LINEAR ALGEBRA

1. VECTOR SPACES & SUBSPACES
2. LINEAR DEPENDENCE AND INDEPENDENCE
3. BASIS & DIMENSIONS
4. RANGE SPACE & NULL SPACE, RANK & NULLITY

   *ALGEBRA OF LTs
5. TO FIND MATRIX OF LT
6. TO FIND LT WHEN MATRIX IS GIVEN

1. VECTOR SPACES & SUBSPACES

1. (1a) 2020

Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. Give examples of two distinct 2×2 magic squares. 10

2. (1b) 2017

Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where

(i) W contains all 2×2 matrices with zero determinant.

(ii) W consists of all 2×2 matrices A such that $A^2 = A$. 8

3. (1a) 2014

Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1, 2, 3) and (1, −1, 1). 10

4. (2a(i)) 2012

(i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer. 8

5. (1b) 2012 IFoS

(b) Show that the set of all functions which satisfy the differential equation

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$ is a vector space. 8

2. LINEAR COMBINATION, LINEAR DEPENDENCE AND INDEPENDENCE

1. (4b(ii)) 2020

(ii) Express the vector (1, 2, 5) as a linear combination of the vectors (1, 1, 1), (2, 1, 2) and (3, 2, 3), if possible. Justify your answer. *9+6=15*

2. (1b) 2018

Express basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ as linear combinations of $\alpha_1 = (2, -1)$ and $\alpha_2 = (1, 3)$. 10

3. (4d) 2018 IFoS

If (n + 1) vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set. *10*

4. (3b) 2017 IFoS

) Given that the set {u, v, w} is linearly independent, examine the sets

(i) {u + v, v + w, w + u}

(ii) {u + v, u − v, u − 2v + 2w}

for linear independence. *10*

5. (1a) 2015

The vectors $V_1 = (1, 1, 2, 4)$, $V_2 = (2, -1, -5, 2)$, $V_3 = (1, -1, -4, 0)$ and $V_4 = (2, 1, 1, 6)$ are linearly independent. Is it true ? Justify your answer. 10

6. (2c(ii)) 2013

Show that the vectors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. 8

7. (2c(i)) 2011

(c) (i) Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T : \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z,\ x + 2y + 5z,\ 2x + 4y + 6z).$$

Show that the images of above vectors under T are linearly dependent. Give the reason for the same. 10

8. (4a(i)) 2010

(i) In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n - e_1\}$$

is linearly independent.

3. BASIS & DIMENSIONS

1. 2c 2021

Show that $S = \{(x, 2y, 3x) : x, y \text{ are real numbers}\}$ is a subspace of $R^3(R)$. Find two bases of S. Also find the dimension of S. 15

2. 2a 2021 IFoS

Express the polynomial $f(x) = x^2 + 4x - 3$ over R as linear combination of polynomials $e_1 = x^2 - 2x + 5$, $e_2 = 2x^2 - 3x$, $e_3 = x + 3$. Also, show that the set $\{e_1, e_2, e_3\}$ forms a basis of all quadratic polynomials over R. *10*

3. (3c) 2019

Let

$$A = \begin{pmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{pmatrix}$$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,\middle|\, A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$

15+5=20

4. (3a) 2019 IFoS

3. *(a)* Consider the vectors $x_1 = (1, 2, 1, -1)$, $x_2 = (2, 4, 1, 1)$, $x_3 = (-1, -2, 0, -2)$ and $x_4 = (3, 6, 2, 0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1, x_2, x_3, x_4\}$ is a subspace of $\mathbb{R}^4$, defined as

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : 2\xi_1 - \xi_2 = 0,\ 2\xi_1 - 3\xi_3 - \xi_4 = 0\}$$

Can this subspace be written as $\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathbb{R}\}$? What is the dimension of this subspace? 15

5. (2d) 2018 IFoS

Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$. *10*

6. (2d) 2017

Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim $V = 6$. Find the possible dimensions of subspace $U \cap W$. 10

7. (1b(ii)) 2016

If

$$W_1 = \{(x, y, z) \mid x + y - z = 0\}$$
$$W_2 = \{(x, y, z) \mid 3x + y - 2z = 0\}$$
$$W_3 = \{(x, y, z) \mid x - 7y + 3z = 0\}$$

then find $\dim(W_1 \cap W_2 \cap W_3)$ and $\dim(W_1 + W_2)$. 3

8. (4b) 2015

Find the dimension of the subspace of R^4, spanned by the set

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)\}$$

Hence find its basis. 12

9. (2a) 2015 IFoS

) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$. 10

10. (2a) 2014

Let V and W be the following subspaces of R^4:

$$V = \{(a, b, c, d) : b - 2c + d = 0\} \text{ and}$$

$$W = \{(a, b, c, d) : a = d, b = 2c\}.$$

Find a basis and the dimension of (i) V, (ii) W, (iii) $V \cap W$. 15

11. (1a) 2014 IFoS

Show that $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Express $(5, 3, 4)$ in terms of u_1, u_2 and u_3. 8

12. (2a(ii)) 2013

Let V be an n-dimensional vector space and $T: V \to V$ be an invertible linear operator. If $\beta = \{X_1, X_2, \ldots, X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, \ldots, TX_n\}$ is also a basis of V. 8

13. (1a) 2013

Find the dimension and a basis of the solution space W of the system

$x + 2y + 2z - s + 3t = 0, \quad x + 2y + 3z + s + t = 0, \quad 3x + 6y + 8z + s + 5t = 0$

8

14. (1c) 2012

(c) Prove or disprove the following statement : 12

If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for $\mathbb{R}^5$ and V is a two-dimensional subspace of $\mathbb{R}^5$, then V has a basis made of just two members of B.

15. (2a(ii) 2012

(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation : 12

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 &= 0 \\ 3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 &= 0 \end{aligned}$$

16. (1a) 2012

Let $V = \mathbb{R}^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. 8

17. (2b(i)) 2011

Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1, 1, -1), (1, 0, 1)\}$ and $\{(1, 2, -3), (5, 2, 1)\}$ are identical. Also find the dimension of this subspace. 10

18. (1a) 2011 IFoS

Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbf{R}$. Let
$W = \{A \in V \mid \text{Trace } A = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W. *10*

19. (2a) 2011 IFoS

Let $V = \{(x, y, z, u) \in \mathbf{R}^4 : y + z + u = 0\}$,
$W = \{(x, y, z, u) \in \mathbf{R}^4 : x + y = 0, z = 2u\}$
be two subspaces of $\mathbf{R}^4$. Find bases for V, W, V + W and $V \cap W$. *10*

20. (1a) 2010 IFoS

Show that the set

$$P[t] = \{at^2 + bt + c \,/\, a, b, c \in \mathbf{R}\}$$

forms a vector space over the field $\mathbf{R}$. Find a basis for this vector space. What is the dimension of this vector space ? *8*

21. (2a) 2010 IFoS

Show that the vectors

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

form a basis for $\mathbf{R}^3$. Find the components of $(1, 0, 0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. *10*

4. RANGE SPACE & NULL SPACE, RANK AND NULLITY

1. (1b) 2020

Let $M_2(R)$ be the vector space of all 2×2 real matrices. Let $B=\begin{bmatrix}1 & -1\\ -4 & 4\end{bmatrix}$. Suppose $T: M_2(R)\to M_2(R)$ is a linear transformation defined by $T(A)=BA$. Find the rank and nullity of T. Find a matrix A which maps to the null matrix. 10

2. (3b) 2020

Let F be a subfield of complex numbers and T a function from $F^3\to F^3$ defined by $T(x_1, x_2, x_3)=(x_1+x_2+3x_3, 2x_1-x_2, -3x_1+x_2-x_3)$. What are the conditions on a, b, c such that (a, b, c) be in the null space of T? Find the nullity of T. 15

3. (3a) 2017

Consider the matrix mapping $A: R^4\to R^3$, where $A=\begin{pmatrix}1 & 2 & 3 & 1\\ 1 & 3 & 5 & -2\\ 3 & 8 & 13 & -3\end{pmatrix}$. Find a basis and dimension of the image of A and those of the kernel A. 15

4. (2d) 2014 IFoS

Show that the mapping $T: V_2(\bar{R})\to V_3(\bar{R})$ defined as $T(a, b)=(a+b, a-b, b)$ is a linear transformation. Find the range, rank and nullity of T. 10

5. (2a) 2013 IFoS

) Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$.

Let $F: V\to V$ be the linear map defined by $F(A)=MA$. Find a basis and the dimension of

(i) the kernel of W of F

(ii) the image U of F. 10

6. (3c) 2013 IFoS

Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T ? Find the nullity of T. 10

7. (1d) 2012

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\alpha, \beta, \gamma) = (\alpha + 2\beta - 3\gamma,\ 2\alpha + 5\beta - 4\gamma,\ \alpha + 4\beta + \gamma)$$

Find a basis and the dimension of the image of T and the kernel of T. 12

8. (2b(ii)) 2011

(ii) Find the nullity and a basis of the null space of the linear transformation $A : R^{(4)} \to R^{(4)}$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

10

9. (1b) 2011 IFoS

Find the linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$ which has its range the subspace spanned by (1, 0, −1), (1, 2, 2). *10*

10. (1b) 2010

(b) What is the null space of the differentiation transformation

$$\frac{d}{dx} : P_n \to P_n$$

where P_n is the space of all polynomials of degree $\leq n$ over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the null space of the kth derivative? 12

*ALGEBRA OF LTs

1. (4a(ii)) 2010

(ii) Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible. 20

5. TO FIND MATRIX OF A LT

1. 1b 2021

Find the matrix associated with the linear operator on $V_3(R)$ defined by $T(a, b, c) = (a + b, a - b, 2c)$ with respect to the ordered basis $B = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$. 10

2. (2a) 2020

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x, -3y, x + y)$, and $B_1 = \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\}$ be a basis of $\mathbb{R}^3$. Find the matrix representation of T relative to the basis B_1. 10

3. (1c) 2019

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2, 1) = (5, 7)$ and $T(1, 2) = (3, 3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2, then find Rank (A). 10

4. (1a) 2019 IFoS

1. (a) Let $T : R^3 \to R^3$ be a linear operator on R^3 defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix of T in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 8

5. (3c) 2018 IFoS

Let $T : V_2(R) \to V_2(R)$ be a linear transformation defined by $T(a, b) = (a, a + b)$. Find the matrix of T, taking $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1), (1, -1)\}$ as a basis for the range. 10

6. (2a(i)) 2016

If $M_2(R)$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T : M_2(R) \to P_2(x)$, such that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + c + (a - d)x + (b + c)x^2$, with respect to the standard bases of $M_2(R)$ and $P_2(x)$. Further find the null space of T. 10

7. (2a(ii)) 2016

If $T: P_2(x) \to P_3(x)$ is such that $T(f(x)) = f(x) + 5\int_0^x f(t)\,dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T. 6

8. (1a) 2016 IFoS

Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y,\ 2x + z,\ x + 2z,\ x + y + z)$.

Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (i.e., (1, 0, 0), (0, 1, 0), etc.). Examine if T is a linear map. 8

9. (2d) 2016 IFoS

Let T be a linear map such that $T: V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis $B_1[(1, 1, 0)\ (1, 0, 1)\ (0, 1, 1)]$ and $B_2[(1, 1)\ (1, -1)]$. Obtain the matrix T_1 relative to B_1 and B_2. 10

10. (3a) 2015

Let $V = \mathbb{R}^3$ and $T \in A(V)$, for all $a_i \in A(V)$, be defined by

$$T(a_1, a_2, a_3) = (2a_1 + 5a_2 + a_3, -3a_1 + a_2 - a_3, -a_1 + 2a_2 + 3a_3)$$

What is the matrix T relative to the basis

$V_1 = (1, 0, 1)$ $\quad V_2 = (-1, 2, 1)$ $\quad V_3 = (3, -1, 1)$? 12

11. (1b) 2015 IFoS

Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$ 8

12. 4d 2014 IFoS

Consider the linear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given as $F(x, y) = (3x + 4y, 2x - 5y)$ with usual basis.

Find the matrix associated with the linear transformation relative to the basis $S = \{u_1, u_2\}$ where $u_1 = (1, 2)$, $u_2 = (2, 3)$. 10

13. (2a(i)) 2013

Let P_n denote the vector space of all real polynomials of degree atmost n and $T : P_2 \to P_3$ be a linear transformation given by

$$T(p(x)) = \int_0^x p(t)\,dt, \qquad p(x) \in P_2.$$

Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. 10

14. (2b(i)) 2012

(i) Consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (3x + 4y, 2x - 5y)$$

Find the matrix A relative to the basis $\{(1, 0), (0, 1)\}$ and the matrix B relative to the basis $\{(1, 2), (2, 3)\}$. 12

15. (2a) 2012 IFoS

Let $f : \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a + b, 0)$.

Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0)$, $e_2' = (0, 1, 1)$, $e_3' = (1, 1, 1)$.

Also, show that there exists an invertible matrix P such that

$$B = P^{-1}AP$$ 10

16. (3a) 2012 IFoS

Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as

$T(a, b, c) = (2b + c, a - 4b, 3a)$

corresponding to the basis

$B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 10

6. TO FIND LT WHEN MATRIX IS GIVEN

1. (2c) 2016

If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a linear transformation $T: P_2(x) \to P_2(x)$ with respect to the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$, then find T. 18

2. (2a) 2010

2. (a) Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ so that M is the matrix of T with respect to the basis

$$\beta = \{v_1 = (1, 0, 0),\ v_2 = (1, 1, 0),\ v_3 = (1, 1, 1)\}$$

of $\mathbb{R}^3$ and

$$\beta' = \{w_1 = (1, 0),\ w_2 = (1, 1)\}$$

of $\mathbb{R}^2$. Also find $T(x, y, z)$. 20