4

Rigid Body Motion

INCLUDING

(MOTION IN THREE DIMENSIONS)

4·1. Degrees of freedom.

The number of co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

Ex. 1. *A particle moving freely in space requires 3 co-ordinates ; e.g. (x, y, z) to specify the position. Thus the number of degrees of freedom is 3.*

Ex. 2. *A system (rigid body)* consisting of N particles moving freely in space requires $3N$ co-ordinates to specify its position. Thus the number of degrees of freedom is $3N$.*

However, since the body is a rigid one, the actual number of degree of freedom is greatly reduced because of the presence of the constraints. Theoretically, the number of degrees of freedom can be obtained by subtracting the number of independent constraints *that there exist six degrees of freedom for a rigid body.* This further implies that there are only $3N-6$ constraints independent of each other.

Ex. 3. *Find the number of degrees of freedom for a rigid body which (a) can move freely in three dimensional space, (b) has one point fixed but can move in space about this point.*

Solution. (a) If three non-colinear points of a rigid body are fixed in space, then the rigid body is also fixed in space. Let the co-ordinates of these points be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively. But the body is rigid, so we have $(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=\text{constant}$, $(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2=\text{constant}$; $(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2=\text{constant}$. This implies that 3 co-ordinates can be expressed in terms of the remaining six. Hence six independent co-ordinates describe the motion implying that the number of degrees of freedom is six.

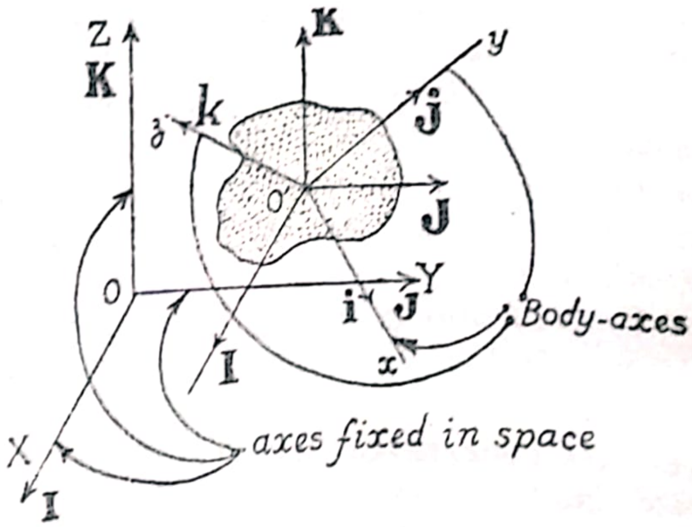
*Rigid body is understood to be a system of particles such that the distance between the particles do not vary.

(b) Let the fixed point be regarded as origin and let the co-ordinates of other two point be (x_2, y_2, z_2), (x_3, y_3, z_3). The body is rigid so we must have

$(x_2-0)^2+(y_2-0)^2+(z_2-0)^2=\text{constant}$, $(x_3-0)^2+(y_3-0)^2+(z_3-0)^2=\text{constant}$, $(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2=\text{constant}$. This shows that 3 co-ordinates can be obtained in term of the remaining three and hence are three degrees of freedom.

From above (case a) it is evident that only six generalised co-ordinates are needed to fix the configuration of a rigid body. They can also be seen from the following consideration.

The configuration of a rigid body w.r.t. some cartesian co-ordinate system fixed in space is completely determined provided the position of the origin and the cartesian co-ordinate system (fixed in the body) are known. In order to fix the origin of the axes fixed in the body, we need three co-ordinates. The axes fixed in the body can be specified by using the direction cosines of the body-axis w.r.t. the axis fixed in the space.



Let $(\alpha_r, \beta_r, \gamma_r)$ $(r=1, 2, 3)$ be the direction consines of the lines $O'x$, $O'y$ and $O'z$, then by the adjoining diagram, we have

$$\left.\begin{array}{l} \mathbf{i}=\alpha_1\mathbf{I}+\beta_1\mathbf{J}+\gamma_1\mathbf{K} \\ \mathbf{j}=\alpha_2\mathbf{I}+\beta_2\mathbf{J}+\gamma_2\mathbf{K} \\ \mathbf{k}=\alpha_3\mathbf{I}+\beta_3\mathbf{J}+\gamma_3\mathbf{K} \end{array}\right\} \qquad \ldots(1)$$

and

where

$$\alpha_r^2+\beta_r^2+\gamma_r^2=1; \; (r=1, 2, 3).$$

Also

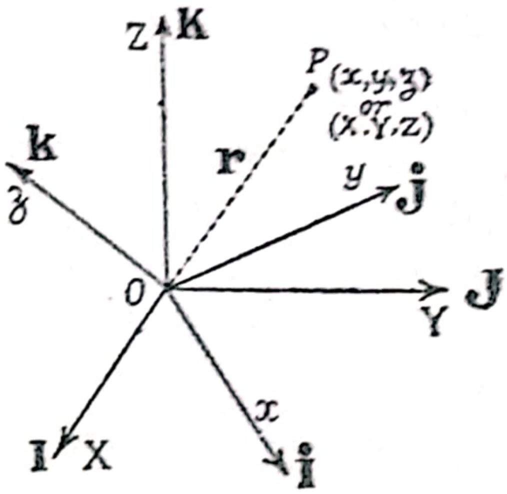
$$\left.\begin{array}{l} \mathbf{i}\cdot\mathbf{j}=0 \Rightarrow \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0 \\ \mathbf{j}\cdot\mathbf{k}=0 \Rightarrow \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3=0 \\ \mathbf{k}\cdot\mathbf{i}=0 \Rightarrow \alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1=0 \end{array}\right\} \qquad \ldots(2)$$

Equations (1) and (2) yield six relations amongst the nine direction cosines. Hence only three direction cosines are independent which further implies that there exist only six degree of freedom; three for fixing the origin and three for specifying the orientation.

But these direction cosines are not independent to each other so we cannot form Lagrange's equations using these as generalised co-ordinates. For doing this we must use some set of three independent functions of these direction cosines. This can be done by a number of methods. Eulerian angles method being of a great use. But before proceeding to this method, we shall discuss briefly the idea of orthogonal transformation and matrix algebra.

4·2. Orthogonal Transformations.

Let the origin of the two co-ordinate system (Space axis and Body axis) be the same and let P be any point such that $\mathbf{OP}=\mathbf{r}$.



Now projection of OP on Ox is given by

$$x=X\alpha_1+Y\beta_1+Z\gamma_1,$$

where $(\alpha_r, \beta_r, \gamma_r)$ $(r=1, 2, 3)$ are the d.s.c. of Ox, Oy and Oz respectively. ...(1)

Similarity projections on other axis are given by

$$y=X\alpha_2+Y\beta_2+Z\gamma_2 \quad ...(2) \quad \text{and} \quad z=X\alpha_3+Y\beta_2+Z\gamma_2 \quad ...(3)$$

To write the matrix relations, we write (x_1, x_2, x_3) in place of (X, Y, Z) so as to get

$$x=\alpha_1x_1+\beta_1x_2+\gamma_1x_3 \quad ...(4)$$

$$y=\alpha_2x_1+\beta_2x_2+\gamma_2x_3 \quad ...(5)$$

and

$$z=\alpha_3x_1+\beta_3x_2+\gamma_3x_3 \quad ...(6)$$

Equations (4), (5) and (6) can be further written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad ...(7)$$

The above relation being the particular case of

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad ...(8)$$

Further writting (x_1', x_2', x_3') in place of (x, y, z) ; relation (8) gives

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad ...(9)$$

which may be further abberviated as

$$P = AQ \qquad ...(10)$$

The matrix A is known as the matrix of transformation. The magnitude of $\mathbf{OP} = \sqrt{(x^2+y^2+z^2)} = (X^2+Y^2+Z^2)$

$$= \sqrt{(x_1'^2+x_2'^2+x_3'^2)} = \sqrt{(x_1^2+x_2^2+x_3^2)}.$$

This means that OP remains unchanged irrespective of the primed or the unprimed system. Further we know already that the matrix A^T is obtained by interchanging the rows and columns of A and is termed as transpose of A, *i.e.* $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Also, we have $\quad P^T = (x_1' \; x_2' \; x_3')$

and $\quad P^T . P = (x_1' \; x_2' \; x_3') \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$

$$= (x_1'^2 + x_2'^2 + x_3'^2) = \sum_{i=1}^{3} x_i'^2 \qquad ...(11)$$

Further also $\quad P = AQ \Rightarrow P^T = (AQ)^T = Q^T T^T \qquad ...(12)$

and $\quad P^T P = (Q^T A^T)(AQ) \Rightarrow \sum_{i=1}^{3} x_i'^2 = Q^T(A^T A)\, Q \qquad ...(13)$

Now let $\quad A^T = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^T$, then (15) yields

$$\sum_{i=1}^{2} x_i'^2 = Q^T I Q = Q^T Q = \sum_{i=1}^{3} x_i^2$$

$$\left[\because \quad Q^T Q = (x_1 \; x_2 \; x_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right]$$

where I is the unit matrix.

When the matrix A satisfies the relation $A^T A = I = AA^T$, it is said to be an orthogonal matrix and $A^T A = I = AA^T$ is termed as the condition of orthogonality.

To obtain second relation.

Solving equations (4), (5), (6) for x_1, x_2, x_3 we get

$$\left.\begin{aligned} x_1 &= a_{11}'x + a_{12}'y + a_{13}'z \\ x_2 &= a_{21}'x + a_{22}'y + a_{23}'z \\ x_3 &= a_{31}'x + a_{32}'y + a_{33}'z \end{aligned}\right\} \qquad \ldots(15)$$

Replacing x, y, z by (x_1', x_2', x_3'), we get

$$\left.\begin{aligned} x_1 &= a_{11}'x_1' + a_{12}'x_2' + a_{13}'x_3' \\ x_2 &= a_{21}'x_1' + a_{22}'x_2' + a_{23}'x_3' \\ x_3 &= a_{31}'x_1' + a_{32}'x_2' + a_{33}'x_3' \end{aligned}\right\} \qquad \ldots(16)$$

and

In matrix form, we have $Q = BP$

Expand form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

where B is formed by the constant coefficients a'_{ij} in (18).

$$\Rightarrow Q = BAQ \qquad (\because \quad P = AQ) \qquad \ldots(17)$$

which is true, if and only if $BA = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ...(18)

Hence the matrix B is such that $BA = I$, which implies that B is the *inverse of A, *viz.* $B = A^{-1}$ *i.e.* $Q = A^{-1} P$ ...(19)

Now comparing (19) and the condition of orthogonality, one can say that the transpose and the inverse of a matrix in orthogonal transformations are equal.

Second order Orthogonal matrix in One parameter.

Let the second order orthogonal matrix is given by

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

$$\Rightarrow \quad AA^T = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} = \begin{pmatrix} a_1^2 + a_2^2 & a_1 a_3 + a_2 a_4 \\ a_3 a_1 + a_4 a_2 & a_3^2 + a_3^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} \qquad \text{(by orthogonality}$$

$$\therefore \quad a_1^2 + a_3^2 = 1,\ a_1 a_3 + a_2 a_4 = 0,\ a_3 a_1 + a_2 a_4 = 0,\ a_3^2 + a_4^2 = 1 \qquad \ldots(21)$$

Let us choose $a_1 = \cos\theta$, $a_2 = \sin\theta$ so that $a_1^2 + a_2^2 = 1$ is fully satisfied and the third gives

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*Inverse of a matrix exists if it is non singular *viz.* determinant of the matrix is non-zero.

$$\frac{a_3}{a_4}=-\frac{a_2}{a_1}=-\tan\theta \Rightarrow a_3=-a_4\tan\theta \Rightarrow a_4^2\tan^2\theta+a_4^2=1$$

$$\Rightarrow a_4=\pm\cos\theta \text{ so that } a_3=\mp\sin\theta \qquad ...(22)$$

If we choose the upper sign, then we get

$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \qquad ...(23)$$

where a choosing the lower sign, we get

$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \qquad ...(24)$$

As a special case putting $\theta=0$ in (23) and (24), we get

$$A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Operating these matrices by the matrix $\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ and calling the new matrix by $\begin{pmatrix}x_1'\\ x_2'\end{pmatrix}$, we get

$$\begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \text{ and } \begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x_1\\ x_2'\end{pmatrix}=\begin{pmatrix}x_1\\ x_2\end{pmatrix} \text{ and } \begin{pmatrix}x_1'\\ x_2'\end{pmatrix}=\begin{pmatrix}x_1\\ -x_2\end{pmatrix}$$

$$\Rightarrow x_1'=x_1,\ x_2'=x_2 \text{ and } x_1'=x_1,'=x_2'-x_2$$

$\Rightarrow$ (transformation is an identity transformation)

and (transformation corresponds to the reflection through the origin as x_1 is unchanged and x_2 reverse the sign)

On the same lines, we can prove that the matrix,

$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix} \text{ represents a reflection.}$$

Further, if the transformation is orthogonal, we have

$$AA^T=I \Rightarrow \det(A).\det(A^T)=1$$

$$\Rightarrow \det A.\det A=1 \Rightarrow \det A=\pm 1 \qquad ...(25)$$

This shows that det A is either $+1$ or -1 provided A is an orthogonal matrix.

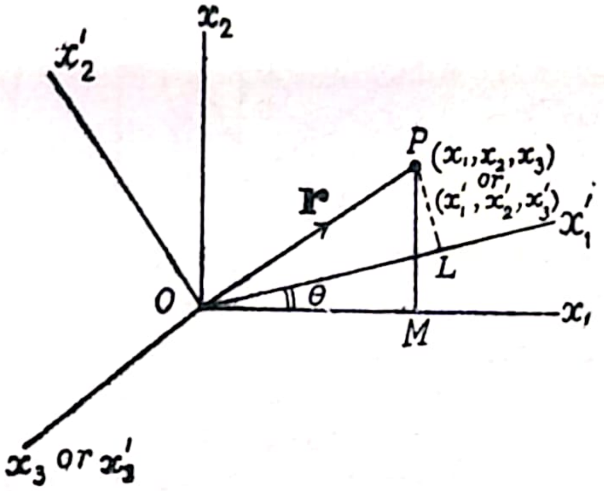
Now let us consider

$$A_+=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

and

$$A_-=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & -1\end{bmatrix} \qquad ...(26)$$

Also, let (x_1, x_2, x_3), (x_1', x_2', x_3') be the co-ordinates of the same point P w.r.t. the two co-ordinate systems, where the axis x_1' and x_2' have been turned through an angle θ in the counter-clockwise direction. Then we have



$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A_+ . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \\ x_3 \end{bmatrix}$$

$$\Rightarrow x_1' = x_1 \cos\theta + x_2 \sin\theta,\ x_2' = -x_1 \sin\theta + x_2 \cos\theta,\ x_3' = x_3 \qquad \text{...(27)}$$

whereas the relation $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A_- \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow x_1' = x_1 \cos\theta + x_2 \sin\theta,\ x_2' = -x_1 \sin\theta + x_2 \cos\theta$$
$$x_3' = -x_2 \qquad \text{...(28)}$$

Relations (27) and (28) are called *proper* and *improper* rotations, but we are mainly concerned with proper rotations.

It is worth noticing that an orthogonal transformation in n-dimensions is given by

$$A = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & \ldots & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ldots & \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \qquad \text{...(29)}$$

$$\text{where } A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & \dots & 0 \\ \sin\theta & \cos\theta & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & 0 & \dots & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \qquad \dots(30)$$

$$\text{Obviously } AA^T = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & & \\ 0 & 0 & 0 & \dots & \dots & 1 \end{bmatrix} \qquad \dots(31)$$

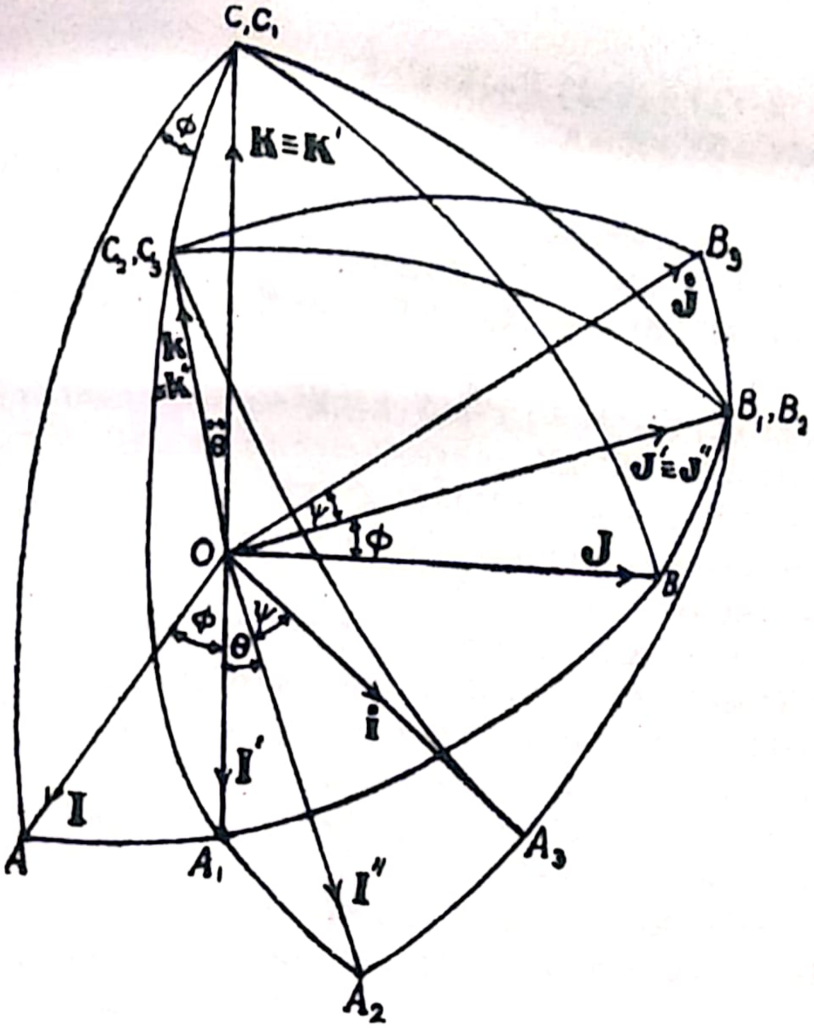
4·3. Eulerian Angles.

In order to describe the rotation of a rigid body about a point (say O), we use three angular co-ordinates θ, ϕ, ψ which are called *Euler angles* describe below. Suppose a rigid body rotates in any manner about a fixed point O. Then the motion of the rigid body as a whole is known if that of a spherical octant $OABC$ (or the triad **I**, **J**, **K**) is known. The figure on p. 177, shows two spherical octants $OA_3B_3C_3$ and $OABC$ (or the two unit orthogonal right handed triads **i**, **j**, **k** and **I**, **J**, **K** at O). The triad (**i**, **j**, **k**) is fixed in the rigid body which is turning about O but the triad (**I**, **J**, **K**) is fixed in space. The first Eulerian angle θ is the angle between **k** and **K**, the second Eulerian angle ϕ is the angle between the plane (**k**, **K**) and the plane (**K**, **I**) whereas the third Eulerian angle ψ is the angle of rotation about **k**. Let us take an initial position in which the octant $OA_3B_3C_3$ (the triad **i**, **j**, **k**) coincide with the octant $OABC$ (**I**, **J**, **K**). We can bring the body to the general position shown in the adjoining diagram, by applying the following rotations in order :

(*i*) *Rotate the octant $OA_3B_3C_3$ initially coincide with $OABC$, about OC through an angle ϕ to assume the new configuration $OA_1B_1C_1$ (the triad* (**I**′, **J**′, **K**′) *where $C_1 \equiv C$;*

(*ii*) *Rotate the octant $OA_1B_1C_1$ (the triad* **I**′, **J**′, **K**′≡**K**), *about OB_1 through an angle θ to assume the new configuration $OA_2B_2C_2$ (the triad* **I**″. **J**″≡**J**′, **K**″) *where $B_2 \equiv B_1$:*

(*iii*) *Rotate the octant $OA_2B_2C_2$ (the triad* **I**″, **J**′≡**J**, **K**″) *about OC_2 through an angle ψ to assume the configuration $OA_3B_3C_3$ (the triad* **i**, **j**, **k**≡**K**″) *where $C_3 \equiv C_2$. This rotatian brings the triad* (**i**, **j**, **k**), *initially along* (**I**, **J**, **K**), *into the required final position.*



A little through will convince the reader that starting from any given configuration $OABC$ one can obtain any other configuration $OA_3B_3C_3$ by suitably choosing θ, ϕ, ψ. Thus the motion of the octant $OA_3B_3C_3$ of the rigid body and hence that of the body as a whole, is known relative to some initial configuration $OABC$ (**I**, **J**, **K**), if we know θ, ϕ, ψ. They are called *Eulerian angles.*

To obtain the relationships between the unit vectors (a) I, J, K and I′, J′, K′ ; (b) I′, J′, K′ and I″, J″, K″ ; (c) I″, J″, K″ and i, j, k ; (d) I, J, K and i, j; k.

(a) From figure, we have

$$\mathbf{I}=(\mathbf{I}\cdot\mathbf{I}')+(\mathbf{I}\cdot\mathbf{J}')\,\mathbf{J}'+(\mathbf{I}\cdot\mathbf{K}')\,\mathbf{K}'$$

$$=\cos\phi\,\mathbf{I}'+\cos\left(\tfrac{1}{2}\pi+\phi\right)\mathbf{J}'+(\mathbf{I}\cdot\mathbf{K})\,\mathbf{K} \qquad (\because\ \mathbf{K}'\equiv\mathbf{K})$$

$$=\cos\phi\,\mathbf{I}'-\sin\phi\,\mathbf{J}' \qquad \text{.. (1)}$$

$$\mathbf{J}=(\mathbf{J}\cdot\mathbf{I}')\,\mathbf{I}'+(\mathbf{J}\cdot\mathbf{J}')+(\mathbf{J}\cdot\mathbf{K}')\,\mathbf{K}'$$

$$=\cos\left(\tfrac{1}{2}\pi-\phi\right)\mathbf{I}'+\cos\phi\,\mathbf{J}'+(\mathbf{J}\cdot\mathbf{K})\,\mathbf{K} \qquad (\because\ \mathbf{K}'\equiv\mathbf{K})$$

$$=\sin\phi\,\mathbf{I}'+\cos\phi\,\mathbf{J}' \qquad \text{...(2)}$$

$$\mathbf{K}=(\mathbf{K}\cdot\mathbf{I}')\,\mathbf{I}'+(\mathbf{K}\cdot\mathbf{J}')\,\mathbf{J}'+(\mathbf{K}\cdot\mathbf{K}')\,\mathbf{K}'$$
$$=0\mathbf{I}'+0\mathbf{J}'+\mathbf{K}'=\mathbf{K}'. \qquad \ldots(3)$$
$$(\because\ \mathbf{K}'\equiv\mathbf{K})$$

(b) From figure, we have

$$\mathbf{I}'=(\mathbf{I}'\cdot\mathbf{I}'')\,\mathbf{I}''+(\mathbf{I}'\cdot\mathbf{J}'')\,\mathbf{J}''+(\mathbf{I}'\cdot\mathbf{K}'')\,\mathbf{K}''$$
$$=\cos\theta\mathbf{I}''-0\mathbf{J}''+\cos\left(\tfrac{1}{2}\pi-\theta\right)\mathbf{K}''=\cos\theta\mathbf{I}''+\cos\left(\tfrac{1}{2}\pi-\theta\right)\mathbf{K}''$$
$$=\cos\theta\mathbf{I}''+\sin\theta\mathbf{K}'' \qquad \ldots(4)$$
$$\mathbf{J}'=(\mathbf{J}'\cdot\mathbf{I}'')\,\mathbf{I}''+(\mathbf{J}'\cdot\mathbf{J}'')\,\mathbf{J}''+(\mathbf{J}'\cdot\mathbf{K}'')\,\mathbf{K}''$$
$$=0\mathbf{I}''+\mathbf{J}''+0\mathbf{K}''=\mathbf{J}'' \qquad \ldots(5)$$
$$\mathbf{K}'=(\mathbf{K}'\cdot\mathbf{I}'')\,\mathbf{I}''+(\mathbf{K}'\cdot\mathbf{J}'')\,\mathbf{J}''+(\mathbf{K}'\cdot\mathbf{K}'')\,\mathbf{K}''$$
$$=\cos\left(\tfrac{1}{2}\pi+\theta\right)\mathbf{I}''+0\mathbf{J}''+\cos\theta\mathbf{K}''$$
$$=-\sin\theta\mathbf{I}''+\cos\theta\mathbf{K}''. \qquad \ldots(6)$$

(c) From figure, we have

$$\mathbf{I}''=(\mathbf{I}''\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{I}''\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{I}''\cdot\mathbf{k})\,\mathbf{k}=\cos\psi\mathbf{i}+\cos\left(\tfrac{1}{2}\pi+\psi\right)\mathbf{j}. \qquad \ldots(7)$$
$$\mathbf{J}''=(\mathbf{J}''\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{J}''\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{J}''\cdot\mathbf{k})\,\mathbf{k}=\cos\left(\tfrac{1}{2}\pi-\psi\right)\mathbf{i}+\cos\psi\mathbf{j} \qquad \ldots(8)$$
$$\mathbf{K}''=(\mathbf{K}''\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{K}''\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{K}''\cdot\mathbf{k})\,\mathbf{k}=\mathbf{k}. \qquad \ldots(9)$$
$$\therefore\ \mathbf{I}'=\cos\theta\mathbf{I}''+\sin\theta\mathbf{K}''=\cos\theta\,(\cos\psi\mathbf{i}-\sin\psi\mathbf{j})+\sin\theta\mathbf{k} \qquad \ldots(10)$$
$$\mathbf{J}'=\mathbf{J}''=\sin\psi\mathbf{i}+\cos\psi\mathbf{j} \qquad \ldots(11)$$
$$\mathbf{K}'=-\sin\theta\,(\cos\psi\mathbf{i}-\sin\psi\mathbf{j})+\cos\theta\,(\mathbf{k}) \qquad \ldots(12)$$

and $\mathbf{I}=(\cos\phi\mathbf{I}'-\sin\phi\mathbf{J}')=\cos\phi\,(\cos\theta\cos\psi\mathbf{i}-\cos\theta\sin\psi\mathbf{j}$
$$+\sin\theta\mathbf{k})-\sin\phi\,(\sin\psi\mathbf{i}+\cos\psi\mathbf{j})$$
$$=\cos\theta\cos\phi\cos\psi\mathbf{i}-\cos\phi\cos\theta\sin\psi\mathbf{j}+\cos\phi\sin\theta\mathbf{k}$$
$$-\sin\phi\sin\psi\mathbf{i}-\sin\phi\cos\psi\mathbf{j}. \qquad \ldots(13)$$
$$\mathbf{J}=\sin\phi\mathbf{I}'+\cos\phi\mathbf{J}'=\sin\phi\,(\cos\theta\mathbf{I}''+\sin\theta\mathbf{k})$$
$$+\cos\phi\,(\sin\psi\mathbf{i}+\cos\psi\mathbf{j})$$
$$=\sin\phi\cos\theta\,(\cos\psi\mathbf{i}-\sin\psi\mathbf{j})+\sin\phi\sin\theta\mathbf{k}$$
$$+\cos\phi\sin\psi\mathbf{i}+\cos\phi\sin\psi\mathbf{j}$$
$$=(\sin\phi\cos\theta\cos\psi+\cos\phi\sin\psi)\,\mathbf{i}+(\cos\phi\cos\psi$$
$$-\cos\theta\sin\phi\sin\psi)\,\mathbf{j}+\sin\theta\sin\phi\mathbf{k} \qquad \ldots(14)$$
$$\mathbf{K}=\mathbf{K}'=-\sin\theta\cos\psi\mathbf{i}+\sin\theta\sin\psi\mathbf{j}+\cos\theta\mathbf{k}. \qquad \ldots(15)$$

To obtain the spin components about OA_3, OB_3, OC_3

$$\vec{\omega}=\omega_\phi\mathbf{K}+\omega_\theta\mathbf{J}+\omega_\psi\mathbf{k}=\dot{\phi}\mathbf{K}+\dot{\theta}\mathbf{J}'+\dot{\psi}\mathbf{k}$$
$$=\dot{\phi}\,(-\sin\theta\cos\psi\mathbf{i}+\sin\theta\sin\psi\mathbf{j}+\cos\theta\mathbf{k})$$
$$+\dot{\theta}\,(\sin\psi\mathbf{i}+\cos\psi\mathbf{j})+\dot{\psi}\mathbf{k}$$
$$=(\sin\psi\dot{\theta}-\sin\theta\cos\psi\dot{\phi})\,\mathbf{i}+(\cos\psi\dot{\theta}+\sin\theta\sin\psi\dot{\phi})\,\mathbf{j}$$
$$+(\cos\theta\dot{\phi}+\dot{\psi})\,\mathbf{k}. \qquad \ldots(16)$$

Now let ω_1, ω_2, ω_3 be the spin components about **i**, **j**, **k**, then we get

$$\vec{\omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k} \qquad \ldots(17)$$

Whence (16) and (17),

$$\Rightarrow \omega_1=\sin\psi\dot{\theta}-\sin\theta\cos\psi\dot{\phi} \quad ...(18)$$
$$\omega_2=\cos\psi\dot{\theta}+\sin\theta\sin\psi\dot{\phi} \quad ...(19)$$
$$\omega_3=\cos\theta\dot{\phi}+\dot{\psi} \quad ...(20)$$

Equations (18), (19) and (20) are known as **Euler's Geometrical Equations.**

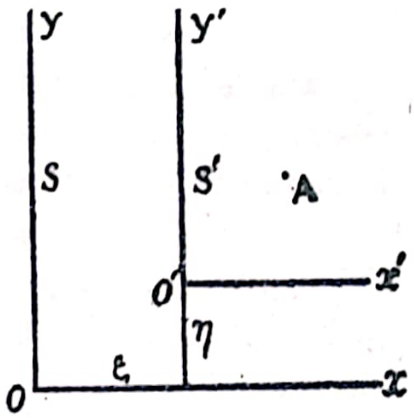
Note 1. In an infinitestimal time $d\cdot$, the body receives the motion $\vec{\omega}\, dt = dn$ where $dn=(\sin\psi\, d\theta-\sin\theta\cos\psi\, d\phi)\,\mathbf{i}+(\cos\psi\, d\theta+\sin\theta\sin\psi\, d\phi)\,\mathbf{j}$
$+(\cos\theta d\phi+d\psi)\,\mathbf{k}$.

Note 2. These equations can also be deduced by using the property of orthogonal transformations as discussed in 3-2.

4·4. Moving frames of Reference.

(a) Frames of Reference with uniform translation velocity.

Let S be a *Newtonian frame of reference and S' another frame of reference which has relative to S, a uniform translation motion. In S, consider axes Oxy and in S' consider parallel axes $O'\,x'\,y'$. Let ξ, η be the co-ordinates of O' relative to O, then we have $\dot{\xi}=u_1$ and $\dot{\eta}=u_2$, where u_1, u_2 are the constant components of the velocity of S' relative to S' relative of S. Now consider A to be the position of any moving particle ; it has co-ordinates (x, y) relative to Oxy and co-ordinates (x', y') relative to $O'\,x'\,y'$. The co-ordinates are connected by the relations



$$x=x'+\xi,\ y=y'+\eta \Rightarrow \dot{x}=\dot{x}'+\dot{\xi},\ \dot{y}=\dot{y}'+\dot{\eta}... \quad ...(1)$$

If we denote v to be the velocity of A relative to S and $\mathbf{v}'$ to the velocity of A relative of S' then equation (1) may be expressed in the form $\mathbf{v}=\mathbf{v}'+\mathbf{v}_0$ where $\mathbf{v}_0$ is the velocity of S' relative to S. This is called the law of *composition of velocities* and is shown graphically in the adjoining diagram.

Differentiating (1) again w.r.t. "t" we get

$$\ddot{x}=\ddot{x}',\ \ddot{y}=\ddot{y}' \quad ...(2)$$

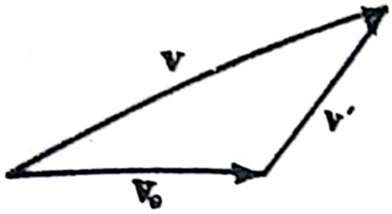
($\because$ $\dot{\xi}$, $\dot{\eta}$ are constants)

*Newtonian frame of reference is the frame of reference relative to which bodies moves in accordance with the Newtonian laws.

which shows that the acceleration relative to S' is equal to the acceleration relative to S. In vector, may say $\mathbf{a}'=\mathbf{a}$ and hence the motion $m\mathbf{a}=\mathbf{F}$ may also be written as

$$m\mathbf{a}'=\mathbf{F} \qquad \ldots(3)$$

which shows that Newton's Law of motion holds in S' as well as in S



Then we draw an important result :

Given one Newtonian frame of reference S, we can find an infinity of other Newtonian frames of reference which have a uniform motion of translation relative to S.

(b) Frames of reference with translation motion under constant acceleration.

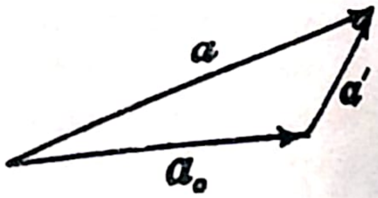
Let $\ddot{\xi}=\alpha_0$ and $\ddot{\eta}=\beta_0$ say then we have

$$\ddot{x}=\ddot{x}'+\alpha_0 \text{ and } \ddot{y}=\ddot{y}'+\beta_0. \qquad \ldots(4)$$

Now let $\mathbf{a}$, $\mathbf{a}'$ denote respectively, the acceleration of A relative to S and S', and let $\mathbf{a}_0$ denote the acceleration of S' relative to S; then equation (4) may be written as :

$$\mathbf{a}=\mathbf{a}'+\mathbf{a}_0.$$

This is called the *law of composition of accelerations* and is shown graphically in the adioining diagram.



$\therefore$ Equation of motion $(m\mathbf{a})=\mathbf{F}$

$$\Rightarrow m(\mathbf{a}'+\mathbf{a}_0)=\mathbf{F}$$

$$\Rightarrow m\mathbf{a}'=\mathbf{F}-m\mathbf{a}_0 \qquad \ldots(5)$$

which shows that Newtonian Law of motion does not hold relative to S' but this law holds provided that we add to the true force $\mathbf{F}$, a *fictitious* force $-m\mathbf{a}_0$.

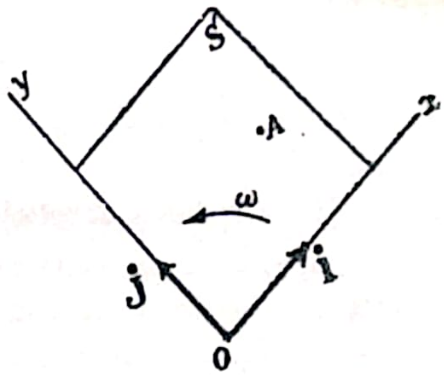
(c) Frames of reference rotating with constant angular velocity.

Let S be the Newtonian frame of reference and S' another frame of reference rotating about a point O of S with constant angular velocity ω. Now let $\mathbf{i}$, $\mathbf{j}$ be perpendicular unit vectors. fixed is S' and let A be the moving particle, then taking axes Oxy in S' in directions of $\mathbf{i}$ and $\mathbf{j}$, the position vector of A is given by

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}$$

$$\Rightarrow \quad \dot{\mathbf{r}}=x\frac{d\mathbf{i}}{dt}+\mathbf{i}\dot{x}+y\frac{d\mathbf{j}}{dt}+\mathbf{j}\frac{dy}{dt} \qquad \ldots(6)$$

But $\frac{di}{dt}=\omega \mathbf{j}$ and $\frac{dj}{dt}=-\omega \mathbf{i}$



$\therefore$ (6) $\Rightarrow$ $\mathbf{r}=x\omega\mathbf{j}$
$+\mathbf{i}\dot{x}-y\omega\mathbf{i}+\mathbf{j}\dot{y}$

or $\dot{\mathbf{r}}=(\dot{x}-y\omega)\ \mathbf{i}+(\dot{y}+y\omega)\ \mathbf{j}=\mathbf{v}$ ...(7)

Another differentiation gives

$$\mathbf{a}=\dot{\mathbf{v}}=(\ddot{x}-2\omega\dot{y}-\omega^2x)\ \mathbf{i}+(\ddot{y}+2\omega\dot{x}-\omega^2y)\ \mathbf{j}.$$

Hence if X, Y are the components of true force in the direction of $\mathbf{i}$, $\mathbf{j}$ respectively, we have the equations of motion

$$m\ (\ddot{x}-2\dot{y}\omega-\omega^2x)=X. \quad ...(8)$$

and $$m\ (\ddot{y}+2\omega\dot{x}-\omega^2y)=Y. \quad ...(9)$$

Equations (8) and (9) can be re-written as

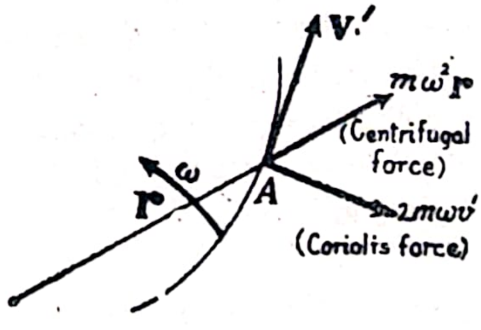
$$m\ddot{x}=X+X'+X'' \quad ...(10)$$

and $$m\ddot{y}=Y+Y'+Y'' \quad ...(11)$$

where $X'=2m\dot{y}\omega$, $X''=m\omega^2x$, $Y'=-2m\omega\dot{x}$ and $X''=m\omega^2y$. ...(12)

This shows that Newtonian Law of motion does not hold relative to S′ but this law holds provided that we add to the true Force F the two ficititious forces, (X', Y') and (X'', Y'').

Note 1. The force (X', Y') is called **Coriolis force.** Obviously $\sqrt{(X'^2+Y'^2)}=2m\omega v'$, which shows that the magnitude of the above said force is proportional to the angular velocity of S' and to the speed v' of the particle relative to S', its direction is perpendicular to the velocity v' relative to S'.



2. The force (X'', Y'') is called the **Centrifugal force.** It magnitude is proportional to the square of the angular velocity of S' and to the distance of the particle from the centre of rotation. This force is directed radially outward from the centre of rotations.

3. The frame of reference which we use in our daily life is the earth. Earth rotates relative to the astronomial frame with an angular velocity of 2π radians per sidereal day where one sidereal day contains 86,164·09 sec., the angular velocity of the

earth is $7{\cdot}29\times10^{-5}$ radians per second. The angular velocity is very small, hence the corioli force and centrifugal force due to the rotation of the earth are not roticeable in our daily lives. These two forces are important geographically because centrifugal forces causes the equatorial bulge on the earth but the coriolis force gives r se to trade winds.

(d) Resolution of velocities and accelerations.

Many times, we need to connect the velocities and accelerations of a particle relative to two different frames of reference S and S'. Here consider that case in which there is no relative rotation of toe frames. Now, let O be the point fixed in one frame S and Q' another point fixed in other frame S'. Further, let the *p.v.* of A, [w.r.t. "S"] w.r.t. O be $\mathbf{r}=\mathbf{OA}$ and w.r.t. "O" [w.r.t. S'] be $\mathbf{r}'=\mathbf{O'A}$.

They are connected by $\mathbf{r}=\mathbf{r}_0+\mathbf{r}'$ where $\mathbf{r}_0=\mathbf{OO}'$. ...(13)

Differentiating (13), w.r.t. "t"

we get $\mathbf{v}=\mathbf{v}_0+\mathbf{v}'$. ... (14)

Again differentiation gives

$\mathbf{a}=\mathbf{a}_0+\mathbf{a}'$ .. (15)

Composion Laws

where $\mathbf{v}$, $\mathbf{a}$=velocity and acceleration of A relative to S.

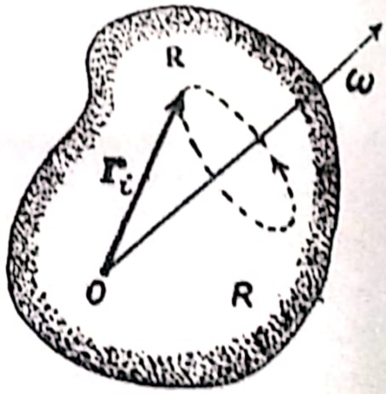
$\mathbf{v}'$, $\mathbf{a}'$=velocity and acceleration of A relative to S'.

$\mathbf{v}_0$, $\mathbf{a}_0$=velocity and acceleration of S' relative to S.

4·5. Kinematics of a rigid body.

(a) Motion of a rigid body with fixed point.

Suppose that point O of the rigid body R is fixed. Let t_1, t_2 be two instants then during the time interval t_2-t_1, the rigid body receives a displacement which is equivalent to a rotation $\mathbf{n}$ about O. If t_1 is kept fixed and, $t_2\to t_1$, the direction of $\mathbf{n}$ will tend to some limiting direction say $\mathbf{i}$. The ratio of the angle of rotation n to the time interval t_2-t_1 also approaches to a limiting value ω.



At the instant t_1 the vector $\vec{\omega}=\omega\mathbf{i}$ is called the *angular velocity* of the body, and the body is rotating about a line through O in the direction of $\vec{\omega}$; the line is called the *instantaneous axis of rotation.*

In time dt. the body receives an infinitesimal rotation $\vec{\omega}\,dt$ and hence the displacement of A is given by

$d\mathbf{r}_i = \omega\, dt \times \mathbf{r}_i$ where $OA = \mathbf{r}_i$. This further implies

$$\frac{d\mathbf{r}_i}{dt} = \vec{\omega} \times \mathbf{r}_i \Rightarrow \mathbf{v}_i = \vec{\omega} \times \mathbf{r}_i.$$

When the body turns about O, the instantaneous axis determined by $\vec{\omega}$ will occupy different positions in the body and are such that they always pass through O. The locus of instantaneous axis of rotation is a cone with vertex O and is called as body cone (or *polhode cone*). Similarly, the locus of instantaneous axis of rotation in space is a *space cone* (or *herpolhode cone*).

Particular case :

When a rigid body is moving parallel to a fundamental plane then we can regard it as a body turning about a point at infinity and in this case the body cone and the space cone become cylinders.

(b) . Components of angular velocity is terms of the Eulerian angles.

We have already defined Eulerian angles θ, ϕ, ψ which describe (relative to a fixed trviad **I, J, K**) the position of a triad of unit orthogonal vectors **i, j, k** fixed in a rigid body turning about O. When θ, ϕ, ψ are known as functions of time, we can determine easily the motion of the rigid body. But the motion of the rigid body can also be described by the angular velocity $\vec{\omega}(t)$.

In an infinitesimal time dt the body receives the rotation $\vec{\omega}\, dt$ and is given by $(\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi)\,\mathbf{i} + (\cos\psi\, d\theta + \sin\theta\sin\psi\, d\phi)\,\mathbf{j}$

$$+ (\cos\theta\, d\phi + d\psi)\,\mathbf{k} = d\mathbf{n} \text{ say}$$

where $d\mathbf{n} = dn_1\mathbf{i} + dn_2\mathbf{j} + dn_3\mathbf{k} = (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})\, dt.$

$$\therefore \quad (\omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k})\, dt = (\sin\psi\, d\theta - \sin\theta\cos\psi\, d\phi)\,\mathbf{i}$$
$$+ (\cos\psi\, d\theta + \sin\theta\sin\psi\, d\phi)\,\mathbf{j} + (\cos\theta\, d\phi + d\phi)\,\mathbf{k}$$

$$\Rightarrow \quad (\omega_1\mathbf{i} + \omega_1\mathbf{j} + \omega_2\mathbf{k}) = (\sin\psi\dot{\theta} - \sin\theta\cos\psi\dot{\phi})\,\mathbf{i}$$
$$+ (\cos\psi\dot{\theta} + \sin\theta\sin\psi\dot{\phi})\,\mathbf{j} + (\cos\theta\dot{\phi} + \dot{\phi})\,\mathbf{k}$$

$$\Rightarrow \quad \omega_1 = \sin\psi\dot{\theta} - \sin\theta\cos\psi\dot{\phi}$$
$$\omega_2 = \cos\psi\dot{\theta} + \sin\theta\sin\psi\dot{\phi}$$
$$\omega_3 = \cos\theta\dot{\phi} + \dot{\psi}.$$

When the motion is known i.e. when θ, ϕ, ψ are known as functions of time, we can determine ω_1, ω_2, ω_3. Conversely, when the compount ω_1, ω_2, ω_3 (of $\vec{\omega}$) are known at any time t, we can solve the three equations to obtain θ, ϕ, ψ as functions of t.

(c) General motion of the rigid body.

Such general motion is composed of a translation of a fixed point of the body (usually the centre of mass) plus rotation about an axis through the fixed point which is not necessarily restricted in direction.

Consider the motion of a rigid body in a general manner and select a particle (of mass m_A say) at A which is regarded as the base point. Let its velocity be denoted by $\mathbf{v}_A$.-Now the displacement of any particle B of the body is $\mathbf{v}_A\, dt + d\mathbf{n} \times \mathbf{r}$ when $\mathbf{AB} = \mathbf{r}$.

$\therefore$ velocity of B is given by $\mathbf{v} = \mathbf{v}_A + \vec{\omega} \times \mathbf{r}$

$$\left(\because \frac{d\mathbf{n}}{dt} = \vec{\omega}\right) \quad \ldots(1)$$

$\Rightarrow$ $\mathbf{v}_B$ = the velocity $\mathbf{v}_A$ of the point

+ the velocity of B relative to A *viz.* $\vec{\omega} \times \mathbf{r}$.

It is quite obvious that velocity of B relative to A is precisely the same as if the body were turning about A (regarded as a fixed point) with angular velocity $\vec{\omega}$. If we change the base point A, the translation $\mathbf{v}_A\, dt$ is changed, but $d\mathbf{n}$ remains the same or in other words the vector $\vec{\omega}$ is free vector as it does not depend on our choice of base point. From equation (1) it is clear, that, if $\vec{\omega}$ and $\mathbf{v}_A$ are known than velocity $\mathbf{v}$ of any point of the body can be determined and hence $\vec{\omega}$ and $\mathbf{v}_A$ completely describe the motion.

Differentiating (1), w.r.t. "t", we get

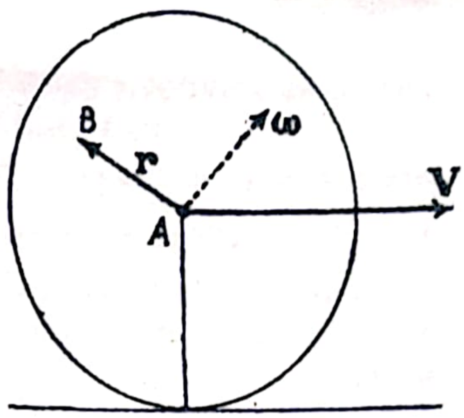
$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\vec{\omega}}{dt} \times \mathbf{r} + \vec{\omega} \times \frac{d\mathbf{r}}{dt} \Rightarrow \mathbf{a} = \mathbf{a}_A \times \frac{d\vec{\omega}}{dt} \times \mathbf{r} + \vec{\omega} \times (\vec{\omega} \times \mathbf{r}). \quad \ldots(2)$$

$$\left[\because \frac{d\mathbf{r}}{dt} = \vec{\omega} \times \mathbf{r}\right]$$

where $\mathbf{a}_A$ is the acceleration of the base point and depends solely on the motion of A and not on $\vec{\omega}$.

Ex. 1. *Consider a circular wheel rolling with constant speed V along a straight level track.*

Solution. Let the centre of the wheel be taken as base point and let its velocity be denoted by $\mathbf{V}$ which is a constant vector as it lies in the plane of the wheel and is horizontal. The angular velocity vector $\vec{\omega}$ of the wheel is a vector perpendicular to its plane. Now 4·5 (c) equations (1) and (2) give



$$\mathbf{v}=\mathbf{v}_A+\vec{\omega}\times\mathbf{r}$$

and $\mathbf{a}=\mathbf{a}_A+\dfrac{d\vec{\omega}}{dt}\times\mathbf{r}+\vec{\omega}\times(\vec{\omega}\times\mathbf{r})$

$\Rightarrow\quad \mathbf{v}=\mathbf{V}+{}^*\vec{\omega}\times\mathbf{r}$

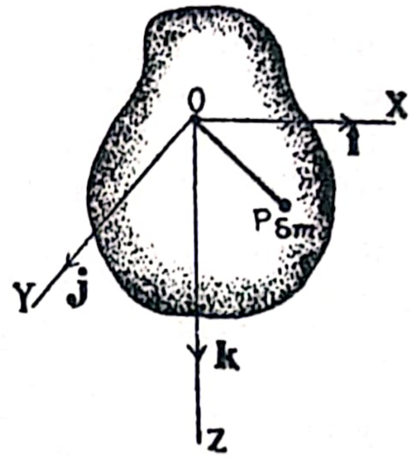
and $\mathbf{a}=0+0\times\mathbf{r}+\vec{\omega}\times(\vec{\omega}\times\mathbf{r})$

$=\vec{\omega}\,(\vec{\omega}.\mathbf{r})-\mathbf{r}\omega^2=-\omega^2\mathbf{r}$ ..(3)

which implies that each particle of the wheel has an acceleration of magnitude $\omega^2\mathbf{r}$, directed towards A.

4·6. Kinetic energy of Rigid Body with a fixed point.

Consider a rigid body turning about a fixed point O with angular velocity $\vec{\omega}$. A particle P of this body, with velocity v and mass δm, has kinetic energy $\frac{1}{2}\delta m\ \mathrm{v}^2$ and hence the kinetic energy of the whole body is $T=\frac{1}{2}\Sigma\delta m\ v^2$, where the summation extends over all particles of the body. Now we want to find the value of T in terms of $\vec{\omega}$ and the principal moments of inertia at O.



Now $\mathbf{r}=\mathbf{OP}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\vec{\omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$; where ω_1, ω_2, ω_3 are the components of the angular velocity $\vec{\omega}$ in the directions of OX, OY and OZ respectively.

$$\mathbf{v}=\vec{\omega}\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}=(\omega_2 z-\omega_3 y)\,\mathbf{i}+(\omega_3 x-\omega_1 z)\,\mathbf{j}+(\omega_1 y-\omega_2 x)\,\mathbf{k}$$

$$\begin{aligned}\text{Thus } 2T&=\Sigma\,\delta m\,[(\omega_2 z-\omega_3 y)^2+(\omega_3 x-\omega_1 z)^2+(\omega_1 y-\omega_2 x)^2]\\ &=\omega_1^2\,\Sigma\,(y^2+z^2)\,\delta m+\omega_2^2\,\Sigma\,(z^2+x^2)\,dm\\ &\quad+\omega_3^2\,\Sigma\,(x^2+y^2)\,dm-2\omega_2\omega_3\,\Sigma\,yz\,\delta m\\ &\quad-2\omega_3\omega_1\,\Sigma\,zx\,dm-2\omega_1\omega_2\,\Sigma\,xy\,dm\end{aligned}$$

---

*Since the vector $\vec{\omega}\times\mathbf{r}$ is perpendicular to ω, this velocity lies in the plane of the wheel—a fact which is intuitively obvious.

$$= A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_2\omega_3 - 2G\omega_3\omega_1 - 2H\omega_1\omega_2, \quad ...(1)$$

where A, B, C, F, G, H are the moments and products of inertia for $OXYZ$. When the axes OX, OY, OZ are the principal axis, then we obviously have $F = G = H = 0$

$$\Rightarrow T = \tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$$

where A, B, C are now principal moments of inertia. ...(2)

4·7. Kinetic energy of body in general.

Here we shall find the kinetic energy T of a rigid body moving quite generally in space. Now applying the theorem of *Koning, we have

$$T = \tfrac{1}{2}mv_0^2 + T' \quad ..(1)$$

where m = mass of the body.
v_0 = speed of mass centre
T' = kinetic energy of motion relative to mass centre.

But the mass centre can be regarded as a base point, in the body and therefore the motion relative to the mass centre, T', is that a body turning about the fixed point. Thus T' is equal to $\tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$; where A, B, C are the principal moments of inertia at the mass centre and $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity $\vec{\omega}$ in directions of principal axes of inertia at the mass centre.

We therefore have $T = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$ ...(2)

But, if Ox, Oy, Oz are not principal axes of the system, then we have

$$T = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_2\omega_3 - 3G\omega_3\omega_1 - 2H\omega_1\omega_2)$$

$$\Rightarrow 2T = mv_0^2 + (A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_2\omega_3 - 2G\omega_3\omega_1 - 2H\omega_1\omega_2) \quad ...(3)$$

4·8. Angular momentum of a particle and of a system of particles.

Consider a particle of mass m, moving with velocity **v** relative to some frame reference S, then we define the angular momentum **h**, about any point O, as the moment of linear momentum (m**v**), by **h**=**r**×m**v** where . is the position vector of the particle relative to O. It is quite obvious that **h** depends on S used in the

*Theorem of Koning. The K.E. of a moving system is equal to the sum of
(i) the kinetic energy of a fictitious particle moving with the mass centre and having a mass equal to the total mass of the system.
(ii) the kinetic energy of the motion relative to the mass centre.

cles. Let m_i, $\mathbf{r}_i$, $\mathbf{v}_i$ denote the mass, position vector relative to O and velocity of ith particle respectively, then angular momentum about O is

$$\mathbf{h} = \sum_{i=1}^{N} (\mathbf{r}_i \times m_i \mathbf{v})$$

where N is the number of particles in the system. ...(1)

Now $\mathbf{r}_i \times \mathbf{v}_i = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_i & y_i & z_i \\ \dot{x}_i & \dot{y}_i & \dot{z}_i \end{vmatrix} = (y_i \dot{z}_i - z_i \dot{y}_i)\,\mathbf{i} + (z_i \dot{x}_i - x_i \dot{z}_i)\,\mathbf{j} + (x_i \dot{y}_i - y_i \dot{x}_i)\,\mathbf{k}$ and $\mathbf{h} = h_1 \mathbf{i} + h_2 \mathbf{j} + h_3 \mathbf{k}$

$$\therefore \quad h_1 = \sum_{i=1}^{N} m_i (y_i \dot{z}_i - z_i \dot{y}_i), \quad h_2 = \sum_{i=1}^{N} m_i (z_i \dot{x}_i - x_i \dot{z}_i)$$

$$h_3 = \sum_{i=1}^{N} m_i (x_i \dot{y}_i - y_i \dot{x}_i)$$

where we have assumed that O is fixed in the frame of reference.

Now, we want to find the effect of changing the frame of reference.

Let S be some frame of reference and S' be a new frame, having a velocity $\mathbf{v}_0$ of translation relative to S, then we have

$$\mathbf{v}_i = \mathbf{v}_0 + \mathbf{v}_i',$$

where $\mathbf{v}_i$ is the velocity of the ith particle relative to S and $\mathbf{v}_i'$ is the velocity of the same particle relative to S'.

Now angular momenta about O is

$$\mathbf{h} = \sum_{i=1}^{N} (\mathbf{r}_i \times m_i \mathbf{v}_i) \text{ [for } S] \text{ and } \mathbf{h}' = \sum_{i=1}^{N} (\mathbf{r}_i - m_i \mathbf{v}_i') \text{ [for } S'].$$

$$\therefore \quad \mathbf{h} = \sum_{i=1}^{N} \mathbf{r}_i \times m_i (\mathbf{v}_0 + \mathbf{v}_i') = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_0 + \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i'$$

$$\Rightarrow \quad \mathbf{h} = \left(\sum_{i=1}^{N} m_i \mathbf{r}_i \right) \times \mathbf{v}_0 + \mathbf{h}'. \qquad \text{...(2)}$$

If we assume **O** to be the mass centre, then we have

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = 0 \quad \Rightarrow \quad \mathbf{h} = \mathbf{h}'.$$

$\Rightarrow$ *Angular momentum about the mass centre is the same for all frames of reference in relative translation motion.*

Note. Generally it is most convenient to use a frame of reference in which the mass centre is fixed. In future, by the angular momentum about a point O, we shall mean a frame of reference in which O is fixed.

4·9. Angular momentum of a rigid body.

Angular momentum of a rigid body about O, is given by

$$\mathbf{h}=\Sigma\,(\mathbf{r}\times\delta m\mathbf{v})$$

where δm is the mass of a typical particle, $\mathbf{r}$ its p.v., and $\mathbf{v}$ its velocity and the summation extends over all particles in the body.

Now, if $\vec{\omega}$ is the angular velocity of the body then *we have

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\vec{\omega}\times\mathbf{r}$$

$$\Rightarrow\quad \mathbf{h}=\Sigma\,[\mathbf{r}\times\delta m\,(\vec{\omega}\times\mathbf{r})]=\Sigma\,\delta m\,[\mathbf{r}\times(\omega\times\mathbf{r})]$$

$$=\delta m\,[\vec{\omega}r^2-\mathbf{r}\,(\vec{\omega}\cdot\mathbf{r})] \qquad \ldots(1$$

$$\Rightarrow\quad h_1\mathbf{i}+h_2\mathbf{j}+h_3\mathbf{k}$$
$$=\Sigma\,\delta m\,[(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k})\,(x^2+y^2+z^2)$$
$$-(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\,(\omega_1x+\omega_2y+\omega_3z)]$$

$$\Rightarrow\quad h_1\mathbf{i}+h_2\mathbf{j}+h_3\mathbf{k}$$
$$=\Sigma\,\delta m\,[\omega_1\mathbf{i}\,(x^2+y^2+z^2)+\omega_2\mathbf{j}\,(x^2+y^2+z^2)+\omega_3\mathbf{k}\,(x^2+y^2+z^2)$$
$$-x\mathbf{i}\,(\omega_1x+\omega_2y+\omega_3z)-y\mathbf{j}\,(\omega_1x+\omega_2y-\omega_3z$$
$$-z\mathbf{k}\,(\omega_1x+\omega_2y+\omega_3z)]$$

$$\Rightarrow\quad h_1=\Sigma\,\delta m\,[\omega_1\,(x^2+y^2+z^2)-x\,(\omega_1x+\omega_2y+\omega_3z)]$$
$$h_2=\Sigma\,\delta m\,[\omega_2\,(x^2+y^2+z^2)-y\,(\omega_1x+\omega_2y+\omega_3z)]$$
$$h_3=\Sigma\,\delta m\,[\omega_3\,(x^2+y^2+z^2)-z\,(\omega_1x+\omega_2y+\omega_3z)]$$

$$\Rightarrow\quad h_1=\omega_1\,\Sigma\,\delta m\,(y^2+z^2)-\omega_3\,\Sigma\,\delta m\,xy-\omega_3\,\Sigma\,\delta m\,zx$$
$$=A\omega_1-H\omega_2-G\omega_3$$
$$h_2=\omega_2\,\Sigma\,\delta m\,(z^2+x^2)-\omega_3\,\Sigma\,\delta m\,yz-\omega_1\,\Sigma\,\delta m\,xy$$
$$=-H\omega_1+B\omega_1-F\omega_3$$
$$h_3=\omega_3\,\Sigma\,\delta m\,(x^2+y^2)-\omega_1\,\Sigma\,\delta m\,zx-\omega_2\,\Sigma\,\delta m\,yz$$
$$=-G\omega_1-F\omega_2-C\omega_3$$

where $A=\Sigma\,\delta m\,(z^2+y^2)$, $B=\Sigma\,\delta m\,(z^2+x^2)$, $C=\Sigma\,\delta m\,(x^2+y^2)$,
$F=\Sigma\,\delta m\,yz$, $G=\Sigma\,\delta m\,zx$, $H=\Sigma\,\delta m\,xy$.

These equation can also be written as

$$\begin{bmatrix} h_1\\ h_2\\ h_2\end{bmatrix}_{3\times1}=\begin{bmatrix} A & -H & -G\\ -H & +B & -F\\ -G & -F & +C\end{bmatrix}_{3\times1}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}_{3\times1}$$

The matrix $\begin{pmatrix} A & -H & -G\\ -H & B & -F\\ -G & -F & C\end{pmatrix}_{3\times3}$ is called a moment of inertia tensor. This is the simplest example of the concept which

* We have $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}.\mathbf{c})\,\mathbf{b}-(\mathbf{a}.\mathbf{b})\,\mathbf{c}$.

has played such an important part in the theory of relativity The diagonal elements are known as the moments of inertia coefficients, *viz.* A, B, C.

If **i**, **j**, **k** are principal axes of inertia at **O**, then $F=G=H=0$, and these formulas are greatly simplified and reduce to

$$h_1=A\omega_1,\ h_2=B\omega_2,\ h_3=C\omega_3, \qquad \ldots(2)$$

where A, B, C are principal moments of inertia at **O**.

The moment of inertia tensor can also be written as

$$\begin{pmatrix} \Sigma\ \delta m(y^2+z^2) & -\Sigma\ \delta mxy & -\Sigma\ \delta mxz \\ -\Sigma\ \delta myx & \Sigma\ \delta m\ (z^2+x^2) & -\Sigma\ \delta myz \\ -\Sigma\ \delta m\ zx & -\Sigma\ \delta mzy & \Sigma\ \delta m\ (x^2+y^2) \end{pmatrix}=I \text{ say.}$$

In case the system of particles is a bounded continuous solid medium of density ρ, then the components of inertia tensor are given by

$$A=\iiint(y^2+z^2)\ \rho\ dx\ dy\ dz,\quad B=\iiint(z^2+x^2)\ \rho\ dx\ dy\ dz,$$

$$C=\iiint(x^2+y^2)\ \rho\ dx\ dy\ dz,\quad F=\iiint yz\ \rho\ dx\ dy\ dz,$$

$$G=\iiint zx\ \rho\ dx\ dy\ dz,\qquad H=\iiint xy\ \rho\ dx\ dy\ dz.$$

It is worth noticing that I is a symmetrical tensor and it can be reduced to a diagonal form by an appropriate choice of the direction of the axes x, y, z just like any other tensor of second rank. These axes are called as the principal axes of inertia and the corresponding values of the diagonal components of the tensor are called the principal moments of inertia. As a matter of fact principal moments of inertia are the *eigen values* of the inertia tensor and can be obtained as the roots of the equation

$$\begin{pmatrix} A-I & -H & -G \\ -H & B-I & -F \\ -G & -F & C-I \end{pmatrix}=0.$$

4 10. Motion of a system.

1. Principle of linear momentum ; Motion of mass centre.

Let m_i and $\mathbf{v}_i$ denote the mass and velocity of ith particle of the system, then the linear momentum is $\mathbf{M}=\sum_{i=1}^{N} m_i\ \mathbf{v}_i$, where N is the number of particles If $\mathbf{F}$ is the vector sum of the external forces, then we have

$$^{*}\dot{\mathbf{M}}=\mathbf{F} \Rightarrow \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{v}_i\right)=\mathbf{F}$$ which can be interpreted as under :

The rate of incr. ase of linear momentum of a system of particles is equal to the vector sum of the external forces.

(Principle of linear momentum)

Now let $\bar{\mathbf{r}}$ be the position of the C.G. of the system, then we have

$$\bar{\mathbf{r}}=\frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} \Rightarrow \sum_{i=1}^{N} m_i \mathbf{r}_i=m\bar{\mathbf{r}} \Rightarrow m\dot{\bar{\mathbf{r}}}=\sum_{i=1}^{N} m_i \mathbf{v}_i$$

$$\Rightarrow m\bar{\mathbf{v}} \Rightarrow \sum_{i=1}^{N} m_i \mathbf{v}_i.$$ where $m=\Sigma m_i$, is the total mass

So we have $\dot{\mathbf{M}}=\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{v}_i\right)=\frac{d}{dt}(m\bar{\mathbf{v}})=m\bar{\mathbf{a}}$

$$\Rightarrow m\bar{\mathbf{a}}=\mathbf{F}. \qquad (\because \dot{\mathbf{M}}=\mathbf{F})$$

This is the equation of motion for a single particle of mass m under a force F, and so we have the following result :

The mass centre of a system moves like a particle, having a mass equal to the mass of the system ; acted on by a force equal to the vector sum of the external forces acting on the system.

(Motion of mass centre)

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*We have $\mathbf{M}=\sum_{i=1}^{N} m_i\mathbf{v}_i \Rightarrow \dot{\mathbf{M}}=\sum_{i=1}^{N} m_i\dot{\mathbf{v}}_i=\sum_{i=1}^{N} m_i\mathbf{a}_i$, where $\mathbf{a}_i$ is the acceleration of the ith particle.

Again if $\mathbf{P}_i$ is the external force on the ith particle and $\mathbf{P}'_i$ the internal force on it, then we have $\sum_{i=1}^{N} m_i\mathbf{a}_i=\sum_{i=1}^{N}(\mathbf{P}_i+\mathbf{F}_i')$. But from the equality of action and reaction, we see that $\sum_{i=1}^{N}\mathbf{P}'_i=0 \Rightarrow \sum_{i=1}^{N} m_i\mathbf{a}_i=\sum_{i=1}^{N}\mathbf{P}'=\mathbf{F}$ say, where F is the vector sum of the external forces.

This implies $\dot{\mathbf{M}}=\mathbf{F}$.

2. Principle of angular momentum.

Let **h** be the angular momentum of the system of particles about O, then we have

$$\mathbf{h}=\sum_{i=1}^{N}(\mathbf{r}_i \times m_i\mathbf{v}_i) \qquad \ldots(1)$$

where m_i = mass of the ith particle

$\mathbf{r}_i = p.v$ of the ith particle relative to **O**

N=number of particles in the system.

In what follows we shall assume 0 to be either a fixed point in Newtonian frame of refrence of the mass centre of the system.

$$(1) \quad \Rightarrow \dot{\mathbf{h}}=\sum_{i=1}^{N}(\dot{\mathbf{r}}_i \times m_i\mathbf{v}_i \times m_i\dot{\mathbf{v}}_i)$$

$$=\sum_{i=1}^{N}(\mathbf{v}_i \times m_i\mathbf{v}_i+\mathbf{r}_i \times m_i\dot{\mathbf{v}}_i) \qquad (\because \quad \mathbf{v}_i=\dot{\mathbf{r}}_i)$$

$$=\sum_{i=1}^{N}(\mathbf{r}_i \times m_i\mathbf{a}_i) \qquad (\because \quad \mathbf{a}_i=\dot{\mathbf{v}}_i) \quad \ldots(2)$$

Now there arise two cases.

Case I. *When 0 is a fixed point.*

In this case we have $m_i\mathbf{a}_i=\mathbf{P}_i+\mathbf{P}_i'$; where $\mathbf{P}_i$, $\mathbf{P}_i'$; are respectively, the external and internal forces on the ith particle.

$$\therefore \quad (2) \quad \Rightarrow \dot{\mathbf{h}}=\sum_{i=1}^{N}[\mathbf{r}_i \times m_i\mathbf{a}_i]=\sum_{i=1}^{N}[\mathbf{r}_i \times (\mathbf{P}_i+\mathbf{P}_i')]$$

$$=\sum_{i=1}^{N}\mathbf{r}_i \times \mathbf{P}_i+\sum_{i=1}^{N}\mathbf{r}_i \times \mathbf{P}_i'.$$

The second summation vanishes, since the internal forces have no moment about any point. Hence

$\dot{\mathbf{h}}=\mathbf{G}$, where $\mathbf{G}=\sum_{i=1}^{N}\mathbf{r}_i \times \mathbf{P}_i$ is the total moment of the external forces about the fixed point 0. ...(3)

Case II. *When 0 is the mass centre.*

The acceleration of the ith particle relative to a Newtonian frame S is $\mathbf{a}_0+\mathbf{a}_i$ where $\mathbf{a}_0$ is the acceleration of 0 relative to S.

$\therefore$ equation of motion for the ith particle is given by

$$m_i(\mathbf{a}_0+\mathbf{a}_i)=\mathbf{P}_i+\mathbf{P}_i' \Rightarrow m_i\mathbf{a}_i=\mathbf{P}_i+\mathbf{P}_i'-m_i\mathbf{a}_0 \qquad \ldots(4)$$

$$\Rightarrow \quad \dot{\mathbf{h}}=\sum_{i=1}^{N}(\mathbf{r}_i \times m_i\mathbf{a}_i)=\sum_{i=1}^{N}[\mathbf{r}_i \times (\mathbf{P}_i+\mathbf{P}_i'-m_i\mathbf{a}_0)]$$

$$=\sum_{i=1}^{N}(\mathbf{r}_i\times\mathbf{P}_i)+\sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{P}_i'-\left(\sum_{i=1}^{N}m_i\mathbf{r}_i\right)\times\mathbf{a}_0. \quad ...(5)$$

But $\sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{P}_i'=0$ and $\sum_{i=1}^{N}m_i\,\mathbf{r}_i=0$

[$\because$ C.G. is the origin of vectors]

$\therefore$ $\dot{\mathbf{h}}=\sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{P}_i=\mathbf{G}$, where $\mathbf{G}$ is now the total moment of the external forces about the mass centre. *Thus we may sum up the principle of angular momentum as follows :*

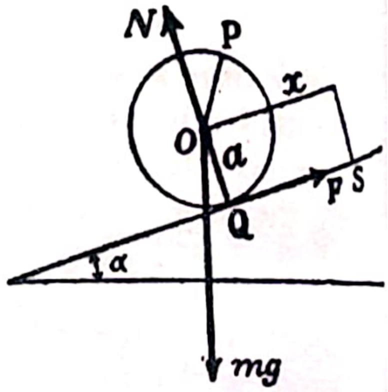
The rate of change of angular momentum of a system about a point, either fixed or moving with the mass centre, is equal to the total moment of the external forces about point ; in symbols

$$\dot{\mathbf{h}}=\mathbf{G}. \quad ...(6)$$

Ex. 2. *Consider a cylinder rolling down an inclined plane.*

Solution. The mass centre moves in a vertical plane, and so vector $\vec{v}$ and F lie in this plane. Resolving them along and perpendicular to the inclined plane, we get

$$m\ddot{x}=mg\sin\alpha-F \quad ...(7)$$
$$0=mg\cos\alpha-N \quad ...(8)$$



The angular velocity $\vec{\omega}$ is parallel to the axis of the cylinder, and the angular momentum about the mass centre is $\mathbf{h}=I\,\vec{\omega}$, where I is the moment of inertia of the body about the axis of the cylinder.

Since $\mathbf{h}$ has got a fixed direction, equation (6) gives a single scalar equation $mk^2\ddot{\theta}=Fa$, where a is the radius of the cylinder.

3. Principle of Energy.

(*i*) (*for a particle of mass m*).

Let T be the K.E. of a particle of mass m, then we have

$$T=\tfrac{1}{2}mv^2=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2):$$

$$\because\ \frac{d}{dt}(T)=m\,(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})=m\,(\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z})$$
$$=(m\ddot{x})\,\dot{x}+(m\ddot{y})\,\dot{y}+(m\ddot{z})\,\dot{z}$$
$$=X\dot{x}+Y\dot{y}+Z\dot{z}$$

$$\Rightarrow\left[T\right]_{t_0}^{t_1}=\int_{t_0}^{t_2}\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt=\int(Xdx+Xdy+Zdz)$$

$$=\int_{t=t_0}^{t=t_1} dW = W \qquad \ldots(7)$$

where X, Y, Z are the components of the force acting on the particle, (t_0, t_1) is the time interval and W is the amount of work done during the time interval.

Equation (7) $\Rightarrow$ $W = T_1 - T_0$ ...(8)

where $T_1 - T_0$ is the increase in kinetic energy in the time interval (t_0, t_1).

This establishes the principle of energy :

The increase in kinetic energy is equal to the work done by the force.

Differentiating (8), w.r.t. "t_1" and then dropping the subscript we have, $\dot{T} = \dot{W}$

$\Rightarrow$ *the rate of increase of kinetic energy equals to the rate of working of the force.*

When the field is conservative with potential energy V, the we have,

$$X = -\frac{\partial V}{\partial x},\ Y = -\frac{\partial V}{\partial y},\ Z = -\frac{\alpha V}{\partial z}$$

and
$$\int_{t=t_0}^{t=t_1} \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt} \right) dt$$

$$= -\int_{t_0}^{t_1} \left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dy}{dt} \right) dt$$

$$= -\int_{t_0}^{t_1} \frac{dV}{dt} dt = -\int dV = -\Big[V \Big]_0^1 = -V_1 + V_0$$

where V_1, V_0 are the potential energies at times t_1, t_0 respectively.

This implies $W - T = -V_1 + V_0$

i.e. $T_0 + V_0 = T + V = \text{constant} = E$, say.

Thus the sum of the kinetic and potential energies is constant. This is called the principle of the conservation of energy.

(*ii*) (*for a system of particles*). Here we also have $T + V = E$ = total energy of the system.

The two most common systems in mechanics are the particle and the rigid body. For each of these systems, the principle of energy is not independent of the particle of linear and angular momentum ; it may however be used in place of any one of the scalar equations deduced from $m\dot{\mathbf{v}} = \mathbf{F}$ and $\dot{\mathbf{h}} = \mathbf{G}$

4·10. Moving frames of Reference (continued).

(a) Frame of reference with translation motion (*motion of a particle*).

Let S be a Newtonian frame of reference and S' be another frame of reference which has, relation to S, a motion of translation only. Then for a moving particle, we have

$$\mathbf{a}=\mathbf{a}_0+\mathbf{a}'$$

where $\mathbf{a}_0$ is the acceleration of S' relative S.

Now, if m is the mass of a particle and $\mathbf{F}$ is the force acting on it, then the law of motion is $m\mathbf{a}=\mathbf{F}$

($\because$ S is the Newtonian frame of reference)

The motion of the particle w.r.t. S' is given by

$$m\mathbf{a}'=m\,(\mathbf{a}-\mathbf{a}_0)=m\mathbf{a}-m\mathbf{a}_0=\mathbf{F}-m\mathbf{a}_0$$

which shows that the motion of S' gives rise to the fictitous force $-m\mathbf{a}_0$ hence we can regard S' as Newtonian frame of reference, provided we add to the actual forces a fictitous force $-m\mathbf{a}_0$ on each particle.

(b) Rotating frame ; Rate of change of a vector.

Let S be the Newtonian frame of reference and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be a triad of unit orthogonal vector in S', which rotates with angular velocity $\vec{\Omega}$ w.r.t. "S". Any velocity $\mathbf{P}$ can be expressed as $\mathbf{P}=P_1\mathbf{i}+P_2\mathbf{j}+P_3\mathbf{k}$. We wish to find the rate of change of $\mathbf{P}$ as estimated by an observer in S.

Now differentiating $\mathbf{P}=P_1\mathbf{i}+P_2\mathbf{j}+P_3\mathbf{k}$ w.r.t. "t", we get

$$\frac{d\mathbf{P}}{dt}=\frac{dP_1}{dt}\,\mathbf{i}+\frac{dP_2}{dt}\,\mathbf{j}+\frac{dP_3}{dt}\,\mathbf{k}+P_1\frac{d\mathbf{i}}{dt}+P_2\frac{d\mathbf{j}}{dt}+P_3\frac{d\mathbf{k}}{dt} \quad \ldots(10)$$

But $\mathbf{i}$ is a vector fixed in a rigid body (S'), which rotates with angular velocity $\vec{\Omega}$. We may think of $\mathbf{i}$ as the position vector of a particle B of this body relative to a base point A, then origin of $\mathbf{i}$, obviously $\frac{d\mathbf{i}}{dt}$ is the velocity of B relative to A and hence it can be written as

$$\frac{d\mathbf{i}}{dt}=\vec{\Omega}\times\mathbf{i} \quad \ldots(11) \qquad \text{Similarly} \quad \frac{d\mathbf{j}}{dt}=\vec{\Omega}\times\mathbf{j} \quad \ldots(12)$$

and

$$\frac{d\mathbf{k}}{dt}=\vec{\Omega}\times\mathbf{k}. \quad \ldots(13)$$

Substituting the values in (10), we get

$$\frac{d\mathbf{P}}{dt}=\left(\frac{dP_1}{dt}\,\mathbf{i}+\frac{dP_2}{dt}\,\mathbf{j}+\frac{dP_3}{dt}\,\mathbf{k}\right)+P_1\vec{\Omega}\times\mathbf{i}+P_2\vec{\Omega}\times\mathbf{j}+P_3\vec{\Omega}\times\mathbf{k}$$

$$=\frac{\delta \mathbf{P}}{\delta t}+\vec{\Omega}\times(P_1\mathbf{i}+P_2\mathbf{j}+P_3\mathbf{k})$$

where $\dfrac{\delta \mathbf{P}}{\delta t}=\left(\dfrac{dP_1}{dt}\,\mathbf{i}+\dfrac{dP_2}{dt}\,\mathbf{j}+\dfrac{dP_3}{dt}\,\mathbf{k}\right)$. ...(14)

and we use the symbol $(\delta/\delta t)$ to denote a partial differentiation in which $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are kept fixed.

Now $\dfrac{d\mathbf{P}}{dt}=\dfrac{\delta \mathbf{P}}{\delta t}+\vec{\Omega}\times\mathbf{P}$ $\quad(\because \quad \mathbf{P}=P_1\mathbf{i}+P_2\mathbf{j}+P_3\mathbf{k})$ ...(15)

Obviously (d/dt) (P) consists of two parts.

(*i*) *First part* $(\delta\mathbf{P}/\delta t)$ *which is the rate of change of* $\mathbf{P}$ *as measured by an observer moving with S'. Usually it is called as the rate of growth ; since in calculating it, we think of the vector* $\mathbf{P}$ *as changing as growth, whereas* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *remain constant.*

(*ii*) *Second part* $\vec{\Omega}\times\mathbf{P}$ *is due to the rotation of the triad* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *and is called as the* **rate of transport.**

Hence (15) can be interpreted as :

Rate of change of vector = Rate of growth + Rate of transport. ...(16)

(c) **Motion of a particle relative to a rotating frame.**

Let S' be a frame of reference which rotates with angular velocity $\vec{\Omega}$ about a point O which is fixed in the Newtonian frame S. Hence the velocity of a moving particle A is given by

$\vec{v}=\dfrac{d\mathbf{r}}{dt}=\dfrac{\delta\mathbf{r}}{\delta t}+\vec{\Omega}\times\mathbf{r}\left(\because \quad \dfrac{d\mathbf{P}}{dt}=\dfrac{\delta\mathbf{P}}{\delta t}+\vec{\Omega}\times\mathbf{r}\right)$, where $\mathbf{OA}=\mathbf{r}$ ...(17)

Also, the acceleration $\mathbf{a}=\dfrac{d\mathbf{v}}{dt}$ is given by $\mathbf{a}=\dfrac{d\mathbf{v}}{dt}=\dfrac{\delta\mathbf{v}}{\delta t}+\vec{\Omega}\times\mathbf{v}$. ...(18)

$$\Rightarrow \mathbf{a}=\frac{\delta\mathbf{v}}{\delta t}+\vec{\Omega}\times\left\{\frac{\delta\mathbf{r}}{\delta t}+\vec{\Omega}\times\mathbf{r}\right\}=\frac{\delta}{\delta t}\left(\frac{\delta\mathbf{r}}{\delta t}+\vec{\Omega}\times\mathbf{r}\right)+\vec{\Omega}\times\left\{\frac{\delta\mathbf{r}}{\delta t}+\vec{\Omega}\times\mathbf{r}\right\}$$

$$=\frac{\delta^2\mathbf{r}}{\delta t^2}+\left(\frac{\delta\vec{\Omega}}{\delta t}\times\mathbf{r}+\vec{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}\right)+\vec{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}+\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})$$

$$=\frac{\delta^2\mathbf{r}}{\delta t^2}+\frac{\delta\vec{\Omega}}{\delta t}\times\mathbf{r}+2\vec{\Omega}\times\frac{\delta\mathbf{r}}{\delta t}+\overset{*}{\vec{\Omega}}\,(\vec{\Omega}.\mathbf{r})-\mathbf{r}\vec{\Omega}^2 \qquad ...(19)$$

Let $\mathbf{v}'$ and $\mathbf{a}'$ denote respectively, the velocity and acceleration of the particle relative to S, so that

* $\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})=\vec{\Omega}\,(\vec{\Omega}.\mathbf{r})-\mathbf{r}\Omega^2$.

$$\mathbf{v}'=\frac{\delta \mathbf{r}}{\delta t},\ \mathbf{a}'=\frac{\delta^2 \mathbf{r}}{\delta t^2}. \qquad \ldots(20)$$

We also have

$$\frac{d\vec{\Omega}}{dt}=\frac{\delta\vec{\Omega}}{\delta t}+\vec{\Omega}\times\vec{\Omega} \Rightarrow \frac{d\vec{\Omega}}{dt}=\frac{\delta\vec{\Omega}}{\delta t} \qquad \ldots(21)$$

$\therefore$ equation (19) $\Rightarrow$ $\mathbf{a}=\mathbf{a}'+\mathbf{a}_t+\mathbf{a}_c$ ...(22)

where $\mathbf{a}_t=\frac{d\vec{\Omega}}{dt}\times\mathbf{r}+\vec{\Omega}\,(\vec{\Omega}.r)-\mathbf{r}\vec{\Omega}^2$ and $\mathbf{a}_c=2\,\vec{\Omega}\times\mathbf{v}'$. ...(23)

When the particle is fixed in S', we have

$$\mathbf{v}=0 \Rightarrow \mathbf{a}'=0 \text{ and } \mathbf{a}_c=0.$$

$$\therefore \quad \mathbf{a}=\mathbf{a}_t.$$

For this reason $\mathbf{a}_t$, may in general case be called as the *acceleration of transport*, whereas the acceleration $\mathbf{a}_c$ is called the *acceleration of coriolis* or the *complementary acceleration*. It is easy to see that $\mathbf{a}_c=2\,\vec{\Omega}\times\mathbf{v}'$ and hence we can say that the *complementary acceleration* is perpendicular to both $\vec{\Omega}$ and $\mathbf{v}'$.

To obtain the equation of motion.

For a particle of mass m acted on by a force $\mathbf{F}$, the law of motion w.r.t. "S" is given by, $m\mathbf{a}=\mathbf{F}$. ...(24)
whereas the law of motion w.r.t. "S" gives

$$m\mathbf{a}'=\mathbf{F}-m\mathbf{a}_t-m\mathbf{a}_c. \qquad \ldots(25)$$

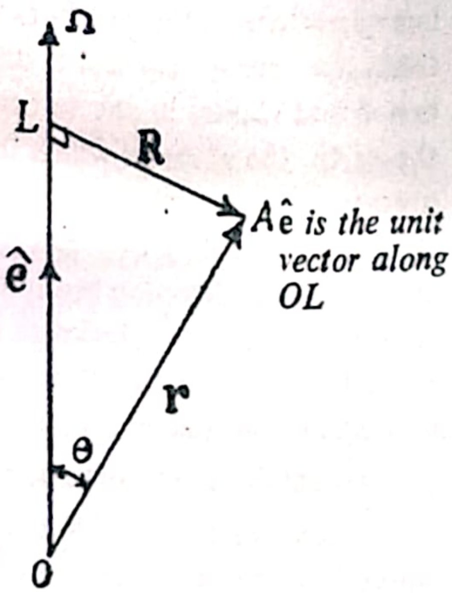
Equation (25) $\Rightarrow$ *that the rotation of S' gives rise to two ficitious forces* :

(i) $-m\mathbf{a}_t$: (ii) $-m\mathbf{a}_c$.

When these two forces are added to $\mathbf{F}$, *the law of motion of a particle in S' is precisely the Newtonian Law. In this case we say that S' is reduced to rest by the introduction of two ficitious forces. The force $-m\mathbf{a}_t$ is intimately related to the force usually known as centrifugal force and the force $-m\mathbf{a}_c$ is known as coriolis force, while $\vec{\omega}\ m\times(\vec{\omega}\times\mathbf{r})$ is known as centrifugal force, and $\vec{\omega}\times(\vec{\omega}\times\mathbf{r})$ is known as centrifugal acceleration.*

(d) Frame (S') rotating with constant angular velocity $\vec{\Omega}$.

Since $\vec{\Omega}$ is a constant vector, it gives a fixed axis of rotation through O. Now let AL be the perpendicular from the position of the particle A to the axis of rotation, then we have $\vec{\Omega}\cdot\mathbf{r}$ $\Omega r = \cos\theta$



$$\therefore \quad \mathbf{a}_t = \frac{d\vec{\Omega}}{dt}\times\mathbf{r} + \vec{\Omega}\,(\vec{\Omega}\cdot\mathbf{r}) - \mathbf{r}\Omega^2$$

$$= 0\times\mathbf{r} + \vec{\Omega}\,\Omega\, r\cos\theta - \mathbf{r}\Omega^2$$

($\because$ $\vec{\Omega}$ is a constant vector)

$$= \vec{\Omega}\,\Omega r\cos\theta - \mathbf{r}\Omega^2$$

$$= \Omega^2\,\hat{\mathbf{e}}\,(r\,(\cos\theta) - (\mathbf{OL}+\mathbf{LA})\Omega^2$$

$$= \Omega^2\,\hat{\mathbf{e}}\,(OL) - \mathbf{OL}\,\Omega^2 - \mathbf{R}\,\Omega^2 = \Omega^2\,\mathbf{OL} - \mathbf{OL}\,\Omega^2 - \mathbf{R}\Omega^2 = -\mathbf{R}\,\Omega^2$$

$\therefore$ Equation of motion is

$$m\mathbf{a}' = \mathbf{F} - m\,(-\mathbf{R}\,\Omega^2) - 2(m\vec{\Omega}\times\mathbf{v}')$$

$$= \mathbf{F} + m\mathbf{R}\,\Omega^2 - 2m\vec{\Omega}\times\mathbf{v}'. \qquad \ldots(26)$$

When the particle is at rest in S', then we have

$$\mathbf{v}' = 0 \text{ and } \mathbf{a}' = 0.$$

$$\therefore \quad (26) \Rightarrow \mathbf{F} + m\mathbf{R}\,\Omega^2 = 0. \qquad \ldots(27)$$

(*Condition for relative equilibrium*)

(e) Motion of particle relative to earth.

Let m be the mass of the particle (assumed constant) and $\mathbf{F}$ be the resultant of all forces acting on the particle as viewed in Newtonian frame of reference, then equation of motion $m\mathbf{a} = \mathbf{F}$ gives

$$m\,(\mathbf{a}' + \mathbf{a}_t + \mathbf{a}_c) = \mathbf{F} \Rightarrow m\mathbf{a}' = \mathbf{F} - m\mathbf{a}_t - m\mathbf{a}_c \qquad \ldots(A)$$

$$\Rightarrow m\mathbf{a}' = \mathbf{F} - \left(m\frac{d\vec{\Omega}}{dt}\times\mathbf{r} + m\,\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})\right) - m\,(2\,\vec{\Omega}\times\mathbf{v}').$$

$$\mathbf{F} - m\,(\vec{\Omega}\times\mathbf{r}) - 2m\,(\vec{\Omega}\times\mathbf{v}') - m\,[(\vec{\Omega}\times\vec{\Omega}\times\mathbf{r})],$$

where $\vec{\Omega}$ is the angular velocity of the earth.

Ex. 3. *Calculate the angular speed of the earth about its axis.*

Solution. Since the earth makes one revolution (2π radions) about its axis in 86, 164·09 second so the angular speed is given by

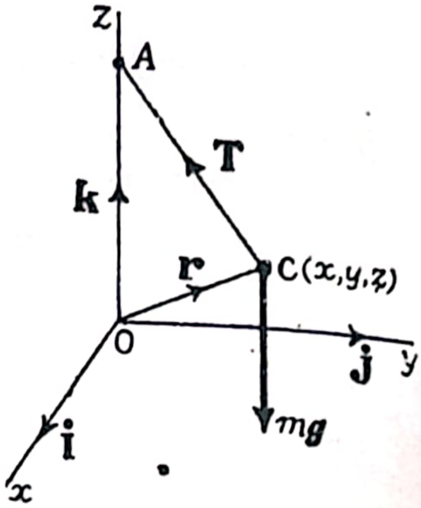
$$\omega = \frac{2\pi}{86,\,164{\cdot}09} = 7{\cdot}29\times10^{-5} \text{ radians/sec.}$$

(f) Focault's Pendulum

Consider a simple pendulam consisting of a long string and heavy particle suspended from a frictionless support. Now assume that the heavy partical is displaced from the position of equilibrium and rotates in any vertical plane. Owing to the rotation of the earth, the plane in which the pendulum swings will gradually precess about a vertical axis. If we look down at the earth surface then in the southern hemisphere the precession would be in the anti-clockwise direction but in the northern hemisphere the precession would be in the clockwise direction. This type of pendulum is used to detect the rotation of the earth and was first employed by *Focault's pendulum.*

To obtain the equation of motion for such a pendulum.

Under such circumstances, the equation of motion is given by



$$m\mathbf{a}'=\mathbf{F}-2m\,(\vec{\Omega}\times\mathbf{v}')-m\,[\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})]$$

$$\Rightarrow\ m\ddot{\mathbf{r}}=(\mathbf{T}+m\mathbf{g})-2m\,(\vec{\Omega}\times\mathbf{v}')$$

$$-m\,[\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})] \qquad \ldots(28)$$

$$\because\ (\mathbf{F}=\mathbf{T}+m\mathbf{g})$$

Neglecting last term in (28), we

get $m\ddot{\mathbf{r}}={}^{\dagger}\mathbf{T}+m\mathbf{g}-2m\,{}^{*}\vec{\Omega}\times\mathbf{v}'$

$$\Rightarrow\ m\,(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k})$$

$$=\left\{-T\frac{x}{l}\,\mathbf{i}-T\frac{y}{l}\,\mathbf{j}+T\,\frac{(l-z)}{l}\,\mathbf{k}\right\}$$

$$-mg\mathbf{k}-2m\,\vec{\Omega}\times\mathbf{v} \qquad (\because\ \mathbf{g}=-g\mathbf{k}).$$

Equating the coefficients of like vectors on both sides (using the value of $(\vec{\Omega}\times\mathbf{v})$, we get

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$${}^{\dagger}\mathbf{T}=(\mathbf{T}.\mathbf{i})\,\mathbf{i}+(\mathbf{T}.\mathbf{j})\,\mathbf{j}+(\mathbf{T}.\mathbf{k})\,\mathbf{k}=T\cos\alpha\mathbf{i}+T\cos\beta\mathbf{j}+T\cos\gamma\mathbf{k}$$

$$=-T\frac{x}{l}\,\mathbf{i}-T\frac{y}{l}\,\mathbf{j}+T\,\frac{(l-z)}{l}\,\mathbf{k}$$

$${}^{*}\vec{\Omega}\times\mathbf{v}'=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \Omega_1 & \Omega_2 & \Omega_3\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}=(\Omega_2\dot{z}-\Omega_3\dot{y})\,\mathbf{i}+(\Omega_3\dot{x}-\Omega_1\dot{z})\,\mathbf{j}+(\Omega_1\dot{y}-\Omega_2\dot{x})\,\mathbf{k}$$

$$=-\Omega\cos\lambda\,\dot{y}\mathbf{i}+(\Omega\cos\lambda\dot{x}+\Omega\sin\lambda\,\dot{z})\,\mathbf{j}-\Omega\sin\lambda\,\dot{y}\,\mathbf{k}$$

$$(\because\ \Omega_1=-\Omega\sin\lambda,\ \Omega_2=0 \text{ and } \Omega_3=\Omega\cos\lambda).$$

$$m\ddot{x}=-T\frac{x}{l}+2m\,\Omega\dot{y}\cos\lambda \quad ...(29)$$

$$m\ddot{y}=-T\frac{y}{l}-2m\Omega\,(\dot{x}\cos\lambda+\dot{z}\sin\lambda) \quad ...(30)$$

and $$m\ddot{z}=T\{(l-z)/l\}-mg+2m\,\Omega\,\dot{y}\sin\lambda \quad ...(31)$$

Ex. 4. *By assuming that the particle in previous example undergoes small oscillations about the equilibrium position so that its motion can be assumed to take place in a horizontal plane simplify the equations of motion.*

Solution. Let the motion of the bob take place in a horizontal plane which amounts to assuming that $\ddot{z}$ and $\dot{z}$ are zero. When vibrations are very small, we observe that $\frac{l-z}{l}\to 1$.

$\therefore$ Equation (31), part (f)

$\Rightarrow$ $0=T-mg+2m\,\Omega\dot{y}\sin\lambda$ *i.e.* $T=mg-2m\,\Omega\dot{y}\sin\lambda$. ...(32)

Thus $$\ddot{x}=-\frac{gx}{l}+2\frac{\Omega}{l}x\dot{y}\sin\lambda+2\Omega\dot{y}\cos\lambda,$$

$$\ddot{y}=-\frac{gy}{l}+\frac{2\Omega\,y\dot{y}\sin\lambda}{l}-2\Omega\,\dot{x}\cos\lambda$$

[using equation (29) and (30)].

But $x\dot{y}$ and $y\dot{y}$ are negligible compared with the other since, Ω, x and y are small.

Neglecting them, we obtain $\ddot{x}=-\frac{g}{l}x+2\Omega\,\dot{y}\cos\lambda$

and $$\ddot{y}=-\frac{gy}{l}-2\Omega\,\dot{x}\cos\lambda.$$

General motion of the frames of reference.

When **O** is moving, the expressions $\mathbf{v}=\frac{\delta\mathbf{r}}{\delta t}+\Omega\times\mathbf{r}$ and $\mathbf{a}=\frac{\delta\mathbf{v}}{\delta t}+\dot{\Omega}\times\mathbf{v}$ give merely the velocity and acceleration of A relative to O, So in order to obtain the complete expression for velocity and acceleration we have to add $\mathbf{v}_0$ and $\mathbf{a}_0$ (the velocity and acceleration of O). Thus, relative to a frame S' moving in a general manner, the motion of the particle is governed by the equation

$$m\mathbf{a}'=\mathbf{F}-m\mathbf{a}_0-m\mathbf{a}_t-m\mathbf{a}_c$$

where $\mathbf{a}_t$=acceleration of transport

$\mathbf{a}_c$=acceleration of coriolis

$\mathbf{a}_0$=acceleration of base point in S' [w.r.t. S]

$\mathbf{F}$=Force applied to the particle.

4·12 Motion of a Rigid Body (Continued).

(a) Rigid body with a fixed point.

Consider the motion of a rigid body about a fixed point, say O, then angular momentum of the body about O is given by

$$\mathbf{h}=A\omega_1\mathbf{i}+A\omega_2\mathbf{j}+C\omega_2\mathbf{k}$$

where A, B, C=principal moments of inertia at O

$\omega_1, \omega_2, \omega_3$=Components of the angular velocity $\vec{\omega}$ of the rigid body in the directions **i, j, k,** ; **i, j, k** being the unit vectors in the directions of principal axes of inertia at O.

In general, the principal axes at O are fixed in the body; in that case the triad **i, j, k** has the angular velocity $\vec{\omega}$ but if A, B, C. are not all different, we may use a principal traid which is neither fixed in the body and nor in the space. Taking all possibilities in mind, we shall assume the angular velocity of the triad as $\vec{\Omega}$; of course $\vec{\Omega}=\vec{\omega}$, when triad is fixed in the body.

But we know that $\dfrac{d\mathbf{P}}{dt}=\dfrac{\delta \mathbf{P}}{\delta t}+\vec{\Omega}\times\mathbf{P}$

$$\therefore \quad \frac{d\mathbf{h}}{dt}=\dot{\mathbf{h}}=\frac{\delta \mathbf{h}}{\delta t}+\vec{\Omega}\times\mathbf{h}$$

$$\Rightarrow \quad \dot{\mathbf{h}}=\frac{\delta}{\delta t}(A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+C\omega_3\mathbf{k})+(\Omega_1\mathbf{i}+\Omega_2\mathbf{j}+\Omega\omega_3\mathbf{k})$$
$$\times(A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+C\omega_3\mathbf{k})$$

$$\because \quad \vec{\Omega}=(\Omega_1\mathbf{i}+\Omega_2\mathbf{j}+\Omega_3\mathbf{k}),\ \mathbf{h}=(A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+C_3\mathbf{k})$$

$$\Rightarrow \quad \dot{\mathbf{h}}=\dot{\omega}_1 A\mathbf{i}+\dot{\omega}_2 B\mathbf{j}+\dot{\omega}_3 C\mathbf{k}+\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_1 & \Omega_2 & \Omega_3 \\ A\omega_1 & B\omega_2 & C\omega_3 \end{vmatrix}$$

$$\Rightarrow \quad \dot{\mathbf{h}}=\dot{\omega}_1+\dot{\omega}_2 B\mathbf{j}+\dot{\omega}_3 C\mathbf{k}+\mathbf{i}\,(\Omega_2 C\omega_3-C\omega_2\,\Omega_3)$$
$$+\mathbf{j}\,(\Omega_3\,A\omega_1-C\omega_3\,\Omega_1)+\mathbf{k}\,(\Omega_1\,B\omega_2-A\omega_1\,\Omega_2)$$

$$\Rightarrow \quad \dot{\mathbf{h}}=(A\dot{\omega}_1-B\omega_2\,\Omega_3+C\omega_3\,\Omega_2)\,\mathbf{i}+(B\dot{\omega}_2-C\omega_3\,\Omega_1+A\omega_1\,\Omega_3)\,\mathbf{j}$$
$$+(C\dot{\omega}-A\omega_1\,\Omega_2+B\omega_2\,\Omega_1)\,\mathbf{k}. \qquad \text{...(1)}$$

But, if **G** is the moment of the external force about O; then we have

$$\dot{\mathbf{h}}=\mathbf{G}=G_1\mathbf{i}+G_2\mathbf{j}+G_3\mathbf{k}. \qquad \text{...(2)}$$

where G_1, G_2, G_3 are the components of **G** along **i, j, k**.

$\therefore$ Equations (1) and (2),

$$\Rightarrow \quad A\dot{\omega}_1 - B\omega_2\,\Omega_3 + C\omega_3\,\Omega_2 = G_1, \qquad \text{...(3)}$$

$$B\dot{\omega}_2 - C\omega_2\,\Omega_1 + A\omega_1\,\Omega_3 = G_2. \qquad \text{...(4)}$$

$$C\dot{\omega}_3 - A\omega_1\,\Omega_2 + B\omega_2\,\Omega_1 = G_3. \qquad \text{...(5)}$$

When the triad is fixed in the body, then we have $\vec{\Omega} = \vec{\omega}$

$$\Rightarrow \quad (\Omega_1, \Omega_2, \Omega_3) = (\omega_1, \omega_2, \omega_3).$$

$\therefore$ (3). (4) and (5),

$$\Rightarrow \quad A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 = G_1, \qquad \text{...(6)}$$

$$B\dot{\omega}_2 = (C-A)\,\omega_2\omega_1 = G_2, \qquad \text{...(7)}$$

$$C\dot{\omega}_3 - (A-B)\,\omega_1\omega_2 = G_3 \qquad \text{...(8)}$$

These are called Euler's equatians of motion for a rigid body with a fixed point.

The special case of motion about O under no forces is of importance. In this case dynamical equations are

$$A\dot{\omega}_1 - (B-C)\;\omega_2\omega_3 = 0. \qquad \text{...(9)}$$

$$B\dot{\omega}_2 - (C-A)\;\omega_3\omega_1 = 0, \qquad \text{...(10)}$$

$$C\dot{\omega}_3 - (A-B)\;\omega_1\omega_2 = 0. \qquad \text{.(11)}$$

Multiplying these three equations by ω_1, ω_2, ω_3 and adding, we get

$$A\omega_1\dot{\omega}_1 + B\omega_2\dot{\omega}_2 + C\omega_3\dot{\omega}_3 = 0$$

$$\Rightarrow \quad A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \text{constant}. \qquad \text{...(12)}$$

But we have $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \quad \therefore \quad T = \text{constant}$

i.e., the kinetic energy remains constant during the motion. It is obvious since no work is being done on the body, there being no external forces.

Another integral of equations X, Y, Z can be obtained by multiplying them by $A\omega_1$, $B\omega_2$, $C\omega_3$ and adding to give

$$A^2\omega_1\dot{\omega}_1 + B^2\dot{\omega}_2\omega_2 + C\omega_3\dot{\omega}_3 = 0$$

$$\Rightarrow \quad A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \text{constant}. \qquad \text{...(13)}$$

Equation (13) $\Rightarrow$ *the constancy of the magnitude of the angular momentum* **h** *during the motion. Physically this is attributable to the zero moment of forces about O.*

General solution of Euler's Equations.

Equations X, Y, Z are given by

$$\left.\begin{aligned} A\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 &= 0 \\ B\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 &= 0 \\ C\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 &= 0 \end{aligned}\right\} \qquad \text{...(14)}$$

For a tentative solution, let us assume

$\omega_1 = a\sqrt{(1-k^2\sin^2\phi)}$, $\omega_2 = b\sin\phi$ and $\omega_2 = c\cos\phi$

$\Rightarrow \quad \dot{\omega}_1 = \dfrac{-ak^2 \sin\phi \cos\phi}{\sqrt{(1-k^2 \sin^2\phi)}}\,\dot{\phi},\ \dot{\omega}_2 = b\cos\phi\,\dot{\phi}$ and $\dot{\omega}_3 = -c\sin\phi\,\dot{\phi}$.

Substituting these values in Euler's equations, we obtain

$$\left.\begin{aligned} &\frac{-Aak^2\dot{\phi}}{\sqrt{(1-k^2\sin^2\phi)}} = bc\,(B-C),\ \frac{Bb\dot{\phi}}{\sqrt{(1-k^2\sin^2\phi)}} \\ &= ca\,(C-A),\ \frac{-Cc\dot{\phi}}{\sqrt{(1-k^2\sin^2\phi)}} = ab\,(A-B) \end{aligned}\right\} \quad ...(15)$$

Let us again assume consistent with the requirements of equations (15)],

$$\frac{B-C}{A}\,\frac{bc}{ak^2} = -\lambda,\ \frac{C-A}{B}\,\frac{ca}{b} = \lambda,\ \frac{A-B}{C}\,\frac{ab}{c} = -\lambda \quad ...(16)$$

so that $\dot{\phi} = \lambda\sqrt{(1-k^2\sin^2\phi)} \Rightarrow \displaystyle\int \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = \lambda t + \mu$, ...(17)

The integral involved in (17) is the elliptical integral of the first kind, so we can define $\phi = am\,(\lambda t + \mu)$

[by the terminology of elliptic functions]

Thus ω_1, ω_2, ω_3 in terms of elliptic functions are given by

$\omega_1 = a d_n\,(\lambda t + \mu)$, $\omega_2 = b S_n\,(\lambda t + \mu)$

and $\omega_3 = c\,C_n\,(\lambda t + \mu)$.

This gives the required solution in terms of elliptic functions.

Ex. 5. *Show that if a rigid body rotates about a fixed point under no forces, once the instantaneous axis of rotation, coincides with a principal axis, rotation about this axis persists.*

(Principle of the gyroscopic compass)

Solution. Let $\vec{\omega}$ be the angular velocity of the body, then we have

$$\frac{d\mathbf{h}}{dt} = \frac{\delta\mathbf{h}}{\delta t} + \vec{\omega} \times \mathbf{h}$$

where $\mathbf{h}$ is the angular momentum about the fixed point.

But $\dot{\mathbf{h}} = 0$, so we have $\dfrac{\delta\mathbf{h}}{\delta t} + \vec{\omega} \times \mathbf{h} = 0 \Rightarrow \dfrac{\delta\mathbf{h}}{\delta t} = 0$

($\because$ rotation ensues about a principal axis implying $\vec{\omega} \times \mathbf{h} = 0$).

This signifies that **h** *is fixed relative to the frame to principal axis through the point of rotation.* **Hence a wheel set rotating about its axis tends to continue in this motion. This is the principle of the gyroscopic compass.**

Note 1. When the force system in conservative then we can use, in place of any one of the three equations (6), (7), (8) [or (9), (10), (11)] the following equation obtained by the principle of energy :

$$T+V=E, \text{ where } T=\tfrac{1}{2}\,(A\omega_1^2+B\omega_2^2+C\omega_3^2)$$

Note 2. If the unit vectors **i**, **j**, **k** are not in the directions of principal axis, then we have $\mathbf{h}=h_1\mathbf{i}+h_2\mathbf{j}+h_3\mathbf{k}$,

where $h_1=A\omega_1-H\omega_2-G\omega_3$, $h_2=-H\omega_1+B\omega_2-F\omega_3$,

$$h_3=-G\omega_1-F\omega_2+C\omega_3.$$

$$\therefore\quad \frac{\delta\mathbf{h}}{\delta t}=\frac{dh_1}{dt}\,\mathbf{i}+\frac{dh_2}{dt}\,\mathbf{j}+\frac{dh_3}{dt}\,\mathbf{k}.$$

Thus the equation $\dot{\mathbf{h}}=\mathbf{G}$

$$\Rightarrow\quad \frac{\delta\mathbf{h}}{\delta t}+\vec{\Omega}\times\mathbf{h}=\mathbf{G}$$

i.e.,

$$\left\{\dot h_1\mathbf{i}+\dot h_2\mathbf{j}+\dot h_3\mathbf{k}\right\}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \Omega_1 & \Omega_2 & \Omega_3\\ h_1 & h_2 & h_3\end{vmatrix}=G_1\mathbf{i}+G_2\mathbf{j}+G_3\mathbf{k}$$

Equating coefficients of like vector on both sides, we get

$$\dot h_1-h_2\Omega_3+h_3\Omega_2=G_1,$$
$$\dot h_2-h_3\Omega_1+h_1\Omega_3=G_2,$$
$$\dot h_3-h_1\Omega_2+h_2\Omega_1=G_3.$$

If the triad **i**, **j**, **k** is fixed in the body, we have $\vec{\Omega}=\vec{\omega}$

$$\therefore\quad \dot h_1-h_2\omega_3+h_3\omega_2=G_1,\qquad \dot h_2-h_3\omega_1+h_1\omega_3=G_2,$$

and

$$\dot h_3-h_1\omega_2+h_2\omega_1=G_3.$$

(b) Properties of rigid body motion under no forces.

We know already that the scalars, T, H are constants for rotation about a fixed point under no forces. Further since $\dot{\mathbf{h}}=\mathbf{0}$, $\mathbf{h}$ is a constant vector and so $\hat{\mathbf{h}}$ is also constant. Also we have $\mathbf{h}\cdot\vec{\omega}=2T$,

$\therefore\ \vec{\omega}\cdot\hat{\mathbf{h}}=\dfrac{2T}{h}=$ constant, showing that the projection of $\vec{\omega}$ on the constant direction $\hat{\mathbf{h}}$ is constant. Thus, if O is the point of rotation and if $\mathbf{OP}=\vec{\omega}$ then the locus of P is an *invariable plane* perpendicular to **h**.

Further we observe that the locus of the instantaneous axis of rotation in a space is a cone (not necessarily right circular) having O as vertex. This is called a *herpolhode*.

Also the locus of the instantaneous axis relative to the body is also a cone of vertex O and is called *polhode* whose equation can be obtained as below :

Referred to the moving frame of principal axes **i, j, k**, the equation of the instantaneous axis of rotation is given by

$$\frac{x}{\omega_1}=\frac{y}{\omega_2}=\frac{z}{\omega_3}.$$

But we have $A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2=h^2=\text{constant}.$

$$(\because \quad \mathbf{h}=A\omega_1\mathbf{i}+A\omega_2\mathbf{j}+C\omega_3\mathbf{k})$$

and $$A\omega_1^2+B\omega_2^2+C\omega_3^2=2T=\text{constant}.$$

From these three equations, we derive the equation

$$2T\,(A^2x^2+B^2y^2+C^2z^2)=h^2\,(Ax^2+By^2+Cz^2).$$

Obviously this is a cone of second degree in x, y, z. The polhode and herpolhode touch along the instantaneous axis of rotation. This signifies that rigid body motion about a fixed point is equivalent to the rolling of one-cone on another.

We shall now establish the following theorem :

Motion of a rigid body about a fixed point may be represented by the rolling of an ellipsoid fixed in the body upon a plane fixed in space.

Proof. Let i, j, k be the direction of the principal axes of the body and let (x, y, z) be the co-ordinates w.r.t., these lines, fixed in the body.

Now consider the ellipsoid $Ax^2+By^2+Cz^2=2T.$ ...(i)

The instantaneous axis is

$$\frac{x}{\omega_1}+\frac{y}{\omega_2}+\frac{z}{\omega_3}=\frac{r}{\omega}$$

and this meets the ellipsoid at $r=\omega$ [*i.e.* at the point $\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$ $\equiv(\omega_1, \omega_2, \omega_3)$]. Now the equation of the tangent plane to the elliposed at $(\omega_1, \omega_2, \omega_3)$ is $A\omega_1x+B\omega_2y+C\omega_3z=2T.$ ...(ii)

The tangent plane is at a distance p from the origin (the point of rotation) where $p=2T/\sqrt{\{A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2\}}=(2T/h)=\text{const.}$ $\Rightarrow$ that the tangent plane at $[\omega_1, \omega_2, \omega_3]$ is the invariable plane and so the ellipsoid touches the invariable plane.

The point of contact of the ellipsoid and the invariable plane lies on the instaneous axis of rotation and hence no slipping can occur at this point. In this way the ellipsoid rolls on the invariable plane.

4·13. General motion of a rigid body.

Let **F** be the total external forces and **G** the total moment of the external forces about the centre of mass, then the acceleration **a** of the centre of mass (relative to a Newtonian frame) is given by

$$^{*}m\mathbf{a}=\mathbf{F} \qquad \ldots(1)$$

where m is the mass of the body.

Also the motion relative to the centre of mass is given by

$$\dot{\mathbf{h}}=\mathbf{G}, \qquad \ldots(2)$$

where **h** is the angular momentum of the body about the centre of mass. The equation is the same as if the centre of mass were fixed and hence can be tackled by the methods given above.

Now $$\mathbf{a}=\frac{\delta \mathbf{v}}{\delta t}+\vec{\Omega}\times\mathbf{v} \qquad \ldots(3)$$

and $$\mathbf{v}=v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}. \qquad \ldots(4)$$

Substituting these value of **a** and **v** in (1) and (2), we get

$$m\left[\frac{\delta}{\delta t}(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \Omega_1 & \Omega_3 & \Omega_3\\ v_1 & v_2 & v_3\end{vmatrix}\right]=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k} \qquad \ldots(5)$$

and $$\Sigma\,(A\dot{\omega}_1-B\omega_2\Omega_3+C\omega_3\Omega_2)\,\mathbf{i}=\Sigma\, G_1\mathbf{i} \qquad \ldots(6)$$

or we have $$m(\dot{v}_1-v_2\Omega_3+v_3\Omega_2)=F_1 \qquad \ldots(7)$$

$$m(\dot{v}_2-v_3\Omega_1+v_1\Omega_3)=F_2 \qquad \ldots(8)$$

$$m(\dot{v}_3-v_1\Omega_2+v_2\Omega_1)=F_3 \qquad \ldots(9)$$

$$A\dot{\omega}_1-B\omega_2\Omega_3+C\omega_3\Omega_3=G_1 \qquad \ldots(10)$$

$$B\dot{\omega}_2-C\omega_3\Omega_1+A\omega_1\Omega_3=G_2 \qquad \ldots(11)$$

$$C\dot{\omega}_3-A\omega_1\Omega_2+A\omega_2\Omega_1=G_2 \qquad \ldots(12)$$

where the symbols have their usual meanings.

Note 1. For any one of the six equations (7 to 12), we can substitute the law of conservation of energy

$$T+V=E \text{ i.e. } \tfrac{1}{2}m(v_1^2+v_2^2+v_3^2)+\tfrac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2)+V=E$$

provided the external forces are conservative.

Note 2. If the unit vectors **i, j, k** are not in the directions of principal axes, then the above six equations reduce to

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*For the sake of simplicity, we have dropped the subscripts from x^0 and a_0, the velocity and acceleration of the centre of mass.

$$m(\dot{v}_1 - v_2\Omega_3 + v_3\Omega_2) = F_1 \qquad \ldots(7')$$

$$m(\dot{v}_2 - v_3\Omega_1 + v_1\Omega_3) = F_2 \qquad \ldots(8')$$

$$m(\dot{v}_3 - v_1\Omega_2 + v_2\Omega_1) = F_3 \qquad \ldots(9')$$

$$\dot{h}_1 - h_2\Omega_3 + h_3\Omega_2 = G_1 \qquad \ldots(10')$$

$$\dot{h}_2 - h_3\Omega_1 + h_1\Omega_3 = G_2 \qquad \ldots(11')$$

$$\dot{h}_3 - h_1\Omega_2 + h_2\Omega_1 = G_3. \qquad \ldots(12')$$

If the triad **i**, **j**, **k** is fixed in the body, we have $\vec{\Omega} = \vec{\omega}$. Hence the above six equations reduce to

$$m(\dot{v}_1 - v_2\omega_3 + v_3\omega_2) = F_1 \qquad \ldots(7'')$$

$$m(\dot{v}_2 - v_3\omega_1 + v_1\omega_3) = F_2 \qquad \ldots(8'')$$

$$m(\dot{v}_3 - v_1\omega_2 + v_2\omega_1) = F_3 \qquad \ldots(9'')$$

$$\dot{h}_1 - h_2\omega_3 + h_3\omega_2 = G_1 \qquad \ldots(10'')$$

$$\dot{h}_2 - h_3\omega_1 + h_1\omega_3 = G_2 \qquad \ldots(11'')$$

$$\dot{h}_3 - h_1\omega_2 + h_2\omega_1 = G_3. \qquad \ldots(12'')$$

4·14. General equations of impulsive motion.

If **v** denotes the velocity of the centre of mass and m the total mass, then we have $m\dot{\mathbf{v}} = \mathbf{F}$ ...(13)
where **F** is the external force.

Also the principle of angular momentum (under finite system) gives $\dot{\mathbf{h}} = \mathbf{G}$. ...(14)

Integration of the equations (9) and (10) from time t_0 to time t_1, gives

$$\delta(m\mathbf{v}) = \int_{t_0}^{t_1} \mathbf{F}\, dt\,; \qquad \ldots(15)$$

and

$$\delta\mathbf{h} = \int_{t_0}^{t_1} \mathbf{G}\, dt \qquad \ldots(16)$$

where δ denotes an increment in the time interva.,

In words equations (15) and (16) $\Rightarrow$

(*i*) *The increment in the linear momentum of a system is equal to the total impulse of the external forces.*

(Principle of linear momentum)

(*ii*) *The increment in the angular momentum about the point O (either a fixed point in Newtonian frame or the centre of mass* *of the system is equal to the time integral of total moment about O of the external forces i.e. the total angular impules about O.*

(Principle of angular momentum)

The very short interval of time ($t_1=t_0$) in which the changes occur is regarded an infinitesimal. Any finite force **F** will then contribute to the total impulsive force

$$\mathbf{F}^*=\lim_{t_1\to t_0}\int_{t_0}^{t_1}\mathbf{F}\,dt.$$

On the other hand, a force **P**, for which $\mathbf{P}^*=\lim_{t_1\to t_0}\int_{t_0}^{t_1}\mathbf{P}\,dt$, is finite, contributes the *impulsive force* $\mathbf{P}^*$ to $\mathbf{F}^*$. Now let **r** be the point O about which the angular momentum is calculated, then we have

$$\lim_{t_1\to t_0}\int_{t_0}^{t_1}(\mathbf{r}\times\mathbf{P})\,dt=\mathbf{r}\times\lim_{t_1\to t_0}\int_{t_0}^{t_1}\mathbf{P}\,dt=\mathbf{r}\times\mathbf{P}^* \qquad ...(17)$$

($\because$ **r** does not change by a finite amount in the infinitesimal time t_1-t_0).

Also thc force **P** contributes the impulse moment $\mathbf{r}\times\mathbf{P}^*$ to the total impulsive moment

$$\mathbf{G}^*=\lim_{t_1\to t_0}\int_{t_0}^{t_1}\mathbf{G}\,dt. \qquad ...(18)$$

$\therefore$ (15) and (16) $\Rightarrow$ $\delta\,(m\mathbf{v})=\mathbf{F}^*$; $\qquad ...(19)$

$$\delta\,(\mathbf{h})=\mathbf{G}^* \qquad ...(20)$$

where $\mathbf{F}^*$ = vector sum of the external impulsive forces

$\mathbf{G}^*$ = total moment of external impulsive forces.

Equations (19) and (20) are known as the general equations of impulsive motion.

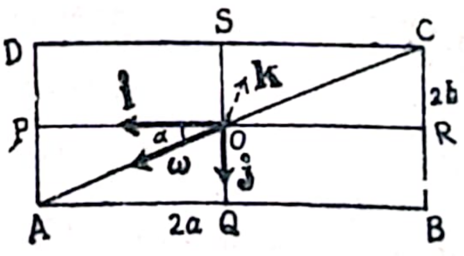
Ex. 6. *The rectangular plate spins which constant angular velocity ω about a diagonal. Find the couple which must act on the plate in order to maintain this motion.*

Solution. O is the mass centre of the rectangular plate $ABCD$ and **i**, **j**, **k** are the unit vectors along the principal axes of inertia at O. Vectors **i** and **j** lie in the plane but **k** is normal to the plane of the plate.

Now M. I. of the body about $OP=\frac{1}{3}mb^2$,

M. I. of the body about $OQ=\frac{1}{3}ma^2$ and M. I. of the body about a line passing throught O and perpendicular to its plane

$$=\frac{m}{3}(a^2+b^2)$$



where m is the mass of the plate, $2a$ its length and $2b$ its breadth.

Obviously, we have

$$\vec{\omega}=\omega \cos \alpha \mathbf{i}+\omega \sin \alpha \mathbf{j}+0\mathbf{k} \text{ where } \tan \alpha=\frac{b}{a}$$

$$\Rightarrow (\omega_1, \omega_2, \omega_3)=(\omega \cos \alpha, \omega \sin \alpha, 0)$$

$$\Rightarrow \omega_1=\omega \cos \alpha, \omega_2=\omega \sin \alpha, \omega_3=0.$$

Here **i**, **j**, **k** are fixed in the body so Euler's equations of motion are :

$$A\dot{\omega}_1=(B-C)\,\omega_2\omega_3=G_1 \Rightarrow \tfrac{1}{3} mb^2 \dot{\omega}_1=G_1 \Rightarrow G_1=0$$

$$(\because \vec{\omega} \text{ is a constant vector}) \qquad ...(1)$$

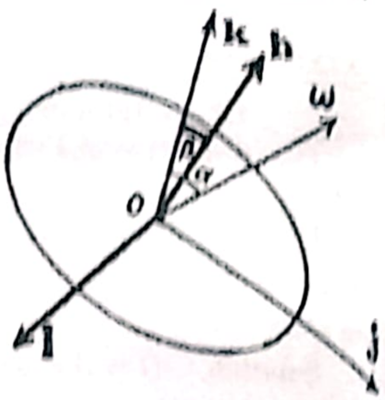
$$B\dot{\omega}_2-(C-A)\,\omega_3\omega_1=G_2 \Rightarrow \tfrac{1}{3} ma^2 \dot{\omega}_2=G_2 \Rightarrow G_2=0 \qquad ...(2)$$

$$C\dot{\omega}_3-(A-B)\,\omega_1\omega_2=G_3 \Rightarrow \tfrac{1}{3}m(a^2-b^2)\,\omega^2 \sin \alpha \cos \alpha=G_3 \quad ...(3)$$

Thus we have obtained the values of G_1, G_2, G_3 *i.e.* **G** is known.

Ex. 7. *A circular disc of radius a and mass m is supported on a needle, points at its centre; it is set spinning with angular velocity ω_0 about a line making an angle α with the normal to the disc. Find the angular velocity of the disc at any subsequent time.*

Solution. Unit vector **k** is normal to plane of the disc at the point O, the mass centre of the circular disc, and **i**, **j**, are fixed in the plane of the disc. New assume that the initial angular velocity lies in the plane of **j** and **k**. After a time t, the angular velocity of the disc is given by



$$\vec{\omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$$

At time $t=0$, we have

$$\omega_1=0,\ \omega_2=\omega_0 \sin \alpha,$$
$$\omega_3=\omega_0 \cos \alpha.$$

Also we have $A=B=\frac{1}{4} ma^2$ and $C=\dfrac{ma^2}{2}$.

But the external forces (the reaction at O and the weight of the disc) have no moment about O, hence Euler's equations are :

$$A\dot{\omega}_1-(A-C)\,\omega_2\omega_3=0 \qquad ...(1)$$

$$A\dot{\omega}_2-(C-A)\,\omega_3\omega_1=0 \qquad ...(2)$$

$$C\dot{\omega}_3=0 \Rightarrow \omega_3=\text{constant}=\omega_0 \cos \alpha \qquad ...(3)$$

$\therefore$ (1) and (2) $\Rightarrow A(\dot{\omega}_1+i\dot{\omega}_2)+(C-A)(\omega_2\omega_3-i\omega_3\omega_1)=0$

$\Rightarrow A\dot{\xi}-i(C-A)\,\omega_0\cos\alpha\,\xi=0$ where $\xi=\omega_1+i\omega_2$

$\Rightarrow A\dot{\xi}-Ai\omega_0\cos\alpha\,\xi=0 \qquad ...(\because \; C=2A)$

$\Rightarrow \dot{\xi}-i\omega_0\cos\alpha\,\xi=0 \Rightarrow \xi=\xi_0\,e^{i\omega_0 t\cos\alpha}$

where ξ_0 is a constant.

Initially we have

$$\xi_0=(\omega_1+i\omega_2)_t+i\omega_0\sin\alpha=i\omega_0\sin\alpha$$
$$=0$$

$$\therefore \quad \xi=i\omega_0\cos\alpha\,e^{i\omega_0 t\cos\alpha}$$
$$=i\omega_0\sin\alpha\,\{\cos(\omega_0 t\cos\alpha)+i\sin(\omega_0\cos\alpha)\}$$

$\Rightarrow \xi=i\omega_0\sin\alpha\cos(\omega_0 t\cos\alpha)-\omega_0\sin\alpha\sin(\omega_0 t\cos\alpha)$

$\Rightarrow \omega_1+i\omega_2=i\omega_0\sin\alpha\cos(\omega_0 t\cos\alpha)-\omega_0\sin\alpha\sin(\omega_0 t\cos\alpha)$

$\Rightarrow \omega_1=-\omega_0\sin\alpha\sin(\omega_0 t\cos\alpha)$

and $\qquad \omega_2=\omega_0\sin\alpha\cos(\omega_0 t\cos\alpha)$.

Ex. 8. *Find the motion in space of the disc considered in the previous example.*

Solution. See the figure of the previous example.

The vector **h** is the angular momentum, $\vec{\omega}$ the angular velocity vector, **k** a unit vector normal to the disc and **i**, **j** are the unit vectors in the plane of the disc, but not fixed in it.

Now the following facts are obvious,

(*i*) *External forces have no moment about O. Therefore* **G**=0. *This implies* $\dot{\mathbf{h}}=0$ *i.e.* **h** *is a constant vector meaning thereby it has a fixed direction in space determined by the initial conditions.*

(ii) $\mathbf{h}=A\omega_1\mathbf{i}+A\omega_2\mathbf{j}+A\omega_3\mathbf{k}$ *and* $h_1=0$. *Hence* $\omega_1=0$ which *also implies that* $\vec{\omega}$ *lies in the plane of* **j** *and* **k**.

(*iii*) *Triad* **i**, **j**, **k** *is not fixed in the disc so* $\vec{\omega}$ *is different from* $\vec{\omega}$. *But* **k** *is fixed both in the triad and in the disc. The extremity of* **k** *has velocity* $\vec{\Omega}\times\mathbf{k}$, *as a point of the disc, it has a velocity* $\vec{\omega}\times\mathbf{k}$ *thus implying*

$$(\Omega_1\mathbf{i}+\Omega_2\mathbf{j}+\Omega_3\mathbf{k})\times\mathbf{k}=(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k})\times\mathbf{k}$$

i.e. $\qquad \Omega_1=\omega_1,\ \Omega_2=\omega_2,\ \Omega_3=\omega_3.$

But $\Omega_1=0$, *so we have* $\omega_1=0$. *Now the equations of motion of a rigid body are*

to be fixed in space (i.e. it is an intertial system). The angular velocity of the xyz system relative to the XYZ system is given by $\vec{\Omega}=2\mathbf{i}-t^2\mathbf{j}+(2t+4)\,\mathbf{k}$ *where t is the time. The position vector of a particle at time t as observed in the xyz system is given by* $\mathbf{r}=(t^2+1)\,\mathbf{i}-6t\mathbf{j}+4t^3\mathbf{k}$. *Find (a) the apparent velocity and (b) the true velocity at time* $t=1$.

Solution. (a) We have $\mathbf{v}=\mathbf{v}'+\vec{\Omega}\times\mathbf{r}$

where $\mathbf{v}'=\dfrac{\delta\mathbf{r}}{\delta t}=2t\mathbf{i}-6\mathbf{j}+12t^2\mathbf{k} \Rightarrow (\mathbf{v}')_{t=1}=2\mathbf{i}-6\mathbf{j}+12\mathbf{k}.$

(*b*) $$\vec{\Omega}\times\mathbf{r}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & -t^2 & 2t+4 \\ t^2+1 & -6t & 4t^3 \end{vmatrix}$$

$$\Rightarrow \left(\vec{\Omega}\times\mathbf{r}\right)_{t=1}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix}$$

$$\therefore\quad (\mathbf{v})_{t=1}=2\mathbf{i}-6\mathbf{j}+12\mathbf{k}+\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix}=34\mathbf{i}-2\mathbf{j}-2\mathbf{k}.$$

Ex. 11. *Find (a) the apparent acceleration and (b) the true acceleration of the particle in Ex. 10.*

Solution. (a) We have $\mathbf{a}'=\dfrac{\delta^2\mathbf{r}}{\delta t^2}$.

$\therefore\quad (\mathbf{a}')_{t=1}=(2\mathbf{i}+6\times 2^2t\mathbf{k})_{t=1}=2\mathbf{i}+24\mathbf{k}.$

(b) $\mathbf{a}=\mathbf{a}'+\mathbf{a}_t+\mathbf{a}_c$

where $\mathbf{a}'=2\mathbf{i}+24t\mathbf{k},\ \mathbf{a}_t=\dfrac{d\Omega}{dt}\times\mathbf{r}+\vec{\Omega}\,(\vec{\Omega}.\mathbf{r})-\mathbf{r}\Omega^2$

$$=(2\mathbf{i}-2t\mathbf{j}+2\mathbf{k})\times[(t^2+1)\,\mathbf{i}-6t\mathbf{j}+4t^2\mathbf{k}]$$
$$+[\,\mathbf{i}t-t^2\mathbf{j}+(2t+4)\,\mathbf{k}]\,[2t\,(t^2+1)+6t^3+4t^3\,(2t+4)]$$
$$-[(t^2+1)\,\mathbf{i}-6t\mathbf{j}+4t^3\mathbf{k}]\,[4t^2+t^4+(2t+4)^2]$$

$$\Rightarrow (\mathbf{a}_t)_{t=1}=(\,\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(2\mathbf{i}-5\mathbf{j}+4\mathbf{k})$$
$$+[(2\mathbf{i}-\mathbf{j}+6\mathbf{k})\,(34)-(2\mathbf{i}-6\mathbf{j}+4\mathbf{k})\,41].$$

$$\text{Also } \mathbf{a}_c=2\vec{\Omega}\times\mathbf{v}'=2\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2t & -t^2 & 2t+4\\ 2t & -6 & 12t^2\end{vmatrix}$$

$$\Rightarrow (\mathbf{a}_c)_{t=1}=2\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 6\\ 2 & -6 & 12\end{vmatrix}=48\mathbf{i}-24\mathbf{j}-20\mathbf{k}$$

$$\therefore \quad (\mathbf{a}_c)_{t=1}=(\mathbf{a}')_{t=1}+(\mathbf{a}_t)_{t=1}+(\mathbf{a}_c)_{t=1}$$

$$=(2\mathbf{i}+24\mathbf{k})+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -2 & 2\\ 2 & -6 & 4\end{vmatrix}$$

$$+(68\mathbf{i}-34\mathbf{j}+204\mathbf{k}-82\mathbf{i}+246\mathbf{j}-64\mathbf{k})+(48\mathbf{i}-24\mathbf{j}-20\mathbf{k})$$

$$=(2\mathbf{i}+24\mathbf{k})+(4\mathbf{i}-4\mathbf{j}-8\mathbf{k})+(68\mathbf{i}-34\mathbf{j}+204\mathbf{k}-82\mathbf{i}+246\mathbf{j}-164\mathbf{k})$$
$$+(48\mathbf{i}-24\mathbf{j}-20\mathbf{k})$$

$$=(2\mathbf{i}+24\mathbf{k})+(4\mathbf{i}-4\mathbf{j}-8\mathbf{k})+(-14\mathbf{i}+212\mathbf{j}+40\mathbf{k})$$
$$+(48\mathbf{i}-24\mathbf{j}-24\mathbf{k})$$

$$=40\mathbf{i}+184\mathbf{j}+36\mathbf{k}.$$

Ex. 32. *Find (a) coriolis acceleration, (b) the centripetal acceleration and (c) their magnitudes at time $t=1$.*

Solution. (*a*) We have coriolis acceleration

$$=\mathbf{a}_c=2\vec{\Omega}\times\mathbf{v}'=2\ (2\mathbf{i}-\mathbf{j}+6\mathbf{k})\times(2\mathbf{i}-6\mathbf{j}+12\mathbf{k})$$
$$=48\mathbf{i}-24\mathbf{j}-20\mathbf{k}.$$

(b) Centripetal acceleration.

$$=\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})=(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\times(32\mathbf{i}+4\mathbf{j}-10\mathbf{k})$$
$$=10\mathbf{i}+200\mathbf{j}+20\mathbf{k}.$$

(*c*) Magnitude of coriolis acceleration

$$=[(48)^2+(-24)^2+(-20)^2]=\sqrt{(205)}.$$

Magnitude of coriolis acceleration

$$=\sqrt{[100+(200)^2+(30)^2]}=10\sqrt{(410)}.$$

Ex. 13. *Find the equation of motian of a particle relative to an observer on the earth's surface.*

Solution. Let the angular speed of the earth be Ω.

Now by equations (*A*) on P. 196 we have

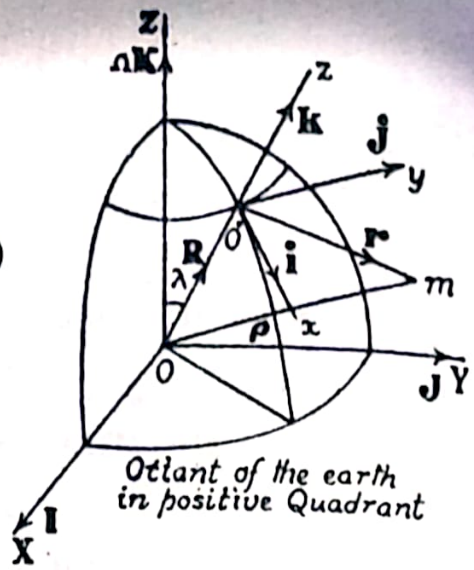
$$m\mathbf{a}'=\mathbf{F}-m\mathbf{a}_0-m\mathbf{a}_t-m\mathbf{a}_c \qquad \ldots(A)$$

where $\quad \mathbf{a}'=\dfrac{d^2\mathbf{r}}{dt^2},\ \mathbf{a}_t=\dot{\vec{\Omega}}\times\mathbf{r}+\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r}),\ \mathbf{a}_0=\ddot{\vec{R}}$

and $\quad \mathbf{a}_c=2\vec{\Omega}\times\mathbf{v}'$;

[$\mathbf{v}'$=apparent velocity=$\dot{\vec{\mathbf{r}}}$]

For the case of earth we have



$\dot{\vec{\Omega}}=0,$ ...(1)

$\mathbf{a}_t=\vec{\Omega}\times(\vec{\Omega}\times\mathbf{R}),$ ...(2)

$\mathbf{F}=-\dfrac{GMm}{\rho^2}\hat{\rho}=-\dfrac{GMm}{\rho^3}\vec{\rho},$ (3)

and $\ddot{\mathbf{R}}=\vec{\Omega}\times(\vec{\Omega}\times\mathbf{R})$ ...(4)

$\therefore$ $(A)\Rightarrow$

$m\dfrac{d^2\mathbf{r}}{dt^2}=-\dfrac{GMm}{\rho^3}\vec{\rho}-m\ddot{\mathbf{R}}-m\vec{\Omega}$

$\times(\vec{\Omega}+\mathbf{r})-2m\vec{\Omega}\times\mathbf{v}'$ ...(5)

$\Rightarrow \ddot{\mathbf{r}}=-\dfrac{GM}{\rho^3}\vec{\rho}-\vec{\Omega}\times(\vec{\Omega}\times\mathbf{R})-2\vec{\Omega}\times\mathbf{v}'-\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})$ ...(6)

where it has been assumed that air resistance etc. have on effect up on the motion.

Let us define $\mathbf{g}=-\dfrac{GM}{\rho^3}\vec{\rho}+\vec{\Omega}\times(\vec{\Omega}\times\mathbf{R})$ ...(7)

as the acceleration due to gravity, so that (6) gives

$$\ddot{\mathbf{r}}=\mathbf{g}-2\vec{\Omega}\times\mathbf{v}'-\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r}) \qquad ...(8)$$

On the surface of the earth we can neglect $\vec{\Omega}\times(\vec{\Omega}\times\mathbf{r})$, so that to a high degree of approximation,

$$\ddot{\mathbf{r}}=\mathbf{g}-2\,(\vec{\Omega}\times\mathbf{v}'). \qquad ...(9)$$

This is the equation of motion of particle relative to an observer on the earth's surface.

Ex. 14. *Show that if the particle of Ex. 13 P. 213 moves near earth's surface, then the equations of motion are given by*

$\ddot{x}=2\Omega\cos\lambda\,\dot{y}$

$\ddot{y}=-2\,(\Omega\cos\lambda\,\dot{x}+\Omega\sin\lambda\,\dot{z})$

$\ddot{z}=-g+2\Omega\sin\lambda\,\dot{y}$

where the angle λ *is the colatitude,* $90°-\lambda$ *is the latitude.*

Solution. By the figure of Ex. 13, we have

$\mathbf{K}=(\mathbf{K}\cdot\mathbf{i})\,\mathbf{i}+(\mathbf{K}\cdot\mathbf{j})\,\mathbf{j}+(\mathbf{K}\cdot\mathbf{k})\,\mathbf{k}=\cos\left(\dfrac{\pi}{2}+\lambda\right)\mathbf{i}+0\mathbf{j}+\cos\lambda\mathbf{k}$

$=-\sin\lambda\,\mathbf{i}+\cos\lambda\mathbf{k}.$

$\therefore \quad \vec{\Omega}=\Omega \mathbf{K}=-\Omega \sin \lambda \mathbf{i}+\Omega \cos \lambda \mathbf{k}.$

Also $\vec{\Omega}\times \mathbf{v}'=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\Omega \sin \lambda & 0 & \Omega \cos \lambda \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$

where $\mathbf{v}'=\dot{x}\mathbf{i}+\dot{y}\mathbf{j}+\dot{z}\mathbf{k}$

$=(-\Omega \cos \lambda\dot{y})\ \mathbf{i}+(\Omega \cos \lambda\dot{x}+\Omega \sin \lambda\dot{z})\ \mathbf{j}$

$-(\Omega \sin \lambda\dot{y})\ \mathbf{k}$

Substituting these values in equation

$\ddot{\mathbf{r}}=\mathbf{g}-2\vec{\Omega}\times \mathbf{v}$, we get

$\ddot{\mathbf{r}}=g\mathbf{k}+2\Omega \cos \lambda\dot{y}\mathbf{i}-2\ (\Omega \cos \lambda+\Omega \sin \lambda)\ \dot{z}\mathbf{j}+2\Omega \sin \lambda\dot{y}\mathbf{k}.$

Now equating coefficients of like vectors on both sides, we get

$\ddot{x}=2\Omega \cos \lambda\dot{y},\ \ddot{y}=-2\ (\Omega \cos \lambda\ \dot{x}+\Omega \sin \lambda\dot{z})$

and $\ddot{z}=-g+2\Omega \sin \lambda\ \dot{y}.$ ...(1)

Ex. 15. *An object of mass "m" initial at rest is dropped to the earth's surface from a height which is small compared with the earth's radius. Assuming that the angular speed of the earth about its axis is constant Ω, prove that after time t, the object is deflected east of the vertical by the amount of $\frac{1}{3}\ \Omega\ gt^3 \sin \lambda$.*

Solution. We assume that the object is located on the z-axis at $x=0,\ y=0,\ z=h$.

Now by previous example, we get

$\ddot{x}=2 \cos \lambda\dot{y}$ and $\ddot{y}=-2\ (\Omega \cos \lambda\dot{x}+\Omega \sin \lambda\dot{z}).$

Integrating, we obtain $\dot{x}=2\Omega \cos \lambda y+C_1$

and $\dot{y}=-2\ (\Omega \cos \lambda x+\Omega \sin \lambda z)+C_2.$ ...(2)

Initially, we have $t=0,\ \dot{x}=0,\ \dot{y}=0,\ x=0=y,\ z=h,$

$\therefore \quad C_1=0$ and $C_2=2\Omega h \sin \lambda.$

Thus $\dot{x}=2\Omega \cos \lambda y,\ \dot{y}=-2\ (\Omega \cos \lambda\cdot x+\Omega \sin \lambda\cdot z)$

$+2\Omega \sin \lambda\cdot h$ ...(3)

So we have

$\ddot{z}=-g+2\Omega \sin \lambda,\ \ddot{y}=-g-4\Omega^2 \sin \lambda\ \{x \cos \lambda+(z-h) \sin \lambda\}$

As Ω is very small compared with g, we obtain

$\ddot{z}=-g.$ ...(4)

Integration yields $\dot{z}=-gt+C_3$ ...(5)

Initially, $t=0,\ \dot{z}=0 \Rightarrow C_3=0 \quad \therefore \quad \dot{z}=-gt$ ...(6)

Now $\ddot{y}=(-2\Omega \cos \lambda)\ (2\Omega \cos \lambda y)+(-2\Omega \sin \lambda)\ (-gt)$

$=-4\Omega^2 \cos \lambda^2 y+2\Omega \sin \lambda gt)$

$\Rightarrow \quad \ddot{y}=2\Omega \sin \lambda\cdot gt \qquad (\because \quad \Omega^2$ is neglected) ...(7)

Integration yields $\dot{y}=\Omega g \sin \lambda t^2+C_4$

Initially $t=0$, $\dot{y}=0 \Rightarrow C_4=0$ and $\dot{y}=\Omega g \sin \lambda t^2$ ...(8)

Integrating (8), again, we get $y=(\Omega/3)\, g \sin \lambda t^3+C_5$

Initially $t=0$, $y=0 \Rightarrow C_5=0$

and $y=\frac{1}{3}g\, \Omega t^3 \sin \lambda$.

Ex. 16. *Referring to previous example, show that an object dropped from a height h above the earth's surface hits the earth at a point east of the vertical at a distance* $\frac{2}{3}\Omega\, h \sin \lambda \sqrt{\left(\frac{2h}{g}\right)}$.

Solution. By previous example, we have $\dot{z}=-gt$ (eqn. 6)

$\because \quad z=-\frac{1}{2}gt^2+C_6$

Initially $t=0$, $z=h \Rightarrow C_6=h$ and $z=h-\frac{1}{2}gt^2$

When $z=0$, we have $t=\sqrt{\left(\frac{2h}{g}\right)}$ ...(10)

But $y=\frac{1}{3}\Omega g \sin \lambda t^3$ [see eqn. (9) of the previous example].

$$\therefore \quad y=\frac{1}{3}\, \Omega g \sin \lambda \, \frac{2h}{g} \sqrt{\left(\frac{2h}{g}\right)}=\frac{2}{3}\, \Omega h \sin \lambda \sqrt{\left(\frac{2h}{g}\right)}.$$

Ex. 17. *If the earth is assumed to be an oblate spheroid such that a=b while c differs slightly from a or b. Prove that to a high degree of approximation,* $\frac{C-A}{A}=\frac{1-c}{a}$ *where A, B, C are the principal moments of inertia at the centre of the earth.*

Solution. Principal moments of inertia at the centre of the ellipsoid are

$$A=(M/5)\,(b^2+c^2),\ B=(M/5)\,(c^2+a^2) \text{ and } C=(M/5)\,(a^2+b^2).$$

Putting $a=b$, the principal moments of inertia at the centre of the earth are

$$A=(M/5)\,(a^2+c^2),\ B=(M/5)\,(a^2+c^2) \text{ and } C=(M/5)\,(2a^2).$$

$$\therefore \quad \frac{C-A}{A}=\frac{(2a^2-a^2-c^2)}{a^2+c^2}=\frac{a^2-c^2}{a^2-c^2} \quad ...(1)$$

If c differs slightly from a, then $a+c \approx 2a$ and $a^2+c^2 \approx 2a^2$.

Thus $\frac{C-A}{A}=(a-c)\,\frac{2a}{2a^2}=\frac{a-c}{a}\ 1-\left(\frac{c}{a}\right)$ approximately.

But the polar diameter or distance between north and south poles is very nearly 7900 miles while the equatorial dimeter is very nearly 7926 miles, then taking the polar axis as "z" axis, we have

$$2c=7900,\ 2a=7926 \Rightarrow c=3950 \text{ and } a=3963.$$

$$\therefore \quad \frac{C-A}{A}=1-\frac{3950}{3963}=\frac{3963-3950}{3963}=.00328.$$

Ex. 18. *A rigid body which is symmetric about an axis has one point fixed on this axis. Discuss the rotational motion of the body, assuming that there are no forces acting other than the reaction force at the fixed point.*

Solution. Let us assume that the axis of sysmmetry coincident with one of the principal axes, say the one having direction **k**. Then $A=B$ and Euler's equation give ;

$$A\dot{\omega}_1+(C-A)\,\omega_3\omega_2=0 \qquad ...(1)$$

$$A\dot{\omega}_2+(A-C)\,\omega_3\omega_1=0 \qquad ...(2)$$

$$C\dot{\omega}_3=0. \qquad ...(3)$$

(1) $\Rightarrow$ $\omega_3=\text{constant}=K$ say so that (1) and (2) become after dividing by A,

$$\omega+\left(\frac{C-A}{A}\right)K\omega_2=0 \qquad ...(4)$$

and $$\dot{\omega}_2+\left(\frac{A-C}{A}\right)K\omega_1=0 \qquad ...(5)$$

$$\therefore\ \ddot{\omega}_2=-\left(\frac{A-C}{A}\right)K\dot{\omega}_1=\left(\frac{A-C}{A}\right)K^3\left(\frac{C-A}{A}\right)\omega_2$$

$$\Rightarrow\ \ddot{\omega}_2+\kappa^2\omega_2=0 \qquad ...(6)$$

where $$\kappa=\left|\frac{C-A}{A}\right|K.$$

Solving (6), we obtain $\omega_2=\alpha\cos\kappa t+\beta\sin\kappa t$. ...(7)

Now, have the time scale in such a way that $t=0 \Rightarrow \omega_2=0$.

$\therefore$ (7) gives $\omega_2=\beta\sin\kappa t$ $\quad(\because\ \alpha=0.)$ ...(8)

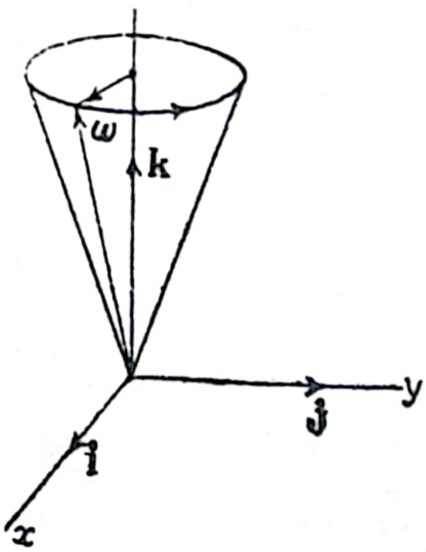
Then from (5), we have $\dot{\omega}_2=\left(\frac{C-A}{A}\right)K\omega_1=\kappa\omega_1$

$\Rightarrow\ \beta\kappa\cos\kappa t=\kappa\,\omega_1$

$\Rightarrow\ \omega_1=\beta\cos\kappa t$ ...(9)

$\therefore\ \boldsymbol{\omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$

$=\beta\cos\kappa t\,\mathbf{i}+\beta\sin\kappa t\,\mathbf{j}+K\mathbf{k}$ ...(10)



and $\omega=|\vec{\omega}|=\sqrt{(\beta^2+K^2)}$
$=\text{constant}.$

This show that the angular velocity is constant in magnitude equal to $\omega=\sqrt{(\beta^2+K^2)}$ and precesses round the **k** axis with frequency.

$$f=\frac{\kappa}{2\pi}=\frac{|C-A|}{2\pi A}\ K$$

as shown in the adjoining diagram.

Ex. 19. *Calculate the precession frequency of the previous example in the case of rotating earth.*

Solution. We know that the earth rotates about its axis once in a day, so we have $\omega_3=K=2\pi$ radians/day.

$\therefore$ precessional frequency is given by

$$f=\frac{\kappa}{2\pi}=\frac{1}{2\pi}\left(\frac{C-A}{A}\right)K=\frac{1}{2\pi}\left(1-\frac{c}{a}\right)K=\frac{1}{2\pi}(00328)(2\pi)$$

$$=\cdot 00328 \text{ radians per day.}$$

Thus period of precession$=\frac{1}{f}$ 305 days.

Ex. 20. *Describe the rotation of the earth about its axis in terms of the space and body cones.*

(Invariable line and plane ; Polhode, Herpolhode, Space and Body cones).

Solution. We have

$$\mathbf{h}=A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+C\omega_3\mathbf{k} \qquad ...(1)$$

and $\vec{\omega}=\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}$

where $\omega_1=\beta\cos\kappa t$, $\omega_2=\beta\sin\kappa t$ and $\omega_3=K$.

Now let γ be the angle between $\vec{\omega_3}=\omega_3\mathbf{k}=K\mathbf{k}$ and $\mathbf{h}$, then we have

$$\vec{\omega_3}.\mathbf{h}=\omega_3 h\cos\gamma=K\sqrt{(A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2)}\cos\gamma=(\omega_3)\,C\omega_3$$

$$\Rightarrow \quad \cos\gamma=\left\{\frac{K\sqrt{(A^2\omega_1^2+B^2\omega_2^2+C^2\omega_2^2)}}{C\omega_3^2}\right\}^{-1}$$

$$=\frac{CK^2}{K\sqrt{(A^2\beta^2\cos^2\kappa t+A^2\beta^2\sin^2\kappa t+C^2K^2)}}$$

$$(\because \ B=A \text{ for earth})$$

$$\Rightarrow \quad \cos\gamma=\frac{CK}{\sqrt{(A^2\beta^2+C^2K^2,}}. \qquad ...(2)$$

$$\therefore \quad \sin\gamma=\frac{AB}{\sqrt{(A^2\beta^2+C^2K^2)}}. \qquad ...(2')$$

Similarly, let δ be the angle between $\vec{\omega_3}$ and $\vec{\omega}$, then we have

$$\vec{\omega_3}.\vec{\omega}=\omega_3\sqrt{(\omega_1^2+\omega_2^2+\omega_3^2)}\cos\delta=K\sqrt{(\beta^2+K^2)}\cos\delta$$

$$\Rightarrow \quad K^2 = K\sqrt{(\beta^2+K^2)}\cos\delta \Rightarrow \cos\delta = \frac{K}{\sqrt{(\beta^2+K^2)}} \qquad ...(3)$$

and
$$\sin\delta = \frac{\beta}{\sqrt{(\beta^2+K^2)}} \qquad ...(3')$$

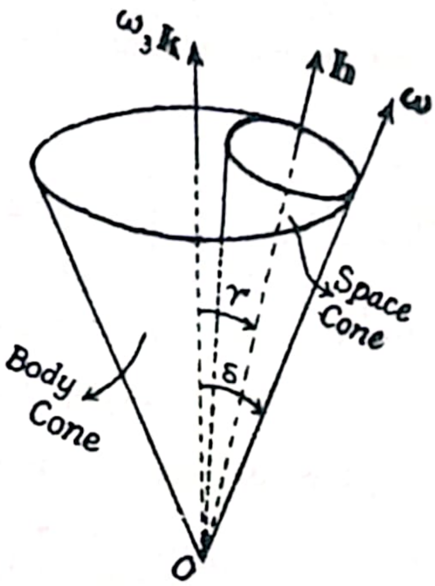
$$\therefore \quad \tan\gamma = \frac{AB}{CK} \text{ and } \tan\delta = \frac{\beta}{K}. \qquad ...(4)$$

The implies
$$\frac{\tan\gamma}{\tan\delta} = \frac{AB}{CK}\cdot\frac{K}{\beta} = \frac{A}{C}. \qquad ...(5)$$

But for the earth (or any oblate spheroid flattened at the poles). We have $A < C$.

This implies
$$\tan\gamma < \tan\delta \Rightarrow \gamma < \delta.$$

This situation can be described geometrically by the adjoining diagram. The cone with axis in the direction **h** is fixed in space as the cone having ω_3**k** as axis (being fixed in the earth) is known as Body cone and this cone rolls on the space cone so that $\vec{\omega}$ is the common element.



$$\text{Now } \vec{\omega_3}\times\vec{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & K \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}$$

$$= KA\beta\cos\kappa t\mathbf{j} - KA\beta\sin\kappa t\mathbf{i}$$

$$\Rightarrow \quad \mathbf{h}.(\vec{\omega_3}\times\vec{\omega}) = (A\omega_1\mathbf{i}+B\omega_2\mathbf{j}+CK\mathbf{k}).(KA\beta\cos\kappa t\mathbf{j} - KA\beta\sin\kappa t\mathbf{i})$$
$$= A\omega_1(-KA\beta\sin\kappa t) + B\omega_2(KA\beta\cos\kappa t)$$
$$= -KA^2\beta\omega_1\sin\kappa t + KA^2\beta\omega_2\cos\kappa t = 0 \quad (\because B = A)$$

$\Rightarrow$ **h**, $\vec{\omega_3}$ and $\vec{\omega}$ lie in one plane.

Ex. 21. *A rigid body is free to rotate about its C.G., the principal moments of inertia at which are 7,25,32 units respectively. The body is given an angular velocity Ω about a line through G whose direction ration are 4 : 0 : 3. Show that after time t the components of angular velocity about the principal axes of inertia at G are*

$$\tfrac{4}{5}\,\Omega\cos\phi,\ \tfrac{4}{5}\sin\phi,\ \tfrac{3}{5}\,\Omega\cos\phi$$

where $\qquad \tan\left(\frac{\phi}{2}\right)=\tanh\left(\frac{3\Omega t}{10}\right).$

Deduce that ultimately the body rotates about the principal a of intermediate moment of inertia.

Solution. The body is rotating about G, hence there is ze moment about this point. Assuming, $A=7$, $B=25$, $C=32$ a ω_1, ω_2, ω_3 to be the angular velocities about the principal axes time t, Euler's dynamical equations imply

$$7\dot{\omega}_1-(25-32)\,\omega_2\omega_3=0 \Rightarrow \dot{\omega}_1+\omega_2\omega_3=0 \qquad \dots($$

$$25\dot{\omega}_2-(32-7)\,\omega_3\omega_1=0 \Rightarrow \dot{\omega}_2-\omega_3\omega_1=0 \qquad \dots($$

$$32\dot{\omega}_3-(7-25)\,\omega_1\omega_2=0 \Rightarrow 16\dot{\omega}_3+9\omega_1\omega_2=0 \qquad \dots($$

Multiplying (1), (2) by ω_1, ω_2, respectively, adding, integrati and using initial values gives

$$\omega_1^2+\omega_2^2=\frac{16\Omega}{25}$$

$$\left[\because \text{ initial values of the } a\text{-}v\text{'s are } \frac{4\Omega}{5},\, 0,\, \frac{3\Omega}{5}\right]$$

Multiplying (2), (3) by $9\omega_2$, ω_3 respectively, adding, integr ting and using initial conditions we obtain

$$9\omega_2^2+16\omega_3^2=\frac{144\Omega}{25}. \qquad \dots($$

From equations (2), (4), (5), we have

$$dt=\frac{d\omega_2}{\omega_3\omega_1}=\frac{4}{3}\,\frac{d\omega_2}{\left\{\left(\frac{2\Omega}{5}\right)^2-\omega_2^2\right\}} \qquad \dots($$

Integration provides

$$t=\frac{5}{3\Omega}\tanh^{-1}\left(\frac{5\omega_2}{4\Omega}\right) \Rightarrow \omega_2=\left(\frac{4\Omega}{5}\right)\tanh\left(\frac{3\Omega t}{5}\right)=\frac{4\Omega}{5}\sin\phi.$$

Substituting this value of ω_2 into (4), (5) ; we get

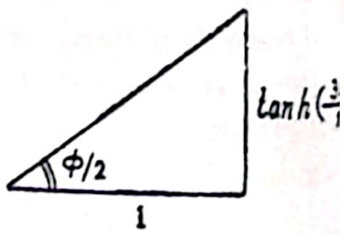
$$\omega_1=\frac{4\Omega}{5}\cos\phi,\ \omega_3=\frac{3\Omega}{5}\cos\phi.$$

2nd Part.

As $t\to\infty$, $\tanh\left(\frac{3\Omega t}{10}\right)\to 1$

i.e. $\frac{\phi}{2}\to\frac{\pi}{4}$ i.e. $\phi\to\frac{\pi}{2}$.

$\therefore\ \omega_1=0,\ \omega_2=\frac{4\Omega}{5},\ \omega_3=0.$

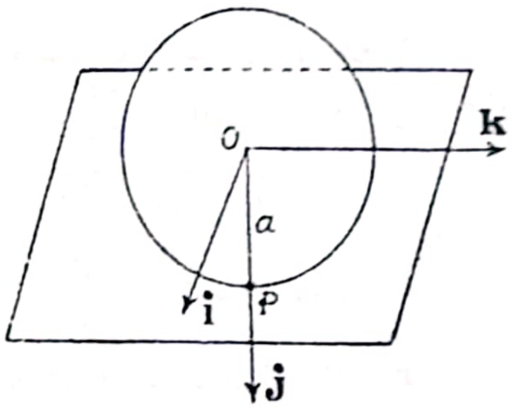


Thus ultimately rotation about the axis of intermediate moment of inertia ensues.

Ex. 22. Non-rotating axes.

Set up the general equations of motion of a uniform sphere on a fixed rough horizontal plane under a given system of external forces through its centre. Show that if the sphere rolls without slipping its translation is equivalent to that of a particle of equal mass on a smooth horizontal plane subject to external forces in the same directions but having magnitudes equal to $\frac{2}{7}$ of those acting on the sphere.

Sol. The adjoining figure shows the sphere, centre O, radius a moving generally on the fixed horizontal plane. The vectors $\mathbf{k}$, $\mathbf{i}$, are chosen in the horizontal plane whereas the vector $\mathbf{j}$ is taken vertically downwards such that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ etc. Let the externally applied force through O be $X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ and that the friction at P be $F_1\mathbf{i}+F_2\mathbf{j}+F\mathbf{k}$. At time t,



let the angular velocity of the sphere be $\vec{\omega}=[\omega_1, \quad \omega_2, \quad \omega_3]$. The moment of momentum about O is $\mathbf{h}=\frac{2}{5}Ma^2(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k})$, since the principal axes for the sphere are $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. But the axes are not rotating so we have

$$\dot{\mathbf{h}}=\tfrac{2}{5}a^2(\dot{\omega}_1\mathbf{i}+\dot{\omega}_2\mathbf{j}+\dot{\omega}_2\mathbf{k}).$$

The moment of external forces about O

$$=\mathbf{OP}\times(F_3\mathbf{k}+F_1\mathbf{i})=a\mathbf{j}\times(F_3\mathbf{k}+F_1\mathbf{i})$$
$$=(-aF_1\mathbf{k}+F_3a\mathbf{i}).$$

But O is the C.G. of the sphere so it can be treated as a fixed point, and so

$$a\mathbf{j}\times(F_3\mathbf{k}+F_1\mathbf{i})=\tfrac{2}{5}Ma^2(\dot{\omega}_1+\dot{\omega}_2\mathbf{j}+\dot{\omega}_3\mathbf{k})$$

$$\Rightarrow(-aF_1\mathbf{k}+aF_2\mathbf{i})=\tfrac{2}{5}Ma^2(\dot{\omega}_1\mathbf{i}+\dot{\omega}_2\mathbf{j}+\dot{\omega}_3\mathbf{k}).$$

$$\therefore\quad \dot{\omega}_1=\frac{5aF_3}{2Ma^2},\ \dot{\omega}_2=0,\ \dot{\omega}_3=-\frac{5aF_1}{2Ma^2}.$$

When $\dot{\omega}_2=0$ we have $\omega_2=\text{constant}=n$ say.

Now let $-R\mathbf{j}$ be the normal reaction at P, then the total for acting on the sphere is

$$(X\mathbf{i}+F_1\mathbf{i})+(Z+F_3)\,\mathbf{k}+(Mg-R)\,\mathbf{j}.$$

$\therefore$ equation of motion is given by

$$M\,(\ddot{z}\mathbf{k}+\ddot{x}\mathbf{i})=(X+F_1)\,\mathbf{i}+(Mg-R)\,\mathbf{j}+(Z+F_3)\,\mathbf{k}$$

$$\Rightarrow \quad M\ddot{z}=X+F_1,\ Mg-R=0,\ M\ddot{z}=Z+F_3.$$

The condition for pure rolling is that P is instantaneously rest.

The velocity of P is $(\dot{x}\mathbf{i}+\dot{z}\mathbf{k})+\vec{\omega}\times a\mathbf{j}$

$$=\dot{x}\mathbf{j}+\dot{z}\mathbf{k}+\omega_1 a\mathbf{k}-\omega_3 a\mathbf{j}$$

Hence we get $\dot{z}+a\omega_1=0,\ \dot{x}-a\omega_3=0.$

Thus $\ddot{z}+a\omega_1=0,\ \ddot{x}-a\omega_3=0$

$$\Rightarrow \quad \ddot{z}+\frac{5F_3}{2M}=0,\ \ddot{x}+\frac{5F_1}{2M}=0 \Rightarrow F_3=-\tfrac{2}{5}M\ddot{z},\ F_1=-\tfrac{2}{5}M\ddot{x}$$

$$\therefore \quad X=M\ddot{x}-F_1=-\frac{5F_1}{2}-F_1=-\tfrac{7}{2}F_1=(-\tfrac{7}{2})\,(-\tfrac{2}{5}M\ddot{x})=\tfrac{7}{5}M$$

$$Z=M\ddot{z}-F_3=-\frac{5F_3}{2}-F_3=-\tfrac{7}{2}F_3=\tfrac{7}{5}M\ddot{z}.$$

Ex. 23. Non-rotating axes.

A uniform solid sphere rolls without slipping on a rough horizon plane which is rotating with uniform angular velocity about vertical axis. If there are no forces acting on the sphere say weight and the friction at the contact; prove that the locus of t centre of the sphere is a circle.

Sol. The plane is having a uniform angular velocity Ω. The axes $\mathbf{i}, \mathbf{j}, \mathbf{k}$ through the moving point O are again fixed direction.

Let $\vec{\omega}=[\omega_1,\omega_2,\omega_3]$ be the angular velocity of the sphere.

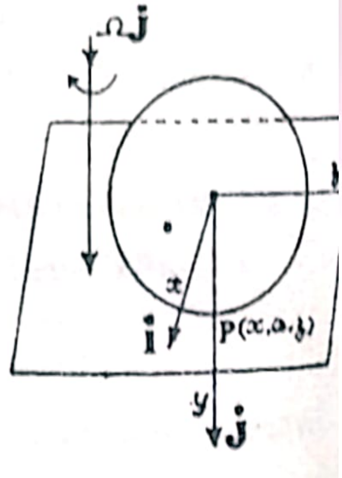
Now taking moments about O, we obtain as before

$$F_3=\tfrac{2}{5}Ma\dot{\omega}_1,\ F_1=-\tfrac{2}{5}Ma\dot{\omega}_3,\ \dot{\omega}_2=0. \quad \ldots(1)$$

Also the linear motion of the sphere is given by

$$F_3\mathbf{k}+F_1\mathbf{i}=M\,(\ddot{x}\mathbf{i}+\ddot{z}\mathbf{k})$$

$$\Rightarrow \quad F_1=M\ddot{x},\ F_3=M\ddot{z}, \quad \ldots(2)$$



Thus velocity of P on the plane is equal to

$$\Omega \mathbf{j} \times (x\mathbf{i} + a\mathbf{j} + z\mathbf{k}) = \Omega(-x\mathbf{k} + z\mathbf{i}).$$

Also, the velocity of P on the sphere is

$$\dot{z}\mathbf{k} + \dot{x}\mathbf{i} + \vec{\omega} \times a\mathbf{j} = (\dot{z} + a\omega_1)\,\mathbf{k} + (\dot{x} - a\omega_3)\,\mathbf{i}$$

$\therefore$ Conditions for no slipping at P are

$$\dot{z} + a\omega_1 = -\Omega x, \; \dot{x} - a\omega_3 = \Omega z.$$

Using (1) and (2), we obtain

$$\tfrac{2}{5} M a\dot{\omega}_1 = M\ddot{z}, \; -\tfrac{2}{5} M a\dot{\omega}_3 = M\ddot{x} \;\Rightarrow\; \dot{\omega}_1 = \frac{5\ddot{z}}{2a}, \; \dot{\omega}_3 = \frac{5\ddot{x}}{2a} \qquad \ldots(4)$$

Now differentiating equations (3), we get

$$\ddot{z} + a\dot{\omega}_1 = -\Omega\dot{x}, \; \ddot{x} - a\dot{\omega}_3 = \Omega\dot{z}.$$

Using (4), we get $\ddot{z} + \dfrac{5\ddot{z}}{2} = -\Omega$, $\ddot{x} + \dfrac{5\ddot{x}}{2} = \Omega\dot{z}$.

$$\ddot{z} + \tfrac{2}{7}\Omega\dot{x} = 0, \; \ddot{x} - \tfrac{2}{7}\Omega\dot{z} = 0 \qquad \ldots('5)$$

[(i)] [(ii)]

Multiplying (ii) by i and adding it to (i), we get

$$\ddot{z} + \ddot{x}i + \tfrac{2}{7}\Omega\dot{x} - \tfrac{2}{7}i\Omega\dot{z} = 0 \;\Rightarrow\; D\,(D - i\,\tfrac{2}{7}\Omega)\,(z + ix) = 0$$

where $\qquad D = \dfrac{d}{dt}$

General solution of this is given by

$$z + ix = A + B \exp.\,(\tfrac{2}{7}i\Omega t).$$

Taking $A = \alpha + i\beta$, $B = \lambda + i\mu$, where $\lambda, \mu, \alpha, \beta$ are all real constants, we obtain

$$z = \alpha + \lambda\cos(\tfrac{2}{7}\Omega t) - \mu\sin(\tfrac{2}{7}\Omega t),$$
$$x = \beta + \mu\cos(\tfrac{2}{7}\Omega t) + \lambda\sin(\tfrac{2}{7}\Omega t)$$

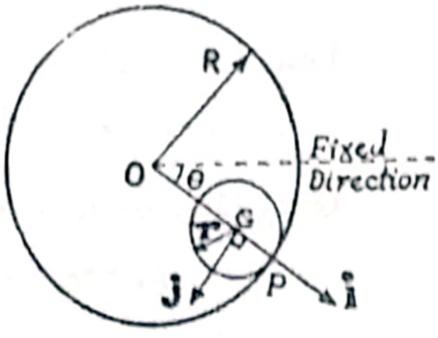
which $\qquad \Rightarrow (z - \alpha)^2 + (x - \beta)^2 = \lambda^2 + \mu^2, \; y = 0.$

This is the required locus of the centre of the sphere.

Ex. 24. Rotating axes.

A uniform sphere rolls without slipping on the rough interior of a fixed vertical cylinder of greater radius. Discuss the motion of the sphere.

Sol. Let R be the radius of the cylinder, r that of sphere, $\mathbf{i}$ the unit vector along OG where O and G are the centres of the horizontal sections of the cylinder and the sphere through the point of contact P, $\mathbf{k}$ the unit vector in the downward vertical



and $\mathbf{j}=\mathbf{k}\times\mathbf{i}$, $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$.

Let $\vec{\omega}=(\omega_1, \omega_2, \omega_3)$ be the angular velocity of the sphere, than the moment of momentum about O is

$$\mathbf{h}=\tfrac{2}{5}Mr^2\,(\omega_1\mathbf{i}+\omega_2\mathbf{j}+\omega_3\mathbf{k}),$$

where **i**, **j**, **k** are the principal axes for the sphere and M is the mass of the sphere. If OG makes an angle θ with a fixed horizontal direction, then $\dot{\theta}\mathbf{k}$ is the angular velocity of the frame **i**, **j**, **k** and hence

$$\frac{d\mathbf{h}}{dt}=\frac{\delta\mathbf{h}}{\delta t}+(\dot{\theta}\mathbf{k})\times(\mathbf{h})=\tfrac{2}{5}Mr^2\,\{(\dot{\omega}_1-\omega_2\dot{\theta})\,\mathbf{i}+(\dot{\omega}_2+\omega_1\dot{\theta}\mathbf{j})+\dot{\omega}_3\mathbf{k}\}$$

Let $p=R-r$, hence the acceleration of G is $(-p\dot{\phi}^2, p\ddot{\phi}, \ddot{z})$ where z is the depth of G below some fixed horizontal plane.

Now moment of the rate of change of momentum about P

$$=\frac{d\mathbf{h}}{dt}+(-r\mathbf{i})\times M\,(-p\dot{\theta}^2\mathbf{i}+p\ddot{\theta}\,\mathbf{j}+\ddot{z}\mathbf{k})$$

$$=\tfrac{2}{5}Mr^2\,\{(\dot{\omega}_1-\omega_1\dot{\theta})\,\mathbf{i}+(\dot{\omega}^2+\omega_1\dot{\theta})\,\mathbf{j}+\dot{\omega}_3\mathbf{k}\}$$
$$+Mr\ddot{z}\mathbf{j}-Mrp\ddot{\theta}\mathbf{k} \qquad \ldots(1)$$

Also moment of the forces about P

$$=-r\mathbf{i}\times Mg\mathbf{k}=Mgr\mathbf{j}. \qquad \ldots(2)$$

Equating (1) and (2), we get

$$\dot{\omega}_1-\omega_2\dot{\theta}=0 \qquad \ldots(3)$$

$$\frac{2r}{5}\,(\dot{\omega}_2+\omega_1\dot{\theta})+\ddot{z}=g \qquad \ldots(4)$$

$$\frac{2r}{5}\,\dot{\omega}_3-p\ddot{\theta}=0. \qquad \ldots(5)$$

Now velocity of P relative to G

$$=(\omega_2\mathbf{j}+\omega_3\mathbf{k})\times r\mathbf{i}$$
$$=r\omega_3\mathbf{j}-r\omega_2\mathbf{k}, \qquad \ldots(6)$$

and velocity of $G=p\dot{\theta}\mathbf{j}+\dot{z}\mathbf{k}$. $\qquad \ldots(7)$

$\therefore$ total velocity of $P=(p\dot{\theta}+r\omega_3)\,(+\dot{z}-r\omega_1)\,\mathbf{k}$ $\qquad \ldots(8)$

During pure rolling, the point of contact is momentarily at rest, so we have

$$p\dot{\theta}+r\omega_3=0, \qquad \ldots(9)$$
$$\dot{z}-r\omega_2=0. \qquad \ldots(10)$$

From (5) and (9), we obtain $\dot{\omega}_3=0$, $\ddot{\theta}=0$

$$\Rightarrow\ \omega_3=n \text{ say},\ \dot{\theta}=-\frac{rn}{p} \text{ where } n \text{ is constant.} \qquad \ldots(11)$$

Using (3), (10), (11), we get

$$\dot{\omega}_1=\dot{\theta}\omega_2=-\frac{n\dot{z}}{p}\ \Rightarrow\ \omega_1=C\,\frac{nz}{p} \qquad \ldots(12)$$

$$\therefore \quad (4) \Rightarrow \tfrac{2}{5}\,[\ddot{z}+\{C-(nz/p)\}\,(-r^2n/a)]+\ddot{z}=g$$

$$\Rightarrow \tfrac{7}{5}\ddot{z}+\tfrac{2}{5}\,(rn/p)^2\,z=g+\frac{2r^2}{5}\frac{nc}{p}=\text{constant},$$

$$\Rightarrow \ddot{z}=-\tfrac{2}{7}\left(\frac{r^2n^2}{p}\right)z+\text{cont.} \qquad \text{...(13)}$$

Equation (13) show that the vertical motion of G is a S.H.M. of period $2\pi\,(7/2)^{1/2}\,(R-r)/rn$, $(\because\ p=R-r)$.

Ex. 25. Rotating axes.

A hoop of radius a, rolls without slipping on a rough horizontal surface. Obtain the equations of its motion in the form

$$3\ddot{\theta}-\dot{\phi}^2\sin\theta\cos\theta+4\omega\dot{\phi}\sin\theta=-2\,(g/a)\cos\theta.$$

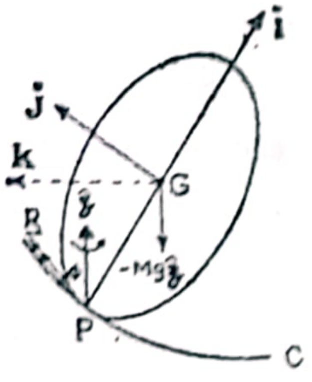
$$\frac{d}{dt}(\dot{\phi}\sin^2\theta)-2\omega\dot{\theta}\sin\theta=0,$$

$$2\dot{\omega}-\dot{\theta}\dot{\phi}\sin\theta=0,$$

where θ is the angle which the plane of the hoop makes with the horizontal, $\dot{\phi}$ in the angular velocity of the vertical plane containing the diameter through the paint of contact, and ω is the angular velocity of the hoop about its axis of symmetry.

If the hook rolls along a straight line with its plane vertical, show that the motion is stable if $\omega>\frac{1}{2}\,(g/a)^{1/2}$.

Solution. As obvious from the figure, the vector **i** is the unit vector through G in the direction PG, **j** is the unit vector in the plane of the hoop at right angles to **i** and **k** is the unit vector normal to the hoop so that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ etc. Let $\mathbf{z}$ be the unit vector in the upward vertical, **R** the total reaction at the point of contact P.



Now $\hat{\mathbf{z}}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{k}$ and the resultant spin of the frame is given by

$$\vec{\Omega}=\dot{\phi}\hat{\mathbf{z}}=\dot{\theta}\mathbf{j}=\dot{\phi}\sin\theta\,\mathbf{i}-\dot{\theta}\mathbf{j}+\dot{\phi}\cos\theta\,\mathbf{k}. \qquad \text{...(1)}$$

The point P is instantaneously at rest, so the velocity at G is given by

$$\mathbf{v}_G=\vec{\Omega}\times a\mathbf{i}=a\dot{\phi}\cos\theta\,\mathbf{j}+a\dot{\theta}\mathbf{k}. \qquad \text{...(2)}$$

Now equation of motion (of translation) of the hoop is given by

$$\mathbf{R}-Mg\hat{\mathbf{z}}=M\frac{d\mathbf{v}_G}{dt}=M\dot{\mathbf{v}}_G. \qquad \text{...(3)}$$

Also, moment of momentum about G is

$$\mathbf{h}_G=\tfrac{1}{2}Ma^2\dot\phi\sin\theta\mathbf{i}+\tfrac{1}{2}Ma^2(-\dot\theta)\,\mathbf{j}+Ma^2\dot\phi\cos\theta\mathbf{k} \quad ...(4)$$

But $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the principal axes for the hoop at G and $A=\frac{1}{2}Ma^2$, $B=\frac{1}{2}Ma^2$; $C=Ma^2$; hence we get

$$\dot{\mathbf{h}}=\frac{\delta\mathbf{h}_G}{\delta t}+\vec{\Omega}\times\mathbf{h}_G=A\dot\Omega_1\mathbf{i}+B\dot\Omega\mathbf{j}+C\dot\Omega_3\mathbf{k}.$$

$$+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \Omega_1 & \Omega_2 & \Omega_3\\ A\Omega_1 & B\Omega_2 & C\Omega_3\end{vmatrix}$$

$$=\tfrac{1}{2}Ma^2\{(\ddot\phi\sin\theta+\dot\phi\dot\theta\cos\theta)\,\mathbf{i}-\ddot\theta\mathbf{j}+2(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{k}\}$$

$$+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dot\phi\sin\theta & -\dot\theta & \dot\phi\cos\theta\\ A\dot\phi\sin\theta & -A\dot\theta & 2A\dot\phi\cos\theta\end{vmatrix}$$

$$=\tfrac{1}{2}Ma^2\{\ddot\phi\sin\theta\mathbf{i}-(\ddot\theta+\dot\phi^2\sin\theta\cos\theta)\mathbf{j}+2(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{k}\} \quad ...(5)$$

Now taking moments about G, we get

$$\mathbf{G}=-a\mathbf{i}\times\mathbf{R}. \quad ...(6)$$

$\therefore$ equation of motion of the hoop about G gives

$$\dot{\mathbf{h}}_G=\mathbf{G} \Rightarrow \tfrac{1}{2}Ma^2\{\ddot\phi\sin\theta\mathbf{i}-(\ddot\theta+\dot\phi^2\sin\theta\cos\theta)\,\mathbf{j}+2(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{k}\}$$

$$=-a\mathbf{i}\times\mathbf{R}$$

i.e. $\frac{1}{2}Ma^2\{\ddot\phi\sin\theta\mathbf{i}-(\ddot\theta+\dot\phi^2\sin\theta\cos\theta)\,\mathbf{j}+2(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{k}\}$

$$=-a\mathbf{i}\times\left(Mg\hat{\mathbf{z}}+M\frac{d}{dt}\mathbf{v}_G\right) \quad ...(7)$$

But $\quad \hat{\mathbf{z}}=\sin\theta\mathbf{i}+\cos\theta\mathbf{k} \quad ...(8)$

and

$$\dot{\mathbf{v}}_G=\frac{\delta}{\delta t}\mathbf{v}_G+\vec{\Omega}\times\mathbf{v}_G$$

$$=\{(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{j}+\ddot\theta\mathbf{k}\}$$

$$+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dot\phi\sin\theta & -\dot\theta & \dot\phi\cos\theta\\ 0 & a\dot\phi\cos\theta & a\dot\theta\end{vmatrix}$$

$$=a\{-(\dot\theta^2+\dot\phi^2\cos^2\theta)\,\mathbf{i}+(\ddot\phi\cos\theta-2\dot\phi\dot\theta\sin\theta)\,\mathbf{j}+(\ddot\theta+\dot\phi^2\sin\theta\cos\theta)\,\mathbf{k}\} \quad ...(9)$$

Using (8), (9); equation (7) $\Rightarrow$

$\frac{1}{2}Ma^2\{\ddot\phi\sin\theta\mathbf{i}-(\ddot\theta+\dot\phi^2\sin\theta\cos\theta)\,\mathbf{j}+2(\ddot\phi\cos\theta-\dot\phi\dot\theta\sin\theta)\,\mathbf{k}\}$

$$=Mga\,[\{g\cos\theta+(a\ddot{\theta}+a\dot{\phi}^2\sin\theta\cos\theta)\}\,\mathbf{j}$$
$$-(a\ddot{\phi}\cos\theta-2a\dot{\phi}\dot{\theta}\sin\theta\}\,\mathbf{k}]$$

$$[\because\ -a\mathbf{i}\times(Mg\hat{\mathbf{z}}+M\dot{\mathbf{v}}_G)=Mga[\{g\cos\theta+a(\ddot{\theta}+\dot{\phi}^2\sin\theta\cos\theta)\}\,\mathbf{j}$$
$$-a\,(\ddot{\phi}\cos\theta-2\dot{\phi}\dot{\theta}\sin\theta)\,\mathbf{k}]$$

This implies

$$\ddot{\phi}\sin\theta=0 \qquad \text{...(10)}$$

$$\ddot{\theta}+\dot{\phi}^2\sin\theta\cos\theta=\frac{2g}{a}\cos\theta \qquad \text{...(11)}$$

$$2\ddot{\phi}\cos\theta=3\dot{\phi}\dot{\theta}\sin\theta. \qquad \text{...(12)}$$

Thus the magnitude of the angular velocity of the body is given by

$$\omega=\vec{\Omega}\cdot\mathbf{k}=\dot{\phi}\cos\theta \qquad \text{...(13)}$$

$$\therefore\quad \dot{\omega}=\ddot{\phi}\cos\theta-\dot{\phi}\dot{\theta}\sin\theta, \qquad \text{...(14)}$$

$$\Rightarrow\ 2\dot{\omega}=\dot{\theta}\dot{\phi}\sin\theta\ [\text{using (12)}]$$

$$\text{But } \frac{d}{dt}(\dot{\phi}\sin^2\theta)-2\omega\dot{\theta}\sin\theta=\ddot{\phi}\sin^2\theta+2\dot{\theta}\sin\theta\,(\dot{\phi}\cos\theta-\omega)$$

$$=0\ [\text{using (10), (13)}] \qquad \text{...(15)}$$

From (11), we get

$$3\ddot{\theta}-\dot{\phi}^2\sin\theta\cos\theta+4\dot{\phi}\sin\theta\cos\theta=-\frac{2g}{a}\cos\theta$$

i.e.

$$3\ddot{\theta}-\dot{\phi}^1\sin\theta\cos\theta+4\omega\sin\theta=-\frac{2g}{a}\cos\theta \qquad \text{...(16)}$$

(14), (15), (16) are the required equations.

Equation (15) can be re-written as

$$\ddot{\phi}\sin\theta+2\dot{\phi}\dot{\theta}\cos\theta\ 2\omega\dot{\theta}=0.$$

Stability of the hoop.

In order to test stability of the hoop when rolling commences along the straight line with its plane vertical; we allow the hoop to become slightly pertuabed from this motion so that

$$\theta=\frac{\pi}{2}+\epsilon,\ \omega=\omega_0+\eta$$

where ϵ, η, ϕ are small, ω_0 being the steady value of ω for $\theta=\frac{\pi}{2}$.

To the first order of smallness equations (15), (16) $\Rightarrow$

$$\ddot{\phi}-3\omega_0\epsilon=0 \qquad \text{...(17)}$$

$$3\ddot{\epsilon}+4\omega_0\dot{\phi}=\frac{2g}{a}\epsilon \qquad \text{...(18)}$$

Now (18) $\Rightarrow$ [using (17)]

$$3\ddot{\epsilon}+\left(8\omega_0^2-\frac{2g}{a}\right)\epsilon=0 \quad \ldots(19)$$

$$\Rightarrow \ddot{\epsilon}=-\frac{1}{3}\left(8\omega_0^2-\frac{2g}{a}\right)\epsilon. \quad \ldots(20)$$

This is S.H.M. provided $8\omega_0^2>\dfrac{2g}{a}$ *i.e.* $\omega_0>\dfrac{1}{2}\left(\dfrac{g}{a}\right)^{1/2}$.

Hence the condition of stability is that $\omega_0>\dfrac{1}{2}\left(\dfrac{g}{a}\right)^{1/2}$.

Ex. 26. *A body turns about a fixed point and*

$$2T=A\omega_1^2+B\omega_2^2+C\omega_3^2-2F\omega_2\omega_3-2G\omega_3\omega_1-2H\omega_1\omega_2.$$

show that, if the axes are fixed in the body, but are not necessarily the principal axes. Euler's equations of motion may be written in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega_1}\right)-\frac{\partial T}{\partial \omega_2}\omega_3+\frac{\partial T}{\partial \omega_3}\omega_2=G_1$$

with two similar expressions.

Solution. If the axes are not the principal axes, the equations of motion are $\dot{h}_1-h_2\omega_3+h_3\omega_2=G_1$ etc.

where $h_1=A\omega_1-H\omega_2-G\omega_3=\dfrac{\partial T}{\partial \omega_1}$, $h_2=\dfrac{\partial T}{\partial \omega_2}$ and $h_3=\dfrac{\partial T}{\partial \omega_3}$.

Now Euler's dynamical equations are :

$$\dot{h}_1-h_2\omega_3+h_3\omega_2=G_1$$

$$\dot{h}_2-h_3\omega_1+h_1\omega_3=G_2$$

$$\dot{h}_3-h_1\omega_2+h_2\omega_1=G_3$$

i.e. $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \omega_1}\right)-\dfrac{\partial T}{\partial \omega_2}\omega_3+\dfrac{\partial T}{\partial \omega_3}\omega_2=G_1$, with two similar expressions.

Ex. 27. Motion of a Billiard ball on a Billiard table.

A homogeneous billiard ball, spining abaut any axis, moves on a billiard imperfectly rough table. Show that the path of the centre is at first an arc of a parabola and then a straight line. Find also the time and place when pure rolling commences.

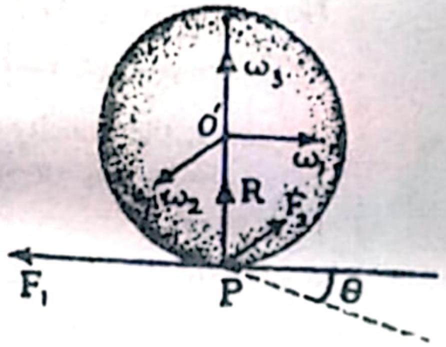
Solution. Consider the initial position of the point of contact as origin, z-axis vertically upwards and x-axis the initial direction of motion of the point of contact.

Let the initial velocities be u, v, 0 parallel to co-ordinate axes and ω_1, ω_2, ω_3 be the a.v's about the axes. At the interval of time t, let the co-ordinates of the centre be (x, y, a) and the a.v's components be ω_1, ω_2, ω_3; F_1, F_2, are the frictional components and R is the reaction.

Now equations of motion of the centre are : ...(1)

$m\ddot{x}=-F_1,\ m\ddot{y}=-F_2,\ 0=R-mg.$

Since every axis at O' is the principal axis, so the angular momentum about O' is



$$\mathbf{h}=mk^2\vec{\omega} \text{ where } k^2=\frac{2a^2}{5}.$$

$\therefore$ equations of motion relative to O' are $[\mathbf{h}=\mathbf{G}]$.

$$\tfrac{2}{5}ma^2\dot{\omega}_1=-F_2a,\ \tfrac{2}{5}ma_2\dot{\omega}_2=F_1a,$$

$$\tfrac{2}{5}ma\dot{\omega}_3=0. \qquad ...(2)$$

The last gives ω_3=constant.

Due to sliding, the resultant friction is opposite to the direction of motion at the point of contact and equals to $\mu R=\mu mg$ where μ is the coefficient of friction.

Now, velocity of the point of contact in the x-direction

$$=\dot{x}-a\omega_2$$

velocity of the point of contact in the y-direction

$$=\dot{y}+a\omega_1.$$

$$\frac{F_2}{F_1}=\frac{\dot{y}+a\omega_1}{\dot{x}-a\omega_2} \qquad ...(3)$$

and $$F_1{}^2+F_2{}^2=\mu^2m^2g^2.$$

Now from the equations of motion, we have

$$\frac{F_2}{F_1}=\frac{\ddot{y}}{\ddot{x}}=-\frac{a\dot{\omega}_1}{a\dot{\omega}_2}=\frac{\ddot{x}_1-a\omega_1}{\ddot{x}-a\omega_1} \qquad ...(4)$$

$$\therefore \quad (3) \Rightarrow \frac{\ddot{y}+a\dot{\omega}_1}{\dot{y}+a\omega_1}=\frac{\ddot{x}-a\dot{\omega}_2}{\dot{x}-a\omega_2}$$

$$\Rightarrow \log(\dot{y}+a\omega_1)=\log(\dot{x}-a\omega_2)+\text{const.}$$

$$\Rightarrow \frac{\dot{y}+a\omega_1}{\dot{x}-a\omega_2}=\text{const.}=\frac{v+a\Omega_1}{u-a\Omega_2}. \qquad ...(5)$$

Since the initial velocity of the point of contract parallel to the axis of y is zero, we have $v+a\Omega_1=0$ ...(6)

$$\therefore \quad \frac{\dot{y}+a\omega_1}{\dot{x}-a\omega_2}=0 \qquad ...(7)$$

$$\Rightarrow \dot{y}+a\omega_1=0. \qquad ...(8)$$

Thus (3) $\Rightarrow F_2=0$ given $F_1=\mu mg$ ...(9)

$$\therefore \quad \ddot{x}=-\mu g,\ \ddot{y}=0,\ \dot{\omega}_1=0,\ \dot{\omega}_2=5\,\frac{\mu g}{2a},$$

[(i)] [(ii)] [(iii)] [(iv)]

Equations (i) and (ii) $\Rightarrow$ that the motion of the centre is along a parabolic path so long as there is sliding.

Now integrating, we get

$$\dot{x}=u-\mu g t,\ \dot{y}=v,\ \omega_1=\Omega_1\ \omega_2=\Omega_2+\frac{5\mu g}{2a}\ t.$$

At time t, the velocity of the point of contact $\parallel$ to Ox.

$$=\dot{x}-a\omega_2=u-a\Omega_2-\tfrac{7}{2}\mu g t. \qquad \text{...(10)}$$

Pure Rolling.

When (10) becomes zero, sliding vanishes ; therefore pure rolling begins at $t=\frac{2}{7}(u-a\Omega_2)/\mu g$.

When this happens, we have $\dot{x}=u-\frac{2}{7}(u-a\Omega_2)$

$$=\frac{5u}{7}+\frac{2a}{7}\Omega_2=\tfrac{1}{7}(5u+2a\Omega_2) \text{ and } \dot{y}=v.$$

Let θ be the angle which the direction of motion of the centre makes with x-axis when pure rolling begins, then we have

$$\tan\theta=\frac{\dot{y}}{\dot{x}}=7v/(5u+2a\Omega_2).$$

On integrating $\dot{x}=-\mu g t+u,\ \dot{y}=v$ we see that pure rolling commences at the point whose co-ordinates are

$$\frac{2(u-a\Omega_2)(6u+a\Omega_2)}{49\mu g} \text{ and } \frac{2v(u-a\Omega_2)}{7\mu g}$$

This shows that the centre now moves in a straight line.

Ex. 28. *A homogenous circular cylinder is divided by an axial plane and kept in shape by a band round it. If the cylinder is placed on a smooth horizontal plane with the plane of separation vertical and the band is then cut, prove that the pressure on the plane is instantaneously reduced by the fraction* $\frac{32}{27\pi^2}$ *of itself.*

Ex. 29. *A uniform sphere is projected horizontally on an inclined plane the surfaces of which is perfectly rough ; show that its centre describes a parabola.*

Ex. 30. Motion of a heavy sphere inside the cylinder.

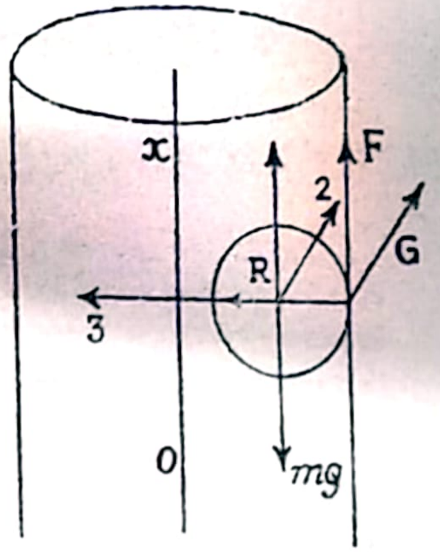
To find the motion of a heavy sphere on the inner surface of a rough vertical circular cylinder.

Sol. Let a and b be the radii of the sphere and the cylinder, C the centre of the sphere. Take axes through the centre C of the sphere, 1 vertically upwards, 3 to intersect the axis of the cylinder and 2 horizontal forming a right handed screw system. Let θ be the azimuthal angle *i.e.* the angle which the plane through the axis of the cylinder and the centre of the sphere makes with a fixed vertical plane, then obviously the angular velocities of the axes are

$$\Omega_1,\ \Omega_2,\ \Omega_3=\dot{\theta},\ 0,\ 0 \qquad \ldots(1)$$

If x be measured vertical upwards along the axis of the cylinder the components of velocity of C are

$$u_1,\ u_2,\ u_3=\dot{x},\ (b-a)\,\dot{\theta}. \qquad \ldots(2)$$



Let ω_1, ω_2, ω_3 be the angular velocities of the sphere about these axes, then since the sphere does not slide, we have

$$u_1-a\omega_2=0, u_2+a\omega_1=0 \Rightarrow a\omega_2=\dot{x},$$
$$a\omega_1=-(b-a)\,\dot{\theta}, \qquad \ldots(3)$$

Now in order to avoid the forces of friction at the point of contact, we take moments about axes parallel to 1, 2, 3 through the point of contact and thus obtain

$$-ma\,(\dot{u}_2-u_3\Omega_1+u_1\Omega_3)+A\,(\dot{\omega}_1-\omega_2\Omega_3+\omega_3\Omega_2)=0 \qquad \ldots(4)$$

$$ma\,(\dot{u}_1-u_2\Omega_3+u_3\Omega_2)+A\,(\dot{\omega}_2-\omega_3\Omega_1+\omega_1\Omega_3)=-mga \qquad \ldots(5)$$

$$A\,(\dot{\omega}-\omega_1\Omega_2+\omega_2\Omega_1)=0, \text{ where } A=\tfrac{2}{5}ma^2 \qquad \ldots(6)$$

Substituting from (1), (2) and (3) in (4), (5) and (6), the latter reduces to

$$\ddot{\theta}=0 \qquad \ldots(4') \Rightarrow \dot{\theta}=\text{constant}=n \text{ say} \qquad \ldots(4'')$$

$$\tfrac{7}{5}\ddot{x}-\tfrac{2}{5}a\omega_3\dot{\theta}=-g \quad \ldots(5') \Rightarrow \frac{7\ddot{x}}{5}=\frac{2}{5}\,a\omega_3 n-g \qquad \ldots(5'')$$

$$\text{and } a\dot{\omega}_3+\ddot{x}\dot{\theta}=0 \quad \ldots(6') \Rightarrow a\omega_3+nx=\text{const.}=a\mu \text{ say} \qquad \ldots(6)$$

$$\text{Thus } (5'') \Rightarrow \ddot{x}+\frac{2n^2}{7}\,x=\frac{2}{7}\,a\mu n-\frac{5g}{7} \qquad \ldots(7)$$

$$\Rightarrow \ddot{x}=-\frac{2n^2}{7}\,x+\left(\frac{2}{7}\,a\mu n-\frac{5g}{7}\right). \qquad \ldots(8)$$

Equation (8) shows that the sphere rises and falls in the cylinder with a vertical simple harmonic motion of period

$$\frac{2\pi}{\sqrt{\left(\dfrac{2n^2}{7}\right)}}=\frac{\sqrt{(14)}}{n}\,\pi. \qquad \ldots(9)$$

If initially $x=0$, $\dot{x}=0$, $\dot{\theta}=n$ and $\omega_3=\mu$, then (8)

$\Rightarrow x=\left(\frac{a\mu}{n}-\frac{5g}{2n^2}\right).[1-\cos\{\sqrt{(\frac{2}{7})}nt\}]$. so that the sphere rises above or descends below its intial levels according as

$$a\mu n > \text{ or } < \frac{5g}{2}.$$

Ex. 31. *A uniform solid sphere rolls on the inside of a rough circular cylinder whose axis is vertical. Show that if the cylinder be made to rotate with constant angular velocity ω about a generating line in the plane through the axis from which the azimuth θ is measured, the azimuthal motion of the centre of sphere relative to the cylinder is given $bx(b-a)\ddot{\theta}=5b\omega^2 \sin\theta$, where a, b, are the radii of the sphere and the cylinder.*

Ex. 32. *A uniform solid sphere of radius a rolls without sliding inside a fixed circular cylinder of radius b whose axis is inclined at an angle α to the horizon. Obtain the equations of motion.*

$$(b-a)\dot{\theta}^2=\tfrac{10}{7}g\cos\alpha\cos\theta+\text{const.},$$

$$\dot{x}^2=\tfrac{2}{7}a^2\omega_3{}^2-\tfrac{10}{7}gx\sin\alpha+\text{const.},$$

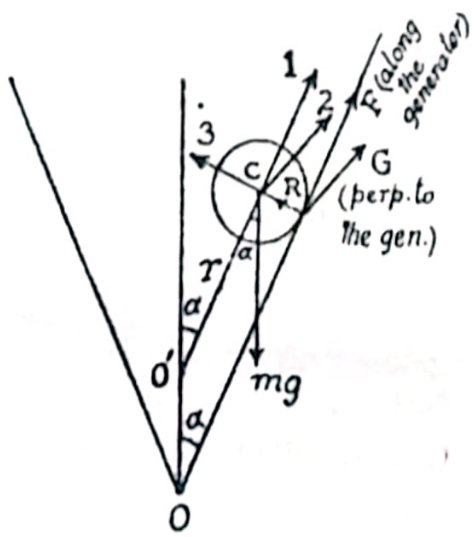
$$a\dot{\theta}=\left(\frac{d^2\omega_3}{dt^2}+\frac{2}{7}\omega_3\right)g\sin\alpha.,$$

where x is the distance the sphere has moved parallel to the axis of the cylinder, θ is the angle between the axial plane through the centre and the vertial axial plane and ω_3 is the angular velocity of the sphere about the radius at the point of contact.

Ex. 33. Motion of a heavy sphere inside a cone.

To find the motion of heavy sphere on the inner surface of a light circular cone with its axis vertical.

Solution. Obviously the centre of the sphere, C, moves on an equal co-axis cone of angle 2α. Let O' be the vertex of this cone and let r be the distance of C from O'. Taking moving axes (1, 2, 3) at C, we have



$$\left.\begin{array}{l}\Omega_1,\ \Omega_2,\ \Omega_3=\dot{\psi}\cos\alpha,\ 0,\ \dot{\psi}\sin\alpha\\ u_1,\ u_2,\ u_3=\dot{r},\ r\dot{\psi}\sin\alpha,\ 0\end{array}\right\} \quad ...(1)$$

where ψ is the azimuthal angle.

The conditions for no sliding are

$$u_1 - a\omega_2 = 0,\ u_2 + a\omega_1 = 0 \Rightarrow \underset{[(i)]}{a\omega_2 = \dot{r}},\ \underset{[(ii)]}{a\omega_1 = -r\dot{\psi}\sin\alpha} \qquad \ldots(2)$$

Now taking moments about the lines parallel to 1, 2, 3 through the point of contact, we obtain

$$-ma\,(\dot{u}_2 - u_3\Omega_1 + u_1\Omega_3) + (A\,\dot{\omega}_1 - \omega_2\Omega_3 + \omega_3\Omega_2) = 0 \qquad \ldots(3)$$

$$ma\,(\dot{u}_1 - u_2\Omega_3 + u_3\Omega_2) + A\,(\dot{\omega}_2 - \omega_3\Omega_1 + \omega_1\Omega_3) = -mga\cos\alpha \qquad \ldots(4)$$

$$A\,(\dot{\omega}_3 - \omega_1\Omega_2 + \omega_2\Omega_1) = 0, \qquad \ldots(5)$$

where $A = 2ma^2/5$.

Substituting from (1) and (2) in (3), (4) and (5), we get

$$(r\ddot{\psi} + 2\dot{r}\dot{\psi})\sin\alpha = 0 \Rightarrow r^2\dot{\psi}\sin\alpha = \text{const.} = h, \text{ say} \qquad \ldots(6)$$

$$\tfrac{7}{5}\,(\ddot{r} - r\dot{\psi}^2\sin^2\alpha) - \tfrac{2}{5}a\omega_3\dot{\psi}\cos\alpha = -g\cos\alpha \qquad \ldots(7)$$

and

$$a\dot{\omega}_3 + \dot{r}\dot{\psi}\cos\alpha = 0. \qquad \ldots(8)$$

Putting the value of $\dot{\psi}$ from (6) in (8), we obtain

$$a\dot{\omega}_3 + \frac{h\dot{r}}{r^2}\cot\alpha = 0 \Rightarrow a\omega_3 = \frac{h\cot\alpha}{r} + \text{const.} \qquad \ldots(9)$$

Hence the equations are reduced to

$$r^2\dot{\psi}\sin\alpha = h \qquad \ldots(10)$$

$$\ddot{r} - r\dot{\psi}^2\sin^2\alpha = f(r)\ [i.e.\text{ a function of } r]. \qquad \ldots(11)$$

To obtain F and G.

From the above figure it is obvious that

$$m\,(\dot{u}_1 - u_3\Omega_3) = F - mg\cos\alpha$$

and

$$m\,(\dot{u}_2 - u_1\Omega_3) = G.$$

Cor. 1. *If the cone be made to rotate about its axis with "a.v" ω.*

The equations of no sliding are

$$u_1 - a\omega_2 = 0,\ u_2 + a\omega_1 = \omega\,(r\sin\alpha + a\cos\alpha)$$

and rest of the equations are as before.

Cor. 2. *If the cone be free to rotate about its axis.*

Here ω is not constant and we have an additional equation

$$I\dot{\omega} = -G\,(r\sin\alpha + a\cos\alpha),$$

where I is the $m.i.$ of the cone about its axis.

Ex. 34. *A perfectly rough right circular cone of semi-vertical angle α has its axis vertical and vertex upwards and is made to rotate about its axis with uniform angular velocity Ω. A uniform* *solid sphere of radius a is gently placed on a point of the cone at a distance a cot α from the vertex. Show that the axial plane through the centre of the sphere rotates with angular velocity* $\frac{1}{7}\Omega$ *and that the distance of the point of contact of the sphere from the vertex of the cone at time t is given be the equation*

$$r=a\cot\alpha+\tfrac{5}{7}g\cos\alpha\,(1-\cos\mu t)/\mu^2,$$

where $343\mu^2=\Omega^2(2+5\sin^2\alpha)$.

Ex. 35. *A sphere of radius a rolls inside a cone of semi-vertical angle α, which is made to rotate about a vertical axis with angular velocity Ω. If* ω_3 *is the spin of the sphere about the radius to the point of contact, show that*

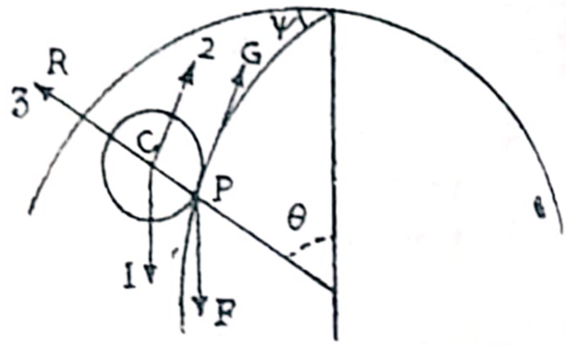
$$a\omega_3+\frac{\cos\alpha}{7}\,\Omega r=\frac{A}{r}+B,$$

where $r+a\cot\alpha$ *is the distance of the point of contact from the vertex and A, B are constants.*

Ex. 36. Motion of a sphere on a rough surface of revolution.

To find the motion of a sphere on a rough surface of revolution whose axis is vertical.

Solution. Let P be the point of contact of the sphere and the surface of revolution. C is the centre of the sphere of radius a. Let the moving axes through C be 1, 2, 3 ; 3 is along the common normal outwards, 2 is parallel to the tangent to the meridian curve at P, and 1 is parallel to the tangent to the circular section through P. If θ is the angle which the common normal makes with the axis and ψ the azimuthal angle, then we have



$$\Omega_1,\ \Omega_2,\ \Omega_3=\dot\theta,\ \dot\psi\sin\theta,\ \dot\psi\cos\theta \qquad \ldots(1)$$

The condition of no sliding ⇒

$$u_1-a\omega_2=0,\ u_2+a\omega_1=0\ ;\ \text{also } u_3=0. \qquad \ldots(2)$$

Now the components of acceleration of C are

$$\dot u_1-u_2\Omega_3+u_3\Omega_2,\ \dot u_2-u_3\Omega_1+u_1+u_1\Omega_3,\ \dot u_3-u_1\Omega_2+u_2\Omega_1$$

i.e. $a\dot\omega_2+a\omega_1\dot\psi\cos\theta,\ -a\dot\omega_1+a\omega_2\dot\psi\cos\theta,\ -a\omega_3\dot\psi\sin\theta-a\omega_1\dot\theta \ldots(3)$

If G_1, G_2, G_3 are the moments of the external forces about lines through P parallel to the axes 1, 2, 3 then moment equations about these lines imply.

$$mk^2(\dot{\omega}_1-\omega_2\dot{\psi}\cos\theta+\omega_3\dot{\psi}\sin\theta)-ma(-a\dot{\omega}_1+a\omega_2\dot{\psi}\cos\theta)=G_1 \quad ...(4)$$

$$mk^2(\dot{\omega}_2-\omega_3\dot{\theta}+\omega_1\dot{\phi}\cos\theta)+ma(a\dot{\omega}_2+a\omega_1\dot{\psi}\cos\theta)=G_3 \quad ...(5)$$

and $mk^2(\dot{\omega}_3-\omega_1\dot{\psi}\sin\theta+\omega_2\dot{\theta})=G_2$ ...(6)

where mk^2 is the moment of inertia of the sphere about a diameter.

When the axis is vertical and gravity is the only force, then we have

$$G_1=mga\sin\theta,\ G_2=0,\ G_3=0.$$

For a steady possible motion in which C, the centre of the sphere describes a horizontal circle of radius r, we have

$$\theta=\alpha, \text{ a constant, and } \dot{\psi}=\omega, \text{ a constant.}$$

$$\because\ a\omega_1=-u_2=0,\ a\omega_2=u_1=r\omega.$$

Now (6) $\Rightarrow$ $\dot{\omega}_2=0$ *i.e.* $\omega_3=\text{constant}=n$ say ...(7)

and (4) $\Rightarrow$ $(k^2+a^2)\,r\omega^2\cot\alpha-k^2\,an\omega+a^2\,g=0$ ...(6′)

i.e. $7r\omega^2\cot\alpha-2an\omega+5g=0$ ...(4″)

while (5) is satisfied identically. $(\because\ k^2=\frac{2}{5}a^2)$

For real values of ω to exist, we must have

$$a^2n^2\geqslant 35\,rg\cot\alpha.$$

Ex. 37. *To find the motion of a sphere rolling and spinning on a fixed sphere.*

Solution. Here the surface of revolution is a sphere of radius b.

$$\therefore\ u_1=a\omega_2=(a+b)\,\dot{\phi}\sin\theta \text{ and } u_2=-a\omega_1=-(a+b)\,\dot{\theta} \quad ...(1)$$

Considering gravity to be the only external forces, we have (on using 4, 5, 6)

$$(k^2+a^2)\,(\dot{\omega}_1-\omega_2\dot{\phi}\cos\theta)+k^2\,\omega_3\dot{\psi}\sin\theta=ag\sin\theta \quad ...(2)$$

$$(k^2+a^2)\,(\dot{\omega}_2+\omega_1\dot{\psi}\cos\theta)-k^2\,\omega_3\dot{\theta}=0 \quad ...(3)$$

and $\dot{\omega}_2=0\ \Rightarrow\ \omega_3=\text{constant}=n$, say. ...(4)

Substituting from (1) in (2) and (3), we get

$$(a+b)\,(\ddot{\theta}-\dot{\psi}^2\sin\theta\cos\theta)+\frac{k^2}{a^2+k^2}\,an\dot{\psi}\sin\theta=\frac{a^2g}{a^2+k^2}\sin\theta \quad ...(5)$$

and $$(a+b)\,\frac{1}{\sin\theta}\frac{d}{dt}\,(\dot{\psi}\sin^2\theta)-\frac{k^2}{a^2+k^2}\,an\dot{\theta}=0. \quad ...(6)$$

Now integrating (6), we get

$$(a+b)\,\dot\psi\,\sin^2\theta+\frac{k^2}{a^2+k^2}\,an\cos\theta=\text{const.}=H \text{ say.} \quad ...(7)$$

Again multiplying (5) by $2\dot\theta$ and (6) by $2\dot\psi\sin\theta$ and adding provides on integrating

$$(a+b)\,(\dot\theta^2+\dot\psi^2\,\sin^2\theta)=\frac{2a^2g}{a^2+k^2}\,(1-\cos\theta)+\text{const.} \quad ...(8)$$

Equation (7) and (8) determine the motion.

Ex. 38. *To find the motion of a sphere rolling and spinning on a sphere which is free to rotate about its centre.*

Solution. Let I be the *m.i.* of the latter sphere about a diameter and let $\Omega_1, \Omega_2, \Omega_3$ be the components of angular velocity about axes through its centre parallel to the axes 1, 2, 3 as defined in Ex. 36 P. 234. Let F, G be the forces of friction in the senses 1, 2 on the sphere of radius a, and in the contrary sense on the sphere of radius b; and R the normal reaction.

The condition of no sliding

$$\Rightarrow u_1-a\omega_2=b\Omega_2,\ u_2+a\omega_1=-b\Omega_1,\ u_3=0. \quad ...(1)$$

As before we also have

$$u_1=(a+b)\,\dot\psi\,\sin\theta,\ u_2=-(a+b)\,\dot\theta \quad ...(2)$$

Now dynamical equations are given by

$$\left.\begin{aligned} m\,(\dot u_1-u_2\,\dot\psi\cos\theta)&=F\\ m\,(\dot u_2+u_1\,\dot\psi\cos\theta)&=G-mg\sin\theta\\ m\,(-u_1\dot\psi\sin\theta-u_2\dot\theta)&=R-mg\cos\theta \end{aligned}\right\} \quad ...(3)$$

$$\left.\begin{aligned} mk^2\,(\dot\omega_1-\omega_2\dot\psi\cos\theta+\omega_3\dot\psi\sin\theta)&=Ga\\ mk^2\,(\dot\omega_2-\omega_3\dot\theta+\omega_1\dot\psi\cos\theta)&=-Fa\\ mk^2\,(\dot\omega_3-\omega_1\dot\psi\sin\theta+\omega_2\dot\theta)&=0 \end{aligned}\right\} \quad ...(4)$$

and

$$\left.\begin{aligned} I\,(\dot\Omega_1-\Omega_2\dot\psi\cos\theta+\Omega_3\dot\psi\sin\theta)&=Gb\\ I\,(\dot\Omega_2-\Omega_3\dot\theta+\Omega_1\dot\psi\cos\theta)&=-Fb\\ I\,(\dot\Omega_3-\Omega_1\dot\psi\sin\theta+\Omega_2\dot\theta)&=0. \end{aligned}\right\} \quad ...(5)$$

Now using last equations (4) and (5), we get

$$a\dot\omega_3+b\dot\Omega_3-(a\omega_1+b\Omega_1)\,\dot\psi\sin\theta+(a\omega_2+b\Omega_2)\,\dot\theta=0,$$

but (1) and (2) $\Rightarrow$ $a\omega_1+b\Omega_1=(a+b)\,\dot\theta,\ a\omega_2+b\Omega_2=(a+b)\,\dot\psi\sin\theta$.

$\therefore\ a\dot\omega_3+b\dot\Omega_3=0$ i.e. $a\omega_3+b\Omega_3=\text{const.}=cn$ say.

To eliminate G, we proceed as follows :

Equations (4) and (5) imply

$$a\dot\omega_1+b\dot\Omega-(a\omega_2+b\Omega_2)\,\dot\theta\cos\theta+cn\,\dot\psi\sin\theta=G\left(\frac{a^2}{mk^2}+\frac{b^2}{I}\right).$$

Using (1), this gives

$$-\dot{u}_2-u_1\dot{\psi}\cos\theta+cn\dot{\psi}\sin\theta=G\left(\frac{a^2}{mk^2}+\frac{b^2}{I}\right).$$

Putting the valve of G from (3) and that of u_1, u_2 from (2) in the above equation, we obtain

$$\left(1+\frac{a^2}{k^2}+\frac{mb^2}{I}\right)(a+b)(\ddot{\theta}-\dot{\psi}^2\sin\theta\cos\theta)+cn\dot{\psi}\sin\theta$$
$$=\left(\frac{a^2}{k^2}+\frac{mb^2}{I}\right)g\sin\theta. \quad ...(6)$$

similarly eliminating F, we obtain

$$\left(1+\frac{a^2}{k^2}+\frac{mb^2}{I}\right)\frac{a+b}{\sin\theta}\cdot\frac{d}{dt}(\dot{\psi}\sin^2\theta)-cn\dot{\theta}=0. \quad ...(7)$$

Ex. 39. *A body moves under no forces about a point O, the principal movements of inertia at O being 6A, 3A and A. Initially the angular velocity of the body has components* $\omega_1=n$, $\omega_2=0$ *and* $\omega_3=3n$ *about the principal axes. Show that at any later time* $\omega_2=-\sqrt{5}n\tanh(\sqrt{5}nt)$ *and ultimately, the body rotates about the mean axis.*

Solution. We have $A\omega_1^2+B\omega_2^2+C\omega_3^2=\text{constant}$, ...(1)

and $A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2=\text{constant}$. ...(2)

$\therefore\quad 6\omega_1^2+3\omega_2^2+\omega_3^2=\text{const.}=6n^2+0+9n^2=15n^2$, ...(1′)

and $36\omega_1^2+9\omega_2^2+\omega_3^2=\text{const.}=36n^2+0+9n^2=45n^2$. ...(2′)

Euler's second equation gives $[B\dot{\omega}_2-(C-A)\omega_3\omega_1=0]$

$$3\dot{\omega}_2=\omega_3\omega_1=0. \quad ...(3)$$

Substituting (1) from (2′), we get $30\omega_1^2+6\omega_2^2=30n^2$

$$\Rightarrow 5\omega_1^2=5n^2-\omega_2^2. \quad ...(4)$$

$$\therefore (1') \Rightarrow 6\left(\frac{5n^2-\omega_2^2}{5}\right)+3\omega_2^2+\omega_3^2=15n^2$$

$$\Rightarrow 30n^2-6\omega_2^2+15\omega_2^2+\omega_3^2=75n^2$$

$$\Rightarrow 9\omega_2^2+5\omega_3^2=45n^2 \Rightarrow \omega_3^2=\tfrac{9}{5}(5n^2-\omega_2^2). \quad ..(5)$$

Using (4) and (5) equation (3)

$$\Rightarrow 3\dot{\omega}_2+5\sqrt{(\tfrac{9}{5})}.\sqrt{(5n^2-\omega_2^2)}.\frac{\sqrt{(5n^2-\omega_2^2)}}{\sqrt{5}}=0$$

$$\Rightarrow 3\dot{\omega}_2+3(5n^2-\omega^2)=0 \Rightarrow \dot{\omega}_2=-(5n^2-\omega_2^2)$$

i.e. $$-\frac{d\omega_2}{5n^2-\omega_2^2}=dt, \quad ...(6)$$

Integrating (6) and applying the initial condition, we get

$$\omega_2=-\sqrt{5}n\tanh(\sqrt{5}nt). \quad ...(7)$$

At $t\to\infty$, $\omega_2^2\to 5\omega \Rightarrow 5\omega_1^2\to 0$ i.e. $\omega_1\to 0$.

Also $\omega_3\to 0$.

Hence the body rotates about mean axis.

Ex. 40. *A uniform cube has its centre fixed and is free to turn about it, it is struck by a blow along one of its edges; find the instantaneons axis.*

Solution. Here, we have

$$A=B=C=\tfrac{2}{3}ma^2,\quad \text{and } F=G=H=0;$$

while $\qquad G_1=-aP,\ G_2=-aP,\ G_3=0.$

$\therefore\quad \tfrac{2}{3}ma^2\omega_x'=-aP,\ \tfrac{2}{3}ma^2\omega_y'=-aP$ and $\tfrac{2}{3}ma^2\omega_z'=0.$

Thus, D.c.'s of the instantaneous axis are proportional to 1, 1, 0.

Ex. 41. *A solid cube is in motion about an angular point which is fixed. If there are no external forces and $\omega_1, \omega_2, \omega_3$ are the angular velocities about the edges through the fixed point, prove that $\omega_1+\omega_2+\omega_3$ and $\omega_1^2+\omega_2^2+\omega_3^2$ are each constant.*

Solution. Edges are not principal axes.

Here $\qquad A=B=C,\quad F=G=H.$

$$A(\omega_1^2+\omega_2^2+\omega_3^2)-2F(\omega_1\omega_2+\omega_2\omega_3+\omega_3\omega_1)=\text{const.} \qquad ...(1)$$

while constancy of the magnitude of the angular momentum gives

$$h_1^2+h_2^2+h_3^2=\text{constant} \qquad ...(2)$$

where
$$h_1=A\omega_1-H\omega_2-G\omega_3$$
$$h_2=-H\omega_1+B\omega_2-F\omega_3$$
$$h_3=-G\omega_1-F\omega_2+C\omega_3.$$

So, we obtain

$$(2F^2+A^2)(\omega_1^2+\omega_2^2+\omega_3^2)+(2F^2-4AF)(\omega_1\omega_2+\omega_2\omega_3+\omega_1\omega_3)=\text{constant}$$

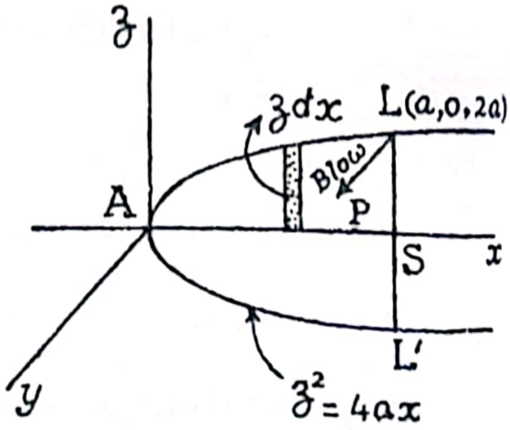
$\therefore\quad \omega_1^2+\omega_2^2+\omega_3^2=\text{constant}$, and $(\omega_1\omega_2+\omega_2\omega_3+\omega_3\omega_1)=\text{const.}$

Hence $\quad \omega_1+\omega_2+\omega_3=\text{constant}.$

$$[\because\quad (\Sigma\omega_1)^2=\Sigma\,\omega_1^2+2\Sigma\,\omega_1\omega_2]$$

Ex. 42. *A disc, in the form of a parabola bounded by its latusrectum and its axis, has vertex fixed, and is struck by a blow through the end of its latus rectum perpendicular to its plane. Show that the disc starts revolving about a line through A inclined at an angle $\theta=\tan^{-1}\left(\frac{14}{25}\right)$ to the axis.*

Solution. We have



$$A=\int_0^a z\,dx\,\rho.\frac{z^2}{3}=\frac{16}{15}a^4\rho$$

$$C=\int_0^a \rho z\,dx.x^2=\frac{4}{7}a^4\rho$$

$$B=A+C=\frac{172}{105}\rho a^4$$

$$F=0=H,\ G=\int_0^a \rho z\,dx.x.\frac{z}{2}$$

$$=\tfrac{2}{3}a^4\rho.$$

$$\therefore \quad \frac{16}{15} Pa^6 \omega_x' - \frac{4}{7} a^4 P\omega_z' = -P.2a.$$

$$\frac{173}{105} \rho a^4 \omega_y' = 0, \quad \frac{4}{7} Pa^4 \omega_z' - \frac{2}{3} a^4 \omega_x' = Pa$$

Eliminating P, we get $\frac{\omega_z'}{\omega_x'} = \frac{14}{25}$.

$$\therefore \quad \tan \theta = \frac{\omega_z'}{\omega_x'} = \frac{14}{25}.$$

Ex. 43. *Prove that if a rectangular parallelopiped (edges 2a, 2b) rotates about its centre of gravity, its angular velocity about one principal axis is constant and about the other principal axis is periodic the period being to the period about the first mentioned principal axis as $(b^2+a^2) : (b^2-a^2)$.*

Solution. We have $A = \frac{m}{3}(a^2+b^2) = B$, $C = \frac{2}{3} ma^2$.

Euler's dynamical equations are

$$\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{const.} = n, \text{ say} \qquad \ldots(1)$$

$$(a^2+b^2)\dot{\omega}_1 - (b^2-a^2)\omega_2 n = 0 \Rightarrow (a^2+b^2)\dot{\omega}_1 = (b^2-a^2)\omega_2 n, \quad \ldots(2)$$

and $(a^2+b^2)\dot{\omega}_2 - (a^2-b^2) n\omega_1 = 0 \Rightarrow (a^2+b^2)\dot{\omega}_2 = -(b^2-a^2) n\omega_1.$

Thus, $(a^2+b^2)^2 \ddot{\omega}_1 = -(b^2-a^2)^2 n^2 \omega_1.$

$$\therefore \quad \text{Periodic time} = \frac{2\pi (a^2+b^2)}{n (b^2-a^2)} = T_2, \text{ say.}$$

But $T_1 = \frac{2\pi}{n}$. $\therefore \frac{T_2}{T_1} = \frac{a^2+b^2}{b^2-a^2}$.

Ex. 44. *If p, q, r be the d.c.'s of OZ with regard to the axes OA, OB, OC, show that two of Euler's geometrical equations may be put in the symmetrical forms*

$$\frac{dp}{dt} - q\omega_3 + r\omega_2 = 0, \quad \frac{dq}{dt} - r\omega_1 + q\omega_3 = 0, \quad \frac{dr}{dt} = p\omega_3 + q\omega_1 = 0.$$

Ex. 45. *A body is turning about a fixed point O and has all its principal movement of inertia at O equ.l. If θ, ϕ, ψ be the Eulerian co-ordinate of the axes OA, OB, OC fixed in the body, show that the angular moment about the axes fixed in the space are respectivel y.*

$$h_1 = A(\sin\psi\dot{\theta} + \sin\theta\cos\psi\dot{\phi}), \quad h_2 = A(\cos\psi\dot{\theta} + \sin\theta\sin\psi\dot{\phi}),$$

$$h_3 = A(\dot{\psi} + \cos\theta\dot{\phi})$$ **(Agra 1981)**

Ex. 46. *If $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$ and G be the moment of the impressed forces about the instantaneous axis, Ω the insultant angular velocity, then prove that*

$$\frac{dT}{dt} = G\Omega.$$

Ex. 47. *If the earth be regarded as a solid of revolution whose principal moments of inertia at its centre of gravity are A, A, C; show that its axis of rotation describes a cone of very small angle about its axis of figure in the period $\frac{A}{(C-A)}$ sideral days.*

(Agra 1982)

SUPPLEMENTARY PROBLEMS

1. (a) Determine the number of degrees of freedom in each of following case ;
   (i) A partical moving on a plane curve :
   (ii) Two particles moving on a space curve and having constant distance between them ;
   (iii) Three particles moving in space so that the distance between any two of them is always constant.
   Ans. (i) 1, (ii) 1, (iii) 6.
   (b) Find the number of degree of freedom for a system consisting of a rigid rod which can move freely in space and a particle which is constrained to move on the rod.
   Ans. 4.
   (c) A particle moves with constant relative speed v round a rim of wheel of radius b ; the wheel rolls along fixed as a straight line with uniform velocity V. Taking the wheel as farme of reference, find the coriolis force and the centrifugal force. Indicate them in a diagram.
2. (a) Find the components of velocity and acceleration along the parametric lines of spherical polar co-ordinates r, θ, ϕ, for a particle moving in space.
   Check your formula by adplying them to the following spacial cases : (i) ϕ=constant (ii) $\theta=\frac{1}{2}\pi$.
   (b) A time t, the position of a moving particle relative to axes $Oxyz$ is given by $x=5\cos 2t,\ y=5\sin 2t,\ z=4t$
   At $t=2$, determine the acceleration and find its components along the tangent and principal normal to the path.
   Ans. The acceleration is always directed along the principal normal and has magnitude 20.
3. A plane, is fixed in space. Co-ordinate axes $Oxyz$ rotates about O. Their angular velocity has components $\omega_1, \omega_2, \omega_3$, along them. If the equations of the plane at any instant is
   $$Ax+By+Cz=1$$
   prove that $\frac{dA}{dt}=B\omega_3-C\omega_2,\ \frac{dB}{dt}=C\omega_1-A\omega_3\ \frac{dC}{dt}=A\omega_2-B\omega_1.$
4. A governer consists of two equal spheres, of mass m and radius a. They are fixed to the ends of equal light rods, each of lenght $c-a$, which are hinged to a collar on a vertical axis By means of a light linkage and sliding collar, the equality of the inclinations to the vertical of

the two rods is ensured. If the angle of inclination is θ and the singular velocity of the governer about its vertical side is ω: show that the kinetic energy is

$$A(\dot{\theta}^2+\omega^2 \sin^2 \theta)+C\omega^2 \cos^2 \theta$$

where $$A=m\left(\frac{2}{5}a^2+c^2\right), \quad C=\frac{2}{5}ma^2.$$

5. Prove that the angular momentum of a moving system about a point O, is the sum of the following parts :

(i) The angular momentum about O of a particle moving with the masscentre and having a mass equal to the total mass of the system :

(ii) The angular momentum of the system about the mass centre.

6. An equilateral triangle is formed of three rods, each of mass m and length $2a$. It hangs from one vertex, about which it is free to turn. A blow P^* is struck on one of the lower vertices in a direction perpendicular to the plane of the triangle, prove that the impulsive reaction on the point of support has a magnitude $1/5\ P^*$.

7. A uniform circular disc of mass M and radius a is also mounted that it can turn freely about its centre which is fixed. It is spinning with angular velocity ω about the perpendicular to its plane at the centre, the plane being horizontal. A particle of mass m, falling vertically, hits the disc near the edge and adheres to it. Prove that immediately afterward the particle is moving in a direction in inclined to the horizontal at an angle α given by $\tan \alpha = u \dfrac{m(M+2m)}{M(M+4m)} \dfrac{v}{a\omega}$

where v is the speed of the particle just before impact.

8. Explain how a man standing on a smooth sheet of ice can turn round by moving his arms.

9. An xyz co-ordinate system moves with angular velocity $\vec{\omega}=2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$ relative to a fixed or internal XYZ co-ordinates system having the same origin. If a vector velocity to the xyz system is given as a function of time t by $\mathbf{A}=\sin t\mathbf{i}-\cos t\mathbf{j}+e^{-t}\mathbf{k}$, find (a) $\dfrac{d\mathbf{A}}{dt}$ relative to the fixed system.

(b) $\dfrac{d\mathbf{A}}{dt}$ relative to the moving system.

Ans. (a) $(6\cos t-3e^{-t})\,\mathbf{j}+(6\sin t-2e^{-t})\,\mathbf{j}+(3\sin t-2\cos t-e^{-t})\,\mathbf{k}$.

(b) $(\cos t\mathbf{i}+\sin t\mathbf{j}-e^{-t}\,\mathbf{k})$

10. Find $\dfrac{d^2\mathbf{A}}{dt^2}$ for the vector $\mathbf{A}$ of the above problem relative to (a) the fixed system and (b) the moving system.

Ans. (a) $(6\cos t-45\sin t+16e^{-t})\,\mathbf{i}+(40\cos t-11e^{-t})\,\mathbf{j}$
$+(10\sin t-23\cos t+16e^{-t})+\mathbf{k}$.

(b) $-\sin t\mathbf{i}+\cos t\mathbf{j}+e^{-t}\mathbf{k}$.

11. An xyz co-ordinate system is rotating with angular velocity $\vec{\omega}=5\mathbf{i}-4\mathbf{j}-10\mathbf{k}$ relative to a fixed XYZ co-ordinate system having the same origin. Find the velocity of a particle fixed in the xyz system at

the point (3, 1, —2) as seen by an observer fixed in the XYZ system

Ans. 18i—20j+17k.

12. A ball is thrown horizontally in the northern hemisphere,

(a) Would the path of the ball, if the coriolis force is taken into account be to the right or to the left of the path when it is not taken into account as viewed by the person throwing the ball ?

(b) What would be your answer to (a) if the ball were thrown in the southern hemisphere ?

Ans. (a) to the right (b) to the left.

13. Prove that the centrifugal force acting on a particle of mass m on the earth's surface is a vector (a) directed away from the earth and perpendicular to the velocity vector $\vec{\omega}$ and (b) of magnitude $m\,\vec{\omega}^2\,R\sin\lambda$ where λ is colatitude.

14. An object is dropped at the equator from a height of 400 metres. If air resistance is neglected how far will the point where it hits the earth's surface be from the point vertically below the initial position.

Ans. 17·6 cm, towards the east.

15. If an object is dropped to the earth's surface prove that its path is a semicubical parabola.

16. An xyz co-ordinate system rotates about the z-axis with angular velocity $\vec{\omega}=\cos t\mathbf{i}+\sin t\mathbf{j}$ relative to a fixed XYZ co-ordinate system where t is the time, the origin of the xyz system has position vector $\mathbf{R}=t\mathbf{i}-\mathbf{j}+t^2\mathbf{k}$ with respect to the XYZ system. If the position vector of a particle is given by $\mathbf{r}=(3t+1)\,\mathbf{i}-2t\mathbf{j}+5\mathbf{k}$ relative to the moving system, find the (a) apparent velocity and (b) true velocity at any time.

17. Determine (a) the apparent acceleration and (b) the true acceleration the particle in the exercise (16).

18. A mass, attached to string which is suspended from a fixed point moves in a horizontal circle having centre vertically below the fixed point wiof speed of 20 revolutions per minute. Find the distance of the centre of the circle below the fixed point.

Ans. 2·23 metres.

19. A uniform heavy square lamina, free to turn about its centre which is fixed is set rotationg with angular velocity Ω about an axis inclined at angle 45° to its plane. Show that in the subsequent motion, the axis of figure of the lamina describes, with uniform angular velocity, a right circular cone whose axis is inclined at $\tan^{-1}(\frac{1}{3})$ with the invariable line. Find the semiangle of the cone and the uniform angular velocity with which it is described.

Ans. Semi angle=45°; angular velocity=Ω.

20 A uniform solid rectangular parallelopided is free to turn about its centre of gravity which is fixed. The edges of the solid have lengths $2a$, $2a$, a. Initially the solid has angular velocity Ω about a diagonal. Find the components parallel to the edges the solid, and its angular

velocity after a time t. Hence or otherwise show that the resolute of the angular velocity of the solid about the same diagonal after time t is

$$\frac{\Omega}{9}\left[1+8\cos\left(\frac{\Omega t}{5}\right)\right].$$

21. Prove that the first component of the angular momentum of a rigid body rotating with one point fixed is $A\omega_1-F\omega_2-G\omega_3$ in the usual notations. A uniform square lamina is rotating under no impressed forces about one corner O, which is fixed. If ω_1, ω_2, ω_3 are the components of the angular velocity about the principal axis at O (in the order of increasing moments of inertia), show that $\omega_1^2+\omega_3^2$ and $3\omega_3^2+4\omega_1^2$ are constant.

22. A rigid body is free in space under no external forces and its motion consists of a pure rotation about an axis fixed in the body. The principal moments of inertia A, B, C at the centre of mass are all unequal. Establish that :

(i) The angular velocity is constant in magnitude ;

(ii) The axis of rotation is also fixed in space ;

(iii) The centre of mass lies on the axis of rotation ;

(iv) The axis of rotation coincides with a principal axis.

Given that $A < B < C$, determine the principal axis about which the rotation is stable.