206/250

Date: 14/10/20

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
INSTITUTE OF MATHEMATICAL SCIENCES

MAINS TEST SERIES-2020

(JULY *to* DEC.-2020)

IAS/IFoS

MATHEMATICS

Under the guidance *of* K. Venkanna

PDE, NUMERICAL ANALYSIS & COMP. PROG. AND MECHANICS & FD

TEST CODE: TEST-4: IAS(M)/02-AUG.-2020

Time: 3 Hours | Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 31 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Divyanshu Choudhary

Roll No. 1367

Test Centre: Jaipur

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate
Divyanshu

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

SECTION - A

1. (a) Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation. [10]

We have

Equation of cones having vertex at origin is given by

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad (1)$$

Differentiating (1) w.r.t. x & y, we get

$$2ax + 2czp + 2fyp + 2gxp + 2gz + 2hy = 0 \quad (2)$$

$$2by + 2czq + 2fyq + 2fz + 2gqx + 2hx = 0 \quad (3)$$

$$\left[p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right]$$

Multiplying (2) by x & (3) by y and adding,

$$ax^2 + by^2 + px(cz + fy + gx) + qy(cz + fy + gx) + gxz + hxy + fyz + hxy = 0 \quad (4)$$

Substituting value of $ax^2 + by^2 + 2hxy$ from (1) in (4)

$$\Rightarrow (px + qy - z)(cz + fy + gx) = 0$$

$\because cz + fy + gx \neq 0 \Rightarrow px + qy = z$ is the differential equation

08

Now, $yz + zx + xy = 0$

$\Rightarrow \cancel{p = \frac{\partial z}{\partial x}} = yp + px + z + y = 0$ & $\Rightarrow p = \frac{-y-z}{x+y}$.

$yq + z + qx + x = 0$

$q = \frac{-x-z}{x+y} \Rightarrow \left(\frac{-y-z}{x+y}\right)x - \left(\frac{x+z}{x+y}\right)y = z$

$\therefore yz + xz + xy$ satisfies above equation

1. (b) Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$. [10]

We have

First we find CF of above equation

$\Rightarrow D^2 - DD' - 2D'^2 = 0 \Rightarrow (D-2D')(D+D') = 0$

$\Rightarrow CF = \phi_1(y+2x) + \phi_2(y-x) = 0$

Now we find PI of above function.

$\Rightarrow \dfrac{1}{D^2 - DD' - 2D'^2}\left[(2x-y)(x+y)\sin xy - \cos xy\right]$

$\Rightarrow \dfrac{1}{(D-2D')(D+D')}\left[(2x-y)(x+y)\sin xy - \cos xy\right]$

$\Rightarrow \dfrac{1}{(D-2D')}\int\left[\{(2x-c-x)(x+c+x)\sin x(c+x) - \cos x(c+x)\right] dx$

$[y = c+x]$

$\Rightarrow \dfrac{1}{D-2D'}\int\left[\{(x-c)(2x+c)\sin(cx+x^2) - \cos(cx+x^2)\right] dx$

$\Rightarrow \dfrac{1}{D-2D'}\left[(x-c)\overset{-\cos(cx+x^2)}{\sin(cx+x^2)} + \int \cos(cx+x^2)\,dx - \int \cos(cx+x^2)\,dx\right]$

$\Rightarrow \dfrac{-1}{D-2D'}\left[(2x-y)\cos xy\right]$

$\Rightarrow \int (y-2x)\cos xy\,dx$ & $y = c'-2x$

$\Rightarrow \int (c'-2x-2x)\cos x(c'-2x)\,dx = \sin x(c'-2x)$

$= \sin xy$

$\Rightarrow$ Total solution $= \boxed{\phi_1(y+2x) + \phi_2(y-x) + \sin xy}$

Scanned by TapScanner

1. (c) By using Newton's forward interpolation formula, find the number of men getting wages between Rs. 10 and 15 from the following date. [10]

Wages in Rs.	0 – 10	10 – 20	20 – 30	30 – 40
No. of Men	9	30	35	42

We have

Wages	y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
10	9			
20	39	30		
30	74	35	7	
40	116	42	7	0

08

Here we ~~cal~~ have to calculate $f(15)$

Now $h = \frac{15-10}{10} = \frac{5}{10} = 0.5$

$\Rightarrow f(15) = f(10) + 0.5\,\Delta y_0 + \frac{(0.5)(0.5-1)}{2}\Delta^2 y_0$

$\Rightarrow f(15) = 9 + 0.5(30) - \frac{0.25}{2}(7)$

$= 24 - 0.875 = 23.125$

$\Rightarrow f(15) \simeq 24$

So no. of men getting wages b/w ₹ 10 & 15 are $24 - 9 = 15$

1. (d) Give a Boolean expression for the following statements:
(i) Y is a 1 only if A is a 1 and B is a 1 or if A is a 0 and B is a 0.
(ii) Y is a 1 only if A, B and C are all 1s or if only one of the variables is a 0.**[10]**

(i) We have

$AB + \bar{A}\bar{B}$

$$Y = AB + \bar{A}\bar{B} = A \oplus B$$

where $\oplus$ represents XOR function

(ii) $Y = ABC + \bar{A}BC + A\bar{B}C + AB\bar{C}$

$\Rightarrow Y = ABC + ABC + ABC + \bar{A}BC + A\bar{B}C + AB\bar{C}$

$[\because ABC + ABC + ABC = ABC]$

$\Rightarrow Y = AB(C+\bar{C}) + AC(B+\bar{B}) + BC(A+\bar{A})$

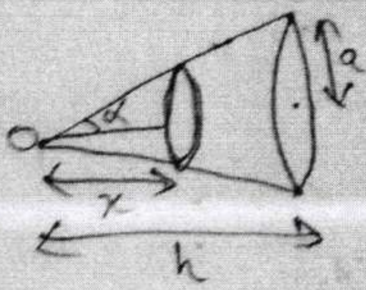
$\Rightarrow Y = AB + BC + CA$

$[\because A+\bar{A} = B+\bar{B} = C+\bar{C} = 1]$

08

1. (e) Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis. [10]

To find the given MI we construct a ring of thickness δx at a distance x from vertex O.



Now $M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha$

where ρ is density of cone & $a = h\tan\alpha$

Now $\delta m = \rho\pi x^2 \tan^2\alpha\, \delta x$

$$\Rightarrow MI = \int_0^h \rho\pi x^2 \tan^2\alpha \cdot x^2 \tan^2\alpha\, \delta x$$

$$\Rightarrow MI = \int_0^h \rho\pi x^4 \tan^4\alpha\, \delta x = \frac{3}{10} Ma^2$$

Hence MI of given cone is $\frac{3}{10} Ma^2$

08

4. (a) Find the integral surface of the partial differential equation $(x-y)p + (y-x-z)q = z$ through the circle $z = 1$, $x^2 + y^2 = 1$. [10]

We have

$$(x-y)p + (y-x-z)q = z$$

$$\Rightarrow \frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z}$$

08

$$\Rightarrow \frac{dx+dy+dz}{(x-y)+(y-x-z)+z} = 0 \Rightarrow x+y+z = c_1 \quad (1)$$

$$\Rightarrow x+z = c_1 - y$$

$$\Rightarrow \frac{dy}{y-(c_1-y)} = \frac{dz}{z} \Rightarrow \frac{dy}{2y-c_1} = \frac{dz}{z}$$

$$\Rightarrow \frac{2y-c_1}{z^2} = c_2 \Rightarrow \frac{y-x-z}{z^2} = c_2 \quad (2)$$

So general solution is given by

$$\phi\left[(x+y+z), \left(\frac{y-x-z}{z^2}\right)\right] = 0$$

Now by given conditions

$$z = 1,\ x = t,\ y = \sqrt{1-t^2}$$

Substituting in (1) & (2), we get

$$t + \sqrt{1-t^2} + 1 = c_1 \ \& \ \sqrt{1-t^2} - t - 1 = c_2$$

$$x+y = c_1 - 1 \ \& \ y - x = c_2 + 1$$

$$\Rightarrow (x+y)^2 + (y-x)^2 = (c_1-1)^2 + (c_2+1)^2$$

$$\Rightarrow 2 = c_1^2 - 2c_1 + c_2^2 + 2c_2 + 2$$

$$\Rightarrow c_1^2 - 2c_1 + c_2^2 + 2c_2 = 0$$

$$\Rightarrow (x+y+z)^2 - 2(x+y+z) + \left(\frac{y-x-z}{z^2}\right)^2 + 2\left(\frac{y-x-z}{z^2}\right) = 0$$

$$\Rightarrow z^4(x+y+z)^2 + (y-x-z)^2 - 2z^4(x+y+z) + 2z^2(y-x-z) = 0$$

4. (b) Convert the following to the base indicated against each:
(i) $(266.375)_{10}$ to base 8
(ii) $(341.24)_5$ to base 10
(iii) $(43.3125)_{10}$ to base 2
(iv) Draw the circuit diagram for
$\bar{F} = A\bar{B}C + \bar{C}B$
using NAND to NAND logic long. **[12]**

(i)

8	266	
8	33	2
8	4	1
	0	4

$0.375 \times 8 = 3$

$\Rightarrow (266.375)_{10} = (412.3)_8$

(ii) $(341.24)_5$ to base 10

$\Rightarrow 3\times5^2 + 4\times5 + 1\times5^0 + 2\times5^{-1} + 4\times5^{-2} = (96.56)_{10}$

(iii)

2	43	
2	21	1
2	10	1
2	5	0
2	2	1
2	1	0
	0	1

$0.3125 \times 2 = 0.625$ 0
$0.625 \times 2 = 1.25$ 1
$0.25 \times 2 = 0.5$ 0
$0.5 \times 2 = 1$ 1

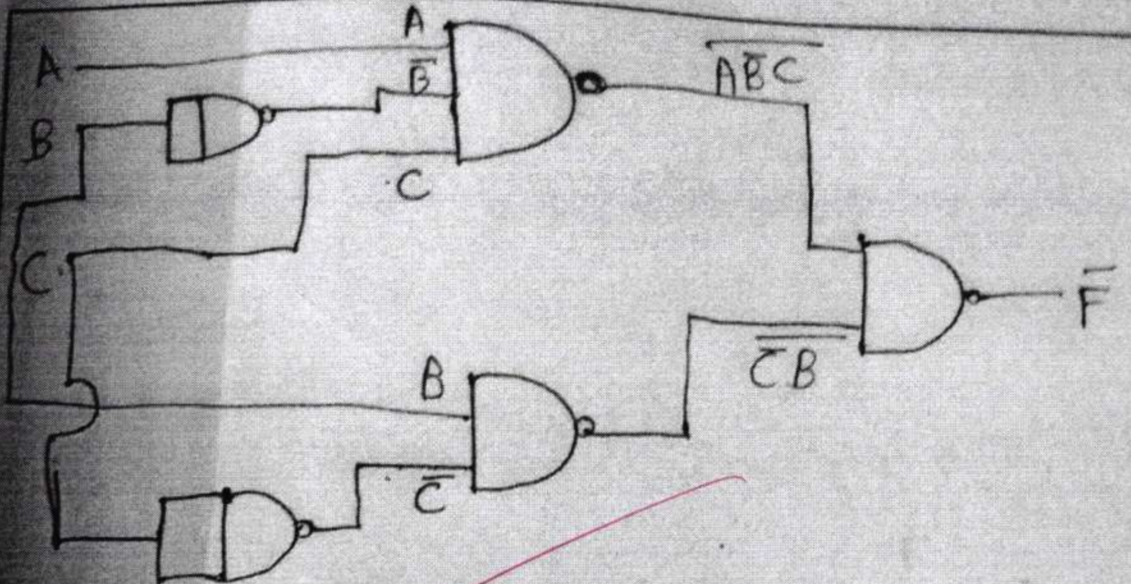
$\Rightarrow (43.3125)_{10} = (101011.0101)_2$

10

(iv) We have

$\bar{F} = A\bar{B}C + \bar{C}B$

$\Rightarrow \bar{\bar{F}} = \overline{A\bar{B}C + \bar{C}B} = \overline{A\bar{B}C} \cdot \overline{\bar{C}B}$



4. (c) For given equidistant values u_{-1}, u_0, u_1 and u_2, a value is interpolated by Lagrange's formula. Show that it may be written in the form

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0,$$

where $x + y = 1$. **[13]**

We have

Given condition is $u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1}$

$+ \frac{x(x^2-1)}{3!}\Delta^2 u_0$ & $x+y=1$ — (1)

Now $\Delta u_0 = u_1 - u_0$ & $\Delta u_{-1} = u_0 - u_{-1}$

$\Rightarrow \Delta^2 u_0 = \Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0$ &

$\Delta^2 u_{-1} = \Delta u_0 - \Delta u_{-1} = u_1 - 2u_0 + u_{-1}$

Substituting these values in (1), we get

$u_x = (1-x)u_0 + xu_1 + (1-x)\frac{[(1-x)^2-1]}{3!}\begin{bmatrix} u_1 - 2u_0 \\ + u_{-1} \end{bmatrix}$

$+ \frac{x(x^2-1)}{3!}[u_2 - 2u_1 + u_0]$

$$\Rightarrow u_x = (1-x)u_0 + xu_1 + \frac{(1-x)}{6}(x^2-2x)(u_1 - 2u_0 + u_{-1}) + \frac{x(x^2-1)}{6}(u_2 - 2u_1 + u_0)$$

Collecting like terms together, we get

$$u_x = \left[(1-x) - \frac{(1-x)}{3}(x^2-2x) + \frac{x(x^2-1)}{6}\right]u_0$$

$$+ \left[x + \frac{(1-x)}{6}(x^2-2x) - \frac{1}{3}x(x^2-1)\right]u_1$$

$$+ \frac{x(x^2-1)}{6}u_2 + \left[\frac{(1-x)}{6}(x^2-2x)\right]u_{-1}$$

$$\Rightarrow u_x = \frac{x(x-1)(x-2)}{-6}u_{-1} + \frac{(x+1)(x-1)(x-2)}{2}u_0$$

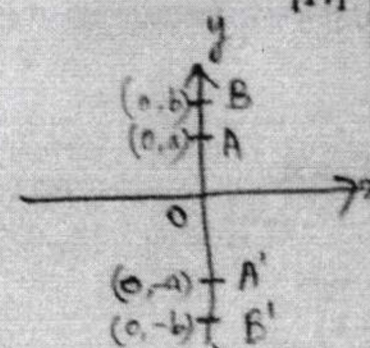
$$+ \frac{(x+1)(x)(x-2)}{-2}u_1 + \frac{(x+1)(x)(x-1)}{6}u_2$$

which is the Lagrange interpolation formula

Hence the given equation ① ~~can also~~ is also one of the forms of Lagrange interpolation formula

4. (d) If the fluid fills the region of space on the positive side of x-axis, is a rigid boundary, and if there be a source + m at the point (0, a), and an equal sink at (0, b), and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m^2 (a-b)^2/ab(a+b)$, where ρ is the density of the fluid. **[17]**

By the given figure, we calculate complex potential of the system



$\Rightarrow W = -m\log(z-ia) - m\log(z+ia)$
$+ m\log(z+ib) + m\log(z-ib)$

$\Rightarrow W = -m\log(z^2+a^2) + m\log(z^2+b^2)$

Now = $q = \dfrac{dW}{dz}$

$\Rightarrow q = \dfrac{-m}{z^2+a^2}(2z) + \dfrac{m}{z^2+b^2}(2z)$

$\Rightarrow q = \dfrac{2mz\,(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$ — ①

By Bernoulli's Theorem

$\dfrac{p}{\rho} + \dfrac{1}{2}q^2 = 0$ [where p is total pressure & ρ is density of fluid]

By initial conditions

$\dfrac{p-p_0}{\rho} + \dfrac{1}{2}q^2 = 0$ $\Rightarrow$ Total pressure (P) $= \int \frac{1}{2}\rho q^2\, dz$

$$\Rightarrow p = \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2 z^2 (a^2-b^2)^2}{(z^2+a^2)^2(z^2+b^2)^2}\, dz$$

[Substituting value of q from ①]

$$\Rightarrow p = 2\rho m^2 (a^2-b^2)^2 \int_{-\infty}^{\infty} \frac{z^2}{(z^2+a^2)^2(z^2+b^2)^2}$$

$$= 4\rho m\, (a^2-b^2)^2 \int_{0}^{\infty} \frac{z^2}{(z^2+a^2)^2(z^2+b^2)^2}\, dz$$

$$\Rightarrow \boxed{p = \frac{\pi \rho m^2 (a-b)^2}{ab(a+b)}}$$ is the resultant pressure on the boundary

SECTION – B

5. (a) Use Lagrange's method to solve the equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0$$

where z = z (x, y). **[10]**

-08

We have

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ p & q & -1 \end{vmatrix} = 0 \Rightarrow x(-\beta - q\gamma) + y(p\gamma + \alpha) + z(q\alpha - p\beta) = 0$$

$$\left[p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}\right] \Rightarrow p(y\gamma - z\beta) + q(z\alpha - x\gamma) = x\beta - y\alpha$$

which is of form $pP + qQ = R$

$$\Rightarrow \frac{dx}{y\gamma - z\beta} = \frac{dy}{z\alpha - x\gamma} = \frac{dz}{x\beta - y\alpha}$$

$$\Rightarrow \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{\alpha y\gamma - \alpha z\beta + z\alpha\beta - x\beta\gamma + x\beta\gamma - y\alpha\gamma} = \frac{\alpha\,dx + \beta\,dy + \gamma\,dz}{0}$$

$\Rightarrow \alpha x + \beta y + \gamma z = C_1$ is one solution

Also $\dfrac{x\,dx + y\,dy + z\,dz}{xy\gamma - xz\beta + yz\alpha - xy\gamma + xz\beta - yz\alpha} =$

$\dfrac{x\,dx + y\,dy + z\,dz}{0} \Rightarrow x^2 + y^2 + z^2 = C_2$ is another solution

So solutions of above given equation is given by $\phi\left[(\alpha x + \beta y + cz),\ (x^2 + y^2 + z^2)\right] = 0$

5. (b) Find a complete integral of $(x^2 + y^2)(p^2 + q^2) = 1$. **[10]**

We have

$(x^2 + y^2)(p^2 + q^2) - 1 = 0$ —— ①

Applying Charpit's equations, we get

$$\frac{f_p}{2xp^2 + 2xq^2} = \frac{f_q}{2yp^2 + 2yq^2} = \frac{f_x}{-[2x^2p + 2y^2p]} = \frac{f_y}{-[2x^2q + 2y^2q]}$$

$\Rightarrow \dfrac{x f_p + y f_q + p f_x + q f_y}{0} = 0$

$\Rightarrow d(px) + d(qy) = 0 \Rightarrow d(px + qy) = 0$

$\Rightarrow px + qy = a \Rightarrow p = \dfrac{a - qy}{x}$ —— ①

Now $dz = p\,dx + q\,dy$ —②

Substituting value of p from ① in ②, we get

Let $x = r\cos\theta$
$y = r\sin\theta$

$dz = \frac{a - qy}{x}dx + q\,dy$ $\Rightarrow x^2 + y^2 = r^2$

$\Rightarrow \frac{dz - q\,dy}{a - qy} = \frac{dx}{x}$ & $\theta = \tan^{-1}(y/x)$

$\Rightarrow p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\times\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}$

$\Rightarrow q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$

Substituting in ①, we get

$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = 1 \Rightarrow P^2 + Q^2 = 1$

$\Rightarrow \boxed{Z = a/2 \log(x^2+y^2) + 1/a\tan^{-1}(y/x) + C}$

0/8

5. (c) Solve the following system of equations by Gauss-Jordan elimination method :

$x_1 + x_2 + x_3 = 3$
$2x_1 + 3x_2 + x_3 = 6$
$x_1 - x_2 - x_3 = -3$ **[10]**

By applying Gauss-Jordan elimination method, we find inverse of matrix of coefficients of x_1, x_2 & x_3

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right]$$

$\begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$ $(R_3 \to R_3 + 2R_2)$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right]$$

$(R_1 \to R_1 - R_2)$ $\qquad$ $(R_1 \to 2R_1 + R_3)$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3/4 & 1/2 & -1/4 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3/4 & 1/2 & -1/4 \\ 0 & 0 & 1 & 5/4 & -1/2 & -1/4 \end{array}\right]$$

~~$(R_2 \to R_3 - 4R_2)$~~

$(R_2 \to R_2 - \frac{1}{4}R_3)$ $\qquad$ $[R_3 \to R_3/-4]$

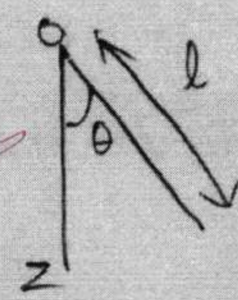
$$\Rightarrow [x_1, x_2, x_3] = \begin{bmatrix} 1 & 0 & 1 \\ -3/4 & 1/2 & -1/4 \\ 5/4 & -1/2 & -1/4 \end{bmatrix} [3 \; 6 \; -3]$$

$$= [0, 3/2, 3/2]$$

So $\boxed{x_1 = 0 \quad x_2 = 3/2 \quad x_3 = 3/2}$

5. (d) Use Hamilton's equations to find the equation of motion of the simple pendulum **[10]**

Let l be length & M be mass of simple pendulum



$$\Rightarrow K.E.\,(T) = \frac{1}{2} M (l\dot{\theta})^2$$

$$= \frac{1}{2} M l^2 \dot{\theta}^2$$

$$\Rightarrow P.E.\,(V) = Mgl(1-\cos\theta)$$

$$\Rightarrow L = T - V = \frac{1}{2} M l^2 \dot{\theta}^2 - Mgl(1-\cos\theta) \quad \text{—} (1)$$

$$\Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{Ml^2}$$

Now by Hamilton equation

$\Rightarrow H = T + V = \frac{1}{2} M l^2 \dot{\theta}^2 + Mgl(1-\cos\theta)$ — ②

Substituting $\dot{\theta}$ from ① in ②, we get

$\Rightarrow H = \frac{1}{2} Ml^2 \left(\frac{p_\theta}{Ml^2}\right)^2 + Mgl(1-\cos\theta)$

$\Rightarrow \dot{p}_\theta = \frac{-\partial H}{\partial \theta} = -Mgl\sin\theta$ — ③

$\Rightarrow p_\theta = Ml^2\dot{\theta} \Rightarrow \dot{p}_\theta = Ml^2\ddot{\theta}$ — ④

$\Rightarrow$ Equating ③ & ④, we get

$Mgl\sin\theta = -Ml^2\ddot{\theta} \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$

~~If~~ ~~Since~~ θ is small $\Rightarrow \sin\theta \sim \theta$

$\Rightarrow \boxed{\ddot{\theta} = -g/l\,\theta}$

5. (e) Show that the velocity potential $\phi = (a/2)\times(x^2 + y^2 - 2z^2)$ satisfies the Laplace equation. Also determine the streamlines. [10]

We have

$\phi = a/2 \times (x^2 + y^2 - 2z^2)$

Laplace equation is given by $\nabla^2\phi = 0$

$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$

98

$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}(ax) = a, \quad \frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}(ay) = a$

$\& \frac{\partial^2\phi}{\partial z^2} = \frac{\partial}{\partial z}(-2az) = -2a$

$\Rightarrow a + a - 2a = 0 \Rightarrow \phi$ satisfies Laplace equation

Now $u = \frac{-\partial\phi}{\partial x} = -ax, \quad v = \frac{-\partial\phi}{\partial y} = -ay$

$\& z = \frac{-\partial\phi}{\partial z} = 2az$

Now streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2z}$$

Considering first 2 fractions, we get

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \boxed{x = c_1 y} \text{ (1)} \quad (c_1 \text{ is a constant})$$

Considering last 2 fractions, we get

$$\frac{dy}{y}(-2) = \frac{dz}{z} \Rightarrow \boxed{\frac{1}{y^2} = c_2 z} \text{ (2)} \quad (c_2 \text{ is a constant})$$

So (1) & (2) give the required streamlines

6. (a) Obtain the partial differential equation governing the equations
$\phi(u, v) = 0$, $u = xyz$,
$v = x + y + z$. [06]

We have

$\phi(u, v) = 0$ where $u = xyz$, $v = x+y+z$

Differentiating wrt x & y, we get

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial x} + p\frac{\partial u}{\partial z}\right] = 0$$

$\left[\text{Here } p = \frac{\partial z}{\partial x} \text{ \& } q = \frac{\partial z}{\partial y}\right]$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left[yz + pxy\right] + \frac{\partial \phi}{\partial v}\left[1 + p\right] = 0 \quad \text{— (1)}$$

Similarly

$$\frac{\partial \phi}{\partial u}\left[\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right] + \frac{\partial \phi}{\partial v}\left[\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial z}\right] = 0$$

$\Rightarrow \frac{\partial \phi}{\partial u}[xz+qxy] + \frac{\partial \phi}{\partial v}[1+q] = 0$ — ②

Calculating values of $\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}}$ from ① & ②

& equating them, we get

05

$\frac{1+p}{yz+pxy} = \frac{1+q}{xz+qxy} \Rightarrow xz+pxz+\cancel{pqxy}+qxy = yz+yqz+pxy+\cancel{pqxy}$

$\Rightarrow p(xz-xy) + q(xy-yz) = \cancel{-z(x+y)}\ z(y-x)$

$\Rightarrow px(z-y) + qy(x-z) = \cancel{-z(x+y)}\ z(y-x)$

6. (b) Solve $(D^2 - DD' - 2D)z = \sin(3x + 4y) - e^{2x+y} + x^2y$. **[13]**

We have

First we calculate CF of above function

$\Rightarrow D^2 - DD' - 2D = 0$

$\Rightarrow$ General solution is of form $z = \sum Ae^{hx+ky}$

$\Rightarrow h^2 - hk - 2k = 0 \Rightarrow k = \frac{h^2}{h+2}$

So $CF = \sum Ae^{hx+\frac{h^2}{h+2}y}$

11

Now to find PI of above ~~so~~ equation

$\Rightarrow \frac{1}{D^2-DD'-2D}\left[\sin(3x+4y) - e^{2x+y} + x^2y\right]$

$= \frac{1}{D^2-DD'-2D}\sin(3x+4y) - \frac{1}{D^2-DD'-2D}e^{2x+y} + \frac{1}{D^2-DD'-2D}x^2y$

$$\Rightarrow \frac{1}{D^2-DD'-2D}\sin(3x+4y) = \frac{1}{-9+12-2D}\sin(3x+4y)$$

$$= \frac{1}{3-2D}\sin(3x+4y) = \frac{3+2D}{9-4D^2}\sin(3x+4y)$$

$$= \frac{3\sin(3x+4y)}{45} + \frac{6\cos(3x+4y)}{45}$$

$$= \frac{1}{15}\left[\sin(3x+4y) + 2\cos(3x+4y)\right]$$

$$\Rightarrow \frac{-1}{D^2-DD'-2D}e^{2x+y} = e^{2x+y}\frac{-1}{(D+2)^2-(D+2)(D'+1)-2(D+2)}$$

$$= -e^{2x+y}\frac{1}{D^2+D-DD'-2D'-2}e^{0x+0y} = \frac{e^{2x+y}}{2}$$

$$\Rightarrow \frac{1}{D^2-DD'-2D}x^2y = \frac{-1}{2D}\left[1-\left(\frac{D}{2}-\frac{D'}{2}\right)\right]^{-1}x^2y$$

$$= \frac{-1}{2D}\left[1+\frac{D-D'}{2}+\frac{(D-D')^2}{4}+\frac{(D-D')^3}{8}\right]x^2y$$

$$= \frac{-1}{2D}\left[x^2y+\frac{2xy-x^2}{2}+\frac{2y-4x}{4}+\left(\frac{-3}{8}\right)(2)\right]$$

$$= \frac{-x^3y}{6} - \frac{1}{4}\left(\cancel{\frac{x^2y}{2}}\; x^2y - \frac{x^3}{3}\right) - \frac{1}{8}\left(2xy-2x^2\right) + \frac{3}{8}x$$

So total solution is

$$Z = CF + PI = \sum A e^{hx+\frac{h^2}{h+2}y} + \frac{1}{15}\left[\sin(3x+4y) + 2\cos(3x+4y)\right]$$

$$- \frac{e^{2x+y}}{2} - \frac{x^3y}{6} - \frac{1}{4}\left(x^2y-\frac{x^3}{3}\right) - \frac{1}{8}\left(2xy-2x^2\right) + \frac{3}{8}x$$

6. (c) Reduce $r + 2xs + x^2 t = 0$ to canonical form [13]

We have, $r + 2xs + x^2 t = 0$ — ①

For conversion to canonical form, we find roots of characteristic question

$$\Rightarrow \lambda^2 + 2x\lambda + x^2 = 0 \Rightarrow (\lambda + x)^2 = 0 \Rightarrow \lambda = -x, -x$$

$$\Rightarrow \frac{dy}{dx} - x = 0 \Rightarrow y - \frac{x^2}{2} = c_1$$ & other root can be taken as $y + \frac{x^2}{2} = c_2$ as $W \neq 0$

$$\Rightarrow \alpha = y - \frac{x^2}{2} \text{ \& } \beta = y + \frac{x^2}{2}$$

$$\Rightarrow p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x\left(\frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta}\right)$$

$$\Rightarrow q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta}$$

$$\Rightarrow r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[x\left(\frac{\partial z}{\partial \beta} - \frac{\partial z}{\partial \alpha}\right)\right] = \frac{\partial z}{\partial \beta} - \frac{\partial z}{\partial \alpha} +$$
$$x\left[\frac{\partial}{\partial \alpha}\frac{\partial z}{\partial \beta}\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} - \frac{\partial}{\partial \alpha}\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} - \frac{\partial}{\partial \beta}\frac{\partial z}{\partial \alpha}\frac{\partial \beta}{\partial x}\right] = x^2\left[\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} - 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right] + \frac{\partial z}{\partial \beta} - \frac{\partial z}{\partial \alpha}$$

$$\Rightarrow t = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta}\right] = \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + 2\frac{\partial^2 z}{\partial \alpha \partial \beta}$$

08

$$\Rightarrow s = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta}\right] = x\left[\frac{\partial^2 z}{\partial \beta^2} - \frac{\partial^2 z}{\partial \alpha^2}\right]$$

Substituting these values in ①, we get

$$x^2\left[\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} - 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right] + 2x^2\left[\frac{\partial^2 z}{\partial \beta^2} - \frac{\partial^2 z}{\partial \alpha^2}\right] +$$
$$x^2\left[\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + 2\frac{\partial^2 z}{\partial \alpha \partial \beta}\right] = 0 \Rightarrow \frac{\partial^2 z}{\partial \beta^2} = 0$$

$$\Rightarrow \frac{\partial z}{\partial \beta} = \phi_1(\beta) + c \Rightarrow z = \int \phi_1(\beta) + c\phi_2\beta + c_1$$

(where c, c_1 are const.)

6\. (d) A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity v given by $v = kx$ if $0 \le x \le l/2$ and $v = k(l-x)$ if $l/2 \le x \le l$. Find the displacement function $y(x, t)$. [18]

We have,

string function is given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial x^2}$$

Let $y(x,t) = X(x)\,T(t)$

$$\Rightarrow \cancel{X(x)}\ X(x)T''(t) = \frac{1}{c^2} X''(x)T(t)$$

$$\Rightarrow \frac{1}{c^2}\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = K \text{ (say)}$$

$$\Rightarrow X''(x) - c^2 K X(x) = 0 \ \& \ T''(t) - KT(t) = 0$$

Solving these equations simultaneously & removing unwanted conditions, we obtain

$$y(x,t) = \sum_{n=0}^{\infty}\left[E_n \cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

Now by given conditions

$v = kx,\ 0 \le x \le l/2$ and $v = k(l-x)$, $l/2 \le x \le l$

So by ~~Lagrange's~~ these equations, $E_n = 0$

& we calculate F_n by

-14

$$\Rightarrow F_n = \frac{2}{n\pi C}\left[\int_0^{l/2} kx \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^{l} k(l-x) \sin\left(\frac{n\pi x}{l}\right) dx\right]$$

$$\Rightarrow F_n = \frac{2}{n\pi C}\left[\int_0^{l/2} \frac{kl}{n\pi} \cos\left(\frac{n\pi x}{l}\right) dx - \int_{l/2}^{l} \frac{kl}{n\pi} \cos\left(\frac{n\pi x}{l}\right) dx\right]$$

$$\Rightarrow F_n = \frac{2}{n\pi C}\left[\frac{kl^2}{n^2\pi^2}(-1)^n + \frac{kl^2}{n^2\pi^2}(-1)^n\right]$$

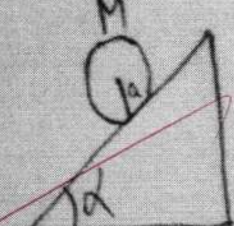
$$\Rightarrow F_n = \frac{4kl^2}{n^3\pi^3 C} \quad \text{when } n = (2m-1)$$

So $$\boxed{y(x,t) = \frac{4kl^2}{n^3\pi^3 C}\sum_{n=1}^{\infty} \sin\frac{(2m-1)\pi ct}{l} \times \sin\left[\frac{(2m-1)\pi x}{l}\right]}$$

5\. (a) A sphere of radius a and mass M rolls down a rough plane inclined at an angle α to the horizontal.

If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equation. [16]

We have



Kinetic energy of sphere (T)

$= \frac{1}{2} I \omega^2 + \frac{1}{2} a^2 \dot{\theta}^2$

where I is moment of inertia of sphere & ω is angular velocity

$\Rightarrow T = \frac{1}{2} M \left(\frac{2}{5} a^2 \dot{\theta}^2\right) + \frac{1}{2} M a^2 \dot{\theta}^2 = \frac{7}{10} M \dot{x}^2$

$[\because \dot{x} = a\dot{\theta}]$

$\Rightarrow V = -Mgx \sin\alpha$

$\Rightarrow L = T - V = \frac{7}{10} M\dot{x}^2 + Mgx \sin\alpha$

Scanned by TapScanner

Now by Hamilton's equation

$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5} M\dot{x} \Rightarrow \dot{x} = \frac{5p_x}{7M}$ — ①

Now by Hamilton's equation

$H = T + V = \frac{7}{10} M\dot{x}^2 - Mgx\sin\alpha$

Substituting value of $\dot{x}$ from ①, we get

$H = \frac{7}{10} M \left(\frac{5p_x}{7M}\right)^2 - Mgx\sin\alpha$

$\Rightarrow H = \frac{5}{14} \frac{p_x^2}{M} - Mgx\sin\alpha$

$\Rightarrow \dot{p}_x = \frac{-\partial H}{\partial x} = Mg\sin\alpha$ — ②

From ①, we get

$p_x = \frac{7}{5} M\dot{x} \Rightarrow \dot{p}_x = \frac{7}{5} M\ddot{x}$ — ③

Equating ② & ③, we get

$\frac{7}{5} M\ddot{x} = Mg\sin\alpha$

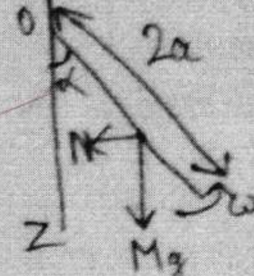
$\Rightarrow \ddot{x} = \frac{5}{7} g\sin\alpha$

So acceleration of sphere ($\ddot{x}$) is given by $\boxed{\frac{5}{7} g\sin\alpha}$

Scanned by TapScanner

(b) A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, find the value of α. **[16]**

Here we first assume mass δm & ~~them~~ of rod & then calculate torque at O to find α



$\Rightarrow$ Balancing torque at O, we get

$$\int_0^{2a} \frac{M}{2a}\delta x\, \omega^2 x^2 \sin\alpha \cos\alpha - Mga\sin\alpha = 0$$

$\left[\text{where } \delta m = \frac{M}{2a}\delta x \text{ \& MI about OZ} = \frac{M}{2a}\delta x \sin\alpha\cos\alpha\, x^2\right]$

$$\Rightarrow \frac{M}{2a}\omega^2 \sin\alpha\cos\alpha\left[\frac{x^3}{3}\right]_0^{2a} - Mga\sin\alpha = 0$$

$$\Rightarrow \sin\alpha\left[\frac{M}{2a}\omega^2\cos\alpha \times \frac{8a^3}{3} - Mga\right] = 0$$

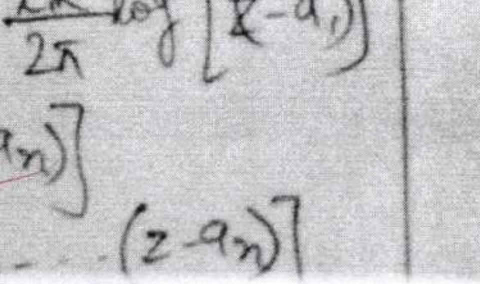
$$\Rightarrow \sin\alpha = 0 \text{ or } \frac{4Ma^2}{3}\omega^2\cos\alpha = Mga$$

$$\Rightarrow \sin\alpha = 0 \text{ or } \cos\alpha = \frac{3g}{4a\omega^2}$$

$$\therefore \boxed{\alpha = 0 \text{ or } \cos^{-1}\left(\frac{3g}{4a\omega^2}\right)}$$

8. (c) If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity at any point of the liquid. [18]

We first calculate complex potential of the entire system



$$\Rightarrow W = \frac{ik}{2\pi}\log\left[(z-a_0)\right] + \frac{ik}{2\pi}\log\left[(z-a_1)\right]$$

$$+ \cdots \frac{ik}{2\pi}\log\left[(z-a_n)\right]$$

$$\Rightarrow W = \frac{ik}{2\pi}\log\left[(z-a_0)(z-a_1)\cdots(z-a_n)\right]$$

$$= \frac{ik}{2\pi}\log\left[z^n - a^n\right]$$

16

$\Rightarrow W' = W - \frac{ik}{2\pi}\log(z-a_0) = \frac{ik}{2\pi}\log\left(\frac{z^n-a^n}{z-a_0}\right)$

Since $W' = \phi' + i\psi'$

$\Rightarrow \psi' = \frac{k}{4\pi}\Big[\log\{(r^n\cos n\theta - a^n)^2 + (r^n\sin n\theta)^2\}$
$- \log\{(r\cos\theta - a_0)^2 + (r\sin\theta)^2\}\Big]$

Now differentiating ψ wrt r & θ, we get

$\Rightarrow \frac{d\psi'}{dr} = \frac{k}{4\pi}\left[\left\{\frac{1\times 2[nr^{n-1}\cos n\theta + nr^{n-1}\sin n\theta]}{(r^n\cos n\theta - a^n)^2 + (r^n\sin n\theta)^2}\right\} - \left\{\frac{1\times 2[\cos\theta + \sin\theta]}{(r\cos\theta - a_0)^2 + (r\sin\theta)^2}\right\}\right] = \frac{k(n-1)}{4\pi a}$

$\Rightarrow \frac{d\psi'}{d\theta} = 0$ [By L Hospital rule]

Now $q = \sqrt{\left(\frac{d\psi'}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi'}{\partial\theta}\right)^2}$

$\Rightarrow q = \frac{\partial\psi'}{\partial r} = \frac{k(n-1)}{4\pi a}$

$\Rightarrow$ Time required $= \frac{2\pi a}{q}$

$$\Rightarrow \boxed{t = \frac{2\pi a}{\frac{k(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{k(n-1)}}$$