The Inverse Square Law
(Planetary Motion)

§ 1. **Newton's Law of Gravitation.** According to Newton's law of gravitation, "*Every particle of matter attracts every other particle of matter with a force proportional to the product of the masses of the two particles concerned and inversely proportional to the square of the distance between them.*" Thus

$$F=G\cdot\frac{m_1m_2}{r^2},$$

where m_1, m_2 are the masses of the particles, r the distance between them, F the gravitational force of attraction and G a constant called the constant of gravitation or the universal constant.

This law holds good in the case of the motion of all planets in the solar system. In particular, the motion of the earth about the sun is governed by this law. Here in this chapter we shall discuss the case of central orbits when the force is an attraction varying inversely as the square of the distance from the centre of force.

§ 2. **Motion under the Inverse Square Law.** *To show that the path of a particle which is moving so that its acceleration is always directed to a fixed point and is equal to $\mu/(\text{distance})^2$ is a conic section and to distinguish between the three cases that arise.*

[Rohilkhand 1980; Allahabad 76, 77]

Here the force is always directed to a fixed point, so it is a case of central orbit.

Also given that the central acceleration $P=\mu/r^2$.

The differential equation of the path (in pedal form) is

$$\frac{h^2}{p^3}\frac{dp}{dr}=P=\frac{\mu}{r^2}.$$

Multiplying both sides by -2, we have

$$\frac{-2h^2}{p^3}\,dp=\frac{-2\mu}{r^2}\,dr.$$

Integrating, we have

$$v^2=\frac{h^2}{p^2}=\frac{2\mu}{r}+B, \tag{1}$$

where B is a constant. [Note that in a central orbit, $v=h/p$.]

We know that referred to the focus as pole the pedal equations of ellipse, parabola and hyperbola (that branch which is nearer to the focus taken as pole) are

$$\frac{b^2}{p^2}=\frac{2a}{r}-1,\ p^2=ar \text{ and } \frac{b^2}{p^2}=\frac{2a}{r}+1$$

respectively, where

in the case of ellipse $2a$ and $2b$ are the lengths of major and minor axes,

in the case of parabola $4a$ is the length of latus rectum,

and in the case of hyperbola $2a$ and $2b$ are the lengths of transverse and conjugate axes.

Now since the equation (1) can be of any of the above three forms, three cases arise here.

Case I. Elliptic path.

Comparing (1) with $\frac{b^2}{p^2}=\frac{2a}{r}-1$, the pedal equation of an ellipse, we have

$$\frac{h^2}{b^2}=\frac{\mu}{a}=\frac{B}{-1}.$$

$$\therefore \quad h^2=\frac{\mu b^2}{a} \text{ and } B=-\frac{\mu}{a}.$$

Substituting in (1), for elliptic path, we have

$$v^2=\frac{2\mu}{r}-\frac{1}{a}=\mu\left(\frac{2}{r}-\frac{1}{a}\right).$$

Obviously here $v^2<\frac{2\mu}{r}$.

Case II. Parabolic Path. Comparing (1) with $p^2=ar$, the pedal equation of a parabola, we have

$$\frac{h^2}{1}=\frac{2\mu}{1/a}=\frac{B}{0}.$$

$\therefore \quad h^2=2\mu a$ and $B=0$.

Substituting in (1), for parabolic path, we have

$$v^2=\frac{2\mu}{r}.$$

Case III. Hyperbolic path. Comparing (1) with $\frac{b^2}{p^2}=\frac{2a}{r}+1$, the pedal equation of a hyperbola, we have

$$\frac{h^2}{b^2}=\frac{\mu}{a}=\frac{B}{1}.$$

$$\therefore \quad h^2=\frac{\mu b^2}{a} \quad \text{and} \quad B=\frac{\mu}{a}.$$

Substituting in (1), for hyperbolic path, we have

$$v^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right).$$

Obviously here $v^2>\frac{2\mu}{r}.$

Thus from the above three cases, we conclude that the equation (1) always represents a conic section whose focus is at the centre of force. Further the path of the particle is an ellipse, parabola or hyperbola according as B is $-$ive, zero or $+$ive. The sign of the value of the constant B depends upon the magnitude of the velocity of the particle at any point. We have found that

if $\quad v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right)$ or $v^2<\frac{2\mu}{r}$, then the path is elliptic,

if $\quad v^2=\frac{2\mu}{r}$ then the path is parabolic,

and if $\quad v^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$ or $v^2>\frac{2\mu}{r}$, then the path is hyperbolic.

It is to be noted that in each of the three cases the magnitude of the velocity at any point is independent of the direction of the velocity at that point.

Also we have found that

$h^2=\mu b^2/a=\mu l$ in the case of elliptic path,

$h^2=2\mu a=\mu l$ in the case of parabolic path,

and $h^2=\mu b^2/a=\mu l$ in the case of hyperbolic path.

Thus in all the three cases

$h=\sqrt{(\mu l)}$, where l is the length of the semi-latus rectum.

Cor. 1. From the above discussion we see that if a particle is projected from a point at a distance R from the centre of force with velocity V in any direction, then the path is elliptic, parabolic or hyperbolic, according as

$$V^2< \text{ or } = \text{ or } >\frac{2\mu}{R}.$$

If V_1 is the velocity acquired in falling from infinity to a distance R under the same law of force $P=\mu/r^2$, then as in § 6 of the chapter on 'Central Orbits',

$$V_1^2=-2\int_\infty^R P\,dr=-2\int_\infty^R \frac{\mu}{r^2}dr$$

$$=2\mu\left[\frac{1}{r}\right]_{\infty}^{R}=\frac{2\mu}{R}.$$

Hence *the path of the particle will be elliptic, parabolic or hyperbolic, according as the velocity at any point < or = or > the velocity from infinity to that point.*

Cor. 2. If V_2 is the velocity for the description of a circle of radius R, then

$$\frac{V_2^2}{R}=\frac{\mu}{R^2} \text{ (normal acceleration).}$$

$$\therefore \quad V_2=\sqrt{\left(\frac{\mu}{R}\right)}=\frac{1}{\sqrt{2}}\sqrt{\left(\frac{2\mu}{R}\right)}=\frac{1}{\sqrt{2}}V_1, \qquad \text{[from cor. 1]}$$

where V_1 is the velocity from infinity.

Thus the velocity for the description of a circle of radius R =(1/√2). (velocity from infinity to distance R from the centre of force).

§ 3. **Kepler's Laws of Planetary Motion.** **[Allahabad 1978]**

The laws according to which the planets move with reference to the sun were discovered by the astronomer **John Kepler.** These laws are as follows :

(i) *Each planet describes an ellipse having the sun as one of its foci.*

(ii) *The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.*

(iii) *The squares of the periodic times of the various planets are proportional to the cubes of the semi-major axes of their orbits.*

§ 4. **Deductions from Kepler's Laws.**

(i) In the chapter on central orbits, we have proved that if the orbit is an ellipse under an acceleration towards one of its foci, then the law of acceleration is that of the inverse square of the distance from the focus (centre of force). Hence from the first Kepler's law we conclude that '*the acceleration of each planet towards the sun varies inversely as the square of its distance from the sun*'.

(ii) According to the Kepler's second law the rate of description of sectorial area is constant which is true only in central orbits. Hence from the second Kepler's law we conclude that '*the acceleration of the planet and therefore the force on it is directed towards the sun.*'

(iii) The periodic time of a closed central orbit under inverse square law is given by

$$T=\frac{\text{area of the ellipse (i.e., the curve described)}}{\text{rate of description of the sectorial area}}$$

$$=\frac{\pi ab}{\frac{1}{2}h}=\frac{\pi ab}{\frac{1}{2}\sqrt{(\mu l)}}=\frac{2\pi ab}{\sqrt{\{\mu(b^2/a)\}}}=\frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

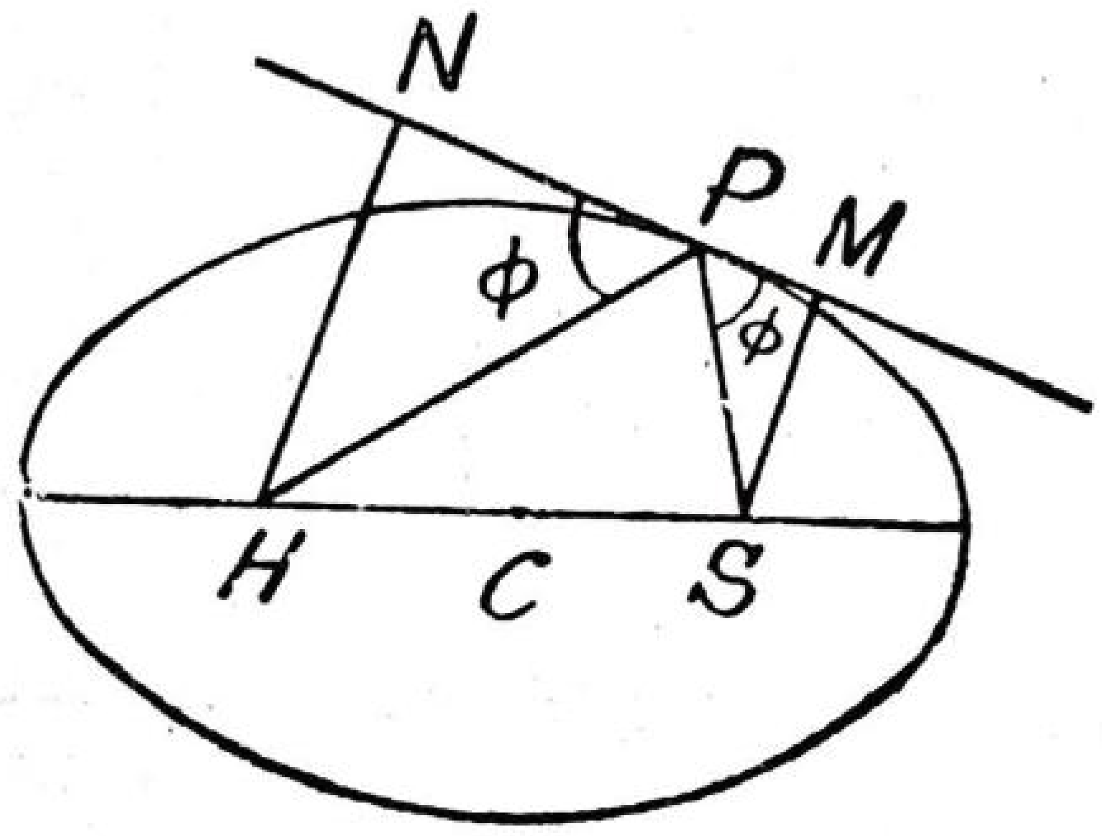
or $$T^2=\frac{4\pi^2a^3}{\mu}.$$

But according to the third Kepler's law $T^2 \propto a^3$, from which it follows that μ is a constant.

Hence from the Kepler's third law we conclude that '*the absolute acceleration μ is the same for all the planets.*'

§ 5. Some important geometrical properties of an ellipse.

In the adjoining figure S and H are the two foci of the ellipse and C is the centre of the ellipse.



1. The product of the perpendiculars drawn from the foci on the tangent at any point of an ellipse is constant and is equal to the square of the semi-minor axis of the ellipse *i.e.*,

$$SM\,.\,HN=b^2.$$

2. The sum of the focal distances of any point on an ellipse is equal to $2a$, where $2a$ is the length of the major axis of the ellipse. Thus $SP+HP=2a$.

3. The length of the latus rectum of an ellipse is $2(b^2/a)$, where

$$b^2=a^2(1-e^2),$$

e being the eccentricity of the ellipse.

4. The tangent at any point P of an ellipse is equally inclined to the focal radii SP and HP of that point.

Illustrative Examples.

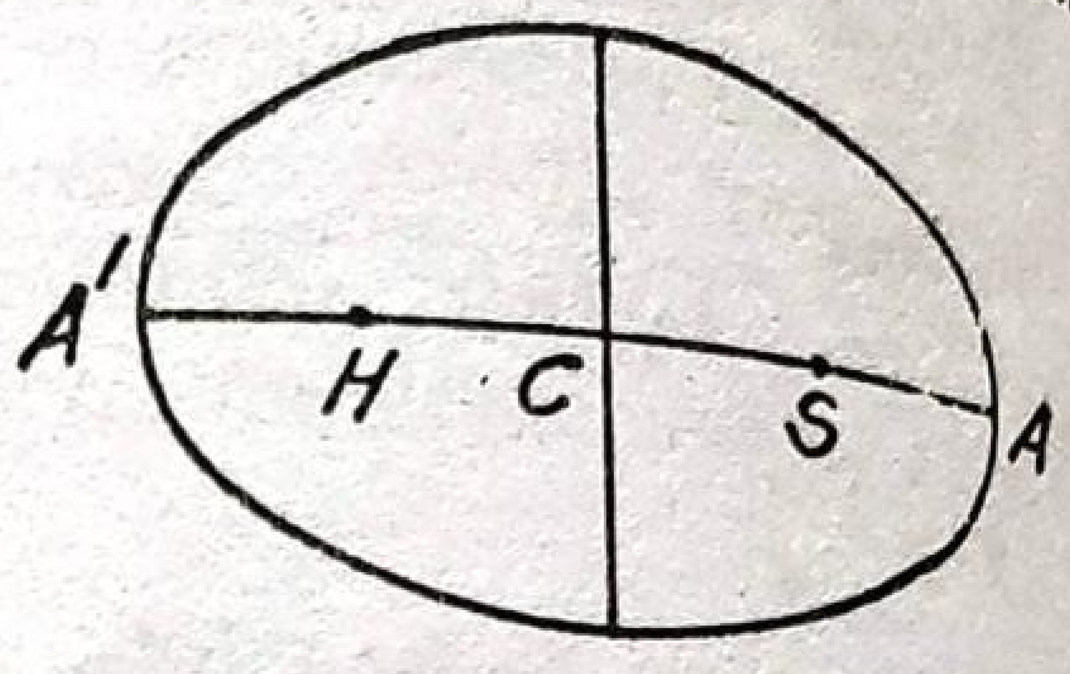
Ex. 1. *If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that*

$$(1-e)v_1=(1+e)v_2.$$

Sol. The path of a planet is an ellipse with the sun at its focus. Therefore the velocity v of the planet at a distance r from the focus S (the sun) is given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right). \qquad ...(1)$$

Let v_1 and v_2 be the velocities of the planet at the points A and A' which are nearest and farthest from the sun at S. Then at A,
$r=SA=CA-CS=a-ae$, $v=v_1$
and at A', $r=SA'=CS+CA'$
$=ae+a$, $v=v_2$.



Substituting these values in (1), we have

$$v_1^2=\mu\left(\frac{2}{a-ae}-\frac{1}{a}\right)=\mu\left\{\frac{2-(1-e)}{a(1-e)}\right\}=\mu\frac{(1+e)}{a(1-e)}$$

and $$v_2^2=\mu\left\{\frac{2}{ae+a}-\frac{1}{a}\right\}=\mu\left\{\frac{2-(1+e)}{a(1+e)}\right\}=\mu\frac{(1-e)}{a(1+e)}.$$

Dividing, we have

$$\frac{v_1^2}{v_2^2}=\frac{(1+e)^2}{(1-e)^2} \quad \text{or} \quad \frac{v_1}{v_2}=\frac{1+e}{1-e}$$

or $$(1-e)v_1=(1+e)v_2.$$

Ex. 2. *The greatest and least velocities of a certain planet in its orbit round the sun are 30 km./sec. and 29·2 km./sec. respectively. Find the eccentricity of the orbit.*

Sol. Refer figure of Ex. 1.

The path of the planet round the sun is an ellipse with the sun as the focus S. Therefore the velocity of the planet at a point distant r from S is given by

$$v^2=\mu(2/r-1/a). \quad \text{...(1)}$$

From (1), it is evident that the velocity of the planet is greatest or least according as r is least or greatest. Thus the velocity is greatest at A and least at A'. Therefore according to the question,

$v=30$ km./sec. when $r=SA=CA-CS=a-ae=a(1-e)$

and $v=29{\cdot}2$ km./sec. when $r=SA'=CA'+CS=a+ae=a(1+e)$.

Putting these values in (1), we have

$$30^2=\mu\left[\frac{2}{a(1-e)}-\frac{1}{a}\right]=\frac{\mu(1+e)}{a(1-e)}$$

and $$(29{\cdot}2)^2=\mu\left[\frac{2}{a(1+e)}-\frac{1}{a}\right]=\frac{\mu(1-e)}{a(1+e)}.$$

Dividing, we have $\left(\frac{1+e}{1-e}\right)^2=\left(\frac{30}{29{\cdot}2}\right)^2$

or $(1+e)/(1-e)=30/29{\cdot}2$ or $(29{\cdot}2)(1+e)=30(1-e)$

or $e(29{\cdot}2+30)=30-29{\cdot}2$

or $e=(0{\cdot}8)/(59{\cdot}2)=1/74.$

Ex. 3. *A particle is projected from the earth's surface with velocity v. Show that if the diminution of gravity is taken into account, but the resistance of the air neglected, the path is an ellipse, of major axis $2ga^2/(2ga-v^2)$, where a is the earth's radius.*

Sol. If $2a_1$ is the major axis of the ellipse described by the particle, then its velocity V at a distance r from the centre of the earth is given by $V^2=\mu(2/r-1/a_1)$. ...(1)

But on the surface of the earth, we have

r=the radius of the earth$=a$.

Also the particle has been projected with velocity v from the earth's surface. Therefore putting $r=a$ and $V=v$ in (1), we have

$$v^2=\mu\left(\frac{2}{a}-\frac{1}{a_1}\right). \quad ...(2)$$

Now for a particle on the surface of the earth the acceleration 'g' due to gravity is given by $g=\mu/a^2$ so that $\mu=a^2g$.

Substituting the value of μ in (2), we have

$$v^2=ga^2(2/a-1/a_1)$$

or $2/a-1/a_1=v^2/ga^2$ or $1/a_1=2/a-v^2/ga^2=(2ga-v^2)/ga^2$

or $a_1=ga^2/(2ga-v^2)$.

Hence the length of the major axis of the ellipse described

$$=2a_1=2ga^2/(2ga-v^2).$$

Ex. 4. *A parlicle describes an ellipse under a force $\mu/(\text{distance})^2$ towards the focus. If it was projected wtth velocity V from a point dtstant r from the centre of force, show that its periodic time is $(2\pi/\sqrt{\mu}).[2/r-V^2/\mu]^{-3/2}$.* **[Meerut 1976]**

Sol. Let a be the length of the semi-major axis of the ellipse described by the particle. Then the velocity V at a point distant r from the centre of force is given by

$$V^2=\mu[2/r-1/a].$$

$\therefore$ $V^2/\mu=2/r-1/a$ or $1/a=2/r-V/\mu$

or $a=[2/r-V^2/\mu]^{-1}$.

$\therefore$ the periodic time$=\dfrac{2\pi a^{3/2}}{\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}\left[\dfrac{2}{r}-\dfrac{V^2}{\mu}\right]^{-3/2}$

Ex. 5. *Show that an unresisted particle falling to the earth's surface from a great distance would acquire a velocity $\sqrt{(2ga)}$, where a is the radius of the earth.*

Sol. When the particle is at a distance x from the centre of the earth, its acceleration due to the attraction of the earth is μ/x^2 and is directed towards the centre of the earth. On the surface of the earth $x=a$ and the acceleration due to gravity is g. Therefore $\mu/a^2=g$ or $\mu=a^2g$.

Now a particle falls unresisted to the earth's surface from a great distance. The only force acting on the particle is the attraction of the earth. If v is the velocity of the particle at a distance x from the centre of the earth, we have

$$v\,(dv/dx)=-\mu/x^2=-a^2g/x^2.$$

$$\therefore\quad v\,dv=-(a^2g/x^2)\,dx.$$

If V is velocity acquired by the particle on the surface of the earth, we have

$$\int_0^V v\,dv=-\int_\infty^a \frac{a^2g}{x^2}\,dx$$

or $\left[\frac{1}{2}v^2\right]_0^V=a^2g\left[\frac{1}{x}\right]_\infty^a$

or $V^2/2=a^2g/a$ or $V^2=2ag$ or $V=\sqrt{(2ag)}$.

Ex. 6. *If the velocity of the earth at any point of its orbit, assumed to be circular, were increased by about one-half, prove that it would describe a parabola about the sun as focus.*

Show also that, if a body was projected from the earth with a velocity exceeding 7 miles per second, it will not return to the earth and may even leave the solar system.

Sol. Let a be the radius of the earth's orbit (supposed to be circular) with sun as centre. If v_1 is the velocity in a circular central orbit at a distance a, we have

v_1^2/a (*i.e.*, normal acceleration)$=\mu/a^2$ (*i.e.*, the central acceleration). ...(1)

$$\therefore\quad v_1^2=\mu/a.$$

If v_2 is the velocity in a parabolic path at a distance $r=a$ from the focus, then from § 2 of this chapter, $v_2^2=2\mu/a$. ...(2)

From (1) and (2), we have $v_2^2=2v_1^2$ or $v_2=\sqrt{2}v_1$ ...(3)

or $v_2=v_1+(\sqrt{2}-1)v_1=v_1+\frac{1}{2}v_1$ (approximately).

$[\because\ \sqrt{2}-1=\frac{1}{2}$ (approx.)$]$

Thus the velocity in a parabolic path is $\frac{3}{2}$ times the velocity at the same distance in the circular path. Hence if the velocity of the earth in circular path be increased by about one half of itself at the same distance, then it would describe a parabola about the sun as the focus.

Second part. Let V be the least velocity of projection from the surface of the earth so that the body will not return to the earth. Then for this velocity of projection the path of the body is a parabola with focus at the earth's centre. Therefore if R be the radius of the earth, then from (2), this velocity V on the surface of the earth is given by $V=\sqrt{(2\mu/R)}$.

Also on the surface of the earth the acceleration $g=\mu/R^2$.

$\therefore \quad \mu = R^2 g.$

$\therefore \quad V = \sqrt{\left(\dfrac{2R^2 g}{R}\right)} = \sqrt{(2Rg)}.$

But $R = 4000$ miles $= 4000 \times 1760 \times 3$ ft. and $g = 32$ ft./sec^2.

$\therefore \quad V = \sqrt{[2 \times 4000 \times 1760 \times 3 \times 32]}$ ft./sec.

$$= \frac{\sqrt{(2 \times 4000 \times 1760 \times 3 \times 32)}}{1760 \times 3} \text{ miles/sec.}$$

$$= \sqrt{\left(\frac{2 \times 4000 \times 32}{1760 \times 3}\right)} \text{ miles/sec.}$$

$= 7$ miles/sec approximately.

Hence if a body is projected from the earth's surface with a velocity exceeding 7 miles per second, it will not return to the earth.

Also we know that the velocity of the earth, say v_1, is $18{\cdot}5$ miles/sec. nearly. Therefore, using the result (3), if it is changed to $v_2 = \sqrt{2} v_1 = (18{\cdot}5)\sqrt{2}$ miles/sec., then it will describe a parabolic path.

But $v_2 = (18{\cdot}5)\sqrt{2}$ miles/sec. $= 26$ miles/sec. nearly

$= (v_1 + 7{\cdot}5)$ miles/sec. nearly.

Hence if a body were projected from the earth's surface in the direction of the earth's velocity with a velocity $7{\cdot}5$ miles/sec. more than the velocity of the earth, it would describe a parabola with the sun as its focus. But the parabola is an open curve and so the body will go to infinity and will leave the solar system.

Ex. 7. *Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to the point from rest at a distance from the sun equal to the length of the major axis.*

Sol. If V is the velocity of a planet in the elliptic path at a distance r from the sun, then $V^2 = \mu(2/r - 1/a)$. ...(1)

Now let the planet fall from rest at a distance $2a$ (length of the major axis of the elliptic path) from the sun. If at any time t the planet is at a distance x from the sun and v is its velocity there, then the equation of motion of the planet is

$$v\,(dv/dx) = -\mu/x^2.$$

$\therefore \quad v\,dv = -(\mu/x^2)\,dx.$

Integrating, we get

$\frac{1}{2}v^2 = \mu/x + C$, where C is a constant.

But initially when $x = 2a$, $v = 0$. Therefore $C = -\mu/2a$.

$\therefore \quad \frac{1}{2}v^2 = \mu/x - \mu/2a$ or $v^2 = \mu(2/x - 1/a)$. ...(2)

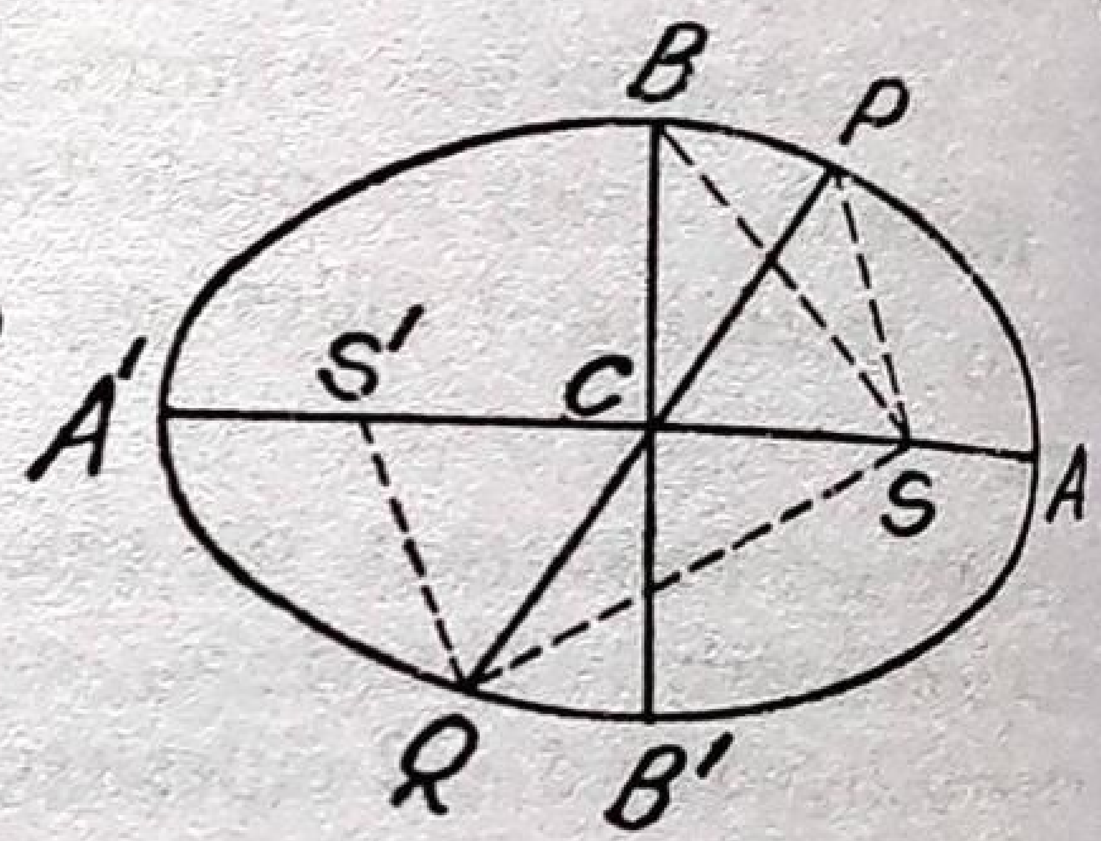
If v_1 is the velocity of the planet in this case at a distance r from the sun, then putting $v=v_1$ and $x=r$ in (2), we get

$$v_1^2=\mu(2/r-1/a).$$

From (1) and (3), we observe that $V=v_1$ ...(3)

Ex. 8 (a). *A particle describes an ellipse as a central orbit about the focus. Prove that the velocity at the end of the minor axis is the geometric mean between the velocities at the ends of any diameter.* **[Meerut 1974]**

Sol. Let AA' and BB' be the major and minor axes of the ellipse.



Let S, S' be the foci and PQ any diameter of the ellipse.

The velocity v of the particle at any point of the ellipse at a distance r from the focus S is given by $v^2=\mu(2/r-1/a)$, ...(1)

where $2a$ is the length of the major axis of the ellipse.

We have $SB+S'B=2a$ and $SB=S'B$.

$\therefore$ $SB=a$.

Let V, V_1 and V_2 be the velocities of the particle at the points B, P and Q respectively. Then at B, $r=SB=a$, $v=V$; at P, $r=SP$, $v=V_1$; and at Q, $r=SQ$, $v=V_2$.

$\therefore$ from (1), we have, $V^2=\mu\left(\frac{2}{a}-\frac{1}{a}\right)=\frac{\mu}{a}$, ...(2)

$$V_1^2=\mu\left(\frac{2}{SP}-\frac{1}{a}\right) \quad \text{and} \quad V_2^2=\mu\left(\frac{2}{SQ}-\frac{1}{a}\right).$$

Now $V_1^2 V_2^2=\mu^2\left(\frac{2}{SP}-\frac{1}{a}\right)\left(\frac{2}{SQ}-\frac{1}{a}\right)$

$$=\mu^2\left[\frac{4}{SP.SQ}-\frac{2}{a}\left(\frac{1}{SP}+\frac{1}{SQ}\right)+\frac{1}{a^2}\right]$$

$$=\mu^2\left[\frac{4}{SP.SQ}-\frac{2}{a}\cdot\left(\frac{SQ+SP}{SP\cdot SQ}\right)+\frac{1}{a^2}\right] \quad ...(3)$$

We have $SP+SQ=QS'+SQ$ $[\because QS'=SP]$

$=2a$ [by a property of the ellipse]

Substituting in (3), we have

$$V_1^2V_2^2=\mu^2\left[\frac{4}{SP.SQ}-\frac{2}{a}\cdot\frac{2a}{SP.SQ}+\frac{1}{a^2}\right]=\frac{\mu^2}{a^2}$$

or $V_1V_2=\mu/a$ or $V_1V_2=V^2$ [from (2)]

or $V=\sqrt{(V_1V_2)}$.

Hence V (*i e.*, the velocity at the end of the minor axis) is equal to the geometric mean between V_1 and V_2 (*i.e.*, the velocites, at the ends of any diameter).

Ex. 8. (b). *A particle describes an ellipse under a force μ/r^2 to a focus. Show that the velocity at the end of the minor axis is a geometric mean between the greatest and the least velocities.* **[Meerut 1974]**

Sol. Proceed as in Ex. 8 (a).

Ex. 9. *A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without any change in the direction of motion. Prove that the particle proceeds to describe a parabola.*

Sol. Refer figure of Ex. 8.

The velocities v and V at a distance r from the focus S in an elliptic and parabolic path are respectively given by

$$v^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right), \quad \text{...(1)}$$

and

$$V^2=\frac{2\mu}{r}. \quad \text{...(2)}$$

If v_1 is the velocity of the particle (describing the elliptic path) at the point B (an end of the minor axis), then from (1), we have

$$v_1^2=\mu\left(\frac{2}{SB}-\frac{1}{a}\right)=\mu\left(\frac{2}{a}-\frac{1}{a}\right) \quad [\because SB=a]$$

$$=\frac{\mu}{a}. \quad \text{...(3)}$$

If v_2 is the velocity of the particle when its kinetic energy is doubled at B, then

$\frac{1}{2}mv_2^2=2(\frac{1}{2}mv_1^2)$ where m is the mass of the particle.

$\therefore\ v_2^2=2v_1^2$ or $v_2^2=2\mu/a$ or $v_2^2=2\mu/SB$.

Also from (2) the velocity at B for a parabolic path is given by

$$V^2=2\mu/SB.$$

Since $v_2^2=V^2$ *i.e.*, $v_2=V$, therefore the subsquent path of the particle at B is a parabola.

Ex. 10. *A particle moves with a central acceleration $\mu/(distance)^2$; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is* $\sin^{-1}[\mu/\{VR\sqrt{(V^2-2\mu/R)}\}]$.

Sol. If the particle describes a hyperbola under the central acceleration $\mu/(\text{distance})^2$, then the velocity v of the particle at a distance r from the centre of force is given by ...(1)

$$v^2=\mu(2/r+1/a),$$

where $2a$ is the transverse axis of the hyperbola.

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2=\mu\left(\frac{2}{R}+\frac{1}{a}\right) \quad \text{or} \quad \frac{\mu}{a}=V^2-\frac{2\mu}{R}. \quad \text{...(2)}$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h=vp$, we have $h=Vp=VR \sin \alpha$ ...(3)

[$\because$ $p=r \sin\phi$ and initially $r=R$, $\phi=\alpha$.]

Also $h=\sqrt{(\mu l)}=\sqrt{\{\mu.(b^2/a)\}}=\sqrt{(\mu a)}$ ...(4)

[$\because$ $b=a$ for a rectangular hyperbola]

From (3) and (4), we have

$$VR \sin\alpha=\sqrt{(\mu a)}$$

or $$\sin\alpha=\frac{\sqrt{(\mu a)}}{VR}=\frac{\mu\sqrt{a}}{VR\sqrt{\mu}}=\frac{\mu}{VR\sqrt{(\mu/a)}}.$$

Substituting for μ/a from (2), we have

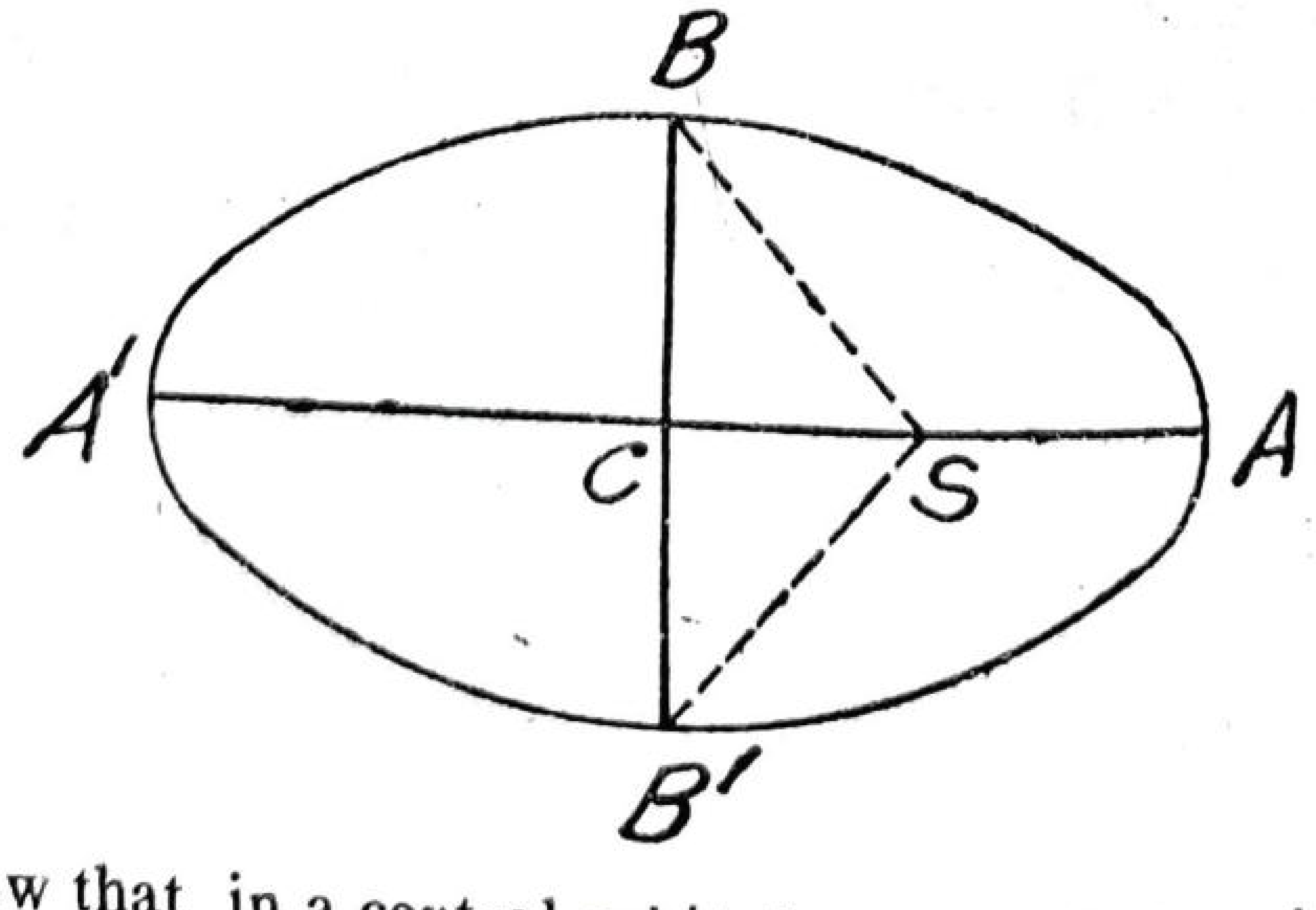
$$\sin \alpha=\mu/\{VR\sqrt{(V^2-2\mu/R)}\}$$

or $$\alpha=\sin^{-1}[\mu/\{VR\sqrt{(V^2-2\mu/R)}\}],$$

which is the required angle of projection.

Ex. 11. *Prove that the time taken by the earth to travel over half its orbit, remote from the sun, separated by the minor axis is two days more than half the year. The eccentricity of the orbit is 1/60.*

Sol. In the figure the path of the earth round the sun is an ellipse, the sun being at the focus S. BB' is the minor axis of the ellipse and C the centre of the ellipse. Remote half from the sun separated by the minor axis is the arc $B\ A'\ B'$.



We know that in a central orbit the rate of description of sectorial area is constant and is equal to $h/2$. The whole area of the ellipse is $\pi\ ab$ and the earth takes one year time to describe the elliptic orbit round the sun.

$\therefore$ one year $= \dfrac{\pi ab}{h/2}$. ...(1)

The sectorial area traced out by the earth while describing the arc $BA'B'$

$=$ the area $SBA'B'S$

$= \frac{1}{2}$ area of the ellipse $+$ area of the $\triangle SBB'$

$= \frac{1}{2}\pi ab + \frac{1}{2}BB' \cdot CS = \frac{1}{2}\pi ab + \frac{1}{2} \cdot 2b \cdot ae = \frac{1}{2}\pi ab + abe.$

If t be the time taken by the earth to describe the arc $BA'B'$, then

$$t = \frac{\text{the sectorial area } SBA'B'S}{h/2}$$

$$\therefore \quad t = \frac{\frac{1}{2}\pi ab + abe}{h/2}. \quad \text{...(2)}$$

Dividing (2) by (1), we have

$$\frac{t}{\text{one year}} = \frac{\frac{1}{2}\pi ab + abe}{\pi ab} = \frac{1}{2} + \frac{e}{\pi}.$$

$$\therefore \quad t = \left(\frac{1}{2} + \frac{e}{\pi}\right) \times \text{one year} = \frac{1}{2} \text{ year} + \frac{e}{\pi} \text{ year}$$

$$= \frac{1}{2} \text{ year} + \frac{1}{60\pi} \text{ year} \qquad \left[\because \; e = \frac{1}{60}\right]$$

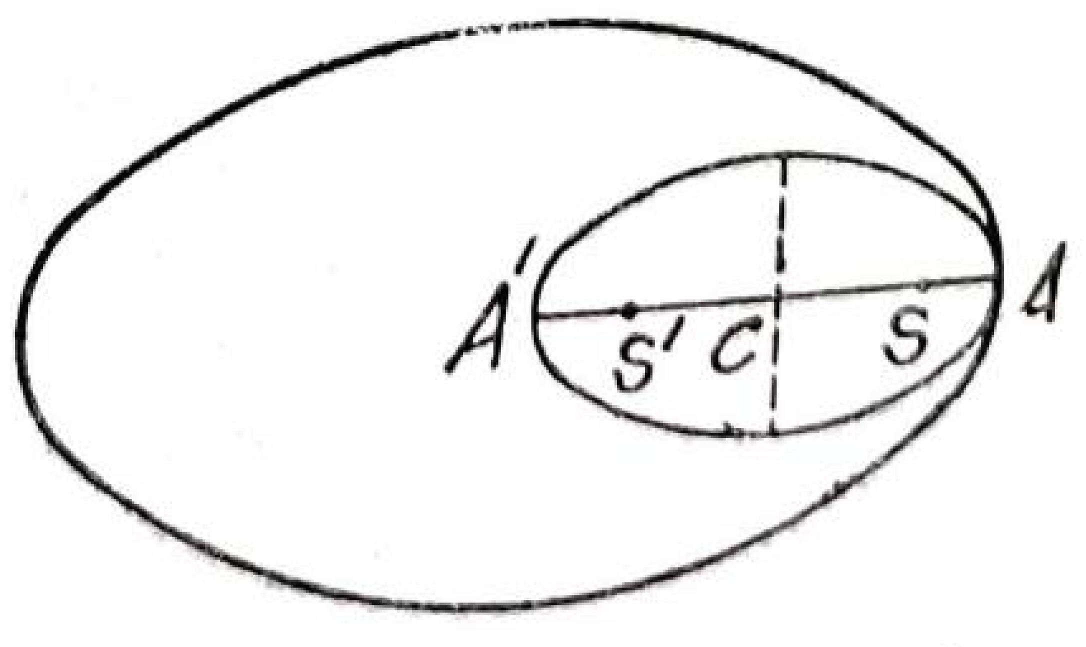
$$= \frac{1}{2} \text{ year} + \left[\frac{1}{60} \times \frac{7}{22} \times 365\right] \text{ days} \quad \left[\because \; \text{one year} = 365 \text{ days}\right]$$

$= \frac{1}{2}$ year $+ 2$ days nearly.

Hence the time of describing half of the orbit remote from the sun is two days more than half the year.

Ex. 12. *A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is* $e(3+e)/(1-e)$.

Sol. Let S and S' be the foci of an ellipse of eccentricity e and major axis of length $2a$ described by the body under the action of a force tending to the focus S. When S is the centre of force, A is the nearer apse.



The velocity v of the body at a distance r from S is given by

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]. \quad \text{...(1)}$$

If V is the velocity of the body at A, when S is the centre of force, then from (1), we have

$$V^2 = \mu\left[\frac{2}{SA} - \frac{1}{a}\right] \qquad [\because \; \text{at } A, r = SA]$$

But $SA = CA - CS = a - ae = a(1-e)$. ...(2)

$$\therefore \quad V^2 = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right] = \frac{\mu(1+e)}{a(1-e)} \qquad \text{...(3)}$$

When the body is at A and the centre of force is transferred to the other focus S', the body will describe a new elliptic orbit with the centre of force S' as a focus. Since the velocity of the body at A is not changed, therefore if $2a'$ is the length of the major axis of the new ellipse, then the velocity V at A is given by

$$V^2 = \mu\left[\frac{2}{S'A} - \frac{1}{a'}\right] = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a'}\right] \qquad \text{...(4)}$$

$[\because \; S'A = CS' + CA = ae + a = a(1+e)]$

From (3) and (4), we have

$$\frac{\mu(1+e)}{a(1-e)} = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a'}\right]$$

or
$$\frac{(1+e)}{a(1-e)} = \frac{2}{a(1+e)} - \frac{1}{a'} \qquad \text{...(5)}$$

Since the direction of the velocity of the body at A, is also not changed, therefore for the new elliptic orbit also the point A is an apse. If e' is the eccentricity of the new ellipse, then cororresponding to the result (2) for the original ellipse, we have for the new ellipse $S'A = a'(1-e')$.

But $S'A = a(1+e)$, from the original ellipse, as mentioned above.

$\therefore \quad a(1+e) = a'(1-e')$ or $a' = a(1+e)/(1-e')$.

Substituting this value of a' in (5), we have

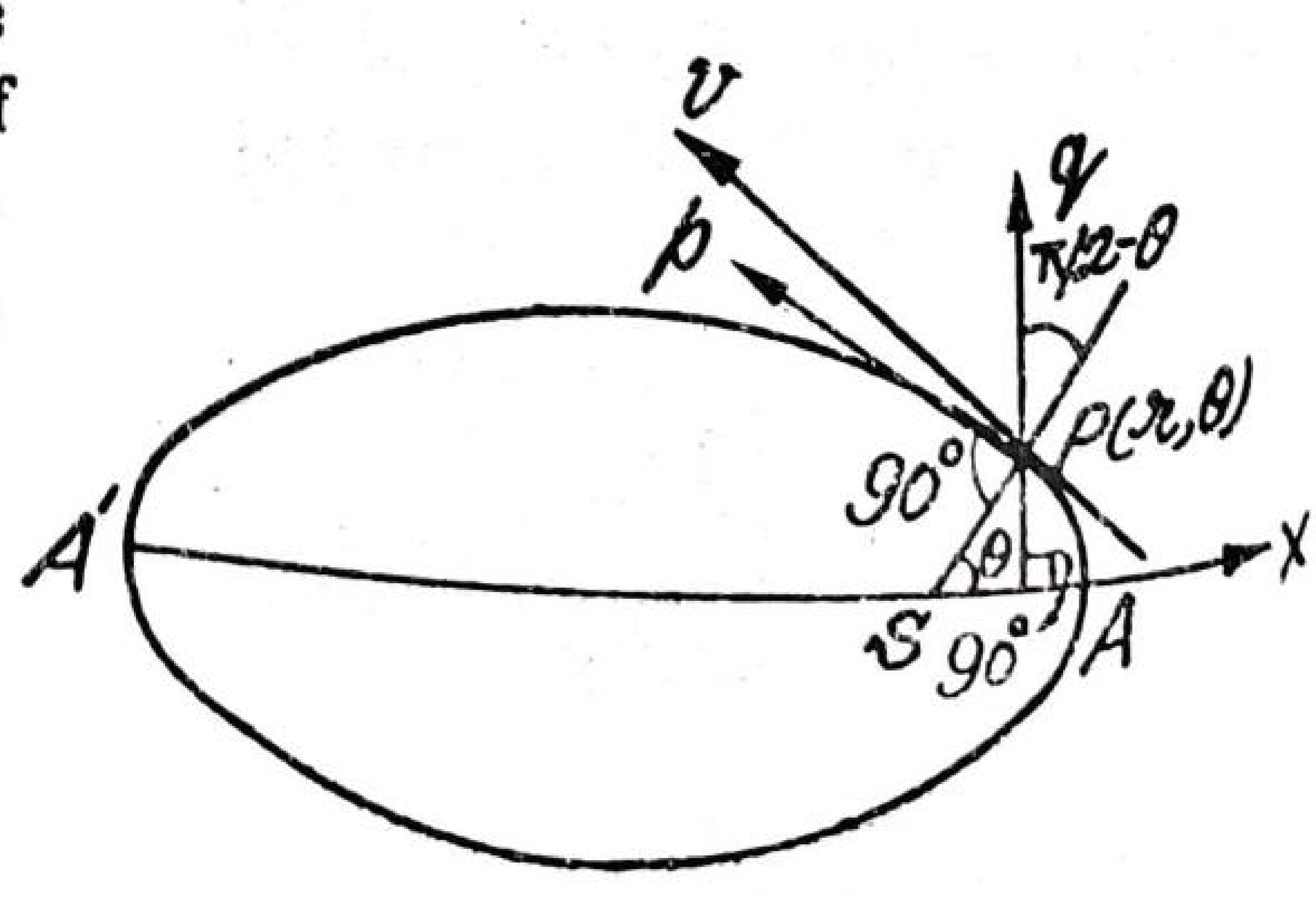
$$\frac{(1+e)}{a(1-e)} = \frac{2}{a(1+e)} - \frac{(1-e')}{a(1+e)}$$

or
$$\frac{1+e}{a(1-e)} = \frac{2-(1-e')}{a(1+e)} \quad \text{or} \quad 1+e' = \frac{(1+e)^2}{(1-e)}$$

or
$$e' = \frac{(1+e)^2}{(1-e)} - 1 = \frac{(1+e)^2 - (1-e)}{(1-e)} = \frac{3e+e^2}{(1-e)} = \frac{e(3+e)}{1-e}.$$

Ex. 13. *Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities μ/h perpendicular to the radius and $\mu e/h$ perpendicular to the major axis.*

Sol. Referred to the focus S (*i.e.*, the centre of force) as pole, let the equation of the elliptic orbit be $l/r = 1 + e\cos\theta$, ...(1)
where l is the semi-latus rectum of the ellipse.



Let P (r, θ) be the position of the particle at any time t. The resultant velocity v of the particle at P is along the tangent to the ellipse at P. Suppose the velocity v is the resultant of two velocities p and q where p is perpendicular to the radius vector SP and q is perpendicular to the major axis AA'. Resolving the velocities p and q at P along and perpendicular to the radius vector SP, we have

the radial velocity $dr/dt = q \cos(\frac{1}{2}\pi - \theta) = q \sin\theta$, ...(2)

and the transverse velocity $r\,(d\theta/dt) = p + q \sin(\frac{1}{2}\pi - \theta) = p + q \cos\theta$. ...(3)

From (2), $q = \dfrac{1}{\sin\theta} \dfrac{dr}{dt}$. ...(4)

Differentiating both sides of (1) w.r.t. 't', we have

$$-\frac{l}{r^2}\frac{dr}{dt} = -e \sin\theta \frac{d\theta}{dt}.$$

$$\therefore \frac{dr}{dt} = \frac{e}{l} \sin\theta\, r^2 \frac{d\theta}{dt}$$

$$= \frac{eh}{l} \sin\theta. \qquad \left[\because \text{ in a central orbit, } r^2 \frac{d\theta}{dt} = h\right]$$

Substituting the value of dr/dt in (4), we get

$$q = \frac{1}{\sin\theta} \cdot \frac{eh}{l} \sin\theta = \frac{eh}{l}$$

$$= \frac{eh}{(h^2/\mu)} \qquad \left[\because h^2 = \mu l\right]$$

$$= e\mu/h = \text{constant}.$$

This gives one desired result.

Again from (3), we have

$$p = r\frac{d\theta}{dt} - q \cos\theta$$

$$= \frac{h}{r} - \frac{e\mu}{h}\cos\theta \qquad \left[\because r^2 \frac{d\theta}{dt} = h \text{ and } q = \frac{e\mu}{h}\right]$$

$$= \frac{h}{r} - \frac{\mu}{h}\left(\frac{l}{r} - 1\right) \qquad \left[\because \text{ from (1), } e\cos\theta = \frac{l}{r} - 1\right]$$

$$= \frac{h}{r} - \frac{\mu l}{hr} + \frac{\mu}{h}$$

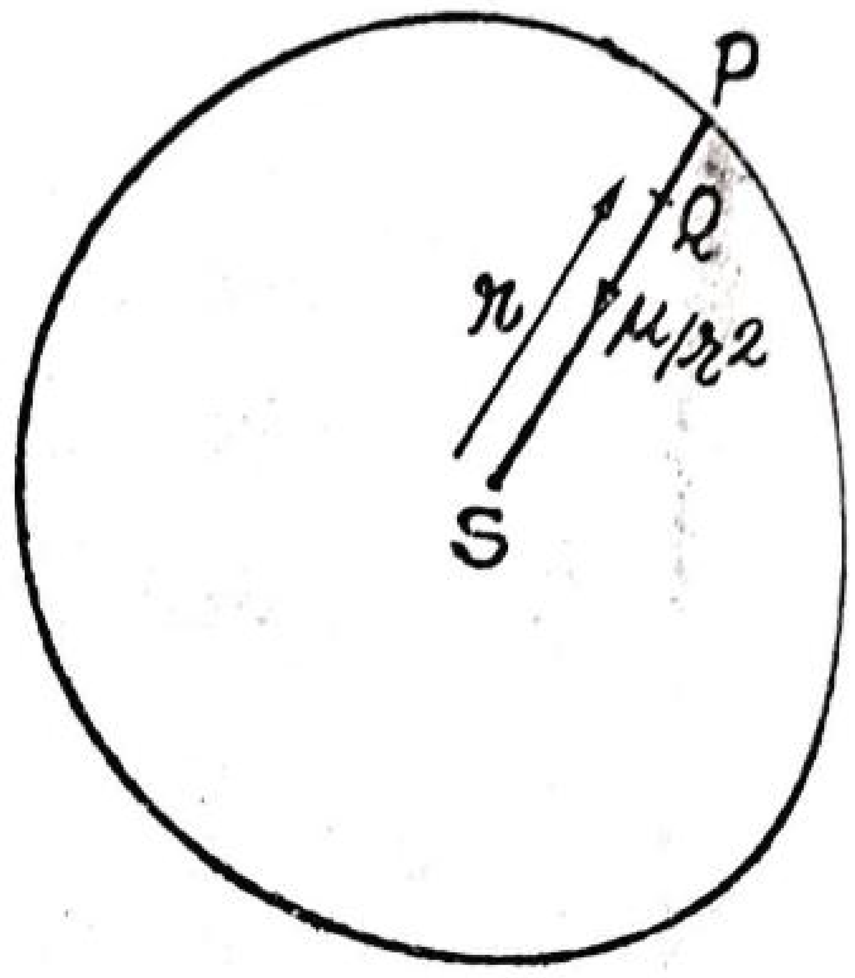
$$= \frac{h}{r} - \frac{h^2}{hr} + \frac{\mu}{h} \qquad \left[\because h^2 = \mu l\right]$$

$$= \mu/h = \text{constant}.$$

This gives the other desired result.

Ex 14. *If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is $\sqrt{2}/8$ times the period of the planet's revolution.* **Meerut 1980**

Sol. Let a planet describing a circular path of radius a and centre S (the sun) be stopped suddenly at the point P of its path. Then it will begin to move towards S along the straight line PS under the acceleration $\mu/(\text{distance})^2$.



If Q is the position of the planet at time t such that $SQ=r$, then the acceleration at Q is μ/r^2 directed towards S.

$\therefore$ the equation of motion of the planet at Q is

$$v\frac{dv}{dr}=-\frac{\mu}{r^2}$$ (—ive sign is taken as the acceleration at Q is in the direction of r decreasing)

or $$vdv=-\frac{\mu}{r^2}\,dr.$$

Integrating, $\dfrac{v^2}{2}=\dfrac{\mu}{r}+A$, where A is a constant.

But at P, $r=SP=a$ and $v=0$. [Note that the planet begins to move along PS with zero velocity at P]

$$\therefore\quad 0=\frac{\mu}{a}+A \quad\text{or}\quad A=-\frac{\mu}{a}.$$

$$\therefore\quad \frac{v^2}{2}=\frac{\mu}{r}-\frac{\mu}{a}=\frac{\mu(a-r)}{ar}$$

or $$v=\frac{dr}{dt}=-\sqrt{(2\mu/a)}\cdot\sqrt{\left(\frac{a-r}{r}\right)}$$

(—ive sign is taken because r decreases as t increases)

or $$dt=-\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\sqrt{\left(\frac{r}{a-r}\right)}\,dr. \quad ...(1)$$

If t_1 is the time taken by the planet from P to S, then integrating (1), we have

$$\int_0^{t_1}dt=-\sqrt{\left(\frac{a}{2\mu}\right)}\int_{r=a}^{0}\sqrt{\left(\frac{r}{a-r}\right)}\,dr$$

or $$t_1=\sqrt{\left(\frac{a}{2\mu}\right)}\int_0^{\pi/2}\sqrt{\left(\frac{a\cos^2\theta}{a-a\cos^2\theta}\right)}\cdot 2a\cos\theta\sin\theta\,d\theta,$$

putting $r=a\cos^2\theta$, so that $dr=-2a\cos\theta\sin\theta\,d\theta$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\int_0^{\pi/2} 2\cos^2\theta\, d\theta = a\sqrt{\left(\frac{a}{2\mu}\right)}\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta$$

$$=a\sqrt{\left(\frac{a}{2\mu}\right)}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}=\frac{\pi a^{3/2}}{2\sqrt{(2\mu)}}.$$

But the time period T of the planet's revolution is given by

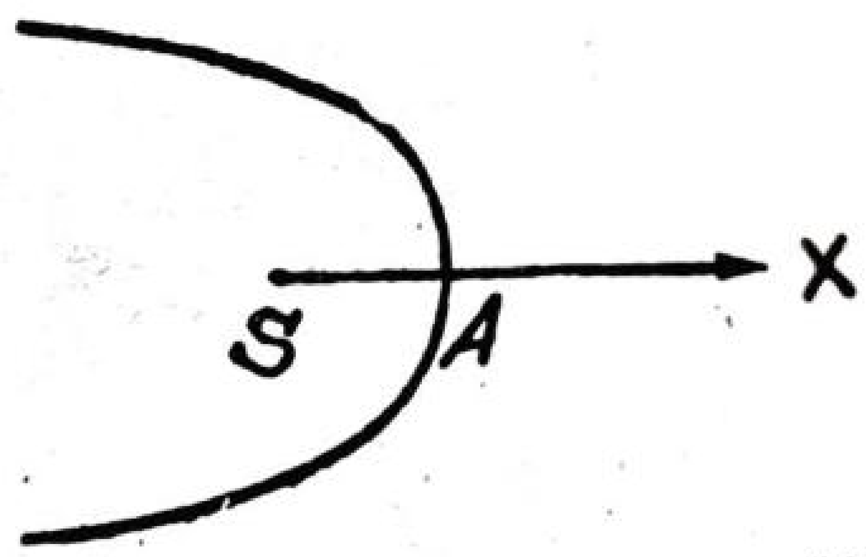
$$T=\frac{2\pi\, a^{3/2}}{\sqrt{\mu}}.$$

$$\therefore\ \frac{t_1}{T}=\frac{1}{4\sqrt{2}}=\frac{\sqrt{2}}{8}$$

or $\quad t_1=(\sqrt{2}/8)\, T$

i.e., the time taken by the planet from P to S is $\sqrt{2}/8$ times the period of the planet's revolution.

Ex. 15. *A comet describing a parabola about the sun, when nearest to it suddenly breaks up, without gain or loss of kinetic energy into two equal portions one of which describes a circle; prove that the other will describe a hyperbola of eccentricity* 2.



Sol. Let a comet describe a parabola of latus rectum $4a$ about the sun S as the focus. The velocity v of the comet at a distance r from the sun (*i.e.*, the focus) is given by

$$v^2=2\mu/r. \qquad \text{...(1)}$$

If V is the velocity of the comet at the point nearest to the sun *i.e.*, at the vertex A, then from (1), we have

$$V^2=2\mu/a. \qquad \text{.. (2)}$$

[$\because\ r=SA=a$, at the vertex A]

Let m be the mass of the comet which breaks into two equal parts $m/2$ and $m/2$ at A and let their velocities at A be v_1 and v_2. It is given that on account of explosion of mass there is no loss or gain of kinetic energy. Therefore

$$\tfrac{1}{2}\, m\, V^2=\tfrac{1}{2}\left(\tfrac{1}{2}\, mv_1^2\right)+\tfrac{1}{2}\left(\tfrac{1}{2}\, mv_2^2\right)$$

or $\quad v_1^2+v_2^2=2V^2=2\cdot(2\mu/a),$ from (2)

or $\quad v_1^2+v_2^2=4\mu/a. \qquad$ ...(3)

If the portion of the comet with velocity v_1 at A describes a circle about S, we have

$$\frac{v_1^2}{a}=\frac{\mu}{a^2} \quad \text{or} \quad v_1^2=\frac{\mu}{a}. \qquad \text{...(4)}$$

Substituting this value of v_1^2 in (3), we have

$$v_2^2=\frac{4\mu}{a}-\frac{\mu}{a}=\frac{3\mu}{a}.$$

Now $v_2^2 = \frac{3\mu}{a} > \frac{2\mu}{a}$ i.e., at A, $v_2^2 > \frac{2\mu}{r}$.

Therefore the portion of the comet with velocity v_2 at A describes a hyperbola of transverse axis $2a_1$ (say).

In a hyperbolic orbit with transverse axis of length $2a_1$ the velocity v at a distance r from the focus (i.e., the sun) is given by

$$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a_1}\right]$$

Here at the point A, $v = v_2$ and $r = a$.

$$\therefore \quad v_2^2 = \mu\left[\frac{2}{a} + \frac{1}{a_1}\right]$$

or $$\frac{3\mu}{a} = \mu\left[\frac{2}{a} + \frac{1}{a_1}\right] \quad \text{or} \quad \frac{1}{a} = \frac{1}{a_1} \quad \text{or} \quad a_1 = a.$$

Thus the length of the transverse axis of the hyperbola described is $2a$. Let $2b$ be the length of the conjugate axis of this hyperbola and e be its eccentricity.

Now we know that in a hyperbolic orbit $vp = h = \sqrt{(\mu l)}$.

Here at the point A, $v = v_2$ and $p = a$.

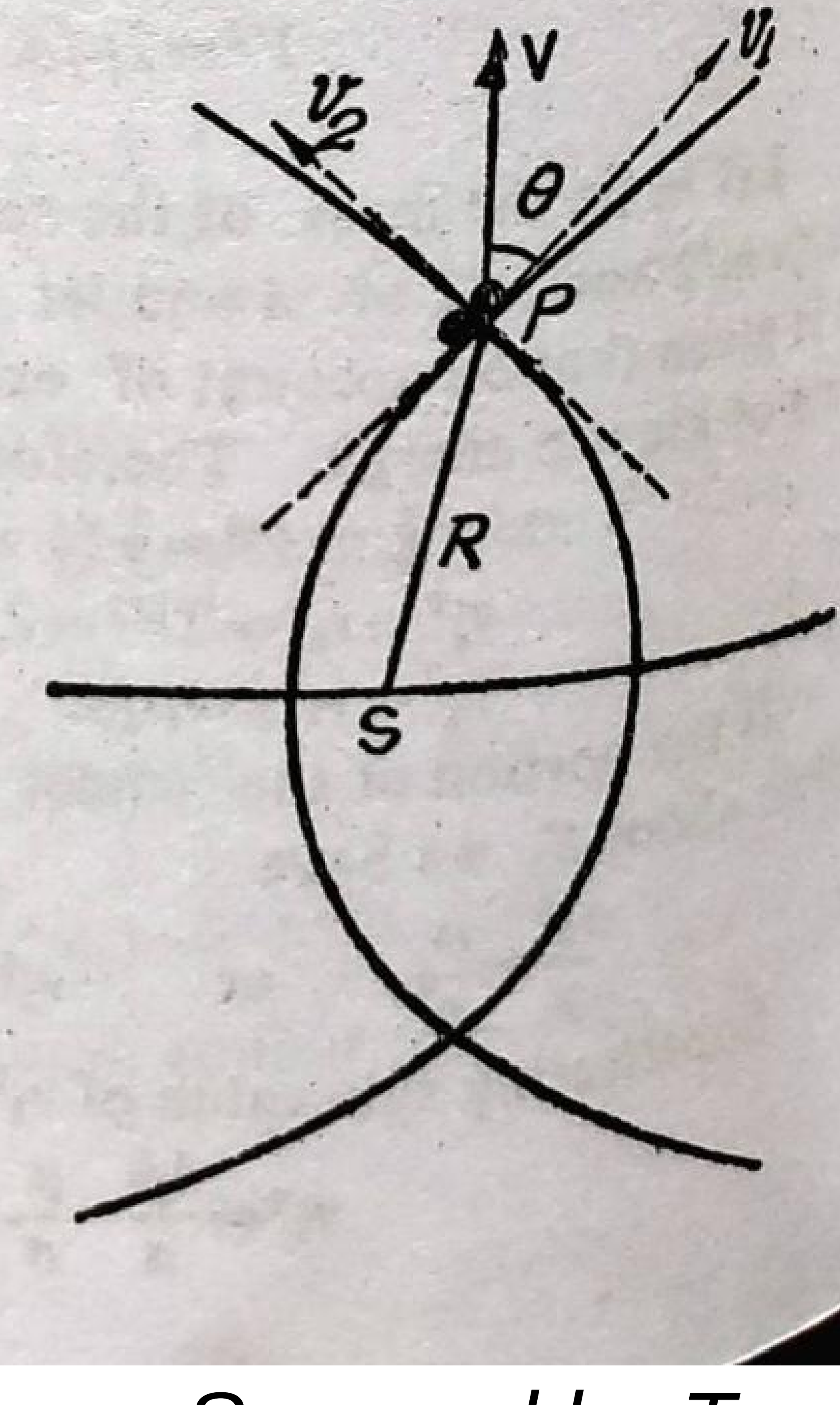
$$\therefore \quad v_2 a = \sqrt{(\mu l)} = \sqrt{\{\mu\,(b^2/a)\}} \qquad [\because \ l = b^2/a]$$

or $$\sqrt{\left(\frac{3\mu}{a}\right)} \cdot a = \sqrt{\left\{\frac{\mu a^2\,(e^2 - 1)}{a}\right\}} \qquad [\because \text{ for the hyperbola } b^2 = a^2\,(e^2 - 1)]$$

or $$3 = e^2 - 1 \quad \text{or} \quad e^2 = 4 \quad \text{or} \quad e = 2.$$

Ex. 16. *Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun, collide at right angles and coalesce when their common distance from the sun is R. Show that the subsequent path of the combined particle is an ellipse of major axis*

$$\frac{(m_1 + m_2)^2\, R}{2 m_1 m_2}.$$



Sol. Let the two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun S (as focus) collide at right angles at the common point P at a distance R from S.

The velocity v of a particle in a parabolic path at a distance r from the focus S is given by

$$v^2 = 2\mu/r.$$

Let v_1 and v_2 be the velocities of m_1 and m_2 respectively at the time of collision at P. Then

$$v_1^2 = \frac{2\mu}{R} \text{ and } v_2^2 = \frac{2\mu}{R}.$$

$$\therefore \quad v_2^2 = v_1^2. \quad \ldots(1)$$

Let the two particles coalesce into a single body of mass (m_1+m_2) after collision at P and let this single mass move with velocity V at an angle θ with the direction of the velocity v_1 of the mass m_1.

By the principle of conservation of momentum in the direction of v_1 and perpendicular to it (*i.e.*, along the direction of v_2), we have

$$(m_1+m_2)\, V \cos\theta = m_1 v_1 + m_2 . 0,$$

and

$$(m_1+m_2)\, V \sin\theta = m_1 . 0 + m_2 v_2.$$

Squaring and adding, we have

$$(m_1+m_2)^2\, V^2 = m_1^2 v_1^2 + m_2^2 v_2^2 = m_1^2 v_1^2 + m_2^2 v_1^2 \quad [\because\ v_2^2 = v_1^2]$$

or

$$V^2 = \frac{(m_1^2 + m_2^2)\, v_1^2}{(m_1+m_2)^2}. \quad \ldots(2)$$

Since $(m_1+m_2)^2 > m_1^2 + m_2^2$, therefore

$$V^2 < v_1^2 \qquad \textit{i.e.,} \qquad V^2 < \frac{2\mu}{R}.$$

Therefore the path of the combined body after collision at P is an ellipse with S as focus. If $2a_1$ is the length of the major axis of this ellipse, then the velocity at any point of this elliptic orbit at a distance r from S is given by

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a_1}\right].$$

But at the point P on this elliptic orbit, $r = R$ and $v = V$.

$$\therefore \quad V^2 = \mu\left[\frac{2}{R} - \frac{1}{a_1}\right]. \quad \ldots(3)$$

From (2) and (3), we have

$$\frac{(m_1^2+m_2^2) v_1^2}{(m_1+m_2)^2} = \mu\left[\frac{2}{R} - \frac{1}{a_1}\right]$$

or

$$\frac{m_1^2+m_2^2}{(m_1+m_2)^2} \cdot \frac{2\mu}{R} = \mu\left[\frac{2}{R} - \frac{1}{a_1}\right] \qquad \left[\because\quad v_1^2 = \frac{2\mu}{R}\right]$$

or

$$\frac{1}{a_1} = \frac{2}{R} - \frac{(m_1^2+m_2^2)}{(m_1+m_2)^2} \cdot \frac{2}{R}$$

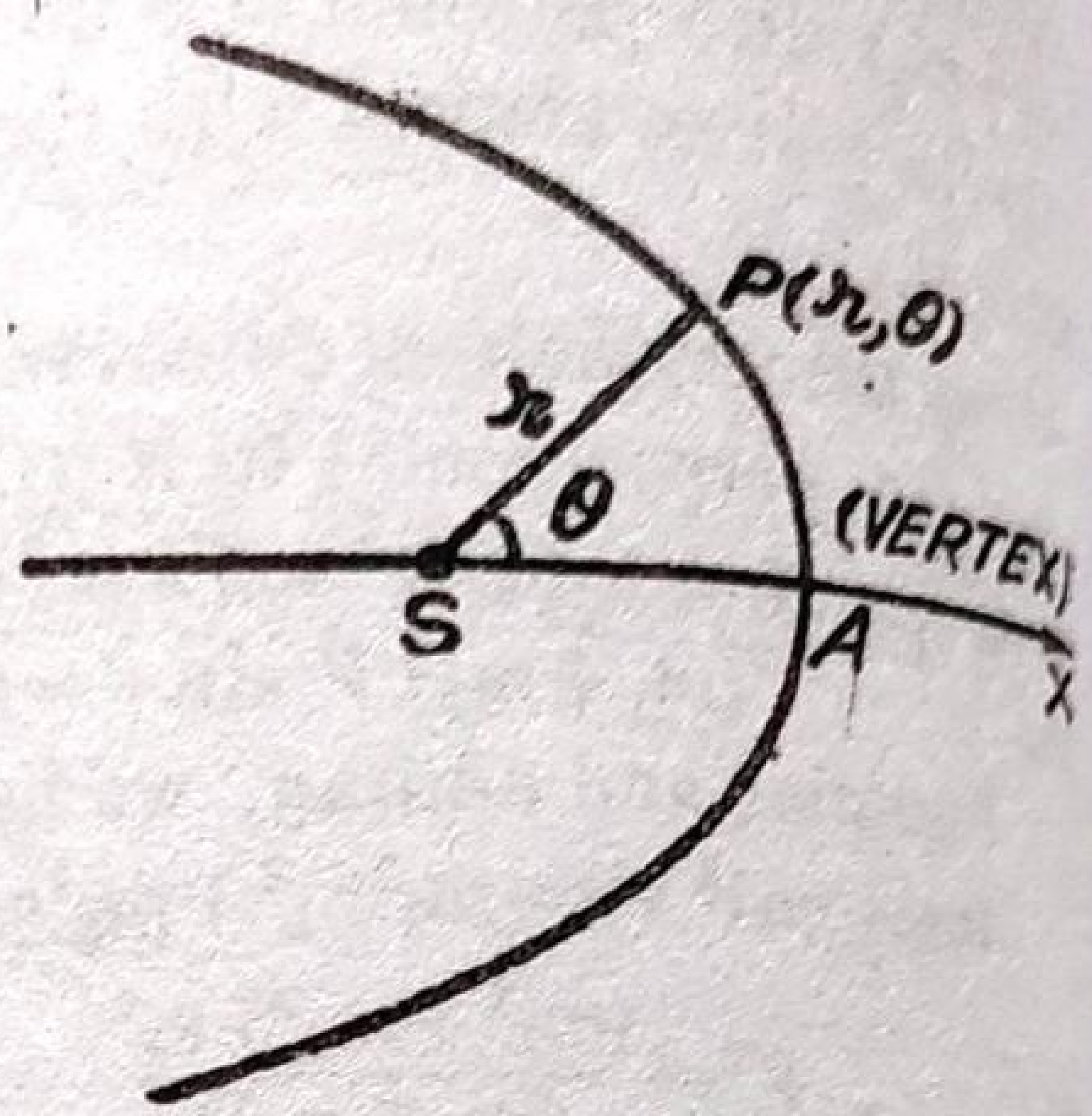
$$= \frac{2}{R}\left[1 - \frac{(m_1^2+m_2^2)}{(m_1+m_2)^2}\right] = \frac{2}{R} \cdot \frac{2m_1 m_2}{(m_1+m_2)^2}.$$

$$\therefore \quad 2a_1 = \frac{(m_1+m_2)^2}{2m_1 m_2} \cdot R,$$

which is the required length of the major axis.

§ 6. **Time of description of an arc of a central orbit.** The time of passing from one point of a central orbit to another is usually determined by the equation $\qquad r^2\,(d\theta/dt) = h.$

§ 7. **To find the time of description of a given arc of a parabolic orbit starting from the vertex.**



The polar equation of a parabola of latus rectum $4a$ referred to the focus S as the pole and the axis as the initial line is

$$2a/r=1+\cos\theta=2\cos^2\tfrac{1}{2}\theta$$

or $$r=a\sec^2\tfrac{1}{2}\theta.$$

But we have $h=r^2\dfrac{d\theta}{dt}$.

$$\therefore\quad dt=\frac{r^2}{h}\,d\theta=\frac{a^2\sec^4\frac{1}{2}\theta}{\sqrt{(\mu l)}}\,d\theta \qquad [\because\ h^2=\mu l]$$

$$=\frac{a^2}{(\sqrt{\mu}\,2a)}\sec^4\tfrac{1}{2}\theta\,d\theta. \qquad [\because\ l=2a]$$

Integrating, the time taken from the vertex A (*i.e.*, $\theta=0$) to the point $P(r,\theta)$ is given by

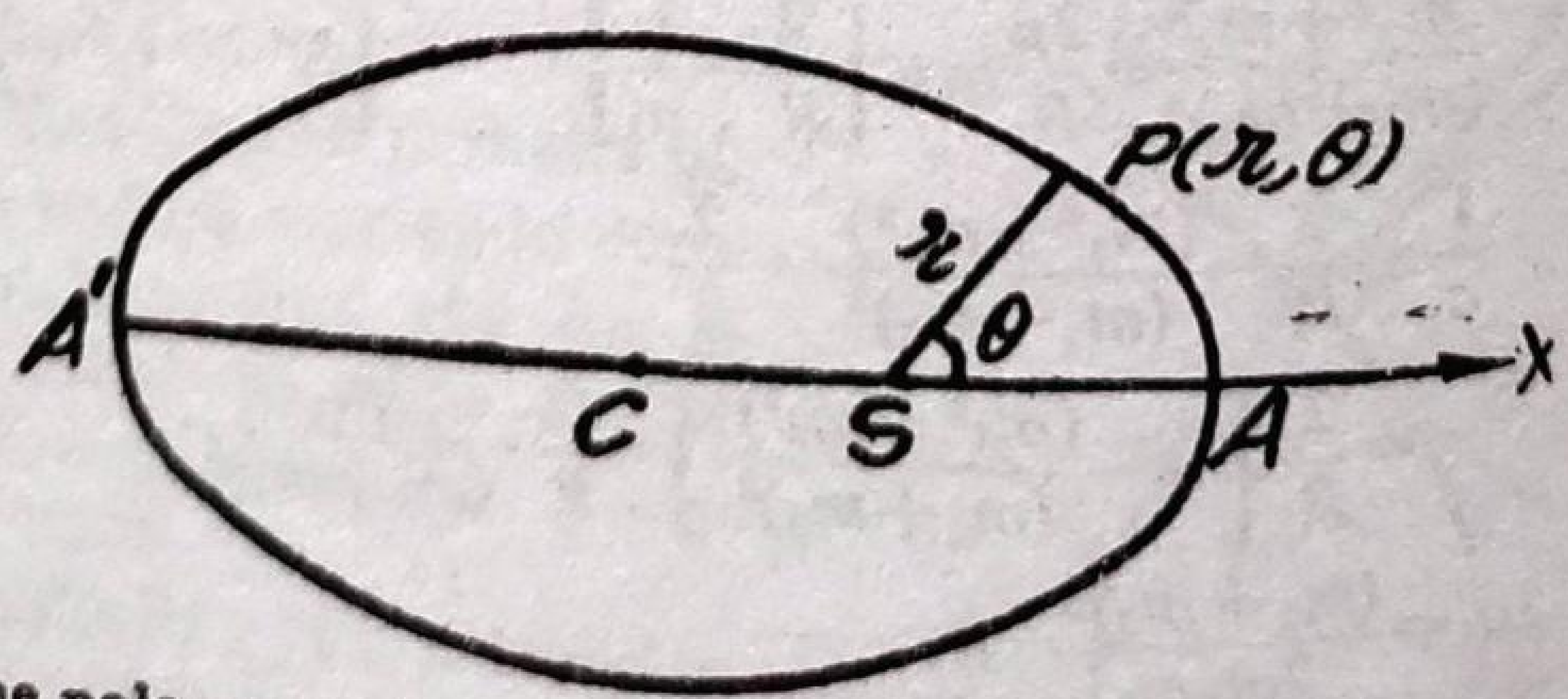
$$t=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\int_{\theta=0}^{\theta}\sec^4\tfrac{1}{2}\theta\,d\theta$$

$$=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\int_0^{\theta}(1+\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\,d\theta$$

$$=\sqrt{\left(\frac{a^3}{2\mu}\right)}.2\left[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta\right]_0^{\theta}$$

or $$t=\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right].$$

§ 8. **To find the time of description of a given arc of an elliptic orbit starting from the nearer end of the major axis.**



The polar equation of an ellipse referred to its focus S as pole and SA as initial line is

$$l/r=1+e\cos\theta, \qquad ...(1)$$

where $e<1$.

Also we have $\quad h=r^2 \dfrac{d\theta}{dt}$

$$\therefore \quad dt=\frac{r^2}{h}\,d\theta=\frac{l^2}{h}\cdot\frac{d\theta}{(1+e\cos\theta)^2}.$$

Integrating, the time from A to P is given by

$$t=\frac{l^2}{h}\cdot\int_{\theta=0}^{\theta}\frac{d\theta}{(1+e\cos\theta)^2}. \qquad (2)$$

To evaluate the integral we proceed as follows :

We have $\quad \dfrac{d}{d\theta}\left(\dfrac{\sin\theta}{1+e\cos\theta}\right)=\dfrac{\cos\theta\,(1+e\cos\theta)-\sin\theta\cdot(-e\sin\theta)}{(1+e\cos\theta)^2}$

$$=\frac{e+\cos\theta}{(1+e\cos\theta)^2}=\frac{e^2+e\cos\theta}{e\,(1+e\cos\theta)^2}$$

$$=\frac{(1+e\cos\theta)-(1-e^2)}{e\,(1+e\cos\theta)^2}$$

$$=\frac{1}{e\,(1+e\cos\theta)}-\frac{(1-e^2)}{e\,(1+e\cos\theta)^2}.$$

$$\therefore \quad \frac{1-e^2}{e\,(1+e\cos\theta)^2}=\frac{1}{e\,(1+e\cos\theta)}-\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right).$$

Integrating both sides, we have

$$\frac{(1-e^2)}{e}\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{1}{e}\int\frac{d\theta}{(1+e\cos\theta)}-\frac{\sin\theta}{1+e\cos\theta}. \quad ...(3)$$

Now $\displaystyle\int\frac{d\theta}{1+e\cos\theta}=\int\frac{d\theta}{1+e\,\{(1-\tan^2\frac{1}{2}\theta)/(1+\tan^2\frac{1}{2}\theta)\}}$

$$=\int\frac{(1+\tan^2\frac{1}{2}\theta)\,d\theta}{(1+e)+(1-e)\tan^2\frac{1}{2}\theta}$$

$$=\frac{1}{(1-e)}\int\frac{\sec^2\frac{1}{2}\theta\,d\theta}{\{(1+e)/(1-e)\}+\tan^2\frac{1}{2}\theta}$$

$$=\frac{2}{1-e}\int\frac{dz}{\{(1+e)/(1-e)\}+z^2}$$

[putting $\tan\frac{1}{2}\theta=z$, so that $\frac{1}{2}\sec^2\frac{1}{2}\theta\,d\theta=dz$]

$$=\frac{2}{1-e}\cdot\frac{1}{\sqrt{\{(1+e)/(1-e)\}}}\cdot\tan^{-1}\left[\frac{z}{\sqrt{\{(1+e)/(1-e)\}}}\right]$$

$$=\frac{2}{\sqrt{(1-e^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}.$$

$\therefore$ from (3), we have

$$\frac{1-e^2}{e}\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{2}{e\sqrt{(1-e^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-\frac{\sin\theta}{1+e\cos\theta}$$

or $$\int \frac{d\theta}{(1+e\cos\theta)^2} = \frac{2}{(1-e^2)^{3/2}} \tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)} \tan\frac{\theta}{2}\right\} - \left(\frac{e}{1-e^2}\right) \frac{\sin\theta}{1+e\cos\theta}.$$

$\therefore$ from (2), we have

$$t = \frac{l^2}{h} \cdot \left[\frac{2}{(1-e^2)^{3/2}} \tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)} \tan\frac{\theta}{2}\right\} - \left(\frac{e}{1-e^2}\right) \frac{\sin\theta}{1+e\cos\theta}\right],$$

$$= \frac{l^2}{\sqrt{(\mu l)}} \cdot \frac{1}{(1-e^2)^{3/2}} \left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)} \tan\frac{\theta}{2}\right\} - e\sqrt{(1-e^2)} \cdot \frac{\sin\theta}{1+e\cos\theta}\right]$$

$$= \frac{l^{3/2}}{\sqrt{\mu}} \cdot \frac{1}{(1-e^2)^{3/2}} \left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)} \tan\frac{\theta}{2}\right\} - e\sqrt{(1-e^2)} \frac{\sin\theta}{1+e\cos\theta}\right].$$

But $l = b^2/a = \{a^2(1-e^2)\}/a = a(1-e^2)$.

$$\therefore \quad t = \frac{a^{3/2}}{\sqrt{\mu}} \left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)} \tan\frac{\theta}{2}\right\} - e\sqrt{(1-e^2)} \cdot \frac{\sin\theta}{1+e\cos\theta}\right].$$

§ 9. To find the time of description of a given arc of a hyperbolic orbit starting from the vertex.

The polar equation of a hyperbola referred to its focus S as pole and SA (where A is the vertex) as initial line is

$$l/r = 1 + e\cos\theta, \qquad \ldots(1)$$

where $e > 1$.

Also, we have $\quad h = r^2 \dfrac{d\theta}{dt}$.

$$\therefore \quad dt = \frac{r^2}{h} d\theta = \frac{l^2}{h} \cdot \frac{d\theta}{(1+e\cos\theta)^2}.$$

Integrating, the time from the vertex A (*i.e.*, $\theta = 0$) to any point $P(r, \theta)$ is given by

$$t = \frac{l^2}{h} \int_0^\theta \frac{d\theta}{(1+e\cos\theta)^2}. \qquad \ldots(2)$$

Now proceeding as in § 8, we have

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right) = \frac{1}{e(1+e\cos\theta)} + \frac{e^2-1}{e(1+e\cos\theta)^2}.$$

[$\because$ here $e > 1$]

$$\therefore \quad \frac{e^2-1}{e(1+e\cos\theta)^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right) - \frac{1}{e(1+e\cos\theta)}.$$

Integrating both sides, we have

$$\left(\frac{e^2-1}{e}\right)\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{e}\int\frac{d\theta}{(1+e\cos\theta)} \quad \ldots(3)$$

Now $$\int\frac{d\theta}{(1+e\cos\theta)}=\int\frac{d\theta}{1+e\{(1-\tan^2\frac{1}{2}\theta)/(1+\tan^2\frac{1}{2}\theta)\}}$$

$$=\int\frac{(1+\tan^2\frac{1}{2}\theta)\,d\theta}{(e+1)-(e-1)\tan^2\frac{1}{2}\theta}$$

$$=\frac{1}{(e-1)}\int\frac{\sec^2\frac{1}{2}\theta\,d\theta}{\{(e+1)/(e-1)\}-\tan^2\frac{1}{2}\theta}$$

$$=\frac{2}{(e-1)}\int\frac{dz}{\{(e+1)/(e-1)\}-z^2},$$

putting $\tan\frac{1}{2}\theta=z$ so that $\frac{1}{2}\sec^2\frac{1}{2}\theta\,d\theta=dz$

$$=\frac{2}{(e-1)}\cdot\frac{1}{2\sqrt{\{(e+1)/(e-1)\}}}\log\left[\frac{\sqrt{\{(e+1)/(e-1)\}}+z}{\sqrt{\{(e+1)/(e-1)\}}-z}\right]$$

$$=\frac{1}{\sqrt{(e^2-1)}}\log\left[\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right].$$

$\therefore$ from (3), we have

$$\left(\frac{e^2-1}{e}\right)\int\frac{d\theta}{(1+e\cos\theta)^2}$$

$$=\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{e}\cdot\frac{1}{\sqrt{(e^2-1)}}\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}$$

or $$\int\frac{d\theta}{(1+e\cos\theta)^2}$$

$$=\left(\frac{e}{e^2-1}\right)\cdot\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{3/2}}\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}.$$

$\therefore$ from (2), we have

$$t=\frac{l^2}{\sqrt{(\mu l)}}\left[\frac{e}{e^2-1}\cdot\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{3/2}}\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}\right]_0^\theta$$

$$=\frac{l^{3/2}}{\sqrt{\mu}}\cdot\frac{1}{(e^2-1)^{3/2}}\left[e\sqrt{(e^2-1)}\cdot\frac{\sin\theta}{1+e\cos\theta}-\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}\right].$$

But $l=b^2/a=\{a^2(e^2-1)\}/a=a(e^2-1)$.

$$\therefore \quad t=\frac{a^{3/2}}{\sqrt{\mu}}\left[e\sqrt{(e^2-1)}\,\frac{\sin\theta}{1+e\cos\theta}\right.$$
$$\left.-\log\left\{\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right\}\right]$$

Illustrative Examples

Ex. 17. *Prove that in a parabolic orbit the time taken to move from the vertex to a point distant r from the focus is*

$$\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{(2r-l)},$$

where 2l is the latus rectum.

Sol. For figure refer § 7, page 20.

The polar equation of a parabola of latus rectum $2l$ referred to the focus S as the pole and the axis SA, where A is the vertex, as the initial line is $\quad l/r=1+\cos\theta=2\cos^2\frac{1}{2}\theta$

or $\quad r=\frac{1}{2}l\sec^2\frac{1}{2}\theta. \qquad \ldots(1)$

But we have $r^2(d\theta/dt)=h$.

$$\therefore \quad dt=\frac{r^2}{h}\,d\theta=\frac{\frac{1}{4}l^2\sec^4\frac{1}{2}\theta}{\sqrt{(\mu l)}}\,d\theta \qquad [\because \quad h^2=\mu l]$$
$$=\tfrac{1}{4}(l^{3/2}/\sqrt{\mu})\sec^4\tfrac{1}{2}\theta\,d\theta.$$

Integrating, the time taken from the vertex (*i.e.*, $\theta=0$) to the point $P(r,\theta)$ is given by

$$t=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^\theta \sec^4\tfrac{1}{2}\theta\,d\theta$$
$$=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^\theta (1+\tan^2\tfrac{1}{2}\theta)\sec^2\tfrac{1}{2}\theta\,d\theta$$
$$=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\int_0^\theta [\sec^2\tfrac{1}{2}\theta+2(\tan^2\tfrac{1}{2}\theta)(\tfrac{1}{2}\sec^2\tfrac{1}{2}\theta)]\,d\theta$$
$$=\frac{1}{4}\frac{l^{3/2}}{\sqrt{\mu}}\left[2\tan\tfrac{1}{2}\theta+2\cdot\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta\right]_0^\theta$$
$$=\tfrac{1}{2}(l^{3/2}/\sqrt{\mu})[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta]$$
$$=\tfrac{1}{6}(l^{3/2}/\sqrt{\mu})\tan\tfrac{1}{2}\theta.(3+\tan^2\tfrac{1}{2}\theta).$$

But from (1), $\sec^2\frac{1}{2}\theta=2r/l$.

$\therefore \quad 1+\tan^2\frac{1}{2}\theta=2r/l \quad$ or $\quad \tan^2\frac{1}{2}\theta=(2r/l)-1=(2r-l)/l.$

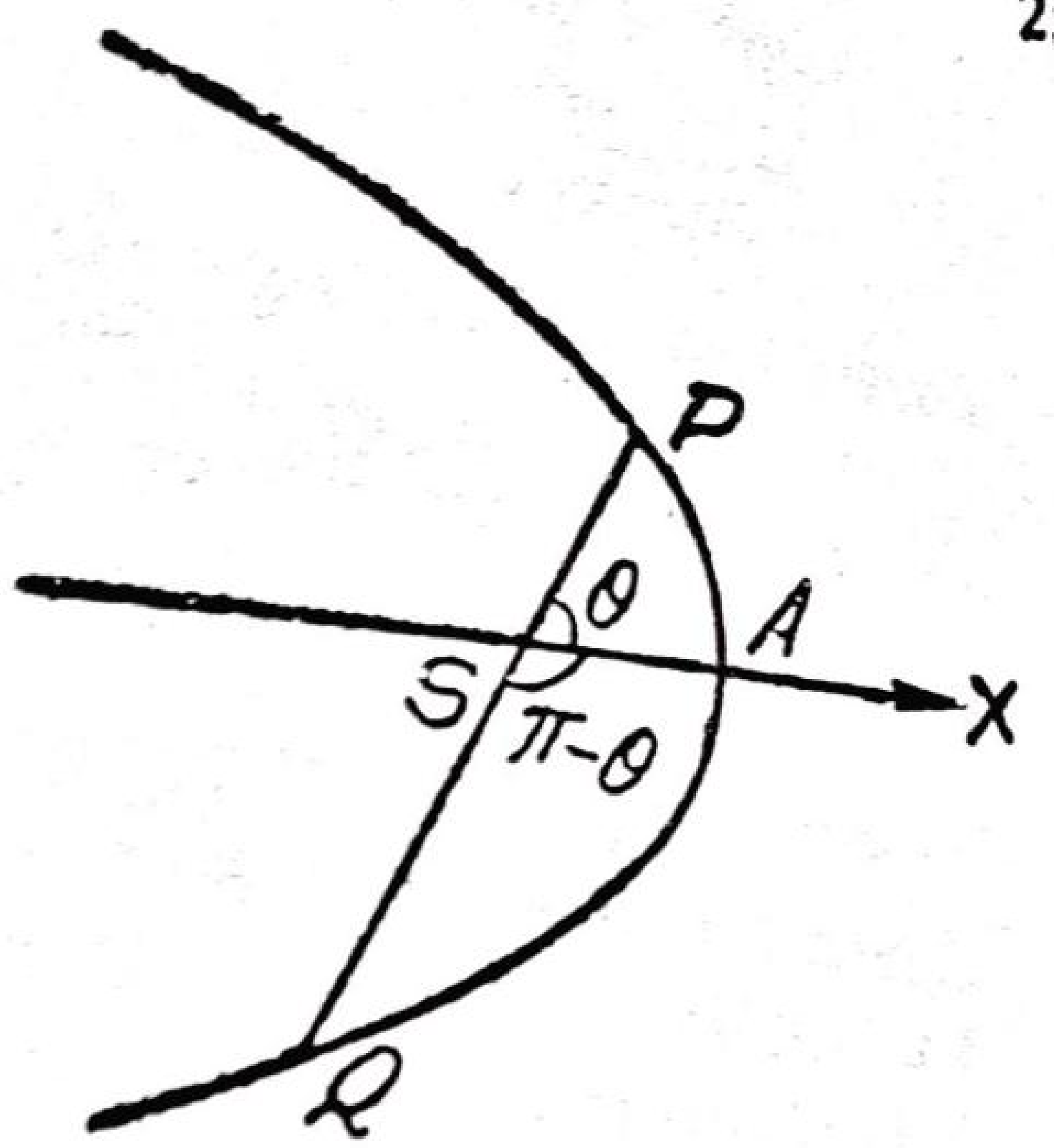
$$\therefore \quad t=\tfrac{1}{6}(l^{3/2}/\sqrt{\mu})\cdot\sqrt{\{(2r-l)/l\}}\cdot\{3+(2r-l)/l\}$$
$$=\frac{1}{6}\,\frac{l^{3/2}\sqrt{(2r-l)}\cdot(2l+2r)}{\sqrt{\mu}\;l^{3/2}}=\frac{1}{3\sqrt{\mu}}(r+l)\sqrt{(2r-l)}.$$

Ex. 18. *Show that if the time of describing an arc bounded by a focal chord of a parabolic orbit under Newtonian law be T, then*

$$T\propto(\text{focal chord})^{3/2}.$$

Sol. Let PSQ be a focal chord of a parabolic orbit described by a particle under Newtonian law. The equation of the parabola referred to the focus S as pole is

$$2a/r=1+\cos\theta. \quad \ldots(1)$$



If θ is the vectorial angle of the point P, then the vectorial angle of the point Q (the other end of the focal chord PSQ) can be taken as $-(\pi-\theta)$.

Proceeding as in § 7, the time T for describing the arc QAP bounded by the focal chord QSP is given by

$$T=\sqrt{\left(\frac{2a^3}{\mu}\right)}\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}\right]_{-(\pi-\theta)}^{\theta}$$

$$=\sqrt{\left(\frac{2a^3}{\mu}\right)}\cdot\left[\tan\frac{\theta}{2}+\frac{1}{3}\tan^3\frac{\theta}{2}-\left\{\tan\left(\frac{-(\pi-\theta)}{2}\right)+\tfrac{1}{3}\tan^3\left(\frac{-(\pi-\theta)}{2}\right)\right\}\right]$$

$$=\sqrt{(2a^3/\mu)}\,[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta+\tan(\tfrac{1}{2}\pi-\tfrac{1}{2}\theta)+\tfrac{1}{3}\tan^3(\tfrac{1}{2}\pi-\tfrac{1}{2}\theta)]$$

$$[\because \ \tan(-\theta)=-\tan\theta]$$

$$=\sqrt{(2a^3/\mu)}\,[\tan\tfrac{1}{2}\theta+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta+\cot\tfrac{1}{2}\theta+\tfrac{1}{3}\cot^3\tfrac{1}{2}\theta]$$

$$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\,[\tan^3\tfrac{1}{2}\theta+\cot^3\tfrac{1}{2}\theta+3\tan\tfrac{1}{2}\theta\cot\tfrac{1}{2}\theta\,(\tan\tfrac{1}{2}\theta+\cot\tfrac{1}{2}\theta]$$

$$[\because \ \tan\tfrac{1}{2}\theta\cot\tfrac{1}{2}\theta=1]$$

$$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\tan\tfrac{1}{2}\theta+\cot\tfrac{1}{2}\theta)^3$$

$$[\because \ (a+b)^3=a^3+b^3+3ab\,(a+b)]$$

$$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\left[\frac{\sin^2\frac{1}{2}\theta+\cos^2\frac{1}{2}\theta}{\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta}\right]^3$$

$$=\tfrac{1}{3}\sqrt{(2a^3/\mu)}\cdot\frac{8}{(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta)^3}$$

$$=\frac{8}{3}\sqrt{\left(\frac{2a^3}{\mu}\right)}\frac{1}{\sin^3\theta}. \quad \ldots(2)$$

Now from (1), we have $r=2a/(1+\cos\theta)$.

Since the vectorial angles of the points P and Q are θ and $-(\pi-\theta)$ respectively, therefore

$$SP=\frac{2a}{1+\cos\theta} \text{ and } SQ=\frac{2a}{1+\cos\{-(\pi-\theta)\}}=\frac{2a}{1-\cos\theta}.$$

$\therefore$ length of the focal chord $QSP = SP + SQ$

$$= \frac{2a}{1+\cos\theta} + \frac{2a}{1-\cos\theta} = 2a\,\frac{2}{1-\cos^2\theta} = \frac{4a}{\sin^2\theta}.$$

Thus the focal chord $PQ = \dfrac{4a}{\sin^2\theta}$.

$$\therefore \quad \frac{1}{\sin^2\theta} = \frac{\text{chord } PQ}{4a} \quad \text{or} \quad \frac{1}{\sin^3\theta} = \frac{(\text{chord } PQ)^{3/2}}{(4a)^{3/2}}.$$

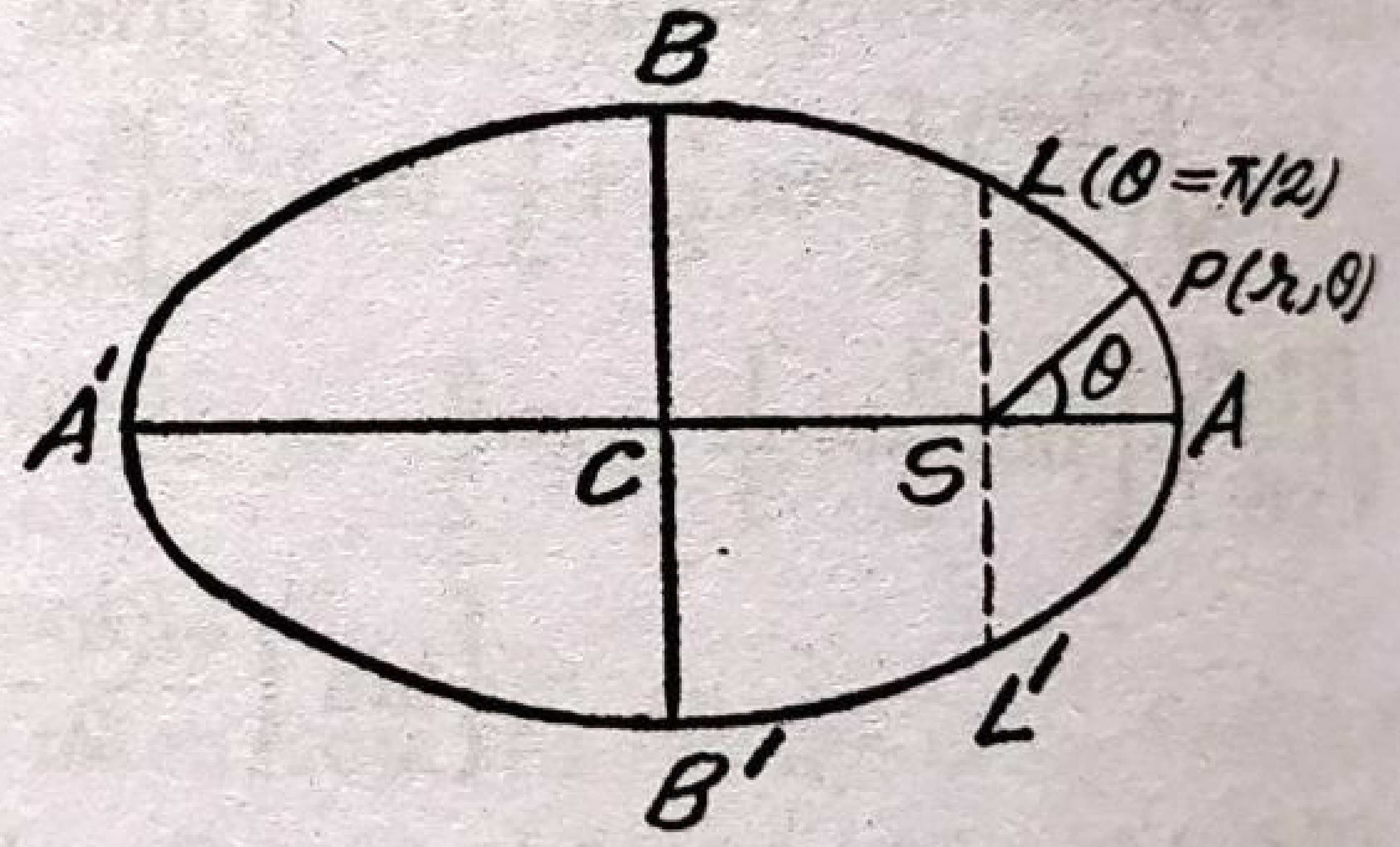
Substituting the value of $1/\sin^3\theta$ in (2), we get

$$T = \tfrac{8}{3}\sqrt{\left(\frac{2a^3}{\mu}\right)}\frac{(\text{chord } PQ)^{3/2}}{(4a)^{3/2}} = \tfrac{1}{3}\sqrt{\left(\frac{2}{\mu}\right)}.(\text{chord } PQ)^{3/2}.$$

Hence $T \propto$ (focal chord)$^{3/2}$.

Ex. 19. *Prove that the times taken to describe two portions into which an ellipse is divided by the latus rectum through the centre of force are in a ratio*

$$\{\cos^{-1} e - e\sqrt{(1-e^2)}\} : \{\pi - \cos^{-1} e + e\sqrt{(1-e^2)}\}.$$



Sol. In § 8 we have proved that the time of description t of an arc extending from the vertex A to the point $P(r, \theta)$ is given by

$$t = \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\} - e\sqrt{(1-e^2)}\cdot\frac{\sin\theta}{1+e\cos\theta}\right] \quad ...(1)$$

Let LL' be the latus rectum through the centre of force S. At the point L, $\theta = \pi/2$.

Substituting $\theta = \pi/2$ in (1), the time t_1 from A to L is given by

$$t_1 = \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{4}\pi\right\} - e\sqrt{(1-e^2)}\cdot\frac{\sin\frac{1}{2}\pi}{1+e\cos\frac{1}{2}\pi}\right]$$

$$= \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\right\} - e\sqrt{(1-e^2)}\right].$$

Now $2\tan^{-1}\sqrt{\left(\dfrac{1-e}{1+e}\right)} = 2\tan^{-1}\sqrt{\left(\dfrac{1-\cos\alpha}{1+\cos\alpha}\right)}$

by putting $e = \cos\alpha$

$$= 2\tan^{-1}\tan\tfrac{1}{2}\alpha = 2\times(\tfrac{1}{2}\alpha) = \alpha = \cos^{-1} e.$$

$\therefore$ $t_1 = (a^{3/2}/\sqrt{\mu})\,[\cos^{-1} e - e\sqrt{(1-e^2)}]$.

$\therefore$ the time T_1 taken to describe the arc

$$L'AL = 2t_1 = (2a^{3/2}/\sqrt{\mu})\,[\cos^{-1} e - e\sqrt{(1-e^2)}].$$

Also the time T taken to describe the whole elliptic path

$$= (2\pi a^{3/2})/\sqrt{\mu}.$$

$\therefore$ the time T_2 taken to describe the arc $LA'L'$

$$=T-T_1=\frac{2\pi a^{3/2}}{\sqrt{\mu}}-\frac{2a^{3/2}}{\sqrt{\mu}}[\cos^{-1}e-e\sqrt{(1-e^2)}]$$

$$=\frac{2a^{3/2}}{\sqrt{\mu}}[\pi-\cos^{-1}e+e\sqrt{(1-e^2)}].$$

$\therefore$ the required ratio of the times taken to describe the two portions (arc $L'AL$ and arc $LA'L'$) into which an ellipse is divided by the latus rectum through the centre of force S is

$$T_1:T_2=\{\cos^{-1}e-e\sqrt{(1-e^2)}\}:\{\pi-\cos^{-1}e+e\sqrt{(1-e^2)}\}.$$

Ex. 20. *If the period of a planet be 365 days and eccentricity be 1/60, then show that the times of describing two halves of the orbit bounded by the latus rectum through the centre of force are* $\frac{365}{2}\left[1\pm\frac{1}{15\pi}\right]$ *nearly.* **[Meerut 1971, 75]**

Sol. Refer figure of Ex. 19.

Let S be the focus and LSL' the latus rectum of the ellipse.

From § 8, the time of description t of the arc AP extending from the vertex A to any point P is given by

$$t=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{\theta}{2}\right\}-e\sqrt{(1-e^2)}\cdot\frac{\sin\theta}{1+e\cos\theta}\right] \quad \text{..(1)}$$

Putting $\theta=\pi/2$ in (1), the time of description t_1 of the arc AL is given by

$$t_1=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\tfrac{1}{4}\pi\right\}-e\sqrt{(1-e^2)}\cdot\frac{\sin\frac{1}{2}\pi}{1+e\cos\frac{1}{2}\pi}\right]$$

$$=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\left(1-\frac{1}{60}\right)^{1/2}\left(1+\frac{1}{60}\right)^{-1/2}\right\}-\frac{1}{60}\cdot\left\{1-\left(\frac{1}{60}\right)^2\right\}^{1/2}\cdot 1\right]$$

$$=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\left(1-\frac{1}{2}\cdot\frac{1}{60}\right)\left(1-\frac{1}{2}\cdot\frac{1}{60}\right)-\frac{1}{60}\cdot\left\{1-\frac{1}{2}\cdot\left(\frac{1}{60}\right)^2\right\}\right\}\right],$$

neglecting higher powers of (1/60)

$$=\frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left(1-\frac{1}{60}\right)-\frac{1}{60}\right], \quad \text{...(2)}$$

neglecting second and higher powers of 1/60 as 1/60 is small.

Now since $\tan^{-1}1=\pi/4$, therefore $\tan^{-1}(1-1/60)=\frac{1}{4}\pi-z$, where z is small.

Then $1-1/60=\tan(\frac{1}{4}\pi-z)=(1-\tan z)/(1+\tan z)$

$$=(1-\tan z)(1+\tan z)^{-1}$$

$$=(1-\tan z)(1-\tan z),$$

neglecting higher powers of $\tan z$

$$=1-2\tan z,$$ neglecting second and higher powers of $\tan z$.

$\therefore$ $\tan z=\frac{1}{120}$.

But if z is very small, then $\tan z = z$.

$\therefore \quad z = \frac{1}{120}$ (approximately).

$\therefore \quad \tan^{-1}(1 - \frac{1}{60}) = \frac{1}{4}\pi - \frac{1}{120}$.

Also given that the periodic time $\dfrac{2\pi a^{3/2}}{\sqrt{\mu}} = 365$ days.

$$\therefore \quad \frac{a^{3/2}}{\sqrt{\mu}} = \frac{365}{2\pi}.$$

Substituting in (2), the time of description t_1 of the arc AL is given by $\quad t_1 = \dfrac{365}{2\pi} \cdot \left[2\left(\dfrac{\pi}{4} - \dfrac{1}{120}\right) - \dfrac{1}{60}\right] = \dfrac{365}{4}\left[1 - \dfrac{1}{15\pi}\right]$.

$\therefore$ the time of description of the arc $L'AL$

$$= 2t_1 = \frac{365}{2}\left[1 - \frac{1}{15\pi}\right].$$

Also the time of description of the remaining arc $LA'L$

$= 365$—time of description of the arc $L'AL$

$$= 365 - \frac{365}{2}\left[1 - \frac{1}{15\pi}\right] = \frac{365}{2}\left[1 + \frac{1}{15\pi}\right].$$

Hence the required times are

$$\frac{365}{2}\left[1 \pm \frac{1}{15\pi}\right] \text{ nearly.}$$