Vector Analysis

Paper I

(1) $\frac{d\bar{v}}{dt} = \frac{d\bar{v}}{ds}\frac{ds}{dt}$, $\left|\frac{d\bar{v}}{ds}\right| = 1$, $\left|\frac{d\bar{v}}{dt}\right| = \frac{ds}{dt}$

(2) $\int_C M\,dx + N\,dy = \int d(F) = F_2 - F_1$ if exact, $M_y = N_x$
$F = \nabla\phi$ or $\nabla \times F = 0$ thus.

(3) $\iint_S \bar{F}\cdot\hat{n}\, ds = \iint_R \bar{F}\cdot\hat{n}\,\frac{dx\,dy}{n\cdot k}$ R is orthogonal projection on xy plane

(4) Cylindrical coordinate $dA = r\,dr\,d\theta\,dz$

(5) Green's theorem in plane $\iint_R (N_x - M_y)\,dx\,dy = \oint_C M\,dx + N\,dy$

(6) Area bounded by simple closed curve
$= \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta$

(7) Divergence theorem $\iiint_V \nabla\cdot F\, dV = \iint_S \bar{F}\cdot\hat{n}\, ds$

(8) Stoke's theorem $\oint_C \bar{F}\cdot d\bar{r} = \iint_S (\nabla\times\bar{F})\cdot\hat{n}\, ds$
C → +ive direction

(9) $\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$

(10) $\nabla\times(A\times B) = (B\cdot\nabla)A - B\,\mathrm{div}\,A - (A\cdot\nabla)B + A\,\mathrm{div}\,B$

(11) $\nabla(A\cdot B) = (B\cdot\nabla)A + (A\cdot\nabla)B + B\times(\nabla\times A) + A\times(\nabla\times B)$

(12) $\nabla\cdot(\phi\bar{A}) = (\nabla\phi)\cdot\bar{A} + \phi(\nabla\cdot\bar{A})$

(13) $\nabla\times(\phi A) = (\nabla\phi)\times A + \phi(\nabla\times A)$

(14) $\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$

(15) Spherical Coordinates (r,θ,ϕ)
~~$x = r\sin\theta\sin\phi$~~
~~$y = r\sin\theta\cos\phi$~~
$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$
$\theta: 0 \to \pi$
$\phi: 0 \to 2\pi$
~~$r: 0 \to a$~~
above xy plane $\theta: 0 \to \pi/2$

(16) Walli's formula
$\int_0^{\pi/2} (\sin\theta)^m (\cos\theta)^n\, d\theta = \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots 3\cdot 1}{(m+n)(m+n-2)\cdots 4, 2}\cdot K$
$K = \pi/2$ for m, n both even
$K = 1$ otherwise

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(17) Volume of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4\pi}{3}abc$

(18) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

(19) $\oint_c F\cdot dr$, Line integral is independent of path

$F=\nabla\phi$, $\nabla\times F=\bar{0}$, conservative force

or $\oint_c F\cdot dr=0$ for every closed path

(20) $\nabla(fg)=f\nabla g+g\nabla f$ ($f, g \to$ scalar)

(21) Tangent plane to the surface at (x_1,y_1) $xy \to \frac{xy_1+x_1y}{2}$

(22) if $f = f_1\hat{i}+f_2\hat{j}+f_3\hat{k}$, $\nabla\cdot f = \nabla f_1\cdot\hat{i}+\nabla f_2\cdot\hat{j}+\nabla f_3\cdot\hat{k}$

(23) Directional derivative $\frac{df}{ds}=\nabla f\cdot\hat{a}$, max value $=|\nabla f|$

(24) $\nabla f = \frac{df}{dn}\hat{n}$

(25) Solenoidal $\nabla\cdot F=0$, Irrotational $\nabla\times F=\bar{0}$.

(26) $\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$

$\nabla\cdot(f\nabla g) = f\nabla^2 g+\nabla f\cdot\nabla g$

(27) Tangent plane at surface ϕ, $(\bar{R}-\bar{r})\cdot\nabla\phi=0$

(28) curvature $K=\frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3}$, torsion $\tau=\frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$

(29)

(30) Frenet–Serret formula

$\frac{dT}{ds}=K\bar{N}$, $\frac{dN}{ds}=\tau\bar{B}-K\bar{T}$, $\frac{dB}{ds}=-\tau\bar{N}$, $\bar{B}=\bar{T}\times\bar{N}$

(31) $\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n x\,dx=\frac{(n-1)(n-3)\cdots 3\cdot 1}{n(n-2)\cdots 4\cdot 2}\boxed{\cdot\pi/2}$

(32) Dirichlet's theorem $x+y+z\le 1$, $x,y,z\ge 0$

$\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz=\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$

(33) tangent plane $(\bar{R}-\bar{r})\cdot\nabla f=0$ to a level surface f

normal $(\bar{R}-\bar{r})\times\nabla f=0$

(34) $\frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{dt}$, $\hat{t} = \frac{d\bar{r}}{ds}$ $(|\hat{t}| = 1)$

(35) For problem like $\oint_c f\, d\bar{r} = \iint_s d\bar{s} \times \nabla f$

constant vector $\bar{C}$ use divergence and stokes theorem with certain manipulation.

$\boxed{\bar{A} = \bar{B} \times \bar{C}}$ const

(36) Curvature vector dT/ds

(37) Frenet-Serret formula

$\frac{d\bar{T}}{ds} = \kappa \bar{N}$, $\frac{d\bar{N}}{ds} = \tau\bar{B} - \kappa\bar{T}$ $\frac{d\bar{B}}{ds} = -\tau\bar{N}$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = r(t)$, dr/dt, $ds/dt = |dr/dt|$

~~$dT/ds = dT/d$~~ $T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt}$.

$\frac{dT}{ds} = \frac{dT/dt}{ds/dt}$. $\frac{dT}{ds} = \kappa N \Rightarrow \kappa$ ||

$N = \frac{1}{\kappa} \frac{dT}{ds}$

$B = T \times N \Rightarrow B$ $\frac{dB}{ds} = \frac{dB/dt}{ds/dt} \Rightarrow \tau$

(38) curve $r(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ lies in a plane

if $\tau = 0$ or $\bar{B} = $ const

(39)

Ordinary Differential Equations (ODE)

1. $d(\log xy) = \dfrac{x\,dy + y\,dx}{xy}$

2. $d\left(\tan^{-1}\frac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

3. $d(\log y/x) = \dfrac{x\,dy - y\,dx}{xy}$, $d\left(\log \frac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{xy}$

4. $d\left(\frac{1}{2}\log(x^2+y^2)\right) = \dfrac{x\,dx + y\,dy}{x^2+y^2}$

5. $d\left(-\frac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2y^2}$

6. Subtangent $= y x'$

Subnormal $= y y'$

$PG = y\sqrt{1+y'^2}$

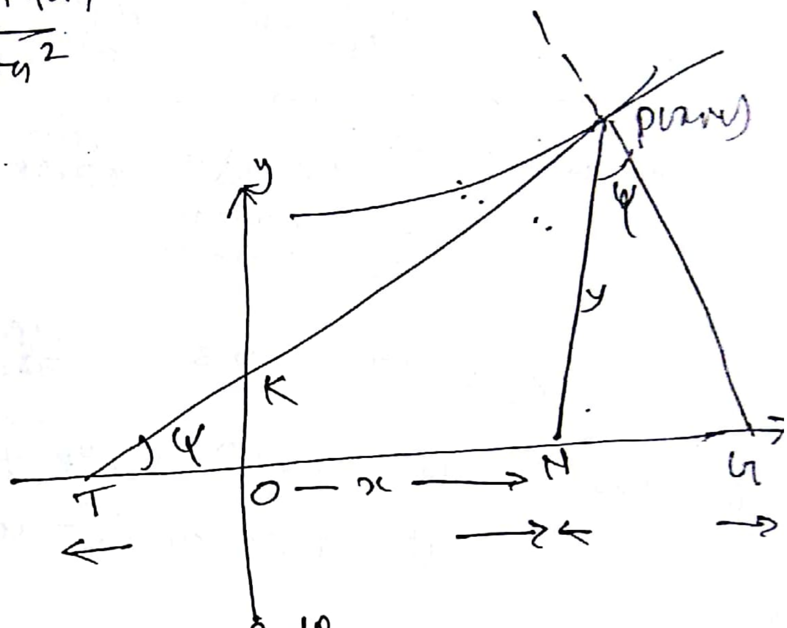
$PT = y\sqrt{1+x'^2}$

$OT = x - y x'$

$OK = y - x y'$

$OG = x + y y'$

$OL = y + x x'$



Polar subtangent $= r^2 \dfrac{d\theta}{dr}$

7. $\int \dfrac{dx}{\sqrt{a^2+x^2}} = \log\left(x + \sqrt{x^2+a^2}\right)$

Radius of curvature $= ds/d\psi$ $\quad x^2+y^2 = r^2$

8. $x = r\cos\theta$, $y = r\sin\theta$ $\quad \theta = \tan^{-1}(y/x)$

9. $x = X+h$, $y = Y+K$, $\quad x^{\alpha} y^{\beta}(my\,dx + nx\,dy) + x^{\alpha'} y^{\beta'}(m'y\,dx + n'x\,dy) = 0$

10. $Mdx + Ndy = 0$ $\quad$ I.F. $= x^{km-1-\alpha}\, y^{kn-1-\alpha}$ $\quad km-1-\alpha = k'm'-1-\alpha'$

$kn-1-\beta = k'n'-1-\beta'$

I.F. $= x^h y^k \rightarrow$ exact cond^n

Substitution for Clairaut's form

11. 1) $x^2 = u$, $y^2 = v \Rightarrow p = py/x$, $\quad y^2 = \left(\frac{py}{x}\right)x^2 + f(py/x)$

ii) $e^{ax} = u$, $e^{by} = v$ $\quad e^{by}(a - bp) = f\left(b\, e^{by - ax}\right)$

iii) $y^2 = v$ $\quad y = 2px + y^2p^3$

$y^2(y - px) = x^4p^2$

iv) $x = \frac{1}{u}$, $y = \frac{1}{v}$

v) $x + y = u$, $xy = v$, $p = \dfrac{px+y}{p+1}$, $\quad (px^2 + y^2)(px + y) = (p+1)^2$

vi) $x+y=u,\ x^2+y^2=v$ $p=\frac{2(x+yp)}{1+p}$, $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)-1(x+yp)^2=0$

vii) $x^2=u,\ y-x=v$ $xp^2-2yp+x+2y=0$

viii) $y=u,\ xy=v$ $\rightarrow$ $xp^2+yp(2x+y)+y^2=0$

ix) $x=u^2$ $y=2px+f(xp^2)$

x) $x+y=u,\ xy-1=v,$ $(2x^2+1)p^2+(x^2+y^2+4xy+1)p+xy^2+1=0$

xi) $x=1/u$ $\rightarrow$ $y=-px+x^4p^2$

(12) $y''+Py'+Qy=R$, $u=$ particular PI of $y''+Py'+Qy=0$

$v''+P_1\frac{dv}{dx}=R_1$, $P_1=P+\frac{2}{u}\frac{du}{dx}$, $R_1=R/u$

$\frac{dv}{dx}=p$ I.F. $=u^2 e^{\int P dx}$ general soln

$\frac{dp}{dx}+P_1p=R_1 \rightarrow p \rightarrow \frac{dv}{dx} \rightarrow v \rightarrow \boxed{uv}$

$\boxed{y=uv}$

Finding P.I. e^{mx} $m^2+mP+Q=0$

x^m $m(m-1)+Pmx+Qx^2=0$

(13) Set $P_1=P+\frac{2}{u}\frac{du}{dx}=0 \Rightarrow u=e^{-\frac{1}{2}\int P dx}$

Removal of first derivative $Q_1=Q-\frac{P^2}{4}-\frac{1}{2}\frac{dP}{dx}$, $R_1=R/u$

$y''+Py'+Qy=R \rightarrow v''+Q_1v=R_1$

$\boxed{\text{PI/CN}}$

(14) Extraneous loci (singular solution) $\boxed{\text{tp/cn}}$

	singular soln	tac locus	node locus	cusp locus
• p-dis	once	squared	Absent	once
c-dis	once	absent	squared	cube
whether not satisfy diff eqn	yes, always	not, nogen.	Not, nogen	not, inger

(15) $\frac{1}{f(D)}(xV)=x\frac{1}{f(D)}V+\frac{d}{dD}\left(\frac{1}{f(D)}\right)V$

(16) $\frac{1}{f(D)}(e^{ax}V)=e^{ax}\frac{1}{f(D+a)}(V)$

(17). $\frac{1}{f(D)}(e^{ax}) = \frac{1}{f(a)} e^{ax}$, if $f(a) \neq 0$, $\frac{1}{f(D)} x^{m} = \frac{1}{f(m)} x^{m}$ $f(m) \neq 0$

(18) $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$, $f(a) = 0$, $\frac{1}{(D_1 - a)^n} x^m = \frac{(\log x)^n}{n!} x^m$ $f(m) = 0$

(19) method of undetermined coefficient

(20) variation of parameters $y' + py = Q \Rightarrow y = u\{c + \int Q/u \, dx\}$

(21) $y'' + py' + Qy = R$, $y = u\{c_1 + f(x)\} + v(c_2 + g(x))$

$f(x) = \int -\frac{vR}{W} dx$, $g(x) = \int \frac{uR}{W} dx$

(22) exact differential eqn

$P_0 y^{(n)} + P_1 y^{(n-1)} + \dots \quad P_n y = \phi(x)$

If $P_n - P_{n-1}' + P_{n-2}'' \quad - - = 0 \quad$ I.F. $= x^m$

$Q_{n-1} = P_{n-1} - P_{n-2}' + P_{n-3}''$

(23) $\frac{dv}{dx} = q$, $\frac{dq}{dx} + (p + \frac{2}{u}\frac{du}{dx}) q = R/u$ $\quad v = \int \frac{e^{-\int q/p \, dx}}{u^2} dx$

(24) general soln $y = c_1 u + c_2 uv$,

for $py'' + qy' + ry = 0$

(25) Bernoulli's eqn $y' + py = Qy^n$

(26) oblique trajectory $y' \to \frac{y' + \tan\alpha}{1 - y'\tan\alpha}$

(27) solvable for x, $x = F(y, p)$, $\frac{1}{p} = \phi(y, p, dp/dy) \to \psi(y, p, c) = 0$

eliminate p

(28) Lagrange eqn $y = xF(p) + f(p) \Rightarrow \frac{dx}{dp} - \frac{xF'(p)}{p - F(p)} = \frac{f'(p)}{p - F(p)}$

(29) clairaut form $y = px + f(p)$

(30) $\frac{1}{D - \alpha} X = e^{\alpha x} \int X e^{-\alpha x} dx$, $\frac{1}{D_1 - \alpha} X = x^{\alpha} \int x^{-\alpha - 1} X dx$

e^{ax}

(31) 2nd order $a^2 + ap + Q = 0$

$m(m-1) + pmx + Qx^2 = 0$ $\quad x^m$

(32) $py'' + qy' + ry = 0$, u is one soln, $v = \int \frac{e^{-\int q/p \, dx}}{u^2} dx$

general soln $= c_1 u + c_2 uv$, 2nd soln $= uv$

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(33) $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, $s>0$, $t\geq 0$

(34) (i) $u_c(t) \rightarrow e^{-cs}/s \quad s>0$

(ii) $u_c(t) f(t-c) \rightarrow e^{-cs} F(s)$

(iii) $f(ct) \rightarrow \frac{1}{c} F(s/c)$ | $\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f(t/a)$

(iv) $\int_0^t f(t-\tau) g(\tau)\,d\tau \Rightarrow F(s)G(s)$ convolution

(v) $\delta(t-c) \rightarrow e^{-cs}$

(vi) $f^{(n)}(t) \rightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots f^{(n-1)}(0)$

(vii) $(-t)^n f(t) \rightarrow F^{(n)}(s)$

(viii) $e^{at} f(t) \rightarrow F(s-a)$

(35) $\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s)$ Laplace transform of integral

(36) unit step function, t-shifting $u(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$

$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$ $\quad f(t-a)\,u(t-a) = \begin{cases} 0 & t<0 \\ f(t-a) & t>0 \end{cases}$

(37) second shifting theorem

$\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as} F(s)$.

(38) short impulses, dirac delta function

$f_k(t-a) = \begin{cases} 1/k & a \leq t \leq a+k \\ 0 & \text{else} \end{cases}$

$\mathcal{L}\{f_k(t-a)\} = e^{-as}$

(39) Laplace transform of differentiation

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$ $\quad \mathcal{L}^{-1}[F(s)]$

OR $\quad \mathcal{L}^{-1}[F^{(n)}(s)] = (-1)^n t^n$

(40) Laplace transform of integration

$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

(41) Inverse Laplace transform

(i) $\ln \frac{s-a}{s-b} \rightarrow \frac{1}{t}(e^{bt} - e^{at})$

(ii) $\ln \frac{s^2+w^2}{s^2} \rightarrow \frac{2}{t}(1-\cos wt)$

(iii) $\ln \left(\frac{s^2-a^2}{s^2}\right) \rightarrow \frac{2}{t}(1-\cosh at)$

(iv) $\frac{1}{s}\ln s \rightarrow -\ln t - \gamma \quad (\gamma = .5772)$

(42) $\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$

Bernoulli's formula

(43) $\int \frac{dx}{x\sqrt{x^2+a^2}} = \left(\frac{-1}{a}\right)\ln\left(\frac{a+\sqrt{x^2+a^2}}{x}\right) = \frac{1}{a}\log\left(\frac{x}{a+\sqrt{x^2+a^2}}\right)$

(44) Method of variation of parameter $\quad y = Au + Bv$

$A_1u + B_1v = 0$

$A_1u_1 + B_1v_1 = R \quad \Rightarrow A_1 \Rightarrow A$

$B_1 \Rightarrow B$

(45) Radius of curvature

$R = \frac{(1+y'^2)^{3/2}}{|y''|}$

polar (r,θ), $\quad R = \frac{\left[r^2 + (dr/d\theta)^2\right]^{3/2}}{\left| r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$ (?)

(46) $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

(47) method of variation of parameter for third order

$y''' + Py'' + Qy' + Ry = S$, $\quad u, v, w$ Homogeneous

$CF = c_1u + c_2v + c_3w$, ~~$PI = u f(x) + v g(x) +$~~

$PI = u f(x) + v g(x) + w h(x)$

$W = \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$

$\frac{df}{dx} = \frac{S}{W}\begin{vmatrix} v & w \\ v' & w' \end{vmatrix}$, $\quad \frac{dg}{dx} = \frac{-S}{W}\begin{vmatrix} u & w \\ u' & w' \end{vmatrix}$

$\frac{dh}{dx} = \frac{S}{W}\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} \quad \Rightarrow f(x), g(x), h(x)$

(48) substitution $xy = u$ e-g $x^3 y'' + xy = \sqrt{1 - x^2 y^2}$

(49) $\frac{1}{D^2 + a^2} \sin ax = \frac{x}{2} \int \sin ax \, dx$

(50) $\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2} \int \cos ax \, dx$

(51) $L^{-1}[F(s)] = \cancel{\frac{1}{2\pi i} \int_c F(s) e^{st} ds} = \frac{1}{2\pi i} \int_c F(s) e^{st} ds$

(52) $L^{-1}[F(s)] = \frac{1}{2\pi i} \int_c F(s) e^{st} ds$

(53) Bernoulli's eqn, dx/dy or dy/dx selectively

(54) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(x/a)$

(55) ~~eta~~ $P_0 p^n + P_1 p^{n-1} + \dots P_n = 0$ parametric form $x =$ $y =$

$F_1(x, y, c) \, F_2(x, y, c) \dots = 0$

(i) Solvable for p,

(ii) Solvable for x, $\underline{x = F(y, p)}$, $\frac{1}{p} = \phi(y, p, \frac{dp}{dy}) \rightarrow \underline{\psi(y, p, c) = 0}$

(iii) Solvable for y $\underline{y = F(x, p)}$, $p = \phi(x, p, \frac{dp}{dx}) \rightarrow \underline{\phi(x, p, c) = 0}$

(iv) Lagrange's eqn $y = x F(p) + f(p)$ $\frac{dx}{dp} - x =$

(v) Clairaut's form

(vi) Reducible to Clairaut's form

(56) $W(\phi_1, \phi_2)(x) = W(\phi_1, \phi_2)(x_0) \cdot e^{-\int_{x_0}^{x} p_1(x) dx}$

of $y'' + p_1(x) y' + p_2(x) y = 0$

(57)

Linear Algebra

① $\dim(U) + \dim(V) = \dim(U \cap V) + \dim(U+V)$

② Vectorspace $u, v, w \in V$, $k \in K$ (scalar)

$u+v \in V$, $ku \in V$ (closure)

A1 $u+(v+w) = (u+v)+w$

A2 $u+0 = 0+u = u$

A3 $u+(-u) = (-u)+u = 0$

A4 $u+v = v+u$

M1 $k(u+v) = ku+kv$

M2 $(a+b)u = au+bu$

M3 $(ab)u = a(bu)$

M4 $1u = u$

③ Linear transformation $T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$

④ $\ker F = \{u \in V : F(u) = 0\}$

⑤ $\operatorname{Im} F = \{u \in U : \exists\, u \in V$ for $F(u) = u\}$

⑥ Diagonalisation $D = P^{-1}AP$, $A = PDP^{-1}$

⑦ Unitary $A^{\theta}A = I$, nilpotent $A^{k} = 0$, idempotent $A^2 = A$

⑧ eigenvalues: Hermitian → real

Skew Hermitian → zero / purely imaginary

⑨ If A is diagonalisable → 3 L.I. eigenvectors, characteristic polynomial

$(x-1)(x-2)$ e.g.

non diagonalisable $(x-1)(x-2)^2$

⑩ $T^{3\times3}$ is diagonalisable ⇒ 3 L.I. eigenvectors.

⑪ Subspace: Let V be a vector space and let $W \subset V$ be a subset. Then W is a subspace iff it is closed under addition and scalar multiplication and $0 \in W$ (0 vector)

⑫ If W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is also subspace of V. However $W_1 \cup W_2$ will be subspace of V iff $W_1 \subset W_2$ or $W_2 \subset W_2$

⑬ ~~If~~

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(14) Similar matrices $B = P^{-1}AP$ OR $A = PBP^{-1}$, same eigenvalues

(15) Diagonalizable matrix, $D = P^{-1}AP$, n = L.I. eigenvectors

(16) Eigenvectors of distinct eigenvalue of a real symmetric matrix are orthogonal. Real symmetric $P^TAP = P^{-1}AP = \text{dia}(d_1, d_2, \dots d_n)$ orthogonal P

(17) Hermitian $U^TAU = U^{-1}AU = \text{dia}(d_1, d_2, \dots d_n)$ unitarily similar

(18) $\det(A) = |A|$ = product of eigenvalues $= d_1 d_2 d_3 \dots d_n$

(19) Quadratic form $Q = x^TAx$ — $D = P^TAP$

$x = Py$ $\begin{pmatrix} \square & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & \square \end{pmatrix} = (\ )(\ )(\ )$

Diag form $= d_1y_1^2 + d_2y_2^2 + d_3y_3^2$

rank = no. of +ve terms

sign = no. of +ve terms − no. of negative terms

(20) For a normal matrix C, there is a unitary matrix U such that U^*AU is diagonal.

A is normal if $A^*A = AA^*$

(21) System of homogeneous linear eqns for which the space of solns is exactly the subspace of R^4 spanned by $\alpha_1, \alpha_2, \alpha_3$ $T[\ ] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ - & - & 1 & 0 \\ - & - & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ $\begin{pmatrix} \alpha_1' \end{pmatrix}$ $\begin{pmatrix} \alpha_2' \end{pmatrix}$

r.s.t $\alpha_1' \alpha_2'$ Kernel

(22) Same row/column space → exactly same vectors in basis

(i) use row transformation to get same rows.

(23) Normal form $[I_r | 0]$, $\left[\frac{I_r}{0}\right]$, $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$ etc.

(24) $F[x]$ = vector space of all polynomials over field F.

infinite dimensional $S = \{1, x, x^2, \dots x^n, \dots\}$

(25)

Analytic Geometry

① Skew lines: not parallel and not intersecting

② Coplanar $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$ plan $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

eqn of plane through given line and ∥ to another

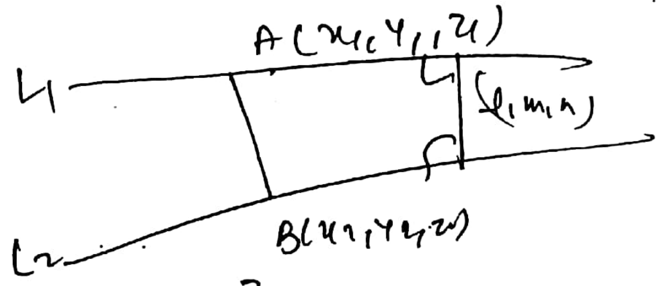
③ Three mutually ⊥ tangent lines from any point $SS_1 = T^2$

④ Coordinate of intersection

$$(x_1, y_1, z_1) = \left\{ \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \pm c\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \right.$$

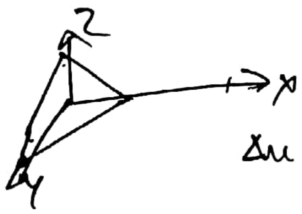
⑤ Shortest distance
projection method

$S.D. = \Sigma l(x_2-x_1)$



⑥ $\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$ projection

$\cdot \Delta_x =$ projected area on xy plane



$\Delta_x = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

⑦ Three planes will intersect in a

(i) point $\Delta \neq 0$

(ii) line $\Delta_4 = 0$ and all $\Delta_1 = \Delta_2 = \Delta_3 = 0$

(iii) Triangular prism $\Delta_4 = 0$ and any of Δ_1 or Δ_2 or $\Delta_3 \neq 0$

Sphere

⑧ Plane of contact

$\alpha x + \beta y + \gamma z + u(x+\alpha) + v(y+\beta) + w(z+\gamma) + d = 0$

Locus of points of contact of the tangent planes which passes through a given point (α, β, γ).

⑨ Pole and Polar plane $\frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ}$ (write tangent plane at A)

Ⓐ locus of R

⑩ Polar line $AB \to CD \Rightarrow lx + my + nz = 0 = \alpha x + \beta y + \gamma z - a^2$ for $x^2+y^2+z^2$

(11) Angle of intersection of two spheres $\cos\theta = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}$

(12) Length of the tangent $= x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

(13) Radical plane $S_1-S_2=0$

(14) Coaxial system of spheres $S_1+\mu(S_1-S_2)=0$

(15) Limiting point of a coaxial system of spheres.

aight line

(16) Symmetric form $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ①

(17) General form to symmetric form $P_1=0=P_2$
find (l,m,n) and $(x,y,0)$ etc.

(18) Condition for line ① to lie in a plane = valid for all r

(19) Plane through given line and $\parallel$ to another line

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

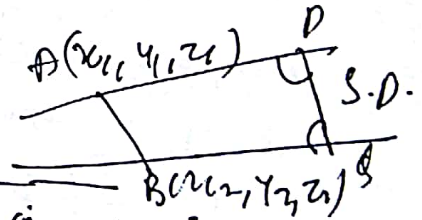
Coplanar lines
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \quad \underline{OR} \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}=0$$

Condition for three lines to be coplanar $\equiv \leftarrow \frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Intersection of three planes

Shortest distance

(23) ① Projection method: symmetrical form



$$S.D. = \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}$$

Coplanar

shortest distance eqn
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l & m_1 & n_1 \\ l & m & n \end{vmatrix}$$

(ii) general coordinate r_1, r_2

(iii) Symmetrical & general L_1 $u_1+\lambda u_2=0$ find λ, $\perp$ distance from $A(x_1,y_1,z_1)$

(iv) General & general form $u_1+\lambda_1 v_1=0$, $u_2+\lambda_2 v_2=0$ $\parallel$ planes

(25) Equation of plane containing line (1) and (2)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(26) Relation bet d-c's of three mutually ⊥ line

$l_1^2+m_1^2+n_1^2=1$, $l_1l_2+m_1m_2+n_1n_2=0$

~~27~~ plane

(27) plane through a point $P(x_1,y_1,z_1)$ and ⊥ to line whose d.r.'s are l,m,n $\quad l(x-x_1)+m(y-y_1)+n(z-z_1)=0$

(28) plane through 3 points

coplanarity of 4 points

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

⊥ to x axis $x=a$ (plane)

(29) Equation of plane ∥ to yz plane $x=a$

(30) angle between two planes

(31) plane through intersection of two planes $P_1+\lambda P_2=0$

(32) combined eqn of plane $\sum(ax^2+2fyz)=0$, cond^n $\Delta = abc+2fgh-\ldots$

angle beth plane $\tan\theta = \dfrac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$ $\quad\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ $\quad = \dfrac{\sqrt{\sum(m_1n_2-m_2n_1)^2}}{l_1l_2+m_1m_2+n_1n_2}$

(33) projection on a plane $A^2=A_{xy}^2+A_{yz}^2+A_{zx}^2$

(34) Area of triangle $\Delta^2=\Delta_{xy}^2+\Delta_{yz}^2+\Delta_{zx}^2$

(35) angle between line (d.r. l,m,n) and plane $ax+by+cz+d=0$

$\sin\theta = \dfrac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

∥ $al+bm+cn=0$

⊥ $\dfrac{a}{l}=\dfrac{b}{m}=\dfrac{c}{n}$

(36) Equation of planes bisecting the angle between two planes

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

(37) Converting line from general form $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+\ldots$ to symmetrical form $\dfrac{x-x_1}{b_1c_2-b_2c_1}=\dfrac{y-y_1}{c_1a_2-c_2a_1}=\dfrac{z-z_1}{a_1b_2-a_2b_1}$

$a_1b_2-a_2b_1\neq 0$

$(x_1,y_1,z_1)=\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}, \dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}, 0\right)$, choose other if $=0$

(38) Straight line intersecting two lines

$u_1 = 0 = v_1$ and $u_2 = 0 = v_2$ —:

$u_1 + \mu_1 v_1 = 0 = u_2 + \mu_2 v_2$

(39) Enveloping cone from a point (x_1, y_1, z_1). $SS_1 = T^2$

(40) section with given centre $(\alpha, \beta, \gamma) \Rightarrow T = S_1$

(41) Locus of point from which 3 mutually $\perp$ tangent lines can be drawn. $SS_1 = T^2 \rightarrow$ coeff(x^2) + coeff(y^2) + coeff$(z^2) = 0$

(42) Any curve through intersection of two curves S_1 and S_2

$S_1 + \lambda S_2 = 0$ (key words)

(43) cone $\Sigma(ax^2 + 2fyz) = 0$, generator $x/l = y/m = z/n$ satisfies cone

normal plane to cone $\frac{b-c}{l}x + \frac{c-a}{m}y + \frac{a-b}{n}z = 0$

line through origin $\perp$ to normal plane $\frac{x}{(b-c)/l} = \frac{y}{(c-a)/m} = \frac{z}{(a-b)/n}$

Sphere

(44) General form $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

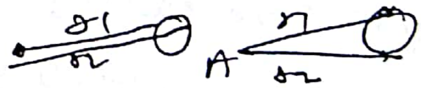
(45) four point form $\begin{vmatrix} x & y & z & x^2+y^2+z^2 & 1 \\ x_1 & y_1 & z_1 & x_1^2+y_1^2+z_1^2 & 1 \\ - & - & - & & \end{vmatrix} = 0$

(46) plane section of sphere → circle, great circle

(47) Intersection of sphere. $S_1 - S_2 = 0$ (plane)

(48) System of sphere through given circle $S + \lambda P = 0$ r $S_1 + \lambda S_2 = 0$

(49) Intersection of straight line and sphere



(50) power of point $= r_1 r_2$

(51) tangent plane, tangent line

(52) Plane of contact : locus of point of contact of tangent planes which passes through a given point (not on ◯)

(53) Pole and polar plane $\frac{2}{AR} = \frac{1}{AQ} + \frac{1}{AP}$, polar line $AB \rightarrow CD$

(54) Length of tangent, touching spheres.

orthogonal sphere $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

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(58) Radical plane $S_1 - S_2 = 0$ radical line (axis), radical centre

(56) Coaxal system of spheres: for all sphere any two of them have the same radical plane

eqn $S_1 + \mu(S_1 - S_2) = 0$ or $S + \lambda P = 0$

limiting point

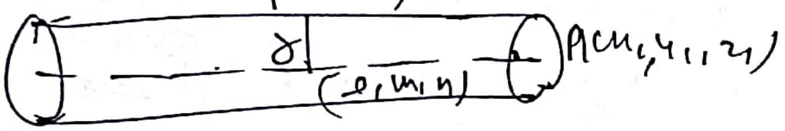
Cylinder

(57) cylinder whose generator are ∥ to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and intersect conic $= 0$, $z = 0$ ⇒ locus of vertex (x_1, y_1, z_1)

(58) :. Right circular cylinder

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2 + \left(\frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right)^2$$

$p(x, y, z)$



(59) tangent plane

(60) Enveloping cylinder of sphere whose generator are ∥ to

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, locus of $P(\alpha, \beta, \gamma) \rightarrow B^2 = 4AC$

(61) Cylinder intersecting two curves and generator ∥ to x-axis

⇒ eliminate x

Cone

Cone through origin $\Sigma(ax^2 + 2fyz) = 0$

(62) Cone passing through coordinate axes $\Sigma xy + yz + zx = 0$

(63) Cone with given vertex and given conic $f(x,y) = 0$, $z = 0$

(64) Cone with 0 D^2

(65) Cond^n for cone $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$ $\Sigma(ax^2 + 2fyz) + 2(ux + 2vy...$

(66) $F(x, y, z, t)$ = homogeneous, $F_x, F_y, F_z, F_t = 0$, $t = 1$ cond

(67) tangent line to cone $\Sigma(ax^2 + 2fyz) = 0$ cond^n from (α, β, γ)

(68) Tangent plane to $\Sigma(ax^2 + 2fyz) = 0$, $x\frac{\partial F}{\partial \alpha} + y\frac{\partial F}{\partial \beta} + z\frac{\partial F}{\partial \gamma} = 0$

$F = F(\alpha, \beta, \gamma)$

(19) Reciprocal cone of $\Sigma(ax^2+2fyz)=0$ is $\Sigma(Ax^2+2Fyz)=0$

$$\Delta=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2, \quad A=\frac{\partial\Delta}{\partial a}, \quad F=\frac{1}{2}\frac{\partial\Delta}{\partial f}$$

(20) Angle between the lines in which plane $ux+vy+wz=0$ cuts the cone $\Sigma(ax^2+2fyz)=0$

$$\tan\theta=\frac{2D\sqrt{u^2+v^2+w^2}}{(a+b+c)(u^2+v^2+w^2)-F(u,v,w)}$$

(21) Angle between generators/line of

$$\tan^2\theta=\frac{\Sigma(l_1m_2-l_2m_1)^2}{(l_1l_2+m_1m_2+n_1n_2)^2}$$

Cure it

(22) three mutually ⊥ generators of $F\equiv\Sigma(ax^2+2fyz)=0$ $\boxed{a+b+c=0}$

$\boxed{A=bc-f^2}$ A, B, C reciprocal $\Sigma(ax^2+\ldots)=\Sigma(\text{coef } x^2+\ldots)=0$

(23) 3 mutually ⊥ tangent plane $bc+ca+ab=f^2+g^2+h^2$ (reciprocal)

(24) Right circular cone $\cos\theta=\frac{\Sigma l(x-\alpha)}{\sqrt{l^2+m^2+n^2}\sqrt{\Sigma(x-\alpha)^2}}$ $A(\alpha,\beta,\gamma)$ (l,m,n) θ $P(x,y,z)$

(25) enveloping cone at (x_1,y_1,z_1) $SS_1=T^2$ or $B^2=4AC$

(26) Hyperboloid $\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1}$, $A(a\cos\alpha, b\sin\alpha, 0)$, $B(a\cos\beta, b\sin\beta, 0)$

$P, Q=$

$$P,Q=\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \pm c\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

generator thr (A)

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}$$

eqns of two systems of generating line

(i) $\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right)$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right)$

(ii) $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$, $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$

(27) Coordinates of the point of intersection of any two generating lines of two different system of hyperbola are given

$$x=\frac{a(1+\lambda\mu)}{\lambda+\mu}, \quad y=\frac{b(1-\lambda\mu)}{\lambda+\mu}, \quad z=\frac{c(1-\lambda\mu)}{\lambda+\mu}$$

(77) Equations of the two systems of the generators of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ → tangency condn of intersection $a^2l^2 - b^2m^2 + 2pn =$

(i) $\frac{x}{a} - \frac{y}{b} = \lambda z$, $\frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}$

(ii) $\frac{x}{a} - \frac{y}{b} = 2\mu$, $\frac{x}{a} + \frac{y}{b} = \frac{z}{\mu}$

(78) plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in $\perp$ generators if

$(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$

(79) $\det(A) = |A| =$ product of eigenvalues

(80) (l, m, n) ⇒ (l, m) quadratic ‖ $B^2 - 4AC = 0$

(81) Three concurrent lines will be coplanar if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

(82) plane bisecting two plane $a_1x + b_1y + c_1z + d_1 = 0$ P_1 → same sign
$a_2x + b_2y + c_2z + d_2 = 0$ P_2

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$\tan\theta = \sec 2\theta - 1$

$\tan\theta < 1$ acute

$\tan\theta > 1$ obtuse

P_1 2θ P_2

θ angle betn bisecting plane and any of the two planes.

(83) Straight line bisecting two line.

$$\frac{x}{l_1 \pm l_2} = \frac{y}{m_1 \pm m_2} = \frac{z}{n_1 \pm n_2}$$

(84) General form to symmetrical form $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$

point $z = 0$ ⇒ $(x_1, y_1, 0)$

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

→ given line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

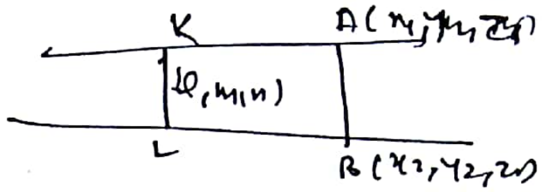
(85) plane through given line $a(x - x_1) + b(y - y_1) + c(z - z_1)$ st. $al + bm + cn = 0$

(86) Three lines passing through origin will be coplanar if they are $\perp$ to a line through origin OR

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

S.D. projection method

(106) l, m, n $\quad ll_1+mm_1+nn_1=0$
$\quad ll_2+mm_2+nn_2=0$



$S.D=\sum l(x_2-x_1)$

(107) $\lambda\mu$ system $\quad a(1-\lambda^2),\ 2b\lambda,\ c(1+\lambda^2)$

$$\frac{a^2\sin\theta}{\sin\frac{\alpha+\beta}{2}}=\frac{b^2\cos\theta}{\cos\frac{\alpha+\beta}{2}}=\frac{c^2}{\cos\frac{\alpha-\beta}{2}}=\frac{1}{K}$$

(108) point (θ,ϕ) on hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

$(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$

(109) two system of generators of paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$

(I) $\frac{x}{a}-\frac{y}{b}=\lambda z,\quad \frac{x}{a}+\frac{y}{b}=2/\lambda$

(II) $\frac{x}{a}-\frac{y}{b}=2\mu;\quad \frac{x}{a}+\frac{y}{b}=\frac{z}{\mu}$

(110) $\tan\alpha=by/ax$, equation of shortest distance.

(111) $\sum(\cos t\, x^2+\cot y^2+\cot z^2=0$ $\Rightarrow$ cone has 3 mutually $\perp$ generators

(112)

Volume of tetrahedron

(87) $V = \frac{1}{6}[a\ b\ c] = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$ $\boxed{=0}$ Four points coplanar

$\frac{1}{6}\, a\cdot(b\times c)$

(88) relations betn d.c. of three mutually $\perp$ lines

$l_1^2+m_1^2+n_1^2=1$ etc. $l_1l_2+m_1m_2+n_1n_2=0$ etc.

central conicoid

(89) $ax^2+by^2+cz^2=1$

ellipsoid / hyperboloid of one and two sheet

(90) tangent plane $axx_1+byy_1+czz_1=1$

(91) condn of tangency $lx+my+nz=p$, $p^2 = \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$

(92) Director sphere: locus of point of intersection of 3 mutually $\perp$ tangent plane to conicoid $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ ellipse $=a^2+b^2+c^2$

(93) Plane through (α,β,γ) $l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$

(94) Polar plane $A(\alpha,\beta,\gamma)$ $a\alpha x+b\beta y+c\gamma z=1$

pole $(\alpha,\beta,\gamma) = \left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

(98) Polar line of AB is line PQ s.t. polar planes of all points on AB passes through PQ

$a\alpha x+b\beta y+c\gamma z-1=0 = alx+bmy+cnz$

(99) Locus of chord bisected at a given point $T=S_1$

(100) normal to conicoid at (α,β,γ)

$\frac{x-\alpha}{a\alpha p}=\frac{y-\beta}{b\beta p}=\frac{z-\gamma}{c\gamma p}$ ellipsoid $\frac{x-\alpha}{\alpha/a^2}=\frac{y-\beta}{\beta/b^2}=\frac{z-\gamma}{\gamma/c^2}$

(101) Number of normal through (x_1,y_1,z_1) 6

$\frac{\alpha}{a}=\frac{ax_1}{\lambda+a^2}$, $\frac{\beta}{b}=\frac{by_1}{\lambda+b^2}$, $\frac{\gamma}{c}=\frac{cz_1}{\lambda+c^2}$

(102) Equation of cone through six concurrent normals from a point (x_1,y_1,z_1) (not on ellipse) to an ellipsoid

$\Sigma \frac{(b^2-c^2)x_1}{x}=0$

(103) angle between two line $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$

(104) angle bisector

PI $>45°$ obtuse $c < 45°$ c $\to$ acute

PI

P2

(105) line of greatest slope in plane (0,0,0) $\to l, m, n$ $\to l', m', n'$

Calculus + Real Analysis

① If $y = mx + c$ is an asymptote to the curve $y = f(x)$, then

$$m = \lim_{x \to \infty} \frac{y}{x}, \quad c = \lim_{x \to \infty} (y - mx)$$

② Put $x = 1$, $y = m$ highest degree $\phi_n(m) = 0 \Rightarrow m_1, m_2, \ldots$

$x = 1, y = m$ $\quad$ $\cancel{\phi_n(m)}\ \phi_{n-1}(m)$ $\quad$ $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

$y = m_1 x + c_1$ etc

③ Non existence of asymptotes

$\phi_n(m) = 0 \Rightarrow m$ for which ~~ϕ_n ...~~ $\phi_n'(m) = 0 \Rightarrow c = \infty$ non existent

④ Parallel asymptotes :-

(i) two ∥ asymptotes $\quad \phi_n(m) = 0$ repeated m two times.

and $\phi_{n-1}(m) = 0$ for that m

there c is given by $\dfrac{c^2}{2!}\phi_n''(m) + \dfrac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

(ii) three ∥ asymptotes $\phi_n(m) = 0 \Rightarrow 3$ times repeated roots

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$$

⑤ Asymptotes ∥ to y-axis: obtained by equating to zero the coefficient of the highest power of y in the eqn of the curve. In case coefficient is constant or factors are imaginary there are no asymptotes parallel to y-axis.

Similarly for asymptotes ∥ to x-axis.

⑥ No. of asymptotes, real or imaginary of the nth degree curve can't exceed n.

⑦ Asymptotes by expansion: $y = mx + c$ is an asymptote of

$y = mx + c + A/x + B/x^2 + C/x^3 + \ldots$

where the series $A/x + B/x^2 + C/x^3 + \ldots$ convergent

⑧ Asymptotes of polar curves

if α be the root of eqn $f(\theta) = 0$, then

$$r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

is an asymptote of the curve $\frac{1}{r} = f(\theta)$ or $u = f(\theta)$ $(u = \frac{1}{r})$

⑨

Jacobians

(10) $J(u,v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$, $J(u,v,w) = \begin{vmatrix} - & - & - \\ - & - & - \\ - & - & - \end{vmatrix}$

(11) $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$J(x,y,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

(12) functions of functions + generalized here u_1, u_2 are fn of

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \frac{\partial(u_1,u_2)}{\partial(y_1,y_2)} \cdot \frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}$$

y_1, y_2 and y_1, y_2 are function of x_1, x_2

(13) Implicit functions

$F_1(u_1,u_2,u \cdots u_n, x_1, x_2 \cdots x_n) = 0$

$F_2(u_1,u_2 \cdots u_n, x_1, x_2 \cdots x_n) = 0$

$\cdots$

$F_n(u_1,u_2 \cdots u_n, x_1, x_2 \cdots x_n) = 0$

$$\frac{\partial(u_1,u_2,u_3 \cdots u_n)}{\partial(x_1,x_2 \cdots x_n)} = (-1)^n \frac{\dfrac{\partial(F_1,F_2 \cdots F_n)}{\partial(x_1,x_2 \cdots x_n)}}{\dfrac{\partial(F_1,F_2, \cdots F_n)}{\partial(u_1,u_2, \cdots u_n)}}$$

(14) If $u_1, u_2, \cdots u_n$ be functions of n independent variables $x_1, x_2, \cdots x_n$. In order that these n functions may not be independent i.e. there may exist betn these n functions a reln $F(u_1,u_2 \cdots u_n) = 0$, it is necessary and sufficient that Jacobian $\frac{\partial(u_1,u_2 \cdots u_n)}{\partial(x_1,x_2 \cdots x_n)}$ should vanish identically. → Find relation →

(15) $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

(16) Leibniz theorem: nth order differential coefficient of product of two functions

$D^n(uv) = (D^n u)\cdot v + {}^nC_1(D^{n-1}u)Dv + \cdots {}^nC_r(D^{n-r}u)D^r v + \cdots u(D^n v)$

(17) Taylor series: $f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots$

(18) Maclaurin series $f(x) = f(0) + x\frac{f'(0)}{1!} + \frac{x^2}{2!}f''(0) + \cdots$

we use it to find formula for $\sin x$, $\cos x$ etc

(19) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $(x<1)$

(20) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $(x<1)$

(21) $(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots$

(22) Euler's theorem for homogeneous function

$f(tx, ty) = t^n f(x,y) \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

where $f = x^n f(y/x)$

generalised $x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots x_n\frac{\partial f}{\partial x_n} = nf$

(23) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$

(24) $\lim_{x\to 0}(1+x)^{1/x} = e$, $\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^x = e$

(25) radius of curvature $\rho = \frac{ds}{d\psi} = \frac{(1+y'^2)^{3/2}}{y''}$, $ds = \sqrt{dx^2+dy^2}$

$\rho = r\frac{dr}{dp}$ (pedal)

(26) radius of curvature $\rho^2 = \left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2$

(27) Radius of curvature at origin when curve passes through origin and x-axis is the tangent at origin

$\rho = \lim_{y\to 0}\frac{x^2}{2y}$ Similarly, $\rho = \lim_{x\to 0}\frac{y^2}{2x}$

(28) Radius of curvature at pole $= \lim_{\theta\to 0}\left(\frac{r}{2\theta}\right)$

(29) chord of curvature $\parallel$ to x axis $= 2\rho\sin\psi$

(30) chord of curvature $\parallel$ to y axis $= 2\rho\cos\psi$

(31) change of independent variable

(i) $(1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + y = 0$, $x = \tan z$

$(1+x^2)\frac{d}{dx}\left((1+x^2)\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right)$ etc

(ii) $x^4\frac{d^2y}{dx^2} + 2x^3\frac{dy}{dx} + n^2y = 0$, $x = 1/z$

$x^2\frac{dy}{dx} = -\frac{dy}{dz}\cdots$

(iii) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$, $x = \sin z$

(iv) $\frac{d^2y}{dx^2} + \cdots\frac{dy}{dx} + \cdots = 0$, $x = \tan^{-1}z$

(v) $\sin^2 z \, (\frac{d^2y}{dz^2}) + \sin 4z \, (\frac{dy}{dz}) + 4y = 0$

$\tan z = e^x$

(vi) $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2\cos^3 x \, y = 2\cos^5 x$

$z = \sin x$

(32) Limit: A function $f(x)$ is said to tend to the limit l as x tends to a, if corresponding to any $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ for $0 < |x - a| < \delta$

$\therefore \lim_{x \to a} f(x) = l$

LHL | RHL

(33) Continuity: A function $f(x)$ ~~is said~~ defined for $x = a$ is said to be continuous at $x = a$ if (i) $f(a)$ is finite (ii) the limit of the function $f(x)$ as $x \to a$ exist and equal to $f(a)$ at $x = a$

(34) Differentiability: function $f(x)$ is said to be differentiable at $x = a$ if

$Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, $h > 0$ and $Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$ $h > 0$

exist and have common value (finite (#infinite)..

(35) Rolle's theorem

$f(x)$ continuous on $[a, b]$, differentiable on (a, b), $f(a) = f(b)$ $a < c < b$

then $f'(c) = 0$

(36) Lagrange MVT: continuous in $[a, b]$, differentiable in (a, b)

$a < x < b$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

(37) Cauchy's MVT / 2nd MVT

$f(x), g(x)$ continuous on $[a, b]$, differentiable on (a, b)

$g'(x) \neq 0$ in (a, b) $a < c < b$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

(38) $\int \frac{dx}{\sqrt{a^2+x^2}} = \log(x+\sqrt{x^2+a^2}) = \sinh^{-1}(x/a)$

(39) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}) = \cosh^{-1}(x/a)$ cosh... y = ...

(40) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log(x+\sqrt{x^2+a^2})$

(41) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$

(42) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log(x+\sqrt{x^2-a^2})$

(43) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(44) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

(45) $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(46) $\Gamma n+1 = n\Gamma n$, $\Gamma\frac{1}{2} = \sqrt{\pi}$

(47) for $\int_0^1 x^m(1-x^2)^n dx$, use $x = \sin t$, $\sin^2 t$ etc.

Definite Integrals

$\int_a^b f(x)\,dx = F(b) - F(a)$

(48) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(49) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(50) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx$

(51) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(52) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ according $f(x)$ odd, even

$\int_{-a}^a f(x)\,dx = 0$ or $2\int_0^a f(x)\,dx$

(53) $\int_{-a}^a f(x)\,dx = 0$ or $2\int_0^a f(x)\,dx$

(54) $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$

and $\int_0^{2a} f(x)\,dx = 0$ if $f(2a-x) = -f(x)$

$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

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(55) definite integral as the limit of a sum

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f(r/n)$$

Also $$\lim_{n\to\infty} \frac{1}{n} \sum_{r=p}^{n+q} f(r/n) = \int_0^1 f(x)\,dx$$

(56) Area betn x axis and curve $y=f(x)$, $A = \int_a^b y\,dx$

(57) Area of curves in polar eqn $A = \frac{1}{2}\int_{\theta=\theta_1}^{\theta_2} r^2\,d\theta$
of $r = f(\theta)$

(58) $\text{Area} = \frac{1}{2}\oint x\,dy - y\,dx = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$

Length of curves

(59) $S = \int_a^b \sqrt{1+y'^2}\,dx = \int_c^d \sqrt{1+x'^2}\,dy$

(60) $S = \int_{t_1}^{t_2} \sqrt{\dot{x}^2+\dot{y}^2}\,dt$

(61) polar form $r = f(\theta)$, $S = \int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

for $\theta = f(r)$ $S = \int_{r_1}^{r_2} \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr$

Volumes of Solids of revolution

(62) About x-axis $y=f(x)$
$V = \int_a^b \pi y^2\,dx$,

about y-axis
$V = \int_c^d \pi x^2\,dy$

(63) Parametric form $V = \int_a^b \left[\pi (y(t))^2 \frac{dx}{dt}\right] dt$

(64) Polar coordinate $V = \int_{\theta=\alpha}^{\beta} \left(\pi y^2 \frac{dx}{d\theta}\right) d\theta$

(65) $r = f(\theta)$, $\theta=\theta_1$ to $\theta=\theta_2$

(i) about initial line $\theta=0$, $V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin\theta\,d\theta$

(ii) about line $(\theta=\gamma)$

$$V = \int_{\theta_1}^{\theta_2} \frac{2\pi}{3} r^3 \sin(\theta-\gamma)\,d\theta$$

(66) axis of rotation being any line $V = \int_{OC}^{OD} \pi (PM)^2\,d(OM)$

Asymptote axis of rotation CD, O is on CB. PM $\perp$ from P to CD.

67 Surfaces of solids of revolution

(i) about x-axis $S=\int_a^b 2\pi y\, ds=\int_a^b 2\pi y\sqrt{1+y'^2}\, dx$

(ii) about y-axis $S=\int_a^b 2\pi x\, ds=\int_a^b 2\pi x\sqrt{1+x'^2}\, dy$

(iii) Parametric form (x axis) $S=\int 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt$

(iv) Polar equations

$$S=\int_{\theta=\alpha}^{\theta=\beta} 2\pi(r\sin\theta)\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(v) about any axis ~~$S=\int 2\pi y \frac{ds}{dx} dx$~~

$S=2\pi\int PM\, ds$

68 Pappus theorem

Volume of rev. $=A\cdot 2\pi\bar{y}$

surface area $=l\cdot 2\pi\bar{y}$

69 Beta function.

$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$, $m,n>0$

if $m,n\in N$ $B(m,n)=\frac{(m-1)!\,(n-1)!}{(m+n-1)!}$ $B(m,n)=B(n,m)$

70 $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta=\frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$, $p>-1, q>-1$

71 Alternate form of beta function $B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$, $m>0, n>0$

72 Gamma function $\Gamma n=\int_0^\infty e^{-x}x^{n-1}\, dx$, $n>0$

$\Gamma(n+1)=n\Gamma n$, $n>0$ and if $n\in N$ $\Gamma n=(n-1)!$, $\Gamma\frac{1}{2}=\sqrt{\pi}$ 80

73 $B(m,n)=\frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$, $m,n>0$

74 $\int_0^{\pi/2}\cos^m\theta\sin^n\theta\, d\theta=\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$, $m,n>-1$ 81

75 $\int_0^\infty e^{-x^2/2}\, dx=\frac{\sqrt{\pi}}{2}$ $\boxed{\Gamma n\,\Gamma(1-n)=\pi/\sin n\pi}$ take

76 Dirichlet's theorem.

$\iiint x^{l-1}y^{m-1}z^{n-1}\, dx\,dy\,dz=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+m+n+l)}$, $l,m,n>0$ $x+y+z\leq 1$ (2) • Use it

generalised for n-variable

(77) Liouville's extension of Dirichlet's theorem — use it

$x, y, z > 0$ s.t. $h_1 \le x+y+z \le h_2$, then

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

(78) Convergence of improper integrals

infinite interval, bounded functions, monotone function, proper integral, improper integral = first kind (limit) / second kind (range)

(78) First kind

$$\int_a^\infty f(x)\,dx = \lim_{x\to\infty} \int_a^x f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{x\to\infty} \int_{-x}^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{x_1\to\infty} \int_{-x_1}^c f(x)\,dx + \lim_{x_2\to\infty} \int_c^{x_2} f(x)\,dx$$

(79) (2nd kind) Unbounded

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon'\to 0} \int_{c+\varepsilon'}^b f(x)\,dx$$

$a < c < b$ — both ends

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$\int_a^\infty f(x)\,dx$ at b

(80) Convergence of improper integral

$\left|\int_a^b f(x)\,dx - I\right| < \varepsilon$ for all $b \ge N$

(81) Comparison test: $f(x), g(x)$ bounded + integrable, $g(x) \ge 0$

$|f(x)| \le g(x)$ convergent / divergent $|f(x)| \ge g(x)$

limit form $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ = definite number ($\neq 0$) converge or diverge together

take $g(x) = \frac{1}{x^n}$, $\int_a^\infty \frac{dx}{x^n}$ is comparison integral

(82) $\int_a^\infty \frac{dx}{x^n}$; $a > 0$ is convergent when $n > 1$ and divergent when $n \le 1$

(83) M-test: $f(x)$ bounded + integrable in (a, ∞), $a > 0$
then if (i) $M > 1$ s.t. $\lim_{x \to \infty} x^M f(x)$ exist then $\int_a^\infty f(x) dx$ convergent

(ii) $M \le 1$ s.t. $\lim_{x \to \infty} x^M f(x)$ exist and $(\neq 0)$, $\to$ divergent

(84) Abel's test: $\int_a^\infty f(x) dx$ converges + $\phi(x)$ bounded and monotonic for $a > 0$ then $\int_a^\infty f(x) \phi(x) dx$ is convergent

(85) Dirichlet's test: $f(x)$ bounded and monotonic and $\lim_{x \to \infty} f(x) = 0$ then $\int_a^\infty f(x) \phi(x) dx$ converges provided $\left| \int_a^x \phi(x) dx \right|$ bounded as x takes all finite values.

(86) Absolute convergence: Infinite integral $\int_a^\infty f(x) dx$ is said to be absolutely convergent iff $\int_a^\infty |f(x)| dx$ is convergent

(87) comparison integral $\int_a^b \frac{dx}{(x-a)^n}$, $g(x) = \frac{1}{(x-a)^n}$

M-test $\quad$ $\underline{(x-a)^M f(x)}$ exist $\quad 0 < M < 1$ converges, $\underline{M \ge 1}$ divergent

(88) comparison integral $\int_a^b \frac{dx}{(x-a)^n}$ dx is convergent if $n < 1$, divergent if $n \ge 1$

(89) Gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$

converges iff $n > 0$

(90) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \pi^2/6$

(91) Leibniz rule

$$\frac{d}{dx} \int_{A(x)}^{B(x)} F(x, \xi) d\xi = \int_{A(x)}^{B(x)} \frac{\partial}{\partial x} F(x, \xi) d\xi + F(x, \xi) \frac{d}{dx}(B(x)) \Big|_{\text{at } \xi = B(x)} - F(x, \xi) \frac{d}{dx}(A(x)) \Big|_{\xi = A(x)}$$

(92) $f(h, k) - f(0, 0) = Ah + Bk + \sqrt{h^2 + k^2}\, g(h, k)$

differentiability of $f \Rightarrow g(h, k)$ must exist

(93) If f_x and f_y are both differentiable at a point (a, b) of the domain of f then $f_{xy}(a, b) = f_{yx}(a, b)$

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(94) $e^x > x^{n+1}$

(95) $\Gamma n = (n-1)\Gamma n-1$ for $n > 1$ $\quad \Gamma\frac{1}{2} = \sqrt{\pi}$

(96) $\Gamma(n) = (n-1)!$ $n \in N$

(97) $\Gamma\frac{1}{2} = \sqrt{\pi}$, $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2} = \int_{-\infty}^0 e^{-x^2}dx$

(98) $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2}\frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{q+1}{2}}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}, \quad p > -1, q > -1$$

(99) sign (signum) function

$$\operatorname{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$\operatorname{sgn}(x) = \frac{x}{|x|}$

$$\operatorname{sgn}(\cos x) = \begin{cases} 1 & \cos x > 0 & -\frac{\pi}{2} < x < \pi/2 \\ 0 & \cos x = 0 & x = \pi/2, -\pi/2 \\ -1 & \cos x < 0 & \frac{\pi}{2} < x < 3\pi/2 \end{cases}$$

(100) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m,n > 0$ $\quad$ put $x = \frac{z}{1+z}$, $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$ $\quad m,n > 0$

$B(m,n) = B(n,m)$

(101) $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ $\quad m,n > 0$

(102) Area betn curve $r = a(\sec\theta + \cos\theta)$ and its asymptote

$r = a\sec\theta$ or $x = a$ is $\quad 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$

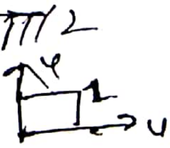
(103) $a\sin x = b\cos x \tan\theta \Rightarrow e^{\frac{x-y}{x+y}}$ or $\cos\frac{x-y}{x+y} \rightarrow$ integral

transformation

(104) $x = r\cos^2\theta$, $y = r\sin^2\theta$

(105) $\Gamma\frac{1}{n}\,\Gamma\frac{2}{n}\,\Gamma\frac{3}{n} \cdots \Gamma\frac{n}{n}$ or $\frac{n-1}{n}$ $= \frac{(2\pi)^{\frac{n-1}{2}}}{n^{1/2}}$

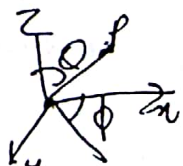
(106) $f(x) = x - \sin x(\cos x)^{-1/3}$, $0 < x < \pi/2$



(107) $x+y = u$, $y = uv$

(108) $x = a\cos^2\theta + y\sin^2\theta$, $\quad \pi/2 \to 0$, $0 \to a$

(109) $\iint f\,ds$ or sphere, $\quad ds = |r_\theta \times r_\phi|\,d\theta\,d\phi$



$r(\theta,\phi) = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

(110)

Statics

① Stable equilibrium $\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$, h: height of CG from point of contact.

② Work function and potential energy test for stable and unstable equilibrium

③ stable eqm $h < \frac{\rho_1 \rho_2 \cos\alpha}{\rho_1+\rho_2}$

④ CG of hemispherical shell $= r/2$

⑤ common catenary

$T\cos\psi = T_0 = wc$

$T\sin\psi = ws$

$T = wy$

$s = c\tan\psi$ (intrinsic eqn)

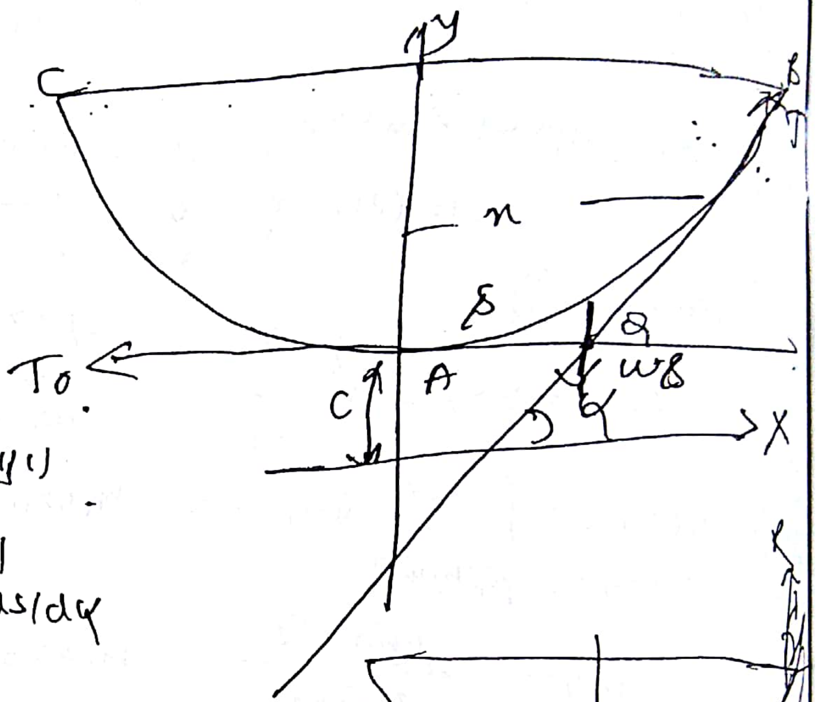
$y^2 = c^2+s^2$,

$y = c\sec\psi$

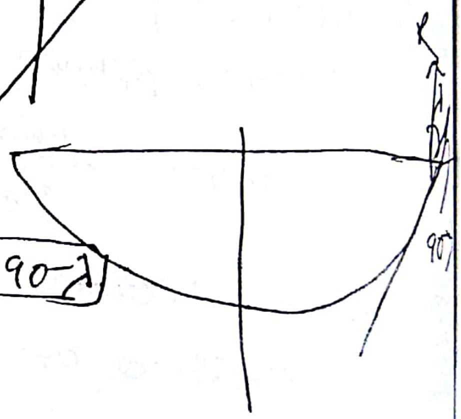
$s = c\sinh(x/c)$, $y = c\cosh(x/c)$

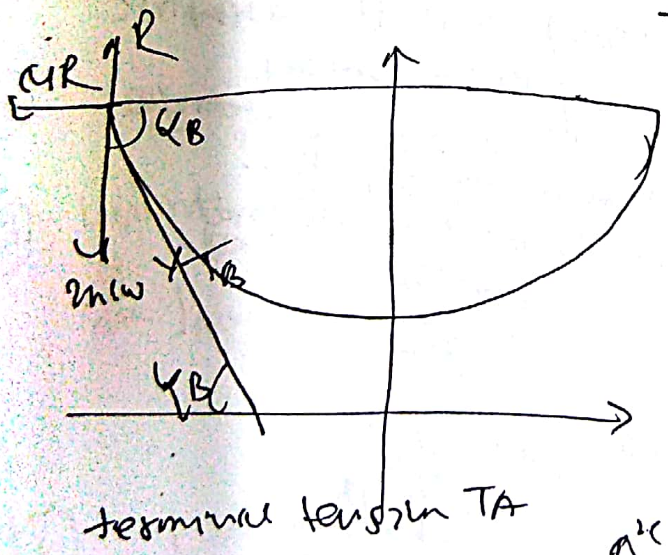
$x = c\log|\sec\psi+\tan\psi|$

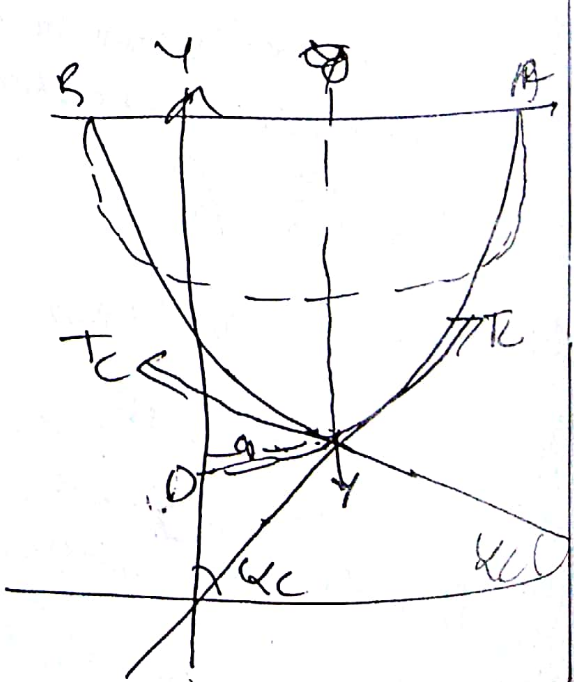
$\rho = c\sec^2\psi = ds/d\psi$



$\tan\lambda = \mu$, $\boxed{\psi = 90-\lambda}$

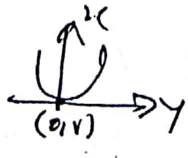






terminal tension TA

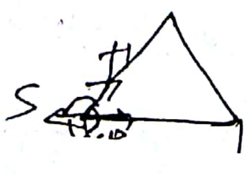
radius of curvature at origin $= \lim_{x\to 0}(y^2/2x)$



⑥ virtual work

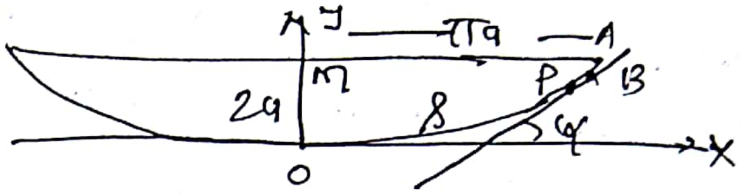
⑦ $T = \lambda\frac{(x-a)}{a}$

⑧ Lami's theorem



Dynamics

① Cycloidal motion

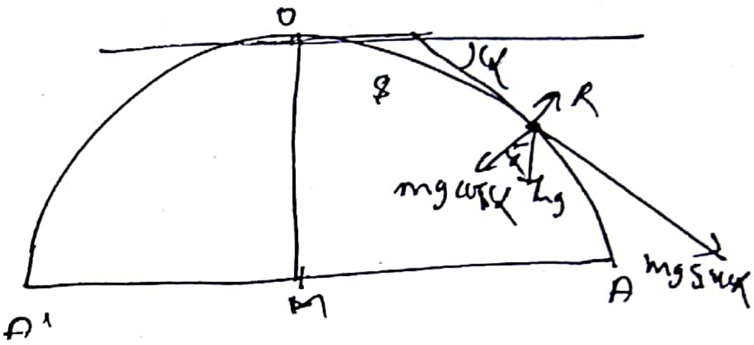


$x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$ $\quad \theta = 2\psi$

$s = 4a\sin\psi$

$T = 4\pi\sqrt{a/g}$, $\quad s^2 = 8ay$, $\quad OA = 4a$, $\quad OM = 2a$



← Inverted

$m\frac{d^2s}{dt^2} = mg\sin\psi$

$m\frac{v^2}{\rho} = mg\cos\psi - R$

$s = 4a\sin\psi$

② Central orbit $\quad \frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$, $\quad P = \frac{h^2}{p^3}\frac{dp}{dr}$, $\quad v = h/p = \frac{ds}{dt}$

angular momentum $\quad h = r^2\frac{d\theta}{dt} = \text{const}$ $\quad \frac{dr}{ds} = \cos\phi$

$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right)$ $\quad p = r\sin\phi$

$v^2 = -2\int_\infty^a P\,dr$ ← velocity from infinity

$h = \sqrt{\mu l}$, $\quad l = b^2/a$
$b = a$ for hyperbola

Steps

(i) use $\frac{P}{u^2} = h^2\left(u + \frac{d^2u}{d\theta^2}\right)$

(ii) multiply by $2\frac{du}{d\theta}$ and integrate

$v^2 = h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = \int \frac{P}{u^2}\cdot 2\,du + c$

(iii) use initial condn at $u = \frac{1}{a}$, $\frac{du}{d\theta} = 0$, $u = \frac{a}{P}$, $\Rightarrow h, v^2$

(iv) Find $du/d\theta$ → manipulate and integrate $\boxed{\text{take either of } + - \text{ sign}}$

(v) use initial condn
sometimes are $\quad du/d\theta = ~~\ldots~~ -\frac{1}{r^2}\frac{dr}{d\theta}$

use $h = r^2\frac{d\theta}{dt}$ ⇒ time

rate of description of sectoral area $= h/2$

Law of force $\quad v^2 = \begin{cases} \mu\left(\frac{2}{r} - \frac{1}{a}\right) & \text{elliptic} \\ 2\mu/r & \text{parabolic} \\ \mu\left(\frac{2}{r} + \frac{1}{a}\right) & \text{hyperbolic} \end{cases}$

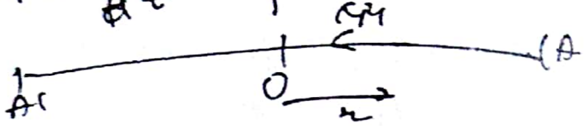
$h = \sqrt{\mu l}$, $\quad l = \frac{b^2}{a}$

③ Radial and transverse velocity $\bar{v} = \dot{r} e_r + r\dot{\theta} e_\theta$

④ Radial and transverse acceleration $\bar{a} = (\ddot{r} - r\dot{\theta}^2) e_r + (2\dot{r}\dot{\theta} + r\ddot{\theta}) e_\theta$

⑤ tangential and normal acceleration $\bar{a} = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$

⑥ SHM $v^2 = \mu(a^2 - x^2)$, $x = a\cos\sqrt{\mu}\,t$

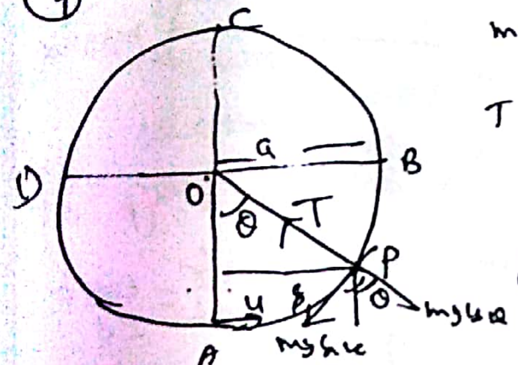


⑦ Hookes law $T = \lambda \frac{x-l}{l}$

⑧ Inverse square law $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$, $T = 2\pi\sqrt{\frac{a^3}{2\mu}}$

particle falling on earth (outside)

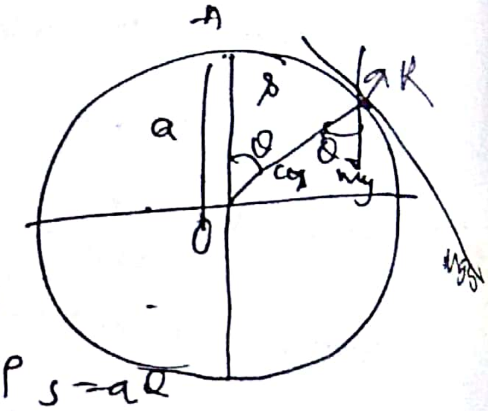
⑨ Motion in a vertical circle



$m\frac{d^2s}{dt^2} = -mg\sin\theta$

$T - mg\cos\theta = \frac{mv^2}{r}$

$s = a\theta$



$m\frac{d^2s}{dt^2} = mg\sin\theta$

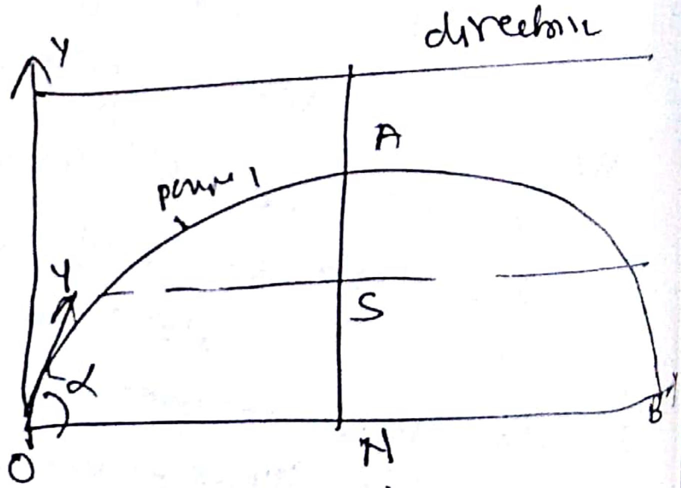
$mg\cos\theta - R = mv^2/\rho$, $s = a\theta$

⑩ Simple pendulum $T = 2\pi\sqrt{l/g}$

⑪ Projectile $y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{u^2\cos^2\alpha}$

$\left(x - \frac{u^2\sin 2\alpha}{2g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{g}\right)$

$T = \frac{2u\sin\alpha}{g}$, $R = \frac{u^2\sin 2\alpha}{g}$, $h = \frac{u^2\sin^2\alpha}{2g}$



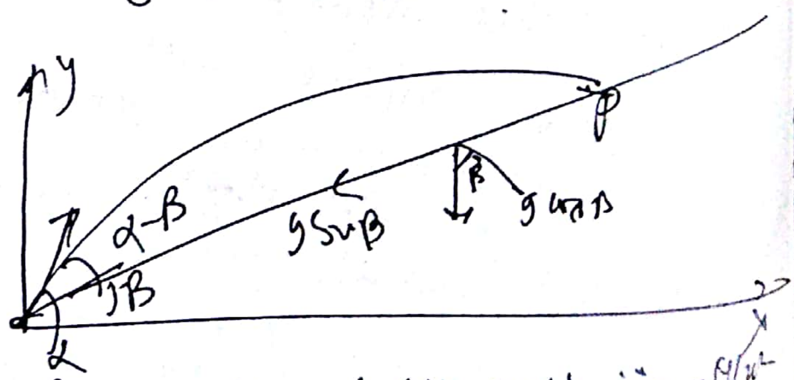
Inclined plane

$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$

$R = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$

max $R = \frac{u^2}{g(1+\sin\beta)}$

For down the plane $\beta \to -\beta$



motion outside earth $\ddot{x} = -\mu/x^2$

inside $\ddot{x} = -\lambda x$

at $x=a$, $\ddot{x} = -g$

⑫ work $= \int \bar{F}\cdot d\bar{r}$