EXADEMY

ONLINE NATIONAL TEST

Course: UPSC – CSE - Mathematics Optional

Test 2

Subject: **LINEAR ALGEBRA** Time: **3 Hours**

Total Questions: **30** Total Marks: **(100)**

Q1. Suppose u and v belong to a vector space V. Simplify each of the following expressions:

i. $E_1 = 3(2u - 4v) + 5u + 7v$
ii. $E_2 = 3u - 6(3u - 5v) + 7u$
iii. $E_3 = 2uv + 3(2u + 4v)$
iv. $E_4 = 5u - (3/v) + 5u$

2 Marks

Q2. Show that

a) $k(u - v) = ku - kv$
b) $u + u = 2u$

2 Marks

Q3. a) Express v = (1, -2, 5) in R3 as a linear combination of the vectors

u1 = (1, 1, 1), u2 = (1, 2, 3), u3 = (2, -1, 1)

b) Express v = (2, -5, 3) in R^3 as a linear combination of the vectors

$u_1 = (1, -3, 2)$, $u_2 = (2, -4, -1)$, $u_3 = (1, -5, 7)$

2 Marks

Q4. Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

a) Find the Latent values of Matrix A
b) Find Latent Vectors for each Latent Value of Matrix A
c) Diagonalize the Matrix A
d) Diagonalize the matrix $A^3 - 5A^2 + 3A + I$, where I is the 2 x 2 identity matrix
e) Calculate A^{100}
f) Calculate $(A^3 - 5A^2 + 3A + I)^{100}$

12 Marks

Q5. Express the polynomial $v = t^2 + 4t - 3$ in P(t) as a linear combination of the polynomials

$p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$, $p_3 = t + 1$

3 Marks

Q6. Express M as a linear combination of the matrices A, B, C where

$$M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}, and\ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

3 Marks

Q7. Let V = P(t), the vector space of real polynomials. Determine whether or not W is a subspace of V, where

a) W consists of all polynomials with integral coefficients.
b) W consists of all polynomials with degree ≥ 6 and the zero polynomial
c) W consists of all polynomials with only even powers of t.

3 Marks

Q8. Let V be the vector space of functions f : R → R. Show that W is a subspace of V, where

a) W = {f(x): f(1) = 0}, all functions whose value at 1 is 0.
b) W = {f(x): f(3) = f(1)}, all functions assigning the same value to 3 and 1.
c) W = {f(t): f(-x) = -f(x) = -f(x)}, all odd functions.

3 Marks

Q9. Determine whether or not u and v are linearly dependent, where

a) u = (1, 2), v = (3, -5)
b) u = (1, -3), v = (-2, 6)
c) u = (1, 2, -3), v = (4, 5, -6)
d) u = (2, 4, -6), v = (3, 6, -12)

4 Marks

Q10. Determine whether or not u and v are linearly independent, where

a) u = $2t^2 + 4t - 3$, v = $4t^2 + 8t - 6$,
b) u = $2t^2 - 3t + 4$, v = $4t^2 - 3t + 2$
c) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}, v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$
d) $u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, v = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

4 Marks

Q11. Determine whether or not the vectors u = (1, 1, 2), v = (2, 3, 1), w = (4, 5, 5) in R^3 are linearly dependent.

3 Marks

Q12. Determine whether or not each of the following lists of vectors R^3 is linearly independent:

a) $u_1 = (1, 2, 5)$, $u_2 = (1, 3, 1)$, $u_3 = (2, 5, 7)$, $u_4 = (3, 1, 4)$
b) $u = (1, 2, 5)$, $v = (2, 5, 1)$, $w = (1, 5, 2)$
c) $u = (1, 2, 3)$, $v = (0, 0, 0)$, $w = (1, 5, 6)$

3 Marks

Q13. Show that the functions $f(t) = \sin t$, $g(t) = \text{cost } t$, $h(t) = t$ from R into R are linearly independent.

3 Marks

Q14. Suppose the vectors u, v, w are linearly independent. Show that the vectors $u + v$, $u - v$, $u - 2v + w$ are also linearly independent.

3 Marks

Q15. Show that the vectors $u = (1 + i, 2i)$ and $w = (1, 1 + i)$ in C^2 are linearly independent over the complex field C but linearly independent over the real field R.

3 Marks

Q16. Determine whether or not each of the following form a basis of R^3:

a) (1, 1, 1), (1, 0, 1)
b) (1, 2, 3), (1, 3, 5), (1, 0, 1), (2, 3, 0)
c) (1, 1, 1), (1, 2, 3), (2, -1, 1)
d) (1, 1, 2), (1, 2, 50, (5, 3, 4)

3 Marks

Q17. Find a basis and dimension of the subspace W of R^3 where

a) W = {(a. b. c): a + b + c = 0}
b) W = {(a, b, c): (a = b= c)}

3 Marks

Q18. Let W be the subspace of R^4 spanned by the vectors

$u_1 = (1, -2, 5, -3)$, $u_2 = (2, 3, 1, -4)$, $u_3 = (3, 8, -3, -5)$

a) Find the basis and dimension of W
b) Extend the basis of W to a basis of R^4.

3 Marks

Q19. Let W be the subspace of R^5 spanned by $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$, $u_5 = (2, 7, 3, 3, 9)$. Find a subset of the vectors that form a basis of W.

3 Marks

Q20. Find the rank and basis of the row space of each of the following matrices:

a) $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

4 Marks

Q21. Show that U = W, where U and W are the following subspaces of R^3:

U = span(u_1, u_2, u_3) = span (1, 1, -1), (2, 3, -1), (3, 1, -5)

W = span(w_1, w_2, w_3) = span(1, -1, -3), (3, -2, -8), (2, 1, -3)

3 Marks

Q22. Find the dimension and a basis of the solution space W of each homogeneous system:

a) $x + 2y + 2z - s + 3t = 0$
$x + 2y + 3z + s + t = 0$
$3x + 6y + 8z + s + 5t = 0$

b) $x + 2y + z - 2t = 0$
$2x + 4y + 4z - 3t = 0$
$3x + 6y + 7z - 4t = 0$

c) $x + y + 2z = 0$
$2x + 3y + 3z = 0$
$X + 3y + 5z = 0$

3 Marks

Q23. Suppose $F: R^3 \rightarrow R^2$ is defined by $F(x, y, z) = (x + y + z, 2x - 3y + 4z)$. Show that F is linear.

3 Marks

Q24. Show that the following mappings are not linear:

a) $F: R^2 \rightarrow R^2$ defined by $F(x, y) = (xy, x)$
b) $F: R^2 \rightarrow R^3$ defined by $F(x, y) = (x + 3, 2y, x + y)$
c) $F: R^3 \rightarrow R^2$ defined by $F(x, y) = (|x|, y + z)$

3 Marks

Q25. Let V be the vector space of n-square real matrices. Let M be an arbitrary but fixed matrix in V. Let F: V → V be defined by F(A) = AM + MA, where A is any matrix in V. Show that F is linear.

3 Marks

Q26. Let F: $R^4 \rightarrow R^3$ be the linear mapping defined by

F(x, y, z, t) = (x – y + z + t, x + 2z – t, x + y + 3z – 3t)

Find a basis and the dimension of

a) The image of F
b) The kernel of F

3 Marks

Q27. Let F: $R^3 \rightarrow R^3$ be the linear mapping defined by

G(x, y, z, t) = (x + 2y - z, y + z, x + y – 2z)

Find a basis and the dimension of

a) The image of G
b) The kernel of G

3 Marks

Q28. Determine whether or not each of the following linear maps is non singular. If not, find a non zero vector v whose image is 0.

a) F: $R^2 \rightarrow R^2$ defined by F(x, y) = (x – y, x – 2y)
b) G: $R^2 \rightarrow R^2$ defined by G(x, y) = (2x – 4y, 3x – 6y)

3 Marks

Q29. Consider the linear mapping $F: R^2 \rightarrow R^2$ defined by

$F(x, y) = (3x + 4y, 2x - 5y)$ and the following bases of R^2:

$E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$ and $S = \{u_1, u_2\} = \{(1, 2), (2, 3)\}$

a) Find the matrix A representing F relative to the basis E.
b) Find the matrix B representing F relative to the basis S.

3 Marks

Q30. Consider the following linear operator G on R^2 and basis S:

$G(x, y) = (2x - 7y, 4x + 3y)$ and $S = \{u_1, u_2\} = \{(1, 3), (2, 5)\}$

a) Find the matrix representation $[G]_S$ of G relative to S.
b) Verify $[G]_S[v]_S = [G(v)]_S$ for the vector $v = (4, -3)$ in R^2

4 Marks

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