UPSC MATHEMATICS 2020

PAPER 1 SOLUTIONS

(1a) Consider set V of $n\times n$ real magic ~~sq~~ squares.
Show V is vector space over $\mathbb{R}$.
Give example of two distinct 2×2 magic square

What is Magic Square ?

→ It is a square with numbers
→ Distinct
→ positive or negative
→ Sum of numbers in any horizontal
= Sum of numbers in any vertical
= Sum of numbers in diagonal

For any $p \leq n$ $\quad \sum_{i=1}^{i=n} a_{ip} = \sum_{j=1}^{j=n} a_{pj} = \sum_{k=1}^{k=n} a_{kk}$

What is Vector Space ?

V is non empty set
F is field
$\alpha \in F$ $\quad (v \in V, u \in V)$ $\quad \beta \in F$

Vector Space property

→ $(V +)$ must be abelian
- (a) Closure
- (b) Associative
- (c) Inverse
- (d) Identity
- (e) Commutative

→ $\alpha(u+v) = \alpha u + \alpha v$
→ $(\alpha+\beta)u = \alpha u + \beta u$
→ $(\alpha\beta)u = \alpha(\beta u)$
→ $1v = v$

Let M_1, M_2 are Magic magic, $(n\times n)$, real, square

$\alpha, \beta \in R$,

$M_1, M_2 \in V$

$M_1 \to \sum a_{ip} = \sum a_{pj} = \sum a_{kk}$

$M_2 \to \sum b_{ip} = \sum b_{pj} = \sum b_{kk}$

(a) <u>Closure</u>: $M_1 \in V$, $M_2 \in V$

$M_1 + M_2$ is Matrix addition

Let $M_1 + M_2 = M_3$

Clearly in M_3 $\sum a_{ip} + \sum b_{ip}$

$= \sum a_{pj} + \sum b_{pj}$

$= \sum a_{kk} + \sum b_{kk}$

Magic property is satisfied by M_3

So $M_2 \in V$

Closure property satisfied

(b) <u>Associativity</u> $M_1 \in V$, $M_2 \in V$, $M_3 \in V$

$(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$

Clearly satisfied

$(\sum a_{ip} + \sum b_{ip}) + \sum c_{ip} = \sum a_{ip} + (\sum b_{ip} + \sum c_{ip})$

$(\sum a_{pj} + \sum b_{pj}) + \sum c_{pj} = \sum a_{pj} + (\sum b_{pj} + \sum c_{pj})$

$(\sum a_{kk} + \sum b_{kk}) + \sum c_{kk} = \sum a_{kk} + (\sum b_{kk} + \sum c_{kk})$

Magic property also satisfied

(c) 0 is a Magic Matrix (Existence of Identity

$M_1 + 0 = 0 + M_1 = M$

(d) Existence of Inverse

If $M_1 \in V$

$(-1)M_1 = -M_1$ also has Magic property

$-M_1 \in V$

$M_1 + (-M_1) = 0$

So $-M_1$ is inverse of M_1

(e) Commutative

$(M_1+M_2) = (M_2+M_1)$

$\Sigma a_{ip} + \Sigma b_{ip} = \Sigma b_{ip} + \Sigma a_{ip}$

$\Sigma a_{pj} + \Sigma b_{pj} = \Sigma b_{pj} + \Sigma a_{pj}$

$\Sigma a_{kk} + \Sigma b_{kk} = \Sigma b_{kk} + \Sigma a_{kk}$

—×—

M is Abelian over R

(f) $\alpha(M_1+M_2) = \alpha M_1 + \alpha M_2$

M_1+M_2 is Magic : Matrix Addition

It is Matrix addition & multiplication by scalar

RHS is also a Magic square

Satisfy property

(h) $(\alpha+\beta)M_1 = \alpha M_1 + \beta M_1$

clearly Magic property and Matrix addition satisfied

(i) $\alpha(\beta M_1) = (\alpha\beta) M_1$

(j) $1 \in R$

$1 . M_1 = M_1$

All vector space property satisfied

Hence V is vector space over R.

Note: students, can avoid explaining each and every step, as everything is Self-Explanatory.

Qn is for 10 Marks. Allote Max 5 minutes to Explain steps

Examples of 2×2 magic

2	2
2	2

-6	-6
-6	-6

(1b) $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is LT $T(A) = BA$

$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$

Find (a) Rank of T

(b) Nullity of T

(c) Find matrix A which maps to Null matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\to T(A) = BA = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{pmatrix} a-c & b-d \\ -4a+4c & -4b+4d \end{pmatrix}$

$\to$ First Find Null Space $T(A) = 0$

$\Rightarrow a-c=0 \Rightarrow a=c$

$b-d=0 \Rightarrow b=d$

Let $\left.\begin{matrix} c=\alpha \\ d=\beta \end{matrix}\right\} \Rightarrow \begin{matrix} a=\alpha \\ b=\beta \end{matrix}$

Null Space $= \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

Clearly Nolity is 2

$\to$ Rank-Nolity Theoram $\Rightarrow$ Rank + Nolity $= 4$ (4 elements)

$\Rightarrow$ Rank $= 4-2 = 2$

$\to$ Find A which maps to Null matrix

$T(A) = 0 \Rightarrow a=c \quad b=d$

$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$ example $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}$

(1-c) Evaluate $\underset{x\to\pi/4}{Lt}(\tan x)^{\tan 2x}$

$$y=(\tan x)^{\tan 2x}$$

$$\log y = \tan 2x \log \tan x$$

$$\underset{x\to\pi/4}{Lt} \log y = \underset{x\to\pi/4}{Lt} \tan 2x \log \tan x$$

$$= \underset{x\to\pi/4}{Lt} \frac{\log \tan x}{\cot 2x} \left(\frac{0}{0}\right)$$

$$= \underset{x\to\pi/4}{Lt} \frac{\frac{1}{\tan x}\sec^2 x}{-2\operatorname{cosec}^2 2x}$$

$$= \frac{(\sqrt{2})^2}{-2} = -1$$

$$\boxed{y=e^{-1}}$$

This Qn is present in Question Bank of SuccessClap Qn No 39

UPSC-2020

SuccessClap

(1d) Find asymptotes of $(2x+3)y=(x-1)^2$

$$(2x+3)y=(x-1)^2$$

$$2xy+3y = x^2-2x+1$$

$$(x^2-2xy)+(-2x-3y)+1=0$$

degree 2 - max

$\rightarrow \phi_2(m) = (x^2-2xy)_{\substack{x=1\\y=m}} = 1-2m$

$\rightarrow \phi_1(m) = (-2x-3y)_{\substack{x=1\\y=m}} = -2-3m$

$\rightarrow \phi_2'(m) = -2$

$\rightarrow \phi_2(m)=0 \Rightarrow 1-2m=0 \Rightarrow m=1/2$

$$\rightarrow c = \frac{-\phi_1(m)}{\phi_2'(m)} = \frac{-(-2-3m)_{m=1/2}}{-2}$$

$$= \frac{-2-3\left(\frac{1}{2}\right)}{2} = \frac{-7}{4}$$

$\rightarrow$ Asymptote is $y=mx+c$

$$y=\left(\frac{1}{2}\right)x+\left(\frac{-7}{4}\right)$$

(1e) Find tangent planes to $2x^2+6y^2+3z^2=27$ which pass through $x-y-z=0=x-y+2z-9$

$$\boxed{\begin{array}{c} lx+my+nz=p \text{ touch } ax^2+by^2+cz^2=1 \\ \text{if } \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2 \end{array}}$$

→ Plane through line is given by

$$(x-y-z)+\lambda(x-y+2z-9)=0$$

$$\underset{l}{(1+\lambda)}x+y\underset{m}{(-1-\lambda)}+z\underset{n}{(-1+2\lambda)}=\underset{p}{9\lambda}$$

→ $2x^2+6y^2+3z^2=27 \Rightarrow \left(\frac{2}{27}\right)x^2+\left(\frac{6}{27}\right)y^2+\left(\frac{3}{27}\right)z^2=1$

$ax^2+by^2+cz^2=1$

→ For tangent condition

$$\frac{(1+\lambda)^2}{2/27}+\frac{(-1-\lambda)^2}{6/27}+\frac{(2\lambda-1)^2}{3/27}=(9\lambda)^2=81\lambda^2$$

$$\frac{(\lambda+1)^2}{2}+\frac{(\lambda+1)^2}{6}+\frac{(2\lambda-1)^2}{3}=3\lambda^2 \qquad \frac{81}{27}=3$$

↓ solve

$$6\lambda^2+3=9\lambda^2 \Rightarrow 3\lambda^2=3 \Rightarrow \lambda=\pm 1$$

Tangent planes are

$$\boxed{(x-y-z)+(\pm 1)(x-y+2z-9)=0}$$

$$(x-y-z)+(x-y+2z-9)=0 \quad \quad (x-y-z)-(x-y+2z-9)=0$$

(2a) Evaluate $\int_0^1 \tan^{-1}\left(1-\frac{1}{x}\right)dx$

$$I = \int_0^1 \tan^{-1}\left(1-\frac{1}{x}\right)dx = \int_0^1 \tan^{-1}\left(\frac{x-1}{x}\right)dx$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^1 \tan^{-1}\left(\frac{1-x-1}{1-x}\right)dx = \int_0^1 \tan^{-1}\left(\frac{-x}{1-x}\right)dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x}{x-1}\right)dx$$

Add $I + I = 2I = \int_0^1 \tan^{-1}\left(\frac{x-1}{x}\right) + \int_0^1 \tan^{-1}\left(\frac{x}{x-1}\right)$

$$= \int_0^1 \left(\tan^{-1}A + \tan^{-1}B\right)$$

$A = \frac{x-1}{x}$

$B = \frac{x}{x-1} = \frac{1}{A}$

$$= \int_0^1 \tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$AB = 1$

$1 - AB = 0$

$$= \int_0^1 \tan^{-1}[-\infty] = \int_0^1 \left[\frac{-\pi}{2}\right]dx$$

$$= \frac{\pi}{2}$$

$$I = \left(\frac{-\pi}{2}\right)\left(\frac{1}{2}\right) = \frac{-\pi}{4}$$

Note:

$A+B = \frac{x-1}{x} + \frac{x}{x-1}$

$= \frac{(x-1)^2 + x^2}{x(x-1)}$

In 0 to 1

A+B is -ve negative

(2b) $n\times n$ matrix $A = I - 2u.u^T$ u is unit column vector

(i) Examine A is symetrix

(ii) Examine if A is orthogonal

(iii) Show Trace $A = n-2$

(iv) Find $A_{3\times3}$ $u = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$

Note : u is unit column matrix

$$u = \begin{bmatrix} \frac{a_1}{|a|} \\ \frac{a_2}{|a|} \\ \frac{a_3}{|a|} \\ \vdots \\ \frac{a_n}{|a|} \end{bmatrix}$$

$|a| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$

Now u is unit matrix

$$u_T = \begin{bmatrix} \frac{a_1}{|a|} & \frac{a_2}{|a|} & \frac{a_3}{|a|} & \cdots & \frac{a_n}{|a|} \end{bmatrix}$$

$$uu_T = \begin{bmatrix} \frac{a_1^2}{|a|^2} & \frac{a_1a_2}{|a|^2} & \frac{a_1a_3}{|a|^2} & \cdot & \frac{a_1a_n}{|a|^2} \\ \frac{a_1a_2}{|a|^2} & \frac{a_2^2}{|a|^2} & \frac{a_2a_3}{|a|^2} & & \frac{a_2a_n}{|a|^2} \\ \frac{a_1a_n}{|a|^2} & \frac{a_2a_n}{|a|^2} & & & \frac{a_n^2}{|a|^2} \end{bmatrix}$$

$A = I - 2uu^T$

$$= \begin{bmatrix} 1-\frac{2a_1^2}{|a|^2} & -\frac{2a_1a_2}{|a|^2} & -\frac{2a_1a_3}{|a|^2} & -\frac{2a_1a_n}{|a|^2} \\ -\frac{2a_1a_2}{|a|^2} & 1-\frac{2a_2^2}{|a|^2} & -\frac{2a_2a_3}{|a|^2} & -\frac{2a_2a_n}{|a|^2} \\ -\frac{2a_1a_3}{|a|^2} & -\frac{2a_2a_2}{|a|^2} & 1-\frac{2a_3^2}{|a|^2} & \cdots \\ -\frac{2a_1a_n}{|a|^2} & -\frac{2a_2a_n}{|a|^2} & & 1-\frac{2a_n^2}{|a|^2} \end{bmatrix}$$

→ Clearly $a_{ij} = a_{ji}$

So it is symetric matrix

(b) To show Trace :

Diagonal element sum = Trace

$$= \left[1 - \frac{2a_1^2}{|a|^2}\right] + \left[1 - \frac{2a_2^2}{|a|^2}\right] + \cdots \left[1 - \frac{2a_n^2}{|a|^2}\right]$$

$$= n - \frac{2}{|a|^2}(a_1^2 + a_2^2 + \cdot\, a_n^2)$$

But we have $|a|^2 = a_1^2 + a_2^2 + a_n^2$

$= n - 2$

(b) To show orthogonal

Orthogonal $\Rightarrow AA^T = I$

$A = I - 2uu^T$

$A^T = (I - 2uu^T)^T = I^T - 2(uu^T)^T$

$= I - 2(uu^T)^T$

Since $[uu^T]$ as a combined matrx is symetric $(uu^T)^T = uu^T$

Note : Donot do $(uu^T)^T = u^T(u^T)^T$

$u \to n \times 1$ $\quad u^T u \to (1 \times n) \times (n \times 1) = 1 \times 1$

$u^T \to 1 \times n$

Consider as combined matrix

$A^T = I - 2(uu^T)$

$AA^T = (I - 2uu^T)(I - 2uu^T)$

$= I - \cancel{2uu^T} - \cancel{2uu^T} + \cancel{4uu^T}$

$= I$

orthogonal proved

(iv) $u = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$

$a_1 = 1$
$a_2 = 2$
$a_3 = 2$

$|a| = \sqrt{1^2+2^2+2^2}$
$= 3$

$= \begin{bmatrix} a_1/|a| \\ a_2/|a| \\ a_3/|a| \end{bmatrix}$

Put in A

$$A = \begin{vmatrix} 1-\frac{2.1^2}{3^2} & -\frac{2.1.2}{3^2} & -\frac{2.1.2}{3^2} \\ -\frac{2.1.2}{3^2} & 1-\frac{2.2^2}{3^2} & -\frac{2.2.2}{3^2} \\ -\frac{2.1.2}{3^2} & -\frac{2.2.2}{3^2} & 1-\frac{2.2^2}{3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 1-2/9 & -4/9 & -4/9 \\ -4/9 & 1-\frac{8}{9} & -\frac{8}{9} \\ -4/9 & -8/9 & 1-8/9 \end{vmatrix}$$

$$= \begin{vmatrix} 7/9 & -4/9 & -4/9 \\ -4/9 & 1/9 & -8/9 \\ -4/9 & -8/9 & 1/9 \end{vmatrix}$$

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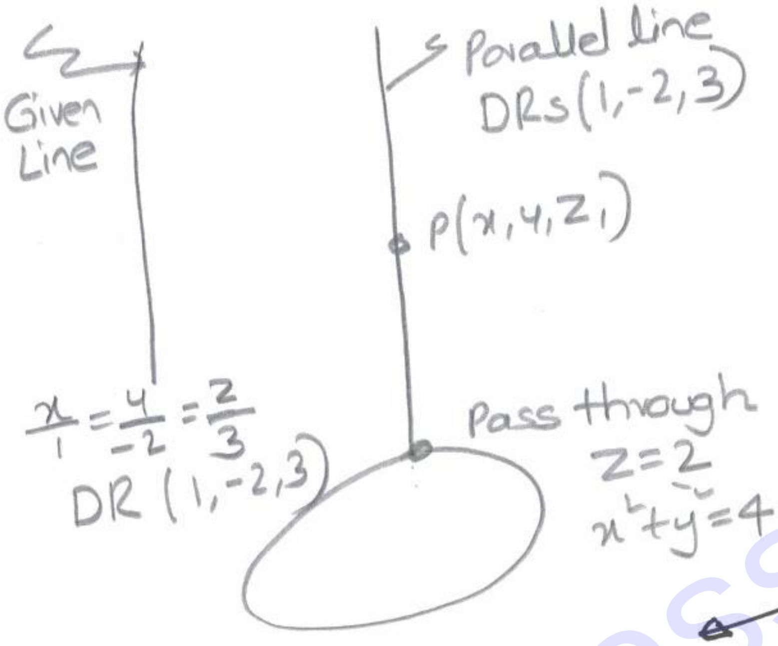
SuccessClap

(2c) Find eqn of Cylinder whose generators are parallel to $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ whose guiding curve $x^2+y^2=4$, $z=2$



Parallel line eqn

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$$

Pass throug $z=2$

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{2-z_1}{3}$$

Any point is $\left(x_1 + \frac{2}{3} - \frac{z_1}{3},\ y_1 - \frac{4}{3} + \frac{2z_1}{3},\ 2\right)$

3 points

This lie on $x^2+y^2=4$

$$\left(x_1 + \frac{2}{3} - \frac{z_1}{3}\right)^2 + \left(y_1 - \frac{4}{3} + \frac{2z_1}{3}\right)^2 = 4$$

~~Lous~~ Locus is replace $x_1 \to x$, $y_1 \to y$, $z_1 \to z$

$$\boxed{\left(x + \frac{2}{3} - \frac{z}{3}\right)^2 + \left(y - \frac{4}{3} + \frac{2z}{3}\right)^2 = 4}$$

This is eqn of Cylinder

(You can simplify it, if you want)

(3a) $f(x) = \int_0^x (t^2-5t+4)(t^2-5t+6)\,dt$

a) Find Critical points

$f'(x)=0 \quad f'(x) = \underbrace{(x^2-5x+4)}_{(x-1)(x-4)}\underbrace{(x^2-5t+6)}_{(x-2)(x-3)}$

$=(x-1)(x-2)(x-3)(x-4)$

$f'(x)=0 \Rightarrow$ Critical points $x=1,2,3,4$

b) Find local Minima, (c) Find local maxima

$f''(x) = [(x-2)(x-3)(x-4)] + [(x-1)(x-3)(x-4)] + [(x-1)(x-2)(x-4)] + [(x-1)(x-2)(x-3)]$

$x=1 \quad f''(1) = (-1)(-2)(-3)+0+0+0 = -6$ (Maxima)

$x=2 \quad f''(2) = 0+(1)(-1)(-2) = 2$ (Minima)

$x=3 \quad f''(3) = 0+0+2\times1(-1) = -2$ (Maxima

$x=4 \quad f''(4) = 0+0+3(2)(1) = 6$ (Minima)

Minima at $x=2,4$ Maxima at $x=1,3$)

(c) Find number of zeroes in [0 5] : What is α s.t $f(\alpha)=0$

$f(x) = \int_0^x [\,]dx = \frac{x^5}{5} - \frac{10x^4}{4} + \frac{35x^3}{3} - \frac{50x^2}{2} + 24x$

$f(1) = \frac{251}{30}$ positive Max value

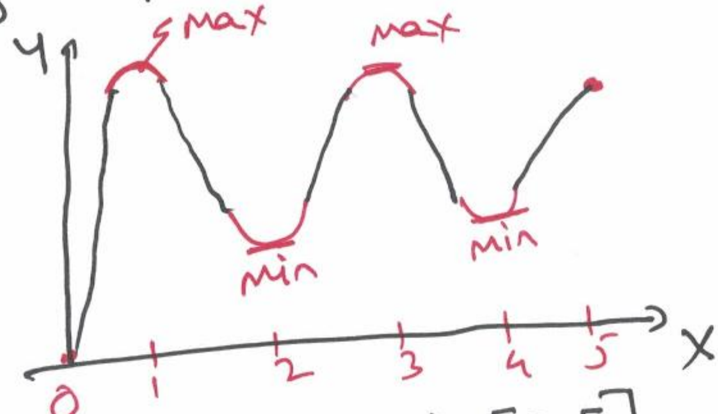
$f(2) = \frac{116}{15}$ positive Min value

$f(3) = \frac{112}{15}$ positive Max value

$f(4) = \frac{112}{15}$ positive Min value

$f(5) = 95/16$ positive

$f(0)=0$



Only one zero in [0 5]

(3b) $T(x_1, x_2 x_3) = (x_1 + x_2 + 3x_3,\ 2x_1 - x_2,\ -3x_1 + x_2 - x_3)$

What are conditions on a, b, c, such that

(a) (a,b,c) is null space of T?

(b) Find nullity of T

Soln: $T(a,b,c) = (a+b+3c,\ 2a-b,\ -3a+b-c)$

→ (a,b,c) is Null Space $\Rightarrow T(a,b,c)=0$
$= (0,0,0)$

$a+b+3c=0$

$2a-b=0 \qquad \Rightarrow 2a=b$

$-3a+b-c=0$

Let $a=\alpha \Rightarrow [2a=b \Rightarrow b=2\alpha$

Put in eqn $a+b+3c=0 \Rightarrow \alpha+2\alpha+3c=0$
$\Rightarrow c=-\alpha$

Put in $-3a+b-c=0 \Rightarrow -3\alpha+2\alpha+\alpha=0$
Satisfy
So our ep is
Correct

→ $\left\{ \begin{array}{l} a=\alpha \\ b=2\alpha \\ c=-\alpha \end{array} \right\} \Rightarrow b=a-c$
Combined Condition

or $b=2a,\ c=-a,\ b=-2c$

→ Nulity of T : $T(a,b,c)=0 \Rightarrow$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\alpha \\ -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Null space (1,2,−1)
↳ Nulity is 1

(3c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one of three mutually perpendicular generators of cone $5yz-8zx-3xy=0$ Find other two

Repeat Question : Asked earlier in UPSC

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one set, then other two are line of intersection of cone by plane through vertex $(0,0,0)$ and perpendicular to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

ie plane $1.x+2y+3z=0$

→ Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be line of intersection

→ $\left.\begin{array}{l} 5mn-8nl-3lm=0 \\ l+2m+3n=0 \end{array}\right]$

→ Eliminate $l \Rightarrow 5mn-8n(-2m-3n)-3(-2m-3n)m=0$

$m^2+5mn+4n^2=0 \Rightarrow (m+4n)(m+n)=0$

→ $m+4n=0 \Rightarrow m=-4n \Rightarrow l-5n=0 \quad l=5n$

$4l=-5m=20n \Rightarrow \frac{l}{5}=\frac{m}{-4}=\frac{n}{1}$

→ $m+n=0 \Rightarrow m=-n \Rightarrow l+n=0 \quad l=-n$

$l=m=-n \Rightarrow l/1=m/1=n/-1$

other two generators are

$\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$

Check Answer $5.1+(-4)(1)+(1)(-1)=0$

$5.1+(-4)(2)+1(3)=0 \qquad 1.1+1.2+(-1).3=0$

(4a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ $B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$

(a) Find AB

Simple Matrix Multiplication $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= I$

(b) Find det A , det B

$\det A = 1$

$\det B = 1$

(c) Solve $x + 2z = 3$ $2x - y + 3z = 3$ $4x + y + 8z = 14$

This is $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 14 \end{bmatrix}$

$A \; S = \cancel{B} Z$

we have $AB = I \Rightarrow A^{-1} = B$

$\cancel{AS = B \Rightarrow S = A^{-1}}$ $AS = Z \Rightarrow S = A^{-1} Z$

$= BZ$

$S = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$x = 1$ $y = 2$ $z = 1$

4b) Find locus of point of intersection of the perpendicular generators of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$

Eqn of generators to λ-system are

$\frac{x}{a}-\frac{y}{b}=\lambda z$ & $\frac{x}{a}+\frac{y}{b}=\frac{2}{\lambda}$

$\Rightarrow \frac{x}{a}-\frac{y}{b}-\lambda z=0$ $\frac{x}{a}+\frac{y}{b}+0.z-\frac{2}{\lambda}=0$

Let $(l_1, m_1 n_1)$ be DR of generator, then

$\frac{l_1}{a}-\frac{m_1}{b}-\lambda n_1=0$ & $\frac{l_1}{a}+\frac{m_1}{b}+0.n_1=0$

Solving $\frac{l_1}{\lambda/b}=\frac{m_1}{-\lambda/a}=\frac{n_1}{2/ab} \Rightarrow \frac{l_1}{a\lambda}=\frac{m_1}{-b\lambda}=\frac{n_1}{2}$

Similarly if $(l_2 m_2 n_2)$ be DRs $\frac{x}{a}-\frac{y}{b}=\frac{2}{\mu}$ and

$\frac{x}{a}+\frac{y}{b}=\mu z$

Solving gives

$\frac{l_2}{a\mu}=\frac{m_2}{b\mu}=\frac{n_2}{2}$

→ Since two generators are perpendicular

$l_1 l_2+m_1 m_2+n_1 n_2=0$

$a^2\lambda\mu-b^2\lambda\mu+4=0 \Rightarrow (a^2-b^2)\lambda\mu+4=0$

$(a^2-b^2)\left(\frac{2}{z}\right)+4=0$

→ pt of intersection $\Rightarrow x=a\frac{\lambda+\mu}{\lambda\mu}$ $y=b\frac{\mu-\lambda}{\lambda\mu}$ $z=\frac{2}{\lambda\mu}$

$a^2-b^2+2z=0$

→ Required locus is point of intersection of hyperbolic paraboloid and plane $a^2-b^2+2z=0$

(4-C) Find extreme value of $u = x^2+y^2+z^2$ condition $2x+3y+5z=30$ by using Lagrange

$u = x^2+y^2+z^2 \quad \phi = 2x+3y+5z-30$

Lagrange: $\quad 2x+$

$F = u + \lambda\phi$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + 2\lambda = 0$ —(1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + 3\lambda = 0$ —(2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda(5) = 0$ —(3)

Multiply (1) by x, (2) by y, (3) by z & add

$2(x^2+y^2+z^2) + \lambda(2x+3y+5z) = 0$

$2u + \lambda(30) = 0 \Rightarrow \lambda = \frac{-2u}{30}$

(1) $\Rightarrow x = -\lambda \quad$ (2) $\Rightarrow y = -\frac{3}{2}\lambda \quad$ (3) $\Rightarrow z = -\frac{5}{2}\lambda$

$\underbrace{x^2+y^2+z^2}_{u} = \lambda^2\left(1+\frac{3^2}{2^2}+\frac{5^2}{2^2}\right) = \underbrace{\lambda^2}_{\left(-\frac{2u}{30}\right)^2}\left(\frac{2^2+3^2+5^2}{2^2}\right)$

$u = \frac{2^2u^2}{30^2}\left(\frac{2^2+3^2+5^2}{2^2}\right)$

$\boxed{u = \frac{30^2}{2^2+3^2+5^2}}$

Correct Answer

(5a) Solve $x\cos\frac{y}{x}(y\,dx+x\,dy) = y\sin\left(\frac{y}{x}\right)(x\,dy-y\,dx)$

Rearrage in (y/x)

$$\frac{dy}{dx} = \frac{\left[\cos y/x + (y/x)\sin y/x\right](y/x)}{\left[y/x \sin y/x - \cos y/x\right]}$$

$\rightarrow \frac{y}{x} = v \quad y = vx \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\rightarrow \quad v + x\frac{dv}{dx} = \frac{v(\cos v + v\sin v)}{v\sin v - \cos v}$

$$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\frac{2dx}{x} = \left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \left(\frac{\sin v}{\cos v} - \frac{1}{v}\right)dv$$

| Integrate

$$2\log x = -\log\cos v - \log v + \log c$$

$$\log x^2 = \log\left(\frac{c}{v\cos v}\right)$$

$$x^2 = \frac{c}{v\cos v} \qquad v\,x^2\cos v = c$$

$$\boxed{xy\cos\frac{y}{x} = c}$$

(5b) Find orthogonal trajectory of family of circles passing through $(0,2)$ & $(0,-2)$

→ Eqn of Circle $x^2+y^2+2gx+2fy+d=0$

→ Pass through $(0,2) \Rightarrow 0+4+0+4f+d=0$

$(0,-2) \Rightarrow 0+4-4f+d=0$

$$\left.\begin{array}{l} 4+4f+d=0 \\ 4-4f+d=0 \end{array}\right\} \text{clearly } f=0$$

~~Als~~ Then $d=-4$

→ Eqn becomes $x^2+y^2+2gx-4=0$

→ Differentiate $2x+2y\frac{dy}{dx}+2g=0 \Rightarrow g=-\left(x+y\frac{dy}{dx}\right)$

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)-4=0$$

$$y^2-x^2-2xy\frac{dy}{dx}-4=0$$

→ Orthogonal $\frac{dy}{dx} \to -\frac{dx}{dy}$ $\Rightarrow y^2-x^2+2xy\frac{dx}{dy}-4=0$

→ $2xy\,dx+(y^2-x^2-4)\,dy=0$

$M\,dx+N\,dy=0$ $\quad \frac{\partial M}{\partial y}=2x \quad \frac{\partial N}{\partial x}=-2x$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{4x}{2xy}=\frac{2}{y} \qquad IF=e^{\int \frac{2}{y}dy}=e^{2\ln y}=y^2$$

$$2xy^3dx+(y^4-x^2y^2-4y^2)dy=0$$

Soln is

$$\int 2xy^3dx+\int_{\text{remove } x} (y^4-4y^2)dy=0$$

$$\boxed{x^2y^3+\frac{y^5}{5}-\frac{4y^3}{3}=C}$$

(5c) Find (a, b, c) s.t

$$V = (-4x - 3y + az)i + (bx + 3y + 5z)j + (4x + cy + 3z)\hat{k}$$

is irrotational

Express V as gradient of scalar function ϕ.

Determine ϕ.

→ Irrotational $\nabla \times V = 0$

$$\nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4x-3y+az & bx+3y+5z & 4x+cy+3z \end{vmatrix}$$

$$\nabla \times V = i(c-5) - j(4-a) + k(b+3) = 0 \Rightarrow$$

$a = 4$

$b = -3$

$c = 5$

→ $V = \nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$$\frac{\partial \phi}{\partial x} = -4x - 3y + 4z \Rightarrow \phi = -2x^2 - 3xy + 4xz + \phi_1(y, z)$$

$$\frac{\partial \phi}{\partial y} = -3x + 3y + 5z \Rightarrow \phi = -3xy + \frac{3}{2}y^2 + 5yz + \phi_2(x, z)$$

$$\frac{\partial \phi}{\partial z} = 4x + 5y + 3z \Rightarrow \phi = 4xz + 5yz + \frac{3}{2}z^2 + \phi_3(x, y)$$

Combine all

$$\phi = -2x^2 + \frac{3}{2}y^2 + \frac{3}{2}z^2 - 3xy + 4xz + 5yz + c$$

5(c)(ii) A particle starts at a great distance with velocity V. Let p be the length of perpendicular from centre of a star on the tangent to initial path of particle. Show that least distance of particle from the centre of star is λ, where

$$v^2\lambda = \sqrt{\mu^2+p^2V^4}-\mu \quad , \mu \text{ is constant}$$

In

Starts at infinite / great distance

Total energy $= \frac{1}{2}mv^2$

At least distance Total energy $= \frac{1}{2}mv'^2 + \frac{k}{\lambda}$

$PE = \frac{k}{\lambda}$ k is constant, λ is distance at least

Energy Conservation $\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{k}{\lambda}$

Angular momentum Conservation

$mvp = mv'\lambda \Rightarrow v' = \frac{vp}{\lambda}$ Put

$$\frac{1}{2}m(v^2-v'^2) = \frac{k}{\lambda} \Rightarrow \frac{1}{2}m\left(v^2-\frac{v^2p^2}{\lambda^2}\right) = \frac{k}{\lambda} \quad -(1)$$

Given $v^2\lambda = \sqrt{\mu^2+p^2V^4}-\mu$ <u>To show</u>

$$\mu + v^2\lambda = \sqrt{\mu^2+p^2V^4}$$

Square

$$\mu^2+v^4\lambda^2+2\mu v^2\lambda = \mu^2+p^2V^4$$

$$v^4\lambda^2 + 2Hv^2\lambda = p^2v^4$$

$$\lambda^2v^2 + 2H\lambda = p^2v^2$$

$$(p^2-\lambda^2)v^2 = 2H\lambda$$

$$\frac{(p^2-\lambda^2)v^2}{2\lambda} = \mu \quad \text{To show this exists}$$

$$① \Rightarrow \frac{1}{2}m\left(v^2 - \frac{v^2p^2}{\lambda^2}\right) = \frac{k}{\lambda}$$

$$\frac{1}{2}m\frac{v^2(\lambda^2-p^2)}{\lambda^2} = \frac{k}{\lambda}$$

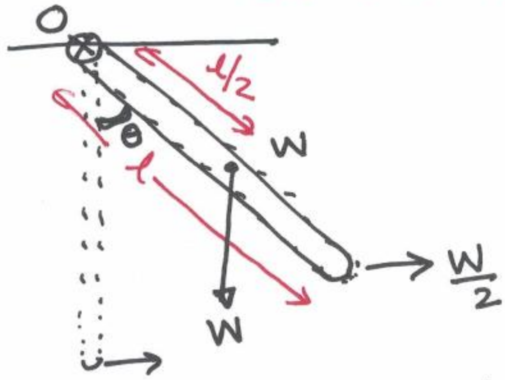
$$\frac{(p^2-\lambda^2)v^2}{2\lambda} = \frac{-k}{m}$$

k is constant
m is const] If we take $\frac{-k}{m} = \mu$

then we get $\frac{(p^2-\lambda^2)v^2}{2\lambda} = \mu$

Hence our eqn is satisfied

(5d) A uniform rod (in vertical position, can turn freely about one of its ends and is pulled aside from vertical by horizontal force acting at other end of the rod and equal to half its weight. At what inclination to vertical will be rod red



Rod weight W
Length L

To

Force $= \frac{W}{2}$ applied $= F$

Two forces act

(a) weight W cause rod to rotate Clockwise

(b) ~~we~~ Force $\frac{W}{2}$ cause rod to rotate Anti Clock

Take moment at O (Equilibrium)

$$W\left(\frac{l}{2}\sin\theta\right) = F\,(l\cos\theta)$$

$$= \frac{W}{2}\,l\cos\theta \qquad F = \frac{W}{2}$$

$$\sin\theta = \cos\theta$$

$$\tan\theta = 1 \quad \theta = 45^\circ$$

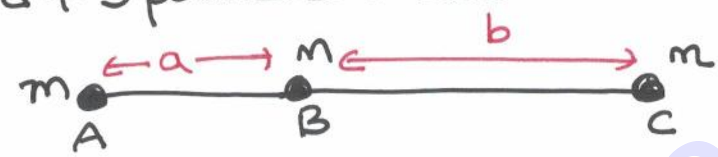
(5e) A rigid rod ABC has 3 particles each of mass m attached to it at A, B, C. The rod is struck by a blow P at right angles to it at a point distance from A equal to BC.

Prove kinetic energy set up is $\frac{P^2}{2m}\,\frac{a^2-ab+b^2}{a^2+ab+b^2}$

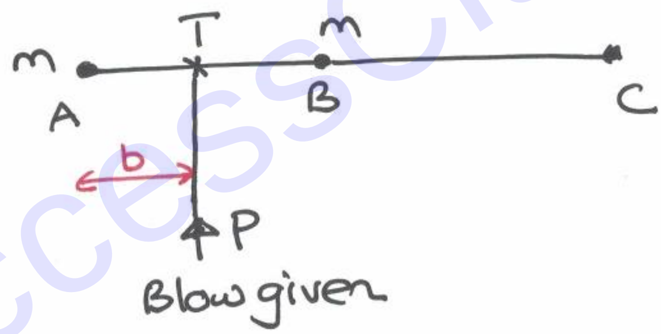
AB = a BC = b

→ Given rod + 3 particles mass m attached

→ Given AB = a BC = b



→ Blow P given, Let say given at T



→ Given
AT = BC
AT = b

→ To find Total energy ie KE

KE = Translational energy TE + Rotational energy RE

$$TE = \frac{1}{2}(\text{Total mass})(V_{cm}^2)$$

V_{cm} = velocity of Centre of mass

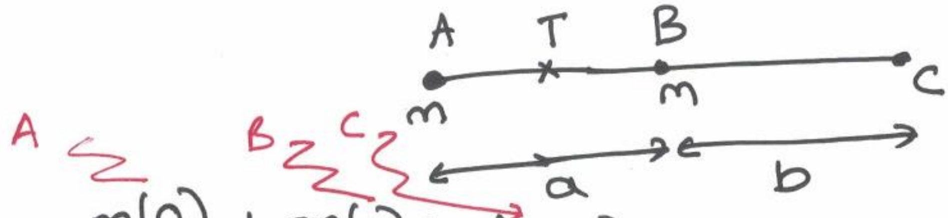
$$RE = \frac{1}{2} I \omega^2$$

I = Moment of Inertia at Centre of mass

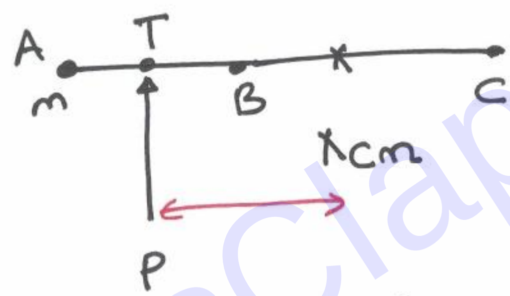
ω = Angular velocity

→ First find Centre of mass

Find from A (why? Given $AT = b$) So To find from A



$$X_{cm} = \frac{m(0) + m(a) + m(a+b)}{m+m+m} = \frac{2a+b}{3}$$



→ Blow gives Linear momentum & Angular momentum

→ Linear momentum conservation

$$P = (3m) V_{cm} \Rightarrow V_{cm} = \frac{P}{3m}$$

$$TE = \frac{1}{2}(3m) V_{cm}^2 = \frac{1}{2} \times 3m \times \left(\frac{P}{3m}\right)^2 = \frac{P^2}{6m}$$

→ Angular momentum conservation

distance $TX_{cm} = AX_{cm} - AT$

$$= X_{cm} - b$$

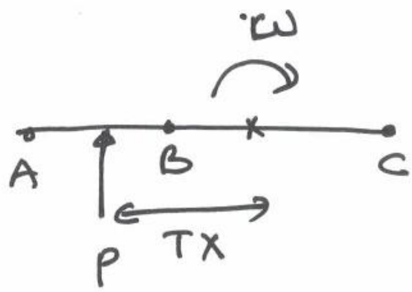
$$= \frac{2a+b}{3} - b$$

$$= \frac{2(a-b)}{3}$$

Angular momentum conservation gives

$$I_{cm}\,\omega = P \times (TX)$$

$$= P\left[\frac{2}{3}(a-b)\right]$$



$$I_{cm} = m\left[AX_{cm}\right]^2 + m\left[BX_{cm}\right]^2 + m\left[CX_{cm}\right]^2$$

$$= m\left(\frac{2a+b}{3}\right)^2 + m\left[\frac{2a+b}{3} - a\right]^2$$

$$+ m\left[(a+b) - \left(\frac{2a+b}{3}\right)\right]^2$$

$$= \frac{2}{3}m\left(a^2+b^2+ab\right)$$

$$RE = \frac{1}{2}I_{cm}\omega^2 = \frac{(I_{cm}\omega)^2}{2I_{cm}} = \frac{\left[P\,\frac{2}{3}(a-b)\right]^2}{2 \times \frac{2}{3}m(a^2+b^2+ab)}$$

$$= P^2 \times \frac{4}{9}\,\frac{(a-b)^2}{4m(a^2+b^2+ab)} \times 3$$

$$= \frac{P^2\,(a^2-2ab+b^2)}{3m(a^2+b^2+ab)}$$

$$TrE = \frac{P^2}{6m} = \text{Translation energy}$$

$$\text{Total energy} = \frac{P^2}{3m}\,\frac{(a^2-2ab+b^2)}{(a^2+b^2+ab)} + \frac{P^2}{6m}$$

$$= \frac{P^2}{3m}\left[\frac{a^2-2ab+b^2}{a^2+b^2+ab} + \frac{1}{2}\right]$$

$$= \frac{P^2}{3m}\,\frac{3(a^2-ab+b^2)}{2(a^2+b^2+ab)} = \frac{P^2\,(a^2-ab+b^2)}{2m(a^2+ab+b^2)}$$

(6 a) Use method of variation of parameter, Solve $y'' + (1-\cot x)y' - y\cot x = \sin^2 x$, if y^{-x} is one soln of CF

→ Question is Ambiguous.

→ If we have to solve by variation of parameter then one solution $y = e^{-x}$ should NOT be given we have to derive u, v & then find Wronskian

→ If $y = e^{-x}$ is one soln, then we have to follow Inspection method

$$\frac{dy}{dx} + Py + Q = R \qquad P = 1-\cot x$$
$$Q = \cot x$$

Clearly $1 - P + Q = 0$

So $y = e^{-x}$ is one integral

→ Let $y = v(e^{-x})$

$$\frac{dy}{dx} = -ve^{-x} + \frac{dv}{dx}e^{-x}$$

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}e^{-x} - 2e^{-x}\frac{dv}{dx} + ve^{-x}$$

Eqn becomes

$$\left(\frac{d^2v}{dx^2}e^{-x} - 2e^{-x}\frac{dv}{dx} + ve^{-x}\right) + (1-\cot x)\left\{\frac{dv}{dx}e^{-x} - ve^{-x}\right\}$$
$$- ve^{-x}\cot x = \sin^2 x$$

$$\frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} = \frac{\sin^2 x}{e^{-x}}$$

$p = \frac{dv}{dx}$

$$\frac{dp}{dx} - (1+\cot x)p = e^{x}\sin^2 x$$

$$IF = e^{-\int (1+\cot x)dx} = e^{-(x+\log \sin x)}$$

$$= \frac{e^{-x}}{\sin x}$$

Soln is $p\left(\frac{e^{-x}}{\sin x}\right) = \int e^{x}\sin^2 x\left(\frac{e^{-x}}{\sin x}\right)dx + c_1$

$$= -\cos x + c_1$$

$$p = \frac{dv}{dx} = -e^{x}\sin x\cos x + c_1 e^{x}\sin x$$

$$v = -\frac{1}{2}\int e^{x}\sin 2x\,dx + c_1\int e^{x}\sin x\,dx + c_2$$

$$= -\frac{1}{2\times 5}e^{x}(\sin 2x - 2\cos 2x) + c_1\cdot\frac{e^{x}}{2}(\sin x - \cos x) + c_2$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$y = ve^{-x} = -\frac{1}{10}(\sin 2x - 2\cos 2x) + \frac{c_1}{2}(\sin x - \cos x) + c_2 e^{-x}$$

6(b) $A = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$

Calculate $\int_C A.dr$ from $(0,0,0)$ to $(1,1,1)$ along

(i) $x=t,\ y=t^2,\ z=t^3$

$$dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\boxed{A.dr = (3x^2+6y)dx - 14yz\,dy + 20xz^2 dz}$$

$x=t \quad y=t^2 \quad z=t^3$

$\Rightarrow dx=dt \quad dy=2t\,dt \quad dz=3t^2dt$

$$A.dr = (3t^2+6t^2)dt - 14t^5(2t\,dt) + 20t^7.3t^2dt$$

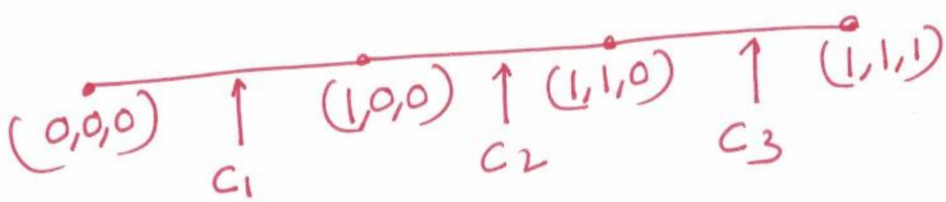
$$= (9t^2 - 28t^6 + 60t^9)dt$$

$$I = \int A.dr = \int_{t=0}^{t=1} \left[9t^2 - 28t^6 + 60t^9\right]dt$$

$$= 3t^3 - 4t^7 + 6t^{10}\Big|_{t=0}^{t=1}$$

$$= 3-4+6 = 5$$

(ii) Straight Line joining $(0,0,0)$ to $(1,0,0)$ then to $(1,1,0)$ and then to $(1,1,1)$



C_1 $(0,0,0)$ to $(1,0,0)$

$y=0, \quad z=0 \quad \Rightarrow dy=0, \, dz=0$

$A.dr = 3x^2 dx$

$$I_1 = \int_{x=0}^{x=1} 3x^2 dx = \frac{3x^3}{3}\Big|_0^1 = 1$$

C_2 $(1,0,0)$ to $(1,1,0)$

$x=1, \, z=0 \quad \Rightarrow dx=0 \quad dz=0$

$A.dr = 0 \qquad I_2 = 0$

C_3 $(1,1,0)$ to $(1,1,1)$

$x=1 \quad y=1 \quad \Rightarrow dx=0 \quad dy=0$

$A.dr = 20z^2 dz$

$$I_3 = \int_{z=0}^{z=1} 20z^2 dz = \frac{20z^3}{3}\Big|_0^1 = \frac{20}{3}$$

$$I = I_1 + I_2 + I_3 = 1 + \frac{20}{3} = \frac{23}{3}$$

(c) Straight line $(0,0,0)$ to $(1,1,1)$

Line through $(0,0,0)$ to $(1,1,1)$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)} \qquad t: [0 \text{ to } 1]$$

$x=t \quad y=t, \, z=t$

$$A.dr = (3t^2+6t)dt - 14t^2 dt + 20t^3 dt$$
$$= [3t^2+6t-14t^2+20t^3]dt$$

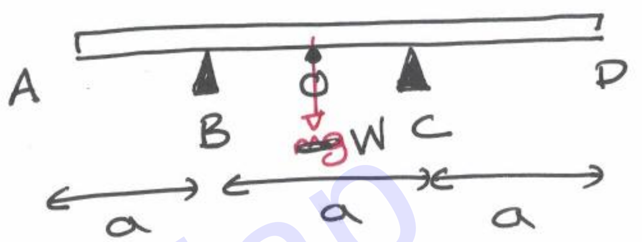
$$I = \int A.dr = \int_{t=0}^{t=1} [3t^2+6t-14t^2+20t^3] dt$$
$$= \frac{13}{3}$$

---

why not same : $\nabla \times A \neq 0$ Non Conservative

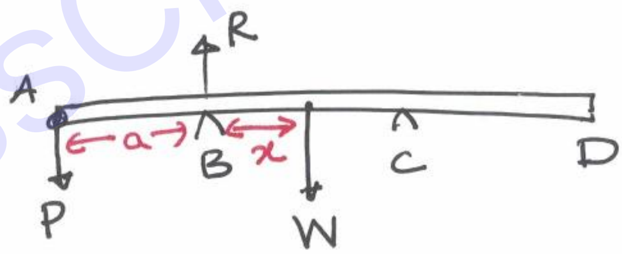
(6c) A beam AD rests on two supports B and C, where AB = BC = CD. It is found that the beam will tilt when a weight of p kg is hung from A or when a weight of q kg is hung from D. Find weight of beam.

Given beam AD
Rests on B, C
$AB = BC = CD = a$
weight of beam W at O



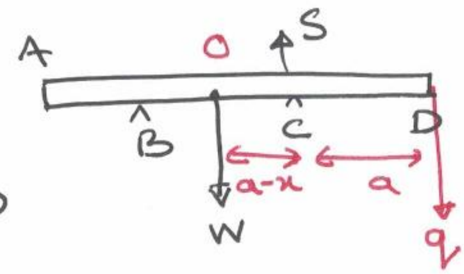
Case 1 : weight p is hung at A

Not given that beam AD is Uniform
→ So weight W (Centre of mass) will lie on some place (Not equal to half distance of BC)
Let $BO = x$



when p is placed, beam at C will break, just tilt
Reaction R is at B
Take moment at B

$$(p)\cdot(a) = W(x)$$



Case 2 : weight q is hung at D
$OC = a-x$
Similar q placed at D. For just tilt,
Take moment at C : $q(a) = W(a-x)$

$$\left.\begin{aligned} pa &= Wx \\ qa &= W(a-x) \end{aligned}\right]$$

Solve : Add it $(p+q)a = W(x+a-x) = Wa$

$$\boxed{W = p+q}$$

7(b) Use laplace to solve $ty''+2ty'+2y=2$
$y(0)=1$ $y'(0)$ is arbit.
Does it have unique solution

$$ty''+2ty'+2y=2 \quad y(0)=1 \quad y'(0) \text{ is arbit}$$

$$L\{ty''\} = -\frac{d}{ds}[L(y'')]$$

$$= -\frac{d}{ds}(s^2\bar{y} - sy(0) - y'(0)]$$

$$= -\frac{d}{ds}(s^2\bar{y} - s) \qquad y(0)=1$$

$$= -2s\bar{y} - s^2\frac{d\bar{y}}{ds} + 1$$

$$L\{2ty'\} = (2)\left\{-\frac{d}{ds}(s\bar{y} - y(0)\right\}$$

$$= (2)\left\{-s\frac{d\bar{y}}{ds} - \bar{y}\right\}$$

$$L\{2y\} = 2\bar{y}$$

Taking laplace on both side of eqn

$$L(2) = \frac{2}{s}$$

$$\left[-2s\bar{y} - s^2\frac{d\bar{y}}{ds} + 1\right]\left[-2s\frac{d\bar{y}}{ds} - \cancel{2\bar{y}}\right] + \left[\cancel{2\bar{y}}\right] = \frac{2}{s}$$

$$\frac{d\bar{y}}{ds}\left[s^2+2s\right] + \bar{y}(2s) = 1 - \frac{2}{s}$$

$$\frac{d\bar{y}}{ds} + \left(\frac{2s}{s^2+2s}\right)\bar{y} = \frac{(s-2)}{s^2(s+2)}$$

$$\downarrow$$

$$\frac{2}{(s+2)}$$

$\frac{dy}{dx} + Py = Q$ type

$$IF = e^{\int \frac{2}{s+2}ds} = e^{2\ln(s+2)} = e^{\ln(s+2)^2}$$

$$= (s+2)^2$$

Soln

$$\bar{y}(s+2)^2 = \int \frac{s-2}{s^2(s+2)} \times (s+2)^2 ds$$

$$= \int \frac{(s-2)(s+2)}{s^2} ds = \int \frac{s^2-4}{s^2}$$

$$= \int \left(1 - \frac{4}{s^2}\right) ds$$

$$= s + \frac{4}{s} + c$$

$$\bar{y} = \frac{s}{(s+2)^2} + \frac{4}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$= \frac{s^2+4}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$= \frac{(s+2)^2 - 4s}{s(s+2)^2} + \frac{c}{(s+2)^2}$$

$$\bar{y} = \frac{1}{s} - \frac{4}{(s+2)^2} + \frac{c}{(s+2)^2}$$

$$= \frac{1}{s} + \frac{(c-4)}{(s+2)^2} \qquad \text{Let } c-4=d$$

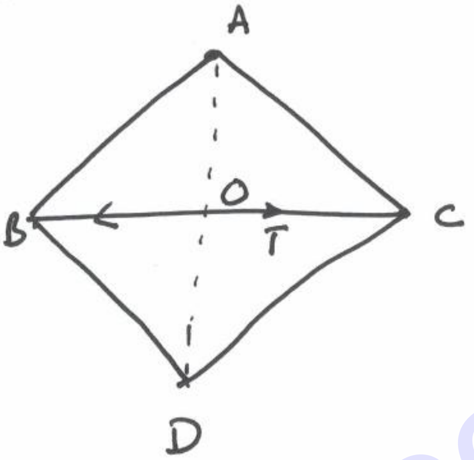
$$= \frac{1}{s} + \frac{d}{(s+2)^2}$$

Take inverse laplace

$$y(t) = 1 + d\,te^{-2t} \qquad d \text{ is constant}$$

7 (c) (i) A square frame formed by rods of equal weight w joined together is hung by one corner. A weight w is suspended from each of three lower corners, and shape is preserved by a light rod along horizontal diagonal.

Find thrust of rod.



Square frames AB, AC, BD, BC each of weight W

→ weight of 4 rods + Two weigh at B & C, which is $4w + 2w = 6W$

is equavalent to 6W at O

→ There is weight at D

→ Thrust exist

Eqn of Virtual work

$$6W\,\delta(OA) + w\,\delta(AD) + T\,\delta(BC) = 0$$

Let $AB = a$ $AD = 2OA = 2a\cos\theta$ $BC = 2BO = 2a\sin\theta$

$$6W\,\delta(a\cos\theta) + w\,\delta(2a\cos\theta) + T\,\delta(2a\sin\theta) = 0$$

$$w\,\delta\theta(-6\sin\theta - 2\sin\theta] + 2T\cos\theta\,\delta\theta = 0$$

$\delta\theta \neq 0$

$$T = 4W\tan\theta$$

Equilibrium $\theta = 45$

$$T = 4W\tan 45$$
$$= 4W$$

8(a)(i) Solve $(x+1)^2y''-4(x+1)y'+6y=6(x+1)^2+\sin\log(1+x)$

$\rightarrow$ Let $\ln(1+x)=z$ $\quad (1+x)=e^z$

$D_1=\frac{d}{dz}$ $\quad (1+x)D=D_1$ $\quad (1+x)^2D^2=D_1(D_1-1)$

$\rightarrow$ Eqn bcomes $\left(D_1(D_1-1)-4D_1+6\right)y=6e^{2z}+\sin z$

$\rightarrow$ AE $D_1^2-5D_1+6=0 \Rightarrow (D_1-2)(D_1-3)=0 \quad D_1=2,3$

$CF=c_1e^{2z}+c_2e^{3z}=c_1(x+1)^2+c_2(x+1)^3$

$\rightarrow$ $PI=\dfrac{6e^{2z}}{(D_1-2)(D_1-3)}+\dfrac{\sin z}{(D_1-2)(D_1-3)}$

$=\dfrac{-6e^{2z}}{D_1-2}+\dfrac{\sin z}{D_1^2-5D_1+6} \qquad D_1^2=-1$

$=-6e^{2z}\cdot\dfrac{z}{1!}+\dfrac{\sin z}{5(1-D_1}$

$=-6ze^{2z}+\dfrac{(1+D_1)}{5(1-D_1^2)}\sin z$

$=-6ze^{2z}+\frac{1}{5}\left[\dfrac{\sin z}{1-D_1^2}+\dfrac{\cos z}{1-D_1^2}\right]$

$=-6ze^{2z}+\frac{1}{5}\left[\dfrac{\sin z}{1+1}+\dfrac{\cos z}{1+1}\right] \qquad D_1^2=-1$

$=-6ze^{2z}+\frac{1}{10}(\sin z+\cos z)$

$z=\log(1+x)$

$\rightarrow y=CF+PI$

8a (ii) Find General & Singular solutions of
$9p^2(2-y)^2 = 4(3-y)$

$\rightarrow$ Solve p $\Rightarrow p = \pm \frac{2}{3} \frac{(3-y)^{1/2}}{2-y} = \frac{dy}{dx}$

$\rightarrow \quad dx = \pm \frac{3}{2} \frac{2-y}{(3-y)^{1/2}} dy \qquad 2-y = (3-y)-1$

$= \pm \frac{3}{2} \left[\frac{3-y}{(3-y)^{1/2}} - \frac{1}{(3-y)^{1/2}} \right] dy$

$\downarrow$ Integrate

$x + c = \pm \frac{3}{2} \int \left[(3-y)^{1/2} - (3-y)^{-1/2} \right] dy$

$= \pm \frac{3}{2} \left[-\frac{2}{3}(3-y)^{3/2} + 2(3-y)^{1/2} \right]$

$= \pm (3-y)^{1/2} \left[-(3-y) + 3 \right]$

$x + c = \pm y(3-y)^{1/2}$

$(x+c)^2 = y^2(3-y)$ General solution

$\rightarrow$ Singular Soln : $9p^2(2-y)^2 + 0.p - 4(3-y) = 0$

Quadratic $B^2 - 4AC = 0$ p-disc

$0^2 - 4.9(2-y)^2(-4(3-y)) = 0$

$(2-y)^2(3-y) = 0 \qquad \Rightarrow y = 2, y = 3$

$y = 2 \Rightarrow p = \frac{dy}{dx} = 0$ put $p = 0, y = 0$ in eqn do not satisfy

$y = 3 \Rightarrow p = \frac{dy}{dx} = 0$ put in eqn satisfy

$y = 3$ is Singular Solution

8(b) Evaluate Surface Integral $\int_S (\nabla \times F) \cdot n\, ds$

$F = yi + (x - 2xz)j - xy\hat{k}$

S is surface of sphere $x^2+y^2+z^2=a^2$ above xy plane

Use Stoke theorem

$$I \quad \int_S (\nabla \times F) \cdot n\, ds = \int_C F \cdot dr$$

$$= \int_C \left[yi + (x-2xz)j - xy\hat{k} \right] \cdot \left[i\,dx + j\,dy + k\,dz \right]$$

$$= \int_C y\,dx + (x - 2xy)\,dy - xy\,dz$$

C : Boundg : $z = 0$, $x^2+y^2=a^2$

$x = a\cos t$

$y = a\sin t$

$$= \int_C y\,dx + x\,dy$$

$$= \int_0^{2\pi} (a\sin t)(-a\sin t)\,dt + a\cos t\,(a\cos t)\,dt$$

$$= a^2 \int_0^{2\pi} (\cos^2 t - \sin^2 t)\,dt$$

$$= a^2 \int_0^{2\pi} \cos 2t\,dt$$

$$= a^2 \left. \frac{\sin 2t}{2} \right|_0^{2\pi} = 0$$