MAINS TEST SERIES-2021

TEST-7 (BATCH-II) &

TEST-17 (BATCH-I)

FULL SYLLABUS (PAPER-I)

Answer Key

1.(a) → Let T be the set of columns of the matrix B below. Define $W = \langle T \rangle$. Find a set R so that (i) R has 3 vectors, (ii) R is a subset of T, and (iii) $W = \langle R \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Solⁿ: Let $T = \{w_1, w_2, w_3, w_4\}$. The vector $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a solution to the homogeneous system with the matrix B as the coefficient matrix. By Theorem SLSLC it provides the scalars for a linear combination of the columns of B (the vectors in T) that equals the zero vector, a relation of linear dependence on T,

$$2w_1 + (-1)w_2 + (1)w_4 = 0$$

We can rearrange this equation by solving for w_4,

$$w_4 = (-2)w_1 + w_2$$

This equation tells us that the vector w_4 is superfluous in the span construction that creates W. So $W = \langle \{w_1, w_2, w_3\} \rangle$.

The requested set is $R = \{w_1, w_2, w_3\}$.

(1)(b) Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & -1 \end{bmatrix}$$

<u>Sol^n</u>

$$A \simeq \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 2 & 3 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & -1 \end{bmatrix} (R_1 \leftrightarrow R_4)$$

$$\xrightarrow[\substack{R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1}]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 2 & 2 & 2 & 1 \\ 0 & -1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 \to R_4 + R_2 \\ R_5 \to R_5 - R_2}]{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow[R_4 \to R_4 - \frac{R_3}{2}]{R_5 \to R_5 - \frac{R_3}{2}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -5 & -9/2 \\ 0 & 0 & 0 & 1 & -3/2 \end{bmatrix} \xrightarrow{R_5 \to 5R_5 + R_4} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -5 & -9/2 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix}$$

which is echelon form of matrix.

Rank of echelon form of matrix
= Number of non zero rows

= 5

We know Rank + Nullity = dim(A)

$\Rightarrow$ 5 + Nullity = 5

$\Rightarrow$ $\boxed{\text{Nullity} = 0}$

Thus rank of matrix is 5 and Nullity is zero.

1.(C) Show that the function

$$f(x,y)=\begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2\neq 0 \\ 0, & \text{when } x^2+y^2=0 \end{cases}$$

is continuous but not differentiable at $(0,0)$.

Solⁿ: Putting $x=r\cos\theta$, $y=r\sin\theta$ we get

$$|f(x,y)-f(0,0)|=\left|\frac{r^2\cos^2\theta\cdot r\sin\theta}{r^2}-0\right|$$

$$=r|\cos\theta|\,|\cos\theta|\,|\sin\theta|$$

$$\leq r=\sqrt{x^2+y^2}.$$

Let $\epsilon>0$ be given. choose $\delta=\epsilon$. Then

$$|f(x,y)-f(0,0)|<\epsilon \text{ if } \sqrt{x^2+y^2}<\delta.$$

Hence f is continuous at the origin.

Now $f_x(0,0)=\lim\limits_{h\to 0}[f(h,0)-f(0,0)]/h$

$$=\lim_{h\to 0}\frac{0-0}{h}=0.$$

Similarly $f_y(0,0)=0$.

Let, if possible, f be differentiable at $(0,0)$.

Then

$$f(h,k)-f(0,0)=Ah+Bk+\sqrt{h^2+k^2}\,g(h,k), \quad (1)$$

where $A=f_x(0,0)$, $B=f_y(0,0)$ and $g(h,k)\to 0$ as $(h,k)\to(0,0)$.

$$\therefore \quad \frac{h^2 k}{h^2+k^2} = \sqrt{h^2+k^2}\, g(h,k)$$

$$\Rightarrow \quad g(h,k) = \frac{h^2 k}{(h^2+k^2)^{3/2}}$$

Now $\lim_{h\to 0} g(h, mh) = \frac{m}{(1+m^2)^{3/2}}$ $(k = mh)$

$$\therefore \quad \lim_{(h,k)\to(0,0)} g(h,k) = \frac{m}{(1+m^2)^{3/2}},$$

which depends on m and so the limit does not exist. This contradicts ①. Hence f is not differentiable at $(0,0)$.

1(d) If $u = \sin^{-1}\left\{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right\}^{1/2}$, then show that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{12}\tan u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144}(13+\tan^2 u)$.

Sol'n: From the given relation, we can write

$$\sin u = \left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)^{1/2} = \frac{x^{1/6}}{x^{1/4}}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]^{1/2}$$

$$= x^{-1/12} f(y/x).$$

Thus $z = \sin u$ is a homogeneous function of x and y of degree $-1/12$ and so by Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{12}z, \quad \text{where } z = \sin u.$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = -\frac{1}{12}\sin u$$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{12}\tan u$

(ii) Differentiating (1) partially w.r.t x and y, respectively

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad \text{—— (2)}$$

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = -\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

multiply (2) by x, (3) by y and add to get

$$\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= -\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \quad \text{—— (4)}$$

from (1) and (4), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left(1+\frac{1}{12}\sec^2 u\right)$$

$$= \frac{\tan u}{144}(12+1+\tan^2 u)$$

$$= \frac{\tan u}{144}(13+\tan^2 u).$$

1.(e) The sections of the enveloping cone of the surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ whose vertex is $P(x_1, y_1, z_1)$ by the plane $z=0$ is
(i) Rectangular hyperbola, (ii) a parabola

sol^n: For the given surface we have

$S = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1$; $S_1 = x_1^2/a^2 + y_1^2/b^2 + z_1^2/c^2 - 1$.

and $T = (xx_1/a^2) + (yy_1/b^2) + (zz_1/c^2) - 1$

$\therefore$ The enveloping cone is $SS_1 = T^2$

i.e. $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2$$

Its section by the plane $z=0$ is given by

$$z=0,\ \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2 \quad \text{——— (1)}$$

(i) If the equations (A) represent a rectangular hyperbola then the sum of the coefficients of x^2 and y^2 should be zero.

i.e. $$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2b^2}+\frac{1}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z_1^2=\frac{1}{a^2}+\frac{1}{b^2}$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2+b^2}+\frac{z_1^2}{c^2}=1, \text{ dividing each term by } a^2+b^2.$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1.$

(ii) If the equations (1) represent a parabola, then we should have $h^2 = ab$

Here 'a' = coefficient of $x^2 = \frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$

$b' =$ Coeff. of $y^2 = \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$ & $h' =$ coeff. of $2xy = \frac{x_1 y_1}{a^2b^2}$.

$\therefore$ If the equations (1) represent a parabola, then $h'^2 = a'b'$

i.e. $\frac{x_1^2 y_1^2}{a^4b^4} = \frac{1}{a^2b^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$

$\Rightarrow \frac{x_1^2}{a^2}\left(\frac{z_1^2}{c^2}-1\right)+\frac{y_1^2}{b^2}\left(\frac{z_1^2}{c^2}-1\right)+\left(\frac{z_1^2}{c^2}-1\right)^2=0$

$\Rightarrow \left(\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0$

$\Rightarrow \frac{z_1^2}{c^2}-1=0$, since $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1 \neq 0$ as $P(x_1, y_1, z_1)$ does not lie on the given surface.

$\Rightarrow z_1^2 = c^2 \Rightarrow z_1 = \pm c$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $z = \pm c$.

2.a(i) → Suppose that v_1 and v_2 are any two vectors from $\mathbb{C}^m$. Prove the following set equality.

$$\langle\{v_1, v_2\}\rangle = \langle\{v_1+v_2, v_1-v_2\}\rangle$$

Sol^n: This is an equality of sets, so Definition SE applies.

The "easy" half first. Show that

$$X = \langle\{v_1+v_2, v_1-v_2\}\rangle \subseteq \langle\{v_1, v_2\}\rangle = Y.$$

Choose $x \in X$. Then $x = a_1(v_1+v_2) + a_2(v_1-v_2)$ for some scalars a_1 and a_2. Then,

$$\begin{aligned} x &= a_1(v_1+v_2) + a_2(v_1-v_2) \\ &= a_1v_1 + a_1v_2 + a_2v_1 - a_2v_2 \\ &= (a_1+a_2)v_1 + (a_1-a_2)v_2 \end{aligned}$$

which qualifies x for membership in Y, as it is a linear combination of v_1, v_2.

Now show the opposite inclusion,

$$Y = \langle\{v_1, v_2\}\rangle \subseteq \langle\{v_1+v_2, v_1-v_2\}\rangle = X.$$

Choose $y \in Y$. Then there are scalars b_1, b_2 such that $y = b_1v_1 + b_2v_2$.

Rearranging, we obtain,

$$\begin{aligned} y &= b_1v_1 + b_2v_2 \\ &= \frac{b_1}{2}[(v_1+v_2)+(v_1-v_2)] + \frac{b_2}{2}[(v_1+v_2)-(v_1-v_2)] \\ &= \frac{b_1+b_2}{2}(v_1+v_2) + \frac{b_1-b_2}{2}(v_1-v_2) \end{aligned}$$

This is an expression for y as a linear combination of v_1+v_2 and v_1-v_2, earning y membership in X. Since X is a subset of Y, and vice versa, we see that $X=Y$, as desired.

(2)(a)(ii) For the matrix A below find a set of vectors T meeting the following requirements : (i) the span of T is the column space of A, that is

$\langle T \rangle = C(A)$

(ii) T is linearly independent

(iii) The elements of T are columns of A.

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 & 2 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

Sol^n Row reduced form of A –

$$\begin{bmatrix} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Pivot column are $\{1, 2, 4, 5\}$

Then $T = \{A_1, A_2, A_4, A_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

has the required properties.

2. b(i) Determine $\lim (\cot x)^{1/\log x}$, $x \to 0$.

Solⁿ: Let $y = (\cot x)^{1/\log x}$

$$\log y = \frac{1}{\log x} \log(\cot x)$$

$$\Rightarrow \lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log \cot x}{\log x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to 0} \frac{\frac{-\operatorname{cosec}^2 x}{\cot x}}{1/x}$$

$$= \lim_{x\to 0} -\frac{x}{\sin x} \cdot \frac{1}{\cos x}$$

$$= -1$$

$$\Rightarrow \log \lim_{x\to 0} y = -1$$

$$\Rightarrow \lim_{x\to 0} y = e^{-1} = 1/e.$$

2.(b)(ii) Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2}$.

Sol^n: put $a\sin x = b\cos x \tan\theta$.

$\Rightarrow a\tan x = b\tan\theta$.

$\Rightarrow a\sec^2 x\, dx = b\sec^2\theta\, d\theta$

Hence $\int_0^{\pi/2} \frac{dx}{(a^2\sin^2 x + b^2\cos^2 x)^2} = \frac{b}{a}\int_0^{\pi/2} \frac{\cos^2 x \sec^2\theta\, d\theta}{b^4\cos^4 x(\tan^2\theta + 1)^2}$

$= \frac{1}{ab^3}\int_0^{\pi/2} \frac{\cos^2\theta}{\cos^2 x}\, d\theta$

$= \frac{1}{ab^3}\int_0^{\pi/2} \cos^2\theta\left(1 + \frac{b^2}{a^2}\tan^2\theta\right) d\theta$

$= \frac{1}{a^3b^3}\int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta$

$= \frac{1}{a^3b^3}\left[a^2 \frac{1}{2}\frac{\pi}{2} + b^2\frac{1}{2}\cdot\frac{\pi}{2}\right]$

$= \frac{\pi(a^2+b^2)}{4a^3b^3}$.

2.C(i) → Find the equation of the two planes through the origin which are parallel to the line

$$\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-1}{-2} \text{ and distance } \frac{5}{3} \text{ from it.}$$

Solⁿ: The equation of any plane through origin is

$$Ax+By+Cz=0. \quad \text{——— (1)}$$

If this plane is parallel to the given line, whose direction ratios are 2, −1, −2, then the normal to this plane (1) must be perpendicular to the given line

i.e., $A\cdot 2+B\cdot(-1)+C(-2)=0$

or $2A-B-2C=0$ ——— (2)

Also the plane (1) is at a distance (5/3) from the given line i.e., at a distance 5/3 from the point (1, −3, −1) on this line.

$$\therefore \frac{A(1)+B(-3)+C(-1)}{\sqrt{(A^2+B^2+C^2)}}=\frac{5}{3}$$

or $9(A-3B-C)^2=25(A^2+B^2+C^2)$

or $9(A^2+9B^2+C^2-6AB+6BC-2AC)=25(A^2+B^2+C^2)$

or $8A^2-28B^2+8C^2+27AB-27BC+9CA=0$ ——— (3)

From (2) we have $B=2(A-C)$. Substituting this in (3) we get

$$8A^2-28\{4(A-C)^2\}+8C^2+54A(A-C)-54C(A-C)+9CA=0$$

or $\quad -50A^2 - 50C^2 + 125AC = 0$

or $\quad 2A^2 + 2C^2 - 5AC = 0$

or $\quad (2A-C)(A-2C) = 0$

or $\quad A = \frac{1}{2}C, \; 2C$

$\therefore$ From (2) we have

$$B = 2(A-C) = 2\left(\tfrac{1}{2}C - C\right), \text{ if } A = \tfrac{1}{2}C$$

$$\text{or } 2(2C-C) \text{ if } A = 2C$$

ie, $B = -C, \; 2C$

Thus we have two cases

$$A = \tfrac{1}{2}C, \; B = -C \text{ and } A = 2C = B$$

$\therefore$ From (1) the required equations are

$$\tfrac{1}{2}Cx - Cy + Cz = 0 \text{ and } 2Cx + 2Cy + Cz = 0$$

or $\quad x - 2y + 2z = 0$ and $2x + 2y + z = 0$

2. C(ii) Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ and find the point of contact.

Solⁿ: If the plane $2x-2y+z+12=0$ —— ①

touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$ —— ②

then the length of the perpendicular from the centre $(1,2,-1)$ of the sphere ② to the plane ① must be equal to the radius

$$\sqrt{[(-1)^2+2^2+(-1)^2-(-3)]}=\sqrt{9}=3 \text{ of the sphere ②}$$

i.e. $$\frac{2(1)-2(2)+1(-1)+12}{\sqrt{[2^2+(-2)^2+1^2]}}=3 \quad \text{or} \quad 9/3=3,$$

which being true the plane ① touches the sphere ②.

Also if C be the centre of the sphere and P the required point of contact, then the d.r.'s of the line CP are same as those of the normal to the plane ① i.e. $2,-2,1$. Also C is $(1,2,-1)$.

Hence the equation of the line CP is

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r \text{ (say)}$$

If $CP=r$, the coordinates of P are $(2r+1,-2r+2,r-1)$ and P lies on ①, so we have

$$2(2r+1)-2(-2r+2)+(r-1)+12=0$$

or $$9r+9=0$$

or $$r=-1.$$

$\therefore$ The coordinates of P are

$$[2(-1)+1,\ -2(-1)+2,\ -1-1]$$

or $(-1,4,-2)$

(3)(a)(i) Consider the matrix A below. Show that A is diagonalisable by computing the geometric multiplicities of eigen values and quoting the relevant theorem. Find the diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

Sol^n $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 18-\lambda & -15 & 33 & -15 \\ -4 & 8-\lambda & -6 & 6 \\ -9 & 9 & -16-\lambda & 9 \\ 5 & -6 & 9 & -4-\lambda \end{vmatrix} = 0$$

Solving we get, $\lambda = 3, 2, 2, -1$

For $\underline{\lambda = 3}$ $\Rightarrow$ $AX = \lambda X$ or $(A - \lambda I)X = 0$

$$\begin{bmatrix} 15 & -15 & 33 & -15 \\ -4 & 5 & -6 & 6 \\ -9 & 9 & -19 & 9 \\ 5 & -6 & 9 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

solving we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \alpha$$ where α is arbitrary constant

Therefore Algebric and geometric multiplicity for $(\lambda = 3) = 1$

Similary $\Rightarrow$ $\lambda = 2$ (Algebric multiplicity $= 2$)

Solving $(A - 2I)X = 0$

we get, two linearly Independent vector -

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Therefore geometric multiplicity for $\lambda = 2$ is 2.

Similarly $d = -1$ (Algebric multiplicity $= 1$)

$(A - dI)X = 0 \Rightarrow$ we get

$$\left\{ \begin{bmatrix} 6 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$ $\Rightarrow$ Therefore geometric multiplicity $= 1$

So for all eigen values

Algebric multiplicity = Geometric multiplicity $\Rightarrow$ A is diagonalizable.

Also

$$S = \begin{bmatrix} -1 & 0 & -3 & 6 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Taking eigen vector as column vector of S.

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

(3)(a)(ii) Suppose that A and B are similar matrices. Prove that A^3 and B^3 are similar matrices. Generalise

Sol^n A and B are similar, so there exist nonsingular matrix S so that $A = S^{-1}BS$

Then $A^3 = (S^{-1}BS)^3$

$= (S^{-1}BS)(S^{-1}BS)(S^{-1}BS)$

$= S^{-1}B(SS^{-1})B(SS^{-1})BS$

$= S^{-1}B(I_3)\,B(I_3)\,BS$

$= S^{-1}BBBS$

$= S^{-1}B^3S$

$\Rightarrow A^3$ is similar to B^3.

Generally, if A is similar to B & m is nonnegative integer then A^m is similar to B^m since

$A^m = S^{-1}B^mS$.

3(C) If Q is a point on the normal to the ellipsoid $\sum\left(\frac{x^2}{a^2}\right)=1$ at the point P, such that $3PQ = PG_1 + PG_2 + PG_3$, where G_1, G_2, G_3 are the points where the normal at P meets the yz, zx and xy planes, then the locus of Q is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2}+\frac{b^2y^2}{(2b^2-c^2-a^2)^2}+\frac{c^2z^2}{(2c^2-a^2-b^2)^2}=\frac{1}{9}.$$

Sol^n: Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)}=\frac{y-\beta}{(p\beta/b^2)}=\frac{z-\gamma}{(p\gamma/c^2)}=r \text{ (say)} \quad \text{—— (1)}$$

where $$\frac{1}{p^2}=\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4} \quad \text{—— (2)}$$

$\therefore$ The coordinates of any point Q on the normal (1) are

$$\left(\alpha+\frac{p\alpha}{a^2}r,\ \beta+\frac{p\beta}{b^2}r,\ \gamma+\frac{p\gamma}{c^2}r\right), \quad \text{—— (3)}$$

where r is the distance of Q from P.

If Q lies on the given ellipsoid i.e. PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha+\frac{p\alpha}{a^2}r\right)^2+\frac{1}{b^2}\left(\beta+\frac{p\beta}{b^2}r\right)^2+\frac{1}{c^2}\left(\gamma+\frac{p\gamma}{c^2}r\right)^2=1$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2rp\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)=1$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2rp\left(\frac{1}{p^2}\right)=0, \text{ from (2) } \& \sum\frac{\alpha^2}{a^2}=1$$

as $P(\alpha,\beta,\gamma)$ lies on the given conicoid.

$$r=\frac{-2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)} = \text{length of normal chord PQ} \quad \text{—— (4)}$$

Given $PQ = \frac{1}{3}(PG_1 + PG_2 + PG_3)$

$\Rightarrow r = \frac{1}{3}\left(-\frac{a^2}{p} - \frac{b^2}{p} - \frac{c^2}{p}\right)$, from the normal at any point $P(\alpha, \beta, \gamma)$ to the conicoid meets the three principal planes at G_1, G_2, G_3: then $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.

$$\Rightarrow pr = -\frac{1}{3}(a^2 + b^2 + c^2) \quad \text{——— (5)}$$

$\therefore$ from (3) & (5), we have

$$x_1 = \alpha + \frac{\alpha}{a^2}\left[-\frac{1}{3}(a^2+b^2+c^2)\right] = \frac{\alpha(2a^2-b^2-c^2)}{3a^2}$$

$$\frac{\alpha}{a} = \frac{3ax_1}{2a^2-b^2-c^2} \quad \text{——— (6)}$$

Similarly from (3) and (5), we can get

$$\frac{\beta}{b} = \frac{3by_1}{2b^2-c^2-a^2}, \quad \frac{\gamma}{c} = \frac{3cz_1}{2c^2-a^2-b^2} \quad \text{— (7)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the ellipsoid $\Sigma(x^2/a^2) = 1$, so we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1 \Rightarrow \left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2 + \left(\frac{\gamma}{c}\right)^2 = 1$$

$$\Rightarrow \left(\frac{3ax_1}{2a^2-b^2-c^2}\right)^2 + \left(\frac{3by_1}{2b^2-c^2-a^2}\right)^2 + \left(\frac{3cz_1}{2c^2-a^2-b^2}\right)^2 = 1$$

$\therefore$ The locus of $Q(x_1, y_1, z_1)$ is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)} + \frac{b^2y^2}{(2b^2-c^2-a^2)^2} + \frac{c^2z^2}{(2c^2-a^2-b^2)^2} = 1/9.$$

(4)(a) Let $T: U \to V$ & $S: V \to W$ be two Linear maps. Then

(i) If S & T are nonsingular, then ST is nonsingular & $(ST)^{-1} = T^{-1}S^{-1}$

(ii) If ST is one-one then T is one-one

(iii) If ST is onto then S is onto

(iv) If ST is nonsingular, then T is one-one and S is onto

(v) If U, V, W are of the same finite dimension and ST is nonsingular, then both S & T are nonsingular.

Sol^n (i) Since S is nonsingular, S^{-1} is defined and $SS^{-1} = I_W$ & $S^{-1}S = I_V$

Since T is nonsingular, T^{-1} is defined & $TT^{-1} = I_V$, $T^{-1}T = I_U$

$$(ST)(T^{-1}S^{-1}) = S(T(T^{-1}S^{-1})) = S((TT^{-1})S^{-1})$$
$$= S(I_V S^{-1}) = SS^{-1} = I_W$$

Similarly, $(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}(ST))$
$$= T^{-1}((S^{-1}S)T)$$

$= T^{-1}(I_V T) = T^{-1}T = I_U$

Therefore ST is nonsingular and $(ST)^{-1} = T^{-1}S^{-1}$.

(ii) Let $u \in N(T)$ then $T(u) = 0_V$.

So $S(T(u)) = 0_W$ i.e. $(ST)(u) = 0_W$

But ST is one-one

Therefore $u = 0_U$ i.e. $N(T) = \{0_U\}$

Thus, T is one-one.

(iii) Let $w \in W$. Since ST is onto, there exists a vector $u \in U$ s.t. $(ST)(u) = w$. Therefore $S(T(u)) = w$

~~ther~~ Hence, there exist a vector $v = T(u) \in V$ s.t. $S(u) = W$

Thus ~~t~~ S is onto.

4.(b) The sphere $x^2+y^2+z^2=a^2$ is pierced by the cylinder $(x^2+y^2)^2=a^2(x^2-y^2)$; prove that the volume of the sphere that lies inside the cylinder is $\frac{8}{3}\left(\frac{\pi}{4}+\frac{5}{3}-\frac{4\sqrt{2}}{3}\right)a^3$.

Solⁿ: Here the limits of z are from $-\sqrt{(a^2-x^2-y^2)}$ to $\sqrt{(a^2-x^2-y^2)}$ and therefore

The volume $=\iiint dx\,dy\,dz = 2\iint \sqrt{(a^2-x^2-y^2)}\,dx\,dy$.

Now the equation of the cylinder is

$$(x^2+y^2)^2 = a^2(x^2-y^2) \quad \text{—— (1)}$$

Putting $x=r\cos\theta,\ y=r\sin\theta$ in (1), we get

$$(r^2)^2 = a^2r^2\cos 2\theta$$

i.e. $r^2 = a^2\cos 2\theta$.

$\therefore$ The limits of r are from $-a\sqrt{(\cos 2\theta)}$ to $a\sqrt{(\cos 2\theta)}$ and limits of θ are from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$.

$\therefore$ The required volume

$$= 2\int_{-\pi/4}^{\pi/4}\int_{-a\sqrt{(\cos 2\theta)}}^{a\sqrt{(\cos 2\theta)}} \sqrt{(a^2-r^2)}\,r\,d\theta\,dr$$

$$= 8\int_0^{\pi/4} \frac{1}{2}\left[-\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{8}{3}a^3\int_0^{\pi/4}\left[1-(1-\cos 2\theta)^{3/2}+1\right]d\theta$$

$$= \frac{8a^3}{15}\int_0^{\pi/4}(1-2^{3/2}\sin^3\theta)\,d\theta$$

$$= \frac{8}{3}a^3 \int_0^{\pi/4} \left[1 - 2^{3/2}(1-\cos^2\theta)\sin\theta\right] d\theta$$

$$= \frac{8}{3}a^3 \left[\theta - 2^{3/2}\left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\right]_0^{\pi/4}$$

$$= \frac{8}{3}a^3 \left[\frac{\pi}{4} - 2^{3/2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{3\times 2\sqrt{2}} + 1 + \frac{1}{3}\right)\right]$$

$$= \frac{8}{3}a^3 \left[\frac{\pi}{4} + \frac{5}{3} - \frac{4\sqrt{2}}{3}\right].$$

4.(c) Prove that in general two generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse $(x^2/a^6)+(y^2/b^6)=c^4/(a^4b^4)$.

Sol^n We know for the given hyperboloid, the generator belonging to λ-system is given by –

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \text{ and } \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right) \quad \text{—(i)}$$

or $$\frac{x}{a}+\frac{\lambda y}{b}-\frac{z}{c}=\lambda \text{ and } \lambda\frac{x}{a}-\frac{y}{b}+\lambda\frac{z}{c}=1$$

$\therefore$ If l_1, m_1, n_1 be the dr.'s of the generator (i)

then $$\frac{l_1}{a}+\frac{\lambda m_1}{b}-\frac{n_1}{c}=0 \text{ and } \frac{\lambda l_1}{a}-\frac{m_1}{b}+\frac{\lambda n_1}{c}=0$$

Solving these simultaneously, we get

$$\frac{l_1/a}{\lambda^2-1}=\frac{m_1/b}{-\lambda-\lambda}=\frac{n_1/c}{-1-\lambda^2}$$

or $$\frac{l_1}{-a(\lambda^2-1)}=\frac{m_1}{2\lambda b}=\frac{n_1}{c(1+\lambda^2)} \quad \text{— (ii)}$$

Similarly the direction ratio l_2, m_2, n_2 of the generator belonging to μ-system viz.

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right) \text{ and } \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad \text{—(iii)}$$

are given by $$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)} \quad \text{— (iv)}$$

If these two generators given by (i) and (iii) are perpendicular then, $-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(\mu^2+1)=0$ —(v)

Now if λ-generator is given, then λ is constant and (v) will be a quadratic equation in μ which gives two values of μ and this shows that there will be two generators of μ-system which will be perpendicular to a generator of λ-system.

Now let the generators of μ-system meet the plane $z=0$ in the points $P(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$

$\therefore$ The generator of the μ-system through these points are given by

$$\frac{x-a\cos\alpha}{a\sin\alpha} = \frac{y-b\sin\alpha}{-b\cos\alpha} = \frac{z}{c} \quad \text{——(vi)}$$

and $$\frac{x-a\cos\beta}{a\sin\beta} = \frac{y-b\sin\beta}{-b\cos\beta} = \frac{z}{c} \quad \text{——(vii)}$$

These two generators intersect at right angles a generator of λ-system through any point $(a\cos\theta, b\sin\theta, 0)$ say whose equations are

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{-c} \quad \text{——(viii)}$$

As (vi) and (vii) are both perpendicular to (viii), so

$$a^2\sin\alpha\sin\theta + b^2\cos\alpha\cos\theta - c^2 = 0$$

and $$a^2\sin\beta\sin\theta + b^2\cos\beta\cos\theta - c^2 = 0$$

Solving these simultaneously for $a^2\sin\theta$, $b^2\cos\theta$ and $-c^2$, we get

$$\frac{a^2\sin\theta}{\cos\alpha-\cos\beta} = \frac{b^2\cos\theta}{\sin\beta-\sin\alpha} = \frac{-c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

or $\dfrac{a\sin\theta}{2\sin\frac{\alpha+\beta}{2}.\sin\frac{\beta-\alpha}{2}} = \dfrac{b^2\cos\theta}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}} = \dfrac{-c^2}{\sin(\alpha-\beta)}$

$$= \frac{-c^2}{2\sin\frac{\alpha-\beta}{2}.\cos\frac{\alpha-\beta}{2}}$$

$\Rightarrow \dfrac{a^2\sin\theta}{c^2} = \dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$ $\quad \dfrac{b^2\cos\theta}{c^2} = \dfrac{\cos\frac{(\alpha+\beta)}{2}}{\cos\frac{\alpha-\beta}{2}}$ ——— (ix)

Also equation of the joining P and Q is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}, \quad z=0$$

or $\dfrac{x}{a}\left(\dfrac{b^2\cos\theta}{c^2}\right) + \dfrac{y}{b}\left(\dfrac{a^2\sin\theta}{c^2}\right) = 1, \quad z=0$ ——(x)

using the results of (ix).

Now in order to find its envelope, we should differentiate (x) with respect to θ and then eliminate θ.

Differentiating (x) w.r.to θ, we get

$$-\frac{xb^2}{ac^2}\sin\theta + \frac{ya^2}{bc^2}\cos\theta = 0, \quad z=0 \quad \text{——(xi)}$$

Squaring and adding (x) and (xi), θ is eliminated and we get the required envelope of PQ as

$\dfrac{x^2b^4}{a^2c^4} + \dfrac{y^2a^4}{b^2c^4} = 1, \ z=0$ or $\dfrac{x^2}{a^6} + \dfrac{y^2}{b^6} = \dfrac{c^4}{a^4b^4}, z=0$

which represents an ellipse on the plane $z=0$

Hence proved.

5.(a) → Find the orthogonal trajectories of cardioids $r = a(1-\cos\theta)$, a being parameter.

Sol'n: The given family of cardioids is

$$r = a(1-\cos\theta) \quad \text{---} ①$$

Taking logarithm of bothsides of ②, we get

$$\log r = \log a + \log(1-\cos\theta) \quad \text{---} ②$$

Differentiating ② with respect to 'θ', we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} \quad \text{---} ③$$

Hence ③ is the differential equation of the given family ①

Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in ③, the differential equation of the required orthogonal trajectories is $\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta}$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$\Rightarrow \frac{1}{r}dr = -\tan(\theta/2)\,d\theta$, on separating variables

Integrating, $\log r = 2\log\cos\theta/2 + \log c$

$$\log r = \log(c\cos^2\theta/2)$$

$$\Rightarrow r = c/2\,(1+\cos\theta)$$

$$\Rightarrow r = b(1+\cos\theta) \quad \text{---} ④$$

where $b(=c/2)$ is arbitrary constant.

④ gives another family of cardioids.

5.(b) → Apply the method of variation of parameters to solve
$$x^2y''-2xy'+2y = x\log x,\ x>0$$

Sol^n: Re-writing, given equation is
$$y''-(2/x)y'+(2/x^2)y = (1/x)\log x \quad \text{——— ①}$$
or $\{D^2-(2/x)D+(2/x^2)\}y = (1/x)\log x$,

where $D \equiv d/dx$

comparing ① with $y''+Py'+Qy=R$

here $R=(1/x)\log x$

Consider $\{D^2-(2/x)D+(2/x^2)\}y=0$

or $\{x^2D^2-2xD+2\}y=0$ ——— ②

Let $x=e^z$ or $\log x = z$.

Also let $D_1 \equiv d/dz$ ——— ③

Then $xD = D_1$ and $x^2D^2 = D_1(D_1-1)$ and so ② reduces to
$$\{D_1(D_1-1)-2D_1+2\}y=0$$
or $(D_1^2-3D_1+2)y=0$

Whose auxiliary equation is $D_1-3D_1+2=0$ giving $D_1=1,2$

$\therefore$ C.F. of ① $= c_1e^z+c_2e^{2z}$

$= c_1e^z + c_2(e^z)^2 = c_1x+c_2x^2$, ——— ④

c_1 and c_2 being arbitrary constants.

Let $u=x$ and $v=x^2$ Also, here $R=(1/x)\log x$ ——— ⑤

Here $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2-x^2 = x^2 \neq 0$, ——— ⑥

$\therefore$ P.I. of (1) $= uf(x) + vg(x)$, ——— (7)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^2 \log x}{x^2 \times x} dx$

$= -\int \log x \cdot \frac{1}{x} dx = -\frac{(\log x)^2}{2}$, using (5) and (6)

and $g(x) = \int \frac{uR}{W} dx = \int \frac{\log x}{x^2} dx$

$= \int \log x \cdot x^{-2} dx$, by (5) and (6)

$= \log x \times \frac{x^{-1}}{(-1)} - \int \frac{1}{x} \times \frac{x^{-1}}{(-1)} dx$, on Integrating by parts

$= -\frac{\log x}{x} + \int x^{-2} dx = -\frac{\log x}{x} + \frac{x^{-1}}{(-1)}$

$= -\frac{1}{x}(1+\log x)$

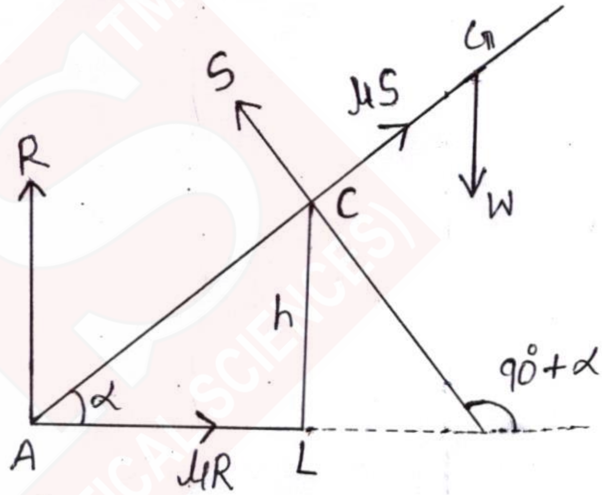
$\therefore$ using (7), P.I. of (1) $= x \times (-1/2) \times (\log x)^2 + x^2 \times (-1/x) \times (1+\log x)$

$= -(x/2) \times (\log x)^2 - x(1+\log x)$

$\therefore$ The solution of (1) is

$$y = C_1 x + C_2 x^2 - (x/2) \times (\log x)^2 - x(1+\log x)$$

5.(c) → A straight uniform beam of length $2h$ rest in limiting equilibrium in contact with a rough vertical wall of height h, with one end on a rough horizontal plane and with the other end projecting beyond the wall. If both the wall the plane be equally rough; Prove that λ, the angle of friction is given by $\sin 2\lambda = \sin\alpha \sin 2\alpha$, where α is the inclination of the beam to the horizone.



Soln: Let μ be the co-efficient of friction, then $\mu = \tan\lambda$.

Taking moments about A,

$$S \cdot AC = W \cdot AG \cos\alpha$$

or $S \cdot h \operatorname{cosec}\alpha = W \cdot h \cos\alpha$

$$\therefore \quad S = W \sin\alpha \cos\alpha = \frac{W}{2}\sin 2\alpha \quad \text{——— (1)}$$

Resolving horizontally,

$$\mu R + \mu S \cos\alpha = S \sin\alpha \quad \text{——— (2)}$$

Resolving vertically,

$$R + \mu S \sin\alpha + S\cos\alpha = W \quad \text{——— (3)}$$

Multiplying (3) by μ and subtracting (2) from the result

$$\mu^2 S \sin\alpha = \mu W - S\sin\alpha$$

or $\mu^2 \cdot \frac{W}{2}\sin 2\alpha \sin\alpha = \mu W - \frac{W}{2}\sin 2\alpha \cdot \sin\alpha$ $\quad (\because$ of (1))

or $(1+\mu)^2 \cdot \frac{1}{2}\sin 2\alpha \sin\alpha = \mu$

or $\sin 2\alpha \sin\alpha = \frac{2\mu}{1+\mu^2} = \frac{2\tan\lambda}{1+\tan^2\lambda}$

or $\sin 2\alpha \sin\alpha = \sin 2\lambda$.

5.(d) → A particle is moving with S.H.M. of amplitude a and periodic time T. Prove that $\int_0^T v^2 dt = \frac{2\pi^2 a^2}{T}$.

Sol^n: We know that

$$v^2 = \mu(a^2 - x^2) \quad \text{——— (1)}$$

$$x = a\sin\sqrt{\mu}t \quad \text{——— (2)}$$

$$T = 2\pi/\sqrt{\mu} \quad \text{——— (3)}$$

From (3), $\mu = \frac{4\pi^2}{T^2}$

From (1) and (2), we have

$$v^2 = \frac{4\pi^2}{T^2}(a^2 - a^2\sin^2\sqrt{\mu}t)$$

$$= \frac{4\pi^2 a^2}{T^2}\cos^2\sqrt{\mu}t = \frac{4\pi^2 a^2}{T^2}\cos^2\frac{2\pi t}{T}$$

$$\Rightarrow \int_0^T v^2 dt = \int_0^T \frac{4\pi^2 a^2}{T^2}\cos^2\frac{2\pi t}{T}\,dt$$

$$= \frac{2\pi^2 a^2}{T^2}\int_0^T 2\cos^2\frac{2\pi t}{T}\,dt$$

$$= \frac{2\pi^2 a^2}{T^2}\int_0^T \left(1 + \cos\frac{4\pi t}{T}\right)dt$$

$$= \frac{2\pi^2 a^2}{T^2}\left[t + \frac{T}{4\pi}\sin\frac{4\pi t}{T}\right]_0^T$$

$$= \frac{2\pi^2 a^2}{T^2}[T] = \frac{2\pi^2 a^2}{T}.$$

5(e) → Find the value of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Sol^n: The directional derivative is maximum along the normal ie along grad ϕ.

$$\text{grad}\,\phi = (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)K.$$

$$= (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)K \text{ at } (1,2,-1)$$

But directional derivative is maximum along z-axis hence coefficient of $\hat{i}$ and $\hat{j}$ should be zero

$$4a+3c=0 \text{ and } 4a-b=0$$

$\therefore$ grad $\phi = (2b-2c)K$ Also maximum value of directional derivative $= |\text{grad}\,\phi| \Rightarrow 2(b-c)=64$

$$\Rightarrow (b-c) = 32.$$

Ans.

6.(a)(i) Sol^n: Given eq^n $(2xy^4e^y + 2xy^3+y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ —— ①

Comparing ① with $M\,dx + N\,dy = 0$, we get

$M = 2xy^4e^y + 2xy^3 + y$ & $N = x^2y^4e^y - x^2y^2 - 3x$ —— ②

Here $\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$ & $\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$.

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2x\cdot y^3e^y + 2xy^2 + 1) = \frac{-4}{y}(2xy^4e^y + 2xy^3 + y) = \frac{-4M}{y}$

$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}$, which is a function of y alone.

$\Rightarrow$ I.F of ① $= e^{\int(-4/y)\,dy}$

$= e^{-4\log y} = 1/y^4$

Multiplying ① by $1/y^4$, we have

$\left\{2xe^y + 2x/y + 1/y^3\right\}dx + \left\{x^2e^y - x^2/y^2 - 3(x/y^4)\right\}dy = 0$

whose solution as usual is

$\int\left\{2xe^y + (2x/y) + (1/y^3)\right\}dx = c$

(Treating y as constant)

$\Rightarrow x^2e^y + (x^2/y) + (x/y^3) = c$.

6.(a)(ii) → Solve and examine for singular solution of $x^2(y-xp) = yp^2$.

Sol^n: given ~~that~~ $x^2(y-xp) = yp^2$ —— (1)

$$\Rightarrow y = \frac{x^3 p}{x^2 - p^2} \text{ —— (2)}$$

clearly it is solvable for y.

Diff. it w.r.t x, we get

$$\frac{dy}{dx} = \frac{(x^2-p^2)\left(3x^2 p + x^3\frac{dp}{dx}\right) - (x^3 p)\left(2x - 2p\frac{dp}{dx}\right)}{(x^2-p^2)^2}$$

$$\Rightarrow p = \frac{(x^2-p^2)3x^2 p - 2x^4 p + \left[(x^2-p^2)x^3 + 2x^3p^2\right]\frac{dp}{dx}}{(x^2-p^2)^2}$$

$$\Rightarrow p = \frac{x^4 p - 3x^2p^3 + \left[x^5 + x^3p^2\right]\frac{dp}{dx}}{(x^2-p^2)^2}$$

$$\Rightarrow \cancel{x^4 p} - 3x^2p^3 - p(\cancel{x^4} + p^4 - 2x^2p^2) = (x^5 + x^3p^2)\frac{dp}{dx}$$

$$\Rightarrow -x^2p^3 - p^5 = (x^5 + x^3p^2)\frac{dp}{dx}$$

$$\Rightarrow -p^3(x^2+p^2) = x^3(x^2+p^2)\frac{dp}{dx}$$

$$\Rightarrow -p^3 = x^3\frac{dp}{dx}$$

$$\Rightarrow \frac{1}{x^3}dx + \frac{1}{p^3}dp = 0$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{p^2} = c_1 \text{ —— (3)}$$

The g.s of given ϵ ty

The equations (2) & (3) together

let us examine the singular solution:

first of all, we find p-discriminant:

$$① \equiv yp^2 + x^3 p - x^2 y = 0 \quad — ④$$

p-discri is $b^2 - 4ac = 0$

$$(x^3)^2 + 4y^2x^2 = 0.$$

$$x^2(x^4 + 4y^2) = 0.$$

$$\Rightarrow x = 0, \quad x^4 + 4y^2 = 0. \quad — ⑤$$

– Clearly $x=0$ does not satisfy the given ODE.

We have
$$x^4 + 4y^2 = 0$$
$$\Rightarrow 4x^3 + 8y\frac{dy}{dx} = 0$$
$$\Rightarrow x^3 = -2y\frac{dy}{dx} \Rightarrow x^3 = -2yp$$
$$\Rightarrow \boxed{p = -\frac{x^3}{2y}}$$

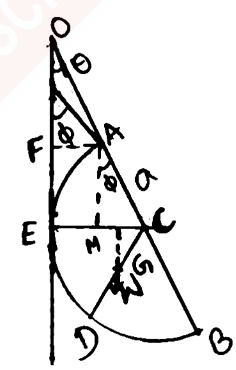
$$② \equiv \frac{x^3 p}{x^2 - p^2} = \frac{x^3\left(\frac{-x^3}{2y}\right)}{x^2 - \frac{x^6}{4y^2}} = \frac{-x^6}{2y} \times \frac{4y^2}{x^2 4y^2 - x^6}$$
$$= \frac{-2x^6 y}{4x^2y^2 - x^6}$$
$$= \frac{-2x^4 y}{4y^2 - x^4}.$$
$$\neq y.$$

$\therefore$ $x^4 + 4y^2 = 0$ also does not the given ODE.

Clearly there exists no singular and there does not require to find c-discri.

6.(b). A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that
$\tan\phi = \frac{3}{8} + \tan\theta$.

Sol'n: 'O' is a fixed point in the wall to which one end of the string has been attached. Let l be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C. The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that $CG = 3/8\, a$.



The hemisphere touches the wall at E.
we have $\angle OEC = 90°$ so that EC is horizontal.
The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below O $= OF + AM + NG$

$$= l\cos\theta + a\cos\phi + 3/8\, a\sin\phi$$

[Note that $\angle NCG = 90° - \angle ACM = 90° - (90° - \phi) = \phi$]

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed, the length of the string AO does not change so that the workdone by its tension is zero & the point G is slightly displaced. The $\angle OEC$ remains $90°$.

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above. The equation of virtual work is

$$W\delta\left(l\cos\theta + a\cos\phi + \tfrac{3}{8}a\sin\phi\right) = 0$$

$$\Rightarrow -l\sin\theta\,\delta\theta - a\sin\phi\,d\phi + \tfrac{3}{8}a\cos\phi\,\delta\phi = 0$$

$$\Rightarrow l\sin\theta\,\delta\theta = a\left(\tfrac{3}{8}\cos\phi - \sin\phi\right)d\phi \quad \text{---} (1)$$

from the fig, $EC = a$

Also $EC = EM + MC = FA + MC$

$$= l\sin\theta + a\sin\phi$$

$$\therefore a = l\sin\theta + a\sin\phi$$

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,d\phi$

$$\Rightarrow -l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi \quad \text{---} (2)$$

Dividing (1) by (2), we get

$$-\tan\theta = \tfrac{3}{8} - \tan\phi$$

$$\tan\phi = \tfrac{3}{8} + \tan\theta.$$

(6)(c) Find T, N, B, k, τ for the space curve

$r(t) = 3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4t\,\hat{k}$.

Soln $r(t) = 3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4t\,\hat{k}$

$$\frac{d\vec{r}}{dt} = 3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 4\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -3\sin t\,\hat{i} - 3\cos t\,\hat{j} \cancel{+\hat{k}} + 0\hat{k}$$

~~we know~~ $$\frac{d^3\vec{r}}{dt^3} = -3\cos t\,\hat{i} + 3\sin t\,\hat{j}$$

we know $T = \dfrac{d\vec{r}}{dt} \Big/ \left|\dfrac{d\vec{r}}{dt}\right|$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (4)^2}$$

$$= \sqrt{9(\cos^2 t + \sin^2 t) + 16} = 5$$

$$T = \frac{3\cos t\,\hat{i} - 3\sin t\,\hat{j} + 4\hat{k}}{5}$$

$$K = \frac{\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|}{\left|\dfrac{dr}{dt}\right|^3}, \quad \tau = \frac{\left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right]}{\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|^2}$$

$$\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \end{vmatrix}$$

$$= \hat{i}(12\cos t) - \hat{j}(12\sin t) + \hat{k}(-9\cos^2 t - 9\sin^2 t)$$

$$= 12\cos t\,\hat{i} - 12\sin t\,\hat{j} - 9\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(12\cos t)^2 + (-12\sin t)^2 + (-9)^2}$$

$$= 13$$

$$K = \frac{13}{(5)^3} = \frac{13}{125}$$

$$\tau = \frac{\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d^3\vec{r}}{dt^3}}{(13)^2} = \frac{(12\cos t)(-3\cos t) + (-12\sin t)(3\sin t) + 0}{(13)^2}$$

$$= \frac{-36(\sin^2 t + \cos^2 t)}{(13)^2} = \frac{-36}{169}.$$

$$N = \frac{1}{K}\frac{dT}{ds} = \frac{1}{K}\, {}^{dT}/_{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$$

$$= \frac{1}{(13/125)} \times \frac{1}{5}\,\frac{(-3\sin t\,\hat{i} - 3\cos t\,\hat{j})}{5}$$

$$= \frac{1}{13}(-3\sin t\,\hat{i} - 3\cos t\,\hat{j})$$

$$B = T\times N = \frac{1}{13}(12\cos t\,\hat{i} - 12\sin t\,\hat{j} - 9\hat{k})$$

7.(a) → solve $\left(\frac{d^2y}{dx^2}\right) - \cot x\left(\frac{dy}{dx}\right) - (1-\cot x)y = e^x \sin x$.

Sol^n: comparing the given equation with

$$y'' + Py' + Qy = R, \text{ we get}$$

$$P = -\cot x, \quad Q = -1+\cot x, \quad R = e^x \sin x$$

Here $1+P+Q=0$, showing that $y=u=e^x$ is a part of C.F of the given equation.

Let the required general solution by $y=uv$.

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(-\cot x + \frac{2}{e^x}e^x\right)\frac{dv}{dx} = \sin x$$

$$\Rightarrow \frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = \sin x \quad \text{——} \quad (1)$$

Let $\frac{dv}{dx} = t$ so that $\frac{d^2v}{dx^2} = \frac{dt}{dx}$

$\therefore$ from (1)

$$\frac{dt}{dx} + (2-\cot x)t = \sin x$$

which is linear in t & x

$$\text{I.F} = e^{\int(2-\cot x)dx} = e^{2x-\log\sin x}$$

$$= e^{2x}(\sin x)^{-1}$$

$$\therefore t\frac{e^{2x}}{\sin x} = \int\left(\sin x\frac{e^{2x}}{\sin x}\right)dx + c_1$$

$$= \frac{1}{2}e^{2x} + c_1$$

$$\Rightarrow t = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x}\sin x$$

$$\Rightarrow dv = \left(\frac{1}{2}\sin x + c_1 e^{-2x}\sin x\right)dx$$

Integrating,

$$v = -\frac{1}{2}\cos x + c_1\int e^{-2x}\sin x\, dx$$

$$= -\frac{\cos x}{2} + \frac{c_1}{1+4}(-2\sin x - \cos x) + c_2$$

$$v = -\frac{\cos x}{2} - \frac{c_1}{5}(2\sin x + \cos x) + c_2$$

$\therefore$ The required solution is $y = uv$

$$\text{i.e. } y = e^{x}\left[-\frac{\cos x}{2} + \frac{c_1}{5}e^{-2x}(2\sin x + \cos x) + c_2\right]$$

$$\Rightarrow y = -\frac{1}{2}e^{x}\cos x + c_2 e^{x} + c_1' e^{-x}(2\sin x + \cos x)$$

where $c_1' = c_1/5$

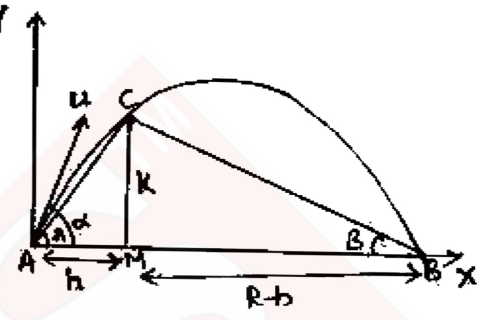
7.(b)(i) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, Prove that

$$\tan\alpha = \tan A + \tan B.$$

Sol^n: Let A be the point of projection, u the velocity of projection and α the angle of projection.

The particle while grazing over the vertex C falls at the point B.

If $AB = R$, then $R = \dfrac{2u^2 \sin\alpha\cos\alpha}{g}$ ——(1)



Take the horizontal line AB as the x-axis and the vertical line AY as the y-axis. Let the coordinates of the vertex C be (h, k). Then the point (h,k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore\ k = h\tan\alpha - \frac{1}{2}g\frac{h^2}{u^2\cos^2\alpha}$$

$$= h\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$$= h\tan\alpha\left[1 - \frac{h}{R}\right] \quad \text{by (1)}$$

$$\therefore\ \frac{k}{h} = \tan\alpha\left(\frac{R-h}{R}\right) \qquad \left[\because \text{from } \Delta CAM,\ \tan A = \frac{k}{h}\right]$$

$$\Rightarrow \tan A = \tan\alpha\left(\frac{R-h}{R}\right)$$

$$\therefore\ \tan\alpha = \tan A\left(\frac{R}{R-h}\right)$$

$$= \tan A\left(\frac{(R-h)+h}{R-h}\right)$$

$$= \tan A\left[1 + \frac{h}{R-h}\right] = \tan A + \tan A\left(\frac{h}{R-h}\right)$$

$$= \tan A + \frac{k}{h}\cdot\frac{h}{R-h} \qquad \left[\because \tan A = \frac{k}{h}\right]$$

$$= \tan A + k/(R-h)$$

But from the ΔCMB, $\tan B = k/(R-h)$

$$\therefore\ \tan\alpha = \tan A + \tan B$$

7(b)(ii). A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{gm^2}{\mu^2}\left[e^{-(\mu/m)t}-1+\frac{\mu t}{m}\right]$$

Sol: Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e., in the direction of x decreasing. The weight mg of the particle acts vertically downwards i.e., in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=mg-\mu v$$

or, $$\frac{dv}{dt}=g-\frac{\mu}{m}v, \qquad \left[\because \frac{d^2x}{dt^2}=\frac{dv}{dt}\right]$$

or, $$dt=\frac{dv}{g-(\mu/m)v}$$

Integrating, we have

$$t=-\frac{m}{\mu}\log\left(g-\frac{\mu}{m}v\right)+A,$$

where A is a constant.

But initially when $t=0$, $v=0$;

$\therefore \; A = (m/\mu) \log g$

$\therefore \; t = -\frac{m}{\mu} \log\left(g - \frac{\mu}{m} v\right) + \frac{m}{\mu} \log g$

or, $t = -\frac{m}{\mu} \log\left\{\frac{g - (\mu/m)v}{g}\right\}$

or, $-\frac{\mu t}{m} = \log\left(1 - \frac{\mu}{gm} v\right)$

or, $1 - \frac{\mu}{gm} v = e^{-\mu t/m}$

or, $v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right)$

or, $dx = \frac{gm}{\mu}\left(1 - e^{-\mu t/m}\right) dt.$

Integrating, we have

$$x = \frac{gm}{\mu}\left[t + \frac{m}{\mu} e^{-\mu t/m}\right] + B \quad \text{--- (1)}$$

where B is a constant.

But initially when $t=0$, $x=0$.

$$\therefore \; 0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B \quad \text{--- (2)}$$

subtracting (2) from (1), we have

$$x = \frac{gm}{\mu}\left[\frac{m}{\mu} e^{-\mu t/m} - \frac{m}{\mu} + t\right.$$

$$= \frac{gm^2}{\mu^2}\left[e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right]$$

7.(C) Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 3 and if the force field is given by

$$F=(2x-y+z)i+(x+y-z^2)j+(3x-2y+4z)K$$

Solⁿ: In the plane $z=0$,

$$F=(2x-y)i+(x+y)j+(3x-2y)K \text{ and}$$

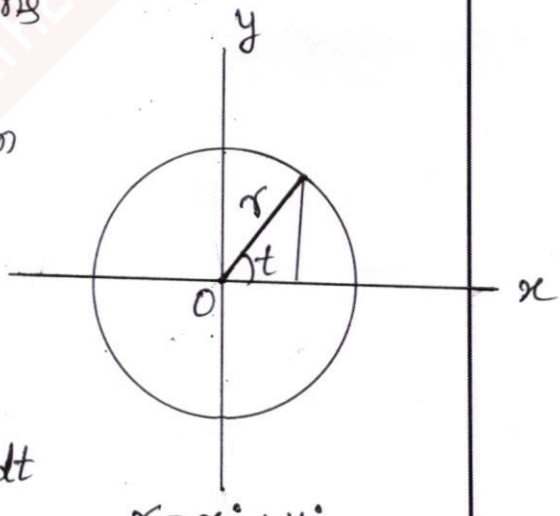
$$dr=dxi+dyj$$

So that the work done is

$$\int_C F.dr=\int_C [(2x-y)i+(x+y)j+(3x-2y)K]\cdot[dxi+dyj]$$

$$=\int_C (2x-y)dx+(x+y)dy$$

choose the parametric equations of the circle as $x=3\cos t$, $y=3\sin t$, where t varies from 0 to 2π. Then the line integral equals



$r=xi+yj$
$=3\cos t\, i+3\sin t\, j$

$$\int_{t=0}^{2\pi}[2(3\cos t)-3\sin t][-3\sin t]dt+[3\cos t+3\sin t][3\cos t]dt$$

$$=\int_0^{2\pi}(9-9\sin t\cos t)dt$$

$$=9t-\frac{9}{2}\sin^2 t\Big|_0^{2\pi}=18\pi$$

7.(d) If $A = (2y+3)i + xz\,j + (yz - x)K$, evaluate $\int_C A \cdot dr$ along the following paths C:

(i) $x = 2t^2,\ y = t,\ z = t^3$ from $t = 0$ to $t = 1$,

(ii) the straight lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$, and then to $(2,1,1)$,

(iii) the straight line joining $(0,0,0)$ and $(2,1,1)$.

Soln: Please do yourself.

ANS: (i) $288/35$

(ii) 10

(iii) 8.

8. a(i) Evaluate $L^{-1}\{e^{-4s}/(s-3)^4\}$

Sol^n: Let $f(s) = 1/(s-3)^4$ and $F(t) = L^{-1}\{f(s)\}$ ——— ①

$$\therefore F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{(s-3)^4}\right\} = e^{3t}L^{-1}\left\{\frac{1}{s^4}\right\}$$

[Using first shifting theorem]

or $F(t) = e^{3t}(t^3/3!) = (1/6)t^3e^{3t}$ ——— ②

Hence by second shifting theorem, we have

$$L^{-1}\{e^{-4s}f(s)\} = \begin{cases} F(t-4), & t>4 \\ 0, & t<4 \end{cases}$$

or
$$L^{-1}\left\{e^{-4s}\frac{1}{(s-3)^4}\right\} = \begin{cases} (1/6)(t-4)^4e^{3(t-4)}, & t>4 \\ 0, & t<4 \end{cases}$$

using ① and ②

$$= (1/6)(t-4)^4e^{3(t-4)}H(t-4),$$

interms of Heaviside unit step function.

8.(a)(ii) By using Laplace transform solve $(D^2+m^2)x = a\sin nt$, $t>0$ where x, Dx equal to x_0 and x_1, when $t=0$, $n\neq m$.

Sol^n: Re-writing the given equation and conditions, we get $x'' + m^2 x = a\sin nt$ — ①

with initial conditions: $x(0) = x_0$ and $x'(0) = x_1$ — ②

Taking Laplace transform of bothsides of ①, we get

$$L\{x''\} + m^2 L\{x\} = aL\{\sin nt\}$$

$$\Rightarrow s^2 L\{x\} - s x(0) - x'(0) + m^2 L\{x\} = \frac{an}{(s^2+n^2)}$$

$$\Rightarrow (s^2+m^2) L\{x\} - s x_0 - x_1 = \frac{an}{(s^2+n^2)}, \text{ using ②}$$

$$\Rightarrow L\{x\} = \frac{x_0 s}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{an}{(s^2+m^2)(s^2+n^2)}$$

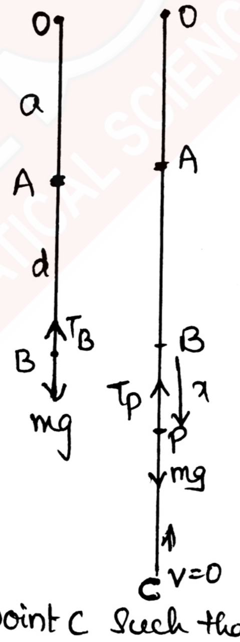
$$\therefore L\{x\} = \frac{x_0 s}{s^2+m^2} + \frac{x_1}{s^2+m^2} + \frac{an}{(s^2+m^2)(s^2+n^2)}$$

Taking inverse Laplace transform of bothsides, we get

$$x = x_0 \cos mt + \frac{x_1}{m}\sin mt + \frac{an}{m^2-n^2}\left[\frac{1}{n}\sin nt - \frac{1}{m}\sin mt\right]$$

8.(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\frac{a}{g}}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, where a is the natural length of the string.

Sol'n: Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law, $T_B=\lambda\frac{OB-OA}{OA}=\lambda\frac{d}{a}$



where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, we have

$$mg=T_B=\lambda\frac{d}{a}=mg\frac{d}{a}\quad[\because \lambda=mg, \text{ as given}]$$

$$\therefore d=a$$

Now the particle is pulled down to a point C such that $OC=4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP=x$

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\frac{a+x}{a}=\frac{mg}{a}(a+x)$ in the string OP acting in the direction PO, i.e. in the direction of x decreasing

(ii) The weight mg of the particle acting vertically downwards i.e. in the direction of x increasing.

Hence by Newton's second law of motion ($P = mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = mg - \frac{mg}{a}(a+x) = -\frac{mgx}{a}$$

Thus $\frac{d^2x}{dt^2} = -\frac{g}{a}x$ ——— (1)

which is the equation of S.H.M with Centre at the origin B and the amplitude $BC = 2a$ which is greater than $AB = a$. Multiplying bothsides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + k, \quad \text{where } k \text{ is a constant.}$$

At the point C, $x = BC = 2a$, and the velocity $dx/dt = 0$

$$\therefore k = \frac{g}{a}4a^2$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2) \quad \text{——— (2)}$$

Taking square root of (2), we have

$$dx/dt = -\sqrt{(g/a)}\sqrt{4a^2 - x^2}$$

the $-$ve sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\sqrt{a/g}\,\frac{dx}{\sqrt{(4a^2 - x^2)}} \quad \text{——— (3)}$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1} dt = -\sqrt{(a/g)}\int_{2a}^{-a}\frac{dx}{\sqrt{4a^2 - x^2}}$$

$$\Rightarrow t_1 = \sqrt{a/g}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a} = \sqrt{\frac{a}{g}}\left[\cos^{-1}\left(-\tfrac{1}{2}\right) - \cos^{-1}(1)\right] = \sqrt{\frac{a}{g}}\cdot\frac{2\pi}{3}$$

Let v_1 be the velocity of the particle at A. Then at A

$x = -a$ and $(dx/dt)^2 = v_1^2$

So from (2), we have $v_1^2 = (g/a)(4a^2 - a^2)$

$\Rightarrow v_1 = \sqrt{3ag}$, the direction of v_1 being vertically upwards. Thus the velocity at A is $\sqrt{3ag}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{3ag}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by $0 = \sqrt{3ag} - gt_2$. So that $t_2 = \sqrt{\frac{3a}{g}}$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle return to

$C = 2(t_1 + t_2)$

$$= 2\left[\sqrt{a/g}\cdot\frac{2\pi}{3} + \sqrt{\frac{3a}{g}}\right] = \sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right].$$

Q(C) Verify stoke's theorem for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

Sol^n The boundary C of S is a circle in the xy-plane of radius unity and centre origine

Suppose $x = \cos t$, $y = \sin t$, $z=0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C [(2x-y)dx - yz^2dy - y^2zdz]$$

$$= \oint_C (2x-y)\,dx, \text{ since } z=0 \text{ and } dz=0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} \left[\sin 2t - \frac{1}{2}(1-\cos 2t)\right]dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\left[\left(-\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}(\pi-0) + \frac{1}{4}(0-0)\right]$$

$$= \pi \quad \text{——— (1)}$$

Also find $(\nabla \times \vec{F})$

and $(\nabla \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$

$= (-2yz+2yz)\hat{i} - (0-0)\hat{j} + (0+1)\hat{k}$

$= \hat{k}$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

$\therefore$ by an application of Gauss divergence theorem, we have,

$$\iint_{S'} \text{Curl } \vec{F} \cdot \hat{n}\, ds = 0$$

or $\iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds + \iint_{S_1} \text{Curl } \vec{F} \cdot \hat{n}\, ds = 0$

($\because$ S' consists of S and S_1)

or $\iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds - \iint_{S_1} \text{Curl } \vec{F} \cdot \hat{k}\, ds = 0$

[$\because$ on S_1, $\hat{n} = -\hat{k}$]

or $\iint_S \text{Curl } \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{Curl } \vec{F} \cdot \hat{k}\, ds$

$\therefore \iint_S \text{Curl.} \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{Curl } \vec{F} \cdot \hat{k}\, ds$

$= \iint_{S_1} \hat{k} \cdot \hat{k}\, ds = \iint_{S_1} ds = S_1 = \pi$ —— (2)

Note that S_1 = area of a circle of radius 1

$= \pi(1)^2 = \pi$, Hence from (1) and (2) Stok's theorem verified.