5 Partial Differential Equations

1. Formulation of P.D.E.

1.1 Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that this equation is satisfied by the surface $yz + zx + xy = 0$.

(2009 : 12 Marks)

Solution:

The equation cone having vertex at origin

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \quad \text{...(i)}$$

where a, b, c, f, g, h are parameters.

Differentiating w.r.t. x and y, we get

$$2ax + 2hy + 2gz + 2gxp + 2zcp + 2fyp = 0$$

And $$2by + 2czq + 2hx + 2fyq + 2fz + 2gxq = 0$$

So, $$ax + hy + qz + p(gx + zc + fy) = 0 \times x$$

$$by + hx + fz + q(cz + fy + gx) = 0 \times y$$

$\Rightarrow$ $$ax^2 + hxy + gzx + p(gx^2 + czx + fyx) = 0$$

$$by^2 + hxy + fzy + q(czy + fy^2 + gxy) = 0$$

On adding,

$\Rightarrow$ $$ax^2 + by^2 + 2hxy + gzx + fzy + px + qy[cz + fy + gx] = 0$$

$\Rightarrow$ $$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0$$

$\Rightarrow$ $$(cz + fy + gx)(px + qy - z) = 0$$

Clearly, $px + qy - z = 0$ is required differential equation.

Given surface is $yz + zx + xy = 0$ (*)

Differentiating (*) w.r.t. x and y, we get

$$yp + z + px + y = 0 \quad \text{...(2)}$$

$$z + yq + xq + x = 0 \quad \text{...(3)}$$

So, we get $$p = \frac{-(z+y)}{(x+y)}, q = \frac{-(x+z)}{(x+y)}$$

$$px + qy - z = \frac{-(z+y)x}{(x+y)} - \frac{(x+z)}{(x+y)}y - z$$

$$= \frac{-(z+y)x - (x+z)y - z(x+y)}{(x+y)}$$

$$= \frac{-xz - xy - xy - zy - zx - zy}{(x+y)}$$

$$= \frac{-2(xy + yz + zx)}{x+y} = \frac{-20}{x+y} = 0$$

1.2 From the partial differential equation by eliminating the arbitrary function f given by :

$$f(x^2 + y^2, z - xy) = 0$$

(2009 : 6 Marks)

Solution:

The function is

$$z = xy + F(x^2 + y^2) \quad \text{...(1)}$$

Now differentiating partially (1) w.r.t. x we get

$$\frac{\partial z}{\partial x} = y + F'(x^2 + y^2)2x$$

So,
$$\frac{p-y}{2x} = F'(x^2 + y^2) \quad \text{...(2)}$$

Now, differentiating partially (1) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = x + F'(x^2 + y^2).2y$$

So,
$$\frac{q-x}{2y} = F'(x^2 + y^2) \quad \text{...(3)}$$

Equating (2) and (3), we get

$$\frac{p-y}{2x} = \frac{q-x}{2y}$$

So, $py - qx = y^2 - x^2$ is linear PDE.

1.3 Find the surface satisfying the P.D.E. $(D^2 - 2DD' + D'^2)z = 0$ and the conditions that $bz = y^2$ when $x = 0$ and $az = x^2$ when $y = 0$.

(2010 : 12 Marks)

Solution:

Given, the equation is

$$(D^2 - 2DD' + D'^2)z = 0$$

$$\Rightarrow \quad (D - D')^2 z = 0$$

The auxiliary eqn. for above eqn. is

$$(m-1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$\therefore$ The solution of above eqn. is

$$z = \phi_1(y + x) + x\phi_2(y + x)$$

Given, at $x = 0$, $bz = y^2 \Rightarrow$
$$z = \frac{y^2}{b}$$

i.e.,
$$\frac{y^2}{b} = \phi_1(y) + 0 \Rightarrow \phi_1(y) = \frac{y^2}{b}$$

$$\Rightarrow \quad \phi_1(x + y) = \frac{(y + x)^2}{b}$$

at $y = 0$, $az = x^2 \Rightarrow$
$$z = \frac{x^2}{a}$$

i.e.,
$$\frac{x^2}{a} = \phi_1(x) + x\phi_2(x)$$

$$\Rightarrow \quad \frac{x^2}{a} = \frac{x^2}{b} + x\phi_2(x)$$

$$\Rightarrow \quad x\phi_2(x) = x^2\left(\frac{1}{a} - \frac{1}{b}\right) \Rightarrow \phi_2(x) = x\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\Rightarrow \qquad \phi_2(y+x) = (y+x)\left(\frac{1}{a}-\frac{1}{b}\right)$$

∴ Putting these values of ϕ_1 and ϕ_2 in the solution, we get

$$z = \frac{(y+x)^2}{b} + x(y+x)\left(\frac{1}{a}-\frac{1}{b}\right)$$

1.4 Find the surface satisfying $\frac{\partial^2 z}{\partial x^2} = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.

(2011 : 20 Marks)

Solution:

Given :
$$\frac{\partial^2 z}{\partial x^2} = 6x + 2$$

$$\Rightarrow \qquad \frac{\partial P}{\partial x} = 6x + 2 \text{ where } P = \frac{\partial z}{\partial x} \qquad \text{...(i)}$$

Integrating (i) w.r.t. x,

$$P = 3x^2 + 2x + f(y)$$

$$\Rightarrow \qquad \frac{\partial z}{\partial x} = 3x^2 + 2x + f(y) \qquad \text{...(ii)}$$

Integrating (ii) w.r.t. x,

$$z = x^3 + x^2 + xf(y) + F(y) \qquad \text{...(iii)}$$

where $f(y)$ and $F(y)$ are arbitrary functions.

The given surface is

$$z = x^3 + y^3 \qquad \text{...(iv)}$$

and the given plane is

$$x + y + 1 = 0 \qquad \text{...(v)}$$

Since (iii) and (iv) touch each other, along their section by (v), the values of p and q at any point on (v) must be equal. Thus, we must have

$$3x^2 + 2x + f(y) = 3x^2 \qquad \text{...(vi)}$$

and
$$xf'(y) + F'(y) = 3y^2 \qquad \text{...(vii)}$$

From (v) and (vi),

$$f(y) = -2x = 2(y+1) \qquad \text{...(viii)}$$

$$\Rightarrow \qquad f'(y) = 2$$

∴ from (vii),
$$2x + F'(y) = 3y^2$$

$$\Rightarrow \qquad F'(y) = 3y^2 - 2x$$
$$= 3y^2 + 2(y+1) \qquad (\because \text{ of (v)}) \qquad \text{...(ix)}$$

∴ on integration,
$$F(y) = y^3 + y^2 + 2y + C,$$

where C is an arbitrary constant.

From (viii) and (ix), and using (iii), we get,

$$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + C \qquad \text{...(x)}$$

At the point of contact of (iv) and (x) values of z must be the same and hence, we have

$$x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + C = x^3 + y^3 \qquad \text{...(xi)}$$

Using $y = -x - 1$ from (v), (xi) gives

$$C = 1$$

Putting $C = 1$ in (x), the required surface is

$$z = x^3 + y^3 + 2x(y+1) + y^3 + y^2 + 2y + 1$$
$$= x^3 + y^3 + (x+y+1)^2$$

1.5 Form a partial differential equation by eliminating the arbitrary functions f and g from $z = yf(x) + xg(y)$.
(2013 : 10 Marks)

Solution:

$$z = yf(x) + xg(y)$$

Differentiating partially with respect to x and y

$$\frac{\partial z}{\partial x} = yf'(x) + g(y); \quad \frac{\partial z}{\partial y} = f(x) + xg'(y)$$

and
$$\frac{\partial^2 z}{\partial y \partial x} = f'(x) + g'(y)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy[f'(x) + g'(y)] + xg(y) + yf(x)$$

$$= xy\frac{\partial^2 z}{\partial y \partial z} + z$$

$\therefore$
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - xy\frac{\partial^2 z}{\partial x \partial y} - z = 0$$

is the required partial differential equation.

1.6 Find the surface which intersects the surface of the system

$$z(x + y) = C(3z + 1) \ (C \text{ being a constant})$$

Orthogonally and which passes through the circle $x^2 + y^2 = 1$, $z = 1$.

(2013 : 15 Marks)

Solution:

$$f(x, y, z) \equiv \frac{z(x+y)}{(3z+1)} = C$$

is the given surface.

$$\frac{\partial f}{\partial x} = \frac{z}{3z+1}$$

$$\frac{\partial f}{\partial y} = \frac{z}{3z+1}$$

$$\frac{\partial f}{\partial z} = \frac{(x+y)(3z+1) - 3z(x+y)}{(3z+1)^2} = \frac{(x+y)}{(3z+1)^2}$$

$\therefore$ Differential equation of orthogonal surface is

$$\frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{dz}{\frac{\partial f}{\partial z}}$$

$\Rightarrow$
$$\frac{dx}{\frac{z}{(3z+1)}} = \frac{dy}{\frac{z}{(3z+1)}} = \frac{dz}{\frac{(x+y)}{(3z+1)^2}}$$

From first two fractions

$$dx = dy \Rightarrow x - y = C_1$$

Choosing $\frac{x}{(3z+1)}, \frac{y}{(3z+1)}, -z$ are multipliers each fraction is equal to

$$= \frac{\frac{x}{(3z+1)}dx + \frac{y}{(3z+1)}dy - zdz}{0}$$

$$\Rightarrow \qquad xdx + ydy - z(3z+1)dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} - \left(z^3 + \frac{z^2}{2}\right) = \frac{C_2}{2} \Rightarrow x^2 + y^2 - (2z^3 + z^2) = C_2$$

$\therefore$ Any surface orthogonal to $f(x, y, z) = C$ has the form.

$$x^2 + y^2 - (2z^3 + z^2) = \phi(x - y) \qquad \text{...(i)}$$

Since it passes through $x^2 + y^2 = 1$, $z = 1$

At $z = 1$, (i) becomes

$$x^2 + y^2 - 3 = f(x - y)$$

$$\Rightarrow \qquad 1 - 3 = f(x - y) \Rightarrow f(x - y) = -2$$

$\therefore$ The requirement surface is

$$x^2 + y^2 - (2z^3 + z^2) = -2$$

1.7 Find the partial differential equation of family of all tangent planes to the ellipsoid : $x^2 + 4y^2 + 4z^2 = 4$, which are not perpendicular to xy plane.

(2018 : 10 Marks)

Solution:

Given, ellipsoid is $\qquad x^2 + 4y^2 + 4z^2 = 4$

Let equation of tangent plane is

$$lx + my + nz = p \qquad \text{...(i)}$$

and ellipsoid is $\qquad \frac{x^2}{4} + y^2 + z^2 = 1$

$\therefore$ by condition of tangency,

$$4l^2 + m^2 + n^2 = p^2$$

So, equation of tangent plane is (by (i))

$$lx + my + nz = \pm\sqrt{4l^2 + m^2 + n^2} \qquad \text{...(ii)}$$

$\because \qquad n \neq 0$ (given as it is not perpendicular to xy plane)

$\therefore$ from (ii)

$$\left(\frac{l}{n}\right)x + \left(\frac{m}{n}\right)y + z = \pm\sqrt{4\left(\frac{l}{n}\right)^2 + \left(\frac{m}{n}\right)^2 + 1}$$

$$\Rightarrow \qquad \alpha x + \beta y + z = \pm\sqrt{4\alpha^2 + \beta^2 + 1} \qquad \text{...(iii)}$$

Differentiating (iv) partially w.r.t. 'x' and 'y', we get

$$\alpha + \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\alpha \Rightarrow p = -\alpha$$

$$\beta + \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\beta \Rightarrow q = -\beta$$

Putting these values of α and β in (iii), we get

$$-px - qy + z = \pm\sqrt{4p^2 + q^2 + 1}$$

or $\qquad (px + qy - z)^2 = 4p^2 + q^2 + 1$ is the required differential equation.

1.8 Form a partial differential equation of the family of surfaces given by the following expression :

$$\psi(x^2 + y^2 + 2z^2, y^2 - 2zx) = 0$$

(2019 : 10 Marks)

Solution:

Given :

$$y(x^2 + y^2 + 2z^2, y^2 - 2zx) = 0$$

$$4 = x^2 + y^2 + z^2, v = y^2 - 2zx$$

$$\frac{\partial\phi}{\partial y}\left(\frac{\partial y}{\partial x}+p\frac{\partial y}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0$$

$\therefore$
$$\frac{\partial y}{\partial x} = 2x;\ \frac{\partial y}{\partial z} = 4z;\ \frac{\partial v}{\partial x} = -2z;\ \frac{\partial v}{\partial z} = 2x$$

$$\frac{\partial\phi}{\partial y}(2x+4pz)+\frac{\partial\phi}{\partial v}(-2z+p(-2x)) = 0$$

$$\frac{\partial\phi}{\partial y}(2x+4pz)-\frac{\partial\phi}{\partial v}(-2z+2xp) = 0$$

$$\frac{\partial\phi}{\partial y}(x+2pz)-\frac{\partial\phi}{\partial v}(z+xp) = 0$$

$$\frac{\partial\phi}{\partial y}(x+2pz) = \frac{\partial\phi}{\partial v}(z+xp) \quad ...(1)$$

$$\frac{\partial y}{\partial y} = 2y;\ \frac{\partial y}{\partial z} = 4z;\ \frac{\partial v}{\partial y} = 2y;\ \frac{\partial v}{\partial z} = -2x$$

$$\frac{\partial\phi}{\partial y}(2y+4zq)+\frac{\partial\phi}{\partial v}(2y-2qx) = 0$$

$$\frac{\partial\phi}{\partial y}(y+2zq)+\frac{\partial\phi}{\partial v}(y-xq) = 0$$

$$\frac{\partial\phi}{\partial y}(y+2zq) = (xq-y)\frac{\partial\phi}{\partial v} \quad ...(2)$$

Divide eqn. (1) by (2), we get

$$\frac{(x+2pz)}{(y+2zq)} = \frac{(z+px)}{(xq-y)}$$

$\therefore$ $(xq - y)(x + 2pz) = (y + 2zq)(z + px)$

$\Rightarrow$ $x^2q + 2xpqz - xy - 2pyz = 2y + 2z^2q + pxy + 2xpqz$

$\Rightarrow$ $x^2q - yx - 2pyz - zy - 2z^2q + (-pxy) = 0$

$\Rightarrow$ $(x^2 - 2z^2)q - (x + z)y - (xy + 2yz)p = 0$

Hence, $(x^2 - 2z^2)q - (x + z)y - (xy + 2yz)p = 0$ (Required Solution)

2. Solution of Quasilinear Partial Differential Equations of 1st Order, Lagrange's Auxiliary Equations and Charpit's Auxiliary Equations

2.1 Find the integral surface of :

$$x^2p + y^2q + z^2 = 0$$

which passes through the curve :

$$xy = x + y, z = 1$$

(2009 : 6 Marks)

Solution:

Given $$x^2p + y^2q + z^2 = 0 \text{ or } x^2p + y^2q = z^2 \quad ...(1)$$

Given curve is $$xy = x + y,\ z = 1 \quad ...(2)$$

Here Lagrange's auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(-z^2)} \quad ...(3)$$

Taking the first and third fractions,

$$x^2dx + z^2dz = 0$$

Integrating $$-\left(\frac{1}{x}\right) - \left(\frac{1}{z}\right) = -c_1$$

or $$\frac{1}{x} + \frac{1}{z} = c_1 \quad ...(4)$$

$$y^2dy + z^2dz = 0$$

Integrating $$\left(\frac{1}{y}\right) + \left(\frac{1}{z}\right) = -c_2 \text{ or } \frac{1}{y} + \frac{1}{z} = c_2 \quad ...(5)$$

Adding (4) and (5), $$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = c_1 + c_2$$

or $$\frac{x+y}{xy} + \frac{2}{z} = c_1 + c_2$$

or $$(x + y)(xy) + 2 = c_1 + c_2, \text{ using (2)}$$

or $$c_1 + c_2 = 3 \quad ...(6)$$

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$$\frac{1}{x} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} = 3 \text{ or } yz + 2xy + xz = 3xyz$$

2.2 Solve the PDE :

$$(x + 2z)\frac{\partial z}{\partial x} + (4zx - y)\frac{\partial z}{\partial y} = 2x^2 + y$$

(2011 : 12 Marks)

Solution:

The given partial differential equation is

$$(x + 2z)\frac{\partial z}{\partial x} + (4zx - y)\frac{\partial z}{\partial y} = 2x^2 + y$$

Putting $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$,

$$(x + 2z)p + (4zx - y)q = 2x^2 + y$$

It is of the form of

$$P_p + Q_q = R$$

The Lagrange's auxilliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y} \quad ...(i)$$

Using $2x, -1, -1$ as multipliers, each fraction in (i) is equal to

$$\frac{2xdx - dy - dz}{2x^2 + 4xz - 4zx + y - 2x^2 - y} = \frac{2xdx - dy - dz}{0}$$

$$\Rightarrow \qquad x^2 - y - z = C_1$$

Taking $y, x, -2z$ as multipliers, we have,

$$\frac{ydx + xdy - 2zdz}{xy + 2yz + 4zx^2 - xy - 4x^2z - 2zy} = \frac{ydx + xdy - 2zdz}{0}$$

$$\Rightarrow \qquad xy - z^2 = C_2$$

$$\therefore \qquad \phi(x^2 - y - z, xy - z^2) = 0$$

is the required solution of the given partial differential equation.

2.3 Solve the partial differential equation $px + qy = 3z$.

(2012 : 20 Marks)

Solution:

The given equation is

$$px + qy = 32 \qquad ...(i)$$

The Lagrange's Auxilliary equation is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \qquad ...(ii)$$

Taking first two fractions of equation (ii),

$$\frac{dx}{x} = \frac{dy}{y}$$

On integrating, $\log x = \log y + \log C_1$

$$\Rightarrow \qquad x = yC_1 \Rightarrow \frac{x}{y} = C_1$$

Taking last two fractions of equation (ii),

$$\frac{dy}{y} = \frac{dz}{3z} \Rightarrow 3\log y = \log z + \log C_2$$

$$\Rightarrow \qquad \frac{y^3}{z} = C_2$$

So, required general solution of (i) is

$$\phi(C_1, C_2) = 0$$

$$\phi\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0$$

2.4 Solve the partial differential equation

$$(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$$

where $$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$$

(2015 : 10 Marks)

Solution:

Given equation is :

$$(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$$

$$\Rightarrow \qquad (y^2 + z^2 - x^2)p - 2xyq = -2xz$$

Lagrange's auxiliary equation are :

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xy}$$

Consider $$\frac{dy}{-2xy} = \frac{dz}{-2xz} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

Integrating it,

$\Rightarrow$ $$\log y = \log z + \log(a), \text{ where } a \text{ is a constant.}$$

$\Rightarrow$ $$\log\left(\frac{y}{z}\right) = \log a$$

$\Rightarrow$ $$y = az \quad \text{...(i)}$$

Also, using x, y and z as multipliers, we get

$\Rightarrow$ $$\frac{xdx+ydy+zdz}{x(y^2+z^2-x^2)-x(2y^2)-x(2z^2)} = \frac{dy}{-2xy}$$

$\Rightarrow$ $$\frac{xdx+ydy+zdz}{x(y^2+z^2-x^2-2y^2-2z^2)} = \frac{dy}{xy}$$

$\Rightarrow$ $$\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dy}{y}$$

Integrating both sides, we get

$$\log(x^2+y^2+z^2) = \log y + \log b; \text{ where } b \text{ is a constant.}$$

$\Rightarrow$ $$x^2+y^2+z^2 = by \quad \text{...(ii)}$$

From (i) and (ii), solution is given by

$$\phi(y-az, x^2+y^2+z^2-by) = 0$$

2.5 **Solve for the general solution $p\cos(x+y) + q\sin(x+y) = z$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.**

(2015 : 15 Marks)

Solution:

Given equation is :

$$p\cos(x+y) + q\sin(x+y) = z$$

We first write Lagrange's auxiliary equation for given equation

i.e., $$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$

Consider $$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}$$

$\Rightarrow$ $$\frac{dx+dy}{\sqrt{2}\times\frac{1}{\sqrt{2}}\{\cos(x+y)+\sin(x+y)\}} = \frac{dz}{z}$$

$\Rightarrow$ $$\frac{dx+dy}{\sqrt{2}\left\{\frac{\cos(x+y)}{\sqrt{2}}+\frac{\sin(x+y)}{\sqrt{2}}\right\}} = \frac{dz}{z}$$

$\Rightarrow$ $$\frac{dx+dy}{\sqrt{2}\left\{\cos(x+y)\sin\left(\frac{\pi}{4}\right)+\sin(x+y)\cos\left(\frac{\pi}{4}\right)\right\}} = \frac{dz}{z}$$

$$\Rightarrow \quad \frac{d\left(x+y+\frac{\pi}{4}\right)}{\sqrt{2}\sin\left(x+y+\frac{\pi}{4}\right)} = \frac{dz}{z}$$

$$\Rightarrow \quad \operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)d\left(x+y+\frac{\pi}{4}\right) = \frac{dz}{z}$$

Integrating both sides, we get

$$\Rightarrow \quad -\log\left[\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)+\cot\left(x+y+\frac{\pi}{4}\right)\right] = \frac{dz}{z}\log z+\log a;$$

where a is a constant.

$$\Rightarrow \quad \log z+\log\left\{\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)+\cot\left(x+y+\frac{\pi}{4}\right)\right\}+\log a = 0$$

$$\Rightarrow \quad z\left(\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)+\cot\left(x+y+\frac{\pi}{4}\right)\right) = \frac{1}{a} \quad \text{...(i)}$$

Now, consider

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\Rightarrow \quad \frac{\cos(x+y)-\sin(x+y)}{\sin(x+y)+\cos(x+y)}d(x+y) = d(x-y)$$

Integrating both sides, we get

$\log\{\sin(x+y)+\cos(x+y)\} = x-y+b$, where b is a constant.

$$\Rightarrow \quad \log\{\sin(x+y)+\cos(x+y)\}-x+y = b \quad \text{...(ii)}$$

From (i) and (ii), solution of given equation is

$$\phi\left[z\left\{\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)+\cot\left(x+y+\frac{\pi}{4}\right)\right\}, \log\{\sin(x+y)+\cos(x+y)\}\right]$$

2.6 Find the general equation of surfaces orthogonal to the family of spheres given by $x^2+y^2+z^2=cz$.

(2016 : 10 Marks)

Solution:

Given, the equation of family of sphere is $x^2+y^2+z^2=cz$.

$$\Rightarrow \quad c = \frac{x^2+y^2+z^2}{z} = f$$

Now,

$$\frac{\partial f}{\partial x} = \frac{2x}{z},\ \frac{\partial f}{\partial y} = \frac{2y}{z}$$

$$\frac{\partial f}{\partial z} = \frac{2z}{z}-\frac{x^2+y^2+z^2}{z^2} = \frac{z^2-x^2-y^2}{z^2}$$

$\therefore$ equation of orthogonal surface is

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

where,

$$p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$$

$\therefore$ differential equation becomes

$$p\left(\frac{2x}{z}\right)+q\left(\frac{2y}{z}\right) = \frac{z^2-x^2-y^2}{z^2}$$

$$\Rightarrow \quad 2xzp + 2yzq = z^2 - x^2 - y^2$$

So, Lagrange's auxiliary equations are

$$\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{z^2 - x^2 - y^2}$$

Consider,
$$\frac{dx}{2xz} = \frac{dy}{2yz}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \quad \log x = \log y + \log C_1$$ (C_1 is a constant, integrate both sides)

$$\Rightarrow \quad \log\left(\frac{x}{y}\right) = \log C_1$$

$$\Rightarrow \quad \frac{x}{y} = C_1 \quad \text{...(i)}$$

Also,
$$\frac{xdx + ydy + zdz}{2x^2z + 2y^2z + z^3 - x^2z - y^2z} = \frac{dx}{2xz}$$

$$\Rightarrow \quad \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)\cdot z} = \frac{dx}{xz}$$

Integrating both sides, we get

$$\log(x^2 + y^2 + z^2) = \log x + \log C_2$$ (C_2 is a constant)

$$\Rightarrow \quad \log\left(\frac{x^2 + y^2 + z^2}{x}\right) = \log C_2$$

$$\Rightarrow \quad \frac{x^2 + y^2 + z^2}{x} = C_2 \quad \text{...(ii)}$$

From (i) and (ii), the solution can be written as

$$\phi\left(\frac{x}{y} - C_1, \frac{x^2 + y^2 + z^2}{x} - C_2\right) = 0$$

2.7 Find the general integral of the partial differential equation :

$$(y + zx)p - (x + yz)q = x^2 - y^2$$

(2016 : 10 Marks)

Solution:

Given equation is :

$$(y + zx)p - (x + yz)q = x^2 - y^2$$

The Lagrange's auxiliary equations are

$$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2}$$

Consider,
$$\frac{ydx + xdy}{y^2 + xyz - x^2 - xyz} = \frac{dz}{x^2 - y^2}$$

$$\Rightarrow \quad \frac{xdy + ydx}{-(x^2 - y^2)} = \frac{dz}{x^2 - y^2}$$

$$\Rightarrow \quad d(xy) = -dz$$

Integrating both sides, we get

$$xy = -z + C_1 \quad (C_1 \text{ is a constant})$$

$$\Rightarrow \quad xy + z = C_1 \quad \ldots(i)$$

Also,
$$\frac{dx + dy}{y + zx - x - y^2} = \frac{dz}{x^2 - y^2}$$

$$\Rightarrow \quad \frac{d(x+y)}{(y-x) + z(x-y)} = \frac{dz}{(x-y)(x+y)}$$

$$\Rightarrow \quad \frac{d(x+y)}{(x-y)(z-1)} = \frac{dz}{(x-y)(x+y)}$$

$$\Rightarrow \quad (x+y)d(x+y) = (z-1)dz$$

Integrating both sides, we get

$$\frac{(x+y)^2}{2} = \frac{z^2}{2} - z + C_2, \quad \ldots\text{(ii) where } C_2 \text{ is a constant.}$$

So, general solution of given equation is

$$\phi\left(xy + z, \frac{(x+y)^2}{2} - \frac{z^2}{2} + z\right) = 0 \quad \text{(from (i) \& (ii))}$$

2.8 Find a complete integral of the PDE

$$2(pq + yp + qx) + x^2 + y^2 = 0$$

(2017 : 15 Marks)

Solution:

Charpit's auxiliary equations are :

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}}$$

Here,
$$f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0 \quad \ldots(i)$$

$$\frac{dp}{2q + 2x} = \frac{dq}{2p + 2y} = \frac{dz}{-p(2q+2y) - q(2p+2x)}$$

$$= \frac{dx}{-(2q+2y)} = \frac{dy}{-(2p+2x)}$$

$$= \frac{dp + dq + dx + dy}{(2q+2x) + (2p+2y) - (2q+2y) - (2p+2x)}$$

$$= \frac{d(p+q+x+y)}{0}$$

So that
$$(p + x) + (q + y) = a \quad \ldots(ii)$$

Re-writing (i)

$$2(p + x)(q + y) + (x - y)^2 = 0$$

or
$$(p + x)(q + y) = \frac{-(x+y)^2}{2} \quad \ldots(iii)$$

Now,
$$(p + x) - (q + y) = \sqrt{[(p+x)+(q+y)]^2 - 4(p+x)(q+y)}$$

By (ii) and (iii)
$$= \sqrt{a^2 + 2(x-y)^2} \quad \ldots(iv)$$

(ii) + (iv) $\Rightarrow$
$$2(p + x) = a + \sqrt{a^2 + 2(x-y)^2}$$

(ii) − (iv) $\Rightarrow$
$$2(q + y) = a - \sqrt{a^2 + 2(x-y)^2}$$

From these, we get p & q as

$$p = -x + \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$$

$$q = -y + \frac{a}{2} - \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$$

$\therefore$ $dz = pdx + qdy$ becomes

$$= -(xdx + ydy) + \frac{a}{2}\times(dx+dy) + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}\times(dx-dy)$$

or
$$dz = -\frac{1}{2}d(x^2+y^2) + \frac{a}{2}d(x+y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}\times(x-y)^2}d(x-y) \quad \text{...(v)}$$

Put $x - y = t$ so that $d(x-y) = dt$

$\therefore$
$$dz = -\frac{1}{2}d(x^2+y^2) + \frac{a}{2}d(x+y) + \frac{1}{2}\sqrt{\left(\frac{a^2}{2}\right) + t^2}\,dt$$

$\therefore$
$$z = -\frac{x^2+y^2}{2} + a\frac{x+y}{2} + \frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2}\right] +$$

$$\frac{\left(\frac{a}{\sqrt{2}}\right)^2}{2}\log\left[t + \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2}\right] + b$$

Putting back the value of t, the required complete integral is

$$z = -\frac{(x^2+y^2)}{2}2 + \frac{a(x+y)}{2} + \frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2} + (x-y)^2} + \frac{a^2}{2}\log\left\{x - y + \sqrt{\frac{a^2}{2} + (x-y)^2}\right\}\right] + b$$

2.9 Find the general solution of the partial differential equation :

$$(y^3x - 2x^4)_p + (2y^4 - x^3y)_q = 9z(x^3 - y^3)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and find its integral surface that passes through the curve $x = t$, $y = t^2$, $z = 1$.

(2018 : 15 Marks)

Solution:

Lagrange's auxiliary equations are :

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}$$

Now,
$$\frac{\frac{dx}{x} + \frac{dy}{y}}{y^3 - 2x^3 + 2y^3 - x^3} = \frac{\frac{dz}{9z}}{x^3 - y^3}$$

$\Rightarrow$
$$\frac{\frac{dx}{x} + \frac{dy}{y}}{3(y^3 - x^3)} = \frac{-dz}{9z(y^3 - x^3)}$$

$\Rightarrow$
$$\log xy = \frac{-1}{3}\log z + \log c_1$$

$$\Rightarrow \quad \log xy = \log \frac{c_1}{z^{1/3}} \qquad (c_1 \text{ is a constant})$$

$$\Rightarrow \quad xy = \frac{c_1}{z^{1/3}} \qquad \text{...(i)}$$

Also,
$$\frac{x^3dx + y^2dy}{y^3x^3 - 2x^6 + 2y^6 - x^3y^3} = \frac{dz}{9z(x^3 - y^3)}$$

$$\Rightarrow \quad \frac{x^2dx + y^2dy}{-2(x^6 - y^6)} = \frac{dz}{9z(x^3 - y^3)}$$

$$\Rightarrow \quad \frac{x^2dx + y^2dy}{-2(x^3 - y^3)(x^3 + y^3)} = \frac{dz}{9z(x^3 - y^3)}$$

$$\Rightarrow \quad \frac{3x^2dx + 3y^2dy}{-2(x^3 + y^3)} = \frac{dz}{3z} \Rightarrow \frac{d(x^3 + y^3)}{-2(x^3 + y^3)}$$

$$\Rightarrow \quad \log(x^3 + y^3) = \frac{-2}{3}\log z + \log c_2, \text{ where } c_2 \text{ is a constant.}$$

$$\Rightarrow \quad x^3 + y^3 = \frac{c_2}{z^{2/3}} \qquad \text{...(ii)}$$

From (i) and (ii), general solution is

$$\phi(xyz^{1/3}, (x^3 + y^3)z^{2/3}) = 0$$

Now, it passes through $x = t$, $y = t^2$, $z = 1$ (given)

i.e.,
$$y = x^2 \text{ and } z = 1$$

$\therefore$ from (i),
$$x \cdot x^2 = \frac{c_1}{1} \Rightarrow c_1 = x^3$$

from (ii),
$$x^3 + y^3 = \frac{c_2}{1^{2/3}}$$

$$\Rightarrow \quad x^3 + (x^2)^3 = c_2$$

$$\Rightarrow \quad c_2 = x^3 + (x^3)^2 = c_1 + c_1^2$$

or
$$c_2 = c_1 + c_1^2$$

$\therefore$ from (i) and (ii), we get

$$(x^3 + y^3)z^{2/3} = xyz^{1/3} + x^2y^2z^{2/3}$$

2.10 $x\dfrac{\partial u}{\partial x} + (u - x - y)\dfrac{\partial u}{\partial y} = x + 2y$ in $x > 0, -\infty < y < \infty$ with $u = 1 + y$ on $x = 1$.

(2019 : 15 Marks)

Solution:

Given quasilinear p.d.e. of first order

$$x = \frac{\partial u}{\partial x} + (u - x - y)\frac{\partial u}{\partial y} = x + 2y \qquad \text{...(1)}$$

$$x > 0, -\infty < y < \infty$$

With $u = 1 + y$ on $x = 1$

Clearly it is in the form of

$$Pp + Qq = R$$

where $P = x$, $Q = u - x - y$, $R = x + 2y$

$$p = \frac{\partial u}{\partial x}, \; q = \frac{\partial u}{\partial y}$$

Here, the lagrange's auxiliary equations of (1) are :

$$\frac{dx}{x} = \frac{dy}{4-x-y} = \frac{du}{x+2y} \quad \text{...(2)}$$

Taking (1, 1, 1) multiplies for above equations, we get

$$\frac{dx+dy+du}{x+u-x-y+x-2y} = \frac{d(x+y+u)}{x+y+u}$$

$$\frac{dx}{x} = \frac{d(x+y+u)}{x+y+u}$$

Integrating both sides

$$\log x + \log C_1 = \log(x+y+u)$$

$$C_1 = \frac{x+y+u}{x} \quad \text{...(3)}$$

Taking (0, 1, 1) as multiplies for each fraction of (2), we get

$$\frac{d(y+u)}{y+u} = \frac{d(x+y+u)}{x+y+u}$$

Integrating both sides, we get

$$\log C_2 + \log(y+u) = \log(x+y+u)$$

$$C_2(y+u) = x+y+u$$

$$C_2 = \frac{x+y+u}{y+u} \quad \text{...(4)}$$

Now, put the values $x = 1$ and $u = 1 + y$ in (3) and (4), we get

$$C_1 = 2y+2 \text{ and } C_2 = \frac{2y+2}{2y+1} \quad \text{...(5)}$$

Solving (5), we get

$$C_2 = \frac{C_1}{C_1-1}$$

$$(C_1-1)C_2 = C_1 \quad \text{...(6)}$$

Now, put the value of C_1 and C_2 in (6) from (3) and (4), we get

$$\left[\frac{x+y+u}{x}-1\right]\left[\frac{x+y+u}{y+u}\right] = \left[\frac{x+y+u}{x}\right]$$

3. Cauchy's Method of Characteristics

3.1 Find the characteristics of :

$$y^2r - x^2t = 0$$

where r and t have their usual meanings.

(2009 : Marks)

Solution:

$$y^2r - x^2r = 0 \quad \text{...(1)}$$

Comparing (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

We get $R = y^2$, $T = -x^2$, $S = 0$

The characteristic equation is $S^2 - 4RT$

$$= -4(y^2)(-x^2) = 4x^2y > 0$$

And hence (1) is hyperbolic everywhere except on the coordinate axis $x = 0$, $y = 0$.

The quadratic equation is $R\lambda^2 + S\lambda + T = 0$, i.e.,

$$y^2\lambda^2 + 0\lambda - x^2 = 0$$

$$\Rightarrow \quad y^2\lambda^2 = x^2$$

$$\Rightarrow \quad \lambda = \frac{\pm x}{y}$$

2 distinct roots $\quad \lambda_1 = \frac{x}{y}, \lambda_2 = \frac{-x}{y}$

2 positive real roots

$$\frac{dy}{dx} + \lambda_1 = 0, \frac{dy}{dx} + \lambda_2 = 0$$

$$\frac{dy}{dx} + \frac{x}{y} = 0 \text{ and } \frac{dy}{dz} - \frac{x}{y} = 0$$

$$\Rightarrow \quad ydy + xdx = 0 \text{ and } ydy - xdx = 0$$

On integrating, $\quad \frac{y^2}{2} + \frac{x^2}{2} = c_1, \frac{y^2}{2} - \frac{x^2}{2} = c_2$

So. $\quad y^2 + x^2 = c_1$ and $y^2 - x^2 = c_2$

are the required family of characteristics. Hence, these are families of clearly circles and hyperbola's respectively.

3.2 Solve the following partial differential equation

$$zp + yq = x, x_0(s) = s, y_0(s) = 1, z_0(s) = 2s$$

by method of characteristics.

(2010 : 20 Marks)

Solution:

Given, the equation is $\quad zp + yq = x$

or $\quad zp + yq - x = 0 = F(x, y, z, p, q)$

$$x_0 = s, y_0 = 1, z_0 = 2s$$

Now, $\quad f_3'(s) = p_0 f_1'(s) + q_0 f_2'(s)$

$$\Rightarrow \quad \frac{d^2s}{ds} = p_0 \times \frac{ds}{s} + q_0 \times \frac{d(1)}{ds}$$

$$\Rightarrow \quad 2 = p_0 + 0 \Rightarrow p_0 = 2$$

By eqn., $\quad z_0p_0 + y_0q_0 = x_0$

$$\Rightarrow \quad 2s \times 2 + 1 \times q_0 = s$$

$$\Rightarrow \quad q_0 = -3s$$

$$\therefore \quad x_0 = s, y_0 = 1, z_0 = 2s, p_0 = 2, q_0 = -3s \quad \text{...(1)}$$

Now, characteristics are :

$$\frac{dx}{dt} = F_p = z \quad \text{...(2)}$$

$$\frac{dy}{dt} = F_q = y \quad \text{...(3)}$$

$$\frac{dz}{dt} = pz + qy = x \quad \text{...(4)}$$

$$\frac{dp}{dt} = F_x - pF_z = 1 - p^2 \quad \text{...(5)}$$

$$\frac{dq}{dt} = -F_y - qF_z = -q - pq = -q(1+p) \qquad ...(6)$$

From (3)

$$\frac{dy}{dt} = y \Rightarrow \frac{dy}{y} = dt$$

Integrating both sides, we get

$$y = c_1 e^t \text{ (c_1 is a constant)}$$

At $t = 0$, $\quad y_0 = 1 \qquad$ (from (1))

$\Rightarrow \quad 1 = c_1$

$\therefore \quad y = e^t \qquad ...(7)$

From (5)

$$\frac{dp}{dt} = 1 - p^2 \Rightarrow \frac{dp}{1-p^2} = dt$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{1-p} + \frac{1}{1+p}\right)dp = dt$$

Integrating both sides, we get

$$\Rightarrow \quad \frac{1}{2}\ln\left(\frac{1+p}{1-p}\right)dp = t + c_2 \qquad \text{(c_2 is constant)}$$

$$\Rightarrow \quad \frac{1+p}{1-p} = c_3 e^{2t} \qquad \text{(c_3 is constant)}$$

At $t = 0$, $\quad p_0 = 2 \qquad$ (from (1))

$$\Rightarrow \quad \frac{3}{-1} = c_3 \Rightarrow c_3 = -3$$

$$\therefore \quad \frac{1+p}{1-p} = -3e^{2t}$$

$$\Rightarrow \quad p = \frac{3e^{2t}+1}{3e^{2t}-1} \qquad ...(8)$$

From (6)

$$\frac{dq}{dt} = -q(1+p) = -q\left(1 + \frac{3e^{2t}+1}{3e^{2t}-1}\right) \qquad \text{...(from (8))}$$

$$\Rightarrow \quad \frac{dq}{dt} = -q \times \frac{6e^{2t}}{(3e^{2t}-1)}$$

$$\Rightarrow \quad \frac{dq}{q} = -\frac{6e^{2t}}{3e^{2t}-1}dt$$

Integrating both sides, we get

$$\Rightarrow \quad q = \frac{c_4}{3e^{2t}-1} \qquad \text{(c_4 is a constant)}$$

Now at $t = 0$, $q_0 = -3s$ $\qquad$ ...(from (1))

$$\Rightarrow \quad -3s = \frac{c_4}{3-1} \Rightarrow c_4 = -6s$$

$$\Rightarrow \quad q = \frac{-6s}{3e^{2t}-1} \qquad ...(9)$$

Now, from (2) and (4)

$$\frac{dx}{z} = \frac{dz}{x}$$

$\Rightarrow$ $$xdx = zdz$$

Integrating both sides, we get

$$x^2 - z^2 = c_5 \quad (c_5 \text{ is a constant})$$

At $t = 0$, $x_0 = 5$, $z_0 = 2s$ ...(from (1))

$\therefore$ $$s^2 - 4s^2 = c_5$$

$\Rightarrow$ $$c_5 = -3s^2$$

$$x^2 - z^2 = -3s^2 \quad ...(10)$$

Now, by eqn., $zp + yq = x$

Using values from (7), (8), (9), we get

$$z\cdot\left(\frac{3e^{2t}+1}{3e^{2t}-1}\right) + y\cdot\left(\frac{-6s}{3e^{2t}-1}\right) = x$$

$\Rightarrow$ $$z\cdot\frac{(3y^2+1)}{3y^2-1} + y\cdot\frac{-6s}{3y^2-1} = x$$

$\Rightarrow$ $$\frac{-6sy}{3y^2-1} = \frac{x - z(3y^2+1)}{3y^2-1}$$

$\Rightarrow$ $$-6sy = x(3y^2-1) - z(3y^2+1)$$

$\Rightarrow$ $$s = \frac{3y^2x - 3y^2z - x - z}{-6y} \quad ...(11)$$

Putting this value of s in (10), we get

$$x^2 - z^2 = \frac{-3\times(3y^2x - 3y^2z - x - z)^3}{36y^2}$$

$\Rightarrow$ $$x^2 - z^2 = \frac{-\{3y^2(x-z) - (x+z)\}^2}{12y^2}$$

which is the required solution.

3.3 **Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0$, $y = 0$.**

(2016 : 15 Marks)

Solution:

Given equation is

$$z = p^2 - q^2 \text{ or } p^2 - q^2 - z = 0 = f \quad ...(i)$$

The curve is $4z + x^2 = 0$ and $y = 0$.

Let characteritic variables are λ and t.

Now, let $$x_0 = \lambda,\ y = 0,\ z_0 = \frac{-x^2}{4} = \frac{-\lambda^2}{4}$$

$\therefore$ $$f_1(\lambda) = \lambda = x_0 \quad ...(ii)$$

$$f_2(\lambda) = 0 = y_0 \quad ...(iii)$$

$$f_3(\lambda) = \frac{-\lambda^2}{4} = z_0 \quad ...(iv)$$

Now, $$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$$

$$\Rightarrow \quad \frac{-\lambda}{2} = p_0 \cdot 1 + q_0 \cdot 0$$

$$\Rightarrow \quad p_0 = \frac{-\lambda}{2} \quad \text{...(v)}$$

From (i), using this value of p_0, we get

$$p_0^2 - q_0^2 - z_0 = 0 \Rightarrow \frac{\lambda^2}{4} - q_0^2 + \frac{\lambda^2}{4}$$

$$\Rightarrow \quad q_0^2 = \frac{\lambda^2}{2}$$

$$\Rightarrow \quad q_0 = \frac{\lambda}{\sqrt{2}} \quad \text{...(vi)}$$

Now, Charpit's equation for f are

$$\frac{dx}{dt} = 2p, \frac{dy}{dt} = -2q, \frac{dz}{dt} = 2p^2 - 2q^2, \frac{dp}{dt} = p, \frac{dq}{dt} = q \quad \text{...(vii)}$$

Now, $\frac{dp}{dt} = p \Rightarrow \frac{dp}{p} = dt$

$\Rightarrow \quad \log p = t + \log C_1$ (C_1 is a constant)

$\Rightarrow \quad p = C_1 e^t$

Now, at $t = 0$, $p_0 = -\frac{\lambda}{2}$ (from (v))

$\therefore \quad -\frac{\lambda}{2} = C_1$

$\therefore \quad p = -\frac{\lambda}{2} e^{-t}$ ...(viii)

Also, $\frac{dq}{dt} = q \Rightarrow \frac{dq}{q} = dt$

$\Rightarrow \quad \log q = t + \log C_2$ (C_2 is a constant)

$\Rightarrow \quad q = C_2 e^t$

At $t = 0$, $q_0 = \frac{\lambda}{\sqrt{2}}$ (from (vi))

$\frac{\lambda}{\sqrt{2}} = C_2$

$\therefore \quad q = \frac{\lambda}{\sqrt{2}} e^t$ ...(ix)

Now, $\frac{dx}{dt} = 2p = 2x - \frac{\lambda}{2} e^t$ (from (viii))

$\Rightarrow \quad \frac{dx}{dt} = -\lambda e^t$

$\Rightarrow \quad dx = -\lambda e^t dt$

$\Rightarrow \quad x = -\lambda e^t + C_3$ (C3 is a constant)

At $t = 0$, $x_0 = \lambda$ (from (ii))

$\therefore \quad \lambda = -\lambda + C_3 \Rightarrow C_3 = 2\lambda$

$\therefore \quad x = -\lambda e^t + 2\lambda = \lambda(2 - e^t)$ ...(x)

Also, $$\frac{dy}{dt} = -2q = -2\times\frac{\lambda}{\sqrt{2}}e^t = -\sqrt{2}e^t\cdot\lambda \quad \text{(from (ix))}$$

$$\Rightarrow \quad dy = -\sqrt{2}\lambda e^t dt$$

$$\Rightarrow \quad y = -\sqrt{2}\lambda e^t + C_4 \quad (C_4 \text{ is a constant})$$

From (iii), at $t = 0$, $y_0 = 0$

$$\Rightarrow \quad 0 = -\sqrt{2}\lambda e^t + C_4 \Rightarrow C_4 = \sqrt{2}\lambda$$

$$\therefore \quad y = -\sqrt{2}\lambda e^t + \sqrt{2}\lambda = \sqrt{2}\lambda(1-e^t) \quad \text{...(xi)}$$

$$\frac{dz}{dt} = 2p^2 - 2q^2 = 2\times\frac{\lambda^2}{4}e^{2t} - 2\times\frac{\lambda^2}{2}e^{2t} \quad \text{(from (viii) and (ix))}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{-\lambda^2}{2}e^{2t}$$

$$\Rightarrow \quad dz = \frac{-\lambda^2}{2}e^{2t}dt$$

$$\Rightarrow \quad z = \frac{-\lambda^2}{4}e^{2t} + C_5 \quad (C_5 \text{ is a constant})$$

At $t = 0$, $$z_0 = \frac{-\lambda^2}{4} \quad \text{(from (iv))}$$

$$\frac{-\lambda^2}{4} = \frac{-\lambda^2}{4} + C_5$$

$$\Rightarrow \quad C_5 = 0$$

$$\therefore \quad z = \frac{-\lambda^2}{4}e^{2t} \quad \text{(xii)}$$

From (x), (xi) and (xii), characteristics are

$$x = \lambda(2-e^t)$$
$$y = \sqrt{2}\lambda(1-e^t)$$
$$z = \frac{-\lambda^2}{4}e^{2t}$$

To find the equation of integral surface, we eliminate λ and e^t from (x) and (xi) and put their values in (xii).

$$e^t = \frac{\sqrt{2}(x-\sqrt{2}y)}{\sqrt{2}x-y}$$

and $$\lambda = \frac{\sqrt{2}x-y}{\sqrt{2}}$$

Using these values of λ and e^t, we get

$$z = \frac{-\lambda^2}{4}e^{2t} = -\frac{(\sqrt{2}x-y)^2}{2}\times\frac{(\sqrt{2}x-2y)^2}{(\sqrt{2}x-y)^2}$$

$$\Rightarrow \quad z = -\frac{2}{z}(x-\sqrt{2}y)^2$$

$$\Rightarrow \quad z = -(x-\sqrt{2}y)^2 \text{, which is the required surface.}$$

4. Linear P.D.E. of Second Order with Constant Coefficient

4.1 Solve :

$$(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$$

where D and D' represent $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

(2009 : 15 Marks)

Solution:

Clearly the Auxilliary equation is

$$m^2 - m - 2 = 0$$

$$\Rightarrow \quad m^2 - 2m + m - 2 = 0$$

$$\Rightarrow \quad m(m-2) + 1(m-2) = 0$$

$$\Rightarrow \quad m = -1, 2$$

$$Z_c = \phi_1(y+2x) + \phi_2(y-x)$$

where ϕ_1 and ϕ_2 are Arbitrary function.

Particular Integral

$$= \frac{1}{(D+D')(D-2D')}[(2x^2 + xy - 2y^2)\sin xy - \cos xy]$$

Putting $y - x = c$ we get $y = c + x$

$$= \frac{1}{(D-2D')}\int [2x^2 + x(c+x) - 2(c+x)^2 \sin x(x+c) - \cos x(x+c)]dx$$

$$= \frac{1}{(D-2D')}\int [2x^2 + (c+x)[x - 2x - 2x \mathrm{l} \sin(x^2 + cx) - \cos(x^2 + cx)]dx$$

$$= \frac{1}{(D-2D')}\int [2x^2 + (c+x)(-x-2c)\sin(x^2 + cx) - \cos(x^2 + cx)]dx$$

$$= \frac{1}{(D-2D')}\int [2x + c)(x-c)\sin(x^2 + cx) - \cos(x^2 + cx)]dx$$

$$= \frac{1}{(D-2D')}\int -(2x+c)(x-c)\frac{\cos(x^2 + cx)}{(2x+c)} + \int \cos(x^2 + cx)dx - \int \cos(x^2 + cx)dx$$

$$= \frac{1}{(D-2D')}(y-2x)\cos xy \int (c' - 2x - 2x)\cos x(c' - 2x)dx - c = y + 2x$$

$$\simeq \int (c' - 4x)\cos(c'x - 2x^2)dx$$

$$\simeq \frac{[\sin(c'x - 2x^2)]}{(c' - 4x)}(c' - 4x)$$

$$= \sin[(y + 2x)x - 2x^2]$$

$$= \sin xy$$

Clearly, the required solution is $z = z_1 + z_2$

4.2 Solve the PDE :

$$(D^2 - D')(D - 2D')z = e^{2x+y} + xy$$

(2010 : 12 Marks)

Solution:

Given, the equation is

$$(D^2 - D')(D - 2D')z = e^{2x+y} + xy$$

Complementary Function :

The auxiliary equation is :

$$(m^2-1)(m-2) = 0$$

$\Rightarrow$ $$m = \pm 1, 2$$

$\therefore$ Solution is $$z_c = \phi_1(y+x)+\phi_2(y-x)+\phi_3(y+2x)$$

Particular INtegral :

$$z_p = \frac{1}{(D^2-D')(D-2D')}e^{2x+y}+\frac{1}{(D^2-D')(D-2D')}xy$$

$$\Rightarrow \quad z_p = \frac{1}{(2^2-1)(D-2D')}e^{2x+y}+\frac{1}{(D^2-D')\cdot D}\times\frac{1}{\left(1-\frac{2D'}{D}\right)}xy$$

$$\Rightarrow \quad z_p = \frac{1}{3}\frac{1}{(D-2D')}e^{2x+y}+\frac{1}{(D^2-D')}\cdot\frac{1}{D}\left(1+\frac{2D'}{D}\right)xy$$

$$\Rightarrow \quad z_p = \frac{1}{3}\frac{x}{1\cdot 1!}e^{2x+y}+\frac{1}{(D^2-D')}\cdot\frac{1}{D}\left(xy+x^2\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{1}{(D^2-D')}\times\left(\frac{x^2y}{2}+\frac{x^3}{3}\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{1}{D^2}\times\left(1-\frac{D'}{D^2}\right)^{-1}\left(x^2y+\frac{x^3}{3}\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{1}{D^2}\times\left(1+\frac{D'}{D^2}\right)\left(x^2y+\frac{x^3}{3}\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{1}{D^2}\times\left(x^2y+\frac{x^3}{3}+\frac{x^4}{12}\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{1}{D}\left(\frac{x^3y}{3}+\frac{x^4}{12}+\frac{x^5}{60}\right)$$

$$\Rightarrow \quad z_p = \frac{x}{3}e^{2x+y}+\frac{x^4y}{12}+\frac{x^5}{60}+\frac{x^6}{360}$$

Now, $$z = z_c+z_p$$

$$\Rightarrow \quad z = \phi_1(y+x)+\phi_2(y-x)+\phi_3(y+2x)+\frac{x}{3}e^{2x+y}+\frac{x^4y}{12}+\frac{x^5}{60}+\frac{x^6}{360}$$

4.3 Solve the PDE

$$(D^2-D'^2+D+3D'-2)z = e^{(x-y)}-x^2y$$

(2011 : 12 Marks)

Solution:

The given partial differential equation is

$$(D^2-D'^2+D+3D'-2)z = e^{x-y}-x^2y$$

$$\Rightarrow \quad (D-D'+2)(D+D'-1)z = e^{x-y}-x^2y \quad \text{...(i)}$$

$\therefore$ the complementary function of (i) is

$$Z_C = e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x) \quad \text{...(ii)}$$

The particular integral of (i) is

$$Z_P = \frac{1}{(D-D'+2)(D+D'-1)}(e^{x-y}-x^2y)$$

$$= \frac{1}{(D-D'+2)(D+D'-1)}e^{x-y} - \frac{1}{(D-D'+2)(D+D'-1)}\cdot x^2y$$

$$= \frac{1}{(1+1+2)(1-1-1)}e^{x-y} - \frac{1}{D^2-D'^2+D+3D'-2}\cdot x^2y$$

$$= \frac{-1}{4}e^{x-y} + \frac{1}{2\left[1-\frac{D^2-D'^2+D+3D'}{2}\right]}\cdot x^2y$$

$$= \frac{-1}{4}e^{x-y} + \frac{1}{2}\left[1+\left(\frac{D^2-D'^2+D+3D'}{2}\right)+\left(\frac{D^2-D'^2+D+3D'}{2}\right)^2+\ldots\right]x^2y$$

$$= \frac{-1}{4}e^{x-y} + \frac{1}{2}\left[1+\frac{D}{2}+\frac{3D'}{2}+\left(\frac{1}{2}+\frac{1}{4}\right)D^2+\left(\frac{9}{4}-\frac{1}{2}\right)D'^2+\frac{3}{2}DD'+\frac{21}{8}D^2D'-\frac{23}{8}DD'^2\ldots\right]x^2y$$

$$= \frac{-1}{4}e^{x-y} + \frac{1}{2}\left[x^2y+xy+\frac{3}{2}x^2+\frac{3}{2}y+3x+\frac{21}{4}\right]$$

$$\therefore Z = Z_C + Z_P = e^{-2x}\phi_1(y+x)+e^x\phi_2(y+x)-\frac{1}{4}e^{x-y}+\frac{1}{2}\left[x^2y+xy+\frac{3}{2}x^2+\frac{3}{2}y+3x+\frac{21}{4}\right]$$

4.4 Solve the partial differential equation :

$$(D-2D')(D-D')^2z = e^{x+y}$$

(2011 : 12 Marks)

Solution:

The given equation is

$$(D-2D')(D-D')^2z = e^{x+y}$$

Replace D by m and D' by 1,

$$(m-2)(m-1)^2 = 0$$

$$\Rightarrow \quad m = 2, 1, 1$$

$\therefore$ Complementary function $= \phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary function.

$$\text{Particular integral} = \frac{1}{(D-D')^2}\cdot\frac{1}{(1-2)}\int e^u du,\ u = x+y$$

$$= \frac{1}{(D-D')^2}(-1)(e^{x+y})$$

$$= \frac{-x}{2(D-D')}e^{x+y} = \frac{-x^2}{2}e^{x+y}$$

$\therefore$ Complete solution is

$$z = \phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)-\frac{x^2}{2}e^{x+y}$$

4.5 Solve :

$$(D^2+DD'-6D'^2)z = x^2\sin(x+y)$$

where D and D' denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

(2013 : 15 Marks)

Solution:

$$(D^2 + DD' - 6D'^2)z = 0 \text{ is the homogeneous part.}$$

$\Rightarrow$ $$(D + 3D')(D - 2D')z = 0$$

Solution of $(D + 3D')z = 0$ is

$$\phi_1(y - 3x) = C_1$$

and that of $(D + 2D')z = 0$

is $$\phi_2(y + 2x) = C_2$$

where ϕ_1 and ϕ_2 are arbitrary function.

$\therefore$ Complementary function

$$\text{C.F.} = C_1\phi_1(y - 3x) + C_2\phi_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D + 3D')(D - 2D')}x^2 \sin(x + y)$$

$$= \frac{1}{D + 3D'}\left(\frac{1}{D - 2D'}x^2 \sin(x + y)\right)$$

$$= \frac{1}{D + 3D'}\int x^2 \sin(x + c - 2x)dx \text{ where } y + 2x = c$$

$$= \frac{1}{D + 3D'}\int x^2 \sin(c - x)dx$$

$$= \frac{1}{D + 3D'}\left[x^2 \cos(c - x) - \int 2x\cos(c - x)dx\right]$$

$$= \frac{1}{D + 3D'}\left[x^2 \cos(c - x) - \left\{-2x\sin(c - x) - \int -2\sin(c - x)dx\right\}\right]$$

$$= \frac{1}{D + 3D'}\left[(x^2 - 2)\cos(c - x) + 2x\sin(c - x)\right]$$

$$= \frac{1}{D + 3D'}\left[(x^2 - 2)\cos(x + y) + 2x\sin(x + y)\right] \text{ as } c = y + 2x$$

$$= \int[(x^2 - 2)\cos(x + c + 3x) + 2x\sin(x + c + 3x)]dx \text{ where } y - 3x = c$$

$$= \frac{1}{4}(x^2 - 2)\sin(c + 4x) - \frac{1}{4}\int 2x\sin(c + 4x)dx + \int 2x\sin(c + 4x)dx$$

$$= \frac{1}{4}(x^2 - 2)\sin(c + 4x) + \frac{3}{2}\int x\sin(c + 4x)dx$$

$$= \frac{1}{4}(x^2 - 2)\sin(c + 4x) + \frac{3}{2}\left[-\frac{x}{4}\cos(c + 4x) - \left(-\frac{1}{4}\right)\int \cos(c + 4x)dx\right]$$

$$= \frac{1}{4}(x^2 - 2)\sin(c + 4x) - \frac{3}{8}x\cos(c + 4x) + \frac{3}{32}\sin(c + 4x)$$

$$= \frac{1}{32}(8x^2 - 13)\sin(c + 4x) - \frac{3}{8}x\cos(c + 4x)$$

$$= \frac{1}{32}(8x^2 - 13)\sin(x + y) - \frac{3}{8}x\cos(x + y)$$

$\therefore$ Complete solution

$$y = C_1\phi_1(y + 2x) + C_2\phi_2(y - 3x) + \frac{1}{32}(8x^2 - 13)\sin(x + y) - \frac{3}{8}x\cos(x + y)$$

4.6 Solve the partial differential equation $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

(2014 : 10 Marks)

Solution:

The auxiliary of the given equation is

$$2m^2 - 5m + 2 = 0$$

$$\Rightarrow \quad m = \frac{1}{2}, 2$$

$$\therefore \quad \text{C.F.} = \phi_1(y + 2x) + \phi_2(2y + x)$$

ϕ_1, ϕ_2 being arbitrary functions.

Now,

$$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 24(y - x)$$

$$= 24 \times \frac{1}{2D^2 - 5DD' + 2D'^2}(y - x)$$

$$= \frac{24}{2(-1)^2 - 5(-1)(2) + 2 \times (2)^2} \iint v\,dv\,dv \text{ where } v = y - x$$

$$= \frac{24}{20}\int \frac{v^2}{2} dv$$

$$= \frac{6}{5}\left(\frac{v^3}{6}\right)$$

$$= \frac{1}{5}v^3 = \frac{1}{5}(y - 2)^3$$

Hence, the required general solution is

$$z = \phi_1(y + 2x) + \phi_2(2y + x) + \frac{1}{5}(y - x)^3$$

4.7 Solve : $(D^2 + DD' - 2D'^2)u = e^{x+y}$, where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$.

(2015 : 10 Marks)

Solution:

We first find solution for complementary function and then particular integral. Final solution would be sum of both.

Given, the equation is

$$(D^2 + DD' + 2D'^2)u = e^{x+y}$$

C.F. :

$$(D^2 + DD' - 2D'^2)u = 0$$

Auxiliary equation is

$$m^2 + m - 2m = 0$$

$$\Rightarrow \quad m^2 + 2m - m - 2m = 0$$

$$\Rightarrow \quad m(m + 2) - 1(m + 2) = 0$$

$$\Rightarrow \quad (m - 1)(m + 2) = 0$$

$$\therefore \quad m = 1$$

$$m = 2$$

So, solution is

$$u_c = \phi_1(y + x) + \phi_2(y - 2x)$$

P.I. :

$$u_p = \frac{1}{D^2 + DD' - 2D'} e^{x+y}$$

$$\Rightarrow \quad u_p = \frac{1}{(D+2D')(D-D')}e^{x+y}$$

$$= \frac{1}{(D-D')}\times\frac{1}{(D+2D')}e^{x+y}$$

$$= \frac{1}{(D-D')}\times\frac{1}{(1+2\times 1)}\int e^{x+y}d(x+y)$$

$$= \frac{1}{3}\times\frac{1}{D-D'}\times\int e^{x+y}d(x+y) = \frac{1}{3}\times\frac{1}{D-D'}e^{x+y}$$

$$\Rightarrow \quad u_p = \frac{1}{3}\times\frac{x^1}{1^1\cdot 1!}e^{x+y} = \frac{xe^{x+y}}{3}$$

$$\therefore \quad u = u_c + u_p = \phi_1(y+x)+\phi_2(y-2x)+\frac{xe^{x+y}}{3}$$

4.8 Solve the partial differential equation :

$$\frac{\partial^3 z}{\partial x^3}-\frac{2\partial^3 z}{\partial x^2\partial y}-\frac{\partial^3 z}{\partial x\partial y^2}+\frac{2\partial^3 z}{\partial y^3} = e^{x+y}$$

(2016 :15 Marks)

Solution:

Given equation is

$$\frac{\partial^3 z}{\partial x^3}-\frac{2\partial^3 z}{\partial x^2\partial y}-\frac{\partial^3 z}{\partial x\partial y^2}+\frac{2\partial^3 z}{\partial y^3} = e^{x+y}$$

or $\quad (D^3-2D^2D'-DD'^2+2D'^3)z = e^{x+y}$

where $\quad D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

$$\Rightarrow \quad (D^3-2D^2D'-DD'^2+2D'^3)z = e^{x+y}$$

$$\Rightarrow \quad \{D^2(D-2D')-D'^2(D-2D')\}z = e^{x+y}$$

$$\Rightarrow \quad (D-2D')(D^2-D'^2)z = e^{x+y}$$

$$\Rightarrow \quad (D-2D')(D-D')(D+D')z = e^{x+y}$$

C.F. : Auxiliary equation is

$$(m-2)(m-1)(m+1) = 0$$

$$\Rightarrow \quad m = -1, 1, 2$$

$$\therefore \quad z_c = \phi_1(y-x)+\phi_2(y+x)+\phi_3(y+2x)$$

P.I. :

$$z_p = \frac{1}{(D-2D')(D-D')(D+D')}\cdot e^{x+y}$$

$$\Rightarrow \quad z_p = \frac{1}{(D-D')(D-2D')(1+1)}\int e^{x+y}d(x+y)$$

$$\Rightarrow \quad z_p = \frac{1}{2(D-D')(1-2)}\int e^{x+y}d(x+y)$$

$$\Rightarrow \quad z_p = \frac{-1}{2(D-D')}e^{x+y}$$

$$\Rightarrow \quad z_p = \frac{-x^1}{2\cdot 1^1\cdot 1!}e^{x+y} = \frac{-xe^{x+y}}{2}$$

$\therefore$
$$z = z_c + z_p$$

$\Rightarrow$
$$z = \phi_1(y-x)+\phi_2(y+x)+\phi_3(y+2x)-\frac{xe^{x+y}}{2}$$

4.9 Solve : $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3 + \sin 2x$, where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial^2}{\partial x^2}$, $D'^2 = \frac{\partial^2}{\partial y^2}$.

(2017 : 10 Marks)

Solution:

Given equation can be written as :

$$(D-D')^2 z = e^{x+2y} + x^3 + \sin 2x \qquad \text{...(i)}$$

Its auxiliary equation $\Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore$
$$\text{C.F.} = \phi_1(y+x) + x\phi_2(y+x)$$

where ϕ_1 and ϕ_2 are arbitrary functions.

PI corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}e^{x+2y} = e^{x+2y}$$

PI corresponding to x^3

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2\left(1-\frac{D'}{D}\right)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3$$

$$= \frac{1}{D^2}\left(1+2\frac{D'}{D}+.....\right)x^3 = \frac{1}{D^2}x^3 = \frac{1}{D}\frac{x^4}{4} = \frac{x^5}{20}$$

PI corresponding to $\sin 2x$

$$= \frac{1}{(D-D')^2}\sin 2x = \frac{1}{(D-D')^2}\sin(2x+0y)$$

$$= \frac{1}{(2-0)^2}\iint \sin V dV dV, \text{ where } V = 2x + 0y$$

$$= -\frac{1}{4}\int \cos V dV = -\frac{1}{4}\sin V = -\frac{1}{4}\sin 2x$$

General Solution :
$$Z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y} + \frac{x^5}{20} - \frac{1}{4}\sin 2x$$

4.10 Solve the Partial Differential Equation

$$(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

where
$$D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

(2018 : 15 Marks)

Solution:

Given equation is

$$(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

C.F. : The auxiliary equation is :

$$(2D^2 - 5DD' + 2D'^2)z = 0$$

or
$$2m^2 - 5m + 2 = 0$$

$$\Rightarrow \quad 2m^2 - 4m - m + 2 = 0$$

$$\Rightarrow \quad (2m-1)(m-2) = 0$$

$$\Rightarrow \quad m = \frac{1}{2} \text{ or } m = 2$$

$$\therefore \quad z_c = \phi_1(y+2x) + \phi_2\left(y + \frac{1}{2}x\right)$$

P.I. :

$$z_p = \frac{1}{(2D-D')(D-2D')} \times [5\sin(2x+y) + 24(y-x) + e^{3x+4y}]$$

Now,

$$\frac{1}{2D-D'} \times \frac{1}{D-2D'} \times 5\sin(2x+y) = \frac{5}{(D-2D')} \times \frac{1}{2D-D'} \times \sin(2x+y)$$

$$= \frac{5}{D-2D'} \times \frac{1}{2\times 2-1} \times \int \sin(2x+y)d(2x+y)$$

$$= \frac{-5}{3(D-2D')} \times \cos(2x+y) = \frac{-5}{3} \cdot \frac{x'}{1!1'} \cos(2x+y)$$

$$= \frac{-5x}{3}\cos(2x+y) \qquad \text{...(i)}$$

Now,

$$\frac{1}{(2D-D')(D-2D')} 24(y-x) = \frac{24}{+3\times +3} \times \frac{(y-x)^3}{6}$$

$$= \frac{4}{9}(y-x)^3 \qquad \text{...(ii)}$$

Also,

$$\frac{1}{(2D-D')(D-2D')} e^{5x+4y} = \frac{1}{2\times -5} \times e^{3x+4y} = \frac{-1}{10} e^{3x+4y} \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$z_p = \frac{-5x}{3}\cos(2x+y) + \frac{4}{9}(y-x)^3 - \frac{1}{10}e^{3x+4y}$$

$$\therefore \quad z = z_c + z_p$$

$$\Rightarrow \quad z = \phi_1(y+2x) + \phi_2\left(y + \frac{x}{2}\right) - \frac{5x}{3}\cos(2x+y) + \frac{4}{9}(y-x)^3 - \frac{1}{10}e^{3x+4y}$$

5. Canonical Form

5.1 Reduce the following 2nd order partial differential equation into canonical form and find its general solution.

$$xu_{xx} + 2x^2 u_{xy} - u_x = 0$$

(2010 : 20 Marks)

Solution:

Let the characteristic variables be m and n.

Now, the given equation is

$$xu_{xx} + 2x^2 u_{xy} - 4_x = 0$$

$\therefore$ λ-characteristic equation is

$$x\lambda^2 + 2x^2\lambda = 0$$

$$\Rightarrow \quad \lambda(x\lambda + 2x^2) = 0$$

$$\Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = -2x$$

For $\lambda = 0$: $\quad \dfrac{dy}{dx} + 0 = 0$

$\Rightarrow \quad y = \text{Constant} = m$

For $\lambda = -2x$: $\quad \dfrac{dy}{dx} - 2x = 0$

$\Rightarrow \quad dy = 2x\,dx$

$\Rightarrow \quad y = x^2 + c \qquad$ (c is a constant)

$\Rightarrow \quad y - x^2 = c = n$

$\therefore \quad m = y$ and $n = y - x^2$

Now,
$$u_{xx} = \frac{\partial^2 u}{\partial m^2}\times\left(\frac{\partial m}{\partial x}\right)^2 + \left(\frac{\partial^2 u}{\partial x^2}\right)\times\left(\frac{\partial m}{\partial x}\right)^2 + \frac{2\partial^2 u}{\partial m\partial n}\times\frac{\partial m}{\partial x}\times\frac{\partial n}{\partial x} + \frac{\partial u}{\partial m}\times\frac{\partial^2 m}{\partial x^2} + \frac{\partial u}{\partial x}\times\frac{\partial^2 n}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial m^2}\times 0 + \frac{\partial^2 u}{\partial n^2}\times 4x^2 + \frac{2\partial^2 u}{\partial m\partial n}\times 0 + \frac{\partial u}{\partial m}\times 0 + \frac{\partial u}{\partial n}\times(-2)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 4x^2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} = u_{xx} \qquad \ldots(i)$$

$$u_{xy} = \frac{\partial^2 u}{\partial m^2}\times\frac{\partial m}{\partial x}\times\frac{\partial m}{\partial y} + \frac{\partial^2 u}{\partial x^2}\times\frac{\partial n}{\partial x}\times\frac{\partial n}{\partial y} + \frac{\partial u}{\partial m}\times\frac{\partial^2 m}{\partial x\partial y} + \frac{\partial u}{\partial n}\times\frac{\partial^2 n}{\partial x\partial y} + \frac{\partial^2 y}{\partial m\partial x}\left(\frac{\partial m}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial m}{\partial y}\frac{\partial y}{\partial x}\right)$$

$$= 0 + \frac{\partial^2 u}{\partial n^2}\times(-2x) + 0 + 0 + \frac{\partial^2 u}{\partial m\partial n}\times 1\times(-2n)$$

$$\Rightarrow \quad u_{xy} = -2x\frac{\partial^2 u}{dn^2} - 2x\frac{\partial^2 u}{\partial m\partial n} \qquad \ldots(ii)$$

$$u_x = \frac{\partial u}{\partial m}\times\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\times\frac{\partial n}{\partial x} = 0 - 2x\frac{\partial u}{\partial n} = -2x\frac{\partial u}{\partial n} \qquad \ldots(iii)$$

Using values in equations (i), (ii) and (iii), we get

$$\Rightarrow \quad \frac{\partial^2 u}{\partial m\partial n} = 0,$$ which is the canonical form of the equation.

Now,
$$\frac{\partial^2 u}{\partial m\partial n} = 0$$

$$\frac{\partial u}{\partial x} = \phi(x)$$

$$u = \int \phi(n)dx$$

$$= \int \phi(y - x^2)d(y - x^2)$$

which is the required solution.

5.2 Reduce the equation :

$$y\frac{\partial^2 z}{\partial x^2} + (x+y)\frac{\partial^2 z}{\partial x\partial y} + x\frac{\partial^2 z}{\partial y^2} = 0$$

to its canonical form where $x \neq y$.

(2013 : 10 Marks)

Solution:

Comparing the equation to

$$Rr + Ss + Tt = 0 \qquad \ldots(i)$$

$$R = y,\ S = (x+y),\ T = x$$

$$S^2 - 4RT = (x+y)^2 - 4xy = (x-x)^2 > 0 \text{ for } x \neq y$$

So, the equation is hyperbolic in form.

The λ-quadratic is

$$\lambda^2 R + S\lambda + T = 0 \Rightarrow \lambda^2 y + (x+y)\lambda + x = 0$$

$$\Rightarrow \quad (\lambda y + x)(\lambda + 1) = 0 \Rightarrow \lambda = -\frac{x}{y}, -1$$

So, the corresponding quadratic equations are :

$$\frac{dy}{dx} + \lambda = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{y} = 0 \text{ and } \frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \quad \frac{y^2}{2} - \frac{x^2}{z} = C_1 \text{ and } y - x = C_2$$

Let

$$u = \frac{1}{2}(y^2 - x^2);\ v = y - x$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\frac{\partial z}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial^2 z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial v}{\partial x}\right] - \left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]$$

$$= -\frac{\partial z}{\partial u} + x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$S = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-x\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -x\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial v}{\partial y}\right] - \left[\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right]$$

$$= -xy\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$$

$$= \frac{\partial z}{\partial u} + y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting in (i), we have

$$\{4xy - (x+y)^2\}\frac{\partial^2 z}{\partial u \partial v} - y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial u} = 0$$

$$(y-x)^2 \frac{\partial^2 z}{\partial u \partial v} + (y-x)\frac{\partial z}{\partial u} = 0$$

which is the canonical form.

[**Note** : In hyperbolic form only coefficient of $\frac{\partial^2 z}{\partial u \partial v}$ remains and this can be used].

5.3 **Reduce the equation $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to Canonical form.**

(2014 : 15 Marks)

Solution:

Let, the given equation,

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^2 \partial^2 z}{\partial y^2}$$

Re-writing the given equation becomes $r - x^2 t = 0$ ...(i)

comparing (i) with

$$Rx + 5s + T_t + f(x, y, z, p, q) = 0$$

We have, $R = 1, S = 0, T = -x^2$

Now, the λ–quadratic equation,

$$R\lambda^2 + 5\lambda + 7 = 0$$

$$\lambda^2 - x^1 = 0$$

$$\Rightarrow \qquad \lambda = Ix$$

Here, $\lambda_1 = x$ and $\lambda_2 = -x$ (real & distinct roots)

Hence, the characteristic equations.

$$\frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0$$

$$\frac{dy}{dx} + x = 0 \text{ and } \frac{dy}{dx} - x = 0$$

Integrating these two equations, we get

$$y + \left(\frac{x^2}{2}\right) = c_1$$

$$y - \left(\frac{x^2}{2}\right) = c_2$$

Hence in order to reduce (i) to canonical form, we change x, y to u, v by taking.

$$u = u + \frac{x^2}{2} \text{ and } v = y - \frac{x^2}{2} \qquad \text{...(ii)}$$

Now, $$P = \frac{\partial z}{\partial z} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x} = \frac{x\partial z}{\partial u} - x\frac{\partial z}{\partial v} \qquad \text{from (ii)}$$

$$P = x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \qquad \text{...(iii)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \text{from (ii), ...(iv)}$$

$$\therefore \qquad pq = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$pq = x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right) \qquad \text{from (iii)}$$

$$= x^2\left[\frac{\partial^2 z}{\partial u^2}-\frac{2\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = r$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) \qquad \text{from (iv)}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2} \qquad \text{from (iv)}$$

Put, the values of r & t in equation (i)

$$x^2\left[\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}-x^2\left[\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right] = 0$$

or
$$\frac{\partial^2 z}{\partial u\partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

or
$$\frac{\partial^2 z}{\partial u\partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

Which is the required canonical form of the given equation.

5.4 Reduce the second order partial differential equation :

$$x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$$

into canonical form. Hence, find its general solution.

(2015 : 15 Marks)

Solution:

Given : $x^2\frac{\partial^2 u}{\partial x^2}-2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 0$ ...(i)

Consider the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{...(ii)}$$

where,
$$r = \frac{\partial^2 u}{\partial x^2},\ s = \frac{\partial^2 u}{\partial x\partial y},\ t = \frac{\partial^2 u}{\partial y^2},\ p = \frac{\partial u}{\partial x},\ q = \frac{\partial u}{\partial y}$$

Comparing (i) and (ii), we get

$$R = x^2,\ s = -2xy,\ T = y^2,\ f(x, y, z, p, q) = x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

λ-characteristic equation for eqn. (i) is

$$\lambda^2 R + \lambda S + T = 0$$

$\Rightarrow$
$$x^2\lambda^2 - 2xy\lambda + y^2 = 0$$

$\Rightarrow$
$$(x\lambda - y)^2 = 0$$

$\Rightarrow$
$$x\lambda = y \Rightarrow \lambda = \frac{y}{x}$$

$\therefore$
$$\frac{dy}{dx}+\lambda = 0 \Rightarrow \frac{dy}{dx}+\frac{y}{x} = 0$$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x}$$

Integrating both the sides, we get

$$\log y = -\log x + \log a$$

$$\Rightarrow \quad xy = a, \text{ where } a \text{ is a constant.}$$

As roots of λ are equal, we consider b such that a and b are independent.

$$\therefore \quad b = \frac{x}{y}$$

So, $\quad a = xy$ and $b = \dfrac{x}{y}$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\times\frac{\partial a}{\partial x}+\frac{\partial u}{\partial b}\times\frac{\partial b}{\partial x} = \frac{\partial u}{\partial a}\times\frac{\partial(xy)}{\partial x}+\frac{\partial u}{\partial b}\times\frac{\partial(x/y)}{\partial x}$$

$$= y\frac{\partial u}{\partial a}+\frac{2}{y}\frac{\partial u}{\partial b} \quad \text{...(iii)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a}\times\frac{\partial a}{\partial y}+\frac{\partial u}{\partial b}\times\frac{\partial b}{\partial y} = x\frac{\partial u}{\partial a}+\left(\frac{-x}{y^2}\right)\times\frac{\partial u}{\partial b}$$

$$= x\frac{\partial u}{\partial a}-\frac{x}{y^2}\frac{\partial u}{\partial b} \quad \text{...(iv)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial a^2}\times\left(\frac{\partial a}{\partial x}\right)^2+\frac{\partial^2 u}{\partial b^2}\times\left(\frac{\partial b}{\partial x}\right)^2+\frac{\partial u}{\partial a}\times\frac{\partial^2 a}{\partial x^2}+\frac{\partial u}{\partial b}\times\frac{\partial^2 b}{\partial x^2}+2\frac{\partial^2 u}{\partial a\partial b}\times\left(\frac{\partial a}{\partial x}\right)\left(\frac{\partial b}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial a^2}\times y^2+\frac{1}{y^2}\frac{\partial^2 y}{\partial b^2}+0+0+y\times\frac{1}{y}\times 2\frac{\partial^2 u}{\partial a\partial b}$$

$$\frac{\partial^2 u}{\partial x^2} = y^2\frac{\partial^2 u}{\partial a^2}+\frac{1}{y^2}\frac{\partial^2 y}{\partial b^2}+2\frac{\partial^2 u}{\partial a\partial b} \quad \text{...(v)}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial a^2}\times\left(\frac{\partial a}{\partial x}\right)\left(\frac{\partial a}{\partial y}\right)+\frac{\partial^2 u}{\partial b^2}\times\left(\frac{\partial b}{\partial x}\right)\left(\frac{\partial b}{\partial y}\right)+\frac{\partial u}{\partial a}\times\frac{\partial^2 a}{\partial x\partial y}+$$

$$\frac{\partial u}{\partial b}\times\frac{\partial^2 b}{\partial x\partial y}+\frac{\partial^2 u}{\partial a\partial b}\left(\frac{\partial a}{\partial x}\cdot\frac{\partial b}{\partial y}+\frac{\partial a}{\partial y}\cdot\frac{\partial b}{\partial x}\right)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial a^2}\times xy+\frac{\partial^2 u}{\partial b^2}\times\frac{1}{4}\times\frac{-x}{y^2}+$$

$$\frac{\partial u}{\partial a}\times 1+\frac{\partial u}{\partial b}\times\frac{-1}{y^2}\times\frac{\partial^2 u}{\partial a\partial b}\times\left(x\times\frac{1}{y}-y\times\frac{x}{y^2}\right)$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x\partial y} = xy\frac{\partial^2 u}{\partial a^2}-\frac{x}{y^3}\frac{\partial^2 u}{\partial b^2}+\frac{\partial u}{\partial a}-\frac{1}{y^2}\frac{\partial u}{\partial b} \quad \text{...(vi)}$$

$$\frac{\partial^2 y}{\partial x\partial y} = \frac{\partial^2 u}{\partial a^2}\times\left(\frac{\partial a}{\partial y}\right)^2+\frac{\partial^2 u}{\partial b^2}\times\left(\frac{\partial b}{\partial y}\right)^2+\frac{\partial u}{\partial a}\times\frac{\partial^2 a}{\partial y^2}+\frac{\partial y}{\partial b}\times\frac{\partial^2 b}{\partial y^2}+$$

$$\frac{2\partial^2 y}{\partial a\partial b}\left(\frac{\partial a}{\partial y}\cdot\frac{\partial b}{\partial y}\right)$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial y^2} = x^2\frac{\partial^2 y}{\partial a^2}+\frac{x^2}{y^4}\frac{\partial^2 y}{\partial b^2}+0+\frac{2x}{y^3}\frac{\partial y}{\partial b}+\frac{2\partial^2 y}{\partial a\partial b}\times\frac{-x^2}{y^2}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial y^2} = x^2\frac{\partial^2 y}{\partial a^2}+\frac{x^2}{y^4}\frac{\partial^2 y}{\partial b^2}+\frac{2x}{y^3}\frac{\partial y}{\partial b}-\frac{2x^2}{y^2}\cdot\frac{\partial^2 u}{\partial a\partial b} \qquad \text{...(vii)}$$

Putting values from (iii), (iv), (v), (vi), (vii) in (i), we get

$$\left(y^2\frac{\partial^2 y}{\partial a^2}+\frac{1}{y^2}\frac{\partial^2 y}{\partial b^2}+\frac{2\partial^2 y}{\partial a\partial b}\right)-2xy\left(xy\frac{\partial^2 y}{\partial a^2}-\frac{x}{y^3}\frac{\partial^2 y}{\partial b^2}+\frac{\partial y}{\partial x}-\frac{1}{y^2}\frac{\partial y}{\partial b}\right)+$$

$$y^2\left(x^2\frac{\partial^2 y}{\partial a^2}+\frac{x^2}{y^4}\cdot\frac{\partial^2 u}{\partial b^2}+\frac{2x}{y^3}\cdot\frac{\partial y}{\partial b}-\frac{2x^2}{y^2}\cdot\frac{\partial^2 u}{\partial a\partial b}\right)+x\left(y\frac{\partial y}{\partial a}+\frac{1}{y}\cdot\frac{\partial u}{\partial b}\right)+y\left(x\frac{\partial y}{\partial a}-\frac{x}{y^2}\cdot\frac{\partial u}{\partial b}\right)=0$$

$$\Rightarrow x^2y^2\frac{\partial^2 y}{\partial a^2}+\frac{x^2}{y^2}\frac{\partial^2 y}{\partial b^2}+2x^2\frac{\partial^2 u}{\partial a\partial b}-2x^2y^2\frac{\partial^2 y}{\partial b^2}+\frac{2x^2}{y^2}\frac{\partial^2 y}{\partial b^2}-2xy\frac{\partial y}{\partial a}+\frac{2x}{y}\frac{\partial u}{\partial b}+x^2y^2\frac{\partial^2 y}{\partial a^2}+\frac{x^2}{y^2}\frac{\partial^2 y}{\partial b^2}+$$

$$\frac{2x}{y}\frac{\partial u}{\partial b}-2x^2\frac{\partial^2 u}{\partial a\partial b}+xy\frac{\partial u}{\partial a}+\frac{x}{y}\frac{\partial u}{\partial b}+xy\frac{\partial u}{\partial a}-\frac{x}{y}\frac{\partial u}{\partial b}=0$$

$$\Rightarrow \frac{\partial^2 y}{\partial a^2}(x^2y^2-2x^2y^2+x^2y^2)+\frac{\partial^2 y}{\partial b^2}\left(\frac{x^2}{y^2}+\frac{2x^2}{y^2}+\frac{x^2}{y^2}\right)+\frac{\partial^2 y}{\partial a\partial b}(2x^2-2x^2)+\frac{\partial u}{\partial a}(-2xy+xy+xy)$$

$$+\frac{\partial u}{\partial b}\left(\frac{2x}{y}+\frac{2x}{y}+\frac{x}{y}-\frac{x}{y}\right)=0$$

$$\Rightarrow \quad \frac{4x^2}{y^2}\frac{\partial^2 u}{\partial b^2}+\frac{4x}{y}\frac{\partial u}{\partial b} = 0$$

$$\Rightarrow \quad \frac{x^2}{y^2}\frac{\partial^2 u}{\partial b^2}+\frac{x}{y}\frac{\partial u}{\partial b} = 0$$

$$\Rightarrow \quad \frac{x^2}{y^2}\frac{\partial^2 u}{\partial b^2} = \frac{-x}{y}\frac{\partial y}{\partial b}$$

$$\Rightarrow \quad \frac{x}{y}\frac{\partial^2 y}{\partial b^2} = -\frac{\partial y}{\partial b}$$

$$\Rightarrow \quad b\cdot\frac{\partial}{\partial b}\left(\frac{\partial u}{\partial b}\right)+\frac{\partial y}{\partial b} = 0$$

Let $\dfrac{\partial u}{\partial b} = t$

$\therefore$ equation becomes

$$b\frac{\partial t}{\partial b}+t = 0 \Rightarrow \frac{\partial t}{\partial b} = \frac{-t}{b}$$

$$\Rightarrow \quad \frac{\partial t}{t} = \frac{-\partial b}{b}$$

Integrating both sides, we get

$$\Rightarrow \quad \log t = -\log b+\log\phi(a)$$

where $\phi(a)$ is an arbitrary function.

$$\therefore \quad u = \phi(a)\log b+\psi(a)$$

$$\Rightarrow \quad u = \phi(xy)\log b+\psi(xy)$$

$$\Rightarrow \quad u = \log\left(\frac{x}{y}\right)\cdot\phi(xy)+\psi(xy)$$

which is the required solution.

5.5 Reduce the equation :

$$y^2 \cdot \frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canomical form and hence solve it.

(2017 : 15 Marks)

Solution:

Given equation can be rewritten as :

$$y^2 r - 2xyS + x^2 t - \frac{y^2}{x}p - \frac{x^2}{y}q = 0 \qquad \text{...(ii)}$$

Comparing (ii) with

$$Rr + Ss + Tt + f(r, y, z, p, q) = 0$$

Here, $R = y^2$, $S = -2xy$, $T = x^2$

$$S^2 - 4RT = 0 \Rightarrow \text{PDE (i) is parabolic.}$$

The λ-quadratic equation, $R\lambda^2 + S\lambda + T = 0$ reduces to

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0$$

or

$$(y\lambda - x)^2 = 0 \Rightarrow \lambda = \frac{x}{y}, \frac{x}{y}$$

The corresponding characteristic equation is :

$$\frac{dy}{dx} + \frac{x}{y} = 0 \text{ or } xdx + ydy = 0$$

So that,

$$\frac{x^2}{2} + \frac{y^2}{2} = C_1,\ C_1 \text{ being arbitrary constant.}$$

Let us choose,

$$u = \frac{x^2}{2} + \frac{y^2}{2} \text{ and } v = \frac{x^2}{2} - \frac{y^2}{2} \qquad \text{...(iii)}$$

In such a way that u and v are independent functions of x, y as varified by Jacobian.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} x & y \\ x & -y \end{vmatrix}$$

$$= -2xy \neq 0$$

Now,

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ using (2)} \qquad \text{...(iv)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \qquad \text{...(v)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\}$$

$$= 1\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + x\cdot\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 5\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \text{ (by 3)} \quad ...(vi)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - \frac{2\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad ...(vii)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) \quad ...(viii)$$

Using above values in (ii) and simplifying

$$4x^2y^2\left(\frac{\partial^2 z}{\partial v^2}\right) = 0 \Rightarrow \frac{\partial^2 z}{\partial v^2} = 0 \quad ...(ix)$$

which is required canonical form.

Integrating (ix) partially w.r.t. v,

$$\frac{\partial z}{\partial v} = \phi(u)$$

Again integrating partially w.r.t. v,

$$z = v\cdot\phi(u) + \phi(u)$$

$$\therefore \quad z = \frac{(x^2 - y^2)}{2}\phi\left(\frac{x^2+y^2}{2}\right) + \psi\left(\frac{x^2+y^2}{2}\right)$$

5.6 Reduce the following second order partial differential equation to canonical form and find the general solution :

$$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} + 12x$$

(2019 : 20 Marks)

Solution:

Given second order partial differential equation :

$$\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y} + 12x \quad ...(1)$$

Let
$$u = \frac{x^2}{2} + y\,;\ v = x$$

$$p = \frac{\partial z}{\partial u}x + \frac{\partial z}{\partial v}(1) \quad ...(2)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + 0 \quad ...(3)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]$$

$$= \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] +$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u} + x\left[x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right] + \frac{\partial^2 z}{\partial u \partial v}x + \frac{\partial^2 z}{\partial v^2}$$

$$r = \frac{\partial z}{\partial u} + \frac{x^2 \partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ...(4)$$

$$S = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}$$

$$S = x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2} \qquad ...(6)$$

from (1)

$$\frac{\partial^2 z}{\partial x^2} - 2x\frac{\partial^2 z}{\partial x \partial y} + x^2\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y} + 12x$$

$$= \frac{\partial z}{\partial u} + \frac{x^2 \partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - 2x\left[x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right] + x^2\frac{\partial^2 z}{\partial u^2}$$

$$= \frac{\partial z}{\partial u} + 12x$$

$$2x^2\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - 2x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial v \partial u} = 12x$$

$$\frac{\partial^2 z}{\partial v^2} = 12x \Rightarrow \frac{\partial^2 z}{\partial v^2} = 12v \qquad [\because v = x]$$

$$D^2 = 12v \qquad ...(7)$$

$$D = \frac{\partial}{\partial}$$

Now for the general solution

$$D^2 - 12v = 0$$

$$(D^2 - 12)v = 0$$

$$D^2 - 12 = 0$$

$$D2 = 12 \text{ Þ } D = \pm\sqrt{12} \text{ or } 2\sqrt{3}$$

$\therefore$ $$\text{C.F.} = \phi_1(y + 2\sqrt{3}v) + \phi_2(y - 2\sqrt{3}v)$$

Since the R.H.S. is zero

$$\text{P.I.} = 0$$

$\therefore$ General solution

$$z = \text{C.F.} + \text{P.I.}$$

$$z = \phi_1(y + 2\sqrt{3}v + \phi_2(y - 2\sqrt{3}v)$$

or

$$u = \phi_1(y + 2\sqrt{3}x + \phi_2(y - 2\sqrt{3}x) \qquad [\because v = x]$$

which is the required general solution.

6. Application of PDE

6.1 A tightly stretched string has its end fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at time $t > 0$.

(2009 : 30 Marks)

Solution:

The one dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{...(1)}$$

Boundary condition (BC) are

$$y(0, t) = y(l, t) = 0 \quad \text{...(2)}$$

Initial condtion (IC) are :

$$y(x, 0) = f(x) = \mu x(l - x) \quad \text{...(3)}$$

$$y_t(x, 0) = 0 \quad \text{...(4)}$$

Let $Y(x, t)' = X(x)T(t)$ be the trial solution where ...(5)

$X(x)$ is a function of x only.

$T(t)$ is a function of t only.

Then clearly by (1)

$$X''T = XT''$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''}{T} = \eta \text{ (say)}$$

$$X'' - \eta X = 0 \quad \text{...(6)}$$

and $$T'' - \eta t = 0 \quad \text{...(7)}$$

Using (2) and (5), we get

$$y(0, t) = 0 \text{ and } y(l, t) = 0$$

Now, $$y(x, t) = X(x)T(t)$$

As $$X(0)T(t) = 0 \text{ and } X(l)T(t) = 0 \quad \text{...(8)}$$

Since $T(t) = 0$ leads to $y = 0$

So suppose that $T(t) \neq 0$

Then clearly equation (8) gives us that $X(0) = 0$ and $X(l) = 0$ ...(9)

Then (9) is modified boundary condition

Now solving (6) and (9), three cases arise clearly :

Case 1 : $\eta = 0$ then clearly solution (6) will be $X'' = 0$

So. $$\lambda''(x) = A$$

$$\Rightarrow \quad X(x) = Ax + B$$

where A and B are arbitrary constants.

Clearly, $X(0) = 0$ and $X(t) = 0$

So, $\Rightarrow B = 0$ and $Al = 0 \Rightarrow A = 0$ $(\lambda \neq 0)$

So, $$X(x) = 0$$

This leads to $X(x) = 0$ which does not satisfy (3) and (4).

So, reject $\eta = 0$ case.

Case 2 : η is positive, i.e., $\eta > 0$

Clearly, $\eta = \eta^2$: $(\lambda \neq 0)(\lambda > 0)$

So clearly, $$X'' - \lambda^2 X = 0$$

So. $$m^2 - \lambda^2 = 0$$

$\Rightarrow \quad m = \pm\lambda$

So, $\quad X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

Using (9), $X(0) = 0$ and $X(l) = 0$

$\Rightarrow \quad A + B = 0$ and $Ae^{\lambda l} + Be^{-\lambda l} = 0$

Now, $\quad B = -A$ as $-A(e^{\lambda l} - e^{-\lambda l}) = 0$

As $\quad e^{\lambda l} - e^{-\lambda l} = 0$

$\Rightarrow \quad A = 0$

So $\quad B = 0$

Reject this case, as again we get a trivial solution.

Case 3 : $\eta < 0$

So, $\quad \eta = -\lambda^2$ then

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

Now, $\quad X(0) = 0$ and $X(l) = 0$

So, $A = 0$ and $A\cos\lambda l + B\sin\lambda l = 0$

As clearly $B \neq 0$

$\because$ otherwise $X = 0$ which does not satisfy initial condition.

So $\quad \sin\lambda l = 0$

$\Rightarrow \quad \sin\lambda l = \sin n\pi$

$\Rightarrow \quad l = \dfrac{n\pi}{l}$

So, $\quad X(x) = B\sin\dfrac{n\pi}{l}x;\ n = 1, 2, ...$

Hence, non-zero solution of (6) is clearly given by

$$X_n(x) = B_a\frac{\sin n\pi x}{l}$$

Then $\quad T' - \eta c^2 T = 0$

$\Rightarrow \quad T' + \lambda^2 c^2 T = 0$

So, $\quad T' + \dfrac{n^2\pi^2}{l^2}c^2 T = 0$

So, $\quad T_n(t) = C_0\cos\dfrac{n\pi ct}{l} + D_0\dfrac{\sin n\pi ct}{l}$

$\therefore \quad X_n(x, t) = X_n(x)T_n(t)$

$$= B_n\frac{\sin n\pi x}{l}\left[C_n\frac{\cos n\pi ct}{l} + D_n\frac{\sin n\pi ct}{l}\right]$$

General solution

$$Y_i(x, t) = \sum_{n=1}^{\infty}\frac{\sin n\pi x}{l}\left[E_a\frac{\cos n\pi ct}{l} + F_a\frac{\sin n\pi ct}{l}\right] \quad ...(6)$$

Now differentiating (6) w.r.t. t, we get

$$Y_i(x, t) = \sum_{n=1}^{\infty}\left[-E_n\sin\frac{n\pi ct}{l}\left(\frac{n\pi c}{l}\right) + F_n\frac{\cos n\pi ct}{l}\left(\frac{n\pi c}{l}\right)\right]\frac{\sin n\pi x}{l}$$

Now $\quad Y_i(x, 0) = 0$

$\Rightarrow \quad 0 = \displaystyle\sum_{n=1}^{\infty}F_n\frac{n\pi c}{l}\frac{\sin n\pi c}{l}$

$\Rightarrow \quad F_n = 0$

So, clearly $$Y(x,t) = \sum_{n=1}^{\infty} E_n \frac{\cos n\pi ct}{l} \frac{\sin n\pi c}{l}$$

Now, $$Y(x,0) = \mu x(l-x)$$

So, $$E_n = \frac{2}{l}\int_0^l \mu x(l-x)\frac{\sin n\pi x}{l}dx$$

$$= \frac{2\mu}{l}\left\{(lx-x^2)\frac{-\frac{\cos n\pi x}{l}}{\frac{n\pi}{l}}\Bigg|_0^l + \frac{l}{n\pi}\int_0^l (l-2x)\frac{\cos n\pi x}{l}dx\right.$$

$$= \frac{2\mu}{n\pi}\left[(l-2x)\frac{\frac{\sin n\pi x}{l}}{\frac{n\pi}{l}}\Bigg| + \frac{4\mu l^2}{n^2\pi^2}\cdot\frac{l}{n\pi}\cdot - \frac{\cos n\pi x}{l}\right]_0^l dx$$

$$= \frac{-4\mu l^3}{n^2\pi^3}[\cos n\pi - 1]$$

$$= \begin{cases} 0 & \text{if } n = 2m \\ \dfrac{8\mu l^3}{(2m-1)^3\pi^3} & \text{if } n = 2m-1 \end{cases}$$

So, clearly we have

$$y(x,t) = \sum_{m-1}^{\infty}\frac{8\mu l^3}{(2m-1)^3\pi^3}\cdot\frac{\cos(2m-1)\pi ct}{l}\cdot\frac{\sin(2m-1)\pi x}{l}$$

6.2 Solve the following heat equation

$$u_t - u_{xx} = 0, \qquad 0 < x < 2,\ t > 0$$
$$u(0,t) = u(2,t) = 0 \qquad t > 0$$
$$u(x,0) = x(2-x), \qquad 0 \le x \le 2$$

(2010 : 20 Marks)

Solution:

Given, the equation is $u_t - u_{xx} = 0$ popularly known as "Heat Equation".

Let, $$u = X(n)\,T(t)$$

$\therefore$ Equation is $$XT' - X''T = 0$$

$\Rightarrow$ $$X'T = X''T$$

$\Rightarrow$ $$\frac{T'}{T} = \frac{X''}{X} = \mu$$

Case-1:

$$\mu = 0$$

In this case, $$\frac{X''}{X} = 0$$

$\Rightarrow$ $$X = Ax + B$$

Given, $$u(0,t) = u(2,t) = 0$$

$\therefore$ $B = 0,\ A = 0 \Rightarrow X = 0 \Rightarrow u = 0$ which is not possible. $\therefore\ \mu \neq 0$.

Case-2:

$$\mu = \lambda^2,\ \lambda \in R$$

$\therefore$ $$\frac{X''}{X} = \lambda^2$$

$\Rightarrow$ $$X'' - \lambda^2 X = 0$$

$\Rightarrow$ $$\frac{\partial^2 x}{\partial t^2} - \lambda^2 x = 0$$

$\Rightarrow$ $$x = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \text{ where } c_1, c_2 \text{ are constants.}$$

Given, $u(0, t) = u(2, t) = 0$

$\Rightarrow$ $$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{2\lambda} + c_2 e^{-2\lambda} = 0$$

$\Rightarrow$ $$c_1 = c_2 = 0$$

$\therefore$ $$x = 0 \Rightarrow u = 0, \text{ which is not possible.}$$

$\therefore$ $$\mu \neq \lambda^2$$

Case-3:

$$\mu = -\lambda^2, \quad \lambda \in R$$

$\therefore$ $$\frac{X''}{X} = -\lambda^2 \quad \Rightarrow X'' + \lambda^2 x = 0$$

$\Rightarrow$ $$\frac{\partial^2 x}{\partial x^2} + \lambda^2 x = 0$$

$\Rightarrow$ $$x = c_3 e^{i\lambda x} + c_4 e^{-i\lambda n} \qquad (c_3, c_4 \text{ are constant})$$

$$= c_5 \cos\lambda x + c_6 \sin\lambda x \qquad (c_5, c_6 \text{ are constant})$$

Given, $u(0, t) = u(2, t) = 0$

$\therefore$ $$c_5 = 0$$

and $$c_5 \cos 2\lambda + c_6 \sin 2\lambda = 0 \quad \Rightarrow c_6 \sin 2\lambda = 0$$

$\Rightarrow$ $$\sin 2\lambda = 0$$

$\Rightarrow$ $$2\lambda = n\pi$$

$\Rightarrow$ $$\lambda = \frac{n\pi}{2}$$

$\therefore$ $$X = c_6 \sin\left(\frac{n\pi x}{2}\right)$$

Also, $$\frac{T'}{T} = -\lambda^2 \quad \Rightarrow \frac{dT}{T} = -\lambda^2 dt$$

$\Rightarrow$ $$T = c_7 e^{-\lambda^2 t} \qquad (c_7 \text{ is constant})$$

$\therefore$ $$u = XT = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right) e^{-\left(\frac{n\pi}{2}\right)^2 t}$$

which is a Fourier series.

Now, $$u(x, 0) = x(2 - x) \qquad \text{(Given)}$$

$\therefore$ $$x(2 - x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{2}\right) \cdot 1$$

$\therefore$ $$E_n = \frac{2}{2} \int_0^2 x(2 - x) \sin\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^2 2x \sin\left(\frac{n\pi x}{2}\right) dx - \int_0^2 x^2 \sin\left(\frac{n\pi x}{2}\right) dx$$

$$\therefore \quad E_n = \int_0^2 2x\sin(\lambda x)dx - \int_0^2 x^2\sin(\lambda x)\,dx$$

$$\therefore \quad E_n = \left[\frac{-2x\cos(\lambda x)}{\lambda}\right]_0^2 + \int_0^2 \frac{2\cos\lambda x}{\lambda}dx - \left[\frac{-x^2\cos\lambda x}{\lambda}\right]_0^2 - \int_0^2 \frac{2x\cos\lambda x}{\lambda}dx$$

$$E_n = -\frac{-4}{\lambda}\cos 2\lambda + \frac{2}{\lambda^2}\sin\lambda x\Big|_0^2 + \frac{4}{\lambda}\cos 2\lambda - \frac{2x}{\lambda^2}\sin\lambda x\Big|_0^2 + \int_0^2 \frac{2}{\lambda^2}\sin\lambda x\,dx$$

$$\Rightarrow \quad E_n = -\frac{2}{\lambda^3}\cos\lambda x\Big|_0^2 = \frac{-2}{\lambda^3}\left[\cos\frac{n\pi}{2}\times 2 - 1\right] = \frac{-2}{\lambda^3}[\cos n\pi - 1]$$

$$\Rightarrow \quad E_n = \begin{cases} \dfrac{32}{(2m-1)^3\pi^3} & \text{If } n = 2m-1,\ m \in N \\ 0 & \text{If } n = 2m,\ m \in N \end{cases}$$

$$\therefore \quad u = \sum_{m=1}^{\infty} \frac{32}{(2m-1)^3\pi^3}\sin\left\{\frac{(2m-1)\pi x}{2}\right\} e^{-\left\{\frac{(2m-1)\pi}{2}\right\}^2 t}$$

Which is the required solution.

6.3 Solve : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, $0 \le x \le a$, $0 \le y \le b$ satisfying the boundary conditions

$$u(0, y) = 0,\ u(x, 0) = 0,\ u(x, b) = 0$$

$$\frac{\partial u}{\partial x}(a, y) = T\sin^3\frac{\pi y}{a}$$

(2011 : 20 Marks)

Solution:

Given : $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 \le x \le a,\ 0 \le y \le b \qquad \text{...(i)}$$

with boundary conditions.

$$u(0, y) = 0,\ u(x, 0) = 0,\ u(x, b) = 0$$

$$\frac{\partial u}{\partial x}(a, y) = T\sin^3\frac{\pi y}{a}$$

Suppose (i) has a solution of the form

$$u(x, y) = X(x)Y(y) \qquad \text{...(ii)}$$

where $X(x)$ and $Y(y)$ are functions of x and y respectively.

$\therefore$ From (i), we have

$$X''Y + YX'' = 0$$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} \qquad \text{...(iii)}$$

Since x and y are independent variables, therefore (iii) can only be true if each side is equal to the same constant μ (say).

$\therefore$ from (iii), $$X'' - \mu X = 0$$
$$Y'' + \mu Y = 0$$

Using the boundary conditions, (ii) gives

$$X(x)Y(0) = 0 \qquad \text{(using } u(x, 0) = 0\text{)}$$

$$X(x)Y(b) = 0 \qquad \text{(using } u(x, b) = 0\text{)}$$

$\Rightarrow$
$$Y(0) = 0 = Y(b) \qquad \text{...(iv)}$$

We have taken $X(x) \not\equiv 0$, since otherwise $u \equiv 0$, which does not satisfy the other boundary conditions.

Now, three cases arise :

Case I :
$$\mu = 0$$

$\therefore$
$$Y'' + \mu Y = 0 \Rightarrow Y'' = 0$$

$\Rightarrow$
$$Y(y) = Ay + B$$

Using (iv), we get
$$A = B = 0$$

$\Rightarrow$
$$Y(y) = 0$$

This leads to $u \equiv 0$, which does not satisfy the BCs.

Case II :
$$\mu = -\lambda^2, \lambda \neq 0$$

$\therefore$
$$Y'' + \mu Y = 0 \Rightarrow Y'' - \lambda^2 Y = 0$$

$\Rightarrow$
$$Y(y) = Ae^{\lambda y} + Be^{-\lambda y}, A \text{ and } B \text{ are arbitrary constants.}$$

Using $Y(0) = 0 = Y(b)$, we have
$$A + B = 0 \text{ and } Ae^{by} + Be^{-by} = 0$$

$\Rightarrow$
$$A = B = 0$$

$\therefore$ We reject this case also.

Case 3 :
$$\mu = \lambda^2, \lambda \neq 0$$

Then
$$Y'' + \mu Y = 0 \Rightarrow Y'' + \lambda^2 Y = 0$$

$\Rightarrow$
$$Y(y) = A\cos\lambda y + B\sin\lambda y$$

Using $Y(0) = 0 = Y(b)$, we have
$$A = 0 \text{ and } \sin\lambda b = 0$$

$\Rightarrow$
$$\lambda b = n\pi \Rightarrow \lambda = \frac{n\pi}{b}, n = 1, 2, 3, \ldots..$$

Hence, all the non-zero solutions of $Y'' + \mu Y = 0$ are given by
$$Y_n(y) = B_n \sin\left(\frac{n\pi y}{b}\right), n = 1, 2, 3, \ldots..$$

Again,
$$\mu = \lambda^2 \Rightarrow \mu = \frac{n^2\pi^2}{b^2}, n = 1, 2, 3, \ldots..$$

$\therefore$
$$X'' - \mu X = 0 \text{ becomes}$$
$$X'' - \frac{n^2\pi^2}{b^2}X = 0$$

whose general solution is
$$X_n(x) = C_n \cos h\left(\frac{n\pi x}{b}\right) + D_n \sin h\left(\frac{n\pi x}{b}\right), n = 1, 2, 3, \ldots..$$

$\therefore$
$$u_n(x, y) = X_n(x)\frac{1}{n}(y)$$
$$= \sin\left(\frac{n\pi y}{b}\right)\left[E_n \cos h\left(\frac{n\pi x}{b}\right) + F_n \sin h\left(\frac{n\pi x}{b}\right)\right], n = 1, 2, 3, \ldots..$$

where $E_n = B_n C_n$ and $F_n = B_n D_n$ are arbitrary constants.

$\therefore$
$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$$
$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right)\left[E_n \cos h\left(\frac{n\pi x}{b}\right) + F_n \sin h\left(\frac{n\pi x}{6}\right)\right] \qquad \text{...(v)}$$

Using $u(0, y) = 0$ in (v), we have

$$0 = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi y}{b}\right) \qquad [\because \sin h\, 0 = 0, \cos h\, 0 = 1]$$

which is Fourier Sine series. Hence, the constants E_n are given by

$$E_n = \frac{2}{b}\int_0^b 0 \cdot \sin\left(\frac{n\pi y}{b}\right) dy = 0$$

From (v), we have

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right)\left[\frac{n\pi F_n}{b}\cos h\left(\frac{n\pi x}{b}\right)\right]$$

Using $$\frac{\partial y}{\partial x}(0, y) = T\sin^3\frac{\pi y}{a}, \text{ we have}$$

$$T\sin^3\frac{\pi y}{a} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right)\cdot\frac{n\pi}{b}\cdot F_n \cos h\left(\frac{n\pi a}{b}\right)$$

which is Fourier Sine series.

$$\therefore \quad \frac{n\pi}{b}\cdot F_n \cos h\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b T\sin^3\frac{\pi y}{a}\cdot\sin\frac{n\pi y}{b}\, dy$$

$$\Rightarrow \quad F_n = \frac{2}{n\pi}\cdot\frac{T}{\cos h\left(\frac{n\pi a}{b}\right)}\cdot\int_0^b \sin^3\frac{\pi y}{a}\cdot\sin\frac{n\pi y}{b}\, dy, \; n = 1, 2, 3 \qquad \text{...(vi)}$$

Hence, the required solution is

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{b}\right)\cdot F_n \sin h\left(\frac{n\pi x}{b}\right) \text{ where } F_n \text{ is given by (vi).}$$

6.4 Obtain temperature distribution $y(x, t)$ in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Also it is given that $y(x, 0) = 1 - x$, $0 < x < 1$.

(2011 : 20 Marks)

Solution:

Let us place the bar along the x-axis with its one end (which is at 10°C) at origin and the other end at $x = 1$ (which is insulated so that flux $-k\frac{\partial y}{\partial x}$ is zero there, k being the thermal conductivity).

Here, we have to solve heat equation

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2} \qquad \text{...(i)}$$

With boundary condition

$$y_x(1, t) = 0, \; y(0, t) = 10 \qquad \text{...(ii)}$$

and initial conditions

$$u(x, 0) = 1 - x, \; 0 < x < 1 \qquad \text{...(iii)}$$

Let $$y(x, t) = u(x, t) + 10$$

$$\Rightarrow \quad u(x, t) = y(x, t) - 10 \qquad \text{...(iv)}$$

Using (iv), (i), (ii) and (iii) reduces to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{...(v)}$$

and
$$u_x(1, t) = 0, u(0, t) = 0 \quad \text{...(vi)}$$
$$u(x, 0) = y(x, 0) - 10) = -(x + 9) \quad \text{...(vii)}$$

Suppose that (v) has solutions of the form
$$u(x, t) = X(x)T(t) \quad \text{...(viii)}$$

∴ from (v),
$$XT' = kX''T$$

⇒
$$\frac{X''}{X} = \frac{T'}{kT} \quad \text{...(ix)}$$

Since x and t are independent variables, (ix) can only be true if each side is equal to the same constant, say μ.

∴ from (ix),
$$X'' - \mu X = 0 \quad \text{...(x)}$$

and
$$T' = \mu kT \quad \text{...(xi)}$$

Using (vi), (viii) gives
$$X'(1)T(t) = 0$$

and
$$X(0)T(t) = 0 \quad \text{...(xii)}$$

Since $T(t) = 0$ leads to $u \equiv 0$, so we suppose that $T(t) \neq 0$.

∴ from (xii),
$$X'(1) = 0$$

and
$$X(0) = 0 \quad \text{...(xiii)}$$

Now three cases arise :

Case 1 :
$$\mu = 0$$

Then from (x),
$$X'' = 0$$

⇒
$$X(x) = Ax + B \quad \text{...(xiv)}$$

⇒
$$X'(x) = A$$

Using (xiii), we have $A = 0$ and $B = 0$.

∴ from (xiv), $X(x) \equiv 0$ which leads to $u \equiv 0$.

So, we reject the case $\mu = 0$.

Case 2 : Let $\mu = \lambda^2, \lambda \neq 0$

Then from (x),
$$X'' - \lambda^2 X = 0$$

⇒
$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

⇒
$$X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$$

Using (xiii), we have
$$0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda}$$

and
$$0 = A + B$$

⇒
$$A = 0, B = 0$$

⇒
$$X(x) = 0 \Rightarrow u \equiv 0$$

So, we reject this case.

Case 3 :
$$\mu = -\lambda^2, \lambda \neq 0$$

∴ from (x),
$$X''(x) + \lambda^2 X(x) = 0$$

⇒
$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{...(xv)}$$

⇒
$$X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$$

Using (xiii), we have
$$0 = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$$

and
$$0 = A$$

⇒
$$A = 0, \cos\lambda = 0$$

⇒
$$\lambda = (2n-1)\frac{\pi}{2}, \; n = 1, 2, 3, ...$$

(We have taken $B \neq 0$, since otherwise $X(x) \equiv 0 \Rightarrow u \equiv 0$)

∴
$$\mu = -\lambda^2 = -(2n-1)^2\frac{\pi^2}{4} \quad \text{...(xvi)}$$

Hence, non-zero solutiosn $X_n(x)$ of (xv) are

$$X_n(x) = B_n \sin\left[(2n-1)\frac{\pi x}{2}\right]$$

Using (xvi), (xi) reduces to

$$\frac{T'}{T} = -\frac{(2n-1)^2\pi^2 k}{4} = -C_n^2 \quad \text{...(xvii)}$$

where
$$C_n^2 = \frac{(2n-1)^2\pi^2 k}{4} \quad \text{...(xvii)'}$$

Solving (xvii), we have

$$T_n(t) = D_n e^{-C_n^2 t}$$

$$\therefore \quad u_n(x, t) = X_n T_n$$

$$= E_n \sin\left[(2n-1)\frac{\pi x}{2}\right] \cdot e^{-C_n^2 t}$$

are solutions of (v), where $E_n = B_n D_n$ is another arbitrary constant.
In order to obtain a solution also satisfying (vii), we take a more general solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin(2n-1)_{\frac{\pi x}{2}} e^{-C_n^2 t} \quad \text{...(xviii)}$$

Putting $t = 0$ in (xviii), and using (vii), we get

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin(2n-1)_{\frac{\pi x}{2}}$$

$$\Rightarrow \quad -\int_0^1 (x+9)\sin(2m-1)_{\frac{\pi x}{2}}\,dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin(2n-1)_{\frac{\pi x}{2}} \cdot \sin(2m-1)_{\frac{\pi x}{2}}\,dx \quad \text{...(xix)}$$

But $\int_0^1 \sin(2x-1)_{\frac{\pi x}{2}} \cdot \sin(2m-1)_{\frac{\pi x}{2}}\,dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$

$$\therefore \quad -\int_0^1 (x+9)\sin(2n-1)_{\frac{\pi x}{2}}\,dx = E_n$$

$$\Rightarrow \quad E_n = -2\left[(x+9)\left(\frac{-\cos(2n-1)_{\frac{\pi x}{2}}}{(2n-1)_{\frac{\pi x}{2}}}\right) - 1\left(\frac{-\sin(2n-1)_{\frac{\pi x}{2}}}{(2n-1)^2\frac{\pi^2 x^2}{4}}\right)\right]_0^1$$

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \quad \text{...(xx)} \quad \left\{\begin{array}{l} \because \cos(2n-1)_{\frac{\pi}{2}} = 0 \\ \text{and } \sin(2n-1)_{\frac{\pi}{2}} = (-1)^{n-1} \end{array}\right\}$$

$\therefore$ The required solution is given by

$$y(x, t) = 10 + \sum_{n=1}^{\infty} E_n \sin(2n-1)_{\frac{\pi x}{2}} \cdot e^{-C_n^2 t}$$

where C_n and E_n are given by (xvii)' and (xx).

6.5 A string of length l is fixed at its ends. The string from the mid-point is pulled upto a height k and then released from rest. Find the deflection $y(x, t)$ of the vibrating string.

(2012 : 20 Marks)

Solution:

The one dimensional wave equation is :

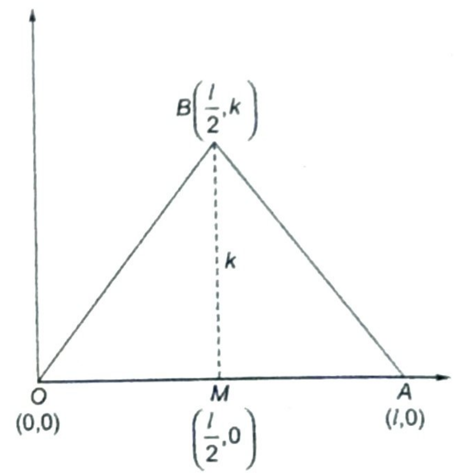
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 y}{\partial t^2} \quad \text{...(i)}$$

The given boundary conditions are :

$$y(0, t) = y(1, t) = 0 \;\forall\; t \geq 0 \quad \text{...(ii)}$$

Initial position of the string at $t = 0$ is made up of two straight line segments OB and BA and the string is released from rest.

The equation of OB is



$$y - 0 = \frac{k-0}{\frac{l}{2}-0}(x-0)$$

$$\Rightarrow \quad y = \frac{2k}{l}x,\; 0 \leq x \leq \frac{l}{2}$$

The equation of BA is

$$y - 0 = \frac{k-0}{\left(\frac{l}{2}-l\right)}(x-l)$$

$$\Rightarrow \quad y = \frac{2k(l-x)}{l},\; \frac{l}{2} \leq x \leq l$$

Hence, the initial displacement is given by

$$y(x, t) = f(x) = \begin{cases} \frac{2kx}{l}, & 0 \leq x \leq \frac{l}{2} \\ \frac{2k(l-x)}{l}, & \frac{l}{2} \leq x \leq l \end{cases}$$

And initial velocity $\left.\frac{\partial y}{\partial t}\right|_t = 0$

Taking $y(x, t) = X(x)T(t)$ as a trial solution and using the boundary conditions in (ii), the modified boundary conditions are : $X(0) = 0$, $X(l) = 0$ as $T(t) = 0$ leads to $y = 0$ which contradicts (iii) and (iv).

Also, $$\frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = \mu \quad \text{...(v)}$$

We get, $X'' - \mu X = 0$ and $T'' - C^2\mu T = 0$.

As $\mu = 0$ and $\mu > 0$ gives trivial solution, we reject it.

When $\mu < 0$, let $\mu = -\lambda^2, \lambda > 0$

From (v), $X(x) = A\cos\lambda x + B\sin\lambda x$

Now $X(0) = 0$ and $X(l) = 0 \Rightarrow A = 0$

and $B\sin\lambda l = 0$, as $b \neq 0 \Rightarrow \sin\lambda l = 0$

$\Rightarrow$ $\sin\lambda l = \sin n\pi$

$$\Rightarrow \quad \lambda = \frac{n\pi}{l}$$

$$\therefore \quad X_n = B_n \sin\left(\frac{n\pi x}{l}\right)$$

Again, from (v)

$$T'' - C^2\mu T = 0$$

$$\Rightarrow \quad T'' + C^2\lambda^2 T = 0$$

$$\Rightarrow \quad T'' + \frac{C^2n^2\pi^2T}{l^2} = 0$$

$$\Rightarrow \quad T_n(t) = C_n\cos\left(\frac{n\pi Ct}{l}\right) + D_n\sin\left(\frac{n\pi Ct}{l}\right)$$

where C_n and D_n are arbitrary constants.

$$\therefore \quad Y_n(x, t) = X_n(x)T_n(t)$$

$$\Rightarrow \quad Y_n(x, t) = \sum_{n=1}^{\infty}\left[E_n\cos\left(\frac{n\pi Ct}{l}\right) + F_n\sin\left(\frac{n\pi Ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right) \quad \text{...(v)}$$

As $\quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = g(x)$

Differentiating (v) w.r.t. t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left[E_n\left(-\sin\left(\frac{n\pi Ct}{l}\right)\right)\right]\frac{n\pi C}{l} +$$

$$F_n\left[\cos\left(\frac{n\pi Ct}{l}\right)\cdot\frac{n\pi C}{l}\right]\cdot\sin\frac{n\pi x}{l} \quad \text{...(vi)}$$

Putting $t = 0$ in (vi),

$$0 = \sum_{n=1}^{\infty}F_n\cdot\frac{n\pi C}{l}\sin\frac{n\pi x}{l}$$

By Fourier Series,

$$F_n = \frac{2}{n\pi C}\int_0^1 g(x)\sin\frac{n\pi x}{l}\,dx = 0 \qquad (\because g(x) = 0)$$

Now putting $t = 0$ in (v),

$$y(x, 0) = \sum_{n=1}^{\infty}E_n\cdot\sin\frac{n\pi x}{l}$$

where $\quad E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}2k(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0 & \text{if } n = 2m \\ \dfrac{8k}{(2m-1)^2\pi^2} & \text{if } n = (2m-1) \end{cases}$$

$\therefore$ the required solution is

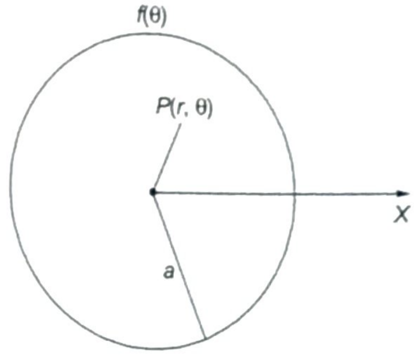
$$y(x, t) = \frac{8k}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m+1)^2}\sin\frac{(2m-1)\pi x}{l}\cos\left(\frac{(2m+1)\pi x}{l}\right)$$

6.6 The edge $r = a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution in steady state.

(2012 : 20 Marks)

Solution:

Consider a thin circular plate with insulated surface and radius 'a'.



The Laplace's equation in polar co-ordinates is

$$r^2\left(\frac{\partial^2 u}{\partial r^2}\right)+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \text{...(i)}$$

The steady state temperature $u(r, \theta)$ is the solution of equation (i) under the boundary condition,

$$u(a, \theta) = f(\theta), 0 \le \theta \le q\pi \qquad \text{...(ii)}$$

The required solution $u(r, \theta)$ must be periodic in θ and is finite when $r \to 0$.

Suppose, $u(r, \theta) = R(r)\,\Theta(\theta)$ ...(iii)

is a solution of (i), where R and Θ are functions of r and θ respectively.

$\therefore$ From (i), $r^2R''\Theta + rR'\Theta + R\Theta'' = 0$

or

$$\frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} \qquad \text{...(iv)}$$

Since the L.H.S. of (iv) is a function of r only and R.H.S. is a function of θ only, the two sides of (iv) must be equal to the same constant, say μ.

$\therefore$ From (iv),

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\mu R = 0 \qquad \text{...(v)}$$

and

$$\frac{d^2\Theta}{d\theta^2}+\mu\Theta = 0 \qquad \text{...(vi)}$$

From (v),

$$(r^2D^2 + rd - \mu)R = 0, D \equiv \frac{d}{dr} \qquad \text{...(vii)}$$

Let $r = e^z \Rightarrow$ $\quad z = \log r, D_1 = \dfrac{d}{dz}$

Then, $rD = D_1 \Rightarrow$ $\quad r^2D^2 = D_1(D_1 - 1)$

$\therefore$ from (vii), $\quad (D_1^2 - \mu)R = 0$ ...(viii)

Also let $D_2 \equiv \dfrac{d}{d\theta}$

$\therefore$ from (vi), $\quad (D_2^2 + \mu)\Theta = 0$ ...(ix)

Case I : $\mu = 0$

From (viii) and (ix),

$$\frac{d^2R}{dz^2} = 0 \text{ and } \frac{d^2\Theta}{d\theta^2} = 0$$

After solving, $\quad R(r) = A_1z + B_1 = A_1 \log r + B_1$

and $\quad \Theta(\theta) = C_1\theta + D_1$

$\therefore$ from (iii), a solution of (i) is

$$u(r,\theta) = (A_1 \log r + B_1)(C_1\theta + D_1) \quad \text{...(x)}$$

Since $u(r, \theta)$ is periodic in θ and is finite when $r \to 0$.

$$\therefore \quad A_1 = 0 \text{ and } C_1 = 0$$

$\therefore$ from (x),

$$u(r,\theta) = B_1D_1 = \frac{E_0}{2} \text{ (say)} \quad \text{...(xi)}$$

Case II : $\mu = \lambda^2, \lambda \neq 0$

$\therefore$ from (viii) and (ix),

$$(D_1^2 - \lambda^2)R = 0 \text{ and } (D_2^2 + \lambda^2)(-) = 0 \quad \text{...(xii)}$$

After solving, we get

$$R(r) = A_2e^{\lambda z} + B_2e^{-\lambda z} = A_2r^{\lambda} + B_2r^{-\lambda} \quad \text{...(xiii)}$$

and

$$(-)(\theta) = C_2 \cos \lambda\theta + D_2 \sin \lambda\theta$$

Again, since $u(r, \theta)$ is periodic in θ with perido 2π, we must take $\lambda = n$, $n = 1, 2, 3, \ldots$

$\therefore$ Using (xiii), (iii) $\Rightarrow$

$$u(r,\theta) = (A_2r^n + B_2r^{-n})(C_2 \cos n\theta + D_2 \sin \theta),\ n = 1, 2, 3 \quad \text{(xiv)}$$

Since $u(r, \theta)$ is finite when $r \to 0$, we must take $B_2 = 0$.

$\therefore$ From (xiv),

$$u(r,\theta) = A^2r^n(C_2 \cos n\theta + D_2 \sin n\theta)$$

$$\Rightarrow \quad u(r,\theta) = r^n(E_n \cos n\theta + F_n \sin n\theta) \quad \text{...(xv)}$$

where $E_n = A_2C_2$ and $F_n = A_2D_2$ are arbitrary constants.

From (xi) and (xv), the most general solution of (i) is

$$u(r,\theta) = \frac{E_0}{2} + \sum_{n=1}^{\infty} r^n(E_n \cos n\theta + F_n \sin n\theta) \quad \text{...(xvi)}$$

Taking $r = a$ and using (ii), we get

$$F(\theta) = \frac{E_0}{2} + \sum_{n=1}^{\infty} r^n(E_na^n \cos n\theta + F_na^n \sin n\theta) \quad \text{...(xvii)}$$

which is usual expansion of $f(\theta)$ as Fourier series in $(0, 2\pi)$

$$\therefore \quad E_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$E_n = \frac{1}{a^n\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

and

$$F_n = \frac{1}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta \quad \text{...(xviii)}$$

Hence, the required temperature is given by (xvi) wherein the constants E_0, E_n and F_n are given by (xviii).

Note : We cannot choose $\mu = -\lambda^2$ because it will lead to $(D_2^2 - \lambda^2)(-) = 0$ whose solution will not contain trigonometric functions and hence peridoic nature of $u(r, \theta)$ will not be attained.

6.7 **A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in equilibrium position. If it is set vibrating by giving each point a velocity $\lambda x(l - x)$ find the displacement of the string at any distance x from one end at any time.**

(2013 : 20 Marks)

Solution:

Let $u(x, t)$ be the displacement of particle at distance x from the end $x = 0$ at time t.

The motion of string is governed by one dimensional wave equation, i.e.,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{...(i)}$$

Subject to boundary conditions

$$u(0, t) = u(l, t) = 0 \quad \text{...(ii)}$$

and initial conditions

$$u(x, 0) = 0 \quad \text{...(iii)}$$

$$u + (x, 0) = \lambda x(l - x) \quad \text{...(iv)}$$

By separation of variables let

$$u(x, t) = X(x)T(t)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = X''T \text{ and } \frac{\partial^2 u}{\partial y^2} = XT''$$

Using this (i) becomes

$$X''T = \frac{1}{C^2}XT'' \Rightarrow \frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T}$$

as both sides are functions of independent variables they can only be equal if

$$\frac{X''}{X} = \frac{1}{C^2}\frac{T''}{T} = \mu \text{ (constant)}$$

Also $\quad X(0)T(t) = 0$ and $X(l)T(t) = 0$ (from (ii))

$\Rightarrow \quad X(0) = 0$ and $X(l) = 0$ as use $T(t) = 0$

which implies $u(x, t) = 0$ which does not satisfy (iv).

Three cases arises :

Case 1 : $\mu = 0$

$$X'' = 0 \Rightarrow X = Ax + B$$

$$X(0) = 0 \Rightarrow B = 0$$

and $\quad X(l) = 0 \Rightarrow A \cdot l = 0 \Rightarrow A = 0$

$\therefore \quad X(x) = 0$

But this is not possible as else $u(x, t) = 0$ which does not satisfy (iv).

So this case gives no solution.

Case 2 : $\mu = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X = Ae^{\lambda x} + Be^{-\lambda x}$$

$$X(0) = 0 \Rightarrow A + B = 0$$

$$X(l) = 0 \Rightarrow Ae^{\lambda l} + Be^{-\lambda l} = 0$$

which gives $A = B = 0 \Rightarrow X(x) = 0$ which is not possible.

So, again this case gives no solution.

Case 3 : $\mu = -\lambda^2$

$$X'' + \lambda^2 X = 0 \Rightarrow X = A\cos\lambda x + B\sin\lambda x$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(l) = 0 \Rightarrow B\sin\lambda l = 0 \Rightarrow \lambda l = n\pi$$

as if $B = 0 \Rightarrow X(x) = 0$ which is not possible.

$\therefore \quad \lambda = \frac{n\pi}{l}$

and $\quad X_n(x) = B_n \sin\frac{n\pi}{l}x$

Using $\quad \mu_n = -\lambda_n^2 = -\frac{n^2\pi^2}{l^2}$

$$T'' + C^2\lambda_n^2 T = 0$$

$$\Rightarrow \quad T_n(t) = C_n\cos\frac{Cn\pi t}{l} + D_n\sin\frac{Cn\pi t}{l}$$

From (iii), $X(x)T(0) = 0 \Rightarrow T(0) = 0$

as $X(x) = 0 \Rightarrow u(x, t) = 0$

$\therefore$ $T_n(0) = 0 \Rightarrow C_n = 0$

$\therefore$ $$T_n(t) = D_n \sin\frac{Cn\pi}{l}t$$

$\therefore$ $$u_n(x, t) = B_n D_n \sin\frac{n\pi x}{l}\sin\frac{Cn\pi t}{l}$$

$$= F_n \sin\frac{n\pi x}{l}\sin\frac{Cn\pi t}{l}$$

By principal of super-position

$$u(x, t) = \sum u_n(x,t)$$

$$= \sum F_n \sin\frac{n\pi x}{l}\sin\frac{Cn\pi t}{l} \text{ is a solution.}$$

$$\frac{\partial u}{\partial t} = \sum F_n \frac{Cn\pi}{l}\sin\frac{n\pi x}{l}\cos\frac{Cn\pi t}{l}$$

$$\frac{\partial u}{\partial t}(x,0) = \sum F_n \frac{n\pi C}{l}\sin\frac{n\pi x}{l}$$

$$= \lambda x(l-x)$$

$\Rightarrow$ $$F_n\frac{n\pi C}{l} = \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}dx$$

$$F_n = \frac{2\lambda}{n\pi C}\int_0^l (l\cdot x - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2\lambda}{n\pi C}\left\{\left[(lx-x^2)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l}\right]_0^l - \int (l-2x)\left(\frac{-l}{n\pi}\right)\cos\frac{n\pi x}{l}dx\right\}$$

$$= \frac{2\lambda}{n\pi C}\left\{\left[(l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l - \int(-2)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}dx\right\}$$

$$= \frac{2\lambda}{n\pi C}\left\{\frac{2l^2}{n^2\pi^2}\left(\frac{-l}{n\pi}\right)\left[\cos\frac{n\pi x}{l}\right]_0^l\right\}$$

$$= -\frac{4\lambda l^3}{n^4\pi^4 C}[(-1)^n - 1] = \frac{4\lambda l^3}{n^4\pi^4 C}[1-(-1)^n]$$

$$= \begin{cases} \dfrac{8\lambda l^3}{n^4\pi^4 C} & n \text{ is odd} \\ 0 & n \text{ is odd} \end{cases}$$

$\therefore$ $$u(x, t) = \frac{8\lambda l^3}{\pi^4 C}\sum\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi Ct}{l}$$

6.8 Find the deflection of a vibrating string (length = π, ends fixed, $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ corresponding to zero initial velocity and initial deflection

$$f(x) = k(\sin x - \sin 2x).$$

(2014 : 15 Marks)

Solution:

The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

As the end points of the string are fixed for all time

B.C. : $u(0, t) = 0$ and $u(p, t) = 0$ ...(ii)

I.C. : Initial velocity $= \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ (for $0 \le x \le \pi$) ...(iii)

and initial displacement $= u(x, 0) = K(\sin x - \sin 2x)$ ...(iv)

Suppose that (i) has the solution of the form

$$u(x, t) = X(x)T(t) \quad ...(v)$$

Substituting this value of u in (i), we have

$$XT'' = X''T$$

$\Rightarrow$ $\frac{X''}{X} = \frac{T''}{T} = \mu$ (say)

$\Rightarrow$ $X'' - \mu X = 0$ ...(vi)

and $T'' - \mu T = 0$ ...(vii)

Using (ii), (v) gives

$$X(0)T(t) = 0 \text{ and } X(\pi)T(t) = 0 \quad ...(viii)$$

Since $T(t) = 0$ leads to $u \equiv 0 \forall f$

So suppose that $T(t) \neq 0$.

Then (viii) gives $X(0) = 0$ and $X(\pi) = 0$...(ix)

We now solve (vi) under boundary conditions (ix). Three cases arise.

Case (i) : Let $\mu = 0$

Then solution of (ix) is

$$X(x) = Ax + B \quad ...(x)$$

Using B.C. (ix) and (x) gives $B = 0$, $A = 0$

$\Rightarrow$ $X(x) = 0$

This leads to $u \equiv 0$ which does not satisfy I.C. (iii) and (iv), so we reject $\mu = 0$.

Case (ii) : Let $\mu = \lambda^2$, $\lambda \neq 0$. Then the solution of (vi) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad ...(xi)$$

Using B.C. (ix), (xi) gives $A = 0$, $B = 0$

$\Rightarrow$ $X(x) = 0$

This leads to $u \equiv 0$ which does not satisfy (iii) and (iv). So reject $\mu = \lambda^2$.

Case (iii) : Let $\mu = -\lambda^2, \lambda \neq 0$

The solution of (vi) is $X(x) = A\cos\lambda x + B\sin\lambda x$ ...(xii)

Using B.C. (ix), (xii) gives

$$X(0) = 0 = A(t) + B(0) \Rightarrow A = 0$$

and $X(\pi) = 0 = 0 + B\sin\lambda\pi \Rightarrow B\sin\lambda\pi = 0$

$\Rightarrow$ $\sin\lambda\pi = 0$ $(\because B \neq 0)$

$$\Rightarrow \quad \sin\lambda\pi = 0$$
$$\Rightarrow \quad \lambda\pi = n\pi$$
$$\Rightarrow \quad \lambda = n,\ n = 1, 2, \ldots..$$

From (xii), we have

$$X(x) = B\sin nx,\ n = 1, 2, \ldots..$$

Hence, non-zero solution $X_n(x)$ of (vi) are given by

$$X_n(x) = B_n \sin nx \quad \ldots(\text{xiii})$$

From (vii), $\quad T'' - \mu T = 0 \Rightarrow T'' + \lambda^2 T = 0 \quad (\because \mu = -\lambda^2)$

$$\Rightarrow \quad T'' + n^2 T = 0 \quad (\because \lambda = x)$$

whose general solution is

$$T_n(t) = C_n \cos(nt) + D_n \sin(nt)$$

$$\therefore \quad u_n(x, t) = X_n(t)\, T_n(t)$$
$$= B_n \sin(nx)\,[C_n \cos(nt) + D_n \sin(nt)]$$
$$= (E_n \cos nt + F_n \sin nt)\sin(nx) \quad \ldots(\text{xiv})$$

are solutions of (i) satisfying (ii).

Here, $\quad E = B_n C_n$ and $F_n = B_n D_n$

In order to obtain a solution also satisfying (iii) and (iv), we consider more general solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x,t)$$

i.e.,
$$u(x, t) = \sum_{n=1}^{\infty} (E_n \cos nt + F_n \sin nt)\sin nx \quad \ldots(\text{xv})$$

Differentiating (xv) partially w.r.t. t, we get

$$\frac{\partial u}{\partial t} = \sum_{h=1}^{\infty} (-nE_n \sin nt + nF_n \cos nt)\sin nx \quad \ldots(\text{xvi})$$

Putting $t = 0$ in (xv) and (xvi) and using initial conditions (iii) and (iv), we get

$$(\text{xv}) \equiv u(x, 0) = K(\sin x - \sin 2x) = \sum_{h=1}^{\infty} E_n \sin nx \quad \ldots(\text{xvii})$$

$$(\text{xvi}) \equiv \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 = \sum_{n=1}^{\infty} nE_n \sin nx$$

where
$$F_n = \frac{2}{n}\int_0^{\pi} \sin nx\,dx = 0$$

From (xvii), we have

$$K(\sin x - \sin 2x) = \sum_{n=1}^{\infty} E_n \sin nx$$

Comparing the coefficient of like terms on both sides, we have

$$E_1 = 1,\ E_2 = -k \text{ and } E_3 = E_4 = \ldots\ldots = 0$$

$\therefore$ From (xv), we have

$$u(x, t) = E_1 \sin x \cos t - E_2 \sin 2x \cos 2t$$
$$= K\cos t \sin x - K\cos 2t \sin 2x$$
$$u(x, t) = K(\cos t \sin x - \cos 2t \sin 2x)$$

which is the required solution.

6.9 Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$, given that

(i) $u(x, 0) = 0$, $0 \le x \le 1$

(ii) $\frac{\partial u}{\partial t}(x,0) = x^2$, $0 \le x \le 1$

(iii) $u(0, t) = u(1, t) = 0$ for all t.

(2014 : 15 Marks)

Solution:

The vibration of the string is given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{...(i)}$$

As the ends points of the string are fixed for all time.

B.C. : $$u(0, t) = u(1, t) = 0 \quad \text{...(ii)}$$

I.C. : $$u(x, 0) = 0, 0 \le x \le 1 \quad \text{...(iii)}$$

and $$\frac{\partial u}{\partial t}(x,0) = x^2, 0 \le x \le 1 \quad \text{...(iv)}$$

Suppose that (i) has the solution of the form

$$u(x, t) = X(x)T(t) \quad \text{...(v)}$$

Substituting this value of u in (i), we have

$$XT'' = X''T$$

$\Rightarrow$ $$\frac{X''}{x} = \frac{T''}{T} = \mu \text{ (say)}$$

$\Rightarrow$ $$X'' - \mu X = 0 \quad \text{...(vi)}$$

$$T'' - \mu T = 0 \quad \text{(vii)}$$

Using (ii), (v) gives

$$X(0)T(t) = 0 \text{ and } X(1)T(t) = 0 \quad \text{...(viii)}$$

Since $T(t) = 0$ leads to $y \equiv 0 \ \forall t$

So suppose that $T(t) \neq 0$

Then (viii) gives $$X(0) = 0 \text{ and } X(1) = 0 \quad \text{...(ix)}$$

which are boundary conditions.

We now solve (vi) under boundary conditions (ix). Three cases arise.

Case (i) : Let $\mu = 0$

Then solution of (vi) is

$$X(x) = Ax + B \quad \text{...(x)}$$

Using B.C. (ix), (x) gives $B = 0$, $A = 0$

$\Rightarrow$ $$X(x) = 0$$

This leads to $u \equiv 0$ which does not satisfy I.C. (iii) and (iv). So we reject $\mu = 0$.

Case (ii) : Let $$\mu = \lambda^2, \lambda \neq 0,$$

then the solution of (vi) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{...(xi)}$$

Using B.C. (ix), (xi) gives $A = 0$, $B = 0$

$\Rightarrow$ $$X(x) = 0$$

This leads to $u \equiv 0$ which does not satisfy (iii) and (iv). So reject $\mu = \lambda^2$.

Case (iii) : Let $$\mu = -\lambda^2, \lambda \neq 0$$

The solution of (vi) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{...(xii)}$$

Using B.C. (xi), (xii) gives

$$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$$

and

$$X(1) = 0 = 0 + B\sin\lambda(1)$$

$$\Rightarrow B\sin\lambda = 0$$

$$\Rightarrow \sin\lambda = 0 \qquad (\because B \neq 0)$$

$$\Rightarrow \lambda = n\pi,\ n = 1, 2, 3, \ldots$$

From (xii), we have

$$X(x) = B\sin n\pi x,\ n = 1, 2, \ldots$$

Hence, non-zero solution $X_n(x)$ of (vi) are given by

$$X_n(x) = B_n\sin(n\pi x) \qquad \ldots\text{(xiii)}$$

From (vii),

$$T'' - \mu T = 0$$

$$\Rightarrow T'' + n^2\pi^2 T = 0$$

whose general solution is

$$T_n(t) = C_n\cos n\pi t + D_n\sin n\pi t$$

$$\therefore \quad u_n(x, t) = X_n(x)\,T_n(t)$$

$$= B_n\sin(n\pi x)[C_n\cos n\pi t + D_n\sin n\pi t]$$

$$= [E_n\cos n\pi t + F_n\sin n\pi t]\sin n\pi x \qquad \ldots\text{(xiv)}$$

are solutions of (i) satisfying (ii).

Here $E_n = B_nC_n$ and $F_n = B_nD_n$

In order to obtain a solution also satisfying (iii) and (iv), we consider more general solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x, t) = \sum_{n=1}^{\infty}(E_n\cos n\pi t + F_n\sin n\pi t)\sin n\pi x \qquad \ldots\text{(xv)}$$

Differentiating (xv) partially w.r.t. t, we get

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}(-E_n n\pi\sin n\pi t + F_n n\pi\cos n\pi t)\sin n\pi x \qquad \ldots\text{(xvi)}$$

Putting $t = 0$ in (xv) and (xvi) and using I.C. (iii) and (iv), we get

$$\text{(xv)} \equiv u(x, 0) = 0 = \sum_{n=1}^{\infty} E_n\sin n\pi x$$

where

$$E_n = \frac{2}{1}\int_0^1 (0)\sin(n\pi x)dr = 0$$

$$\text{(xvi)} \equiv \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} n\pi F_n\sin n\pi x = x^2$$

$$n\pi F_n = \frac{2}{1}\int_0^1 x^2\sin n\pi x\,dx$$

$$\Rightarrow \quad F_n = \frac{2}{n\pi}\int_0^1 x^2\sin(n\pi x)dx$$

$$= \frac{2}{n\pi}\left[x^2\left(\frac{-\cos n\pi x}{n\pi}\right) + \int 2x\frac{\cos n\pi x}{n\pi}dx\right]_0^1$$

$$= \frac{2}{n\pi}\left[-\frac{1}{n\pi}\cos n\pi + \frac{2}{n\pi}\left(x\cdot\frac{\sin n\pi x}{n\pi} + \frac{\cos n\pi x}{(n\pi)^2}\right)\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[-\cos n\pi + 2\left(0 + \frac{\cos n\pi}{n^2\pi^2} - 0 - \frac{1}{n\pi^2}\right)\right]$$

$$= \frac{2}{n^2\pi^2}\left[(-1)^{n+1} + 2\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]\right]$$

$$= \begin{cases} \frac{2}{n^2\pi^2}\left(1 - \frac{4}{n^2\pi^2}\right), & \text{if } n \text{ is odd} \\ -\frac{2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

$$F_n = \begin{cases} \frac{2}{(2m-1)^2\pi^2}\left(1 - \frac{4}{(2m-1)^2\pi^2}\right) & \text{if } n = 2m-1,\ m = 1, 2, \ldots \\ -\frac{2}{(2m)^2\pi^2}, & \text{if } n = 2m,\ m = 1, 2, \ldots \end{cases} \quad \ldots\text{(xvii)}$$

∴ The required displacement given by

$$u(x, t) = \sum_{n=1}^{\infty} F_n \sin n\pi t \sin n\pi x \text{ where } F_n \text{ is given by (xvii)}$$

6.10 Find the solution of initial value boundary problem

$$u_t - u_{xx} + u = 0;\ 0 < x < l,\ t > 0$$
$$u(0, t) = u(l, t) = 0;\ t \geq 0$$
$$u(x, 0) = x(l - x);\ 0 < x < l$$

(2015 : 15 Marks)

Solution:

Given equation is

$$u_t - u_{xx} + u = 0$$

and
$$u(0, t) = u(l, t) = 0 \quad \ldots\text{(i)}$$
$$u(x, 0) = x(l - x) \quad \ldots\text{(ii)}$$

Now, let
$$u = X(x)T(t)$$

∴ the equation becomes

$$XT' - X''T + XT = 0$$

⇒
$$X(T' + T) = X''T$$

⇒
$$\frac{T'}{T} + 1 = \frac{X''}{X} = \mu = \text{Constant} \quad \ldots\text{(iii)}$$

Case 1 : $\mu = 0$

⇒
$$\frac{X''}{X} = 0 \quad \text{(from (iii))}$$

Integrating it, we get

$$\frac{X'}{X} = A \quad (A \text{ is a constant})$$

Again integrating it, we get

$$X = Ax + B$$

∴
$$u = (Ax + B)T$$

Now, from (i),
$$u(0, t) = u(l, t) = 0$$

⇒
$$(A \times 0 + B)T = 0 \Rightarrow B = 0$$

Also,
$$(A \times l + B)T = 0 \Rightarrow A = 0$$

$\therefore$ $u = 0$ for $\forall$ x, t which is not true.

$\therefore$ $\mu \neq 0$ or $u = 0$ is not possible.

Case 2 : $\mu = \lambda^2$ where λ is a constant.

$\therefore$ equation becomes

$$\frac{X''}{X} = \lambda^2 \Rightarrow X'' - \lambda^2 X = 0$$

which gives $X = ae^{\lambda x} + be^{-\lambda x}$, where a & b are constants.

From (i), $a \cdot 1 + b \cdot 1 = 0$ at $(0, t)$

and $ae^{\lambda l} + be^{-\lambda l} = 0$ at (l, t)

which gives $a = b = 0 \Rightarrow u = 0 \Rightarrow$ not possible.

Case 3 : $\mu = -\lambda^2$

So, equation becomes

$$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X^2 = 0 \Rightarrow (D^2 + \lambda^2)X = 0$$

This gives $X = C\cos\lambda x + D\sin\lambda x$ ...(iv), C and D are constants.

Now, from (i)

$$u(0, t) = 0$$

$$\Rightarrow \quad 0 = C \times 1 + D \times 0 \Rightarrow C = 0$$

$$u(l, t) = 0$$

$$\Rightarrow \quad 0 = C \times \cos\lambda l + D\sin\lambda l$$

$$\Rightarrow \quad D\sin\lambda l = 0 \text{ as } C = 0$$

$$\Rightarrow \quad \sin\lambda l = 0$$

$$\lambda l = n\pi$$

or
$$\lambda = \frac{n\pi}{l} \quad \text{...(v)}$$

Also, from (iii)

$$\frac{T'}{T} + 1 = \mu = -\lambda^2$$

$$\Rightarrow \quad \frac{T'}{T} = -\lambda^2 - 1$$

$$\Rightarrow \quad \frac{\alpha T}{T} = (-\lambda^2 - 1)dt$$

Integrating both the sides, we get

$$T = Ee^{-(\lambda^2+1)t} \quad \text{...(v), where E is a constant}$$

$$\therefore \quad u = X(x)T(t)$$

$$\Rightarrow \quad u_n = (C\cos\lambda x + D\sin\lambda x)Ee^{-(\lambda^2+1)t} \quad \text{(from (iv) and (vi))}$$

$$\Rightarrow \quad u = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n^2\pi^2}{l^2}+1\right)t} \quad \text{...(vii)}$$

which is a fourier series.

Now, given $u(x, 0) = x(l - x)$

$$\Rightarrow \quad x(l - x) = \sum E_n \sin\left(\frac{n\pi x}{l}\right)$$

$\therefore$ $$E_n = \frac{2}{l}\int_0^l x(l-x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$\Rightarrow$ $$E_n = \frac{2}{l}\left[l\int_0^l x\sin\frac{n\pi x}{x}dx - \int_0^l x^2\sin\frac{n\pi x}{l}dx\right] \quad \text{...(viii)}$$

Let $$I_1 = \int_0^l x\sin\frac{n\pi x}{l}dx = \left.\frac{-x\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]_0^l + \int_0^l \frac{\cos\frac{n\pi x}{l}dx}{\frac{n\pi}{l}}$$

$$= \left[\frac{-l\cos\left(\frac{n\pi^2}{l}\right)-0}{\left(\frac{n\pi}{l}\right)}\right] + \frac{l}{n\pi}\frac{\left[\sin\frac{n\pi x}{l}\right]_0^l}{\frac{n\pi}{l}}$$

$$= \frac{-l^3}{n\pi}\cos n\pi + \frac{2l}{n\pi}\frac{\left[x\sin\frac{n\pi x}{l}\right]_0^l}{\frac{n\pi}{l}} - \frac{2l^2}{n^2\pi^2}\int_0^l \sin\frac{n\pi x}{l}dx$$

$\Rightarrow$ $$I_2 = \frac{-l^3}{n\pi}\cos n\pi + 0 + \frac{2l^2}{n^2\pi^2}\frac{\left[\cos\frac{n\pi x}{l}\right]_0^l}{\left(\frac{n\pi}{l}\right)}$$

$$I_2 = \frac{-l^3}{n\pi}\cos n\pi + \frac{2l^2}{n^2\pi^3}(\cos n\pi - 1)$$

$$E_n = \frac{2}{l}[lI_1 - I_2] \quad \text{(from (viii))}$$

$$= \frac{2}{l}\left[\frac{-l^3}{2\pi}\cos n\pi + \frac{l^3}{n\pi}\cos n\pi - \frac{2l^3}{n^3\pi^3}(\cos(n\pi)-1)\right]$$

$$= \frac{-4l^2}{n^3\pi^3}(\cos n\pi - 1)$$

If n is even, then $$E_n = 0$$

If n is odd, then $$E_n = \frac{8l^2}{n^3\pi^3}$$

Let $n = (2m-1)$ where $m \in N$

$\therefore$ $$E_n = \frac{8l^2}{(2m-1)^3\pi^3}$$

Putting this value in eqn. (vii), we get

$$u = \sum_{m=1}^{\infty}\frac{8l^2}{(2m-1)^3\pi^3}\sin\frac{(2m-1)\pi n}{l}e^{-\left\{\frac{(2m-1)^2\pi^2}{l^2}+1\right\}t}$$

or $$u = \frac{8l^2}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\left(\frac{(2m-1)\pi x}{l}\right)e^{-\left\{\frac{(2m-1)^2\pi^2}{l^2}+1\right\}t}$$

which is the required solution.

6.11 Find the temperature $u(x, t)$ in a bar of silver of length 10 cm and constant cross-section of area 1 cm^2. Let density $\rho = 10.6$ g/cm^3, thermal conductivity $K = 1.04$ cal/(cm sec °C) and specific heat $\sigma = 0.056$ cal/g °C. The bar is perfectly isolated laterally, with ends kept at 0°C and initial temperature $f(x) = \sin(0.1\,\pi x)$°C. Note that u(x, t) follows the heat equation $u_t = c^2 u_{xx}$, where $c^2 = K/(\rho\sigma)$.

(2016 : 20 Marks)

Solution:

Given, heat equation is

$$u_t = C^2 u_{xn}$$

where $u(x, t)$ is temperature distribution.

and $$C^2 = \frac{K}{\rho\sigma}$$

and $$u(0, t) = 0 \quad \ldots(i)$$

$$u(10, t) = 0 \quad \ldots(ii)$$

Let $u = XT$, where X is function of X and T is function of t.

$\therefore$ equation becomes $$XT' = C^2 X'' T$$

$$\Rightarrow \quad \frac{T'}{C^2 T} = \frac{X''}{X} = \mu \quad \ldots(iii)$$

Case 1 : Let $\mu = 0$

$\therefore$ from (iii) $$\frac{X''}{X} = 0$$

$$\Rightarrow \quad X = Ax + B \quad \text{(where A and B are constants)}$$

From (i) and (ii), $$0 = A\cdot 0 + B$$

and $$0 = A\cdot 10 + B$$

$$\Rightarrow \quad A = B = 0 \Rightarrow X = 0$$

$\therefore u = 0 \ \forall\ x, t$ which is not possible.

So, $$\mu \neq 0$$

Case 2 : Let $$\mu = \lambda^2 \quad (\lambda \neq 0)$$

$\therefore$ from (iii) $$\frac{X''}{X} = \lambda^2 \Rightarrow X'' - \lambda^2 X = 0$$

$$\Rightarrow \quad X = C_1 e^{\lambda x} + C_2 e^{-\lambda x} \quad (C_1, C_2 \text{ are constants})$$

From (i) and (ii)

$$0 = C_1 + C_2$$

$$0 = C_1 e^{10\lambda} + C_2 e^{-10\lambda}$$

$$\Rightarrow \quad C_1 = C_2 = 0 \Rightarrow X = 0$$

$\Rightarrow u = 0 \ \forall\ (x, t)$ which is not possible.

So, $$\mu \neq \lambda^2$$

Case 3 : Let $$\mu = -\lambda^2 \quad (\lambda \neq 0)$$

from (iii), $$\frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0$$

$$\Rightarrow \quad X = C_3 \cos \lambda x + C_4 \sin \lambda x \quad \text{(where } C_3 \text{ and } C_4 \text{ are constants)}$$

From (i) and (ii)

$$0 = C_3\cdot 1 + C_4\cdot 0 \Rightarrow C_3 = 0$$

and $$0 = C_3 \cos \lambda \cdot 10 + C_4 \sin \lambda \cdot 10$$

$$\Rightarrow \quad C_4 \sin(\lambda\cdot 10) = 0$$

$$\Rightarrow \quad 10\lambda = n\pi$$

$$\Rightarrow \qquad \lambda = \frac{n\pi}{10}$$

$$\therefore \qquad X = C_4 \sin\left(\frac{n\pi x}{10}\right) \qquad \text{...(iv)}$$

Also,
$$\frac{T'}{C^2 T} = -\lambda^2 \qquad \text{(from (iii))}$$

$$\Rightarrow \qquad \frac{T'}{T} = -\lambda^2 C^2$$

$$\Rightarrow \qquad T = C_5 e^{-C^2\lambda^2 t} \qquad \text{...(v) (where } C_5 \text{ is a constant)}$$

From (iv) and (v)

$$u = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) e^{-C^2\lambda^2 t} \qquad \text{...(vi)}$$

Now, given at $t = 0$

$$u(x, 0) = f(x) = \sin(0.1\pi x)$$

$$\therefore \qquad \sin(0.1\pi x) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right), \text{ which is a foureir series.}$$

$$\sin(0.1\pi x) = E_1 \sin(0.1\pi x) + \sum_{n=2}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right)$$

Comparing LHS and RHS, we get

$$E_1 = 1,\ E_n = 0 \text{ for } x \geq 2$$

$$\therefore \qquad u(x, t) = \sin(0.1\pi x) e^{-C^2 \times (0.1\pi)^2 t} \qquad \left(\lambda = \frac{n\pi}{10}\right)$$

$$C^2 = \frac{K}{\rho\sigma} = \frac{1.04}{0.056 \times 10.6} = 1.752$$

$$\therefore \qquad u(x, t) = \sin(0.1\pi x) e^{-1.752 \times (0.1\pi)^2 t}$$

$$\Rightarrow \qquad u(x, t) = e^{-0.1729t} \sin(0.1\pi x)$$

6.12 Given the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2};\ t > 0 \text{ where } C^2 = \frac{T}{m} \qquad \text{...(i)}$$

T is the constant tension in the string and m is the mass per unit length of the string.

(i) Find the appropriate solution of the above wave equation.

(ii) Find also the solution under the conditions $y(0, t) = 0$, $y(l, t) = 0$, for all t and $\left.\frac{dy}{dt}\right|_{t=0} = 0$, $y(x, 0) = a \sin\frac{\pi x}{l}$, $0 < x < l$, $a > 0$.

(2017 : 20 Marks)

Solution:

(i) Assume that the solution of (i) is of the form

$$y(x, t) = X(t)T(t)$$

Then,
$$\frac{\partial^2 y}{\partial t^2} = X \cdot T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X''T$$

Substituting in (i), $XT'' = C^2X''T$

i.e., $$\frac{X''}{X} = \frac{1}{C^2}\cdot\frac{T''}{T} \quad \text{...(ii)}$$

Clearly, the L.H.S. of (ii) is a function of x only and the RHS is a function of t only. Since x and t are independent variables, (ii) can hold good if each side is equal to a constant k (say).

Then, (ii) leads to the ODEs :

$$\frac{d^2y}{dx^2} - kX = 0 \quad \text{...(iii)}$$

$$\frac{d^2T}{dt^2} - kC^2T = 0 \quad \text{...(iv)}$$

Solving (iii) and (iv), we get

(i) When $k > 0$ and equal to p^2, then

$$X = C_1e^{px} + C_2e^{-px};\ T = C_3e^{Cpt} + C_4e^{-Cpt}$$

(ii) When $k < 0$ and equal to $-p^2$, then

$$X = C_5 \cos px + C_6 \sin px$$
$$T = C_7 \cos Cpt + C_8 \sin Cpt$$

(iii) When k = 0, $X = C_9x + C_{10};\ T = C_{11}t + C_{12}$

Thus, various possible solutions of wave equation (i) are

$$y = (C_1e^{px} + C_2e^{-px})(C_3e^{Cpt} + C_4e^{-Cpt}) \quad \text{...(v)}$$
$$y = (C_3 \cos px + C_6 \sin px)(C_7 \cos Cpt + C_8 \sin Cpt) \quad \text{...(vi)}$$
$$y = (C_9x + C_{10})(C_{11}t + C_{12}) \quad \text{...(vii)}$$

Of these three solutions, we choose that solution which is consistent with the physical nature of the problem. As the problem deals with vibrations, y must be a periodic function of x and t.

Hence, required solution is :

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt) \quad \text{...(viii)}$$

(ii) Given boundary conditions :

$$y(0, t) = 0 \quad \text{...(ii)}$$
$$y(l, t) = 0 \quad \text{...(iii)}$$
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{...(iv)}$$
$$y(x, 0) = a \sin\frac{\pi x}{l} \quad \text{...(v)}$$
$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt)$$

(ii) gives, $y(0, t) = C_1(C_3 \cos Cpt + C_4 \sin Cpt) = 0$

For this to be true for all time, $C_1 = 0$

$$y(x, t) = C_2 \sin px(C_3 \cos Cpt + C_4 \sin Cpt) \quad \text{...(vi)}$$
$$\frac{\partial y}{\partial t} = C_2 \sin px[C_3(-Cp \sin Cpt) + C_4(p \cos Cpt)]$$

By (iv) $\Rightarrow$ $$\left.\frac{\partial y}{\partial t}\right|_{t=0} = C_2 \sin px \cdot (C_4Cp) = 0$$

$\Rightarrow$ $C_2C_4Cp = 0$

If $C_2 = 0$, (vi) will give trivial solution, $y(x, t) = 0$.

$\therefore$ the only possibility is $C_4 = 0$.

(vi) becomes, $$y(x, t) = C_2C_3 \sin px \cos Cpt \quad \text{...(vii)}$$

By (iii), $y(l, t) = C_2C_3 \sin pl \cos Cpt = 0 \ \forall\ t$

Since C_2 and $C_3 \neq 0$, $\therefore \sin pl = 0$

$$\Rightarrow \quad pl = n\pi$$

$$\therefore \quad p = \frac{n\pi}{l}, n \in .....$$

Hence, (iii) reduces to

$$y(x, t) = C_2C_3 \sin\frac{n\pi x}{l}\cos\frac{n\pi Ct}{l}$$

By condition (v), $$y(x, 0) = C_2C_3 \sin\frac{n\pi x}{l} = a\sin\frac{\pi x}{l}$$

which will get satisfied if we take $C_2C_3 = a$ and $n = 1$.

Hence, the required solution is

$$y(x, t) = a\sin\frac{\pi x}{l}\cdot\cos\frac{\pi Ct}{l}$$

6.13 A thin annulus occupies the region $0 < a \le r \le b$, $0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0°C while along the outer edge, the temperature held at $T = k\cos\theta/2$, where k is a constant. Determine the temperature distribution in the annulus.

(2018 : 20 Marks)

Solution:

Let temperature distribution = $T(r, \theta)$ at any point $P(r, \theta)$ in the annular region.

Heat equation for annulus is given by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \theta^2}\right) = 0 \quad ...(i)$$

or $$r^2\frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0 \quad ...(i)$$

$$\therefore \quad T(a, \theta) = 0 \text{ and } T(b, \theta) = k\cos\theta/2 \text{ (given)} \quad ...(ii)$$

Clearly, $T(r, \theta)$ is periodic of period 2π.

Now, let $$T(r, \theta) = R(r)H(\theta)$$

where, $R(r)$ and $H(\theta)$ and functions of r and θ respectively (principle of superposition).

Putting value of T in eq. (i), we get

$$r^2R''H + rR'H + RH'' = 0$$

$$\Rightarrow \quad \frac{r^2R'' + rR'}{R} = \frac{-H''}{H} = \mu = \text{Constant} \quad ...(iii)$$

or $$r^2R'' + rR' - \mu R = 0 \text{ and } H'' + \mu H = 0$$

or $$(r^2D^2 + rD - \mu)R = 0 \text{ and } (D_1^2 + \mu)H = 0$$

Here, $$D = \frac{d}{dr} \text{ and } D_1 = \frac{d}{d\theta}$$

Consider $$(r^2D^2 + rD - \mu)R = 0$$

Let $$r = e^z$$

$\therefore$ equation becomes

$$(D_2(D_2 - 1) + D_2 - \mu)R = 0,\ D_2 = \frac{d}{dz}$$

$$\Rightarrow \quad (D_2^2 - \mu)R = 0 \quad ...(iv)$$

Consider $$H'' + \mu H = 0$$

or $$(D_1^2 + \mu)H = 0 \quad ...(v)$$

Case 1 : $\mu = 0$

Solving (iv), we get $R = c_1 z + c_2$ (c_1, c_2 are constants)

and $H = c_3\theta + c_4$ (c_3, c_4 are constants)

$\therefore$ $T = RH = (c_1 z + c_2)(c_3\theta + c_4)$

as T is periodic in θ, $\therefore$ $c_3 = 0$

So, the equation becomes

$$T = (c_1 \log r + c_2)c_4 = \frac{1}{2}(a_0 \log r + b_0) \quad \text{...(vi)} (a_0, b_0 \text{ are constants})$$

Case 2 : $\mu = -\lambda^2, \lambda \neq 0$

Equation (v) becomes $(D_1^2 - \lambda^2)H = 0$

but this solution gives H which is not peridoic. So $\mu = -\lambda^2$ is not possible.

Case 3 : $\mu = \lambda^2, \lambda \neq 0$

then eqn. (iv) gives $(D_1^2 - \lambda^2)R = 0$

$$\Rightarrow \quad R = c_5 e^{\lambda z} + c_6 e^{-\lambda z} \quad (c_5, c_6 \text{ are constants})$$

$$= c_5 r^{\lambda} + c_6 r^{-\lambda}$$

Eqn. (v) gives $H = c_7 \cos\lambda\theta + c_8 \sin\lambda\theta$ (c5, c7 are constants)

As T is periodic with period 2π, $\lambda = n = 1$

$$\therefore \quad T(r, \theta) = (c_5 r^n + c_6 r^{-n})(c_7 \cos n\theta + c_8 \sin n\theta) \quad \text{...(vii)}$$

From case (1), (2) and (3), T forms Fourier series.

$$T(r, \theta) = \frac{1}{2}(a_0 \log r + b_0) + \sum_{n=1}^{\infty}[(a_n r^n + b_n r^{-n})\cos n\theta + (c_n r^n + d_n r^{-n})\sin n\theta \quad \text{...(viii)}$$

where a_n, b_n, c_n, d_n are constants.

Using conditions in (ii), we get

$$0 = \frac{(a_0 \log a + b_0)}{2} + \sum_{n=1}^{\infty}[(a_n a^n + b_n a^{-n})\cos n\theta + (c_n a^n + d_n a^{-n})\sin n\theta] \quad \text{...(ix)}$$

and

$$k\cos\frac{\theta}{2} = \frac{1}{2}[a_0 \log b + b_0) + \sum_{n=1}^{\infty}[(a_n b^n + b_n b^{-n})\cos n\theta + (c_n b^n + d_n b^{-n})\sin n\theta] \quad \text{...(x)}$$

From (ix) and (x)

$$a_0 \log a + b_0 = \frac{1}{\pi}\int_0^{2\pi} 0\, d\theta = 0$$

And $a_0 \log b + b_0 = 0$

$$\therefore \quad a_0 = b_0 = 0$$

Also,

$$a_n a^n + b_n a^n = \frac{1}{\pi}\int_0^{2\pi} 0(\cos n\theta)d\theta = 0 \Rightarrow b_n = -a_n a^{2n} \quad \text{...(xi)}$$

and

$$a_n b^n + b_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} k\cos\frac{\theta}{2}\cos n\theta\, d\theta$$

$$= \frac{k}{2\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)^q}{n+\frac{1}{2}} + \frac{\sin\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= 0$$

$$\Rightarrow \quad a_n b^n + b_n b^{-n} = 0$$

or $$a_n b^n - a_n a^{2n} b^{-n} = 0 \quad \text{(from (xi))}$$

$\Rightarrow$ $$a_n = 0 = b_n$$

Now, $$c_n b^n + dn b^{-n} = \frac{k}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)\theta + \sin\left(n-\frac{1}{2}\right)\theta\right]d\theta$$

$$= \frac{-k}{2\pi}\left[\frac{\cos\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\frac{2n}{\left(n^2-\frac{1}{4}\right)}$$

$\therefore$ $$c_n b^n + d_n b^{-n} = \frac{8kn}{\pi(4n^2-1)} \quad \text{...(xii)}$$

Now, we have $$c_n a^n + d_n a^{-n} = 0$$

$\Rightarrow$ $$d_n = -c_n a^{2n} \quad \text{...(xiii)}$$

$\therefore$ from (xii)

$$c_n b^n - c_n a^{2n} b^{-n} = \frac{8kn}{\pi(4n^2-1)}$$

$\Rightarrow$ $$c_n[b^n - b^{-n}a^{2n}] = \frac{8kn}{\pi(4n^2-1)}$$

$\Rightarrow$ $$c_n = \frac{8kn}{\pi(4n^2-1)} \times \frac{1}{(b^n - b^{-n}a^{2n})}$$

$$T(r,\theta) = \sum_{n=1}^{\infty}(c_n r^n + d_n r^{-n})\sin n\theta$$

$$= \sum_{n=1}^{\infty}(c_n r^n - c_n a^{2n} r^{-n})\sin n\theta \quad \text{(from (xiii))}$$

$\Rightarrow$ $$T(r,\theta) = \sum_{n=1}^{\infty} c_n(r^n - a^{2n}r^{-n})\sin n\theta$$

$$= \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\cdot\frac{[r^n - a^{2n}r^{-n}]}{b^n - b^{-n}a^{2n}}\sin n\theta$$

$\therefore$ $$T(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\cdot\frac{[r^n - a^{2n}r^{-n}]}{(b^n - b^{-n}a^{2n})}\sin(n\theta)$$

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