# ANALYTICAL CONDITIONS OF EQUILIBRIUM OF CO-PLANAR FORCES

CHAPTER 5

## 5.1. EQUILIBRIUM OF THREE FORCES ACTING AT A POINT

### 5.1.1. Theorem

***If three forces acting on a rigid body keep it in equilibrium, they must be coplanar.*** **[M.D.U. 2017; K.U. 2001]**

**Proof.** Let the three given forces P, Q, R act along the lines $a, b, c$ respectively keeping the rigid body in equilibrium.

Take a point A on the line of action $a$ of P and a point B on the line of action $b$ of Q. Then, as the forces P and Q meet a straight line AB, their moments about AB separately vanish. Also, since P, Q, R are in equilibrium, the sum of their moments about AB is zero. Hence the moment of R about AB is zero.

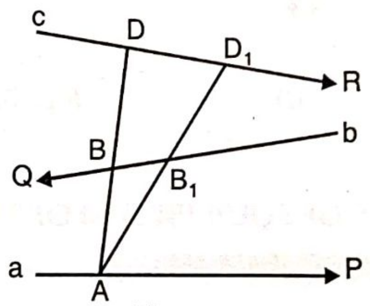


*Fig. 5.1*

Hence, either $R = 0$ or its line of action $c$ meets AB or is parallel to AB.

But $R \neq 0$ and since AB is any arbitrary line, $c$ cannot in general, be parallel to AB. Hence the line $c$ meets AB in a point D (say).

Similarly, if $B_1$ is another point on the line of action $b$ of Q, then $c$ also meets $AB_1$ in some point $D_1$ (say).

Thus the lines $b$ and $c$ are coplanar, lying in the plane determined by AB and $AB_1$.

But A is any point on the line $a$. Hence the plane containing $b, c$ contains all the points of $a$, *i.e.*, it contains $a$. Hence $a, b, c$ are coplanar.

### 5.1.2. Theorem

***If three forces, acting in one plane upon a rigid body, keep it in equilibrium, they must either meet in a point or be parallel.***

**[M.D.U. 2018; K.U. 2012]**

**Proof.** Let P, Q, R be the three forces acting on the rigid body and are in equilibrium. Therefore, the forces are coplanar.

**Case I.** If all the three forces are not parallel, then atleast two of them say P and Q must meet in a point O and therefore the resultant of P and Q passes through O. Since P, Q, R are in equilibrium, therefore R must be equal and opposite to the resultant of P and Q and so R must also pass through O.

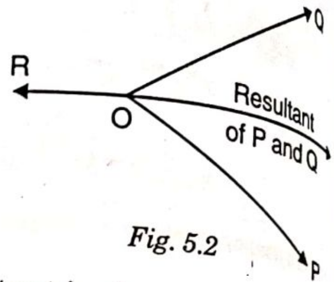
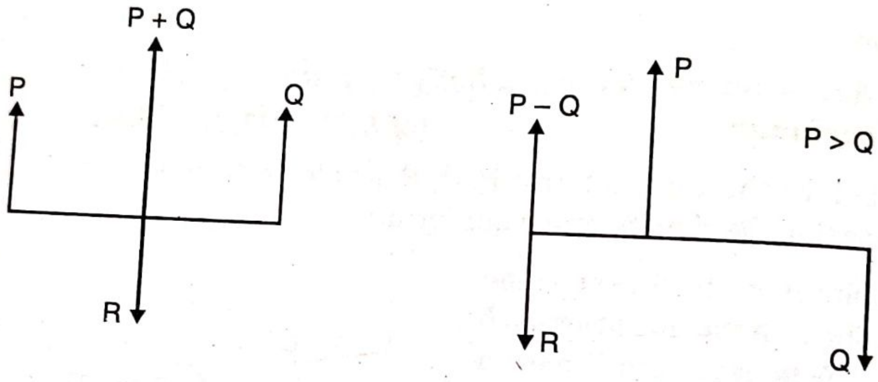


Fig. 5.2

Hence the three forces meet in a point.

**Case II.** If two of the forces P and Q are parallel, their resultant is also parallel to them. Since R balances this resultant, therefore direction of R must coincide with the direction of the resultant of P and Q. Hence the third force R is also parallel to P and Q.



(i) Fig. 5.3 (ii)

## 5.2. CONDITIONS OF EQUILIBRIUM OF THREE FORCES ACTING ON A RIGID BODY

It follows from the above two results that if a rigid body is in equilibrium under the action of three forces, then

(i) they must be coplanar

(ii) they must be concurrent or parallel.

If the three coplanar forces are concurrent, then the following result is used which is the **Lami's theorem.**

*If three coplanar forces acting at a point are in equilibrium, then each is proportional to the sine of the angle between the other two.*

If the three coplanar forces are parallel, then we use analogue of Lami's theorem. Its statement is as under :

*If three parallel forces acting on a rigid body are in equilibrium, then each is proportional to the distance between the other two.*

We shall now illustrate the above by means of a few examples, but before that we shall give two useful trigonometrical formulae and some important points as notes, which will be of considerable help in the solution of a certain class of problems.

## 5.3. TRIGONOMETRICAL THEOREM

**If G is a point in the side AB of a triangle OAB and if OG divides AB into two parts '$m$' and '$n$' and the angle AOB into two parts $\alpha$ and $\beta$, then**

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta \quad ...(1)$$

**and**

$$(m+n)\cot\theta = n\cot A - m\cot B, \quad ...(2)$$

**where $\theta$ is the angle OGB.**

**Proof.** We have

$$\frac{m}{n} = \frac{AG}{GB} = \frac{AG}{GO}\cdot\frac{GO}{GB}$$

$$= \frac{\sin\alpha}{\sin(\theta-\alpha)}\cdot\frac{\sin(\theta+\beta)}{\sin\beta}$$

[*Sine formulae*]

$$= \frac{\sin\alpha}{\sin\beta}\cdot\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\theta\cos\alpha-\cos\theta\sin\alpha}$$

$$= \frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$$

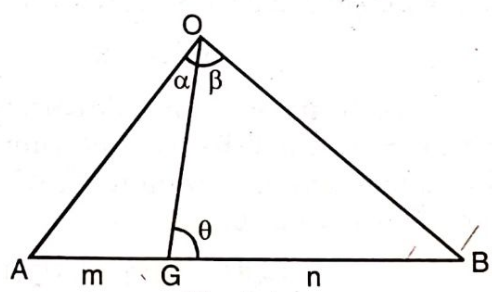


*Fig. 5.4*

$\therefore \quad m(\cot\alpha-\cot\theta) = n(\cot\beta+\cot\theta)$

Hence, $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$.

Again, $$\frac{m}{n} = \frac{AG}{GB} = \frac{AG}{GO}\cdot\frac{GO}{GB} = \frac{\sin(\theta-A)}{\sin A}\cdot\frac{\sin B}{\sin(\theta+B)}$$

$$= \frac{\sin B}{\sin A}\cdot\frac{\sin\theta\cos A-\cos\theta\sin A}{\sin\theta\cos B+\cos\theta\sin B}$$

$$= \frac{\cot A-\cot\theta}{\cot B+\cot\theta}$$

$\therefore \quad m(\cot B+\cot\theta) = n(\cot A-\cot\theta)$

Hence, $(m+n)\cot\theta = n\cot A - m\cot B$.

*Note :*

1. The centre of gravity of a heavy uniform bar, rod or beam is its middle point.
2. Weight of a body, unless it is given to be light, always acts vertically downwards through its C.G.
3. If a string is attached to a body, the force of tension acts along the string in the direction away from the point. *Tension always pulls.*
4. If a body is in contact with some other smooth body, then the reaction acts along the normal to the common surface.

## SOLVED EXAMPLES

### Example 1.

***A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall and the other against a smooth plane inclined to the wall at an angle θ. Prove that if α be the inclination of the rod to the horizontal, then 2 tan α = tan θ.***

[M.D.U. 2014, 10, 2000]

**Solution.** Let $2a$ be the length of the uniform rod AB of weight W with end A in contact with the plane and end B in contact with the wall.

$$\therefore \quad \angle ACB = \theta$$

Let the lines of normal reactions S at A and R at B meet in O. Let G be the mid point of AB. Therefore, the line of action of weight W, which is acting at G vertically passes through O.

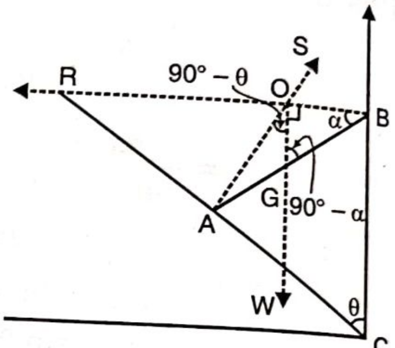



*Fig. 5.5*

$$\therefore \quad \angle BOG = 90°; \ \angle AOG = 90° - \theta;$$

$$\angle ABO = \alpha \quad \text{and} \quad \angle OGB = 90° - \alpha$$

By '$m - n$' theorem in Δ AOB, we get

$$(a + a) \cot (90° - \alpha) = a \cot (90° - \theta) - a \cot 90°$$

or $$2a \tan \alpha = a \tan \theta$$

or $$2 \tan \alpha = \tan \theta.$$

### Example 2.

***A uniform rod AB of length 'a' hangs with one end A against a smooth vertical wall, being supported by a string of length l, attached to the other end of the rod and to a point C of the wall vertically above A. Show that if the rod rests inclined to the wall at an angle θ, then $\cos^2 \theta = \dfrac{l^2 - a^2}{3a^2}$. Also find the limits for 'l' and 'a' for equilibrium to be possible.***

**Solution.** Let G be the middle point of the rod through which the weight W of the rod is acting vertically downwards. Let O be the point of intersection of the line of action of the weight W and the normal reaction R between the wall and the rod. Since these two forces pass through O, the third force *i.e.*, tension of the string, must also pass through O.

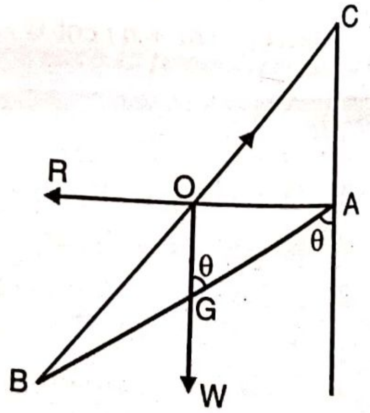



*Fig. 5.6*

Now in Δ ABC, through the middle point of AB, the line GO is drawn parallel to the base AC, therefore it must bisect BC and also must be half of AC.

Hence $OC = \frac{1}{2} BC = \frac{1}{2} l$, where $l$ is the length of the string BC

and
$$AC = 2.OG = 2.AG \cos\theta = a\cos\theta$$

Also,
$$OA = AG\sin\theta = \frac{a}{2}\sin\theta$$

From the right-angled triangle AOC, we have
$$OC^2 = OA^2 + AC^2$$

or
$$\frac{l^2}{4} = \frac{a^2}{4}\sin^2\theta + a^2\cos^2\theta$$

or
$$l^2 = a^2\sin^2\theta + 4a^2\cos^2\theta = a^2 + 3a^2\cos^2\theta$$

$$\therefore \quad \cos^2\theta = \frac{l^2 - a^2}{3a^2}$$

In order that $\theta$ may be real, $\cos^2\theta$ must be positive

*i.e.,*
$$\frac{l^2 - a^2}{3a^2} > 0 \quad \text{or} \quad l > a$$

Also
$$\cos^2\theta < 1$$

$$\therefore \quad \frac{l^2 - a^2}{3a^2} < 1 \quad \text{or} \quad l^2 < 4a^2$$

*i.e.,*
$$l < 2a.$$

Hence, $l$ must lie between $a$ and $2a$.

**Example 3.**

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***A uniform rod AB of weight w and length 2a is hinged at A to a fixed point by means of a smooth hinge. It is kept at rest inclined at an angle α to the vertical by means of a force F applied horizontally at B. Find the magnitude and direction of the reaction at the hinge on the rod and show that the magnitude of F is $\frac{1}{2} w \tan\alpha$.***

[K.U. 2016]

**Solution.** Let the lines of action of $w$ and F meet in O. Join OA. Since two of the forces, F and $w$ meet in O, the reaction R at A must also pass through O and therefore acts along OA. Let $\angle AOG = \theta$. Then by Lami's theorem, we have

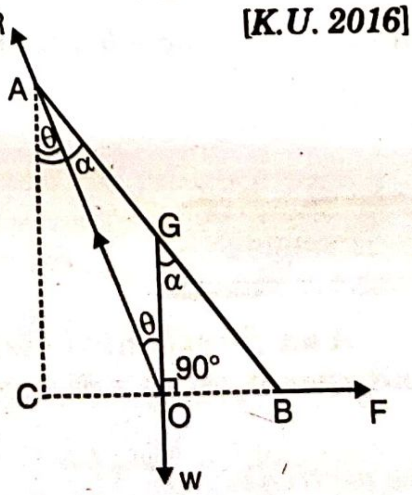


*Fig. 5.7*

$$\frac{R}{\sin 90°} = \frac{F}{\sin(180° - \theta)} = \frac{w}{\sin(90° + \theta)}$$

*i.e.,*
$$\frac{R}{\sin 90°} = \frac{F}{\sin\theta} = \frac{w}{\cos\theta}$$

$$\therefore \quad R = w\sec\theta \quad \text{and} \quad F = w\tan\theta$$

Let the vertical through A meet the horizontal through B in C. Then O is the middle point of BC, so that

$$\tan\theta = \tan CAO = \frac{OC}{AC} = \frac{\frac{BC}{2}}{AC} = \frac{1}{2}\tan\alpha$$

$\therefore$
$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\frac{1}{4}\tan^2\alpha}$$

Hence
$$R = w\sqrt{1+\frac{1}{4}\tan^2\alpha} \quad \text{and} \quad F = \frac{w}{2}\tan\alpha.$$

**Example 4.**

***A beam whose centre of gravity divides it into two portions a and b is placed inside a smooth sphere. Show that if θ be its inclination to the horizon in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere, then***

$$\tan\theta = \frac{b-a}{b+a}\tan\alpha.$$

*[K.U. 1997]*

**Solution.** Let AB be the beam and G be its centre of gravity. Let C be the centre of smooth sphere such that the reactions R and S at A and B will act along the normals to the sphere and so pass through the centre C. The third force W must also pass through C. Hence CG is vertical.

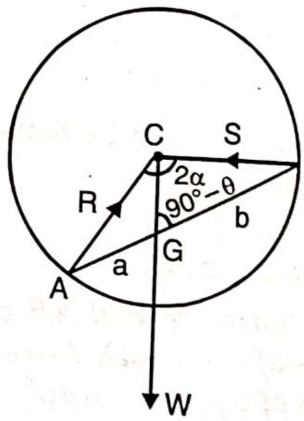



*Fig. 5.8*

Now, $\angle ACB = 2\alpha$

$\therefore$ $\angle CAB = \angle CBA = 90° - \alpha$

Also $\angle CGB = 90° - \theta$

$\therefore$ Using trigonometrical $m-n$ theorem, we have

$$(a+b)\cot\angle CGB = b\cot\angle CAG - a\cot\angle CBG$$

or
$$(a+b)\cot(90°-\theta) = b\cot(90°-\alpha) - a\cot(90°-\alpha)$$

Hence,
$$\tan\theta = \frac{b-a}{b+a}\tan\alpha.$$

**Example 5.**

***A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a peg at a distance 'b' from the wall. Show that the inclination of the beam to the vertical is*** $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$.

*[M.D.U. 2013; K.U. 2011]*

**Solution.** Let AB be the beam of length $2a$ resting against the wall at A and over the peg at C. The weight W of the beam acts vertically downwards through its mid-point G. At A, the reaction S is along the normal to the wall and so acts in a horizontal direction. At point C, the reaction R is normal to the beam AB.

Thus the beam is in equilibrium under the action of three forces W, R and S.

If the lines of action of the forces W and S meet at O, then the direction of R also passes through O.

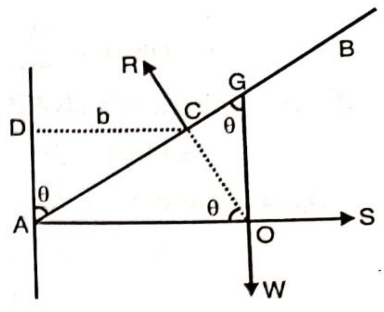



*Fig. 5.9*

Let CD = $b$, be the horizontal distance of C from the wall.

Let the beam make an angle θ with the vertical.

$$\therefore \quad \angle DAC = \angle AGO = \angle COA = \theta$$

$$\therefore \text{ From } \Delta\, CDA, \qquad \sin\theta = \sin\angle DAC = \frac{CD}{AC} = \frac{b}{AC} \qquad ...(1)$$

$$\text{From } \Delta\, AOG, \qquad \sin\theta = \sin\angle AGO = \frac{AO}{AG} = \frac{AO}{a} \qquad ...(2)$$

$$\text{From } \Delta\, AOC, \qquad \sin\theta = \sin\angle COA = \frac{AC}{AO} \qquad ...(3)$$

Multiplying (1), (2) and (3), we get

$$\sin^3\theta = \frac{b}{AC} \times \frac{AO}{a} \times \frac{AC}{AO} = \frac{b}{a}$$

$$\therefore \qquad \sin\theta = \left(\frac{b}{a}\right)^{1/3}$$

$$\text{Hence} \qquad \theta = \sin^{-1}\left(\frac{b}{a}\right)^{1/3}.$$

**Example 6.**

***Equal weights P and P are attached to two strings AC and BC passing over a smooth peg C. AB is a heavy beam of weight W whose centre of gravity is 'a' ft. from A and 'b' ft. from B. Show that AB is inclined to the horizon at an angle***

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

**Solution.** The peg being smooth, the tensions in both the strings are equal to P and their resultant will bisect the angle ACB. This resultant at C is balanced by the weight W acting vertically downwards through G.

$\therefore \quad W = 2P\cos\alpha$, where $\angle ACB = 2\alpha$

$$\therefore \quad \cos\alpha = \frac{W}{2P} \quad \ldots(1)$$

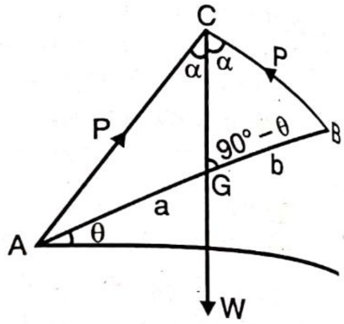


*Fig. 5.10*

If θ is the angle which AB makes with the horizontal, then

$$\angle CGB = 90° - \theta$$

$\therefore$ By trigonometrical theorem in $\Delta$ ACB, we have

$$(a+b)\cot(90° - \theta) = a\cot\alpha - b\cot\alpha$$

$$\therefore \quad \tan\theta = \frac{a-b}{a+b}\cot\alpha = \frac{a-b}{a+b}\tan\left(\frac{\pi}{2} - \alpha\right)$$

$$= \frac{a-b}{a+b}\tan\left[\frac{\pi}{2} - \cos^{-1}\frac{W}{2P}\right] \quad \text{[From (1)]}$$

$$= \frac{a-b}{a+b}\tan\left[\sin^{-1}\frac{W}{2P}\right] \quad \left[\because \sin^{-1}A = \frac{\pi}{2} - \cos^{-1}A\right]$$

Hence, $$\theta = \tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

**Example 7.**

***A cylinder of radius 'r' whose axis is fixed horizontally touches a vertical wall along a generating line. A flat beam of uniform material of length 2l and weight W rests with its extremities in contact with the wall and the cylinder, making an angle of 45° with the vertical. Show that in the absence of friction $\frac{l}{r} = \frac{\sqrt{5}-1}{\sqrt{10}}$; that the pressure on the wall is $\frac{W}{2}$ and the reaction of the cylinder is $\frac{1}{2}W\sqrt{5}$.***

**Solution.** Ref. fig. 5.11, which represents the vertical section through the beam AB. The beam AB is in equilibrium under three forces *i.e.*, its weight W acting through the mid point G of AB, the normal reactions R and S at A and B respectively. They must meet in a point (say C).

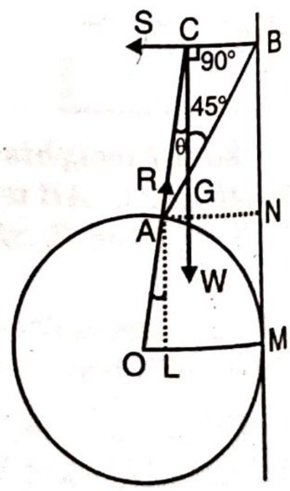


*Fig. 5.11*

$\therefore \quad \angle CGB = 45°, \quad \angle BCG = 90°$

Let $\quad \angle ACG = \theta$.

Using trigonometrical theorem,

$$(AG + GB)\cot CGB = AG\cot ACG - BG\cot BCG$$

or $$2l\cot 45° = l\cot\theta - BG\cot 90°$$

$\Rightarrow$ $$\cot\theta = 2$$

$\therefore$ $$\cos\theta = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{5}}.$$

Using Lami's theorem,

$$\frac{R}{\sin 90°} = \frac{S}{\sin(180° - \theta)} = \frac{W}{\sin(90° + \theta)}$$

$\therefore$ $$R = \frac{W}{\cos\theta} = \frac{W}{2}\sqrt{5}$$

and $$S = W\tan\theta = \frac{W}{2}$$

These give the reactions of the cylinder and the pressure on the wall respectively.

Now, $$OM = OL + LM = OL + AN$$

*i.e.*, $$r = r\sin\theta + AB\cos 45° = \frac{r}{\sqrt{5}} + 2l \cdot \frac{1}{\sqrt{2}}$$

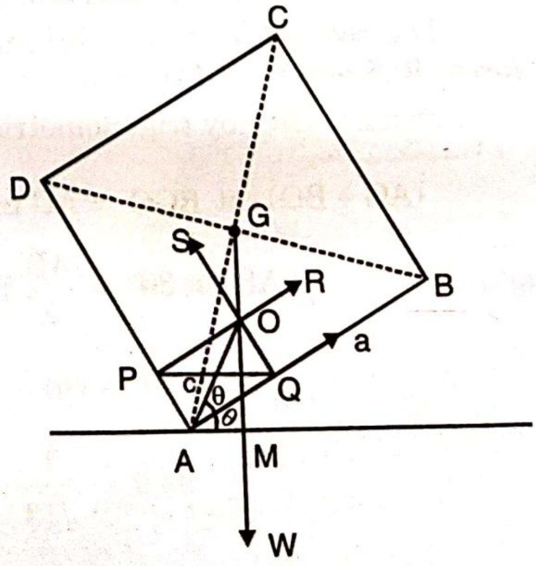
or $$r\sqrt{5} = r + \sqrt{10}\,l$$

or $$r(\sqrt{5} - 1) = l\sqrt{10}$$

$\therefore$ $$\frac{l}{r} = \frac{\sqrt{5} - 1}{\sqrt{10}}.$$

**Example 8.**

***A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance c apart. Show that it will be in equilibrium when the inclination of one of its sides to the horizontal is either 45° or*** $\frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right)$. **[K.U. 2015, 14]**




**Solution.** Let ABCD be the square, resting on two pegs P and Q where PQ is horizontal. Reactions R and S at the pegs are perpendicular to the edges. Let they meet in a point O, then a third force of weight W of the square must also pass through O. Let $\theta$ be the inclination of AB to the horizontal.

Join OA. Now APOQ is rectangle

$\therefore$ $$AO = PQ = c$$

and $$\angle OAQ = \angle PQA = \theta.$$

Let the line of action of W meet the horizontal through A in M.

$\therefore \quad \angle OAM = 2\theta$ and $AM = OA\cos 2\theta = c\cos 2\theta$ ...(1)

Also, $\quad AM = AG \,.\, \cos M\hat{A}G = a\sqrt{2}\cos(45° + \theta)$ ...(2)

[Since $\angle GAQ = 45°$; $AG = a\sqrt{2}$]

Equating the two values of AM, we have

$$c\cos 2\theta = a\sqrt{2}\cos(45° + \theta)$$

or
$$c\cos 2\theta = a\sqrt{2}\left[\cos\theta \,.\, \frac{1}{\sqrt{2}} - \sin\theta \,.\, \frac{1}{\sqrt{2}}\right] = a(\cos\theta - \sin\theta)$$

$$\therefore \quad c(\cos^2\theta - \sin^2\theta) - a(\cos\theta - \sin\theta) = 0$$

or
$$(\cos\theta - \sin\theta)[c(\cos\theta + \sin\theta) - a] = 0$$

$\therefore$ either $\cos\theta - \sin\theta = 0$ or $c(\cos\theta + \sin\theta) - a = 0$

*i.e.* either $\theta = 45°$ or $c(\cos\theta + \sin\theta) = a$

$$\Rightarrow \quad c^2(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) = a^2 \qquad \text{[Squaring]}$$

$$\Rightarrow \quad c^2(1 + \sin 2\theta) = a^2 \Rightarrow \sin 2\theta = \frac{a^2 - c^2}{c^2}$$

Hence, $\quad \theta = \dfrac{1}{2}\sin^{-1}\left(\dfrac{a^2 - c^2}{c^2}\right).$

**Example 9.**

***A smooth hemispherical bowl of diameter 'a' is placed so that its edge touches a smooth vertical wall. A heavy uniform rod is in equilibrium inclined at 60° to the horizontal with one end resting on the inner surface of the bowl and the other end against the wall. Show that the length of rod is*** $\left(a + \dfrac{a}{\sqrt{13}}\right)$.

**Solution.** Let the bowl touch the vertical wall at P and O be the centre of the bowl. Let AB be the uniform rod of weight W and C.G. at G. Let the reactions at A and B be R and S; R is normal to the surface of bowl and S is normal to the wall.

The rod AB is in equilibrium under the action of three forces R, S and W and they meet in the point C.

From $\Delta$ ABC, by trigonometrical theorem, we have

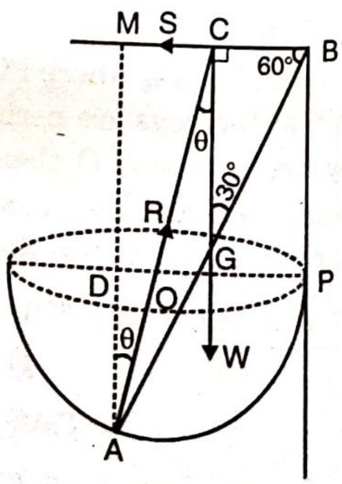



*Fig. 5.13*

$$(AG + BG)\cot B\hat{G}C = AG\cot A\hat{C}G - BG\cot B\hat{C}G$$

*i.e.*,
$$AB\cot 30° = \frac{AB}{2}\cot\theta - \frac{AB}{2}\cot 90°$$

$$\therefore \quad 2\cot 30° = \cot\theta \Rightarrow \cot\theta = 2\sqrt{3}$$

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{13}}.$$

From $\Delta$ AOD; $\quad OD = OA \sin\theta = \dfrac{a}{2}\cdot\dfrac{1}{\sqrt{13}}$

$\therefore \quad BM = DP' = OP + OD = \dfrac{a}{2} + \dfrac{a}{2}\cdot\dfrac{1}{\sqrt{13}}$

Now from $\Delta$ ABM, $\quad AB = BM \sec 60^\circ = \left(\dfrac{a}{2} + \dfrac{a}{2}\cdot\dfrac{1}{\sqrt{13}}\right)\sec 60^\circ$

$$= 2\left(\frac{a}{2} + \frac{a}{2}\cdot\frac{1}{\sqrt{13}}\right) = a + \frac{a}{\sqrt{13}}$$

Hence, length of the rod $= a + \dfrac{a}{\sqrt{13}}$.

**Example 10.**

***A rod rests wholly within a smooth hemispherical bowl of radius r, its centre of gravity dividing the rod into two portions of lengths a and b. Show that if $\theta$ be the inclination of the rod to the horizon in the position of equilibrium, then sin $\theta = \dfrac{b-a}{2\sqrt{r^2 - ab}}$ and find the reactions between the rod and the bowl.***

**Solution.** Let AB be the rod and G the centre of gravity, so that AG = $a$, GB = $b$.

Let O be the centre of sphere of which the given hemispherical bowl is a part and OD be perpendicular to AB.

The reactions R and S at A and B and the vertical through G must meet at O. Since the rod is inclined at an angle $\theta$ to the horizontal

$\therefore \quad \angle OGD = 90^\circ - \theta.$

Now, $\quad GD = AD - AG$

$$= \frac{a+b}{2} - a = \frac{b-a}{2}$$

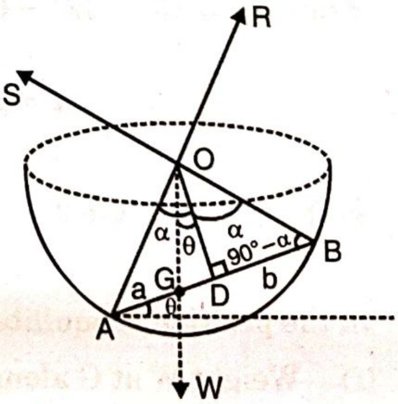


*Fig. 5.14*

and $\quad OD^2 = OA^2 - AD^2 = r^2 - \left(\dfrac{a+b}{2}\right)^2$

$\therefore \quad OG^2 = OD^2 + GD^2$

$$= r^2 - \left(\frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2 = r^2 - ab$$

$\therefore \quad \sin\theta = \sin G\hat{O}D = \dfrac{GD}{OG} = \dfrac{b-a}{2\sqrt{r^2 - ab}}.$

To find reactions, we use Lami's theorem :

If $\angle AOD = \alpha = \angle BOD$, then

$$\frac{R}{\sin(\alpha - \theta)} = \frac{S}{\sin(\alpha + \theta)} = \frac{W}{\sin 2\alpha}$$

$\therefore$
$$R = W\frac{\sin(\alpha - \theta)}{\sin 2\alpha}$$

and
$$S = W\frac{\sin(\alpha + \theta)}{\sin 2\alpha}.$$

**Example 11.**

***A hemisphere of radius 'a' and weight W is placed with its curved surface on a smooth table and a string of length l (< a) is attached to a point on its rim and to a point on the table. Find the position of equilibrium and prove that the tension of the string is*** $\dfrac{3W}{8}\cdot\dfrac{a-l}{\sqrt{(2al-l^2)}}$.

[M.D.U. 2005]

**Solution.** Let C be the centre of the base and G the centre of gravity of the hemisphere. Then $CG = \dfrac{3a}{8}$.

Let A be the point of the hemisphere in contact with the smooth table and T be the tension in the string BM whose length is $l$.

The reaction R at the point A is vertical and passes through the centre C.

From fig. 5.15, $AC = CL + LA = CL + BM$

*i.e.,*
$$a = a\sin\theta + l \Rightarrow \sin\theta = \frac{a-l}{a}$$

so that,
$$\tan\theta = \frac{a-l}{\sqrt{(2al-l^2)}}.$$

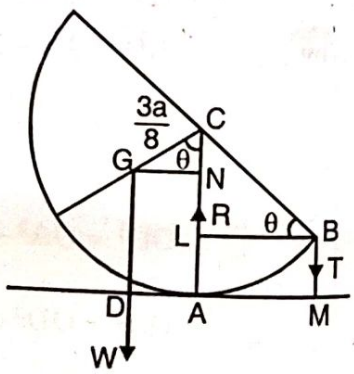


*Fig. 5.15*

In the position of equilibrium, the forces acting are

(*i*) Weight W at G along GD.

(*ii*) Tension T in the string along BM

(*iii*) The reaction R at A along AC.

Taking moment about A, we have

$$T \,.\, AM = W \,.\, AD$$

*i.e.,*
$$T \,.\, BL = W . NG$$

or $$T.\, a \cos\theta = W \,.\, \frac{3a}{8} \sin\theta$$

or $$T = \frac{3W}{8} \tan\theta = \frac{3W}{8} \cdot \frac{a-l}{\sqrt{(2al-l^2)}}.$$

### Example 12.

***Two uniform rods AB, BC rigidly joined at B so that angle ABC is a right angle, hang freely in equilibrium from a fixed point A. The length of the rods are a and b and their weights are 'aw' and 'bw'. Prove that if AB makes an angle θ with the vertical, then***

$$\tan\theta = \frac{b^2}{a^2 + 2ab}$$ [C.D.L.U. 2013; M.D.U. 2007]

**Solution.** The rods being rigidly joined at B, their mutual actions at B neutralize. Then the forces left are :

(*i*) weight $aw$ acting through D, the mid-point of AB

(*ii*) weight $bw$ acting through E, the mid-point of BC

(*iii*) the action R at the fixed point A.

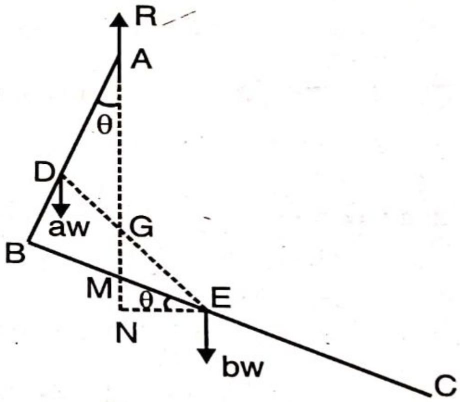


*Fig. 5.16*

The three forces are in equilibrium, therefore common centre of gravity of $aw$ and $bw$ *i.e.*, the point G is vertically below A; hence AG is vertical.

Taking moments about the point A, we have

$$aw\,.\,AD\sin\theta = bw\,.\,NE = bw\,.\,ME\cos\theta$$
$$= bw\,.\,(BE - BM)\cos\theta$$

*i.e.*, $$aw\,\frac{a}{2}\sin\theta = bw\,.\left(\frac{b}{2} - a\tan\theta\right)\cos\theta$$

*i.e.*, $$a^2\sin\theta = b\,(b\cos\theta - 2a\sin\theta)$$

or $$(a^2 + 2ab)\sin\theta = b^2\cos\theta$$

$$\therefore \quad \tan\theta = \frac{b^2}{a^2 + 2ab}.$$

## EXERCISE 5.1

**1. A uniform rod has its lower end fixed to a hinge and its other end attached to a string which is tied to a point vertically above the hinge. Show that the direction of the reaction at the hinge bisects the string.**

2. A heavy uniform rod, 15 cms long is suspended from a fixed point by strings attached to its ends, their lengths being 9 and 12 cms. Show that the rod is inclined to the vertical at an angle $\sin^{-1}\left(\frac{24}{25}\right)$. [M.D. U. 2006]

3. A heavy uniform rod of mass 7 kg and length 5 metres is suspended from a fixed point by inextensible strings fastened to its ends, their lengths being 3 and 4 metres. Find the tensions in the strings.

4. A uniform rod can turn freely about one of its ends and is pulled aside from the vertical by a horizontal force acting at the other end of the rod and equal to half its weight. At what inclination to the vertical will the rod rest ? [M.D.U. 2015]

5. A uniform rod AB can turn freely about a hinge at A and to the other end B is attached a string which passes over a smooth pulley at C, vertically above A and supports a mass of weight P at its free end. Prove that in the position of equilibrium, $BC = \frac{2P}{W} \cdot AC$.

6. A uniform rod AB, movable about a hinge at A rests with one end in contact with a smooth wall. If $\alpha$ be the inclination of the rod to the vertical, prove that the reaction at the hinge is $\frac{1}{2} W\sqrt{3+\sec^2\alpha}$. Also find its direction. [C.D.L.U. 2012; M.D.U. 2007]

7. A rod is movable in a vertical plane about a hinge at one end and at the other end is fastened a weight equal to half the weight of the rod and this end is fastened by a string of length $l$, to a point at a height $c$ vertically over the hinge. Show that the tension of the string is $\frac{lW}{c}$, where W is the weight of the rod. [M.D.U. 2005]

8. A uniform rod rests with its ends in contact with two smooth planes inclined at angles $\alpha$ and $\beta$ to the horizon. If $\theta$ be the inclination of the beam to the horizon, show that $2\tan\theta = \cot\alpha - \cot\beta$. Also find the reactions at the ends of the rod. [M.D.U. 2011]

9. A heavy uniform rod of length $2l$ is resting in contact with a smooth sphere of radius $r$ and the lower end of the rod presses against a smooth vertical wall. If the distance of the centre of the sphere from the wall is $c$ and the rod makes an angle $\alpha$ with the wall, prove that $l\sin^3\alpha + r\cos\alpha = c$. [M.D.U. 2015, 12, 07]

10. A heavy uniform rod of length $2a$ rests partly within and partly outside a fixed smooth hemi-spherical bowl of radius $r$. The rim of the bowl is horizontal and one point of the rod is in contact with the rim. If $\theta$ is the inclination of the rod with the horizontal, prove that $2r\cos 2\theta = a\cos\theta$. Also, show that the greatest inclination of the rod that can thus rest is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$. [M.D.U. 2016]

11. A triangular lamina ABC is suspended from a point O by light strings attached to points A and B and hangs so that BC is vertical. If $\alpha$, $\beta$ be the angles which the strings AO, BO make with the vertical, prove that : $2\cot\alpha - \cot\beta = 3\cot B$. [K.U. 2015]

12. A heavy uniform beam is hung from a fixed point by two strings attached to its extremities. If the lengths of the strings and beam be as 2 : 3 : 4, show that the tensions of the strings and the weight of the beam are as $2 : 3 : \sqrt{10}$. [C.D.L.U. 2016]

13. One end of a string of length $2l$ is attached to one end A of a smooth uniform rod of length $2a$ and the other end of the string is attached to a light ring C which slides on the rod. The string passes over a smooth pulley O. Prove that in the position of equilibrium, the inclination $\theta$ of the rod to the horizontal is given by $a \cos^3 \theta = l \sin \theta$.

14. A solid cone of height $h$ and semi-vertical angle $\alpha$ is placed with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall. Show that the greatest possible length of the string is $h\sqrt{1+\frac{16}{9}\tan^2\alpha}$.

[K.U. 2016, 13, 10, 08, 05; M.D.U. 2012, 08]

15. Two uniform rods AB, BC of length $2a$, $2b$ respectively are rigidly joined at B and are suspended freely from A. If they rest inclined at an angle $\theta$, $\phi$ respectively to the vertical, prove that

$$\frac{\sin\theta}{\sin\phi} = \frac{b^2}{a(a+2b)}.$$ [K.U. 2012]

16. A rigid wire, without weight, in the form of an arc of a circle subtending an angle $\alpha$ at its centre and having two weights P and Q at its extremities, rests with its convexity downwards upon a horizontal plane. Show that if $\theta$ be the inclination to the vertical of the radius to the end at which P is suspended, then $\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$.

17. A square uniform plate is suspended at one of its vertices and a weight equal to half that of the plate is suspended from the adjacent vertex of the square. Find the position of equilibrium of the plate.

## ANSWERS

3. $\frac{21}{5}$ kg and $\frac{28}{5}$ kg.

4. 45°

6. $\tan^{-1}\left(\frac{1}{2}\tan\alpha\right)$ with vertical

8. $R = \frac{W\sin\beta}{\sin(\alpha+\beta)}$; $S = \frac{W\sin\alpha}{\sin(\alpha+\beta)}$

17. Adjacent side is inclined at an angle $\tan^{-1}\left(\frac{1}{2}\right)$ to vertical.

## 5.4. CONDITIONS OF EQUILIBRIUM OF ANY NUMBER OF CO-PLANAR FORCES

There are three different forms of finding the conditions of equilibrium of a system of coplanar forces acting on a rigid body which are given below.

**5.4.1. First Form :** *The necessary and sufficient conditions that a system of coplanar forces acting on a rigid body be in equilibrium are*

(*i*) *the algebraic sum of the resolved parts of the forces along a direction OX is zero*

(*ii*) *the algebraic sum of the resolved parts of the forces along a direction OY is zero*

(*iii*) *the algebraic sum of the moments of the forces about any point in their plane is zero.*

[*M.D.U.* 2008, 05]

Let O be any point in the plane of the forces and OX and OY be any two perpendicular lines through O as the co-ordinate axes. Then the system can be reduced to a single force R acting at O together with a couple whose moment G is equal to the algebraic sum of the moment of the forces about O.

Now, $R^2 = X^2 + Y^2$, where X and Y are the algebraic sums of the resolved parts of all the forces along OX and OY respectively.

If the given system of forces be in equilibrium then the force R and the couple G must both vanish because a force and a couple cannot balance each other. Hence for equilibrium, the necessary conditions are $R = 0$ and $G = 0$.

But, since $R^2 = X^2 + Y^2$ [sum of the squares of two real quantities X and Y], therefore if $R = 0$, then $X = 0$, $Y = 0$.

Hence the necessary conditions are $\mathbf{X = 0}$, $\mathbf{Y = 0}$ and $\mathbf{G = 0}$.

**Conversely.** Let the algebraic sums of the resolved parts of all the forces along two perpendicular directions vanish separately and also the algebraic sum of the moments of all the forces about a point in their plane be zero.

*i.e.*, $X = 0$, $Y = 0$ and $G = 0$. This gives $R = 0$ and $G = 0$.

Thus the system is in equilibrium. This proves the sufficiency of the given conditions.

Hence the necessary and sufficient conditions that a system of coplanar forces be in equilibrium are $\mathbf{X = 0}$, $\mathbf{Y = 0}$, $\mathbf{G = 0}$.

**Remark :**

It may be noted that it is not necessary that the two directions may be perpendicular. All that is required is that there should be two different directions along which the algebraic sum of the resolved parts of all the forces vanish separately. For, let OA and OB be any two direction along which the algebraic sum of the resolved parts of all the forces vanish separately. Then the resolved parts of R along OA and OB are also zero. Since resolved part of R along OA is zero, therefore R must either be zero or perpendicular to OA. Similarly R must either be zero or perpendicular to OB. But R cannot be perpendicular to two different directions, therefore R must be zero. Hence the result.

### 5.4.2. Second Form

*A system of co-planar forces shall be in equilibrium if the algebraic sum of the moments of all the forces about any three non-collinear points in their plane vanish separately.*

[C.D.L.U. 2016]

Let A, B, C be three non-collinear points about which the algebraic sum of the moments of all the forces vanish separately. Since any system of coplanar forces can be reduced to either a single force or to a single couple or it is in equilibrium. Here the system cannot reduce to a single couple because the algebraic sum of the moments of all the forces about any point in the plane is zero.

Therefore either the system reduces to a single resultant force or it is in equilibrium.

Suppose, if possible, that the system reduces to a single force R. Since the algebraic sum of the moments of all the forces about A is zero, therefore the moment of their resultant R about A is also zero. Hence R is either equal to zero or it passes through A.

Similarly either R is zero or it passes through B and C. Since a force cannot pass through three non-collinear points, it follows that R must be equal to zero.

Hence the system must be in equilibrium.

### 5.4.3. Third Form

*A system of coplanar forces shall be in equilibrium if the algebraic sum of the moments of all the forces about any two points in their plane vanish separately and the algebraic sum of their resolved parts be zero in any direction not perpendicular to the line joining the two points.*

[C.D.L.U. 2013; K.U. 2005]

Any system of coplanar forces can be reduced to either a single force or to a single couple. Here the system cannot reduce to a couple only as the algebraic sum of the moments of all the forces about a point in their plane is given to be zero.

Therefore the system must either reduce to a single resultant force or be in equilibrium.

Let A and B be the two points about which the sums of the moments are zero and let $\theta$ be the angle made with AB be the line along which the sum of the resolved parts is zero. Then by hypothesis, $\theta$ is not equal to a right angle and hence $\cos\theta \neq 0$.

If possible, let the system reduce to a single resultant force R. Since the algebraic sum of the moments of all the forces about A is zero, therefore the moment of the resultant R about A is also zero. Hence R must either be zero or it passes through A.

Similarly R must either be zero or it must pass through B. Hence R must act along AB.

Resolving along the direction in which the algebraic sum of the resolved parts of all the forces is zero, we get $R\cos\theta = 0$. Since $\theta \neq 90°$ therefore, $\cos\theta \neq 0$ and hence it follows that R must be equal to zero. Therefore, the system cannot reduce to a single resultant force also.

Hence the system must be in equilibrium.

## 5.5. GENERAL WORKING RULE FOR SOLUTION OF PROBLEMS

The following remarks regarding the solution of problems should be noted :

(i) *A neat and clear figure should be drawn, carefully marking all the forces acting on the body.*

(ii) *Write down any geometrical relations that are provided in the figure.*

(iii) *Take two suitable perpendicular directions as axes OX and OY. Generally the horizontal and vertical directions are found quite convenient. In the case of problems where inclined planes are involved, the directions along and perpendicular to the plane are sometimes found more suitable.*

(iv) *Equate to zero the sum of the resolved parts of the force along OX as well as along OY.*

(v) *Equate to zero the algebraic sum of the moments of the forces about a suitable point (i.e., on point through which the maximum number of forces pass).*

(vi) *If a rod is placed on a peg, the reaction will be perpendicular to the rod.*

(vii) *If a rod is placed on the floor or against a wall, the reaction will be perpendicular to the floor or the wall.*

(viii) *If a rod is placed inside the sphere, the reaction at the ends pass through the centre of the sphere.*

(ix) *Reaction at the hinge is of the adjusting type. If the body is in equilibrium under three coplanar forces and the two forces are intersecting, then the reaction at the hinge also passes through the point of intersection. In case the two forces are parallel, the reaction at the hinge is also parallel to them.*

## SOLVED EXAMPLES

### Example 1.

***A uniform ladder rests at an angle α to the horizon, with its ends resting on a smooth floor and against a smooth vertical wall, the lower end being attached by a string to the junction of the wall and the floor. Find the tension of the string.***

***Find also the tension of the string when a man, whose weight is one-half that of the ladder, has ascended the ladder two-third of its length.***

**Solution.** Let AB be the ladder of length $2a$ making an angle $\alpha$ with the horizontal and G be its centre of gravity. Let T be the tension of the string AO; R, S be the reactions of the wall and the floor and W, W′ the weights of the rod and the man acting vertically at G and M respectively.

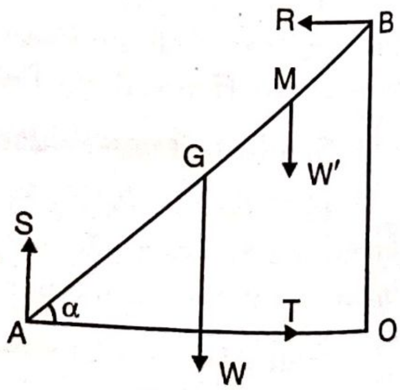


*Fig. 5.17*

Resolving horizontally and vertically,

$$T = R \quad ...(1)$$

and

$$W + W' = S \quad ...(2)$$

Taking moments about A, we have

$$W \, . \, AG \cos \alpha + W' \, . \, AM \cos \alpha = R \, . \, OB$$

*i.e.*, $$W \, . \, a \cos \alpha + W' \, . \, \frac{2}{3} \, . \, 2a \cos \alpha = R \, . \, 2a \, . \, \sin \alpha$$

*i.e.*, $$3W + 4W' = 6 R \tan \alpha$$

*i.e.*, $$R = \frac{1}{6} (3W + 4W') \cot \alpha = T \quad ...(3)$$

For the first case, $W' = 0$

$\therefore$ From (3), $$T = \frac{1}{2} W \cot \alpha$$

For the second case, $$W' = \frac{1}{2} W$$

$$\therefore \quad T = \frac{5}{6} W \cot \alpha \, .$$

**Example 2.**

***A uniform square plate ABCD can turn freely in a vertical plane about a hinge at A. Forces equal to W and 2W act respectively along the sides BC and CD where W is the weight of the plate. Find the force along DB which will keep the plate at rest with AB horizontal. Also, find the reaction at the hinge.*** [*M.D.U. 2006*]

**Solution.** Let P be the force which acting along DB keeps the plate in equilibrium.

Let X, Y be the horizontal and vertical components of the reaction at A.

Taking moments about A, we have

$$P \, . \, a\sqrt{2} = W \, . \, 2a + 2W \, . \, 2a + W \, . \, a,$$

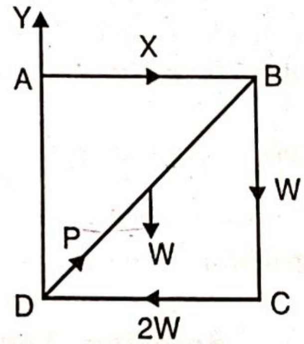


*Fig. 5.18*

where $2a$ is the side of the square.

$$\therefore \quad P = \frac{7}{2} W \sqrt{2}$$

Resolving horizontally and vertically, we get

$$X + P \cos 45° - 2W = 0$$

$$\Rightarrow \quad X = -\frac{3}{2} W$$

and $$Y + P \sin 45° - W - W = 0$$

$$\Rightarrow \quad Y = -\frac{3}{2} W$$

Hence the reaction at the hinge $= \sqrt{X^2 + Y^2} = \sqrt{\frac{9}{4} W^2 + \frac{9}{4} W^2} = \frac{3\sqrt{2}}{2} W \, .$

## Example 3.

***One end of a uniform beam, of weight W, is placed on a smooth horizontal plane; the other end to which a string is fastened rests against another smooth inclined plane inclined at an angle α to the horizon. The string passing over a pulley at the top of the inclined plane, hangs vertically and supports a weight P. Show that the beam will rest in all positions if 2P = W sin α.***

**Solution.** Let AB be the beam inclined at an angle $\theta$ to the horizontal, its weight W acting at the middle point G. As the string passes over the pulley at D, the tension in the two parts of the string will be the same *i.e.*, P.

Therefore the force at B on the beam is P along BD.

Let R and S be normal reactions at A and B. Taking moments about B, we have $W. l \cos\theta = R \cdot 2l \cos\theta$, where $2l$ is the length of the beam.

$$\therefore \qquad R = \frac{W}{2}$$

Resolving along the plane CD, we get

$$P + R\cos(90° - \alpha) = W\cos(90° - \alpha)$$

*i.e.*,
$$P + \frac{W}{2}\sin\alpha = W\sin\alpha$$

Hence,
$$2P = W\sin\alpha.$$

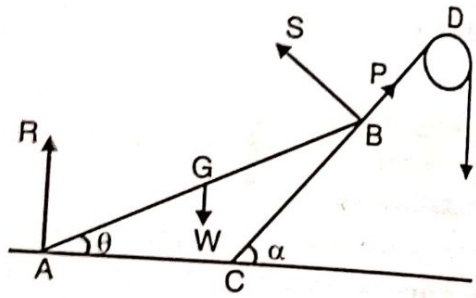



*Fig. 5.19*

## Example 4.

***A step ladder in the form of the letter 'A' with each of its legs inclined at an angle α to the vertical, is placed on a horizontal floor and is held up by a chord connecting the middle point of its legs, there being no friction anywhere. Show that when a weight W is placed on one of the steps at a height from the floor equal to $\frac{1}{n}$ of the height of the ladder, the increase in the tension of the chord is $\frac{1}{n}$ W tan α.***

**Solution.** Let AB, AC be the two parts of the ladder inclined at an angle α to the vertical, held up by the string DE.

Let $2a$ be the length and $w$ be the weight of each of the parts.

Let W be the weight placed at P, where

$$BP = \frac{1}{n}BA = \frac{2a}{n}.$$

Let T be the tension in the string and R, S the normal reactions at B and C.

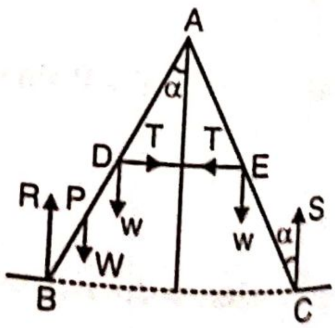



*Fig. 5.20*

Taking moments about B for the whole ladder, we have

$$S \,.\, BC = w \,.\, \frac{3}{4} BC + w \,.\, \frac{1}{4} BC + W \,.\, \frac{1}{2n} BC$$

$$\therefore \qquad S = w + \frac{W}{2n}.$$

Taking moments about A for the ladder AC, we have

$$w \,.\, a \sin \alpha + T \,.\, a \cos \alpha = S \,.\, 2a \sin \alpha$$

$$\therefore \qquad T = (2S - w) \tan \alpha = \left( w + \frac{W}{n} \right) \tan \alpha \qquad ...(1)$$

When there is no extra weight in the ladder, let $T_0$ be the tension. Then putting $W = 0$ in (1), we get $T_0 = w \tan \alpha$.

$$\text{Hence increase in tension} = T - T_0 = \left( w + \frac{W}{n} \right) \tan \alpha - w \tan \alpha$$

$$= \frac{1}{n} W \tan \alpha \,.$$

**Example 5.**

***Three uniform rods AB, BC and CD whose weights are proportional to their lengths are joined at B and C and are in a horizontal position resting on two pegs P and Q. Find the actions at the joints B and C and show that the distance between the pegs is***

$$\frac{a^2}{2a+b} + \frac{c^2}{2c+b} + b$$

***where a, b, c are the lengths of the rods.***

**Solution.** Let AB, BC, CD be three uniform rods with $G_1$, $G_2$ and $G_3$ as their mid-points where the weights $aw$, $bw$ and $cw$ act vertically downwards. Let the pressures at the pegs P and Q be R and S respectively.

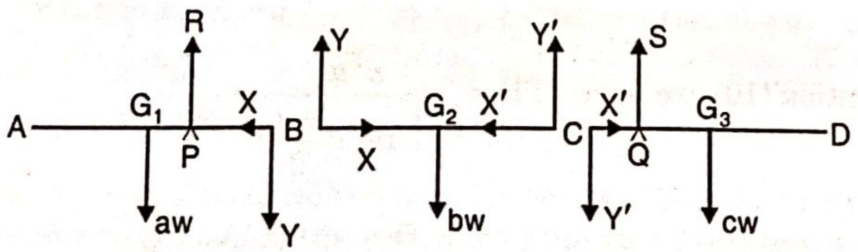



*Fig. 5.21*

Resolving the forces horizontally, we have $X = 0$ ...(1)

Resolving the forces vertically, we have $Y + aw = R$ ...(2)

Taking moments about point P, we get

$$aw\,.\,G_1P - Y\,.\,PB = 0 \quad \text{...(3)}$$

**For rod BC :**

Resolving the forces horizontally, we have $\quad X = X' \quad$ ...(4)

Resolving the forces vertically, we have $\quad Y + Y' = bw \quad$ ...(5)

Taking moments about $G_2$, we get $\quad Y = Y' \quad$ ...(6)

**For rod CD :**

Resolving the forces horizontally, we have $\quad X' = 0 \quad$ ...(7)

Resolving the forces vertically, we have $\quad Y' + cw = S \quad$ ...(8)

Taking moments about Q, we get

$$Y'CQ - cw\,.\,G_3Q = 0 \quad \text{...(9)}$$

From equations (5) and (6), we have

$$2Y' = bw \quad \Rightarrow \quad Y' = \frac{bw}{2}$$

$\therefore$

$$Y = Y' = \frac{bw}{2} \quad \text{...(10)}$$

From equations (2) and (10), we have $\quad R = \dfrac{bw}{2} + aw \quad$ ...(11)

From equations (8) and (10), we have $\quad S = \dfrac{bw}{2} + cw \quad$ ...(12)

Equations (11) and (12) give the pressure at the pegs P and Q.

Now, distance between the pegs = PQ = PB + BC + CQ ...(13)

From equation (3), we have. $aw\,.\,G_1P - Y\,.\,PB = 0$

*i.e.,*

$$aw\left(\frac{a}{2} - PB\right) - YPB = 0 \Rightarrow PB = \frac{a^2w}{2\,(aw + Y)}$$

Using equation (10), we have $\quad PB = \dfrac{a^2w}{2\left(aw + \dfrac{bw}{2}\right)} = \dfrac{a^2}{2a + b} \quad$ ...(14)

Now, from equation (9), we have $\quad Y\,.\,CQ = cw\,(CG_3 - CQ)$

or

$$Y\,.\,CQ = cw\left(\frac{c}{2} - CQ\right) \quad \Rightarrow \quad CQ = \frac{c^2w}{2\,(Y + cw)}$$

Using equation (10), we have $CQ = \dfrac{c^2 w}{2\left(\dfrac{bw}{2} + cw\right)} = \dfrac{c^2}{2c + b}$ ...(15)

From equations (13), (14) and (15), we have $PQ = \dfrac{a^2}{2a + b} + \dfrac{c^2}{2c + b} + b$.

**Example 6.**

***Two equal heavy rods of weight W and length 2a are freely hinged together and placed symmetrically over a smooth fixed sphere of radius r. Show that the inclination of each rod to the horizontal is given by $r(\tan^3\theta + \tan\theta) = a$.***

[K.U. 2015, 06; M.D.U. 2011, 06]

**Solution.** Let AB, AC be two rods symmetrically placed with respect to the sphere with centre O.

Due to symmetry, the reactions at the points of contact D, E of the rods with the sphere will be equal. Let each be equal to R. Also the reactions at A balance each other.

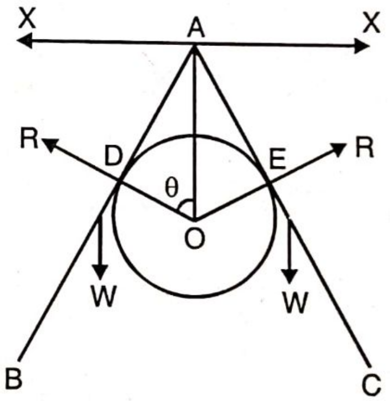



*Fig. 5.22*

Now, the external forces acting on the rods are :

(*i*) the weight W, acting vertically downwards at the middle point of each rod.

(*ii*) the normal reactions R; R passing through O.

Resolving the forces vertically, we have

$$2W = 2R\cos\theta \quad \Rightarrow \quad W = R\cos\theta \qquad ...(1)$$

Taking moments about A for the rod AB, we have

$$W \,.\, a\cos\theta = R \,.\, AD \qquad ...(2)$$

But $AD = r\tan\theta$

$\therefore$ From (1) and (2), $\quad a\cos\theta = r\tan\theta \,.\, \sec\theta$

or
$$r\tan\theta\sec^2\theta = a \quad \Rightarrow \quad r(\tan^3\theta + \tan\theta) = a.$$

**Example 7.**

***One end of a uniform rod is attached to a hinge and the other end is supported by a string attached to the extremity of the rod; the rod and the string are inclined at the same angle θ to the horizontal. If W be the weight of the rod, show that the reaction at the hinge is $\frac{1}{4} W\sqrt{8 + \text{cosec}^2\theta}$. Also find the tension of the string.***

[K.U. 2014, 07; M.D.U. 2014, 08, 01]

**Solution.** Let AB be the uniform rod, A the hinge and BC the string so that AB and BC make the angle $\theta$ with the horizontal AC.

Let R be the reaction at the hinge and X, Y, its horizontal and vertical components.

Let $2a$ be the length of the rod and T, the tension in the string.

Resolving horizontally and vertically, we have

$$X + T\cos\theta = 0 \qquad ...(1)$$

and
$$Y - W + T\sin\theta = 0 \qquad ...(2)$$

$$\angle ABC = 180° - 2\theta$$

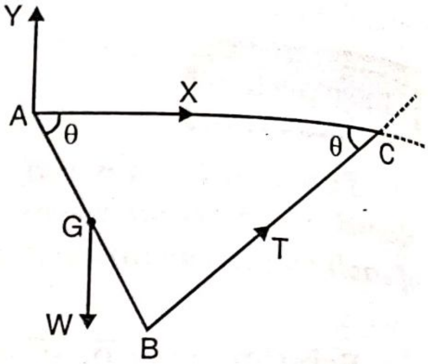


*Fig. 5.23*

Now taking moments about A,

$$W.AG\cos\theta = T.AB\sin(180° - 2\theta),$$

or
$$W.a\cos\theta = T.2a\sin 2\theta$$

$$T = \frac{W\cos\theta}{2\sin 2\theta} = \frac{W}{4\sin\theta} = \frac{W}{4}\text{cosec}\,\theta \qquad ...(3)$$

$\therefore$ From (1), $X = -T\cos\theta = -\dfrac{W}{4}\cot\theta$

and from (2), $Y = W - T\sin\theta = W - \dfrac{W}{4} = \dfrac{3W}{4}$

$$\therefore \quad R = \sqrt{X^2 + Y^2} = \frac{W}{4}\sqrt{9 + \cot^2\theta} = \frac{W}{4}\sqrt{8 + 1 + \cot^2\theta}$$

$$= \frac{W}{4}\sqrt{8 + \text{cosec}^2\,\theta}\,.$$

## EXERCISE 5.2

1. A ladder of length 40 m and mass 100 kg rests with one end on a smooth floor and the other against a smooth vertical wall. It is kept in equilibrium by a horizontal string attached to its lower end. Find the tension of the string when the ladder is inclined at 30° to the horizontal.

2. A uniform ladder of weight W rests with one end against a smooth vertical wall and with the other end resting on a smooth floor. If the inclination of the ladder to the horizon be 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from sliding down.

   *[M.D.U. 2008, 05]*

3. A uniform ladder of weight 20 kg rests with one end against a smooth vertical wall and the other end resting on a smooth floor. If the inclination of the ladder to the horizontal is 30°, find the horizontal force that must be applied to the lower end to prevent the ladder from sliding down. Also find the reactions at the floor and the wall.

4. A uniform ladder whose weight is 150 kg and whose length is 20m rests with one end against a smooth vertical wall and the other against a smooth floor, the lower end being joined by a string of length 10 m to the junction of the wall and floor. Find the tension of the string and the reactions of the floor and the wall.

5. A uniform beam AB, 17 m long whose mass is 120 kg rests with one end against a smooth vertical wall and the other end on a smooth horizontal floor, this end being tied by a chord 8 m long to a peg at the bottom of the wall. Find the tension of the chord.

[*M.D.U. 2016, 09*]

6. A uniform rod AB of weight W rests with its ends A and B on two smooth planes inclined to the horizontal at angles of 45° and 30° respectively, in such a way that the vertical plane through the rod is perpendicular to both the inclined planes. Find what weight fixed to the rod at a quarter of its length from the end B will suffice to enable the rod to rest horizontally.

7. A ladder of length $2a$ and weight W with its centre of gravity $\frac{3}{8}$th of the way up it, stands on a smooth horizontal plane resting against a smooth vertical wall and the middle point is tied to a point in the wall by a horizontal rope of length $l$. Find the tension of the rope and the reactions of the wall and the horizontal plane. [*K.U. 2016; M.D.U. 2013*]

8. A ladder of length $2b$ and weight W has its centre of gravity at a distance $\frac{3}{4}b$ way up the ladder, stands on a smooth horizontal plane resting against a smooth vertical wall and middle point is tied to a point in the wall by a horizontal rope of length $l$. Find the tension in the rope and the reactions due to wall and the horizontal plane. [*M.D.U. 2007*]

9. A beam of weight W is divided by its centre of gravity C into two portions AC and BC whose lengths are $a$ and $b$ respectively. The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and to the beam at a point P. If T be the tension of the string and $\theta$ and $\phi$ be the inclinations of the beam and string respectively to the horizon, show that $T = W \cdot \dfrac{a\cos\theta}{(a+b)\sin(\theta-\phi)}$.

10. Two equal beams AB, AC each of weight W connected by a hinge at A are placed in a vertical plane with their extremities B, C resting on a horizontal plane. They are kept from falling by strings connecting B and C with the middle points of opposite beams. Show that the tension of each string is $\frac{W}{8}\sqrt{9\cot^2\theta+1}$, $\theta$ being the inclination to the horizon of each beam. [*M.D.U. 2011*]

11. Two equal uniform rods AB and AC are freely joined at A. The ends B and C are connected by a fine string. The rods are suspended from B by a string. If in the position of equilibrium the angle between the rods is $2\alpha$, then show that the tension in the string is $\dfrac{3W\sin\alpha}{\sqrt{1+3\sin^2\alpha}}$, where 2W is the weight of either rod. [*K.U. 2016; M.D.U. 2004*]

12. Two equal uniform rods AB and AC each of weight W, are smoothly hinged at A and placed in a vertical plane with the ends B and C resting in a smooth horizontal plane. Equilibrium is preserved by a string which attaches C to the middle point of AB. Show that the tension of the string and the reaction of the rods at A are both equal to $\frac{1}{4} W \sqrt{1 + 9 \cot^2 \alpha}$ and that each is inclined at an angle $\tan^{-1}\left(\frac{1}{3} \tan \alpha\right)$ to the horizon, where $\alpha$ is the inclination of either rod to the horizon.

13. A uniform rod AB of length '$2a$' rests in a vertical plane with its upper end A against a smooth wall, the other end being kept at a distance '$2b$' from the wall by a string fastened to B and to a nail in the wall. Prove that the thrust against the wall is $\frac{Wb}{2\sqrt{a^2 - b^2}}$, where 'W' is the weight of the rod. [C.D.L.U. 2012; M.D.U. 2009]

14. Two equal uniform heavy rods each of length $2l$ are smoothly jointed together at an end and are placed so as to rest symmetrically over a smooth cylinder of radius $r$ fixed with its axis horizontal. Show that in the position of equilibrium the angle $\theta$ between each rod and the vertical is given by $l \sin^3 \theta = r \cos \theta$. [K.U. 2014]

## ANSWERS

1. $50\sqrt{3}$ kg

2. $\frac{W}{2\sqrt{3}}$ kg

3. $F = 10\sqrt{3}$ kg; $R = 20$ kg and $S = 10\sqrt{3}$ kg.

4. $T = S = 25\sqrt{3}$ kg; $R = 150$ kg.

5. 32 kg

6. $\frac{2}{\sqrt{3}} W_1$

7. $S = T = \frac{3}{4} \cdot \frac{Wl}{\sqrt{a^2 - l^2}}$ and $R = W$

8. $S = T = \frac{3}{4} \cdot \frac{Wl}{\sqrt{b^2 - l^2}}$ and $R = W$.