1(a), P-II 2018. Let R be an integral domain with unit element. Show that any unit in $R[x]$ is a unit in R.

Sol^n Let a_0 be a unit of R. Then a_0 divides 1

i.e., $a_0 / 1$

i.e. there exists some $b_0 \in R$ such that

$$a_0 b_0 = 1$$

Let $f(x) = a_0 + 0x + 0x^2 + \cdots$

$g(x) = b_0 + 0x + 0x^2 + \cdots$

Then $f(x), g(x) \in R[x]$ and

$$f(x)g(x) = a_0 b_0 + 0x + 0x^2 + \cdots$$

or $f(x)g(x) = 1 \quad (\because a_0 b_0 = 1)$

$\Rightarrow f(x)/1$ (i.e. $f(x)$ divides 1) in $R[x]$

$\Rightarrow f(x)$ is a unit in $R[x]$

Hence $a_0 = f(x)$ is a unit in $R[x]$.

Conversely, let $f(x)$ be a unit of $R[x]$.

Then there exists some $g(x) \in R[x]$ such that

$$f(x)g(x) = 1 = 1 + 0\cdot x + 0\cdot x^2 \quad \text{—— (1)}$$

$\Rightarrow \deg(f(x)g(x)) = \deg(1 + 0x + 0x^2 + \cdots)$
$= 0$

$\Rightarrow \deg f(x) + \deg g(x) = 0 \quad (\because R \text{ is ID. } \deg(f(x)g(x)) = \deg f(x) + \deg g(x))$

$\Rightarrow \deg f(x) = 0$ and $\deg g(x) = 0$

$\Rightarrow$ $f(x)$ and $g(x)$ are constant polynomials

Say $f(x) = \alpha$ $(\alpha \neq 0 \in R)$, $g(x) = \beta$ $(\beta \neq 0 \in R)$

$\Rightarrow \alpha\beta = 1$ by (1)

$\Rightarrow \alpha \mid 1$ (i.e. α divides 1) in R

Hence $f(x) = \alpha$ is a unit of R.

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1(b) IAS 2018 P-II

Prove the inequality: $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx < \frac{2\pi^2}{9}$.

Sol^n

$1 \le \frac{1}{\sin x} \le 2$ for all $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Therefore $x \le \frac{x}{\sin x} \le 2x$ for all $x \in [\pi/6, \pi/2]$

Let $f(x) = \frac{x}{\sin x}$

$\phi(x) = x$

$\psi(x) = 2x$, $x \in [\pi/6, \pi/2]$.

f and ϕ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \ge \phi(x)$ for all $x \in [\pi/6, \pi/2]$

Also f and ϕ are both continuous at $\pi/3$ and $f(\pi/3) > \phi(\pi/3)$.

Hence, $\int_{\pi/6}^{\pi/2} f(x)\,dx > \int_{\pi/6}^{\pi/2} \phi(x)\,dx$

$= \int_{\pi/6}^{\pi/2} x\,dx$

$= \frac{\pi^2}{9}$.

f and ψ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \le \psi(x)$ for all $x \in [\pi/6, \pi/2]$

Also f and ψ are both continuous at $\pi/3$ and $f(\pi/3) < \psi(\pi/3)$.

Hence $\int_{\pi/6}^{\pi/2} f(x)\,dx < \int_{\pi/6}^{\pi/2} \psi(x)\,dx$

$= 2\int_{\pi/6}^{\pi/2} x\,dx$

$= \frac{2\pi^2}{9}$

Consequently, $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx < \frac{2\pi^2}{9}$.

1(c) → prove that the function $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Sol Given that $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ —— (1)

$\therefore \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$; $\frac{\partial^2 u}{\partial x^2} = 6(x-1)$

$\frac{\partial u}{\partial y} = -6xy + 6y$; $\frac{\partial^2 u}{\partial y^2} = -6x+6 = -6(x-1)$.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$\therefore u(x,y)$ is a harmonic function.

clearly $u(x,y)$ be a real part of analytic function $f(z) = u + iv$.

$\because f(z)$ is an analytic function

$\therefore u_x = v_y$; $u_y = -v_x$.

$v_x = -u_y = -\frac{\partial u}{\partial y} = -(-6xy + 6y) = 6xy - 6y$.

$\therefore \frac{\partial v}{\partial x} = 6xy - 6y$

$\Rightarrow v(x,y) = 3x^2y - 6xy + f_1(y)$ —— (1)

$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 6x + f_1'(y)$

$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 6x + f_1'(y)$ (by C-R equations)

$\Rightarrow 3(x-1)^2 - 3y^2 = 3x^2 - 6x + f_1'(y)$

$\Rightarrow 3(x^2+1-2x) - 3y^2 = 3x^2 - 6x + f_1'(y)$

$\Rightarrow \cancel{3x^2} - \cancel{6x} + 3 - 3y^2 = \cancel{3x^2} - \cancel{6x} + f_1'(y)$

$\Rightarrow f_1'(y) = 3 - 3y^2$

$\Rightarrow \boxed{f_1(y) = 3y - y^3 + C}$

$\therefore (1) \equiv v(x,y) = 3x^2y - 6xy + 3y - y^3 + C$.

Now let us find $f(z)$ in terms of z.

$\because \dfrac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 = \phi_1(x,y)$ say.

and $\dfrac{\partial u}{\partial y} = -6xy + 6y = \phi_2(x,y)$ say.

$\therefore$ By Milne's Method,

$$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$$

$$= 3(z-1)^2 - 3(0)^2 - i\left[6(z)(0) + 6(0)\right]$$

$$= 3(z-1)^2$$

$$\therefore f(z) = (z-1)^3 + C.$$

1(d) → IAS 2018 P-II

find the range of $(p>0)$ for which the series:

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots\cdots, \; a>0 \text{ is}$$

(i) absolutely convergent and

(ii) conditionally convergent.

Sol^n →

Let $\sum u_n$ be the given series and $v_n = |u_n|$.

Then $\sum v_n$ is a series of positive real numbers and $v_n = \frac{1}{(n+a)^p}$.

Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$

By comparision test $\sum v_n$ is convergent if $p>1$

$\sum v_n$ is divergent if $0<p\le 1$.

Case-1. $p>1$

In this case $\sum u_n$ is an alternating series and $\sum |u_n|$ is convergent.

Therefore $\sum u_n$ is absolutely convergent.

Case-2 $0<p\le 1$

In this case $\{v_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim v_n = 0$

By Leibnitz's test, $\sum (-1)^{n+1} v_n$

i.e. $\sum u_n$ is convergent.

Since $\Sigma|u_n|$ is divergent, Σu_n is conditionally convergent.

Let Σu_n be a series of positive real numbers and let.

$$p_n = u_n \text{ if } u_n > 0 \quad , q_n = 0 \text{ if } u_n \geq 0$$
$$= 0 \text{ if } u_n \leq 0 \quad , \quad = u_n \text{ if } u_n < 0$$

Then Σp_n is a series of positive real numbers along with some 0's and Σq_n is a series of negative real numbers along with some 0's.

for example, for the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$\Sigma p_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots$ and

$\Sigma q_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \cdots$

$$p_n = \frac{u_n + |u_n|}{2} \quad , \quad q_n = \frac{u_n - |u_n|}{2}$$

and $\quad u_n = p_n + q_n$

1(e) An agricultural firm has 180 tons of nitrogen fertilizer, 250 tons of phosphate and 220 tons of potash. It will be able to sell a mixture of these substances in their respective ratio 3:3:4 at a profit of Rs. 1500 per ton and a mixture of these substances in the ratio 2:4:2 at a profit of Rs. 1200 per ton. Pose a linear programming problem to show how many tons of these two mixtures should be prepared to obtain the max. profit.

Sol.

	Nitrogen	Phosphate	Potash	Rs/ton
(x) Mix 1	3/10	3/10	4/10	₹1500
(y) Mix 2	2/8 = 1/4	4/8 = 1/2	2/8 = 1/4	₹1200
Tonnes	180	250	220	

Let x tons of mix 1 and y tons of mix 2 should be prepared to obtain the max. profit.

Thus, LPP can be formulated as:

$$\text{Max } z = 1500x + 1200y$$

$$\frac{3x}{10} + \frac{y}{4} \le 180 \quad \Rightarrow \quad 6x + 5y \le 3600$$

$$\frac{3x}{10} + \frac{y}{2} \le 250 \quad \Rightarrow \quad 3x + 5y \le 2500$$

$$\frac{4x}{10} + \frac{y}{4} \le 220 \quad \Rightarrow \quad 8x + 5y \le 4400$$

$$\forall\, x, y \ge 0$$

Adding the slack variables,

$$\text{Max } z = 1500x + 1200y + 0S_1 + 0S_2 + 0S_3$$

$$6x + 5y + S_1 + 0S_2 + 0S_3 = 3600$$

$$3x + 5y + 0S_1 + S_2 + 0S_3 = 2500$$

$$8x + 5y + 0S_1 + 0S_2 + S_3 = 4400$$

where, $S_1, S_2, S_3 \geq 0$ are slack variables and $x, y \geq 0$.

Now the initial basic feasible solution is given by, setting $x = y = 0$ (non-basic)

$S_1 = 3600,\ S_2 = 2500,\ S_3 = 4400$

$\therefore$ The IBFS is $(0, 0, 3600, 2500, 4400)$ for which $z = 0$,

c_B	c_j Basis	1500 x	1200 y	0 S_1	0 S_2	0 S_3	b	θ
0	S_1	6	5	1	0	0	3600	600
0	S_2	3	5	0	1	0	2500	2500/3
0	S_3	$\boxed{8}$	5	0	0	1	4400	550 →
	$z_j = \Sigma c_B a_{ij}$	0	0	0	0	0	0	
	$c_j^* = c_j - z_j$	1500 ↑	1200	0	0	0	—	
0	S_1	0	$\boxed{5/4}$	1	0	$-3/4$	300	240 →
0	S_2	0	25/8	0	1	$-3/8$	850	272
1500	x	1	5/8	0	0	1/8	550	880
	z_j	1500	937.5	0	0	187.5	825,000	
	$c_j - z_j$	0	262.5 ↑	0	0	-187.5	—	
1200	y	0	1	4/5	0	$-3/5$	240	
0	S_2	0	0	$-5/2$	1	3/2	100	
1500	x	1	0	$-1/2$	0	1/2	400	
	z_j	1500	1200	210	0	30	888,000	
	$c_j - z_j$	0	0	-210	0	-30	—	

$\therefore$ As all $c_j - z_j \leq 0$, thus the optimal cond^n has been obtained.

Thus, $x = 400$, $y = 240$ tonnes of mix 1 and mix 2 are required for max. profit [Could also be solved by GRAPHICAL METHOD

2(c) → Show that the quotient group of $(\mathbb{R},+)$ modulo $\mathbb{Z}$ is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers.

Sol: let $\frac{\mathbb{R}}{\mathbb{Z}} = \{ \mathbb{Z}+a / a \in \mathbb{R} \}$ be a quotient group w.r.t + modulo $\mathbb{Z}$.

let T be a group of all complex numbers on the unit circle in the complex plane.

then $T = \{ z \in \mathbb{C} / |z| = 1 \}$

$= \{ x+iy \in \mathbb{C} / x^2+y^2 = 1 \}$

$= \{ e^{i2\pi a} / a \in \mathbb{R} \}$

To show that $\frac{\mathbb{R}}{\mathbb{Z}}$ is an isomorphic to T.

i.e $\frac{\mathbb{R}}{\mathbb{Z}} \cong T$.

Define a mapping $f: \mathbb{R} \to T$ such that

$f(a) = e^{2a\pi i}$ $\forall a \in \mathbb{R}$ — (1)

To s.T $f: \mathbb{R} \to T$ is well-defined:

let $a, b \in \mathbb{R}$ such that

$a = b$

$\Rightarrow 2a\pi i = 2b\pi i$

$\Rightarrow e^{2a\pi i} = e^{2b\pi i}$

$\Rightarrow f(a) = f(b)$

$\therefore f$ is well-defined.

To show that $f: \mathbb{R} \to T$ is onto:

for $e^{2a\pi i} \in T$ there exists $a \in \mathbb{R}$

$\therefore$ by definition, $f(a) = e^{2a\pi i}$.

$\therefore$ f is onto

To show that $f: \mathbb{R} \to T$ is a homomorphism:

let $a, b \in \mathbb{R}$ s.t. $f(a) = e^{2a\pi i}$, $f(b) = e^{2b\pi i}$

we have

$$f(a+b) = e^{2(a+b)\pi i} = e^{2a\pi i} \cdot e^{2b\pi i}$$

$$= f(a) \cdot f(b)$$

$\therefore$ f is homomorphism.

$\therefore$ By fundamental theorem of homomorphism

$$\frac{\mathbb{R}}{K} \cong T$$

where $K = \ker f = \{a \in \mathbb{R} / f(a) = 1\}$

$$= \{a \in \mathbb{R} / e^{2a\pi i} = 1\}$$

$$= \{a \in Z / e^{2a\pi i} = 1\}$$

$$= Z.$$

$$\therefore \frac{\mathbb{R}}{Z} \cong T$$

1(b) Solve the following LPP by Big M-method and show that the problem has finite optimal solutions. Also find the value of the obj. function:

$$\text{Min } z = 3x_1 + 5x_2$$

subject to
$$x_1 + 2x_2 \geq 8$$
$$3x_1 + 2x_2 \geq 12$$
$$5x_1 + 6x_2 \leq 60$$
$$x_1, x_2 \geq 0$$

Ans. The objective function of the given LPP is of minimization type.

Thus, $\text{Max } z' = \text{Min}(-z)$
$$= -3x_1 - 5x_2$$

Thus, LPP in standard form can be written as:

$$\text{Max } z' = -3x_1 - 5x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$$

subject to

$$x_1 + 2x_2 - S_1 + 0S_2 + 0S_3 + A_1 + 0A_2 = 8$$
$$3x_1 + 2x_2 + 0S_1 - S_2 + 0S_3 + 0A_1 + A_2 = 12$$
$$5x_1 + 6x_2 + 0S_1 + 0S_2 + S_3 + 0A_1 + 0A_2 = 60$$
$$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$$

where, S_1, S_2 are surplus var., S_3 is slack var. and $A_1 A_2$ are artificial variables.

IBFS is given by,

$$x_1 = x_2 = 0, \; S_1 = S_2 = 0, \; A_1 = 8, \; A_2 = 12, \; S_3 = 60$$

$\Rightarrow (x_1, x_2, S_1, S_2, A_1, A_2, S_3)$

$$= (0, 0, 0, 0, 8, 12, 60)$$

		-3	-5	0	0	0	-M	-M		
C_B	Basis	x_1	x_2	S_1	S_2	S_3	A_1	A_2	b	θ
-M	A_1	1	2	-1	0	0	1	0	8	8
-M	A_2	$\boxed{3}$	2	0	-1	0	0	1	12	4 →
0	S_3	5	6	0	0	1	0	0	60	60/5 = 12
	z_j	-4M	-4M	M	M	0	-M	-M	-20M	
	$C_j = c_j - z_j$	-3+4M ↑	-5+4M	-M	-M	0	0	0		
-M	A_1	0	$\boxed{4/3}$	-1	1/3	0	1	$\cancel{1}$	4	3 →
-3	x_1	1	2/3	0	-1/3	0	0	$\cancel{1/3}$	4	6
0	S_3	0	8/3	0	5/3	1	0	–	40	15
	z_j	-3	$-2-\frac{4M}{3}$	M	$-\frac{M}{3}+1$	0	-M	–	-4M-12	
	$c_j - z_j$	0	$-3+\frac{4M}{3}$ ↑	-M	$\frac{M}{3}-1$	0	0	–		
-5	x_2	0	1	-3/4	1/4	0	3/4	–	3	
-3	x_1	1	0	1/2	-1/2	0	-1/2	–	2	
0	S_3	0	0	2	1	1	–	–	32	
	z_j	-3	-5	2	1/4	0	–	–	-21	
	$c_j - z_j$	0	0	-2	-1/4	0	–	–		

As all $c_j - z_j \geq 0$, thus the optimality has been reached and, $x_1 = 2$, $x_2 = 3$

$\therefore$ Max $z' = -21$

$\Rightarrow$ Min $z = 21$

Thus, the value of the objective function is 21.

2(c) Show that if a function f defined on an open interval (a, b) of $\mathbb{R}$ is convex, then f is continuous. Show by example if the condition of open interval is dropped, then the ~~continuous~~ convex function need not be cts.

Ans. Suppose f is convex on (a,b) and let $[c,d] \subseteq (a,b)$.

Choose c_1 and d_1 such that

$$a < c_1 < c < d < d_1 < b$$

If $x, y \in [c, d]$ with $x < y$. Then, we know that,

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(x)-f(c)}{x-c} \geq \frac{f(c)-f(c_1)}{c-c_1}$$

and,

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(d)-f(y)}{d-y} \leq \frac{f(d_1)-f(d)}{d_1-d}$$

showing the set

$$\left\{ \left| \frac{f(y)-f(x)}{y-x} \right| : c \leq x < y \leq d \right\}$$

is bounded by $M > 0$. It follows that

$$|f(y)-f(x)| \leq M\,|y-x|$$

and therefore f is uniformly continuous on $[c, d]$.

Recalling that uniformly cts $\Rightarrow$ continuity.

$\Rightarrow f$ is cts on $[c, d]$.

Since the interval $[c, d]$ was arbitrary, f is continuous on (a, b).

Example: Suppose $a > 0$ and define $f: [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{for } a \leq x \leq c \\ (x-c)^2 + c^2 & \text{for } c \leq x \leq b \end{cases}$$

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Clearly f is cts at $x=c$ as

$f(c)=c^2$

$\lim_{h\to 0} f(x$

$\lim_{x\to c^-} f(x+0) = \lim_{x\to c^-} f(x) = \lim_{x\to c^+} c^2 = c^2$

$\lim_{x\to c^+} f(x) = \lim_{x\to c^+} (x-c)^2+c^2 = c^2$

$\Rightarrow f$ is cts on $[a,b]$.

but f is not convex on $[a,b]$

as, $f^-(x) = 2x]_{x=c} = 2c$

$f^+(x) = 2(x-c)]_{x=c} = 0$

Now, $f^-(x) \nleq f^+(x)$ $(\because c>0)$

$\forall x\in(a,b)$

$\therefore$ f is not convex on $[a,b]$ as left hand derivative of $f(x)$ is not less than or equal to $f^+(x)$.

3(a) Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 resp.

Ans. First we shall show that $\mathbb{Z}_{13} = \{1, 2, 3, ---, 12\}$ under $\times^n$ is cyclic with generator 2.

Since 13 is prime, $\mathbb{Z}_{13}^* = \{1, 2, ---, 12\}$ is a cyclic group.

($\because$ $\mathbb{Z}_p$ is cyclic iff p is prime)

Now, 2 is a generator of $\mathbb{Z}_{13}$ as:

$2^1 = 2$	$2^7 = 11$
$2^2 = 4$	$2^8 = 9$
$2^3 = 8$	$2^9 = 5$
$2^4 = 16 = 3$	$2^{10} = 10$
$2^5 = 32 = 6$	$2^{11} = 7$
$2^6 = 12$	$2^{12} = 1$

$\Rightarrow \mathbb{Z}_{13} = \langle 2 \rangle$ is cyclic of order 12, with generator 2,

Now, by Fundamental Theorem of cyclic Groups, Order of every subgroup divides the order of the group (by Lagrange)

Also, subgroup of the cyclic group is also cyclic.

Since the divisors of 12 are 1, 2, 3, 4, 6 and 12.

Also, every subgroup of the cyclic group of order n is a divisor of n and there is exactly one subgroup for each divisor. This is called the fundamental theorem of cyclic groups.

Now, the subgroups of Z_{13} are,

$\langle 1\rangle = \langle 2^{12}\rangle$, $\langle 2^6\rangle = \langle 12\rangle$, $\langle 2^4\rangle = \langle 3\rangle$,

$\langle 2^3\rangle = \langle 8\rangle$

$\langle 2^2\rangle = \langle 4\rangle$

$\langle 2\rangle = \langle 2\rangle = Z_{13}$.

Thus, the proper subgroups are:

$\langle 12\rangle$

$\langle 3\rangle$

$\langle 8\rangle$ Ans.

$\langle 4\rangle$

3 (b) Show by applying the residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}$, $a>0$

Ans. Consider $\int_C \frac{dz}{(z^2+a^2)^2} = \int_C f(z)\,dz$,

where $f(z) = \frac{1}{(z^2+a^2)^2}$, C is the contour consisting of a large semi-circle Γ of radius R together with part of real axis from $x=-R$ to R.

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{dx}{(a^2+x^2)^2} + \int_\Gamma \frac{dz}{(a^2+z^2)^2}$$

$$= 2\pi i \sum R^+ \qquad \text{---(1)}$$

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{(a^2+z^2)^2}$$

$$= \lim_{z\to\infty} \frac{z}{z^4\left(\frac{a^2}{z^2}+1\right)^2} = 0$$

$$\therefore \lim_{R\to\infty} \int_\Gamma \frac{dz}{(a^2+z^2)^2} = 0$$

$$\text{and } \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2}$$

Taking $R\to\infty$ in (1), we get,

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} + 0 = 2\pi i \sum R^+ \qquad \text{---(2)}$$

Poles of $f(z) = \dfrac{1}{(a^2+z^2)^2}$ are given by $(a^2+z^2)^2 = 0$

$\Rightarrow z = \pm ai$ (twice) out of which only pole $z = ai$ (order 2) lies inside C.

$\therefore$ Residue at $ai = \dfrac{\phi'(ai)}{\lfloor 1} = \phi'(ai)$

Here, $\phi(z) = (z-ai)^2 f(z)$

$$= (z-ai)^2 \times \frac{1}{(a^2+z^2)^2}$$

$$= \cancel{(z-ai)^2} \times \frac{1}{\cancel{(z-ai)^2}(z+ai)^2} = \frac{1}{(z+ai)^2}$$

$$\therefore \phi'(z) = -\frac{2}{(z+ai)^3}$$

Therefore, residue at $ai = \phi'(ai) = \dfrac{-2}{(ai+ai)^3}$

$$= \frac{+2}{8a^3 i} = \frac{1}{4a^3 i}$$

$\therefore$ from (2), we have,

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} = 2\pi i \times \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{2a^3}$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \ a>0$$

3 (c) How many basic solutions are there in the following linearly independent set of equations? Find all of them.

Ans. Here, $2x_1 - x_2 + 3x_3 + x_4 = 6$

$4x_1 - 2x_2 - x_3 + 2x_4 = 10$

The given system of eq's can be written in the matrix form as $Ax = b$

where $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & -1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

Since the rank of A is 2 (i.e. m=2), the maximum number of linearly indep. columns of A is 2.

Thus, we consider any of the 2×2 submatrices as basic matrix B.

$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$, $\begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$,

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$

Let $B = \begin{bmatrix} +2 & -1 \\ 4 & -2 \end{bmatrix}$

A basic soln to the system is obtained by taking $x_3 = x_4 = 0$ and solving the system

$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

$\Rightarrow 2x_1 - x_2 = 6$

$4x_1 - 2x_2 = 10$

As, $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = -4+4 = 0$

$\Rightarrow B$ is not LI.

If, $B = \begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}$, then $x_1 = x_4 = 0$

$$\begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$-x_2 + 3x_3 = 6 \quad \Rightarrow -2x_2 + 6x_3 = 12$$

$$-2x_2 - x_3 = 10$$

$$7x_3 = 2 \Rightarrow x_3 = 2/7$$

$$\therefore x_2 = 3x_3 - 6 = \frac{6}{7} - 6 = \frac{-36}{7}$$

$\therefore \left(0, -\frac{36}{7}, \frac{2}{7}, 0\right)^T$ is one of the basic soln.

If $B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $x_1 = x_2 = 0$

$$\therefore \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$3x_3 + x_4 = 6 \Rightarrow 6x_3 + 2x_4 = 12$$

$$-x_3 + 2x_4 = 10$$

$$7x_3 = 2 \Rightarrow x_3 = 2/7$$

$$\Rightarrow x_4 = 6 - 3x_3 = 6 - \frac{6}{7} = \frac{36}{7}$$

$\therefore \left(0, 0, \frac{2}{7}, \frac{36}{7}\right)^T$ is one of the basic soln.

If $B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $x_2 = x_4 = 0$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\Rightarrow 2x_1 + 3x_3 = 6$$

$$4x_1 - x_3 = 10 \quad \Rightarrow 12x_1 - 3x_3 = 30$$

$$14x_1 = 36 \Rightarrow x_1 = \frac{18}{7}$$

$$x_3 = 4x_1 - 10 = \frac{72}{7} - 10 = \frac{2}{7}$$

$\therefore \left(\frac{18}{7}, 0, \frac{2}{7}, 0\right)^T$ is one of the basic soln

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If $B = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$,

$\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = -2+2=0$

$\Rightarrow B$ is not LI

If $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ then $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4-4 = 0$

$\therefore B$ is not LI

Thus, $\left(0, 0, \frac{2}{7}, \frac{36}{7}\right)^T$, $\left(\frac{18}{7}, 0, \frac{2}{7}, 0\right)^T$

and $\left(0, -\frac{36}{7}, \frac{2}{7}, 0\right)^T$

are the basic solutions.

4(a) Suppose $\mathbb{R}$ be the set of all real numbers and $f: \mathbb{R} \to \mathbb{R}$ is a function such that the following equation hold for all $x, y \in \mathbb{R}$.

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(xy) = f(x) f(y)$

show that $\forall\, x \in \mathbb{R}$ either $f(x) = 0$ or $f(x) = x$.

Sol^n Let us consider that $f(x) \neq 0$, then we shall show that $f(x) = x$

As $f(x+x) = f(x) + f(x) = 2f(x)$ —— ①

$f(2x) = 2f(x) \quad \forall\, x, y \in \mathbb{R}$.

from ① –

$f(2) f(x) = 2f(x)$

$\Rightarrow f(2) f(x) - 2f(x) = 0$

$\therefore f(x)(f(2) - 2) = 0$

$\because f(x) \neq 0 \Rightarrow f(2) = 2$

Similarly $\forall\, k \in \mathbb{R}$, $f(kx) = f(x + \cdots + x)$

$= f(x) + \cdots + f(x) = k f(x)$

$\Rightarrow f(k) f(x) = k f(x)$ —— ①

$(\because f(x) \neq 0)$

$\therefore f(k) = k \quad \forall\, k \in \mathbb{R}$.

$\Rightarrow f(x) = x \quad \forall\, x \in \mathbb{R}$

Now let us assume that $f(x) \neq x$, for some x, then we need to show that $f(x) = 0$

As, $f(k)f(x) = k f(x)$ (from ①)

$\Rightarrow f(x)[f(k) - k] = 0$

Since, $f(x) = 0$ or $f(k) = k$

but $f(k) \neq k \Rightarrow f(x) = 0$

Hence we are done.

4(b) Find the Laurent's series which represent the function $\frac{1}{(1+z^2)(z+2)}$ when.

(i) $|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$

Sol^n Solving $f(z)$ into partial fractions we obtain $f(z) = \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$

(i) for $|z|<1$, $f(z)$ is analytic and so we have $f(z) = \frac{1}{5}\cdot\frac{1}{2}\left[1+\frac{z}{2}\right]^{-1} - \frac{1}{5}(z-2)(1+z^2)^{-1}$

$$= \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}+\cdots+(-1)^n\frac{z^n}{2^n}+\cdots\right]$$

$$-\frac{z-2}{5}\left[1-z^2+z^4-z^6+\cdots(-1)^n z^{2n}+\cdots\right]$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \frac{(z-2)}{5}\sum_{n=0}^{\infty}(-1)^n z^{2n}$$

This series being in the +ve powers of z represents Taylor's expansion for $f(z)$.

(ii) $1<|z|<2$, we have.

$$f(z) = \frac{1}{5}\cdot\frac{1}{2}\left(1+\frac{1}{2}z\right)^{-1} - \frac{z-2}{5}\cdot\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right] - \frac{z-2}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right]$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} - \frac{z-2}{5z^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}}.$$

$\left(\because |z|>1, \Rightarrow \frac{1}{|z|}<1 \Rightarrow \frac{1}{|z|^2}<1\right.$

So that binomial expansion of $\left(1+\frac{1}{z^2}\right)^{-1}$ is valid.

The above series being in positive and negative powers of z represents Laurent's expansion for $f(z)$ in the region $1<|z|<2$.

(iii) for $|z|>2$ we have

$$f(z)=\frac{1}{5}\cdot\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{1}{5}(z-2)\frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$=\frac{1}{5z}\left[1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+\cdots\right]-\frac{z-2}{5z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\cdots\right]$$

$$=\frac{1}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{z-2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}}.$$

4(c) →

Machine

	M_1	M_2	M_3	M_4	M_5
O_1	24	29	18	32	19
O_2	17	26	34	22	21
O_3	27	16	28	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

In a factory there are five operators O_1, O_2, O_3, O_4, O_5 and five machines m_1, m_2, m_3, m_4, m_5. The operating costs are given when the O_i operator operates the m_j machine $(i, j = 1, 2, \cdots, 5)$. But there is a restriction that O_3 cannot be fallowed to operate the third machine m_3 and O_2 cannot be allowed to operate the fifth machine m_5. The cost matrix is given above. Find the optimal assignment and the optimal assignment cost also.

Sol^n →

Assigning 'M' to the restricted combinations or introducing M, where M is a large value.

P.T.O.

	M_1	M_2	M_3	M_4	M_5
O_1	24	29	18	32	19
O_2	17	26	34	22	M
O_3	27	16	M	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

Reducing the matrix row wise

	M_1	M_2	M_3	M_4	M_5
O_1	6	11	0	14	1
O_2	0	9	17	5	M
O_3	11	0	M	1	9
O_4	4	0	10	12	6
O_5	12	0	15	8	11

Reducing column wise.

	M_1	M_2	M_3	M_4	M_5
O_1	6	11	0	13	0
O_2	0	9	17	4	M
O_3	11	0	M	0	8
O_4	4	0	10	11	5
O_5	12	0	15	7	10

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All zeroes to be covered (since the no. of lines covered are less than 5, thus).

Subtracting smallest element 4 from the uncovered elements and adding at the inthal.

	M_1	M_2	M_3	M_4	M_5
O_1	6	15	0	13	0
O_2	0	13	17	4	M
O_3	11	4	M	0	8
O_4	0	0	6	7	1
O_5	8	0	11	3	6

Again $4 < 5$

	M_1	M_2	M_3	M_4	M_5
O_1	7	16	0	13	0
O_2	0	13	16	3	M
O_3	12	5	M	0	8
O_4	0	0	5	6	0
O_5	8	0	10	2	5

Now the zero's are covered.

Thus optional assignment is –

O_1M_3, O_2M_1, O_3M_4, O_4M_5, O_5M_2.

and cost is $18+17+17+24+16$

$= 92$.

5(a) Find the partial differential equation of the family of all tangent planes to the ellipsoid: $x^2+4y^2+4z^2=4$, which are not perpendicular to the xy-plane.

Sol'n: Given that $x^2+4y^2+4z^2=4$ —— (1)

Its tangent plane at a point $P(x_1,y_1,z_1)$ is

$$xx_1+4yy_1+4zz_1=4 \quad \text{—— (2)}$$

Let the plane be $lx+my+nz=p$ —— (3)

then $\dfrac{x_1}{l}=\dfrac{4y_1}{m}=\dfrac{4z_1}{n}=\dfrac{4}{p}$

$$\Rightarrow x_1=\frac{4l}{p},\quad y_1=\frac{m}{p},\quad z_1=\frac{n}{p}$$

$$(1)\equiv \frac{16l^2}{p^2}+\frac{4m^2}{p^2}+\frac{4n^2}{p^2}=4$$

$$\Rightarrow 4l^2+m^2+n^2=p^2$$

$$\therefore (3)\equiv lx+my+nz=\pm\sqrt{4l^2+m^2+n^2} \quad \text{—— (4)}$$

Since it is not perpendicular to xy-plane,

$\therefore n\neq 0$

$$\therefore (4)\equiv \frac{l}{n}x+\frac{m}{n}y+z=\pm\sqrt{4\left(\frac{l}{n}\right)^2+\left(\frac{m}{n}\right)^2+1}$$

$$\Rightarrow \alpha x+\beta y+z=\pm\sqrt{4\alpha^2+\beta^2+1} \quad \text{—— (5)}$$

which is the required tangent plane

Differentiating (4) partially w.r.t 'x' & 'y'

we get

$$\alpha+\frac{\partial z}{\partial x}=0\Rightarrow \frac{\partial z}{\partial x}=-\alpha\Rightarrow p=-\alpha\Rightarrow \alpha=-p$$

$$\beta+\frac{\partial z}{\partial y}=0\Rightarrow \frac{\partial z}{\partial y}=-\beta\Rightarrow q=-\beta\Rightarrow \beta=-q$$

$\therefore (5)\equiv$

$$-px-qy+z=\pm\sqrt{4p^2+q^2+1}$$

$$\Rightarrow (px+qy-z)^2=(4p^2+q^2+1)$$

which is the required partial Differential Equation.

5.b) Using Newton's Forward difference formula find the lowest degree polynomial u_x when it is given that $u_1 = 1,\ u_2 = 9,\ u_3 = 25,\ u_4 = 55$ and $u_5 = 105$.

Soln:

Here,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
2	9	8			
3	25	16	8	6	
4	55	30	14	6	0
5	105	50	20		

Here,

$$x_0 = 1,\ h = 1$$

$$\therefore\ u = \frac{x - x_0}{h} = \frac{x-1}{1} = (x-1)$$

Now, using Newton's forward formula,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$= 1 + 8u + \frac{u(u-1)}{2}\times 8 + \frac{u(u-1)(u-2)}{6}\times 6 + 0$$

$$= 1 + 8(x-1) + (x-1)(x-2)4 + (x-1)(x-2)(x-3)$$

$$= 1 + 8x - 8 + 4(x^2 - 3x + 2) + (x^2 - 3x + 2)(x-3)$$

$$= 8x - 7 + 4x^2 - 12x + 8 + x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$$

$$= x^3 - 2x^2 + 7x - 5$$

$$= x^3 - 2x^2 + 7x - 5$$

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Q.5(c) For an incompressible fluid flow, two components of velocity (u, v, w) are given by

$u = x^2 + 2y^2 + 3z^2$; $v = x^2y - y^2z + zx$.

Determine the third component 'w' so that they satisfy the equation of continuity. Also, find the z-component of acceleration.

Sol. For an incompressible flow, the equation of continuity can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{---} (1)$$

Given; $u = x^2 + 2y^2 + 3z^2$

$v = x^2y - y^2z + zx$

Hence;

$$\frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial v}{\partial y} = x^2 - 2yz$$

Substituting $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial y}$ in eq^n (1)

$$2x + x^2 - 2yz + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = -x^2 - 2x + 2yz$$

Integrate w.r.t z, we have

$$w = -x^2z - 2xz + yz^2 + f(x, y)$$

There are infinite number of possible 'z' components of, since $f(x, y)$ is arbitrary, the simplest one would be found by setting $f(x, y) = 0$

z - component of accelaration

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

and $W = -x^2z - 2xz + yz^2 + f(x,y)$

$$\frac{\partial w}{\partial x} = -2xz - 2z + \frac{\partial f}{\partial x}$$

$$\frac{\partial w}{\partial y} = z^2 + \frac{\partial f}{\partial y}$$

$$\frac{\partial w}{\partial z} = -x^2 - 2x + 2yz$$

thus, substituting the values of u, v, w from the Question, we get the required z-component of acceleration.

$$a = (x^2+2y^2+3z^2)\left(-2xz-2z+\frac{\partial f}{\partial x}\right) + (x^2y - y^2z + 2zx)\left[z^2 + \frac{\partial f}{\partial y}\right]$$
$$+ \left[-x^2z - 2xz + yz^2 + f(x,y)\right]\left[-x^2 - 2x + 2yz\right]$$

5(d)

Time (Minutes)	2	4	6	8	10	12	14	16	18	20
Speed (km/hr)	10	18	25	29	32	20	11	5	2	8.5

Starting from rest in the beginning, the speed (in km/hr) of a train at different times (in minutes) is given by the above table:

Using Simpson's 1/3rd rule, find the approximate distance travelled (in km) in 20 minutes from the beginning.

Sol'n:

Time (minutes)	0	2	4	6	8	10	12	14	16	18	20
Speed (km/hr)	0	10	18	25	29	32	20	11	5	2	8.5

$h=2$

If D km is the distance covered in time t minutes then

$$\frac{dD}{dt} = v$$

$$D = \int_0^{20} v\,dt$$

$v_0 = 0,\ v_1 = 0.17,\ v_2 = 0.30,\ v_3 = 0.42,\ v_4 = 0.48,\ v_5 = 0.53$

$v_6 = 0.33,\ v_7 = 0.18,\ v_8 = 0.08,\ v_9 = 0.03,\ v_{10} = 0.14$

Applying Simpson's 1/3rd rule:

$$\int_0^{20} v\,dt = \frac{h}{3}\Big[(v_0+v_{10}) + 4(v_1+v_3+v_5+v_7+v_9) + 2(v_2+v_4+v_6+v_8)\Big]$$

$$= \frac{2}{3}\Big[(0+0.14) + 4(0.17+0.42+0.53+0.18+0.03) + 2(0.30+0.48+0.33+0.08)\Big]$$

$$= 2/3\,[0.14 + 4(1.33) + 2(1.19)]$$

$$= 2/3\,[0.14 + 5.32 + 2.38]$$

$$\int_0^{20} v\,dt = 5.2267 \text{ Kms}$$

5(e) Write down the basic algorithm for solving the equation : $xe^x - 1 = 0$ by bisection method, correct to 4 decimal places.

Sol:- Basic Algorithm of Bisection Method:

1. Start
2. Read $x1, x2, e$

   * Here $x1$ and $x2$ are initial guesses
   e is the absolute error
   i.e. the desired of accuracy

   * Here $f(x) = xe^x - 1 = 0$
   or
   $f(x) = x*e^\wedge x - 1 = 0$

3. Compute: $f1 = f(x1)$ and $f2 = f(x2)$
4. If $(f1*f2) > 0$, then display initial guesses are wrong and goto (11).
   Otherwise continue.
5. $X = (x1+x2)/2$
6. If $([(x1-x2)/x] < e)$, then display x and goto (11).

   * Here [ ] refers to the modulus sign.*

7. Else ; $f = f(x)$
8. If $((f*f1)>0)$, then $x1 = x$ and $f1 = f$.
9. Else ; $x2 = x$ and $f1 = f$
10. Goto (5)

    * Now the loop continues with new values upto correct to 4 decimal places *

11. Stop.

Its iterations are; when we enter the values.

Iteration No. 1 $x = 0.5$

Iteration No. 2 $x = 0.75$

Iteration No. 3 $x = 0.625$

Iteration No. 4 $x = 0.5625$

Iteration No. 5 $x = 0.59375$

Iteration No. 6 $x = 0.5781$

Iteration No. 7 $x = 0.5703$

Iteration No. 8 $x = 0.5664$

Iteration No. 9 $x = 0.5684$

Iteration No. 10 $x = 0.5674$

Iteration No. 11 $x = 0.5669$

Iteration No. 12 $x = 0.5672$

Iteration No. 13 $x = 0.5671$ = Ans

Q.6(a) Solve $(y^3x-2x^4)p+(2y^4-x^3y)q=9z(x^3-y^3)$.

Sol: Here Lagrange's Auxillary equations are given by

$$\frac{dx}{y^3x-2x^4}=\frac{dy}{2y^4-x^3y}=\frac{dz}{9z(x^3-y^3)} \quad \text{---(1)}$$

Taking first two fractions of (1), we have

$$(2y^4-x^3y)dx=(y^3x-2x^4)dy.$$

$$\left(\frac{2y}{x^3}-\frac{1}{y^2}\right)dx=\left[\frac{1}{x^2}-\frac{2x}{y^3}\right]dy$$

[$\because$ By diving it by x^3y^3]

or $\left[\frac{1}{x^2}dy-\frac{2y}{x^3}dx\right)+\left(\frac{1}{y^2}dx-\frac{2x}{y^3}dy\right)=0$

or $d\left(\frac{y}{x^2}\right)+d\left(\frac{x}{y^2}\right)=0$

Integrating, we get

$$(y/x^2)+(x/y^2)=c_1$$

$$\boxed{x^3+y^3=x^2y^2c_1} \quad \text{---(2)}$$

Choosing $(1/x),(1/y),(1/3z)$ as multiplier of each fraction of (1).

$$=\frac{1/x\,dx+1/y\,dy+1/3z\,dz}{(y^3-2x^3)+(2y^3-x^3)+3(x^3-y^3)}$$

$$=\frac{1/x\,dx+1/y\,dy+1/3z\,dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{3\cdot z}dz=0$$

so that $\log x + \log y + \frac{1}{3}\log z = \log c_2$

$$\log x + \log y + \log z^{1/3} = \log c_2$$

$$\log(xyz^{1/3}) = \log c_2$$

$$\boxed{xyz^{1/3} = c_2} \quad \text{— (3)}$$

from (2) & (3), the required general solution is

$$\boxed{\phi\left(xyz^{1/3},\ y/x^2 + x/y^2\right) = 0}$$

ϕ being the arbitrary function.

6(b) Find the equivalent of numbers given in a specified number system to the system mentioned against them.

(i) $(111011.101)_2$ to decimal system.

(ii) $(1000111110000.00101100)_2$ to hexadecimal system.

(iii) $(C4F2)_{16}$ to decimal system.

(iv) $(418)_{10}$ to binary system.

Soln (i) $(111011.101)_2$ to $(x)_{10}$

$= 1\times2^5 + 1\times2^4 + 1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0 . 1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3}$

$= 32+16+8+0+2+1 . 0.5+0+0.125$

$x = (59.625)_{10}$

(ii) $(1000111110000.00101100)_2$ to $(x)_{16}$

0001 0001 1111 0000 . 0010 1100

$(1 \quad 1 \quad F \quad 0 . 2C)_{16}$

$x = (11F0.2C)_{16}$.

(iii) $(C4F2)_{16}$ to decimal.

$C - 12$
$F - 15$

$(C \times 16^3 + 4 \times 16^2 + F \times 16^1 + 2 \times 16^0)_{10}$

$(12 \times 16^3 + 4 \times 16^2 + 15 \times 16 + 2 \times 1)_{10}$

$= (49152 + 1024 + 240 + 2)$

$= (50,418)_{10}$.

(iv) $(418)_{10} \longrightarrow (x)_2$

2	418	0
2	209	1
2	104	0
2	52	0
2	26	0
2	13	1
2	6	0
2	3	1
	1	

$(110100010)_2$

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6(c) Suppose the Lagrangian of a mechanical system is given by $L = \frac{1}{2}m(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{1}{2}k(ax^2 + 2bxy + cy^2)$ where $a, b, c, m (>0), k (>0)$ are constants and $b^2 \neq ac$. Write down the Lagrangian equations of motion and identify the system.

Sol^n: Given $L = \frac{1}{2}m(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{1}{2}k(ax^2 + 2bxy + cy^2)$

To find Lagrangian equation of motion

Here variables involved $x, y, \dot{x}, \dot{y}$

$$\therefore \frac{\partial L}{\partial x} = -\frac{1}{2}k(2ax + 2by) = -k(ax + by) \quad \text{---} (1)$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m(2a\dot{x} + 2b\dot{y}) = m(a\dot{x} + b\dot{y}) \quad \text{---} (2)$$

and

$$\frac{\partial L}{\partial y} = -\frac{1}{2}k(2bx + 2cy) = -k(bx + cy) \quad \text{---} (3)$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}m(2b\dot{x} + 2c\dot{y}) = m(b\dot{x} + c\dot{y}) \quad \text{---} (4)$$

Thus, Lagrangian equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \text{---} (5)$$

and,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \quad \text{---} (6)$$

$$\therefore (5) \Rightarrow m(a\ddot{x} + b\ddot{y}) + k(ax + by) = 0 \quad \text{---} (7)$$

$$(6) \Rightarrow m(b\ddot{x} + c\ddot{y}) + k(bx + cy) = 0 \quad \text{---} (8)$$

For equations of motion

$(7) \times c - (8) \times b$ gives

$$m(ac - b^2)\ddot{x} + k(ac - b^2)x = 0$$

$$\Rightarrow \boxed{\ddot{x} = -(k/m)x} \quad \text{as } b^2 - ac \neq 0 \quad \text{---} (A)$$

Similarly (7) × b − (8) × a gives.

$$m(b^2-ac)\ddot{y} + k(b^2-ac)y = 0.$$

$$\Rightarrow \boxed{\ddot{y} = -(k/m)\,y} \text{ as } b^2-ac \neq 0 \quad \text{— (B)}$$

from (A) & (B) are the required equations of motion, and the system is a 2-D Harmonic oscillator.

7(a) Solve the partial differential Equation

$$(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

where; $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Sol: Given P.DE is

$$(2D^2 - 5DD' + 5D'^2)z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

The auxillary of the given equation is

$$2m^2 - 5m + 2 = 0$$

$$(2m-1)(m-1) = 0$$

$$m = 1/2, 1$$

$$\therefore C.F = \phi_1(y + x/2) + \phi_2(y+2x)$$

$$= \phi_1[1/2(2y+x)] + \phi_2(y+2x).$$

$$\boxed{C.F = \phi_1(2y+x) + \phi_2(y+2x)}$$

where, ϕ_1, ϕ_2 being arbitrary functions.

Now $P.I = \frac{1}{2D^2 - 5DD' + 2D'^2}[5\sin(2x+y) + 24(y-x) + e^{3x+4y}]$

$$P.I = \underbrace{\frac{1}{2D^2-5DD'+2D'^2}[5\sin(2x+y)]}_{P.I\cdot 1} + \underbrace{\frac{1}{2D^2-5DD'+2D'^2}24(y-x)}_{PI\cdot 2} + \underbrace{\frac{1}{2D^2-5DD'+2D'^2}\cdot e^{3x+4y}}_{P\cdot I\cdot 3}.$$

$$P.I_2 = \frac{1}{2D^2-5DD'+2D'^2}[24(y-x)]$$

$$= 24 \cdot \frac{1}{2D^2-5DD'+2D'^2}(y-x)$$

$$= 24 \cdot \frac{1}{2(-1)^2 - 5\times1\times-1 + 2\times(1)^2}\iint v\,dv\,dv \quad [\because v = y-x]$$

$$= \frac{24}{20}\ \frac{24}{20}\int\frac{v^2}{2}dv$$

$$P.I_2 = \frac{24}{20}\cdot\left(\frac{v^3}{6}\right) = \frac{1}{5}(y-x)^3$$

$$P.I_3 = \frac{1}{2D^2-5DD'+2D'^2}\cdot e^{2x+3y}$$

$$= e^{2x+3y}\cdot\frac{1}{2(2)^2-5\times2\times3+2\times9}$$

$$= e^{2x+3y}\cdot\frac{1}{8-30+18} = \frac{-e^{2x+3y}}{4}$$

$$P.I_1 = \frac{1}{2D^2-5DD'+2D'^2}[5\sin(2x+y)]$$

$$= 5\left[\frac{1}{2D^2-5DD'+2D'^2}\sin(2x+y)\right]$$

$$= 5\cdot\frac{1}{(D-2D')}\left[\frac{1}{(2D-D')}\sin(2x+y)\right]$$

$$= 5\cdot\frac{1}{(D-2D')}\left[\frac{1}{2\cdot2-1}\int\sin v\,dv\right] \quad \text{where } v=2x+y$$

$$= 5\cdot\frac{1}{(D-2D')}\times\frac{1}{3}(-\cos v)$$

$$PI_1 = -\frac{5}{3}\cdot\frac{1}{D-2D'}\cos v = -\frac{5}{3}\cdot\frac{x}{1!\cdot1!}\cos(2x+y)$$

$$\therefore P.I = PI_1 + PI_2 + PI_3$$

$$P.I = -\frac{5}{3}x\cos(2x+y) + \frac{1}{5}(y-x)^3 - \frac{e^{2x+3y}}{4}$$

$$\therefore G.S = y = CF + PI$$

$$y = \phi_1(2y+x) + \phi_2(y+2x) + (-5/3)x\cos(2x+y) + \frac{1}{5}(y-3)^3 - \frac{1}{4}e^{2x+3y}$$

7(b) Find the values of the constants a, b, c such that the quadrature formula

$$\int_0^h f(x)\,dx = h\left[af(0) + bf\left(\tfrac{h}{3}\right) + cf(h)\right]$$ is exact for polynomials of as high degree as possible, and hence find the order of the truncation error.

Sol^n: Making the method exact for polynomials of degree upto 2, we obtain

$f(x)=1$: $h = h(a+b+c)$ (or) $a+b+c=1$

$f(x)=x$: $\frac{h^2}{2} = h\left(\frac{bh}{3}+ch\right)$ (or) $\frac{1}{3}b+c=\frac{1}{2}$

$f(x)=x^2$: $\frac{h^3}{3} = h\left(\frac{bh^2}{9}+ch^2\right)$ (or) $\frac{1}{9}b+c=\frac{1}{3}$.

Solving the above equations, we get $a=0$, $b=3/4$ and $c=\frac{1}{4}$.

Hence the required formula is

$$\int_0^h f(x)\,dx = \frac{h}{4}\left[3f(h/3)+f(h)\right]$$

The truncation error of the formula is given by

$$T.E = \frac{C}{3!} f'''(\xi),\quad 0<\xi<h$$

where $C = \int_0^h x^3\,dx - h\left[\frac{bh^3}{27}+ch^3\right] = -\frac{h^4}{36}$

Hence we have.

$$TE = -\frac{h^4}{216} f'''(\xi) = O(h^4).$$

7(c). The Hamiltonian of a mechanical system is given by, $H = p_1 q_1 - a q_1^2 + b q_2^2 - p_2 q_2$, where a, b are the constants, solve the Hamiltonian equations and show that

$$\frac{p_2 - b q_2}{q_1} = \text{constant}.$$

Sol^n: Let us consider the generalised coordinates $q_1, q_2 \ldots q_n$ and the generalised components of momentum $p_1, p_2 \ldots p_n$.

Now we have the Hamiltonian equation

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

where $i = 1, 2 \ldots$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}$$

$$\dot{q}_1 = q_1 \Rightarrow \frac{\partial q_1}{\partial t} = q_1 \Rightarrow q_1 = c_1 e^t$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -[p_1 - 2aq_1] = 2aq_1 - p_1$$

$$\frac{\partial p_1}{\partial t} = 2ac_1 e^t - p_1$$

$$\frac{\partial p_1}{\partial t} + p_1 = 2ac_1 e^t$$

Integrating factor $= e^t$

$$p_1 e^t = \int 2ac_1 e^t . e^t \, dt$$

$$p_1 e^t = 2ac_1 \int e^{2t} dt$$

$$= ac_1 e^{2t} + c_2$$

$$p_1 = ac_1e^t + c_2e^{-t}$$

$$\dot{q}_2 = -q_2 \Rightarrow q_2 = c_3e^{-t}$$

$$\frac{\partial p_2}{\partial t} = -\left[2bq_2 - p_2\right] = p_2 - 2bq_2$$

$$\frac{\partial p_2}{\partial t} - p_2 = -2bq_2 = -2bc_3e^{-t}$$

Integrating factor $= e^{-t}$

$$p_2 \times e^{-t} = 2bc_3\int e^{-2t}dt = bc_3e^{-2t} + c_4$$

$$\Rightarrow p_2 = bc_3e^{-t} + c_4e^{+t}$$

Consider

$$\frac{p_2 - bq_2}{q_1} = \frac{\cancel{bc_3e^{-t}} + c_4e^t - \cancel{bc_3e^{-t}}}{c_1e^t}$$

$$= \frac{c_4\cancel{e^t}}{c_1\cancel{e^t}}$$

$$= \frac{c_4}{c_1}$$

$= \text{constant}$

8(a) Simplify the boolean expression: $(a+b)\cdot(\bar{b}+c)+b\cdot(\bar{a}+\bar{c})$ by using the law of boolean algebra. From its truth table write it in minterm normal form.

Sol^n

$(a+b)(\bar{b}+c)+b(\bar{a}+\bar{c})$

$= a\bar{b}+ac+b\bar{b}+bc+b\bar{a}+b\bar{c}$

$= a\bar{b}+ac+0+b[c+\bar{c}]+b\bar{a} \quad [\because b\bar{b}=0]$

$= a\bar{b}+ac+b[1]+b\bar{a} \quad [\because c+\bar{c}=1]$

$= a\bar{b}+ac+b[1+\bar{a}]$

$= a\bar{b}+ac+b \quad [\because 1+\bar{a}=1]$

a	b	c	$\bar{b}$	$a\bar{b}$	ac	$a\bar{b}+ac$	$z=a\bar{b}+ac+b$	Minterm
0	0	0	1	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
0	0	1	1	0	0	0	0	$\bar{a}\bar{b}c$
0	1	0	0	0	0	0	1	$\bar{a}b\bar{c}$
0	1	1	0	0	0	0	1	$\bar{a}bc$
1	0	0	1	1	0	1	1	$a\bar{b}\bar{c}$
1	0	1	1	1	1	1	1	$a\bar{b}c$
1	1	0	0	0	0	0	1	$ab\bar{c}$
1	1	1	0	0	1	1	1	abc

From Truth Table, Minterm normal form is

$\bar{a}b\bar{c}+\bar{a}bc+a\bar{b}\bar{c}+a\bar{b}c+ab\bar{c}+abc$

8(b) For a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$. Determine the velocity components along the directions x and y. Also, determine the stream function ψ and check wether ϕ represents a possible case of flow or not.

Sol^n Given velocity potential $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$

The knowledge of velocity potential (ϕ) immediately gives the velocity components. Its gradient gives rise to the velocity vector (i.e.)

$$\vec{V} = \nabla\phi$$

(or) $$u\vec{i} + v\vec{j} + \omega\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

thus, the velocity components can be written as.

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad \omega = \frac{\partial\phi}{\partial z}$$

Then, $$u = \frac{\partial}{\partial x}\left[x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)\right]$$

$$= 2xy - y^2 + x^2$$

$$v = \frac{\partial}{\partial y}\left[x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)\right]$$

$$= x^2 - 2xy + \tfrac{1}{3}(-3y^2) = x^2 - y^2 - 2xy$$

Stream function ψ: Since $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

$$x^2 - y^2 - 2xy = \frac{\partial\psi}{\partial y}$$

$$\Rightarrow \psi = x^2y - y^3/3 - xy^2 + f(x)$$

ϕ will represent a case of flow if it satisfies the Laplace equation.

i.e. $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$

$$\frac{\partial\phi}{\partial x}=2xy-y^2+\tfrac{1}{3}(3x^2)\Rightarrow\frac{\partial^2\phi}{\partial x^2}=2y+2x$$

Similarly $\dfrac{\partial\phi}{\partial y}=x^2-2xy+\tfrac{1}{3}(-3y^2)\Rightarrow\dfrac{\partial^2\phi}{\partial y^2}=-2x-2y$

Since $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$

Thus ϕ is represent a possible case of flow.

8(c) A thin annulus occupies the region $0 < a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at $0°$, while along the outer edge the temperature is held at $T = K\cos\theta/2$, where K is a constant. Determine the temperature distribution in the annulus.

Solⁿ:

We are given a circular annulus whose inner and outer radii are a and b respectively.

The steady state temperature $T(r,\theta)$ at any point $P(r,\theta)$ of the annulus is the solution of the Laplace's equation in polar co-ordinates (r,θ) namely

$$r^2\left(\frac{\partial^2 T}{\partial r^2}\right) + r\left(\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{—(1)}$$

Since the temperatures along the inner $(r=a)$ and outer boundary $(r=b)$ are maintained at $0°$ and $K\cos\theta/2$ respectively

i.e. $T(a,\theta) = 0$ and $T(b,\theta) = K\cos\theta/2$; $0 \leq \theta \leq 2\pi$ —(2)

Clearly the temperature function $T(r,\theta)$ must be periodic in θ of period 2π. Accordingly, we now proceed to solve (1)

Suppose (1) has a solution of the form

$$T(r,\theta) = R(r)\,H(\theta) \quad \text{---} (3)$$

where R and H are functions of r and θ respectively

Using (3), (1) reduces to

$$r^2R''H + rR'H + RH'' = 0$$

$$\Rightarrow (r^2R'' + rR')H = -RH''$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{H''}{H} \quad \text{---} (4)$$

Since L.H.S of (4) is a function of r only and the R.H.S is a function of θ only, the two sides of (4) must be equal to the same constant say μ.

Then (4) gives

$$r^2R'' + rR' - \mu R = 0 \quad \text{---} (5)$$

$$\text{and} \quad H'' + \mu H = 0 \quad \text{---} (6)$$

As usual, we first reduce linear homogeneous differential equation (5) into a linear differential equation with constant coefficients.

Re-writing (5), $(r^2D^2 + rD - \mu)R = 0 \quad \text{---} (7)$

where $D \equiv \frac{d}{dr}$

Let $r = e^z \Rightarrow z = \log r$ and $D_1 = \frac{d}{dz}$

Then w.r.t $rD = D_1$ and $r^2D^2 = D_1(D_1 - 1)$.

(7) $\equiv$ $[D_1(D_1-1) + D_1 - \mu]R = 0$

$\Rightarrow (D_1^2 - \mu)R = 0$ —— (8)

Again, let $D_2 = \frac{d}{d\theta}$. Then (6) may be re-written as $(D_2^2 + \mu)H = 0$ —— (9)

The solutions of (8) and (9) depend on μ.

Consider following cases:

Case(i): Let $\mu = 0$. Then (8) and (9) reduces to

$$\frac{d^2R}{dz^2} = 0 \text{ and } \frac{d^2H}{d\theta^2} = 0.$$

solving these,

$$R(r) = c_1 z + c_2 = c_1 \log r + c_2$$

$$\text{and } H = c_3\theta + c_4.$$

Hence, from (3), solution of (1) is of the form

$$T(r,\theta) = (c_1 \log r + c_2)(c_3\theta + c_4) \text{ —— (10)}$$

Since $T(r,\theta)$ is periodic in θ, we must take $c_3 = 0$. Then equation (10) becomes

$$T(r,\theta) = (c_1 \log r + c_2)\, c_4$$

$$= \frac{1}{2}(a_0 \log r + b_0) \text{ —— (11)}$$

where $a_0 = 2c_1c_4$ and $b_0 = c_2c_4$ are new arbitrary constants.

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Case (ii): Let $\mu=\lambda^2$, where $\lambda\neq 0$. Then (8) and (9) become

$$(D_1^2-\lambda^2)R=0 \text{ and } (D_2^2+\lambda^2)H=0 \quad (12)$$

[Note that we cannot choose $\mu=-\lambda^2$ because it will lead to $(D_2^2-\lambda^2)H=0$ whose solution will not contain trigonometric functions and hence periodic nature of $T(r,\theta)$ will not be attained]

Solving (12)

$$R(r)=C_5e^{\lambda z}+C_6e^{-\lambda z}$$
$$=C_5(e^z)^\lambda+C_6(e^z)^{-\lambda}=C_5r^\lambda+C_6r^{-\lambda}$$

and $H(\theta)=C_7\cos\lambda\theta+C_8\sin\lambda\theta$.

Hence, from (3), a solution of (1) is of the form

$$T(r,\theta)=(C_5r^\lambda+C_6r^{-\lambda})(C_7\cos\lambda\theta+C_8\sin\lambda\theta) \quad (13)$$

Since $T(r,\theta)$ is periodic in θ with period 2π, we must take $\lambda=n$, where $n=1,2,3\ldots$

Hence (13) takes the form

$$T(r,\theta)=(C_5r^n+C_6r^{-n})(C_7\cos n\theta+C_8\sin n\theta) \quad (14)$$
$$n=1,2,3,\ldots$$

With the help of (11) and (14), the most general solution of (1) is

$$T(r,\theta) = \frac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \cos n\theta + (c_n r^n + d_n r^{-n}) \sin n\theta \quad \text{(15)}$$

which holds for $a \le r \le b$.

Here $a_n = c_5 c_7$, $b_n = c_6 c_7$, $c_n = c_5 c_8$, $d_n = c_6 c_8$ are new arbitrary constants.

Putting $r = a$ and $r = b$ by turn in (15) and B.C (2) we have

$$0 = \frac{a_0 \log a + b_0}{2} + \sum_{n=1}^{\infty} (a_n a^n + b_n a^{-n}) \cos n\theta + (c_n a^n + d_n a^{-n}) \sin n\theta \quad \text{(16)}$$

$$k \cos\frac{\theta}{2} = \frac{a_0 \log b + b_0}{2} + \sum_{n=1}^{\infty} (a_n b^n + b_n b^{-n}) \cos n\theta + (c_n b^n + d_n b^{-n}) \sin n\theta \quad \text{(17)}$$

(16) and (17) are usual expansions of $f_1(\theta) = 0$ and $f_2(\theta) = k\cos\theta/2$ as Fourier series in $(0, 2\pi)$.

Hence we have

$$a_0 \log a + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_1(\theta)\, d\theta \;\&\; a_0 \log b + b_0 = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\, d\theta$$

Solving these we get

$a_0 \log a + b_0 = 0$ and $a_0 \log b + b_0 = 0$ (18)

$\Rightarrow \boxed{a_0 = 0 \text{ and } b_0 = 0}$

$$a_n a^n + b_n a^{-n} = \frac{1}{\pi}\int_0^{2\pi}(0)\cos n\theta = 0 \Rightarrow b_n = -a_n a^{2n}$$

& $$a_n b^n + b_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} k\cos\frac{\theta}{2}\cos n\theta\, d\theta$$

$$a_n b^n + b_n b^{-n} = \frac{k}{2\pi}\int_0^{2\pi}\left[\cos\left(n+\tfrac{1}{2}\right)\theta + \cos\left(n-\tfrac{1}{2}\right)\theta\right]d\theta$$

$$= \frac{k}{2\pi}\left[\frac{\sin(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\sin(n-\frac{1}{2})\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= 0$$

$$\Rightarrow a_n b^n - a_n a^{2n} b^{-n} = 0$$

$$\Rightarrow a_n\left[b^n - a^{2n} b^{-n}\right] = 0$$

$$\Rightarrow a_n = 0 \text{ and } b_n = 0$$

$$c_n a^n + d_n a^{-n} = \frac{1}{\pi}\int_0^{2\pi}(0)\sin n\theta\, d\theta = 0$$

& $$c_n b^n + d_n b^{-n} = \frac{1}{\pi}\int_0^{2\pi} k\cos\frac{\theta}{2}\sin n\theta\, d\theta$$

$$c_n b^n + d_n b^{-n} = \frac{k}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\tfrac{1}{2}\right)\theta + \sin\left(n-\tfrac{1}{2}\right)\theta\right]d\theta$$

$$= -\frac{k}{2\pi}\left[\frac{\cos(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\cos(n-\frac{1}{2})\theta}{(n-\frac{1}{2})}\right]_0^{2\pi}$$

$$= \frac{-k}{2\pi}\left[\frac{-1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right]$$

$$= \frac{k}{\pi}\left[\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right]$$

$$= \frac{k}{\pi}\frac{2n}{(n^2-\frac{1}{4})}$$

$$c_n b^n + d_n b^{-n} = \frac{8kn}{\pi(4n^2-1)} \quad \text{—— (20)}$$

Now we have

$$c_n a^n + d_n a^{-n} = 0$$

$$\Rightarrow d_n = -c_n a^{2n}$$

$\therefore$ from (20)

$$c_n b^n - c_n a^{2n} b^{-n} = \frac{8kn}{\pi(4n^2-1)}$$

$$c_n\left[b^n - b^{-n} a^{2n}\right] = \frac{8kn}{\pi(4n^2-1)}$$

$$c_n = \frac{8kn}{\pi(4n^2-1)} \frac{1}{(b^n - b^{-n}a^{2n})}$$

$\therefore$ From (1)

$$T(r,\theta) = \sum_{n=1}^{\infty}\left(c_n r^n + d_n r^{-n}\right)\sin n\theta$$

$$= \sum_{n=1}^{\infty}\left(c_n r^n - c_n a^{2n} r^{-n}\right)\sin n\theta$$

$$(\because d_n = -c_n a^{2n})$$

$$= \sum_{n=1}^{\infty} c_n\left[r^n - a^{2n} r^{-n}\right]\sin n\theta$$

$$= \frac{8k}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2-1}\frac{\left[r^n - a^{2n} r^{-n}\right]}{(b^n - b^{-n}a^{2n})}$$

$$T(r,\theta) = \frac{8K}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)}\left[\frac{(r/a)^n-(a/r)^n}{(b/a)^n-(a/b)^n}\right]\sin n\theta$$

which is the required temperature distribution in the given annulus.

NOTE: Q.No. 8.(c)
"STUDENTS ARE ADVICED TO WRITE CONCISE ANSWER IN EXAM"

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