Solutions to Practice Problems

Exercise 8.8
(a) Show that if $\{a_n\}_{n=1}^{\infty}$ is Cauchy then $\{a_n^2\}_{n=1}^{\infty}$ is also Cauchy.
(b) Give an example of a Cauchy sequence $\{a_n^2\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty}$ is not Cauchy.

Solution.
(a) Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, it is convergent. Since the product of two convergent sequences is convergent the sequence $\{a_n^2\}_{n=1}^{\infty}$ is convergent and therefore is Cauchy.
(b) Let $a_n = (-1)^n$ for all $n \in \mathbb{N}$. The sequence $\{a_n\}_{n=1}^{\infty}$ is not Cauchy since it is divergent. However, the sequence $\{a_n^2\}_{n=1}^{\infty} = \{1, 1, \cdots\}$ converges to 1 so it is Cauchy ■

Exercise 8.9
Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence such that a_n is an integer for all $n \in \mathbb{N}$. Show that there is a positive integer N such that $a_n = C$ for all $n \geq N$, where C is a constant.

Solution.
Let $\epsilon = \frac{1}{2}$. Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, there is a positive integer N such that if $m, n \geq N$ we have $|a_m - a_n| < \frac{1}{2}$. But $a_m - a_n$ is an integer so we must have $a_n = a_N$ for all $n \geq N$ ■

Exercise 8.10
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that satisfies

$$|a_{n+2} - a_{n+1}| < c^2|a_{n+1} - a_n| \text{ for all } n \in \mathbb{N}$$

where $0 < c < 1$.
(a) Show that $|a_{n+1} - a_n| < c^n|a_2 - a_1|$ for all $n \geq 2$.
(b) Show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution.
(a) See Exercise 1.10.

(b) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} c^n = 0$ we can find a positive integer N such that if $n \geq N$ then $|c|^n < (1-c)\epsilon$. Thus, for $n > m \geq N$ we have

$$\begin{aligned}
|a_n - a_m| \leq & |a_{m+1} - a_m| + |a_{m+2} - a_{m+1}| + \cdots + |a_n - a_{n-1}| \\
< & c^m|a_2 - a_1| + c^{m+1}|a_2 - a_1| + \cdots + c^{n-1}|a_2 - a_1| \\
< & c^m(1 + c + c^2 + \cdots)|a_2 - a_1| \\
= & \frac{c^m}{1-c}|a_2 - a_1| < \epsilon
\end{aligned}$$

It follows that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence ■

Exercise 8.11
What does it mean for a sequence $\{a_n\}_{n=1}^{\infty}$ to not be Cauchy?

Solution.
A sequence $\{a_n\}_{n=1}^{\infty}$ is not a Cauchy sequence if there is a real number $\epsilon > 0$ such that for all positive integers N there exist $n, m \in \mathbb{N}$ such that $n, m \geq N$ and $|a_n - a_m| \geq \epsilon$ ■

Exercise 8.12
Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences. Define $c_n = |a_n - b_n|$. Show that $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Solution.
Let $\epsilon > 0$ be given. There exist positive integers N_1 and N_2 such that if $n, m \geq N_1$ and $n, m \geq N_2$ we have $|a_n - a_m| < \frac{\epsilon}{2}$ and $|b_n - b_m| < \frac{\epsilon}{2}$. Let $N = N_1 + N_2$. If $n, m \geq N$ then $|c_n - c_m| = ||a_n - b_n| - |a_m - b_m|| \leq |(a_n - b_n) + (a_m - b_m)| \leq |a_n - a_m| + |b_n - b_m| < \epsilon$. Hence, $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence ■

Exercise 8.13
Explain why the sequence defined by $a_n = (-1)^n$ is not a Cauchy sequence.

Solution.
We know that every Cauchy sequence is convergent. We also know that the given sequence is divergent. Thus, it can not be Cauchy ■

Exercise 8.14
Show that every subsequence of a Cauchy sequence is itself a Cauchy sequence.

Solution.
Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. By Exercise 8.7, the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent. By Exercise 7.4, $\{a_{n_k}\}_{k=1}^{\infty}$ is convergent and hence Cauchy $\blacksquare$

Exercise 8.15
Prove that if a subsequence of a Cauchy sequence converges to L, then the full sequence also converges to L.

Solution.
Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$ converging to L. By Exercise **??**, the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent say to a limit L'. By Exercise **??**, we must have $L = L'$ $\blacksquare$

Exercise 8.16
Prove directly from the definition that the sequence

$$a_n = \frac{n+3}{2n+1}, \; n \in \mathbb{N}$$

is a Cauchy sequence.

Solution.
Let $\epsilon > 0$ be given. Let N be a positive integer to be chosen. Suppose that $n, m \geq N$. We have

$$\begin{aligned}
|a_n - a_m| = &\left|\frac{n+3}{2n+1} - \frac{m+3}{2m+1}\right| = 3\frac{|m-n|}{(2n+1)(2m+1)} \\
\leq &\frac{2m+2n}{(2n+1)(2m+1)} = \frac{(2n+1)+(2m+1)-2}{(2n+1)(2m+1)} \\
= &\frac{1}{2m+1} + \frac{1}{2n+1} - \frac{2}{(2n+1)(2m+1)} \\
\leq &\frac{1}{2m+1} + \frac{1}{2n+1} \\
\leq \frac{2}{2N+1} &
\end{aligned}$$

Choose N so that $\frac{2}{2N+1} < \epsilon$. That is $N > \frac{2-\epsilon}{2\epsilon}$. In this case,

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$. That is, $\{\frac{n+3}{2n+1}\}_{n=1}^{\infty}$ is Cauchy $\blacksquare$

Exercise 8.17
Consider a sequence defined recursively by $a_1 = 1$ and $a_{n+1} = a_n + (-1)^n n^3$ for all $n \in \mathbb{N}$. Show that such a sequence is not a Cauchy sequence. Does this sequence converge?

Solution.
We will show that there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exist m and n such that $m, n \geq N$ but $|a_m - a_n| \geq \epsilon$. Note that $|a_{n+1} - a_n| = n^3 \geq 1$. Let $\epsilon = 1$. Let $N \in \mathbb{N}$. Choose $m = N + 1$ and $n = N$. In this case, $|a_m - a_n| = N^3 \geq 1 = \epsilon$. Hence, the given sequence is not a Cauchy sequence. Since every convergent sequence must be Cauchy, the given sequence is divergent ■