Maines Test Series -2021
Test-11 - Paper-I
Answer key (Batch I)
full length Test.

1(a) For the matrix A below, compute the dimension of the null space of A, dim (N(A)).

$$A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}$$

Sol'n: Given $A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}$

Row reduce A

$$A \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ (Row reduced echelon form)

So $r = 3$ for this matrix.

Then

$$\dim(N(A)) = n(A)$$
$$= (n(A) + r(A)) - r(A)$$
$$= 5 - r(A)$$
$$= 5-3$$
$$= 2$$

1(b) (i) Suppose that A is a square matrix. Prove that the constant term of the characteristic polynomial of A is equal to the determinant of A.

(ii) Suppose that A is a square matrix. Prove that a single vector may not be an eigen vector of A for two different eigen values.

Sol^n: (i) Suppose that the characteristic polynomial of A is

$$P_A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Then
$$a_0 = a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n$$
$$= P_A(0)$$
$$= \det(A - 0I_n)$$
$$= \det(A)$$

(ii) Suppose that the vector $x \neq 0$ is an eigen vector of A for the two eigen values λ and ρ, where $\lambda \neq \rho$. Then $\lambda - \rho \neq 0$, and we also have

$$0 = Ax - Ax$$
$$= \lambda x - \rho x$$
$$= (\lambda - \rho)x$$
$$\Rightarrow \lambda - \rho = 0 \text{ or } x = 0$$

which are both contradictions.

1.(c) Use the transformation $x = u+\frac{1}{2}v$, $y=v$ to evaluate the integral

$$\int_0^2\int_{y/2}^{(y+4)/2} y^3(2x-y)\,e^{(2x-y)^2}\,dx\,dy.$$

Sol: we have $x=u+\frac{v}{2}$, $y=v$.

$\therefore \frac{\partial x}{\partial u}=1$, $\frac{\partial x}{\partial v}=\frac{1}{2}$, $\frac{\partial y}{\partial u}=0$, $\frac{\partial y}{\partial v}=1$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} 1 & 1/2 \\ 0 & 1\end{vmatrix} = 1$$

$\therefore dx\,dy = J\,du\,dv = 1\,du\,dv = du\,dv.$

Here the region of integration is bounded by the lines $x=y/2$, $x=\frac{y+4}{2}$, $y=0$ and $y=2$

Changing these equations to new variables u and v by using the relations

$x=u+\frac{v}{2}$ and $y=v$

we have

Limits $x=\frac{y}{2}$; $x=\frac{y+4}{2}$

$\Rightarrow u+\frac{v}{2}=\frac{y}{2}=\frac{v}{2}$ and $u+\frac{v}{2}=\frac{y+4}{2}$

$\Rightarrow u=0$ $\qquad\qquad =\frac{v+4}{2}$

$\Rightarrow u=2$

$\therefore u=0$ to 2

$y=v=0$ to 2.

$\therefore$ Hence for the given region v varies from 0 to 2 and u varies from 0 to 2

further $y^3(2x-y)\,e^{(2x-y)^2} = v^3(2u+v-v)\,e^{(2u)^2}$

$= v^3(2u)\,e^{4u^2}$

$\therefore$ Changing the variables to u, v the given integral becomes

$$\int_0^2 \int_{y/2}^{y+4/2} y^3(2x-y)e^{(2x-y)^2} = \int_0^2 \int_0^2 2v^3 u e^{4u^2} du\,dv$$

$$= \int_0^2 2\left[\frac{v^4}{4}\right]_0^2 u e^{4u^2} du$$

$$= \frac{16}{2} \int_0^2 u e^{4u^2} du$$

$$= 8\int_0^2 u e^{4u^2} du \qquad \text{Let } 4u^2 = t,\ 8u\,du = dt$$

$$= \int_0^{16} e^t dt$$

$$= e^{16} - 1.$$

1(d)» If $V = \log_e \sin\left\{\dfrac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$, find the value $x\dfrac{\partial V}{\partial x}+y\dfrac{\partial V}{\partial y}+z\dfrac{\partial V}{\partial z}$ when $x=0,\ y=1,\ z=2$.

Sol'n: Given

$$V = \log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow e^V = \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}$$

$$\Rightarrow \sin^{-1}e^V = \frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}} \quad (=u) \text{ say} \quad \text{—— (1)}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = nu \quad \text{—— (2)}$$

where $n = 1-\frac{2}{3}=\frac{1}{3}$

But from (1)

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial y} \text{ and } \frac{\partial u}{\partial z} = \frac{1}{\sqrt{1-e^{2V}}}e^V\frac{\partial V}{\partial z}$$

∴ from (2)

$$\frac{e^V}{\sqrt{1-e^{2V}}}\left[x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}\right] = \frac{1}{3}(\sin^{-1}e^V)$$

$$\Rightarrow x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \frac{1}{3}(\sin^{-1}e^V)\frac{\sqrt{1-e^{2V}}}{e^V} \quad \text{—— (3)}$$

when $(x,y,z)=(0,1,2)$

$$V = \log_e \sin\left\{\frac{\pi(1)^{1/2}}{2(4+4)^{1/3}}\right\}$$

$$= \log_e \sin\left[\frac{\pi}{2(8)^{1/3}}\right]$$

$$= \log_e \sin\left(\pi/4\right)$$

$$v = \log_e\left(1/\sqrt{2}\right)$$

$$\Rightarrow e^v = \frac{1}{\sqrt{2}}$$

and $u = \sin^{-1} e^v = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi/4$

$\therefore$ from (3)

$$\therefore x\frac{\partial v}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3}\left(\pi/4\right)\frac{\sqrt{1-1/2}}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\pi}{12}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right)$$

$$= \pi/12$$

1.(e) If O be the centre of a sphere of radius unity and A, B be two points in a line with O such that $OA \cdot OB = 1$, and if P be a variable point on the sphere, show that $PA : PB =$ constant.

Sol^n: The equation of the sphere is

$$x^2+y^2+z^2=1 \quad \text{———} (1)$$

Let A be the point (x_1, y_1, z_1) then the direction ratios of OA are x_1, y_1, z_1 so that the equation of line OA is $x/x_1 = y/y_1 = z/z_1$.

If B be the point on this line at a distance r from O, then the coordinates of B are (rx_1, ry_1, rz_1).

Given that the $OA \cdot OB = 1$, which reduces to

$$\sqrt{(x_1^2+y_1^2+z_1^2)} \cdot \sqrt{(r^2x_1^2+r^2y_1^2+r^2z_1^2)} = 1$$

or $$r(x_1^2+y_1^2+z_1^2) = 1 \quad \text{———} (2)$$

If the coordinates of any variable point P on the sphere be (x_2, y_2, z_2), then from (1). we get

$$x_2^2+y_2^2+z_2^2=1 \quad \text{———} (3)$$

Now $$\frac{PA^2}{PB^2} = \frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{(rx_1-x_2)^2+(ry_1+y_2)^2+(rz_1-z_2)^2}$$

$$= \frac{(x_1^2+y_1^2+z_1^2)-2(x_1x_2+y_1y_2+z_1z_2)+(x_2^2+y_2^2+z_2^2)}{r^2(x_1^2+y_1^2+z_1^2)-2r(x_1x_2+y_1y_2+z_1z_2)+(x_2^2+y_2^2+z_2^2)}$$

$$= \frac{(1/r) - 2(x_1x_2 + y_1y_2 + z_1z_2) + 1}{r^2(1/r) - 2r(x_1x_2 + y_1y_2 + z_1z_2) + 1} \text{, from (2) \& (3)}$$

$$= \frac{[1 - 2r(x_1x_2 + y_1y_2 + z_1z_2) + r]/r}{r - 2r(x_1x_2 + y_1y_2 + z_1z_2) + 1}$$

$$= \frac{1}{r} = \text{constant.}$$

Hence PA : PB is constant.

2(a)(i) Let $A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 2 & -1 & 1 & 0 & 1 \\ 1 & 2 & -1 & -2 & 1 \\ 1 & 3 & 2 & 1 & 2 \end{bmatrix}$ and let $T: \mathbb{C}^5 \to \mathbb{C}^4$ be given by $T(X) = AX$. Is T injective?

(ii) Let $T: \mathbb{C}^3 \to \mathbb{C}^3$ be given by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+2c \\ 2c \\ a+b+c \end{bmatrix}.$$

Find a basis of $R(T)$. Is T surjective?

Sol^n: (i) If a linear transformation $T: U \to V$ is injective, then $\dim(U) \le \dim(V)$.

In this case, $T: \mathbb{C}^5 \to \mathbb{C}^4$.

and $5 = \dim(\mathbb{C}^5) > \dim(\mathbb{C}^4) = 4$

Thus, T cannot possibly be injective.

(ii) The range of T is

$$R(T) = \left\{ \begin{bmatrix} a+b+2c \\ 2c \\ a+b+c \end{bmatrix} \middle/ a, b, c \in \mathbb{C} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \middle/ a, b, c \in \mathbb{C} \right\}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\rangle$$

Since the vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\\1\end{bmatrix}$ are linearly independent.

$\therefore$ basis of $R(T)$ is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}\right\}$.

Since the dimension of the range is 2 and the dimension of the codomain is 3.

T is not surjective.

2(b) → Prove that the function $f(x,y)=\sqrt{|xy|}$ is not differentiable at the point $(0,0)$, but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

Sol^n: Now at $(0,0)$

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0}{k}=0$$

If the function is differentiable at $(0,0)$ then by definition

$$f(h,k)-f(0,0)=0h+0k+h\phi+k\psi$$

where ϕ and ψ are functions of h and k, and tend to zero as $(h,k)\to(0,0)$

putting $h=\rho\cos\theta$, $k=\rho\sin\theta$ and dividing by ρ, we get

$$|\cos\theta\sin\theta|^{1/2}=\phi\cos\theta+\psi\sin\theta$$

Now for arbitrary θ, $\rho\to 0$ implies that $(h,k)\to(0,0)$

Taking the limits as $\rho\to 0$, we get

$$|\cos\theta\sin\theta|=0$$

which is impossible for all arbitrary θ.

Hence, the function is not differentiable at $(0,0)$ and consequently the partial derivatives f_x, f_y cannot be continuous at $(0,0)$, for otherwise the function would be differentiable there at,

Let us now see that it is actually so

For $(x,y)\neq(0,0)$

$$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{|x+h|\,|y|}-\sqrt{|x|\,|y|}}{h}$$

$$= \lim_{h\to 0} \sqrt{|y|} \frac{|x+h| - |x|}{h\left[\sqrt{|x+h|} - \sqrt{|x|}\right]}$$

Now as $h \to 0$, we can take $x+h>0$, i.e. $|x+h| = x+h$, when $x>0$ and $x+h<0$ (or) $|x+h| = -(x+h)$, when $x<0$

$$\therefore \quad f_x(x,y) = \begin{cases} \frac{1}{2}\sqrt{\frac{|y|}{|x|}}, & \text{when } x>0 \\ -\frac{1}{2}\sqrt{\frac{|y|}{|x|}}, & \text{when } x<0 \end{cases}$$

Similarly

$$f_y(x,y) = \begin{cases} \frac{1}{2}\sqrt{\frac{|x|}{|y|}}, & \text{when } y>0 \\ -\frac{1}{2}\sqrt{\frac{|x|}{|y|}}, & \text{when } y<0 \end{cases}$$

which are, obviously, not continuous at the origin.

2.c(i) → Prove that the locus of a line which meets the lines $y=\pm mx$, $z=\pm c$ and the circle $x^2+y^2=a^2$, $z=0$ is

$$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.$$

Solⁿ: Given lines are $y-mx=0,\ z-c=0$ —— (1)

$$y+mx=0,\ z+c=0 \quad \text{—— (2)}$$

and the circle is $x^2+y^2=a^2,\ z=0$ —— (3)

Any line intersecting (1) and (2) is

$$(y-mx)+k_1(z-c)=0,\ (y+mx)+k_2(z+c)=0 \quad \text{—— (4)}$$

If it meets the circle (3), then we are to eliminate k_1, k_2 from (3) and (4).

Putting $z=0$ in (4) we get

$$(y-mx)-k_1c=0,\ (y+mx)+k_2c=0$$

or $\quad mx-y+k_1c=0,\ mx+y+k_2c=0$

Adding and subtracting these, we get

$$x=-\frac{(k_1+k_2)c}{2m},\quad y=\frac{c(k_1-k_2)}{2}$$

Substituting these values of x and y in (3), we get

$$\frac{(k_1+k_2)^2c^2}{4m^2}+\frac{(k_1-k_2)^2c^2}{4}=a^2$$

or $\quad \left[(k_1+k_2)^2+m^2(k_1-k_2)^2\right]c^2=4a^2m^2$ —— (5)

or $$\left[\left\{\left(\frac{mx-y}{z-c}\right)+\left(-\frac{mx+y}{z+c}\right)\right\}^2+ m^2\left\{\left(\frac{mx-y}{z-c}\right)-\left(-\frac{mx+y}{z+c}\right)\right\}^2\right]c^2$$

$$= 4a^2m^2 \text{ from } (4)$$

Now simplify and get the result.

2.C(ii) The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right) + zx\left(\frac{c}{a}+\frac{a}{c}\right) + xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

Solⁿ: The plane ABC is

$$x/a + y/b + z/c = 1 \quad \text{——— (i)}$$

It meets the axes at A(a,0,0), B(0,b,0) and C(0,0,c). The equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0 \quad \text{——— (ii)}$$

The required cone is generated by the lines drawn from O to meet the circle ABC [given by (i) and (ii) together] and will be homogeneous. So making (ii) homogeneous with the help of (i), we get the required equation as

$$x^2+y^2+z^2-(ax+by+cz)(x/a+y/b+z/c)=0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Proved.

3(a)(i) Suppose U and W are two-dimensional subspace of $\mathbb{R}^3$. Show that $U \cap W \neq \{0\}$. In particular, find the possible dimensions of $U \cap W$.

Soln: Suppose $U = W$.

Then $U \cap W = U = W$.

and hence $\dim(U \cap W) = 2$

$(\because \dim U = \dim W = 2)$

Suppose $U \neq W$.

Then $U + W$ properly contains U (and W).

Hence $\dim(U+W) > \dim U = 2$.

But $U + W \subseteq \mathbb{R}^3$, which has dimension 3.

$\therefore \dim(U+W) = 3$.

$\therefore$ By using the theorem

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$$
$$= 2+2-3$$
$$= 1.$$

i.e, $U \cap W$ is a line through the origin

3.(a) (ii) Show that the real field R is a vector space of infinite dimension over the rational field Q.

Sol^n: We claim that, for any n, $\{1, \pi, \pi^2, \ldots \pi^n\}$ is linearly independent over Q.

For suppose
$$a_0 + a_1\pi + a_2\pi^2 + \ldots + a_n\pi^n = 0,$$
where $a_i \in Q$, and not all the a_i are zero.

Then π is a root of the non-zero polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ over Q.

This is impossible, since π is a transcendental number
i.e, π is not a root of any non-zero polynomial over Q.

Accordingly, the $(n+1)$ real numbers $1, \pi, \pi^2, \ldots \pi^n$ are linearly independent over Q.

Thus for any finite n, R cannot be of dimension n over Q.
i.e., R is of infinite dimension over Q.

3.(a) (iii) Let V be the vector space of ordered pairs of complex numbers over the real field $\mathbb{R}$. Show that V is of dimension 4.

Sol^n: We claim that $B = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis of V.

Suppose $v \in V$.

Then $v = (z, w)$, where z, w are complex numbers

and so $v = (a+bi, c+di)$ where a, b, c, d are real numbers.

Then $v = a(1,0) + b(i,0) + c(0,1) + d(0,i)$.

Thus B generates V.

The proof is complete if we show that B is independent.

Suppose $x_1(1,0) + x_2(i,0) + x_3(0,1) + x_4(0,i) = 0$ where $x_1, x_2, x_3, x_4 \in \mathbb{R}$

Then

$(x_1 + x_2 i, x_3 + x_4 i) = (0,0)$

and so $x_1 + x_2 i = 0$
$x_3 + x_4 i = 0$

$\Rightarrow x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$

and so B is independent

3.(b) The ellipsoid with equation $x^2+2y^2+z^2=4$ is heated so that its temperature at (x,y,z) is given by $T(x,y,z)=70+10(x+z)$. Find the hottest and coldest points on the ellipsoid.

Sol^n: Given that the ellipsoid $x^2+2y^2+z^2=4$ — ①

is heated so that its temperature at a point (x,y,z) is given by

$$T(x,y,z)=70+10(x+z) \quad \text{— ②}$$

we have to find the hottest and coldest points on the ellipsoid.

Now, by using Lagranges multiplier method.

Consider the function

$$F=70+10(x+z)+\lambda(x^2+y^2+z^2-4)$$

$$dF=[10+2x\lambda]dx+4y\lambda dy+(10+2z\lambda)dz$$

For stationary values $F_x=F_y=F_z=0$.

$$\Rightarrow 10+2x\lambda=0 \quad \text{— ③}$$

$$4y\lambda=0$$

$$10+2z\lambda=0 \quad \text{— ④}$$

Now $10+2x\lambda=0 \Rightarrow \lambda\neq 0$

and so $0=4y\lambda \Rightarrow y=0$.

from ③ & ④, we have

$$-2x\lambda=-2z\lambda$$

$$\Rightarrow x=z$$

Substituting $x=z$ & $y=0$ in equation ①, we have

$$x^2 + 0 + x^2 = 4$$
$$2x^2 = 4$$
$$x = \pm\sqrt{2}$$

$\therefore$ The points are $(\sqrt{2}, 0, \sqrt{2})$ and $(-\sqrt{2}, 0, -\sqrt{2})$

At $(\sqrt{2}, 0, \sqrt{2})$;

$$T(x, y, z) = 70 + 10(2\sqrt{2})$$
$$= 70 + 20\sqrt{2}$$
$$= 70 + 28.28 = 98.28 \approx 98$$

At $(-\sqrt{2}, 0, -\sqrt{2}) = 70 - 20\sqrt{2}$
$$= 70 - 28.28$$
$$= 41.72$$
$$\approx 42$$

Thus the hottest point on the ellipsoid is $(\sqrt{2}, 0, \sqrt{2})$ and the coldest point on the ellipsoid is $(-\sqrt{2}, 0, -\sqrt{2})$.

3.(c) Prove that $z(ax+by+cz)+\alpha x+\beta y=0$ represents a paraboloid and the equations to the axis are $ax+by+2cz=0$, $(a^2+b^2)z+a\alpha+b\beta=0$.

There is a small correction in question, in place of c put a

Sol^n: Given equation is

$$cz^2+byz+azx+\alpha x+\beta y=0$$

$\therefore$ Here 'a'$=0$, 'b'$=0$, 'c'$=c$, 'f'$=b/2$, 'g'$=a/2$, 'h'$=0$, 'u'$=\alpha/2$, 'v'$=\beta/2$, 'w'$=0$ and 'd'$=0$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & a/2 \\ 0 & 0-\lambda & b/2 \\ a/2 & b/2 & c-\lambda \end{vmatrix}=0 \quad \text{—— (1)}$$

or $-\lambda[-\lambda(c-\lambda)-(b^2/4)]+(a/2)[a\lambda/2]=0$

or $4\lambda^3-4a\lambda^2-(a^2+b^2)\lambda=0$

or $\lambda[4\lambda^2-4a\lambda-a^2-b^2]=0$

or $\lambda=0$ and $\lambda=\dfrac{4a\pm\sqrt{(16a^2+16a^2+16b^2)}}{8}$

$$=\frac{a\pm\sqrt{(2a^2+b^2)}}{2}$$

Let $\lambda_1=\frac{1}{2}\left[a+\sqrt{(2a^2+b^2)}\right]$, $\lambda_2=\frac{1}{2}\left[a-\sqrt{(2a^2+b^2)}\right]$, $\lambda_3=0$.

Now putting $\lambda=0$ in the determinant given by (1) and associating each row with l_3, m_3, n_3 we have

$$0\cdot l_3+0\cdot m_3+(a/2)n_3=0,$$
$$0\cdot l_3+0\cdot m_3+(b/2)n_3=0$$
and $$(a/2)l_3+(b/2)m_3+cn_3=0$$

These gives $n_3 = 0$ and $al_3 + bm_3 = 0$

i.e. $\frac{l_3}{b} = \frac{m_3}{-a} = \frac{n_3}{0} = \frac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(b^2 + a^2 + 0)}} = \frac{1}{\sqrt{(a^2+b^2)}}$

$\therefore l_3 = b/\sqrt{(a^2+b^2)}, \; m_3 = -a/\sqrt{(a^2+b^2)}, \; n_3 = 0$

Now $K = ul_3 + vm_3 + wn_3$

$$= \frac{\alpha}{2} \cdot \frac{b}{\sqrt{(a^2+b^2)}} + \frac{\beta}{2}\left[\frac{-a}{\sqrt{(a^2+b^2)}}\right] + 0$$

$$= \frac{b\alpha - a\beta}{2\sqrt{(a^2+b^2)}} \neq 0$$

$\therefore$ Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2Kz = 0$

or $\frac{1}{2}\left[a + \sqrt{(2a^2+b^2)}\right]x^2 + \frac{1}{2}\left[a - \sqrt{(2a^2+b^2)}\right]y^2 +$

$$\frac{b\alpha - a\beta}{\sqrt{(a^2+b^2)}} z = 0 \quad \text{——— (2)}$$

Now as $a + \sqrt{(2a^2+b^2)} > 0$ and $a - \sqrt{(2a^2+b^2)} < 0$,

So (2) represents a hyperbolic paraboloid.

Also if $F(x,y,z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations

$$\frac{\partial F/\partial x}{l_3} = \frac{\partial F/\partial y}{m_3} = \frac{\partial F/\partial z}{n_3} = 2K$$

and $K(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

i.e. any two of the equations

$$\frac{\alpha + az}{b/\sqrt{(a^2+b^2)}} = \frac{\beta + bz}{-a/\sqrt{(a^2+b^2)}} = \frac{ax + by + 2cz}{0}$$

$$= \frac{2(b\alpha - a\beta)}{2\sqrt{(a^2+b^2)}}$$

and $K\left[\frac{bx}{\sqrt{(a^2+b^2)}} - \frac{ay}{\sqrt{(a^2+b^2)}} + 0\right] + \frac{\alpha}{2}x + \frac{\beta}{2}y = 0,$

on substituting the values.

Thus we have $\frac{\alpha + az}{b} = \frac{\beta + bz}{-a}$, $ax+by+2cz=0$.

and $\frac{b\alpha - a\beta}{2\sqrt{(a^2+b^2)}}\left[\frac{bx-ay}{\sqrt{(a^2+b^2)}}\right] + \frac{1}{2}(\alpha x + \beta y) = 0$

i.e. $(a^2+b^2)z + a\alpha + b\beta = 0$, $ax+by+2cz=0$

and $(b\alpha - a\beta)(bx-ay) + (\alpha x + \beta y)(a^2+b^2) = 0$

Now if (x_1, y_1, z_1) be the vertex of the paraboloid then x, y, z satisfies above three eqn

i.e. $$(a^2+b^2)z_1 + a\alpha + b\beta = 0 \quad \text{——} (3)$$

$$ax_1 + by_1 + 2cz_1 = 0 \quad \text{——} (4)$$

and $$(b\alpha - a\beta)(bx - ay) + (\alpha x + \beta y)(a^2+b^2) = 0 \quad \text{——} (5)$$

And the equations of the axis are

$$\frac{x-x_1}{l_3} = \frac{y-y_1}{m_3} = \frac{z-z_1}{n_3}$$

or $$\frac{x-x_1}{b/\sqrt{(a^2+b^2)}} = \frac{y-y_1}{-a/\sqrt{(a^2+b^2)}} = \frac{z-z_1}{0} \quad \text{——} (6)$$

Substituting values of l_3, m_3, n_3 These give $z - z_1 = 0$ or $z = z_1 = -\frac{a\alpha + b\beta}{(a^2+b^2)}$, from (3)

or $$(a^2+b^2)z + a\alpha + b\beta = 0 \quad \text{——} (7)$$

Again from first two fractions of (6), we get $a(x-x_1) + b(y-y_1) = 0$

or $\quad ax+by = ax_1+by_1 = -2cz_1$, from (4)

$$= -2c\left[-\frac{a\alpha+b\beta}{a^2+b^2}\right], \text{ from (3)}$$

$$= -2cz, \text{ from (7)}$$

or $\quad ax+by+2cz=0$ —— (8)

Hence from (7) and (8) the equations of the axis of the paraboloid are

$(a^2+b^2)z+a\alpha+b\beta=0,\ ax+by+2cz=0.$

4(a)(i) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$. Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is characteristic vector of T.

sol^n: The Characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}-9-\lambda & 4 & 4\\ -8 & 3-\lambda & 4\\ -16 & 8 & 7-\lambda\end{vmatrix}=0$$

$C_1 \to C_1+C_2+C_3$

$$\Rightarrow \begin{vmatrix}-1-\lambda & 4 & 4\\ -1-\lambda & 3-\lambda & 4\\ -1-\lambda & 8 & 7-\lambda\end{vmatrix}=0$$

$$\Rightarrow (-1-\lambda)\begin{vmatrix}1 & 4 & 4\\ 1 & 3-\lambda & 4\\ 1 & 8 & 7-\lambda\end{vmatrix}=0$$

$R_2 \to R_2-R_1$
$R_3 \to R_3-R_1$

$$\Rightarrow (1+\lambda)\begin{vmatrix}1 & 4 & 4\\ 0 & -1-\lambda & 0\\ 0 & 4 & 3-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda)=0$$

$$\Rightarrow \lambda=-1,-1,3.$$

The characteristic roots of A are $-1,-1,3$

The eigen vectors x of A corresponding to the characteristic root -1 are given by

$[A-(-1)I]\,X=0$

$$\Rightarrow \begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$R_2 \to R_2 - R_1,\quad R_3 \to R_3 - 2R_1$$

$$\sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix = 1

$\therefore$ The equations have 3-1 = 2 linearly independent solutions.

$\therefore$ we have

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$

Let $x_2 = k_1$ and $x_3 = k_2$; k_1, k_2 are arbitrary constants.

$$\therefore x_1 = \frac{k_1 + k_2}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1+k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k_1}{2} + \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X = k_1 X_1 + k_2 X_2$$

Here $X_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$ are LI vectors of A corresponding to characteristic root -1.

$\therefore$ The geometric multiplicity of eigen value is equal to its algebraic multiplicity.

Now the eigen vectors X of A corresponding to the eigen value 3 are given by

$(A-3I)X=0$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \to R_1 - R_3$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 4R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix $= 2$

$\therefore$ The equations have $3-2=1$ L.I. solution.

$\therefore$ we have $4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$

$-8x_2 + 4x_3 = 0 \quad \frac{8}{2}x_2 = x_3$

Taking $x_3 = 2$

then $x_2 = 1$, and $x_1 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3.

$\therefore$ The geometric multiplicity of eigen value 3 is 1 and its algebraic multiplicity is also 1.

$\therefore$ A is similar to diagonal matrix.

$\therefore$ A is diagonalizable matrix.

Let $P = [X_1 \ X_2 \ X_3]$

$$= \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The columns of P are LI eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively. The matrix P will transform A to diagonal form D is given by the relation $P^{-1}AP = D$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The transforming matrix $P = \begin{bmatrix} 1/2 & 1/3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4(a)(ii) Suppose that A is a 2×2 matrix with real entries which is symmetric $(A^T=A)$. Prove that A is similar over $\mathbb{R}$ to a diagonal matrix.

Sol Let $A=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a symmetric matrix over $\mathbb{R}$ (as $A^T=A$).

Now the characteristic equation of A is $|A-\lambda I|=0$

$\Rightarrow \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0 \Rightarrow (a-\lambda)(c-\lambda)-b^2=0$

$\Rightarrow (\lambda-a)(\lambda-c)-b^2=0$

$\Rightarrow \lambda^2-(a+c)\lambda+(ac-b^2)=0$ —— (1)

The discriminant of (1) is

$\Delta=(a+c)^2-4(1)(ac-b^2)=$

$a^2+2ac+c^2-4ac+4b^2$

$=a^2-2ac+c^2+4b^2=(a-c)^2+(2b)^2$

$\therefore \Delta=(a-c)^2+(2b)^2$. —— (2)

$\therefore \Delta \geq 0$.

$\therefore$ The characteristic roots of A are all real.

Since $b\neq0$ $\therefore \Delta>0$.

i.e The characteristic roots

of A are real and distinct.
Since the matrix has two distinct
roots characteristic roots,
. hence it is diagonalizable.

4(b) (i) If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$.

(ii) show that $\int_0^{\pi} \log(1+\cos x)\,dx = -\pi \log 2$.

Sol^n: (ii) Let $I = \int_0^{\pi} \log(1+\cos x)\,dx$ —— (1)

then $I = \int_0^{\pi} \log[1+\cos(\pi-x)]\,dx$

$\Rightarrow I = \int_0^{\pi} \log(1-\cos x)\,dx$ —— (2).

$\therefore$ Adding (1) and (2), we get

$$2I = \int_0^{\pi} [\log(1+\cos x) + \log(1-\cos x)]\,dx$$

$$= \int_0^{\pi} \log(1-\cos^2 x)\,dx$$

$$= \int_0^{\pi} \log \sin^2 x\,dx$$

$$= 2\int_0^{\pi} \log \sin x\,dx = 4\int_0^{\pi/2} \log \sin x\,dx$$

$$= -4\cdot\frac{\pi}{2}\log 2.$$

$$I = -\pi \log 2$$

(i) Let $z = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ —— (1) $= \dfrac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{y/x}\right]} = x^{1/2}\left(\dfrac{1+\frac{y}{x}}{1+\sqrt{y/x}}\right)$

z is a homogeneous function of x and y of degree $\frac{1}{2}$.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad \text{—— (2)}$$

from (1), $\dfrac{\partial z}{\partial x} = -\sin u \dfrac{\partial u}{\partial x}$

and $\dfrac{\partial z}{\partial y} = -\sin u \dfrac{\partial u}{\partial y}$

from (2)

$$-x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

$$\Rightarrow\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$$

4.c → If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid
$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Sol^n: Let r be the length of each semi-conjugate diameter. Then $OP = OQ = r$, where P, Q, R are the extremities of the semi-conjugate diameters and their coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Also we know
$$OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$$
or $$3r^2 = a^2 + b^2 + c^2$$
or $$r^2 = \frac{1}{3}(a^2+b^2+c^2) \quad \text{——— (1)}$$

Let l, m, n be the direction cosines of OP, then the coordinates of P can be taken as (lr, mr, nr)

$\because$ $P(lr, mr, nr)$ lies on the given ellipsoid,

So we have $$\frac{l^2r^2}{a^2} + \frac{m^2r^2}{b^2} + \frac{n^2r^2}{c^2} = 1$$

or $$r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = 1 = l^2 + m^2 + n^2$$

or $$\frac{1}{3}(a^2+b^2+c^2)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = l^2 + m^2 + n^2, \text{ from (1)}$$

or $$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = 3\frac{(l^2+m^2+n^2)}{(a^2+b^2+c^2)} \quad \text{——— (2)}$$

Also the equations of the line OP are
$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{——— (3)}$$

Eliminating l, m, n between (2) and (3) we get the locus of OP as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{(a^2+b^2+c^2)}$$

or $$x^2\left[\frac{1}{a^2}-\frac{3}{a^2+b^2+c^2}\right]+y^2\left[\frac{1}{b^2}-\frac{3}{a^2+b^2+c^2}\right]+z^2\left[\frac{1}{c^2}-\frac{3}{a^2+b^2+c^2}\right]=0$$

or $$\frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}(2c^2-a^2-b^2)=0$$

which is a cone.

5.(a) Solve $(D^3+1)y = e^{2x}\sin x + e^{x/2}\sin(\sqrt{3}x/2)$.

Solⁿ: Given $(D^3+1)y = e^{2x}\sin x + e^{x/2}\sin(\sqrt{3}x/2)$.

The auxiliary equation is

$$D^3+1=0 \quad \text{or} \quad (D+1)(D^2-D+1)=0$$

giving $D=-1,\ \{1\pm(1-4)^{1/2}\}/2$

or $D=-1,\ 1/2 \pm i(\sqrt{3}/2)$

$$\therefore\ C.F. = c_1 e^{x} + e^{x/2}\left[c_2\cos(x\sqrt{3}/2) + c_3\sin(x\sqrt{3}/2)\right]$$

c_1, c_2 and c_3 being arbitrary constants.

P.I. corresponding to $e^{2x}\sin x$

$$= \frac{1}{D^3+1}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3+1}\sin x$$

$$= e^{2x}\frac{1}{D^3+3D^2\cdot 2+3D\cdot 2^2+2^3+1}\sin x$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$$

$$= e^{2x}\frac{1}{-D-6+12D+9}\sin x = e^{2x}\frac{1}{11D+3}\sin x$$

$$= e^{2x}(11D-3)\frac{1}{121D^2-9}\sin x = e^{2x}\frac{(11D-3)}{-121-9}\sin x$$

$$= -(1/130)\cdot e^{2x}(11\cos x - 3\sin x).$$

P.I. corresponding to $e^{x/2}\sin(x\sqrt{3}/2)$

$$= \frac{1}{D^3+1} e^{x/2} \sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2} \frac{1}{(D+1/2)^3+1} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^3+3D^2\cdot(1/2)+3D\cdot(1/2)^2+(1/2)^3+1} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^3+(3/2)D^2+(3/4)D+(9/8)} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{[D^2+(3/4)][D+(3/2)]} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

[As denominator is zero when $D^2 = -3/4$, factorize denominator]

$$= e^{x/2} \frac{1}{D^2+(3/4)} \left(D-\frac{3}{2}\right) \cdot \frac{1}{D^2-(9/4)} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{D^2+(3/4)} \left(D-\frac{3}{2}\right) \cdot \frac{1}{(-3/4)-(9/4)} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= -\frac{e^{x/2}}{3} \cdot \frac{1}{D^2+(3/4)} \cdot \left(D-\frac{3}{2}\right) \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= -\frac{e^{x/2}}{3} \cdot \frac{1}{D^2+(3/4)} \left[\frac{\sqrt{3}}{2} \cos\left(\frac{x\sqrt{3}}{2}\right) - \frac{3}{2} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6} \left[\sqrt{3} \frac{1}{D^2+(\sqrt{3}/2)^2} \cos\left(\frac{x\sqrt{3}}{2}\right) - 3 \frac{1}{D^2+(\sqrt{3}/2)^2} \sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6} \left[\sqrt{3} \frac{x}{2(\sqrt{3}/2)} \sin\left(\frac{x\sqrt{3}}{2}\right) + 3 \cdot \frac{1}{2(\sqrt{3}/2)} \cos\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{xe^{x/2}}{6} \left[\sin\left(\frac{x\sqrt{3}}{2}\right) + \sqrt{3} \cos\left(\frac{x\sqrt{3}}{2}\right)\right].$$

$\therefore$ Required solution is:

$$y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)\right]$$
$$- (x/6)\cdot e^{x/2}\left[\sin(x\sqrt{3}/2) + \sqrt{3}\cos(x\sqrt{3}/2)\right].$$

5.(b) → Find the equation of the system of orthogonal trajectories of the parabolas $r = 2a/(1+\cos\theta)$, where a is the parameter.

Solⁿ: From the given equation, we get

$$\log r = \log 2a - \log(1+\cos\theta) \text{ ——— } (1)$$

differentiating (1) w.r.t. θ, we get

$$(1/r)(dr/d\theta) = (\sin\theta)/(1+\cos\theta) \text{ ——— } (2)$$

equation (2) is the differential equation of the given system of parabolas. Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (2), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}(-r^2)\frac{d\theta}{dr} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

or $\dfrac{dr}{r} = -\cot\dfrac{\theta}{2}\,d\theta.$

Integrating,

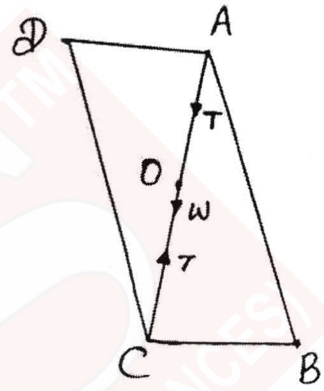
$$\log r = -2\log\sin(\theta/2) + \log C,$$

or $\quad r = C/\sin^2(\theta/2)$

or $\quad r = 2C/(1-\cos\theta)$, as $\sin^2(\theta/2) = (1-\cos\theta)/2$

Q. 5.C) Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.

Sol: ABCD is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC. Let T be the tension in the string AC. The total weight W of all the four rods.



AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC = 2x$.

Give the system a small displacement in which x changes to $x+\delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes.

We have, $AO = x$

By the principle of virtual work, we have

$$-T\delta(AC) + W\delta(AO) = 0$$

or $\quad -T\delta(2x) + W\delta(x) = 0$

or $\quad -2T\delta x + W\delta x = 0$

or $\quad [-2T+W]\delta x = 0$ or $-2T+W=0$ $[\because \delta x \neq 0]$

or $\quad T = \frac{1}{2}W = \frac{1}{2}$ (total weight of all the four rods).

5.(d) Prove that for the curve $\vec{r} = a(3t-t^3)\hat{i} + 3at^2\hat{j} + a(3t+t^3)\hat{k}$, the curvature equals torsion.

Sol: Given $\vec{r} = a(3t-t^3)\hat{i} + 3at^2\hat{j} + a(3t+t^3)\hat{k}$

$$\frac{d\vec{r}}{dt} = a(3-3t^2)\hat{i} + 6at\hat{j} + a(3+3t^2)\hat{k}$$

$$\frac{d^2r}{dt^2} = -6at\hat{i} + 6a\hat{j} + 6at\hat{k}$$

$$\frac{d^3r}{dt^3} = -6a\hat{i} + 6a\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = 3a\sqrt{(1-t^2)^2+(2t)^2+(1+t^2)^2} = 3a\sqrt{2t^4+4t^2+1}$$
$$= 3a\sqrt{2}(1+t^2)$$

$$\frac{dr}{dt}\times\frac{d^2r}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(3-3t^2) & 6at & a(3+3t^2) \\ -6at & 6a & 6at \end{vmatrix}$$

$$= 18a^2\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (1-t^2) & 2t & (1+t^2) \\ -t & 1 & t \end{vmatrix}$$

$$= 18a^2\left[(t^2-1)\hat{i} - 2t\hat{j} + (1+t^2)\hat{k}\right]$$

$$\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right| = 18a^2\sqrt{2t^4+4t^2+2} = 18a^2\sqrt{2}(t^2+1)$$

$$\left[\frac{d\vec{r}}{dt},\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] = 18a^2\left((t^2-1)(-6a) + (t^2+1)6a\right) = \underline{216a^3}$$

Curvature $\kappa = \left|\frac{d\vec{r}}{dt}\times\frac{d^2r}{dt^2}\right| \Big/ \left|\frac{dr}{dt}\right|^3 = \dfrac{18a^2\sqrt{2}(t^2+1)}{(3a\sqrt{2}(t^2+1))^3}$

$$= \frac{1}{3a(t^2+1)^2}$$

Torsion $\tau = \left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] \Big/ \left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|^2$

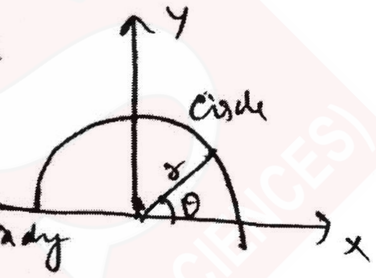
$$= \frac{216a^3}{(18\sqrt{2}a^2(t^2+1))^2} = \frac{1}{3a(t^2+1)^2}$$

$\therefore$ Torsion equals to Curvature.

5.(e) Apply Green's theorem to evaluate $\int (2x^2-y^2)dx+(x^2+y^2)dy$ where c is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2+y^2=a^2$.

Sol: By greens theorem

$$\int_c [(2x^2-y^2)dx+(x^2+y^2)\,dy] =$$



$$= \iint \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy$$

$$= 2\iint_A (x+y)\,dx\,dy$$

we have
$x = r\cos\theta$
$y = r\sin\theta$

$$= 2\int_0^a \int_0^\pi (r\sin\theta + r\cos\theta)\, r\,d\theta\,dr$$

$$= 2\int_0^r r^2 dr . \int_0^\pi (\sin\theta+\cos\theta)\,d\theta$$

$$= 2 . \frac{a^3}{3} . (1+1)$$

Area $= \frac{4}{3}a^3$

6.(a)(i) Solve $(d^3y/dx^3) - 3(d^2y/dx^2) + 4(dy/dx) - 2 = e^x + \cos x$.

Solⁿ: Given $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x$, —— (1)

where $D \equiv d/dx$

Its auxiliary equation is

$$D^3 - 3D^2 + 4D - 2 = 0$$

or $$D^2(D-1) - 2D(D-1) + 2(D-1) = 0$$

or $$(D-1)(D^2 - 2D + 2) = 0$$

giving $D = 1,\ (2 \pm \sqrt{4-8})/2$

i.e. $D = 1,\ 1 \pm i$

$\therefore$ C.F. $= c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$

where c_1, c_2, c_3 being arbitrary constants.

P.I. corresponding to $e^x = \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)} e^x = \frac{1}{D-1} \frac{1}{1-2+2} e^x$$

$$= \frac{1}{D-1} e^x \cdot 1 = e^x \frac{1}{(D+1)-1} \cdot 1$$

$$= e^x \frac{1}{D} \cdot 1$$

$$= x e^x$$

P.I. corresponding to $\cos x = \dfrac{1}{D^3-3D^2+4D-2}\cos x$

$$= \frac{1}{D^2\cdot D-3D^2+4D-1}\cos x$$

$$= \frac{1}{(-1)^2 D-3(-1)^2+4D-2}\cos x = \frac{1}{3D+1}\cos x$$

$$= (3D-1)\frac{1}{9D^2-1}\cos x = (3D-1)\frac{1}{9(-1)^2-1}\cos x$$

$$= -\frac{1}{10}(3D\cos x-\cos x) = -\frac{1}{10}(-3\sin x-\cos x)$$

$\therefore$ Required solution is:

$$y = e^x(c_1+c_2\cos x+c_3\sin x)+xe^x+(3\sin x+\cos x)/10$$

6.a(ii) Find the general and singular solution of $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$.

Sol^n: Given $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$

or $(py)^2 - 2(py)x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0$

$$\therefore\ py = \frac{2x\tan^2\alpha \pm \sqrt{\{4x^2\tan^4\alpha - 4(y^2\sec^2\alpha - x^2\tan^2\alpha)\}}}{2}$$

or $py = x\tan^2\alpha \pm \sqrt{\{x^2\tan^2\alpha(\tan^2\alpha+1) - y^2\sec^2\alpha\}}$

or $y(dy/dx) = x\tan^2\alpha \pm \sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}$

or $y\,dy - x\tan^2\alpha\,dx = \pm\sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}\,dx$

or $$\pm\frac{x\tan^2 dx - y\,dy}{\sqrt{(x^2\tan^2\alpha - y^2)}} = -\sec\alpha\,dx.$$

Integrating, $\pm\sqrt{(x^2\tan^2\alpha - y^2)} = c - x\sec\alpha$,
'c' being an arbitrary constant.

Squaring, $x^2\tan^2\alpha - y^2 = c^2 - 2cx\sec\alpha + x^2\sec^2\alpha$

or $\boxed{x^2 + y^2 + 2cx\sec\alpha + c^2 = 0}$

Now we have

$c^2 + 2(x\sec\alpha)x + x^2 + y^2 = 0$,
where 'c' being an arbitrary constant.

This is quadratic in c. So here c-discriminant relation is

$$4x^2\sec^2\alpha - 4\cdot 1\cdot(x^2+y^2) = 0$$

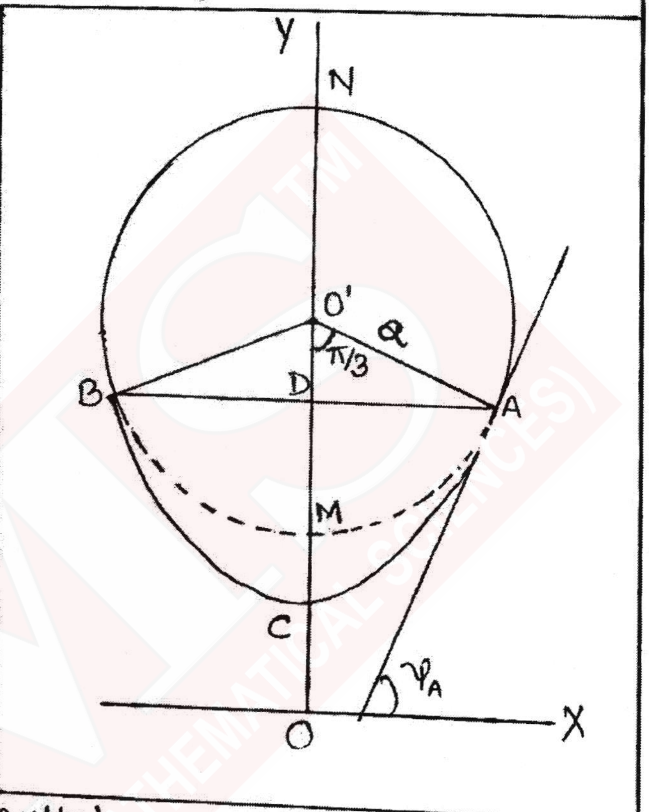
or $x^2(\sec^2\alpha - 1) - y^2 = 0$

or $y^2 - x^2\tan^2\alpha = 0$

or $(y - x\tan\alpha)(y + x\tan\alpha) = 0$

Now, $y = x\tan\alpha$ gives $p = dy/dx = \tan\alpha$.
Substitution of $p = \tan\alpha$ and $y = x\tan\alpha$ in the given equation satisfies it. Hence $y = x\tan\alpha$ is a singular solution.
Similarly, we easily verify that $y = -x\tan\alpha$ is also a singular solution.

6(b) → Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with the two thirds of the circumference of the pulley.



Solution:-

Let ANBMA be the circular pulley of radius 'a' and ANBCA the endless chain hanging over it.

Since, the chain is in contact with the $2/3^{rd}$ of the circumference of the pulley, hence the length of this portion ANB of the chain.

$= \frac{2}{3}$(circumference of the pulley).

$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a.$

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e., the vertex, CO'N the axis and OX the directrix of this catenary.

Let $OC = c$ = the parameter of the catenary.

The tangent at A will be perpendicular of the radius O'A.

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left[\frac{1}{3}\times 2\pi\right) = \frac{1}{3}\pi = \frac{\pi}{3}$$

From the triangle AO'D, we have

$$DA = O'A \sin\frac{\pi}{3} = a\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}a}{2}.$$

$\therefore$ from $x = c\log(\tan\psi + \sec\psi)$; for the point A.

we have; $\quad x = DA = c\log(\tan\psi_A + \sec\psi_A)$

or $\quad a\cdot\frac{\sqrt{3}}{2} = c\log\left[\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right]$

$$a\frac{\sqrt{3}}{\cancel{2}} = c\cdot\log(\sqrt{3}+2)$$

$$\therefore \boxed{c = \frac{a\sqrt{3}}{2\log(\sqrt{3}+2)}} \quad \text{——— (A)}$$

From $s = c\tan\psi$ applied for the point A, we have

$$\text{arc } CA = c\tan\psi_A = c\tan\frac{\pi}{3} = c\cdot\sqrt{3}$$

$$\boxed{\text{arc } CA = \frac{3a}{2\log(\sqrt{3}+2)}} \quad \text{———— [from (A)] ——— (B)}$$

Hence, the total length of the chain = arc ABC + length of chain in the contact with pulley.

$$= 2(\text{arc. } CA) + \frac{4}{3}\pi a.$$

$$= \cancel{2}\cdot\frac{3a}{\cancel{2}\log(\sqrt{3}+2)} + \frac{4}{3}\pi a.$$

$$\boxed{\therefore \text{Total length of Chain} = a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4}{3}\pi\right\}}$$

Required Result.

6(c)(i) The temperature of points in space is given by $T(x,y,z) = x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction that it will get warm as soon as possible. In what direction should it move ?

Sol^n: Directional derivative is minimum along the normal.

$$|\nabla(T)| = |\nabla(x^2+y^2-z)|$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\therefore (\nabla(T))_{(1,1,2)} = 2(1)\hat{i} + 2(1)\hat{j} - \hat{k}$$

$\therefore$ Normal vectors at $(1,1,2)$ is $2\hat{i}+2\hat{j}-\hat{k}$

$\therefore$ Unit normal vector $= \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$

$\therefore$ Along $\left(\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$ the fly will get warm as soon as possible.

6(c)(iii) Show that the vector field given by $A = 3x^2y\hat{i} + (x^3 - 2yz^2)\hat{j} + (3z^2 - 2y^2z)\hat{k}$ is irrotational but not solenoidal. Also find $\phi(x,y,z)$ such that $\nabla\phi = A$.

Sol^n: Given $A = 3x^2y\hat{i} + (x^3 - 2yz^2)\hat{j} + (3z^2 - 2y^2z)\hat{k}$ —— (1)

Curl $\vec{A} = 0$ then A is irrotational.

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2y & x^3 - 2yz^2 & 3z^2 - 2y^2z \end{vmatrix}$$

$$= \hat{i}(-4yz + 4yz) - \hat{j}(0-0) + \hat{k}(3x^2 - 3x^2)$$

$$= \hat{0}$$

$\therefore$ A is irrotational.

If div $A = 0$ then A is solenoidal

$$\text{div} A = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{A}$$

$$= 6xy - 2z^2 + 6z - 2y^2 \neq 0$$

$\therefore$ A is not solenoidal

To find $\nabla\phi = A$

As $\nabla \times \phi = 0$,

$\therefore \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \vec{A}$

$\frac{\partial\phi}{\partial x} = 3x^2y \Rightarrow \phi = x^3y + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = x^3 - 2yz^2 \Rightarrow \phi = x^3y - y^2z^2 + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = 3z^2 - 2y^2z \Rightarrow \phi = z^3 - y^2z^2 + f_3(x,y)$.

$\therefore$ from above equations.

$\boxed{\phi = x^3y - y^2z^2 + z^3}$ Required solution.

8(a)(i) Find $L\{F(t)\}$, where $F(t) = \begin{cases} \sin(t-\pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases}$

Sol^n: Let $\phi(t) = \sin t$

$$\therefore F(t) = \begin{cases} \phi(t-\pi/3), & t > \pi/3 \\ 0, & t < \pi/3. \end{cases}$$

we have $L(\phi(t)) = L\{\sin t\} = \frac{1}{p^2+1} = f(p)$

$\therefore$ from second shifting theorem, we have

$$L\{F(t)\} = e^{-(\pi/3)p} \cdot f(p) = e^{-\pi p/3} \frac{1}{p^2+1}, \ p>0.$$

(or)

we have $L\{F(t)\} = \int_0^\infty e^{-pt} F(t)\,dt$

$$= \int_0^{\pi/3} e^{-pt} \cdot 0\,dt + \int_{\pi/3}^\infty e^{-pt} \cdot \sin(t-\pi/3)\,dt$$

$$= \int_{\pi/3}^\infty e^{-pt} \sin(t-\pi/3)\,dt$$

$$= \int_0^\infty e^{-p[x+\pi/3]} \cdot \sin x\,dx$$, putting $t-\pi/3 = x$; so that $dt = dx$

$$= e^{-p(\pi/3)} \int_0^\infty e^{-px} \sin x\,dx$$

$$= e^{-p(\pi/3)} \int_0^\infty e^{-pt} \sin t\,dt$$

$$= e^{-p\pi/3} \cdot L\{\sin t\} = e^{-\pi p/3} \cdot \frac{1}{p^2+1}, \ p>0.$$

8(a)(ii) By using Laplace transform find the solution of initial value problem $(D^2+1)y = \sin t \sin 2t, t>0$ $y=1$, $Dy=0$ when $t=0$.

Sol^n: Given equation is

$$(D^2+1)y = \sin t \sin 2t$$

i.e. $y'' + y = \sin t \sin 2t$ —— ①.

Taking the Laplace transform of bothsides of equation ①, we get

$$L\{y''\} + L\{y\} = L\{\sin t \cdot \sin 2t\}$$

$$= \frac{1}{2} L\{\cos t - \cos 3t\}$$

$$p^2 L\{y(t) - py(0) - y'(0) + L\{y(t)\}\} = \frac{1}{2}\left[L(\cos t) - L(\cos 3t)\right]$$

$$\Rightarrow p^2 L\{y(t)\} - py(0) - y'(0) + L\{y(t)\} = \frac{1}{2}\frac{p}{p^2+1} - \frac{p}{2(p^2+9)} \quad \text{—— ②}$$

Using the given condition $y(0)=1$ and $y'(0)=0$ equation ② reduces to

$$p^2 L\{y(t)\} - p(1) - 0 + L\{y(t)\} = \frac{p}{2(p^2+1)} - \frac{p}{2(p^2+9)}$$

$$\Rightarrow (p^2+1) L\{y(t)\} - p = \frac{p}{2(p^2+1)} - \frac{p}{2(p^2+9)}$$

$$\Rightarrow L\{y(t)\} = \frac{p}{2(p^2+1)^2} - \frac{p}{2(p^2+1)(p^2+9)} + \frac{p}{p^2+1}$$

$$\Rightarrow y(t) = \frac{1}{2} L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} - \frac{1}{2} L^{-1}\left\{\frac{p}{(p^2+9)(p^2+1)}\right\} + L^{-1}\left(\frac{p}{p^2+1}\right) \quad \text{—— ③}$$

Now

$$L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = L^{-1}\left\{-\frac{1}{2}\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right\}$$

$$= (-1)t\left(-\frac{1}{2}\right)L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \frac{t}{2}\sin t$$

and

$$L^{-1}\left\{\frac{p}{(p^2+1)(p^2+9)}\right\} = L^{-1}\left\{\frac{1}{8}\left(\frac{-p}{p^2+9}+\frac{p}{p^2+1}\right)\right\}$$

$$= \frac{1}{8}\left\{L^{-1}\left(\frac{-p}{p^2+9}\right)+L^{-1}\left(\frac{p}{p^2+1}\right)\right\}$$

$$= \frac{1}{8}\left[-\cos 3t + \cos t\right]$$

$\therefore$ from ③

$$y(t) = \frac{1}{2}\,\frac{t}{2}\sin t - \frac{1}{2}\left\{\frac{1}{8}\left(-\cos 3t + \cos t\right)\right\} + \cos t$$

$$= \frac{t}{4}\sin t + \frac{1}{16}\cos 3t - \frac{1}{16}\cos t + \cos t$$

$$= \frac{t}{4}\sin t + \frac{1}{16}\cos 3t + \frac{15}{16}\cos t$$

7 a(i) → Solve $dy/dx = (x-2y+5)/(2x+y-1)$

Sol^n: Let $x = X+h$, $y = Y+K$

So that $dy/dx = \frac{dY}{dX}$ ——— (1)

Then given equation becomes

$$\frac{dY}{dX} = \frac{X-2Y+h-2K+5}{2X+Y+2h+K-1} \quad \text{——— (2)}$$

choose h and K so that

$$h-2K+5=0 \quad \text{and} \quad 2h+K-1=0 \quad \text{——— (3)}$$

(3) ⇒ $h = -3/5$, $K = 11/5$ so by (1)

$$X = x+3/5 \quad \text{and} \quad Y = y-11/5 \quad \text{——— (4)}$$

using equation (3), (2) becomes

$$\frac{dY}{dX} = \frac{X-2Y}{2X+Y} = \frac{1-2(Y/X)}{2+(Y/X)} \quad \text{——— (5)}$$

Putting $Y = Xv$ and $dY/dX = v + X(dv/dX)$, equation (5) gives

$$v + X\frac{dv}{dX} = \frac{1-2v}{2+v}$$

$$\text{or} \quad \frac{dX}{X} + \frac{1}{2}\,\frac{2v+4}{v^2+4v-1}\,dv = 0$$

Integrating, $\log X = (1/2)\log(v^2+4v-1) = (1/2)\log C$

or $x^2(v^2+4v-1)=c$

or $x^2(y^2/x^2+4y/x-1)=c$, as $v=y/x$

or $y^2+4xy-x^2=c$

or $(y-11/5)^2+4(x+3/5)(y-11/5)-(x+3/5)^2=c$

or $x^2-y^2-4xy+10x+2y=c_1$,

where 'c_1' is another arbitrary constant.

7. a(ii) → Solve $(x+1)(d^2y/dx^2) - 2(x+3)(dy/dx) + (x+5)y = e^x$.

Sol^n: Dividing by $(x+1)$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2(x+3)}{x+1}\frac{dy}{dx} + \frac{x+5}{x+1}y = \frac{e^x}{x+1} \qquad (1)$$

Comparing (1) with $y'' + Py' + Qy = R$ we get

$P = -2(x+3)/(x+1)$,

$Q = (x+5)/(x+1)$,

$R = e^x/(x+1)$ ——— (2)

Here $1+P+Q = 1 - \frac{2x+6}{x+1} + \frac{x+5}{x+1}$

$$= \frac{x+1-(2x+6)+x+5}{x+1} = 0,$$

Showing that $u = e^x$ ——— (3)

Let the general solution of (1) be

$y = uv$. ——— (4)

Then v is given by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left(-\frac{2x+6}{x+1} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{e^x}{e^x(x+1)}$$

or $$\frac{d^2v}{dx^2} + \left(2 - \frac{2x+6}{x+1}\right)\frac{dv}{dx} = \frac{1}{x+1}$$

or $\dfrac{d^2v}{dx^2} - \dfrac{4}{x+1}\dfrac{dv}{dx} = \dfrac{1}{x+1}$ ——— (5)

Let $dv/dx = q$ so that

$$d^2v/dx^2 = dq/dx \quad \text{——— (6)}$$

Then (5) becomes

$\dfrac{dq}{dx} - \dfrac{4}{x+1}q = \dfrac{1}{x+1}$, which is linear in q and x.

its integrating factor I.F. =

$$e^{-\int[4/(x+1)]dx} = e^{-4\log(x+1)} = (x+1)^{-4}.$$

and solution is

$$q(x+1)^{-4} = \int \frac{1}{x+1}\cdot(x+1)^{-4}dx + c_1$$

$$= \int (x+1)^{-5}dx + c_1$$

or $\dfrac{dv}{dx} = -(1/4) + c_1(x+1)^4$

or $dv = [-(1/4) + c_1(x+1)^4]dx.$

Integrating, $v = -(1/4)x + (c_1/5)(x+1)^5 + c_2$ ——— (7)

From (3), (4) and (7), the required general solution is:

$$y = uv = e^x[-(1/4)x + (c_1/5)(x+1)^5 + c_2]$$

or $y = c_1' e^x (x+1)^5 + c_2 e^x - (1/4)x e^x$, where $c_1' = c_1/5$.

7.(b) → A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$, $a>b$ and is projected from an apse at distance $(a+b)$ with velocity $\frac{\sqrt{\mu}}{(a+b)}$. Show that the equation of its path is $r = a+b\cos\theta$.

Solution :

Here, the central acceleration

$$P = \mu\{3au^4 - 2(a^2-b^2)u^5\}.$$

∴ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\{3au^4 - 2(a^2-b^2)u^5\}$$

$$\Rightarrow h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\{3au^2 - 2(a^2-b^2)u^3\}.$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left\{au^3 - 2(a^2-b^2)\frac{u^4}{4}\right\} + A$$

$$\Rightarrow v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\{2au^3 - (a^2-b^2)u^4\} + A$$

where A is constant. —— (1)

But initially at an apse, $r = a+b$, $u = 1/(a+b)$, $du/d\theta = 0$ and $v = \sqrt{\mu}/(a+b)$.

∴ from (1), we have,

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right] + A$$

$\therefore\ h^2 = \mu$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left\{2au^3 - (a^2-b^2)u^4\right\}$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = -u^2 + 2au^3 - (a^2-b^2)u^4 \qquad (2)$$

But $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$.

Substituting in (2), we have,

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2 + 2ar - (a^2-b^2)\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2$$

$$= b^2 - (r^2 - 2ar + a^2)$$

$$= b^2 - (r-a)^2$$

$$\therefore \frac{dr}{d\theta} = \sqrt{\{b^2 - (r-a)^2\}}$$

$$\Rightarrow d\theta = \frac{dr}{\sqrt{\{b^2 - (r-a)^2\}}}$$

Integrating, $\theta = \sin^{-1}\left(\frac{r-a}{b}\right) + B' \qquad (3)$

But initially, when $r = a+b$, let us take $\theta = 0$.

Then from (3), $B' = -\sin^{-1}(1) = -\pi/2$.

Substituting in (3), we have

$$\theta + \pi/2 = \sin^{-1}\left(\frac{r-a}{b}\right) \Rightarrow r - a = b\sin\left(\frac{\pi}{2}+\theta\right)$$

$\Rightarrow \boxed{r = a + b\cos\theta}$, which is the required equation of the path. Hence, proved.

7(c) → Evaluate $\iint_S (\nabla \times F)\cdot \hat{n}\, ds$, where $F = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ and S is the surface of the paraboloid $z = 4-(x^2+y^2)$ above the xy-plane.

Sol^n: The surface $z = 4-(x^2+y^2)$ meets the plane $z=0$ in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane S_1, we have $n = -\hat{k}$.

Note that $\hat{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl} F)\cdot n\, dS = \iiint_V \text{div}(\text{curl} F)\, dV$$

$$= 0 \qquad (\because \text{div curl} F = 0)$$

$$\therefore \iint_S \text{curl} F\cdot n\, ds + \iint_{S_1} \text{curl} F\cdot \hat{n}\, ds = 0$$

$(\because S'$ consists of S and $S_1)$

$\Rightarrow \iint_S (\text{curl} F)\cdot \hat{n}\, ds - \iint_{S_1} (\text{curl} F)\cdot \hat{k}\, ds = 0$ $(\because$ on S_1, $\hat{n} = -k)$

$\Rightarrow \iint_S (\text{curl} F)\cdot n\, ds = \iint_{S_1} (\text{curl} F)\cdot k\, ds$

Now $\text{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$= 0\hat{i} - 2z\hat{j} + (3y-1)k$

$\therefore (\text{curl} F)\cdot k = [-2z\hat{j} + (3y-1)k]\cdot k$

$= 3y-1$ over the surface S_1 bounded by the circle $x^2+y^2=4,\ z=0$

Hence $\iint_S (\text{curl} F)\cdot \hat{n}\, ds = \iint_{S_1} (3y-1)\, ds$

$= \int_{-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (3y-1)\, dx\, dy$

$= 2\int_{x=-2}^{2} \int_{y=0}^{\sqrt{4-x^2}} (-1)\, dx\, dy$ $(\because 3y$ is an odd function of $y)$

$= -2\int_{-2}^{2} [y]_0^{\sqrt{4-x^2}}$

$= -2\int_{-2}^{2} \sqrt{4-x^2}\, dx = -4\int_0^2 \sqrt{4-x^2}\, dx$

$= -4\left[\frac{x}{2}\left(\sqrt{4-x^2}\right) + 2\sin^{-1}\frac{x}{2}\right]$

$= -4\left[2\cdot\frac{\pi}{2}\right] = -4\pi$ Ans.

8.(b) → If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are $\alpha, \alpha-\beta, \alpha-2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that $v_3 t_1 = v_1 t_2$ and

$$\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}.$$

Solution:-

Since, the horizontal velocity of a projectile remains constant throughout the motion, therefore.

$$v_1 \cos\alpha = v_2 \cos(\alpha-\beta) = v_3 \cos(\alpha-2\beta) \quad \text{---} (1)$$

considering the verticle motion from P to Q and then from Q to R and using the formula

$v = u + ft$, we get

$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - gt_1 \quad \text{---} (2)$$

and $$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - gt_2 \quad \text{---} (3)$$

from (2) and (3), we have

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - v_2 \sin(\alpha-\beta)}{v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - \dfrac{v_1\cos\alpha}{\cos(\alpha-\beta)} \cdot \sin(\alpha-\beta)}{v_3 \dfrac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)} \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

[Substituting suitably for v_2 from (1)]

$$\Rightarrow \frac{t_1}{t_2} = \frac{v_1\left[\sin\alpha\cos(\alpha-\beta) - \cos\alpha\sin(\alpha-\beta)\right]}{v_3\left[\sin(\alpha-\beta)\cos(\alpha-2\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)\right]}$$

$$\Rightarrow \frac{t_1}{t_2} = \frac{v_1\sin\{\alpha-(\alpha-\beta)\}}{v_3\sin\{(\alpha-\beta)-(\alpha-2\beta)\}}$$

$$\Rightarrow \boxed{\frac{t_1}{t_2} = \frac{v_1\sin\beta}{v_3\sin\beta} = \frac{v_1}{v_3}}$$

$$\therefore \boxed{v_3 t_1 = v_1 t_2}$$

This proves the first result.

Again from ①, we have

$$\frac{1}{v_1} = \frac{1}{v_2}\cdot\frac{\cos\alpha}{\cos(\alpha-\beta)} \quad \text{and} \quad \frac{1}{v_3} = \frac{1}{v_2}\cdot\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

$$\therefore \frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\left[\frac{\cos\alpha + \cos(\alpha-2\beta)}{\cos(\alpha-\beta)}\right]$$

$$\frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\left[\frac{2\cos(\alpha-\beta)\cos\beta}{\cos(\alpha-\beta)}\right]$$

$$\boxed{\frac{1}{v_1} + \frac{1}{v_3} = \frac{2\cos\beta}{v_2}}$$

This proves the second result.

8.(c) (i) Find div grad r^m and verify that $\nabla\times\nabla r^m = 0$

(ii) If $\vec{f} = \nabla(\vec{a}\cdot\nabla\vec{r})$, show that div $\vec{f}=0$ and $\vec{f} = \text{curl}\,\vec{g}$ where $\vec{g} = -\vec{a}\times\nabla\vec{r}$.

Sol:
$$\text{div grad } r^m = \text{div}(m r^{m-1}\,\text{grad } r)$$
$$= \text{div}\left(m r^{m-1}\tfrac{1}{r}\vec{r}\right)$$
$$= \text{div}\left(m r^{m-2}\vec{r}\right)$$
$$= m r^{m-2}\,\text{div}\,\vec{r} + \vec{r}\cdot(\text{grad } m r^{m-2})$$
$$= 3m r^{m-2} + \vec{r}\cdot\left[m(m-2)r^{m-3}\,\text{grad } r\right]$$
$$= 3m r^{m-2} + \vec{r}\cdot\left[m(m-2)r^{m-3}\tfrac{1}{r}\vec{r}\right]$$
$$= 3m r^{m-2} + \vec{r}\cdot\left[m(m-2)r^{m-4}\vec{r}\right]$$
$$= 3m r^{m-2} + m(m-2)r^{m-4}(\vec{r}\cdot\vec{r})$$
$$= 3m r^{m-2} + m(m-2)r^{m-2}$$
$$= m r^{m-2}(3+m-2)$$
$$= m(m+1)r^{m-2}.$$

$$\therefore \text{div}(\text{grad } r^m) = m(m+1)r^{m-2}$$

To find $\nabla\times\nabla r^m$:

we have $\nabla r^m = \frac{\partial r^m}{\partial x}\hat{i} + \frac{\partial r^m}{\partial y}\hat{j} + \frac{\partial r^m}{\partial z}\hat{k}$.

$$= mr^{m-2}x\hat{i} + mr^{m-2}y\hat{j} + mr^{m-2}z\hat{k}.$$

$$\nabla\times\nabla r^m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ mr^{m-2}x & mr^{m-2}y & mr^{m-2}z \end{vmatrix}$$

$$= \hat{i}\left[m(m-2)\,r^{m-3}(yz - yz)\right]$$
$$+\hat{j}\left[m(m-2)\,r^{m-3}(xz - xz)\right]$$
$$+\hat{k}\left[m(m-2)\,r^{m-3}(xy - yx)\right]$$
$$= \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$
$$= 0.$$

we have $\nabla r^{-1} = \nabla\left(\frac{1}{r}\right)$

$$= -\frac{1}{r^2}\nabla r$$
$$= -\frac{1}{r^2}\frac{1}{r}\vec{r} = -\frac{1}{r^3}\vec{r}$$

$$\Rightarrow \vec{a}\cdot\nabla r^{-1} = \vec{a}\cdot\left(-\frac{1}{r^3}\vec{r}\right) = -\frac{\vec{a}\cdot\vec{r}}{r^3}$$

Now $f = \nabla\left(\vec{a}\cdot\nabla r^{-1}\right)$

$$= \nabla\left(-\frac{\vec{a}\cdot\vec{r}}{r^3}\right)$$
$$= \sum \hat{i}\frac{\partial}{\partial x}\left(-\frac{\vec{a}\cdot\vec{r}}{r^3}\right)$$
$$= \sum \hat{i}\left\{-\frac{1}{r^3}\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$
$$= \sum \hat{i}\left\{-\frac{1}{r^3}\left(\vec{a}\cdot\frac{\partial\vec{r}}{\partial x}\right) + 3(\vec{a}\cdot\vec{r})r^{-4}\frac{\partial r}{\partial x}\right\}$$
$$= \sum \hat{i}\left\{-\frac{\vec{a}\cdot\hat{i}}{r^3} + \frac{3x}{r^5}(\vec{a}\cdot\vec{r})\right\} \quad \left(\because \frac{\partial\vec{r}}{\partial x} = \hat{i} \;\&\; \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$
$$= \sum\left\{-\frac{1}{r^3}(\vec{a}\cdot\hat{i})\hat{i} + \frac{3}{r^5}(\vec{a}\cdot\vec{r})\,x\hat{i}\right\}$$

$$= -\frac{1}{r^3}\vec{a} + \frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r} \longrightarrow (1)$$ $\left(\because \sum(a\cdot\hat{i})\hat{i} = \vec{a} \text{ \& } \sum x\hat{i} = \vec{r}\right)$

Now $\nabla\cdot f = \nabla\cdot\left\{-\frac{1}{r^3}\vec{a} + \frac{3}{r^5}(\vec{a}\cdot\vec{r})\vec{r}\right\}$

$$= \nabla\cdot\left\{-\frac{1}{r^3}\vec{a}\right\} + 3\nabla\cdot\left\{\frac{1}{r^5}(\vec{a}\cdot\vec{r})\vec{r}\right\}$$

$$= -\left[\nabla\left(\frac{1}{r^3}\right)\cdot\vec{a} + \frac{1}{r^3}(\nabla\cdot\vec{a})\right] + 3\left[(\nabla\cdot\vec{r})\left(\frac{\vec{a}\cdot\vec{r}}{r^5}\right) + \vec{r}\cdot\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^5}\right)\right]$$

$$= -\left[-3\left(\frac{\vec{r}\cdot\vec{a}}{r^5}\right) + 0\right] + 3\left[3\frac{(\vec{a}\cdot\vec{r})}{r^5} + \vec{r}\cdot\sum i\frac{\partial}{\partial x}\left(\frac{\vec{a}\cdot\vec{r}}{r^5}\right)\right]$$

$$= 3\frac{(\vec{r}\cdot\vec{a})}{r^5} + 9\frac{(\vec{r}\cdot\vec{a})}{r^5} + 3\vec{r}\cdot\sum i\left[\frac{1}{r^5}\frac{\partial}{\partial x}(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\frac{\partial}{\partial x}\left(\frac{1}{r^5}\right)\right]$$

$$= \frac{12(\vec{a}\cdot\vec{r})}{r^5} + 3\vec{r}\cdot\sum i\left[\frac{1}{r^5}\left(\vec{a}\cdot\frac{\partial\vec{r}}{\partial x}\right) - \frac{5(\vec{a}\cdot\vec{r})}{r^6}\frac{\partial r}{\partial x}\right]$$

$$= \frac{12(\vec{a}\cdot\vec{r})}{r^5} + 3\vec{r}\cdot\sum i\left[\frac{1}{r^5}(\vec{a}\cdot\hat{i}) - \frac{5}{r^7}(\vec{a}\cdot\vec{r})x\right]$$

$$= \frac{12(\vec{a}\cdot\vec{r})}{r^5} + 3\vec{r}\cdot\left[\sum\frac{1}{r^5}(\vec{a}\cdot\hat{i})\hat{i} - \frac{5}{r^7}(\vec{a}\cdot\vec{r})x\hat{i}\right]$$

$$= \frac{12(\vec{a}\cdot\vec{r})}{r^5} + 3\vec{r}\cdot\left[\frac{\vec{a}}{r^5} - \frac{5}{r^7}(\vec{a}\cdot\vec{r})\vec{r}\right]$$

$$= \frac{12(\vec{a}\cdot\vec{r})}{r^5} + 3\left(\frac{\vec{r}\cdot\vec{a}}{r^5}\right) - \frac{15}{r^7}(\vec{a}\cdot\vec{r})(\vec{r}\cdot\vec{r})$$

$$= \frac{15(\vec{a}\cdot\vec{r})}{r^5} - \frac{15(\vec{a}\cdot\vec{r})}{r^5} = 0$$

$\therefore\ \text{div} f = 0$

<u>To show that</u> $f = \text{curl } g$:

$$\text{curl } g = \nabla\times g = \nabla\times(\vec{a}\times\nabla r^{-1}) = \nabla\times\left(\vec{a}\times\frac{1}{r^3}\vec{r}\right)$$

$$= \nabla\times\left[\frac{1}{r^3}(\vec{a}\times\vec{r})\right]$$

$$= \nabla\left(\frac{1}{r^3}\right)\times(\vec{a}\times\vec{r}) + \frac{1}{r^3}\left[\nabla\times(\vec{a}\times\vec{r})\right]$$

$\left(\because \nabla\times\phi\vec{A} = \nabla\phi\times\vec{A} + \phi(\nabla\times\vec{A})\right)$

$$= -\frac{3}{r^4}\left(\frac{\vec{r}}{r}\right) \times (\vec{a} \times \vec{r})$$
$$+ \frac{1}{r^3}\left[\vec{a}(\nabla \cdot \vec{r}) - \vec{r}(\nabla \cdot \vec{a}) + (\vec{r} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{r}\right]$$

$$= -\frac{3\vec{r}}{r^5} \times (\vec{a} \times \vec{r}) + \frac{1}{r^3}(3\vec{a} - 0 + 0 - \vec{a})$$

$$\left(\because \nabla \cdot \vec{a} = 0,\ (\vec{r} \cdot \nabla)\vec{a} = 0 \ \&\ (\vec{a} \cdot \nabla)\vec{r} = \vec{a}\right)$$

$$= -\frac{3}{r^5}\left[(\vec{r} \cdot \vec{r})\vec{a} - (\vec{r} \cdot \vec{a})\vec{r}\right] + \frac{2\vec{a}}{r^3}$$

$$= \frac{-3r^2\vec{a}}{r^5} + \frac{3(\vec{r} \cdot \vec{a})\vec{r}}{r^5} + \frac{2\vec{a}}{r^3}$$

$$= \frac{-3\vec{a}}{r^3} + \frac{3(\vec{r} \cdot \vec{a})}{r^5}\vec{r} + \frac{2\vec{a}}{r^3}$$

$$= \frac{-\vec{a}}{r^3} + \frac{3}{r^5}(\vec{r} \cdot \vec{a})\vec{r}$$

$$= f \quad (\text{by } ①)$$

$\therefore f = \text{curl } g$

Hence the proof.