5(a). Solve the PDE $(D-2D')(D-D')^2 z = e^{x+y}$

SOLUTION

Auxiliary equation for the given PDE $(m-2)(m-1)^2 = 0$

$\Rightarrow$ m = 2, 1, 1.

$\therefore$ C.F. = $\phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)$

$$\text{Particular integral} = \frac{e^{x+y}}{(D-2D')(D-D')^2}$$

$$= \frac{1}{(D-D')^2}\frac{e^{x+y}}{(1-2)}$$

$$= \frac{-e^{x+y}}{(D-D')^2}$$

$$\text{P.I.} = \frac{-x^2}{2}e^{x+y}$$

$\therefore$ General solution $\quad y = \text{C.F.} + \text{P.I.}$

$$y = \phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)-\frac{x^2}{2}e^{x+y}$$

Where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

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6(a). Solve the PDE px + qy = 3z

SOLUTION

Lagranges auxiliary equation of (1) is given by $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{3z}$

$$(2)\Rightarrow \frac{dx}{x}=\frac{dy}{y}\Rightarrow ln\, x = l\, ny + lnc_1$$

$$x = c_1 y$$
$$x/y = c_1$$

$$(2)\Rightarrow \frac{dy}{y}=\frac{dz}{3z} \Rightarrow lny = \frac{lnz}{3}+lnc_2$$

$$\frac{y^3}{z}=c_2$$

$\therefore\ \phi\left(x/y, y^3/z\right)=0$ ϕ being arbitrary functions.

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6(b). A string of length 'l' is fixed at its ends. The string from the mid point is pulled up to a height k and then released from rest. Find the deflection y(x, t) of the vibrating string.

SOLUTION

Given string of length 'l' pulled upwards by k units at $n = \frac{l}{2}$.

∴ Initial Conditions:

$$u(x,0) = k\frac{x}{l/2} = \frac{2kx}{l} \quad \text{for } 0<x<l/2$$

$$= -k/_{l/2}(x-l) = \frac{2k(l-x)}{l} \quad \text{for } l/2 < x < l$$

$$u_t(x,\ 0) = 0$$

Boundary conditions

$$u(0,\ t) = 0$$
$$v(l,\ t) = 0$$

PDE of transverse vibrations of the given elastic string is given by

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}} \quad .....(1)$$

Let us suppose y(x, t) = X(x) T(t)

From boundary conditions we have

$$X(0)T(t) = 0$$
$$X(l)\ T(t) = 0$$

T(t) ≠ 0 for some t > 0.

∴ $$X(0) = X(l) = 0 \quad .....(2)$$

∴ From (1) $$X''T = 1/c^2\ XT''$$

Let $$\frac{X''}{X} = \frac{T''}{c^2T} = \mu$$

∴ $X'' - \mu X = 0$ solve using the boundary conditions.

Case(i)

$\mu = 0 \Rightarrow X'' = 0$

∴ $$X = X = Ax = B$$

From(2)⇒ X(0) = 0; X(l) = 0 ⇒ A = 0 and B = 0

∴ We reject μ = 0.

Case(ii) $$\mu = +\lambda^2,\ \mu \neq 0$$

$$X'' - \lambda^2 X = 0$$

Solving $$X = Ae^{\lambda x} + B\,e^{-\lambda x}$$

from(2) ⇒ X(0) = 0; X(l) = 0

$A + B = D;\ Ae^{\lambda l} + Be^{-\lambda l} = 0$

⇒ A = 0 and B = 0

∴ We reject $\mu = \lambda^2$

Case(iii) $$\mu = -\lambda^2 \Rightarrow X''+\lambda^2 X = 0$$

solving $$X = A\cos\lambda x + B\sin\lambda x$$

$X(0) = 0;\ X(l) = 0 \Rightarrow A = 0,\ \sin\lambda l = 0$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

∴ $$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$$

we have $$T''+\lambda^2c^2T = 0$$

solving $$T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

$$T_n(t) = C\cos\left(\frac{n\pi ct}{l}\right) + D\sin\left(\frac{n\pi ct}{l}\right)$$

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$$\therefore \quad \mu(x, t) = \nabla(x)\tau(t)$$

$$= \sum_{n=1}^{\infty}\left[E_n Cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

Applying initial conditions

$$u_t(x,0) = 0$$

$$\Rightarrow \left[-E_n \frac{n\pi c}{l}.\sin\frac{n\pi ct}{l} + F_n \frac{n\pi c}{l}\cos\left(\frac{n\pi ct}{l}\right)\right]\sin\frac{n\pi x}{l} = u_t(x,t)$$

$$\therefore \quad u_t(x,0) \Rightarrow F_n = 0$$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi ct}{l}\right).\sin\left(\frac{n\pi x}{l}\right)$$

We have $u(x, 0) = \frac{2kx}{l} \; 0 < x < l/2$

$$= \frac{2k(l-x)}{l} \quad l/2 < x < l.$$

$$E_n = \frac{2}{l}\int_0^l u(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$E_n = \frac{2}{l}\left[\int_0^{l/2}\frac{2kx}{l}.\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}\frac{2k(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{4k}{l^2}\left[\frac{x\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{n\pi/l} - \frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right]_0^{l/2} + \frac{4k}{l^2}\left[\frac{(l-x)\cos\frac{n\pi x}{l}}{n\pi/l} - \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right]_{l/2}^{l}$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{2n\pi} - \cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2}(-1) - \frac{l^2}{n^2\pi^2}\sin n\pi + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$E_n = \begin{cases} \frac{8k}{\pi^2}\frac{(-1)^{n+1}}{(2m-1)^2}, & \text{if } n = 2m-1 \text{(odd)} \;\&\, m = 1,2,3,\ldots \\ 0 & , \quad \text{if } n = 2m \quad \text{(even)} \,\&\, m = 1,2,3,.. \end{cases}$$

Substituting the above value of E_n in ( ), the required displacement function is given by

$$\therefore \quad \boxed{u(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{l}.\sin\frac{(2n-1)\pi x}{l}}$$

Required solution

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7(b) IAS-2012 P-II.

The edge $r=a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface. Find the temperature distribution.

Sol^n: Here we have to take the solution in polar coordinates.

The solution is $u=(c_1\cos p\theta+c_2\sin p\theta)(c_3r^p+c_4r^{-p})$ —— (1)

Since the temperature remains finite at $r=0$

$\therefore c_4=0$

Also, if we increase θ by 2π, we arrive at the same point, so the solution (1) should be periodic with period 2π.

$\therefore p=n$, an integer.

Hence we may write the general solution as

$$u=\sum_{n=0}^{\infty}(c_1\cos n\theta+c_2\sin n\theta)c_3r^n$$

$$=\sum(A_n\cos n\theta+B_n\sin n\theta)r^n$$

where $c_1c_3=A_n$, $c_2c_3=B_n$ (say)

Applying to this, the condition $u=f(\theta)$ for $r=a$, we get

$$f(\theta)=\sum(A_n\cos n\theta+B_n\sin n\theta)a^n$$

$$\therefore a^nA_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta.d\theta$$

$$a^nB_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta.d\theta$$

Hence the result.

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