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UPSC Mathematics (Optional)

STUDY MATERIAL

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MECHANICS

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Exam Oriented Study Material

Last Update 25-06-2020 (We Update Regularly)

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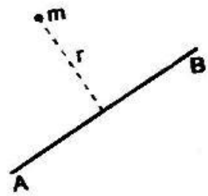
Moment Of Inertia

0·01 Definitions.

(a) Rigid Body : A rigid body is the system of particles such that the mutual distance of every pair of specified particles in it is invariable and the body does not expand or contract or change its shape in any way. *i.e. the rigid body has invariable size and shape and the distance between any two particles remains always same.*

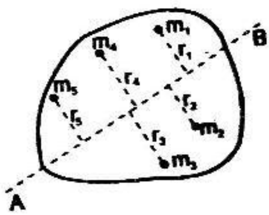
(b) Moment of inertia of a particle : Consider a particle of mass m and a line a line AB, then the moment of inertia of the particle of mass m about the line AB is defined as $I = mr^2$, where r is the perp. distance of the particle from the line.



(c) Moment of inertia of a system of particles: Let there be a number of particles $m_1, m_2, m_3, \ldots, m_p$, and let $r_1, r_2, r_3, \ldots, r_p$ be the perp. distances of these masses from the given line AB, then the moment of inertia of the system is defined as

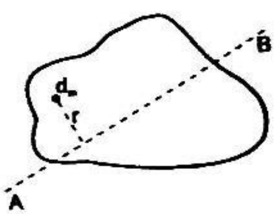
$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_pr_p^2$$

$$= \sum_{p=1}^{n} m_pr_p^2$$



(d) Moment of inertia of a continuous distribution of mass : Consider a rigid body and let dm be mass of the elementary portion of the body which is at a perpendicular distance r from the given line AB, then the moment of inertia of the whole body is defined as $I = \int r^2\, dm$,

where the integration is taken over the whole body.



(e) Radius of Gyration : The moment of inertia of a system of particles about the line AB is

$$I = \sum_{p=1}^{n} m_pr_p^2.$$

Let the total mass of the system of particles be M, then

$M = \sum_{p=1}^{n} m_p$ and further define a quantity K such that

$$I = MK^2 \Rightarrow K^2 = \left(\frac{1}{M}\right) = \frac{\sum_{p=1}^{n} m_p r_p^2}{\sum_{p=1}^{n} m_p}$$

Then K is called the **radius** of **gyration** of the system about AB. In the case of *continuous mass distribution*, we similarly have

$$K^2 = \left(\frac{I}{M}\right) = \left[\frac{\int r^2\, dm}{\int dm}\right]$$

where the integration is taken over the whole body.

(f) Product of inertia : If $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots (x_p, y_p)$ be the respective coordinates of the particles of masses $m_1, m_2, m_3, \ldots m_p$, referred to two mutually perpendicular lines OX and OY, then the product of inertia of the system of particles with respect to the lines OX and OY, is defined as,

$$P = m_1 x_1 y_1 + m_2 x_2 y_2 + m_3 x_3 y_3 \ldots + m_p x_p y_p = \sum_{p=1}^{n} m_p x_p y_p.$$

If mutually perpendicular axes OX, OY, OZ be taken in space and $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_p, y_p, z_p)$ be the respective co-ordintes of the particles of masses $m_1, m_2, \ldots, m_p$, then we have, product of inertia of the system with respect to the axes OX and $OY = \sum_{p=1}^{n} m_p x_p y_p$

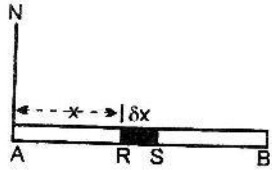
Product of inertia of the system with respect to the axes OY and $OZ = \sum_{p=1}^{n} m_p y_p z_p$.

Product of inertia of the system with respect to the axes OZ and $OX = \sum_{p=1}^{n} m_p x_p z_p$.

0·02 Moment of inertia in some simple cases.

(A) (i) *Moment of inertia of a rod of length 2a and mass M about a line through one of its extremities perp. to its length.*

Consider an element RS of breadth δx of the rod AB at distance x from the line AN, where AN is perp. to AB, M.I. of the element RS about $AN \doteq \frac{M}{2a}\delta x\, x^2$ where $(M/2a)\,\delta x =$ mass of the element.



$\therefore$ M.I. of the whole rod

$$= \int_0^{2a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = M.\frac{4a^2}{3}.$$

(ii) *Moment of inertia of a rod of length 2a and of mass M about a line through its centre perpendicular to its length.*

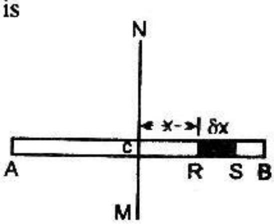
Consider an element RS of breadth δx at distance x from the centre C.

$\therefore$ M.I. of the element RS about NCM is

$$= \frac{M}{2a}\,\delta x.\, x^2$$

$\Rightarrow$M.I. of the whole rod about

$$MN = \int_{-a}^{a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = M.\frac{a^2}{3}.$$



(B) Rectangular Lamina.

(i) *Moment of inertia of a rectangular lamina about a line through its centre and parallel to one of its edges.*

Consider the strip $RSPQ$ of breadth δx of the rectangular lamina $ABCD$ such that $AB = 2a$ and $AD = 2b$. Let M be the mass of the rectangular lamina. Then mass per unit area $= \dfrac{M}{4ab} = \rho$ (*say*).

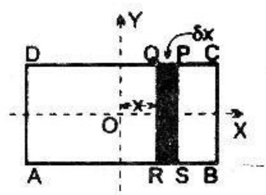
$\therefore$ Mass of the strip $RSPQ = 2b\,\delta x\,\rho = \dfrac{M}{4ab}(2b\,.\delta x)$

Now using [A case (ii)], we get M.I. of the strip about

$$OX = \frac{M}{4ab} 2b\,\delta x\left(\frac{b^2}{3}\right) = \frac{M}{2a}.\frac{b^2}{3}\,\delta x$$

$\therefore$ M.I. of the rectangular lamina about OX

$$= \int_{-a}^{a} \frac{M}{2a}.\frac{b^2}{3}\, dx = \frac{1}{3} M b^2.$$

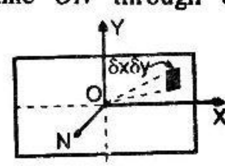


Similarly M.I. of the rectangular lamina about OY is $\frac{1}{3} M a^2$.

(ii) *Moment of inertia of a rectangular lamina about a line through its centre and perp. to its plane.*

Consider an elementary area $\delta x\,\delta y$ of the lamina at a distance $\sqrt{(x^2+y^2)}$ from O. Mass of the elementary area$= \dfrac{M}{4ab}\,\delta x.\,\delta y$.

M.I. of this elementary area about the line ON through O and perpendicular to the plane of the rectangular lamina$= \dfrac{M}{4ab}\,\delta x.\,\delta y\,(x^2+y^2)$



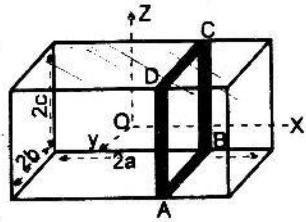
$\therefore$ M.I. of the rectangular lamina about ON is

$$= \frac{M}{4ab}\int_{x=-a}^{a}\int_{y=-b}^{b}(x^2+y^2)\,dx\,dy = \frac{M}{4ab}4\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dx\,dy$$

$$= \frac{M}{ab}\left[\frac{1}{3}b\,x^3 + \frac{1}{3}b^3x\right]_0^a = \frac{M}{3}(a^2+b^2).$$

(iii) Rectangular Parallelopiped :

Let O be the centre and $2a, 2b, 2c$ the lengths of the edges of the parallelopiped and further let OX, OY, OZ, be the axes of reference, parallel to the edges of lengths $2a$, $2b$ and $2c$ respectively. Divide the parallelopiped into thin rectangular slices perp. to OX, $ABCD$ being one such slice at a distance x. Let the width of the slice be δx.



$\therefore$ M.I. of the rectangular slice about OX

$$= \text{mass} \times \frac{b^2+c^2}{3} = 2b\,.2c.\,\rho.\,\delta x.\,\frac{b^2+c^2}{3} = 4\,b\,c\,\rho\,\frac{b^2+c^2}{3}\,\delta x$$

[mass of the slice $ABCD = 2b\;2c\;\delta x\,\rho$]

$\Rightarrow$M.I. of the parallelopiped about OX

$$= 4bc\,\rho\,\frac{b^2+c^2}{3}\int_{-a}^{a} dx = 8abc\,\rho\,\frac{b^2+c^2}{3}$$

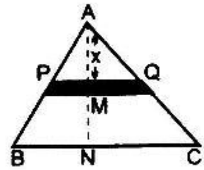
$$= M\,\frac{b^2+c^2}{3}$$ [$\because$ mass of the parallelopiped $= 2a\,2b.2c\rho = 8abc\,\rho$]

Similarly, M.I. of the parallelopiped about $OY = M\,\dfrac{c^2+a^2}{3}$ and M.I. of the parallelopiped about $OZ = M\,\dfrac{a^2+b^2}{3}$.

Note : M.I. of the cube of side $2a$ about any of its axis is $\frac{2}{3}Ma^2$.

(c) ***Moment of inertia of a uniform triangular lamina about one side.***

Let us divide the lamina ABC by strips parallel to BC. Let PQ be one of such strips of breadth δx at distance x from A and let p be the length of perpendicular AN.



Now $\dfrac{PQ}{a} = \dfrac{x}{p}$ *i.e.* $PQ = \dfrac{xa}{p}$. If M is the mass of the triangular lamina, then mass per unit area$= [M/(\frac{1}{2}a.p)] = \rho\,(say)$.

$\therefore$ Mass of the strip $= \dfrac{M}{\frac{1}{2}ap}\,PQ\,\delta x = \dfrac{2M}{p^2}\,x^2\,\delta x$

M.I. of the strip about $BC = \dfrac{2M}{p^2} x\,\delta x\,(p-x)^2$

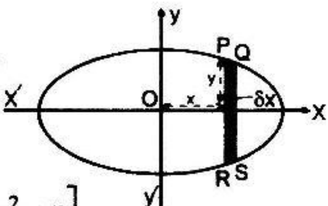
$\therefore$ M.I. of the triangle about $BC = \dfrac{2M}{p^2}\displaystyle\int_0^p (p-x)^2\,x\,dx = \frac{1}{6} M p^2$

(D) Elliptic disc : *Moment of inertia of an elliptic disc about its major axis.*

Let $PRSQ$ be an elementary strip of breadth δx at a distance x from O, where O is the centre of the disc. M.I. of the strip about

$OX = 2y.\,\delta x.\,\rho.\dfrac{y^2}{3}$, where ρ is the mass per unit area.

$\therefore$ M.I. of the elliptic lamina about OX



$$= \int_{-a}^{a} 2y\rho.\frac{y^2}{3}\,dx = \frac{2\rho}{3}\int_{-a}^{a} y^3\,dx$$

$$= \frac{2\rho}{3} b^3 \int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2} dx$$

$$\left[\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow y = b\left\{1-\frac{x^2}{a^2}\right\}^{1/2}\right]$$

Put $x = a\sin\phi$, so that $dx = a\cos\phi\,d\phi$

$\therefore$ M.I. of the elliptic lamina about OX

$$= \frac{2\rho b^3}{3}\int_{-\pi/2}^{\pi/2}\cos^3\phi\,.a\cos\phi\,d\phi = \frac{4\rho b^3 a}{3}\int_0^{\pi/2}\cos^4\phi\,d\phi$$

$$= \frac{4\rho b^3 a}{3}.\frac{3\pi}{16} = \frac{\pi a b^3\rho}{4}.$$

Again mass of the elliptic lamina

$$M = \rho\int_{-a}^{a} 2y\,dx = 2\rho b\int_{-a}^{a}\left\{1-\frac{x^2}{a^2}\right\}^{1/2} dx$$

$$= 2\rho b\int_{-\pi/2}^{\pi/2}\cos\phi\,.\,a\cos\phi\,d\phi = \pi a b\,\rho \Rightarrow \rho = \frac{M}{\pi a b}.$$

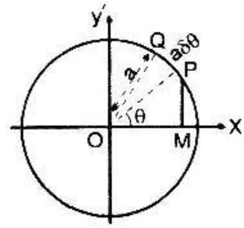
Hence from (1), M.I. of the elliptic lamina about OX *i.e.* about major axes $= \dfrac{\pi a b^3}{4}.\dfrac{M}{\pi a b} = \dfrac{1}{4} M b^2$

Similarly M.I. of the elliptic lamina about OY *i.e.* about minor axes $= \dfrac{1}{4} M a^2.$

(E) Hoop or Circumference of a circle.

(i) *Moment of inertia of a hoop about a diameter.*

Consider an element PQ of the hoop and let it subtend an angle $\delta\theta$ at its centre O i.e. $P\hat{O}Q = \delta\theta$ where $P\hat{O}X = \theta$.



By the figure it is obvious that arc $PQ = a\,\delta\theta$, where a is the radius of the hoop. Now M.I. of the element PQ about $OX = (a\,\delta\theta)\,\rho . a^2 \sin^2\theta$,

where ρ is the mass per unit length of the hoop.

M.I. of the hoop about $OX = \int_{\theta=0}^{2\pi} (a\,d\theta)\,\rho . a^2 \sin^2\theta$

$$= \frac{Ma^2}{4\pi} \int_0^{2\pi} (1 - \cos 2\theta)\, d\theta = \tfrac{1}{2} M a^2$$

(ii) *Moment of inertia of a hoop about a line through its centre and perp. to its plane.*

M.I. of the hoop about a line through O and perp. to its plane

$$= (a.\,\delta\theta)\,\rho .\, OP^2 = \frac{M}{2\pi} a^2\,\delta\theta \qquad (\because\ OP = a, M = 2\pi a \rho)$$

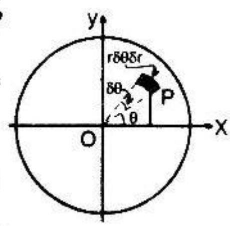
$\therefore$ M.I. of the hoop about a line through O and perp. to its plane

$$= \frac{M a^2}{2\pi} \int_0^{2\pi} d\,\theta = \frac{M a^2}{2\pi} [\theta]_0^{2\pi} = M a^2.$$

(F) Circular Disc.

(ii) *Moment of inertia of a circular disc of radius a about its diameter.*

Consider an element $r\,\delta\theta\,\delta r$ of the disc at P such that OP makes an angle θ with the axis OX. The perp. distance of P from OX is $r \sin\theta$.



Let ρ be the mass per unit area of the disc. M.I. of this element about $OX = r\,\delta\theta\,\delta r .\, \rho .(r \sin\theta)^2$.

$\therefore$ M.I. of this element about the diameter OX

$$= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r^3 \rho \sin^2\theta\, d\theta\, dr = \frac{M}{\pi a^2} \int_{r=0}^{a} \int_{\theta=0}^{2\pi} r^3 \sin^2\theta\, d\theta\, dr \quad (\because\ M = \pi a^2 \rho)$$

$$= \frac{M}{2\pi a^2} \int_0^a r^3 \left[\theta - \tfrac{1}{2} \sin 2\theta \right]_0^{2\pi} dr = \frac{M a^2}{4}.$$

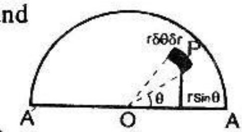
(ii) *Moment of inertia of a circular disc of radius a about a line through its centre perp. to its plane.*

M.I. of the element $r\delta\theta\,\delta r$ about a line through O and perp. to the plane of the disc.$= (r\delta\theta\,\delta r).\,\rho.OP^2 = \dfrac{M\,r^3}{\pi\,a^2}\,d\,\theta\,dr$ $(\because\ \pi a^2\rho = M)$

M.I. of the circular disc about a line through O and perp. to the plane of the disc $= \displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M\,r^3}{\pi\,a^2}\,d\,\theta\,dr = \frac{M\,a^2}{2}$

(G) Solid Sphere.

If a semi-circular area is revolved about its bounding diameter then the solid so generated is called sphere. Now consider an element of area $r\delta\theta\,\delta r$ at P such that $OP = r$ and makes an angle θ with the diameter. When this area is revolved about the diameter $A'A$, it will generate a ring of cross-section $r\delta\theta\,\delta r$ and radius $r\sin\theta$.



$\therefore$ Mass of this elementary ring

$= 2\pi r\sin\theta.\, r\delta\theta\,\delta r.\,\rho$

M.I. of the elementary ring about $A'A$

$= (2\pi r\sin\theta . r\delta\theta\,\delta r.\,\rho)\,(r\sin\theta)^2$ **[refer E, (ii)]**

$= 2\pi\,\rho r^4(\sin^3\theta)\,\delta\theta\,\delta r.$

M.I. of the solid sphere about the diameter $A'A$

$$= 2\pi\,\rho\int_0^{\pi}\int_0^{a} r^4\sin^3\theta\,d\theta\,dr = 4\pi\,\rho\int_0^{\pi/2}\int_0^{a} r^4\sin^3\theta\,d\theta\,dr$$

$$= 4\pi\,\rho\left[\frac{r^5}{5}\right]_0^a.\frac{2}{3} = 4\pi\,\rho.\frac{a^5}{5}.\frac{2}{3} = \frac{8\pi a^5\rho}{15} = I \text{ say}$$

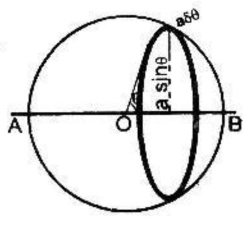
But mass of the sphere, $M = \dfrac{4}{3}\pi a^3\rho \Rightarrow \rho = \dfrac{3M}{4\pi\,a^3}$ ...(1)

$$\Rightarrow I = \frac{8}{15}\pi a^5.\frac{3M}{4\pi a^3} = \tfrac{2}{5}(M\,a^2)$$

(H) Hollow sphere.

If a semicircular arc is revolved about its diameter, then the surface so formed is known as hollow sphere.

Consider an elementary arc $a\,\delta\theta$. This arc $a\,\delta\theta$ will generate a circular ring of radius $a\sin\theta$ when revolved about the diameter AB. Now mass of the elementary ring $= 2\pi\,a\sin\theta.\,a\,\delta\theta.\,\rho$.



M.I. of the elementary ring about AB

$= (2\pi a\sin\theta.\,a\,\delta\theta.\,\rho).\,a^2\sin^2\theta$

$= 2\pi a^4 \rho \sin^3\theta\, \delta\theta.$ **[refer E, (iii)]**

$\Rightarrow$M.I. of the hollow sphere about the diameter AB

$$= 2\pi a^4 \rho \int_0^{\pi} \sin^3\theta\, d\theta = 2\pi a^4 . \frac{M}{4\pi a^2} \int_0^{\pi} \sin^3\theta\, d\theta \quad (\because\ M = 4\pi a^2 \rho)$$

$$= \frac{M a^2}{2} . 2\int_0^{\pi/2} \sin^3\theta\, d\theta = M a^2 \int_0^{\pi/2} \sin^3\theta\, d\theta = M a^2 . \frac{2}{3} = \frac{2}{3} M a^2.$$

(I) Ellipsoid. Consider an elementary volume $\delta x\, \delta y\, \delta z$ in the positive octant of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$. **[Kanpur 92]**

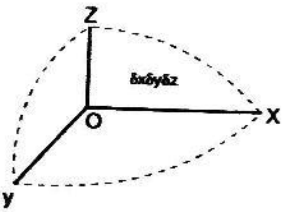
Let ρ be mass per unit volume then mass of the elementary volume

$= \rho.(\delta x.\ \delta y.\ \delta z).$

Distance of this element from $OX = \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of the ellipsoid about OX

$= 8 \int\int\int \rho\, dx\, dy\, dz\, (y^2 + z^2)$, the integration being taken over the positive octant of the ellipsoid and



$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1.$

Putting $(x^2/a^2) = u$, $(y^2/b^2) = v$ and $(z^2/c^2) = w$, we get,

$x = au^{1/2}, dx = \frac{1}{2} au^{-1/2}\, du;\ y = bv^{1/2}, dy = \frac{1}{2} bv^{-1/2}\, dv;$

$z = cw^{1/2}, dz = \frac{1}{2} cw^{-1/2}\, dw.$

Now, M.I. of the ellipsoid about OX

$$= 8 \int\int\int \frac{\rho}{8} abc\, (b^2 v + c^2 w) u^{-1/2} v^{-1/2} w^{-1/2} du\, dv\, dw$$

where $u + v + w \le 1$

$$= abc\,\rho \iiint (b^2 u^{-1/2} v^{-1/2} w^{-1/2} + c^2 u^{-1/2} v^{-1/2} w^{-1/2})\, du\, dv\, dw$$

$$= abc\rho \iiint (b^2 u^{1/2-1} v^{3/2-1} w^{1/2-1} + c^2 u^{1/2-1} v^{1/2-1} w^{3/2-1})\, du\, dv\, dw$$

$$= abc\,\rho \left[b^2 . \frac{\Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} + c^2 \frac{\Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} \right]$$

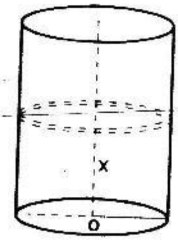
(using Dirichlet's theorem)

$$= abc\,\rho\, (b^2 + c^2) \frac{\pi}{(\frac{5}{2})(\frac{3}{2})} = \frac{4abc\,\rho\,\pi}{3} . \frac{b^2 + c^2}{5}$$

$$= M . \frac{b^2 + c^2}{5}$$ where $M = \frac{4}{3} \pi abc\, \rho$

(J) Right circular cylinder.

Let there be a right circular cylinder of radius a and height h. Consider a circular disc of thickness δx at a distance x from O the centre of base. Mass of the disc $= \pi a^2 \delta x . \rho$, where ρ is mass per unit volume.



$\therefore$M.I. of the disc about the axes perp. to the plane of the disc

$$= \pi a^2 \, \delta x \, \rho . \frac{1}{2} a^2 \qquad \text{[refer F (ii)]}$$

$$\Rightarrow \text{M.I. of the cylinder} = \frac{\pi a^4 \rho}{2} \int_0^h dx = \frac{\pi a^4 h \rho}{2}$$

$$= \frac{1}{2} M a^2 \qquad [\because \; M = \pi a^2 h \rho]$$

Reference table : The following table shows the moments of inertia of various rigid bodies considered above. In all cases it is assumed that the body has uniform density.

Rigid Body	Moments of inertia
(1) *Uniform rod of length 2a and mass M.*	
(i) About an axis perp. to the rod through the centre of mass.	$\frac{1}{3} M a^2$
(ii) About a line perp. to the rod through an end.	$\frac{4}{3} M a^2$
(2) *Rectangular plate of sides 2a, 2b and mass M*	
(i) About an axis perp. to the plate through the centre of mass.	$\frac{M}{3} (a^2 + b^2)$
(ii) About a line through centre parallel to the side 2*a*.	$\frac{1}{3} M b^2$
(3) *Rectangular parallelopiped of edges 2a, 2b, 2c.* About a line parallel to the edge 2a, through the centre	$\frac{M}{3} (b^2 + c^2)$
(4) *Circular plate of radius a and mass M.*	
(i) About its diameter.	$\frac{1}{4} M a^2$
(ii) About a line perp. to the plate through the centre.	$\frac{1}{2} M a^2$
(5) *Elliptic disc of axes 2a and 2b and mass M.*	
(i) About the axis 2a.	$\frac{1}{4} M b^2$

(ii) About a line perp. to the disc through its centre.	$\frac{M}{4}(a^2+b^2)$
(6) *Circular ring of radius a and mass M.*	
(i) About a diameter.	$\frac{1}{2}Ma^2$
(ii) About a line perp. to the plate and through the centre.	Ma^2
(7) *Solid sphere of radius a and mass M.* About a diameter.	$\frac{2}{5}Ma^2$
(8) *Hollow sphere of radius a and mass M.*	$\frac{2}{3}Ma^2$
About a diameter (thickness negligible)	
(9) *Ellipsoid of axes 2a, 2b and 2c.* about the axis 2a.	$\frac{M}{5}(b^2+c^2)$

Routh's Rule : For remembering the moment of inertia of symmetric rigid bodies. M.I. about an axis of symmetry

$$= \text{Mass} \times \frac{\text{Sum of the squares of perp. semi axes}}{3, 4 \text{ or } 5}$$

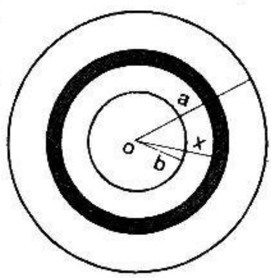
The denominator is 3, 4 or 5 according as the body is rectangular (including rod) elliptical (including circular) or ellipsoid (including sphere).

ILLUSTRATIVE EXAMPLES

Ex.1. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being a and b.*

Sol. Consider a hollow sphere, the external radius of which is a and the internal radius is b. Take a spherical shell in it of radius x and of width δx.

$\therefore$ Moment of inertia of this shell about diameter $= \frac{2}{3}$ (Mass of the shell) $\times$ (radius of the shell)^2 [refer H]



$$= \frac{2}{3} \cdot 4\pi x^2 \rho \delta x . x^2 = \frac{8\pi\rho}{3} x^4 \delta x.$$

Hence M.I. of the given hollow sphere

$$= \int_0^a \frac{8\pi\rho}{3} x^4\, dx = \frac{8\pi\rho}{3}\left[\frac{x^5}{5}\right]_0^a = \frac{8}{15}\pi\rho(a^5 - b^5)$$

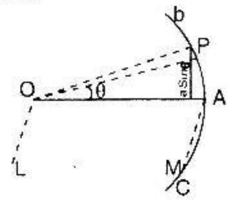
$$= \frac{8}{15}\pi . \frac{3M}{4\pi(a^3-b^3)} . (a^5-b^5) \quad \left[\because M = \frac{4}{3}\pi(a^3-b^3)\rho\right]$$

$$= \frac{2M}{5}\frac{a^5 - b^5}{a^3 - b^3}.$$

Ex.2. *Find the moment of inertia of the arc of a circle about (i) the diameter bisecting the arc,*
(ii) an axes through the centre perp. to its plane. **[Kanpur 91]**
(iii) an axes through its middle point perp. to its plane. **[Kanpur 91]**

Sol. Consider the arc BC, such that $\angle BOC = 2\alpha$, where O is the centre of the circular arc.

Let OA be the semi diameter bisecting the arc. Take an elementary arc $a\,\delta\theta$ at P.
Mass of this element $= \rho\, a\, \delta\theta$.
(i) Distance of the element from $OA = a \sin\theta$



$\therefore$ M.I. of the element about $OA = \rho a\, \delta\theta\, (a \sin\theta)^2 = \rho a^3 \sin^2\theta\, \delta\theta$
$\Rightarrow$M.I.of the whole arc about

$$OA = \rho a^3 \int_{-\alpha}^{\alpha} \sin^2\theta\, d\theta$$

$$= 2a^3\rho \int_0^{\alpha} \sin^2\theta\, d\theta = a^3\rho \int_0^{\alpha} (1 - \cos 2\theta)\, d\theta = a^3\rho(\alpha - \sin\alpha.\cos\alpha)$$

$$= a^3 . \frac{M}{2\alpha a}(\alpha - \sin\alpha\cos\alpha) = \frac{M a^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha) \quad [\because M = 2\alpha a\rho]$$

(*ii*) Let OL be a line through centre O and perp. to the plane of the arc. Distance of the element from $OL = a$

$\therefore$ M.I. of elementary arc about $OL = (\rho a\, \delta\theta) a^2 = \rho\, a^3\, \delta\theta$.
Now M.I. of the whole arc about OL is given by

$$I = \int_{-\alpha}^{\alpha} \rho a^3\, d\theta = 2a^3\alpha\rho = 2a\,\alpha\,\rho a^2 = M a^2 \qquad [\because M = 2\alpha\, a\, \rho]$$

(*iii*) Let AM be the line through the middle point A of the arc and perp. to the plane of the arc, then distance of the element from

$$AM = AP = \{(AN)^2 + (NP)^2\}^{1/2}$$
$$= \{(OA - ON)^2 + (NP)^2\}^{1/2} = \{(a - a\cos\theta)^2 + (a\sin\theta)^2\}^{1/2}$$
$$= \{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\}^{1/2} = 2a \sin\frac{\theta}{2}.$$

M.I. of the elementary arc about AM is given by

$$I_1 = \rho a\, \delta\theta (2a \sin\tfrac{1}{2}\theta)^2 = 4a^3\, \rho \sin^2\frac{\theta}{2}\, \delta\theta.$$

M.I. of the whole arc about AM

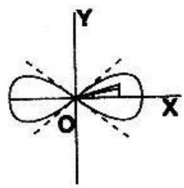
$$= \int_{-\alpha}^{\alpha} 4a^3\rho \sin^2\tfrac{1}{2}\theta\, d\theta = 4a^3\rho . 2\int_0^{\alpha} \sin^2\frac{\theta}{2}\, d\theta = 4a^3\rho \int_0^{\alpha} 2\sin^2\frac{\theta}{2}\, d\theta$$

$$= \frac{2Ma^2}{\alpha}(\alpha - \sin\alpha) \qquad \left[\because \rho = \frac{M}{2a\alpha}\right]$$

Ex.3. *Show that the moment of inertia of the area bounded by* $r^2 = a^2 \cos 2\theta$. **(i)** *about its axis is* $\frac{Ma^2}{16}(\pi - \frac{8}{3})$. **(ii)** *about a line through the origin, in its plane and perp. to its axis is* $\frac{Ma^2}{16}(\pi + \frac{8}{3})$. **(iii)** *about a line through the origin and perp. to its plane is* $\frac{Ma^2\pi}{8}$.

Sol. The curve consists of two loops and the variation of θ is from $-(\pi/4)$ to $(\pi/4)$. Let OX be its axis and OY a line in the plane of the curve perp. to OX. Consider an elementary area $r\delta\theta\,\delta r$, then its mass $= r\delta\theta\,\delta r\,\rho$.

$\therefore$ Mass of the whole area (both loops)



$$M = 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\,d\theta\,dr$$

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\,d\theta\,dr$$

$$= 4\rho\int_0^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{(\cos 2\theta)}} d\theta = 4\rho\int_0^{\pi/4}\frac{a^2\cos 2\theta}{2}\,d\theta$$

$$= 2\rho a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = 2\rho a^2\int_0^{\pi/2}\frac{1}{2}\cos\phi\,d\phi, \text{ Putting } 2\theta = \phi$$

$$= \rho a^2\int_0^{\pi/2}\cos\phi\,d\phi = \rho a^2, \quad \rho = \frac{M}{a^2} \qquad \ldots(1)$$

(i) Distance of the elementary area $r\delta\theta\,\delta r$ from $OX = r\sin\theta$

$\therefore$ M.I. of the elementary area about $OX = \rho\, r\delta\theta\,\delta r.\,(r\sin\theta)^2$

$$\Rightarrow \text{M.I. of whole area} = 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\,d\theta\,dr\,(r\sin\theta)^2$$

$$= 2\rho\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} r^3\sin^2\theta\,d\theta\,dr = 4\rho\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}}\sin^2\theta\,d\theta$$

$$= a^4\rho\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\,d\theta = \frac{a^4\rho}{2}\int_0^{\pi/4}(\cos^2\theta)(1-\cos 2\theta)\,d\theta$$

$$= \frac{a^4\rho}{4}\int_0^{\pi/2}\cos^2\phi(1-\cos\phi)\,d\phi, \text{ putting } 2\theta = \phi \text{ i.e. } d\theta = \tfrac{1}{2}d\phi$$

$$= \frac{1}{4}a^4\rho \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{M a^2}{16}[\pi - \tfrac{8}{3}] \qquad \text{[using (1)]}$$

(ii) Distance of the elementary area $r\delta\theta\,\delta r$ from $OX = r\cos\theta$

$\therefore$ Its M.I. about $OY = \rho.r\delta\theta\,\delta r(r\cos\theta)^2 = \rho r^3\cos^2\theta\,\delta\theta\,\delta r$

Hence M.I. of the whole area about $OY = 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\cos^2\theta\, d\theta\, dr$

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\cos^2 d\theta\, dr = 4\rho\int_0^{\pi/4}\left(\frac{r^4}{4}\right)_0^{a\sqrt{(\cos 2\theta)}} \cos^2\theta\, d\theta$$

$$= 4\rho\int_0^{\pi/4}\frac{a^4\cos^2 2\theta}{4}\cos^2\theta\, d\theta = \rho a^4\int_0^{\pi/4}\cos^2 2\theta.\tfrac{1}{2}(1+\cos 2\theta)\, d\theta,$$

putting $2\theta = \phi$ and $d\theta = \frac{1}{2}d\phi$

$$= \frac{\rho a^4}{4}\int_0^{\pi/2}\cos^2\phi\,(1+\cos\phi)\, d\phi = \int_0^{\pi/2}(\cos^2\phi + \cos^3\phi)\, d\phi$$

$$= \frac{\rho a^4}{4}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{2}{3}\right] = \frac{M a^2}{16}(\pi + \tfrac{8}{3}), \qquad \left[\because\ \rho = \frac{M}{a^2}\right]$$

(iii) Distance of the element $r\delta\theta\,\delta r$ from the line which passes through the origin and is perp. to the plane of the curve is r.

$\therefore$ M.I. of the element $= \rho r\delta\theta\,\delta r.r^2 = \rho r^3\,\delta\theta\,\delta r.$

Hence required M.I. about the given line

$$= 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\, d\theta\, dr = 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\, d\theta\, dr$$

$$= 4\rho\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta = \rho a^4\int_0^{\pi/4}\cos^2 2\theta\, d\theta$$

$$= \frac{\rho a^4}{2}\int_0^{\pi/2}\cos^2\phi\, d\phi, \text{ where } 2\theta = \phi \text{ etc.} = \frac{M a^2\pi}{8}.$$

Ex. 4. *From a uniform sphere of radius a, a spherical sector of vertical angle 2α is removed. Show that the moment of inertia of the remainder of mass M about the axis of symmetry is*

$\frac{1}{5}M a^2(1+\cos\alpha)(2-\cos\alpha).$

Sol. Let the spherical sector that has been removed be $OADB$. Let M be the mass of the sphere after removing the portion $OADB$.

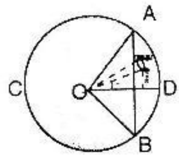
$\therefore M =$ mass of the sphere $-$ mass of the sector

$$= \frac{4}{3}\pi a^3\rho - \int_0^{a}\int_0^{\alpha}\rho(2\pi r\sin\theta)r\, d\theta\, dr$$

$$= \frac{4}{3}\pi a^3\rho - \frac{2}{3}\pi a^3\rho(1-\cos\alpha)$$

$$= \frac{2\pi a^3\rho}{3}(1+\cos\alpha)$$

Thus we have $\rho = \dfrac{3M}{(1+\cos\alpha)\,2\pi a^3}$ ...(1)



$\therefore$ Required M.I. of the portion $OACB$ about the axis of symmetry, COD = M.I. of the sphere $-$M.I. of the sector $(OADB)$

$$= \frac{2}{5}(\textit{mass of sphere}) \times (\textit{radius})^2 - \int_0^a\int_0^\alpha \rho(2\pi r\sin\theta)\times r^2\sin^2\theta\, d\theta\, dr$$

$$= \frac{2}{5}\left(\frac{4}{3}\pi a^3\rho\right)a^2 - 2\pi\rho\int_0^\alpha \left(\frac{r^5}{5}\right)_0^a \sin^3\theta\, d\theta$$

$$= \frac{8\pi a^5\rho}{15} - \frac{2\pi a^5\rho}{5}\int_0^\alpha\left(\frac{3\sin\theta - \sin 3\theta}{4}\right)d\theta$$

$$= \frac{8\pi a^5\rho}{15} - \frac{\pi a^5\rho}{10}\left[-3\cos\alpha + \frac{1}{3}\cos 3\alpha + 3 - \frac{1}{3}\right]$$

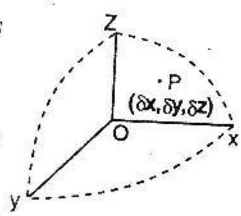
$$= \frac{2\pi a^5\rho}{15}\left[2 + 3\cos\alpha - \cos^3\alpha\right]$$

$$= \frac{2\pi a^5\rho}{15}\cdot\frac{3M}{2\pi a^3(1+\cos\alpha)}(1+\cos\alpha)^2(2-\cos\alpha) \qquad \text{[from (1)]}$$

$$= \frac{Ma^2}{5}(1+\cos\alpha)(2-\cos\alpha).$$

Ex.5. *Find the moment of inertia about the x-axis of the portion of the ellipsoid* $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ *which lies in the positive octant, supposing the law of volume density to be* $\rho = \mu xyz$.

Sol. Take an elementary volume $\delta x\,\delta y\,\delta z$ around the point $P(x, y, z)$, inside the octant $OXYZ$. Perpendicular distance of this element from *x-axis* $= \sqrt{(y^2+z^2)}$.



$\therefore$ M.I. of the element about *x-axis*

$= \rho(y^2+z^2)\,\delta x\,\delta y\,\delta z$

$= \mu\, xyz\,(y^2+z^2)\,\delta x\,\delta y\,\delta z.$

Hence M.I. of the positive octant of the ellipsoid $= \iiint \mu\, xyz\,(y^2+z^2)\,\delta x\,\delta y\,\delta z$

where $(x^2/a^2)+(y^2/b^2)+(z^2/c^2) \le 1.$

Substituting $(x^2/a^2) = u$, $(y^2/b^2) = v$, $(z^2/c^2) = w$
$\Rightarrow x\,dx = \frac{1}{2} a^2\,du$, $y\,dy = \frac{1}{2} b^2\,dv$, and $z\,dz = \frac{1}{2} c^2\,dw$, we get

$$\text{M.I.} = \mu \iiint \tfrac{1}{8} a^2b^2c^2(b^2v + c^2w)\,du\,dv\,dw, \qquad \text{where } u+v+w \le 1$$

$$= \tfrac{1}{8}\mu a^2b^2c^2 \iiint (b^2u^0v^1w^0 + c^2u^0v^0w^1)\,du\,dv\,dw$$

$$= \tfrac{1}{8}\mu a^2b^2c^2 \iiint (b^2u^{1-1}v^{2-1}w^{1-1} + c^2u^{1-1}v^{1-1}w^{2-1})\,du\,dv\,dw$$

$$= \tfrac{4}{8}\mu a^2b^2c^2\left[\frac{b^2\,\Gamma(1)\,\Gamma(2)\,\Gamma(1)}{\Gamma(1+2+1+1)} + \frac{c^2\,\Gamma(1)\,\Gamma(1)\,\Gamma(2)}{\Gamma(1+1+2+1)}\right]$$

(using Dirichlet's theorem)

$$= \tfrac{1}{8}\mu a^2b^2c^2\left[b^2\frac{\Gamma(2)}{\Gamma(5)} + c^2\frac{\Gamma(2)}{\Gamma(5)}\right] = \frac{1}{192}\mu a^2b^2c^2(b^2+c^2)$$

Now, M = mass of the positive octant of the ellipsoid
$OXYZ = \iiint \mu xyz\,dx\,dy\,dz$

under the condition $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1$.

$$= \tfrac{1}{8}\mu\, a^2b^2c^2 \iiint du\,dv\,dw$$

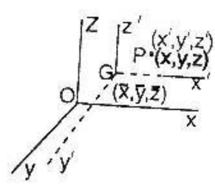
[under the condition $u+v+w \le 1$, and $u = (x^2/a^2)$ *etc.*]

$$= \tfrac{1}{8}\mu a^2b^2c^2\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)} = \tfrac{1}{48}\mu a^2b^2c^2; \qquad \therefore \mu = \frac{48M}{a^2b^2c^2}$$

Substituting this value of μ, we get the required moment of inertia $= \frac{1}{4} M\,(b^2 + c^2)$.

0·03. Theorem of Parallel Axes

The Moment Of Inertia And The Products Of inertia about axes through the centre of gravity are given, to find the moments and products of inertia about parallel axes.



Let OX, OY, OZ be a set of co-ordinate axes through any point O, parallel to a set of co-ordinate axes GX', GY', GZ' through G, the centre of gravity. Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of G with regard to co-ordinate axes OX, OY, OZ. Let the co-ordinates of any element of mass m situated at the point P with regard to axes OX, OY, OZ be (x, y, z) and with regard to parallel axes through G be (x', y', z')
$\therefore x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$

M.I. of the body about $OX = \Sigma m\,(y^2 + z^2)$

$= \Sigma m[(\bar{y}+y')^2+(\bar{z}+z')^2]$

$= \Sigma m[(\bar{y}^2+\bar{z}^2+2y'\bar{y}+2z'\bar{z}+y'^2+z'^2]$

$= \Sigma m(\bar{y}^2+\bar{z}^2)+\Sigma m(y'^2+z'^2)+2\bar{y}\,\Sigma my'+2\bar{z}\,\Sigma mz'$

Now referred to G as origin, the co-ordinates of the centre of the gravity of the body

$$\frac{\Sigma mx'}{\Sigma m}=0, \frac{\Sigma my'}{\Sigma m}=0, \frac{\Sigma mz'}{\Sigma m}=0.$$

$\therefore \Sigma mx'=0, \Sigma my'=0, \Sigma mz'=0.$

Hence M.I. of the body about $OX = \Sigma m\,(\bar{y}^2+\bar{z}^2)+\Sigma m(y'^2+z'^2)$

$= (\bar{y}^2+\bar{z}^2)\,\Sigma m +$ M.I. of the body about GX'

$= M\,(\bar{y}^2+\bar{z}^2)+$ M.I. about GX'

$=$ M.I. of mass M placed at G about $OX +$ M.I. about GX'.

Again product of inertia of the body about OX and OY.

$= \Sigma mxy = \Sigma m(x'+\bar{x})\,(y'+\bar{y})$

$= \Sigma mx'y'+\Sigma mx'\bar{y}+\Sigma m\bar{x}y'+\Sigma m\bar{x}\bar{y}+\Sigma mx'y'+\bar{y}\,\Sigma mx'$
$\qquad +\bar{x}\,\Sigma my'+\bar{x}\bar{y}\Sigma m$

$= \Sigma mx'y'+M\,\bar{x}\bar{y} =$ The product of inertia about $(GX'+GY')$

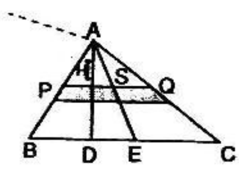
$+$ Product of inertia of mass M placed at G about the axes OX and OY.

ILLUSTRATIVE EXAMPLES

Ex.6. *Find the moment of inertia of the triangle ABC abot a perp. to its plane through A.*

Sol. Let AE be the median and AD the perp. through A on BC.

Let $AD = h$ and $AE = r$, then we have



$AE^2 = AD^2+DE^2 = AD^2+(BE-BD)^2$

$= (AD^2+BD^2)+BE^2-2BE.\,BD$

$= AB^2+(\frac{1}{2}BC)^2-2.\frac{1}{2}BC\,AB\cos B$

$= c^2+(a^2/4)-ac.\dfrac{a^2+c^2-b^2}{2ac}$

$= \dfrac{2b^2+2c^2-a^2}{4}, \therefore r^2 = [(2b^2+2c^2-a^2)/4]$ ...(1)

Take an elementary strip PQ of thickness δx at a distance x from A and parallel to BC. If S is the mid point of the strip PQ, then we get

$$\frac{x}{AD}=\frac{AS}{AE}\Rightarrow\frac{x}{h}=\frac{AS}{r}\Rightarrow AS=\frac{xr}{h}.$$

Also $\dfrac{x}{h}=\dfrac{PQ}{BC}\Rightarrow PQ=\dfrac{xa}{h}$

Let AL be the line through A perp. to the plane of the triangle, then M.I. of the strip about AL.

$= $ M.I. of the strip about a line through the mid points of the strip parallel to $AL +$ (mass of this strip) $\times AS^2$

$$= \frac{1}{3} \cdot \frac{ax}{h} \delta x \rho . \left(\frac{ax}{2h}\right)^2 + \frac{ax}{h} \delta x \rho \left(\frac{xr}{h}\right)^2 = \frac{1}{12} \frac{\rho x^3 a}{h^3} \delta x \left[a^2 + 12r^2\right]$$

$\therefore$ Required M.I. of the whole triangle about AL

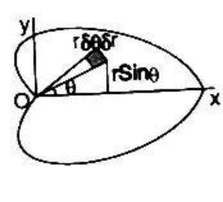
$$= \frac{1}{12}(a^2 + 12r^2) \frac{\rho a}{h^3} \int_0^h x^3 \, dx = \frac{1}{12}(a^2 + 12r^2) \frac{\rho a}{h^3} \cdot \frac{1}{4} h^4$$

$$= \frac{1}{48}\left[a^2 + 3(2b^2 + 2c^2 - a^2)\right] \rho a h \quad \text{[using (1)]}$$

$$= \frac{M}{12}(3b^2 + 3c^2 - a^2), \text{ since } M = \frac{1}{2} a h \rho.$$

Ex.7: *A solid body, of density* ρ, *is in the shape of the solid formed by the revolution of the cardiod* $r = a(1 + \cos\theta)$ *about the initial line; show that its moment of inertia about a straight line through the pole perp. to the initial line is* $\left(\frac{352}{105}\right) \pi \rho a^5$. **[Meerut 1989]**

Sol. Let OX be the initial line and OY a line through the pole perp. to the initial line. Consider an elementary area $r\delta\theta\,\delta r$. This area when revolved about OX generates a circular ring of radius $r \sin\theta$. Its mass $= 2\pi r \sin\theta \, r\delta\theta\,\delta r \rho$. M.I. of the ring about a diameter parallel to



$$OY = (2\pi r \sin\theta \, r\delta\theta\,\delta r \rho) \frac{r^2 \sin^2\theta}{2}$$

[$\therefore$ M.I. of the ring about a diameter $= \frac{M a^2}{2}$]

$\therefore$ M.I. of the ring about $OY =$ M.I. of the ring about a diameter parallel to $OY +$ mass of ring $\times (r\cos\theta)^2$

$$= (2\pi r \sin\theta \, r\delta\theta\,\delta r \rho)\left(\frac{r^2 \sin^2\theta}{2} + r^2\cos^2\theta\right)$$

Hence moment of inertia of the whole solid of revolution about

$$OY = 2\pi\rho \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^4 \sin\theta \left(\frac{\sin^2\theta}{2} + \cos^2\theta\right) d\theta \, dr$$

$$= \pi\rho \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^4 \sin\theta \, (1 + \cos^2\theta) \, d\theta \, dr$$

$$= \pi\rho \int_0^{\pi} \left(\frac{r^5}{5}\right)_0^{a(1+\cos\theta)} \sin\theta \, (1 + \cos^2\theta) \, d\theta$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi} (1+\cos\theta)^5 \sin\theta\,(1+\cos^2\theta)\,d\theta$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^5 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times\left[1+\left(2\cos^2\frac{\theta}{2}-1\right)^2\right]d\theta$$

Putting $2t=\theta$ so that $2\,dt = d\theta$, we get

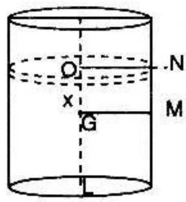
M.I. of the whole solid about OY

$$= \frac{256\,\pi\rho a^5}{5}\int_0^{\pi/2}[\cos^{11}t\sin t + 2\cos^{15}t\sin t - 2\cos^{13}t\sin t\,]\,dt$$

$$= \frac{256\,\pi\rho a^5}{5}\left[-\frac{\cos^{12}t}{12} - 2\frac{\cos^{16}t}{16} + 2\frac{\cos^{14}t}{14}\right]_0^{\pi/2} = (352\pi\rho a^5/105).$$

Ex.8. *Find the moment of inertia of a right circular cylinder about* **(i)** *its axis* **(ii)** *a straight line through its centre of gravity perp. to its axis.*

Sol. (i). Let a be the radius of the base, h the height of the cylinder. Take any elementary disc of breadth δx at a distance x from the centre of gravity G. M.I. of this disc about OL, where O is the centre of the disc and L the centre of the base of the cylinder $= (\rho\pi a^2\delta x)\,\dfrac{a^2}{2}$



$\therefore$ M.I. of the cylinder about the axis OGL

$$= \int_{-h/2}^{h/2} \rho\pi a^2\,dx\,\frac{a^2}{2} = \frac{\rho\pi a^4}{2}\Big[x\Big]_{-h/2}^{h/2}$$

$$= \frac{\rho\pi a^4}{2}h = \frac{M a^2}{2} \quad (\because \; M = \pi a^2 h\rho)$$

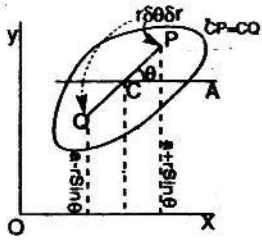
(ii) M.I. of the elementary disc about GM (line through CG and perp. to the axis) $=$ M.I. of the disc about $ON +$ M.I. of the mass of the disc placed at O, about $GM = (\rho\pi a^2\delta x)\dfrac{a^2}{4} + (\rho\pi a^2\delta x)x^2 = \rho\pi a^2\left[\dfrac{a^2}{4}+x^2\right]\delta x$

Hence M.I. of the whole cylinder about GM

$$= \pi a^2\rho\int_{-h/2}^{h/2}\left(\frac{a^2}{4}+x^2\right)dx = \frac{M}{4}\left(a^2+\frac{h^2}{3}\right).$$

0·04, Theorem : *A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the moment of inertia of the solid of revolution so formed about OX is equal to* $M\,(a^2+3K^2)$, *where M is the mass of the solid generated, a is the distance from OX of the centre C of the curve, and K is the radius of gyration of the curve about a line through C parallel to OX.*

Proof : Let CA be a line parallel to OX at a distance a from OX. If M is the mass of the solid of revolution formed about OX, then $M = 2\pi a\, \rho S$ where S is the area of the closed curve (using *Pappus theorem*). Take an element $r\delta\theta\,\delta r$ at a distance r from C, which makes an angle θ with CA. Now corresponding to the element $r\delta\theta\,\delta r$ there will be an equal element for the same value θ in the opposite direction at the point Q. The distance of these elements from OX are $a + r\sin\theta$ and $a - r\sin\theta$ respectively. Now $S = \iint 2r\,d\theta\,dr$ ...(i)



where integration is taken to cover the upper half of the area. But moment of inertia of the area S about $CA = S\rho . K^2$

$\therefore S\,\rho K^2 = \iint \rho . 2r\,d\theta\,dr . r^2\sin^2\theta$ ...(ii)

Further M.I. of the solid of revolution about OX

$$= \iint r\,d\theta\,dr\,\rho\{2\pi(a + r\sin\theta)\,(a + r\sin\theta)^2 + 2\pi(a - r\sin\theta)(a - r\sin\theta)^2\}$$

$$= 4\pi\rho \iint r(a^3 + 3ar^2\sin^2\theta)\,d\theta\,dr$$

$$= 2\pi\rho a^3 \iint 2r\,d\theta\,dr + 2\pi\rho a^3 \iint 2\,d\theta\,dr\,r^2\sin^2\theta$$

$$= M(a^2 + 3K^2) \qquad \text{[using (i) and (ii)]}$$

0·05. Theorem : *Prove a theorem similar to the one proved previously for the moment of inertia of the surface generated by the arc of the curve.*

Proof : Let l be the length of the whole curve, so that we have

$l = 2\int ds$ ...(1)

Also mass of the surface of revolution $= 2\pi a\rho l = M$ *(say)* ...(2)

Let K be the radius of gyration of the arc of the curve about CA, then it is evident that $lK^2 =$ M.I. of the arc about CA {Ref. figure of the Previous theorem

$= 2\int \rho r^2\sin^2\theta\,ds$ ...(3)

Hence M.I. of thesurface of revolution about OX

$$= \int \rho[2\pi(a + r\sin\theta)^3 + 2\pi(a - r\sin\theta)^3]\,ds$$

$$= 4\pi\rho \int (a^3 + 3ar^2\sin^2\theta)\,ds$$

$$= 2\pi\rho a^3 \int 2\,ds + 6\pi\rho a \int 2r^2\sin^2\theta\,ds = Ma^2 + 3MK^2$$

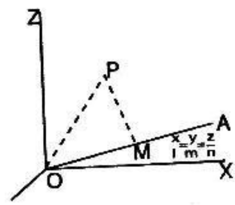
[using (1), (2) and (3)]

$$= M(a^2 + 3K^2).$$

0·06. Moment of Inertia about a line : *To find the moment of inertia about any axis through the meeting point of three perp. axes, the moments and products of inertia about these three axes being known.*

Proof : Let OX, OY, OZ be a set of three mutually perp. axes.

Let $A =$ M.I. about OX,
$B =$ M.I. about OY, $C =$ M.I. about OZ,
$D =$ Product of inertia w.r.t. axes of y and z. $E =$ Product of inertia w.r.t. axes of z and x and $F =$ Product of inertia w.r.t. axes of x and y. Now if m' is the mass of the element at P whose co-ordinates are (x, y, z), then we easily have



$$\left.\begin{array}{c} A = \Sigma m'(y^2+z^2), B = \Sigma m(x^2+z^2)\,;\, C = \Sigma m'(x^2+y^2) \\ D = \Sigma m'\,yz\,, E = \Sigma m'\,zx, F = \Sigma m'\,xy \end{array}\right\}$$

Let OA be a line with direction cosines (l, m, n). From P draw $PM \perp$ to OA, then $PM^2 = OP^2 - OM^2 = (x^2+y^2+z^2) - (lx+my+nz)$

$$[\because\ OP = (x^2+y^2+z^2), ON = (lx+my+nz)]$$

$$= x^2(1-l^2)+y^2(1-m^2)+z^2(1-n^2) - 2mnyz - 2lnzx - 2lmxy$$
$$= x^2(m^2+n^2)+y^2(l^2+n^2)+z^2(l^2+m^2) - 2mnyz - 2lnzx - 2lmxy$$
$$[\text{using } l^2+m^2+n^2 = 1]$$
$$= l^2(y^2+z^2)+m^2(x^2+z^2)+n^2(x^2+y^2) - 2mnyz - 2lnzx - 2lmxy$$

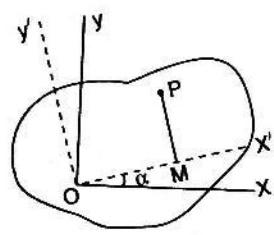
$\therefore$ Moment of inertia of the body abou OA,

$$= \Sigma m'\,PM^2 = l^2\,\Sigma m'(y^2+z^2)+m^2\,\Sigma m'(x^2+z^2)+n^2\,\Sigma m'(x^2+y^2)$$
$$- 2mn\,\Sigma m'yz - 2ln\,\Sigma m'zx - 2lm\,\Sigma m'xy$$
$$= Al^2+Bm^2+Cn^2-2Dmn-2Eln-2Flm \qquad [\text{using } (1)]$$

0·07. Theorem : *If the moments and products of inertia of a plane lamina about two perp. axes in the plane are known, to find the moment of inertia about any other axis through their point of intersection.*

Proof : Consider an elementary mass m at the point P whose co-ordinates with reference to the axes OX, OY be (x, y). If A and B are the moments of inertia and F the product of inertia of the plane lamina about these axes in the plane, then we have,

$$A = \Sigma my^2, B = \Sigma mx^2, F = \Sigma mxy \qquad ...(1)$$



Now let OX' be a line inclined at an angle α to OX about which the moment of inertia is required. Let OX', OY' be the new system of co-ordinates axes and let (x', y') be the co-ordinates of the point P with respect to the new axes. Then we will have the relation

$$x' = x\cos\alpha + y\sin\alpha\ ,\ \ y' = y\cos\alpha - x\sin\alpha$$

So M.I. of the lamina about $OX' = \Sigma my'^2 = \Sigma m(y\cos\alpha - x\sin\alpha)^2$

$$= \cos^2\alpha\,\Sigma my^2 - 2\sin\alpha\cos\alpha\,\Sigma mxy + \sin^2\alpha\,\Sigma mx^2$$

$= A\cos^2\alpha - 2F\sin\alpha\cos\alpha + B\sin^2\alpha = A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha$

Further product of inertia of the lamina about OX', OY'

$= \Sigma m x' y' = \Sigma m(x\cos\alpha + y\sin\alpha)(y\cos\alpha - x\sin\alpha)$

$= \cos\alpha\sin\alpha(\Sigma m y^2 - \Sigma m x^2) + (\cos^2\alpha - \sin^2\alpha)\,\Sigma mxy$

$= (A - B)\sin\alpha\cos\alpha + F\cos 2\alpha \equiv \frac{1}{2}(A - B)\sin 2\alpha + F\cos 2\alpha$

0·08. Some Elementary Theorems.

Theorem I. *If A, B, C are the moments and D, E, F; the product of inertia about the axes, then the sum of any two of them is greater than the third.*

Proof : $A + B - C = \Sigma m(y^2 + z^2) + \Sigma m(x^2 + z^2) - \Sigma m(x^2 + y^2)$

$= 2\Sigma m z^2 = +$ *ive* and hence $A + B > C$.

Theorem II. *The sum of moments of inertia about any three axes (rectangular) meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

Proof : $A + B + C = 2\Sigma m(x^2 + y^2 + z^2) = 2\Sigma m r^2$

$\Rightarrow A + B + C$ is independent of the direction of the axes.

Theorem III. *The sum of the moments of inertia of a body with reference to any plane through a given point and its normal at that point is constant and is equal to the moment of inertia of the body with reference to that point.*

Proof : Choose the given point as origin and plane as the plane of xy. Now let R' be the moment of inertia w.r.t. xy plane and R the moment of inertia about the normal at origin (z-axis) then we (on applying Theorem II) get $R' + R = \Sigma m r^2 = \frac{1}{2}(A + B + R)$

which is independent of the direction of axis

$\Rightarrow R' = \frac{1}{2}(A + B - R)$

Hence if P', Q', R' are the moments of inertia with reference to the planes of (y, z), (z, x) and (x, y) then we easily obtain

$$P' = \frac{1}{2}(B + R - A),\ Q' = \frac{1}{2}(A + R - B),\ R' = \frac{1}{2}(A + B - R)$$

Theorem IV. *To prove that* $A > 2D$, $B > 2E$ and $C > 2F$ *where the meanings have their usual significance.*

Proof : $(y^2 + z^2) > 2yz$ *etc.* ($\because$ $AM > G.M.$)

$\Rightarrow A > 2D$, similarly other results follow.

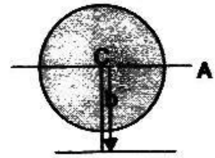
Ex.9. *The moment of inertia about its axis of a solid rubber tyre of mass M and circular cross section of radius a is* $(M/4)(4b^2 + 3a^2)$ *where b is the radius of the curve. If the tyre be hollow and of small uniform thickness, show that the corresponding result is* $(M/2)(2b^2 + 3a^2)$

Sol. The tyre solid/hollow is formed by the revolution of a circular area/circular ring about an axis.

Case I. For Solid Tyre . Moment of inertia of circular area about $CA = mass \times (a^2/4) = m K^2 \Rightarrow K^2 = (a^2/4)$.

By theorem 0-04 P.21, we have required

M.I.$= M(b^2 + 3a^2/4) = (M/4)(4b^2 + 3a^2)$



Case II. Hollow Tyre. We know that the moment of inertia of circular arc about $CA = mass.(a^2/2) = m K^2 \Rightarrow K^2 = (a^2/2)$

By theorem 0-05, P.22, we have M.I. of the hollow tyre

$$= M(b^2 + 3a^2/2) = (M/2)(2b^2 + 3a^2).$$

Ex.10. *Show that the moment of inertia of an elliptic area of mass M and semi axes a and b about a diameter of length r is* $\frac{1}{4} M \frac{a^2b^2}{r^2}$.

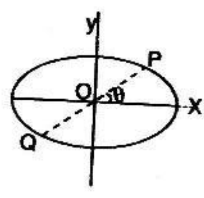
Sol. Consider an elliptic area with its centre at the origin (i) its moment of inertia about major axis $OX = (Mb^2)/4$ (ii) Its moment of inertia about $OY = (Ma^2)/4$ (iii) Its product of inertia about $OX, OY = 0$. Consider a diameter PQ making an angle θ with the axis OX, so that moment of inertia of the ellipse about the diameter

$$PQ = (Mb^2/4)\cos^2\theta + (Ma^2/4)\sin^2\theta + 0 = (M/4)(b^2\cos^2\theta + a^2\sin^2\theta) \quad ....(1)$$

If $OP = r$, then co-ordinates of P are $(r\cos\theta, r\sin\theta)$. Since P lies on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

$\therefore (r^2\cos^2\theta/a^2) + (r^2\sin^2\theta/b^2) = 1$

$\Rightarrow b^2\cos^2\theta + a^2\sin^2\theta = (a^2b^2/r^2)$



Hence moment of inertia of the elliptic area about $OP = (Ma^2b^2/4r^2)$

Ex.11 *Show that the moment of intertia of a right solid cone whose height is h and radius of whose base is a, is* $\frac{3Ma^2}{20}\left\{\frac{6h^2 + a^2}{h^2 + a^2}\right\}$ *about a slant side and* $(3M/80)(h^2 + 4a^2)$ *about a line through the centre of gravity of the cone perpendicular to its axis.* **[Agra 1988]**

Sol. Consider an elementary disc of thickness δx at a distance x from A, then we have radius of the disc $= x\tan\alpha$, mass of the disc

$$= \pi x^2 \tan^2\alpha\, \delta x\, \rho.$$

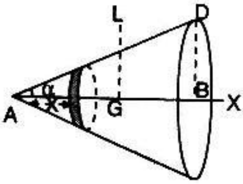
From the figure , $\tan\alpha = \frac{a}{h}$

Now M.I. of the disc about

$$ab = \pi x^2 \tan^2 \alpha \rho \frac{x^2 \tan^2 \alpha}{2} \delta x.$$

$$= \frac{1}{2} \rho \pi \tan^4 \alpha x^4 \delta x$$

Hence M.I. of the cone about



$$AB = \frac{\rho \pi \tan^4 \alpha}{2} \int_0^h x^4 \, dx$$

$$= \frac{\pi \tan^4 \alpha \rho h^5}{10} \quad \text{...(1)}$$

Now M.I. of the cone about a line through the vertex A and $\perp$ to AB.

$$= \int_0^h \rho \pi x^2 \tan^2 \alpha \, dx \left[\frac{x^2 \tan^2 \alpha}{4} + x^2 \right] = \rho \pi \tan^2 \alpha \int_0^h \left(\frac{\tan^2 \alpha}{4} + 1 \right) x^4 dx$$

$$= \rho \pi \tan^2 \alpha \left(\frac{\tan^2 \alpha}{4} + 1 \right) \int_0^h x^4 dx = \frac{\rho \pi \tan^2 \alpha h^5}{5} \left(\frac{\tan^2 \alpha}{4} + 1 \right) \quad \text{...(2)}$$

Now product of inertia of the cone about AB and AK.

= P.I. of the cone about GX and GL + P.I. of the mass of the cone (being concentrated at G) about AG and $AK = 0 + M \cdot 0\,(AG) = 0$.

Again M.I. of the cone about slant side AD

$$= \frac{\pi \tan^4 \alpha \rho h^5}{10} \cos^2 \alpha + \frac{\pi \tan^2 \alpha \rho h^5}{5} \left(\frac{\tan^2 \alpha}{4} + 1 \right) \sin^2 \alpha + 0. \text{ (using 0·06)}$$

$$= \frac{\pi \tan^4 \alpha \rho h^5}{20} \left[2\cos^2 \alpha + \sin^2 \alpha \right] + \frac{\pi \tan^2 \alpha \rho h^5}{5} \sin^2 \alpha$$

$$= \frac{\pi a^4 \rho h}{20} \left[1 + \frac{h^2}{a^2 + h^2} \right] + \frac{\pi a^2 \rho h^3}{5} \cdot \frac{a^2}{a^2 + h^2} \qquad \text{since } h \tan \alpha = a$$

$$= \frac{\pi a^4 \rho h}{20} \left[\frac{a^2 + 2h^2 + 4h^2}{a^2 + h^2} \right] = \frac{3Ma^2}{20} \left[\frac{6h^2 + a^2}{a^2 + h^2} \right], \text{ since } M = \frac{1}{3} \pi a^2 h \rho.$$

(ii) To find the moment of inertia about a line GL (through centre of gravity and perp. to the axis).

M.I. about the line through A and $\perp$ AB = M.I. about GL + M.I. of mass placed at G, about the line through A and $\perp$ AB.

$\therefore$ M.I. ab L = M.I. about the line through $A \perp AB$ − M.I. of mass M placed about the line through A and $\perp$ AB

$$= \frac{\pi a^2 \rho h^3}{5} \left[\frac{a^2}{4h^2} + 1 \right] - M \frac{9h^2}{16} = \frac{3M}{5} \left(\frac{a^2}{4} + h^2 \right) - M \frac{9h^2}{16}$$

$$= \frac{3M}{80} [4a^2 + 16h^2 - 15h^2] = \frac{3M}{80} [h^2 + 4a^2]$$

Ex.12. *Show that the moment of inertia of an ellipse of mass M and semi-axes a and b, about a tangent is* $(5M/4)p^2$, *where p is the perp. from the centre on the tangent.*

Sol. Let the equation of ellipse be $(x^2/a^2)+(y^2/b^2)=1$. Then equation of the tangent is $y = mx + \sqrt{[(a^2m^2+b^2)]}$ where m is the slope of the tangent. If the tangent is inclined at an angle θ to the x-axis then $= \tan\theta$.

$\therefore$ equation of the tangent is given by

$$y = x\tan\theta + \sqrt{\{a^2\tan^2\theta + b^2\}} \Rightarrow x\tan\theta - y + \sqrt{[a^2\tan^2\theta + b^2]} = 0$$

$p =$ distance of the perp. from the centre $(0, 0)$ to the tangent

$$= [\sqrt{\{(a^2\tan^2\theta + b^2)\}}/\sqrt{(1+\tan^2\theta)}] = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}$$

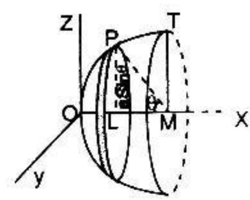
$\therefore$ Moment of inertia about a diameter parallel to the given tangent will be $= (M/4)b^2\cos^2\theta + (M/4)a^2\sin^2\theta + 0 = (M/4)(b^2\cos^2\theta + a^2\sin^2\theta)$

$$= (M/4)p^2.$$

Hence moment of inertia about the tangent $= (M/4)p^2 + Mp^2 = \frac{5}{4}Mp^2$.

Ex.13. *Show that for a thin hemi-spherical shell of mass M and radius a, the moment of inertia about any line through the vertex is* $\frac{2}{3}Ma^2$.

Sol. Let O be the vertex and OX the axis of symmetry of the semi-spherical shell. Take the two perp. lines OY and OZ each perp. to OX, so that OX, OY, OZ form the axes of reference. The quadrant of the circle when revolved about OX will generate the hemi-spherical shell. $\therefore$ M.I. about



OX *i.e.* $A = \int_0^{\pi/2} (\rho\, 2\pi a\sin\theta)\, a\, d\theta\, a^2\sin^2\theta$

$$= 2\pi\rho a^4 \int_0^{\pi/2} \sin^3\theta\, d\theta = 2\pi\rho a^4\, \frac{2}{3}$$

$$= \frac{4\pi\rho a^4}{3} = \frac{2}{3}Ma^2 \quad (\because\; M = 2\pi a^2\rho)$$

and $B =$ M.I. about $OY = \Sigma($ M.I. of circular disc about a line through L and parallel to $OY +$ mass of the circular disc $\times\, OL^2)$

$$= \int_0^{\pi/2} 2\pi a\rho\sin\theta\, a\, d\theta[a^2\sin^2\theta/2 + (a - a\cos\theta)^2]$$

$$= \pi\rho a^4 \int_0^{\pi/2} \sin\theta\,(3 - 4\cos\theta + \cos^2\theta)\, d\theta$$

$$= \pi\rho a^4\,[3.2 + \tfrac{1}{3}] = 4\pi a^4\rho/3 = (\tfrac{2}{3})\,Ma^2$$

Also M.I. about OZ *i.e* $C = B = (\frac{2}{3})\, M\, a^2$

Clearly the products of inertia D, E, F will vanish about these axes. Since the co-ordinates of $C.G.$ are $(a/2, 0, 0)$

Now if $[l, m, n]$ are Direction cosines of a line through O, then M.I. about this line $= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm$

$$\equiv (\tfrac{2}{3})\, M\, a^2\, (l^2 + m^2 + n^2) = (\tfrac{2}{3})\, M\, a^2$$

Ex.14. *If K_1, K_2 be the radii of gyration of a elliptic lamina about two conjugate diameters, then* $(1/K_1^2) + (1/K_2^2) = 4\,[(1/a^2) + (1/b^2)]$

Sol. Let OP and OQ be the conjugate semi diameters of the ellipse.

Let $OP = r_1, OQ = r_2$, then $M\,K_1^2 = \dfrac{M\,a^2 b^2}{4\,r_1^2}$ and $M\,K_2^2 = \dfrac{M\,a^2 b^2}{4\,r_2^2}$

$\therefore (1/K_1^2) + (1/K_2^2) = (4/a^2b^2)\,(r_1^2 + r_2^2)$

$= (4/a^2b^2)\,[(a^2\cos^2\theta + b^2\sin^2\theta) + \{a^2\cos^2(\pi/2 + \theta) + b^2\sin^2(\pi/2 + \theta)\}]$

$$= (4/a^2b^2)(a^2 + b^2) = 4[(1/a^2) + (1/b^2)].$$

Ex.15. *Show that the sum of moments of inertia of an elliptic area about any two tangents at right angles is always the same.*

Sol. Proceeding similarly as in Ex.12, we have

Moment of inertia about a tangent inclined at an

angle $\theta = (\frac{3}{4})M\,p^2 = (\frac{5}{4})\,M\,(a^2\sin^2\theta + b^2\cos^2\theta)$

Also moment of inertia about a tangent perp. to the first tangent [by changing θ to $\{(\pi/2) + \theta\}$]$= (\frac{5}{4})\; m(a^2\cos^2\theta + b^2\sin^2\theta)$

Hence sum of the moments of inertia about two perp. tangents

$= (\frac{5}{4})\,M\,(a^2 + b^2)$, this being independent of θ, remains constant.

Ex.16. *Show that the moment of inertia of an elliptic area of mass M and equation $ax^2 + 2hxy + by^2 + 2gx + 2yf + c = 0$ about a diameter parallel to the axis of x is $-\ \{aM\Delta/4(ab - h^2)^2\}$, where*

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Sol. Equation of the ellipse is given to be

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

Shift the origin to the centre of the ellipse then the equation of the ellipse takes the form $ax^2 + 2hxy + by^2 + \{\Delta/(ab - h^2)\} = 0$ ...(1)

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Putting $y = 0$ in (1), we get $ax^2 = -\ \{\Delta/(ab - h^2)\}$

Now if the length of the semi diameter parallel to the axis of x is r then , we have $r^2 = \{(\Delta/(ab - h^2)\}$

Now Putting $\{\Delta/(ab-h^2)\} = c'$, the equation of the ellipse becomes as $ax^2+2hxy+c'=0$ or $-(ax^2/c')-(2hxy/c')-(by^2/c')=1$,
which is of the standard form $Ax^2+2Hxy+By^2=1$. If α, β are the semi axes of ellipse , then α, β are the values of R in the equation $\{A-(1/R^2)\}/\{B-(1/R^2)\}=H^2$
or $(-a/c'-1/R^2)(-b/c'-1/R^2)=(-h/c')^2$
$\therefore (1/R^4)+(1/R^2)[(a/c')+(b/c')]+[(ab-h^2)/c^2]=0$

$$\therefore \frac{1}{\alpha^2\beta^2}=\frac{ab-h^2}{c'^2}=\frac{(ab-h^2)^3}{\Delta^2}$$

M.I. about the diameter is $= (M/4)(\alpha^2\beta^2/r^2)$
$=(-M/4)\{\Delta^2/(ab-h^2)^3\}.\{a(ab-h^2)/\Delta\}=-\{aM\Delta/4(ab-h^2)^2\}$

0·09. Method of Differentiation : We know that if y is a function of x and if δx and δy are small changes in the value of x and y , then

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}\Rightarrow \delta y=\frac{dy}{dx}\delta x$$

For example (i) if area of the circle is $A=\pi r^2$, then

$$\delta A=\frac{d}{dr}(\pi r^2)\,\delta r=2\pi r\,\delta r= \text{ circumference of the circle} \times \delta r$$

(ii) If V is the volume of sphere of radius r, then

$$V=(\tfrac{4}{3})\pi r^3 \Rightarrow \delta V=\frac{d}{dr}(\tfrac{4}{3}\pi r^3).\delta r=4\pi r^2\delta r$$

= surface of a sphereical shell of radius r and thickness δr.

Ex. 17. *Show that the moment of inertia of a thin homogeneous ellipsoidal shell (bounded by similar and similarly situated concentric ellipsoids) about an axis is $M\{(b^2+c^2)/3\}$, where M is the mass of the shell.*

[Agra 1986]

Sol. Let a, b, c be the length of the semi axes and ρ the density of the uniform solid ellipsoid , then moment of inertia of the solid ellipsoid about x- *axis* $= \{(\frac{4}{3})\pi abc\,\rho\}.\{(b^2+c^3)/5\}$

Now let the ellipsoid increase indefinitely small in size .

$$\therefore \text{ M.I. of the enclosed ellipsoidal shell} = d\left\{\tfrac{4}{3}\pi abc\,\rho.\frac{b^2+c^2}{5}\right\} \quad \text{... (1)}$$

Since the concentric bounded ellipsoids are similar ellipsoid then we have $(a/a')=(b/b')=(c/c')$
$\therefore\ b=(b'/a')\,a$ and $c=(c'/a')\,a \Rightarrow b=pa$ and $c=qa$
$\therefore$ M.I. of the shell $d\,[(\frac{4}{3})\pi\rho\, p q\ (p^2+q^2)\,a^2/5]$ [Using (1)]

$$=(\tfrac{4}{3})\pi\rho\, p\, q\,(p^2+q^2)\,a^4 da.$$

Now mass of the solid ellipsoid $= (\frac{4}{3})\,\pi abc\,\rho = (\frac{4}{3})\,\pi\,\rho pq\,a^3$.

$\therefore$ Mass of the ellipsoidal shell $M = d\,[\frac{4}{3}\pi\rho pqa^3] = 4\pi\,\rho pqa^2 da$.

Hence M.I. of the ellipsoidal shall $= \frac{4}{3}\pi\,\rho pq\,(p^2+q^2)a^4 \cdot \dfrac{M}{4\pi\,\rho pqa^2}$

$$= \frac{1}{3}M\,(p^2+q^2)a^2 = \frac{1}{3}M(p^2a^2+q^2a^2) = \frac{1}{3}M(b^2+c^2).$$

0· 10. Moment of Inertia of Heterogeneous Bodies :

In the case of a heterogeneous body whose boundary is a surface of uniform density, the method of differentiation can be successfully used in finding the moment of inertia of the body, the method is as follows:

(i) Suppose the M.I. of a homogeneous solid body of density ρ *is known*

(ii) Let this M.I. be expressed as a function of single parameter α *(say) i.e.*

$$\text{M.I.} = \rho\phi\,(\alpha).$$

Then the M.I. of a shell which is considered to be made of a layer of a uniform density $\rho = \rho\phi'(\alpha)\,d\alpha$. ...(1)

In case the density is not uniform and the variable density is given to be σ then we have, $M.I. = \int \sigma\phi'(\alpha)\,d\alpha$ ...(2)

Ex. 18. *The moment of inertia of a heterogeneous ellipse about minor axis, the strata of uniform density being confocal ellipse and density along minor axis varying as the distance from the centre is* $\dfrac{3M}{20} \cdot \dfrac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}$

[Agra 1981]

Sol. Equation of the confocal ellipse may be written as $\dfrac{x^2}{b^2+c^2} + \dfrac{y^2}{b^2} = 1$

with the condition $a^2 = b^2+c^2$. Now mass of ellipse of uniform density $\rho = \pi\,ab\,\rho = \pi b\sqrt{(b^2+c^2)}\,\rho =$ function of (b).

Mass of stratum of density $\rho = \dfrac{d}{db}\{\pi b\sqrt{[(b^2+c^2)]}\,\rho\}\,db$

$$= \pi\rho\,\sqrt{[(b^2+c^2)} + b^2/\sqrt{(b^2+c^2)}]\,db$$

$$= \pi\rho\,\{(b^2+c^2+b^2)/\sqrt{(b^2+c^2)}\}\,db = \pi\rho\,\{2b^2+c^2/\sqrt{(b^2+c^2)}\}.db$$

Now the density $= \lambda b$ (given)

$\therefore$ Mass of the heterogeneous ellipse, i.e.

$$M = \int_0^b \pi\lambda b\,\frac{2b^2+c^2}{\sqrt{(b^2+c^2)}}\,db = \pi\lambda\int_0^b \left[2\sqrt{(b^2+c^2)} - \frac{c^2}{\sqrt{(b^2+c^2)}}\right] db$$

$$= \pi\lambda\left[\int_0^b (b^2+c^2)^{1/2}2b\,db - c^2\int_0^b \frac{b\,db}{\sqrt{(b^2+c^2)}}\right]$$

$$= \pi\lambda\left\{\frac{2}{3}(b^2+c^2)^{3/2} - c^2(b^2+c^2)^{1/2}\right\}_0^b$$

$$= \pi\lambda[\tfrac{2}{3}\pi\,\lambda(a^3-c^3) - c^2(a-c)] \qquad \text{Using } a^2 = b^2+c^2$$

$$\therefore M = \tfrac{1}{3}\pi\,\lambda(2a^3+c^3-3ac^2) \qquad ...(1)$$

Now M.I. of the elliptic disc of uniform density ρ about its minor axis

$$= \pi ab\,\rho\,\frac{a^2}{4} = \frac{\pi a^3}{4}\,b\,\rho = \rho\pi\,\frac{b}{4}(b^2+c^2)^{3/2} = \frac{1}{4}\pi\,\rho b(b^2+c^2)^{3/2}$$

which is a function of b as c is constant.

M.I. of an elliptic stratum of density ρ is

$$\frac{d}{db}[\tfrac{1}{4}\pi\rho b(b^2+c^2)^{3/2}]\,db = \frac{1}{4}\pi\rho[\sqrt{(b^2+c^2)}(4b^2+c^2)\,db]$$

$$= \frac{1}{4}\pi\lambda\,[b\sqrt{(b^2+c^2)}\,(4b^2+c^2)\,db] \quad \text{putting } \rho = \lambda b.$$

Now moment of inertia of heterogeneous elliptic disc about minor axis

$$= \frac{1}{4}\pi\lambda\int_0^b b\sqrt{(b^2+c^2)}\,(4b^2+c^2)\,db$$

$$= \frac{1}{4}\pi\lambda\int_0^b b\sqrt{\{(b^2+c^2)}.[(4b^2+c^2)-3c^2]\}\,db$$

$$= \frac{1}{4}\pi\lambda\int_0^b [4b\,(b^2+c^2)^{3/2} - 3bc^2\,(b^2+c^2)^{1/2}]\,db$$

$$= \frac{1}{4}\pi\lambda\left[\frac{4}{5}(b^2+c^2)^{5/2} - c^2(b^2+c^2)^{3/2}\right]_0^b$$

$$= \frac{1}{4}\pi\lambda[\tfrac{4}{5}(b^2+c^2)^{5/2} - c^2(b^2+c^2)^{3/2} - (\tfrac{4}{5}c^5 - c^2c^3)]$$

$$= \frac{1}{20}\pi\lambda[4a^5+c^5-5c^2a^3] = \frac{3}{20}M\,\frac{4a^5+c^5-5c^2a^3}{2a^3+c^3-3ac^2}$$

$$\text{Since } M = \frac{1}{3}\pi\lambda(2a^3+c^3-3ac^2)$$

Ex.19. *Show that the M.I. of a heterogeneous ellipsoid about the major axis is $\frac{2}{9}M(b^2+c^2)$, the starta of uniform density being similarly concentric ellipsoids and density along the major axis varying as the distance from the centre.*

Sol. Since the bounding surfaces are similar ellipsoids, so

$(a'/a) = (b'/b) = (c'/c) \Rightarrow (b'/a') = (b/a) = p$ and $(c'/a') = (c/a) = q$

$\therefore b = ap$ and $c = aq$.

Mass of the ellipsoid $= (4/3)\,\pi\rho abc = (4/3)\,\pi\rho pq\,a^3$.

Also M.I. of the homogeneous ellipsoid about x- axis = (mass of the ellipsoid) $\times \dfrac{b^2+c^2}{5} = \dfrac{4}{3}\pi\rho pqa^3 \dfrac{p^2+q^2}{5} a^2 = \dfrac{4}{15}\pi\rho pqa^5 (p^2+q^2)$,

which is a function of single parameter a .

Now the variable density , $\rho = \lambda a$ (given)

$\therefore$ Moment of inertia of the ellipsoidal shell

$$= \frac{d}{da}\left[\frac{4}{15}\pi\rho pq\,(p^2+q^2)\,a^5\right] da = \frac{4}{3}\pi\rho pq\,(p^2+q^2)a^4 da.$$

$\therefore$ M.I. of ellipsoid $= \displaystyle\int_0^a \frac{4}{3}\lambda\pi pqa^5 (p^2+q^2)da.$ (putting $\rho = \lambda a$).

$$= \frac{4}{3}\lambda\pi pq\,(p^2+q^2)\frac{a^6}{6} = \frac{2}{9}\lambda pq\,(p^2+q^2)a^6 \qquad ...(1)$$

Again mass of the shell $\dfrac{d}{da}\left[\frac{4}{3}\pi pq\,\rho a^3 da\right] = 4\pi pq\,\lambda a^3 da$

$\Rightarrow M =$ Mass of given ellipsoid $= 4\pi\lambda pq \displaystyle\int_0^a a^3 da = \pi\lambda pqa^4.$

Required M.I. $= \frac{2}{9}M\,(p^2+q^2)a^2 = \frac{2}{9}M\,(b^2+c^2)$.

0·11. Momental Ellipsoid : *If A, B, C, D, E, F, are the moments and products of inertia about the axes then the moment of inertia about as line OQ whose direction cosines are* $[l, m, n]$ *is given by*

$A l^2 + Bm^2 + Cn^2 - Dmn - 2Enl - 2Flm.$

Let a length $OP = (r)$ along the line OQ be such that the moment of inertia of the body about OQ may be inversely proportional to OP^2, so that $(Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Fml) \propto \dfrac{1}{OP^2} = \dfrac{Constant}{OP^2}$

$\Rightarrow Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm = (MK^4/r^2)$

$\Rightarrow Al^2r^2 + Bm^2r^2 + Cn^2r^2 - 2Dmnr^2 - 2Enlr^2 - 2Flmr^2 = MK^4$

where M is mass of the body and K is suitable constant .

$$\Rightarrow Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = MK^4 \qquad ...(1)$$

Since A, B, C are essentially positive, so this equation represents an ellipsoid . This is called the momental ellipsoidal of the body at the point O, We know from Solid Geometry with suitable change of axis, the equation of the ellipsoid is transformed into the form

$$A_1x^2 + B_1y^2 + C_1z^2 = MK^4 \qquad ...(2)$$

Obviously the product of inertia with respect to the new axes vanish . These three new axes will be called the principal axes of the body at the point O.

Momental Sphere. *This forms a particular case of the above result. If the three principal moments of inertia at the point O are equal to each other then the ellipsoid becomes a sphere. In this case every diameter is a principal diameter and all radii vectors are equal.*

0·12. Momental Ellipse : *We know that in the case of a plane lamina, where A, B are moments of inertia about the axes and F the product of inertia about them, the moment of inertia if lamina about a line OQ which makes an angle θ with OX is given by,*

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta. \qquad \textbf{[Meerut 1981. Agra 1981]}$$

Take a length OP along OQ such that this moment of inertia is inversely proportional to the square of OP. If $OP = r$, then we have

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta = (Mk^4/r^2)$$

$$\Rightarrow A r^2\cos^2\theta - 2F\sin\theta\cos\theta\, r^2 + B r^2\sin^2\theta = Mk^4$$

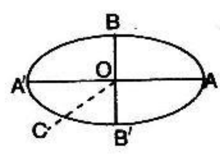
$$\Rightarrow Ax^2 - 2Fxy + By^2 = Mk^4$$

This equation represents an ellipse, because A and B are always + ive being sum of a number of squares. This ellipse is called the momental ellipse at the point O.

Note. The momental ellipse is the section of the momental ellipsoid at O by the plane of the lamina.

Ex. 20. *Show that the momental ellipsoid at the centre of an elliptic plate is* $(x^2/a^2) + (y^2/b^2) + z^2[(1/a^2) + (1/b^2)] =$ constant.

Sol. Let $ABA'B'$ be the elliptic plate. Take the major axis OA, minor axis OB and a perpendicular line OC as the axes of x, y and z respectively. Then we will have A = moment of ineretia about $OA = \frac{1}{4}Mb^2$



B = Moment of inertia about $OB = \frac{1}{4}Ma^2$

C = moment of inertia about $OC = \frac{1}{4}M(a^2 + b^2)$

Clearly $D = E = F = 0$ where D, E and F are products of inertia about the axes y and z, z and x, x and y respectively.

Hence the equation of the momental ellipsoid at O is

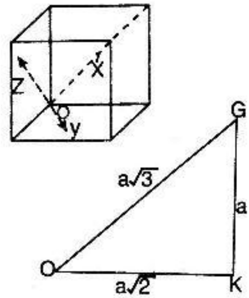
$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$$\Rightarrow (1/4)\,Mb^2x^2 + (1/4)\,Ma^2y^2 + (1/4)\,M(a^2 + b^2)\,z^2 = \text{constant}$$

$$\Rightarrow (x^2/a^2) + (y^2/b^2) + z^2[(1/a^2) + (1/b^2)] = \text{constant}.$$

Ex. 21. *Show that the equation of the momental ellipsoid at the corner of a cube of side 2a referred to its principal axes is given by* $2x^2 + 11(y^2 + z^2) = c$, *where c is constant.*

Sol. Let O be the corner of the cube and G its centre of gravity . Take the line OX passing through G, the centre of gravity of the cube as x- axis and the two mutually perpendicular lines OY and OZ as the axes of y and z. We know that moment of inertia of the cube of the side $2a$ about any axis through $G = (2/3)\, Ma^2$.



Now the products of inertia of the cube about any two mutually perpendicular lines through G are zero. So the products of inertia about the axes OX, OY, OZ taken in pairs will be zero. It implies OX, OY, OZ are the principal axes of the momental ellipsoid at O. $\therefore A = M.I.$ about $OX = \frac{2}{3} Ma^2$

$B =$ M.I. about $OY =$ M.I. about a line through $G \parallel$ to $OY +$ M.I. of mass M placed at G about $OY = (2/3)M\, a^2 + M(3a^2) = (11/3)\, Ma^2$,

since $OG = a\sqrt{3}$

By symmetry, $C =$ M.I. about $OZ = (11/3)\, Ma^2$

But $D = E = F = 0$. Hence equation of the momental ellipsoid is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$\Rightarrow (2/3)Ma^2x^2 + (11/3)\, Ma^2y^2 + (11/3)\, Ma^2z^2 = c'(\text{constant})$

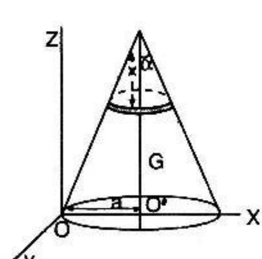
$\Rightarrow 2x^2 + 11(y^2 + z^2) = c\,(\text{constant})$.

Ex.22 *Prove that the equation of the momental ellipsoid at a point on the circular edge of a solid cone is*

$(3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)\, y^2 + 26a^2z^2 - 10ahxz =$ *constant. where* h *is the height of the cone and* a *is the radius of the base.*

[Vikram 63,Nagpur 64,Agra 66]

Sol. Consider a cone with vertex at V, semi vertical angle α and height h. Let O be the point on the circular edge of the cone where we want to determine the momental ellipsoid. Now take a disc of breadth δx at a depth x from the vertex V of the cone.



$\therefore A =$ Moment of inertia of the cone about $OX = \Sigma$ mass of the circular disc $[(\text{radius})^2/4) + (O'K)^2]$

$$= \int_0^h \pi x^2 \tan^2\alpha\, \rho\, [(x^4 \tan^2\alpha)/4 + (h - x)^2]\, dx$$

$$= \pi \tan^2\alpha\, \rho \int_0^h [(x^4 \tan^2\alpha)/4 + h^2x^2 - 2hx^3 + x^4]\, dx$$

$$= \pi \tan^2\alpha\, \rho \Big[(x^5 \tan^2\alpha/20) + (h^2x^3/3) - (hx^4/2) + (x^5/5) \Big]_0^h$$

$$= \pi \tan^2\alpha\, \rho\ [(h^5 \tan^2\alpha/20) + (h^5/3) - (h^5/2) + (h^5/5)]$$

$$= (\pi a^2/h^2)\, \rho h^5\, [(a^2/20h^2) + (1/30)]$$

$$= 3\,Mh^2 \Big[a^2/20h^2 + (1/30) \Big] \qquad [\because\ M = (1/3)\,\pi a^2 h\rho]$$

$$= (M/20)\,[3a^2 + 2h^2]$$

$B =$ M.I. about $OY =$ M.I. about a line parallel to axis of y through $O' + M\,a^2 = (M/20)\,(3a^2 + 2h^2) + M\,a^2 = (M/20)(23a^2 + 2h^2)$

$C =$ M.I. about a line parallel to OZ through O'

$M\,a^2 = (3/10) M\,a^2 + M\,a^2 = (13/10)\,M\,a^2.$

Since the co-ordinates of the centre of gravity G are $(a, 0, h/4)$

$\therefore D = F = 0$ and $E = M\,a(1/4)h = (\frac{1}{4})\ Mah$

Hence the equation of the momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Exz = \text{constant}$$

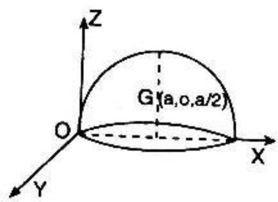
$$\Rightarrow (M/20)(3a^2 + 2h^2)x^2 + (M/20)(23a^2 + 2h^2)y^2 + (13/10)M\,a^2z^2 - 2(1/4)\,Mahxz = \text{constant.}$$

$$= (3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)y^2 + 26a^2z^2 - 10ahxz = \text{constant}$$

Ex.23. *Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemi-spherical shell is*

$$2x^2 + 5\,(y^2 + z^2) - 3zx = \text{constant.}$$

Sol. Let O be the point on the circular edge of the hemispherical shell. Take OX the diameter as *x-axis*, the line OY in the plane of the base perp. OX as *y-axis*, the line $OZ \perp$ to the plane of the base as *z-axis*.



$A =$ M.I. of the hemispherical shell about $OX = (\frac{2}{3})\,M\,a^2$

$B =$ M.I. about $OY =$ M.I. about a line parallel to OY and passing through C, the centre of the circular base $+\ M\,a^2 = (\frac{2}{3})M\,a^2 + M\,a^2 = (\frac{5}{3})\,Ma^2$

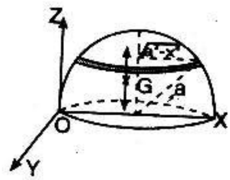
Similarly $C = (\frac{5}{3})\,M\,a^2$. Since co-ordinates of centre of gravity G are $(a, 0, a/2)$; $D = F = 0, E = M\,a(a/2) = (1/2)\,M\,a^2.$

Hence equation of momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{ constant}$$
$$\Rightarrow (\tfrac{2}{3})\, M\, a^2x^2 + (\tfrac{5}{3})\, M\, a^2y^2 + (\tfrac{5}{3})\, M\, a^2z^2 - 2(\tfrac{1}{2})\, M\, a^2zx = \text{ constant}$$
$$\Rightarrow 2x^2 + 5(y^2 + z^2) - 3zx = \text{ constant.}$$

Ex.24. *Show that the momental ellipsoid at a point on the rim of a hemisphere is* $2x^2 + 7(y^2 + z^2) - (\frac{15}{4})xz =$ constant. **[Nagpur 1990]**

Sol. Let O be the point on the rim of hemisphere and the **diameter** OX through O, the axis of x. Take OY a line in the plane of the base and perp. to OX as $y-$ *axis* and a line OZ perp. to the plane of base as $z-$ *axis*.
If G is centre of gravity of the hemisphere then the co-ordinates of G are $[a, 0, 3a/8]$. Let A, B, C be the moments and D, E, F the products of inertia of the hemisphere about these axis. Now take an elementary disc of width δx at a distance x from OX.



The radius of the disc $= \sqrt{(a^2 - x^2)}$

M.I. of the disc about $OX = \pi(a^2 - x^2)\, \rho\, dx\, [(\frac{1}{4})(a^2 - x^2) + x^2]$

$$\therefore A = \frac{1}{4}\pi\rho \int_0^a (a^2 - x^2)(a^2 + 3x^2)\, dx$$
$$= \frac{1}{4}\pi\rho \int_0^a (a^4 + 2a^2x^2 - 3x^4)\, dx$$
$$= \frac{1}{4}\pi\,\rho\, [a^5 + \frac{2}{3}a^5 - \frac{3}{5}a^5] = \frac{4\pi\,\rho\, a^5}{15} = \frac{2}{5}M\, a^2 \quad (\because\ M = \frac{2}{3}\pi a^3\, \rho)$$

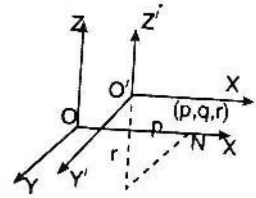
Similarly $B = C = (\frac{2}{5})\, M\, a^2 + M\, a^2 = (\frac{7}{5})\, M\, a^2$

Also $D = F = 0$ and $E = M\, [a.\, (\frac{3}{8}a)] = (3M\, a^2/8)$.

Hence the equation of the required momental ellipsoid at O is
$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{ constant}$$
$$\Rightarrow M\, a^2[(\tfrac{2}{5})x^2 + (\tfrac{7}{5})y^2 + (\tfrac{7}{5})z^2 - (\tfrac{3}{4})xz] = \text{ constant.}$$
$$\Rightarrow 2x^2 + 7(y^2 + z^2) - (\tfrac{15}{4})xz = \text{ constant..}$$

Ex.25. *The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point* (p, q, r) *is*
$$\left(\frac{A}{M} + q^2 + r^2\right)x^2 + \left(\frac{B}{M} + p^2 + r^2\right)y^2 + \left(\frac{C}{M} + q^2 + p^2\right)z^2$$
$$- 2qryz - 2rpzx - 2pqxy = \text{ constant.}$$
when referred to its centre of gravity as origin.

Sol. Let GX, GY, GZ be the principal axes through G, the C.G. of the body. Further let A, B, C be the moments of inertia of the body about the principal axes GX, GY, GZ and D, E, F the products of inertia about these axes taken two at a time. Then we will have $D = E = F = 0$



Now if we take A', B', C' as the moment of inertia and D', E', F' the products of inertia about parallel axes through $O' \equiv (p, q, r)$ then we will have

$$A' = A + M(q^2 + r^2) = M\left(\frac{A}{M} + q^2 + r^2\right)$$

Similarly $B' = B + M(r^2 + p^2) = M\left(\frac{B}{M} + r^2 + p^2\right)$

$$C' = C + M(q^2 + p^2) = M\left(\frac{C}{M} + q^2 + p^2\right)$$

Also $D' = D + M\,qr, E' = E + M\,rp, F' = F + M\,pq = Mpq$

Hence the equation of momental ellipsoid at O' is

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{constant}$$

$$\Rightarrow \left(\frac{A}{M} + q^2 + r^2\right)x^2 + \left(\frac{B}{M} + p^2 + r^2\right)y^2 + \left(\frac{C}{M} + q^2 + p^2\right)z^2$$
$$- 2qryz - 2rpzx - 2pqxy = \text{constant.}$$

Ex.26 (a) *If* $S \equiv A\,x^2 + B\,y^2 + C\,z^2 - 2D\,yz - 2E\,zx - 2F\,xy = \text{constant}$ *be the equation of momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that the momental ellipsoid at the point* (p, q, r) *is*

$$S + M[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{constant}$$

where M is the mass of the body.

Sol. If A', B', C' are the moments of inertia and D', E', F' are the products of inertia of the body with reference to a set of parallel axes through (p, q, r), then we have

$$A' = A + M(q^2 + r^2), B' = B + M(r^2 + p^2), C' = C + M(q^2 + p^2),$$

Also $D' = D + M\,qr, E' = E + M\,rp, F' = D + M\,pq.$

Hence the momental ellipsoid at (p, q, r) is given by

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{constant}$$

Substituting the values of A', B', C' etc. and simplifying, we get

$$S + M[\Sigma(qz - ry)^2] = \text{constant.}$$

0·13. Equimomental Bodies : *Two systems or bodies are said to be equimomental or kinetically equivalent when moments and products of inertia of one system or body about the axes are each correspondingly equal* *to moments and products of inertia of the other system or body about the same axes.*

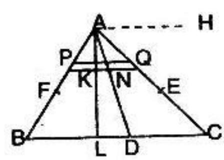
Necessary and sufficient conditions. *The two systems will be equimomental, iff the following conditions are satisfied.*

(i) The centres of gravity of the two system should coincide.

(ii) Both the systems should have the same mass.

(iii) The two systems should have the same principal axes and same principal moments about centre of gravity. **[Rajasthan 1991]**

0·14 *To show that moments and products of inertia of a uniform triangle about any lines are the same as the moments and products of inertia about the same lines, of three particles placed at the mid points of the sides, each equal to one third of the mass of the triangle.* **[Agra 84, Rajasthan 83]**



Let PQ be an elementary strip of breadth δx at a distance x from the vertex A. If the height of triangle $AL = h$, then from similar triangles ABC and APQ, we have

$$\frac{BC}{PQ} = \frac{h}{x} \Rightarrow PQ = \frac{ax}{h} \quad (\because \ BC = a)$$

Now draw the line AH in the plane of the lamina and parallel to BC. M.I. of the lamina about

$$AH = \int_0^h \left[\frac{ax}{h}\rho\, dx\right] x^2 = \frac{a\rho}{h}\int_0^h x^2\, dx = \frac{a\rho h^4}{4h} = \frac{a\rho h^3}{4} = \frac{1}{2} M h^2$$

Since $M = \dfrac{a\rho h}{2}$

M.I. of the lamina about AL

$$= \int_0^h \left[\frac{ax}{h}\rho\, dx\right]\left[NK^3 + \frac{1}{3}\left(\frac{ax}{2h}\right)^2\right] = \frac{a\rho}{h}\int_0^h \left[x\left(\frac{x}{h}LD\right)^2 + \frac{1}{3}\left(\frac{ax}{2h}\right)^2\right] dx$$

$$= \frac{a\rho}{h}\int_0^h \left[\frac{LD^2}{h^2} + \frac{a^2}{12h^2}\right] x^3\, dx = \frac{a\rho}{h}\left[\frac{LD^2}{h^2} + \frac{a^2}{12h^2}\right]\frac{h^4}{4}$$

$$= \frac{1}{4}a\rho h\left[LD^2 + \frac{a^2}{12}\right] = \frac{1}{4}a\rho h\left[(BD - BL)^2 + \frac{a^2}{12}\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{BC}{2} - BL\right)^2 + \frac{a^2}{12}\right] = \frac{1}{4}a\rho h\left[\left(\frac{a}{2} - C\cos b\right)^2 + \frac{a^2}{12}\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{b\cos C + c\cos B}{2} - c\cos B\right)^2 + \left(\frac{b\cos C + c\cos B}{2\sqrt{3}}\right)^2\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{b\cos C - c\cos B}{2}\right)^2 + \left(\frac{b\cos C + c\cos B}{2\sqrt{3}}\right)^2\right]$$

$$= \frac{1}{48} a\rho h\left[3(b\cos C - c\cos B)^2 + (b\cos C + c\cos B)^2\right]$$

$$= \frac{M}{6}\left[b^2\cos^2 C + c^2\cos^2 B - bc\cos B.\cos C\right] \quad ...(2)$$

The products of inertia about AL and AH

$$= \int_0^h \left(\frac{ax}{h}\rho x\right) x.NK = \int_0^h \frac{a\rho}{h} x^2 \frac{x}{h}.LD = \frac{a\rho}{h^2}\int_0^h x^3 (BD - BL)\,dx$$

$$= \frac{a\rho}{h^2}\int_0^h x^3 \left(\tfrac{1}{2}BC - BL\right) dx = \frac{a\rho}{h^2}\left[\tfrac{1}{2}(b\cos C + c\cos B) - c\cos B\right]\frac{h^4}{4}$$

$$= \frac{a\rho}{h^2}\frac{1}{2}(b\cos C - c\cos B)\frac{h^4}{4} = \frac{a\rho h^2}{8}(b\cos C - c\cos B)$$

$$= \frac{1}{4} Mh(b\cos C - c\cos B). \quad ...(3)$$

Now let three particles each of mass $\frac{M}{3}$ be placed at the mid points D, E, F of the sides respectively. Then we have M.I. of the three particles about $AH = \frac{M}{3}h^2 + \frac{M}{3}\left(\frac{h}{2}\right)^2 + \frac{M}{3}\left(\frac{h}{2}\right)^2 = \frac{1}{2}M h^2$ ...(4)

M.I. of three particles about AL

$$= \frac{M}{3}LD^2 + \frac{M}{3}\left(\frac{b}{2}\cos C\right)^2 + \frac{M}{3}\left(\frac{1}{2}c\cos B\right)^2$$

$$= \frac{M}{3}\left[(a/2 - c\cos B)^2 + \left(\tfrac{1}{2}b\cos C\right)^2 + \left(\tfrac{1}{2}c\cos B\right)^2\right]$$

$$= \frac{M}{6}\left[b^2\cos^2 C + c^2\cos^2 B - bc\cos B\cos C\right] \quad ...(5)$$

Product of inertia of three particles about AL and AH

$$= \frac{M}{3}AL.LD - \frac{M}{3}\frac{1}{2}AL.\left(\tfrac{1}{2}BL\right) + \frac{M}{3}\frac{1}{2}AL.\left(\tfrac{1}{2}CL\right)$$

$$= \frac{Mh}{3}\left[LD - \tfrac{1}{4}BL + \tfrac{1}{4}CL\right]$$

$$= \frac{Mh}{3}\left[(a/2 - c\cos B) - \tfrac{1}{4}c\cos B + \tfrac{1}{4}b\cos C\right]$$

$$= \frac{Mh}{3}\left[\frac{b\cos C + c\cos B}{2} - c\cos B - \frac{1}{4}c\cos B + \frac{1}{4}b\cos C\right]$$

$$= \frac{Mh}{3}\left[\tfrac{3}{4}b\cos C - \tfrac{3}{4}c\cos B\right] = \frac{Mh}{4}\left[b\cos C - c\cos B\right] \quad ...(6)$$

Comparing the above results, we conclude that moments and products of inertia of the triangle about AH and AL are the same as those of three particles each of mass $\frac{1}{3}M$ placed at the mid points of the sides. Therefore moments and poducts of inertia about any two perp. lines will be the same.

Hence a triangle of mass M is kinetically equivalent to three particles each of mass $\frac{1}{3}M$ placed at the mid points of the side.

Ex.26.(b) *Show that a uniform triangular lamina of mass m is equimomental with three particles, each of mass (1/12)m placed at the angular points and a particle of mass (3/4)m placed at the centre of inertia of the triangle.*

Sol. Clearly C.G. of the four particles is the same as the C.G. of the triangular lamina. Also mass of the four particles

$$= (3/12)m + (3/4)m = m = \text{mass of the triangle.}$$

Now if the distances of the vertices of the triangle from a line are α, β, γ respectively and the distance of its C.G. from this line is h, then M.I. of the four particles about the line

$$= (1/12)m\alpha^2 + (1/12)m\beta^2 + (1/12)\gamma^2 + (3/4)mh^2$$

$$= (1/12)m[\alpha^2 + \beta^2 + \gamma^2 + 9h^2] \qquad [h = (\alpha + \beta + \gamma)/3]$$

$$= (1/12)\, m[\alpha^2 + \beta^2 + \gamma^2 + 9\{(\alpha + \beta + \gamma)/3\}^2]$$

$$= (1/6)\, m[\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta]$$

= M.I. of the triangle about the same line.

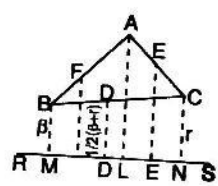
Hence the systems are equi-momental.

Ex.27. *If α, β, γ be the distances of the vertices of a triangle of mass m from any straight line in its plane, show that the moment of inertia of the triangle about this line is* $(1/6)\, m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$.

Hence deduce that if h be the distance of the centre of inertia of the triangle from the line, then M.I. about this line

$$= (1/12)\, m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2).$$

Sol. The triangle of mass m is equimomental to three particles each of mass $\frac{1}{3}m$ placed at the middle points of the sides. Let ABC be the triangle and D, E, F the middle points of its sides. Also let RS be the line in the plane of the triangle. Draw perpendiculars from A, B, C to the line RS.



If $AL = \alpha$, $BM = \beta$, $CN = \gamma$ then the lengths of the perpendiculars from D, E, and F are as given below.

$$DD' = (\tfrac{1}{2})(\beta + \gamma),\ EE' = (\tfrac{1}{2})(\alpha + \gamma),\ FF' = (\tfrac{1}{2})(\alpha + \beta)$$

$$\therefore \text{M.I.} = (\tfrac{1}{3})\, m\{(\beta + \gamma)/2\}^2 + (\tfrac{1}{3})\, m\{(\beta + \alpha)/2\}^2 + (\tfrac{1}{3})\, m\{(\alpha + \beta)/2\}^2$$

$$= (1/12)\, m(2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta)$$

$$= (1/6)\, m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$$

If h is the distance of C.G. from the line RS, then we have

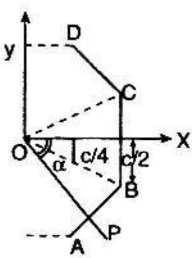
$h = (\frac{1}{3})(\alpha + \beta + \gamma)$

$\therefore$ M.I. $= (\frac{1}{12}) m[\alpha^2 + \beta^2 + \gamma^2 + (\alpha + \beta + \gamma)^2]$

$= (m/12)(\alpha^2 + \beta^2 + \gamma^2 + 9h^2)$.

Ex.28. *Show that the moment of inertia of a regular polygon of n sides about any straight line through its centre is* $\frac{Mc^2}{24} \cdot \frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}$, *where n is the number of sides and C is the length of each side.*

Sol. Consider a polygon $ABCDEF$ of n sides, each of length c. Let O be the centre of the polygon. The polygon can be divided into n equal isosceles triangles. If M is the mass of polygon then mass of each of the n isosceles triangles $= (M/n)$. Consider one of the isosceles triangle, say $\Delta\, OBC$.



Let the right bisector of BC be the *x-axis* and a line through O perp. to OX in the plane of the polygon be the *y– axis*. Clearly $\angle BOC = (2\pi/n)$ and $\angle COX = (\pi/n)$. Now the triangle BOC of mass (M/n) placed at the middle point of its sides.

$\therefore$ M.I. of the triangle about

$OX = (M/3n)[(\frac{1}{4}c)^2 + (\frac{1}{4}c)^2] = (M/24n)\, c^2$.

[The length of the perp. from the mid point of OB on $OX = (c/4)$].

M.I. of the triangle about OY

$= (M/3n)[(\frac{1}{2}c \cot \pi/n)^2 + 2(\frac{1}{4}c \cot \pi/n)^2 = (M c^2/8n)\cot^2 \pi/n$

If OP is line making an angle α with OX about which we want to determine moment of inertia then M.I. of the triangle OBC about OP

$$= \left(\frac{M}{24n}c^2\right)\cos^2\alpha + \left(\frac{Mc^2}{8n}\cot^2 \pi/n\right)\sin^2\alpha$$

Taking the other triangles one by one, we get the M.I. of the polygon

about $OP = \frac{Mc^2}{24n}\left\{\cos^2\alpha + \cos^2\left(\alpha + \frac{2\pi}{n}\right) + \cos^2\left(\alpha + \frac{4\pi}{n}\right) + \ldots n \text{ terms}\right\}$

$$+ \frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\left\{\sin^2\alpha + \sin^2\left(\alpha + \frac{2\pi}{n}\right) + \sin^2\left(\alpha + \frac{4\pi}{n}\right) + \ldots\ldots n \text{ terms}\right\}$$

$$= \frac{Mc^2}{24n}\frac{1}{2}[1 + \cos 2\alpha + 1 + \cos(2\alpha + 4\pi/n) + \ldots]$$

$$+ \frac{Mc^2}{8n}\cot^2(\pi/n) \cdot \frac{1}{2} \cdot [1 - \cos 2\alpha + 1 - \cos(2\alpha + 4\pi/n) \ldots.]$$

$$= \frac{Mc^2}{24n}\frac{1}{2}n + \frac{Mc^2}{8n}\cot^2\left(\frac{\pi}{n}\right)\frac{1}{2}n + \frac{Mc^2}{48n}\left[\cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \text{....}n\ terms\right]$$

$$- \frac{Mc^2}{16n}\cot^2\left(\frac{\pi}{n}\right)\left[\ [\cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \text{....}n\ terms\right]$$

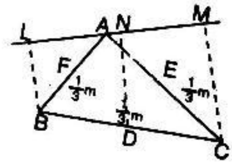
$$= \frac{Mc^2}{48}\left[1 + 3\cot^2\left(\frac{\pi}{n}\right)\right] + 0 - 0 = \frac{Mc^2}{48}\left\{\frac{\sin^2(\pi/n) + 3\cos^2(\pi/n)}{2\sin^2\pi/n}\right\}$$

$$= \frac{Mc^2}{24}\left[\frac{\sin^2(\pi/n) + \cos^2(\pi/n) + 2\cos^2(\pi/n)}{2\sin^2\pi/n}\right]$$

$$= \frac{Mc^2}{24}\left[\frac{1 + \{1 + \cos(2\pi/n)\}}{1 - \cos(2\pi/n)}\right] = \frac{Mc^2}{24}\left[\frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}\right]$$

Ex.29. *Obtain the moment of inertia of a triangular lamina ABC about a straight line through A (or any vertex) in the plane of the triangle.*

Sol. Let LAM be the line about which moment of inertia is to be determined, and let β, γ be the respective distance of two vertices B and C of the triangle from the straight line LAM. If m is the mass of the triangle, then the triangle is equimomental to three particles, each of mass $\frac{1}{3}m$, placed at the mid points D, E, F of the sides BC, CA, AB. Length of the perp. from F on $LAM = (\beta/2)$



Length of perp. from D on $LAM = (\gamma + \beta)/2$

Length of perp. from E on $LAM = (\gamma/2)$

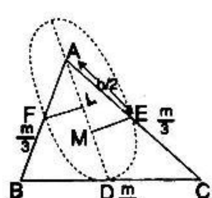
$\therefore$ M.I. of the triangle about LAM

$$= (1/3)\,m[(\beta/2)^2 + \{(\beta + \gamma)/2\}^2 + (\gamma/2)^2] = (1/6)\,m[\beta^2 + \beta\gamma + \gamma^2].$$

Ex.30. *Show that there is a momental ellipse at an angular point of a triangular area which touches the opposite side at its middle point and bisects at the adjacent sides.* **[Delhi Hons. 64]**

Sol. We know that the momental ellipsoid at A passes through D, if moment of inertia about $AD = (mK^4/AD^2)$ where K is some constant.

Similarly the momental ellipsoid will pass through E and F if moment of inertia about $AE = (mK^4/AE^2)$ etc. Now replace the triangle of mass m by three particles, each of mass $(1/3)\,m$ at D, E and F.



M.I. of the triangle about

$A = m/3\,[FL^2 + EM^2]$

$= m/3\,[\{(c/2)\sin BAD\}^2 + \{(b/2)\sin CAD\}^2]$

$= m/12\,[c^2\sin^2 BAD + b^2\sin^2 CAD]$

Using Lami's theorem in $\Delta s\, ABD$ and ACD, we get

$$\frac{\sin BAD}{(a/2)} = \frac{\sin B}{AD} \text{ and } \frac{\sin CAD}{(a/2)} = \frac{\sin C}{AD}$$

$$\therefore \sin BAD = (a/2)\frac{\sin B}{AD} \text{ and } \sin CAD = (a/2)\frac{\sin C}{AD}$$

$$\therefore \text{M.I. about } AD \text{ becomes} = \frac{m}{12}\left(\frac{1}{4}a^2c^2\sin^2 B + \frac{1}{4}a^2b^2\sin^2 C\right)\frac{1}{AD^2}$$

$$= \frac{m}{12}\left[\Delta^2 + \Delta^2\right]\frac{1}{AD^2} = \left(\frac{m\Delta^2}{6}\right)\frac{1}{AD^2}$$

M.I. of the triangle about AF

$$= \frac{m}{3}\left[\left(\frac{b}{2}\sin A\right)^2 + \left(\frac{b}{2}\sin A\right)^2\right] = \frac{mb^2}{6}\sin^2 A = \frac{m}{6}\frac{1}{c^2}b^2c^2\sin^2 A$$

$$= \frac{m}{6c^2}4\Delta^2 = \frac{m}{6}\frac{4\Delta^2}{(2AF)^2} = \left(\frac{M\Delta^4}{6}\right)\frac{1}{AF^2}$$

Similarly M.I. of the triangle about $AE = \left(\dfrac{m\Delta^2}{6}\right)\dfrac{1}{AE^2}$.

Thus we see that the momental ellipse at A passes through D, E and F and AD is the diameter of the ellipse. The tangent at D will be parallel to the chord EF which is bisected by the diameter and obviously BC is parallel to EF. Hence BC is tangent to the ellipse at D.

Ex. 31. *Show that any lamina is dynamically equivalent to the three particles each one third of the mass of the lamina, placed at the centre of a maximum triangle inscribed in the ellipse, whose equation referred to the principal axes at the centre of inertia is* $(x^2/B') + (y^2/A') = 2$

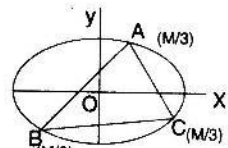
where mA' and mB' *are the principal moments of inertria about* OX and OY *and* m *is the mass*.

Sol. If a maximum triangle is inscribed in an ellipse, then the eccentric angles of its angular points will be of the form

$\phi, \phi + (2\pi/3), \phi + (4\pi/3)$.

Let ABC be the maximum triangle inscribed in the ellipse $(x^2/B') + (y^2/A') = 2$

$\Rightarrow (x^2/2B') + (y^2/2A') = 1$ then the co-oridinates of A, B, C, will be



$[\sqrt{(2B')}\cos\phi, \sqrt{(2A')}\sin\phi]$,

$\left[\sqrt{(2B')}\cos\left(\phi + \dfrac{2\pi}{3}\right),\right.$

$\left[\sqrt{(2A')}\sin\left(\phi+\frac{2\pi}{3}\right)\right];\left[\sqrt{(2B')}\cos\left(\phi+\frac{4\pi}{3}\right),\sqrt{(2A')}\sin\left(\phi+\frac{4\pi}{3}\right)\right]$

Now if $(\bar{x},\bar{y})$ be co- ordinates of C.G. of the three particles placed at *A, B, C* and each of mass m/3 then

$$\bar{x}=\frac{\frac{m}{3}\sqrt{(2B')}}{m}\left\{\cos\phi+\cos\left(\phi+\frac{2\pi}{3}\right)+\cos\left(\phi+\frac{4\pi}{3}\right)\right\}$$
$$=\frac{\sqrt{(2B')}}{\cdot 3}\left\{\cos\phi+2\cos(\phi+\pi).\cos\frac{\pi}{3}\right\}$$
$$=\frac{\sqrt{(2B')}}{3}\left\{\cos\phi+2\cos\phi.\frac{1}{2}\right\}=0,\text{ similarly }\bar{y}=0.$$

Thus the centres of inertia of the two system coincide. Also masses of the two systems are equal. M.I. of the particles each of mass m/3 placed at *A, B, C about OX* .

$$=\frac{m}{3}2A'\left[\sin^2\phi+\sin^2\left(\phi+\frac{2\pi}{3}\right)+\sin^2\left(\phi+\frac{4\pi}{3}\right)\right\}$$
$$=\frac{1}{3}mA'\left\{3-\cos 2\phi-\left\{\cos\left(2\phi+\frac{4\pi}{3}\right)+\sin\left(2\phi+\frac{8\pi}{3}\right)\right\}\right]$$
$$=\frac{1}{3}mA'\left[3-\cos 2\phi-2\cos(2\phi+2\pi)\cos\frac{2\pi}{3}\right]$$
$$=\frac{1}{3}mA'\,[3-\cos 2\phi+\cos 2\phi]=mA'=\text{M.I. of the lamina about }OX.$$

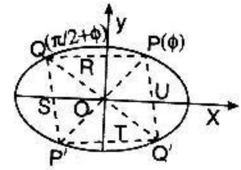
Similarly M.I. about $OY=mB'=$ M.I. of the lamina about OY. Thus M.I. of the three particles about OX and OY are the same as that of lamina. Product of inertia of the three particles about OX, OY.

$$=\frac{m}{3}\sqrt{(2B')}.\sqrt{(2A')}\left[\cos\phi\sin\phi+\cos\left(\phi+\frac{2\pi}{3}\right)\sin\left(\phi+\frac{2\pi}{3}\right)\right.$$
$$\left.+\cos\left(\phi+\frac{4\pi}{3}\right)\sin\left(\phi+\frac{4\pi}{3}\right)\right]$$
$$=\frac{m}{3}\sqrt{(A'B')}\left[\sin 2\phi+\sin(2\phi+\tfrac{4}{3}\pi)+\sin\left(2\phi+\frac{8\pi}{3}\right)\right]$$
$$=\frac{m}{3}\sqrt{(A'B')}\left[\sin 2\phi+2\sin(2\phi+2\pi).\cos\frac{2\pi}{3}\right]$$
$$=\frac{1}{3}m\sqrt{(A'B')}\;[\sin 2\phi+2\sin 2\phi\,(-\tfrac{1}{2})]=0.$$

Thus the two systems are dynamically equivalent as the two systems have same mass, same C.G., same M.I. and same product of inertia at their C.G.

Ex.32. *Particles each equal to one quarter of the mass of an elliptic area are placed at the middle points of the chords joining the extremities of a pair of conjugate diameters . Prove that these four particles are equimomental to the elliptic area.*

Sol. Let POP' and QOQ' be the conjugate diameters of the elliptic area of mass m. If ϕ is the eccentric angle of P then eccentric angle of Q is $\{(\pi/2)+\phi\}$. The co-ordinates of P, Q respectively are $(a\cos\phi,\ b\sin\phi)$, $(-a\sin\phi, b\cos\phi)$ and that of P', Q' are $(-a\cos\phi, -b\sin\phi)$ respectively.



If (x_1, y_1) be the co-ordinates of R, the mid point of PQ then

$x_1 = (\frac{1}{2})\, a(\cos\phi - \sin\phi)$; $y_1 = (\frac{1}{2})\, b\,(\sin\phi + \cos\phi)$.

Similarly if (x_2, y_2) are co-ordinates of S then

$x_2 = -(a/2)(\sin\phi + \cos\phi)$, $y_2 = (b/2)(\cos\phi - \sin\phi)$

Also if (x_3, y_3) and (x_4, y_4) are the co-ordinates of the mid points T and U then $x_3 = -(a/2)(\cos\phi - \sin\phi)$, $y_3 = -(b/2)(\sin\phi + \cos\phi)$,

and $x_4 = (a/2)(\sin\phi + \cos\phi)$, $y_4 = (b/2)(\sin\phi - \cos\phi)$.

Now if the co-ordinates of the C.G. of the four particles, each of mass (m/4) placed at the mid points R, S, T and U be $(\overline{x}, \overline{y})$ then

$$\overline{x} = \frac{(\frac{1}{4})m\,(x_1 + x_2 + x_3 + x_4)}{(\frac{1}{4})m + (\frac{1}{4})m + (\frac{1}{4})m + (\frac{1}{4})m}$$

$$= \frac{(m/4)(a/2)[(\cos\phi - \sin\phi) - (\sin\phi + \cos\phi) - (\cos\phi - \sin\phi) + (\sin\phi + \cos\phi)]}{m} = 0.$$

Similarly $\overline{y} = 0$. Thus the C.G. of the particles is the same as the C.G. of the elliptic area. M.I. of the four particles about the major axis OX

$$= \frac{m}{4}[y_1^2 + y_2^2 + y_3^2 + y_4^2] = \frac{m}{4}\left[\frac{b^2}{4}\{(\sin\phi + \cos\phi)^2 + (\cos\phi - \sin\phi)^2 + (\sin\phi + \cos\phi)^2 + (\sin\phi - \cos\phi)^2\}\right]$$

$$= \frac{mb^2}{4} = \text{M.I. of the elliptic area about } OX.$$

Similarly M.I. of the particles about $OY = \dfrac{ma^2}{4}$ = M.I. of the elliptic plate about OY. Now product of inertia about OX, OY

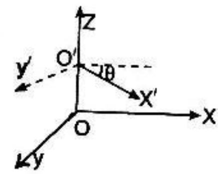
$$= \frac{m}{4}[x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4] = 0$$

= Product of inertia of the elliptic area about OX, OY. Hence the four particles are equimomental to the elliptic area.

0·15. Principal Axes : *To find whether a given straight line is, at any point of its length, a principal axis of a material system. And if the line is principal axis, then to determine the other two principal axis.*

Take the given straight line OZ as y-axis and two perpendicular lines OX, OY as x axis and y- axis respectively, through any point O on the given line.

Now suppose that OZ is a principal axis of the system at a point O' where $OO'=h$. Assume $O'X', O'Y'$ to be the other two principal axes, such that $O'Y'$ is inclined at an angle θ to a line parallel to OX. Now consider a particle of mass m in the material system. If (x, y, z) be the co-ordinates of that particle with reference to axes OX, OY, OZ and (x', y', z') its co-ordinates with reference to $O'X', O'Y', O'Z'$ as axes, then we will have



$x' = x\cos\theta + \sin\theta, y' = -x\sin\theta + y\cos\theta, z' = z - h.$

The necessary and sufficient condition, that the new axes $O'X', O'Y', O'Z'$ become the principal axes of the system, is, that the product of inertia with reference to these axes must vanish,

i.e., $\Sigma\, my'z' = 0, \Sigma\, mz'x' = 0, \Sigma\, mx'y' = 0.$

Now $\Sigma\, my'z' = \Sigma\, m(-x\sin\theta + y\cos\theta)\,(z-h)$

$= \Sigma\, m\,(yz\cos\theta - xz\sin\theta + hx\sin\theta - hy\cos\theta)$

$= \cos\theta\,\Sigma myz - \sin\theta\,\Sigma mzx + h\sin\theta\,\Sigma mx - h\cos\theta\,\Sigma\, my$

$= D\cos\theta - E\sin\theta + h\sin\theta\, M\bar{x} - h\cos\theta\, M\bar{y}$

$= D\cos\theta - E\sin\theta + Mh\,(\bar{x}\sin\theta - \bar{y}\cos\theta)$

Similarly $\Sigma mz'x' = \Sigma m\,[yz\sin\theta + xz\cos\theta - hx\cos\theta - hy\sin\theta]$

$= D\sin\theta + E\cos\theta - Mh\,(\bar{x}\cos\theta + \bar{y}\sin\theta)$

and $\Sigma mx'y' = \Sigma m\,[-x^2\sin\theta\cos\theta + xy\,(\cos^2\theta - \sin^2\theta) + y^2\sin\theta.\cos\theta]$

$= \frac{1}{2}(A - B)\sin 2\theta + F\cos 2\theta.$

Now taking $\Sigma mx'y' = 0$, we get

$$\tan 2\theta = \frac{2F}{B-A} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} \qquad ...(1)$$

taking $\Sigma\, my'z' = 0$, we get $Mh = \dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta}$

and taking $\Sigma mz'x' = 0$, we have $Mh = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$\therefore\ Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}.$$

Now $\dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$= \frac{(E\sin\theta - D\cos\theta)\sin\theta + (D\sin\theta + E\cos\theta)\cos\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)\sin\theta + (\bar{x}\cos\theta + \bar{y}\sin\theta)\cos\theta} = \frac{E}{\bar{x}} \qquad ...(2)$$

and $\dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$= \frac{(E\sin\theta - D\cos\theta)(-\cos\theta) + (D\sin\theta + E\cos\theta)\sin\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)(-\cos\theta) + (\bar{x}\cos\theta + \bar{y}\sin\theta)\sin\theta} = \frac{D}{y} \quad ...(3)$$

so, we have $Mh = \dfrac{E}{\bar{x}} = \dfrac{D}{\bar{y}}$. ...(4)

Thus the condition that the line OZ may be principal axis of the system at some point of its length is $\dfrac{E}{\bar{x}} = \dfrac{D}{\bar{y}}$ and then from (4) we get that point at which the line OZ is a principal axis and (1) gives the co-ordinates of the other principal axes at O'.

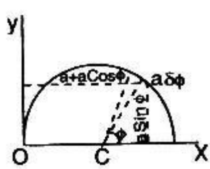
Cor. I. *If an axis passes through the centre of gravity of a body and it is a principal axis at any point of its length, then it is a principal axis at all points of its length.* **[Meerut 1978,Raj.62]**

If z-axis is a principal axis at O, then $D = E = 0$ and further $h = D/M\bar{y} = E/M\bar{x}$ implies that $h = 0$ which means that there is no such other point as O'. But if $\bar{x} = 0$, $\bar{y} = 0$ and $D = E = 0$, then in this case the value of h becomes indeterminate.

So in a case where the axis passes through C.G. of the body and is a principal axis at any point of its length, it is a principal axis at all the points of its length.

Ex. 33. *A wire is in the form of a semi-circle of radius a, show that at an end of its diameter the principal axes in its plane are inclined to the diameter at angles* $(\frac{1}{2})\tan^{-1}(4/\pi)$ *and* $(\pi/2) + (\frac{1}{2})\tan^{-1}(4/\pi)$.

Sol. Let O be an end of the diameter of the semicircular wire. Now choose OX and OY as the axes of reference.



Consider an elementary arc $a\,\delta\phi$, where a is the radius of the circle, then we easily have A = $M.I.$ of the wire about OX.

$$= \int_0^{\pi} (\rho a\, d\phi)(a\sin\phi)^2$$

$$= \int_0^{\pi/2} 2a^3\rho\sin^2\phi\, d\phi = 2a^3\rho\int_0^{\pi/2}\sin^2\phi\, d\phi$$

$$= 2a^3\rho.\frac{1}{2}.\frac{\pi}{2} = (\pi a^3\rho/2) = (Ma^2/2), \qquad [\because\ M = \tfrac{1}{2}(2\pi a\rho) = \pi a\rho]$$

$$B = M.I. \text{ about } OY = \int_0^{\pi} \rho a(a + a\cos\phi)^2 d\phi$$

$$= \rho a^3\int_0^{\pi}(1 + 2\cos\phi + \cos^2\phi)\, d\phi$$

$$= \rho a^3 (\pi + 2 . \tfrac{1}{2} . \pi/2) = \frac{3 \rho a^3 \pi}{2} = \frac{3Ma^2}{2}$$

$F =$ Product of inertia about OX, OY

$$= \int_0^{\pi} \rho a (a \sin \phi)(a + a \cos \phi)\, d\phi = \rho a^3 \int_0^{\pi} (\sin \phi + \sin \phi \cos \phi)\, d\phi$$

$$= \rho a^3 \left[-\cos \phi + \frac{1}{2} \sin^2 \phi \right]_0^{\pi} = 2 \rho a^3 = 2Ma^2/\pi.$$

Now let θ be the inclination of one of the principal axes to the diametter OX then, we have

$$\tan 2\theta = \frac{2F}{B - A} \Rightarrow \theta = \tfrac{1}{2} \tan^{-1} \frac{2F}{B - A} = \tfrac{1}{2} \tan^{-1} \left\{ \frac{(4Ma^2/\pi)}{\tfrac{3}{2}Ma^2 - \tfrac{1}{2}Ma^2} \right\}$$

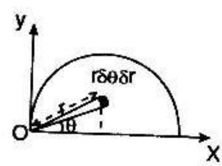
$$= \tan^{-1}(4/\pi).$$

The other principal axis being perpendicular to the above axis will make an angle $[(\pi/2) + (\tfrac{1}{2}) \tan^{-1}(4/\pi)]$ with OX.

Ex. 34. *Show that at extremity of the boundary diameter of a semi-circular lamina, the principal axis makes an angle* $\tfrac{1}{2} \tan^{-1}(8/3\pi)$ *to the diameter.*

Sol. We know that the equation of the circle reffered to O (the extremity of the diameter OX) as pole and OX as the initial line is $r = 2a \cos \theta$ Consider an element $r\delta\theta\delta r$ at P, then M.I. of the lamina$O$ OX is given

$$\text{by } A = \rho \int_0^{\pi/2} \int_0^{2a \cos \theta} r\, d\theta\, dr (r^2 \sin^2 \theta)$$



$$= \rho \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^{2a \cos \theta} \sin^2 \theta\, d\theta$$

$$= \frac{1}{4} \rho (2a)^4 \int_0^{\pi/2} \cos^4 \theta \sin^2 \theta\, d\theta = \tfrac{1}{3} \pi \rho a^4$$

$$B = M.I. \text{ about } OY = \rho \int_0^{\pi/2} \int^{2a \cos \theta} r\, d\theta\, dr\, (r^2 \cos^2 \theta)$$

$$= \rho \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^{2a \cos \theta} \cos^2 \theta\, d\theta = \frac{1}{4} \rho\, (2a)^4 \int_0^{\pi/2} \cos^6 \theta\, d\theta$$

$$= 4 \rho a^4 \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{5}{3} \pi \rho a^4$$

and $F =$ Product of inertia about OX, OY

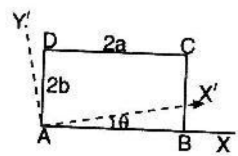
$$= \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr (r\sin\theta)(r\cos\theta)$$

$$= \frac{1}{4}\rho (2a)^4 \int_0^{\pi/2} \cos^5\theta \sin\theta\, d\theta = 4\rho a^4 \frac{1}{6} = \frac{2}{3}\rho a^4$$

$\Rightarrow \phi = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} = \frac{1}{2}\tan^{-1}(8/3\pi)$ where ϕ is the angle that a principal axis at O makes with OX.

Ex.35. *The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}[3ab/2(a^2-b^2)]$.* **[Agra 1989,Delhi Hons.66]**

Sol. Let AB be the axis of x and AD the axis of y and a line through A perpendicular to the plane of the rectangle, the axis of z.



Then $B =$ M.I. of the rectangle about $AD = \frac{1}{3}Ma^2 + Ma^2$

$= \frac{4}{3}Ma^2$. Similarly $A = \frac{4}{3}Mb^2$

$F =$ product of inertia about $AB, AD = Mab$

Now if θ is the inclination of a principal axis to AB, then we have

$$\tan 2\theta = \frac{2F}{B-A} = \frac{2Mab}{\frac{4}{3}Ma^2 - \frac{4}{3}Mb^2} = \frac{3ab}{2(a^2-b^2)}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left[\frac{3ab}{2(a^2-b^2)}\right].$$

Ex. 36. *A uniform square lamina is bounded by the axes of x and y and the lines $x = 2c$, $y = 2c$ and a corner is out off by the line $x/a + y/b = 2$. Show that the principal axes at the centre of the square are inclined to the axes of x at angles given by*

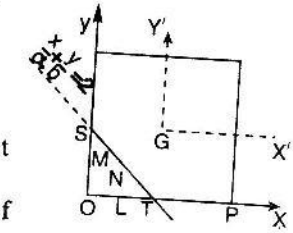
$$\tan 2\theta = \frac{ab - 2(a+b)c + 3c^2}{(a-b)(a+b-2c)}.$$ **[Lucknow 1990]**

Sol. Let OST be the triangular lamina cut off from the square and let L, M, N be the mid points of its sides. This triangular lamina of mass m can be replaced by three particles each of mass $m/3$ placed at L, M and N. Let ST be the line whose equation is

$(x/a) + (y/b) = 2 \Rightarrow (x/2a) + (y/2b) = 1$

Then $OT = 2a$ and $OS = 2b$, also $OP = 2c$ (given). Let $m_1 =$ mass of the square and $m =$ mass of the triangle OST. Let G be the centre of the square and GX', GY' be the new axes of reference.

With reference to the new axes the co-ordinates of L are $[-(c-a), -c]$, coordinates *of* M *are* $[-c, -(c-b)]$ and co-ordinates of N are $[-(c-a), -(c-b)]$



Then $A =$ M.I. of the remaining area about GX'

$= $ M.I. of the whole square - M.I. of the $\Delta\, OST$.

$=$ M.I. of the square $-$ M.I. of the three particles each of mass $\frac{1}{3}m$ placed at the points L, M, N

$$= \frac{1}{3}m_1c^2 - \frac{1}{3}m[c^2 + (c-b)^2 + (c-b)^2]$$

$B =$ M.I. about $GY =$ M.I. of whole square - M.I. of $\Delta\, OST$

$$= \frac{1}{3}mc^2 - \frac{m}{3}[c^2 + (c-a)^2 + (c-a)^2]$$

and $F =$ Product of inertia about GX', GY' of the remaining area

$=$ P.I. of the whole square $-$ P.I. of $\Delta\, OST$

$$= 0 - \frac{1}{3}m\,[(c-a)\,c + c(c-b) + (c-a)(c-b)]$$

$$= -\frac{1}{3}\{3c^2 - 2c\,(a+b) + ab\}$$

$$\therefore \tan 2\theta = \frac{2F}{B-A} = \frac{-\frac{2}{3}\{3c^2 - 2c\,(a+b) + ab\}}{\frac{1}{3}[2\,(c-b)^2 - 2\,(c-a)^2]}.$$

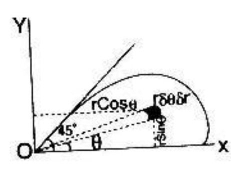
$$= \frac{-\{3c^2 - 2c\,(a+b) + ab\}}{(a-b)(2c-a-c)} = \frac{ab - 2c\,(a+b) + 3c^2}{(a-b)\,(a+b-2c)}.$$

Ex. 37. *Show that the principal axes at the node of a half loop of the leminiscate* $r^2 = a^2 \cos 2\theta$ *are inclined to the initial line at angles* $\frac{1}{2}\tan^{-1}\frac{1}{2}$ and $(\pi/2) + \frac{1}{2}\tan^{-1}\frac{1}{2}$. **[Agra 1986,Delhi Hons.86]**

Sol. The equation of the lemniscate is $r^2 = a^2 \cos 2\theta$. Consider an element of area $r\,\delta\theta\ \ \delta r$ at P.

Then $A =$ M.I. of half of the loop about OX.

$$= \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr(r^2 \sin^2\theta)$$



$$= (\tfrac{1}{4})\,\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \sin^2\theta\ d\theta.$$

$$= (\tfrac{1}{8})\,\rho a^4 \int_0^{\pi/4} \cos^2 2\theta\,(1-\cos 2\theta)\, d\theta$$

[Put $2\theta = t$ so that $2d\theta = dt$]

$$\therefore A = \frac{1}{16}\rho a^4 \int_0^{\pi/2} (\cos^2 t - \cos^3 t)\, dt = \frac{\rho a^4}{192}(3\pi - 8)$$

$B =$ M.I. of half of the loop about OY.

$$= \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr . r^2 \cos^2\theta = \frac{1}{4} a^4 \int_0^{\pi/4} \cos^2 2\theta \cos^2\theta\, d\theta$$

$$= \frac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta (1 + \cos 2\theta)\, d\theta \qquad (\text{Putting } 2\theta = t)$$

$$= \frac{1}{16}\rho a^4 \int_0^{\pi/2} (\cos^2 t + \cos^3 t) dt = \frac{\rho a^4}{192}(3\pi + 8)$$

and $F =$ Product of inertia of the half loop about OX, OY

$$= \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr (r \cos\theta)(r \sin\theta)$$

$$= \frac{1}{4}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \sin\theta . \cos\theta\, d\theta$$

$$= \frac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta . \sin 2\theta\, d\theta = \frac{1}{16}\rho a^4 \int_0^{\pi/2} \cos^2 t \sin t\, dt = (\tfrac{1}{48})\rho a^4$$

If θ is the angle which the principle axis at O makes with OX,

then $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2F}{B - A}\right) = \frac{1}{2}\tan^{-1}\left\{\frac{8}{(3\pi + 8) - (3\pi - 8)}\right\}$

$$= \frac{1}{2}\tan^{-1}(\tfrac{8}{16}) = \frac{1}{2}\tan^{-1}(\tfrac{1}{2}).$$

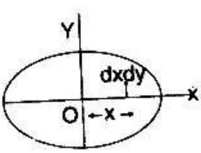
The other principal axis being at right angles to this principal axis will make an angle $(\pi/2) + \frac{1}{2}\tan^{-1}(\frac{1}{2})$ with OX.

Ex.38. *Show that at the centre of a quadrant of an ellipse, the principal axes in its plane are inclined at angle* $(\frac{1}{2})\tan^{-1}\left(\frac{4ab}{\pi(a^2 - b^2)}\right)$ *to the axis.*

Sol. Let the equation of the ellipse be $(x^2/a^2) + (y^2/b^2) = 1$.

Consider an element $\delta x\, \delta y$ at the point (x, y), then

$A =$ M.I. of the quadrant of an ellipse about OX.



$$= \int_0^a \int_0^{(b/a)\sqrt{(a^2 - x^2)}} y^2 \rho\, dx\, dy$$

$$= \rho \int_0^a \left| \frac{y^3}{3} \right|_0^{(b/a)\sqrt{(a^2-x^2)}} dx$$

$$= \frac{1}{3}\rho \int_0^a \frac{b^3}{a^3}(a^2-x^2)^{3/2}\, dx = \frac{1}{3}\frac{\rho b^3}{a^3}\int_0^{\pi/2} a^3 \cos^3\theta \; a\cos\theta \, d\theta$$

[Putting $x = a \sin\theta \Rightarrow dx = a \cos\theta \, d\theta$]

$$= \frac{1}{3}\rho\; ab^3 \; \frac{3.1}{4.2}.\frac{\pi}{2} = \frac{1}{16}\rho a\pi b^3$$

$$= \frac{1}{4} M b^2 \text{ where } M = \text{ mass of the quadrant } = \frac{\pi ab\,\rho}{4}$$

Similarly $B =$ M.I. of the quadrant of an ellipse about $OY = \frac{1}{4}Ma^2$

and $F =$ P.I. about OX , OY

$$= \int_0^a \int_0^{(b/a)\sqrt{(a^2-x^2)}} xy\,\rho\, dx\, dy = \int_0^a \left| y^2 \right|_0^{(b/a)\sqrt{(a^2-x^2)}} dx$$

$$= \frac{\rho}{2}\int_0^a x . \frac{b^2}{a^2}(a^2-x^2)\, dx = \frac{\rho}{2} . \frac{b^2}{a^2}\left[a^2 . \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 \right]_0^a$$

$$= \frac{\rho\, b^2}{2\,a^2} . \tfrac{1}{4}a^4 = \tfrac{\rho}{8} a^2 b^2 = \tfrac{1}{2}\left(\tfrac{1}{4}\pi ab\,\rho\right)\frac{ab}{\pi} = \frac{Mab}{2\pi}$$

Now if θ be the inclination of principal axis with OX , then

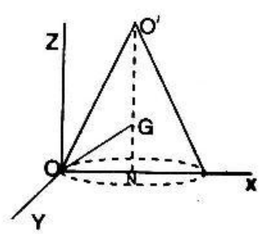
$$\tan\, 2\theta = \frac{2F}{B-A)} = \frac{4ab}{(a^2-b^2)\,\pi} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{4ab}{\pi(a^2-b^2)} \right)$$

Ex.39. *Find the principal axes of a right circular cone at a point on the circumference of the base , and show that one of them will pass through its C G . if the vertical angle of the cone is* $2\tan^{-1}\left(\frac{1}{2}\right)$.

Sol. We know that in the case of a cone of height h and semi vertical angle α . $A =$ M.I. about $OX = \frac{M}{20}(3a^2 + 2h^2)$, where a is the radius of the base ,

$B =$ M.I. about $OY = \frac{M}{20}(3a^2 + 2h^2)$,

$C =$ M.I. about $OZ = \frac{13}{10}Ma^2$.



Also
$D = F = 0, E = Mah/4$, since $D = F = 0$; the axis OY as shown in figure is the principal axis at O . Let one of the other two

principal axes make an angle θ with OX. Then in that case we have

$$\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{1}{2}Mah}{\frac{13}{10}Ma^2 - \frac{1}{20}M(3a^2+2h^2)} = \frac{10ah}{23a^2-2h^2} \quad ...(1)$$

Now if the axis OG is to pass through G the C.G. of the cone, then

$$\tan\theta = \frac{GN}{ON} = \frac{(h/4)}{a} = h/(4a)$$

$$\text{Now } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{h/2a}{1-(h^2/16a^2)} = \frac{8ah}{16a^2-h^2} \quad ...(2)$$

From (1) and (2), we get $\dfrac{10ah}{23a^2-2h^2} = \dfrac{8ah}{16a^2-h^2} \Rightarrow h = 2a$

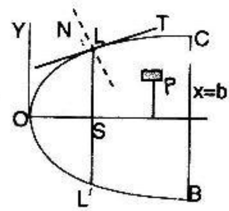
But $\tan\alpha = a/h = a/2a = \frac{1}{2} \Rightarrow \alpha = \tan^{-1}(\frac{1}{2})$

$2\alpha = 2\tan^{-1}(\frac{1}{2})$ [2α being the vertical angle of cone].

Ex.40. *A uniform lamina is bounded by a parabolic arc of latus rectum $4a$, and a double ordinates at a distance b from the vertex. If $b = (a/3)(7+4\sqrt{7})$. Show that two of the principal axes at the end of a latus rectum are the tangent and normal there.*

Sol. Let $y^2 = 4ax$ be the parabola and LSL' the latus rectum. The co-ordinates of L are $(a, 2a)$. Now slope of the tangent at

$L = \left(\dfrac{dy}{dx}\right)_{(a,2a)}$. But $\dfrac{dy}{dx} = \sqrt{\left(\dfrac{a}{x}\right)}$



$\therefore$ Slope $= \left(\dfrac{dy}{dx}\right)_{(a,2a)} = \sqrt{\left(\dfrac{a}{a}\right)} = 1$

$\therefore$ Equation of the tangent at L is given by

$y - 2a = 1(x-a) \Rightarrow y - x - a = 0$

Again slope of the normal at $L = -1$

$\therefore$ Equation of the normal is

$y - 2a = -1(x-a) \Rightarrow y + x - 3a = 0$

Consider an element $\delta x\,\delta y$ at the point $P \equiv (x, y)$, then distance of the element from the tangent at $L = \dfrac{y-x-a}{\sqrt{2}} = PK$ and distance of the element from the normal at $L = \dfrac{y+x-3a}{\sqrt{2}} = PH$.

If the tangent and normal at L are the principal axes, then the product of inertia about these axes vanish. Now P. I. about the tangent and normal at L.

$$= \int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} \left(\frac{y+x-a}{\sqrt{2}}\right) . \left(\frac{y-x-3a}{\sqrt{2}}\right) dx\, dy$$

$$= \frac{1}{2}\int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} (y^2 - x^2 + 2ax - 4ay + 3a^2)\, dx\, dy$$

$$= \int_0^b \int_0^{2\sqrt{(ax)}} (y^2 - x^2 + 2ax + 3a^2)\, dx\, dy - \frac{1}{2}\int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} 4ay\, dx\, dy$$

$$= \int_0^b \left\{ \left| (2ax - x^2 + 3a^2)y \right|_0^{2\sqrt{(ax)}} + \left| \frac{y^3}{3} \right|_0^{2\sqrt{(ax)}} \right\} dx - 0$$

$$= \int_0^b [\tfrac{1}{3}\ \{2\sqrt{(ax)}\}^3 + (3a^2 + 2ax - x^2)\, 2\sqrt{(ax)}\ ]dx$$

$$= \int_0^b [\tfrac{8}{3} a^{3/2} x^{3/2} + 4a^{3/2} x^{3/2} + 6a^{5/2} x^{1/2} - 2a^{1/2} x^{5/2}]\, dx$$

$$= \frac{4}{21} a^{1/2} b^{3/2} (14ab + 21a^2 - 3b^2)$$

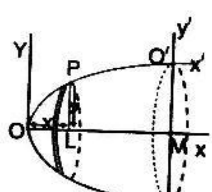
If this P.I. is zero then , we have $21a^2 + 14ab - 3b^2 = 0$

$$\Rightarrow 3b^2 - 14ab - 21a^2 = 0 \Rightarrow b = \frac{14a \pm \sqrt{[196a^2 + 12 \times 21a^2]}}{6}$$

$= \frac{1}{3} a\ (7 + 4\sqrt{7})$, which is equal to the given value of b.

Ex. 41. *The length of the axis of a solid parabola of revolution is equal to the latus rectum of the generating parabola . Prove that one principal axis at a point on the circular rim meets the axis of revolution at an angle* $\frac{1}{2}\tan^{-1}(\frac{2}{3})$.

Sol. Let the equation of the generating parabola be $y^2 = 4ax$.
Length of the axis $OM = 4a$ given.
Let O' be a point on the circular rim . Also let $O'X'$ and $O'Y'$ the new axes . Now Consider a circular disc of breath δx at a distance x from the vertex O' then



A = M.I. about $O'X'$

$$= \int_0^{4a} (\pi y^2 \rho dx)\ (y^2/2 + O'M^2)$$

Evidently
$O'M^2 = 4a \,.\, 4a$ (giving) $O'M = 4a$

$$\therefore A = \pi\rho \int_0^{4a} 4ax\,[\,2ax + (4a^2)\,]dx$$

$$= 8a^2\pi\rho \left[\, x^3/3 + 8a\,x^2/2 \right]_0^{4a} = \tfrac{1}{3}.\,8.64.4a^5\pi\rho$$

$B =$ M.I. about $O'Y'$

$$= \int_0^{4a} \pi y^2\,\rho\,dx \left[\frac{y^2}{4} + LM^2\right] = \pi\rho \int_0^{4a} 4ax\,[ax + (4a - x)^2\,]dx$$

$$= 4a\pi\rho \int_0^{4a} (16a^2x^2 - 7ax^2 + x^3)dx$$

$$= 4a\pi\rho \left[\, 8a^2x^2 - (7/3)\,ax^3 + \tfrac{1}{4}x^4 \right]_0^{4a} = (2/3) \times 4 \times 64a^5\pi\,\rho$$

and $F =$ product of inertia about $O'X'$ and $O'Y'$

$$= \int_0^{4a} \pi y^2 \rho d\,x\,LM.O'M = \pi\rho \int_0^{4a} 4ax\,(4a - x)\,4adx$$

$$= 16a^2\pi\rho \int_0^{4a} (4ax - x^2)\,dx = \tfrac{1}{3} \times 16 \times 32a^5\pi\rho$$

Let the principal axis at O' make an angle ϕ with $O'X'$, then we have

$$\tan 2\phi = \frac{2F}{B - A} = \frac{2 \times \frac{1}{3} \times 16 \times 32a^5\pi\rho}{\frac{2}{3} \times 4 \times 64 \times a^6\pi\rho - \frac{1}{3} \times 8 \times 64 \times 4a^5\pi\rho} = -\frac{2}{3}$$

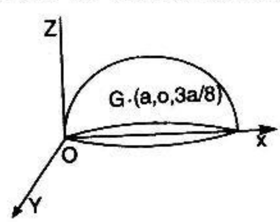
$\Rightarrow \phi = \frac{1}{2}\tan^{-1}(\frac{2}{3})$ (numerically).

Ex. 42. *Show that one of the principal axes at a point on the circular rim of the solid hemisphere is inclined at an angle* $\tan^{-1}(\frac{1}{3})$ *to the radius through the point .*

Sol. Let O be the point on the base of the solid hemisphere . Let OX, OY, OZ be the axes of reference , then we know in this case

$A = \frac{2}{5}Ma^2,\; B = \frac{7}{5}Ma^2,$

$C = \frac{7}{5}Ma^2, D = F = 0\;, E = \frac{3}{8}Ma^2$



If θ is angle which the principal axis makes with OX , then we have

$$\tan 2\theta = \frac{2E}{C - A}$$

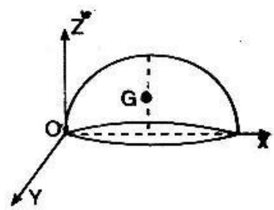
$$= \frac{(\frac{3}{4})\, Ma^2}{(\frac{7}{5})\, Ma^2 - (\frac{2}{5})\, Ma^2} = \frac{3}{4}$$

$$\Rightarrow \frac{2 \tan\theta}{1 - \tan^2\theta} = \frac{3}{4} \Rightarrow (\tan\theta + 3)\,(3\tan\theta - 1) = 0$$

$\Rightarrow \tan\theta = (\frac{1}{3})$, the other value of θ being inadamissible.

Ex. 43. *Show that one of the principal axes at any point on the edge of the circular base of a thin hemispherical shell is inclined at an angle $\pi/8$ to the radius through the point .*

Sol. Let O be the point on the circular rim of the hemispherical shell. Let OX , OY , OZ as shown in the fig. be the axes of reference , then we have



$A =$ M.I. about $OX = \frac{2}{3} ma^2$

$B =$ M.I. about $OY =$ M.I. about a line through the C.G. parallel to OY

$$= \frac{2}{3} Ma^2 + Ma^2 = \frac{5}{3} Ma^2 .$$

Similarly $C = \frac{5}{3} Ma^2$. If G is the C.G. of the shell then co-ordinates of G are $(a, 0, a/2)$. Then we have

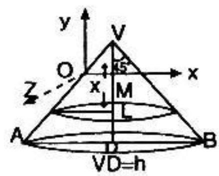
$D = F = 0$ and $E = Ma\ \frac{1}{2} a = \frac{1}{2} Ma^2$

Now if θ is the angle which the principal axis makes with OX then we have $\tan 2\theta = \frac{2F}{C - A} = \frac{Ma^2}{(\frac{5}{3})\, Ma^2 - (\frac{2}{3})\, Ma^2} = 1 = \tan(\pi/4)$

$\therefore 2\theta = (\pi/4) \Rightarrow \theta = (\pi/8)$.

Ex. 44. *If the vertical angle of the cone is 90°. The point at which a generator is a principal axis divides the generator in the ratio 3 : 7 .*

Sol. Consider $\Delta\ VAB$ as the section of the cone through the generator VA and *axis* VD of the cone. In this plane section let OX and OY be the axes of x and y and a line OZ perpendicular to this section as *z-axis*. Obviously the *z*-coordinates of C.G. is zero. $D = E = 0 \Rightarrow$ *z-axis* is a principal axis at O .



Now M.I. of the cone about OZ

$$= \int_0^h \pi x^2 \rho\ [\tfrac{1}{4} x^2 + ML^2]\, dx$$

$$= \int_0^h \pi x^2 \rho\, [\tfrac{1}{4} x^2 + (x - VM)^2\, dx$$

$$= \pi\rho \int_0^h [\tfrac{5}{4}x^4 - 2VMx^3 + VM^2x^2]\,dx$$

$$= \pi\rho\left[\frac{h^5}{4} - \tfrac{1}{2}VMh^4 + \tfrac{1}{3}VM^2h^3\right] = A\ (say)$$

and M.I. about $OY = \int_0^h \pi x^2 \rho\left[\frac{x^2}{2} + OM^2\right]dx$

$$= \pi\rho\int_0^h\left[\frac{x^4}{2} + VM^2x^2\right]dx \qquad \left[\because \frac{OM}{VM} = \tan\pi/4 = 1\right]$$

$$= \pi\rho\left[\frac{1}{10}h^5 + \frac{1}{3}VM^2h^3\right] = B\ (say)$$

But VA is principal axis and it makes an angle $\frac{1}{4}\pi$ with *x-axis* at the point O. Thus we have

$$\tan 2\theta = \frac{2F}{B-A} \Rightarrow \tan 2.\frac{\pi}{4} = \frac{2F}{B-A}$$

$$\Rightarrow \tan\frac{\pi}{2} = \frac{2F}{B-A} \text{ but } \tan\frac{\pi}{2} = \infty \therefore B - A = 0 \Rightarrow A = B$$

$$\therefore\ \pi\rho\left[\frac{h^5}{4} - \tfrac{1}{2}VMh^4 + \tfrac{1}{3}VM^2h^3\right] = \pi\rho\left[\tfrac{1}{10}h^5 + \tfrac{1}{2}VM^2h^3\right]$$

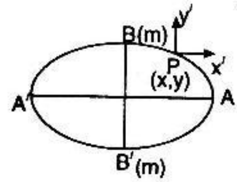
$\Rightarrow \frac{VM}{h} = \frac{3}{10}$. By similarity of triangles, we have

$$\frac{VO}{VA} = \frac{3}{10} \text{ or } \frac{VO}{VA - VO} = \frac{3}{7}, \therefore \frac{VO}{OA} = \frac{3}{7}.$$

Ex.45. *Two particles each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse if* $\frac{m}{M} = \frac{5}{8}\cdot\frac{e^2}{1-2e^2}$

Sol. Let P be a point on the circumference of the ellipse $x^2/a^2 + y^2/b^2 = 1$ whose co-ordinates are (x, y).

Let PX' and PY' be a set of parallel axes through P. Let particles, each of mass m be placed at the extremities B and B' of the minor axis.



$A =$ moment of the inertia of the elliptic lamina and the two particles about PX'

$$= M(\tfrac{1}{4}b^2 + y^2) + m\{(b-y)^2 + (b+y)^2\}$$

$$= \tfrac{1}{4}M(b^2 + 4y^2) + 2m(b^2 + y^2)$$

$B =$ M.I. of the elliptic lamina and the two particles about PY'

$$= M\left(\tfrac{1}{4}a^2 + x^2\right) + m\left(x^2 + x^2\right) = \tfrac{1}{4}M\left(a^2 + 4x^2\right) + 2mx^2$$

$F =$ Product of inertia about PX' and PY'

$$= M(-x)(-y) + m\{(-x)(b-y) + (-x)(-b-y)\}$$

$$= (M + 2m)xy.$$

$$\therefore \tan 2\theta = \frac{2F}{B-A}$$

$$= \frac{2(M+2m)xy}{\{(M/4)(a^2 + 4x^2) + 2mx^2\} - \{(M/4)b^2 + 4y^2) + 2m(b^2 + y^2)\}}$$

$$= \frac{8(M+2m)xy}{M(4x^2 - 4y^2 + a^2 - b^2) + 8m(x^2 - y^2 - b^2)} \qquad ...(1)$$

Now equation of the tangent at P is $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$

$\therefore$ Slope of the tangent $= -\dfrac{b^2x}{a^2y}$.

If the tangent at P is the principal axis, then

$\tan\theta = -(b^2x/a^2y)$

$$\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{-(2b^2x/a^2y)}{1-(b^4x^2/a^4y^2)} = \frac{-2xya^2b^2}{a^4y^2 - b^4x^2} \qquad ...(2)$$

Hence (1) and (2) $\Rightarrow \dfrac{8(M+2m)xy}{M(4x^2 - 4y^2 + a^2 - b^2) + 8m(x^2 - y^2 - b^2)}$

$$= \frac{-2xya^2b^2}{a^4y^2 - b^4x^2} \Rightarrow 5(a^2 - b^2)M = 8m(2b^2 - a^2)$$

$$\Rightarrow \frac{m}{M} = \frac{5}{8}\cdot\frac{a^2 - b^2}{2b^2 - a^2} = \frac{5}{8}\cdot\frac{e^2}{1-2e^2}$$

Ex. 46. *A uniform lamina bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has an elliptic hole (semi-axes c, d) in it whose major axis lies in the line $x = y$, the centre being at a distance r from the origin, prove that if one of the principal axis at the point (x,y) makes an angle θ with x-axis then*

$$\tan 2\theta = \frac{8abxy - cd\{4(x\sqrt{2} - r)(y\sqrt{2} - r) - (c^2 - d^2)\}}{ab\{4(x^2 - y^2) + a^2 - b^2\} - 2cd\{(x\sqrt{2} - r)^2 - (y\sqrt{2} - r)^2\}}$$

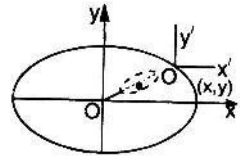
Sol. Consider C to be the centre of the elliptic hole whose **major axis** lies along the line $y = x$ *i.e.* $y = (\tan 45)\,x$. Let O' be any **point** (x, y) and let $O'X'$, $O'y'$ be the set of parallel axes through o' **then**

$oc = r$ and $\angle COX = \pi/4$

Mass of the elliptic plate = $\pi ab\,\rho$.

Its M.I. about $O'X'$

$$= \pi ab\,\rho\,(b^2/4 + y^2).$$



Mass of the elliptic hole = $\pi c\,d\,\rho$

($\because$ its semi axes are c and d)

Its M.I. about $O'X'$

$$= \pi cd\rho \int \left\{(d^2/4)\cos^2 45 + (c^2/4)\sin^2 45 + \left(y - \frac{r}{\sqrt{2}}\right)^2\right\}$$

$\therefore A =$ M.I. of the remainder about $O'X'$

$$= \pi\rho ab\left(\frac{b^2}{4} + y^2\right) - \frac{1}{8}\pi\,\rho cd\,\{d^2 + c^2 + 4(\sqrt{2}y - r)^2\} \qquad ...(1)$$

Now M.I. of elliptic plate about $O'Y' = \pi ab\,\rho\left(\frac{a^2}{4} + x^2\right)$

M.I. of the remainder hole about $O'Y'$

$$= \pi\rho ab\left[\frac{d^2}{4}\sin^2 45 + \frac{c^2}{4}\cos^2 45 + \left(x - \frac{r}{\sqrt{2}}\right)^2\right]$$

M.I. of the remainder about $O'Y'$

$$= \pi\rho ab\left[\frac{a^2}{4} + x^2\right] - \frac{1}{8}\pi\rho cd\,\{d^2 + c^2 + 4(\sqrt{2}x - r)^2\} = B \text{ (say)}$$

Further product of inertia of the elliptic plate about $(O'X', O'Y') = \pi ab\,\rho xy$

and product of inertia of the elliptic hole about $O'X', O'Y'$

$$= \pi cd\rho\left\{\frac{1}{2}\,\frac{d^2 - c^2}{4}\sin 90 + \left(x - \frac{r}{\sqrt{2}}\right)\left(x - \frac{r}{\sqrt{2}}\right)\right\}$$

$$= \frac{1}{8}\pi cd\,\rho\,\{d^2 - c^2 + 4(x\sqrt{2} - r)(y\sqrt{2} - r)\}$$

$\therefore$ P.I. of the remainder about $O'X', O'Y'$

$$= \pi ab\,\rho\,xy - \frac{1}{8}\pi cd\,\rho\,\{d^2 - c^2 + 4(x\sqrt{2} - r)(y\sqrt{2} - r)\} = F$$

(say).

Let θ be the angle, which the principal axis makes with *x-axis* then we have $\tan 2\theta = \dfrac{2F}{B - A}$

$$= \frac{8ab\,xy - cd\,\{4(x\sqrt{2} - r)(y\sqrt{2} - r) - (c^2 - d^2)\}}{ab\,\{4(x^2 - y^2) + a^2 - b^2\} - cd\,\{2(x\sqrt{2} - r)^2 - 2(y\sqrt{2} - r)^2\}}$$

0·16. Principal Moments : *Moments of inertia of any body about its principal axes at any point are called its principal moments at that point.*

The equation of ellipsoid at any point is given by

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = MK^4$

This equation when referred to principal axes as co-ordinate axes takes the from as $A'x^2 + B'y^2 + C'z^2 = MK^4$, where A' B', C' are principal moments and are the roots of the cubic-equation given below :

$$\begin{vmatrix} A-\lambda & H & G \\ H & B-\lambda & F \\ G & F & C-\lambda \end{vmatrix} = 0. \qquad \textbf{Reduction Cubic.}$$

Ex.47. *Three rods AB, BC, CD each of mass m and length $2a$ are such that each is perpendicular to the other two. Show that the principal moments of inertia at the centre of mass are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.*

Sol. Draw a line BY, parallel to CD and let BA, BY and BC be the axes of x, y and z respectively.

Let L, M, N be the mid points of the rod. Then their co-ordinates are given by $(a, 0, 0)$, $(0, 0, a)$, $(0, a, 2a)$.

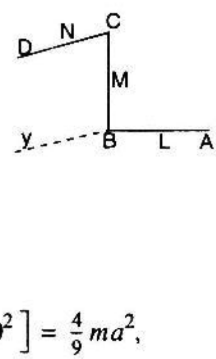
Further let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of $C.G.$ 'G' of the three rods then we easily obtain

$$\bar{x} = \frac{m.a + m.0 + m.0}{m+m+m} = \frac{a}{3}, \bar{y} = \frac{m.0 + m.0 + m.a}{m+m+m} = \frac{a}{3},$$

$$\bar{z} = \frac{m.0 + m.a + m.2a}{m+m+m} = a.$$

i.e. $(\bar{x}, \bar{y}, \bar{z}) = (a/3, a/3, a)$.

Now GX, GY, GZ be the set of parallel axes through G, then the new co-ordinates of L, M, N referred to G as origin are $(2a/3, -a/3, -a)$, $(-a/3, -a/3, 0)$, $(-a/3, 2a/3, a)$



$\therefore$ M.I. of AB about GX

$$= m\left[\left(\frac{-a}{3}\right)^2 + (-a)^2\right] = \frac{10}{9}ma^2$$

$$\text{M.I. of } BC \text{ about } GX = \frac{1}{3}ma^2 + m\left[\left(\frac{-a}{3}\right)^2 + 0^2\right] = \frac{4}{9}ma^2,$$

$$\text{M.I. of } CD \text{ about } GX = \frac{1}{3}ma^2 + m\left[\left(\tfrac{2}{3}a\right)^2 + a^2\right] = \frac{16ma^2}{9}$$

$\Rightarrow$ M.I. of three rods **about** $GX = ma^2\left[\frac{16}{9} + \frac{4}{9} + \frac{10}{9}\right] = \frac{10}{3}ma^2$

$= A_1$, say .

M.I. of the three rods **about** GY

$$= \frac{1}{3}ma^2 + m\left[\left(\tfrac{2}{3}a\right)^2 + (-a)^2\right] + \frac{1}{3}ma^2 + m\left[(-a/3)^2 + 0^2\right]$$

$$+ m\left[(-a/3)^2 + a^2\right]$$

$= \frac{10}{3} ma^2 = B_1$, say .

Similarly M.I. of the three rods about GZ

$$= \frac{1}{3} ma^2 + [(2a/3)^2 + (a/3)^2] + m[(-a/3)^2 + (-a/3)^2] + \frac{1}{3} ma^2$$
$$+ m[(-a/3)^2 + (2a/3)^2]$$

$= 2ma^2 = G$, say. Now if D_1, E_1, F_1 are products of inertia , about the parallel axes through G, then we have

$D_1 = \Sigma m y_1 z_1 = m(-a)(-a/3) + m(-a/3)\,0 + m(\frac{2}{3}a)\,a = -ma^2$

$E_1 = \Sigma m z_1 x_1 = m(-a)(\frac{2}{3}a) + m(0)(-a/3) + m(a)(-a/3) = -ma^2$

$F = m(\frac{2}{3}a)(-a/3) + m(-a/3)(-a/3) + m(-a/3)(2a/3) = -\frac{1}{3}ma^2$

$(\because\ F_1 = \Sigma m x_1 y_1)$

Hence the momental ellipsoid at G is given by

$$\frac{10}{3} ma^2x^2 + \frac{10}{3} ma^2y^2 + 2ma^2z^2 - 2ma^2\,yz + 2ma^2\,zx + 2.\frac{1}{3}\,ma^2xy$$
$$= 3\,mK^4$$

$\Rightarrow \frac{1}{3} ma^2 (10x^2 + 10y^2 + 6z^2 - 6yz + 6zx + 2xy) = 3\,mK^4$

Whence the discriminiating cubic is

$$\lambda^3 - (a+b+c)\lambda^2 + (ab+bc+ca-f^2-g^2-h^2)\lambda$$
$$-(abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

Reducing $10x^2 + 10y^2 + 6z^2 - 6yz + 6zx + 2xy$ by means of reducing cubic , we get

$$\lambda^3 - (10+10+6)\lambda^2 + \lambda(100+60+60-9-9-1) - 396 = 0$$

$\Rightarrow \lambda^3 - 26\lambda^2 + 201\lambda - 396 = 0$

$\Rightarrow (\lambda - 3)\,\lambda - 11\,(\lambda - 12) = 0$

$\Rightarrow \lambda = 3, 11, 12$. Hence the equation of the momental ellipsoid referred to principal axes through G is $\frac{1}{3} ma^2 (3x^2 + 11y^2 + 12z^2) = 3mK^4$

$\Rightarrow ma^2x^2 + \frac{11}{3} ma^2y^2 + 4ma^2z^2 = 2mK^4 =$ Constant.

Or in other words , the principal moments are the coefficients of x^2, y^2, z^2 i.e. $ma^2, \frac{11}{3}ma^2, 4ma^2$

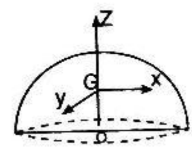
Ex.48. *Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are*

$$\frac{5}{12}Ma^2,\ \frac{5}{12}Ma^2,\ \frac{2}{3}Ma^2.$$

Sol. Consider GX, GY and GZ as co-ordinate axes where G is the C.G.of the body. Now M.I. of the hemispherical shell

about $GX = \frac{2}{3}Ma^2 - M(GO)^2$

$$= \frac{2}{3}Ma^2 - \frac{Ma^2}{4} = \frac{5}{12}Ma^2 = A \text{ (say)}$$



($\because$ G bisects the central radius OZ)

Similarly M.I. about $GY = \frac{5}{12}Ma^2 = B$ (say)

and M.I. about $GZ = \frac{2}{3}Ma^2 = C$ (say)

Again $D = E = F = O \Rightarrow$ that GX, GY, GZ are principal axes at G . So the principal moments at the C.G. are

$$\frac{5}{12}Ma^2, \frac{5}{12}Ma^2, \frac{2}{3}Ma^2.$$

Ex.49. *The principal axes at the C.G. being the axes of reference , obtain the equation of the ellipsoid at the point (p, q, r) and show that the principal M.I., at this point are the roots of*

$$\begin{vmatrix} \{(I-A)/M\} - q^2 - r^2 & pq & rp \\ pq & \{(I-B)/M\} - r^2 - p^2 & qr \\ rp & qr & \{(I-C)/M\} - p^2 - q^2 \end{vmatrix} = 0$$

where I, M, A, B, C have their usual meanings.

Sol. Let A', B', C' be the moments of inertia about parallel axes through (p, q, r) and D', E', F', the products of inertia about them , then we easily obtain

$A' = A + M(q^2 + r^2)$, $B' = B + M(r^2 + p^2)$, $C' = C + M(p^2 + q^2)$,

$D' = D + Mqr = Mqr$, $E' = Mrp$, $F' = Mpq$

$\therefore$ Equation of the momental ellipsoid at (p, q, r) will be

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{Constant} .$$

$$\Rightarrow \Sigma[\{(A/M) + q^2 + r^2\}x^2 - 2\Sigma\, qryz = K^4 \text{ (say)} \qquad \text{....(1)}$$

Now if (l, m, n) are the direction cosines of one of the principal axes of the momental ellipsoid (1) and R the length of principal axis then the equation of the tangent plane at the end of the principal axis will be

$lx + my + nz = R.$ ...(2)

If I is the M.I. about that principal axis then $I = (MK^4/R)$. ...(3)

Co-ordinates of the end of the principal axis will be given by

(lR, mR, nR)

Equation of tangent plane to (1) at the end of the principal axis is also given by $x\dfrac{\partial F}{\partial \alpha}+y\dfrac{\partial F}{\partial \beta}+z\dfrac{\partial F}{\partial \gamma}+t\dfrac{\partial F}{\partial t}=0$

where $\alpha=lR,\ \beta=mR,\ \lambda=nR$

$\Rightarrow x[\{(A/M)+q^2+r^2\}lR-pqmR-pmR]+y[\{(B/M)+r^2+p^2\}mR$
$-qmR-pq\,lR]+z[\{(C/M)+p^2+q^2\}nR-rplR-qrmR]=K^4$ ...(4)

Equation (4) and (2) represent the same tangent plane, therefore comparing the coeffcients of x, y, z and constant term, we get

$$\frac{\{(A/M)+q^2+r^2\}\,lR-pqmR-pmR}{l}$$
$$=\frac{\{(B/M)+r^2+p^2\}\,mR-pq\,lR-qmR}{m}$$
$$=\frac{\{(C/M)+p^2+q^2\}\,nR-rplR-qrmR}{n}=\frac{K^4}{R}=\frac{RI}{M}$$

$$\left.\begin{aligned}\therefore\ lR\,[\{(I-A)/M\}-q^2-r^2]+mR\,.\,pq+nR\,.\,pr&=0\\ lR\,.\,pq+mR\,[\{(I-B)/M\}-r^2-p^2]-nR\,.\,qr&=0\\ lR\,.\,rp+mR\,.\,qr+nR\,[\{(I-C)/M\}-p^2-q^2]&=0\end{aligned}\right\}$$

Eliminating lR, mR, nR from the equation (5) we get

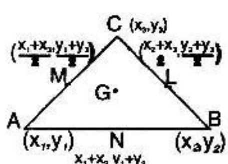
$$\begin{vmatrix}\{(I-A)/M\}-q^2-r^2 & pq & rp\\ pq & \{(I-B)/M\}-r^2-p^2 & qr\\ rp & qr & \{I-C)/M\}-p^2-q^2\end{vmatrix}=0.$$

Ex.50. *Prove that the principal radii of gyration at the C.G. of a triangle are the roots of the equation* $x^4-\dfrac{a^2+b^2+c^2}{30}x^2+\dfrac{\Delta^2}{108}=0$, *where Δ is area of the triangle.*

Sol. Let G be the $C.G.$ of the $\Delta\ ABC$ and let the principal axes through G be taken as the co-ordinate axes.

Let the co-ordinates of A, B, C be $(x_1, y_1); (x_2, y_2); (x_3, y_3)$; respectively then the co-ordinates of the mid point L, M, N, are

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),\left(\frac{x_3+x_1}{2},\frac{y_3+y_1}{2}\right),$$
$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).$$



Now co-ordinates of G are also given as $\left(\dfrac{x_1+x_2+x_3}{3},\dfrac{y_1+y_2+y_3}{3}\right)$ and they are evidently $(0,0)$.

$\therefore x_1 + x_2 + x_3 = 0,\ y_1 + y_2 + y_3 = 0$ ...(1)

$\Rightarrow (x_1 + x_2 + x_3)^2 = 0 \Rightarrow x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1 = 0$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 = -2(x_1x_2 + x_2x_3 + x_3x_1)$

Similarly we get $y_1^2 + y_2^2 + y_3^2 = -2(y_1y_2 + y_2y_3 + y_3y_1)$

The length of the side $AB = c = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$

or $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

Similarly $\left.\begin{aligned} a^2 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ b^2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \end{aligned}\right\}$ ...(2)

$\therefore a^2 + b^2 + c^2 = 2(x_1^2 + x_2^2 + x_3^2) + 2(y_1^2 + y_2^2 + y_3^2) - 2(x_1x_2 + x_2x_3 + x_3x_1)$
$- 2(y_1y_2 + y_2y_3 + y_3y_1)$

$= 3(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2)$

Again a triangular lamina of mass m is equimomental to three particles of mass $(m/3)$ each placed at the mid points L, M, N.

$\therefore A =$ M.I. of three particles about *x-axis*

$= \frac{m}{3}\left[\left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_2 + y_3}{2}\right)^2 + \left(\frac{y_3 + y_1}{2}\right)^2\right]$

$= \frac{m}{12}[2(y_1^2 + y_2^2 + y_3^2) + 2(y_1y_2 + y_2y_3 + y_3y_1)] = \frac{m}{12}(y_1^2 + y_2^2 + y_3^2)$

$B =$ M.I. about y-axis

$= \frac{m}{3}\left[\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{x_2 + x_3}{2}\right)^2 + \left(\frac{x_3 + x_1}{2}\right)^2\right]$

$= \frac{m}{12}[2(x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_2x_3 + x_3x_1)] = \frac{m}{12}(x_1^2 + x_2^2 + x_3^2)$.

Further we have taken the co-ordinate axes as the principal axes, therefore the P.I. about these axes is zero *i.e.*

$$\frac{m}{3}\left[\frac{x_1 + x_2}{2} \cdot \frac{y_1 + y_2}{2} + \frac{x_2 + x_3}{2} \cdot \frac{y_2 + y_3}{2} + \frac{x_3 + x_1}{2} \cdot \frac{y_3 + y_1}{2}\right] = 0$$

$\Rightarrow (x_1 + x_2)(y_1 + y_2) + (x_2 + x_3)(y_2 + y_3) + (x_3 + x_1)(y_3 + y_1) = 0$

$\Rightarrow 2(x_1y_1 + x_2y_2 + x_3y_3) + x_1(y_2 + y_3) + x_2(y_3 + y_1) + x_3(y_1 + y_2) = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 + (y_1 + y_2 + y_3)(x_1 + x_2 + x_3) = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$ ...(3)

Now suppose $A = mK_1^2, B = mK_2^2$, then we have

$m(K_1^2 + K_2^2) = A + B$

$A + B = \frac{m}{12}[(x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2)]$

$$= \frac{m}{12} \cdot \frac{a^2+b^2+c^2}{3} = \frac{m\,(a^2+b^2+c^2)}{36}$$

Again $m^2 K_1^2 K_2^2 = A \times B = \frac{m^2}{144}(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)$

$$= \frac{m^2}{144}[x_1^2y_1^2+x_2^2y_2^2+x_3^2y_3^2+ x_1^2(y_2^2+y_3^2)+x_2^2(y_3^2+y_1^2)+ x_3^2(y_1^2+y_2^2)]$$

$$= \frac{m}{144}[(x_1y_1+x_2y_2+x_3y_3)^2+(x_1y_2-x_2y_1)^2+(x_1y_3-x_3y_1)^2 + (x_2y_3-x_3y_2)^2]$$

$$= \frac{m^2}{144}[(x_1y_2-x_2y_1)^2+(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2] \quad \text{[Using (3)]}$$

$$= \frac{m^2}{144}\left[\frac{4\Delta^2}{9}+\frac{4\Delta^2}{9}+\frac{4\Delta^2}{9}\right] = \frac{m^2\Delta^2}{108}$$

$[\Delta = \frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\} = \frac{3}{2}(x_1y_2-x_2y_2),$
when $x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0]$

Let K_1^2, K_2^2 are the roots of the equation $x^4-(K_1^2+K_2^2)x^2+K_1^2\times K_2^2=0$

$$\Rightarrow x^4 - \frac{a^2+b^2+c^2}{36}x^2 + \frac{\Delta^2}{108} = 0$$

Supplementary Problems

1. Show that the moment of inertia of a semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2\left(\frac{5}{3}-\frac{8}{3}\pi\right)$, where a is the radius and M is the mass of the lamina.

2. Show that the M.I. of the paraboloid of revolution about its axis is $\frac{M}{3}\times$ the square of the radius of its base.

3. Find the product of inertia of a semi-circular wire about diameter and tangent at its extremity.

4. Find the moment of inertia of a truncoted cone about its axis, the radii of its ends being a and b.

5. Show that the moment of inertia of a parabolic area (of latus rectum 4a) cut off by an ordinate of distance h from the vertex is $\frac{3}{7}Mh^2$ about the tangent at the vertex and $\frac{4}{5}Mah$ about the axis.

6. Show that M.I. of a rectangle of mass M and sides $2a, 2b$ about a diagonal is

$\frac{2M}{3} \cdot \frac{a^2b^2}{a^2 + b^2}$.

7. A closed shell of total mass M , made of thin uniform sheet metal is in the form of a right circular cone of slant height l and base radius **r**. Prove that M.I. of the shell about its axis of symmetry is $\frac{1}{2}Mr^2$ and that about a line through the vertex perpendicular to the axis is $\frac{M}{4}(2l^2 + 2rl - 3r^2)$

8. Find the momenental ellipsoid at any point O of a material straight rod PQ *of mass* M and *length* $2a$.

9. $ABCD$ is a uniform parallelogram of mass **m**. At the middle points of the four sides are placed particles , each equal to $\frac{m}{6}$ and at the intersection of the diagonals a particle of mass $\frac{m}{3}$. Show that these five particles and the parallelogram are equimomental system.

10. Show that a uniform rod of mass m, is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and at its middle point the masses of the particles being $\frac{m}{6}, \frac{m}{6}$ and $\frac{2}{3}m$.

11. Show that there is a momental ellipse at the centre of inertia of a uniform triangle which touches the side of the triangle at the middle points.

12. At the vertex C of a triangle ABC , which is right angle at C , the principal axes are, a perpendicular to the plane and two others inclined to the sides at an angle

$\frac{1}{2}\tan^{-1}\frac{ab}{a^2 - b^2}$.

13. A uniform rectangular plate whose sides are of lengths $2a, 2b$ has a portion cut out in the form of a square whose centre is the centre of rectangle and whose mass is half the mass of the plate. Show that the axes of greastes and least moment of inertia at a corner of the rectangle makes angle $\theta, (\pi/2) + \theta$ with a sidewhere $2\theta = \frac{6ab}{5(a^2 - b^2)}$.

14. ABC is a triangular area and AD is perpendicular to BC and AE is a medim, O is the middle point of DE, show that B is a principal axis of the triangle at O .

15. A uniform circular solid cone of semi-vertical angle α and height h is cut in half by a plane through its axis. Show that the principal moments of inertia at the vertex for one of the halves are $\frac{3}{5}Mh^2(1 + \frac{1}{4}\tan^2\alpha)$ and $\frac{3}{10}Mh^2(1 + \frac{3}{4}\tan^2\alpha)$

D'Alembert's Principle

1·.01. Motion of a particle and a rigid body.

Motion of a particle . The motion of a single particle under the action of given forces is determined by the Newton's second law of motion, which states that the rate of change of momentum in any direction is proportionate to the applied force in that direction .From this law it is deduced that $P = mf$ where f is the acceleration of particle m in the direction of the force P. This mf is called the *effective force* and P the applied force . If (x, y, z) be the co-ordinates of a moving particle of mass m at any time t referred to three rectangular axes fixed in space and X, Y, Z, be the components of the forces acting on the particle in directions parallel to the axes of x, y, z respectively , the motion is found by solving the following three simultaneous equations.

$$m\ddot{x} = X,\ m\ddot{y} = Y,\ m\ddot{z} = Z$$

Motion of a rigid body..If the rigid body is considered as the collection of material particles. we can write the equation of motion of all particles according to the above law but here the external forces include, over and above the applied forces, the mutual actions between the particles. As regards mutual actions between any two particles we assume that (1). The mutual action between two particles is along the line which joins them (2). The action and reaction beetwen them are equal and opposite. In order to find the motion of a rigid body or bodies, D' Alembert gave a method by which all the necessary equations may be obtained of the body. In doing so only the following consequence of the laws of motion is kept in view :

The intemal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

1· 02. Impressed and effective forces.

Impressed forces. The external forces acting on a rigid body are termed as impressed forces *e.g.* weight of the body. If the body is tied to the string, then tension in the string is the impressed force on the body.

Effective forces. When a rigid body is in motion, each particle of it is acted upon by the external impressed forces and also by the molecular reactions of the other particles. If we assume that particle is separated from the rest of the body, and all these forces are removed, there is some force which would make it to move in the same direction as before. This force is termed as *effective force* on the particle, it is the resultant of the impressed and molecular forces on the particle.

1.03. D ' Alemberts Principle. *The reversed effective forces acting on each particle of the body and the external forces of the system are in equilibrium.*

[Meerut 1995, 88, 90, Agra 89, Nagpur 84, Delhi Hons 82]

Let (x, y, z) be the co-ordinates of a particle of mass m, of a rigid body at any time t. Let f be the resultant of its component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$, so that the effective force on m is mf. Let F be the resultant of the impressed forces on m and R be the resultant of mutual actions, then mf is resultant of F and R.

In case mf is reversed, the mf (reversed), F and R are in equilibrium. So for all the other particles of the body. Thus the reversed effective forces $\Sigma(mf)$ acting on each particle of the body, the external forces (ΣF) and the internal actions and reactions (ΣR) of the rigid body form a system of forces in equilibrium.

But ΣR *i.e.* the internal actions and reactions of the body are itself in equilibrium i.e. $\Sigma R = 0$ Hence the forces ΣF and Σmf (reversed are in equilibrium

i.e. $\Sigma - (mf) + \Sigma F = 0$

Hence the reversed effective forces acting at each point of the system and the impressed (external) forces on the system are in equilibrium.

Note. This principle reduces the dynamical problem to the statical one.

Vector Method. Consider a rigid body in motion. Let at any time t, $\boldsymbol{r}$ be the position vector of a particle of mass m and F and $\boldsymbol{R}$ be the external and internal forces respectively acting on it. Now by Newton's second law $m(d^2r/dt^2) = \boldsymbol{F} + \boldsymbol{R}$ or $\boldsymbol{F} + \boldsymbol{R} - m\,(d^2r/dt^2) = 0$

i.e. the three forces, namely $\boldsymbol{F}, \boldsymbol{R}$ and $-\,m(d^2r/dt^2)$ are in equilibrium. Now applying the same argument to every particle of the rigid body, the forces $\Sigma\boldsymbol{F}, \Sigma\boldsymbol{R}$ and $\Sigma\left(-m\,\dfrac{d^2\boldsymbol{r}}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

Since the internal forces acting on the rigid body form pairs of equal and opposite forces, thus their vector sum must be zero

i.e. $\Sigma\boldsymbol{R} = \boldsymbol{0}$

$\Rightarrow$ The forces $\Sigma\boldsymbol{F}$ and $-\,\Sigma m\,(d^2r/dt^2)$ are in equilibrium. This proves the D' Alembert's Principle.

1.04. Angular momentum of a system of particles. If $\boldsymbol{r}$ be the position vector of a particle of mass m relative to a point O then the vector sum

$\boldsymbol{H} = \Sigma\boldsymbol{r} \times m\boldsymbol{v} = \Sigma m\boldsymbol{r} \times \boldsymbol{v}$

is called *angular momentum (or moment of momentum)* of the system *about O.*

1.05. General equation of motion. *To deduce the general equation of motion of rigid body from D' Alembert's Principle.*

Cartesian method. Let (x, y, z) be the coordinates of a particle of mass m at any time t referred to a set of rectangular axes fixed in space. Let X, Y, Z represent the components, parallel to the axes of the external forces acting on it.

By D' Alembert's Principle of the forces

$$X - m\ddot{x}\ ,\ Y - m\ddot{y}\ ,\ Z - m\ddot{z}$$

together with similar forces acting on each particle of the body will be in equilibrium.

Hence as in statics, the six conditions of equilibrium are

$$\Sigma\,(X - m\ddot{x}\,) = 0, \Sigma(Y - m\ddot{y}\,) = 0, \Sigma(Z - m\ddot{z}\,) = 0.$$

$$\Sigma[y(Z - m\ddot{z}\,) - z\,(Y - m\ddot{y}\,)] = 0$$

$$\Sigma[z\,(X - m\ddot{x}\,) - x\,(Z - m\ddot{z}\,)] = 0$$

and $\Sigma[x(Y - m\ddot{y}\,) - y\,(X - m\ddot{x}\,)] = 0$

where summations are to be taken over all the particles of the body.

These equations give

$$\Sigma m\ddot{x} = \Sigma X,\ \Sigma m\ddot{y} = \Sigma Y,\ \Sigma\, m\ddot{z} = \Sigma Z$$

$$\Sigma m(y\ddot{z} - z\ddot{y}\,) = \Sigma(yZ - zY\,)$$

$$\Sigma m(z\ddot{x} - x\ddot{z}\,) = \Sigma(zX - xZ\,)$$

and $\Sigma m(x\ddot{y} - y\ddot{x}\,) = \Sigma(xY - yX\,)$

These are the six equations of motion of any rigid body.

The first three equations can be written as

$$\frac{d}{dt}\Sigma\, m\dot{x} = \Sigma X\,,\ \frac{d}{dt}\Sigma m\dot{y} = \Sigma Y,\ \frac{d}{dt}\Sigma m\dot{z} = \Sigma Z$$

and the other three equations are written as

$$\frac{d}{dt}\ \Sigma\, m(y\dot{z} - z\dot{y}) = \Sigma(yZ - zY)$$

$$\frac{d}{dt}\ \Sigma\, m\,(z\dot{x} - x\dot{z}) = (zX - xZ)$$

and $\dfrac{d}{dt}\ \Sigma\, m\,(x\dot{y} - y\dot{x}) = \Sigma\,(xY - yX)$

The first three equations show that the rate of change of linear momentum in any direction is equal to the total external force in that direction and the rest three equations express that the rate of change of the angular momentum about any given axis is equal to the total moment of all the external forces about that axis.

Vector Method. At time t let $\boldsymbol{r}$ be the position vector of a particle of mass m and $\boldsymbol{F}$ be the external force acting on it, then by D' Alembert's *Principle*

$$\Sigma\left(-m\frac{d^2\boldsymbol{r}}{dt^2}\right)+\Sigma\boldsymbol{F}=\boldsymbol{0} \text{ or } \Sigma m\frac{d^2\boldsymbol{r}}{dt^2}=\Sigma\boldsymbol{F} \quad ...(1)$$

Taking cross product by $\boldsymbol{r}$, we get

$$\Sigma\boldsymbol{r}\times m\frac{d^2\boldsymbol{r}}{dt^2}=\Sigma\boldsymbol{r}\times\boldsymbol{F} \quad ...(2)$$

Equations (1) and (2) are in general, vector equations of motion of a rigid body.

Again $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ ...(3) and $\boldsymbol{F}=X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$ ...(4)

where X, Y, Z are the components of $\boldsymbol{F}$.

From (3) $(d^2\boldsymbol{r}/dt^2)=(d^2x/dt^2)\boldsymbol{i}+(d^2y/dt^2)\boldsymbol{j}+(d^2z/dt^2)\boldsymbol{k}$ ...(5)

Putting for $\boldsymbol{r}$, $\boldsymbol{F}$ and $(d^2\boldsymbol{r}/dt^2)$ from (3), (4) and (5) respectively in (1) and (2), we get

$\Sigma m\,[(d^2x/dt^2)\boldsymbol{i}+(d^2y/dt^2)\boldsymbol{j}+(d^2z/dt^2)\boldsymbol{k}\,]=\Sigma(X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k})$

and $\Sigma m(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times[(d^2x/dt^2)\boldsymbol{i}+(d^2y/dt^2)\boldsymbol{j}+(d^2z/dt^2)\boldsymbol{k}\,]$

$$=\Sigma[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times(X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k})]$$

Equating the coefficients of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we get the six conditions of equilibrium as obtained earlier.

1.06. Linear Momentum. *The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.*

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the C.G. of the system and M the whole mass, then

$M\bar{x}=\Sigma mx$, $M\bar{y}=\Sigma my$ and $M\bar{z}=\Sigma mz$.

Differentiating these relations, we get

$M\bar{x}=\Sigma\,mx$ etc. Hence the result.

1.07. Motion of the centre of inertia. *To prove that the centre of inertia (C.G.) of a body moves as if the whole mass of the body were collected at it, and as if all the external forces were acting at it in directions parallel to those in which they act.* **[Meerut 88, 80]**

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the C.G. of the body of mass M then

$M\bar{x}=\Sigma mx$, so that $M\ddot{\bar{x}}=\Sigma m\ddot{x}$.

But from the general equation of motion, we have $\Sigma\,mx=\Sigma X$

Therefore, $M\ddot{\bar{x}}=\Sigma X$ ...(1)

Similarly we have, $M\ddot{\bar{y}}=\Sigma Y$ ...(2) and $M\ddot{\bar{z}}=\Sigma Z$ ...(3)

The equation (1) is the equation of motion of a particle of mass M (placed at the centre of inertia) acted on by a force ΣX parallel to the original directions of the forces on different particles. Similarly the equations (2) and (3) can be interpreted.

Vector Method. Let $\bar{r}$ be the position vector of the centre of inertia and r be the position vector of mass m of a rigid body whose mass is M. Now by the definition of centroid, we have

$$\bar{r}=\frac{\Sigma m r^{*}}{\Sigma m} \therefore \frac{d^2 \bar{r}}{d t^2}=\frac{\Sigma m \dfrac{d^2 r}{d t^2}}{\Sigma m}=\frac{\Sigma m \dfrac{d^2 r}{d t^2}}{M} \qquad ...(1)$$

Again vector equation of motion of a rigid body is

$$\Sigma m \frac{d^2 r}{d t^2}=\Sigma F. \qquad ...(2)$$

Putting the value of $\dfrac{d^2 r}{d t^2}$ from (1) in (2), we get $M \dfrac{d^2 \bar{r}}{d t^2}=\Sigma F.$

If the components of $\bar{r}$ be $(\bar{x}, \bar{y}, \bar{z})$ then $\bar{r}=\bar{x} i+\bar{y} j+\bar{z} k$;

$$\therefore \frac{d^2 \bar{r}}{d t^2}=\frac{d^2 \bar{x}}{d t^2} i+\frac{d^2 \bar{y}}{d t^2} j+\frac{d^2 \bar{z}}{d t^2} k$$

Equation (3) reduces to

$$M\left[\frac{d^2 \bar{x}}{d t^2} i+\frac{d^2 \bar{y}}{d t^2} j+\frac{d^2 \bar{z}}{d t^2} k\right]=\Sigma(X i+Y j+Z k),$$

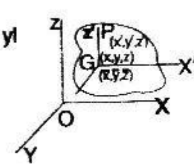
Equating coefficient of i, j, k, we get

$$M \frac{d^2 \bar{x}}{d t^2}=\Sigma x, M \frac{d^2 \bar{y}}{d t^2}=\Sigma y, M \frac{d^2 \bar{z}}{d t^2}=\Sigma z$$

1.08. Motion relative to centre of interia. *The motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body.* **[Agra 1990, Meerut 90]**

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of gravity G of the body with reference to the rectangular axes through a fixed point, say O.

Let (x', y', z') be the coordinates of a particle of mass m with G (centre of inertia) as original axes parallel to the original axes and (x, y, z) be its coordinates with reference to original axes. Then



$x=\bar{x}+x', y=\bar{y}+y'$ and $z=\bar{z}+z'$

Now consider the fourth equation of the general equation of motion of rigid body,

viz $\Sigma m(y \ddot{z}-z \ddot{y})=\Sigma(y Z-z Y)$. ...(i)

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* If r is position vector of any particle of mass m of the system relative to a point O, the origin of vectors then the point with position vector $\bar{r}=(\Sigma m r / \Sigma m)$ is defined as the centroid of the system.

Again, $(y\ddot{z} - z\ddot{y}) = (\bar{y} + y')(\ddot{\bar{z}} + \ddot{z}') - (\bar{z} + z')(\ddot{\bar{y}} + \ddot{y}')$

$$(\because \ddot{x} = \ddot{\bar{x}} + \ddot{x}' \text{ etc.})$$

Therefore from (1), we get

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m\bar{y}\ddot{\bar{z}} + \Sigma m\bar{y}\ddot{z}' + \Sigma m y'\ddot{\bar{z}} + \Sigma m y'.\ddot{z}'$$

$$- \Sigma m\bar{z}\ddot{\bar{y}} - \Sigma m\bar{z}\ddot{y}' - \Sigma m z'\ddot{\bar{y}} - \Sigma m z'\ddot{y}' \qquad ...(2)$$

As G (the centre of inertia) is the origin of coordinates w. r. t. the new axis.

$\therefore \Sigma m x' = \Sigma m y' = \Sigma m z' = 0 \quad \left(\because \dfrac{\Sigma m x'}{\Sigma m} = 0 \text{ etc}.\right)$

Therefore $\Sigma m\ddot{x}' = 0 = \Sigma m\ddot{y}' = \Sigma m\ddot{z}'$, also $\Sigma m = M =$ total mass of the body. Again $\bar{x}, \bar{y}, \bar{z}$ and their differential coefficients are common to all particles of the body, so we can take them outside the sigma sign. Hence equation (2)

$$\Rightarrow \Sigma m(y\ddot{z} - z\ddot{y}) = M\bar{y}\ddot{\bar{z}} - M\bar{z}\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - \ddot{z}'y')$$

$\therefore$ Equation (1) becomes

$$M\bar{y}\ddot{\bar{z}} - M\bar{z}\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma\{(\bar{y} + y')Z - (\bar{z} + z')Y\}$$

$$= \Sigma\bar{y}Z + \Sigma y'Z - \Sigma\bar{z}Y - \Sigma z'Y.$$

Again by **1.07**, we know that $M\ddot{\bar{z}} = \Sigma Z$, $M\ddot{\bar{y}} = \Sigma Y$.

Hence $\Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma(y'Z - z'Y)$.

Similarly, we get other two equations.

But these equations are the same as would have been obtained had we regarded the C.G. to be a fixed point and same forces acted on the body.

Vector Method. Let $\boldsymbol{r}$ be the position vector of the centre of inertia of the rigid body at any time t (say). Mass of the rigid body is M and *let* m be the mass of particle whose position vector with respect to centre of inertia be $\boldsymbol{r}$ and position vector referred to a fixed origin be $\boldsymbol{r}'$ then $\boldsymbol{r} = \bar{\boldsymbol{r}} + \boldsymbol{r}'$.

$$\therefore \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2\bar{\boldsymbol{r}}}{dt^2} + \frac{d^2\boldsymbol{r}'}{dt^2}.$$

Now moment vector equation of motion for a rigid body is

$$\Sigma m\boldsymbol{r} \times \frac{d^2\boldsymbol{r}}{dt^2} = \Sigma\boldsymbol{r} \times \boldsymbol{F}$$

$$\text{or } \Sigma m\left(\bar{\boldsymbol{r}} + \boldsymbol{r}'\right) \times \left(\frac{d^2\bar{\boldsymbol{r}}}{dt^2} + \frac{d^2\boldsymbol{r}'}{dt^2}\right) = \Sigma\left(\bar{\boldsymbol{r}} + \boldsymbol{r}'\right) \times \boldsymbol{F}$$

or $\bar{r}\times M\,\dfrac{d^2\bar{r}}{dt^2}+\bar{r}\,\Sigma mr\,\dfrac{d^2r'}{dt^2}+(\Sigma mr')\times\dfrac{d^2\bar{r}}{dt^2}+\Sigma mr'\times\dfrac{d^2r'}{dt^2}$

$= \bar{r}\times\Sigma F+\Sigma r'\times F$ ...(1)

Again $\dfrac{\Sigma mr'}{\Sigma m}=\mathbf{0}$ (Since it is the position vector of the centre of inertia G referred to G as origin)

$\therefore\ \Sigma mr'=\mathbf{0}$, so $\Sigma m\,\dfrac{d^2r'}{dt^2}=\mathbf{0}$

Again we know that $M\,\dfrac{d^2\bar{r}'}{dt^2}=\Sigma F$.

$\therefore\ \bar{r}\times M\dfrac{d^2\bar{r}}{dt^2}=\bar{r}\times\Sigma F$

Thus equation (1) reduces to $\Sigma\, mr'\times\dfrac{d^2r'}{dt^2}=\Sigma r'\times F$ ...(2)

Equation (2) is the vector equation of motion of a rigid body when centre of gravity is regarded as a fixed point.

Again $r'=x'i+y'j+z'k$

$$\therefore\quad \frac{d^2r'}{dt^2}=\frac{d^2x'}{dt^2}i+\frac{d^2y'}{dt^2}j+\frac{d^2z'}{dt^2}k.$$

Thus equation (2) becomes

$$\Sigma m(x'i+y'j+z'k)\times\left(\frac{d^2x'}{dt^2}i+\frac{d^2y'}{dt^2}j+\frac{d^2z'}{dt^2}k\right)$$

$$=\Sigma\,(x'i+y'j+z'k)\times(Xi+Yj+Zk)$$

$$\text{or } \Sigma\Big[\,m\left(y'\frac{d^2z'}{dt^2}-z'\frac{d^2y'}{dt^2}\right)i+m\left(z'\frac{d^2x'}{dt^2}-x'\frac{d^2z'}{dt^2}\right)j$$
$$+\,m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)k\Big]$$

$$=\Sigma[(y'Z-z'Y)\,i+(z'X-x'Z)\,j+(x'Y-y'X)\,k\,]$$

Equating coefficient of i, we get

$$\Sigma m\,(y'\ddot{z}'-z'\ddot{y}')=\Sigma(y'Z-z'Y)$$

Similarly, we get other two equations on equating the coefficients of j and k.

Note. 1. The two important properties discussed in **1.07** and **1.08** are respectively called the principle of conservation of motion of translation and rotation and together called the principle of independence of translation and rotation.

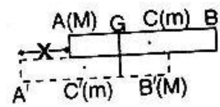
Note 2. 1.07 states that the motion of the C.G. is the same as if the whole mass were collected at the point and is therefore independent of rotation.

Note 3. 1.08 states that the motion round the C.G. is the same as if that point were fixed and is therefore independent of the motion of that point.

IllUSTRATIVE EXAMPLES

Ex 1. *A rough uniform board, of mass m and length 2a, rests on a smooth horizontal plane, and a boy, of mass M, walks on it from one end to the other, show that the distance through which the board moves in this time is* $2\ Ma/(m+M)$. **[Meerut 88]**

Sol. Here the weight of the boy and the board are downwards, the actions and reactions between the boy and the board vanish for the system. The reaction of the smooth plane is acting vertically upwards. Thus there are no external forces on the system in the horizontal direction. Thus by D ' Alembert's principle the C.G. of the system does not move. As the boy goes to left, the board comes to the right.



Let $\bar{x}$ be the distance of the C.G. of the system and x be the distance through which the board moves, when the boy goes from one end to the other.

Now in the initial position, $(M+m)\bar{x} = M\ 2a + ma$

in the final position, $(M+m)\bar{x} = Mx + m(a+x)$.

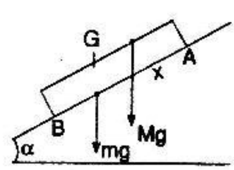
Therefore $M.2a + ma = Mx + m(a+x)$,

or $x = 2Ma/(M+m)$.

Ex.2. *A plank, of mass m and length 2a, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle* α *to the horizon, and a man; of mass M, starting from the upper end walks downs the plank so that it does not move, show that he will reach the other end in time.*

$$\left[\frac{4Ma}{(m+M)g\sin\alpha}\right]^{1/2}.$$ **[Meerut 1990]**

Sol. Suppose that the man has come down a distance x in time t, starting from the end A of the plank. Since the plank does not move, its centre is fixed. If $\bar{x}$ be the distance of the C.G. of the system from A, then $(M+m)\bar{x} = am + Mx$.



This gives $(M+m)\ddot{\bar{x}} = M\ddot{x}$. ...(1)

Again the motion of the C.G. of the system is given by

$(M+m)\ddot{\bar{x}}$ Ext. forces acting parallel to the plank

$= \Sigma X = (m+M)g\sin\alpha$ ...(2)

From (1) and (2), we get

$$M\ddot{x} = (m+M)g\sin\alpha \text{ or } \ddot{x} = \frac{(m+M)g\sin\alpha}{M}.$$

Integrating twice and applying the condition that when we have

$$x = \frac{(m+M)\,g\sin\alpha}{M}\cdot\frac{1}{2}t^2.$$

Putting $x = 2a$, we get the time to reach the other end as

$$\left[\frac{4Ma}{(m+M)\,g\sin\alpha}\right]^{1/2}.$$

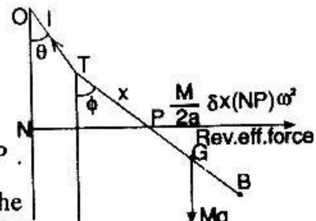
Ex.3. *A rod, of length 2a, is suspended by a string of length l, attached to one end, if the string and rod revolve about the vertical with uniform angular velocity, and their inclination to the vertical be* θ and ϕ *respectively, show that* $\dfrac{3l}{a} = \dfrac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}$. **[Meerut 96,95,94]**

Sol. Take a small element δx of the rod AB at a distance x from A. Let ω be the uniform angular velocity of the rod. Mass of the element

$= \dfrac{M}{2a}\delta x$. Its reversed effective force

$= \dfrac{M}{2a}\delta x\, NP\ \omega^2$, along NP

$= \dfrac{M}{2a}\delta x\ (l\sin\theta + x\sin\phi)\,\omega^2$ along NP.



The external forces on the rod are (1) the tension T of the string and (2) of the weight Mg of the rod.

Resolving horizontally, vertically, and taking moments about A, we have

$$T\sin\theta = \frac{M}{2a}\omega^2\,\Sigma NP\ \delta x = \frac{M}{2a}\omega^2\int_0^{2a}(l\sin\theta + x\sin\phi)\,dx$$

$$= \frac{M}{2a}\omega^2\,(2al\sin\theta + 2a^2\sin\phi) \qquad [\textit{Horizontally}] \qquad ...(1)$$

$$T\cos\theta = Mg \qquad [\textit{Vertically}] \qquad ...(2)$$

and $Mga\sin\phi = \dfrac{M}{2a}\omega^2\ \Sigma NP.\,\delta x.x\cos\phi$ [*Moment equation*]

$$= \frac{M}{2a}\omega^2\int_0^{2a}(l\sin\theta + x\sin\phi)\,x\cos\phi\,dx$$

$$= \frac{M}{2a}\omega^2\left(l\sin\theta\cos\phi\,2a^2 + \frac{8a^3}{3}\sin\phi\cos\phi\right)$$

or $$\omega^2 = \frac{3g\sin\phi}{(3l\sin\theta + 4a\sin\phi)\cos\phi}. \qquad ...(3)$$

Diviiding (1) by (2), we have $\dfrac{\sin\theta}{\cos\theta} = \omega^2\dfrac{(l\sin\theta + a\sin\phi)}{g}$. ...(4)

Putting value of ω^2 from (3) in (4), we get

$$\frac{\sin\theta}{\cos\theta}=\frac{3\sin\phi\,(l\sin\theta+a\sin\phi)}{(3l\sin\theta+4a\sin\phi)\cos\phi}$$

or $\quad \sin\theta\cos\phi\,(3l\sin\theta+4a\sin\phi)=\sin\phi\cos\theta\,(l\sin\theta+a\sin\phi)$

or $\quad 3l\sin\theta\,(\sin\theta\cos\phi-\sin\phi\cos\theta)$

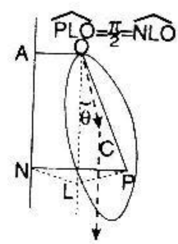
$$=a\sin\phi\,(3\sin\phi\cos\theta-4\sin\theta\cos\phi)$$

or $\quad \dfrac{3l}{a}=\dfrac{\sin\phi\,(3\sin\phi\cos\theta-4\sin\theta\cos\phi)}{\sin\theta\,(\sin\theta\cos\phi-\sin\phi\cos\theta)}$

$$=\frac{\sin\phi\,(3\tan\phi-4\tan\theta)}{\sin\theta\,(\tan\theta-\tan\phi)}=\frac{(4\tan\theta-3\tan\phi)\sin\phi}{(\tan\phi-\tan\theta)\sin\theta}.$$

Ex.4. *A thin circular disc of mass M and radius a, can turn freely about a thin axis OA, which is perp. to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity ω about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}(g/a\omega^2)$ unless $\omega^2 < g/a$ and then θ is zero.*

Sol. Consider the circular disc in the vertical plane so that the axis OA about which it turns is horizontal. when the axis OA moves horizontally round A, the disc will be raised in its vertical plane and its radius OC makes an angle θ with the vertical. Consider an element δm at P. Let PL be perpendicular to the vertical through O and LN be perpendicular from L to the vertical through A so that PN is perp. to AN. Now P discribes a circle of radius PN with a constant angular velocity ω about N. Thus the reversed effective force along NP is $\delta m NP\omega^2$



Again $\overrightarrow{NP}=\overrightarrow{NL}+\overrightarrow{LP}$.

$\therefore\ \delta m.\,\omega^2\,\overrightarrow{NP}=\delta m\,\omega^2\,\overrightarrow{NL}+\delta m\,\omega^2\,\overrightarrow{LP}$ i.e. the force $\delta m\,\omega^2\,\overrightarrow{NP}$ is equivalent to forces $\delta m\,\omega^2\,LP$ and *other* $\delta m\,\omega^2 NL$ along NL. The external forces on the disc are its weight Mg and the reaction at O.

By D'Alemberts Principle, Rev. effective forces along with external forces form the system in equlibrium. Hence moment of Rev. effective forces + moment of external forces = 0 i.e. moment of effective forces about OA = moment of external forces (1).

In order to avoid reaction at O, we take moment about the line OA. Since NL and OA lie in one plane (they are parallel also) the shortest distance between them is zero.

$\therefore$ Moment of the force $\delta m\omega^2\times NL$ about OA is zero. Further the shortest distance between OA and LP is OL and the shortest distance between OA and the vertical through C is $a \sin\theta$. Hence moment of the force $\delta m\omega^2 LP$ about OA is given by $\delta m\omega^2 LP \times OL$. Taking moments about OA, we get $Mga\sin\theta = \Sigma\delta m\omega^2 LP \times OL$

or $aMg\sin\theta \equiv \omega^2\Sigma(\delta m\, LP\,.\,OL)$. But, $\Sigma(\delta m\, LP\, OL) =$ Product of inertia of the disc about OL and horizontal line through $O =$ Product of inertia about the parallel lines through $C + Mx'y'$, where x', y' are the co-ordinates of C with respect of the vertical and horizontal through O $= 0 = Ma^2\sin\theta\cos\theta$.

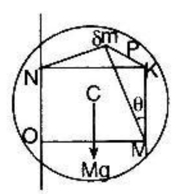
$\Rightarrow aMg\sin\theta = \omega^2 Ma^2\sin\theta\cos\theta \Rightarrow \sin\theta = 0$ or $\cos\theta = (g/a\omega^2)$,

where $a\,\omega^2 > g$.

But, $\omega^2 < (g/a) \Rightarrow \cos\theta > 1$, which is impossible and hence in this case $\cos\theta = 1$ *i.e.* $\theta = 0$.

Ex.5. *A thin heavy disc can turn freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity* ω *about a fixed point on itself. Show that the inclination* θ *of the plane of the disc to the vertical is given by* $\cos\theta = (gh/k^2\omega^2)$ *where h is the distance of the centre of inertia of the disc from the axis and k is the radius of the gyration of the disc about the axis. If* $\omega^2 < gh/k^2$, *prove that the plane of the disc is vertical.*

Sol. Let OM be the horizontal axis in the plane of the disc which, rotates about O so that the vertical line ON is the axis of rotation of the system. Consider an element of mass δm at P. Draw PN perp. to this vertical axis ON then effective force for δm is $\delta m\,\omega^2 PN$. Here PN is not in the plane of the disc. From P draw PM perp. to OM, here PM is in the plane of the disc. Through N draw NK perp. to OM and from P draw PK perp. to NK so that PK is perp. to KM. thus if $\angle PMK = \theta, \theta$ is the inclination of the disc to the vertical, KM being vertical.



Again $\overrightarrow{PN} = \overrightarrow{PK} + \overrightarrow{KN}$. Therefore $\delta m\,\omega^2\overrightarrow{PN} = \delta m\omega^2\overrightarrow{PK} + \delta m\,\omega^2\,\overrightarrow{KN}$. Thus the effective force on δm are $\delta m\,\omega^2\overrightarrow{PK}$ and $\delta m\,\omega^2\overrightarrow{KN}$. Since KN is parallel to OM, the moment of the force $\delta m\,\omega^2\overrightarrow{KN}$ about OM will be zero and the moment of $\delta m\omega^2\,\overrightarrow{PK}$ about OM is $\delta m\omega^2\, PK\,.\,KM$. The moment of Mg about OM is $Mgh\sin\theta$. Hence taking moments about OM, we get $Mgh\sin\theta = \Sigma\delta m\,\omega^2\, PK\, KM$

$$= \Sigma\delta m\,\omega^2(PM\sin\theta)(PM\cos\theta) = \omega^2\sin\theta\cos\theta\,\Sigma\delta m\, PM^2$$

But $\Sigma\delta m\,PM^2 =$ M.I. of the disc about $OM = M\,k^2$, where k is the radius of gyration. $\therefore Mgh\sin\theta = \omega^2\sin\theta\cos\theta\,Mk^2$.

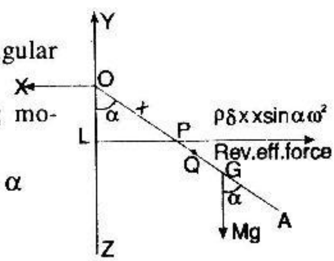
Hence either $\sin\theta = 0$ *i.e.* $\theta = 0$ or $\cos\theta = \dfrac{gh}{\omega^2k^2}$

If $\omega^2 < \dfrac{gh}{k^2}$, as in that case $\cos\theta > 1$ the only possible value of θ is zero and then plane of the disc is vertical.

Ex.6. *A uniform rod OA, of length 2a, free to turn about its end O, resolves with uniform angular velocity ω about a vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or* $\cos^{-1}(3g/4a\omega^2)$. **[Meerut 84,75,73]**

Sol. Consider a small element $PQ = \delta x$ at a distance x from O. The point P will move in a horizontal circle whose radius is $PL = x\sin\alpha$. Here only effective force on the element PQ is

$\rho\,\delta x\,PL\omega^2 = \rho\delta x\,.\,x\sin\alpha\,\omega^2$



where ρ is the density of the rod and angular velocity ω is constant.

Reversing the effective force and taking moments about O, we have

$$\Sigma(\rho\,\delta x\,.\,x\sin\alpha\,\omega^2)x\cos\alpha = Mga\sin\alpha$$

or $\rho\omega^2\sin\alpha\cos\alpha\displaystyle\int_0^{2a} x^2\,dx = Mga\sin\alpha$

or $(M/2a)\omega^2\sin\alpha\cos\alpha(8a^3/3) = Mga\sin\alpha \quad (\because\ 2a\rho = M)$

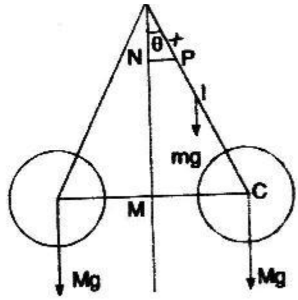
or $\sin\alpha\left(g - \dfrac{4a\omega^2\cos\alpha}{3}\right) = 0$. It implies either $\sin\alpha = 0$ *i.e.* $\alpha = 0$

or $\cos\alpha = (3g/4a\,\omega^2)$ *i.e.* $x = \cos^{-1}(3g/4a\,\omega^2)$.

Ex.7. *Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two uniform thin rods, each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Governor of a steam-Engine, about a vertical line through O with the angular velocity ω. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ given by the equation*

$$\cos\theta = \frac{g}{\omega^2}\cdot\frac{M(l+a)+\frac{1}{2}ml}{M(l+a)^2+\frac{1}{3}ml^2}.$$

Sol. Take an element δx in one of the rods at a distance x from O. Let PN, CM be the perpendiculars on the vertical line through O. Here C is the centre of one of the spheres. The reversed effective force on the rod at P is $\delta x \frac{m}{l} \omega^2 x \sin\theta$ along NP and the reversed effective force on the sphere is $M\omega^2(a+l)\sin\theta$ along MC. On taking moments about O for the system of a rod and a sphere on one side of the vertical OM, we have



$$\Sigma\{\delta x(m/l)\omega^2 x\sin\theta\, x\cos\theta\} + M\omega^2(a+l)\sin\theta(a+l)\cos\theta$$
$$= Mg(a+l)\sin\theta + mg(l/2)\sin\theta$$

or $$\int_0^l \frac{m}{l}\omega^2 \sin\theta\cos\theta\, x^2\,dx + M\omega^2(a+l)^2\sin\theta\cos\theta$$
$$= Mg(a+l)\sin\theta + \frac{1}{2}mgl\sin\theta$$

or $$\omega^2\{\tfrac{1}{3}ml^2 + M(a+l)^2\}\cos\theta = g(\tfrac{1}{2}ml + M(l+a))\}$$

or $$\cos\theta = \frac{g}{\omega^2}\cdot\frac{\frac{1}{2}ml + M(l+a)}{\frac{1}{3}ml^2 + M(l+a)^2}$$

1·09. Impulsive Forces : When the forces acting on a body are very large and act for a very short time, then their effects are measured by impulses. Let a particle of mass 'm' be acted upon by a force F always in the same direction, the equation of motion is $m\,(dv/dt) = F$. ...(1)
where v is the velocity of the particle at time t. If τ be the time during which the force F acts and v_1, v_2 be the velocities before and after the action of the force, then on integrating (1), we have

$$m(v_2 - v_1) = \int_0^\tau F\,dt \qquad ...(2)$$

Now if F increases indefinitely while τ decreases indefinitely, then the integral on the right hand side of (2) may have a definite finite limit. Let this finite limit be I then equation (2) may be written as

$$m\,(v_2 - v_1) = I \qquad ...(3)$$

The velocity during the time τ has increased or decreased from v_1 to v_2. Supposing that the velocity have remained finite, let v be the greatest velocity during the interval. Then the space described is less than $v\tau$. Since $v\tau \to 0$ as $\tau \to 0$, hence we conclude that the particle has not moved during the action of the force F. It could not have time to move, but its velocity has been changed from v_1 to v_2.

Thus in the case of finite forces which act on a body for indefinitely short *time, the change of place is zero and the change of velocity is the measure of these forces. A force so measured is called an impulse. We can define impulse as the limit of a force which is indefinitely greater but acts only for an indefinitely short time e.g. the below of a hammer is a force of this kind. In fact an impulsive force is measured by the whole momentum generated by the impulse.*

1·10. When impulsive force acts, the finite forces acting on the body may be neglected in calculating the effect.

Let F be the impulsive force and f a finite force acting simultaneously on the body. Then $m(v_1 - v_2) = \int_0^\tau F\,dt + \int_0^\tau f\,dt = P + f\tau$.

But since $f\tau \to 0$ as $\tau \to 0$, f may be neglected in forming the equations.

1·11. Application of D'Alembert's principle to impulsive forces, general equation of motion.

Scalar Method. let u, v, w be the velocities parallel to co-ordinate axes before the action of impulsive forces and u', v', w' be the velocities after the action of these forces. Let X', Y', Z' be the resolved parts of the impulsive forces parallel to the axes. Then from $\Sigma m\ddot{x} = \Sigma X$, on integrating with respect to t, we get

$$\left[\Sigma m \frac{dx}{dt}\right]_0^\tau = \int_0^\tau \Sigma X\,dt = \Sigma \int_0^\tau X\,dt = \Sigma X'$$

or $\Sigma m(u' - u) = \Sigma X'$.

Similarly, $\Sigma m(v' - v) = \Sigma Y'$ and $\Sigma m(w' - w) = \Sigma Z'$

Thus the change in the momentum parallel to any of the axes of the whole mass M, supposed collected at the centre of inertia and moving with it is equal to the impulse of the external forces parallel to the corresponding axis. Again we have the moment equation

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m(yZ - zY)$$

Integrating this we have $\left[\Sigma m(y\dot{z} - z\dot{y})\right]_0^\tau = \Sigma\left[y\int_0^\tau Z\,dt - z\int_0^\tau Y\,dt\right]$

Since the interval τ is so short that the body has not moved during this period, we may take x, y, z as constants, thus the above equation becomes

$$\Sigma m\{y(w' - w) - z(v' - v)\} = \Sigma(yZ' - zY')$$

Similarly we have other two equations

$$\Sigma m\{x(v' - v) - y(u' - u)\} = \Sigma(xY' - yX')$$

and $\Sigma m\{z(u' - u) - x(w' - w)\} = \Sigma(zX' - xZ')$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Ex.8. *A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity* of the ball *and cannon at the moment when it leaves the cannon is* **u**. *Show that the cannon will recoil a distance* $\left(\dfrac{mu}{M+m}\right)^2 \dfrac{1}{2\mu g}$ *along the plane, m being the mass of the ball.* **[Raj.88]**

Sol. Let I be the impulse between the cannon and the ball and V, v be their velocities. Since their relative velocity is u, we have

$V + v = u$ ...(1) and $mv = I = MV$ ...(2)

From (1) and (2), we have $(MV/m) + V = u$ or $V = \{mu/(m+M)\}$

Again on the rough plane, for the cannon the equation is

$M\ddot{x} = -\mu R = -\mu Mg$, where x is the distance cannon has moved.

$\therefore \ddot{x} = -\mu g$, , Multiplying by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = -2\mu g x + C$$

When $x = 0, \dot{x} = V$, so that $C = V^2$, $\quad \dot{x}^2 = V^2 - 2\mu g x$

when the cannon comes to rest $\dot{x} = 0$, $\quad \therefore x = (V^2/2\mu g)$

or $x = (mu/M+m)^2 (1/2\mu g)$ $\qquad [\because \ V = (mu/M+m)]$

Supplementary Problems

1. A light rod **OAB** can turn freely in a vertical plane about a smooth fixed hinge at **O**; two heavy particles of mass **m** and m' are attached to the rod A and B and oscillate with it. Find the motion.

2. Find the motion of the rod **OAB** with two masses **m** and m' attached it, when it moves round the vertical as a conical pendulum with uniform angular velocity, the angle θ which the rod makes with the vertical being constant.

3. A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts. What is the subsequent motion of the two parts ?

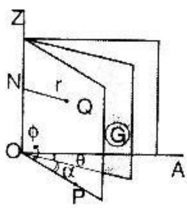
4. A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes cone of semi vertical angle α ; show that $\omega^2 = (3g/4a)\cos\alpha$. Prove also that the direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}[(3/4)\tan\alpha]$. **[Agra 1971]**

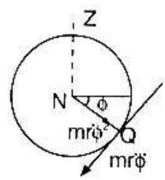
5. Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons, of mass M throws a ball of mass m towards the other which reaches him in time t. Prove that the first person will begin to slide along the plane with velocity (ma/Mt).

Motion About A Fixed Axis

2·01. *A rigid body is rotating about a fixed axis. To find the moment of the effective forces about the axis of rotation.*

Let the axis of rotation be OZ, perpendicular to the plane of the paper. Take a plane AOZ through OZ and fixed in space, cutting the plane of the paper along OA. Let this plane be taken as the plane of reference. Let θ be the angle, which another plane ZOG through the axis fixed in the body makes with the plane AOZ.





Take a particle of mass m at Q and let the plane through OZ and Q cut the plane of the paper along OP. Let the angle between ZOP and ZOG be α. When body rotates about OZ, α remains constant. Let the angle between the plane ZOP and the plane ZOA be ϕ. Now

$$\theta + \alpha = \phi \;\therefore\; \dot{\theta} = \dot{\phi} \text{ and } \ddot{\theta} = \ddot{\phi}$$

The accelerations of the particle of mass m are $r\dot{\phi}^2$ and $r\ddot{\phi}$ along QN and perpendicular to QN respectively.

Therefore effective forces on the particles are $mr\dot{\phi}^2$ and $mr\ddot{\phi}$ in the above said directions. Again $r\dot{\phi}^2 = r\dot{\theta}^2$ and $r\ddot{\phi} = r\ddot{\theta}$

The moment of the force $mr\dot{\phi}^2$ about OZ is zero and moment of the force $mr\ddot{\phi}$ about OZ (& NZ) is $r.mr\ddot{\phi} = mr^2\ddot{\phi} = mr^2\ddot{\theta}$

Hence the moment of the effective forces of the whole body about OZ is $\Sigma mr^2\ddot{\theta} = \ddot{\theta}\,\Sigma mr^2 = Mk^2\ddot{\theta}$, where k is the radius of gyration of the body about OZ.

Moment of momentum about the axis of rotation.

Velocity of the particle m is $r\dot{\phi}$ perpendicular to QN. Therefore the moment of momentum of the particle about OZ is $mr^2\dot{\phi}$ or $mr^2\dot{\theta}$.

Hence the moment of momentum of the whole body about OZ is

$\Sigma m r^2\dot{\phi} = (\Sigma m r^2)\dot{\theta} = \dot{\theta}\,\Sigma m r^2 = Mk^2\dot{\theta}$

2·02. Kinetic Energy : The kinetic energy of the particles of mass m is $\frac{1}{2} m r^2 \dot{\phi}^2$.

Hence K.E. of the whole body is

$$\Sigma \frac{1}{2} m r^2 \dot{\phi}^2 = \Sigma \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{1}{2}\dot{\theta}^2 \,\Sigma m r^2 = \frac{1}{2} Mk^2\dot{\theta}^2.$$

2·03. Equation of motion :

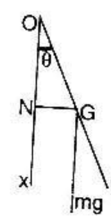
The impressed forces include besides the external forces, the reactions on the axis of rotation OZ. We take moment about OZ, so that this reaction could be avoided i.e. the moment of the effective forces about OZ will be equal to the moment of the external forces about OZ. Thus

$Mk^2\ddot{\theta} = L$, where L represents the moment of all external forces about OZ. Above equation is called the equation of motion of the body. In the case of impulsive forces if ω_1 and ω_2 be angular velocities of the body just before and just after the action of the impulses, L the moment of the impulses then we have $Mk^2(\omega_2 - \omega_1) = L$.

ILLUSTRATIVEEXAMPLES

Ex.1. *A straight uniform rod can turn freely about one end O, hangs from O vertically. Find the least angular velocity with which it must begin to move so that it may perform complete revolution in a vertical plane.*

Sol. Let the rod OA at any instant t make an angle θ with the initial vertical position OX. Let G be the centre of gravity and GN perp. to OX. Let $OA = a$ and mass of the rod be m. The equation of motion is



$$mk^2\ddot{\theta} = -mg\left(\frac{a}{2}\right)\sin\theta$$

$\because$ Moment of effective forces about the axis of rotation $= mk^2\ddot{\theta}$ and moment of external forces about the axis of rotation $= -mg(a/2)\sin\theta$

$$\Rightarrow 2a\ddot{\theta} = -3g\sin\theta \left(\because\ k^2 = \frac{a^2}{3}\right)$$

Multiplying the above equation by $\dot{\theta}$ and integrating, we get

$$a\dot{\theta}^2 = 3g\cos\theta + C \qquad ...(1)$$

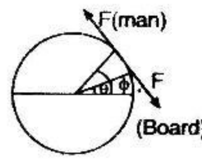
Let $\dot{\theta} = \omega$ when $\theta = 0$ $\therefore a\omega^2 = 3g + C$ ...(2)

Hence from (1) and (2), we get $a\dot{\theta} = a\omega^2 - 3g(1-\cos\theta)$

we require that $\dot{\theta} = 0$ when $\theta = \pi$ $\therefore 0 = a\omega^2 - 6g \Rightarrow \omega = \sqrt{(6g/a)}$.

Ex.2. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through the centre. A man whose weight is equal to that of the board walks on and around it at the edge. when he has completed the circuit, what will be his position in space.*

Sol. Let any time t, θ and ϕ be the angles described by the board and man respectively and let F be the action between the feet of the man and the board. Equation of motion



for the man is $m\,a\ddot{\phi} = F$ ...(1)

Equation of motion for the board is

$$m\,k^2\ddot{\theta} = -F\,a \quad ...(2)$$

On eliminating F between (1) and (2), we get

$$a^2\ddot{\phi} + k^2\ddot{\theta} = 0 \Rightarrow 2\ddot{\phi} + \ddot{\theta} = 0 \; (\because\; k^2 = \frac{a^2}{2})$$

Integrating twice the above equation and considering that initially both man and the board were at rest, we get $2\phi + \theta = 0$

Therefore when $\phi - \theta = 2\pi$ (after completing the circuit)

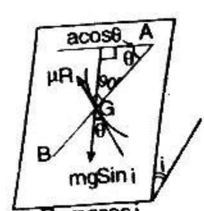
we get, $3\phi = 2\pi \Rightarrow \phi = 2\pi/3$.

This is the angle in space described by the man.

Ex.3. *A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the bar is held in the horizontal position in the plane and allowed to fall from this position, if θ be the angle through which it falls from rest show that* $(\sin\theta/\theta) = \mu\cot i$.

[Agra 89,Raj. 80,Meerut 94,95]

Sol. Let any instant t, the position of the rod be AB, making an angle θ with the initial horizontal position. The external forces acting on the rod, perp. to the plane, are the normal reaction R and resolved part of its weight i.e. $mg\cos i$. External forces acting on the rod in the plane are, (i) the resolved part of its weight, $mg\sin i$ acting down the line of greatest slope through G (centre of gravity). (ii) the friction $\mu R = \mu\,mg\cos i$ acting perp. to AB through G; (iii) the reaction at A. We take moments about A to avoid reaction, so



$$m\,k^2\ddot{\theta} = mg\sin i\,.a\cos\theta - \mu\,mg\cos i\,a$$

$\Rightarrow k^2\ddot{\theta} = ga(\sin i\cos\theta - \mu\cos i)$, where $2a$ is length of the rod.

Multiplying the above equation by $2\dot{\theta}$ and integrating, we get

$$k^2\dot{\theta}^2 = 2ag\sin i\sin\theta - 2\mu ag\theta\cos i + D.$$

When $\theta = 0, \dot{\theta} = 0 \therefore D = 0$. Hence $k^2\dot{\theta}^2 = 2ag\sin i\sin\theta - 2\mu ag\theta\cos i$

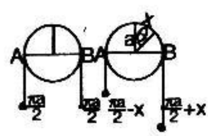
Rod will come to rest when $\dot{\theta} = 0$

$\therefore 0 = 2ag\sin i\sin\theta - 2\mu ag\cos i \Rightarrow (\sin\theta/\theta) = \mu\cot i.$

Ex.4. *A uniform vertical circular plate of radius a, is capable of revolving about a smooth horizontal axis through its centre; a rough perfectly flexible chain, whose mass is equal to that of the plate and whose length is equal to its circumference hangs over its rim in equilibrium, if one end be slightly displaced show that the velocity of the chain when the other end reaches the plate is* $\left(\dfrac{\pi ag}{6}\right)^{1/2}$ **[Meerut 1996,94,92,90]**

Sol. Let x be the distance described in time t. Let v be the velocity of the string and θ be the angular velocity of the plate, then $v = \dot{x} = a\dot{\theta}$

Let m be the mass of the plate and that of the string, then K.E. of the string $= \frac{1}{2}mv^2$. K.E. of the plate



$$= \frac{1}{2}mk^2\dot{\theta}^2 = \frac{1}{2}mk^2\frac{v^2}{a^2}$$

$$= \frac{1}{2}m\frac{a^2}{2}\cdot\frac{v^2}{a^2} = \frac{1}{4}mv^2$$

$$\left[\because k^2 = \frac{a^2}{2}\right]$$

Hence, the total K.E. generated $= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2.$

At time t, length of the string hanging to the right is $\left(\dfrac{\pi a}{2} + x\right)$ and hanging to the left is $\left(\dfrac{\pi a}{2} - x\right)$. the weights of these two portion are respectively $\dfrac{mg}{2\pi a}\left(\dfrac{\pi a}{2} + x\right)$ and $\dfrac{mg}{2\pi a}\left(\dfrac{\pi a}{2} - x\right)$.

The depths of the C.G.'s of these portions below AB are

$$\frac{1}{2}\left(\frac{\pi a}{2} + x\right) \text{ and } \frac{1}{2}\left(\frac{\pi a}{2} - x\right).$$

Hence when x is the displacement, work function on the right is

$$W_1 = \frac{mg}{2\pi a}\left(\frac{\pi a}{2} + x\right)\cdot\frac{1}{2}\left(\frac{\pi a}{2} + x\right).$$

Work function of the left is $W_2 = \frac{mg}{2\pi a}\left(\frac{\pi a}{2} - x\right) \cdot \frac{1}{2}\left(\frac{\pi a}{2} - x\right)$

$\therefore$ Total work function

$$W = W_1 + W_2 = \frac{mg}{4\pi a}\left(\frac{\pi a}{2} + x\right)^2 + \frac{mg}{4\pi a}\left(\frac{\pi a}{2} - x\right)^2 \qquad ...(1)$$

In the initial position i.e. when $x = 0$

$$W_0 = 2\frac{mg}{4\pi a}\frac{\pi^2 a^2}{4} = \frac{1}{8} mg\pi a \qquad \text{[from (1)]}$$

Hence total work done

$$= W - W_0 = \frac{mg}{4\pi a}\left[\left(\frac{\pi a}{2} + x\right)^2 + \left(\frac{\pi a}{2} - x\right)^2 - \frac{1}{2}\pi^2 a^2\right] = \frac{mgx^2}{2\pi a}.$$

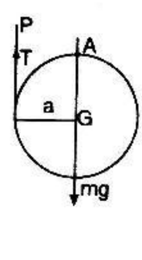
Therefore energy equation gives $\frac{3}{4} mv^2 = \frac{mgx^2}{2\pi a} \Rightarrow v^2 = \frac{2gx^2}{3\pi a}$.

When $x = \frac{\pi a}{2}$ (i.e. when other end reaches the plate)

$$v^2 = \frac{1}{6}\pi a g \Rightarrow v = \left(\frac{1}{6}\pi ag\right)^{1/2}$$

Ex.5. *One end of a light string is fixed to a point of the rim of a uniform circular disc of radius a and mass m and the string is wounded several times round the rim. The free end is attached to a fixed point and the disc is held so that the part of the string not in contact with it, is vertical. If the disc be let go, find the acceleration and the tension of the string.*

Sol. Let the free end be attached to the fixed point P. Let A be the initial position of the centre of gravity G. Let T be the tension of the string. There being no horizontal force the C.G. will move vertically downwards. Let x be the distance moved by G in time t and during this period, θ be the angle turned through some radius.



$$\therefore mg - T = m\ddot{x} \qquad ...(1)$$

and $$Ta = mk^2\ddot{\theta} = m\frac{a^2}{2}\ddot{\theta} \qquad ...(2)$$

Again $x = a\theta$, $\therefore \ddot{x} = a\ddot{\theta}$ ...(3)

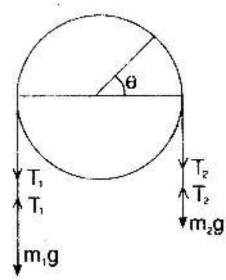
On eliminating T and $\ddot{\theta}$ from (1), (2) and (3), we get

$$mga = ma\ddot{x} + m\frac{a}{2}\ddot{x} \Rightarrow \ddot{x} = \frac{2g}{3}$$

Substituting this value in (1), we get $T = \frac{1}{3}mg$.

Ex.6. *Two unequal masses m_1 and m_2 $(m_1 > m_2)$ are suspended by a light* *string over a circular pulley of mass M and radius a. There is no slipping and the friction of the axis may be neglected. If f be the acceleration; show that this is constant, and if k^2 be the radius of gyration of the pulley about the axle, show that* $k^2 = \dfrac{a^2}{Mf}\left[(g-f)m_1 - (g+f)m_2\right]$

Sol. Let in time t, m_1 move a distance x downwards and m_2 move a distance x upwards. Let θ be the angle through which the pulley has rotated in time t. Since $x = a\,\theta$, $\therefore \ddot{x} = a\ddot{\theta}$



Equations of motion of m_1 and m_2 are

$$m_1\ddot{x} = m_1 g - T_1 \quad ...(1)$$

and $$m_2\ddot{x} = T_2 - m_2 g \quad ...(2)$$

Equation of motion of the pulley is

$$Mk^2\ddot{\theta} = T_1 a - T_2 a$$

(Moment is taken about the axle)

$$\Rightarrow M\frac{k^2}{a^2}\ddot{x} = T_1 - T_2 \quad \left(\because \ddot{\theta} = \frac{\ddot{x}}{a}\right) \quad ...(3)$$

Adding (1), (2) and (3), we get $\ddot{x}\left(m_1 + m_2 + M\dfrac{k^2}{a^2}\right) = m_1 g - m_2 g$

$$\Rightarrow \ddot{x} = f = \frac{(m_1 - m_2)g}{m_1 + m_2 + M\dfrac{k^2}{a^2}}, \text{ which is constant.}$$

From above we get $f(m_1 + m_2) + \dfrac{Mk^2}{a^2}f = (m_1 - m_2)g$

$$\Rightarrow k^2 = \frac{a^2}{Mf}\left[(m_1 - m_2)g - (m_1 + m_2)f\right] = \frac{a^2}{Mf}\left[(g-f)m_1 - (g+f)m_2\right].$$

Pressure on the pulley $= T_1 + T_2$.

Again on subtracting (1) from (2), we get

$$(m_2 - m_1)\ddot{x} = T_2 + T_1 - (m_1 + m_2)g$$

$$\Rightarrow T_2 + T_1 = (m_2 - m_1)\ddot{x} + (m_1 + m_2)g = (m_2 - m_1)f + (m_1 + m_2)g.$$

Ex.7. *A fine string has two masses M and M′ tied to its ends and passes over a rough pulley, of mass m, whose centre is fixed. If the string does not slip over the pulley, show that M will descend with acceleration*

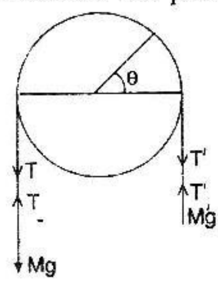
$$\frac{M - M'}{M + M' + (mk^2/a^2)}g$$

[Agra 85,89]

where a is the radius and k the radius of gyration of the pulley.
If the pulley be not sufficiently rough to prevent sliding, and M be the descending mass, show that its acceleration is $\dfrac{M - M' e^{\mu\pi}}{M + M' e^{\mu\pi}} g$ and that pulley will now spin with an angular acceleration equal to

$$\frac{2M M' ga(e^{\mu\pi} - 1)}{m k^2 (M + M' e^{\mu\pi})}$$ **[Agra, 90]**

Sol. First part, when the pulley is rough enough to prevent sliding.
Proceeding like **Ex.6** the equations of motion of masses and pulley are



$$M\ddot{x} = Mg - T \qquad ...(1)$$

and $\quad M'\ddot{x} = T' - M'g \qquad ...(2)$

and moment of effective forces about the axis of rotation $= mk^2\ddot{\theta} = (T - T')a \qquad ...(3)$

Again $x = a\theta, \ddot{x} = a\ddot{\theta}$,

$$\therefore m k^2 \frac{\ddot{x}}{a^2} = T - T' \qquad ...(4)$$

Adding (1), (2) and (4), we get $\ddot{x}\,[M + M' + (mk^2/a^2)] = (M - M')g$

$\Rightarrow$Acceleration $= \ddot{x} = \dfrac{(M - M')g}{M + M' + (mk^2/a^2)}$.

Second Part. When the pulley is not sufficiently rough to prevent sliding, then we can not take $x = a\theta$. In this case, from Statics, we have

$T = T' e^{\mu\pi}$ ...(5). Solving (1), (2) and (5), we have

$$T = \frac{2MM' g e^{\mu\pi}}{M + M' e^{\mu\pi}}, T' = \frac{2MM' g}{M + M' e^{\mu\pi}}$$

and $\ddot{x} = \dfrac{M - M' e^{\mu\pi}}{M + M' e^{\mu\pi}} g$.

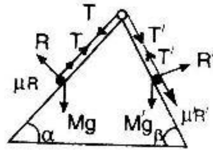
Further putting above values of T and T' in (3), we get

$$\ddot{\theta} = \frac{2ga(e^{\mu\pi} - 1)}{m k^2} \cdot \frac{M M'}{M + M' e^{\mu\pi}}.$$

Ex.8. Two *unequal masses, M and M' rest on two rough planes inclined at an angles α and β to the horizon: they are connected by a fine string passing over a small pulley, of mass m and radius a, which is placed at the common vertex of the two planes; show that the acceleration of either mass is* $\dfrac{g\,[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)}{M + M' + (m k^2/a^2)}$. **[Agra 1991]**

where μ *and* μ' *are the coefficients of friction, k is the radius of gyration of the pulley about its axis and M is the mass which moves downwards.*

Sol. Suppose in time t, the mass M moves a distance x downwards, and also M' moves a distance x upwards. Let the pulley turn through an angle θ, in the same time t.



$\therefore x = a\theta, \dot{x} = a\dot{\theta}, \ddot{x} = a\ddot{\theta}$. The equation of motion of the masses M and M' are

$$M\ddot{x} = Mg\sin\alpha - Mg\mu\cos\alpha - T \qquad ...(1)$$

$$M'\ddot{x} = T' - M'g\sin\beta - M'g\mu'\cos\beta \qquad ...(2)$$

Equation of motion of pulley is $mk^2\ddot{\theta} = (T - T')a$

$$\Rightarrow \frac{mk^2\ddot{x}}{a^2} = T - T' \qquad \left(\because \ddot{\theta} = \frac{\ddot{x}}{a}\right) \qquad ...(3)$$

Adding (1), (2) and (3), we get

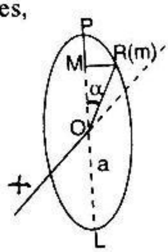
$$\left(\frac{mk^2}{a^2} + M + M'\right)\ddot{x} = g[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta - \mu'\cos\beta)]$$

$$\Rightarrow \ddot{x} = \frac{g[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)]}{M + M' + \dfrac{mk^2}{a^2}}.$$

Ex.9. *A uniform circular disc is free to turn about a horizontal axis through its centre perp. to its plane. A particle of mass m is attached to a point in the edge of the disc. If the motion starts from the position in which radius to the particle makes an angle* α *with the upward vertical, find the angular velocity when m is in its lowest position. Take the mass of the disc as M.* **[Meerut 1973]**

Sol. The circular disc is turning about the fixed horizontal axis OX, through its centre O. Let ω be the angular velocity when m is in its lowest position, say L then energy principle gives,

Change in K.E. = work done by forces.



$$\Rightarrow \left(\frac{1}{2}ma^2\omega^2 + \frac{1}{2}M\frac{a^2}{2}\omega^2\right) - 0$$

$$= mg(a + a\cos\alpha)$$

or $a\omega^2(2m + M) = 4g(1 + \cos\alpha)$

or $\omega = 2\dfrac{\sqrt{2}}{\sqrt{(2m + M)}}\sqrt{(g/a)}\cos\dfrac{\alpha}{2}$.

Remark : The weight of the disc does not work as its C.G. is fixed.

Supplementary Problems

1. *A uniform rod, of mass m and length 2a, can turn freely about one end which is fixed, it is started with angular velocity ω from the position in which it hangs vertically; find its angular velocity at any instant.*

Ans. $\dot{\theta}^2 = \omega^2 - (3g/2a)(1 - \cos\theta)$, where θ *is the angle which the rod makes with downward vertical.*

2. *In solved example 8 show that to prevent slipping the coefficient of friction must be greater than* $\frac{1}{\pi}\log_e \frac{M(2M'a^2 + mk^2)}{M'(2Ma^2 + mk^2)}$.

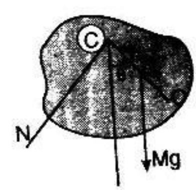
3. *A uniform string of length 20 feet and mass 40 lbs. hangs in equal length over a circular pulley, of mass 10 lbs, and small radius, the axis of the pulley being horizontal, masses of 40 and 35 lbs. are attached to the ends of the string and motion takes place. Show that the time taken by the smaller mass to reach the pulley is* $\frac{1}{4}\sqrt{(15)}\log_e(9 + 4\sqrt{5})$ *seconds.*

[Raj. 1982, Delhi Hons. 83]

2·04. The Compound Pendulum :

To determine the motion of a body acted on by the force of gravity only and moving about a fixed horizontal axis.

Let us take plane of the paper as the plane through the centre of gravity G of the body and perpendicular to the fixed axis. Let the plane meet the axis in C. Let θ be the angle between the vertical and CG *i.e.* θ is the angle between a plane fixed in space and a plane fixed in the body.



Let $CG = h$. The forces on the body are :

(i) its weight Mg acting downward through G.

(ii) the reaction at C of the fixed axis.

We take moments about the fixed axis to eliminate this reaction.

The equation of motion is $Mk^2\ddot{\theta} = -Mgh\sin\theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\sin\theta = -\frac{gh}{k^2}\theta, \text{ (θ being small)} \qquad ...(1)$$

Equation (1) shows that motion is S.H.M. Hence the time of complete oscillation of compound pendulum is $2\pi\sqrt{\left(\frac{k^2}{gh}\right)}$.

Simple Equivalent Pendulum. We know that equation of motion of a particle of any mass suspended by a string of length l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\theta \text{ (θ being small)}$$

The time of complete oscillation is $2\pi\sqrt{\left(\dfrac{l}{g}\right)}$.

If $2\pi\sqrt{(l/g)} = 2\pi\sqrt{\{k^2/(gh)\}}$, then $l = (k^2/h)$.

This length (k^2/h) in the case of a compound pendulum is called the length of the simple equivalent pendulum.

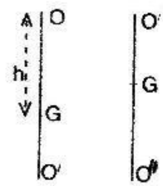
2·05. Centre of Suspension : **[Meerut 1995,94]**

Through C, if a line be drawn perpendicular to the axis of rotation cutting it at C, then C is called the **Centre of suspension.**

Centre of Oscillation. If O is the point on CG produced such that $CO = l = \dfrac{k^2}{h}$ (the length of the simple equivalent pendulum) then the point O is called the centre of oscillation.

2·06. To show that the centres of suspension and oscillation are convertible. **[Meerut 1993,85,83, Agra 86]**

Let us take O and O' as the centre of suspension and oscillation respectively. $\therefore OO' = \dfrac{k^2}{h}$ where $OG = h$, and k is radius of gyration of the body about the axis through O.



Now if K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then

$M k^2 = M K^2 + M . OG^2$

$\Rightarrow M k^2 = M K^2 + M h^2 \Rightarrow k^2 = K^2 + h^2$

$$\therefore OO' = \frac{K^2 + h^2}{h} = \frac{K^2 + OG^2}{OG}$$

$\Rightarrow OO'.OG = K^2 + OG^2 \Rightarrow K^2 = OG\,(OO' - OG) = OG.O'G.$ ...(1)

Let O'' be the centre of oscillation when the body rotates about a parallel axis through O'. We can show as above that

$$K^2 = O'G.O''G \qquad \text{...(2)}$$

From (1) and (2), we observe that O'' is simply the point O. Thus if the body were suspended from a parallel axis through O', O is the centre of oscillation. This proves the theorem.

2·07. Minimum time of oscillation of a compound pendulum.

[Meerut 1980]

If K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then $k^2 = K^2 + h^2$.

Therefore length of the simple equivalent pendulum is

$$l = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = \frac{K^2}{h} + h.$$

The time of oscillation of a compound pendulum will be least when the length of the simple equivalent pendulum is minimum. For that

$$\frac{dl}{dh}=\frac{d}{dh}\left(h+\frac{K^2}{h^2}\right)=0\Rightarrow 1-\frac{K^2}{h^2}=0\Rightarrow h=K.$$

The length of simple equivalent pendulum in this case

$$l=\frac{K^2+h^2}{h}=\frac{K^2+K^2}{K}=2K.$$

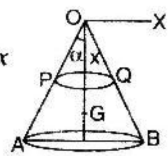
In case $h=0$ or ∞ i.e. if the axis of suspension either passes through G or be at infinite, the corresponding simple equivalent pendulum is of infinite length, thus the time of oscillaion is infinite.

Ex.10. *A solid homogeneous cone of height h and vertical angle 2 α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is* $\frac{1}{5}h(4+\tan^2\alpha)$. **[Meerut 1990,93]**

Sol. Let OX be the horizontal axis through the vertex O. Let us take a circular disc PQ of thickness δx at distance x from O. Moment of Inertia of disc about OX

$=(\rho\,\pi x^2\tan^2\alpha\,\delta x)\,(\frac{1}{4}x^2\tan^2\alpha+x^2)$.

Therefore M.I. of whole cone about OX



$$Mk^2=\rho\pi\tan^2\alpha\,(1+\tfrac{1}{4}\tan^2\alpha)\int_0^h x^4\,dx$$

$$=\rho\pi\tan^2\alpha\,(1+\tfrac{1}{4}\tan^2\alpha)\,\tfrac{1}{5}h^5$$

$$=\frac{1}{20}\rho\,\pi\tan^2\alpha\,(\tan^2\alpha+4)\,h^5$$

$$=\frac{3}{20}M(\tan^2\alpha+4)\,h^2 \qquad (\because\ M=\tfrac{1}{3}\pi h^3\tan^2\alpha\,\rho)$$

$\therefore k^2=\frac{3}{20}(\tan^2\alpha+4)\,h^2$. Again $OG=\frac{3}{4}h$.

Therefore the length of the simple equivalent pendulum i.e.

$$l=\frac{k^2}{OG}=\frac{1}{5}(\tan^2\alpha+4)\,h.$$

Ex.11. *A solid homogeneous cone of height h and semi-vertical angle α, oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is* $\frac{1}{5}h(2+3\tan^2\alpha)$.

Sol. Referring to the fig. of the **Example 10,** we observe that M.I. of the cone about AB

$$=\int_0^h \rho\,\pi x^2\tan^2\alpha\,dx\left[\frac{x^2\tan^2\alpha}{4}+(h-x)^2\right]$$

$$=\int_0^h \frac{\rho\,\pi\tan^2\alpha}{4}\,[x^4\tan^2\alpha+4x^2(h-x)^2]\,dx$$

$$= \frac{\rho \pi \tan^2\alpha}{4} \int_0^h (x^4 \tan^2\alpha + 4x^4 - 8hx^3 + 4h^2x^2)\, dx$$

$$= \tfrac{1}{4} \rho \pi \tan^2\alpha \left(\frac{h^5}{5} \tan^2\alpha + 4\frac{h^5}{5} - 8h\frac{h^4}{4} + 4h^2 \frac{h^3}{3} \right)$$

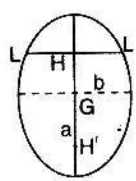
$$= \tfrac{1}{4} \rho \pi \tan^2\alpha\, h^5 \left[\frac{1}{5} \tan^2\alpha + \frac{2}{15} \right] = \frac{1}{60} \rho \pi h^4 \tan^2\alpha\, [3 \tan^2\alpha + 2]$$

$$= \frac{1}{20} M h^2 (3 \tan^2\alpha + 2), \qquad \text{since } M = \tfrac{1}{3} \pi h^3 \tan^2\alpha \,.\, \rho$$

$\therefore M k^2 = \dfrac{1}{20} M h^2 (3 \tan^2\alpha + 2) \Rightarrow k^2 = \dfrac{1}{20} h^2 (3 \tan^2\alpha + 2)$, where k is the radius of gyration of the cone about AB. Hence length of the simple equivalent pendulum

$$= \frac{k^2}{\text{distance of } G \text{ from } AB} = \frac{k^2}{(h/4)} = \frac{1}{5} h (3 \tan^2\alpha + 2).$$

Ex.12. *An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, prove that the eccentricity is* $\frac{1}{2}$.



Sol. When one of the focii say H, is the centre of suspension then the other focus H' is the centre of oscillation. LHL' is the latus rectum (horizontal axis) about which the elliptic lamina oscillates.

$\because$ The length of simple equivalent pendulum

$$l = HH' = 2ae, \qquad \ldots(1)$$

Also $HG = ae$ and $M k^2 =$ Moment of Inertia of the body about the axis of the rotation

$$\text{'}LHL'\text{'} = M \left[\frac{a^2}{4} + a^2e^2 \right] \Rightarrow k^2 = \tfrac{1}{4} a^2 (1 + 4e^2)$$

$$\therefore l = \frac{k^2}{HG} = \frac{1}{4} \frac{a^2 (1 + 4e^2)}{ae} \qquad \ldots(2)$$

From (1) and (2), we get $2ae = \dfrac{1}{4} \dfrac{a^2 (1 + 4e^2)}{ae}$

$\Rightarrow 8e^2 = 1 + 4e^2 \Rightarrow 4e^2 = 1 \Rightarrow e = \frac{1}{2}$.

Ex.13. *A uniform elliptic board swings about a horizontal axis at right angles to the plane of the board and passing through one focus. If the centre of oscillation be the other focus prove that its eccentricity is* $\sqrt{(2/5)}$.

Sol. Refer fig. **Ex.12,** here $M k^2 = M [\frac{1}{4} (a^2 + b^2) + a^2e^2]$

$\therefore$ length of simple equivalent pendulum

$$l = \frac{k^2}{HG} \Rightarrow \frac{k^2}{ae} = \frac{1}{4ae}(a^2 + b^2 + 4a^2e^2) \qquad ...(1)$$

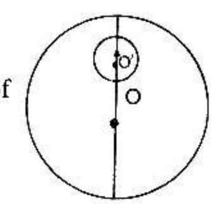
Also $l = 2ae \therefore 2ae = \dfrac{1}{4ae}(a^2 + b^2 + 4a^2e^2)$

$\Rightarrow 8a^2e^2 = a^2 + b^2 + 4a^2e^2 = a^2 + (1 - e^2)\, a^2 + 4a^2e^2$

$\Rightarrow 5a^2e^2 = 2a^2 \Rightarrow e = \sqrt{(2/5)}.$

Ex.14. *A flat circular disc of radius a has a hole in it of radius b whose centre is at a distance c from the centre of the disc ($c < a - b$). The disc is free to oscillate in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Show that the length of the equivalent pendulum is* $c + \dfrac{1}{2}\dfrac{a^4 - b^4}{a^2c}$.

Sol. Let O' be the centre of the hole in the disc whose centre is O. $OO' = c$ (*given*). The disc is oscillated in a vertical plane about a smooth horizontal circular rod of radius b passing through O'.



If h be the depth of C.G. of the body from O',

then $h = \dfrac{\rho\pi a^2 c - \rho\pi b^2 .0}{\rho\pi a^2 - \rho\pi b^2} = \dfrac{a^2c}{a^2 - b^2}$

Let k be the radius of gyration about the axis of rotation, then we have

$$(\rho\pi a^2 - \rho\pi b^2)\, k^2 = \rho\pi a^2\left(\frac{a^2}{2} + c^2\right) - \rho\pi b^2 . \frac{b^2}{2}$$

$$\Rightarrow k^2 = \frac{a^4 + 2a^2c^2 - b^4}{2\,(a^2 - b^2)}$$

$$\therefore l = \frac{k^2}{h} = \left[\frac{a^4 + 2a^2c^2 - b^4}{2\,(a^2 - b^2)}\right] \Big/ \left[\frac{a^2c}{(a^2 - b^2)}\right]$$

$$= \frac{a^4 + 2a^2c^2 - b^4}{2\,a^2c} = c + \frac{1}{2}\frac{a^4 - b^4}{a^2c}.$$

Ex.15. *A bent lever, whose arms are of length a and b, the angle between them being α, makes small oscillations in its own plane about the fulcrum, show that the length of the corresponding simple pendulum is*

$$\frac{2}{3}\frac{a^3 + b^3}{\sqrt{(a^4 + 2a^2b^2\cos\alpha + b^4)}}. \qquad \textbf{[Meerut 1993,85,84]}$$

Sol. Let G_1 and G_2 be the centres of gravity of the arms OA and OB of the lever. Let $OA = a$ and $OB = b$. Also let OA be the axis of x and a perp. line OY the axis of y. Then the co-ordinates of G_1 and G_2 will be $(\frac{1}{2}a, 0)$ and $(\frac{1}{2}b\cos\alpha, \frac{1}{2}b\sin\alpha)$ respectively.

Now if $(\bar{x}, \bar{y})$ is the C.G. of the lever, then

$$\bar{x} = \frac{a\omega \cdot \frac{1}{2}a + b\omega \cdot \frac{1}{2} b\cos\alpha}{a\omega + b\omega} = \frac{1}{2}\frac{a^2 + b^2 \cos\alpha}{a+b};$$

where ω is the weight of unit length of the rod.

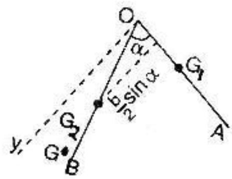
$$\bar{y} = \frac{a\omega \cdot 0 + b\omega \cdot \frac{1}{2} b\sin\alpha}{a\omega + b\omega} = \frac{1}{2}\frac{b^2 \sin\alpha}{a+b}$$

Also the distance of C.G. $(\bar{x}, \bar{y})$ from $O\ (0, 0)$ is

$$\sqrt{(\bar{x}^2 + \bar{y}^2)} = \frac{1}{2(a+b)} \sqrt{\{a^4 + 2a^2b^2\cos\alpha + b^4\}}$$

Now if k is the radius of gyration about the axis of rotation through O, then we have $(a+b)\,\omega k^2 = a\,\omega.\frac{4}{3}(\frac{1}{2}a)^2 + b\,\omega.\frac{4}{3}(\frac{1}{2}b)^2 \Rightarrow k^2 = \dfrac{a^3+b^3}{3(a+b)}$.

Hence the length of the simple pendulum



$$= \frac{k^2}{\text{Dist. of C.G. of the lever from } O}$$

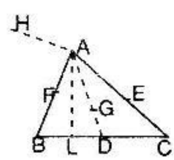
$$= \frac{1}{3}\frac{a^3+b^3}{a+b} \cdot \frac{2(a+b)}{(a^4 + 2a^2b^2\cos\alpha + b^4)^{1/2}}$$

$$= \frac{2}{3}\frac{a^3+b^3}{(a^4 + 2a^2b^2\cos\alpha + b^4)^{1/2}}.$$

Ex.16. *A uniform triangular lamina can oscillate in its own plane about the angle A. Prove that the length of the simple equivalent pendulum is* $\dfrac{3(b^2+c^2) - a^2}{4\sqrt{\{2(b^2+c^2) - a^2\}}}$, *the axis through A being horizontal.*

[Agra 1990]

Sol. Let AH be perpendicular to the plane of the lamina so that it oscillates in its own plane about AH. Instead of the triangular lamina of mass m, we can have three particles each of mass $\frac{1}{3}m$ placed at the mid points D, E, F of the sides respectively. Distance of D from AH is



$$AD = [AL^2 + LD^2]^{1/2} = [AL^2 + (BD - BL)^2]^{1/2}$$

$$= [AL^2 + BD^2 + BL^2 - 2BD \cdot BL]^{1/2}$$

$$= [(AL^2 + BL^2) + BD^2 - 2BD \cdot BL]^{1/2}$$

$$= [(AB)^2 + (\tfrac{1}{2}BC)^2 - 2(\tfrac{1}{2}BC) \cdot AB\cos B]^{1/2}$$

$$= \left(c^2 + \frac{a^2}{4} - ac\cos B\right)^{1/2} = \left(c^2 + \frac{a^2}{4} - ac.\ \frac{a^2 + c^2 - b^2}{2ac}\right)^{1/2}$$

$$= \left(\frac{2b^2 + 2c^2 - a^2}{4}\right)^{1/2}$$

Distance of E from $AH = EA = b/2$

Distance of F from $AH = FA = c/2$

M.I. of the triangle about AH

$$= \frac{1}{3}m\left(\frac{2b^2 + 2c^2 - a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}\right) = \frac{1}{12}m(3b^2 + 3c^2 - a^2)$$

$$\therefore\ mk^2 = \frac{m}{12}[3b^2 + 3c^2 - a^2] \Rightarrow k^2 = \frac{3b^2 + 3c^2 - a^2}{12}$$

Hence length of the simple equivalent pendulum

$$= \frac{k^2}{\text{Dist. of C.}G\text{. from } AH} = \frac{k^2}{AG} = \frac{k^2}{\frac{2}{3}AD}$$

$$= \frac{k^2}{\frac{2}{3}\cdot\frac{1}{2}(2b^2 + 2c^2 - a^2)^{1/2}} = \frac{3k^2}{\sqrt{(2b^2 + 2c^2 - a^2)}}$$

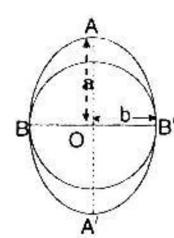
$$= \frac{3(3b^2 + 3c^2 - a^2)}{12\sqrt{(2b^2 + 2c^2 - a^2)}} = \frac{3(b^2 + c^2) - a^2}{4\sqrt{(2b^2 + 2c^2) - a^2)}}.$$

Ex. 17. *An ellipse of axis a, b and a circle of radius b are cut from the same sheet of thin uniform metal and are superposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of the major axis. Show the length of the equivalent simple pendulum is* $\dfrac{5a^2 - ab + 2b^2}{4a}$.

Sol. Mass of the circle $= \pi b^2\rho$

Mass of the ellipse $= \pi ab\,\rho$, where ρ is the mass of the sheet per unit area.

Mass of the system $= \pi b^2\rho + \pi ab\,\rho$



Now taking k to be the radius of gyration of the body about a line through A perpendicular to lamina, we have

$$(\pi ab + \pi b^2)\rho k^2$$

$$= \pi b^2\rho.\left(a^2 + \frac{b^2}{2}\right) + \pi ab\,\rho\left(a^2 + \frac{a^2 + b^2}{4}\right)$$

$$\Rightarrow 4\pi\ b\ (a + b)\ \rho k^2 = \pi b^2\rho(4a^2 + 2b^2) + \pi ab\,\rho(5a^2 + b^2).$$

$$\Rightarrow k^2 = \frac{b(2b^2 + 4a^2) + a(5a^2 + b^2)}{4(a + b)} = \frac{5a^3 + 4a^2b + ab^2 + 2b^3}{4(a + b)}$$

$$= \frac{5a^2(a+b) - ab(a+b) + 2b^2(a+b)}{4(a+b)}$$

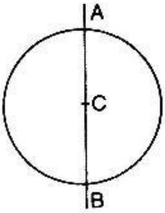
$$= \frac{(a+b)(5a^2 - ab + 2b^2)}{4(a+b)} = \frac{1}{4}(5a^2 - ab + 2b^2)$$

Hence length of the equivalent simple pendulum

$$= \frac{k^2}{\text{Dist. of C.G. of the system from } A} = \frac{k^2}{a} = \frac{5a^2 - ab + 2b^2}{4a}.$$

Ex. 18. *A uniform rod of mass m and length 2a can oscillate about a horizontal axis through one end. A circular disc of mass 24 m and radius $\frac{1}{3}a$ can have its centre clamped to any point of the rod and its plane contains the axis of rotation. Show that for oscillations under gravity the length of the simple equivalent pendulum lies between* $(a/2)$ and $2a$.

Sol. Let AB be the rod axis of rotation pass through A. Let the centre C of the disc, be clamped at a distance x from A.



The distance of C.G. of the system i.e. of the rod and the disc together,

$$h = \frac{ma + 24m.x}{m + 24m} = \frac{2a^2 + 24x^2}{25}.$$

then if k is the radius of gyration then

$$(m + 24m)k^2 = m\frac{4}{3}a^2 + 24m \times \left[\left(\frac{1}{4}\cdot\frac{a}{3}\right)^2 + x^2\right]$$

$$\Rightarrow k^2 = \frac{4a^2 + 2a^2 + 72x^2}{3 \times 2} = \frac{2a^2 + 24x^2}{25}.$$

Hence length of the simple equivalent pendulum.

$$l = \frac{k^2}{h} = \left(\frac{2a^2 + 24x^2}{25}\right) / \left(\frac{a + 24x}{25}\right) \Rightarrow l = \frac{2a^2 + 24x^2}{a + 24x} \qquad ...(1)$$

For maximum or minimum of l, $\frac{dl}{dx} = 0$

$$\Rightarrow \frac{dl}{dx} = \frac{48x(a + 24x) - 24(2a^2 + 24x^2)}{(a + 24x)^2} = 0$$

$$\Rightarrow (24x^2 + 2ax - 2a^2) = 0 \Rightarrow 24x^2 + 8ax - 6ax - 2a^2 = 0$$

$$\Rightarrow 8x(3x + a) - 2a(3x + a) = 0 \Rightarrow (3x + a)(8x - 2a) = 0$$

$$\Rightarrow x = \frac{a}{4} \text{ or } x = -\frac{a}{3}. \text{ Since } x \neq -\frac{a}{3}, \quad \text{we have } x = \frac{a}{4}.$$

When $x = \frac{a}{4}$, we get $l = \frac{a}{2}$ [from (1)]

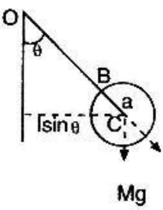
The other extreme value of l (*i.e.* $2a$) is given by putting $x = 0$ or $x = 2a$ in (1). Hence the length of the simple equivalent pendulum lies between $\frac{a}{2}$ and $2a$.

Ex.19. *A sphere of radius a, is suspended by a fine wire from a fixed point at a distance l from its centre. Show that the time of a small oscillation is given by* $2\pi\left(\frac{5l^2+2a^2}{5lg}\right)^{1/2}\left[1+\frac{1}{4}\sin^2\left(\frac{\alpha}{2}\right)\right]$

where α *represents the amplitude of the vibration.* **[Meerut 1988]**

Sol. Suppose that the axis of rotation is passing through O, where $OC = l$. Moment of inertia of sphere of mass M about the axis of rotation is $M\left(\frac{2}{5}a^2+l^2\right)$. Equation of motion is $M\left(\frac{2}{5}a^2+l^2\right)\ddot{\theta} = -M\,gl\sin\theta$

$$\Rightarrow \ddot{\theta} = -\frac{5gl}{2a^2+5l^2}\sin\theta.$$



Integrating, we get $\dot{\theta}^2 = \frac{10\,gl}{2a^2+5l^2}\cos\theta+\lambda$ ...(1)

Let when $\theta=\alpha$, $\dot{\theta}=0$.
Hence (1) reduces to

$$\dot{\theta}^2 = \frac{10gl}{2a^2+5l^2}(\cos\theta-\cos\alpha)$$

$$\Rightarrow \frac{d\theta}{dt} = -\sqrt{\left(\frac{10gl}{5a^2+5l^2}\right)}\sqrt{(\cos\theta-\cos\alpha)}$$

($\because$ Sphere is coming in the direction of θ decreasing)

$$= -\sqrt{\left(\frac{10gl}{2a^2+5l^2}\right)}\sqrt{\left(1-2\sin^2\frac{\theta}{2}-1+2\sin^2\frac{\alpha}{2}\right)}$$

$$= -\sqrt{\left(\frac{10gl}{2a^2+5l^2}\right)}\sqrt{2}.\sqrt{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)}.$$

It t be the time from one extreme to the lowest point, then

$$t = -\frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10gl}\right)}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\{\sin^2(\alpha/2)-\sin^2(\theta/2)\}}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10gl}\right)}\int_{0}^{\alpha}\frac{d\theta}{\sqrt{\{\sin^2(\alpha/2)-\sin^2(\theta/2)\}}}$$

Putting $\sin(\theta/2) = \sin(\alpha/2)\sin\phi$,

i.e. $\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{1}{2}\alpha\cos\phi\,d\phi$, we get

$$t = \sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_{0}^{\pi/2}\frac{d\phi}{\cos(\theta/2)}$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \int_0^{\pi/2} \frac{d\phi}{\sqrt{\{(1 - \sin^2 \frac{\alpha}{2} . \sin^2 \phi)\}}}$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \int_0^{\pi/2} \left[1 + \frac{1}{2} \sin^2 \frac{\alpha}{2} \sin^2 \phi + \ldots\right] d\phi$$

$$\left[\because \quad (1 - x)^{-1/2} = 1 + \tfrac{1}{2} x + \ldots\right]$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[\frac{\pi}{2} + \frac{1}{2} \sin^2 \frac{\alpha}{2} . \frac{\pi}{4} + \ldots\right]$$

$$\because \int_0^{\pi/2} \sin^2 \phi \, d\phi = (\pi/4)$$

$$= (\pi/2) \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[1 + \frac{1}{4} \sin^2 \frac{\alpha}{2}\right]$$

neglecting higher powers of $\sin \frac{\alpha}{2}$, since α is small.

$\therefore$ Time for one small oscillation is

$$4t = 2\pi \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[1 + \frac{1}{4} \sin^2 \frac{\alpha}{2}\right].$$

Ex.20. *Three equal particles are attached to a weightless rod at equal distances a apart. The system is suspended, and is free to turn about a point of the rod distant x from the middle particle. Find the time of a small oscillation and show that it is least when $x = \cdot 82a$ nearly.*

Sol. Let the three particles each of mass m, be attached to the rod at the points A, B and C such that $AB = BC = a$.

Again let the system rotate about ON such that $OB = x$. Then M.I. of the three particles about ON

$$= m(a - x)^2 + mx^2 + m(a + x)^2$$

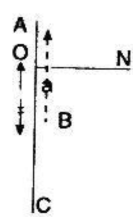
$$\Rightarrow 3mk^2 = m(a - x)^2 + mx^2 + m(a + x)^2$$

$$\Rightarrow k^2 = \frac{3x^2 + 2a^2}{3},$$

where k is the radius of gyration of the system about ON. Now if l is the length of the equivalent simple pendulum then we have

$$l = \frac{k^2}{\text{Dist of C.G. of the system from } O} = \frac{k^2}{x}$$

$$= \frac{3x^2 + 2a^2}{3x} = x + \frac{2a^2}{3x}.$$



$$\therefore \frac{dl}{dx} = 1 - \frac{2a^2}{3x^2}$$

For max. or min. of l, we have $\frac{dl}{dx} = 0$

i.e. $1-\dfrac{2a^2}{3x^2}=0 \Rightarrow x=\dfrac{a}{3}\sqrt{6}=\cdot 816a=\cdot 82a$ nearly.

Further $\dfrac{d^2l}{dx^2}=\dfrac{4a^2}{3x^3}$, which is positive for $x=\cdot 82a$

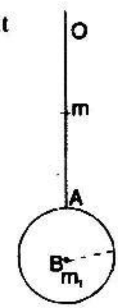
Hence min. value of l is given by $x=\cdot 82a$

Ex.21. *Find the time of oscillation of a compound pendulum consisting of a rod of mass m and length a, carrying at one end a sphere of mass m_1 and diameter 2b , the other end of the rod being fixed.*

Sol. Let $OA=a$ be the rod of mass m, and a sphere of mass m_1 be attached to it at A.

If h is the distance of the C.G. of the system from O, then

$$h=\frac{m.\dfrac{a}{2}+m_1(a+b)}{m+m_1} \quad ...(1)$$



Also if k is the radius of gyration of the system about the axis through O, we have

$$(m+m_1)k^2=m.\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]$$

$$\Rightarrow k^2=\frac{m\dfrac{a^2}{3}+m_1\left[\dfrac{2}{5}b^2+(a+b)^2\right]}{m+m_1}$$

Hence length of equivalent simple pendulum

$$=\frac{k^2}{h}=\frac{m\dfrac{a^2}{3}+m_1\left[\dfrac{2}{5}b^2+(a+b)^2\right]}{m+m_1}\cdot\frac{m+m_1}{m\dfrac{a}{2}+m_1(a+b)}$$

$$=\frac{m\dfrac{a^2}{3}+m_1\left[\dfrac{2}{5}b^2+(a+b)^2\right]}{m\dfrac{a}{2}+m_1(a+b)}$$

and the time of complete oscillation is

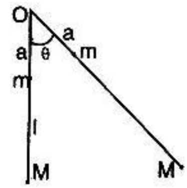
$$=2\pi\left(\frac{k^2}{gh}\right)^{1/2}=\frac{2\pi}{\sqrt{g}}\left[\frac{m\dfrac{a^2}{3}+m_1\left\{\dfrac{2}{5}b^2+(a+b)^2\right\}}{m\dfrac{a}{2}+m_1(a+b)}\right]^{1/2}$$

Ex. 22. *A simple circular pendulum is formed of a* mass *M suspended from a fixed point by a weightless wire of length l, if a mass m, very small compared with M , be knotted on to the wire at a distance a from the point* *of suspension, show that the time of a small vibration of the pendulum is approximately diminished by*

$\frac{m}{2M} \cdot \frac{a}{l}\left(1-\frac{a}{l}\right)$ *of itself.* **[Meerut 1993, Agra 84, 83]**

Sol. Let t be the period of simple pendulum before knotting the mass m, then $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$

Let k be the radius of gyration when mass m is attached to the wire at a distance a from the point of suspension O.



Then $(m+M)k^2 = Ml^2 + ma^2$

or $$k^2 = \frac{Ml^2 + ma^2}{M+m}$$

Distance of C.G. of the system from O is

$$h = \frac{Ml + ma}{M+m}$$

If t' be period for the compound pendulum consisting of masses M and m, then

$$t' = 2\pi\sqrt{\left(\frac{k^2}{gh}\right)} = 2\pi\left[\frac{l}{g}\left(\frac{ml^2+ma^2}{M+m} \cdot \frac{M+m}{Ml+ma}\right)\right]^{1/2}$$

$$= 2\pi\left(\frac{Ml^2+ma^2}{g(Ml+ma)}\right)^{1/2} = 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{Ml^2}\right]^{1/2}\left[1+\frac{ma}{Ml}\right]^{-1/2}$$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{2Ml^2}\right]\left[1-\frac{ma}{2Ml}\right],$$

neglecting higher powers of $\frac{m}{M}$.

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right] = t\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right]$$

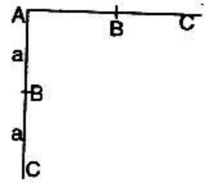
$$\Rightarrow t - t' = \frac{ma}{2Ml}\left(1-\frac{a}{l}\right)t.$$

Ex. 23. *A weightless straight rod ABC of length 2a is movable about the end A which is fixed and carries two particles of the same mass, one fastened to the middle point B and the other to the end C of the rod. If the rod be held in a horizontal position and then let go, show that its angular velocity when vertical is* $\left(\frac{6g}{5a}\right)^{1/2}$ and that $\frac{5a}{3}$ *is the length of the simple equivalent pendulum.*

Sol. Let v, v' be the velocities of the masses at B and C when in vertical position. Let ω be the angular velocity of the rod in this position.
Then we have energy equation as

$$\frac{1}{2}mv^2+\frac{1}{2}mv'^2=mg.a+mg.2a$$

Also $v=a\omega$ and $v'=2a\omega$



$$\therefore\ \frac{1}{2}m\,(a^2+4a^2)\,\omega^2=mga+2mga$$

$$\Rightarrow \omega=\left(\frac{6g}{5a}\right)^{1/2}$$

Again $(m+m)\,k^2=ma^2+m\,(2a)^2 \Rightarrow k^2=\dfrac{5a^2}{2}$

Distance of C.G. from A

$$h=\frac{m.a+m.2a}{m+m}=\frac{3a}{2}\ \therefore\ l=\frac{k^2}{h}=\frac{\frac{5a^2}{2}}{\frac{3a}{2}}=\frac{5a}{3}$$

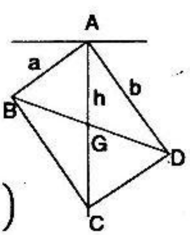
Ex.24. *A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of the diagonal.*

[Meerut 1989]

Sol. Let k be the radius of gyration of the plate about the axis, through A and perpendicular to its plane; then we have

$$mk^2=m\,\frac{a^2+b^2}{4.3}+mh^2\ \text{[by parallel axis theorem]}$$

$$=\frac{mh^2}{3}+mh^2\ =\frac{4mh^2}{3}\Rightarrow k^2=\frac{4h^2}{3}$$



$BG=GD$. Further, distance of C.G. from A

$$=AG=h=\frac{1}{2}\sqrt{(a^2+b^2)}$$

$$\therefore\ \text{period}=2\pi\sqrt{\left(\frac{k^2}{hg}\right)}=2\pi\sqrt{\left(\frac{4h^2}{3gh}\right)}=4\pi\sqrt{\left(\frac{h}{3g}\right)}$$

But period $=1 \Rightarrow 4\pi\sqrt{\left(\dfrac{h}{3g}\right)}=1$ or $h=\dfrac{3g}{16\pi^2}$

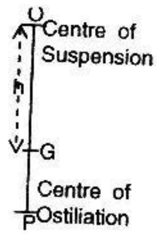
$\therefore$ Length of the diagonal $=2h=\dfrac{3g}{8\pi^2}$.

Ex.25. *A pendulum is supported at O, and P is the centre of oscillation. Show that, if an additional weight is rigidly attached at P, the period of oscillation is unaltered.* **[Meerut 1986, 84]**

Sol. Let m be the mass of the body forming the compound pendulum and let h be the depth of its C.G. below the point of suspension O. Also let k be its radius of gyration about the horizontal axis through O; then we easily obtain

$$OP=(k^2/h)$$

$\Rightarrow$ Period of Oscillation $= 2\pi\sqrt{\left(\dfrac{k^2/h}{g}\right)} = T$, say.



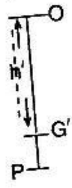
Let an additional weight M be knotted at P, then if k' is the radius of gyration about the horizontal axis through O, we immediately have

$(M+m)\,k'^2 = mk'^2 + M.OP^2$

$$= mk^2 + M.\left(\frac{k^2}{h}\right)^2 = k^2\left(m + M\frac{k^2}{h^2}\right) \qquad ...(1)$$

and by well known C.G. formula,

$$(M+m)\,h' = mh + M.OP = mh + M.\frac{k^2}{h} = h\left(m + M.\frac{k^2}{h^2}\right) \qquad ...(2)$$



(1) and (2) $\Rightarrow \dfrac{k'^2}{h'} = \dfrac{k^2}{h} \Rightarrow T'$

i.e. $2\pi\sqrt{\left(\dfrac{(k'^2/h')}{g}\right)} = 2\pi\sqrt{\left(\dfrac{(k^2/h)}{g}\right)} = T$

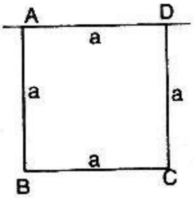
$\Rightarrow$ Period of oscillation is unaltered.

Ex. 26. *Three uniform rods AB, BC, CD each of length a, are freely jointed at B* **and** *C and suspended from the points A* and *D which are in the same horizontal line and a distance a apart. Prove that when the rods move in a vertical plane, the length of simple equivalent pendulum is $\dfrac{5a}{6}$.*

[Meerut 1990, 84]

Sol. The system form a compound pendulum swinging about the horizontal AD. The figure is self explanatory.

Let m be the mass of the each rod.



Let h be the depth of C.G. of the system from AD and k the radius of gyration of the system about the horizontal axis AD, then we easily obtain $3mk^2 =$ sum of the moments of inertia of three rods about AD

$$= m\frac{a^2}{3} + m\frac{a^2}{3} + ma^2 = \frac{5ma^2}{3} \Rightarrow k^2 = (5a^2/9)$$

$$\text{and } h = \frac{\left(m\dfrac{a}{2} + m\dfrac{a}{2} + ma\right)}{3m} = \frac{2ma}{3m} = \frac{2a}{3} \qquad ...(2)$$

$\Rightarrow (k^2/h) = (5a^2/9)/(2a/3) = \dfrac{5a}{6}$

$\Rightarrow$ length of simple equivalent pendulum $= \dfrac{5a}{6}$.

Supplementary Problems

1. Find the length of the equivalent simple pendulum in the following cases, the axis being horizontal :

(i) Circular disc; axis a tangent to it. **Ans.** $(5a/4)$

(ii) Hemisphere; axis a diameter of the base. **Ans.** $(16a/15)$

(a) a diagonal of one face, (b) an edge.

Ans. (a) $(5a/3)$, **(b)** $4\sqrt{(2a/3)}$

2. Find the length of the simple equivalent pendulum for an elliptic lamina when the axis is a latus rectum. **[Meerut 74]**

Ans. $a\,[e + (\frac{1}{4}e)]$

3. A uniform wire, in the form of an arc of a circle of given radius; is swinging about a horizontal axis through the middle point of the arc perpendicular to the plane of the arc. Show that the time of a small oscillation is independent of the length of the arc, and length of an equivalent simple pendulum is equal to the diameter of the circle.

[Agra 1976, 69, Punjab 54]

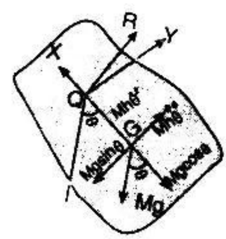
4. Two bodies can move freely and independently under the action of gyration about the same horizontal axis, their masses are m, m' and the distance of their centres of gravity from the axis are h and h'. If the lengths of their equivalent simple pendulum be l, l', Prove that when fastened togther the length of the equivalent simple pendulum will be

$$\frac{mhl + m'h'l}{mh + m'h'}.$$

2·08. Reactions of the axis of rotation.

A body moves about a fixed axis under the action of forces and both the body and the forces are symmetrical with respect to the plane through the C.G. perpendicular to the axis, find the reactions of the axis of rotation.

Let O be the point where the plane through G perpendicular to the axis of rotation meets this axis. By symmetry the actions on the axis reduce to a single force at O, the centre of suspension. Let the components of this single force be X and Y along and perpendicular to GO respectively.



Now G describes a circle round O as centre, its acceleration along and perpendicular to GO are $h\dot{\theta}^2$ and $h\ddot{\theta}$. Equations of motion of C.G. are

$$Mh\,\dot{\theta}^2 = X - Mg\cos\theta \qquad \text{...(1)}$$

$$Mh\,\ddot{\theta} = Y - Mg\sin\theta \qquad \text{...(2)}$$

By taking moments about O, $Mk^2\ddot{\theta} = -Mgh\sin\theta$ ...(3)

where k is the radius of gyration about the axis.

Y is obtained by eliminating $\ddot{\theta}$ from (2) amd (3), By integrating (3) and determining the constant from the initial conditions, and then from (1), we can find X.

Resultant reaction $R = \sqrt{(X^2 + Y^2)}$ and $\tan\phi = (X/Y)$ where ϕ is the angle which the direction of R makes with GO.

Note : On resolving X and Y horizontally and vertically.

The horizontal reaction $= X \sin\theta - Y \cos\theta$

Vertical reaction $= X \cos\theta + Y \sin\theta$

ILLUSTRATIVE EXAMPLES

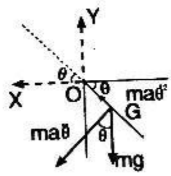
Ex.27. *A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of $45°$ to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.*

[Meerut 1995]

Sol. Let $OA = 2a$, and let the rod make an angle θ with the horizontal after time t. Equations of motion of G along and perpendicular to GO are

$$ma\dot{\theta}^2 = Y \sin\theta + X \cos\theta - mg \sin\theta \qquad ...(1)$$

$$ma\ddot{\theta} = -Y \cos\theta + X \sin\theta + mg \cos\theta \qquad ...(2)$$



Since $k^2 = a^2 + \frac{a^2}{3} = \frac{4}{3}a^2$,

$\therefore$ moment equation about O is

$$m.\frac{4}{3}a^2\ddot{\theta} = mg.a \cos\theta \Rightarrow \ddot{\theta} = \frac{3g}{4a}\cos\theta. \qquad ...(3)$$

Intergrating (3), we get $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta + C$

when $\theta = 0$, $\dot{\theta} = 0$ $\therefore C = 0$, $\therefore$ $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta$.

Putting this value of $\dot{\theta}^2$ in (1), we get

$$\frac{3}{2}mg \sin\theta = Y \sin\theta + X \cos\theta - mg \sin\theta$$

$$\Rightarrow Y \sin\theta + X \cos\theta = \frac{5}{2}mg \sin\theta \qquad ...(4)$$

With the help of (3), the equation (2) becomes as

$$-Y \cos\theta + X \sin\theta + mg \cos\theta = \frac{3mg}{4}\cos\theta$$

$$\Rightarrow -Y \cos\theta + X \sin\theta = -\frac{1}{4}mg \cos\theta \qquad ...(5)$$

Multiplying (4) by $\cos\theta$ and (5) by $\sin\theta$ and adding, we get

$X = (\frac{5}{2} - \frac{1}{4})\, mg \sin\theta \cos\theta = \frac{9}{4} mg \sin\theta \cos\theta = \frac{9}{8} mg \sin 2\theta$.

Similarly, we have $Y = mg\,(\frac{5}{2}\sin^2\theta + \frac{1}{4}\cos^2\theta)$.

We observe that X is maximum when $\sin 2\theta = 1$

i.e. when $2\theta = \frac{\pi}{2}$ or $\theta = \frac{\pi}{4}$.

when $\theta = (\pi/4)$, we have $Y = mg\,[\frac{5}{2}\sin^2(\pi/4) + \frac{1}{4}\cos^2(\pi/4)]$

$$= mg\,[\tfrac{5}{2}\cdot\tfrac{1}{2} + \tfrac{1}{4}\cdot\tfrac{1}{2}] = \tfrac{11}{8}\,mg = \tfrac{11}{8} \text{ times the weight of the rod.}$$

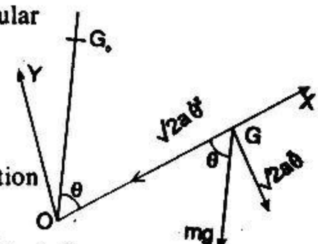
Ex.28. *A heavy homogeneous cube of weight W, can swing about an edge which is horizontal, it starts from rest being displaced from its unstable position of equilibrium. When the perpendicular from the centre of gravity upon the edge has turned through an angle θ, show that the components of the action at the hinge along and at right angles to this perpendicular are $\frac{1}{2}W(3 - 5\cos\theta)$ and $\frac{1}{4}W\sin\theta$.*

Sol. Let G_0 be the initial position of C.G. and G be the position of C.G. when the edge has turned through an angle θ.

$OG_0 = OG = \sqrt{(OL^2 + LG_0^2)} = \sqrt{(a^2 + a^2)} = a\sqrt{2}$,

where $2a$ is the length of the edge.

Equation of motion of G along and perpendicular to GO are



$$Ma\sqrt{2}\dot{\theta}^2 = mg\cos\theta - X \quad \text{...(1)}$$

and $ma\sqrt{2}\ddot{\theta} = mg\sin\theta - Y$ ...(2)

where X, Y are the components of the reaction of the axis in this position.

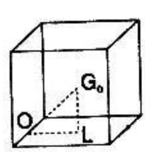
Moment equation about O is $mk^2\ddot{\theta} = mga\sqrt{2}\sin\theta$

$$\Rightarrow m\,(2a^2 + \tfrac{2}{3}a^2)\,\ddot{\theta} = \sqrt{2}amg\sin\theta \Rightarrow \ddot{\theta} = \frac{3}{8}\cdot\frac{\sqrt{2}}{a}\,g\sin\theta \quad \text{...(3)}$$

Integrating, we get $\dot{\theta}^2 = -\frac{3}{4a}\sqrt{2}g\cos\theta + C$.

Initally $\dot{\theta} = 0$, when $\theta = 0$

$$\therefore \dot{\theta}^2 = \frac{3\sqrt{2}g}{4a}(1 - \cos\theta). \quad \text{...(4)}$$



From (1) and (4), we have

$$\frac{3}{2}mg(1 - \cos\theta) = mg\cos\theta - X$$

$$\Rightarrow X = mg(\tfrac{3}{2}\cos\theta + \cos\theta - \tfrac{3}{2}) = \frac{mg}{2}(5\cos\theta - 3) = -\frac{mg}{2}(3 - 5\cos\theta)$$

$$= -\tfrac{1}{2}W(3 - 5\cos\theta) \qquad [\because mg = W],$$

where negative sign of X shows its opposite direction.

From (2) and (3), we have $\frac{3}{4} mg \sin\theta = mg \sin\theta - Y$

$\Rightarrow Y = mg\sin\theta - \frac{3}{4} mg\sin\theta = \frac{1}{4} mg\sin\theta.$

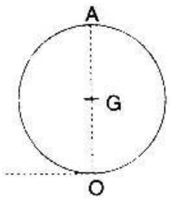
Ex.29. *A circular area can turn freely about a horizontal axis which passes through a point O of its circumference and is perp. to its plane. If motion commences when the diameter through O is vertically above, show that when the diameter has turned through an angle θ the components of the strain at O along and perp. to this diameter are respectively* $\frac{1}{3} W(7\cos\theta - 4)$ *and* $\frac{1}{3} W\sin\theta$. **[Agra 1986]**

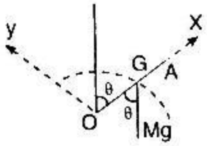
Sol. Initially, when the diameter through O is vertically above O.

M.I. of the disc about an axis through O perp. to the disc

$$= M\frac{a^2}{2} + Ma^2$$

$$= \frac{3Ma^2}{2}.$$





If k is the radius of gyration, then

$$Mk^2 = \frac{3Ma^2}{2} \Rightarrow k^2 = \frac{3a^2}{2}$$

After time t, let the diameter OA makes an angle θ with the vertical. In this position we will have

$$Mk^2 \frac{d^2\theta}{dt^2} = Mgh\sin\theta.$$

where h = distance of C.G. of the disc from $O = a$.

$$\therefore Mk^2\frac{d^2\theta}{dt^2} = Mga\sin\theta \Rightarrow \frac{3a^2}{2}\frac{d^2\theta}{dt^2} = ga\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} = \frac{2g}{3a}\sin\theta.$$

Multiplying by $2\dot{\theta}$ on both sides and integrating it, we get

$$(d\theta/dt)^2 = -\frac{4g}{3a}\cos\theta + c.$$

Initially $\theta = 0, (d\theta/dt) = 0. \quad \therefore 0 = -\frac{4g}{3a} + a \Rightarrow c = \frac{4g}{3a}$

Hence $\left(\frac{d\theta}{dt}\right)^2 = \frac{4g}{3a}(1 - \cos\theta).$ ...(2)

Now considering the motion of C.G., we have

$$M\,a\left(\frac{d\theta}{dt}\right)^2 = Mg\cos\theta - X \qquad ...(3)$$

and $$M\,a\frac{d^2\theta}{dt^2} = M\,g\sin\theta - Y \qquad ...(4)$$

where X, Y are the components of the reaction and perp. to GO.
Solving equation (3), we get

$$X = Mg\cos\theta - Ma\frac{4g}{3a}(1-\cos\theta) \Rightarrow X = \frac{Mg}{3}(7\cos\theta - 4)$$

$$= \frac{1}{3}W(7\cos\theta - 4) \qquad ...(5)$$

Similarly, solving equations (1) and (4), we get

$$Y = Mg\sin\theta - Ma\frac{2g}{3a}\sin\theta = \frac{Mg}{3}\sin\theta = \frac{1}{3}W\sin\theta \qquad ...(6)$$

Ex.30. *A circular disc of weight W can turn freely about a horizontal axis perp. to its plane which passes through a point O on its circumference. If it starts from rest with the diameter vertically above O, show that the resultant pressure on the axis when that diameter is horizontal and vertically below O are respectively $\frac{1}{3}\sqrt{(17)}\,W$ and $\frac{11}{3}W$, Further prove that the axis must be able to bear at least $\frac{11}{3}$ times the weight of the disc.*

Sol. This question is a particular case of the previous example.
When the diameter is horizontal, viz $\theta = \frac{\pi}{2}$, we have

$$X = \frac{W}{3}(0-4) = -\frac{4W}{3}, \quad Y = \frac{W}{3} \qquad \left(\because \sin\frac{\pi}{2} = 1\right)$$

Hence resultant pressure in this case $= \sqrt{\left(\frac{16}{9}W^2 + \frac{W^2}{9}\right)} = \frac{W}{3}\sqrt{(17)}$

When the diameter is vertically below.

$$\theta = \pi, \quad \therefore X = \frac{W}{3}(-7-4) = -\frac{11W}{3}, \quad Y = \frac{1}{3}W\sin\pi = 0$$

Resultant pressure in this case $= \left\{\left(\frac{11W}{3}\right)^2 + 0\right\}^{1/2} = \frac{11}{3}W.$

in general, we have

$$\sqrt{(X^2+Y^2)} = \left[\left\{\frac{W}{3}(7\cos\theta - 4)\right\}^2 + \left\{\frac{W}{3}\sin\theta\right\}^2\right]^{1/2}$$

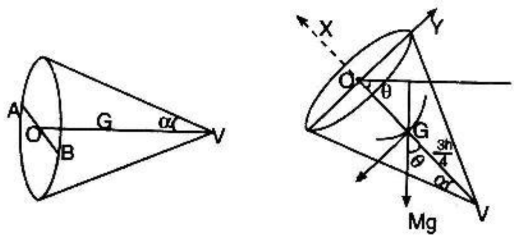
$$= \left\{\frac{W^2}{9}\left(48\cos^2\theta - 56\cos\theta + 17\right)\right\}^{1/2}$$

This is maximum when $\theta = \pi$ and its value is $\frac{11}{3}W$, which implies that the maximum pressure, that the axis must be able to bear is at least $\frac{11}{3}$ times the weight of the disc.

Ex.31 *A right cone of angle 2α can turn freely about an axis passing through the centre of its base and perpendicular to the axis; if the cone starts from rest with its axis horizontal, show that, when the axis is vertical, the thrust on the fixed axis is to the weight of the cone as* $1+\frac{1}{2}\cos^2\alpha : 1-\frac{1}{3}\cos^2\alpha$.

[Agra 1971]

Sol. Let initially the cone be as shown in fig.(i). After any time t, let the cone take the position as shown in fig.(ii). If the height of the cone *i.e.* $OV = h$ then $OG = \frac{1}{4}h$ where G denotes the centre of gravity of the cone. Now since the C.G. of the cone i.e. point G is describing a circle of radius $h/4$, the equations of motion of G are



$M.\frac{1}{4}h\,\dot{\theta}^2 = X - Mg\sin\theta$ ...(1) ; $M.\frac{1}{4}h\,\ddot{\theta} = Mg\cos\theta - Y$ ...(2)

where X and Y denote the components of reaction at O along and perp. to OX. Taking moments about O, we have

$$Mk^2\ddot{\theta} = Mg\frac{1}{4}h\cos\theta \quad \text{...(3)}$$

Also $Mk^2 =$ M.I. of the cone about $AB = M.\frac{1}{20}(2h^2 + 3h^2\tan^2\alpha)$

$$\Rightarrow k^2 = \frac{h^2}{20}(2+3\tan^2\alpha) \quad \text{...(4)}$$

Substituting this value of k^2 in (3), we get

$$h\ddot{\theta} = \frac{5}{2+3\tan^2\alpha}g\cos\theta \quad \text{...(5)}$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$h\dot{\theta}^2 = \frac{10g}{2+3\tan^2\alpha}\sin\theta + C$$

Initially $\dot{\theta} = 0$, when $\theta = 0$, giving there by the constant $C = 0$.

Therefore, we have $h\dot{\theta}^2 = \dfrac{10g}{2+3\tan^2\alpha}\sin\theta$ ...(6)

Using (6) in (1), we get $M.\frac{1}{4}\dfrac{10g}{2+3\tan^2\alpha}\sin\theta = X - Mg\sin\theta$

$$\Rightarrow X = Mg \sin\theta \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right)$$

Also using (5) in (2), we get $Y = Mg\cos\theta \left[\frac{3 + 6\tan^2\alpha}{8 + 12\tan^2\alpha} \right]$

When the axis is vertical i.e. when $\theta = \pi/2$, we have

$$X = Mg \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right), Y = 0.$$

$\therefore$ Resultant pressure

$$= \sqrt{(X^2 + Y^2)} = X = Mg \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right) = Mg \left(\frac{9\cos^2\alpha + 6\sin^2\alpha}{4\cos^2\alpha + 6\sin^2\alpha} \right)$$

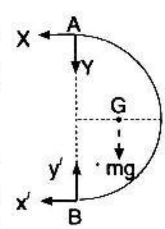
$$\Rightarrow \frac{X}{Mg} = \frac{6 + 3\cos^2\alpha}{6 - 2\cos^2\alpha} = \frac{1 + \frac{1}{2}\cos^2\alpha}{1 - \frac{1}{3}\cos^2\alpha}.$$

Note : If $2\alpha = \pi/2$ then in that case, we have

$$\frac{X}{Mg} = \frac{1 + \frac{1}{2}\cos^2(\pi/4)}{1 - \frac{1}{3}\cos^2(\pi/4)} = \frac{1 + \frac{1}{4}}{1 - \frac{1}{6}} = \frac{3}{2}.$$

Ex.32. *A uniform semi-circular arc, of mass m and radius a, is fixed at its ends to two points in the same vertical line and is rotating with constant angular velocity* ω. *Show that the horizontal thrust on the upper end is* $m\frac{g + \omega^2 a}{\pi}$. **[Meerut 1993]**

Sol. Let the uniform semi-circular arc with centre at O rotate about AB with constant angular velocity ω. If G is the C.G. of the arc, then $OG = \frac{2a}{\pi}$. As the arc rotate, the point G will describe a circle of radius $\frac{2a}{\pi}$ about the point O.



Let X and Y be the horizontal and vertical components of reactions at the point A and X' and Y' the horizontal and vertical reactions at the lower end B. Now since the arc is rotating with constant angular velocity ω about AB, the only effective force on it is $m\frac{2a}{\pi}\omega^2$ along GO.

Taking moments about the point B, we have

$$m\frac{2a}{\pi}\omega^2 a = -mg\frac{2a}{\pi} + X.2a$$

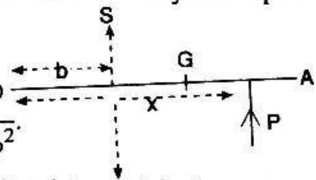
[$\because$ moment of the effective forces = moment of external forces].

$\Rightarrow X = w\dfrac{(g+a\,\omega^2)}{\pi}$.

Ex.33. *A uniform rod OA of mass M and length 2a rests on a smooth table and is free to turn about a smooth pivot at its end O; in contact with it at distance b from O is an inelastic particle of mass m, a horizontal blow of impulse P, is given to the rod at a distance x from O in a direction perp. to the rod ; find the resulting instantaneous angular velocity of the rod and the impulsive action at O and on the particle.* **[Agra 1994]**

Sol. Let OA be the rod of length $2a$ and let a horizontal blow of impulse P be given at a distance x from O. Further let S be the impulse of the action between the rod and inelastic particle of mass m. Then the moment equation about A is $M\,\frac{4}{3}a^2\omega = P\,x - S\,b$ ...(1)

But $S = m\,b\,\omega$,(since velocity $b\,\omega$ is generated in mass m by the impulse S).



$\therefore M\frac{4}{3}a^2\omega = P\,x - m\,b^2\omega$

$\Rightarrow \omega = \dfrac{P\,x}{\frac{4}{3}M\,a^2 + m\,b^2}$ and $S = \dfrac{m P\,b\,x}{\frac{4}{3}M\,a^2 + m b^2}$.

Now since the change in the motion of C.G. of the rod is the same as if all the impulsive forces were applied there, so $M\,a\,\omega = P - S - X$, where X is the impulsive action at O.

$\therefore X = P - (M\,a + m\,b)\,\omega = P\,[1 - (m\,b + M\,a)\,x]/(M\frac{4}{3}a^2 + m\,b^2)$.

2·09. Motion about a fixed axis : Impulsive forces.

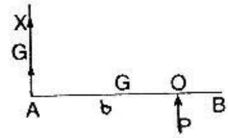
Consider a rigid body under the effect of impulsive forces. Let ω and ω' be the angular velocities about the axis just before and just after the action of impulsive forces. Now change in moment of momentum about the axis $= M\,k^2(\omega' - \omega)$. Also let L the moment of external impulses about the axis of rotation, then we have $M\,k^2(\omega' - \omega) = L$ (since change in moment of momentum of the body about the axis is equal to the moment of the impulsive forces about it).

Ex.34. *A rod, of mass m and length 2a, which is capable of free motion about one end A falls from a vertical position and when it is horizontal strikes a fixed inelastic obstacle at a distance b from the end A. Show that the impulse of the blow is* $m\,\dfrac{2a}{b}\sqrt{(2ga/3)}$ *and that the impulse of the reaction at A is* $m\sqrt{(3ga/2)}\left[1 - \dfrac{4a}{3b}\right]$ *vertically upwards.*

Sol. If ω is the angular velocity just before striking the obstacle then we have the energy equation as $\frac{1}{2}m\,.\,\frac{4}{3}a^2\omega^2 - 0 = m g a$

[Change of K.E. = work done]. $\therefore \omega = \sqrt{(3g/2a)}$

Let the rod AB strike the inelastic obstacle at O such that $AO = b$ and the impulse of the blow be P and the impulsive reaction at A be X. Since the rod reduces to rest after striking the obstacle, therefore we get on taking moment about A



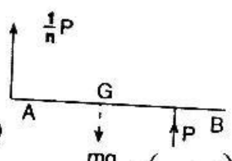
$$m\frac{4}{3}a^2(0-\omega) = -Pb$$

$$\Rightarrow P = \frac{4ma^2\omega}{3b} = \frac{2a}{b}m\,.\sqrt{(2ga/3)}$$

Also for G, we have $m\,(0-a\omega) = -P - X \Rightarrow X = m\sqrt{(3ga/2)}\left[1-\frac{4a}{3b}\right]$.

Ex.35. *A uniform beam AB can turn about its end A is in equlibrium; find the point of its length where a blow must be applied to it so that the impulses at A may be in each case $\frac{1}{n}$ th of that of the blow.*

Sol. Let AB be the uniform rod of mass m and length $2a$. Let an impulse P be applied at a distance x from A so as to produce an impulsive action $\frac{1}{n}P$ at A. If the angular velocity produced is ω, then the equations of motion are



$$m\,k^2\omega = Px \Rightarrow m\frac{4}{3}a^3\omega = Px \qquad ...(1)$$

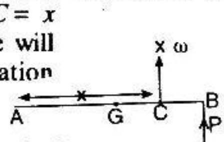
and $$m\,a\,\omega = P + \frac{1}{n}P = \frac{n+1}{n}P. \qquad ...(2)$$

Eliminating P from these two equations, we get $x = \frac{4}{3}\left(\frac{n+1}{n}\right)a$

Note. If the direction of the impulsive action is opposite to that as shown in the fig., then in that case we will have $x = \frac{4}{3}\left(\frac{n-1}{n}\right)a$.

Ex.36. *A rod of mass nM is lying in a horizontal table and has one end fixed; a particle of mass M is in contact with it. The rod receives a horizontal blow at its free end; find the position of the particle so that it may start moving with the maximum velocity. In this case show that the kinetic energies communicated to the rod and mass are equal.*

Sol. Let AB be the rod, the end A of which is fixed. Let an impulse P be applied to the rod at the end B so as to give an angular velocity ω, if the particle of mass M is at C where $AC = x$ then the velocity V acquired by the particle will be $V = x\omega$. Thus we get the moment equation as $nM\frac{4}{3}a^2\omega + Mx\omega\,.\,x = P\,.\,2a$



$$\Rightarrow \omega = \frac{2aP}{M\frac{4}{3}(a^2n + x^2)}\,. \qquad \therefore V = x\omega = \frac{2aPx}{M\frac{4}{3}(na^2 + x^2)}$$

For maximum V, we must have $\dfrac{dV}{dx} = 0 \Rightarrow \dfrac{2aP}{M}\left[\dfrac{\frac{4}{3}na^2 + x^2 - 2x^2}{(\frac{4}{3}na^2 + x^2)^2}\right] = 0$

$\Rightarrow \frac{4}{3}na^2 - x^2 = 0 \Rightarrow x = 2a\sqrt{(n/3)}$

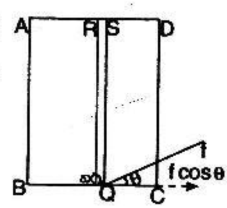
Also K.E. of the rod $= \frac{1}{2}nM \cdot \frac{4}{3}a^2\omega^2 = \frac{2}{3}nMa^2\omega^2$ ...(1)

and K.E. of the particle $= \frac{1}{2}Mx^2\omega^2 = \frac{1}{2}M\dfrac{4a^2n}{3}\omega^2 = \frac{2}{3}Mna^2\omega^2$. ...(2)

From (1) and (2), we observe that kinetic energies of the rod and mass are equal.

Ex.37. *The door of a railway carriage stands upon at right angles to the length of the train when the latter starts to move with an acceleration f; the door being supposed to be smoothly hinged to the carriage and to be uniform and of breadth 2a, show that its angular velocity when it has turned through an angle θ is* $\sqrt{\left(\dfrac{3f}{2a}\sin\theta\right)}$.

Sol. Let $ABCD$ be the door which can rotate about AB. If the train moves with acceleration f, then every element of the door will have the same acceleration f parallel to the rails. Now consider an elementary strip $PQRS$ at a distance x from AB. Mass of the strip $= \dfrac{M}{2a}\delta x$, where M is the mass of the door. Hence moment equation about AB gives



$$M\frac{4}{3}a^2\ddot{\theta} = \int_0^{2a}\frac{m}{2a}dx\, f\cos\theta\, x = m\,a\,f\cos\theta$$

$$\Rightarrow \ddot{\theta} = \frac{3f}{4a}\cos\theta$$

Multiplying both sides by $2\dot{\theta}$ and integrating it, we get

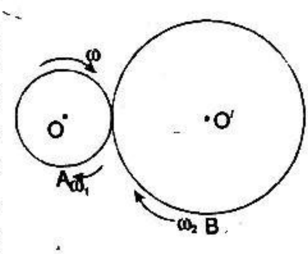
$$\dot{\theta}^2 = \frac{3f}{2a}\sin\theta + \lambda. \text{ Initially } \dot{\theta} = 0 \text{ when } \theta = 0 \;\therefore\; \lambda = 0$$

Hence $\dot{\theta} = \sqrt{\left(\dfrac{3f}{2a}\sin\theta\right)}$.

Ex.38. *Two wheels on spindles in fixed bearings suddenly engage so that their angular velocities become inversely proportional to their radii and in opposite directions. One wheel, of radius a and moment of inertia* I_1 *has angular velocity* ω *initially, the other of radius b and moment of inertia* I_2 *is initially at rest. Show that their new angular velocities are*

$$\frac{I_1 b^2}{I_1 b^2 + I_2 a^2}\omega \text{ and } \frac{I_1 ab\,\omega}{I_1 b^2 + I_2 a^2}.$$

Sol. Let A and B be the two wheels. The wheel A is of radius a and moment of inertia I_1 whereas the wheel B is of radius b and moment of inertia I_2. Initially A was rotating with angular velocity ω and the wheel B was at rest. Now let ω_1 and ω_2 be the angular velocity of A and B after the impact. Since the velocity of the point of contact is the same for each wheel, we have $a\,\omega_1 = b\,\omega_2$ ...(1)



Also $I_1(\omega - \omega_1) = R \times a$ (for the wheel A) ...(2)

$I_2(\omega - 0) = R \times b$ (for the wheel B) ...(3)

where R is the impulsive force.

From the last two equations, we get $I_1(\omega - \omega_1)\,b = I_2\,a\,\omega_2$ ...(4)

Now substituting the value of ω_2 from (1) in (4), we get

$$\omega_1 = \frac{I_1 b^2}{I_1 b^2 + I_2 a^2}\,\omega$$

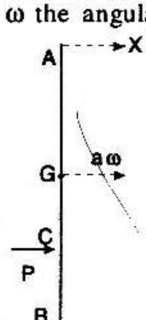
Substituting the value of ω_1 in (1), we have $\omega_2 = \dfrac{I_1\,a\,b}{I_1 b^2 + I_2 a^2}\,\omega.$

2·10. Centre Of Percussion : **[Meerut 1990,94,93,82,80,76]**

If a body , rotating about a given axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of action of the force is called a *centre of percussion*. If the line of action of the blow is known, the axis about which the body begins to turn is called the axis of *spontaneous rotation*. Obviously this combines with the position of the fixed axis in the first case.

2·11. Centre of Percussion of a rod : **[Meerut 1996,75]**

Consider a rod AB of length $2b$. Let it be suspended freely from one end A. Let a horizontal blow of impulse P be applied to it at the point C where $AC = x$. If X is the impulsive action at A and ω the angular velocity communicated to the rod, then the equations of motion are



$$M k^2 \omega = P x \quad (moment\ eq^n) \qquad ...(1)$$

$$M(a\,\omega - 0) = P + X \qquad ...(2)$$

where $a\,\omega$ is the velocity with which G moves.

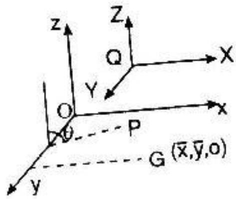
Now if the blow has been given through the centre of percussion then $X = 0$ and equation (2) becomes $M\,a\omega = P$.

Substituting this value of P in (1), we get

$x = \dfrac{k^2}{a} =$ length of the equivalent simple pendulum.

2·12. General Case of Centre of Percussion :

Let us take the fixed axis as the axis of y. Also let centre of gravity G lie in the xy– plane, so that coordinates of G are $(\bar{x}, \bar{y}, 0)$.



If Q is the point where the blow is applied then take a plane through Q and perp. to xy -plane as the xz-plane so that coordinates of Q may be $(\xi, 0, \zeta)$. Now consider any other point P of mass m of the body at a distance r from Oy at any angle θ with z-axis. The coordinates of P will be

$$x = r\sin\theta,\ y = const.,\ z = r\cos\theta.$$

If before the blow, angular velocity is ω and the velocity component along the axes are u, v, w respectively, then we have

$$\dot{x} = u = r\cos\theta, \dot{\theta} = z\,\omega, \dot{y} = v = 0, \dot{z} = w = -r\sin\theta\,.\,\dot{\theta} = -x\,\omega.$$

If after the blow, the angular velocity is ω and velocity component along the axes becomes as u', v', w', then

$$u' = z\,\omega', v' = 0, w' = -x\,\omega'.$$

If X, Y, Z are the components of the blow at the point Q, then equations of motion will be

$$X = \Sigma m(u' - u) = \Sigma m z(\omega' - \omega) = (\omega' - \omega)\Sigma m z$$
$$= (\omega' - \omega)\bar{z}\,\Sigma m = M(\omega' - \omega)\,\bar{z} = 0 \quad (\text{since } \bar{z} = 0) \qquad ...(1)$$
$$Y = \Sigma m\,(v' - v) = 0 \qquad (\text{since } v' = 0 \text{ and } v = 0) \qquad ...(2)$$
$$Z = \Sigma m(w' - w) = -(\omega' - \omega)\Sigma m x = -(\omega' - \omega)\,\bar{x}\,\Sigma m$$
$$= -M(\omega' - \omega)\,\bar{x} \qquad ...(3)$$
$$-Y\zeta = \Sigma m\,\{y(w' - w) - z(v' - v)\} = -(\omega' - \omega)\Sigma m x y$$
$$= -(\omega' - \omega)\,F \qquad ...(4)$$
$$\Rightarrow F = 0 \qquad (\because\ Y = 0)$$
$$\zeta X - \xi Z = \Sigma m\,\{z(u' - u) - x(w' - w)\}$$
$$= -(\omega' - \omega)\,\Sigma m\,(z^2 + x^2) = M k^2(\omega' - \omega) \qquad ...(5)$$

[$M k^2$ is the M.I. of the body about y– *axis*]

$$\xi Y = \Sigma m\,\{x(v' - v) - y(u' - u)\} = -(\omega' - \omega)\,\Sigma m z x = -(\omega' - \omega)\,D$$
$$\Rightarrow D = 0 \qquad [\because\ Y = 0] \qquad ...(6)$$

Thus we get $X = 0, Y = 0$, which implies that blow has no components parallel to the axes of x and y. Hence the blow must be perp. to xy– plane which contains the fixed axis and the instantaneous position of the centre of gravity. Also we see that $F = 0$ and $D = 0$ which implies that the y– *axis* which is also the axis of the body is a principal axis at the point where the plane through the line of action of the blow perp. to the fixed axis cuts it. This is a necessary condition for the existence of the centre of percussion. So if the fixed axis is not a principal axis at some point, then there is no centre of percussion.

Using equation (3) and (5), we get $\xi = \dfrac{k^2}{x}$ ...(7)

The obvious conclusion from the relation (7) is that the distance of the centre of percussion from the fixed axis is the same as that of the centre of oscillation.

Points to remember in finding out the centre of percussion of a body for fixed axis.

(i) *Find the point where the fixed axis is principal axis.*

(ii) *Take a distance* $\dfrac{k^2}{x}$.

(iii) *Draw an axis perp. to the plane containing the fixed axis and C.G. at a distance* $\dfrac{k^2}{x}$ *below the point where fixed axis is principal axis.*

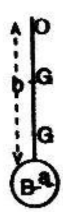
(iv) *Any point on this line is a centre of percussion of the body for the fixed axis.*

Ex.39. *A pendulum is constructed of a solid sphere of mass M and radius a which is attached to the end of a rod of mass m and length b. Show that there will be no strain on the axis if the pendulum be struck at a distance* $\left[M.\left\{\frac{2}{5}a^2+\left(a+b\right)^2\right\}+\frac{1}{3}mb^2\right] \div \left[M\left(a+b\right)+\frac{1}{2}mb\right]$ *from the axis.*

Sol. Let $OA = b$ be the rod fixed at the point O. Let a sphere of radius a and mass M be attached to the other end A of the rod.

Distance of the C.G. of the pendulum from O

$$h = \frac{m\,(b/2) + M\,(b+a)}{m+M} \quad ...(1)$$



Let k be the radius of gyration of the pendulum about O, then we have

$$(m+M)\,k^2 = M\,[(b+a)^2 + \tfrac{2}{5}a^2] + \tfrac{4}{3}m\left(\frac{b}{2}\right)^2$$

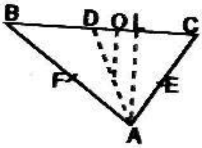
$$\Rightarrow k^2 = \frac{M}{m+M}\left[\,(b+a)^2 + \frac{2}{5}a^2\right] + \frac{4m}{3(m+M)}\left(\frac{k}{2}\right)^2$$

$\therefore$ Distance of centre of percussion from $O = \dfrac{k^2}{h}$

$$= \frac{M\,[\frac{2}{5}a^2 + (a+b)^2] + \frac{1}{3}mb^2}{\frac{1}{2}mb + M\,(a+b)}.$$

Ex.40. *Find the centre of percussion of a triangle ABC which is free to move about its side BC.* **[Meerut 1983,80,75]**

Sol. To find out the point where BC is a principal axis. Let us proceed like this. Draw AD, the median and AL the perp. from A on BC. Let O be the mid point of DL. Then, by the elementary knowledge of M.I. and P.I., BC is a principal axis at the point O. Let the mass of the ΔABC be m. The triangle of mass



m is kinetically equivalent to the particles each of mass $\frac{m}{3}$ placed at the mid points D, E, and F. Let $AL = p$, then

$$m k^2 = \frac{m}{3}\left(\frac{1}{2}p\right)^2 + \frac{m}{3}\left(\frac{1}{2}p\right)^2 + \frac{m}{3}(0) = \frac{1}{6} m p^2 \Rightarrow k^2 = \frac{1}{6} p^2.$$

But the depth of C.G. below $BC = h = \frac{1}{3}p$.

Hence depth of the centre of percussion below BC along a vertical through $O = (k^2/h) = \frac{1}{2}p$.

Particular Case : If the triangle ABC is an equilateral triangle, then the point D and O coincide. In this case $k^2 = \frac{1}{6}p^2, h = \frac{1}{3}p$

Hence the depth of the centre of percussion below BC along the median bisecting BC is $= \frac{k^2}{h} = \frac{1}{2}p$.

Ex.41. *Find how an equilateral lamina must be struck that it may commence to rotate about a side.* **[Meerut 1974,73]**

Sol. Refer fig. Ex.40. The triangle ABC rotate about the side BC. The blow should be given at the centre of percussion when BC is the axis of rotation of the lamina. Here BC is the principal axis of triangle at its middle point (points D, O, L will coincide).

Again $k^2 = \frac{1}{6}p^2$; $h = \frac{1}{3}p$ where p is the height of the triangle.

$\therefore$ Depth of the centre of percussion below BC along the median bisecting BC is $\frac{k^2}{h}$ i.e. $\frac{1}{2}p$. Hence the blow should be given at the middle point of the median bisecting the side about which the lamina rotates.

Ex.42. *Find the position of the centre of percussion of a sector of a circle, axis in the plane of the sector, perp. to its symmetrical radius and passing through the centre of the circle.*

Sol. Consider the sector AOB of a circle of radius a. Let $\angle AOB = 2\alpha$. Let a line OY perp. to the plane of the sector be the axis of rotation.

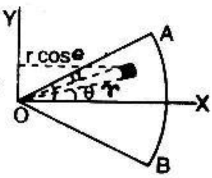
Then M.I. of the sector AOB about $OY = 2\int_0^a \int_0^\alpha \rho\, r^2 \cos^2\theta\, r\, d\theta\, dr$

$$= \rho \cdot \frac{a^4}{4} \int_0^{\alpha} (1 + \cos 2\theta)\, d\theta = \rho \cdot \frac{a^4}{4} (\alpha + \sin \alpha \cos \alpha)$$

$$= \frac{M a^2}{4\alpha} (\alpha - \sin \alpha \cos \alpha), \text{ since mass of the sector } M = \rho \cdot \alpha a^2$$

$$\therefore M k^2 = \frac{M a^2}{4\alpha} (\alpha + \sin \alpha \cos \alpha) \Rightarrow k^2 = \frac{a^2}{4\alpha} (\alpha + \sin \alpha \cos \alpha)$$

Distance of C.G. from $O = h = \frac{2a}{3} \cdot \frac{\sin \alpha}{\alpha}$



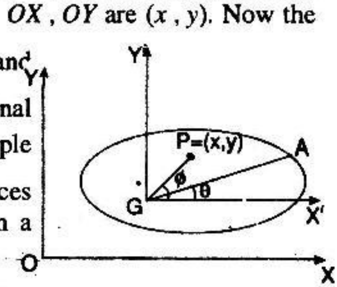
Hence distance of centre of percussion from O

$$= \frac{k^2}{h} = \frac{3a}{8} \left(\frac{\alpha + \sin \alpha \cos \alpha}{\sin \alpha} \right)$$

Motion in Two Dimensions

(Under finite Forces)

3·01. Dynamical Equations of Motion . *To determine dynamical equations of motion in two dimensions when the forces acting on the body are finite.*
The motion of a rigid body consists of two independent motions *viz.*,
(i) the motion of centre of gravity, and
(ii) the motion about the centre of gravity.
Motion of Centre of Gravity.
Cartesian Method
Motion of *C.G. states that the motion of centre of gravity is such that the* ***total mass*** *M of the rigid body is allowed to act at the C.G. and all the* ***external forces*** *are transferred parallel to themselves to act at the C.G. of the body.*
Consider a particle m of the rigid body at point P whose coordinates referred to two axes fixed in space of two dimensions, OX , OY are (x , y). Now the effective forces acting on the particle are $m\ddot{x}$ and $m\ddot{y}$, let X , Y be the components of the external forces acting at P. By D' Alembert's principle $(X - m\ddot{x}),(Y - m\ddot{y})$ together with similar forces acting on all other particles of the body form a system in statical equilibrium, thus we have



$$\Sigma (X - m\ddot{x}) = 0 , \Sigma (m\, Y - m\ddot{y}) = 0$$

and $$\Sigma [x (Y - m\ddot{y}) - y ((X - m\ddot{x})] = 0$$

and $$\left.\begin{array}{l}\Rightarrow \Sigma\, m\ddot{x} = \Sigma\, X , \Sigma\, m\ddot{y} = \Sigma Y \\ \Sigma m\, (x\ddot{y} - y\,\ddot{x}) = \Sigma\, (xY - yX)\end{array}\right\} \qquad ...(1)$$

Let (x_G , y_G) be the co-ordinates of the centre of gravity refered to axes OX and OY and (x' , y') be the co-ordinates of the point P referred to parallel axes GX' and GY' through G.

$\therefore\ x = x_G + x' , y = y_G + y'$

then $Mx_G = \Sigma\, mx , My_G = \Sigma\, my$ (where $\Sigma\, m = M$)

$\Rightarrow \ddot{x}\, M_G = \Sigma m\ddot{x}$ and $M\,\ddot{y}_G = \Sigma m\ddot{y}$.

Thus the first two equations of (1) reduces to

$$M\,\ddot{x}_G = \Sigma X \text{ and } M\,\ddot{y}_G = \Sigma Y \qquad ...(2)$$

Motion Relative to Centre of gravity

Third equation of (1) gives

$$\Sigma m\,[(x_G + x')(\ddot{y}_G + \ddot{y}') - (y_G + y')(\ddot{x}_G + \ddot{x}')]$$
$$= \Sigma\,[(x_G + x')\,Y - (y_G + y')\,X]$$

or $$(x_G\,\ddot{y}_G - y_G\,\ddot{x}_G)\,\Sigma m + x_G\,\Sigma m\ddot{y}' + \ddot{y}_G\,\Sigma mx'$$
$$- y_G\,\Sigma m\ddot{x}' - \ddot{x}_G\,\Sigma my' + \Sigma m\,(x'\ddot{y}' - y'\ddot{x}')$$
$$= x_G\,\Sigma Y - y_G\,\Sigma X + \Sigma\,(x'\,Y - y'X) \qquad ...(3)$$

where $\Sigma m = M$.

By (2), first term on L.H.S. of (3) cancels the first two terms **on the R.H.S.** of (3).

Again $\dfrac{\Sigma mx'}{\Sigma m}$ and $\dfrac{\Sigma my'}{\Sigma m}$ give the coordinates of G with respect to **axes** GX' and GY'

i.e. $\Sigma\,mx' = 0,\ \Sigma\,my' = 0 \quad \Rightarrow \Sigma\,m\ddot{x}' = 0,\ \Sigma\,m\ddot{y}' = 0$

Thus (3) reduces to

$$\Sigma m\,(x'\,\ddot{y}' - y'\,\ddot{x}') = \Sigma\,(x'Y - y'X) \qquad ...(4)$$

$$\frac{d}{dt}\,\Sigma m\,(x'\,\dot{y}' - y'\,\dot{x}') = \Sigma\,(x'Y - y'X) \qquad ...(5)$$

Let GA be a line fixed on the body which makes an angle θ with GX

Let $GP = r$ and $\angle PGX' = \phi$.

$$\phi = \theta + \angle AGP.$$

Since the body turns about G, $\angle AGP$ remains constant.

$\Rightarrow \dot{\phi} = \dot{\theta}$ and $\ddot{\phi} = \ddot{\theta}$.

Again the velocity of m at point P is $r\,\dot{\phi}$ perpendicular the GP, its **moment** about G is $r\,\dot{\phi}.\ r = r^2\,\dot{\phi}$

$$\therefore\ \Sigma\,m\,(x'\dot{y}' - y'\dot{x}') = \Sigma\,mr^2\,\dot{\phi}$$

or $$\Sigma\,m\,(x'\dot{y}' - y'\dot{x}') = \Sigma\,mr^2\,\dot{\theta} = \dot{\theta}\ \Sigma\,mr^2 = Mk^2\,\dot{\theta}$$

where Mk^2 is the moment of inertia of the body about G.

Hence equation (5) may be put as

$$\frac{d}{dt}\,(Mk^2\,\dot{\theta}\,) = \Sigma\,(x'Y - y'X) \quad \text{or } Mk^2\,\ddot{\theta}\; = L \qquad ...(6)$$

where L is the moment of the external forces about G.

Thus the equations of motion of the body are $M\,\ddot{x}_g = \Sigma\,X$, $M\,\ddot{y}_g = \Sigma\,Y$ and are known as equations of motion of the centre of gravity

and $\qquad \Sigma\,m\,(x\,\ddot{y}\,' - y\,\ddot{x}\,') = \Sigma\,(x'\,Y - y'\,X)$

known as equation of motion about the centre of gravity, this can also be put as $\quad Mk^2\,\ddot{\theta}\; = L$

where L is the moment of external forces about G.

This states that the sum of the moments, of the effective forces about the centre of gravity G, is equal to the sum of the moments of the external forces about G.

Vector Method .

Let $\mathbf{r}_G$ be the position vector of the **C.G.** and **F** the external forces acting at any particle m of the body, then we have $M\,\dfrac{d^2\mathbf{r}_G}{dt^2} = \Sigma\mathbf{F}$.

But $\qquad \mathbf{r}_G = x_G\,\mathbf{i} + y_G\,\mathbf{j}$ and $\mathbf{F} = X\;\mathbf{i} + Y\;\mathbf{j}$

where $(x_g\,, y_g)$ are the co-ordinates of **C.G.** and X, Y the components of the forces **F** parallel of the axes.

$\therefore\quad$ (1) gives ; $M\left[\dfrac{d^2x_G}{dt^2}\;\mathbf{i} + \dfrac{d^2y_G}{dt^2}\;\mathbf{j}\right] = \Sigma\,(X\;\mathbf{i} + Y\;\mathbf{j})$.

Equating coefficients of **i** and **j** on both sides, we get

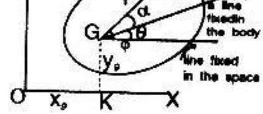
$$M\frac{d^2x_G}{dt^2} = \Sigma X \qquad ...(2) \quad \text{and} \quad M\frac{d^2y_G}{dt^2} = \Sigma Y \qquad ...(3)$$

These are the equations of the centre of gravity.

Let $\mathbf{r}'$ be the position vector, of the particle m at P, relative to G, and **F** the external forces acting on it, then we have

$$\Sigma\mathbf{r}' \times \frac{d^2\mathbf{r}'}{dt^2} = \Sigma\mathbf{r}' \times \mathbf{F} \;\Rightarrow \frac{d}{dt}\;\Sigma m\mathbf{r}' \times \frac{d\,\mathbf{r}'}{dt} = \Sigma\mathbf{r}' \times \mathbf{F} \qquad ...(4)$$

Now let θ be the angle that a line GA fixed in the body makes with a line GB fixed in the space, and let ϕ be the angle which the line joining P to G makes with the line GB (fixed in the space), then as obvious from the adjoining figure, we have $\quad \phi = \theta + \angle AGP = \theta + \alpha$.

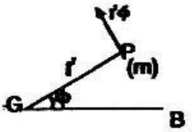


$\therefore\quad \dfrac{d\,\phi}{dt} = \dfrac{d\,\theta}{dt}$, $\quad [\because\;\angle AGP = \alpha$ is constant]

Let $Gm = r'$

$\therefore$ the velocity of m relative to G

$= r' \dfrac{d\phi}{dt}$ in a direction perpendicular to r' in the plane AGP.



If $\hat{e}_1$, $\hat{e}_2$ be the unit vectors along and perpendicular to $\mathbf{r}'$ in the AGP plane, then we have

$$\mathbf{r}' = r'\hat{e}_1 \quad \text{and} \quad \frac{d\mathbf{r}'}{dt} = \mathbf{r}' \frac{d\phi}{dt}\hat{e}_2$$

$$\Rightarrow \Sigma m\mathbf{r}' \times \frac{d\mathbf{r}}{dt} = \Sigma m(r'\hat{e}_1) \times \mathbf{r}' \frac{d\phi}{dt}\hat{e}_2$$

$$= \Sigma m r'^2 \frac{d\theta}{dt}\hat{e}_1 \times \hat{e}_2 \quad \left[\because \frac{d\phi}{dt} = \frac{d\theta}{dt} + \frac{d\alpha}{dt} = \frac{d\theta}{dt}\right]$$

$= \dfrac{d\theta}{dt} \Sigma m r'^2 \,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector normal to the plane AGP.

$$= \frac{d\theta}{dt}(\Sigma m r'^2)\,\hat{\mathbf{n}}$$

$= \dfrac{d\theta}{dt}(Mk^2)\,\mathbf{n}$ where k is the radius of gyration of the body about G

$$= \left(Mk^2 \frac{d\theta}{dt}\right)\hat{\mathbf{n}}.$$

Also we have, moment of the forced $\mathbf{F}$ about $G = \Sigma \mathbf{r}' \times \mathbf{F}$

$= \Sigma p' F\,\hat{\mathbf{n}}$ where p is the lwength of the perpendicular from G upon the direction of the force $\mathbf{F}$

$\therefore$ Equation (4) reduces to $\dfrac{d}{dt}\left(Mk^2 \dfrac{d\theta}{dt}\right)\hat{\mathbf{n}} = (\Sigma p' F)\hat{\mathbf{n}}$ ...(5)

Equating coefficients of $\hat{\mathbf{n}}$ on both sides, we get

$$\frac{d}{dt}\left(Mk^2 \frac{d\theta}{dt}\right) = \Sigma p' F \quad \text{...(6)} \qquad \Rightarrow Mk^2 \frac{d^2\theta}{dt^2} = \Sigma p' F \quad \text{...(7)}$$

Let (x', y') be the co-ordinates of P relative to G and X, Y the components of $\mathbf{F}$ in the directions of the axes, scalar moment , of the force $\mathbf{F}$ about G is $p' F$ which is equivalent to $(x'Y - y'X)$,

$\therefore$ (6th) equation may be written as $\dfrac{d}{dt}\left(Mk^2 \dfrac{d\theta}{dt}\right) = \Sigma(x'Y - y'X)$

$$\Rightarrow Mk^2 \frac{d^2\theta}{dt^2} = \Sigma(x'Y - y'X). \quad \text{...(8)}$$

Equations (2), (3) and (7) are the dynamical equations of motion of rigid body moving in two dimensions, under finite forces.

3·02. Kinetic Energy. *When a body is moving in two dimensions, then to express the kinetic energy in terms of the motion of the centre of inertia and the motion relative to the centre of inertia.*

(Meerut 84; Agra 81, 89; Raj 85; Patna 83)

At any time t, let $\mathbf{r}_G$ be the position vector of the centre of gravity of G of the rigid body, referred to an origin O; and let $\mathbf{r}$ be the position vector of a particle m, referred to an origin O, then we have $\mathbf{r} = \mathbf{r}_G + \mathbf{r}'$

where $\mathbf{r}'$ is the $p.v$ of the particle of mass m w.r.t. C. G.

Now let T be the kinetic energy of the body, then we get

$$T = \tfrac{1}{2}\Sigma m\, \dot{\mathbf{r}}^2 \quad ...(1) \qquad = \tfrac{1}{2}\Sigma m\, (\dot{\mathbf{r}}_G + \dot{\mathbf{r}}')^2$$

$$= \tfrac{1}{2}\Sigma m\, \dot{\mathbf{r}}_G^{\ 2} + \tfrac{1}{2}\Sigma m\, \dot{\mathbf{r}}'^2 + \Sigma m\, \dot{\mathbf{r}}_G \,.\, \dot{\mathbf{r}}'$$

$$= \tfrac{1}{2}\dot{\mathbf{r}}_G^{\ 2}\Sigma m + \tfrac{1}{2}\Sigma m\, \dot{\mathbf{r}}'^2 + \dot{\mathbf{r}}_G \,.\, \Sigma m \dot{\mathbf{r}}'$$

But $\dfrac{\Sigma m\mathbf{r}'}{\Sigma m} = 0,$

[$\because$ $\mathbf{r}'$ is the position vector of the centroid relative to the centroid itself.]

$\therefore \quad \Sigma m\mathbf{r}' = 0$, and so $\Sigma m\dot{\mathbf{r}}' = 0$,

$$\therefore \quad T = \tfrac{1}{2} M \dot{\mathbf{r}}_g'^2 + \tfrac{1}{2}\Sigma m\dot{\mathbf{r}}'^2 \quad [\because \Sigma m = M] \qquad ...(2)$$

Another form. Let $\mathbf{v}_G$ be the velocity of centre of gravity and let $\hat{\mathbf{e}}_2$ be the unit vector perpendicular to the direction of $\mathbf{r}'$ then we readily obtain

$$\mathbf{v}_G = \frac{d\mathbf{r}_g}{dt} = \dot{\mathbf{r}}_G$$

and $\dot{\mathbf{r}}'^2 = \left(\mathbf{r}' \dfrac{d\phi}{dt}\hat{\mathbf{e}}_2\right)^2 = \mathbf{r}'^2\left(\dfrac{d\theta}{dt}\right)^2 \left[\because \dfrac{d\phi}{dt} = \dfrac{d\theta}{dt} \text{ and } \hat{\mathbf{e}}_2^{\,2} = \hat{\mathbf{e}}_2 \,.\, \hat{\mathbf{e}}_2 = 1\right]$

$$\therefore (2) \Rightarrow T = \tfrac{1}{2} M \mathbf{v}_G^{\ 2} + \tfrac{1}{2}\Sigma m r'^2\left(\frac{d\theta}{dt}\right)^2$$

$$= \tfrac{1}{2} M v_G^{\ 2} + \tfrac{1}{2}\left(\frac{d\theta}{dt}\right)^2 \Sigma m r'^2 \qquad [\because \mathbf{v}_G^2 = v_G^2]$$

$$= \tfrac{1}{2} M v_G^{\ 2} + \tfrac{1}{2} Mk^2\left(\frac{d\theta}{dt}\right)^2 \qquad ...(3)$$

where k is the radius of gyration of the body about the centre of intertia.
Hence equation (3) expresses that ;

The total kinetic energy of a rigid body moving in two dimensions is equal to the kinetic energy of a particle of mass M placed at the centre of inertia *and moving with it together with the kinetic energy of the body relative to the centre of inertia.*

Equation (3) can also be put as

K.E. of the body =(K.E. due to translation)+ (K.E. due to rotation) ...(4)

3·03. Moment of the Momentum . *To find the moment of momentum of the body about the fixed origin O, when the body is moving in two dimensions.*

At any time t , let $\mathbf{r}_G$ be the position vector of the centre of gravity G of the body referred the origin O, and let $\mathbf{r}$ be the position vector of a particle of mass m, referred to the origin O,

then we have $\mathbf{r}=\mathbf{r}_G+\mathbf{r}'$; where $\mathbf{r}'$ is the position vector of the particle of mass m w.r.t. G.

Now let $\mathbf{H}$ be the moment of momentum (or angular momentum) of the body about O, then we have $=\Sigma\mathbf{r}\times m\dot{\mathbf{r}}$

$$=\Sigma m\times\dot{\mathbf{r}}=\Sigma m\,(\mathbf{r}_G+\mathbf{r}')\times(\dot{\mathbf{r}}_G+\dot{\mathbf{r}}')$$

$$=\Sigma m\mathbf{r}_G\times\dot{\mathbf{r}}_G+\Sigma m\mathbf{r}_G\times\dot{\mathbf{r}}'+\Sigma m\mathbf{r}'\times\dot{\mathbf{r}}_G+\Sigma m\mathbf{r}'\times\dot{\mathbf{r}}'$$

$$=\mathbf{r}_G\times\dot{\mathbf{r}}_G\,\Sigma m+\mathbf{r}_G\times\Sigma m\dot{\mathbf{r}}'+(\Sigma m\mathbf{r}')\times\dot{\mathbf{r}}_G{}'+\Sigma m\mathbf{r}'\times\dot{\mathbf{r}}' \qquad ...(1)$$

But $\dfrac{\Sigma m\mathbf{r}'}{\Sigma m}=\mathbf{0}$, being position vector of C.G. relative to C.G.

$\therefore\ \Sigma m\mathbf{r}'=\mathbf{0}$ and so $\Sigma m\dot{\mathbf{r}}'=\mathbf{0}$

$\therefore\ (1)\Rightarrow\ \mathbf{H}=\mathbf{r}_G\times\dot{\mathbf{r}}_G\ \Sigma m+\Sigma m\mathbf{r}'\times\dot{\mathbf{r}}'$

$$=\mathbf{r}_G\times M\,\dot{\mathbf{r}}_G+\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}' \qquad [\because\ \Sigma m=M]$$

$$=\mathbf{r}_G\times M\,\mathbf{v}_G+\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}' \qquad ...(2)$$

Another form. Let $\hat{\mathbf{n}}$ be the unit vector parallel to $\mathbf{H}$, then we get

$\mathbf{r}_G\times M\,\mathbf{v}_G=M\,\mathbf{r}_G\times\mathbf{v}_G$

$$=(M\,v_G\ p)\,\hat{\mathbf{n}}$$

[using the definition of moment ; p is the length of the perpendicular from the origin O on the direction of the velocity $\mathbf{v}_g$ of centre of gravity].

But we have $\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}'=\left(Mk^2\,\dfrac{d\theta}{dt}\right)\hat{\mathbf{n}}$ **[3.01]**

and $\mathbf{H}=H\,\hat{\mathbf{n}}$

$\therefore\ (2)\Rightarrow\ H\hat{\mathbf{n}}=M\,v_g\,p\,\hat{\mathbf{n}}+Mk^2\dfrac{d\theta}{dt}\,\hat{\mathbf{n}}$

Equating coefficients of $\hat{\mathbf{n}}$ on both sides, we get

$$H = M v_g p + Mk^2 \frac{d\theta}{dt} \quad ...(3)$$

This equation expresses that the moment of momentum (or angular momentum) of a rigid body about a fixed point O is equal to the angular momentum about O of a single particle of mass M (equal to mass of the body concentrated at its C.G. and moving with the centroid's velocity), together with the angular momentum of the body in motion relative to the C.G.

Equation (3) can also be written as.

Angular momentum of the rigid body

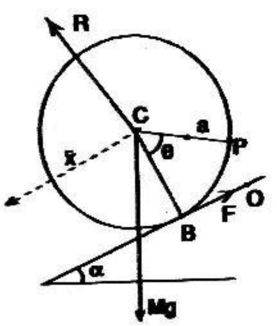
= Angular momentum of centre of inertia

+ angular momentum relative to the centre of inertia.

3·04. *A uniform sphere rolls down an inclined plane, rough enough to prevent any sliding ; to discuss the motion.*

(Meerut 83, Agra 88, Ranchi 83)

Initially, the sphere was at rest with its points P in contact with O. During the motion, after any time t, let the centre "C" of the sphere describes a distance x on the inclined plane and θ is the angle through which the sphere turns. Thus CP a line fixed in the body, makes an angle θ with the normal to the plane, a line fixed in the space.



Let F be the frictional force and R the normal reaction at the point of contact B, then equations of motion of C.G. of the body are

$$M \frac{d^2x}{dt^2} = Mg \sin\alpha - F.$$

...(1)

Since there is no motion perpendicular to the plane, we have

$$M\ddot{y} = 0 = Mg\cos\alpha - R \quad \text{or } Mg\cos\alpha = R. \quad ...(2)$$

Also equation of motion about the centre of gravity is

$$Mk^2 \frac{d^2\theta}{dt^2} = F.a. \quad ...(3)$$

Since there is no sliding, so we have $OB = \text{arc } PB$

$$\Rightarrow x = a\theta, \dot{x} = a\dot{\theta} \text{ and } \ddot{x} = a\ddot{\theta}, \quad ...(4)$$

$\therefore$ (3) gives $M.\dfrac{k^2}{a^2}\dfrac{d^2x}{dt^2} = F.a$ $\quad [\because \ddot{x} = a\ddot{\theta}]$

Substituting the value of F from here in (1), we readily get

$$\frac{d^2x}{dt^2}\left(1+\frac{k^2}{a^2}\right)=g\ \sin\alpha \quad \text{or} \quad \frac{d^2x}{dt^2}=\frac{a^2 g\sin\alpha}{a^2+k^2} \qquad \text{...(5)}$$

i.e. the sphere rolls down with a constant acceleration $\dfrac{a^2 g\ \sin\alpha}{a^2+k^2}$

(5) $\Rightarrow \dfrac{dx}{dt}=\dfrac{a^2 g\ \sin\alpha}{a^2+k^2}\ t+C$; and C,

the constant of intergration vanishes as t and $\dot{x}$ vanish together.

Intergrating again , $x=\frac{1}{2}\ \dfrac{a^2 g\ \sin\alpha}{a^2+k^2}\ t^2$;

because constant of intergration again vanishes as x and t vanish simultaneously.

Now we shall discuss various cases :

(i) If the body be a solid sphere, $k^2=\frac{2}{5}\ a^2$ and then equation (5) implies,

$$\ddot{x}=\tfrac{5}{7}\ g\ \sin\alpha.$$

(ii) If the body be hollow sphere, $k^2=\frac{2}{3}\ a^2$ $\quad\therefore\ \ddot{x}=\frac{3}{5}\ g\ \sin\alpha.$

(iii) If the body be circular disc, $k^2=\frac{1}{2}\ a^2$ $\quad\therefore\ \ddot{x}=\frac{2}{3}\ g\ \sin\alpha.$

(iv) If the body be circular ring , $k^2=a^2$ $\quad\therefore\ \ddot{x}=\frac{1}{2}\ g\ \sin\alpha.$

Pure rolling : Eliminating $\dfrac{d^2x}{dt^2}$ from (5), and (1), we get

$$F=Mg\ \sin\ \alpha-\tfrac{5}{7}\ Mg\ \sin\ \alpha=\tfrac{2}{7}\ Mg\ \sin\ \alpha\ \left(\because\ k^2=\frac{2a^2}{5}\right)$$

Also from (2) $R=Mg\ \cos\ \alpha.$

In order that there may be no sliding $\dfrac{F}{R}$ must be less than μ *i.e.* for pure rolling $F<\mu R$ *i.e.* $\mu>\dfrac{F}{R}=\frac{2}{7}\tan\ \alpha.$

ILLUSTRATIVE EXAMPLES

Ex. 1. *A uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is* α *, show that the least coefficient of friction between it and the plane, so that it may roll and not slide, is* $\frac{1}{3}\tan\alpha$ *. If the cylinder be hollow, and of small thickness, the least value is* $\frac{1}{2}\tan\alpha$ *.*

Sol. At any time t, let the axis of the cylinder describe a distance x and θ be the angle turned Arguing as in 3.04, we have

$$x = a\theta. \qquad [\because \text{ there is no sliding}]$$

Also the equations of a C.G. are given by

$$M \frac{d^2x}{dt^2} = Mg \sin\alpha - F \quad ...(1) \text{ and } \quad 0 = Mg \cos\alpha - R \qquad ...(2)$$

Again taking moments abouts the axis through G, the centre of gravity of the body, we have

$$Mk^2 \frac{d^2\theta}{dt^2} = F \times a \Rightarrow M \frac{k^2}{a} \cdot \frac{d^2x}{dt^2} = F \times a \qquad ...(3)$$

whence elimination of $M \frac{d^2x}{dt^2}$ in between (1) and (3), we get

$$\frac{k^2}{a} (Mg \sin\alpha - F) = F \times a \Rightarrow F = \frac{k^2}{a^2 + k^2} Mg \sin\alpha \qquad ...(4)$$

But $\qquad R = Mg \cos\alpha \qquad ...(5)$

$\therefore$ For pure rolling, $\mu > \frac{F}{R} = \frac{k^2}{a^2 + k^2} \tan\alpha$.

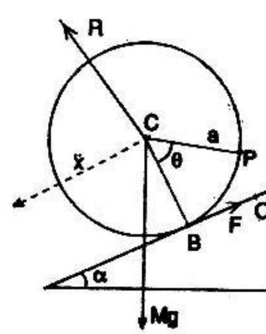
But when cylinder is solid, we have $k^2 = \frac{1}{2} a^2, \Rightarrow \mu > \frac{\frac{1}{2} a^2}{a^2 + \frac{1}{2} a^2} \tan\alpha$

$$= \tfrac{1}{3} \tan\alpha$$

In case of hollow cylinder, we have

$$k^2 = a^2, \Rightarrow \mu > \frac{a^2}{a^2 + a^2} \tan\alpha = \tfrac{1}{2} \tan\alpha.$$

Ex. 2. *A cylinder rolls down a smooth plane whose inclination to the horizontial is α, unwrapping, as it goes, a fine string fixed to the highest point of the plane ; find its acceleration and the tension of the string.*



Sol. When the cylinder has rolled down a distance x along the plane, let T be the tension in the string and in this time (say t), let θ be the angle turned by the cylinder, then as the string is tight, the motion is of pure rolling

i.e. $\quad$ arc $BP = OB \Rightarrow x = a\theta \qquad ...(1)$

$\therefore \dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

equations of motion of the centre of gravity of the cylinder are

$$M \frac{d^2x}{dt^2} = Mg \sin\alpha - T \qquad ...(2)$$

and $M\dfrac{d^2y}{dt^2}=0=Mg\cos\alpha-R.$ ...(3)

Now taking moments about the centre , we have

$$Mk^2\ddot{\theta}=T\times \text{ i.e. } M.\tfrac{1}{2}a^2\ddot{\theta}=T\times a$$

or $\frac{1}{2}M\ddot{x}=T.$ [$\because\ \ddot{x}=a\ddot{\theta}$] ...(4)

$\therefore$ (5) and (2), gives $\frac{3}{2}M\ddot{x}=Mg\sin\alpha$ i.e. $\ddot{x}=\frac{2}{3}g\sin\alpha$

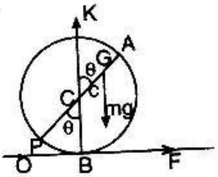
$\Rightarrow T=\frac{1}{2}m\ddot{x}=\frac{1}{2}M(\frac{2}{3}g\sin\alpha)=\frac{1}{3}Mg\sin\alpha$

Ex. 3. *A circular cylinder, whose centre of inertia is at a distance c from axis, rolls on a horizontal plane. If it be just started from a position of unstable equilibrium. Show that the normal reaction of the plane when the centre of mass is in its lowest position is* $\left[1+\dfrac{4c^2}{(a-c)^2+k^2}\right]$ *times its weight, where k is the radius of gyration about an axis through the centre of mass.*

Sol. Initially the point of contact P of the cylinder was at O when its centre of gravity was vertically above the centre of the figure .

At any time t let the radius through G turn through an angle θ.

Referred to O as origin and horizontal and vertical line as axes, the co-ordinates (x, y) of G fiven by $x=a\theta+c\sin\theta, y=a+c\cos\theta,$ [$\because$ $CG=c$.]



Equations of motion of C.G. are

$$m\frac{d^2x}{dt^2}=m\frac{d^2}{dt^2}(a\theta+c\sin\theta)=F \quad ...(1)$$

and $$m\frac{d^2y}{dt^2}=m\frac{d^2}{dt^2}(a+c\cos\theta)=R-mg . \quad ...(2)$$

Also energy equation gives

$$\tfrac{1}{2}m[(\dot{x}^2+\dot{y}^2)+k^2\dot{\theta}^2]=\text{work done by the forces .}$$

i.e. $\frac{1}{2}m[(a\dot{\theta}+c\cos\theta\dot{\theta})^2+(-c\sin\theta\dot{\theta})^2]$

$$+\tfrac{1}{2}mk^2\dot{\theta}^2=mg(c-c\cos\theta)$$

Let ω be the angular velocity when G is in its lowest position

i.e. $\dot{\theta}=\omega$ when $\theta=\pi$; thus we have

$\frac{1}{2}\, m\ [(a-c)^2 + k^2]\, \omega^2 = 2mgc \;\Rightarrow\; \omega^2 = \dfrac{4gc}{k^2 + (a-c)^2}$.

Now (2) gives $R = mg - mc\,(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2)$

$$= mg - mc\,\cos\pi\,.\,\omega^2$$

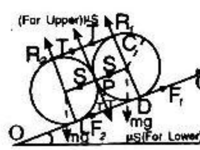
(since in the lowest position $\theta = \pi;\ \dot{\theta} = \omega$)

$$= mg + mc\,\frac{4cg}{k^2 + (a-c)^2} = mg\left[1 + \frac{4c^2}{k^2 + (a-c)^2}\right].$$

Ex. 4. *Two equal cylinders, of mass m, are bound together by an elastic string, whose tension is T, and roll with their axes horizontal down a rough plane of inclination* α*. Show that their acceleration is*

$\frac{2}{3}\, g\, \sin\alpha \left[1 - \dfrac{2\mu T}{mg\,\sin\alpha}\right]$, *where* μ *is the coefficient of friction between the cylinders .*

Sol. Let R_1, F_1 be the normal reaction and friction on the upper cylinder and R_2, F_2 be the normal reaction and friction on the lower cylinder due to the plane. Let S be the normal reaction between the two cylinders at P. The force μS acts away from the plane for upper cylinder and towards the plane for the lower cylinder .



At any time t let the cylinders move through a distance z along the plane, and θ be the angle turned by them .

Then as there is no slipping, we have

$z = a\theta \Rightarrow \ddot{z} = a = a\,\ddot{\theta}$. ...(1)

Equations of motion of the upper cylinder are given by

$$m\ddot{z} = mg\,\sin\alpha + 2T - F_1 - S \qquad ...(2)$$

$$0 = R_1 - mg\,\cos\alpha + \mu S \qquad ...(3)$$

and $\quad mk^2\,\ddot{\theta} = F_1 \times a - \mu S \times a.$ ...(4)

where as the equations of motion for the lower cylinder are given by

$$m\ddot{z} = mg\,\sin\alpha - 2T - F_2 + S \qquad ...(5)$$

$$0 = R_2 - mg\,\cos\alpha - \mu S \qquad ...(6)$$

and $\quad mk^2\,\ddot{\theta} = F_2 \times a - \mu S \times a$ ...(7)

Comparing (4) and (7), we have $F_1 = F_2$.

Subtracting (2) from (5), we have $S = 2T$. ...(8)

Also from (4), $F_1 = \frac{mk^2}{a}\ddot{\theta} + \mu S$, **where** $k^2 = a^2$

$= \frac{1}{2} m\ddot{z} + 2\mu T$ **[From (1) and (8)]**

$\therefore$ (2) $\Rightarrow m\ddot{z} = mg \sin \alpha + 2T - (\frac{1}{2} m\ddot{z} + 2\mu T) - 2T$ $[\because S = 2T]$

or $$\ddot{z} = \frac{2}{3} g \sin \alpha \left[1 - \frac{2\mu T}{mg \sin \alpha}\right]$$

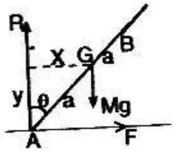
3·05. Slipping of rods.

A unoform rod is held in a vertical position with one end resting upon a perfectly rough table and when released rotates about the end in contact with the table. To discuss the motion.

(Meerut 1984 ; Agra 86, 88)

Let AB be the rod having length $2a$ and mass M.

Let the rod which is rotating about A makes an angle θ with the vertical at any time t.



Taking A point as the origin and horizontal and vertical lines as axes, the coordinate (x, y) of centre of mass G are given by

$$x = a \sin \theta, y = a \cos \theta$$

$$\therefore \dot{x} = a \cos \theta\dot{\theta}, \dot{y} = -a \sin \theta\dot{\theta}$$

and $\ddot{x} = -a \sin \theta\dot{\theta}^2 + a \cos \theta\ddot{\theta}$, $\ddot{y} = -a \cos \theta\dot{\theta}^2 - a \sin \theta\ddot{\theta}$.

Let F be the frictional force and R the normal reaction at A. Now the equation of motion of C.G. are

$$M \frac{d^2x}{dt^2} = M[a \cos \theta\ddot{\theta} - a \sin \theta\dot{\theta}^2] = F \quad ...(1)$$

$$M \frac{d^2y}{dt^2} = M[-a \sin \theta\ddot{\theta} - a \cos \theta\dot{\theta}^2] = R - Mg \quad ...(2)$$

Again energy of the rod $= \frac{1}{2} M[(\dot{x}^2 + \dot{y}^2 + \frac{1}{3} a^2 \dot{\theta}^2)]$

$[\because v^2 = \dot{x}^2 + \dot{y}^2, k^2 = \frac{1}{3} a^2]$

$$= \frac{1}{2} m\left[(a\dot{\theta})^2 + \frac{1}{3} a^2 \dot{\theta}^2\right] = \frac{2}{3} Ma^2 \dot{\theta}^2$$

and work done by the forces $= Mg(a - a \cos \theta)$

Hence from energy equation, we have

$$\tfrac{2}{3}\ Ma^2\dot{\theta}^2 = Mg\ (a - a\cos\theta) \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}\ (1-\cos\theta)^* \qquad \text{...(3)}$$

Differentiating (3) with respect to t, we have $\ddot{\theta} = \dfrac{3g}{4a}\sin\theta$ ...(4)

Putting the values of $\dot{\theta}$ and $\ddot{\theta}$ from (3) and (4) in (1) and (2), we get $F = \dfrac{3}{4}\ Mg\ \sin\theta\,(3\cos\theta - 2)$ and $R = \dfrac{1}{4}\ Mg\ (1 - 3\cos\theta)^2$

We observe that R does not change its sign and vanishes when $\cos\theta = \frac{1}{3}$. Hence the end **A does not leave the plane.**

From the value of F, we see that F changes its sign as θ passes through the angle $\cos^{-1}\left(\frac{2}{3}\right)$; thus its direction is then reversed.

At $\cos\theta = \frac{1}{3}$, $R = 0$, hence the ratio $\dfrac{F}{R}$ **becomes infinite where $\cos\theta = \frac{1}{3}$**, hence unless the plane be infinitely rough there will be sliding at this value of θ. In practice the end A of the rod begins to slip for some value of θ less then $\cos^{-1}\left(\frac{1}{3}\right)$. The end A will slip backwards or forward according as the slipping takes place before of after the

inclination of the rod is $\cos^{-1}\left(\frac{2}{3}\right)$.

ILLUSTRATIVE EXAMPLES

Ex. 1. *A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ. If it be then released show that it will commence to slide if*

$$\mu < \left(\frac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}\right) \qquad \textbf{(Agra 91)}$$

Sol. Let AB be the rod having length $2a$ and mass m. Let F be the force

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*Equation (3) can also be obtained by taking moments about G, then

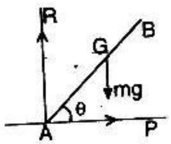
$$M\frac{a^2}{3}\ \ddot{\theta} = Ra\sin\theta - Fa\cos\theta = Mg\sin\theta - Ma^2\ddot{\theta}. \text{ [From (1) and (2)]}$$

or $\quad \ddot{\theta} = \dfrac{3g}{4a}\sin\theta$

Multiplying by $2\dot{\theta}$ and integrating, we get $\dot{\theta}^2 = -\dfrac{3g}{2a}\cos\theta + C$

When $\theta = 0$, $\dot{\theta} = 0 \Rightarrow C = \dfrac{3g}{2a}$. $\therefore\ \dot{\theta}^2 = \dfrac{3g}{2a}\ (1-\cos\theta)$

of friction sufficient to prevent sliding and R the normal reaction. With reference to point A as the origin, the coordinated of point G *i.e.* C.G.are($a\cos\theta$, $a\sin\theta$). the coordinates of point G before the motion begins are ($a\cos\alpha$, $a\sin\alpha$).



Thus the vertical distance moved by the C.G. is ($a\sin\alpha - a\sin\theta$).

Equations of motion of C.G. are

$$m\frac{d^2x}{dt^2} = m\,[-a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}\,] = F \quad \text{...(1)}$$

and. $$m\frac{d^2y}{dt^2} = m\,[-a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}\,] = R - mg \quad \text{...(2)}$$

The equation of energy gives

$$\tfrac{1}{2}\,m\,[(\dot{x}^2+\dot{y}^2)+\tfrac{1}{3}\,a^2\dot{\theta}^2] = mg\,(a\sin\alpha - a\sin\theta)$$

$$\Rightarrow \tfrac{1}{2}\,m\,(a^2\dot{\theta}^2+\tfrac{1}{3}\,a^2\dot{\theta}^2) = amg\,(\sin\alpha - \sin\theta)$$

$$\Rightarrow \tfrac{2}{3}\,a^2\dot{\theta}^2 = ga\,(\sin\alpha-\sin\theta) \quad \Rightarrow \dot{\theta}^2 = \frac{3g}{2a}\,(\sin\alpha - \sin\theta) \quad \text{...(3)}$$

Differentiating (3) w.r.t., to t, we get $\ddot{\theta} = \dfrac{-3g}{4a}\cos\theta$ ...(4)

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (3) and (4) in (1) and (2), we get

$$F = m\left[-a\cos\theta\,.\,\frac{3g}{2a}\,(\sin\alpha-\sin\theta) - a\sin\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{3}{4}\,mg\cos\theta\,(3\sin\theta - 2\sin\alpha) = \frac{3}{4}\,mg\cos\alpha\sin\alpha,$$

when $\theta = \alpha$

and $$R = mg + m\left[-a\sin\theta\,.\,\frac{3g}{2a}\,(\sin\alpha-\sin\theta) + a\cos\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{1}{4}\,mg\,[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$$

$$= \tfrac{1}{4}\,mg\,(4 - 3\cos^2\alpha), \qquad \text{when } \theta = \alpha$$

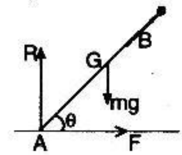
$$= \tfrac{1}{4}\,mg\,[1 + 3\,(1-\cos^2\alpha)] = \tfrac{1}{4}\,mg\,(1+3\sin^2\alpha)$$

The end A will commence to slide if $\mu < \dfrac{F}{R}$ *i.e.* $\mu < \dfrac{3\sin\alpha\cos\alpha}{1+3\sin^2\alpha}$.

Ex. 2. *The lower end of a uniform rod, inclined intially at an angle α to the horizon is placed on a smooth horizontal table. A horizontal force is applied to its lower end of such a magnitude that the rod rotates in vertical* *plane with constant angular velocity ω. Show that when the rod is inclined at an angle θ to the horizon the magnitude of the force is*

$mg\ \cot\theta - ma\ \omega^2 \cos\theta$ *where m is the mass of the rod .*

Sol. Let the horizontal force applied at the lower end A of the rod be F. Let at any time t, θ be the angle that the rod makes with the horizontal. Since the rod rotates with a uniform angular velocity ω $\therefore$ $\dot{\theta} = \omega$ (const).



$\Rightarrow \ddot{\theta} = 0$ ...(2)

The equation of motion of G along the vertical

$$R - mg = m\frac{d^2}{dt^2}(a\sin\theta) = ma\,(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta})$$

$$= -ma\sin\theta\,.\,\omega^2 \quad \text{from (1) and (2)} \qquad ...(3)$$

Since the end A is not fixed, the equation of horizontal motion of C.G. is not written .

Again taking moments about G, we have

$mk^2\ddot{\theta} = Fa\sin\theta - Ra\cos\theta \Rightarrow F = R\cot\theta$ $\{\because \ddot{\theta} = 0$ from (2)$\}$

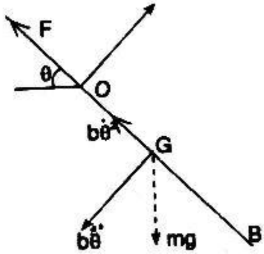
$\Rightarrow F = (mg - ma\sin\theta\,.\,\omega^2)\cot\theta$ from (3)

$\Rightarrow F = mg\cot\theta - ma\,\omega^2\cos\theta.$

Ex. 3. *A rough uniform rod , of length 2a , is placed on a rough table at right angles to its edge ; if its centre of gravity be initially at distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle* $\dfrac{\mu a^2}{a^2 + 9b^2}$ *where μ is the coefficient of fricton .*

(Kanpur 89)

Sol. Initially the rod was at right angles to the edge of the rough table, now it has turned through an angle θ.



Let there be no sliding when the rod has turned through this angle. Let A and R be the normal reaction and the force of friction on the rod. Acceleration of G along and perpendicular to GO are respectively $b\,\dot{\theta}^2$ and $b\,\ddot{\theta}$. Equations of motion of centre of gravity G are

$mb\,\ddot{\theta} = mg\cos\theta - R$ ...(1)

and $\quad mb\,\dot{\theta}^2 = F - mg \sin\theta$ ....(2)

Taking moments about O, the point of contact of the rod and table, we have $mk^2\ddot{\theta} = mg\,b\cos\theta$, $\Rightarrow m\left(b^2+\dfrac{a^2}{3}\right)\ddot{\theta} = mg\,b\cos\theta$

$$\Rightarrow \ddot{\theta} = \frac{3gb}{a^2+3b^2}\cos\theta \qquad ...(3)$$

Multiplying (3) by $2\dot{\theta}$ and integrating, we get $\dot{\theta}^2 = \dfrac{6gb}{a^2+3b^2}\sin\theta$

The constant of integration vanishes as initially when $\theta = 0$, $\dot{\theta} = 0$. Putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ in (1) and (2) from (3) and (4), we have

$$R = -gb\,.\frac{3bg}{a^2+3b^2}\cos\theta + mg\cos\theta = \frac{mga^2}{a^2+3b^2}\cos\theta$$

and $F = mg\sin\theta + mb\dfrac{6gb}{a^2+3b^2}\sin\theta = mg\dfrac{a^2+9b^2}{a^2+3b^2}\sin\theta.$

The sliding commences when

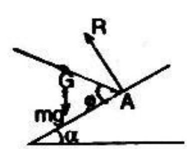
$F = \mu R$ *i.e.* when $\quad mg\dfrac{a^2+9b^2}{a^2+3b^2}\sin\theta = \mu\dfrac{mg\,a^2}{a^2+3b^2}\cos\theta$

or when $\quad \tan\theta = \dfrac{\mu a^2}{a^2+9b^2}.$

Ex. 4. *A uniform rod of mass m, is placed at right angle to a smooth plane of inclination α with one end in contact with it. The rod is then released. Show that when the inclination to the plane is φ, the reaction of the plane will be*

(Meerut 1980)

$$mg\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha$$



Sol. As there is no force acting along the plane, so initially there is no motion along the plane. The C.G. *i.e.* point G moves perpendicular to the plane.

Let ϕ be the angle which the rod makes with the plane after time t. Taking A as the origin, the plane as x-axis and a line perpendicular to to the plane as y-axis, the co-ordinates of G are

$$x = a\cos\phi,\quad y = a\sin\phi.$$

Equations of motion of point G are

$$m\ddot{y} = m(a\cos\phi\,\ddot{\phi} - a\sin\phi\,\dot{\phi}^2) = R - mg\cos\alpha \qquad \text{...(1)}$$

and $$m\frac{a^2}{3}\ddot{\phi} = -R\,.\,a\cos\phi \qquad \text{...(2)}$$

Also from energy equation, we have

$$\tfrac{1}{2} ma^2\cos^2\phi\,\dot{\phi}^2 + \tfrac{1}{2}\frac{ma^2}{3}\dot{\phi}^2 = \text{work done by gravity}$$

$$= mg\,a\cos\alpha(1-\sin\phi)$$

or $$\dot{\phi}^2 = \frac{6g(1-\sin\phi)}{a(1+3\cos^2\phi)}\cos\alpha \qquad \text{...(3)}$$

Differentiating (3) w.r.t. t, we get

$$\dot{\phi}\,\ddot{\phi} = \frac{3g\cos\alpha}{a}\left[\frac{-\cos\phi}{(1+3\cos^2\phi)} + \frac{6\cos\phi\sin\phi(1-\sin\phi)}{(1+3\cos^2\phi)^2}\right]\dot{\phi}$$

$$= -\frac{3g}{a}\cos\alpha\left[\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\right]\cos\phi\,.\,\dot{\phi}$$

or $$\ddot{\phi} = -\frac{3g}{a}\cos\phi\cos\alpha\left[\frac{1+3(1-\sin\phi)^2}{(1+3\cos^2\phi)^2}\right]$$

Putting the value of $\ddot{\phi}$ in (2), we get

$$R = mg\,\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\cos\alpha.$$

Ex. 5. *A uniform rod is held nearly vertically with one end resting on an imperfectly rough plane. It is released from rest and falls forward. The inclination to the vertical at any instant is* θ. *Prove that*

(i) *If the coefficient of friction is less than a certain finite amount, the lower end of the rod will slip backward before* $\sin^2(\theta/2) = \left(\frac{1}{6}\right)$.

(ii) *However great the coefficient of friction may be, the lower end will begin to slip forward at a value of* $\sin^2(\theta/2)$ *between* $\frac{1}{6}$ *and* $\frac{1}{3}$.

Sol. (i) Proceeding in the same way as in **3.05**, we get

$$F = \tfrac{3}{4}Mg\sin\theta(3\cos\theta - 2) \quad \text{and} \quad R = \tfrac{1}{4}mg(1-3\cos\theta)^2$$

Obviously $F = 0$ if $\sin\theta = 0$ or $3\cos\theta - 2 = 0$

i.e., if $\theta = 0$ or $\cos\theta = \frac{2}{3}$

i.e. if $\theta = 0$ or $1 - 2\sin^2(\theta/2) = \frac{2}{3}$ or $\sin^2(\theta/2) = \frac{1}{6}$.

The value of F is positive when θ takes all intermediate values between $\theta = 0$ and $\theta = \cos^{-1}\frac{2}{3}$ and is continuous function of θ, hence between these two values of θ where F vanishes, F has a maximum value for some

θ. Let F_1 be the maximum value. We observe that for $0 \le \theta \le \cos^{-1} \frac{2}{3}$ the value of $R \le Mg$.

Thus there is a finite value of μ for which $F_1 > \mu R$ and therefore for this value of μ, sliding will take place before $\cos^{-1} \frac{2}{3}$ *i.e.* before $\sin^2 \frac{\theta}{2} = \frac{1}{6}$. Since F is positive (in the forward direction) hence the slipping will start in the backward direction.

(ii) We observe from the value of F that if $\cos\theta > 3/2$, F changes its sign, *i.e.* the direction of the friction is reversed if

$$F' = -F = \tfrac{3}{4}\, mg\, (2 - 3\cos\theta)$$

Now the slipping may start when $F' > \mu R$

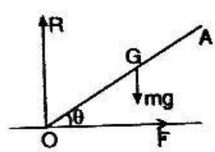
i.e. when $3\sin\theta\,(2 - 3\cos\theta) > \mu\,(1 - 3\cos\theta)^2$ ...(1)

As θ increases from $\cos^{-1}\frac{2}{3}$ to $\cos^{-1}\frac{1}{3}$, the term on the left hand side increases while the right hand side term decreases from 1 to 0. Therefore for some value of θ between $\cos^{-1}\frac{2}{3}$ and $\cos^{-1}\frac{1}{3}$ *i.e.* for $\sin^2(\theta/2)$ between $\frac{1}{6}$ and $\frac{1}{2}$ the condition (1) is satisfied and the slipping will then start in the forward direction.

Ex. 6. *A uniform rod is placed with one end in contact with a horizontal table, and is then at an inclination α to the horizon and is allowed to fall. when it becomes horizontal, show that its angular velocity is $\left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$ whether the plane is perfectly smooth or perfectly rough. Show also that the end of the rod will not leave the plane in either case.*

Sol. Let at any instant t the rod makes an angle θ with the horizontal. Let R and F be the normal reaction and friction at the instant with O as origin, the co-ordinates of C.G. are $x = a\cos\theta$, $y = a\sin\theta$.

Case I. *When plane is perfectly rough and O is fixed.*



Then energy equation gives

$\frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} mk^2\dot{\theta}^2 =$ work done by gravity

$$\Rightarrow \tfrac{1}{2} m\,(a^2\dot{\theta}^2 + \tfrac{1}{3} a^2\theta^2) = mga\,(\sin\alpha - \sin\theta)$$

$$\dot{\phi}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta). \qquad ...(1)$$

When the rod becomes horizontal *i.e.* when $\theta = 0$, the angular velocity $\dot{\theta} = \omega$ (say) is given by $\omega^2 + \frac{3g}{2a}\sin\alpha$ or $\omega = \left(\frac{3g}{2a}\sin\alpha\right)^{1/2}$

Differentiating (1) w.r.t. 't' we get $\ddot{\theta} = \frac{-3g}{4a} \cos\theta$. ...(2)

The equation of motion of C.G. is

$$R - mg = m \frac{d^2}{dt^2}(a \sin\theta) = ma(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta})$$

$$\Rightarrow R = mg + ma\left[-\sin\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) + \cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

[substituting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (1) and (2)]

$$= \tfrac{1}{4} mg\,(4 - 6\sin\alpha\sin\theta + 6\sin^2\theta - 3\cos^2\theta)$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 - 9\sin^2\alpha\sin^2\theta + 9\sin^2\theta]$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\,(1 - \sin^2\alpha)]$$

$$= \tfrac{1}{4} mg\,[(1 - 3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\cos^2\alpha]$$

This shows that R is always positive, therefore the end O of the rod never leaves the plane.

Case II. *When the plane is perfectly smooth.*

In this case there is no horizontal forces, hence C.G. descends in a vertical line *i.e.* the only velocity of G being along the vertical direction

$y = a\sin\theta$, $\dot{y} = a\cos\theta\dot{\theta}$.

The energy equation gives $\frac{1}{2} m\dot{y}^2 + \frac{1}{2} mk^2\dot{\theta}^2 =$ work done by gravity

i.e. $\frac{1}{2} m(a^2\cos^2\theta\dot{\theta}^2 + \frac{1}{3}a^2\dot{\theta}^2) = mg(a\sin\alpha - a\sin\theta)$

or $\dot{\theta}^2(\cos^2\theta + \frac{1}{3}) = \left(\frac{2g}{a}\right)(\sin\alpha - \sin\theta)$ ...(1)

when the rod becomes horizontal *i.e.* when $\theta = 0$, the angular velocity

$\dot{\theta} = \omega$ (say) is given by

$$\omega^2(1 + \tfrac{1}{3}) = \frac{2g}{a}\sin\alpha \Rightarrow \omega^2 = \frac{3g}{2a}\sin\alpha \Rightarrow \omega = \left(\frac{3g}{2a}\sin\alpha\right)^{1/2}.$$

This gives the required result in the case of plane being smooth.

Differentiating (1), we have

$$\ddot{\theta}\,(\cos^2\theta + \tfrac{1}{3}) - \dot{\theta}^2\sin\theta\cos\theta = -\left(\frac{g}{a}\right)\cos\theta$$

$$\Rightarrow \ddot{\theta}\,(\cos^2\theta + \tfrac{1}{3}) - \sin\theta\cos\theta\left[\frac{(2g/a)(\sin\alpha - \sin\theta)}{\cos^2\theta + \frac{1}{3}}\right] = -\left(\frac{g}{a}\right)\cos\theta$$

$$\Rightarrow \ddot{\theta}\,(\cos^2\theta + \tfrac{1}{3})^2 = -\left(\frac{g}{a}\right)\cos\theta\,[\sin^2\theta - 2\sin\alpha\sin\theta + \tfrac{4}{3}]$$

$$= -(g/a)\cos\theta\,[(\sin\theta - \sin\alpha)^2 + \tfrac{1}{3} + \cos^2\alpha] \qquad ...(3)$$

Again taking moments about G, we have

$$m\,\frac{a^2}{3}\,\ddot{\theta} = -Ra\cos\theta \text{ or } R = -\tfrac{1}{3}a\sec\theta\,.\,m\ddot{\theta}$$

$$\Rightarrow R = \frac{mg}{3}\left[\frac{(\sin\theta - \sin\alpha)^2 + \frac{1}{3} + \cos^2\alpha}{(\cos^2\theta + \frac{1}{3})^2}\right]$$

from (2) by putting the value of $\ddot{\theta}$

$$\Rightarrow R = mg\left[\frac{1 + 3\cos^2\alpha + 3(\sin\theta - \sin\alpha)^2}{(1 + 3\cos\theta)^2}\right]$$

we observe that R is positive for every value of α and θ. Hence the end never leaves the plane.

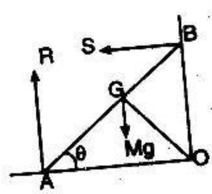
3·06 . *A uniform straight rod slides down in a vertical plane its end being in contact with two smooth planes, one horizontal and the other vertical. If it started from rest at an angle α with the horizontal ; to discuss the motion.* **(Meerut 1987, 84, 83)**

Let at any instant t, the rod makes an angle θ with the horizontal. Let R and S be the reactions at the ends A and B of the rod AB whose length is $2a$ and mass M .

With reference to point O as origin, the co-ordinates of G *i.e.* centre of gravity are $x = a\cos\theta,\ y = a\sin\theta$

$$\therefore\ \ddot{x} = -a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta},$$

$$\ddot{y} = -a\sin\theta\,\dot{\theta}^2 + a\cos\theta\ddot{\theta}$$



The equations of motion of C.G. are $M\ddot{x} = S$

$$\Rightarrow M(-a\cos\theta\dot{\theta}^2 - a\sin\theta\,\ddot{\theta}) = S \qquad ...(1)$$

and $M\ddot{y} = R - Mg$

$$\Rightarrow M(-a\sin\theta\dot{\theta}^2 + a\cos\theta\,\ddot{\theta}) = R - Mg$$

Energy equation gives

$$\tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}Mk^2\dot{\theta}^2 = \text{work done by the gravity}.$$

$$\Rightarrow \tfrac{1}{2}M(a^2\dot{\theta}^2 + \tfrac{1}{3}a^2\dot{\theta}^2) = Mga(\sin\alpha - \sin\theta)$$

$$\Rightarrow \dot{\theta}^2 = \left(\frac{3g}{2a}\right)(\sin\alpha - \sin\theta). \qquad ...(3)$$

Differentiating (3) w.r.t. t, we get $\ddot{\theta} = -\left(\dfrac{3g}{4a}\right)\cos\theta \qquad ...(4)$

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ in (1) and (2), we have

$$S = M\left[-a\cos\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{3}{4} Mg \cos\theta\,(3\sin\theta - 2\sin\alpha) \qquad ...(5)$$

$$R = Mg + M\left[-a\sin\theta \cdot \frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right]$$

$$= \tfrac{1}{4} Mg\,[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$$

$$= \tfrac{1}{4} Mg\,[1 - 6\sin\theta\sin\alpha + 9\sin^2\theta]$$

$$= \tfrac{1}{4} Mg\,[1 - \sin^2\alpha + \sin^2\alpha - 6\sin\theta\sin\alpha + 9\sin^2\theta]$$

$$= \tfrac{1}{4} Mg\,[(3\sin\theta - \sin\alpha)^2 + \cos^2\alpha] \qquad ...(6)$$

From (5), we observe that $S = 0$ when $\sin\theta = \frac{2}{3}\sin\alpha$ and S will be negative when this value of θ is reached. Hence the end B leaves the wall when $\sin\theta = \frac{2}{3}\sin\alpha$.

Again from (6), we observe that R is always positive *i.e.* the end A never leaves the plane

Further when the end B leaves the plane $\sin\theta = \frac{2}{3}\sin\alpha$ and $S = 0$ thus equations of motion (1), (2), (3) and (4) cease to hold good for further motion.

Putting $\sin\theta = \frac{2}{3}\sin\alpha$ in (3), the angular velocity of the rod now becomes $\left(\frac{g}{2a}\sin\alpha\right)^{1/2}$, this will be the initial angular velocity for the next part of the motion.

Second part of the motion.

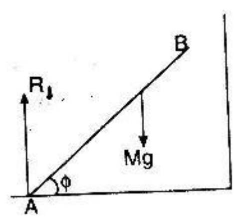
When the end B leaves the wall, let R_1 be the normal reaction at A. Let the rod be inclined at angle ϕ to the horizontal.

The equations of motion are

$$M\ddot{x} = 0 \qquad ...(1)$$

$$M\ddot{y} = R_1 - Mg \qquad ...(2)$$

and $$M\frac{a^2}{3}\ddot{\phi} = -R_1 a\cos\phi \qquad ...(3)$$



As $y = a\sin\phi$, $\therefore \ddot{y} = -a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi}$

Hence from (2) and (3), we get

$$\left(\tfrac{1}{3} + \cos^2\phi\right)\left(\frac{d^2\phi}{dt^2}\right) - \sin\phi\cos\phi\left(\frac{d\phi}{dt}\right)^2 = -\frac{g}{a}\cos\phi \qquad ...(4)$$

Integrating it, we get $\left(\frac{1}{3}+\cos^2\phi\right)\left(\frac{d\phi}{dt}\right)^2 = -\frac{2g}{a}\sin\phi + C$ ...(5)

when $\sin\phi = \frac{2}{3}\sin\alpha$, $\frac{d\phi}{dt} = \sqrt{\left(\frac{g}{2a}\sin\alpha\right)}$

$\therefore\ \frac{g\sin\alpha}{2a}\left[\frac{1}{3}+1-\frac{4}{9}\sin^2\alpha\right] = -\frac{2g}{a}\cdot\frac{2}{3}\sin\alpha + C$

or $$C = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right)$$

Hence from (5), we have

$$\left(\frac{1}{3}+\cos^2\phi\right)\left(\frac{d\phi}{dt}\right)^2 = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right) - \frac{2g}{a}\sin\phi. \quad ...(6)$$

When $\phi = 0$ *i.e.* when rod reaches the horizontal plane, let its angular velocity be Ω, then

$$\Omega^2\left(\frac{1}{3}+1\right) = \frac{2g\sin\alpha}{a}\left(1-\frac{\sin^2\alpha}{9}\right) \Rightarrow \Omega^2 = \frac{3g}{2a}\left(1-\frac{\sin^2\alpha}{9}\right)\sin\alpha. \quad ...(7)$$

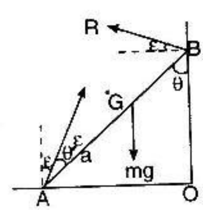
Ex. 7. *A heavy rod, of length 2a is placed in a vertical plane with its ends in contact with a rough vecrtical wall and an equally rough horizontal plane. the coefficient of friction being* $\tan\varepsilon$. *Show that it will begin to slip down if its initial inclination to the vartical is greater that* 2ε. *Prove also that the inclination* θ *of the rod to the vertical at any time is given by* $\ddot{\theta}\,(k^2 + a^2\cos 2\varepsilon) - a^2\dot{\theta}^2\sin 2\varepsilon = ag\,\sin(\theta - 2\varepsilon)$

Sol. Let AB be the rod of length $2a$ and mass m. When AB makes an angle θ with the vertical and let R and S be the resultant reactions at B and A respectively .

Writting equations of motion of centre of mass G , we have

$$m\frac{d^2}{dt^2}(a\sin\theta) = -S\sin\varepsilon + R\cos\varepsilon \quad ...(1)$$

and $$m\frac{d^2}{dt^2}(a\cos\theta) = R\sin\varepsilon + S\cos\varepsilon - mg \quad ...(2)$$



Taking moments about G, we have

$$mk^2\ddot{\theta} = Sa\sin(\theta-\varepsilon) - Ra\cos(\theta-\varepsilon) \quad ...(3)$$

From (2), we have $ma(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) = R\cos\varepsilon - S\sin\varepsilon$ ...(4)

From (2), we have

$$ma\,(\sin\,\theta\ddot{\theta} + \cos\,\theta\dot{\theta}^2) = mg - R\,\sin\varepsilon - S\,\cos\varepsilon \qquad ...(5)$$

On solving equations (4) and (5), we have

$$R = mg\,\sin\,\varepsilon + ma\,\cos(\theta+\varepsilon)\,\ddot{\theta} - ma\,\sin(\theta+\varepsilon)\,\dot{\theta}^2 \qquad ...(6)$$

$$S = mg\,\cos\,\varepsilon - ma\,\sin(\theta+\varepsilon)\,\ddot{\theta} - ma\,\cos(\theta+\varepsilon)\,\dot{\theta}^2 \qquad ...(7)$$

Putting the values of R and S in (3), we have

$$mk^2\ddot{\theta} = a\,\sin(\theta-\varepsilon)\,[mg\,\cos\,\varepsilon - ma\,\sin(\theta-\varepsilon)\,\ddot{\theta} - ma\,\cos(\theta+\varepsilon)\,\dot{\theta}^2]$$

$$-\,a\,\cos(\theta-\varepsilon)\,[mg\,\sin\varepsilon + ma\,\cos(\theta+\varepsilon)\,\ddot{\theta} - ma\,\sin(\theta+\varepsilon)\,\dot{\theta}^2]$$

$$= m\,ga\,\sin(\theta-2\varepsilon) - ma^2\,\ddot{\theta}\,\cos\,2\varepsilon + ma^2\dot{\theta}^2\sin 2\varepsilon]$$

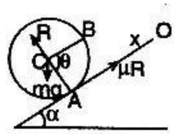
or $\ddot{\theta}\,(k^2 + a^2\,\cos 2\varepsilon) - a^2\,\dot{\theta}^2\sin\,2\varepsilon = ag\,\sin(\theta-2\varepsilon)$, which gives θ.

If $\theta > 2\varepsilon$, it is obvious that $\ddot{\theta}$ is positive and hence the rod starts slipping if $\theta > 2\varepsilon$.

3·07. When rolling and sliding are combined .

An imperfectly rough sphere moves from rest down a plane inclined at an angle α to the horizon, to determine the motion .

Let C be the centre of sphere whose radius is a . Let in time t the sphere have turned through an angle θ *i.e.* let CB be a radius (a line fixed in the body) which was initially normal to the plane, makes an angle θ with the normal CA during this period .



Let us suppose that the friction is not sufficient to produce pure rolling therefore the sphere slides as well as turns . So the maximum friction μR acts up the plane, μ being the coefficient of friction . Let x be the distance described by the centre of gravity C parallel to the inclined plane in time t , and θ the angle through which the sphere turns .

As there is no motion perpendicular to the plane, so the C.G. of the sphere always moves parallel to the plane. The equations of motion are

$$m\ddot{x} = mg\,\sin\,\alpha - \mu R \quad ...(1) \qquad 0 = R - mg\,\cos\,\alpha \qquad ...(2)$$

and $m\,\frac{2}{5}\,a^2\,\ddot{\theta} = \mu Ra$ ...(3)

From (1) and (2), we have $\ddot{x} = g\,(\sin\,\alpha - \mu\,\cos\,\alpha)$ ...(4)

Integrating (4) w.r.t. 't' we get $\dot{x} = g\,(\sin\,\alpha - \mu\,\cos\,\alpha)\,t$ ...(5)

Integrating (5) again, we get $x = g\,(\sin\,\alpha - \mu\,\cos\,\alpha)\,\dfrac{t^2}{2}$ ...(6)

Constants of integration vanish as $\dot{x} = 0$, $x = 0$ when $t = 0$

From (2) and (3), we get $a\ddot{\theta} = \frac{5}{2}\mu g \cos\alpha$.

Integrating it, we get $a\dot{\theta} = \frac{5}{2}\mu g\, t \cos\alpha$.

Integrating it again, we get $\theta = \frac{5\mu g}{4} t^2 \cos\alpha$

The constants of integration vanish as $\dot{\theta} = 0$, $\theta = 0$ when $t = 0$.
The velocity of the point of contact A down the plane
= velocity of C, the centre of sphere, + velocity of A relative to C,

$$= \dot{x} - a\dot{\theta}$$
$$= g(\sin\alpha - \mu\cos\alpha)\, t - \tfrac{5}{2}\mu g t \cos\alpha$$
$$= \tfrac{1}{2} g(2\sin\alpha - 7\mu\cos\alpha). \qquad \text{...(8)}$$

Equation (8) gives rise to the following three cases :
First case. If $2\sin\alpha > 7\mu\cos\alpha$ *i.e.* if $\mu < \frac{2}{7}\tan\alpha$.

In this case, velocity of the point of contact is positive for all values of t *i.e.* it does not vanish, hence the point of contact always slides down and the maximum friction μR acts. The sphere never rolls. The equations of motion established above govern the entire motion.
Second case. If $2\sin\alpha = 7\mu\cos\alpha$ *i.e.* if $\mu = \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is zero for all values of t and therefore motion of the sphere is that of pure rolling throughout and the maximum friction μR is always exerted.
Third case. $2\sin\alpha < 7\mu\cos\alpha$ *i.e.* if $\mu > \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is negative *i.e.* if the maximum friction μR were allowed to act, the point of contact will slide up the plane which is impossible because that amount of friction will only act which is just sufficient to keep the point of contact at rest. Hence in this case the motion is of pure rolling from the very start and remains the same throughout and the maximum friction μR is not exerted. Therefore in this case the equations of motion established above do not hold good.
Let F be the frictional force now in play, then equations of motion are

$$m\ddot{x} = mg\sin\alpha - F \quad \text{...(9)} \qquad 0 = R - mg\cos\alpha \quad \text{...(10)}$$

and $m\frac{2}{5}a^2\ddot{\theta} = Fa$

Beacause the point of contacty is at rest, we have

$$\dot{x} - a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta} \qquad \text{...(12)}$$

From (9), (11) and (12), we have $\ddot{x} = a\dot{\theta} = \frac{5}{7} g\sin\alpha$

Integrating above, we get $\dot{x} = a\dot{\theta} = \frac{5}{7}\, g\, t \sin\alpha$

Again integrating above, we get $x = a\theta = \dfrac{5}{14}\, gt^2 \sin\alpha,$ ...(14)

the constants of integration vanish as $\dot{x} = 0, x = 0,$ when $t = 0$

ILLUSTRATIVE EXAMPLES

Ex. 1. *A hoop is projected with velocity V down on inclined plane of inclination* α *, the coefficient of friction being* μ ($>\tan\alpha$). *It has initialy such a backward spin* Ω *that after a time* t_1 *it starts moving uphill and continues to do so for a time* t_2 *after which it once more descends. The motion being in a vertical at right angles to the given inclined plane, show that* $(t_1 + t_2)\, g \sin\alpha = a\Omega - V.$

Sol. Let C be the centre of the hoop and CB its radius (a line fixed in the body) makes an angle θ with CA which is normal to the plane (CA is a line fixed in space), after time t . Initially CB was normal to the plane. Initially the velocity of the point of contact A down the plane

= velocity of centre C + velocity of A relative to $C = V + a\Omega$, which is a positive quantity

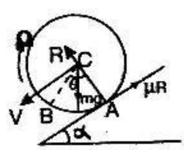
Hence the point of contact slides down and friction μR acts up the plane.

The equations of motion are

$m\ddot{x} = mg \sin\alpha - \mu R$ ...(1)

$0 = R - mg\cos\alpha$ ...(2) and

$ma^2\ddot{\theta} = -\mu Ra$ ...(3)



From (1) and (2), we have

$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$, integrating it, we get

$\dot{x} = g(\sin\alpha - \mu\cos\alpha)\, t +$ constant

when $\dot{x} = V, t = 0.$ $\therefore$ constant $= V$

Therefore $\dot{x} = g(\sin\alpha - \mu\cos\alpha)\, t + V$ ...(4)

From (1) and (3), we get

$a\ddot{\theta} = -\mu g\cos\alpha$, integrating it, we get

$a\dot{\theta} = -\mu gt\cos\alpha +$ constant ; When $t = 0, \dot{\theta} = \Omega$ constant $= a\Omega$

Therefore $a\dot{\theta} = -\mu gt\cos\alpha + a\Omega$ ...(5)

The hoop will cease to move downwards, when $\dot{x} = 0$ *i.e.* from (4),

$$t_1 = \frac{V}{g\,(\mu \cos \alpha - \sin \alpha)} \qquad \text{...(6)}$$

Obviously the velocity of the point of contact is $\dot{x} + a\dot{\theta}$, even when $\dot{x} = 0$, for the hoop to move uphill $a\dot{\theta}$ should be positive. It follows that throughout the downward motion $\dot{x} + a\dot{\theta}$ is always positive. Therefore when moving downward pure rolling does not take place. Thus the equations established above are true throughout the downward motion.

Putting the value of t_1 from (6) in (5), we get

$a\dot{\theta} = a\Omega - \dfrac{\mu V \cos \alpha}{(\mu \cos \alpha - \sin \alpha)}$ since $a\dot{\theta}$ is positive, the hoop begins to move uphill.

When the hoop starts moving uphill. The initial velocity of the centre is zero and $a\dot{\theta}$ is positive with the sense of the direction as θ.

Initial velocity of the point of contact $= 0 - a\dot{\theta}$ which is negative.
Thus initially the velocity of the point of contact is in the downward direction hence the friction μR acts upwards. Equations of motion are

$$m\ddot{y} = -mg \sin \alpha + \mu R \quad \text{...(1)} \qquad 0 = R - mg \cos \alpha \qquad \text{...(2)}$$

$$ma^2 \ddot{\phi} = -\mu Ra \qquad \text{...(3)}$$

on eliminating R, we get $\ddot{y} = (\mu \cos \alpha - \sin \alpha)\, g$ and $a\ddot{\phi} = -\mu g \cos a$
Intetgrating these two equations, with the initial conditions, we get

$$\dot{y} = g\,(\mu \cos \alpha - \sin \alpha)\, t \qquad \text{...(4)}$$

and $a\dot{\phi} = -\mu g t \cos \alpha + a\Omega - \dfrac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}$ ...(5)

$$\left[\because \text{ when } t = 0,\ \dot{y} = 0,\ a\dot{\phi} = a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}\right]$$

Rolling commences when the velocity of the point of contact is zero *i.e.*

$\dot{y} - a\dot{\phi} = 0 \Rightarrow \dot{y} = a\dot{\phi}$

$\Rightarrow g\,(\mu \cos \alpha - \sin \alpha)\, t' = -\mu g t' \cos \alpha + a\Omega - \dfrac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}$

$\Rightarrow g t'\,(2\mu \cos \alpha - \sin \alpha) = a\Omega - \dfrac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha}$

This gives value of t'

$\therefore$ At this time $\dot{y} = g t'\,(\mu \cos \alpha - \sin \alpha)$ from (4).

When Rolling commences. Equations of motion are

$$m\ddot{z} = F - mg \sin \alpha \quad ...(1) \qquad ma^2 \ddot{\psi} = -Fa \qquad ...(2)$$

and $\quad \dot{z} - a\dot{\psi} = 0 \qquad ...(3)$

Solving these equations, we get $\quad F = \frac{1}{2} mg \sin \alpha$

Since $\mu > \tan \alpha \Rightarrow \mu R > \tan \alpha \, mg \cos \alpha \Rightarrow \mu R > mg \sin \alpha$.

We observe that $F < \mu R$, so the condition of pure rolling is satisfied, and hence the equations of motion holds good for the motion .

From (1), we have , $m\ddot{z} = F - mg \sin \alpha = \frac{1}{2} mg \sin \alpha - mg \sin \alpha$.

i.e. $\ddot{z} = -\frac{1}{2} g \sin \alpha$; integrating it, we get $\dot{z} = -\frac{1}{2} gt \sin \alpha + K$

when $t = 0, \dot{z} = \dot{y} = gt'(\mu \cos \alpha - \sin \alpha)$, $\therefore K = gt'(\mu \cos \alpha - \sin \alpha)$

Therefore $\quad \dot{z} = -\frac{1}{2} gt \sin \alpha + gt'(\mu \cos \alpha - \sin \alpha)$

The hoop ceases to move up the hill if $\dot{z} = 0$. Let this happen after time t''.

$$\therefore 0 = -\frac{1}{2} gt'' \sin \alpha + gt'(\mu \cos \alpha - \sin \alpha)$$

or $\quad t'' = 2 \dfrac{(\mu \cos \alpha - \sin \alpha) t'}{\sin \alpha}$

$$\therefore t_2 = t' + t'' = t' + 2 \frac{(\mu \cos \alpha - \sin \alpha)}{\sin \alpha} t' = \left(\frac{2\mu \cos \alpha - \sin \alpha}{\sin \alpha} \right) t'$$

$$= \left(\frac{2\mu \cos \alpha - \sin \alpha}{\sin \alpha} \right) . \frac{1}{g (2\mu \cos \alpha - \sin \alpha)} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

$$= \frac{1}{g \sin \alpha} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

Hence the total time is $t_1 + t_2$

$$= \frac{V}{g (\mu \cos \alpha - \sin \alpha)} + \frac{1}{g \sin \alpha} \left(a\Omega - \frac{\mu V \cos \alpha}{\mu \cos \alpha - \sin \alpha} \right)$$

$$= \frac{1}{g \sin \alpha} \left[a\Omega - \frac{\mu V \cos \alpha - V \sin \alpha}{\mu \cos \alpha - \sin \alpha} \right] = \frac{1}{g \sin \alpha} (a\Omega - V)$$

or $\qquad (t_1 + t_2) \, g \sin \alpha = a\Omega - V$

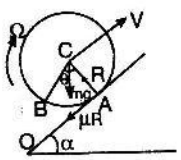
Ex. 2. *A sphere, of radius a is projected up an inclined plane with a velocity V and angular velocity Ω in the sense which would cause it to roll up , $V > a\Omega$, and the coefficient of friction $\frac{2}{7} \tan \alpha$; show that the sphere will cease to ascend at the end of a time* $\dfrac{5V + 2a\Omega}{5 g \sin \alpha}$, *where α is the inclination of the plane.* **(Meerut 81)**

Sol. Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA normal to the plane (CA is a line fixed in space. Initial CB was normal to the plane. Initial velocity of the point of contact A up the plane

= Velocity of the centre C + velocity of A relative to C

$= V - a\Omega > 0$ as $V > a\Omega$.

Hence the friction μR acts down the plane, implying, that the sphere slides as well as turns .



Equations of motion are $m\ddot{x} = mg \sin\alpha - \mu R$ ...(1)

$0 = R - mg \cos\alpha$ ...(2) and $m \cdot \dfrac{2a^2}{5} \ddot{\theta} = \mu Ra$ ...(3)

Eliminating R from (1) and (2), we have

$\ddot{x} = -g(\sin\alpha + \mu \cos\alpha)$, integrating it, we get

$\dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + K$.

Now when $t = 0, \dot{x} = V$, $\therefore K = V$

Therefore $\dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + V$ ...(4)

Similarly, we have $a\ddot{\theta} = \frac{5}{2}\mu g t \cos\alpha$, integrating it with intial conditions

i.e. when $t = 0$, $\dot{\theta} = \Omega$, we get $a\dot{\theta} = \frac{5}{2}\mu g t \cos\alpha + a\Omega$ ...(5)

The velocity of the point contact $= \dot{x} - a\dot{\theta}$. Rolling commences, say after time t_1 when $\dot{x} - a\dot{\theta} = 0$.

or $-g(\sin\alpha + \mu\cos\alpha)t_1 + V - \dfrac{5\mu}{2} g t_1 \cos\alpha - a\Omega = 0$

or $t_1 = \dfrac{2V - 2a\Omega}{g(7\mu\cos\alpha + 2\sin\alpha)}$

Putting this value of $t = t_1$ in (4), we get

$$\dot{x} = V - g(\sin\alpha + \mu\cos\alpha)\left[\frac{2V - 2a\Omega}{g(7\mu\cos\alpha + 2\sin\alpha}\right]$$
$$= \frac{5\mu V\cos\alpha + 2a\Omega(\sin\alpha + \mu\cos\alpha)}{7\mu\cos\alpha + 2\sin\alpha} = V_1 \text{ (say)}.$$

When rolling begins *i.e.* when the point of contact has been brought to rest, let F be the friction which is sufficient for pure rolling. Because the point of contact is at rest, so friction will try to keep it at rest if possible , hence the friction F acts upwards.

Equations of motion are $m\ddot{y} = -mg\sin\alpha + F$ ...(1)

and $\quad m.\dfrac{2a^2}{5}\ddot{\phi} = -Fa$ ...(2)

Since, throughout the motion the point of contact is at rest so

$\dot{y} - a\dot{\phi} = 0 \quad$ or $\quad \dot{y} = a\dot{\phi} \Rightarrow \ddot{y} = a\ddot{\phi}$

Solving equations (1) and (2), we get $F = \frac{2}{7}.mg \sin\alpha$

Again $\mu R = \mu.mg\cos\alpha > \frac{2}{7}\tan\alpha.mg\cos\alpha$ *i.e.* $> \frac{2}{7}mg\sin\alpha$.

Therefore the condition $F < \mu R$ is satisfied.

Putting the value of F in (1), we get $\ddot{y} = -\frac{5}{7}g\sin\alpha$.

Integrating it with intial conditions *i.e.* when $t = 0$, $\dot{y} = V_1$, we get

$\dot{y} = -\frac{5}{7}gt\sin\alpha + V_1$.

The sphere will cease to ascend when $\dot{y} = 0$, let this happen after time t_2.

$\therefore\; 0 = -\dfrac{5}{7}gt_2\sin\alpha + V_1 \quad$ or $\quad t_2 = \dfrac{7V_1}{5g\sin\alpha}$

$\therefore$ The total time of ascent $= t_1 + t_2$

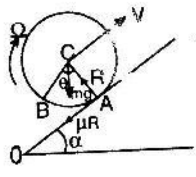
$$= \frac{2V - 2a\Omega}{g(7\mu\cos\alpha + 2\sin\alpha)} + \frac{7}{5g\sin\alpha} \times \left\{\frac{5\mu V\cos\alpha + 2a\Omega(\sin\alpha + \mu\cos\alpha)}{7\mu\cos\alpha + 2\sin\alpha}\right\}$$

$$= \frac{10(V - a\Omega)\sin\alpha + 35\mu V\cos\alpha + 14a\Omega(\sin\alpha + \mu\cos\alpha)}{5g\sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V(7\mu\cos\alpha + 2\sin\alpha) + 2a\Omega(7\mu\cos\alpha + 2\sin\alpha)}{5g\sin\alpha(7\mu\cos\alpha + 2\sin\alpha)}$$

$$= \frac{5V + 2a\Omega}{5g\sin\alpha}.$$

Ex. 3. *If a sphere be projected up an inclined plane, for which $\mu = \frac{1}{7}\tan\alpha$, with velocity V and an initial angular velocity Ω (in the direction in which it would roll up), and if $V > a\Omega$, show that the friction acts downwards at first and upwards afterwards, and prove that the whole time during which the sphere rises is* $\dfrac{17V + 4a\Omega}{18g\sin\alpha}$.



Sol. Let C be the centre of the sphere and CB a radius which is a line fixed in the body makes an angle θ after time t with CA, the normal to the plane (CA is a line fixed in the space). Initially CB was normal to the plane.

Initial velocity of the point of contact A up the plane

$= $ Velocity of the centre $C +$ velocity of A relative to C

$= V - a\Omega > 0$, since $V > a\Omega$.

Hence the velocity of the point of contact A is up the plane, thus the friction μR acts down the plane.

The sphere therefore slides as well as turns.

Equations of motion are

$$m\ddot{x} = -mg \sin \alpha - \mu R \quad ...(1) \quad 0 = R - mg \cos \alpha \quad ...(2)$$

and $$m\frac{2a^2}{5} \ddot{\theta} = \mu Ra \quad ...(3)$$

Eilminating R from (1) and (2), we get

$$m\ddot{x} = -mg \sin \alpha - \mu (mg \cos \alpha) = -mg \sin \alpha - \tfrac{1}{7} \tan \alpha . mg \cos \alpha$$

$$= -\tfrac{8}{7} mg \sin \alpha \quad (\mu = \tfrac{1}{7} \tan \alpha)$$

or $\dot{x} = -\frac{8}{7} g \sin \alpha$.

Similarly, we have $m\frac{2a}{5} \ddot{\theta} = \mu R = \frac{1}{7} \tan \alpha \; mg \cos \alpha = \frac{1}{7} mg \sin \alpha$

or $$a\ddot{\theta} = \frac{5}{14} g \sin \alpha \quad ...(5)$$

Integrating (4) and (5) with intial conditions *i.e.* when $t = 0$, $\dot{x} = V$ and

$\dot{\theta} = \Omega$, we get $$\dot{x} = -\tfrac{8}{7} gt \sin \alpha + V \quad ...(6)$$

and $$a\dot{\theta} = \frac{5}{14} gt \sin \alpha + a\Omega \quad ...(7)$$

Let the velocity of the point of contact *i.e.* $\dot{x} - a\dot{\theta}$ be zero after time t_1 (then the point of contact is brought to rest)

i.e. $\dot{x} - a\dot{\theta} = 0 \Rightarrow \dot{x} = a\dot{\theta}$

$\Rightarrow -\frac{8}{7} gt_1 \sin \alpha + V = \frac{5}{14} gt \sin \alpha + a\Omega$ (Putting the values of $\dot{x}$ and $\dot{\theta}$)

$\Rightarrow t_1 = \dfrac{2(V - a\Omega)}{3g \sin \alpha}$. Putting this value of t_1 in (6), we get

$$\dot{x} = V - \frac{16}{21}(V - a\Omega) = \frac{5V + 16a\Omega}{21} = V_1 \text{ (say)}.$$

When the point of contact has been brought to rest, the pure rolling will commence if there is enough friction to keep the point of contact at rest. Let F be the force of friction sufficient for pure rolling. Equations of motion are $m\ddot{y} = -mg \sin \alpha + F$, $m\frac{2a^2}{5} \dot{\phi} = -Fa$ Also $\dot{y} - a\dot{\phi} = 0$

Solving these equations, we get $F = \frac{2}{7} mg \sin \alpha$

while $\quad \mu R = \frac{1}{7} \tan \alpha \; mg \cos \alpha = \frac{1}{7} mg \sin \alpha$.

Hence we observe that $F > \mu R$.

From this we conclude that the friction required for pure rolling is more than the maximum friction that can be exerted by the plane, so the pure rolling is impossible .

Inspite of exerting the maximum friction μR upwards, the friction cannot keep the point of contact at rest . Hence the sphere slides as well as turns.

The equations of motion are $m\ddot{y} = -mg \sin \alpha + \mu R$ ...(i)

$0 = R - mg \cos \alpha$ ...(ii) and $m \dfrac{2a^2}{5} \ddot{\phi} = -\mu Ra$ ...(iii)

i.e. $m\dot{y} = -mg \sin \alpha + \frac{1}{7} \tan \alpha . mg \cos \alpha = -\frac{6}{7} mg \sin \alpha$

$\dot{y} = -\frac{6}{7} gt \sin \alpha + V_1$

The sphere will cease to ascend when $y^{\cdot} = 0$, let this happen after time t_2 . $\therefore 0 = -\frac{6}{7} gt_2 \sin \alpha + V_1$ or $t_2 = (7V_1/6g \sin \alpha)$.

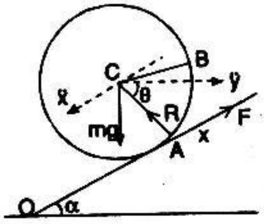
Hence the whole time of ascent

$$= t_1 + t_2 = \frac{2(V - a\Omega)}{3g \sin \alpha} + \frac{7}{6g \sin \alpha} - \left(\frac{5V + 16a\Omega}{21}\right)$$

$$= \frac{12(V - a\Omega) + 5V + 16a\Omega}{18g \sin \alpha} = \frac{17V + 4a\Omega}{18g \sin \alpha}.$$

Ex. 4. *An inclined plane of mass M is capable of moving freely on a smooth horizontal plane . A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity . If y be the horizontal distance advanced by the inclined plane and x the part of the plane rolled over by the sphere, prove that* $(M + m) y = mx \cos \alpha$ *and* $\frac{7}{5} x - y \cos a = \frac{1}{2} gt^2 \sin \alpha$, *where* α *is the inclination of the plane to the horizon .*

Sol. There are two accelerations of the centre C , one $\ddot{x}$ down the plane and other $\ddot{y}$ in a horizontal direction .



The actual acceleration of C parallel to the plane $= \ddot{x} - \ddot{y} \cos \alpha$.

Equations of motion of the sphere are

$$m(\ddot{x} - \ddot{y} \cos \alpha) = mg \sin \alpha - F \quad ...(1)$$

$$m\ddot{y} \sin \alpha = mg \cos \alpha - R \quad ...(2)$$

and $m \dfrac{2a^2}{5} \ddot{\theta} = Fa$ ...(3)

Since it is a case of pure rolling $\quad x = a\theta \Rightarrow \dot{x} = a\dot{\theta}$ ...(4)
Equation of motion of the plane is given by

$M\ddot{y} = R \sin\alpha - F\cos\alpha$ ...(5)
From (1) and (3) on adding, we have

$\frac{7}{5}\ddot{x} - \ddot{y}\cos\alpha = g\sin\alpha$ {from (4) $\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta}$ }

Integrating above, we get $\frac{7}{5}\dot{x} - \dot{y}\cos\alpha = gt\sin\alpha$

Integrating again $\quad \frac{7}{5}x - y\cos\alpha = \frac{1}{2}gt^2\sin\alpha$.

The constants of integrating vanish since initiailly all $\dot{x}, \dot{y}, x$ and y are zero. Equation (5) is $M\ddot{y} = R\sin\alpha - F\cos\alpha$

$= (mg\cos\alpha - m\ddot{y}\sin\alpha)\sin\alpha + (m\ddot{x} - m\ddot{y}\cos\alpha - mg\sin\alpha)\cos\alpha$
[Putting the values of F and R from (1) and (2)]

or $\quad M\ddot{y} = -m\ddot{y}(\cos^2\alpha + \sin^2\alpha) + m\ddot{x}\cos\alpha$

$= -m\ddot{y} + m\ddot{x}\cos\alpha$

$\Rightarrow (M+m)\ddot{y} = m\ddot{x}\cos\alpha$

Integrating, we get $\quad (M+m)\dot{y} = m\dot{x}\cos\alpha$.
Again integrating, we get $(M+m)y = mx\cos\alpha$.

The contants of integrating vanish as initiailly $\dot{x}, \dot{y}, x$ and y are all zero.

Ex. 5. *A uniform sphere, of radius a, is rotating about a horizontal diameter with angular velocity Ω and is gently placed on a rough plane which is inclined at an angle α to the horizontal, the sense of rotation being such as to tend to cause the sphere to move up the plane along the line of greatest slope. Show that, if the coefficient of friction be* $\tan\alpha$, *the centre of the sphere will remain at rest for a time* $\frac{2a\Omega}{5g\sin\alpha}$ *and will then move downwards with acceleration* $\frac{5}{7}\sin\alpha$. *If the body be a thin circular hoop instead of sphere, show that the time is* $\frac{a\Omega}{g\sin\alpha}$ *and the acceleration* $\frac{1}{2}g\sin\alpha$.

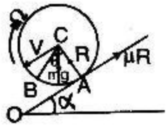
Sol. The sphere before being placed gently on the inclined plane was rotating with an angular velocity Ω about the horizontal diameter. Hence initially the velocity of the centre is zero .
The sense of rotation at the time of placing the sphere on inclined plane is such that it tends to cause the sphere to move up the plane, that means

sense of Ω is as shown in the figure. The initial velocity of the point of contact A down the plane

$=$ Velocity of the centre $C +$ velocity of A relative to C

$= 0 + a\Omega$, which is a positive quantity .

Hence the initial velocity of the point of contact is down the plane, so the friction μR acts up the plane.



Equations of motion are $m\ddot{x} = mg \sin \alpha - \mu R$ ...(1)

$0 = R - mg \cos \alpha$ ...(2) and $mk^2\ddot{\theta} = -\mu Ra$ ...(3)

where $\mu = \tan \alpha$

Eilminating R from (1) and (2), we get

$m\ddot{x} = mg \sin \alpha - \tan \alpha \,.\, mg \cos \alpha = 0 \Rightarrow \ddot{x} = 0 \Rightarrow \dot{x} = 0$ ...(4)

From (2) and (3), we get (Initially when $t = 0 , \dot{x} = 0$)

$mk^2\ddot{\theta} = -\tan \alpha\,(mg \cos \alpha)\, a = -mga \sin \alpha$ or $k^2\ddot{\theta} = -ga \sin \alpha$.

Integrating it, we get $k^2\dot{\theta} = -gat \sin \alpha + k^2\Omega$

From equation (4) and (5), we observe that the centre of the sphere does not move at all , but the sphere goes on revolving .

The sphere will cease to rotate when $\dot{\theta} = 0$

$\therefore$ From (5), we get $0 = -gat \sin \alpha + k^2\Omega$ or $t = \dfrac{k^2\Omega}{ga \sin \alpha}$

For sphere $k^2 = \frac{2}{5}a^2$, and for the hoop $k^2 = a^2$, hence the sphere will remain at rest for a time $\dfrac{2}{5}\,\dfrac{a\Omega}{g \sin \alpha}$ and for the hoop this time will be $\dfrac{a\Omega}{g \sin \alpha}$.

Now when $\dot{x}$ and $a\dot{\theta}$ become zero, the velocity of the point of contact $(\dot{x} + a\dot{\theta})$ becomes zero , therefore pure rolling may commence provided the friction is sufficient for pure rolling. Let F be the value of friction sufficient for pure rolling .

The equations of motion are $m\ddot{y} = mg \sin \alpha - F$ ...(i)

$mk^2\ddot{\phi} = Fa$ ...(ii) a nd $\dot{y} - a\dot{\phi} = 0$ ...(iii)

as $\dot{y} - a\dot{\phi} = 0 \Rightarrow \dot{y} = a\dot{\phi} \Rightarrow \ddot{y} = a\ddot{\phi}$)

Solving (i) and (ii) with the help of (iii) we get $F = \dfrac{mg \sin \alpha}{1 + (a^2/k^2)}$ which is obviously less then $mg \sin \alpha$.

When $F < \mu R$, the rolling continues and the equations (i),(ii) and (iii) hold good .

From (ii) we get $k^2 \ddot{\phi} = \dfrac{ag \sin \alpha}{1 + (a^2/k^2)}$ or $\ddot{y} = \dfrac{ga^2 \sin \alpha}{(a^2 + k^2)}$ $(\because \; a\ddot{\phi} = \ddot{y})$

Putting $k^2 = \frac{2}{5} a^2, \ddot{y}$ *i.e.* acceleration in case of sphere is $\frac{5}{7} g \sin \alpha$.

Putting $d^2 = a^2, \ddot{y}$ *i.e.* acceleration in case of hoop is $\frac{1}{2} g \sin \alpha$.

Ex. 6. *A homogeneous sphere of radius a , rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $\dfrac{2 \omega a}{7 \mu g}$ and that then the sphere will roll with angular velocity* $(2 \omega/7)$. **(Agra 1987, 86 ; Garhwal 90)**

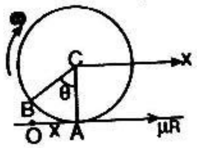
Sol. Since the sphere is gently placed on the table, the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω .

Initial velocity of the point contact=initial velocity of the centre C + Initial velocity of the point of contact with respect to C.

$= \quad 0 + a\omega$ in direction $\leftarrow$

Hence the point of contact will slip in the direction ($\leftarrow$), therefore full friction μR acts in the direction ($\rightarrow$).

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns, then at any time t, equations of motion are,



$$m\ddot{x} = \mu R \quad ...(1) \quad (\text{Here } R = mg)$$

and $$m \frac{2a^2}{5} \ddot{\theta} = -\mu R a \quad ...(2)$$

Therefore from (1) $\ddot{x} = \mu g$ and from (2)

$\frac{2}{5} a \ddot{\theta} = -\mu g$

Integrating these equations, we get

$$\dot{x} = \mu g t \quad ...(3) \quad \text{and} \quad a\dot{\theta} = -\tfrac{5}{2} \mu g t + a\omega \quad ...(4)$$

Since initially when $t = 0 , \dot{x} = 0 , \dot{\theta} = \omega$.

Velocity of the point contact $= \dot{x} - a\dot{\theta}$

Hence the point of contact will come to rest when $\dot{x} - a\dot{\theta} = 0$

i.e. when $\mu g t - (-\frac{5}{2}\mu g t + a\omega) = 0$ or when $t = \dfrac{2a\omega}{7\mu g}$

Therefore after time $\dfrac{2a\omega}{7\mu g}$ the slipping will stop and pure rolling will commence . Putting this value of t in (4), we get $\dot{\theta} = \dfrac{2\omega}{7}$

when rolling commences , the equations of motions are

$$m\ddot{x} = F \qquad \text{...(i)}, \qquad m\,\frac{2a^2}{5}\,\ddot{\theta} = -Fa \qquad \text{...(ii)}$$

and $\dot{x} - a\dot{\theta} = 0$ ...(iii)

From (i) and (ii) with the help of (iii) , we get

$ma\ddot{\theta} = F$ and $\frac{2}{5}ma\,\ddot{\theta} = -F$ $\quad (\dot{x} = a\dot{\theta} \Rightarrow \ddot{x} = a\ddot{\theta})$

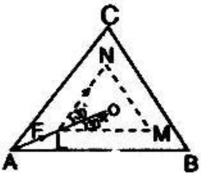
Adding these two equations , we get

$$\tfrac{7}{5}\,ma\ddot{\theta} = 0 \text{ or } \ddot{\theta} = 0 \Rightarrow \dot{\theta} = \text{const.} = \frac{2\omega}{7}$$

Ex. 7. *Three uniform spheres, each of radius a and of mass m attract one another according to the law of the inverse square of the distance. Initially they are placed on a perfectly rough horizontal plane with their centres forming a triangle whose sides are each of length 4a. Show that the velocity of their centres when they collide is* $\left(\gamma\dfrac{5m}{14a}\right)^{1/2}$ *where γ is the constant of graviation .* **(Agra 1986)**

Sol. Let A , B and C be the points of contact of the spheres with the horizontal plane, when they are initially at rest. ABC is an equilateral triangle of side $4a$. Let O be the centre of the triangle ABC .

Due to the symmetry of the attraction , the spheres will move in the way that their points of contact with the horizontal plane always form equilateral triangle



Let L, M, N be the new positions of the points of contact with the horizontal plane after time t.

Let $OL = x$,

By geometry, we observe that

$$OL = x\frac{LM}{\sqrt{3}} \left(\because \tfrac{1}{2}\,\frac{LM}{x} = \cos 30^\circ\right)$$

Therefore initially $x = \left(\dfrac{4a}{\sqrt{3}}\right)$ because initially the side of the triangle is $4a$.

Now when the spheres collide, $x = \left(\frac{2a}{\sqrt{3}},\right)$ because in this case the side of the triangle will become $2a$ (As radius of each sphere is a, so the distance between their centres will be $2a$)

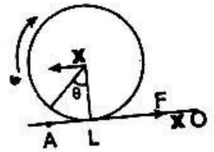
Let L be the point of contact of the first sphere with horizontal plane at time t.

Force of attraction on this sphere due to other two spheres is

$$= \left(\frac{\gamma m^2}{LM^2}\cos 30^\circ + \frac{\gamma m^2}{LN^2}\cos 30^\circ\right) \text{ in the direction } LO$$

$$= \frac{\gamma m^2}{3x^2}\cdot\frac{\sqrt{3}}{2} + \frac{\gamma m^2}{3x^2}\cdot\frac{\sqrt{3}}{2} \qquad (LM = LN = x\sqrt{3})$$

$$= \frac{\gamma m^2}{\sqrt{3}\,x^2} \text{ in the direction } LO \text{. i.e. towards}$$



x decreasing.

As the plane is perfectly rough, there is pure rolling thus the force of friction at the point of contact is F and acts opposite to the tendency of the motion of the point of contact, *i.e.* F acts towards x decreasing.

The equations of motion of the first sphere are

$$m\ddot{x} = -\left(\frac{\gamma m^2}{x^2\sqrt{3}}\right) - F \qquad \text{...(1)}$$

$$m\left(\frac{2a^2}{5}\right)\ddot{\theta} = -Fa \qquad \text{...(2)}$$

Since there is no slipping, the velocity of the point of contact $\dot{x} + a\dot{\theta}$ is zero. *i.e.* $\dot{x} = -a\dot{\theta} \Rightarrow \ddot{x} = -a\ddot{\theta}$ ...(3)

From (1), (2) and (3) on eliminating F and $a\ddot{\theta}$, we have $\ddot{x} = -\frac{5\gamma m}{7x^2\sqrt{3}}$

Integrating, we get $(\dot{x})^2 = \frac{10\gamma m}{7\sqrt{3}x} + K$.

Now, when $x = \frac{4a}{\sqrt{3}}, \dot{x} = 0, \quad \therefore K = -\frac{10\gamma m}{7\sqrt{3}}\cdot\frac{\sqrt{3}}{4a}$

$$\therefore (\dot{x})^2 = \frac{10\gamma m}{\sqrt{3}}\left(\frac{1}{x} - \frac{\sqrt{3}}{4a}\right) \qquad \text{...(4)}$$

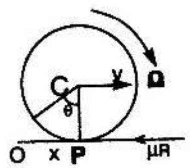
When the spheres collide *i.e.* when $x = \frac{2a}{\sqrt{3}}$; from (4), the velocity at that time is $(\dot{x})^2 = \frac{10\gamma m}{7\sqrt{3}}\left(\frac{\sqrt{3}}{2a} - \frac{\sqrt{3}}{4a}\right)$ or $\dot{x} = \left(\gamma\frac{5m}{14a}\right)^{1/2}$

3·08 . *A uniform circular disc is projected with its plane vertical along a rough horizontal plane with a velocity v of translation and an angular velocity Ω about the centre . Find the motion .*

(Agra 1987)

Case I. When $v \rightarrow, \Omega \downarrow.$ **and** $v > a\Omega$.

In this case initial velocity of the point of contact P is given by $v - a\Omega$, hence its direction is $\rightarrow$ as $v > a\Omega$, so the friction μR acts in the direction $\leftarrow$. When the centre has moved through a distance x and θ is the angle through which the disc has turned , the equations of motion are given by



$$m\ddot{x} = -\mu R = -\mu mg \quad i.e. \quad \ddot{x} = -\mu g \quad ...(1)$$

and $$m\frac{a^2}{2}\ddot{\theta} = \mu Ra = \mu mga \; i.e. \; a\ddot{\theta} = 2\mu g \quad ...(2)$$

Integrating (1) and (2) and making use of initial conditions

i.e. $t = 0$, $\dot{x} = v$ and $\dot{\theta} = \Omega$, we have $\dot{x} = -\mu gt + v$ ...(3)

and $a\dot{\theta} = 2\mu gt + a\Omega$ ...(4)

Now rolling commences when $\dot{x} - a\dot{\theta} = 0$. Let this happen after time t_1,

then $\dot{x} - a\dot{\theta} = -\mu gt_1 + v - 2\mu gt_1 - a\Omega = 0$ or $t_1 = \dfrac{v - a\Omega}{3\,\mu g}$.

Putting this value of t_1 in (3), we observe that at this time velocity of the centre *i.e.* $\dot{x} = \dfrac{2v + a\Omega}{3}$ . ...(5)

When rolling commences equations of motion reduce to

$$m\ddot{x} = -F \quad \text{and} \quad \frac{ma^2}{2}\dot{\phi} = Fa \quad ...(6)$$

Since there is no sliding, $\dot{x} = a\dot{\phi}$ or $\ddot{x} = a\ddot{\phi}$.

Solving these equations, we have $F = 0$.

Thus we observe that no friction is required throughout the motion for pure rolling, so equations for this motion are

$$m\ddot{x} = 0 \;\; i.e. \; \ddot{x} = 0 \quad \text{and} \quad m\frac{a^2}{2}\ddot{\phi} = 0 \;\; i.e. \; a\ddot{\phi} = 0 \quad ...(7)$$

Integrating (7), we get $\dot{x}$ = constant $= \dfrac{2v + a\omega}{3}$, from (5).

The disc therefore continues to roll with a constant velocity $\dfrac{2v + a\omega}{3}$ which is less then its initial velocity .

Case II. when $v \rightarrow, \Omega \downarrow$ **and** $v < a\Omega$.

In this case initial velocity of the point of contant is $v - a\Omega < 0$, so its direction is $\leftarrow$, hence friction μR acts in the direction $\rightarrow$.
Now the equations of motion are

$$m\ddot{x} = \mu R = \mu mg \quad i.e.\ \ddot{x} = \mu g \qquad ...(1)$$

and $m\dfrac{a^2}{2}\ddot{\theta} = -\mu Ra = -\mu mga \quad i.e.\ a\ddot{\theta} = -2\mu g$ ...(2)

Integrating these equations and making use of initial conditions to evaluate constants, we get

$\dot{x} = \mu gt + v$ ...(3) and $a\dot{\theta} = -2\mu gt + a\Omega$ ...(4)

Pure rolling commences when $\dot{x} - a\dot{\theta} = 0$, let this happen after time t_1 then from (3) and (4), $\mu gt_1 + v + 2\mu gt_1 - a\Omega = 0$ or $t_1 = \dfrac{a\Omega - v}{3\mu g}$

Putting this value of t_1 in (3), we get $\dot{x} = \dfrac{2v + a\Omega}{3}$

When pure rolling begins, equations of motion are same as in **case I** by which $F = 0$, so the disc continues rolling with constant velocity $= \dfrac{2v + a\omega}{3}$.

Case III when $v \rightarrow, \Omega \uparrow$

In this case, initial velocity of the point of contact is $v + a\Omega$, so its direction is $\rightarrow$, hence μR acts in the direction $\leftarrow$.
Equations of motion are

$$m\ddot{x} = -\mu R = -\mu mg \quad i.e.\ \ddot{x} = -\mu g \qquad ...(1)$$

and $m\dfrac{a^2}{2}\ddot{\theta} = -\mu Ra = -\mu mga \quad i.e.\ a\ddot{\theta} = -2\mu g$ ...(2)

Integrating these equations and making use of initial conditions to determine the constants, we get $\dot{x} = -\mu gt + v$ ...(3) and $a\dot{\theta} = -2\mu gt + a\Omega$ ...(4)

The velocity of the point of contact is $\dot{x} + a\dot{\theta}$ ($\dot{x}$ and $\dot{\theta}$ are in the same direction). Pure rolling begins when $\dot{x} + a\dot{\theta} = 0$, let this happen after time t_1 then from (3) and (4), we get $-\mu gt_1 + v + (-2\mu gt_1 + a\Omega) = 0$ or $t_1 = \dfrac{v + a\Omega}{3\mu g}$. Putting this value of t_1 in (3) and (4), we get $\dot{x} = a\dot{\theta} = \dfrac{2v - a\Omega}{3}$.

If $2v > a\Omega$ the velocity of the centre is positive , so the motion is of pure rolling with uniform velocity $\frac{2v - a\Omega}{3}$.

If $2v < a\Omega$ the velocity of the centre is negative (backward). In this case we observe from equation (3) that velocity of the centre becomes zero when $t = \frac{v}{\mu g}$, and at that time from equation (4) we observe that

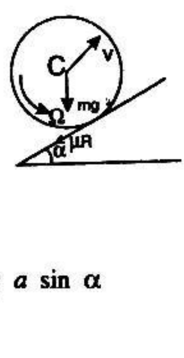
$a\dot{\theta} = -2v + a\Omega$. which is positive since $2v < a\Omega$. Hence when $2v < a\Omega$, the disc begins to move backward before pure rolling begins .

ILLUSTRATIVE EXAMPLES

Ex. 1. *A thin napkin ring, of radius a is projected up a plane inclined at angle α to the horizontal with velocity v , and an initial angular velocity Ω in the sense which would cause the ring to move down the plane. If $v > 5a\Omega$ and $\mu = \frac{1}{4} \tan \alpha$, show that the ring will never roll and will cease to ascend at the end of a time $\frac{4(2v - a\Omega)}{9g \sin \alpha}$ and will slide back to the point of projection .*

Sol. Initial velocity of the point of contact is $v + a\Omega$, which is up the plane , hence the friction μR acts down the plane.

The equations of motion are



$$m\ddot{x} = -mg \sin \alpha - \mu R$$
$$= -mg \sin \alpha - \mu mg \cos \alpha$$

or $\ddot{x} = -g(\sin \alpha + \mu \cos \alpha)$
$$= -g(\sin \alpha + \tfrac{1}{4} \tan \alpha \cos \alpha)$$

or $\ddot{x} = -\frac{5}{4} g \sin \alpha$ ...(1)

and

$$ma^2\ddot{\theta} = -\mu Ra = -\frac{1}{4} \tan \alpha . mg \cos \alpha . a = -\frac{1}{4} mg\, a \sin \alpha$$

$$\therefore \; a\ddot{\theta} = -\frac{1}{4} g \sin \alpha. \qquad ...(2)$$

Integrating (1) and (2) and applying initial conditions that at $t = 0$, $\dot{x} = v$ and $\dot{\theta} = \Omega$, we get

$\ddot{x} = -\frac{5}{4} g \sin \alpha . t + v$ ...(3) and $a\ddot{\theta} = -\frac{1}{4} g \sin \alpha . t + a\Omega$ ...(4)

From (3), we observe that velocity of the centre is zero after time $\frac{4v}{5g \sin \alpha}$.

The velocity of the point of contact at any time $= \dot{x} + a\dot{\theta}$

$$= -\frac{5}{4} g \sin\alpha + v - \frac{1}{4} g \sin\alpha . t + a\Omega \quad \{\text{From (3) and (4)}\}$$

$$= v + a\Omega - \frac{3}{2} g t \sin\alpha$$

Hence the point of contact will come to rest after time

$$\frac{2(v + a\Omega)}{3g \sin\alpha} \quad (\because \ \dot{x} + a\dot{\theta} = 0).$$

It can be seen that $\frac{2(v + a\Omega)}{3g \sin\alpha} < \frac{4v}{5g \sin\alpha}$ as $v > 5a\Omega$.

$\therefore$ pure rolling may begin before the upward motion ceases if the friction is sufficient for pure rolling.

At this time $\dot{x} = \frac{v - 5a\Omega}{6}$ which is positive and $\dot{\theta} = \frac{5a\Omega - v}{6a}$ which is $-ve$ ($\because \ v > 5a\Omega$)

or $\dot{\theta} = \frac{v - 5a\Omega}{6a}$ in clockwise direction .

when pure rolling commences, and rotation is in the , clockwise direction, the equations of motion are $m\ddot{y} = -mg \sin\alpha + F$

$ma^2\ddot{\phi} = -Fa$, $\dot{y} = a\dot{\phi}$ and $\ddot{y} = a\ddot{\phi}$.

Solving these equations, we get $F = \frac{1}{2} mg \sin\alpha$

But $\mu R = \frac{1}{4} \tan\alpha . mg \cos\alpha = \frac{1}{4} mg \sin\alpha$; hence friction is not sufficient for pure rolling . Hence the sliding persists and pure roling is not possible. The above equations of motion now become

$$m\ddot{y} = -mg \sin\alpha + \mu R = -mg \sin\alpha + \tfrac{1}{4} \tan\alpha . mg \cos\alpha$$

$$= -\tfrac{3}{4} mg \sin\alpha \quad \text{or} \quad \ddot{y} = -\frac{3}{4} g \sin\alpha \qquad \text{...(i)}$$

and $ma^2\ddot{\phi} = -\mu R a = -\frac{1}{4} \tan\alpha . mg \cos\alpha . a = -\frac{1}{4} mg\, a \sin\alpha$

$$\text{or} \quad a\ddot{\phi} = -\tfrac{1}{4} g \sin\alpha \qquad \text{...(ii)}$$

Integrating (i) and (ii) and applying the initial conditions when

$t = 0$, $\dot{y} = \frac{v - 5a\Omega}{6}$ and $a\dot{\phi} = \frac{5a\Omega - v}{6}$, we get

$$\dot{y} = -\frac{3}{4} g \sin\alpha . t + \frac{v - 5a\Omega}{6} \qquad \text{...(iii)}$$

and $\quad a\dot{\phi} = -\frac{1}{4} g \sin \alpha \,.\, t + \dfrac{5a\Omega - v}{6}$ ...(iv)

We observe that $\dot{y} = 0$ after time $\dfrac{2(v - 5a\Omega)}{9g \sin \alpha}$

Putting this value of time in (iv), we get $a\dot{\phi} = \dfrac{2(5a\,\Omega - v)}{9}$

Therefore total time of upwards motion

$$= 2\,\frac{(v + a\Omega)}{3g \sin \alpha)} + \frac{2(v - 5a\Omega)}{9g \sin \alpha} = \frac{4(2v - a\Omega)}{9g \sin \alpha}$$

Again, when the upward motion ceases, we have $a\dot{\phi} = \frac{2}{9}(5a\Omega - v)$ which is negative since $v > 5a\Omega$, hence the ring returns,

The velocity of the point of contact

= velocity of the centre + velocitry relative to the centre

$= \dot{y} - a\dot{\phi} = 0 - \frac{2}{9}(5a\Omega) = \frac{2}{9}\,(v - 5a\Omega)$

= a positive quantity as $v > 5a\Omega$

i.e. the velocity of the point of contact is up the plane ; therefore friction μR acts downwards ; hence the equations of motion are

$m\ddot{z} = mg \sin \alpha + \mu R = mg \sin \alpha + \frac{1}{4} \tan \alpha \cdot mg \,.\, \cos \alpha$

i.e. $\quad \ddot{z} = \dfrac{5}{4} g \sin \alpha$ ...(1)

and $\; ma^2\ddot{\psi} = -\mu R a = -\frac{1}{4} \tan \alpha \; mg \cos \alpha \,.\, a$

i.e. $a\ddot{\psi} = -\frac{1}{4} g \sin \alpha$ ...(2)

Integrating (1) and (2) and applying the initial condition that when $t = 0$, $\dot{z} = 0$ and $a\,\dot{\psi} = \frac{2}{9}(5a\Omega - v)$, we get

$\dot{z} = \frac{5}{4} gt \sin \alpha, \; a\dot{\psi} = -\frac{1}{4} gt \sin \alpha + \frac{2}{9}\,(5a\Omega - v)$

Hence the velocity of the point of contact down the plane $= \dot{z} - a\dot{\psi}$

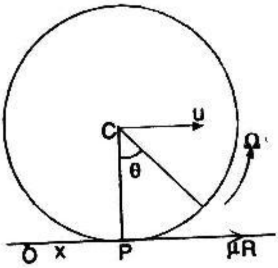
$= \frac{5}{4} gt \sin \alpha - [-\frac{1}{4}\, gt \sin \alpha + \frac{2}{9}\,(5a\Omega - v)]$

$= \frac{2}{9}\,(v - 5a\Omega) + \frac{3}{2}\, gt \sin \alpha$

which is positive ($v > 5a\Omega$) ; hence the ring slides back to the point of projection .

Ex. 2. *A napkin ring, of radius a, is projected forward on a rough horizontal table with a linear velocity u and a backward spin Ω which is $> \dfrac{u}{a}$. Find the motion and show that the ring will return to the point of projection in* *time* $\dfrac{(\mu + a\Omega)^2}{4\mu g\,(a\Omega - u)}$ *where* μ *is the coefficient of friction . What happens if* $u > a\Omega$? **(Agra 1985)**

Sol. Initially $u \rightarrow, \Omega \uparrow$ and $u < a\Omega$. This initial velocity of the point of contact is $u + a\Omega$ and is in the direction $(\rightarrow)$. Hence the friction μR acts in the direction $(\rightarrow)$. For this forward motion, equations of motion are



$$m\ddot{x} = -\mu R = -\mu mg \quad i.e. \quad \ddot{x} = -\mu g \quad ...(1)$$

and $ma^2\ddot{\theta} = -\mu Ra = -\mu mg$ *i.e.* $a\ddot{\theta} = -\mu g$...(2)

Integrating (1) and (2) and applying the initial conditions that when $t = 0$, $\dot{x} = u$ and $\dot{\theta} = \Omega$, we get

$$\dot{x} = -\mu gt + u \quad ...(3)$$

and

$$a\dot{\theta} = -\mu gt + a\Omega \quad ...(4)$$

The ring ceases to move forward if $\dot{\theta} = 0$, let this happen after time t_1, then from (3) $t_1 = \dfrac{u}{\mu g}$

Again integrating (4) and applying the condition that when $x = 0, t = 0$, we get $x = -\frac{1}{2}\mu gt^2 + ut$ ...(5)

Thus the distance traversed by the ring in time $t = \dfrac{u}{\mu g}$ is found by putting $t = \dfrac{u}{\mu g}$ in (6), *i.e.* $x = -\frac{1}{2}\mu g\left(\dfrac{u^2}{\mu^2 g^2}\right) + u\left(\dfrac{u}{\mu g}\right) = \dfrac{u^2}{2\mu g}$ ...(6)

and then $a\dot{\theta} = a\Omega - u$ which is in the direction $\uparrow$ ($\because u < a\Omega$).

Hence the ring returns .

When the ring returns .

Initial velocity of the point of contact is in the direction $(\rightarrow)$ hence the friction μR will act in the direction $(\leftarrow)$

For this motion equations are $m\ddot{y} = \mu R = \mu mg$ *i.e.* $\ddot{y} = \mu g$

and $ma^2\ddot{\phi} = -\mu Ra = -\mu mga$ *i.e.* $a\ddot{\phi} = -\mu g$ ...(ii)

Integrating (i) and (ii) and applying the initial condition *i.e.* when $t = 0, \dot{y} = 0$ and $a\dot{\phi} = a\Omega - u$, we get

$\dot{y} = \mu gt$ ...(iii) and $a\dot{\phi} = -\mu gt + a\Omega - u$ ...(iv)

These equations hold good until pure rolling commence *i.e.* until $\dot{y} = a\dot{\phi}$ (the velocity of the point of contact) is zero. Let this occur after time t_2 then from (iii) and (iv), we have

$$\mu g t_2 + \mu g t_2 - a\Omega + u = 0 \quad i.e. \quad t_2 = \frac{a\Omega - u}{2\mu g}$$

Hence from (iii) $\dot{y} = \dfrac{a\Omega - u}{2}$ ...(v)

Integrating (iii) again, we get $y = \frac{1}{2}\mu g t^2$ ($\because$ at $t = 0, y = 0$).

putting $t = \dfrac{a\Omega - u}{2\mu g}$, we get $y = \frac{1}{2}\mu g \left(\dfrac{a\Omega - u}{2\mu g}\right)^2 = \dfrac{(a\Omega - u)^2}{8\mu g}$ ...(vi)

when rolling begins, equations of motion are

$$m\ddot{z} = F \quad \text{and} \quad ma^2\ddot{\psi} = -Fa.$$

Since there is no sliding, hence $\dot{z} = a\dot{\psi} \Rightarrow \ddot{z} = a\ddot{\psi}$.

On solving these equations, we get $F = 0$, hence no friction is required then $\ddot{z} = 0$ *i.e.* $\dot{z}$ = constant $= \left\{ \because \text{at } t = 0, \dot{z} = \dfrac{a\Omega - u}{2} \text{ from } (v) \right\}$

i.e. when pure rolling commences (in return motion) the ring continues to move with its initial constant velocity $\dfrac{a\Omega - u}{2}$

Again the point where pure rolling commences is from the point of projection at distance $= \dfrac{u^2}{2\mu g} - \dfrac{(a\Omega - u)^2}{8\mu g}$ {from (6) and (v i)}

Therefore the time taken to traverse this distance is

$$t_3 = \left\{\frac{u^2}{2\mu g} - \frac{(a\Omega - u)^2}{8\mu g}\right\} \div \left(\frac{a\Omega - u}{2}\right)$$

or $\quad t_3 = \dfrac{u_2}{\mu g\,(a\Omega - u)} - \dfrac{a\Omega - u}{4\mu g}$

Hence the total time when the ring returns to the point of projection is

$$t_1 + t_2 + t_3 = \frac{u}{\mu g} + \frac{a\Omega - u}{2\mu g} + \left\{\frac{u_2}{\mu g\,(a\Omega - u)} - \frac{a\Omega - u}{4\mu g}\right\}$$

$$= \frac{u}{\mu g} + \frac{a\Omega - u}{4\mu g} + \frac{u_2}{\mu g\,(a\Omega - u)} = \frac{(a\Omega + u)^2}{4\mu g\,(a\Omega - u)}$$

Second Part. What happens when $u > a\Omega$?

To know this, we should consider the motion in the forward direction already discussed in the beginning.

In that case velocity of the point of contact is

$$\dot{x} + a\dot{\theta} = (-\mu g t + u) + (-\mu g t + a\Omega) \text{ [From (3) and (4)]}$$

$$= -2\mu g t + a\Omega + u$$

Rolling will commences when $\dot{a} + a\dot{\theta} = 0$ *i.e.* when $t = \dfrac{u + a\Omega}{2\mu g}$

Again it is proved that the ring ceases to move forward after a time $\dfrac{u}{\mu g}$ for the moment of projection .

Hence the rolling commence before the forward motion has ceased *i.e.* if $\dfrac{u + a\Omega}{2\mu g} < \dfrac{u}{\mu g}$ *i.e.* if $u > a\Omega$.

In other words we say that $u > a\Omega$, the rolling will commence before the forward motion ceases .

3·09 . *When two bodies are in contact ; then to determine whether the relative motion involves sliding or rolling at the point of contact .*

Let P be the point of contact of a moving body placed over the other, assume that the initial velocity of the point of contact is zero. To find whether the relative motion is of sliding or rolling we make use of following two methods .

In the first method, assume that the body rolls and suppose F is the force of friction sufficient to keep P (the point of contact) at rest. Hence F is unknown. Again write the equations of motion along with the geometrical equation to express the condition that the tangential velocity of the point P is zero. Solve these equation and find $\dfrac{F}{R}$

In case $\dfrac{F}{R} < \mu$, the necessary friction can be called into play to keep the point P at rest. Thus the body rolls and will remain so long as $\dfrac{F}{R} \le \mu$, but when $\dfrac{F}{R} > \mu$, the point of contact will slide. When this happens the equations of motion discussed before will not hold good, and we apply the following method .

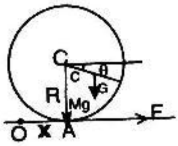
In this method write the equations of motion on the supposition that the point of contact slids *i.e.* the friction is μR instead of F and there is no geometrical equations . On solving these equations we find the tangential velocity of the point of contact P. In case this velocity is not zero and is in the direction opposite to the direction in which μR acts (μ has a proper sign), the body will slide at P and will remain so long as the velocity at P does not vanish , when velocity at P vanishes, we again apply the first method .

3·10. *A sphere, of radius a whose centre of gravity G is at a distance c from its centre C, placed on a rough plane so that CG is horizontal ; show that it will begin to roll or slide according as the coefficient of friction*

$\mu >$ *or* $< \dfrac{ac}{k^2+a^2}$, *where k is the radius of gyration about a horizontal axis through G ; if* μ *is equal to this value, what happens ?*

When CG is inclined at an angle θ to the horizontal, let A, the point of contact have moved through a horizontal distance x from its initial position O, and let $OA = x$. Assume that the sphere rolls and F be the force of friction sufficient for pure rolling. Since the motion is of pure rolling so $x = a\theta$ and the point of contact A is at rest

$\therefore\ \dot{x} = a\dot{\theta}$ ...(1)



The coordinates of G (the centre of gravity) with reference to O the fixed point as origin and horizontal and vertical lines through O as coordinate axes are $(x + c\cos\theta, a - c\sin\theta)$.

Equations of motion of the sphere are

$$F = M\frac{d^2}{dt^2}(x + c\cos\theta) = M\frac{d^2}{dt^2}(a\theta + c\cos\theta)$$

$$= M\,[a\ddot{\theta} - c\sin\theta\,\ddot{\theta} - c\cos\theta\,\dot{\theta}^2] \qquad ...(2)$$

$$R - Mg = M\frac{d^2}{dt^2}(a - c\sin\theta) = M\,[-c\cos\theta\,\ddot{\theta} + c\sin\theta\,\dot{\theta}^2] \qquad ...(3)$$

$$\text{and}\quad Rc\cos\theta - F(a - c\sin\theta) = Mk^2\ddot{\theta} \qquad ...(4)$$

We only want the initial motion when $\theta = 0$ and $\dot{\theta}$ is zero but $\ddot{\theta}$ is not zero. The equations (2), (3), (4) then give

$F = ma\ddot{\theta};\ R = mg - Mc\ddot{\theta};\ Rc - Fa = Mk^2\ddot{\theta}.$ **for the initial values**

On eliminating R, and F, we get $\ddot{\theta} = \dfrac{gc}{k^2+a^2+c^2}$, then

$$\frac{F}{M} = g\frac{ac}{k^2+a^2+c^2} \quad\text{and}\quad \frac{R}{M} = g\frac{k^2+a^2}{k^2+a^2+c^2};\ \frac{F}{M} = \frac{ac}{k^2+a^2}$$

The sphere will roll or slide according as

$F <$ or $> \mu R$ or as $\mu >$ or $< \dfrac{F}{R}$ *i.e.* as $\mu >$ or $< \dfrac{ac}{k^2+c^2}$.

Critical case. If $\mu = \dfrac{ac}{k^2+c^2}$. In this case we shall consider whether $\dfrac{F}{R}$ is a little greater or little less than μR when θ is small but not absolutely zero.

From equation (1), (2) and (3) on eliminating F and R, we get

$$(k^2+a^2+c^2-2ac\sin\theta)\ddot{\theta}-ac\cos\theta\,\dot{\theta}^2=gc\cos\theta \quad ...(4)$$

Integrating it, we get $(k^2+a^2+c^2-2ac\sin\theta)\,\dot{\theta}^2=2gc\sin\theta$ ...(5)
As θ is small, $\sin\theta$ can be replaced by θ and $\cos\theta$ by unity, neglecting squares and higher powers of θ, $\sin\theta\dot{\theta}^2$ is also neglected.

Thus (5) reduces to $(k^2+a^2+c^2)\,\dot{\theta}^2=2gc\,\theta$ ...(6)

and then from (4), $(k^2+a^2+c^2-2ac\,\theta)\,\ddot{\theta}=gc\left(1+\dfrac{a\,\dot{\theta}^2}{g}\right)$

$$=gc\left(1+\frac{2ac\,\theta}{k^2+a^2+c^2}\right) \quad \text{[from (6)]}$$

or $\quad (k^2+a^2+c^2)\left(1-\dfrac{2ac\,\theta}{k^2+a^2+c^2}\right)\ddot{\theta}=gc\left(1+\dfrac{2ac\,\theta}{k^2+a^2+c^2}\right)$

or $\quad (k^2+a^2+c^2)\,\ddot{\theta}=gc\left(1+\dfrac{2ac\,\theta}{k^2+a^2+c^2}\right)\left(1-\dfrac{2ac\,\theta}{k^2+a^2+c^2}\right)^{-1}$

$$=gc\left(1+\frac{2ac\,\theta}{k^2+a^2+c^2}\right)\left(1+\frac{2ac\,\theta}{k^2+a^2+c^2}\right)$$

$$=gc\left(1+\frac{4ac\,\theta}{k^2+a^2+c^2}\right), \text{ approximately}$$

From (1) and (2), we have $\dfrac{F}{R}=\dfrac{(a-c\sin\theta)\,\ddot{\theta}-c\cos\theta\,\dot{\theta}^2}{g-c\cos\theta\,\ddot{\theta}+c\sin\theta\,\dot{\theta}^2}$

$$=\frac{(a-c\,\theta)\,\ddot{\theta}-c\,\dot{\theta}^2}{g-c\,\ddot{\theta}} \quad \text{neglecting } \theta^2, \theta^3 \text{ etc. and also } \sin\theta\,\dot{\theta}^2$$

$=\dfrac{ac}{k^2+c^2}\left[1-c\,\dfrac{(3k^2-a^2)\,\theta}{a\,(k^2+a^2)}\right]$ by putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ as found above

If $\quad k^2>\dfrac{a^2}{3}$, then $\dfrac{F}{R}$ is less than $\dfrac{ac}{k^2+a^2}$

i.e. $\dfrac{F}{R}$ is less than μ or $F<\mu R$ and the sphere rolls .

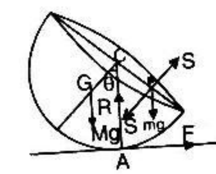
If $\quad k^2<\dfrac{a^3}{3}$, then $\dfrac{F}{R}>\dfrac{ac}{k^2+a^2}$ *i.e.* $\dfrac{F}{R}>\mu$ or $F>\mu R$ and the sphere slides.

ILLUSTRATIVE EXAMPLES

Ex 1. *A homogeneous solid hemispphere , of mass M and radius a, rests with its vertex in contact with a rough horizontal plane and a particle, of* *mass m , is placed on its base ; which is smooth , at a distance c from the centre. Show that the hemisphere will commence to roll or slide according as the coefficient of friction is greater or less than*

$$\frac{25\ mac}{26\,(M+m)\ a^2+40\ mc^2}$$ **(Agra 1984)**

Sol. Let C be the centre of the base and G the centre of gravity of the hemisphere . At point P, distant c from the centre, a particle of mass m is placed . When CG is inclined at an angle θ to the vertical, let A the point of contact have moved through a horizontal distance x from its initial position O, *i.e.* $OA = x$. Assume that the hemisphere rolls, and the point of contact A is at rest, so $x = a\theta$, hence,



$\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$

The co-ordinates of G, referred to O as origin are $\left(a\theta - \frac{3}{8}\,a\sin\theta,\, a - \frac{3a}{8}\sin\theta\right)$

The equations of motion of the hemisphere are

$$F - S\sin\theta = M\,\frac{d^2}{dt^2}\left(a\theta - \frac{3a}{8}\sin\theta\right)$$

$$= M\left[a\ddot{\theta} - \frac{3a}{8}\,(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2)\right] \qquad ...(1)$$

$$R - Mg - S\cos\theta = M\,\frac{d^2}{dt^2}\left(a - \frac{3a}{8}\cos\theta\right)$$

$$= M\left(\frac{3a}{8}\sin\theta\,\ddot{\theta} + \frac{5a}{8}\cos\theta\,\dot{\theta}^2\right) \qquad ...(2)$$

Taking moments about G,

$$Sc - F\left(a - \frac{3a}{8}\cos\theta\right) - R\,\frac{3a}{8}\sin\theta = Mk^2\ddot{\theta} \qquad ...(3)$$

The co-ordinates of particle P are $(a\theta + c\cos\theta,\, a - c\sin\theta)$, where $GP = c$.

The equation of motion of the particle is

$$S\cos\theta - mg = m\,\frac{d^2}{dt^2}\,(a - c\sin\theta) = m\,(-c\cos\theta\ddot{\theta} + c\sin\theta\,\dot{\theta}^2) \quad ...(4)$$

As the initial motion is required *i.e.* when $\theta = 0$, $\dot{\theta} = 0$ but $\ddot{\theta} \neq 0$, we have from (1), (2), (3) and (4)

$$\left.\begin{aligned} &F=\frac{5}{8}\ Ma\ddot{\theta}\ ,\ R=Mg+S \\ &Sc-\frac{5a}{8}\ F=Mk^2\ddot{\theta}\ \text{ and }\ S=mg-mc\ \ddot{\theta} \end{aligned}\right]\text{for the initial values}$$

Eliminating F and S from first, third and fourth of the late for above equations, we get $\left(Mk^2+\frac{25}{64}\ Ma^2+mc^2\right)\ddot{\theta}=mgc$ ...(5)

But $$Mk^2=\frac{2}{5}\ Ma^2-M\left(\frac{3a}{8}\right)^2=\frac{83}{320}\ Ma^2.$$

Hence (5) reduces to $\left(\frac{83}{320}\ Ma^2+\frac{25}{64}\ Ma^2+mc^2\right)\ddot{\theta}=mgc$

or $\ddot{\theta}=\dfrac{20\ mgc}{13\ Ma^2+20mc^2}$. Then $F=\dfrac{5}{8}\ Ma\cdot\dfrac{20\ mgc}{13\ Ma^2+20mc^2}$

and $R=Mg+mg-mc\cdot\dfrac{20\ mgc}{13\ Ma^2+20mc^2}=\dfrac{13\ Ma^2\ (M+m)+20\ Mmc^2}{13\ Ma^2+20mc^2}g$

$\therefore\ \dfrac{F}{R}=\dfrac{25\ mac}{26\ (M+m)\ a^2+40\ mc^2}$

The hemispher will commence to roll or slide

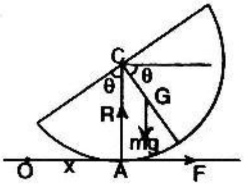
If $F<$ or $>\mu R$ *i.e.* if $\mu>$ or $<\dfrac{F}{R}$

or $\mu>$ or $<\dfrac{25\ mac}{26\ (M+m)\ a^2+40\ mc^2}$

Ex. 2. *If a uniform semi-circular wire be placed in a vertical plane with one extremity on a rough horizontal plane, and the diameter through that extremity vertical, show that the semi-circle will begin to roll or slide according as* μ *be greater or less than* $\dfrac{\pi}{\pi^2-2}$. *If* μ *has this value, prove that the wire will roll.*

Sol. Let C be the centre of the base of the semi-circular wire and G be its centre of gravity, then $CG=\dfrac{2a}{\pi}$.

Let as assume that the wire rolls. When CG is inclined at an angle θ to the horizontal, let the point of contact A have moved through a distance x from its initial position O, *i.e.* $OA=x$. Since the motion is assumed to be of pure rolling, therefore, $x=a\theta$.



$\therefore\ \dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$.

The co-ordinates of the centre of gravity G with reference to O as origin are $\left(x+\frac{2a}{\pi}\cos\theta, a-\frac{2a}{\pi}\sin\theta\right)$

Equations of motion of the wire are :

$$F=m\frac{d^2}{dt^2}\left(x+\frac{2a}{\pi}\cos\theta\right)=m\frac{d^2}{dt^2}\left(a\theta+\frac{2a}{\pi}\cos\theta\right)$$

$$=m\left(a\ddot{\theta}-\frac{2a}{\pi}\sin\theta\ddot{\theta}-\frac{2a}{\pi}\cos\theta\dot{\theta}^2\right) \qquad ...(1)$$

$$R-mg=m\frac{d^2}{dt^2}\left(a-\frac{2a}{\pi}\sin\theta\right)=m\left(-\frac{2a}{\pi}\cos\theta\ddot{\theta}+\frac{2a}{\pi}\sin\theta\dot{\theta}^2\right) \qquad ...(2)$$

and $R\frac{2a}{\pi}\cos\theta-F\left(a-\frac{2a}{\pi}\sin\theta\right)=mk^2\ddot{\theta}$ ...(3)

Since we want only initial motion, when $\theta=0;\dot{\theta}=0$, but $\ddot{\theta}\neq 0$. The equations (1), (2) and (3) give us

$F=ma\ddot{\theta}$; $R=mg-m\frac{2a}{\pi}\ddot{\theta}$; $R\frac{2a}{\pi}-Fa=mk^2\ddot{\theta}$. **For the initial values**

On eliminating F and R between these equations, we get

$$\left(k^2+\frac{4a^2}{\pi^2}+a^2\right)\ddot{\theta}=\frac{2a}{\pi}g \qquad ...(4)$$

But $mk^2=ma^2-m\left(\frac{2a}{\pi}\right)^2$ or $k^2=a^2-\frac{4a^2}{\pi^2}$

Thus (4) gives us. $\left(a^2-\frac{4a^2}{\pi^2}+\frac{4a^2}{\pi^2}+a^2\right)\ddot{\theta}=\frac{2a}{\pi}g$ or $\ddot{\theta}=\frac{g}{\pi a}$

Then $F=ma\ddot{\theta}=ma\cdot\frac{g}{\pi a}=\frac{mg}{\pi}$

$$R=mg-m\frac{2a}{\pi}\ddot{\theta}=mg-m\frac{2a}{\pi}\cdot\frac{g}{\pi a}=mg\left(1-\frac{2}{\pi^2}\right)=mg\frac{\pi^2-2}{\pi^2}$$

$\therefore \frac{F}{R}=\frac{mg}{\pi}\cdot\frac{\pi^2}{mg(\pi^2-2)}=\frac{\pi}{\pi^2-2}$

Hence the wire will roll or slide according as

$F <$ or $> \mu R$ or $\mu >$ or $< \frac{F}{R}$ or $\mu >$ or $< \frac{\pi}{\pi^2-2}$

If μ has this value then the wire will commence to roll

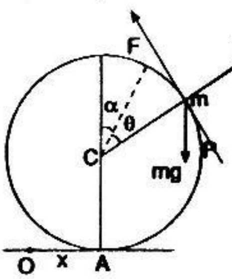
if $k^2>\frac{a^2}{3}$ *i.e.* if $a^2-\frac{4a^2}{\pi^2}>\frac{a^2}{3}$ *i.e.* if $\frac{2a^2}{3}>\frac{4a^2}{\pi^3}$

i.e. if $\pi^2 > 6$, which is true.

Hence for $\mu = \frac{\pi}{\pi^2 - 2}$, the wire rolls

Ex. 3. *A heavy uniform sphere, of mass M, is resting on a perfectly rough horizontal plane, and a particle, of mass m, is gently placed on it at an angular distance α from its highest point, Show that the particle will at once slip on the sphere if* $\mu < \frac{\sin\alpha\,\{7M + 5m(1+\cos\alpha)\}}{7M\cos\alpha + 5m(1+\cos\alpha)^2}$, *where μ is the coefficient of friction between the sphere and the particle.*

Sol. Let C be the centre of the sphere. The horizontal plane is perfectly rough. So if the sphere rolls on the plane, the particle of mass m remains at rest placed at point P, such that CP is inclined at an angle $(\alpha+\theta)$ to the vertical. Let the distance of the point of contact A be x from the initial position O i.e. $OA = x$. Since the sphere rolls, $x = a\theta$ and the point of contact is at rest, hence $\dot{x} = a\dot{\theta}$.



Let R and F be the reaction and friction at the point P.

With point O as the origin and the horizontal and vertical lines through O as co-ordinate axes, the co-ordinates of point P are given by

$$x = a\theta + a\sin(\alpha+\theta),\ y = a + a\cos(\alpha+\theta)$$

$$\therefore\ \ddot{x} = a\ddot{\theta} + a\cos(\alpha+\theta)\,\ddot{\theta} - a\sin(\alpha+\theta)\,\dot{\theta}^2$$

$$\ddot{y} = -a\sin(\alpha+\theta)\,\ddot{\theta} - a\cos(\alpha+\theta)\dot{\theta}^2$$

Equations of motion of the particle m are

$$mg\sin(\alpha+\theta) - F = \ddot{x}\,m\cos(\alpha+\theta) - m\ddot{y}\sin(\alpha+\theta)$$

$$= m\{a + a\cos(\alpha+\theta)\}\,\ddot{\theta} \qquad ...(1)$$

$$R - mg\cos(\alpha+\theta) = m\ddot{x}\sin(\alpha+\theta) + m\ddot{y}\cos(\alpha+\theta)$$

$$= ma\sin(\alpha+\theta)\,\ddot{\theta} - ma\ddot{\theta} - ma\dot{\theta}^2 \qquad ...(2)$$

The energy equation gives

$$\frac{1}{2}\left[M\frac{2a^2}{5}\dot{\theta}^2 + Ma^2\dot{\theta}^2 + m(\dot{x}^2 + \dot{y}^2)\right] = \text{work done by gravity}$$

$$= mga\{\cos\alpha - \cos(\alpha+\theta)\}$$

i.e. $\frac{7}{10}Ma^2\dot{\theta}^2 + ma^2\{1+\cos(\alpha+\theta)\}\,\dot{\theta}^2 = mga\{\cos\alpha - \cos(\alpha+\theta)\}$

of $[7Ma^2 + 10ma^2\{1+\cos(\alpha+\theta)\}]\,\dot{\theta}^2 = 10mga\{\cos\alpha - \cos(\alpha+\theta)\}$

Differentiating w.r.t. to 't' and dividing by $2\dot{\theta}$, we have

$$[7Ma^2 + 10ma^2 \{1 + \cos(\alpha + \theta)\}]\ddot{\theta} - 5ma^2 \sin(\alpha + \theta)\dot{\theta}^2 = 5mga \sin(\alpha + \theta) \quad ...(3)$$

As we only want initial motion, when $\theta = 0$, $\dot{\theta} = 0$ but $\ddot{\theta} \neq 0$ equations (1), (2) and (3) reduce to

$$\left.\begin{aligned} F &= mg \sin\alpha - ma(1 + \cos\alpha)\ddot{\theta} \\ R &= mg \cos\alpha + ma \sin\alpha\ddot{\theta} \\ [7Ma^2 + 10ma^2(1 + \cos\alpha)]\ddot{\theta} &= 5mga \sin\alpha \end{aligned}\right] \text{These equations give the}$$

initial values of F, R and $\ddot{\theta}$.

On solving these equations, we get

$$F = mg \sin\alpha - ma(1 + \cos\alpha)\frac{5mga \sin\alpha}{7Ma^2 + 10ma^2(1 + \cos\alpha)}$$

$$= g \sin\alpha \frac{7M + 5m(1 + \cos\alpha)}{7M + 10m(1 + \cos\alpha)}$$

and $R = mg \cos\alpha + ma \sin\alpha \dfrac{5mga \sin\alpha}{7Ma^2 + 10ma^2(1 + \cos\alpha)}$

$$= g\frac{7M\cos\alpha + 5m(1 + \cos\alpha)^2}{7M + 10m(1 + \cos\alpha)}$$

$$\therefore \frac{F}{R} = \sin\alpha\left[\frac{7M + 5m(1 + \cos\alpha)}{7M\cos\alpha + 5m(1 + \cos\alpha)^2}\right]$$

The particle will slip on the sphere if $F > \mu R$ or if $\mu < \dfrac{F}{R}$

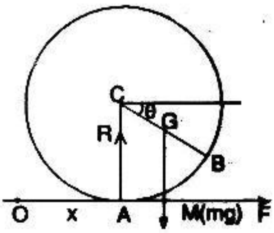
i.e. if $\mu < \dfrac{\sin\alpha\{7M + 5m(1 + \cos\alpha)\}}{7M\cos\alpha + 5m(1 + \cos\alpha)^2}$.

Ex. 4. *A homogeneous sphere, of mass M, is placed on an imperfectly rough table, and a particle, of mass m, is attached to the end of a horizontal diameter, Show that the sphere will begin to roll or slide according as μ is greater or less than $\dfrac{5(M+m)m}{7M^2 + 17Mm + 5m^2}$. If μ be equal to this value, show that the sphere will begin to roll if $5m^2 < M^2 + 11Mm$.*

Sol. Let the radius of the sphere be a and mass M. B is the point at which a particle of mass m is attached .Let in time t the sphere have turned through an angle θ and the point of contact have moved through a distance x from its initial position O. *i.e.* $OA = x$.

Let G be the common centre of gravity of two masses, such that $CG = c$, then $Mc = m(a - b) \Rightarrow c(M + m) = ma$

i.e. $c = \dfrac{ma}{M+m}$ $\quad\therefore\ BG = a - c = \dfrac{aM}{M+m}$.



Assume that the sphere rolls and F be the force of friction sufficient for pure rolling. Since the motion is of pure solling.

$\therefore\ x = a\theta$; and $\dot{x} = a\dot{\theta}$. Also

$$(M+m)\ k^2 = M \cdot \frac{2a^2}{5} + Mc^2 + m\,(a-c)^2$$

$$= 2\ \frac{Ma^2}{5} + \frac{Mm^2a^2}{(M+m)^2} + \frac{ma^2M^2}{(M+m)^2} = \frac{2Ma^2}{5} + \frac{Mma^2}{(M+m)^2}$$

$$\therefore\ k^2 = \frac{Ma^2\,(2M+7m)}{5\,(M+m)^2} \qquad ...(1)$$

Referred to O as origin, the coordinates of C.G, are $(x + c\ \cos\ \theta\ ,\ a - c\ \sin\ \theta)$.

The equations of motions are

$$F\,(M+m)\ \frac{d^2}{dt^2}\ (x + c\ \cos\theta) = (M+m)\ \frac{d^2}{dt^2}\ (a\theta + c\ \cos\theta) \qquad ...(2)$$

$$= (M+m)\ [(a - c\ \sin\ \theta)\,\ddot{\theta}\ - c\ \cos\ \theta\,\dot{\theta}^2], \qquad ...(3)$$

and $\quad R - (M+m)\,g = (M+m)\ \dfrac{d^2}{dt^2}\ [(a - c\ \sin\ \theta)]$

$$= (M+m)\ [-c\ \cos\ \theta\ddot{\theta}\ + c\sin\ \theta\dot{\theta}^2] \qquad ...(4)$$

and $\quad Rc\ \cos\ \theta - F\,(a - c\ \sin\ \theta) = (M+m)\,k^2\,\ddot{\theta} \qquad ...(5)$

As we discuss only the initial motion when $\theta = 0$, and $\dot{\theta}$ is zero but $\ddot{\theta} \neq 0$, equations (3), (4) (5); become

$$\left.\begin{aligned} &F = (M+m)\ a\ddot{\theta} \\ &R = (M+m)\ g - (M+m)\ c\,\ddot{\theta}\ , \\ &Rc - Fa = Mm\,k^2\ddot{\theta}\ , \end{aligned}\right]$$ **For the initial values of F, R and $\ddot{\theta}$**

solving these equations, we get $\quad \ddot{\theta} = \dfrac{gc}{k^2 + a^2 + c^2}$

putting for in above equations, we have

$$\frac{R}{(M+m)} = \frac{k^2 + a^2}{k^2 + a^2 + c^2}\ g\ ;\ \frac{F}{(M+m)} = \frac{gca}{k^2 + c^2 + a^2}$$

The sphere will commence to slide or roll according as $F >$ or $< \mu R$

i.e. if $\quad \dfrac{gca}{k^2 + a^2 + c^2} > \quad$ or $\ < \mu\ \dfrac{k^2 + a^2}{k^2 + c^2 + a^2} - g$

i.e. if $\mu <$ or $> \dfrac{ac}{(k^2 + a^2)}$

i.e. if $\mu <$ or $< \dfrac{5m\,(M+m)}{7M^2 + 17mM + 5m^2}$ (putting the value of c)

Critical Case. Suppose $\mu = \dfrac{5m\,(M+m)}{7M^2 + 17mM + 5m^2}$

We have proved $k^2 = \dfrac{a^2 M\,(7m + 2M)}{5\,(M+m)^2}$ in (2) and the sphere will roll if $k^2 > (a^2/3)$ proved in **3.10**

i.e. if $\dfrac{a^2 M\,(7m+2M)}{5\,(M+m)^2} > (a^2/3)$ or $3M\,(7m + 2M) > 5\,(M+m)^2$

or $21\,Mm + 6M^2 > 5\,(M^2 + 2Mm + m^2)$ *i.e.* $5m^2 < M^2 + 11Mm.$

3·11. One of the bodies fixed .

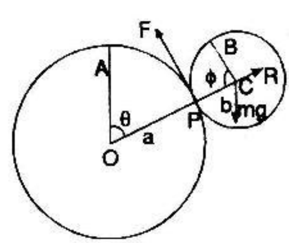
A solid homogeneous sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down it. Show that it will slip when the common normal makes with the vertical an angle θ *given by the equation* $2\sin(\theta - \lambda) = 5\sin\lambda\,(3\cos\theta - 2)$ *where* λ *is the angle of friction .*
Also prove that the upper sphere will leave when $\theta = \cos^{-1}(10/17)$.

(Garhwal 1988; Agra 1988)

Sol. Let O be the centre of the fixed sphere whose highest point is A. Let CB be the position at any point t, of the radius of the upper sphere (moving sphere) which was originally vertical.
So if P is the point of contact, then arc AP = arc BP

i.e. $a\theta = b\phi$, then $a\dot{\theta} = b\dot{\phi}$ ...(1)

where a and b are the radii of the lower and upper sphere respectively, θ and ϕ are the angle which the common normal OC makes with the vertical and CB, a line fixed in the moving sphere respectively.



Let R and F be the normal reaction and the friction acting on the upper sphere.
Since C describe a circle of radius $(a+b)$ about O, its acceleration are $(a+b)\,\dot{\theta}^2$ and $(a+b)\,\ddot{\theta}$ along and perpendicular to CO.

Hence $m\,(a+b)\,\dot{\theta}^2 = mg\cos\theta - R$, ...(2)

and $m\,(a+b)\,\ddot{\theta} = mg\sin\theta - F$ ...(3)

Referred the O as the origin, the coordinates of the centre C are $\{(a+b)\sin\theta, (a+b)\cos\theta\}$.

This energy equation gives us

$$\frac{1}{2} m(\dot{x}^2+\dot{y}^2)+\frac{1}{2} mk^2(\dot{\phi}+\dot{\theta})^2 = \text{work done by gravity}$$
$$= mg(a+b)(1-\cos\theta)$$

or $\quad (a+b)^2\dot{\theta}^2+\frac{2b^2}{5}\left(\frac{a+b}{b}\right)\dot{\theta}^2 = 2g(a+b)(1-\cos\theta)\left(\because \dot{\phi}=\frac{a}{b}\dot{\theta}\right)$

or $\quad \frac{7}{5}\dot{\theta}^2=\frac{2g}{a+b}(1-\cos\theta)$ $\quad$ or $\quad \dot{\theta}^2=\frac{10g}{7(a+b)}(1-\cos\theta)$ ...(4)

Differentiating w.r.t. to 't' and dividing by $2\dot{\theta}$, we get

$$\ddot{\theta}=\frac{5g}{7(a+b)}\sin\theta \quad ...(5)$$

From (2) and (4), we get $\quad R=mg\cos\theta-m\frac{10g}{7}(1-\cos\theta)$

$$=\frac{mg}{7}(17\cos\theta-10) \quad ...(6)$$

From (5) and (3), we get $\quad F=mg\sin\theta-\frac{5}{7}mg\sin\theta=\frac{2}{7}mg\sin\theta.$

Hence the sphere will slip if $\quad F=\mu R$

i.e. $\quad$ if $\frac{2}{7}mg\sin\theta=\tan\lambda.\frac{(17\cos\theta-10)mg}{7}$

or $\quad$ if $\quad 2\sin\theta\cos\lambda=(17\cos\theta-10)\sin\lambda.$

or $\quad$ if $\quad 2(\sin\theta\cos\lambda-\cos\theta\sin\lambda)=5(3\cos\theta-2)\sin\lambda$

or $\quad$ if $\quad 2\sin(\theta-\lambda)=5\sin\lambda(3\cos\theta-2).$

The upper sphere will leave the lower one when $R=0$, hence from (6) $(17\cos\theta-10)=0$ *i.e.* $\theta=\cos^{-1}\left(\frac{10}{17}\right)$.

When both the spheres are smooth. In this case $F=0$, so the energy equation becomes $\frac{1}{2}m(a+b)^2\dot{\theta}^2=mg(a+b)(1-\cos\theta)$

i.e. $\quad \dot{\theta}^2=\frac{2g}{(a+b)}(1-\cos\theta)$

Further equation (2) remains unchanged,

$\Rightarrow R=mg\cos\theta-2mg(1-\cos\theta)=mg(3\cos\theta-2).$

The upper sphere will leave the lower if $R=0$

i.e. $\quad$ if $mg(3\cos\theta-2)=0$ $\quad$ or $\quad \theta=\cos^{-1}\left(\frac{2}{3}\right)$.

Ex. 1. *A solid sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down. If the plane through their axes makes an angle α with the vertical when first cylinder is at rest, show that it will slip when the common normal makes with the vertical an angle given by*

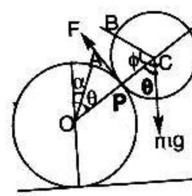
$k^2 \sin\theta = \mu\,[(k^2 + 3b^2)\cos\theta - 2b^2\cos\alpha]$, where b is radius of the moving sphere and k is the radius of gyration.
The upper sphere will leave the fixed sphere
if $\theta = \cos^{-1}\left(\dfrac{2b^2\cos\alpha}{k^2 + 3b^2}\right)$.

Sol. Let CB a radius fixed in the moving sphere makes an angle ϕ with the vertical, initially B coincided with A. Let R and F be the reaction and friction respectively.

Since there is no slipping between the two spheres, therefore

arc AP = arc BP, *i.e.* $a(\theta - \alpha) = b(\phi - \theta)$

or $a\dot{\theta} = b\dot{\phi} - b\dot{\theta}$ or $b\dot{\phi} = (a+b)\dot{\theta}$ ...(1)



Referring to O as the origin and the horizontal and vertical lines through O as coordinate axes, the coordinates of C are

$x = (a+b)\sin\theta$ and $y = (a+b)\cos\theta$ ...(2)

The energy equation gives

$$\tfrac{1}{2} m\,[k^2\dot{\phi}^2 + (\dot{x}^2 + \dot{y}^2)] = mg\,(a+b)(\cos\alpha - \cos\theta) \quad ...(3)$$

or $$\tfrac{1}{2}\,[k^2\dot{\phi}^2 + (a+b)^2\dot{\theta}^2] = (a+b)(\cos\alpha - \cos\theta)$$

or $$\tfrac{1}{2}\left[\frac{k^2}{b^2}(a+b)^2\,\dot{\theta}^2 + (a+b)^2\dot{\theta}^2\right] = g\,(a+b)(\cos\alpha - \cos\theta)$$

or $$\dot{\theta}^2 = \frac{-2b^2 g}{(k^2+b^2)(a+b)}(\cos\alpha - \cos\theta) \quad ...(4)$$

Differentiating (3) and dividing by $2\dot{\theta}$, we get $\ddot{\theta} = \dfrac{gb^2\sin\theta}{(a+b)(k^2+b^2)}$ ...(5)

As C describes a circle of radius $(a+b)$ about O, its acceleration are $(a+b)\,\dot{\theta}^2$ and $(a+b)\,\ddot{\theta}$ along and perpendicular to O. Therefore the equations of motion of the sphere are

$$m\,(a+b)\,\dot{\theta}^2 = mg\cos\theta - R\,, \quad ...(6)$$

and $m\,(a+b)\,\ddot{\theta} = mg\sin\theta - F$ ...(7)

Hence from (6) and (4), we have

$$R = \left[mg\cos\theta - \frac{2b^2 mg}{(b^2+k^2)}(\cos\alpha - \cos\theta)\right]$$

$$=\frac{mg}{k^2+b^2}\,[(k^2+3b^2)\cos\theta-2b^2\cos\alpha].$$

and from (7) and (5), we have

$$F=mg\sin\theta-\frac{mgb^2}{b^2+k^2}\sin\theta=\frac{mgk^2\sin\theta}{(k^2+b^2)}$$

$$\therefore\quad \frac{F}{R}=\frac{k^2\sin\theta}{(k^2+3b^2)\cos\theta-2b^2\cos\alpha}$$

The sphere will slip when $F=\mu R$ *i.e.*

if $k^2\sin\theta=\mu\,[(k^2+3b^2)\cos\theta-2b^2\cos\alpha]$.

The upper sphere will leave the fixed sphere if $R=0$ *i.e.* if

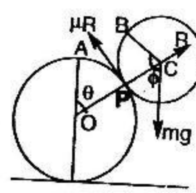
$(k^2+3b^2)\cos\theta=2b^2\cos\alpha$ *i.e.* $\theta=\cos^{-1}\left(\frac{2b^2\cos\alpha}{k^2+3b^2}\right)$

Ex. 2. *A homogenous sphere rolls down on imperfectly rough fixed sphere, starting from rest at the highest point. If the spheres separate when the line joining their centres makes an angle θ with the vertical, prove that $\cos\theta+2\mu\sin\theta=Ae^{2\mu\theta}$ where A is the function of μ only.*

Sol. As the fixed sphere is imperfectly rough so the moving sphere rolls as well as slide on it thus friction μR acts upwards. Let a be radius of moving sphere .

Equation of motion are

$$mv\frac{dv}{ds}=mg\sin\theta-\mu R \qquad ...(1)$$



and $\frac{mv^2}{a}=mg\cos\theta-R \qquad ...(2)$

Eliminating R from (1) and (2), we get

$$\tfrac{1}{2}\frac{dv^2}{ds}-\mu\frac{v^2}{a}=g(\sin\theta-\mu\cos\theta)$$

or $\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}-2\mu\frac{v^2}{a}=2g(\sin\theta-\mu\cos\theta)$

or $\frac{dv^2}{d\theta}-2\mu v^2=2ag(\sin\theta-\mu\cos\theta)\quad\left(\because s=a\theta\Rightarrow\frac{ds}{d\theta}=a\right)$

Above is a linear differential equation, its solution is

$$v^2e^{-2\mu\theta}=C+2ag\int e^{-2\mu\theta}(\sin\theta-\cos\theta)\,d\theta$$

$$=C+\frac{2age^{-2\mu\theta}}{1+4\mu^2}\,[(-2\mu\sin\theta-\cos\theta)-\mu(-2\mu\cos\theta+\sin\theta)]$$

or $v^2e^{-2\mu\theta}=C+\frac{2ag}{1+4\mu^2}\,e^{-2\mu\theta}\,[-3\mu\sin\theta-(1-2u^2)\cos\theta]$.

Again when $\theta = 0$, $v = 0$ $\therefore$ $C = \dfrac{2ag}{1+4\mu^2}(1-2\mu^2)$

Therefore $v^2 e^{-2\mu\theta} = \dfrac{2ag}{1+4\mu^2} e^{-2\mu\theta}[-3\mu \sin\theta - (1-2u^2)\cos\theta]$

$$+\frac{2ga}{1+4\mu^2}(1-2\mu^2)$$

or $v^2 = \dfrac{2ag}{1+4\mu^2}[-3\mu \sin\theta - (1-2u^2)\cos\theta] + \dfrac{2ga}{1+4\mu^2}(1-2\mu^2)e^{2\mu\theta}$

The sphere separtes where $R = 0$, thus from (2), we have $v^2 = ag\cos\theta$

or $\dfrac{2ag}{1+4\mu^2}[-3\mu \sin\theta - (1-2u^2)\cos\theta]$

$$+\frac{2ga}{1+4\mu^2}(1-2\mu^2)e^{2\mu\theta} = ag\cos\theta$$

or $\quad 2[-3\mu\sin\theta - (1-2\mu^2)\cos\theta] + 2(1-2\mu^2)e^{2\mu\theta} = (1+4\mu^2)\cos\theta$

or $\quad 6\mu\sin\theta + 3\cos\theta = 4(1-2\mu^2)e^{2\mu\theta}$

or $\quad \cos\theta + 2\mu\sin\theta = \frac{4}{3}(1-2\mu^2)e^{2\mu\theta}$

or $\quad \cos\theta + 2\mu\sin\theta = Ae^{2\mu\theta}$, where $A = \frac{4}{3}(1-2\mu^2)$

Ex. 3. *A rough solid circular cylinder rolls down a second rough cylinder which is fixed with its axis horizontal. If the plane through their axes make an angle* α *with the vertical when first cylinder is at rest, show that the bodies will separate when this angle of friction is* $\cos^{-1}\left(\dfrac{4\cos\alpha}{7}\right)$

Sol. Refer figure of Ex. 1, page 169.

Let CB a radius fixed in the moving cylinder make an angle ϕ with the vertical, initially B coincided with A. Let c and b be the radii of fixed and moving cylinder respectively. As there is no slipping between the two cylinders, therefore arc $AP =$ arc BP *i.e.* $a(\theta - \alpha) = b(\phi - \theta)$

$\therefore a\dot{\theta} = b(\dot{\phi} - \dot{\theta})$ or $b\dot{\phi} = (a+b)\dot{\theta}$.

Referring to O as the origin and horizontal and vertical lines through O as coordinates axes the coordinates of C are $\{(a+b)\sin\theta, (a+b)\cos\theta\}$.

The energy equation gives

$\frac{1}{2}m[k^2\dot{\phi}^2 + (\dot{x}^2 + \dot{y}^2)] = mg(a+b)(\cos\alpha - \cos\theta)$ where m is the mass of the rolling cylinder and k, the radius of gyration.

or $\quad \frac{1}{2}m\left[\dfrac{b^2}{2}\dot{\phi}^2 + (a+b)\dot{\theta}^2\right] = mg(a+b)(\cos\alpha - \cos\theta)$

or $\quad \frac{1}{2}(a+b)^2\dot{\theta}^2 + (a+b)^2\dot{\theta}^2 = 2g(a+b)(\cos\alpha - \cos\theta)$

$$\{\because b\dot{\phi} = (a+b)\dot{\theta}\}$$

or $\quad (a+b)\,\dot{\theta}^2 = \dfrac{4g}{3}(\cos\alpha - \cos\theta)$ ...(1)

The centre C describes the circle of radius $(a+b)$ about O

$\therefore\; m\,(a+b)\,\dot{\theta}^2 = mg\cos\theta - R$ ...(2)

From (1) and (2),

$$R = mg\cos\theta - \frac{4mg}{3}(\cos\alpha - \cos\theta) = \frac{mg}{3}(7\cos\theta - 4\cos\alpha)$$

The bodies will saprate when $R = 0$

i.e. when $\quad 7\cos\theta - 4\cos\alpha = 0 \quad$ or $\quad \cos\theta = \frac{4}{7}\cos\alpha$

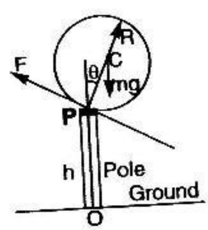
or $$\theta = \cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$$

Ex. 4. *A uniform sphere of radius a is gently placed on the top of a thin vertical pole of height $h\,(>a)$ and then allowed to fall over. Show that however rough the pole may be the sphere will slip on the pole before it finally falls off it.* **(Meerut 81, Agra 84)**

Sol. Let OP be a fixed vertical pole of height h and a sphere is gently placed at top P and then displaced. Let us assume that friction is sufficient to keep the point of contact at rest, so the sphere turns about P without slipping.

Let at any time t the angle turned by the sphere be θ and F be the force sufficient to keep the point of contact at rest.

Equations of motion of C.G. of the sphere are



$$ma\ddot{\theta} = mg\sin\theta - F \qquad ...(1)$$

and $$ma\dot{\theta}^2 = mg\cos\theta - R \qquad ...(2)$$

Energy equation gives $\quad \dfrac{1}{2}m\left(\dfrac{2a^2}{5}\dot{\theta}^2 + a^2\dot{\theta}^2\right) = mg\,(a - a\cos\theta)$

or $\quad a\dot{\theta}^2 = \frac{10}{7}g\,(1-\cos\theta)$ ...(3)

Differentiating (3) we get $\quad a\ddot{\theta} = \frac{5}{7}g\sin\theta$ ...(4)

From (1) and (4), we have $F = mg\sin\theta - ma\ddot{\theta}$

or $\quad F = mg\sin\theta - \frac{5}{7}mg\sin\theta = \frac{2}{7}mg\sin\theta.$

From (2) and (3), we have $\quad R = mg\cos\theta - ma\dot{\theta}^2$

or $\quad R = mg\cos\theta - \frac{10}{7}mg\,(1-\cos\theta) = \frac{1}{7}mg\,(17\cos\theta - 10)$

The sphere finally falls of when $R = 0$
i.e. when $17 \cos \theta - 10 = 0$ or $\cos \theta = \frac{10}{17}$.

Also then sphere will slip when $F \geq \mu R$ or $\mu \leq \dfrac{F}{R}$

or $\quad \mu \leq \dfrac{2 \sin \theta}{17 \cos \theta - 10}$ we observe that if μ is not negative, then

$\mu = 0$ when $\theta = 0$ (*i.e.* when motion just begins)

and $\mu = \infty$ when $\cos \theta = \frac{10}{17}$ (*i.e.* when particle falls off)

Thus sphere will slip between $\theta = 0$ and $\theta = \cos^{-1} \frac{10}{17}$ if μ lies between 0 and ∞.

Thus we observe that however rough the pole may be, the sphere will slip on the pole before it finally falls over.

Ex. 5. *A uniform beam of mass M and length l stands upright on perfectly rough ground ; on the top of it which is flat rests a weight of mass m, the coefficient of friction between the beam and the weight being μ. If the beam is allowed to fall to the ground , its inclination θ to the vertical when the weight slips is given by* $(\frac{4}{3} M + 3m) \cos \theta - (M/6\mu) \sin \theta = M + 2m$

(Rachi 83)

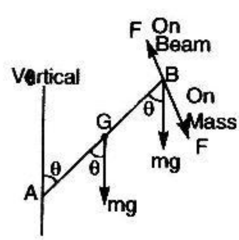
Sol. Let at any time t, the rod AB make an angle θ with the vertical with m resting on the top, B.

Now, taking moments about A for the beam, we get $M \cdot \frac{1}{3} l^2 \ddot{\theta} = M \; \frac{1}{2} l \sin \theta - F \cdot l$

Further equations of motion for mass m are

$$ml\ddot{\theta} = mg \sin \theta + F \quad ...(2)$$

$$M\, l\, \dot{\theta}^2 = mg \cos \theta - R \quad ...(3)$$



Whence eliminating F between (1) and (2), we obtain $(M + 3m)\, l\, \ddot{\theta} = \frac{3}{2} (M + 2m)\, g \sin \theta$ ...(4)

Again Multiplying both sides by $2\dot{\theta}$ and integrating, we get $(M + 3m)\, l\, \dot{\theta}^2 = -3\,(M + 2m)\, g \cos \theta + c$.

when $\theta = 0$, $\dot{\theta} = 0 \Rightarrow c = 3g\,(M + 2m)$

$$\Rightarrow (M + 3m)\, l\, \dot{\theta}^2 = 3g\,(M + 2m)\,(1 - \cos \theta) \quad ...(5)$$

$\Rightarrow F = ml\ddot{\theta} - mg \sin \theta \quad$ [using (2)]

$$= \frac{3m\,(M + 2m)\, g \sin \theta}{2\,(M + 3m)} - mg \sin \theta \text{ by (4)}$$

$$\Rightarrow F = \frac{m\,MG \sin\theta}{2\,(M+3m)} \qquad \text{...(6)}$$

Further $R = mg \cos\theta - l\,\dot{\theta}^2$, using (3)

$$= mg\cos\theta - \frac{3mg\,(M+2m)\,(1-\cos\theta)}{M+3m} \qquad \text{by (5).}$$

$$\Rightarrow R = \frac{mg\,(4M+9m)\cos\theta - 3\,(M+2m)}{M+3m} \qquad \text{...(7)}$$

$$\Rightarrow \frac{F}{R} = \frac{1}{2}\,\frac{M\sin\theta}{(4M+9m)\cos\theta - 3\,(M+2m)}$$

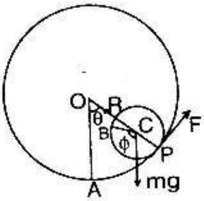
But $F = \mu R \Rightarrow \mu = F/R$ when the weight slips.

$$\Rightarrow \mu = \frac{1}{2}\,\frac{M\sin\theta}{(4M+9m)\cos\theta - 3\,(M+2m)}.$$

$$\Rightarrow (\tfrac{4}{3}M + 3m)\cos\theta - \left(\frac{M}{6\mu}\right)\sin\theta = M + 2m.$$

3·12. *A hollow cylinder, of radius a is fixed with its axis horizontal, in side it moves a solid cylinder, of radius b, whose velocity in its lowest position is given ; if the friction between the cylinders be sufficient to prevent any sliding, find the motion.* **(Patna 1962)**

Let C be the centre of the moving cylinder and let ϕ be the angle which a line CB fixed in the moving cylinder makes with the vertical, a line fixed in space. Initially B coincided with A. Let a be the radius of the fixed cylinder whose centre is O and b that of the movable cylinder.



Since there is no slipping between the two cylinders therefore arc AP = arc BP

or $a\theta = b\,(\phi + \theta)$

or $b\phi = (a-b)\,\theta$; $\therefore\ b\dot{\phi} = (a-b)\,\dot{\theta}$. ...(1)

Let R and F be the normal reaction and friction at P. Since C describes a circle of radius $(a-b)$ about O, the equations of motion of the cylinder are $m\,(a-b)\,\dot{\theta}^2 = R - mg\cos\theta$ ...(2)

and $m\,(a-b)\,\ddot{\theta} = F - mg\sin\theta$ ...(3)

The co-ordinates of C with respect to O as origin and the vertical and horizontal lines as axes through O are $\{(a-b)\sin\theta, (a-b)\cos\theta\}$.

$$\therefore\ (\text{its velocity})^2 = (\dot{x}^2 + \dot{y}^2) = (a-b)^2\cos^2\theta\dot{\theta}^2 + (a-b)^2\sin^2\theta\dot{\theta}^2$$

$$= (a-b)^2\,\dot{\theta}^2.$$

So kinetic energy of the moving cylinder at any time 't' is

$= \frac{1}{2}\, mk^2\dot{\phi}^2 + \frac{1}{2} m\ (a-b)^2\,\dot{\theta}^2 = \frac{1}{2} m \frac{b^2}{2}\ \dot{\phi}^2 + \frac{1}{2}\ m\,(a-b)^2\,\dot{\theta}^2 \left(\because k^2 = \frac{b^2}{2}\right)$

$= \frac{1}{2}\ m \frac{(a-b)^2}{2}\ \dot{\theta}^2 = \frac{1}{2} m\,(a-b)^2\,\dot{\theta}^2 = \frac{3}{4}\ m\,(a-b)^2\,\dot{\theta}^2$

$\{\ \because\ b\dot{\phi} = (a-b)\,\dot{\theta}$ from (1)$\}$

$\therefore$ Kinetic energy at the time of projection $= \frac{3}{4}\ m\,(a-b)^2\,\Omega^2$

($\because\ \dot{\theta} = \Omega$ initiallly)

Therefore equations gives

$\frac{3}{4}\ m\ (a-b)^2\,\dot{\theta}^2 - \frac{3}{4}\ m\ (a-b)^2\,\Omega^2 = -\,mg\,(a-b)\,(1-\cos\ \theta)$

i.e. $\quad (a-b)^2\,\dot{\theta}^2 = (a-b)\,\Omega^2 - \frac{4}{3} g\ (1-\cos\ \theta).$ ...(4)

Differentiating (4) w.r.t. θ and dividing be $2\dot{\theta}$, we get

$f(a-b)\,\ddot{\theta} = -\frac{2}{3} g\ \sin\ \theta.$ ...(5)

Again from (2), $R = mg\cdot\cos\ \theta + m\,(a-b)\,\dot{\theta}^2$

$= mg\ \cos\ \theta + m\,(a-b)\,\Omega^2 - \frac{4}{3} mg\,(1-\cos\ \theta)$ from (4)

or $\ R = m\,(a-b)\,\Omega^2 + \dfrac{mg}{3}\ (7\ \cos\ \theta - 4).$ ...(6)

From (3), $F = mg\ \sin\ \theta + m\,(a-b)\,\dot{\theta} = mg\ \sin\ \theta - \frac{2}{3}\ mg\ \sin\ \theta$ from (5)

or $\qquad F = \frac{1}{3} mg\ \sin\ \theta$ ...(7)

Equation (4), (6) and (7) determine the motion .

Case 1. *In order that the cylinder may roll down completely, R should be zero at the highest point* *i.e.* $R = 0$ when $\theta = \pi$.

$\therefore\ 0 = m\,(a-b)\,\Omega^2 + \frac{1}{3} mg\ (-7-4)$ or $\Omega = \sqrt{\left[\dfrac{11g}{3\,(a-b)}\right]}$

Case 2. *The moving cylinder will leave the fixed cylinder if*

$R = 0$ *i.e.,* $m\,(a+b)\,\Omega^2 + \frac{1}{3}\ mg\ (7\ \cos\ \theta - 4) = 0$

or $\quad \cos\ \theta = \left[\frac{4}{3} g - (a-b)\,\Omega^2\right] \dfrac{3}{7g}$

or $\quad \cos\ \theta = \dfrac{1}{7g}\,[4g - 3\,(a-b)\,\Omega^2].$

This gives the position when the two bodies separate.

Case 3. *If the rolling cylinder makes small oscillations about the the lowest point of the fixed cylinder, then θ is always small, hence equation (5). gives on taking θ for* sin θ

$$\ddot{\theta} = \frac{-2g}{3(a-b)}\theta$$

Hence time of small oscillation is $\dfrac{2\pi}{\left[\dfrac{2g}{3(a-b)}\right]^{1/2}} = 2\pi\left[\dfrac{3(a-b)}{2g}\right]^{1/2}$

ILLUSTRATIVE EXAMPLES

Ex. 1. *A circular plate rolls down the inner circumference of a rough circle under the action of gravity, the planes of both the plate and the circle being vertical. When the line joining their centres is inclined at an angle* θ *to the vertical, show that the friction between the bodies is* $\frac{1}{3}\sin\theta$ *times the weight of the plate.*

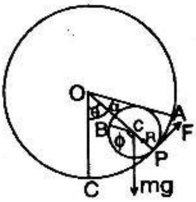
Sol. Let O be the centre of the fixed circle whose radius is a and C be the centre of circular plate that rolls down and its radius is b.

Let at any instant the radius CB (a line fixed in the body) make an angle ϕ with the vertical a line fixed in space. Initially, B coincided with A, a fixed point on fixed circle. OA is inclined at an angle α to the vertical OC.

As there is no slipping between the bodies

$\therefore$ Arc AP = are BP

[upper side in the figure]



i.e. $a(\alpha-\theta) = b\{2\pi - (\theta+\phi)\}$

or $\quad -a\dot{\theta} = -b(\dot{\theta}+\dot{\phi})$

or $\quad b\dot{\phi} = (a-b)\dot{\theta}, \quad \therefore\ b\ddot{\phi} = (a-b)\ddot{\theta} \quad \ldots(1)$

Equations of motion of the plate are

$$m(a-b)\ddot{\theta} = F - mg\sin\theta \qquad \ldots(2)$$

and $$m\frac{b^2}{2}\ddot{\phi} = -F.b \qquad \ldots(3)$$

On eliminiating $\ddot{\theta}$ and $\ddot{\phi}$ from (1), (2) and (3), we get

$-2F = F - mg\sin\theta$ or $3F = mg\sin\theta$ or $F = \frac{1}{3}mg\sin\theta$

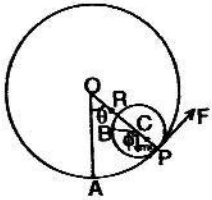
i.e. $F = \dfrac{\sin\theta}{3}(mg) = \dfrac{\sin\theta}{3}$ (times the weight).

Ex. 2. *A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through their axes moves in a circular pendulum of length*

$(b-a)\left(1+\dfrac{k^2}{a^2}\right)$ **(Vikram 1992)**

Sol. Let θ be the angle through which the plane of axes turn and let ϕ be the angle which CB a line fixed in the moving cylinder makes with the vertical.

The outer cylinder is fixed. Equations of motion of the inner cylinder are



$$m(b-a)\ddot{\theta} = F - mg \sin \theta \qquad ...(1)$$

and $$mk^2\ddot{\phi} = -Fa. \qquad ...(2)$$

Again there is no slipping .

$\therefore$ arc $AP =$ arc BP

or $b\theta = a(\theta + \phi)$ or $a\ddot{\phi} = (b-a)\ddot{\theta}$ ...(3)

Eliminating F and $\ddot{\phi}$ between (1), (2) and (3), we get

$$m(b-a)\ddot{\theta} = -\frac{mk^2}{a}\ddot{\phi} - mg \sin \theta = -\frac{mk^2}{a^2}(b-a)\ddot{\theta} - mg \sin \theta \text{or}$$

$$(b-a)\left(1+\frac{k^2}{a^2}\right)\ddot{\theta} = -g \sin \theta \text{ or } \ddot{\theta} = -\frac{g}{(a-b)\left(1+\frac{k^2}{a^2}\right)}$$

as θ is small.

Therefore length of the simple equivalent pendulum is $(b-a)\left(1+\frac{k^2}{a^2}\right)$

Ex. 3. *A disc rolls on the inside of a fixed hollow circular cylinder whose axis is horizontal, the plane of the disc being vertical and perpendicular to the axis of cylinder ; if , when in the lowest position, its centre is moving with a velocity* $\left[\frac{8g}{3(a-b)}\right]^{1/2}$, *show that the centre of the disc will describe an angle* ϕ *about the centre of the cylinder in time*

$$\left[\frac{3(a-b)}{2g}\right]^{1/2} . \log \tan \left(\frac{\pi}{4}+\frac{\phi}{4}\right).$$ **(Vikram 1984; Punjab 87)**

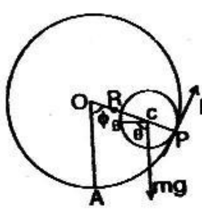
Sol. Let C be the centre of the disc and O be the centre of the fixed hollow cylinder whose radius is a. Let a line CB (fixed in the body) which was initially in a vertical position and coincided with OA makes an angle θ with the vertical.

Assume that the disc rolls, so that the

arc $AP =$ arc BP

or $a\phi = b(\theta + \phi)$ or $b\theta = (a-b)\phi$

or $b\ddot{\theta} = (a-b)\ddot{\phi}$. ...(1)



Referring to O as the origin and vertical and horizontal lines through O as axes, the coordinates of centre C are

$\{(a-b)\sin\phi, (a-b)\cos\phi\}$.

Kinetic energy of the disc at any time t is

$$= \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} mk^2\dot{\theta}^2 = \frac{1}{2} m\left[(a-b)^2\dot{\phi}^2 + \frac{b^2}{2}\dot{\theta}^2\right]$$

$$= \frac{1}{2} m\left[(a-b)^2\dot{\phi}^2 + \frac{b^2}{2}\frac{(a-b)^2}{b^2}\dot{\phi}^2\right]\left\{\because \dot{\theta} = \left(\frac{a-b}{b}\right)\dot{\phi}\right\}.$$

$$= \tfrac{3}{4} m (a-b)^2\dot{\phi}^2.$$

It follows that the initial K.E. of the disc $= \frac{3}{4} m \frac{8}{3} g(a-b) = 2mg(a-b)$

Since at $t = 0$, $\dot{\phi}^2 = \left[\frac{8g}{3(a-b)}\right]$

Therefore the energy equation gives

$$\tfrac{3}{4} m (a-b)^2\dot{\phi}^2 - 2mg(a-b) = \text{the work done by gravity}$$
$$= -mg(a-b)(1-\cos\phi)$$

or $\frac{3}{4} m (a-b)^2\dot{\phi}^2 = mg(a-b)(1+\cos\phi) = 2mg(a-b)\cos^2\frac{\phi}{2}$

or $\dot{\phi}^2 = \frac{8g}{3(a-b)}\cos\frac{\phi}{2}$ or $\frac{d\phi}{dt} = \left[\frac{8g}{3(a-b)}\right]^{1/2}\cos^2\frac{\phi}{2}$

or $\int dt = \left[\frac{3(a-b)}{8g}\right]^{1/2}\int_0^{\phi}\sec\frac{\phi}{2}\,d\phi$

or $t = \left[\frac{3(a-b)}{2g}\right]^{1/2}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$

Ex. 4. *A solid homogeneous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane. Show that the smaller sphere will make complete revolution if, when it is in its lowest position, the pressure on it is greater that $\frac{34}{7}$ times its own weight.*

Sol. Let O be the centre of the fixed hollow sphere whose radius is a and C the centre of the moving solid sphere whose radius is b. Let CP be a radius (a line fixed in the body) makes an angle ϕ with the vertical (a line fixed in space) initially B coincided with A.

Let θ be the angle that the line of centres make with the vertical at any time t.

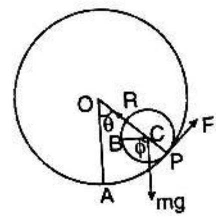
As there is no slipping between the two bodies, therefore

arc AP = arc BP or $a\theta = b(\phi+\theta)$ or $b\dot{\phi} = (a-b)\ddot{\theta}$. ...(1)

C describes a circle of radius $(a-b)$ about O.

Equation of motion of the sphere is $m(a-b)\dot{\theta}^2 = R - mg\cos\theta$, ...(2)

Taking the horizontal and vertical lines through O as coordinate axes. Coordinates of the centre C are $\{(a-b)\sin\theta, (a-b)\cos\theta\}$.



So at any time t, the $(\text{velocity})^2$ of the

centre $C = \{(a-b)\cos\theta\dot{\theta}\}^2$

$+\{-(a-b)\sin\theta\dot{\theta}\}^2 = (a-b)^2\dot{\theta}^2$

$\therefore$ At any time t, kinetic energy of the sphere

$$= \tfrac{1}{2} m \frac{2b^2}{5}\dot{\phi}^2 + \tfrac{1}{2} m(a-b)^2\dot{\theta}^2$$

$$= \tfrac{1}{2} m \cdot \tfrac{2}{5} m(a-b)^2\dot{\theta}^2 + \tfrac{1}{2} m(a-b)^2\dot{\theta}^2 \text{ from (1)}$$

$$= \frac{7m}{10}(a-b)^2\dot{\theta}^2,$$

$\therefore$ Initially K .E. of the sphere $= \dfrac{7m}{10}(a-b)^2\Omega^2$

where Ω is the initial angular velocity.

Hence energy equation gives, .

$$= \frac{7m}{10}(a-b)^2\dot{\theta}^2 - \frac{7m}{10}(a-b)^2\Omega^2 = \text{work done by the gravity}$$

$$= -mg[(a-b)-(a-b)\cos\theta]$$

i.e. $(a-b)\dot{\theta}^2 = (a-b)\Omega^2 - \dfrac{10g}{7}(1-\cos\theta).$ ...(3)

Again from (2), $R = mg\cos\theta + m(a-b)\dot{\theta}^2$

$$= mg\cos\theta + m(a-b)\Omega^2 - \frac{10mg}{7}(1-\cos\theta).$$

The sphere will make complete revolutions if $R=0$ when $\theta=\pi$

i.e. $0 = -mg + m(a-b)\Omega^2 - \dfrac{20mg}{7}$ *i.e.* $\Omega^2 = \dfrac{27g}{7(a-b)}$.

This gives least value of Ω for making complete revolution.

Again, to know the value of R in the lowest position put $\theta = 0$ and $\dot{\theta} = \Omega$ in equation (2); then

R (in lowest position) $= mg\cos 0 + m(a-b)\Omega^2$

$$= mg + \frac{27mg}{7}\left\{\because \Omega^2 = \frac{27g}{7(a-b)}\right\}$$

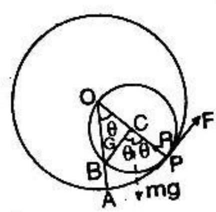
$$= \frac{34}{7}mg = \frac{34}{7} \text{ times the weight .}$$

Ex. 5. *A cylinder, of radius a, lies within a rough fixed cylindrical cavity of radius 2a. The centre of gravity of the cylinder is at a distance c from the axis, and the initial state is that of stable equilibrium at the lowest* *point of the cavity. Show that the smallest angular velocity with which the cylinder must be started that it may roll right round the cavity is given by*

$\Omega^2\ (a+c) = g\left\{1 + \dfrac{4\,(a+c)^2}{(a-c)^2 + k^2}\right\}$ *where k is the radius of gyration about the centre of gravity.*

Find also the normal reaction between the cylinders at any position

Sol. Let O be the centre of fixed cylindrical cavity whose radius is given $2a$, C the centre of the moving cylinder whose radius is given as a. At time t let CB (a line fixed in the moving body) makes an angle θ with the vertical (a line fixed in space). By geometry each of the other angles are also equal to θ as marked. Initially B coincided with A; it can be easily derived that B lies on the vertical line OA. Taking the horizontal and vertical lines through the fixed point O as co-ordinate axes, the coordinates of the centre of gravity G are



$$x = (a-c)\ \sin\ \theta, \qquad y = (a+c)\cos\ \theta, \text{ where } CG = c$$

so that $\dot{x} = (a-c)\ \cos\ \theta\dot{\theta},\ \ \dot{y} = -(a+c)\ \sin\ \theta\dot{\theta},$

$$\ddot{x} = (a-c)\ \cos\ \theta\ddot{\theta} - (a-c)\sin\ \theta\dot{\theta}^2.$$

$$\ddot{y} = -(a+c)\ \sin\ \theta\ddot{\theta} - (a+c)\cos\ \theta\dot{\theta}^2.$$

$$\therefore\ \dot{x}^2 + \dot{y}^2 = (a-c)^2\ \cos^2\ \theta\dot{\theta}^2 + (a+c)^2\sin^2\ \theta\dot{\theta}^2$$

$$= (a^2 + c^2 - 2ac\ \cos\ 2\theta)\,\dot{\theta}^2.$$

So at any time t, (velocity)^2 of $C = (a^2 + c^2 - 2ac\ \cos\ 2\theta)\,\dot{\theta}^2$.

$\therefore$ at the lowest point (velocity)^2 of $C = (a-c)^2\,\Omega^2$

(when $\theta = 0, \dot{\theta} = \Omega$)

Hence energy equation gives

$$\tfrac{1}{2}m\ [k^2 + a^2 + c^2 - 2ac\ \cos\ 2\theta]\,\dot{\theta}^2 - \tfrac{1}{2}m\,[k^2 + (a-c)^2]\,\Omega^2$$

$$= -mg\,[(a+c) - (a+c)\cos\theta]. \qquad \text{...(1)}$$

Equations of motion of the cylinder is

$$R - mg\ \cos\ \theta = -m\ddot{x}\ \ \sin\ \theta - m\ddot{y}\ \ \cos\ \theta$$

$$= -m\,(a-c)\,(\cos\ \theta\ddot{\theta} - \sin\ \theta\dot{\theta}^2)\sin\ \theta$$

$$+ m\,(a+c)\,(\sin\ \theta\ddot{\theta} + \cos\ \theta\dot{\theta}^2)\sin\ \theta$$

$$= m\,[c\ \sin\ 2\theta\ddot{\theta} + (a + c\cos\ 2\theta)\,\dot{\theta}^2].$$

or $\quad R - mg\ \cos\ \theta = m\,[c\ \sin\ 2\theta\ddot{\theta} + (a + c\cos\ 2\theta)\,\dot{\theta}^2].$ ...(2)

The cylinder will roll round the cavity if $R = 0$ when $\theta = \pi$;

then from (2), $\quad g = (a + c)\,\dot{\theta}^2$ ...(3)

and from (1), $[k^2 + (a - c)^2]\,\dot{\theta}^2 - [k^2 + (a - c)^2]\,\Omega^2 = -4g\,(a + c)$ ...(4)

Eliminating $\dot{\theta}^2$ between (3) and (4), we have

$$[k^2 + (a - c)^2]\ \frac{g}{a + c} - [k^2 + (a - c)^2]\,\Omega^2 = -4g\,(a + c).$$

or $\quad [k^2 + (a - c)^2]\,(a + c)\,\Omega^2 = g\,[k^2 + (a - c)^2 + 4\,(a + c)^2]$

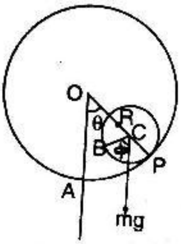
or $\quad (a + c)\,\Omega^2 = g\left[1 + \dfrac{4\,(a + c)^2}{k^2 + (a - c)^2}\right]$

which is the required result.

Ex. 6. *A solid spherical ball rests in limiting equilibrium at the bottom of a fixed spherical globe whose inner surface is perfectly 'ough. The ball is struck a horizontal blow of such a magnitude that he initial speed of its centre is v; prove that is v lies between*

$\sqrt{\left\{\left(\frac{10}{7}\ gd\right)\right\}}$ *and* $\sqrt{\left\{\left(\frac{27}{7}\ gd\right)\right\}}$ *the ball would leave the globe, d being the difference of the radii of the ball and the globe.*

Sol. Refer Ex. 4. and with the same figure, we



have $a\theta = b\,(\theta + \phi)$ or $b\,\dot{\phi} = (a - b)\,\dot{\theta} = d\,\dot{\theta}$
where $d = a - b.$ (1)

Initial velocity of the centre is given by

$\therefore\ d^2\,\dot{\theta}^2 = v^2$; **for initial value.** ...(2)

At any time t, the K.E. of the ball is given by

$$T_1 = \frac{1}{2}\ m\left(\frac{2b^2}{5}\ \dot{\phi}^2 + d^2\,\dot{\theta}^2\right)$$
$$= \frac{1}{2}\ m\left(\frac{2}{5}\ d^2\,\dot{\theta}^2 + d^2\dot{\theta}^2\right) = \frac{7m}{10}\ d^2\,\dot{\theta}^2$$

$\therefore$ At the time of projection, K.E. of the ball will be $= \dfrac{1}{2}\ m.\ \dfrac{7}{5}\,v^2$ using (2)

Again, energy equation gives $\dfrac{1}{2}\,m\,\dfrac{7}{5}\ d^2\,\dot{\theta}^2 - \dfrac{1}{2}\ m\,\dfrac{7v^2}{5} = -\,mg\,(d - d\cos\theta)$

$$\Rightarrow d\,\dot{\theta}^2 = -\,\frac{10}{7}\ g\,(1 - \cos\theta) + \frac{v^2}{d}.$$

Again centre C describes a circle of [radius a about O, so we obtain

$md\ \dot{\theta}^2 = R - mg\cos\theta.$ ...(4)

Eliminating $d\,\dot{\theta}^2$ between (3) and (4), we readily get

$$R = mg\cos\theta - \frac{10}{7}\,mg\,(1-\cos\theta) + m\,\frac{v^2}{d}$$

$$\Rightarrow \quad R = \frac{1}{7}\,mg\left[17\cos\theta - \left(10 - \frac{7v^2}{gd}\right)\right]$$

Now, the ball would leave the globe when $R = 0$

$$\Rightarrow 17\cos\theta - \left(10 - \frac{7v^2}{gd}\right) = 0$$

$$\Rightarrow \cos\theta = \frac{10\,gd - 7v^2}{27\,gd} = -\frac{7v^2 - 10\,gd}{17\,gd}. \qquad ...(4)$$

But $\cos\theta$ is to be numerically less than 1 $\therefore\ 7v^2 - 10\,gd < 17\,gd.$

$$\Rightarrow v < \left(\left(\frac{27}{7}\,gd\right)\right)^{1/2}$$

Again when θ is obtuse, we have $\cos\theta = -\,ive$

i.e. $7v^2 - 10 \nprec$ positive i.e. $v -> \surd(10\,gd/17)$

i.e. $\left(\left(\frac{10\,gd}{7}\right)\right)^{1/2} < v < \left(\left(\frac{27\,gd}{7}\right)\right)^{1/2}$

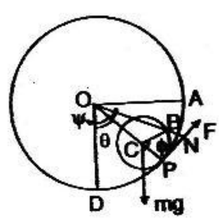
3·13. Motion of one body on another ; when the lower body is free to turn about its axis.

Ex. 1. *A thin hollow cylinder of radius a and mass M is free to turn about its axis which is horizontal and a smaller cylinder of radius b and mass m rolls inside it without slipping, the axes of the two cylinders being parallel. Show that when the plane of the two axes is inclined at an angle θ to the vertical, angular velocity of the large cylinder is given by*

$$a^2\,(M+m)\,(2M+m)\,\Omega^2 = 2gm^2\,(a-b)\,(\cos\ \theta - \cos\ \alpha)$$

provided both the cylinders are at rest when $\theta = \alpha$.

Sol. Let O be the centre of the outer cylinder and C the centre of the inner cylinder. The figure is the vertical section of the system through O and C.



Let CB be the line fixed in the inner cylinder and ON be the line fixed in the outer cylinder. Initially ON and CB coincided with OA *i.e.* initially B coincided with N.

After time t when the line OC makes an angle θ with the vertical, let ON and CB make angles ψ and ϕ with the vertical.

Since there is no slipping, $\therefore$ Arc NP = arc BP

i.e. $a(\psi-\theta)=b(\phi-\theta)$, $\quad \therefore\ b\phi=a\psi-(a-b)\theta$

or $\quad b\dot{\phi}=a\dot{\psi}-(a-b)\dot{\theta}$ ...(1)

Considering the motion of the cylinders and taking moments about their centres of gravity, we get

$mb^2\ddot{\phi}=Fb$ (for smaller) ...(2) ; and $Ma^2\ddot{\psi}=-Fa$ (for larger) ...(3)

From (2) and (3) we have $mb\ddot{\phi}=-Ma\ddot{\psi}$.

Integrating, we get $mb\dot{\phi}=-Ma\dot{\psi}$ ...(4)

(initially $\dot{\phi}$ and $\dot{\psi}$ are both zero)

From (1) and (4) on eliminating $\dot{\phi}$, we get $-Ma\dot{\psi}=ma\dot{\psi}-m(a-b)\dot{\theta}$

or $a(M+m)\dot{\psi}=m(a-b)\dot{\theta}$ or $(a-b)\dot{\theta}=\dfrac{a(M+m)}{m}\dot{\psi}$ ...(5)

The coordinates of the centre of gravity C of the smaller cylinder with reference to O which is at rest are $\{(a-b)\sin\theta, (a-b)\cos\theta\}$.
Hence energy equation gives

$$\tfrac{1}{2}Ma^2\dot{\psi}^2+\tfrac{1}{2}m\left[b^2\dot{\phi}^2+(a-b)^2\dot{\theta}^2\right]=mg(a-b)(\cos\theta-\cos\alpha) \quad ...(6)$$

In (6) putting the values of $b\dot{\phi}$ and $(a-b)\dot{\theta}$ from (4) and (5) respectively, we get

$$Ma^2\dot{\psi}^2+m\left[\frac{M^2}{m^2}a^2\dot{\psi}^2+\frac{(M+m)^2}{m^2}a^2\dot{\psi}^2\right]=2mg(a-b)(\cos\theta-\cos\alpha)$$

or $\quad a^2\left[M+\dfrac{M^2}{m}+\dfrac{(M+m)^2}{m^2}\right]\dot{\psi}^2=2mg(a-b)(\cos\theta-\cos\alpha)$

or $\quad a^2\left[M(m+M)+(M+m)^2\right]\dot{\psi}^2=2m^2g(a-b)(\cos\theta-\cos\alpha)$

or $\quad a^2(m+M)(2M+m)\dot{\psi}^2=2m^2g(a-b)(\cos\theta-\cos\alpha)$

or $\quad a^2(m+M)(2M+m)\Omega^2=2m^2g(a-b)(\cos\theta-\cos\alpha)$;

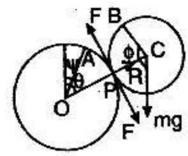
which is the required result, if $\dot{\psi}=\Omega$.

Ex. 2. *A uniform circular cylinder of mass M is free to rotate about its axis which is smooth and horizontal and about which its radius of gyration is equal to its radius. A uniform solid sphere of mass m is placed with its lowest point in contact with the highest generator of the cylinder, both sphere and cylinder being initially at rest. The sphere is then slightly disturbed and rolls down the cylinder. Show that the slipping takes place before the sphere leaves the cylinder, and begins when*

$$2M\sin\theta=\mu\{(17M+6m)\cos\theta-(10M+4m)\}$$

where θ is the inclination to the vertical of the plane through their axes and μ the coefficient of friction.

Sol. Let O be the centre of the cylinder whose radius is a and C the centre of the sphere whose radius is b.



Let the cylinder have turned through an angle ψ to the vertical and CB, a line fixed in the sphere make an angle ϕ with the vertical , a line fixed in space.

Initially B coincided with A and OA and CB were vertical.

Since there is no slipping, hence arc AP = arc BP

i.e. $a(\theta - \psi) = a(\phi - \theta)$ or $b\dot{\phi} + a\dot{\psi} = (a+b)\dot{\theta}$ ...(1)

Considering the motion of the cylinder and the sphere respectively and taking moments about their centres, we get

$Ma^2\ddot{\psi} = Fa$ (for the cylinder) ; and $m\dfrac{2b^2}{m}\ddot{\phi} = Fb$ (for the sphere)

$$\therefore\ Ma\ddot{\psi} = \frac{2mb}{5}\ddot{\phi} \quad i.e.\ b\ddot{\phi} = \frac{5M}{2m}a\ddot{\psi}$$

Integrating, we get $b\dot{\phi} = \dfrac{5M}{2m}a\dot{\psi}$ $\left\{\begin{array}{c}\text{Initially } \dot{\phi} \text{ and } \dot{\psi} \text{ zero}\\ \text{so constant vanishe}\end{array}\right\}$ ...(2)

Putting the value of $\dot{\phi}$ from (2), we get

$$\frac{5M}{2m}a\dot{\psi} + a\dot{\psi} = (a+b)\dot{\theta}$$

i.e. $\dfrac{5M+2m}{2m}a\dot{\psi} = (a+b)\dot{\theta}$ or $a\dot{\psi} = \dfrac{2m}{5M+2m} = (a+b)\dot{\theta}$

Then $b\dot{\phi} = \dfrac{5M}{5M+2m} = (a+b)\dot{\theta}$

The coordinates of C, the centre of the sphere with reference to O as the origin and vertical and horizontal through O as axes are $\{(a+b)\sin\theta, (a+b)\cos\theta\}$,

$$\therefore\ (\text{velocity})^2 \text{ of } C = \{(a+b)\cos\theta\dot{\theta}\}^2$$
$$+\{-(a+b)\sin\theta\dot{\theta}\}^2 = (a+b)^2\dot{\theta}^2$$

Therefore energy equation gives

$$\tfrac{1}{2}Ma^2\dot{\psi}^2 + \tfrac{1}{2}m\left[\frac{2b^2}{5}\dot{\phi}^2 + (a+b)^2\dot{\theta}^2\right] = mg\{(a+b) - (a+b)\cos\theta\}$$

or $\quad \dfrac{1}{2}M\dfrac{4m^2}{(5M+2m)^2}(a+b)^2\dot{\theta}^2$

$$+\tfrac{1}{2}\, m\left[\frac{2}{5}\cdot\frac{25M^2}{(5M+2m)^2}\,(a+b)^2\,\dot{\theta}^2+(a+b)^2\,\dot{\theta}^2\right]=mg\,(a+b)\,(1-\cos\,\theta)$$

or $$\left[\frac{M\,(5M+2m)}{(5M+2m)^2}+\frac{1}{2}\right](a+b)\,\dot{\theta}^2=g\,(1-\cos\,\theta)$$

or $$\left(\frac{M}{5M+2m}+\frac{1}{2}\right)(a+b)\,\dot{\theta}^2=g\,(1-\cos\,\theta)$$

or $$\frac{7M+2m}{5M+2m}\,(a+b)\,\dot{\theta}^2=2g\,(1-\cos\,\theta)$$

or $$(a+b)\,\dot{\theta}^2=\frac{10M+4m}{7M+2m}=g\,(1-\cos\,\theta)$$

Differentiating above and dividing by $2\dot{\theta}$, we get

$$(a+b)\,\ddot{\theta}=\frac{5M+2m}{7M+2m}\,g\,\sin\,\theta.$$

Equations of motion are $m\,(a+b)\,\dot{\theta}^2=mg\,\cos\,\theta-R$

and $m\,(a+b)\,\ddot{\theta}\;=mg\,\sin\,\theta-F.$

$$\therefore\; R=mg\,\cos\,\theta-m\,(a+b)\,\dot{\theta}^2=mg\cos\,\theta-\frac{10M+4m}{7M+2m}\,mg\,(1-\cos\,\theta)$$

$$=mg\cos\,\theta\left(1+\frac{10M+4m}{7M+2m}\right)-\frac{10M+4m}{7M+2m}\,mg$$

$$=\frac{mg}{7M+2m}\,[(17M+6m)\,\cos\,\theta-(10M+4m)]$$

and $F=mg\,\sin\,\theta-m\,(a+b)\,\ddot{\theta}$

$$=mg\sin\,\theta-\frac{5M+2m}{7M+2m}\,mg\,\sin\,\theta=\frac{2Mmg\sin\,\theta}{7M+2m}$$

$$\therefore\;\frac{F}{R}=\frac{2M\sin\,\theta}{(17M+6m)\,\cos\,\theta-(10M+4m)}$$

Slipping begins when $F=\mu R$

i.e. $2M\,\sin\,\theta=\mu\,[(17M+6m)\,\cos\,\theta-(10M+4m)]$.

Above equation gives the value of θ, when slipping begins where $\theta<\pi$.

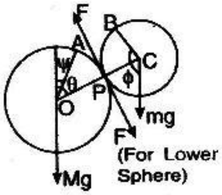
Now $R=\dfrac{F}{\mu}=\dfrac{2\,Mmg\sin\,\theta}{\mu\,(7M+2m)}$ which is obviously positive for all values of θ lying between 0 and π.

Hence the slipping begins before the sphere leaves the cylinder.

Ex. 3. *The mass of a sphere is $\frac{1}{5}$ of that of another sphere of the same material which is free to move about its centre as a fixed point, the first sphere rolls down the second from rest at the highest point, the coefficient* *of friction being μ. Prove that sliding will begin when the angle θ which the line of centres makes with the vertical is given by*

$$\sin\theta = 2\mu(5\cos\theta - 3)$$

Sol. Let the mass of the lower and upper sphere be M and m respectively so that $M = 5m$. The lower sphere is free to move. Let a be the radius of the lower sphere and b that of upper sphere.



Let the lower sphere have turned through an angle ψ such that OA, a line fixed in the lower sphere make an angle ψ with the vertical and the line CB (a line fixed in the upper sphere) an angle ϕ with the vertical. Initially OA and CB were vertical and B coincided with A, OC the line joining the centres makes an angle θ with the vertical.

Since there is no slipping between the spheres, so

arc AP = arc BP *i.e.* $a(\theta - \psi) = b(\phi - \theta)$

or $\quad a\dot{\psi} + b\dot{\phi} = (a+b)\dot{\theta} = c\dot{\theta}$ $\quad$ {Here $c = (a+b)$} ...(1)

The equation of motion for the lower sphere is $M\dfrac{2a^2}{5}\ddot{\psi} = Fa$ ...(2)

Equations of motion for the upper sphere are

and $\quad mc\dot{\theta}^2 = mg\cos\theta - R$ ...(3) ; $\quad mc\ddot{\theta} = mg\sin\theta - F$, ...(4)

since C describes a circle about O of radius $(a+b) = c$.

Hence F is the friction sufficient for pure rolling .

Also $\quad m\dfrac{2b^2}{5}\ddot{\phi} = Fb$ ...(5)

From (2) and (5), we have $\dfrac{a\ddot{\psi}}{m} = \dfrac{b\ddot{\phi}}{M}$ or $\dfrac{a\dot{\psi}}{m} = \dfrac{b\dot{\phi}}{M} = \dfrac{a\dot{\psi} + b\dot{\phi}}{m+M} = \dfrac{c\dot{\theta}}{m+M}$

$\therefore\ a\dot{\psi} = \dfrac{m}{m+M}c\dot{\theta}$ and $b\dot{\phi} = \dfrac{M}{m+M}c\dot{\theta}$

Energy equation gives

$$\tfrac{1}{2}M\frac{2a^2}{5}\dot{\psi}^2 + \tfrac{1}{2}m\left(\frac{2b^2}{5}\dot{\phi}^2 + c^2\dot{\theta}^2\right) = mg(c - c\cos\theta)$$

or $$\tfrac{1}{2}M\cdot\tfrac{2}{5}\frac{m^2}{(m+M)^2}c^2\dot{\theta}^2 + \tfrac{1}{2}m\left[\frac{2}{5}\cdot\frac{M^2}{(m+M)^2}c^2\dot{\theta}^2 + c^2\dot{\theta}^2\right] = mgc(1-\cos\theta)$$

or $$\left[\frac{2}{5}\cdot\frac{Mm}{M+m} + m\right]c^2\dot{\theta}^2 = 2mgc(1-\cos\theta)$$

or $\quad \left(\frac{2}{5}\cdot\frac{5m^2}{6m}+m\right)c\,\dot{\theta}^2=2mg\ (1-\cos\ \theta)\ (\because\ M=5m)$

or $\quad c\,\dot{\theta}^2=\frac{3}{2}\ g\ (1-\cos\ \theta).$ ...(6)

Differentiating (6) w.r.t. 't' and dividing by $2\dot{\theta}$, we get

$c\,\ddot{\theta}=\frac{3}{4}\ g\ \sin\ \theta.$ ...(7)

From (3), we have

$$R=mg\ \cos\ \theta-mc\,\dot{\theta}^2=mg\ \cos\ \theta-\tfrac{3}{2}mg\ (1-\cos\ \theta) \qquad \{\text{from (6)}\}$$
$$=mg\left(\frac{5\ \cos\ \theta-3}{2}\right)$$

From (4), we have

$$F=mg\ \sin\ \theta-mc\,\ddot{\theta}=mg\ \sin\ \theta-\tfrac{3}{4}\ mg\ \sin\ \theta\ =\ \tfrac{1}{4}\ mg\sin\ \theta\ \{\text{from (7)}\}$$

$$\therefore\ \frac{F}{R}=\tfrac{1}{4}\ mg\ \sin\ \theta\,.\frac{2}{mg\,(5\ \cos\ \theta-3)}=\frac{\sin\theta}{2\,(5\ \cos\ \theta-3)}$$

Sliding will begin when $F=\mu R$ or $\dfrac{F}{R}=\mu$

i.e. when $\dfrac{\sin\theta}{2\,(5\ \cos\ \theta-3)}=\mu$ or $\sin\ \theta=2\mu\,(5\ \cos\ \theta-3)$

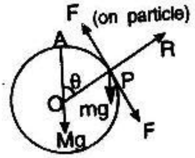
Ex. 4. *A rough cylinder, of mass M, is capable of motion about its axis which is horizontal ; a particle of mass m is placed on it vertically above the axis and the system is slightly disturbed. Show that the particle will slip on the cylinder when it has moved through an angle* θ *given by* $\mu\,(M+6m)\ \cos\ \theta-M\sin\ \theta=4m\mu.$ *where* μ *is the coefficient of friction.*

Sol. Assume that F is the force of friction which keeps the particle at rest the radius OP makes an angle θ with the vertical.

Reffered to O as the origin, the co-ordinates of particle are $(a\ \sin\ \theta, a\ \cos\ \theta)$.

Energy of the particle

$$=\frac{1}{2}\,m\,(\dot{x}^2+\dot{y}^2)=\frac{1}{2}\ ma^2\,\dot{\theta}^2$$



Energy of the cylinder $=\frac{1}{2}M\ \dfrac{a^2}{2}\ \dot{\theta}^2$ due to rotation.

The energy equation gives

$$\tfrac{1}{2}M\ \frac{a^2}{2}\ \dot{\theta}^2+\tfrac{1}{2}\ ma^2\dot{\theta}^2=\text{Work done by gravity}\ =mga\,(1-\cos\ \theta)$$

or $\quad a\,(M+2m)\,\dot{\theta}^2=4mg\ (1-\cos\ \theta)$ ...(1)

Differentiating above and dividing by $2\dot{\theta}$, we get

$$a(M+2m)\ddot{\theta} = 2mg \sin \theta \qquad \text{...(2)}$$

The particle m describes a circle about O, therefore

$$ma\dot{\theta}^2 = mg \cos \theta - R \quad \text{...(3)} \qquad \text{and} \quad ma\ddot{\theta} = mg \sin \theta - F. \qquad \text{...(4)}$$

Hence $R = mg \cos \theta - ma\dot{\theta}^2 = mg \cos\theta - \dfrac{4m^2 g}{M+m}(1-\cos\theta)$ [from (1)]

$$= \frac{mg}{M+2m}[(M+2m)\cos\theta - 4m(1-\cos\theta)]$$

$$= \frac{mg}{M+2m}[(M+6m)\cos\theta - 4m]$$

and $F = mg \sin\theta - ma\ddot{\theta} = mg \sin\theta - \dfrac{2m^2 g \sin\theta}{M+2m}$ From (2)

$$= \frac{mg\sin\theta}{M+2m}(M+2m-2m) = \frac{mMg\sin\theta}{M+2m}. \qquad \text{...(4)}$$

From (3) and (4) $\quad \dfrac{F}{R} = \dfrac{M\sin\theta}{(M+6m)\cos\theta - 4m}$

The particle slips from the cylinder when $F = \mu R$ *i.e.* when $\dfrac{F}{R} = \mu$.

i.e. when $\quad \dfrac{M\sin\theta}{(M+6m)\cos\theta - 4m} = \mu$

or when $\quad \mu(M+6m)\cos\theta - 4m\mu = M\sin\theta$

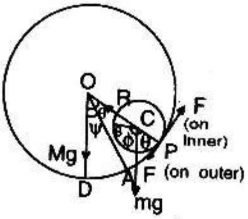
or when $\quad \mu(M+6m)\cos\theta - M\sin\theta = 4m\mu$

which is the required result.

Ex. 5. *A circular cylinder of radius a and of radius of gyration k rolls without slipping inside a hollow cylinder of radius b which is free to move about its axis. Show that the plane through their axis will move like a simple circular pendulum of length* $(b-a)(1+n)$

where $\quad n = \dfrac{(k^2/a^2)}{1 + \dfrac{b^2}{a^2}\cdot\dfrac{mk^2}{mK^2}}$ *where k and K are the radii of gyration, of the inner and outer cylinders respectively, about their axes; and m and M their masses.*

(Vikram 1862)



Sol. Adjoining figure is the vertical section through the centres of gravity of the two cylinders. The centre O remains fixed and the outer cylinder turns about it, let ψ be the angle turned by it when the plane of the axis makes an angle θ with the vertical. Let CB a line fixed in the inner cylinder makes an angle ϕ with the vertical a line fixed in space. Since there is no slipping so Arc AP = Arc PB]

i.e. $b(\theta - \psi) = a(\phi + \theta)$

or $\quad b\ddot{\psi} + a\ddot{\phi} = (b-a)\ddot{\theta}$ ...(1)

Equations of motion are $Mk^2\ddot{\psi} = -Fb$ ...(2) for outer cylinder

$$\left.\begin{array}{ll} mk^2\ddot{\phi} = -Fa & \text{...(3)} \\ \text{and } m(b-a)\ddot{\theta} = F - mg\sin\theta & \text{...(4)} \end{array}\right\} \text{ for inner cylinder.}$$

From (2) and (3), we have

$$-F = \frac{MK^2\ddot{\psi}}{b} = \frac{mk^2\ddot{\phi}}{a}$$

or
$$-F = \frac{b\ddot{\psi}}{(b^2/MK^2)} = \frac{a\ddot{\phi}}{(a^2/mk^2)} = \frac{b\ddot{\psi} + a\ddot{\phi}}{(b^2/MK^2) + (a^2/mk^2)}$$

$$= \frac{(b-a)\ddot{\theta}}{(b^2/MK^2) + (a^2/mk^2)}. \text{ (By virtue of (1))}$$

Therefore
$$F = -\frac{(b-a)\ddot{\theta}}{(b^2/MK^2) + (a^2/mk^2)} = -m(b-a).\frac{(k^2/a^2)}{1 + \frac{b^2}{a^2}\cdot\frac{mk^2}{MK^2}}\ddot{\theta}$$

$$= -m(b-a)\,n\ddot{\theta} \quad \text{where } n = \frac{(k^2/a^2)}{1 + \frac{b^2}{a^2}\cdot\frac{mk^2}{MK^2}}$$

Putting this value of F in (4), we get

$$m(b-a)\ddot{\theta} = -m(b-a)\,n\ddot{\theta} - mg\sin\theta$$

or $(b-a)(1+n)\ddot{\theta} = -g\sin\theta$ or $\quad \ddot{\theta} = -\dfrac{g}{(b-a)(1+n)}\theta$

(θ is small taking θ for $\sin\theta$)

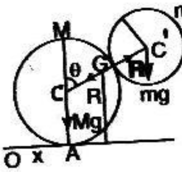
Thus length of simple equivalent pendulum is $(b-a)(1+n)$.

3·14 *Motion of one body on another when both the bodies are free to move.*

ILLUSTRATIVE EXAMPLES

Ex. 1. *Two unequal smooth spheres, one placed on the top of the other are in unstable equlibrium, the lower sphere resting on a smooth table. The system is slightly disturbed; show that the sphere will separate when the lines joining their centres make an angle θ with the vertical given by the equation $m\cos^3\theta = (M+m)(3\cos\theta - 2)$, where M is the mass of the lower, and m that of the upper sphere.*

Sol. Let C and C' be the centres a and b the radius of the lower and upper sphere respectively and their masses are M and m respectively. Let after time t the lower sphere have moved through a distance x on the table when CC' the line joining their centre makes an angle θ with the vertical.



As both the sphere are given to be smooth there are no forces acting on them to turn either sphere about its centre *i.e.* there is no rotation.

The co-ordinaters of centres of gravity of both spheres with reference to O as origin are (x, a) (for the lower sphere)

and $X = x + (a+b)\sin\theta,\ \ Y = (a+b)\cos\theta$ (for the upper sphere)

There is no horizontal force on the system, since the sphere and the planes are smooth. Thus $\dfrac{d}{dt}\,[M\dot{x} + m\ \{\dot{x} + (a+b)\cos\theta\,\dot{\theta}\ \}] = 0.$

Integrating above we get $(M+m)\,\dot{x} + m\,(a+b)\cos\ \theta\,\dot{\theta} = 0.$ ....(2)

(Initially $\dot{x} = 0 = \dot{\theta}$, so that constant $= 0$)

or $$\dot{x} = -\frac{m}{M+m}(a+b)\cos\theta\,\dot{\theta}. \quad ....(3)$$

Energy equation gives

$$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\,(\dot{X}^2 + \dot{Y}^2) = mg\ \{(a+b) - (a+b)\cos\theta)\}$$

or $$\tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\ [\dot{x}^2 + (a+b)^2\dot{\theta}^2 + 2\,(a+b)\,\dot{\theta}\,\dot{x}\ \cos\theta]$$
$$= mg\ (a+b)\ (1-\cos\theta).$$

Putting for $\dot{x}^2$ from (3) we get

$$\frac{1}{2}\left[\frac{(M+m)\,m^2}{(M+m)^2}\,(a+b)^2\cos^2\theta\,\dot{\theta}^2 + m\,(a+b)^2\dot{\theta}^2 - 2\,(a+b)^2\,\frac{m^2}{(M+m)}\cos^2\theta\,\dot{\theta}^2\right] = mg\,(a+b)\,(1-\cos\theta)$$

or $$\left[-\frac{m\,(a+b)}{(M+M)}\cos^2\theta + (a+b)\right]\dot{\theta}^2 = 2g\ (1-\cos\theta)$$

or $$(M + m\ \sin^2\theta)\,\dot{\theta}^2 = \frac{2g}{(a+b)}\,(M+m)\,(1-\cos\theta) \quad ...(4)$$

Differentiating (4) with respect to t, we have

$$(M+m\ \sin^2\theta)\,\ddot{\theta}\ +m\cos\theta\ \sin\theta\,\dot{\theta}^2=\frac{M+m}{a+b}\,g\ \sin\theta. \qquad ...(5)$$

Let R be the reaction between the two spheres.

Considering the horizontial motion of the lower sphere, we have

$$-R\ \sin\theta=M\ddot{x}=M\left\{-\frac{m}{M+m}\ (a+b)\right\}(\cos\theta\,\ddot{\theta}\ -\sin\ \theta\,\dot{\theta}^2)$$

or $$R\ \sin\ \theta=\frac{Mm}{M+m}\ (a+b)\,(\cos\ \theta\ddot{\theta}\ -\sin\ \theta\dot{\theta}^2). \qquad ...(6)$$

By (6), R vanishes *i.e.* sphere separate, when $\cos\ \theta\ddot{\theta}\ =\sin\ \theta\dot{\theta}^2$ ...(7)

On eliminating $\dot{\theta}^2$ from (5) and (7), we get

$$(M+m)\,\ddot{\theta}\ =\frac{M+m}{(a+b)}\,g\ \sin\theta \qquad \text{or} \qquad \ddot{\theta}\ =\frac{g}{a+b}\ \sin\ \theta.$$

Thus from (7), $\cos\ \theta.\left(\dfrac{g}{a+b}\right)\sin\ \theta=\sin\ \theta\,\dot{\theta}^2$ or $\dot{\theta}^2=\dfrac{g\cos\theta}{(a+b)}$

Putting this values of $\dot{\theta}^2$ in (4), we get

$$(M+m\ \sin^2\theta)\,\frac{g\cos\theta}{(a+b)}=\frac{2g}{(a+b)}\,(M+m)\,(1-\cos\theta)$$

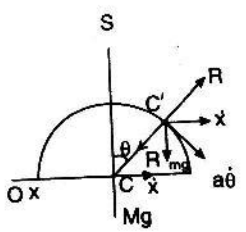
or $\{M+m\ (1-\cos^2\theta\ \}\cos\theta=2\,(M+m)\,(1-\cos\theta)$

or $m\ \cos^3\theta=(M+m)\,(3\ \cos\theta-2)$ which is the required result.

Ex. 2. *A hemisphere of mass M is free to slide with its base on a smooth horizontal table. A particle of mass m is placed on the hemisphere at an angular distance α from the vertex, show that the radius to the point of contact at which the particle leaves the surface, makes with the vertical an angle θ given by the equation*

$m\ \cos^3\theta-(M+m)\,(3\ \cos\theta-2\cos\alpha)=0.$ **(Agra 91, 86, 82)**

Sol. Let in time t the centre of the hemisphere have moved through distance x on the place and its velocity be $\dot{x}$ while CC' makes an angle θ with the vertical where C in the centre of the hemisphere and C' is point where particle is placed. Let $\dot{x}$ and $a\dot{\theta}$ be the horizontal and tangential velocities of the particle .



With reference to O as the origin, the coordinates of centre of gravity of particle are $X=x+a+a\sin\theta,\ Y=a\cos\theta$

As there in no horizontal force on the system, so

$\frac{d}{dt}\{M\dot{x}+m(\dot{x}+a\cos\theta\,\dot{\theta}\}=0$. Intergrating above equation, we get

$M\dot{x}+m(\dot{x}+\cos\theta\dot{\theta})=0$ (since $\dot{x}=0=\dot{\theta}$ initially, so constant $=0$)

or $$\dot{x}=-\frac{ma\cos\theta\,\dot{\theta}}{(M+m)} \quad ...(1)$$

Kinetic energy of the hemisphere is $\frac{1}{2}M\dot{x}^2$ and that of the particle is $\frac{1}{2}m(\dot{X}^2+\dot{Y}^2)$. Since there are no-forces to turn the hemisphere, so there is no rotational energy. Hence the energy equation gives

$$\tfrac{1}{2}M\dot{x}^2+m(\dot{x}^2+a^2\dot{\theta}^2+2a\,\dot{\theta}\dot{x}\cos\theta)=mga(\cos\alpha-\cos\theta) \quad ...(2)$$

Putting for $\dot{x}$ from (1) in (2), we get

$$\frac{m(M+m)a^2\cos^2\theta\,.\,\dot{\theta}^2}{(M+m)^2}+a^2\dot{\theta}^2-\frac{2a^2m\cos^2\theta}{(M+m)}\dot{\theta}^2=2ga(\cos\alpha-\cos\theta)$$

or $$\left(1-\frac{m}{M+m}\cos^2\theta\right)a^2\dot{\theta}^2=2ga(\cos\alpha-\cos\theta)$$

or $$\{(M+m)-m\cos^2\theta\}a\dot{\theta}^2=2g(M+m)(\cos\alpha-\cos\theta) \quad ...(3)$$

Considering horizontal motion of the hemisphere, we have $M\ddot{x}=-R\sin\theta$

or $M\frac{d}{dt}(\dot{x})=-R\sin\theta$ or $-\frac{M\,ma}{(M+m)}\frac{d}{dt}(\dot{\theta}\cos\theta)=-R\sin\theta$

[from (1)]

The particle leaves the hemisphere of $R=0$,

i.e. if $\frac{d}{dt}(\dot{\theta}\cos\theta)=0$ or $\ddot{\theta}\cos\theta=\dot{\theta}^2\sin\theta$ ...(4)

Equation (3) may be written as

$a(M+m)\sin^2\theta)\theta^2=2g(M+m)(\cos\alpha-\cos\theta)$

Differentiating it with regard to 't' and dividing by 2θ, we get

$$(M+m\sin^2\theta)\ddot{\theta}+m\sin\theta\cos\theta\dot{\theta}^2=\frac{(M+m)g}{a}\sin\theta \quad ...(5)$$

Thus from (4) and (5), we get $\ddot{\theta}=\frac{g}{a}\sin\theta$, $\therefore$ $\dot{\theta}^2=\frac{g}{a}\cos\theta$

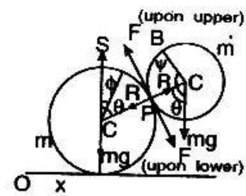
Putting this value of $\dot{\theta}^2$ in (3), we get

$$\{(M+m)-m\cos^2\theta\}a\left(\frac{g}{a}\cos\theta\right)=2g(M+m)(\cos\alpha-\cos\theta)$$

or $m\cos^3\theta-(M+m)(3\cos\theta-2\cos\alpha)=0$ which is the required result.

Ex. 3. *Two homogeneous sphere of equal radii and masses m and m′ rest on a smooth horizontal plane with m′ on the highest point of m. If the system be disturbed show that the inclination θ of their common normal to the vertical is given by* $a\dot{\theta}^2(7m+5m'\sin^2\theta)=5g(m+m')(1-\cos\theta)$.

Sol. Let C and C' be the centres of the two spheres whose masses are m and m' respectively. CA and $C'B$ of are the redii (lines fixed in the bodies) which were initially vertical. Let in time t the lower sphere have moved through a distance x on the table while the line of their centres CC' make an angle θ with the vertical and the bodies have turned though angle ϕ and ψ in space. As there is no sliding, hence Arc AP = Arc BP.



i.e. $a(\theta-\phi)=a(\psi-\theta)$ or $\theta-\phi=\psi-\theta$

or $\psi+\phi=2\theta$ ...(1)

Considering the motion of the spheres and taking moments about their centres C and C', we have

$m'\dfrac{2a^2}{5}\ddot{\psi}=Fa$ (for the upper sphere); $m'\dfrac{2a^2}{5}\ddot{\phi}=Fa$ (for the lower sphere)

$\therefore\ m'\ddot{\psi}=m\ddot{\phi}$ or $\dfrac{\ddot{\psi}}{m}=\dfrac{\ddot{\phi}}{m'}$

Integrating it, we get $\dfrac{\dot{\psi}}{m}=\dfrac{\dot{\phi}}{m'}=\dfrac{\dot{\psi}+\dot{\phi}}{(m+m')}=\dfrac{2\dot{\theta}}{(m+m')}$

(By componendo and dividend)

$\therefore\ \dot{\psi}=\dfrac{2m\dot{\theta}}{(m+m')}$ and $\dot{\phi}=\dfrac{2m'\dot{\theta}}{(m+m')}$ (Initially $\dot{\phi}=0=\dot{\psi}$)

The coordinate of C and C' with respect to O as origin, are (x, a) and $(x+2a\sin\theta,\ a+2a\cos\theta)$ respectively.

Since there is no horizontal force on the system, we have

$$\frac{d}{dt}\left\{m\dot{x}+m'(\dot{x}+2a\cos\theta\dot{\theta})\right\}=0$$

Integrating it, we get $m\dot{x}+m'(\dot{x}+2a\cos\theta\dot{\theta})=0$

(Initially $\dot{x}=0=\dot{\theta}$, so constant $=0$)

or $(m+m')\dot{x}=-2am'\cos\theta\dot{\theta}$ or $\dot{x}=\dfrac{-2am'}{m+m'}\cos\theta\dot{\theta}$ ...(2)

The energy equation gives

$$\tfrac{1}{2}m\left(\dot{x}^2+\frac{2a^2}{5}\dot{\phi}^2\right)+\tfrac{1}{2}m'\left(\dot{x}^2+4a^2\dot{\theta}^2+4a\cos\theta\dot{\theta}\dot{x}+\frac{2a^2}{5}\dot{\psi}^2\right)=2am'g(1-\cos\theta) \quad ...(3)$$

Putting for $\dot{x}$, ϕ and in $\dot{\psi}$ (3), we get

$$\left[\frac{4a^2mm'^2}{(m+m')^2}\cos^2\theta\dot{\theta}+\frac{2a^2}{5}\frac{4mm'^2}{(m+m')^2}\dot{\theta}^2+\frac{4m'^3a^2}{(m+m')^2}\cos^2\theta\dot{\theta}^2\right.$$

$$\left.+4a^2m'\dot{\theta}^2-\frac{8a^2m'^2\cos^2\theta\dot{\theta}^2}{(m+m')}+\frac{2a^2}{5}\cdot\frac{4m^2m'\dot{\theta}^2}{(m+m')^2}\right]=4am'g(1-\cos\theta)$$

$$\text{or}\left[\frac{m'(m+m')}{(m+m')^2}\cos^2\theta+\frac{2m(m+m')}{5(m+m')^2}+1-\frac{2m'\cos^2\theta}{m+m'}\right]a\dot{\theta}^2=g(1-\cos\theta)$$

$$\text{or}\left[\frac{2}{5}\frac{m}{(m+m')}+1-\frac{m'\cos^2\theta}{(m+m')}\right]a\dot{\theta}^2=g(1-\cos\theta)$$

$$\text{or}\quad(2m+5m+5m'-5m'\cos^2\theta)\,a\dot{\theta}^2=5(m+m')g(1-\cos\theta)$$

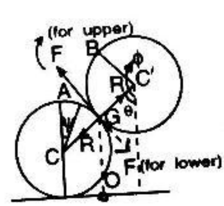
or $(7m+5m'\sin^2\theta)\,a\dot{\theta}^2=5(m+m')g(1-\cos\theta)$ which is the required **result.**

Ex. 4. *A uniform solid cylinder rests on a smooth horizontal plane and on it placed a second equal cylinder touching it along its hightes generator, if there is no slipping between the cylinders and system moves from rest, show that the cylinders separate when the plane of either axes makes an angle θ with the vertical given by the equation*

$2\cos^3\theta+4\cos^2\theta-35\cos\theta+20=0.$

Also show that until the cylinder separate the same generators remain in contact.

Sol. The adjoining figure is the vertical section of the system through the centre of gravity of the cylinders. Let C and C' be the centres of two cylinders, CA and $C'B$ the lines fixed in the cylinders making angles ψ and ϕ with the vertical at time t. Initially CA and $C'B$ were vertical and B coincided with A .



Since there is no slipping, between the two cylinder, hence arc AG = are BG where G is their point of contact. The cylinders being equal (given)

$\therefore\quad \angle ACG=\angle BC'G.$

Considering motion of the two cylinders and taking moments about C and C', we have

$m\dfrac{a^2}{2}\ddot{\psi} = Fa$ **(for the lower cylinder)** ; and $m\dfrac{a^2}{5}\ddot{\phi} = Fa$

(for the upper cylinder)

Integrating, we get $\dot{\phi} = \dot{\psi}$	The constants vanish when initially
Again integrating, $\phi = \psi$	$\dot{\psi}$, $\dot{\phi}$, ψ and ϕ are all zero.

Again $\angle ACG = \angle BC'G$ *i.e.* $\theta - \psi = \phi - \theta$ *i.e.* $\psi + \phi = 2\theta$
i.e. the same generators remain in contact until the cylinders separate.

Since there is no horizontal force on the two cylinders considered combined together therefore the comman centre for gravity G (which is the point of contact) will descend vertically. Let the vertical through G cut thr horizontal plane in O, then 0 is a fixed point. With O as origin and harizontal and vertical lines through O as coordinate axes, the coordinates of C' and C are $(a \sin\theta\, a + 2a\cos\theta)$, the $(-a\sin\theta, a)$ respectively.

Energy equation gives

$$\tfrac{1}{2}m\left[\frac{a^2}{2}\dot{\theta}^2 + a^2\cos^2\theta\dot{\theta}^2\right] + \tfrac{1}{2}m\left[\frac{a^2}{2}\dot{\theta}^2 + (a^2\cos^2\theta\dot{\theta}^2 + 4a^2\sin^2\theta\dot{\theta}^2)\right]$$

$$= mg\,[2a - 2a\cos\theta] \qquad \text{(Here we have taken } \dot{\phi} = \dot{\theta}\text{)}$$

or $\quad a\,(3 + 2\sin^2\theta)\,\dot{\theta}^2 = 4g\,(1 - \cos\theta)$ or

or $\quad a\,(5 - 2\cos^2\theta)\;\dot{\theta}^2 = 4g\,(1 - \cos\theta)$. ...(1)

Differentiating and dividing by $2\dot{\theta}$, we get

$$a\,(5 - 2\cos^2\theta)\;\ddot{\theta} + 2a\sin\theta\cos\theta\dot{\theta}^2 = 2g\sin\theta \qquad \text{...(2)}$$

Now consider the horizontal motion of the upper cylinder.

$$R\sin\theta - F\cos\theta = m\frac{d^2}{dt^2}(a\sin\theta) = ma\,(\cos\theta\ddot{\theta} - \sin\theta\,\dot{\theta}^2 0 \qquad \text{...(3)}$$

and also $\quad m\dfrac{a^2}{2}\ddot{\phi} = Fa,\quad$ ....(4) $\qquad$ (taking moment about C'.)

Eliminating F between (3) and (4), we get

$$R\sin\theta = ma(\cos\theta\,\ddot{\theta} - \sin\theta\,\ddot{\theta}^2) + \frac{ma\,\ddot{\theta}}{2}\cos\theta$$

i.e. $\quad R\sin\theta = \dfrac{ma}{2}(3\cos\theta\,\ddot{\theta} - 2\sin\theta\,\dot{\theta}^2)$

The cylinders will separate if $R = 0$, *i.e.* if $3\cos\theta\,\ddot{\theta} = 2\sin\theta\,\dot{\theta}^2$.

Now we eliminate $\ddot{\theta}$ and $\dot{\theta}^2$ between (1), (2) and (5).

Putting the value of $\ddot{\theta}$ from (5) in (2), we get

$$a(5-2\cos^2\theta)\frac{2\sin\theta}{3\cos\theta}\dot{\theta}^2+2a\sin\theta\cos\theta\,\dot{\theta}^2=2g\sin\theta$$

or $\quad a(5+\cos^2\theta)\dot{\theta}^2=3g\cos\theta,$

Putting this value of $\dot{\theta}^2$ in (1), we have

$$(5-2\cos^2\theta)\frac{3g\cos\theta}{5+\cos^2\theta}=4g(1-\cos\theta)$$

i.e. $\quad 3(5-2\cos^2\theta)\cos\theta=4(1-\cos\theta)(5+\cos^2\theta)$

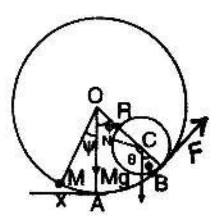
i.e. $\quad 15\cos\theta-6\cos^3\theta=20-20\cos\theta+4\cos^2\theta-4\cos^3\theta.$

i.e. $2\cos^3\theta+4\cos^2\theta-35\cos\theta+20=0,$ which is the required result.

Ex. 5. *A uniform rough ball is at rest within a hollow cylindrical garden roller, and the roller is then drown along a level-path with uniform velocity V. If $V^2>\frac{27}{7}g(b-a)$, show that the ball will roll completely round the inside of the roller ; a, b, being the radii of the ball and roller.*

(Raj. 92)

Sol. Let O be the centre of the roller and C the centre of the spherical ball moving inside the cylindrical roller. Let CN be the radius of the ball which was vertical when it was in its lowest position. When the roller has moved through a distance x, let CN have turned through an angle θ. The line joining the centre makes an angle ϕ with the vertical and the ball has turned through an angle θ. As there is no sliding,



$$\text{arc } BM=\text{arc } BN$$

i.e. $\quad b(\phi+\psi)=a(\theta+\phi)$

or $\quad (b-a)\phi=a\theta-b\psi \qquad ...(1)$

Again the velocity of the roller is constant

i.e. $\quad \dot{x}=b\dot{\psi}=V.$

Then $\qquad \ddot{x}=b\ddot{\psi}=0. \qquad ...(2)$

Let R and F be the normal reaction and friction. As C describes a circle of radius $(b-a)$ about O, so accelerations along CO and perpendicular to CO are $(b-a)\dot{\phi}^2$ and $(b-a)\ddot{\phi}$ respectively.

Thus equation of motion are $\quad m(b-a)\dot{\phi}^2=R\;\; mg\cos\phi, \qquad ...(3)$

$$m\,(b-a)\,\ddot{\phi} = F - mg\sin\phi, \qquad \text{...(4)}$$

and $$m\,\frac{2a^2}{5}\,\ddot{\theta} = -F\,.\,a, \qquad \text{...(5)}$$

Eliminating F between (4) and (5), we get

$$(b-a)\,\ddot{\phi} = -\frac{2a}{5}\,\ddot{\theta} - g\sin\phi \quad \text{or} \quad (b-a)\,\ddot{\phi} + \tfrac{2}{5}(b-a)\,\ddot{\phi} = -g\sin\phi$$

[since $(b-a)\,\ddot{\phi} = a\ddot{\theta} - a\ddot{\psi} = a\ddot{\theta}$ by virtue of (2)]

or $$\tfrac{7}{5}(b-a)\,\ddot{\phi} = -g\sin\phi. \qquad \text{...(6)}$$

Integrating it, we get $$\tfrac{7}{5}(b-a)\,\dot{\phi}^2 = 2g\cos\phi + A. \qquad \text{...(7)}$$

Initially the velocity of the C.G. is $\dot{x} + (b-a)\,\dot{\phi} = 0$,

i.e. $(b-a)\,\dot{\phi} = -\dot{x} = -V$. $\quad \therefore \quad A = \dfrac{7V^2}{5\,(b-a)} - 2g.$

Hence the equation (7) gives

$$\tfrac{7}{5}(b-a)\,\dot{\phi}^2 = -2g\,(1-\cos\phi) + \frac{7V^2}{5\,(b-a)}. \qquad \text{...(8)}$$

Substituting for $\dot{\phi}^2$ from (8) in (3), we get

$$\frac{R}{m} = g\cos\phi + \frac{V^2}{b-a} - \frac{10}{7}\,(-\cos\phi) = \frac{1}{7}\left(17g\cos\phi - 10g + \frac{7V^2}{b-a}\right).$$

The necessary condition that the ball should roll completely round the fixed cylinder is that R is positive when $\phi = \pi$, and if R is positive in this position, when it will be positive in all positions.

Hence $$\left\{\frac{7\,V^2}{b-a} - 10 + 17g\cos\phi\right\}_{\phi=\pi} > 0$$

or $$\frac{7\,V^2}{b-a} > 27g \quad \text{or} \quad V^2 > \frac{27g\,(b-a)}{7}.$$

Revision at a Glance

(i) $M\ddot{x}_G = \Sigma X$, $M\ddot{y}_G = \Sigma Y$, $Mk^2\ddot{\theta} = L$ where L is the moment of external forces about G.

(ii) K .E. of the body = K .E. due to translation $(\frac{1}{2}\,Mv_G^{\,2})$

+ K. E. due to rotation $(\frac{1}{2}\,Mk^2\dot{\theta}^2)$

(iii) Moment of momentum about the fixed origin $O = Mv_{GP} + Mk^2\dot{\theta}$.

PROBLEM SET

1. One end of thread, which is wound on a reel is fixed and the reel falls in a vertical line, its axis being horizontal and the unwound part of the thread being vertical. If the reel be a solid cylinder of radius a and weight W, show that the acceleration of the centre of the reel is $\frac{2}{3}g$ and the tension of the thread is $\frac{1}{3}W$.

2. A uniform rod is held at an inclination at an 45° to the vertical with one end in contact with a horizontal table whose coefficient of friction is μ. If it is then released, that it will commence to slide if $\mu < \frac{3}{5}$. **Hint :** Put $\alpha = 45$ in Ex. 1, Page 128.

3. A unifrom beam lies on a rough horizontal table at right angles to the edge and is held so that one third of its length is in contact with the table. Prove that after it is released it will begin to slide, over the edge of the table when it has turned through an angle $\tan^{-1}\left(\frac{1}{2}\mu\right)$, μ being the coefficient of friction between the table and the beam.

Hint : Proceed like Ex. 3, Page 130.

4. A sphere is projected with underhand spin up a slope of angle α, show that if the velocity V of projection be large, the sphere will turn back after a time $\dfrac{5V - 2a\Omega}{5g \sin\alpha}$; where Ω is angular velocity in the sense which would cause the sphere to roll up.

Hint : proceed like Ex. 2, Page 142. after changing the sign of Ω

5. A sphere is projected with an underhand twist down a rough inclined plane ; show that it will turn back in the course of its motion if $2a\omega(\mu - \tan\alpha) > 5\mu u$, where u, ω are the initial liner and angular velocities of the sphere, μ is the coefficient of friction and α is the inclination of the -plane.

6. A sphere of radius a, whose centre of gravity is not at its centre C is placed on a rough horizontal table so that C.G. is inclined at an angle α to upward drawn vertical, show that it will commence to slide along the table if coefficient of friction μ be less than $\dfrac{c \sin\alpha\,(a + c\cos\alpha)}{k^2 + (a + c\cos\alpha)^2}$, where C.G. $= c$ and k is the radius of gyration about a horizontal axis through G.

7. A solid uniform sphere resting on the top of another fixed sphere is slightly displayed and begins to roll down. Show that it will slip when the common normal makes with the vertical an angle given by $2\sin\theta = \mu\,[17\cos\theta - 10\cos\alpha]$.

Hint : Proceed as in Ex. 1, Page 169, put $k^2 = (2b^2/5)$.

8. A uniform sphere is placed on the top of a fixed rough cylinder whose generators are horizontal. Show that, if slightly displaced, it will roll on the cylinder untill it reaches a plane where the inclination of the tangent plane to the horizon is given by $2\sin\theta = \mu\,[17\cos\theta - 10]$ μ being the coefficient of friction.

Hint : Proceed an in Ex. 1, Page 169, put $\alpha = 0$, $k^2 = (2b^2/5)$.

9. A hollow cylinder of radius a is fixed the axis horizontal, inside it moves a solid cylinder, of radius b, whose angular velocity in its lowest position is Ω. If the friction between the cylinder be sufficient to prevent any sliding, prove that the small cylinder will go round the inner suface if $\Omega = \left[\left(\dfrac{11g}{3(a-b)}\right)\right]^{\frac{1}{2}}$.

Hence deduce the time for a small oscillation of the inner cylinder. Further when Ω has less than this value, find the condition that the two cylinder separete. **(Agra 70)**

Hint : 3·12 Page 176.

10. A rough cylinder of mass $2nm$, capable of motion about its horizontal axis has a particle of mass m and the coefficient of friction μ placed on it vertically above the axis. The system is then slightly disturbed. Show that the particle will slip on the cylinder after it has moved through an angle θ given by $(n+3)\cos\theta - 2 = \dfrac{n \sin\theta}{\sqrt{(\mu)}}$.

Hint : It is exactly the Ex. 4, Page 188. Put $M = 2nm$.

11. Two equal perfectly rough sphere are placed in unstable equilibrium, one on the top of the other, the lower resting on a perfectly smooth table. A slight disturbance being given, show that the sphere will continue to touch each other at the same points and that, if θ be the inclination to the vertical of the straight line joining the centres, then

$(k^2 + a^2 + a^2 \sin^2\theta)\,\dot{\theta}^2 = 2ga\,(1-\cos\theta)$. Where k is the radius of gyration of each sphere about an axis through its centre.

Hint : Proceed like Ex. 3, Page 194, put $k^2 = (2a^2/5)$ and $m = m'$.

Lagrange's Equations of Motion Small Oscillations : Normal Co-Ordinates

7·01. Generalised Co-ordinates. **(Meerut 91)**

Suppose that a particle or a system of N-particles moves subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates then needed to specify the motion. These co-ordinates denoted by $q_1, q_2, \ldots, q_n$ *are called generalised co-ordinates. These co-ordinates may be distances, angles or quantities relating to them.*

7·02. Degrees of freedom. **(Meerut 91)**

The number of independent co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

Ex. 1. *A particle moving freely in space require 3 co-ordinates, e.g.* (x, y, z), *to specify its position. Thus the number of degrees of freedom is 3.*

Ex. 2. *A system containing of N-particles moving freely in space require 3N co-ordinates to specify the position. The number of degrees of freedom is 3N.*

A rigid body which can move freely in space has 6 degrees of freedom *i.e.* 6 co-ordinates are required to specify the position.

Let 3 non-collinear points of a rigid body be fixed in space, then the rigid body also fixed in space. Let these points have co-ordinates (x_1, y_1, z_1) ; (x_2, y_2, z_2) ; (x_3, y_3, z_3) respectively, a total of 9. Since the body is rigid, we must have

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = \text{constant.}$$

$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = \text{constant.}$$

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = \text{constant.}$$

Hence 3 co-ordinates can be expressed in terms of the remaining six. Thus six independent co-ordinates are needed to describe the motion *i.e* there exit six degrees of freedom.

7·03. Transformation equation.

Let $\boldsymbol{r}_v = x_v\ \boldsymbol{i} + y_v\ \boldsymbol{j} + z_v\ \boldsymbol{k}$ be the position vector of v-th particle with respect to xyz co-ordinates system. The relationships of the genaralised co-ordinates $q_1, q_2, \dots, q_n$ the position co-ordinates are given by the *transformation equations.*

$$\left.\begin{aligned} x_v &= x_v\ (q_1, q_2, \dots, q_n\ ; t) \\ y_v &= y_v\ (q_1, q_2, \dots, q_n\ ; t) \\ z_v &= z_v\ (q_1, q_2, \dots, q_n\ ; t) \end{aligned}\right\} \qquad \dots(1)$$

where t denotes the time. In vector (1) can be written as

$$\boldsymbol{r}_v = \boldsymbol{r}_v\ (q_1, q_2, \dots q_n\ ; t) \qquad \dots(2)$$

where the functions in (1) or (2) are continuous and have continuous derivatives.

7·04. Classification of Mechanical systems.

(a) Scleronomic system.

The mechanical system in which t, the time, does not enter explicity in equations (1) or (2) *is called a scleronomic system.*

(b) Rheonomic system.

The mechanical system in which the moving constraints are involved and the time t does enter explicity is called a Rheonomic system.

(Meerut 1989)

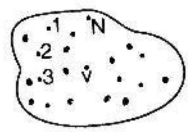
(c) Holonomic system and Non Holonomic system.

Let $q_1, q_2, \dots, q_n$ denote the generalised co-ordinates describing a system and let t denote the time. If all the constraints of the system can be expressed as equations having the form $(q_1, q_2, \dots, q_n\ ; t) = 0$ or their equivalent, then the system is said to be Holonomic otherwise it is be Non-Holonomic system.

(d) Conservative and non-conservative system.

If the forces acting on the system are derivable from a potential function [or potential energy] V, then the system is called conservative, otherwise it is non-conservative.

7·05. Kinetic energy and generalised velocities.



The K.E of the system is $T = \frac{1}{2} \sum_{v=1}^{n} m_v\ \dot{\boldsymbol{r}}_v^2$.

The K.E of the system can be written as a quadratic form in the generalised co-ordinates q_α. If the system is independent of time explicitly i.e. Scleronomic then the quadratic form has only terms of the type $a_{\alpha\beta}\ q_\alpha\, q_\beta$. In case the system is Rheonomic, linear terms in q_α are also present.

7·06. Generalised Forces. **(Meerut 1989)**

If W is the total work done on a system of particles by forces $\mathbf{F}_v$ *acting on the v-th particle, then*

$$dW = \sum_{\alpha=1}^{n} \phi_\alpha \, dq_\alpha \text{ where } \phi_\alpha = \sum_{v=1}^{N} F_v \bullet \frac{\partial r_v}{\partial q_\alpha}$$

is called the **generalised force** *associated with generalised co-ordinates* q_α.

Suppose that a system undergoes increments $dq_1, dq_2, \ldots, dq_n$ of the generalised co-ordinates $q_1, q_2, \ldots, q_n$, then the v-th particle undergoes a displacement.

$$dr_v = \sum_{\alpha=1}^{n} \frac{\partial r_v}{\partial q_\alpha} dq_\alpha \qquad \ldots(4)$$

$\therefore$ Total work done is given by

$$dW = \sum_{v=1}^{N} F_v \bullet dr_v = \sum_{v=1}^{N} \left\{ \sum_{\alpha=1}^{n} F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \right\} dq_\alpha \qquad \ldots(5)$$

Now, let $\phi_\alpha = \sum_{v=1}^{N} F_v \bullet \dfrac{\partial r_v}{\partial q_\alpha}$

then (5) $dW = \sum_{\alpha=1}^{N} \left(\sum_{v=1}^{N} F_v \bullet \dfrac{\partial r_v}{\partial q_\alpha} \right) d\, q = \sum_{\alpha=1}^{n} \phi_\alpha d\, q$

...(6)

We have $dW = \sum_{\alpha=1}^{N} \dfrac{\partial W}{\partial q_\alpha} dq_\alpha$, $\therefore \dfrac{\partial W}{\partial q_\alpha} = \phi_\alpha$ ...(7)

Note. **(i)** α varies from (1) to n, the number of degrees of freedom.

(ii) v aries from 1 to N, the number of particles in the system.

7·07. Lagrange's equations. (Meerut 1993, 95)

Let F be the net external force acting on the Vth particle of a system, then by Newton"s second law applied to Vth particle, we have

$$m_v \ddot{r}_v = F_v$$

$$\Rightarrow m_v \ddot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} = F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \qquad \ldots(8)$$

$$\Rightarrow \sum_{v=1}^{N} m_v \ddot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} = \sum_{v=1}^{N} F_v \bullet \frac{\partial r_v}{\partial q_\alpha} \qquad \ldots(9)$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{v=1}^{N} m_v \dot{r}_v \bullet \frac{\partial r_v}{\partial q_\alpha} \right] - \sum_{v=1}^{N} m_v \dot{r}_v \bullet \frac{d}{dt}\left(\frac{\partial r_v}{\partial q_\alpha} \right) = \sum_{v=1}^{N} F_v \bullet \frac{\partial r_v}{\partial q_\alpha}$$

But $r_v = r_v \, (q_1, q_2, \ldots, q_n; t)$ ...(10)

$$\therefore \dot{r}_v = \frac{\partial r_v}{\partial q_1} \dot{q}_1 + \frac{\partial r_v}{\partial q_2} \dot{q}_2 + \ldots + \frac{\partial r_v}{\partial q_n} \dot{q}_n + \frac{\partial r_v}{\partial t} \qquad \ldots(11)$$

$$\Rightarrow \frac{\partial \dot{r}_v}{\partial \dot{q}_\alpha} = \frac{\partial r_v}{\partial q_\alpha} \qquad \textbf{[Cancellation law of the dots]} \qquad \ldots(12)$$

Also, $\frac{\partial}{\partial q_\alpha}(\dot{r}_v) = \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial q_1}\dot{q}_1 + \frac{\partial r_v}{\partial q_2}\dot{q}_2 + \ldots + \frac{\partial r_v}{\partial q_n}\dot{q}_n + \frac{\partial r_v}{\partial t}\right)$

$$= \frac{\partial^2 r_v}{\partial q_\alpha \partial q_1}\dot{q}_1 + \frac{\partial^2 r_v}{\partial q_\alpha \partial q_1}\dot{q}_2 + \ldots + \frac{\partial^2 r_v}{\partial q_\alpha \partial q_n}\dot{q}_n + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial t}\right)$$

$$= \frac{\partial}{\partial q_1}\left(\frac{\partial r_v}{\partial q_\alpha}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial r_v}{\partial q_\alpha}\right)\dot{q}_2 + \ldots + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_v}{\partial q_n}\right)\dot{q}_n + \frac{\partial}{\partial t}\left(\frac{\partial r_v}{\partial q_\alpha}\right)$$

or $\frac{\partial}{\partial q_\alpha}\left(\frac{dr_v}{dt}\right) = \frac{d}{dt}\left(\frac{\partial r_v}{\partial q_\alpha}\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right) \equiv \frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right)$ ...(13)

[interchange law of the order of operators]

Now $$\frac{d}{dt}\left\{\sum_{v=1}^{N} m_v\, \dot{r}_v \cdot \frac{\partial r_v}{\partial q_\alpha}\right\} - \sum_{v=1}^{N} m_v\, \dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial q_\alpha} = \sum_{v=1}^{N} F_v \cdot \frac{\partial r_v}{\partial q_\alpha}$$
...(14)

and $T = \frac{1}{2}\sum_v m_v\, \dot{r}_v^2 = \frac{1}{2}\sum_v m_v\, (\dot{r}_v \cdot \dot{r}_v)$ ...(15)

$\Rightarrow \frac{\partial T}{\partial q_\alpha} = \sum_v m_v\, \dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial q_\alpha}$ ...(16)

and $\frac{\partial T}{\partial \dot{q}_\alpha} = \sum_v m_v\, \dot{r}_v \cdot \frac{\partial \dot{r}_v}{\partial \dot{q}_\alpha} = \sum m_v\, \dot{r}_v \cdot \frac{\partial r_v}{\partial q_\alpha}$ [using (12)] ...(17)

$\therefore$ (14) $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \sum_v F_v \cdot \frac{\partial r_v}{\partial q_\alpha}$

or $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha = \frac{\partial W}{\partial q_\alpha}$ using (7) ...(18)

Note. The quantity $p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$ is called the generalised momentum associated with the general co-ordinates q_α.

7·08. Lagrangian function.

If the forces are derivable from a potential function V, then

$$\phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha},$$

since the potential, or potential energy is a function of q's only (and possibly the name t) then, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}(T - V)\right] - \left(\frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0, \text{ where } L = T - V \qquad ...(19)$$

The function L defined by L = T − V is said to be Lagrangian function.

7·09 . Generalised momentum.

We defined $p_\alpha = \dfrac{\partial T}{\partial \dot{q}_\alpha}$ *to be the generalised momentum associated with generalised co-ordinates* q_α *or the conjugate momentum.*

In case the system is conservative, we have

$$T = L + V \Rightarrow (\partial T/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha) + (\partial V/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha)$$

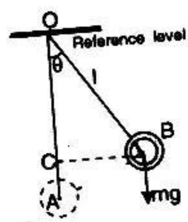
because V, the P. E. of the system does not depend upon $\dot{q}_\alpha$

$$\therefore\ p_\alpha = (\partial L/\partial q_\alpha).$$

7·10. Illustrative Examples.

Ex. 1. **(i)** *Set up the Lagrangian for a simple pendulum, and* **(ii)** *obtain an equation describing its motion.*

Sol. **(i)** Choose as generalised coordinates, the angle θ made by the string OB of the pendulum and the vertical OA . Let l be the length of OA, then K. E. is given by



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\theta})^2 = \frac{1}{2} m l^2 \dot{\theta}^2$$

where m is the mass of the bob.

The potential energy of mass m is given by

$$V = mg(OA - OC) = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

$$\therefore L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

(ii) Hence Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(m l^2 \dot{\theta}) - (-mgl\sin\theta) = 0$$

$$\Rightarrow l\,\ddot{\theta} = -g\sin\theta \Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta,$$

which is the required equation of motion.

7·11. Kinetic energy as a Quadratic function of velocities.

If at time t, the position of the ν^{th} particle (mass m), of a holonomic system is defined by $\mathbf{r}_\nu$, then K. E. is given by

$$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{\mathbf{r}}_\nu^2, \text{ where } \mathbf{r}_\nu = \mathbf{r}_\nu(q_1, ...\, q_n; t) \qquad ....(1)$$

so that $\dot{r}_v = \dot{q}_1 \frac{\partial r_v}{\partial q_1} + \dot{q}_2 \frac{\partial r_v}{\partial q_2} + ... + \dot{q}_n \frac{\partial r_v}{\partial q_n} + \frac{\partial r_v}{\partial t}$

$$\Rightarrow T = \frac{1}{2}\sum_{v=1}^{N} m_v \left(\dot{q}_1 \frac{\partial r_v}{\partial q_1} + \dot{q}_2 \frac{\partial r_v}{\partial q_2} + ... + \dot{q}_n \frac{\partial r_v}{\partial q_n} + \frac{\partial r_v}{\partial t} \right)^2$$

$$= \frac{1}{2} [(a_{11} \dot{q}_1^2 + a_{22} \dot{q}_2^2 + ... + a_{nn} \dot{q}_n^2 + 2a_{12} \dot{q}_1\dot{q}_2 + 2a_{1n} \dot{q}_1\dot{q}_n + \dots)$$

$$+ 2(a_1\dot{q}_1 + a_2\dot{q}_2 + . + a_n\dot{q}_n) + a] ...(2)$$

$$\left.\begin{array}{c} \text{where } \; 5a_{rs} = \sum_{v=1}^{N} m_v (\partial r_v / \partial q_r) \cdot (\partial r_v / \partial q_s) \; (s \geq r) \\ a_{rr} = \sum_{v=1}^{N} m_v (\partial r_v / \partial q_r)^2, a = \sum_{v=1}^{N} m_v (\partial r_v / \partial t)^2, a_r = \sum_{v=1}^{N} m_v \left(\frac{\partial r_v}{\partial q_r} \right) \cdot \left(\frac{\partial r_v}{\partial t} \right) \end{array}\right\}$$

From (2), we see that T is a quadratic function of the generalised velocities.

The case t is not explicity involved, is of considerable importance. Hence we have $\frac{\partial r_v}{\partial t} = 0$ and therefore (2) implies that

$$T = \frac{1}{2} (a_{11} \dot{q}_1^2 + a_{22} \dot{q}_2^2 + ... + a_{nn} \dot{q}_n^2 + 2a_{12} \dot{q}_1 \dot{q}_2 + \dots) \quad ...(3)$$

$$= \frac{1}{2} \sum_{s=1}^{n} \sum_{r=1}^{n} a_{rs} \dot{q}_r \dot{q}_s \text{ where } a_{rs} = a_{sr} . \quad ...(4)$$

Now using Euler's theorem for homogeneous functions, we get

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + ... + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T$$

$$\Rightarrow 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha$$

i.e $\quad 2T = p_1 \dot{q}_1 + p_1 \dot{q}_2 + ... + p_n \dot{q}_n$.

7·12. To deduce the principle of energy from The Lagrange's equations (Conservative field).

Lagrange's equations are :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha} \right) - \frac{\partial T}{\partial q_\alpha} = - \frac{\partial V}{\partial q_\alpha}; \qquad (\alpha = 1, 2, ... n) \qquad ...(1)$$

Also by 7·11. equation we know that

$$T = \frac{1}{2} (a_{11} \dot{q}_1^2 + a_{22} \dot{q}_2^2 + .. + a_m \dot{q}_n^2 + 2a_{12} \dot{q}_1 \dot{q}_2 + ...),$$

that is, T can be expressed as a quadratic expression in generalised velocities. Hence applying Euler's theorem. we get

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T \qquad ...(2)$$

Also, $$\frac{dT}{dt} = \sum_{\alpha=1}^{n} \frac{\partial T}{\partial q_\alpha} \dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha} \ddot{q}_\alpha \qquad ...(3)$$

Now multiplying the n equations of (1) by $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$ respectively and then adding, we get

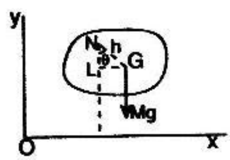
$$\left\{\dot{q}_1 \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right] + ... + \dot{q}_n \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right]\right\} - \left\{\dot{q}_1 \frac{\partial T}{\partial q_1} + ... + \dot{q}_n \frac{\partial T}{\partial q_n}\right\}$$
$$= -\left\{\dot{q}_1 \frac{\partial V}{\partial q_1} + ... + \dot{q}_n \frac{\partial V}{\partial q_n}\right\}$$

$$\Rightarrow \frac{d}{dt}\left\{\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \left(\sum_{\alpha=1}^{n} \ddot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right) - \left\{\sum_{\alpha=1}^{n} \dot{q}_n \frac{\partial T}{\partial q_\alpha}\right\} = -\left(\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial V}{\partial q_\alpha}\right)$$

$$\Rightarrow \frac{d}{dt}(2T) - \frac{dT}{dt} = -\frac{dV}{dt} \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0 \quad \Rightarrow \frac{d}{dt}(T+V) = 0 \Rightarrow T + V =$$ constant.

Ex. 2. *Use Lagrange's equations to find the differantial equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.*

Sol. Let the plane of oscillation be represented by xy-plane, where N is its intersection with the axis of rotation and G is the centre of gravity. Let the mass of the pendulum be M and let its moment of interia about the axis of rotation be MK^2. Then potential energy relative to the horizontal plane through N is $V = -Mgh\cos\theta$.



Also $T = \frac{1}{2}Mk^2\dot{\theta}^2 \quad \therefore L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta \qquad ...(1)$

$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = Mk^2\theta$ and $\frac{\partial L}{\partial \theta} = -Mgh\sin\theta \qquad ...(2)$

Now Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(Mk^2\dot{\theta}) + Mgh\sin\theta = 0$$

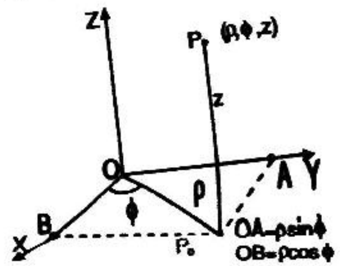
i.e. $Mk^2\ddot{\theta} + Mgh\sin\theta = 0 \Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta$

When θ is small, we have $D^2\theta = -\frac{gh}{k^2}\theta \qquad (\because \sin\theta = \theta)$

or $$\left(D^2 + \frac{gh}{k^2}\right)\theta = 0.$$

This is the differential equation of the pendulum.

Ex. 3. *A particle of mass m moves in a conservative force field. Find* **(a)** *the Lagrangian function,* **(b)** *the equations of motion in cylindrical co-ordinates* (ρ, ϕ, z). **[Meerut 95]**



Sol. we have $OP = OP_0 + P_0P = OA + AP_0 + P_0P = \vec{\rho}$ (say)

$\therefore \vec{\rho} = \rho \sin\phi\, \boldsymbol{j} + \rho\cos\phi\, \boldsymbol{i} + z\boldsymbol{k}$ where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are the unit vector along OX, OY and OZ respectively.

Hence the unit vector along the direction of ρ increasing is

given by $\quad \vec{\rho_1} = \dfrac{\partial \vec{\rho}}{\partial \rho} \Big/ \left|\dfrac{\partial \vec{\rho}}{\partial \rho}\right| = \sin\phi\, \boldsymbol{j} + \cos\phi\, \boldsymbol{i}$

Similarly $\quad \vec{\phi_1} = \dfrac{\partial \vec{\rho}}{\partial \phi} \Big/ \left|\dfrac{\partial \vec{\rho}}{\partial \phi}\right|$

$$= \frac{\rho\cos\phi\, \boldsymbol{j} - \rho\sin\phi\, \boldsymbol{i}}{\rho} = -\sin\phi\, \boldsymbol{i} + \cos\phi\, \boldsymbol{j}$$

Now $\boldsymbol{v} = \dfrac{d\vec{\rho}}{dt} = \dfrac{d}{dt}(\rho\sin\phi\, \boldsymbol{j} + \rho\cos\phi\, \boldsymbol{i} + z\boldsymbol{k})$

$$= \rho\cos\phi\dot{\phi}\, \boldsymbol{j} + \dot{\rho}\sin\phi\, \boldsymbol{j} - \rho\sin\phi\dot{\phi}\, \boldsymbol{i} + \dot{\rho}\cos\phi\, \boldsymbol{i} + \boldsymbol{k}\dot{z}$$

$$= \dot{\rho}\cos\phi\, \boldsymbol{i} + \dot{\rho}\sin\phi\, \boldsymbol{j} + \rho\dot{\phi}(\cos\phi\, \boldsymbol{j} - \sin\phi\, \boldsymbol{i}) + \dot{z}\boldsymbol{k}$$

$$= \dot{\rho}\vec{\rho_1} + \rho\dot{\phi}\vec{\phi_1} + \dot{z}\boldsymbol{k}$$

$$\therefore T = \frac{1}{2}m\{\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\} \text{ and } V = V(\rho, \phi, z)$$

(a) Hence the Lagrangian function is

$$L = T - V = \frac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2] - V(\rho, \phi, z)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0 \text{ i.e. } \frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 \frac{\partial V}{\partial \rho}\right) = 0$$

i.e. $m\ (\ddot{\rho}-\rho\dot{\phi}^2) = -\dfrac{\partial V}{\partial \rho}$ ...(1)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \text{ i.e. } \frac{d}{dt}(m\rho^2\dot{\phi}) + \frac{\partial V}{\partial \phi} = 0,$$

or $\dfrac{d}{dt}\left(\rho^2\dot{\phi}\right) = -\dfrac{\partial V}{\partial \phi}$ ...(2)

and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{z}}\right) - \dfrac{\partial L}{\partial z} = 0$ i.e. $\dfrac{d}{dt}(m\dot{z}) + \dfrac{\partial V}{\partial z} = 0$ or $m\ddot{z} = -\dfrac{\partial V}{\partial z}$ ...(3)

Ex. 4. *A particle Q moves on a smooth horizontal circular wire of radius a which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the $\angle QCO = \theta$, show that*

$$a\ddot{\theta} + \dot{\omega}\ (a - c\cos\theta) = c\,\omega^2 \sin\theta.$$

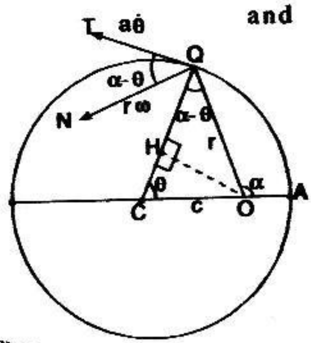
where ω is the angular velocity of the wire.

[Meerut 1995,1993,84,86; Garhwal 1982]

Sol. Let $OQ = r$, and $\angle AOQ = \alpha$

$\Rightarrow r^2 = a^2 + c^2 - 2ac\cos\theta$ ...(1) and

$r\cos(\alpha - \theta) = a - c\cos\theta$. ...(2)



The particle Q moves on circle of radius a, so **its velocity along the tangent QT will be $a\dot{\theta}$ but Q revolves about O with angular velocity ω, which causes a velocity $a\omega$ at the right angles to OQ.**

$\Rightarrow v_Q^2 =$ (velocity)^2 of the particle at Q

$$= a^2\dot{\theta}^2 + r^2\omega^2 + 2ar\dot{\theta}\cos(\alpha - \theta)\,\omega$$

$\angle AOQ = \alpha$, $\angle CQO = \alpha - \theta$

Now $T = \frac{1}{2} m v_Q^2 = \frac{1}{2} m\ [a^2\dot{\theta}^2 + r^2\omega^2 + 2a r\omega\dot{\theta}\cos(\alpha - \theta)]$

$\left[\ \angle NQT = \alpha - \theta,\ OQ = r \quad HQ = a - c\cos\theta\right.$

NOTE :- *If **r** is the position vector of the particle at any time t, then $\partial r/\partial r$ is the vector tangent to the curve θ = constant i.e. a vector in the direction of r (increasing r). A unit vector in this direction is thus given by $r_1 = \dfrac{\partial r}{\partial r} \Big/ \left|\dfrac{\partial r}{\partial r}\right|$.

Similerly, $\partial r/\partial\theta$ is the vector tangant to the curve r=constant, A unit vector in this direction is given by $\vec{\theta_1} = \dfrac{\partial r}{\partial \theta} \Big/ \left|\dfrac{\partial r}{\partial \theta}\right|$.

$$= \frac{1}{2} m [a^2 \dot{\theta}^2 + (a^2 + c^2 \qquad\qquad = r \cos (\alpha - \theta)]$$

$$- 2ac \cos \theta) \omega^2 + 2a \omega \dot{\theta} (a - c \cos \theta)$$

and work function = 0 ($\because$ weight does no work)

$\therefore$ Lagrange's θ equation $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$.

$$\Rightarrow \frac{d}{dt} [a^2 \dot{\theta} + a \omega (a - c \cos \theta)] - ac \omega^2 \sin \theta - ac \omega \dot{\theta} \sin \theta = 0$$

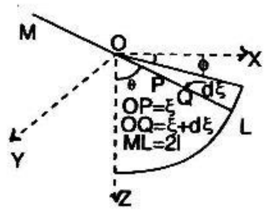
$$\Rightarrow a^2 \ddot{\theta} + a \dot{\omega}(a - c \cos \theta) + a\omega c \dot{\theta} \sin \theta - ac \omega^2 \sin \theta - ac \omega \dot{\theta} \sin \theta = 0$$

$$\Rightarrow a\ddot{\theta} + \dot{\omega}(a - c \cos \theta) = c \omega^2 \sin \theta.$$

Ex. 5. *A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to* $\sqrt{\left(\frac{2ng}{l}\right)}$ *show that the heavy end of the rod will fall till the inclination of the rod to the vertical is* $\cos^{-1} [\sqrt{(n^2 + 1)} - n]$ *and will then rise again.* **[Meerut 91,93,95; Agra 1992]**

Sol. The mass m is attached at L. On the rod ML, take a point p such that

$OP = \xi$, the element $PQ = d\xi$.



Further at any time t, let the plane through it and the vertical have turned through nn angle ϕ from its initial position and let the rod be inclined at an angle θ to the vertical. Taking O, the mid point of the rod, as the origin and OX, OY (a line perpendicular to the plane of the paper) and OZ as axes of refrence, then co-ordinates of the point P on the rod are:

$$x = \xi \sin \theta \cos \phi, \; y = \xi \sin \theta \sin \phi, \; z = \xi \cos \theta$$

$$\therefore \dot{x} = \xi \cos \theta \cos \phi \, \dot{\theta} - \xi \sin \phi \, \dot{\phi} \sin \theta,$$

$$\dot{y} = \xi \cos \theta \sin \phi \, \dot{\theta} + \xi \sin \theta \cos \phi \, \dot{\phi}, \; \dot{z} = -\xi \sin \theta \dot{\theta}. \text{ Thus,}$$

$$v_P^2 = (\text{velocity})^2 \text{ of } P = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \xi^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta).$$

$$\therefore v_L^2 = l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) = (\text{velocity})^2 \text{ of mass } m,$$

Now mass of the element $PQ = \frac{3m}{2l} d\xi = dm$, say.

$\therefore$ Its kinetic energy

$= \frac{1}{2} dm.v_p^2 = \frac{1}{2} \cdot \frac{3m}{2l} d\xi (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \xi^2$

$= \frac{3m}{4l} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \xi^2 \, d\xi$

and K.E. of the rod $= \frac{3m}{4l} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \int_{-l}^{l} \xi^2 d\xi$

$= \frac{1}{2} m (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) l^2.$

Again, (velocity)^2 of the particle $m = l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

$\therefore$ Kinetic energy of the particle of mass $m = \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

$\therefore$ Total K.E.$= T =$ K.E. of the rod + K.E. of the particle

$= \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$ *i.e.*

$T = ml^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

Also the work function is given by $W = mgl \cos\theta + C$.

Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

which gives $\frac{d}{dt} (2ml^2 \dot{\phi} \sin^2\theta) = 0.$

Integrating it, we get $\dot{\phi} \sin^2\theta = K$ (constant).

Initially, $\theta = \frac{\pi}{2}$ and $\dot{\phi} = \sqrt{\left(\frac{2ng}{l}\right)}$. $\therefore K = \sqrt{\left(\frac{2ng}{l}\right)}$.

$\therefore \dot{\phi} \sin^2\theta = \sqrt{\left(\frac{2ng}{l}\right)}$. ...(1)

and Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt} (2ml^2 \dot{\theta}) - 2ml^2 \dot{\phi}^2 \sin\theta \cos\theta = -mgl \sin\theta$

or $2l\ddot{\theta} - 2l\dot{\phi}^2 \sin\theta \cos\theta = -g \sin\theta$. ...(2)

Substituting value of $\dot{\phi}$ from (1) in (2), we have

$2l\ddot{\theta} - 4ng \cot\theta \operatorname{cosec}^2\theta = -g \sin\theta$. ...(2′)

Integration provides us $2l\dot{\theta}^2 + 4ng \cot^2\theta = 2g\cos\theta + k$.

Initially $\theta = \frac{\pi}{2}, \dot{\theta} = 0, \therefore k = 0$.

$$\therefore\ 2l\dot{\theta}^2 + 4ng\cot^2\theta = 2g\cos\theta. \qquad ...(3)$$

The rod will fall till $\dot{\theta} = 0$.

i.e. $4ng\cot^2\theta = 2g\cos\theta$ or $2n\cos^2\theta - \cos\theta\sin^2\theta = 0$.

$\therefore$ **either** $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ **which gives initial position.**

as $2n\cos\theta - \sin^2\theta = 0 \Rightarrow \cos^2\theta + 2n\cos\theta - 1 = 0.$

Solving it, $\cos\theta = \dfrac{-2n \pm \sqrt{(4n^2+4)}}{2} = \left\{-n + \sqrt{(n^2+1)}\right\}$

[the other value being inadmissible because θ can not be obtuse]

or $\theta = \cos^{-1}[-n + \sqrt{(n^2+1)}]$. This proves the required result.

If we substitute this value of θ in equation (2′), then we find that $\ddot{\theta}$ comes out to be positive. Hence at that time the rod begins to rise.

Ex. 6. *A bead, of mass M, slides on a smooth fixed wire, whose inclination to the vertical is α, and has hinged to it a rod, of mass m and length 2l, which can move freely in the vertical plane through the wire. If the system starts from rest with the rod hanging vertically, show that*

$$\{4M + m(1 + 3\cos^2\theta)\}\, l\dot{\theta}^2 = 6(M+m)\, g\sin\alpha\,(\sin\theta - \sin\alpha)$$

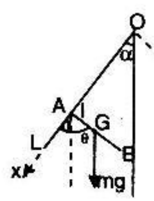
where θ is the angle between the rod and the lower part of the wire.

(Meerut 95, Agra 1993, 91)

Sol. Let OL be the fixed wire. At any time t, let the bead of mass M be at A where $OA = x$, also let θ be the angle which the rod AB makes with the lower part of the fixed wire.

Take O as origin and the fixed wire OL as x axis and a line through O and prep. to OL as y axis; the co-ordinates of G, the C.G. of the rod AB, are $\{x + l\cos\theta,\ l\sin\theta\}$

i.e. $x_G = (x + l\cos\theta)$ and $y_G = l\sin\theta$.



$$\dot{x}_G = (\dot{x} - l\sin\theta\,\dot{\theta}) \quad ; \quad \dot{y}_G = l\cos\theta\,\dot{\theta}$$

$\therefore$ (velocity)^2 of $G = v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (\dot{x} - l\sin\theta\,\dot{\theta})^2 + (l\cos\theta\,\dot{\theta})^2$.

Now let T be the kinetic energy and W the work function of the system.

Then we easily get

Total energy $= T =$ K.E. of the bead + K.E. of the rod

$$= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[\frac{l^2}{3}\dot{\theta}^2 + (\dot{x} - l\sin\theta\,\dot{\theta})^2 + (l\cos\theta\,\dot{\theta})^2\right]$$

$$= \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{x}\,\dot{\theta}\sin\theta + \tfrac{2}{3}ml^2\dot{\theta}^2.$$

Also, the work function is given by

$$W = Mgx\cos\alpha + mg\{x\cos\alpha + l\cos(\theta-\alpha)\}$$
$$= (M+m)gx\cos\alpha + mgl\cos\cos(\theta-\alpha)$$

$\therefore$ Lagrange's x-equation gives, $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \dfrac{\partial W}{\partial x}$

i.e. $\dfrac{d}{dt}[(M+m)\dot{x} - ml\dot{\theta}\sin\theta] = (M+m)g\cos\alpha$

or $(M+m)\ddot{x} - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta = (M+m)g\cos\alpha$. ...(i)

Again Lagrang's θ– equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e $\dfrac{d}{dt}[-ml\dot{x}\sin\theta + \tfrac{4}{3}ml^2\dot{\theta}] + ml\dot{x}\dot{\theta}\cos\theta$

$$= -mgl\sin(\theta-\alpha),$$

or $-\ddot{x}ml\sin\theta - \dot{x}\dot{\theta}ml\cos\theta + \tfrac{4}{3}ml^2\ddot{\theta} + \dot{x}\dot{\theta}ml\cos\theta$

$$= -mgl\sin(\theta-\alpha).$$

or $-\ddot{x}\sin\theta + \tfrac{4}{3}l\ddot{\theta} = -g\sin(\theta-\alpha)$ ...(ii)

Eliminating $\ddot{x}$ between (i) and (ii), we get

$$\ddot{\theta}[-ml\sin^2\theta + \tfrac{4}{3}(M+m)l] - ml\dot{\theta}^2\sin\theta\cos\theta$$
$$= (M+m)g[\cos\alpha\sin\theta - \sin(\theta-\alpha)]$$

or $l\ddot{\theta}[3M + m + 3m\cos^2\theta] - 3ml\dot{\theta}^2\sin\theta\cos\theta$

$$= 3(M+m)g\cos\theta\sin\alpha.$$

Whence on integrating, we get

$$l\dot{\theta}^2[4M + m + 3m\cos^2\theta] = 6(M+m)g\sin\alpha\sin\theta + C \quad \text{...(iii)}$$

When $\theta = \alpha$, $\dot{\theta} = 0$, $\therefore C = -6(M+m)g\sin^2\alpha$.

Putting the value of C in (iii), we get

$$l\dot{\theta}^2(4M + m + 3m\cos^2\theta) = 6(M+m)g\sin\alpha(\sin\theta - \sin\alpha).$$

7·13. Small Oscillations. *To explain how Lagrange's equations are used in case of small oscillations.* **[Meerut 1990]**

(*when there exist three generalised co-ordinates*).

In order to investigate the theory of small oscillations by the use of Lagrange's equation about the position of equilibrium, the generalised co-ordinates q_1, q_2, q_3 must be chosen in such a way that they vanish in the position of equilibrium.

But the system makes small oscillations about the position of equilibrium, so q_1, q_2, q_3 and their differentials $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ will remain small during the whole motion. Hence we shall reject all the powers of small quantites except the lowest one.

Now, let T be kinetic energy, and W the work-function of the system; then we have

$$T = A_{11}\dot{q}_1^2 + A_{22}\dot{q}_2^2 + A_{33}\dot{q}_3^2 + 2A_{12}\dot{q}_2\dot{q}_1 + 2A_{23}\dot{q}_2\dot{q}_3$$

$$+ 2A_{13}\dot{q}_1\dot{q}_3 \text{ [by 7·11]} \quad ...(1)$$

$$W = C + B_1 q_1 + B_2 q_2 + B_3 q_3 + B_{11} q_1^2 + B_{22} q_2^2 + B_{33} q_3^2 \quad ...(2)$$

Now assume that q_1, q_2, q_3 can be expressed in terms of X, Y, Z by the equations of the form

$$q_1 = \lambda_1 X + \lambda_2 Y + \lambda_3 Z\ ; \quad q_2 = \mu_1 X + \mu_2 Y + \mu_3 Z\ ;$$

$$q_3 = \nu_1 X + \nu_2 Y + \nu_3 Z\ ;$$

and further assume λ_1, λ_2, λ_3, μ_1, μ_2, μ_3, ν_1, ν_2, ν_3 in such a way that if we put value of q_1, q_2, q_3, in (1) and (2), then T does not involve $\dot{Y}\dot{Z}$, $\dot{Z}\dot{X}$, $\dot{X}\dot{Y}$ and W does not involve YZ, ZX, XY. Then X, Y, Z, are called the **normal or Principal Co-ordinates.**

Thus equation (1) and (2) $\Rightarrow$

$$T = A'_{11}\dot{X}^2 + A'_{22}\dot{Y}^2 + A'_{33}\dot{Z}^2$$

$$W = C' + B'_1 X + B'_2 Y + B'_3 Z + B'_{11} X^2 + B'_{22} Y^2 + B'_{33} Z^2$$

Now, using Lagrange's equations,

$$viz.\ \frac{d}{dt}\left(\frac{dT}{d\dot{X}}\right) - \left(\frac{dT}{dX}\right) = \left(\frac{dW}{dX}\right) \text{ etc., we obtain}$$

$$2A'_{11}\ddot{X} = B'_1 + 2B'_{11} X\ ; \qquad 2A'_{22}\ddot{Y} = B'_2 + 2B'_{22} Y\ ;$$

$$2A'_{33}\ddot{Z} = B'_3 + 2B'_{33} Z\ ,$$

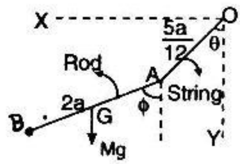
which can be put in the forms

$$\ddot{X} = -\omega_1^2 X\ , \quad \ddot{Y} = -\omega_3^2 Y\ . \quad \ddot{Z} = -\omega_3^2 Z \text{ etc.}$$

which represent simple Harmonic Motions, Or in other words they give small oscillations about the position of equilibrium.

Ex. 7. *A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string, of length $\frac{5}{12}a$, performing small oscillations in a vertical plane about its position of equillibrium. Find the* *position at any time, and show that the period of its principal oscallations are* $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\frac{a}{3g}\right)}$.

Sol. Figure is self explanatory. At any time t, let the string and the rod be inclined at θ and ϕ to the vertical OY.



Co-ordinates of G are given by

$$x_G = \frac{5}{12}a\sin\theta + a\sin\phi,$$

$$y_G = \frac{5}{12}a\cos\theta + a\cos\phi.$$

$$\dot{x}_G = \frac{5a}{12}a\cos\theta\dot{\theta} + a\cos\phi\,\dot{\phi}.$$

$$\dot{y}_G = -\left(\frac{5a}{12}\sin\theta\dot{\theta} + a\sin\phi\,\dot{\phi}\right)$$

$$\therefore\ \dot{x}_G^2 + \dot{y}_G^2 = (\text{velocity})^2 \text{ of G}$$

$$= \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5a^2}{6}\dot{\theta}\dot{\phi}\cos(\theta+\phi) = \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}$$

[$\because$ θ and ϕ are small so $\cos(\theta+\phi) = 1$]

Again let T, be the kinetic energy and W, the work function of the system, then we easily get

$$T = \frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2 + \left(\frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}\right)\right]$$

$$= \frac{ma^2}{288}[25\dot{\theta}^2 + 192\dot{\phi}^2 + 120\dot{\theta}\dot{\phi}]$$

and $$W = mg\left[\frac{5}{12}a\cos\theta + a\cos\phi\right]$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta} \Rightarrow \frac{d}{dt}\left\{\frac{ma^2}{144}(25\dot{\theta} + 60\dot{\phi})\right\} = -\frac{5mga}{12}\theta$$

{ $\sin\theta = \theta$ as θ is small }

$$\Rightarrow 5\ddot{\theta} + 12\ddot{\phi} = -\frac{12g}{a}\theta. \qquad ...(1)$$

and Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{ma^2}{144}(192\dot{\phi} + 60\dot{\theta})\right\} = -mga\phi \Rightarrow 5\ddot{\theta} + 16\ddot{\phi} = -12\frac{g}{a} \qquad ...(2)$$

Equations (1) and (2) $\Rightarrow (5D^2 + 12c)\theta + 12D^2\phi = 0$ ...(3)

and $5D^2\theta + (6D^2 + 12c)\phi = 0$ where $(g/a) = c$. ...(4)

Now eliminating ϕ between these two equations, we get

$$[(5D^2 + 12c)(16D^2 + 12c) - 60D^4]\theta = 0$$

or $(5D^4 + 63cD^2 + 36c^2)\,\theta = 0$. ...(5)

Let $\theta = A\cos(pt + B) \quad \therefore \quad D\theta = -pA\sin(pt + B)$,

$D^2\theta = -p^2 A\cos(pt + B) = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting these values in (5), we get

$(5p^4 - 63c^2 - 36c^2)\,\theta = 0 \Rightarrow (5p^4 - 63cp^2 + 36c^2) = 0 \quad (\because \theta \neq 0)$

$\Rightarrow (5p^2 - 3c)(p^2 - 12c) = 0 \Rightarrow \left(5p^2 - \frac{3g}{a}\right)\left(p^2 - \frac{12g}{a}\right) = 0$

$\therefore p_1^2 = \frac{3g}{5a}$ and $p_2^2 = \frac{12g}{a}$

The periods of oscillations are $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$

i.e. $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $2\pi\sqrt{\left(\frac{a}{12g}\right)}$ *i.e.* $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\frac{a}{3g}\right)}$.

Ex. 8. *A uniform rod, of mass 5m and length 2a, turns freely about one end which is fixed, to its other extremity is attached one end of a light string, of length 2a, which carries at its other end a particle of mass m, show that the periods of the small oscillations in a vertical plane are the same as those of simple pendulums of length* $\frac{2a}{3}$ and $\frac{20a}{7}$.

[Meerut 74, 81]

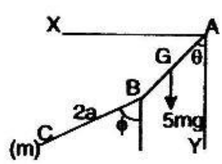
Sol. *Let the string BC and the rod AB make angles ϕ and θ with the vertical at any time t. The particle of mass m is tied to the end C of the string.*

Now $x_c = 2a\sin\theta + 2a\sin\phi$,

$\dot{x}_c = 2a(\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi})$

$y_c = 2a\cos\theta + 2a\cos\phi$.

$\dot{y}_c = -2a(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi})$



$\therefore$ (velocity)^2 of $m = \dot{x}_c^2 + \dot{y}_c^2$

$= 4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$.

Again co-ordinates of G are $(a\sin\theta, a\cos\theta)$.

$\therefore$ (velocity)^2 of $G = a^2\dot{\theta}^2$

Now let T be the kinetic energy, and W the work function of the system, then we have

Total K.E. = K.E. of rod + K.E. of particle of mass m.

$T = \frac{1}{2}\,5m\left(\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \frac{1}{2}m\,.\,4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

$= ma^2\left(\frac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\right)$

and $W = 5mga\cos\theta + mg\,.\,2a(\cos\theta + \cos\phi)$

$$= 7mga\cos\theta + 2mga\cos\phi = 7mag\left(1 - \frac{\theta^2}{2}\right) + 2mag\left(1 - \frac{\phi^2}{2}\right)$$

$\therefore$ Lagrange's θ equation is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{32}{3}\dot\theta + 4\dot\phi\right) = -\frac{7g}{a}\theta \Rightarrow 32\ddot\theta + 12\ddot\phi = -21\frac{g}{a}\theta \qquad ...(1)$$

Lagrange's ϕ equation is given by

$$\frac{d}{dt}(4\dot\phi + 4\dot\theta) = -\frac{2g}{a}\phi \Rightarrow i.e.\ 2\ddot\theta + 2\ddot\phi = -\frac{g}{a}\phi \qquad ...(2)$$

$\therefore$ (1) and (2) $\Rightarrow (32D^2 + 21c)\theta + 12D^2\phi = 0$ ...(3)

and $2D^2\theta + (2D^2 + c)\phi = 0$ where $\frac{g}{a} = c$.

Now eliminating 'ϕ' between (3) and (4), we get

$$[(32D^2 + 21c)(2D^2 + c) - 24D^2]\theta = 0$$

or $[40D^4 + 74cD^2 + 21c^2]\theta = 0$ ...(5)

Now let $\theta = A\cos(pt + B) \Rightarrow D\theta = -pA\sin(pt + B)$.

$$D^2\theta = -p^2A\cos(pt + B) = -p^2\theta \text{ and } D^4\theta = p^4\theta.$$

Substituting these in (5), we get

$$(40p^4 - 74cp^2 + 21c^2)\theta = 0 \text{ i.e. } 40p^4 - 74cp^2 + 21c^2 = 0 \quad \text{as } \theta \neq 0$$

or $(2p^2 - 2c)(20p^2 - 7c) = 0$

i.e. $\left(2p^2 - \frac{3g}{a}\right)\left(20p^2 - 7\frac{g}{a}\right) = 0. \Rightarrow p_1^2 = \frac{3g}{2a}$ and $p_2^2 = \frac{7g}{20a}$.

*Hence length of equivalent pendulums are

$$\frac{g}{p_1^2} \text{ and } \frac{g}{p_2^2} \text{ i.e. } \frac{2a}{3} \text{ and } \frac{20}{7}a.$$

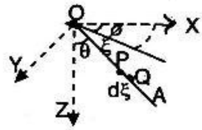
Ex. 9. *A uniform rod, of length 2a, can turn freely about one end, which is fixed. Initially it is inclind at an angle α, to the down-ward drawn vertical and it is set rotating about a vertical axis through its fixed end with angular velocity ω. Show that, during the motion, the rod is always inclined to the vertical at an angle which is > or < α, according as $\omega^2 >$ or $< \frac{3}{4a\cos\alpha}$ and that in each case its motion is inclined between the inclination α and*

$$\cos^{-1}[-n + \sqrt{(1 - 2n\cos\alpha + n^2)}], \text{ when } n = \frac{a\omega^2\sin^2\alpha}{3g}$$

If it be slightly disturbed when revolving steadily at a constant angle α, show that the time of a small oscillation is

$$2\pi\sqrt{\left[\frac{4a\cos\alpha}{3g(1 + 3\cos^2\alpha)}\right]}$$

Sol. The rod OA is turning about the end O. Take a point P on the rod such that $OP = \xi$, and the element $PQ = d\xi$.



$\therefore$ mass of element $PQ = \dfrac{m}{2a} d\xi$,

where m is the mass of the rod

Further at any time t, let the rod be inclined at an angle θ to the vertical and let the plane through the rod and the vertical have turned through angle ϕ from its initial position OX, then co-ordinates of the point P are $x_p = \xi \sin\theta \cos\phi$, $y_p = \xi \sin\theta \sin\phi$, $z = \xi \cos\theta$.

$\therefore v_P^2 =$ (velocity)^2 of $P = \dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2 = \xi^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

and kinetic energy of the element $PQ = \frac{1}{2}\frac{m}{2a} v_P^2$

$$= \frac{1}{2}\frac{m}{2a} d\xi\, (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\, \xi^2$$

Now, let T, *be the K.E. of the rod OA*, then we have

$$T = \frac{1}{2}\frac{m}{2a} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \int_0^{2a} \xi^2 d\xi = \frac{2ma^2}{3} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

or $T = \dfrac{2ma^2}{3} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

Also the work function, $W = mga \cos\theta + C$.

Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\phi} \sin^2\theta)\right) = 0 \text{ i.e. } \frac{d}{dt}[\dot{\phi} \sin^2\theta] = 0 \qquad ...(1)$$

$\Rightarrow \dot{\phi} \sin^2\theta = K$ (constant), ...(2)

Initially $\theta = \alpha$, $\dot{\phi} = \omega$, $\therefore K = \omega \sin^2\alpha$

Thus (2) gives $\dot{\phi} \sin^2\theta = \omega \sin^2\alpha$, ...(3)

and Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) - \dfrac{\partial T}{\partial \theta} = 0$.

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*When $\theta = A\,(pt + B)$, the period of motion is given by $T = \dfrac{2\pi}{p}$. If l is the length of the simple equivalent pendulum, we have

$T = 2\pi\sqrt{(l/g)} \Rightarrow l = \dfrac{g}{p^2}$

$$\Rightarrow \frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\theta}\right) - \frac{2ma^2}{3}\dot{\phi}^2 .2 \sin\theta\cos\theta = -\ mga \sin\theta.$$

$$\Rightarrow \ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta = -\frac{g}{4a}\sin\theta. \qquad ...(4)$$

Eliminating $\dot{\phi}$ between (4) and (3), we have

$$\ddot{\theta} - \frac{\omega^2 \sin^4\alpha}{\sin^3\theta}\cos\dot{\theta} = -\frac{3g}{4a}\sin\theta. \qquad ...(5)$$

$$\Rightarrow \dot{\theta}^2 + \frac{\omega^2 \sin^4\alpha}{\sin^3\theta} = \frac{3g}{4a}\cos\theta + A. \qquad ...(6)$$

Initially $\theta = \alpha$, $\dot{\theta} = 0$, $\quad \therefore\ A = \omega^2 \sin^2\alpha - \frac{3g}{2a}\cos\alpha.$

Substituting this value of A in (6), we get

$$\dot{\theta}^2 + \frac{\omega^2 \sin^4\alpha}{\sin^2\theta} = \frac{3g}{2a}\cos\theta + \omega^2 \sin^2\alpha - \frac{3g}{2a}\cos\alpha,$$

$$\text{or}\quad \dot{\theta}^2 = \omega^2 \sin^2\alpha\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a}(\cos\theta - \cos\alpha)$$

$$= \frac{3ng}{a}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a}(\cos\theta - \cos\alpha)\ \left[\therefore\ n = \frac{a\,\omega^2 \sin^2\alpha}{3g}\right]$$

$$= \frac{3g}{2a}\cdot\frac{\cos\alpha - \cos\theta}{\sin^2\theta}\,[2n(\cos\alpha + \cos\theta) - \sin^2\theta]$$

$$\text{i.e.}\quad \dot{\theta}^2 = \frac{3g}{2a}\cdot\frac{\cos\alpha - \cos\theta}{\sin^2\theta}\,[(\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1)] \qquad ...(7)$$

From (7), we see that $\dot{\theta} = 0$, when

$$(\cos\alpha - \cos\theta)\,[(\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1)] = 0$$

i.e. if either $\cos\alpha - \cos\theta$ *i.e.* $\theta = \alpha$ (the initial position)

or $\cos^2\theta + 2n\cos\theta + 2n\cos\alpha - 1 = 0$

$$\text{i.e.}\quad \cos\theta = \frac{-2n \pm \sqrt{[4n^2 + 4(1 - 2\cos\alpha)]}}{2}$$

$$\text{or}\quad \cos\theta = -n + \sqrt{(1 - 3n\cos\alpha + n^2)} \qquad ...(8)$$

(the other value being inadmissible because that gives value of $\cos\theta$ numerically greater than unity.

Hence the motion is included between $\theta = \alpha$ and $\theta = \theta_1$ where

$$\cos\theta_1 = \{\sqrt{(1 - 2n\cos\alpha + n^2)} - n\}$$

The rod will move above or below its initial position, if $\theta_1 >$ or $< \alpha$

or if $\cos\theta_1 <$ or $> \cos\alpha$

i.e. if $1 - 2n\cos\alpha + n^2 <$ or $> (n + \cos\alpha)^2$

i.e. if $\dfrac{3ng}{a\,\omega^2} <$ or $> 4n \cos\alpha$ *i.e.* if $\omega^2 >$ or $< \dfrac{3g}{4a\cos\alpha}$.

2nd part.

Small oscillations about the steady motion :— The motion will be steady if the rod goes round, inclined at the same angle α with the vertical or mathematically if $\theta = \alpha$ (throughout the motion), then $\dot{\theta} = 0$.

Making these substitutions in (5), we get,

$$-\frac{\omega^2 \sin^4\alpha}{\sin^3\theta}\cos\theta = -\frac{3g}{4a}\sin\theta \text{ i.e. } \omega^2 = \frac{3g}{4a\cos\alpha}$$

When ω^2 has this value and there are small oscillations about the position $\theta = \alpha$, then putting $\theta = \alpha + \psi$ in equation (5) we get

$$\ddot{\psi} = \frac{3g}{4a\cos\alpha}\frac{\sin^4\alpha}{\sin^3(\alpha+\psi)}\cos(\alpha+\psi) - \frac{3g}{4a}\sin(\alpha+\psi)$$

$$= \frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha\cos\psi - \sin\alpha\sin\psi)}{\cos\alpha(\sin\alpha\cos\psi + \cos\alpha\sin\psi)^3} - (\sin\alpha\cos\psi + \cos\alpha\sin\psi)\right]$$

$$= \frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha - \psi\sin\alpha)}{\cos\alpha(\sin\alpha + \psi\cos\alpha)^3} - (\sin\alpha + \psi\cos\alpha)\right], \text{ approximately}$$

$$= -\frac{3g\sin\alpha}{4a}[(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3} - (1+\psi\cot\alpha)], \text{ approx.}$$

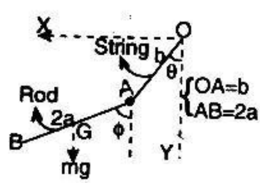
$$= -\frac{3g\sin\alpha}{4}(4\cot\alpha + \tan\alpha)\psi, \text{ app.} = \frac{-3g(1+3\cos^2\alpha)}{4a\cos\alpha}\psi = -\mu\psi \text{ say}$$

$\therefore$ time of small oscillation $= \dfrac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left(\dfrac{4a\cos\alpha}{3g(1+3\cos^2\alpha)}\right)}$.

Ex. 10. *A uniform bar of length 2a is hung from a fixed point by a string of length b fastened to one end of the bar. Show that when the system makes small normal oscillations in a vertical plane, the length l of the equivalent simple pendulum is a root of the quadratic,*

$$l^2 - \left(\frac{4}{3}a + b\right)l + \frac{ab}{3} = 0.$$

Sol. Figure is self explanatory.



At any time t, let the string OA and the rod AB make angles θ and ϕ with the vertical.

$x_G = b\sin\theta + a\sin\phi$.

$y_G = b\cos\theta + a\cos\phi$.

$\therefore \dot{x}_G^2 + \dot{y}_G^2 =$ (velocity)^2 of G

$= b^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2ab\,\dot{\theta}\dot{\phi}\cos(\theta - \phi)$

$$= b^2\dot\theta^2 + a^2\dot\phi^2 + 2ab\,\dot\theta\dot\phi \qquad [\because\ \theta \text{ and } \phi \text{ are small}]$$

Now let T be the kinetic energy and W the work function of the system, then we easily obtain

$$W = mg\,[b\cos\theta + a\cos\phi]$$

and $$T = \tfrac{1}{2}m\left[\frac{a^2}{3}\dot\phi^2 + b^2\dot\theta^2 + a^2\dot\phi^2 + 2ab\,\dot\theta\,\dot\phi\right]$$

$$= \tfrac{1}{2}m\left[\frac{4a^2}{3}\dot\phi^2 + b^2\dot\theta^2 + 2ab\,\dot\theta\,\dot\phi\right]$$

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot\theta}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

$$\Rightarrow \frac{d}{dt}\{m\,(b^2\dot\theta + ab\,\dot\phi)\} = -\,mgb\,\theta. \qquad \{\because\ \sin\theta = 0\}$$

$$\Rightarrow b\,\ddot\theta + a\,\dot\phi = -\,g\,\theta \qquad \text{...(1)}$$

Lagrange's ϕ-equation is given by $\dfrac{d}{dt}\left\{m\left(\dfrac{4}{3}a^2\dot\phi + ab\,\theta\right)\right\} = -\,mag\,\phi$

$$\Rightarrow 4a\,\ddot\phi + 3b\,\ddot\theta = -\,3g\,\phi\,. \qquad \text{...(2)}$$

Equations (1) and (2) again can be written as

$$(b\,D^2 + g)\ \theta + a\,D^2\ \phi = 0 \qquad \text{...(3)}$$

and $$3bD^2\,\theta + (4aD^2 + 3g)\ \phi = 0\,. \qquad \text{...(4)}$$

Eliminating ϕ between these equations, we obtain

$$[(b\,D^2 + g)\ (4a\,D^2 + 3g) - 3ab\,D^4]\,\theta = 0$$

i.e. $$[ab\,D^4 + (4a + 3b)\ gD^2 + 3g^2]\,\theta = 0. \qquad \text{...(5)}$$

Now let $\theta = A\cos\left[\sqrt{\left(\dfrac{g}{l}\right)}\,t + B\right]$

where l is the length of the simple equivalent pendulum.

Then $D\,\theta = -\sqrt{\left(\dfrac{g}{l}\right)}\,A\sin\left[\sqrt{\left(\dfrac{g}{l}\right)}\,t + B\right]$

$$D^2\,\theta = -\,\frac{g}{l}A\cos\left[\sqrt{\left(\frac{g}{l}\right)}\,t + B\right] = -\,\frac{g}{l}\,\theta \text{ and } D^4\,\theta = \frac{g^2}{l^2}\,\theta,$$

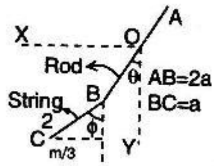
$$\therefore\ (5) \Rightarrow \left[ab\frac{g^2}{l^2} - (4a + 3b)\frac{g^2}{l} + 3g^2\right]\theta = 0$$

$$\Rightarrow 3l^2 - (4a + 3b)\ l + ab = 0 \qquad \{\because\ \theta \neq 0\}$$

$$\Rightarrow l^2 - \left(\frac{4}{3}a + b\right) l + \frac{ab}{3} = 0\,.$$

Ex. 11. *A uniform straight rod of length 2a, is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string, of length a, to one end of the rod ; show that one period of principle oscillation is* $(\sqrt{5} + 1)\,\pi\sqrt{\left\{\dfrac{a}{g}\right\}}$.

Sol. Figure is self explanatory.



At time t, let θ and ϕ be the inclinations of the rod and the string to the vertical.

Co-ordinates of C are

$x_C = a \sin\theta + a \sin\phi$ and

$y_C = a \cos\theta + a \cos\phi$.

$\therefore\ \dot{x}_C = a \cos\theta\,\dot{\theta} + a \cos\phi\,\dot{\phi}$. and $\dot{y}_C = -a \sin\theta\,\dot{\theta} - a \sin\phi\,\dot{\phi}$

$\Rightarrow \dot{x}_C^2 + \dot{y}_C^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2 \cos(\theta - \phi)\,\dot{\theta}\dot{\phi}$

$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$

[neglecting higher powers of small quantites]

$\therefore$ (velocity)^2 of the particle C $= v_C^2 = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$.

And velocity of the C.G. of the rod *i.e.* of O, is zero.

Now let T, be the kinetic energy and W, the work function of the system, then we easily get $W = \dfrac{mg}{3}(a\cos\theta + a\cos\phi) + C$

and $T = \dfrac{1}{2} m \dfrac{a^2}{3}\dot{\theta}^2 + \dfrac{1}{2}\left(\dfrac{m}{3}\right)[a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}]$

$= \dfrac{ma^2}{6}[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}]$.

$\therefore$ Lagrange's θ-equation is given by

$$\frac{d}{dt}\left(\frac{2ma^2}{3}\dot{\theta} + \frac{ma^2}{3}\dot{\phi}\right) = -\frac{mga}{3}\theta \Rightarrow 2\ddot{\theta} + \ddot{\phi} = -\frac{g}{a}\theta \qquad ...(1)$$

While Lagrange's ϕ-equation gives $\dfrac{d}{dt}\left[\dfrac{ma^2}{3}\dot{\phi} + \dfrac{ma^2}{3}\dot{\theta}\right] = -\dfrac{mga}{3}\phi$

i.e $\ddot{\theta} + \ddot{\phi} = -\dfrac{g}{a}\phi$. ...(2)

Equations (1) and (2) again give

$(2D^2 + c)\theta + D^2\phi = 0$ ...(3) and $D^2\theta + (D^2 + c)\phi = 0$ ...(4)

where $c = \dfrac{g}{a}$.

Eliminating ϕ in between (3) and (4), we get

$\{(D^2 + c)(2D^2 + c) - D^4\}\theta = 0$ *i.e.* $[D^4 + 3cD^2 + c^2]\theta = 0$. ...(5)

To solve (5), let $\theta = A\cos(pt + B) \therefore D\theta = -pA\sin(pt + B)$,

$D^2\theta = -p^2 A\cos(pt + B) = -p^2\theta$ and $D^4\theta = p^4\theta$.

With these substitutions, (5) gives

$(p^4 - 3cp^2 + c^2)\theta = 0 \Rightarrow p^4 - 3cp^2 + c^2 = 0$ $(\because\ \theta \neq 0)$

$$\therefore\ p^2 = \frac{3c \pm \sqrt{(9c^2 - 4c^2)}}{2} = \left(\frac{3 \pm \sqrt{5}}{2}\right) c = \frac{(3 \pm \sqrt{5})}{2} \cdot \frac{g}{a}$$

$$\therefore\ p_1^2 = \frac{3 - \sqrt{5}}{2} \cdot \frac{g}{a} \text{ and } p_2^2 = \frac{3 + \sqrt{5}}{2} \cdot \frac{g}{a}.$$

$\therefore$ one period of principal oscillations corresponding to p_1, is given by

$$\frac{2\pi}{p_1} = 2\pi \sqrt{\left(\frac{2}{3 - \sqrt{5}} \cdot \frac{a}{g}\right)} = 2\pi \sqrt{\left(\frac{a}{g}\right)} \sqrt{\left\{\frac{2(3 + \sqrt{5})}{9 - 5}\right\}}$$

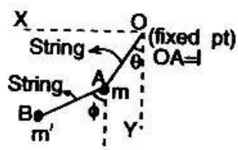
$$= 2\pi \sqrt{\left(\frac{a}{g}\right)} \sqrt{\left(\frac{6 + 2\sqrt{5}}{4}\right)} = 2\pi \sqrt{\left(\frac{a}{g}\right)} \sqrt{\left\{\frac{(\sqrt{5} + 1)^2}{4}\right\}}$$

$$= (\sqrt{5} + 1)\pi \sqrt{\left(\frac{a}{g}\right)}.$$

Ex. 12. *A mass m hangs from a fixed point by a light string of length l and a mass m′ hangs from m by a second string of length l′. For oscillations in a vertical plane, show that the periods of the principal oscillations are the values of* $\frac{2\pi}{n}$ *where n is given by the equation*

$$n^4 - gn^2 \frac{m + m'}{m}\left(\frac{1}{l} + \frac{1}{l'}\right) + g^2 \frac{m + m'}{ml'l} = 0.$$

Sol. At any time t, let the strings be inclined at angles θ and ϕ to the vertical. Co-ordinates of m are $(l \sin\theta, l \cos\theta)$.



$\therefore$ (velocity)^2 of $m = l^2 \dot{\theta}^2$

while co-ordinates of m' are

$$x_B = l \sin\theta + l' \sin\phi, \Rightarrow$$

$$\dot{x}_B = l \cos\theta\, \dot{\theta} + l' \cos\phi\, \dot{\phi}$$

$$y_B = l \cos\theta + l' \cos\phi \Rightarrow$$

$$\dot{y}_B = -(l \sin\theta\, \dot{\theta} + l' \sin\phi\, \dot{\phi})$$

$$\therefore\ (\text{velocity})^2 \text{ of } m' = \dot{x}_B^2 + \dot{y}_B^2 = l^2 \dot{\theta}^2 + l'^2 \dot{\phi}^2 + 2ll'\dot{\theta}\dot{\phi}$$

[$\because$ θ and ϕ are small]

Now let T, be the kinetic energy and, W the work function, of the system, then we have

$$W = mgl \cos\theta + m'g(l \cos\theta + l' \cos\phi)$$
$$= gl(m + m') \cos\theta + m'gl' \cos\phi$$

$$\text{and } T = \tfrac{1}{2} m l^2 \dot{\theta}^2 + \tfrac{1}{2} m' [l^2 \dot{\theta}^2 + l'^2 \dot{\theta}^2 + 2ll' \dot{\theta}\dot{\phi}]$$

$$= \tfrac{1}{2} [(m + m') l^2 \dot{\theta}^2 + m'l'^2 \dot{\theta}^2 + 2m'll' \dot{\theta}\dot{\phi}]$$

Lagrange's θ-equation is given by $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) - \dfrac{\partial T}{\partial \theta} = \dfrac{\partial W}{\partial \theta}$

$$\Rightarrow \frac{d}{dt}\left[(m+m')\, l^2 \dot{\theta} + m'\, ll'\, \dot{\phi}\right] = -\, gl\,(m+m')\,\theta.$$

$$\Rightarrow \{m+m'\}\, l\ddot{\theta} + m'l'\,\ddot{\phi} = -\, g\,(m+m')\,\theta \qquad ...(1)$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left[m'l'^2\,\dot{\phi} + m'\, ll'\,\dot{\theta}\right] = -\, m'\, gl'\,\phi$$

$$\Rightarrow l'\,\ddot{\phi} + l\,\ddot{\theta} = -\, g\,\phi \qquad ...(2)$$

Equation (1) and (2) again give

$$(m+m')\,(lD^2+g)\,\theta + m'\,l'\,D^2\phi = 0 \qquad ...(3)$$

$$lD^2\theta + (l'\,D^2+g)\,\phi = 0 \qquad ...(4)$$

Eliminating ϕ, we get $[(m+m')\,(lD^2+g)\,(l'D^2+g) - m'll'D^4]\,\theta = 0$

i.e. $[m\,ll'\,D^4 + (m+m')\,(l+l')\,gD^2 + (m+m')\,g^2]\,\theta = 0 \qquad ...(5)$

Now let $\theta = A\cos(nt+B)$; $\therefore$ $D\theta = -\,nA\sin(nt+B)$

$$D^2\theta = -\,n^2 A\cos(nt+B) = -\,n^2\theta \text{ and } D^4\theta = n^4\theta \qquad ...(6)$$

$\therefore$ (5) and (6) give $\quad mll'\,n^4 - (m+m')\,(l+l')\,gn^2 + (m+m')\,g^2 = 0$

$$\text{or } n^4 - \frac{m+m'}{m}\left(\frac{1}{l}+\frac{1}{l'}\right)gn^2 + \frac{(m+m')\,g^2}{mll'} = 0. \qquad ...(7)$$

Ex. 13. (a) *A mass M hangs from a fixed point at the end of a very long string whose length l is a, to M is suspended a mass m by a string whose length l is small compound with a; prove that the time of a small oscillation of m is* $\quad 2\pi\sqrt{\left(\dfrac{M}{M+m}\cdot\dfrac{l}{g}\right)}$

Sol. Here, we have $m = M,\ m' = m,\ l = a,\ l' = l$.

$$\therefore\ n^4 - \frac{M+m}{M}\left(\frac{l}{a}+\frac{1}{l}\right)gn^2 + \frac{(M+m)\,g^2}{Mal} = 0$$

$$\text{i.e. } n^4 - \frac{M+m}{M}\left(\frac{l}{a}+1\right)\frac{g}{l}\,n^2 + \frac{(M+m)\,g^2}{M\,l^2}\cdot\frac{l}{a} = 0 \qquad ...(8)$$

But a is large compared to l $\therefore$ $\dfrac{l}{a}\to 0$

Hence the equation (8), gives

$$n^4 - \frac{M+m}{M}\cdot\frac{g}{l}\cdot n^2 = 0 \text{ i.e } n^2 = \frac{M+m}{M}\cdot\frac{g}{l}$$

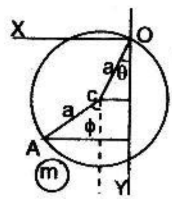
$$\therefore\ \text{Time of a small oscillation} = \frac{2\pi}{n} = 2\pi\sqrt{\left\{\frac{M}{M+m}\cdot\frac{l}{g}\right\}}.$$

Ex. 13. (b) *At the lowest point of a smooth circular tube, of mass M and radius a, is placed a particle of mass M', the tube hangs in a vertical plane from its highest point, which is fixed, and can turn freely in its own plane* *about this point. If the system be slightly displaced, show that the periods of the two independent oscillations of the system are*

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{Mag^{-1}}{M+M'}\right)}. \qquad \textbf{(Meerut 1983)}$$

And that for one principal mode of oscillations, the particle remains at rest relative to the tube and for the other, the centre of gravity of the particle and the tube remain at rest.

Sol. Let C be the centre of the tube and A the position of the particle M' at time t when OC and CA make angle 3 θ and ϕ with the vertical



$\therefore\ x_A = a \sin\theta + a\sin\phi$,

$y_A = a\cos\theta + a\cos\phi$.

and

$(\text{velocity})^2$ of $A = \dot{x}_A^{\,2} + \dot{y}_A^{\,2}$

$$= (a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi})^2 + (-a\sin\theta\dot{\theta} - a\sin\phi\dot{\phi})^2$$

$$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi) \quad = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$$

[neglecting small quantities of the higher order].

Also $C \equiv (a\sin\theta,\ a\cos\theta)$

$(\text{velocity})^2$ of $C = (a\cos\theta\dot{\theta})^2 + (-a\sin\theta\dot{\theta})^2 = a^2\dot{\theta}^2$

Now let T, be the kinetic energy and W the work function of the system then we readily obtain $W = Mga\cos\theta + M'g(a\cos\theta + a\cos\phi) + K$
$= (M+M')\,ga\cos\theta + M'\,ga\cos\phi + K$

$T =$ K.E. of circular tube + K.E. of particle

$$= \frac{1}{2}M\,(a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \frac{1}{2}M\,(a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi})$$

$$= \frac{2M+M'}{2}a^2\dot{\theta}^2 + \frac{1}{2}M'a^2\dot{\phi}^2 + M'a^2\dot{\theta}\dot{\phi}. \qquad \text{...(2)}$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left[(2M+M')\,a^2\dot{\theta} + M'a^2\dot{\phi}\right] = -(M+M')\,ga\theta.$$

$$\Rightarrow (2M+M')\,\ddot{\theta} + M'\ddot{\phi} = -(M+M')\frac{g}{a}\theta. \qquad \text{...(3)}$$

Also Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left[M'a^2\dot{\phi} + M'a^2\dot{\theta}\right] = -M'ga\ddot{\phi} \Rightarrow \phi + \ddot{\theta} = -\frac{g}{a}\phi \qquad \text{...(4)}$$

Equations (3) and (4) can be re-written as

$[(2M+M')D^2+(M+M')c]\theta+M'D^2\phi=0$ ...(5)

and $D^2\theta+(D^2+c)\phi=0$ where $c=\dfrac{g}{a}$. ...(6)

Eliminating ϕ between these two equations, we get

$$[\{(2M+M')D^2+(M+M')c\}(D^2+c)-M'D^4]\theta=0$$

i.e. $[2MD^4+c(3M+2M')D^2+c^2(M+M')]\theta=0$. ...(7)

To solve (7),

let $\theta=A\cos(pt+B)$; $D\theta=-pA\sin(pt+B)$

$D^2\theta=-p^2A\cos(pt+B)=-p^2\theta$ and $D^4\theta=p^4\theta$ ...(8)

$\therefore$ (7) and (8) give $[2Mp^4-c(3M+2M')p^2+c^2(M+M')]\theta=0$

i.e. $2Mp^4-c(3M+2M')p+c^2(M+M')=0$. $[\because \theta\neq 0]$

which again gives $(2p^2-c)\,[Mp^2-c(M+M')]=0$

$\therefore$ $p_1^2=\dfrac{c}{2}$ and $p_2^2=\dfrac{c(M+M')}{M}$

i.e. $p_1^2=\dfrac{g}{2a}$ and $p_2^2=\dfrac{M+M'}{M}\dfrac{g}{a}$. $\left(\because c=\dfrac{g}{a}\right)$

Hence periods of oscillations are given by

$\dfrac{2\pi}{p_1}$ and $\dfrac{2\pi}{p_2}$ *i.e.* by $2\pi\sqrt{(2a/g)}$ and $2\pi\sqrt{\left(\dfrac{M}{(M+M')}\dfrac{a}{g}\right)}$.

Multiplying (6) by λ and adding to (5), we have

$D^2\{(2M+M'+\lambda)\theta+(M'+\lambda)\phi\}=-\{(M+M')\theta+\lambda\phi\}$ ....(9)

Now chose λ such that

$$\frac{2M+M'+\lambda}{M'+\lambda}=\frac{M+M'}{\lambda}\Rightarrow\lambda=M' \text{ and } \lambda=-(M+M').$$

taking $\lambda=M'$, equation (9) reduces to

$$D^2\{(M+M')\theta+M'\phi\}=-\tfrac{1}{2}c\,\{(M+M')\theta+M'\phi\}$$

and when $\lambda=-(M+M')$, equation (9) reduces to

$$D^2(\theta-\phi)=-\frac{M+M'}{M}c(\theta-\phi)$$

$\therefore$ Principal co-ordinates are $\theta-\phi$ and $\{(M+M')\theta+M'\phi\}$.

For the first mode, $\theta-\phi=0$. *i.e.* $\theta=\phi$. This shows that the particle is at rest relative to the tube. For the second mode we have $(M+M')\theta+M'\phi=0$.

Further, the x-coordinates of C.G. of the particle and the tube

$$=\frac{Ma\sin\theta+M'(a\sin\theta+a\sin\phi)}{M+M'}$$

$$=\frac{a}{M+M'}\{M\theta+M'(\theta+\phi)\}$$ (since θ and ϕ are small, $\therefore$ $\sin\theta=\theta$ and $\sin\phi=\phi$)

$$=\frac{a}{M+M'}\{(M+M')\theta+M'\phi\}=0$$ [using above results]

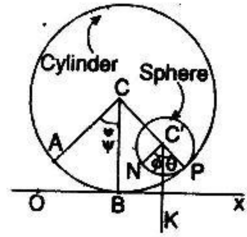
$\Rightarrow$ The common C.G. of the particle and the tube remains at rest.

Ex. 14. *A perfectly rough sphere lying inside a hollow cylinder, which rests on a perfectly rough plane, is slightly displaced from its position of equilibrium. Show that the time of a small oscillation is*

$$2\pi\sqrt{\left(\frac{a-b}{g}\cdot\frac{4M}{10M+7m}\right)}$$

where a is the radius of the cylinder, b that of the sphere, and M, m are the masses of the cylinder and the sphere.

Sol. Let CA be a line fixed in the cylinder and $C'N$, a line fixed in the sphere, which were initially vertical. Let these lines, after a time t, make angles ψ and ϕ with the vertical. Further let θ be the angle which the line joining centre makes with the vertical.



Initially O was the point of contact with the horizontal plane, which is regarded as origin. There is no slipping ; between the cylinder, horizontal plane and between the cylinder, sphere ; therefore, we get

$OB =$ arc $AB = a\psi$ and arc $AP =$ arc NP

i.e. $a(\theta+\psi) = b(\theta+\phi)$, *i.e.* $b\dot{\phi} = (a-b)\dot{\theta} + a\dot{\psi}$.

Referred to the horizontal and vertical through O as co-ordinate axes, we have

$x_{C}' = a\psi + a(a-b)\sin\theta,\ y_{C}' = a-(a-b)\cos\theta.$

(co-ordinates of C')

$\therefore$ (velocity)^2 of C'

$$\dot{x}_{C}'^2 + \dot{y}_{C}^2 = \{a\dot{\psi} + (a-b)\cos\theta\dot{\theta}\}^2 + man\{(a-b)\sin\theta\,\dot{\theta}\}^2$$

$$= a^2\dot{\psi}^2 + (a-b)^2\dot{\theta}^2 + 2a(a-b)\dot{\psi}\dot{\theta}\cos\theta$$

$$= a^2\dot{\psi}^2 + (a-b)^2\dot{\theta}^2 + 2a(a-b)\dot{\psi}\dot{\theta}\ . \qquad ...(1)$$

Also co-ordinates of C are $(a\psi, a)$.

$\therefore$ (velocity)^2 of $C = (a\dot{\psi})^2 = a^2\dot{\psi}^2$. ...(2)

Now let T be the kinetic energy and W, the work function. Then we have

$$W = (a-b)\,mg\cos\theta + C \qquad ...(3)$$

and $T = \frac{1}{2}M\left(v_C^2 + a^2\dot{\psi}^2\right) + \frac{1}{2}m\left(v_{C'}^2 + \frac{2b^2}{5}\dot{\phi}^2\right)$

$\Rightarrow T = \frac{1}{2}M\,(a^2\dot{\psi}^2 + a^2\dot{\psi}^2)$

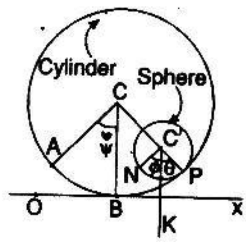
$\Rightarrow$ The common C.G. of the particle and the tube remains at rest.

Ex. 14. *A perfectly rough sphere lying inside a hollow cylinder, which rests on a perfectly rough plane, is slightly displaced from its position of equilibrium. Show that the time of a small oscillation is*

$$2\pi\sqrt{\left(\frac{a-b}{g}\cdot\frac{4M}{10M+7m}\right)}$$

where a is the radius of the cylinder, b that of the sphere, and M, m are the masses of the cylinder and the sphere.

Sol. Let CA be a line fixed in the cylinder and $C'N$, a line fixed in the sphere, which were initially vertical. Let these lines, after a time t, make angles ψ and ϕ with the vertical. Further let θ be the angle which the line joining centre makes with the vertical.



Initially O was the point of contact with the horizontal plane, which is regarded as origin. There is no slipping ; between the cylinder, horizontal plane and between the cylinder, sphere ; therefore, we get $OB=$ arc $AB = a\psi$ and arc $AP=$ arc NP

i.e. $a(\theta+\psi)=b(\theta+\phi)$, *i.e.* $b\dot{\phi}=(a-b)\dot{\theta}+a\dot{\psi}$.

Referred to the horizontal and vertical through O as co-ordinate axes, we have

$x_{C}'=a\psi+a(a-b)\sin\theta,\ y_{C}'=a-(a-b)\cos\theta.$

(co-ordinates of C')

$\therefore$ (velocity)^2 of C'

$$\dot{x}_{C}'^2+\dot{y}_{C}'^2=\{a\dot{\psi}+(a-b)\cos\theta\dot{\theta}\}^2+ man/\{(a-b)\sin\theta\,\dot{\theta}\}^2$$

$$=a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}\cos\theta$$

$$=a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}\ . \quad ...(1)$$

Also co-ordinates of C are $(a\psi, a)$.

$$\therefore \text{(velocity)}^2 \text{ of } C=(a\dot{\psi})^2=a^2\dot{\psi}^2. \quad ...(2)$$

Now let T be the kinetic energy and W, the work function. Then we have

$$W=(a-b)\,mg\cos\theta+C \quad ...(3)$$

and $T=\frac{1}{2}M\left(v_C^2+a^2\dot{\psi}^2\right)+\frac{1}{2}m\left(v_{C'}^2+\frac{2b^2}{5}\dot{\phi}^2\right)$

$\Rightarrow T=\frac{1}{2}M\,(a^2\dot{\psi}^2+a^2\dot{\psi}^2)$

$$+\frac{1}{2}m\left[\frac{2b^2}{5}\dot{\phi}^2+a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}\right]$$

$$=Ma^2\dot{\psi}^2+\frac{1}{2}m\ [\tfrac{2}{5}\{(a-b\dot{\theta}+a\dot{\psi}\}^2+a^2\dot{\psi}^2$$

$$+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}]$$

$$=\frac{10M+7m}{10}a^2\dot{\psi}^2+\frac{7m}{10}[(a-b)^2\theta^2+2a(a-b)\dot{\theta}\dot{\psi}] \qquad ...(4)$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left[\frac{7m}{5}\{(a-b)^2\dot{\theta}+a(a-b).\dot{\psi}\}\right]=-(a-b)mg\sin\theta$$

$$\Rightarrow 7(a-b)\ddot{\theta}+5a\ddot{\psi}=-5g\theta \text{ (as } \theta \text{ is small).} \qquad ...(5)$$

Also Lagrange's ψ-equation gives

$$\frac{d}{dt}\left[\frac{10M+7m}{5}a^2\dot{\psi}+\frac{7m}{5}a(a-b)\ \theta\right]=0$$

$$\Rightarrow (10M+7m)a\ddot{\psi}+7m(a-b)\ddot{\theta}=0$$

$$\Rightarrow 7m(a-b)\ddot{\theta}+(10M+7m)a\ddot{\psi}=0. \qquad ...(6)$$

Eliminating $\ddot{\psi}$ between (5) and (6), we get

$$[7(10M+7m)-49m](a-b)\ddot{\theta}=-5(10M+7m)g\theta$$

or $\quad 70M(a-b)\ddot{\theta}=-5(10M+7m)g\theta$

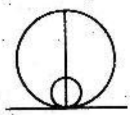
or $\quad \ddot{\theta}=-\dfrac{(10M+7m)}{14M}\cdot\dfrac{g}{a-b}\theta=-\mu\theta$ say,

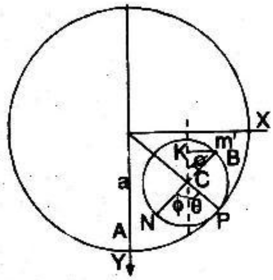
$\therefore$ time of a small oscillation $=\dfrac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left(\dfrac{a-b}{g}\cdot\dfrac{14M}{10M+7m}\right)}.$

Ex. 15. *A perfectly rough sphere, of mass m and radius b, rests at the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length*

$$(a-b)\frac{4m'+\frac{7}{5}m}{m+m'\left(2-\dfrac{a}{b}\right)}$$ **(Agra 1995, 89)**

Sol. O is the centre of the fixed cavity, C the centre of the sphere and at B, a particle of m′ is attached. Initially NB was vertical, and N conicided with A.





After the time t, let BN or CN, a line fixed in the sphere make an angle ϕ with the vertical. But there is no slipping between the cylinder and the sphere.

$\therefore$ arc $AP =$ arc NP, *i.e.* $a\theta = b(\theta + \phi)$

i.e. $b\phi = (a-b)\theta \Rightarrow b\dot{\phi} = (a-b)\dot{\theta}$

Now co-ordinates of B (i.e. of m') are

$x_B = (a-b)\sin\theta + b\sin\phi$, $y_B = (a-b)\cos\theta - b\cos\phi$,

and co-ordinates of the centre C are

$\{(a-b)\sin\theta, (a-b)\cos\theta\}$

$\therefore$ (velocity)^2 of $B = \dot{x}_B^2 + \dot{y}_B^2$

$$= \{(a-b)\cos\theta\dot{\theta} + b\cos\phi\dot{\phi}\}^2 + \{-(a-b)\sin\theta\dot{\theta} + b\sin\phi\dot{\phi}\}^2$$

$$= (a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2b(a-b)\dot{\theta}\dot{\phi}$$

[neglecting small quantities of second and higher orders]

and (velocity)^2 of $C = \{(a-b)\cos\theta\dot{\theta}\}^2 + \{-(a-b)\sin\theta\dot{\theta}\}^2$

$$= (a-b)^2\dot{\theta}^2$$

Now let T, be the kinetic energy and W the work function of the system, then we readily obtain

$$W = mg(a-b)\cos\theta + m'g\{(a-b)\cos\theta - b\cos\phi\} + D$$

$$= (a-b)(m+m')g\cos\theta - m'gb\cos\phi + D$$

$$= (a-b)(m+m')g\cos\left(\frac{b}{a-b}\phi\right) - m'gb\cos\phi + D$$

$$\{\because \quad (a-b)\theta = b.\phi\}$$

and $T = \frac{1}{2}m\left[\frac{2b^2\dot{\phi}^2}{5} + (a-b)^2\dot{\theta}^2\right] + \frac{1}{2}m'\{(a-b)^2\dot{\theta}^2 + b^2\dot{\phi}^2$

$$+ 2b(a-b)\dot{\theta}\dot{\phi}\}$$

$$= \frac{1}{2}m\left[\frac{2b^2\dot{\phi}^2}{5} + b^2\dot{\phi}^2\right] + \frac{1}{2}m'[b^2\dot{\phi}^2 + b^2\dot{\phi}^2 + 2b^2\dot{\phi}^2]$$

$$[\because \quad (a-b)\,\dot{\theta} = b\,\dot{\phi}\,]$$

$$= \frac{b^2}{10}(7m + 20m')\dot{\phi}^2.$$

Further Lagrange's ϕ-equation gives $\dfrac{d}{dt}\left[\dfrac{b^2}{5}(7m + 20m')\,\dot{\phi}\right]$

$$= -(a-b)(m+m')g\,\frac{b}{a-b}\sin\left(\frac{b}{a-b}\phi\right) + m'\,g\,b\sin\phi$$

$$= -(m+m')bg\sin\left(\frac{b\,\phi}{a-b}\right) + m'\,g\,b\sin\phi$$

$$\Rightarrow \frac{b^2}{5}(7m + 20m')\,\ddot{\phi} = -(m+m')\,bg\,.\frac{b}{a-b}\phi + m'\,g\,b\,\phi \text{ app.}$$

$$\Rightarrow \left(4m' + \frac{7m}{5}\right)\ddot{\phi} = -\frac{g}{a-b}\left[m + m'\left(2 - \frac{a}{b}\right)\right]\phi, \text{ app.}$$

$$\Rightarrow \ddot{\phi} = -\frac{g}{a-b}\cdot\frac{m + m'\left(2-\frac{a}{b}\right)}{4m' + \frac{7}{5}m}\phi = -\mu\,\phi \qquad [\text{form } \ddot{x} = -\mu x]$$

$\therefore$ Length of the simple equivalent pendulum $= \dfrac{g}{\mu}$

$$= (a-b)\frac{4m' + \frac{7}{5}m}{m + m'\left(2 - \frac{a}{b}\right)}.$$

Ex. 16. *A perfectly rough sphere rests at the lowest point of a fixed spherical cavity of double its own radius. To the highest point of the movable sphere is attached a particle of mass $\frac{7}{20}$ times that of the sphere and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length $\frac{14}{5}$ times the radius of the sphere.* **[Agra 91]**

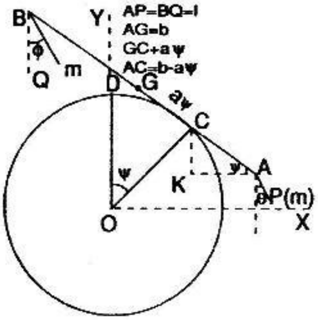
Sol. Just like EX. 15.

Put $m' = \dfrac{7m}{20}$ in Ex. 15 and proceed similarly to get the required result.

Ex. 17. *A plank, of mass M, radius of gyration k and length 2b, can swing like a sea-saw across a perfectly rough fixed cylinder of radius a. At its ends hang two particles each of mass m, by strings of length l. Show that, as the system swings, the lengths of its simple equivalent pendulum are*

l and $\dfrac{Mk^2 + 2mb^2}{(M+2m)\,a}$.

Sol. Let the strings be AP and BQ such that their free extremities are having particles of mass m. Initially G was at D and AP, BQ were vertical.



After a time t, when the plank has turned through an angle ψ with the horizontal. let the strings be inclined at angle θ and ϕ to the vertical.
But there is no slipping between the plank and the cylinder, so we have $CG =$ arc $DC = a\psi$

Now $x_P = a\sin\psi + (b - a\psi)\cos\psi + l\sin\theta = b + l\theta - \frac{1}{2}b\psi^2$ ...(1)

$y_P = a\cos\psi - (b - a\psi)\sin\psi - l\cos\theta$

$= a(1 - \frac{1}{2}\psi^2) - (b - a\psi)\psi - l(1 - \frac{1}{2}\theta^2)$

neglecting higher powers of θ and ψ.

$= a - l - b\psi + \frac{1}{2}a\psi^2 - \frac{1}{2}l\theta^2$ ...(2)

$\therefore \dot{x}_P = l\dot{\theta} - b\psi\dot{\psi}$ and $\dot{y}_P = -b\dot{\psi} + a\psi\dot{\psi} + l\theta\dot{\theta}$

$\therefore$ (velocity)^2 of $P = \dot{x}_P^2 + \dot{y}_P^2 = l^2\dot{\theta}^2 + b^2\dot{\psi}^2$

Thus kinetic energy of m at $P = \frac{1}{2}m\,(l^2\dot{\theta}^2 + b^2\dot{\psi}^2)$

*Smilarly kinetic energy of m at $Q = \frac{1}{2}m\,(l^2\dot{\phi}^2 + b^2\dot{\psi}^2)$

Also y-co-ordinate of $Q = a - l + b\psi + \frac{1}{2}a\psi^2 + \frac{1}{2}l\phi^2 = y_Q$ say

[put $-\psi$ for ψ and ϕ for θ in (2)]

the co-ordinates of G are $\{a\sin\psi - a\psi\cos\psi, a\cos\psi + a\psi\sin\psi\}$

(neglecting higher powers of ψ as it is small)

i.e. $\left\{\dfrac{a\psi^2}{2}, a + \dfrac{a\psi^2}{2}\right\}$.

$\therefore$ (velocity)^2 of $G = a^2\psi^2\dot{\psi}^2$ $[\because \dot{x} = a\dot{\psi}\psi, \dot{y} = a\psi\dot{\psi}]$

(neglecting smaller, quantities). Thus kinetic energy of the plank

$= \frac{1}{2}M\,(k^2\dot{\psi}^2 + a^2\psi^2\dot{\psi}^2) = \frac{1}{2}Mk^2\dot{\psi}^2$, approximately

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*To obtain the corresponding quantities for the point Q, we shall write $-\psi$ for ψ and ϕ for θ.

Now let T be the kinetic energy and W, the work function of the system, then we obtain $W = -mg\left(a - l - b\,\psi + \dfrac{a\psi^2}{2} + l\dfrac{\theta^2}{2}\right)$

$$- mg\left(a - l + b\,\psi + \frac{a\psi^2}{3} + l\frac{\phi^2}{2}\right) - mg\left(a + \frac{a\psi^2}{2}\right)$$

$$= K - \tfrac{1}{2}a\,(M + 2m)\,g\,\psi^2 - \tfrac{1}{2}mgl\,(\theta^2 - \phi^2)$$

and $\quad T = \frac{1}{2}m\,(l^2\dot\theta^2 + b^2\dot\psi^2) + \frac{1}{2}m\,(l^2\dot\phi^2 + b^2\dot\psi^2) + \frac{1}{2}MK^2\dot\psi^2$

$$= \frac{1}{2}(MK^2 + 2mbq^2)\,\dot\psi^2 + \tfrac{1}{2}Ml^2\dot\theta^2 + \tfrac{1}{2}ml^2\dot\phi^2$$

Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot\theta}\right) - \dfrac{\partial T}{\partial \theta} = \dfrac{\partial W}{\partial \theta}$

$$\Rightarrow \frac{d}{dt}(ml^2\dot\theta^2) = -\,mgl\,\theta \Rightarrow \ddot\theta = -\frac{g}{l}\theta,$$

$\therefore$ length of the simple equivalent pendulum $= l$

Also Lagrange's $\psi-$ eq^n gives

$$\frac{d}{dt}\{(Mk^2 + 2mb^2)\,\dot\psi\} = -\,a\,(M + 2m)\,g\,\psi \;\Rightarrow \ddot\psi = -\,ag\frac{M + 2m}{Mk^2 + 2mb^2}\psi$$

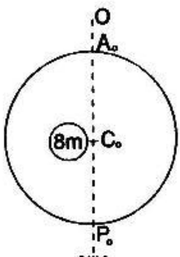
$\therefore$ length of the other equivalent pendulum $= \dfrac{Mk^2 + 2mb^2}{(M + 2m)\,a}$.

Ex. 18. *A smooth circular wire, of mass 8m and radius a, swings in a vertical plane, being suspended by an inxtensible string of length a attached to one point of it, a particle of mass m can slide on the wire. Prove that the periods of small oscillations are*

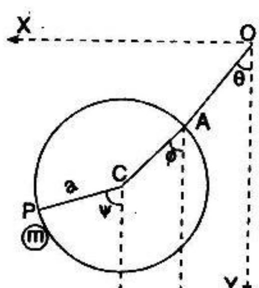
$2\pi\sqrt{\left(\dfrac{8a}{3g}\right)},\ 2\pi\sqrt{\left(\dfrac{a}{3g}\right)},\ 2\pi\sqrt{\left(\dfrac{8a}{9g}\right)}$. **(Agra 91)**

Sol. At any time t, let the string OA, and the radius AC be inclined at angles θ and ϕ with the vertical and further let the radius to the particle (m) be inclined at an angle ψ with the vertical.

Now co-ordinates of C are



[(i)]
Rest position



[(ii)]
displaced position

$(a\sin\theta + a\sin\phi,\ a\cos\theta + a\cos\phi)$.

$\therefore$ (velocity)^2 of $C = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\cos(\theta - \phi)$

$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$ approximately

$$\left[\because \begin{array}{l} \dot{x}_C = a\cos\theta\,\dot{\theta} + a\cos\phi\,\dot{\phi} \\ \dot{y}_C = -a(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi}) \end{array}\right]$$

Also co-ordinates of the particle m (*i.e.* of the pt. P) are

$x_P = a(\sin\theta + \sin\phi + \sin\psi)$,

$y_P = a(\cos\theta + \cos\phi + \cos\psi)$.

$\because$ (velocity)^2 of $m = a^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta})$ app,

Let T, be the kinetic energy and W, the work function of the system, then we readily get

$$W = 8mg(a\cos\theta + a\cos\phi) + mg(a\cos\theta + a\cos\phi + a\cos\psi)$$

$$= mga[9\cos\theta + 9\cos\phi + \cos\psi] \quad ...(1)$$

and $T = \frac{1}{2}\cdot 8m[a^2\dot{\phi}^2 + (a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi})]$

$$+ \frac{1}{2}ma^2[\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}]$$

i.e. $T = \frac{1}{2}m[9\dot{\theta}^2 + 17\dot{\phi}^2 + \dot{\psi}^2 + 18\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}]$ ...(2)

$\therefore$ Lagrange's θ, and ψ equations give

$9\ddot{\theta} + 9\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\theta$ ...(3) $9\ddot{\theta} + 17\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\phi$ ...(4)

and $\ddot{\theta} + \ddot{\phi} + \ddot{\psi} = -\frac{g}{a}\psi$, ...(5)

which can be rewritten as

$$(9D^2 + 9c)\theta + 9D^2\phi + D^2\psi = 0 \quad ...(6)$$

$$9D^2\theta + (17D^2 + 9c)\phi + D^2\psi = 0 \quad ...(7)$$

and $D^2\theta + D^2\phi + (D^2 + c)\psi = 0$ ...(8)

Eliminating ϕ and ψ in (6), (7) and (8), we get

$$\begin{vmatrix} 9D^2 + 9c & 9D^2 & D^2 \\ 9D^2 & 17D^2 + 9c & D^2 \\ D^2 & D^2 & D^2 + c \end{vmatrix}\theta = 0$$

i.e. $(8D^2 + 9c)[9c(2D^2 + c) + D^2(8D^2 + 9c)]\theta = 0$

i.e. $(8D^2 + 9c)[8D^4 + 27cD^2 + 9c^2]\theta = 0$

i.e. $[(8D^2 + 9c)(8D^2 + 3c)(D^2 + 3c)]\theta = 0$ ...(9)

Now let $\theta = A \cos(pt + B)$, then

$D\theta = -pA \sin(pt + B), D^2\theta = -p^2 A \cos(pt + B) = -p^2\theta$ ...(10)

$\therefore$ (9) gives $(8p^2 - 9c)(8p^2 - 3c)(p^2 - 3c) = 0$ $[\because \theta \neq \theta]$

$$\Rightarrow \left(8p^2 - \frac{9g}{a}\right)\left(8p^2 - \frac{3g}{a}\right)\left(p^2 - \frac{3g}{a}\right) = 0.$$

i.e. $p_1^2 = \frac{9g}{8a}, p_2^2 = \frac{3g}{8a}, p_3^2 = \frac{3g}{a}$

Thus periods of small oscillations are $\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \frac{2\pi}{p_3}$

i.e. $2\pi\sqrt{\left(\frac{8a}{9g}\right)}, 2\pi\sqrt{\left(\frac{8a}{3g}\right)}, 2\pi\sqrt{\left(\frac{a}{9g}\right)}$

Ex. 19. *A plank, 2a feet long, is placed symmetrically across a light cylinder, of radius a, which rests and is free to roll on a perfectly rough horizontal plane. A heavy particle whose mass is n times that of plank is embedded in the cylinder at its lowest point. If the system is sligthly displaced, show that its periods of oscillation are the values of*

$$\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)}$$

given by the equation $4p^4 - (n + 12)p^2 + 3(n - 1) = 0$.

Sol. C is the centre of the cylinder whose radius is a.

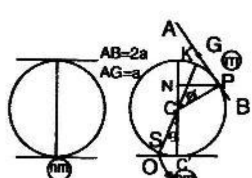
Let the mass of the plank be m, therefore mass of the particle embedded at the lowest point of the cylinder is nm.

Suppose θ and ϕ are the angles through which the cylinder and the plank have turned in time t, from their initial positions.

Initially G was coinciding with K which was at the highest point of the cylinder.

There is no slipping between the plank and the cylinder, hence

Further there is no sliding between the cylinder and the horizontal plane,



$$PG = \text{arc } KP = a(\phi - \theta)$$

hence $OC_1 = \text{arc } C_1S = a\theta$, where O was the initial point of contact of the cylinder and the horizontal plane.

Taking O as origin and the horizontal and vertical through O as co-ordinates axes, we obtain $x_G = a\theta + a\sin\phi - a(\phi - \theta)\cos\phi$

$$= a\theta + a\phi - a(\phi - \theta) = 2a\theta$$

and $y_G = a + a\cos\phi + a(\phi - \theta)\sin\phi = 2a - a\theta\phi + a\dfrac{\phi^2}{2}$

up to first approximations.

$\therefore\ \dot{x}_G^2 + \dot{y}_G^2 = (2a\dot{\theta})^2 + (-a\theta\dot{\phi} - a\phi\dot{\theta} + a\phi\dot{\phi})^2 = 4a^2\dot{\theta}^2$

Further co-ordinates of the particle nm (*i.e.* of the point S)

arc $\equiv \{a\theta - a\sin\theta, a - a\cos\theta\}$

$\therefore$ (velocity)^2 of $(nm) = (a\dot{\theta} - a\cos\theta\dot{\theta})^2 + (a\sin\theta\dot{\theta})^2$,

$$= (a^2 + a^2 - 2a^2\cos\theta)\dot{\theta}^2.$$

Now let T, be the kinetic energy and W, the work function of the system, then we readily obtain

$$W = -n\,mg\,(a - a\cos\theta) - mg\left(2a - a\theta\phi + a\frac{\phi^2}{2}\right)$$

$$= -n\,mga\frac{\theta^2}{2} - mga\left(\frac{\phi^2}{2} - \theta\phi\right) - 2mga$$

and $T = \frac{1}{2}m\left[\dfrac{a^2}{3}\dot{\phi}^2 + 4a^2\dot{\theta}^2\right] + \frac{1}{2}nm\,[2a^2 - 2a^2\cos\theta]\,\dot{\theta}^2$

$$= \frac{1}{2}m\left(\frac{a^2}{3}\dot{\phi}^2 + 4a^2\dot{\theta}^2\right) + \frac{1}{2}nm.\,2a^2\dot{\theta}^2\frac{\theta^2}{2}$$

$$= \frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2 + 4a^2\dot{\theta}^2\right], \text{ approximately,}$$

(neglecting the second part)

$\therefore$ Lagrange's θ-equation is given by $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

$$\Rightarrow \frac{d}{dt}(4ma^2\dot{\theta}) = -n\,mga\,\theta + mga\,\phi \Rightarrow 4\ddot{\theta} = -\frac{g}{a}(n\theta - \phi) \quad ...(1)$$

Further Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left(\frac{ma^2}{3}\dot{\phi}\right) = -mg(\phi - \theta) \Rightarrow \ddot{\phi} = -\frac{3g}{a}(\phi - \theta) \quad ...(2)$$

Equations (1) and (2) can be re-written a $(4D^2 + nc)\,\theta - c\phi = 0$ ...(3)

and $3c\theta - (D^2 + 3c)\,\phi = 0$ where $c = (g/a)$ ...(4)

Eliminating ϕ in between (3) and (4), we get

$$[(4D^2 + nc)(D^2 + 3c) - 3c^2]\,\theta = 0$$

$$\Rightarrow [4D^2 + c(n + 12)D^2 + 3(n - 1)c^2]\,\theta = 0 \quad ...(5)$$

Now if the periods of oscillation are the values of

$$\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)} \text{ i.e. values of } \frac{2\pi}{\sqrt{\left(\frac{g}{a}p^2\right)}},$$

then the solution of the above equation must be of the form

$$\theta = A\cos\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$\therefore D\theta = -\sqrt{\left(\frac{g}{a}\right)}pA\sin\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$D^2\theta = -\left(\frac{g}{a}\right)p^2A\cos\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$= -\left(\frac{g}{a}\right)p^2\theta = -cp^2\theta \text{ and } D^4\theta = c^2p^4\theta.$$

Substituting these values in (5) we get

$$[4c^2p^4 - c^2(n+12)p^2 + 3(n-1)c^2]\theta = 0$$

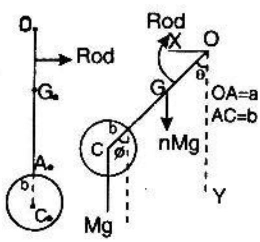
$$\Rightarrow 4p^4 - (n+12)p^2 + 3(n-1) = 0 \qquad (\because \theta \neq 0)$$

Ex. 20. *To a point of a solid homogeneous sphere, of mass M, is freely hinged one end of a homogeneous rod, of mass nM ; and the other end is freely hinged to a fixed point. If the system makes small oscillations under gravity about the position of equilibrium, the centre of a sphere and the rod being always in a vertical plane passing through the fixed point, show that the periods of the principal oscillations are the values of* $\frac{2\pi}{p}$ *given by the equation*

$$2ab(6+7n)p^4 - p^2g\{10a(3+n) + 21b(2+n)\} + 15g^2(2+n) = 0,$$

where a is the length of the rod and b is the radius of the sphere.

Sol. At any time t, let the rod and the sphere have turned through



angles θ and ϕ to vertical, where $OA = a$, $AC = b$.
Now co-ordinates of C are

$$x_C = a\sin\theta + b\sin\phi,\ y_C = a\cos\theta + b\cos\phi$$

$$\therefore (\text{velocity})^2 \text{ of } C = \dot{x}_C^2 + \dot{y}_C^2 = a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$

$$= a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi} \quad [\because \theta \text{ and } \phi \text{ are small}]$$

Also the co-ordinates of G are $\left(\frac{a}{2}\sin\theta, \frac{a}{2}\cos\theta\right)$

$\therefore$ (velocity)^2 of $G = \frac{a^2}{4}\dot{\theta}^2$

Now let T be, the kinetic energy and W, the work function of the system, then we readily obtain $W = n\,Mg\frac{a}{2}\cos\theta + Mg\,(a\cos\theta + b\cos\phi)$

$$= \tfrac{1}{2}a\,(n+2)\,Mg\cos\theta + b\,Mg\cos\phi$$

and $$T = \tfrac{1}{2}M\left[\frac{2b^2}{5}\dot{\phi}^2 + (a^2\dot{\theta}^2 + b^2\dot{\theta}^2 + 2ab\,\dot{\theta}\dot{\phi})\right] + \tfrac{1}{2}n.M\left[\frac{a^2}{12}\dot{\theta}^2 + \frac{a^2}{4}\dot{\theta}^2\right]$$

$$= \frac{1}{6}a^2\,(n+3)\,M\dot{\theta}^2 + \frac{7b^2}{10}M\dot{\phi}^2 + ab\,M\dot{\theta}\dot{\phi}$$

$\therefore$ Lagrange's θ-equation is given by $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$$\Rightarrow \frac{d}{dt}\{\tfrac{1}{3}a^2\,(n+3)\,M\dot{\theta} + ab\,M\dot{\phi}\} = -\tfrac{1}{2}a\,(n+2)\,Mg\theta \text{ app.}$$

$$\Rightarrow 2a\,(n+3)\,\ddot{\theta} + 6b\,\ddot{\phi} = -3\,(n+2)\,g\theta \qquad \text{...(1)}$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{7b^2}{5}M\dot{\phi} + ab\,M\dot{\theta}\right\} = -b\,Mg\phi$$

$$\Rightarrow 7b\,\ddot{\phi} + 5a\,\ddot{\theta} = -5g\phi \qquad \text{...(2)}$$

These can be re-written as

$$\{2a\,(n+3)\,D^2 + 3\,(n+2)\,g\}\,\theta + 6b\,D^2\phi = 0 \qquad \text{...(3)}$$

and $$5a\,D^2\theta + (7b\,D^2 + 5g)\,\phi = 0 \qquad \text{...(4)}$$

Eliminating ϕ between (3) and (4), we get

$$[2ab\,(7n+6)\,D^4 + \{10a\,(n+3) + 21b\,(n+2)\}\,gD^2 + 15\,(n+2)\,g^2]\,\theta = 0 \qquad \text{...(5)}$$

To solve (5), let us put

$$\theta = A\cos(pt+B), \qquad \therefore D\theta = -pA\sin(pt+B).$$

$$D^2\theta = -p^2A\cos(pt+B) = -p^2\theta \text{ and } D^4\theta = p^4\theta.$$

Substituting these values in (5), we have

$$2ab\,(7n+6)\,p^4 - \{10\,(n+3)\,a + 21b\,(n+2)\}\,p^2g + 15\,(n+2)\,g^2 = 0.$$

Ex. 21. *A hollow cylindrical garden roller is fitted with a counterpoise which can turn on the axis of the cylinder; the system is placed on the*

rough horizontal plane and oscillates under gravity, if $\frac{2\pi}{p}$ *be the time of a small ascillation, show that p is given by the equation* $p^2[(2M+M')k^2 - M'h^2] = (2M+M')gh,$
where M and M' are the masses of the roller and counterpoise, k is the radius of gyration of M' about the axis of the cylinder and h is the distance of its centre of mass from the axis.

Sol. Suppose O' is the centre of the cylinderical roller and G is the centre of gravity of the counterpoise, whose mass is M'. Obviously the line $O'K$ is a line fixed in the roller and the line $O'G$ is fixed in the counterpoise which were initially vertical. Suppose they make angles θ and ϕ with the vertical at the time t. Initially K was at O.

There is no slipping between the cylinder and the horizontal plane, hence $OB =$ arc $KB = a\theta$. Now assuming O as origin and the horizontal and vertical through O as co-ordinates axes, we have co-ordinates of G :-

$$x_G = a\theta + h\sin\phi\ ,\ y_G = a - h\cos\phi,$$

$\therefore$ (velocity)^2 of $G = \dot{x}_G^2 + \dot{y}_G^2 = (a\dot{\theta} + h\cos\phi\dot{\phi})^2 + (h\sin\phi\ \dot{\phi})^2$

$$= a^2\dot{\theta}^2 + h^2\dot{\phi}^2 + 2ah\dot{\theta}\dot{\phi} \qquad \text{[neglecting higher powers of } \phi]$$

But radius of gyration of the counterpoise about O' is k, where $O'G = h$. Hence the square of radius of gyration about G is $(k^2 - h^2)$.

Also (velocity)^2 of $O' = (a\dot{\theta})^2 = a^2\dot{\theta}^2$.

[$\because$ co-ordinates of O' are $(a\theta, a)$]

Now let T, be the kinetic energy and W, the work function of the system then we get

$$T = \frac{1}{2}M(a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \frac{1}{2}M'[(k^2 - h^2)\dot{\phi}^2 + a^2\dot{\theta}^2 + h^2\dot{\phi}^2 + 2ah\dot{\theta}\dot{h}]$$

$$= \frac{M+M'}{2}a^2\dot{\theta}^2 + \frac{1}{2}M'k^2\dot{\phi}^2 + M'ah\dot{\theta}\dot{\phi}.$$

$$W = M'gh\cos\phi + D.$$

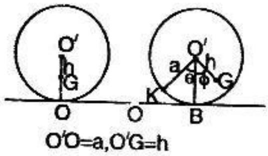
$\therefore$ Lagrange's θ-equation gives $\frac{d}{dt}[(2M+M')a^2\dot{\theta} + M'ah\dot{\phi}] = 0$

$$\Rightarrow (2M+M')a\ddot{\theta} + M'h\ddot{\phi} = 0 \qquad ...(1)$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}[M'k^2\dot{\phi} + M'ah\dot{\theta}] = -M'gh\phi \Rightarrow k^2\ddot{\phi} + ah\ddot{\theta} = -gh\phi$$

$$\Rightarrow ah\ddot{\theta} + k^2\ddot{\phi} = -gh\phi, \qquad ...(2)$$



$$O'O = a, \quad O'G = h$$

Now eliminating $\ddot{\theta}$ between (1) and (2), we get

$$[Mh^2 - (2M + M')k^2]\ddot{\phi} = (2M + M')gh\phi.$$

or $\quad [(2M + M')k^2 - M'h^2]\ddot{\phi} = -(2M + M')gh\phi \qquad ...(3)$

But, if $\dfrac{2\pi}{p}$ is the time of small oscillation, then we assume $\phi = A\cos(pt + b)$

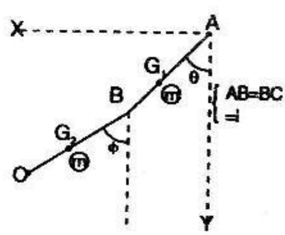
$\therefore\ \dot{\phi} = -pA\sin(pt + B)$ and $\ddot{\phi} = -p^2A\cos(pt + B) = -p^2\phi$

$\therefore$ Equation (3) gives

$$-p^2[(2M + M')k^2 - M'h^2] = -(2M + M')gh \qquad [\because\ \phi \neq 0]$$

Ex. 22. *Two equal rods AB and BC; each of length l, smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are*

$$\frac{2\pi}{n} \text{ where } n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}.$$



Sol. At any time t, let the rod AB and BC be inclined at angles θ and ϕ to the vertical.

$\therefore$ Co-ordinates of G_1 are $\left(\dfrac{l}{2}\sin\theta, \dfrac{l}{2}\cos\theta\right)$

$$\therefore\ v_{G_1}^2 = \left(\frac{l}{2}\cos\theta\,\dot{\theta}\right)^2 + \left(-\frac{l}{2}\sin\theta\,\dot{\theta}\right)^2 = \frac{l^2}{4}\dot{\theta}^2$$

$$X_{G_2} = l\sin\theta + \frac{l}{2}\sin\phi, \quad y_{G_2} = l\cos\theta + \frac{l}{2}\cos\phi$$

$$\therefore\ v_{G_2}^2 = l^2\dot{\theta}^2 + \frac{1}{2}l^2\dot{\phi}^2 + l^2\dot{\theta}\dot{\phi}\cos(\theta - \phi) = l^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\phi}^2 + l^2\dot{\theta}\dot{\phi}$$

$[\because\ \theta$ and ϕ are small$]$

now, let T, be the kinetic energy and W, the work function, of the system, then we have $W = mg\frac{l}{2}\cos\theta + mg\left(l\cos\theta + \frac{l}{2}\cos\phi\right)$

$$= \frac{1}{2}mgl(3\cos\theta + \cos\phi)$$

and $T = \frac{1}{2}m\left[\frac{1}{12}l^2\dot{\theta}^2 + \frac{1}{2}l^2\dot{\theta}^2\right] + \frac{1}{2}m\left\{\frac{1}{12}l^2\dot{\phi}^2 + l^2\left(\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)\right\}$

$$= \frac{1}{2}ml^2\left(\frac{4}{3}\dot{\theta}^2 + \frac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$$

$\therefore$ Lagrange's θ-equations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta} \Rightarrow \frac{d}{dt}\left(\frac{4}{3}l\dot{\theta} + \frac{1}{2}l\dot{\phi}\right) = -\frac{3}{2}g\theta$$

$$\Rightarrow 8\ddot{\theta} + 3\ddot{\phi} = -\frac{9g}{l}\theta \qquad ...(1)$$

While Lagrange's ϕ equation gives

$$\frac{d}{dt}\left(\frac{1}{3}l\dot{\phi} + \frac{1}{2}l\dot{\theta}\right) = -\frac{1}{2}g\phi \Rightarrow 3\ddot{\theta} + 2\ddot{\phi} = -\frac{3g}{l}\phi \qquad ...(2)$$

Equations (1) and (2) can be written as

$$(8D^2 + 9c)\theta + 3D^2\phi = 0 \qquad ...(3)$$

and $3D^2\theta + (2D^2 + 3c)\phi = 0$ where $c = \left(\frac{g}{l}\right)$ ...(4)

Eliminating ϕ between (3) and (4), we get

$$[(8D^2 + 9c)(2D^2 + 3c) - 9D^4]\theta = 0$$

i.e $\quad (7D^2 + 42cD^2 + 27c^2)\theta = 0$ ...(5)

Let us assume, $\theta = A\cos(nt + B) \Rightarrow D\theta = nA\sin(nt + B)$

$$D^2\theta = -n^2A\cos(nt + B) = -n^2\theta \text{ and } D^4\theta = n^4\theta$$

Substituting these values in (5), we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$ i.e. $7n^4 - 42cn^2 - 27c^2 = 0 \quad [\because \theta \neq 0]$

i.e. $n^2 = \dfrac{42c \pm \sqrt{[\{(42c)^2 - 4(7)(27c^2)\}]}}{14}$

$$= \left[3 \pm \frac{6}{14}\sqrt{(28)}\right]c = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left\{3 \pm \frac{6}{\sqrt{7}}\right\}\frac{g}{l}.$$

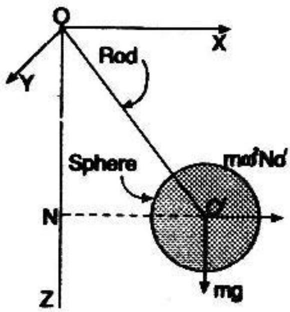
Ex. 23. *A solid uniform sphere has a light rod rigidly attached to it which passes through its centre. This rod is so jointed to a fixed vertical axis that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve constantly with uniform angular velocity, show that the equation of motion is of the form*

$$\dot{\theta}^2 = n^2(\cos\theta - \cos\beta)(\cos\alpha - \cos\theta)$$

Show also that the total energy imparted to the sphere as θ increases from θ_1 to θ_2 varies as $\cos^2\theta_1 - \cos^2\theta_2$. **[Meerut 1989, Agra 86]**

Sol. First of all, we reduce the revolving axis OZ to rest by introducing an additional force $m\,\omega^2\, NO'$, along NO' i.e. a force $m\omega^2 l\sin\theta$ along NO'.

After the interval of time t. let the rod make an angle θ with the vertical ONZ (taken as Z-axis. (Further let ϕ be the angle through which the sphere turn in the horizontal plane, then the co-ordinates of C.G. of the sphere, are given by



$$x_{0'} = l\sin\theta\cos\phi.\ y_{0'} = l\sin\theta\sin\phi,\ z_{0'} = l\cos\theta$$

$$\therefore\ \dot{x}_0' = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi,$$

$$\dot{y}_0' = l\dot{\theta}\cos\theta\cos\phi + l\dot{\phi}\sin\theta\cos\phi,\quad \dot{z}_0' = -l\dot{\theta}\sin\phi$$

$$\Rightarrow \dot{x}_0'^2 + \dot{y}_0'^2 + \dot{z}_0'^2 = l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2$$

$$\therefore\ T = \frac{1}{2}m\left[\frac{2a^2}{5}\dot{\theta}^2 + \dot{x}_{0'}^2 + \dot{y}_{0'}^2 + \dot{z}_{0'}^2\right]$$

$$= \frac{1}{2}m\left[\frac{2a^2}{5}\dot{\theta}^2 + l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta\right]$$

$$= \frac{1}{2}m\left[\left(\frac{2a^2}{5}l^2\right)\dot{\theta}^2 + l^2\omega^2\sin^2\theta\right] \qquad (\because\ \dot{\phi} \neq \omega \text{ given })$$

and $W^* = mgl\cos\theta + m\omega^2 l\sin\theta.\, l\sin\theta = mgl\cos\theta + m\omega^2 l^2\sin^2\theta$ ...(2)

$\because$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left\{m\left(\frac{2a^2}{5} + l^2\right)\dot{\theta}\right\} - \frac{1}{2}ml^2\omega^2.2\sin\theta\cos\theta$$

$$= -mgl\sin\theta + 2m\omega^2 l^2\sin\theta\cos\theta$$

$$\Rightarrow \left(\frac{2a^2}{5} + l^2\right)\ddot{\theta} = -mgl\sin\theta + 3m\,\omega^2 l^2\sin\theta\cos\theta$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$\left(\tfrac{2}{5}a^2 + l^2\right)\dot{\theta}^2 = 2gl\cos\theta - 3l^2\omega^2\cos^2\theta + D$$

$\Rightarrow \dot{\theta}^2 = n^2(\cos\theta - \cos\beta)(\cos\alpha - \cos\theta)$

where n is constant and α, β are the values of θ for which $\dot{\theta}$ vanish.

SECOND PART

**K.E+ Potential Energy = $T + K$, when $P.E.$ is K and is given by

$$K = T - W + C = \tfrac{1}{2}m\left[\left(\frac{2a^2}{5} + l^2\right)\dot{\theta}^2 + l^2\omega^2\sin^2\theta\right] - mgl\cos\theta - m\omega^2 l^2\sin^2\theta + C$$

$$= \tfrac{1}{2}m\left[\{2gl\cos\theta - 3l^2\omega^2\cos^2\theta + D\} + l^2\omega^2\sin^2\theta\right] - mgl\cos\theta - ml^2\omega^2\sin^2\theta + C$$

$= -\tfrac{1}{2}m\omega^2 l^2\cos^2\theta - \tfrac{1}{2}m\omega^2 l^2 + A$ where A is constant.

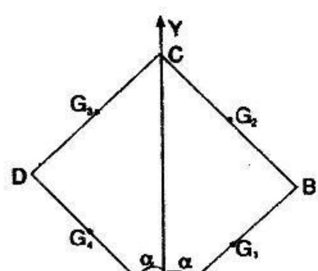
$= -\tfrac{1}{2}m\omega^2 l^2\cos^2\theta + B$ where B is constant

$\therefore$ Total Energy imparted

$$= \left[-\tfrac{1}{2}m\omega^2 l^2\cos^2\theta + B\right]_{\theta_1}^{\theta_2} = \tfrac{1}{2}m\omega^2 l^2(\cos^2\theta_1 - \cos^2\theta_2)$$

which varies as $(\cos^2\theta_1 - \cos^2\theta_2)$.

Ex. 24. *Four uniform rods, each of length 2a, are hinged at their ends so as to form a rhombus ABCD. The angles B and D are connected by an elastic string and the lowest end A rests on a horizontal plane while the end C slides on a smooth vertical wire passing through A; in the position of equlibrium the string is strected to twice its natural length and the angle BAD is 2α. Show that the time of a small oscillation about this position is* $2\pi\left\{\dfrac{2a(1 + 3\sin^2\alpha)}{3g\cos 2\alpha}\cos\alpha\right\}^{1/2}$



Sol. In the position of equilibrium, rods are making angles α with the vertical. When the system is slightly displaced from the position of equilibrium, let the rods make angle $(\alpha + \theta)$ with the vertical θ being a small displacement. Now assuming the fixed end A as origin and the horizontal and vertical lines through it as co-ordinate axes, the co-ordinates of G_2 are $\{a\sin(\alpha + \theta), 3a\cos(\alpha + \theta)\}$

$\therefore$ (velocity)^2 of $G_2 = \{a\cos(\alpha+\theta)\dot{\theta}\}^2 + (-3a\sin(\alpha+\theta)\dot{\theta}\}^2$

$$= a^2[(1+8\sin^2(\alpha+\theta]\dot{\theta}^2.$$

Co-ordinates of G_1 are $\{a\sin(\alpha+\theta),\ a\cos(\alpha+\theta)\}$

$\therefore$ (velocity)^2 of $G_1 = a^2\dot{\theta}^2$

$\therefore$ Kinetic energy of the four rods taken together is

$$T = 2.\tfrac{1}{2}m.\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right] + 2.\tfrac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\{1+8\sin^2(\alpha+\theta)\}\dot{\theta}^2\right]$$

$$= \frac{8ma^2}{3}\left[1+\sin^2(\alpha+\theta)\right]\dot{\theta}^2 \quad (\because\ v_{G_1} = v_{G_4} \text{ and } v_{G_2} = v_{G_3})$$

The work function W is given by $W = 2\{-mga\cos(\alpha+\theta)\}$

$$+ 2\{-mg\,3a\cos(\alpha+\theta)\} - 2\int_o^{2a\sin(\theta+\alpha)} \lambda\left(\frac{y-c}{c}\right)dy$$

$$= -8mga\cos(\alpha+\theta) - \frac{\lambda}{c}\{2a\sin(\alpha+\theta) - c\}^2$$

Lagrange's θ equation gives

$$\frac{d}{dt}\left[\frac{16ma^2}{3}\{1+3\sin^2(\alpha+\theta)\}\dot{\theta} - 16ma^2\sin(\alpha+\theta)\cos(\alpha+\theta)\dot{\theta}^2\right.$$

$$= 8mga\sin(\alpha+\theta) - \frac{4\lambda}{c}a\cos(\alpha+\theta)\{2a\sin(\alpha+\theta) - c\}$$

$$\Rightarrow \frac{16ma^2}{3}(1+3\sin^2(\alpha+\theta)\}\ddot{\theta}$$

$$= 8mga\sin(\alpha+\theta) - \frac{4\lambda a}{c}\cos(\alpha+\theta)\{2a\sin(\alpha+\theta) - c\} \quad ...(1)$$

Initially when $\theta = 0$, $\dot{\theta} = 0$, $\ddot{\theta} = 0$, $c = a\sin\alpha$, hence (1) gives

$$\lambda = \frac{2mgc}{a\cos\alpha}.$$

Putting this value of λ in equation (1), we get

$$\frac{16ma^2}{3}\ \ +3\sin^2(\alpha+\theta)\}\ddot{\theta}$$

$$= mga\,8\sin(\alpha+\theta) - \frac{8mg\cos(\alpha+\theta)}{\cos\alpha}\{2a\sin(\alpha+\theta) - c\}$$

---

*The force $m\omega^2 l\sin\theta$ also contributes to W. The distance of the point of applicat ion at O' of this force from the vertical OZ is equal to $l\sin\theta$, hence the contribution $m\omega^2 l^2\sin^2\theta$ to W is as given in (2).

** If W is the work function of the system, then P.E. $= C - W$.

i.e. $\dfrac{16\,ma^2}{3}\{1+3\sin^2\alpha\}\ddot{\theta}=8\,mga(\sin\alpha+\theta\cos\alpha)$

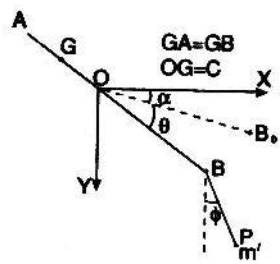
$$-\frac{8mg}{\cos\alpha}(\cos\alpha-8\sin\alpha)\{2a(\sin\alpha+\theta\cos\alpha)-a\sin\alpha].$$

$$=-\frac{8\,mga\cos 2\alpha}{\cos\alpha}\theta\ \text{app.}\Rightarrow\ddot{\theta}=-\frac{3g\cos 2\alpha}{2a\cos\alpha(1+3\sin^2\alpha)}\theta\ \text{app.}$$

$\therefore$ Time of a small oscillation about the position of equilibrium is given by $2\pi\sqrt{\left\{\dfrac{2a\cos\alpha(1+3\sin^2\alpha)}{3g\cos 2\alpha}\right\}}$.

Ex. 25. *A uniform rod AB, of length 2a can turn freely about a point distance c from its centre, and is at rest at an angle α to the horizon when a particle is hung by a light string of length l from one end. If the particle be displaced slightly in the vertical plane of the rod show that it will oscillate in the same time as a simple pendulum of length* $l\dfrac{a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha}{a^2+3ac}$. **[Meerut 95]**

Sol. Let M be the mass and G the centre of gravity of the rod. Further let O be the point about which it can turn where $OG=c$. Let BP be a string of length l and m the mass of the particle tied to the end P of the string BP.



Initially rod was making an angle α with the horizontal and the string was vertical.

Let B_0 be the initial position of B so that $\angle B_0OX=\alpha$, After a time t let the rod make an angle θ with OB_0 i.e. an angle $(\alpha+\theta)$ with the horizontal and let the string be inclined at an angle θ with the vertical.

Initially the system was at rest, hence taking moments about O, (in this position), we get $Mc=m(a-c)$.

Assuming O as origin and the horizontal and the vertical through O as co-ordinates axes, we obtain the following results

$$x_p=(a-c)\cos(\alpha+\theta)+l\sin\phi.\quad y_p=(a-c)\sin(\alpha+\theta)+l\cos\phi.$$

$$\therefore\ (\text{velocity})^2\text{ of }P=\{-(a-c)\sin(\alpha+\theta)\dot{\theta}+l\cos\phi\dot{\phi}\}^2$$

$$+\{(a-c)\cos(\alpha+\theta)\dot{\theta}-l\sin\phi\dot{\phi}\}^2$$

$$=(a-c)^2\dot{\theta}^2+l^2\dot{\phi}^2-2l(a-c)\dot{\theta}\dot{\phi}\sin(\alpha+\theta+\phi)$$

$$= (a-c)^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2l(a-c)\dot{\theta}\dot{\phi}\sin\alpha$$

(neglecting small quantities of the higher order)

Also co-ordinates of G (the C.G. of the rod) are

$$\{-c\cos(\alpha+\theta), -c\sin(\alpha+\theta)\}$$

$\therefore$ (velocity)^2 of $G = \{c\sin(\alpha+\theta)\dot{\theta}\}^2 + \{-c\cos(\alpha+\theta)\dot{\theta}\}^2 = c^2\dot{\theta}^2$.

Now let T, be the kinetic energy and W, the work function of the system, then we easily obtain

$$W = mg[(a-c)\sin(\alpha+\theta) + l\cos\phi] - Mgc\sin(\alpha+\theta) + D$$

$$= mgl\cos\phi + D$$

[the first and the last terms cancel, because $m(a-c) = Mc$]

and $T = \frac{1}{2}M\left[\frac{a^2}{3}\dot{\theta}^2 + c^2\dot{\theta}^2\right] + \frac{1}{2}m[(a-c)^2\dot{\theta}^2$

$$+ l^2\dot{\phi}^2 - 2l(a-c)\dot{\theta}\dot{\phi}\sin\alpha]$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left[\left\{M\left(\frac{a^2}{3}+c^2\right) + m(a-c)^2\right\}\dot{\theta} - ml(a-c)\dot{\phi}\sin\alpha\right] = 0$$

$$\Rightarrow \left\{M\left(\frac{a^2}{3}+c^2\right) + m(a-c)^2\right\}\ddot{\theta} - ml(a-c)\ddot{\phi}\sin\alpha = 0.$$

$$\Rightarrow \left\{M\left(\frac{a^2}{3}+c^2\right) + Mc(a-c)\right\}\ddot{\theta} - Mcl\ddot{\phi}\sin\alpha = 0 \quad [\because m(a-c) = Mc]$$

$\Rightarrow (a^2+3ac)\ddot{\theta} - 3lc\sin\alpha\,\ddot{\phi} = 0.$ While Lagrange's ϕ equation gives

$$\frac{d}{dt}\left\{ml^2\dot{\phi} - ml(a-c)\dot{\theta}\sin\alpha\right\} = -mgl\phi$$

$$\Rightarrow l\ddot{\phi} - (a-c)\ddot{\theta}\sin\alpha = -g\phi \qquad \text{...(2)}$$

Eliminating θ in (1) and (2), we have $l\ddot{\phi} - (a-c)\sin\alpha\dfrac{3lc\sin\alpha}{a^2+3ac}\ddot{\phi} = -g\phi$

i.e. $\Rightarrow \dfrac{a^2+3ac-3c(a-c)\sin^2\alpha}{a^2+3ac}\,l\ddot{\phi} = -g\phi$

$$\Rightarrow \frac{a^2+3ac\cos^2\alpha+3c\sin^2\alpha}{a^2+3ac}\,l\ddot{\phi} = -g\phi$$

$$\Rightarrow \ddot{\phi} = -\frac{g}{l(a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha)}\phi = -\mu\phi \text{ (say)}$$

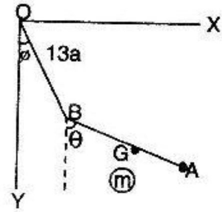
$\therefore$ Length of the simple equivalent pendulum $= \dfrac{g}{\mu}$

$$= \frac{a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha}{a^2+3ac}\,l.$$

Ex. 26. *A uniform rod AB of length 8a is suspended from a fixed point O by means of light inextensible string, of length 13a, attached to B. If the system is slightly displaced in a vertical plane, show that* $(\theta + 3\phi)$ and $(12\theta - 13\phi)$ *are principal co-ordinates where* θ and ϕ *are the angles which the rod and string respectively make with the vertical. Also show that periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{52a}{3g}\right)}.$$

Sol. We have



$x_G = 13a\sin\phi + 4a\sin\theta$

and $y_G = 13a\cos\phi + 4a\cos\theta$

$\therefore\ \dot{x}_G^2 + \dot{y}_G^2 = 169a^2\dot{\phi}^2 + 16a^2\dot{\theta}^2$

$$+ 104a^2\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$

Thus, $T = \frac{1}{2}m\,[k^2\dot{\theta}^2 + (\dot{x}_G^2 + \dot{y}_G^2)] = \frac{1}{2}ma^2\left[\frac{64}{3}\dot{\theta}^2 + 169\dot{\phi}^2 + 104\dot{\theta}\dot{\phi}\right]$

and the work function

$W = mg(13a\cos\phi + 4a\cos\theta)$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} \Rightarrow 61\ddot{\theta} + 39\ddot{\phi} = -\frac{3g}{a}\theta \qquad ...(1)$$

While Lagrange's ϕ-equatic: gives $4\ddot{\theta} + 13\ddot{\phi} = -\frac{g}{a}\phi$ ...(2)

Equatious (1) and (2) $\Rightarrow D^2(\theta + 3\phi) = -\frac{3g}{52a}(\theta + 3\phi)$

and $D^2(12\theta - 13\phi) = -\frac{g}{a}(12\theta - 13\phi)$

Now putting $\theta + 3\phi = X$ and $12\theta - 13\phi = Y$ in these equations, we get $D^2X = -\frac{3g}{52a}X$ and $D^2Y = -\frac{g}{a}.Y$;

which obviously represent two independent simple harmonic motions. Hence X and Y are principal co-ordinates, that is

$(\theta + 3\phi)$ and $(12\theta - 13\phi)$ are principal co-ordinates.

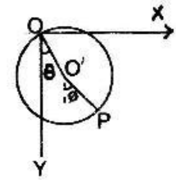
Also, periods of small oscillations are given by

$$2\pi\sqrt{\left(\frac{52a}{3g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{g}\right)}$$

Ex. 27. *A ring slides on a smooth circular hoop of equal mass and of radius a which can turn in a vertical plane about a fixed point O in its circumference. If* θ and ϕ *be the inclinations to the vertical of the radius through O and of the radius through the ring, prove that principal* *co-ordinates are* $(2\theta+\phi)$ and $(\phi-\theta)$, *and the periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{2g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{2a}{g}\right)}.$$ **[Agra 1994]**

Sol. Let the mass of the hoop and the ring be m. In a displaced position, let O' be the centre of the hoop and P the position of the ring which slides on the hoop.



$\therefore$ Co-ordiantes of the point P are given by

$$x_P = a(\sin\theta+\sin\phi)$$
$$y_P = a(\cos\theta+\cos\phi)$$

$\Rightarrow \dot{x}_P^2+\dot{y}_P^2 = a^2(\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi})$

when θ and $\phi$are small.

Also co-ordinates of O' are $(a\sin\theta, a\cos\theta)$

$\therefore$ (velocity)^2 of $O' = a^2\dot{\theta}^2$

Thus T= Energy of circular hoop+ Energy of ring

$$=\frac{1}{2}m\,(a^2\dot{\theta}^2+a^2\dot{\theta}^2)+\frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi})=\frac{1}{2}ma^2(3\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi})$$

Also the work function

$$W = mga\cos\theta+mg\,(a\cos\theta+a\cos\phi) = mga\,(2\cos\theta+\cos\phi).$$

$\therefore$ Lagrange's θ-equation gives $\quad 3\ddot{\theta}+\ddot{\phi}=-\dfrac{2g}{a}\theta$ ...(1)

Lagrange's ϕ-equation gives $\ddot{\theta}+\ddot{\phi}=-\dfrac{g}{a}\phi$ ...(2)

Now adding (1) and (2) and multiplying (2) by 2 and subtracting (1) from that, we get $D^2(2\theta+\phi)=-\dfrac{g}{2a}(2\theta+\phi)$

and $D^2(\phi-\theta)=-\dfrac{2g}{a}(\phi-\theta)$

Taking $2\theta+\phi=X$ and $\phi-\theta=Y$, we get

$$D^2X=-\frac{g}{2a}X \text{ and } D^2Y=-\frac{2g}{a}Y\,;$$

which obviously represent two independent simple harmonic motions. Thus X and Y are principal co-ordinates, that is $(2\theta+\phi)$ and $(\phi-\theta)$ are principal co-ordinates.

Also, periods of small oscillations are given by

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{2g}\right)}.$$

7·14. Lagrange's Equations with impulsive forces **[Meerut 1995]**

When the forces are finite, we have by **7·07**

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha \text{ where } \phi_\alpha = \sum_v \boldsymbol{F}_v \bullet \frac{\partial \boldsymbol{r}_v}{\partial q_\alpha}$$

Integrating both sides of (1) w.r.t "t" form $t = 0$ to $t = \tau$, we get

$$\left[\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)\right]_{t=0}^{t=\tau} - \int_0^\tau \frac{\partial T}{\partial q_\alpha}\, dt = \int_{t=0}^{\tau} \phi_\alpha\, dt = \sum_v \left\{\left(\int_{t=0}^{\tau} \boldsymbol{F}_v\, dt\right) \bullet \frac{\partial \boldsymbol{r}_v}{\partial q_\alpha}\right\}$$

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0} - \int_0^\tau \frac{\partial T}{\partial q_\alpha} dt = \sum_v \left\{\left(\int_0^\tau \boldsymbol{F}_v\, dt\right) \bullet \frac{\partial \boldsymbol{r}_v}{\partial q_\alpha}\right\} \qquad \text{...(2)}$$

Taking the limit as $\tau \to 0$, we get

$$\lim_{\tau \to 0}\left\{\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0}\right\} - \lim_{\tau\to 0}\int_0^\tau \frac{\partial T}{\partial q_\alpha} dt$$

$$= \sum_v \left\{\left(\lim_{\tau\to 0}\int_0^\tau \boldsymbol{F}_v\, dt\right) \bullet \frac{\partial \boldsymbol{r}_v}{\partial q_\alpha}\right\}$$

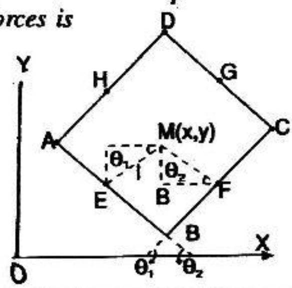
$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_0 = \Sigma^* I_v \bullet \frac{\partial \boldsymbol{r}_v}{\partial q_\alpha} = {}^{**}P_\alpha \text{ (say)} \qquad \text{...(3)}$$

where subscripts 0 and 1 denote respectively quantities before and after the application of the implusive force.

These equations are known as Lagrange's equations under implusive forces.

Ex. 28. *A square ABCD formed by four rods each of length 2l and mass hinged at their ends, rests on a horizontal frictionless table. An implusive of magnitube I is applied to the vertex A in the direction AD Find the equations of motion, and prove that tha K.E. devloped immediately after the application of the impulsive forces is*

$$T = (I^2/2m)$$



Sol. When the square is struck, its shape will in general be a rhombus. Suppose that at any time t, the angles made by sides AD (or BC) and BA (or CD) with the x-axis are θ_1 and θ_2 respectively, while the co-ordinates of the centre M are (x, y).

---

*Suppose that the force $\mathbf{F}_v$ acting on a system are such that $\lim_{\tau\to 0}\int_0^\tau \mathbf{F}_v\, dt = \mathbf{I}_v$

where τ represents a time interval, Then we call $\mathbf{F}_v$ impulsive forces and $\mathbf{I}_v$ are called Impulsse.

** If we call P_α to be the generalised impules, (3) can be written as

Hence x, y, θ_1, θ_2 are the generalised co-ordinates.

From the adjoining diagram, we see that the position vectors of the centre E, F, G, H of the rods are given respectively by

$$\mathbf{r}_E = (x - l\cos\theta_1)\,\mathbf{i} + (y - l\sin\theta_1)\,\mathbf{j},$$
$$\mathbf{r}_F = (x + l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_2)\,\mathbf{j},$$
$$\mathbf{r}_G = (x + l\cos\theta_1)\,\mathbf{i} + (y + l\sin\theta_1)\,\mathbf{j},$$

and $\mathbf{r}_H = (x - l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1)\,\mathbf{j}$.

$$\because \quad \mathbf{v}_E = \dot{\mathbf{r}}_E = (\dot{x} + l\sin\theta_1\,\dot{\theta}_1)\,\mathbf{i} + (\dot{y} - l\cos\theta_1\,\dot{\theta}_1)\,\mathbf{j},$$

$$\mathbf{v}_F = \dot{\mathbf{r}}_F = (\dot{x} - l\sin\theta_2\,\dot{\theta}_2)\,\mathbf{i} + (\dot{y} - l\cos\theta_2\,\dot{\theta}_2)\,\mathbf{j},$$

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = (\dot{x} - l\sin\theta_1\,\dot{\theta}_1)\,\mathbf{i} + (\dot{y} + l\cos\theta_1\,\dot{\theta}_1)\,\mathbf{j},$$

$$\mathbf{v}_H = \dot{\mathbf{r}}_H = (\dot{x} + l\sin\theta_2\,\dot{\theta}_2)\,\mathbf{i} + (\dot{y} + l\cos\theta_2\,\dot{\theta}_2)\,\mathbf{j}.$$

Now, K.E. of the rod $AB = \frac{1}{2}m\,\dot{\mathbf{r}}_E^2 + \frac{1}{3}m\,l^2\dot{\theta}_2^2\cdot\frac{1}{2} = T_{AB}$ say

K.E. of the rod $CB = \frac{1}{2}m\,\dot{\mathbf{r}}_F^2 + \frac{1}{3}m\,l^2\dot{\theta}_1^2\cdot\frac{1}{2} = T_{BC}$ say

K.E. of the rod $CD = \frac{1}{2}m\,\dot{\mathbf{r}}_G^2 + \frac{1}{3}m\,l^2\dot{\theta}_2^2\cdot\frac{1}{2} = T_{CD}$ say

K.E. of the rod $DA = \frac{1}{2}m\,\dot{\mathbf{r}}_H^2 + \frac{1}{3}m\,l^2\dot{\theta}_1^2\cdot\frac{1}{2} = T_{DA}$ say

$$\therefore \text{ K.E. of system} = \frac{1}{2}m\,(\dot{\mathbf{r}}_E^2 + \dot{\mathbf{r}}_F^2 + \dot{\mathbf{r}}_G^2 + \dot{\mathbf{r}}_H^2) + \frac{2}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)\cdot\frac{1}{2} = T \text{ say}$$

or $\quad T = \frac{1}{2}m\,(4\dot{x}^2 + 4\dot{y}^2 + 2l^2\dot{\theta}_1^2 + 2l^2\dot{\theta}_2^2) + \frac{ml^2}{3}(\dot{\theta}_1^2 + \dot{\theta}_2^2)$.

$$= 2m\,(\dot{x}^2 + \dot{y}^2) + \frac{4}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2). \qquad \text{...(1)}$$

Let us now assume that initially the rhombus is a square at rest with its sides parallel to the co-ordinate axes and its centre located at the origin.

Then, we get $x = 0 = y$, $\theta_1 = \frac{\pi}{2}$, $\theta_2 = 0$, $\dot{x} = 0$, $\dot{y} = 0$, $\dot{\theta}_1 = 0$, $\dot{\theta}_2 = 0$.

Now if we use the notation, $(\ )_1$ and $(\ )_2$ to denote quantities before and after the impulse is applied, we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 = (4m\dot{x})_1 = 0 \quad \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = (4m\dot{y})_1 = 0,$$

$$\left(\frac{\partial T}{\partial \dot\theta}\right)_1 = \left(\frac{8}{3}\, ml^2\,\dot\theta_1\right) = 0, \left(\frac{\partial T}{\partial \dot\theta_2}\right)_1 = \left(\frac{8}{3} ml^2\,\dot\theta_2\right) = 0$$

and $\left(\frac{\partial T}{\partial \dot x}\right)_2 = (4m\dot x)_2 = 4m\dot x, \left(\frac{\partial T}{\partial \dot y}\right)_2 = (4m\dot y)_2 = 4m,$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 = \left(\frac{8}{3} ml^2\,\dot\theta_1\right)_2 = \frac{8}{3} ml^2\,\dot\theta_1, \left(\frac{\partial T}{\partial \dot\theta_2}\right)_2 = \frac{8}{3} ml^2\,\dot\theta_2\,.$$

Hence Lagrange's equations are

$$\left(\frac{\partial T}{\partial \dot x}\right)_2 - \left(\frac{\partial T}{\partial \dot x}\right)_1 = P_x \Rightarrow 4m\dot x = P_x \qquad ...(2)$$

$$\left(\frac{\partial T}{\partial \dot y}\right)_2 - \left(\frac{\partial T}{\partial \dot y}\right)_1 = P_y \Rightarrow 4m\dot y = P_y \qquad ...(3)$$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 - \left(\frac{\partial T}{\partial \dot\theta_1}\right)_1 = P_{\theta_1} \Rightarrow \frac{8}{3}\, ml^2\,\dot\theta_1 = P_{\theta_1}. \qquad ...(4)$$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 - \left(\frac{\partial T}{\partial \dot\theta_2}\right)_1 = P_{\theta_1} \Rightarrow \frac{8}{3}\, ml^2\,\dot\theta_2 = P_{\theta_2}. \qquad ...(5)$$

Now we shall find the value of P_x, P_y, P_{θ_1} and P_{θ_2}

We have $P_\alpha = \sum_v \mathbf{I}_v \bullet \frac{\partial \mathbf{r_v}}{\partial q_\alpha}$ where $\mathbf{I}_v$ are the impulsive forces.

$$\therefore\ P_x = \mathbf{I}_A \bullet \frac{\partial \mathbf{r_A}}{\partial x} + \mathbf{I}_B \bullet \frac{\partial \mathbf{r_B}}{\partial x} + \mathbf{I_C} \bullet \frac{\partial \mathbf{r_C}}{\partial x} + \mathbf{I_D} \bullet \frac{\partial \mathbf{r_D}}{\partial x} \qquad ...(6)$$

$$P_y = \mathbf{I}_A \bullet \frac{\partial \mathbf{r_A}}{\partial y} + \mathbf{I}_B \bullet \frac{\partial \mathbf{r_B}}{\partial y} + \mathbf{I_C} \bullet \frac{\partial \mathbf{r_C}}{\partial y} + \mathbf{I_D} \bullet \frac{\partial \mathbf{r_D}}{\partial y} \qquad ...(7)$$

$$P_{\theta_1} = \mathbf{I}_A \bullet \frac{\partial \mathbf{r_A}}{\partial \theta_1} + \mathbf{I}_B \bullet \frac{\partial \mathbf{r_B}}{\partial \theta_1} + \mathbf{I_C} \bullet \frac{\partial \mathbf{r_C}}{\partial \theta_1} + \mathbf{I_D} \bullet \frac{\partial \mathbf{r_D}}{\partial \theta_1} \qquad ...(8)$$

$$P_{\theta_2} = \mathbf{I}_A \bullet \frac{\partial \mathbf{r_A}}{\partial \theta_2} + \mathbf{I}_B \bullet \frac{\partial \mathbf{r_B}}{\partial \theta_2} + \mathbf{I_C} \bullet \frac{\partial \mathbf{r_C}}{\partial \theta_2} + \mathbf{I_D} \bullet \frac{\partial \mathbf{r_D}}{\partial \theta_2} \qquad ...(9)$$

where

$$\mathbf{r}_A = (x - l\cos\theta_1 - l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_1 + l\sin\theta_2)\,\mathbf{j} \qquad ...(10)$$

$$\mathbf{r}_B = (x - l\cos\theta_1 + l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_1 - l\sin\theta_2)\,\mathbf{j} \qquad ...(11)$$

$$\mathbf{r}_C = (x + l\cos\theta_1 + l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1 - l\sin\theta_2)\,\mathbf{j} \qquad ...(12)$$

$$\mathbf{r}_D = (x + l\cos\theta_1 - l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1 + l\sin\theta_2)\,\mathbf{j} \qquad ...(13)$$

But initially the impulsive force at A is in the direction of the positive y-axis, we have

$\mathbf{I}_A = I\,\mathbf{j}$. ...(14)

$\therefore$ Equation (6), (7), (8) and (9) $\Rightarrow$

$P_x = 0, P_y = \mathrm{I}, P_{\theta_1} = -\mathrm{I}l\cos\theta_1$ and $P_{\theta_2} = \mathrm{I}l\cos\theta_2$. ...(15)

Thus equations (2) and (3), (4) and (5) give

$$4m\dot{x} = 0, 4m\dot{y} = 1, \tfrac{8}{3}ml^2\dot{\theta}_1 = -\mathrm{I}l\cos\theta_1, \tfrac{8}{3}ml^2\dot{\theta}_2 = \mathrm{I}l\cos\theta_2. \quad ...(16)$$

Second Part.

We have $\dot{x} = 0$, $\dot{y} = \dfrac{1}{4m}$, $\dot{\theta}_1 = -\dfrac{3\mathrm{I}}{8ml}\cos\theta_1$ and $\dot{\theta}_2 = \dfrac{3\mathrm{I}}{8ml}\cos\theta_2$.

$$\therefore\ T = 2m\ (\dot{x}^2 + \dot{y}^2) + \tfrac{4}{3}ml^2\ (\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})$$

$$= 2m\left(0 + \frac{\mathrm{I}^2}{16m^2}\right) + \tfrac{4}{3}ml^2\left[\frac{9\mathrm{I}^2}{64m^2l^2}\cos^2\theta_1 + \frac{9\mathrm{I}^2}{64m^2l^2}\cos^2\theta_2\right] \quad ...(17)$$

But immediately after the application of the implusive forces,

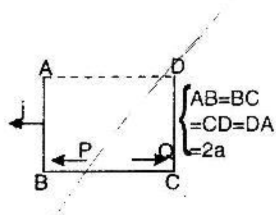
$\theta_1 = \frac{1}{2}\pi$ and $\theta_2 = 0$ approximately, so (17) gives $T = \dfrac{I^2}{2m}$.

Ex. 29. *Three equal uniform rods, AB, BC, CD are freely joined at B and C and the ends A and B are fastened to smooth, fixed pivots whose distance apart is equal to the length of either rod. The frame being at rest in the form of the square, a blow J is given perpendicular to AB at its middle point and in the plane of square. Show that the energy set up is $\dfrac{3J^2}{40m}$ where m is the mass of each rod.*

Find also the blows at the joints B and C.

Sol. The blow J is given at G_1 the C.G. of AB, at right angles to AB

The rods AB and CD will turn through the same angle θ but the rod **BC** will remain parallel to AD.



Now let T be the K.E. of the system, then we have

$$T = 2.\tfrac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right] + \tfrac{1}{2}m\,(2a\dot{\theta})^2 = \frac{10ma^2}{3}\dot{\theta}^2.$$

$\therefore$ Lagrange's θ-equation gives

$$\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \left(\frac{\partial T}{\partial\dot{\theta}}\right)_1 = J \Rightarrow \frac{20ma^2}{3}\dot{\theta} = Ja$$

i.e. $\dot{\theta} = \dfrac{3J}{20ma}$ $\quad\therefore\ T = \dfrac{10ma^2}{3}\cdot\dfrac{9J^2}{400m^2a^2} = \dfrac{3J^2}{40m}$.

SECOND PART.

Let P and Q be the impulses at B and C respectively. with directions, as shown in the figure, then considering the motion of AB and taking moments about A, we get

Change in the angular momentum about the axis through A

= moments of the impulses about this axis

i.e. $\quad m\,\dfrac{4a^2}{3}\,\dot{\theta} = Ja - P\;2a \Rightarrow P = \frac{1}{2}J - \dfrac{2ma}{3}$

$$\therefore\; P = \frac{1}{2}J - \frac{2ma}{3}\,\frac{3J}{20ma} = \frac{2J}{5}$$

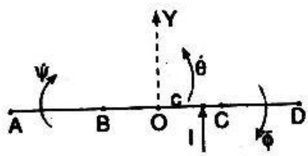
Again considering the motion of CD and taking moments about D, we obtain

$$m\,\frac{4a^2}{3}\,\dot{\theta} = Q\;2a \Rightarrow Q = \frac{2ma}{3}\,\dot{\theta} = \frac{2ma}{3}\cdot\frac{3J}{20ma} = \frac{1}{10}J.$$

Ex. 30. *Three equal uniform rods AB, BC, CD, each of mass m and length $2a$, are at rest in a straight line smoothly jointed at B and C. A blow I is given to the middle rod at a distance c from the centre O in a direction perpendicular to it, show that the initial velocity of O is $\dfrac{2I}{3T}$, and that the initial angular velocities of the rods are*

$\dfrac{(5a+9c)\,I}{10ma^2}, \dfrac{6cI}{5ma^2}$ and $\dfrac{(5a-9c)\,I}{10ma^2}$ **(Nagpur 1993)**

Sol. Just after the blow let



$\dot{y}$ be the linear velocity of O, and let θ, ϕ, ψ be the inclination of BC, CD and AB to their initial positions. Now let T be K.E. of the system, then we have

$$T = \frac{1}{2}\left(\frac{a^2}{3}\dot{\theta}^2 + \dot{y}^2\right) + \frac{1}{2}m\left\{\frac{a^2}{3}\dot{\psi}^2 + (\dot{y} - a\dot{\theta} + a\dot{\psi})^2\right\}$$
$$+\frac{1}{2}m\left\{\frac{a^2}{3}\dot{\phi}^2 + (\dot{y} + a\dot{\theta} + a\dot{\phi})^2\right\}$$

$$= \frac{1}{2}m\left[3\dot{y}^2 + \frac{7a^2}{3}\dot{\theta}^2 + \frac{4a^2}{3}\dot{\psi}^2 + \frac{4}{3}a^2\dot{\phi}^2 + 2a\dot{y}\dot{\psi} + 2a\dot{y}\dot{\phi} + 2a\dot{\theta}\dot{\phi} - 2a\dot{\theta}\dot{\psi}\right]$$

Also just before the action of the impluse, we have $T = 0$

$\therefore$ Lagrange's equation for blows gives

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2 - \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = I \Rightarrow 3m\dot{y} + ma\dot{\phi} + ma\dot{\psi} = I \;\Rightarrow 3\dot{y} + a\dot{\theta} + a\dot{\psi} = \frac{I}{m}.$$

Similarly Lagrange's θ, ϕ, ψ equations give

$$\frac{7}{3}a\dot{\theta}+a\dot{\phi}-a\dot{\psi}=\frac{Ic}{ma},\ \dot{y}+\frac{4}{3}a\dot{\phi}+a\dot{\theta}=0,\ \dot{y}-a\dot{\theta}+\frac{4}{3}a\dot{\psi}=0.$$

Last two equations $\Rightarrow 3\dot{y}+2a\dot{\phi}+2a\dot{\psi}=0$ $3\dot{y}=\frac{2I}{m}$, or $\dot{y}=\frac{2I}{3m}$

Also $2a\dot{\theta}=\frac{4a}{3}(\dot{\psi}-\dot{\phi})$, *i.e.* $\frac{3}{2}a\dot{\theta}+a\dot{\phi}-a\dot{\psi}=0$

$\therefore$ $\frac{5a}{6}\dot{\theta}=\frac{Ic}{ma}$ *i.e.* $\dot{\theta}=\frac{6Ic}{5ma^2}$.

Substituting now the values of $\dot{y}$ and $\dot{\theta}$ in the equation

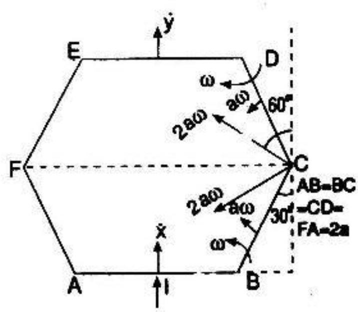
$\dot{y}+\frac{4}{3}a\dot{\phi}+a\dot{\theta}=0$ we get $\frac{2I}{3m}+\frac{4}{3}a\dot{\phi}+\frac{6Ic}{5ma}=0$ or $\dot{\phi}=\frac{(5a-9c)}{10ma^2}$.

Also substituting values of $\dot{y}$ and $\dot{\theta}$ in equation $\dot{y}-a\dot{\theta}+\frac{4}{3}a\dot{\psi}=0$,

we get $\frac{I}{3m}-\frac{6Ic}{5ma}+\frac{4}{3}a\dot{\psi}=0$ or $\dot{\psi}=\frac{(5a-9c)I}{10ma^2}$.

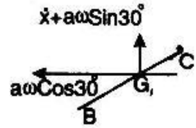
Ex. 31. *Six equal uniform rods form a regular hexagon, loosely jointed at the angular points, and rest on a smooth table ; a blow is given perpendicular to one of them at its middle point, find the resulting motion and show that the opposite rod begins to move with one tenth of the velocity of the rod that is struck.* **(Meerut 94; Rajasthan 1983; Agra 81, 89)**

Sol. The impluse is given at the middle point of AB in the direction as shown in the figure so the ensuing motion of BC, AF and CD, EF will be symmertrical.



Therefore they have after impluse same angular velocity say ω.

Also the rods AB and DE will begin to move at right angles to themselves. Now let $\dot{x}$ and $\dot{y}$ be their velocities just after the action of impluse.



$\therefore$ Velocity of C in the direction of $\dot{x}$

= Velocity of B + velocity of C relative to B

$=\dot{x}+2a\omega\sin 30^\circ=\dot{x}+a\omega$ ...(1)

and velocity of C in the same direction (with respect to the rod CD)

= Velocity of D + velocity of C relative to D,

$= \dot{y} - 2a\omega \sin 30^\circ = \dot{y} - a\omega$ ...(2)

$\therefore$ (1) and (2) $\Rightarrow \dot{x} + a\omega = \dot{y} - a\omega \Rightarrow a\omega = \frac{1}{2}(\dot{y} - \dot{x})$

Now actual velocity of the C.G of the rod BC is $a\omega \cos 30^\circ$ parallel to AB and $\dot{x} + a\omega \sin 30^\circ$ at right angles to AB

$\therefore$ (velocity)^2 of $G_1 = (a\omega \cos 30^\circ)^2 + (\dot{x} + a\omega \sin 30^\circ)^2$

Thus K.E. of BC

$$= \frac{1}{2}m\left[\frac{1}{3}a^2\omega^2 + \{(a\omega \cos 30)^2 + (\dot{x} + a\omega \sin 30)^2\}\right]$$

$= \frac{1}{2}m\left(\frac{4}{3}a^2\omega^2 + \dot{x}^2 + a\omega\dot{x}\right) = \frac{1}{12}m\left[5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}\right] = T_1$ say $\left[\because \; a\omega = \frac{1}{2}(\dot{y} - \dot{x})\right]$

Also Kinetic energy of CD

$$= \frac{1}{2}m\left[\frac{1}{3}a^2\omega^2 + \{(a\omega \cos 30)^2 + (\dot{y} - a\omega \sin 30)^2\}\right]$$

$$= \frac{1}{2}m\left[\frac{4}{3}a^2\omega^2 + \dot{y}^2 - a\omega\dot{y}\right] = \frac{1}{12}m\left[2\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y}\right) = T_2 \text{ say}$$

$\therefore$ Total Kinetic Energy T is given by $T = 2T_1 + 2T_2 + T_3 + T_4$ where T_3 and T_4 are K. E.'s of AB and DE respectively.

$$\therefore \; T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + 2.(m/12)\left[5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}\right]$$

$$+ 2.(m/12)\left[5\dot{y}^2 + 2\dot{x}^2 - \dot{x}\dot{y}\right]$$

or $\quad T = \frac{1}{3}m\left(5\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y}\right) \quad \therefore$ Lagrange's x-equation is

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 - \left(\frac{\partial T}{\partial \dot{x}}\right)_1 = I \Rightarrow \frac{1}{3}m\left(10\dot{x} - \dot{y}\right) = I$$

Lagrange's y-equation gives $\frac{1}{3}m\left(10\dot{y} - \dot{x}\right) = 0$

Solving (1) and (2), we get $\dot{x} = \frac{10}{33}\frac{I}{m}$ and $\dot{y} = \frac{I}{33m}$

Also $\dot{y} = (x/10)$ and $a\,\omega = \frac{1}{2}(\dot{y} - \dot{x}) \Rightarrow \omega = -\frac{3I}{22m}a$

Revision at a Glance

(1) Generalised Co-ordinates

Suppose that a particle or a system of N-particles moves subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates then needed to specify the motion. These co-ordinates denoted by q_1, q_2, ..., q_n are called generalised *co-ordinates. These co-ordinates may be distances, angles or quantities to them.*

(2) Degrees of freedom.

The number of co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

(3) Transformation equations.

Let $\boldsymbol{r}_v = x_v \boldsymbol{i} + y_v \boldsymbol{j} + z_v \boldsymbol{k}$ be the position vector of vth particle with respect to xyz co-ordinate system. The relationships of the generalised co-ordinates $q_1, q_2 \ldots, q_n$ to the position co-ordinates are given by the transformation equations.

$$\left.\begin{aligned} x_v &= x_v(q_1, q_2, \ldots, q_n, t) \\ y_v &= y_v(q_1, q_2, \ldots, q_n, t) \\ z_v &= z_v(q_1, q_2, \ldots, q_n\ t) \end{aligned}\right\} \quad \ldots(1)$$

where t denotes the time. In vector, (1) can be written as

$$\boldsymbol{r}_v = \boldsymbol{r}_v(q_1, q_2, \ldots, t) \quad \ldots(2)$$

where the functions in (1) or (2) are continuous and have continuous derivatives.

4. Classification of Mechanical systems.

(a) Scleronmic system.

The mechanical system in which t, the time, does not enter explicitly in equations (1) or (2) is called a sceleronomic system.

(b) Rhenomic system

The mechanical system in which the moving constraints are involved and the time t does enter explicitly is called a Rhenomic system.

(c) Holonomic system and Non Holonomic system.

(Meerut 1983 (P))

Let $q_1, q_2, \ldots, q_n$ denote the generalised co-ordinates describing a system and let t denote the time. If all the constraints of the sys.emm can be expressed as equations having the form $f(q_1, q_2 \ldots, q_n\ t) = 0$ or their equivalent, then the system is siad to be Holonomic otherwise it is be Non-Holonomic system.

(d) Conservative and non conservative system.

If the forces acting on the system are dervable from a potential function [or potential energy V], then the system is called conservative, otherwise it is non-conservative.

5. Kinetic energy and genralised velocities

The K.E. of the system is $T = \frac{1}{2} \sum_{v=1}^{n} m_v \dot{r}_v^2$.

The K.E. of the system can be written as a quadratic form in the generalised $\dot{q}_\alpha$ If the system is independent of time explicitly i.e. Sclemomic then the system is Rheonomic, linear terms in q_α are also present.

6. Generalised Forces.

If W is the total work done on a system of paticles by forces $\mathbf{F}_v$ *acting on the vth particle, then*

$$dW = \sum_{\alpha=1}^{n} \phi_\alpha dq_\alpha \text{ where } \phi_\alpha = \sum_{v=1}^{N} \mathbf{F}_v \cdot \frac{\partial \mathbf{r}_v}{\partial q_\alpha}$$

is called the generalised force associated with generalised coordinates q_α.

7. Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_1 = \frac{\partial W}{\partial q_\alpha}$$

For $\alpha = 1, 2, ..., n$, *we have n different equations which are called Lagrange's equations.*

Note. The quantity $q_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$ is called the generalised momenttum associated with the general co-ordinate q_α.

8. If the forces are derivable from a potential V, then

$$\phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha},$$

since the potential, or potential energy is a function of q's only (and possible the time t) then, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}(T - V)\right] - \left(\frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0, \text{ where } L = T - V \qquad ...(19)$$

The function L defined by $L = T - V$ *is said to be Lagrangian function.*

9. Generalising momentum.

We defined $p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$ *to be the generalised momentum associated with generalised co-ordinate* q_α. *We usually refer* p_α *as the momentum conjugate to* q_α, *or the conjugate momentum.*

In case the system is conservative, we have

$$T = L + V \Rightarrow (\partial T/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha) + (\partial V/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha)$$

because V, the P.E. system does not depend upon q_α

$$\therefore\ q_\alpha = (\partial L/\partial \dot{q}_\alpha).$$

10. Lagrange's Equation with Impulsive forces.

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_2 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 = \Sigma\, \mathbf{I}_v \cdot \frac{\partial \mathbf{r}_v}{\partial q_\alpha} = P_\alpha \text{ (say)}$$

where subscripts 1 and 2 denote respective quantities before and after the application of the i plusive forces.

These equations are known as Lagrange's equatious under impuisive forces.

Hamiltonian Formulation and Variational Principles

9·00. Hamilton's form of the equations of motion.

Here we shall obtain the differential equations of motion of a conservative holonomic dynamical system in a form which constitutes the basis of most of the advanced theory of dynamics. Let $(q_1, q_2 ...., q_n)$ be the generalised co-ordinates and let $L(q_1, q_2, ..., q_n ; \dot{q}_1, \dot{q}_2 ..., \dot{q}_n ; t)$, the kinetic potential of the system, so that the equations of motion in the Lagrangian form are

$$\frac{d}{dt}(\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0 ; \quad (i = 1, 2, ..., n) \qquad ...(1)$$

writting $p_i = (\partial L/\partial \dot{q}_i)$ we get $\dot{p}_i = (\partial L/\partial q_i)$ $(i = 1, 2, ..., n)$ ...(2)

hence from the former of these sets of equations we can regard either of the sets of quantities $(q_1, q_2 ...., q_n)$ or $(p_1, p_2, ..., p_n)$ as functions of the other set.

Now, let δ denote the increment in any function of the variables $(q_1, q_2 ...., q_n ; p_1, p_2 ...., p_n)$ or $(q_1, q_2 ...., q_n ; \dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$; then we get

$$dL = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) + \frac{\partial L}{\partial t} \quad \text{(when } L \text{ contains } t \text{ explicitly)}$$

$$= \sum_{i=1}^{n} (\dot{p}_1 \, dq_i + p_i \, d\dot{q}_i) + \frac{\partial L}{\partial t} dt$$

$$= d\left(\sum_{i=1}^{n} p_i \dot{q}_i\right) + \sum_{i=1}^{n} (\dot{p}_i \, dq_i - \dot{q}_i \, dp_i) + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow d\left[\sum_{i=1}^{n} (p_i \dot{q}_i) - L\right] = \sum_{i=1}^{n} (\dot{q}_i \, dp_i - p_i \, dq_i) - (\partial L/\partial t) \, dt.$$

Thus if the quantity $\sum_{i=1}^{n} (p_i \dot{q}_i - L)$ when expressed in terms of $(q_1, q_2 ...., q_n ; p_1, p_2 ...., p_n ; t)$ be denoted by H, we have

$$dH = \sum_{i=1}^{n} (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt = \sum_{i=1}^{n} (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i}, \; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If H does not contain t explicity (i.e. does not contain t explicity) we have $\dot{p}_i = -(\partial H/\partial q_i)$ and $\dot{q}_i = (\partial H/\partial p_i)$. ...(4)

These equations are called as Hamilton's equations, or Hamilton's canoncial equations and the function H is called Hamiltonian.

The total order of Hamilton's equations is the same as the total order of Lagrange's equations, nameiy $2n$. But whereas Lagrange's equations present us with n equations each of the second order. Hamilton's equations are $2n$ equations, each of the first order. Hamilaton's equations can also be written as $\dfrac{dp_i}{-(\partial H/\partial q_i)} = \dfrac{dq_i}{(\partial H/\partial p_i)} = dt.$

9·01. Physical significance of the Hamiltonian. **[Meerut 1995]**

If the Hamiltonian H is independent of t explicity, prove that it is **(a)** *constant and* **(b)** *equal to the total energy of the system.*

Proof. **(a)** We have $\dfrac{dH}{dt} = \sum_{i=1}^{n} \dfrac{\partial H}{\partial q_i} \dfrac{dq_i}{dt} + \sum_{i=1}^{n} \dfrac{\partial H}{\partial p_i} \dfrac{dp_i}{dt}$

$$= \sum_{i=1}^{n} -(\dot{p}_i)\,\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \, \dot{p}_i \left(\because \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i} \right) = 0$$

$\Rightarrow H =$ constant, say E.

(b) By Euler's theorem on homogeneous function, we have

$\Sigma \dot{q}_i \dfrac{\partial T}{\partial \dot{q}_i} = 2T$, where T is the K.E. of the system.

But $L = T - V$, $\therefore \dfrac{\partial L}{\partial \dot{q}_i} = \dfrac{\partial (T-V)}{\dot{q}_i} = \dfrac{\partial T}{\partial \dot{q}_i}$ (V does not depend on $\dot{q}_i$)

or $\Sigma \dot{q}_i \dfrac{\partial L}{\partial \dot{q}_i} = 2T \Rightarrow \Sigma \dot{q}_i \, p_i = 2T \left(\because \quad p_i = \dfrac{\partial L}{\partial \dot{q}_i} \right)$

$$\therefore \quad H = \Sigma p_i \dot{q}_i - L = 2T - (T - V) = T + V = E.$$

9·02. Passage from the Hamiltanian to the Lagrangian.

Suppose that we are given a function $^*H(q, p, t)$ and are told that the motion of the system satisfies the canoncial equations

$\dot{p}_i = -(\partial H/\partial q_i)$ and $\dot{q}_i = (\partial H/\partial p_i)$ ...(1)

Then we want to find a function

$$L\ (p_1, p_2, \ldots, p_n\ ;\ q_1, q_2, \ldots, q_n\ ; t),\ i.e.\ L\ (p, q, t)$$

such that the motion also satisfies the equations

$$(d/dt)\ (\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0. \qquad \ldots(2)$$

Solve the first set of equations in (1) for the p's in terms of the q's the $\dot{q}$'s and t.

Then write $L = \sum_{i=1}^{n} \dot{q}_i\ p_i - H$ and express L as a function of the q's, the $\dot{q}$'s and t. This is the required Lagrangian.

$$\therefore\ (\partial L/\partial \dot{q}_i) = p_i \left(\text{using } L = \sum_{i=1}^{n} p_i\,\dot{q}_i - H \right)$$

$$\Rightarrow (d/dt)\ (\partial L/\partial \dot{q}_i) = \dot{p}_i \text{ and } (\partial L/\partial q_i) = -\ (\partial H/\partial q_i)$$

$$\Rightarrow \frac{d}{dt}\ (d\,L/\partial \dot{q}_i) - (\partial L/\partial q_i) = \dot{p}_i + (\partial H/\partial q_i) = 0$$

i.e. L satisfies (2) assuming (1).

9·03. Variational Methods.

9·03·1. Techniques of Calculus of Variations. (Meerut 1985, 90, 93)

The calculus of variations arose out of the quest for the mathematical requirments in the solution of problems like the study of :

(i) The path followed by a body falling freely under gravity, (brachistochrone) first studied by Newton,

(ii) the equilibrium shape of a freely hanging homogeneous flexible cord between two horizontal points (catenary), first studied by Bernoulli.

Presently the technique of calculus of variations, serves as a mathamatical preliminary (in the form of Euler-Lagrange's equation) in the study of a wide range of physical problems such as geodesics and minimal surface in Riemannian and differential geometrics and the various *variational (minimal)* principles in the different branches of physics, The technique is co-ordinates invariant and there in lies its great power. First we develop here this technique in a purely mathematical form.

Suppose A and B are fixed points (x_1, y_1), (x_2, y_2) in a cartesian plane. Also suppose that (x, y, y') is known functional form of the variables $x, y, y'\ \{= dy/dx\}$. Then if C is a curve joining A and B and having equation $y = y(x)$, Then the integral

$$I = \int_{x_1}^{x_2} f\left[x, y, y'\right] dx \qquad \ldots(1)$$

has a definite value whenever the function $y(x)$ is prescribed. The value of I will change as we vary the form of the curve C through A, B. Consequently we may consider that in general there will be some curve

---

*$H\ (q_1, q_2, \ldots, q_n, p_1, \ldots, p_n, t)$ is also written as $H\ (q, p, t)$.

C through these fixed points such that the value of I *taken along it is stationary (in general a* maximum or minimum) compared with the value along neighbouring paths C. Calculus of variations, the branch of mathematics of concerned amongst other things with finding the form of $y(x)$ for which this stationary property holds. Let the path $\overline{C}$ have the equation $y = \overline{y}(x)$ and let the equation of a neighbouring curve C be $y = \overline{y}(x) + \varepsilon\,\eta(x)$, where ε is small and $\eta(x)$ is an arbitrary continuous differentiable function through A and B.

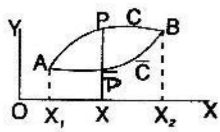
Now we have $PX = \overline{y}(x) + \varepsilon\,\eta(x)$, and $\overline{P}X = \overline{y}(x) \Rightarrow P\overline{P} = \varepsilon\,\eta(x)$.

Hence the values of I taken along is thus a function of ε of the form

$$I(\varepsilon) = \int_{x_1}^{x_2} f\,[x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta']\,dx = \int_{x_1}^{x_2} f\,[x, y, y']\,dx \qquad \ldots(2)$$

where $y = \overline{y} + \varepsilon\eta$ and $y' = \overline{y}' + \varepsilon\eta'$

But the curve $\overline{C}$, for which $\varepsilon = 0$, makes I stationary, so we must have $I'(0) = 0$. Now differentiating (2) w.r.t. ε, we get



$$I'(\varepsilon) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \varepsilon}\right] dx$$

$$\left[\because \quad f = f\,[x, y, y']\right] \qquad \ldots(3)$$

$$\Rightarrow I'(\varepsilon) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\,\eta(x) + \frac{\partial f}{\partial y'}\,\eta'(x)\right] dx$$

$$\left[\because \quad y = \overline{y} + \varepsilon\,\eta(x)]\right]$$

$$= \int_{x_1}^{x_2} [f_y \cdot \eta(x) + f_{y'} \cdot \eta'(x)]\,dx$$

$$I'(\varepsilon) = \int_{x_1}^{x_2} [\eta f_y(x, \overline{y} + \varepsilon\,\eta, \overline{y}' + \varepsilon\,\eta')]$$

$$+ \eta' f_{y'}(x, \overline{y} + \varepsilon\,\eta, \overline{y}' + \varepsilon\,\eta')]\,dx \qquad \ldots(4)$$

where $f_y\,(x, \overline{y} + \varepsilon\,\eta, \overline{y}' + \varepsilon\,\eta')$

$= (\partial/\partial y)\,f(x, y, y')$ when $y = \overline{y} + \varepsilon\,\eta$, $y' = \overline{y}' + \varepsilon\,\eta$,

with a similar meaning for $f_{y'}\,(x, \overline{y} + \varepsilon\,\eta, \overline{y}' + \varepsilon\,\eta')$. Hence the stationary condition gives $I'(0) = 0$.

$$\Rightarrow \int_{x_1}^{x_2} [\eta f_{y'}(x, \overline{y}, \overline{y}') + \eta' f_{y'}(x, \overline{y}, \overline{y}')]\,dx = 0 \qquad \ldots(5)$$

Now $\displaystyle\int_{x_1}^{x_2} \eta' f_{y'}(x, \overline{y}, \overline{y}')\,dx$

$$= \left[\eta f_{y'}(x, \bar{y}, \bar{y}') \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x, \bar{y}, \bar{y}')\, dx$$

$$= - \int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x, \bar{y}, \bar{y}')\, dx \quad (\because \quad \eta(x_1) = \eta(x_2) = 0) \qquad ...(6)$$

Using (6), equation (5) implies

$$\int_{x_1}^{x_2} \eta(x) \left[f_y(x, \bar{y}, \bar{y}') - \frac{d}{dx} f_{y'}(x, \bar{y}, \bar{y}') \right] dx = 0 \qquad ...(7)$$

[using (5)]

But $\eta(x)$ is arbitrary, subject to its being differentiable and vanishing at A, B ; (7) implies that

$$f_y(x, \bar{y}, \bar{y}') - (d/dx) f_{y'}(x, \bar{y}, \bar{y}') = 0 \qquad ...(8)$$

or $$(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0. \qquad ...(9)$$

This is called **Euler's Lagrange's equation**. Equation (9) can be proved from (7) by using the following lemma.

Lemma. *If* x_1 and $x_2 (> x_1)$ *are fixed constants and* $\phi(x)$ *is a particular continuous function for* $x_1 \le \phi \le x_2$ and *if*

$$\int_{x_1}^{x_2} \eta(x)\, \phi(x)\, dx = 0 \qquad ...(10)$$

for every choice of continuously differentiable function $\eta(x)$ *for which* $\eta(x_1) = \eta(x_2) = 0$ *then* $\phi(x) = 0$ *indentically in* $x_1 \le x \le x_2$.

Proof. Let the lemma be not true for all x *in* $x_1 \le x \le x_2$. Let x' be a point where $\phi(x') \ne 0$ and > 0. But $\phi(x)$ is continuous in $x_1 \le x \le x_2$ and in particular is continuous at $x = x'$. Hence there exists an interval $x_1' \le x \le x_0'$ around x' where $\phi(x) > 0$. For the other x's $\phi(x)$ may not vanish ; then the equation $\int_{x_1}^{x_2} \eta(x)\, \phi(x)\, dx = 0$

could be written as $\int_{x_1'}^{x_2'} \eta(x)\, \phi(x)\, dx = 0$.

But $\eta(x)$ is at our disposal ; we could choose it as greater than zero for $x_1' \le x \le x_2'$ and $\eta(x) = 0$ for other values of x. Thus $\eta(x)\, \phi(x) > 0$ for $x_1' \le x \le x_2'$; consequently integrand cannot be zero. Thus we arrive at the contradiction. Hence $\phi(x) = 0$ in $x_1 \le x \le x_2$.

Certain remarks about Euler-Lagrange's equation.

(A) We have $(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0$

where $f = f(x, y, y')$

which can be re-written as

$(\partial f/\partial y) - (dp/dx) = 0$ where $p = (\partial f/\partial y')$ ...(11)

But $\left(\dfrac{dp}{dx}\right) = \left(\dfrac{\partial p}{\partial x}\right) + \left(\dfrac{\partial p}{\partial y}\right)\left(\dfrac{dy}{dx}\right) + \left(\dfrac{\partial p}{\partial y'}\right)\left(\dfrac{dy'}{dx}\right) = \left(\dfrac{\partial^2 f}{\partial x \partial y'}\right) +$
$(\partial^2 f/\partial y\ \partial y')\ (dy/dx) + (\partial^2 f/\partial y'^2)\ (d^2y/dx^2)$. Hence (9) gives

$$(\partial f/\partial y) - [(\partial/\partial x)\ (\partial f/\partial y') + (\partial/\partial y)\ (\partial f/\partial y')\ (dy/dx) + (\partial/\partial y')\ (\partial f/\partial y')\ (dy'/dx)] = 0$$

or $$(\partial f/\partial y) - [(\partial^2 f/\partial x/\partial y') + (\partial^2 f/\partial y\, \partial y')\ (dy/dx) + (\partial^2 f/\partial y'^2)\ (d^2y/dx^2)] = 0$$

or $$(\partial^2 f/\partial y'^2)\ (d^2y/dx^2) + (\partial^2 f/\partial y\ \partial y')\ (dy/dx) - (\partial f/\partial y) + (\partial^2 f/\partial x\ \partial y') = 0 \quad ...(12)$$

This is a second order differential equation for determining y as a function of x. The solution will contain two arbitrary constants which could be determined from the conditions :

$$y = y_1 \text{ at } x = x_1 \text{ and } y = y_2 \text{ at } x = x_2 \quad ...(13)$$

(B) Consider $\dfrac{d}{dx}\left[f - y'\dfrac{\partial f}{\partial y'}\right] = \dfrac{\partial f}{\partial y'}$ ...(14)

i.e. $\dfrac{dQ}{dx} = \dfrac{\partial f}{\partial x}$ which $\dot{Q} = f - y'\dfrac{\partial f}{\partial y'}$ and f is a function of x, y, y'

$$\Rightarrow \frac{\partial}{\partial x}\left[f - y'\frac{\partial f}{\partial y'}\right] + \frac{\partial}{\partial y}\left[f - y'\frac{\partial f}{\partial y'}\right]\frac{dy}{dx} + \frac{\partial}{\partial y'}\left[f - y'\frac{\partial f}{\partial y'}\right]\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x}$$

$$\Rightarrow - y'\frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial f}{\partial y}\frac{dy}{dx} - y'\frac{\partial^2 f}{\partial y \partial y'}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{d^2y}{dx^2} - \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x^2} - y'\frac{\partial^2 f}{\partial y'^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow y'\left[\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2}\right] = 0$$

$$\Rightarrow f_y - f_{xy'} - y'\ f_{yy'} - y''\ f_{y'y'} = 0 \text{ (if } y' \neq 0) \quad ...(15)$$

which is same as (12) and shows that if Euler's equation is satisfied equation (14) is also satisfied but if (14) is satisfied, Euler's equation may not be true.

(C) If f does not cotain x explicity, *i.e.* $I = \int_{x_1}^{x_2} f(y, y')\, dx$,

(Meerut 1985, 91)

then from (14), we have

$$(d/dx)\ \{f - y'\ (\partial f/\partial y')\} = 0 \qquad \{\because\ (df/x) = 0\}$$

$$\Rightarrow f - y'\ (\partial f/\partial y') = \text{constant} = C \text{ (say)}. \quad ...(16)$$

Obviously (14) is a first order differential equation.

(D) If f does not contain y explicity, then we have $I = \int_{x_1}^{x_2} f(x, y')\, dx$.

$\therefore$ from the equation

$(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0$, we obtain $(d/dx)(\partial f/\partial y') = 0$

$$\Rightarrow (\partial f/\partial y') = C \text{ (say)}. \quad \ldots(17)$$

(E) If we assume that the end points are not fixed.

i.e. $\eta(x_1) \neq 0$ and $\eta(x_2) \neq 0$.

Euler's equation still holds in this case, if $(\partial f/\partial y') = 0$ when $x = x_1, x_2$

This becomes the boundary condition of the problem.

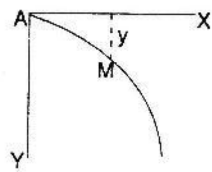
(F) Euler's equation gives only a necessary condition ; even if it is satisfied, there may be no extremum.

(G) The conditions to be imposed on f are such that its partial derivaties are continuous and differentiable.

(H) The conditions to be imposed on the curves are that they should be continuous curves.

9·03–2. Brachistochrone Problem.* **(Meerut 1994)**

Given two poits A and B in a vertical plane, to find for the moveable particle M, the path AMB, descending along which by its own gravity and beginning to be urged from the point A, it may in the shortest time reach the point B. It is a tacit in the statement that the particle decends without friction.



Let A be the origin, then velocity of the particle of mass M at any time t is given by

$$v = (ds/dt) = \sqrt{(2gy)}$$

$$\Rightarrow dt = (ds / \sqrt{(2gy)}$$

$$\Rightarrow t = \int_A^B \frac{ds}{\sqrt{(2gy)}} = \int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{(2gy)}}\, dx$$

$$= \frac{1}{\sqrt{(2g)}} \int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\, dx = \frac{1}{\sqrt{(2g)}} \int_A^B f(y, y')\, dx,$$

$$\text{where } f(y, y') = \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}$$

As x is absent in f, so we have $f - y'(\partial f/\partial y') = A$

$$\Rightarrow \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y' \cdot \frac{y'}{\sqrt{\{y(1+y'^2)\}}} = A$$

$$\Rightarrow \frac{1}{\sqrt{\{y(1+y'^2)\}}} = \text{constant} \Rightarrow y(1+y'^2) = \text{constant} = 2c, \text{ (say)}$$

But $(dy/dx) = \tan\psi \Rightarrow y' = \tan\psi$

$\therefore y \sec^2\psi = 2c$ or $y = 2c\cos^2\psi = c(1+\cos 2\psi)$

Also $dx = \cot\psi\, dy = -2c\sin 2\psi \cot\psi\, d\psi$

$$= -4c\cos^2\psi\, d\psi = -2c(1+\cos 2\psi)\, d\psi$$

$$\Rightarrow x = a - 2c\left(\psi + \frac{\sin 2\psi}{2}\right) = a - 2c\,(\psi + \sin\psi\cos\psi)$$

Hence the required path is

$$\mathbf{x = a - c\,(2\psi + \sin 2\psi)\,,\ y = c\,(1 + \cos 2\psi)\ (cycloid)}$$

9·03·3. Extension of the variational method.

Suppose the n-co-ordinates $q_1, q_2, \ldots, q_n$ are each functions of an independent variable t and that we require the solution of the variational problem $I = \int_{t_1}^{t_2} f(q_1, q_2, \ldots, q_n\,;\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\,;t)\,dt$

$$= \int_{t_2}^{t_1} f\,[q, \dot{q}, t]\,dt = \text{stationary value}$$

where f is of known functional form, t_1 and t_2 are fixed and each q_i is to be determined.

Let $q_i = q_i(t)$ give the stationary value to I and let $q_i = q_i(t) + \varepsilon_i\eta_i(t)$ be the neighbouring path.

$$\Rightarrow I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \int_{t_1}^{t_2} f\{q_i(t) + \varepsilon_i\eta_i(t)\,;\ \dot{q}_i(t) + \varepsilon_i\eta_i(t), t\}\,dt$$

$$= \int_{t_1}^{t_2} f(q_1, \dot{q}_i\,;\,t)\,dt + \Sigma\varepsilon_i\int_{t_1}^{t_2}\left\{\eta_i(t)\frac{\partial f}{\partial q_i} + \dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\}dt + 0(\varepsilon_i^2)$$

First variation $\quad \delta I = \Sigma\varepsilon_i\displaystyle\int_{t_1}^{t_2}\left\{\eta_i(t)\frac{\partial f}{\partial q_i} + \dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\}dt$

$$= \Sigma\varepsilon_i\int_{t_1}^{t_2}\eta_i(t)\frac{\partial f}{\partial q_i}\,dt + \Sigma\varepsilon_i\int_{t_1}^{t_2}\dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\,dt$$

$$= \Sigma\varepsilon_i\int_{t_1}^{t_2}\eta_i(t)\frac{\partial f}{\partial q_i}\,dt + \Sigma\varepsilon_i\left[\left\{\eta_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2}\eta_i(t)\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)dt\right]$$

$$= \Sigma\varepsilon_i\int_{t_1}^{t_2}\eta_i(t)\left[\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)\right]dt \qquad \{\because\ 0 = \eta_i(t_1) = \eta_i(t_2)\}.$$

As η_i's are arbitrary and I is stationary, so we have

$$(\partial f/\partial q_i) - (d/dt)\,(\partial f/\partial \dot{q}_i) \qquad (i = 1, 2, \ldots, n)$$

9·03·4. Hamilton's variational principle. **[Meerut 1991, 94]**

(Proof based on calculus of variation)

Variational (mimimal) principles has been the greatest fascination of old generation physicists and in this pursuit several and varied variational principles have been put forth in the varius branches of physics, such as, in optics, the Fermat's principles of *'lest time'*, in machanics, the Gauss's principles of *'least constraint'*, the Hertz's principle of *'lest curvature'* and most important of all the Hamilton's *variational principle'* which will be our main centre of attraction here in mechanics. The great value of these variational principles; lies in thier extreme economy of expression.

Let T, V be the kinetic and potential energies of a. Conservative holonomic dynamical system defined by n generalised co-ordinates $q_1, q_2, \ldots, q_n$ at time t. Writing $L = T - V$, we know that Lagrange's equations for the motion of the system are :

$$(\partial L/\partial q_i) - (d/dt)(\partial L/\partial \dot{q}_i) = 0 \qquad (i = 1, 2, \ldots, n)$$

Now applying previous article we can say that these are the n Euler-Lagrange's equation arising from the variational problem,

$$\int_{t_1}^{t_2} L(q_1, q_2, \ldots, q_n ; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt = \text{stationary} \quad (\textit{i.e.}\ \textbf{stat})$$

where t_1, t_2 are fixed. Thus we have established that

During the motion of a 'conservative holonomic dynamical system over a fixed time interval, the time integral over that interval of the difference between the kinetic and potential energies is stationary. This is Hamilton's principle.

9·03·5. Derivation of Hamilton equations from the variational principle.

Hamilton's principle implies

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0 \quad \text{or} \quad \delta \int_{t_1}^{t_2} [\sum_i p_i \cdot \dot{q}_i - H(q, p, t)]\, dt = 0$$

$$[\because \quad L = \sum_i p_i\, \dot{q}_i - H(q, p, t)]$$

This implies $\delta \sum_i \int_{q_1}^{q_2} p_i\, dq_i - \delta \int_{t_1}^{t_2} H\, dt = 0.$ ...(1)

Equation (1) is called the modified Hamilton's principle.

Now $\delta I = \dfrac{\partial I}{\partial \alpha} d\alpha = d\alpha \dfrac{\partial}{\partial \alpha} \displaystyle\int_{t_1}^{t_2} (\sum_i p_i\, \dot{q}_i - H(q, p, t)\, dt = 0$

where $\delta = d\alpha(\partial/\partial_\alpha)$

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*This problem was set by the Johan Bernoulli June, 1696 before the scholars of his time Although Newton had earlier considered at least one problem falling within. the province of the Calculus of variations, the proposal of Bernoulli brachistochrone problem marked the real beginning of general interest in the subject. Actually the term *Brachicstochrone* derives from the greek Brachistos, shortest and chronos, time.

$$\Rightarrow d\alpha \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha}[\sum_i p_i \dot{q}_i - H\ (q, p, t)]\, dt = 0$$

[$\because$ the times t_1, t_2 are not varied and so they are not functions of α, thus the differentiation can be interchanged].

$$\Rightarrow d\alpha \int_{t_1}^{t_2} \sum_i \left[\frac{\partial p_i}{\partial \alpha}\dot{q}_i + \frac{\partial \dot{q}_i}{\partial \alpha_i} p_i - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha}\right] dt = 0 \qquad \text{...(2)}$$

Further, we have $\int_{t_1}^{t_2} \frac{\partial \dot{q}_i}{\partial \alpha} p_i\, dt = \int_{t_1}^{t_2} p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right) dt = p_i\left[\frac{\partial q_i}{\partial \alpha}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt.$

But all the varied paths have the same end points. Hence $\frac{\partial \dot{q}_i}{\partial \alpha}$ vanishes for t_1 and t_2, $\Rightarrow \int_{t_4}^{t_2} \frac{\partial q_i}{\partial \alpha} p_i\, dt = -\int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt = \int_{t_2}^{t_1} p_i \frac{\partial q_i}{\partial \alpha}\, dt.$

Making this substitution in equation (2), we have

$$d\alpha \int_{t_1}^{t_2} \Sigma\left[\frac{\partial p_i}{\partial \alpha}\dot{q}_i - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha} - \dot{p}_i \frac{\partial q_i}{\partial \alpha}\right] dt = 0 \qquad \text{...(2)}$$

$$\Rightarrow \int_{t_1}^{t_2} \Sigma \dot{q}_i \frac{\partial \dot{q}_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha} d\alpha - p_i \frac{\partial q_i}{\partial \alpha} d\alpha\, dt = 0 \qquad \text{...(3)}$$

But $\delta p_i = d\alpha . (\partial p_i / \partial \alpha)$ and $\delta q_i = (\partial q_i / \partial \alpha)\, d\alpha$

$$\therefore\ (3) \Rightarrow \int_{t_1}^{t_2} \Sigma\left[\delta p_i\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) + \delta q_i\left(\frac{\partial H}{\partial q_i} - \dot{p}_i\right)\right] dt = 0. \qquad \text{...(4)}$$

The variation δq_i and δp_i are independent to each other, hence equation (4) holds good only when the coefficients of δp_i and δq_i vanish separately.

This (4) $\Rightarrow \dot{q}_i = (\partial H/\partial p_i)$, $\dot{p}_i = -(\partial H/\partial q_i)$. These are Hamilton's equations.

9·03·6. Principal of Least Action.

Principle of least action states that if T is kinetic energy, at time t, of a conservative, holonomic dynamical system specified by the generalised co-ordinates, then the integral $I = \int_{t_1}^{t_2} 2T\, dt$

has necessary an extreme value, minimum or maximum, on actual path as compared with varied path as the systems passes from one configuration at time t_0 to another configuration at time t_1. **[Meerut 1992,93,94]**

We know that $L = T - V$, i.e. Lagrangian = K.E.– P.E.
and $T + V = E$ (const), since system is conservative.
But by Hamilton's principle, we know that

$$\int_{t_0}^{t_1} \delta L\, dt = 0 \Rightarrow \int_{t_0}^{t_1} \delta(T - V)\, dt = 0 \quad \int_{t_0}^{t_1} \delta\,(2T - E)\, dt = 0$$

$$\Rightarrow \int_{t_0}^{t_2} \delta\,(2T)\, dt = 0 \quad \Rightarrow \delta \int_{t_0}^{t_1} (2T)\, dt = 0. \qquad ...(1)$$

[$\because$ $\delta E = 0$ as E, the total energy is const.]

Result (1), is known as **Principle of least action.**

Equation (1) can also be written as $\delta A = 0$, where $A = \int_{t_0}^{t_1} 2T\, dt$

and is defined by action as follows :
This implies that principal of least action states that the action in the actual path is minimum compared with the varied path, as the system passes from one configuration to another.

9·03– 7. Distinction between Hamilton's Principle and Principle of Least Action. **(Meerut 92,93,94)**

In Hamilton's principle, the time of description $t_1 - t_0$ is prescribed (fixed) as the body moves from one configuration to another configuration, while in the principle of least action there is no such restriction on the time $t_1 - t_0$, but the total energy between the end points A and B is prescribed.

Deduction of Lagrange's Equation from Hamilton's Principle.

(Meerut 93)

By Hamilton's principle, we have $\int_{t_0}^{t_1} \delta L\, dt = 0$

$$\therefore\ (1) \Rightarrow \int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\frac{L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i\, dt + \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i)\, dt = 0$$

Now integrating by parts, we have

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i\, dt + \left\{ \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right\}_{t_0}^{t_1} - \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{n} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right\} dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} \right\} \delta q_i \right] dt = 0$$

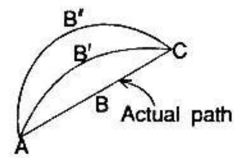
[$\because$ middle term vanishes as all $\delta q_i = 0$ at t_0 and t_1]

$$\Rightarrow \sum_{i=1}^{n}\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}\right\}\delta q_i = 0.$$

Here δq_i's are arbitrary and independent to each other, so equating to zero their coefficients, we readily obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

9·03·8. Deduction of Hamilton's principle (*Proof based on D'Alembert's principle*). Assume that the conservative holonomic dyamical system moves from A to C, where A and C, are the initial and final configurations of the system at time t_1 and t_2 respectively. Let ABC be the actual path and $AB'C.AB''C$ the two neighbouring paths out of infinite number of possibilities. *In order to deduce the principle, the following two conditions must be satisfied.*



1. δt **must be equal to zero at** A **and** B *i.e.* at t_1 the particle must be at A, at t_2 the particle must be at C.
2. δr **must be equal to zero at** A and $C \Rightarrow$ that the points A, and C are fixed in space.

Now assume that the system is acted upon by a number of forces represented by $\mathbf{F}$.

Let the ith particle of system be acted upon by a force

$\mathbf{F}_i = m_i\ddot{r}_i$ where $\ddot{r}_i$ is acceleration vector.

Agian by D'Alembert's principle, we have

$$\sum_i(\mathbf{F}_i - m\ddot{\mathbf{r}}_i)\bullet\delta\mathbf{r}_i = 0,\ i.e.\ \sum_i \mathbf{F}_i\bullet\delta\mathbf{r}_i - \sum m_i\ddot{\mathbf{r}}_i\bullet\delta\mathbf{r}_i = 0 \quad ...(1)$$

If there is a little variation along the actual and neighbouring paths, then $\delta\mathbf{r}_i = \mathbf{r}_i' - \mathbf{r}_i$ (say).

$$\Rightarrow (d/dt)(\delta r_i) = (d/dt)(\mathbf{r}_i' - \mathbf{r}_i) = (d\mathbf{r}_i'/dt) - (d\mathbf{r}_i/dt)$$

$$= \delta(d\,\mathbf{r}_i/dt) = \delta(d\dot{\mathbf{r}}_i). \quad ...(2)$$

where dashes have been used for neighbouring paths.

$$\text{But}\ \ddot{\mathbf{r}}_i\bullet\delta_1 = (d/dt)(\dot{\mathbf{r}}_i\bullet\delta\mathbf{r}_i) - \dot{\mathbf{r}}_i\bullet(d/dt)(\delta\mathbf{r}_i). \quad ...(3)$$

$$\text{Using (2), (3)} \Rightarrow \ddot{\mathbf{r}}_i\bullet\delta\mathbf{r}_i = (d/dt)(\dot{\mathbf{r}}_i\bullet\delta\mathbf{r}_i) - \dot{\mathbf{r}}_i\,\delta(\dot{\mathbf{r}}_i) \quad ...(4)$$

$$\therefore (1) \Rightarrow \sum_i \mathbf{F}_i\bullet\delta\mathbf{r}_i - \sum_i m_i[(d/dt)(\mathbf{r}_i\bullet\delta\mathbf{r}_i) - \dot{\mathbf{r}}_i\bullet\delta\dot{\mathbf{r}}_i] = 0$$

or $\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i - \sum_i m_i [(d/dt)(\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i) - \frac{1}{2}\delta(\ddot{\mathbf{r}}_i^2)] = 0$

or $\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i + \sum_i \frac{1}{2} m_i \,\delta(\dot{\mathbf{r}}_i^2) = \sum_i (d/dt)(m_i \ddot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i)$

or $\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i + \delta(\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2) = \sum_i (d/dt)(m_i \mathbf{r}_i \bullet \delta\mathbf{r}_i)$ ...(5)

But $\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$ = kinetic energy of the system $= T$

and $\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i$ = work done by the forces $\mathbf{F}_i$ during displacement $\delta\mathbf{r}_i = \delta W$ (say).

$\therefore$ equation (5) $\Rightarrow \delta W + \delta T = \sum_i (d/dt)(m_i \dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i)$ ...(6)

Intergrating (6) between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i)\, dt = \sum \int_{t_1}^{t_2} d\,(m_i \mathbf{r}_i \bullet \delta\mathbf{r}_i)$$

$$= \sum \left[m^2 \mathbf{r}_i^2 \bullet \delta\mathbf{r}^2 \right]_A^C = 0 \text{ [since } \delta\mathbf{r}^2 = 0 \text{ at the end points } A \text{ and } C.$$

But we know that, for a conservative system,

$\delta W = -\delta V$ where V is potential energy

$$\Rightarrow \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0, \text{ i.e. } \delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

i.e. $\delta \int_{t_1}^{t_2} (T - V)\, dt = 0$ or $\delta \int_{t_1}^{t_2} L\, dt = 0$

or $\int_{t_1}^{t_2} L\, dt =$ stat $=$ extremum. This is Hamilton's principle.

Ex. 1. *Use Hamilton's principle to find the equation of motion of one-dimensional harmonic oscillator.*

Sol. The kinetic energy of harmonic oscillator is given by $T = m\dot{x}^2$ and the potential energy of the harmonic oscillator is given by $V = -\int F dx = \int kx\, dx = \frac{1}{2}kx^2$.

$\therefore$ The Lagrangian of the system $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

Using Hamilton's Principle, we have $\delta \int_{t_1}^{t_2} L\, dt = 0.$

$$\Rightarrow \delta \int_{t_1}^{t_2} \left(\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2\right) dt = 0 \Rightarrow \int_{t_1}^{t_2} \delta \left(\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2\right) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (m\dot{x}\,\delta\dot{x} - kx\,\delta x)\, dt = 0 \qquad \text{But } \delta\dot{x} = (d/dt)(\delta x). \qquad ...(1)$$

$$\therefore (1) \Rightarrow \int_{t_1}^{t_2} m\dot{x} \frac{d}{dt}(\delta x)\, dt - \int_{t_1}^{t_2} kx\,\delta x\, dt = 0$$

$$\text{i.e., } \Big[m\dot{x}\,\delta x\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x})\, dt - \int_{t_1}^{t_2} kx\,\delta x\, dt = 0 \qquad ...(2)$$

But $\Big[m\dot{x}\,\delta x\Big]_{t_1}^{t_2} = 0$ [$\because$ $\delta x = 0$ at fixed points, *i.e.*, at instants t_1 and t_2]

$$\therefore (2) \Rightarrow -\int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x})\,\delta x\, dt - \int_{t_1}^{t_2} kx\,\delta x\, dt = 0 \Rightarrow \int_{t_1}^{t_2} (m\dot{x} + kx)\,\delta x\, dt = 0$$

But δx is arbitrary, hence the above equation is only satisfied if

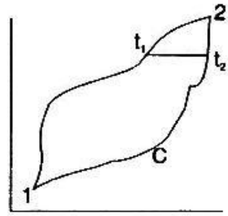
$m\dot{x} + kx = 0$. This is the equation of motion for one dimensional harmonic oscillator.

9·03·9. Extension of Hamiltion's principle to non-conservative and non-holonomic system.

We can generalise Hamilton's principle to include non-conservative forces as well, so that an alternative form of the Lagrange's equation can be obtained. the extension of the Hamilton's principle is

$$\delta I = \delta \int_1^2 (T + W)\, dt = 0 \quad ...(1); \quad \text{where} \quad W = \Sigma\, \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(2)$$

We know that the variations δq_i or $\delta \mathbf{r}_i$ are identical with virtual displacements of the co-ordinates as there is no variation of time. Thus we can consider the varied path in configuration space as built up by a succession of virtual displacements from the actual path of motion C. Again each virtual displacements takes place at some definite time and at that time the forces acting on the body have definite values. δW repre-



sents the amount of work done by the forces on the system during the period of virtual displacement from the actual to the varied path. Thus the Hamilton's principle given by (1) can be said as *the integral of the variation of kinetic energy togehter with the amount of virtual work involved in the variation must be zero.*

We can evaluate the variations $\delta \mathbf{r}_i$ in terms of δq_i by making use of the equations of transformation between r and q, where each q depends on the path chosen through a parameter say α :

$$\mathbf{r}_i = \mathbf{r} \ \{q_k(\alpha, t)t\}$$

We can abbreviate the process by using the equivalance of $\delta \mathbf{r}_i$ with a virtual displacement. We know that

$$\sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i = \sum_k Q_k \ \delta q_k.$$

thus (1) may be put as $\delta \int_1^2 T\, dt + \int_1^2 \sum_k Q_k \ \delta q_k \ dt = 0.$ ...(3)

Now it can be shown easily that (3) reduces to ordinary form of Hamilton's principle, in case Q_k's are derivable form the generalised potential. Therefore the integral of virtual work, under these conditions becomes,

$$\int_1^2 \sum_k Q_k \ \delta q_k \ dt = -\int_1^2 \sum_k \delta q_k \left(\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_k} \right) dt.$$

Now reversing by integration-paths procedure; the above integral can be put as $-\int_1^2 \sum_k \left(\frac{\partial V}{\partial q_k}\delta q_k + \frac{\partial V}{\partial \dot{q}_k}\delta \dot{q}_k \right) dt = -\ \delta \int_1^2 V\ dt$

Thus (3) reduces to

$$\delta \int_1^2 T\,dt = \int_1^2 V\,dt = \delta \int_1^2 (T - V)\,dt = \delta \int_1^2 L\,dt = 0,$$

which is Hamilton's principle.

For more general problem, the variation of the first integral in (3) can be written at once, as T like L for conservative systems is a function of q_k and $\dot{q}_k$

$$\therefore \ \delta \int_1^2 T\,dt = \int_1^2 \sum_b \left(\frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} \right) \delta q_k\, dt \qquad \text{...(4)}$$

$$\Rightarrow \int_1^2 \sum_k \left(\frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} + Q_k \right) \delta q_k\, dt = 0 \qquad \text{...(5)}$$

Further it is assumed that the constraints are holonomic, so the integral (5) can vanish if and only if the seperate coefficients vanish, *i.e.*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$$

so that (1) represents the proper extension of Hamilton's principle, which yields that from of Lagrange's equation, in case the forces are not derived from a potential.

We can also extend the Hamilton's principle to cover certain cetogories of non-holonomic systems as well. While deriving Lagrange's equations from Hamilton's or D'Alemberts principle, application of holonomic constraints is made only in last step when the variation q_k are considered as independent of each other.

While in non-holonomic system the generalized co-ordinates are not independent to each other so they cannot be reduced further by means of equations of constrints of the form

$f\,[q_1, q_2, ... q_n\,; t] = 0$ Thus q_k's can not be as all independent.

We can further treat non-holonomic system, provided the equations of constriant can be puttin the form

$$\sum_j a_{ij}\, da_j + a_{ij}\, dt = 0 \quad \left[j = 1, 2, - \ldots m\right] \qquad ...(7)$$

which is a relation connecting the differentials of q's Since. in variation process used in Hamilton's principle, the time for each point on the path is taken constant, therefore the virtual displacements occurring in the variation must satisfy the equations of constraint of the form

$$\sum_j a_{ij}\, \delta q_j = 0, \ (j = 1, 2, ... m) \qquad ...(8)$$

The equations (8) can very well be used to reduce the number of virtual displacement to independent ones. The method used for eliminating these extra virtual displacements is that of Lagrange's *undetermined multipliers.*

Also from (8), we get

$$\lambda_l \sum_j a_{ij}\, \delta q_j = 0, \ (j = 1, 2, ... m) \qquad ...(9)$$

where λ_l are some undetermined constants, which in general, are functions of time.

Now first of all summing the equations (9) over l and then integrating the resulting equation from 1 to 2, we get

$$\int_1^2 \sum_{j,i} \lambda_l a_{ij}\, \delta q_j\, dt = 0 \qquad ...(10)$$

Again the equations corresponding to (5) in the case of conservative system are given as

$$\int_1^2 dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j = 0 \qquad ...(11)$$

Combining the equations (10) with (11), we get

$$\int_1^2 dt \sum_j^n \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{ij} \right) \delta q_j = 0 \qquad ...(12)$$

It is to be noted here that δq_j's are still not independent. They are connected by the m relations given by (8.) This is, while the first $n - m$ of these equations may be chosen independently, the last m are then fixed by the equations (8). Since the values of λ_l's are at our disposal, we choose them to be such that

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{ij} = 0 , (j = n - m + 1 ,...n) \qquad ...(13)$$

which are in the nature of equations of motion for the last m of the q_j variable.

Using the value of λ_l obtained from (13) in (12), we get

$$\int_1^2 dt \sum_{j=1}^{n-m} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} \right) \delta q_j = 0 \qquad ...(14)$$

Since δq_j 's involved here are independent ones, therefore, we have

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} = 0 \, (j = 1 , 2 ,...n - m) \qquad ...(15)$$

Now combining (13) with (15), we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j x} - \frac{\partial L}{\partial q_j} = \sum_l \lambda_l a_{lj} , (j = 1 , 2 ,...n) \qquad ...(16)$$

which are known as *Lagrange's equations for non holonomic systems.*

We now observe that (16) gives us a total of only n equations, where as the unknowns involved are $(n + m)$ in number, namely then n co-ordinates q_j and the m , λ_l 's. The additional equations required are exactly the equations of constraint i.e.

$$\sum_j a_{ij} \dot{q}_j + a_{ij} + 0 \qquad ...(17)$$

The equations (16) together with (17) constitute $(n + m)$ equations for $n + m$ unknowns.

It is to be noted here that equation (7) is not the most general type of non-holonomic constraint. For example, it does not include equations of constraints in the form of inequalities. More over it includes holonomic constraints. An equation of the form

$f \, (q_1 , q_2 , q_3 ,...q_n \, ; t) = 0$ is known as a holonomic equation of constraint.

This is equivalent to a differential equation

$$\sum_j \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial t} dt = 0 \qquad ...(18)$$

It is of the same form as the equation (7) with the coefficients

$$a_{lj} = \frac{\partial f}{\partial q_j} \text{ and } a_{ji} = \frac{\partial f}{\partial t} \qquad \ldots(19)$$

It means that method of Lagrange's undetermined multipliers can be used also for holonomic constraints, when it is not easy to reduce all the q's to independent co-ordinates.

Illustrative Examples

Ex. 2. *A particle moves in the xy-plane under the influence of a central force depending only on its distance from the origin.*

(a) *Set up the Hamiltonian for the system.*

(b) *Write Hamilton's equations of motion.*

Sol. **(a)** Let the potential due to the central force be $V(r)$. Then, we have

$$T = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m \ [(\text{radial velocity})^2 + (\text{transverse velocity}^2)]$$

$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2),$$

$$L = T - V = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r).$$

$$\therefore \ p_r = (\partial L / \partial \dot{r}) = m\dot{r}, p_\theta = (\partial L / \partial \dot{r}) = mr^2 \dot{\theta}$$

$$\Rightarrow \dot{r} = (p_r/m), \dot{\theta} = (p_\theta / mr^2)$$

Thus $\quad H = \Sigma p_i \dot{q}_i - L = p_r \dot{q}_r + p_\theta \dot{q}_\theta - L$

$$= p_r \cdot \dot{r} + p_\theta \dot{\theta} - \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r)$$

$$= p_r (p_r/m) + p_\theta (p_\theta / mr^2)$$

$$- \tfrac{1}{2} m \{(p_r^2 / m^2) + r^2 \cdot (p_\theta^2 / m^2 r^4)\} - V(r)\}$$

$$= (p_r^2 / 2m) + (p_\theta^2 / 2m_r^2) + V(r) = \text{total energy of the system.}$$

(b) Hamilton's equations are

$$\dot{p}_i = - (\partial H / \partial q_i), \dot{q}_i = (\partial H / \partial p_i)$$

$$\Rightarrow \dot{r} = (\partial H / \partial p_r) = (p_r / m), \dot{\theta} = (\partial H / \partial p_\theta) = (p_\theta / mr^2)$$

$$\dot{p}_i = - (\partial H / \partial r) = (p_\theta^2 / mr^3) - V(r), \dot{p}_\theta = - (\partial H / \partial \theta) = 0.$$

Ex. 3. *A particle of mass m moves in a force field of potential V. Write*
(a). the Hamiltonian and
(b) Hamilton's equations in spherical polar co-ordinates.

Sol. **(a)** K.E. is given by

$$T = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) \qquad \ldots(1)$$

$$\therefore\ L = T - V = \tfrac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2) - V \qquad \text{...(2)}$$

We have $p_r = (\partial L/\partial \dot{r}) = m\dot{r}$, $p_\theta = (\partial L/\partial \dot{\theta}) = mr^2\,\dot{\theta}$,

$$p_\phi = (\partial L/\partial \dot{\phi}) = mr^2 \sin^2\theta\,\dot{\phi}$$

$$\Rightarrow \dot{r} = (p_r/m),\ \dot{\theta} = (p_\theta/mr^2),\ \dot{\phi} = p_\phi/(mr^2\sin^2\theta). \qquad \text{...(3)}$$

Now Hamiltonian is given by

$$H = \Sigma p_i\,\dot{q}_i - L = p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\phi\,\dot{\phi} - L$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\theta^2}{2mr^2\sin^2\theta} + V(r, \theta, \phi)$$

= total energy of the system.

(b) Hamilton's equations are given by

$$\dot{q}_i = (\partial H/\partial p_i),\ \dot{p}_i = -(\partial H/\partial q_i)$$

i.e. $\dot{r} = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m}$ $\qquad \dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{mr^3} + \dfrac{p_\phi^2}{mr^3\sin^2\theta} - \dfrac{\partial V}{\partial r}$

$\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2}$ $\qquad \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} - \dfrac{\partial V}{\partial \theta}$

$\dot{\phi} = \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{mr^2\sin^2\theta}$ $\qquad \dot{p}_\phi = -\dfrac{\partial H}{\partial \phi} = -\dfrac{\partial V}{\partial \phi}$

Ex. 4. *If H is the Hamiltonian, prove that if f is any function depending on position, momenta and time, then*

$$(df/dt) = (\partial f/\partial t) + [H, f].$$

Sol. We have

$$(df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(dq_i/dt) + (\partial f/\partial p_i)(dp_i/dt)\}$$

$$\Rightarrow (df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(\partial H/\partial p_i) - (\partial f/\partial p_i)(\partial H/\partial q_i)\}$$

$\{\because$ By Hamilton's equations $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -(\partial H/\partial q_i)\}$

$\Rightarrow (df/dt) = (\partial f/\partial t) + [H, f]$ where $[H, f]$ is the Poisson Bracket)

Ex. 5. *A particle of mass m moves in a force field of potential V.*

(a) *Write the Hamiltonian and*

(b) *Hamilton's equations in cartesian co-ordinates.*

Sol. (a) We have

$$T = \tfrac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\Rightarrow L = \tfrac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \qquad \text{...(1)}$$

$$\because \quad p_x = (\partial L/\partial \dot{x}) = m\dot{x},\ p_y = (\partial L/\partial \dot{y}) = m\dot{y};\ p_z = (\partial L/\partial \dot{z}) = m\dot{z}$$

$$\Rightarrow \dot{x} = (p_x/m),\ \dot{y} = (p_y/m),\ \dot{z} = (p_z/m).$$

Thus $H = \Sigma p_x \dot{q}_x - L = \dot{p}_x x + \dot{p}_y y + \dot{p}_z z - \tfrac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z)$

$$= p_x (p_x/m) + p_y (p_y/m) + p_z \cdot (p_z/m)$$

$$- \tfrac{1}{2} m\,[(p_x^2/m^2) + (p_y^2/m^2) + (p_z^2/m^2)] + V(x, y, z)$$

$$= (p_x^2/2m) + (p_y^2/2m) + (p_z^2/2m) + V(x, y, z)$$

= total energy of the system.

(b) Hamilton's equations are :

$$\dot{p}_x = -(\partial H/\partial x); \quad \dot{p}_y = -(\partial H/\partial y); \quad \dot{p}_z = -(\partial H/\partial z) \text{ and}$$

$$\dot{x} = (\partial H/\partial p_x); \quad \dot{y} = (\partial H/\partial p_y); \quad \dot{z} = (\partial H/\partial p_z).$$

$$\Rightarrow \dot{p}_x = -(\partial V/\partial x),\ \dot{p}_y = -(\partial V/\partial y),\ \dot{p}_z = -(\partial V/\partial z)$$

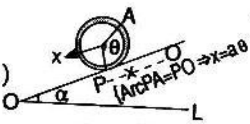
$$\dot{x} = (p_x/m),\ \dot{y} = (p_y/m),\ \dot{z} = (p_z/m).$$

Ex. 6. *A sphere rolls down a rough included plane ; if x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration.* **(Rajasthan 95; Meerut 93)**

Sol. We have $T = \tfrac{1}{2} m\,(\dot{x}^2 + k^2 \dot{\theta}^2) = \tfrac{1}{2} m\,(\dot{x}^2 + \tfrac{2}{5} a^2 \dot{\theta}^2) \quad (\because \ k^2 = \tfrac{2}{5} a^2)$

$$= \tfrac{1}{2} m\,(\dot{x}^2 + \tfrac{2}{5}\dot{x}^2) = \tfrac{7}{10} m\dot{x}^2; \qquad \text{...(1)} \qquad V = -\,mgx \sin\alpha. \qquad \text{...(2)}$$

$$\therefore\ L = T - V = = \tfrac{7}{10} m\dot{x}^2 + mgx \sin\alpha. \qquad \text{...(3)}$$



Now $p_x = (\partial L/\partial \dot{x}) = \tfrac{7}{5} m\dot{x} \Rightarrow \dot{x} = (5p_x/7m)$

Thus $H = -\,L + p_x \dot{x} = -\,\tfrac{7}{10} m\dot{x} - mgx \sin\alpha + p_x \cdot (5p_x/7m)$

$$= \tfrac{5}{14}(p_x^2/m) - mgx \sin\alpha. \qquad \text{...(4)}$$

$\therefore$ One of Hamilton's equations gives

$$\dot{p}_x = -(\partial H/\partial x) = mg \sin\alpha \Rightarrow \tfrac{7}{5} m\ddot{x} = mg \sin\alpha \Rightarrow \ddot{x} = \tfrac{5}{7} g \sin\alpha$$

Ex. 7. *If the Hamiltonian H is independent of time explicily, prove that it is*

(a) *a constant, and* **(b)** *equal to the total energy of the system.*

Sol. (*a*) $(dH/dt) = \sum_{i=1}^{n} (\partial H/\partial p_i)\, \dot{p}_i + \sum_{i=1}^{n} (\partial H/\partial q_i)\dot{q}_i$

$\sum_{i=1}^{n} \dot{q}_i\; \dot{p}_i + (-\,\dot{p}_i)\,\dot{q}_i = 0 \quad [\because \quad (\partial H/\partial p_i) = \dot{q}_i\,, (\partial H/\partial q_i) = -\,\dot{p}_i\,]$

$\Rightarrow H = \text{constant} = E$ say.

(b) By Euler's theorem on homogeneous functions, we have

$$\sum_{i=1}^{n} \dot{q}_i\;\; (\partial T/\partial \dot{q}_i\,) = 2T. \qquad ...(2)$$

Put $p_i = (\partial L/\partial \dot{q}_i\,) = \{\partial(T - V)/\partial \dot{q}_i\,\} = (\partial T/\partial \dot{q}_i\,) - (\partial V/\partial \dot{q}_i\,) = (\partial T/\partial \dot{q}_i\,)$

$\{\because \quad (\partial V/\partial \dot{q}_i\,) = 0$ as V is independent of $\dot{q}_i\,\}$

$\therefore$ (2) $\Rightarrow \sum_{i=1}^{n} \dot{q}_i\;\; p_i = 2T$

Thus $H = \sum_{i=1}^{n} p_i\,\dot{q}_i - L = 2T - L = 2T - (T - V) = T + V = E.$

Ex. 8. *Write the Hamiltonian and equation of motion for a simple pendulum.*

Sol. We have $T = \frac{1}{4}\, ml^2\, \dot{\theta}^2$ and $V = mgl\,(1 - \cos\theta)$,

$$\therefore\; L = T - V = \tfrac{1}{2}\, ml^2\, \dot{\theta}^2 - mgl\,(1 - \cos\theta) \qquad ...(1)$$

$\Rightarrow H = \Sigma p_i\,\dot{q}_i - L = p_\theta \dot{\theta} - L = ml^2\,\dot{\theta}^2 - \{\tfrac{1}{2}\, ml^2 \dot{\theta}^2 - mgl\,(1 - \cos\theta)\}$

$= \tfrac{1}{2}\, ml^2 \dot{\theta}^2 + mgl\,(1 - \cos\theta) = T + V =$ total energy.

Now $p_\theta = (\partial L/\partial \dot{\theta}\,) = ml^2 \dot{\theta}^2 = (p_\theta / ml^2)$

$\therefore\; H = \tfrac{1}{2}\, ml^2\,(p_\theta / ml^2)^2 + mgl\,(1 - \cos\theta) = (p_\theta^2 / 2ml^2) + mgl\,(1 - \cos\theta)$

$\Rightarrow (\partial H/\partial p_\theta) = (p_\theta / ml^2), (\partial H/\partial \theta) = mgl \sin\theta$

Now Hamilton's equation of motion for θ and p_θ are

$\dot{\theta} = (\partial H/\partial p_\theta), \dot{p}_\theta = -\,(\partial H/\partial \theta) \Rightarrow \dot{\theta} = (p_\theta / ml^2)$ and $\dot{p}_\theta = -\,mgl\,\sin\theta,$

These represent Hamilton's equations for a simple pendulum.

From above, we have $p_\theta = ml^2 \dot{\theta}$, *i.e.* $\dot{p}_\theta = ml^2\, \ddot{\theta}$

$\therefore\; ml^2\, \ddot{\theta} = -\, mgl \sin\theta \Rightarrow \ddot{\theta} + (g/l) \sin\theta = 0$

This gives the equation of motion of the simple pendulum.

Ex. 9. *Write the Hamiltonian function and equation of motion for a compound pendulum.* **(Meerut 1995)**

Sol. We have $L = \frac{1}{2} I \dot{\theta}^2 + mgh \cos\theta \Rightarrow p_\theta = (\partial L/\partial\theta) = I\theta$.
where $I = mk^2$

$\therefore H = \Sigma p_i \dot{q}_i - L = p_\theta \dot{\theta} - L = I\dot{\theta}\dot{\theta} - \frac{1}{2} I \dot{\theta}^2 - mgh \cos\theta$

$= \frac{1}{2} I \dot{\theta}^2 - mgh \cos\theta$

$\Rightarrow \frac{1}{2} H = (p_\theta/I)^2 - mgh \cos\theta = (p_\theta^2/2I) - mgh \cos\theta \quad \{\because \dot{\theta} = (p_\theta/I)\}$

$\therefore (\partial H/\partial p_\theta) = (p_\theta/I), (\partial H/\partial\theta) = mgh \sin\theta$

Thus the Hamilton's equations for $\dot{\theta}$ and $\dot{p}_\theta$ are given by

$$\dot{\theta} = (\partial H/\partial p_\theta), \dot{p}_\theta = -(\partial H/\partial\theta)$$

i.e. $\dot{\theta} = (p_\theta/I)$ and $\dot{p}_\theta = -mgh \sin\theta$. But $p_\theta = I\theta \Rightarrow \dot{p}_\theta = I\ddot{\theta}$.

$\therefore I\ddot{\theta} = -mgh \sin\theta \Rightarrow \ddot{\theta} + \frac{mgh}{I} \sin\theta = 0.$

This is exactly the same as obtained prevoiusly using Lagrange's equations.

Ex. 10. *Obtain Euler's equations from Hamilton's equations.*

Sol. We know that $2T = (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$,

$\Rightarrow L = T - V = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) - V$ ...(1)

Also Euler's geometrical relations give

$\omega_1 = \dot{\theta} \sin\psi - \dot{\phi} \sin\theta \cos\psi;$

$\omega_2 = \dot{\theta} \cos\psi + \dot{\phi} \sin\theta \sin\psi$; and

$\omega_3 = \psi + \phi \cos\theta.$

Now $H = T + V = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + V$

Again, $p_\phi = \frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\omega_1}\frac{\partial\omega_1}{\partial\dot{\theta}} + \frac{\partial L}{\partial\omega_2}\frac{\partial\omega_2}{\partial\dot{\theta}} + \frac{\partial L}{\partial\omega_3}\frac{\partial\omega_3}{\partial\dot{\theta}}$.

$= A\omega_1 \sin\psi + B\omega_2 \cos\psi + C\omega_3 . 0$ ...(1)

$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = -A\omega_1 \sin\theta \cos\psi + B\omega_2 \sin\theta \sin\psi + C\omega_3 \cos\theta$ ...(2)

and $p_\psi = (\partial L/\partial\dot{\psi}) = C\omega_3$

Solving the three equations for $\omega_1, \omega_2, \omega_3$ we have

$$\omega_1 = \frac{1}{A}\left[p_\theta \sin\psi + (p_\psi \cos\theta - p_\phi)\frac{\cos\psi}{\sin\theta}\right]$$

$$\omega_2 = \frac{1}{A}\left[p_\theta \cos\psi - (p_\psi \cos\theta - p_\phi)\frac{\sin\psi}{\sin\theta}\right]; \text{ and } \omega_3 = \frac{1}{C}\cdot p_\psi$$

Also, Hamilton's equations are $\dot{p}_\psi = -\dfrac{\partial H}{\partial \psi}$ and $\dot{\psi} = \dfrac{\partial H}{\partial p_\psi}$

Now $\dot{p}_\psi = -\dfrac{\partial H}{\partial \psi}$

$$\Rightarrow C\,\dot{\omega}_3 = -\left[\frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial \psi}\right] - \frac{\partial V}{\partial \psi}$$

$$= -\left[A\,\omega_1 \cdot \frac{1}{A} B\,\omega_2 + B\,\omega_2\left(\frac{A}{B}\omega_1\right) + C\,\omega_3 \,.\, 0\right] - \frac{\partial V}{\partial \psi}$$

$$= (A - B)\,\omega_1\,\omega_2 - \frac{\partial V}{\partial \psi}$$

$$\Rightarrow C\frac{d\omega_1}{dt} - (A - B)\,\omega_1\,\omega_2 = N\left(\because\ -\frac{\partial V}{\partial \phi} = N\right)$$

This is Euler's third familiar dynamical equation

Also, $\dot{\psi} = (\partial H/\partial p_\psi) = \dfrac{\partial H}{\partial \omega_1}\dfrac{\partial \omega_1}{\partial p_\psi} + \dfrac{\partial H}{\partial \omega_2}\dfrac{\partial \omega_2}{\partial p_\psi} + \dfrac{\partial H}{\partial \omega_3}\dfrac{\partial \omega_3}{\partial p_\psi}$

$$= (\omega_1\ \cos\psi - \omega_2 \sin\psi)\cot\theta + \omega_3 = -\ \phi\ \sin\theta\cot\theta + \omega_3$$

i.e. $\dot{\psi} = -\ \dot{\phi}\cos\theta + \omega_3 \Rightarrow \omega_3 = \dot{\psi} + \dot{\phi}\cos\theta$

This is Euler's third geometrical equation.

On the same lines, we can deduce Euler's other equations (dynamical and geometrical).

Ex. 11. *Prove that*

$$\left(\frac{dH}{dt}\right) = \left(\frac{\partial H}{dt}\right) \text{ where } H \text{ is the Hamilton's function.}$$

Sol. Let $q_1, q_2 ..., q_n$ be the generalised co-ordinates then Hamilton's equation are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i}\ (i = 1, 2, ..., n) \qquad ...(1)$$

But Hamiltonian H is a function of q's and ps'

$$\therefore \frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial H}{\partial t} + \sum_{i=1}^{n}(-\dot{p}_i)\,\dot{q}_i + \sum_{i=1}^{n}\dot{q}_i\,\dot{p}_i = \frac{\partial H}{\partial t}. \qquad \text{[using (1)]}$$

Ex. 12. *Use Hamilton's equations to find the equations of motion of a projectile in space.*

Solution. Let (x, y, z) be the co-ordinates of the projectile in space at time t, then we have

$$T = \tfrac{1}{2}m\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), V = mgz$$

$$\therefore\ L = T - V = \tfrac{1}{2}m\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

$$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x},\ p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y},\ p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

But L does not involve t explicity therefore Hamiltonian H is given by

$$H = T + V = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$= \tfrac{1}{2}m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2}\right) + mgz = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

Now Hamilton's equations are given by

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \quad ...(1), \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad ...(2),$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \quad ...(3), \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad ...(4),$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \quad ...(5), \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad ...(6)$$

Using (1) and (2), we have $\ddot{x} = 0$ ...(7)

Using (3) and (4), we have $\ddot{y} = 0$ ...(8)

Again making use of (6) and (5), we have

$$m\ddot{z} = \dot{p}_z = -mg \text{ or } \ddot{z} = -g. \quad ...(9)$$

These (7, 8, 9) are the equations of motion of the projectile in space.

Ex. 13. *Using cylindrical coordinates* (ρ, ϕ, z) *write the Hamiltonian and Hamilton's equations for a particle of mass m moving in a force field of potential* $V(\rho, \phi, z)$.

Sol. In cylindrical coordinates, co-ordinates of any points are
$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$ ...(1)

$$\therefore\ T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) \quad ...(2)$$

$$\Rightarrow L = T - V = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z) \quad ...(3)$$

$$\Rightarrow p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho},\ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \text{ and } p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Evidently, L does not involve t explicity, therefore Hamiltonian H is given

by $H = T + V = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) + V$

$$= \tfrac{1}{2}m\left[\frac{p_\rho^2}{m^2} + \frac{p_\phi^2}{m^2\rho^2} + \frac{p_z^2}{m^2}\right] + V = \frac{1}{2m}\left[p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right] + V$$

Hence, Hamilton's are given by :

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\phi^2}{m\rho^2} - \frac{\partial V}{\partial \rho};\ \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m_\phi}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = -\frac{\partial V}{\partial \phi}\,;\ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}\,;\ \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

Ex. 14. *Using cylindrical coordinates, write the Hamiltonian* ***and*** *Hamilton's equations for a particle of mass moving on the* ***inside of a*** *frictionless cone* $x^2 + y^2 = z^2 \tan^2 \alpha$

Sol. Like previous example, we have

$$T = \tfrac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \tfrac{1}{2} m\,(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \rho^2 \cot^2 \alpha)$$

[$\because\ x = \rho \cos \phi\,,\ y = \rho \sin \phi\,,\ z = \rho \cot \alpha$]

$$= \tfrac{1}{2} m\,(\dot{\rho}^2 \operatorname{cosec}^2 \alpha + \dot{\rho}^2 \dot{\phi}^2) \qquad ...(1)$$

and $V = -W = -mgz = mg\rho \cot \alpha,$

[$\because$ the particle is above the vertex **(origin)**].

$$\Rightarrow L = T - V = \tfrac{1}{2} m\,(\rho^2 \operatorname{cosec}^2 \alpha + \rho^2 \dot{\phi}^2) - mg\,\rho \cot \alpha \qquad ...(2)$$

This gives, $p_\rho = \dfrac{\partial L}{\partial \dot{\rho}} = m\rho\ \operatorname{cosec}^2 \alpha\,, p_\phi = \dfrac{\partial L}{\partial \dot{\phi}} = m\rho^2\,\dot{\phi}$ ...(3)

Again, L does not involve t explicitly, therefore Hamiltonian H is given by

$$H = T + V = \tfrac{1}{2} m\,(\dot{\rho}^2 \operatorname{cosec}^2 \alpha + \rho^2\dot{\phi}^2) + mg\,\rho \cot \alpha$$

$$= \tfrac{1}{2} m \left[\frac{p_\rho^2}{m^2 \operatorname{cosec}^2 \alpha} + \frac{p_\phi^2}{m^2 \rho^2} \right] + mg\rho \cot \alpha + \frac{1}{2m} \left[\frac{p_\rho^2}{\operatorname{cosec}^2 \alpha} + \frac{p_\phi^2}{\rho^2} \right] + mg\,\rho \cot \alpha$$

Thus Hamilton's equations are given by :

$$\dot{p}_\rho = -\frac{\partial H}{\partial \rho} = \frac{p_\phi^2}{m\rho^2} - mg\,\cot \alpha\,;\ \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m\,\operatorname{cosec}^2 \alpha}\,.$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \qquad ;\ \dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}\,.$$

Ex. 15. *Use the variational method to show that the shortest curve joining two fixed is a straight line.* **(Meerut 1981, 90)**

Sol. We have

$\dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$ for any curve, joining two fixed points say $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$.

$$\Rightarrow s=\int_{x_1}^{x_2} \sqrt{(1+y'^2)}dx= \text{ length of the curve joining A to B.}$$

$$=\int_{x_1}^{x_2} f(y')\,dx \text{ where } f(y')=\sqrt{(1+y'^2)} \qquad ...(1)$$

In equation (1), y is absent from f, therefore for s to be stationary (here minimum), we have from Euler-Lagrange's equation

$$\frac{\partial f}{\partial y'}=\text{(constant)} \Rightarrow \frac{\partial}{\partial y'}[(\sqrt{1+y'^2})]=\text{const.} \because \quad f=(1+y'^2)^{1/2}$$

$$\Rightarrow \frac{y'}{\sqrt{(1+y'^2)}}=\text{(constant) } i.e.\ y'=b \text{ where } b \text{ is constant.}$$

$$\Rightarrow \frac{dy}{dx}=b \text{ or } y=a+bx \text{ where } a \text{ is a constant.} \quad ...(2)$$

$\Rightarrow$ the shortest path joining A to B is the curve whose equation is $y=a+bx$ and this is evidently a line.

Ex. 16. *Show that the area of the surface of revolution of a curve* $y=y(x)$ *is*

$$2\pi\int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\,dx.$$

Hence show that for this to be a minimum, the curve must be a catenary.

(Meerut 81 (P), 82 (P), 84 (P))

Sol. Consider any curve, and let $A\equiv(x_1, y_1)$ and $B\equiv(x_2, y_2)$ be its extremities, then if this curve revolves about x-axis the surface of revolution is given by

$$S=\int 2\pi y\,ds=2\pi\int y\frac{ds}{dx}dx=2\pi\int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\,dx$$

$$\Rightarrow S=2\pi\int_{x_1}^{x_2} f(y, y')dx, \text{ where } f=f(y, y')=y\sqrt{(1+y'^2)}$$

Evidently, x is absent from f, so for S to be stationary (here a minimum) we have :

$$f-y'\frac{\partial f}{\partial y'}=\text{(constant) } i.e.\ y\sqrt{(1+y'^2)}-y'\frac{yy'}{\sqrt{(1+y')}}=\text{const.}$$

$$\Rightarrow \frac{y}{\sqrt{(1+y'^2)}}=c \text{ where } c \text{ is constant } i.e.\ y=c\sqrt{(1+\tan^2\psi)}$$

$$[\because \quad y'=\tan\psi]$$

$\therefore$ $y=c\sec\psi$ and this represents catenary.

Ex. 17. *A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is* $\phi(r)$ *at a distance r from the origin.*

Deduce from the principle of energy and least action that the differential equation of the path is

$$c^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]=r^4\,[h-\phi(r)]. \qquad \textbf{(Meerut 1995)}$$

Sol. Let T and V be the kinetic are potential energies, then we have $T+V=h \Rightarrow T=h-V$

$$= h-\phi(r), \because \quad V=\phi(r) \text{ given}$$

$\Rightarrow \frac{1}{2}v^2=h-\phi(r)$ where v is velocity $(\because \quad T=\frac{1}{2}mv^2=\frac{1}{2}v^2)$

$\Rightarrow v=\sqrt{2}\,\{h-\phi(r)\}^{1/2}$

Whence, the action $A=\int_{t_0}^{t_1} 2T\,dt$, by definition given earlier

$$=\int_{t_0}^{t_1} 2.\tfrac{1}{2}v^2\,dt=\int_{t_0}^{t_1} v\,ds \text{ since } v=(ds/dt)=\sqrt{2}\int_{t_0}^{t_1}\{h-\phi(r)\}^{1/2}\,ds$$

$$=\sqrt{2}\int_{t_2}^{t_1}\{h-\phi(r)\}^{1/2}\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}dr \because \quad \frac{ds}{dr}=\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}$$

$$=\sqrt{2}\int_{t_0}^{t_1}\left[\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}\right]dr \text{ where } \theta'=\frac{d\theta}{dr}$$

$$=\sqrt{2}\int_{t_0}^{t_1} f(\theta',r)\,dr, \text{ where } f(\theta',r)=\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}$$

Evidently, θ is absent from f, therefore we must have

$\dfrac{\partial f}{\partial\theta'}=c$ (const.) $\Rightarrow \dfrac{\partial}{\partial\theta'}\,[\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}]=c$

$$\Rightarrow \{h-\phi(r)\}^{1/2}\frac{r^2}{\left\{r^2+\left(\dfrac{dr}{d\theta}\right)^2\right\}^{1/2}}=c$$

$$\Rightarrow c^2\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}=r^4\,\{h-\phi(r)\}.$$

This is the required equation.

Ex. 18. *A projectile is launched in a vertical plane with a velocity whose horizontal and vertical components are* v_x and v_y *respectively. Calculate the value of the integral* $\int_0^{t_0} L\,dt$, where $t_0=\dfrac{n\pi}{\omega}$.

Evaluate this integral for the varied path given by the equations

$$x = v_x t,\ y = v_y t - \tfrac{1}{2} g t^2 + \varepsilon \sin \omega t,$$

where ε is a small constant quantity.

Show that the integral $\int_0^{t_0} L\, dt$ is greater for the varied path than for the actual path, but the result is in agreement with Hamilton's principle.

(Meerut 1994)

Sol. The path of the projectile is given by

$$x = v_x t,\ y = v_y t - \tfrac{1}{2} g t^2.$$

Also $V = mgy$, where V denotes the potential,

whence $L = T - V = \frac{1}{2} m\ (\dot{x}^2 + \dot{y}^2) - mgy$

$$= \tfrac{1}{2} m\ [v_x^2 + (v_y - gt)^2] - mg\,(v_y t - \tfrac{1}{2} g t^2)$$

$$= \tfrac{1}{2} m\ [v_x^2 + v_y^2) - 4gt\, v_y + 2g^2 t^2] \qquad \text{...(1)}$$

$$\Rightarrow \int_0^{t_0} L\, dt = \int_0^{t_0} (T - V)\, dt = \tfrac{1}{2} m \int_0^{t_0} [(v_x^2 + v_y^2) - 4gt\, v_y + 2g^2 t^2] dt$$

$$= \tfrac{1}{2} m\ [(v_x^2 + v_y^2)\, t_0 - 2t_0^2 v_y + \tfrac{2}{3} g^2 t_0^3] \qquad \text{...(1)}$$

Also the varied path is given by : $x = v_x t,\ y = v_y t - \frac{1}{2} g t^2 + \varepsilon \sin \omega t$

Now after obtaining $\dot{x}, \dot{y}$; we have $T = \frac{1}{2} m\ [v_x^2 + (v_y - gt + \varepsilon\omega \cos \omega t)^2]$

and $V = mg\,(v_y t - \frac{1}{2} g t^2 + \varepsilon \sin \omega t) \Rightarrow \int_0^{t_0 = n\pi/\omega} L\, dt = \int_0^{t_0} (T - V)\, dt$

$$\int_0^{t_0} [\tfrac{1}{2} m\ \{v_x^2 + (v_y - gt + \varepsilon\omega \cos \omega t)^2\} - mg\,(v_y t - \tfrac{1}{2} g t^2 + \varepsilon \sin \omega t)\, dt$$

$$= \tfrac{1}{2} m\ [(v_x^2 + v_y^2)\, t_0 - 2t_0^2 g v_y + \tfrac{2}{3} g^2 t_0^3 + \tfrac{1}{2} \varepsilon^2 \omega^2 t_0] \ (\because \text{ other integrals vanish})$$

$$= \tfrac{1}{2} m\ [(v_x^2 + v_y^2)\, t_0 - 2t_0^2 g v_y + \tfrac{2}{3} g^2 t_0^3] + m\pi n\, \omega\, \varepsilon^2 \qquad \text{...(2)}$$

From equations (1) and (2), we see that the integarl $\int_0^{t_0} L\, dt$ is greater for the varied path then for the actual path and is minimum when $\varepsilon = 0$. Evidently, the minimum value in this case, the varied *path coincides with actual path.*

Ex. 19. *A particle moves in a straight line with central acceleration $\omega^2 x$ between two fixed points x_0* and *x_1 in the prescribed time $t_1 - t_0$.*

Show that Hamilton's principal function S is

$$\frac{\omega}{2\sin\omega(t_1-t_0)}\left[(x_1^2+x_0^2)\cos\omega(t_1-t_0)-2x_1x_0\right]$$

(Meerut 92, 93; Agra 90)

Sol. Equation of motion is, $\ddot{x}=\omega^2 x$ ...(1)

Solving (1), the path is $x=A\cos\omega t+B\sin\omega t$ ...(2)

Now, $\dot{x}=\omega[-A\sin\omega t+B\cos\omega t]\Rightarrow T=\frac{1}{2}\dot{x}^2$.

Also, we have potential $V=$ work done against the force

$$=\int_0^{t_0}\omega^2 x\,dx=\tfrac{1}{2}\omega^2x^2 \qquad ...(3)$$

Again, we also have

$$A\cos\omega t_0+B\sin\omega t_0-x_0=0 \qquad (t=t_0;\ x=x_0)$$

and $A\cos\omega t_1+B\sin\omega t_1-x_1=0 \qquad (t=t_1;\ x=x_1)$

Solving these,

$$A=\frac{x_0\sin\omega t_1-x_1\sin\omega t_0}{\sin\omega(t_1-t_0)},\ B=\frac{x_1\cos\omega t_0-x_0\cos\omega t_1}{\sin\omega(t_1-t_0)}$$

$$\Rightarrow B^2-A^2=[x_1^2\cos 2\omega t_0+x_0^2\cos 2\omega t_1-2x_1x_0$$
$$-\cos\omega(t_1+t_0)]\frac{1}{\sin^2\omega(t_1-t_0)} \qquad ...(4)$$

and $2AB=-[x_1^2\sin 2\omega t_0+x_0^2\sin 2\omega t_1$

$$-2x_1x_0\sin\omega(t_1+t_0)]\frac{1}{\sin^2\omega(t_1-t_0)}$$

$\therefore\ S=$ Hamilton's principal function

$$=\int_{t_0}^{t_1}L\,dt=\int_{t_0}^{t_1}(T-V)\,dt\ (\because\ L=T-V)$$

$$=\int_{t_0}^{t_1}(\tfrac{1}{2}\dot{x}^2-\tfrac{1}{2}\omega^2x^2)\,dt \qquad \text{[from above]}$$

$$=\tfrac{1}{2}\omega^2\int_{t_0}^{t_1}[(-A\sin\omega t+B\cos\omega t)^2-(A\cos\omega t+B\sin\omega t)^2]\,dt$$

$$=\tfrac{1}{2}\omega\left[(B-A)^2\sin 2\omega t+2AB\cos 2\omega t\right]_{t_0}^{t_1}$$

$$=\tfrac{1}{2}\omega[(B^2-A^2)\cos\omega(t_1+t_0)\sin\omega(t_1-t_0)$$
$$-2AB\sin\omega(t_1-t_0)\sin\omega(t_1-t_0)]$$

Substituting for $B^2 - A^2$ and $2AB$, we obtain

$$S = \frac{\omega}{2\sin\omega(t_1 - t_0)}[(x_1^2\cos 2\omega t_0 + x_0^2\cos 2\omega t_1 - 2x_1x_0$$

$$\cos\omega(t_1+t_0)\}\cos\omega(t_1+t_0) + \{x_1^2\sin 2\omega t_0 + x_0^2\sin 2\omega t_1$$

$$- 2x_1x_0\sin\omega(t_1+t_0)\}\sin\omega(t_1+t_0)]$$

$$= \frac{\omega}{2\sin\omega(t_1 - t_0)}[x_1^2\cos\omega(t_1-t_0) + x_0^2\cos\omega(t_1-t_0) - 2x_1x_0]$$

$$= \frac{\omega}{2\sin\omega(t_1 - t_0)}[(x_1^2 + x_0^2)\cos\omega(t_1-t_0) - 2x_1x_0]$$

This is the requried result.

Ex. 20. *A particle moves in a plane curve, under the central acceleration $\omega^2 r$, between two fixed points (x_0, y_0) and (x_1, y_1) in the prescribed time $t_1 - t_0$; prove that Hamilton's principle function S is*

$$= \frac{\omega}{2\sin\omega(t_1 - t_0)}[(x_1^2 + y_1^2 + x_0^2 + y_0^2)\cos\omega(t_1-t_0) - 2(x_1x_0 + y_1y_0)]$$

(Meerut 1992; Agra 91)

Sol. Equations of motion of the particle are given by

$$\ddot{x} = -\omega^2 x \text{ and } \ddot{y} = -\omega^2 y$$

Solving these, we obtain

$$x = A\cos\omega t + B\sin\omega t \quad ; \quad y = C\cos\omega t + D\sin\omega t \qquad ...(1)$$

Now, kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$

$$= \tfrac{1}{2}\omega^2[(-A\sin\omega t + B\cos\omega t)^2 + (-C\sin\omega t + D\cos\omega t)^2]$$

Potential $V = \frac{1}{2}\omega^2 r^2 = \frac{1}{2}\omega^2(x^2+y^2)$

Hence Hamilton's principal tunction S is given by

$$S = \int_{t_0}^{t_1} L\,dt = \int_{t_0}^{t_1}(T - V)\,dt \text{ since } L = T - V$$

$$= \int_{t_0}^{t_1}[\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}\omega^2(x^2+y^2)]\,dt$$

$$= \int_{t_0}^{t_1}(\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2x^2)\,dt + = \int_{t_0}^{t_1}(\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}\omega^2y^2)\,dt \qquad ...(2)$$

But $\int_{t_0}^{t_1}(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2x^2)\,dt = \frac{\omega}{2\sin\omega(t_1 - t_0)}[(x_1^2 + x_0^2)\cos\omega(t_1-t_0) - 2x_1x_0]$