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Euler's Integrals
(Beta and Gamma Functions)

§ 1. Beta Function. Definition. **(Meerut 1988; Kanpur 87, 90)**

The definite integral

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ for } m>0, n>0$$

is called the **Beta function** *and is denoted by* **B** (m, n) [*read as "Beta m, n"*]. *Thus*

$$\mathrm{B}(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

where m, n are positive numbers, integral or fractional. Beta function is also called the Eulerian integral of the first kind.

§ 2. Some Simple properties of Beta function.

(i) Symmetry of Beta function i.e. B (m, n) = B (n, m). **(Agra 1987)**

We have $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, by the def. of the Beta function

$$=\int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}\,dx,$$

$$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$=\int_0^1 (1-x)^{m-1}x^{n-1}\,dx=\int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$=\mathrm{B}(n,m)$, by the def. of Beta function.

Hence $\mathrm{B}(m,n)=\mathrm{B}(n,m)$.

(ii) If m or n is a positive integer, B (m, n) can be evaluated in an explicit form.

Case I. When n is a positive integer. If $n=1$, the result is obvious because $\mathrm{B}(m,1)=\int_0^1 x^{m-1}(1-x)^{1-1}\,dx=\int_0^1 x^{m-1}\,dx$

$$=\left[\frac{x^m}{m}\right]_0^1=\frac{1}{m}.$$

So let us take $n>1$. We have

$$\mathrm{B}(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx=\int_0^1 (1-x)^{n-1}x^{m-1}\,dx$$

$$= \left[(1-x)^{n-1} \cdot \frac{x^m}{m} \right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1) \cdot \frac{x^m}{m}\, dx,$$

integrating by parts taking x^{m-1} as the second function

$$= 0 + \frac{n-1}{m} \cdot \int_0^1 x^m (1-x)^{n-2}\, dx, \quad [\because n > 1]$$

$$= \frac{n-1}{m} \cdot \int_0^1 x^{(m+1)-1} (1-x)^{(n-1)-1}\, dx$$

$$= \frac{n-1}{m} \mathrm{B}(m+1, n-1).$$

By the repeated application of this process, we get

$$\mathrm{B}(m,n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \mathrm{B}(m+n-1, 1)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2}(1-x)^0\, dx$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2}\, dx$$

$$= \frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)} \cdot \left[\frac{x^{m+n-1}}{m+n-1} \right]_0^1.$$

$$\therefore \quad \mathrm{B}(m,n) = \frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)(m+n-1)}.$$

Case II. When m is a positive integer. Since the Beta function is symmetrical in m and n *i.e.*, $\mathrm{B}(m,n) = \mathrm{B}(n,m)$, therefore by case I, we conclude that

$$\mathrm{B}(m,n) = \frac{(m-1)!}{n(n+1)(n+2)\cdots\cdots(n+m-2)(n+m-1)}.$$

(iii) If both m and n are positive integers, then

$$\mathbf{B(m, n)} = \frac{\mathbf{(m-1)!\,(n-1)!}}{\mathbf{(m+n-1)!}}.$$

From (ii), we have

$$\mathrm{B}(m,n) = \frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)(m+n-2)\cdots\cdots(m+1)\,m\,(m-1)!},$$

writing the denominator in the reversed order and multiplying the Nr and Dr by $(m-1)!$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}.$$

Solved Examples

Ex. 1. *Express the following integrals in terms of Beta function :*

(i) $\int_0^1 x^m (1-x^2)^n \, dx, m > -1, n > -1$;

(ii) $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} \, dx$.

Sol. (i) We have

$\int_0^1 x^m (1-x^2)^n \, dx = \int_0^1 x^{m-1} (1-x^2)^n . x \, dx$ (Note)

$= \int_0^1 y^{(m-1)/2} (1-y)^n \cdot \frac{dy}{2}$, putting $x^2 = y$ so that $2x \, dx = dy$

$= \frac{1}{2} \int_0^1 y^{(m-1)/2} (1-y)^n \, dy = \frac{1}{2} \int_0^1 y^{[(m+1)/2]-1} (1-y)^{(n+1)-1} \, dy$

$= \frac{1}{2} \mathrm{B} \left(\frac{1}{2}(m+1), n+1\right)$.

(ii) We have $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} \, dx = \int_0^1 x^2 (1-x^5)^{-1/2} \, dx$

$= \int_0^1 x^2 \cdot \frac{1}{x^4} (1-x^5)^{-1/2} . x^4 \, dx = \int_0^1 x^{-2} (1-x^5)^{-1/2} x^4 \, dx$

$= \int_0^1 y^{-2/5} (1-y)^{-1/2} \cdot \frac{1}{5} dy$, putting $x^5 = y$ so that $5x^4 \, dx = dy$

$= \frac{1}{5} \int_0^1 y^{-2/5} (1-y)^{-1/2} \, dy = \frac{1}{5} \int_0^1 y^{(3/5)-1} (1-y)^{(1/2)-1} \, dy$

$= \frac{1}{5} \mathrm{B} \left(\frac{3}{5}, \frac{1}{2}\right)$.

Ex. 2. *Prove that*

$$\int_0^a (a-x)^{m-1} . x^{n-1} \, dx = a^{m+n-1} \mathrm{B}(m, n).$$

Sol. We have

$$\int_0^a (a-x)^{m-1} x^{n-1} \, dx$$

$= \int_0^1 (a-ay)^{m-1} (ay)^{n-1} a \, dy$, putting $x = ay$

$= \int_0^1 a^{(m-1)+(n-1)+1} (1-y)^{m-1} y^{n-1} \, dy$

$= a^{m+n-1} \int_0^1 y^{n-1} (1-y)^{m-1} \, dy$

$= a^{m+n-1} \mathrm{B}(n, m) = a^{m+n-1} \mathrm{B}(m, n)$,

$[\because \mathrm{B}(m, n) = \mathrm{B}(n, m)]$.

Ex. 3. *Show that if m, n are positive, then*

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx = (b-a)^{m+n-1} . \mathrm{B}(m, n).$$

Sol. The given integral is

$$\int_a^b (x-a)^{m-1} (b-x)^{n-1} \, dx.$$

Put $x = a + (b-a) y$ so that $dx = (b-a) \, dy$.

Also when $x = a, y = 0$ and when $x = b, y = 1$.

$$\therefore \int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx$$

$$= \int_0^1 [(b-a)y]^{m-1}[b-a-(b-a)y]^{n-1}.(b-a)\,dy$$

$$= \int_0^1 (b-a)^{m-1}.y^{m-1}.(b-a)^{n-1}.(1-y)^{n-1}.(b-a)\,dy$$

$$= (b-a)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$= (b-a)^{m+n-1}\,\mathrm{B}(m,n).$$

Ex. 4. *Show that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \frac{1}{(a+b)^m.a^n}\,\mathrm{B}(m,n).$$

Sol. The given integral

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1}\cdot\left(\frac{1-x}{a+bx}\right)^{n-1}\cdot\frac{1}{(a+bx)^2}\,dx. \qquad \textbf{[Note]}$$

Put $\dfrac{x}{a+bx} = \dfrac{y}{a+b}$ so that $\dfrac{(a+bx).1-x.b}{(a+bx)^2}\,dx = \dfrac{dy}{a+b}$

i.e., $\dfrac{1}{(a+bx)^2}\,dx = \dfrac{dy}{a(a+b)}$.

Further

$$\frac{1-x}{a+bx} = \frac{1}{a}\frac{a-ax}{a+bx} = \frac{1}{a}\left[\frac{a+bx-ax-bx}{a+bx}\right] = \frac{1}{a}\left[1-\frac{x(a+b)}{a+bx}\right]$$

$$= \frac{1-y}{a}.$$

Also when $x = 0, y = 0$ and when $x = 1, y = 1$.

$$\therefore \quad I = \int_0^1 \left(\frac{y}{a+b}\right)^{m-1}\left(\frac{1-y}{a}\right)^{n-1}\cdot\frac{dy}{a(a+b)}$$

$$= \frac{1}{(a+b)^m.a^n}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \frac{\mathrm{B}(m,n)}{(a+b)^m.a^n}.$$

Ex. 5. *Show that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\mathrm{B}(m,n)}{a^n(1+a)^m}.$$

Sol. Proceed exactly as in Ex. 4. Here we have $b = 1$.

***Ex. 6.** *Show that*

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2},\frac{q+1}{2}\right), p > -1, q > -1.$$

Deduce that $\int_0^2 x^4 (8 - x^3)^{-1/3} dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$.

Sol. We have $\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta$

$$= \int_0^{\pi/2} \sin^{p-1} \theta \cos^{q-1} \theta \sin \theta \cos \theta \, d\theta \qquad \textbf{(Note)}$$

$$= \int_0^{\pi/2} \sin^{p-1} \theta . (1 - \sin^2 \theta)^{\frac{1}{2}(q-1)} \sin \theta \cos \theta \, d\theta.$$

Put $\sin^2 \theta = x$ so that $2 \sin \theta \cos \theta \, d\theta = dx$.

Also when $\theta = 0, x = 0$ and when $\theta = \frac{1}{2}\pi, x = 1$.

$\therefore$ the given integral

$$= \int_0^1 x^{(p-1)/2} . (1-x)^{(q-1)/2} . \tfrac{1}{2} dx$$

$$= \tfrac{1}{2} \int_0^1 x^{\{(p+1)/2\} - 1} (1-x)^{\{(q+1)/2\} - 1} dx$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), \qquad \text{...(1)}$$

where $\frac{p+1}{2} > 0$ *i.e.*, $p > -1$ and $\frac{q+1}{2} > 0$ *i.e.*, $q > -1$.

Second part. We have

$$I = \int_0^2 x^4 (8 - x^3)^{-1/3} dx = \int_0^2 x^2 (8 - x^3)^{-1/3} . x^2 dx. \qquad \textbf{(Note)}$$

Put $x^3 = 8 \sin^2 \theta$ so that $3x^2 dx = 16 \sin \theta \cos \theta \, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 2, \theta = \frac{1}{2}\pi$.

$$\therefore \quad I = \int_0^{\pi/2} (8 \sin^2 \theta)^{2/3} (8 - 8 \sin^2 \theta)^{-1/3} . \tfrac{16}{3} \sin \theta \cos \theta \, d\theta$$

$$= \int_0^{\pi/2} 4 \sin^{4/3} \theta \cdot \tfrac{1}{2} \cos^{-2/3} \theta \cdot \tfrac{16}{3} \sin \theta \cos \theta \, d\theta$$

$$= \tfrac{32}{3} \int_0^{\pi/2} \sin^{7/3} \theta \cos^{1/3} \theta \, d\theta$$

$$= \frac{32}{3} \cdot \frac{1}{2} B\left(\frac{\frac{7}{3} + 1}{2}, \frac{\frac{1}{3} + 1}{2}\right), \qquad \left[\text{from (1); here } p = \frac{7}{3}, q = \frac{1}{3}\right]$$

$$= \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right).$$

Ex. 7. *Prove that if* p, q *are positive, then*

(*i*) $\frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}$;

(*ii*) $B(p, q) = B(p+1, q) + B(p, q+1)$.

Sol. (i) We have $\frac{B(p, q+1)}{q} = \frac{1}{q} B(p, q+1)$

$$=\frac{1}{q}\int_0^1 x^{p-1}(1-x)^{(q+1)-1}\,dx=\frac{1}{q}\int_0^1 (1-x)^q\,.\,x^{p-1}\,dx$$

(Note)

$$=\frac{1}{q}\left[\left\{(1-x)^q\cdot\frac{x^p}{p}\right\}_0^1-\int_0^1 q\,(1-x)^{q-1}(-1)\cdot\frac{x^p}{p}\,dx\right],$$

(integrating by parts)

$$=\frac{1}{q}\left[0+\frac{q}{p}\int_0^1 x^{(p+1)-1}(1-x)^{q-1}\,dx\right]=\frac{1}{p}\,\mathbf{B}\,(p+1,q).\quad ...(1)$$

Again $\dfrac{\mathbf{B}\,(p,q+1)}{q}=\dfrac{1}{p}\,\mathbf{B}\,(p+1,q)$, from (1)

$$=\frac{1}{p}\int_0^1 x^p(1-x)^{q-1}\,dx,$$ by the def. of Beta function

$$=\frac{1}{p}\int_0^1 x^{p-1}.x(1-x)^{q-1}\,dx$$ **(Note)**

$$=\frac{1}{p}\int_0^1 x^{p-1}[1-(1-x)](1-x)^{q-1}\,dx$$ **(Note)**

$$=\frac{1}{p}\int_0^1 x^{p-1}(1-x)^{q-1}\,dx-\frac{1}{p}\int_0^1 x^{p-1}(1-x)^q\,dx$$

$$=\frac{1}{p}\,\mathbf{B}\,(p,q)-\frac{1}{p}\,\mathbf{B}\,(p,q+1).$$

$$\therefore\quad \frac{\mathbf{B}\,(p,q+1)}{q}+\frac{\mathbf{B}\,(p,q+1)}{p}=\frac{\mathbf{B}\,(p,q)}{p}$$

or $$\frac{\mathbf{B}\,(p,q+1)}{q}=\frac{\mathbf{B}\,(p,q)}{p+q}.\quad ...(2)$$

Now from (1) and (2), we have

$$\frac{\mathbf{B}\,(p,q+1)}{q}=\frac{\mathbf{B}\,(p+1,q)}{p}=\frac{\mathbf{B}\,(p,q)}{p+q}.\quad ...(3)$$

(Proved.)

From (3), we have

$$\mathbf{B}\,(p,q+1)=\frac{q}{p+q}\,\mathbf{B}\,(p,q)\text{ and }\mathbf{B}\,(p+1,q)=\frac{p}{p+q}\,\mathbf{B}\,(p,q).$$

Adding, we get

$$\mathbf{B}\,(p+1,q)+\mathbf{B}\,(p,q+1)$$

$$=\frac{p}{p+q}\,\mathbf{B}\,(p,q)+\frac{q}{p+q}\,\mathbf{B}\,(p,q)=\frac{p+q}{p+q}\,\mathbf{B}\,(p,q)=\mathbf{B}\,(p,q).$$

Ex. 8. *Prove that*

$$\frac{B\,(m+1,n)}{B\,(m,n)}=\frac{m}{m+n}.$$ **(Meerut 1991 P)**

Sol. We have

$$B(m+1, n) = B(n, m+1)$$

[by the symmetry of Beta function]

$$= \int_0^1 x^{n-1} (1-x)^{(m+1)-1} dx = \int_0^1 (1-x)^m x^{n-1} dx \quad \textbf{(Note)}$$

$$= \left[(1-x)^m \cdot \frac{x^n}{n}\right]_0^1 - \int_0^1 m(1-x)^{m-1}(-1) \cdot \frac{x^n}{n} dx,$$

(integrating by parts)

$$= 0 + \frac{m}{n} \int_0^1 x^{n-1} \cdot x (1-x)^{m-1} dx$$

$$= \frac{m}{n} \int_0^1 x^{n-1} [1-(1-x)] (1-x)^{m-1} dx$$

$$= \frac{m}{n} \left[\int_0^1 x^{n-1} (1-x)^{m-1} dx - \int_0^1 x^{n-1} (1-x)^m dx\right]$$

$$= \frac{m}{n} \left[B(n, m) - B(n, m+1)\right] = \frac{m}{n} B(m, n) - \frac{m}{n} B(m+1, n)$$

or $\quad \left(1 + \dfrac{m}{n}\right) B(m+1, n) = \dfrac{m}{n} B(m, n)$

or $\quad (n+m) B(m+1, n) = m B(m, n)$

or $\quad \dfrac{B(m+1, n)}{B(m, n)} = \dfrac{m}{m+n}.$

**§ 3. Another form of Beta Function.

$$\mathbf{B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx, \; m > 0, n > 0.}$$

(Meerut 1994, 96 BP; Rohilkhand 84; Agra 78)

Proof. By the definition of Beta function, we have

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

Put $x = \dfrac{1}{1+y}$ so that $dx = -\dfrac{1}{(1+y)^2} dy.$

Also when $x \to 0, y \to \infty$ and when $x = 1, y = 0.$

$$\therefore \quad B(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}} \cdot \left[1 - \frac{1}{1+y}\right]^{n-1} \cdot \left[-\frac{1}{(1+y)^2}\right] dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1+2}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx, \quad \text{[by a property of definite integrals]} \quad \ldots(1)$$

Again since Beta function is symmetrical in m and n, we have

$$B(m,n) = B(n,m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx, \qquad \text{by (1).}$$

Thus $$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx,$$
$$m > 0, n > 0.$$

Ex. 9. *Prove that*

$$\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = 0, \; m > 0, n > 0.$$

Sol. The given integral is

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$
$$= B(m,n) - B(n,m) = B(m,n) - B(m,n) = 0.$$

Ex. 10. *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx$ *in terms of Beta function where* $m > 0, n > 0, a > 0, b > 0.$

Sol. In the given integral put $bx = ay$ i.e., $x = (a/b)y$ so that $dx = (a/b)\,dy$. When $x = 0, y = 0$ and when $x \to \infty, y \to \infty$.

$$\therefore \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}\,dx = \int_0^\infty \left(\frac{a}{b}y\right)^{m-1} \cdot \frac{1}{(a+ay)^{m+n}} \cdot \frac{a}{b}\,d$$
$$= \int_0^\infty \frac{a^{m-1} y^{m-1} a}{b^{m-1} \cdot a^{m+n} (1+y)^{m+n} b}\,dy = \frac{1}{a^n b^m} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}$$
$$= \frac{1}{a^n b^m} B(m,n), \text{ [by § 3].}$$

§ 4. Gamma Function. Definition.

(Gorakhpur 1982, 87; Rohilkhand 80, 77; Agra 78, 76; Meerut 88, 90)

The definite integral

$\int_0^\infty e^{-x} x^{n-1}\,dx$, *for* $n > 0$

is called the **Gamma Function** and is denoted by $\Gamma(n)$ [read as "Gamma n"]. Thus

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx, \text{ for } n > 0.$$

Gamma function is also called **Eulerian integral of the second kind.**

§ 5. Fundamental property of Gamma function.

To prove that

(i) $\Gamma(n+1) = n\,\Gamma(n)$, *where* $n > 0$ **(Rohilkhand 1980, 77)**

and (ii) $\Gamma(n) = (n-1)!$, *where n is a positive integer.*

Proof. By the definition of gamma function, we have

$$\Gamma(n+1)=\int_0^\infty e^{-x}x^{(n+1)-1}\,dx=\int_0^\infty x^n e^{-x}\,dx$$

$$=\left[-e^{-x}x^n\right]_0^\infty+\int_0^\infty e^{-x}.\,nx^{n-1}\,dx, \quad \text{...(1)}$$

integrating by parts taking e^{-x} as the second function.

Now $\lim_{x\to\infty}\dfrac{x^n}{e^x}=\lim_{x\to\infty}\dfrac{x^n}{1+x+(x^2/2!)+\ldots+(x^n/n!)+\ldots}$

$$=\lim_{x\to\infty}\frac{1}{\dfrac{1}{x^n}+\dfrac{1}{x^{n-1}}+\ldots+\dfrac{1}{n!}+\dfrac{x}{(n+1)!}+\ldots\ldots}=\frac{1}{\infty}=0.$$

$\therefore$ from (1), we get

$$\Gamma(n+1)=0+n\int_0^\infty e^{-x}x^{n-1}\,dx, \quad [\because n>0]$$

$=n\,\Gamma(n)$, which proves the result (i).

(ii) We have $\Gamma(n)=\Gamma[(n-1)+1]=(n-1)\,\Gamma(n-1)$,

$[\because \Gamma(n+1)=n\,\Gamma(n)]$.

Similarly $\Gamma(n-1)=(n-2)\,\Gamma(n-2)$,... etc.

Hence if n is a +ive integer, then proceeding as above, we get

$$\Gamma(n)=(n-1)(n-2)\ldots 2.1\,\Gamma(1).$$

But $\Gamma(1)=\int_0^\infty e^{-x}x^{1-1}\,dx=\int_0^\infty e^{-x}x^0\,dx=\int_0^\infty e^{-x}.\,1\,dx$

$$=\left[\frac{e^{-x}}{-1}\right]_0^\infty=-\left[\lim_{x\to\infty}\frac{1}{e^x}-e^0\right]=-[0-1]=1.$$

Hence $\Gamma(n)=(n-1)(n-2)\ldots 2.1.1=(n-1)!$

if n is a + ive integer.

Remember. $\Gamma(n)=(n-1)\,\Gamma(n-1)$, where $n>1$ and $\Gamma(1)=1$.

Also it may be remarked that $\Gamma(0)=\infty$ and $\Gamma(-n)=\infty$ where n is a positive integer.

§ 6. Some transformations of Gamma function.

We have $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$. ...(1)

(i) Put $x=ay$ so that $dx=a\,dy$; when $x=0, y=0$ and when $x\to\infty, y\to\infty$.

$\therefore \quad \Gamma(n)=\int_0^\infty e^{-ay}a^n y^{n-1}\,dy$.

Hence $\int_0^\infty e^{-ay}y^{n-1}\,dy=\dfrac{\Gamma(n)}{a^n}$. **(Remember)**

(Gorakhpur 1982; Agra 78)

(ii) In (1) if we put $x=\log(1/y)$ or $y=e^{-x}$ so that

$$dy = -e^{-x}\,dx,$$

then $\Gamma(n) = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$

(Rohilkhand 1980, 77)

(iii) In (1) if we put $x^n = y$ so that $nx^{n-1}\,dx = dy$, we get

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-(y)^{1/n}}\,dy$$

or $$\int_0^\infty e^{-(y)^{1/n}}\,dy = n\,\Gamma(n) = \Gamma(n+1).$$

§ 7. Relation between Beta and Gamma Functions.

To prove that

$$\mathbf{B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}, \text{ where } \mathbf{m > 0, n > 0}.$$

(Agra 1980, 84, 86; Gorakhpur 87, 89; Meerut 91 P, 95, 97)

Proof. We have

$$\frac{\Gamma(m)}{z^m} = \int_0^\infty e^{-zx} x^{m-1}\,dx. \qquad \text{[See § 6, part (ii)]}$$

$$\therefore \quad \Gamma(m) = z^m \int_0^\infty e^{-zx}.x^{m-1}\,dx = \int_0^\infty z^m e^{-zx} x^{m-1}\,dx.$$

Multiplying both sides by $e^{-z} z^{n-1}$, we get

$$\Gamma(m)\,e^{-z} z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1}\,dx. \qquad ...(1)$$

Now integrating both sides of (1) with respect to z from 0 to ∞, we get

$$\Gamma(m)\int_0^\infty e^{-z} z^{n-1}\,dz = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1}\,dx\right] dz$$

or $$\Gamma(m)\,\Gamma(n) = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1}\,dz\right] x^{m-1}\,dx$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1}\,dx, \qquad \text{[by § 6, part (ii)]}$$

$$= \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \text{(Agra 1976)}$$

$$= \Gamma(m+n).B(m, n), \text{ by § 3}.$$

$$\therefore \quad B(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Thus remember that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

Cor. $\mathbf{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$, **where** $\mathbf{0 < n < 1}$.

(Meerut 1991 P, 92, 96 P; Rohilkhand 78; Agra 77)

Proof. We know that $B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$, [See § 3]

and $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where $m > 0$ and $n > 0$.

$$\therefore \quad \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Putting $m+n=1$ or $m = 1-n$ in the above relation, we get

$$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\,dx, \text{ where } 0 < n < 1.$$

[Note that $m > 0 \Rightarrow 1-n > 0 \Rightarrow n < 1$.]

But $\Gamma(1) = 1$. Also

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}. \qquad \textbf{[Remember it]}$$

$$\therefore \quad \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

§ 8. The value of $\Gamma(\frac{1}{2})$.

To prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(Meerut 1991; Rohilkhand 84, 87, 91; Agra 80)

Proof. We know that $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ ...(1)

If we take $m = \frac{1}{2}$, $n = \frac{1}{2}$, then from (1), we have

$$B(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{[\Gamma(\frac{1}{2})]^2}{\Gamma(1)} = [\Gamma(\tfrac{1}{2})]^2. \qquad [\because \Gamma(1) = 1]$$

Thus $[\Gamma(\frac{1}{2})]^2 = B(\frac{1}{2},\frac{1}{2})$

$$= \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx,$$

by the definition of Beta function

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$$

Now put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$$\therefore \quad [\Gamma(\tfrac{1}{2})]^2 = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\Big[\theta\Big]_0^{\pi/2}$$

$$= 2[\tfrac{1}{2}\pi - 0] = \pi.$$

Taking square root of both the sides, we get

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (Remember)

Important Deduction. *To prove that* $\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

(Gorakhpur 1985; Rohilkhand 79; Meerut 96 P)

Proof. Let $I = \int_0^\infty e^{-x^2} dx$.

Put $x^2 = z$ so that $2x\,dx = dz$

or $$dx = \frac{1}{2x} dz = \frac{1}{2\sqrt{z}} dz = \tfrac{1}{2} z^{-1/2} dz.$$

Also when $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \quad I = \int_0^\infty e^{-z} \tfrac{1}{2} z^{-1/2} dz = \tfrac{1}{2} \int_0^\infty e^{-z} z^{1/2-1} dz = \tfrac{1}{2} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

Hence $\displaystyle\int_0^\infty \mathbf{e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$. (Remember)

**** § 9. To prove that for all values of m and n such that m > − 1, n > − 1,**

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

Proof. Put $\sin^2 \theta = x$ so that $2 \sin \theta \cos \theta \, d\theta = dx$

or $$2 \sin \theta . \sqrt{(1 - \sin^2 \theta)} \, d\theta = dx$$

or $$2x^{1/2} . \sqrt{(1 - x)} \, d\theta = dx.$$

$$\therefore \quad d\theta = \frac{dx}{2x^{1/2} (1 - x)^{1/2}}.$$

Also when $\theta = 0, x = 0$ and when $\theta = \frac{1}{2}\pi, x = 1$.

$$\therefore \quad \int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \int_0^{\pi/2} (1 - \sin^2 \theta)^{m/2} . \sin^n \theta \, d\theta$$

$$= \int_0^1 (1 - x)^{m/2} . x^{n/2} . \frac{dx}{2x^{1/2} (1 - x)^{1/2}}$$

$$= \tfrac{1}{2} \int_0^1 x^{(n-1)/2} (1 - x)^{(m-1)/2} dx$$

$$= \tfrac{1}{2} \int_0^1 x^{\{(n+1)/2\} - 1} (1 - x)^{\{(m+1)/2\} - 1} dx$$

$$= \tfrac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right), \quad \text{provided } m > -1 \text{ and } n > -1$$

$$= \tfrac{1}{2} \frac{\Gamma \frac{1}{2}(m+1) \Gamma \frac{1}{2}(n+1)}{\Gamma \frac{1}{2}(m + 1 + n + 1)}, \quad \left[\because B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\Gamma \frac{1}{2}(m+1)\,\Gamma \frac{1}{2}(n+1)}{2\,\Gamma \frac{1}{2}(m+n+2)}.$$

§ 10. Some Important Transformations of Beta Function.

Beta function can be transformed into many other forms. A few of them are given below.

(i) We know that $\displaystyle\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \mathrm{B}(m,n)$.

Now

$$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$

Making the substitution $y = 1/x$ in the last integral, we get

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}}.$$

$$\therefore \quad \mathrm{B}(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Hence $\displaystyle\int_0^1 \frac{\mathbf{x}^{m-1}+\mathbf{x}^{n-1}}{(1+\mathbf{x})^{m+n}}\,\mathbf{dx} = \mathbf{B}(\mathbf{m},\mathbf{n}) = \frac{\Gamma(\mathbf{m})\,\Gamma(\mathbf{n})}{\Gamma(\mathbf{m}+\mathbf{n})}$.

(ii) We know that $\displaystyle\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \mathrm{B}(m,n)$.

If we put $x = \dfrac{ay}{b}$, so that $dx = \dfrac{a}{b}\,dy$, we get

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy.$$

$$\therefore \quad \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy = \frac{1}{a^m b^n}\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{1}{a^m b^n}\,\mathrm{B}(m,n).$$

Hence $\displaystyle\int_0^\infty \frac{\mathbf{y}^{m-1}}{(\mathbf{ay}+\mathbf{b})^{m+n}}\,\mathbf{dy} = \frac{\Gamma(\mathbf{m})\,\Gamma(\mathbf{n})}{\mathbf{a}^m\mathbf{b}^n\,\Gamma(\mathbf{m}+\mathbf{n})}$.

Again putting $y = \tan^2\theta$ *i.e.*, $dy = 2\tan\theta\sec^2\theta\, d\theta$ in the integral just obtained, we get

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma(m)\cdot\Gamma(n)}{2a^m b^n\,\Gamma(m+n)}.$$

(iii) We know that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = B(m,n)$.

Putting $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\, d\theta$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta.$$

$$\therefore \quad \int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{B(m,n)}{2} = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}.$$

(Kanpur 1987)

This result may also be written in the form

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\dfrac{p+1}{2}\right)\cdot\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q+2}{2}\right)},$$

by putting $2m-1=p$ and $2n-1=q$.

(iv) We know that $\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = B(m,n)$.

Putting $y = \dfrac{x-b}{a-b}$, so that $dy = \dfrac{dx}{a-b}$, we have

$$\int_0^1 y^{m-1}\cdot(1-y)^{n-1}\,dy = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1}\left(\frac{a-x}{a-b}\right)^{n-1}\cdot\frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx.$$

$$\therefore \quad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

or $$\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}B(m,n)$$

$$= (a-b)^{m+n-1}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

***§ 11. Duplication formula.**

To prove that $\Gamma(m)\,\Gamma(m+\frac{1}{2}) = \dfrac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$, where $m > 0$.

(Meerut 1990, 93 P, 98; Kanpur 81; Agra 80, Gorakhpur 89; U.P. P.C.S. 96)

Proof. We know that

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \text{ where } m>0, n>0.$$

If we take $n = m$, then

$$B(m, m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)}. \quad \text{...(1)}$$

Again by the definition of Beta function, we have

$$B(m, m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx.$$

Let us put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.
Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

Then $B(m, m) = \int_0^{\pi/2} \sin^{2(m-1)}\theta \,.\cos^{2(m-1)}\theta \,.2\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,.\cos^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi \cdot \frac{d\phi}{2}, \text{ putting } 2\theta = \phi \text{ so that } d\theta = \tfrac{1}{2}d\phi$$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi = \frac{1}{2^{2m-1}} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi$$

(Note)

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1}\phi\,.\cos^0\phi\,d\phi$$ (Note)

$$= \frac{1}{2^{2m-2}} \frac{\Gamma\frac{1}{2}(2m-1+1)\,\Gamma\frac{1}{2}(0+1)}{2\,\Gamma\frac{1}{2}(2m-1+0+2)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

$$= \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}. \quad \text{...(2)}$$

$[\because \Gamma(\frac{1}{2}) = \sqrt{\pi}]$

Now equating the two values of $B(m, m)$ obtained in (1) and (2), we get

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}} \cdot \frac{\Gamma(m)\,.\sqrt{\pi}}{\Gamma(m+\frac{1}{2})}$$

or

$$\Gamma(m)\,\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m). \quad \textbf{(Remember)}$$

§ 12. To find the value of

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right),$$

where n is a positive integer. **(Meerut 1983)**

Proof. Let $A = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$. ...(1)

Writing the above expression in the reverse order, we have

$$A = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\dots\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right). \quad \dots(2)$$

Multiplying (1) and (2), we get

$$A^2 = \Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\dots\dots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

$$= \frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\sin\frac{n-1}{n}\pi}. \quad \text{[See corollary of § 7]}$$

To calculate this expression, we factorize $1 - x^{2n}$.

Now the roots of the equation $x^{2n} - 1 = 0$ are given by

$$x = (1)^{1/2n} = (\cos 2r\pi + i\sin 2r\pi)^{1/2n}$$

$$= \cos\frac{r\pi}{n} + i\sin\frac{r\pi}{n}, \text{ where } r = 0, 1, 2, \dots, 2n-1.$$

Hence, we have $1 - x^{2n}$

$$= (1-x)(1+x)\left(x - \cos\frac{\pi}{n} - i\sin\frac{\pi}{n}\right)\left(x - \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)\dots$$

$$\dots\left(x - \cos\frac{n-1}{n}\pi - i\sin\frac{n-1}{n}\pi\right)\left(x - \cos\frac{n-1}{n}\pi + i\sin\frac{n-1}{n}\pi\right)$$

$$= (1-x^2)\left(1 - 2x\cos\frac{\pi}{n} + x^2\right)\left(1 - 2x\cos\frac{2\pi}{n} + x^2\right)\dots$$

$$\dots\left(1 - 2x\cos\frac{n-1}{n}\pi + x^2\right).$$

$$\therefore \quad \frac{1-x^{2n}}{1-x^2} = \left(1 - 2x\cos\frac{\pi}{n} + x^2\right)\left(1 - 2x\cos\frac{2\pi}{n} + x^2\right)\dots$$

$$\dots\left(1 - 2x\cos\frac{n-1}{n}\pi + x^2\right).$$

Putting $x = 1$ and $x = -1$ respectively, we have in the limit,

$$n = \left(2 - 2\cos\frac{\pi}{n}\right)\left(2 - 2\cos\frac{2\pi}{n}\right)\dots\left(2 - 2\cos\frac{n-1}{n}\pi\right)$$

and $$n = \left(2 + 2\cos\frac{\pi}{n}\right)\left(2 + 2\cos\frac{2\pi}{n}\right)\dots\left(2 + 2\cos\frac{n-1}{n}\pi\right).$$

Multiplying these, we get

$$n^2 = 2^{2n-2}\sin^2\frac{\pi}{n}\cdot\sin^2\frac{2\pi}{n}\dots\sin^2\frac{n-1}{n}\pi$$

or $$n = 2^{n-1}\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\dots\sin\frac{n-1}{n}\pi.$$

Hence, from (3), we get

$$A^2 = \frac{\pi^{n-1}}{n/2^{n-1}} = \frac{(2\pi)^{n-1}}{n}$$

or
$$A = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}.$$

Remark. The value of $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n}\pi$ can also be found by using the trigonometrical identity

$$\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin \left(\theta + \frac{\pi}{n}\right) \sin \left(\theta + \frac{2\pi}{n}\right) \sin \left(\theta + \frac{3\theta}{n}\right) \ldots$$
$$\ldots \sin \left(\theta + \frac{n-1}{n}\pi\right).$$

[Hobson, Trigonometry, page 117]

From the above identity, we have

$$\frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin \theta} \cdot n = 2^{n-1} \sin \left(\theta + \frac{\pi}{n}\right) \sin \left(\theta + \frac{2\pi}{n}\right) \ldots$$
$$\ldots \sin \left(\theta + \frac{n-1}{n}\pi\right).$$

Taking limit as $\theta \to 0$, we get

$$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \ldots \sin \frac{n-1}{n}\pi.$$

§ 13. To find the values of the integrals

(i) $\int_0^\infty e^{-ax} \cos bx . x^{m-1}\, dx$

and (ii) $\int_0^\infty e^{-ax} \sin bx . x^{m-1}\, dx$.

(Agra 1976)

We have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1}\, dx = \int_0^\infty e^{-(a-ib)x} x^{m-1}\, dx = \frac{\Gamma(m)}{(a-ib)^m}$$

[See § 6, part (i)]

$$= (a-ib)^{-m}\, \Gamma(m). \quad \ldots(1)$$

Let us first separate $(a-ib)^{-m}$ into real and imaginary parts.

Put $a = k\cos\alpha$ and $b = k \sin \alpha$ so that

$$\alpha = \tan^{-1}(b/a) \text{ and } k = \sqrt{(a^2+b^2)}.$$

Then $(a-ib)^{-m} = [k(\cos\alpha - i\sin\alpha)]^{-m}$

$= k^{-m}(\cos\alpha - i\sin\alpha)^{-m}$

$= k^{-m}(\cos m\alpha + i\sin m\alpha)$, by De-Moivre's theorem.

Now from (1), we have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1}\, dx = k^{-m}(\cos m\alpha + i \sin m\alpha)\,\Gamma(m)$$

or
$$\int_0^\infty e^{-ax}(\cos bx + i\sin bx)\, x^{m-1}\, dx$$

$$= \frac{\Gamma(m)}{k^m}(\cos m\alpha + i \sin m\alpha),$$

[$\because e^{i\theta} = \cos\theta + i\sin\theta$, by Euler's theorem]

or $\int_0^\infty e^{-ax}\cos bx \,.\, x^{m-1}\,dx + i\int_0^\infty e^{-ax}\sin bx \,.\, x^{m-1}\,dx$

$$= \frac{\Gamma(m)}{k^m}\cos m\alpha + i\frac{\Gamma(m)}{k^m}\sin m\alpha. \quad ...(2)$$

Equating real and imaginary parts in (2), we get

$$\int_0^\infty e^{-ax}\cos bx \,.\, x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}\cos m\alpha,$$

and $$\int_0^\infty e^{-ax}\sin bx \,.\, x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}\sin m\alpha,$$

where $k = \sqrt{(a^2+b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Deductions. **(i)** If we put $a = 0$, then $\alpha = \pi/2$ and $k = b$.

Hence $$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)}{b^m}\cos\frac{m\pi}{2}$$

and $$\int_0^\infty x^{m-1}\sin bx\,dx = \frac{\Gamma(m)}{b^m}\sin\frac{m\pi}{2}.$$

(ii) If we put $m = 1$, then

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{\Gamma(1)}{k}\cos\alpha = \frac{k\cos\alpha}{k^2} = \frac{a}{a^2+b^2}$$

and, $$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{\Gamma(1)}{k}\sin\alpha = \frac{k\sin\alpha}{k^2} = \frac{b}{a^2+b^2}.$$

Solved Examples

Ex. 11. *Evaluate the following integrals.*

(i) $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx,$ **(ii)** $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx,$ **(Kanpur 1985)**

(iii) $\displaystyle\int_0^\infty \frac{x\,dx}{1+x^6}$ **(Rohilkhand 1990; Agra 85)**

Sol. **(i)** We have

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8\,dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx \quad \text{[by § 3]}$$

$= B(9, 15) - B(15, 9)$,

$= B(9, 15) - B(9, 15)$, by symmetry of Beta function

$= 0$.

(ii) We have

$$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4\,dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9\,dx}{(1+x)^{15}}$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= B(5, 10) + B(10, 5) = B(5, 10) + B(5, 10)$$

$$= 2B(5, 10) = 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}$$

$$= 2 \cdot \frac{4.3.2.1}{14.13.12.11.10} = \frac{1}{5005}.$$

(iii) Let $I = \int_0^\infty \frac{x\,dx}{1+x^6}$.

Put $x^6 = y$ or $x = y^{1/6}$, so that $dx = \frac{1}{6}y^{-5/6}\,dy$.

$$\therefore \quad I = \tfrac{1}{6}\int_0^\infty \frac{y^{1/6}.y^{-5/6}}{1+y}\,dy = \tfrac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1+y}\,dy$$

$$= \tfrac{1}{6}\int_0^\infty \frac{y^{(1/3)-1}}{(1+y)^{(1/3)+(2/3)}}\,dy = \tfrac{1}{6}B\left(\tfrac{1}{3}, \tfrac{2}{3}\right), \text{ by § 3}$$

$$= \tfrac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \tfrac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)}{\Gamma 1} = \tfrac{1}{6} \cdot \frac{\pi}{\sin\frac{1}{3}\pi},$$

$$\left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \tfrac{1}{6} \cdot \frac{\pi}{(\sqrt{3}/2)} = \tfrac{1}{6} \cdot \frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Ex. 12. *Show that*

$$\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)}.$$ **(Agra 1975)**

Sol. Let $x^n = \sin^2\theta$ *i.e.*, $x = \sin^{2/n}\theta$ so that

$$dx = \frac{2}{n}\sin^{(2/n)-1}\theta\cos\theta\,d\theta.$$

Then $$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{2}{n}\int_0^{\pi/2} \frac{\sin^{(2/n)-1}\theta\cos\theta\,d\theta}{\cos\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta\cos^0\theta\,d\theta$$

$$= \frac{2}{n} \cdot \frac{\Gamma(1/n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma\left(1/n+\frac{1}{2}\right)}.$$

Ex. 13. *Evaluate* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$.

Sol. We have

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}\,dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}} \qquad \ldots(1)$$

Now in the second integral on the R.H.S. of (1), we put $x = 1/y$ so that $dx = -(1/y^2)\,dy$; also when $x \to 0, y \to \infty$ and when $x = 1, y = 1$.

$$\therefore \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_\infty^1 \frac{(1/y)^{n-1}}{(1+1/y)^{m+n}}\left(-\frac{1}{y^2}\,dy\right)$$

$$= -\int_\infty^1 \frac{y^{m+n}\,dy}{(1+y)^{m+n}.\,y^{n-1}.\,y^2} = \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \qquad \textbf{(Note)}$$

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx, \qquad \left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy\right]$$

Now from (1), we have

$$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$, by a property of definite integrals

$$= B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$$

[Refer § 3 page 268 and § 7 page 271].

Ex. 14. *Prove that*

$$B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Ex. 15. *Show that*

$$\int_0^{\pi/2} (\tan x)^n\,dx = \frac{\pi}{2}\sec\frac{n\pi}{2}, \text{ where } -1 < n < 1.$$

Sol. We have

$$\int_0^{\pi/2} \tan^n x\,dx = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x}\,dx = \int_0^{\pi/2} \sin^n x\cos^{-n} x\,dx$$

$$= \frac{\Gamma\frac{1}{2}(n+1).\Gamma\frac{1}{2}(-n+1)}{2\,\Gamma\frac{1}{2}(n-n+2)},$$

where $-n+1>0$ i.e., $n<1$ and $n+1>0$ i.e., $n>-1$

$$= \tfrac{1}{2}\Gamma\tfrac{1}{2}(n+1)\,\Gamma\tfrac{1}{2}(1-n) = \tfrac{1}{2}\Gamma\tfrac{1}{2}(n+1)\,\Gamma\left[1-\tfrac{1}{2}(n+1)\right]$$

$$= \tfrac{1}{2}\frac{\pi}{\sin\frac{1}{2}(n+1)\pi},$$

$\left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right.$, by cor. to § 7, page 271$\left.\right]$

$$= \frac{\pi}{2} \cdot \frac{1}{\sin\left(\frac{1}{2}\pi + \frac{1}{2}n\pi\right)} = \frac{\pi}{2} \cdot \frac{1}{\cos\left(\frac{1}{2}n\pi\right)} = \frac{\pi}{2} \sec \frac{n\pi}{2},$$

where $-1 < n < 1$.

***Ex. 16.** *Show that*

$$\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$$

(Agra 1980, 83, 88; Meerut 98)

Sol. The given integral is

$$= \int_0^{\pi/2} \sin^{-1/2}\theta\, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta \qquad \textbf{(Note)}$$

$$= \frac{\Gamma\{\frac{1}{2}(-\frac{1}{2}+1)\}\Gamma\{\frac{1}{2}(0+1)\}}{2\Gamma\{\frac{1}{2}(-\frac{1}{2}+0+2)\}} \times \frac{\Gamma\{\frac{1}{2}(\frac{1}{2}+1)\}\Gamma\{\frac{1}{2}(0+1)\}}{2\Gamma\{\frac{1}{2}(\frac{1}{2}+0+2)\}}$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})} \times \frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{4})} = \frac{\Gamma\frac{1}{4}.\sqrt{\pi}.\sqrt{\pi}}{2.2 \cdot \frac{1}{4}\Gamma(\frac{1}{4})} = \pi.$$

Ex. 17. *Prove that*

(i) $\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$; **(Agra 1975; Gorakhpur 82; Meerut 96)**

(ii) $\int_0^\infty x^{2n-1} e^{-ax^2}\, dx = \frac{\Gamma(n)}{2a^n}$;

(iii) $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n)$;

(Meerut 1994, 97; Rohilkhand 80, 87)

(iv) $\int_0^\infty x^m e^{-ax^n}\, dx = \frac{\Gamma[(m+1)/n]}{na^{(m+1)/n}}$; **(Rohilkhand 1976)**

(v) $\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$. **(Agra 1980; Meerut 96 BP)**

(vi) $\Gamma(m)\Gamma(n) = \Gamma(m+n) \int_0^\infty \frac{x^{n-1}\, dx}{(1+x)^{m+n}}$. **(Agra 1976)**

Sol. (i) Let $I = \int_0^\infty e^{-ax} x^{n-1}\, dx$.

Put $ax = y$ so that $a\, dx = dy$. When $x = 0, y = 0$ and when $x \to \infty, y \to \infty$.

$$\therefore \quad I = \int_0^\infty e^{-y} \left(\frac{y}{a}\right)^{n-1} \cdot \frac{1}{a} dy = \frac{1}{a^n} \int_0^\infty e^{-y} y^{n-1}\, dy$$

$= \frac{1}{a^n}\Gamma(n)$, by the definition of Gamma function.

(ii) Let $I = \int_0^\infty x^{2n-1} e^{-ax^2} dx = \int_0^\infty x^{2n-2} e^{-ax^2} x\, dx$.

Put $ax^2 = z$ so that $2ax\, dx = dz$. When $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{z}{a}\right)^{n-1} e^{-z} \frac{1}{2a} dz = \frac{1}{2a^n} \int_0^\infty e^{-z} z^{n-1} dz$$

$= \frac{1}{2a^n} \Gamma(n)$, by definition of Gamma function.

(iii) Let $I = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$.

Put $\log(1/x) = y$ i.e., $1/x = e^y$ i.e., $x = e^{-y}$ so that $dx = -e^{-y} dy$.
Also when $x \to 0, y \to \infty$ and when $x = 1, y = 0$.

[Note that $\log \infty = \infty$]

$$\therefore \quad I = -\int_\infty^0 y^{n-1} e^{-y} dy = \int_0^\infty e^{-y} y^{n-1} dy = \Gamma(n).$$

(iv) Let $\quad I = \int_0^\infty x^m e^{-ax^n} dx = \int_0^\infty \frac{x^m}{x^{n-1}} e^{-ax^n} x^{n-1} dx$

(Note)

$$= \int_0^\infty x^{m-n+1} e^{-ax^n} x^{n-1} dx.$$

Put $ax^n = t$ so that $na\, x^{n-1} dx = dt$. Also when $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{t}{a}\right)^{(m-n+1)/n} e^{-t} \cdot \frac{1}{na} dt,$$

$$\left[\because ax^n = t \Rightarrow x = \left(\frac{t}{a}\right)^{1/n}\right]$$

$$= \frac{1}{na \cdot a^{(m-n+1)/n}} \int_0^\infty t^{\{(m+1)/n\}-1} e^{-t} dt$$

$$= \frac{1}{na^{(m+1)/n}} \Gamma\{(m+1)/n\},$$

by the definition of Gamma function.

(v) Let

$$I = \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_0^1 \frac{dx}{\sqrt{\{\log(1/x)\}}} = \int_0^1 \left(\log \frac{1}{x}\right)^{-1/2} dx.$$

Put $\log(1/x) = y$ i.e., $1/x = e^y$ i.e., $x = e^{-y}$ so that $dx = -e^{-y} dy$.
Also when $x \to \infty, y \to \infty$ and when $x = 1, y = 0$.

$$\therefore \quad I = -\int_\infty^0 y^{-1/2} e^{-y} dy = \int_0^\infty e^{-y} y^{1/2-1} dy = \Gamma(\tfrac{1}{2}),$$

by the def. of Gamma function

$$= \sqrt{\pi}.$$

(vi) We have

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\, dx. \qquad \text{[See § 6, part (i)]}$$

$$\therefore \quad \Gamma(n) = z^n \int_0^\infty e^{-zx} . x^{n-1}\, dx = \int_0^\infty z^n e^{-zx} x^{n-1}\, dx.$$

Multiplying both sides by $e^{-z} z^{m-1}$, we get

$$\Gamma(n)\, e^{-z} z^{m-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1}\, dx.$$

Now proceed as in § 7.

Ex. 18. *Show that*

$$\int_0^1 \frac{x^2\, dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}.$$

(Agra 1986)

Sol. In the first integral put $x^2 = \sin\theta$ so that $2x\, dx = \cos\theta\, d\theta$ and the corresponding limits for θ are 0 to $\pi/2$.

$$\therefore \quad \int_0^1 \frac{x^2\, dx}{(1-x^4)^{1/2}} = \int_0^1 \frac{\frac{1}{2} x . 2x\, dx}{(1-x^4)^{1/2}} = \frac{1}{2} \int_0^{\pi/2} \frac{\sin^{1/2}\theta . \cos\theta\, d\theta}{(1-\sin^2\theta)^{1/2}}$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{\sqrt{(\sin\theta)} . \cos\theta\, d\theta}{\cos\theta} = \frac{1}{2} \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta.$$

Again, in 2nd integral put $x^2 = \tan\theta$ so that

$$2x\, dx = \sec^2\theta\, d\theta$$

and the corresponding limits for θ are 0 to $\pi/4$.

$$\therefore \quad \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sqrt{(\tan\theta)} \sec\theta}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin\theta \cos\theta)}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{(\sin 2\theta)}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{1}{\sqrt{(\sin\phi)}}\, d\phi, \text{ where } 2\theta = \phi.$$

Hence the given integral becomes

$$\int_0^1 \frac{x^2\, dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$$

$$= \frac{1}{2} \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(\sin\phi)}}$$

$$= \frac{1}{4\sqrt{2}} \cdot \int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}}$$

$$= \frac{1}{4\sqrt{2}} \cdot \pi, \qquad \text{[as proved in Ex. 16; prove it here].}$$

Ex. 19. *Show that*

$$\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}.$$ (Meerut 1984, 98)

Sol. In the given integral put $x^n = a^n \sin^2\theta$

i.e., $x = a \sin^{2/n}\theta\, d\theta$ so that $dx = (2a/n)\sin^{(2/n)-1}\theta \cos\theta\, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = a, \theta = \pi/2$.

$$\therefore \quad \int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{2a}{n}\int_0^{\pi/2} \frac{\sin^{(2/n)-1}\theta\cos\theta\, d\theta}{a\cos^{2/n}\theta}$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta\cos^{1-(2/n)}\theta\, d\theta = \frac{\frac{2}{n}\Gamma\frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)}{2\Gamma 1}$$

$$= \frac{1}{n}\cdot\Gamma\frac{1}{n}\Gamma\left(1-\frac{1}{n}\right) = \frac{1}{n}\cdot\frac{\pi}{\sin(\pi/n)}.$$

$$\left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right].$$

Ex. 20 (a). *Show that*

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta = 4\int_0^\infty \frac{x^2\, dx}{1+x^4} = \pi\sqrt{2}.$$

Sol. Let $I = 4\displaystyle\int_0^\infty \frac{x^2\, dx}{1+x^4} = 4\int_0^\infty \frac{x \cdot x\, dx}{1+x^4}$.

Put $x^2 = \tan\theta$ so that $2x\, dx = \sec^2\theta\, d\theta$. Also when $x = 0, \theta = 0$ and when $x \to \infty, \theta \to \pi/2$.

$$\text{Then } I = 4\int_0^{\pi/2} \frac{\sqrt{(\tan\theta)}\cdot\frac{1}{2}\sec^2\theta\, d\theta}{(1+\tan^2\theta)} = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta \quad \text{...(1)}$$

$$= 2\int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\, d\theta$$

$$= 2\frac{\Gamma\{\frac{1}{2}(\frac{1}{2}+1)\}\Gamma\{\frac{1}{2}(-\frac{1}{2}+1)\}}{2\Gamma\{\frac{1}{2}(\frac{1}{2}-\frac{1}{2}+2)\}} = 2\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{2\Gamma 1}$$

$$= \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) \quad \text{...(2)}$$

$$= \Gamma\left(\tfrac{1}{4}\right)\Gamma\left(1-\tfrac{1}{4}\right) = \frac{\pi}{\sin\frac{1}{4}\pi}, \quad \left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}. \quad \text{...(3)}$$

From (1), (2) and (3), the required result follows.

Ex. 20 (b). *Evaluate the integral*

$$\int_a^b (x-a)^p (b-x)^q\, dx,$$

where p and q are positive integers.

Sol. Let $I = \displaystyle\int_a^b (x-a)^p (b-x)^q\, dx$.

Put $x = a\cos^2\theta + b\sin^2\theta$ so that

$dx = -2a\cos\theta\sin\theta\,d\theta + 2b\sin\theta\cos\theta\,d\theta$

i.e., $dx = 2(b-a)\cos\theta\sin\theta\,d\theta$.

Also $x - a = a\cos^2\theta + b\sin^2\theta - a = b\sin^2\theta - a(1-\cos^2\theta)$

$= b\sin^2\theta - a\sin^2\theta = (b-a)\sin^2\theta$

and $b - x = b - a\cos^2\theta - b\sin^2\theta = b(1-\sin^2\theta) - a\cos^2\theta$

$= (b-a)\cos^2\theta$.

To find the limits for θ, when $x = a$, we have

$a = a\cos^2\theta + b\sin^2\theta$

i.e., $(b-a)\sin^2\theta = 0$ *i.e.*, $\sin^2\theta = 0$ as $a \neq b$

i.e., $\theta = 0$

and when $x = b$, we have $b = a\cos^2\theta + b\sin^2\theta$

i.e., $(a-b)\cos^2\theta = 0$ *i.e.*, $\cos^2\theta = 0$ as $a \neq b$

i.e., $\theta = \pi/2$.

Thus the new limits for θ are 0 to $\pi/2$. Hence the given integral

$$I = \int_0^{\pi/2} (b-a)^p\sin^{2p}\theta\,.(b-a)^q\cos^{2q}\theta\,.2(b-a)\cos\theta\sin\theta\,d\theta$$

$$= 2(b-a)^{p+q+1}\int_0^{\pi/2}\sin^{2p+1}\theta\cos^{2q+1}\theta\,d\theta$$

$$= 2(b-a)^{p+q+1}\frac{\Gamma\{\frac{1}{2}(2p+1+1)\}\,\Gamma\{\frac{1}{2}(2q+1+1)\}}{2\,\Gamma(2p+1+2q+1+2)},$$

provided $2p+1 > -1$ and $2q+1 > -1$
i.e., $p > -1$ and $q > -1$ which is so because
p and q are given to be +ive integers

$$= (b-a)^{p+q+1}\frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(p+q+1+1)}$$

$$= (b-a)^{p+q+1}\frac{p\,!\,q\,!}{(p+q+1)\,!},$$

because $\Gamma(n+1) = n\,!$ if n is a positive integer.

Ex. 21 (a). *Prove that* $\mathbf{B}(m,m)\,.\,\mathbf{B}(m+\frac{1}{2}, m+\frac{1}{2}) = \dfrac{\pi\, m^{-1}}{2^{4m-1}}$.

Sol. We have $B(m,m)\,.\,B(m+\frac{1}{2}, m+\frac{1}{2})$

$$= \frac{\Gamma(m)\,.\Gamma(m)}{\Gamma(m+m)}\cdot\frac{\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{1}{2}+m+\frac{1}{2})},$$

$$\left[\because B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{[\Gamma(m)\,\Gamma(m+\frac{1}{2})]^2}{\Gamma(2m)\,.\Gamma(2m+1)} = \frac{[\Gamma(m)\,\Gamma(m+\frac{1}{2})]^2}{\Gamma(2m)\,.2m\,\Gamma(2m)},$$

$$[\because \Gamma(p+1) = p\,\Gamma(p)]$$

$$=\frac{1}{2m}\left[\frac{\Gamma(m)\,\Gamma(m+\frac{1}{2})}{\Gamma(2m)}\right]^2=\frac{1}{2m}\cdot\left[\frac{\sqrt{\pi}}{2^{2m-1}}\right]^2 \qquad \textbf{(Note)}$$

$\left[\because\right.$ by Duplication formula, (§ 11 page 275),

$$\left.\frac{\Gamma(m)\,\Gamma(m+\frac{1}{2})}{\Gamma(2m)}=\frac{\sqrt{\pi}}{2^{2m-1}}\right]$$

$$=\frac{1}{2m}\cdot\frac{\pi}{2^{4m-2}}=\frac{\pi m^{-1}}{2^{4m-1}}.$$

Ex. 21 (b). *Show that*

$$\mathbf{B}(m,n)=\mathbf{B}(m+1,n)+\mathbf{B}(m,n+1),\ \textit{for } m>0, n>0.$$

(Meerut 1991; Agra 88; U.P. P.C.S. 95)

Sol. We know that $\mathbf{B}(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$$\therefore\quad \mathbf{B}(m+1,n)=\frac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+1+n)}$$

and $\mathbf{B}(m,n+1)=\dfrac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$.

Adding, we get $\mathbf{B}(m+1,n)+\mathbf{B}(m,n+1)$

$$=\frac{\Gamma(m+1)\,\Gamma(n)+\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$=\frac{m\,\Gamma(m)\,\Gamma(n)+\Gamma(m)\,.\,n\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}$$

$$=\frac{(m+n)\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma(m+n)}=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=\mathbf{B}(m,n),$$

provided $m>0$ and $n>0$.

Ex. 22 (a). *Find the value of*

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right).$$

Sol. We know that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}},$$

where n is a positive integer.

Putting $n=9$ in the above relation, we get

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)=\frac{(2\pi)^{(9-1)/2}}{9^{1/2}}=\frac{(2\pi)^4}{3}$$

$$=(16/3)\,\pi^4.$$

Ex. 22 (b). *Show that*

$$\Gamma(0\cdot 1)\,\Gamma(0\cdot 2)\,\Gamma(0\cdot 3)\ldots\Gamma(0\cdot 9)=\frac{(2\pi)^{9/2}}{\sqrt{10}}.$$

Sol. Proceed as in Ex. 22 (a). Put $n = 10$.

Ex. 23. *Show that*

(i) $2^n \Gamma(n + \frac{1}{2}) = 1.3.5.... (2n - 1)\sqrt{\pi}$, *where n is a +ive integer,*

(ii) $\Gamma\left(\frac{3}{2} - x\right)\Gamma\left(\frac{3}{2} + x\right) = \left(\frac{1}{4} - x^2\right)\pi \sec \pi x$,

provided $-1 < 2x < 1$.

Sol. (i) We have

$$\Gamma(n + \tfrac{1}{2}) = (n - \tfrac{1}{2})\Gamma(n - \tfrac{1}{2})$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\Gamma\left(n - \frac{3}{2}\right)$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\left(n - \frac{5}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$= \frac{1}{2^n}(2n-1)(2n-3)(2n-5)\ldots 3.1\,.\sqrt{\pi}.$$

$$\therefore \quad 2^n \Gamma(n + \tfrac{1}{2}) = 1.3.5.\ldots(2n - 1)\sqrt{\pi}.$$

(ii) We have $\Gamma\left(\frac{3}{2} - x\right)\Gamma\left(\frac{3}{2} + x\right)$

$$= (\tfrac{1}{2} - x)\Gamma(\tfrac{1}{2} - x).(\tfrac{1}{2} + x)\Gamma(\tfrac{1}{2} + x)$$

$$= \left(\frac{1}{4} - x^2\right)\Gamma\left(\frac{1 - 2x}{2}\right)\Gamma\left(\frac{1 + 2x}{2}\right)$$

$$= \left(\frac{1}{4} - x^2\right)\Gamma\left(\frac{1 - 2x}{2}\right)\Gamma\left(1 - \frac{1 - 2x}{2}\right)$$

$$= \left(\frac{1}{4} - x^2\right)\frac{\pi}{\sin\left(\frac{1-2x}{2}\pi\right)} = \left(\frac{1}{4} - x^2\right)\cdot\frac{\pi}{\sin\left(\frac{1}{2}\pi - x\pi\right)}$$

$$= \left(\frac{1}{4} - x^2\right)\cdot\frac{\pi}{\cos x\pi} = \left(\frac{1}{4} - x^2\right)\cdot\pi\sec\pi x.$$

Ex. 24. *With certain restrictions on the values of a, b, m and n, prove that*

$$\int_0^\infty\int_0^\infty e^{-(ax^2 + by^2)}x^{2m-1}y^{2n-1}\,dx\,dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}.$$

Sol. Let us denote the given integral by I. Then

$$I = \int_0^\infty e^{-ax^2}x^{2m-1}\,dx \times \int_0^\infty e^{-by^2}y^{2n-1}\,dy = I_1 \times I_2.$$

To evaluate I_1, put $ax^2 = t$ so that $2ax\,dx = dt$.

$$\therefore \quad I_1 = \int_0^\infty e^{-t}(t/a)^{(2m-1)/2} \cdot \frac{dt}{2\sqrt{(at)}}$$

$$= \frac{1}{2a^m}\int_0^\infty e^{-t} t^{m-1}\,dt = \frac{\Gamma(m)}{2a^m}, \text{ provided } a \text{ and } m \text{ are +ive.}$$

Similarly, $I_2 = \dfrac{\Gamma(n)}{2b^n}$, provided b and n are +ive.

Hence $I = \dfrac{\Gamma m \,\Gamma n}{4a^m b^n}$.

Ex. 25. *Show that the sum of the series*

$$\frac{1}{n+1} + m\frac{1}{n+2} + \frac{m(m+1)}{2!}\cdot\frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!}\cdot\frac{1}{n+4} + \ldots$$

is $\dfrac{\Gamma(n+1)\Gamma(1-m)}{\Gamma(n-m+2)}$, *where* $-1 < n < 1$.

Sol. We have

$$\frac{\Gamma(n+1)\Gamma(1-m)}{\Gamma(n-m+2)} = B(n+1, 1-m)$$

$$= \int_0^1 x^n (1-x)^{-m}\,dx$$

$$= \int_0^1 x^n \left[1 + mx + \frac{m(m+1)}{2!}x^2 + \frac{m(m+1)(m+2)}{3!}x^3 + \ldots\right] dx$$

$$= \int_0^1 \left[x^n + mx^{n+1} + \frac{m(m+1)}{2!}x^{n+2} + \frac{m(m+1)(m+2)}{3!}x^{n+3} + \ldots\right] dx$$

$$= \left[\frac{x^{n+1}}{n+1} + m\frac{x^{n+2}}{n+2} + \frac{m(m+1)}{2!}\frac{x^{n+3}}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{x^{n+4}}{n+4} + \ldots\right]_0^1$$

$$= \frac{1}{n+1} + m.\frac{1}{n+2} + \frac{m(m+1)}{2!}\cdot\frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{1}{n+4} + \ldots$$

Ex. 26. *Prove that* $\displaystyle\int_0^\infty \frac{\sin bz}{z}\,dz = \frac{\pi}{2}$.

Sol. We have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\,dx\,dz$$

$$= \int_0^\infty \left[\frac{e^{-xz}}{-z}\right]_0^\infty \sin bz\, dz, \text{ on first integrating w.r.t. } x$$

$$= \int_0^\infty \frac{\sin bz}{z}\, dz. \qquad \text{...(1)}$$

Again on first integrating w.r.t. z, we have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz = \int_0^\infty \left[\int_0^\infty e^{-xz} \sin bz\, dz\right] dx$$

$$= \int_0^\infty \frac{b}{b^2 + x^2}\, dx, \qquad \text{[See § 13, Deductions (ii)]}$$

$$= \left[\tan^{-1} \frac{x}{b}\right]_0^\infty = \frac{\pi}{2}. \qquad \text{...(2)}$$

Hence equating the two values (1) and (2) of I, we have

$$\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}.$$

Ex. 27. *Show that* $\int_0^\infty \cos (bz^{1/n})\, dz = \frac{1}{b^2} \Gamma (n + 1) .\cos \frac{n\pi}{2}$.

(Meerut 1996)

Sol. Put $z^{1/n} = x$ *i.e.*, $z = x^n$, so that $dz = nx^{n-1}\, dx$.

$$\therefore \int_0^\infty \cos(bz^{1/n})\, dz = \int_0^\infty \cos (bx) .nx^{n-1}\, dx$$

$$= n \int_0^\infty x^{n-1} \cos (bx)\, dx$$

$$= \text{real part of } n \int_0^\infty e^{-ibx} x^{n-1}\, dx$$

$$= \text{real part of } n \frac{\Gamma (n)}{(ib)^n}$$

$$= \text{real part of } \frac{n\, \Gamma (n)}{b^n} \cdot (\cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi)^{-n}$$

$$= \text{real part of } \frac{\Gamma (n + 1)}{b^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}\right)$$

$$= \frac{1}{b^n} \cdot \Gamma (n + 1) .\cos \left(\frac{n\pi}{2}\right).$$

Ex. 28. *Prove that*

(i) $\int_0^2 (8 - x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$

(ii) $\int_0^1 x^m (1 - x^n)^p\, dx = \frac{1}{n} B \left(\frac{m+1}{n}, p + 1\right)$.

(iii) $\int_0^1 (1-x^n)^{1/n}\, dx = \dfrac{1}{n} \dfrac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{2\,\Gamma\left(\frac{2}{n}\right)}$.

(iv) $\int_0^1 x^{m-1}(1-x^2)^{n-1}\, dx = \frac{1}{2} B\left(\frac{m}{2}, n\right)$. **(Rohilkhand 1985)**

Sol. (i) Let

$$I = \int_0^2 (8-x^3)^{-1/3}\, dx = \int_0^2 x^{-2}(8-x^3)^{-1/3} x^2\, dx.$$

Put $x^3 = 8y$ so that $3x^2\, dx = 8\, dy$.

When $x = 0, y = 0$ and when $x = 2, y = 1$.

$$\therefore \quad I = \int_0^1 (8y)^{-2/3}(8-8y)^{-1/3} \cdot \frac{8}{3} dy$$

$$= \int_0^1 8^{-2/3} y^{-2/3} . 8^{-1/3} (1-y)^{-1/3} . \frac{8}{3} dy$$

$$= \frac{1}{3} \int_0^1 y^{-2/3}(1-y)^{-1/3}\, dy$$

$$= \frac{1}{3} \int_0^1 y^{(1/3)-1}(1-y)^{(2/3)-1}\, dy = \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}{\Gamma 1} = \frac{1}{3} \cdot \frac{\pi}{\sin \frac{1}{3}\pi}$$

$$= \frac{\pi}{3} \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}.$$

(ii) Let $I = \int_0^1 x^m (1-x^n)^p\, dx$.

Put $x^n = y$ or $x = y^{1/n}$.

Then $dx = \dfrac{1}{n} y^{(1/n)-1}\, dy$.

When $x = 0, y = 0$ and when $x = 1, y = 1$.

$$\therefore \quad I = \int_0^1 (y^{1/n})^m (1-y)^p \cdot \frac{1}{n} y^{(1/n)-1}\, dy$$

$$= \frac{1}{n} \int_0^1 y^{(m/n)+(1/n)-1}(1-y)^{(p+1)-1}\, dy$$

$$= \frac{1}{n} \int_0^1 y^{\{(m+1)/n\}-1}(1-y)^{(p+1)-1}\, dy$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right).$$

(iii) Let $I = \int_0^1 (1-x^n)^{1/n}\, dx$.

Put $x^n = y$ or $x = y^{1/n}$. Then $dx = (1/n)\, y^{(1/n)-1}\, dy$.

When $x = 0, y = 0$ and when $x = 1, y = 1$.

$$\therefore \quad I = \int_0^1 (1-y)^{1/n} \cdot \frac{1}{n} y^{(1/n)-1}\, dy$$

$$= \frac{1}{n}\int_0^1 y^{(1/n)-1}(1-y)^{\{(1/n)+1\}-1}\, dy$$

$$= \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{n}+1\right) = \frac{1}{n}\frac{\Gamma(1/n)\,\Gamma\{(1/n)+1\}}{\Gamma\{(2/n)+1\}}$$

$$= \frac{1}{n}\cdot\frac{\Gamma(1/n)\,.(1/n)\,\Gamma(1/n)}{(2/n)\,.\Gamma(2/n)} = \frac{1}{n}\frac{[\Gamma(1/n)]^2}{2\,\Gamma(2/n)}.$$

(iv) Let

$$I = \int_0^1 x^{m-1}(1-x^2)^{n-1}\, dx = \int_0^1 x^{m-2}(1-x^2)^{n-1}\, x\, dx.$$

Put $x^2 = y$ so that $2x\, dx = dy$.
When $x = 0, y = 0$ and when $x = 1, y = 1$.

$$\therefore \quad I = \int_0^1 y^{(m-2)/2}(1-y)^{n-1}\cdot\frac{1}{2}\, dy$$

$$= \frac{1}{2}\int_0^1 y^{(m/2)-1}(1-y)^{n-1}\, dy$$

$$= \tfrac{1}{2} B\left(\frac{m}{2}, n\right).$$

Ex. 29. *Show that, if $m > -1$, then*

$$\int_0^\infty x^m e^{-n^2x^2}\, dx = \frac{1}{2n^{m+1}}\Gamma\left(\frac{m+1}{2}\right).$$

(Agra 1984)

Sol. Let $I = \int_0^\infty x^m e^{-n^2x^2}\, dx = \int_0^\infty x^{m-1} e^{-n^2x^2}\, x\, dx.$

Put $n^2x^2 = t$, so that $2n^2 x\, dx = dt$. Also $x = t^{1/2}/n$.
When $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{t^{1/2}}{n}\right)^{m-1} e^{-t}\cdot\frac{1}{2n^2}\, dt$$

$$= \frac{1}{2n^{m-1}\,.\,n^2}\int_0^\infty e^{-t}\, t^{(m-1)/2}\, dt$$

$$= \frac{1}{2n^{m+1}}\int_0^\infty e^{-t}\, t^{\{(m+1)/2\}-1}\, dt$$

$= \dfrac{1}{2n^{m+1}}\Gamma\left(\dfrac{m+1}{2}\right)$, by def. of Gamma function, provided $m+1 > 0$ *i.e.*, $m > -1$.

Ex. 30. *Prove that*

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}[\Gamma(\tfrac{1}{4})]^2.$ **(Meerut 1994 P, 95)**

(ii) $\int_0^{\pi/2} \sin^p \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right)$.

Sol. (i) Let $I = \int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$.

Put $x^4 = \sin^2 \theta$ *i.e.*, $x = \sin^{1/2} \theta$ so that
$$dx = \tfrac{1}{2} \sin^{-1/2} \theta \cos \theta \, d\theta.$$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{\frac{1}{2} \sin^{-1/2} \theta \cos \theta \, d\theta}{\cos \theta} = \tfrac{1}{2} \int_0^{\pi/2} \sin^{-1/2} \theta \cos^0 \theta \, d\theta$$

$$= \tfrac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)} = \tfrac{1}{4} \frac{[\Gamma\left(\frac{1}{4}\right)]^2 . \sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{[\Gamma\left(\frac{1}{4}\right)]^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{[\Gamma\left(\frac{1}{4}\right)]^2}{\pi/(\sin \frac{1}{4}\pi)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{[\Gamma\left(\frac{1}{4}\right)]^2}{\pi . \sqrt{2}} = \frac{\sqrt{2}}{8\sqrt{\pi}} [\Gamma\left(\tfrac{1}{4}\right)]^2.$$

(ii) We have $I = \int_0^{\pi/2} \sin^p \theta \, d\theta = \int_0^{\pi/2} \sin^p \theta \cos^0 \theta \, d\theta$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{p+2}{2}\right)} = \tfrac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \tfrac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$$

Alternative Method. Put $\sin^2 \theta = y$ or $\sin \theta = y^{1/2}$.

Then $\cos \theta \, d\theta = \frac{1}{2} y^{-1/2} \, dy$

or $$d\theta = \tfrac{1}{2} \frac{y^{-1/2} \, dy}{\sqrt{(1-\sin^2 \theta)}} = \tfrac{1}{2} (1-y)^{-1/2} y^{-1/2} \, dy.$$

When $\theta = 0, y = 0$ and when $\theta = \pi/2, y = 1$.

$$\therefore \quad I = \int_0^1 y^{p/2} . \tfrac{1}{2} (1-y)^{-1/2} y^{-1/2} \, dy$$
$$= \tfrac{1}{2} \int_0^1 y^{(p-1)/2} (1-y)^{-1/2} \, dy$$
$$= \tfrac{1}{2} \int_0^1 y^{\{(p+1)/2\}-1} (1-y)^{(1/2)-1} \, dy$$
$$= \tfrac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \tfrac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$$

Ex. 31. *Show that the perimeter of a loop of the curve* $r^n = a^n \cos n\theta$ *is* $\frac{a}{n} \cdot 2^{(1/n)-1} \cdot \frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}$.

Sol. The given curve is $r^n = a^n \cos n\theta$

The curve (1) is symmetrical about the initial line. We have $r = 0$ when $\cos n\theta = 0$ *i.e.*, when $n\theta = -\pi/2, \pi/2$ or $\theta = -\pi/2n$, $\pi/2n$. Therefore one loop of the curve lies between the lines $\theta = -\pi/2n$ and $\theta = \pi/2n$. This loop is symmetrical about the initial line and for the portion of this loop lying above the initial line θ varies from 0 to $\pi/2n$.

Taking log of both sides of (1), we have

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n \sin n\theta) \text{ or } \frac{dr}{d\theta} = -r \tan n\theta.$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{(r^2 + r^2 \tan^2 n\theta)} = r \sec n\theta$$

or $\quad ds = r \sec n\theta \, d\theta = a \cos^{1/n} n\theta \sec n\theta \, d\theta = a (\cos n\theta)^{(1/n)-1} d\theta.$

$\therefore$ The perimeter of a loop of the curve (1)

$$= 2 \int_0^{\pi/2n} a (\cos n\theta)^{(1/n)-1} d\theta$$

$$= 2a \int_0^{\pi/2} (\cos t)^{(1/n)-1} \frac{dt}{n},$$

putting $n\theta = t$ so that $n \, d\theta = dt$

$$= \frac{2a}{n} \int_0^{\pi/2} (\cos t)^{(1/n)-1} \sin^0 t \, dt$$

$$= \frac{2a}{n} \cdot \frac{\Gamma(1/2n)\,\Gamma(\frac{1}{2})}{2\,\Gamma\{(1/2n) + \frac{1}{2}\}} = \frac{a}{n} \cdot \frac{\sqrt{\pi}\,.\,[\Gamma(1/2n)]^2}{\Gamma(1/2n)\,\Gamma\{(1/2n) + \frac{1}{2}\}}$$

$$= \frac{a}{n} \cdot \sqrt{\pi} \frac{[\Gamma(1/2n)]^2}{\dfrac{\sqrt{\pi}}{2^{(1/n)-1}} \Gamma(1/n)}$$

$$\left[\because \Gamma(m)\,\Gamma(m + \tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m), \text{ by § 11}\right]$$

$$= \frac{a}{n} \cdot 2^{(1/n)-1} \frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

Ex. 32. *Show that*

(i) $\displaystyle\int_0^\infty x e^{-\alpha x} \cos \beta x \, dx = \frac{(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2}$

and (ii) $\displaystyle\int_0^\infty x e^{-\alpha x} \sin \beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$

Sol. We have $\displaystyle\int_0^\infty x e^{-\alpha x} \cos \beta x \, dx + i \int_0^\infty x e^{-\alpha x} \sin \beta x \, dx$

$$= \int_0^\infty x e^{-\alpha x} (\cos \beta x + i \sin \beta x) \, dx = \int_0^\infty x e^{-\alpha x} e^{i\beta x} \, dx$$

$$= \int_0^{\infty} x e^{-(\alpha - i\beta)x} dx = \int_0^{\infty} e^{-(\alpha - i\beta)x} x^{2-1} dx$$

$$= \frac{\Gamma(2)}{(\alpha - i\beta)^2} = \frac{1}{(\alpha - i\beta)^2} = \frac{(\alpha + i\beta)^2}{[(\alpha - i\beta)(\alpha + i\beta)]^2} = \frac{(\alpha^2 - \beta^2) + 2i\alpha\beta}{(\alpha^2 + \beta^2)^2}$$

$$= \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2} + i \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}. \qquad \text{...(1)}$$

Equating real and imaginary parts in (1), we get

$$\int_0^{\infty} x e^{-\alpha x} \cos \beta x \, dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2}$$

and $$\int_0^{\infty} x e^{-\alpha x} \sin \beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

Ex. 33. *Evaluate*

$\int_0^{\infty} x^{m-1} \cos bx \, dx$ *and* $\int_0^{\infty} x^{m-1} \sin bx \, dx$. **(Meerut 1993)**

Sol. We have

$$\int_0^{\infty} x^{m-1} \cos bx \, dx - i \int_0^{\infty} x^{m-1} \sin bx \, dx$$

$$= \int_0^{\infty} x^{m-1} (\cos bx - i \sin bx) \, dx = \int_0^{\infty} x^{m-1} e^{-ibx} \, dx$$

$$= \frac{\Gamma(m)}{(ib)^m} = \frac{\Gamma(m)}{b^m i^m} = \frac{\Gamma(m)}{b^m (\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi)^m}$$

$$= \frac{\Gamma(m)}{b^m} (\cos \tfrac{1}{2}\pi + i \sin \tfrac{1}{2}\pi)^{-m} = \frac{\Gamma(m)}{b^m} \left(\cos \frac{m\pi}{2} - i \sin \frac{m\pi}{2} \right). \qquad \text{...(1)}$$

Equating real and imaginary parts in (1), we get

$$\int_0^{\infty} x^{m-1} \cos bx \, dx = \frac{\Gamma(m)}{b^m} \cos \frac{m\pi}{2}$$

and $$\int_0^{\infty} x^{m-1} \sin bx \, dx = \frac{\Gamma(m)}{b^m} \sin \frac{m\pi}{2}.$$

Ex. 34. *Prove that* $\int_{-\infty}^{\infty} \cos (\frac{1}{2}\pi x^2) \, dx = 1$. **(Agra 1985)**

Sol. We have

$$I = \int_{-\infty}^{\infty} \cos (\tfrac{1}{2}\pi x^2) \, dx = 2 \int_0^{\infty} \cos (\tfrac{1}{2}\pi x^2) \, dx.$$

Put $\frac{1}{2}\pi x^2 = t$ i.e., $x = \sqrt{(2/\pi)}\, t^{1/2}$,

so that $dx = \frac{1}{2} \sqrt{(2/\pi)}\, t^{-1/2} dt$.

$$\therefore \quad I = 2 \cdot \tfrac{1}{2} \cdot \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} t^{-1/2} \cos t \, dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} t^{-1/2} \cos t \, dt$$

$$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} t^{-1/2} e^{-it} dt, \quad [\because e^{-it} = \cos t - i \sin t]$$

$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty t^{(1/2)-1} e^{-it}\, dt$

$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Gamma(\frac{1}{2})}{(i)^{1/2}}$

$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sqrt{\pi}}{(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)^{1/2}}$

$= \text{real part in } \sqrt{2}.(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)^{-1/2}$

$= \text{real part in } \sqrt{2}.(\cos\frac{1}{4}\pi - i\sin\frac{1}{4}\pi)$

$= \sqrt{2}.\cos\frac{\pi}{4} = \sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1.$

Ex. 35. *Prove that*

$$\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{2a^m b^n \Gamma(m+n)}.$$

Sol. We have $I = \int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}$

$= \int_0^{\pi/2} \frac{\tan^{2m-1}\theta \sec^2\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}}$,

dividing the Nr. and Dr. by $(\cos^2\theta)^{m+n}$ *i.e.*, by $(\cos\theta)^{2m+2n}$

$= \int_0^{\pi/2} \frac{\tan^{2m-2}\theta \tan\theta \sec^2\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}}$

$= \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1} \tan\theta \sec^2\theta\, d\theta}{(b + a\tan^2\theta)^{m+n}}.$

Put $a\tan^2\theta = by$ so that $2a\tan\theta\sec^2\theta\, d\theta = b\, dy$.

$\therefore \quad I = \int_0^\infty \frac{(by/a)^{m-1}}{(b+by)^{m+n}} \cdot \frac{b}{2a} dy = \frac{b^m}{b^{m+n}.a^m} \cdot \frac{1}{2} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$

$= \frac{1}{2a^m b^n} B(m, n) = \frac{1}{2a^m b^n} \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$

Ex. 36. *Prove that*

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{B(m,n)}{a^n(1+a)^m}.$$

(Meerut 1992; Rajasthan 80; Agra 78)

Sol. We know that $B(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}\, dy$. ...(1)

Put $y = \frac{(1+a)x}{a+x} = (1+a)\left(1 - \frac{a}{a+x}\right)$ so that $dy = \frac{a(1+a)}{(a+x)^2} dx$.

When $y = 1, x = 1$ and when $y = 0, x = 0$.

Also $1-y=1-\dfrac{x(1+a)}{a+x}=\dfrac{a(1-x)}{a+x}$.

$$\therefore \quad B(m,n)=\int_0^1 \frac{(1+a)^{m-1}x^{m-1}}{(a+x)^{m-1}}\cdot\frac{a^{n-1}(1-x)^{n-1}}{(a+x)^{n-1}}\cdot\frac{a(1+a)}{(a+x)^2}\,dx$$

$$=(1+a)^m a^n\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx.$$

$$\therefore \quad \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx=\frac{B(m,n)}{a^n(1+a)^m}=\frac{\Gamma(m)\,\Gamma(n)}{a^n(1+a)^m\,\Gamma(m+n)}.$$

Ex. 37. *Prove that*

$$\int_0^\infty e^{-x^2}x^{2n}\,dx=\sqrt{\pi}\cdot\frac{1.3.5....(2n-1)}{2.4.6....2n}\int_0^\infty e^{-x^2}x^{2n+1}\,dx.$$

Sol. Let $U=\dfrac{\int_0^\infty e^{-x^2}x^{2n}\,dx}{\int_0^\infty e^{-x^2}x^{2n+1}\,dx}=\dfrac{\int_0^\infty e^{-x^2}.x^{2n-1}x\,dx}{\int_0^\infty e^{-x^2}.x^{2n}x\,dx}$.

Put $x^2=t$ so that $2x\,dx=dt$ and $x=t^{1/2}$.

$$\therefore \quad U=\frac{\int_0^\infty e^{-t}(t^{1/2})^{2n-1}\,dt}{\int_0^\infty e^{-t}(t^{1/2})^{2n}\,dt}=\frac{\int_0^\infty e^{-t}t^{n-(1/2)}\,dt}{\int_0^\infty e^{-t}t^n\,dt}$$

$$=\frac{\int_0^\infty e^{-t}t^{\{n+(1/2)\}-1}\,dt}{\int_0^\infty e^{-t}t^{(n+1)-1}\,dt}=\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}=\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}$$

$$=\frac{\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2})}{n(n-1)(n-2)\ldots 2.1\,\Gamma(1)}$$

$$=\sqrt{\pi}\cdot\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)}.$$

$$\therefore \quad \int_0^\infty e^{-x^2}x^{2n}\,dx=\sqrt{\pi}\cdot\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)}\int_0^\infty e^{-x^2}x^{2n+1}\,dx.$$

Ex. 38. *Prove that*

$$\int_0^p x^m(p^q-x^q)^n\,dx=\frac{p^{qn+m+1}}{q}B\left(n+1,\frac{m+1}{q}\right),$$

if $p>0, q>0, m+1>0, n+1>0$.

Sol. Let $I=\int_0^p x^m(p^q-x^q)^n\,dx$.

Put $x^q=p^q y$ or $x=py^{1/q}$, so that $dx=p.(1/q)\,y^{(1/q)-1}\,dy$.
When $x=0, y=0$ and when $x=p, y=1$.

$$\therefore \quad I = \int_0^1 (py^{1/q})^m (p^q - p^q y)^n \cdot \frac{p}{q} y^{(1/q)-1}\, dy$$

$$= p^m . p^{qn} \cdot \frac{p}{q} \int_0^1 y^{(m/q)+(1/q)-1} (1-y)^n\, dy$$

$$= \frac{p^{qn+m+1}}{q} \int_0^1 y^{[(m+1)/q]-1} (1-y)^{(n+1)-1}\, dy$$

$$= \frac{p^{qn+m+1}}{q} B\left(\frac{m+1}{q}, n+1\right) = \frac{p^{qn+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right),$$

if $m+1>0$ and $n+1>0$.

Ex. 39. *Prove that*

$$\int_0^\pi \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} \cdot B\left(\frac{n}{2}, \frac{n}{2}\right), a > b.$$

Sol. Let $I = \int_0^\pi \frac{\sin^{n-1} x}{(a+b\cos x)^n} dx$

$$= \int_0^\pi \frac{(2\sin\frac{1}{2}x\cos\frac{1}{2}x)^{n-1}}{[a(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x)+b(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x)]^n} dx$$

$$= \int_0^\pi \frac{2^{n-1}(\sin\frac{1}{2}x)^{n-1}(\cos\frac{1}{2}x)^{n-1}}{[(a+b)\cos^2\frac{1}{2}x+(a-b)\sin^2\frac{1}{2}x]^n} dx$$

$$= \int_0^\pi \frac{2^{n-1}(\tan\frac{1}{2}x)^{n-1}\sec^2\frac{1}{2}x\, dx}{[(a+b)+(a-b)\tan^2\frac{1}{2}x]^n},$$

dividing the Nr. and Dr. by $(\cos^2\frac{1}{2}x)^n$ *i.e.*, $(\cos\frac{1}{2}x)^{2n}$

$$= 2^{n-1} \int_0^\pi \frac{(\tan\frac{1}{2}x)^{n-2}\tan\frac{1}{2}x\sec^2\frac{1}{2}x\, dx}{[(a+b)+(a-b)\tan^2\frac{1}{2}x]^n}.$$

Put $(a-b)\tan^2\frac{1}{2}x = (a+b)y$ so that

$$(a-b).(2\tan\tfrac{1}{2}x\sec^2\tfrac{1}{2}x).\tfrac{1}{2}dx = (a+b)\,dy.$$

When $x=0, y=0$ and when $x \to \pi, y \to \infty$.

$$\therefore \quad I = 2^{n-1} \int_0^\infty \frac{\{(a+b)y/(a-b)\}^{(n-2)/2}}{[(a+b)+(a+b)y]^n} \frac{a+b}{a-b} dy$$

$$= 2^{n-1} \int_0^\infty \frac{(a+b)^{(n-2)/2}(a+b)}{(a+b)^n(a-b)^{(n-2)/2}(a-b)} \cdot \frac{y^{(n-2)/2}}{(1+y)^n} dy$$

$$= \frac{2^{n-1}}{(a+b)^{n/2}(a-b)^{n/2}} \int_0^\infty \frac{y^{(n/2)-1}}{(1+y)^{(n/2)+(n/2)}} dy$$

$$= \frac{2^{n-1}}{(a^2-b^2)^{n/2}} B\left(\frac{n}{2}, \frac{n}{2}\right).$$

Ex. 40. *Prove that*

$$\int_0^{\pi} \frac{\sqrt{(\sin x)}}{(5+3\cos x)^{3/2}}\,dx = \frac{[\Gamma(\frac{3}{4})]^2}{2\sqrt{(2\pi)}}.$$

Sol. Proceed as in Ex. 39.

Ex. 41. *Show that* $\displaystyle\int_0^{\infty} \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 1.$

Sol. We have $\displaystyle I = \int_0^{\infty} \frac{x^c}{c^x}\,dx = \int_0^{\infty} \frac{x^c}{e^{\log c^x}}\,dx$

$$= \int_0^{\infty} \frac{x^c}{e^{x\log c}}\,dx = \int_0^{\infty} e^{-x\log c}\, x^c\,dx.$$

Put $x\log c = y$ so that $(\log c)\,dx = dy$.
When $x = 0$, we have $y = 0$
and when $x \to \infty, y \to \infty$ because $c > 1 \Rightarrow \log c > 0$.

$$\therefore \quad I = \int_0^{\infty} e^{-y}\left(\frac{y}{\log c}\right)^c \frac{dy}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}} \int_0^{\infty} e^{-y}\, y^{(c+1)-1}\,dy$$

$$= \frac{1}{(\log c)^{c+1}}\,\Gamma(c+1),$$

provided $c + 1 > 0$ which is so because $c > 1$.

Ex. 42. *Prove that* $\displaystyle\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1} dx = \frac{\Gamma m}{n^m}, (m, n > 0).$

(Meerut 1993 P)

Sol. Let $\displaystyle I = \int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1} dx.$

Put $\log(1/x) = y$ i.e., $1/x = e^y$ i.e., $x = e^{-y}$ so that $dx = -e^{-y}\,dy$.
Also when $x \to 0, y \to \infty$ and when $x = 1, y = 0$.

[Note that $\log\infty = \infty$]

$$\therefore \quad I = -\int_{\infty}^{0} (e^{-y})^{n-1}\, y^{m-1}\, e^{-y}\,dy$$

$$= \int_0^{\infty} (e^{-y})^{n-1+1}\, y^{m-1}\,dy = \int_0^{\infty} e^{-ny}\, y^{m-1}\,dy$$

$= \dfrac{\Gamma m}{n^m}$, provided $m > 0$ and $n > 0$. [See § 6 part (i)]

Ex. 43. *Prove that* $\displaystyle B(n,n) = \frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\,\Gamma\left(n+\frac{1}{2}\right)}.$

(Meerut 1995 BP; U.P. P.C.S. 90)

Sol. By the definition of Beta function, we have

$$B(n,n) = \int_0^1 x^{n-1}(1-x)^{n-1}\,dx, \quad n > 0.$$

Now put $x = \sin^2 \theta$ and proceed as in § 11 to get the required result.

Ex. 44. *Evaluate the following integrals :*

(i) $\int_0^{\pi/2} \sqrt{(\tan \theta)}\, d\theta$ **(Meerut 1994 P)**

(ii) $\int_0^{\pi/2} \sqrt{(\cot \theta)}\, d\theta$ **(Meerut 1994)**

(iii) $\int_0^{\infty} \frac{dy}{1+y^4}$. **(Meerut 1995 BP)**

Sol. (i) We have $I = \int_0^{\pi/2} \sqrt{(\tan \theta)}\, d\theta$

$$= \int_0^{\pi/2} \frac{\sin^{1/2} \theta}{\cos^{1/2} \theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta\, d\theta$$

$$= \frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+1\right)\right\} \Gamma\left\{\frac{1}{2}\left(-\frac{1}{2}+1\right)\right\}}{2\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}+2\right)\right\}} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma 1}$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{2} = \frac{1}{2} \cdot \frac{\pi}{\sin \frac{1}{4}\pi} = \frac{\pi}{2.(1/\sqrt{2})} = \frac{\pi}{\sqrt{2}}.$$

(ii) We have $\int_0^{\pi/2} \sqrt{(\cot \theta)}\, d\theta$

$$= \int_0^{\pi/2} \cos^{1/2} \theta \sin^{-1/2} \theta\, d\theta = \frac{\pi}{\sqrt{2}},$$

proceeding as in part (i) of this exercise.

(iii) Let $I = \int_0^{\infty} \frac{dy}{1+y^4}$.

Put $y = \tan \theta$ *i.e.*, $y = \sqrt{(\tan \theta)}$ so that $dy = \frac{1}{2} \tan^{-1/2} \theta \sec^2 \theta\, d\theta$.

When $y = 0, \theta = 0$ and when $y \to \infty, \theta \to \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{1}{1+\tan^2 \theta} \cdot \frac{1}{2} \tan^{-1/2} \theta \sec^2 \theta\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \tan^{-1/2} \theta\, d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta\, d\theta$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}},$$ proceeding as in part (i) of this exercise.

Ex. 45. *Evaluate the following integrals :*

(i) $\int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\, dy$ **(Meerut 1994)**

(ii) $\int_0^{\infty} 4x^4 e^{-x^4}\, dx$ **(Meerut 1995)**

(iii) $\int_{-\infty}^{\infty} e^{-k^2x^2}\,dx$

(Meerut 1995 BP)

Sol. (i) Let $I = \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$.

Put $y = \frac{1}{x} - 1 = \frac{1-x}{x}$ so that $dy = -\frac{1}{x^2}\,dx$.

Also $\frac{1}{x} = y + 1$ or $x = \frac{1}{y+1}$.

When $y = 0$, we have $x = 1$ and when $y \to \infty, x \to 0$.

$$\therefore \quad I = \int_1^0 \left(\frac{1-x}{x}\right)^{q-1} . x^{p+q} . \left(-\frac{1}{x^2}\right) dx$$

$$= \int_0^1 x^{p+q} (1-x)^{q-1} . \frac{1}{x^{q-1}} \cdot \frac{1}{x^2}\,dx$$

$$= \int_0^1 x^{p-1} (1-x)^{q-1}\,dx$$

$= B(p, q)$, provided $p > 0$ and $q > 0$

$$= \frac{\Gamma p\,\Gamma q}{\Gamma(p+q)}.$$

(ii) Let $I = \int_0^{\infty} 4x^4 e^{-x^4}\,dx$.

Put $x^4 = t$ i.e., $x = t^{1/4}$ so that $dx = \frac{1}{4} t^{-3/4}\,dt$.

Then $I = \int_0^{\infty} 4 . t . e^{-t} . \frac{1}{4} t^{-3/4}\,dt$

$$= \int_0^{\infty} e^{-t} t^{1/4}\,dt = \int_0^{\infty} e^{-t} t^{(5/4)-1}\,dt$$

$$= \Gamma(5/4) = \tfrac{1}{4}\Gamma\left(\tfrac{1}{4}\right).$$

(iii) Let $I = \int_{-\infty}^{\infty} e^{-k^2x^2}\,dx$.

Since $e^{-k^2x^2}$ is an even function of x, therefore

$$I = 2\int_0^{\infty} e^{-k^2x^2}\,dx.$$

Now put $k^2x^2 = t$ i.e., $kx = t^{1/2}$ so that $k\,dx = \frac{1}{2} t^{(1/2)-1}\,dt$.

When $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$\therefore \quad I = 2\int_0^{\infty} e^{-t} . \frac{1}{2k} t^{(1/2)-1}\,dt$$

$$= \frac{1}{k}\int_0^{\infty} e^{-t} t^{(1/2)-1}\,dt = \frac{1}{k}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{k} \cdot \sqrt{\pi}.$$

§ 14. Dirichlet's theorem for three variables.

If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le 1$.

(Meerut 1991; Agra 79; Rohilkhand 87, Gorakhpur 88)

Proof. Let us first consider the double integral

$$I_2 = \iint x^{l-1} y^{m-1}\, dx\, dy,$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x+y \le 1$.

Obviously the region of integration of I_2, in the 2-dimensional Euclidean space, is bounded by the straight lines $x=0, y=0$ and $x+y=1$. The limits of integration for this region can be expressed as $0 \le x \le 1, 0 \le y \le 1-x$.

$$\therefore \quad I_2 = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1} y^{m-1}\, dx\, dy$$

$$= \int_0^1 x^{l-1} \left[\frac{y^m}{m}\right]_0^{1-x} dx = \int_0^1 \frac{1}{m} x^{l-1} (1-x)^m\, dx$$

$$= \frac{1}{m} \int_0^1 x^{l-1} (1-x)^{m+1-1}\, dx = \frac{1}{m} \mathrm{B}(l, m+1),$$

by the def. of Beta function

$$= \frac{1}{m} \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = \frac{1}{m} \frac{\Gamma(l)\,.\,m\,\Gamma(m)}{\Gamma(l+m+1)}, \quad [\because \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}. \qquad \textbf{(Remember)} \qquad \text{...(1)}$$

This is Dirichlet's theorem for two variables.

Now consider the double integral

$$U_2 = \iint x^{l-1} y^{m-1}\, dx\, dy,$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x+y \le h$.

We have $x+y \le h \Rightarrow \dfrac{x}{h} + \dfrac{y}{h} \le 1$.

So putting $x/h = u$ and $y/h = v$ so that $dx = h\, du$ and $dy = h\, dv$, the integral U_2 becomes

$$U_2 = \iint (hu)^{l-1} (hv)^{m-1} h^2\, du\, dv$$

$$= h^{l+m} \iint u^{l-1} v^{m-1}\, du\, dv, \text{ where } u+v \le 1$$

$$= h^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}, \text{ by (1).} \qquad \text{...(2)}$$

Now we consider the triple integral

$$I_3 = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

subject to the condition $x+y+z \le 1$ i.e., $y+z \le 1-x$ and $0 \le x \le 1$.

We have

$$I_3 = \int_{x=0}^{1} \left[\iint y^{m-1} z^{n-1}\, dy\, dz\right] x^{l-1}\, dx, \text{ where } y+z \le 1-x$$

$$= \int_0^1 (1-x)^{m+n} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} x^{l-1}\, dx, \qquad \text{by using (2)}$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n+1-1}\, dx$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \mathrm{B}(l, m+n+1)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}, \text{ which proves the required result.}$$

Remark. The triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = h^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le h$.

Alternative proof of Dirichlet's theorem for three variables.

Let $I_3 = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$,

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le 1$.

Obviously the region of integration, in the 3-dimensional Euclidean space, is the volume bounded by the coordinate planes $x=0, y=0, z=0$ and the plane $x+y+z=1$. After a little geometric consideration, we observe that the limits of integration for this region can be expressed as

$$0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y.$$

Hence the triple integral I_3 may be written as

$$I_3 = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{n} \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n\, dx\, dy$$

$$= \frac{1}{n} \int_0^1 x^{l-1} \left[\int_0^{1-x} y^{m-1} \{(1-x)-y\}^n\, dy\right] dx.$$

To integrate w.r.t. y, put $y = (1-x)\,t$ so that $dy = (1-x)\,dt$; also when $y = 0, t = 0$ and when $y = 1 - x, t = 1$.

$\therefore$ the required integral

$$I_3 = \frac{1}{n}\int_0^1 x^{l-1}\left[\int_0^1 (1-x)^{m-1}\,t^{m-1}\,\{(1-x)^n\,(1-t)^n\}\,(1-x)\,dt\right]dx$$

$$= \frac{1}{n}\int_0^1\int_0^1 x^{l-1}(1-x)^{m+n}\,t^{m-1}(1-t)^n\,dx\,dt$$

$$= \frac{1}{n}\int_0^1 x^{l-1}(1-x)^{m+n}\,dx \times \int_0^1 t^{m-1}.(1-t)^n\,dt$$

$$= \frac{1}{n}B\,(l, m+n+1)\,B\,(m, n+1),$$

(by the definition of Beta function)

$$= \frac{1}{n}\cdot\frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}\cdot\frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)},$$

$[\because \Gamma(n+1) = n\,\Gamma(n)]$.

Note. Dirichlet's theorem holds good even if the condition is taken as $x + y + z < 1$ in place of $x + y + z \le 1$.

Cor. *Evaluate without using Dirichlet's theorem*

$$\iiint x^p y^q z^r\,dx\,dy\,dz,$$

where x, y, z are always positive and $x + y + z \le 1$. **(Agra 1982)**

§ 15. Dirichlet's theorem for n variables.

The theorem states that

$$\iint \dots \int x_1^{l_1-1}\,x_2^{l_2-1}\dots x_n^{l_n-1}\,dx_1\,dx_2\dots dx_n$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2)\dots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\dots+l_n)},$$

where the integral is extended to all positive values of the variables $x_1, x_2, \dots, x_n$ *subject to the condition* $x_1 + x_2 + \dots + x_n \le 1$.

(Meerut 1980)

Proof. We shall prove the theorem by mathematical induction.

To start the induction we shall first show that the theorem is true for two variables *i.e.*, for $n = 2$.

So let us consider the integral

$$I_2 = \iint x_1^{l_1-1}\,x_2^{l_2-1}\,dx_1\,dx_2$$

subject to the condition $x_1 + x_2 \le 1$.

Now proceeding as in § 14, show that

$$I_2 = \frac{\Gamma(l_1)\,\Gamma(l_2)}{\Gamma(1+l_1+l_2)}. \qquad \dots(1)$$

The result (1) shows that the theorem is true for two variables *i.e.*, for $n = 2$.

Now assume as our induction hypothesis that the theorem is true for n variables *i.e.*, assume that

$$I_n = \iint \ldots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \ldots x_n^{l_n - 1} \, dx_1 \, dx_2 \ldots dx_n$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2) \ldots \Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \ldots + l_n)}, \qquad \ldots(2)$$

subject to the condition $x_1 + x_2 + \ldots + x_n \le 1$.

If the condition be $x_1 + x_2 + \ldots + x_n \le h$, then putting

$\dfrac{x_1}{h} = u_1, \dfrac{x_2}{h} = u_2, \ldots, \dfrac{x_n}{h} = u_n$, so that

$dx_1 = h\,du_1,\ dx_2 = h\,du_2, \ldots, dx_n = h\,du_n$, we have

$$\iint \ldots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \ldots x_n^{l_n - 1} \, dx_1 \, dx_2 \ldots dx_n$$

$$= h^{l_1 + l_2 + \ldots + l_n} \iint \ldots \int u_1^{l_1 - 1} u_2^{l_2 - 1} \ldots u_n^{l_n - 1} \, du_1 \, du_2 \ldots du_n$$

subject to the condition $u_1 + u_2 + \ldots + u_n \le 1$

$$= h^{l_1 + l_2 + \ldots + l_n} \frac{\Gamma(l_1)\,\Gamma(l_2) \ldots \Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \ldots + l_n)}, \qquad \ldots(3)$$

using the assumed result (2).

Now for $n + 1$ variables the condition is

$$x_1 + x_2 + \ldots + x_n + x_{n+1} \le 1$$

i.e., $x_2 + x_3 + \ldots + x_n + x_{n+1} \le 1 - x_1$, and $0 \le x_1 \le 1$.

We then have

$$\iint \ldots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \ldots x_n^{l_n - 1} x_{n+1}^{l_{n+1} - 1} \, dx_1 \, dx_2 \ldots dx_n \, dx_{n+1},$$

where $x_1 + x_2 + \ldots + x_{n+1} \le 1$

$$= \int_{x_1 = 0}^{1} x_1^{l_1 - 1} \left[\iint \ldots \int x_2^{l_2 - 1} \ldots x_{n+1}^{l_{n+1} - 1} \, dx_2 \ldots dx_{n+1} \right] dx_1$$

$$= \int_{x_1 = 0}^{1} x_1^{l_1 - 1} \cdot \frac{\Gamma(l_2)\,\Gamma(l_3) \ldots \Gamma(l_{n+1})}{\Gamma(1 + l_2 + l_3 + \ldots + l_n + l_{n+1})} \cdot (1 - x_1)^{l_2 + l_3 + \ldots + l_{n+1}} \, dx_1,$$

using (3)

$$= \frac{\Gamma(l_2)\,\Gamma(l_3) \ldots \Gamma(l_{n+1})}{\Gamma(1 + l_2 + \ldots + l_n + l_{n+1})} \cdot \int_0^1 x_1^{l_1 - 1} (1 - x_1)^{(1 + l_2 + l_3 + \ldots + l_{n+1}) - 1} \, dx_1$$

$$= \frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(1+l_2+\ldots+l_n+l_{n+1})}\cdot\frac{\Gamma(l_1)\Gamma(1+l_2+\ldots+l_{n+1})}{\Gamma(1+l_1+l_2+\ldots+l_n+l_{n+1})}$$

$$= \frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_{n+1})}{\Gamma(1+l_1+l_2+\ldots+l_{n+1})}. \qquad \ldots(4)$$

The result (4) shows that the theorem holds for $(n+1)$ variables if it holds for n variables. But we have seen that the theorem is true for two variables. Hence by mathematical induction the theorem is true for all values of n.

Solved Examples

Ex. 1. *Evaluate* $\iint x^{2l-1}y^{2m-1}\,dx\,dy$ *for all positive values of* x *and* y *such that* $x^2+y^2\le c^2$.

Sol. Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of x and y subject to the condition

$$\left(\frac{x}{c}\right)^2+\left(\frac{y}{c}\right)^2\le 1.$$

Put $(x/c)^2=u$ i.e., $x=cu^{1/2}$, so that $dx=\frac{1}{2}cu^{-1/2}\,du$,

and $(y/c)^2=v$ i.e., $y=cv^{1/2}$, so that $dy=\frac{1}{2}cv^{-1/2}\,dv$.

Then the required integral

$I=\iint (cu^{1/2})^{2l-1}(cv^{1/2})^{2m-1}\cdot\frac{1}{2}cu^{-1/2}\cdot\frac{1}{2}cv^{-1/2}\,du\,dv$

$=\frac{1}{4}c^{2l+2m}\iint u^{l-1}v^{m-1}\,du\,dv$, where u, v take all +ive values subject to the condition $u+v\le 1$

$=\frac{1}{4}c^{2l+2m}\cdot\dfrac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)}$, by Dirichlet's theorem.

Ex. 2. *Find the value of*

$$\iint\ldots\int dx_1\,dx_2\ldots dx_n$$

extended to all positive values of the variables, subject to the condition

$$x_1^2+x_2^2+\ldots+x_n^2<R^2.$$

Sol. Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of $x_1, x_2, \ldots, x_n$ subject to the condition

$$\frac{x_1^2}{R^2}+\frac{x_2^2}{R^2}+\ldots+\frac{x_n^2}{R^2}<1.$$

Put $(x_1/R)^2=u_1$ i.e., $x_1=Ru_1^{1/2}$, so that $dx_1=\frac{1}{2}Ru_1^{-1/2}\,du_1$,
$(x_2/R)^2=u_2$ i.e., $x_2=Ru_2^{1/2}$, so that $dx_2=\frac{1}{2}Ru_2^{-1/2}\,du_2$, and so on.

Then the required integral

$$I=\iint\ldots\int (\tfrac{1}{2})^n R^n u_1^{-1/2}u_2^{-1/2}\ldots u_n^{-1/2}\,du_1\,du_2\ldots du_n$$

$$= \left(\frac{R}{2}\right)^n \iint \dots \int u_1^{(1/2)-1} u_2^{(1/2)-1} \dots u_n^{(1/2)-1} \, du_1 \, du_2 \dots du_n,$$

subject to the condition $u_1 + u_2 + \dots + u_n < 1$

$$= \left(\frac{R}{2}\right)^n \frac{\{\Gamma(\frac{1}{2})\}^n}{\Gamma(1 + n \cdot \frac{1}{2})}, \qquad \text{by Dirichlet's theorem}$$

$$= \left(\frac{R}{2}\right)^n \cdot \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2}n)}. \qquad [\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}]$$

Ex. 3. *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1. \qquad \textbf{(Agra 1974)}$$

Sol. Since the equation $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ does not change by putting $-x$ for x, $-y$ for y and $-z$ for z, therefore the surface represented by this equation is symmetrical in all the eight octants.

So the volume of the solid surrounded by this surface $= 8 \times$ the volume of the portion of this solid lying in the positive octant.

Now the volume of a small element situated at any point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ the volume of the solid in the positive octant

$$= \iiint dx\, dy\, dz,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} \le 1.$$

Now put $(x/a)^{2/3} = u$, $(y/b)^{2/3} = v$, $(z/c)^{2/3} = w$

i.e., $x = au^{3/2}, y = bv^{3/2}, z = cw^{3/2}$

so that $dx = \frac{3}{2} au^{1/2} du$, $dy = \frac{3}{2} bv^{1/2} dv$, $dz = \frac{3}{2} cw^{1/2} dw$.

$\therefore$ the volume in the positive octant

$$= \iiint \frac{27}{8} abc\, u^{(3/2)-1} v^{(3/2)-1} w^{(3/2)-1} \, du\, dv\, dw,$$

where $u + v + w \le 1$

$$= \frac{27}{8} abc \frac{[\Gamma(3/2)]^3}{\Gamma(\frac{9}{2} + 1)} = \frac{27}{8} abc \cdot \frac{(\frac{1}{2} \cdot \sqrt{\pi})^3}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}$$

$$= \frac{27}{8} abc \cdot \frac{\pi}{8} \cdot \frac{32}{27.35} = \frac{\pi abc}{8} \cdot \frac{4}{35}.$$

Hence the required volume $= 8 \cdot \dfrac{\pi abc}{8} \cdot \dfrac{4}{35} = \dfrac{4\pi abc}{35}$.

Ex. 4. *Find the volume enclosed by the surface*

$$(x/a)^{2n} + (y/b)^{2n} + (z/c)^{2n} = 1. \qquad \textbf{(Agra 1981)}$$

Sol. The given surface is symmetrical in all the eight octants.

The volume V in the positive octant

$= \iiint dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$(x/a)^{2n} + (y/b)^{2n} + (z/c)^{2n} \le 1.$$

Now put $(x/a)^{2n} = u, (y/b)^{2n} = v, (z/c)^{2n} = w$

i.e. $x = au^{1/2n}, y = bv^{1/2n}, z = cw^{1/2n}$

so that $dx = \frac{a}{2n} u^{(1/2n) - 1} du$, etc.

$$\therefore \quad V = \frac{abc}{8n^3} \iiint u^{(1/2n) - 1} v^{(1/2n) - 1} w^{(1/2n) - 1} du\, dv\, dw,$$

where $u + v + w \le 1$

$$= \frac{abc}{8n^3} \cdot \frac{[\Gamma(1/2n)]^3}{\Gamma\{(3/2n) + 1\}} = \frac{abc}{8n^3} \cdot \frac{[\Gamma(1/2n)]^3}{(3/2n) . \Gamma(3/2n)}$$

$$= \frac{abc}{12n^2} \cdot \frac{[\Gamma(1/2n)]^3}{\Gamma(3/2n)}.$$

Hence the total volume encolsed by the given surface

$$= 8V = \frac{2}{3} \cdot \frac{abc}{n^2} \cdot \frac{[\Gamma(1/2n)]^3}{\Gamma(3/2n)}.$$

Ex. 5. *Show that the integral* $\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$ *integrated over the region in the first octant below the surface* $(x/a)^p + (y/b)^q + (z/c)^r = 1$ *is*

$$\frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p)\Gamma(m/q)\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}.$$

(Rohilkhand 1987; Meerut 88, 90; Agra 80)

Or

Evaluate the integral $\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$, *where* x, y, z *are all positive but limited by the condition*

$$(x/a)^p + (y/b)^q + (z/c)^r \le 1.$$

Sol. The required integral $= \iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y and z subject to the condition

$$(x/a)^p + (y/b)^q + (z/c)^r \le 1.$$

Put $(x/a)^p = u$ *i.e.*, $x = au^{1/p}$ so that $dx = (a/p)\, u^{(1/p) - 1} du$,

$(y/b)^q = v$ *i.e.*, $y = bv^{1/q}$ so that $dy = (b/q)\, v^{(1/q) - 1} dv$,

and $(z/c)^r = w$ *i.e.*, $z = cw^{1/r}$ so that $dz = (c/r)\, w^{(1/r) - 1} dw$.

Then the required integral

$$= \iiint (a^{l-1} u^{(l-1)/p})(b^{m-1} v^{(m-1)/q})(c^{n-1} w^{(n-1)/r}) .$$

$$\frac{a}{p} u^{(1/p) - 1} . \frac{b}{q} v^{(1/q) - 1} . \frac{c}{r} w^{(1/r) - 1} du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{(l/p)-1} v^{(m/q)-1} w^{(n/r)-1} \, du \, dv \, dw,$$

where $u + v + w \leq 1$

$$= \frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)},$$ by Dirichlet's integral.

Ex. 6. *Show that if l, m, n are all positive*

$$\iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)}$$

where the triple integral is taken throughout the part of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, which lies in the positive octant.

Sol. The required integral

$$= \iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1.$$

Put $x^2/a^2 = u$ i.e., $x = au^{1/2}$ so that $dx = \frac{1}{2} au^{-1/2} du$,

$y^2/b^2 = v$ i.e., $y = bv^{1/2}$ so that $dy = \frac{1}{2} bv^{-1/2} dv$,

and $z^2/c^2 = w$ i.e., $z = cw^{1/2}$ so that $dz = \frac{1}{2} cw^{-1/2} dw$.

Then the required integral

$$= \iiint a^{l-1} u^{(l-1)/2} b^{m-1} v^{(m-1)/2} c^{n-1} w^{(n-1)/2} \times \tfrac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2} \, du \, dv \, dw$$

$$= \frac{a^l b^m c^n}{8} \iiint u^{(l/2)-1} v^{(m/2)-1} w^{(n/2)-1} \, du \, dv \, dw,$$

where $u + v + w \leq 1$

$$= \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)},$$ by Dirichlet's integral.

Ex. 7 (a). *Evaluate*

$$\iiint dx \, dy \, dz, \text{ where } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

Sol. Here we are to evaluate the given integral over the whole volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ which is symmetrical in all the eight octants. Obviously the required integral

$$= 8 \iiint dx \, dy \, dz,$$

where the integral is to be evaluated throughout the volume of the ellipsoid which lies in the positive octant.

Let $I = \iiint dx \, dy \, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1.$$

Put $x^2/a^2 = u$ i.e., $x = au^{1/2}$ so that $dx = \frac{1}{2} au^{-1/2} du$,

$y^2/b^2 = v$ i.e., $y = bv^{1/2}$ so that $dy = \frac{1}{2} bv^{-1/2}\, dv$,

and $z^2/c^2 = w$ i.e., $z = cw^{1/2}$ so that $dz = \frac{1}{2} cw^{-1/2}\, dw$.

Then $I = \iiint \frac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$

$$= \iiint \frac{1}{8} abc\, u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw,$$

where $u + v + w \leq 1$

$$= \frac{abc}{8} \frac{\Gamma(\frac{1}{2}) . \Gamma(\frac{1}{2}) . \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1)},$$ by Dirichlet's integral

$$= \frac{abc}{8} \frac{(\sqrt{\pi})^3}{\Gamma(\frac{5}{2})} = \frac{abc}{8} \frac{\pi . \sqrt{\pi}}{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{\pi abc}{6}.$$

Hence the required integral

$$= 8 \cdot \frac{\pi abc}{6} = \frac{4}{3} \pi abc.$$

Ex. 7 (b). *Find the volume of the ellipsoid*

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1.$$

(Rohilkhand 1991; Agra 87; Kanpur 86; Vikram 84; Meerut 94 P, 95 BP)

Sol. The given ellipsoid is symmetrical in all the eight octants. Therefore the volume of the given ellipsoid $= 8 \times$ the volume of the part of the ellipsoid lying in the positive octant.

Now the volume of a small element situated at any point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ the volume of the given ellipsoid

$$= 8 \iiint dx\, dy\, dz,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1.$$

Now to evaluate this integral proceed as in Ex. 7 (a).

Hence the required volume $= 8 \cdot \dfrac{\pi abc}{6} = \dfrac{4}{3} \pi abc$.

Ex. 7 (c). *Find the volume in the positive octant of the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(Agra 1978)**

Sol. Proceed as in Ex. 7 (a). The answer is $\pi abc/6$.

Ex. 7 (d). *Find the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(Meerut 1994, 95)

Sol. The given sphere $x^2 + y^2 + z^2 = a^2$ is symmetrical in all the eight octants.

$\therefore$ the volume of the given sphere $x^2 + y^2 + z^2 = a^2$

$$= 8 \iiint dx\, dy\, dz,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} \le 1$.

Now put $x = au^{1/2}, y = av^{1/2}, z = aw^{1/2}$ and proceed as in Ex. 7 (a).

The required volume $= \frac{4}{3}\pi a^3$.

Ex. 7 (e). *Evaluate* $\iiint xyz\, dx\, dy\, dz$ *for all positive values of the variables throughout the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol. We are to evaluate the integral $\iiint xyz\, dx\, dy\, dz$, for all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1.$$

Put $x^2/a^2 = u, y^2/b^2 = v, z^2/c^2 = w$ so that

$$x\, dx = \tfrac{1}{2} a^2\, du, y\, dy = \tfrac{1}{2} b^2\, dv, z\, dz = \tfrac{1}{2} c^2\, dw.$$

Then the required integral

$= \frac{1}{8} a^2b^2c^2 \iiint du\, dv\, dw$, where $u + v + w \le 1$

$= \frac{1}{8} a^2b^2c^2 \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$, where $u + v + w \le 1$

$= \frac{1}{8} a^2b^2c^2 \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)}$, by Dirichlet's integral

$$= \frac{a^2b^2c^2}{8} \frac{1}{\Gamma(4)} = \frac{a^2b^2c^2}{8} \times \frac{1}{3.2.1} = \frac{a^2b^2c^2}{48}.$$

Ex. 8 (a). *Find the volume of the tetrahedron bounded by the plane* $x/a + y/b + z/c = 1$ *and the coordinate planes.* **(Kanpur 1976)**

Sol. The volume of a small element situated at any point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ the volume of the given tetrahedron

$= \iiint dx\, dy\, dz$, where the integral is extended throughout the volume enclosed by the coordinate planes and the plane $x/a + y/b + z/c = 1$

$= \iiint dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition $x/a + y/b + z/c \le 1$.

Put $x/a = u, y/b = v, z/c = w$ so that

$$dx = a\, du, dy = b\, dv, dz = c\, dw.$$

Then the required volume $= \iiint abc\, du\, dv\, dw$, where $u + v + w \le 1$

$= abc \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$

$= abc \frac{[\Gamma(1)]^3}{\Gamma(1+1+1+1)}$, by Dirichlet's integral

$$= abc \cdot \frac{1}{\Gamma(4)} = \frac{abc}{3.2.1} = \frac{abc}{6}.$$

Ex. 8 (b). *The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in the points A, B, C. Use Dirichlet's integral to evaluate the mass of the tetrahedron $OABC$, the density at any point (x, y, z) being $kxyz$.*

(Kanpur 1989, Agra 76)

Sol. The mass of a small element of volume $dx\, dy\, dz$ situated at any point (x, y, z)

$= \rho \,.\, dx\, dy\, dz$, where ρ is the density per unit volume

$= k\, xyz\, dx\, dy\, dz$.

$\therefore$ the mass of the tetrahedron $OABC$

$= \iiint k\, xyz\, dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition $(x/a) + (y/b) + (z/c) \le 1$.

Put $x/a = u, y/b = v, z/c = w$ so that

$dx = a\, du, dy = b\, dv, dz = c\, dw$.

Then the required mass

$= \iiint k \,.au \,.\, bv \,.\, cw \,.abc\, du\, dv\, dw$, where $u + v + w \le 1$

$= k\, a^2b^2c^2 \iiint u^{2-1} v^{2-1} w^{2-1}\, du\, dv\, dw$

$$= k\, a^2b^2c^2 \frac{[\Gamma(2)]^3}{\Gamma(2+2+2+1)}, \text{ by Dirichlet's integral}$$

$$= k\, a^2b^2c^2 \cdot \frac{1}{\Gamma(7)}, \qquad [\because \Gamma(2) = 1\, \Gamma(1) = 1]$$

$$= \frac{k\, a^2b^2c^2}{6.5.4.3.2.1} = \frac{k\, a^2b^2c^2}{720}.$$

§ 16. Liouville's Extension of Dirichlet's Theorem.

If the variables x, y, z are all positive such that

$$h_1 \le x + y + z \le h_2,$$

then the triple integral

$$\iiint \mathbf{f}(\mathbf{x} + \mathbf{y} + \mathbf{z})\, \mathbf{x}^{\mathbf{l}-1} \mathbf{y}^{\mathbf{m}-1} \mathbf{z}^{\mathbf{n}-1}\, \mathbf{dx\, dy\, dz}$$

$$= \frac{\Gamma(\mathbf{l})\, \Gamma(\mathbf{m})\, \Gamma(\mathbf{n})}{\Gamma(\mathbf{l} + \mathbf{m} + \mathbf{n})} \int_{h_1}^{h_2} \mathbf{f}(\mathbf{u})\, \mathbf{u}^{l+m+n-1}\, \mathbf{du}.$$

Proof. Let

$$I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

integrated over some region.

Subject to the condition $x + y + z \le u$, we have by Dirichlet's theorem

$$I = u^{l+m+n} \frac{\Gamma(l)\, \Gamma(m)\, \Gamma(n)}{\Gamma(l + m + n + 1)}. \qquad \text{...(1)}$$

If the condition be $x + y + z \le u + \delta u$, then

$$I = (u + \delta u)^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}. \qquad \ldots(2)$$

Therefore the value of the integral I extended to all such positive values of the variables as make the sum of the variables lie between u and $u + \delta u$ is

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} [(u+\delta u)^{l+m+n} - u^{l+m+n}],$$

[subtracting (2) from (1)]

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n} \left[\left(1 + \frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n} \left[1 + (l+m+n)\frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} (l+m+n)\, u^{l+m+n-1}\,\delta u,$$

to the first order of approximation

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} u^{l+m+n-1}\,\delta u.$$

Now consider the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

subject to the condition $h_1 \le x + y + z \le h_2$.

If $x + y + z$ lies between u and $u + \delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

which arises from supposing the sum of the variables to lie between u and $u + \delta u$ is ultimately equal to

$$\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} f(u)\,.\,u^{l+m+n-1}\,\delta u.$$

Therefore the whole integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

where $h_1 \le x + y + z \le h_2$, is equal to

$$\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\,.\,u^{l+m+n-1}\, du.$$

Remark. The above theorem holds good even if we take the condition as $h_1 < x + y + z < h_2$ in place of $h_1 \le x + y + z \le h_2$.

Ex. 9. *Find the value of* $\iiint \log(x+y+z)\, dx\, dy\, dz$, *the integral extending over all positive values of* x, y, z *subject to the condition* $x + y + z < 1$. **(Kanpur 1984; Agra 80, 85)**

Sol. Here the integral is to be extended for all positive values of x, y and z such that $0 < x + y + z < 1$.

$\therefore$ the required integral

$= \iiint \log(x+y+z)\,dx\,dy\,dz$, where $0 < x+y+z < 1$

$= \iiint \log(x+y+z)\,x^{1-1}y^{1-1}z^{1-1}\,dx\,dy\,dz$ **(Note)**

$= \dfrac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 (\log u)\,u^{1+1+1-1}\,du$,

by Liouville's extension of Dirichlet's theorem

$= \dfrac{1}{\Gamma(3)} \int_0^1 u^2 \log u\,du$, $[\because \Gamma(1) = 1]$

$= \dfrac{1}{2.1}\left[\left((\log u)\cdot\dfrac{u^3}{3}\right)_0^1 - \int_0^1 \dfrac{1}{u}\cdot\dfrac{u^3}{3}\,du\right]$,

integrating by parts taking u^2 as the second function

$= \frac{1}{2}\left[0 - \frac{1}{3}\lim_{u\to 0} u^3 \log u - \frac{1}{3}\int_0^1 u^2\,du\right]$

$= -\frac{1}{6}\left[\dfrac{u^3}{3}\right]_0^1$, $\left[\because \lim_{u\to 0} u^3 \log u = 0\right]$

$= -\dfrac{1}{18}$.

Note. $\underset{u\to 0}{\text{Lim}}\, u^3 \log u = \lim_{u\to 0} \dfrac{\log u}{1/u^3}$

$= \lim_{u\to 0} \dfrac{1/u}{-3/u^4} = \lim_{u\to 0} -\frac{1}{3}u^3 = 0.$

Ex. 10. *Evaluate* $\iiint e^{x+y+z}\,dx\,dy\,dz$ *taken over the positive octant such that* $x+y+z \le 1$.

Sol. Here the integral is to be extended for all positive values of x, y and z such that $0 \le x+y+z \le 1$.

$\therefore$ the required integral

$\iiint e^{x+y+z}\,dx\,dy\,dz = \iiint e^{x+y+z}x^{1-1}y^{1-1}z^{1-1}\,dx\,dy\,dz$,

where $0 \le x+y+z \le 1$

$= \dfrac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 e^u u^{3-1}\,du$,

by Liouville's extension of Dirichlet's integral

$= \dfrac{1.1.1}{2.1}\left[\left\{u^2 . e^u\right\}_0^1 - \int_0^1 2u . e^u\,du\right]$,

integrating by parts taking e^u as the second function

$= \frac{1}{2}\left[e - 2\left\{(e^u . u)_0^1 - \int_0^1 1.e^u\,du\right\}\right]$

$= \frac{1}{2}\left[e - 2e + 2\left\{e^u\right\}_0^1\right] = \frac{1}{2}[e - 2e + 2e - 2] = \dfrac{(e-2)}{2}$.

Ex. 11. *Evaluate*

$\iiint x^{-1/2} y^{-1/2} z^{-1/2} (1-x-y-z)^{1/2}\, dx\, dy\, dz$ *extended to all positive values of the variables subject to the condition* $x+y+z<1$.

Sol. The given condition is

$$0 < x+y+z < 1.$$

$\therefore$ the required integral

$$\iiint x^{-1/2} y^{-1/2} z^{-1/2} (1-x-y-z)^{1/2}\, dx\, dy\, dz$$

$$= \iiint x^{1/2-1} y^{1/2-1} z^{1/2-1} \{1-(x+y+z)\}^{1/2}\, dx\, dy\, dz,$$

where $0 < x+y+z < 1$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_0^1 (1-u)^{1/2}\, u^{1/2+1/2+1/2-1}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)} \int_0^1 (1-u)^{3/2-1}\, u^{(3/2)-1}\, du = \frac{(\sqrt{\pi})^3}{\frac{1}{2}\cdot\sqrt{\pi}} \cdot \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma(\frac{3}{2}+\frac{3}{2})}.$$

$$= 2\pi \cdot \frac{\frac{1}{2}\sqrt{\pi}\,.\frac{1}{2}\sqrt{\pi}}{2.1} = \frac{\pi^2}{4}.$$

Ex. 12. *Prove that* $\displaystyle\iiint \frac{dx\, dy\, dz}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8}$, *the integral being extended to all positive values of the variables for which the expression is real.* **(Kanpur 1983; Meerut 98)**

Sol. The given expression $1/\sqrt{(1-x^2-y^2-z^2)}$ is real if

$$x^2+y^2+z^2 < 1.$$

Therefore the given integeral is to be extended for all positive values of the variables x, y and z such that

$$0 < x^2+y^2+z^2 < 1.$$

Now put $x^2 = u_1$ *i.e.*, $x = u_1^{1/2}$, so that $dx = \frac{1}{2} u_1^{-1/2}\, du_1$,

$y^2 = u_2$ *i.e.*, $y = u_2^{1/2}$, so that $dy = \frac{1}{2} u_2^{-1/2}\, du_2$,

and $z^2 = u_3$ *i.e.*, $z = u_3^{1/2}$, so that $dz = \frac{1}{2} u_3^{-1/2}\, du_3$.

With these substitutions the given condition reduces to

$$0 < u_1 + u_2 + u_3 < 1$$

and the required integral becomes

$$= \iiint \frac{(\frac{1}{2})^3 u_1^{-1/2} u_2^{-1/2} u_3^{-1/2}\, du_1\, du_2\, du_3}{\sqrt{(1-u_1-u_2-u_3)}},$$

for $0 < u_1 + u_2 + u_3 < 1$

$$= \frac{1}{8} \iiint \frac{u_1^{1/2-1} u_2^{1/2-1} u_3^{1/2-1}\, du_1\, du_2\, du_3}{\sqrt{\{1-(u_1+u_2+u_3)\}}}$$

$$= \frac{1}{8} \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_0^1 u^{3/2-1} \cdot \frac{1}{\sqrt{(1-u)}}\, du,$$

[by Liouville's Extension of Dirichlet's Theorem]

$$= \frac{1}{8} \cdot \frac{[\sqrt{\pi}]^3}{\frac{1}{2}\sqrt{\pi}} \int_0^{\pi/2} \frac{\sin\theta \,.2 \sin\theta \cos\theta \, d\theta}{\sqrt{(1-\sin^2\theta)}}, \text{ putting } u = \sin^2\theta \text{ etc.}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}.$$

Ex. 13. *Prove that*

$$\iiint \frac{dx\, dy\, dz}{\sqrt{(a^2 - x^2 - y^2 - z^2)}} = \frac{\pi^2 a^2}{8},$$

the integral being extended for all positive values of the variables for which the expression is real. **(Agra 1979)**

Sol. The given expression is real when $x^2 + y^2 + z^2 < a^2$.

Therefore the required integral is to be extended to all positive values of x, y and z such that

$$0 < x^2 + y^2 + z^2 < a^2$$

i.e., $\quad 0 < x^2/a^2 + y^2/a^2 + z^2/a^2 < 1.$

Put $(x^2/a^2) = u_1$, $(y^2/a^2) = u_2$ and $(z^2/a^2) = u_3$

i.e., $\quad x = au_1^{1/2}, y = au_2^{1/2}$ and $z = au_3^{1/2}$

so that $\quad dx = \frac{1}{2} a u_1^{-1/2} du_1, dy = \frac{1}{2} a u_2^{-1/2} du_2$ and $dz = \frac{1}{2} a u_3^{-1/2} du_3$.

With these substitutions the given condition reduces to

$$0 < u_1 + u_2 + u_3 < 1$$

and the required integral becomes

$$= \iiint \frac{(\frac{1}{2})^3 . a^3 u_1^{-1/2} u_2^{-1/2} u_3^{-1/2} du_1\, du_2\, du_3}{a\sqrt{\{1-(u_1+u_2+u_3)\}}}$$

$$= \frac{a^2}{8} \iiint \frac{u_1^{1/2-1} u_2^{1/2-1} u_3^{1/2-1} du_1\, du_2\, du_3}{\sqrt{\{1-(u_1+u_2+u_3)\}}}$$

$$= \frac{a^2}{8} \cdot \frac{[\Gamma(1/2)]^3}{\Gamma(\frac{3}{2})} \cdot \int_0^1 u^{3/2-1} \cdot \frac{1}{\sqrt{(1-u)}}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2}{8} \cdot \frac{[\sqrt{\pi}]^3}{\frac{1}{2} \cdot \sqrt{\pi}} \cdot \int_0^{\pi/2} \frac{\sin\theta \,.2 \sin\theta \cos\theta \, d\theta}{\sqrt{(1-\sin^2\theta)}}, \text{ putting } u = \sin^2\theta \text{ etc.}$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi a^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^2}{8}.$$

Ex. 14. *Evaluate*

$$\iiint_R (x+y+z+1)^2 \, dx\, dy\, dz,$$

where R is defined by $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1$.

Sol. As given x, y, z are all positive such that

$$0 \le x + y + z \le 1,$$

$$\therefore \quad \iiint (x + y + z + 1)^2\, dx\, dy\, dz$$

$$= \iiint x^{1-1} y^{1-1} z^{1-1} \{(x + y + z) + 1\}^2\, dx\, dy\, dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1 + 1 + 1)} \int_0^1 (u + 1)^2 \,.\, u^{1+1+1-1}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{1}{2} \int_0^1 (u^2 + 2u + 1)\, u^2\, du = \frac{1}{2}\left[\frac{u^5}{5} + \frac{2u^4}{4} + \frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right] = \frac{1}{2}\,\frac{(6 + 15 + 10)}{5 \times 2 \times 3} = \frac{1}{2} \cdot \frac{31}{30} = \frac{31}{60}.$$

Ex. 15. *Show that*

$$\iiint \left(\frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right)^{1/2} dx\, dy = \frac{\pi}{8}(\pi - 2)$$

over the positive quadrant of the circle $x^2 + y^2 = 1$. **(Kanpur 1976)**

Sol. Here the given integral is to be extended to all positive values of x and y such that

$$0 \le x^2 + y^2 \le 1. \qquad \ldots(1)$$

Put $x^2 = u, y^2 = v$ *i.e.*, $x = u^{1/2}, y = v^{1/2}$ so that

$$dx = \tfrac{1}{2} u^{-1/2}\, du,\ dy = \tfrac{1}{2} v^{-1/2}\, dv.$$

With these substitutions the condition (1) becomes

$$0 \le u + v \le 1.$$

Hence the required integral

$$= \iint \left[\frac{1 - (u + v)}{1 + (u + v)}\right]^{1/2} \tfrac{1}{4} u^{-1/2} v^{-1/2}\, du\, dv$$

$$= \frac{1}{4} \iint \left[\frac{1 - (u + v)}{1 + (u + v)}\right]^{1/2} u^{(1/2)-1} v^{(1/2)-1}\, du\, dv,$$

where $0 \le u + v \le 1$

$$= \frac{1}{4}\,\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(\frac{1}{2} + \frac{1}{2})} \int_0^1 \left[\frac{1 - h}{1 + h}\right]^{1/2} h^{(1/2)+(1/2)-1}\, dh,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{1}{4}\,\frac{\sqrt{\pi}\,.\sqrt{\pi}}{\Gamma(1)} \int_0^1 \frac{1 - h}{\sqrt{(1 - h^2)}}\, dh$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{(1 - \sin\theta)}{\cos\theta} \cos\theta\, d\theta, \text{ putting } h = \sin\theta \text{ so that}$$

$$dh = \cos\theta\, d\theta$$

$$= \frac{\pi}{4}\left[\theta + \cos\theta\right]_0^{\pi/2} = \frac{\pi}{4}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{8}(\pi - 2).$$

Ex. 16. *Prove that when x and y are positive and $x + y < h$,*

$$\iint f'(x+y)\,x^{l-1}y^{-l}\,dx\,dy = \frac{\pi}{\sin l\pi}[f(h)-f(0)].$$

Sol. The given integral

$I = \iint f'(x+y)\,x^{l-1}y^{(1-l)-1}\,dx\,dy$, where $0 < x+y < h$

$$= \frac{\Gamma(l)\Gamma(1-l)}{\Gamma(l+1-l)}\int_0^h f'(u)\,u^{l+(1-l)-1}\,du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(l)\Gamma(1-l)}{\Gamma(1)}\int_0^h f'(u)\,du$$

$$= \frac{\pi}{\sin \pi l}\Big[f(u)\Big]_0^h = \frac{\pi}{\sin \pi l}[f(h)-f(0)].$$

Ex. 17. *Evaluate* $\iiint x^{\alpha}y^{\beta}z^{\gamma}(1-x-y-z)^{\lambda}\,dx\,dy\,dz$ *over the interior of the tetrahedron formed by the coordinate planes and the plane* $x+y+z=1$. **(Meerut 1988)**

Sol. Here the region of integration is bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$. So the variables x, y, z take all positive values subject to the condition

$$0 < x+y+z < 1.$$

Hence the given integral

$$= \iiint x^{(\alpha+1)-1}y^{(\beta+1)-1}z^{(\gamma+1)-1}[1-(x+y+z)]^{\lambda}\,dx\,dy\,dz$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\cdot\int_0^1 u^{\alpha+1+\beta+1+\gamma+1-1}(1-u)^{\lambda}\,du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\cdot\int_0^1 u^{(\alpha+\beta+\gamma+3)-1}(1-u)^{(\lambda+1)-1}\,du$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}B(\alpha+\beta+\gamma+3,\lambda+1)$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\cdot\frac{\Gamma(\alpha+\beta+\gamma+3)\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}.$$

Ex. 18. *Evaluate*

$\iiint \sqrt{(a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2)}\,dx\,dy\,dz$ *taken throughout the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol. The given ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is symmetrical in all the eight octants. Let us first evaluate the given integral over the region of the ellipsoid which lies in the positive octant *i.e.*, where x, y, z are all positive.

Put $x^2/a^2 = u$, $y^2/b^2 = v$, $z^2/c^2 = w$.

Then $x = au^{1/2}$, $dx = \frac{1}{2} au^{-1/2}\, du$ etc.

Now the given integral extended over the positive octant of the given ellipsoid is

$$I = abc \iiint \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)}\, dx\, dy\, dz,$$

where $0 < x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$

$$= abc \iiint \sqrt{(1 - u - v - w)} \cdot \tfrac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

where $0 < u + v + w \le 1$

$$= \frac{a^2b^2c^2}{8} \iiint u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} \sqrt{\{1 - (u + v + w)\}}\, du\, dv\, dw$$

$$= \frac{a^2b^2c^2}{8} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)} \int_0^1 \sqrt{(1-t)} \,.\, t^{1/2 + 1/2 + 1/2 - 1}\, dt,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2b^2c^2}{8} \cdot \frac{(\sqrt{\pi})^3}{\frac{1}{2} \cdot \sqrt{\pi}} \int_0^1 (1-t)^{(3/2)-1} t^{(3/2)-1}\, dt$$

$$= \frac{a^2b^2c^2}{8} \cdot 2\pi \cdot \frac{\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(3)} = \frac{\pi^2 a^2b^2c^2}{32}.$$

Hence if the integration is extended throughout the ellipsoid, the given integral $= 8I = 8 \cdot \dfrac{\pi^2 a^2b^2c^2}{32} = \dfrac{\pi^2 a^2b^2c^2}{4}$.

Ex. 19. *Evaluate* $\displaystyle\iint \sqrt{\left\{\frac{1 - x^2/a^2 - y^2/b^2}{1 + x^2/a^2 + y^2/b^2}\right\}}\, dx\, dy$

where $x^2/a^2 + y^2/b^2 \le 1$. **(Meerut 1983, 91P; Kanpur 79)**

Sol. The ellipse $x^2/a^2 + y^2/b^2 = 1$ is symmetrical in all the four quadrants.

Let us first evaluate the given integral over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant *i.e.*, where x and y are both positive.

Put $x^2/a^2 = u$, $y^2/b^2 = v$.

Then $x = au^{1/2}$, $dx = \frac{1}{2} au^{-1/2}\, du$,

$y = bv^{1/2}$, $dy = \frac{1}{2} bv^{-1/2}\, dv$.

$\therefore$ the given integral extended over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant is given by

$$I = \iint \sqrt{\left(\frac{1 - u - v}{1 + u + v}\right)} \cdot \tfrac{1}{2} ab u^{-1/2} v^{-1/2}\, du\, dv,$$

where $0 < u + v \le 1$

$$= \frac{ab}{4} \iint \sqrt{\left\{\frac{1-(u+v)}{1+(u+v)}\right\}} u^{(1/2)-1} v^{(1/2)-1} \, du \, dv$$

$$= \frac{ab}{4} \cdot \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} \int_0^1 \sqrt{\left(\frac{1-t}{1+t}\right)} \cdot t^{1/2+1/2-1} \, dt$$

$$= \frac{\pi ab}{4} \cdot \int_0^1 \frac{1-t}{\sqrt{(1-t^2)}} dt = \frac{\pi ab}{4} \int_0^{\pi/2} \frac{1-\sin\theta}{\cos\theta} \cos\theta \, d\theta,$$

putting $t = \sin\theta$ so that $dt = \cos\theta \, d\theta$

$$= \frac{\pi ab}{4} \int_0^{\pi/2} (1-\sin\theta) \, d\theta = \frac{\pi ab}{4} \left[\theta + \cos\theta\right]_0^{\pi/2}$$

$$= \frac{\pi ab}{4} \left[\left(\frac{\pi}{2}+0\right) - (0+1)\right] = \frac{\pi ab}{4} \left(\frac{\pi}{2} - 1\right).$$

Hence the given integral extended over the whole region of the ellipse $x^2/a^2 + y^2/b^2 = 1 = 4.I = \pi ab\,(\frac{1}{2}\pi - 1)$.

Ex. 20. *Prove that $I = \iiiint dx\,dy\,dz\,dw$, for all positive values of the variables for which $x^2 + y^2 + z^2 + w^2$ is not less than a^2 and not greater than b^2 is $\pi^2(b^4 - a^4)/32$.*

Sol. We have to evaluate I subject to the condition

$$a^2 < x^2 + y^2 + z^2 + w^2 < b^2.$$

Putting $x^2 = u_1$ i.e., $x = u_1^{1/2}$, $dx = \frac{1}{2} u_1^{-1/2} du_1$ etc., we get

$$I = \iiiint \tfrac{1}{2} u_1^{-1/2} . \tfrac{1}{2} u_2^{-1/2} . \tfrac{1}{2} u_3^{-1/2} . \tfrac{1}{2} u_4^{-1/2} \, du_1 \, du_2 \, du_3 \, du_4$$

subject to the condition $a^2 < u_1 + u_2 + u_3 + u_4 < b^2$

or $\quad I = \frac{1}{16} \iiiint u_1^{(1/2)-1} u_2^{(1/2)-1} u_3^{(1/2)-1} u_4^{(1/2)-1} \, du_1 \, du_2 \, du_3 \, du_4$

$$= \frac{1}{16} \cdot \frac{[\Gamma(\frac{1}{2})]^4}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_{a^2}^{b^2} t^{1/2+1/2+1/2+1/2-1} \, dt,$$

by Liouville's theorem

$$= \frac{(\sqrt{\pi})^4}{16\,\Gamma(2)} \int_{a^2}^{b^2} t \, dt = \frac{\pi^2}{16} \cdot \left[\frac{t^2}{2}\right]_{a^2}^{b^2} = \frac{\pi^2}{32}(b^4 - a^4).$$

Ex. 21. *Find the value of*

$$\iiint xyz \sin(x+y+z) \, dx \, dy \, dz,$$

the integral being extended to all positive values of the variables subject to the condition $x + y + z \le \pi/2$. **(Agra 1983)**

Sol. Here $0 < x + y + z \le \pi/2$.

$\therefore$ the required integral

$$I = \iiint \sin(x+y+z)\, x^{2-1} y^{2-1} z^{2-1} \, dx \, dy \, dz,$$

where $0 < x + y + z \le \pi/2$

$$= \frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2)} \int_0^{\pi/2} (\sin u)\, u^{2+2+2-1}\, du,$$

by Liouville's theorem

$$= \frac{1}{\Gamma(6)} \int_0^{\pi/2} u^5 \sin u \, du.$$

Applying successive integration by parts, we have

$$I = \frac{1}{5!} \left[u^5(-\cos u) - (5u^4)(-\sin u) + (20u^3)(\cos u)\right.$$

$$\left. - (60u^2)(\sin u) + (120u)(-\cos u) - 120(-\sin u)\right]_0^{\pi/2}.$$

In the above expression all the terms vanish for $u = 0$ and all those which involve $\cos u$ vanish for $u = \pi/2$.

$$\therefore \quad I = \frac{1}{120}\left[-5(\pi/2)^4 . (-1) - 60(\pi/2)^2 . (1) - 120 . (-1)\right]$$

$$= \frac{1}{120}\left[\frac{5\pi^4}{16} - 15\pi^2 + 120\right] = \frac{1}{384}\left[\pi^4 - 48\pi^2 + 384\right].$$

Ex. 22. *Evaluate* $\iiint \sqrt{\left(\frac{1 - x^2 - y^2 - z^2}{1 + x^2 + y^2 + z^2}\right)} \, dx\, dy\, dz$ *integral being taken over all positive values of x, y, z such that*

$$x^2 + y^2 + z^2 \le 1.$$

(Agra 1981)

Sol. Put $x^2 = u$, $y^2 = v$, $z^2 = w$.

Then $x = u^{1/2}$, $dx = \frac{1}{2} u^{-1/2} du$; $y = v^{1/2}$, $dy = \frac{1}{2} v^{-1/2} dv$;

$z = w^{1/2}$, $dz = \frac{1}{2} w^{-1/2} dw$.

$\therefore$ the given integral

$$= \frac{1}{8} \iiint \sqrt{\left\{\frac{1 - (u + v + w)}{1 + (u + v + w)}\right\}} \, u^{1/2 - 1} v^{1/2 - 1} w^{1/2 - 1} \, du\, dv\, dw,$$

where u, v, w are all +ive and $0 < u + v + w \le 1$

$$= \frac{1}{8} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{3}{2})} \int_0^1 \sqrt{\left(\frac{1 - t}{1 + t}\right)} \, t^{1/2 + 1/2 + 1/2 - 1} \, dt$$

$$= \frac{\pi}{4} \int_0^1 \frac{1 - t}{\sqrt{(1 - t^2)}} \, t^{1/2} \, dt.$$

Put $t^2 = z$ or $t = z^{1/2}$ so that $dt = \frac{1}{2} z^{-1/2} dz$.

$$\therefore \quad I = \frac{\pi}{4} \int_0^1 (1 - z)^{-1/2} [1 - z^{1/2}] \, z^{1/4} . \tfrac{1}{2} z^{-1/2} \, dz$$

$$= \frac{\pi}{8} \int_0^1 (1 - z)^{-1/2} (z^{-1/4} - z^{1/4}) \, dz$$

$$= \frac{\pi}{8} \int_0^1 [z^{(3/4) - 1} (1 - z)^{(1/2) - 1} - z^{(5/4) - 1} (1 - z)^{(1/2) - 1}] \, dz$$

$$= \frac{\pi}{8} \left[B\left(\frac{3}{4}, \frac{1}{2}\right) - B\left(\frac{5}{4}, \frac{1}{2}\right)\right]. \qquad \square$$