IFOS-2016
Real Analysis

(1.)(b) Examine the Uniform Convergence of $f_n(x) = \frac{\sin(nx+n)}{n}$.

Sol:- For uniform continuity, we apply M_n-test.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx+n)}{n}$$

$$= 0$$

$$M_n = \sup\left\{ \left| \frac{\sin(nx+n)}{n} - 0 \right| \right\}$$

$$= \sup\left\{ \left| \frac{1}{n} \right| \right\}, \text{ as } \sin(nx+n) \le 1 \ \forall n, x.$$

$$= \frac{1}{n}.$$

$$\lim_{n\to\infty} M_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, as stated by M_n-test, $\frac{\sin(nx+n)}{n}$ is uniformly cgt.

1(c) Find the maximum & minima of $f(x,y) = x^3+y^3-3x-12y+20$.

$f_x = 3x^2-3 = 0 \Rightarrow x = \pm 1$

$f_y = 3y^2-12 = 0 \Rightarrow y = \pm 2$

$(1,2)$; $(-1,2)$, $(1,-2)$ $(-1,-2)$ are stationary points

$f_{xx}f_{yy} - f_{xy}^2 > 0$ for extremums.

$6x(6y) - 0 > 0$ $\forall (x,y) = (1,2)$
$(x,y) = (-1,-2)$.

$f_{xx} > 0$, $f_{yy} > 0$ $(x,y) = (1,2)$.
$\downarrow$
minima

$f_{xx} < 0$, $f_{yy} < 0$ $(x,y) = (-1,-2)$
$\downarrow$
maxima

$f(x,y)\big|_{1,2} = 1+8-3-24+20 = 2$ - minima value

$f(x,y)\big|_{-1,-2} = -1-8+3+24+20$
$= 38$ — maxima value.

3.(a) If $f_n(x) = \frac{3}{x+n}$, $0 \le x \le 2$, state with reason whether $\{f_n\}_n$ cgs. uniformly on $[0,2]$ or not.

Sol^n Again, we use M_n-test.

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{3}{x+n} = 0$$

$$M_n = \sup_{x\in[0,2]} \left\{ |f_n(x) - f(x)| \right\}$$

$$= \sup_{x\in[0,2]} \left\{ \left| \frac{3}{x+n} \right| \right\} = \lim_{n\to\infty} \left| \frac{3}{n} \right| = 0$$

Hence, the function is cgt. uniformly.

3.(b) Examine continuity at $(0,0)$

$$f(x,y) = \begin{cases} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x,y) \ne (0,0) \\ \frac{1}{2}, & (x,y) = (0,0) \end{cases}$$

$\sin x < x$.

$\sin^{-1} x > x$.

For continuity, $\cancel{|f(x,y) - f(0,0)| < \varepsilon}$

$f(a,b) = LHL = RHL$

$f(a,b) = \frac{1}{2}$.

$$\lim_{h,k\to 0^+} \frac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} \quad \lim_{h,k\to 0} \frac{(x+2y)}{2x+4y}$$
$$= \frac{1}{2}.$$

$f(a,b) = LHL = RHL$

Hence continuous at $(0,0)$

Scanned by CamScanner

③ (c.)

$$u = \cos^{-1}\left\{\frac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$$

$$Z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = x^{1/2}\left[\frac{1+y/x}{1+\sqrt{y/x}}\right].$$

Using Euler's Eqn.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}\times z$$

$$-x\sin u\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\tan u.$$

4.(a.)

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$$

Put $\sqrt{x} = t$.

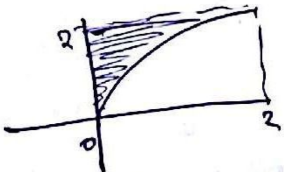
$$\frac{dx}{2\sqrt{x}} = dt.; \quad 2.\int_0^\infty \frac{dt}{(t^2+1)}.$$

$$= 2\int_0^\infty \frac{dt}{t^2+1} = 2.\tan^{-1}t.\Big|_0^\infty$$

$$2.\left(\frac{\pi}{2}-0\right) = \pi.$$

(b)

$$\int_0^2\int_0^{y^2/2} \frac{y}{(x^2+y^2+1)^{1/2}}\,dx\,dy$$



Changing the order of integration

$$\int_0^2\int_{\sqrt{2x}}^2 \frac{y}{(x^2+y^2+1)^{1/2}}\,dy\,dx.$$

$$\int_0^2 \left[x^2+y^2+1\right]^{1/2}\Big|_{\sqrt{2x}}^{2} dx.$$

$$\int_0^2 (x^2+5)^{1/2} - (x^2+2x+1)^{1/2}\,dx.$$

$$= \int_0^2 (x^2+5)^{1/2} - x - 1\,dx.$$

$$I = \left[\frac{1}{2}x\sqrt{x^2+5} + \frac{5}{2}\ln\left|x+\sqrt{x^2+5}\right| - \frac{x^2}{2} - x\right]_0^2$$

$$= 3 + \frac{5}{2}\ln 5 - 2 - 2 - \frac{5}{2}\ln\sqrt{5}$$

$$= \boxed{-1 + \frac{5}{4}\ln 5}$$

Scanned by CamScanner

Real Analysis

1.)(b) $f(x) = x^2 \sin\frac{1}{x}$, $0 < x < \infty$

$f:(0,\infty) \to \mathbb{R}$, show that there is a diff. function $g:\mathbb{R} \to \mathbb{R}$ that extends f.

$x^2 \sin\frac{1}{x}$ is differentiable at $(0,\infty)$ as $\sin\frac{1}{x}$ & x^2 are both differentiable in $(0,\infty)$

Similarly $x^2 \sin\frac{1}{x}$ is differentiable for $(-\infty,0)$ if we extend it from $(0,-\infty)$.

Now, we need an extension of the function such that $f(x)$ is diff. at $x=0$.

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin\frac{1}{h} - f(0,0)}{h}$$

$$f'(x) = 2x\sin\frac{1}{x} - x^2\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)$$

$$f'(0) = \lim_{h\to 0} h\sin\frac{1}{h} - \frac{f(0,0)}{h}$$

$$f(0,0) = 0$$

for

LHD = RHD.

Hence, extension of the function is

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \in \mathbb{R}-\{0\} \\ 0 & x = 0. \end{cases}$$

(1) (c) $x_1 = \frac{1}{2}$, $y_1 = 1$

$$x_n = \sqrt{x_{n-1}y_{n-1}} \quad , n = 2,3,4,\ldots$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right) \quad , n = 2,3,4,\ldots$$

Prove that both the sequences converge to the same limit l, where $\frac{1}{2} < l < 1$.

If $0 < a < b$, then geometric mean $G = \sqrt{ab}$ and the harmonic mean

$$H = \left[\frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)\right]^{-1}$$

Also, $a < H < G < b$

we are given that

~~$a < H < G < b$~~

$\frac{1}{2} = x_1 < y_1 = 1$

On the assumption $x_{n-1} < y_{n-1}$

we have

$x_{n-1} < x_n < y_{n-1}$ $(\because x_n = \sqrt{x_{n-1}y_{n-1}})$

because y_n is the H.M. of x_n & y_{n-1}.

It follows, that (by induction)

$x_{n-1} < x_n < y_n < y_{n-1}$, $n = 2,3,\ldots$

The sequences $\{x_n\}$ increases and is bdd. above by $y_1 = 1$.

$\{y_n\}$ decreases and is bdd. by $x_1 = \frac{1}{2}$

Hence, both sequences converge.

$\lim_{n\to\infty} x_n \to l$ then $l^2 = lm$

$\lim_{n\to\infty} y_n \to m$. & $\frac{1}{m} = \frac{1}{2}\left(\frac{l+m}{lm}\right)$.

Both the sequences yield

$l = m$.

Scanned by CamScanner

② (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \text{ is conditionally cgt.}$$

Leibnitz Test on Alternating Series:

$\sum (-1)^{n+1} u_n$ is cgt. if

(i) $u_n > u_{n+1} \quad \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$.

(i) $u_n = \frac{1}{n+1}$; $u_{n+1} < u_n$.

(ii) $\lim_{n\to\infty} \frac{1}{n+1} = 0$.

It is conditionally convergent.

Proof of Leibnitz Test :- Google it!

③ (b) Find the relative max & min values of the function

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$f_x = 4x^3 - 4x + 4y = 4(x^3 - x + y)$

$f_y = 4y^3 - 4y + 4x = 4(y^3 - y + x)$.

$f_x = f_y = 0 \Rightarrow (x,y) = (0,0), (\sqrt{2},\sqrt{2}), (\sqrt{2},-\sqrt{2})$

$f_{xx} = 12x^2 - 4$; $f_{xx}f_{yy} - f_{xy}^2 > 0$

$(12x^2-4)(12y^2-4) - 16 > 0$

$f_{yy} = 12y^2 - 4$

~~$f_{xx}f_{yy}$~~

$f_{xy} = 4$

$f_{yx} = 4$.

$f_{xx} > 0$, $f_{yy} > 0$ at $x,y = (\sqrt{2},\sqrt{2})$
$x,y = (\sqrt{2},-\sqrt{2})$.

min

$f(\sqrt{2},\sqrt{2}) = 4 + 4 - 2\times 2 + 4\times 2 - 2\times 2$.
$= 8$

$f(\sqrt{2},-\sqrt{2}) = 4 + 4 - 2\times 2 - \cancel{4\times 2} - 2\times 2$.

max at $(\sqrt{2},\sqrt{2}) = 8$

min at $(\sqrt{2},-\sqrt{2}) = -8$

Prove that every ~~Po~~ f is uniformly cont on $\mathbb{R}$ where $f: \mathbb{R}\to\mathbb{R}$ is a cont. function such that $\lim_{x\to+\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$.

~~The function~~

Let $\epsilon > 0$, there is a positive number M such that

$$|f(x) - a| < \frac{\epsilon}{2} \quad \forall x > M \text{ or } x < -M$$

The function is continuous on the finite interval ~~[0, M+1]~~ $[-M-1, M+1]$; hence f is also uniformly cont. on this compact interval

Consequently there's a +ve number $\delta < 1$

s.t. $|f(x_1) - f(x_2)| < \epsilon \quad \forall x_1, x_2 \in [-M-1, M+1]$ with $|x_1 - x_2| < \delta$. ②

Let x_1 & x_2 be numbers $|x_1 - x_2| < \delta$.

Then $|x_1 - x_2| < 1$ and thus both the numbers belong to ~~[0, M+1]~~ $[-M-1, M+1]$ or greater than $M+1$ in magnitude,

in the latter case

$$|f(x_1) - f(x_2)| = |f(x_1) - a + a - f(x_2)|$$
$$\le |f(x_1) - a| + |f(x_2) - a|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \text{ ③}$$

So, either ② and ③ are always ~~is~~.

Scanned by CamScanner