# LINEAR ALGEBRA

## : CSE - 2015 :

① Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$

→ Let $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$. Then $A^2 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}$

$$A^3 = \begin{bmatrix} x^2 & xy+yz \\ 0 & z^2 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^3 & x^2y+xyz+yz^2 \\ 0 & z^3 \end{bmatrix} = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}.$$

$\therefore x^3 = 8,\ z^3 = 27,\quad x^2y + xyz + yz^2 = -57$

$\Rightarrow x = 2,\ z = 3,\quad 4y + 6y + 9y = -57$

$y = -3$

$\therefore A = \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}.$

② Let G be a linear operator on $\mathbb{R}^3$ define as $G(x,y,z) = (2y+z,\ x-4y,\ 3x)$. Find the matrix representation of G relative to the basis $S = \{(1,1,1),(1,1,0),(1,0,0)\}$

→ Let $(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$

$\Rightarrow (x,y,z) = (a+b+c,\ a+b,\ a) \quad \Rightarrow \quad a = z$ — ①

$a+b = y$

$b = y - z$ — ②

$a+b+c = x \Rightarrow c = x - a - b$

$\Rightarrow c = x - z - y + z$ [from ① & ②]

$\Rightarrow c = x - y.$

$\therefore (x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

$G(1,1,1) = (3,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 6(1,0,0)$

$G(1,1,0) = (2,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 5(1,0,0)$

$G(1,0,0) = (0,1,3) = 3(1,1,1) + (-2)(1,1,0) + (-1)(1,0,0)$

$\therefore$ Matrix of G is $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

(3) Suppose U and W are distinct four dimensional subspaces of a vectorspace V, where dim V=6. Find the possible dimensions of $U \cap W$.

$\longrightarrow$ $\dim U = 4$, $\dim W = 4$. But U and W are distinct. Hence, $\dim U \cap W \neq 4$

$\dim(U \cap W) \leq 4$.

$\therefore \dim(U \cap W) < 4$ — (1)

Now: $\dim U + \dim W - \dim U \cap W = \dim U+W$

$\Rightarrow \quad 4 + 4 - \dim U \cap W = \dim U+W$

$\Rightarrow \quad \dim U \cap W = 8 - \dim U+W$ — (2)

WKT $\quad \dim U \leq \dim U+W \leq \dim V$

$\Rightarrow \quad 4 \leq \dim U+W \leq 6$

$\Rightarrow \quad -4 \geq -\dim U+W \geq -6$

$\Rightarrow \quad 8-4 \geq 8 - \dim U+W \geq 8-6$

$\Rightarrow \quad 4 \geq \dim U \cap W \geq 2$

But $\dim(U \cap W) < 4$ from (1).

$\therefore 2 \leq \dim U \cap W < 4 \quad \Rightarrow \quad \dim(U \cap W) = 2 \text{ or } 3$

(4) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$x+2y-3z=a, \quad 2x+6y-11z=b, \quad x-2y+7z=c$

→ Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Then, the given system of equations can be written as $AX = B$.

Now:

① For unique solution: Rank $(A)$ = Rank $(A|B)$ = No of unknowns

② For infinitely many solutions: Rank $(A)$ = Rank$(A|B)$ < No. of unknown

Aug. matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{array}\right]$

Converting it into echelon form:

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$ $\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{array}\right]$ $\sim$ $R_3 \to R_3 + 2R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & c-a+2(b-2a) \end{array}\right]$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & -5a+2b+c \end{array}\right]$

For the solution to exist:

$\rho(A) = \rho(A|B)$ which is only possible iff

$-5a + 2b + c = 0 \Rightarrow \boxed{5a = 2b + c}$ which is the required condition.

⑤ Find the minimal polynomial of the matrix $A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

→ Characteristic equation of $A$ is given by $|A - \lambda I| = 0$.

$$\begin{vmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)\left[(3-\lambda)(-3-\lambda) + 8\right] + 2(6(3-\lambda) - 12) + 2(-12 + 3(3+\lambda)) = 0$$

$$\Rightarrow (4-\lambda)[\lambda^2 - 1] + 2[6 - 6\lambda] + 2[-3 + 3\lambda] = 0$$

$$\Rightarrow 4\lambda^2 - \lambda^3 - 4 + \lambda + 6 - 6\lambda = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$\Rightarrow (\lambda - 2)(\lambda - 1)^2 = 0$$

Possibilities for minimal polynomials are:

$\lambda - 2 = 0$, $\lambda - 1 = 0$, $\lambda^2 - 3\lambda + 2 = 0$, $(\lambda - 1)^2 = 0$ and

$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$

$(A-2I) \neq 0$, $(A-I) \neq 0$. $(A-I) = \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}$.

$$(A-I)^2 = \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 4 \\ 3 & -2 & 2 \end{bmatrix} \neq 0.$$

$$(A^2-3A+2I) = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} - 3\begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ Minimal polynomial is $x^2-3x+2=0$

(6) Find a 3x3 orthogonal matrix whose first two rows are $[1/3, 2/3, 2/3]$ and $[0, 1/\sqrt{2}, -1/\sqrt{2}]$.

$\rightarrow$ Let $A = [1/3, 2/3, 2/3]$, $B = [0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$

Let the third row be C. Then C is given by

$C = \dfrac{A \times B}{|A \times B|}$ since C is normal to A and B and has unit length.

$$A \times B = \begin{vmatrix} i & j & k \\ 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{vmatrix} = \hat{i}\left[\frac{-2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}}\right] + \hat{j}\left[\frac{1}{3\sqrt{2}}\right] + \hat{k}\left[\frac{1}{3\sqrt{2}}\right]$$

$$= \frac{-4}{3\sqrt{2}}\hat{i} + \frac{1}{3\sqrt{2}}\hat{j} + \frac{1}{3\sqrt{2}}\hat{k}$$

$$|A \times B| = \sqrt{\left(\frac{-4}{3\sqrt{2}}\right)^2 + \left(\frac{1}{3\sqrt{2}}\right)^2 + \left(\frac{1}{3\sqrt{2}}\right)^2} = \sqrt{\frac{16}{18} + \frac{1}{18} + \frac{1}{18}} = \pm 1$$

$$\therefore C = \pm \frac{\left[-\frac{4}{3\sqrt{2}}\hat{i} + \frac{1}{3\sqrt{2}}\hat{j} + \frac{1}{3\sqrt{2}}\hat{k}\right]}{1} = \pm\left[\frac{-4}{3\sqrt{2}} \quad \frac{1}{3\sqrt{2}} \quad \frac{1}{3\sqrt{2}}\right]$$

$\therefore$ Required matrix $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -4/3\sqrt{2} & 1/3\sqrt{2} & 1/3\sqrt{2} \end{bmatrix}$ or $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 4/3\sqrt{2} & -1/3\sqrt{2} & -1/3\sqrt{2} \end{bmatrix}$