Mathematics Mains Test -5

Paper 1 (Full)

Time : 3 hours Maximum marks: 250

Instructions

1. Question paper contains **8** questions out of this candidate need to answer **5** questions in the following pattern
2. Candidate should attempt question No's **1** and **5** compulsorily and any **three** of the remaining questions selecting **atleast one** question from each section.
3. The number of marks carried by each question is indicated at end of each question.
4. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/Notations carry their usual meanings, unless otherwise indicated.

Section- A

1.

a) Show that the set $\{(1,0,0),(0.1.0),(1,1,0),(1,2,3)\} \subseteq V_3(R)$ is not a basis of $V_3(R)$. (5 marks)

b) If α, β, γ are linearly independent vectors of $V(F), where\ F$ is a field of complex numbers, the show that $\alpha+\beta, \beta+\gamma, \alpha+\gamma$ are also linearly independent. (10 marks)

c) If F is the field of complex numbers, prove that the vectors $(a_1, a_2) and\ (b_1.b_2)$ in $V_2(F)$ are linearly dependent if $a_1b_2 - a_2b_1 = 0$ (5 marks)

d) If $2r$ is the distance between two parallel tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, prove that the line through the origin perpendicular to the planes lies on the cone $x^2(a^2 - r^2) + y^2(b^2 - r^2) + z^2(c^2 - r^2) = 0.$ (10 marks)

e) Examine the continuity of the function f defined by $f(x) = \lim_{n\to\infty} \frac{e^x - x^n sinx}{1+x^n}, 0 \le x \le \frac{\pi}{2}$, at $x = 1$. Explain why the function f does not vanish anywhere on $[0, \frac{\pi}{2}]$ although $f(0) and\ f(\frac{\pi}{2})$ have opposite signs. (10 marks)

f) Prove that $\frac{2\pi}{3} \le \int_0^{2\pi} \frac{dx}{10+3cosx} \le \frac{2\pi}{7}$. (10 marks)

2.

a) Let W be subspace of R^5 spanned by following vectors $u_1 = (1,2,1,3,2),\ u_2 = (1,3,3,5,3),\ u_3 = (3,8,7,13,8),\ u_4 = (1,4,6,9,7),\ u_5 = (5,13,13,25,19)$. Find Basis of W consisting of original given vectors and dimension of W. Extended the basis of W to basis of R^4 .

(15 marks)

b) Let $F: R^4 \to R^3$, be a linear mapping defined by
$F(x, y, z, t) = (x - y + z + t, 2x - 2y + 3z + 4t, 3x - 3y + 4z + 5t)$.
   I. Find basis and dimension of image of F
   II. Find basis and dimension of Kernel of F. (15 marks)

c) Let $T: R^3 \to R^3$, be linear transformation defined by $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, 3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$. Then find dimension of range space of T^2. Also find dimension of null space of T^3.

(15 marks)

d) If A is Hermition matrix and P is unitary matrix then show that $P^{-1}AP$ is Hermition. (5 Marks)

3.

a. The function f defined by $f(x) = \begin{cases} x^2 + 3x + a, if\ x \leq 1 \\ bx + 2, \qquad if\ x\ > 1 \end{cases}$ is given to be derivable for every x. Find $a\ and\ b$. (15 marks)

b. Show that the plane $ax + by + cz + d = 0$ touches the surfaces $px^2 + qy^2 + 2z = 0$ if $\frac{a^2}{p} + \frac{b^2}{q} + 2cd = 0$. (15 marks)

c. In a plane triangle ABC, find the maximum value of $cosAcosBcosC$. (15 marks)

d. Investigate the continuity of the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ \quad 0, \quad at\ origin \end{cases}$$ (5 marks)

4.

a) Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ whose generators are parallel to the lines $\frac{x}{0} = \frac{y}{\pm\sqrt{a^2-b^2}} = \frac{z}{c}$ meet the plane $z = 0$ in circles. (15 marks)

b) Find the equation of the cylinder whose generators are parallel to the line $y = mx, z = nx$ and which intersect the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$. (10 marks)

c) Show that three mutuallyperpendicular tangent lines can be drawn to sphere $x^2 + y^2 + z^2 = r^2$ from any point on the sphere $2(x^2 + y^2 + z^2) = 3r^2$. (10 marks)

d) Show that the equation $2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$, represents a cone whose vertex is $(\frac{-7}{6}, \frac{1}{3}, \frac{5}{6})$. (10 marks)

e) If r_1, r_2 are the radii of two orthogonal spheres, then find the area of the circle of their intersection. (5 marks)

Section- B

5.

a) Find constants a, b, c so that the vector $A = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrotational.Also find $\emptyset$. Also find $\emptyset$ such that $A = \nabla\emptyset$ (10 marks)

b) Find the work done in moving a particle, force field $F = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{k}$ along a straight line from $(0,0,0) to\ (2,1,3)$. (10 marks)

c) Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x sinx$ (10 marks)

d) Solve $(2x^2y - 3y^2)dx + (2x^3 - 12xy + logy)dy = 0$ (10 marks)

e) A particle starts from rest at a distance $'a'$ from the center of force which attracts inversely as the distance. Prove that the time of arriving at the center is $a\sqrt{\frac{\pi}{2\mu}}$. (10 marks)

6.

a) If $F = (x + y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$.Evaluate $\int_S F.N\ ds\ where\ S$ is the surface of the plane $2x + y + 2z = 6$ in the first octant. (10 marks)

b) By converting the surface integral to volume integral show that $\iint_S x^3dydz + y^3dzdx + z^3dxdy = \frac{12\pi a^5}{5}\ where\ S$ is the surface of the sphere $x^2 + y^2 + z^2 = a^2$. (10 marks)

c) Find the curvature of the Helix $r(t) = acost\hat{\imath} + asint\hat{\jmath} + bt\hat{k}$.Also find N (10 marks)

d) Verify Gauss divergence theorem for $A = 4x\hat{\imath} - 2y^2\hat{\jmath} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0, z = 3$. (20 marks)

7.

a) Solve, if $y = x\ and\ y = xe^{2x}$ are linear independent solutions of the homogenous equation corresponding to $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(x+1)y = x^3$. (10 marks)

b) Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$ (10 marks)

c) Use the transformation $u = x^2\ and\ v = y^2$ to solve $axyp^2 + (x^2 - ay^2 - b)p - xy = 0$. (10 marks)

d) Solve $\frac{dz}{dx} + \frac{z}{x}logz = \frac{z}{x^2}(logz)^2, z > 0\ and\ x > 0$. (10 marks)

e) A curve is such that the length of the perpendicular from origin on the tangent at any point P on the curve is equal to the abscissa of P. Prove that the differential equation of the curve $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$. hence find the curve. (10 marks)

8.

a) A particle is free to move on a smooth vertical circular wire of radius $'a'$. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{a/g}\,.\log(\sqrt{6} + \sqrt{5})$. (20 marks)

b) Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height. (20 marks)

c) If a pendulum of length l makes n complete oscillations in a given time, show that if g is changed to $(g + g')$, the number of oscillations gained is $ng'/2g$. (10 marks)