MODERN ALGEBRA
CSE PYQs

2019

1. 1a

Let G be a finite group, H and K subgroups of G such that $K \subset H$. Show that $(G:K) = (G:H)(H:K)$. 10

2. 2a

If G and H are finite groups whose orders are relatively prime, then prove that there is only one homomorphism from G to H, the trivial one. 10

3. 2b

Write down all quotient groups of the group Z_{12}. 10

4. 3d

Let a be an irreducible element of the Euclidean ring R, then prove that $R/(a)$ is a field. 10

2018

5. 1a

Let R be an integral domain with unit element. Show that any unit in $R[x]$ is a unit in R. 10

6. 2a

Show that the quotient group of $(\mathbb{R}, +)$ modulo $\mathbb{Z}$ is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. 15

7. 3a

Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 respectively. 20

2017

8. 1b

Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n. *10*

9. 2c

Let F be a field and F[X] denote the ring of polynomials over F in a single variable X. For f(X), g(X) $\in$ F[X] with g(X) $\neq$ 0, show that there exist q(X), r(X) $\in$ F[X] such that degree (r(X)) < degree (g(X)) and

$$f(X) = q(X) \cdot g(X) + r(X).$$ *20*

10. 3a

Show that the groups $\mathbb{Z}_5 \times \mathbb{Z}_7$ and $\mathbb{Z}_{35}$ are isomorphic. *15*

2016

11. 1a

Let $\mathbb{K}$ be a field and $\mathbb{K}[X]$ be the ring of polynomials over $\mathbb{K}$ in a single variable X. For a polynomial $f \in \mathbb{K}[X]$, let (f) denote the ideal in $\mathbb{K}[X]$ generated by f. Show that (f) is a maximal ideal in $\mathbb{K}[X]$ if and only if f is an irreducible polynomial over $\mathbb{K}$. *10*

12. 2b

Let p be a prime number and $\mathbb{Z}_p$ denote the additive group of integers modulo p. Show that every non-zero element of $\mathbb{Z}_p$ generates $\mathbb{Z}_p$. *15*

13. 3a

Let K be an extension of a field F. Prove that the elements of K, which are algebraic over F, form a subfield of K. Further, if $F \subset K \subset L$ are fields, L is algebraic over K and K is algebraic over F, then prove that L is algebraic over F. *20*

14. 4a

Show that every algebraically closed field is infinite. *15*

2015

15. 1a

How many generators are there of the cyclic group G of order 8? Explain.

Taking a group $\{e, a, b, c\}$ of order 4, where e is the identity, construct composition tables showing that one is cyclic while the other is not.

5+5=10

16. 1b

Give an example of a ring having identity but a subring of this having a different identity. 10

17. 2a

If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R', prove that $\phi(1)$ is the unit element of R'. 15

18. 4a

Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields : 5+6+4=15

(i) $b\sqrt{2}$ के रूप की संख्याओं का समुच्चय, जहाँ b परिमेय संख्या है

The set of numbers of the form $b\sqrt{2}$ with b rational

(ii) सम पूर्णांकों का समुच्चय

The set of even integers

(iii) धनात्मक पूर्णांकों का समुच्चय

The set of positive integers

2014

19. 1a

Let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz \neq 0$. Show that G is a group under matrix multiplication. Let N denote the subset $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}$. Is N a normal subgroup of G? Justify your answer. 10

20. 2a

Show that $\mathbb{Z}_7$ is a field. Then find $([5] + [6])^{-1}$ and $(-[4])^{-1}$ in $\mathbb{Z}_7$. 15

21. 3a

Show that the set $\{a + b\omega : \omega^3 = 1\}$, where a and b are real numbers, is a field with respect to usual addition and multiplication. 15

22. 4a

Prove that the set $\mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in \mathbb{Q}\}$ is a commutative ring with identity. 15

2013

23. 1a

Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \; a, b \in \mathbb{R} \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f : \mathbb{C} \to S$ defined by $f(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.) *10*

24. 1b

Give an example of an infinite group in which every element has finite order. *10*

25. 2a

What are the orders of the following permutations in S_{10} ?

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9 \end{pmatrix}$ and $(1\,2\,3\,4\,5)\,(6\,7)$. *10*

26. 2b

What is the maximal possible order of an element in S_{10} ? Why ? Give an example of such an element. How many elements will there be in S_{10} of that order ? *13*

27. 3a

Let $J = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers (subring of $\mathbb{C}$). Which of the following is J : Euclidean domain, principal ideal domain, unique factorization domain ? Justify your answer. *15*

28. 3b

(b) Let R^C = ring of all real valued continuous functions on [0, 1], under the operations

$$(f + g)\, x = f(x) + g(x)$$

$$(fg)\, x = f(x)\, g(x).$$

Let $M = \left\{ f \in R^C \;\middle|\; f\left(\frac{1}{2}\right) = 0 \right\}$.

Is M a maximal ideal of R ? Justify your answer. *15*

2012

29. 1a

1. (a) How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$. 12

30. 2a

2. (a) How many conjugacy classes does the permutation group S_5 of permutations 5 numbers have ? Write down one element in each class (preferably in terms of cycles). 15

31. 3a

3. (a) Is the ideal generated by 2 and X in the polynomial ring $\mathbb{Z}[X]$ of polynomials in a single variable X with coefficients in the ring of integers $\mathbb{Z}$, a principal ideal ? Justify your answer. 15

32. 4a

4. (a) Describe the maximal ideals in the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. 20

2011

33. 1a

1. (a) Show that the set

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$$

of six transformations on the set of Complex numbers defined by

$f_1(z) = z$, $f_2(z) = 1 - z$

$f_3(z) = \dfrac{z}{(z-1)}$, $f_4(z) = \dfrac{1}{z}$

$f_5(z) = \dfrac{1}{(1-z)}$ and $f_6(z) = \dfrac{(z-1)}{z}$

is a non-abelian group of order 6 w.r.t. composition of mappings. 12

34. 1e

(e) (i) Prove that a group of Prime order is abelian. 6

(ii) How many generators are there of the cyclic group $(G, \cdot)$ of order 8 ? 6

35. 2a

2. (a) Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic. 15

36. 3a

3. (a) Let F be the set of all real valued continuous functions defined on the closed interval [0, 1]. Prove that (F, +, $\cdot$) is a Commutative Ring with unity with respect to addition and multiplication of functions defined pointwise as below :

$$\left.\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ \text{and}\quad (f\,.\,g)(x) &= f(x)\,.\,g(x)\end{aligned}\right\}\ x\in[0,1]$$

where f, g $\in$ F. 15

37. 4a

4. (a) Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4a$.

Find the order of ab, and express its inverse in each of the forms a^mb^n and b^ma^n. 20

2010

38. 1a

(a) Let $G = \mathbb{R} - \{-1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a * b = a + b + ab$. Show $(G, *)$ is a group and it is abelian 12

39. 1b

(b) Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this ? Justify. 12

40. 2a

2. (a) Let $\left(\mathbb{R}^*, \cdot\right)$ be the multiplicative group of non-zero reals and $(GL(n, \mathbb{R}), X)$ be the multiplicative group of $n \times n$ non-singular real matrices. Show that the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ and $\left(\mathbb{R}^*, \cdot\right)$ are isomorphic where

$SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) / \det A = 1\}$.

What is the centre of $GL(n, \mathbb{R})$? 15

41. 2b

(b) Let $C = \{f : I = [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.

Show C is a commutative ring with 1 under pointwise addition and multiplication.

Determine whether C is an integral domain. Explain. 15

42. 3a

3. (a) Consider the polynomial ring $Q[x]$. Show $p(x) = x^3 - 2$ is irreducible over Q. Let I be the ideal in $Q[x]$ generated by $p(x)$. Then show that $Q[x]/I$ is a field and that each element of it is of the form $a_0 + a_1 t + a_2 t^2$ with a_0, a_1, a_2 in Q and $t = x + I$. 15

43. 3b

(b) Show that the quotient ring $\mathbb{Z}[i]/(1 + 3i)$ is isomorphic to the ring $\mathbb{Z}/10\mathbb{Z}$ where $\mathbb{Z}[i]$ denotes the ring of Gaussian integers. 15

2009

44. 1a

(a) If $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers, show that $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and $\mathbb{R}_+$ under multiplication $(\mathbb{R}_+, \cdot)$ are isomorphic. Similarly if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers, are $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, \cdot)$ isomorphic ? Justify your answer. 4+8=12

45. 1b

(b) Determine the number of homomorphisms from the additive group $\mathbb{Z}_{15}$ to the additive group $\mathbb{Z}_{10}$. ($\mathbb{Z}_n$ is the cyclic group of order n). 12

46. 2a

2. (a) How many proper, non-zero ideals does the ring Z_{12} have ? Justify your answer. How many ideals does the ring $\mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$ have ? Why ?

2+3+4+6=15

47. 2b

(b) Show that the alternating group on four letters A_4 has no subgroup of order 6. 15

48. 3a

Show that $\mathbb{Z}[X]$ is a unique factorization domain that is not a principal ideal domain ($\mathbb{Z}$ is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain ? ($\mathbb{Z}$ [X] is the ring of polynomials in the variable X with integer.) 15

49. 3b

(b) How many elements does the quotient ring $\frac{\mathbb{Z}_5[X]}{(X^2+1)}$ have ? Is it an integral domain ? Justify yours answers. 15