6 Functions of a Single Variable (II)

1. THE DERIVATIVE

In this chapter we shall study the derivative, its existence and applications. We shall be concerned mainly with the real valued functions of a real variable, *i.e.*, the domains and the ranges of the functions considered here will be sets of real numbers.

1.1 Derivability at a Point

Let f be a real valued function defined on an interval $I = [a, b] \subseteq \mathbf{R}$. It is said to be derivable at an *interior point* c (where $a < c < b$) if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ or } \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

The limit in case it exists, is called the *Derivative* or the *Differential Coefficient* of the function f at $x = c$, and is denoted by $f'(c)$. The limit exists when the left-hand and the right-hand limits exist and are equal.

$\lim_{x \to c-0} \frac{f(x) - f(c)}{x - c}$ is called the *Left-hand Derivative* and is denoted by

$$f'(c-0),\ f'(c-) \text{ or } Lf'(c),$$

while $\lim_{x \to c+0} \frac{f(x) - f(c)}{x - c}$ is called the *Right-hand Derivative* and is denoted by

$$f'(c+0),\ f'(c+) \text{ or } Rf'(c).$$

Thus, the derivative $f'(c)$ exist when

$$Lf'(c) = Rf'(c)$$

1.2 Derivability in an Interval

A function f defined on $[a, b]$ is *derivable at the end point a, i.e.*, $f'(a)$ exists if,

$$\lim_{x \to a+0} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

In other words,

$$f'(a) = \lim_{x \to a+0} \frac{f(x) - f(a)}{x - a}$$

Similarly, it is *derivable at the end point b*, if $\lim\limits_{x \to b-0} \dfrac{f(x) - f(b)}{x - b}$ exists.

If a function is derivable at all points of an interval except the end points, it is said to be derivable in the *open interval*.

A function is derivable in the *closed interval* $[a, b]$, if it is derivable in the open interval $]a, b[$ and also at the end points a and b.

If f is not differentiable at $x = a$, then the upper and lower one-sided limits

$x \to a+$ or $x \to a-$ of $\dfrac{f(x) - f(a)}{x - a}$ will exist (possibly ∞ or $-\infty$).

These are denoted by $D^+ f, D_+ f, D^- f, D_- f$ respectively, and are called the *Dini* derivatives at a. For example,

$$D^+ f(a) = \overline{\lim_{x \to a+}} \frac{f(x) - f(a)}{x - a} = \overline{\lim_{h \to 0+}} \frac{f(a+h) - f(a)}{h}$$

$$D^+ f(a) = \underline{\lim_{x \to a+}} \frac{f(x) - f(a)}{x - a} = \underline{\lim_{h \to 0+}} \frac{f(a+h) - f(a)}{h}$$

If $D^+ f(a) = D_+ f(a) = l$, then we say that the right hand derivative at a exists and its value is l. Similarly, if $D^- f(a) = D_- f(a) = m$, then we say that the left hand derivative exists and is equal to m, where

$$\overline{\lim_{x \to a}} f(x) = \inf_{\delta > 0} \sup \{f(x) : 0 < |x - a| < \delta\}, \text{ and}$$

$$\underline{\lim_{x \to a}} f(x) = \sup_{\delta > 0} \inf \{f(x) : 0 < |x - a| < \delta\}$$

for f to be defined, bounded and real in

$$]a - \delta, a + \delta[, \delta > 0$$

If f is unbounded above in $]a - \delta, a + \delta[$, then $\overline{\lim} f(x) = +\infty$ and for f to be unbounded below

$$\underline{\lim} f(x) = -\infty.$$

Example 1. Show that the function $f(x) = x^2$ is derivable on $[0, 1]$.

■ Let x_0 be any point of $]0, 1[$, then

$$f'(x_0) = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0$$

At the end points, we have

$$f'(0) = \lim_{x\to 0+0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{x^2}{x} = \lim_{x\to 0+} x = 0$$

$$f'(1) = \lim_{x\to 1-0} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1-} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1-} (x+1) = 2$$

Thus, the function is derivable in the closed interval $[0, 1]$.

Example 2. A function f is defined on **R** by

$$\begin{aligned} f(x) &= x \quad \text{if } 0 \le x < 1 \\ &= 1 \quad \text{if } x \ge 1 \end{aligned}$$

- Consider the derivability at $x = 1$.

$$Lf'(1) = \lim_{x\to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1-} \frac{x-1}{x-1} = 1$$

$$Rf'(1) = \lim_{x\to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1+} \frac{1-1}{x-1} = 0$$

$\therefore$ $$Lf'(1) \neq Rf'(1)$$

Thus, $\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$ does not exist, *i.e.*, $f'(1)$ does not exist.

Example 3. Consider the derivability of the function $f(x) = |x|$ at the origin.

- $$\text{Left hand derivative} = \lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0-} \frac{|x|}{x} = \lim_{x\to 0-} \frac{-x}{x} = -1$$

$$\text{Right hand derivative} = \lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = 1$$

Thus,

$$f'(0-) \neq f'(0+)$$

Hence, the function is not derivable at $x = 0$.

Example 4. A function f defined as :

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is derivable at $x = 0$ but $\lim_{x\to 0} f'(x) \neq f'(0)$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x}$$

$$= \lim_{x \to 0} (x \sin 1/x) = 0$$

From elementary calculus we know that for $x \neq 0$,

$$f'(x) = 2x \sin (1/x) - \cos (1/x)$$

Clearly $\lim_{x \to 0} f'(x)$ does not exist and therefore, there is no possibility of $\lim_{x \to 0} f'(x)$ being equal to $f'(0)$.

Thus, $f'(x)$ is not continuous at $x = 0$ but $f'(0)$ exists.

2. CONTINUOUS FUNCTIONS

In this section we shall consider a relation between derivability and continuity, viz.,

derivability at a point $\Rightarrow$ *continuity at that point*

Thus, we shall prove that continuity at a point is a necessary condition for derivability at that point.

Theorem 1. *A function which is derivable at a point is necessarily continuous at that point.*

Let a function f be derivable at $x = c$.

Hence, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

Now

$$f(x) - f(c) = \frac{f(x) - f(c)}{(x - c)} (x - c), (x \neq c)$$

Taking limits as $x \to c$, we have

$$\lim_{x \to c} \{f(x) - f(c)\} = \lim_{x \to c} \left\{\frac{f(x) - f(c)}{x - c} (x - c)\right\}$$

$$= \lim_{x \to c} \left\{\frac{f(x) - f(c)}{x - c}\right\} \cdot \lim_{x \to c} (x - c)$$

$$= f'(c) \cdot 0 = 0$$

so that $\lim_{x \to c} f(x) = f(c)$, and therefore, f is continuous at $x = c$.

2.1 It is to be clearly understood that while continuity is a necessary condition for derivability at a point, it is not a sufficient condition. We come across functions which are continuous at a point without being derivable there at, and still many more functions may be constructed.

Consider the function f defined by

$$f(x) = |x|, \ \forall \ x \in \mathbf{R}$$

$f(x)$ is continuous at $x = 0$, for

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0 = f(0)$$

But as shown in example 3, $f'(0)$ does not exist. Thus, the function is continuous but not derivable at the origin.

However, it was the genius of German mathematician Weierstrass, who gave a function which is continuous everywhere but not derivable anywhere, viz.,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cos(3^n x), \ \forall\, x \in \mathbf{R}$$

Proof, however, is beyond the scope of the present book.

Some Counter Examples

1. $f(x) = |x| + |x-1|, \ \forall\, x \in \mathbf{R}$

   Continuous but not derivable at $x = 0$ and $x = 1$.

2. $f(x) = |x - \alpha|$

   Continuous but not derivable at $x = \alpha$.

3. $f(x) = x \sin 1/x \quad$ if $x \neq 0$

   $\quad = 0 \quad$ if $x = 0$

   Continuous but not derivable at the origin.

4. $f(x) = 0$ if $x \leq 0$

   $\quad = x$ if $x > 0$

   Continuous but not derivable at $x = 0$.

2.2 The existence of the derivative of a function at a point depends on the existence of a limit, viz., $\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c}$. Therefore, keeping in view the corresponding theorems on limits, one can easily establish the following *fundamental theorem on derivatives.*

If the functions f, g are derivable at c, then the functions $f+g$, $f-g$, $f\cdot g$ *and* f/g $(g(c) \neq 0)$ *are also derivable at c, and*

$$(f \pm g)'(c) = f'(c) \pm g'(c)$$

$$(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$(f/g)'(c) = \{f'(c)g(c) - f(c)g'(c)\}/g''(c), \text{ if } g(c) \neq 0$$

To illustrate the procedure, we prove the following theorem.

Theorem 2. *If f is derivable at c and* $f(c) \neq 0$ *then the function* $1/f$ *is also derivable there at, and*

$$(1/f)'(c) = -f'(c)/\{f(c)\}^2$$

Since f is derivable at c, it is also continuous there at. Again since $f(c) \neq 0$, there exists a neighbourhood N of c wherein f does not vanish.

Now

$$\frac{1/f(x) - 1/f(c)}{x - c} = -\frac{f(x) - f(c)}{x - c} \cdot \frac{-1}{f(x)\, f(c)}, \quad x \in N$$

Proceeding to limits when $x \to c$, we get

$$\left(\frac{1}{f}\right)'(c) = \lim_{x \to c} \frac{1/f(x) - 1/f(c)}{x - c}$$

$$= -f'(c) \cdot \frac{1}{f(c) \cdot f(c)} = -\frac{f'(c)}{\{f(c)\}^2}$$

Thus, the limits exists and are equals, $-f'(c)/\{f(c)\}^2$.

Note: If f and g be two functions having the same domain D and $f \pm g$ or $f \cdot g$ be derivable at $c \in D$, then f and g are not necessarily derivable at c. Consider, for instance

(*i*) $f(x) = \begin{cases} 1/x & x \neq 0 \\ \text{and } g(x) = -f(x) \\ 0 & x = 0 \end{cases}$

$f + g$ is derivable at the origin but f and g are not.

(*ii*) $f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ \text{and } g(x) = x \\ 0 & x = 0 \end{cases}$

Here fg is derivable at the origin, whereas f is not.

(*iii*) $f(x) = |x|$ and $g(x) = -|x|$

$f(x) \cdot g(x) = -x^2$, so that $f \cdot g$ is derivable at the origin but the functions f and g are not derivable. Similarly many more functions may be constructed.

EXERCISE

1. Find the derivatives of the following functions at the indicated points:

   (*i*) $f(x) = K$, a constant, at $x = c$

   (*ii*) $f(x) = x$ at $x = 0$

   (*iii*) $f(x) = \sqrt{x}$ at $x = 4$

   (*iv*) $f(x) = e^x$ at $x = x_0$

2. Show that the function

   $$f(x) = |x| + |x - 1|$$

   is derivable at all points except 0 and 1.

3. $f(x)=\begin{cases} x^3 \sin 1/x, & x \neq 0 \\ 0, & x=0 \end{cases}$

Prove that $f(x)$ has a derivative at $x=0$ and that $f(x)$ and $f'(x)$ are continuous at $x=0$.

4. $f(x)=(x-a)\sin\dfrac{1}{x-a}, \quad x \neq a$

$\quad = 0, \quad x=a$

Show that $f(x)$ is continuous but not derivable at $x=a$.

5. Discuss the derivability of the following functions:

(i) $f(x)=\begin{cases} 2, & x \le 1 \\ x, & x > 1 \end{cases}$ at $x=1$

(ii) $f(x)=\begin{cases} 1, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$ at $x=1$

(iii) $f(x)=\begin{cases} 2x-3, & 0 \le x \le 2 \\ x^2-3, & 2 < x \le 4 \end{cases}$ at $x=2, 4$

6. Show that the function $f(x)=x|x|$ is derivable at the origin.

7. Find the derivative of f at $x=0$, where $f(x)=x^2|x|$.

8. $f(x)=|x|$ and $g(x)=3|x|$ $x \in R$, show that f, g is not derivable at the origin but $\lim\left[f(x)/g(x)\right]$ exists and is equal to $\lim\left[f'(x)/g'(x)\right]$, when $x \to 0$.

9. Find $Lf'(0)$ and $Rf'(0)$ for the following functions:

(i) $f(x)=\begin{cases} x\tan^{-1} 1/x, & x \neq 0 \\ 0, & x=0 \end{cases}$

(ii) $f(x)=\begin{cases} \dfrac{x(e^{1/x}-1)}{(e^{1/x}+1)}, & x \neq 0 \\ 0, & x=0 \end{cases}$

10. If $f(x)=\begin{cases} \dfrac{x(e^{1/x}-e^{-1/x})}{(e^{1/x}+e^{-1/x})}, & x \neq 0 \\ 0, & x=0 \end{cases}$

Show that f is continuous but not derivable at $x=0$ and $Lf'(0)=-1$, $Rf'(0)=1$.

11. Examine the function f, where $f(x)=\begin{cases} x^m \sin 1/x, & x \neq \theta \\ 0, & x=0 \end{cases}$ for derivability at the origin. Also determine m where f' is continuous at the origin.

12. If functions f and g are defined on $[0, \infty[$ by

$$f(x) = \lim_{n \to \infty} \frac{x^n - 1}{x^n + 1} \text{ and } g(x) = \int_0^x f(t)\, dt\,;$$

then prove that g is continuous but not derivable at $x = 1$.

ANSWERS

5. (*i*) Not derivable (*ii*) Not derivable, $Lf'(1) = 0$, $Rf'(1) = 1$.

(*iii*) Not derivable ; at $x = 2$, $f'(4) = 8$.

7. 0. 9. (*i*) $-\pi/2, \pi/2$ (*ii*) $-1, 1$.

11. Derivable if $m > 1$; $m > 2$.

3. INCREASING AND DECREASING FUNCTIONS

A function f is said to be increasing or decreasing at a point $x = c$ according as the value of the function increases or decreases at that point with increase in x. Thus for any x in the neighbourhood $]c-\delta, c+\delta[, \delta>0$ of c,

$$f(c-\delta) \le f(x) \le f(c+\delta) \text{ for an increasing function.}$$
$$f(c-\delta) \ge f(x) \ge f(c+\delta) \text{ for a decreasing function.}$$

A function is increasing or decreasing in the interval $[a, b]$ according as

$$f(x_2) \ge f(x_1) \text{ or } f(x_2) \le f(x_1),\ \forall\, x_2 \ge x_1 \wedge x_1, x_2 \in [a, b].$$

The function is *strictly increasing* or *strictly decreasing* if the strict inequality holds in the above relations, *i.e.*,

$$f(x_2) > f(x_1) \text{ for a strictly increasing function,}$$
$$f(x_2) < f(x_1) \text{ for a strictly decreasing function}$$
$$\forall\, x_2 > x_1 \wedge x_1, x_2 \in [a, b]$$

A function is said to be *monotone* or *monotonic* in an interval I if it is either increasing in I or decreasing in I.

It is said to be *strictly monotone* in I if it is either strictly increasing or strictly decreasing in I.

3.1 Sign of the Derivative

Let c *be an interior point* of the domain $[a, b]$ of a function f and let $f'(c)$ exist and be positive.

By definition,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c),\ x \ne c$$

Thus depending on the choice of a positive ε however small, $\exists$ a positive number δ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon, \text{ when } |x - c| < \delta,\ x \ne c$$

or

$$f'(c) - \varepsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \varepsilon, \text{ when } x \in]c - \delta, c + \delta[, x \neq c$$

If $\varepsilon > 0$ is selected less than $f'(c)$, $\dfrac{f(x) - f(c)}{x - c}$ lies between two positive numbers and is therefore itself positive.

Hence,

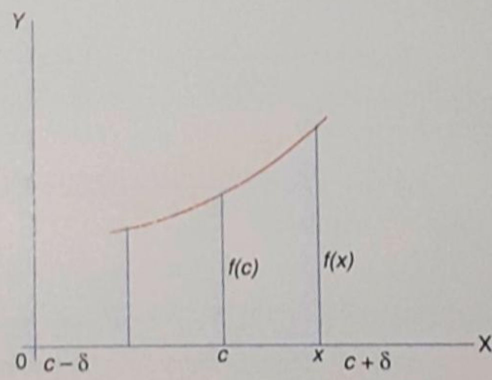
$$\frac{f(x) - f(c)}{x - c} > 0, \text{ when } x \in]c - \delta, c + \delta[, x \neq c.$$

Thus

(*i*) $f(x) - f(c) > 0$ or $f(x) > f(c)$ and $x \in]c, c+\delta[$, and

(*ii*) $f(x) < f(c)$, when $x \in]c - \delta, c[$.

From (*i*) and (*ii*), we see that $f(x)$ is increasing at $x = c$.



Hence, the function is increasing at c if $f'(c) > 0$.

Similarly, it can be shown that the function is decreasing at $x = c$ if $f'(c) < 0$.

Let us now consider the *end points*.

(*a*) At the *end point* a, $\exists$ an interval $[a, a + \delta[$, such that

$$f'(a) > 0 \Rightarrow f(x) > f(a), \text{ for } x \in]a, a + \delta[$$
$$f'(a) < 0 \Rightarrow f(x) < f(a), \text{ for } x \in]a, a + \delta[$$

(*b*) At the *end point* b, $\exists$ an interval $]b - \delta, b]$, such that

$$f'(b) > 0 \Rightarrow f(x) < f(b), \text{ for } x \in]b - \delta, b[$$
$$f'(b) < 0 \Rightarrow f(x) > f(b), \text{ for } x \in]b - \delta, b[$$

Example 5. Show that $\log(1 + x)$ lies between

$$x - \frac{x^2}{2} \text{ and } x - \frac{x^2}{2(1 + x)}, \ \forall x > 0$$

- Consider

$$f(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$$

$$\therefore \qquad f'(x) = \frac{1}{1+x} - (1-x) = \frac{x^2}{1+x} > 0, \ \forall\, x > 0$$

Hence, $f(x)$ is an increasing function for all $x > 0$.
Also

$$f(0) = 0$$

Hence for $x > 0, f(x) > 0.$
Thus

$$\log(1+x) > x - \frac{x^2}{2}, \text{ for } x > 0$$

Similarly by considering the function

$$F(x) = x - \frac{x^2}{2(1+x)} - \log(1+x),$$

it can be shown that

$$\log(1+x) < x - \frac{x^2}{2(1+x)}, \ \forall\, x > 0.$$

Ex. 1. Show that

$$x^3 - 6x^2 + 15x + 3 > 0, \ \forall\, x > 0$$

Ex. 2. Show that

(*i*) $\dfrac{x}{1+x} < \log(1+x) < x, \ \forall\, x > 0$

(*ii*) $\dfrac{x}{1+x^2} < \tan^{-1} x < x, \ \forall\, x > 0$

Ex. 3. Show that

(*i*) $\tan x > x, 0 < x < \pi/2$

(*ii*) $\dfrac{2}{\pi} \le \dfrac{\sin x}{x} < 1, 0 < |x| \le \pi/2.$

Ex. 4. Show that

$$2x < \log\frac{1+x}{1-x} < 2x\left(1 + \frac{x^2}{3(1-x^2)}\right), 0 < x < 1.$$

Ex. 5. Show that

$$2/(2x+1) < \log(1+1/x) < 1/\sqrt{x(x+1)}, \ \forall\, x > 0.$$

4. DARBOUX'S THEOREM

If a function f is derivable on a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs then there exists at least one point c between a and b such that $f'(c) = 0$.

For the sake of definiteness, let us take $f'(a) < 0$ and $f'(b) > 0$.

Since $f'(a)$ is negative, therefore, there exists a positive number δ_1 such that

$$f(x) < f(a), \ \forall\, x \in\,]a, a+\delta_1[\qquad \ldots(1)$$

Again, since $f'(b)$ is positive, there exists a positive number δ_2 such that

$$f(x) < f(b), \ \forall\, x \in\,]b-\delta_2, b[\qquad \ldots(2)$$

Also, since f is derivable in $[a, b]$, it is continuous in the closed interval $[a, b]$. Being continuous in the closed interval, it is bounded and attains its bounds. Thus if m is the infimum ($g.\ l.\ b.$), $\exists$ a point $c \in [a, b]$ such that

$$f(c) = m$$

It is clear from (1) and (2) that c cannot coincide with a or b for otherwise there would exist points where the values of the function would be less than $f(c)$. Thus, c is an interior point of $[a, b]$.

Now, we proceed to show that it is this point where $f'(c) = 0$.

If $f'(c) < 0$, then $\exists$ an interval $]c, c+\delta_3[, \delta_3 > 0$ such that $\forall\, x \in\,]c, c+\delta_3[,\ f(x) < f(c) = m$, which contradicts the fact that m is infimum of f.

Again, if $f'(c) > 0$, $\exists$ an interval $]c-\delta_4, c[, \delta_4 > 0$ for every point x of which $f(x) < f(c) = m$, which again is a contradiction.

Hence, the only possibility, $f'(c) = 0$.

Note: If $f'(a) > 0$ and $f'(b) < 0$, then proceeding as above, it can be shown that $f'(d) = 0$ where $d \in\,]a, b[$ is the point where the function attains the supremum.

Intermediate value theorem for derivatives. *If a function f is derivable on a closed interval $[a, b]$ and $f'(a) \neq f'(b)$ and k, a number lying between $f'(a)$ and $f'(b)$ then $\exists$ at least one point $c \in\,]a, b[$ such that $f'(c) = k$.*

Let $g(x) = f(x) - kx, x \in [a, b]$.

Clearly $g(x)$ is derivable on $[a, b]$ and

$$g'(a) = f'(a) - k,\ g'(b) = f'(b) - k$$

Since k lies between $f'(a)$ and $f'(b)$.

$\therefore$ $g'(a)$ and $g'(b)$ are of opposite signs.

$\therefore$ $g(x)$ satisfies the conditions of Darboux's theorem.

Thus there exists at least one point $c \in]a, b[$ such that

$$g'(c) = 0 \text{ or } f'(c) = k.$$

Note: The derivative f' (not necessarily continuous) satisfies the intermediate-value property, and so f' has no discontinuities of the first kind or removable. As a consequence, it follows that, *monotone derivatives are necessarily continuous.*

5. ROLLE'S THEOREM

If a function f defined on $[a, b]$ *is*

(i) *continuous on* $[a, b]$,

(ii) *derivable on* $]a, b[$, *and*

(iii) $f(a) = f(b)$,

then there exists at least one real number c between a and b $(a < c < b)$ *such that* $f'(c) = 0$.

Since the function is continuous on the closed interval $[a, b]$ it is bounded and attains its bounds. Thus, if m and M are the infimum (*g. l. b.*) and the supremum (*l.u.b.*) respectively of the function f, then $\exists$ points c and d of $[a, b]$ such that

$$f(c) = m \text{ and } f(d) = M.$$

There are two possibilities: either $m = M$ or $m \neq M$.

If $m = M$, then f is constant over $[a, b]$ and therefore, its derivatives

$$f'(x) = 0, \ \forall x \in [a, b].$$

When $m \neq M$, both of these cannot be equal to the same quantity $f(a)$. At least one of these, say, m is different from $f(a)$ or $f(b)$, so that

$$f(c) = m \neq f(a) \Rightarrow c \neq a$$
$$f(c) = m \neq f(b) \Rightarrow c \neq b$$

This means that c lies in the open interval $]a, b[$.

We shall now show that c is the point where $f'(c) = 0$.

If $f'(c) < 0$, then $\exists$ an interval $]c, c + \delta_1[, \delta_1 > 0$ for every point x of which $f(x) < f(c) = m$; which contradicts the fact that m is the infimum.

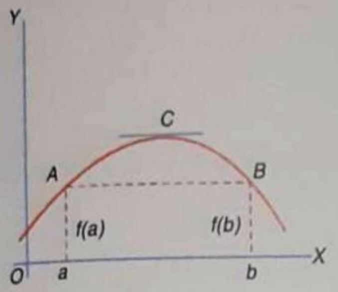
If $f'(c) > 0, \exists$ an interval $]c - \delta_2, c[, \delta_2 > 0$ for every point x of which $f(x) < f(c) = m$; which is also a contradiction.

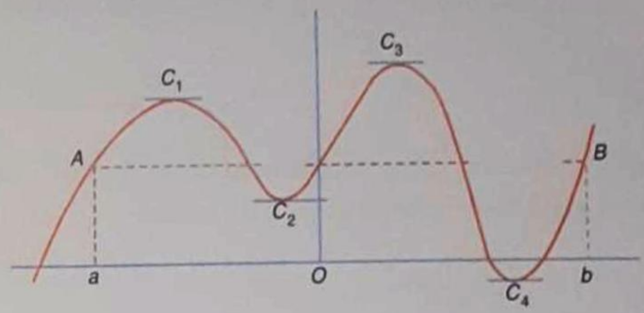
Hence, the only possibility is $f'(c) = 0$.

5.1 Interpretation of Rolle's Theorem

Geometric. Let the curve $y = f(x)$, which is continuous on $[a, b]$ and derivable on $]a, b[$ be drawn.

The theorem simply states that between two points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to x-axis.





Algebraic. Between two zeros a and b of $f(x)$ (*i.e.*, between two roots a and b of $f(x) = 0$) there exists at least one zero of $f'(x)$.

Ex. 1. Show that between two consecutive zeros of $f'(x)$ there lies at the most one zero of $f(x)$.

Ex. 2. Show that, for any real number k, the polynomial

$f(x) = x^3 + x + k$ has exactly one real root.

6. LAGRANGE'S MEAN VALUE THEOREM
(First mean value theorem of differential calculus)

If a function f defined on $[a, b]$ *is*

(*i*) *continuous on* $[a, b]$, *and*

(*ii*) *derivable on* $]a, b[$,

then there exists at least one real number c between a and b $(c \in]a, b[)$ *such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Let us consider a function

$$\phi(x) = f(x) + Ax, x \in [a, b],$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$\therefore \qquad A = -\frac{f(b) - f(a)}{b - a}$$

Now the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself

(*i*) continuous on $[a, b]$,

(*ii*) derivable on $]a, b[$, and

(*iii*) $\phi(a) = \phi(b)$.

Therefore, by Rolle's Theorem $\exists$ a real number $c \in]a, b[$ such that $\phi'(c) = 0$.
But

$$\phi'(x) = f'(x) + A$$

$$0 = \phi'(c) = f'(c) + A$$

$\therefore$

or

$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}$$

Another Statement. If in the statement of the theorem, b is replaced by $a + h$, then the number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus,

$$f(a + h) - f(a) = hf'(a + \theta h)$$

or

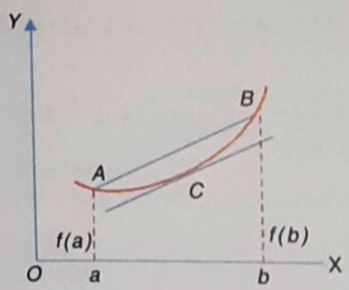
$$f(a + h) = f(a) + hf'(a + \theta h), \text{ where } 0 < \theta < 1.$$

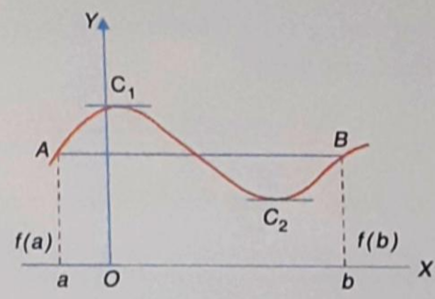
6.1 Deductions

1. If a function $f(x)$ satisfies the conditions of the Mean Value Theorem and $f'(x) = 0$ for all $x \in]a, b[$, then $f(x)$ is constant on $[a, b]$.
   Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$.
   Hence by Lagrange's Mean Value Theorem,
   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = 0, \text{ where } x_1 < c < x_2$$
   Thus,
   $$f(x_2) = f(x_1)$$
   Hence, the function keeps the same value and is therefore constant on $[a, b]$.
2. If two functions have equal derivatives at all points of $]a, b[$, then they differ only by a constant.
3. If a function f is (*i*) continuous on $[a, b]$, (*ii*) derivable on $]a, b[$, and (*iii*) $f'(x) > 0$, $\forall x \in]a, b[$, then f is strictly increasing on $[a, b]$.
   Let x_1, x_2 (where $x_1 < x_2$) be any two distinct points of $[a, b]$, then by Lagrange's Mean Value Theorem
   $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0, \text{ for } x_1 < c < x_2$$
   or
   $$f(x_2) - f(x_1) > 0 \Rightarrow f(x_2) > f(x_1), \text{ for } x_2 > x_1$$
   Thus, f is strictly increasing on $[a, b]$.
4. If f' exists and is bounded on some interval I, then f is uniformly continuous on I.

6.2 Geometrical Interpretation

The theorem simply states that between two points A and B of the graph of f there exists at least one point where the tangent is parallel to the chord AB.





7. CAUCHY'S MEAN VALUE THEOREM
(Second mean value theorem)

If two functions f, g defined on $[a, b]$ *are*

(*i*) *continuous on* $[a, b]$,

(*ii*) *derivable on* $]a, b[$, *and*

(*iii*) $g'(x) \neq 0$, *for any* $x \in]a, b[$,

then there exists at least one real number c between a and b $(c \in]a, b[)$ *such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

At the outset we notice that $g(a) \neq g(b)$, for otherwise, in view of the conditions (*i*) and (*ii*) of the theorem, $g(x)$ would satisfy all the conditions of Rolle's Theorem and its derivative $g'(x)$ would vanish for some $x \in]a, b[$ contrary to (*iii*).

Consider the function

$$\phi(x) = f(x) + Ag(x), \; x \in [a, b],$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$\therefore$
$$A = -\frac{f(b)-f(a)}{g(b)-g(a)}$$

Now the function $\phi(x)$, being the sum of two continuous and derivable functions, is itself:

(*i*) *continuous on* $[a, b]$,

(*ii*) *derivable on* $]a, b[$, and

(*iii*) $\phi(a) = \phi(b)$.

Therefore, by Rolle's Theorem $\exists$ a real number $c \in]a, b[$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x) + Ag'(x)$

$\Rightarrow$
$$0 = \phi'(c) = f'(c) + Ag'(c)$$

$\therefore$
$$\frac{f'(c)}{g'(c)} = -A = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Another Statement. If two functions f, g defined on $[a, a+h]$ are continuous on $[a, a+h]$, derivab on $]a, a+h[$ and $g'(x) \neq 0$ for any $x \in]a, a+h[$, then there exists at least one real number between 0 and 1 such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1.$$

Corollary. *Lagrange's Mean Value Theorem* may be deduced as a particular case, for $g(x) = x$.

> **Note:** Cauchy's Mean Value Theorem cannot be deduced by applying Lagrange's Mean Value Theorem separately to the two functions and then dividing, for, then we get
>
> $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c_1)}{g'(c_2)}$$
>
> where c_1 and c_2 may not be equal.

Geometrical (Physical) Interpretation. We may write

$$\frac{\{f(b)-f(a)\}/(b-a)}{\{g(b)-g(a)\}/(b-a)} = \frac{f'(c)}{g'(c)}$$

Hence, the ratio of the mean rates of increase of two functions in an interval is equal to the ratio of the actual rates of increase of the functions at some point within the interval.

Example 6. Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}, \text{ if } 0<u<v$$

and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

■ Let $f(x) = \tan^{-1} x$, then for $u < x < v$,

$$f'(x) = \frac{1}{1+x^2}$$

Applying the Mean Value Theorem to f, we get

$$\frac{\tan^{-1} v - \tan^{-1} u}{v-u} = \frac{1}{1+\xi^2}, \text{ for } u<\xi<v$$

But

$$\xi > u \Rightarrow \frac{1}{1+\xi^2} < \frac{1}{1+u^2}$$

and

$$\xi < v \Rightarrow \frac{1}{1+\xi^2} > \frac{1}{1+v^2}$$

$\therefore$

$$\frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2}$$

or

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2}$$

The other result follows by taking $v = \dfrac{4}{3}$ and $u = 1$.

Example 7. Show that

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}$$

- Let $f(x) = \sin x$ and $g(x) = \cos x$, for $x \in [\alpha, \beta]$.

$\therefore$

$$f'(x) = \cos x \text{ and } g'(x) = -\sin x$$

Functions f and g are both continuous and differentiable, therefore by Cauchy's Mean Value Theorem on $[\alpha, \beta]$,

$$\frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}, \alpha < \theta < \beta$$

or

$$\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \alpha < \theta < \beta.$$

Example 8. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$, for $a < c < b$. Prove that there is at least one value ξ between a and b for which $f''(\xi) < 0$.

- Let us consider the function f on $[a, b]$.

Since f'' exists, f' and f both exist and are continuous on $[a, b]$. Since c is a point between a and b, applying Lagrange's Mean Value Theorem to f on the intervals $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1), \ a < \xi_1 < c$$

and

$$\frac{f(b) - f(c)}{b - c} = f'(\xi_2), \ c < \xi_2 < b$$

But

$$f(a) = f(b) = 0$$

$\therefore$

$$f'(\xi_1) = \frac{f(c)}{c-a} \text{ and } f'(\xi_2) = -\frac{f(c)}{b-c}$$

where $a < \xi_1 < c < \xi_2 < b$.

Again $f'(x)$ is continuous and derivable on $[\xi_1, \xi_2]$. Therefore, by Mean Value Theorem,

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\xi), \text{ where } \xi_1 < \xi < \xi_2$$

Substituting the values of $f'(\xi_2)$ and $f'(\xi_1)$, we get

$$f''(\xi) = \frac{-f(c)}{\xi_2 - \xi_1}\left(\frac{1}{b-c} + \frac{1}{c-a}\right)$$

$$= \frac{-(b-a)f(c)}{(\xi_2 - \xi_1)(b-c)(c-a)} < 0.$$

Example 9. If a function f is such that its derivative f' is continuous on $[a, b]$ and derivable on $]a, b[$, then show that there exists a number c between a and b such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$$

■ Clearly the functions f and f' are continuous and derivable on $[a, b]$.

Consider the function

$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - (b-x)^2 A$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$.

$$\therefore \quad f(b) - f(a) - (b-a)f'(a) - (b-a)^2 A = 0 \qquad \ldots(1)$$

Now $\phi(x)$, being the sum of continuous and derivable functions, is itself continuous on $[a, b]$ and derivable on $]a, b[$ and also $\phi(a) = \phi(b)$.

Thus $\phi(x)$ satisfies all the conditions of Rolle's Theorem and therefore $\exists\, c \in]a, b[$ such that $\phi'(c) = 0$.

Now,

$$\phi'(x) = -f'(x) + f'(x) - (b-x)f''(x) + 2(b-x)A$$

$$\therefore \quad -(b-c)f''(c) + 2(b-c)A = \phi'(c) = 0$$

But

$$b - c \neq 0$$

$$\therefore \quad A = \frac{1}{2}f''(c) \qquad \ldots(2)$$

Hence, from (1) and (2)

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$$

Note: Motivation for $\phi(x)$ is attained by replacing a by x and transposing all the terms to the right in the result to be proved.

Example 10. Assuming f'' to be continuous on $[a, b]$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - \frac{c-a}{b-a}f(b) = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in $[a, b]$.

- We have to show that

$$(b-a)f(c)-(b-c)f(a)-(c-a)f(b)=\frac{1}{2}(b-a)(c-a)(c-b)f''(\xi)$$

Consider the function, for $x \in [a, b]$ defined by

$$\phi(x)=(b-a)f(x)-(b-x)f(a)-(x-a)f(b)-(b-a)(x-a)(x-b)A$$

where A is a constant to be determined such that $\phi(c)=0$.

$$\therefore \quad (b-a)f(c)-(b-c)f(a)-(c-a)f(b)-(b-a)(c-a)(c-b)A=0 \qquad \ldots(1)$$

Clearly $\phi(a)=0=\phi(b)$, and $\phi(x)$ is differentiable in $[a, b]$.

The function ϕ satisfies all the conditions of Rolle's Theorem on each of the intervals $[a, c]$ and $[c, b]$ and therefore $\exists$ two numbers ξ_1, ξ_2 in $]a, c[$ and $]c, b[$ respectively, such that $\phi'(\xi_1)=0$ and $\phi'(\xi_2)=0$.

Again

$$\phi'(x)=(b-a)f'(x)+f(a)-f(b)-(b-a)\{2x-(a+b)\}A$$

which is continuous on $[a, b]$ and derivable on $]a, b[$ and in particular on $[\xi_1, \xi_2]$.

Also $\phi'(\xi_1)=\phi'(\xi_2)=0$

Therefore by Rolle's Theorem $\exists\, \xi \in]\xi_1, \xi_2[$ such that $\phi''(\xi)=0$.

Now $\phi''(x)=(b-a)f''(x)-2(b-a)A$

so that

$$f''(\xi)-2A=0,\ b \neq a$$

or

$$A=\frac{1}{2}f''(\xi), \text{ where } a<\xi_1<\xi<\xi_2<b \qquad \ldots(2)$$

From (1) and (2), the result follows.

Example 11. Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, for $0<x<\dfrac{\pi}{2}$

- We have thus to show that

$$\frac{\tan x}{x}-\frac{x}{\sin x}>0 \text{ or } \frac{\sin x\tan x - x^2}{x\sin x}>0, \text{ for } 0<x<\frac{\pi}{2}.$$

Since $x\sin x>0$, for $0<x<\pi/2$, it will therefore suffice to show that $\sin x\tan x - x^2>0$

Let $f(x)=\sin x\tan x - x^2$ then for $0<x<\dfrac{\pi}{2}$,

$$f'(x)=\cos x\tan x+\sin x\sec^2 x-2x=\sin x+\sin x\sec^2 x-2x$$

We cannot decide about the sign of $f'(x)$ mainly because of the presence of the $2x$ term.

The function $f'(x)$ is continuous and derivable on $]0, \pi/2[$.

$$\therefore \quad f''(x)=\cos x+\cos x\sec^2 x+2\sin x\sec^2 x\tan x-2$$

$$=\left(\sqrt{\sec x}-\sqrt{\cos x}\right)^2+2\tan^2 x\sec x>0, \text{ for } 0<x<\pi/2$$

Since the derivative $f''(x)$ of $f'(x)$ is positive, the function $f'(x)$ is an increasing function. Further since $f'(0)=0$, therefore the function $f'(x)>0$ for $0<x<\pi/2$.

Again, since $f'(x)>0$, $f(x)$ is an increasing function and because $f(0)=0$, the function $f(x)>0$, for $0<x<\pi/2$.

Thus it follows that

$$\frac{\tan x}{x}>\frac{x}{\sin x}, \text{ for } 0<x<\pi/2.$$

Note: The above inequality can be put in the form:

$$\cos x<\left(\frac{\sin x}{x}\right)^2, 0<x<\frac{\pi}{2}.$$

Ex. Show that $\cos x<\left(\frac{\sin x}{x}\right)^3$, for $0<x<\frac{\pi}{2}$.

$$\left[\textit{Hint}: \text{Take } f(x)=x-\sin x\cos^{-1/3}x, 0\le x<\frac{\pi}{2}\right].$$

EXERCISE

1. Examine the validity of the hypothesis and the conclusion of Rolle's Theorem:
   (i) $f(x)=x^3-4x$ on $[-2,2]$.
   (ii) $f(x)=(x-a)^m(x-b)^n$, where m and n are positive integers on $[a,b]$.
   (iii) $f(x)=1-(x-1)^{2/3}$ on $[0,2]$.
   (iv) $f(x)=|x|$ on $[-1,1]$.
   (v) $f(x)=1-|x-1|$ on $[0,2]$.
2. Examine the validity of the hypothesis and the conclusion of Lagrange's Mean Value Theorem:
   (i) $f(x)=|x|$ on $[-1,1]$.
   (ii) $f(x)=\log x$ on $\left[\frac{1}{2},2\right]$.
   (iii) $f(x)=x(x-1)(x-2)$ on $\left[0,\frac{1}{2}\right]$.
   (iv) $f(x)=x^{1/3}$ on $[-1,1]$.
   (v) $f(x)=2x^2-7x+10$ on $[2,5]$.
3. Deduce Lagrange's Mean Value Theorem by considering the derivable function

$$\phi(x)=f(x)-f(a)-A(x-a), x\in[a,b].$$

4. Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.
[*Hint*: Apply Rolle's Theorem to the function $e^{-x} - \sin x$.]

5. A function f is continuous on $[a-h, a+h]$ and derivable on $]a-h, a+h[$, show that
$$f(a-h) - 2f(a) + f(a+h) = h[f'(a+\theta h) - f'(a-\theta h)], 0 < \theta < 1$$
[*Hint*: Use Mean Value Theorem for the function
$$\phi(t) = f(a+th) + f(a-ht) \text{ on } [0, 1].$$

6. If a function f is twice derivable on $[a, a+h]$, then show that
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(a+\theta h), 0 < \theta < 1$$

7. If f', g' are continuous and differentiable on $[a, b]$, then show that for $a < c < b$.
$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$
[*Hint*: Apply Rolle's Theorem to the function
$$\phi(x) = f(x) + (b-x)f'(x) + A\{g(x) + (b-x)g'(x)\}]$$

8. If f', ϕ and ψ are continuous on $[a, b]$ and derivable on $]a, b[$, then show that there is a value c lying between a and b such that
$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \psi(a) & \psi(b) & \psi'(c) \end{vmatrix} = 0.$$
[*Hint*: Apply Rolle's Theorem to the function
$$g(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ \phi(a) & \phi(b) & \phi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}].$$

9. A function f is such that its second derivative is continuous on $[a, a+h]$ and derivable on $]a, a+h[$. Show that there exists a number θ between 0 and 1 such that
$$f(a+h) - f(a) - \frac{1}{2}h\{f'(a) + f'(a+h)\} + \frac{h^3}{12}f'''(a+\theta h) = 0.$$

10. If $f(0) = 0$ and $f''(x)$ exists for all $x > 0$, then show that
$$f'(x) - \frac{f(x)}{x} = \frac{1}{2}xf''(\xi), 0 < \xi < x.$$
Also deduce that if $f''(x)$ is positive for positive values of x, then $f(x)/x$ strictly increases as x increases.

11. If $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$, and if $g'(x)$ does not vanish anywhere on $]a, b[$. Then prove that for some c between a and b,
$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$
[*Hint*: Apply Rolle's Theorem to the function $fg - gf(a) - fg(b)$].

12. Apply Lagrange's Mean Value Theorem to the function $\log(1+x)$ to show that
$$0 < [\log(1+x)]^{-1} - x^{-1} < 1, \ \forall x > 0$$

13. If $|f(x)-f(y)| \le (x-y)^2$, for all real numbers x and y. Prove that f is a constant function.

14. Applying Lagrange's Mean Value Theorem to the function

$$f(x)=\tan^{-1}x,\ x \in \mathbf{R}.$$

Show that f and f' both are uniformly continuous on **R**.

15. Find the range of the values of x for which the function $x^3-6x^2-36x+7$, increases with x.

16. Establish the following inequalities :

(i) $x-\dfrac{x^2}{2}+\dfrac{x^3}{3(1+x)}<\log(1+x)<x-\dfrac{x^2}{2}+\dfrac{x^3}{3},\ x>0$

(ii) $\dfrac{x^2}{2(1+x)}<x-\log(1+x)<\dfrac{x^2}{2},\ x>0$

(iii) $\dfrac{x^2}{2}<x-\log(1+x)<\dfrac{x^2}{2(1+x)},\ -1<x<0$

(iv) $1-x<-\log x<\dfrac{1}{x}-1,\ 0<x<1$

(v) $x<-\log(1-x)<x(1-x)^{-1},\ 0<x<1$

(vi) $x<\sin^{-1}x<\dfrac{x}{\sqrt{1-x^2}},\ 0<x<1$

(vii) $(1-x)<e^{-x}<1-x+\dfrac{x^2}{2},\ x>0$

(viii) $1+x \le e^x \le 1+xe^x,\ \forall x$

(ix) $x/(1+x) \le \log(1+x) \le x,\ x>-1$

17. If $g(x)=0$ has two equal roots, show that $g'(x)=0$ has one root equal to either.

ANSWERS

1. (i), (ii) Both valid, (iii), (iv) and (v) not valid.

2. (i) Not valid, (ii) valid, (iii) valid, $c=(6-\sqrt{21})/6$,

(iv) Hypothesis not valid but the conclusion is valid,

(v) valid, $c=$ [illegible].

15. $x<-2$ and $x>6$.

8. HIGHER ORDER DERIVATIVES

We know that the existence of the derivative f' of a function f at a point c implies the existence and continuity of the function in a neighbourhood of c. The derivative of the function f' at c in case it exists, is called the second derivative of f at c and denoted by $f''(c)$. Evidently the existence of $f''(c)$ implies the existence and continuity of f' in a neighbourhood of c.

Higher order derivatives can be similarly defined. The derivative of f^{n-1} at c, in case, it exists is called the nth derivative of f at c and is denoted by $f^{n}(c)$.

8.1 Taylor's Theorem

If a function f defined on $[a, a+h]$, is such that (i) the $(n-1)$th derivative f^{n-1} is continuous on $[a, a+h]$, and (ii) the nth derivative f^{n} exists on $]a, a+h[$, then $\exists$ at least one real number θ between 0 and 1 $(0<\theta<1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \dots$$

$$\dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}f^{n}(a+\theta h) \qquad \dots(1)$$

where p is a given positive integer.

First of all we observe that the condition (i) in the statement implies that all the derivatives $f', f'', \dots, f^{n-1}$ exist and are continuous on $[a, a+h]$.

Consider the function ϕ defined on $[a, a+h]$ as

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \dots$$

$$\dots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

where A is a constant to be determined such that $\phi(a+h) = \phi(a)$.

$\therefore$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$

$$\dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^p \qquad \dots(2)$$

Now,

(i) $f, f', f'', \dots, f^{n-1}$ being all continuous on $[a, a+h]$, the function $\phi(x)$ is continuous on $[a, a+h]$;

(ii) the functions $f, f', \dots, f^{n-1}$ and $(a+h-x)^r$ for all r being derivable in $]a, a+h[$, the function $\phi(x)$ is derivable in $]a, a+h[$; and

(iii) $\phi(a+h) = \phi(a)$.

Thus, the function $\phi(x)$ satisfies all the conditions of Rolle's Theorem and hence $\exists$ at least one real number θ between 0 and 1 such that $\phi'(a+\theta h) = 0$.

But

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - Ap(a+h-x)^{p-1}$$

$\therefore$
$$0 = \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - Aph^{p-1}(1-\theta)^{p-1}$$

$\Rightarrow$
$$A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta n), \quad h \neq 0, \theta \neq 1$$

Substituting A from (3) in (2), we get the required result.

Forms of Remainder after n Terms

(*i*) The term

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]} f^n(a+\theta h)$$

which occurs after n terms, is known as Taylor's remainder after n terms. The the with this form of remainder is known as Taylor's Theorem with *Schlömilch and R form of remainder*.

(*ii*) For $p = 1$, we get

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

called *Cauchy's form of remainder*.

(*iii*) For $p = n$, we get

$$R_n = \frac{h^n}{n!} f^n(a+\theta h)$$

called *Lagrange's form of remainder*.

EXERCISE

1. Prove Taylor's Theorem with Lagrange's form of remainder by considering the function

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots$$
$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + A(a+h-x)^n$$

2. Prove Taylor's Theorem with Cauchy's form of remainder by taking the last term of $\phi(x)$ as $A(a+h-x)$

Second form of Taylor's Theorem. If f satisfies the conditions of Taylor's Theorem in $[a, a+$ and x is any point of $[a, a+h]$ then it satisfies the conditions in the interval $[a, x]$ also. Replacing $a+h$ by x or h by $(x-a)$ in (1), we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+\frac{(x-a)^n}{p[n-1]!}(1-\theta)^{n-p}f^n(a+\theta(x-a)) \qquad \ldots(4)$$

where $0 < 6 < 1$.

The remainder after n terms can thus be written as

$$R_n = \frac{(x-a)^n(1-\theta)^{n-p}}{p[(n-1)!]}f^n(c)$$

where c lies between a and x and depends on the selection of x.

8.2 Maclaurin's Theorem

Putting $a = 0$ in (4), we have for $x \in]0, h[$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0)$$

$$\frac{x^n(1-\theta)^{n-p}}{p[(n-1)!]}f^n(\theta x)$$

is called Maclaurin's Theorem with *Schlömilch* and *Röche* form of remainder.

Cauchy's form of remainder (for $p = 1$):

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x)$$

Lagrange's form of remainder (for $p = n$):

$$R^n = \frac{x^n}{n!}f^n(\theta x)$$

We have thus proved *Maclaurin's Theorem*. Thus Maclaurin's Theorem with Lagrange's form of remainder may be stated as:

If f^{n-1} is continuous in $[0, h]$ and is derivable in $]0, h[$, then for each $x \in [0, h]$, there exists a number 6 between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n}{n!}f^n(\theta x).$$

8.3 Generalised Mean Value Theorem (*Taylor's theorem*)

Deduction of Taylor's Theorem from the Mean Value Theorem

Let a function f be such that $(n-1)$th derivative f^{n-1} is continuous in $[a, a+h]$ and its nth derivative f^n exists in $]a, a+h[$. Consequently, the functions $f, f', f'' \ldots f^{n-1}$ exist and are continuous in $[a, a+h]$ while f^n exists in $]a, a+h[$.

Consider the function

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \dots$$

$$\dots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)$$

which, being the sum of continuous and derivable functions, is itself continuous in $[a, a+h]$ and derivable in $]a, a+h[$. Therefore by Lagrange's Mean Value Theorem $\exists$ a positive number θ between 0 and 1 such that

$$\phi(a+h) = \phi(a) + h\phi'(a+\theta h)$$

Now

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)$$

$$\therefore \quad \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$$

Also

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

and

$$\phi(a+h) = f(a+h)$$

$$\therefore \quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)$$

where $0 < \theta < 1$, which is *Taylor's Theorem with Cauchy's form of remainder*.

Note: For $n = 1$, the theorem reduces to the Mean Value Theorem. For this reason Taylor's Theorem is also called General Mean Value Theorem.

8.4 Taylor's Infinite Series and Power Series Expansions

We have seen that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + R_n \qquad \dots(5)$$

where R_n is the remainder after n terms.

The result can be interpreted in two ways:

(i) The value $f(a+h)$ of the function at a point may be approximated by a summation of the terms like $\frac{h^r}{r!}f^r(a)$ involving values of the function and its derivatives at some other point of the domain of definition.

(ii) The value $f(a+h)$ of the function may be expanded in powers of h.

The natural question as to how far the R.H.S. of (5) correctly represents the L.H.S. is answered if the series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots$$

converges to $f(a+h)$.

Let $S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$, so that

$$f(a+h) = S_n + R_n$$

Thus if $R_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_n = f(a+h)$$

i.e., the infinite series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^n(a) + \ldots$$

converges to $f(a+h)$.

Thus we have proved that if a function f possesses derivatives of every order in $[a, a+h]$ and Taylor's remainder $R_n \to 0$ as $n \to \infty$, then

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots = f(a+h)$$

The infinite series

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots \qquad \ldots(6)$$

is called *Taylor's series*. It can also be looked upon as expansion of $f(a+h)$ in powers of h.

Similarly for $x \in [a, a+h]$, when $\lim_{n\to\infty} R_n = 0$, we have from (4),

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots \qquad \ldots(7)$$

which is the expansion of $f(x)$ in powers of $(x-a)$.

6.5 Maclaurin's Infinite Series

We may easily deduce from (5) or (6) that if f possesses derivatives of every order in $[0, h[$ and $\lim_{n\to\infty} R_n = 0$, then for all $x \in [0, h]$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots$$

which is *Maclaurin's infinite series* expansion of $f(x)$ in powers of x.

Note: In the above discussion the remainder R_n can be in any of the forms.

Example 12. Show that the number θ which occurs in the Taylor's Theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n+1)$ as h approaches zero, provided that $f^{n+1}(x)$ is continuous and different from zero at $x = a$.

- Applying Taylor's Theorem with remainders after n terms and $n+1$ terms successively, we get

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

$$f(a+h) = f(a) + hf'(a) + \dots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h)$$

These give

$$\frac{h^n}{n!} f^n(a+\theta h) = \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta' h)$$

or

$$f^n(a+\theta h) - f^n(a) = \frac{h}{n+1} f^{n+1}(a+\theta' h)$$

Applying Lagrange's Mean Value Theorem to the left hand side, we have

$$\theta h f^{n+1}(a+\theta''\theta h) = \frac{h}{n+1} f^{n+1}(a+\theta' h)$$

or

$$\theta = \frac{1}{n+1} \frac{f^{n+1}(a+\theta' h)}{f^{n+1}(a+\theta''\theta h)}$$

Taking the limit when $h \to 0$, we get

$$\lim_{h\to 0} \theta = \frac{1}{n+1}$$

8.6 To illustrate the applications of these theorems we consider series *expansion of the functions* e^x, $\cos x$, $\log(1+x)$, $(1+x)^m$ by Maclaurin's Theorem.

1. Let $f(x) = e^x$ so that $f^n(x) = e^x$, $\forall\, n$.
   Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x.
   Let us now consider the limit of the remainder R_n.
   Taking Lagrange's form of the remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!} e^{\theta x}, 0 < \theta < 1$$

$$\therefore \qquad \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} e^{\theta x} = \left(\lim_{n\to\infty} \frac{x^n}{n!}\right) e^{\theta x} = 0$$

Thus the conditions of Maclaurin's infinite expansion are satisfied. Now $f(0) = 1$ and $f^n(0) = 1$ for all integral values of n.

Substituting these values in the Maclaurin's infinite series, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \;\forall\, x \in \mathbf{R}$$

2. Let $f(x) = \cos x$, so that $f^n(x) = \cos\left(\frac{1}{2}n\pi + x\right), \forall\, n.$

Evidently $f(x)$ and all its derivatives exist and are continuous for every real value of x. Taking Lagrange's form of the remainder,

$$R_n = \frac{x^n f^n(\theta x)}{n!} = \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right)$$

$$\Rightarrow \qquad |R_n| = \left|\frac{x^n}{n!}\right| \cdot \left|\cos\left(\frac{n\pi}{2} + \theta x\right)\right| \le \left|\frac{x^n}{n!}\right|$$

$$\therefore \qquad \lim_{n\to\infty} R_n = 0, \;\forall\, x$$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

Now

$$f(0) = 1,\; f'(0) = 0,\; f''(0) = -1, \dots,\; f^n(0) = \cos(n\pi/2)$$

which is zero for n odd, and alternately $+1, -1$ for n even.

Substituting in the equation

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \;\forall x \in R$$

3. Let $f(x) = \log(1 + x)$, for $-1 < x \le 1$, then

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

Evidently $f(x)$ and all its derivatives exist and are continuous for $|x| < 1$.

Taking the Lagrange's form of remainder, we have

$$R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{(-1)^{n-1} x^n}{n(1+\theta x)^n} = (-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

(a) When $0 \le x \le 1$, then $0 < \theta x < x \le 1$ and

$$|R_n| = \frac{x^n}{n}\left(\frac{1}{1+\theta x}\right)^n \le \frac{x^n}{n} \le \frac{1}{n}.$$

Also $\frac{1}{n} \to 0$ as $n \to \infty$

$\therefore$ $$R_n \to 0 \text{ as } n \to \infty, \text{ for } 0 \le x \le 1.$$

Thus, the conditions of Maclaurin's infinite series expansion are satisfied for $0 \le x \le 1$.

(b) When $-1 < x < 0$.

In this case x may or may not be numerically less than $1 + \theta x$, so that nothing can be said about the limit of $\left(\frac{x}{1+\theta x}\right)^n$ when $n \to \infty$. Thus we fail to draw any definite conclusion from Lagrange's form of remainder. Let us now see if the other form of the remainder is of any help.

Cauchy's form of remainder,

$$R_n = \frac{x^n(1-\theta)^{n-1} f^n(\theta x)}{(n-1)!} = \frac{(-1)^{n-1} x^n (1-\theta)^{n-1}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{(1+\theta x)}$$

Now $(1-\theta) < 1 + \theta x$ so that $\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also

$$x^n \to 0 \text{ as } n \to \infty \text{ and } \frac{1}{1+\theta x} < \frac{1}{1-|x|}$$

and moreover it is independent of n.

Thus, $R_n \to 0$ as $n \to \infty$.

Hence, the conditions of Maclaurin's series expansion are satisfied also when $-1 < x < 0$.

Thus substituting the values $f(0) = 0$, $f^n(0) = (-1)^{n-1}[(n-1)!]$ in Maclaurin's infinite expansion, we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \text{ for } -1 < x \le 1$$

4. Let $f(x) = (1+x)^m$.

Two cases arise according as m is or not a positive integer.

(a) When m is a positive integer, $f(x)$ possesses continuous derivatives of all orders upto m. Derivatives of order higher than m vanish identically. Consequently for $n > m$, $R_n \to 0$ identically so that the conditions of Maclaurin's expansion are satisfied. On substituting, the