P-2 IFS 2019

DATE

4(a) Show that the smallest subgroup V of A_4 containing $(1,2)(3,4)$, $(1,3)(2,4)$ and $(1,4)(2,3)$ is isomorphic to the Klein's 4-group. (10)

Consider

$$V = \{e', (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

$(1,2)(3,4)(1,2)(3,4) = e \quad \therefore o((1,2)(3,4)) = 2$

$(1,3),(2,4)(1,3)(2,4) = e \quad \therefore o((1,3),(2,4)) = 2$

$(1,4)(2,3)(1,4),(2,3) = e \quad \therefore o((1,4),(2,3)) = 2$

Klein's four Group

$$K = \{e, a, b, c\}$$

where $a^2 = e$, $b^2 = e$, $c^2 = e$

Since, number of elements in V and K are same and each ~~ele~~ non-identity element has order 2.

$\therefore$ Groups V and K have same group structure, hence V and K are isomorphic.

Isomorphism can be defined as

$\phi : K \to V$ s.t.

$\phi(e) = e'$, $\phi(a) = (1,2)(3,4)$

$\phi(b) = (1,3),(2,4)$, $\phi(c) = (1,4),(2,3)$.

4(b) Classify the singular point $z=0$ of the function $f(z) = \frac{e^z}{z+\sin z}$ and obtain the principal part of the Laurent series expansion of $f(z)$. (15)

$$\lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} \frac{z \cdot e^z}{z+\sin z}$$

$$= \lim_{z\to 0} \frac{e^z}{1+\frac{\sin z}{z}} = \frac{1}{1+1} = \frac{1}{2}$$

$\therefore$ $z=0$ is a pole of order 1.

Laurent series Expansion :-

$$f(z) = \frac{e^z}{z+\sin z} = \frac{1+z+\frac{z^2}{2!}+\cdots}{z+\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)}$$

$$= \frac{1+z+\frac{z^2}{2!}+\cdots}{2z\left(1-\frac{z^2}{12}+\frac{z^5}{240}-\cdots\right)}$$

$$= \frac{1}{2z}\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1-\frac{z^2}{12}+\frac{z^5}{240}-\cdots\right)^{-1}$$

$$= \frac{1}{2z}\left(1+z+\frac{z^2}{2!}+\cdots\right)\left(1+\frac{z^2}{12}+O(z^3)\right)$$

$$= \frac{1}{2z}\left(1+z+\frac{7z^2}{12}+O(z^3)\right)$$

$$= \frac{1}{2z}+\frac{1}{2}+\frac{7z}{24}+O(z^2)$$

Principal Part $= \frac{1}{2z}$

IFoS 2019

Q. A Salesman wants to visit cities C_1, C_2, C_3 & C_4. He does not want to visit any cities twice beforing completing the tour of all the cities and wishes to return to his home city, the starting station. Cost of going from one city to another in rupees is given below in the table. Find the least cost route.

To city

From city	C_1	C_2	C_3	C_4
C_1	0	30	80	50
C_2	40	0	140	30
C_3	40	50	0	20
C_4	70	80	130	0

Sol. Let the starting city be C_1. Also to restrict movement within the city we assign cost M (where M is sufficiently large) to cost of going to city to itself. Now we have cost matrix as follows.

	C_1	C_2	C_3	C_4
C_1	M	30	80	50
C_2	40	M	140	30
C_3	40	50	M	20
C_4	70	80	130	M

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Subtract 30 (Minimum in row 1) from Row 1, 30 (minimum in row 2) from Row 2, 20 (Minimum in Row 3) Row 3 & 70 (Minimum in Row 4) from Row 4 we get cost Matrix as

	C_1	C_2	C_3	C_4
C_1	M-30	0	50	20
C_2	10	M-30	110	0
C_3	20	30	M-20	0
C_4	0	10	60	M-70

Subtract minimum of column from corresponding columns we get cost matrix as

	C_1	C_2	C_3	C_4
C_1	M-30	0	0	20
C_2	10	M-30	60	0
C_3	20	30	M-70	0
C_4	0	10	10	M-70

Cover all the zeros of the matrix with minimum no. of horizontal or vertical lines

	C_1	C_2	C_3	C_4
~~C_1~~	~~M-30~~	~~0~~	~~0~~	~~20~~
C_2	10	M-30	60	0
C_3	20	30	M-70	0
C_4	0	10	10	M-70

Since minimal no. of lines is less than 4, optimal assignment is not reached

Now note that 10 is the smallest entry not covered by any line. Subtract 10 from all element not covered by any line & add 10 to elements at intersection of

lines. As a result cost Matrix becomes

	C_1	C_2	C_3	C_4
C_1	M-20	0	0	30
C_2	10	M-40	50	0
C_3	20	20	M-80	0
C_4	0	0	0	M-70

Now cover all zeros of matrix with minimum no. of horizontal or vertical lines

	C_1	C_2	C_3	C_4
~~C_1~~	~~M-20~~	~~0~~	~~0~~	~~30~~
C_2	10	M-40	50	0
C_3	20	20	M-80	0
~~C_4~~	~~0~~	~~0~~	~~0~~	~~M-70~~

Since minimum lines is less than 4, optimal assignment is not reached.

Now note, 10 is the smallest entry not covered by a line Subtract 10 from all uncovered entries & add 10 to entries at intersection of lines.

	C_1	C_2	C_3	C_4
C_1	M-20	0	0	40
C_2	0	M-50	40	0
C_3	10	10	M-90	0
C_4	0	0	0	M-60

Cover all zeros with minimum no. of horizontal & vertical lines

	C_1	C_2	C_3	C_4
~~C_1~~	~~M-20~~	~~0~~	~~0~~	~~40~~
~~C_2~~	~~0~~	~~M-50~~	~~40~~	~~0~~
C_3	10	10	M-90	0
~~C_4~~	~~0~~	~~0~~	~~0~~	~~M-60~~

Since the minimal no. of lines is 4, an optimal assignment of zeros is possible.

	C_1	C_2	C_3	C_4
C_1	M-20	$\cancel{0}$	$\boxed{0}$	40
C_2	$\boxed{0}$	M-50	40	$\cancel{0}$
C_3	10	10	M-90	$\boxed{0}$
C_4	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	M-60

So assignment is given as $C_1 \xrightarrow[80]{} C_3 \xrightarrow[20]{} C_4 \xrightarrow[80]{} C_2 \xrightarrow[40]{} C_1$

i.e. Minimum cost route is $\boxed{C_1 \to C_3 \to C_4 \to C_2 \to C_1}$ with min cost $= 80+20+80+40 = Rs\ 220$