IFOS - 2014

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① a] If G is a Group in which $(ab)^4 = a^4 \cdot b^4$, $(ab)^5 = a^5 \cdot b^5$ and $(ab)^6 = a^6 \cdot b^6$ for all $a, b \in G$, then prove that G is abelian. ⑧

Sol) Given that

$$(ab)^4 = a^4 \cdot b^4 \quad \text{--- ①}$$
$$(ab)^5 = a^5 \cdot b^5 \quad \text{--- ②}$$
$$(ab)^6 = a^6 \cdot b^6 \quad \text{--- ③}$$

consider,

$$(ab)^6 = a^6 \cdot b^6$$
$$(ab)(ab)^5 = a^6 \cdot b^6$$
$$(ab)(a^5 \cdot b^5) = a^6 \cdot b^6 \quad \text{--- using ②}$$
$$\therefore ba^5 = a^5 \cdot b \quad \text{--- ④ using cancellation}$$

Similarly. $(ab)^5 = a^5 \cdot b^5$

$$(ab)(ab)^4 = a^5 \cdot b^5$$
$$(ab)(a^4 b^4) = a^5 b^5 \quad \text{--- using ①}$$
$$ba^4 = a^4 \cdot b \quad \text{--- using cancellation}$$

post multiplying by a, we get

$$ba^5 = a^4 \cdot ba$$
$$\Rightarrow a^5 b = a^4 \cdot ba \quad \text{--- using ④}$$
$$\therefore ab = ba \quad \text{--- using cancellation.}$$

$\therefore G$ is abelian.

2] a] let J_n be the set of integer mod n. then prove that J_n is ring under the operation of addition and multiplic^n modn. under what condition of n, J_n is a field. Justify your ans. (10)

Sol^n Let J_n be the set of integer mod n. then J_n has n distinct elements.

Thus $J_n = \{[0], [1], [2] \ldots [n-1]\}$

let $[a], [b] \in J_n$

the we define addition and multiplic^n of modulo as follows.

$[a]+[b] = [a+b]$

$[a]\cdot[b] = [a\cdot b]$

$\because [a+b]$ and $[a\cdot b]$ are both modulo n

$\therefore J_n$ is closed with respect to addition and multiplication.

now let $[a],[b],[c]$ be any elements of J_n. then we observe

commutativity of addition:-

$[a+b] = [a+b]$ ... by def^n of Residue

$= [b+a]$ ... $\{$integers are commut

$= [b]+[a]$

Associativity of addition:-

$([a]+[b])+[c] = [a+b]+[c]$

$= [(a+b)+c] = [(a)+(b+c)]$

$= [a]+([b+c])$

Additive identity:- we have $[0] \in J_n$ if $[a] \in J_n$

then $[a]+[0] = [a+0]$

$= [a]$

Additive inverse:- let $[a] \in J_n$ then $[-a] \in J_n$

where $[-a] = [n-a]$

$\therefore [a]+[-a] = [a-a] = [0]$

$\therefore [-a] = [n-a]$ is additive inverse.

Associative of multiplication:-

$([a][b])[c] = [ab][c]$

$= [(ab)c]$

$= [a(bc)]$

$= [a][bc]$

$= [a]([b][c])$

commutative:-

$[a][b] = [ab] = [ba] = [b][a]$

Distributive:-

$[a]([b]+[c]) = [a][b+c]$

$= [a(b+c)]$

$= [ab+ac]$

$= [ab]+[ac]$

$= [a][b]+[a][c]$

Thus J_n is commutative Ring.

If J_n is finite ring having n elements. if n is prime. then to prove that J_n is field.

let $[a], [b] \in J_n$

$\therefore [a]\cdot[b] = [0]$

$= [a\cdot b] = [0]$

$\Rightarrow n$ is divisor of ab : $n | ab$

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but n is prime
$\therefore n \mid a$ or $n \mid b$
$= [a] = 0$ or $[b] = 0$
$\therefore J_n$ is integral domain.
but we know that finite integral domains are field.
$\therefore J_n$ is field.

3] a) let R be an integral domain with unity. prove that the units of R and $R[x]$ are same. (10) IAS 2018

Solⁿ: Given R is an integral domain with unity,
$\therefore 1 \in R$
$\therefore 1 + 0x + 0x^2 + \cdots + 0x^n + \cdots$ is unity in $R[x]$.
let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is unit in $R[x]$
$\therefore \exists\ q(x) = b_0 + b_1 x + b_2 x^2 + \cdots \ni$
$p(x) \cdot q(x) = 1 + 0x + 0x^2 + \cdots$

$\therefore (a_0 b_0) + (a_0 b_1 + b_0 a_1) x + \cdots = 1 + 0x + \cdots$
but two polynomials are equal iff $a_i = b_i \ \forall\ i \in \mathbb{Z}$

$\therefore a_0 b_0 = 1$ & $a_1 = a_2 = \cdots a_n = b_1 = b_2 \cdots b_n = 0$
$\therefore$ ~~neither~~ $a_0 \neq 0$ and $b_0 \neq 1$
$\because R$ is I.D.
$\therefore P(x) = a_0$ is unit in $R[x]$
if a_0 is unit in R.