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J.N. Sharma
A.R. Vasishtha

CONTENTS

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Multiple Products

§ 1. Triple Products.

We know that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is itself a vector quantity. Therefore we can multiply it by another vector $\mathbf{c}$ both scalarly and vectorially. The product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called **scalar triple product**, which is a pure number. On the other hand the product $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is called **vector triple product**, which is again a vector quantity.

Note. Since $\mathbf{a} \cdot \mathbf{b}$ is a scalar quantity, therefore the products $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ and $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ are meaningless. Moreover in the product $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ we can omit the parentheses and we can simply write it as $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$. Obviously the product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ has meaning only if we regard it as the product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

§ 2. Scalar Triple Product.

[Meerut 1982; Kerala 74; Guru Nanak 81]

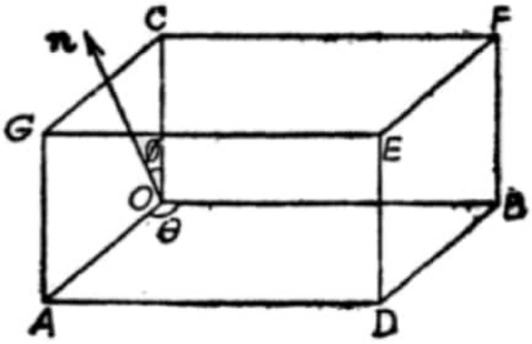
The scalar product of two vectors one of which is itself the vector product of two vectors is a scalar quantity called a "Scalar Triple Product". Thus if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the scalar triple product of these three vectors.

Since the scalar triple product involves both the signs of 'cross' and 'dot' therefore it is sometimes also called the **mixed product.**

***Geometrical Interpretation of Scalar Triple Product.**

[Meerut 1982; Kerala 74; Allahabad 80]

Let us consider a parallelopiped whose coterminous edges OA, OB, OC have the lengths and directions of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively. Let V be the volume of this parallelopiped. We shall regard V, as necessarily positive.



Let $\mathbf{a}\times\mathbf{b}=\mathbf{n}$. Then from our definition of vector product, the vector $\mathbf{n}$ is perpendicular to the face $OADB$, and its modulus n is the measure of the area of the parallelogram $OADB$. Also, by definition, the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ form a right handed triad.

Let ϕ denote the angle between the directions of the vectors $\overrightarrow{OC}$ and $\mathbf{n}$. Then the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ will form a right handed or a left handed triad according as ϕ is acute or obtuse.

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=|(\mathbf{a}\times\mathbf{b})|\,|\mathbf{c}|\cos\phi=|\mathbf{n}|\,|\mathbf{c}|\cos\phi$
$=$(area of the parallelogram $OADB$).($OC\cos\phi$)
$[\because\ |\mathbf{c}|=OC]$.

Now $OC\cos\phi$ will be positive or negative according as ϕ is acute or obtuse. Its absolute value will give us the length of the perpendicular from C to the plane of the parallelogram $OADB$.

Now the volume V of the parallelopiped=(Area of the parallelogram $OADB$)$\times$length of the perpendicular from C on this parallelogram. Therefore $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=+V$, if ϕ is acute *i.e.* if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right handed triad and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=-V$ if ϕ is obtuse *i.e.* if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a left handed triad.

Now we know that if the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right handed triad, then the vector triads $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}$ and $\mathbf{c}$, $\mathbf{a}$, $\mathbf{b}$ are also right handed. Hence each of the products $(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}$ and $(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$ will have the same value $+V$ or $-V$ according as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right handed or a left handed triad. Thus we conclude that in all cases $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}$.

Now $\quad \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ and $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

$$\begin{aligned}\therefore\quad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}&=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\\&=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})\\&=-(\mathbf{b}\times\mathbf{a})\cdot\mathbf{c}=-\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})=-(\mathbf{c}\times\mathbf{b})\cdot\mathbf{a}\\&=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})=-(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b}=-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}).\end{aligned}$$

From this we conclude that the *value of a scalar triple product depends on the cyclic order of the factors and is independent of the position of the dot and cross. These may be interchanged at pleasure. However, an anticyclic permutation of the three factors changes the value of the product in sign but not in magnitude.* **[Important]**

Notation. In view of the properties discussed above, the scalar triple product is usually written as $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=[\mathbf{abc}]$ or $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. *This notation takes into consideration only the cyclic order of the three vectors and disregards the unimportant positions of dot and cross.* Thus $[\mathbf{abc}]=[\mathbf{bca}]=[\mathbf{cab}]=-[\mathbf{cba}]$ etc.

The signs of dot and cross can be inserted at pleasure

i.e. $\quad [\mathbf{a}, \mathbf{b}, \mathbf{c}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \quad$ or $\quad =(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

Note 1. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ consitute an orthogonal right handed triad of unit vectors, then $[\mathbf{i}, \mathbf{j}, \mathbf{k}]=(\mathbf{i}\times\mathbf{j})\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{k}=1$.

Note 2. The scalar triple product $[\mathbf{abc}]$ is positive or negative according as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed or a left handed triad of vectors.

**§ 3. Distributive Law for Vector Product.

To prove that $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$, *where* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are any three vectors.* **[Delhi 1978; Allahabad 75]**

Let $\mathbf{r}\equiv\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}$ ...(1).

Now forming the scalar product of both sides of (1) with an arbitrary vector $\mathbf{d}$, we get

$$\mathbf{d}\cdot\mathbf{r}=\mathbf{d}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}] \quad ...(2)$$

or
$$\mathbf{d}\cdot\mathbf{r}=\mathbf{d}\cdot[\mathbf{a}\times(\mathbf{b}+\mathbf{c})]-\mathbf{d}\cdot(\mathbf{a}\times\mathbf{b})-\mathbf{d}\cdot(\mathbf{a}\times\mathbf{c})$$

[Since scalar product is distributive]

Now in a scalar triple product the positions of dot and cross càn be interchanged without affecting its value. Therefore from (2), we get

$$\mathbf{d}\cdot\mathbf{r}=(\mathbf{d}\times\mathbf{a})\cdot(\mathbf{b}+\mathbf{c})-(\mathbf{d}\times\mathbf{a})\cdot\mathbf{b}-(\mathbf{d}\times\mathbf{a})\cdot\mathbf{c}$$
$$=(\mathbf{d}\times\mathbf{a})\cdot\mathbf{b}+(\mathbf{d}\times\mathbf{a})\cdot\mathbf{c}-(\mathbf{d}\times\mathbf{a})\cdot\mathbf{b}-(\mathbf{d}\times\mathbf{a})\cdot\mathbf{c}$$

[Since scalar product is distributive]

$$=0.$$

Therefore either $\mathbf{d}=\mathbf{0}$, or $\mathbf{r}=\mathbf{0}$ or $\mathbf{d}$ is perpendicular to $\mathbf{r}$. But the vector $\mathbf{d}$ is arbitrary. Therefore we can take it to be non-zero and not perpendicular to $\mathbf{r}$.

Hence $\mathbf{r}=\mathbf{0}$ *i.e.* $\mathbf{a}\times(\mathbf{b}+\mathbf{c})-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}=\mathbf{0}$

i.e.
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}.$$

§ 4. Properties of Scalar triple product.

(*i*) *The value of a scalar triple product, if two of its vectors are equal, is zero.* **[Agra 1973]**

We have $[\mathbf{aab}]=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})$.

Now $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$.

Therefore $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$.

(*ii*) *The value of a scalar triple product, if two of its vectors are parallel, is zero.*

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that $\mathbf{a}$ and $\mathbf{b}$ are parallel *i.e.* $\mathbf{b}=t\mathbf{a}$, where t is some scalar.

Now $[\mathbf{abc}]=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=(\mathbf{a}\times t\mathbf{a})\cdot\mathbf{c}=t\,(\mathbf{a}\times\mathbf{a})\cdot\mathbf{c}$
$$=t\,(\mathbf{0}\cdot\mathbf{c}) \qquad [\because\ \mathbf{a}\times\mathbf{a}=\mathbf{0}]$$
$$=0.$$

**(*iii*) *The necessary and sufficient condition that three non-parallel and non-zero vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be coplanar is that* $[\mathbf{abc}]=0$.

[Lucknow 1977; Rohilkhand 79; Meerut 83]

Let **a**, **b**, **c** be three coplanar vectors. Now $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane of **a** and **b**. Since **a**, **b**, **c** are coplanar, therefore $\mathbf{a}\times\mathbf{b}$ is also perpendicular to **c**. Now the dot product of two perpendicular vectors is equal to zero. Hence $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$

i.e. $[\mathbf{abc}]=0$. **Therefore the condition is necessary.**

The condition is also sufficient. Because if $[\mathbf{abc}]=0$ *i.e.*, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$, then **c** is perpendicular to $\mathbf{a}\times\mathbf{b}$. But $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to the plane of **a** and **b**. Since **c** is perpendicular to $\mathbf{a}\times\mathbf{b}$, therefore **c** is parallel to the plane of **a** and **b**.

Hence **a**, **b**, **c** are coplanar.

(*iv*) *Since the distributive law holds for both scalar and vector products, it holds also for the scalar triple product.*

Thus $[\mathbf{a},\ \mathbf{b}+\mathbf{d},\ \mathbf{c}+\mathbf{r}]=[\mathbf{abc}]+[\mathbf{abr}]+[\mathbf{adc}]+[\mathbf{adr}]$, the cyclic order of the factors being maintained in each term.

***§ 5. To express the value of the scalar triple product [abc] in terms of rectangular components of the vectors.**

[Allahabad 1980]

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},\ \mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

Now $\mathbf{b}\times\mathbf{c}=(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\times(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=(b_2c_3-b_3c_2)\mathbf{i}-(b_1c_3-b_3c_1)\,\mathbf{j}+(b_1c_2-b_2c_1)\,\mathbf{k}.$$

$$\therefore\quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot[(b_2c_3-b_3c_2)\mathbf{i}-(b_1c_3-b_3c_1)\mathbf{j}+(b_1c_2-b_2c_1)\,\mathbf{k}]$$

$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$

$$[\because\ \mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1 \text{ and } \mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0]$$

$$\therefore\quad [\mathbf{abc}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \qquad \ldots(1)$$

Also $$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

showing that the value of a scalar triple product is independent of the positions of dot and cross.

Note. If OA, OB, OC be three concurrent edges of a parallelopiped and if (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) be the rectangular coordinates of A, B, C referred to O as origin, then the determinant (1) gives the volume of that parallelopiped.

§ 6. To express the scalar triple product [a, b, c] in terms of any three non-coplanar vectors l, m, n. **[Agra 1988]**

Let $\mathbf{a}=a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n}$,

$\mathbf{b}=b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n}$,

and $\mathbf{c}=c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n}$.

Now
$$\mathbf{b}\times\mathbf{c}=(b_1\mathbf{l}+b_2\mathbf{m}+b_3\mathbf{n})\times(c_1\mathbf{l}+c_2\mathbf{m}+c_3\mathbf{n})$$
$$=b_1c_1\mathbf{l}\times\mathbf{l}+b_1c_2\mathbf{l}\times\mathbf{m}+b_1c_3\mathbf{l}\times\mathbf{n}+b_2c_1\mathbf{m}\times\mathbf{l}+b_2c_2\mathbf{m}\times\mathbf{m}$$
$$+b_2c_3\mathbf{m}\times\mathbf{n}+b_3c_1\mathbf{n}\times\mathbf{l}+b_3c_2\mathbf{n}\times\mathbf{m}+b_3c_3\mathbf{n}\times\mathbf{n}$$
$$=(b_2c_3-b_3c_2)\,\mathbf{m}\times\mathbf{n}-(b_1c_3-b_3c_1)\,\mathbf{n}\times\mathbf{l}$$
$$+(b_1c_2-b_2c_1)\,\mathbf{l}\times\mathbf{m}$$
$$[\because\ \mathbf{l}\times\mathbf{l}=\mathbf{0}\text{ and }\mathbf{l}\times\mathbf{m}=-\mathbf{m}\times\mathbf{l}\text{ etc.}]$$

$$\therefore\ \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(a_1\mathbf{l}+a_2\mathbf{m}+a_3\mathbf{n})\cdot[(b_2c_3-b_3c_2)\,\mathbf{m}\times\mathbf{n}$$
$$-(b_1c_3-b_3c_1)\mathbf{n}\times\mathbf{l}+(b_1c_2-b_2c_1)\,\mathbf{l}\times\mathbf{m}]$$
$$=a_1(b_2c_3-b_3c_2)[\mathbf{lmn}]-a_2(b_1c_3-b_3c_1)[\mathbf{lmn}]$$
$$+a_3(b_1c_2-b_2c_1)[\mathbf{lmn}].$$

Since $[\mathbf{lmn}]=[\mathbf{mnl}]=[\mathbf{nlm}]$ and all the scalar triple products of the type $[\mathbf{lml}]$ in which two vectors are equal vanish,

$$\therefore\ [\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\mathbf{lmn}].$$

Note. Since $[\mathbf{i}, \mathbf{j}, \mathbf{k}]=1$, therefore § 5 is particular case of § 6.

Solved Examples

Ex. 1. *Define scalar triple product of three vectors* **a, b, c** *and interpret the same geometrically.* **[Meerut 1982]**

Sol. Scalar triple product. Definition. The scalar products $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ are called scalar triple products of the vectors **a, b, c**.

Geometrical interpretation of scalar triple product. Geometrically the scalar triple product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ represents the volume of a parallelopiped whose three coterminous edges are represented by the vectors **a, b** and **c**.

For complete discussion refer § 2.

Ex. 2. *Define scalar triple product of* **a**, **b**, **c**. *Prove that the value of the scalar triple product of* **a**, **b**, **c** *remains unchanged if the cyclic order of the vectors is maintained i.e., prove that*

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$ [Madras 1975]

Sol. For definition of scalar triple product see solved example 1 above.

Let $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k},\ \mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k},$
$\mathbf{c}=c_1\,\mathbf{i}+c_2\,\mathbf{j}+c_3\,\mathbf{k}.$

Proceeding as in § 5, we have

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)$$ [Do it here]

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=-\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ interchanging } R_1 \text{ and } R_2$$

$$=\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}). \qquad \ldots(1)$$

Again
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=-\begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix},$$
interchanging R_1 and R_3

$$=\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \text{ interchanging } R_2 \text{ and } R_3$$

$$=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}). \qquad \ldots(2)$$

From (1) and (2), we have

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

Ex. 4. *Prove that* $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}.$ [Meerut 1983, 86]

Sol. We have $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$= \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ [$\because$ scalar triple product is unchanged if the cyclic order of the vectors is maintained]

$= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ [$\because$ dot product of two vectors is commutative]

$= \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

Ex. 5. *Show that* $\mathbf{i} \cdot \mathbf{j} \times \mathbf{k} = 1$.

Sol. We have $\mathbf{i} \cdot \mathbf{j} \times \mathbf{k} = \mathbf{i} \cdot (\mathbf{j} \times \mathbf{k})$

$= \mathbf{i} \cdot \mathbf{i}$ [$\because$ $\mathbf{j} \times \mathbf{k} = \mathbf{i}$]

$= 1$.

Ex. 6. *Show that* $[\lambda \mathbf{a} + \mu \mathbf{b}, \mathbf{c}, \mathbf{d}] = \lambda [\mathbf{a}, \mathbf{c}, \mathbf{d}] + \mu [\mathbf{b}, \mathbf{c}, \mathbf{d}]$.

Sol. We have $[\lambda \mathbf{a} + \mu \mathbf{b}, \mathbf{c}, \mathbf{d}]$

$= (\lambda \mathbf{a} + \mu \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

$= \lambda \mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}) + \mu \mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})$, by distributive law for dot product

$= \lambda [\mathbf{a}, \mathbf{c}, \mathbf{d}] + \mu [\mathbf{b}, \mathbf{c}, \mathbf{d}]$.

Ex. 7. *Prove that* $[\mathbf{i} - \mathbf{j}, \mathbf{j} - \mathbf{k}, \mathbf{k} - \mathbf{i}] = 0$.

Sol. We have $[\mathbf{i} - \mathbf{j}, \mathbf{j} - \mathbf{k}, \mathbf{k} - \mathbf{i}]$

$= (\mathbf{i} - \mathbf{j}) \cdot \{(\mathbf{j} - \mathbf{k}) \times (\mathbf{k} - \mathbf{i})\}$

$= (\mathbf{i} - \mathbf{j}) \cdot (\mathbf{j} \times \mathbf{k} - \mathbf{j} \times \mathbf{i} - \mathbf{k} \times \mathbf{k} + \mathbf{k} \times \mathbf{i})$

$= (\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + \mathbf{k} + \mathbf{j})$

$= \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{k} + \mathbf{i} \cdot \mathbf{j} - \mathbf{j} \cdot \mathbf{i} - \mathbf{j} \cdot \mathbf{k} - \mathbf{j} \cdot \mathbf{j}$

$= 1 + 0 + 0 - 0 - 0 - 1 = 0$.

Ex. 8. *Find the volume of the parallelopiped whose edges are represented by*

(*i*) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

(*ii*) $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{j} + \mathbf{k}$.

Alternative form of the question. *Find the volume of the parallelopiped whose three coterminous edges are the vectors* (2, −3, 4), (1, 2, −1) *and* (3, −1, 2). **(Meerut 1991S)**

Sol. (i) The required volume of the parallelopiped is equal to the absolute value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

We have $[\mathbf{abc}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$=2(4-1)+3(2+3)+4(-1-6)$,

expanding the determinant along R_1

$=2.3+3.5+4.(-7)=6+15-28=-7$.

Neglecting the negative sign, we get the volume of the parallelopiped$=7$ cubic units.

(ii) The required volume$=\begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$

$=\begin{vmatrix} 1 & -2 & 3 \\ 0 & 5 & -7 \\ 0 & 1 & 1 \end{vmatrix}$, by R_2-2R_1

$=1.(5+7)=12$ cubic units.

Ex. 9. *Show that the vectors* $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $-2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{i}-3\mathbf{j}+5\mathbf{k}$ *are coplanar.*

Sol. Let $\mathbf{a}=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{c}=\mathbf{i}-3\mathbf{j}+5\mathbf{k}$.

The vectors **a**, **b**, **c** are coplanar if their scalar triple product is zero, otherwise they are non-coplanar.

We have $[\mathbf{abc}]=\begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$

$=\begin{vmatrix} 1 & -2 & 3 \\ 0 & -1 & 2 \\ 0 & -2 & 2 \end{vmatrix}$, by R_2+2R_1 and R_3-R_1

$=1.(-2+2)=1.0=0$.

Hence the given vectors are coplanar.

***Ex. 10.** *Find the constant p such that the vectors* $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}+2\mathbf{j}-3\mathbf{k}$, $\mathbf{c}=3\mathbf{i}+p\mathbf{j}+5\mathbf{k}$ *are coplanar.* **[Agra 1978]**

Sol. If the vectors **a**, **b**, **c** are coplanar, then we should have $[\mathbf{abc}]=0$.

Now $[\mathbf{abc}]=\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix}$

$$=2\,(10+3p)+1\,(5+9)+1\,(p-6)=7p+28.$$

$\therefore$ [**abc**] will be zero if $7p+28=0$

or $\qquad p=-4.$

Hence for the given vectors to be coplanar, we should have $p=-4$.

****Ex. 11.** *Prove that the four points* $4\mathbf{i}+5\mathbf{j}+\mathbf{k}$, $-(\mathbf{j}+\mathbf{k})$, $(3\mathbf{i}+9\mathbf{j}+4\mathbf{k})$ *and* $4(-\mathbf{i}+\mathbf{j}+\mathbf{k})$ *are coplanar.*

[Meerut 1989, 90P; Kanpur 79; Delhi 77]

Sol. Let A, B, C, D be the four given points whose position vectors referred to some origin O are

$$4\mathbf{i}+5\mathbf{j}+\mathbf{k},\ -(\mathbf{j}+\mathbf{k}),\ (3\mathbf{i}+9\mathbf{j}+4\mathbf{k}) \text{ and } 4(-\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

If the four points A, B, C, D are coplanar, then the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ should also be coplanar.

We have $\overrightarrow{AB}=$ position vector of $B-$position vector of A

$$=-(\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-4\mathbf{i}-6\mathbf{j}-2\mathbf{k}=\mathbf{a} \text{ (say)}.$$

Similarly $\overrightarrow{AC}=(3\mathbf{i}+9\mathbf{j}+4\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=-\mathbf{i}+4\mathbf{j}+3\mathbf{k}$
$=\mathbf{b}$ (say),

and $\overrightarrow{AD}=4(-\mathbf{i}+\mathbf{j}+\mathbf{k})-(4\mathbf{i}+5\mathbf{j}+\mathbf{k})=8\mathbf{i}-\mathbf{j}+3\mathbf{k}=\mathbf{c}$ (say).

Now the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ will be coplanar if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$.

$$\text{Now } [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$=-4(12+3)+6(-3+24)-2(1+32)$$
$$=-60+126-66=0.$$

$\therefore$ the points A, B, C, D are coplanar.

****Ex. 12.** *Show that the four points* $-\mathbf{a}+4\mathbf{b}-3\mathbf{c}$, $3\mathbf{a}+2\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+8\mathbf{b}-5\mathbf{c}$ *and* $-3\mathbf{a}+2\mathbf{b}+\mathbf{c}$ *are coplanar.* **[Meerut 1981]**

Sol. Let A, B, C and D be the points whose position vectors are respectively $-\mathbf{a}+4\mathbf{b}-3\mathbf{c}$, $3\mathbf{a}+2\mathbf{b}-5\mathbf{c}$, $-3\mathbf{a}+8\mathbf{b}-5\mathbf{c}$ and $-3\mathbf{a}+2\mathbf{b}+\mathbf{c}$.

We have $\overrightarrow{AB}=$position vector of B—position vector of A

$$=(3\mathbf{a}+2\mathbf{b}-5\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})=4\mathbf{a}-2\mathbf{b}-2\mathbf{c},$$

$\overrightarrow{AC}$ = position vector of C − position vector of A

$=(-3\mathbf{a}+8\mathbf{b}-5\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})=-2\mathbf{a}+4\mathbf{b}-2\mathbf{c}$,

and $\overrightarrow{AD}=(-3\mathbf{a}+2\mathbf{b}+\mathbf{c})-(-\mathbf{a}+4\mathbf{b}-3\mathbf{c})=-2\mathbf{a}-2\mathbf{b}+4\mathbf{c}$.

Now the scalar triple product of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$$=[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}]=\begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \quad \text{[Refer § 6]}$$

$$=\{4(16-4)+2(-8-4)-2(4+8)\}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$=(48-24-24)[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0.$$

Since the scalar triple product of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ is zero, therefore these vectors are coplanar. Hence the points A, B, C and D are coplanar.

Ex. 13. *If* **a, b, c** *are the position vectors of* A, B, C *prove that* $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$ *is a vector perpendicular to the plane of* ABC.

Sol. We have $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$, $\overrightarrow{BC}=\mathbf{c}-\mathbf{b}$ and $\overrightarrow{CA}=\mathbf{a}-\mathbf{c}$.
Let $\mathbf{d}=\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$.

Now $\mathbf{d}\cdot\overrightarrow{AB}=\mathbf{d}\cdot(\mathbf{b}-\mathbf{a})=(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})$

$=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}-(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}+(\mathbf{b}\times\mathbf{c})\cdot\mathbf{b}-(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}+(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}-(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}$

$=[\mathbf{abb}]-[\mathbf{aba}]+[\mathbf{bcb}]-[\mathbf{bca}]+[\mathbf{cab}]-[\mathbf{caa}]$

$=-[\mathbf{bca}]+[\mathbf{cab}]$, since $[\mathbf{abb}]=0$ etc.

$=-[\mathbf{bca}]+[\mathbf{bca}]$, since $[\mathbf{cab}]=[\mathbf{bca}]$

$=0$.

Therefore vector $\mathbf{d}$ is perpendicular to $\overrightarrow{AB}$. Similarly, we can show that $\mathbf{d}$ is perpendicular to $\overrightarrow{BC}$.

Now since $\mathbf{d}$ is perpendicular to two lines in the plane ABC, hence it is perpendicular to the plane ABC.

****Ex. 14.** *Prove that* $[\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}]=2[\mathbf{abc}]$.

[Meerut 1984, 86P, 88P, 90; Rohilkhand 76; Agra 80]

Sol. L.H.S. $=(\mathbf{a}+\mathbf{b})\cdot[(\mathbf{b}+\mathbf{c})\times(\mathbf{c}+\mathbf{a})]$

$=(\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}]$

$=(\mathbf{a}+\mathbf{b})\cdot[\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{b}\times\mathbf{a}]$,

since $\mathbf{c}\times\mathbf{c}=\mathbf{0}$

$$=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})+\mathbf{a}\cdot(\mathbf{b}\times\mathbf{a})$$
$$+\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})+\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})+\mathbf{b}\cdot(\mathbf{b}\times\mathbf{a})$$
$$=[\mathbf{abc}]+[\mathbf{aca}]+[\mathbf{aba}]+[\mathbf{bbc}]+[\mathbf{bca}]+[\mathbf{bba}]$$
$$=[\mathbf{abc}]+[\mathbf{bca}],$$

since all the scalar triple products in which two vectors are equal vanish.

But $[\mathbf{abc}]=[\mathbf{bca}]$.

Hence the L.H.S. $=2\,[\mathbf{abc}]$.

**Ex. 15. *Prove that*

$$[\mathbf{lmn}]\,[\mathbf{abc}]=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c}\end{vmatrix}:$$

[Lucknow 1981; Meerut 82, 86; Agra 87; Rohilkhand 80]

Sol. Let $\mathbf{l}=l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k}$, $\mathbf{m}=m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k}$, $\mathbf{n}=n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$; $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

Now L.H.S. $=[\mathbf{lmn}]\,[\mathbf{abc}]$

$$=\begin{vmatrix}l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\begin{vmatrix}l_1a_1+l_2a_2+l_3a_3 & l_1b_1+l_2b_2+l_3b_3 & l_1c_1+l_2c_2+l_3c_3\\ m_1a_1+m_2a_2+m_3a_3 & m_1b_1+m_2b_2+m_3b_3 & m_1c_1+m_2c_2+m_3c_3\\ n_1a_1+n_2a_2+n_3a_3 & n_1b_1+n_2b_2+n_3b_3 & n_1c_1+n_2c_2+n_3c_3\end{vmatrix},$$

by the rule for the multiplication of determinants of the same order.

Now $\mathbf{l}\cdot\mathbf{a}=(l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k})\cdot(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=l_1a_1+l_2a_2+l_3a_3$, etc.

$$\text{Hence the L.H.S.}=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\cdot\mathbf{c}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\cdot\mathbf{c}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\cdot\mathbf{c}\end{vmatrix}.$$

*Ex. 16. *Prove that if* **l, m, n** *be three non-coplanar vectors, then*

$$[\mathbf{lmn}]\,(\mathbf{a}\times\mathbf{b})=\begin{vmatrix}\mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l}\\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m}\\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n}\end{vmatrix}.$$

[Meerut 1982, 91P]

Sol. Let $\mathbf{l}=l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k}$, $\mathbf{m}=m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k}$, $\mathbf{n}=n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k}$,

and $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$.

Now $$[\mathbf{lmn}]=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

and $$(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

$$\therefore \quad [\mathbf{lmn}]\,(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k} & l_1a_1+l_2a_2+l_3a_3 & l_1b_1+l_2b_2+l_3b_3 \\ m_1\mathbf{i}+m_2\mathbf{j}+m_3\mathbf{k} & m_1a_1+m_2a_2+m_3a_3 & m_1b_1+m_2b_2+m_3b_3 \\ n_1\mathbf{i}+n_2\mathbf{j}+n_3\mathbf{k} & n_1a_1+n_2a_2+n_3a_3 & n_1b_1+n_2b_2+n_3b_3 \end{vmatrix}.$$

Now $\mathbf{l}\cdot\mathbf{a}=(l_1\mathbf{i}+l_2\mathbf{j}+l_3\mathbf{k})\cdot(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=l_1a_1+l_2a_2+l_3a_3$ etc.

$$\therefore \quad [\mathbf{lmn}]\,(\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{l} & \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} \\ \mathbf{m} & \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} \\ \mathbf{n} & \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} \end{vmatrix}=\begin{vmatrix} \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l} \\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m} \\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n} \end{vmatrix}.$$

Ex. 17. *Show that the vectors* $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$, $\mathbf{a}+\mathbf{b}-2\mathbf{c}$ *and* $\mathbf{a}+\mathbf{b}-3\mathbf{c}$ *are non-coplanar where* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are non-coplanar vectors.*

Sol. Let $\mathbf{A}=2\mathbf{a}-\mathbf{b}+3\mathbf{c}$, $\mathbf{B}=\mathbf{a}+\mathbf{b}-2\mathbf{c}$, $\mathbf{C}=\mathbf{a}+\mathbf{b}-3\mathbf{c}$.

The vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are non-coplanar if their scalar triple product is not equal to zero. We have

$$[\mathbf{A}\ \mathbf{B}\ \mathbf{C}]=\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \qquad [\text{See § 6}]$$

$$=\{2\,(-3+2)+1\,(-3+2)+3\,(1-1)\}\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$=(-2-1)\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=-3\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}].$$

Since **a**, **b**, **c** are non-coplanar, therefore $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$.

Hence $[\mathbf{A}\ \mathbf{B}\ \mathbf{C}] \neq 0$ and so the vectors **A**, **B**, **C** are non-coplanar.

Ex. 18. *Prove that the four points* $6\mathbf{a}-4\mathbf{b}+10\mathbf{c}$, $-5\mathbf{a}+3\mathbf{b}-10\mathbf{c}$, $4\mathbf{a}-6\mathbf{b}-10\mathbf{c}$ *and* $2\mathbf{b}+10\mathbf{c}$ *are coplanar.* **[Meerut 1987]**

Sol. Let A, B, C and D be the points whose position vectors are respectively $6\mathbf{a}-4\mathbf{b}+10\mathbf{c}$, $-5\mathbf{a}+3\mathbf{b}-10\mathbf{c}$, $4\mathbf{a}-6\mathbf{b}-10\mathbf{c}$ and $2\mathbf{b}+10\mathbf{c}$.

We have $\overrightarrow{AB}=$ position vector of $B-$ position vector of A

$$=-5\mathbf{a}+3\mathbf{b}-10\mathbf{c}-6\mathbf{a}+4\mathbf{b}-10\mathbf{c}=-11\mathbf{a}+7\mathbf{b}-20\mathbf{c},$$

$\overrightarrow{AC}=$ position vector of $C-$ position vector of A

$$=4\mathbf{a}-6\mathbf{b}-10\mathbf{c}-6\mathbf{a}+4\mathbf{b}-10\mathbf{c}=-2\mathbf{a}-2\mathbf{b}-20\mathbf{c},$$

and $\overrightarrow{AD}=2\mathbf{b}+10\mathbf{c}-6\mathbf{a}+4\mathbf{b}-10\mathbf{c}=-6\mathbf{a}+6\mathbf{b}$.

Now the scalar triple product of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$$=[\overrightarrow{AB},\ \overrightarrow{AC},\ \overrightarrow{AD}]=\begin{vmatrix} -11 & 7 & -20 \\ -2 & -2 & -20 \\ -6 & 6 & 0 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$=\{-6\,(-140-40)-6\,(220-40)\}\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, expanding the determinant along the third row

$$=-6\,\{-180+180\}\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0.$$

Since the scalar triple product of the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ is zero, therefore these vectors are coplanar. Hence the points A, B, C and D are coplanar.

Ex. 19. *Find p in order that the points* $A\,(3, 2, 1)$, $B\,(4, p, 5)$, $C\,(4, 2, -2)$ *and* $D\,(6, 5, -1)$ *may be coplanar.* **[Meerut 1991]**

Sol. We have $\overrightarrow{AB}=$ position vector of $B-$ position vector of A

$$=(4-3,\ p-2,\ 5-1)=(1,\ p-2,\ 4),$$

$$\overrightarrow{AC}=(4-3,\ 2-2,\ -2-1)=(1,\ 0,\ -3),$$

and $$\overrightarrow{AD}=(3,\ 3,\ -2).$$

The points A, B, C and D are coplanar if the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar *i.e.*, if $[\overrightarrow{AB},\ \overrightarrow{AC},\ \overrightarrow{AD}]=0$ *i.e.*, if

$$\begin{vmatrix} 1 & p-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}=0$$

or $$\begin{vmatrix} 1 & p-2 & 7 \\ 1 & 0 & 0 \\ 3 & 3 & 7 \end{vmatrix}=0, \text{ applying } C_3+3C_1$$

or $$-1\cdot\begin{vmatrix} p-2 & 7 \\ 3 & 7 \end{vmatrix}=0, \text{ expanding the determinant along the second row}$$

or $$-(7p-14-21)=0$$

or $$7p-35=0 \text{ or } 7p=35 \text{ or } p=5.$$

Ex. 20. *Show that the vectors* **a, b, c** *are coplanar if* **b**+**c**, **c**+**a**, **a**+**b** *are coplanar.* **[Kanpur 1983]**

Sol. Let **A**=**b**+**c**, **B**=**c**+**a** and **C**=**a**+**b**.

Then proceeding as in solved example 14, we have

[**A B C**]=2 [**a b c**]. [Do it here]

$\therefore$ [**a b c**]=$\frac{1}{2}$ [**A B C**]. ...(1)

Now if the vectors **A**, **B** and **C** are coplanar then [**A B C**]=0. So from (1), we have [**a b c**]=0 which means that the vectors **a**, **b** and **c** are also coplanar.

Hence the vectors **a, b, c** are coplanar if **b**+**c**, **c**+**a**, **a**+**b** are coplanar.

Ex. 21. *Prove that the four points with position vectors* **a, b, c, d** *are coplanar if and only if*

[**b, c, d**]+[**c, a, d**]+[**a, b, d**]=[**a, b, c**].

[Meerut 1992; Rohilkhand 90: Agra 88]

Sol. Let A, B, C and D be the four points whose position vectors are **a, b, c** and **d** respectively.

We have $\vec{AB}$=position vector of B−position vector of A

=**b**−**a**,

$\vec{AC}$=**c**−**a** and $\vec{AD}$=**d**−**a**.

Now the scalar triple product of the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$

$$=[\vec{AB}, \vec{AC}, \vec{AD}]=\vec{AB}\cdot(\vec{AC}\times\vec{AD})$$

$$= (\mathbf{b}-\mathbf{a})\cdot\{(\mathbf{c}-\mathbf{a})\times(\mathbf{d}-\mathbf{a})\}$$
$$= (\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}\times\mathbf{d}-\mathbf{c}\times\mathbf{a}-\mathbf{a}\times\mathbf{d}+\mathbf{a}\times\mathbf{a})$$
$$= (\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}\times\mathbf{d}-\mathbf{c}\times\mathbf{a}-\mathbf{a}\times\mathbf{d}) \qquad [\because \ \mathbf{a}\times\mathbf{a}=\mathbf{0}]$$
$$= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})-\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{d})-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{d})+\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a}) +\mathbf{a}\cdot(\mathbf{a}\times\mathbf{d})$$
$$= [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]-[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]-[\mathbf{b}\ \mathbf{a}\ \mathbf{d}]-[\mathbf{a}\ \mathbf{c}\ \mathbf{d}] \qquad [\because \ \mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})=0=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{d})]$$
$$= [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}].$$

Now the points A, B, C and D are coplanar if and only if the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar

or if and only if $[\vec{AB},\ \vec{AC},\ \vec{AD}]=0$

or if and only if $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=0$

or if and only if $[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

**§ 7. Vector triple product.

The vector product of two vectors one of which is itself the vector product of two vectors is a vector quantity called a "Vector Triple Product". Thus if **a**, **b** and **c** be three vectors, the products of the form $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ etc. are called **"Vector Triple products"**.

****Theorem.** *To prove that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\ \mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\ \mathbf{c}$.

[Meerut 1975, 79, 80, 84, 86P; Lucknow 81; Gorakhpur 88; Rohilkhand 90; Allahabad 79; Delhi 80]

Let $\mathbf{r}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and $\mathbf{b}\times\mathbf{c}=\mathbf{d}$.

Since $\mathbf{b}\times\mathbf{c}=\mathbf{d}$, therefore **d** is a vector perpendicular to the plane containing **b** and **c**. Also $\mathbf{r}=\mathbf{a}\times\mathbf{d}$. Therefore **r** is a vector perpendicular to both **a** and **d**. Now the vector **r** is perpendicular to the vector **d**, whereas the vector **d** is perpendicular to the plane containing **b** and **c**. Therefore the vector **r** must lie in the plane containing **b** and **c** Hence the vector **r** can be expressed linearly in terms of **b** and **c** in the form

$$\mathbf{r}=l\mathbf{b}+m\mathbf{c} \qquad \ldots(1), \text{ where } l \text{ and } m \text{ are scalars.}$$

Since **r** is perpendicular to **a**, therefore $\mathbf{r}\cdot\mathbf{a}=0$.

$\therefore \ (l\mathbf{b}+m\mathbf{c})\cdot\mathbf{a}=0$ or $l(\mathbf{b}\cdot\mathbf{a})+m(\mathbf{c}\cdot\mathbf{a})=0$.

$$\therefore \ \frac{l}{\mathbf{c}\cdot\mathbf{a}}=\frac{-m}{\mathbf{b}\cdot\mathbf{a}}=\lambda \text{ (say)}.$$

Putting the values of l and m in (1), we get

$$\mathbf{r}=\lambda(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-\lambda(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}$$

$$=\lambda[(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}]. \quad \ldots(2)$$

Now we are to find the value of λ.

Consider unit vectors $\mathbf{j}$ and $\mathbf{k}$, the first parallel to $\mathbf{b}$ and the second perpendicular to it in the plane containing $\mathbf{b}$ and $\mathbf{c}$. Then we may write

$$\mathbf{b}=b_2\mathbf{j}$$

and

$$\mathbf{c}=c_2\mathbf{j}+c_3\mathbf{k}.$$

In terms of $\mathbf{j}$ and $\mathbf{k}$ and the other unit vector $\mathbf{i}$ of the right handed system, the remaining vector $\mathbf{a}$ may be written as

$$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}.$$

Now $\mathbf{b}\times\mathbf{c}=b_2\mathbf{j}\times(c_2\mathbf{j}+c_3\mathbf{k})=b_2c_2\mathbf{j}\times\mathbf{j}+b_2c_3\mathbf{j}\times\mathbf{k}$

$$=b_2c_3\mathbf{i} \qquad [\because\ \mathbf{j}\times\mathbf{j}=\mathbf{0} \text{ and } \mathbf{j}\times\mathbf{k}=\mathbf{i}]$$

$$\therefore\quad \mathbf{r}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_2c_3\mathbf{i})$$

$$=a_1b_2c_3\mathbf{i}\times\mathbf{i}+a_2b_2c_3\mathbf{j}\times\mathbf{i}+a_3b_2c_3\mathbf{k}\times\mathbf{i}$$

$$=a_3b_2c_3\mathbf{j}-a_2b_2c_3\mathbf{k} \quad [\because\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} \text{ and } \mathbf{k}\times\mathbf{i}=\mathbf{j}] \quad \ldots(3)$$

Also $\mathbf{r}=\lambda[(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}]=\lambda[\{(c_2\mathbf{j}+c_3\mathbf{k})\cdot(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\}\,b_2\mathbf{j}$

$$-(b_2\mathbf{j})\cdot(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\,(c_2\mathbf{j}+c_3\mathbf{k})\}$$

$$=\lambda\,[c_2a_2b_2\mathbf{j}+c_3a_3b_2\mathbf{j}+b_2a_2c_2\mathbf{j}+b_2a_2c_3\mathbf{k}]$$

$$[\because\ \mathbf{i}\cdot\mathbf{i}=1,\ \mathbf{i}\cdot\mathbf{j}=0 \text{ etc.}]$$

$$=\lambda\,[a_3b_2c_3\mathbf{j}-a_2b_2c_3\mathbf{k}]. \quad \ldots(4)$$

Now from (3) and (4) we conclude that $\lambda=1$.

Hence $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}$

$$=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \qquad [\because\ \mathbf{c}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{c}]$$

Corollary. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=-[\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]$

$$=-[(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}]$$

$$=(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}.$$

Rule to remember $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$. It is a vector to be expressed linearly in terms of $\mathbf{b}$ and $\mathbf{c}$ which are the vectors within the brackets. Also

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=$[Dot product of $\mathbf{a}$ and $\mathbf{c}$] $\mathbf{b}$
$-$[Dot product of $\mathbf{a}$ and $\mathbf{b}$] $\mathbf{c}$.

Similarly we may remember $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

§ 8. Vector triple product is not associative.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors, then $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ gives a vector which lies in the plane of $\mathbf{b}$ and $\mathbf{c}$ and which is perpendicular to $\mathbf{a}$. Moreover $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ gives a vector which lies in the plane of $\mathbf{a}$ and $\mathbf{b}$ and which is perpendicular to $\mathbf{c}$. Hence, in general, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. Thus in the case of vector triple product the position of brackets cannot be, in general, changed without altering the value of the product.

Solved Examples

***Ex. 1.** *Prove that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\mathbf{0}.$

[Meerut 1975, 80, 82, 86S; Kanpur 80; Allahabad 75; Gorakhpur 88; Delhi 81; Agra 79]

Sol. We have $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$,

$$\mathbf{b}\times(\mathbf{c}\times\mathbf{a})=(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a},$$

and
$$\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}.$$

Adding these three expressions, we get

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$$
$$=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}$$
$$=\mathbf{0}. \qquad [\because\ \mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a},\ \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a},\ \mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{b}].$$

Ex. 2. *Show that the vectors* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ *are coplanar* **[Karnatak 1971]**

Sol. Let $\mathbf{r}_1=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, $\mathbf{r}_2=\mathbf{b}\times(\mathbf{c}\times\mathbf{a})$, $\mathbf{r}_3=\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$.

Now first prove that $\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3=\mathbf{0}$ as we have done in the previous exercise.

Since there exists a linear relation between the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ therefore any of these vectors can be expressed as a linear combination of the other two. Hence these three vectors are coplanar.

Ex. 3. *Evaluate* $(\mathbf{b}\times\mathbf{c})\times\mathbf{a}$ *where* $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-5\mathbf{k}$,

$\mathbf{b}=-\mathbf{i}+\mathbf{j}+\mathbf{k}$, *and* $\mathbf{c}=4\mathbf{i}+2\mathbf{j}+6\mathbf{k}$. **[Meerut 1984]**

Sol. We have $(\mathbf{b}\times\mathbf{c})\times\mathbf{a}=(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}$

$$=[(-\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-5\mathbf{k})]\,(4\mathbf{i}+2\mathbf{j}+6\mathbf{k})$$
$$-[(4\mathbf{i}+2\mathbf{j}+6\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}-5\mathbf{k})]\,(-\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$=(-2+3-5)\,(4\mathbf{i}+2\mathbf{j}+6\mathbf{k})-(8+6-30)\,(-\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$=-4\,(4\mathbf{i}+2\mathbf{j}+6\mathbf{k})+16\,(-\mathbf{i}+\mathbf{j}+\mathbf{k})$$

$$=-16\mathbf{i}-8\mathbf{j}-24\mathbf{k}-16\mathbf{i}+16\mathbf{j}+16\mathbf{k}$$
$$=-32\mathbf{i}+8\mathbf{j}-8\mathbf{k}=8\,(-4\mathbf{i}+\mathbf{j}-\mathbf{k}).$$

Ex. 4. *Verify the formula for vector triple product*

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

by taking $\mathbf{a}=\mathbf{i}+\mathbf{j},\ \mathbf{b}=-\mathbf{i}+2\mathbf{k},\ \mathbf{c}=\mathbf{j}+\mathbf{k}$.

Sol. We have $\mathbf{b}\times\mathbf{c}=(-\mathbf{i}+2\mathbf{k})\times(\mathbf{j}+\mathbf{k})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 0 & 2\\ 0 & 1 & 1\end{vmatrix}$$

$$=(0-2)\,\mathbf{i}-(-1-0)\,\mathbf{j}+(-1-0)\,\mathbf{k}=-2\mathbf{i}+\mathbf{j}-\mathbf{k}.$$

$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{i}+\mathbf{j})\times(-2\mathbf{i}+\mathbf{j}-\mathbf{k})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ -2 & 1 & -1\end{vmatrix}=(-1-0)\,\mathbf{i}-(-1-0)\,\mathbf{j}+(1+2)\,\mathbf{k}$$

$$=-\mathbf{i}+\mathbf{j}+3\mathbf{k}. \qquad \ldots(1)$$

Again $(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

$$=[(\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}+\mathbf{k})]\,(-\mathbf{i}+2\mathbf{k})-[(\mathbf{i}+\mathbf{j})\cdot(-\mathbf{i}+2\mathbf{k})]\,(\mathbf{j}+\mathbf{k})$$
$$=(0+1+0)\,(-\mathbf{i}+2\mathbf{k})-(-1+0+0)\,(\mathbf{j}+\mathbf{k})$$
$$=1\,(-\mathbf{i}+2\mathbf{k})+(\mathbf{j}+\mathbf{k})=-\mathbf{i}+\mathbf{j}+3\mathbf{k}. \qquad \ldots(2)$$

From (1) and (2), we see that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}.$$

Ex 5. *Prove that* $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}$.

[Meerut 1983 S]

Sol. Let $\mathbf{b}\times\mathbf{c}=\mathbf{d}$.

Then $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=\mathbf{d}\times(\mathbf{c}\times\mathbf{a})$

$$=(\mathbf{d}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{d}\cdot\mathbf{c})\,\mathbf{a}=[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c}-[(\mathbf{b}\times\mathbf{c}\cdot\mathbf{c}]\,\mathbf{a}$$
$$=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]\,\mathbf{c}-[\mathbf{b}\ \mathbf{c}\ \mathbf{c}]\,\mathbf{a}$$
$$=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c},\ \text{since}\ [\mathbf{b}\ \mathbf{c}\ \mathbf{c}]=0\ \text{and}\ [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}].$$

Hence $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{c}$.

Ex. 6. *Prove that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})=(\mathbf{a}\times\mathbf{b})\times\mathbf{a}$.

Sol. We have $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})=(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a} \qquad \ldots(1)$

Again $(\mathbf{a}\times\mathbf{b})\times\mathbf{a}=-\{\mathbf{a}\times(\mathbf{a}\times\mathbf{b})\}$ $[\because \ \mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u}]$

$$=-\{(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}\}$$
$$=-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}+(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}$$
$$=(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{a})=(\mathbf{a}\times\mathbf{b})\times\mathbf{a}.$$

***Ex. 7.** *If* $\mathbf{a}=\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$, *find* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

Sol. We have $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

$$=[(\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+2\mathbf{j}-\mathbf{k})]\,(2\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$-[(\mathbf{i}-2\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}+\mathbf{k})]\,(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$
$$=(1-4-1)\,(2\mathbf{i}+\mathbf{j}+\mathbf{k})-(2-2+1)\,(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$
$$=(-8\mathbf{i}-4\mathbf{j}-4\mathbf{k})-(\mathbf{i}+2\mathbf{j}-\mathbf{k})=-9\mathbf{i}-6\mathbf{j}-3\mathbf{k}.$$

Ex. 8. *Show that* $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})=2\mathbf{a}$.

[Nagpur 1973; Lucknow 78; Delhi 81; Meerut 82, 87S, 88 P, 90 P]

Sol. We have $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})=(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}$

$$=\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i} \qquad [\because \ \mathbf{i}\cdot\mathbf{i}=1]$$
$$\mathbf{j}\times(\mathbf{a}\times\mathbf{j})=(\mathbf{j}\cdot\mathbf{j})\,\mathbf{a}-(\mathbf{j}\cdot\mathbf{a})\,\mathbf{j}$$
$$=\mathbf{a}-(\mathbf{j}\cdot\mathbf{a})\,\mathbf{j} \qquad [\because \ \mathbf{j}\cdot\mathbf{j}=1]$$

and $\mathbf{k}\times(\mathbf{a}\times\mathbf{k})=(\mathbf{k}\cdot\mathbf{k})\,\mathbf{a}-(\mathbf{k}\cdot\mathbf{a})\,\mathbf{k}=\mathbf{a}-(\mathbf{k}\cdot\mathbf{a})\,\mathbf{k}$ $[\because \ \mathbf{k}\cdot\mathbf{k}=1]$

Adding these three expressions, we get

$$\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})$$
$$=3\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}-(\mathbf{j}\cdot\mathbf{a})\,\mathbf{j}-(\mathbf{k}\cdot\mathbf{a})\,\mathbf{k}$$
$$=3\mathbf{a}-[(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{a}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{a}\cdot\mathbf{k})\,\mathbf{k}] \qquad [\because \ \mathbf{a}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{a} \text{ etc.}]$$

Now we shall show that

$$\mathbf{a}=(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{a}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{a}\cdot\mathbf{k})\,\mathbf{k}.$$

Let $\mathbf{a}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Taking dot product of both sides with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ successively, we get

$$x=\mathbf{a}\cdot\mathbf{i},\ y=\mathbf{a}\cdot\mathbf{j},\ z=\mathbf{a}\cdot\mathbf{k}.$$

$$\therefore \quad \mathbf{a}=(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{a}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{a}\cdot\mathbf{k})\,\mathbf{k}.$$

Hence $\mathbf{i}\times(\mathbf{a}\times\mathbf{i})+\mathbf{j}\times(\mathbf{a}\times\mathbf{j})+\mathbf{k}\times(\mathbf{a}\times\mathbf{k})=3\mathbf{a}-\mathbf{a}=2\mathbf{a}$.

***Ex. 9.** *Show that* $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})=\mathbf{0}$. **[Rohilkhand 1979]**

Sol. We have $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})=\mathbf{i}\times\mathbf{i}$ $[\because \ \mathbf{j}\times\mathbf{k}=\mathbf{i}]$

$$=\mathbf{0}. \qquad [\because \ \mathbf{i}\times\mathbf{i}=\mathbf{0}]$$

Ex. 10. *Show that* $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$, *and express the result by means of determinants.*

[Meerut 1981, 86S, 87, 88, 89P; Kanpur 79; Lucknow 81; Gorakhpur 87; Allahabad 80]

Sol. We have $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]$

$$=(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})].$$

Let us first find the value of $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})$.

Let $\mathbf{b}\times\mathbf{c}=\mathbf{d}$.

Then $(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=\mathbf{d}\times(\mathbf{c}\times\mathbf{a})$

$$=(\mathbf{d}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{d}\cdot\mathbf{c})\,\mathbf{a}$$

$$=[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c}-[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a}$$

$$=[\mathbf{bca}]\,\mathbf{c}-[\mathbf{bcc}]\,\mathbf{a}$$

$$=[\mathbf{abc}]\,\mathbf{c},\ \text{since } [\mathbf{bcc}]=0$$

and $[\mathbf{bca}]=[\mathbf{abc}]$.

$\therefore\ [\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=(\mathbf{a}\times\mathbf{b})\cdot[\mathbf{abc}]\,\mathbf{c}$

$$=[\mathbf{abc}]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=[\mathbf{abc}]\,[\mathbf{abc}]$$

$$=[\mathbf{abc}]^2.$$

Second part. Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k},\ \mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k},$

$$\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}.$$

We have
$$[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Again
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=(a_2b_3-b_2a_3)\,\mathbf{i}+(b_1a_3-a_1b_3)\,\mathbf{j}+(a_1b_2-a_2b_1)\,\mathbf{k}.$$

Similarly
$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=(b_2c_3-b_3c_2)\,\mathbf{i}+(c_1b_3-b_1c_3)\,\mathbf{j}+(b_1c_2-c_1b_2)\,\mathbf{k}$$

and $$\mathbf{c}\times\mathbf{a}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$=(c_2a_3-a_2c_3)\,\mathbf{i}+(a_1c_3-a_3c_1)\,\mathbf{j}+(c_1a_2-a_1c_2)\,\mathbf{k}.$$

$$\therefore\quad [\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=\begin{vmatrix} a_2b_3-b_2a_3 & b_1a_3-a_1b_3 & a_1b_2-a_2b_1 \\ b_2c_3-b_3c_2 & c_1b_3-b_1c_3 & b_1c_2-b_2c_1 \\ c_2a_3-c_3a_2 & a_1c_3-a_3c_1 & c_1a_2-c_2a_1 \end{vmatrix}.$$

$$=\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

where the capital letters A_1, A_2, A_3 etc. denote the cofactors of the corresponding small letters a_1, a_2, a_3 etc. in the determinant .

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Since $[\mathbf{abc}]^2=[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]$,

$$\therefore\quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Ex. 11. *Prove that for any three vectors* **A**, **B** *and* **C**,
$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})=(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})^2$. **[Meerut 1974, 76]**

Sol. This question is the same as is the first part of solved example 10 above.

Ex. 12. *Prove that for any three vectors* **a**, **b** *and* **c**,
$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$. **[Meerut 1986S, 87]**

Hence show that the vectors **a**, **b**, **c** *are non-coplanar if and only if the vectors* $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ *are non-coplanar.* **[Delhi 1980]**

Sol. For the solution of the first part of this question see solved example 10 above.

Second part. As proved in the first part of this question, we have

$$[\mathbf{abc}]^2=[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}].$$

$\therefore\quad [\mathbf{abc}]=0$ if and only if $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=0$

or $[\mathbf{abc}]\neq 0$ if and only if $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]\neq 0$.

Now $[\mathbf{abc}]\neq 0$ if and only if the vectors, **a**, **b**, **c** are non-coplanar.

Hence the vectors **a**, **b**, **c** are non-coplanar if and only if the vectors $\mathbf{a}\times\mathbf{b}$, $\mathbf{b}\times\mathbf{c}$, $\mathbf{c}\times\mathbf{a}$ are non-coplanar.

Ex. 13. *Show that*

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}=[\mathbf{abc}]^2.$$

Sol. First prove that $[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c},\ \mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$. For complete solution see solved example 10.

Now to prove the second part of the question, let

$\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$, $\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}$, $\mathbf{c}=c_1\,\mathbf{i}+c_2\,\mathbf{j}+c_3\,\mathbf{k}$.

Then $[\mathbf{abc}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\therefore\quad [\mathbf{abc}]^2=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\cdot\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3 \\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3 \\ c_1a_1+c_2a_2+c_3a_3 & c_1b_1+c_2b_2+c_3b_3 & c_1^2+c_2^2+c_3^2 \end{vmatrix}$$

by the row-by-row multiplication rule for the product of two determinants of the same order

$$=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

because $\mathbf{a}\cdot\mathbf{a}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})$
$=a_1^2+a_2^2+a_3^2$, etc.

From the two relations proved above we get the relation required to be proved in the question.

Ex. 14. *Prove that* $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})\cdot\mathbf{d}=(\mathbf{a}\cdot\mathbf{d})\,[\mathbf{abc}]$.

Sol. Let $\mathbf{a}\times\mathbf{b}=\mathbf{r}$.

Then $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})=\mathbf{r}\times(\mathbf{a}\times\mathbf{c})=(\mathbf{r}\cdot\mathbf{c})\,\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{c}$

$$=[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,\mathbf{a}-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}]\,\mathbf{c}$$

$$=[\mathbf{abc}]\,\mathbf{a}-[\mathbf{aba}]\,\mathbf{c}$$

$$=[\mathbf{abc}]\,\mathbf{a},\text{ since }[\mathbf{aba}]=0.$$

Therefore $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})\cdot\mathbf{d}=[\mathbf{abc}]\,\mathbf{a}\cdot\mathbf{d}$

$$=(\mathbf{a}\cdot\mathbf{d})\,[\mathbf{abc}].$$

***Ex. 15.** *If* $\mathbf{a},\mathbf{b},\mathbf{c}$ *be three unit vectors such that* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{1}{2}\mathbf{b}$ *find the angles which* $\mathbf{a}$ *makes with* $\mathbf{b}$ *and* $\mathbf{c}$, $\mathbf{b}$ *and* $\mathbf{c}$ *being non-parallel.* **[Rajasthan 1975; Rohilkhand 78; Kanpur 86]**

Sol. It is given that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\frac{1}{2}\mathbf{b}$.

$$\therefore\quad(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=\tfrac{1}{2}\mathbf{b}.$$

$$\therefore\quad(\mathbf{a}\cdot\mathbf{c}-\tfrac{1}{2})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=\mathbf{0}.\qquad\ldots(1)$$

Since $\mathbf{b}$ and $\mathbf{c}$ are non-parallel, therefore for the existence of relation (1) the coefficients of $\mathbf{b}$ and $\mathbf{c}$ should vanish separately. Therefore, we get

$$\mathbf{a}\cdot\mathbf{c}-\tfrac{1}{2}=0,\ i.e.\ \mathbf{a}\cdot\mathbf{c}=\tfrac{1}{2}\text{ and }\mathbf{a}\cdot\mathbf{b}=0.$$

Let θ and ϕ be the angles which $\mathbf{a}$ makes with $\mathbf{b}$ and $\mathbf{c}$ respectively. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are unit vectors, therefore

$$\mathbf{a}\cdot\mathbf{b}=\cos\theta=0\qquad\therefore\quad\theta=90^\circ,$$

$$\mathbf{a}\cdot\mathbf{c}=\cos\phi=\tfrac{1}{2}\qquad\therefore\quad\phi=60^\circ.$$

Ex. 16. *Prove that* $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, *if and only if* $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$. **[Meerut 1992; Rohilkhand 79; Gorakhpur 88]**

Sol. We have $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

if and only if $(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$

i.e., if and only if $-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}=-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$, since $\mathbf{c}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{c}$

i.e., if and only if $(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=\mathbf{0}$

i.e., if and only if $(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}=\mathbf{0}$, since $\mathbf{c}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}$ and $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

i.e., if and only if $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$.

Note. $(\mathbf{c}\times\mathbf{a})\times\mathbf{b}=\mathbf{0}$ is possible when (i) $\mathbf{a}$ and $\mathbf{c}$ are collinear because then $\mathbf{c}\times\mathbf{a}=\mathbf{0}$ or (ii) $\mathbf{b}$ is parallel to $\mathbf{c}\times\mathbf{a}$ *i.e.* $\mathbf{b}$ is perpendicular to both $\mathbf{c}$ and $\mathbf{a}$ or (iii) at least one of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is a null vector. **[Meerut 1988P]**

Products of four vectors

§ 9. Scalar product of four vectors.

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are four vectors, the products $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$,

$(\mathbf{a}\times\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c})$ etc. are called scalar products of four vectors.

***Theorem.** **To prove that**

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$$

[Mysore 1971; Meerut 83, 84, 89P, 90P; Rohilkhand 92]

Let $\mathbf{a}\times\mathbf{b}=\mathbf{r}$. Then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\mathbf{r}\cdot(\mathbf{c}\times\mathbf{d})$.

Now in a scalar triple product the position of dot and cross may be interchanged without altering the value of the product. Therefore, $\mathbf{r}\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{r}\times\mathbf{c})\cdot\mathbf{d}$.

$$\therefore\quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]\cdot\mathbf{d}$$
$$=[(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}]\cdot\mathbf{d}$$
$$=(\mathbf{c}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{d})$$
$$=(\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})$$
$$=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}.$$

This relation is known as **Lagrange's Identity**.

§ 10. Vector product of four vectors.

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be four vectors. Consider the vector product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. This product can be written as $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ and is called the vector product of four vectors. It is a vector perpendicular to $\mathbf{a}\times\mathbf{b}$ and, therefore coplanar with $\mathbf{a}$ and $\mathbf{b}$. Similarly it is a vector coplanar with $\mathbf{c}$ and $\mathbf{d}$. Hence this vector must be parallel to the line of intersection of a plane parallel to $\mathbf{a}$ and $\mathbf{b}$ with another plane parallel to $\mathbf{c}$ and $\mathbf{d}$.

Theorem. **To prove that**

(i) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}$

[Allahabad 1980; Gorakhpur 87]

(ii) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=[\mathbf{acd}]\,\mathbf{b}-[\mathbf{bcd}]\,\mathbf{a}$.

[Lucknow 1980; Gorakhpur 88]

$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ is a vector which can be either expressed in terms of $\mathbf{c}$ and $\mathbf{d}$ or in terms of $\mathbf{a}$ and $\mathbf{b}$. To express it in terms of $\mathbf{c}$ and $\mathbf{d}$, let us put $\mathbf{a}\times\mathbf{b}=\mathbf{l}$. Then

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{l}\times(\mathbf{c}\times\mathbf{d})=(\mathbf{l}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{l}\cdot\mathbf{c})\,\mathbf{d}$$
$$=[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}]\,\mathbf{c}-[(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,\mathbf{d}=[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}.$$

Again to express $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ in terms of **a** and **b**, let us put $\mathbf{c}\times\mathbf{d}=\mathbf{m}$. Then

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\times\mathbf{b})\times\mathbf{m}=-\mathbf{m}\times(\mathbf{a}\times\mathbf{b})$$
$$=-[(\mathbf{m}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{m}\cdot\mathbf{a})\,\mathbf{b}]=(\mathbf{m}\cdot\mathbf{a})\,\mathbf{b}-(\mathbf{m}\cdot\mathbf{b})\,\mathbf{a}$$
$$=[(\mathbf{c}\times\mathbf{d})\cdot\mathbf{a}]\,\mathbf{b}-[(\mathbf{c}\times\mathbf{d})\cdot\mathbf{b}]\,\mathbf{a}$$
$$=[\mathbf{cda}]\,\mathbf{b}-[\mathbf{cdb}]\,\mathbf{a}=[\mathbf{acd}]\,\mathbf{b}-[\mathbf{bcd}]\,\mathbf{a}.$$

Linear Relation connecting four vectors. Equating the above two expressions for the value of $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$, we get

$$[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}=[\mathbf{acd}]\,\mathbf{b}-[\mathbf{bcd}]\,\mathbf{a}$$

or $$[\mathbf{bcd}]\,\mathbf{a}-[\mathbf{acd}]\,\mathbf{b}+[\mathbf{abd}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{d}=0, \quad \ldots(1)$$

[Meerut 1984]

which is the required linear relation connecting the four vectors **a, b, c, d.**

***To find an expression for any vector r, in space, as a linear combination of three non-coplanar vectors a, b, c.**

[Karnatak 1971; Allahabad 77; Kanpur 88]

Replacing **d** by **r** in the relation (1) just established, we get

$$[\mathbf{bcr}]\,\mathbf{a}-[\mathbf{acr}]\,\mathbf{b}+[\mathbf{abr}]\,\mathbf{c}-[\mathbf{abc}]\,\mathbf{r}=0$$

or $$[\mathbf{abc}]\,\mathbf{r}=[\mathbf{bcr}]\,\mathbf{a}-[\mathbf{acr}]\,\mathbf{b}+[\mathbf{abr}]\,\mathbf{c}. \quad \ldots(2)$$

Since **a, b, c** are non-coplanar, therefore $[\mathbf{abc}]\neq 0$.

Therefore dividing both sides of (2) by $[\mathbf{abc}]$, we get

$$\mathbf{r}=\frac{[\mathbf{bcr}]\,\mathbf{a}-[\mathbf{acr}]\,\mathbf{b}+[\mathbf{abr}]\,\mathbf{c}}{[\mathbf{abc}]},$$

or $$\mathbf{r}=\frac{[\mathbf{bcr}]\,\mathbf{a}+[\mathbf{car}]\,\mathbf{b}+[\mathbf{abr}]\,\mathbf{c}}{[\mathbf{abc}]}, \text{ since } [\mathbf{acr}]=-[\mathbf{car}]$$

or $$\mathbf{r}=\frac{[\mathbf{rbc}]\,\mathbf{a}+[\mathbf{rca}]\,\mathbf{b}+[\mathbf{rab}]\,\mathbf{c}}{[\mathbf{abc}]}, \quad \ldots(3)$$

which is the required expression for **r**.

*§ 11. Reciprocal system of vectors.

If **a, b, c** *be any three non-coplanar vectors so that* $[\mathbf{abc}]\neq 0$, *then the three vectors* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *defined by the equations*

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

are called reciprocal system of vectors to the vectors **a, b, c.**

(*i*) *To show that* $\mathbf{a}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{b}'=\mathbf{c}\cdot\mathbf{c}'=1$.

We have $\mathbf{a}\cdot\mathbf{a}'=\mathbf{a}\cdot\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\dfrac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{abc}]}=\dfrac{[\mathbf{abc}]}{[\mathbf{abc}]}=1.$

Similarly $\mathbf{b}\cdot\mathbf{b}'=\mathbf{b}\cdot\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}=\dfrac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{abc}]}=\dfrac{[\mathbf{bca}]}{[\mathbf{abc}]}=\dfrac{[\mathbf{abc}]}{[\mathbf{abc}]}=1,$

and $\mathbf{c}\cdot\mathbf{c}'=\mathbf{c}\cdot\dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}=\dfrac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{abc}]}=\dfrac{[\mathbf{cab}]}{[\mathbf{abc}]}=\dfrac{[\mathbf{abc}]}{[\mathbf{abc}]}=1.$

Note. The reason for the name reciprocal lies in the relations $\mathbf{a}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{b}'=\mathbf{c}\cdot\mathbf{c}'=1.$

**(*ii*) *The scalar product of any other pair of vectors, one from each system, is zero i.e.*

$\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\mathbf{c}'=\mathbf{b}\cdot\mathbf{a}'=\mathbf{b}\cdot\mathbf{c}'=\mathbf{c}\cdot\mathbf{a}'=\mathbf{c}\cdot\mathbf{b}'=0.$

We have $\mathbf{a}\cdot\mathbf{b}'=\mathbf{a}\cdot\dfrac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}=\dfrac{\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{abc}]}=\dfrac{[\mathbf{aca}]}{[\mathbf{abc}]}$

$=0$, since $[\mathbf{aca}]=0$.

Similarly we can prove the other results.

(*iii*) *The scalar triple product* $[\mathbf{abc}]$ *formed from three non-coplanar vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *is the reciprocal of the scalar triple product* $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']$ *formed from the reciprocal system* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *i.e.* $[\mathbf{abc}]\,[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']=1.$ **[Lucknow 1977; Rohilkhand 80; Meerut 88]

We have $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']=\mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}')$

$$=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}\cdot\left\{\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}\times\frac{(\mathbf{a}\times\mathbf{b})}{[\mathbf{abc}]}\right\}$$

$$=\frac{(\mathbf{b}\times\mathbf{c})\cdot[(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})]}{[\mathbf{abc}]^3}.$$

Now expanding $(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})$ by vector triple product treating $\mathbf{c}\times\mathbf{a}$ as one vector, we get

$$\begin{aligned}(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})&=[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}]\,\mathbf{a}-[(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}]\,\mathbf{b}\\&=[\mathbf{cab}]\,\mathbf{a}-[\mathbf{caa}]\,\mathbf{b}\\&=[\mathbf{abc}]\,\mathbf{a},\text{ since }[\mathbf{caa}]=0\\&\qquad\text{and }[\mathbf{cab}]=[\mathbf{abc}].\end{aligned}$$

$$\therefore\quad[\mathbf{a}'\mathbf{b}'\mathbf{c}']=\frac{(\mathbf{b}\times\mathbf{c})\cdot[\mathbf{abc}]\,\mathbf{a}}{[\mathbf{abc}]^3}=\frac{[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,[\mathbf{abc}]}{[\mathbf{abc}]^3}$$

$$=\frac{[\mathbf{bca}]\,[\mathbf{abc}]}{[\mathbf{abc}]^3}$$

$$=\frac{[\mathbf{abc}]^2}{[\mathbf{abc}]^3}=\frac{1}{[\mathbf{abc}]}.$$

$\therefore\quad[\mathbf{a}'\mathbf{b}'\mathbf{c}']\,[\mathbf{abc}]=1.$

Note 1. Since $[\mathbf{abc}] \neq 0$, therefore from the relation, $[\mathbf{a'b'c'}]\,[\mathbf{abc}] = 1$, we conclude that $[\mathbf{a'b'c'}] \neq 0$.

Hence the vectors $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{c'}$ are also non-coplanar.

Note 2. The symmetry of results proved in properties (i), (ii) and (iii) suggest that if $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{c'}$ is the reciprocal system to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is also the reciprocal system to $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{c'}$.

Note 3. The relation $[\mathbf{abc}]\,[\mathbf{a'\ b'\ c'}] = 1$ shows that the scalar triple products $[\mathbf{abc}]$ and $[\mathbf{a'\ b'\ c'}]$ are either both positive or both negative. Hence the two system of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{c'}$ are either both right handed or both left handed.

(*iv*) *The orthonormal vector triads* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *form a self reciprocal system.*

Let $\mathbf{i'}$, $\mathbf{j'}$, $\mathbf{k'}$ be the system of vectors reciprocal to the system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Then by definition $\mathbf{i'} = \dfrac{\mathbf{j} \times \mathbf{k}}{[\mathbf{i\ j\ k}]} = \dfrac{\mathbf{i}}{1} = \mathbf{i}$.

Similarly $\mathbf{j'} = \mathbf{j}$ and $\mathbf{k'} = \mathbf{k}$.

Hence the result.

§ 13. *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be three non-coplanar vectors and* $\mathbf{a'}$, $\mathbf{b'}$, $\mathbf{c'}$ *consitute the reciprocal system of vectors, then prove that any vector* $\mathbf{r}$ *can be expressed qs* $\mathbf{r} = (\mathbf{r} \cdot \mathbf{a'})\,\mathbf{a} + (\mathbf{r} \cdot \mathbf{b'})\,\mathbf{b} + (\mathbf{r} \cdot \mathbf{c'})\,\mathbf{c}$.

[Karanatak 1971]

Let $\mathbf{r}$ be expressed as a linear combination of the non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the form

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} \qquad \dots(1)$$

where x, y, z are some scalars.

Multiplying both sides of (1) scalarly with $\mathbf{b} \times \mathbf{c}$, we get

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) &= x\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + y\mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) + z \cdot \mathbf{c}(\mathbf{b} \times \mathbf{c}) \\ &= x\,[\mathbf{abc}] + y\,[\mathbf{bbc}] + z\,[\mathbf{cbc}] \\ &= x\,[\mathbf{abc}], \text{ since } [\mathbf{bbc}] = 0 = [\mathbf{cbc}]. \end{aligned}$$

$$\therefore \quad x = \frac{\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]} = \mathbf{r} \cdot \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} = \mathbf{r} \cdot \mathbf{a'}, \text{ since } \mathbf{a'} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]}.$$

Similarly multiplying both sides of (1) scalarly with $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$, we can show that

$$y = \mathbf{r} \cdot \mathbf{b'} \text{ and } z = \mathbf{r} \cdot \mathbf{c'}.$$

Putting the values of x, y and z in (1), we get

$$\mathbf{r} = (\mathbf{r} \cdot \mathbf{a'})\,\mathbf{a} + (\mathbf{r} \cdot \mathbf{b'})\,\mathbf{b} + (\mathbf{r} \cdot \mathbf{c'})\,\mathbf{c}. \qquad \dots(2)$$

Note 1. In a similar manner, we can prove that

$$\mathbf{r}=(\mathbf{r}\cdot\mathbf{a})\,\mathbf{a}'+(\mathbf{r}\cdot\mathbf{b})\,\mathbf{b}'+(\mathbf{r}\cdot\mathbf{c})\,\mathbf{c}'.$$

Note 2. Since the system of vectors **i**, **j**, **k** is self-reciprocal, therefore from (2) we conclude that

$$\mathbf{r}=(\mathbf{r}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{r}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{r}\cdot\mathbf{k})\,\mathbf{k}.$$

Solved Examples

Ex. 1. *Find a set of vectors reciprocal to the set* $2\mathbf{i}+3\mathbf{j}-\mathbf{k},\ \mathbf{i}-\mathbf{j}-2\mathbf{k},\ -\mathbf{i}+2\mathbf{j}+2\mathbf{k}.$ **[Kanpur 1980; Agra 79]**

Sol. Let $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-\mathbf{k},\ \mathbf{b}=\mathbf{i}-\mathbf{j}-2\mathbf{k},\ \mathbf{c}=-\mathbf{i}+2\mathbf{j}+2\mathbf{k}.$

Let $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ be the set of vectors reciprocal to the set **a**, **b**, **c**. Then by definition,

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}.$$

Now $$[\mathbf{abc}]=\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}=2\,(2)-3\,(0)-1\,(1)=3$$

and $$\mathbf{b}\times\mathbf{c}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}=2\mathbf{i}+0\mathbf{j}+\mathbf{k}=2\mathbf{i}+\mathbf{k}.$$

$$\therefore\quad \mathbf{a}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]}=\frac{2\mathbf{i}+\mathbf{k}}{3}=\left(\frac{2}{3}\,\mathbf{i}+\frac{\mathbf{k}}{3}\right).$$

Similarly $\mathbf{b}'=\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$

$$=\tfrac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix}=\frac{-8\mathbf{i}+3\mathbf{j}-7\mathbf{k}}{3}$$

and $\mathbf{c}'=\dfrac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$

$$=\tfrac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix}=\frac{-7\mathbf{i}+3\mathbf{j}-5\mathbf{k}}{3}.$$

Ex. 2. *Obtain a set of vectors reciprocal to the three vectors*

$-\mathbf{i}+\mathbf{j}+\mathbf{k},\ \mathbf{i}-\mathbf{j}+\mathbf{k},\ \mathbf{i}+\mathbf{j}+\mathbf{k}.$

Sol. Proceed as in solved example 1.

Ans. $-\frac{1}{2}\,\mathbf{i}+\frac{1}{2}\,\mathbf{k},\ -\frac{1}{2}\,\mathbf{j}+\frac{1}{2}\,\mathbf{k},\ \frac{1}{2}\,\mathbf{i}+\frac{1}{2}\,\mathbf{j}.$

Ex. 3. *Prove the identity* $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]=(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{b}\times\mathbf{a}).$

Sol. We have $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]=\mathbf{a}\times[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}]$

$=\mathbf{a}\times[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}]-\mathbf{a}\times[(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}]$, by dist. law for cross product

$=(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{a}\times\mathbf{a})-(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{a}\times\mathbf{b})$ $\quad[\because\ \mathbf{a}\times(m\mathbf{b})=m(\mathbf{a}\times\mathbf{b})]$

$=-(\mathbf{a}\cdot\mathbf{a})\,(\mathbf{a}\times\mathbf{b})$ $\quad[\because\ \mathbf{a}\times\mathbf{a}=\mathbf{0}]$

$=(\mathbf{a}\cdot\mathbf{a})\,\mathbf{b}\times\mathbf{a}$ $\quad[\because\ \mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times(\mathbf{a}]$

***Ex. 4.** *Prove that*

$$\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\times\mathbf{d}).$$

Hence expand $\mathbf{a}\times[\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}].$

[Kanpur 1977; Rohilkhand 76]

Sol. First part. We have

$$\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{a}\times\{(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{d}\}$$
$$=(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\times\mathbf{d}).$$

Second part. We have

$$\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}=\mathbf{b}\times\{(\mathbf{c}\cdot\mathbf{e})\,\mathbf{d}-(\mathbf{c}\cdot\mathbf{d})\,\mathbf{e}\}$$
$$=(\mathbf{c}\cdot\mathbf{e})\,(\mathbf{b}\times\mathbf{d})-(\mathbf{c}\cdot\mathbf{d})\,(\mathbf{b}\times\mathbf{e}).$$

$\therefore\quad \mathbf{a}\times[\mathbf{b}\times\{\mathbf{c}\times(\mathbf{d}\times\mathbf{e})\}]=\mathbf{a}\times[(\mathbf{c}\cdot\mathbf{e})\,(\mathbf{b}\times\mathbf{d})-(\mathbf{c}\cdot\mathbf{d})\,(\mathbf{b}\times\mathbf{e})]$

$$=(\mathbf{c}\cdot\mathbf{e})\,[\mathbf{a}\times(\mathbf{b}\times\mathbf{d})]-(\mathbf{c}\cdot\mathbf{d})\,[\mathbf{a}\times(\mathbf{b}\times\mathbf{e})]$$
$$=(\mathbf{c}\cdot\mathbf{e})\,[(\mathbf{a}\cdot\mathbf{d})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{d}]$$
$$-(\mathbf{c}\cdot\mathbf{d})\,[(\mathbf{a}\cdot\mathbf{e})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{e}].$$

Ex. 5. *Prove that* $\mathbf{d}\cdot[\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}]=(\mathbf{b}\cdot\mathbf{d})\,[\mathbf{acd}].$

Sol. $\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{a}\times\{(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\,\mathbf{d}\}$

$$=(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\times\mathbf{d}).$$

$\therefore\quad \mathbf{d}\cdot[\mathbf{a}\times\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}]=\mathbf{d}\cdot[(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\times\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\times\mathbf{d})]$

$$=(\mathbf{b}\cdot\mathbf{d})\,[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{c})]-(\mathbf{b}\cdot\mathbf{c})\,[\mathbf{d}\cdot(\mathbf{a}\times\mathbf{d})]$$
$$=(\mathbf{b}\cdot\mathbf{d})\,[\mathbf{dac}]-(\mathbf{b}\cdot\mathbf{c})\,[\mathbf{dad}]$$

$=(\mathbf{b}\cdot\mathbf{d})\,[\mathbf{acd}]$, since $[\mathbf{dad}]=0$ and $[\mathbf{dac}]=[\mathbf{acd}].$

Ex. 6. *If the four vectors* $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}$ *are coplanar, show that*

$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})=\mathbf{0}.$ [Andhra 1975]

Sol. $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Similarly $\mathbf{c} \times \mathbf{d}$ is a vector perpendicular to the plane containing $\mathbf{c}$ and $\mathbf{d}$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are all coplanar, therefore the vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are perpendicular to the same plane. Therefore $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are parallel.

Now we know that the vector product of two parallel vectors is equal to a zero vector, therefore $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.

Ex. 7. *Prove that*

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = -2\,[\mathbf{bcd}]\,\mathbf{a}.$$

Sol. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{l} \times (\mathbf{c} \times \mathbf{d})$, where $\mathbf{l} = \mathbf{a} \times \mathbf{b}$

$$= (\mathbf{l} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{l} \cdot \mathbf{c})\,\mathbf{d}$$

$$= [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d}]\,\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]\,\mathbf{d}$$

$$= [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d} \qquad \ldots(1)$$

Again $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{c}) \times \mathbf{m}$, where $\mathbf{m} = \mathbf{d} \times \mathbf{b}$

$$= (\mathbf{m} \cdot \mathbf{a})\,\mathbf{c} - (\mathbf{m} \cdot \mathbf{c})\,\mathbf{a}$$

$$= [(\mathbf{d} \times \mathbf{b}) \cdot \mathbf{a}]\,\mathbf{c} - [(\mathbf{d} \times \mathbf{b}) \cdot \mathbf{c}]\,\mathbf{a}$$

$$= [\mathbf{dba}]\,\mathbf{c} - [\mathbf{dbc}]\,\mathbf{a}$$

$$= -[\mathbf{abd}]\,\mathbf{c} - [\mathbf{bcd}]\,\mathbf{a}, \qquad \ldots(2)$$

since $[\mathbf{dba}] = -[\mathbf{abd}]$, as we have changed the cyclic order of the vectors and $[\mathbf{dbc}] = [\mathbf{bcd}]$, as the cyclic order has been maintained.

Also $(\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{d}) \times \mathbf{n}$, where $\mathbf{n} = \mathbf{b} \times \mathbf{c}$

$$= (\mathbf{n} \cdot \mathbf{a})\,\mathbf{d} - (\mathbf{n} \cdot \mathbf{d})\,\mathbf{a} = [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\,\mathbf{d} - [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}]\,\mathbf{a}$$

$$= [\mathbf{bca}]\,\mathbf{d} - [\mathbf{bcd}]\,\mathbf{a} = [\mathbf{abc}]\,\mathbf{d} - [\mathbf{bcd}]\,\mathbf{a}. \qquad \ldots(3)$$

Adding (1), (2) and (3), we get

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = -2\,[\mathbf{bcd}]\,\mathbf{a}.$$

Ex. 8. *Prove that*

$$[\mathbf{a} \times \mathbf{p}, \mathbf{b} \times \mathbf{q}, \mathbf{c} \times \mathbf{r}] + [\mathbf{a} \times \mathbf{q}, \mathbf{b} \times \mathbf{r}, \mathbf{c} \times \mathbf{p}] + [\mathbf{a} \times \mathbf{r}, \mathbf{b} \times \mathbf{p}, \mathbf{c} \times \mathbf{q}] = 0.$$

[Rohilkhand 1990; Kanpur 86]

Sol. We have $[\mathbf{a} \times \mathbf{p}, \mathbf{b} \times \mathbf{q}, \mathbf{c} \times \mathbf{r}]$

$$= (\mathbf{a} \times \mathbf{p}) \cdot [(\mathbf{b} \times \mathbf{q}) \times (\mathbf{c} \times \mathbf{r})]$$

$$= (\mathbf{a} \times \mathbf{p}) \cdot [\{(\mathbf{b} \times \mathbf{q}) \cdot \mathbf{r}\}\,\mathbf{c} - \{(\mathbf{b} \times \mathbf{q}) \cdot \mathbf{c}\}\,\mathbf{r}]$$

$$= (\mathbf{a} \times \mathbf{p}) \cdot \{[\mathbf{bqr}]\,\mathbf{c} - [\mathbf{bqc}]\,\mathbf{r}\}$$

$$= [\mathbf{apc}]\,[\mathbf{bqr}] - [\mathbf{apr}]\,[\mathbf{bqc}]. \qquad \ldots(1)$$

Again $[\mathbf{a}\times\mathbf{q},\ \mathbf{b}\times\mathbf{r},\ \mathbf{c}\times\mathbf{p}]=[\mathbf{b}\times\mathbf{r},\ \mathbf{c}\times\mathbf{p},\ \mathbf{a}\times\mathbf{q}]$

$$=(\mathbf{b}\times\mathbf{r})\cdot[(\mathbf{c}\times\mathbf{p})\times(\mathbf{a}\times\mathbf{q})]$$

$$=(\mathbf{b}\times\mathbf{r})\cdot\{[(\mathbf{c}\times\mathbf{p})\cdot\mathbf{q}]\ \mathbf{a}-[(\mathbf{c}\times\mathbf{p})\cdot\mathbf{a}]\ \mathbf{q}\}$$

$$=[\mathbf{bra}]\ [\mathbf{cpq}]-[\mathbf{brq}]\ [\mathbf{cpa}]. \qquad ...(2)$$

And $[\mathbf{a}\times\mathbf{r},\ \mathbf{b}\times\mathbf{p},\ \mathbf{c}\times\mathbf{q}]=[\mathbf{c}\times\mathbf{q},\ \mathbf{a}\times\mathbf{r},\ \mathbf{b}\times\mathbf{p}]$

$$=(\mathbf{c}\times\mathbf{q})\cdot[(\mathbf{a}\times\mathbf{r})\times(\mathbf{b}\times\mathbf{p})]$$

$$=(\mathbf{c}\times\mathbf{q})\cdot\{[(\mathbf{a}\times\mathbf{r})\cdot\mathbf{p}]\ \mathbf{b}-[(\mathbf{a}\times\mathbf{r})\cdot\mathbf{b}]\ \mathbf{p}\}$$

$$=[\mathbf{cqb}]\ [\mathbf{arp}]-[\mathbf{cqp}]\ [\mathbf{arb}]. \qquad ...(3)$$

Adding (1), (2) and (3), we get

$$[\mathbf{a}\times\mathbf{p},\ \mathbf{b}\times\mathbf{q},\ \mathbf{c}\times\mathbf{r}]+[\mathbf{a}\times\mathbf{q},\ \mathbf{b}\times\mathbf{r},\ \mathbf{c}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{r},\ \mathbf{b}\times\mathbf{p},\ \mathbf{c}\times\mathbf{q}]$$

$$=[\mathbf{apc}]\ [\mathbf{bqr}]-[\mathbf{apr}]\ [\mathbf{bqc}]+[\mathbf{bra}]\ [\mathbf{cpq}]-[\mathbf{brq}]\ [\mathbf{cpa}]$$
$$+[\mathbf{cqb}]\ [\mathbf{arp}]-[\mathbf{cqp}]\ [\mathbf{arb}]$$

$$=[\mathbf{apc}]\ [\mathbf{bqr}]-[\mathbf{apr}]\ [\mathbf{bqc}]+[\mathbf{bra}]\ [\mathbf{cpq}]-[\mathbf{bqr}]\ [\mathbf{apc}]$$
$$+[\mathbf{bqc}]\ [\mathbf{apr}]-[\mathbf{cpq}]\ [\mathbf{bra}]$$

$=0$, since $[\mathbf{brq}]=-[\mathbf{bqr}]$, $[\mathbf{cpa}]=-[\mathbf{apc}]$ etc.

Ex. 9. *Prove that*

$$[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]=[\mathbf{abd}]\ [\mathbf{cef}]-[\mathbf{abc}]\ [\mathbf{def}]$$
$$=[\mathbf{abe}]\ [\mathbf{fcd}]-[\mathbf{abf}]\ [\mathbf{ecd}]$$
$$=[\mathbf{cda}]\ [\mathbf{bef}]-[\mathbf{cdb}]\ [\mathbf{aef}].$$

[Rohilkhand 1992]

Sol. We have $[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]$

$$=(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{c}\times\mathbf{d})\times(\mathbf{e}\times\mathbf{f})]$$

$=(\mathbf{a}\times\mathbf{b})\cdot[\mathbf{l}\times(\mathbf{e}\times\mathbf{f})]$, where $\mathbf{l}=\mathbf{c}\times\mathbf{d}$

$$=(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{l}\cdot\mathbf{f})\ \mathbf{e}-(\mathbf{l}\cdot\mathbf{e})\ \mathbf{f}]$$

$$=(\mathbf{a}\times\mathbf{b})\cdot[\{(\mathbf{c}\times\mathbf{d})\cdot\mathbf{f}\}\ \mathbf{e}-\{(\mathbf{c}\times\mathbf{d})\cdot\mathbf{e}\}\ \mathbf{f}]$$

$$=[\mathbf{cdf}]\ [\mathbf{abe}]-[\mathbf{cde}]\ [\mathbf{abf}]$$

$=[\mathbf{abe}]\ [\mathbf{fcd}]-[\mathbf{abf}]\ [\mathbf{ecd}]$, since $[\mathbf{cdf}]=[\mathbf{fcd}]$ etc.

Again $[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]=[\mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f},\ \mathbf{a}\times\mathbf{b}]$

$$=(\mathbf{c}\times\mathbf{d})\cdot[(\mathbf{e}\times\mathbf{f})\times(\mathbf{a}\times\mathbf{b})]$$

$$=(\mathbf{c}\times\mathbf{d})\cdot[\{(\mathbf{e}\times\mathbf{f})\cdot\mathbf{b}\}\ \mathbf{a}-\{(\mathbf{e}\times\mathbf{f})\cdot\mathbf{a}\}\ \mathbf{b}]$$

$$=[\mathbf{cda}]\ [\mathbf{efb}]-[\mathbf{cdb}]\ [\mathbf{efa}]$$

$$=[\mathbf{cda}]\ [\mathbf{bef}]-[\mathbf{cdb}]\ [\mathbf{aef}].$$

And $[\mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d},\ \mathbf{e}\times\mathbf{f}]=[\mathbf{e}\times\mathbf{f},\ \mathbf{a}\times\mathbf{b},\ \mathbf{c}\times\mathbf{d}]$

$$=(\mathbf{e}\times\mathbf{f})\cdot[(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})]$$

$$=(\mathbf{e}\times\mathbf{f})\cdot[\{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{d}\}\ \mathbf{c}-\{(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\}\ \mathbf{d}]$$

$$=[\mathbf{efc}]\ [\mathbf{abd}]-[\mathbf{efd}]\ [\mathbf{abc}]$$

$$=[\mathbf{abd}]\ [\mathbf{cef}]-[\mathbf{abc}]\ [\mathbf{def}].$$

Ex. 10. *Prove that*

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0.$$

[Meerut 1969; Gorakhpur 87; Rohilkhand 77; Delhi 80]

Sol. We have

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})=\begin{vmatrix}\mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{d}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{d}\end{vmatrix}=(\mathbf{b}\cdot\mathbf{a})\,(\mathbf{c}\cdot\mathbf{d})-(\mathbf{c}\cdot\mathbf{a})\,(\mathbf{b}\cdot\mathbf{d}) \quad ...(1)$$

Similarly $$(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})=\begin{vmatrix}\mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{d}\\ \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{d}\end{vmatrix}=(\mathbf{c}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{c}\cdot\mathbf{d}) \quad ...(2)$$

and $$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}=(\mathbf{a}\cdot\mathbf{c})\,(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d}). \quad ...(3)$$

Adding (1), (2) and (3), we get

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=0$$

since $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ etc.

Ex. 11. *Establish the identity*

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{d}=[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{a}+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]\ \mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c}$$

for any four vectors **a**, **b**, **c**, **d**. **[Burdwan 1975]**

Hence show that any vector **r** *can always be expressed as a linear combination of three non-coplanar vectors.*

Sol. For complete solution of this question refer § 10.

Equating the two expressions for the value of

$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$, we get

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c}-[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{d}=[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{b}-[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{a}$$

or $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{d}=[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{a}-[\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c}$$

$$=[\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\ \mathbf{a}+[\mathbf{c}\ \mathbf{a}\ \mathbf{d}]\ \mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c}. \quad ...(1)$$

Now let **a**, **b**, **c** be three given non-coplanar vectors and **r** be any vector. Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\neq 0$. Replacing **d** by **r** in (1), we get

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\,\mathbf{r}=[\mathbf{b}\ \mathbf{c}\ \mathbf{r}]\ \mathbf{a}+[\mathbf{c}\ \mathbf{a}\ \mathbf{r}]\ \mathbf{b}+[\mathbf{a}\ \mathbf{b}\ \mathbf{r}]\ \mathbf{c}$$

or $$\mathbf{r}=\frac{[\mathbf{r}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\,\mathbf{a}+\frac{[\mathbf{r}\ \mathbf{c}\ \mathbf{a}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\,\mathbf{b}+\frac{[\mathbf{r}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\,\mathbf{c},$$

which is the required expression for **r** as a linear combination of three non-coplanar vectors **a**, **b**, **c**.

Ex. 12. *If* **a**, **b**, **c** *be a set of non-coplanar vectors and*

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \mathbf{c}'=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

then prove that

$$\mathbf{a}=\frac{\mathbf{b}'\times\mathbf{c}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{b}=\frac{\mathbf{c}'\times\mathbf{a}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']}\ \textit{and}\ \mathbf{c}=\frac{\mathbf{a}'\times\mathbf{b}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']}.$$

[Meerut 1987 S]

Sol. First prove that $[\mathbf{a\ b\ c}]\,[\mathbf{a'\ b'\ c'}]=1$.

[For its complete solution see § 11, part (iii)]

Now $$\mathbf{b'}\times\mathbf{c'}=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a\ b\ c}]}\times\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a\ b\ c}]}=\frac{(\mathbf{c}\times\mathbf{a})\times(\mathbf{a}\times\mathbf{b})}{[\mathbf{a\ b\ c}]^2}$$

$$=\frac{\{(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}\}\,\mathbf{a}-\{(\mathbf{c}\times\mathbf{a})\cdot\mathbf{a}\}\mathbf{b}}{[\mathbf{a\ b\ c}]^2}$$

$$=\frac{[\mathbf{c\ a\ b}]\,\mathbf{a}-[\mathbf{c\ a\ a}]\,\mathbf{b}}{[\mathbf{a\ b\ c}]^2}$$

$$=\frac{[\mathbf{a\ b\ c}]\,\mathbf{a}}{[\mathbf{a\ b\ c}]^2},\text{ since }[\mathbf{c\ a\ a}]=0$$

$$=\frac{\mathbf{a}}{[\mathbf{a\ b\ c}]}.$$

$\therefore\quad \mathbf{a}=[\mathbf{a\ b\ c}]\,(\mathbf{b'}\times\mathbf{c'})=\dfrac{\mathbf{b'}\times\mathbf{c'}}{[\mathbf{a'\ b'\ c'}]}.$

[$\because$ $[\mathbf{a\ b\ c}]\,[\mathbf{a'\ b'\ c'}]=1$].

Similarly prove that $\mathbf{b}=\dfrac{\mathbf{c'}\times\mathbf{a'}}{[\mathbf{a'\ b'\ c'}]}$ and $\mathbf{c}=\dfrac{\mathbf{a'}\times\mathbf{b'}}{[\mathbf{a'\ b'\ c'}]}$.

Ex. 13. *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *and* $\mathbf{a'}, \mathbf{b'}, \mathbf{c'}$ *are reciprocal system of vectors, prove that*

(*i*) $\mathbf{a}\times\mathbf{a'}+\mathbf{b}\times\mathbf{b'}+\mathbf{c}\times\mathbf{c'}=\mathbf{0}$,

[Rohilkhand 1990; Meerut 89; Agra 87]

(*ii*) $\mathbf{a'}\times\mathbf{b'}+\mathbf{b'}\times\mathbf{c'}+\mathbf{c'}\times\mathbf{a'}=\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{a\ b\ c}]}$, **[Agra 1988]**

and (*iii*) $\mathbf{a}\cdot\mathbf{a'}+\mathbf{b}\cdot\mathbf{b'}+\mathbf{c}\cdot\mathbf{c'}=3$. **[Rohilkhand 1979]**

Sol. Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a'}, \mathbf{b'}, \mathbf{c'}$ are reciprocal system of vectors, therefore

$$\mathbf{a'}=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a\ b\ c}]},\ \mathbf{b'}=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a\ b\ c}]},\ \mathbf{c'}=\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a\ b\ c}]}.$$

(i) We have $\mathbf{a}\times\mathbf{a'}+\mathbf{b}\times\mathbf{b'}+\mathbf{c}\times\mathbf{c'}$

$$=\frac{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})}{[\mathbf{a\ b\ c}]}$$

$$=\frac{(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b}}{[\mathbf{a\ b\ c}]}$$

$$=\frac{1}{[\mathbf{a\ b\ c}]}\,\mathbf{0}\ [\because\ \mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}\text{ etc.}]$$

$$=\mathbf{0}.$$

(ii) We have $$\mathbf{a'}\times\mathbf{b'}=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a\ b\ c}]}\times\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a\ b\ c}]}=\frac{(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})}{[\mathbf{a\ b\ c}]^2}$$

$$=\frac{\{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}\}\,\mathbf{c}-\{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}\}\,\mathbf{a}}{[\mathbf{a\ b\ c}]^2}=\frac{[\mathbf{b\ c\ a}]\,\mathbf{c}-[\mathbf{b\ c\ c}]\,\mathbf{a}}{[\mathbf{a\ b\ c}]^2}$$

$$=\frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2},\ \text{since } [\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \text{ and } [\mathbf{b}\ \mathbf{c}\ \mathbf{c}]=0$$

$$=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ \mathbf{c}.$$

Similarly we can show that

$$\mathbf{b}'\times\mathbf{c}'=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ \mathbf{a} \text{ and } \mathbf{c}'\times\mathbf{a}'=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ \mathbf{b}.$$

$$\therefore\quad \mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'=\frac{1}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

(iii) We have $\mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}'+\mathbf{c}\cdot\mathbf{c}'$

$$=\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}+\mathbf{b}\cdot\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}+\mathbf{c}\cdot\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$=\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}+\frac{\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}+\frac{\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$=\frac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]+[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$=\frac{3[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ \text{since } [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]=[\mathbf{c}\ \mathbf{a}\ \mathbf{b}]$$

$$=3.$$