IFoS 2016.

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4(a) Examine whether the real quadratic form
$4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$

is positive definite or not. Reduce it to its diagonal form and determine its signature.

Let $q = 4x^2 + xy - 2xz$
$+ yx - y^2 - yz$
$- 2zx - zy + 2z^2$

Matrix Representation

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

To reduce it to diagonal form, we apply the congruent operations—

$$\begin{matrix} R_2 \to -4R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 4 & 1 & -2 \\ 0 & 5 & 2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{matrix} C_2 \to -4C_2 + C_1 \\ C_3 \to 2C_3 + C_1 \end{matrix} \sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & 4 \\ 0 & 4 & 4 \end{bmatrix}$$

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$R_3 \to 5R_3 + R_2$ $\sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & 4 \\ 0 & 0 & 24 \end{bmatrix}$

$C_3 \to 5C_3 + C_2$ $\sim \begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 120 \end{bmatrix}$

r = Rank = Non-zero element in diagonal form = 3

p = Index = No. of positive elements in diagonal form = 2

s = signature = (No. of positive terms) − (No. of negative terms)

$= p - (\cancel{r} - p) = 2p - r$

$= 2 \times 2 - 3 = 1.$

Since, all the diagonal entries are not positive, hence the given quadratic form is not positive definite.

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4(b) show that the integral $\int_0^\infty e^{-x} x^{\alpha-1}\, dx,\ \alpha>0$ exists by separately taking ~~the cases for $\alpha \geq 1$ and $0<\alpha<1$.~~

Let

$$f(x) = e^{-x} x^{\alpha-1} = \frac{x^{\alpha-1}}{e^x}$$

$$\int_0^\infty e^{-x} \cdot x^{\alpha-1}\, dx = \int_0^1 e^{-x} \cdot x^{\alpha-1}\, dx + \int_1^\infty e^{-x} \cdot x^{\alpha-1}\, dx$$

$$I = I_1 + I_2.$$

Case-I : Let $\alpha \geq 1$ i.e. $\alpha-1 \geq 0$.

Then $f(x) = e^{-x} \cdot x^{\alpha-1}$ will be bounded over interval $(0,1)$ and

$$\lim_{x\to 0} e^{-x} \cdot x^{\alpha-1} = 0 \quad \text{as } \alpha-1 \geq 0$$

$$\lim_{x\to\infty} e^{-x} \cdot x^{\alpha-1} = \lim_{x\to\infty} \frac{x^{\alpha-1}}{e^x} = 0. \quad \text{(Continuous use of L-Hospital)}$$

Hence I_1 is proper integral

$\therefore$ I_1 is convergence.

Case-II : When $0<\alpha<1$ i.e. $1-\alpha>0$

$$f(x) = \frac{e^{-x}}{x^{1-\alpha}}$$ will be unbounded at $x=0$.

By μ-test

$$I_1 = \lim_{a\to 0^+} \int_a^1 e^{-x} \cdot x^{\alpha-1}\, dx.$$

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$$\lim_{x \to a^+} (x-a)^{\mu} f(x)$$

$$= \lim_{x \to 0^+} x^{\mu} \cdot e^{-x} \cdot x^{\alpha-1}$$

$$= \lim_{x \to 0^+} x^{\mu+\alpha-1} \cdot e^{-x}$$

$$= 1 \quad \text{if} \quad \mu+\alpha-1 = 0$$

ie $\mu = 1-\alpha$

So, when $0 < \mu < 1$ ie $0 < \alpha < 1$ then I_1 is convergent.

and if $\mu > 1 \Rightarrow \alpha \leq 0$, $\Rightarrow I_1$ is divergent.

$\therefore I_1$ is convergent if $\alpha > 0$.

Now, we take $I_2 = \int_1^{\infty} e^{-x} \cdot x^{\alpha-1} \, dx$

Here $f(x) = e^{-x} x^{\alpha-1}$ is bounded over integral $(1, \infty)$ as

$$\lim_{x \to \infty} e^{-x} \cdot x^{\alpha-1} = \lim_{x \to \infty} \frac{x^{\alpha-1}}{e^{x}} \to 0$$

(as $\alpha > 0 \therefore \alpha - 1 > -1$)

$\therefore I_2$ is convergent

$\Rightarrow I = I_1 + I_2$ is convergent.

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4(c) Prove that

$$\Gamma 2z = \frac{2^{2z-1}}{\sqrt{\pi}} \cdot \Gamma z \, \Gamma\left(z+\frac{1}{2}\right)$$

Duplication formula —

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\therefore \quad \beta(m,m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx$$

Put $x = \sin^2\theta \quad \therefore \quad dx = 2\sin\theta\cos\theta\,d\theta$

If $x=0,\ \theta=0$ & $x=1 \Rightarrow \theta=\pi/2$

$$\therefore \quad \beta(m,m) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{m-1} \times 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \frac{(2.\sin\theta\cos\theta)^{2m-1}}{2^{2m-1}}\,d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1}\,d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\pi} (\sin t)^{2m-1}\,\frac{dt}{2} \qquad \left(\text{let } 2\theta = t,\ d\theta = \frac{dt}{2}\right)$$

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$$= \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} (\sin t)^{2m-1} \cdot 1\,dt$$

$$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin t)^{2m-1} \cdot (\cos t)^0\,dt$$

$$= \frac{2}{2^{2m-1}} \cdot \frac{1}{2} \frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2m-1+1}{2}+\frac{0+1}{2}\right)}$$

$$\left[\because \int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}\right]$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2m+1}{2}\right)}$$

$$\Rightarrow \frac{\cancel{\Gamma(m)}\,\Gamma(m)}{\Gamma(m+m)} = \frac{1}{2^{2m-1}} \frac{\cancel{\Gamma(m)}\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\left[\because B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$\therefore \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m).$$

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4(c). Let the eqn of plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=p \quad \text{--- (1)}$$

It meets the axis at points
$A(ap,0,0)$, $B(0,bp,0)$, $C(0,0,cp)$

We find eqn of sphere passing through origin $O(0,0,0)$ and A, B, C.

$$x^2+y^2+z^2-apx-bpy-cpz=0. \quad \text{--- (2)}$$

Eqn (1) and (2) together gives the equation of circle ABC.

If we homogenize eqn (2) with help of eqn (1), we will get the eqn of cone with ~~centre~~ vertex, at origine.

$$x^2+y^2+z^2-(apx+bpy+cpz)\left(\frac{x}{ap}+\frac{y}{bp}+\frac{z}{cp}\right)=0$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\left(\cancel{x^2}+\frac{b}{a}xy+\frac{c}{a}zx+\frac{a}{b}xy+\cancel{y^2}+\frac{c}{b}yz\right.$$

$$\left.+\frac{a}{c}xz+\frac{b}{c}zy+\cancel{z^2}\right)=0.$$

$$\therefore\ yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

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