UPSC Civil Services Main 1979 - Mathematics Complex Analysis

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July 19, 2010

Question 1(a) *If a function $f(z)$ is analytic and bounded in the whole plane, show that $f(z)$ reduces to a constant. Hence show that every polynomial has a root.*

Solution. See 1989, question 2(b) for the first part. See 1996 question 2(a) for the second part. ∎

Question 1(b) *Evaluate the following integrals by the method of residues.*

1.
$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta \qquad (a > b > 0)$$

2.
$$\int_0^{\infty} \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\, dx$$

Solution.

1. Let
$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos 2\theta}{a + b\cos\theta}\, d\theta$$

   Let $I_1 = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta$. Put $z = e^{i\theta}$ so that
$$I_1 = \frac{1}{2}\int_{|z|=1} \frac{dz}{iz(a + \frac{b}{2}(z + \frac{1}{z}))} = \frac{1}{i}\int_{|z|=1} \frac{dz}{bz^2 + 2az + b}$$

The integrand $\dfrac{1}{bz^2+2az+b}$ has two simple poles at $z_1 = \dfrac{-a+\sqrt{a^2-b^2}}{b}$, $z_2 = \dfrac{-a-\sqrt{a^2-b^2}}{b}$. Since $a > b > 0$, $|z_2| > 1$, but $|z_1 z_2| = 1$ so $|z_1| < 1$ i.e. the pole at $z = z_1$ lies within $|z| \le 1$.

Residue at z_1 is $\displaystyle\lim_{z\to z_1} \frac{z-z_1}{bz^2+2az+b} = \frac{1}{2bz_1+2a} = \frac{1}{2\sqrt{a^2-b^2}}$. Thus $I_1 = 2\pi i \dfrac{1}{i}\dfrac{1}{2\sqrt{a^2-b^2}} = \dfrac{\pi}{\sqrt{a^2-b^2}}$. Let

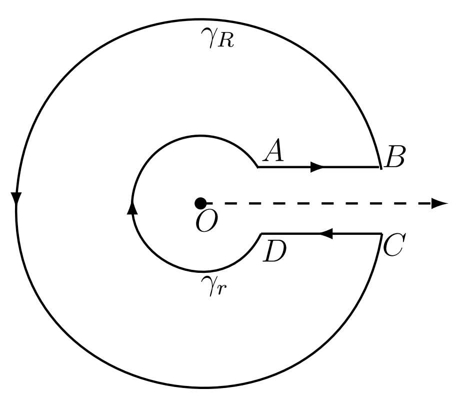
$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{a+b\cos\theta}\,d\theta = \frac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\frac{e^{2i\theta}\,d\theta}{a+b\cos\theta} \\
&= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{i}\int_{|z|=1}\frac{2z^2\,dz}{bz^2+2az+b} \\
&= \mathrm{Re}\,\frac{1}{i}\times 2\pi i \text{ Residue of } \frac{z^2}{bz^2+2az+b} \text{ at } z = z_1 \\
&= 2\pi\frac{1}{b}\frac{z_1^2}{z_1-z_2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_1 - I_2 &= \frac{2\pi}{b(z_1-z_2)} - \frac{2\pi z_1^2}{b(z_1-z_2)} \\
&= \frac{2\pi}{2\sqrt{a^2-b^2}}(1-z_1^2) \\
&= \frac{\pi}{\sqrt{a^2-b^2}}\Big(1 - \frac{a^2-2a\sqrt{a^2-b^2}+(a^2-b^2)}{b^2}\Big) \\
&= \frac{\pi}{\sqrt{a^2-b^2}}\Big(2\sqrt{a^2-b^2}\frac{a-\sqrt{a^2-b^2}}{b^2}\Big)
\end{aligned}
$$

Thus $I = \dfrac{2\pi}{a+\sqrt{a^2-b^2}}$.

2. Let $f(z) = \frac{z^{\frac{1}{6}}\log z}{(1+z)^2}$ and the contour C as shown. γ_r is a circle of radius r oriented clockwise, and γ_R a circle of radius R oriented anticlockwise. AB is along x-axis on which $z = x$, CD is the line on which $z = xe^{2\pi i}$. To avoid the branch point of the multiple valued function $\log z$, we consider $\mathbb{C}-$ positive side of the x-axis. We choose the branch of $\log z$ for which $\log z = \log|z| + i\theta, 0 < \theta \le 2\pi$.



(a) Clearly $f(z)$ has a double pole at $z=-1$. Residue of $f(z)$ at $z=-1$ is

$$
\begin{aligned}
& \frac{1}{1!}\,\frac{d}{dz}\left[\frac{(z+1)^2 z^{\frac{1}{6}}\log z}{(z+1)^2}\right]_{at\ z=1} \\
= & \left[\frac{z^{\frac{1}{6}}}{z}+\frac{1}{6}z^{-\frac{5}{6}}\log z\right]_{at\ z=-1=e^{i\pi}} \quad = \quad \frac{\log z+6}{6z^{\frac{5}{6}}}\ at\ z=e^{i\pi} \\
= & \frac{\log e^{i\pi}+6}{6e^{\frac{5i\pi}{6}}} \quad = \quad \frac{i\pi+6}{6}\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right) \\
= & \frac{i\pi+6}{6}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) \quad = \quad -\frac{1}{12}(6+i\pi)(\sqrt{3}+i)
\end{aligned}
$$

(b) On γ_R, $z=Re^{i\theta}$, $|z+1|\geq|z|-1=R-1$ and $|\log z|=|\log Re^{i\theta}|=|\log R+i\theta|\leq \log R+\theta\leq\log R+2\pi$ as $0\leq\theta\leq 2\pi$. Thus

$$
\left|\int_{\gamma_R}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz\right|\leq\int_0^{2\pi}\frac{R^{\frac{1}{6}}(\log R+2\pi)}{(R-1)^2}R\,d\theta=2\pi\frac{R^{\frac{7}{6}}}{(R-1)^2}(\log R+2\pi)
$$

Clearly $\displaystyle\lim_{R\to\infty}\left[\frac{R^{\frac{7}{6}}\log R}{(R-1)^2}+\frac{2\pi R^{\frac{7}{6}}}{(R-1)^2}\right]=0$, and therefore

$$
\lim_{R\to\infty}\int_{\gamma_R}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz=0
$$

(c) On γ_r, $z=re^{i\theta}$, $|z+1|\geq 1-|z|=1-r$ and $|\log z|=|\log re^{i\theta}|=|\log r+i\theta|\leq \log r+\theta\leq\log r+2\pi$ as $0\leq\theta\leq 2\pi$. Thus

$$
\left|\int_{\gamma_r}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz\right|\leq\int_0^{2\pi}\frac{r^{\frac{1}{6}}(\log r+2\pi)}{(1-r)^2}r\,d\theta=2\pi\frac{r^{\frac{7}{6}}}{(1-r)^2}(\log r+2\pi)
$$

But $\displaystyle\lim_{r\to 0}\left[\frac{r^{\frac{7}{6}}\log r}{(1-r)^2}+\frac{2\pi r^{\frac{7}{6}}}{(1-r)^2}\right]=0$, and therefore

$$
\lim_{r\to 0}\int_{\gamma_r}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz=0
$$

By Cauchy's residue theorem, using $1,2,3$, we get

$$
\lim_{\substack{R\to\infty\\ r\to 0}}\int_C f(z)\,dz=\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx+\int_\infty^0\frac{(xe^{2\pi i})^{\frac{1}{6}}\log(xe^{2\pi i})}{(1+x)^2}\,dx
$$

because on AB, $z=x$ and on CD, $z=xe^{2\pi i}$. Therefore

$$
\begin{aligned}
\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx-\int_0^\infty\frac{x^{\frac{1}{6}}e^{\frac{2\pi i}{6}}(\log x+2\pi i)}{(1+x)^2}\,dx \quad &= \quad -\frac{2\pi i}{12}(6+i\pi)(\sqrt{3}+i) \\
\Rightarrow\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx-\int_0^\infty\frac{x^{\frac{1}{6}}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)2\pi i}{(1+x)^2}\,dx \quad &= \quad -\frac{\pi}{6}\Big[-(6+\pi\sqrt{3})+i(6\sqrt{3}-\pi)\Big]
\end{aligned}
$$

Equating real and imaginary parts, we get

$$
\begin{aligned}
\frac{1}{2}\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx + \sqrt{3}\pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx &= \frac{\pi}{6}(6+\pi\sqrt{3}) \qquad (1)\\
-\frac{\sqrt{3}}{2}\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx - \pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx &= \frac{\pi}{6}(\pi-6\sqrt{3}) \qquad (2)
\end{aligned}
$$

Multiplying (1) by $\sqrt{3}$ and adding

$$-\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx = \frac{\pi}{6}\left[6+\pi\sqrt{3}+\sqrt{3}\pi-18\right] = \frac{\pi}{6}\left[2\pi\sqrt{3}-12\right]$$

Thus

$$\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx = 2\pi - \frac{\pi^2}{\sqrt{3}}$$

In addition, multiplying (2) by $\sqrt{3}$ and adding, we get

$$2\pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{\pi}{6}\left[6\sqrt{3}+3\pi+\pi-6\sqrt{3}\right]$$

giving us

$$\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{2\pi}{3}$$

$\blacksquare$

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June 20, 2010

Question 1(a) *Find the expansion in powers of z of* $\dfrac{1}{z(z-1)(z+3)}$ *in the region* $0 < |z| < 4$.

Solution. It can easily be seen that

$$f(z) = \frac{1}{z(z-1)(z+3)} = -\frac{1}{3z} + \frac{1}{4(z-1)} + \frac{1}{12(z+3)}$$

1. Region $0 < |z| < 1$.

$$\begin{aligned} f(z) &= -\frac{1}{3z} - \frac{1}{4}(1-z)^{-1} + \frac{1}{36}\Big(1+\frac{z}{3}\Big)^{-1} \\ &= -\frac{1}{3z} - \frac{1}{4}\sum_{n=0}^{\infty} z^n + \frac{1}{36}\sum_{n=0}^{\infty}\Big(\frac{z}{3}\Big)^n(-1)^n \\ &= -\frac{1}{3z} - \frac{2}{9} + \sum_{n=1}^{\infty} z^n\Big(-\frac{1}{4} + \frac{(-1)^n}{3^n}\Big) \end{aligned}$$

   This is the Laurent expansion of $f(z)$ in the region $0 < |z| < 1$. The given function satisfies the requirements of Laurent's theorem.

2. Region $1 < |z| < 3$.

$$\begin{aligned} f(z) &= -\frac{1}{3z} + \frac{1}{4z}\Big(1-\frac{1}{z}\Big)^{-1} + \frac{1}{36}\Big(1+\frac{z}{3}\Big)^{-1} \\ &= -\frac{1}{3z} + \frac{1}{4z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{36}\sum_{n=0}^{\infty}\Big(\frac{z}{3}\Big)^n(-1)^n \\ &= \frac{1}{36}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z}{3}\Big)^n - \frac{1}{12z} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{z^{n+1}} \end{aligned}$$

This is again the Laurent expansion valid in the annular region $1 < |z| < 3$.

3. Region $|z| > 3$

$$\begin{aligned} f(z) &= -\frac{1}{3z} + \frac{1}{4z}\Big(1 - \frac{1}{z}\Big)^{-1} + \frac{1}{3z}\Big(1 + \frac{3}{z}\Big)^{-1} \\ &= -\frac{1}{3z} + \frac{1}{4z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{3z}\sum_{n=0}^{\infty}\Big(\frac{3}{z}\Big)^n(-1)^n \\ &= \frac{1}{4z} + \sum_{n=1}^{\infty}\frac{1}{z^{n+1}}\Big(\frac{1}{4} + 3^{n-1}(-1)^n\Big) \end{aligned}$$

The is Taylor's expansion of $f(z)$ around ∞.

∎

Question 1(b) *Evaluate by contour integration*

1. $\displaystyle\int_0^{\infty}\frac{dx}{x^4+1}$

2. $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5+4\sin\theta}\,d\theta$

Solution.

1. See 2001 question 2(b).

2. The given integral is the real part of

$$I = \int_0^{2\pi}\frac{e^{2i\theta}\,d\theta}{5+4\sin\theta}$$

Put $z = e^{i\theta}, dz = ie^{i\theta}\,d\theta$ so that

$$I = \int_{|z|=1}\frac{z^2}{5+\frac{4}{2i}(z-\frac{1}{z})}\,\frac{dz}{iz} = \int_{|z|=1}\frac{z^2\,dz}{5iz+2z^2-2}$$

The integrand $\dfrac{z^2}{5iz+2z^2-2}$ has two simple poles, which are given by $2z^2+5iz-2=0$ or $2(z+\frac{i}{2})(z+2i)=0$. Out of the two poles $z=-2i, -\frac{i}{2}$, only $z=-\frac{i}{2}$ is inside the unit disc $|z|\le 1$. Residue at this pole is given by $\dfrac{(\frac{i}{2})^2}{2(-\frac{i}{2}+2i)} = \frac{i^2}{4(3i)} = \frac{i}{12}$. Thus by Cauchy's residue theorem

$$\int_{|z|=1}\frac{z^2\,dz}{5iz+2z^2-2} = \int_0^{2\pi}\frac{e^{2i\theta}}{5+4\sin\theta}\,d\theta = 2\pi i\frac{i}{12} = -\frac{\pi}{6}$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\sin\theta}\, d\theta = -\frac{\pi}{6}, \quad \int_0^{2\pi} \frac{\sin 2\theta}{5 + 4\sin\theta}\, d\theta = 0$$

∎

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Question 1(a) *State and prove Cauchy's integral formula.*

Solution. See 1986 question 1(a). ∎

Question 1(b) *Evaluate*

1. $\displaystyle\int_0^\infty \frac{x^{-k}}{x+1}\,dx,\ 0 < k < 1.$

2. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

Solution.

1. We shall show that for $0 < a < 1$

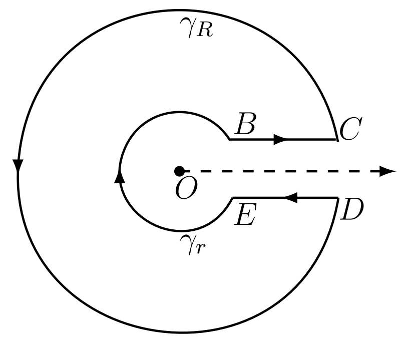
$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}$$

Now let $a-1 = -k$ so that $a = 1-k$ and $0 < a < 1 \Leftrightarrow 0 < k < 1$. Thus for $0 < k < 1$,

$$\int_0^\infty \frac{x^{-k}}{x+1}\,dx = \frac{\pi}{\sin(1-k)\pi} = \frac{\pi}{\sin k\pi}$$

We consider first of all $\mathbb{C} - \{\text{positive real axis}\}$ i.e. there is a cut along the real axis for which $x \geq 0$ to make $\log z$ single valued. We choose that branch of $\log z$ for which $\log z = \log x$ when $z = x, x > 0$.

For $\int_0^\infty \frac{x^{a-1}}{1+x}\,dx$ we take $f(z) = \frac{z^{a-1}}{1+z}$ and the contour C as shown in the figure. γ_r is a circle of radius r oriented clockwise, and γ_R is a circle of radius R oriented anticlockwise. BC is the line joining $(r,0)$ to $(R,0)$, so is DE. We finally make $r \to 0$ and $R \to \infty$. Note that on BC $z^{a-1} = x^{a-1}$ and on DE $z^{a-1} = (xe^{2\pi i})^{a-1}$.



(a) Clearly $f(z) = \dfrac{z^{a-1}}{1+z}$ has a simple pole at $z = -1$ inside the contour. Residue at $z = -1 = e^{\pi i}$ of $f(z)$ is $(e^{\pi i})^{a-1}$. Thus

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_C \frac{z^{a-1}}{1+z}\,dz = 2\pi i(-e^{\pi i a})$$

Note that $z = 0$ is excluded by the cut.

(b)

$$\left|\int_{\gamma_R} \frac{z^{a-1}}{1+z}\,dz\right| = \left|\int_0^{2\pi} \frac{R^{a-1}e^{i\theta(a-1)}}{1+Re^{i\theta}} Rie^{i\theta}\,d\theta\right| \le \frac{R^{a-1}R}{R-1}2\pi$$

Here we use $|z+1| \ge |z| - 1$. Thus $\lim\limits_{R\to\infty} \displaystyle\int_{\gamma_R} \frac{z^{a-1}}{1+z}\,dz = 0$ as $a < 1$.

(c) Similarly

$$\left|\int_{\gamma_r} \frac{z^{a-1}}{1+z}\,dz\right| \le \frac{r^a}{1-r}2\pi$$

because $|z+1| \ge 1 - |z|$. Thus $\lim\limits_{r\to 0} \displaystyle\int_{\gamma_r} \frac{z^{a-1}}{1+z}\,dz = 0$.

Thus

$$\begin{aligned}
\lim_{\substack{R\to\infty \\ r\to 0}} \int_C \frac{z^{a-1}}{1+z}\,dz &= \lim_{\substack{R\to\infty \\ r\to 0}} \int_{BC} \frac{x^{a-1}}{1+x}\,dx + \lim_{\substack{R\to\infty \\ r\to 0}} \int_{DE} \frac{x^{a-1}e^{2\pi i(a-1)}}{1+x}\,dx \\
&= \int_0^\infty \frac{x^{a-1}}{1+x}\,dx + \int_\infty^0 \frac{x^{a-1}}{1+x}e^{2\pi i a}\,dx
\end{aligned}$$

as on BC, $z = x$ and on DE, $z = xe^{2\pi i}$. Thus

$$\int_0^\infty \frac{x^{a-1}}{1+x}(1 - e^{2\pi i a})\,dx = -2\pi i e^{\pi i a}$$

or

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = -2\pi i \frac{e^{\pi i a}}{1 - e^{2\pi i a}} = \pi \frac{-2i}{e^{-\pi i a} - e^{\pi i a}} = \frac{\pi}{\sin a\pi}$$

Alternate proof: This avoids the use of multiple valued functions. In 1991, question 2(c), we proved

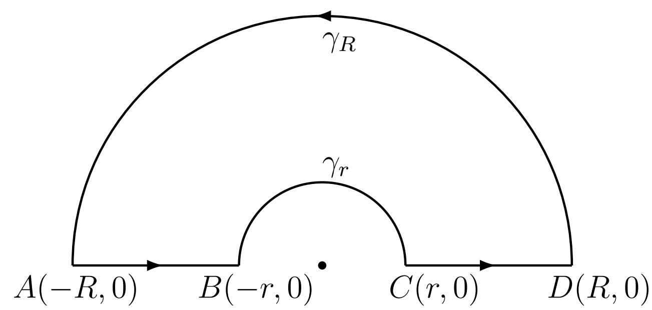
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi} \text{ for } 0 < a < 1$$

Put $e^x = t$, then

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \int_0^{\infty} \frac{t^a}{1+t}\frac{dt}{t} = \int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt$$

Thus $\displaystyle\int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt = \frac{\pi}{\sin a\pi},\ 0 < a < 1.$

2. Clearly $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$ and $\dfrac{\sin^2 x}{x^2}$ is the real part of $\dfrac{1-e^{2ix}}{2x^2}$, therefore we take $f(z) = \dfrac{1-e^{2iz}}{2z^2}$ and the contour C as shown. Finally we let $R \to \infty, r \to 0$.



(a) On γ_R, $z = Re^{i\theta}$ and

$$|1-e^{2iz}| = |1-e^{2i(R\cos\theta+iR\sin\theta)}| \le 1 + |e^{2i(R\cos\theta+iR\sin\theta)}| \le 2$$

because $|e^{2iR\cos\theta}| = 1$ and $|e^{-2R\sin\theta}| \le 1$ as $\sin\theta > 0$ for $0 < \theta < \pi$. Therefore

$$\left|\int_{\gamma_R} \frac{1-e^{2iz}}{2z^2}\,dz\right| \le \frac{2}{2R^2}\pi R = \frac{\pi}{R}$$

and hence $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} \frac{1-e^{2iz}}{2z^2}\,dz = 0.$

(b) Residue of $f(z)$ at $z = 0$: Note that $z = 0$ is a simple pole, so the residue is $\displaystyle\lim_{z\to 0} z\frac{1-e^{2iz}}{2z^2} = \lim_{z\to 0}\frac{1-e^{2iz}}{2z} = \lim_{z\to 0}\frac{-2ie^{2iz}}{2} = -i$. Thus

$$\lim_{r\to 0}\int_{\gamma_r} \frac{1-e^{2iz}}{2z^2}\,dz = i(-i)(0-\pi) = -\pi$$

Here we have used the following property: If $f(z)$ has a simple pole at $z = a$ and γ_r is a circular arc (part of a circle with center a and radius r), from θ_1 to θ_2, then

$$\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = ia_{-1}(\theta_2 - \theta_1)$$

where a_{-1} is the residue of $f(z)$ at $z = a$. See 1985, question 1(c) for more details and proof.

Thus

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_C \frac{1-e^{2\pi iz}}{2z^2}\,dz = \int_{-\infty}^{\infty} \frac{1-e^{2\pi ix}}{2x^2}\,dx - \pi = 0$$

as there is no singularity inside C. Taking real parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi \Longrightarrow \int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

∎

Question 1(c) *Obtain the Laurent expansion in powers of z of*

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2}$$

Solution.

1. $\frac{1}{z-1}$ is analytic in the annular region $0 \le |z| < 1$, so we have the Taylor series for $\frac{1}{z-1}$ valid in $0 \le |z| < 1$. In fact for $|z| < 1$,

$$\frac{1}{z-1} = -(1-z)^{-1} = -\sum_{n=0}^{\infty} z^n$$

2. $\frac{\sinh z}{z^2}$ has a simple pole at $z = 0$ and is analytic everywhere else. We have Laurent series valid in $|z| > 0$:

$$\frac{\sinh z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Note that $\sinh z = \dfrac{e^z + e^{-z}}{2}$, which gives us the desired expansion.

Thus

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2} = z - \sum_{n=0}^{\infty} z^n + \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

or

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2} = \frac{1}{z} - 1 + \frac{z}{3!} + \sum_{n=1}^{\infty} z^{2n+1}\left(\frac{1}{(2n+3)!} - 1\right) - \sum_{n=1}^{\infty} z^{2n}$$

and this expansion is valid in $0 < |z| < 1$. ∎

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Question 1(a) *Evaluate by contour integration*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx, \quad 0 < a < 1$$

Solution. See 1991, question 2(c). ∎

Question 1(b) *Find the function $f(z)$, holomorphic within the unit circle, which takes the values*

$$\frac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$$

on the circle.

Solution. See 1997, question 2(c). ∎

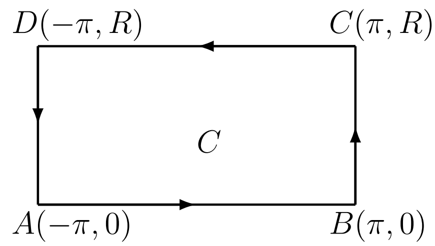
Question 1(c) *Find by contour integration the value of*

$$\int_0^{\pi} \frac{x\sin x\,dx}{a^2 - 2a\cos x + 1}$$

if $a > 1$.

Solution. Note: Even though the question restricts us to $a > 1$, we shall also consider the case $0 < a < 1$ for completeness.

We take $f(z) = \frac{z}{1-ae^{-iz}}$ and the contour C is the rectangle $ABCD$ where $A = (-\pi, 0), B = (\pi, 0), C = (\pi, R), D = (-\pi, R)$ oriented in the anticlockwise direction. We let $R \to \infty$ eventually.



Clearly $f(z)$ has simple poles at points z given by $e^{iz} = a = e^{\log a + 2n\pi i}$, $n \in \mathbb{Z}$. Thus $z = -i\log a + 2n\pi, n \in \mathbb{Z}$.

Thus when $a > 1$ i.e. $\log a > 0$, $f(z)$ has no pole in the vertical strip bounded by $x = -\pi, x = \pi, y > 0$. When $0 < a < 1$, $f(z)$ has a simple pole at $z = -i\log a$ inside C.

Residue at $z = -i\log a$ is given by $\lim_{z \to -i\log a} \frac{(z+i\log a)z}{1-ae^{-iz}} = \frac{-i\log a}{aie^{-\log a}} = -\log a$.

Thus by Cauchy's residue theorem,

$$\lim_{R\to\infty} \int_C \frac{z}{1-ae^{-iz}}\,dz = \begin{cases} 0, & \text{when } a > 1 \\ -2\pi i \log a, & \text{when } 0 < a < 1 \end{cases}$$

1. On CD, $z = x + iR$, x varies from π to $-\pi$.

$$\left|\int_{CD} \frac{z}{1-ae^{-iz}}\,dz\right| = \left|\int_{\pi}^{-\pi} \frac{x+iR}{1-ae^{-i(x+iR)}}\,dx\right|$$

Now $|x + iR| \le |x| + |R|, |1 - ae^{-ix}e^R| \ge |ae^R e^{-ix}| - 1$ and therefore

$$\left|\int_{CD} \frac{z}{1-ae^{-iz}}\,dz\right| \le \int_{-\pi}^{\pi} \frac{|x|+|R|}{ae^R-1}\,dx \le \int_{-\pi}^{\pi} \frac{\pi+|R|}{ae^R-1}\,dx$$

Now $\lim_{R\to\infty} \dfrac{2\pi(\pi+|R|)}{ae^R-1} = 0$, so $\int_{CD} f(z)\,dz = 0$.

2.
$$\int_{AB} f(z)\,dz = \int_{-\pi}^{\pi} \frac{x\,dx}{1-ae^{-ix}} = \int_{-\pi}^{0} \frac{x\,dx}{1-ae^{-ix}} + \int_{0}^{\pi} \frac{x\,dx}{1-ae^{-ix}}$$

Changing x to $-x$ in the first integral, we get

$$\begin{aligned} \int_{AB} f(z)\,dz &= \int_0^{\pi} \frac{x\,dx}{1-ae^{-ix}} - \int_0^{\pi} \frac{x\,dx}{1-ae^{ix}} \\ &= \int_0^{\pi} \frac{ax(-e^{ix}+e^{-ix})}{1-a(e^{ix}+e^{-ix})+a^2}\,dx \\ &= \int_0^{\pi} \frac{-2iax\sin x}{1-2a\cos x+a^2}\,dx \end{aligned}$$

3. On BC, $z = \pi + iy$ and on DA, $z = -\pi + iy$, $dz = i\,dy$, and

$$\begin{aligned}
&\lim_{R\to\infty}\left[\int_{BC} f(z)\,dz + \int_{DA} f(z)\,dz\right] \\
= \;& i\int_0^\infty \frac{\pi + iy}{1 - ae^{y-i\pi}}\,dy + i\int_\infty^0 \frac{-\pi + iy}{1 - ae^{y+i\pi}}\,dy \\
= \;& i\int_0^\infty \left[\frac{\pi + iy}{1 + ae^{y}} - \frac{-\pi + iy}{1 + ae^{y}}\right] dy \\
= \;& 2\pi i\int_0^\infty \frac{e^{-y}}{e^{-y} + a}\,dy \;=\; -2\pi i\log(e^{-y} + a)\Big]_0^\infty \;=\; 2\pi i\log\frac{a+1}{a}
\end{aligned}$$

Thus

$$\lim_{R\to\infty}\int_C f(z)\,dz = \int_0^\pi \frac{-2iax\sin x}{1 - 2a\cos x + a^2}\,dx + 2\pi i\log\frac{a+1}{a} = \begin{cases} 0, & \text{when } a > 1 \\ -2\pi i\log a, & \text{when } 0 < a < 1 \end{cases}$$

showing that when $a > 1$,

$$\int_0^\pi \frac{x\sin x}{1 - 2a\cos x + a^2}\,dx = \frac{-2\pi i\log\frac{a+1}{a}}{-2ia} = \frac{\pi}{a}\log\left(1 + \frac{1}{a}\right)$$

and when $0 < a < 1$,

$$\int_0^\pi \frac{x\sin x}{1 - 2a\cos x + a^2}\,dx = \frac{-2\pi i\log a - 2\pi i\log\frac{a+1}{a}}{-2ia} = \frac{\pi}{a}\log(1 + a)$$

∎

UPSC Civil Services Main 1983 - Mathematics Complex Analysis

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Question 1(a) *Obtain the Taylor and Laurent series expansions which represent the function* $\dfrac{z^2-1}{(z+2)(z+3)}$ *in the regions*
(i) $|z| < 2$ *(ii)* $2 < |z| < 3$ *(iii)* $|z| > 3$.

Solution. The only singularities of the function are at $z = -2$ and $z = -3$.

1. $|z| < 2$. In this region $f(z)$ is analytic and therefore will have Taylor series. It can be checked easily using partial fractions that

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Therefore

$$\begin{aligned} f(z) &= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \\ &= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n \\ &= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n \end{aligned}$$

is the required Taylor series valid in $|z| < 2$.

2. $2 < |z| < 3$: In this case we shall have a Laurent series.

$$\begin{aligned}
f(z) &= 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1} \\
&= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n \\
&= 3\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - \frac{5}{3} - \frac{8}{3}\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{3^n}
\end{aligned}$$

This is valid in $2 < |z| < 3$.

3. $|z| > 3$. We have a Taylor series around ∞ given by

$$f(z) = 1 + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}(3 \cdot 2^n - 8 \cdot 3^n)$$

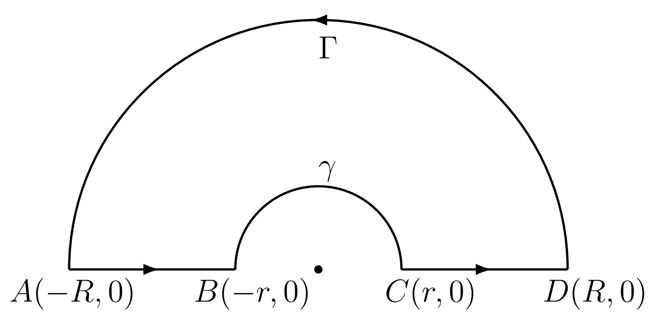
∎

Question 1(b) *Use the method of contour integration to evaluate*

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx, \qquad 0 < a < 2$$

Solution.

We take $f(z) = \frac{z^{a-1}}{1+z^2}$ and the contour C as shown in the figure. We choose the branch of z^{a-1} which results in $f(x) = \frac{x^{a-1}}{1+x^2}$ on the real axis. The only pole of $f(z)$ inside C is at $z = i$. The residue at $z = i$ is $\displaystyle\lim_{z\to i}\frac{(z-i)z^{a-1}}{1+z^2} = \frac{i^{a-1}}{2i} = \frac{1}{2i}(e^{\frac{\pi i}{2}})^{a-1} = \frac{1}{2i}\left(\cos\frac{\pi(a-1)}{2} + i\sin\frac{\pi(a-1)}{2}\right)$.



Now
$$\left|\int_\Gamma \frac{z^{a-1}}{1+z^2}\,dz\right| \le \int_0^\pi \frac{R^{a-1}}{R^2-1}R\,d\theta \le \frac{\pi R^a}{R^2-1} \to 0 \text{ as } R \to \infty \because 0 < a < 2$$
and
$$\left|\int_\gamma \frac{z^{a-1}}{1+z^2}\,dz\right| \le \int_0^\pi \frac{r^{a-1}}{1-r^2}r\,d\theta \le \frac{\pi r^a}{r^2-1} \to 0 \text{ as } r \to 0 \because a > 0$$

Here we use $|1+z^2| \geq 1-|z|^2 = 1-r^2$. Thus

$$\begin{aligned}
\lim_{\substack{R\to\infty\\ r\to 0}} \int_C f(z)\,dz &= \int_{-\infty}^{0} f(xe^{i\pi})(-\,dx) + \int_0^\infty f(x)\,dx \\
&= \int_0^\infty \frac{x^{a-1}e^{i\pi(a-1)}}{1+x^2}\,dx + \int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx \\
&= \int_0^\infty \frac{x^{a-1}}{1+x^2}\big(1+\cos\pi(a-1) + i\sin\pi(a-1)\big)\,dx \\
&= 2\pi i \cdot \frac{1}{2i}\Big(\cos\frac{\pi(a-1)}{2} + i\sin\frac{\pi(a-1)}{2}\Big)
\end{aligned}$$

Equating the real parts on both sides,

$$\big(1+\cos\pi(a-1)\big)\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx = \pi\cos\frac{\pi(a-1)}{2}$$

or

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx = \pi\sec\frac{\pi(a-1)}{2}$$

Equating the imaginary parts also gives us the same answer. ∎

Alternate solution: In 1984, question 1(b), we obtained

$$\begin{aligned}
2\sin^2\frac{\pi a}{2}\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt + \pi\sin\pi a\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\
-\sin\pi a\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt - \pi\cos\pi a\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\sin\frac{\pi a}{2}
\end{aligned}$$

Multiplying the first by $\cos\frac{\pi a}{2}$ and the second by $\sin\frac{\pi a}{2}$ and adding gives us

$$\begin{aligned}
\Big(\pi\sin\pi a\cos\frac{\pi a}{2} - \pi\cos\pi a\sin\frac{\pi a}{2}\Big)\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\Big(\cos^2\frac{\pi a}{2} + \sin^2\frac{\pi a}{2}\Big) \\
\implies \quad \pi\sin\Big(a\pi - \frac{a\pi}{2}\Big)\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2} \\
\implies \quad \int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi}{2}\frac{1}{\sin\frac{a\pi}{2}} = \frac{\pi}{2\cos\big(\frac{\pi}{2}-\frac{a\pi}{2}\big)} = \frac{\pi}{2}\sec(a-1)\frac{\pi}{2}
\end{aligned}$$

as calculated before.

Note: In this solution the advantage is that we avoid the use of the multiple valued function $\log z$, however it is much longer.

UPSC Civil Services Main 1984 - Mathematics Complex Analysis

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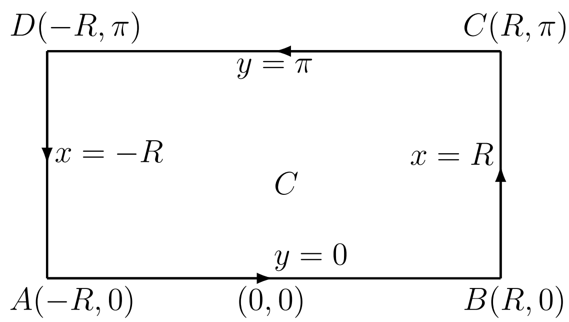
Question 1(a) *Evaluate by contour integration method* $\displaystyle\int_0^\infty \frac{x \sin mx}{x^4 + a^4}\, dx.$

Solution. See 1998 question 2(b). ∎

Question 1(b) *Evaluate by contour integration method* $\displaystyle\int_0^\infty \frac{x^{a-1}\log x}{1 + x^2}\, dx,\ 0 < a < 2.$

Solution.

We take $f(z) = \dfrac{ze^{az}}{1 + e^{2z}}$ and the contour C as the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, \pi), D = (-R, \pi)$ oriented positively.



1. On $BC, z = R + iy$ and therefore

$$\left|\int_{BC} f(z)\, dz\right| = \left|\int_0^\pi \frac{(R + iy)e^{aR+iay}}{1 + e^{2(R+iy)}} i\, dy\right| \leq \int_0^\pi \frac{(R + y)e^{aR}}{e^{2R} - 1}\, dy \leq \frac{\pi(R + \pi)e^{aR}}{e^{2R} - 1}$$

because $|1 + e^{2R}| \geq e^{2R} - 1$ and $R + y \leq R + \pi$ on $0 \leq y \leq \pi$.

Since $\displaystyle\lim_{R\to\infty} \frac{(R + \pi)e^{aR}}{e^{2R} - 1} = \lim_{R\to\infty} \frac{(R + \pi)ae^{aR} + e^{aR}}{2e^{2R}} = \lim_{R\to\infty} \frac{(R + \pi)a + 1}{2e^{2R-aR}} = 0$ if $2-a > 0$ i.e. $a < 2$, it follows that $\displaystyle\lim_{R\to\infty} \int_{BC} f(z)\, dz = 0$.

2.

$$\left|\int_{DA} f(z)\,dz\right| = \left|\int_0^\pi \frac{(-R+iy)e^{-aR+iay}}{1+e^{2(-R+iy)}} i\,dy\right| \le \int_0^\pi \frac{(-R+y)e^{-aR}}{1-e^{-2R}}\,dy \le \frac{\pi(R+\pi)e^{-aR}}{1-e^{-2R}}$$

But $\lim_{R\to\infty} Re^{-aR} = 0$ (note that $e^{-aR} \le \frac{1}{a^2R^2}$), therefore $\lim\limits_{R\to\infty}\int_{DA} f(z)\,dz = 0$.

3. $\lim\limits_{R\to\infty}\int_{AB} f(z)\,dz = \int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx.$

4. $\lim\limits_{R\to\infty}\int_{CD} f(z)\,dz = \int_{\infty}^{-\infty} \frac{(x+i\pi)e^{a(x+i\pi)}}{1+e^{2x+2i\pi}}\,dx$ as $z = x+i\pi$.

Thus

$$\begin{aligned}
\lim_{R\to\infty}\int_C f(z)\,dz &= \int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx - \int_{-\infty}^{\infty} \frac{(x+i\pi)e^{a(x+i\pi)}}{1+e^{2x}}\,dx \\
&= \int_{-\infty}^{\infty} \frac{xe^{ax}(1-e^{i\pi a})}{1+e^{2x}}\,dx - i\pi\int_{-\infty}^{\infty} \frac{e^{ax}e^{i\pi a}}{1+e^{2x}}\,dx
\end{aligned}$$

The poles of $f(z)$ are given by $e^{2z} = e^{(2n+1)\pi i}$. The only pole in the strip $0 \le y \le \pi$ is $z = \frac{\pi i}{2}$ and it is a simple pole.

Residue at $z = \frac{\pi i}{2}$ is $\lim\limits_{z\to\frac{\pi i}{2}} \frac{ze^{az}(z-\frac{\pi i}{2})}{1+e^{2z}} = \frac{\frac{\pi i}{2}e^{\frac{\pi i a}{2}}}{2e^{\pi i}} = -\frac{\pi i}{4}e^{\frac{\pi i a}{2}}$. Thus

$$\int_{-\infty}^{\infty} \frac{xe^{ax}(1-e^{i\pi a})}{1+e^{2x}}\,dx - i\pi\int_{-\infty}^{\infty} \frac{e^{ax}e^{i\pi a}}{1+e^{2x}}\,dx = 2\pi i\left(-\frac{\pi i}{4}e^{\frac{\pi i a}{2}}\right) \tag{1}$$

Equating the real part of both sides, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{xe^{ax}(1-\cos\pi a)}{1+e^{2x}}\,dx + \pi\sin a\pi\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx &= \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\
\Rightarrow 2\sin^2\frac{\pi a}{2}\int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx + \pi\sin\pi a\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx &= \frac{\pi^2}{2}\cos\frac{\pi a}{2}
\end{aligned}$$

Putting $e^x = t$ so that $x = \log t, dx = dt/t$, we get

$$2\sin^2\frac{\pi a}{2}\int_0^{\infty} \frac{t^{a-1}\log t}{1+t^2}\,dt + \pi\sin\pi a\int_0^{\infty} \frac{t^{a-1}}{1+t^2}\,dt = \frac{\pi^2}{2}\cos\frac{\pi a}{2} \tag{2}$$

Equating the imaginary parts in (1), we get

$$-\sin\pi a\int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx - \pi\cos\pi a\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx = \frac{\pi^2}{2}\sin\frac{\pi a}{2}$$

Put $e^x = t$ as before, to get

$$-\sin \pi a \int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt - \pi \cos \pi a \int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt = \frac{\pi^2}{2}\sin\frac{\pi a}{2} \tag{3}$$

Multiplying (2) by $\cos \pi a$ and (3) by $\sin \pi a$ and adding we get

$$\begin{aligned}
\left(2\sin^2\frac{\pi a}{2}\cos\pi a - \sin^2 \pi a\right)\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt &= \frac{\pi^2}{2}\left[\cos\pi a\cos\frac{\pi a}{2} + \sin\pi a\sin\frac{\pi a}{2}\right] \\
&= \frac{\pi^2}{2}\cos\left(\pi a - \frac{\pi a}{2}\right) = \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\
\text{Now}\quad 2\sin^2\frac{\pi a}{2}\cos\pi a - \sin^2\pi a &= 2\sin^2\frac{\pi a}{2}\left[\cos\pi a - 2\cos^2\frac{\pi a}{2}\right] \\
&= 2\sin^2\frac{\pi a}{2}\left[2\cos^2\frac{\pi a}{2} - 1 - 2\cos^2\frac{\pi a}{2}\right] \\
&= -2\sin^2\frac{\pi a}{2} \\
\Rightarrow \quad \int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt &= -\frac{\pi^2}{2}\cos\frac{\pi a}{2}\Big/ 2\sin^2\frac{\pi a}{2} \\
&= -\frac{\pi^2}{4}\cot\frac{\pi a}{2}\csc\frac{\pi a}{2}
\end{aligned}$$

as required. ∎

Question 1(c) *Distinguish clearly between a pole and an essential singularity. If $z = a$ is an essential singularity of a function $f(z)$, prove that for any positive numbers η, ρ, ϵ there exists a point z such that $0 < |z - a| < \rho$ for which $|f(z) - \eta| < \epsilon$.*

Solution. If $f(z)$ has an isolated singularity at z_0, which is not a removable singularity, then $f(z)$ has a pole at $z = z_0$ if $\lim_{z\to z_0} f(z) = \infty$. In this case if $f(z)$ has a pole of order k at $z = z_0$, then

$$f(z) = a_{-k}(z-z_0)^{-k} + \ldots + a_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

and this Laurent expansion is valid in some deleted neighborhood $0 < |z - z_0| < \delta$ of z_0.

If $\lim_{z\to z_0} f(z)$ does not exist, then $f(z)$ has an essential singularity at $z = z_0$. (Note that $\lim_{z\to z_0} f(z)$ is not finite as z_0 is not a removable singularity). In this case

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

and $a_{-n} \neq 0$ for infinitely many n. Again this Laurent expansion is valid in some deleted neighborhood $0 < |z - z_0| < \delta$ of z_0.

The second part is Casorati-Weierstrass theorem. Let $f(z)$ be analytic in some deleted neighborhood N of a. Suppose that there exists $\epsilon > 0$ such that $|f(z) - \eta| < \epsilon$ is not satisfied for any $z \in N$i.e. $|f(z) - \eta| \geq \epsilon$ for every $z \in N$. Let $g(z) = \frac{1}{f(z)-\eta}$. Then $g(z)$ is analytic in N and $g(z)$ is bounded in N, therefore $g(z)$ has a removable singularity at a. Since $g(z)$ is not constant as $f(z)$ is not constant, either $g(a) \neq 0$ or $g(z)$ has a zero of order $k > 0$ at $z = a$. This means that $f(z) - \eta$ is either analytic at $z = a$ or $f(z) - \eta$ has a pole of order k at $z = a$. But this is not true, because $f(z)$ has an essential singularity at $z = a$. Thus our assumption is false i.e. we must have $z \in N$ for which $|f(z) - \eta| < \epsilon$. Note that we could take our deleted neighborhood N of the type $0 < |z - a| < \delta \leq \rho$. ∎

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Question 1(a) *Prove that every power series represents an analytic function within its circle of convergence.*

Solution. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ have R as its radius of convergence. We shall show that for any z in the region $C = \{z \ : \ |z| < R\}$, $f'(z) = \displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$. We first of all note that the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R as $\displaystyle\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Let $z \in C$ and $|z| < \rho < R$ and let h be chosen so small that $|z| + |h| \leq \rho < R$. Thus

$$\left|\frac{(z+h)^n - z^n}{(z+h) - z}\right| \leq (|z| + |h|)^{n-1} + |z|(|z| + |h|)^{n-2} + \ldots + |z|^{n-1} \leq n\rho^{n-1} \tag{1}$$

Since the series $\displaystyle\sum_{n=1}^{\infty} n a_n \rho^{n-1}$ is convergent, given $\epsilon > 0$ $\exists N_1 > 0$ such that

$$\left|\sum_{r=n+1}^{\infty} r|a_r|\rho^{r-1}\right| < \frac{\epsilon}{3} \quad \text{for } n \geq N_1$$

and in particular $\displaystyle\sum_{r=N_1+1}^{\infty} r|a_r|\rho^{r-1} < \frac{\epsilon}{3}$. $\qquad$ (2)

Since $\displaystyle\lim_{h\to 0}\left[a_n\frac{(z-h)^n - z^n}{h} - n a_n z^{n-1}\right] = 0$, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{n=1}^{N_1}\left[a_n\frac{(z-h)^n - z^n}{h} - n a_n z^{n-1}\right]\right| < \frac{\epsilon}{3} \text{ for } |h| < \delta \tag{3}$$

Now

$$
\begin{aligned}
&\left|\frac{f(z+h)-f(z)}{h}-\sum_{n=1}^{\infty} n a_n z^{n-1}\right| \\
\leq\ & \left|\sum_{n=1}^{N_1}\left[a_n \frac{(z+h)^n-z^n}{h}-n a_n z^{n-1}\right]\right|+\sum_{n=N_1+1}^{\infty} \frac{\left|a_n\left((z+h)^n-z^n\right)\right|}{h}+\sum_{n=N_1+1}^{\infty}\left|n a_n z^{n-1}\right| \\
=\ & \frac{\epsilon}{3}+\sum_{n=N_1+1}^{\infty}\left|a_n\right| n \rho^{n-1}+\sum_{n=N_1+1}^{\infty}\left|a_n\right| n \rho^{n-1} \quad \text{for } |h|<\delta \\
<\ & \epsilon
\end{aligned}
$$

Thus $\lim\limits_{h \rightarrow 0} \dfrac{f(z+h)-f(z)}{h}=\sum_{n=1}^{\infty} n a_n z^{n-1}=f'(z)$, so $f(z)$ is analytic in C. ∎

Question 1(b) *Prove that the derivative of a function analytic in a domain is itself an analytic function.*

Solution. Cauchy's integral formula states that if $f(z)$ is analytic within and on a simple closed countour C oriented positively and if z_0 is any interior point of C, then $f(z_0)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{z-z_0}$.

Let $f(z)$ be differentiable in a domain D and $z_0 \in D$. Let C be a circle with center z_0, the boundary of which is positively oriented, such that $f(z)$ is differentiable within and on C, and C along with its interior lies in D. Then by Cauchy's integral formula, $f(z_0)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{z-z_0}$.

Let $h \in \mathbb{C}$ be so small that z_0+h also lies in the interior of C.

$$
\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2\pi i h}\int_C\left(\frac{f(z)}{z-z_0-h}-\frac{f(z)}{z-z_0}\right) dz \\
&= \frac{1}{2\pi i h}\int_C \frac{h f(z)\,dz}{(z-z_0-h)(z-z_0)} \\
&= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0-h)(z-z_0)}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^2} \\
=\ & \frac{1}{2\pi i}\int_C\left(\frac{f(z)}{(z-z_0-h)(z-z_0)}-\frac{f(z)}{(z-z_0)^2}\right) dz \\
=\ & \frac{1}{2\pi i}\int_C \frac{h f(z)\,dz}{(z-z_0-h)(z-z_0)^2}
\end{aligned}
$$

Let $M = \sup_{z\in C} |f(z)|, l =$ length of $C, d = \min_{z\in C} |z - z_0|, d > 0$. Since we are interested in $h \to 0$, we could have assumed in the beginning itself that $0 < |h| < d$. Thus we get

$$\left| \frac{f(z_0+h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^2} \right| \le \frac{M|h|l}{2\pi d^2(d-|h|)}$$

Since the right hand side of the above inequality tends to 0 as $h \to 0$, it follows that

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^2}$$

i.e. $f(z)$ is differentiable at z_0 and since z_0 is an arbitrary point of D, it follows that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2}$$

where C is any positively oriented circle containing z in its interior.

We shall now prove that

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^3}$$

where z_0, C are as chosen above. Let h be also chosen as above. Then

$$\begin{aligned}
& \frac{f'(z_0+h) - f'(z_0)}{h} - \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^3} \\
= \; & \frac{1}{2\pi i h} \int_C f(z) \left[\frac{1}{(z-z_0-h)^2} - \frac{1}{(z-z_0)^2} - \frac{2h}{(z-z_0)^3} \right] dz \\
= \; & \frac{1}{2\pi i h} \int_C f(z) \frac{(z-z_0)^3 - (z-z_0-h)^2(z-z_0) - 2h(z-z_0-h)^2}{(z-z_0-h)^2(z-z_0)^3}\, dz
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad & (z-z_0)^3 - (z-z_0-h)^2(z-z_0) - 2h(z-z_0-h)^2 \\
= \; & (z-z_0)[(z-z_0)^2 - (z-z_0-h)^2] - 2h[(z-z_0)^2 - 2h(z-z_0) + h^2] \\
= \; & (z-z_0)h[2(z-z_0) - h] - 2h(z-z_0)^2 + 4h^2(z-z_0) - 2h^3 \\
= \; & h[2(z-z_0)^2 - h(z-z_0) - 2(z-z_0)^2 + 4h(z-z_0) - 2h^2] \\
= \; & h^2[3(z-z_0) - 2h]
\end{aligned}$$

Thus we get

$$\left| \frac{f'(z_0+h) - f'(z_0)}{h} - \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^3} \right| \le \frac{M|h|(3\rho + 2|h|^2)l}{2\pi d^3(d-|h|)^2}$$

where M, d, ρ are as before. Since the right hand side of the above inequality tends to 0 as $h \to 0$, it follows that

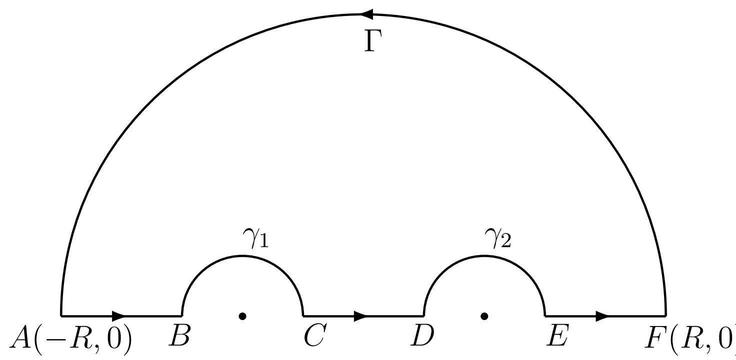
$$f''(z_0) = \lim_{h\to 0} \frac{f'(z_0+h) - f'(z_0)}{h} = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^3}$$

i.e. $f'(z)$ is also analytic in D. ∎

Question 1(c) *Evaluate by the method of contour integration* $\displaystyle\int_0^\infty \frac{x \sin ax}{x^2 - b^2}\, dx.$

Solution. We take $f(z) = \frac{ze^{iaz}}{z^2-b^2}$ and the contour C consisting of the following

1. The line AB joining $A = (-R, 0)$ and $B = (-b - r_1, 0)$.
2. γ_1, the semicircle $(x+b)^2+y^2 = r_1^2$ lying in the upper half plane.
3. Line CD joining $C = (-b + r_1, 0)$ and $D = (b - r_2, 0)$.
4. γ_2, the semicircle $(x-b)^2+y^2 = r_2^2$ lying in the upper half plane.
5. Line EF joining $E = (b + r_2, 0)$ and $F = (R, 0)$.
6. Γ, the semicircle $x^2 + y^2 = R^2$ lying in the upper half plane.



Eventually we will let $R \to \infty, r_1, r_2 \to 0$. Now the integrand has no pole in the upper half plane, therefore

$$\lim_{\substack{R\to\infty \\ r_1\to 0 \\ r_2\to 0}} \int_C \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = 0$$

1. On Γ,

$$\left| \int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)} \right| \leq \left| \int_0^\pi \frac{Re^i\theta e^{iaRe^{i\theta}}}{R^2 - b^2} Rie^{i\theta}\, d\theta \right|$$

because of Γ, $|z^2 - b^2| \geq |z|^2 - b^2 = R^2 - b^2$.

$$\left| \int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)} \right| \leq \frac{R^2}{R^2 - b^2} \int_0^\pi e^{-aR\sin\theta}\, d\theta = \frac{2R^2}{R^2 - b^2} \int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\, d\theta$$

(We can double the integral and halve the limit, because $\sin(\pi - \theta) = \sin\theta$). Using Jordan's inequality $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ we get

$$\left| \int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)} \right| \leq \frac{2R^2}{R^2 - b^2} \int_0^{\frac{\pi}{2}} e^{-aR\frac{2\theta}{\pi}}\, d\theta = \frac{2R^2}{R^2 - b^2} \left(\frac{1 - e^{-aR}}{2aR/\pi} \right) = \frac{\pi R(1 - e^{-aR})}{a(R^2 - b^2)}$$

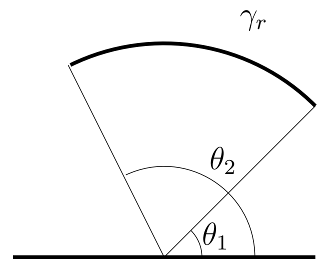
showing that $\displaystyle\lim_{R\to\infty} \int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = 0.$

2.

To get the value of the integral along γ_1, γ_2 we observe that if $f(z)$ has a simple pole at $z = a$ and γ_r is a part of a circle of radius r with center a, then

$$\lim_{r \to 0} \int_{\gamma_r} f(z)\, dz = i a_{-1}(\theta_2 - \theta_1)$$

where a_{-1} is the residue of $f(z)$ at a.



Proof: Let

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots = \frac{a_{-1}}{z-a} + \phi(z)$$

where $\phi(z)$ is analytic in the circle $|z-a| \leq r$. Thus

$$\left| \int_{\gamma_r} \phi(z)\, dz \right| \leq Mr(\theta_2 - \theta_1)$$

where $M = \sup_{|z-a|=r} |\phi(z)|$. Thus $\lim_{r \to 0} \int_{\gamma_r} \phi(z)\, dz = 0$ and

$$\lim_{r \to 0} \int_{\gamma_r} f(z)\, dz = \int_{\gamma_r} \frac{a_{-1}\, dz}{z-a} = i \int_{\theta_1}^{\theta_2} a_{-1}\, d\theta = i a_{-1}(\theta_2 - \theta_1)$$

Now the residue of $\dfrac{ze^{iaz}}{z^2 - b^2}$ at $z = b$ is $\frac{1}{2}e^{iab}$, and the residue at $z = -b$ is $\frac{1}{2}e^{-iab}$. Thus $\lim_{r_1 \to 0} \int_{\gamma_1} f(z)\, dz = \frac{1}{2} i e^{-iab}(0-\pi) = -\frac{i\pi}{2} e^{-iab}$ and $\lim_{r_2 \to 0} \int_{\gamma_2} f(z)\, dz = \frac{1}{2} i e^{iab}(0-\pi) = -\frac{i\pi}{2} e^{iab}$.

Using the above data we get

$$0 = \lim_{\substack{R \to \infty \\ r_1 \to 0 \\ r_2 \to 0}} \int_C \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = \int_{-\infty}^{\infty} \frac{xe^{iax}\, dx}{(x^2 - b^2)} - \frac{i\pi}{2} e^{-iab} - \frac{i\pi}{2} e^{iab}$$

or

$$\int_{-\infty}^{\infty} \frac{xe^{iax}\, dx}{(x^2 - b^2)} = \pi i \cos(ab)$$

Taking imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x \sin ax\, dx}{(x^2 - b^2)} = \pi \cos(ab)$$

or

$$\int_0^{\infty} \frac{x \sin ax\, dx}{(x^2 - b^2)} = \frac{\pi \cos(ab)}{2}$$

∎

UPSC Civil Services Main 1986 - Mathematics Complex Analysis

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July 19, 2010

Question 1(a) *Let $f(z)$ be single valued and analytic within and on a simple closed curve C. If z_0 is any point in the interior of C, then show that*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0}$$

where the integral in taken in the positive sense around C.

Solution. This is known as the Cauchy integral formula. We shall show that given $\epsilon > 0$

$$\left| \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0} - f(z_0) \right| < \epsilon$$

which implies the result as ϵ is arbitrary.

Since $f(z)$ is analytic at z_0, it is continuous at z_0, therefore given $\epsilon > 0$ as above, there exists a $\delta > 0$ such that $|z - z_0| \leq \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon$. We choose $\delta > 0$ so small that the disc $|z - z_0| < \delta$ lies within the interior of C. Then by Cauchy-Goursat's theorem (See 1987, 1(b)) we have

$$\frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{z - z_0} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z - z_0}$$

where γ is the circle $|z - z_0| = \rho < \delta$ and is positively oriented.

Now put $z - z_0 = \rho e^{i\theta}$ to get

$$\int_\gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}} = 2\pi i$$

Therefore

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\right| &= \left|\frac{1}{2\pi i}\left[\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_\gamma \frac{dz}{z-z_0}\right]\right| \\
&= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \\
&\leq \frac{1}{2\pi\rho}\epsilon\int_\gamma |dz| = \frac{\epsilon}{2\pi\rho}\text{length of }\gamma = \epsilon
\end{aligned}$$

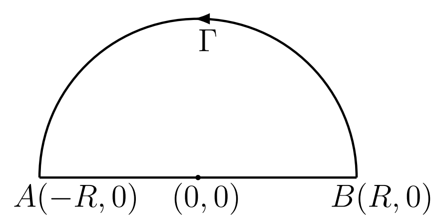
Thus $\displaystyle\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0) = 0$ and the proof is complete. ∎

Question 1(b) *By the contour integration method show that*

1. $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^2}$ *where* $a > 0$.
2. $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Solution.

1. We take $f(z) = \frac{1}{z^4+a^4}$ and the contour C consisting of Γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane, and the line AB joining $(-R,0)$ and $(R,0)$. C is positively oriented.



   (a) Poles of $f(z)$ are given by $z = \pm a e^{\pm\frac{\pi i}{4}} = \pm a[\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}]$, out of which $z = ae^{\frac{\pi i}{4}}, z = -ae^{-\frac{\pi i}{4}}$ are in the upper half plane.

   Residue at $z = ae^{\frac{\pi i}{4}}$ is $\displaystyle\frac{1}{4a^3 e^{\frac{3\pi i}{4}}} = \frac{1}{4a^3}\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]^{-1}$.

   Residue at $z = -ae^{-\frac{\pi i}{4}}$ is $\displaystyle\frac{-1}{4a^3 e^{-\frac{3\pi i}{4}}} = \frac{-1}{4a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]^{-1}$.

   Sum of residues is $\displaystyle\frac{\sqrt{2}}{4a^3}\left[\frac{1}{i-1} + \frac{1}{1+i}\right] = -\frac{i\sqrt{2}}{4a^3}$. Thus

$$\lim_{R\to\infty}\int_C \frac{dz}{z^4+a^4} = \frac{2\pi i \cdot -i\sqrt{2}}{4a^3} = \frac{2\sqrt{2}\pi}{4a^3} = \frac{\pi}{\sqrt{2}a^3}$$

   (b)

$$\left|\int_\Gamma \frac{dz}{z^4+a^4}\right| \leq \left|\int_0^\pi \frac{Rie^{i\theta}}{R^4-a^4}\right| \leq \frac{\pi R}{R^4-a^4}$$

   because on Γ $|z^4+a^4| \geq |z^4| - a^4 = R^4 - a^4$. Thus $\displaystyle\lim_{R\to\infty}\int_\Gamma \frac{dz}{z^4+a^4} = 0$.

(c) $\displaystyle \lim_{R\to\infty} \int_{AB} \frac{dz}{z^4+a^4} = \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\int_0^{\infty} \frac{dx}{x^4+a^4}.$

Using (a), (b) and (c) we get

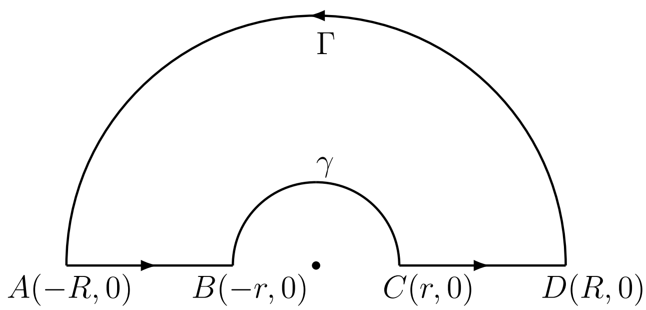
$$\lim_{R\to\infty} \int_C \frac{dz}{z^4+a^4} = 2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{\sqrt{2}a^3}$$

Thus

$$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3} = \frac{\pi\sqrt{2}}{4a^3}$$

as required.

2. We take $f(z) = \frac{e^{iz}}{z}$ and the contour C consisting of the line AB joining $(-R, 0)$ to $(-r, 0)$, the semicircle γ of radius r with center $(0, 0)$, the line CD joining $(r, 0)$ to $(R, 0)$ and Γ a semicircle of radius R with center $(0, 0)$. The contour lies in the upper half plane and is oriented anticlockwise. We took γ as part of the contour to avoid the pole at $(0, 0)$.

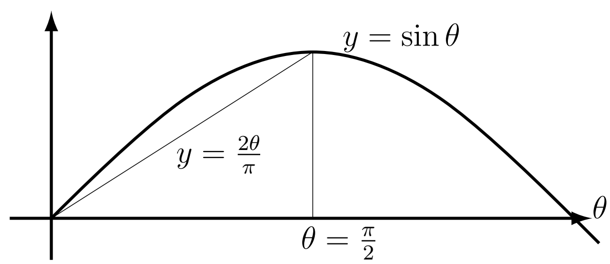


We will eventually make $R \to \infty$ and $r \to 0$.

(a) Since the integrand $\frac{e^{iz}}{z}$ has no poles in the upper half plane, it follows that

$$\lim_{R\to\infty, r\to 0} \int_C \frac{e^{iz}}{z}\, dz = 0$$

(b) In order to prove that $\displaystyle \lim_{R\to\infty} \int_\Gamma \frac{e^{iz}}{z}\, dz = 0$, we use Jordan inequality, which states that $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ — compare the graphs as shown in the figure.



$$\begin{aligned} \left| \int_\Gamma \frac{e^{iz}}{z}\, dz \right| &\leq \int_0^{\pi} \frac{e^{-R\sin\theta}}{R} R\, d\theta \;=\; 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \qquad (\because \sin(\pi-\theta) = \sin\theta) \\ &\leq 2\int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}}\, d\theta \;=\; 2\cdot\frac{\pi}{2R}[1-e^{-R}] \end{aligned}$$

showing that $\displaystyle \lim_{R\to\infty} \int_\Gamma \frac{e^{iz}}{z}\, dz = 0$ as $\displaystyle \lim_{R\to\infty} \frac{1-e^{-R}}{R} = 0.$

(c) Now $\frac{e^{iz}}{z} = \frac{1}{z} + \phi(z)$ where $\phi(z)$ is analytic at $z = 0$. Thus given $\epsilon > 0, \exists \delta > 0$ such that $|z| < \delta \Rightarrow |\phi(z)| < \epsilon$. Thus $|\int_\gamma \phi(z)\,dz| \le \epsilon \cdot (\text{length of } \gamma) \Rightarrow \lim_{r\to 0} \int_\gamma \phi(z)\,dz = 0$.

$$\int_\gamma \frac{dz}{z} = \int_\pi^0 \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = -i\pi$$

(d)

$$\lim_{R\to\infty, r\to 0} \int_{AB} \frac{e^{iz}}{z}\,dz = \int_{-\infty}^0 \frac{e^{ix}}{x}\,dx, \quad \lim_{R\to\infty, r\to 0} \int_{CD} \frac{e^{iz}}{z}\,dz = \int_0^\infty \frac{e^{ix}}{x}\,dx$$

Using these results, we get

$$0 = \lim_{R\to\infty, r\to 0} \int_C \frac{e^{iz}}{z}\,dz = \int_{-\infty}^0 \frac{e^{ix}}{x}\,dx - i\pi + \int_0^\infty \frac{e^{ix}}{x}\,dx \qquad (*)$$

Since

$$\int_{-\infty}^0 \frac{e^{ix}}{x}\,dx = -\int_0^\infty \frac{e^{-iy}}{y}\,dy$$

we get

$$\int_0^\infty \frac{e^{ix}}{x}\,dx - \int_0^\infty \frac{e^{-ix}}{x}\,dx = i\pi$$

or

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{2i} \frac{1}{x}\,dx = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Note that in (*) we cannot write $\displaystyle\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx = \pi i$ and conclude that $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi$, $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x}\,dx = 0$, because $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x}\,dx$ has convergence problem at $x = 0$.

∎

UPSC Civil Services Main 1987 - Mathematics Complex Analysis

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Question 1(a) *By considering the Laurent series for* $f(z) = \dfrac{1}{(1-z)(z-2)}$ *prove that if* C *is a closed contour oriented in the counter clockwise direction, then* $\displaystyle\int_C f(z)\,dz = 2\pi i$.

Solution. Laurent's theorem states that if $f(z)$ is analytic throughout the annular region $R_1 < |z - z_0| < R_2$ and C is any positively oriented simple closed curve lying in the annular region and having z_0 in its interior, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

Thus $\displaystyle\int_C f(z)\,dz = 2\pi i a_{-1}$.

Now in our case: $R_1 = 1, R_2 = 2, z_0 = 0$ i.e. $f(z)$ is analytic in $1 < |z| < 2$. Moreover

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

Since $|\frac{1}{z}| < 1$ and $|\frac{z}{2}| < 1$, we get $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ as the Laurent expansion of $f(z)$ valid in $1 < |z| < 2$. Thus if C is a simple closed contour lying in $1 < |z| < 2$ with the origin in its interior, then $\displaystyle\int_C f(z)\,dz = 2\pi i$, since $a_{-1} = 1$.

Note that in this question, the curve C has not been clearly specified. If C is in the region $1 < |z| < 2$, but does not contain the origin, then $\displaystyle\int_C f(z)\,dz = 0$.

Note: What about the other cases — C lies in $|z| < 1$ or $|z| > 2$ ∎

Question 1(b) *State and prove Cauchy's residue theorem.*

Solution. Statement: If C is a simple closed contour oriented anticlockwise and $f(z)$ is a complex valued function which is analytic on and within the interior of C except for a finite number of poles $z_1, \dots z_n$ in the interior of C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } z = z_r$$

Proof: We enclose each $z_j, 1 \leq j \leq n$ is a small disc C_j such that C_j along with its boundary lies within C and, and these discs are small enough that C_j and C_k, $j \neq k$ do not overlap, or even touch on their boundaries. We now use Cauchy-Goursat theorem:

Theorem (Caucy-Goursat): If C is a simple closed positively oriented contour and $C_1, \dots C_n$ are simple closed positively oriented contours which lie within C, and whose interiors have no points in common, and if $f(z)$ is a function which is analytic within and on C except for the interiors of C_j, $1 \leq j \leq n$, then

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz$$

Using this theorem we see that in our case

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz$$

But in the last question we have seen that

$$\int_{C_j} f(z)\, dz = 2\pi i \times \text{Residue of } f(z) \text{ at } z = z_j$$

(This is Laurent's theorem.) Thus

$$\int_C f(z)\, dz = 2\pi i \sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } z = z_r$$

■

Question 1(c) *By the method of contour integration show that*

$$\int_0^\infty \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi e^{-a}}{2a}, \ a > 0$$

Solution. See 2002, question 2(b). ■

UPSC Civil Services Main 1988 - Mathematics Complex Analysis

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July 19, 2010

Question 1(a) *By evaluating* $\int \frac{dz}{z+2}$ *over a suitable contour* C *prove that*

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$$

Solution. By using the unit circle $|z| = 1$ as contour, and integrating $\int_{|z|=1} \frac{dz}{z+2}$, we have proved $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$ — see 1997, question 1(b). Now in $\int_{\pi}^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$ put $\theta = 2\pi - \phi$ so that

$$\int_{\pi}^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \int_{\pi}^{0} \frac{1+2\cos(2\pi-\phi)}{5+4\cos(2\pi-\phi)}\,(-d\phi) = \int_0^{\pi} \frac{1+2\cos\phi}{5+4\cos\phi}\,d\phi$$

Thus $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 2\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$ showing that

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$$

. ■

Note: If the contour was not prescribed, we could have put $z = e^{i\theta}$ to get

$$\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = \frac{1}{i}\int_{|z|=1} \frac{z^2+z+1}{z(5z+2z^2+2)}\,dz$$

The integrand has two poles at $z = 0, z = -\frac{1}{2}$ inside $|z| = 1$, which are simple poles. The residue at $z = 0$ is $\frac{1}{2}$ and the residue at $z = -\frac{1}{2}$ is $-\frac{1}{2}$, so we get

$$\int_{|z|=1} \frac{z^2 + z + 1}{z(5z + 2z^2 + 2)}\, dz = 0 \Rightarrow \int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$$

Question 1(b) *If $f(z)$ is analytic in $|z| \leq R$ and x, y lie inside the disc, evaluate the integral $\displaystyle\int_{|z|=R} \frac{f(z)\, dz}{(z-x)(z-y)}$ and deduce that a function analytic and bounded for all finite z is a constant.*

Solution. Cauchy's integral formula states that if $f(z)$ is analytic on and within the disc $|z| \leq R$, then for any ζ which lies within the disc

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)\, dz}{\zeta - z}$$

Thus

$$\int_{|z|=R} \frac{f(z)\, dz}{(z-x)(z-y)} \;=\; \frac{1}{x-y}\left[\int_{|z|=R} \frac{f(z)\, dz}{z-x} - \int_{|z|=R} \frac{f(z)\, dz}{z-y}\right] \;=\; \frac{2\pi i}{x-y}\left[f(x) - f(y)\right]$$

We now prove the remaining part, which is Liouville's theorem.

Let $|f(z)| \leq M$ for every z. Clearly $|z - x| \geq |z| - |x| = R - |x|$ and similarly $|z - y| \geq R - |y|$ on $|z| = R$, and therefore

$$\left|\int_{|z|=R} \frac{f(z)\, dz}{(z-x)(z-y)}\right| \leq \frac{M \cdot 2\pi R}{(R - |x|)(R - |y|)}$$

Thus $|f(x) - f(y)| \leq \left|\dfrac{1}{2\pi i}\right| \dfrac{|x-y| \cdot M \cdot 2\pi R}{(R - |x|)(R - |y|)}$. Since $\dfrac{R}{(R - |x|)(R - |y|)} \to 0$ as $R \to \infty$, it follows that $|f(x) - f(y)| = 0$ or $f(x) = f(y)$, so f is constant. ∎

Question 1(c) *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $0 < r < R$, prove that*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Solution.

$$|f(z)|^2 = f(z) \cdot \overline{f(z)} = \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} \overline{a_m}\, \overline{z}^m = \sum_{n=0}^{\infty} \sum_{p+q=n} a_p \overline{a_q} z^p \overline{z}^q$$

We know that if a power series has a radius of convergence R, then it is uniformly and absolutely convergent in $|z| \leq r$ where $0 < r < R$, therefore

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\, d\theta = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \sum_{p+q=n} a_p \overline{a_q} r^p r^q e^{i(p-q)\theta}\, d\theta$$

Since $\int_0^{2\pi} e^{i(p-q)\theta}\, d\theta = 0$ when $p \neq q$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

(Note: This shows that if $|f(z)| \leq M$ on $|z| = r$, then $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq M^2$.) ∎

Question 2(a) *Evaluate* $\displaystyle\int_C \frac{ze^z\, dz}{(z-a)^3}$ *if a lies inside the closed contour C.*

Solution. Clearly the only pole of $\dfrac{ze^z}{(z-a)^3}$ is of order 3 at $z = a$. The residue at this pole is

$$\frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{(z-a)^3 ze^z}{(z-a)^3}\right)_{z=a} = \frac{1}{2}\frac{d}{dz}\Big(ze^z + e^z\Big)_{z=a} = \frac{1}{2}\Big(ze^z + e^z + e^z\Big)_{z=a} = e^a\Big(1 + \frac{a}{2}\Big)$$

Thus by Cauchy's residue theorem,

$$\int_C \frac{ze^z\, dz}{(z-a)^3} = 2\pi i \cdot e^a\Big(1 + \frac{a}{2}\Big) = \pi i e^a(2 + a)$$

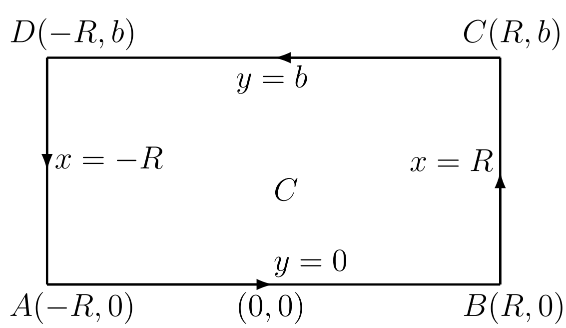
∎

Question 2(b) *Prove*

$$\int_0^{\infty} e^{-x^2}\cos(2bx)\, dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \qquad (b > 0)$$

by integrating e^{-z^2} along the boundary of the rectangle $|x| \leq R, 0 \leq y \leq b$.

Solution.



Let the rectangle be $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, b), D = (-R, b)$ oriented positively. Since e^{-z^2} has no pole inside $ABCD$, we get $\displaystyle\lim_{R\to\infty}\int_{ABCD} e^{-z^2}\, dz = 0.$

(Note that e^{-z^2} has no pole in the entire complex plane.)

1. On BC, $z = R + iy$ and $0 \le y \le b$, therefore
$$\left|\int_{BC} e^{-z^2}\,dz\right| = \left|\int_0^b e^{-R^2}e^{-2Riy}e^{-i^2y^2}i\,dy\right| \le e^{-R^2}\int_0^b e^{y^2}\,dy = (\text{constant})e^{-R^2}$$
Clearly $e^{-R^2} \to 0$ as $R \to \infty$, so $\lim_{R\to\infty}\int_{BC} e^{-z^2}\,dz = 0$.

2. On DA, $z = -R + iy$ and $0 \le y \le b$, therefore
$$\left|\int_{DA} e^{-z^2}\,dz\right| = \left|\int_b^0 e^{-R^2}e^{2Riy}e^{-i^2y^2}i\,dy\right| \le e^{-R^2}\int_0^b e^{y^2}\,dy$$
Thus $\lim_{R\to\infty}\int_{DA} e^{-z^2}\,dz = 0$.

3. On AB, $z = x$ so $\lim_{R\to\infty}\int_{AB} e^{-z^2}\,dz = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

4. On CD, $z = x + ib$, therefore
$$\lim_{R\to\infty}\int_{CD} e^{-z^2}\,dz = \int_{\infty}^{-\infty} e^{-x^2}e^{-i^2b^2}e^{-2ibx}\,dx = -e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}[\cos 2bx - i\sin 2bx]\,dx$$

Using the above calculations, we get
$$0 = \lim_{R\to\infty}\int_C e^{-z^2}\,dz = \sqrt{\pi} - e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}[\cos 2bx - i\sin 2bx]\,dx$$

Equating real and imaginary parts,
$$\int_{-\infty}^{\infty} e^{-x^2}\sin 2bx\,dx = 0$$
and
$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = \sqrt{\pi}e^{-b^2}$$
Thus
$$\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}e^{-b^2}}{2}$$

∎

Question 2(c) *Prove that the coefficients c_n of the expansion*

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy $c_n = c_{n-1} + c_{n-2}, n \geq 2$. Determine c_n.

Solution. $z^2 + z - 1 = 0 \Rightarrow z = \frac{-1\pm\sqrt{5}}{2}$. Let $\lambda = \frac{-1+\sqrt{5}}{2}, \mu = \frac{-1-\sqrt{5}}{2}$. Thus $f(z) = \frac{1}{1-z-z^2}$ is analytic in the disc $|z| < \lambda$ as both the singularities at $z = \lambda$ and $z = \mu$ lie outside it. Thus $f(z)$ has Taylor series expansion with center $z = 0$.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then $(1 - z - z^2) \sum_{n=0}^{\infty} c_n z^n = 1$. Equating coefficients of like powers we get

$$\begin{aligned}
c_0 &= 1 \\
c_1 - c_0 &= 0 \\
c_2 - c_1 - c_0 &= 0 \\
&\dots \\
c_n - c_{n-1} - c_{n-2} &= 0
\end{aligned}$$

Thus $c_n = c_{n-1} + c_{n-2}, n \geq 2$. The c_n's are Fibonacci numbers.

Now

$$\begin{aligned}
f(z) &= \frac{-1}{(z-\lambda)(z-\mu)} \\
&= \frac{-1}{\lambda - \mu}\left[\frac{1}{z-\lambda} - \frac{1}{z-\mu}\right] \\
&= \frac{-1}{\sqrt{5}}\left[-\frac{1}{\lambda}\left(1 - \frac{z}{\lambda}\right)^{-1} - \frac{-1}{\mu}\left(1 - \frac{z}{\mu}\right)^{-1}\right]
\end{aligned}$$

If we confine z to the disc $|z| < \lambda$, then $|\frac{z}{\lambda}| < 1, |\frac{z}{\mu}| < 1$ and we have

$$f(z) = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty} \frac{z^n}{\lambda^{n+1}} - \sum_{n=0}^{\infty} \frac{z^n}{\mu^{n+1}}\right] = \sum_{n=0}^{\infty} c_n z^n$$

where c_n are given as above. But the Taylor series of a function is unique, therefore we have

$$\begin{aligned}
c_n &= \frac{1}{\sqrt{5}}\left[\frac{1}{\lambda^{n+1}} - \frac{1}{\mu^{n+1}}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{2}{\sqrt{5}-1}\right)^{n+1} - \left(\frac{-2}{\sqrt{5}+1}\right)^{n+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{2(\sqrt{5}+1)}{5-1}\right)^{n+1} - \left(\frac{-2(\sqrt{5}-1)}{5-1}\right)^{n+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} + (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\right]
\end{aligned}$$

∎

UPSC Civil Services Main 1989 - Mathematics
Complex Analysis

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Question 1(a) *Evaluate the integral*

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3}$$

if

1. *the point* 0 *lies inside* C *and the point* 1 *lies outside* C.
2. *both* 0 *and* 1 *lie inside* C.
3. *the point* 1 *lies inside* C *and the point* 0 *lies outside* C.

Solution. The only possible poles of $\dfrac{e^z}{z(z-1)^3}$ are $z=0$ and $z=1$. Clearly $z=0$ is a simple pole, and residue at $z=0$ is $\displaystyle\lim_{z\to 0}\frac{ze^z}{z(1-z)^3} = 1$.

We have a triple pole at $z=1$, and the residue at $z=1$ is $\displaystyle\frac{1}{2!}\frac{d^2}{dz^2}\Big(\frac{(z-1)^3e^z}{z(1-z)^3}\Big)_{z=1} = -\frac{1}{2}\frac{d}{dz}\Big(\frac{ze^z-e^z}{z^2}\Big)_{z=1} = -\frac{1}{2}\Big(\frac{(z^2(ze^z+e^z-e^z)-2z(ze^z-e^z)}{z^4}\Big)_{z=1} = -\frac{e}{2}$.

By Cauchy's residue theorem, $\displaystyle\frac{1}{2\pi i}\int_C\frac{e^z\,dz}{z(1-z)^3}$ = Sum of the residues at poles of integrand within C.

1. The only pole inside C is 0, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 0 = 1$$

2. Both poles are in C, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 0 + \text{Residue at } 1 = 1 - \frac{e}{2}$$

3. The only pole inside C is 1, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 1 = -\frac{e}{2}$$

∎

Question 1(b) *Let f have the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R$ and let $s_n(z) = \sum_{k=0}^{m} a_k z^k$. If $0 < r < R$ and if $|z| < r$ show that*

$$s_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\;\frac{w^{n+1}-z^{n+1}}{w-z}\,dw$$

where γ is the circle $|w| = r$ oriented positively.

Solution. Since $\dfrac{w^{n+1}-z^{n+1}}{w-z} = w^n + w^{n-1}z + \ldots + wz^{n-1} + z^n = \sum_{k=0}^{n} w^{n-k}z^k$, it follows that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\;\frac{w^{n+1}-z^{n+1}}{w-z}\,dw &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\left(\sum_{k=0}^{n} w^{n-k}z^k\right)dw \\
&= \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=0}^{n} z^k \frac{f(w)}{w^{k+1}}\right)dw \\
&= \frac{1}{2\pi i}\sum_{k=0}^{n} z^k \int_\gamma \frac{f(w)}{w^{k+1}}\,dw
\end{aligned}$$

But from Cauchy's integral formula we know that

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{k+1}}\,dw$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\;\frac{w^{n+1}-z^{n+1}}{w-z}\,dw = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}z^k$$

Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and the series is uniformly and absolutely convergent within $|z| \leq r$, we can differentiate it termwise. Thus we obtain $f(0) = a_0, f'(0) = a_1, f''(0) = 2a_2, \ldots, f^{(k)}(0) = k!a_k, \ldots$. Substituting above, we get

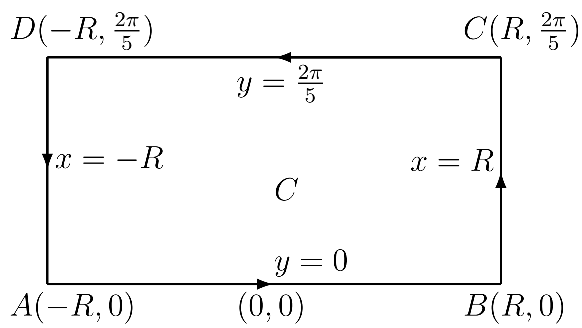
$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w^{n+1}} \frac{w^{n+1} - z^{n+1}}{w - z} \, dw = \sum_{k=0}^{n} a_k z^k = s_n(z)$$

$\blacksquare$

Question 1(c) *By integrating* $\dfrac{1}{1+z^5}$ *around a suitable contour, prove that*

$$\int_0^{\infty} \frac{dx}{1+x^5} = \frac{\pi}{5} \Big/ \sin\frac{\pi}{5}$$

Solution. We shall present two proofs.



Proof 1: Let $f(z) = \frac{e^z}{1+e^{5z}}$ and the contour be C, the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, \frac{2\pi}{5}), D = (-R, \frac{2\pi}{5})$ oriented positively. We let $R \to \infty$ eventually. The only pole in the strip bounded by $y = 0$ and $y = \frac{2\pi}{5}$ is $z = \frac{\pi i}{5}$ and it is a simple pole.

Residue of $f(z)$ at $z = \frac{\pi i}{5}$ is $\displaystyle\lim_{z \to \frac{\pi i}{5}} \frac{(z - \frac{\pi i}{5})e^z}{1 + e^{5z}} = \frac{e^{\frac{\pi i}{5}}}{5e^{i\pi}} = -\frac{e^{\frac{\pi i}{5}}}{5}$. Thus

$$\lim_{R \to \infty} \int_C \frac{e^z \, dz}{1 + e^{5z}} = -\frac{2\pi i e^{\frac{\pi i}{5}}}{5}$$

Now we evaluate the integral on all 4 sides of the rectangle.

1. $$\left| \int_{BC} \frac{e^z \, dz}{1 + e^{5z}} \right| = \left| \int_0^{\frac{2\pi}{5}} \frac{e^{R+iy}}{e^{5z} + 1} i \, dy \right| \leq \int_0^{\frac{2\pi}{5}} \frac{e^R}{e^{5R} - 1} \, dy \leq \frac{2\pi}{5} \frac{e^R}{e^{5R} - 1}$$
   because $|e^{5z} + 1| \geq |e^{5z}| - 1 = |e^{5R+5iy}| - 1 = e^{5R} - 1$. as on BC, $z = R + iy$. Thus $\displaystyle\lim_{R \to \infty} \int_{BC} \frac{e^z \, dz}{1 + e^{5z}} = 0$.

2. On DA, $z = -R + iy$ and therefore $|e^{5z} + 1| \geq 1 - |e^{5z}| = 1 - e^{-5R}$. This shows that
   $$\left| \int_{DA} \frac{e^z \, dz}{1 + e^{5z}} \right| \leq \frac{2\pi}{5} \frac{e^{-R}}{1 - e^{-5R}}$$
   As $\dfrac{e^{-R}}{1 - e^{-5R}} \to 0$ as $R \to \infty$, it follows that $\displaystyle\lim_{R \to \infty} \int_{DA} \frac{e^z \, dz}{1 + e^{5z}} = 0$.

3. On AB, $z = x$ so

$$\lim_{R\to\infty}\int_{AB}\frac{e^z\,dz}{1+e^{5z}} = \int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}}$$

4. On CD, $z = x + \frac{2\pi i}{5}$, so

$$\lim_{R\to\infty}\int_{CD}\frac{e^z\,dz}{1+e^{5z}} = \int_{\infty}^{-\infty}\frac{e^x e^{\frac{2\pi i}{5}}\,dx}{1+e^{5x}}$$

Using the above, we get

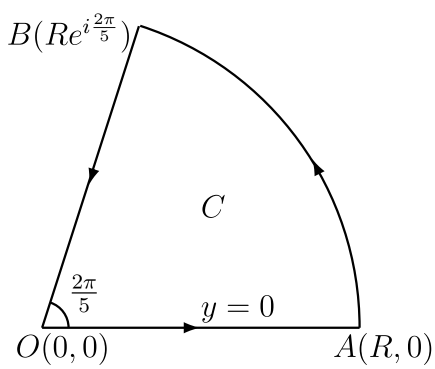
$$\lim_{R\to\infty}\int_{C}\frac{e^z\,dz}{1+e^{5z}} = \int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} - e^{\frac{2\pi i}{5}}\int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} = -\frac{2\pi i e^{\frac{\pi i}{5}}}{5}$$

or

$$\int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} = -\frac{2\pi i}{5}\,\frac{e^{\frac{\pi i}{5}}}{1-e^{\frac{2\pi i}{5}}} = \frac{\pi}{5}\,\frac{2i}{e^{\frac{\pi i}{5}}-e^{\frac{-\pi i}{5}}} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}$$

We now put $e^x = t$ to get $\displaystyle\int_0^{\infty}\frac{dt}{1+t^5} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}$ as desired.

Proof 2: Let $f(z) = \frac{1}{1+z^5}$ and the contour be C, the angular region $OABO$ where OA is the line joining $(0,0),(R,0)$, AB is the arc of the circle $|z| = R$ and B is on the circle such that angle $\angle AOB = \frac{2\pi}{5}$. C is oriented positively. We let $R\to\infty$ eventually. The only pole in the sector is $z = \frac{\pi i}{5}$ and it is a simple pole.



Using Cauchy's residue theorem, we get

$$\lim_{R\to\infty}\int_C\frac{dz}{1+z^5} = 2\pi i\times\text{Residue at } e^{\frac{\pi i}{5}} = 2\pi i\lim_{z\to e^{\frac{\pi i}{5}}}\frac{z-e^{\frac{\pi i}{5}}}{1+z^5} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}}$$

1. On AB, $z = Re^{i\theta}$, $|z^5+1|\ge |z|^5-1 = R^5-1$, $0\le\theta\le\frac{2\pi}{5}$ and therefore

$$\left|\int_{AB}\frac{dz}{1+z^5}\right| \le \left|\int_0^{\frac{2\pi}{5}}\frac{Rie^{i\theta}\,d\theta}{R^5-1}\right| \le \frac{2\pi}{5}\,\frac{R}{R^5-1}$$

showing that $\displaystyle\lim_{R\to\infty}\int_{AB}\frac{dz}{1+z^5} = 0.$

2. On OA, $z = x$ and therefore $\displaystyle\lim_{R\to\infty}\int_{OA}\frac{dz}{1+z^5} = \int_0^{\infty}\frac{dx}{1+x^5}.$

3. On BO, $z = Re^{\frac{2\pi i}{5}}$ and R varies from ∞ to 0. Therefore

$$\lim_{R\to\infty} \int_{BO} \frac{dz}{1+z^5} = \int_\infty^0 \frac{e^{\frac{2\pi i}{5}}\, dR}{1+(Re^{\frac{2\pi i}{5}})^5} = -e^{\frac{2\pi i}{5}} \int_0^\infty \frac{dR}{1+R^5}$$

Thus

$$\begin{aligned}
\lim_{R\to\infty} \int_C \frac{dz}{1+z^5} &= \int_0^\infty \frac{dx}{1+x^5} - e^{\frac{2\pi i}{5}} \int_0^\infty \frac{dR}{1+R^5} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}} \\
\Rightarrow \int_0^\infty \frac{dx}{1+x^5} &= \frac{2\pi i}{5e^{\frac{4\pi i}{5}}} \frac{1}{1-e^{\frac{2\pi i}{5}}} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}} \frac{e^{-\frac{\pi i}{5}}}{e^{-\frac{\pi i}{5}} - e^{\frac{\pi i}{5}}} \\
&= \frac{\pi}{5} \frac{2i}{e^{\frac{\pi i}{5}} - e^{-\frac{\pi i}{5}}} = \frac{\pi}{5} \Big/ \sin\frac{\pi}{5}
\end{aligned}$$

Note: We have provided both proofs because sometimes the examiner prescribes the contour.

■

Question 2(a) *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be analytic for $|z| < 1+\delta, (\delta > 0)$. Prove that the polynomial $p_k(z)$ of degree k which minimizes the integral*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta}) - p_k(e^{i\theta})|^2 \, d\theta$$

is $p_k(z) = \sum_{n=0}^k a_n z^n$. Prove that the minimum value is given by $\sum_{n=k+1}^\infty |a_n|^2$.

Solution. On $|z| = 1, z = e^{i\theta}$ and

$$\int_0^{2\pi} f(z)\overline{f(z)}\, d\theta = \int_0^{2\pi} \sum_{n,m=0}^\infty a_n \overline{a_m} e^{i(n-m)\theta}\, d\theta$$

Now termwise integration is justified because the series $\sum_{n=0}^\infty a_n z^n$ is uniformly convergent in $|z| \le 1$ as the given series is convergent in $|z| < 1+\delta$ with $\delta > 0$. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 \, d\theta = \frac{1}{2\pi} \sum_{n,m=0}^\infty a_n \overline{a_m} \int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = \sum_0^\infty |a_n|^2$$

as $\int_0^{2\pi} e^{i(n-m)\theta}\, d\theta = 0$ or 2π according as $n \ne m$ or $n = m$.

Let $p_k(z) = \sum_{n=0}^k b_n z^n$, then as above

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z) - p_k(z)|^2 \, d\theta = \sum_{n=0}^k |a_n - b_n|^2 + \sum_{n=k+1}^\infty |a_n|^2$$

Clearly the right hand side is minimum if and only if $\sum_{n=0}^k |a_n - b_n|^2 = 0 \Rightarrow a_n = b_n$ for $n = 1, \dots, k$, as all terms in the sum are non-negative. Thus $p_k(z) = \sum_{n=0}^k a_n z^n$ and the minimum value of the integral is $\sum_{n=k+1}^\infty |a_n|^2$. ■

Question 2(b) *If f is regular in the whole plane and the values of $f(z)$ do not lie in the disc with center w_0 and radius δ, show that f is constant.*

Solution. Liouville's Theorem: If $f(z)$ is entire, i.e. regular in the whole plane, and bounded, then $f(z)$ is constant.

Consider the function $F(z) = \frac{1}{f(z)-w_0}$. Since $f(z)$ is entire and $f(z) \neq w_0$ (note that if $f(z) = w_0$ for some z then one of its values would lie inside the disc with center w_0 and radius δ). it follows that $F(z)$ is an entire function. Since $|f(z) - w_0| > \delta$ for every z, $|F(z)| < \frac{1}{\delta}$ for every z, thus by Liouville's theorem $F(z) \equiv c$ a constant, and therefore $f(z)$ is a constant.

Proof of Liouville's theorem: From Cauchy's integral formula, we have for any z_0 and ρ however large

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=\rho} \frac{f(z)\, dz}{(z-z_0)^2}$$

Now $f(z)$ is bounded, say $|f(z)| \leq M$ and $|z - z_0| = \rho$, so let $z - z_0 = \rho e^{i\theta}, dz = \rho i e^{i\theta}\, d\theta$ which gives us

$$|f'(z_0)| \leq \frac{M}{2\pi\rho^2} 2\pi\rho = \frac{M}{\rho}$$

Letting $\rho \to \infty$, we get $f'(z_0) = 0$ for any z_0, thus $f'(z) = 0$ so f is a constant. ∎

Question 2(c) *Find the singularities of* $\sin(\frac{1}{1-z})$ *in the complex plane.*

Solution. Since $\frac{1}{1-z}$ is analytic everywhere except $z = 1$, $\sin(\frac{1}{1-z})$ is regular everywhere except $z = 1$. At $z = 1$ the function has an essential singularity — Clearly $\sin(\frac{1}{1-z}) = 0 \Leftrightarrow \frac{1}{1-z} = n\pi, n \neq 0 \Leftrightarrow z = 1 - \frac{1}{n\pi}$, $n \in \mathbb{Z}, n \neq 0$. Thus 1 is a limit point of zeros of $\sin(\frac{1}{1-z})$ and therefore $\sin(\frac{1}{1-z})$ has an essential singularity at $z = 1$.

Note that $\sin(\frac{1}{1-z})$ is regular at ∞ as $\sin(\frac{\zeta}{1-\zeta})$ is regular at $\zeta = 0$. ∎

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March 5, 2010

Question 1(a) *Let f be regular for $|z| < R$, prove that, if $0 < r < R$,*

$$f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta)\exp(-i\theta)\,d\theta$$

where $u(\theta) = \operatorname{Re} f(re^{i\theta})$.

Solution. Using Cauchy's integral formula, it is easily deduced that for any z in the interior of $\{C_R \ : \ |z| = R\}$, we have

$$\frac{f^{(t)}(z)}{t!} = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{(\zeta - z)^{t+1}}\,d\zeta$$

In particular, $f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_{C_R}\frac{f(\zeta)}{\zeta^2}\,d\zeta$.

Putting $\zeta = Re^{i\theta}, d\zeta = Rie^{i\theta}\,d\theta$, we get

$$f'(0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})}{R^2e^{2i\theta}}Rie^{i\theta}\,d\theta = \frac{1}{2\pi R}\int_0^{2\pi} f(Re^{i\theta})e^{-i\theta}\,d\theta \qquad (1)$$

We now consider the integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\bar{z}}{R}\zeta)^{t+1}}\,d\zeta$$

By Cauchy's residue theorem, the above integral is equal to $2\pi i$(sum of residues of the integrand within C_R). If $t \geq 1$, the only possibility of a pole could be at the point $\zeta = \frac{R^2}{\bar{z}}$, but $|z| = |\overline{z}| < R$, therefore $|\frac{R^2}{\overline{z}}| > \frac{R^2}{R} = R$, so $\frac{R^2}{\overline{z}}$ lies outside C_R and hence the integrand has no pole inside C_R, so

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\overline{z}}{R}\zeta)^{t+1}}\, d\zeta = 0 \quad \text{for } t \geq 1$$

In particular, taking $t = 1, z = 0$,

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{R^2}\, d\zeta = 0$$

Thus we get

$$\begin{aligned} 0 &= \frac{1}{2\pi R}\int_0^{2\pi} f(Re^{i\theta})e^{i\theta}\, d\theta \\ \Rightarrow 0 &= \frac{1}{2\pi R}\int_0^{2\pi} \overline{f}(Re^{i\theta})e^{-i\theta}\, d\theta \qquad (2) \end{aligned}$$

Adding $(1), (2)$, we get

$$f'(0) = \frac{1}{2\pi R}\int_0^{2\pi} (f(Re^{i\theta}) + \overline{f}(Re^{i\theta}))e^{-i\theta}\, d\theta = \frac{1}{\pi R}\int_0^{2\pi} u(\theta)\exp(-i\theta)\, d\theta$$

as required.

Note 1: To get the desired form, we could have considered the integral over $\{C_r : |z| = r < R\}$ instead of C_R and in that case $\zeta = re^{i\theta}$ and instead of R, we would have got r i.e.

$$f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta)\exp(-i\theta)\, d\theta$$

Note 2: The integral

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\overline{z}}{R}\zeta)^{t+1}}\, d\zeta$$

plays an important role in questions of this type, and has to be kept in mind. ∎

Question 1(b) *Prove that the distance from the origin to the nearest zero of* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is at least* $\dfrac{r|a_o|}{M + |a_0|}$ *where* r *is any number not exceeding the radius of convergence of the series, and* $M = M(r) = \sup_{|z|=r} |f(z)|$.

Solution. By Cauchy's integral formula,

$$f(z) - f(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta}$$

where $|z| < r \leq R$, R is the radius of convergence. If $f(z) = 0$, then

$$|f(0)| \leq \frac{1}{2\pi} M \left|\int_{|\zeta|=r} \Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big)\, d\zeta\right| \leq \frac{M}{2\pi}|z| \int_0^{2\pi} \left|\frac{rie^{i\theta}\, d\theta}{re^{i\theta}(r - |z|)}\right| = \frac{M|z|}{r - |z|}$$

because $|\zeta - z| \geq |\zeta| - |z| = r - |z|$ on $|\zeta| = r$. Thus $r|f(0)| \leq |z|(M + |f(0)|) \Rightarrow |z| \geq \frac{|f(0)|r}{M + |f(0)|}$. Here $f(0) = a_0$, and the result follows. ∎

Question 1(c) *If $f = u + iv$ is regular throughout the complex plane, and $au + bv - c \geq 0$ for suitable constants a, b, c then f is constant.*

Solution. Theorem: If $f(z) = u + iv$ is entire, and $u \leq 0$, then f is constant.

Proof: Consider $F(z) = e^{f(z)}$, then $F(z)$ is also entire. Moreover

$$|F(z)| = |e^{u+iv}| = |e^u| \leq 1 \because u \leq 0$$

Thus $F(z)$ is entire and bounded, hence is a constant by Liouville's theorem. Now $F'(z) = f'(z)e^{f(z)} = 0 \Rightarrow f'(z) = 0$ because $e^{f(z)} \neq 0$, so $f(z)$ is constant.

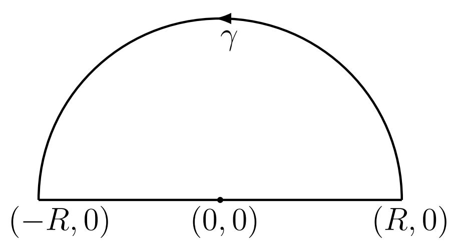
Corollary: If $f(z) = u + iv$ is entire, and $u \geq 0$, then f is constant. Proof: Consider $-f(z) = -u - iv$, then $-u \leq 0$ and $-f(z)$ is constant.

Now consider $F(z) = (a - ib)f(z) - c = (au + bv - c) + i(av - bu)$. Now $F(z)$ is entire, and $\operatorname{Re} F(z) = au + bv - c \geq 0$, so $F(z)$ is constant, hence $f(z)$ is constant. ∎

Question 2(a) *Prove that* $\displaystyle\int_{-\infty}^{\infty} \frac{x^4\,dx}{1+x^8} = \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{8}$ *using residue calculus.*

Solution.

We take $f(z) = \frac{z^4}{1+z^8}$ and the contour C consisting of γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane, and the line joining $(-R, 0)$ and $(R, 0)$. Finally we will let $R \to \infty$.



By Cauchy's residue theorem

$$\begin{aligned}\lim_{R\to\infty} \int_C \frac{z^4\,dz}{1+z^8} &= \int_{-\infty}^{\infty} \frac{x^4\,dx}{1+x^8} + \lim_{R\to\infty} \int_\gamma \frac{z^4\,dz}{1+z^8} \\ &= 2\pi i(\text{sum of residues at poles of } f(z) \text{ in the upper half plane})\end{aligned}$$

Now

$$\left|\int_\gamma \frac{z^4\,dz}{1+z^8}\right| \leq \left|\int_0^\pi \frac{R^4 e^{4i\theta} Rie^{i\theta}\,d\theta}{R^8 - 1}\right| \leq \frac{\pi R^5}{R^8 - 1}$$

because $|z^8 + 1| \geq |z^8| - 1 = R^8 - 1$ on $|z| = R$. Therefore

$$\lim_{R\to\infty} \int_\gamma \frac{z^4\,dz}{1+z^8} = 0$$

$f(z)$ has poles at zeros of $z^8 + 1 = 0 \Rightarrow z^8 = -1 \Rightarrow z^8 = e^{(2n+1)\pi i} \Rightarrow z = e^{\frac{(2n+1)\pi i}{8}}, n \in \mathbb{Z}$. Clearly $z = e^{\frac{\pi i}{8}}, e^{\frac{3\pi i}{8}}, e^{\frac{5\pi i}{8}}, e^{\frac{7\pi i}{8}}$ are the only poles of $f(z)$ in the upper half plane and all these are simple poles. The residue at any simple pole z_0 is $\frac{z_0^4}{8z_0^7} = \frac{1}{8z_0^3}$,

$$
\begin{aligned}
& \text{sum of residues at poles of } f(z) \text{ in the upper half plane} \\
= \; & \frac{1}{8}\big(e^{-3\pi i/8} + e^{-9\pi i/8} + e^{-15\pi i/8} + e^{-21\pi i/8}\big) \\
= \; & \frac{1}{8}\big(e^{-3\pi i/8} - e^{-\pi i/8} + e^{\pi i/8} - e^{3\pi i/8}\big) \\
= \; & \frac{1}{8}\big(2i\sin\frac{\pi}{8} - 2i\sin\frac{3\pi}{8}\big) \\
= \; & \frac{i}{4}\big(\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\big) \\
= \; & \frac{i\sqrt{2}}{4}\big(\cos\frac{\pi}{4}\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\sin\frac{\pi}{4}\big) \\
= \; & -\frac{i}{2\sqrt{2}}\sin\frac{\pi}{8}
\end{aligned}
$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{1+x^8} = 2\pi i\left(-\frac{i}{2\sqrt{2}}\sin\frac{\pi}{8}\right) = \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{8}$$

as required. ∎

Question 2(b) *Derive a series expansion of* $\log(1+e^z)$ *in powers of* z.

Solution. Let $f(z) = \log(1+e^z)$, then

$$f'(z) = \frac{e^z}{1+e^z} = \frac{1}{2}e^{\frac{z}{2}}\frac{2}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} = \frac{1}{2}e^{\frac{z}{2}}\frac{1}{\cosh\frac{z}{2}}$$

Let $g(z) = \cosh\frac{z}{2}$, then

$$g^{(n)}(z) = \begin{cases} \frac{1}{2^n}\sinh\frac{z}{2}, & n \text{ odd} \\ \frac{1}{2^n}\cosh\frac{z}{2}, & n \text{ even} \end{cases}$$

In particular, $g^{(n)}(0) = 0$ when n is odd, and $g^{(n)}(0) = \frac{1}{2^n}$ when n is even. Moreover

$$f'(z)\cosh\frac{z}{2} = f'(z)g(z) = \frac{1}{2}e^{\frac{z}{2}}$$

Using Leibnitz rule for the derivative of the product of two functions, we get

$$\frac{d^n}{dz^n}\left(\frac{1}{2}e^{\frac{z}{2}}\right) = \frac{e^{\frac{z}{2}}}{2^{n+1}} = \sum_{p=0}^{n}\binom{n}{p} g^{(n-p)}(z) f^{(p+1)}(z)$$

Thus when $z = 0$, we get

$$\sum_{p=0}^{n}\binom{n}{p}\frac{\epsilon_{n-p}}{2^{n-p}}f^{(p+1)}(0) = \frac{1}{2^{n+1}} \quad \text{where } \epsilon_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

and therefore

$$2^{n+1}f^{(n+1)}(0) = 1 - \sum_{p=0}^{n-1} \binom{n}{p} 2^{p+1}\epsilon_{n-p} f^{(p+1)}(0)$$

Case (1) : When n is even

$$2^{n+1}f^{(n+1)}(0) = 1 - \binom{n}{0} 2f'(0) - \sum_{p=1}^{n-2} \binom{n}{p} 2^{p+1}\epsilon_{n-p} f^{(p+1)}(0)$$

Note that odd p do not contribute anything to the summation, as $\epsilon_{n-p} = 0$ for odd p. Now we can see by induction that $f^{(n)}(0) = 0$ whenever n is odd and $n > 1$. $f'(0) = \frac{1}{2}$. $2^3 f^{(3)}(0) = 1 - 2 \cdot \frac{1}{2} = 0$. Assume by induction hypothesis that $f^{(3)}(0) = f^{(5)}(0) = \ldots = f^{(2m-1)}(0) = 0$, then letting $n = 2m$ in the above formula,

$$2^{2m+1}f^{(2m+1)}(0) = -\sum_{p=1}^{m-1} \binom{2m}{2p} 2^{2p+1} f^{(2p+1)}(0) = 0$$

Case (2): When n is odd: The terms with even p in the formula above do not make any contribution. Thus letting $n = 2m + 1$,

$$2^{2m+2}f^{(2m+2)}(0) = 1 - \sum_{r=0}^{m-1} \binom{2m+1}{2r+1} 2^{2r+2} f^{(2r+2)}(0) = 1 - \sum_{r=1}^{m} \binom{2m+1}{2r-1} 2^{2r} f^{(2r)}(0) \qquad (*)$$

We can now see that $f''(0) = \frac{1}{4}, f^{(4)}(0) = -\frac{1}{8}, f^{(6)}(0) = \frac{1}{4}$.

Thus

$$\begin{aligned} \log(1 + e^z) &= \log 2 + \frac{z}{2} + \frac{1}{4}\frac{1}{2!}z^2 - \frac{1}{8}\frac{1}{4!}z^4 + \frac{1}{4}\frac{1}{6!}z^6 + \ldots \\ &= \log 2 + \frac{z}{2} + \sum_{n=1}^{\infty} \frac{f^{(2n)}(0)z^{2n}}{(2n)!} \end{aligned}$$

where $f^{(2n)}(0)$ is given by $(*)$ for $n \geq 1$. ∎

Note: We now present an alternative solution, where we use Leibnitz rule for the n-th derivative of the quotient of two functions. It is a good exercise in itself and is usually missing from textbooks.

Theorem: Let $y = \frac{u}{v}$, where u, v are functions with derivatives up to order n. Then

$$y_n = \frac{1}{v^{n+1}} \begin{vmatrix} v & 0 & 0 & \ldots & u \\ v_1 & v & 0 & \ldots & u_1 \\ v_2 & \binom{2}{1}v_1 & v & \ldots & u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_n & \binom{n}{1}v_{n-1} & \binom{n}{2}v_{n-2} & \ldots & u_n \end{vmatrix}$$

Here the determinant is $(n+1)\times(n+1)$, and $y_n = \dfrac{d^n y}{dx^n}$.

Proof: $vy = u$, therefore, by taking successive derivatives using Leibnitz product rule we get

$$\begin{aligned}
vy &= u\\
v_1 y + v y_1 &= u_1\\
v_2 y + 2v_1 y_1 + v y_2 &= u_2\\
\ldots & \quad \ldots\\
v_n y + \tbinom{n}{1} v_{n-1} y_1 + \ldots + v y_n &= u_n
\end{aligned}$$

These are $n+1$ equations in $n+1$ unknowns $y, y_1, \ldots, y_n$, and the determinant of the coefficient matrix is v^{n+1}. Thus by Cramer's rule

$$y_n = \frac{1}{v^{n+1}} \begin{vmatrix} v & 0 & 0 & \ldots & u\\ v_1 & v & 0 & \ldots & u_1\\ v_2 & \binom{2}{1} v_1 & v & \ldots & u_2\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ v_n & \binom{n}{1} v_{n-1} & \binom{n}{2} v_{n-2} & \ldots & u_n \end{vmatrix}$$

as required.

Now $f(z) = \log(1+e^z), f(0) = \log 2$. $f'(z) = \frac{e^z}{1+e^z}, f'(0) = \frac{1}{2}$. Let $u = e^z, v = 1+e^z$. Then $u_n(0) = 1$ for every n, and $v(0) = 2, v_n(0) = 1$ for $n \geq 1$. Let $F(z) = \frac{u}{v}$, then

$$F^{(n)}(0) = f^{(n+1)}(0) = \frac{1}{2^{n+1}} \begin{vmatrix} 2 & 0 & 0 & \ldots & 0 & 1\\ 1 & 2 & 0 & \ldots & 0 & 1\\ 1 & 2 & 2 & \ldots & 0 & 1\\ \ldots & & & \ldots & & \ldots\\ 1 & \binom{n}{1} & \binom{n}{2} & \ldots & \binom{n}{n-1} & 1 \end{vmatrix}$$

$$\begin{aligned}
F^{(1)}(0) &= f^{(2)}(0) = \frac{1}{4}\begin{vmatrix} 2 & 1\\ 1 & 1\end{vmatrix} = \frac{1}{4}\\
F^{(2)}(0) &= f^{(3)}(0) = \frac{1}{8}\begin{vmatrix} 2 & 0 & 1\\ 1 & 2 & 1\\ 1 & 2 & 1\end{vmatrix} = 0\\
F^{(3)}(0) &= f^{(4)}(0) = \frac{1}{16}\begin{vmatrix} 2 & 0 & 0 & 1\\ 1 & 2 & 0 & 1\\ 1 & 2 & 2 & 1\\ 1 & 3 & 3 & 1\end{vmatrix} = \frac{1}{16}\begin{vmatrix} 2 & 0 & 0 & 1\\ -1 & 2 & 0 & 0\\ -1 & 2 & 2 & 0\\ -1 & 3 & 3 & 0\end{vmatrix} = \frac{-2}{16} = -\frac{1}{8}\\
F^{(4)}(0) &= f^{(5)}(0) = \frac{1}{32}\begin{vmatrix} 2 & 0 & 0 & 0 & 1\\ 1 & 2 & 0 & 0 & 1\\ 1 & 2 & 2 & 0 & 1\\ 1 & 3 & 3 & 2 & 1\\ 1 & 4 & 6 & 4 & 1\end{vmatrix} = 0
\end{aligned}$$

Thus $\log(1+e^z)$ has the expansion as given above.

Question 2(c) *Determine the nature of singular points of* $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ *and investigate its behavior at* $z = \infty$.

Solution.

1. Let $\zeta = \frac{1}{z}$, and $\phi(\zeta) = f(\frac{1}{\zeta}) = \sin\left(\dfrac{1}{\cos\zeta}\right)$. Therefore $\lim\limits_{\zeta\to 0}\phi(\zeta) = \sin 1$, showing that $\phi(\zeta)$ has a removable singularity at $\zeta = 0$. In fact $\phi(\zeta)$ is analytic at $\zeta = 0$ if $\phi(0)$ is defined to be $\sin 1$. Note that

$$\lim_{\zeta\to 0}\frac{\phi(\zeta)-\phi(0)}{\zeta} = \lim_{\zeta\to 0}\frac{\sin(\frac{1}{\cos\zeta}) - \sin 1}{\zeta} = \lim_{\zeta\to 0}\cos\left(\frac{1}{\cos\zeta}\right)\sec\zeta\tan\zeta = 0$$

   Thus $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ is regular at ∞.

2. At all zeros of $\cos\frac{1}{z}$ i.e. $z = \dfrac{2}{(2n+1)\pi}$ the function $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ has essential singularities because $\lim_{x\to\infty}\sin x$ does not exist — if it did, then given $\epsilon > 0$, we would have N such that $x_1 > N, x_2 > N \Rightarrow |\sin x_1 - \sin x_2| < \epsilon$. But for any N we can take $x_1 = 2n\pi + \frac{\pi}{2} > x_2 = 2n\pi > N$, then $|\sin x_1 - \sin x_2| = 1 \not< \epsilon$ if $\epsilon < 1$.

3. $z = 0$ is also an essential singularity of the given function as it is a limit point of essential singularities $z = \dfrac{2}{(2n+1)\pi}$.

∎

UPSC Civil Services Main 1991 - Mathematics Complex Analysis

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March 5, 2010

Question 1(a) *A function $f(z)$ is defined for all finite values of z by $f(0) = 0$ and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy-Riemann equations are satisfied at the origin. Show also that $f(z)$ is not analytic at the origin.*

Solution. Let $f(z) = u + iv$. By definition

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0}\frac{u(x,0) - u(0,0)}{x},\ \ \frac{\partial u}{\partial y}(0,0) = \lim_{y\to 0}\frac{u(0,y) - u(0,0)}{y}$$

Now $u(x,0) = \operatorname{Re} f(x,0) = e^{-x^{-4}}, u(0,y) = \operatorname{Re} f(0,y) = e^{-(iy)^{-4}}$, therefore

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0}\frac{e^{-x^{-4}} - 0}{x} = \lim_{t\to\infty} te^{-t^4} = 0,\ \ \frac{\partial u}{\partial y}(0,0)\lim_{y\to 0}\frac{e^{-y^{-4}}}{y} = 0$$

(Note that $e^{t^4} > t^4 \Rightarrow te^{-t^4} < \frac{1}{t^3} \Rightarrow \lim_{t\to\infty} te^{-t^4} = 0$).

It is obvious that $v(x,0)$ =Imaginary part of $e^{-x^{-4}} = 0$, and $v(0,y)$ =Imaginary part of $e^{-(iy)^{-4}} = 0$, and therefore $v_x(0,0) = v_y(0,0) = 0$. Thus $\frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0), \frac{\partial u}{\partial y}(0,0) = -\frac{\partial v}{\partial x}(0,0)$, i.e. the Cauchy-Riemann equations are satisfied at $(0,0)$.

However $f(z)$ is not analytic at $z = 0$ because it is not even continuous at $z = 0$: if we take $z = re^{\frac{i\pi}{4}}$, then $z \to 0 \Leftrightarrow r \to 0$, but $\lim_{r\to 0} f(re^{\frac{i\pi}{4}}) = \lim_{r\to 0} e^{-r^{-4}e^{\pi i}} = \lim_{r\to 0} e^{r^{-4}} = \infty$, so $\lim_{z\to 0} f(z) \neq f(0)$. ∎

Question 1(b) *If $|a| \neq R$, show that*

$$\int_{|z|=R}\frac{|dz|}{|(z-a)(z+a)|} < \frac{2\pi R}{|R^2 - |a|^2|}$$

Solution. On $|z| = R, z = Re^{i\theta}, 0 \leq \theta \leq 2\pi, |dz| = |Rie^{i\theta}\,d\theta| = R\,d\theta$. $|z^2 - a^2| \geq |z|^2 - |a|^2 = R^2 - |a|^2$ and $|z^2 - a^2| \geq |a|^2 - |z|^2 = |a|^2 - R^2$, showing that $|z^2 - a^2| \geq |R^2 - |a|^2|$, with the strict inequality occurring when $a = |a|e^{i\theta}, z \neq Re^{i\theta}$. Thus

$$\int_{|z|=R} \frac{|dz|}{|(z-a)(z+a)|} < \int_0^{2\pi} \frac{R\,d\theta}{|R^2 - |a|^2|} = \frac{2\pi R}{|R^2 - |a|^2|}$$

as required. ∎

Question 1(c) *If*

$$J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta$$

show that

$$e^{\frac{t}{2}(z-\frac{1}{z})} = J_0(t) + zJ_1(t) + z^2 J_2(t) + \ldots - \frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + \ldots$$

Solution. The function $f(z) = e^{\frac{t}{2}(z-\frac{1}{z})}$ is analytic in $0 < |z| < \infty$ and therefore by Laurent's theorem — If $f(z)$ is analytic is the annular region $D : R_1 < |z - z_0| < R_2$ and if C is any positively oriented simple closed contour lying within the region D, then for any $z \in D$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} b_n(z - z_0)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \text{ where } c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

If we take $f(z) = e^{\frac{t}{2}(z-\frac{1}{z})}, R_1 = 0, R_2 = \infty, z_0 = 0$, then

$$f(z) = e^{\frac{t}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} c_n z^n \text{ where } c_n = \frac{1}{2\pi i} \int_C \frac{e^{\frac{t}{2}(z-\frac{1}{z})}\,dz}{z^{n+1}}$$

We now take C as $|z| = 1$. Then $z = e^{i\theta}$ and

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{t}{2}(e^{i\theta} - e^{-i\theta})}}{e^{i(n+1)\theta}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ti\sin\theta} e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta + \frac{i}{2\pi} \int_0^{2\pi} \sin(-n\theta + t\sin\theta)\,d\theta \end{aligned}$$

But $\int_0^{2\pi} \sin(-n\theta + t\sin\theta)\, d\theta = 0$, because if we put $\theta = 2\pi - \eta$, then $\int_0^{2\pi} \sin(-n\theta + t\sin\theta)\, d\theta = \int_{2\pi}^{0} \sin(-n\eta + t\sin\eta)\, d\eta$. Therefore

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - t\sin\theta)\, d\theta = J_n(t)$$

Hence

$$e^{\frac{t}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Since on replacing z by $-\frac{1}{z}$, the function $f(z)$ remains unaltered, we get $J_{-n}(t) = J_n(t)$ if n is even, and $J_{-n}(t) = -J_n(t)$ if n is odd. Thus

$$e^{\frac{t}{2}(z-\frac{1}{z})} = J_0(t) + zJ_1(t) + z^2 J_2(t) + \ldots - \frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + \ldots$$

as required. ∎

Question 2(a) *Examine the nature of the singularity of e^z at ∞.*

Solution. e^z has an essential singularity at ∞. We examine the nature of the singularity of $e^{\frac{1}{\zeta}}$ at $\zeta = 0$. Taking $\zeta = \frac{1}{n}$, $\lim_{\zeta\to 0} e^{\frac{1}{\zeta}} = \lim_{n\to\infty} e^n = \infty$.

Taking $\zeta = -\frac{1}{n}$, $\lim_{\zeta\to 0} e^{\frac{1}{\zeta}} = \lim_{n\to\infty} e^{-n} = 0$.

Thus $\lim_{\zeta\to 0} e^{\frac{1}{\zeta}}$ does not exist and therefore $e^{\frac{1}{\zeta}}$ has an essential singularity at $\zeta = 0$, proving that e^z has an essential singularity at ∞.

Alternately $e^{\frac{1}{\zeta}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{\zeta^n}$ is the Laurent expansion of $e^{\frac{1}{\zeta}}$ having infinitely many negative powers, showing the same result. ∎

Question 2(b) *Evaluate the residues of the function $\frac{z^3}{(z-2)(z-3)(z-5)}$ at all its singularities and show that their sum is 0.*

Solution. The given function has simple poles at $z = 2, 3, 5$.

Residue at $z = 2$ is $\lim_{z\to 2} \frac{z^3(z-2)}{(z-2)(z-3)(z-5)} = \frac{8}{3}$.

Residue at $z = 3$ is $\lim_{z\to 3} \frac{z^3(z-3)}{(z-2)(z-3)(z-5)} = -\frac{27}{2}$.

Residue at $z = 5$ is $\lim_{z\to 5} \frac{z^3(z-5)}{(z-2)(z-3)(z-5)} = \frac{125}{6}$.

Residue at ∞ is $= -$ coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ around ∞.

$$\begin{aligned}
f(z) &= \left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1}\left(1-\frac{5}{z}\right)^{-1} \\
&= \left(1+\frac{2}{z}+\text{Higher powers of }\frac{1}{z}\right)\left(1+\frac{3}{z}+\text{Higher powers of }\frac{1}{z}\right) \\
&\quad \left(1+\frac{5}{z}+\text{Higher powers of }\frac{1}{z}\right) \\
&= 1+\frac{10}{z}+\text{Higher powers of }\frac{1}{z}
\end{aligned}$$

Thus the residue at ∞ is -10.

Sum of all residues is $\dfrac{16-81+125-60}{6} = 0$.

Note: The function $f(z)$ has no singularity as such at ∞, but the residue at ∞ is always defined as such. The function is actually analytic at ∞ as $f(\frac{1}{z})$ is analytic at $z=0$. ∎

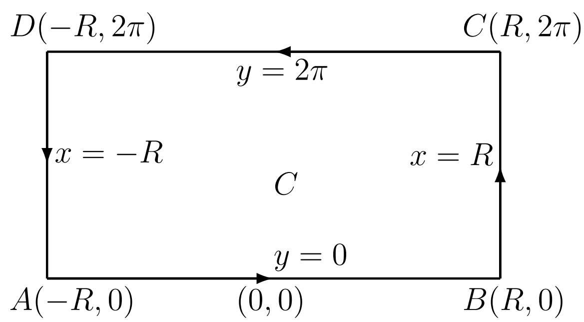
Question 2(c) *By integrating along a suitable contour show that*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi}$$

where $0 < a < 1$.

Solution.

Our $f(z) = \frac{e^{az}}{1+e^z}$ and the contour is C, the rectangle $ABCD$ where $A = (-R,0), B = (R,0), C = (R,2\pi), D = (-R,2\pi)$ oriented in the anticlockwise direction. We let $R \to \infty$ eventually.



The function $f(z)$ has only a simple pole at $z = \pi i$ in the strip bounded by $y = 0$ and $y = 2\pi$. Residue of $f(z)$ at πi is $\displaystyle\lim_{z\to\pi i} \frac{z-\pi i}{1+e^z} e^{az} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}$.

Thus by Cauchy's residue theorem $\displaystyle\lim_{R\to\infty} \int_C \frac{e^{az}\,dz}{1+e^z} = -2\pi i e^{\pi i a}$.

We now evaluate the integral along the four lines.

1. On the line BC i.e. $x = R, z = R+iy, dz = i\,dy$ and
$$\left|\int_{BC} \frac{e^{az}\,dz}{1+e^z}\right| = \left|\int_0^{2\pi} \frac{e^{a(R+iy)} i\,dy}{1+e^{R+iy}}\right| \le \int_0^{2\pi} \frac{e^{aR}\,dy}{e^R-1} = \frac{2\pi e^{aR}}{e^R-1}$$
because $|e^z+1| \ge |e^z| - 1 = |e^{R+iy}| - 1 = e^R - 1$ on BC. Since $\lim_{R\to\infty} \frac{e^{aR}}{e^R-1} = 0$ as $0 < a < 1$ using L'Hospital's Rule, it follows that $\displaystyle\lim_{R\to\infty} \int_{BC} \frac{e^{az}\,dz}{1+e^z} = 0$.

2. On the line DA i.e. $x = -R, z = -R + iy, dz = i\,dy$. Since $|e^z + 1| \geq 1 - |e^z| = 1 - |e^{-R+iy}| = 1 - e^{-R}$

$$\left|\int_{DA} \frac{e^{az}\,dz}{1+e^z}\right| \leq \frac{2\pi e^{-aR}}{1-e^{-R}}(\to 0 \ as \ R \to \infty)$$

thus $\lim\limits_{R\to\infty} \displaystyle\int_{DA} \frac{e^{az}\,dz}{1+e^z} = 0.$

3. On the line AB, $z = x$, so

$$\lim_{R\to\infty} \int_{AB} \frac{e^{az}\,dz}{1+e^z} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$$

4. On the line CD, $z = x + 2\pi i$, so

$$\lim_{R\to\infty} \int_{CD} \frac{e^{az}\,dz}{1+e^z} = \int_{\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{2\pi ia} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$$

Thus

$$\lim_{R\to\infty} \int_{C} \frac{e^{az}\,dz}{1+e^z} = (1-e^{2\pi ia}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = -2\pi i e^{\pi ia}$$

so

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{-2\pi i e^{\pi ia}}{1-e^{2\pi ia}} = \frac{-2\pi i}{e^{-\pi ia}-e^{\pi ia}} = \frac{\pi}{\sin a\pi}$$

as required. ∎

UPSC Civil Services Main 1992 - Mathematics Complex Analysis

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Question 1(a) *If $u = e^{-x}(x \sin y - y \cos y)$, find v such that $f(z) = u + iv$ is analytic. Also find $f(z)$ explicitly as a function of z.*

Solution. See 1993, question 2(b). ∎

Question 1(b) *Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$ and let $re^{i\theta}$ be any point inside C. Prove that*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi$$

Solution. By Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\int_{C_R:|\zeta|=r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad (1)$$

We note that the function $\dfrac{f(\zeta)}{\zeta - \frac{R^2}{\overline{z}}}$ has no singularity within and on C_R, because $f(\zeta)$ is analytic within and on C_R and $(\zeta - \frac{R^2}{\overline{z}})^{-1}$ is also analytic within and on C_R as $\frac{R^2}{\overline{z}}$ lies outside C_R and therefore $\zeta - \frac{R^2}{\overline{z}} \neq 0$ (Note that $R^2 = R \cdot R > R|\overline{z}|$, because $|z| = r < R$, thus $|\frac{R^2}{\overline{z}}| > R$. Thus by Cauchy's theorem

$$0 = \int_{C_R} \frac{f(\zeta)}{\zeta - \frac{R^2}{\overline{z}}}\, d\zeta \qquad (2)$$

Using (1), (2) we get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{\overline{z}}}\right] d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{z - \frac{R^2}{\overline{z}}}{(\zeta - z)(\zeta - \frac{R^2}{\overline{z}})}\right] d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{z\overline{z} - R^2}{(\zeta - z)(\zeta\overline{z} - R^2)}\right] d\zeta \\
\Rightarrow f(re^{i\theta}) &= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{(Re^{i\phi} - re^{i\theta})(rRe^{i(\phi-\theta)} - R^2)}\right] Re^{i\phi} i\, d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{(R - re^{i(\theta-\phi)})(re^{i(\phi-\theta)} - R)}\right] d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{-R^2 - r^2 + rR(e^{i(\theta-\phi)} + e^{i(\phi-\theta)})}\right] d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{R^2 - r^2}{R^2 + r^2 + 2rR\cos(\theta - \phi)}\right] d\phi
\end{aligned}
$$

as required. ∎

Question 1(c) *Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Solution. See 2006 question 2(b). ∎

Question 2(a) *Find the region of convergence of the series whose n-th term is* $\dfrac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$.

Solution. Clearly

$$\left|\frac{\text{Coefficient of the } (n+1)\text{-th term}}{\text{Coefficient of the } n\text{-th term}}\right| = \frac{(2n-1)!}{(2n+1)!} \to 0 \text{ as } n \to \infty$$

Thus $\lim_{n\to\infty} |\text{Coefficient of the } n\text{-th term}|^{\frac{1}{n}} = 0$. So the radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$ is ∞, i.e. the region of convergence is the entire complex plane. ∎

Question 2(b) *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in a Laurent series valid for* (i) $|z| > 3$, (ii) $1 < |z| < 3$, (iii) $|z| < 1$.

Solution. (i) $|z| > 3$.

$$f(z) = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right) = \frac{1}{2z}\left[\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{3}{z}\right)^{-1}\right]$$

Since $|\frac{1}{z}| < \frac{1}{3}, |\frac{3}{z}| < 1$, we have

$$\begin{aligned} f(z) &= \frac{1}{2z}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{z^n}\right] \\ &= \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n(1-3^n)}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(1-3^n)}{2}\frac{1}{z^{n+1}} \end{aligned}$$

(ii) $1 < |z| < 3$.

$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2}\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

Since $|\frac{1}{z}| < 1, |\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} \\ &= \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}z^n}{3^{n+1}}\right] \end{aligned}$$

(iii) $|z| < 1$.

$$f(z) = \frac{1}{2}\left(1+z\right)^{-1} - \frac{1}{2}\frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

As $|z| < 1, |\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{3^{n+1}}\right)z^n \end{aligned}$$

These are the Laurent or Taylor series in the required three cases. ∎

Question 2(c) *By integrating along a suitable contour evaluate* $\displaystyle\int_0^{\infty}\frac{\cos mx}{x^2+1}\,dx$

Solution. See 1995, question 2(a). ∎

UPSC Civil Services Main 1993 - Mathematics Complex Analysis

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Question 1(a) *In the finite plane, show that the function $f(z) = \sec\dfrac{1}{z}$ has infinitely many isolated singularities in a finite interval which includes zero.*

Solution. We know that $\cos\dfrac{1}{z} = 0$ if and only if $\dfrac{1}{z} = (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$, or $z = \dfrac{2}{(2n+1)\pi}$. Moreover all these zeros are simple zeros of $\cos\dfrac{1}{z}$ and are isolated singular points. Thus the given function has infinitely many simple poles at the points $z = \dfrac{2}{(2n+1)\pi}$. Since $\dfrac{2}{(2n+1)\pi} \to 0$ as $n \to \infty$, it follows that any finite interval containing 0 will have all but finitely many points of the type $z = \dfrac{2}{(2n+1)\pi}$. Thus any finite interval containing 0 will have infinitely many isolated singularities (simple poles) of $\sec\dfrac{1}{z}$. ∎

Question 1(b) *Find the orthogonal trajectories of the family of curves in the xy-plane defined by $e^{-x}(x\sin y - y\cos y) = \alpha$, where α is a real constant.*

Solution. If $f(z) = u + iv$ is an analytic function, then $u =$constant, $v =$constant represent families of curves which are orthogonal to each other, because of the Cauchy-Riemann equations:

$$\left(-\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y}\right) \times \left(-\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y}\right) = -1$$

i.e. tangents to the curve $u = c$ and $v = c'$ respectively cut each other at right angles. Thus given $u = e^x(x\sin y - y\cos y)$ we have to find v so that $f = u + iv$ is analytic.

We use Milne Thompson's method. We know

$$f'(z) = \frac{\partial u}{\partial x}\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - i\frac{\partial u}{\partial y}\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

is an identity.

$$f'(z) = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) - ie^{-x}(x\cos y - \cos y + y\sin y)$$

Putting $z = \overline{z} \Rightarrow x = z, y = 0 \Rightarrow f'(z) = -ie^{-z}(z-1)$, or $f(z) = -i\int e^{-z}(z-1)\,dz = ize^{-z}$. Thus

$$\begin{aligned} u + iv &= i(x+iy)e^{-x}(\cos y - i\sin y) \\ &= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y) \end{aligned}$$

Thus $v = e^{-x}(x\cos y + y\sin y) = \beta$ is the required family of curves. ∎

Question 1(c) *Prove by applying Cauchy's integral formula or otherwise that*

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} 2\pi$$

where $n = 1, 2, 3, \ldots$.

Solution. We put $z = e^{i\theta}$ so that $dz = ie^{i\theta}\,d\theta$ and the integral is along the curve $|z| = 1$. We get

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \int_{|z|=1} \frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}}\frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{(1+z^2)^{2n}}{2^{2n}z^{2n+1}}\,dz$$

Thus by Cauchy's residue theorem $\int_0^{2\pi}\cos^{2n}\theta\,d\theta = 2\pi i\frac{1}{2^{2n}i}$(sum of residues at poles of $\dfrac{(1+z^2)^{2n}}{z^{2n+1}}$ inside $|z| = 1$). Clearly $z = 0$ is the only pole of the integrand in $|z| = 1$, and it is of order $2n+1$.

Residue of $\dfrac{(1+z^2)^{2n}}{z^{2n+1}}$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion.

Now $(1+z^2)^{2n}$ =sum of powers of z with exponent $< 2n$ + $\binom{2n}{n}z^{2n}$ + sum of powers with exponent $> 2n$. Thus coefficient of $\frac{1}{z}$ in the Laurent expansion of $\dfrac{(1+z^2)^{2n}}{z^{2n+1}}$ around $z = 0$ is $\binom{2n}{n}$. Thus

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\frac{1}{2^{2n}}\frac{2n!}{n!n!}$$

Now $2^n n! = 2n(2n-2)(2n-4)\ldots\cdot 6\cdot 4\cdot 2$, so $\dfrac{2n!}{2^n n!} = (2n-1)(2n-3)(2n-5)\ldots\cdot 5\cdot 3\cdot 1$. Hence

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} 2\pi$$

as required. ∎

Question 2(a) *If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,1)$ and $(2,3)$, find the value of* $\int_C (12z^2 - 4iz)\, dz$.

Solution. If C_1 is any curve joining $(1,1)$ and $(2,3)$, then C and C_1 form a closed contour. Since $12z^2 - 4iz$ is analytic, by Cauchy's theorem

$$-\int_C (12z^2 - 4iz)\, dz + \int_{C_1} (12z^2 - 4iz)\, dz = 0$$

so the integral is independent of the path between $(1,1)$ and $(2,3)$. Thus

$$\begin{aligned}
\int_C (12z^2 - 4iz)\, dz &= \Big[4z^3 - 2iz^2\Big]_{1+i}^{2+3i} \\
&= 4[(2+3i)^3 - (1+i)^3] - 2i[(2+3i)^2 - (1+i)^2] \\
&= 4[8 + 36i - 54 - 27i - 1 - 3i + 3 + i] - 2i[4 + 12i - 9 - 1 - 2i + 1] \\
&= 4[-44 + 7i] - 2i[-5 + 10i] = -156 + 38i
\end{aligned}$$

Note that calculating $\int_C$ by $\int_C u dx + v dy$ would be more work. ■

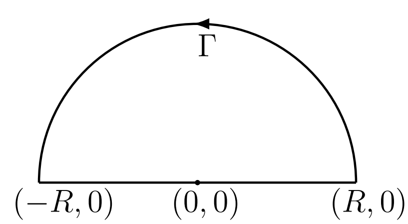
Question 2(b) *Prove that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ *converges absolutely for* $|z| \le 1$.

Solution. Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Now $s_n = \sum_{r=1}^{n}(\frac{1}{r} - \frac{1}{r+1}) = 1 - \frac{1}{n+1}$. Thus $s_n \to 1$ as $n \to \infty$, so $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. Now $\left|\dfrac{z^n}{n(n+1)}\right| \le a_n$ for $|z| \le 1$, therefore by Weierstrass M-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges absolutely (in fact uniformly) in the region $|z| \le 1$. ■

Question 2(c) *Evaluate* $\displaystyle\int_0^{\infty} \frac{dx}{x^6 + 1}$ *by choosing an appropriate contour.*

Solution.

We take $f(z) = \frac{1}{1+z^6}$ and the contour γ consisting of Γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane, and the line joining $(-R,0)$ and $(R,0)$.



By Cauchy's residue theorem $\displaystyle\int_\gamma \frac{dz}{1+z^6} = 2\pi i$(sum of residues at poles of $f(z)$ in the upper half plane).

Clearly $\frac{1}{1+z^6}$ has simple poles at $z = e^{\frac{\pi i}{6}}, z = e^{\frac{\pi i}{2}}$ and $z = e^{\frac{5\pi i}{6}}$ inside the contour.

Residue at $z = \zeta$ is $\dfrac{1}{6\zeta^5}$.

$$\begin{aligned}\text{Sum of residues} &= \frac{1}{6}\left[\frac{1}{e^{\frac{5\pi i}{6}}} + \frac{1}{e^{\frac{15\pi i}{6}}} + \frac{1}{e^{\frac{25\pi i}{6}}}\right] \\ &= \frac{1}{6}[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} - i + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}] = \frac{-2i}{6}\end{aligned}$$

as $\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}, \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$. Thus $\displaystyle\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^6} = 2\pi i\frac{-i}{3} = \frac{2\pi}{3}$.

Now

$$\left|\int_\Gamma \frac{dz}{1+z^6}\right| \le \int_0^\pi \frac{R}{R^6-1}\,d\theta = \frac{\pi R}{R^6-1}$$

on putting $z = Re^{i\theta}$ and using $|z^6+1| \ge R^6 - 1$ on Γ.

Thus $\displaystyle\int_\Gamma \frac{dz}{1+z^6} \to 0$ as $R \to \infty$. Consequently,

$$\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^6} = \int_{-\infty}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

and hence $\displaystyle\int_0^\infty \frac{dx}{1+x^6} = \frac{\pi}{3}$. ∎

UPSC Civil Services Main 1994 - Mathematics Complex Analysis

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Question 1(a) *Suppose that z is the position vector of a particle moving on the ellipse $C : z = a\cos\omega t + ib\sin\omega t$ where ω, a, b are positive constants, $a > b$ and t is time. Determine where*

1. *the velocity has the greatest magnitude.*
2. *the acceleration has the least magnitude.*

Solution. See 1996, question 1(a). ∎

Question 1(b) *How many zeroes does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$ possess (i) in the first quadrant, (ii) in the fourth quadrant.*

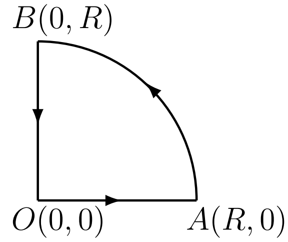
Solution.

1. $p(-1) = 0$. $p(-2) = -2 < 0, p(-3) = 22 > 0$, therefore the intermediate value theorem shows that there exists $x, -3 < x < -2$ such that $p(x) = 0$. Thus we have determined that to zeros of $p(z)$ lie on the negative real axis, and since p is a polynomial of degree 4 and hence has 4 zeros, we are left with the task of locating the the remaining two zeros.

2. $p(z)$ has no zeros on the positive real axis because $p(x) > 0$ when $x \geq 0$.

3. $p(z)$ has has no zero on the imaginary axis because $p(iy) = y^4 + 4 - 2iy^3 + 3iy = 0 \Rightarrow y^4 + 4 = 0, 2y^3 - 3y = 0$, but $y^4 + 4 = 0$ has no real zeros, so $p(iy) \neq 0$.

We now consider the contour $OABO$ where OA is straight line joining $(0,0)$ and $(R,0)$, AB is the arc of the circle $x^2+y^2=R^2$ in the first quadrant, and BO is the line joining $(0,R)$ to $(0,0)$.



By the Argument Principle, the number of zeros of $p(z)$ in the first quadrant $=\frac{1}{2\pi}\times$ (the change in the argument of $p(z)$ when z moves along the contour $OABO$ oriented anticlockwise as $R\to\infty$).

Change in the argument along OA: On OA, $p(z)=x^4+2x^3+3x+4>0\Rightarrow \arg p(z)=0$ for every x on OA. Therefore as z moves from O to A, the change in the argument of $p(z)$ i.e. $\Delta_{OA}\arg p(z)=0$.

Change in the argument along BO: On BO, $z=iy$ and $p(z)=y^4+4+i(3y-2y^3)$. Therefore $\arg p(z)=\tan^{-1}\Big(\dfrac{3y-2y^3}{y^4+4}\Big)$.

$$\Delta_{BO}\arg p(z)=\tan^{-1}\Big(\frac{3y-2y^3}{y^4+4}\Big)\Big]_{\infty}^{0}=0-0=0$$

Change in argument along AB: On arc AB, $z=Re^{i\theta}, 0\le\theta\le\frac{\pi}{2}$, so that

$$p(z)=R^4e^{4i\theta}+2R^3e^{3i\theta}+3Re^{i\theta}+4=R^4e^{4i\theta}\Big[1+\frac{2}{Re^{i\theta}}+\frac{3}{R^3e^{3i\theta}}+\frac{4}{R^4e^{4i\theta}}\Big]\longrightarrow R^4e^{4i\theta}$$

as $R\to\infty$. Thus $\Delta_{AB}\arg p(z)=4\theta\Big]_0^{\frac{\pi}{2}}=2\pi$.[1]

Hence $\Delta_{OABO}\arg p(z)=2\pi$ as $R\to\infty$, so $p(z)$ has exactly one zero in the first quadrant.

Since $p(z)$ is a polynomial with real coefficients, it follows that if ζ is a zero of $p(z)$ and it lies in the first quadrant, then $\overline{\zeta}$ is also a zero of $p(z)$ and it lies in the fourth quadrant.

Thus $p(z)$ has one zero in each of the first and the fourth quadrants. ∎

Question 1(c) *Test for uniform convergence in the region $|z|\le 1$ the series*

$$\sum_{n=1}^{\infty}\frac{\cos nz}{n^3}$$

Solution. By definition

$$\cos nz=\frac{e^{inz}+e^{-inz}}{2}=\frac{e^{-ny}e^{inx}+e^{ny}e^{-inx}}{2}$$

[1]Alternately, $p(z)=z^4\Big(1+\frac{2}{z}+\frac{3}{z^3}+\frac{4}{z^4}\Big)=z^4(1+w)$ where $w=\frac{2}{z}+\frac{3}{z^3}+\frac{4}{z^4}$. Clearly $w\to0$ as $R\to\infty$. Therefore $|1+w-1|<\epsilon$ for $|z|$ large. This means $1+w$ remains inside a circle of radius 1 as z moves along AB and $R\to\infty$. Therefore $\Delta_{AB}\arg(1+w)=0$ and $\Delta_{AB}p(z)=\Delta_{AB}z^4+\Delta_{AB}(1+w)=4\Delta_{AB}z=4\cdot\frac{\pi}{2}=2\pi$.

and therefore

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^3} = \sum_{n=1}^{\infty} \frac{e^{-ny}e^{inx}}{2n^3} + \sum_{n=1}^{\infty} \frac{e^{ny}e^{-inx}}{2n^3}$$

Case 1: $y > 0$.

$$\sum_{n=1}^{\infty} \left| \frac{e^{-ny}e^{inx}}{2n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{2n^3}$$

showing that the first term is absolutely convergent.

But the second term is not convergent, because its n-th term $\left| \frac{e^{ny}e^{-inx}}{2n^3} \right| \not\to 0$ as $n \to \infty$ — in fact $\left| \frac{e^{ny}e^{-inx}}{2n^3} \right| \to \infty$ as $n \to \infty$ when $y > 0$.

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$ is not even convergent when $y > 0$.

Case 2: $y < 0$. This case is entirely analogous to the above case — the first term $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-ny}e^{inx}}{2n^3}$ is not convergent, so $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$ is not convergent.

Case 3: $y = 0$. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$ is uniformly and absolutely convergent, because of Weierstrass M-test, which states that if $\sum_{n=1}^{\infty} f_n(z)$ is a series and there exist positive constants M_n such that $|f_n(z)| < M_n$ for every $z \in \Omega$ and $\sum_n M_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(z)$ is absolutely and uniformly convergent in Ω. Here $M_n = \frac{1}{n^3}$ for all x.

Thus the given series converges uniformly only on the real axis in $|z| \leq 1$. ∎

Question 2(a) *Find the Laurent series for*

1. $\dfrac{e^{2z}}{(z-1)^3}$ *about* $z = 1$.

2. $\dfrac{1}{z^2(z-3)^2}$ *about* $z = 3$.

Solution.

1. The function e^{2z} is analytic everywhere in the complex plane. The Taylor series of e^{2z} with center $z = 1$ is given by

$$e^{2z} = \sum_{n=0}^{\infty} \frac{\frac{d^n e^{2z}}{dz^n} \text{ at } z = 1}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(z-1)^n$$

because $\dfrac{d^n e^{2z}}{dz^n} = 2^n e^{2z}$. Thus

$$\begin{aligned}
\frac{e^{2z}}{(z-1)^3} &= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{4e^2}{2!(z-1)} + \sum_{n=3}^{\infty} \frac{2^n e^2}{n!}(z-1)^{n-3} \\
&= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{4e^2}{2!(z-1)} + \sum_{n=0}^{\infty} \frac{2^{n+3} e^2}{(n+3)!}(z-1)^{n}
\end{aligned}$$

which is the required Laurent series of $\dfrac{e^{2z}}{(z-1)^3}$ with center $z = 1$. It is valid in the ring $1 < |z| < \infty$.

2. Let $f(z) = \frac{1}{z^2}$ then

$$f'(z) = -\frac{2}{z^3},\ f''(z) = \frac{(-2)(-3)}{z^4}, \dots, f^{(n)}(z) = \frac{(-2)(-3)\dots(-n-1)}{z^{n+2}}$$

and therefore

$$f(3) = \frac{1}{3^2}, f'(3) = -\frac{2}{3^3}, \dots, f^{(n)}(3) = \frac{(-1)^n (n+1)!}{3^{n+2}}$$

Thus the Taylor series of $f(z)$ with center $z = 3$ is given by

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{3^{n+2} n!}(z-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{3^{n+2}}(z-3)^n$$

Thus

$$\frac{1}{z^2(z-3)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{3^{n+2}}(z-3)^{n-2} = \frac{1}{3^2(z-3)^2} - \frac{2}{3^3(z-3)} + \sum_{m=0}^{\infty} \frac{(-1)^m (m+3)}{3^{m+4}}(z-3)^m$$

is the required Laurent series of $\frac{1}{z^2(z-3)^2}$ with center $z = 3$ valid in $0 < |z| < 3$.

∎

Question 2(b) *Find the residues of $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.*

Solution. The poles are at zeros of $\sin^2 z$, and $\sin^2 z = 0$ iff $z = n\pi, n \in \mathbb{Z}$, the set of integers. All these poles are double poles.

Residue at $z = n\pi$ of $f(z)$ is $\dfrac{1}{1!}\dfrac{d}{dz}\left(\dfrac{(z-n\pi)^2 e^z}{\sin^2 z}\right)_{z=n\pi}$. Now

$$\begin{aligned}
\frac{d}{dz}\left(\frac{(z-n\pi)^2 e^z}{\sin^2 z}\right) &= \frac{\sin^2 z[(z-n\pi)^2 e^z + 2(z-n\pi)e^z] - (z-n\pi)^2 e^z 2\sin z \cos z}{\sin^4 z} \\
&= \frac{e^z (z-n\pi)}{\sin^3 z}\big((z-n\pi)\sin z + 2\sin z - 2(z-n\pi)\cos z\big)
\end{aligned}$$

Using $\lim_{z\to n\pi} \dfrac{z-n\pi}{\sin z} = \dfrac{1}{\cos n\pi} = (-1)^n$, we get

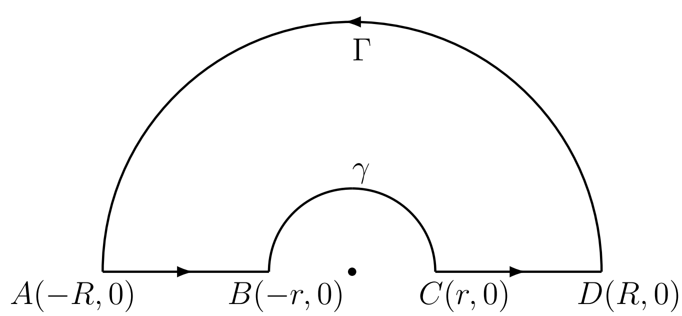
$$\begin{aligned}
\frac{d}{dz}\left(\frac{(z-n\pi)^2 e^z}{\sin^2 z}\right)_{z=n\pi} &= e^{n\pi} \lim_{z\to n\pi} \frac{(z-n\pi)}{\sin^3 z}\big((z-n\pi)\sin z + 2\sin z - 2(z-n\pi)\cos z\big) \\
&= e^{n\pi}(-1)^n \lim_{z\to n\pi} \frac{(z-n\pi)(\sin z - 2\cos z) + 2\sin z}{\sin^2 z} \\
&= e^{n\pi}(-1)^n \lim_{z\to n\pi} \frac{\sin z - 2\cos z + (z-n\pi)(\cos z + 2\sin z) + 2\cos z}{2\sin z\cos z} \\
&= e^{n\pi} \lim_{z\to n\pi} \frac{\sin z + (z-n\pi)(\cos z + 2\sin z)}{2\sin z} \\
&= e^{n\pi} \lim_{z\to n\pi} \frac{\cos z + \cos z + 2\sin z + (z-n\pi)(-\sin z + 2\cos z)}{2\cos z} \\
&= e^{n\pi}
\end{aligned}$$

Thus the residue at $z = n\pi$ of $e^z \csc^2 z$ is $e^{n\pi}$. ∎

Question 2(c) *By means of contour integration evaluate* $\displaystyle\int_0^\infty \frac{(\log_e u)^2}{u^2+1}\,du$.

Solution.

We take $f(z) = \frac{(\log z)^2}{z^2+1}$ and the contour C consisting of the line joining $(-R,0)$ to $(-r,0)$, the semicircle γ of radius r with center $(0,0)$, the line joining $(r,0)$ to $(R,0)$ and Γ a semicircle of radius R with center $(0,0)$. The contour lies in the upper half plane and is oriented anticlockwise. We have avoided the branch point $z=0$ of the multiple valued function $\log z$.



(Eventually we shall let $R\to\infty, r\to 0$).

(1) On Γ, $z = Re^{i\theta}$ and $|1+z^2| \ge |z|^2 - 1 = R^2 - 1$. Thus

$$\begin{aligned}
\left|\int_\Gamma f(z)\,dz\right| &\le \left|\int_0^\pi \frac{\left(\log(Re^{i\theta})\right)^2}{R^2-1} iRe^{i\theta}\,d\theta\right| \\
&\le \int_0^\pi \frac{|\log R + i\theta|^2}{R^2-1} R\,d\theta \\
&= \frac{R}{R^2-1}\int_0^\pi ((\log R)^2 + \theta^2)\,d\theta = \frac{R}{R^2-1}\left(\pi(\log R)^2 + \frac{\pi^3}{3}\right)
\end{aligned}$$

But $\frac{R}{R^2-1}\left(\pi(\log R)^2 + \frac{\pi^3}{3}\right) \to 0$ as $R\to\infty$, therefore

$$\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

(2) On γ, $z = re^{i\theta}$, $|z|^2 + 1 \geq 1 - |z|^2 = 1 - r^2$. Thus

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\pi^0 \frac{(\log r)^2 + \theta^2}{1 - r^2} r\,d\theta = \frac{r}{1-r^2}\Big(\pi(\log r)^2 + \frac{\pi^3}{3}\Big)$$

But the right side $\to 0$ as $r \to 0$, it follows that $\lim_{r\to 0} \int_\gamma f(z)\,dz = 0$.

(3) $f(z)$ has a simple pole at $z = i$ in the the upper half plane (inside C) and the residue at $z = i$ of $f(z)$ is $\dfrac{(\log i)^2}{2i} = \dfrac{1}{2i}\Big(\dfrac{\pi i}{2}\Big)^2 = \dfrac{\pi^2 i}{8}$. Thus

$$\lim_{R\to\infty, r\to 0} \int_C f(z)\,dz = \lim_{R\to\infty, r\to 0} \int_r^R f(x)\,dx + \int_R^r f(xe^{i\pi})\,dxe^{i\pi} = 2\pi i \frac{\pi^2 i}{8}$$

because on the line CD, $z = x$, and on the line AB, $z = xe^{i\pi}$. Hence

$$-\int_\infty^0 \frac{(\log(xe^{i\pi}))^2}{1 + x^2 e^{2\pi i}}\,dx + \int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = -\frac{\pi^3}{4}$$

Now $(\log(xe^{i\pi}))^2 = (\log x)^2 - \pi^2 + 2i\pi \log x$, so

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx - \pi^2 \int_0^\infty \frac{dx}{1+x^2} + 2i\pi \int_0^\infty \frac{\log x}{1+x^2}\,dx = -\frac{\pi^3}{4}$$

Equating real parts, and noting that $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x\Big]_0^\infty = \frac{\pi}{2}$, we get

$$2\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}$$

so that $\displaystyle\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8}$.

Note that by equating imaginary parts, we get $\displaystyle\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0$. ∎

UPSC Civil Services Main 1995 - Mathematics Complex Analysis

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Question 1(a) *Let $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that u is a harmonic function. Find a harmonic function v such that $u + iv$ is an analytic function of z.*

Solution. Clearly

$$\frac{\partial u}{\partial x} = 6xy + 4x \quad , \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$
$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad , \quad \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, showing that u is a harmonic function.

Let $f(z) = u + iv$, where v is to be so determined that $f(z)$ is analytic and v is harmonic. Such a function v along with u would have to satisfy the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Now

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= 6xy + 4x - i(3x^2 - 3y^2 - 4y) \\ &= -3i(x^2 - y^2 + 2ixy) + 4(x + iy) \\ &= -3iz^2 + 4z \end{aligned}$$

Thus

$$\begin{aligned}
f(z) &= 2z^2 - iz^3 \\
&= 2(x+iy)^2 - i(x+iy)^3 \\
&= 2x^2 - 2y^2 + 4ixy - ix^3 + 3x^2y + 3ixy^2 - y^3 \\
&= 3x^2y + 2x^2 - y^3 - 2y^2 + i(4xy - x^3 + 3xy^2)
\end{aligned}$$

Thus $v = 4xy - x^3 + 3xy^2$. Clearly

$$\frac{\partial v}{\partial x} = 4y - 3x^2 + 3y^2 \quad , \quad \frac{\partial v}{\partial y} = 4x + 6xy$$
$$\frac{\partial^2 v}{\partial x^2} = -6x \quad , \quad \frac{\partial^2 v}{\partial y^2} = 6x$$

so that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, showing that v is a harmonic function. ∎

Question 1(b) *Find the Taylor series expansion of* $f(z) = \dfrac{z}{z^4+9}$ *around* $z = 0$*. Find also the radius of convergence.*

Solution. It is obvious that

$$\begin{aligned}
f(z) &= \frac{z}{9}\Big(1 + \frac{z^4}{9}\Big)^{-1} = \frac{z}{9}\Big(1 - \frac{z^4}{9} + \frac{z^8}{81} - \frac{z^{12}}{729} + \ldots\Big) \\
&= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z^4}{9}\Big)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{9^{n+1}}
\end{aligned}$$

provided $|\frac{z^4}{9}| < 1$. This indeed is Taylor's series representation of $f(z)$ which to start with is valid for $|\frac{z^4}{9}| < 1$. The radius of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is given by $\Big(\limsup |a_n|^{\frac{1}{n}}\Big)^{-1}$. In this case the radius of convergence is $\Big(\displaystyle\lim_{n\to\infty}\Big(\frac{1}{9^{n+1}}\Big)^{\frac{1}{4n+1}}\Big)^{-1} = \displaystyle\lim_{n\to\infty} 9^{\frac{n+1}{4n+1}} = 9^{\frac{1}{4}} = \sqrt{3}$.

Note: We did not get the radius of convergence greater than the disc of validity namely $|\frac{z^4}{9}| < 1$ as we have a singularity of $f(z)$ on $|z| = \sqrt{3}$, namely those z for which $z^4 = -9 = i^2 9$ or $z^2 = \pm 3i$. ∎

Question 1(c) *Let* C *be a circle* $|z| = 2$ *oriented counter-clockwise. Evaluate the integral* $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$ *with the aid of residues.*

Solution. By Cauchy's residue theorem, $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = 2\pi i($sum of residues at poles of $\displaystyle\frac{\cosh \pi z}{z(z^2+1)}$ inside $C)$.

The only poles of $\displaystyle\frac{\cosh \pi z}{z(z^2+1)}$ are at $z = 0, \pm i$ all within $|z| = 2$. All these are simple poles.

Residue at $z = 0$ is $\displaystyle\lim_{z\to 0} \frac{z\cosh \pi z}{z(z^2+1)} = 1$.

Residue at $z = i$ is $\displaystyle\lim_{z\to i} \frac{(z-i)\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi i}{i\cdot 2i} = -\frac{\cos\pi}{2} = \frac{1}{2}$.

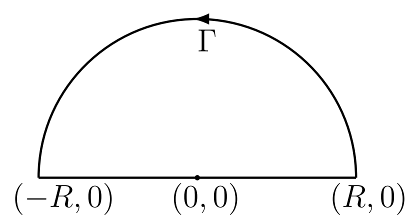
Residue at $z = -i$ is $\displaystyle\lim_{z\to -i} \frac{(z+i)\cosh \pi z}{z(z^2+1)} = \frac{\cosh(-\pi i)}{(-i)\cdot(-2i)} = \frac{1}{2}$.

Thus $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = 2\pi i\Big[1 + \frac{1}{2} + \frac{1}{2}\Big] = 4\pi i$. ∎

Question 2(a) *Evaluate the integral* $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx,\ a \geq 0$.

Solution.

Let $f(z) = \dfrac{e^{iaz}}{z^2+1}$. Let γ be the contour consisting of the line joining $(-R, 0)$ and $(R, 0)$ and Γ, which is the arc of the circle of radius R and center $(0, 0)$ lying in the upper half plane. γ is oriented counter-clockwise.



$$\lim_{R\to\infty}\int_\gamma f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx + \lim_{R\to\infty}\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz$$

Since $|z^2+1| \geq R^2 - 1$ on Γ and $|e^{iaz}| = |e^{iaRe^{i\theta}}| = |e^{-aR\sin\theta}| \leq 1$ because $\sin\theta \geq 0$ in $0 \leq \theta \leq \pi$, so

$$\left|\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1}$$

as $dz = iRe^{i\theta}\,d\theta$, showing that $\displaystyle\lim_{R\to\infty}\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz = 0$.

By Cauchy's residue theorem, $\lim_{R\to\infty}\int_\gamma f(z)\,dz = 2\pi i($ Sum of residues at poles of $\dfrac{e^{iaz}}{z^2+1}$ in the upper half plane). $z = i$ is the only pole of $f(z)$ in the upper half plane, and the residue there is given by $\displaystyle\lim_{z\to i}\frac{(z-i)e^{iaz}}{z^2+1} = \frac{e^{-a}}{2i}$.

Thus $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$, so

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}, \quad \int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx = 0$$

Since $\dfrac{\cos ax}{x^2+1}$ is an even function of x,

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi e^{-a}}{2}$$

∎

Question 2(b) *Let f be analytic in the entire complex plane. Suppose that there exists a constant $A > 0$, such that $|f(z)| \leq A|z|$ for all z. Prove that there is a complex number a such that $f(z) = az$ for all z.*

Solution. We first prove (Cauchy's inequality) that if $f(z)$ is analytic in a domain G and if the disc $|z - z_0| \leq \rho \subseteq G$ then

$$|f^{(n)}(z_0)| \leq \frac{n!M(\rho)}{\rho^n}$$

where $M(\rho) = \max|f(z)|$ on $|z - z_0| = \rho$ — this follows from Cauchy's Integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

and therefore

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\,\frac{M(\rho)}{\rho^{n+1}}\,2\pi\rho = \frac{n!M(\rho)}{\rho^n}$$

We now prove that if $f(z)$ is entire i.e. analytic over the whole complex plane, and $|f(z)| \leq G|z|^m$ for all $|z| > R$, then $f(z)$ is a polynomial of degree $\leq m$.

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be a Taylor series of $f(z)$ around $z = 0$. Then $a_n = \dfrac{f^{(n)}(0)}{n!}$. By Cauchy's inequality proved above, $|a_n| = \left|\dfrac{f^{(n)}(0)}{n!}\right| \leq \dfrac{M(r)}{r^n}$ where $M(r)$ is maximum of $|f(z)|$ on $|z| = r$. Let $r > R$, then $M(r) \leq Gr^m$ and we get $|a_n| \leq \dfrac{Gr^m}{r^n} = \dfrac{G}{r^{n-m}}$. and therefore as $r \to \infty$, $\dfrac{G}{r^{n-m}} \to 0$ for $n > m$ i.e. $|a_n| = 0$ for $n > m$. Hence $f(z) = \sum_{r=0}^m a_r z^r$ i.e. $f(z)$ is a polynomial of degree $\leq m$.

Now we are given $|f(z)| \leq A|z|$. This means that $f(z) = a_0 + a_1 z$. But $0 \leq |f(0)| \leq A \cdot 0 \Rightarrow f(0) = 0 \Rightarrow a_0 = 0$, so $f(z) = a_1 z$, where a_1 is a constant.

Note: An alternative statement of the above question is: If $f(z)$ is an entire transcendental function, then whatever $G > 0, R > 0, m > 0$ are prescribed, there exist points z such that $|f(z)| > G|z|^m$ and $|z| > R$. ∎

Alternate solution: Consider the function $g(z) = \frac{f(z)}{z}, z \neq 0$ and $g(0) = f'(0)$. Note that $|f(z)| \leq A|z| \Rightarrow f(0) = 0$. Then g is continuous at 0, because

$$\lim_{z\to 0} |g(z) - g(0)| = \lim_{z\to 0}\left|\frac{f(z)}{z} - f'(0)\right| = \lim_{z\to 0}\left|\frac{f(z) - f(0)}{z} - f'(0)\right| = 0$$

Let $f = u + iv$, where u, v satisfy the Cauchy Riemann equations, since f is entire. Then

$$g(z) = \frac{u + iv}{x + iy} = \frac{(ux + yv) + i(vx - uy)}{x^2 + y^2}$$

Writing $g(z) = U + iV$, we get $U = \dfrac{ux + yv}{x^2 + y^2}, V = \dfrac{vx - uy}{x^2 + y^2}$. Now it is clear that g is analytic over the entire complex plane except possibly at $z = 0$. We now check the Cauchy Riemann equations for U, V at $z = 0$. Note that $f(0) = 0 \Rightarrow u(0, 0) = v(0, 0) = 0$.

$$\begin{aligned}
\frac{\partial U}{\partial x}(0,0) &= \lim_{h\to 0}\frac{U(h,0) - U(0,0)}{h} = \lim_{h\to 0}\frac{\frac{u(h,0)}{h} - u_x(0,0)}{h} = \lim_{h\to 0}\frac{u(h,0) - hu_x(0,0)}{h^2}\\
&= \lim_{h\to 0}\frac{u_x(h,0) - u_x(0,0)}{2h} = \frac{1}{2}u_{xx}(0,0)\\
\frac{\partial U}{\partial y}(0,0) &= \lim_{k\to 0}\frac{U(0,k) - U(0,0)}{k} = \lim_{k\to 0}\frac{\frac{v(0,k)}{k} - u_x(0,0)}{k} = \lim_{k\to 0}\frac{v(0,k) - ku_x(0,0)}{k^2}\\
&= \lim_{k\to 0}\frac{v_y(0,k) - u_x(0,0)}{2k} = \frac{1}{2}v_{yy}(0,0)\\
\frac{\partial V}{\partial x}(0,0) &= \lim_{h\to 0}\frac{V(h,0) - V(0,0)}{h} = \lim_{h\to 0}\frac{\frac{v(h,0)}{h} - v_x(0,0)}{h} = \lim_{h\to 0}\frac{v(h,0) - hv_x(0,0)}{h^2}\\
&= \lim_{h\to 0}\frac{v_x(h,0) - v_x(0,0)}{2h} = \frac{1}{2}v_{xx}(0,0)\\
\frac{\partial V}{\partial y}(0,0) &= \lim_{k\to 0}\frac{V(0,k) - V(0,0)}{k} = \lim_{k\to 0}\frac{\frac{-u(0,k)}{k} - v_x(0,0)}{k} = \lim_{k\to 0}\frac{-u(0,k) - kv_x(0,0)}{k^2}\\
&= \lim_{k\to 0}\frac{-u_y(0,k) - v_x(0,0)}{2k} = -\frac{1}{2}u_{yy}(0,0)
\end{aligned}$$

Now by the Cauchy Riemann equations for u, v, $u_x = v_y \Rightarrow u_{xx} = v_{xy}$ and $u_y = -v_x \Rightarrow u_{yy} = -v_{yx}$. Hence $U_x(0,0) = \frac{1}{2}u_{xx}(0,0) = \frac{1}{2}v_{xy}(0,0) = -\frac{1}{2}u_{yy}(0,0) = V_y(0,0)$.

Also, $v_x = -u_y \Rightarrow v_{xx} = -u_{xy}$, and $v_y = u_x \Rightarrow v_{yy} = u_{yx}$. So $U_y(0,0) = \frac{1}{2}v_{yy}(0,0) = \frac{1}{2}u_{yx} = -\frac{1}{2}v_{xx}(0,0) = -V_x(0,0)$. Thus the Cauchy Riemann equations hold at $(0,0)$ also, so $g(z)$ is analytic at 0, as it is continuous at 0. Thus $g(z)$ is an entire function.

But $|g(z)| = |\frac{f(z)}{z}| \leq \frac{A|z|}{|z|} = A$, so g is bounded over the complex plane. Hence by Liouville's theorem, g is a constant, say a. Thus $f(z) = az$, as required.

Question 2(c) *Suppose a power series* $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ *converges at a point* $z_0 \neq 0$. *Let* z_1 *be such that* $|z_1| < |z_0|$ *and* $z_1 \neq 0$. *Show that the series converges uniformly in the disc* $\{z \,:\, |z| \leq |z_1|\}$.

Solution. Let $|\frac{z_1}{z_0}| = \rho$, then $\rho < 1$. Since $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent, $a_n z_0^n \to 0$ as $n \to \infty$, therefore there exists M such that $|a_n z_0^n| < M$ for $n \geq 0$. Now let z be any point such that $|z| \leq |z_1|$, then

$$\left|\sum_{n=r}^{r+p} a_n z^n\right| \leq \sum_{n=r}^{r+p} |a_n z^n| = \sum_{n=r}^{r+p} \left|a_n z_0^n \left(\frac{z}{z_0}\right)^n\right| \leq M \sum_{n=r}^{r+p} \left|\frac{z}{z_0}\right|^n = M \sum_{n=r}^{r+p} \rho^n$$

Since the series $\displaystyle\sum_{n=0}^{\infty} \rho^n$ is convergent, given $\epsilon > 0$ there exists N such that $\displaystyle\sum_{n=r}^{r+p} \rho^n < \frac{\epsilon}{M}$ for all $r \geq N$ and $p = 1, 2, \ldots$. Clearly this N is independent of z. Thus given $\epsilon > 0$ there exists N independent of z such that

$$\left|\sum_{n=r}^{r+p} a_n z^n\right| < \epsilon \quad \text{for } n \geq N, p = 1, 2, 3, \ldots$$

i.e. the series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent for all z with $|z| \leq |z_1|$. ∎

UPSC Civil Services Main 1996 - Mathematics Complex Analysis

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February 20, 2010

Question 1(a) *Sketch the ellipse C described in the complex plane by*

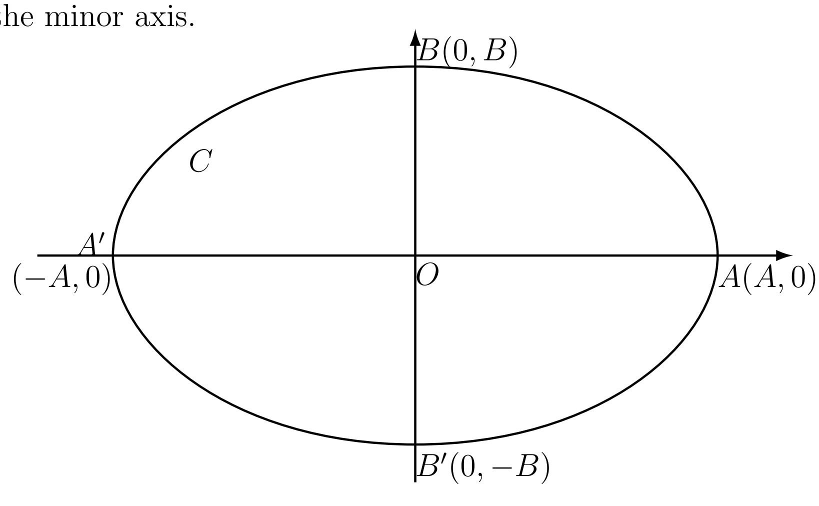
$$z = A\cos\lambda t + iB\sin\lambda t, A > B$$

where t is a real variable and A, B, λ are positive constants.

If C is the trajectory of a particle with $z(t)$ as the position vector of the particle at time t, identify with justification

1. *the two positions where the velocity is minimum.*
2. *the two positions where the acceleration is maximum.*

Solution. We are given that $x = A\cos\lambda t, y = B\sin\lambda t$ which implies that $\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$. Since $A > B$, it follows that it is the standard ellipse with $2A$ as the major axis and $2B$ as the minor axis.



1. The velocity $v = \dfrac{dz}{dt} = -A\lambda \sin \lambda t + iB\lambda \cos \lambda t$.

$$\begin{aligned}\text{Speed} &= \text{magnitude of velocity} = \left|\frac{dz}{dt}\right| \\ &= \sqrt{A^2\lambda^2 \sin^2 \lambda t + B^2\lambda^2 \cos^2 \lambda t} \\ &= \lambda\sqrt{(A^2 - B^2)\sin^2 \lambda t + B^2}\end{aligned}$$

Since $A^2 - B^2 > 0$, the speed is minimum when $\sin^2 \lambda t = 0$ i.e. when $x(t) = \pm A, y(t) = 0$ i.e. when the particle is at the two ends of the major axis, the points A and A' in the figure.

2. Acceleration $= \dfrac{d^2z}{dt^2} = -A\lambda^2 \cos \lambda t - iB\lambda^2 \sin \lambda t$.

Magnitude of acceleration $= \lambda^2\sqrt{A^2 \cos^2 \lambda t + B^2 \sin^2 \lambda t} = \lambda^2\sqrt{(A^2 - B^2)\cos^2 \lambda t + B^2}$. Since $A^2 - B^2 > 0$, acceleration is maximum when $\cos^2 \lambda t = 1 \Rightarrow \cos \lambda t = \pm 1$ i.e. the particle is at either end of the major axis, A or A'. (Note that acceleration is minimum when $\cos^2 \lambda t = 0$ i.e. the particle is at either end of the minor axis).

∎

Question 1(b) *Evaluate* $\displaystyle\lim_{z\to 0} \frac{1 - \cos z}{\sin(z^2)}$.

Solution.

$$\lim_{z\to 0} \frac{1 - \cos z}{\sin(z^2)} = \lim_{z\to 0} \frac{2\sin^2 \frac{z}{2}}{\sin(z^2)} = \lim_{z\to 0} \frac{2}{4}\,\frac{\frac{\sin^2 \frac{z}{2}}{(\frac{z}{2})^2}}{\frac{\sin(z^2)}{z^2}} = \frac{1}{2}$$

Note that $\sin z$ has a simple zero at $z = 0$ and $\sin z = z\phi(z)$ where $\phi(z)$ is analytic and $\phi(0) = 1$, so $\displaystyle\lim_{z\to 0} \frac{\sin z}{z} = 1$. ∎

Question 1(c) *Show that $z = 0$ is not a branch point for the function $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity?*

Solution. We know that $w = \sqrt{z}$ is a multiple valued function and has two branches. Once we fix a branch of $w = \sqrt{z}$, $\sin\sqrt{z}$ is analytic, and

$$\sin\sqrt{z} = \sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} + \ldots$$

or

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \ldots$$

Thus $\lim_{z\to 0} \frac{\sin\sqrt{z}}{\sqrt{z}} = 1$, so $z = 0$ is not a branch point of the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. In fact $z = 0$ is a removable singularity of $f(z)$. In fact

$$F(z) = \begin{cases} \frac{\sin\sqrt{z}}{\sqrt{z}}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

is analytic everywhere once a branch of $\sqrt{z}$ is specified. ∎

Question 2(a) *Prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0, a_n \neq 0, n \geq 1$ has exactly n roots.*

Solution. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$. Suppose, if possible, that $P(z) \neq 0$ for any $z \in \mathbb{C}$. Let $f(z) = \frac{1}{P(z)}$, then $f(z)$ is an entire function i.e. $f(z)$ is analytic in the whole complex plane. We shall now show that $f(z)$ **is bounded**.

$$P(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right)$$

Since $\frac{a_j}{z^{n-j}} \to 0$ as $z \to \infty$, for $0 \leq j < n$, is follows that given $\epsilon = \frac{|a_n|}{2n}$ there exists $R > 0$ such that $|z| > R \Rightarrow \left| \frac{a_j}{z^{n-j}} \right| < \frac{|a_n|}{2n}$ for $0 \leq j < n$. Thus

$$\left| a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right| \geq |a_n| - n \left| \frac{a_n}{2n} \right| = \left| \frac{a_n}{2} \right|$$

and therefore

$$|f(z)| = \left| \frac{1}{P(z)} \right| = \left| \frac{1}{z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right)} \right| \leq \frac{2}{|a_n| R^n} \quad \text{for } |z| > R$$

Since $|z| \leq R$ is a compact set and $f(z)$ is analytic on it, $f(z)$ is bounded on $|z| \leq R$. Consequently $f(z)$ is bounded on the whole complex plane. Now we use Liouville's theorem — If an entire function is bounded on the whole complex plane, then it is a constant. Thus $f(z)$ and therefore $P(z)$ is a constant, which is not true, hence our assumption that $P(z) \neq 0$ for all $z \in \mathbb{C}$ is false. So there is at least one $z_1 \in \mathbb{C}$ where $P(z_1) = 0$. (This result is called the fundamental theorem of algebra.)

We now prove by induction on n that $P(z)$ has n zeros. If $n = 1$, $P(z) = a_0 + a_1 z$ has one zero namely $z = -\frac{a_0}{a_1}$.

Assume as induction hypothesis that any polynomial of degree $n-1$ has $n-1$ zeros. By Euclid's algorithm, , we get $P_1(z)$ and $R(z)$ such that $P(z) = (z - z_1)P_1(z) + R(z)$, where $R(z) \equiv 0$ or $\deg R(z) < 1$ i.e. $R(z)$ is a constant. Putting $z = z_1$ we get $R(z) \equiv 0$, so $P(z) = (z - z_1)P_1(z)$. Since $P_1(z)$ is a polynomial of degree $n-1$, by induction hypothesis it has $n-1$ roots in $\mathbb{C}$, and therefore $P(z)$ has n roots in $\mathbb{C}$.

We now prove that $P(z)$ has exactly n roots. Let $z_1, z_2, \ldots, z_n$ be the (not necessarily distinct) roots of $P(z)$. Let $g(z) = \dfrac{P(z)}{(z-z_1)(z-z_2)\ldots(z-z_n)}$. Clearly $g(z)$ is analytic in the whole complex plane. Since

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{P(z)}{(z-z_1)(z-z_2)\ldots(z-z_n)} = \frac{a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n}}{(1-\frac{z_1}{z})(1-\frac{z_2}{z})\ldots(1-\frac{z_n}{z})} = a_n$$

it follows that given $\epsilon > 0$ there exists R such that $|g(z) - a_n| < \epsilon$ for $|z| > R$, so $g(z)$ is bounded in the region $|z| > R$. The function $g(z)$ being analytic is bounded in the compact region $|z| \le R$. Thus by Liouville's theorem $g(z)$ is a constant, in fact $g(z) = a_n$, and therefore

$$P(z) = a_n(z-z_1)(z-z_2)\ldots(z-z_n)$$

Thus if ζ is a zero of $P(z)$, then $\zeta = z_j$ for some j, $1 \le j \le n$. Thus $P(z)$ has exactly n zeroes. ∎

Alternate Proof: We shall use Rouche's theorem — Let γ be a simple closed rectifiable curve. Let $f(z), g(z)$ be analytic on and within γ. Suppose $|g(z)| < |f(z)|$ on γ, then $f(z)$ and $f(z) \pm g(z)$ have the same number of zeroes inside γ.

Let $f(z) = a_n z^n$ and $g(z) = a_{n-1}z^{n-1} + \ldots + a_0$. Let R be so large that $|g(z)| < |f(z)|$ on $|z| = R$. Then $f(z)$ and $f(z) + g(z) = P(z)$ have the same number of zeroes within $|z| = R$. But whatever $R > 0$ we take, $f(z)$ has exactly n zeroes in $|z| = R$, therefore $P(z)$ has exactly n zeroes in $\mathbb{C}$.

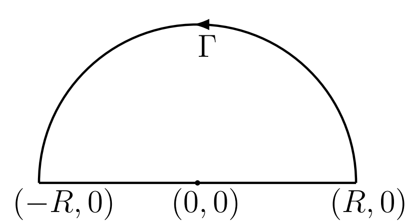
Note: Rouche's theorem follows from the Argument Principle — Note that $\Delta_\gamma(\arg(f(z)+g(z)))$ =change in argument of $f(z)+g(z)$ as z moves along $\gamma = \Delta_\gamma \arg f(z) + \Delta_\gamma \arg(1+\frac{g(z)}{f(z)})$ as $f(z) \ne 0$ along γ. But $\Delta_\gamma \arg(1+\frac{g(z)}{f(z)}) = 0$ because $|\frac{g(z)}{f(z)}| < 1$ and therefore $\frac{g(z)}{f(z)}$ continues to lie in the disc $|w-1| < 1$ as z moves on γ i.e. does not go around the origin.

Question 2(b) *By using the residue theorem, evaluate*

$$\int_0^\infty \frac{\log_e(x^2+1)}{x^2+1}\,dx$$

Solution.

Let $f(z) = \dfrac{\log(z+i)}{1+z^2}$ and we consider $\log(z+i)$ in $\mathbb{C} - \{z \mid z = iy, y \le -1\}$, where it is single-valued. Let γ be the contour consisting of the line joining $(-R,0)$ and $(R,0)$ and Γ, which is the arc of the circle of radius R and center $(0,0)$ lying in the upper half plane. γ is oriented counter-clockwise.



Clearly $f(z)$ has a simple pole at $z=i$ in the upper half plane. The residue at $z=i$ is

$$\lim_{z\to i}\frac{(z+i)\log(z+i)}{1+z^2}=\frac{\log 2i}{2i}=\frac{1}{2i}\log 2e^{\frac{\pi i}{2}}=\frac{1}{2i}\big[\log 2+i\frac{\pi}{2}\big]=\frac{\pi}{4}-\frac{1}{2}i\log 2$$

Thus by Cauchy's residue theorem

$$\lim_{R\to\infty}\int_\gamma\frac{\log(z+i)}{1+z^2}=\lim_{R\to\infty}\int_\Gamma\frac{\log(z+i)}{1+z^2}+\int_{-\infty}^{\infty}\frac{\log(x+i)}{1+x^2}\,dx=2\pi i\big[\frac{\pi}{4}-\frac{1}{2}i\log 2\big]$$

as $z=x$ on the real axis.

We shall now show that $\lim_{R\to\infty}\int_\Gamma\frac{\log(z+i)}{1+z^2}=0$. On Γ, $z=Re^{i\theta}$, so

$$\left|\int_\Gamma\frac{\log(z+i)}{1+z^2}\right|=\left|\int_0^\pi\frac{\log(Re^{i\theta}+i)Rie^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta\right|$$

Now $|R^2e^{2i\theta}+1|\geq R^2-1, \log(Re^{i\theta}+i)=\log Re^{i\theta}+\log(1+\frac{i}{Re^{i\theta}})$. Clearly $|\log Re^{i\theta}|=|\log R+i\theta|\leq\log R+\pi$ and therefore

$$\left|\int_\Gamma\frac{\log(z+i)}{1+z^2}\right|\leq\int_0^\pi\frac{(\pi+\log R)R}{R^2-1}\,d\theta+\int_0^\pi\frac{R|\log(1+\frac{i}{Re^{i\theta}})|}{R^2-1}\,d\theta$$

Since $\frac{(\pi+\log R)R}{R^2-1}\to 0$ and $\frac{R|\log(1+\frac{i}{Re^{i\theta}})|}{R^2-1}\to 0$ as $R\to\infty$, it follows that $\lim_{R\to\infty}\int_\Gamma\frac{\log(z+i)}{1+z^2}=0$.

Thus

$$\int_{-\infty}^{\infty}\frac{\log(x+i)}{1+x^2}\,dx=\pi\log 2+i\frac{\pi^2}{2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty\frac{\log(1+x^2)}{1+x^2}\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(1+x^2)}{1+x^2}\,dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(x+i)+\log(x-i)}{1+x^2}\,dx=\frac{1}{2}[2\pi\log 2]=\pi\log 2$$

∎

Question 2(c) *Find the Laurent expansion of* $f(z)=(z-3)\sin\Big(\frac{1}{z+2}\Big)$ *about the singularity* $z=-2$. *Specify the region of convergence and the nature of the singularity at* $z=-2$.

Solution. It is well known that

$$\begin{aligned}\sin\Big(\frac{1}{z+2}\Big)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1}\\ \Rightarrow (z-3)\sin\Big(\frac{1}{z+2}\Big)&=(z+2)\sin\Big(\frac{1}{z+2}\Big)-5\sin\Big(\frac{1}{z+2}\Big)\\ &=(z+2)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1}-5\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1}\\ &=\sum_{k=0}^{\infty}\frac{a_k}{(z+2)^k},\quad a_{2k-2}=\frac{(-1)^{k-1}}{(2k-1)!},\quad a_{2k-1}=\frac{5(-1)^{k-1}}{(2k-1)!}\end{aligned}$$

The region of convergence of the series is $0 < |z+2| < \infty$. The Laurent expansion shows that the function has an essential singularity at $z = -2$ — this also follows from the fact that $\lim_{z\to 0} \sin \frac{1}{z}$ does not exist. ∎

UPSC Civil Services Main 1997 - Mathematics Complex Analysis

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Question 1(a) *Prove that $u = e^x(x\cos y - y\sin y)$ is harmonic and find the analytic function whose real part is u.*

Solution.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^x(x\cos y - y\sin y) + e^x\cos y \\
\frac{\partial^2 u}{\partial x^2} &= e^x(x\cos y - y\sin y) + 2e^x\cos y \\
\frac{\partial u}{\partial y} &= e^x(-x\sin y - \sin y - y\cos y) \\
\frac{\partial^2 u}{\partial y^2} &= e^x(-x\cos y - 2\cos y + y\sin y)
\end{aligned}$$

Clearly $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, showing that u is harmonic.

Let $f(z) = u\Big(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\Big) + iv\Big(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\Big)$. Then

$$\begin{aligned}
f'(z) &= u_x + iv_x = u_x - iu_y \text{ because of the C-R equations} \\
&= u_x(x,y) - iu_y(x,y) = u_x\Big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\Big) - iu_y\Big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\Big)
\end{aligned}$$

Since the above is an identity, we take $z = \overline{z}$, so $x = z, y = 0$. Thus $f'(z) = u_x(z,0) - iu_y(z,0) = ze^z + e^z$. Then

$$f(z) = \int f'(z)\,dz = \int (z+1)e^z\,dz = ze^z + C$$

Hence $f(z) = ze^z$ is the required function. ∎

Question 1(b) *Evaluate* $\oint_C \frac{dz}{z+2}$ *where* C *is the unit circle. Deduce that* $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$.

Solution. Cauchy's theorem implies that $\oint_C \frac{dz}{z+2} = 0$ because $\frac{1}{z+2}$ has no pole inside $|z| = 1$.

Putting $z = e^{i\theta}$, we get

$$\begin{aligned}
I &= \int_{|z|=1} \frac{dz}{z+2} = \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)\,d\theta}{\cos\theta + 2 + i\sin\theta} \\
&= \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)(\cos\theta + 2 - i\sin\theta)}{(\cos\theta+2)^2 + \sin^2\theta}\,d\theta \\
&= \int_0^{2\pi} \frac{i(\cos^2\theta + \sin^2\theta + 2\cos\theta) - 2\sin\theta}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}\,d\theta \\
&= -\int_0^{2\pi} \frac{2\sin\theta}{5+4\cos\theta}\,d\theta + i\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta
\end{aligned}$$

Since $I = 0$, it follows that

$$\int_0^{2\pi} \frac{2\sin\theta}{5+4\cos\theta}\,d\theta = 0, \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$$

∎

Question 1(c) *If* $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n}$, *find the residue at* a *for* $\frac{f(z)}{z-b}$ *where* $A_1, A_2, \ldots A_n, a, b$ *are constants. What is the residue at infinity?*

Solution. Case (1): $a \neq b$.

$$\frac{f(z)}{z-b} = \frac{A_1(z-a)^{n-1} + A_2(z-a)^{n-2} + \ldots + A_n}{(z-b)(z-a)^n}$$

showing that $\frac{f(z)}{z-b}$ has a pole of order n at $z = a$. The residue at $z = a$ is the coefficient of $\frac{1}{z-a}$ in the Laurent expansion of $\frac{f(z)}{z-b}$ around a. Now

$$\begin{aligned}
\frac{1}{z-b} &= (z-a+a-b)^{-1} = \frac{1}{a-b}\Big(1 + \frac{z-a}{a-b}\Big)^{-1} \\
\frac{f(z)}{z-b} &= \Big[\frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n}\Big]\frac{1}{a-b}\Big[1 - \frac{z-a}{a-b} + \Big(\frac{z-a}{a-b}\Big)^2 - \Big(\frac{z-a}{a-b}\Big)^3 + \ldots\Big]
\end{aligned}$$

Thus the coefficient of $\dfrac{1}{z-a}$ i.e. the residue of $\dfrac{f(z)}{z-b}$ is given by

$$\frac{A_1}{a-b} - \frac{A_2}{(a-b)^2} + \frac{A_3}{(a-b)^3} + \ldots + \frac{(-1)^{n-1}A_n}{(a-b)^n} = -f(b)$$

Note: The same residue could be computed by using the formula — Residue at $z = a$ is $\dfrac{1}{(n-1)!}\dfrac{d^{n-1}}{dz^{n-1}}\left((z-a)^n \dfrac{f(z)}{z-b}\right)$ but this calculation would be much more complicated.

Case (2): $a = b$. In this case $\dfrac{f(z)}{z-b}$ has a pole of order $n+1$ at $z = a$. Residue at $z = a$ is given by $\dfrac{1}{n!}\dfrac{d^n}{dz^n}[(z-a)^n f(z)] = 0$.

Residue at ∞:

$$\begin{aligned} f(z) &= \frac{A_1}{z}\left(1-\frac{a}{z}\right)^{-1} + \frac{A_2}{z^2}\left(1-\frac{a}{z}\right)^{-2} + \ldots + \frac{A_n}{z^n}\left(1-\frac{a}{z}\right)^{-n} \\ \frac{f(z)}{z-b} &= \frac{f(z)}{z}\left(1-\frac{b}{z}\right)^{-1} = \left[\frac{A_1}{z} + \frac{1}{z^2}(\ldots) + \frac{1}{z^3}(\ldots) + \ldots\right]\left[\frac{1}{z} + \frac{b}{z^2} + \ldots\right] \end{aligned}$$

Since the term $\frac{1}{z}$ is not present in the Laurent expansion of $\dfrac{f(z)}{(z-b)}$ the residue at ∞ is 0. ∎

Question 2(a) *Find the Laurent series for the function* $e^{\frac{1}{z}}$ *in* $0 < |z| < \infty$. *Deduce that*

$$\frac{1}{\pi}\int_0^\pi e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

for all $n = 0, 1, 2, \ldots$.

Solution. See 2001 question 2(a). ∎

Question 2(b) *Integrating* e^{-z^2} *along a suitable rectangular contour show that*

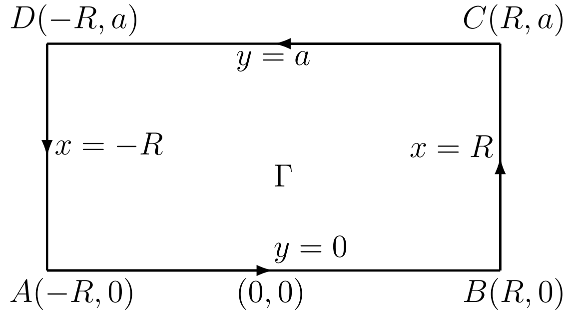
$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

Solution. More generally, we shall prove that

$$\int_0^\infty e^{-\lambda x^2}\cos 2a\lambda x\,dx = \frac{\sqrt{\pi}}{2}\lambda^{-\frac{1}{2}}e^{-\lambda a^2}$$

then $\lambda = 1, a = b$ will give us the desired result.

Our $f(z) = e^{-\lambda z^2}$ and the contour is Γ, the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, a), D = (-R, a)$ oriented in the anticlockwise direction.



Since $e^{-\lambda z^2}$ is an entire function, and therefore has no poles, $\oint_\Gamma e^{-\lambda z^2}\, dz = 0$.

Now we compute the integrals along the four sides.

1.
$$\int_{BC} e^{-\lambda z^2}\, dz = \int_0^a e^{-\lambda(R+iy)^2} i\, dy$$
because $z = R + iy$ on BC and $0 \le y \le a$. Thus
$$\left| \int_{BC} e^{-\lambda z^2}\, dz \right| \le e^{-\lambda R^2} \int_0^a e^{\lambda y^2}\, dy = \text{constant} \times e^{-\lambda R^2}$$
Thus since $e^{-\lambda R^2} \to 0$ as $R \to \infty$, $\int_{BC} e^{-\lambda z^2}\, dz \to 0$ as $R \to \infty$.

2. A similar argument shows that $\int_{DA} e^{-\lambda z^2}\, dz \to 0$ as $R \to \infty$.

3.
$$\lim_{R\to\infty} \int_{AB} e^{-\lambda z^2}\, dz = \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx$$
as $z = x$ on AB.

4.
$$\lim_{R\to\infty} \int_{CD} e^{-\lambda z^2}\, dz = \int_{\infty}^{-\infty} e^{-\lambda(x+ia)^2}\, dx$$
as $z = x + ia$ on CD and orientation is from C to D. Thus
$$\begin{aligned}
\lim_{R\to\infty} \int_{CD} e^{-\lambda z^2}\, dz &= \int_{\infty}^{-\infty} e^{-\lambda(x^2-a^2)} e^{-2ia\lambda x}\, dx \\
&= \int_{-\infty}^{\infty} -e^{-\lambda(x^2-a^2)} \cos(2a\lambda x)\, dx + i \int_{-\infty}^{\infty} e^{-\lambda(x^2-a^2)} \sin(2a\lambda x)\, dx
\end{aligned}$$

Now
$$\begin{aligned}
0 &= \lim_{R\to\infty} \int_\Gamma f(z)\, dz = \lim_{R\to\infty} \int_\Gamma e^{-\lambda z^2}\, dz \\
&= \lim_{R\to\infty} \left[\int_{AB} e^{-\lambda z^2}\, dz + \int_{BC} e^{-\lambda z^2}\, dz + \int_{CD} e^{-\lambda z^2}\, dz + \int_{DA} e^{-\lambda z^2}\, dz \right] \\
&= \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx - e^{\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx + i e^{\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2} \sin(2a\lambda x)\, dx
\end{aligned}$$

Equating real and imaginary parts, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\lambda x^2} \sin(2a\lambda x)\, dx &= 0 \\
\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx &= e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx \\
&\quad \text{(Substituting } X = \sqrt{\lambda}x\text{)} \\
&= 2\int_0^{\infty} e^{-X^2} \lambda^{-\frac{1}{2}} dX = \sqrt{\pi}\lambda^{-\frac{1}{2}} \\
\Rightarrow \int_0^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx \;=\; \frac{1}{2} e^{-\lambda a^2} \lambda^{-\frac{1}{2}} \sqrt{\pi}
\end{aligned}$$

This completes the proof. ∎

Question 2(c) *Find the function $f(z)$ analytic within the unit circle which takes the values $\dfrac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}, 0 \le \theta \le 2\pi$ on the circle.*

Solution. Since $f(z)$ is analytic within $|z| < 1$, the Maclaurin series of $f(z)$ is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!}, \quad \text{where } f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(z)\, dz}{z^{n+1}}$$

We are given that on $|z| = 1$, $f(z) = \dfrac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$ and we know that on $|z| = 1$, $z = e^{i\theta}, dz = ie^{i\theta} d\theta, 0 \le \theta \le 2\pi$, therefore

$$f(z) = \frac{a - \frac{1}{z}}{a^2 - a(z + \frac{1}{z}) + 1} = \frac{a - \frac{1}{z}}{(a - \frac{1}{z})(a - z)} = \frac{1}{a - z} \text{ on } |z| = 1$$

Now we use the Maclaurin series to compute the value of f inside the unit circle.

$$\begin{aligned}
f^{(n)}(0) &= \frac{n!}{2\pi i} \int_{|z|=1} \frac{dz}{(a - z) z^{n+1}} \\
&= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{ie^{i\theta} e^{-(n+1)i\theta}}{a - e^{i\theta}}\, d\theta \\
&= \frac{n!}{2\pi a} \int_0^{2\pi} e^{-ni\theta} \Big(1 - \frac{e^{i\theta}}{a}\Big)^{-1} d\theta \\
&= \frac{n!}{2\pi a} \int_0^{2\pi} e^{-ni\theta} \Big(1 + \frac{e^{i\theta}}{a} + \frac{e^{2i\theta}}{a^2} + \ldots \frac{e^{in\theta}}{a^n} + \ldots\Big)\, d\theta
\end{aligned}$$

Since $\displaystyle\int_0^{2\pi} e^{ik\theta}\, d\theta = \frac{e^{ik\theta}}{ik}\Big]_0^{2\pi} = 0$ for $k \ne 0$, and $\displaystyle\int_0^{2\pi} d\theta = 2\pi$, it follows that

$$f^{(n)}(0) = \frac{n!}{2\pi a} \cdot \frac{1}{a^n} \cdot 2\pi = \frac{n!}{a^{n+1}}$$

Consequently

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}} = \frac{1}{a} + \frac{z}{a^2} + \ldots = \frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1} = \frac{1}{a-z}$$

We need $a > 1$ so that $|\frac{z}{a}| < 1$ on $|z| \leq 1$. $\blacksquare$

UPSC Civil Services Main 1998 - Mathematics Complex Analysis

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Question 1(a) *Show that the function*

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous and C-R conditions are satisfied at $z = 0$, but $f'(z)$ does not exist at $z = 0$.

Solution. Let $f(z) = u + iv$, then $u = \dfrac{x^3 - y^3}{x^2+y^2}, v = \dfrac{x^3+y^3}{x^2+y^2}$ for $z \neq 0$, and $u(0,0) = v(0,0) = 0$.

$$\begin{aligned} \frac{\partial u}{\partial x}(0,0) &= \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1 \\ \frac{\partial u}{\partial y}(0,0) &= \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k\to 0} \frac{\frac{-k^3}{k^2} - 0}{k} = -1 \\ \frac{\partial v}{\partial x}(0,0) &= \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1 \\ \frac{\partial v}{\partial y}(0,0) &= \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k\to 0} \frac{\frac{k^3}{k^2} - 0}{k} = 1 \end{aligned}$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at $(0,0)$, i.e. the Cauchy Riemann equations are satisfied at $(0,0)$.

$f(z)$ is clearly continuous at $z = 0$, because

$$\begin{aligned} |u(x,y) - u(0,0)| &= \left|\frac{x^3 - y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right| \leq 2\sqrt{x^2+y^2} \\ |v(x,y) - v(0,0)| &\leq 2\sqrt{x^2+y^2} \end{aligned}$$

Thus u, v are continuous at $(0,0)$, so $f(z)$ is continuous at $(0,0)$.

If $f(z)$ is to be differentiable at 0, then

$$\lim_{z\to 0}\frac{f(z)-0}{z} = \lim_{x\to 0, y\to 0}\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)} = \lim_{x\to 0, y\to 0}\frac{(x^3+iy^3)(1+i)(x-iy)}{(x^2+y^2)^2}$$

should exist and it should be equal to $\dfrac{\partial u}{\partial x}(0,0)+i\dfrac{\partial v}{\partial x}(0,0)=1+i$.

But if we take the limit along $y=x$, then

$$\lim_{z\to 0}\frac{f(z)-0}{z} = \lim_{x\to 0}\frac{(x^3+ix^3)(1+i)(x-ix)}{(2x^2)^2} = \frac{1+i}{2}$$

Therefore $f(z)$ is not differentiable at $z=0$. ∎

Question 1(b) *Find the Laurent expansion of* $\dfrac{z}{(z+1)(z+2)}$ *about the singularity* $z=-2$. *Specify the region of convergence and nature of singularity at* $z=-2$.

Solution. Clearly

$$\begin{aligned} f(z) &= \frac{z}{(z+1)(z+2)} = \frac{2}{z+2}-\frac{1}{z+1} = \frac{2}{z+2}+\frac{1}{1-(z+2)} \\ &= \frac{2}{z+2}+\sum_{n=0}^{\infty}(z+2)^n \text{ for } |z+2|<1 \qquad (*) \end{aligned}$$

The function satisfies the requirements of Laurent's theorem in the region $0<|z+2|<1$ and the right hand side of $(*)$ represents the Laurent series of $f(z)$, which converges for $|z+2|<1$, because we have a singularity at $z=-1$ which lies on $|z+2|=1$. The Laurent series expansion $(*)$ shows that $f(z)$ has a simple pole at $z=-2$, where its residue is 2. ∎

Question 1(c) *By using the integral representation of* $f^{(n)}(0)$, *prove that*

$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{x^n e^{xz}}{n! z^{n+1}}\,dz$$

Hence show that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}\,d\theta$$

Solution. It is easily deducible from Cauchy's Integral formula that if $f(z)$ is analytic within and on a simple closed contour C and z_0 is a point in the interior of C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}, dz$$

Let $f(z) = e^{xz}$ (Here x is not $\operatorname{Re} x$ but a parameter), then $f(z)$ is an entire function and therefore

$$f^{(n)}(0) = x^n = \frac{n!}{2\pi i} \oint_C \frac{e^{xz}}{z^{n+1}}, dz$$

where C is any closed contour containing 0 in its interior. Hence

$$\left(\frac{x^n}{n!}\right)^2 = \frac{x^n}{(n!)^2} \frac{n!}{2\pi i} \oint_C \frac{e^{xz}}{z^{n+1}}, dz = \frac{1}{2\pi i} \oint_C \frac{x^n e^{xz}}{n! z^{n+1}} \, dz$$

as required.

We take C to be the unit circle for convenience. Then

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \frac{x^n e^{xz}}{n! z^{n+1}} \, dz = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{x^n e^{xz}}{n! z^{n+1}} \, dz$$

Interchange of summation and integral is justified. Thus

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz}}{z} \sum_{n=0}^{\infty} \frac{(\frac{x}{z})^n}{n!} \, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz}}{z} e^{\frac{x}{z}} \, dz$$

Put $z = e^{i\theta}$ so that $dz = ie^{i\theta} \, d\theta$ and

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{x(e^{i\theta} + e^{-i\theta})}}{e^{i\theta}} ie^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta} \, d\theta$$

as required. ∎

Question 2(a) *Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Solution. See 2006 question 2(b). ∎

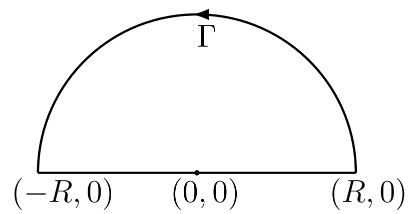
Question 2(b) *By integrating around a suitable contour show that*

$$\int_0^{\infty} \frac{x \sin mx}{x^4 + a^4} \, dx = \frac{\pi}{4b^2} e^{-mb} \sin mb$$

where $b = \frac{a}{\sqrt{2}}$.

Solution.

Let $f(z) = \dfrac{ze^{imz}}{z^4 + a^4}$. We consider the integral $\int_\gamma f(z)\,dz$ where γ is the contour consisting of the line joining $(-R, 0)$ and $(R, 0)$ and Γ, the arc of the circle of radius R and center $(0, 0)$ lying in the upper half plane.



$$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_0^\pi \frac{Re^{i\theta}e^{imR(\cos\theta + i\sin\theta)}}{z^4 + a^4} Rie^{i\theta}\,d\theta\right| \le \frac{R^2}{R^4 - a^4}\pi$$

because $|z^4 + a^4| \ge |z|^4 - |a^4| = R^4 - a^4$ on Γ, and $e^{-mR\sin\theta} \le 1$ as $\sin\theta > 0$ for $0 < \theta < \pi$.

Thus $\displaystyle\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$ and

$$\lim_{R\to\infty}\int_\gamma f(z)\,dz = \int_{-\infty}^\infty \frac{xe^{imx}}{x^4 + a^4}\,dx$$

But by Cauchy's residue theorem $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \times$ (the sum of the residues of poles of $f(z)$ inside γ). The poles of $f(z)$ are simple poles at $\pm ae^{\frac{\pi i}{4}}, \pm ae^{\frac{3\pi i}{4}}$, out of which $ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}$ are inside γ.

Residue at $z = ae^{\frac{\pi i}{4}}$ is $\dfrac{ae^{\frac{\pi i}{4}}e^{ima\,e^{\frac{\pi i}{4}}}}{4a^3e^{\frac{3\pi i}{4}}}$. Residue at $z = ae^{\frac{3\pi i}{4}}$ is $\dfrac{ae^{\frac{3\pi i}{4}}e^{ima\,e^{\frac{3\pi i}{4}}}}{4a^3e^{\frac{9\pi i}{4}}}$.

$$\begin{aligned}
\text{Sum of residues} &= \frac{i}{4a^2}\left[-e^{ima(cos\frac{\pi}{4} + i\sin\frac{\pi}{4})} + e^{ima(cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})}\right] \\
&= \frac{i}{4a^2}\left[-e^{\frac{ma}{\sqrt{2}}(i-1)} + e^{\frac{ma}{\sqrt{2}}(-i-1)}\right] \\
&= \frac{ie^{-\frac{ma}{\sqrt{2}}}}{4a^2}\left(-2i\sin\frac{ma}{\sqrt{2}}\right) = \frac{e^{-\frac{ma}{\sqrt{2}}}}{2a^2}\sin\frac{ma}{\sqrt{2}}
\end{aligned}$$

Thus

$$\int_{-\infty}^\infty \frac{xe^{imx}}{x^4 + a^4}\,dx = 2\pi i\frac{e^{-\frac{ma}{\sqrt{2}}}}{2a^2}\sin\frac{ma}{\sqrt{2}}$$

Taking imaginary parts of both sides, we get

$$\int_{-\infty}^\infty \frac{x\sin mx}{x^4 + a^4}\,dx = 2\int_0^\infty \frac{x\sin mx}{x^4 + a^4}\,dx = \frac{\pi e^{-\frac{ma}{\sqrt{2}}}}{a^2}\sin\frac{ma}{\sqrt{2}} = \frac{\pi e^{-mb}}{2b^2}\sin mb$$

where $b = \frac{a}{\sqrt{2}}$. Thus

$$\int_0^\infty \frac{x\sin mx}{x^4 + a^4}\,dx = \frac{\pi e^{-mb}}{4b^2}\sin mb$$

as required. ∎

Question 2(c) *Using the residue theorem evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Solution. We put $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}(z + \frac{1}{z}), \sin\theta = \frac{1}{2i}(z - \frac{1}{z})$. Thus

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} \\ &= \oint_{|z|=1} \frac{dz}{iz[3 - (z + \frac{1}{z}) + \frac{1}{2i}(z - \frac{1}{z})]} \\ &= 2\oint_{|z|=1} \frac{dz}{6iz - 2iz^2 - 2i + z^2 - 1} \\ &= 2\oint_{|z|=1} \frac{dz}{(1-2i)(z + \frac{i}{1-2i})(z + \frac{5i}{1-2i})} \end{aligned}$$

Clearly $(6iz - 2iz^2 - 2i + z^2 - 1)^{-1}$ has two simple poles $-\frac{i}{1-2i}$ and $-\frac{5i}{1-2i}$ of which only $-\frac{i}{1-2i}$ lies inside $|z| = 1$. The residue at this pole is $\displaystyle\lim_{z \to -\frac{i}{1-2i}} \frac{z + \frac{i}{1-2i}}{(1-2i)(z + \frac{i}{1-2i})(z + \frac{5i}{1-2i})} = \frac{1}{4i}$. Thus by Cauchy's residue theorem

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = 2 \cdot 2\pi i \cdot \frac{1}{4i} = \pi$$

∎

UPSC Civil Services Main 1999 - Mathematics Complex Analysis

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Question 1(a) *Examine the nature of the function*

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0, f(0) = 0$$

in a region including the origin and hence show that the Cauchy-Riemann equations are satisfied at the origin, but $f(z)$ is not analytic there.

Solution.

$$\begin{aligned} u(x,y) &= \operatorname{Re} f(z) = \begin{cases} \frac{x^3y^5}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \\ v(x,y) &= \operatorname{Im} f(z) = \begin{cases} \frac{x^2y^6}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \end{aligned}$$

Now $\dfrac{u(x,0)-u(0,0)}{x} = 0 = \dfrac{v(0,y)-v(0,0)}{y}$, therefore $u_x(0,0) = v_y(0,0) = 0$. Similarly $u_y(0,0) = 0 = -v_x(0,0)$. Thus the Cauchy-Riemann equations are satisfied at $(0,0)$.

However $f(z)$ is not analytic at $(0,0)$ because $\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim\limits_{z\to 0} \dfrac{x^2y^5}{x^4+y^{10}}$ does not exist — when we take $y^5 = mx^2$, then $\lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \dfrac{m}{1+m^2}$ which is different for different values of m. ∎

Additional notes: Let $z \neq 0$. It can be calculated that

$$\frac{\partial u}{\partial x} = \frac{3x^2y^{15} - x^6y^5}{(x^4+y^{10})^2} \qquad \frac{\partial v}{\partial y} = \frac{-4x^2y^{15} + 6x^6y^5}{(x^4+y^{10})^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy^{16} - 2x^5y^6}{(x^4+y^{10})^2} \qquad \frac{\partial u}{\partial y} = \frac{5x^7y^4 - 5x^3y^{14}}{(x^4+y^{10})^2}$$

Now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow 3x^2y^{15} - x^6y^5 = -4x^2y^{15} + 6x^6y^5 \Leftrightarrow x^2y^{15} = x^6y^5 \Leftrightarrow x^4 = y^{10}$ or $x = 0$ or $y = 0$.

Also, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ when $x^4 = y^{10}$ or $x = 0$ or $y = 0$. Thus the Cauchy-Riemann equations are satisfied at all those z for which $x^4 = y^{10}$ or $x = 0$ or $y = 0$. But $f(z)$ is not analytic at any of these points because $f(z)$ is not differentiable in any neighborhood of these points, as we can find points in every neighborhood which are not of this kind, so there are no neighborhoods in which the Cauchy Riemann equations are satisfied everywhere.

Question 1(b) *For the function* $f(z) = \dfrac{-1}{z^2 - 3z + 2}$, *find the Laurent series for the domain (i)* $1 < |z| < 2$ *(ii)* $|z| > 2$.

Show further that $\oint_C f(z)\,dz = 0$ *where* C *is any closed contour enclosing the points* $z = 1$ *and* $z = 2$.

Solution. $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$

(i) $1 < |z| < 2 \Rightarrow |\frac{1}{z}| < 1, |\frac{z}{2}| < 1$.

$$\begin{aligned} f(z) &= \frac{1}{z}\Big(1 - \frac{1}{z}\Big)^{-1} + \frac{1}{2}\Big(1 - \frac{z}{2}\Big)^{-1} \\ &= \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \end{aligned}$$

(ii) $|z| > 2 \Rightarrow |\frac{1}{z}| < 1, |\frac{2}{z}| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z}\Big(1 - \frac{1}{z}\Big)^{-1} - \frac{1}{z}\Big(1 - \frac{2}{z}\Big)^{-1} \\ &= \frac{1}{z}\sum_{n=0}^{\infty} \frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty} \frac{2^n}{z^n} \\ &= \sum_{n=0}^{\infty} \frac{1 - 2^n}{z^{n+1}} \end{aligned}$$

$$\begin{aligned}
\oint_C f(z)\,dz &= \oint_C \Big(\frac{1}{z-1} - \frac{1}{z-2}\Big) dz \\
&= 2\pi i \big[\text{residue of } \tfrac{1}{z-1} \text{ at } z = 1 - \text{residue of } \tfrac{1}{z-2} \text{ at } z = 2\big] \\
&= 2\pi i[1-1] = 0
\end{aligned}$$

■

Question 1(c) *Show that the transformation* $w = \dfrac{2z+3}{z-4}$ *transforms the circle* $x^2+y^2-4x = 0$ *into the straight line* $4u+3=0$ *where* $w = u+iv$.

Solution. The point $z = 4$ goes to the point at ∞, showing that the given circle $0 = x^2+y^2-4x = z\overline{z} - 4(\frac{z+\overline{z}}{2}) = z\overline{z} - 2z - 2\overline{z} = 0$ is mapped onto a line, as $z = 4$ lies on it.

Now $zw - 4w = 2z+3 \Rightarrow zw - 2z = 3+4w \Rightarrow z = \dfrac{3+4w}{w-2}$. Thus the circle $z\overline{z} - 2z - 2\overline{z} = 0$ goes to

$$\begin{aligned}
0 &= \frac{3+4w}{w-2}\,\frac{3+4\overline{w}}{\overline{w}-2} - 2\frac{3+4w}{w-2} - 2\frac{3+4\overline{w}}{\overline{w}-2} = 0 \\
\Rightarrow 0 &= 9 + 12w + 12\overline{w} + 16w\overline{w} - 2(3+4w)(\overline{w}-2) - 2(3+4\overline{w})(w-2) \\
\Rightarrow 0 &= 9 + 12w + 12\overline{w} + 16w\overline{w} - 6\overline{w} + 12 + 16w - 8w\overline{w} - 6w + 12 + 16\overline{w} - 8w\overline{w} \\
&= 33 + 22w + 22\overline{w} \\
0 &= 2(w+\overline{w}) + 3
\end{aligned}$$

Thus $4u+3=0$, as required. ■

Alternate solution: The given circle is $|z-2| = 2 \Rightarrow z = 2+2e^{i\theta}$. Substituting in transformation expression,

$$\begin{aligned}
w &= \frac{2z+3}{z-4} = \frac{4+4e^{i\theta}+3}{2+2e^{i\theta}-4} = \frac{7+4e^{i\theta}}{2(e^{i\theta}-1)} = \frac{(7+4e^{i\theta})(e^{-i\theta}-1)}{2(e^{i\theta}-1)(e^{-i\theta}-1)} \\
&= \frac{7e^{-i\theta} - 4e^{i\theta} - 3}{2(2-e^{i\theta}-e^{-i\theta})} = \frac{7(\cos\theta - i\sin\theta) - 4(\cos\theta + i\sin\theta) - 3}{2(2-2\cos\theta)} \\
&= \frac{3\cos\theta - 3 - 11i\sin\theta}{4(1-\cos\theta)} = -\frac{3}{4} - i\frac{11\sin\theta}{4(1-\cos\theta)}
\end{aligned}$$

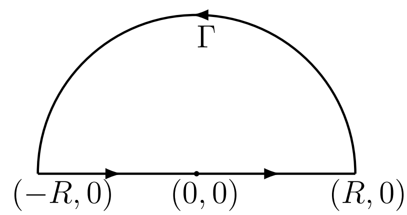
Thus $u = -\frac{3}{4} \Rightarrow 4u+3 = 0$, hence all points on the circle $|z-2| = 2$ are mapped onto the line $4u+3=0$.

Question 2(a) *Using the Residue Theorem show that*

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a \quad (a>0)$$

Solution.

We consider $I = \int_\gamma f(z)\,dz$ where $f(z) = \dfrac{ze^{iaz}}{z^4+4}$ and the contour γ consists of Γ a semi-circle of radius R with center $(0,0)$ lying in the upper half plane bounded by the real axis as shown.



Thus by Cauchy's residue theorem, $\int_\gamma f(z)\,dz = 2\pi i$(sum of residues at poles of $f(z)$ inside γ).

Clearly $f(z)$ has simple poles at $z^4 = 4e^{(2n+1)\pi i}$ for $n = 0,1,2,3$, or $z = \sqrt{2}e^{\frac{\pi i}{4}}, \sqrt{2}e^{\frac{3\pi i}{4}}, \sqrt{2}e^{\frac{5\pi i}{4}}, \sqrt{2}e^{\frac{7\pi i}{4}}$. Out of these only the poles $\sqrt{2}e^{\frac{\pi i}{4}}, \sqrt{2}e^{\frac{3\pi i}{4}}$ lie inside γ.

Residue at $\sqrt{2}e^{\frac{\pi i}{4}}$ is $\left(\dfrac{ze^{iaz}}{\frac{d}{dz}(z^4+4)}\right)$ at $z = \sqrt{2}e^{\frac{\pi i}{4}}$, which is $\dfrac{\alpha e^{ia\alpha}}{4\alpha^3}$ where $\alpha = \sqrt{2}e^{\frac{\pi i}{4}} = 1+i$.

Residue at $\sqrt{2}e^{\frac{3\pi i}{4}}$ is $\dfrac{e^{ia\beta}}{4\beta^2}$ where $\beta = \sqrt{2}e^{\frac{3\pi i}{4}} = -1+i$.

Sum of these residues is

$$\begin{aligned}\frac{1}{4}\Big[\frac{e^{ia\alpha}}{\alpha^2} + \frac{e^{ia\beta}}{\beta^2}\Big] &= \frac{1}{4}\Big[\frac{e^{ia(1+i)}}{2i} + \frac{e^{ia(-1+i)}}{(-2i)}\Big] \\ &= \frac{e^{-a}}{8i}\big[e^{ia} - e^{-ia}\big] = \frac{e^{-a}\sin a}{4}\end{aligned}$$

Thus $\displaystyle\int_\gamma \frac{ze^{iaz}\,dz}{z^4+4} = 2\pi i\frac{e^{-a}\sin a}{4}$. Now

$$\left|\int_\Gamma \frac{ze^{iaz}\,dz}{z^4+4}\right| = \left|\int_0^\pi \frac{Re^{i\theta}e^{iaRe^{i\theta}}}{z^4+4} iRe^{i\theta}\,d\theta\right| \le \frac{R^2}{R^4-4}\int_0^\pi e^{-aR\sin\theta}\,d\theta \le \frac{\pi R^2}{R^4-4}$$

because $|z^4+4| \ge |z^4| - 4 = R^4 - 4$ on Γ, and $e^{-aR\sin\theta} \le 1$ as $\sin\theta \ge 0$ on $[0,\pi]$. Thus $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$. Thus

$$\int_{-\infty}^\infty \frac{xe^{iax}}{x^4+4}\,dx = \lim_{R\to\infty}\int_\gamma \frac{ze^{iaz}}{z^4+4}\,dz = \frac{e^{-a}\sin a}{4}2\pi i$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^\infty \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi e^{-a}\sin a}{2}$$

as required. ∎

Question 2(b) *The function $f(z)$ has a double pole at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If $f(2) = 5$ and $f(-1) = 2$, find $f(z)$.*

Solution. Since $f(z)$ has only poles as singularities in the extended complex plane, it is well known that $f(z)$ has to be a rational function. Since $f(z)$ has a double pole at $z = 0$ and a simple pole at $z = 1$, it has to be of the form $f(z) = \dfrac{\phi(z)}{z^2(z-1)}$. where $\phi(z)$ is a polynomial such that $\phi(0) \neq 0, \phi(1) \neq 0$. Moreover degree of $\phi(z)$ is ≤ 3 as we are given that $f(z)$ is bounded as $z \to \infty$. Let $\phi(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Then

$$f(2) = 5 \quad \Rightarrow \quad \frac{a_0 + 2a_1 + 4a_2 + 8a_3}{4} = 5 \qquad (1)$$

$$f(-1) = 2 \quad \Rightarrow \quad \frac{a_0 - a_1 + a_2 - a_3}{-2} = 2 \qquad (2)$$

Residue of $f(z)$ at $z = 1$ is $\displaystyle\lim_{z \to 1} \frac{(z-1)\phi(z)}{z^2(z-1)} = \phi(1)$. This value is given to be 2, so

$$a_0 + a_1 + a_2 + a_3 \quad = \quad 2 \qquad (3)$$

Residue of $f(z)$ at $z = 0$ is given by $\dfrac{1}{1!}\dfrac{d}{dz}\left(\dfrac{\phi(z)}{z-1}\right)$ at $z = 0$, or $\dfrac{(z-1)(\phi'(z)) - \phi(z)}{(z-1)^2} = -a_1 - a_0$. Since this is given to be 2,

$$-a_0 - a_1 \quad = \quad 2 \qquad (4)$$

Adding $(2), (3)$ we get $2a_0 + 2a_2 = -2 \Rightarrow a_2 = -1 - a_0$. Substituting $a_2 = -1 - a_0, a_1 = -a_0 - 2$ in (1), we get $a_0 - 2a_0 - 4 - 4 - 4a_0 + 8a_3 = 20 \Rightarrow 8a_3 = 5a_0 + 28$. Substituting in (3), we have $a_0 - a_0 - 2 - 1 - a_0 + \frac{5a_0+28}{8} = 2 \Rightarrow -3a_0 + 28 = 40 \Rightarrow a_0 = -4 \Rightarrow a_1 = 2, a_2 = 3, a_3 = 1$.

Hence $f(z) = \dfrac{-4 + 2z + 3z^2 + z^3}{z^2(z-1)}$ is the desired function.

Note: If $f(z)$ has only poles in $\mathbb{C} \cup \infty$, then it is a rational function. If $\phi_1(z), \phi_2(z), \ldots, \phi_r(z)$ are principal parts of $f(z)$ at the poles$z_1, z_2, \ldots, z_r$ and $\psi(z)$ is the principal part of $f(z)$ at ∞, then $f(z) - \sum_{j=1}^r \phi_j(z) - \psi(z)$ being bounded and analytic in $\mathbb{C} \cup \infty$ is constant $\Rightarrow f(z) = \sum_{j=1}^r \phi_j(z) + \psi(z) + C$. Thus $f(z)$ is a rational function, as each $\phi_j(z)$ is a rational function and $\psi(z)$ is a polynomial. ∎

Question 2(c) *What kind of singularities do the following functions have?*

1. $\dfrac{1}{1 - e^z}$ *at* $z = 2\pi i$.
2. $\dfrac{1}{\sin z - \cos z}$ *at* $z = \frac{\pi}{4}$.

3. $\dfrac{\cot \pi z}{(z-a)^2}$ *at $z=a$ and $z=\infty$. What happens when a is an integer (including $a=0$)?*

Solution.

1. Clearly $e^z - 1 = e^{z-2\pi i} - 1 = (z-2\pi i) + \dfrac{(z-2\pi i)^2}{2!} + \dfrac{(z-2\pi i)^3}{3!} + \ldots$, showing that $e^z - 1$ has a simple zero at $z = 2\pi i$. Thus the given function $\frac{1}{1-e^z}$ has a simple pole at $z = 2\pi i$. Now residue at $z = 2\pi i$ is given by

$$\lim_{z\to 2\pi i} \frac{z-2\pi i}{1-e^z} = -1$$

2. $f(z) = \dfrac{1}{\sin z - \cos z}$. We know that

$$\begin{aligned}
\sin z &= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}} + \ldots + \text{Higher powers of } \left(z - \frac{\pi}{4}\right) \\
\cos z &= \frac{1}{\sqrt{2}} - \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\frac{1}{\sqrt{2}} + \ldots + \text{Higher powers of } \left(z - \frac{\pi}{4}\right) \\
\Rightarrow \sin z - \cos z &= \sqrt{2}\left(z - \frac{\pi}{4}\right) - \sqrt{2}\frac{\left(z-\frac{\pi}{4}\right)^3}{3!} + \ldots + \text{Higher powers of } \left(z - \frac{\pi}{4}\right)
\end{aligned}$$

Since $\sin z - \cos z$ has a simple zero at $z = \dfrac{\pi}{4}$, the given function $\dfrac{1}{\sin z - \cos z}$ has a simple pole at $z = \dfrac{\pi}{4}$.

Residue at $z = \dfrac{\pi}{4}$ is given by $\displaystyle\lim_{z\to\frac{\pi}{4}} \frac{z-\frac{\pi}{4}}{\sin z - \cos z} = \frac{1}{\sqrt{2}}$.

3. $f(z) = \dfrac{\cot \pi z}{(z-a)^2}$. $f(z)$ has a simple pole at each $z = n, n \in \mathbb{Z}, n \neq a$, with residue $\frac{1}{(n-a)^2}$. $f(z)$ also has a pole at $z = a$, whose nature is as follows:

   (a) a is not an integer and $a \neq n + \frac{1}{2}$.
   In this case, $\cos \pi a \neq 0, \sin \pi a \neq 0$ and therefore $f(z)$ has a double pole at $z = a$. (The residue at $z = a$ is $\dfrac{d}{dz}[(z-a)^2 f(z)]_{z=a} = -\pi \csc^2 \pi a$.)

   (b) a is not an integer and $a = n + \frac{1}{2}$.
   In this case $\cos \pi z$ has a simple zero at a, and $\sin \pi z = \pm 1$, therefore $f(z)$ has a simple pole at $z = a$. (The residue at $z = a$ is $\displaystyle\lim_{z\to a} \frac{\cos \pi z}{z-a}\frac{1}{\sin \pi a} = \frac{-\pi \sin \pi a}{\sin \pi a} = -\pi$.)

(c) a is an integer.

$\sin \pi z$ has a simple zero at $z = a$ and $\cos \pi a \neq 0$, then $f(z)$ has a **triple pole** at $z = a$. The residue in this case is $-\frac{\pi}{3}$, because

$$
\begin{aligned}
\sin \pi z &= (-1)^a \left[\pi(z-a) - \pi^3 \frac{(z-a)^3}{3!} + \text{Higher powers of}(z-a) \right] \\
\cos \pi z &= (-1)^a \left[1 - \pi^2 \frac{(z-a)^2}{2!} + \text{Higher powers of}(z-a) \right] \\
f(z) &= \frac{1}{(z-a)^2} \, \frac{1 - \pi^2 \frac{(z-a)^2}{2!} + \text{Higher powers of}(z-a)}{\pi(z-a)[1 - \pi^2 \frac{(z-a)^2}{3!} + \text{Higher powers of}(z-a)]} \\
&= \frac{1}{\pi(z-a)^3} \left[1 - \pi^2 \frac{(z-a)^2}{2!} + \ldots \right] \left[1 + \pi^2 \frac{(z-a)^2}{3!} + \ldots \right]
\end{aligned}
$$

The coefficient of $\frac{1}{z-a}$ in the Laurent series of $f(z)$ (formed by multiplying the above series) is $\frac{1}{\pi} \left[-\frac{\pi^2}{2} + \frac{\pi^2}{6} \right] = -\frac{\pi}{3}$, which is the required residue.

(Note that the computation of residues was not required for this problem.)

Finally, $f(z)$ has an essential singularity at ∞, because $f(z)$ has zeros at $z = n + \frac{1}{2}, a \neq n + \frac{1}{2}$ whose limit point is ∞.

■

UPSC Civil Services Main 2000 - Mathematics Complex Analysis

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Question 1(a) *Show that any four given points of the complex plane can be carried by a bilinear transformation to positions $1, -1, k, -k$ where the value of k depends on the given points.*

Solution. It is known that a bilinear transformation mapping z_1, z_2, z_3 to w_1, w_2, w_3 is given by the crossratio(z_1, z_2, z_3, z) = crossratio(w_1, w_2, w_3, w), i.e.

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)}$$

Now $w_1 = 1, w_2 = -1, w_3 = k$, so

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - 1)(-1 - k)}{(2)(k - w)}$$

It will map z_4 to $-k$ provided k is given by

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(-k - 1)(-1 - k)}{(2)(k - (-k)} = \frac{(k + 1)^2}{4k}$$

Clearly k depends on the points z_1, z_2, z_3, z_4. ∎

Question 2(a) *Suppose $f(\zeta)$ is continuous on a circle C. Show that $\displaystyle\int_C \frac{f(s)\,ds}{s - z}$ as z varies inside C is differentiable under the integral sign. Find the derivative. Hence or otherwise derive as integral representation for $f'(z)$ if $f(z)$ is analytic on and inside of C.*

Solution. If $f(z)$ is analytic on and inside C, then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\,ds}{s-z}$$

Let h be a complex number so chosen that $z+h$ also lies in the interior of C. Then

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi i}\left[\frac{1}{h}\int_C \Big[\frac{f(s)}{s-z-h} - \frac{f(s)}{s-z}\Big] ds\right] \\
&= \frac{1}{2\pi i}\left[\frac{1}{h}\int_C \frac{hf(s)}{(s-z)(s-z-h)}\,ds\right] \\
\Longrightarrow \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds &= \frac{1}{2\pi i}\int_C \Big[\frac{f(s)}{(s-z-h)(s-z)} - \frac{f(s)}{(s-z)^2}\Big] ds \\
&= \frac{1}{2\pi i}\int_C \frac{hf(s)}{(s-z-h)(s-z)^2} ds
\end{aligned}$$

Let $M = \sup_{z\in C} |f(z)|$, $l =$ arc length of C, $d = \min_{s\in C} |z-s| > 0$. Since we are interested in $h \to 0$, we can assume that $0 < |h| < d$. Now $|s-z| \geq d$, $|s-z-h| \geq |s-z| - |h| \geq d - |h|$, and therefore

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds\right| \leq \frac{M}{2\pi}\frac{|h|}{d^2(d-|h|)} \cdot l$$

Since the right hand side of the above inequality $\to 0$ as $h \to 0$, it follows that

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds$$

i.e.

$$\frac{d(f(z))}{dz} = f'(z) = \frac{1}{2\pi i}\int_C \frac{d}{dz}\Big(\frac{f(s)}{s-z}\Big)\,ds$$

or

$$\frac{d}{dz}\frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{s-z} = \frac{1}{2\pi i}\int_C \frac{d}{dz}\Big(\frac{f(s)}{s-z}\Big)\,ds$$

i.e. differentiation under the integral sign is valid. The representation for $f'(z)$ is given by

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^2}$$

∎

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Question 1(a) *Prove that the Riemann zeta function ζ defined by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$ and converges uniformly for $\operatorname{Re} z > 1 + \epsilon$ where ϵ is arbitrarily small.*

Solution.

$$\left|\frac{1}{n^z}\right| = \left|\frac{1}{n^x \cdot n^{iy}}\right| = \left|\frac{1}{n^x}\right| \qquad \because \left|\frac{1}{n^{iy}}\right| = \left|\frac{1}{e^{iy\log n}}\right| = 1$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges for $x > 1$, it follows that $\sum_{n=1}^{\infty} n^{-z}$ converges absolutely for $\operatorname{Re} z > 1$.

If $\operatorname{Re} z \geq 1 + \epsilon$, then $\frac{1}{n^x} \leq \frac{1}{n^{1+\epsilon}}$ and

$$\sum_{n=1}^{\infty} |n^{-z}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$$

for $\operatorname{Re} z \geq 1 + \epsilon$. Weierstrass' M-test gives that the given series converges uniformly and absolutely for $\operatorname{Re} z \geq 1 + \epsilon$. ∎

Question 2(a) *Find the Laurent series for the function $e^{\frac{1}{z}}$ in $0 < |z| < \infty$. Using the expansion show that*

$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

$n = 1, 2, \ldots$.

Solution. Clearly $e^{\frac{1}{z}}$ is analytic in $0 < |z| < \infty$ and satisfies requirements of Laurent's expansion, and we have

$$e^{\frac{1}{z}} = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^{\frac{1}{z}}}{z^{n+1}}\,dz \qquad (*)$$

Note — $z = 0$ is an essential singularity, therefore we have infinitely many terms with negative exponents. In the expression for a_n we could have taken any disc, we have taken $|z| = 1$ for convenience.

Put $z = e^{i\theta}$ in $(*)$, $dz = ie^{i\theta}\, d\theta$, we get

$$\begin{aligned}
a_n &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\cos\theta - i\sin\theta}}{e^{i(n+1)\theta}}\, ie^{i\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} e^{-i\sin\theta - in\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\, d\theta
\end{aligned}$$

Let $g(\theta) = e^{\cos\theta}[\sin(\sin\theta + n\theta)]$, then $g(2\pi - \theta) = -e^{\cos\theta}[\sin(\sin\theta + n\theta)] = -g(\theta)$. Thus $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\, d\theta = 0$.

Thus $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\, d\theta$.

In particular, $a_{-n} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\, d\theta$ for $n = 1, 2, \ldots$.

But we know that $e^{\frac{1}{z}} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!z^n}$.

Therefore, comparing the two expansions we get for $n = 1, 2, \ldots$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\, d\theta = \frac{1}{n!}$$

Since $e^{\cos 2\pi - \theta}\cos(\sin(2\pi - \theta) - n(2\pi - \theta)) = e^{\cos\theta}[\cos(\sin\theta - n\theta)]$, we can double the integral and halve the limit to obtain

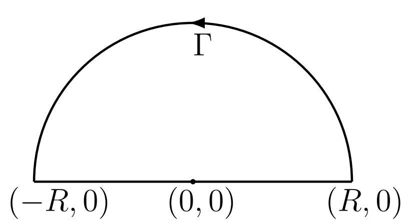
$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}$$

∎

Question 2(b) *Show that* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}$.

Solution.

We take $f(z) = \frac{1}{1+z^4}$ and the contour γ consisting of Γ a semicircle of radius R with center $(0, 0)$ lying in the upper half plane, and the line joining $(-R, 0)$ and $(R, 0)$.



By Cauchy's residue theorem $\displaystyle\int_\gamma \frac{dz}{1+z^4} = 2\pi i$(sum of residues at poles of $f(z)$ in the upper half plane).

Clearly $\frac{1}{1+z^4}$ has two simple poles at $z = e^{\frac{\pi i}{4}}$ and $z = e^{\frac{3\pi i}{4}}$ inside the contour.

Residue at $z = e^{\frac{\pi i}{4}}$ is $\dfrac{1}{\frac{d(z^4+1)}{dz}} = \dfrac{1}{4e^{\frac{3\pi i}{4}}}$.

Residue at $z = e^{\frac{3\pi i}{4}}$ is $\dfrac{1}{4e^{\frac{9\pi i}{4}}} = \dfrac{1}{4e^{\frac{\pi i}{4}}}$.

$$\begin{aligned}
\text{Sum of residues} &= \frac{1}{4}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] \\
&= \frac{1}{4}\left[-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] \\
&= -\frac{i}{4}\frac{2}{\sqrt{2}} = -\frac{i}{2\sqrt{2}}
\end{aligned}$$

Thus $\displaystyle\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^4} = 2\pi i\frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$.

Now

$$\left|\int_\Gamma \frac{dz}{1+z^4}\right| \le \int_0^\pi \frac{R}{R^4-1}\,d\theta = \frac{\pi R}{R^4-1}$$

on putting $z = Re^{i\theta}$ and using $|z^4+1| \ge R^4 - 1$ on Γ.

Thus $\displaystyle\int_\Gamma \frac{dz}{1+z^4} \to 0$ as $R \to \infty$. Consequently,

$$\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^4} = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

as required. ∎

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Question 1(a) *Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f(\frac{1}{n}) = g(\frac{1}{n})$ for $n = 1, 2, 3, \ldots$, then show that $f(z) = g(z)$ for all $z \in \mathbb{C}$.*

Solution. Let $G(z) = f(z) - g(z)$, then $G(\frac{1}{n}) = 0$ for $n = 1, 2, \ldots$. We shall show that $G(z) \equiv 0$ for $z \in \mathbb{C}$ which would prove the result.

Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series of $G(z)$ with center 0 and radius of convergence R, clearly $R > 0$. We shall now prove that $a_n = 0$ for every n.

If $a_n \neq 0$ for some n, let a_k be the first non-zero coefficient. Then

$$G(z) = z^k(a_k + a_{k+1}z + \ldots) = z^k H(z)$$

Clearly $H(z)$ is analytic in $|z| < R$, and $H(0) \neq 0$. We now claim that $H(z) \neq 0$ in a neighborhood $|z| < \delta$ of 0. Let $\epsilon = \frac{|H(0)|}{2}$, then continuity of $H(z)$ at $z = 0$ implies that there exists a $\delta > 0$ such that $|z| < \delta \Rightarrow |H(z) - H(0)| < \epsilon$ or $|H(0)| - \epsilon < |H(z)| < |H(0)| + \epsilon$ for $|z| < \delta$. Thus $|H(z)| > \frac{|H(0)|}{2} > 0$ for $|z| < \delta$. Consequently, $G(z) \neq 0$ for any z in $0 < |z| < \delta$. But this is not possible, as $|z| < \delta$ contains all but finitely many $\frac{1}{n}$, at which $G(z)$ vanishes. Thus our assumption that $a_n \neq 0$ for some n is false, thus $G(z) \equiv 0$ in $|z| < R$.

Let z' be any point in $\mathbb{C}$, and let $r(t), a \leq t \leq b$ be a continuous curve joining 0 and z'. Using uniform continuity of $r(t)$, we get a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ such that $r(t_0) = 0, r(t_1) = z_1, \ldots, r(t_n) = r(b) = z'$, and $|z_j - z_{j-1}| < R$.

Now the disc $K_0 = |z - 0| < R$ contains z_1, the center of disc $K_1 = |z - z_1| < R$. Since $G(z_1) = 0$ as $z_1 \in K_0 \cap K_1$, and $K_0 \cap K_1$ contains a sequence of points y_n such that $y_n \to z_1$ and $G(y_n) = 0$, we can prove as before that $G(z) \equiv 0$ in K_1. Proceeding in this way, in n steps we get $G(z) \equiv 0$ in K_n, or $G(z') = 0$. Since z' is an arbitrary point of $\mathbb{C}$, we get $G(z) \equiv 0$ in $\mathbb{C}$. ∎

Question 2(a) *Show that when* $0 < |z-1| < 2$, *the function* $f(z) = \dfrac{z}{(z-1)(z-3)}$ *has the Laurent series expansion in powers of* $(z-1)$ *as*

$$\frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$$

Solution. Let $\zeta = z - 1$, so that

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{\zeta+1}{\zeta(\zeta-2)} = -\frac{1}{2\zeta} + \frac{3}{2(\zeta-2)}$$

Now for $0 < |\zeta| < 2$, $\dfrac{3}{2(\zeta-2)} = \dfrac{3}{2} \cdot \dfrac{-1}{2} \cdot \left(1 - \dfrac{\zeta}{2}\right)^{-1}$ and $\left|\dfrac{\zeta}{2}\right| < 1$. Consequently,

$$\frac{3}{2(\zeta-2)} = -\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^n$$

and

$$f(z) = -\frac{1}{2\zeta} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{\zeta}{2}\right)^n = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

which is the desired Laurent series expansion. ∎

Question 2(b) *Establish by contour integration*

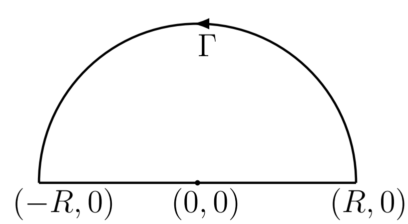
$$\int_0^{\infty} \frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}, \text{ where } a \geq 0$$

Solution. Let I be the given integral. Put $ax = t$, so that

$$I = \int_0^{\infty} \frac{\cos t}{\frac{t^2}{a^2}+1}\,\frac{dt}{a} = a\int_0^{\infty}\frac{\cos t}{t^2+a^2}\,dt$$

We shall now prove that $\displaystyle\int_0^{\infty}\frac{\cos t}{t^2+a^2}\,dt = \frac{\pi}{2a}e^{-a}$, which will show that $I = \frac{\pi}{2}e^{-a}$ as required.

Clearly $\dfrac{\cos x}{x^2+a^2}$ is the real part of $\dfrac{e^{ix}}{x^2+a^2}$. We consider the integral $\int_\gamma f(z)\,dz$ where $f(z) = \dfrac{e^{iz}}{z^2+a^2}$ and γ is the contour consisting of the line joining $(-R,0)$ and $(R,0)$ and Γ, which is the arc of the circle of radius R and center $(0,0)$ lying in the upper half plane.



Clearly on Γ, if we put $z = Re^{i\theta}$, then $0 \leq \theta \leq \pi$ and

$$\left|\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2}\right| = \left|\int_0^\pi \frac{Rie^{i\theta}e^{iRe^{i\theta}}\,d\theta}{R^2e^{2i\theta}+a^2}\right| \leq \int_0^\pi \left|\frac{Rie^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2}\right| d\theta$$

But $|e^{iRe^{i\theta}}| = |e^{iR\cos\theta}e^{-R\sin\theta}| = e^{-R\sin\theta} \leq 1$ as $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$. $|z^2+a^2| \geq |z|^2 - a^2 = R^2 - a^2$. Therefore

$$\left|\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2}\right| \leq \int_0^\pi \frac{R}{R^2-a^2}\,d\theta = \frac{\pi R}{R^2-a^2}$$

Hence $\displaystyle\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2} \to 0$ as $R \to \infty$.

Now $\displaystyle\int_\gamma \frac{e^{iz}\,dz}{z^2+a^2} = 2\pi i$(sum of residues at poles inside γ).

But the only pole in the upper half plane is $z = ia, (a > 0)$ and the residue at $z = ia$ is $\dfrac{e^{i(ia)}}{2ia} = \dfrac{e^{-a}}{2ia}$. Thus

$$\begin{aligned}
\lim_{R\to\infty}\int_\gamma \frac{e^{iz}\,dz}{z^2+a^2} &= \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+a^2} = 2\pi i \cdot \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a} \\
\Longrightarrow \int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2+a^2} &= \frac{\pi e^{-a}}{a}, \int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+a^2} = 0 \\
\Longrightarrow \int_0^{\infty} \frac{\cos x\,dx}{x^2+a^2} &= \frac{\pi e^{-a}}{2a} \qquad \because \cos x = \cos(-x)
\end{aligned}$$

This completes the proof. ∎

UPSC Civil Services Main 2003 - Mathematics Complex Analysis

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Question 1(a) *Determine all the bilinear transformations which transform the unit circle $|z| \leq 1$ into the unit circle $|w| \leq 1$.*

Solution. Let the required transformation be $w = \dfrac{az+b}{cz+d}$. Clearly $z = -\frac{b}{a} \Rightarrow w = 0$ and $z = -\frac{d}{c} \Rightarrow w = \infty$. Since $0, \infty$ are inverse points with respect to the circle $|w| = 1$, then $-\frac{b}{a}, -\frac{d}{c}$ are inverse points with respect to the circle $|z| = 1$ (note that R, S different from 0 are said to be inverse points with respect to $|z| = 1$ if O, R, S are collinear and $OR \cdot OS = 1$). Thus if we set $-\frac{b}{a} = \alpha$, then $-\frac{d}{c} = \frac{1}{\overline{\alpha}}$ and we get

$$w = \frac{a}{c}\, \frac{z-\alpha}{z - \frac{1}{\overline{\alpha}}} = \frac{a\overline{\alpha}}{c}\, \frac{z-\alpha}{\overline{\alpha}z - 1}$$

Since $|z| = 1$ maps onto $|w| = 1$, we take $z = 1$ to get $\left|\dfrac{a\overline{\alpha}}{c}\, \dfrac{1-\alpha}{\overline{\alpha}-1}\right| = 1$. But $|1-\alpha| = |1-\overline{\alpha}|$, therefore $\left|\dfrac{a\overline{\alpha}}{c}\right| = 1$. Let $\dfrac{a\overline{\alpha}}{c} = e^{i\theta}, \theta \in \mathbb{R}$, so that

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z - 1}$$

We now check that when $|z| = 1$, we have $|w| = 1$.

$$\begin{aligned} |w| &= |e^{i\theta}|\left|\frac{z-\alpha}{\overline{\alpha}z-1}\right| \\ &= |\overline{z}|\left|\frac{z-\alpha}{\overline{\alpha}-\overline{z}}\right| \quad (\because z\overline{z} = 1) \\ &= 1 \quad (\because |z-\alpha| = |\overline{\alpha}-\overline{z}|) \end{aligned}$$

Now let $|z| < 1$. Then

$$\begin{aligned}
w\overline{w} - 1 &= e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1} \cdot e^{-i\theta}\frac{\overline{z}-\overline{\alpha}}{\alpha\overline{z}-1} - 1 \\
&= \frac{z\overline{z} - \overline{\alpha}z - \alpha\overline{z} + \alpha\overline{\alpha}}{(\overline{\alpha}z-1)(\alpha\overline{z}-1)} - 1 \\
&= \frac{z\overline{z} - \overline{\alpha}z - \alpha\overline{z} + \alpha\overline{\alpha} - \alpha\overline{\alpha}z\overline{z} + \overline{\alpha}z + \alpha\overline{z} - 1}{(\overline{\alpha}z-1)(\alpha\overline{z}-1)} \\
&= \frac{z\overline{z} + \alpha\overline{\alpha} - \alpha\overline{\alpha}z\overline{z} - 1}{|\overline{\alpha}z-1|^2} \\
&= \frac{(z\overline{z}-1)(1-\alpha\overline{\alpha})}{|\overline{\alpha}z-1|^2}
\end{aligned}$$

Thus if $|\alpha| < 1$, then $|w| < 1$. This shows that the transformation

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}, \theta \in \mathbb{R}, |\alpha| < 1$$

maps the interior of $|z| = 1$ onto the interior of $|w| = 1$ and the boundary of $|z| = 1$ onto the boundary of $|w| = 1$. Thus all bilinear transforms which map $|z| \leq 1$ onto $|w| \leq 1$ are given by

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}, \theta \in \mathbb{R}, |\alpha| < 1$$

Note: If $|\alpha| > 1$, then the interior of $|z| = 1$ would map onto the exterior of $|w| = 1$. The boundary will map onto the boundary, as before. ∎

Question 2(a)

1. *Discuss the transformation* $W = \left(\frac{z-ic}{z+ic}\right)^2$, *c real, showing that the upper half of the* W*-plane corresponds to the interior of a semicircle lying to the right of the imaginary axis in the* z*-plane.*

2. *Using the method of contour integration prove that*

$$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}} \qquad (a > 0)$$

Solution.

1. We need to assume $c > 0$ as otherwise the question is incorrect.

Let $W = U + iV$, so that

$$\begin{aligned}
U + iV &= \left(\frac{x + i(y-c)}{x + i(y+c)}\right)^2 \\
&= \left(\frac{(x + i(y-c))(x - i(y+c))}{x^2 + (y+c)^2}\right)^2 \\
&= \left(\frac{x^2 + y^2 - c^2 - 2icx}{x^2 + (y+c)^2}\right)^2 \\
&= \frac{(x^2 + y^2 - c^2)^2 - 4c^2x^2 - 4icx(x^2 + y^2 - c^2)}{[x^2 + (y+c)^2]^2} \\
\Rightarrow U &= \frac{(x^2 + y^2 - c^2)^2 - 4c^2x^2}{[x^2 + (y+c)^2]^2} \\
V &= \frac{-4cx(x^2 + y^2 - c^2)}{[x^2 + (y+c)^2]^2} = \frac{4cx(c^2 - x^2 - y^2)}{[x^2 + (y+c)^2]^2}
\end{aligned}$$

Thus if z belongs to the interior of the semicircle given by $x^2 + y^2 = c^2, x \geq 0$, then $V > 0$, which means that $U + iV$ is in the upper half plane.

For any point on the line $x = 0$, we have $V = 0$ and $U = \dfrac{(y^2 - c^2)^2}{(y+c)^4} = \left(\dfrac{y-c}{y+c}\right)^2$. Clearly when y changes from $-c$ to c, U changes from ∞ to 0.

As z moves over the circle $x^2 + y^2 = c^2$, we have $V = 0$ and

$$U = \frac{-4c^2x^2}{(x^2 + (y+c)^2)^2} = \frac{-4c^2x^2}{(x^2 + y^2 + c^2 + 2yc)^2} = \frac{-4c^2x^2}{(2c^2 + 2yc)^2} = \frac{-x^2}{(y+c)^2} = -\frac{c^2 - y^2}{(y+c)^2} = -\frac{c-y}{c+y}$$

Let $y = c\cos\theta$, then $U = -\dfrac{1 - \cos\theta}{1 + \cos\theta} = -\tan^2\frac{\theta}{2}$. When y moves from $-c$ to c, i.e.z traverses the boundary of the semicircle, θ varies from π to 0, and U varies from $-\infty$ to 0. Thus the boundary of the semicircle $x^2 + y^2 = c^2$ with $x \geq 0$ is mapped onto the U-axis. Hence the semicircle $x^2 + y^2 = c^2$ with $x \geq 0$ is mapped onto $W = U + iV$ with $V \geq 0$.

2. Let the given integral be I. Then

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \int_0^{\pi} \frac{2a\,d\theta}{2a^2 + (1 - \cos 2\theta)} = \int_0^{2\pi} \frac{a\,d\phi}{2a^2 + 1 - \cos\phi}$$

on putting $2\theta = \phi$. We now let $z = e^{i\phi}$ to obtain

$$I = \int\limits_{|z|=1} \frac{a\,dz}{iz(2a^2 + 1 - \frac{1}{2}(z + \frac{1}{z}))} = \frac{1}{i}\int\limits_{|z|=1} \frac{2a\,dz}{2(2a^2+1)z - (z^2+1)} = \int\limits_{|z|=1} \frac{2ai\,dz}{z^2 - 2(2a^2+1)z + 1}$$

Now $z^2 - 2(2a^2+1)z + 1 = 0 \Rightarrow z = 2a^2 + 1 \pm \sqrt{(2a^2+1)^2 - 1} = 2a^2 + 1 \pm 2a\sqrt{a^2+1}$.

Clearly $|2a^2 + 1 + 2a\sqrt{a^2+1}| > 1$ showing that $|2a^2 + 1 - 2a\sqrt{a^2+1}| < 1$ because the product of the roots is 1. Thus the only pole inside $|z| = 1$ is $z = 2a^2 + 1 - 2a\sqrt{a^2+1}$.

Residue at $z = 2a^2+1-2a\sqrt{a^2+1}$ is $\dfrac{1}{(2a^2+1-2a\sqrt{a^2+1}) - (2a^2+1+2a\sqrt{a^2+1})} = \dfrac{1}{-4a\sqrt{a^2+1}}$.

Thus $I = 2ai \cdot 2\pi i \cdot \dfrac{1}{-4a\sqrt{a^2+1}} = \dfrac{\pi}{\sqrt{a^2+1}}$.

∎

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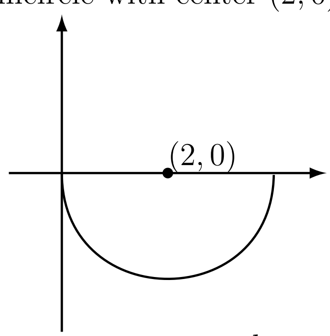
Question 1(a) *Find the image of the line $y = x$ under the mapping $w = \dfrac{4}{z^2+1}$ and draw it. Find the points where this transformation ceases to be conformal.*

Solution. Let $z = x + iy$. Then

$$\begin{aligned} w &= \frac{4}{z^2+1} = \frac{4}{x^2-y^2+1+2ixy} \\ &= \frac{4}{(x^2-y^2+1)^2+4x^2y^2}[x^2-y^2+1-2ixy] \end{aligned}$$

So if $x = y$, $w = \dfrac{4-8ix^2}{1+4x^4}$. Let $u = \dfrac{4}{1+4x^4}, v = \dfrac{-8x^2}{1+4x^4} \Rightarrow u^2+v^2 = 16\dfrac{1}{1+4x^4} = 4u \Rightarrow (u-2)^2+v^2 = 4, v \leq 0$. So the image of the line $x = y$ under the mapping $w = \dfrac{1}{z^2+1}$ is a semicircle with center $(2,0)$, radius 2 and below the x-axis.



Conformality: $\dfrac{dw}{dz} = -\dfrac{8z}{(z^2+1)^2}$ when $z \neq \pm i$. Clearly $\dfrac{dw}{dz} \neq 0$ when $z \neq 0$. Thus the mapping is conformal at all points which are different from $z = 0, \pm i$ (as $\dfrac{dw}{dz}$ does not exist at $\pm i$). ∎

Question 2(a) *If all zeros of a polynomial $P(z)$ lie in a half plane, then show that zeros of the derivative $P'(z)$ also lie in the same half plane.*

Solution. We can assume without loss of generality that the zeros of $P(z)$ lie in the half plane $\operatorname{Re} z < 0$. Let $P(z) = \prod_{j=1}^{n}(z - \alpha_j)$ where $\alpha_j = x_j + iy_j, x_j < 0$.

If $\operatorname{Re} z \geq 0$, then $P(z) \neq 0$ and

$$\begin{aligned} \frac{P'(z)}{P(z)} &= \sum_{j=1}^{n} \frac{1}{z - \alpha_j} \\ &= \sum_{j=1}^{n} \frac{1}{x - x_j + i(y - y_j)} \\ &= \sum_{j=1}^{n} \frac{x - x_j - i(y - y_j)}{(x - x_j)^2 + (y - y_j)^2} \end{aligned}$$

Since $x_j < 0, 1 \leq j \leq n$, it follows that

$$\operatorname{Re}\left(\frac{P'(z)}{P(z)}\right) = \sum_{j=1}^{n} \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2} > 0$$

whenever $\operatorname{Re} z = x \geq 0$. Thus $\dfrac{P'(z)}{P(z)}$ and therefore $P'(z)$ has no zeros in the right half plane $\operatorname{Re} z \geq 0$. Hence all zeros of $P'(z)$ lie in the same half plane in which the zeros of $P(z)$ lie. ∎

Question 2(b) *Using Contour integration, evaluate*

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta, \quad 0 < p < 1$$

Solution. Clearly

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta$$

The integrand is the real part of $\dfrac{1 + e^{i6\theta}}{1 + 2p\cos 2\theta + p^2}$. Put $z = e^{i\theta}$, $dz = ie^{i\theta}\, d\theta$ or $d\theta = \dfrac{dz}{iz}$.

$$\begin{aligned} \int_0^{2\pi} \frac{1 + e^{i6\theta}}{1 + 2p\cos 2\theta + p^2}\, d\theta &= \frac{1}{i}\int_{|z|=1} \frac{1 + z^6}{1 - p(z^2 + \frac{1}{z^2}) + p^2}\, \frac{dz}{z} \\ &= \frac{1}{i}\int_{|z|=1} \frac{z(1 + z^6)}{-pz^4 + z^2(1 + p^2) - p}\, dz \\ &= \frac{1}{i}\int_{|z|=1} \frac{z(1 + z^6)}{(1 - pz^2)(z^2 - p)}\, dz \end{aligned}$$

Now the integrand has simple poles at $z = \pm\sqrt{p}, \pm\frac{1}{\sqrt{p}}$. Since $0 < p < 1$, the only poles inside $|z| = 1$ are $z = \pm\sqrt{p}$. The residue at $z = \sqrt{p}$ is

$$\lim_{z\to\sqrt{p}} \frac{(z-\sqrt{p})z(1+z^6)}{(1-pz^2)(z^2-p)} = \frac{\sqrt{p}(1+p^3)}{(1-p^2)2\sqrt{p}} = \frac{1+p^3}{2(1-p^2)}$$

Similarly residue at $z = -\sqrt{p}$ is

$$\lim_{z\to-\sqrt{p}} \frac{(z+\sqrt{p})z(1+z^6)}{(1-pz^2)(z^2-p)} = \frac{-\sqrt{p}(1+p^3)}{(1-p^2)(-2\sqrt{p})} = \frac{1+p^3}{2(1-p^2)}$$

Thus

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\,d\theta &= \frac{1}{2}\,\mathrm{Re}\left(\int_0^{2\pi} \frac{1+e^{i6\theta}}{1+2p\cos 2\theta+p^2}\,d\theta\right) \\
&= \mathrm{Re}\left(\frac{1}{2i}2\pi i[\text{Sum of residues at } z=\pm\sqrt{p}]\right) \\
&= \pi\frac{1+p^3}{1-p^2} = \pi\frac{1-p+p^2}{1-p}
\end{aligned}$$

∎

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Paper II

Question 1(a) *If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, determine $f(z)$ in terms of z.*

Solution. Let $F(z) = (1+i)f(z) = (1+i)(u+iv) = (u-v) + i(u+v)$. Now

$$\frac{\partial^2(u-v)}{\partial x^2} + \frac{\partial^2(u-v)}{\partial y^2} = e^x(\cos y - \sin y) + e^x(-\cos y + \sin y) = 0$$

Let $F(z) = U + iV$, where $U = u - v$ is harmonic. If $f(z)$ is analytic, then so is $F(z)$ and

$$\frac{dF}{dz} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

as $\dfrac{\partial V}{\partial x} = -\dfrac{\partial U}{\partial y}$ by the Cauchy-Riemann equations. Thus

$$\begin{aligned} F'(z) &= e^x(\cos y - \sin y) - ie^x(-\sin y - \cos y) \\ &= e^x(\cos y + i\sin y) + ie^x(\cos y + i\sin y) \\ &= (1+i)e^x \cdot e^{iy} = (1+i)e^z \end{aligned}$$

Thus $F(z) = (1+i)e^z$ and $f(z) = e^z$, which is the required function. ∎

Question 2(a) *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in Laurent's series which is valid for (i)* $1 < |z| < 3$ *(ii)* $|z| > 3$ *and (iii)* $|z| < 1$.

Solution. Clearly $f(z) = \dfrac{1}{2}\Big(\dfrac{1}{z+1} - \dfrac{1}{z+3}\Big)$

(i) $1 < |z| < 3$. In this region

$$f(z) = \frac{1}{2}\Big[\frac{1}{z}\Big(1+\frac{1}{z}\Big)^{-1} - \frac{1}{3}\Big(1+\frac{z}{3}\Big)^{-1}\Big]$$

Since $|\frac{1}{z}| < 1$ and $|\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z}{3}\Big)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{z^n}{3^{n+1}} \end{aligned}$$

as Laurent's expansion in the region $1 < |z| < 3$.

(ii) $|z| > 3$. In this region

$$f(z) = \frac{1}{2z}\Big[\Big(1+\frac{1}{z}\Big)^{-1} - \Big(1+\frac{3}{z}\Big)^{-1}\Big]$$

Now $|\frac{1}{z}| < 1$ and $|\frac{3}{z}| < 1$, so we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n} - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{1-3^n}{z^{n+1}} \end{aligned}$$

as Laurent's expansion in the region $|z| > 3$. This is Taylor's expansion of $f(z)$ around ∞.

(iii) $|z| < 1$. In this region

$$f(z) = \frac{1}{2}\Big[(1+z)^{-1} - \frac{1}{3}\Big(1+\frac{z}{3}\Big)^{-1}\Big]$$

Now $|z| < 1, |\frac{z}{3}| < 1$, so we get

$$\begin{aligned} f(z) &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\Big(1-\frac{1}{3^{n+1}}\Big)z^n \end{aligned}$$

as Laurent's expansion valid in $|z| < 1$. This has no negative powers of z as $f(z)$ is analytic in $|z| < 1$. ∎

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Question 1(a) *Determine all the bilinear transformations which map the half plane* $\mathrm{Im}(z) \geq 0$ *into the unit circle* $|w| \leq 1$.

Solution. Since the x-axis is to be mapped onto the circle $|w| = 1$, we determine conditions on a, b, c, d where

$$w = \frac{az+b}{cz+d}$$

is the desired bilinear transformation, such that points $z = 0, z = 1, z = \infty$ are mapped onto points with modulus 1.

First of all, $c \neq 0$, because if $c = 0$ then the image of $z = \infty$ would be $w = \infty$, which is not possible.

Clearly $z = \infty$ is mapped onto $w = \frac{a}{c}$, thus we must have $|\frac{a}{c}| = 1$ or $|a| = |c| \neq 0$.

When $z = 0$, $w = \frac{b}{d}$ (note that $d \neq 0$, because otherwise $z = 0$ would be mapped onto ∞), thus $|w| = 1$ gives us $|b| = |d| \neq 0$.

Since $a \neq 0, c \neq 0$, we can write $w = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\left(\dfrac{z+\frac{b}{a}}{z+\frac{d}{c}}\right) = e^{i\alpha}\dfrac{z-z_0}{z-z_1}$ with $e^{i\alpha} = \dfrac{a}{c}, z_0 = -\dfrac{b}{a}, z_1 = -\dfrac{d}{c}$. But $\left|\dfrac{b}{a}\right| = \left|\dfrac{d}{c}\right|$, so $|z_0| = |z_1|$. Thus we have proved that w can be written in the form

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-z_1}\right) \text{ with } \alpha \in \mathbb{R}, |z_0| = |z_1|$$

We now use the fact that the image of $z = 1$ has modulus 1, and get

$$\begin{aligned}
|1 - z_0| &= |1 - z_1| \\
\Rightarrow (1 - z_0)(1 - \overline{z_0}) &= (1 - z_1)(1 - \overline{z_1}) \\
\Rightarrow 1 - z_0 - \overline{z_0} + z_0\overline{z_0} &= 1 - z_1 - \overline{z_1} + z_1\overline{z_1} \\
\Rightarrow z_0 + \overline{z_0} &= z_1 + \overline{z_1} \quad \because |z_0| = |z_1| \Rightarrow z_0\overline{z_0} = z_1\overline{z_1} \\
\Rightarrow \mathrm{Re}(z_0) &= \mathrm{Re}(z_1)
\end{aligned}$$

This gives us $z_0 = z_1$ or $z_0 = \overline{z_1}$ because if $z_0 = x + iy_0, z_1 = x + iy_1$, then $x^2 + y_0^2 = x^2 + y_1^2 \Rightarrow y_0^2 = y_1^2 \Rightarrow y_1 = \pm y_0$. If $z_0 = z_1$, then $w = e^{i\alpha}$, a constant, which is not possible, therefore $z_1 = \overline{z_0}$ and the transformation w can be written as $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \overline{z_0}}\right)$.

This transformation maps z_0 to $w = 0$. Since we require the upper half plane $\mathrm{Im}(z) > 0$ to be mapped onto the interior of $|w| = 1$, we must have $\mathrm{Im}(z_0) > 0$. Thus any transformation which maps the real axis onto $|w| = 1$ and the region $\mathrm{Im}(z) > 0$ to the interior of $|w| = 1$ can be written in the form

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0$$

We now prove the converse — any bilinear transformation $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0$ maps $\mathrm{Im}(z) > 0$ to $|w| \leq 1$.

If z is such that $\mathrm{Im}(z) \geq 0$, then it can be seen easily that $|z - z_0| < |z - \overline{z_0}|$, therefore $|w| < 1$. Similarly if we assume that $\mathrm{Im}(z) < 0$, then $|z - z_0| > |z - \overline{z_0}|$ and therefore $|w| > 1$. Clearly when z lies on the real axis, then $|w| = 1$ as $|z - z_0| = |z - \overline{z_0}|$. This proves the result.

Hence all bilinear transformations of the required type are of the form

$$\left\{w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0\right\}$$

Alternate Solution: Since the x-axis is mapped to the unit circle $|w| = 1$, the reflection of the image of z in $|w| = 1$ is the same as the image of the reflection of z in the x-axis i.e. $\overline{z}$. Thus

$$\begin{aligned}
\overline{\left(1\Big/\frac{az + b}{cz + d}\right)} &= \frac{a\overline{z} + b}{c\overline{z} + d} \\
\Rightarrow \quad \overline{(cz + d)}(c\overline{z} + d) &= \overline{(az + b)}(a\overline{z} + b) \\
\Rightarrow \quad c\overline{c}\overline{z}^2 + (c\overline{d} + \overline{c}d)\overline{z} + d\overline{d} &= a\overline{a}\overline{z}^2 + (a\overline{b} + \overline{a}b)\overline{z} + b\overline{b}
\end{aligned}$$

Comparing coefficients of the powers of $\overline{z}$ we have $|a| = |c|, |b| = |d|, \mathrm{Re}\,(b\overline{a}) = \mathrm{Re}\,(d\overline{c})$, thus since $ad - bc \neq 0$, we have $|a| = |c| \neq 0, |b| = |d| \neq 0$, and $\mathrm{Re}\,(\frac{b}{a}) = \mathrm{Re}\,(\frac{b\overline{a}}{a\overline{a}}) = \mathrm{Re}\,(\frac{d\overline{c}}{c\overline{c}}) = \mathrm{Re}\,(\frac{d}{c})$.

Thus we can write $w = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\left(\dfrac{z+\frac{b}{a}}{z+\frac{d}{c}}\right) = e^{i\alpha}\dfrac{z-z_0}{z-z_1}$ with $e^{i\alpha} = \dfrac{a}{c}, z_0 = -\dfrac{b}{a}, z_1 = -\dfrac{d}{c}$. Thus $\operatorname{Re} z_0 = \operatorname{Re} z_1, |z_0| = |z_1|$, hence $z_0 = z_1$ or $z_0 = \overline{z_1}$. The former is not possible as it would make $ad - bc = 0$, hence $z_0 = \overline{z_1}$. For the same reason, $\operatorname{Im} z_0 \neq 0$. So we have

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right), \quad \alpha \in \mathbb{R}$$

Now in addition, since the upper half plane $\operatorname{Im}(z) > 0$ is mapped onto the interior of $|w| = 1$, and the image of z_0 is 0, and thus inside the unit circle, so z_0 is in the upper half plane, hence $\operatorname{Im}(z_0) > 0$.

Hence the required set of bilinear transformations is

$$\left\{w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \operatorname{Im}(z_0) > 0\right\}$$

The converse is proved as above. ∎

Question 2(a) *With the aid of residues, evaluate*

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta, \quad -1 < a < 1$$

Solution. Let

$$\begin{aligned}
I &= \int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta \\
&= \int_0^\pi \frac{(\cos 2\theta)(1 + 2a\cos\theta + a^2)}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta \\
&= \int_0^\pi \frac{(\cos 2\theta)(1+a^2)}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta + \int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta
\end{aligned}$$

Since $\cos(\pi - \theta) = -\cos\theta$, on puuting $\pi - \theta = \alpha$ we get

$$\int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta = \int_\pi^0 \frac{2a(-\cos\alpha)\cos 2\alpha}{(1+a^2)^2 - 4a^2\cos^2\alpha}\,(-d\alpha)$$

showing that $\displaystyle\int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta = 0$. Thus

$$I = \int_0^\pi \frac{(\cos 2\theta)(1+a^2)}{(1+a^2)^2 - 2a^2(1+\cos 2\theta)}\,d\theta$$

Putting $2\theta = \beta$, we get

$$I = \frac{1}{2}\int_0^{2\pi} \frac{(\cos\beta)(1+a^2)}{(1+a^2)^2 - 2a^2(1+\cos\beta)}\,d\beta = \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos\beta}{1 + a^4 - 2a^2\cos\beta}\,d\beta$$

We now put $z = e^{i\beta}$, so that $dz = iz\,d\beta$ or $d\beta = \dfrac{dz}{iz}$.

$$\begin{aligned}
I &= \frac{1}{2}\int_{|z|=1} \frac{(1+a^2)\frac{1}{2}(z+\frac{1}{z})}{1+a^4-a^2(z+\frac{1}{z})}\,\frac{dz}{iz} \\
&= \frac{1+a^2}{4i}\int_{|z|=1} \frac{z^2+1}{z\big[(1+a^4)z-a^2z^2-a^2\big]}\,dz \\
&= \frac{1+a^2}{4a^2 i}\int_{|z|=1} \frac{-(z^2+1)}{z\big[z^2-(a^2+\frac{1}{a^2})z+1\big]}\,dz \\
&= \frac{(1+a^2)i}{4a^2}\int_{|z|=1} \frac{z^2+1}{z(z-a^2)(z-\frac{1}{a^2})}\,dz
\end{aligned}$$

Clearly the integrand has simple poles at $z = 0, z = a^2, z = \frac{1}{a^2}$, out of which $z = 0$ and $z = a^2$ lie inside $|z| = 1$ as $-1 < a < 1$.

Residue at $z = 0$ is $\displaystyle\lim_{z\to 0} \frac{(z^2+1)z}{z(z-a^2)(z-\frac{1}{a^2})} = 1$.

Residue at $z = a^2$ is $\displaystyle\lim_{z\to 0} \frac{(z^2+1)(z-a^2)}{z(z-a^2)(z-\frac{1}{a^2})} = \frac{a^4+1}{a^4-1}$.

Cauchy's residue theorem (the integral around a curve $= 2\pi i\cdot$ sum of residues at poles inside the curve) now gives us

$$I = \frac{(1+a^2)i}{4a^2}\cdot 2\pi i\left[1+\frac{a^4+1}{a^4-1}\right] = -2\pi\frac{(1+a^2)\cdot 2a^4}{4a^2(a^4-1)} = \frac{\pi a^2}{1-a^2}$$

∎

Question 2(b) *Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Solution. Let $g(z) = z^7 - 5z^3, f(z) = 12$, then

1. On $|z| = 1$, $|g(z)| \le |z^7| + 5|z^3| = 6 < 12 = |f(z)|$.
2. Both $g(z)$ and $f(z)$ are analytic on and within $|z| < 1$
3. Both $g(z)$ and $f(z)$ have no zeros on $|z| = 1$

By Rouche's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ and $f(z)$ have the same number of zeros inside $|z| = 1$. But $f(z) = 12$ has no zeros anywhere and in particular in the region $|z| < 1$, therefore $z^7 - 5z^3 + 12$ has no zeros inside the unit circle.

Now we take $g(z) = 12 - 5z^3$, $f(z) = z^7$.

1. On $|z| = 2$, $|g(z)| \le 12 + 5|z^3| = 52 < 2^7 = |f(z)|$.

2. Both $g(z)$ and $f(z)$ are analytic on and within $|z| < 2$

Therefore by Rouche's theorem, $g(z) + f(z) = z^7 - 5z^3 + 12$ and $f(z)$ have the same number of zeros inside $|z| = 2$. Since $f(z) = z^7$ has 7 zeros ($z = 0$ is a zero of order 7 of $f(z)$) inside $|z| = 2$, the given polynomial has seven zeros inside $|z| = 2$ i.e. all its zeros lie inside $|z| = 2$.

Since $z^7 - 5z^3 + 12$ has no zeros inside and on $|z| = 1$, therefore all zeros lie in the ring $1 < |z| < 2$. ∎