Mains Test Series - 2018

Test - 17, Paper - I

Answer key.

1(a) Let W be the subspace of $\mathbb{R}^3$ generated by $u=(2,1,0)$, $v=(1,-1,2)$, $w=(1,2,-2)$. Find condition on a, b, c so that $(a,b,c)\in W$. Can u, v, w generate $\mathbb{R}^3$? Give reasons.

Sol'n: Let $\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$ be a given vectorspace.

let W be subspace of $\mathbb{R}^3$ generated by

$S=\{u,v,w\}\subseteq W$ where $u=(2,1,0)$

$v=(1,-1,2)$, $w=(1,2,-2)$

then $L(S)=W$

we have $A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 2 & -2 \end{bmatrix} \Rightarrow |A| = 2(-2)-1(-4) = 0$

$\therefore$ u, v, w are Linearly dependent.

$\therefore$ The number of L.I vectors $<\dim(\mathbb{R}^3)$.

The vectors u, v, w donot generate $\mathbb{R}^3$

1(b) Reduce the matrix A to its normal form where

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

hence find the rank of A.

Sol: $$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 - C_1 \\ C_4 \to C_4 - C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 + 3C_2 \\ C_4 \to C_4 + C_2 \end{matrix}$$

$$\sim \left[\begin{array}{c|c} I_{2\times 2} & O \\ \hline O & O \end{array}\right]$$

clearly it is in normal form

$\therefore \rho(A) = 2$

1(c) Find the limiting points of co-axial system of spheres determined by $x^2+y^2+z^2-20x+30y-40z+29=0$ and $x^2+y^2+z^2-18x+27y-36z+29=0$.

Sol: Let $S_1 = x^2+y^2+z^2-20x+30y-40z+29=0$
$S_2 = x^2+y^2+z^2-18x+27y-36z+29=0$
be two given spheres.

then $S_1 - S_2 = -2x+3y-4z=0$

Now the eqn of co-axial system of spheres is

$S_1 + \lambda(S_1 - S_2) = 0$.

$\Rightarrow x^2+y^2+z^2-20x+30y-40z+29+\lambda(-2x+3y-4z)=0$

$\Rightarrow x^2+y^2+z^2+(-20-2\lambda)x+(30+3\lambda)y+(-40-4\lambda)z+29=0$.

Its centre $\left(10+\lambda, -\frac{30-3\lambda}{2}, 20+2\lambda\right)$

and equating its radius to zero we get $u^2+v^2+w^2-d=0$

$(10+\lambda)^2+\left(\frac{30+3\lambda}{2}\right)^2+(20+2\lambda)^2-29=0$

$\Rightarrow (10+\lambda)^2\left[1+\frac{9}{4}+4\right]=29$

$\Rightarrow (10+\lambda)^2\left[\frac{29}{4}\right]=29. \Rightarrow (10+\lambda)^2=4$

$\Rightarrow 10+\lambda=\pm 2 \Rightarrow \lambda=-8,-12$.

for $\lambda=-8$, limiting point is $(2,-3,4)$

for $\lambda=-12$, limiting point is $(-2,3,4)$

1(d) Evaluate the following integral $\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$

Sol^n: Given

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx \quad \text{--- (1)}$$

$$\int_a^b f(x)\,dx = \int_b^a f(a+b-x)\,dx$$

Using this property

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin(\pi/2-x)}\,dx}{\sqrt[3]{\sin(\pi/2-x)}+\sqrt[3]{\cos(\pi/2-x)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\cos x}\,dx}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}} \quad \text{--- (2)}$$

Add (1) & (2)

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}} + \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$$

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$I = \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{1}{2}\cdot\frac{\pi}{6}$$

$$\boxed{I = \frac{\pi}{12}}$$

1(e) → Find the equation of the sphere which passes through the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and has its radius as small as possible.

Sol: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \longrightarrow (i)$$

If it passes through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ then

$$1+2u+c=0,\ 1+2v+c=0,\ 1+2w+c=0$$

$$u=v=w=-\tfrac{1}{2}(1+c) \longrightarrow (ii)$$

∴ If r be the radius of the sphere (i) then

$$r^2=u^2+v^2+w^2-c=R \text{ (say)}$$

$$R=\frac{3}{4}(1+c)^2-c \quad \text{from (ii)}$$

If 'r' is least then $\underline{R}$ least

Now $\dfrac{dR}{dc}=\dfrac{3}{2}(1+c)-1$ and $\dfrac{d^2R}{dc^2}=\dfrac{3}{2}=$ positive

Equating $\dfrac{dR}{dc}$ to zero we get $\dfrac{3}{2}c+\dfrac{1}{2}=0$ (or) $c=-\tfrac{1}{3}$

and $\dfrac{d^2R}{dc^2}$ being positive R is least when $c=-\tfrac{1}{3}$

∴ from (ii) when R i.e. r^2 is least we have

$$u=v=w=-\tfrac{1}{2}\left(1-\tfrac{1}{3}\right)=-\tfrac{1}{3}$$

∴ from (i), the required equation is $x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$

(or) $3(x^2+y^2+z^2)-2(x+y+z)-1=0.$

2(a) Discuss for all values of k the system of equations

$$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

Sol^n: The given system of equations is equivalent to the single matrix equation

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

performing $R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

If the given system of equations is to possess any linearly independent solution, the coefficient matrix A must be of a rank less than 3.

for matrix A to be of rank less than 3, we must have

$$(k-8)(-k+2) + 5k(k-2) = 0$$
$$\Rightarrow 4k^2 - 16 = 0$$
$$\Rightarrow k = \pm 2$$

Now three cases arises

Case 1: When $K \neq \pm 2$, the given system of eqns possesses no linearly independent solution and $x=y=z=0$ is the only solution.

Case (2):

If $K=2$, the equation ① reduces to

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The coefficient matrix being of rank 2, the given system of eqns now possesses $3-2=1$ linearly independent solution. The given system of equations is now equivalent to

$-6y-10z=0$
$x+6y+10z=0$

$\Rightarrow y = -\frac{5z}{3}$, $x=0$

$\Rightarrow x=0,\ y=\frac{-5}{3}C,\ z=C$

Case (3):

If ~~k=2~~, the equation ① reduces to

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The coefficient matrix being of rank 2, hence one linearly independent solution, which is $x=4C,\ y=C,\ z=C$.

2(b) let $R_3[x] = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in R\}$

Define $T : R_3[x] \to R_3[x]$ by $T(f(x)) = \frac{d}{dx} f(x)$ → (1)

for all $f(x) \in R_3[x]$. Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

Solⁿ: let $f(x), g(x) \in R_3[x]$ and $\alpha, \beta \in R$ By (1) we have

$$T\{\alpha f(x) + \beta g(x)\} = \frac{d}{dx}\left[\alpha f(x) + \beta g(x)\right]$$

$$= \alpha \frac{d}{dx}\left[f(x)\right] + \beta \frac{d}{dx}\left[g(x)\right]$$

by rule of differentiation

$$= \alpha T(f(x)) + \beta T(g(x)). \text{ by (1)}$$

Hence T is a linear transformation By (1) we see that

$$T(1) = \frac{d}{dx}\{1\} = 0, \quad T(x) = 1, \quad T(x^2) = 2x.$$

Again $T(1) = 0 = 0.1 + 0.(1+x) + 0.(1+x+x^2)$.

$T(x) = 1 = 1.1 + 0.(1+x) + 0.(1+x+x^2)$.

$T(x^2) = 2x = -2.1 + 2(1+x) + 0.(1+x+x^2)$.

Hence the matrix representation of T with reference to the bases $\{1, x, x^2\}$, $\{1, 1+x, 1+x+x^2\}$

is
$$\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

2(C) (i) Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

(ii) If $z = (x+y) + (x+y)\varphi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

Sol^n^ let r be the radius of the circular base; h, the height; s the surface and v the volume of the open cylinder so that

$$S = \pi r^2 + 2\pi r h \longrightarrow (1)$$

$$V = \pi r^2 h \longrightarrow (2)$$

Here, as given s is constant and v a variable. Also h, r are variables. Substituting the value of h as obtained from (i) in (ii) we get

$$V = \pi r^2\left(\frac{S - \pi r^2}{2\pi r}\right) = \frac{Sr - \pi r^3}{2} \longrightarrow (3)$$

which gives v in terms of one variable r.

As v must be necessarily non-negative, we have

$$Sr - \pi r^3 \geq 0 \Rightarrow \pi r^3 \leq Sr \Rightarrow r \leq \sqrt{S/\pi}$$

Also r is non-negative.

Thus 'r' varies in the interval $\left[0, \sqrt{S/\pi}\right]$

Now $\dfrac{dv}{dr} = \dfrac{S - 3\pi r^2}{2}$

So that $\frac{dv}{dr} = 0$ only when $r = \sqrt{S/3\pi}$, Negative value of r being inadmissible. Thus v has only one stationary value.

Now $V = 0$ for the end points $r = 0$ and $\sqrt{S/\pi}$ and positive for every other admissible value of r. Hence v is greatest for

$$r = \sqrt{S/3\pi}$$

Substituting this value of r in (i) we get

$$h = \frac{S - \pi r^2}{2\pi r} = \frac{S - \pi(S/3\pi)}{2\pi\sqrt{(S/3\pi)}}$$

$$= \frac{2S}{3} \cdot \frac{1}{2\pi}\sqrt{\left(\frac{3\pi}{S}\right)} = \sqrt{\left(\frac{S}{3\pi}\right)}$$

Hence $h = r$ for the cylinder of greatest volume and given surface.

2(d) → Find the two tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$.

Sol Let the given sphere be

$x^2+y^2+z^2-4x+2y-6z+5=0$ —— (1)

given plane

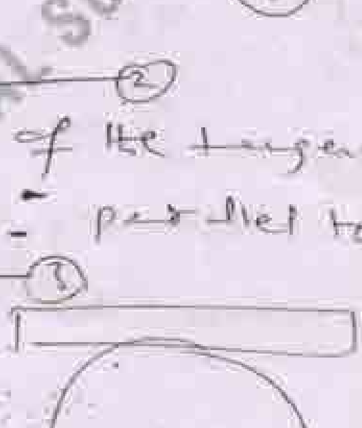
$2x+2y-z=0$ —— (2)

let the equation of the tangent plane be $2x+2y-z+k=0$ parallel to (2). —— (3)

let $C(2,-1,3)$ be the centre of the sphere.

radius $=\sqrt{4+1+9-5}=3$

Since the perpendicular distance from $C(2,-1,3)$ to the plane (3) = radius of the sphere.



$$\frac{|4-2-3+k|}{\sqrt{2^2+2^2+1}}=3.$$

$$\Rightarrow \frac{|-1+k|}{3}=3 \Rightarrow k-1=\pm 9.$$

$$\Rightarrow k=10,-8$$

∴ The required tangent planes:

$2x+2y-z+10=0$!

$2x+2y-z-8=0$.

3(b) A flat circular plate has the shape of the region $x^2+y^2 \le 1$. The plate, including the boundary where $x^2+y^2=1$, is heated so that the temperature at any point (x,y) is

$$T(x,y) = x^2+2y^2-x.$$

Find the hottest and coldest points on the plate, and the temperature at each of these points.

Sol^n: $T(x,y) = x^2+2y^2-x$ — (1)

Given that $x^2+y^2 \le 1$

Let $x^2+y^2 = K$

where K is some +ve number which is <1 (i.e. $0<K\le 1$)

At the boundary $x^2+y^2=1$ —— (2)

putting (2) in (1), we get $T(x) = -x^2 - x + 2$

$\frac{dT(x)}{dx} = -2x-1$

for max or minimum $\frac{dT}{dx} = 0$

$\Rightarrow -2x-1=0$

$\Rightarrow x = -\frac{1}{2}$.

from (2) $y = \pm\sqrt{3}/2$.

Also $\frac{d^2T}{dx^2} = -2 < 0$

$\therefore$ Maxima at $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

In the interior region of plate

$\frac{\partial T}{\partial x} = 2x-1, \quad \frac{\partial T}{\partial y} = 4y$

$\frac{\partial T}{\partial x} = 0$ and $\frac{\partial T}{\partial y} = 0$

$\Rightarrow x = \frac{1}{2}, \ y = 0$

This is the point of minimum value.

from (1) $T\left(\frac{1}{2}, 0\right) = \frac{1}{4} + 0 - \frac{1}{2} = -\frac{1}{4}$

and $T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}$ and $T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4}$.

$\therefore$ Hottest points are $\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $T = \frac{9}{4}$ units

Coldest point is $\left(\frac{1}{2}, 0\right)$ and $T = -\frac{1}{4}$ units

3(a) (i) find the diagonal form D and ~~the~~ diagonalizing matrix P for the following matrix over 'C':

$$A = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

(ii) Let $U = \text{span}\{(1,3,-2,2,3),\ (1,4,-3,4,2),\ (2,3,-1,-2,9)\}$

$W = \text{span}\{(1,3,0,2,1),\ (1,5,-6,6,3),\ (2,5,3,2,1)\}$ be the subspace of $\mathbb{R}^5$.

find the basis and dimension of U, W, $U+W$ and $U \cap W$.

Sol. (i) we have

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)^2 + 16 = 0$$

$$\Rightarrow 9 + \lambda^2 - 6\lambda + 16 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 25 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

Ans. $D = \begin{bmatrix} 3+4i & 0 \\ 0 & 3-4i \end{bmatrix}$

$P = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}$.

Conti. in this way.

(ii) let us construct matrices A and B

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 2 & -6 & 4 & 2 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 3R_2 \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & -1 & 3 & -2 & -1 \end{bmatrix} R_2 \to \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 3R_2 \qquad \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2$$

$$\text{——— (1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 9 & -4 & -2 \\ 0 & 1 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 3R_2 \quad \text{——— (2)}$$

Clearly the above forms are row reduced echelon forms.

The bases of U and W are

$S = \{(1, 0, 1, -4, 0), (0, 1, -1, 2, -1)\}$ and

$T = \{(1, 0, 9, -4, -2), (0, 1, -3, 2, 1)\}$.

$\therefore \dim U = 2, \ \dim W = 2.$

let $C = \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 1 & 0 & 9 & -4 & -2 \\ 0 & 1 & -3 & 2 & 1 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 8 & 0 & -2 \\ 0 & 1 & -3 & 2 & -1 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 8 & 0 & -2 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix} \quad R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 8 & 0 & -2 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & 0 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad R_4 \to R_4 + 4R_3.$$

Clearly it is in echelon form and the number of non-zero rows are 4.

$\therefore$ Basis of $V+W$ is

$$S_1 = \{(1,0,1,-4,0), (0,1,-1,2,-1), (0,0,-2,0,2), (0,0,0,0,0,2)\}$$

and $\dim(V+W) = 4$.

from ① & ②, we have

$V\ \ \ = \{(x, y, x-y, -4x+2y, -y) / x, y \in \mathbb{R}\}$

$W = \{ (a, b, 9a-3b, -4a+2b, -2a+b) / a, b \in \mathbb{R} \}$

We have

$x = a$ — (i), $y = b$ — (ii) ; $x - y = 9a - 3b$ — (iii)

$-4x + 2y = -4a + 2b$ — (iv)

$-y = -2a + b$ — (v)

$(iii) \equiv a - b = 9a - 3b$

$\Rightarrow 8a - 2b = 0$ — (vi)

$(v) \equiv -b = -2a + b$

$\Rightarrow -2a + 2b = 0$ — (vii)

$(vi) + (vii) \equiv 6a = 0$

$\Rightarrow \boxed{a = 0}$

and $\boxed{b = 0}$.

$\therefore U \cap W = \{ (0, 0, 0, 0, 0) \}$

$\dim(U \cap W) = 0$.

$\therefore \dim(U + W) = \dim U + \dim W - \dim(U \cap W)$

is verified

3(c)(i) Prove that the straight lines whose direction cosines are given by relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are $\perp$lar if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$, and parallel if $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)}=0$

Sol^n: Let the d.c's of the two lines be (l_1, m_1, n_1) and (l_2, m_2, n_2)

Eliminating n between the given relations, we get

$fm\left[-(al+bm)/c\right]+gl\left[-(al+bm)/c\right]+hlm=0$

$\Rightarrow -afm-bfm^2-agl^2-bglm+chlm=0$

$\Rightarrow ag(l/m)^2+(af+bg-ch)(l/m)+bf=0$ —— ① dividing each term by m^2

Its roots are l_1/m_1 & l_2/m_2

$\therefore \frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=$ Product of the roots $=\frac{bf}{ag}$

$\Rightarrow \frac{l_1l_2}{bf}=\frac{m_1m_2}{ag} \Rightarrow \frac{l_1l_2}{(f/a)}=\frac{m_1m_2}{(g/b)}=\frac{n_1n_2}{(h/c)}$, by symmetry

If the lines are $\perp$lar, then $l_1l_2+m_1m_2+n_1n_2=0$

$\Longrightarrow f/a+g/b+h/c=0$ Hence proved

If the lines are parallel, then their d.c's must be the same i.e. the roots of ① must be equal, the condition for the same is "$b^2=4ac$"

i.e. $(af+bg-ch)^2=4ag\cdot bf$ —— ②

$\Rightarrow af+bg-ch=\pm 2\sqrt{af}\sqrt{bg}$

$\Rightarrow af+bg\pm 2\sqrt{(af)}\sqrt{(bg)}=ch$

$\Rightarrow\left[\sqrt{(af)}\pm\sqrt{(bg)}\right]^2=ch=\left[\sqrt{(ch)}\right]^2$

$\Rightarrow \sqrt{(af)}\pm\sqrt{(bg)}\pm\sqrt{(ch)}=0$ is the required condition.

Also from ②, we get $a^2f^2+b^2g^2+c^2h^2+2abfg-2acfh-2bcgh=4abfg$.

$\Rightarrow a^2f^2+b^2g^2+c^2h^2-2bcgh-2cafh-2abfg=0$

3(c)(ii) Prove that the condition that the plane $ux+vy+wz=0$ may cut the cone $ax^2+by^2+cz^2=0$ in ⊥lar generators if $(b+c)u^2+(c+a)v^2+(a+b)w^2=0$.

Sol'n: Let $x/l = y/m = z/n$ be one of the lines in which the plane $ux+vy+wz=0$ meets the cone $ax^2+by^2+cz^2=0$, then we have

$$ul+vm+wn=0 \quad \text{——} (1)$$

and $$al^2+bm^2+cn^2=0 \quad \text{——} (2).$$

Eliminating n between (1) & (2), we get

$$al^2+bm^2+c\left[-(ul+vm)/w\right]^2=0$$

$$\Rightarrow (aw^2+cu^2)l^2+2cuvlm+(bw^2+cv^2)m^2=0$$

$$\Rightarrow (aw^2+cu^2)\left(\frac{l^2}{m^2}\right)+2cuv\left(\frac{l}{m}\right)+(bw^2+cv^2)=0$$

If its roots are l_1/m_1 and l_2/m_2, then we have

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \text{Product of the roots} = \frac{bw^2+cv^2}{aw^2+cu^2}$$

$$\Rightarrow \frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{cu^2+aw^2} = \frac{n_1n_2}{av^2+bu^2}, \text{ by symmetry}$$

If the lines are ⊥lar then $l_1l_2+m_1m_2+n_1n_2=0$

i.e. $(bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0$

$$\Rightarrow (b+c)u^2+(c+a)v^2+(a+b)w^2=0$$

Hence Proved.

4(a)(i) Let $H=\begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D=P^T H \bar{P}$ is diagonal.

Sol^n: Let us form the block matrix

$$[H|I]=\left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 1-i & 4 & 2-3i & 0 & 1 & 0 \\ -2i & 2+3i & 7 & 0 & 0 & 1 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 1+i & 2i & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5 & 3 & 2i & 0 & 1 \end{array}\right] \begin{array}{l} R_2\to(-1+i)R_1+R_2 \\ R_3\to 2iR_1+R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 5i & 3 & 2i & 0 & 1 \end{array}\right] \begin{array}{l} C_2\to C_2+(-1-i)C_1 \\ C_3\to C_3+2iC_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -5i & -1+i & 1 & 0 \\ 0 & 0 & -19 & 5+9i & -5i & 2 \end{array}\right] R_3\to 2R_3-5iR_2$$

$$\sim\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1+i & 1 & 0 \\ 0 & 0 & -38 & 5+9i & -5i & 2 \end{array}\right] C_3\to 2C_3+5iC_2$$

clearly H has been diagonalized

set $P=\begin{bmatrix} 1 & -1+i & 5+9i \\ 0 & 1 & -5i \\ 0 & 0 & 2 \end{bmatrix}$ and then $P^T H\bar{P}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -38 \end{bmatrix}$

4(a)(iii). Let A be a non-singular, $n\times n$ square matrix. Show that $A.(\operatorname{adj} A) = |A| I_n$. Hence show that $|\operatorname{adj}(\operatorname{adj} A)| = |A|^{(n-1)^2}$.

Soln: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ then

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix}$$

Now we have

$$(\operatorname{adj} A)A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The $(i,j)^{th}$ element in the matrix $(\operatorname{adj} A)A$ is

$$A_{1i}a_{1j} + A_{2i}a_{2j} + A_{3i}a_{3j} + \cdots + A_{ni}a_{nj}$$

$$= |A|, \text{ if } i=j$$

$$= 0, \text{ if } i\neq j$$

$$\Rightarrow (\operatorname{adj} A)A = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ - & - & \cdots & - \\ - & - & \cdots & - \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= |A| I_n$$

Similarly $A(\text{adj} A) = |A| I_n$

$\therefore (\text{adj} A) A = A(\text{adj} A) = |A| I_n$

$\rightarrow$ Since A is non-singular

$\therefore |A| \neq 0$ and A^{-1} exists.

We know that $A(\text{adj} A) = |A| I_n$

Taking adj A in place of A in ①

we get

$(\text{adj} A)(\text{adj}(\text{adj} A)) = |\text{adj} A| I_n$

$\Rightarrow (\text{adj} A)(\text{adj}(\text{adj} A)) = |A|^{n-1} I_n$ $[\because |\text{adj} A| = |A|^{n-1}]$

Pre multiplying bothsides by A, we get

$(A(\text{adj} A))(\text{adj}(\text{adj} A)) = |A|^{n-1}(A I_n)$

$\Rightarrow (|A| I_n)[\text{adj}(\text{adj} A)] = |A|^{n-1} A$ $(\because A I_n = A)$

$\Rightarrow |A| [I_n \, \text{adj}(\text{adj} A)] = |A|^{n-1} A$

$\Rightarrow |A| [\text{adj}(\text{adj} A)] = |A|^{n-1} A$

Since $|A| \neq 0$

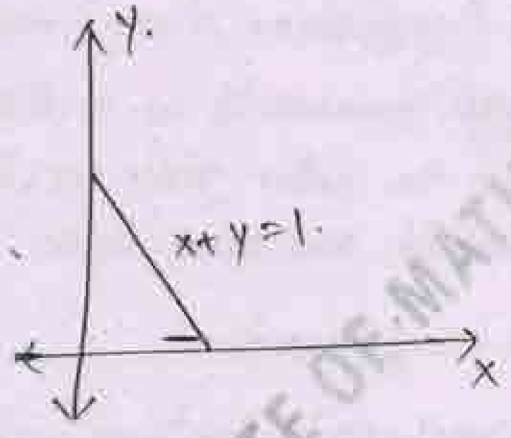
$$\boxed{\text{adj}(\text{adj} A) = |A|^{n-2} A}$$

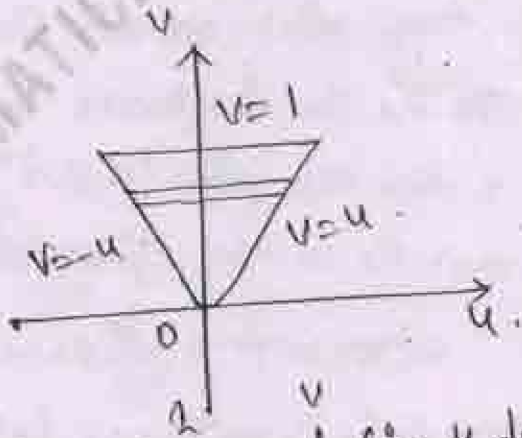
4(b) → Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\,dy$, where E is the region bounded by the Co-ordinate axes and $x+y=1$ in the first Quadrant.

Sol^n. Taking $x-y=u$, $x+y=v$ So that $x=\frac{(u+v)}{2}$

$y=\frac{(v-u)}{2}$ and the Jacobian is $1/2$

$$\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\,dy = \iint_{E_{uv}} \sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv \quad \to (1)$$





Now $\iint_{E_{uv}} \sin\left(\frac{u}{v}\right)\frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^1 dv \int_{-v}^{v} \sin\frac{u}{v}\,du$

$$= \frac{1}{2}\int_0^1 v\{-\cos 1 + \cos(-1)\}\,dv$$

$$= 0. \to (2)$$

Hence from (1) and (2) the required integral is zero.

4(c) Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$ is given by

$$ab(x^2+y^2)-2(a+b)z-1=0$$

Soln: Enveloping cone of the paraboid $ax^2+by^2=2cz$ with vertex at the point (α,β,γ).

The Equations of a line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \rightarrow \text{(i)}$$

Any point on this line is $(\alpha+lr,\beta+mr,\gamma+nr) \rightarrow$ (ii)

If the line (i) meets the given paraboloid at a distance r from the point (α,β,γ) then the point given by (ii) must lie on the given paraboloid and so we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

(or) $r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \rightarrow$ (iii)

If the line (i) is a tangent of the given paraboloid then the line (i) should meet the paraboloid in two coincident points, the condition for the same is that the roots of (iii) are equal i.e "$B^2=4AC$"

(or) $4(al\alpha+bm\beta-cn)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma) \rightarrow$ (iv)

The locus of line (i) which is tangent to the given paraboloid is obtained by eliminating l,m,n between (i) and (iv) and is

$$(a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma))^2$$
$$=\left[a(x-\alpha)^2+b(y-\beta)^2\right]\left[(a\alpha^2+b\beta^2-2c\gamma)\right] \rightarrow (V)$$

If $S\equiv ax^2+by^2-2cz$, $S_1=a\alpha^2+b\beta^2-2c\gamma$ and $P\equiv a\alpha x+b\beta y-c(z+\gamma)$ then eqn (V) can be written as

$$(P-S_1)^2=(S+S_1-2P)\,S_1$$
$$P^2+S_1^2-2PS_1=SS_1+S_1^2-2PS_1 \quad (\text{or}) \quad SS_1=P^2$$

$$(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma)=\left[a(\alpha x)+b\beta y+c(z+\gamma)\right]^2 \rightarrow (VI)$$

the required equation of the enveloping cone of the given paraboloid.

Cor. To find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2cz$.

Here we are to apply the condition that the enveloping cone, of the given paraboloid with vertex at (α,β,γ) may have three mutually perpendicular generators and we know that the condition for the same is that the sum of the co-efficients of x^2, y^2 and z^2 in the equation of the cone is zero.

$\therefore$ from (vi) above we get

$$\left[a(a\alpha^2+b\beta^2-2c\gamma)-a^2\alpha^2\right]+\left[b(a\alpha^2+b\beta^2-2c\gamma)-b^2\beta^2\right]-c^2=0$$

$$ab\beta^2-2ca\gamma+ba\alpha^2-2cb\gamma-c^2=0$$

$$ab(\alpha^2+\beta^2)-2c(a+b)\gamma-c^2=0$$

Hence the required locus of the point (α,β,γ) is

$$ab(x^2+y^2)-2c(a+b)z-c^2=0 \rightarrow \text{(vii)}$$

putting '$c=1$' in the above equation we get the required answer

$$ab(x^2+y^2)-2(a+b)z-1=0$$

5(a) → find the orthogonal trajectories of the following family of curve

$$r^n \sin n\theta = a^n.$$

Soln: Given Equation of family of curve is

$$r^n \sin n\theta = a^n \text{, with 'a' as a parameter} \rightarrow (1)$$

from (1) $n \log r + \log \sin n\theta = \log a^n \rightarrow (2)$

Differentiating (2) w.r.t to θ we get

$$\frac{n}{r}\frac{dr}{d\theta} + n\cot n\theta = 0$$

$$\Rightarrow n\left[\frac{1}{r}\frac{dr}{d\theta} + \cot n\theta\right] = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} + \cot n\theta = 0. \rightarrow (3)$$

which is the differential Equation of ~~the~~ given family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2 d\theta/dr$ in (3) the differential equation of the required othogonal trajectory is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) + \cot n\theta = 0$$

$$-r\frac{d\theta}{dr} + \cot n\theta = 0.$$

$$\cancel{+}r\frac{d\theta}{dr} = \cancel{+}\cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = \frac{dr}{r}$$

Integrating on both sides then

$$\int\frac{d\theta}{\cot n\theta} = \int\frac{dr}{r} \Rightarrow \int \tan n\theta\, d\theta = \int\frac{dr}{r}$$

$$\int\frac{dr}{r} = \int\frac{\sin n\theta}{\cos n\theta}\,d\theta$$

$$+\int\frac{dr}{r} = -\frac{1}{n}\int\frac{-n\sin n\theta}{\cos n\theta}\,d\theta.$$

$$\int\frac{dr}{r} = \frac{-1}{n}\int\frac{-n\sin n\theta}{\cos n\theta}\,d\theta$$

$$\log r = \frac{-1}{n}\log(\cos n\theta) + \frac{1}{n}\log c$$

$$n\log r = -\log(\cos n\theta) + \log c$$

$$n\log r + \log(\cos n\theta) = \log c$$

$$\log r^n + \log(\cos n\theta) = \log c$$

$$\log r^n \cos n\theta = \log c$$

$$\therefore\ r^n\cos n\theta = c$$ Ans.

5(b) → Examine for Singular solution and extraneous loci,

$y+px = x^4p^2$.

Sol'n: The given equation is $y+px = x^4p^2$ —— (1)

Differentiating (1) w.r.t x, we have

$$p+p+x\frac{dp}{dx} = 4x^3p^2+2x^4p\frac{dp}{dx}$$

$$\Rightarrow x\frac{dp}{dx}\left(1-2x^3p\right)+2p\left(1-2x^3p\right)=0$$

$$\Rightarrow (1-2x^3p)\left(x\frac{dp}{dx}+2p\right)=0$$

Here we omit the first factor since it will lead us to singular solution.

For general solution, we have

$$x\frac{dp}{dx}+2p=0 \Rightarrow 2\frac{dx}{x}+\frac{dp}{p}=0$$

Integrating, we get $2\log x+\log p = \log c$

$$\Rightarrow p = c/x^2$$

putting this of p in (1), we get

$$y+(c/x^2)x = x^4(c^2/x^4)$$

$$\Rightarrow y = c^2 - c/x.$$

$$\Rightarrow xc^2 - c - xy = 0 \text{ —— (2)}$$

from (1), the p-disc. relation is

$$x^2(4x^2y+1)=0 \text{ and}$$

from (2), the c-disc relation is

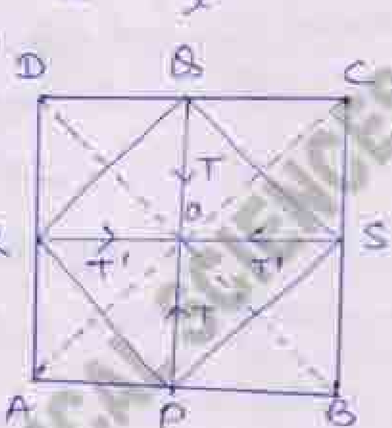
$$4x^2y+1=0$$

∴ $4x^2y+1=0$ is singular since it is occur once in both the discriminants and satisfy the given differential equation.

and $x=0$ is a tac locus.

5(c) The middle points of the opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T' be the tensions in these rods, prove that $\frac{T}{l} + \frac{T'}{l'} = 0$

Sol^n ABCD is a framework in the form of a quadrilateral formed of four light rods. The middle point P and Q of the rods AB and DC are joined by a



light rod in a state of tension T and the middle points R & S of the rod AD and BC are joined by a light rod in a state of Tension T'. Here $PQ = l$ and $RS = l'$

Since P, Q, R, S are middle points of the quadrilateral ABCD, therefore PSQR is a parallelogram. Consequently the diagonals PQ & RS of this parallelogram bisect each other at O.

Replace the rods PQ and RS by two equal and opposite forces T and T' respectively as shown in the figure. Now give the system a small displacement in which PQ changes to $PQ + \delta(PQ)$ and RS changes to $RS + \delta(RS)$. The lengths of rods AB, BC, CD, DA donot change.

The equation of virtual work is

$$-T\,\delta(PQ) - T'\,\delta(RS) = 0$$

$$\Rightarrow \frac{\delta(PQ)}{\delta(RS)} = -\frac{T'}{T} \quad \text{--- (1)}$$

Now let us find a relation between the parameters PQ and RS from the figure.

Since OP is a median of the ΔOAB.

$$\therefore OA^2 + OB^2 = 2(OP)^2 + 2(AP)^2 = 2\left(\tfrac{1}{2}PQ\right)^2 + 2\left(\tfrac{1}{2}AB\right)^2$$

$$= \frac{1}{2}(PQ^2 + AB^2) \quad \text{--- (2)}$$

Similarly from ΔOCD, we have

$$OC^2 + OD^2 = \frac{1}{2}(PQ^2 + CD^2) \quad \text{--- (3)}$$

Adding (2) + (3), we get

$$OA^2 + OB^2 + OC^2 + OD^2 = \frac{1}{2}(2PQ^2 + AB^2 + CD^2) \quad \text{--- (4)}$$

Doing the same thing with ΔOAD and ΔOBC

we get $OA^2 + OB^2 + OC^2 + OD^2 = \frac{1}{2}(2RS^2 + BC^2 + DA^2)$ --- (5)

from (4) + (5), we get

$$\frac{1}{2}[2PQ^2 + AB^2 + CD^2] = \frac{1}{2}[2RS^2 + BC^2 + DA^2]$$

$$\Rightarrow 2(PQ^2 - RS^2) = BC^2 + DA^2 - AB^2 - CD^2$$

$$(PQ^2 - RS^2) = \text{Constant} \quad \text{--- (6)}$$

Since AB, BC, CD, DA are all of fixed length,

Differentiating (6) we get:

$$2PQ\,\delta(PQ) - 2RS\,\delta(RS) = 0 \Rightarrow \frac{\delta(PQ)}{\delta(RS)} = \frac{RS}{PQ} \quad \text{--- (7)}$$

equating the values $\frac{\delta(PQ)}{\delta(RS)}$ from (1) + (7) we get

$$-\frac{T'}{T} = \frac{RS}{PQ} \Rightarrow \frac{T}{PQ} + \frac{T'}{RS} = 0$$

$$\Rightarrow \frac{T}{l} + \frac{T'}{l'} = 0 \quad (\because PQ = l \;\&\; RS = l')$$

5(d) Find the constants a and b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(1,-1,2)$.

Sol: The given surfaces are

$f_1 = ax^2 - byz - (a+2)x = 0$ —(1)

and $f_2 = 4x^2y + z^3 - 4 = 0$ —(2)

The point $(1,-1,2)$ obviously lies on the surface (2). It will also lie on the surface (1) if

$a + 2b - (a+2) = 0 \Rightarrow 2b - 2 = 0$

$\Rightarrow \boxed{b = 1}$

Now grad $f_1 = [2ax - (a+2)]\hat{i} - bz\hat{j} - by\hat{k}$

and grad $f_2 = 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$.

Then n_1 = grad f_1 at the point $(1,-1,2)$

$= (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$

and n_2 = grad f_2 at the point $(1,-1,2)$

$= -8\hat{i} + 4\hat{j} + 12\hat{k}$.

The vectors n_1 and n_2 are along the normals to the surfaces (1) and (2) at the point $(1,-1,2)$.

These surfaces will intersect orthogonally at the point $(1,-1,2)$ if the vectors n_1 and n_2

perpendicular i.e. if $n_1 \cdot n_2 = 0$

$\Rightarrow -8(a-2) - 8b + 12b = 0$

$\Rightarrow b - 2a + 4 = 0$

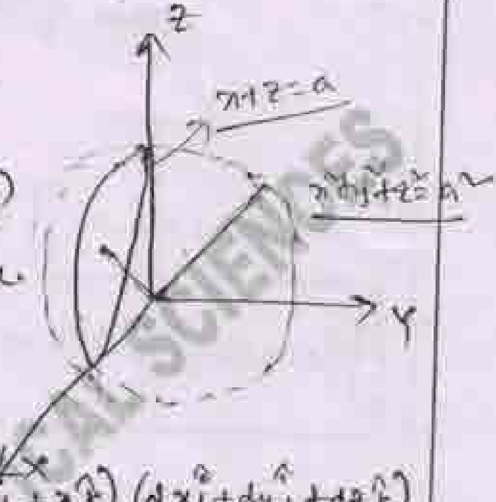
$\Rightarrow 1 - 2a + 4 = 0 \quad (\because b = 1)$

$\Rightarrow \boxed{a = 5/2}$

$\therefore a = 5/2, \; b = 1$

5(c) → Apply Stokes theorem to evaluate $\int y\,dx + z\,dy + x\,dz$ where C is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$.

Sol: Intersection of sphere $x^2+y^2+z^2=a^2$ and line $(x+z=a)$ is a circle lying in the plane.



Given $\int (y\,dx + z\,dy + x\,dz) = \int (y\hat{i} + z\hat{j} + x\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$

here $F = (y\hat{i} + z\hat{j} + x\hat{k})$ $\quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

by stokes theorem

$$\int_C F \cdot d\bar{r} = \iint_S (\nabla \times F) \cdot \hat{n}\, ds$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(\hat{i} + \hat{j} + \hat{k})$$

$$\hat{n} = \frac{\nabla(x+z-a)}{|\nabla(x+z-a)|}$$

$$\hat{n} = \left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)$$

$$\iint (\nabla \times F) \cdot \hat{n}\, ds = (-\hat{i} + -\hat{j} + -\hat{k}) \cdot \left(\frac{\hat{i}+\hat{k}}{\sqrt{2}}\right)$$

$$= -\sqrt{2} \iint s$$

plane is a distance of $\frac{a}{\sqrt{2}}$ from the centre.

$\therefore$ Radius of the circle $\frac{a}{\sqrt{2}}$.

$\therefore$ area of the surface $= \pi (a/\sqrt{2})^2$

$= a^2\pi/2$

$\therefore \oint \bar{F} \cdot d\bar{r} = \iint (\nabla \times \bar{F}) \cdot n \, ds$

$$\boxed{\int (y\,dx + z\,dy + x\,dz) = -\pi a^2/\sqrt{2}}$$

6(a) Justify that a differential equation of the form:

$[y + x f(x^2+y^2)]dx + [y f(x^2+y^2) - x]dy = 0$,

where $f(x^2+y^2)$ is an arbitrary function of (x^2+y^2), is not an exact differential equation and $\frac{1}{x^2+y^2}$ is an integrating factor for it.

Hence solve this differential equation for

$f(x^2+y^2) = (x^2+y^2)^2$

Sol: Given that

$[y + x f(x^2+y^2)]dx + [y f(x^2+y^2) - x]dy = 0$ — (1)

which is in the form of $Mdx + Ndy = 0$

where $M = y + x f(x^2+y^2)$, $N = y f(x^2+y^2) - x$

$\frac{\partial M}{\partial y} = 1 + f'(x^2+y^2) \cdot 2yx$ & $\frac{\partial N}{\partial x} = y f'(x^2+y^2)(2x) - 1$

$= 1 + 2xy f'(x^2+y^2)$ $\quad\quad = -1 + 2xy f'(x^2+y^2)$.

Clearly $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

$\therefore$ Given equation is not an exact differential equation.

To show $\frac{1}{x^2+y^2}$ is an integrating factor,

multiplying equation (1) by $\frac{1}{x^2+y^2}$

Then

$M = \frac{y + x f(x^2+y^2)}{x^2+y^2}$ & $N = \frac{y f(x^2+y^2) - x}{x^2+y^2}$

$\frac{\partial M}{\partial y} = \frac{[1 + 2xy f'(x^2+y^2)](x^2+y^2) - [y + x f(x^2+y^2)](2y)}{(x^2+y^2)^2}$

$= \frac{x^2 - y^2 + 2xy[(x^2+y^2) f'(x^2+y^2) - f(x^2+y^2)]}{(x^2+y^2)^2}$

$$\frac{\partial N}{\partial x} = \frac{\left[-1+2xy\, f'(x^2+y^2)\right](x^2+y^2) - \left[y f(x^2+y^2) - x\right](2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy\left[(x^2+y^2)\, f'(x^2+y^2) - f(x^2+y^2)\right]}{(x^2+y^2)^2}$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\Rightarrow \frac{1}{x^2+y^2}$ is an integrating factor for equation (1)

Given $f(x^2+y^2) = (x^2+y^2)^2$.

Then equation (1) becomes

$$\left[y + x(x^2+y^2)^2\right]dx + \left[y(x^2+y^2)^2 - x\right]dy = 0$$

This can be written as

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\Rightarrow \frac{y\,dx - x\,dy}{x^2+y^2} + (x\,dx + y\,dy)(x^2+y^2) = 0$$

$$\Rightarrow d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) + \frac{x^2+y^2}{2} d(x^2+y^2) = 0$$

Integrating, we get

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\frac{(x^2+y^2)^2}{2} = C$$

which is the required solution.

6(b) → Show that the wronskian of the functions x^2 and $x^2\log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation. If so, determine this differential equation.

Sol'n: Let $y_1(x) = x^2$ and $y_2(x) = x^2\log x$

The wronskian $W(x)$ of y_1 and y_2 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\log x \\ 2x & 2x\log x + x \end{vmatrix}$$

$$= x^2(2x\log x + x) - 2x^3\log x$$

$\therefore W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$

Hence solution y_1 and y_2 can be linearly independent solutions of an ordinary differential equation.

To form the required differential equation:

The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 = Ax^2 + Bx^2\log x \quad \text{—— (1)}.$$

where A & B are arbitrary constants.

Differentiating (1), $y' = 2Ax + B(2x\log x + x)$ —— (2).

Diff. (2) $y'' = 2A + B(2\log x + 2 + 1)$ —— (3)

We now eliminate A and B from (1), (2) and (3).

To this end, we first solve (2) & (3) for A & B.

Multiplying bothsides of (3) by x, we get

$$xy'' = 2Ax + B(3x + 2x\log x) \quad \text{—— (4)}$$

Substracting (2) from (4), $xy'' - y' = 2Bx$

$$\Rightarrow B = (xy'' - y')/2x$$

Substituting this value of B in (3), we have

$$2A = y'' - \frac{1}{2x}(xy'' - y')(3 + 2\log x)$$

$$\Rightarrow A = \frac{1}{4x}\left[2xy'' - (xy'' - y')(3 + 2\log x)\right]$$

Substituting the above values A & B in ①,
we have

$$y = \left(\frac{x}{4}\right)\left[2xy'' - 3xy'' + 3y' - 2xy''\log x + 2y'\log x\right] + \left(\frac{x}{2}\right)\log x\,(xy'' - y')$$

$$4y = x\left[-xy'' + 3y' - 2xy''\log x + 2y'\log x\right] + 2x\log x\,(xy'' - y')$$

$$\Rightarrow \boxed{x^2y'' - 3xy' + 4y = 0}$$

$\therefore$ which is the required equation.

6(c) Solve $\left[(x+1)^2D^2+(x+1)D-1\right]y=\left[\ln(x+1)\right]^2+x-1$ —— (1)

putting $x+1=e^z \Rightarrow z=\log(1+x)$; $1+x>0$.

$\Rightarrow \frac{dz}{dx}=\frac{1}{1+x}$.

$\therefore$ The transformed equation is

$[D_1(D_1-1)+D_1-1]y = z^2+e^z-2$

where $D_1=\frac{d}{dz}$

$\Rightarrow [D_1^2-D_1+D_1-1]y = z^2+e^z-2$

$\Rightarrow [D_1^2-1]y = z^2+e^z-2$. —— (2)

Its AE is

$m^2-1=0$

$\Rightarrow m = \pm 1$

$\therefore y_c(z) = Ae^z+Be^{-z}$

$\Rightarrow y_c(x) = A(x+1)+B\frac{1}{x+1}$ —— (3)

$y_p(z) = \frac{1}{D_1^2-1}(e^z+z^2-2)$

$= \frac{1}{D_1^2-1}e^z + \frac{1}{D_1^2-1}(z^2-2)$

$= \frac{z}{2D_1}e^z - (1-D_1^2)^{-1}(z^2-2)$

$= \frac{z}{2}e^z - (1+D_1^2+D_1^4+\cdots)(z^2-2)$

$= \frac{z}{2}e^z - (z^2-2+2)$

$= \frac{z}{2}e^z - z^2$

$\therefore y_p(x) = \frac{(x+1)}{2}\log(1+x) - [\log(1+x)]^2$

$\therefore$ The general solution is $y(x) = y_c(x) + y_p(x)$

$$\Rightarrow y(x) = A(x+1) + \frac{B}{x+1} + \frac{(x+1)}{2}\log(1+x) - [\log(1+x)]^2$$

6(d) By using Laplace transform, solve the initial value problem $(D^2+m^2)x = a\cos nt$, $t>0$, if x, Dx equal to x_0 and x_1, when $t=0$, $n \neq m$.

Sol^n Re-writing the given equation and conditions, we get $x'' + m^2 x = a\cos nt$ —— ①

with the initial conditions: $x(0) = x_0$ and $x'(0) = x_1$ —— ②

Taking Laplace transform of both sides of ①, we get

$$L\{x''\} + m^2 L\{x\} = aL\{\cos nt\}$$

or $s^2 L\{x\} - s\,x(0) - x'(0) + m^2 L\{x\} = \dfrac{as}{(s^2+n^2)}$

or $(s^2+m^2)L\{x\} - sx_0 - x_1 = as/(s^2+n^2)$ using ②

or $L\{x\} = \dfrac{sx_0}{s^2+m^2} + \dfrac{x_1}{s^2+m^2} + \dfrac{as}{(s^2+m^2)(s^2+n^2)}$

or $L\{x\} = \dfrac{sx_0}{s^2+m^2} + \dfrac{x_1}{s^2+m^2} + \dfrac{a}{m^2-n^2}\left[\dfrac{s}{s^2+n^2} - \dfrac{s}{s^2+m^2}\right]$

Taking inverse Laplace transformation on both sides

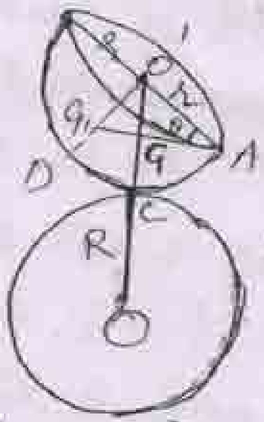
$$x = x_0\cos mt + \frac{x_1}{m}\sin mt + \frac{a}{m^2-n^2}[\cos nt - \cos mt]$$

Here $x_0 = 0$ and $x_1 = 0$ and hence solution is

$$y = a(\cos nt - \cos mt)/(m^2-n^2).$$

7(a) → A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. prove that if $R/r \geq \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

Sol:



Let O' be the centre of the base of the hemispherical shell of radius r. Let weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the spherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre at O at the highest point C.

for equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$ ie, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

i.e, $\frac{1}{h} > \frac{R+r}{Rr}$.

i.e, $h < \frac{rR}{R+r}$ ——— (1)

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is maximum i.e, if $O'G$ is perpendicular to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$.

Then from right angled $\Delta O'G_1$

$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}$.

$\therefore \sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ the minimum value of $O'G$

$= O'A\sin\theta = r\left(\frac{1}{\sqrt{5}}\right) = r/\sqrt{5}$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$= r - \frac{r}{\sqrt{5}} = \frac{r(\sqrt{5}-1)}{\sqrt{5}}$

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if $\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{r+R}$ i.e, if $\frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$

i.e, if $(\sqrt{5}-1)R - (\sqrt{5}-1)r < R\sqrt{5}$

i.e, if $R/r \geq \sqrt{5}-1$

7(b) A particle moves in a straight line, its accelaration directed towards a fixed point 'O' in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance 'x' from 'O'. If it starts from rest at a distance 'a' from 'O', show that it will arrive at 'O' with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.

Sol- Take the centre of force O as origin. Suppose a particle starts from rest at 'A', where OA = a. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time 't', where OP = x.

The accelaration of the particle at 'P' is $\mu a^{5/3} \cdot x^{-2/3}$ directed towards 'O'. Therefore, the equation of motion of the particle is –

$$\frac{d^2x}{dt^2} = -\mu a^{5/3} \cdot x^{-2/3} \quad \text{——— (1)}$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3} \cdot x^{1/3}}{1/3} + K = -6\mu a^{5/3} \cdot x^{1/3} + K$$

where K is a constant.

At A ; $x = a$ and $\frac{dx}{dt} = 0$; so that

$$-6\mu a^{5/3} \cdot a^{1/3} + K = 0 \Rightarrow \boxed{K = 6\mu a^2}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -6\mu a^{5/3} \cdot x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right) \quad \text{—— (2)}$$

which gives the velocity of particle at any distance x from the centre of force. Suppose, the particle arrives at 'O' with the velocity v_1.

Then, at O ; $x=0$ and $\left(\frac{dx}{dt}\right)^2 = v_1^2$

So, from (2), we have

$$v_1^2 = 6\mu a^{5/3}(a^{1/3}-0) = 6\mu a^2$$

or $\quad v_1 = a\sqrt{6\mu}$

Now, taking square root of (2), we get

$$\frac{dx}{dt} = -\sqrt{6\mu a^{5/3}}\sqrt{(a^{1/3}-x^{1/3})}$$

where the -ve sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$\boxed{dt = -\frac{1}{\sqrt{6\mu a^{5/3}}}\cdot\frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}} \qquad (3)$$

Let t_1 be the time from A to O. Then integrating (3) from A to O; we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{6\mu a^{5/3}}}\int_a^0 \frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}$$

$$\int_0^{t_1} = \frac{1}{\sqrt{6\mu a^{5/3}}}\int_0^a \frac{dx}{\sqrt{a^{1/3}-x^{1/3}}}$$

Put $x = a\sin^6\theta$;

So that ; $dx = 6a\sin^5\theta\cos\theta\, d\theta$; when $x=0$ $\theta=0$ and when $x=a$, $\theta=\pi/2$

$$\therefore t_1 = \frac{1}{\sqrt{6\mu a^{5/3}}}\int_0^{\pi/2}\frac{6a\sin^5\theta\cdot\cos\theta}{a^{1/6}\cos\theta}\,d\theta$$

$$\therefore t_1 = \sqrt{\frac{6}{\mu}}\int_0^{\pi/2}\sin^5\theta\, d\theta \Rightarrow t_1 = \sqrt{\frac{6}{\mu}}\left[\frac{4.2}{5.3.1}\right]$$

$$\therefore t_1 = \frac{8}{15}\sqrt{\frac{6}{\mu}}$$

7(c) → Discuss the motion of a particle falling under gravity in a medium whose resistance varies as the velocity.

Solⁿ → Suppose a particle of mass m start at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where OP = x and let v be the velocity of the particle at P.

The force acting on the particle at P are

(i) The force mkv due to the resistance acting vertically upwards i.e against the direction of motion of the particle and

(ii) the weight mg of the particle acting vertically downwards.

By Newton's second law of motion the eqⁿ of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv$$

or $$\frac{d^2x}{dt^2} = g - kv \quad \text{—(1)}$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0 = g - kV \quad \text{or} \quad k = g/V$$

Putting $k = g/V$ in ① we get

$$\frac{d^2x}{dt^2} = g - \frac{g}{V}v = \frac{g}{V}(V-v) \quad \text{— ②}$$

Relation b/w v and x.

The equation ② can be written as

$$v\frac{dv}{dx} = \frac{g}{V}(V-v)$$

or $dx = \frac{V}{g}\frac{v}{V-v}dv = -\frac{V}{g}\frac{-v}{V-v}dv$

$$= -\frac{V}{g}\frac{(V-v)-V}{V-v}dv = -\frac{V}{g}\left[1-\frac{V}{V-v}\right]dv$$

Integrating, $x = -\frac{V}{g}[v + V\log(V-v)] + A$

where A is a constant

But initially at O, $x=0$ and $v=0$

$\therefore \quad A = \frac{V^2}{g}\log V$

$$\therefore \quad x = -\frac{V}{g}v - \frac{V^2}{g}\log(V-v) + \frac{V^2}{g}\log V$$

or $$x = -\frac{V}{g}v + \frac{V^2}{g}\log\frac{V}{V-v} \quad \text{— ③}$$

which gives the velocity of the particle at any position

Relation b/w v and t.

The equation ② can also be written as

$$\frac{dv}{dt} = \frac{g}{V}(V-v)$$

$$dt = \frac{V}{g}\frac{dv}{V-v}$$

Integrating, we have

$t = -\frac{V}{g}\log(V-v)+B$, where B is a constant.

initially at O, $t=0$ and $v=0$

$$B = \frac{V}{g}\log V$$

$$\therefore\ t = -\frac{V}{g}\log(V-v)+\frac{V}{g}\log V$$

or $t = \frac{V}{g}\log\frac{V}{V-v}$ ——— (4)

which gives the velocity of the particle at any time t.

Relation b/w x and t

from (4), we have

$\log\frac{V}{V-v} = \frac{gt}{V}$ or $\frac{V}{V-v} = e^{gt/V}$

or $V-v = Ve^{-gt/V}$, or $v = V[1-e^{-gt/V}]$

or $\frac{dx}{dt} = V[1-e^{-gt/V}]$, or $dx = V[1-e^{-gt/V}]dt$

Integrating we get, $x = Vt+\frac{V^2}{g}e^{-gt/V}+c$ where C is const.

Initially at O, $x=0$ and $t=0$

$$\therefore\ C = -V^2/g$$

$$\therefore\ x = Vt+\frac{V^2}{g}e^{-gt/V}-\frac{V^2}{g}$$

or $x = Vt+\frac{V^2}{g}(e^{-gt/V}-1)$ ——— (5)

which gives the distance fallen through in time t

8(a)(ii) A vector field is given by $\vec{F}=(x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{j}$. Verify that the field $\vec{F}$ is irrotational or not. find the scalar potential

Soln.- Field $\vec{F}$ is irrotational

if, $\nabla\times\vec{F}=0$

Now find $\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2+xy^2 & y^2+x^2y & 0\end{vmatrix}$

$=\hat{i}(0)-\hat{j}(0-0)+\hat{k}(2xy-2xy)$

$=0$

$\therefore \nabla\times\vec{F}=0$, $\vec{F}$ is irrotational.

For scalar potential ϕ -

$F=\nabla\phi$

$F=(x^2+xy^2)\hat{i}+(y^2+x^2y)\hat{j}=\nabla\phi$

$=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

$\Rightarrow \frac{\partial\phi}{\partial x}=x^2+xy^2$, $\frac{\partial\phi}{\partial y}=y^2+x^2y$, $\frac{\partial\phi}{\partial z}=0$

Integrating, $\phi=\frac{x^3}{3}+\frac{x^2y^2}{2}+f_1(y,z)$

$\phi=\frac{y^3}{3}+\frac{x^2y^2}{2}+f_2(x,z)$

$\phi=f_3(x,y)$

We get, $f_1(y,z)=f_2(x,z)=0$

$f_3(x,y)=\frac{x^3}{3}+\frac{y^3}{3}+\frac{x^2y^2}{2}$

So, Comparing $\phi=\left(\frac{x^3}{3}+\frac{y^3}{3}+\frac{x^2y^2}{2}\right)+C$

where C is an arbitrary constant

8(a)(iii) A curve in space is defined by the vector equation $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Determine the angle between the tangents to this curve at the points $t=+1$ and $t=-1$. By using Divergence theorem of Gauss, evaluate the surface integral

Sol^n: Given that $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$

Tangent vector is given by

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

At $t=1$, $\frac{d\vec{r}}{dt} = 2\hat{i} + 2\hat{j} - 3\hat{k} = T_1$ (say)

and at $t=-1$, $\frac{d\vec{r}}{dt} = -2\hat{i} + 2\hat{j} - 3\hat{k} = T_2$ (say)

Angle between the tangents T_1 and T_2 is given by

$$\cos\theta = \frac{T_1 \cdot T_2}{|T_1||T_2|}$$

$$= \frac{(2\hat{i} + 2\hat{j} - 3\hat{k})\cdot(-2\hat{i} + 2\hat{j} - 3\hat{k})}{\sqrt{4+4+9}\,\sqrt{4+4+9}}$$

$$\cos\theta = \frac{-4+4+9}{\sqrt{17}\,\sqrt{17}} = 9/17$$

$$\Rightarrow \theta = \cos^{-1}(9/17)$$

8(b) Find the curvature (κ) and torsion (τ) for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

Sol^n: Ans $\kappa = \tau = \dfrac{1}{(1+t^2)^2}$

8(c) → Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$, given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t=0$.

Sol'n: Given that

$$\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$$

Integrating, we get

$$\frac{d\vec{r}}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} + b$$

b is an arbitrary constant vector

But it is given that when $t=0$, $\frac{dr}{dt} = -\hat{i} - 3\hat{k}$

$$\therefore -i - 3k = -4k + b$$

$$\Rightarrow b = -i + k$$

$$\therefore \frac{dr}{dt} = 3t^2\hat{i} - 8t^3\hat{j} - 4\cos t\,\hat{k} - \hat{i} + \hat{k}$$

$$= (3t^2 - 1)\hat{i} - 8t^3\hat{j} - (-4\cos t + 1)\hat{k}$$

Integrating again, w.r.t t, we get

$$\vec{r} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + c$$

where c is an arbitrary constant vector.

But it is given that when $t=0$, $\vec{r} = 2\hat{i} + \hat{j}$

$$\therefore 2\hat{i} + \hat{j} = 0 + c = c$$

$$\therefore \vec{r} = (t^3 - t)\hat{i} - 2t^4\hat{j} + (t - 4\sin t)\hat{k} + 2\hat{i} + \hat{j}$$

$$\Rightarrow \vec{r} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}$$

is the required solution of the given differential equation.

8(d) Use divergence theorem to evaluate

$\int_S F \cdot ds$ where $F = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2 = a^2$.

Sol Gauss divergence theorem states

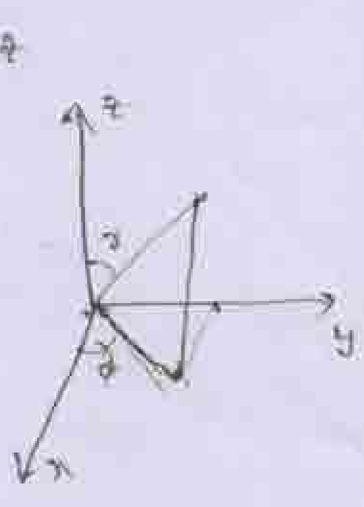
$$\iint_{\text{surface}} \vec{F}\cdot\vec{n}\,ds = \iiint_{\text{Volume of surface enclosed.}} (\nabla\cdot F)\,dx\,dy\,dz$$

$$\therefore \nabla\cdot F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right)$$

$$= 3(x^2+y^2+z^2)$$

$$\therefore \int_S F\cdot ds = \iiint (\nabla\cdot F)\,dV$$

$$= \iiint_{\text{Volume of sphere}} 3(x^2+y^2+z^2)\,dx\,dy\,dz$$



$$= \iiint 3r^2 r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= 3\iiint r^4 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{3a^5}{5}(1+1)(2\pi)$$

$$\int F\cdot ds = 12\pi a^5/5$$

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