Mains Test series – 2018

Test-5 Answer Key (Paper-I)

1(c) Find the values of a and b in order that $\lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3}$ may be equal to 1.

Sol'n: $\underset{x\to 0}{Lt} \frac{x(1+a\cos x)-b\sin x}{x^3}$

It is a $\frac{0}{0}$ form and so by L'Hospital's rule.

$$= \underset{x\to 0}{Lt} \frac{(1+a\cos x)-ax\sin x-b\cos x}{3x^2} \quad \text{— (1)}$$

the denominator of (1) $\to 0$ as $x\to 0$ but

(1) $\to$ a finite limit 1.

$\therefore$ The numerator $(1+a\cos x - ax\sin x - b\cos x)$

must $\to 0$ as $x\to 0$

$\therefore\ 1+a-b=0$ — (2)

Also if the relation (2) holds, then

$$\underset{x\to 0}{Lt} \frac{1+(a-b)\cos x - ax\sin x}{3x^2}$$

$$= \underset{x\to 0}{Lt} \frac{-(a-b)\sin x - a\sin x - ax\cos x}{6x} \quad \left(\tfrac{0}{0}\text{ form}\right)$$

$$= \underset{x\to 0}{Lt} \frac{-(a-b)\cos x - 2a\cos x + ax\sin x}{6}$$

$$= \frac{-a+b-2a}{6} = \frac{-3a+b}{6} = 1 \text{ (given)}$$

$\Rightarrow -3a+b=6$ — (3)

solving (2) & (3), we get

$$\boxed{a=-5/2,\ b=-3/2}$$

1(d) Find the volume of the region lying below the paraboloid with equation $z = 4-x^2-y^2$ and above the xy-plane.

Sol^n: Since the paraboloid intersects the xy-plane when

$$4-x^2-y^2=0,$$

i.e., when $x^2+y^2=4$

V is the volume of the region bounded by $f(x,y)=4-x^2-y^2$ and below by the region $D=\{(x,y): x^2+y^2\le 4\}$

If we describe D as $D=\{x,y): -2\le x\le 2, -\sqrt{4-x^2}\le y\le \sqrt{4-x^2}$,

then we may compute

$$V = \iint_D (4-x^2-y^2)\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2-y^2)\,dy\,dx = \int_{-2}^{2}\left(4y-x^2y-\frac{y^3}{3}\right)\Bigg|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left(8\sqrt{4-x^2}-2x^2\sqrt{4-x^2}-\frac{2}{3}(4-x^2)^{3/2}\right)dx$$

$$= 2\int_{-2}^{2}\left((4-x^2)\sqrt{4-x^2}-\frac{1}{3}(4-x^2)^{3/2}\right)dx$$

$$= \frac{4}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

Using the substitution $x=2\sin\theta$, we have $dx=2\cos\theta\,d\theta$, and so

$$V = \frac{4}{3}\int_{-2}^{2}(4-x^2)^{3/2}\,dx$$

$$= \frac{4}{3}\int_{-\pi/2}^{\pi/2}(4-4\sin^2\theta)^{3/2}\,2\cos\theta\,d\theta = \frac{64}{3}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{16}{3}\int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta$$

$$= \frac{16}{3}\left[\left[\theta\right]_{-\pi/2}^{\pi/2} + \left[\sin 2\theta\right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2}\frac{1+\cos 4\theta}{2}\,d\theta\right]$$

$$= \frac{16}{3}\left(\pi + \left[\frac{\theta}{2}\right]_{-\pi/2}^{\pi/2} + \frac{1}{8}(\sin 4\theta)_{-\pi/2}^{\pi/2}\right)$$

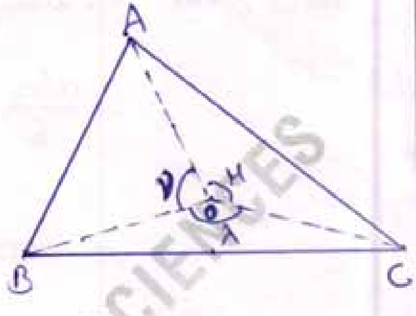
$$= \frac{16}{3}\left(\pi+\frac{\pi}{2}\right) = 8\pi$$

1(a) Find the volume of a tetrahedron in terms of the lengths of the three edges which meet in a point and of the angles which these edges make with each other in pairs.

Sol'n: Let OABC be a tetrahedron.

Let $OA=a$, $OB=b$, $OC=c$.

Let $\angle BOC=\lambda$, $\angle COA=\mu$, $\angle AOB=\nu$



we take 'O' as origin and any system of three mutually $\perp$ lar lines through O as coordinate axes.

Let the direction cosines of the lines OA, OB, OC be l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3

thus, the coordinates of A, B, C are

$(l_1 a, m_1 a, n_1 a)$; $(l_2 b, m_2 b, n_2 b)$; $(l_3 c, m_3 c, n_3 c)$

$\therefore$ the volume of the tetrahedron OABC

$$=\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1 a & m_1 a & n_1 a & 1 \\ l_2 b & m_2 b & n_2 b & 1 \\ l_3 c & m_3 c & n_3 c & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} l_1 a & m_1 a & n_1 a \\ l_2 b & m_2 b & n_2 b \\ l_3 c & m_3 c & n_3 c \end{vmatrix}$$

$$=\frac{abc}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_3 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

Thus, the volume of the tetrahedron OABC

$$= \frac{abc}{6} \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

2(a) Investigate for what values of λ, μ the simultaneous equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (i) no solution, (ii) a unique solution (iii) an infinite number of solutions.

Sol^n: The matrix form of the given system of equations is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A\ B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] R_3 \to R_3 - R_2$$

→ If $\lambda=3$ & $\mu \neq 10$ then $\rho(A|B)=3$ & $\rho(A)=2$

$\therefore \rho(A\ B) \neq \rho(A)$

$\therefore$ the given equations have no solutions.

→ If $\lambda \neq 3$ and $\mu =$ any value then $\rho(A\ B) = \rho(A) = 3 =$ the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

→ If $\lambda=3$ and $\mu=10$ then $\rho(A\ B) = \rho(A) = 2 <$ the number of unknown variables.

$\therefore$ the given equations are consistent and have infinite solutions.

2(b) If α is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of Adj A.

Sol'n: Since α is a characteristic root of a non-singular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A implies that there exists a non-zero vector X such that

$$AX = \alpha X$$

$$\Rightarrow (\text{Adj} A)(AX) = (\text{Adj} A)(\alpha X)$$

$$\Rightarrow [(\text{Adj} A) A]X = \alpha (\text{Adj} A) X$$

$$\Rightarrow |A| I X = \alpha (\text{Adj} A) X \quad [\because (\text{Adj} A) = |A| I]$$

$$\Rightarrow |A| X = \alpha (\text{Adj} A) X \quad [\because IX = X]$$

$$\Rightarrow \frac{|A|}{\alpha} X = (\text{Adj} A) X \quad [\because \alpha \neq 0]$$

$$\Rightarrow (\text{Adj} A) X = \frac{|A|}{\alpha} X.$$

Since X is a non-zero vector, therefore from the relation (1) it is obvious that $\frac{|A|}{\alpha}$ is a characteristic root of the matrix Adj A.

2(C) show that the function f, where

$$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous possesses partial derivations but is not differentiable at the origin.

Sol'n: put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|r(\cos^3\theta - \sin^3\theta)\right| \leq 2|r| = 2\sqrt{x^2+y^2} < \varepsilon,$$

$$x^2 < \frac{\varepsilon^2}{8}, \quad y^2 < \frac{\varepsilon^2}{8}$$

(or), if
$$|x| < \frac{\epsilon}{2\sqrt{2}}, \quad |y| < \frac{\epsilon}{2\sqrt{2}}$$

$$\therefore \left| \frac{x^3 - y^3}{x^2+y^2} - 0 \right| < \epsilon, \text{ when } |x| < \frac{\epsilon}{2\sqrt{2}}, |y| < \frac{\epsilon}{2\sqrt{2}}$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence the function is continuous at $(0,0)$

Again
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

Thus, the function possesses partial derivatives at $(0,0)$.

If the function is differentiable at $(0,0)$, then by definition

$$df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi \quad \text{——} (1)$$

when A and B are constants ($A = f_x(0,0) = 1$, $B = f_y(0,0) = -1$)

and ϕ, ψ tend to zero as $(h,k) \to (0,0)$

Putting $h = \rho\cos\theta$, $k = \rho\sin\theta$, and dividing by ρ, we get

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta \quad \text{——} (2)$$

For arbitrary $\theta = \tan^{-1}(h/k)$, $\rho \to 0$ implies that $(h,k) \to (0,0)$.

Thus we get the limit,

$$\cos^3\theta - \sin^3\theta = \cos\theta - \sin\theta$$

which is plainly impossible for arbitrary θ.

Thus, the function is not differentiable at the origin.

2(d). Show that the projections of the generators of a hyperboloid on any principal plane are tangents to the section of the hyperboloid by the principle plane.

Sol^n: Let the equation of the hyperboloid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \text{—— (1)}$$

Consider a generator

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c} \quad \text{—— (2)}$$

Now consider the coordinate plane $z=0$. The section of the hyperboloid (1) by this plane $z=0$ is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0 \quad \text{—— (3)}$$

The projection of the generator (2) on the plane $z=0$ is given by $\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}, z=0$

which is a plane through the generator $\perp$lar to the plane $z=0$

On simplifying it reduces to $\frac{x}{a\sin\theta}-\frac{\cos\theta}{\sin\theta}=\frac{y}{-b\cos\theta}+\frac{\sin\theta}{\cos\theta}, z=0$

i.e. $\frac{x}{a\sin\theta}+\frac{y}{b\cos\theta}=\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{1}{\sin\theta\cos\theta}, z=0.$

i.e. $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\quad z=0$

which is evidently a tangent to the section (3) of the hyperboloid (1) by the plane $z=0$ at the point $(a\cos\theta, b\sin\theta, 0)$.

Again consider the coordinate plane $x=0$. The section of the hyperboloid (1) by this plane $x=0$ is given by

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ x=0 \quad \text{—— (4)}$$

The projection of the generator ② on the plane $x=0$ is given by $\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$, $x=0$ which is a plane through the generator ⊥ar to the plane $x=0$.

on simplifying it reduces to $\frac{y}{-b\cos\theta}+\frac{\sin\theta}{\cos\theta}=\frac{z}{c}$, $x=0$

$$\Rightarrow \frac{y}{b}+\frac{z\cos\theta}{c}=\sin\theta,\ x=0 \ \text{(or)}$$

$$\Rightarrow \frac{y}{b}\operatorname{cosec}\theta+\frac{z}{c}\cot\theta=1,\ x=0$$

which is evidently a tangent to the section ④ of the hyperboloid ① by the plane $x=0$ at the point $(0, b\operatorname{cosec}\theta, -c\cot\theta)$.

Similarly we can prove the result by considering the plane $y=0$.

3(b)(i) The temperature at a point (x,y) on a metal is $T(x,y) = 4x^2-4xy+y^2$. An ant on the plate walks around the circle of radius 5 centred at the origin. What are the highest and lowest temperatures encountered by the ant?

Sol: Given that the temperature at a point (x,y) on a metal is $T(x,y) = 4x^2-4xy+y^2$

and given the circle of radius is 5 with centre $(0,0)$.

i.e. $x^2+y^2=25$.

we have to find the highest and lowest temperatures encountered by the ant of the function $T(x,y) = 4x^2-4xy+y^2$ — ①

subject to the constraint $x^2+y^2=25$ ②

Consider the function

$F = 4x^2-4xy+y^2+\lambda(x^2+y^2-25)$

$dF = (8x-4y+2\lambda x)\,dx + (-4x+2y+2y\lambda)$

For stationary values $F_x = F_y = 0$

$\Rightarrow 8x-4y+2\lambda x=0 \Rightarrow (4+\lambda)x-2y=0$ — ③

and $-4x+2y+2y\lambda=0 \Rightarrow -2x+(1+\lambda)y=0$ ④

Eliminating x and y from ③ and ④,

we get $\begin{vmatrix} 4+\lambda & -2 \\ -2 & 1+\lambda \end{vmatrix} = 0$

$\Rightarrow (4+\lambda)(1+\lambda)-4=0$

$\Rightarrow \lambda^2+5\lambda=0$

$\Rightarrow \lambda(\lambda+5)=0 \Rightarrow \lambda=0,-5$.

putting $\lambda = 0$ in ③, gives

$4x - 2y = 0 \Rightarrow y = 2x$ —— ⑤

from ② and ⑤, we get

$x^2 + (2x)^2 = 25 \Rightarrow 5x^2 = 25$

$\Rightarrow x = \pm\sqrt{5}$ & $y = \pm 2\sqrt{5}$

putting $\lambda = -5$ in ③, gives

$-x - 2y = 0 \Rightarrow x = -2y$ —— ⑥

$\therefore$ from ② and ⑥, we get

$4y^2 + y^2 = 25 \Rightarrow y = \pm\sqrt{5}$

and $x = \mp 2\sqrt{5}$

$\therefore$ the points are $(\sqrt{5}, 2\sqrt{5}), (-\sqrt{5}, -2\sqrt{5})$

and $(-2\sqrt{5}, +\sqrt{5}), (2\sqrt{5}, -\sqrt{5})$

At $(2\sqrt{5}, -\sqrt{5})$,

$T(x,y) = 80 - 4(-10) + 5 = 125$

At $(-2\sqrt{5}, \sqrt{5}) = 80 + 40 + 5 = 125$

At $(\sqrt{5}, 2\sqrt{5}) = 20 - 40 + 20 = 0$

At $(-\sqrt{5}, -2\sqrt{5}) = 20 - 40 + 20 = 0$

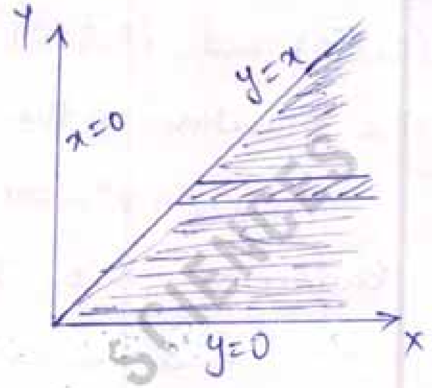
$\therefore$ Maximum $T = 125$ at $\pm(2\sqrt{5}, -\sqrt{5})$

and minimum $T = 0$ at $\pm(\sqrt{5}, 2\sqrt{5})$

3(b)(ii) Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y}\, dx\, dy$ by changing the order of integration.

Sol^n: the limits of integration are given by the straight lines $y=x$, $y=0$, $x=0$ and $x=\infty$

i.e. the region of integration is bounded by $y=0$, $y=x$ and infinite boundary.



Hence taking the strips parallel to x-axis, the limits for y are from 0 to ∞.

from $x=y$ to $x=\infty$.

Hence changing the order of integration we have

$$\int_0^\infty \int_0^x x e^{-x^2/y}\, dx\, dy = \int_{y=0}^{\infty} \int_{x=y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

$$= -\int_{y=0}^{\infty} \int_{-y}^{\infty} \frac{y}{2} e^{t}\, dt\, dy$$

$$= -\int_0^\infty \frac{y}{2} \left(e^{t}\right)_{-y}^{-\infty} dy$$

$$= \int_0^\infty \frac{y}{2} e^{-y}\, dy$$

$$= \frac{1}{2}\left[-ye^{-y} + \int e^{-y}\, dy\right]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^\infty$$

$$= \frac{1}{2}\left[0-(0-1)\right]$$

$$= \frac{1}{2}$$

Let $\frac{x^2}{y} = t$

$-\frac{2x}{y}\, dx = dt$

$\Rightarrow x\, dx = \frac{-y}{2}\, dt$

Limits

when $x=y$

$-\frac{y^2}{y} = t \Rightarrow t = -y$

when $x=\infty$

$t=\infty$

3(c) A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron OABC.

Sol^n: Let coordinates of the points A, B, C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

The equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0.$$

Radius of this sphere is given equal to $2k$.

$$\therefore a^2+b^2+c^2=4(2k)^2=16k^2 \quad \text{——} (1).$$

Let (x,y,z) be the coordinates of the centroid of the tetrahedron OABC: then

$$x=a/4,\ y=b/4,\ z=c/4$$

$$\Rightarrow a=4x,\ b=4y,\ c=4z$$

Eliminating a, b, c from (1), the required locus is

$$\underline{\underline{x^2+y^2+z^2=k^2}}.$$

4(b) (i) If $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u = 0$.

(ii) Show that $\int_0^{\pi}\log(1+\cos x)\,dx = -\pi\log 2$.

Sol^n: (ii) Let $I = \int_0^{\pi}\log(1+\cos x)\,dx$ —— (1)

then $I = \int_0^{\pi}\log[1+\cos(\pi-x)]\,dx$

$\Rightarrow I = \int_0^{\pi}\log(1-\cos x)\,dx$ —— (2).

$\therefore$ Adding (1) and (2), we get

$$2I = \int_0^{\pi}[\log(1+\cos x)+\log(1-\cos x)]\,dx$$

$$= \int_0^{\pi}\log(1-\cos^2 x)\,dx$$

$$= \int_0^{\pi}\log\sin^2 x\,dx$$

$$= 2\int_0^{\pi}\log\sin x\,dx = 4\int_0^{\pi/2}\log\sin x\,dx$$

$$= -4\cdot\frac{\pi}{2}\log 2.$$

$$I = -\pi\log 2$$

(i) Let $z = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\sqrt{y/x}\right]} = x^{1/2}\left(\frac{1+\frac{y}{x}}{1+\sqrt{y/x}}\right)$ —— (1)

z is a homogeneous function of x and y of degree $1/2$.

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{1}{2}z \quad \text{—— (2)}$$

from (1), $\frac{\partial z}{\partial x} = -\sin u\frac{\partial u}{\partial x}$

and $\frac{\partial z}{\partial y} = -\sin u\frac{\partial u}{\partial y}$

from ②

$$-x\sin u\frac{\partial u}{\partial x}-y\sin u\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$$

$$\Rightarrow\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$$

4(c) » Lines are drawn through the origin with direction cosines proportional to $(1,2,2)$, $(2,3,6)$, $(3,4,12)$. Show that the axis of the right circular cone through them has direction cosines $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and that the semi-vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Sol'n: The vertex of the cone is $O(0,0,0)$ and so let the equation of its axis be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}, \text{ where } l,m,n \text{ are direction cosines.} \quad \text{—— (1)}$$

Let θ be the semi-vertical angle of the cone. Then the given lines with direction ratios $(1,2,2)$, $(2,3,6)$, $(3,4,12)$ (or) direction cosines $\left(\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)$, $\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)$, $\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right)$ makes angle θ with (1)

$$\therefore \text{ we have } \cos\theta = l\left(\tfrac{1}{3}\right)+m\left(\tfrac{2}{3}\right)+n\left(\tfrac{2}{3}\right)$$
$$= l\left(\tfrac{2}{7}\right)+m\left(\tfrac{3}{7}\right)+n\left(\tfrac{6}{7}\right)$$
$$= l\left(\tfrac{3}{13}\right)+m\left(\tfrac{4}{13}\right)+n\left(\tfrac{12}{13}\right)$$

from these we have

$$\left(\tfrac{1}{3}\right)l+\left(\tfrac{2}{3}\right)m+\left(\tfrac{2}{3}\right)n=\left(\tfrac{2}{7}\right)l+\left(\tfrac{3}{7}\right)m+\left(\tfrac{6}{7}\right)n$$

and $$\left(\tfrac{1}{3}\right)l+\left(\tfrac{2}{3}\right)m+\left(\tfrac{2}{3}\right)n=\left(\tfrac{3}{13}\right)l+\left(\tfrac{4}{13}\right)m+\left(\tfrac{12}{13}\right)n$$

(or)

$$\left(\tfrac{1}{21}\right)l+\left(\tfrac{5}{21}\right)m-\left(\tfrac{4}{21}\right)n=0$$

and $$\left(\tfrac{4}{39}\right)l+\left(\tfrac{14}{39}\right)m-\left(\tfrac{10}{39}\right)n=0$$

$$\Rightarrow l+5m-4n=0 \quad \text{and} \quad 2l+7m-5n=0$$

Solving these simultaneously we get

$$\frac{l}{-25+28} = \frac{m}{-8+5} = \frac{n}{7-10}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{1} = \frac{n}{1}$$

$\therefore$ Direction cosines of axis of the cone, from (1), are proportional to $-1, 1, 1$ (or) the direction cosines are

$$\frac{-1 \cdot 1 \cdot 1}{\sqrt{(-1)^2+1^2+1^2}} \Rightarrow \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$ Hence proved.

Also from (2) we have

$$\cos\theta = -\frac{1}{\sqrt{3}}(1/3) + \frac{1}{\sqrt{3}}(2/3) + 1/\sqrt{3}(2/3)$$

$$\cos\theta = \frac{1}{3\sqrt{3}}(-1+2+2) = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ Hence proved.

5(a) → Find the orthogonal trajectories of $r = a(1+\cos n\theta)$.

Sol'n: Given family is $r = a(1+\cos n\theta)$, where a is parameter. —— ①

Taking logarithm of bothsides,

$$\log r = \log a + \log(1+\cos n\theta) \quad \text{—— ②}$$

Differentiating ② w.r.t θ

$$\left(\frac{1}{r}\right)\left(\frac{dr}{d\theta}\right) = -(n\sin n\theta)/(1+\cos n\theta) \quad \text{—— ③}$$

which is differential equation of the family of curves ①.

Replacing $dr/d\theta$ by $-r^2\left(\frac{d\theta}{dr}\right)$ in ③, the differential equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta}$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}\,d\theta$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{2\cos^2(n\theta/2)\,d\theta}{2\sin(n\theta/2)\cos(n\theta/2)}$$

$$\Rightarrow n\frac{dr}{r} = \cos(n\theta/2)\,d\theta.$$

Integrating $n\log r = 2/n \times \log \sin(n\theta/2) + (1/n)\log c$, c being parameter

$$\Rightarrow n^2\log r = \log \sin^2(n\theta/2) + \log c$$

$$\Rightarrow r^{n^2} = c\sin^2(n\theta/2)$$

$$\Rightarrow r^{n^2} = \left(\frac{c}{2}\right)(1-\cos n\theta)$$

$$\Rightarrow r^{n^2} = b(1-\cos n\theta), \quad \text{taking } b = c/2.$$

which is the equation of required orthogonal trajectories with b as parameter.

5(b). Use the variation of parameters method to show that the solution of equation $\frac{d^2y}{dx^2}+k^2y=\phi(x)$ satisfying the initial conditions $y(0)=0, y'(0)=0$ is $y(x)=\frac{1}{k}\int_0^x \phi(t)\sin k(x-t)dt$.

Sol^n: Given $y''+k^2y=\phi(x)$, ie, $(D^2+k^2)y=\phi(x)$, $D\equiv d/dx$ — ①.

Comparing ① with $y''+Py'+Qy=R$ here $R=\phi(x)$ — ②

consider $(D^2+k^2)y=0$ whose auxiliary equation is

$D^2+k^2=0$ so that $D=\pm ik$

$\therefore$ C.F of ① $=c_1\cos kx+c_2\sin kx$, c_1 & c_2 being arbitrary constants. — ③

Let $u=\cos kx$ and $v=\sin kx$ — ④

Here $W=$ wronskian of u and $v=\begin{vmatrix} u & v \\ u_1 & v_1\end{vmatrix}$

$$=\begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx\end{vmatrix}=k\neq 0 \quad \text{— ⑤}$$

$\therefore$ P.I of ① $=uf(x)+vg(x)$, where — ⑥

$$\therefore f(x)=-\int\frac{vR}{W}dx=-\int_0^x \frac{\sin kx\,\phi(x)}{k}dx=-\frac{1}{k}\int_0^x\phi(t)\sin kt\,dt \quad \text{— ⑦}$$

$$\text{and } g(x)=\int\frac{uR}{W}dx=\int_0^x\frac{\cos kx\,\phi(x)}{k}dx=\frac{1}{k}\int_0^x\phi(t)\cos kt\,dt \quad \text{— ⑧}$$

using ⑥, ⑦ and ⑧ we have

$$\text{P.I of ①} = -\frac{1}{k}\cos kx\int_0^x\phi(t)\sin kt\,dt+\frac{1}{k}\sin kx\int_0^x\phi(t)\cos kt\,dt$$

$$=\frac{1}{k}\int_0^x\phi(t)(\sin kx\cos kt-\cos kx\sin kt)dt$$

$$=\frac{1}{k}\int_0^x\phi(t)\sin(kx-kt)dt$$

Hence the general solution of ① is $y=$ C.F + P.I

i.e. $y=c_1\cos kx+c_2\sin kx+\frac{1}{k}\int_0^x\phi(t)\sin k(x-t)dt$ — ⑨

Putting $x=0$ in (9) and using the given condition $y(0)=0$, we get $c_1=0$

$\therefore$ (9) $\Rightarrow$ $y = c_2 \sin kx + \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)\,dt$

Differentiating bothsides of (10) w.r.t 'x' and using Leibnitz's rule of differentiation under integral sign, we have

$$y'(x) = c_2 k\cos kx + \frac{1}{k}\left[\int_0^x \frac{\partial}{\partial x}\{\phi(t)\sin k(x-t)\}dt + \phi(x)\sin k(x-x)\frac{dx}{dx} - \phi(0)\sin kx\frac{d0}{dx}\right]$$

$$\Rightarrow y'(x) = c_2 k\cos kx + \int_0^x \phi(t)\cos k(x-t)dt \quad \text{--- (11)}$$

Putting $x=0$ in (11) and using the boundary condition $y'(0)=0$, we get $0 = c_2k+0$ so that $c_2=0$, as $k\neq 0$

putting $c_1=0$ and $c_2=0$ (10), the required solution

is $y = \frac{1}{k}\int_0^x \phi(t)\sin k(x-t)\,dt$.

5(c) → A frame work ABCD consists of four equal, light rods smoothly jointed together to form a square, it is suspended from a peg at A, and a weight W is attached to C, the frame work being kept in shape by a light rod connecting B and D. Determine the thrust in this rod.

Sol'n: Let T be the thrust in the rod and $AB = AD = BC = DC = a$.

Let $\angle BAC = \theta$. Consider a virtual displacement in which θ increases by $\delta\theta$.

Virtual work done by the weight $W = W.\delta(AC)$

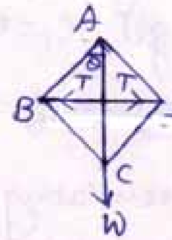
$= W\delta(a\cos\theta + a\cos\theta)$

$= W\delta(2a\cos\theta)$, along AC

Virtual work done by the thrust $T = T\cdot\delta(BD)$

$$= T\delta(a\sin\theta + a\sin\theta)$$

$$= T\delta(2a\sin\theta).$$



Sum of the virtual works done by W and T is $= W\delta(2a\cos\theta) + T\delta(2a\sin\theta)$

By the principle of virtual work, we have

$$W\delta(2a\cos\theta) + T\delta(2a\sin\theta) = 0$$

$$\Rightarrow -2aW\sin\theta\, d\theta + 2aT\cos\theta\,\delta\theta = 0$$

$$\Rightarrow (-W\sin\theta + T\cos\theta)\delta\theta = 0$$

Since $\delta\theta$ is arbitrary, $-W\sin\theta + T\cos\theta = 0.$

$$T = W\tan\theta$$

In equilibrium, $\theta = 45°$. Hence $T = W$.

5(d). A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t.

Sol'n: Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards i.e. in the direction of x decreasing. The weight mg of the particles acts vertically downwards i.e., in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = mg - \mu v$$

$\Rightarrow \frac{dv}{dt} = g - \frac{\mu}{m}v \qquad \left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt}\right]$

$\Rightarrow dt = \frac{dv}{g-\left(\frac{\mu}{m}\right)v}$

Integrating we have

$t = -\frac{m}{\mu}\log\left(g - \frac{\mu}{m}v\right) + A$, where A is constant.

But initially when $t=0$, $v=0$;

$\therefore A = \left(\frac{m}{\mu}\right)\log g.$

$\therefore t = -\frac{m}{\mu}\log\left(g-\frac{\mu}{m}v\right) + \frac{m}{\mu}\log g$

$\Rightarrow t = -\frac{m}{\mu}\log\left\{\frac{g-\left(\frac{\mu}{m}\right)v}{g}\right\}$

$\Rightarrow -\frac{\mu t}{m} = \log\left(1-\frac{\mu}{gm}v\right) \Rightarrow 1 - \frac{\mu}{gm}v = e^{-\mu t/m}$

$\Rightarrow v = \frac{dx}{dt} = \frac{gm}{\mu}\left(1-e^{-\mu t/m}\right)$

$\Rightarrow dx = \frac{gm}{\mu}\left(1-e^{-\mu t/m}\right)dt$

Integrating we have

$x = \frac{gm}{\mu}\left[t + \frac{m}{\mu}e^{-\mu t/m}\right] + B$ —— ① where B is constant.

But initially when $t=0$, $x=0$

$\therefore 0 = \frac{gm}{\mu}\left[\frac{m}{\mu}\right] + B$ —— ②

Subtracting ② from ①, we have

$x = \frac{gm}{\mu}\left[\frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu} + t\right] = \frac{gm^2}{\mu^2}\left[e^{-(\mu t/m)} - 1 + \frac{\mu t}{m}\right]$

5(e) → Represent the vector $A = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates. Thus determine A_ρ, A_ϕ and A_z.

Sol^n: the position vector of any point in cylindrical coordinates is

$$r = x\hat{i} + y\hat{j} + z\hat{k} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$$

The tangent vector to the ρ, ϕ and z curves are given respectively by $\frac{\partial r}{\partial \rho}, \frac{\partial r}{\partial \phi}$ and $\frac{\partial r}{\partial z}$ where

$$\frac{\partial r}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}, \quad \frac{\partial r}{\partial \phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}, \quad \frac{\partial r}{\partial z} = \hat{k}$$

The unit vectors in these directions are

$$e_1 = e_\rho = \frac{\partial r/\partial \rho}{|\partial r/\partial \rho|} = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \quad \text{--- (1)}$$

$$e_2 = e_\phi = \frac{\partial r/\partial \phi}{|\partial r/\partial \phi|} = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi}} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \quad \text{--- (2)}$$

$$e_3 = e_z = \frac{\partial r/\partial z}{|\partial r/\partial z|} = \hat{k} \quad \text{--- (3)}$$

Solving (1) & (2)

$$i = \cos\phi\, e_\rho - \sin\phi\, e_\phi, \quad j = \sin\phi\, e_\rho + \cos\phi\, e_\phi$$

Then $A = z\hat{i} - 2x\hat{j} + y\hat{k}$

$$= z(\cos\phi\, e_\rho - \sin\phi\, e_\phi) - 2\rho\cos\phi(\sin\phi\, e_\rho + \cos\phi\, e_\phi) + \rho\sin\phi\, e_z$$

$$= (z\cos\phi - 2\rho\cos\phi\sin\phi)e_\rho - (z\sin\phi + 2\rho\cos^2\phi)e_\phi + \rho\sin\phi\, e_z$$

and $A_\rho = z\cos\phi - 2\rho\cos\phi\sin\phi$

$A_\phi = -z\sin\phi + 2\rho\cos^2\phi$

$A_z = \rho\sin\phi$

6(a) Solve $(D^2-1)y = \cosh x \cos x + a^x$.

Sol^n: Given $(D^2-1)y = \cosh x \cos x + a^x = \frac{1}{2} \times (e^x+e^{-x})\cos x + a^x$

$$(D^2-1)y = \frac{1}{2}\times e^x \cos x + \frac{1}{2}\times e^{-x}\cos x$$

Here auxiliary equation is $D^2-1=0 \Rightarrow D = 1, -1$.

so $C.F = c_1 e^x + c_2 e^{-x}$.

P.I Corresponding to $\frac{1}{2}(e^x \cos x)$

$$= \frac{1}{2}\frac{1}{D^2-1} e^x \cos x$$

$$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x = \frac{e^x}{2}\frac{1}{D^2+2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{-1^2+2D}\cos x = \frac{e^x}{2}(2D+1)\frac{1}{(2D+1)(2D-1)}\cos x$$

$$= \frac{e^x}{2}(2D+1)\frac{1}{4D^2-1}\cos x$$

$$= \frac{e^x}{2}(2D+1)\frac{1}{-4-1}\cos x = -\frac{e^x}{10}(2D+1)\cos x$$

$$= \left(-\frac{1}{10}\right)\times e^x (2D\cos x + \cos x) = \left(-\frac{1}{10}\right)\times e^x(-2\sin x + \cos x)$$

P.I Corresponding to $\frac{1}{2}e^{-x}\cos x$

$$= \frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x = \frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x$$

$$= \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x = \frac{e^{-x}}{2}\frac{1}{(-1^2-2D)}\cos x$$

$$= -\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$= -\frac{e^{-x}}{2}(2D-1)\frac{1}{4D^2-1}\cos x = -\frac{e^{-x}}{2}(2D-1)\frac{1}{4(-1)^2-1}\cos x$$

$= \frac{1}{10} e^{-x}(2D-1)\cos x = \frac{1}{10} e^{-x}(-2\sin x - \cos x)$

Now P.I corresponding to a^x

$$= \frac{1}{D^2-1} a^x = \frac{1}{D^2-1} e^{x\log a} = \frac{1}{(\log a)^2-1} e^{x\log a}$$

$$= \frac{1}{(\log a)^2-1} a^x$$

$\therefore$ solution is

$$y = c_1 e^x + c_2 e^{-x} + \tfrac{2}{5}\sinh x \sin x - \tfrac{1}{5}\cosh x \cos x + \frac{a^x}{\{(\log a)^2-1\}}$$

6(b) A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$.

Sol^n: A uniform beam AB of length 2a rests in equilibrium against smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b. Suppose the rod makes an angle θ with the wall. i.e. $\angle BAM = \theta$. The weight W of the rod acts at its middle point G. Give the rod a small displacement in which θ changes to $\theta + \delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual work is the weight of the rod acting at G. The reactions at A and C do not work.

we have, the height of G above the fixed point C

$= NM = AM - AN = AG\cos\theta - CN\cot\theta$

$= a\cos\theta - b\cot\theta.$

The equation of virtual work is

$$-W\delta(a\cos\theta - b\cot\theta) = 0$$

$$\Rightarrow \delta(a\cos\theta - b\cot\theta) = 0$$

$$\Rightarrow -a\sin\theta\,\delta\theta + b\,\text{cosec}^2\theta\,\delta\theta = 0$$

$$\Rightarrow (-a\sin\theta + b\,\text{cosec}^2\theta)\delta\theta = 0$$

$$\Rightarrow -a\sin\theta + b\,\text{cosec}^2\theta = 0 \qquad [\because \delta\theta \neq 0]$$

$$\Rightarrow -a\sin\theta = b\,\text{cosec}^2\theta$$

$$\Rightarrow \sin^3\theta = b/a$$

$$\Rightarrow \theta = \sin^{-1}(b/a)^{1/3}$$

giving the inclination of the rod to the vertical in the position of equilibrium.

6(c) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is $\mu\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$.

Sol^n: Let the ends links A & B of a uniform chain slide along a fixed rough horizontal rod. If AB is the maximum span, then A and B are in the state of limiting equilibrium.

Let R be the reaction of the rod at A acting ⊥lar to the rod. Then the fricitonal force μR will act at A along the rod in the outward direction BA. The resultant F of the forces R and μR at A will make

an angle λ (where $\tan\lambda=\mu$) with the direction of R.
For the equilibrium of A the resultant F of R and μR
at A will be equal & opposite to the tension T at A.
Since the tension at A acts along the tangent to
the chain at A, therefore the tangent to the catenary
at A makes an angle $\psi_A = \frac{1}{2}\pi-\lambda$ to the horizontal.
Thus for the point A of the catenary,

we have $\psi = \psi_A = \frac{1}{2}\pi-\lambda$

$\therefore$ the length of the chain

$$= 2s = 2c\tan\psi_A = 2c\tan\left(\tfrac{1}{2}\pi-\lambda\right)$$

$$= 2c\cot\lambda = \frac{2c}{\mu} \quad [\because \tan\lambda=\mu]$$

If (x_A, y_A) are coordinates of the point A, then the maximum span $AB = 2x_A$

$$= 2c\log(\tan\psi_A+\sec\psi_A)$$

$$= 2c\log\left\{\tan\psi_A+\sqrt{(1+\tan^2\psi_A)}\right\}$$

$$= 2c\log\left\{\cot\lambda+\sqrt{(1+\cot^2\lambda)}\right\} \quad \left[\because \psi_A=\tfrac{1}{2}\pi-\lambda\right]$$

$$= 2c\log\left\{\frac{1}{\mu}+\sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\}$$

$$= 2c\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$$

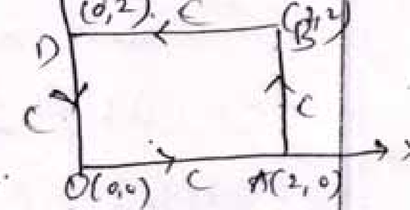
Hence the required ratio

$$= \frac{2x}{2s} = \frac{2c\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}}{(2c/\mu)}$$

$$= \mu\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$$

6(d) verify Stokes theorem for $A=(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}$, where S is the surface of the cube $x=0, y=0, z=0$, $x=2, y=2, z=2$ above the xy-plane.

Sol: The xy-plane cuts the surface of the cube in a square. Thus the curve C bounding the surface S is the square, say OABD, in the xy-plane whose vertices in the xy-plane are points $O(0,0)$, $A(2,0)$, $B(2,2)$, $D(0,2)$.



By Stokes theorem, we have.

$$\iint_S (\nabla\times A)\cdot\hat{n}\,ds = \oint_C A\cdot dr$$

$$\oint_C A\cdot dr = \int_C \left[(y-z+2)\hat{i}+(yz+4)\hat{j}-xz\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int (y-z+2)dx + (yz+4)dy - xz\,dz$$

$$= \int (y+2)dx + 4dy \qquad (\because \text{on } C,\ z=0 \text{ and } dz=0)$$

$$= \int_{OA} (y+2)dx+4dy + \int_{AB} (y+2)dx+4dy + \int_{BD} (y+2)dx+4dy + \int_{DO} (y+2)dx+4dy$$

$$= \int_0^2 2dx + \int_0^2 4dy + \int_2^0 4dx + \int_2^0 4dy$$

($\because$ on OA, $y=0$, $dy=0$ & x varies from 0 to 2
on AB, $x=2$, $dx=0$ & y varies from 0 to 2
on BD, $y=2$, $dy=0$ & x varies from 2 to 0
on DO, $x=0$, $dx=0$ & y varies from 2 to 0)

$= 2[x]_0^2 + 4[y]_0^2 + 4[x]_2^0 + 4[y]_2^0$.

$= 4+8-8-8 = -4$.

Now $\text{curl } A = \nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y-z+2 & yz+4 & -xz \end{vmatrix}$

$= i(0-y) + j(-1+z) + k(0-1)$

$= -y\hat{i} + (z-1)\hat{j} - \hat{k}$

$\hat{n}$ = unit normal vector to $s = \hat{k}$

$\therefore ds = \dfrac{dx\,dy}{|n\cdot k|} = dx\,dy$

$(\nabla\times F)\cdot\hat{n} = [-y\hat{i} + (x+z)\hat{j} - \hat{k}]\cdot\hat{k} = -1$

$\iint(\nabla\times F)\cdot\hat{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{2}(-1)\,dx\,dy$

$= -\int_{x=0}^{2}[y]_0^2\,dx$

$= -\int_{x=0}^{2} 2\,dx$

$= -2[x]_0^2$

$= -2(2) = -4$.

$\therefore \iint_S (\nabla\times F)\cdot\hat{n}\,ds = \int_C f\cdot dr = -4$.

Hence the Stokes theorem is verified.

7(a) Find the general and singular solution of $y^2(y-xp)=x^4p^2$.

Sol^n: Given equation is $y^2(y-xp)=x^4p^2$ —— (1)

putting $x=1/u$, $y=1/v$ so that $dx=-(1/u^2)du$, $dy=-(1/v^2)dv$

we get $\frac{dy}{dx}=\frac{u^2}{v^2}\frac{dv}{du} \Rightarrow p=\frac{u^2}{v^2}P$

where $P=\frac{dv}{du}$ and $p=\frac{dy}{dx}$

$\therefore$ putting $x=1/u$, $y=1/v$, $p=(u^2P)/v^2$ in (1), we have

$$(1/v^2)\{(1/v)-(1/u)(u^2P/v^2)\}=(1/u^4)(u^4P^2/v^4)$$

$$\Rightarrow v=uP+P^2$$

which is in Clairaut's form. Replacing P by c, the required general solution is $v=uc+c^2$

$$\Rightarrow 1/y=c/x+c^2$$

$$\Rightarrow x=cy+c^2xy$$

$$\Rightarrow xyc^2+yc-x=0, \quad \text{—— (2)}$$

which is a quadratic equation in c and so its c- discriminant relation is

$$y^2-4(xy)(-x)=0 \Rightarrow y(y+4x^2)=0$$

Now $y=0$ gives $p=dy/dx=0$. These values satisfy (1). So $y=0$ is a singular solution. Again $y=-4x^2$ gives $p=dy/dx=-8x$. these values satisfy (1). Hence $y+4x^2=0$ is also singular solution.

7(b) (i) solve $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

(ii) Solve $(y+y^3/3+x^2/2)dx+(1/4)x(x+xy^2)dy=0$

Sol^n: (i) comparing the given equation with $Mdx+Ndy=0$

here $M=y^2e^{xy^2}+4x^3$ and $N=2xye^{xy^2}-3y^2$

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 \cdot 2xye^{xy^2} = \frac{\partial N}{\partial x}$$

Hence, the given equation is exact and so its solution is

$$\underset{(y \text{ is constant})}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \int (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = 0$$

$$\Rightarrow y^2 \times \frac{1}{y^2} \times e^{xy^2} + 4 \times \frac{1}{4} \times x^4 - 3 \times y^3/3 = 0$$

$$\Rightarrow e^{xy^2} + x^4 - y^3 = c.$$

(ii) Given $(y + y^3/3 + x^2/2)\,dx + \frac{1}{4} \times (x + xy^2)\,dy = 0$ — ①

Comparing ① with $M\,dx + N\,dy = 0$, $M = y + y^3/3 + x^2/2$

$N = \frac{1}{4} \times (x + xy^2)$

Here $\frac{\partial M}{\partial y} = 1 + y^2$ and $\frac{\partial N}{\partial x} = \frac{1}{4} \times (1 + y^2)$

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x(1+y^2)}\left\{(1+y^2) - \frac{1}{4}(1+y^2)\right\}$$

$$= \frac{4}{x}\left(1 - \frac{1}{4}\right) = 3/x.$$

which is a function of x alone. So $I.F = e^{\int 3/x\,dx}$

$$= e^{3\log x} = e^{\log x^3} = x^3.$$

Multiplying ① with x^3, we have

$\left\{x^3 y + \frac{1}{3}x^3 y^3 + \frac{1}{2}x^5\right\}dx + \left(\frac{1}{4}\right)(x^4 + x^4 y^2)\,dy = 0$ whose solution as usual is

$$\underset{(y \text{ is constant})}{\int \left\{x^3 y + \frac{1}{3}x^3 y^3 + \frac{1}{2}x^5\right\}dx} = c/12$$

$$\Rightarrow \frac{1}{4}x^4 y + \frac{1}{12}x^4 y^3 + \frac{1}{12}x^6 = c/6$$

$\Rightarrow 3x^4 y + x^4 y^3 + x^6 = c$, where c is an arbitrary constant.

7(c) A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is

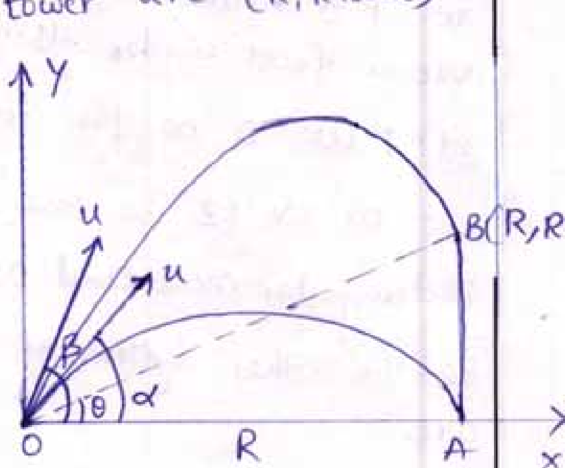
$\frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha \cos\theta)\right]$.

Sol^n: Let AB be the tower and O the point of projection. It is given that $\angle AOB = \theta$.

Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB. Let OA = R

Then $R = \frac{u^2 \sin 2\alpha}{g}$

Referred to the horizontal and vertical lines OX and OY lying in the plane of motion as the coordinate axes, the coordinates of the top B of the tower are $(R, R\tan\theta)$



If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y = x\tan\beta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\beta}$$

$$\therefore R\tan\theta = R\tan\beta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

$$\Rightarrow \tan\theta = \tan\beta - \frac{1}{2}g\frac{R}{u^2\cos^2\beta} \quad [\because R \neq 0]$$

Substituting the value of R from ①, we get

$$\tan\theta = \tan\beta - \frac{1}{2}g\,\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

$$\Rightarrow \tan\theta = \tan\beta - \frac{\sin 2\alpha}{2\cos^2\beta}$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{\sin 2\alpha}{2\cos^2\beta}$$

Multiplying bothsides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta = 2\sin\beta\cos\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow (1+\cos 2\beta)\sin\theta = \sin 2\beta\cos\theta - \cos\theta\sin 2\alpha$$

$$\Rightarrow \sin 2\beta\cos\theta - \cos 2\beta\sin\theta = \sin\theta + \cos\theta\sin 2\alpha$$

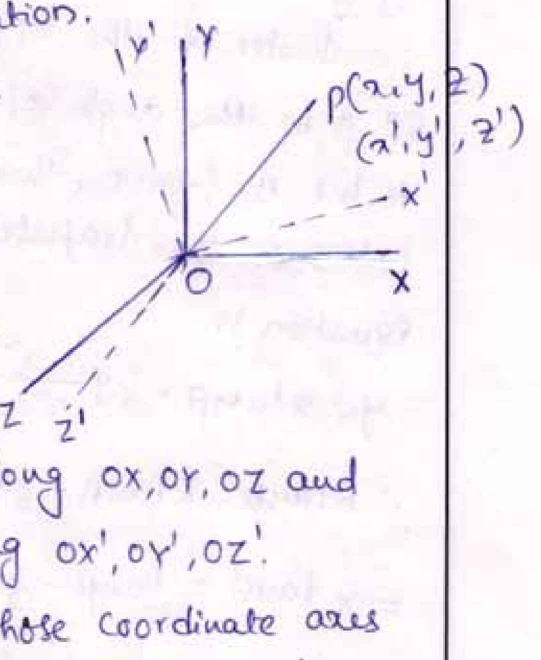
$$\Rightarrow \sin(2\beta-\theta) = \sin\theta + \cos\theta\sin 2\alpha$$

$$\Rightarrow 2\beta-\theta = \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow 2\beta = \theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

$$\Rightarrow \beta = \frac{1}{2}\left[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha\cos\theta)\right]$$

7(d) $\rightarrow$ If $A(x,y,z)$ is an invariant differentiable vector field w.r.t a rotation of axes, Prove that Curl A is invariant vector field under the transformation.



Sol'n: Let O be the fixed origin.

Let OX, OY, OZ be one system of rectangular axes and OX', OY', OZ' be the other system of rectangular axes.

Let i, j, k be the unit vectors along OX, OY, OZ and i', j', k' be the unit vectors along OX', OY', OZ'.

Let P be any point inspace whose coordinate axes are (x,y,z) or (x',y',z') with respect to the two systems of axes.

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction cosines of the lines OX', OY', OZ' w.r.t the Coordinate axes OX, OY, OZ.

The scheme of transformation will be as follows:

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \text{———} (1)$$

Also we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $li + mj + nk$, where i, j, k are unit vectors along coordinate axes.

$$\therefore \left.\begin{aligned} i' &= l_1 i + m_1 j + n_1 k \\ j' &= l_2 i + m_2 j + n_2 k \\ k' &= l_3 i + m_3 j + n_3 k \end{aligned}\right\} \text{———} (2)$$

Now suppose the function $A(x, y, z)$ becomes $A'(x', y', z')$ after rotation of axes.

Then by hypothesis $A(x, y, z) = A'(x', y', z')$.

By chain rule of differentiation, we have

$$\frac{\partial A}{\partial x} = \frac{\partial A'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial A'}{\partial y'}\cdot\frac{\partial y'}{\partial x} + \frac{\partial A'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\frac{\partial x'}{\partial x} = l_1$, $\frac{\partial y'}{\partial x} = l_2$, $\frac{\partial z'}{\partial x} = l_3$

$$\left.\begin{aligned} \therefore \frac{\partial A}{\partial x} &= l_1 \frac{\partial A'}{\partial x'} + l_2 \frac{\partial A'}{\partial y'} + l_3 \frac{\partial A'}{\partial z'} \\ \text{Similarly } \frac{\partial A}{\partial y} &= m_1 \frac{\partial A'}{\partial x'} + m_2 \frac{\partial A'}{\partial y'} + m_3 \frac{\partial A'}{\partial z'} \\ \frac{\partial A}{\partial z} &= n_1 \frac{\partial A'}{\partial x'} + n_2 \frac{\partial A'}{\partial y'} + n_3 \frac{\partial A'}{\partial z'} \end{aligned}\right\} \text{———} (3)$$

Now taking cross product of these three equations by i, j, k respectively, adding and using the results (2), we get

$$i\times\frac{\partial A}{\partial x}+j\times\frac{\partial A}{\partial y}+k\times\frac{\partial A}{\partial z}=(l_1 i+m_1 j+n_1 k)\times\frac{\partial A'}{\partial x'}+(l_2 i+m_2 j+n_2 k)\times\frac{\partial A'}{\partial y'}+(l_3 i+m_3 j+n_3 k)\times\frac{\partial A'}{\partial z'}$$

$$=i'\times\frac{\partial A'}{\partial x'}+j'\times\frac{\partial A'}{\partial y'}+k'\times\frac{\partial A'}{\partial z'}.$$

$$\therefore \nabla\times A=\nabla\times A'$$

$$\Rightarrow \boxed{\text{Curl } A=\text{Curl } A'}$$

8(a) By using Laplace transform method solve the $(D^3-2D^2+5D)y=0$ if $y(0)=0$, $y'(0)=1$, $y(\pi/8)=1$.

Sol^n: Taking the Laplace transform of both sides of given equation, we have

$$L\{y'''\}-2L\{y''\}+5L\{y'\}=0$$

$$\Rightarrow p^3L\{y\}-p^2y(0)-py'(0)-y''(0)-2[p^2L\{y\}-py(0)-y'(0)] + 5[pL\{y\}-y(0)]=0$$

$$\Rightarrow (p^3-2p^2+5p)L\{y\}-p-A-2[-1]+5.0=0 \quad \text{where } y''(0)=A$$

$$\Rightarrow L\{y\}=\frac{A-2+p}{p^3-2p^2+5p}$$

$$=\frac{A-2}{p(p^2-2p+5)}+\frac{1}{p^2-2p+5}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{p-2}{p^2-2p+5}+\frac{1}{p^2-2p+5}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{(p-1)-1}{(p-1)^2+4}+\frac{1}{(p-1)^2+4}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{(p-1)}{(p-1)^2+4}+\frac{A+3}{10}\cdot\frac{2}{(p-1)^2+4}$$

$$\therefore y=\frac{A-2}{5}\cdot L^{-1}\left\{\frac{1}{p}\right\}-\frac{A-2}{5}L^{-1}\left\{\frac{p-1}{(p-1)^2+4}\right\}+\frac{A+3}{10}L^{-1}\left\{\frac{2}{(p-1)^2+4}\right\}$$

$$=\frac{A-2}{5}-\frac{A-2}{5}e^t.\cos 2t+\frac{A+3}{10}e^t\sin 2t$$

Since $y(\pi/8)=1$

$$\therefore 1=\frac{A-2}{5}-\frac{A-2}{5}\cdot e^{\pi/8}\cdot\frac{1}{\sqrt{2}}+\frac{A+3}{10}\cdot e^{\pi/8}\cdot\frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{7-A}{5} = \frac{e^{\pi/8}}{10\sqrt{2}}(-2A+4+A+3)$$

$$\Rightarrow \left(\frac{7-A}{5}\right)\left[1-\frac{e^{\pi/8}}{2\sqrt{2}}\right]=0$$

$$\Rightarrow A=7$$

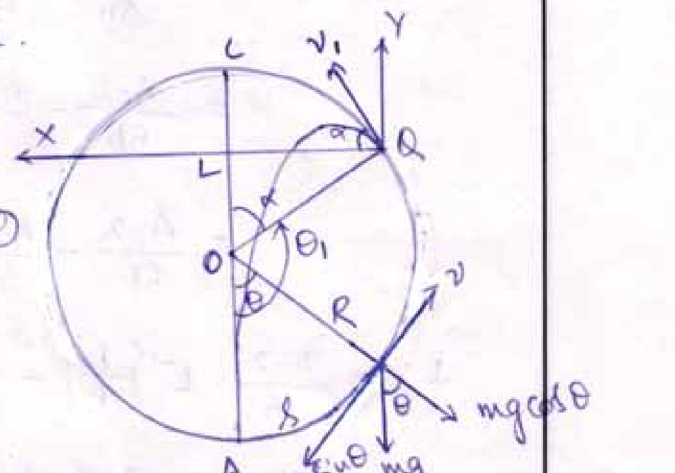
Hence the required solution is

$$y = 1+e^{t}(\sin 2t - \cos 2t)$$

8(b) A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$, Prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.

Sol^n: Take $R=T$ (i.e. the tension in the string)

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are.



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{—— (1)}$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{—— (2)}$$

Also $s = l\theta$

From (1) & (3), we have

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying bothsides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at the point A, $\theta=0$ and $v=2\sqrt{gl}$

$\therefore 4gl = 2lg + A$ so that $A = 2gl$

$\therefore v^2 = 2lg(\cos\theta + 1)$ ——— (4)

from (2) and (4), we have

$$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta + 2) \quad \text{——— (5)}$$

If the cord becomes slack at the point Q, where $\theta = \theta_1$ then from (5), we have

$$T = 0 - mg(3\cos\theta_1 + 2)$$

giving as $\cos\theta_1 = -2/3$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$ and $\cos\alpha = 2/3$

If v_1 is the velocity of the particle at Q, then $v = v_1$ where $\theta = \theta_1$. Therefore from (4), we have

$$v_1^2 = 2lg(1 + \cos\theta_1) = 2lg(1 - 2/3) = 2lg/3$$

Now $OL = l\cos\alpha = \frac{2}{3}l$

Thus the particle leaves the circular path at the point Q at a height $\frac{2}{3}l$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

8(c) Show that $A = (2x^2 + 8xy^2z)\hat{i} + (3x^3y - 3xy)\hat{j} - (4y^2z^2 + 2x^3z)\hat{k}$ is not solenoidal but $B = xyz^2 A$ is solenoidal.

Sol'n: $\nabla \cdot A = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(2x^2 + 8xy^2z)\hat{i} - (3x^3y - 3xy)\hat{j} - (4y^2z^2 + 2x^3z)\hat{k}\right]$

$= \frac{\partial}{\partial x}(2x^2 + 8xy^2z) + \frac{\partial}{\partial y}(3x^3y - 3xy) - \frac{\partial}{\partial z}(4y^2z^2 + 2x^3z)$

$= 4x + \cancel{8y^2z} + 3x^3 - 3x - \cancel{8y^2z} - 2x^3$

$= x^3 + x \neq 0$

$\therefore$ It is not a solenoidal

Now $B = (xyz^2)A$

$= (xyz^2)\left[(2x^2 + 8xy^2z)\hat{i} + (3x^3y - 3xy)\hat{j} - (4y^2z^2 + 2x^3z)\hat{k}\right]$

$= (2x^3yz^2 + 8x^2y^3z^3)\hat{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\hat{j} - (4xy^3z^4 + 2x^4yz^3)\hat{k}$

Now $\nabla \cdot B = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(2x^3yz^2 + 8x^2y^3z^3)\hat{i} + (3x^4y^2z^2 - 3x^2y^2z^2)\hat{j} - (4xy^3z^4 + 2x^4yz^3)\hat{k}\right]$

$= \frac{\partial}{\partial x}(2x^3yz^2 + 8x^2y^3z^3) + \frac{\partial}{\partial y}(3x^4y^2z^2 - 3x^2y^2z^2) - \frac{\partial}{\partial z}(4xy^3z^4 + 2x^4yz^3)$

$= 6x^2yz^2 + 16xy^3z^3 + 6x^4yz^2 - 6x^2yz^2 - 16xy^3z^3 - 6x^4yz^2$

$= 0$

$\therefore$ B is a solenoidal

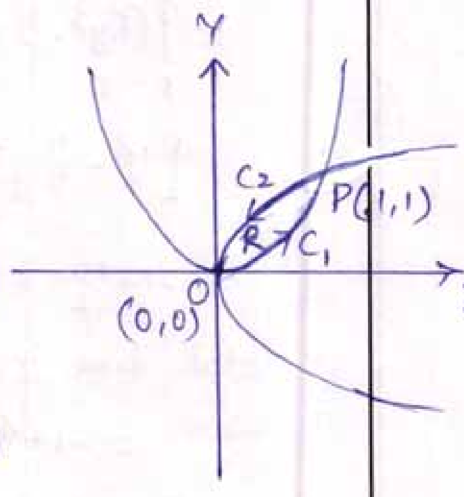
8(d) → Verify Green's theorem in the plane for $\oint_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region defined by: $y=\sqrt{x}$, $y=x^2$.

Sol'n: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = 3x^2-8y^2$, $N = 4y-6xy$

The parabola $y=\sqrt{x}$ i.e. $y^2=x$ and the parabola $y=x^2$ intersect at the points (0,0) and (1,1). The closed curve C consists of the arc C_1 of the parabola $y=x^2$ and the arc C_2 of the parabola $y=\sqrt{x}$. Also R is the region bounded by the closed curve C.



We have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y-6xy) - \frac{\partial}{\partial y}(3x^2-8y^2)\right] dx\,dy$$

$$= \iint_R (-6y+16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dx\,dy \quad [\because \text{for the region R, } x \text{ varies from 0 to 1 \& } y \text{ varies from } x^2 \text{ to } \sqrt{x}]$$

$$= \int_0^1 5\left[y^2\right]_{y=x^2}^{\sqrt{x}} dx = 5\int_0^1 [x-x^4]\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2}-\frac{1}{5}\right] = \frac{15}{10} = 3/2 \quad \text{—— (1)}$$

Now the line integral along the closed curve C.

$$= \oint_C (M\,dx + N\,dy) = \int_{C_1} (M\,dx + N\,dy) + \int_{C_2} (M\,dx + N\,dy)$$

Along C_1, $x^2 = y$, $dy = 2x\,dx$ and x varies from 0 to 1.

$\therefore$ line integral along $C_1 = \int_0^1 [(3x^2 - 8x^4)dx + (4x^2 - 6x^3)2x\,dx]$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)dx = \left[x^3 + 2x^4 - 4x^5\right]_0^1 = 1 + 2 - 4 = -1.$$

Along C_2, $y^2 = x$.

$\therefore dx = 2y\,dy$ and limits for y are 1 to 0.

$\therefore$ line integral along C_2

$$= \int_1^0 [(3y^4 - 8y^2)2y\,dy + (4y - 6y^3)dy]$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)dy$$

$$= \left[y^6 - \frac{11}{2}y^4 + 2y^2\right]_1^0$$

$$= -1 + \frac{11}{2} - 2 = \frac{5}{2}$$

$\therefore$ Total line integral along the closed curve C

$$= -1 + \frac{5}{2} = \frac{3}{2}. \qquad \text{——— (2)}$$

from (1) and (2), we see that Green's theorem is verified.