$\left[\begin{array}{l}\text{Series} \to f_1(x), f_2(x), \dots, f_n(x) \\ \text{sequence} \to f_1(x)+f_2(x)+\dots+f_n(x)\end{array}\right]$

Set – VII

* Uniform Convergence *

[Sequences and Series of functions]

$\left[\begin{array}{l}\text{sequence } \lim_{n\to\infty} f_n(x) \\ \text{series} = \sum_{n=1}^{\infty} f_n(x)\end{array}\right]$

* Just as sequence and series of real numbers play a fundamental role in analysis.

Sequence and series of functions are also important elements of modern analysis.

In many situations, we come across these elements, particularly in connection with convergence.

→ **Sequences of real-valued functions :–**

Let f_n be a real valued function defined on an interval I (or on a subset D of $\mathbb{R}$) and for each $n \in N$.

Then $\{f_1, f_2, f_3, \dots f_n, \dots\}$ is called a sequence of real-valued functions on I. It is denoted by $\{f_n : I \to \mathbb{R}, n \in N\}$ (or) by $\{f_n\}$ (or) $\langle f_n \rangle$ (or) (f_n).

for example :

(i) If f_n is a real valued function defined by $f_n(x) = x^n$, $0 \le x \le 1$

then $\{f_1(x), f_2(x), f_3(x), \dots\}$

$= \{x^1, x^2, x^3, \dots\}$

is a sequence of a real valued functions on $[0,1]$.

Example (2) :–

If f_n is a real-valued function defined by

$$f_n(x) = \frac{\sin nx}{n}, \quad 0 \le x \le 1.$$

then $\{f_1(x), f_2(x), f_3(x), \dots\}$

$= \left\{\sin x, \frac{\sin 2x}{2}, \frac{\sin 3x}{3}, \dots\right\}$ is a sequence of real valued functions on $[0,1]$

→ If $\{f_n\}$ is a sequence of functions defined on I, then for $c \in I$,

$\{f_n(c)\} = \{f_1(c), f_2(c), \dots f_n(c), \dots\}$ is a sequence of real numbers.

For example :

If $\{f_n\}$ is a sequence of functions defined by $f_n(x) = x^n$, $0 \le x \le 1$, then

$\{f_n(\tfrac{1}{2})\} = \{f_1(\tfrac{1}{2}), f_2(\tfrac{1}{2}), f_3(\tfrac{1}{2}), \dots f_n(\tfrac{1}{2}), \dots\}$

$\left\{\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots \frac{1}{2^n}, \dots\right\}$

is a sequence of real numbers corresponding to $\frac{1}{2} \in [0,1]$

∴ to each $x \in I$, we have a sequence of real numbers.

→ **Pointwise convergence of a sequence of functions :–**

Let $\{f_n\}$ be a sequence of functions on I and $c \in I$. Then the sequence of real numbers $\{f_n(c)\}$ may be convergent.

In fact for each $c \in I$, the corresponding sequence of real numbers may be convergent.

If $\{f_n\}$ is a sequence of real-valued functions on I and for each $x \in I$, the corresponding sequence of real numbers is convergent, then we say the sequence $\{f_n\}$ converges pointwise. The limiting values of the sequences of real numbers corresponding to $x \in I$ define a function 'f' called the limit function (or) simply the limit of the sequence $\{f_n\}$ of functions on I.

*** Definition:**

Let $\{f_n\}$ be a sequence of functions on I. If to each $x \in I$ and to each $\epsilon > 0$, there corresponds to positive integer 'm' such that $|f_n(x) - f(x)| < \epsilon \ \forall \ n \geq m$ then we say that $\{f_n\}$ converges pointwise to the function f on I.

Note(1):- $\{f_n\}$ converges pointwise to the function 'f' on $I \Leftrightarrow \lim_{n\to\infty} f_n(x) = f(x) \ \forall x \in I$

$f(x)$ is called the limit function (or) simply the limit (or) the pointwise limit of $\{f_n(x)\}$ on I.

Note(2):- The positive integer 'm' depends on $x \in I$ and given $\epsilon > 0$ i.e. $m = m(x, \epsilon)$

Ex:- ① Let $f_n(x) = x^n, x \in [0,1]$.

Since $\lim_{n\to\infty} x^n = 0$ for $0 \leq x < 1$.

$$\boxed{\therefore \lim_{n\to\infty} f_n(x) = 0 \quad \text{for } 0 \leq x < 1}$$

When $x = 1$, the corresponding sequence $\{f_n(x)\} = \{1, 1, 1, 1, \ldots\}$ is a constant sequence converging to 1.

$$\therefore \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ 1 & \text{when } x = 1 \end{cases}$$

Hence $\{f_n\}$ converges pointwise on $[0,1]$.

$f(x) = \begin{cases} 0 & \text{when } 0 \leq x < 1 \\ 1 & \text{when } x = 1 \end{cases}$ is the limit function of $\{f_n(x)\}$ on $[0,1]$.

$\rightarrow$ Let $\epsilon = \frac{1}{2}$ be given

Then for each $x \in [0,1]$, there exists a +ve integer 'm' such that

$|f_n(x) - f(x)| < \frac{1}{2} \ \forall \ n \geq m$ —①

If $x = 0$, $f(x) = 0$ and $f_n(x) = 0 \ \forall n \in N$

$\therefore |f_n(x) - f(x)| = |0 - 0|$

ϵ

$= 0 < \frac{1}{2} \ \forall n \geq 1$

$\therefore$ ① is true for $m=1$ when $x=0$.

Similarly ① is true for $m=1$ when $x=1$.

If $x = 3/4$, $f(x)=0$ and $f_n(x) = (3/4)^n$

$\therefore |f_n(x) - f(x)| = |(3/4)^n - 0| = (3/4)^n < \frac{1}{2}$

$\forall n \geq 3$.

$\therefore$ ① is true for $m=3$ when $x = 3/4$.

Similarly ① is true for $m=7$ when $x = 9/10$

$\therefore$ There is no single value of 'm' for which ① holds $\forall x \in [0,1]$.

i.e. 'm' is depending on x & ϵ.

Example ②: Let $f_n(x) = \dfrac{x}{1+nx}$, $x \geq 0$.

then for $x>0$, $\lim_{n\to\infty} f_n(x) = 0$

Also $f_n(0) = 0 \ \forall n \in N$ so that $\{f_n(0)\}$ converges to '0'

$\therefore \lim_{n\to\infty} f_n(x) = 0 \ \forall x \geq 0$.

Hence $\{f_n(x)\}$ converges to zero pointwise on $[0,\infty)$ and $f(x)=0$ is the limit function of $\{f_n(x)\}$ on $[0,\infty)$.

Example(3): Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

For $x \neq 0$, $f_n(x) = \dfrac{1/nx}{\frac{1}{n^2x^2}+1} \to 0$ as $n \to \infty$

Also $f_n(0) = 0 \ \forall n \in N$

$\therefore \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in \mathbb{R}$.

Hence $\{f_n(x)\}$ converges to zero pointwise on $\mathbb{R}$ and $f(x)=0$ is the limit function of $\{f_n(x)\}$ on $\mathbb{R}$.

Note (3): For a sequence $\{f_n\}$ of functions, an important question is:

If each function of a sequence $\{f_n\}$ has a certain property such as continuity, differentiability (or) integrability, then to what extent is this property transferred to the limit function f?

In fact pointwise convergence is not strong enough to transfer any of the properties mentioned above from the terms f_n of $\{f_n\}$ to the limit function.

Let us consider a few examples:-

(i) A sequence of continuous functions with a discontinuous limit function.

Consider the sequence $\{f_n\}$ where

$$f_n(x) = \frac{x^{2n}}{1+x^{2n}}, \ x \in \mathbb{R}.$$

Then $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } |x|<1 \text{ i.e. } -1<x<1 \\ \frac{1}{2} & \text{if } |x|=1 \\ 1 & \text{if } |x|>1 \end{cases}$

Here each f_n is continuous on $\mathbb{R}$ but 'f' is discontinuous at $x = \pm 1$.

(ii) A sequence of differentiable functions in which the limit of the derivatives is not equal to the derivative of the limit function.

Consider the sequence $\{f_n\}$ where

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}, \quad x \in \mathbb{R}$$

Then $f(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0 \quad \forall x \in \mathbb{R}$

i.e. $f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in \mathbb{R}$

Now $f'(x) = 0 \quad \forall x \in \mathbb{R} \Rightarrow f'(0) = 0$,

and $f_n'(x) = \sqrt{n}\cos nx$.

$\Rightarrow f_n'(0) = \infty$ as $n \to \infty$ $\left(\because \cos 0 = 1 \text{ and } \sqrt{n} \to \infty\right)$

$\therefore$ At $x = 0$. $\lim_{n\to\infty} f_n'(x) \neq f'(0)$.

iii) A sequence of functions in which the limit of integrals is not equal to the integral of the limit function.

Consider the sequence $\{f_n\}$ where

$$f_n(x) = nx(1-x^2)^n;\ x \in [0,1].$$

Then $f_n(x) = 0$ when $x = 0$ or $x = 1$

Also if $0 < x < 1$ then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x^2)^n \quad \Big|\ \text{form } \infty \times 0.$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x^2)^{-n}} \quad \Big|\ \text{Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x^2)^{-n}\log(1-x^2)} = 0$$

$\therefore f(x) = 0 \quad \forall x \in [0,1]$ $\quad \left[\frac{d}{dx}a^x = a^x \log a,\ \frac{d}{dx}a^{-x} = -a^{-x}\log a\right]$

i.e. $f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1]$.

Now $\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x^2)^n\,dx$

$$= -\frac{n}{2}\int_0^1 (1-x^2)^n(-2x)\,dx$$

$$= -\frac{n}{2}\left[\frac{(1-x^2)^{n+1}}{n+1}\right]_0^1$$

$$= -\frac{n}{2}\left[0 - \frac{1}{n+1}\right]$$

$$= \frac{n}{2(n+1)}$$

$$\Rightarrow \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$$

Also $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

The above few examples show that we need to investigate under what supplementary conditions these or other properties of the terms f_n of $\{f_n(x)\}$ are transformed to the limit function f. A concept of great importance in this respect is that known as uniform convergence.

* Uniform Convergence of Sequence of functions:-

Let $\{f_n\}$ be a sequence of functions on I then $\{f_n\}$ is said to be uniformly convergent to a function f on I, if to each $\epsilon > 0$, there exists a positive integer 'm' (depending on ϵ only) such that $|f_n(x) - f(x)| < \epsilon \ \forall\ n \geq m$ and $\forall\ x \in I$.

The function 'f' is called uniform limit of the sequence $\{f_n\}$ on I.

Ex:- Now consider the sequence $\{f_n\}$ defined by $f_n(x) = \dfrac{x}{1+nx}$, $x \geq 0$.

It converges pointwise to zero i.e. $f(x) = 0 \ \forall\ x \geq 0$.

Now $0 \leq f_n(x) = \dfrac{x}{1+nx} \leq \dfrac{x}{nx} = \frac{1}{n}$

$\therefore$ for any $\epsilon > 0$,

$|f_n(x) - f(x)| = |f_n(x)| \leq \frac{1}{n} < \epsilon$

whenever $n > \frac{1}{\epsilon}$ $\forall\ x \in [0, \infty)$.

If m is a +ve integer $> \frac{1}{\epsilon}$, then

$|f_n(x) - f(x)| < \epsilon \ \forall\ n \geq m$ and $x \in [0, \infty)$

Thus, in this example, we can find an 'm' which depends only on ϵ and not on $x \in [0, \infty)$.

We say that the sequence $\{f_n\}$ is uniformly convergent to 'f' on $[0, \infty)$.

Note(1) :- If a uniform 'm' is found for all x values of I, the sequence $\{f_n\}$ is uniformly converges to 'f' on I.

Note(2) :- A uniformly convergence sequence is a pointwise convergence sequence i.e.

uniform convergence $\Rightarrow$ pointwise convergence.

However, the converse is not true.

For example : If $f_n(x) = x^n \ \forall\ x \in [0,1]$ then the sequence $\{f_n\}$ converges pointwise to the function 'f' on $[0,1]$.

where $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

but $\{f_n\}$ does not converge uniformly on $[0,1]$.

Note(3) :- A sequence $\{f_n\}$ of functions defined on I does not converge uniformly to f on I iff there exists some $\epsilon > 0$ such that there is no positive integer 'm' for which the statement $|f_n(x) - f(x)| < \epsilon \ \forall\ n \geq m$ and $\forall\ x \in I$ holds.

Definition:

* Uniformly Bounded Sequence of functions:-

A sequence $\{f_n\}$ of functions defined on I is said to be uniformly bounded on I if $\exists$ a +ve real number K such that

$|f_n(x)| < K \ \forall\ n \in N$ and $\forall\ x \in I$.

The number K is called a uniform bound for $\{f_n\}$ on I.

Ex:- If $f_n(x) = \sin nx$, $\forall x \in \mathbb{R}$ then.

$|f_n(x)| = |\sin nx| \leq 1$ $\forall n \in \mathbb{N}$ and $x \in \mathbb{R}$

. The sequence $\{f_n\}$ is uniformly bounded on $\mathbb{R}$.

theorem (Cauchy's Criterion for Uniform Convergence):-

A sequence $\{f_n\}$ of functions defined on I is uniformly convergent on I iff for each $\epsilon > 0$ and for all $x \in I$, $\exists$ a +ve integer 'm' such that for any integer $p \geq 1$, $|f_{n+p}(x) - f_n(x)| < \epsilon$ $\forall n \geq m$.

Problems:

→ Show that the sequence $\{f_n\}$ where $f_n(x) = x^n$ is uniformly convergent on $[0,k]$; $k<1$ but only pointwise convergent on $[0,1]$.

Sol'n: Here $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

the sequence $\{f_n\}$ converges pointwise to f on $[0,1]$.

To see whether the sequence $\{f_n\}$ is uniformly convergent.

Let $\epsilon > 0$ be given.

For $0<x<1$, $|f_n(x) - f(x)| = |x^n - 0|$

$= x^n < \epsilon$

whenever $\frac{1}{x^n} > \frac{1}{\epsilon}$;

i.e. whenever $n \log \frac{1}{x} > \log \frac{1}{\epsilon}$

i.e. whenever $n > \dfrac{\log 1/\epsilon}{\log 1/x}$ (Note $0<x<1$ $\Rightarrow \frac{1}{x} > 1$ $\Rightarrow \log \frac{1}{x} > 0$)

the number $\dfrac{\log 1/\epsilon}{\log 1/x}$ increases with x having maximum value $\dfrac{\log 1/\epsilon}{\log 1/k}$ on $(0,k]$, $k<1$.

Choose a positive integer $m > \dfrac{\log 1/\epsilon}{\log 1/k}$,

then $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $0<x<1$

At $x=0$, $|f_n(x) - f(x)| = |0-0| = 0 < \epsilon$ $\forall n \geq 1$.

$\therefore \exists$ a +ve integer 'm' such that $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in [0,k]$, $k<1$.

$\Rightarrow \{f_n\}$ is uniformly convergent on $[0,k]$, $k<1$, when $x \to 1$, the number $\dfrac{\log 1/\epsilon}{\log 1/x} \to \infty$.

Thus it is not possible to find a positive integer 'm' such that

$|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m$ and $\forall x \in [0,1]$. Hence the sequence $\{f_n\}$ is not uniformly convergent on any interval containing 1 and in particular on $[0,1]$.

A Test For Uniform Convergence of Sequences of Functions:—

To determine whether a given sequence $\{f_n\}$ is uniformly convergent (or) not in a given interval, we have been using the definition of uniform convergence. Thus, we find a +ve integer m, independent of x which is not easy in most of the cases. The following test is more convenient in practice and does not involve the computation of m.

* Theorem (M_n Test):—

Let $\{f_n\}$ be a sequence of functions on I such that $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\ x \in I$.

and let $M_n = \text{Sup}\{|f_n(x) - f(x)| / x \in I\}$

Then $\{f_n\}$ converges uniformly on I iff $\lim_{n\to\infty} M_n = 0$.

Note (1): M_n = the maximum value of $|f_n(x) - f(x)|$ for fixed n and $x \in I$.

Note (2): If M_n does not tend to '0'. then the sequence $\{f_n\}$ is not uniformly convergent.

Note (3): $F(x)$ is maximum at $x = c \in I$ if (i) $F'(c) = 0$ and (ii) $F''(c) < 0$.

Problems

→ Show that the sequence of functions $\{f_n\}$ where $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in \mathbb{R}$ converges uniformly on any closed interval $[a,b]$

Sol'n:- Here $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{x}{1+nx^2} = 0\ \forall\ x \in \mathbb{R}$$

Now $|f_n(x) - f(x)| = \left|\dfrac{x}{1+nx^2} - 0\right| = \dfrac{x}{1+nx^2}$ —— ①

(Always keep this term equal to y)

Let $y = \dfrac{x}{1+nx^2}$ then —— ②

$$\frac{dy}{dx} = \frac{(1+nx^2)\cdot 1 - x(2nx)}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{(1+nx^2)^2}$$

For max or min. $\dfrac{dy}{dx} = 0$

$$\Rightarrow 1 - nx^2 = 0$$

$$\Rightarrow x = \frac{1}{\sqrt{n}}$$

Also $$\frac{d^2y}{dx^2} = \frac{(1+nx^2)^2(-2nx) - (1-nx^2)\,2(1+nx^2)(2nx)}{(1+nx^2)^4}$$

$$= \frac{-2nx(1+nx^2) - 4nx(1-nx^2)}{(1+nx^2)^3}$$

Now $$\left[\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}\,(1+1)}{(1+1)^2}$$

$$= -\frac{\sqrt{n}}{2} < 0.$$

$\Rightarrow y$ is maximum when $x = \dfrac{1}{\sqrt{n}}$ and the maximum value of y from ②

$$y = \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}}.$$

$\therefore M_n = \max\limits_{x\in[a,b]} |f_n(x) - f(x)|$

$= \max\limits_{x\in[a,b]} \left|\dfrac{x}{1+nx^2}\right|$ (from ①)

$= \dfrac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$

$\therefore \{f_n\}$ converges uniformly to f on $[a,b]$

→ show that if $f_n(x) = \dfrac{n^2 x}{1+n^4x^2}$, then $\{f_n\}$ converges non-uniformly on $[0,1]$.

Sol:- Here $f(x) = \lim\limits_{n\to\infty} f_n(x)$

$= \lim\limits_{n\to\infty} \dfrac{n^2 x}{1+n^4x^2}$

$= \lim\limits_{n\to\infty} \dfrac{x/n^2}{\frac{1}{n^4}+x^2}$

$= 0 \ \forall x \in [0,1]$

Now $|f_n(x) - f(x)| = \left|\dfrac{n^2x}{1+n^4x^2} - 0\right|$

$= \left|\dfrac{n^2 x}{1+n^4x^2}\right|$ —— ①

Let $y = \dfrac{n^2x}{1+n^4x^2}$ —— ②

$\dfrac{dy}{dx} = \dfrac{(1+n^4x^2).n^2 - n^2x.2n^4x}{(1+n^4x^2)^2}$

$= \dfrac{n^2[1+n^4x^2 - 2n^4x^2]}{(1+n^4x^2)^2}$

$= \dfrac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$

For max or min $\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{n^2(1-n^4x^2)}{(1+n^4x^2)^2} = 0$

$\Rightarrow n^2(1-n^4x^2) = 0$

$\Rightarrow 1 - n^4x^2 = 0$

$\Rightarrow n^4x^2 = 1 \Rightarrow x = 1/n^2$.

Also

$\dfrac{d^2y}{dx^2} = \dfrac{n^2(1+n^4x^2)^2(-2n^4x) - (1-n^4x^2)2(1+n^4x^2).2n^4x}{(1+n^4x^2)^4}$

$= \dfrac{n^2[1+n^4x^2][(1+n^4x^2)(-2n^4x) - 4n^4x(1-n^4x^2)]}{(1+n^4x^2)^4}$

$= \dfrac{n^2(-2n^4x)[1+n^4x^2+2(1-n^4x^2)]}{(1+n^4x^2)^3}$

$= \dfrac{-2n^6x[3-n^4x^2]}{(1+n^4x^2)^3}$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=\frac{1}{n^2}} = \dfrac{-2n^6.\frac{1}{n^2}\left[3-n^4.\frac{1}{n^4}\right]}{\left(1+n^4.\frac{1}{n^4}\right)^3}$

$= \dfrac{-2n^4(3-1)}{(1+1)^3} = \dfrac{-4n^4}{8}$

$= \dfrac{-n^4}{2} < 0.$

$\therefore y$ is max when $x = \frac{1}{n^2}$ and the max value of

$y = \dfrac{n^2.\frac{1}{n^2}}{1+n^4.\frac{1}{n^4}} = \dfrac{1}{1+1} = \dfrac{1}{2}$ [from ②]

$\therefore M_n = \max\limits_{x\in[0,1]} |f_n(x) - f(x)|$

$= \max\limits_{x\in[0,1]} \left|\dfrac{n^2x}{1+n^4x^2}\right| = \frac{1}{2}$

i.e. $M_n \to \frac{1}{2}$ as $n \to \infty$

which does not tend to zero as $n \to \infty$

$\therefore \{f_n\}$ converges non-uniformly on $[0,1]$

→ show that the sequence $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing zero.

$\boxed{x = \frac{1}{n} \to 0 \text{ as } n \to \infty \quad 0 \in \text{any interval}}$

→ show that the sequence $\{f_n\}$, where $f_n(x) = \frac{n^2 x}{1+n^3x^2}$ is not uniformly convergent on $[0,1]$.

→ show that the sequence of functions $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^3x^2}$, $x \in \mathbb{R}$ converges uniformly on any closed interval $[a,b]$.

→ show that the sequence of functions $\{f_n\}$, where $f_n(x) = nx(1-x)^n$, is not uniformly convergent on $[0,1]$.

Sol^n: For $0<x<1$

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n$$

$$= \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} \quad \left[\frac{d}{dx}a^{-x} = -a^{-x}\log a\right]$$

$$= \lim_{n\to\infty} \frac{-x(1-x)^n}{\log(1-x)} = 0$$

Also, when $x=0$, $f_n(x)=0 \ \forall n$

when $x=1$, $f_n(x) = 0 \ \forall n$

$\therefore f(x) = 0 \ \forall x \in [0,1]$

Now $|f_n(x) - f(x)| = |nx(1-x)^n - 0|$

$$= |nx(1-x)^n| \quad \text{--- (1)}$$

Let $y = nx(1-x)^n$ --- (2)

then $\frac{dy}{dx} = n(1-x)^n - n^2x(1-x)^{n-1}$

$$= n(1-x)^{n-1}[1-x-nx]$$

$$= n(1-x)^{n-1}[1-(n+1)x]$$

For max or min $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{n+1}$

Also $\frac{d^2y}{dx^2} = n(n-1)(1-x)^{n-2}[1-(n+1)x] - n(n+1)(1-x)^{n-1}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{n+1}} = 0 - n(n+1)\left(\frac{n}{n+1}\right)^{n-1}$$

$$= -n(n+1)\left(\frac{n}{n+1}\right)^{n-1} < 0.$$

$\Rightarrow y$ is maximum at $x = \frac{1}{n+1}$ and the max value of y is $n\frac{1}{(n+1)}\left(1-\frac{1}{n+1}\right)^n$ (from (2))

$$= \frac{n}{n+1}\left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{n}{n+1}\right)^{n+1} = \left(1-\frac{1}{n+1}\right)^{n+1}$$

$$\therefore M_n = \max_{x\in[0,1]} |f_n(x) - f(x)|$$

$$= \max_{x\in[0,1]} \left|\left(1-\frac{1}{n+1}\right)^{n+1}\right| \quad [(1-x)^{1/x} \text{ form}]$$

$$= \frac{1}{e} \text{ as } n \to \infty$$

$\therefore M_n \to \frac{1}{e}$ as $n \to \infty$

i.e., M_n does not tend to zero as $n\to\infty$.

$\therefore$ the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$.

→ show that the sequence $\{f_n\}$, where $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0,\pi]$.

Sol'n:- Here $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sin nx = 0 \quad \forall\, x\in[0,\pi]$$

Now $|f_n(x) - f(x)| = \left|\frac{\sin nx}{\sqrt{n}} - 0\right| = \left|\frac{\sin nx}{\sqrt{n}}\right|$

Let $y = \frac{\sin nx}{\sqrt{n}}$ ——— (1)

then $\frac{dy}{dx} = \sqrt{n}\cos nx$

For max or min, $\frac{dy}{dx} = 0$

$$\Rightarrow \sqrt{n}\cos nx = 0$$

$$\Rightarrow nx = \pi/2$$

$$\Rightarrow x = \pi/2n$$

Also $\frac{d^2y}{dx^2} = -n^{3/2}\sin nx$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pi/2n} = -n^{3/2}\sin\pi/2$$

$$= -n^{3/2} < 0.$$

$\Rightarrow y$ is max when $x = \pi/2n$

and the max value of y is

$$\frac{\sin\pi/2}{\sqrt{n}} = \frac{1}{\sqrt{n}} \text{ [from (1)]}$$

$$\therefore M_n = \max_{x\in[0,\pi]} |f_n(x) - f(x)|$$

$$= \max_{x\in[0,\pi]} \left|\frac{1}{\sqrt{n}}\right|$$

$$= \frac{1}{\sqrt{n}} \to 0 \text{ as } n\to\infty$$

$\therefore$ The sequence $\{f_n\}$ converges uniformly to '0' on $[0,\pi]$.

* Series of real valued functions

Definition: If $\{f_n\}$ is a sequence of real valued functions on an interval I, then $f_1+f_2+f_3+f_4+f_5+\cdots+f_n+\cdots$ is called a series of real-valued functions defined on I.

This series is denoted by $\sum_{n=1}^{\infty} f_n$ (or) $\sum f_n$.

For example: (i) If $f_n : [0,\infty) \to \mathbb{R}$ is defined by $f_n(x) = \frac{1}{n+x}$, then the series is $\sum f_n = f_1+f_2+f_3+\cdots$

$$= \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \cdots$$

(ii) If $f_n : \mathbb{R}\to\mathbb{R}$ is defined by $f_n(x) = \frac{\sin nx}{\sqrt{n}}$, then the series is

$$\sum f_n = f_1+f_2+f_3+\cdots$$

$$= \sin x + \frac{\sin 2x}{\sqrt{2}} + \frac{\sin 3x}{\sqrt{3}} + \cdots$$

* Convergence (or Pointwise Convergence) of a Series of Functions:-

Let $\sum f_n$ be a series of functions defined on an interval I.

II

Let $S_1 = f_1$, $S_2 = f_1 + f_2 + \cdots$

$S_n = f_1 + f_2 + f_3 + \cdots + f_n$

then the sequence $\{S_n\}$ is a sequence of partial sums of the series Σf_n.

If the ~~series~~ $\{S_n\}$ converges pointwise on I, then the series Σf_n is said to converge pointwise on I. The limit function 'f' of $\{S_n\}$ is called the pointwise sum (or) simply the sum of the series Σf_n and we write $\sum_{n=1}^{\infty} f_n(x) = f(x) \; \forall x \in I$

(or) simply $\Sigma f_n = f$.

For example: Consider the series Σf_n defined by $f_n(x) = x^n, -1 < x < 1$.

then $\Sigma f_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) + \cdots$

where $-1 < x < 1$

changes to

$S_n(x) = x + x^2 + \cdots + x^n$

$= \dfrac{x(1-x^n)}{1-x}$ $\left(\dfrac{a(1-r^n)}{1-r}\right)$

Now $\lim_{n\to\infty} S_n = \dfrac{x}{1-x}$ ($\because -1 < x < 1 \Rightarrow x^n \to 0$ as $n \to \infty$)

$\therefore$ the sequence $\{S_n\}$ of partial sums converges pointwise to $\dfrac{x}{1-x}$ on $(-1, 1)$

$\Rightarrow$ The series Σf_n converges pointwise to $f(x) = \dfrac{x}{1-x}$ on $(-1, 1)$

$\Rightarrow \Sigma f_n(x) = \dfrac{x}{1-x}$ on $(-1, 1)$

* Uniform Convergence of Series of functions:-

Definition: Let Σf_n be a series functions defined on an interval I & $S_n = f_1 + f_2 + f_3 + \cdots + f_n$.

If the sequence $\{S_n\}$ of partial sums converges uniformly on I, then the series Σf_n is said to be uniformly convergent on I.

Thus, a series of functions Σf_n converges uniformly to a function 'f' on an interval I, if for each $\epsilon > 0$ and for each $x \in I$, $\exists$ +ve integer 'm' (depending only on ϵ and not on x) such that

$|S_n(x) - f(x)| < \epsilon \;\; \forall n \geq m$.

The uniform limit function 'f' of $\{S_n\}$ is called the sum of the series Σf_n and we write

$\Sigma f_n = f \;\; \forall x \in I$.

* Theorem: [Cauchy's Criterion for uniform convergence of a series of functions]:-

A series of functions Σf_n is uniformly convergent on an interval I iff for each $\epsilon > 0$ and for all $x \in I$,

$\exists$ a +ve integer 'm' (depending on ϵ) such that $|S_{n+p} - S_n| < \epsilon \;\forall\; n \geq m, p \geq 0$

where $S_n = f_1(x) + f_2(x) + \cdots + f_n(x)$

i.e. $|f_{n+1}(x) + f_{n+2}(x) + \cdots f_{n+p}(x)| < \epsilon$ $\forall\; n \geq m$.

Note(1): Uniform Convergence $\Rightarrow$ Pointwise Convergence

Note(2): The method of testing the uniform convergence of a series $\sum f_n$ by definition, involves finding S_n which is not always easy. The following test avoids S_n.

Theorem [Weierstrass's M-Test]:

A series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely) on an interval I, if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of non-negative terms (i.e. $M_n \geq 0 \;\forall\; n \in N$.) Such that

$|f_n(x)| \leq M_n \;\forall\; n \in N$ and $x \in I$.

$\rightarrow$ show that if $0 < r < 1$, then each of the following series is uniformly Convergent on IR.

(i) $\sum_{n=1}^{\infty} r^n \cos nx$ (ii) $\sum_{n=1}^{\infty} r^n \sin nx$

(iii) $\sum_{n=1}^{\infty} r^n \cos n^2 x$ (iv) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

Sol'n: (i) $\sum_{n=1}^{\infty} r^n \cos nx$

Let $f_n(x) = r^n \cos nx \;\forall\; x \in IR$ then

$|f_n(x)| = |r^n \cos nx|$

$= r^n |\cos nx|$ finding this condition is must

$\leq r^n$ $(\because r > 0)$

$= M_n \;\forall\; x \in IR$ —— (i)

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric Series with $0 < r < 1$, it is Convergent.

Hence by Weierstrass's M-Test, The given Series is Convergent uniformly on IR.

$\rightarrow$ Show that the following series are uniformly convergent for all real 'x'

(i) $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n+2)}$

Sol'n: - (i) Here $f_n(x) = \frac{\sin(x^2+n^2x)}{n(n+2)}$

$|f_n(x)| = \left| \frac{\sin(x^2+n^2x)}{n(n+2)} \right|$

$= \frac{|\sin(x^2+n^2x)|}{|n(n+2)|} \leq \frac{1}{n(n+2)}$ (1)

$\leq \frac{1}{n^2} = M_n$

$\forall\; x \in IR$

Since $\sum_{n=1}^{\infty} M_n = \sum \frac{1}{n^2}$ is convergent (by P-Test).

$\therefore$ By Weierstrass's M-test, the given series is uniformly Convergent for all real 'x'.

ET

→ Show that the following series are uniformly and absolutely convergent for all real values of x and $P>1$.

(i) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^P}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^P}$

$\boxed{2002 \\ P\text{-}II}$

Sol'n (i): Here $f_n(x) = \frac{\sin nx}{n^P}$

$$|f_n(x)| = \left|\frac{\sin nx}{n^P}\right| = \frac{|\sin nx|}{n^P}$$

$$\leq \frac{1}{n^P} = M_n \quad \forall x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^P}$ is convergent for $P>1$ (by P-Test)

∴ By Weierstrass's M-test, the given series converges uniformly and absolutely for all real values of x.

→ Test for uniform convergence the series (i) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ (ii) $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$

Sol'n:- (i) Here $f_n(x) = \frac{x}{(n+x^2)^2}$ —(1)

$$\Rightarrow \frac{df_n(x)}{dx} = \frac{(n+x^2)^2 \cdot 1 - x \cdot 2(n+x^2) \cdot 2x}{(n+x^2)^4}$$

$$= \frac{(n+x^2)\left[n+x^2-4x^2\right]}{(n+x^2)^4}$$

$$= \frac{n-3x^2}{(n+x^2)^3}$$

For max or min $\frac{df_n(x)}{dx} = 0$

$\Rightarrow n-3x^2 = 0$

$\Rightarrow x = \sqrt{n/3}$

Also $\frac{d^2 f_n(x)}{dx^2} = \frac{(n+x^2)^3(-6x) - (n-3x^2) \cdot 3(n+x^2)^2 2x}{(n+x^2)^6}$

$$= \frac{(n+x^2)^2\left[(n+x^2)(-6x) - 6x(n-3x^2)\right]}{(n+x^2)^6}$$

$$= \frac{-6x\left[(n+x^2)+(n-3x^2)\right]}{(n+x^2)^4}$$

$$\frac{d^2 f_n(x)}{dx^2} \text{ at } x = \sqrt{n/3} = \frac{-6\sqrt{n/3}\left(n+\frac{n}{3}+0\right)}{\left(n+\frac{n}{3}\right)^4}$$

$$= \frac{-27\sqrt{3}}{32 n^{5/2}} < 0$$

$\Rightarrow f_n(x)$ is maximum at $x = \sqrt{\frac{n}{3}}$ and the maximum value of $f_n(x)$ is

$$\frac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16 n^{3/2}} \quad (\text{from (1)})$$

$$\Rightarrow |f_n(x)| \leq \frac{3\sqrt{3}}{16 n^{3/2}} < \frac{1}{n^{3/2}} = M_n$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent (by P-Test)

∴ by Weierstrass M-test, the given series is uniformly convergent for all values of x.

(ii) Here $f_n(x) = \frac{x}{n(1+nx^2)}$ —(1)

$$\Rightarrow \frac{df_n(x)}{dx} = \frac{1}{n} \frac{(1+nx^2) \cdot 1 - x \cdot 2nx}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{n(1+nx^2)^2}$$

For max or min $\frac{df_n(x)}{dx} = 0$.

$\Rightarrow 1-nx^2=0$

$\Rightarrow x = \frac{1}{\sqrt{n}}$

Also $\frac{d^2 f_n(x)}{dx^2} = \frac{1}{n}\frac{(1+nx^2)^2(-2nx) - (1-nx^2)\,2(1+nx^2)2nx}{(1+nx^2)^4}$

$= \frac{-2x\left[(1+nx^2)+2(1-nx^2)\right]}{(1+nx^2)^3}$

$\left.\frac{d^2 f_n(x)}{dx^2}\right|_{x=\frac{1}{\sqrt{n}}} = \frac{-2\frac{1}{\sqrt{n}}[1+1+0]}{(1+1)^3}$

$= \frac{-1}{2\sqrt{n}} < 0$

$\Rightarrow f_n(x)$ is max at $x = \frac{1}{\sqrt{n}}$ and the max

Value of $f_n(x)$ is $\frac{\frac{1}{\sqrt{n}}}{n(1+1)} = \frac{1}{2n^{3/2}}$

[from ①]

$\Rightarrow |f_n(x)| \leq \frac{1}{2n^{3/2}} < \frac{1}{n^{3/2}} = M_n$

$\{\because \frac{1}{2n^{3/2}}$ is the max value of $f_n(x)\}$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent

(by P-Test)

$\therefore$ By weiestrass's M-test, the given series is uniformly convergent for all values of x.

→ show that the series $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ converges in $[1,\infty)$.

Sol^n:- Here $f_n(x) = \frac{1}{1+n^2x}$ $[\forall x \geq 1]$

$\therefore |f_n(x)| = \left|\frac{1}{1+n^2x}\right| \leq \frac{1}{1+n^2} < \frac{1}{n^2} = M_n$

$\forall x \in [1,\infty)$.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent,

by weierstrass M-test, the given series is uniformly convergent $\forall x \in [1,\infty)$

→ show that the series (i) $\sum_{n=1}^{\infty}\frac{a_n x^{2n}}{1+x^{2n}}$

(ii) $\sum_{n=1}^{\infty}\frac{a_n x^n}{1+x^{2n}}$ converge uniformly for all real values of x, if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Sol^n:- Here $f_n(x) = \frac{a_n x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}} < 1 \ \forall x \in \mathbb{R}$

$\therefore |f_n(x)| = \left|\frac{a_n x^{2n}}{1+x^{2n}}\right| = |a_n|\frac{x^{2n}}{1+x^{2n}} < |a_n|$

$= M_n \ \forall x \in \mathbb{R}$.

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\therefore \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}|a_n|$ is convergent.

Hence by weiestrass M-test, the given series is uniformly convergent

(ii) Here $f_n(x) = \frac{a_n x^n}{1+x^{2n}}$

Let $y = \frac{x^n}{1+x^{2n}}$

$\Rightarrow \frac{dy}{dx} = \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$

For max or min $\frac{dy}{dx} = 0$

$\Rightarrow x = 1$

Also $\frac{d^2y}{dx^2}$

$= \frac{(1+x^{2n})\left[n(n-1)x^{n-2}(1-x^{2n}) - 2n^2x^{3n-2}\right] - 4n^2x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$

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$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \frac{2[0-2n^2]-0}{(2)^3}$$

$$= \frac{-n^2}{2} < 0$$

$\Rightarrow y = \frac{x^n}{1+x^{2n}}$ is max at $x=1$ and the max value of y is $\frac{1}{2}$.

$$\therefore |f_n(x)| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{x^n}{1+x^{2n}}\right| |a_n|$$

$$\le \frac{1}{2}|a_n| < |a_n| = M_n \quad \forall x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely Convergent.

$\therefore \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent.

Hence by weierstrass M-test, the given Series is uniformly Convergent for all real x.

→ If the series $\sum a_n$ Converges absolutely then Prove that (i) $\sum a_n \cos nx$, and (ii) $\sum a_n \sin nx$ are uniformly Convergent on $\mathbb{R}$.

Sol^n (i) $\sum a_n \cos nx$

Here $f_n(x) = a_n \cos nx$

$$|f_n(x)| = |a_n \cos nx| = |a_n|\,|\cos nx|$$

$$\le |a_n| = M_n$$

$$[\because |\cos nx| \le 1 \ \forall x \in \mathbb{R}]$$

Since $\sum a_n$ is absolutely Convergent

$\therefore \sum M_n = \sum |a_n|$ is Convergent.

Hence by weierstrass's M-test, the series $\sum a_n \cos nx$ is uniformly Convergent.

→ show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \frac{x^{2n}}{1+x^{2n}}$ is absolutely and uniformly Convergent for all real x if $p>1$.

Sol^n:- Here $f_n(x) = \frac{(-1)^n}{n^p} \frac{x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}} < 1 \ \forall x \in \mathbb{R}$

$$\therefore |f_n(x)| = \left|\frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{n^p} = M_n \quad \forall x \in \mathbb{R}$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is Convergent if $p>1$.

$\therefore$ by weierstrass's M-test, the given series is absolutely and uniformly convergent $\forall x \in \mathbb{R}$ if $p>1$.

* Uniform Convergence and Continuity :-

→ Theorem 1: If a sequence of Continuous functions $\{f_n\}$ is uniformly Convergent to a function f on $[a,b]$, then f is continuous on $[a,b]$.

→ Theorem 2: If a Series $\sum_{n=1}^{\infty} f_n$ of Continuous functions is uniformly Convergent to a function f on $[a,b]$, then the Sum function f is also Continuous on $[a,b]$.

Note :- The above theorems converse is not true. i.e. uniform convergence of the sequence $\{f_n\}$ is only a sufficient but not a necessary condition for the continuity of the limit function f, i.e. if the limit function f is continuous on $[a,b]$, then it is not necessary that the sequence $\{f_n\}$ is uniformly convergent on $[a,b]$.

Theorem 1 shows that if the limit function f is discontinuous then the sequence $\{f_n\}$ of continuous functions cannot be uniformly convergent on $[a,b]$. Thus the theorem provides a very good negative test for uniform convergence of a sequence.

Similarly, if the sum function 'f' is discontinuous then the series Σf_n of continuous functions cannot be uniformly convergent.

Problems : (1) Test for uniform convergence and continuity the sequence $\{f_n\}$.

where $f_n(x) = x^n$, $0 \le x \le 1$.

Sol'n: Here $f_n(x) = x^n$, $0 \le x \le 1$.

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (\because f_n(x) \text{ is sequence})$$

$$= \lim_{n\to\infty} x^n$$

$$= \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Clearly 'f' is discontinuous at $x = 1$ and hence 'f' is discontinuous on $[0,1]$.

Also $f_n(x) = x^n$, $0 \le x \le 1$ is continuous on $[0,1]$ $\forall\, n \in N$. {But the f ... is not co...}

Since $\{f_n\}$ is a sequence of continuous functions and its limit function 'f' is discontinuous on $[0,1]$.

$\therefore$ The sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

$\rightarrow$ Test the uniform convergence and continuity of $\{f_n\}$ where

$$f_n(x) = \frac{1}{1+nx}, \quad 0 \le x \le 1.$$

Sol'n:- The limit function 'f' is given by $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx}$

$$= \begin{cases} 0 & \text{for } 0 < x \le 1 \\ 1 & \text{for } x = 0 \end{cases}$$

Clearly 'f' is discontinuous at $x = 0$ and hence 'f' is discontinuous on $[0,1]$.

Also $f_n(x) = \frac{1}{1+nx}$, $0 \le x \le 1$ is continuous on $[0,1]$, $\forall\, n \in N$.

Since $\{f_n\}$ is a sequence of continuous function and its limit

∴ function 'f' is discontinuous on $[0,1]$.

∴ The sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

H.W. → Show that the sequence $\{f_n\}$, where $f_n(x) = \tan^{-1} nx$ is not uniformly convergent on $[0,1]$.

$\left[\text{Hint:- } f(x) = \begin{cases} \pi/2 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}\right.$ is discontinuous at $x=0$.

→ If $f_n(x) = \dfrac{1}{x+n}$, $x \geq 0$ then show that $\{f_n(x)\}$ converges uniformly to the continuous function zero.

Sol'n :- Here $f_n(x) = \dfrac{1}{x+n}$, $x \geq 0$.

which is continuous $\forall\, n \in N$ and $x \geq 0$

Now $f(x) = \lim\limits_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} \frac{1}{x+n} = 0 \quad \forall\, x \geq 0.$$

f being a constant function is continuous for all $x \geq 0$.

But continuity of 'f' is no guarantee for uniform convergence of $\{f_n\}$.

Now $|f_n(x) - f(x)| = \left|\dfrac{1}{x+n} - 0\right|$

$$= \left|\frac{1}{x+n}\right|$$

$$= \frac{1}{x+n}$$

$\therefore |f_n(x) - f(x)| = \dfrac{1}{n+x} < \epsilon$ if $n+x > \dfrac{1}{\epsilon}$

if $n > \dfrac{1}{\epsilon} - x$

choose any +ve integer $\geq \dfrac{1}{\epsilon} - x$

then $|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq m\ \forall\, x \in [0,\infty)$.

$\therefore \{f_n(x)\}$ is uniformly convergent $\forall\, x \geq 0$.

H.W. → Examine for uniform convergence and continuity of the limit function of the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \leq x \leq 1.$$

→ Show that the series $\sum\limits_{n=1}^{\infty} (1-x)x^n$ is not uniformly convergent on $[0,1]$

Sol'n :- $f_n(x) = (1-x)x^n \quad \forall\, x \in [0,1]$

$\Rightarrow f_1(x) = (1-x)x,$

$f_2(x) = (1-x)x^2$

$f_3(x) = (1-x)x^3$

$\vdots$

$f_n(x) = (1-x)x^n$

$\therefore S_n(x) = x(1-x) + x^2(1-x) + \cdots + x^n(1-x)$

$$= (1-x)\left[x + x^2 + x^3 + \cdots + x^n\right]$$

$$= (1-x)\frac{x(1-x^n)}{1-x} = x(1-x^n)$$

$\therefore S(x) = \lim\limits_{n\to\infty} S_n(x) = \lim\limits_{n\to\infty} x(1-x^n)$

$\left\{\because f(x) = \lim\limits_{n\to\infty} f_n(x)\right\}$

$$= \begin{cases} 0 & \text{when } x = 0 \text{ (or) } 1 \\ x & \text{when } 0 < x < 1 \end{cases}$$

Since the sum function $S(x)$ is discontinuous at $x = 0 \in [0,1]$.

∴ The given series not uniformly convergent on $[0,1]$.

* To show that the series $\sum_{n=1}^{\infty} f_n(x)$

where $f_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$

is not uniformly convergent on $[0,1]$ though the sum function is continuous on $[0,1]$.

Sol'n:- Here $f_n(x) = \dfrac{nx}{1+n^2x^2} - \dfrac{(n-1)x}{1+(n-1)^2x^2}$

$$\therefore f_1(x) = \frac{x}{1+x^2} - 0$$

$$f_2(x) = \frac{2x}{1+2^2x^2} - \frac{x}{1+x^2}$$

$$f_3(x) = \frac{3x}{1+3^2x^2} - \frac{2x}{1+2^2x^2}$$

$$\vdots \qquad \vdots$$

$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$$

Now $S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$

$$= \frac{nx}{1+n^2x^2}$$

$$\therefore S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{x/n}{\frac{1}{n^2}+x^2} = 0 \quad \forall x \in [0,1]$$

$\therefore$ The sum $S(x)$ is continuous $\forall x \in [0,1]$

But the continuity of $S(x)$ is no guarantee for uniform convergence of $\sum f_n(x)$.

Now $|S_n(x) - S(x)| = \left|\dfrac{nx}{1+n^2x^2} - 0\right|$

$$= \left|\frac{nx}{1+n^2x^2}\right| \quad \text{—} (1)$$

Let $y = \dfrac{nx}{1+n^2x^2}$ then y is maximum when $x = 1/n$ and the maximum value of $y = 1/2$

(Prove it yourself).

Now $M_n = \max_{x\in[0,1]} |S_n(x) - S(x)|$

$$= \max_{x\in[0,1]} |S_n(x) - S(x)|$$

$$= 1/2$$

$\therefore$ which doesnot tend to '0' as $n \to \infty$

$\therefore$ By M_n - Test for sequences, the sequence $\{S_n(x)\}$ of partial sums is not uniformly convergent.

$\therefore$ The series is not uniformly convergent.

* Uniform Convergence and Integration :-

{whenever something is asked to pro[ve] term by term, then first pro[ve] it for $f(x)$ and $f_n(x)$ by $f(x)=f_n$

Theorem (1) : If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$ then f is integrable on $[a,b]$ and the sequence $\left\{\int_a^b f_n(x)\,dx\right\}$ converges uniformly to $\int_a^b f(x)\,dx$.

Theorem (2) :- If a series of a functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$ and $\sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$ converges uniformly to $\int_a^b f(x)\,dx$.

i.e. $\sum_{n=1}^{\infty}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$

i.e. the series is term by term integrable.

Note (1): the uniform convergence of the sequence $\{f_n\}$ (or series $\sum_{n=1}^{\infty} f_n$) is only sufficient but not a necessary condition for the validity of term by term integration.

Note(2): If $\{f_n\}$ is a sequence of integrable functions converging to f on $[a,b]$ and if $\lim_{n\to\infty}\int_a^b f_n(x)dx \neq \int_a^b f(x)dx$ then $\{f_n(x)\}$ cannot converge uniformly to f.

Problems :-

→ Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$, $n\in N$ is not uniformly convergent on $[0,1]$.

Sol'n: Here $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty}\frac{nx}{e^{nx^2}}$$

$$= \lim \frac{nx}{1+\frac{nx^2}{1!}+\frac{n^2x^4}{2!}+\cdots}$$

$= 0$ for $x\in[0,1]$

Also $\int_0^1 f(x) = 0$

and $\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx^2}dx$

$$= \int_0^n \tfrac{1}{2}e^{-t}\,dt$$

where $t=nx^2$. if $x=0$ then $t=0$ and at $x=1$ then, $t=n$.

$$= -\tfrac{1}{2}\left[e^{-t}\right]_0^n$$

$$= -\tfrac{1}{2}\left[e^{-n}-1\right]$$

$$= \tfrac{1}{2}\left[1-e^{-n}\right]$$

Now $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\tfrac{1}{2}(1-e^{-n})$

$$= \tfrac{1}{2} \neq \int_0^1 f(x)dx.$$

$\Rightarrow$ the sequence $\{f_n\}$ is not uniformly convergent on $[0,1]$ $\because \int_0^1 f(x)dx \neq \lim_{n\to\infty}\int_0^1 f_n(x)dx$

→ Prove that $\int_0^1\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)dx = \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}$

Sol'n:- Let $f_n(x) = \frac{x^n}{n^2}$

$|f_n(x)| = \left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n$ for $0\le x\le 1$.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (by P-Test)

$\therefore$ By weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}$ is uniformly convergent for $0\le x\le 1$.

$\therefore$ The series can be integrated term by term.

$$\Rightarrow \int_0^1\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right)dx = \sum_{n=1}^{\infty}\int_0^1\frac{x^n}{n^2}dx$$

$$= \sum_{n=1}^{\infty}\left[\frac{x^{n+1}}{n^2(n+1)}\right]_0^1$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}$$

→ Show that the series $1-x+x^2-x^3+\cdots$, $0\le x\le 1$, admits of term by term integration on $[0,1]$, though it is not uniformly convergent on $[0,1]$.

Sol'n :- The given series is

$$1-x+x^2-x^3+\cdots\cdots$$

when $x=1$, the series $1-1+1-1+\cdots$ oscillates.

For $0\le x<1$,

$$1-x+x^2-x^3+\cdots=\frac{1}{1-(-x)}=\frac{1}{1+x} \quad \text{--- (1)}$$

∴ The series is not uniformly convergent on $[0,1]$;

However, integrating L.H.S of (1) term by term over the interval $[0,1]$, we have

$$\int_0^1 1\,dx-\int_0^1 x\,dx+\int_0^1 x^2dx-\int_0^1 x^3dx+\cdots$$

$$=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots=\log 2.$$

R.H.S.

Also $\displaystyle\int_0^1 \frac{1}{1+x}\,dx=\Big[\log(1+x)\Big]_0^1$

$$=\log 2$$

i.e. the two sides are equal.

∴ Term by term integration is possible over $[0,1]$, eventhough the given series is not uniformly convergent on $[0,1]$.

→ Test for uniform convergence and term by term integration the series $\displaystyle\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}$. Also show that

$$\int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx=\frac{1}{2}$$

Sol'n :- we know that the series $\displaystyle\sum_{n=1}^{\infty}\frac{x}{(1+x^2)^2}$ is uniformly convergent.

Hence it is integrable term by term between any finite limits.

$$\Rightarrow \int_0^1\left(\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}\right)dx=\lim_{n\to\infty}\int_0^1\sum_{n=1}^{\infty}\frac{x}{(n+x^2)^2}dx$$

$$=\lim_{n\to\infty}\sum_{n=1}^{\infty}\int_0^1 x(n+x^2)^{-2}dx$$

$$=\lim_{n\to\infty}\sum_{n=1}^{\infty}\left[\frac{(n+x^2)^{-1}}{-2}\right]_0^1$$

$$=\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\lim_{n\to\infty}\frac{1}{2}\left[\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{n+1}\right)$$

$$=\frac{1}{2}$$

* Uniform Convergence and Differentiation :-

Theorem 1 :- If a sequence of functions $\{f_n\}$ is such that

(i) each f_n is differentiable on $[a,b]$.

(ii) each f_n' is continuous on $[a,b]$

(iii) $\{f_n\}$ converges uniformly to f on $[a,b]$

(iv) $\{f_n'\}$ converges uniformly to g on $[a,b]$.

then f is differentiable and $f'(x)=g(x)$ $\forall x\in[a,b]$.

Theorem 2:- If a series of functions $\sum_{n=1}^{\infty} f_n$ is such that

(i) each f_n is differentiable on $[a,b]$

(ii) each f_n' is continuous on $[a,b]$

(iii) $\sum_{n=1}^{\infty} f_n$ converges to f on $[a,b]$

(iv) $\sum_{n=1}^{\infty} f_n'$ converges uniformly to g on $[a,b]$.

then f is differentiable on $[a,b]$ and $f'(x)=g(x)$ $\forall x\in[a,b]$

Problems:

IAS 2011 → (1) show that the sequence $\{f_n\}$ where $f_n(x) = \frac{nx}{1+n^2x^2}$, $0\le x\le 1$,

Cannot be differentiated term by term at $x=0$.

Sol'n:- Here $f(x) = \lim_{n\to\infty} f_n(x) = 0$ $\forall x\in[0,1]$

$\therefore f'(0)=0$

Also $f_n'(0) = \lim_{h\to 0}\frac{f_n(0+h)-f_n(0)}{h}$

$= \lim_{h\to 0}\frac{\frac{nh}{1+n^2h^2}-0}{h}$

Remember: Use this formula when we are getting f_n' at a particular point, say at 0.

$= \lim_{h\to 0}\frac{n}{1+n^2h^2} = n$

$\therefore f_n'(0)\to\infty$ as $n\to\infty$

$\Rightarrow f'(0) \ne \lim_{n\to\infty} f_n'(0)$

$\therefore \{f_n\}$ cannot be differentiated term by term at $x=0$.

→ Show that for the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}$ the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true if $x\ne 0$ and false if $x=0$. why so?

Sol'n:- We know that the sequence $\{f_n\}$ converges uniformly to zero for all real x.

$\Rightarrow f(x)=0$ $\forall x\in\mathbb{R}$

$\Rightarrow f'(x)=0$ $\forall x\in\mathbb{R}$ —— (1)

when $x\ne 0$

$f_n'(x) = \frac{(1+nx^2)\cdot 1-2nx\cdot x}{(1+nx^2)^2}$

$= \frac{1-nx^2}{(1+nx^2)^2}$

$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\frac{1-nx^2}{(1+nx^2)^2}$ (form $\frac{\infty}{\infty}$)

$= \lim_{n\to\infty}\frac{-x^2}{2(1+nx^2)\cdot x^2}$

$=0$

$=f'(x)$ (from (1))

So that if $x\ne 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ true, At

$$\therefore f_n'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\frac{h}{1+nh^2} - 0}{h}$$

$$= \underset{h \to 0}{Lt} \frac{1}{1+nh^2} = 1$$

So that $\therefore \underset{n \to \infty}{Lt} f_n'(0) = 1 \neq f'(0)$ [from (1)]

Hence at $x = 0$, the formula

$\underset{n \to \infty}{Lt} f_n'(x) = f'(x)$ is false.

It is so because the sequence

$\{f_n'\}$ is not uniformly Convergent

in any interval Containing zero.

12

Miscellaneous content of sequence and series of real valued functions !

Example 1. *Show that the series* $\sum_{n=1} \frac{x}{n(n+1)}$ *is uniformly convergent in* $(0, b)$, $b>0$ *but is not so in* $(0, \infty)$.

Sol. The given series is

$$\sum_{n=1}^{\infty} \frac{x}{n(n+1)} = \sum_{n=1}^{\infty} f_n(x)$$

so that $$f_n(x) = \frac{x}{n(n+1)} = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$\Rightarrow$
$$f_1(x) = x\,[1-\tfrac{1}{2}]$$
$$f_2(x) = x\,[\tfrac{1}{2}-\tfrac{1}{3}]$$
$$f_3(x) = x\,[\tfrac{1}{3}-\tfrac{1}{4}]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_n(x) = x\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$\therefore \quad S_n(x)=f_1(x)+f_2(x)+\ldots\ldots+f_n(x)$

$$=x\left[1-\frac{1}{n+1}\right]=\frac{nx}{n+1}$$

$$S(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty}\frac{nx}{n+1}=\begin{cases}0, \text{ if } x=0\\ x, \text{ if } x>0\end{cases}$$

$\therefore$ For $x>0$ and for a given $\varepsilon>0$, we have

$$|S_n(x)-S(x)|=\left|\frac{nx}{n+1}-x\right|=\left|\frac{-x}{n+1}\right|$$

$$=\frac{x}{n+1}<\varepsilon$$

if $\quad n+1>\dfrac{x}{\varepsilon}$ or if $n>\dfrac{x}{\varepsilon}-1$

If we choose a positive integer m just $\geq \dfrac{x}{\varepsilon}-1$, then

$$|S_n(x)-S(x)|<\varepsilon \;\forall\; n\geq m \text{ and } x>0$$

Also if $x=0$, $|S_n(x)-S(x)|=0<\varepsilon \;\forall\; n\geq 1$ so that $m=1$ works in this case.

But when $x\to\infty$, $n\to\infty$.

This shows that the same value of m cannot be found which serves uniformly for every x in $(0, \infty)$.

But if the interval is $(0, b)$ where b is any positive number, then the maximum value of $\dfrac{x}{\varepsilon}-1$ is $\dfrac{b}{\varepsilon}-1$ on $(0, b)$.

$\therefore$ If we choose a positive integer m just $\geq \dfrac{b}{\varepsilon}-1$, then the same value of m serves equally for every value of x in $(0, b)$, $b>0$.

Thus, the sequence $<S_n>$ converges uniformly in $(0, b)$ but not in $(0, \infty)$.

Hence the series $\displaystyle\sum_{n=1}^{\infty}\frac{x}{n(n+1)}$ is uniformly convergent in $(0, b)$, $b>0$ but not so in $(0, \infty)$.

Example 2. *Show that $x=0$ is a point of non-uniform convergence of the series*

$$x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\ldots\ldots$$

(13)

Sol. Here $S_n(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\ldots\ldots+\frac{x^2}{(1+x^2)^{n-1}}$ (which is a G.P.)

$$=\frac{x^2\left[1-\frac{1}{(1+x^2)^n}\right]}{1-\frac{1}{1+x^2}}=(1+x^2)\left[1-\frac{1}{(1+x^2)^n}\right]$$

$$=(1+x^2)-\frac{1}{(1+x^2)^{n-1}}$$

$$\therefore\quad S(x)=\lim_{n\to\infty}S_n(x)=\begin{cases}1+x^2, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Now for $x\neq 0$ and for a given $\varepsilon>0$, we have

$$|S_n(x)-S(x)|=\left|(1+x^2)-\frac{1}{(1+x^2)^{n-1}}-(1+x^2)\right|$$

$$=\frac{1}{(1+x^2)^{n-1}}<\varepsilon$$

if $(1+x^2)^{n-1}>\frac{1}{\varepsilon}$

or if $n-1>\dfrac{\log\frac{1}{\varepsilon}}{\log(1+x^2)}$

or if $n>1+\dfrac{\log\frac{1}{\varepsilon}}{\log(1+x^2)}$

This shows that if $x\to 0$, then $n\to\infty$ so that $x=0$ is a point of non-uniform convergence of $<S_n>$ and hence of the given series.

Note. However, if we consider the interval $[a,\infty)$, $a>0$, then the maximum value of $1+\dfrac{\log\frac{1}{\varepsilon}}{\log(1+x^2)}$ is $1+\dfrac{\log\frac{1}{\varepsilon}}{\log(1+a^2)}$.

If we choose a positive integer m just $\geqslant 1+\dfrac{\log\frac{1}{\varepsilon}}{\log(1+a^2)}$, then

$$|S_n(x)-S(x)|<\varepsilon\ \forall\ n\geqslant m \text{ and } \forall\ x\in[a,\infty)$$

Thus the series is uniformly convergent in $[a,\infty)$, $a>0$ and non-uniformly convergent in $[0,\infty)$.

Example 3. *Show that the series*

$$\frac{x}{x+1}+\frac{x}{(x+1)(2x+1)}+\frac{x}{(2x+1)(3x+1)}+\ldots\ldots$$

is uniformly convergent on $[a,\infty)$, $a>0$. Show also that the series is non-uniformly convergent near $x=0$.

Sol. The given series is $\sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1](nx+1)} = \sum_{n=1}^{\infty} f_n(x)$

so that
$$f_n(x) = \frac{x}{[(n-1)x+1](nx+1)}$$
$$= \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$\Rightarrow$
$$f_1(x) = 1 - \frac{1}{x+1}$$
$$f_2(x) = \frac{1}{x+1} - \frac{1}{2x+1}$$
$$f_3(x) = \frac{1}{2x+1} - \frac{1}{3x+1}$$
....
$$f_n(x) = \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$\therefore$
$$S_n(x) = 1 - \frac{1}{nx+1} = \frac{nx}{nx+1}$$
$$S(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{nx+1} = \begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x=0 \end{cases}$$

For $x>0$ and for a given $\varepsilon>0$, we have
$$|S_n(x) - S(x)| = \left|\frac{nx}{nx+1} - 1\right| = \left|\frac{-1}{nx+1}\right|$$
$$= \frac{1}{nx+1} < \varepsilon$$

if $\quad nx+1 > \frac{1}{\varepsilon}$ or if $n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$.

This shows that if $x\to 0$, $n\to\infty$ so that it is not possible to choose a positive integer m such that
$$|S_n(x) - S(x)| < \varepsilon \quad \forall\ n \geq m \text{ and } \forall\ x \in (0, \infty).$$

Thus the convergence is non-uniform near $x=0$.

Since $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ increases as x decreases, if we consider the interval $[a, \infty)$, $a>0$, then the maximum value of $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ is $\frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$. If we choose a positive integer m just $\geq \frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right)$.

then
$$|S_n(x) - S(x)| < \varepsilon \quad \forall\ n \geq m \text{ and } \forall\ x \in [a, \infty).$$

Hence the series is uniformly convergent on $[a, \infty)$.

Example 4. *Show that the series* $\sum_{n=1}^{\infty}\left(\frac{n}{x+n}-\frac{n-1}{x+n-1}\right)$ *is uniformly convergent on any finite interval.*

Sol. Here $f_n(x)=\frac{n}{x+n}-\frac{n-1}{x+n-1}$

$$\therefore \quad f_1(x)=\frac{1}{x+1}-0$$

$$f_2(x)=\frac{2}{x+2}-\frac{1}{x+1}$$

$$f_3(x)=\frac{3}{x+3}-\frac{2}{x+2}$$

....

$$f_n(x)=\frac{n}{x+n}-\frac{n-1}{x+n-1}$$

$$\Rightarrow \quad S_n(x)=\frac{n}{x+n}$$

Now proceed as in Example 4, Illustrative Examples—A.

Example 5. *Show that* $x=0$ *is a point of non-uniform convergence of the series* $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}\right]$.

Sol. Here $f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}$

$$\therefore \quad f_1(x)=\frac{x}{1+x^2}-0$$

$$f_2(x)=\frac{2x}{1+2^2x^2}-\frac{x}{1+x^2}$$

$$f_3(x)=\frac{3x}{1+3^2x^2}-\frac{2x}{1+2^2x^2}$$

....

$$f_n(x)=\frac{nx}{1+n^2x^2}-\frac{(n-1)x}{1+(n-1)^2x^2}$$

$$\Rightarrow \quad S_n(x)=\frac{nx}{1+n^2x^2}$$

Now proceed as in Example 10, Illustrative Examples—A.

Example 6. *Test the series* $\sum_{n=1}^{\infty} x\left[\frac{n}{1+n^2x^2}-\frac{n+1}{1+(n+1)^2x^2}\right]$ *for uniform convergence in* $[0, 1]$.

Sol. Here $f_n(x)=\dfrac{nx}{1+n^2x^2}-\dfrac{(n+1)x}{1+(n+1)^2x^2}$

$\therefore$ $f_1(x)=\dfrac{x}{1+x^2}-\dfrac{2x}{1+2^2x^2}$

$f_2(x)=\dfrac{2x}{1+2^2x^2}-\dfrac{3x}{1+3^2x^2}$

$f_3(x)=\dfrac{3x}{1+3^2x^2}-\dfrac{4x}{1+4^2x^2}$

....

$f_n(x)=\dfrac{nx}{1+n^2x^2}-\dfrac{(n+1)x}{1+(n+1)^2x^2}$

$\Rightarrow$ $S_n(x)=\dfrac{x}{1+x^2}-\dfrac{(n+1)x}{1+(n+1)^2x^2}$

$$\therefore\ S(x)=\lim_{n\to\infty} S_n(x)=\begin{cases}\dfrac{x}{1+x^2}, & \text{if } 0<x\leqslant 1\\ 0, & \text{if } x=0\end{cases}$$

For $0<x\leqslant 1$ and for a given $\varepsilon>0$, we have

$$|S_n(x)-S(x)|=\left|\frac{x}{1+x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}-\frac{x}{1+x^2}\right|$$

$$=\left|-\frac{(n+1)x}{1+(n+1)^2x^2}\right|=\frac{(n+1)x}{1+(n+1)^2x^2}<\varepsilon$$

if $\quad (n+1)^2x^2\varepsilon-(n+1)+\varepsilon>0$

or if $\quad (n+1)>\dfrac{x+\sqrt{x^2-4x^2\varepsilon^2}}{2x^2\varepsilon}$

or if $\quad n>-1+\dfrac{1+\sqrt{1-4\varepsilon^2}}{2x\varepsilon}$

Now if $x\to 0$, then $n\to\infty$ so that it is not possible to choose a positive integer m such that

$$|S_n(x)-S(x)|<\varepsilon \ \forall\ n\geqslant m \text{ and } x\ \forall \in [0, 1]$$

So the convergence is non-uniform in $[0, 1]$. Here $x=0$ is a point of non-uniform convergence.

Example 7. *Test the uniform convergence of the series*

$$\sum_{n=1}^{\infty}\left[\frac{2n^2x^2}{e^{n^2x^2}}-\frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}\right] \text{ in } [0, 1].$$

Sol. Here $$f_n(x)=\frac{2n^2x^2}{e^{n^2x^2}}-\frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

$$\therefore \quad f_1(x)=\frac{2x^2}{e^{x^2}}-0$$

$$f_2(x)=\frac{2\,.\,2^2x^2}{e^{2^2x^2}}-\frac{2x^2}{e^{x^2}}$$

$$f_3(x)=\frac{2\,.\,3^2x^2}{e^{3^2x^2}}-\frac{2\,.\,2^2x^2}{e^{2^2x^2}}$$

....

$$f_n(x)=\frac{2n^2x^2}{e^{n^2x^2}}-\frac{2(n-1)^2x^2}{e^{(n-1)^2x^2}}$$

$$\Rightarrow \quad S_n(x)=\frac{2n^2x^2}{e^{n^2x^2}}$$

$$\therefore \quad S(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty}\frac{2n^2x^2}{e^{n^2x^2}} \qquad \Big| \text{ Form } \frac{\infty}{\infty}$$

$$=\lim_{n\to\infty}\frac{4nx^2}{2nx^2e^{n^2x^2}}$$

$$=\lim_{n\to\infty}\frac{2}{e^{n^2x^2}}=0 \ \forall\ x$$

Now the series $\sum_{n=1}^{\infty} f_n(x)$ will be uniformly convergent in [0, 1] if for a given $\varepsilon>0$, there is always a positive integer m such that

$$|S_n(x)-S(x)|<\varepsilon \ \forall\ n\geq m \text{ and } \forall\ x\in[0,1]$$

i.e., $$\frac{2n^2x^2}{e^{n^2x^2}}<\varepsilon \ \forall\ n\geq m \text{ and } \forall\ x\in[0,1] \qquad ...(1)$$

But, in particular, if we take $x=\frac{1}{n}$ which is a point of [0, 1] for all $n\in N$, then the inequality (1) gives

$$\frac{2}{e}<\varepsilon$$

which shows that if we take $\varepsilon<\frac{2}{e}$, the above inequality will not hold.

Hence the given series is non-uniformly convergent in [0, 1].

Example 8. *The sum to n terms of a series is* $S_n(x)=\frac{n^2x}{1+n^4x^2}$. *Show that the series is non-uniformly convergent on [0, 1].*

Sol. Please try yourself.

[**Hint.** See Example 15, Illustrative Examples—A].

Example 9. *Show that the series*

$$\frac{x}{1+x^2}+\left(\frac{2^2x}{1+2^3x^2}-\frac{x}{1+x^2}\right)+\left(\frac{3^2x}{1+3^3x^2}-\frac{2^2x}{1+2^3x^2}\right)+\ldots$$

does not converge uniformly on $[0, 1]$.

Sol. Here $\quad f_1(x)=\frac{x}{1+x^2}$

$$f_2(x)=\frac{2^2x}{1+2^3x^2}-\frac{x}{1+x^2}$$

$$f_3(x)=\frac{3^2x}{1+3^3x^2}-\frac{2^2x}{1+2^3x^2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_n(x)=\frac{n^2x}{1+n^3x^2}-\frac{(n-1)^2x}{1+(n-1)^3x^2}$$

$$\therefore \quad S_n(x)=\frac{n^2x}{1+n^3x^2}$$

$$S(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty}\frac{n^2x}{1+n^3x^2}=\lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^3}+x^2}$$

$$=0 \ \forall \ x\in[0, 1]$$

$$|S_n(x)-S(x)|=\left|\frac{n^2x}{1+n^3x^2}-0\right|=\frac{n^2x}{1+n^3x^2}$$

Let $\quad y=\frac{n^2x}{1+n^3x^2}$

then $\quad \frac{dy}{dx}=\frac{(1+n^3x^2)\,.\,n^2-n^2x\,.\,2n^3x}{(1+n^3x^2)^2}$

$$=\frac{n^2(1-n^3x^2)}{(1+n^3x^2)^2}$$

For max. or min. $\frac{dy}{dx}=0$

$$\Rightarrow \quad 1-n^3x^2=0 \quad \Rightarrow \quad x=\frac{1}{n^{3/2}}$$

Also $\quad \frac{d^2y}{dx^2}=\frac{n^2[(1+n^3x^2)^2(-2n^3x)-(1-n^3x^2).2(1+n^3x^2).2n^3x]}{(1+n^3x^2)^4}$

$$=\frac{n^2[-2n^3x(1+n^3x^2)-4n^3x(1-n^3x^2)]}{(1+n^3x^2)^3}$$

$$=\frac{-2n^5x[(1+n^3x^2)+2(1-n^3x^2)]}{(1+n^3x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{3/2}}}=\frac{-2n^5\cdot\frac{1}{n^{3/2}}(1+1)}{(1+1)^3}=-\frac{1}{2}n^{7/2}<0$$

$\Rightarrow$ y is maximum when $x=\frac{1}{n^{3/2}}$ and maximum value of y

$$=\frac{n^2\cdot\frac{1}{n^{3/2}}}{1+1}=\frac{1}{2}\sqrt{n}$$

$\therefore$ $M_n=\max\limits_{x\in[0,1]}|S_n(x)-S(x)|=\frac{1}{2}\sqrt{n}\to\infty$ as $n\to\infty$.

Since M_n does not tend to zero as $n\to\infty$, the sequence $<S_n>$ and hence the given series is non-uniformly convergent on $[0, 1]$. Here 0 is a point of non-uniform convergence.

Example 10. *Show that the series*

$$\frac{x^2}{1+x}+\left(\frac{2x^2}{1+2x}-\frac{x^2}{1+x}\right)+\left(\frac{3x^2}{1+3x}-\frac{2x^2}{1+2x}\right)+\ldots\ldots$$

converges uniformly on $[0, 1]$.

Sol. Here $\quad f_1(x)=\frac{x^2}{1+x}$

$$f_2(x)=\frac{2x^2}{1+2x}-\frac{x^2}{1+x}$$

$$f_3(x)=\frac{3x^2}{1+3x}-\frac{2x^2}{1+2x}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f_n(x)=\frac{nx^2}{1+nx}-\frac{(n-1)x^2}{1+(n-1)x}$$

$$\therefore \quad S_n(x)=\frac{nx^2}{1+nx}$$

$$S(x)=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{nx^2}{1+nx}=\begin{cases}x, & \text{if } 0<x\leqslant 1\\ 0, & \text{if } x=0\end{cases}$$

Let $\varepsilon>0$ be given, then for $0<x\leqslant 1$, we have

$$|S_n(x)-S(x)|=\left|\frac{nx^2}{1+nx}-x\right|=\left|\frac{-x}{1+nx}\right|$$

$$=\frac{x}{1+nx}<\varepsilon$$

if $\qquad 1+nx>\frac{x}{\varepsilon}$ or if $nx>\frac{x}{\varepsilon}-1$

or if $\qquad n>\frac{1}{\varepsilon}-\frac{1}{x}$

If we choose a positive integer m just $> \frac{1}{\varepsilon} - \frac{1}{x}$, then

$| S_n(x) - S(x) | < \varepsilon \ \forall \ n \geq m$ and $0 < x \leq 1$

For $x = 0$, $| S_n(x) - S(x) | = 0 < \varepsilon \ \forall \ n \geq 1$.

Hence the series converges uniformly on $[0, 1]$.

Example 11. *Test for uniform convergence the series* $\sum_{n=0}^{\infty} xe^{-nx}$ *in the closed interval* $[0, 1]$.

Sol. Here $S_n(x) = \sum_{n=0}^{n-1} x e^{-nx}$

$$= x + x e^{-x} + x e^{-2x} + \ldots\ldots + xe^{-(n-1)x}$$

which is a geometric series

$$= \frac{x(1 - e^{-nx})}{1 - e^{-x}}$$

$$= \frac{x\left(1 - \frac{1}{e^{nx}}\right)}{1 - \frac{1}{e^x}} = \frac{xe^x}{e^x - 1}\left(1 - \frac{1}{e^{nx}}\right)$$

$$\therefore \quad S(x) = \lim_{n \to \infty} S_n(x) = \begin{cases} \frac{x e^x}{x - 1}, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0 \end{cases}$$

Now for $0 < x \leq 1$ and for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = \left| \frac{xe^x}{e^x - 1}\left(1 - \frac{1}{e^{nx}}\right) - \frac{xe^x}{e^x - 1} \right|$$

$$= \left| \frac{-xe^x}{(e^x - 1)e^{nx}} \right| = \frac{xe^x}{(e^x - 1)e^{nx}} < \varepsilon$$

if $$\frac{(e^x - 1)e^{nx}}{xe^x} > \frac{1}{\varepsilon}$$

or if $$\log(e^x - 1) + nx - \log x - x > \log \frac{1}{\varepsilon}$$

or if $$\log\left\{x + \frac{x^2}{2!} + \ldots\ldots\right\} - \log x + nx - x > \log \frac{1}{\varepsilon}$$

or if $$\log\left\{1 + \frac{x}{2!} + \ldots\ldots\right\} + nx - x > \log \frac{1}{\varepsilon}$$

or if $$n > \frac{\log \frac{1}{\varepsilon} + x - \log\left\{1 + \frac{x}{2!} + \ldots\ldots\right\}}{x}$$

This shows that when $x \to 0$, $n \to \infty$ so that it is not possible to choose a positive integer m such that

$$|S_n(x)-S(x)|<\varepsilon \quad \forall\ n \geq m \text{ and } \forall\ x \in [0, 1].$$

Hence the series is non-uniformly convergent in any interval containing 0.

Example 12. *Show that $x=0$ is a point of non-uniform convergence of the series* $\sum_{n=1}^{\infty} \frac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]}$.

Sol. Here

$$f_n(x)=\frac{-2x(1+x)^{n-1}}{[1+(1+x)^{n-1}][1+(1+x)^n]}$$

$$=\frac{2}{1+(1+x)^n}-\frac{2}{1+(1+x)^{n-1}}$$

$$\Rightarrow \quad f_1(x)=\frac{2}{1+(1+x)}-\frac{2}{1+1}$$

$$f_2(x)=\frac{2}{1+(1+x)^2}-\frac{2}{1+(1+x)}$$

$$f_3(x)=\frac{2}{1+(1+x)^3}-\frac{2}{1+(1+x)^2}$$

....

$$f_n(x)=\frac{2}{1+(1+x)^n}-\frac{2}{1+(1+x)^{n-1}}$$

$$\therefore \quad S_n(x)=\frac{2}{1+(1+x)^n}-1$$

$$S(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty}\left[\frac{2}{1+(1+x)^n}-1\right]$$

$$=\begin{cases} -1 & \text{when } x>0 \\ 0 & \text{when } x=0 \\ 1 & \text{when } x<0 \end{cases}$$

Thus for $x>0$ and for a given $\varepsilon>0$, we have

$$|S_n(x)-S(x)|=\left|\frac{2}{1+(1+x)^n}-1+1\right|=\frac{2}{1+(1+x)^n}<\varepsilon$$

if $\quad 1+(1+x)^n>\frac{2}{\varepsilon}$

or if $\quad n\log(1+x)>\log\left(\frac{2}{\varepsilon}-1\right)$

or if $\quad n>\frac{\log\left(\frac{2}{\varepsilon}-1\right)}{\log(1+x)}$

This shows that if $x \to 0$, $n \to \infty$ so that $x=0$ is a point of non-uniform convergence of the series since no value of m can be chosen such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\ n \geqslant m \text{ and for every } x \text{ near } x=0.$$

Example 13. *Prove that the series*

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots\ldots$$

converges in the interval $[0, k]$, $k>0$ *but the series is not uniformly convergent in* $[0, k]$.

Sol. Here $S_n(x)$ = sum to n terms of the series

$$= x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \ldots + \frac{x^4}{(1+x^4)^{n-1}}$$

$$= \frac{x^4 \left[1 - \dfrac{1}{(1+x^4)^n} \right]}{1 - \dfrac{1}{1+x^4}} = (1+x^4) \left[1 - \frac{1}{(1+x^4)^n} \right]$$

$$\therefore \quad S(x) = \lim_{n \to \infty} S_n(x) = \begin{cases} 1+x^4, & \text{when } 0 < x \leqslant k \\ 0, & \text{when } x=0 \end{cases}$$

As $S(x)$ exists for all values of x in $[0, k]$, $k>0$, the series is convergent in this interval.

To text for uniform convergence, we have for $0 < x \leqslant k$ and for a given $\varepsilon > 0$,

$$| S_n(x) - S(x) | = \left| (1+x^4) \left\{ 1 - \frac{1}{(1+x^4)^n} \right\} - (1+x^4) \right|$$

$$= \frac{1+x^4}{(1+x^4)^n} = \frac{1}{(1+x^4)^{n-1}} < \varepsilon$$

if $\qquad (1+x^4)^{n-1} > \dfrac{1}{\varepsilon}$ or if $(n-1) \log (1+x^4) > \log \dfrac{1}{\varepsilon}$

or if $\qquad n-1 > \dfrac{\log \dfrac{1}{\varepsilon}}{\log (1+x^4)}$

or if $\qquad n > 1 + \dfrac{\log \dfrac{1}{\varepsilon}}{\log (1+x^4)}$

This shows that if $x \to 0$, $n \to \infty$ so that $x=0$ is a point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[h, k]$ where

$| S_n(x) - S(x) | < \varepsilon$ for all $n \geqslant m$ and for every x in $[h, k]$ where m is a positive integer just

$$\geqslant 1 + \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log(1+h^4)}.$$

Example 14. *Discuss the uniform convergence of the series*

$$\sum_{n=1}^{\infty} x^n(1-x) \quad on \quad [0, 1].$$

Sol. Here $S_n(x) = x(1-x) + x^2(1-x) + x^3(1-x) \ldots\ldots + x^n(1-x)$

$$= \frac{x(1-x)(1-x^n)}{1-x} = x(1-x^n)$$

$$\therefore \quad S(x) = \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} x(1-x^n) = \begin{cases} 0 \text{ when } x = 0 \text{ or } 1 \\ x \text{ when } 0 < x < 1 \end{cases}$$

Now if $0 < x < 1$, then for a given $\varepsilon > 0$, we have

$$| S_n(x) - S(x) | = | x(1-x^n) - x | = x^{n+1} < \varepsilon$$

if $$\left(\frac{1}{x}\right)^{n+1} > \frac{1}{\varepsilon} \quad \text{or if } n+1 > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}}$$

or if $$n > \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{x}} - 1$$

This shows that if $x \to 1$, $n \to \infty$. Thus it is not possible to find a positive integer m such that

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\ n \geqslant m \text{ and for every } x \text{ in } [0, 1].$$

Here $x = 1$ is the point of non-uniform convergence of the series.

However, the series is uniformly convergent in $[0, b]$ where $0 < b < 1$ since in this case, we can choose a positive integer m just

$$\geqslant \frac{\log \frac{1}{\varepsilon}}{\log \frac{1}{b}} - 1 \text{ such that}$$

$$| S_n(x) - S(x) | < \varepsilon \quad \forall\ n \geqslant m \text{ and } \forall\ x \text{ in } [0, b].$$

Example 15. *Show that the series* $\sum_{n=1}^{\infty} x^{n-1}(1-x)^2$ *converges uniformly to* $1-x$ *in* $[0, 1]$.

Sol. Please try yourself.

Example 16. *Show that the series* $\sum_{n=1}^{\infty} x^{n-1}$ *converges uniformly to* $\frac{1}{1-x}$ *in* $[0, b]$, $0<b<1$ *but does not converge uniformly on* $[0, 1]$.

Sol. Please try yourself.

Example 17. *Show that if* $0<r<1$, *then each of the following series is uniformly convergent on R :*

(i) $\sum_{n=1}^{\infty} r^n \cos nx$ (ii) $\sum_{n=1}^{\infty} r^n \sin nx$

(iii) $\sum_{n=1}^{\infty} r^n \cos n^2x$ (iv) $\sum_{n=1}^{\infty} r^n \sin a^n x$.

Sol. (i) Here $f_n(x)=r^n \cos nx$

$$|f_n(x)| = |r^n \cos nx| = r^n |\cos nx| \quad (\because \ r>0)$$
$$\leq r^n = M_n \ \forall \ x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n$ is a geometric series with $0<r<1$, it is convergent.

Hence by Weierstrass's M-test, the given series converges uniformly on R.

(ii) Please try yourself.
(iii) Please try yourself.
(iv) Please try yourself.

Example 18. *Show that the following series are uniformly convergent for all real x.*

(i) $\sum_{n=1}^{\infty} \frac{\sin(x^2+n^2x)}{n(n+2)}$ (ii) $\sum_{n=1}^{\infty} \frac{\cos(x^2+n^2x)}{n(n^2+2)}$.

Sol. (i) Here $f_n(x)=\frac{\sin(x^2+n^2x)}{n(n+2)}$

$$|f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+2)}\right| = \frac{|\sin(x^2+n^2x)|}{n(n+2)}$$

$$\leq \frac{1}{n(n+2)} \leq \frac{1}{n^2} = M_n \ \forall \ x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all real x.

(*ii*) Please try yourself.

Example 19. *Show that the following series are uniformly and absolutely convergent for all real values of x and p>1*

(*i*) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ (*ii*) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$.

Sol. (*i*) Here $f_n(x) = \frac{\sin nx}{n^p}$

$$|f_n(x)| = \left|\frac{\sin nx}{n^p}\right| = \frac{|\sin nx|}{n^p}$$

$$\leqslant \frac{1}{n^p} = M_n \ \forall \ x \in R$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p>1$, by Weierstrass's M-test, the given series converges uniformly and absolutely for all real values of x.

(*ii*) Please try yourself.

Example 20. *Show that the series*

$$\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\ldots$$

converges uniformly on R.

Sol. Please try yourself. $\left[\textbf{Hint. } f_n(x) = \frac{\cos nx}{n^2}\right]$

Example 21. *Test for uniform convergence the series*

(*i*) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ (*ii*) $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$.

Sol. (*i*) Here $f_n(x) = \frac{x}{(n+x^2)^2}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n+x^2)^2 \cdot 1 - x \cdot 2(n+x^2) \cdot 2x}{(n+x^2)^4}$$

$$= \frac{(n+x^2) - 4x^2}{(n+x^2)^3} = \frac{n-3x^2}{(n+x^2)^3}$$

For max. or min. $\dfrac{df_n(x)}{dx}=0$

$\Rightarrow \quad n-3x^2=0 \quad \Rightarrow \quad x=\sqrt{\dfrac{n}{3}}$

Also $\dfrac{d^2f_n(x)}{dx^2}=\dfrac{(n+x^2)^3\,.\,(-6x)-(n-3x^2)\,.\,3(n+x^2)^2\,.\,2x}{(n+x^2)^6}$

$$=\frac{-6x[(n+x^2)+(n-3x^2)]}{(n+x^2)^4}$$

$$\left.\frac{d^2f_n(x)}{dx^2}\right]_{x=\sqrt{\frac{n}{3}}}=\frac{-6\sqrt{\frac{n}{3}}\left(n+\frac{n}{3}\right)}{\left(n+\frac{n}{3}\right)^4}=-\frac{27\sqrt{3}}{32n^{5/2}}<0$$

$\Rightarrow \quad f_n(x)$ is maximum at $x=\sqrt{\dfrac{n}{3}}$ and the maximum value of

$f_n(x)$ is $\dfrac{\sqrt{\frac{n}{3}}}{\left(n+\frac{n}{3}\right)^2}=\dfrac{3\sqrt{3}}{16n^{3/2}}$

$\Rightarrow \quad |f_n(x)|\leqslant\dfrac{3\sqrt{3}}{16n^{3/2}}<\dfrac{1}{n^{3/2}}=M_n$

Since $\sum\limits_{n=1}^{\infty} M_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

(ii) Here $f_n(x)=\dfrac{x}{n(1+nx^2)}$

$\Rightarrow \quad \dfrac{df_n(x)}{dx}=\dfrac{1}{n}\cdot\dfrac{(1+nx^2)\,.\,1-x\,.\,2nx}{(1+nx^2)^2}$

$$=\frac{1-nx^2}{n(1+nx^2)^2}$$

For max. or min. $\dfrac{df_n(x)}{dx}=0$

$\Rightarrow \quad 1-nx^2=0 \quad \Rightarrow \quad x=\dfrac{1}{\sqrt{n}}$

Also $\dfrac{d^2f_n(x)}{dx^2}$

$$=\frac{1}{n}\,\frac{(1+nx^2)^2\,.\,(-2nx)-(1-nx^2)\,.\,2(1+nx^2)\,.\,2nx}{(1+nx^2)^4}$$

$$=-\frac{2x[(1+nx^2)+2(1-nx^2)]}{(1+nx^2)^3}$$

$$\left.\frac{d^2f_n(x)}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}}=-\frac{\frac{2}{\sqrt{n}}[1+1]}{(1+1)^3}=-\frac{1}{2\sqrt{n}}<0$$

$\Rightarrow$ $f_n(x)$ is maximum at $x=\frac{1}{\sqrt{n}}$ and the maximum value of

$f_n(x)$ is $\frac{\frac{1}{\sqrt{n}}}{n(1+1)}=\frac{1}{2n^{3/2}}$.

$$\Rightarrow \quad |f_n(x)|\leqslant\frac{1}{2n^{3/2}}<\frac{1}{n^{3/2}}=M_n$$

Since $\sum_{n=1}^{\infty}M_n=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

Example 22. *Show that the series* $\sum_{n=1}^{\infty}\frac{1}{1+n^2x}$ *converges in* $[1, \infty)$.

Sol. Here $f_n(x)=\frac{1}{1+n^2x}$

$$\therefore \quad |f_n(x)|=\left|\frac{1}{1+n^2x}\right|\leqslant\frac{1}{1+n^2}<\frac{1}{n^2}=M_n \ \forall x\in[1, \infty)$$

Since $\sum_{n=1}^{\infty}M_n=\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent, by Weierstrass M-test, the given series is uniformly convergent for all values of $x\in[1, \infty)$.

Example 23. *Show that the series* $\sum_{n=1}^{\infty}\frac{a_nx^{2n}}{1+x^{2n}}$ *is uniformly convergent for all real x if* $\sum_{n=1}^{\infty}a_n$ *is absolutely convergent.*

Sol. Here $f_n(x)=\frac{a_n\,x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}}<1 \ \forall x\in R$

$$\therefore \quad |f_n(x)| = \left|\frac{a_n x^{2n}}{1+x^{2n}}\right| = |a_n| \cdot \frac{x^{2n}}{1+x^{2n}} < |a_n| = M_n$$

for all $x \in R$

Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore,

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$$

is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 24. *Show that the series* $\sum_{n=1}^{\infty} \frac{a_n x^n}{1+x^{2n}}$ *is uniformly convergent for all real x if* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

Sol. Here $f_n(x) = \frac{a_n x^n}{1+x^{2n}}$

Let $$y = \frac{x^n}{1+x^{2n}}$$

Then $$\frac{dy}{dx} = \frac{(1+x^{2n}) \cdot nx^{n-1} - x^n \cdot 2n\, x^{2n-1}}{(1+x^{2n})^2}$$

$$= \frac{nx^{n-1}(1+x^{2n}-2x^{2n})}{(1+x^{2n})^2} = \frac{nx^{n-1}(1-x^{2n})}{(1+x^{2n})^2}$$

For max. or min. $\frac{dy}{dx} = 0 \Rightarrow x = 1$

Also $$\frac{d^2y}{dx^2} = \frac{(1+x^{2n})^2 \cdot [n(n-1)x^{n-2}(1-x^{2n}) + nx^{n-1}(-2nx^{2n-1})] - nx^{n-1}(1-x^{2n}) \cdot 2(1+x^{2n}) \cdot 2nx^{2n-1}}{(1+x^{2n})^4}$$

$$= \frac{(1+x^{2n})[n(n-1)x^{n-2}(1-x^{2n}) - 2n^2 x^{3n-2}] - 4n^2 x^{3n-2}(1-x^{2n})}{(1+x^{2n})^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1} = \frac{2[0-2n^2]-0}{(2)^3} = -\frac{n^2}{2} < 0$$

$\Rightarrow y = \frac{x^n}{1+x^{2n}}$ is maximum at $x=1$ and the maximum value of y is $\frac{1}{2}$.

$$\therefore \quad |f_n(x)| = \left|\frac{a_n x^n}{1+x^{2n}}\right| = \left|\frac{x^n}{1+x^{2n}}\right| |a_n|$$

$$\leq \tfrac{1}{2} |a_n| < |a_n| = M_n \text{ for all real values of } x.$$

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Since $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, therefore, $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |a_n|$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent for all real x.

Example 25. *If the series Σa_n converges absolutely then prove that $\Sigma a_n \cos nx$ and $\Sigma a_n \sin nx$ are uniformly convergent on R.* (M.D.U. 1992)

Sol. Let us test $\Sigma a_n \cos nx$.

Here $f_n(x) = a_n \cos nx$

$$|f_n(x)| = |a_n \cos nx| = |a_n|\,|\cos nx| \leqslant |a_n| = M_n$$ for all real values of x

Since Σa_n is absolutely convergent, therefore, $\Sigma M_n = \Sigma |a_n|$ is convergent. Hence by Weierstrass's M-test, the series $\Sigma a_n \cos nx$ is uniformly convergent on R.

Similarly, $\Sigma a_n \sin nx$ is uniformly convergent on R.

Example 26. *Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$ is absolutely and uniformly convergent for all real x if $p>1$.*

Sol. Here $f_n(x) = \frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}$

Since $\frac{x^{2n}}{1+x^{2n}} < 1 \;\forall\; x \in R$

$$\therefore \quad |f_n(x)| = \left|\frac{(-1)^n}{n^p} \cdot \frac{x^{2n}}{1+x^{2n}}\right| < \frac{1}{n^p} = M_n \text{ for all } x \in R.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$, therefore, by Weierstrass's M-test, the given series is absolutely and uniformly convergent for all real x if $p>1$.

Example 27. *Show that $\sum_{n=1}^{\infty} \frac{1}{n^p \pm n^q x^2}$ is uniformly convergent for all real x and $p>1$.*

Sol. Here $f_n(x) = \frac{1}{n^p + n^q x^2}$

Since $x^2 \geqslant 0$ for all real x.

$\therefore \; n^q x^2 \geqslant 0 \Rightarrow n^p + n^q x^2 \geqslant n^p$

$$\Rightarrow \quad \frac{1}{n^p+n^q x^2} \leqslant \frac{1}{n^p}$$

$$\therefore \quad |f_n(x)| = \left|\frac{1}{n^p+n^q x^2}\right| \leqslant \frac{1}{n^p} = M_n \text{ for all real } x.$$

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p>1$, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real x and $p>1$.

Example 28. *Show that the series* $\sum_{n=1}^{\infty} \frac{x}{n^p+n^q x^2}$ *is uniformly convergent for all real x if $p+q>2$.*

Sol. Here $\quad f_n(x) = \frac{x}{n^p+n^q x^2}$

$$\Rightarrow \quad \frac{df_n(x)}{dx} = \frac{(n^p+n^q x^2) \cdot 1 - x \cdot 2n^q x}{(n^p+n^q x^2)^2}$$

$$= \frac{n^p - n^q x^2}{(n^p+n^q x^2)^2}$$

For max. or min. $\quad \frac{df_n(x)}{dx} = 0$

$$\Rightarrow \quad n^p - n^q x^2 = 0 \quad \Rightarrow \quad x^2 = n^{p-q}$$

$$\Rightarrow \quad x = n^{\frac{p-q}{2}}$$

Also

$$\frac{d^2 f_n(x)}{dx^2} = \frac{(n^p+n^q x^2)^2 \cdot (-2n^q x) - (n^p - n^q x^2) \cdot 2(n^p+n^q x^2) \cdot 2n^q x}{(n^p+n^q x^2)^4}$$

$$= -\frac{2n^q x[(n^p+n^q x^2)+2(n^p-n^q x^2)]}{(n^p+n^q x^2)^3}$$

$$\left.\frac{d^2 f_n(x)}{dx^2}\right]_{x=n^{\frac{p-q}{2}}} = -\frac{2n^q \cdot n^{\frac{p-q}{2}} (n^p+n^p)}{(n^p+n^p)^3}$$

$$= -\frac{1}{2} n^{\frac{q-3p}{2}} < 0$$

(22)

$\Rightarrow$ $f_n(x)$ is maximum at $x=n^{\frac{p-q}{2}}$ and the maximum value

of $f_n(x)$ is $\dfrac{n^{\frac{p-q}{2}}}{n^p+n^p}=\dfrac{1}{2n^{\frac{p+q}{2}}}$

$$\Rightarrow \quad |f_n(x)|=\left|\frac{x}{n^p+n^q x^2}\right|\leqslant\frac{1}{2n^{\frac{p+q}{2}}}<\frac{1}{n^{\frac{p+q}{2}}}=M_n.$$

Since $\sum M_n=\sum \dfrac{1}{n^{\frac{p+q}{2}}}$ is convergent if $\dfrac{p+q}{2}>1$

i.e., if $p+q>2$, therefore, by Weirstrass's M-test, the given series is convergent for all real x if $p+q>2$.

Example 29. *Show that the series*

$$1+\frac{e^{-2x}}{2^2-1}+\frac{e^{-4x}}{4^2-1}+\frac{e^{-6x}}{6^2-1}+\ldots\ldots$$

is uniformly convergent for $x\geqslant 0$.

Sol. Neglecting the first term, we have

$$f_n(x)=\frac{e^{-2nx}}{(2n)^2-1}=\frac{e^{-2nx}}{4n^2-1}$$

For all $x\geqslant 0$, we have

$$e^{2nx}\geqslant 1 \quad \Rightarrow \quad e^{-2nx}\leqslant 1$$

Also $3n^2>1 \ \forall\ n$

$$\Rightarrow \quad 4n^2>n^2+1 \quad \Rightarrow \quad 4n^2-1>n^2$$

$$\therefore \quad |f_n(x)|=\left|\frac{e^{-2nx}}{4n^2-1}\right|<\frac{1}{n^2}=M_n.$$

Since $\sum\limits_{n=1}^{\infty} M_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for $x\geqslant 0$.

Example 30. *Test for uniform convergence the series*

$$\frac{1}{(1+x)^3}+\frac{2}{(2+x)^3}+\frac{3}{(3+x)^3}+\ldots\ldots, \ x\geqslant 0.$$

Sol. The given series is $\sum\limits_{n=1}^{\infty}\dfrac{n}{(n+x)^3}$

Here $f_n(x)=\dfrac{n}{(n+x)^3}$

$$\forall\ x\geq 0,\ |f_n(x)|=\left|\frac{n}{(n+x)^3}\right|=\frac{n}{(n+x)^3}$$

$$<\frac{n}{n^3}=\frac{1}{n^2}=M_n$$

Since $\sum\limits_{n=1}^{\infty} M_n=\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all $x\geq 0$.

Example 31. *Show that the series*

$$\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots\ldots$$

is uniformly convergent in $(-1,\ 1)$.

Sol. The given series is $\sum\limits_{n=1}^{\infty}\dfrac{2^n x^{2^n-1}}{1+x^{2^n}}=\sum\limits_{n=1}^{\infty} f_n(x)$

$$|f_n(x)|=\left|\frac{2^n\ .\ x^{2^n-1}}{1+x^{2^n}}\right|\leq 2^n\ .\ k^{2^n-1}=M_n$$

$$\text{for } |x|\leq k<1 \qquad \ldots(1)$$

Now $M_n=2^n\ .\ k^{2^n-1}\ \Rightarrow\ M_{n+1}=2^{n+1}\ .\ k^{2^{n+1}-1}$

$$\therefore\ \lim_{n\to\infty}\frac{M_n}{M_{n+1}}=\lim_{n\to\infty}\frac{k^{2^n-1}}{2\ .\ k^{2^{n+1}-1}}=\lim_{n\to\infty}\frac{1}{2k^{2^{n+1}-2^n}}$$

$$=\lim_{n\to\infty}\frac{1}{2k^{2^n}}=\infty,\ \text{since } k<1$$

By ratio test, the series $\sum\limits_{n=1}^{\infty} M_n$ is convergent. Hence, by Weierstrass's M-test, the given series is uniformly convergent in $(-1,\ 1)$.

Example 32. *Test the series* $\Sigma f_n(x)$ *for uniform convergence where*

$$f_n(x)=\frac{1}{(x^2+n)(x^2+n+1)}.$$

Sol. $|f_n(x)|=\left|\dfrac{1}{(x^2+n)(x^2+n+1)}\right|<\dfrac{1}{n^2}=M_n\quad \forall\ x\in R$

Since $\Sigma M_n=\sum\dfrac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the given series is uniformly convergent for all real values of x.

Example 33. *Test for uniform convergence the series*

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots+\frac{x^n}{n!}+\ldots\ldots \text{ in } [-1, 1].$$

Sol. Neglecting 1, we have $f_n(x)=\dfrac{x^n}{n!}$

$$|f_n(x)|=\left|\frac{x^n}{n!}\right|\leqslant\frac{1}{n!} \quad \text{for } -1\leqslant x\leqslant 1$$

Since $n!\geqslant 2^n$ for $n>3$, therefore, we have

$$|f_n(x)|\leqslant\frac{1}{2^n}=\left(\frac{1}{2}\right)^n=M_n$$

But we know that $\Sigma M_n=\sum\left(\frac{1}{2}\right)^n$ is convergent. Hence by Weierstrass's M-test, the given series is uniformly convergent in $[-1, 1]$.

Example 34. *Prove that if* δ *is any fixed positive number less than unity, the series* $\sum\limits_{n=1}^{\infty}(n+1)\,x^n$ *converges uniformly in* $(-\delta, \delta)$.

Sol. Here $-\delta<x<\delta$ and $0<\delta<1$

$\Rightarrow$ $|x|<\delta<1$

Also $f_n(x)=(n+1)\,x^n$

$\therefore$ $|f_n(x)|=|(n+1)\,x^n|<(n+1)\,\delta^n=M_n$

$$\lim_{n\to\infty}\frac{M_n}{M_{n+1}}=\lim_{n\to\infty}\frac{(n+1)\,\delta^n}{(n+2)\,\delta^{n+1}}=\frac{1}{\delta}>1$$

$\therefore$ By ratio test, ΣM_n is convergent. Hence the given series is uniformly convergent in $(-\delta, \delta)$.

Example 35. *Show that each of the following series is uniformly convergent for all values of x*

(i) $\sum\dfrac{1}{n^4+n^2x^2}$ (ii) $\sum\dfrac{1}{n^2+n^4x^2}$

(iii) $\sum\dfrac{1}{n^3+n^4x^2}$.

Sol. Please try yourself.

[**Hint.** $\forall\, x\in R$, $n^4+n^2x^2\geqslant n^4$ so that $|f_n(x)|\leqslant\dfrac{1}{n^4}$]

Example 36. *Prove that if k is any fixed positive number less than unity, then each of the following series is uniformly convergent in* $[-k, k]$.

(i) $\sum x^n$ (ii) $\sum\dfrac{x^n}{n+1}$

Sol. Please try yourself.

Example 37. *Prove that if k is any fixed number greater than unity, then each of the following series is uniformly convergent for all $x \geq k$.*

(*i*) $\sum \frac{1}{x^n}$ (*ii*) $\sum \frac{1}{1+x^n}$.

Sol. Please try yourself.

10·14. Abel's Test

If (*i*) $\Sigma f_n(x)$ is uniformly convergent on [a, b],

(*ii*) the sequence $<g_n(x)>$ is monotonic decreasing for all $x \in [a, b]$, and

(*iii*) there exists a positive real number k such that

$$| g_n(x) | < k \quad \forall \ x \in [a, b] \text{ and } n \in N$$

then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on [a, b].

Proof. $\Sigma f_n(x)$ is uniformly convergent on $[a, b]$

$\Rightarrow$ By Cauchy's criterion, for each $\varepsilon > 0$ and $\forall \ x \in [a, b]$, there exists a positive integer m (depending only on ε) such that

$$| f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_{n+p}(x) | < \frac{\varepsilon}{k} \quad \forall \ n \geq m, \ p \in N$$

$$\Rightarrow \quad \left| \sum_{r=n+1}^{n+p} f_r(x) \right| < \frac{\varepsilon}{k} \quad \forall \ n \geq m, \ p \in N$$

Also, the sequence $<g_n(x)>$ is monotonic decreasing on $[a, b]$

and $\quad | g_n(x) | < k \quad \forall \ x \in x[a, b]$ and $n \in N$

$\therefore$ By Abel's lemma (see chapter 8), we get

$$\Rightarrow \quad \left| \sum_{r=n+1}^{n+p} f_r(x) g_r(x) \right| < \frac{\varepsilon}{k} \cdot k = \varepsilon \quad \forall \ n \geq m, \ p \in N$$

and $x \in [a, b]$

$$\Rightarrow \quad | f_{n+1}(x)\, g_{n+1}(x) + f_{n+2}(x) g_{n+2}(x) + \ldots\ldots + f_{n+p}(x)\, g_{n+p}(x) | < \varepsilon$$

$$\forall \ n \geq m, \ p \in N \text{ and } x \in [a, b]$$

Hence ' v Cauchy's criterion, $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

10·15. Dirichlet's Test

If (*i*) there exists a positive real number k such that

$$| S_n(x) | = \left| \sum_{r=1}^{n} f_r(x) \right| < k \quad \forall \ x \in [a, b], \ n \in N \text{ and}$$

(*ii*) $<g_n(x)>$ is a positive monotonic decreasing sequence converging uniformly to zero on $[a, b]$

then the series $\Sigma f_n(x)\, g_n(x)$ is uniformly convergent on $[a, b]$.

Proof. Since $| S_n(x) | < k \ \forall \ x \in [a, b],\ n \in N$

$\therefore \quad \forall \ x \in [a, b],\ n \geqslant m_1,\ p \in N$, we have

$$| S_{n+p}(x) - S_n(x) | \leqslant | S_{n+p}(x) | + | S_n(x) |$$
$$< k + k = 2k$$

$$\Rightarrow \quad \left| \sum_{r=n+1}^{n+p} f_r(x) \right| < 2k \ \forall \ n \geqslant m_1,\ p \in N,\ x \in [a, b]$$

Also, the sequence $<g_n(x)>$ is positive monotonic decreasing on $[a, b]$.

$\therefore$ By Abel's lemma, we get

$$\left| \sum_{r=n+1}^{n+p} f_r(x) g_r(x) \right| < 2k g_{n+1}(x) \ \forall \ n \geqslant m_1,\ p \in N \text{ and } x \in [a, b] \quad \ldots(1)$$

Since $<g_n(x)>$ converges uniformly to zero on $[a, b]$.

$\therefore$ given $\varepsilon > 0$, there exists a positive integer m_2 such that

$$| g_n(x) | < \frac{\varepsilon}{2k} \ \forall \ n \geqslant m_2 \quad \ldots(2)$$

Let $m = \max \{m_1, m_2\}$, then both (1) and (2) hold for $n \geqslant m$.

From (1) and (2), we have

$$\left| \sum_{r=n+1}^{n+p} f_r(x) g_r(x) \right| < 2k \cdot \frac{\varepsilon}{2k} = \varepsilon$$

$$\forall \ n \geqslant m,\ p \in N,\ x \in [a, b]$$

Hence $\Sigma f_n(x) g_n(x)$ is uniformly convergent on $[a, b]$.

ILLUSTRATIVE EXAMPLES

Example 1. *Prove that the series* $\sum \frac{(-1)^{n-1}}{n} x^n$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $f_n(x) = \frac{(-1)^{n-1}}{n}$ and $g_n(x) = x^n$

The series $\Sigma f_n(x)$ is convergent by Leibnitz's test. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Also, for $0 \leqslant x \leqslant 1,\ x^n > x^{n+1} \ \forall \ n \in N$

and $\quad | g_n(x) | = | x^n | = | x |^n \leqslant 1$

$\therefore$ $<g_n(x)>$ is monotonic decreasing and bounded on $[0, 1]$ for all $n \in N$.

Hence by Abel's test, the series

$$\sum f_n(x) g_n(x) = \sum \frac{(-1)^{n-1}}{n} x^n$$

is uniformly convergent on $[0, 1]$.

Example 2. *If Σa_n is convergent, then show that* $\sum \frac{a_n}{n^x}$ *is uniformly convergent on* $[0, 1]$.

Sol. Let $f_n(x) = a_n$ and $g_n(x) = \frac{1}{n^x}$.

The series $\Sigma f_n(x) = \Sigma a_n$ is given to be convergent. Since it is independent of x, it is uniformly convergent on $[0, 1]$.

Since $<n^x>$ increases on $[0, 1]$, $<\frac{1}{n^x}>$ decreases on $[0, 1]$.

Also $| g_n(x) | = \frac{1}{n^x} \leqslant \frac{1}{n^0} = 1$.

$\therefore$ The sequence $<g_n(x)>$ is monotonic decreasing and bounded on $[0, 1]$ for all $n \in N$.

Hence by Abel's test, the series

$$\sum f_n(x) g_n(x) = \sum \frac{a_n}{n^x}$$

is uniformly convergent on $[0, 1]$.

Example 3. *Show that the series* $\sum \frac{(-1)^{n-1}}{n+x^2}$ *is uniformly convergent for all values of* x.

Sol. Let $f_n(x) = (-1)^{n-1}$ and $g_n(x) = \frac{1}{n+x^2}$

$$\text{Now } S_n(x) = \sum_{r=1}^{n} f_r(x) = 1-1+1-1+\ldots\ldots+(-1)^{n-1}$$

$$= \begin{cases} 1, \text{ when } n \text{ is odd} \\ 0, \text{ when } n \text{ is even.} \end{cases}$$

$\Rightarrow$ $S_n(x)$ is bounded for all n and for all x.

Also $<g_n(x)> = <\frac{1}{n+x^2}>$ is a positive monotonic decreasing sequence converging to 0 for all values of x.

Hence by Dirichlet's test, the series

$$\sum f_n(x) g_n(x) = \sum \frac{(-1)^{n-1}}{n+x^2}$$

is uniformly convergent for all x.

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Example 4. *Prove that the series*

$$\frac{1}{1+x^2}-\frac{1}{2+x^2}+\frac{1}{3+x^2}-\frac{1}{4+x^2}+\ldots\ldots$$

is uniformly convergent in any interval.

Sol. Please try yourself. (It is the same as Example 3).

Example 5. *Show that the series*

$$\cos x+\tfrac{1}{2}\cos 2x+\tfrac{1}{3}\cos 3x+\ldots\ldots$$

converges uniformly in $(0, 2\pi)$.

Sol. The given series is $\sum \frac{\cos nx}{n}$

Let $f_n(x)=\cos nx$ and $g_n(x)=\frac{1}{n}$

Now $S_n(x)=\sum\limits_{r=1}^{n} f_r(x)$

$$=\cos x+\cos 2x+\cos 3x+\ldots\ldots+\cos nx$$

$$=\frac{\cos\left[x+\frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}}=\frac{\cos\frac{n+1}{2}x\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore\quad |S_n(x)|=\frac{\left|\cos\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|}\leqslant\frac{1}{\left|\sin\frac{x}{2}\right|}$$

or $$|S_n(x)|\leqslant\left|\operatorname{cosec}\frac{x}{2}\right|$$

But $\operatorname{cosec}\frac{x}{2}$ is bounded for all values of x in $(0, 2\pi)$. Let k be the least upper bound of $\operatorname{cosec}\frac{x}{2}$ in $(0, 2\pi)$, then

$$|S_n(x)|<k \text{ for all } x\in(0, 2\pi).$$

Also $\langle g_n(x)\rangle=\langle\frac{1}{n}\rangle$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series

$$\sum f_n(x)g_n(x)=\sum\frac{\cos nx}{n}$$

converges uniformly in $(0, 2\pi)$.

Example 6. *Test the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ *for uniform convergence on* $[0, 1]$.

Sol. Let $f_n(x)=\sin nx$ and $g_n(x)=\frac{1}{n}$

Now $S_n(x)=\sum_{r=1}^{n} f_r(x)$

$$=\sin x+\sin 2x+\sin 3x+\dots\dots+\sin nx$$

$$=\frac{\sin\left[x+\frac{n-1}{2}x\right]\sin\frac{nx}{2}}{\sin\frac{x}{2}}=\frac{\sin\frac{n+1}{2}x\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$\therefore \quad |S_n(x)|=\frac{\left|\sin\frac{n+1}{2}x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|}\leq\frac{1}{\left|\sin\frac{x}{2}\right|}$$

But $\operatorname{cosec}\frac{x}{2}$ is bounded on $(0, 1]$ which is a subset of $\left(0, \frac{\pi}{2}\right)$.

Let k be the least upper bound of $\operatorname{cosec}\frac{x}{2}$ on $(0, 1]$

When $x=0$, $\quad S_n(x)=0+0+0+\dots\dots+0=0$

$\therefore \quad |S_n(x)|<k \ \forall\ x\in[0, 1]$ and $n\in N$.

Also $\langle g_n(x)\rangle=\langle\frac{1}{n}\rangle$ is a positive monotonic decreasing sequence converging to 0.

Hence by Dirichlet's test, the series $\Sigma f_n(x)g_n(x)=\sum\frac{\sin nx}{n}$ converges uniformly on $[0, 1]$.

Example 7. *Prove that the series* $\sum(-1)^n \frac{x^2+n}{n^2}$ *converges uniformly in every bounded interval, but does not converge absolutely for any value of* x. (M.D.U. 1992)

Sol. Let the bounded interval be $[a, b]$ so that there exists a positive number k such that for all $x\in[a, b]$, $|x|<k$.

Let $\quad f_n(x)=(-1)^n$

and $\quad g_n(x)=\dfrac{x^2+n}{n^2}$

$$S_n(x)=\sum_{r=1}^{n} f_r(x)=-1+1-1+1......+(-1)^n$$

$$=\begin{cases}-1 \text{ when } n \text{ is odd}\\ 0 \text{ when } n \text{ is even}\end{cases}$$

$\Rightarrow$ $S_n(x)$ is bounded for all $x\in[a, b]$ and for all $n\in N$.

Also $\quad g_n(x)=\dfrac{x^2+n}{n^2}<\dfrac{k^2+n}{n^2}$

$\therefore$ $<g_n(x)>$ is a positive monotonic decreasing sequence converging to 0 uniformly for $x\in[a, b]$.

Hence by Dirichlet's test, the series $\Sigma f_n(x)\, g_n(x)$

$=\sum(-1)^n \cdot \dfrac{x^2+n}{n^2}$ converges uniformly on $[a, b]$.

Now $\displaystyle\sum\left|(-1)^2\, \frac{x^2+n}{n^2}\right|=\sum\frac{x^2+n}{n^2}\simeq\sum\frac{1}{n^2}$

which diverges. Hence the given series is not absolutely convergent for any value of x.

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Example 2. *Examine for term by term integration the series* $\sum_{n=1}^{\infty} f_n$, *where* $S_n(x) = \sum_{i=1}^{n} f_i = nx\, e^{-nx^2}$, *over the intervals*

(i) $[0, 1]$ (ii) $[a, 1],\ 0 < a < 1$.

Sol. Here $f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \dfrac{nx}{e^{nx^2}}$

$$= \lim_{n\to\infty} \frac{nx}{1 + \frac{nx^2}{1!} + \frac{n^2x^4}{2!} + \ldots\ldots} = 0$$

for all x

Consider the interval $[0, 1]$

$$\int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0$$

and $$\int_0^1 S_n(x)\, dx = \int_0^1 nx\, e^{-nx^2}\, dx$$

$$= \int_0^n \tfrac{1}{2} e^{-t}\, dt \quad \text{where } t = nx^2$$

$$= \tfrac{1}{2}(1 - e^{-n})$$

$$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx=\lim_{n\to\infty}\tfrac{1}{2}(1-e^{-n})=\tfrac{1}{2}$$

Since $$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx\neq\int_0^1 f(x)\,dx$$

i.e., $$\lim_{n\to\infty}\int_0^1 S_n(x)\,dx\neq\int_0^1\left(\lim_{n\to\infty} S_n(x)\right)dx$$

the series $\sum_{n=1}^{\infty} f_n$ does not admit of term by term integration over the interval [0, 1].

Now consider the interval $[a, 1]$, $0<a<1$

$$\int_a^1 f(x)\,dx=\int_a^1 0\,dx=0$$

and $$\int_a^1 S_n(x)\,dx=\int_a^1 nx\,e^{-nx^2}\,dx$$

$$=\int_{na^2}^{n}\frac{1}{2}e^{-t}\,dt \quad \text{where } t=nx^2$$

$$=\frac{1}{2}\left(e^{-na^2}-e^{-n}\right)$$

$$\lim_{n\to\infty}\int_a^1 S_n(x)\,dx=\lim_{n\to\infty}\frac{1}{2}\left(e^{-na^2}-e^{-n}\right)=0$$

$$=\int_a^1 f(x)\,dx$$

Hence term by term integration is justified over the interval $[a, 1]$ where $0<a<1$.

Example 3. *Prove that*

Examine for term by term integration the series the sum of whose first n terms is $n^2x(1-x)^n$, $0 \leq x \leq 1$.

Sol. Here $S_n(x) = n^2x(1-x)^n$.

When $x = 0$ or 1, $S_n(x) = 0$

When $0 < x < 1$,

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{n^2x}{(1-x)^{-n}} \quad \Big| \text{ Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2nx}{-(1-x)^{-n}\log(1-x)} \quad \Big| \text{ Form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n}[\log(1-x)]^2} = 0$$

$$\therefore \quad f(x) = \lim_{n\to\infty} S_n(x) = 0 \text{ for all } x \in [0, 1]$$

Also $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$

and $\displaystyle\int_0^1 S_n(x)\,dx = \int_0^1 n^2x(1-x)^n\,dx$

Changing x to $1-x$

$$= \int_0^1 n^2(1-x)\,x^n\,dx$$

$$= \int_0^1 n^2(x^n - x^{n+1})\,dx = n^2\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = 1$$

Since $\lim_{n\to\infty} \int_0^1 S_n(x)\, dx \neq \int_0^1 \left(\lim_{n\to\infty} S_n(x) \right) dx$, term by term integration is not justified on [0, 1].

Example 6. *Show that the series for which*

(i) $S_n(x)=\dfrac{1}{1+nx}$ *(ii)* $S_n(x)=nx(1-x)^n$

can be integrated term by term on [0, 1], though they are not uniformly convergent on [0, 1].

Sol. (*i*) Here $S_n(x)=\dfrac{1}{1+nx}$

so that $f(x)=\lim_{n\to\infty} S_n(x)=\lim_{n\to\infty} \dfrac{1}{1+nx}=\begin{cases} 0, \text{ if } 0<x\leqslant 1 \\ 1, \text{ if } x=0 \end{cases}$

For $0<x\leqslant 1$ and for a given $\varepsilon>0$, we have

$$| S_n(x)-f(x) | =\left|\frac{1}{1+nx}-0\right|=\frac{1}{1+nx}<\varepsilon$$

if $$1+nx>\frac{1}{\varepsilon} \quad \text{or} \quad \text{if } n>\frac{\frac{1}{\varepsilon}-1}{x}$$

If $x\to 0$, $n\to\infty$ so that $x=0$ is a point of non-uniform convergence of the series. Thus the series does not converge uniformly on [0, 1].

Now $\int_0^1 f(x)\, dx=\int_0^1 0\, dx=0$

and $$\int_0^1 S_n(x)\, dx=\int_0^1 \frac{dx}{1+nx}=\left[\frac{\log(1+nx)}{n}\right]_0^1$$

$$=\frac{\log(1+n)}{n}$$

$$\lim_{n\to\infty} \int_0^1 S_n(x)\, dx= \lim_{n\to\infty} \frac{\log(1+n)}{n} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1+n}}{1}=0$$

Since $\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty} [S_n(x)]\,dx$, the series is integrable term by term on [0, 1] although $x=0$ is a point of non-uniform convergence of the series.

(*ii*) Here $S_n(x) = nx(1-x)^n$

When $0<x<1$, we have

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}} \qquad \left| \text{Form } \frac{\infty}{\infty} \right.$$

$$= \lim_{n\to\infty} \frac{x}{-(1-x)^{-n}\log(1-x)} = 0$$

Also $S_n(x)=0$ for $x=0$ or 1

$\therefore$ $f(x)=0$ for every x in [0, 1].

$$\text{Now } \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$$

and

$$\int_0^1 S_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$$

Changing x to $1-x$

$$= \int_0^1 n(1-x)\,x^n\,dx = \int_0^1 n(x^n - x^{n+1})\,dx$$

$$= n\left[\frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{n}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \lim_{n\to\infty} \frac{n}{(n+1)(n+2)} = 0$$

$$\text{Since } \lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \int_0^1 \lim_{n\to\infty} [S_n(x)]\,dx$$

the series is integrable term by term on [0, 1] although $x=0$ is a point of non-uniform convergence of the series.

(See Example 7, Illustrative Examples—B)

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Example 3. *Show that the function represented by* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ *is differentiable for every x and its derivative is* $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

(M.D.U. 1991 S)

Sol. Here $f_n(x) = \frac{\sin nx}{n^3}$

$$\therefore \quad f_n'(x) = \frac{\cos nx}{n^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

Since $\left|\frac{\cos nx}{n^2}\right| \leqslant \frac{1}{n^2}$ $\forall x$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all x and hence $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

$$\Rightarrow \quad \left(\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}\right)' = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

Example 4. *Show that the differential co-efficient of*

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$$\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2} \text{ is } -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \text{ for all real } x.$$

Sol. Here $f_n(x) = \frac{1}{n^3 + n^4x^2} = \frac{1}{n^3(1+nx^2)}$

$$\Rightarrow \quad f_n'(x)=\frac{1}{n^3}\left[\frac{-2nx}{(1+nx^2)^2}\right]=-\frac{2x}{n^2(1+nx^2)^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} f_n'(x)=-2x\sum_{n=1}^{\infty}\frac{1}{n^2(1+nx^2)^2}$$

Now $f_n'(x)$ is maximum when $\frac{d}{dx}f_n'(x)=0$

i.e. when $-\frac{2}{n^2}\cdot\frac{(1+nx^2)^2\cdot 1-x\;2(1+nx^2)\cdot 2nx}{(1+nx^2)^4}=0$

or when $\quad 1-3nx^2=0$

or when $\quad x=\frac{1}{\sqrt{3n}}$

$\therefore$ Maximum value of $|f_n'(x)|$

$$=\frac{2\cdot\frac{1}{\sqrt{3n}}}{n^2(1+\frac{1}{3})^2}=\frac{3\sqrt{3}}{8n^{5/2}}$$

$$\Rightarrow \quad |f_n'(x)|\leqslant\frac{3\sqrt{3}}{8n^{5/2}}=\frac{1}{n^{5/2}} \quad \forall\ x$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}$ is convergent, therefore, by Weierstrass's M-test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all real x and hence $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \quad \left(\sum_{n=1}^{\infty} f_n\right)'=\sum_{n=1}^{\infty} f_n'$$

$$\Rightarrow \quad \left(\sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}\right)'=-2x\sum_{n=1}^{\infty}\frac{1}{n^2(1+nx^2)^2}$$

Example 5. *Show that the series for which*

$$S_n(x)=\frac{nx}{1+n^2x^2},\quad 0\leqslant x\leqslant 1$$

cannot be differentiated term by term at $x=0$.

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Sol. Here $f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

for $0 \leqslant x \leqslant 1$

$\therefore \quad f'(0) = 0$

Also $S_n'(0) = \lim_{h\to 0} \frac{S_n(0+h) - S_n(0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{nh}{1+n^2h^2} - 0}{h} = \lim_{n\to\infty} \frac{n}{1+n^2h^2} = n$$

$\Rightarrow \quad \lim_{n\to\infty} S_n'(0) = \infty$

Thus $f'(0) \neq \lim_{n\to\infty} S_n'(0)$

Hence the given series cannot be differentiated term by term.

Example 6. *Given the series* $\sum_{n=1}^{\infty} f_n$ *for which*

$$S_n(x) = \frac{1}{2n^2} \log(1+n^4x^2),\ 0 \leqslant x \leqslant 1.$$

Show that the series $\sum_{n=1}^{\infty} f_n'$ *does not converge uniformly, but the given series can be differentiated term by term.*

Sol. Here $f(x) = \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{\log(1+n^4x^2)}{2n^2}$

| Form $\frac{\infty}{\infty}$

$$= \lim_{n\to\infty} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n}$$

$$= \lim_{n\to\infty} \frac{n^2x^2}{1+n^4x^2} = 0 \text{ for } 0 \leqslant x \leqslant 1$$

$\therefore \quad f'(x) = 0$

Also $\lim_{n\to\infty} S_n'(x) = \lim_{n\to\infty} \left(\frac{1}{2n^2} \cdot \frac{2n^4x}{1+n^4x^2} \right)$

$$= \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2} = 0 \text{ for } 0 \leqslant x \leqslant 1$$

$\therefore \quad f'(x) = \lim_{n\to\infty} S_n'(x)$

Thus term by term differentiation holds.

However, the series $\sum_{n=1}^{\infty} f_n'$ is not uniformly convergent for $0 \leqslant x \leqslant 1$ since the sequence $<S_n'>$ *i.e.*, $<\dfrac{n^2x}{1+n^4x^2}>$ has $x=0$ as a point of non-uniform convergence.

Example 7. *Examine whether the series for which*

$$S_n(x) = \frac{1}{n+n^3x^2}$$

is differentiable term by term.

Sol. Please try yourself. [Ans. Yes]