=9$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is $x^2+y^2+z^2=9$.

→ Sum of the distances of any number of fixed points from a variable plane is zero. Show that the plane passes through a fixed point.

Sol'n:- Let the equation of variable plane be

$lx+my+nz=p$ — ① (normal form)

and let the fixed points be $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_r, y_r, z_r)$ and given that the sum of the distances of these points from the plane ① $=0$.

$\therefore (lx_1+my_1+nz_1-p)+(lx_2+my_2+nz_2-p)+\cdots+(lx_r+my_r+nz_r-p)=0$

$\Rightarrow l(x_1+x_2+\cdots+x_r)+m(y_1+\cdots+y_r)+n(z_1+z_2\cdots+z_r)-rp=0$

$$\Rightarrow l\left(\frac{\Sigma x_1}{r}\right)+m\left(\frac{\Sigma y_1}{r}\right)+n\left(\frac{\Sigma z_1}{r}\right)=p$$

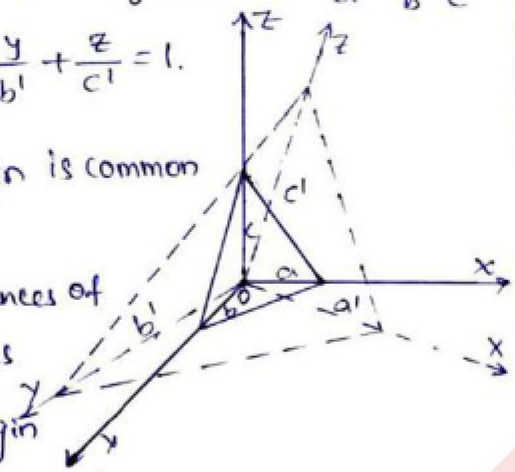
which shows that the plane ① passes through the point.

$$\left(\frac{\Sigma x_1}{r}, \frac{\Sigma y_1}{r}, \frac{\Sigma z_1}{r}\right)$$

which is fixed point. ($\because$ All the points are fixed points).

→ Two systems of rectangular axes have the same origin. If a plane cuts them at a distances a, b, c and a', b', c' respectively from the origin. Prove that $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$

Sol'n:- Let the equations of plane with respect to two systems are $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$.



Since origin is common to both.

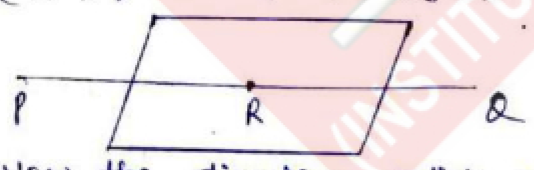
∴ ⊥lar distances of these planes from the origin must be equal.

$$\therefore \frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}} = \frac{1}{\sqrt{\left(\frac{1}{a'}\right)^2+\left(\frac{1}{b'}\right)^2+\left(\frac{1}{c'}\right)^2}}$$

$$\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$$

→ * Find the equation of the plane through the point $(2,-3,1)$ and normal to the line joining the points $P(3,4,-1)$ and $Q(2,-1,5)$.

Sol'n:- Any plane through the point $(2,-3,1)$ is $A(x-2)+B(y+3)+C(z-1)=0$ —(1)



Now the direction ratio's of the line joining the points $P(3,4,-1)$ and $Q(2,-1,5)$ are $2-3, -1-4, 5+1$

$\Rightarrow -1,-5,6$.

Since the plane (1) is normal to PQ whose direction ratio's are A, B, C is ||lel to the line PQ whose direction ratio's as $-1,-5,6$.

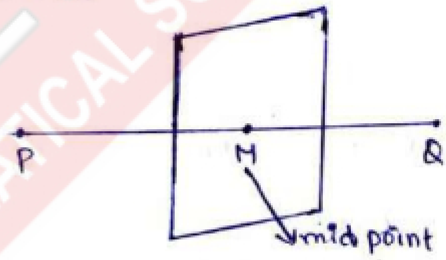
i.e. $\frac{A}{-1}=\frac{B}{-5}=\frac{C}{6}$

$\therefore$ (1) $\equiv -1(x-2)-5(y+3)+6(z-1)=0$

→ A plane passes through the point $(4,-1,2)$ and is ⊥lar to the line joining $(1,-5,10)$ and $(2,3,4)$.

→ Find the equation of the plane that bisects the line joining the points $(1,2,3)$; $(3,4,5)$ at right angles.

Sol'n:- The mid point of the line joining the points $P(1,2,3)$ & $Q(3,4,5)$ is $M(2,3,4)$



Now the equation of plane through the point $M(2,3,4)$ is

$A(x-2)+B(y-3)+C(z-4)=0$ —(1)

Now the d.r's of the line joining the points P & Q are

$3-1,\ 4-2,\ 5-3$

$\Rightarrow 2,2,2$

Since the plane (1) is ⊥lar to PQ whose d.r's are A, B, C is ||lel to the line PQ whose d.rs are 2, 2, 2 i.e. $\frac{A}{2}=\frac{B}{2}=\frac{C}{2}$

i.e. $A=2, B=2, C=2$

(1) $\equiv 2(x-2)+2(y-3)+2(z-4)=0$

$x+y+z-9=0$

→ Find the equation to a plane through $p(a,b,c)$ and perpendicular to OP, where O is the origin.

Sol Given that $p(a,b,c)$ is any point.

Let the required equation of the plane be

$A(x-a)+B(y-b)+C(z-c)=0$ ——— ①.

Since OP will be the normal from the origin to the plane ①.

∴ The dr's of OP are $a-0,\ b-0,\ c-0$ i.e. a, b, c.

∴ The required plane will be

$a(x-a)+b(y-b)+c(z-c)=0$

$\Rightarrow ax+by+cz=a^2+b^2+c^2$

→ A point P moves on a fixed plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. The plane through P perpendicular to OP meets the axes in A, B, C. The planes through A, B, C parallel to YOZ, XOZ, XOY intersect in Q. Show that the locus of Q is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$

Sol Let the point be $P(\alpha,\beta,\gamma)$

Since P lies on

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$\therefore \frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1$ ——— ①

Any plane through P be

$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0$ ——— ②

where A, B, C are d.r's of the normal ②.

Now d.r's of OP are $\alpha-0,\ \beta-0,\ \gamma-0$ i.e. α,β,γ.

Since ② ⊥ to OP.

$\therefore \frac{A}{\alpha}=\frac{B}{\beta}=\frac{C}{\gamma}$

∴ ② becomes

$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$

$\Rightarrow \alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2$.

$\Rightarrow \frac{x}{\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}}+\frac{y}{\frac{\alpha^2+\beta^2+\gamma^2}{\beta}}+\frac{z}{\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}}=1$ ——— ③

Since ③ meets the co-ordinate axes in A, B, C.

∴ The co-ordinates of A, B, C are

$\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha},0,0\right), \left(0,\frac{\alpha^2+\beta^2+\gamma^2}{\beta},0\right)$

$\left(0,0,\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right)$.

The plane through A parallel to YZ-plane is $x=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}$

plane through B, parallel to YZ-plane is
$$y = \frac{\alpha^2+\beta^2+\gamma^2}{\beta}.$$

plane through C parallel to XY-plane is $Z = \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$.

The locus of the point of intersection Q of these planes will be obtained by eliminating α, β, γ from the equations of these planes.

For this, we have
$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$
$$= \frac{1}{\alpha^2+\beta^2+\gamma^2} \quad \text{--- (4)}$$

and we have
$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz} = \frac{\alpha}{a(\alpha^2+\beta^2+\gamma^2)}+\frac{\beta}{b(\alpha^2+\beta^2+\gamma^2)}+\frac{\gamma}{c(\alpha^2+\beta^2+\gamma^2)}$$
$$= \frac{\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}}{\alpha^2+\beta^2+\gamma^2}$$
$$= \frac{1}{\alpha^2+\beta^2+\gamma^2} \text{ (by (1))}. \quad \text{--- (5)}$$

$\therefore$ from (4) & (5),
$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$
which is the reqd locus of Q.

→ The plane $lx+my=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. prove that the equation of the plane in its new position is
$$lx+my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0.$$

Sol:-

The equation of a plane through the line of intersection of the planes $lx+my=0$ (1) and $z=0$ (2) is $(lx+my)+\lambda z=0$ (3)

If this be the equation of plane (1) in its new position, then as given, the angle b/w planes (1) and (3) is α.

$$\therefore \cos\alpha = \frac{l^2+m^2}{\sqrt{(l^2+m^2)}\sqrt{(l^2+m^2+\lambda^2)}}$$
$$\therefore \cos^2\alpha = \frac{(l^2+m^2)^2}{(l^2+m^2)(l^2+m^2+\lambda^2)}$$
$$\lambda = \pm\sqrt{(l^2+m^2)}\tan\alpha$$

Hence, required plane is
$$lx+my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0$$

2008 The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y+4z+5=0$, find the equation of the plane in its new position.

Sol The given planes are $x-2y+3z=0$ —— (1)

$2x+3y-4z+5=0$ —— (2)

Now the equation of any plane through the intersection of (1) and (2) is

$(x-2y+3z)+k(2x+3y-4z+5)=0$ —— (3)

If this be the equation of the plane (1) in its new position, then the planes (1) and (3) are at right angles.

Now the d.r's of the normals to the planes (1) and (3) are

$1, -2, 3$ and $1+2k, -2+3k, 3-4k$.

Since the two planes (1) and (3) are $\perp$,

$\therefore$ Their normals are also $\perp$.

Hence

$1(1+2k)+(-2)(-2+3k)+3(3-4k)=0$

$\Rightarrow 1+2k+4-6k+9-12k=0$

$\Rightarrow 14-16k=0$

$\therefore \boxed{k=\frac{7}{8}}$

From (3), we have

$x-2y+3z+\frac{7}{8}(2x+3y-4z+5)=0.$

$8x-16y+24z+14x+21y-28z+35=0.$

$\Rightarrow 22x+5y-4z+35=0.$

which is the reqd eqn of the plane.

* **Position of the origin with respect to the angle between the two planes :—**

To Prove that the quantity $a_1a_2+b_1b_2+c_1c_2$ is –ve or +ve according as the origin lies in the acute angle (or) obtuse angle between the planes.

$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$

d_1, d_2 being both +ve.

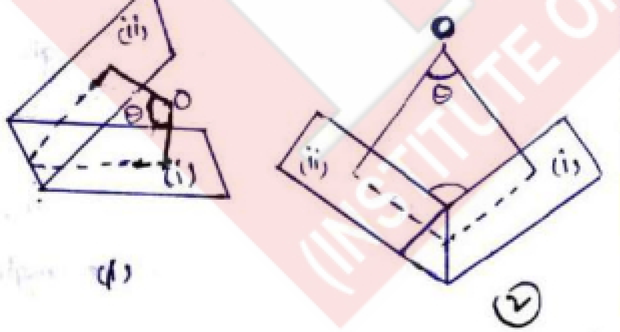
Sol'n :- The given planes are

$a_1x+b_1y+c_1z+d_1=0$ —— ①

$a_2x+b_2y+c_2z+d_2=0$ —— ②

∴ Angle between two planes is

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$



(i) If the origin is in the acute angle between the planes ① & ② then θ, the angle between the normals from the origin to them is obtuse as in figure ①.

∴ $\cos\theta$ is –ve.

i.e. $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}} = -ve$

∴ If the origin is in the acute angle between two planes ① & ② then $a_1a_2+b_1b_2+c_1c_2 = -ve$.

i.e. $a_1a_2+b_1b_2+c_1c_2 < 0$

(ii) If the origin is in the obtuse angle between the planes ① & ② then θ, the angle between the normals from the origin to them is acute as in figure ②.

∴ $\cos\theta$ is +ve.

i.e. $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}} = +ve$

∴ If the origin lies in the obtuse angle between the planes then $a_1a_2+b_1b_2+c_1c_2 = +ve$

i.e. $a_1a_2+b_1b_2+c_1c_2 > 0$.

Problems :—

→ Is the origin in the acute or obtuse angle between the planes $x+y-z=3$ and $x-2y+z=3$

Sol'n Given planes are

$x+y-z-3=0$ and

$x-2y+z-3=0$

$\Rightarrow -x-y+z+3=0$ —① [Here both

$-x+2y-z+3=0$ —② Constants are +ve].

Now $a_1a_2+b_1b_2+c_1c_2$

$$=(-1)(-1)+(-1)(2)+(1)(-1)$$

$$=1-2-1=1-3=-2<0.$$

$\therefore$ The origin lies in the acute angle between the planes.

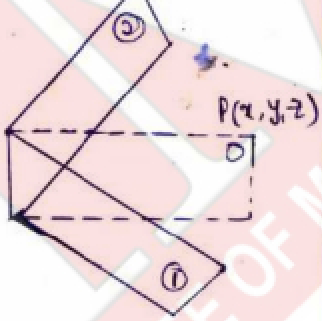
* Bisectors of the angle's between two planes :—

→ To find the equations of the planes bisecting the angles between two given planes:

Sol^n :- Let the two given plane equations be $a_1x+b_1y+c_1z+d_1=0$ —— ①

$a_2x+b_2y+c_2z+d_2=0$ —— ②

Let $P(x,y,z)$ be an arbitrary point on the plane bisecting the angle between two given planes.



Then the ⊥lar distances of this point from both the planes should be equal (in magnitude).

$$\therefore \frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\Sigma a_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\Sigma a_2^2}}$$

(Note: ± sign only on R.H.S)

which gives the required equations of two bisecting planes.

i.e. $$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\Sigma a_1^2}}=\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\Sigma a_2^2}} \quad —— ③$$

& $$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\Sigma a_1^2}}=-\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\Sigma a_2^2}} \quad —— ④$$

out of these two equations, one of them represents the plane bisecting acute angle between planes, while the other one is the bisector of the obtuse angle between the given planes.

Note ①: — Let θ be angle between ③ & ①

find $\cos\theta$ and therefore $\tan\theta$.

Case(i) :- If $\tan\theta<1$ then $\theta<45^\circ$

In this case ③ is the bisector of the acute angle between the given planes.

Case(ii) :- If $\tan\theta>1$ then $\theta>45^\circ$

In this case ③ is the bisector of the obtuse angle between given planes.

(II) If $(a_1a_2+b_1b_2+c_1c_2)<0$ then origin lies in the acute angle.

If $(a_1a_2+b_1b_2+c_1c_2)>0$ then origin lies in the obtuse angle.

(III) If both the constant terms in ① & ② are ~~the~~ then ③ bisects the angle that contains the origin and (4) bisects the angle that doesnot contain the origin.

Problems :-

→ find the equations of the planes bisecting the angle between planes. $x+2y+2z-3=0$, $3x+4y+12z+1=0$ and specify the one which bisects the acute angle.

Sol^n :- The given equations

$$x+2y+2z-3=0 \quad \text{---} (1)$$
$$3x+4y+12z+1=0 \quad \text{---} (2)$$

The equations of the two bisecting planes are.

$$\frac{x+2y+2z-3}{\sqrt{1+4+4}} = \pm\frac{(3x+4y+12z+1)}{\sqrt{9+16+144}}$$

$$\Rightarrow \frac{x+2y+2z-3}{3} = \pm\frac{(3x+4y+12z+1)}{13}$$

$$\Rightarrow 13x+26y+26z-39 = (9x+12y+36z+3)$$

and

$$13x+26y+26z-39 = -(9x+12y+36z+3)$$

$$\Rightarrow 4x+14y-10z-42=0$$

and $22x+38y+62z-36=0$

$$\Rightarrow 2x+7y-5z-21=0 \quad \text{---} (3)$$
$$11x+19y+31z-18=0 \quad \text{---} (4)$$

If θ is the angle between the planes (1) & (3)

$$\cos\theta = \frac{2+14-10}{\sqrt{1+4+4}\sqrt{4+49+25}}$$

$$= \frac{6}{3\cdot\sqrt{78}} = \frac{2}{\sqrt{78}}$$

$$\Rightarrow \boxed{\cos\theta = \frac{2}{\sqrt{78}}}$$

$$\tan^2\theta = \sec^2\theta - 1 = \frac{78}{4} - 1 = \frac{74}{4}$$

$$\Rightarrow \tan\theta = \frac{\sqrt{74}}{2} > 1 \Rightarrow \boxed{\theta > 45^\circ}.$$

∴ The plane (3) bisects the obtuse angle. and then (4) bisects the acute angle.

→ Show that the origin lies in the acute angle between the planes $x+2y+2z=9$ & $4x-3y+12z+13=0$ ~~between them~~ and point out which bisects the acute angle.

Sol^n :- The given planes are

$$-x-2y-2z+9=0 \quad \text{---} (1)$$
$$4x-3y+12z+13=0 \quad \text{---} (2)$$

Now $a_1a_2+b_1b_2+c_1c_2$

$= (-1)(4)+(-2)(-3)+(-2)(12)$

$= -4+6-24 = -22 < 0$.

Hence the origin lies in the acute angle between the given planes.

Now the equations of the planes bisecting the angles between the given planes is

$$\frac{-x-2y-2z+9}{\sqrt{1+4+4}} = \pm\frac{(4x-3y+12z+13)}{\sqrt{16+9+144}}$$

$$\Rightarrow 25x + 17y + 62z - 78 = 0 \quad —③$$

$$x + 35y - 10z - 156 = 0 \quad —④$$

Let θ be the angle between the plane ① & ③ is

$$\tan\theta = \left(\frac{\sqrt{1037}}{61}\right) < 1$$

$$\theta < 45^\circ.$$

$\therefore$ ③ bisects the ~~obtuse~~ acute angle between the given planes. and equation ④ bisects the ~~acute~~ obtuse angle between the given planes.

→ find the bisector of the angle between the planes

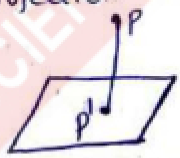
$2x - y - 2z + 3 = 0$, $3x - 2y + 6z + 8 = 0$

→ show that the plane $14x - 8y + 13 = 0$ bisects the obtuse angle between the planes $3x + 4y - 5z + 1 = 0$; $5x + 12y - 13z = 0$.

2000. → find the equations of the planes bisecting the angles between the planes $2x - y - 2z - 3 = 0$ and $3x + 4y + 1 = 0$ and specify the one which bisects the acute angle.

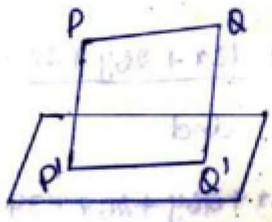
*Orthogonal Projection of a plane:-

→ the projection of a point on a plane is the foot of the ⊥lar drawn from a point to the plane. i.e. P' is the projection of P on the plane.



→ the projection of a segment of a line is the line joining the feet of the ⊥lars from the ends of the segment on the plane.

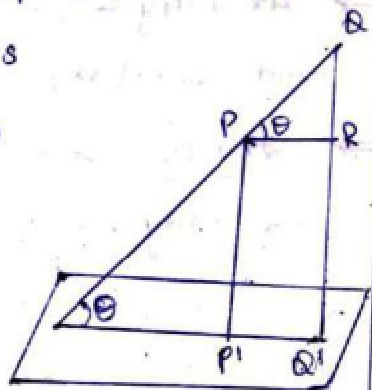
i.e. $P'Q'$ is projection of the line segment PQ on the plane.



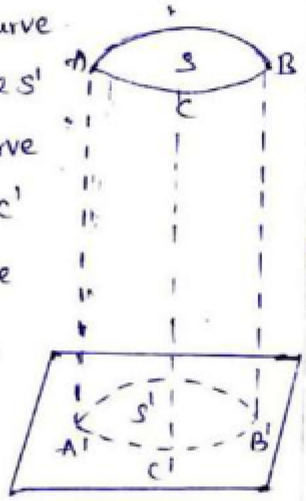
→ the length of projection of a segment PQ is

$$P'Q' = PR = PQ\cos\theta$$

where θ is the angle which PQ makes with the given plane.



→ Let the projection of the area S enclosed by the curve ABC on a plane be S' enclosed by the curve $A'B'C'$ when A', B', C' are the feet of the ⊥lars from A, B, C on the plane of projection.



∴ The area of projection is given by $S' = S\cos\theta$.

when θ is the angle which the plane area S makes with the plane of projection.

* The area of a triangle :-

Let the area of the triangle ABC be represented by Δ.

Let l, m, n be the direction cosines of the normal to the plane of the triangle.

∴ The orthogonal projection of the vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ on YZ – plane are the points $A'(0, y_1, z_1)$; $B'(0, y_2, z_2)$, $C'(0, y_3, z_3)$

∴ the triangle $A'B'C'$ is orthogonal projection of the triangle ABC on the YZ-plane.

If the area of the triangle $A'B'C'$ is denoted by Δx. (i.e. $\Delta x = \Delta\cos\theta$)

$$\therefore \Delta x = l\Delta$$

Similarly $\Delta y = m\Delta$, $\Delta z = n\Delta$

Now squaring and adding these, we get

$$\Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta^2(l^2 + m^2 + n^2)$$

$$\boxed{\Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta^2} \longrightarrow (1)$$

Also from Analytical geometry of two dimensions.

The area of triangle having vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$\therefore \Delta x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}, \quad \Delta y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$\& \ \Delta z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

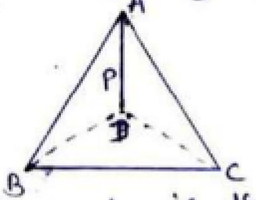
Substituting the values of $\Delta x, \Delta y, \Delta z$ in (1), we obtain the area of the triangle ABC.

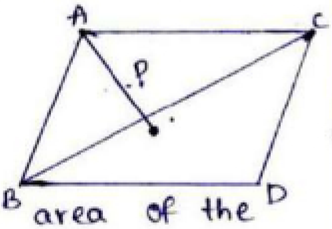
* The Volume of a Tetrahedron :-

→ To find the volume of tetrahedron in terms of the coordinates (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) of its vertices A, B, C, D.

Let V denote volume of the tetrahedron whose vertices are the points $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$, $D(x_4,y_4,z_4)$ then

$$V = \tfrac{1}{3} P \times \Delta \quad \text{—— (I)}$$





where Δ is the area of the triangle BCD and P the $\perp$lar distance from the plane BCD.

Now the equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0 \quad \text{—— (1)}$$

$$\Leftrightarrow x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$-1\begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0 \quad \text{—— (2)}$$

$\therefore$ the length of the $\perp$lar P from the point A is

$$P = \frac{x_1\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}}{\sqrt{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2}} \quad \text{—— (3)}$$

Now numerator of

$$P = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

If Δx, Δy, Δz be the areas of the projections of the triangle on YZ, ZX, XY planes respectively.

we get,

$$2\Delta x = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} = \text{Coefficient of } x \text{ in (2).}$$

$$2\Delta y = \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} = \text{coefficient of } y \text{ in (2).}$$

$$2\Delta z = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \text{Coefficient of } z \text{ in (2)}$$

$\therefore$ The denominator of

$$P = \left(4\left[\Delta x^2 + \Delta y^2 + \Delta z^2\right]\right)^{1/2}$$

$$= 2\Delta$$

$$\therefore \quad \equiv P = \frac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{2\Delta} \quad \text{—— (4)}$$

$\therefore$ Substituting (4) in (I), we get

$$V = \frac{1}{3}\frac{\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}}{2\cancel{\Delta}} \times \cancel{\Delta}$$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

which is the required volume of the tetrahedron.

Problems :-

→ find the area of the triangle whose vertices are the points $(1,2,3)$, $(-2,1,-4)$, $(3,4,-2)$

Sol^n :- Given vertices of the ΔABC are $A(1,2,3)$, $B(-2,1,-4)$, $C(3,4,-2)$

Now the coordinates of the projections of A, B, C on the YZ plane are $(0,2,3)$, $(0,-1,4)$ and $(0,4,-2)$

(Area of the $\Delta ABC = \sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$)

$\therefore \Delta_x$ = area of the projection of ΔABC on YZ-plane

$$= \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 4 & 1 \\ 4 & -2 & 1 \end{vmatrix}$$

Now the coordinates of the projections of A, B, C on the ZX-plane are $A(1,0,3)$, $B(-2,0,4)$, $C(3,0,-2)$

Δ_y = area of the projection of ΔABC on ZX-plane

$$= \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ -2 & 4 & 1 \\ 3 & -2 & 1 \end{vmatrix}$$

Similarly find Δz.

$\therefore$ The required area of the

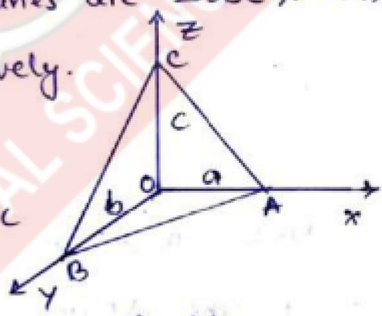
$$\Delta ABC = \sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$$

Proceeding in this way.

→ find the area of the triangle whose vertices are the points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

→ A plane makes intercepts $OA=a$, $OB=b$, $OC=c$ on the axes. find the area of the triangle ABC.

Sol^n :- The projection of ΔABC on the YZ, ZX, XY planes are ΔOBC, ΔOCA, ΔOBA respectively.



If Δ denotes the area of ΔABC then

Δx = Projection of area of ΔABC on YZ-plane

$= \Delta OBC$

$= \frac{1}{2} bc \sin 90^\circ$ / $\frac{1}{2} bc \sin\theta$

$\therefore \boxed{\Delta_x = \frac{1}{2} bc}$

Similarly $\Delta y = \Delta OCA$
$= \frac{1}{2} ca$

and $\Delta z = \Delta OAB$
$= \frac{1}{2} ab$

$$\therefore \Delta = \sqrt{\Delta_x^2+\Delta_y^2+\Delta_z^2}$$

$$= \frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$$

→ A, B, C are $(3,2,1)$, $(-2,0,-3)$, $(0,0,-2)$. find the locus of P if the volume PABC = 5.

Sol'n :- Let $P(x,y,z)$ be any point in the locus.

Given points are $A(3,2,1)$, $B(-2,0,-3)$, $C(0,0,-2)$

∴ the volume of a tetrahedron

$$PABC = \frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ -2 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix} = 5$$

$$\Rightarrow 2x+3y-4z = 38$$

→ The vertices of a tetrahedron are $(0,1,2)$, $(3,0,1)$, $(4,3,6)$, $(2,3,2)$ show that its volume is 6.

→ If the volume of the tetrahedron whose vertices are $(a,1,2)$, $(3,0,1)$, $(4,3,6)$, $(2,3,2)$ is 6. find the value of 'a'.

*

Joint equation of Two planes (or) The equation of pair of planes:-

Sol'n :- Let $ax+by+cz+d=0$ —①

$a_1x+b_1y+c_1z+d_1=0$ —②

be the equations of two planes

Now consider the equation.

$(ax+by+cz+d)(a_1x+b_1y+c_1z+d_1)=0$ —③

Now point $P(x_1,y_1,z_1)$ lies on ①

$\Rightarrow ax_1+by_1+cz_1+d=0$

$\Rightarrow (ax_1+by_1+cz_1+d)(a_1x_1+b_1y_1+c_1z_1+d_1)=0$

A point $P(x_1,y_1,z_1)$ lies on ②

$\Rightarrow a_1x_1+b_1y_1+c_1z_1+d_1=0$

$\Rightarrow (ax_1+by_1+cz_1+d)(a_1x_1+b_1y_1+c_1z_1+d_1)=0$

A point $P(x_1,y_1,z_1)$ lies on ③

$\Rightarrow (ax_1+by_1+cz_1+d)(a_1x_1+b_1y_1+c_1z_1+d_1)=0$

$\Rightarrow ax_1+by_1+cz_1+d=0$ (or) $a_1x_1+b_1y_1+c_1z_1+d_1=0$

∴ The point (x_1,y_1,z_1) lies on the plane ① or ②

∴ A point $P(x_1,y_1,z_1)$ either lies on the plane ① (or) on the plane ②.

$\Leftrightarrow (ax+by+cz+d)(a_1x+b_1y+c_1z+d_1)=0$.

represents the equation of the two planes (or) a pair of planes.

→ To find the condition for the general homogeneous equation of second degree in x, y, z is

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents two planes (a pair of planes) passing through the origin if

$abc+2fgh-af^2-bg^2-ch^2=0$.

Sol'n :- The given equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{—— (1)}$$

Let the equations of two planes represented by (1) be

$$l_1x+m_1y+n_1z=0$$
$$l_2x+m_2y+n_2z=0$$

$\therefore ax^2+by^2+cz^2+2fyz+2gxz+2hxy$

$$\equiv (l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z) \quad \text{—— (2)}$$

Comparing the coefficients of x^2, y^2, z^2, yz, zx and xy we get

$l_1l_2=a, \ m_1m_2=b, \ n_1n_2=c$

$$\left.\begin{aligned} m_1n_2+m_2n_1&=2f\\ n_1l_2+n_2l_1&=2g\\ l_1m_2+l_2m_1&=2h \end{aligned}\right\} \text{—— (3)}$$

The required condition is obtained by eliminating $l_1, m_1, n_1, l_2, m_2, n_2$ from the above six relations given in (3)

Now consider the product of two zero valued determinants.

$$\begin{vmatrix} l_1 & l_2 & 0\\ m_1 & m_2 & 0\\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & m_2 & n_2\\ l_1 & m_1 & n_1\\ 0 & 0 & 0 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} l_1l_2+l_2l_1 & l_1m_2+l_2m_1 & l_1n_2+l_2n_1\\ m_1l_2+m_2l_1 & m_1m_2+m_2m_1 & m_1n_2+n_1m_2\\ n_1l_2+n_2l_1 & n_1m_2+n_2m_1 & n_1n_2+n_2n_1 \end{vmatrix}=0$$

$$\begin{vmatrix} 2a & 2h & 2g\\ 2h & 2b & 2f\\ 2g & 2f & 2c \end{vmatrix}=0 \Rightarrow \begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c \end{vmatrix}=0$$

$\Rightarrow a(bc-f^2)-h(ch-fg)+g(hf-bg)$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$

which is the required condition.

→ Theorem :- If θ is an angle between the pair of planes

$H = ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$

then $\cos\theta = \dfrac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$

Sol'n :- Let $H=0$ be represent the planes.

$$l_1x+m_1y+n_1z=0$$
$$l_2x+m_2y+n_2z=0$$

$\therefore ax^2+by^2+cz^2+2fyz+2gxz+2hxy=$

$$(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z)$$

$\Rightarrow l_1l_2=a, \ m_1m_2=b, \ n_1n_2=c$

$l_1m_2+l_2m_1=2h$

$m_1n_2+m_2n_1=2f, \ n_1l_2+l_1n_2=2g.$

$$\cos\theta = \frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$= \frac{a+b+c}{\sqrt{l_1^2l_2^2+m_1^2m_2^2+n_1^2n_2^2+l_1^2m_2^2+l_1^2n_2^2+m_1^2l_2^2+m_1^2n_2^2+n_1^2l_2^2+n_1^2m_2^2}}$$

$$= \frac{a+b+c}{\sqrt{a^2+b^2+c^2+(l_1m_2+m_1l_2)^2-2l_1l_2m_1m_2+(l_1n_2+n_1l_2)^2-2l_1l_2n_1n_2+(m_1n_2+n_1m_2)^2-2m_1m_2n_1n_2}}$$

$$= \frac{a+b+c}{\sqrt{(a^2+b^2+c^2)+(4h^2-2ab+4g^2-2ac+4f^2-2bc)}}$$

$$= \frac{a+b+c}{\sqrt{a^2+b^2+c^2-2ab-2bc-2ca+4(f^2+g^2+h^2)}}$$

$$\cos\theta = \frac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

* **Theorem** :— Let $H \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a pair of planes.

Then (i) The planes are perpendicular iff $a+b+c=0$

(ii) The planes are coincident iff $f^2=bc$; $g^2=ac$ and $h^2=ab$.

Since $\cos\theta = \dfrac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-ac-bc)}}$

(i) the planes are $\perp$lar

$\therefore \theta = 90^\circ$

$\therefore \boxed{a+b+c=0}$

(ii) the planes are coincident

$\therefore \theta = 0^\circ$

$$\therefore \frac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}=1$$

$$\Rightarrow (a+b+c) = \sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}$$

$$\Rightarrow (a+b+c)^2 = (a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)$$

$$\Rightarrow f^2+g^2+h^2-ab-bc-ca=0$$

$$\Rightarrow (f^2-bc)+(g^2-ac)+(h^2-ab)=0$$

$$\Rightarrow f^2-bc=0;\ g^2-ac=0;\ h^2-ab=0$$

$$\Rightarrow f^2=bc;\ g^2=ac;\ h^2=ab$$

* **Theorem** :— If θ is an angle between the pair of planes $H = ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$

Then $\tan\theta = \dfrac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c}$

Sol'n :- Let θ be the angle between the pair of planes $H=0$

$$\therefore \cos\theta = \frac{a+b+c}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}}$$

Now since $\tan^2\theta = \sec^2\theta - 1$

$$= \frac{1}{\cos^2\theta}-1$$

$$= \frac{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-ca)}{(a+b+c)^2}-1$$

$$\therefore \tan^2\theta = \frac{4(f^2+g^2+h^2-ab-bc-ca)}{(a+b+c)^2}$$

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

Problems :-

→ Prove that the equation

$$2x^2-6y^2-12z^2+18yz+2zx+xy=0$$

represents a pair of planes and find the angle between them.

Sol'n :- Here $a=2,\ b=-6,\ c=-12$

$2f=18;\ 2g=2;\ 2h=2$

$\therefore\ abc+2fgh-af^2-bg^2-ch^2$

$=2(-6)(-12)+2(9)(1)-2(81)+6+12$

$=0$

$\therefore$ the given equation represents a pair of planes.

Now the angle between the planes.

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

$$= \frac{2\sqrt{185}}{2(-16)}$$

$$= \frac{\sqrt{185}}{16}$$

→ show that the following eqn's represent pairs of planes and also find angles between each pair.

(i) $12x^2-2y^2-6z^2-2xy+7yz+6zx=0$

(ii) $2x^2-y^2+4z^2+6xz+2yz+3xy=0$

→ show that the equations

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$$ represents a pair of planes.

Sol'n :- The given equation can be written as

$$a(z-x)(x-y)+b(y-z)(x-y)+c(z-x)(y-z)=0$$

$$\Rightarrow ax^2+by^2+cz^2-(b+c-a)yz-(c+a-b)zx-(a+b-c)xy=0 \quad \text{—(1)}$$

continue in this way.

→ Prove that $\frac{3}{y-z}+\frac{4}{z-x}+\frac{5}{x-y}=0$ represents a pair of planes.

$\boxed{2003}$★ prove that the four planes $my+nz=0,\ nz+lx=0,\ lx+my=0,\ lx+my+nz=p$ form a tetrahedron whose volume is $\frac{2p^3}{3lmn}$.

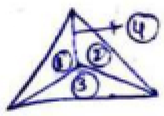
Sol'n :-

The given plane equations are

$my+nz=0$ —(1) $\quad lx+my=0$ —(3)

$nz+lx=0$ —(2) & $lx+my+nz=p$ —(4)

Now solving the above equations, taking three planes at a time, we get the vertices of the tetrahedron.



Now from (1), (2) & (3) $x=y=z=0$

$\therefore$ one vertex of the tetrahedron is $O(0,0,0)$

To solve ①, ② & ④

Now substitute ① & ② in ④ we get,

$lx = p \Rightarrow \boxed{x = \frac{p}{l}}$; $\boxed{y = \frac{p}{m}}$

$① \equiv m\left(\frac{p}{m}\right) + nz = 0$

$\Rightarrow \boxed{z = \frac{-p}{n}}$

$\therefore \left(\frac{p}{l}, \frac{p}{m}, \frac{-p}{n}\right)$ is the second vertex of the tetrahedron.

Similarly $\left(\frac{-p}{l}, \frac{+p}{m}, \frac{p}{n}\right)$ & $\left(\frac{p}{l}, \frac{-p}{m}, \frac{p}{n}\right)$ are the other vertices of the tetrahedron.

$\therefore$ The required volume of the tetrahedron.

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -p/l & p/m & p/n & 1 \\ p/l & -p/m & p/n & 1 \\ p/l & p/m & -p/n & 1 \end{vmatrix}$$

$$= \frac{1}{6}\frac{p}{l}\frac{p}{m}\frac{p}{n}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix}$$

→ find the volume of the tetrahedron formed by planes whose equations are $y+z=0$; $z+x=0$, $x+y=0$ and $x+y+z=1$

→ A, B, C are three fixed points and a variable point 'P' moves so that the volume of the tetrahedron PABC is constant show that the locus of the point 'P' is a plane ||lel to the plane ABC

Sol'n : Let the fixed points be A(a,0,0), B(0,b,0) and C(0,0,c) and P(x,y,z) be the variable point.

Now the equation of the plane ABC (intercept form)

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad —— ①$$

Now the volume of a tetrahedron PABC = Constant

$= \pm \frac{1}{6}k$ (say)

$$\therefore \frac{1}{6}\begin{vmatrix} x & y & z & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = \pm\frac{1}{6}k$$

$R_1 - R_4$; $R_2 - R_4$: $R_3 - R_4$

$$\begin{vmatrix} x & y & z-c & 0 \\ a & 0 & -c & 0 \\ 0 & b & -c & 0 \\ 0 & 0 & c & 1 \end{vmatrix} = \pm k$$

$$\begin{vmatrix} x & y & z-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = \pm k$$

$\Rightarrow x(bc) - y(-ac) + (z-c)(ab) = \pm k$

$\Rightarrow (bc)x + (ac)y + (ab)z - abc = \pm k$

$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \pm k$

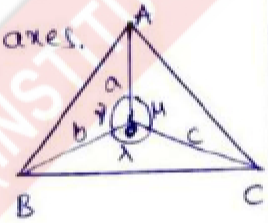
$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \lambda$ (say)

which is a plane ||el to the plane (1)

→ find the volume of a tetrahedron interms of the lengths of the three edges which meet in a point and of the angles which these edges make with each other in pairs. (or)

The lengths of the edges OA, OB, OC of a tetrahedron OABC are a, b, c and angles BOC, COA, AOB are λ, μ, γ; find the volume of the tetrahedron.

Sol'n:- Take O as origin and any three mutually ⊥ lar lines through 'O' as the axes.



Now let OABC be the tetrahedron.

Let OA = a; OB = b; OC = c

Let $\angle BOC = \lambda$ $\angle COA = \mu$ $\angle AOB = \gamma$

Now let the d.c's of the lines OA, OB, OC are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3.

Since OA = a; then the coordinates of A are (l_1a, m_1a, n_1a) $[\because (lr, mr, nr)]$

Similarly the coordinates of B & C are (l_2b, m_2b, n_2b) & (l_3c, m_3c, n_3c).

Also since l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the actual d.c's.

$\therefore l_1^2 + m_1^2 + n_1^2 = 1$; $l_2^2 + m_2^2 + n_2^2 = 1$ and $l_3^2 + m_3^2 + n_3^2 = 1$ —— (1).

Since $\angle BOC = \lambda$; $\angle COA = \mu$ and $\angle AOB = \gamma$.

$$\left.\begin{aligned} \therefore \cos\lambda &= l_2l_3 + m_2m_3 + n_2n_3 \\ \cos\mu &= l_1l_3 + m_1m_3 + n_1n_3 \\ \text{and } \cos\gamma &= l_1l_2 + m_1m_2 + n_1n_2 \end{aligned}\right\} \text{—— (2)}$$

Now the volume of the tetrahedron OABC is

$$\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1a & m_1a & n_1a & 1 \\ l_2b & m_2b & n_2b & 1 \\ l_3c & m_3b & n_3b & 1 \end{vmatrix} = -\frac{1}{6}\begin{vmatrix} l_1a & m_1a & n_1a \\ l_2b & m_2b & n_2b \\ l_3c & m_3c & n_3c \end{vmatrix}$$

$$= -\frac{1}{6}abc\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now we consider

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

$$= \begin{vmatrix} l_1^2+m_1^2+n_1^2 & l_1l_2+m_1m_2+n_1n_2 & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1 & l_2^2+m_2^2+n_2^2 & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1 & l_3l_2+m_3m_2+n_3n_2 & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

From ③ we get the volume of the tetrahedron OABC

$$= \pm \frac{1}{6}abc \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

$$= \frac{1}{6}abc \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

(numerically)

→ Show that the volume of the tetrahedron, the equations of whose faces are $a_r x + b_r y + c_r z + d_r = 0$ $(r = 1,2,3,4)$ is $\frac{\Delta^3}{6 D_1 D_2 D_3 D_4}$,

where Δ is the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$ and

D_1, D_2, D_3, D_4 are the co-factors of d_1, d_2, d_3, d_4 respectively in the determinant Δ.

Sol^n:- The equations of the four faces are

$a_1x + b_1y + c_1z + d_1 = 0$ —— ①

$a_2x + b_2y + c_2z + d_2 = 0$ —— ②

$a_3x + b_3y + c_3z + d_3 = 0$ —— ③

$a_4x + b_4y + c_4z + d_4 = 0$ —— ④.

Solving ②, ③ & ④ equations by determinants, we have.

$$\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}$$

$$= \frac{-1}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}$$

(or) $\frac{x}{A_1} = \frac{-y}{-B_1} = \frac{z}{C_1} = \frac{-1}{-D_1}$

where A_1, B_1, C_1, D_1 are the cofactors of a_1, b_1, c_1, d_1 respectively.

Now: $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

$\therefore x = \frac{A_1}{D_1}$; $y = \frac{B_1}{D_1}$, $z = \frac{C_1}{D_1}$

$\therefore (x, y, z) = \left(\frac{A_1}{D_1}, \frac{B_1}{D_1}, \frac{C_1}{D_1}\right)$ is one of the vertex of the tetrahedron.

Similarly the other vertices are

$\left(\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right)$, $\left(\frac{A_3}{D_3}, \frac{B_3}{D_3}, \frac{C_3}{D_3}\right)$

and $\left(\frac{A_4}{D_4}, \frac{B_4}{D_4}, \frac{C_4}{D_4}\right)$

Now the volume of the tetrahedron.

$$\frac{1}{6}\begin{vmatrix} A_1/D_1 & B_1/D_1 & C_1/D_1 & 1 \\ A_2/D_2 & B_2/D_2 & C_2/D_2 & 1 \\ A_3/D_3 & B_3/D_3 & C_3/D_3 & 1 \\ A_4/D_4 & B_4/D_4 & C_4/D_4 & 1 \end{vmatrix}$$

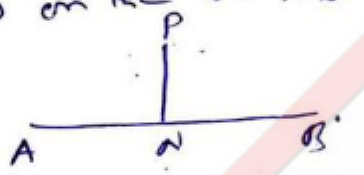
$$= \frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$$= \frac{\Delta^3}{6D_1D_2D_3D_4}$$

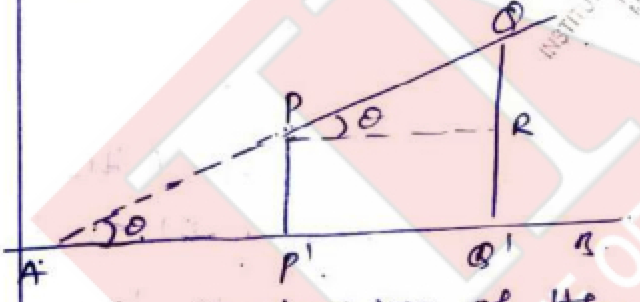
[$\because$ if Δ' is a determinant formed by the cofactors of elements in Δ is equal to $(-1)^{i+j}$ times the determinant of $n-1$ obtained from Δ i.e. $\Delta' = \Delta^{n-1}$].

* Projection of a point on a line:

The projection of a point P on a line AB is the foot N of the perpendicular PN from P on the line AB.



* The projection of a segment of a line on another line:



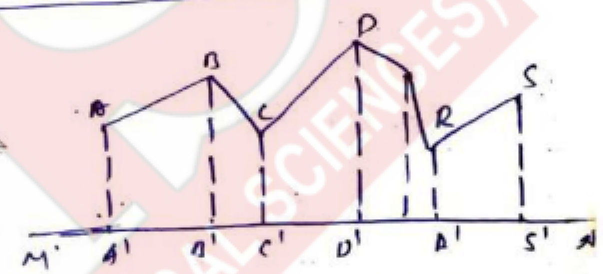
The projection of the line segment PQ of a line on another line AB is the line segment P'Q' of AB, where P' and Q' are the projections of the points P and Q on the line AB.

Length of projection $P'Q' = PR = PQ\cos\theta$

where θ is the angle which the line PQ makes with PR i.e with AB.

Note:– If $PQ \perp AB$, then the projection of PQ on AB = 0 as $\theta = 90°$.

* Projection of a broken line on a given line:



Let A, B, C, D ...... S be any number of points in the space.

Let A', B', C', D' ... S' be their projections on any line MN.

Then as A', B', C', D' ... R', S' lie on the same straight line MN.

We have

$$A'B' + B'C' + C'D' + \cdots + R'S' = A'S' \quad \text{—(1)}$$

But A'B' is the projection of AB on the line MN.

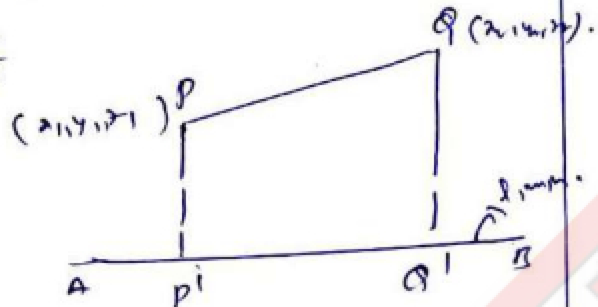
Hence from (1),

we conclude that the sum of projections of AB, BC, CD, ... RS on the line MN = projection A'S' of AS on the line MN.

→ To prove that the projection of the join of two points (x_1, y_1, z_1), (x_2, y_2, z_2) on a line whose direction cosines are l, m, n is

$l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$.

<u>proof</u>



Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given points and AB the line whose d.c's are l, m, n.

Draw PP'; QQ' $\perp$s on AB.

Then $P'Q'$ is the projection of PQ on AB, so that

$P'Q' = PQ \cos\theta$. —— (1)

where θ is the angle which PQ makes with AB.

Now d.r's of PQ are

$x_2-x_1, y_2-y_1, z_2-z_1$.

Dividing each by

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = PQ$.

$\therefore$ The actual d.c's of PQ are

$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$.

Also the d.c's of AB are l, m, n

$\therefore \cos\theta = l\left(\frac{x_2-x_1}{PQ}\right)+m\left(\frac{y_2-y_1}{PQ}\right)+n\left(\frac{z_2-z_1}{PQ}\right)$

$(\because \cos\theta = l_1l_2+m_1m_2+n_1n_2)$

$= \frac{1}{PQ}\left[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\right]$ —— (2)

putting this value of $\cos\theta$ in (1), we have

$\boxed{P'Q' = l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)}$.

→ The projections of a line on the axes are 3, 4, 12. Find the length and d.c's of the line.

<u>Sol</u> Let the ends of the line be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

$3 =$ projection of the line PQ on the x-axis.

$= l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

$= 1(x_2-x_1)+0(y_2-y_1)+0(z_2-z_1)$

$= x_2-x_1$ $(\because$ d.c's of x-axis are $1, 0, 0)$

$\therefore \boxed{x_2-x_1 = 3}$

$4 =$ projection of line PQ on y-axis.

$= ~~y_2~~$

$0(x_2-x_1)+1(y_2-y_1)+0(z_2-z_1)$
($\because$ d.c's of y-axis are 0,1,0)

$\boxed{4 = y_2-y_1}$

also 12 = projection of PQ on z-axis

$= 0(x_2-x_1)+0(y_2-y_1)+1\cdot(z_2-z_1)$

($\because$ the d.c's of the line PQ on z-axis are 0,0,1).

$\boxed{12 = z_2-z_1}$

Now length $PQ = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{9+16+144}$

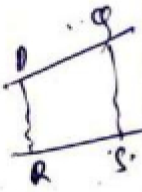
$= 13.$

Now d.r's of PQ are $x_2-x_1, y_2-y_1, z_2-z_1$

i.e $3, 4, 12$.

$\therefore$ Actual d.c's of PQ are $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.

→ P, Q, R, S are the points $(-2,3,4)$, $(-4,4,6)$, $(4,3,5)$ and $(0,1,2)$. Prove by projections that PQ is at right angles to RS.



Sol The four points are $P(-2,3,4)$, $Q(-4,4,6)$, $R(4,3,5)$, $S(0,1,2)$

$\therefore$ d.r's of RS are $0-4, 1-3, 2-5$

$\Rightarrow -4, -2, -3$ or $4, 2, 3$

(21)

(using $x_2-x_1, y_2-y_1, z_2-z_1$).

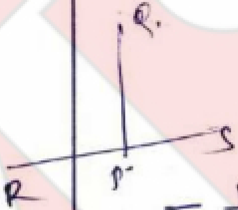
Dividing by $\sqrt{16+4+9} = \sqrt{29}$.

$\therefore$ Actual d.c's of RS are $\frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}$.

$\therefore$ projection of PQ on RS

$= \frac{4}{\sqrt{29}}(-4+2)+\frac{2}{\sqrt{29}}(4-3)+\frac{3}{\sqrt{29}}(6-4).$

(using $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$).



$= \frac{1}{\sqrt{29}}(-8+2+6)$

$= 0$

$\therefore PQ \perp RS$.

PT 2007 → If P, Q, A, B are $(1,2,5)$, $(-2,1,3)$, $(4,4,2)$, $(2,1,-4)$ respectively, then what is the projection of PQ on AB?

(a) 3 (b) $7/2$ (c) 4 (d) $9/2$

PT 2007 → If $\langle l_1, m_1, n_1\rangle$, $\langle l_2, m_2, n_2\rangle$, $\langle l_3, m_3, n_3\rangle$ are direction cosines of three mutually perpendicular lines then what is/are the value(s) of $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$?

(a) 0 (b) ± 1 (c) ± 2 (d) 3

Solⁿ: Given that l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 be the direction cosines of any three mutually $\perp^{lar}$ lines.

Hence we have following two sets of relations.

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1 \\ l_2^2+m_2^2+n_2^2=1 \\ l_3^2+m_3^2+n_3^2=1 \end{array}\right\} \text{—(i)}$$

$$\left.\begin{array}{l} l_1 l_2+m_1 m_2+n_1 n_2=0 \\ l_2 l_3+m_2 m_3+n_2 n_3=0 \\ l_3 l_1+m_3 m_1+n_3 n_1=0 \end{array}\right\} \text{—(ii)}$$

Let $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

Then

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_2 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{(from (i) \& (ii))}$$

$$\Rightarrow \Delta^2 = 1$$

$$\Rightarrow \Delta = \pm 1$$

i.e., $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1$