MAINS TEST SERIES-2019

TEST NUMBER-11

Section-A

Ques: 1) a) Let T be the linear operator on C^2 which is represented in the standard ordered basis by the matrix.

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find the minimal polynomial for T.

Solution:- Given Matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The characteristic polynomial for T is

$$\det(A - xI) = \begin{bmatrix} -x & -1 \\ 1 & -x \end{bmatrix}$$

$$\boxed{|A - xI| = x^2+1}$$

The characteristic values of T are given by $x^2+1=0$ i.e. $x = \pm i$ are the characteristic value of T.

These roots also satisfy the minimal polynomial $p(x)$ for T and so $p(x)$ is divisible by x^2+1.

Hence, $\boxed{p(x) = x^2+1}$ is the minimal polynomial for T.

It is easy to verify that $A^2 + I = 0$.

Ques: 1(b)) If 'A' be an n-rowed non-singular matrix X be an $n \times 1$ matrix, the system of equations $AX = B$ has a unique solution.

Solution:-

Proof:- If A be an n-rowed non-singular matrix, the ranks of the matrices A and [A B] are both 'n'. Therefore, the system of equations $AX = B$ is consistent i.e., possesses a solution.

Pre-multiplying both sides of $AX = B$ by A^{-1} we have;

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

is a solution of the equation $AX = B$.

To show that the solution is unique, let us suppose that X_1 and X_2 be two solutions of

$$AX = B.$$

Then $AX_1 = B$, $AX_2 = B$

$$\Rightarrow AX_1 = AX_2$$

$$\Rightarrow A^{-1}AX_1 = A^{-1}AX_2$$

$$\Rightarrow IX_1 = IX_2$$

$$\Rightarrow \boxed{X_1 = X_2}$$

Hence, the solution is unique.

Ques: 1(c)) Let $f: R^2 \to R$ be defined by setting

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & ; \text{ if } (x,y) \neq (0,0) \\ 0 & ; \text{ if } (x,y) = (0,0) \end{cases}$$

Show that 'f' possesses partial derivatives at (0,0) but is not differentiable at the point.

Solution:-

Given the function $f: \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & ; \text{ if } (x,y) \neq (0,0) \\ 0 & ; \text{ if } (x,y) = (0,0) \end{cases}$$

Then, we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = 0$$

and $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = 0$$

Hence, f possesses partial derivatives at (0,0)

Now, let if possible, f is differentiable at (0,0)

Then;

$$f(h,k) - f(0,0) = Ah + Bk + \left(\sqrt{h^2+k^2}\right)^2 g(h,k)$$

(by definition)

where; $A = f_x(0,0)$

$B = f_y(0,0)$

and $g(h,k) \to 0$ as $(h,k) \to (0,0)$

so, $$\frac{h^2 k}{h^4+k^2} = g(h,k).$$

Putting $k = mh^2$ and taking $h \to 0$, we get

$$\lim_{h\to 0} g(h, mh^2) = \frac{h^2 \cdot h^2 m}{h^4 + m^2 h^4} = \frac{\cancel{h^4} \cdot m}{\cancel{h^4}(1+m^2)}$$

$$\lim_{h\to 0} g(h, mh^2) = \frac{m}{1+m^2}$$

$$\boxed{\therefore \lim_{h\to 0} g(h, k) = \frac{m}{1+m^2}}$$

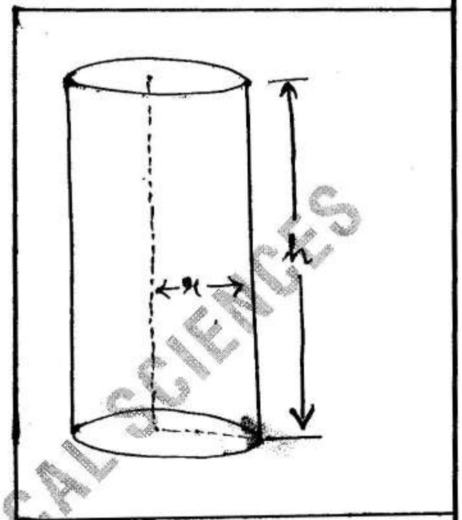
which depends on m.

This is a contradiction.

So, our assumption that 'f' is differentiable is wrong. Thus, we conclude that 'f' is not differentiable at the point $(0,0)$.

Ques: 1(d) Show that the height of an open cylinder of given surface and greatest volume is equal to radius of its base.

Solution:-



Let 'r' be the radius of the cylinder and 'h' be its height.

Again, let 'S' be the surface area and 'V' be the volume of the cylinder.

Then;

$$S = 2\pi r h + \pi r^2 \quad \text{——— (1)}$$

$$h = \frac{S - \pi r^2}{2\pi r} \quad \text{——— (2)}$$

and;

$$V = \pi r^2 h$$

$$V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r} \right) \quad \text{— using (2)}$$

$$\boxed{V = \frac{1}{2}\left[S r - \pi r^3 \right]} \quad \text{——— (3)}$$

Differentiating equation (3) both sides w.r.t 'r', we get

$$\boxed{\frac{dV}{dr} = \frac{1}{2}\left(S - 3\pi r^2 \right)} \quad \text{——— (4)}$$

here S is fixed and

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

Now, for the greatest volume, V, we have

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \quad \frac{1}{2}\left[S - 3\pi r^2\right] = 0 \quad \text{— using (4)}$$

$$\frac{1}{2}S - \frac{3}{2}\pi r^2 = 0$$

$$\Rightarrow \quad \boxed{S = 3\pi r^2}$$

$$\Rightarrow \quad 2\pi r h + \pi r^2 = 3\pi r^2 \quad [\text{using (1)}]$$

$$\cancel{2}\cancel{\pi}\cancel{r}h = \cancel{2}\cancel{\pi}r^{\cancel{2}}$$

$$\boxed{h = r}$$

Also, differentiating eqⁿ (4) both sides w.r.t r, we get

$$\frac{d^2V}{dr^2} = \frac{1}{2}(0 - 6\pi r) = -3\pi r < 0$$

So; V is the greatest volume when $\boxed{h = r}$

Thus, we conclude that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

Ques: 1) e) Show that the straight lines whose direction cosines are given by

$$al+bm+cn=0 \; ; \; fmn+gnl+hlm=0$$

are perpendicular if $f/a+g/b+h/c=0$

are parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch}=0$

Solution:-

Let, the direction cosine's of the two lines be (l_1, m_1, n_1) and (l_2, m_2, n_2).

But, the given relations are

$$al+bm+cn=0 \quad \text{and}$$
$$fmn+gnl+hlm=0$$

Eliminating 'n' between these relations, we get

$$fm[-(al+bm)/c]+gl[-(al+bm)/c]+hlm=0$$

$$\Rightarrow -afml-bm^2f-agl^2-bglm+chlm=0$$

Divide the above equation by m^3

$$\Rightarrow -af\frac{l}{m}-bf-ag\left(\frac{l}{m}\right)^2-bg\left(\frac{l}{m}\right)+ch\left(\frac{l}{m}\right)=0$$

$$\Rightarrow ag\left(\frac{l}{m}\right)^2+(af+bg-ch)\left(\frac{l}{m}\right)+bf=0 \quad \text{—(1)}$$

If the roots are l_1/m_1 and l_2/m_2, then

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \text{Product of the roots} = \frac{bf}{ag}$$

$$\Rightarrow \frac{l_1 l_2}{bf}=\frac{m_1m_2}{ag} \Rightarrow \frac{l_1l_2}{f/a}=\frac{m_1m_2}{g/b}=\frac{n_1n_2}{h/c}$$

(by symmetry).

Now, if the lines are perpendicular, then

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow (f/a) + (g/b) + (h/c) = 0$$

If the lines are parallel, then their DC's must be the same, i.e., the roots of (1) must be equal.

So, the condition for same is "$B^2 = 4AC$"

i.e; $(af + bg - ch)^2 = 4ag \cdot bf$

$$\Rightarrow af + bg - ch = \pm 2\sqrt{ag} \cdot \sqrt{bf}$$

$$\Rightarrow af + bg \pm 2\sqrt{af}\sqrt{bg} = ch = (\sqrt{ch})^2$$

$$= (\sqrt{af} \pm \sqrt{bg})^2 = (\sqrt{ch})^2$$

$$= \sqrt{af} \pm \sqrt{bg} = \pm\sqrt{ch}$$

$$\Rightarrow \boxed{\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0}$$

Hence, proved

Ques:-2(a)(i) Define the linear span, $L(S)$, of a subset 'S' of a vector space $V(F)$. What is $L(S)$ if (i) $S=\{0\}$ if (ii) $S=\phi$.

Let $V=R^3(R)$, $S=\{\alpha_1=(1,1,0), \alpha_2=(0,-1,1), \alpha_3=(1,0,1)\}$.
Prove that $(a,b,c)\in L(S)$ if and only if $a=b+c$.

Solution:-

(i) Definition of the linear span:-

Let $V(F)$ be a vector space and S be any non-empty subset of V. Then the linear span of 'S' is the set of all linear combinations of finite sets of elements of 'S' and is denoted by $L(S)$.
So, we have

$$L(S)=\{a_1\alpha_1+a_2\alpha_2+\cdots\cdots+a_n\alpha_n\}$$

where $\alpha_1,\alpha_2\cdots\cdots\alpha_n$ is any arbitrary finite subset of 'S' and $a_1,a_2\cdots\cdots a_n$ is any arbitrary finite subset of F.

Case I→ If $S=\{0\}$

Then, clearly the linear span of S

$$\boxed{L(S)=L(\{0\})=\{0\}}$$

case II→ If $S=\phi$

Then, the linear span of S

$$\boxed{L(S)=\{0\}}$$

which is the smallest subspace.

Now, given that $V = \mathbb{R}^3(\mathbb{R})$,

and $S = \{\alpha_1 = (1,1,0), \alpha_2 = (0,-1,1), \alpha_3 = (1,0,1)\}$ — ①

Claim: $(a,b,c) \in L(S)$ iff $a = b + c$

Let $(a,b,c) \in L(S)$

Again let $(\alpha, \beta, \gamma) \in V = \mathbb{R}^3(\mathbb{R})$

Then, we can write

$$(a,b,c) = \alpha\alpha_1 + \beta\alpha_2 + \gamma\alpha_3$$

$$(a,b,c) = \alpha(1,1,0) + \beta(0,-1,1) + \gamma(1,0,1)$$

[Using ①]

$$\Rightarrow (a,b,c) = (\alpha+\gamma, \alpha-\beta, \beta+\gamma)$$

So; $\alpha + \gamma = a$ — ②

$\alpha - \beta = b$ — ③

$\beta + \gamma = c$ — ④

Putting the value of β from ③ & ④, we get

$$\boxed{\alpha + \gamma = b + c} \quad \text{——} \ ⑤$$

from ② and ⑤, we can write

$$\boxed{a = b + c}$$

Conversly,

Let $a = b + c$

Suppose (a,b,c) be any element of $V = \mathbb{R}^3(\mathbb{R})$

Then, it can be written in the linear combination. Such that

$(a,b,c) = a\alpha_1 + b\alpha_2 + c\alpha_3$; where $\alpha_1, \alpha_2, \alpha_3 \in S$

$(a,b,c) = a(1,1,0) + b(0,-1,1) + c(1,0,1)$

$$\Rightarrow \boxed{(a,b,c) \in L(S)}$$

Ques: 2(a)ii) Let W be the subspace of $\mathbb{R}^4$ generated by vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$. Find a basis and dimension of W. Extend this basis of W to a basis of $\mathbb{R}^4$.

Solution:-

Given; $W = \{(1,-2,5,-3), (2,3,1,-4), (3,8,-3,-5)\} \subseteq \mathbb{R}^4$

i.e.
$$W = \begin{Bmatrix} [1 & -2 & 5 & 3] \\ [2 & 3 & 1 & -4] \\ [3 & 8 & -3 & -5] \end{Bmatrix}$$

Then, we have.

$$W = \begin{bmatrix} 1 & -2 & 5 & 3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & 3 \\ 0 & 7 & 9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$\therefore$ The basis is $\{(1,-2,5,3)$ and $(0,7;9,2)\}$ and the dimension of $W = 2$.

In particular, the original three given vectors are linearly dependent. Since $\mathbb{R}^4$ is 4-dimensional vector space.

$\therefore$ We require for linear Independent vectors which include the above two vectors.

$\therefore$ The vectors $(1,-2,5,-3)$ $(0,7,-9,2)$ $(0,0,1,0)$ and $(0,0,0,1)$ are linear Independent over $\mathbb{R}^4$. (Since, they form an echelon matrix).

$\therefore$ These vectors from a basis of $\mathbb{R}^4$.

$\therefore$ It is an extension of the basis of W.

Ques: 2 (b) (i) Evaluate: $\iint_D x\sin(x+y)\,dx\,dy$, where D is the region bounded by $0 \le x \le \pi$ and $0 \le y \le \frac{\pi}{2}$.

Solution:-

Given; D is the region bounded by $0 \le x \le \pi$ and $0 \le y \le \pi/2$.

Let $x = u - v$ and $y = v$

so that $x + y = u$.

Then $dx = du$ and $dy = dv$

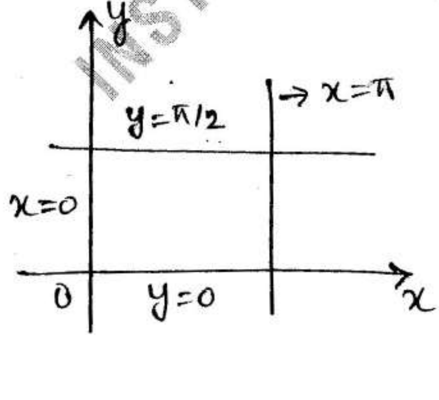
Also, the Jacobian.

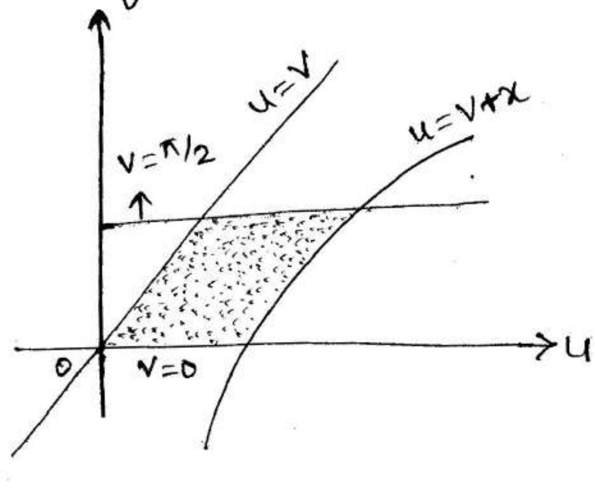
$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow J = 1$$

So, we can write

$$\iint_D x\sin(x+y)\,dx\,dy = \int_{x=0}^{\pi}\int_{y=0}^{\pi/2} x\sin(x+y)\,dx\,dy$$

$$= \int_{v=0}^{\pi/2}\int_{u=v}^{v+\pi} (u-v)\sin u\,du\,dv \quad \text{——} (1)$$





Now,

$$\int_{v=0}^{\pi/2}\left[\int_{u=v}^{v+\pi}(u-v)\sin u\,du\right]dv=\int_{v=0}^{\pi/2}\left[\int_{v}^{v+\pi}(u\sin u-v\sin u)\,du\right]dv$$

$$=\int_{v=0}^{\pi/2}\left\{\left[-u\cos u\right]_{v}^{v+\pi}+\int_{v}^{v+\pi}\cos u\,du+\left[v\cos u\right]_{v}^{v+\pi}\right\}dv$$

$$=\int_{0}^{\pi/2}\Big[-(v+x)\cos(\pi+v)+v\cos v+\sin(\pi+v)-\sin v+v\cos(\pi+v)-v\cos v\Big]\,dv$$

$$=\int_{0}^{\pi/2}\left(\cancel{v\cos v}+\pi\cos v-2\sin v-\cancel{v\cos v}\right)dv$$

$$=\int_{0}^{\pi/2}(\pi\cos v-2\sin v)\,dv$$

$$=\Big[\pi(+\sin v)+2\cos v\Big]_{0}^{\pi/2}$$

$$=\Big[\pi\big[\sin\pi/2-0\big]+2\big[\cos\tfrac{\pi}{2}-\cos 0\big]\Big]$$

$$=\pi-2.\qquad\left[\because \cos 0=1,\ \sin\pi/2=1\right]\text{—②}$$

Thus, from ① and ②, we can write.

$$\boxed{\iint_D x\sin(x+y)\,dx\,dy=\int_{x=0}^{\pi}\int_{y=0}^{\pi/2}x\sin(x+y)\,dx\,dy=\pi-2}$$

which is required result

Ques: 2(b) ii) If $\omega = f[xy/(x^2+y^2)]$ is a differentiable function of $u = f[xy/(x^2+y^2)]$. Show that

$$x\left(\frac{\partial \omega}{\partial x}\right) + y\left(\frac{\partial \omega}{\partial y}\right) = 0$$

Solution:- Given, the equations are.

$$\left.\begin{aligned} \omega &= F[xy/(x^2+y^2)] \quad \text{and} \\ u &= F[xy/(x^2+y^2)] \end{aligned}\right] \text{—} (1)$$

Since, ω is differentiable function.
Hence, differentiating equation (1) partially both sides w.r.t 'x' and 'y' respectively, we get

$$\frac{\partial \omega}{\partial x} = f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{(x^2+y^2)y - xy \cdot 2x}{[x^2+y^2]^2}\right]$$

$$\left[\because \frac{d}{dx}(u/v) = \frac{v \cdot u' - u \cdot v'}{v^2}\right]$$

$$\Rightarrow \frac{\partial \omega}{\partial x} = f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{(x^2+y^2)y - 2x^2y}{(x^2+y^2)^2}\right] \quad \text{—} (2)$$

and.

$$\frac{\partial \omega}{\partial y} = f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{(x^2+y^2)x - 2y \cdot xy}{(x^2+y^2)^2}\right]$$

$$\Rightarrow \frac{\partial \omega}{\partial y} = f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{(x^2+y^2)x - 2xy^2}{(x^2+y^2)^2}\right] \quad \text{—} (3)$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v'}{v^2}\right]$$

Now, multiplying by 'x' in ② and by 'y' in ③ and adding them, we get

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{(x^2+y^2)xy-2x^3y+(x^2+y^2)xy-2xy^3}{(x^2+y^2)^2}\right]$$

$$=f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{2xy(x^2+y^2)-2xy(x^2+y^2)}{(x^2+y^2)^2}\right]$$

$$=f'\left(\frac{xy}{x^2+y^2}\right)\left[\frac{2xy-2xy}{(x^2+y^2)^2}\right]$$

$$\Rightarrow \boxed{x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=0}$$

Thus, we conclude that

$$\boxed{\therefore\ x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=0}$$ required Solution.

Ques: 2(c)) Show that the equation of the plane containing the line $\frac{y}{b}+\frac{z}{c}=1$; $x=0$

and parallel to the line $\frac{x}{a}-\frac{z}{c}=1$; $y=0$

is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d is

the S.D. Show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

Solution:

The equation of the plane containing the line

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0$$

is $$\left(\frac{y}{b}+\frac{z}{c}-1\right)+\lambda x=0$$

$$\Rightarrow \boxed{\lambda x+\frac{1}{b}y+\frac{1}{c}z-1=0} \quad \text{——} \quad (1)$$

If it is parallel to the line

$$\frac{x}{a}-\frac{z}{c}=1 \ ; \ y=0$$

ie $$\frac{x-a}{a}=\frac{y}{0}=\frac{z}{c}$$

then the normal to plane (1) must be perpendicular to the line and so we have

$$\lambda\cdot a+\frac{1}{b}\cdot 0+\frac{1}{c}\cdot c=0$$

$$\Rightarrow \lambda a+1=0$$

$$\Rightarrow \boxed{\lambda=-1/a}$$

$\therefore$ From (1), the equation of the required plane is

$$\left(\frac{y}{b}+\frac{z}{c}-1\right)-\frac{1}{a}x=0$$

$$\Rightarrow \quad \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

Now, any point on the line $\frac{x-a}{a}=\frac{y}{0}=\frac{z}{c}$ is $(a,0,0)$.

Therefore,

2d = S.D. between the given lines

2d = Perpendicular distance of the point $(a,0,0)$ from the plane (1)

$$2d = \frac{a\cdot(1/a)-0\cdot(1/b)-0(1/c)+1}{\sqrt{(1/a)^2+(1/b)^2+(1/c)^2}}$$

$$\cancel{2}d = \frac{\cancel{2}}{\sqrt{a^{-2}+b^{-2}+c^{-2}}}$$

Squaring both side

$$d^2 = \frac{1}{a^{-2}+b^{-2}+c^{-2}}$$

$$\Rightarrow \boxed{d^{-2}=a^{-2}+b^{-2}+c^{-2}}$$ which is required result

Ques:- 3 (a)) (i) Show that the Matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ satisfies Cayley-Hamilton theorem. Hence or otherwise obtained the value of A^{-1} and A^{-2}.

Solution:-

given matrix $\Rightarrow A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

The characteristic equation of the Matrix A is

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda - 5 = 0$$

According to Cayley-Hamilton theorem

$$A^3 + A^2 - 5A - 5I = 0$$

Now, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $A^2 = A \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore\ A^3 + A^2 - 5A - 5I \quad \text{——} ①$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & +1 \end{bmatrix} - 5\begin{bmatrix} +1 & 2 & 0 \\ +2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & +1 \end{bmatrix} + \begin{bmatrix} -5 & -10 & 0 \\ -10 & 5 & 0 \\ 0 & 0 & +5 \end{bmatrix} + \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ from ① $\Rightarrow \quad A^3 + A^2 - 5A - 5I = 0$

$$A^2 + A - 5I - 5A^{-1} = 0$$

$$A^2 + A - 5I = 5A^{-1}$$

$$A^{-1} = \frac{1}{5}\left(A^2 + A - 5I\right) \quad \text{——} ②$$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \frac{1}{\cancel{5}} \times \cancel{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix} + \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & -1/5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & -1/5 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

Multiplying (2) by A^{-1} again, we get

$$A^{-2} = \frac{1}{5}\left(A + I - 5A^{-1}\right)$$

$$A^{-2} = \frac{1}{5}\left[\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - 5\begin{bmatrix}1/5 & 2/5 & 0\\ 2/5 & -1/5 & 0\\ 0 & 0 & -1\end{bmatrix}\right]$$

$$A^{-2} = \frac{1}{5}\left[\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -5\end{bmatrix}\right]$$

$$A^{-2} = \frac{1}{5}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$\boxed{A^{-2} = \begin{bmatrix}1/5 & 0 & 0\\ 0 & 1/5 & 0\\ 0 & 0 & 1\end{bmatrix}}$$ required solution.

Ques: 3(a) ii) Consider the linear transformation $T: R^3 \to R^2$, defined by $T(x,y,z) = (x+y, 2z)$. Find the matrix of T with respect to the base $\{u_1, u_2, u_3\}$ and $\{u_1', u_2'\}$ of $\mathbb{R}^3$ and $\mathbb{R}^2$, where $u_1 = (1,1,0)$, $u_2 = (0,1,4)$, $u_3 = (1,2,3)$ and $u_1' = (1,0)$ & $u_2' = (0,2)$. Use this matrix to find the image of the vector $u = (2,3,5)$.

Solution:-

We find the effect of T on the basis vectors of $\mathbb{R}^3$

$$T(u_1) = T(1,1,0) = (2,0) = 2(1,0) + 0(0,2)$$
$$= 2u_1' + 0u_2'$$
$$T(u_2) = T(0,1,4) = (1,8) = 1(1,0) + 4(0,2)$$
$$= 1u_1' + 4u_2'$$
$$T(u_3) = T(1,2,3) = (3,6) = 3(1,0) + 3(0,2)$$
$$= 3u_1' + 3u_2'$$

The co-ordinate vectors of $T(u_1)$, $T(u_2)$ and $T(u_3)$ are thus $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$. These vectors form the columns of matrix of T.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}$$

Let, us now use A to find the image of the vector $u = (2,3,5)$. We determine the co-ordinate vector of 'u'. It can be shown that —

$u = (2,3,5) = 3(1,1,0) + 2(0,1,4) - (1,2,3)$

$u = (2,3,5) = 3u_1 + 2u_2 + (-1)u_3$

The coordinate vector 'u' is thus $a = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

The co-ordinate vector of $T(u)$ is

$$b = Aa = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Therefore;

$$\boxed{T(u) = 5u_1' + 5u_2' = 5(1,0) + 5(0,2) = (5,10)}$$

We can check this result directly using the definition $T(x,y,z) = (x+y, 2z)$.

For $u = (2,3,5)$ this gives –

$$\boxed{T(u) = T(2,3,5) = (5,10)}$$

which is required solution.

Ques: 3(b)) A flat circular plate has the shape of the region $x^2+y^2\leq 1$. The plate, including the boundary where $x^2+y^2=1$, is heated so that the temperature at any point (x,y) is $T(x,y)=x^2+2y^2-x$. Find the hottest and closest point on the plate, and the temperature at each of these points.

Solution:-

given $T(x,y) = x^2+2y^2-x$ —— ①

given that $x^2+y^2 \leqslant 1$

let $x^2+y^2 = k$

where k is some positive number which is less than or equal to 1 [i.e $0<k\leq 1$]

At the boundary $x^2+y^2=1$ —— ②

putting ② in ①, we get

$$T(x) = -x^2-x+2$$

$$\frac{dT(x)}{dx} = -2x-1$$

for maximum or minimum $\frac{dT}{dx}=0$

$$\Rightarrow -2x-1=0$$

$$\Rightarrow \boxed{x=-1/2}$$

from ② ; $\boxed{y=\pm\sqrt{3}/2}$

Also $\frac{d^2T}{dx^2} = -2 < 0$

$\therefore$ Maxima at $\left(-\frac{1}{2}, \pm\sqrt{3}/2\right)$

In the interior region of plate

$$\frac{\partial T}{\partial x} = 2x - 1$$

$$\frac{\partial T}{\partial y} = 4y.$$

$$\frac{\partial T}{\partial x} = 0 \quad \text{and} \quad \frac{\partial T}{\partial y} = 0$$

$\Rightarrow x = 1/2 \;, y = 0$

This is point of minimum value from (1)

$$T(1/2, 0) = \frac{1}{4} + 0 - \frac{1}{2} = -\frac{1}{4}.$$

and $T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}$

$T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$

$\therefore$ Hottest points are $\left(-\frac{1}{2}, \pm\sqrt{3}/2\right)$ and $T = \frac{9}{4}$ units.

Coldest points are $\left(\frac{1}{2}, 0\right)$ and $T = -\frac{1}{4}$ units.

Ques: 3(c)(i) Find the limiting points of the co-axal system defined by the spheres
$x^2+y^2+z^2+3x-3y+6=0$; $x^2+y^2+z^2-6y-6z+6=0$.

Solution: The equation of the plane of the circle through the two given spheres is

$$3x+3y+6z=0$$

$$x+y+2z=0$$

The equation of the co-axal system determined by the given spheres is

$$x^2+y^2+z^2+3x-3y+6+\lambda(x+y+2z)=0$$

$$\Rightarrow x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0 \quad \text{—(1)}$$

λ being a parameter

The center of (1) is

$$\left[-\frac{(3+\lambda)}{2}, \frac{-(\lambda-3)}{2}, -\lambda\right]$$

and its radius is

$$r=\sqrt{\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6}$$

Equating this radius to zero, we obtain

$$6\lambda^2-6=0 \Rightarrow \cancel{6}\lambda^2=\cancel{6}$$

$$\lambda^2=1 \Rightarrow \boxed{\lambda=\pm 1}$$

The spheres corresponding to these values of λ become point spheres coinciding with their centres are the limiting points of the given system of spheres. The limiting points are $(-1,2,1)$ & $(-2,1,-1)$

Ans

Ques: 3(c) ii)) Show that the locus of the lines of intersection of tangent planes to the cone. $ax^2+by^2+cz^2=0$, which touch along perpendicular generators is the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

Solution:- Let the tangent planes along two perpendicular generators of the cone meet in the line $\boxed{\frac{x}{l}=\frac{y}{m}=\frac{z}{n}}$ ——(1)

Therefore, the equation of the plane containing the two generators is

$$alx+bmy+cnz=0 \quad \text{——(2)}$$

Let, λ, μ, ν be the direction ratios of any one of the two generators so that we have

$$al\lambda+bm\mu+cn\nu=0 \quad \text{——(3)}$$

$$a\lambda^2+b\mu^2+c\nu^2=0 \quad \text{——(4)}$$

Eliminating ν from (3) and (4), we have

$$a(cn^2+al^2)\lambda^2+2ablm\lambda\mu+b(cn^2+bm^2)\mu^2=0$$

If (λ_1,μ_1,ν_1); (λ_2,μ_2,ν_2) be the direction cosines of the two generators, we have

$$\frac{\lambda_1\lambda_2}{\mu_1\mu_2}=\frac{b(cn^2+bm^2)}{a(cn^2+al^2)}$$

$$\Rightarrow \frac{\lambda_1\lambda_2}{(cn^2+bm^2)/a}=\frac{\mu_1\mu_2}{(cn^2+al^2)/b}$$

Hence, by symmetry, we get.

$$\frac{\lambda_1\lambda_2}{(cn^2+bm^2)/a} = \frac{\mu_1\mu_2}{(cn^2+al^2)/b} = \frac{\nu_1\nu_2}{(al^2+bm^2)/c}$$

The generators being at right angle, we have

$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0$$

$$\frac{cn^2+bm^2}{a}+\frac{cn^2+al^2}{b}+\frac{al^2+bm^2}{c}=0$$

$$\Rightarrow a^2(b+c)l^2+b^2(c+a)m^2+c^2(a+b)n^2=0 \quad \text{--- (5)}$$

Eliminating l, m, n from (1) and (5), we obtain

$$\boxed{a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0}$$

as the required locus.

Ques: 4(a)) Let $H = \begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{bmatrix}$, a Hermitian matrix. Find a non-singular matrix P such that $P^T H \bar{P}$ is diagonal.

Solution:-

Given; $H = \begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{bmatrix}$

First form the block matrix (H, I):

$$\left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 1-i & 4 & 2-3i & 0 & 1 & 0 \\ -2i & 2+3i & 7 & 0 & 0 & 1 \end{array}\right]$$

Apply the row operation
$R_2 \to (-1+i)R_1 + R_2$ and
$R_3 \to 2iR_1 + R_3$ to (A, I)

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5i & 3 & 2i & 0 & 1 \end{array}\right]$$

Corresponding "Hermitian Column Operations"
$C_2 \to (-1-i)C_1 + C_2$ and
$C_3 \to -2iC_1 + C_3$ to A to obtain

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5i & 3 & 2i & 0 & 1 \end{array}\right]$$

Next apply the row operation $R_3 \to -5iR_2 + 2R_3$ and the corresponding Hermitian column operation $C_3 \to 5iC_2 + 2C_3$ to obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 0 & -19 & 5+9i & -5i & 2 \end{array}\right]$$ and then

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1+i & 1 & 0 \\ 0 & 0 & -38 & 5+9i & -5i & 2 \end{array}\right]$$

Now, H has been diagonalized. Set

$$P = \begin{bmatrix} 1 & -1+i & 5+9i \\ 0 & 1 & -5i \\ 0 & 0 & 2 \end{bmatrix}$$ and then

$$\boxed{P^T H \bar{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -38 \end{bmatrix}}$$

which is required result.

Ques:-4(b) (i) Examine the convergence of $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

(ii) Compute the volume of the solid bounded by the sphere $x^2+y^2+z^2=4$ and the surface of the paraboloid $x^2+y^2=3z$.

Solution:-

(i) Let $f(x) = \frac{dx}{x\sqrt{x^2+1}}$ (behaves like x^{-2} at α)

and $g(x) = \frac{1}{x^2}$

so that $\lim\limits_{x\to\alpha} \frac{f(x)}{g(x)} = \lim\limits_{x\to\alpha} \frac{x^2}{x\sqrt{x^2+1}} = 1$ (Non-Zero finite).

Hence, the two integrals $\int_0^\infty f\,dx$ and $\int_1^\infty g\,dx$.

behaves alike.

As $\int_1^\infty \frac{dx}{x^2}$ converges therefore $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

also converges.

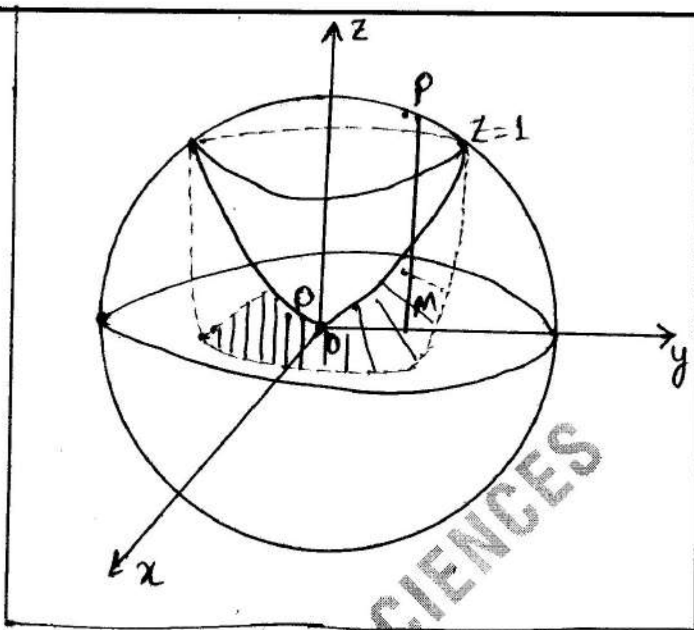
(ii) The two surfaces intersect at $z=1$.

The domain E, under consideration is bounded, above and by the two surfaces

$$z = \sqrt{4-x^2-y^2}$$

and $z = \frac{1}{3}(x^2+y^2)$ and its projection 'D' on the x-y plane is the circle.

$x^2+y^2 \leq 3$



$\therefore$ Volume $= \iiint_E dx\,dy\,dz$

$= \iint_D dx\,dy \int_{\frac{1}{3}\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} dz$

$= \iint_D \left[\sqrt{4-x^2-y^2} - \frac{1}{3}(x^2+y^2)\right] dx\,dy$

changing to polars (O being the circle, $x^2+y^2 \leq 3$)

$= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{4-r^2} - \frac{r^2}{3}\right) r\,dr$

$= \int_0^{2\pi} \left[\int_0^{\sqrt{3}} \left(r\sqrt{4-r^2} - \frac{r^3}{3}\right) dr\right] d\theta$

$= \frac{19}{6}\pi.$

$\therefore$ Volume of solid bounded by the sphere $x^2+y^2+z^2=4$ and surface of paroboloid $x^2+y^2=3z$ is $\frac{19}{6}\pi$

Ques: 4(c)) Prove that $x^2+y^2+z^2-yz-zx-zy-3x-6y-9z+21=0$ represents a paraboloid of revolution and find the co-ordinates of its focus.

Solution:-

given; $x^2+y^2+z^2-yz-zx-zy-3x-6y-9z+21=0$

The discriminating cubic is

$$-4\lambda^3+12\lambda^2-9\lambda=0$$

so that the characteristic roots are

$$\lambda = 0, 3/2, 3/2.$$

Two values of λ being equal, the given quadric is a surface of revolution.

The direction cosines (l, m, n) of the principal direction corresponding to $\lambda=0$ are given any two of these equations.

$$l-\frac{1}{2}m-\frac{1}{2}n=0$$

$$-\frac{1}{2}l+m-\frac{1}{2}n=0$$

$$-\frac{1}{2}l-\frac{1}{2}m+n=0$$

These gives $l:m:n = 1:1:1$

$$\therefore\ l=\frac{1}{\sqrt{3}},\ m=\frac{1}{\sqrt{3}},\ n=\frac{1}{\sqrt{3}}.$$

Now, we have

$$ul+vm+wn = -\frac{3}{2}\cdot\frac{1}{\sqrt{3}}-3\cdot\frac{1}{\sqrt{3}}-\frac{9}{2}\cdot\frac{1}{\sqrt{3}} = -\frac{9}{\sqrt{3}} \neq 0$$

Thus, the quadratic is a paraboloid of revolution and the reduced equation is

$$\frac{3}{2}x^2+\frac{3}{2}y^2-2\cdot\frac{9}{\sqrt{3}}z=0$$

$$\Rightarrow \boxed{x^2+y^2=4\sqrt{3}\,z}$$

The form of the equation shows that the latus rectum of generating parabola is $4\sqrt{3}$.

With respect to the given system of co-ordinates axes, the direction ratios of the axis of the paraboloid which is also the axis of revolution. are 1, 1, 1.

We rewrite the given equations in the form.

$$x^2+y^2+z^2-\frac{1}{2}\left[(x+y+z)^2-(x^2+y^2+z^2)\right]-3x-6y-9z+21=0$$

$$\Rightarrow \frac{3}{2}(x^2+y^2+z^2)-3x-6y-9z+21-\frac{1}{2}(x+y+z)^2=0$$

$$x^2+y^2+z^2-2x-4y-6z+14-\frac{1}{3}(x+y+z)^2=0$$

Thus, the axis of revolution, being the line through the centre of the sphere.

$$x^2+y^2+z^2-2x-4y-6z+14=0$$

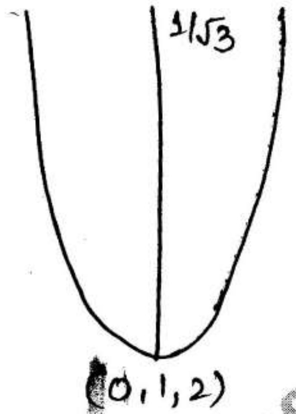
and perpendicular to the plane.

$$x+y+z=0$$

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1} \quad \text{——— (1)}$$

which is the axis of the paraboloid.

The vertex is the point on the axis (1) at a distance $\sqrt{3}$ from $(0,1,2)$.



Rewritting the equations of the axis in the form

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}}$$

we see that the point on the axis at a distance $\sqrt{3}$ from $(0,1,2)$ is $(1,2,3)$.

$\boxed{\text{Thus; } (1,2,3) \text{ is the required focus}}$

Section-B

Ques:-5(a)) Solve $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0.$

Solution:-

Given; $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3+y^2)dy=0$ —(1)

comparing (1) with

$$Mdx+Ndy=0$$

Where; $M = xy^2-x^2$

$N = 3x^2y^2+x^2y-2x^3+y^2$

$\therefore \frac{\partial M}{\partial y} = 2xy$ and

$$\frac{\partial N}{\partial x} = 6xy^2+2xy-6x^2$$

$$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{xy^2-x^2}\left[6xy^2+\cancel{2xy}-6x^2-\cancel{2xy}\right]$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{6xy^2-6x^2}{xy^2-x^2} = \frac{6(xy^2-x^2)}{xy^2-x^2}$$

$$\therefore \boxed{\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = 6}$$

which being a constant can be treated as a function of y alone

$\therefore$ I.F. of (1) $= e^{\int 6dy} = e^{6y}.$

Multiplying (1) by e^{6y}, we have

$$e^{6y}(xy^2-x^2)dx+e^{6y}(3x^2y^2+x^2y-2x^3+y^2)dy=0$$

whose solution is

$$\int e^{6y}(xy^2-x^2)dx+\int e^{6y}\cdot y^2\,dy=c$$

[Treating y as constant]

$$\Rightarrow\ e^{6y}\left[\frac{1}{2}(x^2y^2)-\frac{1}{3}x^3\right]+y^2\cdot\frac{1}{6}e^{6y}-\int 2y\cdot\frac{1}{6}\cdot e^{6y}dy=c$$

$$\Rightarrow\ e^{6y}\left[\frac{1}{2}x^2y^2-\frac{1}{3}x^3\right]+\frac{1}{6}y^2e^{6y}-\frac{1}{3}\left[y\cdot\frac{1}{6}e^{6y}-\int\left(1\times\frac{1}{6}e^{6y}\right)dy\right]=c$$

$$\Rightarrow\ e^{6y}\left(\frac{1}{2}x^2y^2-\frac{1}{3}x^3\right]+\frac{1}{6}y^2e^{6y}-\frac{1}{3}\left[\frac{1}{6}ye^{6y}-\frac{1}{36}e^{6y}\right]=c$$

$$\Rightarrow\ e^{6y}\left[\frac{1}{2}x^2y^2-\frac{1}{3}x^3+\frac{1}{6}y^2-\frac{1}{18}y+\frac{1}{108}\right]=c$$

$$\Rightarrow\ \boxed{\text{General Solution}=e^{6y}\left[\frac{1}{2}x^2y^2-\frac{1}{3}x^3+\frac{1}{6}y^2-\frac{1}{18}y+\frac{1}{108}\right]=c}$$

Required Solution.

Ques: 5(b)) Find the orthogonal trajectories of the family of curves $r = c(\cos\theta + \sin\theta)$, where 'c' is the parameter.

Solution:-

Given, the family of curves

$$r = c[\cos\theta + \sin\theta] \quad \text{——— (1)}$$

where, c is the parameter.

Taking logarithm of (1) both sides, we get

$$\log r = \log c + \log(\cos\theta + \sin\theta] \quad \text{——— (2)}$$

Differentiating equation (2) both sides, w.r.t θ we get,

$$\frac{1}{r}\left(\frac{dr}{d\theta}\right) = 0 + \frac{1}{\cos\theta + \sin\theta}[-\sin\theta + \cos\theta]$$

$$\Rightarrow \quad \frac{1}{r}\left(\frac{dr}{d\theta}\right) = \frac{-\sin\theta + \cos\theta}{\cos\theta + \sin\theta} \quad \text{——— (3)}$$

which is the differential equation of the family of curves (1),

Now, replacing $\frac{dr}{d\theta}$ by $-r^2\left(\frac{d\theta}{dr}\right)$ in eq^n (3), we get

$$\Rightarrow \quad \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-\sin\theta + \cos\theta}{\sin\theta + \cos\theta}$$

$$r\frac{d\theta}{dr} = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$$

$$\frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\,d\theta = \frac{dr}{r}$$

Integrating the above equation

$$\int \frac{dr}{r} = \int \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}\, d\theta + \log c.$$

$$\log r = \log(\sin\theta - \cos\theta) + \log c.$$

[$\log c$ is integrating constant]

$$\log r = \log[\sin\theta - \cos\theta]c.$$

$$\boxed{r = c\cdot[\sin\theta - \cos\theta]}$$

c, be an arbitrary constant.

Hence, $r = c(\sin\theta - \cos\theta)$ is required orthogonal trajectories of the given family of curve (1).

Ques: 5(c)⟩ A particle of mass 'm', is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time 't'.

Solution:- Let a particle of mass 'm' falling under gravity be at a distance 'x' from the starting point, after time 't'. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e., in the direction of 'x' decreasing. The weight mg of the particle acts vertically downwards i.e., in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

or $\frac{dv}{dt} = g - \frac{\mu}{m}v$, $\left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$

or $dt = \dfrac{dv}{g-(\mu/m)v}$

Integrating, we have

$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A$; where A = constant

But initially, when $t=0$, $v=0$;

$\therefore$ $\boxed{A = (m/\mu)\log g}$

$$\therefore \quad t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + \frac{m}{\mu}\log g$$

$$t = -\frac{m}{\mu}\log\left\{\frac{g-(\mu/m)v}{g}\right\}$$

$$-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm}\cdot v\right)$$

$$1 - \frac{\mu}{gm}\cdot v = e^{-\frac{\mu t}{m}}$$

$$\Rightarrow \quad v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)$$

$$dx = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)dt$$

Integrating, we have.

$$\boxed{x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B} \quad \text{——— (1)}$$

B = constant

But initially, when $t=0, x=0$

$$\therefore \quad \boxed{0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B} \quad \text{——— (2)}$$

Subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right]$$

$$\therefore \boxed{x = \frac{gm^2}{\mu^2}\left[e^{-\mu t/m} - 1 + \frac{\mu t}{m}\right]}$$

which is required result

Ques:- 5(d)) A particle moves so that its position vector is given by $r = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$ where ω is a constant ; show that

(i) the velocity of the particle is perpendicular to 'r',

(ii) the accelaration is directed towards the origin and has magnitude proportional to the distance from the origin,

(iii) $r \times \frac{dr}{dt}$ is a constant vector.

Solution:-

given; $r = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

we have

$$r \cdot \frac{dr}{dt} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) \cdot (-i\omega\sin\omega t + \omega\cos\omega t\,\hat{j})$$

$$r \cdot \frac{dr}{dt} = -\omega\cos\omega t \cdot \sin\omega t + \omega\sin\omega t\cos\omega t = 0$$

Therefore, the velocity is perpendicular to 'r'.

(ii) Accelaration of the particle

$$= a = \frac{d^2r}{dt^2} = -\omega^2\cos\omega t\,\hat{i} - \omega^2\sin\omega t\,j$$

$$\Rightarrow \boxed{a = -\omega^2[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}] = -\omega^2 r}$$

$\therefore$ Accelaration is a vector opposite to the direction of 'r' i.e. accelaration is directed towards the origin. Also magnitude of accelaration $= |a| = |-\omega^2 r| = \omega^2 r$, which is proportional to 'r' i.e the distance of the particle from the origin.

(iii) $r \times \frac{dr}{dt} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}) \times (-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})$

$= \omega\cos^2\omega t\,\hat{i} \times j - \omega\sin^2\omega t\,\hat{j} \times i$

$[\because i \times i = 0, \; j \times j = 0]$

$= \omega\cos^2\omega t\,K + \omega\sin^2\omega t\,K$

$= \omega(\cos^2\omega t + \sin^2\omega t)\,\hat{k}$

$= \omega(1)\,\hat{k} = \omega\hat{k}$ a constant vector

$\boxed{\therefore\; r \times \frac{dr}{dt} = \omega\hat{k} \;\text{[a constant vector]}}$

required solution.

Ques:- 5(c)(i)> Prove that $r^n.r$ is an irrotational vector for any value of 'n' but its solenoidal only if $n+3=0$

Solution:-

$\vec{r}$ is a position vector of a point considering $r^n.\vec{r}$ vector.

(i) For irrotational

$$\nabla \times r^n.\vec{r} = 0$$

$$\Rightarrow \quad \Sigma i \frac{\partial}{\partial n} \times r^n.\vec{r} = 0$$

$$\Rightarrow \quad \Sigma i \times \left(\frac{\partial r^n}{\partial n}.\vec{r} + \frac{r^n \partial \vec{r}}{\partial x} \right) = 0$$

$$\Rightarrow \quad \Sigma i \times \left[n.r^{n-1}.\frac{\partial r}{\partial x}.\vec{r} + r^n \hat{k} \right] = 0$$

$$\Rightarrow \quad \Sigma i \times \left[n.r^{n-1}.\frac{2x}{2r}.\vec{r} \right] = 0$$

$$\Rightarrow \quad \Sigma \hat{i} \times [n r^{n-2} \times \vec{r}] = 0$$

$$\Rightarrow \quad \Sigma (i \times \vec{r}) n.r^{n-2} = 0$$

$$\Rightarrow \quad n r^{n-2} \Sigma (y\hat{k} - z\hat{j}) = 0$$

$$\Rightarrow \quad n r^{n-2} \Sigma (xy\hat{k} - z\hat{j}x) = 0 = 0$$

Hence; for real value of 'n' the $r^n\vec{r}$ will be irrotational.

(ii) For solenoidal

$$\nabla\cdot(r^n\cdot\vec{r}) = 0$$

$$\Rightarrow \quad \nabla\cdot r^n\cdot\vec{r} + r^n\,\nabla\cdot\vec{r} = 0$$

$$\Rightarrow \quad n r^{n-1}\nabla r\cdot\vec{r} + r^n\,\nabla\cdot\vec{r} = 0$$

$$\Rightarrow \quad n r^{n-1}\left(\frac{\vec{r}}{r}\right)\cdot\vec{r} + r^n\cdot(3) = 0$$

$$\Rightarrow \quad n r^{n-1}\frac{r^2}{r} + 3r^n = 0$$

$$n r^n + 3r^n = 0$$

$$(n+3)r^n = 0$$

$$n+3 = 0$$

$$\Rightarrow \boxed{n=-3}$$

Ques: 5(e)(ii) If a constant vector, prove that

$$\operatorname{div}\{r^n(a\times\vec{r})\}=0$$

Solution:- We have

$$\operatorname{div}(\phi A)=\phi \operatorname{div} A + A\cdot \operatorname{grad}\phi$$

$$\therefore \operatorname{div}(r^n(a\times\vec{r})) = r^n \operatorname{div}(a\times\vec{r}) + (a\times\vec{r})\cdot \operatorname{grad} r^n$$

$$\Rightarrow r^n \operatorname{div}(a\times\vec{r}) + (a\times\vec{r})\cdot(n r^{n-1} \operatorname{grad}\vec{r})$$

$$= r^n(\vec{r}\cdot\operatorname{curl} a - a\cdot\operatorname{curl}\vec{r}) + (a\times\vec{r})\cdot\left(n r^{n-1}\cdot\frac{1}{r}\cdot\vec{r}\right)$$

$$= r^n(\vec{r}\cdot 0 - a\cdot 0) + n r^{n-2}(a\times\vec{r})\cdot\vec{r}$$

[$\because$ Curl of constant vector is zero and curl $r=0$]

$$= n r^{n-2}[(a\times\vec{r})\cdot\vec{r}]$$

$$=0,$$

Since a scalar triple product having two equal vectors is zero.

Ques: 6(a)) Solve the differential equation $y = x - 2ap + ap^2$. Find the singular solution and interpret it geometrically.

Solution:-

Given that ; $y = x - 2ap + ap^2$ —— ①

where $p = \frac{dy}{dx}$

Differentiating ①, w.r.t 'x',

$$p = 1 - 2a\left(\frac{dp}{dx}\right) + 2ap\left(\frac{dp}{dx}\right)$$

Or $\Rightarrow$ $p - 1 = 2a(p-1)\frac{dp}{dx}$

$$\Rightarrow (p-1)\left\{2a\left(\frac{dp}{dx}\right) - 1\right\} = 0$$

Omitting the first factor since it does not involve dp/dx, we get

$2a(dp/dx) - 1 = 0$ or $dx = 2a\,dp$

Integrating

$$x = 2ap + c$$

so, that $\boxed{p = (x-c)/2a}$ —— ②

Substituting the value of p from ② in ①, general solution of ① is

$$y = \cancel{x} - (\cancel{x} - c) + \frac{1}{4a}(x-c)^2$$

$$\Rightarrow 4a(y-c) = x^2 + c^2 - 2xc$$

$$\Rightarrow c^2 - 2xc + 4ac + x^2 - 4ay = 0$$

$$\Rightarrow \boxed{c^2 - 2c(x-2a) + (x^2 - 4ay) = 0}$$ —— ③

which is a quadratic equation in parameter c.
So, the c-discriminant relation is

$$4(x-2a)^2 - 4(x^2-4ay) = 0$$

$$\boxed{y - x + a = 0} \quad \text{———} \quad (4)$$

Again re-writting (1),

$$\boxed{ap^2 - 2ap + (x-y) = 0} \quad \text{———} \quad (5)$$

which is a quadratic in parameter p.
Hence, the p-discriminant relation is -

$$4a^2 - 4a(x-y) = 0$$

$$\boxed{y - x + a = 0} \quad \text{———} \quad (6)$$

from (4) and (6), we find that
$y-x+a=0$ is present in both p and c discriminant relations.

Further; $y-x+a=0$ gives $y = x-a$

and $p = \frac{dy}{dx} = 1$.

These, satisfy (1). Hence; $y-x+a=0$ is singular solution (1).

Geometrical interpretation of singular solution.

$$y - x + a = 0$$

Re-writting (3); $\boxed{(x-c)^2 = 4a(y-c)}$ ——— (7)

which represents a family of parabolas all of which touch, the line $y-x+a=0$, which is the envelope of this family of parabolas.

Ques: 6 (b) Solve $\left(\frac{d^4y}{dx^4}\right)+6\left(\frac{d^3y}{dx^3}\right)+11\left(\frac{d^2y}{dx^2}\right)+6\left(\frac{dy}{dx}\right) = 20e^{-2x}\sin x.$

Solution:-

given; $\left(\frac{d^4y}{dx^4}\right)+6\left(\frac{d^3y}{dx^3}\right)+11\left(\frac{d^2y}{dx^2}\right)+6\left(\frac{dy}{dx}\right) = 20e^{-2x}\sin x$

Re-writing the given equation -

$$(D^4+6D^3+11D^2+6D)y = 20e^{-2x}\sin x$$

or $D(D+1)(D+2)(D+3)\,y = 20e^{-2x}\sin x$

Its auxillary equation.

$$m(m+1)(m+2)(m+3) = 0$$

Solving it ; $m = 0, -1, -2, -3$

$$\boxed{\therefore C.F = c_1e^{0x}+c_2e^{-x}+c_3e^{-2x}+c_4e^{-3x}}$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

$$P.I. = \frac{1}{D^4+6D^3+11D^2+6D}\cdot 20e^{-2x}\sin x$$

$$= 20e^{-2x}\frac{1}{(D-2)^4+6(D-2)^3+11(D-2)^3+6(D-2)}\sin x$$

$$= 20e^{-2x}\frac{1}{D^4-8D^3+24D^2+32D+16+6(D^3-6D^2+12D-8)}\cdot$$

$$\frac{1}{11(D^2-4D+4)+6(D-2)}\sin x.$$

$$= 20e^{-2x}\frac{1}{D^4-2D^3-2D^2+2D}\sin x.$$

$$P.I. = 20e^{-2x}\frac{1}{(D^2)^2-2D(D^2)-D^2+2D}\sin x$$

$$P.I. = 20e^{-2x}\frac{1}{(-1^2)^2-2(-1)^2D-(-1)^2+2D}\sin x$$

$$P.I. = 20e^{-2x}\frac{1}{1+2D+1+2D}\sin x$$

$$P.I. = \overset{10}{\cancel{20}}\,e^{-2x}\frac{1}{\cancel{2}(1+2D)}.\sin x$$

$$P.I. = 10e^{-2x}(1-2D)\frac{1}{1-4D^2}\sin x$$

$$P.I. = 10e^{-2x}(1-2D).\frac{1}{1-4(-1^2)}\sin x$$

$$P.I. = \overset{2}{\cancel{10}}\,e^{-2x}(1-2D).\frac{1}{\cancel{5}}.\sin x$$

$$P.I = 2e^{-2x}(1-2D).\sin x$$

$$P.I. = 2e^{-2x}(\sin x-2\cos x)$$

$\therefore$ The required solution is

$$y = C.F + P.I$$

$$\boxed{y = C_1+C_2e^{-x}+C_3e^{-2x}+C_4e^{-3x}+2e^{-2x}(\sin x-2\cos x)}$$

required solution.

Ques: 6 (c) Solve $y'' + 3y' + 2y = x + \cos x$ by the method of variation of parameters.

Solution :- Given;
$$y'' + 3y' + 2y = x + \cos x \quad \text{--- (1)}$$

Compare (1) with $y'' + Py' + Qy = R$

here; $R = x + \cos x$ --- (2)

$P = 3, \; Q = 2$

consider $y'' + 3y' + 2y = 0$

$(D^2 + 3D + 2)y = 0 \qquad D \equiv d/dx$ --- (3)

Its auxillary equation

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2.$

$\therefore \boxed{\text{C.F of (1)} = c_1 e^{-x} + c_2 e^{-2x}}$

c_1 & c_2 being arbitrary constants

Let ; $u = e^{-x}$ and $v = e^{-2x}$ --- (4)

Here; $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$

$W = -2e^{-x} \cdot e^{-2x} + e^{-x} \cdot e^{-2x}$

$W = -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0$ --- (5)

Then, P.I of (1) $= uf(x) + vg(x)$, where --- (6)

$$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{e^{-2x}(x + \cos x)}{(-e^{-3x})}\,dx$$

$f(x) = \int e^{x}(x + \cos x)\,dx$ — by (2), (4) & (5)

$$= \int x e^{x} dx + \int e^{x} \cos x \, dx$$

$$= x e^{x} - \int (1 \times e^{x}) dx + \frac{1}{2} \times e^{x} (\cos x + \sin x)$$

$$\left[\because \int e^{ax} \cos bx \, dx = \left\{ \frac{1}{a^2+b^2} \right\} \times e^{ax} (a \cos bx + b \sin bx) \right]$$

Thus; $f(x) = x e^{x} - e^{x} + \frac{1}{2} e^{x} (\cos x + \sin x)$ ——— (7)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{e^{-x}(x+\cos x)}{(-e^{-3x})} dx$

$$g(x) = -\int e^{2x}(x+\cos x) dx = -\int x e^{2x} dx - \int e^{2x} \cos x \, dx$$

$$g(x) = -\left[x \cdot \frac{1}{2} \cdot e^{2x} - \int \left(\frac{1}{1} \times \frac{1}{2} \right) \times e^{2x} dx \right] - \frac{1}{5} e^{2x} (2\cos x + \sin x)$$

$$\therefore g(x) = -\frac{x}{2} \cdot e^{2x} + \frac{1}{4} e^{2x} - \frac{1}{5} e^{2x} (2\cos x + \sin x) \quad \text{——— (8)}$$

from (4), (6), (7) and (8), we have

$$\text{P.I of (1)} = e^{-x} \left\{ x e^{x} - e^{x} + \frac{1}{2} \cdot e^{x} (\cos x + \sin x) \right\}$$
$$+ e^{-2x} \left\{ -(x/2) \times e^{2x} + \frac{1}{4} \times e^{2x} - \frac{1}{5} \times e^{2x} (2\cos x + \sin x) \right\}$$

$$\text{P.I of (1)} = x - 1 + \frac{(\cos x + \sin x)}{2} - \frac{x}{2} + \frac{1}{4} - \frac{(2\cos x + \sin x)}{5}$$

$$\text{P.I of (1)} = \frac{x}{2} - \frac{3}{4} + \frac{1}{10}(3\sin x + \cos x)$$

$\therefore$ Required solution; $y = C.F + P.I$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \frac{x}{2} - \frac{3}{4} + \frac{1}{10}(3\sin x + \cos x)}$$

which is required solution

Ques: 6 (d) (i) Show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$. by using Laplace Transform method.

Solution:-

given; $I = \int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

Let; $F(t) = \sin t$

So that $f(p) = L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1}$

$$L\left\{\frac{1}{t}\cdot \sin t\right\} = \int_0^\infty e^{-pt}\cdot \frac{\sin t}{t} dt = \int_p^\infty f(x)\,dx$$

$$= \int_p^\infty \frac{1}{x^2+1} dx = \left[\tan^{-1} x\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} p.$$

Taking limit as $p \to 0$, we have

$$= \lim_{p\to 0} \frac{\pi}{2} - \tan^{-1} p$$

$$= \frac{\pi}{2}.$$

$\therefore \boxed{\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}}$ Hence proved.

Ques: 6(d)(ii) By using Laplace Transform Method solve the initial value problem

$$(D^3 - D^2 - D + 1)y = 8te^{-t}$$

$y = D^2 y = 0$, $Dy = 1$, when $t = 0$.

Solution:- Taking the Laplace transform of both sides of the given equation, we have—

$$L(y''') - L(y'') - L(y') + L(y) = 8L[te^{-t}]$$

or; $p^3 L\{y\} - p^2 y(0) - py'(0) - y''(0) - [p^2 L\{y\} - py(0) - y'(0)] - [pL\{y\} - y(0)] + L\{y\} = -8\frac{d}{dp}[L\{e^{-t}\}]$

$$\Rightarrow (p^3 - p^2 - p + 1)L\{y\} - p + 1 = -8\frac{d}{dp}\left(\frac{1}{p+1}\right)$$

$$(p-1)^2(p+1)L\{y\} = p - 1 + \frac{8}{(p+1)^2}$$

or $$L\{y\} = \frac{1}{(p-1)(p+1)} + \frac{8}{(p-1)^2(p+1)^3}$$

$$L(y) = \frac{1}{2}\left[\frac{1}{p-1} - \frac{1}{p+1}\right] - \frac{3}{2(p-1)} + \frac{1}{(p-1)^2} + \frac{3}{2(p+1)} + \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3}$$

$$L\{y\} = -\frac{1}{p-1} + \frac{1}{p+1} + \frac{1}{(p-1)^2} + \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3}$$

$$\therefore y = -L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{1}{p+1}\right\} + L^{-1}\left\{\frac{1}{(p-1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^3}\right\}$$

$$y = -e^t + e^{-t} + e^t L^{-1}\left\{\frac{1}{p^2}\right\} + 2e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} + 2e^{-t}L^{-1}\left\{\frac{1}{p^3}\right\}.$$

$$y = -e^{t} + e^{-t} + e^{t}\cdot t + 2e^{-t}\cdot t + 2e^{-t}\cdot \frac{t^2}{2!}$$

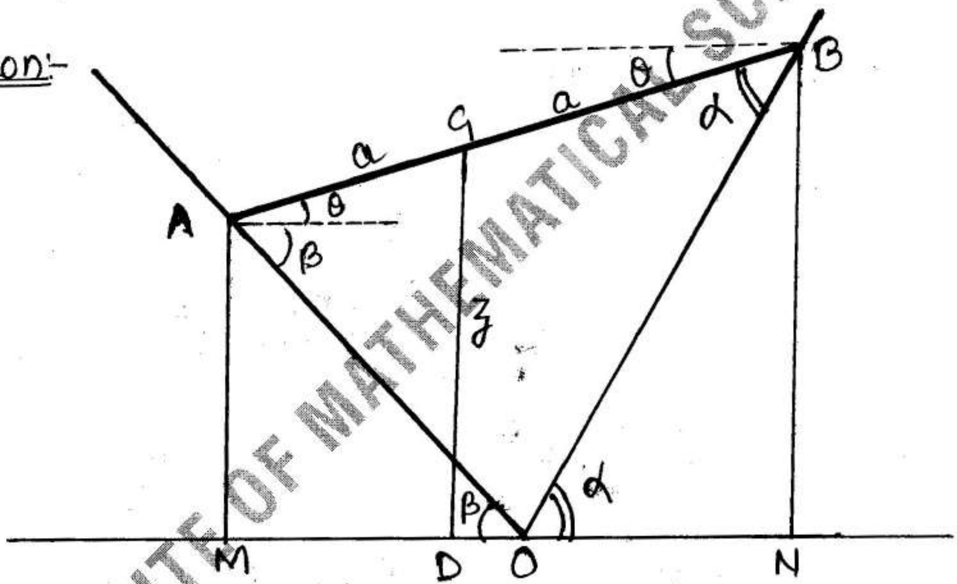
$$y = (1 + 2t + t^2)e^{-t} - (1 - t)e^{t}$$

$$\boxed{\therefore y = (1 + 2t + t^2)e^{-t} + (t-1)e^{t}}$$

is the required solution.

Ques:-7) a) A uniform beam of length 2a rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations to the horizontal are α are β $(\alpha > \beta)$, show that the inclination 'θ' of the horizontal in one of the $\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$ and show that the beam is unstable in this position.

Solution:-



Let AB be a uniform beam of length 2a resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. We have

$$\boxed{\angle AOM = \beta \quad \text{and} \quad \angle BON = \alpha}$$

The centre of gravity of the beam AB is its middle point G.

Let z be the height of G above the fixed horizontal line MN. We shall express z as a function of θ.

We have; $z = GD = \frac{1}{2}(AM+BN)$

$$z = GD = \frac{1}{2}(OA\sin\beta + OB\sin\alpha) \quad \text{--- (1)}$$

Now in the triangle OAB,

$\angle OAB = \beta+\theta$, $\angle OBA = \alpha-\theta$ and

$\angle AOB = \pi-(\alpha+\beta)$

Applying the sine theorem for the ΔOAB, we have

$$\frac{OA}{\sin(\alpha-\theta)} = \frac{OB}{\sin(\beta+\theta)} = \frac{AB}{\sin(\pi-(\alpha+\beta))} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\therefore\ OA = \frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)};\ OB = \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}$$

Substituting for OA and OB in (1), we have

$$z = \frac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$z = \frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta + \sin(\beta+\theta)\sin\alpha\right]$$

$$z = \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha.\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$z = \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta[\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right].$$

$\therefore \Rightarrow \dfrac{dz}{d\theta} = \dfrac{a}{\sin(\alpha+\beta)}\left[\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta\right]$ —— (2)

For equilibrium of the beam, we have

$\dfrac{dz}{d\theta} = 0$

$\Rightarrow \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0$

i.e. $2\sin\theta\cdot\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{2}\left[\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right]$

$\boxed{\tan\theta = \dfrac{1}{2}(\cot\beta - \cot\alpha)}$ —— (3)

This gives the required position of equilibrium of the beam.

Differentiating (2), we have

$\dfrac{d^2z}{d\theta^2} = \dfrac{a}{\sin(\alpha+\beta)}\left[-\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\cos\theta\sin\alpha\sin\beta\right]$

$= \dfrac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\dfrac{1}{2}\tan\theta(\cot\beta - \cot\alpha) + 1\right]$

$\boxed{\dfrac{d^2z}{d\theta^2} = \dfrac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\tan^2\theta + 1\right]}$ — by (3)

$\Rightarrow$ a negative quantity because θ, α and β are all acute angles and $\alpha+\beta < \pi$.

Thus, in the position of equilibrium $\dfrac{d^2z}{d\theta^2}$ is negative, i.e. z is maximum.

Hence, the equilibrium is unstable.

Ques:-7(b)(i) A particle starts from rest at a distance 'a' from the centre of force which attracts inversely as the distance. Find the time of arriving at the centre?

Solution:-

If 'x' is the distance of the particle from the centre of force at time 't', then the equation of motion is

$$\boxed{\frac{d^2x}{dt^2} = -\frac{\mu}{x}}$$

Multiplying both sides by $2(dx/dt)$, and then integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -2\mu\log x + A$$

where A is constant.

But initially at $x=a$, $\frac{dx}{dt}=0$

$$\therefore 0 = -2\mu\log a + A \quad \text{or} \quad A = 2\mu\log a$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2\mu(\log a - \log x) = 2\mu\log(a/x)$$

or $$\frac{dx}{dt} = -\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}}$$

where the -ve sign has been taken since the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\frac{1}{\sqrt{2\mu}}\cdot\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

Integrating from $x=a$ to $x=0$, the required time t_1 to reach the centre is given by

$$t_1 = -\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^{0}\frac{dx}{\sqrt{\{\log(a/x)\}}}$$

Put, $\log\left(\frac{a}{x}\right) = u^2$ i.e. $x = ae^{-u^2}$

so that $dx = -2ae^{-u^2}\cdot u\,du$

When; $x=a$, $u=0$ and when $x\to 0$, $u\to\infty$

$$\therefore\ t_1 = \frac{2}{\sqrt{(2\mu)}}\int_0^{\infty} e^{-u^2}\,du.$$

But $\int_0^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$

$$\therefore\ t_1 = \frac{2a}{\sqrt{2\mu}}\cdot\frac{\sqrt{\pi}}{2}$$

$\boxed{t_1 = a\sqrt{\frac{\pi}{2\mu}}}$ required solution

Ques: 7(b)(ii) If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are $\alpha, \alpha-\beta, \alpha-2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that $v_3 t_1 = v_1 t_2$ and

$$\frac{1}{v_1}+\frac{1}{v_3} = \frac{2\cos\beta}{v_2}.$$

Solution:-

Since, the horizontal velocity of a projectile remains constant throughout the motion, therefore.

$$v_1 \cos\alpha = v_2 \cos(\alpha-\beta) = v_3 \cos(\alpha-2\beta) \quad \text{——} (1)$$

considering the verticle motion from P to Q and then from Q to R and using the formula

$v = u+ft$, we get

$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - gt_1 \quad \text{——} (2)$$

and $$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - gt_2 \quad \text{——} (3)$$

from (2) and (3), we have

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - v_2 \sin(\alpha-\beta)}{v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - \frac{v_1 \cos\alpha}{\cos(\alpha-\beta)}\cdot \sin(\alpha-\beta)}{v_3 \frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)} \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

[Substituting suitably for v_2 from (1)]

$$\Rightarrow \frac{t_1}{t_2} = \frac{v_1[\sin\alpha\cos(\alpha-\beta) - \cos\alpha\sin(\alpha-\beta)]}{v_3[\sin(\alpha-\beta)\cos(\alpha-2\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)]}$$

$$\Rightarrow \frac{t_1}{t_2} = \frac{v_1\sin\{\alpha-(\alpha-\beta)\}}{v_3\sin\{(\alpha-\beta)-(\alpha-2\beta)\}}$$

$$\Rightarrow \boxed{\frac{t_1}{t_2} = \frac{v_1\sin\beta}{v_3\sin\beta} = \frac{v_1}{v_3}}$$

$$\therefore \boxed{v_3 t_1 = v_1 t_2}$$

This proves the first result.

Again from ①, we have

$$\frac{1}{v_1} = \frac{1}{v_2}\cdot\frac{\cos\alpha}{\cos(\alpha-\beta)} \quad \text{and} \quad \frac{1}{v_3} = \frac{1}{v_2}\cdot\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

$$\therefore \frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\left[\frac{\cos\alpha + \cos(\alpha-2\beta)}{\cos(\alpha-\beta)}\right]$$

$$\frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\left[\frac{2\cos(\alpha-\beta)\cos\beta}{\cos(\alpha-\beta)}\right]$$

$$\boxed{\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}}$$

This proves the second result.

Ques: 7(c)) A particle is acted on by a force parallel to the axis of 'y' whose accelaration (always towards the axis of x) is μy^{-2} and when $y=a$, it is projected parallel to the axis of 'x' with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the particle. Here 'μ' is a constant.

Solution:- Here we are given that

$$\frac{d^2y}{dt^2} = -\mu y^{-2} \quad \text{———} \quad (1)$$

the negative sign has been taken because the force is in the direction of y decreasing.

Also, there is no force parallel to the axis of x.

Therefore;
$$\frac{d^2x}{dt^2} = 0 \quad \text{———} \quad (2)$$

Multiplying both sides of (1) by $2\frac{dy}{dt}$ and then integrating w.r.t 't', we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + A \;; \text{ where A is constant}$$

Initially; when $y=a$, $\frac{dy}{dt} = 0$

[Note that initially there is no velocity parallel to the y-axis]

$$\therefore \boxed{A = -\frac{2\mu}{a}}$$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} - \frac{2\mu}{a} = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right) = \frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

Or $\frac{dy}{dt} = -\sqrt{\frac{2\mu}{a}}\sqrt{\left(\frac{a-y}{y}\right)}$ —— (3)

[-ve sign has been taken because the particle is moving in the direction of y decreasing.]

Now, integrating (2), we have

$\frac{dx}{dt} = B$; B is a constant

Initially ; when $y=a$; $\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$

So that $\boxed{B = \sqrt{\frac{2\mu}{a}} = \frac{dx}{dt}}$ —— (4)

By dividing (3) by (4), we have

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$$dx = -\sqrt{\frac{y}{a-y}}\,.dy$$

By integrating both the side

$$x = -\int\sqrt{\frac{y}{a-y}}\,dy + c$$

$$x = 2a\int\frac{\cos\theta}{\sin\theta}.\cos\theta\cdot\sin\theta d\theta + c.$$

[Putting $y = a\cos^2\theta$; so that $dy = -2a\cos\theta\sin\theta d\theta$]

$$x = a\int(1+\cos 2\theta)d\theta + c$$

$$x = a\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$$

$$\boxed{x = \frac{1}{2}a\left(2\theta + \sin 2\theta\right) + c}$$

Let us take $x=0$; when $y=a$

i.e., when $a\cos^2\theta = a$ i.e,

when; $\cos\theta = 1$ i.e $\boxed{\theta = 0}$

Then, $0 = \frac{1}{2}a(0+0)+c$ or $c=0$

$$\therefore x = \frac{1}{2}a(2\theta + \sin 2\theta) \quad \text{——— (5)}$$

Also; $y = a\cos^2\theta = \frac{1}{2}a(1+\cos 2\theta)$ ——— (6)

The equations (5) and (6) give us the path of the particle. But these are the parametric equations of a cycloid.

Hence, the path is a cycloid.

Ques:-8(a)) Find the curvature and torsion of the circular helix $x = a\cos\theta$, $y = a\sin\theta$, $z = a\theta\cot\alpha$.

Solution:- Given, the circular helix.

$$x = a\cos\theta \ , \ y = a\sin\theta \ , \ z = a\theta\cot\alpha \quad \text{--- (1)}$$

Then, we have the following curvature K and Torsion T.

$$K = \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| \Big/ \left|\frac{d\vec{r}}{d\theta}\right|^3$$

and $$T = \left[\frac{d\vec{r}}{d\theta} \cdot \frac{d^2\vec{r}}{d\theta^2} \cdot \frac{d^3\vec{r}}{d\theta^3}\right] \Big/ \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|^2$$

Since, $\vec{r}$ is the position vector at (x, y, z) given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

so; $$\vec{r} = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + \theta a\cot\alpha\,\hat{k} \quad \text{— using (1)}$$

Now; differentiating equation (1) w.r.t 'θ', we get

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\cot\alpha\,\hat{k}$$

$$\Rightarrow \left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\alpha} = \sqrt{a^2 + a^2\cot^2\alpha}$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$\left|\frac{d\vec{r}}{d\theta}\right| = \sqrt{a^2(1 + \cot^2\alpha)} = \sqrt{a^2[\operatorname{cosec}^2\alpha]}$$

$$= a\operatorname{cosec}\alpha. \quad [\because \cot^2\alpha + 1 = \operatorname{cosec}^2\alpha]$$

Again differentiating $dr/d\theta$, w.r.t θ, we get

$$\frac{d^2\vec{r}}{d\theta^2} = -a\cos\theta\,\hat{i} - a\sin\theta\,\hat{j}$$

Also, differentiating equation $\frac{d^2\vec{r}}{d\theta^2}$, both side w.r.t '$\theta$', we get

$$\frac{d^3\vec{r}}{d\theta^3} = a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j}$$

Now, we have,

$$\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & a\cot\alpha \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= i\,(a^2\sin\theta\cot\alpha) - \hat{j}(a^2\cos\theta\cdot\cot\alpha) + \hat{k}(a^2)$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = \sqrt{a^4\sin^2\theta\cot^2\alpha + a^4\cos^2\theta\cot^2\alpha + a^4}$$

$$= \sqrt{a^4\cot^2\alpha(\sin^2\theta + \cos^2\theta) + a^4}$$

$$\therefore \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = \sqrt{a^4\cot^2\alpha + a^4} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\therefore \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = a^2\sqrt{1+\cot^2\alpha} = a^2\sqrt{\operatorname{cosec}^2\alpha}$$

$$[\because 1 + \cot^2\alpha = \operatorname{cosec}^2\alpha]$$

$$\boxed{\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = a^2\operatorname{cosec}\alpha}$$

Also;

$$\left(\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right)\cdot\frac{d^3\vec{r}}{d\theta^3} = \left[(a^2\sin\theta\cot\alpha)i - (a^2\cos\theta\cot\alpha)\hat{j} + a^2\hat{k}\right] \cdot \left[a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j}\right]$$

$$= a^3\sin^2\theta\cdot\cot\alpha + a^3\cos^2\theta\cdot\cot\alpha$$

$$= a^3\cot\alpha\,[\sin^2\theta + \cos^2\theta] = a^3\cot\alpha.$$

Thus, the curvature of the circular helix is

$$K = \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| \Big/ \left|\frac{d\vec{r}}{d\theta}\right|^3$$

$$K = \frac{a^2 \operatorname{cosec}\alpha}{a^3 \operatorname{cosec}^3\alpha} = \frac{1}{a}\cdot\frac{1}{\operatorname{cosec}^2\alpha} = \frac{1}{a}\sin^2\alpha.$$

$$\therefore \boxed{K = \frac{\sin^2\alpha}{a}}$$

Then, torsion of the circular helix is

$$\tau = \left[\frac{d\vec{r}}{d\theta}\cdot\frac{d^2\vec{r}}{d\theta^2}\cdot\frac{d^3\vec{r}}{d\theta^3}\right] \Big/ \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|^2$$

$$\tau = \frac{a^3\cot\alpha}{a^4\operatorname{cosec}^2\alpha} = \frac{1}{a}\cdot\frac{\cos\alpha}{\sin\alpha}\times\sin^2\alpha$$

$$\boxed{\tau = \frac{\cos\alpha\sin\alpha}{a}} \text{ or } \boxed{\tau = \frac{\sin 2\alpha}{2a}}$$

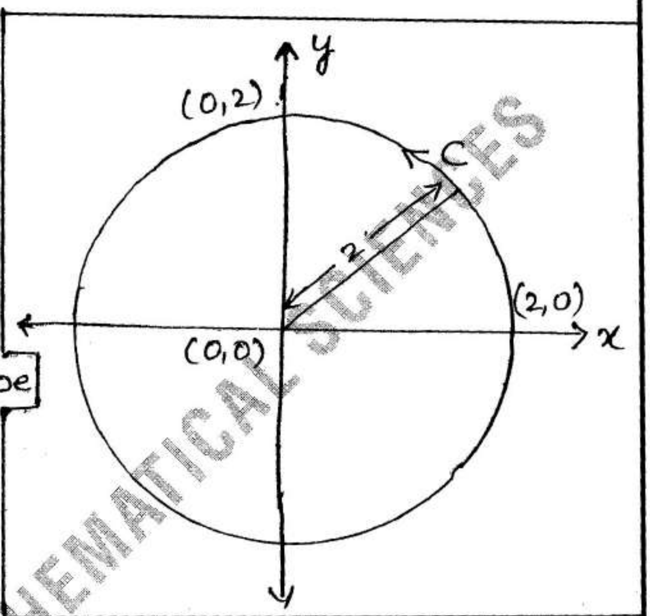
$$[\because \sin 2\theta = 2\sin\theta\cos\theta].$$

Ques: 8(b) Evaluate $\oint_c (x^2+y^2)dx + 3xy^2 dy$ where C, a circle of radius two with centre at the origin of the xy-plane, is traversed in the positive sense.

Solution:-

Since; C is the circle of radius '2' with centre at the origin i.e (0,0)

So, let the parametric equation in x-y plane be



$x = 2\cos t$

$y = 2\sin t \quad , z = 0$

where $0 \le t \le 2\pi$

Then; $\frac{dx}{dt} = -2\sin t \quad ; \quad \frac{dy}{dt} = 2\cos t$

Therefore, we have

$\oint_c (x^2+y^2)dx + 3xy^2 dy$

$$= \int_0^{2\pi} \left[(4\cos^2 t + 4\sin^2 t)\frac{dx}{dt} + 3(2\cos t)\cdot(4\sin^2 t)\frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} [4(-2\sin t) + 24\sin^2 t \cos t \cdot 2\cos t] dt$$

$[\because \cos^2\theta + \sin^2\theta = 1]$

$$= \int_0^{2\pi} [-8\sin t + 48\sin^2 t\cos^2 t] dt$$

$$= 8\int_0^{2\pi} [-8\sin t + 6\sin^2 t\cos^2 t]\,dt$$

$$= 4\int_0^{2\pi} [-2\sin t + 3\sin^2(2t)]\,dt$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= 4\int_0^{2\pi} -2\sin t + \frac{3}{2}(1-\cos 4t)\,dt$$

$$= [8\cos t]_0^{2\pi} + 6\left[t - \frac{\sin 4t}{4}\right]_0^{2\pi}$$

$[\because \int \sin\theta\, d\theta = -\cos\theta]$

$$= [8\cos 2\pi - 8\cos 0] + 6\left[2\pi - \frac{\sin 8\pi}{4} - 0 + 0\right]$$

$$= \cancel{8} - \cancel{8} + 12\pi - 0 = 12\pi$$

$[\because \cos 2\pi = \cos 0 = 1 \;\&\; \sin n\pi = 0]$

Thus;

$$\boxed{\oint_C (x^2+y^2)\,dx + 3xy^2\,dy = 12\pi}$$

required solution.

Ques:-8(C) If $F = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ evaluate $\iint_S (\nabla\times F)\cdot n\,ds$ where 'S' is the surface of the sphere $x^2+y^2+z^2=16$ above the x-y plane.

Solution:-

given; $F = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$

'S' is the surface of sphere $\to x^2+y^2+z^2=16$ above the x-y plane.

The surface $x^2+y^2+z^2=16$ meets the plane $z=0$ in a circle C given by $x^2+y^2=16$, $z=0$.

Let S_1, be the plane region bounded by the circle C. If Sprima and S_1, then S' is a closed surface. Let V be the region bounded by S'.

If 'n' denotes the outward drawn (drawn outside the region 'V') unit normal vector to S', then on the plane surface S_1, we have $\hat{n} = -k$

Note, that 'k' is a unit vector normal to S_1 drawn into the region V.

Now; by application of Gauss divergence theorem, we have,

$$\iint_{S'} \text{curl} F\cdot n\,dS = 0$$

or $\iint_S \text{curl} F\cdot n\,ds + \iint_{S_1} \text{curl} F\cdot n\,ds = 0$

[$\because$ S' consists of S and S_1]

or $\iint_S \text{curl} F\cdot n\,ds - \iint_{S_1} \text{curl} F\cdot k\,ds = 0$

[$\because$ On S_1, $n=-k$]

Or; $\iint_S \text{curl} F\cdot n\, ds = \iint_{S_1} \text{curl } F\cdot K\, ds$

Now; $\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$= 0\hat{i} - 2z\hat{j} + (3y-1)\hat{k}$

$\boxed{\text{Curl } F = -2z\hat{j} + (3y-1)\hat{k}}$

$\therefore \text{Curl } F\cdot k = \{-2z\hat{j} + (3y-1)\hat{k}\}\cdot\hat{k} = 3y-1.$

$\therefore \iint_S \text{curl} F\cdot n\, ds = \iint_{S_1} (3y-1)\, ds$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} (3r\sin\theta - 1) r\, d\theta\, dr.$

[By changing to polar form]

as; $y = \pm\sqrt{16-x^2} \Rightarrow x = \pm 4.$

[Note that S_1 is a circle in x-y plane with centre origin and radius 4].

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} 3r^2\sin\theta\, d\theta\, dr - \int_{\theta=0}^{2\pi}\int_{r=0}^{4} r\, dr\, d\theta$

$= 0 - \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^4 d\theta \quad \left[\because \int_{\theta=0}^{2\pi}\sin\theta\, d\theta = 0\right]$

$= -8\,[\theta]_0^{2\pi} = -8\times 2\pi = -16\pi.$

$\boxed{\therefore \iint_S \text{curl } F\cdot n\, ds = -16\pi}$ required solution.

Ques: 8(d) Verify Stoke's theorem for the vector $A = 3y\hat{i} - xz\hat{j} + yz^2\hat{k}$, where 'S' is the surface of the paraboloid; $2z = x^2+y^2$ bounded by $z=2$ and C is its boundary.

Solution:-

The boundary C of the surface 'S' is the circle in the plane $z=2$, whose equations are $x^2+y^2=4$, $z=2$.

The radius of this circle is 2 and centre $(0,0,2)$.

Suppose $x = 2\cos t$, $y = 2\sin t$, $z=2$; $0 \le t < 2\pi$ are parametric equations of 'C'. By Stoke's theorem

$$\oint_C A\cdot dr = \iint_S (\text{Curl } A)\cdot n \, dS,$$

when 'n' is a unit vector along outward drawn normal to the surface S.

We have;

$$\oint_C A\cdot dr = \oint_C (3y\,\hat{i} - xz\,\hat{j} + yz^2\,\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (3y\,dx - xz\,dy + yz^2 dz)$$

$$= \oint_C (3y\,dx - 2x\,dy),$$

Since, on C; $z=2$ and $dz=0$

$$= \int_{2\pi}^{0} \left(3y\frac{dx}{dt} - 2x\frac{dy}{dt}\right)dt$$

[Note that here the surface 'S' lies below the curve 'C' and so direction of 'C' is positive if 'C' is traversed in clockwise sense].

$$= -\int_0^{2\pi} [3\cdot 2\sin t\cdot(-2\sin t) - 2\cdot 2\cos t\cdot 2\cos t]\,dt$$

$$= -\int_0^{2\pi} [-12\sin^2 t - 8\cos^2 t]\,dt$$

$$= 4\left[12\int_0^{\pi/2}\sin^2 t\,dt + 8\int_0^{\pi/2}\cos^2 t\,dt\right]$$

$$= 4\left[12\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 8\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{4}{4}[12\pi + 8\pi]$$

$$= 20\pi. \quad \text{——} \quad (1)$$

Let, S_1 be the plane region bounded by the circle 'C'. If S' is ~~either~~ surface consisting of the surface S and S_1, then S' is a closed surface.

Let, V be the volume bounded by S'.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } A)\cdot n\,ds = \iiint_V \text{div curl } A\,dV$$

$$= 0; \ [\text{since div curl } A = 0].$$

$$\therefore \iint_S (\text{curl } A)\cdot n\,ds + \iint_{S_1} (\text{curl } A)\cdot n\,ds = 0$$

$[\because S'$ consists of S and $S_1]$

$$\iint_S (\text{curl } A)\cdot n\,ds = -\iint_{S_1} (\text{curl } A)\cdot n\,ds$$

$$= -\iint_{S_1} (\text{curl } A)\cdot K\,ds$$

$[\because$ on S_1, $n = K]$

Now; $\text{Curl } A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y & -xz & yz^2 \end{vmatrix}$

$$\text{Curl } A = i\left[\frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)\right] - j\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial z}(3y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(3y)\right]$$

$$\boxed{\text{Curl } A = (z^2+x)\hat{i} - (z+3)\hat{k}}$$

$$\therefore \iint_S (\text{curl } A)\cdot n\, ds = -\iint_{S_1} [(z^2+x)\hat{i} - (z+3)\hat{k}]\cdot\hat{k}\, ds$$

$$= \iint_{S_1} (z+3)\, ds = \iint_{S_1} 5\, ds \quad [\because \text{ on } S_1,\ z=2]$$

$= 5S_1$; where S_1 is the area of a circle of radius 2.

$$= 5\cdot\pi\cdot(2)^2 = 20\pi.$$

$$\boxed{\therefore \oint_C A\cdot dr = \iint_S (\text{curl } A)\cdot n\, ds = 20\pi}$$

Hence proved.