Set - IV

* VECTOR INTEGRAL *

Introduction :— Integration is the inverse operation of differentiation.

Let $\vec{F}(t)$ be a differentiable vector function of a scalar variable 't' and let $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$ then $\int \vec{f}(t)\,dt = \vec{F}(t)$. —(1) $\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$ w.r.t respect to 't'.

- The function $\vec{f}(t)$ to be integrated is called the integrand.
- If $\vec{c}$ is any arbitrary constant vector independent of 't', then $\frac{d}{dt}\{\vec{F}(t) + \vec{c}\} = \vec{f}(t)$

This is equivalent to $\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{c}$ —(2)

from (2), it is obvious that the integral $\vec{F}(t)$ of $\vec{f}(t)$ is indefinite to the extent of an additive arbitrary constant 'c'. Therefore $\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$.

The constant vector 'c' is called the constant of integration. It can be determined if we are given some initial conditions.

Note: If $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$,

then $\int \vec{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt + C$.

→ If $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$ for all t in the interval $[a, b]$, then the definite integral between the limits $t=a$ and $t=b$ can be written

as $\int_a^b \vec{f}(t)\,dt = \int_a^b \left\{\frac{d}{dt}\vec{F}(t)\right\} dt.$

$$= \left[\vec{F}(t) + c\right]_a^b$$

$$= F(b) - F(a)$$

Some Standard Results:

→ we have $\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}$

$$\therefore \int\left(\frac{d\vec{r}}{dt}\cdot\vec{s} + \vec{r}\cdot\frac{d\vec{s}}{dt}\right)dt = \vec{r}\cdot\vec{s} + c.$$

Hence c is a scalar constant; since the integrand is a scalar.

→ we have $\frac{d}{dt}(\vec{r}^{\,2}) = 2\vec{r}\cdot\frac{d\vec{r}}{dt}.$

$$\therefore \int\left(2\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt = \vec{r}^{\,2} + c.$$

where c is a scalar constant.

→ $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = 2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}.$

$$\therefore \int\left(2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right)dt = \left(\frac{d\vec{r}}{dt}\right)^2 + c$$

where c is a scalar constant

→ we have $\frac{d}{dt}(\vec{r}\times\vec{s}) = \frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}.$

$$\therefore \int\left(\frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}\right)dt = \vec{r}\times\vec{s} + c.$$

Here the constant c is a vector quantity since the integrand is also a vector quantity.

1(i)

→ If $\vec{a}$ is constant vector, we have

$$\frac{d}{dt}(\vec{a}\times\vec{r}) = \vec{a}\times\frac{d\vec{r}}{dt}.$$

$$\therefore \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\int\frac{d\vec{r}}{dt}\,dt$$

$$= \vec{a}\times\vec{r}+c.$$

→ Now $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$$= \vec{r}\times\frac{d^2\vec{r}}{dt^2}.$$

$$\therefore \int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt}+c.$$

→ If $r=|\vec{r}|$, then $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}.$

$$\therefore \int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)dt = \frac{\vec{r}}{r}+c.$$

→ If c is a scalar constant then $\int c\vec{r}\,dt = c\int\vec{r}\,dt$.

→ If $\vec{r}$ and $\vec{s}$ are any two vector functions of a scalar t, then $\int(\vec{r}+\vec{s})\,dt = \int\vec{r}\,dt+\int\vec{s}\,dt$

→ Evaluate $\int_0^1\left(e^t\hat{i}+e^{-2t}\hat{j}+t\hat{k}\right)dt$.

Sol^n: $\int_0^1\left(e^t\hat{i}+e^{-2t}\hat{j}+t\hat{k}\right)dt$

$$= \hat{i}\int_0^1 e^t\,dt+\hat{j}\int_0^1 e^{-2t}\,dt+\hat{k}\int_0^1 t\,dt$$

$$= \hat{i}\left[e^t\right]_0^1+\hat{j}\left(-\frac{1}{2}e^{-2t}\right]_0^1+\hat{k}\left[\frac{t^2}{2}\right]_0^1$$

$$= (e-1)\hat{i}-\frac{1}{2}(e^{-2}-1)\hat{j}+\frac{1}{2}\hat{k}.$$

→ Evaluate $\int_2^3 \vec{f}\cdot\frac{d\vec{f}}{dt}\,dt$ if $\vec{f}(2)=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{f}(3)=4\hat{i}-2\hat{j}+3\hat{k}$.

Sol^n: we know that $\int\left(2\vec{f}\cdot\frac{d\vec{f}}{dt}\right)dt=\vec{f}^{\,2}+c$.

$$\therefore \int_2^3\left(\vec{f}\cdot\frac{d\vec{f}}{dt}\right)dt=\frac{1}{2}\left[\vec{f}^{\,2}\right]_2^3$$

$$=\frac{1}{2}\left[\vec{f}^{\,2}(3)-\vec{f}^{\,2}(2)\right]$$

$$=\frac{1}{2}\left[(4\hat{i}-2\hat{j}+3\hat{k})^2-(2\hat{i}-\hat{j}+2\hat{k})^2\right]$$

$$=\frac{1}{2}\left[(16+4+9)-(4+1+4)\right]$$

$$=\frac{1}{2}(20)=10$$

→ Find the value of $\frac{d\vec{r}}{dt}$ by integrating $\frac{d^2\vec{r}}{dt^2}=-n^2\vec{r}$.

Sol^n: The given equation is $\frac{d^2\vec{r}}{dt^2}=-n^2\vec{r}$.

Taking the dot product with $2\frac{d\vec{r}}{dt}$ both sides and integrating,

we have

$$\int\left(2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right)dt=-n^2\int 2\vec{r}\cdot\frac{d\vec{r}}{dt}$$

$$\left(\frac{d\vec{r}}{dt}\right)^2=n^2\vec{r}^{\,2}+c.$$

where c is any constant vector.

→ If $\vec{f}(t)=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$, find $\int_1^2\left(\vec{f}\times\frac{d^2\vec{f}}{dt^2}\right)dt$.

Sol^n: $$\int_1^2\left(\vec{f}\times\frac{d^2\vec{f}}{dt^2}\right)dt=\left[\vec{f}\times\frac{d\vec{f}}{dt}\right]_1^2$$

1(ii)

Given $\vec{f}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$

$$\therefore \frac{d\vec{f}(t)}{dt} = 10t\hat{i} + \hat{j} - 3t^2\hat{k}.$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

$$\therefore \left[\vec{f} \times \frac{d\vec{f}}{dt}\right]_1^2 = \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2$$

$$= -14\hat{i} + 75\hat{j} - 15\hat{k}.$$

$$\therefore \int_1^2 \left(\vec{f} \times \frac{d^2\vec{f}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

→ If $\vec{F}(t) = (t - t^2)\hat{i} + 2t^3\hat{j} - 3\hat{k}$ find $\int_1^2 F(t)\,dt$.

→ If $\vec{F}(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k}$

find $\int_0^1 f(t)\,dt$.

→ If $\vec{A} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B} = 2t^2\hat{i} + 6t\hat{k}$

find (i) $\int_1^2 (\vec{A}\cdot\vec{B})\,dt$ (ii) $\int_0^2 (\vec{A}\times\vec{B})\,dt$.

→ If $\vec{A} = t\hat{i} - 3\hat{j} + 2t\hat{k}$; $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$; $\vec{C} = 3\hat{i} + t\hat{j} - \hat{k}$

find (i) $\int_1^2 [ABC]\,dt$ (ii) $\int_1^2 [\vec{A} \times (\vec{B} \times \vec{C})]\,dt$

→ If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$ find $\vec{r}$

given that $\vec{r} = 2\hat{i} + \hat{j}$ and

$\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

→ Find $\vec{F}(t)$, given $\frac{dF}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$ and $F(0) = 0$

→ If $\vec{a}, \vec{b}$ and n are constants and

$\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt$.

show that $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = 0$.

→ Given $\frac{d^2\vec{r}}{dt^2} = -k^2\vec{r}$, show that $\left(\frac{d\vec{r}}{dt}\right)^2 = c - k^2\vec{r}^{\,2}$

→ Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$, where $\vec{a}, \vec{b}$ are constant vectors.

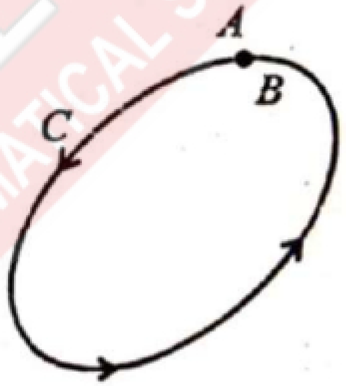
1 (iii)

Set – IV

Green's, Gauss's and Stoke's Theorems

§ 1. Some preliminary concepts.

Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction*; the opposite direction along C is then called the *negative direction*. Suppose A is the initial point and B the terminal point of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve*.



Oriented closed curve

Smooth curve. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t = t_1$ and $t = t_2$ respectively. We know that $\dfrac{d\mathbf{r}}{dt}$ is a tangent vector to this curve at the point $\mathbf{r}$. Suppose the function $\mathbf{r}(t)$ is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a **smooth curve.**

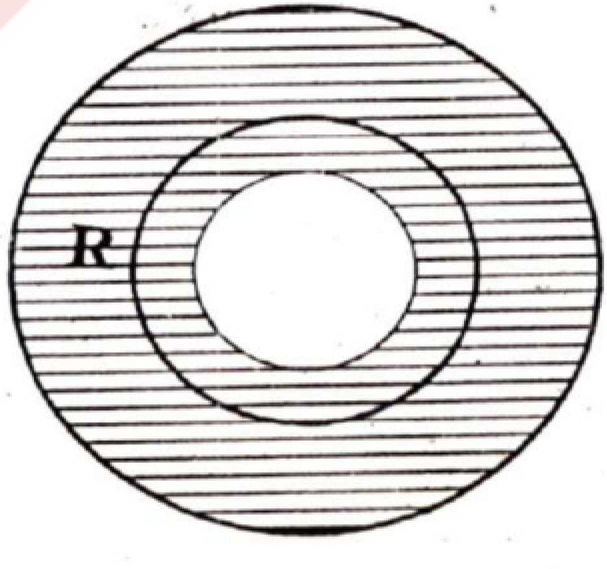
A curve C is said to be **piecewise smooth** if it is composed of a finite number of smooth curves. The curve C in the adjoining figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3. The circle is a smooth closed curve while the curve consisiting of the four sides of a rectangle is a piecewise smooth closed curve.

Piecewise smooth curve

Smooth surface. Suppose S is a surface which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S. Then S is called a **smooth surface.**

If a surface S is not smooth but can be subdivided into a finite number of smooth srufaces, then it is called a **piecewise smooth surface.** The surface of a sphere is smooth while the surface of a cube is piecewise smooth.

Classification of regions. A region R in which every closed curve can be contracted to a point without passing out of the region is called a **simply connected region.** Otherwise the region R is **multiply-connected.** The region interior to a circle is a simply-connected plane region. The region interior to a sphere is a simply-connected region in space. The region between two concentric circles lying in the same plane is a multiply connected plane region.



If we take a closed curve in this region surrounding the inner circle, then it cannot be contracted to a point without passing out of the region. Therefore the region is not simply-connected. However the region between two concentric shperes is a simply-connected region in space. The region between two infinitely long coaxial cylinders is a multiply-connected region in space.

§ 2. Line Integrals. *Any integral which is to be evaluated along a curve is called a line integral.*

Suppose $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z) *i.e.*, $\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, defines a piecewise smooth curve joining two points A and B. Let $t = t_1$ at A and $t = t_2$ at B. Suppose $\mathbf{F}(x, y, z) = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ is a vector point function defined and continuous along C. If s denotes the arc length of the curve C, then $\frac{d\mathbf{r}}{ds} = \mathbf{t}$ is a unit vector along the tangent to the curve C at the point $\mathbf{r}$.

The component of the vector $\mathbf{F}$ along this tangent is $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$. The integral of $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$ along C from A to B written as

$$\int_A^B \left[\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right] ds = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

is an example of a *line integral*. It is called *the tangent line integral* of $\mathbf{F}$ along C.

Since $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, therefore, $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

$$\therefore \mathbf{F} \cdot d\mathbf{r} = (F_1\mathbf{i} + F_2\,\mathbf{j} + F_3\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= F_1\,dx + F_2\,dy + F_3\,dz.$$

Therefore in components form the above line integral is written as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz).$$

The parametric equations of the curve C are $x = x(t)$, $y = y(t)$ and $z = z(t)$.

Therefore we may write

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right] dt.$$

Circulation. *If C is a simple closed curve (i.e. a curve which does not intersect itself anywhere), then the tangent line integral of* $\mathbf{F}$ *around C is called the circulation of* $\mathbf{F}$ *about C. It is often denoted by*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (F_1\,dx + F_2\,dy + F_3\,dz).$$

Work done by a Force. Suppose a force $\mathbf{F}$ acts upon a particle. Let the particle be displaced along a given path C in space. If $\mathbf{r}$ denotes the position vector of a point on C, then $\frac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to C at the point $\mathbf{r}$ in the direction of s increasing. The component of force $\mathbf{F}$ along tangent to C is $\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}$. Therefore the work done by $\mathbf{F}$ during a small displacement ds of the particle along C is $\left[\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right] ds$ *i.e.*, $\mathbf{F} \cdot d\mathbf{r}$. The total work W done by $\mathbf{F}$ in this displacement along C, is given by the line integral

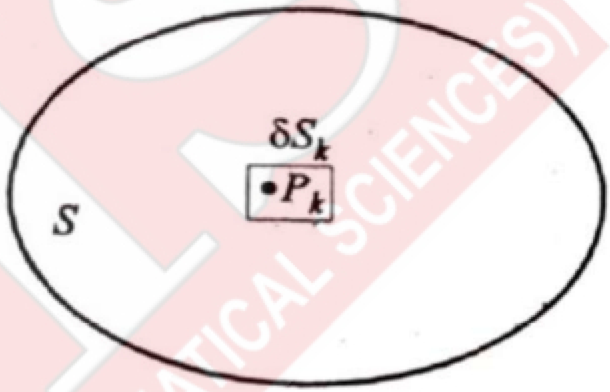
$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

the integration being taken in the sense of the displacement.

§ 3. Surface Integrals.

Any integral which is to be evaluated over a surface is called a surface integral.

Suppose S is a surface of finite area. Suppose $f(x, y, z)$ is a single valued function of position defined over S. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).



We define

$f(P_k) = f(x_k, y_k, z_k)$. Form the sum

$$\sum_{k=1}^{n} f(P_k)\, \delta S_k.$$

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the *surface integral* of $f(x, y, z)$ over S and is denoted by

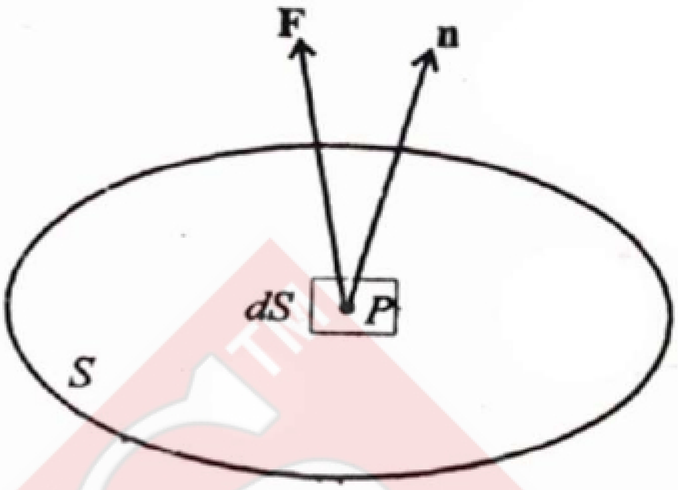
$$\iint_S f(x, y, z)\, dS.$$

It can be shown that if the surface S is piecewise smooth and the function $f(x, y, z)$ is continuous over S, then the above limit exists *i.e.*, is independent of the choice of sub-division and points P_k.

Flux. Suppose S is a piecewise smooth surface and

$\mathbf{F}(x, y, z)$

is a vector function of position defined and continuous over S. Let P be any point on the surface S and let $\mathbf{n}$ be the unit vector at P in the



direction of **outward drawn normal** to the surface S at P. Then $\mathbf{F}\cdot\mathbf{n}$ is the normal component of $\mathbf{F}$ at P. The integral of $\mathbf{F}\cdot\mathbf{n}$ over S is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

It is called the *flux* of $\mathbf{F}$ over S.

Let us associate with the differential of surface area dS a vector $d\mathbf{S}$ (called *vector area*) whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S}=\mathbf{n}\, dS$. Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Suppose the outward drawn normal to the surface S at P makes angles α, β, γ with the positive directions of x, y and z axes respectively. If l, m, n are the direction cosines of the outward drawn normal, then

$$l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma.$$

Also $\mathbf{n} = \cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k} = l\,\mathbf{i}+m\,\mathbf{j}+n\,\mathbf{k}$.

Let $\mathbf{F}(x, y, z) = F_1\,\mathbf{i}+F_2\,\mathbf{j}+F_3\,\mathbf{k}$. Then

$$\mathbf{F}\cdot\mathbf{n} = F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma = F_1 l+F_2 m+F_3 n.$$

Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)\, dS$$

$$= \iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy),$$ if we define

$$\iint_S F_1\cos\alpha\, dS = \iint_S F_1\, dy\, dz,$$

$$\iint_S F_2\cos\beta\, dS = \iint_S F_2\, dz\, dx,$$

$$\iint_S F_3\cos\gamma\, dS = \iint_S F_3\, dx\, dy.$$

Note 1. Other examples of surface integrals are

$$\iint_S f\,\mathbf{n}\, dS,\ \iint_S \mathbf{F}\times d\mathbf{S}$$

where $f(x, y, z)$ is a scalar function of position.

Note 2. Important. In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the co-ordinate plane chosen meets the surface S in no more than one point . If the surface S does not satisfy this condition, then it can be sub-divided into surfaces which do satisfy this condition.

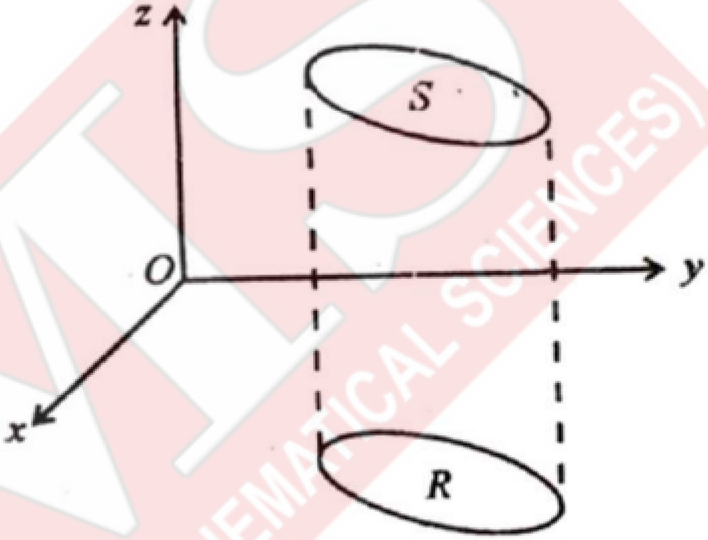
Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form

$$z = h(x, y).$$

Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle which the undirected normal $\mathbf{n}$ at $P(x, y, z)$ to the surface S makes with z-axis, then it can be shown that

$$\cos \gamma \, dS = dx \, dy,$$

where dS is the small element of area of surface S at the point P.



Therefore $dS = \dfrac{dx\,dy}{\cos\gamma} = \dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where $\mathbf{k}$ is the unit vector along z-axis.

Hence $$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}.$$

Thus the surface integral on S can be evaluated with the help of a double integral integrated over R.

§ 4. Volume Integrals.

Suppose V is a volume bounded by a surface S. Suppose $f(x, y, z)$ is a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define $f(P_k) = f(x_k, y_k, z_k)$.

Form the sum

$$\sum_{k=1}^{n} f(P_k)\,\delta V_k.$$

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the volumes δV_k approaches zero. This limit, if it exists, is called the volume integral of $f(x, y, z)$ over V and is denoted by

$$\iiint_V f(x,y,z)\, dV.$$

It can be shown that if the surface is piecewise smooth and the function $f(x,y,z)$ is continuous over V, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinates axes, then $dV = dx\, dy\, dz$ and the above volume integral becomes

$$\iiint_V f(x,y,z)\, dx\, dy\, dz.$$

If $\mathbf{F}(x, y, z)$ is a vector function, then

$$\iiint_V \mathbf{F}\, dV$$

is also an example of a volume integral.

Solved Examples

Ex. 1. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = x^2\,\mathbf{i} + y^3\,\mathbf{j}$ *and curve C is the arc of the parabola* $y = x^2$ *in the x-y plane from* (0, 0) *to* (1, 1).

Sol. We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y = x^2$ from (0, 0) to (1, 1).

Let $x = t$; then $y = t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = t^2\,\mathbf{i} + t^6\,\mathbf{j}$.

At the point (0, 0), $t = x = 0$. At the point (1, 1), $t = 1$.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 (t^2\mathbf{i} + t^6\mathbf{j}) \cdot (\mathbf{i} + 2t\,\mathbf{j})\, dt$$

$$= \int_0^1 (t^2 + 2t^7)\, dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Method 2. In the xy-plane we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$.

$$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

Therefore $\mathbf{F} \cdot d\mathbf{r} = (x^2\,\mathbf{i} + y^3\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) = x^2\, dx + y^3\, dy.$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2\,dx + y^3\,dy).$$

Now along the curve C, $y = x^2$. Therefore $dy = 2x\,dx$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{1} [x^2\,dx + x^6\,(2x)\,dx]$$

$$= \int_0^1 (x^2 + 2x^7)\,dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Ex. 2. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + xy\,\mathbf{j}$ *and curve* C *is the arc of the curve* $y = x^3$ *from* $(0, 0)$ *to* $(2, 8)$.

Sol. The curve C is the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$. Let $x = t$, then $y = t^3$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} = t\,\mathbf{i} + t^3\,\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\,\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = (t^2 - t^6)\,\mathbf{i} + t^4\,\mathbf{j}$.

At the point $(0, 0)$ $t = x = 0$. At the point $(2, 8)$, $t = 2$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_0^2 [(t^2 - t^6)\,\mathbf{i} + t^4\,\mathbf{j}] \cdot (\mathbf{i} + 3t^2\,\mathbf{j})\,dt$$

$$= \int_0^2 [(t^2 - t^6) + 3t^6]\,dt = \int_0^2 [t^2 + 2t^6]\,dt$$

$$= \left[\frac{t^3}{3} + \frac{2t^7}{7}\right]_0^2 = \left[\frac{8}{3} + \frac{256}{7}\right] = \frac{824}{21}.$$

IFoS-2006

Ex. 3. *If* $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* C *is the curve in the* xy*-plane,* $y = 2x^2$, *from* $(0, 0)$ *to* $(1, 2)$.

[Calicut 1983 ; Kanpur 78 ; Agra 76 ; Garhwal 85]

Sol. The parametric equations of the parabola $y = 2x^2$ can be taken as

$$x = t, \; y = 2t^2.$$

At the point $(0, 0)$, $x = 0$ and so $t = 0$. Again at the point $(1, 2)$, $x = 1$ and so $t = 1$.

$$\text{Now} \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$[\because \; \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}, \text{ so that } d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}]$$

$$= \int_C (3xy\,dx - y^2\,dy) = \int_{t=0}^{1} \left(3xy\frac{dx}{dt} - y^2\frac{dy}{dt}\right) dt$$

$$= \int_0^1 (3.t.2t^2.1 - 4t^4.4t)\,dt$$

[$\because$ $x=t,\ y=2t^2$ so that $dx/dt=1$ and $dy/dt=4t$]

$$= \int_0^1 (6t^3 - 16t^5)\,dt = \left[6\cdot\frac{t^4}{4} - 16\cdot\frac{t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Ex. 4. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F}$ *is* $x^2y^2\,\mathbf{i} + y\,\mathbf{j}$ *and* C *is* $y^2 = 4x$ *in the* xy*-plane from* $(0, 0)$ *to* $(4, 4)$. **(Agra 1986, Kanpur 77)**

Sol. In the xy-plane, we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

$$\therefore \quad \mathbf{F}\cdot d\mathbf{r} = (x^2y^2\,\mathbf{i} + y\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= x^2y^2\,dx + y\,dy. \qquad [\because\ \mathbf{i}\cdot\mathbf{i} = 1,\ \mathbf{i}\cdot\mathbf{j} = 0 \text{ etc.}]$$

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x^2y^2\,dx + y\,dy),$$ where C is the given curve $y^2 = 4x$ from $(0, 0)$ to $(4, 4)$

$$= \int_C x^2y^2\,dx + \int_C y\,dy = \int_{x=0}^{4} x^2(4x)\,dx + \int_{y=0}^{4} y\,dy$$

[$\because$ $y^2 = 4x$]

$$= 4\left[\frac{1}{4}x^4\right]_0^4 + \left[\frac{1}{2}y^2\right]_0^4 = 256 + 8 = 264.$$

Ex. 5. *Integrate the function* $\mathbf{F} = x^2\,\mathbf{i} - xy\,\mathbf{j}$ *from the point* $(0, 0)$ *to* $(1, 1)$ *along parabola* $y^2 = x$. **[Rohilkhand 1978]**

Sol. Here the parabola $y^2 = x$ lies in the xy-plane. If $\mathbf{r}$ is the position vector of any point (x, y) on this plane, then

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \text{ so that } d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

Let C be the curve $y^2 = x$ from $(0, 0)$ to $(1, 1)$. The parametric equaions of $y^2 = x$ can be taken as $x = t^2,\ y = t$. At the point $(0, 0)$ we have $t = 0$ and at the point $(1, 1)$ we have $t = 1$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (x^2\,\mathbf{i} - xy\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

6

$$= \int_C (x^2\,dx - xy\,dy), \text{ where } x=t^2,\ y=t$$

$$= \int_{t=0}^{1} (t^4.2t\,dt - t^2.t\,dt) = \int_0^1 (2t^5 - t^3)\,dt$$

$$= 2\left[\frac{1}{6}t^6\right]_0^1 - \left[\frac{1}{4}t^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Ex. 6. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = (x^2+y^2)\,\mathbf{i} + xy\,\mathbf{j}$ *and the curve C is the arc of the parabola $y=x^2$ from $(0, 0)$ to $(3, 9)$ in the xy-plane.*

(Kanpur 1981)

Sol. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [(x^2+y^2)\,\mathbf{i} + xy\,\mathbf{j}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) \quad [\because\ \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}]$$

$$= \int_C [(x^2+y^2)\,dx + xy\,dy] = \int_C (x^2+y^2)\,dx + \int_C xy\,dy$$

$$= \int_{x=0}^{3} (x^2+x^4)\,dx + \int_{y=0}^{9} y^{1/2}.y\,dy \quad [\because\ y^2 = x \text{ and}$$

for the curve C, x varies form 0 to 3 and y varies from 0 to 9]

$$= \left[\frac{1}{3}x^3 + \frac{1}{5}x^5\right]_0^3 + \frac{2}{5}\left[y^{5/2}\right]_0^9$$

IAS – 2008 model.

$$= 9 + \frac{243}{5} + \frac{2}{5}\cdot 243 = \frac{1}{5}[45+243+486]\,] = \frac{774}{5}.$$

Ex. 7. *Evaluate* $\int \mathbf{F}\cdot d\mathbf{r}$ *along the curve* $x^2+y^2=1$, $z=1$ *in the positive direction from* $(0, 1, 1)$ *to* $(1, 0, 1)$ *where*

$$\mathbf{F} = (2x+yz)\,\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\,\mathbf{k}.$$

Sol. Let the given curve be denoted by C and let A and B be points $(0, 1, 1)$ and $(1, 0, 1)$ respectively.

Along the given curve C, we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^B [(2x+yz)\,\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\,\mathbf{k}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_A^B [(2x+yz)\,dx + xz\,dy + (xy+2z)\,dz]. \qquad ...(1)$$

In moving from A to B, x varies from 0 to 1, y varies from 1 to 0 and z remains constant. We have $z=1$ and so $dz=0$.

Hence from (1)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2x+y)\,dx + \int_1^0 x\,dy + 0$$

$$= \int_0^1 [2x+\sqrt{(1-x^2)}]\,dx - \int_0^1 \sqrt{(1-y^2)}\,dy$$

$$= \left[x^2\right]_0^1 = 1,$$

the last two integrals cancel by a property of definite integrals.

Ex. 8. *Find the work done when a force*

$$\mathbf{F} = (x^2-y^2+x)\,\mathbf{i} - (2xy+y)\,\mathbf{j}$$

moves a particle in xy-plane from (0, 0) *to* (1, 1) *along the parabola* $y^2=x$. **(Kanpur 1980)**

Sol. Let C denote the arc of the parabola $y^2=x$ from the point $(0,0)$ to the point $(1,1)$. The parametric equations of the parabola $y^2=x$ can be taken as $x=t^2$, $y=t$. At the point $(0,0)$, $t=0$ and at the point $(1,1)$, $t=1$. The required work done

$$= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \{(x^2-y^2+x)\,\mathbf{i} - (2xy+y)\mathbf{j}\} \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(x^2-y^2+x)\,dx - (2xy+y)\,dy]$$

$$= \int_{t=0}^1 \left[(x^2-y^2+x)\frac{dx}{dt} - (2xy+y)\frac{dy}{dt}\right] dt$$

$$= \int_0^1 [(t^4-t^2+t^2) \,.\, 2t - (2t^3+t) \,.\, 1]\,dt$$

$$= \int_0^1 [2t^5-2t^3-t]\,dt = \left[2 \cdot \frac{t^6}{6} - 2 \cdot \frac{t^4}{4} - \frac{t^2}{2}\right]_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{2} = -\tfrac{2}{3}.$$

Ex. 9. *Find the work done in moving a particle in a force field*

$$\mathbf{F} = 3x^2\,\mathbf{i} + (2xz-y)\,\mathbf{j} + z\,\mathbf{k}$$

along the line joining (0, 0, 0) *to* (2, 1, 3).

Sol. Let C be the straight line joining $(0,0,0)$ to $(2,1,3)$. The parametric equations of this straight line are

$$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{3-0}=t \text{ say}$$

or $\quad x=2t,\ y=t,\ z=3t.$

At the point $(0, 0, 0)$, we have $t=0$ and at the point $(2, 1, 3)$, we have $t=1$.

The required work done

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [3x^2\mathbf{i}+(2xz-y)\,\mathbf{j}+3\,\mathbf{k}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\int_C [3x^2\,dx+(2\,x\,z-y)\,dy+z\,dz]$$

$$=\int_C [3\,(2t)^2 2\,dt+\{2(2t)\,(3t)-t\}\,dt+(3t)\,3\,dt]$$

$$=\int_{t=0}^{1}(24t^2+12\,t^2-t+9t)\,dt=\int_0^1(36t^2+8t)\,dt$$

$$=36\left[\frac{1}{3}t^3\right]_0^1+8\left[\frac{1}{2}t^2\right]_0^1=12+4=16.$$

Ex. 10. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where*

$$\mathbf{F}=c\,[-3a\sin^2 t\cos t\,\mathbf{i}+a\,(2\sin t-3\sin^3 t)\,\mathbf{j}+b\sin 2t\,\mathbf{k}]$$

and C is given by $\mathbf{r}=a\cos t\,\mathbf{i}+a\sin t\,\mathbf{j}+bt\,\mathbf{k}$ *from* $t=\pi/4$ *to* $\pi/2$.

Sol. We have $\frac{d\mathbf{r}}{dt}=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\,\mathbf{k}.$

$$\therefore\ \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{t=\pi/4}^{\pi/2}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$

$$=\int_{\pi/4}^{\pi/2} c\,[(-3a\sin^2 t\cos t)\,(-a\sin t)+a(2\sin t-3\sin^3 t)\,(a\cos t)+(b\sin 2t)\,(b)]\,dt$$

$$=c\int_{\pi/4}^{\pi/2}[3a^2\sin^3 t\cos t+a^2\,(2\sin t\cos t-3\sin^3 t\cos t)+b^2\sin 2t]\,dt$$

$$=c\int_{\pi/4}^{\pi/2}(a^2+b^2)\sin 2t\,dt=c\,(a^2+b^2)\left[-\frac{1}{2}\cos 2t\right]_{\pi/4}^{\pi/2}$$

$$=c\,(a^2+b^2)\left[-\tfrac{1}{2}\{(-1)-0\}\right]=\tfrac{1}{2}c\,(a^2+b^2).$$

Ex. 11. *Find* $\int_C \mathbf{t}\cdot d\mathbf{r}$ *where t is the unit tangent vector and C is the unit circle, in x y-plane, with centre at the origin.*

Sol. For any curve, $\frac{d\mathbf{r}}{ds}$ = unit tangent vector = $\mathbf{t}$.

$$\therefore \quad \int_C \mathbf{t} \cdot d\mathbf{r} = \int_C \left[\mathbf{t} \cdot \left(\frac{d\mathbf{r}}{ds}\right)\right] ds = \int_C (\mathbf{t} \cdot \mathbf{t})\, ds$$

$$= \int_C ds \qquad [\because \ \mathbf{t} \cdot \mathbf{t} = 1, \ \mathbf{t} \text{ being a unit vector}]$$

$$= \int_{s=0}^{2\pi} ds, \text{ since along the unit circle } C, s \text{ goes from } 0 \text{ to } 2\pi$$

$$= \left[s\right]_0^{2\pi} = 2\pi.$$

Ex. 12. *Evaluate* $\int (x\,dy - y\,dx)$ *around the circle* $x^2+y^2=1$.

Sol. Let C denote the circle $x^2+y^2=1$. The parametric equations of this circle are $x=\cos t, y=\sin t$.

To integrate around the circle C we should vary t from 0 to 2π.

$$\therefore \quad \oint_C (x\,dy - y\,dx) = \int_0^{2\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

Ex. 13. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$,

where $\quad \mathbf{F} = \mathbf{i} \cos y - \mathbf{j}\, x \sin y$

and C *is the curve* $y = \sqrt{(1-x^2)}$ *in the* xy*-plane from* $(1, 0)$ *to* $(0, 1)$.

Sol. We have $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$= \int_C (\mathbf{i} \cos y - \mathbf{j}\, x \sin y) \cdot (\mathbf{i}\, dx + \mathbf{j}\, dy)$$

$$= \int_C (\cos y\, dx - x \sin y\, dy)$$

$$= \int_C d\,(x \cos y) = \left[x \cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1.$$

Ex. 14. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i} + (x^2+y^2)\,\mathbf{j}$ *and curve* C *is the arc of* $y = x^2 - 4$ *from* $(2, 0)$ *to* $(4, 12)$. **(Garhwal 1981)**

Sol. We have $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$= \int_C [xy\,\mathbf{i} + (x^2+y^2)\,\mathbf{j}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [xy\,dx + (x^2+y^2)\,dy] = \int_C xy\,dx + \int_C (x^2+y^2)\,dy.$$

Along $C, y = x^2 - 4$ and $x^2 = y + 4$.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=2}^{4} x\,(x^2-4)\,dx + \int_{y=0}^{12} (y+4+y^2)\,dy$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_2^4 + \left[\frac{y^2}{2} + 4y + \frac{y^3}{3}\right]_0^{12} = 732.$$

Ex. 15. *Evaluate* $\int_C xy^3\,ds$, *where* C *is the segment of the line* $y = 2x$ *in the* xy*-plane from* $(-1, -2)$ *to* $(1, 2)$.

Sol. The parametric form of the curve C can be taken as

$$\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} \quad (-1 \le t \le 1).$$

We have $\dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2\,\mathbf{j}$.

Now $\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}$.

$\therefore \left|\dfrac{d\mathbf{r}}{dt}\right| = \left|\dfrac{d\mathbf{r}}{ds}\right|\dfrac{ds}{dt} = \dfrac{ds}{dt}$, because $\dfrac{d\mathbf{r}}{ds}$ is unit vector.

$$\therefore \quad \frac{ds}{dt} = |\mathbf{i} + 2\mathbf{j}| = \sqrt{5}.$$

$$\therefore \int_C xy^3\,ds = \int_C \left(xy^3\frac{ds}{dt}\right) dt = \int_{-1}^{1} t\,(2t)^3 \sqrt{5}\,dt$$

$$= 8\sqrt{5}\int_{-1}^{1} t^4\,dt = \frac{16}{\sqrt{5}}.$$

Ex. 16. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$,

where $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ *and curve* C *is* $\mathbf{r} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, t *varying from* -1 *to* $+1$.

(Tirupati 1989, Rohilkhand 92)

Sol. Along the curve C,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}.$$

$\therefore x=t,\ y=t^2,\ z=t^3$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}$.

$\therefore$ Along the curve C, we have

$$\mathbf{F} = (t\times t^2)\,\mathbf{i}+(t^2\times t^3)\,\mathbf{j}+(t^3\times t)\,\mathbf{k} = t^3\mathbf{i}+t^5\,\mathbf{j}+t^4\,\mathbf{k}.$$

Hence $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$

$$= \int_{-1}^{1}(t^3\,\mathbf{i}+t^5\,\mathbf{j}+t^4\,\mathbf{k})\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}) = \int_{-1}^{1}(t^3+2t^6+3t^6)\,dt$$

$$= \int_{-1}^{1}(t^3+5t^6)\,dt = \int_{-1}^{1}t^3\,dt+5\int_{-1}^{1}t^6\,dt$$

$$= 0+5(2)\int_0^1 t^6\,dt = 10\left[\frac{t^7}{7}\right]_0^1 = \frac{10}{7}.$$

Ex. 17. *If* $\mathbf{F} = (3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}$, *then evaluate* $\int \mathbf{F}\cdot d\mathbf{r}$ *from* $(0, 0, 0)$ *to* $(1, 1, 1)$ *along the curve*

$$x=t,\ y=t^2,\ z=t^3.$$

Sol. Along the given curve C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}. \qquad ...(1)$$

Also from the equations of the given curve we find that the points $(0, 0, 0)$ and $(1, 1, 1)$ correspond to $t=0$ and $t=1$ respectively.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$

$$= \int_C [(3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}]\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k})\,dt,$$

from (1)

$$= \int_{t=0}^{1}[(3x^2+6y)-28yzt+60xz^2t^2]\,dt$$

$$= \int_0^1 [(3t^2+6t^2)-28t^6+60t^9]\,dt,$$

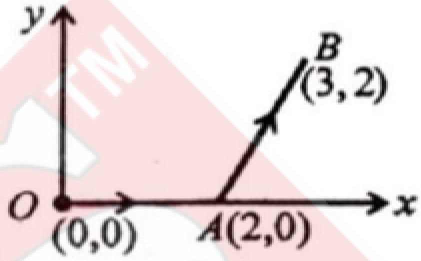
putting $x=t,\ y=t^2,\ z=t^3$

$$= \int_0^1 (9t^2-28t^6+60t^9)\,dt = \left[3t^3-4t^7+6t^{10}\right]_0^1$$

$$= 3-4+6 = 5.$$

Ex. 18. *If* $\mathbf{F} = (2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* C *is the curve in the xy-plane consisting of the straight lines from* $(0,0)$ *to* $(2,0)$ *and then to* $(3,2)$.

Sol. The path of integration C has been shown in the figure. It consists of the straight lines OA and AB.



We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [(2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C [(2x+y)\,dx+(3y-x)\,dy].$$

Now along the straight line OA, $y=0$, $dy=0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y-0 = \frac{2-0}{3-2}(x-2) \text{ i.e., } y=2x-4.$$

$\therefore$ along AB, $y=2x-4$, $dy=2\,dx$ and x varies from 2 to 3.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 [(2x+0)\,dx+0]+\int_2^3 [(2x+2x-4)\,dx + (6x-12-x)\,2dx]$$

$$= \left[x^2\right]_0^2+\int_2^3 (14x-28)\,dx = 4+14\int_2^3 (x-2)\,dx$$

$$= 4+14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4+7 = 11.$$

Ex.19. *If* $\mathbf{F} = (3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* C *is the curve consisting of the straight lines from* $(0,0,0)$ *to* $(1,0,0)$ *then to* $(1,1,0)$ *and then to* $(1,1,1)$.

Sol. We have $\mathbf{F}\cdot d\mathbf{r}$

$$= [(3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$
$$= (3x^2+6y)\,dx-14yz\,dy+20xz^2\,dz.$$

Let C_1 denote the straight line joining $(0,0,0)$ to $(1,0,0)$. Then along C_1, $y=0$, $z=0$ and x goes from 0 to 1. Obviously along C_1, $dy=0$ and $dz=0$.

Let C_2 denote the straight line joining (1, 0, 0) to (1, 1, 0). Then along C_2, $x=1$, $z=0$ and y varies from 0 to 1. Obviously along C_2, $dx=0$ and $dz=0$.

Again let C_3 denote the straight line joining (1, 1, 0) to (1, 1, 1). Along C_3, we have

$x=1$, $y=1$ so that $dx=0$, $dy=0$. Obviously along C_3, z varies from 0 to 1.

Now $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$.

We have $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} [(3x^2+6y)\, dx - 14yz\, dy + 20xz^2\, dz]$

$$= \int_{x=0}^{1} 3x^2\, dx$$

[$\because$ along C_1, $y=0$, $z=0$, $dy=0$, $dz=0$, and x varies from 0 to 1]

$$= 3\left[\frac{1}{3}x^3\right]_0^1 = 1.$$

Again $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{y=0}^{1} -14yz\, dy$

[$\because$ along C_2, $dx=0$, $dz=0$ and y varies from 0 to 1]

$$= \int_{y=0}^{1} -14y \cdot 0\, dy \qquad [\because \text{ along } C_2,\ z=0]$$

$=0$.

Finally $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{z=0}^{1} 20xz^2\, dz$

[$\because$ along C_3, $dx=0$, $dy=0$ and z varies from 0 to 1]

$$= 20\left[\frac{1}{3}z^3\right]_0^1 = \frac{20}{3}.$$

Hence $\int_C \mathbf{F} \cdot d\mathbf{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$.

Ex. 20. *If* $\mathbf{F} = (2y+3)\,\mathbf{i} + xz\,\mathbf{j} + (yz-x)\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where C is the path consisting of the straight lines from* (0, 0, 0) *to* (0, 0, 1) *then to* (0 , 1, 1) *and then to* (2, 1, 1).

Sol. We have $\mathbf{F} \cdot d\mathbf{r}$

$$= [(2y+3)\,\mathbf{i}+xz\,\mathbf{j}+(yz-x)\,\mathbf{k}]\bullet(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$
$$= (2y+3)\,dx+xz\,dy+(yz-x)\,dz.$$

Let C_1 denote the straight line joining $(0, 0, 0)$ to $(0, 0, 1)$, C_2 denote the straight line joining $(0, 0, 1)$ to $(0, 1, 1)$ and C_3 denote the straight line joining $(0, 1, 1)$ to $(2, 1, 1)$.

Along C_1, $x=0$, $y=0$ so that $dx=0$, $dy=0$.

Also along C_1, z varies from 0 to 1.

Along C_2, $x=0$, $z=1$ so that $dx=0$, $dz=0$.

Also along C_2, y varies from 0 to 1.

Along C_3, $y=1, z=1$ so that $dy=0, dz=0$.

Also along C_3, x varies from 0 to 2.

$$\therefore \quad \int_C \mathbf{F}\bullet d\mathbf{r} = \int_{C_1}\mathbf{F}\bullet d\mathbf{r}+\int_{C_2}\mathbf{F}\bullet d\mathbf{r}+\int_{C_3}\mathbf{F}\bullet d\mathbf{r}$$
$$= \int_{z=0}^{1}(0\,.\,z-0)\,dz+\int_{y=0}^{1}(0\,.\,1)\,dy+\int_{x=0}^{2}(2\,.\,1+3)\,dx$$
$$= 0+0+5\left[x\right]_0^2 = 10.$$

Ex. 21. *Evaluate* $\oint_C \mathbf{F}\bullet d\mathbf{r}$ *where* $\mathbf{F} = (x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}$, *curve C is the rectangle in the xy-plane bounded by* $y=0$, $x=a$, $y=b$, $x=0$.

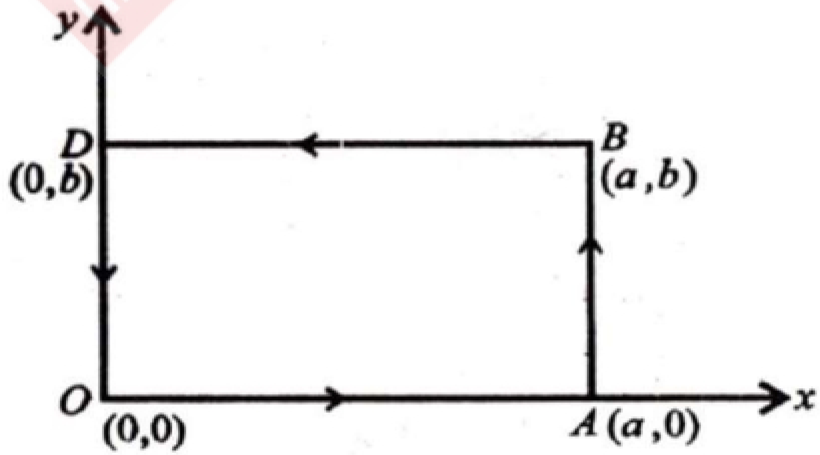
(Andhra 1992 ; Meerut 81 ; Kanpur 79)

Sol. In the xy-plane $z=0$. Therefore

$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}$.

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have $\displaystyle\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C [(x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}]\bullet(dx\,\mathbf{i}+dy\,\mathbf{j})$



$$= \int_C [(x^2+y^2)\,dx - 2xy\,dy].$$

Now on OA, $y=0$, $dy=0$ and x varies from 0 to a,
on AB, $x=a$, $dx=0$ and y varies from 0 to b,
on BD, $y=b$, $dy=0$ and x varies from a to 0,
on DO, $x=0$, $dx=0$ and y varies from b to 0.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}+b^2x\right]_a^0 + 0 = -2ab^2.$$

Ex. 22. *Find the total work done in moving a particle in a force field given by* $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ *along the curve* $x=t^2+1$, $y=2t^2$, $z=t^3$ *from* $t=1$ *to* $t=2$. **[Tirupati 1984, Madras 83, Kanpur 78]**

Sol. Let C denote the arc of the given curve from $t=1$ to $t=2$. Then the total work done

$$= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$

$$= \int_1^2 \left(3xy\frac{dx}{dt} - 5z\frac{dy}{dt} + 10x\frac{dz}{dt}\right) dt$$

$$= \int_1^2 \{3(t^2+1)(2t)^2(2t) - (5t^3)(4t) + 10(t^2+1)(3t^2)\}\,dt$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303.$$

Ex. 23. *Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field* $\mathbf{F}$ *is given by*

$$\mathbf{F} = (2x-y+2z)\,\mathbf{i} + (x+y-z)\,\mathbf{j} + (3x-2y-5z)\,\mathbf{k}.$$

(Kanpur 1979)

Sol. In the xy-plane, we have $z=0$. Therefore

$$\mathbf{F} = (2x-y)\,\mathbf{i} + (x+y)\,\mathbf{j} + (3x-2y)\,\mathbf{k}.$$

The circle C is given by $x^2+y^2=4$ or $x=2\cos t$, $y=2\sin t$.

$$\therefore \quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}.$$

Also $\mathbf{F} = (4\cos t - 2\sin t)\,\mathbf{i} + (2\cos t + 2\sin t)\,\mathbf{j} + (6\cos t - 4\sin t)\,\mathbf{k}$.

In moving round the circle once t will vary from 0 to 2π.

The required work done is $= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt$

$$= \int_0^{2\pi} [-2\sin t\,(4\cos t - 2\sin t) + 2\cos t\,(2\cos t + 2\sin t)]\,dt$$

$$= \int_0^{2\pi} [4(\sin^2 t + \cos^2 t) - 4\sin t\cos t]\,dt$$

$$= \int_0^{2\pi} (4 - 4\sin t\cos t)\,dt = \Big[4t - 2\sin 2t\Big]_0^{2\pi} = 8\pi.$$

Ex. 24. *If* $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* C *is a straight line joining* $(0, 0, 0)$ *to* $(1, 1, 1)$.

(Meerut 1983 ; Bundelkhand 79)

Sol. The equations of the straight line joining $(0, 0, 0)$ and $(1, 1, 1)$ are

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)}.$$

Then along C, $x = t$, $y = t$, $z = t$.

Also $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. $\quad\therefore\ d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt$.

Also along C, $\mathbf{F} = (3t^2 + 6t)\,\mathbf{i} - 14t^2\,\mathbf{j} + 20t^3\,\mathbf{k}$.

At $(0, 0, 0)$, $t = 0$ and at $(1, 1, 1)$, $t = 1$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t=0}^{1} [(3t^2 + 6t) - 14t^2 + 20t^3]\,dt = \frac{13}{3}.$$

Ex. 25. *If* $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *from* $(0, 0)$ *to* $(1, 1)$ *along the following paths* C :

(*a*) *the parabola* $y = x^2$, **(Agra 1973)**

(*b*) *the straight lines from* $(0, 0)$ *to* $(1, 0)$ *and then to* $(1, 1)$.

(*c*) *the straight line joining* $(0, 0)$ *and* $(1, 1)$.

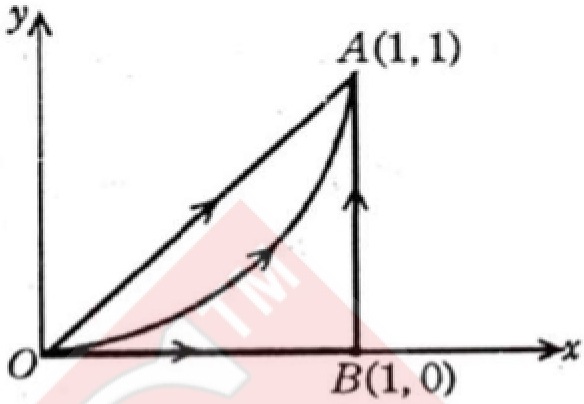
Sol. The three paths of integration have been shown in the figure.

We have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y\,\mathbf{i} - x\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C (y\,dx - x\,dy).$$

(a) C is the arc of parabola $y=x^2$ from (0, 0) to(1, 1).



Here $dy = 2x\,dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [x^2\,dx - x\,(2x)\,dx]$$

$$= \int_0^1 -x^2\,dx = -\frac{1}{3}.$$

(b) C is the curve consisting of straight lines OB and BA.

Along OB, $y=0$, $dy=0$ and x varies from 0 to 1.

Along BA, $x=1$, $dx=0$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 0\,dx + \int_0^1 -1\,dy = -1.$$

(c) C is the straight line OA. The equation of OA is

$$y-0 = \frac{1-0}{1-0}(x-0) \text{ i.e., } y=x.$$

$\therefore dy=dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (x\,dx - x\,dx) = 0.$$

Ex. 26. *Calculate* $\int_C [(x^2+y^2)\,\mathbf{i} + (x^2-y^2)\,\mathbf{j}]\cdot d\mathbf{r}$ *where C is the curve :*

(*i*) $y^2=x$ *joining* (0, 0) *to* (1, 1).

(*ii*) $x^7=y$ *joining* (0, 0) *to* (1, 1).

(*iii*) *consisting of two straight lines joining* (0, 0) *to* (1, 0) *and* (1, 0) *to* (1, 1).

(*iv*) *consisting of three straight lines joining* (0, 0) *to* (2, −2), (2, −2) *to* (0, −1) *and* (0, −1) *to* (1, 1).

Sol. Here $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [(x^2+y^2)\,dx + (x^2-y^2)\,dy] \qquad \ldots(1)$$

(i) Let C be the curve $y^2=x$ from (0, 0) to (1, 1). Then along C, $y^2=x$, x varies from 0 to 1 and y varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C (x^2+y^2)\,dx + \int_C (x^2-y^2)\,dy$$
$$= \int_{x=0}^{1} (x^2+x)\,dx + \int_{y=0}^{1} (y^4-y^2)\,dy$$
$$= \left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1 + \left[\frac{1}{5}y^5-\frac{1}{3}y^3\right]_0^1$$
$$= \left(\frac{1}{3}+\frac{1}{2}\right)+\left(\frac{1}{5}-\frac{1}{3}\right) = \frac{5}{6}-\frac{2}{15} = \frac{21}{30} = \frac{7}{10}.$$

(ii) Let C be the curve $x^7=y$ from $(0,0)$ to $(1,1)$. Then along C, $y=x^7$, $x=y^{1/7}$, x varies from 0 to 1 and y varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=0}^{1} (x^2+x^{14})\,dx + \int_{y=0}^{1} (y^{2/7}-y^2)\,dy$$
$$= \left[\frac{1}{3}x^3+\frac{1}{15}x^{15}\right]_0^1 + \left[\frac{7}{9}y^{9/7}-\frac{1}{3}y^3\right]_0^1$$
$$= \left(\frac{1}{3}+\frac{1}{15}\right)+\left(\frac{7}{9}-\frac{1}{3}\right) = \frac{2}{5}+\frac{4}{9} = \frac{38}{45}.$$

(iii) Let C_1 be the straight line joining $(0,0)$ to $(1,0)$ and C_2 be the straight line joining $(1,0)$ to $(1,1)$.

Along C_1, $y=0$ so that $dy=0$ and x varies from 0 to 1.

Along C_2, $x=1$, $dx=0$ and x varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{C_1} \{(x^2+y^2)\,dx+(x^2-y^2)\,dy\} + \int_{C_2} \{(x^2+y^2)\,dx+(x^2-y^2)\,dy\}$$
$$= \int_{x=0}^{1} (x^2+0)\,dx + \int_{y=0}^{1} (1-y^2)\,dy$$
$$= \frac{1}{3}\left[x^3\right]_0^1 + \left[y-\frac{1}{3}y^3\right]_0^1 = \frac{1}{3}+\left(1-\frac{1}{3}\right) = 1.$$

(iv) Let C_1 be the straight line joining $(0,0)$ to $(2,-2)$, C_2 be the straight line joninig $(2,-2)$ to $(0,-1)$ and C_3 be the straight line joining $(0,-1)$ to $(1,1)$.

The equation of the straight line joining $(0,0)$ and $(2,-2)$ is

$$y=-\frac{2}{2}x \text{ or } y=-x.$$

$\therefore$ along C_1, $y=-x$, $dy=-dx$ and x varies from 0 to 2.

$\therefore$ from (1), $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}=\int_{x=0}^{2}[(x^2+x^2)\,dx+(x^2-x^2)\,(-dx)]$

$$=2\int_0^2 x^2\,dx=2\left[\frac{1}{3}x^3\right]_0^2=\frac{16}{3}\cdot$$

The equation of the straight line joining $(2,-2)$ and $(0,-1)$ is

$y+1=\frac{-2-(-1)}{2-0}(x-0)$ or $y+1=-\frac{1}{2}x$ or $y=-1-\frac{1}{2}x$

or $y=-\frac{1}{2}(x+2)$.

$\therefore$ along C_2, $y=-\frac{1}{2}(x+2)$, $dy=-\frac{1}{2}dx$ and x varies from 2 to 0.

$\therefore$ from (1),

$$\int_{C_2}\mathbf{F}\cdot d\mathbf{r}=\int_{x=2}^{0}\left[\left\{x^2+\frac{1}{4}(x+2)^2\right\}dx+\left\{x^2-\frac{1}{4}(x+2)^2\right\}\left(-\frac{1}{2}dx\right)\right]$$

$$=\int_2^0\left[\left(1+\frac{1}{4}-\frac{1}{2}+\frac{1}{8}\right)x^2+\left(1+\frac{1}{2}\right)x+\left(1+\frac{1}{2}\right)\right]dx$$

$$=\int_2^0\left[\frac{7}{8}x^2+\frac{3}{2}x+\frac{3}{2}\right]dx=\left[\frac{7}{8}\cdot\frac{1}{3}x^3+\frac{3}{4}x^2+\frac{3}{2}x\right]_2^0$$

$$=-\frac{7}{3}-3-3=-\frac{25}{3}\cdot$$

The equation of the straight line joining $(0,-1)$ and $(1,1)$ is

$$y+1=\frac{1+1}{1-0}(x-0) \quad\text{or}\quad y+1=2x \text{ or } y=2x-1.$$

$\therefore$ along C_3, $y=2x-1$, $dy=2\,dx$ and x varies from 0 to 1.

$\therefore$ from (1),

$$\int_{C_3}\mathbf{F}\cdot d\mathbf{r}=\int_{x=0}^{1}\left[\left\{x^2+(2x-1)^2\right\}dx+\left\{x^2-(2x-1)^2\right\}(2\,dx)\right]$$

$$=\int_0^1\left[-x^2+4x-1\right]dx$$

$$=\left[-\frac{1}{3}x^3+2x^2-x\right]_0^1=-\frac{1}{3}+2-1=\frac{2}{3}\cdot$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}$

$$= \frac{16}{3} - \frac{25}{3} + \frac{2}{3} = -\frac{7}{3}.$$

Ex. 27. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ *and* C *is the portion of the curve* $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}$, *from* $t=0$ *to* $t=\pi/2$.

Sol. Along the curve C,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}.$$

$$\therefore \quad x = a\cos t,\ y = b\sin t,\ z = ct.$$

Now $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \int_C (yz\,dx + zx\,dy + xy\,dz) = \int_C d\,(xyz)$$

$$= \Big[xyz\Big]_{t=0}^{t=\pi/2} = \Big[(a\cos t)\,.\,(b\sin t)\,.\,(ct)\Big]_0^{\pi/2}$$

$$= abc\,\Big[t\cos t\sin t\Big]_0^{\pi/2} = abc\,(0-0) = 0.$$

Ex. 28. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ *and* C *is the arc of the curve* $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ *from* $t=0$ *to* $t=2\pi$.

(Agra 1977 ; Garhwal 86)

Sol. The vector equation of the given curve is

$$\mathbf{r} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}. \qquad ...(1)$$

$\therefore$ the parametric equations of (1) are

$$x = \cos t,\ y = \sin t,\ z = t. \qquad ...(2)$$

From (1), $\frac{d\mathbf{r}}{dt} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}.$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$

$$= \int_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k})\cdot[(-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}]\,dt$$

$$= \int_C (-z\sin t + x\cos t + y)\,dt$$

$$= \int_{t=0}^{2\pi} (-t\sin t + \cos^2 t + \sin t)\,dt,$$

putting for x, y, z from (2)

$$= -\int_0^{2\pi} t\sin t\,dt + \tfrac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt + \int_0^{2\pi} \sin t\,dt$$

$$= -\left[\left\{t(-\cos t)\right\}_0^{2\pi} + \int_0^{2\pi} \cos t\, dt\right] + \tfrac{1}{2}\left[t + \tfrac{1}{2}\sin 2t\right]_0^{2\pi} + \left[-\cos t\right]_0^{2\pi}$$

$$= -\left[\left\{2\pi(-1)\right\} + \left\{\sin t\right\}_0^{2\pi}\right] + \tfrac{1}{2}[2\pi] + [-\cos 2\pi + \cos 0]$$

$$= 2\pi + \pi = 3\pi.$$

Ex. 29. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ *and* C *is the arc of the curve* $\mathbf{r} = (a\cos\theta)\,\mathbf{i} + (a\sin\theta)\,\mathbf{j} + a\theta\,\mathbf{k}$ *from* $\theta = 0$ *to* $\theta = \tfrac{1}{2}\pi$.

Sol. The parametric equations of the given curve are

$$x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta. \qquad \text{...(1)}$$

Also $\dfrac{d\mathbf{r}}{d\theta} = (-a\sin\theta)\,\mathbf{i} + (a\cos\theta)\,\mathbf{j} + a\,\mathbf{k}.$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}) \cdot [(-a\sin\theta)\,\mathbf{i} + (a\cos\theta)\,\mathbf{j} + a\,\mathbf{k}]\, d\theta$$

$$= \int_C (-axy\sin\theta + ayz\cos\theta + azx)\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} (-a^3\cos\theta\sin^2\theta + a^3\theta\sin\theta\cos\theta + a^3\theta\cos\theta)\, d\theta,$$

putting for x, y, z from (1)

$$= -a^3\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta + \tfrac{1}{2}a^3\int_0^{\pi/2} \theta\sin 2\theta\, d\theta + a^3\int_0^{\pi/2} \theta\cos\theta\, d\theta$$

$$= -a^3\left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} + \tfrac{1}{2}a^3\left[\left(-\theta\,\frac{\cos 2\theta}{2}\right)_0^{\pi/2} + \tfrac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta\right] + a^3\left[\left(\theta\sin\theta\right)_0^{\pi/2} - \int_0^{\pi/2} \sin\theta\, d\theta\right]$$

$$= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3\left[\tfrac{1}{4}\pi + \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2\theta\right)_0^{\pi/2}\right] + a^3\left[\tfrac{1}{2}\pi + \left(\cos\theta\right)_0^{\pi/2}\right]$$

$$= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3\left[\tfrac{1}{4}\pi\right] + a^3\left[\tfrac{1}{2}\pi - 1\right]$$

$$= a^3\left[-\tfrac{1}{3} + \tfrac{1}{8}\pi + \tfrac{1}{2}\pi - 1\right] = a^3\left[\tfrac{5}{8}\pi - \tfrac{4}{3}\right].$$

Ex. 30. *If* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} - xy\,\mathbf{k}$, *find* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* C *is given by* $x = t,\ y = t^2,\ z = t^3$ *from* $P(0, 0, 0)$ *to* $Q(2, 4, 8)$. **[Madurai 1985]**

Sol. Along the given curve C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}.$$

Also along the given curve C, we have

$$\mathbf{F} = (t^2 \cdot t^3)\,\mathbf{i}+(t^3 \cdot t)\,\mathbf{j}-(t \cdot t^2)\,\mathbf{k} = t^5\,\mathbf{i}+t^4\,\mathbf{j}-t^3\,\mathbf{k}.$$

At the point $P\,(0, 0, 0)$, we have $t=0$ and at the point $Q\,(2, 4, 8)$, we have $t=2$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left[\mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right] dt$$

$$= \int_{t=0}^{2} [(t^5\,\mathbf{i}+t^4\,\mathbf{j}-t^3\,\mathbf{k})\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k})]\,dt$$

$$= \int_0^2 (t^5+2t^5-3t^5)\,dt = \int_0^2 0\,dt = 0.$$

Ex. 31. *Evaluate* $\int_C x^{-1}(y+z)\,ds$, *where C is the arc of the circle* $x^2+y^2 = 4$ *in the xy-plane from* $A\,(2, 0, 0)$ *to* $B\,(\sqrt{2}, \sqrt{2}, 0)$.

Sol. Let $x=2\cos t$, $y=2\sin t$, $z=0$ be the parametric equations of the circle $x^2+y^2 = 4$, $z=0$.

For the point A, $x=2, y=0, z=0$ and so $t=0$ and for the point B, $x=\sqrt{2}$, $y=\sqrt{2}$, $z=0$ and so $t=\pi/4$.

If $\mathbf{r}$ is the position vector of any point (x, y, z) on the circle C, then

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = 2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}+0\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -2\sin t\,\mathbf{i}+2\cos t\,\mathbf{j}.$$

$$\therefore \left(\frac{d\mathbf{r}}{dt}\right)^2 = 4\sin^2 t+4\cos^2 t = 4.$$

But $\left(\frac{d\mathbf{r}}{dt}\right)^2 = \left(\frac{d\mathbf{r}}{ds}\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 \mathbf{t}^2$, where $\mathbf{t} = \frac{d\mathbf{r}}{ds}$ is unit tangent vector

$$= \left(\frac{ds}{dt}\right)^2 \qquad [\because\ \mathbf{t}^2 = \mathbf{t}\cdot\mathbf{t} = 1,\ \mathbf{t} \text{ being unit vector}]$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 4 \text{ or } \frac{ds}{dt} = 2.$$

$$\therefore \int_C x^{-1}(y+z)\,ds = \int_{t=0}^{\pi/4} \frac{y+z}{x}\frac{ds}{dt}\,dt$$

$$= \int_0^{\pi/4} \frac{2\sin t+0}{2\cos t} \cdot 2\,dt = 2\int_0^{\pi/4} \tan t\,dt$$

$$= 2\left[\log \sec t\right]_0^{\pi/4} = 2\log\sqrt{2}$$

$$= 2\cdot\frac{1}{2}\log 2 = \log 2.$$

Ex. 32. *Evaluate* $\int_C (x^2+y^2+z^2)^2\,ds$, *where C is the arc of the circular helix*

$$\mathbf{r}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+3t\,\mathbf{k}$$

from A $(1, 0, 0)$ *to B* $(1, 0, 6\pi)$.

Sol. The equation of the curve C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+3t\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+3\,\mathbf{k}.$$

$$\therefore \left(\frac{d\mathbf{r}}{dt}\right)^2 = \sin^2 t+\cos^2 t+9 = 10.$$

$$\therefore \left(\frac{d\mathbf{r}}{ds}\frac{ds}{dt}\right)^2 = 10 \text{ or } \left(\frac{ds}{dt}\right)^2 \mathbf{t}^2 = 10, \text{ where } \mathbf{t} = \frac{d\mathbf{r}}{ds} \text{ is unit tangent}$$

vector to C at the point 't'

$$\text{or } \left(\frac{ds}{dt}\right)^2 = 10 \qquad [\because\ \mathbf{t}^2 = \mathbf{t}\cdot\mathbf{t} = 1,\ \mathbf{t} \text{ being unit vector}]$$

$$\text{or } \frac{ds}{dt} = \sqrt{(10)}.$$

Parametric equations of C are

$$x = \cos t,\ y = \sin t,\ z = 3t.$$

At the point A, $x=1, y=0, z=0$ and so $t=0$ and at the point B, $x=1$, $y=0$, $z=6\pi$ and so $t=2\pi$.

$$\therefore \int_C (x^2+y^2+z^2)^2\,ds = \int_{t=0}^{2\pi} (\cos^2 t+\sin^2 t+9t^2)^2 \frac{ds}{dt}\,dt$$

$$= \int_0^{2\pi} (1+9t^2)^2\cdot\sqrt{(10)}\,dt = \sqrt{(10)}\int_0^{2\pi} (1+18t^2+81t^4)\,dt$$

$$= \sqrt{(10)}\left[t+6t^3+\frac{81}{5}t^5\right]_0^{2\pi} = \sqrt{(10)}\left[2\pi+6\,(2\pi)^3+\frac{81}{5}(2\pi)^5\right].$$

Ex. 33. *Find the circulation of* **F** *round the curve C, where*

$$\mathbf{F} = e^x \sin y\,\mathbf{i}+e^x \cos y\,\mathbf{j}$$

and C is the rectangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, \frac{1}{2}\pi)$, $(0, \frac{1}{2}\pi)$.

Sol. The rectangle C lies in the xy-plane. If $\mathbf{r}$ is the position vector of any point (x, y) on this plane, then

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j} \text{ so that } d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}.$$

By definition, the circulation of $\mathbf{F}$ round the curve C

$$= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (e^x \sin y\,\mathbf{i}+e^x \cos y\,\mathbf{j})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \oint_C (e^x \sin y\,dx+e^x \cos y\,dy). \qquad ...(1)$$

Draw figure as in solved example 21.

Let O be the point $(0, 0)$, A be the point $(1, 0)$, B be the point $(1, \frac{1}{2}\pi)$ and D be the point $(0, \frac{1}{2}\pi)$.

Now on OA, $y=0, dy=0$ and x varies from 0 to 1,

on AB, $x=1, dx=0$ and y varies from 0 to $\frac{1}{2}\pi$,

on BD, $y=\frac{1}{2}\pi, dy=0$ and x varies from 1 to 0,

on DO, $x=0, dx=0$ and y varies from $\frac{1}{2}\pi$ to 0.

$$\therefore \text{ from (1), } \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} e^x\,.\,0\,dx+\int_{y=0}^{\pi/2} e\cos y\,dy$$

$$+\int_{x=1}^{0} e^x \sin \tfrac{1}{2}\pi\,dx + \int_{y=\pi/2}^{0} \cos y\,dy$$

$$= e\left[\sin y\right]_0^{\pi/2}+\left[e^x\right]_1^0+\left[\sin y\right]_{\pi/2}^0$$

$$= e+1-e+(0-1) = 0.$$

Ex. 34. *If* $\mathbf{F} = \dfrac{-y\,\mathbf{i}+x\,\mathbf{j}}{(x^2+y^2)}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *around any closed path C in the xy-plane.*

Sol. In the xy-plane, $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\frac{-y\,\mathbf{i}+x\,\mathbf{j}}{x^2+y^2}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C \frac{-y\,dx+x\,dy}{x^2+y^2}.$$

We change to polar coordinates by putting $x=r\cos\theta$, $y=r\sin\theta$.

$\therefore \quad dx = -r\sin\theta\, d\theta + \cos\theta\, dr$

and $dy = r\cos\theta\, d\theta + \sin\theta\, dr$.

$$\therefore \quad \frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{1}{r^2}\Big[-r\sin\theta(-r\sin\theta\, d\theta + \cos\theta\, dr) + r\cos\theta(r\cos\theta\, d\theta + \sin\theta\, dr)\Big]$$

$$= \frac{r^2(\cos^2\theta + \sin^2\theta)}{r^2}\, d\theta = d\theta.$$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C d\theta \qquad \text{...(1)}$$

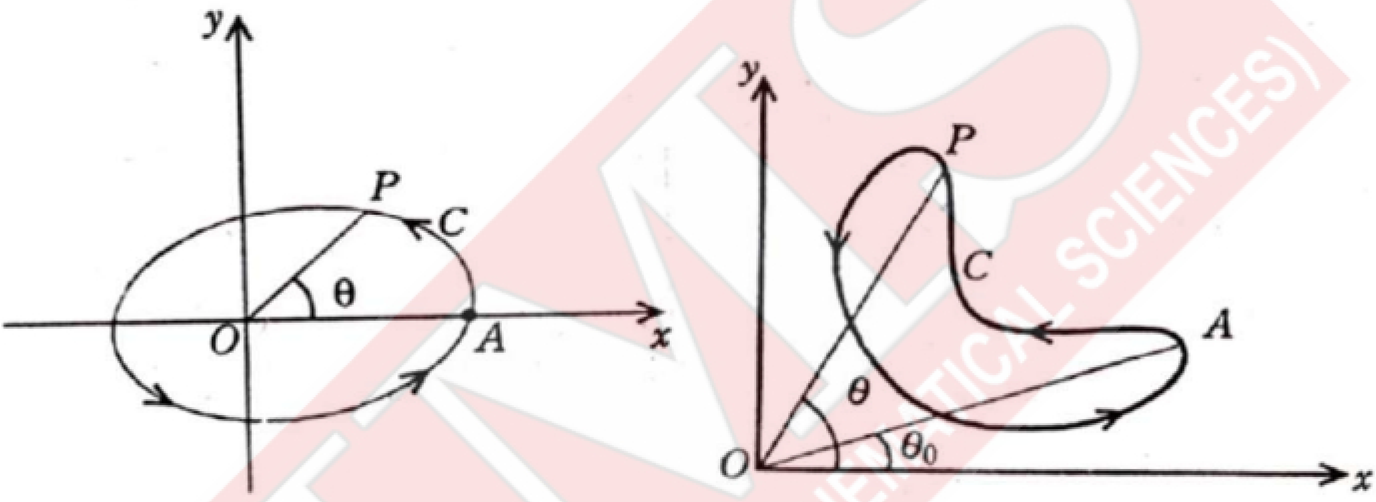


Fig (i) Fig (ii)

Case I. If the origin O lies inside the closed curve C as in fig. (i), then for the curve C at the point A, we have $\theta = 0$ and when after a complete round we come back to A, then at A, $\theta = 2\pi$. So from (1)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\theta=0}^{2\pi} d\theta = 2\pi.$$

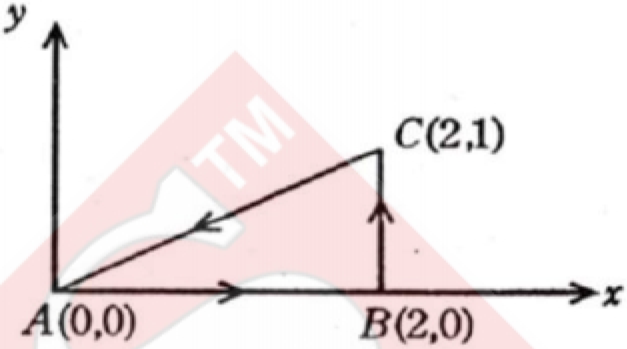
Case II. If the origin O lies outside the closed curve C as in fig. (ii), then for the curve C at the point A, we have $\theta = \theta_0$ and when after a complete round along C we come back to A, then also at A, $\theta = \theta_0$. So from (1)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\theta=\theta_0}^{\theta_0} d\theta = \Big[\theta\Big]_{\theta_0}^{\theta_0} = 0.$$

Ex.35. *If* $\mathbf{F} = (2x^2+y^2)\,\mathbf{i} + (3y-4x)\,\mathbf{j}$, *evaluate* $\int \mathbf{F} \cdot d\mathbf{r}$ *around the triangle ABC whose vertices are* $A\,(0, 0)$, $B\,(2, 0)$ *and* $C\,(2, 1)$.

Sol. Let C_1 denote the curve consisting of the straight lines AB, BC and CA. Then

$\int \mathbf{F}\cdot d\mathbf{r}$ around the triangle



$$ABC = \int_{C_1} [(2x^2+y^2)\,\mathbf{i}+(3y-4x)\,\mathbf{j}] \cdot (dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_{C_1} [(2x^2+y^2)\,dx+(3y-4x)\,dy]$$

$$= \int_{AB} [(2x^2+y^2)\,dx+(3y-4x)\,dy] + \int_{BC} [(2x^2+y^2)\,dx+(3y-4x)\,dy]$$

$$+ \int_{CA} [(2x^2+y^2)\,dx+(3y-4x)\,dy] \quad ...(1)$$

Now along the straight line AB, $y=0$, $dy=0$ and x varies from 0 to 2;

along the straight line BC, $x=2$, $dx=0$ and y varies from 0 to 1;

and along the straight line CA, $y-0=\frac{1-0}{2-0}x$

i.e., $y=\frac{1}{2}x$ or $x=2y$ so that $dx=2\,dy$ and y varies from 1 to 0.

$\therefore$ from (1), $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=0}^{2} 2x^2\,dx + \int_{y=0}^{1} (3y-8)\,dy + \int_{y=1}^{0} [\{2\,.\,(2y)^2+y^2\}(2\,dy) + (3y-8y)\,dy]$$

$$= 2\left[\frac{1}{3}x^3\right]_0^2 + \left[\frac{3y^2}{2}-8y\right]_0^1 + \int_1^0 (18y^2-5y)\,dy$$

$$= \frac{16}{3}+\frac{3}{2}-8+\left[6y^3-\frac{5}{2}y^2\right]_1^0 = \frac{16}{3}+\frac{3}{2}-8-6+\frac{5}{2}$$

$$= \frac{16}{3}-10 = -\frac{14}{3}.$$

Ex. 36. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and* C *is the x-axis from* $x=2$ *to* $x=4$ *and the straight line* $x=4$ *from* $y=0$ *to* $y=12$.

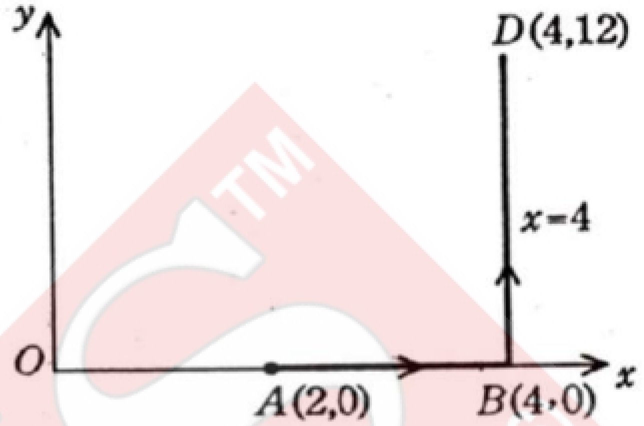
Sol. Here the curve C consists of the straight lines AB and BD where A, B and D are the points (2, 0), (4, 0) and (4, 12) respectively.

We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C [xy\,dx+(x^2+y^2)\,dy]$$

$$= \int_{AB} [xy\,dx+(x^2+y^2)\,dy] + \int_{BD} [xy\,dx+(x^2+y^2)\,dy]. \quad \ldots(1)$$



Along the straight line AB, $y=0$, $dy=0$ and x varies from 2 to 4.

Along the straight line BD, $x=4$, $dx=0$ and y varies from 0 to 12.

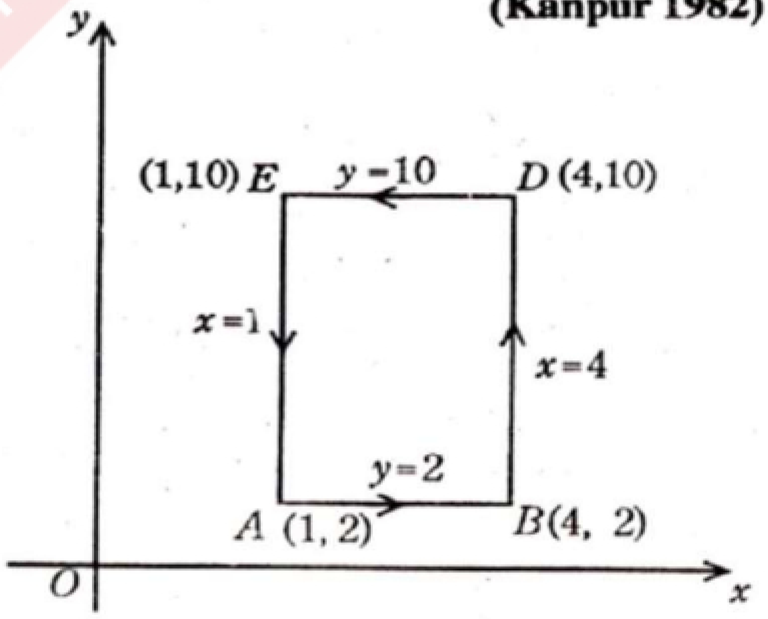
$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=2}^{4} (x\,.\,0)\,dx + \int_{y=0}^{12} (16+y^2)\,dy$$

$$= 0 + \left[16y + \frac{1}{3}y^3\right]_0^{12} = 192+576 = 768.$$

Ex. 37. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and* C *is the rectangle in the xy-plane bounded by the lines* $y=2, x=4, y=10$ *and* $x=1$. **(Kanpur 1982)**

Sol. Here the curve C consists of the four straight lines AB, BD, DE and EA.



Along the line AB, $y=2$, $dy=0$ and x varies from 1 to 4.

Along the line BD, $x=4$, $dx=0$ and y varies from 2 to 10.

Along the line DE, $y=10$, $dy=0$ and x varies from 4 to 1.

Along the line EA, $x=1, dx=0$ and y varies from 10 to 2.

We have $\mathbf{F}\cdot d\mathbf{r} = [xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

$= xy\,dx+(x^2+y^2)\,dy.$

$$\text{Now } \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{AB} \mathbf{F}\cdot d\mathbf{r}+\int_{BD} \mathbf{F}\cdot d\mathbf{r}+\int_{DE} \mathbf{F}\cdot d\mathbf{r}+\int_{EA} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{x=1}^{4} 2x\,dx+\int_{y=2}^{10}(16+y^2)\,dy+\int_{x=4}^{1}10x\,dx+\int_{y=10}^{2}(1+y^2)\,dy$$

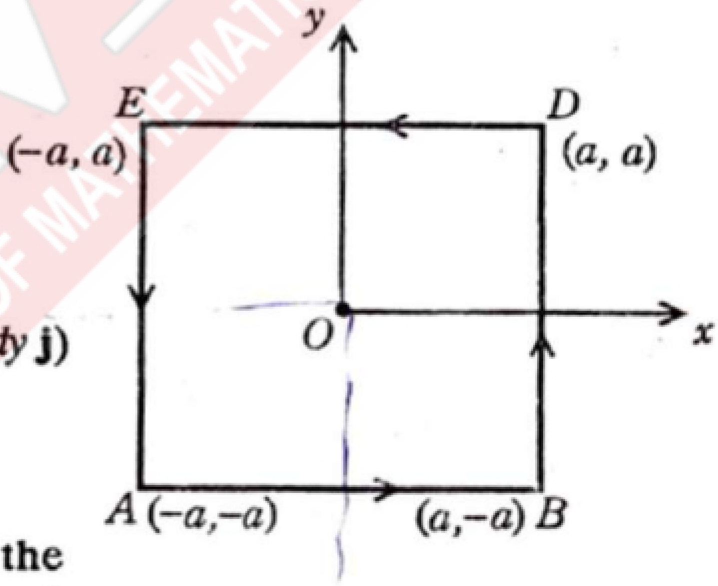
$$= \left[x^2\right]_1^4+\left[16y+\frac{1}{3}y^3\right]_2^{10}+\left[5x^2\right]_4^1+\left[y+\frac{1}{3}y^3\right]_{10}^{2}$$

$$= 15+\left[160+\frac{1000}{3}-32-\frac{8}{3}\right]-75+\left[2+\frac{8}{3}-10-\frac{1000}{3}\right]$$

$$= 15+160-32-75+2-10 = 60.$$

Ex. 38. *Evaluate* $\displaystyle\int_C \frac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot d\mathbf{r}$, *where C is the boundary of the square $x=\pm a$, $y=\pm a$ in the counter clockwise sense.*

Sol. Let $\mathbf{F} = \dfrac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}$.



We have

$$\mathbf{F}\cdot d\mathbf{r}= \frac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot d\mathbf{r}$$

$$= \frac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \frac{-y^3\,dx+x^3\,dy}{(x^2+y^2)^2}.$$

The curve C consists of the four straight lines AB, BD, DE and EA.

Along the line AB, $y=-a, dy=0$ and x varies from $-a$ to a.

Along the line BD, $x=a, dx=0$ and y varies from $-a$ to a.

Along the line DE, $y=a, dy=0$ and x varies from a to $-a$.

Along the line EA, $x=-a, dx=0$ and y varies from a to $-a$.

$$\text{We have } \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{AB} \mathbf{F}\cdot d\mathbf{r}+\int_{BD} \mathbf{F}\cdot d\mathbf{r}+\int_{DE} \mathbf{F}\cdot d\mathbf{r}+\int_{EA} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{x=-a}^{a} \frac{a^3\, dx}{(x^2+a^2)^2} + \int_{y=-a}^{a} \frac{a^3\, dy}{(y^2+a^2)^2} + \int_{x=a}^{-a} \frac{-a^3\, dx}{(x^2+a^2)^2} + \int_{y=a}^{-a} \frac{-a^3\, dy}{(y^2+a^2)^2}$$

$$= 4\int_{-a}^{a} \frac{a^3\, dx}{(x^2+a^2)^2}$$

$$\left[\because \int_a^b f(x)\, dx = \int_a^b f(y)\, dy = -\int_b^a f(x)\, dx\right]$$

$$= 8a^3 \int_0^a \frac{dx}{(x^2+a^2)^2}$$

$$= 8a^3 \int_0^{\pi/4} \frac{a \sec^2\theta\, d\theta}{a^4 \sec^4\theta}, \text{ putting } x = a\tan\theta \text{ so that } dx = a\sec^2\theta\, d\theta$$

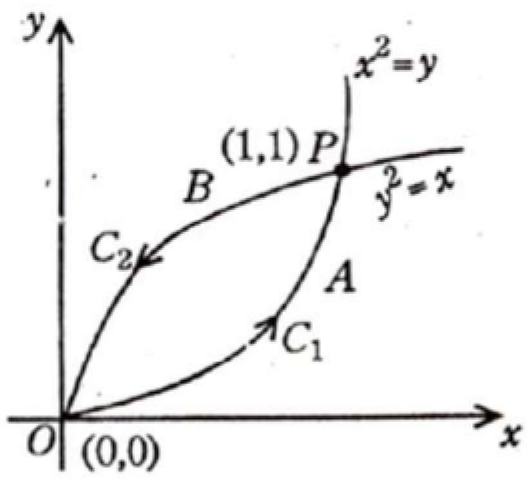
$$= 8\int_0^{\pi/4} \cos^2\theta\, d\theta = 4\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = 4\left[\frac{\pi}{4}+\frac{1}{2}\right] = \pi + 2.$$

Ex. 39. *Find the circulation of* **F** *round the curve C where* $\mathbf{F} = (2x+y^2)\,\mathbf{i} + (3y-4x)\,\mathbf{j}$ *and C is the curve* $y = x^2$ *from* $(0, 0)$ *to* $(1, 1)$ *and the curve* $y^2 = x$ *from* $(1, 1)$ *to* $(0, 0)$.

Sol. Here the closed curve C consists of arcs OAP and PBO.

Let C_1 denote the arc OAP and C_2 denote the arc PBO.



Along C_1, we have $y = x^2$ so that $dy = 2x\, dx$ and x varies from 0 to 1.

Along C_2, we have $x = y^2$ so that $dx = 2y\, dy$ and y varies from 1 to 0.

Also

$$\mathbf{F}\cdot d\mathbf{r} = [(2x+y^2)\,\mathbf{i} + (3y-4x)\,\mathbf{j}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= (2x+y^2)\, dx + (3y-4x)\, dy.$$

Now circulation of **F** round C

$$= \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{C_1} [(2x+y^2)\,dx + (3y-4x)\,dy] + \int_{C_2} [(2x+y^2)\,dx + (3y-4x)\,dy]$$

$$= \int_{x=0}^{1} [(2x+x^4)\,dx + (3x^2-4x)\,2x\,dx]$$

$$+ \int_{y=1}^{0} [(2y^2+y^2)\,2y\,dy + (3y-4y^2)\,dy]$$

$$= \int_0^1 (2x-8x^2+6x^3+x^4)\,dx + \int_1^0 (3y-4y^2+6y^3)\,dy$$

$$= \left[x^2 - \frac{8}{3}x^3 + \frac{3}{2}x^4 + \frac{1}{5}x^5\right]_0^1 + \left[\frac{3}{2}y^2 - \frac{4}{3}y^3 + \frac{3}{2}y^4\right]_1^0$$

$$= 1 - \frac{8}{3} + \frac{3}{2} + \frac{1}{5} - \frac{3}{2} + \frac{4}{3} - \frac{3}{2}$$

$$= 1 - \frac{4}{3} + \frac{1}{5} - \frac{3}{2} = \frac{30-40+6-45}{30} = -\frac{49}{30}.$$

Ex. 40. (a) *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where*

$$\mathbf{F} = (2x+y)\,\mathbf{i} + (3y-x)\,\mathbf{j} + yz\,\mathbf{k}$$

and C is the curve $x=2t^2,\ y=t,\ z=t^3$ *from* $t=0$ *to* $t=1$.

(b) *If* $\mathbf{A}=(2y+3)\,\mathbf{i}+xz\,\mathbf{j}+(yz-x)\,\mathbf{k}$, *evaluate* $\int_C \mathbf{A}\cdot d\mathbf{r}$ *along the curve C* :

$x=2t^2,\ y=t,\ z=t^3$ *from* $t=0$ *to* $t=1$. **(Kakatiya 1992)**

(c) *Evaluate* $\int_C \mathbf{A}\cdot d\mathbf{r}$ *where C is the line joining* (0, 0, 0) *and* (2, 1, 1) *and* $\mathbf{A} = (2y+3)\,\mathbf{i}+xz\,\mathbf{j}+(yz-x)\,\mathbf{k}$. **(Nagarjuna 1991)**

Sol. (a) Along the given curve C, we have

$$\mathbf{r} = 2t^2\,\mathbf{i}+t\,\mathbf{j}+t^3\,\mathbf{k},\quad \frac{d\mathbf{r}}{dt} = 4t\,\mathbf{i}+\mathbf{j}+3t^2\,\mathbf{k}$$

and $\mathbf{F} = (4t^2+t)\,\mathbf{i}+(3t-2t^2)\,\mathbf{j}+t^4\,\mathbf{k}.$

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left[\mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right]dt$$

$$= \int_{t=0}^{1} [4t\,(4t^2+t)+(3t-2t^2)+3t^6]\,dt = \int_0^1 (3t^6+16t^3+2t^2+3t)\,dt$$

$$=\left[\frac{3}{7}t^7+4t^4+\frac{2}{3}t^3+\frac{3}{2}t^2\right]_0^1=\frac{3}{7}+4+\frac{2}{3}+\frac{3}{2}=\frac{277}{42}.$$

(b) Along the given curve C, we have

$$\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=2t^2\,\mathbf{i}+t\,\mathbf{j}+t^3\,\mathbf{k},$$

on putting the values of x, y, z in terms of t.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=4t\,\mathbf{i}+\mathbf{j}+3t^2\,\mathbf{k}.$$

Also $\mathbf{A}=(2t+3)\,\mathbf{i}+2t^5\,\mathbf{j}+(t^4-2t^2)\,\mathbf{k}$.

$$\therefore \int_C \mathbf{A}\cdot d\mathbf{r}=\int_C\left[\mathbf{A}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right]dt$$

$$=\int_{t=0}^{1}[4t\,(2t+3)+1\cdot 2t^5+3t^2\cdot(t^4-2t^2)]dt$$

$$=\int_0^1 (8t^2+12t+2t^5+3t^6-6t^4)\,dt$$

$$=\left[\frac{8}{3}t^3+6t^2+\frac{1}{3}t^6+\frac{3}{7}t^7-\frac{6}{5}t^5\right]_0^1$$

$$=\frac{8}{3}+6+\frac{1}{3}+\frac{3}{7}-\frac{6}{5}=\frac{280+630+35+45-126}{105}=\frac{864}{105}=\frac{288}{35}.$$

(c) The equations of the straight line joining $(0,0,0)$ and $(2,1,1)$ are

$$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{1-0}=t \text{ (say)}.$$

Then along C, $x=2t, y=t, z=t$.

At the point $(0,0,0)$, $t=0$ and at the point $(2,1,1)$, $t=1$.

Along the curve C, we have

$\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=2t\,\mathbf{i}+t\,\mathbf{j}+t\,\mathbf{k}$ and so $d\mathbf{r}=2\,\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Also along C, $\mathbf{A}=(2t+3)\,\mathbf{i}+2t^2\,\mathbf{j}+(t^2-2t)\,\mathbf{k}$.

$$\therefore \int_C \mathbf{A}\cdot d\mathbf{r}=\int_{t=0}^{1}[2\,(2t+3)+1.2t^2+1.(t^2-2t)]dt$$

$$=\int_0^1 [4t+6+2t^2+t^2-2t]dt=\int_0^1 [3t^2+2t+6]dt$$

$$=\left[t^3+t^2+6t\right]_0^1=1+1+6=8.$$

Ex. 41. Evaluate

$$\int_C \{(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy\}$$

where C is the arc of the parabola $2x = \pi y^2$ *from* $(0, 0)$ *to* $(\frac{1}{2}\pi, 1)$.

(Meerut 1977)

Sol. We know that $Mdx+Ndy$ is an exact differential if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Here $M = 2xy^3-y^2\cos x$; $\therefore \frac{\partial M}{\partial y}=6xy^2-2y\cos x$.

Also $N = 1-2y\sin x+3x^2y^2$; $\therefore \frac{\partial N}{\partial x}=-2y\cos x+6xy^2$.

Thus $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Therefore $Mdx+Ndy$ is an exact differential.

Let $\phi(x, y)$ be such that

$$d\phi=(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy.$$

Then $\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy=(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy$.

$\therefore \frac{\partial \phi}{\partial x}=(2xy^3-y^2\cos x)$ which gives $\phi=x^2y^3-y^2\sin x+f_1(y)$ ...(1)

Also $\frac{\partial \phi}{\partial y}=1-2y\sin x+3x^2y^2$ which gives $\phi=y-y^2\sin x$

$+x^2y^3+f_2(x)$. ...(2)

The values of ϕ given by (1) and (2) agree if we take $f_1(y)=y$ and $f_2(x)=0$. Then $\phi=y-y^2\sin x+x^2y^3$.

$\therefore$ The given integral

$$=\int_C d\phi=\int_C d\,(y-y^2\sin x+x^2y^3)$$

$$=\left[y-y^2\sin x+x^2y^3\right]_{(0,0)}^{(\pi/2,\,1)}$$

$$=\left[\left\{1-1\times\sin\frac{\pi}{2}+\frac{\pi^2}{4}\times 1\right\}-0\right]=\frac{\pi^2}{4}.$$

Ex. 42. *Find the circulation of* **F** *round the curve C where*

$$\mathbf{F}=y\,\mathbf{i}+z\,\mathbf{j}+x\,\mathbf{k}$$

and C is the circle $x^2+y^2=1,\ z=0$.

Sol. By definition, the circulation of **F** along the curve C is

$$=\oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$$

$$= \oint_C (y\,\mathbf{i}+z\,\mathbf{j}+x\,\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}) = \oint_C (y\,dx+z\,dy+x\,dz)$$

$$= \oint_C y\,dx \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$= \int_0^{2\pi} \sin\theta\,(-\sin\theta)\,d\theta \qquad [\because \text{ on } C,\ x=\cos\theta,\ y=\sin\theta]$$

$$= -\int_0^{2\pi} \sin^2\theta\,d\theta = -\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= -\tfrac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\pi.$$

Ex. 43. *Find the circulation of* **F** *round the curve C, where* $\mathbf{F} = (x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}$ *and C is the circle* $x^2+y^2 = 4$, $z=0$.

Sol. The parametric equations of the circle $x^2+y^2 = 4$, $z=0$ are $x=2\cos t$, $y=2\sin t$, $z=0$.

By definition, the circulation of **F** along the curve C is

$$= \oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$$

$$= \oint_C [(x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$= \oint_C [(x-y)\,dx+(x+y)\,dy]$$

$$= \int_{t=0}^{2\pi} \left[(x-y)\frac{dx}{dt}+(x+y)\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi} [(2\cos t-2\sin t)\cdot(-2\sin t)+(2\cos t+2\sin t)\cdot 2\cos t]\,dt$$

$$= \int_0^{2\pi} 4(\sin^2 t+\cos^2 t)\,dt = 4\int_0^{2\pi} dt = 4\left[\,t\,\right]_0^{2\pi} = 8\pi.$$

Ex. 44. *Show that*

$$\int_C \left[-\frac{y}{x^2+y^2}\,\mathbf{i}+\frac{x}{x^2+y^2}\,\mathbf{j}\right]\cdot d\mathbf{r} = 2\pi,$$

where C is the circle $x^2+y^2 = 1$ *in the xy-plane described in counter-clockwise sense.*

Sol. The parametric equations of the circle are $x=\cos t$, $y=\sin t$, $z=0$ and along the circle t varies from 0 to 2π.

Along the circle C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j} = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i}+\cos t\,\mathbf{j}.$$

Hence the given integral

$$= \int_{t=0}^{2\pi} [(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}) \cdot (-\sin t\,\mathbf{i}+\cos t\,\mathbf{j})]dt$$

$$= \int_0^{2\pi} (\sin^2 t+\cos^2 t)dt$$

$$= \int_0^{2\pi} dt = \Big[t \Big]_0^{2\pi} = 2\pi.$$

Ex. 45. *If $\phi = 2xyz^2$, and C is the curve $x=t^2, y=2t, z=t^3$ from $t=0$ to $t=1$, evaluate $\int_C \phi\, d\mathbf{r}$.* **(Kanpur 1974)**

Sol. We have $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ so that $d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$.

Now $$\int_C \phi\, d\mathbf{r} = \int_C 2xyz^2(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$= 2\mathbf{i}\int_C xyz^2\,dx+2\mathbf{j}\int_C xyz^2\,dy+2\mathbf{k}\int_C xyz^2\,dz$$

$$= 2\mathbf{i}\int_{t=0}^{1} (t^2)\,(2t)\,(t^3)^2\,2t\,dt+2\mathbf{j}\int_{t=0}^{1} (t^2)\,(2t)\,(t^3)^2\,2\,dt$$

$$+2\mathbf{k}\int_{t=0}^{1} (t^2)\,(2t)\,(t^3)^2\,3t^2\,dt$$

$$[\because\ x=t^2, y=2t; z=t^3]$$

$$= 8\mathbf{i}\int_0^1 t^{10}\,dt+8\mathbf{j}\int_0^1 t^9\,dt+12\mathbf{k}\int_0^1 t^{11}\,dt$$

$$= \frac{8}{11}\mathbf{i}+\frac{4}{5}\mathbf{j}+\mathbf{k}.$$

Ex. 46. *If $\mathbf{F} = xy\,\mathbf{i}-z\,\mathbf{j}+x^2\,\mathbf{k}$ and C is the curve $x=t^2, y=2t, z=t^3$ from $t=0$ to $t=1$, evaluate $\int_C \mathbf{F}\times d\mathbf{r}$.* **(Kanpur 1974)**

Sol. We have $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ so that

$$d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}.$$

$$\therefore \quad \mathbf{F}\times d\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$

$$= -\mathbf{i}\,(z\,dz+x^2\,dy)-\mathbf{j}\,(xy\,dz-x^2\,dx)+\mathbf{k}\,(xy\,dy+z\,dx).$$

Changing in terms of t with the help of parametric equations $x=t^2, y=2t, z=t^3$, we have

$$\mathbf{F}\times d\mathbf{r} = -\mathbf{i}\,(t^3\cdot 3t^2\,dt+t^4\cdot 2\,dt)-\mathbf{j}\,(2t^3\cdot 3t^2\,dt-t^4\cdot 2t\,dt) +\mathbf{k}\,(2t^3\cdot 2\,dt+t^3\cdot 2t\,dt)$$

$$= -\mathbf{i}\,(3\,t^5+2\,t^4)\,dt-\mathbf{j}\,(4\,t^5)\,dt+\mathbf{k}\,(4\,t^3+2\,t^4)\,dt.$$

$$\therefore \int_C \mathbf{F}\times d\mathbf{r} = -\mathbf{i}\int_0^1 (3t^5+2t^4)\,dt-\mathbf{j}\int_0^1 4t^5\,dt+\mathbf{k}\int_0^1 (4t^3+2t^4)\,dt$$

$$= -\mathbf{i}\left[\frac{3}{6}+\frac{2}{5}\right]-\mathbf{j}\left(\frac{4}{6}\right)+\mathbf{k}\left[\frac{4}{4}+\frac{2}{5}\right] = -\frac{9}{10}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{7}{5}\mathbf{k}.$$

Ex. 47. *Let* $\mathbf{r}$ *denote the position vector of any point* (x, y, z) *measured from an origin O and let* $r = |\mathbf{r}|$.

Evaluate $\iint_S \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S}$ *where S denotes the sphere of radius a with centre at the origin.*

Sol. The equation to the sphere S is $x^2+y^2+z^2 = a^2$.

A vector normal to the sphere S at the point (x, y, z) is given by

$$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

If $\mathbf{n}$ denotes the unit vector along the outward drawn normal to the sphere S at the point (x, y, z), then

$$\mathbf{n} = \frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = \frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a},$$

since $x^2+y^2+z^2 = a^2$ on the sphere S.

Again $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$

and $\quad r = |\mathbf{r}| = \sqrt{(x^2+y^2+z^2)}$.

Let $\quad \mathbf{F} = \frac{\mathbf{r}}{r^3}$

Then $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$

$$= \iint_S \frac{\mathbf{r}}{r^3}\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a}\right) dS$$

$$= \iint_S \frac{(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})}{a\,(x^2+y^2+z^2)^{3/2}}\, dS$$

$$= \iint_S \frac{x^2+y^2+z^2}{a\,(x^2+y^2+z^2)^{3/2}}\, dS = \iint_S \frac{1}{a\,(x^2+y^2+z^2)^{1/2}}\, dS$$

$$= \iint_S \frac{1}{a^2}\, dS \qquad [\because \text{ on the sphere } S,\ x^2+y^2+z^2 = a^2]$$

$$= \frac{1}{a^2}\iint_S dS = \frac{1}{a^2}\cdot 4\pi a^2 \qquad [\because \text{ surface of the sphere } S = 4\pi a^2]$$

$$= 4\pi.$$

To change cartesian co-ordinates (x,y) to polar co-ordinates (r,θ) we have

$x = r\cos\theta = \phi(r,\theta)$, $y = r\sin\theta = \psi(r,\theta)$

$J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ (Jacobian)

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$$= \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

Ex. 48. (a) *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, *where* $\mathbf{F} = yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and* S *is that part of the surface of the sphere* $x^2+y^2+z^2 = 1$ *which lies in the first octant.* **(Meerut 1984 ; Agra 74 ; Kanpur 79)**

(b) *If* $\mathbf{F} = y\,\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}$ *and* S *is the surface of the sphere*

$$x^2+y^2+z^2 = a^2,\ 0 \le x, y, z \le a,$$

evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, *where* $\mathbf{n}$ *is a unit vector along the outward drawn normal to the surface* S. **(Osmania 1989)**

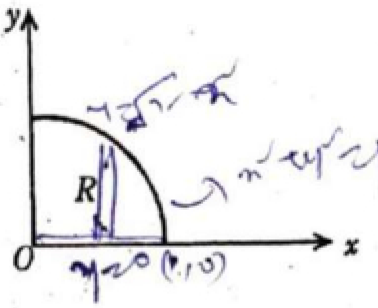
Sol. (a). A vector normal to the surface S is given by

$$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

Therefore $\mathbf{n}$ = a unit normal to any point (x,y,z) of S

$$= \frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k},$$

since $x^2+y^2+z^2=1$ on the surface S.



We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1$, $z=0$.

We have $\mathbf{F}\cdot\mathbf{n} = (yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k})\cdot(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}) = 3xyz.$

Also $\mathbf{n}\cdot\mathbf{k} = (x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot\mathbf{k} = z.$ $\quad\therefore\quad |\mathbf{n}\cdot\mathbf{k}| = z.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{3xyz}{z}\,dx\,dy = 3\iint_R xy\,dx\,dy$

$= 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1}(r\cos\theta)(r\sin\theta)\,r\,d\theta\,dr$, on changing to polars

$$= 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\,d\theta = \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}.$$

(b) A vector normal to the surface S is given by

$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$

$\therefore$ $\mathbf{n}$ = a unit normal to any point (x, y, z) of S

$$= \frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = \frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}),$$

since $x^2+y^2+z^2=a^2$ on the surface S.

Since $0 \le x, y, z \le a$, therefore the surface S is that part of the sphere $x^2+y^2+z^2=a^2$ which lies in the positive octant.

We have $\mathbf{F}\cdot\mathbf{n} = \left[y\,\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}\right]\cdot\frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$

$= \frac{1}{a}\left[xy+y(x-2xz)-xyz\right] = \frac{1}{a}(2xy-3xyz)$

$= \frac{y}{a}(2x-3xz).$

Now $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the zx-plane. The region R is bounded by x-axis, z-axis and the circle $x^2+z^2=a^2$, $y=0$.

We have $\mathbf{n}\cdot\mathbf{j} = \frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot\mathbf{j} = \frac{y}{a}.$ $\quad\therefore\quad |\mathbf{n}\cdot\mathbf{j}| = \frac{y}{a}.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \frac{y}{a}(2x-3xz)\,\frac{dx\,dz}{y/a}$

$$= \iint_R (2x - 3xz)\, dx\, dz$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} [2r\cos\theta - 3(r\cos\theta)(r\sin\theta)]\, r\, d\theta\, dr,$$

on changing to polars

$$= 2\int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a} r^2\, dr\right] \cos\theta\, d\theta - 3\int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a} r^3\, dr\right] \cos\theta \sin\theta\, d\theta$$

$$= \frac{2}{3}a^3 \int_0^{\pi/2} \cos\theta\, d\theta - \frac{3}{4}a^4 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2}{3}a^3 - \frac{3}{4}a^4 \cdot \frac{1}{2} = \frac{2}{3}a^3 - \frac{3}{8}a^4.$$

Ex. 49. *If* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *above the xy-plane.* **(Kanpur 1980; Bundelkhand 79)**

Sol. let $\mathbf{f} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix}$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x - 2xz)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x - 2xz) - \frac{\partial}{\partial y}(y)\right]$$

$$= x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}.$$

A vector normal to the sphere S at the point (x, y, z) is given by

$$\nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

If $\mathbf{n}$ denotes the unit vector along the outward drawn normal to the sphere S at the point (x, y, z), then

$$\mathbf{n} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a},$$

since $x^2 + y^2 + z^2 = a^2$ on the sphere S.

Now $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n} = \dfrac{(x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{a}$

$$= \frac{x^2 + y^2 - 2z^2}{a}.$$

We know that $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \iint_R \mathbf{f} \cdot \mathbf{n} \frac{dx\, dy}{\mathbf{n} \cdot \mathbf{k}}$, where R is the projection of the surface S on the xy-plane. Obviously the region R is the area bounded by the circle $x^2 + y^2 = a^2, z = 0$ in the xy-plane.

We have $\mathbf{n} \cdot \mathbf{k} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{k} = \frac{z}{a}$.

$$\therefore \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{f} \cdot \mathbf{n}\, dS$$

$$= \iint_R (\mathbf{f} \cdot \mathbf{n}) \frac{dx\, dy}{z/a} = \iint_R \frac{x^2 + y^2 - 2z^2}{a} \frac{dx\, dy}{z/a},$$

where $z^2 = a^2 - x^2 - y^2$

$$= \iint_R \frac{x^2 + y^2 - 2(a^2 - x^2 - y^2)}{\sqrt{(a^2 - x^2 - y^2)}} dx\, dy$$

$$= \iint_R \frac{3(x^2 + y^2) - 2a^2}{\sqrt{[a^2 - (x^2 + y^2)]}} dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}} r\, d\theta\, dr, \text{ changing to polar coordinates}$$

$$= 2\pi \int_{r=0}^{a} \frac{r(3r^2 - 2a^2)}{\sqrt{(a^2 - r^2)}} dr, \quad \text{first integrating with respect to } \theta$$

$$= 2\pi \int_0^{\pi/2} \frac{a \sin t\,(3a^2 \sin^2 t - 2a^2)}{a \cos t} a \cos t\, dt,$$

putting $r = a \sin t$ so that $dr = a \cos t\, dt$

$$= 2\pi a^3 \left[3 \int_0^{\pi/2} \sin^3 t\, dt - 2 \int_0^{\pi/2} \sin t\, dt \right]$$

$$= 2\pi a^3 \left[3 \cdot \frac{2}{3 \cdot 1} - 2 \cdot 1 \right] = 2\pi a^3 (2 - 2) = 0.$$

Ex. 50. *If* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *where* S *is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *which lies in the first octant.* **(Kanpur 1974)**

Sol. Proceed as in solved example 49. Here R *i.e.*, the projection of the surface S on the xy-plane is the area in the xy-plane bounded by

x-axis, y-axis and the circle $x^2 + y^2 = a^2, z = 0$. This area is in the form of a quadrant of a circle.

So here $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}} r\, d\theta\, dr$$

$= 0$, proceeding as in solved example 49.

Ex. 51. *Evaluate* $\iint_S \phi \mathbf{n}\, dS$, *where* $\phi = \frac{3}{8} xyz$ *and* S *is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$.

Sol. A vector normal to the surface S *i.e.*, the cylinder $x^2 + y^2 = 16$ is given by $\nabla (x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}, \text{ since } x^2 + y^2 = 16, \text{ on the surface } S.$$

We have $\iint_S \phi \mathbf{n}\, dS = \iint_R \phi \mathbf{n} \dfrac{dx\, dz}{\mathbf{n} \cdot \mathbf{j}}$,

where R is the projection of S on the xz-plane.

We have $\mathbf{n} \cdot \mathbf{j} = \frac{1}{4}(x\,\mathbf{i} + y\,\mathbf{j}) \cdot \mathbf{j} = \frac{1}{4} y$.

$$\therefore \iint_S \phi \mathbf{n}\, dS = \iint_R \left(\frac{3}{8} xyz\right) \left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right) \frac{dx\, dz}{y/4}$$

$$= \frac{3}{8} \int_{z=0}^{5} \int_{x=0}^{4} \left[x^2 z\,\mathbf{i} + xz\sqrt{(16 - x^2)}\,\mathbf{j}\right] dx\, dz,$$

since $y = \sqrt{(16 - x^2)}$ on S

$$= \frac{3}{8} \int_{x=0}^{4} [x^2\,\mathbf{i} + x\sqrt{(16 - x^2)}] \left[\frac{z^2}{2}\right]_0^5 dx,$$

first integrating with respect to z

$$= \frac{3}{8} \cdot \frac{25}{2} \int_0^4 \left[x^2\,\mathbf{i} + \left(-\frac{1}{2}\right)(16 - x^2)^{1/2} (-2x)\,\mathbf{j}\right] dx$$

$$= \frac{75}{16} \left[\frac{x^3}{3}\,\mathbf{i} - \frac{1}{2} \cdot \frac{2}{3} (16 - x^2)^{3/2}\,\mathbf{j}\right]_0^4$$

$$= \frac{75}{16} \left[\frac{64}{3}\,\mathbf{i} - \frac{1}{3}(0 - 64)\,\mathbf{j}\right]$$

$$= \frac{75}{16} \cdot \frac{64}{3} (\mathbf{i} + \mathbf{j}) = 100 (\mathbf{i} + \mathbf{j}).$$

Ex. 52. *Evaluate* $\int_V (2x + y)\, dV$, *where* V *is the closed region bounded by the cylinder* $z = 4 - x^2$ *and the planes* $x = 0, y = 0, y = 2$ *and* $z = 0$. **(Kanpur 1981)**

Sol. The cylinder $z = 4 - x^2$ meets the x-axis (*i.e.*, $y = 0$, $z = 0$) at $x^2 = 4$ or $x = 2$ on the positive side *i.e.*, at the point $(2, 0, 0)$. It meets z-axis (*i.e.*, $x = 0, y = 0$) at $z = 4$ *i.e.*, at the point $(0, 0, 4)$. Therefore the limits of integration for z are from 0 to $4 - x^2$, for x from 0 to 2 and for y from 0 to 2.

Also $dV = dx\, dy\, dz$.

$$\therefore \int_V (2x + y)\, dV = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{4-x^2} (2x + y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y) \Big[z \Big]_{z=0}^{4-x^2} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y)(4 - x^2)\, dx\, dy$$

$$= \int_{x=0}^{2} (4 - x^2) \left[2xy + \frac{y^2}{2} \right]_{y=0}^{2} dx$$

$$= \int_0^2 (4 - x^2)[4x + 2]\, dx = 2 \int_0^2 (4 - x^2)(2x + 1)\, dx$$

$$= 2 \int_0^2 (4 + 8x - x^2 - 2x^3)\, dx$$

$$= 2 \left[4x + 4x^2 - \frac{x^3}{3} - \frac{x^4}{2} \right]_0^2 = 2 \left[8 + 16 - \frac{8}{3} - 8 \right] = \frac{80}{3}.$$

Ex. 53. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\mathbf{k}$ *and* S *is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$. **(Tirupati 1993)**

Sol. A vector normal to the surface S is given by

$$\nabla (x^2 + y^2 sze6) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}, \text{ since } x^2 + y^2 = 16, \text{ on the surface } S.$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now $\mathbf{F}\cdot\mathbf{n} = (z\,\mathbf{i} + x\,\mathbf{j} - 3y^2z\,\mathbf{k})\cdot\left(\dfrac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right) = \frac{1}{4}(xz + xy)$,

$$\mathbf{n}\cdot\mathbf{j} = \left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right)\cdot\mathbf{j} = \frac{y}{4}.$$

Therefore the required surface integral is

$$= \iint_R \frac{xz + xy}{4}\,\frac{dx\,dz}{y/4}$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{(16 - x^2)}} + x\right) dx\,dz, \text{ since } y = \sqrt{(16 - x^2)} \text{ on } S$$

$$= \int_0^5 (4z + 8)\,dz = 90.$$

Ex. 54. *If* $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$, *then evaluate* $\iiint_V \nabla\cdot\mathbf{F}\,dV$ *where* V *is the closed region bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$.

(Osmania 1989, 90; Kanpur 76, 78)

Also Evaluate $\iiint_V \nabla\times\mathbf{F}\,dV$.

Sol. We have $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$.

$$\therefore \nabla\cdot\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x.$$

$$\therefore \iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V 2x\,dx\,dy\,dz \qquad [\because\ dV = dx\,dy\,dz]$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\,dx\,dy\,dz.$$

[Note that we have taken a thin column parallel to z-axis as the elementary volume. It cuts the boundary at $z = 0$ and $z = 4 - 2x - 2y$. Also the projection of the plane $2x + 2y + z = 4$ on the xy-plane is bounded by the axes $y = 0$, $x = 0$ and the line $x + y = 2$. Hence the limits for y are from 0 to $2 - x$ and those for x are from 0 to 2]

$$\therefore \iiint_V \nabla \cdot \mathbf{F}\, dV = 2\int_{x=0}^{2}\int_{y=0}^{2-x} x\left[z\right]_{z=0}^{4-2x-2y} dx\, dy$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x(4-2x-2y)\, dx\, dy$$

$$= 2\int_{x=0}^{2}\left[4xy - 2x^2y - xy^2\right]_{y=0}^{2-x} dx$$

$$= 2\int_0^2 [4x(2-x) - 2x^2(2-x) - x(2-x)^2]\, dx$$

$$= 2\int_0^2 [x^3 - 4x^2 + 4x]\, dx, \text{ on simplifying}$$

$$= 2\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\left[4 - \frac{32}{3} + 8\right] = \frac{8}{3}.$$

Second part. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2 - 3z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2 - 3z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-4+3)\,\mathbf{j} + (-2y)\,\mathbf{k} = \mathbf{j} - 2y\,\mathbf{k}.$$

$$\therefore \iiint_V \nabla \times \mathbf{F}\, dV = \iiint_V (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\mathbf{j} - 2y\,\mathbf{k})(4 - 2x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{2}\left[\mathbf{j}(4y - 2xy - y^2) - 2\,\mathbf{k}(2y^2 - xy^2 - \tfrac{2}{3}y^3)\right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2}\Big[\mathbf{j}(2-x)(4 - 2x - 2 + x)$$

$$-2\,\mathbf{k}\,(2-x)^2\left\{2-x-\tfrac{2}{3}(2-x)\right\}\Big]\,dx$$

$$=\int_0^2\left[(2-x)^2\mathbf{j}-\tfrac{2}{3}(2-x)^3\mathbf{k}\right]dx$$

$$=\int_0^2\left[(x-2)^2\mathbf{j}+\tfrac{2}{3}(x-2)^3\mathbf{k}\right]dx$$

$$=\left[\frac{(x-2)^3}{3}\right]_0^2\mathbf{j}+\left[\frac{2}{3}\frac{(x-2)^4}{4}\right]_0^2\mathbf{k}$$

$$=\frac{8}{3}\mathbf{j}-\frac{8}{3}\mathbf{k}=\frac{8}{3}(\mathbf{j}-\mathbf{k}).$$

Ex. 55. *Evaluate* $\iiint_V \phi\, dV$, *where* $\phi = 45x^2y$ *and* V *is the closed region bounded by the planes* $4x+2y+z=8,\ x=0,\ y=0, z=0.$

Sol. We have

$$\iiint_V \phi\, dv=\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y}45x^2y\,dx\,dy\,dz$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x}x^2y\,\left[z\right]_0^{8-4x-2y}dx\,dy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x}x^2y\,(8-4x-2y)\,dx\,dy$$

$$=45\int_{x=0}^{2}\left[x^2(8-4x)\frac{y^2}{2}-2x^2\frac{y^3}{3}\right]_0^{4-2x}dx$$

$$=45\int_0^2\frac{x^2}{3}(4-2x)^3\,dx=128.$$

Ex. 56. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$

where $\mathbf{F}=(x+y^2)\,\mathbf{i}-2x\,\mathbf{j}+2yz\,\mathbf{k}$ *and* S *is the surface of the plane* $2x+y+2z=6$ *in the first octant.* **(Kanpur 1970)**

Sol. A vector normal to the surface S is given by

$$\nabla(2x+y+2z)=2\,\mathbf{i}+\mathbf{j}+2\,\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x,y,z) of S

$$=\frac{2\,\mathbf{i}+\mathbf{j}+2\,\mathbf{k}}{\sqrt{(4+1+4)}}=\left(\tfrac{2}{3}\mathbf{i}+\tfrac{1}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}\right).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x + y = 6$, $z = 0$.

We have $\mathbf{F}\cdot\mathbf{n} = [(x+y^2)\,\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}]\cdot(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})$

$$= \tfrac{2}{3}(x+y^2) - \tfrac{2}{3}x + \tfrac{4}{3}yz = \tfrac{2}{3}y^2 + \tfrac{4}{3}yz.$$

Also $\quad \mathbf{n}\cdot\mathbf{k} = (\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k})\cdot\mathbf{k} = \frac{2}{3}$.

Hence $\quad \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R [\frac{2}{3}y^2 + \frac{4}{3}yz]\cdot\frac{3}{2}\,dx\,dy$

$$= \iint_R (y^2 + 2yz)\,dx\,dy$$

$$= \iint_R \left[y^2 + 2y\left(\frac{6-2x-y}{2}\right)\right] dx\,dy, \text{ using the fact that}$$

$$z = \frac{6-2x-y}{2} \text{ from the equation of } S$$

$$= \iint_R (y^2 + 6y - 2xy - y^2)\,dx\,dy = 2\iint_R y(3-x)\,dx\,dy$$

$$= 2\int_{y=0}^{6}\int_{x=0}^{(6-y)/2} y(3-x)\,dx\,dy.$$

[Note that R is bounded by x-axis, y-axis and the straight line $2x + y = 6, z = 0$. To evaluate the double integral over R, keep y fixed and integrate with respect to x from $x = 0$ to $x = \dfrac{6-y}{2}$; then integrate with respect to y from $y = 0$ to $y = 6$. In this way R is completely covered].

$$= 2\int_{y=0}^{6} y\left[3x - \frac{x^2}{2}\right]_{x=0}^{(6-y)/2} dy$$

$$= 2\int_0^6 y\left[\frac{3(6-y)}{2} - \frac{(6-y)^2}{8}\right] dy$$

$$= 2\int_0^6 y\left[9 - \frac{3y}{2} - \frac{36}{8} + \frac{12y}{8} - \frac{y^2}{8}\right] dy$$

$$= 2\int_0^6 y\left[\frac{36}{8} - \frac{y^2}{8}\right] dy = \int_0^6 \left[9y - \frac{y^3}{4}\right] dy$$

$$= \left[9\,\frac{y^2}{2} - \frac{y^4}{16}\right]_0^6 = \left[9 \cdot \frac{36}{2} - \frac{36 \times 36}{16}\right] = [162 - 81] = 81.$$

Ex. 57. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}$ *and* S *is the surface of the plane* $2x + y = 6$ *in the first octant cut off by the plane* $z = 4$.

Sol. A vector normal to the surface S is given by

$$\nabla(2x + y) = 2\,\mathbf{i} + \mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\,\mathbf{i} + \mathbf{j}}{\sqrt{(4+1)}} = \frac{1}{\sqrt{5}}(2\,\mathbf{i} + \mathbf{j}).$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n}\, \frac{dx\, dz}{|\mathbf{n} \cdot \mathbf{j}|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to xy-plane.

Now $\mathbf{F} \cdot \mathbf{n} = (y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}) \cdot \left(\frac{2}{\sqrt{5}}\,\mathbf{i} + \frac{1}{\sqrt{5}}\,\mathbf{j}\right) = \frac{2}{\sqrt{5}}y + \frac{2}{\sqrt{5}}x.$

Also $\mathbf{n} \cdot \mathbf{j} = \frac{1}{\sqrt{5}}(2\,\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} = \frac{1}{\sqrt{5}}.$

$\therefore$ The required surface integral is

$$= \iint_R \left(\frac{2}{\sqrt{5}}y + \frac{2}{\sqrt{5}}x\right) \cdot \sqrt{5}\, dx\, dz = \iint_R 2(y + x)\, dx\, dz$$

$$= 2\iint_R [6 - 2x + x]\, dx\, dz, \text{ since } y = 6 - 2x \text{ on } S$$

$$= 2\iint_R (6 - x)\, dx\, dz = 2\int_{z=0}^{4}\int_{x=0}^{3}(6 - x)\, dx\, dz$$

$$= 2\int_{x=0}^{3}(6 - x)\left[z\right]_0^4 dx = 8\left[6x - \frac{x^2}{2}\right]_0^3 = 8\left[18 - \frac{9}{2}\right] = 108.$$

Ex. 58. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ *and* S *is the surface of the plane* $2x + 3y + 6z = 12$ *in the first octant.*

Sol. A vector normal to the surface S *i.e.*, the plane $2x + 3y + 6z = 12$ is given by

$$\nabla(2x + 3y + 6z) = 2\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{(4+9+36)}} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}.$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \frac{dx\, dy}{\mathbf{n} \cdot \mathbf{k}}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x + 3y = 12, z = 0$.

We have $\mathbf{F} \cdot \mathbf{n} = (18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}) \cdot \left(\frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}\right)$

$$= \frac{1}{7}(36z - 36 + 18y) = \frac{18}{7}(2z + y - 2).$$

Also $\mathbf{n} \cdot \mathbf{k} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \cdot \mathbf{k} = \frac{6}{7}$.

Hence $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{18}{7}(2z + y - 2)\frac{dx\, dy}{6/7}$

$$= \iint_R (6z + 3y - 6)\, dx\, dy$$

$$= \iint_R [(12 - 2x - 3y) + 3y - 6]\, dx\, dy,$$

since $6z = 12 - 2x - 3y$ from the equation of S

$$= \iint_R (6 - 2x)\, dx\, dy = 2\iint_R (3 - x)\, dx\, dy$$

$$= 2\int_{x=0}^{6} \int_{y=0}^{(12-2x)/3} (3 - x)\, dx\, dy$$

$$= 2\int_{x=0}^{6} (3 - x)\left[y\right]_{y=0}^{(12-2x)/3} dx$$

$$= 2\int_0^6 (3 - x) \cdot \frac{1}{3}(12 - 2x)\, dx = \frac{4}{3}\int_0^6 (3 - x)(6 - x)\, dx$$

$$= \frac{4}{3}\int_0^6 (18 - 9x + x^2)\, dx = \frac{4}{3}\left[18x - \frac{9}{2}x^2 + \frac{1}{3}x^3\right]_0^6$$

$$= \frac{4}{3}[108 - 162 + 72] = \frac{4}{3} \cdot 18 = 24.$$

Ex. 59. *If* $\mathbf{F} = 2y\,\mathbf{i} - 3\,\mathbf{j} + x^2\,\mathbf{k}$ *and* S *is the surface of the parabolic cylinder* $y^2 = 8x$ *in the first octant bounded by the planes* $y = 4$ *and* $z = 6$, *then evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Sol. A vector normal to the surface S *i.e.*, the cylinder $8x - y^2 = 0$ is given by $\nabla(8x - y^2) = 8\,\mathbf{i} - 2y\,\mathbf{j}$.

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{8\,\mathbf{i} - 2y\,\mathbf{j}}{\sqrt{(64 + 4y^2)}} = \frac{4\,\mathbf{i} - y\,\mathbf{j}}{\sqrt{(16 + y^2)}}.$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \dfrac{dy\,dz}{\mathbf{n} \cdot \mathbf{i}}$,

where R is the projection of S on the yz-plane.

We have $\mathbf{F} \cdot \mathbf{n} = (2y\,\mathbf{i} - 3\,\mathbf{j} + x^2\,\mathbf{k}) \cdot \left[\dfrac{4\,\mathbf{i} - y\,\mathbf{j}}{\sqrt{(16 + y^2)}}\right]$

$$= \frac{8y + 3y}{\sqrt{(16 + y^2)}} = \frac{11y}{\sqrt{(16 + y^2)}}.$$

Also $\mathbf{n} \cdot \mathbf{i} = \left[\dfrac{4\,\mathbf{i} - y\,\mathbf{j}}{\sqrt{(16 + y^2)}}\right] \cdot \mathbf{i} = \dfrac{4}{\sqrt{(16 + y^2)}}.$

Hence $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \dfrac{11y}{\sqrt{(16 + y^2)}} \cdot \dfrac{dy\,dz}{4/\sqrt{(16 + y^2)}}$

$$= \frac{11}{4} \iint_R y\,dy\,dz = \frac{11}{4} \int_{y=0}^{4} \int_{z=0}^{6} y\,dy\,dz$$

$$= \frac{11}{4} \int_{y=0}^{4} y \left[z\right]_{z=0}^{6} dy = \frac{11}{4} \cdot 6 \int_0^4 y\,dy$$

$$= \frac{11}{4} \cdot 6 \cdot \left[\frac{y^2}{2}\right]_0^4 = \frac{11}{4} \cdot 6 \cdot 8 = 132.$$

Ex. 60. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ *over the surface S of the cylinder $x^2 + y^2 = 9$ included in the first octant between $z = 0$ and $z = 4$ where* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - yz\,\mathbf{k}$.

Sol. A vector normal to the surface S *i.e*, the cylinder $x^2 + y^2 = 9$ is given by $\nabla(x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{3}, \quad \text{since } x^2 + y^2 = 9 \text{ on the surface } S.$$

$$\therefore \quad \mathbf{F} \cdot \mathbf{n} = (z\,\mathbf{i} + x\,\mathbf{j} - yz\,\mathbf{k}) \cdot \left[\tfrac{1}{3}(x\,\mathbf{i} + y\,\mathbf{j})\right]$$

$$= \tfrac{1}{3}(xz + xy) = \tfrac{1}{3}x(z + y).$$

Also dS = elementary area on the surface of the cylinder $= 3\, d\theta\, dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \frac{1}{3} x (z + y)\, 3\, d\theta\, dz,$$

where $x = 3\cos\theta$, $y = 3\sin\theta$

$$= \int_{z=0}^{4} \int_{\theta=0}^{\pi/2} 3\cos\theta\, (z + 3\sin\theta)\, d\theta\, dz$$

$$= \int_{\theta=0}^{\pi/2} 3\cos\theta \left[\frac{1}{2} z^2 + z.\, 3\sin\theta\right]_{z=0}^{4} d\theta$$

$$= 3\int_0^{\pi/2} \cos\theta\, [8 + 12\sin\theta]\, d\theta$$

$$= 12\left[2\int_0^{\pi/2} \cos\theta\, d\theta + 3\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta\right]$$

$$= 12\left[2.1 + 3 \cdot \frac{1}{2}\right] = 42.$$

Ex. 61. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ *and S is the closed surface consisting of the cylinder* $x^2 + y^2 = 4$ *and the circular discs* $z = 0$ *and* $z = 3$.

Sol. Here the surface S consists of three surfaces : (i) the surface S_1 of the base *i.e.*, the plane face $z = 0$ of the cylinder (ii) the surface S_2 of the top *i.e.*, the plane face $z = 3$ of the cylinder and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 *i.e.*, $z = 0$, $\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j}$, putting $z = 0$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_1 is obviously $-\mathbf{k}$.

$$\therefore \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (4x\,\mathbf{i} - 2y^2\,\mathbf{j}) \cdot (-\mathbf{k})\, dS$$

$$= \iint_{S_1} 0\, dS = 0.$$

For the surface S_2 *i.e.*, $z = 3$, $\mathbf{F} = 4x\,\mathbf{i} - 2y^2\mathbf{j} + 9\,\mathbf{k}$, putting $z = 3$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_2 is obviously $\mathbf{k}$.

$$\therefore \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + 9\,\mathbf{k})\cdot\mathbf{k}\,dS$$

$$= \iint_{S_2} 9\,dS = 9\iint_{S_2} dS = 9\cdot 2\pi . 2 = 36\pi$$

[$\because$ area of the plane face S_2 of the cylinder $= 2\pi r = 2\pi.2$]

For the convex portion S_3 *i.e.*, $x^2 + y^2 = 4$, a vector normal to S_3 is given by $\nabla(x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

$\therefore$ $\mathbf{n}$ = a unit vector along outward drawn normal at any point of S_3

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{2}, \text{ since } x^2 + y^2 = 4 \text{ on } S_3.$$

$$\therefore \text{ on } S_3,\ \mathbf{F}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k})\cdot\left\{\tfrac{1}{2}(x\,\mathbf{i} + y\,\mathbf{j})\right\}$$

$$= 2x^2 - y^3.$$

Also dS = elementary area on the surface S_3

$= 2\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_3} (2x^2 - y^3)\,2d\theta\,dz,$$

where $x = 2\cos\theta, y = 2\sin\theta$

$$= \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 2\int_{\theta=0}^{2\pi} 8(\cos^2\theta - \sin^3\theta)\Big[z\Big]_{z=0}^{3} d\theta$$

$$= 48\left[\int_0^{2\pi}\cos^2\theta\,d\theta - \int_0^{2\pi}\sin^3\theta\,d\theta\right]$$

$$= 48\left[4\int_0^{\pi/2}\cos^2\theta\,d\theta - 0\right] \quad [\because\ \sin^3(2\pi - \theta) = -\sin^3\theta]$$

$$= 192\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 48\pi.$$

Hence the required surface integral

$$= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS$$

$$= 0 + 36\pi + 48\pi = 84\pi.$$

Ex. 62. *Evaluate* $\iint_S (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k})\cdot\mathbf{n}\,dS$ *where S is the surface of the sphere* $x^2+y^2+z^2=1$ *above the xy-plane.*

Sol. Let $\mathbf{F} = y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k}$.

A vector normal to the surface S *i.e.*, $x^2+y^2+z^2=1$ is given by
$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

Therefore $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

since $x^2+y^2+z^2=1$ on the surface S.

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$, where R is the projection of S on the xy-plane. Obviously the region R is the area of the circle $x^2+y^2=1$, $z=0$ in the xy-plane.

We have $\mathbf{F}\cdot\mathbf{n} = (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k})\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$
$= xy^2z^2 + yz^2x^2 + zx^2y^2$.

Also $\mathbf{n}\cdot\mathbf{k} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot\mathbf{k} = z$.

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{z}$

$$= \iint_R xyz\,(yz+zx+xy)\,\frac{dx\,dy}{z}$$

$$= \iint_R xy\,[xy + z\,(x+y)]\,dx\,dy$$

$$= \iint_R [x^2y^2 + (x^2y+xy^2)\sqrt{\{1-(x^2+y^2)\}}]\,dx\,dy,$$

since $z = \sqrt{(1-x^2-y^2)}$ on S

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} [r^4\cos^2\theta\sin^2\theta + (r^3\cos^2\theta\sin\theta + r^3\cos\theta\sin^2\theta)\cdot\sqrt{(1-r^2)}]\,r\,d\theta\,dr,$$

on changing to polars

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^5\cos^2\theta\sin^2\theta\,d\theta\,dr$$

$$+ \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^4\sqrt{(1-r^2)}\cos^2\theta\sin\theta\,d\theta\,dr$$

$$+ \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^4\sqrt{(1-r^2)}\cos\theta\sin^2\theta\,d\theta\,dr$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^5 \cos^2\theta \sin^2\theta \, d\theta \, dr$$

$$\left[\because \int_{\theta=0}^{2\pi} \cos^2\theta \sin\theta \, d\theta = 0 \text{ and } \int_{\theta=0}^{2\pi} \cos\theta \sin^2\theta \, d\theta = 0\right]$$

$$= 4\int_{\theta=0}^{\pi/2} \cos^2\theta \sin^2\theta \left[\frac{r^6}{6}\right]_{r=0}^{1} d\theta$$

$$= \frac{4}{6}\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta = \frac{2}{3}\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

Ex. 63. *Evaluate* $\iint_S \mathbf{r} \cdot d\mathbf{S}$ *where S is the part of the surface of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lying above the plane $z = 0$, *the normal at any point being directed outwards.*

Sol. A normal vector to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ is given by $\nabla\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = \frac{2x}{a^2}\mathbf{i}+\frac{2y}{b^2}\mathbf{j}+\frac{2z}{c^2}\mathbf{k}.$

Hence the unit vector $\mathbf{n}$ along the outward drawn normal at any point (x, y, z) of the surface S is given by

$$\mathbf{n} = \frac{(2x/a^2)\,\mathbf{i}+(2y/b^2)\,\mathbf{j}+(2z/c^2)\,\mathbf{k}}{\sqrt{[(4x^2/a^4)+(4y^2/b^4)+(4z^2/c^4)]}}$$

$$= \frac{(x/a^2)\,\mathbf{i}+(y/b^2)\,\mathbf{j}+(z/c^2)\,\mathbf{k}}{\sqrt{[\Sigma\,(x^2/a^4)]}}.$$

Now $\iint_S \mathbf{r}\cdot d\mathbf{S} = \iint_S \mathbf{r}\cdot\mathbf{n}\,dS = \iint_R \mathbf{r}\cdot\mathbf{n}\,\frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$,

where R is the projection of S on the xy-plane. Obviously the region R is the area of the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$ lying in the xy-plane.

Now $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$.

$$\therefore \quad \mathbf{r}\cdot\mathbf{n} = \frac{\Sigma\,(x^2/a^2)}{\sqrt{[\Sigma\,(x^2/a^4)]}} = \frac{1}{\sqrt{[\Sigma\,(x^2/a^4)]}}, \quad \text{on the surface } S.$$

Also $\mathbf{n}\cdot\mathbf{k} = \dfrac{z/c^2}{\sqrt{[\Sigma(x^2/a^4)]}} = \dfrac{z}{c^2\sqrt{[\Sigma(x^2/a^4)]}}$.

$$\therefore \iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iint_R \frac{1}{\sqrt{[\Sigma(x^2/a^4)]}} \cdot \frac{dx\, dy}{z/[c^2\sqrt{\{\Sigma(x^2/a^4)\}}]}$$

$$= c^2 \iint_R \frac{dx\, dy}{z} = c^2 \iint_R \frac{dx\, dy}{c\sqrt{[1-(x^2/a^2)-(y^2/b^2)]}},$$

since on S, $z = c\sqrt{[1-(x^2/a^2)-(y^2/b^2)]}$

$$= c\int_{x=-a}^{a}\int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} \frac{b\, dx\, dy}{\sqrt{[b^2\{1-(x^2/a^2)\}-y^2]}}$$

$$= 4bc\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} \frac{dx\, dy}{\sqrt{[b^2\{1-(x^2/a^2)\}-y^2]}}$$

$$= 4bc\int_0^a \left[\sin^{-1}\frac{y}{b\sqrt{(1-x^2/a^2)}}\right]_{y=0}^{b\sqrt{(1-x^2/a^2)}} dx$$

$$= 4bc\int_0^a \frac{\pi}{2}\, dx = 4bc\cdot\frac{\pi}{2}\cdot\Big[x\Big]_0^a = 2\pi abc.$$

Ex. 64. *Evaluate* $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$, *where*

$\mathbf{A} = xy\,\mathbf{i} - x^2\,\mathbf{j} + (x+z)\,\mathbf{k}$, *$S$ is the portion of the plane $2x+2y+z=6$ included in the first octant and* $\mathbf{n}$ *is a unit normal to S.* **(Meerut 1974)**

Sol. A vector normal to the surface S *i.e.*, the plane $2x+2y+z = 6$ is given by

$$\nabla(2x+2y+z) = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}}{\sqrt{(4+4+1)}} = \frac{2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}}{3}.$$

We have $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\dfrac{dx\, dy}{\mathbf{n}\cdot\mathbf{k}}$, where R is the projection of S on the xy-plane. The region R is the area of the triangle in xy-plane bounded by x-axis, y-axis and the straight line $x+y=3$, $z=0$.

We have $\mathbf{A}\cdot\mathbf{n} = [xy\,\mathbf{i} - x^2\,\mathbf{j} + (x+z)\,\mathbf{k}]\cdot[\frac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k})]$

$$= \tfrac{2}{3}xy - \tfrac{2}{3}x^2 + \tfrac{1}{3}(x+z).$$

Also $\mathbf{n}\cdot\mathbf{k} = \frac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k})\cdot\mathbf{k} = \frac{1}{3}$.

Hence $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{3}[2xy - 2x^2 + x + z]\frac{dx\,dy}{1/3}$

$$= \iint_R (2xy - 2x^2 + x + z)\,dx\,dy$$

$$= \iint_R (2xy - 2x^2 + x + 6 - 2x - 2y)\,dx\,dy,$$

since $z = 6 - 2x - 2y$ from the equation of S

$$= \int_{x=0}^{3}\int_{y=0}^{3-x} (2xy - 2x^2 - x - 2y + 6)\,dx\,dy$$

$$= \int_{x=0}^{3}\left[xy^2 - 2x^2y - xy - y^2 + 6y\right]_{y=0}^{3-x} dx$$

$$= \int_{x=0}^{3}[x(3-x)^2 - 2x^2(3-x) - x(3-x) - (3-x)^2 + 6(3-x)]\,dx$$

$$= \int_0^3 (3x^3 - 12x^2 + 6x + 9)\,dx = \left[\frac{3}{4}x^4 - 4x^3 + 3x^2 + 9x\right]_0^3$$

$$= \frac{243}{4} - 108 + 27 + 27 = \frac{243}{4} - 54 = \frac{27}{4}.$$

Ex. 65. *Evaluate* $\iiint_V \mathbf{F}\,dV$ *where* $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ *and* V *is the region bounded by the surfaces* $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = 4$ *and* $z = x^2$. **(Andhra 1992)**

Sol. Here the limits of integration for the region V are $z = x^2$ to $z = 4$, $y = 0$ to $y = 6$ and $x = 0$ to $x = 2$.

We have $\iiint_V \mathbf{F}\,dV = \iiint_V (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\,dx\,dy\,dz$

$$= I_1\,\mathbf{i} + I_2\,\mathbf{j} + I_3\,\mathbf{k}, \text{ say.}$$

Now $I_1 = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[xz\right]_{z=x^2}^{4} dx\,dy$

$$= \int_{x=0}^{2}\int_{y=0}^{6}(4x - x^3)\,dx\,dy = \int_{x=0}^{2}\left[4xy - x^3y\right]_{y=0}^{6} dx$$

$$= \int_0^2 (24x - 6x^3)\,dx = \left[12x^2 - \frac{3}{2}x^4\right]_0^2 = 48 - 24 = 24,$$

$$I_2 = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[yz\right]_{z=x^2}^{4} dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=0}^{6}(4y - yx^2)\,dx\,dy = \int_{x=0}^{2}\left[2y^2 - \tfrac{1}{2}y^2x^2\right]_{y=0}^{6} dx$$

$$= \int_0^2 (72 - 18x^2)\, dx = \left[72x - 6x^3\right]_0^2 = 96$$

and $$I_3 = \int_{x=0}^2 \int_{y=0}^6 \int_{z=x^2}^4 z\, dx\, dy\, dz = \int_{x=0}^2 \int_{y=0}^6 \left[\frac{z^2}{2}\right]_{z=x^2}^4 dx\, dy$$

$$= \int_{x=0}^2 \int_{y=0}^6 \left[8 - \frac{x^4}{2}\right] dx\, dy = \int_{x=0}^2 \left[8y - \frac{x^4}{2} y\right]_{y=0}^6 dx$$

$$= \int_0^2 (48 - 3x^4)\, dx = \left[48x - \frac{3}{5}x^5\right]_0^2 = 96 - \frac{96}{5} = \frac{384}{5}.$$

$$\therefore \iiint_V \mathbf{F}\, dV = I_1\,\mathbf{i} + I_2\,\mathbf{j} + I_3\,\mathbf{k} = 24\,\mathbf{i} + 96\,\mathbf{j} + \frac{384}{5}\mathbf{k}.$$

Ex. 66. *Evaluate* $\int_V \mathbf{F}\, dV$ *for* $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ *where* V *is the region bounded by the surfaces* $x = 0,\ y = 0,\ y = 6,\ z = 4$ *and* $z = x^2$.

Sol. If we put $z = 4$ in $z = x^2$, we get $x^2 = 4$ or $x = \pm 2$.

$\therefore$ the limits of integration for the region V are $z = x^2$ to $z = 4$, $y = 0$ to $y = 6$ and $x = 0$ to $x = 2$.

Now proceed as in Ex. 65.

* Green's Theorem in the plane :-

— we will now see a way of evaluating the line integral of a smooth vector field around a simple closed curve. A vector field $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ is smooth if its component functions $P(x,y)$ and $Q(x,y)$ are smooth.

- we will use Green's theorem (some times called Green's Theorem in the plane) to relate the line integral around a closed curve with double integral over the region inside the curve.

* Statement:

Let R be a region in xy-plane whose boundary is simple closed curve 'C' which is piecewise smooth curve. Let $\vec{f}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$ be a smooth vector field defined on both R and C. Then

$$\oint_C \vec{f}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

i.e $$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

where C is traversed so that R is always on the left side of C.

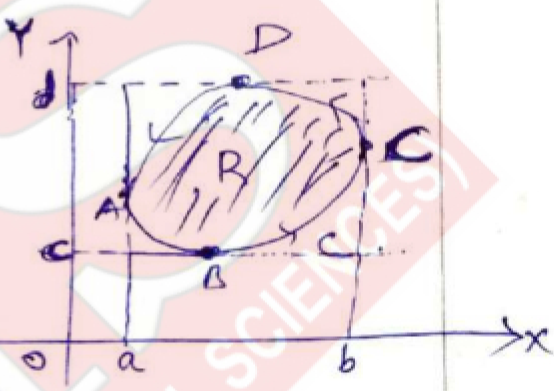
(Or)

If R is a closed region of the xy-plane bounded by a simple closed curve 'C' and if P and Q are continuous functions of x and y having continuous derivatives in R then $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

where 'C' is traversed in the positive (counterclockwise) direction.

Note:— Unless otherwise stated we shall always assume $\oint$ to mean that the integral is described in the positive sense.

proof



If C is closed curve which has the property that any straight line parallel to the co-ordinate axes cuts the curve 'C' in at most two points.

Let the equations of the curves ABC and ADC be $y = Y_1(x)$ and $y = Y_2(x)$ respectively.

If R is the region bounded by C,

we have

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\,dy\right]dx$$

$$= \int_{x=a}^{b} \Big[P(x,y)\Big]_{y=Y_1(x)}^{Y_2(x)}\,dx$$

$$= \int_a^b \left[P(x,Y_2) - P(x,Y_1)\right] dx$$

$$= \int_a^b P(x,Y_2)\,dx - \int_a^b P(x,Y_1)\,dx$$

$$= -\int_a^b P(x, Y_1)\,dx - \int_b^a P(x, Y_2)\,dx$$

$$= -\int_{ABC} P\,dx - \int_{CDA} P\,dx.$$

$$= -\int_{ABCD} P\,dx$$

$$= -\oint_C P\,dx. \quad \text{——— (1)}$$

Similarly let the equations of curves BAD and BCD be $x = x_1(y)$ and $x = x_2(y)$ respectively.

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,dx\right]dy$$

$$= \int_{y=c}^{d}\left[Q(x, y)\right]_{x=x_1(y)}^{x_2(y)}dy$$

$$= \int_{y=c}^{d}\left[Q(x_2(y), y)\right]dy - \int_{y=c}^{d}\left[Q(x_1(y), y)\right]dy$$

$$= \int_{y=c}^{d} Q(x_2(y), y)\,dy + \int_{y=d}^{c} Q(x_1(y), y)\,dy$$

$$= \int_{BCD} Q\,dy + \int_{DAB} Q\,dy$$

$$= \int_{ABCD} Q\,dy = \oint_C Q\,dy \quad \text{——— (2)}.$$

Now $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \oint_C Q\,dy - \left[-\oint_C P\,dx\right]$$

$$= \oint_C (P\,dx + Q\,dy).$$

Green's theorem in the plane in vector notation:

we have $\vec{r} = xi + yj$ so that $d\vec{r} = dx i + dy j$.

Let $\vec{F} = Pi + Qj$

Then $P dx + Q dy = (Pi + Qj)\cdot(dx i + dy j)$

$= \vec{F}\cdot d\vec{r}$.

Also $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= -\frac{\partial Q}{\partial z} i + \frac{\partial P}{\partial z} j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k.$$

$$\therefore (\nabla \times \vec{F})\cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Hence Green's theorem in plane can be written as

$$\iint_R (\nabla \times \vec{F})\cdot k \, dR = \oint_C \vec{F}\cdot d\vec{r}.$$

where $dR = dx\,dy$ and k is unit vector perpendicular to the xy-plane.

If s denotes the arc length of C and t is the unit tangent vector to C, then

$$d\vec{r} = \frac{d\vec{r}}{ds} ds = t\,ds$$

$\therefore$ The above result can also be written as

$$\iint_R (\nabla \times \vec{F})\cdot k \, dR = \oint_C F\cdot t \, ds.$$

1. verify the greens theorem!
$\oint_C (xy+y^2)\,dx + x^2 dy$ where C is the closed curve of the region bounded by $y=x$ and $y=x^2$.

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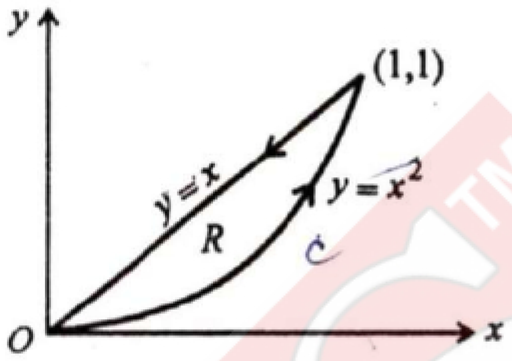
Sol. By Green's theorem in plane, we have



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = xy + y^2$, $N = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. The positive direction in traversing C is as shown in the figure.

We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy = \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\,dy\,dx = \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\,dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Now let us evaluate the line integral along C. Along $y = x^2$, $dy = 2x\,dx$. Therefore along $y = x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2) + x^4\}\,dx + x^2(2x)\,dx]$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}.$$

Along $y = x$, $dy = dx$. Therefore along $y = x$, the line integral equals

$$\int_1^0 [\{(x)(x) + x^2\}\, dx + x^2\, dx] = \int_1^0 3x^4\, dx = -1.$$

Therefore the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Ex. 2. *Verify Green's theorem in a plane for*

$$\oint_C [(x^2 - 2xy)\, dx + (x^2 y + 3)\, dy]$$

where C is the boundary of the region defined by $y^2 = 8x$ *and* $x = 2$.

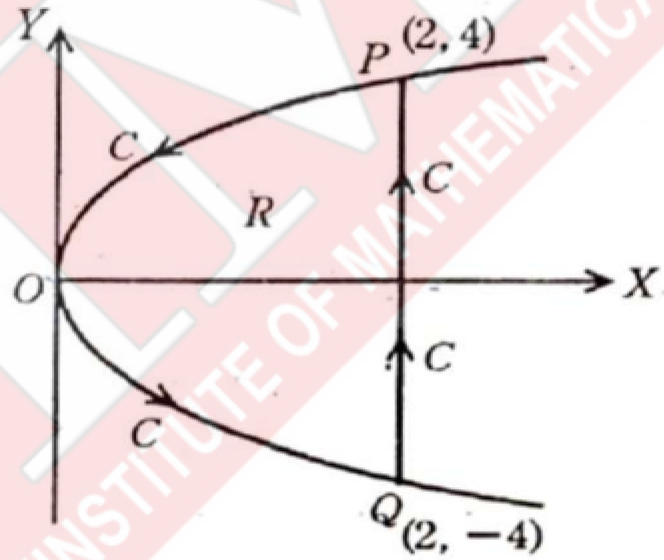
(Osmania 1991)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = x^2 - 2xy,\ N = x^2 y + 3$.

The parabola $y^2 = 8x$ and the straight line $x = 2$ intersect at the points $P(2, 4)$ and $Q(2, -4)$. The positive direction in traversing C is as shown in the figure and R is the region bounded by the curve C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2 y + 3) - \frac{\partial}{\partial y}(x^2 - 2xy)\right] dx\, dy$$

$$= \iint_R (2xy + 2x)\, dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=-\sqrt{(8x)}}^{\sqrt{(8x)}} (2xy + 2x)\, dx\, dy$$

[$\because$ for the region R, x varies from 0 to 2 and y varies from $-\sqrt{(8x)}$ to $\sqrt{(8x)}$]

$$= \int_0^2 2x\left[\frac{y^2}{2}+y\right]_{y=-\sqrt{(8x)}}^{\sqrt{(8x)}} dx, \text{ integrating with respect to } y \text{ regarding } x \text{ as constant}$$

$$= \int_0^2 2x.\left[0+2.\sqrt{(8x)}\right] dx$$

$$= 4\sqrt{8}\int_0^2 x^{3/2}\,dx = 8\sqrt{2}\cdot\frac{2}{5}\left[x^{5/2}\right]_0^2 = \frac{16}{5}\sqrt{2}\cdot 2^{5/2} = \frac{128}{5}. \quad \ldots (1)$$

Now let us evaluate the line integral along C. Along $y^2 = 8x$, we have $x = y^2/8$, $dx = \frac{1}{4}y\,dy$ and y varies from 4 to -4. Therefore along $y^2 = 8x$, the line integral equals

$$\int_{y=4}^{-4}\left[\left(\frac{y^4}{64}-2\cdot\frac{y^2}{8}\cdot y\right)\cdot\frac{1}{4}y\,dy+\left(\frac{y^4}{64}\cdot y+3\right)dy\right]$$

$$= \int_4^{-4}\left[\frac{y^5}{256}+\frac{y^5}{64}-\frac{1}{16}y^4+3\right]dy$$

$$= -\int_{-4}^{4}\left[\frac{y^5}{256}+\frac{y^5}{64}-\frac{1}{16}y^4+3\right]dy$$

$$= -2\int_0^4\left[-\frac{1}{16}y^4+3\right]dy$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = 0 \text{ or } 2\int_0^a f(x)\,dx \text{ according as } f(x) = -f(x) \text{ or } f(-x) = f(x)\right]$$

$$= -2\left[-\frac{1}{16.5}y^5+3y\right]_0^4 = -2\left[-\frac{1}{80}\cdot 4^5+3.4\right]$$

$$= \frac{128}{5} - 24.$$

Along the st. line $x = 2$, we have $dx = 0$ and y varies from -4 to 4. Therefore along $x = 2$, the line integral equals

$$\int_{y=-4}^{4}\left[0+(2^2 y+3)\,dy\right]=\int_{-4}^{4}(4y+3)\,dy$$

$$=3\int_{-4}^{4}dy \qquad \left[\because \int_{-4}^{4} 4y\,dy=0\right]$$

$$=6\int_{0}^{4}dy=6\left[y\right]_0^4=6.4=24.$$

Therefore the total line integral along the curve C *i.e.*,

$$\oint_C (M\,dx+N\,dy)=\frac{128}{5}-24+24=\frac{128}{5}. \qquad \dots (2)$$

From (1) and (2), we see that

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\oint_C (M\,dx+N\,dy),$$

which verifies Green's theorem in plane.

Ex. 3. *Verify Green's theorem in a plane for*

$$\oint_C [(3x^2-8y^2)\,dx+(4y-6xy)\,dy]$$

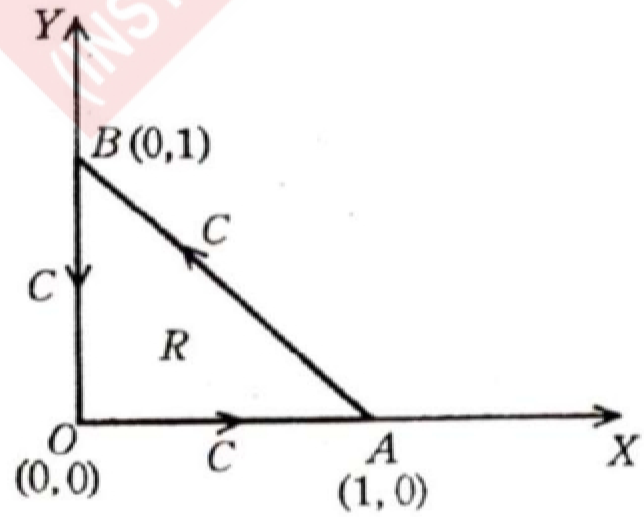
where C is the boundary of the region defined by $x=0$, $y=0$ and $x+y=1$. **(Osmania 1992)**

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy=\oint_C (M\,dx+N\,dy).$$

Here $M=3x^2-8y^2$, $N=4y-6xy$.

The closed curve C consists of the st. line OA, the st. line AB and the straight line BO. The positive direction in traversing C is as shown in the figure and R is the region bounded by C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy$

$= \iint_R [-6y + 16y]\, dx\, dy = 10 \iint_R y\, dx\, dy$

$= 10 \int_{x=0}^{1} \int_{y=0}^{1-x} y\, dx\, dy$ [$\because$ for the region R, x varies from 0 to 1 and y varies from 0 to $1 - x$]

$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx$, iutegrating with respect to y regarding x as constant

$= 5 \int_0^1 (1-x)^2 dx = 5 \int_0^1 (x-1)^2 dx = \frac{5}{3}\left[(x-1)^3\right]_0^1$

$= \frac{5}{3}[0 - (-1)^3] = \frac{5}{3}$. ... (1)

Now let us evaluate the line integral along the curve C.

Along the st. line OA, we have $y = 0$, $dy = 0$ and x varies from 0 to 1.

$\therefore$ line integral along $OA = \int_0^1 3x^2 dx = \left[x^3\right]_0^1 = 1$.

Along the st. line AB, we have $x = 1 - y$, $dx = -dy$ and y varies from 0 to 1.

$\therefore$ line integral along AB

$= \int_0^1 [\{3(1-y)^2 - 8y^2\}(-dy) + \{4y - 6y(1-y)\}\, dy]$

$= \int_0^1 [-3(1 - 2y + y^2) + 8y^2 + 4y - 6y + 6y^2]\, dy$

$= \int_0^1 (11y^2 + 4y - 3)\, dy = \left[\frac{11}{3}y^3 + 2y^2 - 3y\right]_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$.

Along the st. line BO, we have $x = 0$, $dx = 0$ and y varies from 1 to 0.

$\therefore$ line integral along $BO = \int_1^0 4y\, dy = 2\left[y^2\right]_1^0 = -2$.

$\therefore$ total line integral along the closed curve C

$$= 1 + \frac{8}{3} - 2 = \frac{5}{3}. \qquad \ldots (2)$$

From (1) and (2), we see that Green's theorem is verified.

Ex. 4. *Verify Green's theorem in the plane for*

$$\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy],$$

where C is the boundary of the region defined by $y = \sqrt{x},\ y = x^2$.

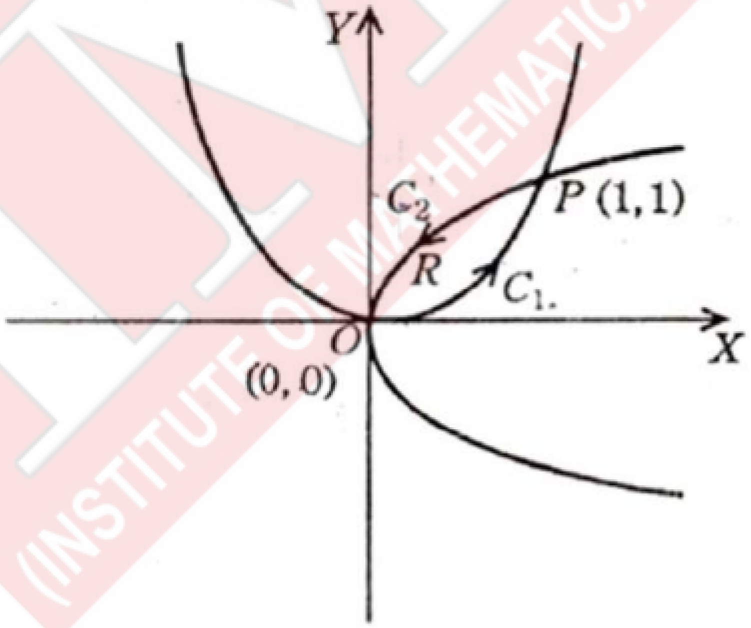
(Kakatiya 1990)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = 3x^2 - 8y^2,\ N = 4y - 6xy$.

The parabola $y = \sqrt{x}$ *i.e.*, $y^2 = x$ and the parabola $y = x^2$ intersect at the points (0, 0) and (1,1). The closed curve C consists of the arc C_1 of the parabola $y = x^2$ and the arc C_2 of the parabola $y = \sqrt{x}$. Also R is the region bounded by the closed curve C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\, dy$$

$$= \iint_R (-6y + 16y)\, dx\, dy = \iint_R 10y\, dx\, dy$$

Along C_2, $y^2 = x$, $dx = 2y\,dy$ and y varies from 1 to 0.

$\therefore$ the line integral along $C_2 = \int_{y=1}^{0} [(2y^3 - y^4)\,2y\,dy + (y^4 + y^2)\,dy]$

$$= \int_1^0 (5y^4 - 2y^5 + y^2)\,dy = \left[y^5 - \frac{y^6}{3} + \frac{y^3}{3}\right]_1^0$$

$$= -1 + \frac{1}{3} - \frac{1}{3} = -1.$$

$\therefore$ total line integral along the closed curve C

$= 1 - 1 = 0.$ ... (2)

From (1) and (2), we see that the two integrals are equal and hence Green's theorem is verified.

Ex. 7. *Evaluate by Green's theorem*

$$\oint_C (x^2 - \cosh y)\,dx + (y + \sin x)\,dy,$$

where C is the rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0,1)$.

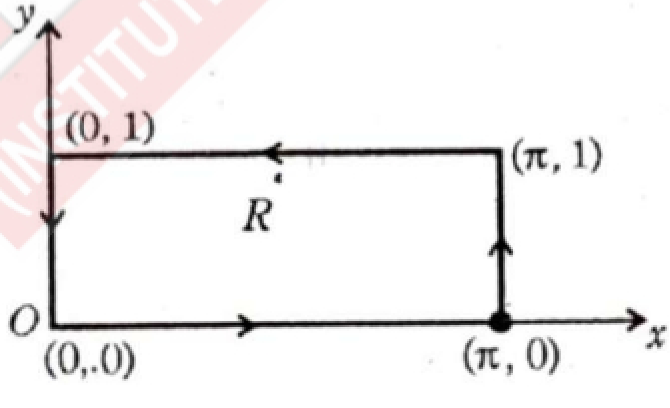
Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = x^2 - \cosh y$, $N = y + \sin x$.

$$\therefore \quad \frac{\partial N}{\partial x} = \cos x, \quad \frac{\partial M}{\partial y} = -\sinh y.$$

Hence the given line integral is equal to



$$\iint_R (\cos x + \sinh y)\,dx\,dy = \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dy\,dx$$

$$= \int_{x=0}^{\pi} \left[y \cos x + \cosh y\right]_{y=0}^{1} dx = \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\,dx$$

$$= \left[\sin x + x \cosh 1 - x \right]_0^{\pi} = (\cosh 1 - 1).$$

Ex. 8. *Evaluate by Green's theorem in plane* $\int_C (e^{-x} \sin y \, dx + e^{-x} \cos y \, dy)$, *where C is the rectangle with vertices* $(0, 0), (\pi, 0), \left(\pi, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Sol. Draw figure as in solved example 7. By Green's theorem in plane

$$\oint_C (M \, dx + N \, dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy.$$

Here $M = e^{-x} \sin y$, $N = e^{-x} \cos y$.

$$\therefore \frac{\partial N}{\partial x} = - e^{-x} \cos y, \quad \frac{\partial M}{\partial y} = e^{-x} \cos y.$$

Hence the given line integral

$$= \iint_R (- e^{-x} \cos y - e^{-x} \cos y) \, dx \, dy,$$

where R is the region enclosed by the rectangle C

$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} - 2e^{-x} \cos y \, dx \, dy$$

$$= \int_{x=0}^{\pi} - 2e^{-x} \left[\sin y \right]_{y=0}^{\pi/2} dx = \int_0^{\pi} - 2e^{-x} \, dx$$

$$= 2 \left[e^{-x} \right]_0^{\pi} = 2 (e^{-\pi} - 1).$$

Ex. 9. *If* $\mathbf{F} = (x^2 - y^2) \mathbf{i} + 2xy \mathbf{j}$ *and* $\mathbf{r} = x \mathbf{i} + y \mathbf{j}$, *find the value of* $\int \mathbf{F} \cdot d\mathbf{r}$ *around the rectangular boundary* $x = 0$, $x = a$, $y = 0$, $y = b$.

(Gauhati 1973)

Sol. Here the four vertices of the rectangle taken in order are $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$. Draw figure as in solved example 7.

Let C be the closed curve traversed in positive direction by the boundary of the rectangle and R be the region bounded by this curve C.

We have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [(x^2 - y^2) \mathbf{i} + 2xy \mathbf{j}] \cdot (dx \, \mathbf{i} + dy \, \mathbf{j})$

$$= \int_C [(x^2 - y^2) \, dx + 2xy \, dx] = \int_C M \, dx + N \, dy,$$

where $M = x^2 - y^2$, $N = 2xy$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy, \quad \text{by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right] dx\, dy$$

$$= \iint_R (2y + 2y)\, dx\, dy = 4 \int_{x=0}^{a} \int_{y=0}^{b} y\, dx\, dy$$

$$= 4 \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{b} dx = 2b^2 \int_0^a dx = 2ab^2.$$

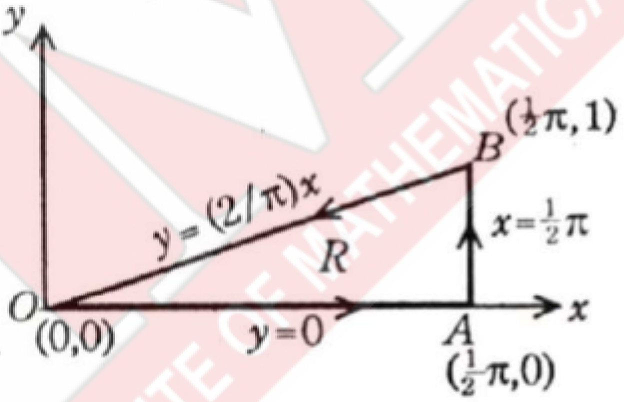
Ex. 10. *Apply Green's theorem in the plane to evaluate*

$$\int_C \{(y - \sin x)\, dx + \cos x\, dy\},$$

where C is the triangle enclosed by the lines $y = 0$, $x = 2\pi$, $\pi y = 2x$.

(Agra 1973)

Sol. Here C is the closed curve traversed in positive direction by ΔOAB and R is the region bounded by this curve C.



We have $\int_C \{(y - \sin x)\, dx + \cos x\, dy\}$

$$= \int_C (M\, dx + N\, dy), \text{ where } M = y - \sin x,\ N = \cos x$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy, \text{ by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y - \sin x)\right] dx\, dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{(2/\pi)x} (-\sin x - 1)\, dx\, dy$$

[$\because$ for the region R, y varies from 0 to $(2/\pi)\,x$ and x varies from 0 to $\pi/2$]

$$= \int_{x=0}^{\pi/2} \left[-y \sin x - y \right]_{y=0}^{(2/\pi)x} dx$$

$$= \int_{0}^{\pi/2} \left[-\frac{2}{\pi} x \sin x - \frac{2}{\pi} x \right] dx$$

$$= -\frac{2}{\pi} \int_{0}^{\pi/2} (x + x \sin x)\, dx$$

$$= -\frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi/2} - \frac{2}{\pi} \left[x(-\cos x) \right]_0^{\pi/2} - \frac{2}{\pi} \int_0^{\pi/2} \cos x\, dx$$

$$= -\frac{2}{\pi} \cdot \frac{\pi^2}{8} - \frac{2}{\pi} \cdot 1 = -\frac{\pi}{4} - \frac{2}{\pi}.$$

Ex. 11. *Evaluate by Green's theorem*

$\oint_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy$, *where C is the circle* $x^2 + y^2 = 1$.

Sol. By Green 's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_C (M dx + N dy).$$

Here $M = \cos x \sin y - xy,\ N = \sin x \cos y.$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x \cos y - x,\ \frac{\partial N}{\partial x} = \cos x \cos y.$$

Hence the given line integral is equal to

$\iint_R x\, dx\, dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \cos\theta\, r\, d\theta\, dr$, changing to polars

$$= \int_{\theta=0}^{2\pi} \left[\frac{r^3}{3} \right]_0^1 \cos\theta\, d\theta = \frac{1}{3} \left[\sin\theta \right]_0^{2\pi} = \tfrac{1}{3}(0) = 0.$$

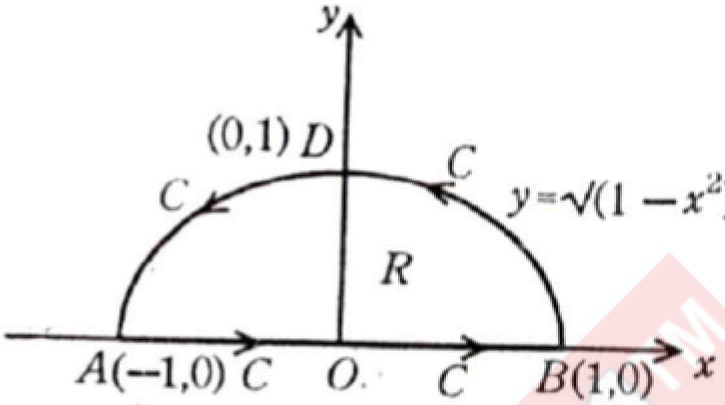
Ex. 12. *Apply Green's theorem in the plane to evaluate*

$$\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy],$$

where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1 - x^2)^{1/2}$.

Sol. Here C is the closed curve traversed in the positive direction by the straight line AOB and the semi-circle BDA. Also R is the region bounded by this curve C.

We have $\int_C [(2x^2 - y^2)\, dx + (x^2 + y^2)\, dy]$



$$= \int_C M\,dx + N\,dy, \text{ where } M = 2x^2 - y^2,\ N = x^2 + y^2$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy, \text{ by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy$$

$$= \iint_R (2x+2y)\,dx\,dy$$

$$= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{(1-x^2)}} 2(x+y)\,dx\,dy, \text{ since for the region } R, y \text{ varies from } 0 \text{ to } \sqrt{(1-x^2)} \text{ and } x \text{ varies from } -1 \text{ to } 1$$

$$= 2\int_{x=-1}^{1} \left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{(1-x^2)}} dx$$

$$= 2\int_{-1}^{1} \left[x\sqrt{(1-x^2)} + \frac{1}{2}(1-x^2)\right] dx$$

$$= 2\int_0^1 (1-x^2)\,dx \quad \left[\because \int_{-1}^{1} x\sqrt{(1-x^2)}\,dx = 0 \text{ and } \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(-x) = f(x)\right]$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}.$$

Ex. 13. *If C is the simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0, \text{ where } \mathbf{F} = \frac{-\mathbf{i}\,y+\mathbf{j}\,x}{x^2+y^2}.$$

Sol. Let R be the region enclosed by the closed curve C.

We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C \left(\frac{-\mathbf{i}\,y+\mathbf{j}\,x}{x^2+y^2}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C \left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right]$$

$$= \int_C (M\,dx+N\,dy), \text{ where } M = -\frac{y}{x^2+y^2},\ N = \frac{x}{x^2+y^2}$$

Since the closed curve C does not enclose origin, therefore both the functions M and N are defined at the origin. So by Green's theorem, we have

$$\int_C (M\,dx+N\,dy) = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \iint_R \left[\frac{1(x^2+y^2)-x\,(2x)}{(x^2+y^2)^2}+\frac{1(x^2+y^2)-y\,(2y)}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint_R \frac{2\,(x^2+y^2)-2\,(x^2+y^2)}{(x^2+y^2)^2}\,dx\,dy = \iint_R 0\,dx\,dy = 0.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

Ex. 14. *Show that the area bounded by a simple closed curve C is given by* $\frac{1}{2}\oint_C (x\,dy-y\,dx)$. *Hence find the area of the ellipse*

$x=a\cos\theta,\ y=b\sin\theta.$ **(Tirupati 1993 ; Agra 74)**

Sol. By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C M\,dx+N\,dy.$$

Putting $M=-y,\ N=x$, we get

$$\oint_C (x\,dy-y\,dx) = \iint_R \left[\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right]dx\,dy$$

$$= 2\iint_R dx\,dy = 2A, \text{ where } A \text{ is the area bounded by } C.$$

Hence $$A = \tfrac{1}{2}\oint_C (x\,dy - y\,dx).$$

The area of the ellipse $= \tfrac{1}{2}\oint_C (x\,dy - y\,dx)$

$$= \tfrac{1}{2}\int_{\theta=0}^{2\pi}\left(a\cos\theta\frac{dy}{d\theta} - b\sin\theta\frac{dx}{d\theta}\right)d\theta$$

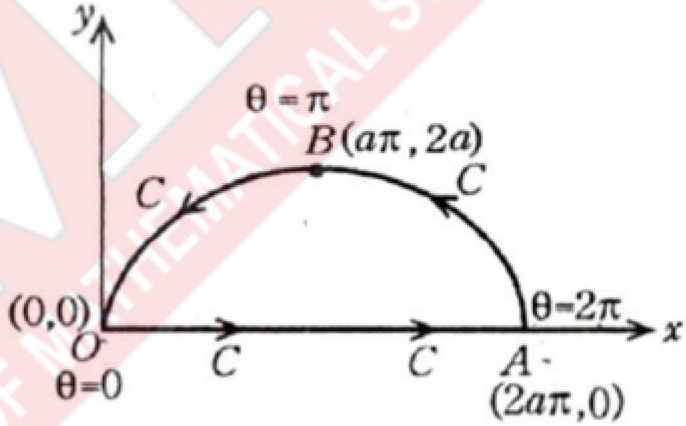
$$= \tfrac{1}{2}\int_0^{2\pi}(ab\cos^2\theta + ab\sin^2\theta)\,d\theta = \tfrac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab.$$

Ex. 15. *Find the area bounded by one arch of the cycloid*

$$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta),\ a>0$$

and the x-axis.

Sol. Let C be the closed curve traversed in positive direction by the straight line OA and the arch ABO of the given cycloid.



At the point O, $\theta = 0$ and at A, $\theta = 2\pi$.

The area bounded by one arch of the given cycloid and the x-axis

= the area enclosed by the simple closed curve C

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx), \text{ by Green's theorem}$$

$$= \frac{1}{2}\int_{OA}(x\,dy - y\,dx) + \frac{1}{2}\int_{\text{arch } ABO}(x\,dy - y\,dx).$$

Now along the straight line OA, we have $y=0$, $dy=0$ and x varies from 0 to $2a\pi$.

$$\therefore\quad \frac{1}{2}\int_{OA}(x\,dy - y\,dx) = 0.$$

Hence the required area $= \dfrac{1}{2}\displaystyle\int_{\text{arch } ABO}(x\,dy - y\,dx)$

$$= \frac{1}{2} \int_{\theta=2\pi}^{0} \left(x \frac{dy}{d\theta} - y \frac{dx}{d\theta} \right) d\theta$$

$$= \frac{1}{2} \int_{2\pi}^{0} [a\,(\theta - \sin\theta) \,.\, a \sin\theta - a\,(1-\cos\theta) \,.\, a\,(1-\cos\theta)]\,d\theta$$

$$= \frac{a^2}{2} \int_{2\pi}^{0} [\theta \sin\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\,d\theta$$

$$= \frac{a^2}{2} \int_{2\pi}^{0} (\theta \sin\theta - 2 + 2\cos\theta)\,d\theta$$

$$= \frac{a^2}{2} \int_{0}^{2\pi} (2 - 2\cos\theta - \theta\sin\theta)\,d\theta$$

$$= \frac{a^2}{2} \Big[2\theta - 2\sin\theta + \theta\cos\theta - \sin\theta \Big]_0^{2\pi}$$

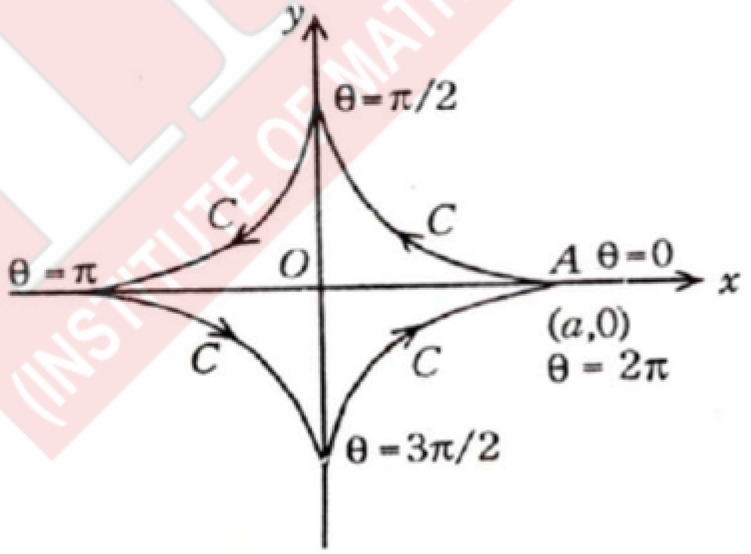
$$= \frac{a^2}{2} [4\pi + 2\pi] = 3\pi a^2.$$

Ex. 16. *Use Green's theorem to find the area bounded by the hypocycloid* $x^{2/3} + y^{2/3} = a^{2/3},\ a > 0.$

Sol. The parametric equations of the given curve

$x^{2/3} + y^{2/3} = a^{2/3}$ can be taken as

$x = a\cos^3\theta,\ y = a\sin^3\theta.$



Here C is the simple closed curve traversed in positive direction by the whole arc of the given hypocycloid.

At the point A, $\theta = 0$ and when after one complete round in anti-clockwise sense along the curve C we come back to A, then at A, $\theta = 2\pi$.

The area bounded by the given hypocycloid

= the area enclosed by the simple closed curve C

$$= \frac{1}{2} \oint_C (x\,dy - y\,dx), \text{ by Green's theorem}$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} \left(x \frac{dy}{d\theta} - y \frac{dx}{d\theta} \right) d\theta, \text{ where } x = a\cos^3\theta,\, y = a\sin^3\theta$$

$$= \frac{1}{2} \int_0^{2\pi} [a\cos^3\theta \,.\, 3a\sin^2\theta\cos\theta - a\sin^3\theta \,.\, (-3a\cos^2\theta\sin\theta)]\,d\theta\,.$$

$$= \frac{3a^2}{2} \int_0^{2\pi} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

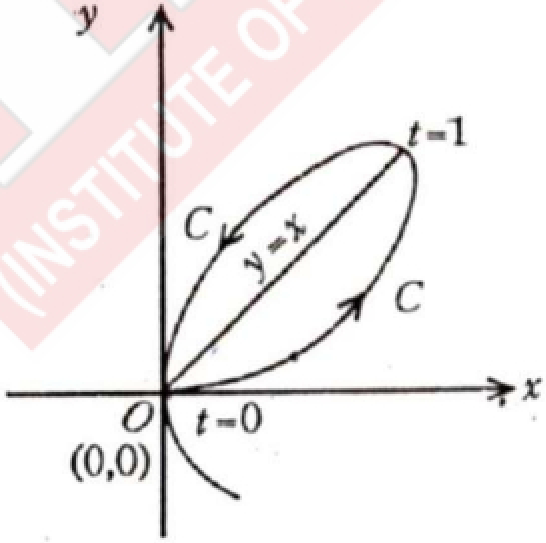
$$= 2 \cdot \frac{3a^2}{2} \int_0^{\pi} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

$$= 4 \cdot \frac{3a^2}{2} \int_0^{\pi/2} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

$$= 6a^2 \left[\frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} + \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} \right] = 6a^2 \cdot \frac{\pi}{16} = \frac{3\pi a^2}{8}.$$

Ex. 17. *Find the area of the loop of the folium* $x^3 + y^3 = 3axy$, $a > 0$.

Sol. Let C be the simple closed curve formed by the loop of the given curve.



By Green's theorem the area of the loop

= the area enclosed by the simple closed curve C

$$= \frac{1}{2} \oint_C (x\,dy - y\,dx), \text{ by Green's theorem}$$

$$= \frac{1}{2} \int_C x^2 \left[\frac{x\,dy - y\,dx}{x^2} \right] = \frac{1}{2} \int_C x^2 \, d\left(\frac{y}{x}\right)$$

$$= \frac{1}{2} \int_C x^2 \, dt, \qquad \text{...(1)}$$

putting $y = tx$.

Putting $y = tx$ in the given equation of the folium, we have

$$x^3 + t^3 x^3 = 3a\,x\,t\,x$$

or $$x^3 (1+t^3) = 3a\,t x^2$$

or $$x = \frac{3at}{1+t^3}.$$

Also for half the loop t varies from 0 to 1.

$\therefore$ from (1), the required area of the loop

$$= \frac{1}{2} \int_C \left(\frac{3at}{1+t^3} \right)^2 dt$$

$$= 2 \cdot \frac{1}{2} \int_{t=0}^{1} \frac{9a^2 t^2}{(1+t^3)^2} \, dt,$$ since the loop is symmetrical about the line $y = x$

$$= 3a^2 \int_0^1 (1+t^3)^{-2} (3t^2) \, dt$$

$$= 3a^2 \left[\frac{(1+t^3)^{-1}}{-1} \right]_0^1, \text{ by power formula}$$

$$= 3a^2 \left[-\frac{1}{1+t^3} \right]_0^1 = 3a^2 \left[-\frac{1}{2} + 1 \right] = \frac{3a^2}{2}.$$

Ex. 18. *Introducing* $\mathbf{A} = N\,\mathbf{i} - M\,\mathbf{j}$, *show that the formula in Green's theorem may be written as*

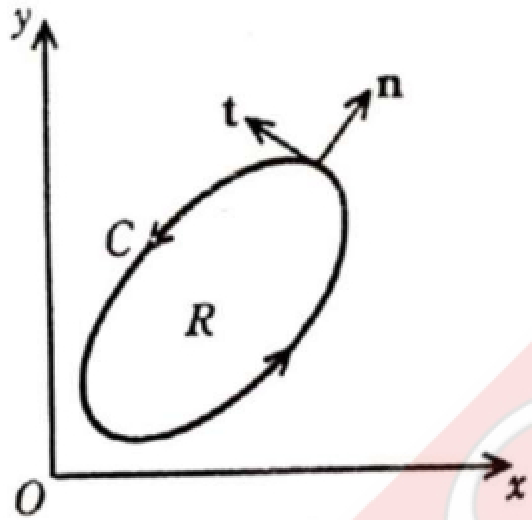
$$\iint_R \text{div}\,\mathbf{A}\,dx\,dy = \oint_C \mathbf{A} \cdot \mathbf{n}\,ds,$$

where $\mathbf{n}$ *is the outward unit normal vector to* C *and* s *is the arc length of* C.

Sol. We have $\mathbf{A} = N\,\mathbf{i} - M\,\mathbf{j}$.

$$\therefore \text{div}\,\mathbf{A} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

$$\therefore \iint_R \text{div}\,\mathbf{A}\,dx\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$



$$= \oint_C (M\,dx + N\,dy), \text{ by Green's theorem.}$$

Now $M\,dx + N\,dy = (M\,\mathbf{i} + N\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) = (M\,\mathbf{i} + N\,\mathbf{j}) \cdot d\mathbf{r}$

$$= \left\{ (M\,\mathbf{i} + N\,\mathbf{j}) \cdot \frac{d\mathbf{r}}{ds} \right\} ds.$$

Now if $\mathbf{t}$ is a unit tangent vector to C, then $\mathbf{t} = \frac{d\mathbf{r}}{ds}$ · Also if $\mathbf{k}$ is a unit vector perpendicular to xy-plane, then $\mathbf{t} = \mathbf{k} \times \mathbf{n}$.

$$\therefore\ M\,dx + N\,dy = [(M\,\mathbf{i} + N\,\mathbf{j}) \cdot \mathbf{t}]\,ds = [(M\,\mathbf{i} + N\,\mathbf{j}) \cdot (\mathbf{k} \times \mathbf{n})]\,ds$$

$$= [(M\,\mathbf{i} + N\,\mathbf{j}) \times \mathbf{k}] \cdot \mathbf{n}\,ds = (M\,\mathbf{i} \times \mathbf{k} + N\,\mathbf{j} \times \mathbf{k}) \cdot \mathbf{n}\,ds$$

$$= (N\,\mathbf{i} - M\,\mathbf{j}) \cdot \mathbf{n}\,ds = \mathbf{A} \cdot \mathbf{n}\,ds.$$

Hence the result.

Note. Putting $\mathbf{A} = \nabla\phi$ in the above result, we get

$$\iint_R \text{div}\,(\nabla\phi)\,dx\,dy = \oint_C (\nabla\phi) \cdot \mathbf{n}\,ds$$

or
$$\iint \nabla^2\phi\,dx\,dy = \oint \frac{\partial\phi}{\partial n}\,ds, \text{ since } \nabla\phi = \frac{\partial\phi}{\partial n}\,\mathbf{n}.$$

* Gauss's Divergence Theorem:

We shall now show that triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely.

This is of practical interest, since in many cases, one of the two kinds of integral is simpler than the other.

The transformation is done by the so-called divergence theorem, which involves the divergence of a vector function F,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Theorem

Divergence theorem of Gauss :-

(transformation b/w volume integrals and surface integrals)

Statement:

Suppose V is the volume bounded by a closed piece-wise smooth surface S. Suppose $\vec{F}(x,y,z)$ is a vector function of position which is continuous and has continuous first partial derivatives in V. Then

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the outwards drawn unit normal vector to S.

Note: Since $\vec{F}\cdot\hat{n}$ is the normal component of vector $\vec{F}$, therefore divergence theorem may also be stated as follows:

The surface integral of the normal component of a vector $\vec{F}$ taken over a closed surface is equal to the integral of the divergence of $\vec{F}$ taken over the volume enclosed by the surface.

* **Cartesian form:**

Let $\vec{F} = F_1 i + F_2 j + F_3 k$.

then $\nabla \cdot \vec{F} = \operatorname{div} \vec{F}$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let α, β, γ be the angles which outward drawn unit normal $\hat{n}$ makes with positive directions of x, y, z-axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\hat{n}$ and

we have $\hat{n} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.

$\therefore \vec{F} \cdot \hat{n} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$.

Hence the divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, ds \quad \text{--- (1)}$$

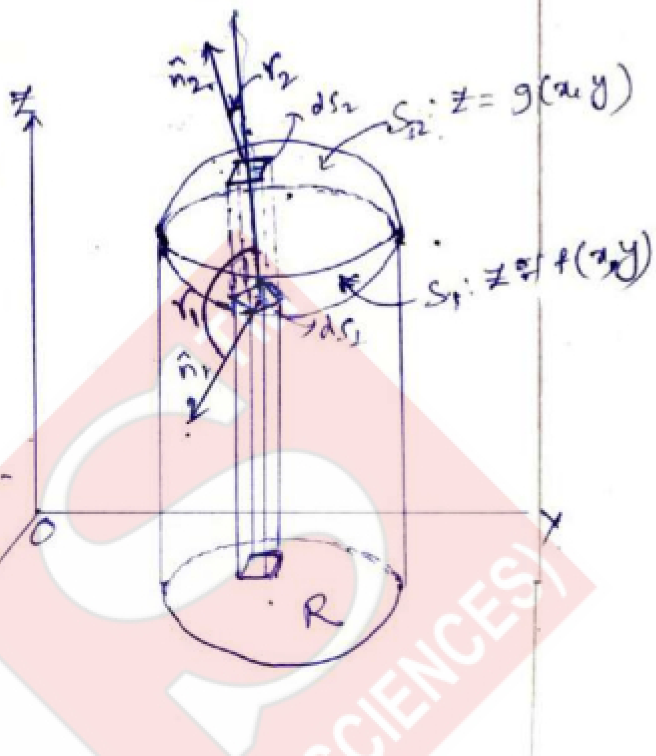
Now the projections of 'S' on xy, yz, zx-planes are $ds = \frac{dx\, dy}{|\hat{n} \cdot k|} = \frac{dx\, dy}{\cos\gamma}$;

$$ds = \frac{dy\, dz}{|\hat{n} \cdot i|} = \frac{dy\, dz}{\cos\alpha} \quad \text{and}$$

$$ds = \frac{dz \cdot dx}{|\hat{n} \cdot j|} = \frac{dz \cdot dx}{\cos\beta}$$

$$\therefore \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz = \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

proof of the divergence theorem



Now we prove the theorem for a special region V, which is bounded by a piece-wise smooth closed surface S and has the property that any straight line parallel to any one of the co-ordinate axes and intersecting V has only one segment (or a single point) in common with 'V'.

If R is the orthogonal projection of 'S' on the xy-plane, then V can be represented in the form $f(x,y) \le z \le g(x,y)$.
where (x,y) varies in R.

Obviously $z = g(x,y)$ represents the upper portion S_2 of S, $z = f(x,y)$ represents the lower portion S_1 of S.

$$\text{Now } \iiint_V \frac{\partial F_3}{\partial z}\,dV = \iiint_V \frac{\partial F_3}{\partial z}\,dx\,dy\,dz = \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\,dz\right]dx\,dy$$

$$= \iint_R \left[F_3(x,y,z)\right]_{z=f(x,y)}^{g(x,y)} dx\,dy$$

$$= \iint_R \left[F_3(x,y,g) - F_3(x,y,f)\right]dx\,dy. \quad \text{——} (A).$$

For the upper portion S_2.

$$dy\,dx = \cos\gamma_2\,dS_2$$
$$= k\cdot\hat{n}_2\,dS_2$$

Since the normal $\hat{n}_2$ to S_2 makes an acute angle γ_2 with k

$$\therefore \boxed{\iint_R F_3(x,y,g)\,dx\,dy = \iint_{S_2} F_3\, \hat{n}_2\cdot k\, dS_2}$$

for the lower curve:

$$dy\,dx = -\cos\gamma_1\, dS_1$$
$$= -k\cdot\hat{n}_1\, dS_1$$

Since the normal n_1 to S_1 makes an obtuse angle γ_1 with k.

$$\therefore \boxed{\iint_R F_3(x,y,f)\,dx\,dy = -\iint_{S_1} F_3\, \hat{n}_1\cdot k\, dS_1}$$

from (A),

$$\iint_R F_3(x,y,g)\,dx\,dy - \iint_R F(x,y,f)\,dx\,dy =$$
$$\iint_{S_2} F_3\,\hat{n}_2\cdot k\, dS_2 + \iint_{S_1} F_3\,\hat{n}_1\cdot k\, dS_1$$

$$= \iint_S F_3\,\hat{n}\cdot k\, dS.$$

$$\therefore \iiint_V \frac{\partial F_3}{\partial z}\,dv = \iint_S F_3\,\hat{n}\cdot k\, dS = \iint_S F_3\, k\cdot\hat{n}\, dS \quad \text{(B)}.$$

Similarly, by projecting S on other co-ordinate planes

$$\iiint_V \frac{\partial F_1}{\partial x}\,dv = \iint_S F_1\,\hat{n}\cdot i\, dS = \iint F_1\, i\cdot\hat{n}\, dS \quad \text{(C)}$$

and $$\iiint_V \frac{\partial F_2}{\partial y}\,dv = \iint_S F_2\,\hat{n}\cdot j\, dS = \iint_S F_2\, j\cdot\hat{n}\, dS \quad \text{(D)}.$$

Adding (B), (C) and (D), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dv = \iint_S (F_1 i + F_2 j + F_3 k)\cdot\hat{n}\, dS$$

$$\Rightarrow \iiint_V (\nabla\cdot F)\,dv = \iint_S F\cdot\hat{n}\, dS.$$

§ 8. Some deductions from divergence theorem.

1. Green's theorem. *Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\, dV = \iint_S (\phi\nabla\psi-\psi\nabla\phi)\cdot\mathbf{n}\, dS.$$

(Meerut 1979 ; Agra 71 ; Gauhati 92 ; Indore 79)

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Putting $\mathbf{F} = \phi\nabla\psi$, we get

$$\nabla\cdot\mathbf{F} = \nabla\cdot(\phi\nabla\psi)$$

$$= \phi(\nabla\cdot\nabla\psi)+(\nabla\phi)\cdot(\nabla\psi) = \phi\nabla^2\psi+(\nabla\phi)\cdot(\nabla\psi).$$

Also $\mathbf{F}\cdot\mathbf{n} = (\phi\nabla\psi)\cdot\mathbf{n}$.

$\therefore$ divergence theorem gives

$$\iiint_V [\phi\nabla^2\psi+(\nabla\phi)\cdot(\nabla\psi)]dV$$

$$= \iint_S (\phi\nabla\psi)\cdot\mathbf{n}\, dS \qquad ...(1)$$

(Osmania 1989, Meerut 70)

This is called *Green's first identity* or *theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2\phi+(\nabla\psi)\cdot(\nabla\phi)]dV$$

$$= \iint_S [\psi\nabla\phi]\cdot\mathbf{n}\, dS \qquad ...(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi-\psi\nabla\phi)\cdot\mathbf{n}\, dS \qquad ...(3)$$

This is called *Green's second identity* or *Green's theorem in symmetrical form.*

Since $\nabla\psi = \dfrac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi = \dfrac{\partial\phi}{\partial n}\mathbf{n}$, therefore

$$(\phi\nabla\psi-\psi\nabla\phi)\cdot\mathbf{n} = \left(\phi\frac{\partial\psi}{\partial n}\mathbf{n} - \psi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}$$

$$= \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS.$$

(Meerut 1980)

Note. Harmonic function. *If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, then ϕ is called harmonic function.* If ϕ and ψ are both harmonic functions, then $\nabla^2\phi = 0$, $\nabla^2\psi = 0$.
Hence from Green's second identity, we get

$$\iint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) dS = 0.$$

2. *Prove that* $\iiint_V \nabla\phi \, dV = \iint_S \phi\mathbf{n} \, dS.$

(Agra 1972 ; Allahabad 77)

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F} dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

Taking $\mathbf{F} = \phi\mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla\cdot(\phi\mathbf{C})\, dV = \iint_S (\phi\mathbf{C})\cdot\mathbf{n}\, dS. \qquad ...(1)$$

Now $\nabla\cdot(\phi\mathbf{C}) = (\nabla\phi)\cdot\mathbf{C} + \phi(\nabla\cdot\mathbf{C})$
$= (\nabla\phi)\cdot\mathbf{C}$, since $\nabla\cdot\mathbf{C} = 0$.

Also $(\phi\mathbf{C})\cdot\mathbf{n} = \mathbf{C}\cdot(\phi\mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C}\cdot(\nabla\phi) dV = \iint_S \mathbf{C}\cdot(\phi\mathbf{n}) dS$$

or $$\mathbf{C}\cdot \iiint_V \nabla\phi \, dV = \mathbf{C}\cdot \iint_S (\phi\mathbf{n})\, dS$$

or $$\mathbf{C}\cdot \left[\iiint_V \nabla\phi\, dV - \iint_S \phi\mathbf{n}\, dS\right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS.$$

3. *Prove that* $\iiint_V \nabla\times\mathbf{B}\, dV = \iint_S \mathbf{n}\times\mathbf{B}\, dS.$

(Gauhati 1974)

Proof. In divergence theorem taking $\mathbf{F} = \mathbf{B}\times\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla\cdot(\mathbf{B}\times\mathbf{C})\, dV = \iint_S (\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}\, dS. \qquad ...(1)$$

Now $\nabla \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot \text{curl } \mathbf{B} - \mathbf{B} \cdot \text{curl } \mathbf{C}$

$= \mathbf{C} \cdot \text{curl } \mathbf{B}$, since curl $\mathbf{C} = \mathbf{0}$.

Also $(\mathbf{B} \times \mathbf{C}) \cdot \mathbf{n} = [\mathbf{B}, \mathbf{C}, \mathbf{n}] = [\mathbf{C}, \mathbf{n}, \mathbf{B}] = \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C} \cdot \text{curl } \mathbf{B})\, dV = \iint_S \mathbf{C} \cdot (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \cdot \iiint_V (\nabla \times \mathbf{B})\, dV = \mathbf{C} \cdot \iint_S (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \cdot \left[\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS \right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Hence we must have

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS = \mathbf{0}$$

or $$\iiint_V (\nabla \times \mathbf{B})\, dV = \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Solved Examples

Ex. 1. *For any closed surface S, prove that*

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = 0.$$

Sol. By divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\text{div curl } \mathbf{F})\, dV,$$ where V is the volume enclosed by S

$=0$, since div curl $\mathbf{F}=0$.

Ex. 2. *Evaluate* $\iint_S \mathbf{r} \cdot \mathbf{n}\, dS$, *where S is a closed surface.*

(Andhra 1992, Madras 83, Rohilkhand 76, Allahabad 75)

Sol. By the divergence theorem, we have

$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = \iiint_V \nabla \cdot \mathbf{r}\, dV$$

$$= \iiint_V 3\, dV, \text{ since } \nabla \cdot \mathbf{r} = \text{div } \mathbf{r} = 3$$

$=3V$, where V is the volume enclosed by S.

Ex. 3. *If* $\mathbf{F} = ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$, a, b, c *are constants show that* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \frac{4}{3}\pi\,(a+b+c)$, *where S is the surface of a unit sphere.*

(Rohilkhand 1992, Allahabad 82, Agra 1980)

Sol. By the divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V (\nabla\cdot\mathbf{F})\,dV,$$

where V is the volume enclosed by S

$$= \iiint_V [\nabla\cdot(ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k})]\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right]dV$$

$$= \iiint_V (a+b+c)\,dV = (a+b+c)\,V = (a+b+c)\,\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi(1)^3$ *i.e.,* $\frac{4}{3}\pi$.

Ex. 4. *If* $\mathbf{n}$ *is the unit outward drawn normal to any closed surface S, show that* $\iiint_V \text{div}\,\mathbf{n}\,dV = S$. **(Andhra 1989)**

Sol. We have by the divergence theorem,

$$\iiint_V \text{div}\,\mathbf{n}\,dV = \iint_S \mathbf{n}\cdot\mathbf{n}\,dS = \iint_S dS = S.$$

Ex. 5. *Prove that*

$$\iiint_V \nabla\phi\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV.$$

Sol. By divergence theorem, we have

$$\iiint_V \nabla\cdot(\phi\mathbf{A})\,dV = \iint_S (\phi\mathbf{A})\cdot\mathbf{n}\,dS \qquad \text{...(1)}$$

Now $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})$.

Also $(\phi\mathbf{A})\cdot\mathbf{n} = \phi(\mathbf{A}\cdot\mathbf{n})$.

Hence (1) gives

$$\iiint_V [(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})]\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS$$

or
$$\iiint_V (\nabla\phi)\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV.$$

Ex. 6. *Prove that* $\int_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = 0.$

Sol. We have $\int_S \nabla\phi \times \nabla\psi \cdot d\mathbf{S} = \int_S (\nabla\phi \times \nabla\psi)\cdot \mathbf{n}\, dS$

$= \int_V \nabla\cdot(\nabla\phi \times \nabla\psi)\, dV$, by divergence throrem

$= 0.$ $\qquad [\because \nabla\cdot(\nabla\phi\times\nabla\psi) = 0]$

Ex. 7. *Prove that*

$$\int_V \nabla\phi\cdot \mathit{curl}\, \mathbf{F}\, dV = \int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S}.$$

Sol. We have $\int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S} = \int_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\, dS$

$= \int_V \nabla\cdot(\mathbf{F}\times\nabla\phi)\, dV$, by divergence theorem applied to the vector function $\mathbf{F}\times\nabla\phi$

$= \int_V (\nabla\phi\cdot \text{curl } \mathbf{F} - \mathbf{F}\cdot\text{curl } \nabla\phi)\, dV$

$[\because \quad \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl } \mathbf{A} - \mathbf{A}\cdot\text{curl } \mathbf{B}]$

$= \int_V \nabla\phi\cdot\text{curl } \mathbf{F}\, dV.$ $\qquad [\because \quad \text{curl } \nabla\phi = 0]$

Ex. 8. *Prove that* $\iiint_V \dfrac{dV}{r^2} = \iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS.$

Sol. $\iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS = \iint_S \left(\dfrac{\mathbf{r}}{r^2}\right)\cdot\mathbf{n}\, dS$

$= \iiint_V \nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right) dV$, by divergence theorem.

Now $\nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right) = \dfrac{1}{r^2}(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\nabla\left(\dfrac{1}{r^2}\right)$

$= \dfrac{3}{r^2} + \mathbf{r}\cdot\left(-\dfrac{2}{r^3}\nabla r\right) = \dfrac{3}{r^2} - \dfrac{2}{r^3}\left(\mathbf{r}\cdot\dfrac{\mathbf{r}}{r}\right) = \dfrac{3}{r^2} - \dfrac{2}{r^4}r^2 = \dfrac{1}{r^2}.$

Hence $\iint_S \dfrac{\mathbf{r}\cdot\mathbf{n}}{r^2}\, dS = \iiint_V \dfrac{dV}{r^2}.$

Ex. 9. *If* $\mathbf{F} = \nabla\phi$ *and* $\nabla^2\phi = 0$, *show that for a closed surface S*

$$\iiint_V \mathbf{F}^2\, dV = \iint_S \phi\mathbf{F}\cdot\mathbf{n}\, dS. \quad \text{(Rohilkhand 1978, 79)}$$

Sol. By divergence theorem, we have

$$\iint_S \phi \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V [\nabla \cdot (\phi \mathbf{F})]\, dV.$$

Now $\nabla \cdot (\phi \mathbf{F}) = (\nabla\phi \cdot \mathbf{F}) + \phi (\nabla \cdot \mathbf{F}) = \mathbf{F} \cdot \mathbf{F} + \phi (\nabla \cdot \nabla\phi)$

$$= \mathbf{F}^2 + \phi \nabla^2 \phi = \mathbf{F}^2, \text{ since } \nabla^2\phi = 0.$$

Hence $\iint_S \phi \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \mathbf{F}^2\, dV.$

Ex. 10. *If* $\mathbf{F} = \nabla\phi, \nabla^2\phi = -4\pi\rho$, *show that*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -4\pi \iiint_V \rho\, dV.$$

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$$

Now $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla\phi) = \nabla^2\phi = -4\pi\rho.$

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (-4\pi\rho)\, dV = -4\pi \iiint_V \rho\, dV.$$

Ex. 11. *If* $\mathbf{C} = \frac{1}{2}\nabla \times \mathbf{B}, \mathbf{B} = \nabla \times \mathbf{A}$, *show that*

$$\frac{1}{2} \iiint_V \mathbf{B}^2\, dV = \frac{1}{2} \iint_S \mathbf{A} \times \mathbf{B} \cdot \mathbf{n}\, dS + \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV.$$

Sol. We have by divergence theorem

$$\frac{1}{2} \iint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}\, dS = \frac{1}{2} \iiint_V \nabla \cdot (\mathbf{A} \times \mathbf{B})\, dV.$$

Now $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B}$

$= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \mathbf{A} \cdot (2\mathbf{C}) = \mathbf{B}^2 - 2(\mathbf{A} \cdot \mathbf{C}).$

Hence $\frac{1}{2} \iint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}\, dS = \frac{1}{2} \iiint_V [\mathbf{B}^2 - 2(\mathbf{A} \cdot \mathbf{C})]\, dV$

$$= \frac{1}{2} \iiint_V \mathbf{B}^2\, dV - \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV$$

or $\frac{1}{2} \iiint_V \mathbf{B}^2\, dV = \frac{1}{2} \iint_S \mathbf{A} \times \mathbf{B} \cdot \mathbf{n}\, dS + \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV.$

Ex. 12. *If* ϕ *is harmonic in* V, *then*

$$\iint_S \frac{\partial \phi}{\partial n}\, dS = 0$$

where S *is the surface enclosing* V. **(Meerut 1972)**

Sol. We have $\iint_S \frac{\partial \phi}{\partial n}\, dS = \iint_S \left(\frac{\partial \phi}{\partial n} \mathbf{n}\right) \cdot \mathbf{n}\, dS$

$$= \iint_S (\nabla\phi)\cdot\mathbf{n}\, dS$$

$$= \iiint_V \nabla\cdot(\nabla\phi)\, dV, \text{ by divergence theorem}$$

$$= \iiint_V \nabla^2\phi\, dV$$

$= 0$, since $\nabla^2\phi = 0$ in V because ϕ is harmonic in V.

Ex. 13. *If ϕ is harmonic in V, then*

$$\iint_S \phi\frac{\partial\phi}{\partial n}dS = \iiint_V |\nabla\phi|^2\, dV.$$

(Meerut 1979 ; Agra 70)

Sol. We have

$$\iint_S \phi\frac{\partial\phi}{\partial n}dS = \iint_S \left(\phi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}\, dS = \iint_S (\phi\nabla\phi)\cdot\mathbf{n}\, dS$$

$$= \iiint_V \nabla\cdot(\phi\nabla\phi)\, dV, \text{ by divergence theorem}$$

$$= \iiint_V [(\nabla\phi\cdot\nabla\phi) + \phi(\nabla\cdot\nabla\phi)]\, dV$$

$$= \iiint_V [(\nabla\phi)^2 + \phi\nabla^2\phi]\, dV$$

$$= \iiint_V |\nabla\phi|^2\, dV, \text{ since } \nabla^2\phi = 0 \text{ and } (\nabla\phi)^2 = |\nabla\phi|^2.$$

Ex. 14. *If ϕ is harmonic in V and $\dfrac{\partial\phi}{\partial n} = 0$ on S, then ϕ is constant in V.*

Sol. Since ϕ is harmonic in V, therefore as in exercise 13, we have

$$\iint_S \phi\frac{\partial\phi}{\partial n}dS = \iiint_V |\nabla\phi|^2\, dV.$$

But $\dfrac{\partial\phi}{\partial n} = 0$ on S. Therefore $\displaystyle\iint_S \phi\frac{\partial\phi}{\partial n}dS = 0.$

$$\therefore \iiint_V |\nabla\phi|^2\, dV = 0.$$

$\therefore |\nabla\phi|^2 = 0$ in V.

$\therefore \nabla\phi = \mathbf{0}$ in V.

$\therefore \phi =$ constant in V.

Ex. 15. *If ϕ and ψ are harmonic in V and $\dfrac{\partial\phi}{\partial n} = \dfrac{\partial\psi}{\partial n}$ on S, then $\phi = \psi + c$ in V, where c is a constant.*

Sol. We have, $\nabla^2\phi = 0, \nabla^2\psi = 0$ in V.

$\therefore \quad \nabla^2(\phi - \psi) = \nabla^2\phi - \nabla^2\psi = 0$ in V.

Therefore $\phi - \psi$ is harmonic in V.

Again on S, $\frac{\partial}{\partial n}(\phi - \psi) = \frac{\partial \phi}{\partial n} - \frac{\partial \psi}{\partial n} = 0.$

Thus $\phi - \psi$ is harmonic in V and on S we have

$$\frac{\partial}{\partial n}(\phi - \psi) = 0.$$

Hence as in exercise 14, we have

$$\phi - \psi = c, \text{ where } c \text{ is a constant}$$

or
$$\phi = \psi + c.$$

Ex. 16. *If div* **F** *denotes the divergence of a vector field* **F** *at a point P, show that*

$$div\ \mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F} \cdot \mathbf{n}\, dS}{\delta V}$$

where δV is the volume enclosed by the surface δS and the limit is obtained by shrinking δV to the point P.

Sol. We have by the divergence theorem,

$$\iiint_{\delta V} \text{div}\ \mathbf{F}\, dV = \iint_{\delta S} \mathbf{F} \cdot \mathbf{n}\, dS. \quad \ldots(1)$$

By the mean value theorem of integral calculus, the left hand side can be written as

$$\overline{\text{div}\ \mathbf{F}} \iiint_{\delta V} dV = \overline{\text{div}\ \mathbf{F}}\, \delta V,$$

where $\overline{\text{div}\ \mathbf{F}}$ is some value intermediate between the maximum and minimum of div **F** throughout δV. Therefore (1) gives

$$\overline{\text{div}\ \mathbf{F}}\, \delta V = \iint_{\delta S} \mathbf{F} \cdot \mathbf{n}\, dS$$

or
$$\overline{\text{div}\ \mathbf{F}} = \frac{\iint_{\delta S} \mathbf{F} \cdot \mathbf{n}\, dS}{\delta V}.$$

Taking the limit as $\delta V \to 0$ such that P is always interior to δV, $\overline{\text{div}\ \mathbf{F}}$ approaches the value div **F** at point P. Hence, we get

$$\text{div}\ \mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F} \cdot \mathbf{n}\, dS}{\delta V}.$$

Ex. 17. *Show that* $\iint_S \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.*

Solution. Let **C** be any arbitrary constant vector. Then

$$\mathbf{C}\cdot\iint_S \mathbf{n}\,dS = \iint_S \mathbf{C}\cdot\mathbf{n}\,dS$$

$$= \iiint_V (\nabla\cdot\mathbf{C})\,dV, \text{ by divergence theorem}$$

$$= 0, \text{ since div } \mathbf{C} = 0.$$

Thus $\mathbf{C}\cdot\iint_S \mathbf{n}\,dS = 0$, where **C** is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n}\,dS = \mathbf{0}$.

Ex. 18. *Prove that* $\iint_S \mathbf{r}\times\mathbf{n}\,dS = \mathbf{0}$ *for any closed surface S.*

Sol. Let **C** be any arbitrary constant vector. Then

$$\mathbf{C}\cdot\iint_S \mathbf{r}\times\mathbf{n}\,dS = \iint_S \mathbf{C}\cdot[(\mathbf{r}\times\mathbf{n})]\,dS$$

$$= \iint_S (\mathbf{C}\times\mathbf{r})\cdot\mathbf{n}\,dS$$

$$= \iiint_V [\nabla\cdot(\mathbf{C}\times\mathbf{r})]\,dV, \text{ by divergence theorem}$$

$$= \iiint_V [\mathbf{r}\cdot\text{curl }\mathbf{C} - \mathbf{C}\cdot\text{curl }\mathbf{r}]\,dV$$

$$= 0, \text{ since curl } \mathbf{C} = \mathbf{0} \text{ and curl } \mathbf{r} = \mathbf{0}.$$

Thus $\mathbf{C}\cdot\iint_S \mathbf{r}\times\mathbf{n}\,dS = 0$, where **C** is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r}\times\mathbf{n}\,dS = \mathbf{0}$.

Ex. 19. *Prove that* $\iint_S (\nabla\phi)\times\mathbf{n}\,dS = \mathbf{0}$ *for a closed surface S.*

Sol. Let **C** be an arbitrary constant vector. Then

$$\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\,dS = \iint_S \mathbf{C}\cdot[(\nabla\phi)\times\mathbf{n}]\,dS$$

$$= \iint_S [\mathbf{C}\times\nabla\phi]\cdot\mathbf{n}\,dS$$

$$= \iiint_V [\nabla\cdot(\mathbf{C}\times\nabla\phi)]\,dV, \text{ by div. theorem}$$

$$= \iiint_V [\nabla\phi\cdot\text{curl }\mathbf{C} - \mathbf{C}\cdot\text{curl }\nabla\phi]\,dV$$

$$= 0, \text{ since curl } \mathbf{C} = \mathbf{0} \text{ and curl } \nabla\phi = \mathbf{0}.$$

Thus $\mathbf{C}\cdot\iint_S (\nabla\phi)\times\mathbf{n}\,dS = 0$, where **C** is an arbitrary vector.

Hence we must have $\iint_S (\nabla\phi) \times \mathbf{n}\, dS = \mathbf{0}$.

Ex. 20. *Prove that* $\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = 2V\mathbf{a}$,

where $\mathbf{a}$ *is a constant vector and* V *is the volume enclosed by the closed surface* S.

Sol. We know that

$$\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$$

[See § 8, part 3 page 232]

Putting $\mathbf{B} = \mathbf{a} \times \mathbf{r}$, we get

$$\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = \iiint_V \nabla \times (\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V \text{curl}\,(\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V 2\mathbf{a}\, dV, \text{ since curl}\,(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

$$= 2\mathbf{a} \iiint_V dV = 2\mathbf{a}V.$$

Ex. 21. *A vector* $\mathbf{B}$ *is always normal to a given closed surface* S. *Show that* $\iiint_V \text{curl}\,\mathbf{B}\, dV = \mathbf{0}$, *where* V *is the region bounded by* S.

Sol. We know that

$$\iiint_V \text{curl}\,\mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$$

Since $\mathbf{B}$ is normal to S, therefore $\mathbf{B}$ is parallel to $\mathbf{n}$. Therefore $\mathbf{n} \times \mathbf{B} = \mathbf{0}$.

$$\therefore \iint_S \mathbf{n} \times \mathbf{B}\, dS = \mathbf{0}.$$

$$\therefore \iiint_V \text{curl}\,\mathbf{B}\, dV = \mathbf{0}.$$

Ex. 22. *Express* $\int_V \{(\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\, \mathbf{v}\}\, dV$, *as a surface integral.* **(Gauhati 1977)**

Sol. From a vector identity we know that

$$\text{div}\,(\rho\mathbf{v}) = (\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\, \mathbf{v}.$$

$$\therefore \int_V \{(\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\, \mathbf{v}\}\, dV = \int_V \text{div}\,(\rho\mathbf{v})\, dV$$

$$= \int_V \nabla \cdot (\rho \mathbf{v})\, dV$$

$$= \int_S (\rho \mathbf{v}) \cdot \mathbf{n}\, dS, \text{ by Gauss divergence theorem}$$

$$= \int_S \rho\, (\mathbf{v} \cdot \mathbf{n})\, dS.$$

Ex. 23 *Using the divergence theorem, show that the volume V of a region T bounded by a surface S is*

$$V = \iint_S x\, dy\, dz = \iint_S y\, dz\, dx = \iint_S z\, dx\, dy$$

$$= \tfrac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

Sol. By divergence theorem, we have

$$\iint_S x\, dy\, dz = \iiint_V \left(\frac{\partial}{\partial x}(x)\right) dV = \iiint_V dV = V$$

$$\iint_S y\, dz\, dx = \iiint_V \left(\frac{\partial}{\partial y}(y)\right) dV = \iiint_V dV = V$$

$$\iint_S z\, dx\, dy = \iiint_V \left(\frac{\partial}{\partial z}(z)\right) dV = \iiint_V dV = V.$$

Adding these results, we get

$$3V = \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

or $$V = \tfrac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

Ex. 24 (a) *Verify divergence theorem for*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped

$$0 \le x \le a,\; 0 \le y \le b,\; 0 \le z \le c.$$

(Andhra 1990, Meerut 76)

Sol. We have div $\mathbf{F} = \nabla \cdot \mathbf{F}$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z.$$

$$\therefore \text{ volume integral} = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V 2(x + y + z)\, dV$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x + y + z)\, dx\, dy\, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dy\, dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2} + ay + az\right] dy\,dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz$$

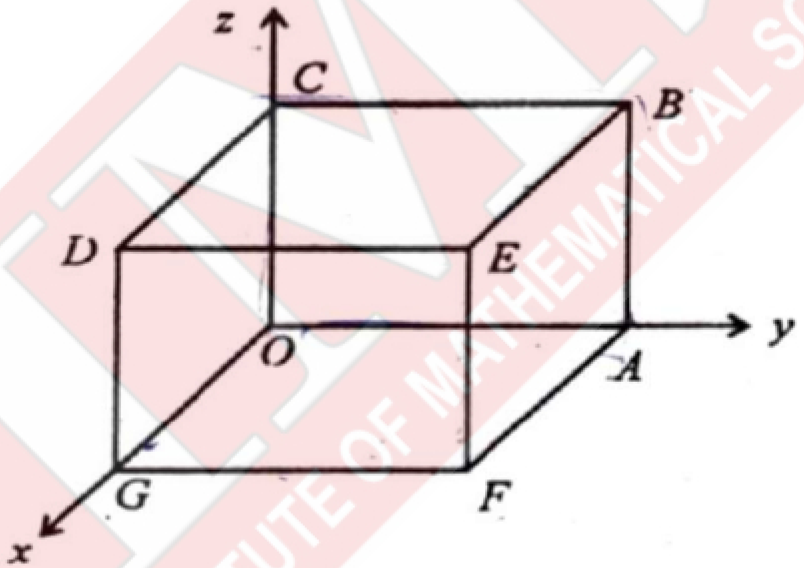
$$= 2\int_{z=0}^{c}\left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right] dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= [a^2bc + ab^2c + abc^2] = abc\,(a + b + c).$$

Surface Integral. We shall now calculate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

over the six faces of the rectangular parallelopiped.



Over the face *DEFG*,

$\mathbf{n} = \mathbf{i}$, $x = a$.

Therefore,

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}[(a^2 - yz)\,\mathbf{i} + (y^2 - za)\,\mathbf{j} + (z^2 - ay)\,\mathbf{k}]\cdot\mathbf{i}\,dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b}(a^2 - yz)\,dy\,dz = \int_{z=0}^{c}\left[a^2y - z\frac{y^2}{2}\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \left[a^2 b - \frac{zb^2}{2}\right] dz = \left[a^2 bz - \frac{z^2}{4} b^2\right]_0^c$$

$$= a^2 bc - \frac{c^2 b^2}{4}.$$

Over the face $ABCO$, $\mathbf{n} = -\mathbf{i}$, $x = 0$. Therefore

$$= \iint_{ABCO} \mathbf{F} \cdot \mathbf{n}\, dS = \iint [(0 - yz)\mathbf{i} + \ldots + \ldots] \cdot (-\mathbf{i})\, dy\, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} yz\, dy\, dz = \int_{z=0}^{c} \left[\frac{y^2}{2} z\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \frac{b^2}{2} z\, dz = \frac{b^2 c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n} = \mathbf{j}$, $y = b$. Therefore

$$\iint_{ABEF} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} [(x^2 - bz)\mathbf{i} + (b^2 - zx)\mathbf{j} + (z^2 - bx)\mathbf{k}] \cdot \mathbf{j}\, dx\, dz$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} (b^2 - zx)\, dx\, dz = b^2 ca - \frac{a^2 c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n} = -\mathbf{j}$, $y = 0$. Therefore

$$\iint_{OGDC} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} zx\, dx\, dz = \frac{c^2 a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n} = \mathbf{k}$, $z = c$. Therefore

$$\iint_{BCDE} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} (c^2 - xy)\, dx\, dy = c^2 ab - \frac{a^2 b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n} = -\mathbf{k}$, $z = 0$. Therefore

$$\iint_{AFGO} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} xy\, dx\, dy = \frac{a^2 b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \left(a^2 bc - \frac{c^2 b^2}{4} + \frac{c^2 b^2}{4}\right) + \left(b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4}\right) + \left(c^2 ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4}\right)$$

$= abc\,(a + b + c)$.

Hence the theorem is verified.

Ex. 24. (b). *If* $\mathbf{F} = 4xz\,\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ *and S is the surface bounded by* $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$, *evaluate*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$ **(Osmania 1990)**

Sol. By Gauss divergence theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV,$$

where V is the volume enclosed by the surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - 2y + y)\,dV = \iiint_V (4z - y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1}(4z - y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left[2z^2 - yz\right]_{z=0}^{1} dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1}(2 - y)\,dx\,dy$$

$$= \int_{x=0}^{1}\left[2y - \frac{y^2}{2}\right]_{y=0}^{1} dx = \int_0^1\left[2 - \frac{1}{2}\right] dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}.$$

Ex. 24. (c). *Evaluate* $\iint_S [4xz\,dy\,dz - y^2\,dz\,dx + yz\,dx\,dy]$ *where S is the surface of the cube bounded by the planes* $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1$ *and* $z = 1$. **(Osmania 1992)**

Sol. By Gauss divergence theorem, the given surface integral is equal to the volume integral

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV,$$

where V is the volume enclosed by the surface S.

Ex. 24. (d). *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *if* $\mathbf{F} = xy\,\mathbf{i} + z^2\,\mathbf{j} + 2yz\,\mathbf{k}$ *over the tetrahedron bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = 1$.

Sol. By Gauss divergence theorem,

$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV$, where V is the volume enclosed by the surface S of the tetrahedron

$$= \iiint_V \left[\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(2yz)\right] dV$$

$$= \iiint_V (y + 0 + 2y)\, dV$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} y\, dx\, dy\, dz = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y\left[z\right]_{z=0}^{1-x-y} dx\, dy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y\,[1-x-y]\, dxdy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} [y - xy - y^2]\, dx\, dy$$

$$= 3\int_0^1 \left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{1-x} dx$$

$$= 3\int_0^1 [\tfrac{1}{2}(1-x)^2 - \tfrac{1}{2}x(1-x)^2 - \tfrac{1}{3}(1-x)^3]\, dx$$

$$= 3\int_0^1 \frac{1}{6}(1-x)^3 dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{3}{24} = \frac{1}{8}.$$

Ex. 24. (e). *Find the value of* $\iint_S (\mathbf{F} \times \nabla\phi)\bullet\mathbf{n}\, dS$

where $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}, \phi = xy + yz + zx,$

S:— $x = \pm 1, y = \pm 1, z = \pm 1.$ **(Nagarjuna 1991)**

Sol. By Gauss divergence theorem,

$$\iint_S (\mathbf{F} \times \nabla\phi)\bullet\mathbf{n}\, dS = \iiint_V \nabla\bullet(\mathbf{F} \times \nabla\phi)\, dV,$$

where V is the volume enclosed by the surface S.

Now from a vector identity, we know that

$\nabla\bullet(\mathbf{A} \times \mathbf{B}) = \mathbf{B}\bullet(\nabla \times \mathbf{A}) - \mathbf{A}\bullet(\nabla \times \mathbf{B}).$

$\therefore \nabla\bullet(\mathbf{F} \times \nabla\phi) = \nabla\phi\bullet(\nabla \times \mathbf{F}) - \mathbf{F}\bullet\{\nabla \times (\nabla\phi)\}$

$= \nabla\phi\bullet(\nabla \times \mathbf{F}).$ $[\because \nabla \times (\nabla\phi) = \text{curl grad}\, \phi = 0]$

Now $\nabla \times \mathbf{F} = \nabla \times (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right]\mathbf{j}$$
$$+ \left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]\mathbf{k}$$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore \nabla \cdot (\mathbf{F} \times \nabla\phi) = (\nabla\phi) \cdot \mathbf{0} = 0.$

Hence the given integral

$$= \iiint_V 0\, dV = 0.$$

Ex. 25. *Verify divergence theorem for* $\mathbf{F} = (2x - z)\mathbf{i} + x^2 y\,\mathbf{j} - xz^2\mathbf{k}$ *taken over the region bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0, z = 1$.

(Rohilkhand 1989; Agra 85)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV,$$

where S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ and V is the volume enclosed by the surface S.

We have $\nabla \cdot \mathbf{F} = \nabla \cdot [(2x - z)\mathbf{i} + x^2 y\mathbf{j} - xz^2\mathbf{k}]$

$$= \frac{\partial}{\partial x}(2x - z) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(-xz^2)$$

$= 2 + x^2 - 2xz.$

$$\therefore \iiint_V (\nabla \cdot \mathbf{F})\, dV = \iiint_V (2 + x^2 - 2xz)\, dV$$

$$= \int_{z=0}^1 \int_{y=0}^1 \int_{x=0}^1 (2 + x^2 - 2xz)\, dx\, dy\, dz$$

$$= \int_{z=0}^1 \int_{y=0}^1 \left[2x + \frac{x^3}{3} - x^2 z\right]_{x=0}^1 dy\, dz$$

$$= \int_{z=0}^1 \int_{y=0}^1 \left(2 + \frac{1}{3} - z\right) dy\, dz = \int_{z=0}^1 \int_{y=0}^1 \left(\frac{7}{3} - z\right) dy\, dz$$

$$= \int_{z=0}^1 \left[\frac{7}{3}y - zy\right]_{y=0}^1 dz = \int_0^1 \left(\frac{7}{3} - z\right) dz$$

$$= \left[\frac{7}{3}z - \frac{1}{2}z^2\right]_0^1 = \frac{7}{3} - \frac{1}{2} = \frac{11}{6}. \quad \ldots(1)$$

We shall now calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ over the six faces of the cube.

Draw figure as in solved example 24 (a). Over the face $OABC$ which lies in the yz-plane, $\mathbf{n} = -\mathbf{i}$, $x = 0$.

Therefore $\displaystyle\iint_{OABC} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{z=0}^{1}\int_{y=0}^{1}(-z\mathbf{i})\cdot(-\mathbf{i})\,dy\,dz$

$$= \int_{z=0}^{1}\int_{y=0}^{1} z\,dy\,dz = \int_{z=0}^{1} z\,\left[y\right]_{y=0}^{1} dz = \int_0^1 z\,dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}.$$

Over the opposite face $DEFG$, $x = 1$, $\mathbf{n} = \mathbf{i}$.

Therefore

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{z=0}^{1}\int_{y=0}^{1}[(2-z)\,\mathbf{i} + y\mathbf{j} - z^2\mathbf{k}]\cdot\mathbf{i}\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}(2-z)\,dy\,dz = \int_{z=0}^{1}(2-z)\,\left[y\right]_{y=0}^{1} dz$$

$$= \int_0^1 (2-z)\,dz = \left[2z - \frac{z^2}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}.$$

Over the face $OGDC$ which lies in the zx-plane, $y = 0$, $\mathbf{n} = -\mathbf{j}$.

$$\therefore \iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{z=0}^{1}\int_{x=0}^{1}\left[(2x - z)\,\mathbf{i} - xz^2\mathbf{k}\right]\cdot(-\mathbf{j})\,dz\,dx$$

$$= \int_{z=0}^{1}\int_{x=0}^{1} 0\,dz\,dx = 0.$$

Over the opposite face $ABEF$, $y = 1$, $\mathbf{n} = \mathbf{j}$.

$$\therefore \iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{z=0}^{1}\int_{x=0}^{1}[(2x - z)\,\mathbf{i} + x^2\mathbf{j} - xz^2\mathbf{k}]\cdot\mathbf{j}\,dz\,dx$$

$$= \int_{z=0}^{1}\int_{x=0}^{1} x^2\,dz\,dx = \int_{z=0}^{1}\left[\frac{x^3}{3}\right]_{x=0}^{1} dz = \int_0^1 \frac{1}{3}\,dz$$

$$= \frac{1}{3}\left[z\right]_0^1 = \frac{1}{3}.$$

Over the face $OGFA$ which lies in the xy-plane, $z = 0$, $\mathbf{n} = -\mathbf{k}$.

$$\therefore \iint_{OGFA} \mathbf{F}\cdot\mathbf{n}\,dS = \int_{x=0}^{1}\int_{y=0}^{1}(2x\,\mathbf{i} + x^2y\,\mathbf{j})\cdot(-\mathbf{k})\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} 0 \, dx \, dy = 0.$$

Over the opposite face $BCDE$, $z = 1$, $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_{BCDE} \mathbf{F} \cdot \mathbf{n} \, dS = \int_{x=0}^{1} \int_{y=0}^{1} [(2x-1)\,\mathbf{i} + x^2 y\,\mathbf{j} - x\,\mathbf{k}] \cdot \mathbf{k} \, dx \, dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} -x \, dx \, dy = \int_{x=0}^{1} -x \left[y \right]_{y=0}^{1} dx = \int_{0}^{1} -x \, dx$$

$$= -\left[\frac{x^2}{2} \right]_0^1 = -\frac{1}{2}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{2} + \frac{3}{2} + 0 + \frac{1}{3} + 0 - \frac{1}{2} = \frac{11}{6}. \qquad \ldots(2)$$

From (1) and (2), we see that

$$\iiint_V \operatorname{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

This verifies Gauss divergence theorem.

Ex. 26. *Verify divergence theorem for* $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ *taken over the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Sol. Proceed as in solved example 25. Here we shall have

$$\iiint_V (\nabla \cdot \mathbf{F}) \, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) \right] dV$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (4z - 2y + y) \, dx \, dy \, dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (4z - y) \, dx \, dy \, dz = \frac{3}{2}.$$

The six surface integrals will come out to be $2, 0, -1, 0, \frac{1}{2}$ and 0. Their sum $= \frac{3}{2}$.

Hence the theorem is verified.

Ex. 27. *Evaluate*

$$\iint_S x^2 \, dy \, dz + y^2 \, dz \, dx + 2z(xy - x - y) \, dx \, dy$$

where S is the surface of the cube

$$0 \le x \le 1, \; 0 \le y \le 1, \; 0 \le z \le 1. \qquad \textbf{(Meerut 1986)}$$

Sol. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z(xy - x - y)\}\right] dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} [2x + 2y + 2xy - 2x - 2y]\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xy\, dx\, dy\, dz = 2\int_{z=0}^{1}\int_{y=0}^{1}\left[\frac{x^2}{2}y\right]_{x=0}^{1} dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\frac{y}{2}\, dy\, dz = \int_{z=0}^{1}\left[\frac{y^2}{2}\right]_{y=0}^{1} dz$$

$$= \int_{z=0}^{1}\tfrac{1}{2}\, dz = \tfrac{1}{2}\left[z\right]_0^1 = \tfrac{1}{2}.$$

Ex. 28. *Evaluate, by Green's theorem in space (i.e., Gauss divergence theorem), the integral*

$$\iint_S 4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy,$$

where S is the surface of the cube bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x = 1$, $y = 1$, $z = 1$. **(Meerut 1974, Kanpur 77)**

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S 4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$= \frac{3}{2}$, as in solved example 26.

Ex. 29. *Use Gauss divergence theorem to show that*

$$\iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\,\mathbf{j} + 2\,\mathbf{k}\}\cdot\mathbf{n}\, dS = \frac{1}{3}a^5,$$

where S denotes the surface of the cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$. **(Rohilkhand 1979, Agra 77)**

Sol. Let V be the volume enclosed by the surface S of the given cube. Then by Gauss divergence theorem, we have

$$\iint_S \{(x^3 - yz)\mathbf{i} - 2x^2y\,\mathbf{j} + 2\,\mathbf{k}\}\cdot\mathbf{n}\, dS$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(2)\right] dV$$

$$= \iiint_V (3x^2 - 2x^2 + 0)\, dV$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} \int_{x=0}^{a} x^2 \, dx \, dy \, dz$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{x^3}{3}\right]_{x=0}^{a} dy \, dz = \int_{z=0}^{a} \int_{y=0}^{a} \frac{a^3}{3} dy \, dz$$

$$= \frac{a^3}{3} \int_{z=0}^{a} \left[y\right]_{y=0}^{a} dz = \frac{a^3}{3} \int_0^a a \, dz = \frac{a^4}{3} \left[z\right]_0^a = \frac{a^5}{3}.$$

Ex. 30. *Evaluate* $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n} \, dS$ *where* S *denotes the surface of the cube bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $x = a, y = a$, $z = a$ *by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly.* **(Agra 1979)**

Sol. Let V be the volume bounded by the surface S of the given cube. Then by Gauss divergence theorem, we have

$$\iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n} \, dS = \iiint_V [\text{div}\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})] \, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV = \iiint_V 3 \, dV$$

$$= 3 \iiint_V dV = 3V = 3a^3, \text{ as } V = a^3 = \text{the volume of the cube}$$

whose each edge is of length a.

To verify our answer we shall evaluate $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n} \, dS$ over the six faces of the cube.

Draw figure as in solved example 24 (a).

Over the face $OABC$ which lies in the yz-plane, $\mathbf{n} = -\mathbf{i}$, $x = 0$.

$$\therefore \iint_{OABC} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n} \, dS = \iint_{OABC} (y\,\mathbf{j} + z\,\mathbf{k}) \cdot (-\mathbf{i}) \, dS = 0.$$

Over the opposite face $DEFG$, $x = a$, $\mathbf{n} = \mathbf{i}$.

$$\therefore \iint_{DEFG} (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{n} \, dS = \iint_{DEFG} (a\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{i} \, dS$$

$$= \iint_{DEFG} a \, dS = a \iint_{DEFG} dS = a. \text{ area of the face } DEFG$$

$$= a.a^2 = a^3.$$

Similarly calculate the other four surface integrals.

Adding the six surface integrals, we get

$$\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot\mathbf{n}\,dS = 0 + a^3 + 0 + a^3 + 0 + a^3 = 3a^3.$$

Since $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot\mathbf{n}\,dS = \iiint_V [\operatorname{div}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})]\,dV$,

therefore our answer is verified.

Ex. 31. (a). *By transforming to a triple integral evaluate*

$$I = \iint_S (x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy)$$

where S is the closed surface bounded by the planes $z = 0$, $z = b$ and the cylinder $x^2 + y^2 = a^2$. **(Meerut 1980)**

Solution. By divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)\right] dV$$

$$= \int_{z=0}^{b} \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (3x^2 + x^2 + x^2)\,dx\,dy\,dz$$

$$= 4 \times 5 \int_{z=0}^{b} \int_{y=0}^{a} \int_{x=0}^{\sqrt{(a^2-y^2)}} x^2\,dx\,dy\,dz$$

$$= 20 \int_{z=0}^{b} \int_{y=0}^{a} \left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy\,dz$$

$$= \frac{20}{3} \int_{z=0}^{b} \int_{y=0}^{a} (a^2 - y^2)^{3/2}\,dy\,dz$$

$$= \frac{20}{3} \int_{y=0}^{a} \left[(a^2 - y^2)^{3/2} z\right]_{z=0}^{b} dy = \frac{20}{3} \int_{y=0}^{a} b\,(a^2 - y^2)^{3/2}\,dy.$$

Put $y = a \sin t$ so that $dy = a \cos t\,dt$.

$$\therefore I = \frac{20}{3} b \int_0^{\pi/2} a^3 \cos^3 t\,(a \cos t)\,dt$$

$$= \frac{20}{3} a^4 b \int_0^{\pi/2} \cos^4 t\,dt = \frac{20}{3} a^4 b\,\frac{3}{4.2}\,\frac{\pi}{2} = \frac{5}{4}\pi a^4 b.$$

Ex. 31. (b). *Evaluate* $\iint_S (zx^2\,dx\,dy + x^3\,dy\,dz + yx^2\,dz\,dx)$

where S is the closed surface consisting of the cylinder $x^2 + y^2 = 4$ and the circular discs $z = 0$ and $z = 3$. **(Osmania 1990)**

Sol. Proceed as in Ex. 31 (a). **(Ans. 60π)**

Ex. 32. *Apply Gauss's divergence theorem to evaluate*

$$\iint_S \left[(x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx + z\, dx\, dy\right]$$

over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$.

Sol. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$
$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$$

Here $F_1 = x^3 - yz,\ F_2 = -2x^2 y,\ F_3 = z.$

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 - 2x^2 + 1 = x^2 + 1.$$

$\therefore$ the given surface integral is equal to the volume integral

$$\int_{z=0}^{a} \int_{y=0}^{a} \int_{x=0}^{a} (x^2 + 1)\, dx\, dy\, dz$$
$$= \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{x^3}{3} + x\right]_{x=0}^{a} dy\, dz$$
$$= \int_{z=0}^{a} \int_{y=0}^{a} \left(\frac{a^3}{3} + a\right) dy\, dz = a^2 \left(\frac{a^3}{3} + a\right).$$

Ex. 33. *If* $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (z^2 - 1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ *where S is the closed surface bounded by the planes* $z = 0,\ z = 1$ *and the cylinder* $x^2 + y^2 = 4$.

(Kanpur 1980)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \text{div}\, \mathbf{F}\, dV.$$

Here $\text{div}\, \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1)$

$$= 1 - 1 + 2z = 2z.$$

$$\therefore \iiint_V \text{div}\, \mathbf{F}\, dV = \int_{z=0}^{1} \int_{y=-2}^{2} \int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} 2z\, dx\, dy\, dz$$
$$= \int_{z=0}^{1} \int_{y=-2}^{2} \left[2zx\right]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} 4z \sqrt{(4-y^2)}\, dy\, dz = \int_{y=-2}^{2} \left[4 \frac{z^2}{2} \sqrt{(4-y^2)} \right]_{z=0}^{1} dy$$

$$= 2 \int_{y=-2}^{2} \sqrt{(4-y^2)}\, dy = 4 \int_{0}^{2} \sqrt{(4-y^2)}\, dy$$

$$= 4 \left[\frac{y}{2} \sqrt{(4-y^2)} + 2 \sin^{-1} \frac{y}{2} \right]_0^2 = 4\,[2 \sin^{-1} 1] = 4\,(2)\,\frac{\pi}{2} = 4\pi.$$

Ex. 34. *Find* $\iint_S \mathbf{A} \cdot \mathbf{n}\, dS$,

where $\mathbf{A} = (2x + 3z)\,\mathbf{i} - (xz + y)\,\mathbf{j} + (y^2 + 2z)\,\mathbf{k}$
and S is the surface of the sphere having centre at $(3, -1, 2)$ *and radius* 3. (Kakatiya 1990, Meerut 74)

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{A} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{A}\, dV.$$

Now $\operatorname{div} \mathbf{A} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}\{-(xz + y)\} + \frac{\partial}{\partial z}(y^2 + 2z)$

$= 2 - 1 + 2 = 3.$

$$\therefore \iint_S \mathbf{A} \cdot \mathbf{n}\, dS = \iiint_V 3\, dV = 3 \iiint_V dV = 3V.$$

But V is the volume of a sphere of radius 3. Therefore
$V = \frac{4}{3}\pi\,(3)^3 = 36\pi.$

$$\therefore \iint_S \mathbf{A} \cdot \mathbf{n}\, dS = 3V = 3 \times 36\pi = 108\pi.$$

Ex. 35. (a). *Apply divergence theorem to evaluate*

$$\iint_S [(x + z)\, dy\, dz + (y + z)\, dz\, dx + (x + y)\, dx\, dy]$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$. (Andhra 1989)

Sol. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x + z) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(x + y) \right] dV$$

$= \iiint_V 2\, dV = 2 \iiint_V dV = 2V$, where V is the volume of the sphere $x^2 + y^2 + z^2 = 4$

$$= 2 \left[\frac{4}{3}\pi\,(2)^3 \right] = \frac{64}{3}\pi.$$

Ex. 35. (b). *By using the Gauss divergence theorem evaluate* $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 4$. **(Osmania 1991)**

Sol. By Gauss divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV, \text{ where } V \text{ is the volume}$$

enclosed by the sphere $x^2 + y^2 + z^2 = 4$

$$= \iiint_V (1 + 1 + 1)\,dV = 3\iiint_V dV = 3V = 3\cdot\left[\frac{4}{3}\pi(2)^3\right]$$

$= 32\pi$.

Ex. 36. *If S is any closed surface enclosing a volume V and* $\mathbf{F} = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$, *prove that*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 6V.$$

(Rohilkhand 1980, Kanpur 79, Agra 78)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV = \iiint_V \operatorname{div}(x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k})\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= \iiint_V (1 + 2 + 3)\,dV = 6\iiint_V dV = 6V.$$

Ex. 37. *Evaluate*

$$\iint_S (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + z^2y^2\,\mathbf{k})\cdot\mathbf{n}\,dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.* **(Agra 1969, Bombay 66)**

Sol. By divergence theorem, we have

$$\iint_S (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + z^2y^2\,\mathbf{k})\cdot\mathbf{n}\,dS$$

$$= \iiint_V \operatorname{div}(y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + z^2y^2\,\mathbf{k})\,dV,$$

where V is the volume enclosed by S

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\,dV = 2\iiint_V zy^2\,dV.$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate this triple integral. In polars $dV = (dr)(r d\theta)(r \sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi$. Also $z = r\cos\theta,\ y = r\sin\theta\sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π. The triple integral is

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (r\cos\theta)(r^2\sin^2\theta\sin^2\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= 2\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^5 \sin^3\theta\cos\theta\sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2\cdot\frac{1}{6}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin^3\theta\cos\theta\sin^2\phi\, d\theta\, d\phi,$$

on integrating with respect to r.

[Note that the order of integration is immaterial because the limits of r, θ and ϕ are all constants].

$$= \frac{1}{3}\cdot\frac{2}{4\cdot 2}\int_0^{2\pi} \sin^2\phi\, d\phi, \text{ on integrating with respect to } \theta$$

$$= \frac{1}{12}\cdot 4\int_0^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{12}.$$

Ex. 38. *By converting the surface integral into a volume integral evaluate*

$$\iint_S (x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy),$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. **(Bombay 1970)**

Sol. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz,$$

where V is the volume enclosed by S.

Here $F_1 = x^3,\ F_2 = y^3,\ F_3 = z^3$.

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3(x^2 + y^2 + z^2).$$

$\therefore$ the given surface integral

$$= \iiint_V 3(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= 3\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^2 r^2 \sin\theta \, dr\, d\theta\, d\phi,$$ changing to polar shperical coordinates

$$= 3 \times 2\pi \times 2 \times \frac{1}{5} = \frac{12\pi}{5}.$$

Ex. 39. *Evaluate by divergence theorem the integral*

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy,$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{(a^2 - x^2 - y^2)}$ and $z = 0$. (Meerut 1974)

Sol. Here S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane and bounded by this plane.

Let V be the volume enclosed by S. By Gauss divergence theorem, we have

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2 y - z^3) + \frac{\partial}{\partial z}(2xy + y^2 z)\right] dV$$

$$= \iiint_V (z^2 + x^2 + y^2)\, dV. \qquad \ldots(1)$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate the triple integral (1). In polars, $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $dV = (dr)\,(r\, d\theta)\,(r\sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi$. Also $x^2 + y^2 + z^2 = r^2$. To cover V the limits of r will be 0 to a, those of θ will be 0 to $\pi/2$ and those of ϕ will be 0 to 2π. Hence the triple integral (1)

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^2 . r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \left[\frac{r^5}{5}\right]_{r=0}^{a} \sin\theta\, d\theta\, d\phi$$

$$= \frac{a^5}{5}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta\, d\phi = \frac{a^5}{5}\int_{\theta=0}^{\pi/2} \sin\theta . \left[\phi\right]_{\phi=0}^{2\pi} d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{\pi/2} \sin\theta\, d\theta = \frac{2\pi a^5}{5}\cdot 1 = \frac{2\pi a^5}{5}.$$

Ex. 40. *By using Gauss divergence theorem, evaluate*

$$\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k})\cdot\mathbf{n}\, dS$$

where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$. **(Agra 1973)**

Sol. Let V the volume enclosed by the closed surface S. Then by Gauss divergence theorem, we have

$$\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k})\cdot\mathbf{n}\, dS$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (2 + 2z)\, dV,$$ where V is the region bounded by the surfaces $z = 0,\ z = 1,\ z^2 = x^2 + y^2$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z}\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} (1 + z)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z} (1 + z)\left[x\right]_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z} (1 + z)\cdot 2\sqrt{(z^2 - y^2)}\, dy\, dz$$

$$= 8\int_{z=0}^{1}\int_{y=0}^{z} (1 + z)\sqrt{(z^2 - y^2)}\, dy\, dz$$

$$= 8\int_{z=0}^{1} (1 + z)\left[\frac{y}{2}\sqrt{(z^2 - y^2)} + \frac{z^2}{2}\sin^{-1}\left(\frac{y}{z}\right)\right]_{y=0}^{z} dz$$

$$= 8\int_0^1 (1 + z)\left[\frac{z^2}{2}\cdot\frac{\pi}{2}\right] dz = 2\pi\int_0^1 (z^2 + z^3)\, dz$$

$$= 2\pi\left[\frac{z^3}{3} + \frac{z^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} + \frac{1}{4}\right] = 2\pi\cdot\frac{7}{12} = \frac{7\pi}{6}.$$

Ex. 41. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ *over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if*

$$\mathbf{F} = 4xz\,\mathbf{i} + xyz^2\,\mathbf{j} + 3z\,\mathbf{k}.$$

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \operatorname{div}\mathbf{F}\, dV,$$

where V is the volume enclosed by S.

Here $\text{div } \mathbf{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z) = 4z + xz^2 + 3.$

Also V i : the region bounded by the surfaces

$$z = 0,\ z = 4,\ z^2 = x^2 + y^2.$$

Therefo e $\iiint_V \text{div } \mathbf{F}\, dV = \iiint_V (4z + xz^2 + 3)\, dx\, dy\, dz$

$$= \int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} (4z + xz^2 + 3)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=0}^{\sqrt{(z^2-y^2)}} (4z + 3)\, dx\, dy\, dz,$$

$$\text{since } \int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} x\, dx = 0$$

$$= 2\int_{z=0}^{4} \int_{y=-z}^{z} (4z + 3)\sqrt{(z^2 - y^2)}\, dy\, dz,$$

on integrating with respect to x

$$= 4\int_{z=0}^{4} \int_{y=0}^{z} (4z + 3)\sqrt{(z^2 - y^2)}\, dy\, dz$$

$$= 4\int_{z=0}^{4} (4z + 3)\left[\frac{y}{2}\sqrt{(z^2 - y^2)} + \frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 4\int_0^4 (4z + 3)\left[\frac{z^2}{2}\sin^{-1} 1\right] dz = \pi\int_0^4 (4z^3 + 3z^2)\, dz$$

$$= \pi\left[z^4 + z^3\right]_0^4 = \pi(256 + 64) = 320\pi.$$

Ex. 42. *Show that* $\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\, dS$ *vanishes where S denotes the surface of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. We have by divergence theorem

$$\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\, dS$$

$= \iiint_V \text{div}(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})\, dV$, where V is the volume enclosed by S

$$= \iiint_V (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= 2\int_{z=-c}^{c} \int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}} \int_{x=-a\sqrt{\{1-y^2/b^2-z^2/c^2\}}}^{a\sqrt{\{1-(y^2/b^2)-(z^2/c^2)\}}} (x + y + z)\, dx\, dy\, dz$$

$$= 4\int_{z=-c}^{c} \int_{y=-b\sqrt{\{1-(z^2/c^2)\}}}^{b\sqrt{\{1-(z^2/c^2)\}}} (y + z)\sqrt{\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)}\, dy\, dz,$$

on integrating with respect to x

[Note that $\int_{-a}^{a} f(x)\, dx = 0$ if $f(-x) = -f(x)$ and $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(-x) = f(x)$]

$$= 8\int_{z=-c}^{c} \int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} z\sqrt{\left(1 - \frac{z^2}{c^2} - \frac{y^2}{b^2}\right)}\, dy\, dz$$

$$= 8\int_{z=-c}^{c} \int_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} \frac{z}{b}\sqrt{\left\{b^2\left(1 - \frac{z^2}{c^2}\right) - y^2\right\}}\, dy\, dz$$

$$= \frac{8}{b}\int_{z=-c}^{c} z\left[\frac{y}{2}\sqrt{\left\{b^2\left(1 - \frac{z^2}{c^2}\right) - y^2\right\}} + \frac{b^2}{2}\left(1 - \frac{z^2}{c^2}\right)\sin^{-1}\frac{y}{b\sqrt{\{1 - (z^2/c^2)\}}}\right]_{y=0}^{b\sqrt{\{1-(z^2/c^2)\}}} dz$$

$$= \frac{8}{b}\int_{z=-c}^{c} z\left[\frac{b^2}{2}\left(1 - \frac{z^2}{c^2}\right)\sin^{-1} 1\right] dz$$

$$= \frac{8}{b}\int_{z=-c}^{c} z\,\frac{b^2}{2}\left(1 - \frac{z^2}{c^2}\right)\frac{\pi}{2}\, dz = 0.$$

Ex. 43. *Use divergence theorem to evaluate*

$$\iint_S [x\, dy\, dz + y\, dz\, dx + z\, dx\, dy],$$

where S is the surface $x^2 + y^2 + z^2 = 1$.

Sol. Let V be the volume bounded by the surface S of the sphere $x^2 + y^2 + z^2 = 1$. Then by Gauss divergence theorem, we have

$$\iint_S [x\, dy\, dz + y\, dz\, dx + z\, dx\, dy]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV$$

$= 3 \iiint_V dV = 3V$, where V is the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$ whose radius is 1

$$= 3 \cdot \tfrac{4}{3}\pi \cdot 1^3 = 4\pi.$$

Ex. 44. *Use divergence theorem to find* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ *for the vector* $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + 2z\,\mathbf{k}$ *over the sphere* $x^2 + y^2 + (z-1)^2 = 1$.

Sol. Here S is the surface of the sphere $x^2 + y^2 + (z-1)^2 = 1$ whose centre is the point $(0, 0, 1)$ and radius 1. Let V be the volume enclosed by S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\operatorname{div} \mathbf{F})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(2z)\right] dV$$

$$= \iiint_V (1 - 1 + 2)\, dV = 2 \iiint_V dV$$

$= 2V$, where V is the volume of the sphere $x^2 + y^2 + (z-1)^2 = 1$ whose radius is 1

$$= 2 \cdot \frac{4}{3}\pi . 1^3 = \frac{8}{3}\pi.$$

Ex. 45. *If* $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, *where a,b,c are constants, show that*

$$\iint_S (\mathbf{n} \cdot \mathbf{F})\, dS = \frac{4\pi}{3}(a + b + c),$$

S being the surface of the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$.

(Gauhati 1971)

Sol. By the divergence theorem, we have

$\iint_S (\mathbf{n} \cdot \mathbf{F})\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV$, where V is the volume enclosed by the sphere S whose radius is 1

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= \iiint_V (a + b + c)\, dV = (a + b + c)\, V = (a + b + c)\,\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is $\frac{4}{3}\pi(1)^3$

i.e., $\frac{4}{3}\pi$

$$= \frac{4}{3}\pi(a+b+c).$$

Ex. 46. *Verify the divergence theorem for*

$$\mathbf{F} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$$

taken over the region bounded by the surfaces $x^2 + y^2 = 4,\ z = 0, z = 3.$

(Allahabad 1978)

Sol. Let S denote the closed surface bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$, $z = 3$. Also let V be the volume bounded by the surface S. By Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \operatorname{div}\mathbf{F}\, dV.$$

We have $\iiint_V \operatorname{div}\mathbf{F}\, dV = \iiint_V \nabla\cdot\mathbf{F}\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\, dV$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (2 - 2y + z)\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=0}^{\sqrt{(4-x^2)}} (2 + z)\, dz\, dx\, dy$$

[$\because$ $2y$ is an odd function of y]

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\left[2y + zy\right]_{y=0}^{\sqrt{(4-x^2)}} dz\, dx$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}[2\sqrt{(4-x^2)} + z\sqrt{(4-x^2)}]\, dz\, dx$$

$$= 4\int_{x=-2}^{2}\left[2z\sqrt{(4-x^2)} + \frac{z^2}{2}\sqrt{(4-x^2)}\right]_{z=0}^{3} dx$$

$$= 4\int_{-2}^{2}\frac{21}{2}\sqrt{(4-x^2)}\, dx = 4\cdot\frac{21}{2}\cdot 2\int_0^2 \sqrt{(4-x^2)}\, dx$$

$$= 84\left[\frac{x}{2}\sqrt{(4-x^2)} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = 84\left[2\cdot\frac{\pi}{2}\right] = 84\pi.$$

Now we shall evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

For evaluating this surface integral give complete solution of solved example 61 on page 197.

Thus $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 84\pi.$

We see that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{F}\,dV.$

This completes the verification of Gauss divergence theorem.

Ex. 47. *Use Gauss divergence theorem to find* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ *and S is the closed surface in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$. **(Kanpur 1976)**

Sol. Let V be the volume enclosed by the closed surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{F}\,dV$$

$$= \iiint_V \nabla\cdot(2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2)\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)\right]dV$$

$$= \iiint_V (4xy - 2y + 8xz)\,dV,$$ where V is the volume in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0,\ x = 2$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\int_{y=0}^{\sqrt{(9-z^2)}} (2xy - y + 4xz)\,dx\,dz\,dy$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[xy^2 - \tfrac{1}{2}y^2 + 4xzy\right]_{x=0}^{\sqrt{(9-z^2)}} dx\,dz$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[x(9-z^2) - \tfrac{1}{2}(9-z^2) + 4xz\sqrt{(9-z^2)}\right]dx\,dz$$

$$= 2\int_{z=0}^{3}\left[\frac{x^2}{2}(9-z^2)-\frac{x}{2}(9-z^2)+2x^2z\sqrt{(9-z^2)}\right]_{x=0}^{2} dz$$

$$= 2\int_{0}^{3}\left[2(9-z^2)-(9-z^2)+8z\sqrt{(9-z^2)}\right] dz$$

$$= 2\int_{0}^{3}\left[9-z^2-4(-2z)(9-z^2)^{1/2}\right] dz$$

$$= 2\left[9z-\frac{z^3}{3}-4\cdot\frac{2}{3}(9-z^2)^{3/2}\right]_0^3$$

$$= 2\left[27-9+\tfrac{8}{3}\cdot 27\right] = 2(18+72) = 180.$$

Ex. 48. *Verify divergence theorem for the function* $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ *over the cylindrical region bounded by* $x^2 + y^2 = a^2$, $z = 0$ *and* $z = h$. **(Kanpur 1975; Allahabad 79)**

Sol. Let S denote the closed surface bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$, $z = h$. Also let V be the volume bounded by the surface S. By Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \operatorname{div}\mathbf{F}\, dV.$$

We have $\displaystyle\iiint_V \operatorname{div}\mathbf{F}\, dV = \iiint_V [\operatorname{div}(y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k})]\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)\right] dV = \iiint_V 2z\, dV$$

$$= 2\int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} z\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}} z\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\,\Big[y\Big]_{y=0}^{\sqrt{(a^2-x^2)}} dz\, dx$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\sqrt{(a^2-x^2)}\, dz\, dx$$

$$= 8\int_{z=0}^{h}\int_{x=0}^{a} z\sqrt{(a^2-x^2)}\, dz\, dx$$

$$= 8 \int_{x=0}^{a} \sqrt{(a^2 - x^2)} \left[\frac{z^2}{2}\right]_{z=0}^{h} dx$$

$$= 8 \int_0^a \frac{h^2}{2} \sqrt{(a^2 - x^2)}\, dx = 4h^2 \int_0^a \sqrt{(a^2 - x^2)}\, dx$$

$$= 4h^2 \left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4h^2 \left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right]$$

$$= \pi a^2 h^2. \qquad \ldots (1)$$

Now we shall evaluate the surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

The surface S consists of three surfaces: (i) the surface S_1 of the base of the cylinder *i.e.*, the plane face $z = 0$, (ii) the surface S_2 of the top face of the cylinder *i.e.*, the plane face $z = h$ and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 *i.e.*, $z = 0$, $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$, putting $z = 0$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_1 is obviously $-\mathbf{k}$.

$$\therefore \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (y\,\mathbf{i} + x\,\mathbf{j}) \cdot (-\mathbf{k})\, dS = 0.$$

For the surface S_2 *i.e.*, $z = h$, $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + h^2\,\mathbf{k}$, putting $z = h$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_2 is given by $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_2} (y\,\mathbf{i} + x\,\mathbf{j} + h^2\,\mathbf{k}) \cdot \mathbf{k}\, dS$$

$$= \iint_{S_2} h^2\, dS = h^2 \iint_{S_2} dS = h^2.\text{ area of the plane face } S_2 \text{ of the cylinder}$$

$$= h^2 . \pi a^2 = \pi a^2 h^2.$$

For the convex portion S_3 *i.e.*, $x^2 + y^2 = a^2$, a vector normal to S_3 is given by

$$\nabla (x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

$\therefore$ $\mathbf{n}$ = a unit vector along outward drawn normal at any point of S_3

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{a}, \text{ since } x^2 + y^2 = a^2 \text{ on } S_3.$$

$$\therefore \text{ on } S_3, \mathbf{F}\cdot\mathbf{n} = (y\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k})\cdot\left[\frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j})\right]$$

$$= \frac{1}{a}xy + \frac{1}{a}xy = \frac{2}{a}xy.$$

Also dS = elementary area on the surface S_3

$= a\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_3} \frac{2}{a}xy\,a\,d\theta\,dz, \text{ where}$$

$$x = a\cos\theta,\ y = a\sin\theta$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} 2a\cos\theta\, a\sin\theta\,d\theta\,dz$$

$$= 2a^2\int_{\theta=0}^{2\pi} \cos\theta\sin\theta\,\Big[z\Big]_{z=0}^{h} d\theta$$

$$= 2a^2h\int_0^{2\pi} \cos\theta\sin\theta\,d\theta = a^2h\int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= a^2h\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} = -\frac{a^2h}{2}[\cos 4\pi - \cos 0] = 0.$$

Hence the total surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 0 + \pi a^2h^2 + 0 = \pi a^2h^2. \quad \ldots(2)$$

From (1) and (2), we see that

$$\iiint_V \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

This verifies divergence theorem.

Ex. 49. *If* $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz + z^2)\,\mathbf{k}$, *evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ *where* S *is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

Sol. The surface $x^2+y^2+z^2=16$ meets the plane $z=0$ in a circle C given by $x^2+y^2=16, z=0$. Let S_1 be the plane region bounded by the circle C. If S prima and S_1, then S' is a closed surface. Let V be the region bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n}=-\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

Now by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS = 0 \qquad \text{[See Ex. 1 page233]}$$

or $$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS = 0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS - \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\, dS = 0$$

[$\because$ on S_1, $\mathbf{n}=-\mathbf{k}$]

or $$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\, dS.$$

Now $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= 0\,\mathbf{i} - z\,\mathbf{j} + (3y-1)\,\mathbf{k} = -z\,\mathbf{j} + (3y-1)\,\mathbf{k}.$$

$$\therefore \text{curl } \mathbf{F}\cdot\mathbf{k} = \{-z\,\mathbf{j} + (3y-1)\,\mathbf{k}\}\cdot\mathbf{k} = 3y-1.$$

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} (3y-1)\, dS$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} (3r\sin\theta - 1)\, r\, d\theta\, dr, \qquad \text{changing to polars}$$

[Note that S_1 is a circle in xy-plane with centre origin and radius 4]

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4} 3r^2\sin\theta\, d\theta\, dr - \int_{\theta=0}^{2\pi}\int_{r=0}^{4} r\, d\theta\, dr$$

$$= 0 - \int_{\theta=0}^{2\pi} \left[\frac{r^2}{2}\right]_0^4 d\theta \qquad \left[\because \int_{\theta=0}^{2\pi} \sin\theta\, d\theta = 0\right]$$

$$= -8\left[\theta\right]_0^{2\pi} = -16\pi.$$

IAS-2012

Ex. 50. *If* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *above the xy-plane.* **(Kanpur 1980)**

Sol. The surface $x^2 + y^2 + z^2 = a^2$ meets the plane $z = 0$ in a circle C given by $x^2 + y^2 = a^2$, $z = 0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If $\mathbf{n}$ denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n} = -\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iiint_V [\text{div}(\text{curl } \mathbf{F})]\, dV$$

$$= 0. \qquad [\because \text{div}(\text{curl } \mathbf{F}) = 0]$$

$$\therefore \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$

$$[\because S' \text{ consists of } S \text{ and } S_1]$$

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS - \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dS = 0$$

$$[\because \text{on } S_1, \mathbf{n} = -\mathbf{k}]$$

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dS.$$

Now $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x - 2xz)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(y)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(x - 2xz) - \frac{\partial}{\partial y}(y)\right]$$

$= x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}.$

$\therefore$ (curl $\mathbf{F}$)$\cdot\mathbf{k} = (x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k})\cdot\mathbf{k}$

$= -2z = 0$ over the surface S_1 bounded by the circle $x^2 + y^2 = a^2$, $z = 0$.

Hence $\iint_S$ (curl $\mathbf{F}$)$\cdot\mathbf{n}\,dS = \iint_{S_1} 0\,dS = 0.$

Ex. 51. *Evaluate* $\iint_S (\nabla \times \mathbf{A})\cdot\mathbf{n}\,dS$, *where* $\mathbf{A} = [xye^z + \log(z+1) - \sin x]\,\mathbf{k}$ *and S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *above the xy-plane.*

Sol. Proceed as in solved example 50.

Here curl $\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & xye^z + \log(z+1) - \sin x \end{vmatrix}$

$$= \mathbf{i}\frac{\partial}{\partial y}\left[xye^z + \log(z+1) - \sin x\right]$$

$$- \mathbf{j}\frac{\partial}{\partial x}\left[xye^z + \log(z+1) - \sin x\right]$$

$= xe^z\,\mathbf{i} - (ye^z - \cos x)\,\mathbf{j}.$

$\therefore$ (curl $\mathbf{A}$)$\cdot\mathbf{k} = [xe^z\,\mathbf{i} - (ye^z - \cos x)\,\mathbf{j}]\cdot\mathbf{k} = 0.$

$\therefore$ (curl $\mathbf{A}$)$\cdot\mathbf{k} = 0$ over the surface S_1.

Hence $\iint_S$ (curl $\mathbf{A}$)$\cdot\mathbf{n}\,dS = \iint_{S_1} 0\,dS = 0.$

Ex. 52. *Evaluate* $\iint_S (\nabla \times \mathbf{A})\cdot\mathbf{n}\,dS$, *where* $\mathbf{A} = (x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$ *and S is the surface of the cone* $z = 2 - \sqrt{(x^2 + y^2)}$ *above the xy-plane.* **(Meerut 1974)**

Solution. The surface $z = 2 - \sqrt{(x^2 + y^2)}$ meets the xy-plane in a circle C given by $x^2 + y^2 = 4$, $z = 0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces

S and S_1, then S' is a closed surface. By application of divergence theorem, we have

$$\iint_{S'} \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS = 0 \qquad \text{[See Ex. 1 page233]}$$

or $$\iint_{S} \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS + \iint_{S_1} \text{curl}\,A\cdot\mathbf{n}\,dS = 0$$

or $$\iint_{S} \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl}\,A\cdot\mathbf{k}\,dS \quad [\because \text{ on } S_1, \mathbf{n} = -\mathbf{k}]$$

Now $$\text{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - z & x^3 + yz & -3xy^2 \end{vmatrix}$$

$$= \mathbf{i}\,(-6xy - y) + \mathbf{j}\,(-1 + 3y^2) + \mathbf{k}\,(3x^2 - 0).$$

$\therefore$ curl $\mathbf{A}\cdot\mathbf{k} = 3x^2$.

$$\therefore \iint_{S} \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} 3x^2\,dS$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\cos^2\theta\, r\,d\theta\,dr, \text{ changing to polars}$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\cos^2\theta\,d\theta\,dr = 3\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2 \cos^2\theta\,d\theta$$

$$= 12\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= 12\times 4\int_0^{\pi/2}\cos^2\theta\,d\theta = 48\times\frac{1}{2}\times\frac{\pi}{2} = 12\pi.$$

Ex. 53. *Evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, *where*

$\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz + z^2)\,\mathbf{k}$ *and* S *is the surface of the paraboloid* $z = 4 - (x^2 + y^2)$ *above the xy-plane.*

Sol. The surface $z = 4 - (x^2 + y^2)$ meets the plane $z = 0$ in a circle C given by $x^2 + y^2 = 4$, $z = 0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If $\mathbf{n}$ denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n} = -\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iiint_V [\text{div curl } \mathbf{F}] \, dV = 0,$$

since div curl $\mathbf{F} = 0$.

$$\therefore \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS + \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = 0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS - \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dS = 0$$

[$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dS.$$

Now $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= \mathbf{i} \left[\frac{\partial}{\partial y} (2xz + z^2) - \frac{\partial}{\partial z} (3xy) \right] - \mathbf{j} \left[\frac{\partial}{\partial x} (2xz + z^2) - \frac{\partial}{\partial z} (x^2 + y - 4) \right] + \mathbf{k} \left[\frac{\partial}{\partial x} (3xy) - \frac{\partial}{\partial y} (x^2 + y - 4) \right]$$

$$= 0\,\mathbf{i} - 2z\,\mathbf{j} + (3y - 1)\,\mathbf{k}.$$

$$\therefore (\text{curl } \mathbf{F}) \cdot \mathbf{k} = [-2z\,\mathbf{j} + (3y - 1)\,\mathbf{k}] \cdot \mathbf{k}$$

$= 3y - 1$ over the surface S_1 bounded by the circle

$x^2 + y^2 = 4$, $z = 0$.

Hence $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_{S_1} (3y - 1) \, dS$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (3y - 1) \, dx \, dy$$

$$= 2 \int_{x=-2}^{2} \int_{y=0}^{\sqrt{(4-x^2)}} (-1) \, dx \, dy$$ [$\because$ $3y$ is an odd function of y]

$$= -2\int_{x=-2}^{2}\left[y\right]_{y=0}^{\sqrt{(4-x^2)}} dx$$

$$= -2\int_{-2}^{2}\sqrt{(4-x^2)}\,dx = -4\int_{0}^{2}\sqrt{(4-x^2)}\,dx$$

$$= -4\left[\frac{x}{2}\sqrt{(4-x^2)} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = -4\left[2\cdot\frac{\pi}{2}\right] = -4\pi.$$

Ex. 54. *Evaluate* $\iint_S (ax^2 + by^2 + cz^2)\,dS$ *over the sphere* $x^2 + y^2 + z^2 = 1$ *using the divergence theorem.*

Solution. Let us first put the integral

$\iint_S (ax^2 + by^2 + cz^2)\,dS$ in the form

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is unit normal vector to S.

The normal vector to $\phi(x, y, z) \equiv x^2 + y^2 + z^2 - 1 = 0$ is

$$= \nabla\phi = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

$$\therefore \quad \mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{[4(x^2 + y^2 + z^2)]}}$$

$$= x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \qquad [\because \; x^2 + y^2 + z^2 = 1, \text{ on } S]$$

Now we are to choose $\mathbf{F}$ such that

$$\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = ax^2 + by^2 + cz^2.$$

Obviously $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$.

Now $\iint_S (ax^2 + by^2 + cz^2)\,dS$

$$= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS, \text{ where } \mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$$

$$= \iiint_V \operatorname{div}\mathbf{F}\,dV, \text{ by divergence theorem}$$

$$= \iiint_V (a + b + c)\,dV \qquad [\because \; \operatorname{div}\mathbf{F} = a + b + c]$$

$$= (a + b + c)\iiint_V dV = (a + b + c)V$$

$= (a + b + c)\frac{4}{3}\pi$, since the volume V enclosed by the sphere S of unit radius is $\frac{4}{3}\pi$.

Ex. 55. *Compute*

(i) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$, *and*

(ii) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol. (i). Let us first put the integral

$$\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS \text{ in the form}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is a unit normal vector to the closed surface S whose equation is $ax^2 + by^2 + cz^2 = 1$.

The normal vector to $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 - 1 = 0$ is $= \nabla\phi = 2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}$.

$$\therefore\ \mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}}{\sqrt{(4a^2x^2 + 4b^2y^2 + 4c^2z^2)}} = \frac{ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}.$$

Now we are to choose $\mathbf{F}$ such that $\mathbf{F} \cdot \mathbf{n} = \sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}$.

Obviously $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$.

Now $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$

$= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$

$= \iiint_V \operatorname{div} \mathbf{F}\, dV$, by divergence theorem; V is the volume enclosed by the closed surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= (a + b + c) \iiint_V dV = (a + b + c)\, V$$

$$= (a + b + c) \cdot \tfrac{4}{3}\pi \cdot \left(\frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}}\right) = \frac{4\pi}{3} \frac{(a + b + c)}{\sqrt{(abc)}}.$$

Note that the equation of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ can be written as $\dfrac{x^2}{1/a} + \dfrac{y^2}{1/b} + \dfrac{z^2}{1/c} = 1$ and the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ is $\tfrac{4}{3}\pi\, abc$.

(ii) Proceed as in part (i) of this question.
Here we are to choose **F** such that
$\mathbf{F}\cdot\mathbf{n} = 1/\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}$ on S.
Obviously $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, because then

$$\mathbf{F}\cdot\mathbf{n} = \frac{ax^2 + by^2 + cz^2}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}} = \frac{1}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}} \text{ on } S.$$

Note that on S, $ax^2 + by^2 + cz^2 = 1$.

Now $\iint_S \frac{1}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}\,dS$

$= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$= \iiint_V (\nabla\cdot\mathbf{F})\,dV$, by divergence theorem; V is the volume enclosed by the surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV = \iiint_V 3\,dV$$

$$= 3V = 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}} = \frac{4\pi}{\sqrt{(abc)}}.$$

Ex. 56. *Evaluate* $\iint_S (x^2 + y^2)\,dS$, *where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.*

Sol. Let S be the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by the planes $z = 0$ and $z = 3$. The plane $z = 3$ cuts the surface $z^2 = 3(x^2 + y^2)$ in the circle $x^2 + y^2 = 3$, $z = 3$. Let S_1 be the plane region bounded by this circle. Let S' be the closed surface consisting of the surfaces S and S_1.

Let us first put the integral

$\iint_S (x^2 + y^2)\,dS$ in the form

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where **n** is a unit vector along the outward drawn normal to the surface S whose equation is $\phi(x, y, z) \equiv 3(x^2 + y^2) - z^2 = 0$.

We have $\mathbf{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{6x\,\mathbf{i} + 6y\,\mathbf{j} - 2z\,\mathbf{k}}{\sqrt{(36x^2 + 36y^2 + 4z^2)}}$

$$= \frac{3x\,\mathbf{i} + 3y\,\mathbf{j} - z\,\mathbf{k}}{\sqrt{[9(x^2+y^2)+z^2]}} = \frac{3x\,\mathbf{i} + 3y\,\mathbf{j} - z\,\mathbf{k}}{\sqrt{(3z^2+z^2)}},$$

since on S, $3(x^2+y^2) = z^2$

$$= \frac{3x\,\mathbf{i} + 3y\,\mathbf{j} - z\,\mathbf{k}}{2}.$$

Now take $\mathbf{F} = \frac{2z}{3}(x\,\mathbf{i} + y\,\mathbf{j})$. Then on S, $\mathbf{F}\bullet\mathbf{n} = x^2 + y^2$.

By Gauss divergence theorem, we have

$$\iint_{S'} \mathbf{F}\bullet\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV, \qquad \dots (1)$$

where V is the volume enclosed by the closed surface S'.

We have $\operatorname{div}\mathbf{F} = \operatorname{div}\left(\frac{2}{3}zx\,\mathbf{i} + \frac{2}{3}zy\,\mathbf{j}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2}{3}zx\right) + \frac{\partial}{\partial y}\left(\frac{2}{3}zy\right) = \frac{2}{3}z + \frac{2}{3}z = \frac{4}{3}z.$$

$\therefore \iiint_V \operatorname{div}\mathbf{F}\,dV = \iiint_V \frac{4}{3}z\,dV,$ where V is the volume bounded by $z = 0$, $z = 3$ and $x^2 + y^2 = z^2/3$

$$= \frac{4}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=-\sqrt{\{(z^2/3)-y^2\}}}^{\sqrt{\{(z^2/3)-y^2\}}} z\,dz\,dy\,dx$$

$$= \frac{4}{3}\cdot 2\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=0}^{\sqrt{\{(z^2/3)-y^2\}}} z\,dz\,dy\,dx$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}} z\left[x\right]_{x=0}^{\sqrt{\{(z^2/3)-y^2\}}} dz\,dy$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}} z\sqrt{\{(z^2/3)-y^2\}}\,dz\,dy$$

$$= 2\cdot\frac{8}{3}\int_{z=0}^{3}\int_{y=0}^{z/\sqrt{3}} z\sqrt{\left(\frac{z^2}{3}-y^2\right)}\,dz\,dy$$

$$= \frac{16}{3}\int_{z=0}^{3} z\left[\frac{y}{2}\sqrt{\left(\frac{z^2}{3}-y^2\right)} + \frac{z^2}{6}\sin^{-1}\left(\frac{y}{z/\sqrt{3}}\right)\right]_{y=0}^{z/\sqrt{3}} dz$$

$$= \frac{16}{3}\int_0^3 z\left[0 + \frac{z^2}{6}\cdot\sin^{-1}1\right]dz = \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\int_0^3 z^3\,dz$$

$$= \frac{16}{3} \cdot \frac{1}{6} \cdot \frac{\pi}{2} \left[\frac{z^4}{4} \right]_0^3 = \frac{16}{3} \cdot \frac{1}{6} \cdot \frac{\pi}{2} \cdot \frac{81}{4} = 9\pi. \qquad \dots (2)$$

Also $\iint_{S'} (\mathbf{F} \cdot \mathbf{n})\, dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS$

[$\because$ S' consists of S and S_1]

$$= \iint_S (x^2 + y^2)\, dS + \iint_{S_1} \frac{2.3}{3} (x\,\mathbf{i} + y\,\mathbf{j}) \cdot \mathbf{k}\, dS,$$

since on S_1, $\mathbf{n} = \mathbf{k}$, $z = 3$

$$= \iint_S (x^2 + y^2)\, dS + 0 = \iint_S (x^2 + y^2)\, dS \qquad \dots (3)$$

From (1), (2) and (3), we have

$$\iint_S (x^2 + y^2)\, dS = 9\pi.$$

Ex. 57. *Prove that*

$$\int_V \mathbf{f} \cdot \mathit{curl}\, \mathbf{F}\, dV = \int_S \mathbf{F} \times \mathbf{f} \cdot d\mathbf{S} + \int_V \mathbf{F} \cdot \mathit{curl}\, \mathbf{f}\, dV.$$

Sol. We have $\int_S \mathbf{F} \times \mathbf{f} \cdot d\mathbf{S} = \int_S (\mathbf{F} \times \mathbf{f}) \cdot \mathbf{n}\, dS$,

where $\mathbf{n}$ is a unit normal vector to the surface S

$$= \int_V [\nabla \cdot (\mathbf{F} \times \mathbf{f})]\, dV, \text{ by Gauss divergence theorem}$$

$$= \int_V (\mathbf{f} \cdot \text{curl}\, \mathbf{F} - \mathbf{F} \cdot \text{curl}\, \mathbf{f})\, dV$$

$$= \int_V \mathbf{f} \cdot \text{curl}\, \mathbf{F}\, dV - \int_V \mathbf{F} \cdot \text{curl}\, \mathbf{f}\, dV.$$

$$\therefore \int_V \mathbf{f} \cdot \text{curl}\, \mathbf{F}\, dV = \int_S \mathbf{F} \times \mathbf{f} \cdot d\mathbf{S} + \int_V \mathbf{F} \cdot \text{curl}\, \mathbf{f}\, dV.$$

Stoke's theorem: Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let F (x, y, z) be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior.

Then $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS = \iint_S (curl\ \mathbf{F})\cdot d\mathbf{S}$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal **n** *to S, has the surface on the left.*

(Meerut 85; Rohilkhand 90; Osmania 89; Kakatiya 90, 92; Tirupati 89, 93)

Note. $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \left(\mathbf{F}\cdot \frac{d\mathbf{r}}{ds}\right) ds = \oint_C (\mathbf{F}\cdot \mathbf{t})\, ds$, where **t** is unit tangent vector to C. Therefore $\mathbf{F}\cdot\mathbf{t}$ is the component of **F** in the direction of the tangent vector of C. Also $(\nabla \times \mathbf{F})\cdot\mathbf{n}$ is the component of curl **F** in the direction of outward drawn normal vector **n** of S. Therefore in words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector **F** *taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of* **F** *taken over any surface S having C as its boundary.*

Cartesian equivalent of Stoke's theorem.

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Let outward drawn normal vector **n** of S make angles α, β, γ with positive directions of x, y, z axes.

Then $\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$.

Also $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore (\nabla \times \mathbf{F})\cdot\mathbf{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma.$$

Also $\mathbf{F}\cdot d\mathbf{r} = (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= F_1\,dx + F_2\,dy + F_3\,dz.$$

$\therefore$ Stoke's theorem can be written as

$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Proof of Stoke's theorem. Let S be a surface which is such that its projections on the xy, yz and zx planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in the forms

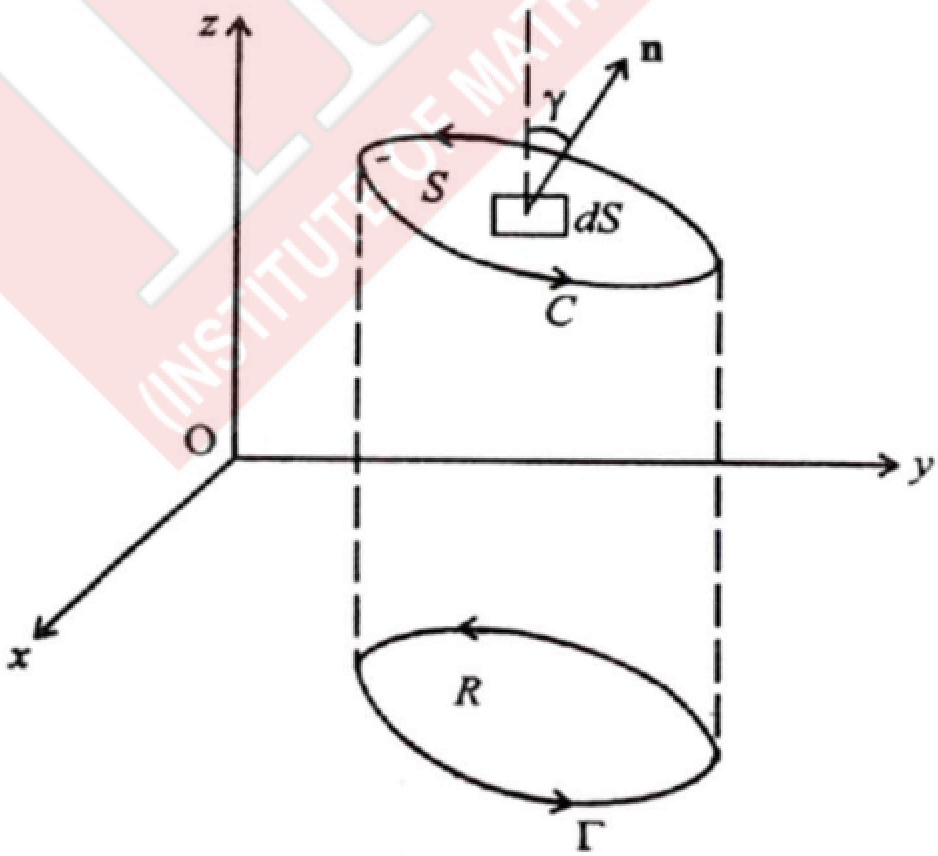
$$z = f(x,y),\ y = g(x,z),\ x = h(z,y)$$

where f, g, h are continuous functions and have continuous first partial derivatives.

Consider the integral $\iint_S [\nabla \times (F_1\mathbf{i})]\cdot\mathbf{n}\,dS.$

We have $\nabla \times (F_1\mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$

$$= \frac{\partial F_1}{\partial z}\mathbf{j} - \frac{\partial F_1}{\partial y}\mathbf{k}.$$



$$\therefore [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n} = \left(\frac{\partial F_1}{\partial z} \mathbf{j} \cdot \mathbf{n} - \frac{\partial F_1}{\partial y} \mathbf{k} \cdot \mathbf{n} \right) = \frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma.$$

$$\therefore \iint_S [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n} \, dS = \iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) dS.$$

We shall prove that

$$\iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) dS = \oint_C F_1 \, dx.$$

Let R be the orthogonal projection of S on the xy-plane and let Γ be its boundary which is oriented as shown in the figure. Using the representation $z = f(x, y)$ of S, we may write the line integral over C as a line integral over Γ. Thus

$$\oint_C F_1 (x, y, z) \, dx = \oint_\Gamma F_1 [x, y, f(x, y)] \, dx$$

$$= \oint_\Gamma \{F_1 [x, y, f(x, y)] \, dx + 0 \, dy\}$$

$$= - \iint_R \frac{\partial F_1}{\partial y} \, dx \, dy, \quad \text{by Green's theorem in plane for the region } R.$$

But $$\frac{\partial F_1 [x, y, f(x, y)]}{\partial y} = \frac{\partial F_1 (x, y, z)}{\partial y} + \frac{\partial F_1 (x, y, z)}{\partial z} \frac{\partial f}{\partial y}$$

$$[\because \quad z = f(x, y)]$$

$$\therefore \oint_C F_1 (x, y, z) \, dx = - \iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial f}{\partial y} \right) dx \, dy \qquad \dots (1)$$

Now the equation $z = f(x, y)$ of the surface S can be written as

$$\phi (x, y, z) \equiv z - f(x, y) = 0.$$

We have grad $\phi = - \dfrac{\partial f}{\partial x} \mathbf{i} - \dfrac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}$.

Let $|\text{grad } \phi| = a$.

Since grad ϕ is normal to S, therefore, we get

$$\mathbf{n} = \pm \frac{\text{grad } \phi}{a}.$$

But the components of both $\mathbf{n}$ and grad ϕ in positive direction of z-axis are positive. Therefore

$$\mathbf{n} = + \frac{\text{grad } \phi}{a}$$

or $\quad \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\mathbf{i} - \frac{1}{a}\frac{\partial f}{\partial y}\mathbf{j} + \frac{1}{a}\mathbf{k}.$

$\therefore \cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}.$

Now $\quad dS = \frac{dx\,dy}{\cos\gamma} = a\,dx\,dy.$

$$\therefore \quad \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$

$$= \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{1}{a}\right] a\,dx\,dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy. \qquad \dots (2)$$

From (1) and (2), we get

$$\oint_C F_1\,dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$

$$= \iint_S [\nabla \times (F_1\,\mathbf{i})]\cdot\mathbf{n}\,dS \qquad \dots (3)$$

Similarly, by projections on the other coordinate planes, we get

$$\oint_C F_2\,dy = \iint_S [\nabla \times (F_2\,\mathbf{j})]\cdot\mathbf{n}\,dS \qquad \dots (4)$$

$$\oint_C F_3\,dz = \iint_S [\nabla \times (F_3\,\mathbf{k})]\cdot\mathbf{n}\,dS \qquad \dots (5)$$

Adding (3), (4), (5), we get

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S [\nabla \times (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})]\cdot\mathbf{n}\,dS$$

or $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS.$$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's theorem will be true provided S can be subdivided into surfaces $S_1, S_2, \dots, S_k$ with boundaries $C_1, C_2, \dots, C_k$ which do satisfy the restrictions. Stoke's theorem holds for each such surface. The sum of surface integrals over $S_1, S_2, \dots, S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, \dots, C_k$ will give us line integral over C.

Note. Green's theorem in plane is a special case of Stoke's theorem. If R is a region in the xy-plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dR = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

This is nothing but a special case of Stoke's theorem because here $\mathbf{k} = \mathbf{n}$ = outward drawn unit normal to the surface of region R.

Solved Examples

Ex. 1. *Prove that* $\oint_C \mathbf{r} \cdot d\mathbf{r} = 0.$ **(Andhra 1989)**

Sol. By Stoke's theorem

$$\oint_C \mathbf{r} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{r}) \cdot \mathbf{n}\, dS = 0, \text{ since curl } \mathbf{r} = \mathbf{0}.$$

Ex. 2. *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}.$

Sol. By Stoke's theorem, we have

$$\oint_C \nabla (\phi\psi) \cdot d\mathbf{r} = \iint_S [\text{curl grad } (\phi\psi)] \cdot \mathbf{n}\, dS$$
$$= 0, \quad \text{since curl grad } (\phi\psi) = \mathbf{0}.$$

But $\nabla (\phi\psi) = \phi \nabla \psi + \psi \nabla \phi$.

$$\therefore \quad \oint_C (\phi \nabla \psi + \psi \nabla \phi) \cdot d\mathbf{r} = 0$$

or

$$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = -\oint_C \psi \nabla \phi \cdot d\mathbf{r}.$$

Ex. 3. (a) *Prove that* $\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n}\, dS.$

Sol. By Stoke's theorem, we have

$$\oint_C \phi \nabla \psi \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi \nabla \psi)] \cdot \mathbf{n}\, dS$$
$$= \iint_S [\nabla \phi \times \nabla \psi + \phi \text{ curl grad } \psi] \cdot \mathbf{n}\, dS$$
$$= \iint_S [\nabla \phi \times \nabla \psi] \cdot \mathbf{n}\, dS, \text{ since curl grad } \psi = \mathbf{0}.$$

Ex. 3 (b) *Show that* $\int_C \phi \nabla \phi \cdot d\mathbf{r} = 0$, *C being a closed curve.*

Sol. Applying Stoke's theorem to the vector function $\phi \nabla \phi$, we have

$$\int_C (\phi \nabla \phi) \cdot d\mathbf{r} = \iint_S [\text{curl } (\phi \nabla \phi)] \cdot \mathbf{n}\, dS$$

$$= \iint_S [\phi \operatorname{curl} \nabla\phi + \nabla\phi \times \nabla\phi] \cdot \mathbf{n}\, dS$$

$$= \iint_S \mathbf{0} \cdot \mathbf{n}\, dS \qquad [\because \operatorname{curl} \nabla\phi = \mathbf{0} \text{ and } \nabla\phi \times \nabla\phi = \mathbf{0}]$$

$$= 0.$$

Ex. 4. *Prove that* $\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$ **(Kanpur 1977)**

Sol. Let $\mathbf{A}$ be any arbitrary constant vector. Let $\mathbf{F} = \phi\mathbf{A}$. Applying Stoke's theorem for $\mathbf{F}$, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi\mathbf{A})] \cdot \mathbf{n}\, dS = \iint_S [\nabla\phi \times \mathbf{A} + \phi \operatorname{curl} \mathbf{A}] \cdot d\mathbf{S}$$

$$= \iint_S (\nabla\phi \times \mathbf{A}) \cdot d\mathbf{S}, \qquad \text{since curl } \mathbf{A} = \mathbf{0}.$$

$$\therefore \oint_C (\phi\mathbf{A}) \cdot d\mathbf{r} = \iint_S \mathbf{A} \cdot (d\mathbf{S} \times \nabla\phi)$$

or $\mathbf{A} \cdot \oint_C \phi\, d\mathbf{r} = \mathbf{A} \cdot \iint_S d\mathbf{S} \times \nabla\phi$ or $\mathbf{A} \cdot \left[\oint_C \phi\, d\mathbf{r} - \iint_S d\mathbf{S} \times \nabla\phi\right] = 0.$

Since $\mathbf{A}$ is an arbitrary vector, therefore we must have

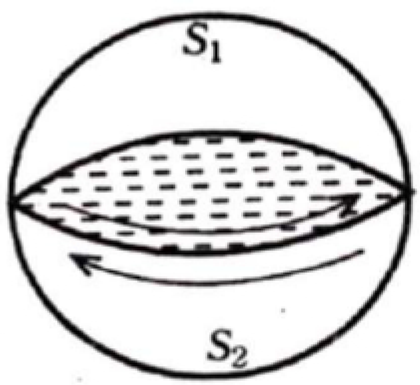
$$\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$$

Ex. 5. *By Stoke's theorem prove that div curl* $\mathbf{F} = 0$.

Sol. Let V be any volume enclosed by a closed surface. Then by divergence theorem

$$\iiint_V \nabla \cdot (\operatorname{curl} \mathbf{F})\, dV$$

$$= \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS.$$



Divide the surface S into two portions S_1 and S_2 by a closed curve C. Then

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS_1$$

$$+ \iint_{S_2} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS_2. \qquad \ldots (1)$$

By Stoke's theorem right hand side of (1) is

$$= \oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \quad \iiint_V \nabla \cdot (\text{curl } \mathbf{F})\, dV = 0.$$

Now this equation is true for all volume elements V. Therefore we have $\nabla \cdot (\text{curl } \mathbf{F}) = 0$

or $\quad$ div curl $\mathbf{F} = 0$.

Ex. 6. *By Stoke's theorem prove that curl grad* $\phi = \mathbf{0}$.

Solution. Let S be any surface enclosed by a simple closed curve C. Then by Stoke's theorem, we have

$$\iint_S (\text{curl grad } \phi) \cdot \mathbf{n}\, dS = \oint_C \text{grad } \phi \cdot d\mathbf{r}.$$

Now $\text{grad } \phi \cdot d\mathbf{r} = \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi.$$

$\therefore \oint_C \text{grad } \phi \cdot d\mathbf{r} = \oint_C d\phi = \left[\phi\right]_A^A$, where A is any point on C

$= 0.$

Therefore we have $\iint_S (\text{curl grad } \phi) \cdot \mathbf{n}\, dS = 0$.

Now this equation is true for all surface elements S.

Therefore we have, curl grad $\phi = \mathbf{0}$.

Ex. 7. (a). *Verify Stoke's theorem for* $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ *where* S *is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and* C *is its boundary.* **(Meerut 1981; Rohilkhand 91; Agra 70; Andhra 92)**

Sol. The boundary C of S is a circle in the xy-plane of radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equation of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (y\mathbf{i} + z\mathbf{j} + x\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$= \oint_C (y\,dx + z\,dy + x\,dz) = \oint_C y\,dx$, since on C, $z = 0$ and $dz = 0$

$$= \int_0^{2\pi} \sin t \frac{dx}{dt} dt = \int_0^{2\pi} -\sin^2 t\, dt$$

$$= -\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt = -\frac{1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\pi. \qquad \ldots (1)$$

Now let us evaluate $\iint_S$ curl $\mathbf{F}\cdot\mathbf{n}\,dS$. We have curl $\mathbf{F}$

$$= \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\,dS \quad \text{[See Ex. 50 page 267]}$$

$$= \iint_{S_1}(-\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot\mathbf{k}\,dS = \iint_{S_1}(-1)\,dS = -\iint_{S_1} dS = -S_1.$$

But S_1 = area of a circle of radius $1 = \pi\,(1)^2 = \pi$.

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS = -\pi. \qquad \ldots (2)$$

Hence from (1) and (2), the theorem is verified.

Ex. 7 (b). *Verify Stoke's theorem for*

$$\mathbf{F} = (x^2+y-4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\,\mathbf{k}$$

where S is the upper half of the sphere $x^2+y^2+z^2=16$ and C is its boundary. **(Osmania 1991)**

Sol. The boundary C of S is the circle $x^2+y^2=16, z=0$ lying in the xy-plane. Suppose $x=4\cos t$, $y=4\sin t$, $z=0$, $0\le t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \{(x^2+y-4)\,\mathbf{i} + 3xy\,\mathbf{j}$$

$$+ (2xz+z^2)\,\mathbf{k}\}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy + (2xz+z^2)\,dz\right]$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy\right], \text{ since on } C,\ z=0 \text{ and } dz=0$$

$$= \int_0^{2\pi}\left[(x^2+y-4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi} \Big[(16\cos^2 t + 4\sin t - 4)(-4\sin t) + 3\cdot 16 \sin t \cos t \cdot 4\cos t\Big]\, dt$$

$$= 128 \int_0^{2\pi} \cos^2 t \sin t\, dt - 16 \int_0^{2\pi} \sin^2 t\, dt + 16 \int_0^{2\pi} \sin t\, dt$$

$$= 128.0 - 16.4 \int_0^{\pi/2} \sin^2 t\, dt + 16.0 = -64 \cdot \tfrac{1}{2} \cdot \frac{\pi}{2} = -16\pi \quad \ldots (1)$$

$$\left[\text{Note that } \int_0^{2a} f(x)\, dx = 0, \text{ if } f(2a - x) = -f(x) \text{ and } = 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

Now let us evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$. We have curl $\mathbf{F}$

$$= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= 0\,\mathbf{i} - z\,\mathbf{j} + (3y - 1)\,\mathbf{k} = -z\mathbf{j} + (3y-1)\,\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of Gauss divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k}\, dS \quad \text{[See Ex. 50 page 267]}$$

$$= \iint_{S_1} \{-z\mathbf{j} + (3y-1)\,\mathbf{k}\} \cdot \mathbf{k}\, dS = \iint_{S_1} (3y - 1)\, dS$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (3r\sin\theta - 1)\, r\, d\theta\, dr, \text{ changing to polars}$$

[Note that S_1 is a circle in xy-plane with centre origin and radius 4]

$$= -16\pi. \quad \ldots (2)$$

[See Ex.49 pag 265]

Hence from (1) and (2), Stoke's theorem is verified.

Ex. 8. *Verify Stoke's theorem for* $\mathbf{F} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}$, *where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.*

(Agra 1960; Rohilkhand 78; Allahabad 78; Kanpur 70; Osmania 89, 91)

Sol. The boundary C of S is a circle in the xy-plane of radius unity and centre origin. Suppose $x = \cos t,\ y = \sin t,\ z = 0,\ 0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C [(2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\,\mathbf{k}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$$

$$= \oint_C (2x-y)\,dx, \text{ since } z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}dt = -\int_0^{2\pi}(2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi}[\sin 2t - \tfrac{1}{2}(1-\cos 2t)]\,dt = -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -[(-\tfrac{1}{2}+\tfrac{1}{2}) - \tfrac{1}{2}(\pi - 0) + \tfrac{1}{4}(0-0)] = \pi. \qquad \ldots (1)$$

$$\text{And } (\nabla\times\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{i} - (0-0)\,\mathbf{j} + (0+1)\,\mathbf{k} = \mathbf{k}$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

$\therefore$ by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = 0 \qquad [\text{See Ex. 1 page 233}]$$

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = 0$$

$$[\because\ S' \text{ consists of } S \text{ and } S_1]$$

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS - \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS = 0$$

$$[\because\ \text{on } S_1,\ \mathbf{n} = -\mathbf{k}]$$

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS.$$

$$\therefore \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS$$

$$= \iint_{S_1} \mathbf{k} \cdot \mathbf{k}\, dS = \iint_{S_1} dS = S_1 = \pi. \qquad \dots (2)$$

Note that S_1 = area of a circle of radius 1

$$= \pi (1)^2 = \pi.$$

Hence from (1) and (2) Stoke's theorem is verified.

Ex. 9. *Verify Stoke's theorem for the function*

$$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$. **(Rohilkhand 1981; Kanpur 78 ; Agra 75)**

Sol. Here the surface S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ lying above the xy-plane. The curve C is the boundary of the surface S and is a circle in the xy-plane of radius 1 and centre origin.

The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (z\,dx + x\,dy + y\,dz) = \oint_C x\,dy, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} \cos t \frac{dy}{dt}\,dt = \int_0^{2\pi} \cos t \,.\, \cos t\,dt = \int_0^{2\pi} \cos^2 t\,dt$$

$$= \tfrac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \tfrac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \tfrac{1}{2} \cdot 2\pi = \pi. \qquad \dots (1)$$

Now let us evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS$.

We have $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial x}(z)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(z)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)\right]$$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

By Gauss divergence theorem,

$$\iint_{S'} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iiint_V (\text{div curl }\mathbf{F})\,dV,$$ where V is the volume enclosed by S'

$= 0$, since div curl $\mathbf{F} = 0$.

$$\therefore \quad \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS + \iint_{S_1} (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = 0$$

[$\because$ S' consists of S and S_1]

$$\text{or} \quad \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS - \iint_{S_1} (\text{curl }\mathbf{F})\cdot\mathbf{k}\,dS = 0$$

[$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

$$\text{or} \quad \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_{S_1} (\text{curl }\mathbf{F})\cdot\mathbf{k}\,dS.$$

$$\therefore \quad \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \iint_{S_1} (\text{curl }\mathbf{F})\cdot\mathbf{k}\,dS$$

$$= \iint_{S_1} (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{k}\,dS = \iint_{S_1} dS$$

$= S_1$, where S_1 is the area of the circle $x^2 + y^2 = 1, z = 0$

$= \pi . 1^2 = \pi$. ... (2)

From (1) and (2), we see that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS.$$

This verifies Stoke's theorem.

Ex. 10. *Verify Stoke's theorem for* $\mathbf{A} = 2y\,\mathbf{i} + 3x\,\mathbf{j} - z^2\,\mathbf{k}$, *where* S *is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 9$ *and* C *is its boundary.* **(Meerut 1975)**

Sol. Proceed as in solved example 9.

Here the parametric equations of the circle are $x = 3\cos t$, $y = 3\sin t$, $z = 0$, $0 \le t < 2\pi$.

We have $$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2y\,\mathbf{i} + 3x\,\mathbf{j} - z^2\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (2y\,dx + 3x\,dy - z^2\,dz) = \oint_C (2y\,dx + 3x\,dy),$$

since on C, $z = 0$ and $dz = 0$

$$= \int_{t=0}^{2\pi} \left(2y \cdot \frac{dx}{dt} + 3x \frac{dy}{dt}\right) dt = \int_0^{2\pi} [6 \sin t \cdot (-3 \sin t) + 9 \cos t \cdot (3 \cos t)] \, dt$$

$$= \int_0^{2\pi} (27 \cos^2 t - 18 \sin^2 t) \, dt$$

$$= 27.4 \int_0^{\pi/2} \cos^2 t \, dt - 18.4 \int_0^{\pi/2} \sin^2 t \, dt$$

$$= 108 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 72 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 9\pi.$$

Again here curl $\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix}$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(3x)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial z}(2y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y)\right]$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + \mathbf{k} = \mathbf{k}.$$

$$\iint_S (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, dS = \iint_{S_1} (\text{curl } \mathbf{A}) \cdot \mathbf{k} \, dS$$

$$= \iint_{S_1} \mathbf{k} \cdot \mathbf{k} \, dS = \iint_{S_1} dS = S_1,$$ where S_1 is the area of the circle $x^2 + y^2 = 9$, $z = 0$

$= \pi . 3^2 = 9\pi$, since radius of the circle is 3.

We see that $\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{A}) \cdot \mathbf{n} \, dS$.

This verifies Stoke's theorem.

Ex. 11. *Verify Stoke's theorem for the vector* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ *taken over the half of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying above the xy-plane.* **(Gauhati 1973)**

Sol. Here let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane and let the curve C be the boundary of this surface. Obviously the curve C is a circle in the xy-plane of radius a and centre origin and its equations are $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a \cos t$, $y = a \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C.

By Stoke's theorem, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \ (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS. \qquad \dots (1)$$

Let us verify (1).

We have $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$= \oint_C (z\,dx + x\,dy + y\,dz) = \oint_C x\,dy,$ since on $C, z = 0$ and $dz = 0$

$$= \int_0^{2\pi} a\cos t\cdot\frac{dy}{dt}\,dt = \int_0^{2\pi} a\cos t\,.\,a\cos t\,dt = a^2\int_0^{2\pi}\cos^2 t\,dt$$

$$= \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)\,dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}\cdot 2\pi = \pi a^2. \qquad \dots (2)$$

Now let us find $\iint_S \ (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS.$

We have $\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$

If $\mathbf{n}$ is a unit vector along outward drawn normal at any point (x, y, z) on the surface S *i.e.*, the surface $\phi(x, y, z) \equiv x^2 + y^2 + z^2 = a^2$, then

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a},$$

since on $S, x^2 + y^2 + z^2 = a^2$.

$$\therefore \iint_S \ (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \iint_S \ (\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot\left(\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}\right)dS$$

$$= \frac{1}{a}\iint_S \ (x + y + z)\,dS.$$

To evaluate it we shall use polar spherical coordinates (r, θ, ϕ). We have $z = r\cos\theta,\ x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi.$

Here $r = a.\quad \therefore\ x = a\sin\theta\cos\phi,\ y = a\sin\theta\sin\phi, z = a\cos\theta.$

Also dS = an elementary area on the surface of the sphere at the point $(a, \theta, \phi) = a\,d\theta.\,a\sin\theta\,d\phi = a^2\sin\theta\,d\theta\,d\phi.$

$$\therefore \iint_S \ (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (a \sin\theta \cos\phi + a \sin\theta \sin\phi + a \cos\theta)\, a^2 \sin\theta \, d\theta \, d\phi$$

$$= a^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (\sin^2\theta \cos\phi + \sin^2\theta \sin\phi + \cos\theta \sin\theta)\, d\theta \, d\phi$$

$$= a^2 \int_{\theta=0}^{\pi/2} \left[\sin^2\theta \sin\phi - \sin^2\theta \cos\phi + \phi \cos\theta \sin\theta\right]_{\phi=0}^{2\pi} d\theta$$

$$= a^2 \int_{\theta=0}^{\pi/2} 2\pi \cos\theta \sin\theta \, d\theta = 2\pi a^2 \cdot \tfrac{1}{2} = \pi a^2. \qquad \ldots (3)$$

From (2) and (3), we see that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} \, dS.$$

This verifies Stoke's theorem.

Ex. 12. *Verify Stoke's theorem for the vector* $\mathbf{A} = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, *where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary.* **(Meerut 1973, 77)**

Sol. The boundary C of the surface S is the circle in the plane $z = 2$ whose equations are $x^2 + y^2 = 4$, $z = 2$. The radius of this circle is 2 and centre $(0, 0, 2)$. Suppose $x = 2\cos t$, $y = 2\sin t$, $z = 2$, $0 \le t < 2\pi$ are parametric equations of C. By Stoke's theorem

$\oint_C \mathbf{A} \cdot d\mathbf{r} = \iint_S (\operatorname{curl} \mathbf{A}) \cdot \mathbf{n} \, dS$, where $\mathbf{n}$ is a unit vector along outward drawn normal to the surface S.

We have $\oint_C \mathbf{A} \cdot d\mathbf{r} = \oint_C (3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \oint_C (3y\,dx - xz\,dy + yz^2\,dz)$$

$$= \oint_C (3y\,dx - 2x\,dy), \text{ since on } C,\ z = 2 \text{ and } dz = 0$$

$$= \int_{2\pi}^{0} \left(3y \frac{dx}{dt} - 2x \frac{dy}{dt}\right) dt$$

[Note that here the surface S lies below the curve C and so direction of C is positive if C is traversed in clockwise sense]

$$= -\int_0^{2\pi} [3 \cdot 2\sin t \cdot (-2\sin t) - 2 \cdot 2\cos t \cdot 2\cos t]\, dt$$

$$= -\int_0^{2\pi} [-12\sin^2 t - 8\cos^2 t]\, dt$$

$$= 4\left[12\int_0^{\pi/2}\sin^2 t\,dt + 8\int_0^{\pi/2}\cos^2 t\,dt\right]$$

$$= 4\left[12\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 8\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = 4\cdot\frac{\pi}{4}\cdot 20 = 20\pi. \qquad \ldots (1)$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = \iiint_V \text{div curl } \mathbf{A}\,dV$$

$$= 0, \text{ since div curl } \mathbf{A} = 0.$$

$$\therefore \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS + \iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = 0$$

$$[\because \ S' \text{ consists of } S \text{ and } S_1]$$

or
$$\iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = -\iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS$$

$$= -\iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{k}\,dS.$$

$$[\because \ \text{on } S_1,\ \mathbf{n} = \mathbf{k}]$$

Now
$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial z}(-xz)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial z}(3y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(3y)\right]$$

$$= (z^2 + x)\,\mathbf{i} - (z+3)\,\mathbf{k}.$$

$$\therefore \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = -\iint_{S_1} [(z^2+x)\,\mathbf{i} - (z+3)\,\mathbf{k}]\cdot\mathbf{k}\,dS$$

$$= \iint_{S_1} (z+3)\,dS = \iint_{S_1} 5\,dS, \text{ since on } S_1,\ z = 2$$

$$= 5S_1, \text{ where } S_1 \text{ is the area of a circle of radius 2}$$

$$= 5.\pi.2^2 = 20\pi. \qquad \ldots (2)$$

From (1) and (2), we see that

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{A})\cdot \mathbf{n}\, dS.$$

This verifies Stoke's theorem.

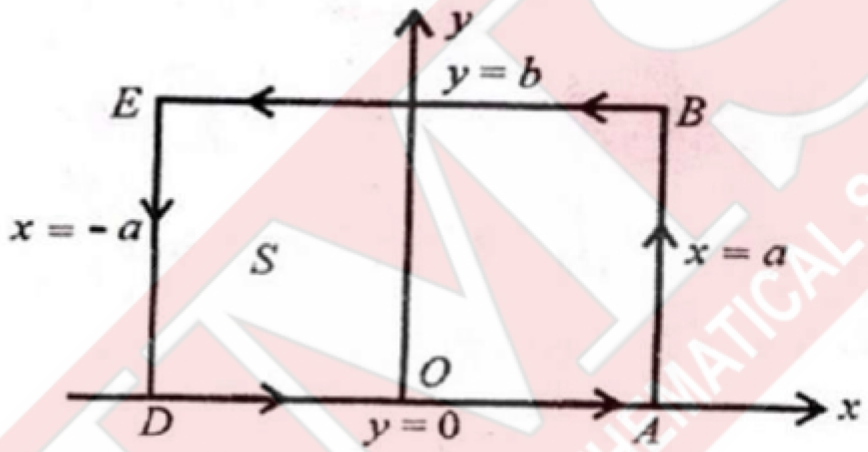
Ex. 13. *Verify Stoke's theorem for*

$$\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$$

taken round the rectangle bounded by

$$x = \pm a,\ y = 0,\ y = b.$$ (Meerut 1967)

Sol. We have



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= (-2y - 2y)\,\mathbf{k} = -4y\mathbf{k}.$$

Also $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y\mathbf{k})\cdot \mathbf{k}\, dx\, dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\, dx\, dy = -4\int_{y=0}^{b} \Big[xy\Big]_{x=-a}^{a} dy$$

$$= -4\int_{y=0}^{b} 2ay\, dy = -4\Big[ay^2\Big]_0^b = -4ab^2.$$

Also $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C [(x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C [(x^2+y^2)\,dx - 2xy\,dy]$$

$$= \int_{DA} [(x^2+y^2)\,dx - 2xy\,dy] + \int_{AB} + \int_{BE} + \int_{ED}.$$

Along DA, $y=0$ and $dy=0$. Along AB, $x=a$ and $dx=0$.
Along BE, $y=b$ and $dy=0$. Along ED, $x=-a$ and $dx=0$.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=-a}^{a} x^2\,dx + \int_{y=0}^{b} -2ay\,dy$$

$$+ \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy$$

$$= \int_{-a}^{a} x^2\,dx - \int_{-a}^{a} (x^2+b^2)\,dx - 4a\int_0^b y\,dy$$

$$= -\int_{-a}^{a} x^2\,dx - 4a\int_0^b y\,dy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS.$

Hence the theorem is verified.

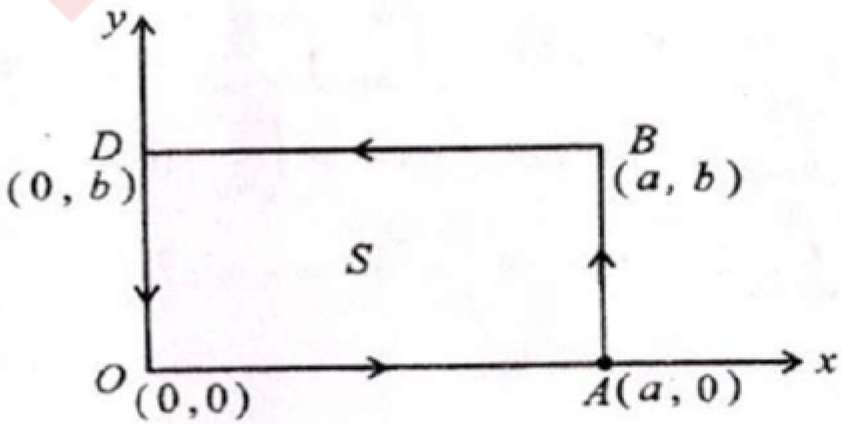
Ex. 14. *Verify Stoke's theorem for the function*

$$\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$$

integrated along the rectangle, in the plane $z=0$, whose sides are along the lines $x=0$, $y=0$, $x=a$ and $y=b$. **(Meerut 1976)**

Sol. We have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$



$$= 0\,\mathbf{i} - 0\,\mathbf{j} + \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]\mathbf{k}$$

$$= y\,\mathbf{k}.$$

The closed curve C is the boundary of the rectangle $OABD$ traversed in anti-clockwise sense. The surface S bounded by C is the area of rectangle $OABD$.

Also $\mathbf{n}$ = unit normal vector to $S = \mathbf{k}$.

By Stoke's theorem, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS.$$

Let us verify it.

We have $\displaystyle\iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \int_{y=0}^{b}\int_{x=0}^{a}(y\,\mathbf{k})\cdot\mathbf{k}\, dx\, dy$

$$= \int_{y=0}^{b}\int_{x=0}^{a} y\, dx\, dy = \int_{x=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{b} dx = \frac{b^2}{2}\int_0^a dx$$

$$= \frac{ab^2}{2} \qquad \ldots (1)$$

Also $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (x^2\mathbf{i} + xy\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C (x^2\, dx + xy\, dy)$$

$$= \int_{OA}(x^2\, dx + xy\, dy) + \int_{AB}(x^2\, dx + xy\, dy) + \int_{BD}(x^2\, dx + xy\, dy) + \int_{DO}(x^2\, dx + xy\, dy).$$

Along OA, $y = 0$ and $dy = 0$, x varies from 0 to a ;
along AB, $x = a$, $dx = 0$, y varies from 0 to b ;
along BD, $y = b$, $dy = 0$, x varies from a to 0 ;
and along DO, $x = 0$, $dx = 0$, y varies from b to 0.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^a x^2\, dx + \int_0^b ay\, dy + \int_a^0 x^2\, dx + \int_b^0 0\, dy$$

$$= \left[\frac{x^3}{3}\right]_0^a + a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}\right]_a^0 + 0$$

$$= \frac{a^3}{3} + \frac{ab^2}{2} - \frac{a^3}{3} = \frac{ab^2}{2} \qquad \ldots (2)$$

From (1) and (2), we see that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS.$$

This verifies Stoke's theorem.

Ex. 15. *Verify Stoke's theorem for the function*

$$\mathbf{F} = x^2 \mathbf{i} + xy\, \mathbf{j},$$

integrated round the square, in the plane $z = 0$, *whose sides are along the lines* $x = 0,\ y = 0,\ x = a,\ y = a.$ **(Bombay 1970)**

Sol. Proceed as in solved example 14.

Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2} a^3 = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS.$

Ex. 16. *Verify Stoke's theorem for a vector field defined by* $\mathbf{F} = (x^2 - y^2)\, \mathbf{i} + 2xy\, \mathbf{j}$ *in the rectangular region in the xy-plane bounded by the lines* $x = 0,\ x = a,\ y = 0$ *and* $y = b.$ **(Kanpur 1975)**

Sol. Proceed as in solved example 14.

Ex. 17. *Verify Stoke's theorem for the function*

$$\mathbf{F}(x, y, z) = xy\, \mathbf{i} + xy^2 \mathbf{j}$$

integrated round the square with vertices $(1, 0, 0),\ (1, 1, 0),\ (0, 1, 0)$ *and* $(0, 0, 0)$, *where* $\mathbf{i}$ *and* $\mathbf{j}$ *are unit vectors along x-axis and y-axis respectively.* **(Meerut 1979)**

Sol. Proceed as in solved example 14.

We observe that the z-coordinate of each vertex of the square is zero. Therefore the square lies in the xy-plane. Its vertices in the xy-plane are the points $(0, 0),\ (1, 0),\ (1, 1)$ and $(0, 1)$. Also here $\mathbf{n} = \mathbf{k}$.

Ex. 18. *Verify Stoke's theorem for* $\mathbf{F} = -y^3 \mathbf{i} + x^3 \mathbf{j}$, *where S is the circular disc* $x^2 + y^2 \le 1,\ z = 0.$

Sol. The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x = \cos t,\ y = \sin t,\ z = 0,\ 0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (-y^3 \mathbf{i} + x^3 \mathbf{j}) \cdot (dx\, \mathbf{i} + dy\, \mathbf{j} + dz\, \mathbf{k})$$

$$= \oint_C (-y^3\, dx + x^3\, dy) = \int_{t=0}^{2\pi} \left\{ -y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt} \right\} dt$$

$$= \int_0^{2\pi} [-\sin^3 t\, (-\sin t) + \cos^3 t\, (\cos t)]\, dt$$

$$= \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, dt = 4 \int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4 \left\{ \frac{3.1}{4.2} \frac{\pi}{2} + \frac{3.1}{4.2} \frac{\pi}{2} \right\} = \frac{3\pi}{2}.$$

Also $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = (3x^2 + 3y^2)\, \mathbf{k}.$

Here $\mathbf{n} = \mathbf{k}$ because the surface S is the xy-plane.

$\therefore (\nabla \times \mathbf{F}) \cdot \mathbf{n} = (3x^2 + 3y^2)\, \mathbf{k} \cdot \mathbf{k} = 3(x^2 + y^2).$

$\therefore \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = 3 \iint_S (x^2 + y^2)\, dS$

$$= 3 \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2\, r\, d\theta\, dr, \text{ changing to polars}$$

$$= \frac{3}{4} \int_0^{2\pi} d\theta = \frac{3}{4}(2\pi) = \frac{3\pi}{2}.$$

Thus $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \frac{3\pi}{2}.$

Hence the theorem is verified.

Ex. 19. *Evaluate by Stoke's theorem*

$$\oint_C (e^x\, dx + 2y\, dy - dz)$$

where C is the curve $x^2 + y^2 = 4$, $z = 2$. **(Meerut 1969; Agra 72)**

Sol. $\oint_C (e^x\, dx + 2y\, dy - dz)$

$$= \oint_C (e^x \mathbf{i} + 2y\, \mathbf{j} - \mathbf{k}) \cdot (dx\, \mathbf{i} + dy\, \mathbf{j} + dz\, \mathbf{k})$$

$$= \oint_C \mathbf{F} \cdot d\mathbf{r}, \text{ where } \mathbf{F} = e^x \mathbf{i} + 2y\, \mathbf{j} - \mathbf{k}.$$

Now curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

79

$$= 0, \quad \text{since curl } \mathbf{F} = \mathbf{0}.$$

Ex. 20. *Evaluate by Stoke's theorem*

$$\oint_C (yz\,dx + xz\,dy + xy\,dz)$$

where C is the curve $x^2 + y^2 = 1,\ z = y^2$. **(Andhra 1989, Kanpur 80)**

Sol. Here $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

$$\therefore \quad \text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

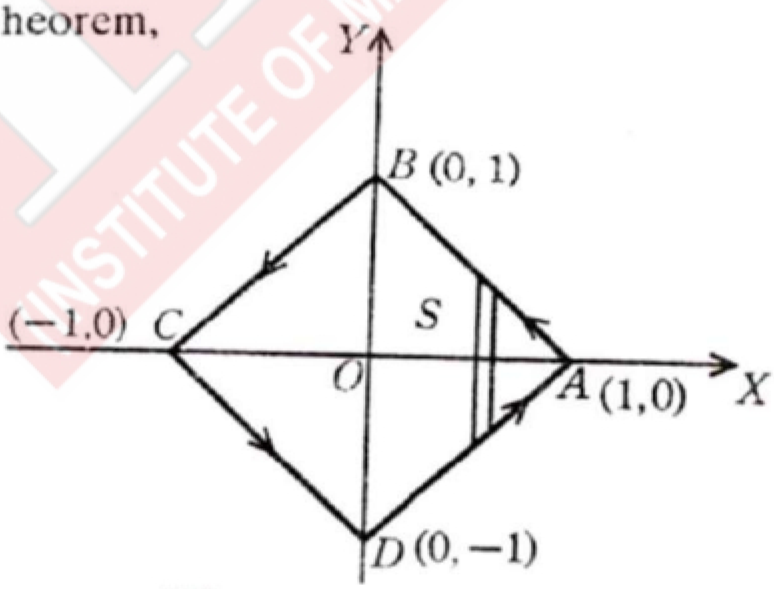
$$= 0, \text{ since curl } \mathbf{F} = \mathbf{0}.$$

Ex. 21. *Evaluate* $\oint_C (xy\,dx + xy^2\,dy)$ *by Stoke's theorem where C is the positively oriented square with vertices* $(1, 0)$, $(-1, 0)$, $(0, 1)$ *and* $(0, -1)$.

Sol. We have $\oint_C (xy\,dx + xy^2\,dy) = \oint_C \mathbf{F} \cdot d\mathbf{r}$, where

$\mathbf{F} = xy\,\mathbf{i} + xy^2\,\mathbf{j}$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$.

By Stoke's theorem,



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is any surface bounded by the square C and $\mathbf{n}$ is unit normal vector to the surface S.

Let us take the surface S as the area bounded by the square C. Since the square lies in the xy-plane, therefore $\mathbf{n} = \mathbf{k}$.

Now curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)\,\mathbf{k}$.

$\therefore$ (Curl $\mathbf{F}$)$\cdot\mathbf{n} = (y^2 - x)\,\mathbf{k}\cdot\mathbf{k} = y^2 - x$.

$\therefore$ The given line integral

$$\oint_C (xy\,dx + xy^2\,dy) = \iint_S (y^2 - x)\,dS,$$ where S is the area of the square $ABCD$.

Equation of the st. line AB is $x + y = 1$ *i.e.*, $y = 1 - x$ and the equation of the st. line BC is $-x + y = 1$ *i.e.*, $y = x + 1$.

$$\therefore \iint_S (y^2 - x)\,dS = \int_{x=-1}^{0}\int_{y=-(x+1)}^{x+1} (y^2 - x)\,dx\,dy + \int_{x=0}^{1}\int_{y=-(1-x)}^{1-x} (y^2 - x)\,dx\,dy$$

$$= 2\int_{x=-1}^{0}\int_{y=0}^{x+1} (y^2 - x)\,dx\,dy + 2\int_{x=0}^{1}\int_{y=0}^{1-x} (y^2 - x)\,dx\,dy$$

[$\because$ the integrand $y^2 - x$ is an even function of y]

$$= 2\int_{x=-1}^{0} \left[\frac{y^3}{3} - xy\right]_{y=0}^{x+1} dx + 2\int_{x=0}^{1} \left[\frac{y^3}{3} - xy\right]_{y=0}^{1-x} dx$$

$$= 2\int_{-1}^{0} \left[\frac{1}{3}(x+1)^3 - x(x+1)\right] dx + 2\int_{0}^{1} \left[\frac{(1-x)^3}{3} - x(1-x)\right] dx$$

$$= 2\left[\frac{1}{12}(x+1)^4 - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{0} + 2\left[-\frac{1}{12}(1-x)^4 - \frac{x^2}{2} + \frac{x^3}{3}\right]_{0}^{1}$$

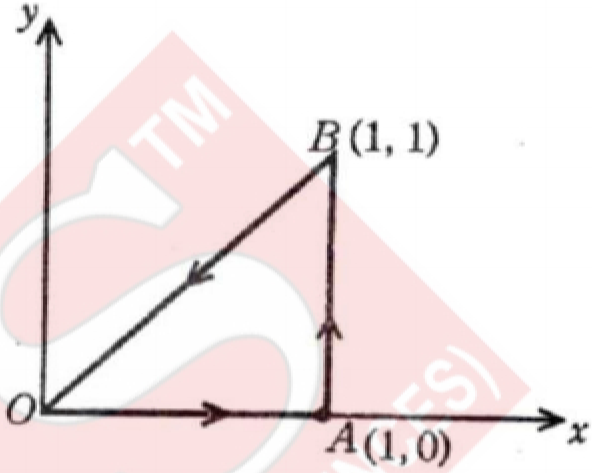
$$= 2\left[\frac{1}{12} - \frac{1}{3} + \frac{1}{2}\right] + 2\left[-\frac{1}{2} + \frac{1}{3} + \frac{1}{12}\right]$$

$$= \frac{2}{12} - \frac{2}{3} + 1 - 1 + \frac{2}{3} + \frac{2}{12} = \frac{4}{12} = \frac{1}{3}.$$

Ex. 22. **(a)** *Evaluate* $\oint_C \mathbf{F}\cdot d\mathbf{r}$ *by Stoke's theorem where* $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} - (x + z)\,\mathbf{k}$ *and* C *is the boundary of the triangle with vertices at* $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$.

Sol. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\mathbf{i} + \mathbf{j} + 2(x-y)\,\mathbf{k}.$$



Also we note that z co-ordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy-plane. So $\mathbf{n} = \mathbf{k}$.

$\therefore$ Curl $\mathbf{F} \cdot \mathbf{n} = [\mathbf{j} + 2(x-y)\mathbf{k}] \cdot \mathbf{k}$
$= 2(x-y)$.

In the figure, we have only considered the x-y plane.

The equation of the line OB is $y = x$.

By Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y)\, dx\, dy = 2\int_{x=0}^{1} \left[xy - \frac{y^2}{2}\right]_{y=0}^{x} dx$$

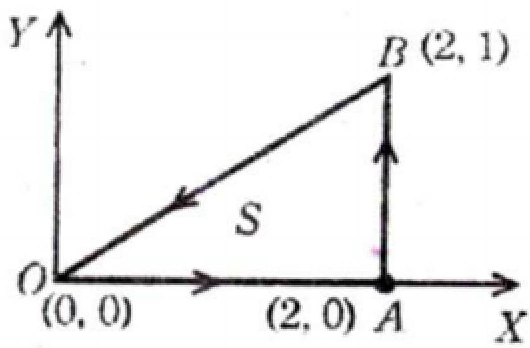
$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2}\right] dx = 2\int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2\, dx = \tfrac{1}{3}.$$

Ex. 22 (b). *If* $\mathbf{F} = (2x^2 + y^2)\mathbf{i} + (3y - 4x)\mathbf{j}$, *evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *where C is the boundary of the triangle with vertices at* (0, 0), (2, 0) *and* (2, 1). **(Andhra 1992)**

Sol. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 + y^2 & 3y - 4x & 0 \end{vmatrix} = 0\,\mathbf{i} + 0\,\mathbf{j} + (-4 - 2y)\,\mathbf{k}$$

$$= -2(2+y)\,\mathbf{k}.$$



By Stoke's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is any surface bounded by the curve C and $\mathbf{n}$ is unit normal vector to the surface S.

Let us take the surface S as the area of the given triangle. Since the triangle lies in the xy-plane, therefore $\mathbf{n} = \mathbf{k}$.

$\therefore$ (curl $\mathbf{F}$)$\cdot\mathbf{n} = -2(2+y)\,\mathbf{k}\cdot\mathbf{k} = -2(2+y)$.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S -2(2+y)\,dS,$$

where S is the area of the triangle OAB.

The equation of the st. line OB is $y = \frac{1}{2}x$.

$$\therefore \iint_S -2(2+y)\,dS$$

$$= \int_{x=0}^{2}\int_{y=0}^{x/2} -2(2+y)\,dx\,dy$$

[$\because$ for the region S, x varies from 0 to 2 and y varies from 0 to $x/2$]

$$= \int_{x=0}^{2}\left[-4y - y^2\right]_{y=0}^{x/2} dx,$$

integrating with respect to y regarding x as constant

$$= \int_0^2 \left[-2x - \frac{x^2}{4}\right] dx = \left[-x^2 - \frac{x^3}{12}\right]_0^2$$

$$= -4 - \frac{8}{12} = -4 - \frac{2}{3} = -\frac{14}{3}.$$

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = -\frac{14}{3}.$$

Ex. 23. *Evaluate by Stoke's theorem*

$$\oint_C (\sin z\,dx - \cos x\,dy + \sin y\,dz)$$

where C is the boundary of the rectangle

$$0 \le x \le \pi,\ 0 \le y \le 1,\ z = 3.$$

Sol. Here $\mathbf{F} = \sin z\,\mathbf{i} - \cos x\,\mathbf{j} + \sin y\,\mathbf{k}$.

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\,\mathbf{i} + \cos z\,\mathbf{j} + \sin x\,\mathbf{k}.$$

Since the rectangle lies in the plane $z = 3$, therefore $\mathbf{n} = \mathbf{k}$.

$\therefore$ curl $\mathbf{F}\cdot\mathbf{n} = (\cos y\,\mathbf{i} + \cos z\,\mathbf{j} + \sin x\,\mathbf{k})\cdot\mathbf{k} = \sin x$.

By Stoke's theorem

81

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \int_{y=0}^{1} \int_{x=0}^{\pi} \sin x\, dx\, dy = \int_{x=0}^{\pi} \sin x\, dx = 2.$$

Ex. 24. *By converting into a line integral evaluate*

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS, \textit{ where } \mathbf{A} = (x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$$

and S is the surface of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ above the xy-plane.

(Meerut 1974)

Sol. The xy-plane cuts the surface S of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ in the circle C whose equations are $x^2 + y^2 = 4$, $z = 0$. Thus the boundary of the surface S is the circle C.

The surface S lies above the circle C. Let the parametric equations of the curve C be $x = 2\cos t, y = 2\sin t, z = 0, 0 \le t < 2\pi$.

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{A}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

$$= \int_C \left[(x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}\right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(x - z)\, dx + (x^3 + yz)\, dy - 3xy^2 dz\right]$$

$$= \int_C (x dx + x^3 dy) \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_{t=0}^{2\pi} \left(x \frac{dx}{dt} + x^3 \frac{dy}{dt}\right) dt$$

$$= \int_0^{2\pi} [2\cos t \cdot (-2\sin t) + 8\cos^3 t \cdot 2\cos t\,]dt$$

$$= -2\int_0^{2\pi} 2\sin t \cos t\, dt + 16\int_0^{2\pi} \cos^4 t\, dt$$

$$= -2\int_0^{2\pi} \sin 2t\, dt + 16 \cdot 4 \int_0^{\pi/2} \cos^4 t\, dt$$

$$= -2\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + 64 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= 0 + 12\pi = 12\pi.$$

Ex. 25. *By converting into a line integral evaluate*

$$\iint_S (\nabla \times \mathbf{F})\bullet\mathbf{n}\,dS$$

where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ *and* S *is the surface of the paraboloid* $z = 4 - (x^2 + y^2)$ *above the xy-plane.*

Sol. The xy-plane cuts the surface S of the paraboloid $z = 4 - (x^2 + y^2)$ in the circle C whose equations are $x^2 + y^2 = 4$, $z = 0$. Thus the boundary of the surface S is the circle C and the surface S lies above the circle C. Let the parametric equations of the curve C be $x = 2\cos t, y = 2\sin t, z = 0, 0 \le t < 2\pi$.

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F})\bullet\mathbf{n}\,dS = \oint_C \mathbf{F}\bullet d\mathbf{r}$$

$$= \int_C \left[(x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}\right]\bullet(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(x^2 + y - 4)\,dx + 3xy\,dy + (2xz + z^2)\,dz\right]$$

$$= \int_C \left[(x^2 + y - 4)\,dx + 3xy\,dy\right], \text{ since on } C, z = 0 \text{ and } dz = 0$$

$$= \int_{t=0}^{2\pi} \left[(x^2 + y - 4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right] d.$$

$$= \int_0^{2\pi} \left[(4\cos^2 t + 2\sin t - 4)(-2\sin t) + 3\cdot 2\cos t\cdot 2\sin t\cdot 2\cos t\,\right] dt$$

$$= -8\int_0^{2\pi} \cos^2 t\sin t\,dt - 4\int_0^{2\pi} \sin^2 t\,dt + 8\int_0^{2\pi} \sin t\,dt + 24\int_0^{2\pi} \cos^2 t\sin t\,dt$$

$$= 8\left[\frac{\cos^3 t}{3}\right]_0^{2\pi} - 4\cdot 2\cdot 2\cdot\int_0^{\pi/2} \sin^2 t\,dt + 8\left[-\cos t\right]_0^{2\pi} - 24\left[\frac{\cos^3 t}{3}\right]_0^{2\pi}$$

$$= 8\cdot 0 - 16\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 8\cdot 0 - \frac{24}{3}\cdot 0 = -4\pi.$$

Ex. 26. *Evaluate* $\iint_S (\nabla \times \mathbf{F})\bullet\mathbf{n}\,dS$, *where* $\mathbf{F} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ *and* S *is the surface of the cube* $x = y = z = 0$, $x = y = z = 2$ *above the xy-plane.*

Sol. The xy-plane cuts the surface of the cube in a square. Thus the curve C bounding the surface S is the square, say $OABD$, in the xy-plane whose vertices in the xy-plane are the points

$$O\,(0,0), A\,(2,0), B\,(2,2), D\,(0,2).$$

[Draw figure as in solved example 14]

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \oint_C \left[(y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}\right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz\right]$$

$$= \int_C \left[(y + 2)\,dx + 4\,dy\right] \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_{OA} \left[(y+2)\,dx + 4\,dy\right] + \int_{AB} \left[(y+2)\,dx + 4\,dy\right]$$

$$+ \int_{BD} \left[(y+2)\,dx + 4\,dy\right] + \int_{DO} \left[(y+2)\,dx + 4\,dy\right]$$

$$= \int_0^2 2\,dx + \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy$$

[$\because$ on OA, $y = 0$, $dy = 0$ and x varies from 0 to 2 ;
on AB, $x = 2$, $dx = 0$ and y varies from 0 to 2 ;
on BD, $y = 2$, $dy = 0$, x varies from 2 to 0 ;
and on DO, $x = 0$, $dx = 0$, y varies from 2 to 0]

$$= 2\left[x\right]_0^2 + 4\left[y\right]_0^2 + 4\left[x\right]_2^0 + 4\left[y\right]_2^0$$

$$= 4 + 8 - 8 - 8 = -4.$$

Ex. 27. *Apply Stoke's theorem to prove that*

$$\int_C (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\pi a^2$$

where C is the curve given by

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$$

and begins at the point $(2a, 0, 0)$ *and goes at first below the z-plane.*

(Meerut 1982 ; Agra 69)

Solution. The centre of the sphere $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ is the point $(a, a, 0)$. Since the plane $x + y = 2a$ passes through the point $(a, a, 0)$, therefore the circle C is great circle of this sphere.

$\therefore$ Radius of the circle C

$$= \text{radius of the sphere} = \sqrt{(a^2 + a^2)} = a\sqrt{2}.$$

Now $\int_C (ydx + zdy + xdz) = \int_C (y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k})\cdot d\mathbf{r}$

$$= \iint_S [\text{curl}\,(y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k})]\cdot\mathbf{n}\,dS,$$

where S is any surface of which circle C is boundary [Stoke's theorem].

Now $\text{curl}\,(y\mathbf{i} + z\mathbf{j} + x\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$$= -\mathbf{i} - \mathbf{j} - \mathbf{k} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Let us take S as the surface of the plane $x + y = 2a$ bounded by the circle C. Then a vector normal to S is grad $(x + y) = \mathbf{i} + \mathbf{j}$.

$\therefore\ \mathbf{n} = \text{unit normal to } S = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$

$\therefore\ \int_C (y\,dx + z\,dy + x\,dz)$

$$= \iint_S -(\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) dS$$

$$= -\frac{2}{\sqrt{2}}\iint_S dS = -\frac{2}{\sqrt{2}}(\text{area of the circle of radius } a\sqrt{2})$$

$$= -\sqrt{2}\,(2\pi a^2).$$

Ex. 28. *Use Stoke's theorem to evaluate* $\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$, *where* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$ *and* S *is the surface of sphere* $x^2 + y^2 + z^2 = a^2$, *above the xy-plane.*

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a\cos t, y = a\sin t, z = 0, 0 \le t < 2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS$$

$$= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C [y\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}]\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= \int_C [y\,dx + (x - 2xz)\,dy - xy\,dz]$$

$$= \int_C (y\,dx + x\,dy) \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} \left(y \frac{dx}{dt} + x \frac{dy}{dt} \right) dt$$

$$= \int_0^{2\pi} [a \sin t\,(-a \sin t) + a \cos t\,(a \cos t)]\, dt$$

$$= a^2 \int_0^{2\pi} (\cos^2 t - \sin^2 t)\, dt = a^2 \int_0^{2\pi} \cos 2t\, dt = a^2 \left[\frac{\sin 2t}{2} \right]_0^{2\pi} = 0.$$

Ex. 29. *Evaluate the surface integral* $\iint_S$ *curl* $\mathbf{F} \cdot \mathbf{n}\, dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$, *and* $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

(Bombay 1979)

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t, y = \sin t, z = 0, 0 \leq t < 2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C (y\mathbf{i} + z\mathbf{j} + x\mathbf{k}) \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz) = \int_C y\,dx + z\,dy + x\,dz$$

$$= \int_C y\,dx \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} y \frac{dx}{dt}\, dt = \int_0^{2\pi} \sin t\,(-\sin t)\, dt = -\int_0^{2\pi} \sin^2 t\, dt$$

$$= -4 \int_0^{\pi/2} \sin^2 t\, dt = -4 \times \tfrac{1}{2} \times \frac{\pi}{2} = -\pi.$$

Ex. 30. *If* $\mathbf{F} = (y^2 + z^2 - x^2)\mathbf{i} + (z^2 + x^2 - y^2)\mathbf{j} + (x^2 + y^2 - z^2)\mathbf{k}$, *evaluate* $\iint$ *curl* $\mathbf{F} \cdot \mathbf{n}\, dS$ *taken over the portion of the surface* $x^2 + y^2 + z^2 - 2ax + az = 0$ *above the plane* $z = 0$, *and verify Stoke's theorem.*

Sol. The surface $x^2 + y^2 + z^2 - 2ax + az = 0$ meets the plane $z = 0$ in the circle C given by $x^2 + y^2 - 2ax = 0$, $z = 0$. The polar equation of the circle C lying in the xy-plane is $r = 2a \cos\theta$, $0 \leq \theta < \pi$. Also the equation $x^2 + y^2 - 2ax = 0$ can be written as

$(x-a)^2+y^2=a^2$. Therefore the parametric equations of the circle C can be taken as

$$x = a + a\cos t, y = a\sin t, z = 0, 0 \le t < 2\pi.$$

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k}\, dS.$$

Now $\text{curl } \mathbf{F}\cdot\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix} \cdot \mathbf{k}$

$$= \left[\frac{\partial}{\partial x}(z^2+x^2-y^2) - \frac{\partial}{\partial y}(y^2+z^2-x^2)\right] \mathbf{k}\cdot\mathbf{k}$$

$[\because \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0]$

$$= 2(x-y).$$

$$\therefore \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k}\, dS = \iint_{S_1} 2(x-y)\, dS$$

$$= 2\int_{\theta=0}^{\pi} \int_{r=0}^{2a\cos\theta} (r\cos\theta - r\sin\theta)\, r\, d\theta\, dr,$$

changing to polar

$$= 2\int_{\theta=0}^{\pi} (\cos\theta - \sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= 2 \times \frac{8a^3}{3}\int_0^{\pi} (\cos\theta - \sin\theta)\cos^3\theta\, d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi} \cos^4\theta\, d\theta \qquad \left[\because \int_0^{\pi} \cos^3\theta \sin\theta\, d\theta = 0\right]$$

$$= 2 \times \frac{16a^3}{3}\int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 2 \times \frac{16a^3}{3} \times \frac{3\times 1}{4\times 2} \times \frac{\pi}{2} = 2\pi a^3. \qquad \dots (1)$$

Also $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y^2+z^2-x^2)\, dx$

$$+ (z^2+x^2-y^2)\, dy + (x^2+y^2-z^2)\, dz$$

$$= \int_C (y^2-x^2)\, dx + (x^2-y^2)\, dy \qquad [\because \text{on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} (x^2 - y^2) \left(\frac{dy}{dt} - \frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} [(a + a\cos t)^2 - a^2 \sin^2 t] (a\cos t + a\sin t)\, dt$$

$$= a^3 \int_0^{2\pi} (1 + \cos^2 t + 2\cos t - \sin^2 t)(\cos t + \sin t)\, dt$$

$$= a^3 \int_0^{2\pi} 2\cos^2 t\, dt, \text{ the other integrals vanish}$$

$$= 2a^3 \times 4 \int_0^{\pi/2} \cos^2 t\, dt = 8a^3 \times \tfrac{1}{2} \times \frac{\pi}{2} = 2\pi a^3. \qquad \ldots (2)$$

Comparing (1) and (2), we see that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Hence Stoke's theorem is verified.

Ex. 31. *Show that*

$$\iint_S \phi \, \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \phi \mathbf{F} \cdot d\mathbf{r} - \iint_S (\operatorname{grad} \phi \times \mathbf{F}) \cdot d\mathbf{S}.$$

Sol. Here C is the closed curve bounding the surface S. Applying Stoke's theorem to the vector $\phi\mathbf{F}$, we have

$$\oint_C (\phi\mathbf{F}) \cdot d\mathbf{r} = \iint_S \operatorname{curl} (\phi\mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \iint_S [\nabla \times (\phi\mathbf{F})] \cdot d\mathbf{S}$$

$$= \iint_S [(\operatorname{grad} \phi) \times \mathbf{F} + \phi \operatorname{curl} \mathbf{F}] \cdot d\mathbf{S}$$

$$[\because \operatorname{curl} (\phi \mathbf{A}) = (\operatorname{grad} \phi) \times \mathbf{A} + \phi \operatorname{curl} \mathbf{A}]$$

$$= \iint_S (\operatorname{grad} \phi \times \mathbf{F}) \cdot d\mathbf{S} + \iint_S \phi \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}.$$

Hence by transposition, we have

$$\iint_S \phi \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \phi \mathbf{F} \cdot d\mathbf{r} - \iint_S (\operatorname{grad} \phi \times \mathbf{F}) \cdot d\mathbf{S}.$$

Ex. 32. *If* $\mathbf{f} = \nabla \phi$ *and* $\mathbf{g} = \nabla \psi$ *are two vector point functions, such that*

$$\nabla^2 \phi = 0, \nabla^2 \psi = 0$$

show that

$$\iint_S (\mathbf{g} \cdot \nabla) \mathbf{f} \cdot d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} + \iint_S (\mathbf{f} \cdot \nabla) \mathbf{g} \cdot d\mathbf{S}.$$

Sol. Here C is the closed curve bounding the surface S. Applying Stoke's theorem to the vector $\mathbf{f} \times \mathbf{g}$, we have

$$\oint_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} = \iint_S \nabla \times (\mathbf{f} \times \mathbf{g}) \cdot \mathbf{n}\, dS$$

$$= \iint_S \text{curl}\,(\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{S}$$

$$= \iint_S \left[(\mathbf{g} \cdot \nabla)\,\mathbf{f} - \mathbf{g}\,\text{div}\,\mathbf{f} - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + \mathbf{f}\,\text{div}\,\mathbf{g}\right] \cdot d\mathbf{S}$$

$$= \iint_S \left[(\mathbf{g} \cdot \nabla)\,\mathbf{f} - \mathbf{g}\,\text{div}\,\nabla\phi - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + \mathbf{f}\,\text{div}\,\nabla\psi\right] \cdot d\mathbf{S}$$

$$[\because \mathbf{f} = \nabla\phi \text{ and } \mathbf{g} = \nabla\psi]$$

$$= \iint_S [(\mathbf{g} \cdot \nabla)\,\mathbf{f} - 0\,\mathbf{g} - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + 0\,\mathbf{f}] \cdot d\mathbf{S}$$

$[\because \text{div}\,\nabla\phi = \nabla \cdot \nabla\phi = \nabla^2\phi = 0$, given

and similarly $\text{div}\,\nabla\psi = \nabla^2\psi = 0$, given$]$

$$= \iint_S (\mathbf{g} \cdot \nabla)\,\mathbf{f} \cdot d\mathbf{S} - \iint_S (\mathbf{f} \cdot \nabla)\,\mathbf{g} \cdot d\mathbf{S}.$$

Hence by transposition, we have

$$\iint_S (\mathbf{g} \cdot \nabla)\,\mathbf{f} \cdot d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} + \iint_S (\mathbf{f} \cdot \nabla)\,\mathbf{g} \cdot d\mathbf{S}.$$

Ex. 33. *Prove that a necessary and sufficient condition that* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every closed curve C lying in a simply connected region R is that* $\nabla \times \mathbf{F} = \mathbf{0}$ *identically.*

Sol. Sufficiency. Suppose R is simply connected and curl $\mathbf{F} = \mathbf{0}$ everywhere in R. Let C be any closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS = 0.$$

Necessity. Suppose $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C and assume that $\nabla \times \mathbf{F} \neq \mathbf{0}$ at some point A.

Then taking $\nabla \times \mathbf{F}$ as continuous, there must exist a region with A as an interior point, where $\nabla \times \mathbf{F} \neq \mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point is in the same direction as

$\nabla \times \mathbf{F}$, *i.e.* $\nabla \times \mathbf{F} = \lambda \mathbf{n}$ where λ is a positive constant. Let C be the boundary of S. Then by Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \lambda\, \mathbf{n} \cdot \mathbf{n}\, dS$$
$$= \lambda S > 0.$$

This contradicts the hypothesis that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Therefore we must have $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere in R.

§ 10. Line integrals Independent of path.

Let $\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}$ be a vector point function defined and continuous in a region R of space. Let P and Q be two points in R and let C be a path joining P to Q. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int (f dx + g dy + h dz) \qquad \ldots (1)$$

is called the line integral of $\mathbf{F}$ along C. In general the value of this line integral depends not only on the end points P and Q of the path C but also on C.

In other words, if we integrate from P to Q along different paths, we shall, in general, get different values of the integral. *The line integral* (1) *is said to be* **independent of path** *in R, if for every pair of end points P and Q in R the value of the integral is the same for all paths C in R starting from P and ending at Q.*

In this case the value of this line integral will depend on the choice of P and Q and not on the choice of the path joining P to Q.

Definition. *The experssion $f\,dx + g\,dy + h\,dz$ is said to be an exact differential if there exists a single valued scalar point function $\phi(x, y, z)$, having continuous first partial derivatives such that*

$$d\phi = f\,dx + g\,dy + h\,dz.$$

It can be easily seen that $f\,dx + g\,dy + h\,dz$ is an exact differential if and only if the vector function

$$\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$$

is the gradient of a single valued scalar function $\phi(x, y, z)$.

Because $\mathbf{F} = \text{grad}\,\phi$

if, and only if $f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$

if, and only if $f = \dfrac{\partial \phi}{\partial x}$, $g = \dfrac{\partial \phi}{\partial y}$, $h = \dfrac{\partial \phi}{\partial z}$

if, and only if $f\,dx + g\,dy + h\,dz = \dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial y}dy + \dfrac{\partial \phi}{\partial z}dz$

if, and only if $f\,dx + g\,dy + h\,dz = d\,\phi$.

Thus $\mathbf{F} = \text{grad}\,\phi$ if, and only if $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Theorem 1. *Let $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ be continuous in a region R of space. Then the line integral*

$$\int (f\,dx + g\,dy + h\,dz)$$

is independent of path in R if and only if the differential form under the integral sign is exact in R. **(Meerut 1968)**

Or

Let $\mathbf{F}(x, y, z)$ be continuous in region R of space. Then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of the path C in R joining P and Q if and only if $\mathbf{F} = \text{grad}\,\phi$ where $\phi(x, y, z)$ is a single-valued scalar function having continuous first partial derivatives in R. **(Kerala 1975)**

Proof. Suppose $\mathbf{F} = \text{grad}\,\phi$ in R. Let P and Q be any two points in R and let C be any path from P to Q in R.

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r}$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = \int_C d\phi$$

$$= \int_P^Q d\phi = [\phi]_P^Q = \phi(Q) - \phi(P).$$

Thus the line integral depends only on points P and Q and not the path joining them. This is true of course only if $\phi(x, y, z)$ is single valued at all points P and Q.

Conversely suppose the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining any two points P and Q in R. Let P be a fixed point (x_0, y_0, z_0) in R and let Q be any point (x, y, z) in R.

Let

$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get

$$\frac{d\phi}{ds} = \mathbf{F} \cdot \frac{d\mathbf{r}}{ds}.$$

But $$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla \phi \cdot \frac{d\mathbf{r}}{ds}.$$

$$\therefore \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} = \nabla \phi \cdot \frac{d\mathbf{r}}{ds} \quad \text{or} \quad (\nabla \phi - \mathbf{F}) \cdot \frac{d\mathbf{r}}{ds} = 0.$$

Now this result is true irrespective of the path joining P to Q *i.e.* this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$$\nabla \phi - \mathbf{F} = \mathbf{0}$$

i.e., $$\nabla \phi = \mathbf{F}.$$

This completes the proof of the theorem.

Definition. *A vector field* $\mathbf{F}(x, y, z)$ *defined and continuous in a region R of space is said to be a conservative vector field if the line integral* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of the path C in R joining P and Q where P and Q are any two points in R.*

By theorem 1, vector field $\mathbf{F}(x, y, z)$ is conservative if and only if $\mathbf{F} = \nabla \phi$ where $\phi(x, y, z)$ is a single valued scalar function having continuous first partial derivatives in R. The function $\phi(x, y, z)$ is called the **scalar potential** *of the vector field* $\mathbf{F}$.

Theorem 2. *Let* $\mathbf{F}(x, y, z)$ *be a vector function defined and continuous in a region R of space. Then the line integral* $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ *is independent of the path joining any two points P and Q in R if and only if* $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ *for every simple closed path in R.*

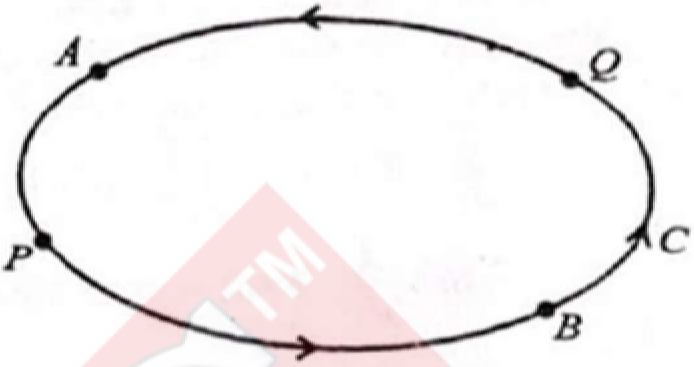
Proof. Let C be any simple closed path in R and let the line integral be independent of path in R. Take two points P and Q on C and subdivide C into two arcs PBQ and QAP. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{PBQAP} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QAP} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}$$

$= 0$, since the integral from P to Q along a path through B is equal to the integral from P to Q along a path through A.



Conversely suppose that the integral under consideration is zero on every simple closed path in R. Let P and Q be any two points in R which join P to Q and do not cross. Then

$$\oint_{PBQAP} \mathbf{F} \cdot d\mathbf{r} = \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QAP} \mathbf{F} \cdot d\mathbf{r} = \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}.$$

But as given, we have $\oint_{PBQAP} \mathbf{F} \cdot d\mathbf{r} = 0.$

$$\therefore \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r} = 0$$

or
$$\int_{PBQ} \mathbf{F} \cdot d\mathbf{r} = \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}.$$

This completes the proof of the theorem.

Theorem 3. *Let* $\mathbf{F}(x, y, z) = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ *be a continuous vector function having continuous first partial derivatives in a region* R *of space. If* $\int f dx + g dy + h dz$ *is independent of path in* R *and consequently* $f dx + g dy + h dz$ *is an exact differential in* R*, then curl* $\mathbf{F} = \mathbf{0}$ *everywhere in* R*. Conversely, if* R *is simply connected and curl* $\mathbf{F} = \mathbf{0}$ *everywhere in* R*, then* $f dx + g dy + h dz$ *is an exact differential in* R *or* $\int f dx + g dy + h dz$ *is independent of path in* R. **(Allahabad 1979)**

Proof. Suppose $\int (f\,dx + g\,dy + h\,dz)$ is independent of path in R. Then $f dx + g dy + h dz$ is an exact differential in R. Therefore

$$\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \text{grad}\,\phi.$$

$$\therefore \quad \text{curl}\,\mathbf{F} = \text{curl}\,(\text{grad}\,\phi) = \mathbf{0}.$$

Conversely suppose R is simply connected and curl $\mathbf{F} = \mathbf{0}$ everywhere in R. Let C be any simple closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = 0.$$

Thus $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is zero for every simple closed path C in R.

Therefore $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of path in R.

Therefore $\mathbf{F} = \nabla \phi$ and consequently $f dx + g dy + h dz$ is an exact differential $d\phi$.

Note. The assumption that R be simply connected is essential and cannot be omitted. It is obvious from the following example.

Example. Let $\mathbf{F} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

Here $\mathbf{F}$ is not defined at origin. In every region R of the xy-plane not containing the origin, we have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & \dfrac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left\{ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \right\} \mathbf{k}$$

$$= \left\{ \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right\} \mathbf{k} = 0\,\mathbf{k} = \mathbf{0}.$$

Suppose R is simply connected. For example let R be the region enclosed by a simple closed curve C not enclosing the origin. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \left(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy \right)$$

$$= \iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) \right] dx\, dy,$$

by Green's theorem in plane

$= 0.$

Suppose R is not simply connected. Let R be the region of the xy-plane contained between concentric circles of radii $\frac{1}{2}$ and $\frac{3}{2}$ and having centre at origin. Obviously R is not simply connected. We have $z = 0$, everywhere in R. Let C be a closed curve in R. The parametric equations of C can be taken as $x = \cos t, y = \sin t, z = 0, 0 \leq t < 2\pi$.

$$\text{We have } \oint_C \mathbf{F} \bullet d\mathbf{r} = \int \left(-\frac{y}{x^2 + y^2} dx + \frac{y}{x^2 + y^2} dy \right)$$

$$= \int_{t=0}^{2\pi} \left[-\frac{\sin t}{\cos^2 t + \sin^2 t} \frac{dx}{dt} + \frac{\cos t}{\cos^2 t + \sin^2 t} \frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

Thus we see that $\oint_C \mathbf{F} \bullet d\mathbf{r} \neq 0.$

Definition. Irrotational vector field. *A vector field* $\mathbf{F}$ *is said to be irrotational if curl* $\mathbf{F} = \mathbf{0}$.

We see that an irrotational field $\mathbf{F}$ is characterised by any one of the three conditions :

(i) $\mathbf{F} = \nabla \phi$,

(ii) $\nabla \times \mathbf{F} = \mathbf{0}$,

(iii) $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed path.

Any one of these conditions implies the other two.

Solved Examples

Ex. 1. *Are the following forms exact ?*

(i) $x dx - y dy + z dz$. (ii) $e^y dx + e^x dy + e^z dz$.

(iii) $yz dx + xz dy + xy dz$. (iv) $y^2 z^3 dx + 2xyz^3 dy + 3xy^2 z^2 dz$.

Sol. (i) We have

$$x dx - y dy + z dz = (x\mathbf{i} - y\mathbf{j} + z\mathbf{k}) \bullet (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \mathbf{F} \bullet d\mathbf{r}, \text{ where}$$

$$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}.$$

We have Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ the given form is exact.

(ii) Here $\mathbf{F} = e^y\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$. We have

Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (e^x - e^y)\,\mathbf{k}.$

Since curl $\mathbf{F} \neq \mathbf{0}$, therefore the given form is not exact.

(iii) Here $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. We have

Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k} = \mathbf{0}.$

Since curl $\mathbf{F} = \mathbf{0}$, therefore the given form is exact.

(iv) Here $\mathbf{F} = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= \mathbf{0}.$$

$\therefore$ the given form is exact.

Ex. 2. *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(i) $x\,dx - y\,dy - z\,dz$. (ii) $dx + z\,dy + y\,dz$.

(iii) $\cos x\,dx - 2yz\,dy - y^2\,dz$.

(iv) $(z^2 - 2xy)\,dx - x^2dy + 2xz\,dz$.

Sol. (i) Here $\mathbf{F} = x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. We have

Curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & -z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$,

or $\quad x\mathbf{i} - y\mathbf{j} - z\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = x \text{ whence } \phi = \frac{x^2}{2} + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = -y \text{ whence } \phi = -\frac{y^2}{2} + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = -z \text{ whence } \phi = -\frac{z^2}{2} + f_3(x, y). \qquad \ldots (3)$$

The constants of integration are functions of the variables not involved in the integration because the derivatives are partial.
(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = -\frac{y^2 + z^2}{2}, \; f_2(x, z) = \frac{x^2 - z^2}{2}, \; f_3(x, y) = \frac{x^2 - y^2}{2}.$$

$\therefore \quad \phi = \dfrac{x^2 - y^2 - z^2}{2}$ to which may be added any constant.

Hence $\phi = \dfrac{x^2 - y^2 - z^2}{2} + C$, where C is a constant.

(ii) Here $\mathbf{F} = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$

or $\quad \mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = 1 \text{ whence } \phi = x + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = z \text{ whence } \phi = zy + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = y \text{ whence } \phi = yz + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = zy, \; f_2(x, z) = x, \; f_3(x, y) = x.$

$\therefore \phi = x + yz$ to which may be added any constant.

$\therefore \phi = x + yz + C.$

(iii) Here $\mathbf{F} = \cos x\mathbf{i} - 2yz\mathbf{j} - y^2\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$$= (-2y + 2y)\,\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$,

or $\quad \cos x\mathbf{i} - 2yz\mathbf{j} - y^2\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = \cos x \text{ whence } \phi = \sin x + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = -2yz \text{ whence } \phi = -y^2 z + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = -y^2 \text{ whence } \phi = -y^2 z + f_3(x, y). \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = -y^2 z$, $f_2(x, z) = \sin x$, $f_3(x, y) = \sin x$.

$\therefore \phi = \sin x - y^2 z$ to which may be added any constant.

$\therefore \phi = \sin x - y^2 z + C$.

(iv) Here $\mathbf{F} = (z^2 - 2xy)\,\mathbf{i} - x^2\mathbf{j} + 2xz\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - 2xy & -x^2 & 2xz \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$

or $\quad (z^2 - 2xy)\,\mathbf{i} - x^2\mathbf{j} + 2xz\,\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = z^2 - 2xy \text{ whence } \phi = z^2 x - x^2 y + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = -x^2 \text{ whence } \phi = -x^2 y + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 2xz \text{ whence } \phi = xz^2 + f_3(x, y). \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = 0$, $f_2(x, z) = xz^2$, $f_3(x, y) = -x^2 y$.

$\therefore \phi = z^2 x - x^2 y$ to which may be added any constant.

$\therefore \phi = z^2 x - x^2 y + C.$

Ex. 3. *Show that*

$$(y^2z^3 \cos x - 4x^3 z)\, dx + 2z^3 y \sin x\, dy + (3y^2z^2 \sin x - x^4)\, dz$$

is an exact differential of some function ϕ *and find this function.*

Sol. Let $\mathbf{F} = (y^2z^3 \cos x - 4x^3z)\,\mathbf{i} + 2z^3y \sin x\,\mathbf{j} + (3y^2z^2 \sin x - x^4)\,\mathbf{k}.$

We have curl $\mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2z^3 \cos x - 4x^3z & 2z^3y \sin x & 3y^2z^2 \sin x - x^4 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(3y^2z^2 \sin x - x^4) - \frac{\partial}{\partial z}(2z^3y \sin x)\right]$$
$$- \mathbf{j}\left[\frac{\partial}{\partial x}(3y^2z^2 \sin x - x^4) - \frac{\partial}{\partial z}(y^2z^3 \cos x - 4x^3z)\right]$$
$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(2z^3y \sin x) - \frac{\partial}{\partial y}(y^2z^3 \cos x - 4x^3z)\right]$$

$$= (6yz^2 \sin x - 6z^2y \sin x)\,\mathbf{i} - \left[(3y^2z^2 \cos x - 4x^3) - (3z^2y^2 \cos x - 4x^3)\right]\mathbf{j} + (2z^3y \cos x - 2yz^3 \cos x)\,\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ there exists a scalar function $\phi(x, y, z)$ such that

$$\mathbf{F} = \nabla\phi.$$

$$\therefore \qquad \mathbf{F} \cdot d\mathbf{r} = \nabla\phi \cdot d\mathbf{r}$$

or $\quad (y^2z^3 \cos x - 4x^3z)\, dx + 2z^3y \sin x\, dy + (3y^2z^2 \sin x - x^4)\, dz = d\phi.$

Hence $\quad (y^2z^3 \cos x - 4x^3z)\, dx + 2z^3y \sin x\, dy + (3y^2z^2 \sin x - x^4)\, dz$

is an exact differential of some function ϕ.

Now $\mathbf{F} = \nabla\phi \Rightarrow \mathbf{F} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}.$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on both sides, we get

$$\frac{\partial\phi}{\partial x} = y^2z^3 \cos x - 4x^3z \text{ whence } \phi = y^2z^3 \sin x - x^4z + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = 2z^3y \sin x \text{ whence } \phi = z^3y^2 \sin x + f_2(z, x) \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = 3y^2z^2 \sin x - x^4 \text{ whence } \phi = y^2z^3 \sin x - x^4z + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z)=0,\ f_2(z,x)=-x^4z,\ f_3(x,y)=0.$

$\therefore \phi = y^2z^3\sin x - x^4z$ to which may be added any constant.

Hence $\phi = y^2z^3\sin x - x^4z + C.$

Ex. 4. *Show that* $\mathbf{F}=(2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\mathbf{k}$ *is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from*

$(1,-2,1)$ *to* $(3,1,4)$.

Sol. The field **F** will be conservative if $\nabla\times\mathbf{F}=\mathbf{0}$.

We have

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+z^3 & x^2 & 3xz^2\end{vmatrix}=\mathbf{0}.$$

Therefore **F** is a conservative force field.

Let $\mathbf{F}=\nabla\phi$

or $\quad (2xy+z^3)\,\mathbf{i}+x^2\mathbf{j}+3xz^2\mathbf{k}=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x}=2xy+z^3 \text{ whence } \phi=x^2y+z^3x+f_1(y,z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y}=x^2 \text{ whence } \phi=x^2y+f_2(x,z) \quad \ldots (2)$$

$$\frac{\partial\phi}{\partial z}=3xz^2 \text{ whence } \phi=xz^3+f_3(x,y) \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z)=0,\ f_2(x,z)=z^3x,\ f_3(x,y)=x^2y.$

$\therefore \phi = x^2y+xz^3$ to which may be added any constant.

$\therefore \phi = x^2y+xz^3+C.$

$$\text{Work done}=\int_{(1,-2,1)}^{(3,1,4)}\mathbf{F}\cdot d\mathbf{r}$$

$$=\int_{(1,-2,1)}^{(3,1,4)}d\phi=\Big[\phi\Big]_{(1,-2,1)}^{(3,1,4)}$$

$$=\Big[x^2y+xz^3\Big]_{(1,-2,1)}^{(3,1,4)}=202.$$

Ex. 5. *Show that the vector field* **F** *given by*

$$\mathbf{F}=(y+\sin z)\,\mathbf{i}+x\,\mathbf{j}+x\cos z\,\mathbf{k}$$

is conservative. Find its scalar potential.

Sol. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin z & x & x \cos z \end{vmatrix} = \mathbf{0}.$$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Let $\mathbf{F} = \nabla \phi$

or $(y + \sin z)\,\mathbf{i} + x\,\mathbf{j} + x \cos z\,\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = y + \sin z \text{ whence } \phi = xy + x \sin z + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = x \quad \text{whence } \phi = xy + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = x \cos z \text{ whence } \phi = x \sin z + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = 0,\ f_2(x, z) = x \sin z,\ f_3(x, y) = xy.$

$\therefore \phi = xy + x \sin z$ to which may be added any constant.

$\therefore \phi = xy + x \sin z + C.$

Ex. 6. *Show that the vector field*

$$\mathbf{F} = (2xy^2 + yz)\,\mathbf{i} + (2x^2y + xz + 2yz^2)\,\mathbf{j} + (2y^2z + xy)\,\mathbf{k}$$

is conservative.

Sol. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + yz & 2x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(2y^2z + xy) - \frac{\partial}{\partial z}(2x^2y + xz + 2yz^2)\right]$$
$$- \mathbf{j}\left[\frac{\partial}{\partial x}(2y^2z + xy) - \frac{\partial}{\partial z}(2xy^2 + yz)\right]$$
$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(2x^2y + xz + 2yz^2) - \frac{\partial}{\partial y}(2xy^2 + yz)\right]$$

$$= [(4yz + x) - (x + 4yz)]\,\mathbf{i} - (y - y)\,\mathbf{j} + [(4xy + z) - (4xy + z)]\,\mathbf{k}$$
$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Ex. 7. *Show that* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *is conservative and find* ϕ *such that* $\mathbf{F} = \nabla \phi$. **(Kanpur 1980)**

Sol. We have Curl $\mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Let $\mathbf{F} = \nabla\phi$

i.e., $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = x \text{ whence } \phi = \frac{1}{2}x^2 + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = y \text{ whence } \phi = \frac{1}{2}y^2 + f_2(z, x) \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = z \text{ whence } \phi = \frac{1}{2}z^2 + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{1}{2}y^2 + \frac{1}{2}z^2, f_2(z, x) = \frac{1}{2}z^2 + \frac{1}{2}x^2,$$

$$f_3(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

$\therefore \phi = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ to which may be added any constant.

Hence $\phi = \frac{1}{2}(x^2 + y^2 + z^2) + C$.

Ex. 8. *show that*

$\mathbf{F} = (\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ *is a conservative vector field and find a function* ϕ *such that*

$\mathbf{F} = \nabla\phi.$ **(Bombay 1966)**

Sol. We have curl $\mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x\cos y - z)\right]$$

$$+ \mathbf{j}\left[\frac{\partial}{\partial z}(\sin y + z) - \frac{\partial}{\partial x}(x - y)\right]$$

$$+\mathbf{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]$$

$= (-1+1)\,\mathbf{i} + (1-1)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Let $\mathbf{F} = \nabla\phi$

i.e., $(\sin y + z)\,\mathbf{i} + (x\cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$

Then $\frac{\partial\phi}{\partial x} = \sin y + z$ whence $\phi = x\sin y + xz + f_1(y,z)$ ... (1)

$\frac{\partial\phi}{\partial y} = x\cos y - z$ whence $\phi = x\sin y - yz + f_2(z,x)$ ... (2)

$\frac{\partial\phi}{\partial z} = x - y$ whence $\phi = xz - yz + f_3(x,y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = -yz,\ f_2(z,x) = xz,\ f_3(x,y) = x\sin y.$

$\therefore \phi = x\sin y + xz - yz.$

$\therefore \phi = x\sin y + xz - yz$ to which may be added any constant.

Hence $\phi = x\sin y + xz - yz + C.$

Ex. 9. *Show that the vector field defined by*

$\mathbf{F} = (2xy - z^3)\,\mathbf{i} + (x^2 + z)\,\mathbf{j} + (y - 3xz^2)\,\mathbf{k}$ *is conservative, and find the scalar potential of* $\mathbf{F}$. **(Bombay 1970)**

Sol. We have curl $\mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^3 & x^2 + z & y - 3xz^2 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y - 3xz^2) - \frac{\partial}{\partial z}(x^2 + z)\right]$$

$$+\mathbf{j}\left[\frac{\partial}{\partial z}(2xy - z^3) - \frac{\partial}{\partial x}(y - 3xz^2)\right]$$

$$+\mathbf{k}\left[\frac{\partial}{\partial x}(x^2 + z) - \frac{\partial}{\partial y}(2xy - z^3)\right]$$

$= (1-1)\,\mathbf{i} + (-3z^2 + 3z^2)\,\mathbf{j} + (2x - 2x)\,\mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ the vector field $\mathbf{F}$ is conservative.

Let $\mathbf{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$\frac{\partial\phi}{\partial x} = 2xy - z^3$ whence $\phi = x^2y - z^3x + f_1(y,z)$ ... (1)

$$\frac{\partial \phi}{\partial y} = x^2 + z \text{ whence } \phi = x^2 y + zy + f_2(z, x) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = y - 3xz^2 \text{ whence } \phi = yz - xz^3 + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = zy,\ f_2(z, x) = -z^3 x,\ f_3(x, y) = x^2 y.$

$\therefore \phi = x^2 y - z^3 x + zy$ to which may be added any constant.

Hence $\phi = x^2 y - z^3 x + zy + C$.

Ex. 10. *Evaluate*

$$\int_C 2xyz^2\, dx + (x^2 z^2 + z \cos yz)\, dy + (2x^2 yz + y \cos yz)\, dz$$

where C is any path from $(0, 0, 1)$ *to* $(1, \frac{1}{4}\pi, 2)$. **(Meerut 1968)**

Sol. We have $\mathbf{F} = 2xyz^2\, \mathbf{i} + (x^2 z^2 + z \cos yz)\, \mathbf{j} + (2x^2 yz + y \cos yz)\, \mathbf{k}$.

$$\therefore \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2 z^2 + z \cos yz & 2x^2 yz + y \cos yz \end{vmatrix}$$

$$= (2x^2 z + \cos yz - yz \sin yz - 2x^2 z - \cos yz + yz \sin yz)\, \mathbf{i} - (4xyz - 4xyz)\, \mathbf{j} + (2xz^2 - 2xz^2)\, \mathbf{k} = \mathbf{0}.$$

$\therefore$ the given line integral is independent of path in space.

Let $\mathbf{F} = \nabla \phi$. Then

$$\frac{\partial \phi}{\partial x} = 2xyz^2 \text{ whence } \phi = x^2 yz^2 + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = x^2 z^2 + z \cos yz \text{ whence } \phi = x^2 z^2 y + \sin yz + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 2x^2 yz + y \cos yz \text{ whence } \phi = x^2 yz^2 + \sin yz + f_3(x, y) \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = \sin yz,\ f_2(x, z) = 0,\ f_3(x, y) = 0.$

$\therefore \phi = x^2 yz^2 + \sin yz$ to which may be added any constant.

The given line integral is therefore

$$\int_C d(x^2 yz^2 + \sin yz) = \Big[x^2 yz^2 + \sin yz \Big]_{(0, 0, 1)}^{(1, \pi/4, 2)}$$

$$= \pi + \sin \tfrac{1}{2}\pi = \pi + 1.$$

Ex. 11. *Evaluate*

$$\int_C yz\,dx + (xz+1)\,dy + xy\,dz,$$

where C is any path from (1, 0, 0) *to* (2, 1, 4). **(Agra 1972 ; Meerut 64)**

Sol. We have $\mathbf{F} = yz\mathbf{i} + (xz+1)\mathbf{j} + xy\mathbf{k}$.

$$\therefore \quad \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz+1 & xy \end{vmatrix}$$

$$= (x-x)\,\mathbf{i} - (y-y)\,\mathbf{j} + (z-z)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ the differential form $yz\,dx + (xz+1)\,dy + xy\,dz$ is exact and the given line integral is independent of path.

Let $\mathbf{F} = \nabla\phi$

or $\quad yz\,\mathbf{i} + (xz+1)\,\mathbf{j} + xy\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = yz \text{ whence } \phi = xyz + f_1(y,z) \qquad \dots(1)$$

$$\frac{\partial\phi}{\partial y} = xz + 1 \text{ whence } \phi = xyz + y + f_2(x,z) \qquad \dots(2)$$

$$\frac{\partial\phi}{\partial z} = xy \text{ whence } \phi = xyz + f_3(x,y) \qquad \dots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = y,\ f_2(x,z) = 0,\ f_3(x,y) = y.$

$\therefore \phi = xyz + y$ to which may be added any constant.

The given line integral is therefore

$$= \int_{(1,0,0)}^{(2,1,4)} d\,(xyz + y) = \Big[xyz + y\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= [8 + 1 - 0 - 0] = 9.$$

Ex. 12. *Show that the form under the integral sign is exact and evaluate*

$$\int_{(0,2,1)}^{(2,0,1)} [z e^x\,dx + 3yz\,dy + (e^x + y^2)\,dz].$$

Sol. Here $\mathbf{F} = ze^x\,\mathbf{i} + 2yz\,\mathbf{j} + (e^x + y^2)\,\mathbf{k}$.

We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ze^x & 2yz & e^x + y^2 \end{vmatrix}$

$$= (2y - 2y)\,\mathbf{i} - (e^x - e^x)\,\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the form under the integral sign is exact and consequently the line integral is independent of path in space.

Let $\mathbf{F} = \nabla\phi$

or $\quad ze^x\,\mathbf{i} + 2yz\,\mathbf{j} + (e^x + y^2)\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = z\,e^x \text{ whence } \phi = z\,e^x + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = 2\,yz \text{ whence } \phi = y^2z + f_2(z, x) \quad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = e^x + y^2 \text{ whence } \phi = e^xz + y^2z + f_3(x, y) \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = y^2z, f_2(x, z) = e^xz, f_3(x, y) = 0.$

$\therefore \phi = ze^x + y^2z$ to which may be added any constant. The given line integral is therefore

$$= \int_{(0,2,1)}^{(2,0,1)} d\,(ze^x + y^2z) = \left[ze^x + y^2z\right]_{(0,2,1)}^{(2,0,1)}$$

$$= [e^2 + 0 - 1 - 4] = e^2 - 5.$$

Ex. 13. *If* $\mathbf{F} = \cos y\,\mathbf{i} - x\sin y\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is the curve* $y = \sqrt{(1 - x^2)}$ *in the x-y plane from* (1, 0) *to* (0, 1).

Sol. We have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (\cos y\,dx - x\sin y\,dy)$

$$= \int_1^0 \cos\sqrt{(1 - x^2)}\,dx - \int_0^1 \sqrt{(1 - y^2)}\sin y\,dy.$$

It is difficult to evaluate the integrals directly. However we observe that

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & -x\sin y & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + (-\sin y + \sin y)\mathbf{k} = \mathbf{0}.$$

$\therefore$ the given line integral is independent of path.

Let $\mathbf{F} = \nabla\phi$

or $\quad \cos y\,\mathbf{i} - x\sin y\,\mathbf{j} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = \cos y \text{ whence } \phi = x\cos y + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = -x\sin y, \text{ whence } \phi = x\cos y + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 0 \text{ whence } \phi = f_3(x, y). \qquad \ldots (3)$$

From (1), (2), (3), we see that $\phi = x \cos y$.

The given line integral is equal to

$$\int_{(1,0)}^{(0,1)} d(x \cos y) = \Big[x \cos y\Big]_{(1,0)}^{(0,1)} = [0 - 1 \cos 0] = -1.$$

Ex. 14. *Show that the vector field* **F** *given by*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ such that $\mathbf{F} = \nabla\phi$.

Sol. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= (-x + x)\,\mathbf{i} - (-y + y)\,\mathbf{j} + (-z + z)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ The vector field **F** is irrotational.

Let $\mathbf{F} = \nabla\phi$

or $\quad (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = x^2 - yz \text{ whence } \phi = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = y^2 - zx \text{ whence } \phi = \frac{y^3}{3} - xyz + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = z^2 - xy \text{ whence } \phi = \frac{z^3}{3} - xyz + f_3(x, y). \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{y^3}{3} + \frac{z^3}{3},\; f_2(x, z) = \frac{x^3 + z^3}{3},\; f_3(x, y) = \frac{x^3 + y^3}{3}.$$

Therefore $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz + C.$

Ex. 15. *Show that the following vector functions* **F** *are irrotational and find the corresponding scalar ϕ such that*

$$\mathbf{F} = \nabla\phi.$$

(i) $\mathbf{F} = (\sin y + z \cos x)\,\mathbf{i} + (x \cos y + \sin z)\,\mathbf{j}$
$\qquad + (y \cos z + \sin x)\,\mathbf{k}$. **(Calcutta 1975)**

(ii) $\mathbf{F} = (y \sin z - \sin x)\,\mathbf{i} + (x \sin z + 2yz)\,\mathbf{j} + (xy \cos z + y^2)\,\mathbf{k}$.

(iii) $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

Sol. (i) We have curl **F**

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z\cos x & x\cos y + \sin z & y\cos z + \sin x \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y\cos z + \sin x) - \frac{\partial}{\partial z}(x\cos y + \sin z)\right]$$

$$+ \mathbf{j}\left[\frac{\partial}{\partial z}(\sin y + z\cos x) - \frac{\partial}{\partial x}(y\cos z + \sin x)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(x\cos y + \sin z) - \frac{\partial}{\partial y}(\sin y + z\cos x)\right]$$

$$= (\cos z - \cos z)\,\mathbf{i} + (\cos x - \cos x)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

$\therefore$ the vector **F** is irrotational.

Let $\mathbf{F} = \nabla\phi$

i.e., $(\sin y + z\cos x)\,\mathbf{i} + (x\cos y + \sin z)\,\mathbf{j} + (y\cos z + \sin x)\,\mathbf{k}$

$= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = \sin y + z\cos x \text{ whence } \phi = x\sin y + z\sin x + f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y} = x\cos y + \sin z \text{ whence } \phi = x\sin y + y\sin z + f_2(z, x) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z} = y\cos z + \sin x \text{ whence } \phi = y\sin z + z\sin x + f_3(x, y) \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = y\sin z, f_2(z, x) = z\sin x, f_3(x, y) = x\sin y.$

$\therefore \phi = x\sin y + z\sin x + y\sin z$ to which may be added any constant.

Hence $\phi = x\sin y + z\sin x + y\sin z + C.$

(ii) Do yourself. **Ans.** $\phi = xy\sin z + \cos x + y^2 z + C.$

(iii) Do yourself. **Ans.** $\phi = \frac{1}{4}(x^4 + y^4 + z^4) + C.$

Ex. 16. *Find a, b, c if* $\mathbf{F} = (3x - 3y + az)\,\mathbf{i} + (bx + 2y - 4z)\,\mathbf{j} + (2x + cy + z)\,\mathbf{k}$ *is irrotational.* **(Calicut 1974)**

Sol. The vector **F** is irrotational if and only if curl **F** = **0**. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x - 3y + az & bx + 2y - 4z & 2x + cy + z \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(2x + cy + z) - \frac{\partial}{\partial z}(bx + 2y - 4z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(3x - 3y + az) - \frac{\partial}{\partial x}(2x + cy + z)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(bx + 2y - 4z) - \frac{\partial}{\partial y}(3x - 3y + az)\right]$$

$= (c + 4)\,\mathbf{i} + (a - 2)\,\mathbf{j} + (b + 3)\,\mathbf{k}.$

Now curl $\mathbf{F} = \mathbf{0}$ if $(c + 4)\mathbf{i} + (a - 2)\,\mathbf{j} + (b + 3)\,\mathbf{k} = \mathbf{0}$

i.e., if $c + 4 = 0,\ a - 2 = 0,\ b + 3 = 0$

i.e., if $a = 2,\ b = -3,\ c = -4$.

Hence the given vector $\mathbf{F}$ is irrotational if $a = 2,\ b = -3, c = -4$.

Ex. 17. *Show that*

$$(2x \cos y + z \sin y)\, dx + (xz \cos y - x^2 \sin y)\, dy + x \sin y\, dz = 0$$

is an exact differential equation and hence solve it.

Sol. The given differential equation is exact if there exists a scalar function $\phi(x, y, z)$ such that

$$(2x \cos y + z \sin y)\, dx + (xz \cos y - x^2 \sin y)\, dy + x \sin y\, dz = d\phi.$$

Let $\mathbf{F} = (2x \cos y + z \sin y)\,\mathbf{i} + (xz \cos y - x^2 \sin y)\,\mathbf{j} + x \sin y\,\mathbf{k}$.

We have curl $\mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x \cos y + z \sin y & xz \cos y - x^2 \sin y & x \sin y \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(x \sin y) - \frac{\partial}{\partial z}(xz \cos y - x^2 \sin y)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(2x \cos y + z \sin y) - \frac{\partial}{\partial x}(x \sin y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(xz \cos y - x^2 \sin y) - \frac{\partial}{\partial y}(2x \cos y + z \sin y)\right]$$

$$= (x \cos y - x \cos y)\,\mathbf{i} + (\sin y - \sin y)\,\mathbf{j} + \left[(z \cos y - 2x \sin y) - (-2x \sin y + z \cos y)\right]\mathbf{k}$$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ the vector $\mathbf{F}$ is conservative.

Hence there exists a scalar function $\phi(x, y, z)$ such that

$$\mathbf{F} = \nabla\phi.$$

$\therefore\ \mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r}$

or $(2x \cos y + z \sin y)\, dx + (xz \cos y - x^2 \sin y)\, dy + x \sin y\, dz = d\phi$.

Hence the given differential equation is exact.

Now $\mathbf{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Therefore

$\frac{\partial \phi}{\partial x} = 2x\cos y + z\sin y$ whence $\phi = x^2\cos y + xz\sin y + f_1(y,z)$ ... (1)

$\frac{\partial \phi}{\partial y} = xz\cos y - x^2\sin y$ whence $\phi = xz\sin y + x^2\cos y + f_2(z,x)$... (2)

$\frac{\partial \phi}{\partial z} = x\sin y$ whence $\phi = xz\sin y + f_3(x,y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = 0, f_2(z,x) = 0, f_3(x,y) = x^2\cos y$.

$\therefore \phi = x^2\cos y + xz\sin y$.

Now the given differential equation reduces to

$d\phi = 0$ whose solution is $\phi = C$

i.e., $x^2\cos y + xz\sin y = C$.

Ex. 18. *If* **F** *is irrotational in a simply connected region R, show that there exists a scalar field ϕ such that* $\mathbf{F} = \text{grad}\,\phi$. **(Calicut 1975)**

Sol. Since **F** is irrotational in a simply connected region R, therefore curl $\mathbf{F} = \mathbf{0}$ in R.

Let C be any simple closed path in R. Then by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS, \text{ where } S \text{ is any surface in } R$$
whose boundary is the closed curve C.

$$= 0, \text{ since curl } \mathbf{F} = \mathbf{0}.$$

$\therefore$ the line integral $\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ is independent of the path joining any two points P and Q in R.

Let P be a fixed point (x_0, y_0, z_0) in R and let Q by any point (x,y,z) in R. Let

$$\phi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_0,y_0,z_0)}^{(x,y,z)} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get

$$\frac{d\phi}{ds} = \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}.$$

But $$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}.$$

$$\therefore \quad \mathbf{F}\cdot\frac{d\mathbf{r}}{ds} = \nabla\phi\cdot\frac{d\mathbf{r}}{ds} \quad \text{or} \quad (\nabla\phi - \mathbf{F})\cdot\frac{d\mathbf{r}}{ds} = 0.$$

Now this result is true irrespective of the path joining P to Q i.e., this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$$\nabla\phi - \mathbf{F} = \mathbf{0}$$

i.e., $\nabla\phi = \mathbf{F}$ *i.e.*, $\mathbf{F} = \text{grad}\,\phi$.