Multiple Integrals: The process of integration for one variable can be extended to the functions of more than one variable.

The generalisation of definite integrals is known as multiple integrals.

— The definite integral $\int_a^b f(x)\,dx$ is defined as the limit sum $f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots + f(x_n)\delta x_n = \sum_{r=1}^{n} f(x_r)$ where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots$ tends to zero. i.e

Double Integrals:

A double integral is the counterpart, in two dimensions, of the definite integral of a function of a single variable. Let A be a finite region of the xy-plane, and let $f(x, y)$ be a function of the independent variables x, y defined at every point in A. Divide the region A into n parts, of areas $\delta A_1, \delta A_2, \ldots \delta A_n$.

Let (x_r, y_r) be any point inside the r^{th} elementary area δA_r.

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Form the sum

$$f(x_1, y_1)\delta A_1 + f(x_2, y_2)\delta A_2 + \ldots + f(x_n, y_n)\delta A_n \quad \text{(1)}$$

i.e $\sum_{r=1}^{n} f(x_r, y_r)\delta A_r$ —— (1)

Increase the number of subdivisions taking smaller and smaller elementary areas. Then the limit of the sum (1), if it exists, as n tends to infinity and the dimension of each subdivision tend to zero, is called the double integral of

$f(x,y)$ over the region A; and is denoted by

$$\iint_A f(x,y)\,dA \qquad \text{—— (2)}$$

Thus $$\iint_A f(x,y)\,dA = \lim_{\substack{n\to\infty \\ \delta A\to 0}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \text{—— (3)}$$

This definition corresponds to the definition

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \delta x\to 0}} \sum_{r=1}^{n} f(x_r)\,\delta x_r \qquad \text{—— (4)}$$

for the definite integral of a single variable.

Just as the definite integral (4) can be interpreted as an area, similarly the double integral (2) can be interpreted as a volume.

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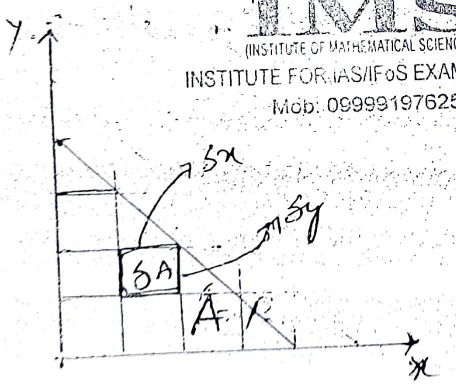
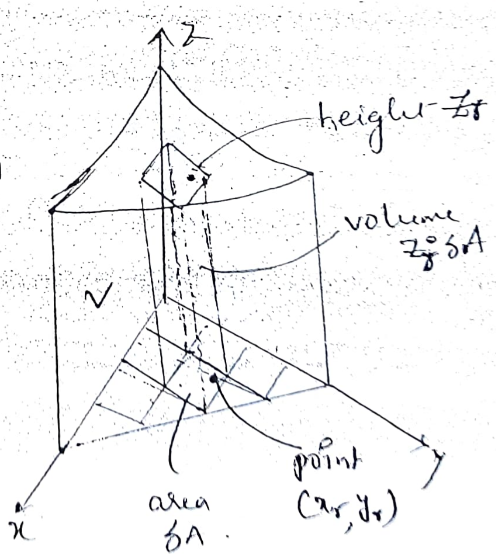


Base 'A' cut into small pieces δA.



Solid 'V' cut into thin sticks $\Delta V = z\,\delta A$

– For single integrals, the interval $[a,b]$ is divided into short pieces of length δx.

– For double integrals, A is divided into small rectangles of area $\delta A = (\delta x)(\delta y)$.

Above the r^{th} rectangle is a thin stick with small volume. That volume is the base area δA times the height above it - except that this height $z = f(x, y)$ varies from point to point. Therefore we select a point (x_r, y_r) in the r^{th} rectangle and compute the volume from the height above that point.

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volume of one stick $= f(x_r, y_r)\,\delta A$

volume of all sticks $= \sum f(x_r, y_r)\,\delta A$.

This is the crucial step for any integral - to see it as a sum of small pieces.

Now take limits $\delta x \to 0$ and $\delta y \to 0$; The height $z = f(x, y)$ is nearly constant over each rectangle (assume that f is continuous function). The sum approaches a limit, which depends only on the base A and the surface above it. The limit is the volume of the solid, and it is the double integral of $f(x, y)$ over A.

$$\iint_A f(x,y)\, dA = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum f(x_r, y_r)\,\delta A.$$

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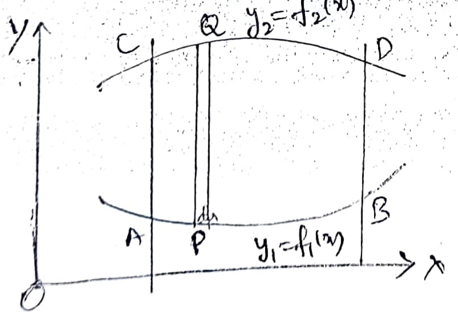
For purpose of evaluation, (2) is expressed as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dx\,dy$.

→ Its value is found as follows:

(i) when y_1, y_2 are functions of x and x_1, x_2 are constants, $f(x,y)$ is first integrated w.r.t y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t x with in the limits x_1, x_2. i.e,

$$I_1 = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x,y)\,dy}\, dx}$$

where integration is carried from the inner to outer rectangle; which is geometrically illustrated as shown below.



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Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$.

PQ is a vertical strip of width dx.

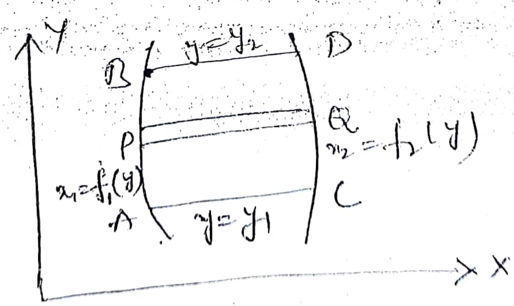
Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant). while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area ABDC.

(ii) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t x keeping 'y' fixed, with in the limits x_1, x_2 and the resulting expression is integrated w.r.t y between the limits y_1, y_2, i.e.,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x,y)\,dx}\,dy}$$

which is geometrically illustrated as shown below.



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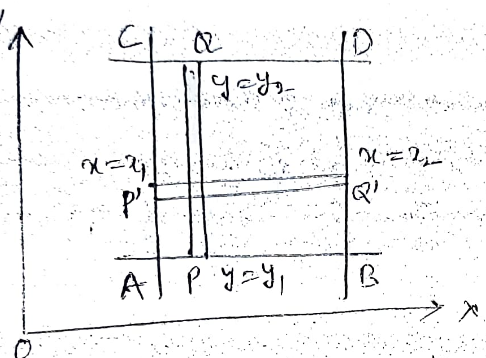
Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. ~~PQ is~~ a horizontal strip of width dy.

The inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area ABDC.

(iii) when both pairs of limits are constants, the region of integration is the rectangle ABDC.



In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip P'Q' and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus for constant limits, it hardly matters whether first integrate w.r.t x and then w.r.t y or vice versa.

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→ Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$.

Sol^n: Let $I = \int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

$$= \int_0^5 dx \int_0^{x^2} (x^3+y^2x)\,dy$$

$$= \int_0^5 \left[x^3y + \frac{xy^3}{3}\right]_0^{x^2} dx$$

$$= \int_0^5 x^5 + \frac{x^7}{3}\,dx$$

$$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5$$

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$$= \frac{5^6}{6} + \frac{5^8}{24} = 5^6\left[\frac{1}{6} + \frac{25}{24}\right]$$

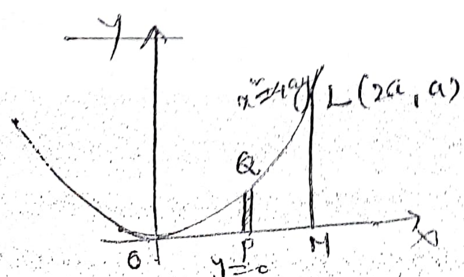
$$= \frac{5^6}{24}\left[\frac{29}{24}\right]$$

—— x ——

→ Evaluate $\iint_A xy\,dx\,dy$, where A is the domain bounded by x-axis, ordinate $x=2a$ and the curve $x^2=4ay$.

Sol^n: The line $x=2a$ and the parabola $x^2=4ay$ intersect at $L(2a,a)$

The figure shows the domain A which is the area OML.



Integrating first over a vertical strip PQ, i.e, w.r.t y from $y=0$ to $y=x^2/4a$ on the parabola and then w.r.t x from $x=0$ to $x=2a$, we have

$$\iint_A xy\,dx\,dy = \int_0^{2a} \left[\int_0^{x^2/4a} xy\,dy\right] dx$$

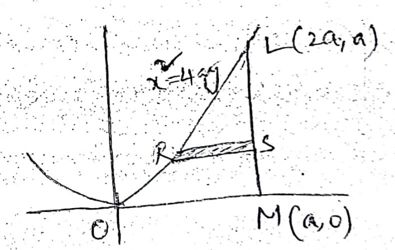
$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{a^4}{3}$$

Otherwise integrating first over a horizontal strip RS, i.e, w.r.t x from $x=2\sqrt{ay}$ on the parabola and then w.r.t y from $y=0$ to $y=a$, we get



$$\iint_A xy\,dx\,dy = \int_0^a dy \int_{2\sqrt{ay}}^{2a} xy\,dx$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{ay}}^{2a} dy$$

$$= \int_0^a \frac{(4a^2-4ay)y\,dy}{2} = 2a\int_0^a (ay-y^2)\,dy$$

$$= 2a\left(\frac{ay^2}{2}-\frac{y^3}{3}\right)_0^a$$

$$= \frac{a^4}{3}$$

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→ Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

Solⁿ:

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2} = \int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$\left[\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}1 - \tan^{-1}0\right) dx$$

$$= \int_0^1 \left(\frac{\pi}{4}-0\right)\frac{1}{\sqrt{1+x^2}}\,dx$$

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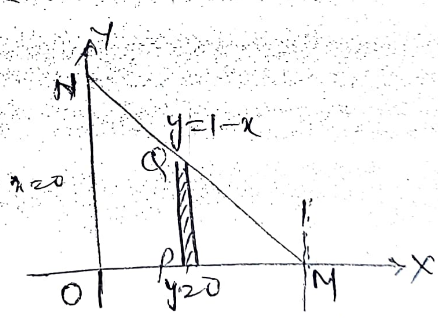
$$= \frac{\pi}{4}\left[\log\{x+\sqrt{1+x^2}\}\right]_0^1$$

$$= \frac{\pi}{4}\log\{1+\sqrt{2}\}.$$

→ Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y\le 1$.

Solⁿ:



The region of integration is the area A bounded by the two axes and the straight line $x+y=1$.

Consider a strip PQ parallel to y-axis.

It has its extremities on $y=0$ and $y=1-x$.

Hence limits of y are from $y=0$ to $y=1-x$.

The limits of x are from $x=0$ to $x=1$.

Hence the given integral

$$\iint_R xy\,dx\,dy = \int_0^1 \int_0^{1-x} xy\,dx\,dy$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{1-x}$$

$$= \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)_0^1$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{1}{24}.$$

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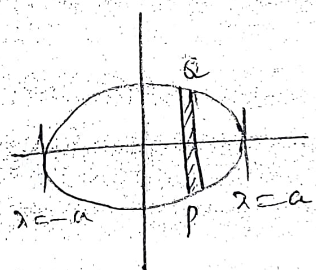
→ Evaluate $\iint (x+y)^2\,dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n:

For the ellipse



$$\frac{y}{b} = \pm\sqrt{\left(1 - \frac{x^2}{a^2}\right)}$$

$$\Rightarrow y = \pm b\sqrt{1 - \frac{x^2}{a^2}}.$$

Integrating first w.r.t y along a vertical strip PQ which extends from $y = -b\sqrt{1-\frac{x^2}{a^2}}$ to $y = +b\sqrt{1 - \frac{x^2}{a^2}}$.

To cover the region, we then integrate w.r.t x from $x=-a$ to $x=a$.

∴ The given integral

$$\iint (x+y)^2 = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} (x^2+2xy+y^2)\,dx\,dy$$

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$$= \int_{-a}^{a}\left[x^2y+2x\frac{y^2}{2}+\frac{y^3}{3}\right]_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= \int_{-a}^{a}\left\{2bx^2\sqrt{1-\frac{x^2}{a^2}} + 0 + \frac{2}{3}b^3\left(\sqrt{1-\frac{x^2}{a^2}}\right)^3\right\}dx$$

$$= 2\int_{-a}^{a}\left\{bx^2\sqrt{1-\frac{x^2}{a^2}}+\frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

$$= 4b\int_{0}^{a}\left\{a^2\sqrt{1-\frac{x^2}{a^2}}+\frac{b^2}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

~~=4b~~ putting $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$

Limits: ~~when~~ $x=0$; $\theta=0$

$x=a$; $\theta=\pi/2$

$$= 4b\int_{0}^{\pi/2}\left\{a^2\sin^2\theta\cos\theta+\frac{1}{3}b^2\cos^3\theta\right\}a\cos\theta\,d\theta$$

$$= 4ab\int_{0}^{\pi/2}\left\{a^2\sin^2\theta\cos^2\theta+\frac{b^2}{3}\cos^4\theta\right\}d\theta.$$

$$= 4ab\left\{\int_{0}^{\pi/2}\frac{a^2}{4}\sin^2 4\theta\,d\theta+\frac{b^2}{3}\int_{0}^{\pi/2}\cos^4\theta\,d\theta\right\}$$

$$= 4ab\left[\int_{0}^{\pi/2}\frac{a^2}{4}\frac{(1-\cos 8\theta)}{2}\,d\theta+\frac{b^2}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= 4ab\left[\frac{a^2}{8}\left(\theta+\frac{\sin 8\theta}{8}\right)_{0}^{\pi/2}+\frac{b^2\pi}{16}\right]$$

$$= 4ab\left[\frac{a^2}{8}\left(\frac{\pi}{2}+0\right)+\frac{\pi b^2}{16}\right]$$

$$= \frac{1}{4}\pi ab\,(a^2+b^2)$$

Ex → Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$ (4)

Find the values of the two integrals.

Sol^n:

$$LHS = \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy$$

$$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy$$

$$= \int_0^1 dx \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy$$

$$= \int_0^1 \left[-\frac{x}{(x+y)^2} + \frac{1}{x+y} \right]_0^1 dx \qquad \left(\because \int \frac{1}{x^n}\, dx = \frac{x^{-n+1}}{-n+1} \right)$$

$$= \int_0^1 \left[-\frac{x}{(1+x)^2} + \cancel{\frac{1}{x}} + \frac{1}{1+x} - \cancel{\frac{1}{x}} \right] dx$$

$$= \int_0^1 \frac{dx}{(1+x)^2}$$

$$= \left[-\frac{1}{1+x} \right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

$$RHS = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \frac{x+y-2y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx$$

$$= \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y} \right] dy$$

$$= -\int_0^1 \frac{dy}{(1+y)^2} = \left[\frac{1}{1+y} \right]_0^1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$\therefore$ LHS $\neq$ RHS

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Evaluate the following double integrals:

(1) $\int_0^1 \int_0^2 (x+2)\,dx\,dy$; Ans: 5

(2) $\int_0^a \int_0^b (x^2+y^2)\,dx\,dy$; Ans: $\frac{1}{3}ab(a^2+b^2)$

(3) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$; Ans: $\log a \log b$

(4) $\int_1^2 \int_0^x \frac{dx\,dy}{x^2+y^2}$; Ans: $\frac{1}{4}\log_e 2$

(5) $\int_1^2 \int_0^{y/2} y\,dy\,dx$; Ans: $\frac{7}{6}$

(6) $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\,dy\,dx$; Ans: -2

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(7) $\int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 y\,dx\,dy$; Ans: $a^5/15$

(8) $\int_0^1 \int_0^{\sqrt{1-y^2}} 4y\,dy\,dx$; Ans: $4/3$

(9) $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$; Ans: $3/35$

(10) $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$; Ans: $1/2$

(11) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$; Ans: $\frac{\pi a^3}{6}$

(12) $\int_0^2 \int_0^{\sqrt{2x-x^2}} x\,dx\,dy$; Ans: $\pi/2$

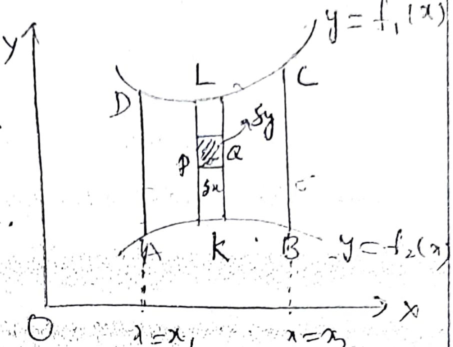
(13) Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2+y^2 \le 1$. Ans: $\pi/24$.

(14) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$. Ans: $1/6$.

(15) Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$. Ans: $1/6$.

Area by double integration (or) Area enclosed by plane curves:

Consider the area enclosed by the curves $y=f_1(x)$ and $y=f_2(x)$ and the ordinates $x=x_1$, $x=x_2$ as shown in the figure.



Divide this area into vertical strips of width δx.

If $P(x,y)$, $Q(x+\delta x, y+\delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x\,\delta y$.

$\therefore$ Area of strip $KL = \lim_{\delta y \to 0} \sum \delta x\,\delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y=f_1(x)$ to $y=f_2(x)$.

$\therefore$ Area of the strip $KL = \delta x \lim_{\delta y \to 0} \sum_{f_1(x)}^{f_2(x)} dy$

$$= \delta x \int_{f_1(x)}^{f_2(x)} dy$$

Now adding up all such strips from $x=x_1$ to $x=x_2$ we get the area ABCD

$$= \lim_{\delta x \to 0} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx\,dy$$

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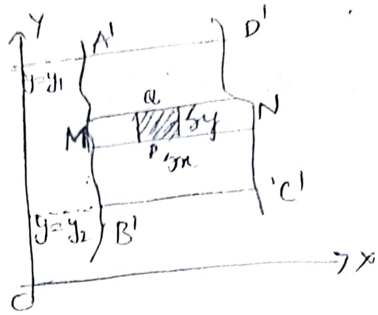
Similarly dividing the area $A'B'C'D'$ into horizontal strips of width δy, we get the area

$$A'B'C'D' = \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx\,dy.$$



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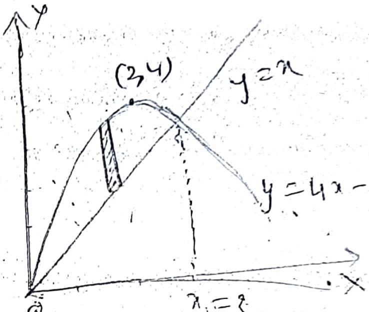
→ find the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.



Sol^n: The equation of the parabola $y = 4x - x^2$ may be written as $(x-2)^2 = -(y-4)$

i.e, this parabola has the vertex at the point $(2,4)$ and its concavity is downwards.

The point of intersection of two curves are given as follows.

$$4x - x^2 = x$$
$$\Rightarrow x^2 - 3x = 0$$
$$\Rightarrow x(x-3) = 0$$
$$\Rightarrow x = 0 \text{ (or) } x = 3.$$

and hence from $y = x$, we get

$y = 0$ at $x = 0$

$y = 3$ at $x = 3$

∴ The points of intersection of the two curves are $(0,0)$, $(3,3)$.

The area can be considered as lying between (9)

the curves $y=x$, $y=4x-x^2$, $x=0$ and $x=3$.

So integrating along a vertical strip first, i.e., y from $y=x$ to $y=4x-x^2$ and then w.r.t x from $x=0$ to $x=3$.

$\therefore$ The required area $= \int_0^3 \int_{y=x}^{4x-x^2} dx\,dy$

$$= \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (4x-x^2-x)dx$$

$$= \int_0^3 (3x-x^2)dx$$

$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

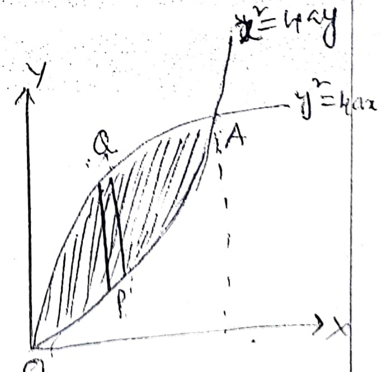
$$= \frac{27}{2}-9 = \frac{9}{2}$$

→ Show that the area between the parabolas $y^2=4ax$ and $x^2=4ay$ is $\frac{16}{3}a^2$.

Sol^n: Solving the equations

$y^2=4ax$ and $x^2=4ay$,

the parabolas intersect at $O(0,0)$ and $A(4a,4a)$. As such for the shaded area between these parabolas (as shown in the figure)



x varies from 0 to $4a$ and y varies from $y=x^2/4a$ to $y=2\sqrt{ax}$.

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Hence the required area

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy$$

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$$= \int_0^{4a}\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{2}{3}x^{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

→ Find by double integration, the area enclosed by the curves $y = 3x/x^2+2$ and $4y = x^2$.

Ans: $\frac{2}{3}\log_e 3 - \frac{2}{3}$

→ Find by double integration the area of the region enclosed by the following curves:

(1) $x^2+y^2 = a^2$ and $x+y = a$ (in the first quadrant)

Ans: $\frac{(\pi-2)a^2}{4}$

(2) $y^2 = x^3$ and $y = x$.

Ans: $\frac{1}{10}$ $\cancel{ab(a^2+b^2)}$)

(3) $9xy = 4$ and $2x+y = 2$.

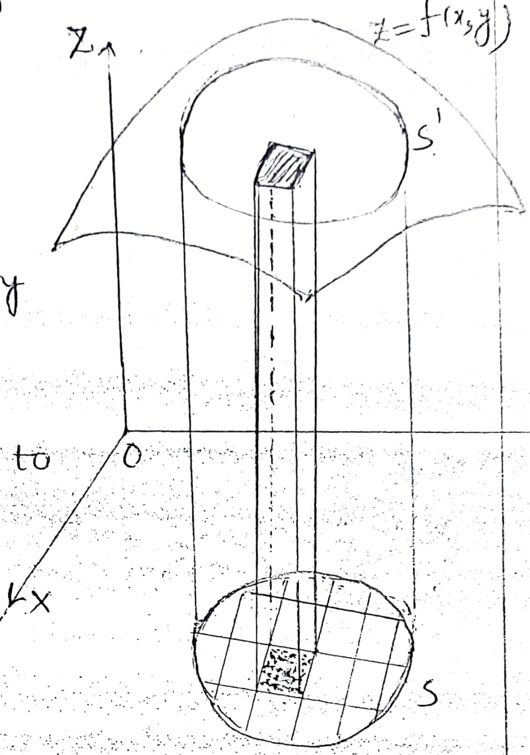
Ans: $\frac{1}{3} - \frac{4}{9}\log_e 2$.

(4) $(x^2+4a^2)\,y = 8a^3$, $2y = x$, and $x = 0$ Ans: $(\pi-1)a^2$.

volume as double integral:

Consider a surface $z = f(x,y)$.

Let the orthogonal projection on xy-plane of its portion S' be the area S.



Divide S into elementary rectangles of area $\delta x \delta y$ by drawing lines parallel to X and Y axes. with each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ volume of this prism between S and the given surface $z = f(x,y)$ is $z\,\delta x\,\delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the z-axis:

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma z\,\delta x\,\delta y$$

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$$= \iint z\,dx\,dy$$

$$= \iint f(x,y)\,dx\,dy$$

where the integration is carried over the area S.

i.e, If the region S may be considered as enclosed by the curves $y = f_1(x)$, $y = f_2(x)$,

$x = a$ and $x = b$, we can write volume as

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dx\, dy.$$

Note: When writing the integral for the volume, the integrand $f(x, y)$ is taken from the top of Surface $z = f(x, y)$ which covers the volume while the limits a, b, f_1, f_2 are taken from the base area S in the xy-plane.

→ Find the volume under the plane $x + y + z = 6$ and above triangle in the xy-plane bounded by $2x = 3y$, $y = 0$, $x = 3$.

Sol: The required volume V

$$= \iint_A z\, dA$$

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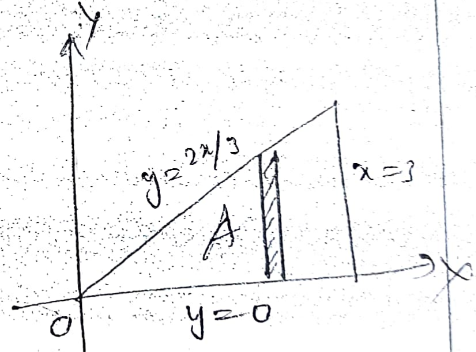
$$= \iint_A (6 - x - y)\, dA,$$



where A is the region shown in the figure.

Integrating along a vertical strip first, we have

$$V = \int_0^3 \int_0^{2x/3} (6 - x - y)\, dx\, dy$$

$$= \int_0^3 \left(6y - xy - \frac{y^2}{2}\right)_0^{2x/3} dx$$

$$= \int_0^3 \left(4x - \frac{2}{3}x^2 - \frac{2}{9}x^2\right) dx$$

$$= \int_0^3 \left(4x - \frac{8}{9}x^2\right) dx.$$

$$= \left(2x^2 - \frac{8}{27}x^3\right)_0^3$$

$$= 18 - 8 = 10.$$

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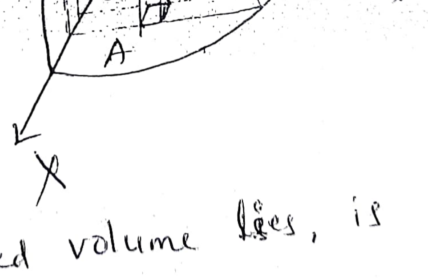
find the volume in the positive octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Soln: The required volume lies between the ellipsoid $z = c\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$ and the plane XOY, and is bounded on the sides by the planes $x=0$, $y=0$.



The given ellipsoid cuts XOY plane in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z=0.$$

Therefore the region A, above which the required volume lies, is bounded by curves

$$y=0, \quad y = b\sqrt{1 - x^2/a^2},$$

$$x=0, \text{ and } x=a.$$

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Hence, the required volume

$$= \iint_A z \, dA$$

$$= \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}\; dx\,dy$$

$$= c\int_0^a \int_0^Y \sqrt{\frac{Y^2}{b^2}-\frac{y^2}{b^2}}\; dx\,dy \quad \text{on putting } \sqrt{1-\frac{x^2}{a^2}} = \frac{Y}{b}$$

$$= \frac{c}{b}\int_0^a \left[\frac{1}{2}y\sqrt{Y^2-y^2}+\frac{1}{2}Y^2\sin^{-1}\frac{y}{Y}\right]_0^Y dx$$

$$= \frac{c}{b}\int_0^a \frac{1}{2}Y^2\,\frac{\pi}{2}\,dx$$

$$= \frac{\pi c}{4b}\int_0^a Y^2\,dx$$

$$= \frac{\pi c}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx \qquad \left(\because Y = b\sqrt{1-\frac{x^2}{a^2}}\right)$$

$$= \frac{1}{4}\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{1}{4}\pi bc\left[a-\frac{a^3}{3a^2}\right]$$

$$= \frac{1}{4}\pi bc\,\frac{2a}{3}$$

$$= \frac{1}{6}\pi abc \quad \text{Ans}$$

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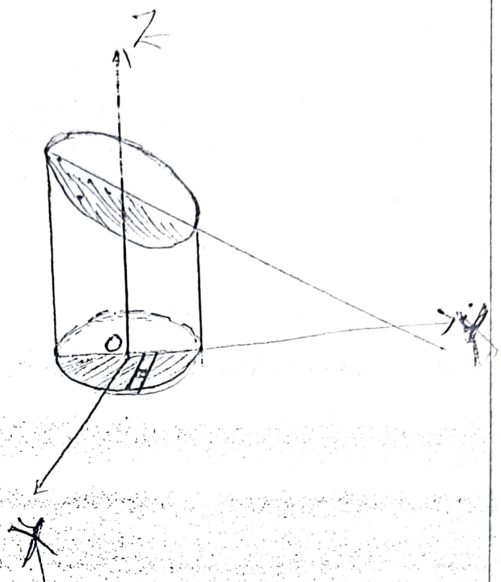
→ Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$ and $z=0$.

Sol^n: From the figure, it is self evident that $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy-plane. To cover the shaded

half of this circle, x varies from 0 to $\sqrt{4-y^2}$ (12)
and y varies from -2 to 2.



$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{-2}^{2} (4-y)\,(x)_0^{\sqrt{4-y^2}}\,dy$$

$$= 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\,dy$$

$$= 2\int_{-2}^{2} 4\sqrt{4-y^2}\,dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\,dy$$

$$= 8\int_{-2}^{2}\sqrt{4-y^2}\,dy$$ (Here the second term vanishes as the integrand is an odd function)

$$= 8\left[\frac{y\sqrt{4-y^2}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right]_{-2}^{2}$$ $\left[\because \int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$$= 8\left[0 + 2\sin^{-1}(1) - 0 - 2\sin^{-1}(-1)\right]$$

$$= 8\left[2\frac{\pi}{2} + 2\frac{\pi}{2}\right]$$

$$= 16\pi.$$

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→ Find the volume of the cylinder $x^2+y^2=ax$ bounded by the planes $z=0$ and $z=x$. Ans: $\pi a^3/8$

→ Find the volume under the plane $x+z=2$, above $z=0$ and within the cylinder $x^2+y^2=4$. Ans: 8π

→ Find the volume under the plane $z = x+y$ and above the area cut from the first quadrant by ellipse $4x^2+9y^2=36$. Ans: 10

→ Find the volume bounded by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Ans: $\frac{abc}{6}$

→ Find the volume bounded by $4z = 16-x^2-y^2$ and the plane $z=0$. Ans: 16π

→ Find the volume enclosed by the cylinders $y^2=x$ and $x^2+y^2=a^2$ and the plane $z=0$. Ans: $\pi a^4/4$.

→ Find the volume in the first octant bounded by the parabolic cylinders $z=9-x^2$, $x=3-y^2$. Ans: $\frac{102\sqrt{3}}{35}$

→ Find the volume in the first octant bounded by $z=x^2+y^2$ and $y=1-x^2$. Ans: $2/7$

→ Find the volume inside the paraboloid $x^2+4z^2+8y=16$ and on the positive side of xz-plane.

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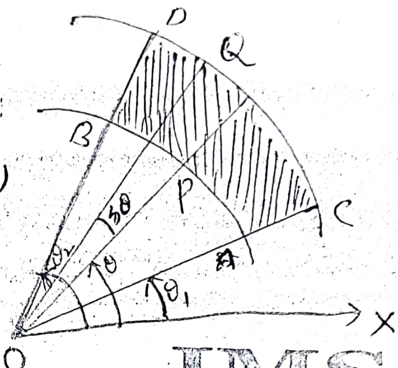
Double integrals in Polar co-ordinates:-

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$, we first integrate w.r.t r between limits $r=r_1$ and $r=r_2$ keeping θ fixed and the resulting expression is integrated w.r.t θ from θ_1 to θ_2. In this integral



Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

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Then $\int_{r_1}^{r_2} f(r,\theta)\,dr$ indicates that the integration is along PQ from P to Q while integration w.r.t θ corresponds to the turning of PQ from AC to BD.

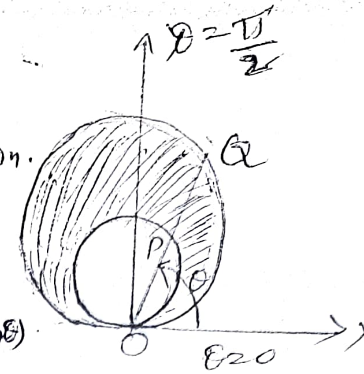
Thus the whole region of integration is the area ACDB. The order of integration may be changed with appropriate changes in the limits

---

→ Calculate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Sol^n: Given circles $r = 2\sin\theta$ and $r = 4\sin\theta$ are as shown in the figure.

The shaded area between these circles is the region of integration. If we integrate first w.r.t r, then its limits are from $P(r=2\sin\theta)$ to $Q(r=4\sin\theta)$ and to cover the whole region θ varies from 0 to π.



Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr$$

$$\int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= \frac{1}{4}\int_0^{\pi} (256-16)\sin^4\theta\, d\theta$$

$$= \frac{240}{4}\int_0^{\pi} \sin^4\theta\, d\theta$$

$$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta\, d\theta$$

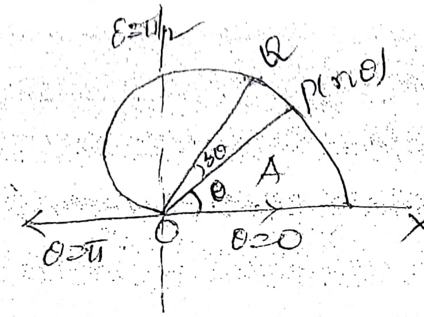
$$= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= 22.5\pi$$

$\int_0^{\pi/2} \sin^n\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2}\cdot\frac{\pi}{2}$

→ Integrate $r\sin\theta$ over the area of the cardioid $r=a(1+\cos\theta)$ above the initial line.

Soln: Here the region of integration A can be covered by



radial strips whose ends are $r=0$ and $r=a(1+\cos\theta)$ i.e.,
The strips start from $\theta=0$ and end at $\theta=\pi$.

Therefore the required integral

$$= \iint_A r\sin\theta \, dA = \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\sin\theta \; r\,d\theta\,dr.$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} \sin\theta\,(1+\cos\theta)^3\, d\theta$$

$$= \frac{1}{3}a^3 \int_0^{\pi} 2\sin\theta/2\,\cos\theta/2\,\left(2\cos^2\theta/2\right)^3 d\theta$$

$$= \frac{16}{3}a^3 \int_0^{\pi} \sin\theta/2\,\cos^7\theta/2\, d\theta$$

putting $\theta = 2\phi \Rightarrow d\theta = 2d\phi$.
Limits: $\phi = 0$ when $\theta = 0$
$\phi = \pi/2$ when $\theta = \pi$

$$= \frac{16}{3}a^3 \int_0^{\pi/2} \sin\phi\,\cos^7\phi\; 2d\phi$$

$$= \frac{32}{3}a^3 \left[-\frac{\cos^8\phi}{8}\right]_0^{\pi/2}$$

$$= \frac{4}{3}a^3.$$

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→ Evaluate $\iint_A r^2\sin\theta\, d\theta\, dr$ over the area of cardioid $r = a(1+\cos\theta)$ above the initial line.

Ans: $\frac{4}{3}a^3$.

→ Evaluate $\int_0^{\pi}\int_0^{a\theta} r^3\, d\theta\, dr$ Ans: $\frac{a^4\pi^5}{20}$

→ Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3 \sin\theta \cos\theta\, d\theta\, dr$ Ans: $\frac{16}{15}a^4$.

→ Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\, d\theta\, dr$ Ans: $\pi a^2/4$

→ Evaluate $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, d\theta\, dr$ Ans: $a^2/6$

→ Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \cos\theta\, d\theta\, dr$ Ans: $\frac{5}{3}\pi a^3$

→ Evaluate $\iint r^2\, d\theta\, dr$ over the area of the circle $r = a\cos\theta$ Ans: $\frac{4a^3}{9}$

→ Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = \frac{2a^2}{3}$, where R is the semi circle $r = 2a\cos\theta$ above the initial line.

→ Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$.   Ans: $\frac{1}{2}(4-\pi)a$

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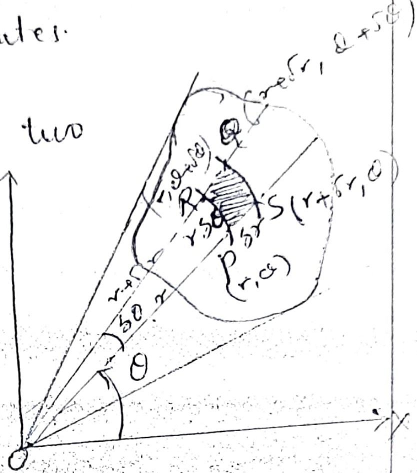
Area enclosed by plane curves.

polar co-ordinates:

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r,\theta)$, $Q(r+\delta r, \theta+\delta\theta)$ be two neighbouring points.



Mark circular areas of radii r and $r+\delta r$ meeting OQ in R and OP in S.

Since arc $PR = r\delta\theta$ $\quad (\because l = r\theta)$

and $PS = \delta r$.

$\therefore$ Area of the curvilinear rectangle PRQS is approximately $= PR \cdot PS$

$= r\delta\theta \cdot \delta r$

If the whole area is divided into such curvilinear rectangles, the sum $\sum\sum r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $A = \lim\limits_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \sum\sum r\delta\theta\delta r = \iint r\, d\theta\, dr.$

where the limits are to be so chosen as to cover the entire area.

→ Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.

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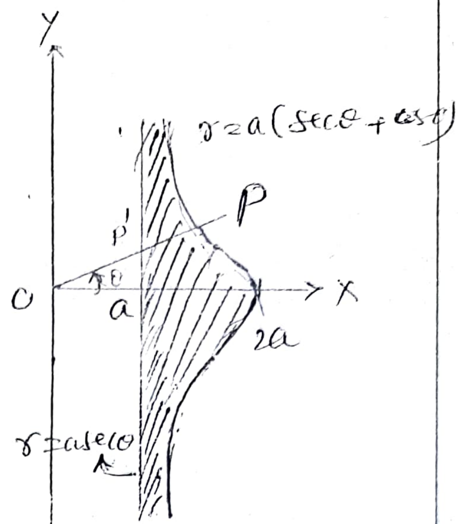
Sol^n: The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$.



Draw any line OP cutting the curve at P and its asymptote at P'.

Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta+\cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$\therefore$ The required area

$$= 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

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$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \frac{2a^2}{2}\int_0^{\pi/2}\left[(\sec\theta+\cos\theta)^2 - \sec^2\theta\right]d\theta$$

$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = a^2\left[2\theta\right]_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{5\pi a^2}{4} \text{ Ans.}$$

→ Find by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Ans:

Change of order of Integration :

The integral $\iint U\,dx\,dy$ is first integrated with respect to the variable 'y', then limits of y are substituted (which in general may be function of x), and the result is integrated with respect to 'x'. But if we want to change $\iint u\,dx\,dy$ to $\iint u\,dy\,dx$ then we have to find the new limits of x as functions of y.

i.e., in a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits.

To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral.

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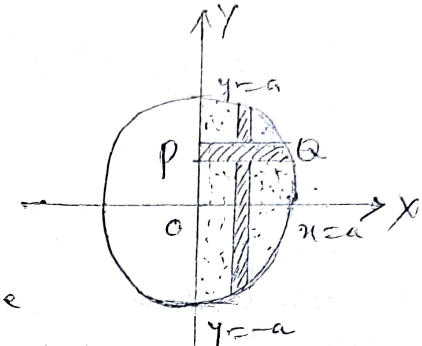
→ Change

→ Change the order of integration in the integral $I = \int_{-a}^{a} \int_{0}^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy$.

Solⁿ:



Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x=0$ to $x=\sqrt{a^2-y^2}$ (i.e, the circle $x^2+y^2=a^2$) and this strip slides from $y=-a$ to $y=a$. This shaded semi-circular area is, therefore, the region of integration.

On changing the order of integration, we first integrate w.r.t y along a vertical strip RS which extends from $R\,[y=-\sqrt{a^2-x^2}]$ to $S\,[y=\sqrt{a^2-x^2}]$. To cover the given region, we then integrate w.r.t x from $x=0$ to $x=a$.

Thus $I = \int_{0}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy$

$$= \int_{0}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy\, dx.$$

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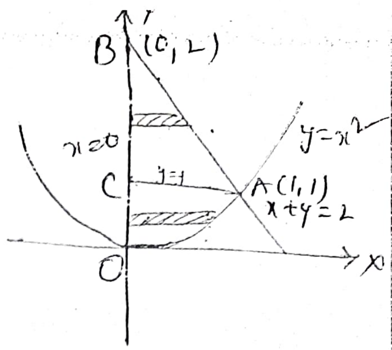
→ Change the order of integration in $I = \int_{0}^{1} \int_{x^2}^{2-x} xy\, dx\, dy$ and hence evaluate the same.

Sol^n:

17.



The given limits show that the region of integration is bounded by the curves

$y=x^2$, $y=2-x$, $x=0$, $x=1$.

The first is a parabola with vertex at the origin and the second the straight line $y=2-x$.

These intersect at the point $(1,1)$.

Therefore the region of integration is OAB.

When we integrate w.r.t x first along a horizontal strip, the strip starts from $x=0$

But some of the strips end on OA while the others end on AB. i.e, at A strips parallel to x-axis change their character.

Hence through the point A, draw a straight line $(y=1)$ CA parallel to the x-axis. This stright line CA divides the region OAB into two parts namely OAC and ABC.

In the region OAC, the strip parallel to x-axis has its extremities on $x=0$ and $y=x^2$.

Hence limits of x are from $x=0$ to $x=\sqrt{y}$

As the point A is $(1,1)$, the limits of y are from $y=0$ to $y=1$

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Again in the region ABC, the strip parallel to x-axis has its extremities on $x=0$ and $y=2-x$

Hence limits of x are from $x=0$ to $x=2-y$

The limits of y are from $y=1$ to $y=2$.

Hence changing the order of integration the given integral is

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 dy\int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy\int_0^{2-y} xy\,dx$$

$$= \int_0^1 \left[\frac{x^2}{2}y\right]_0^{\sqrt{y}} dy + \int_1^2 \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

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$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + 2y^2\right]_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left[4 - \frac{32}{3} + 8 - \left(\frac{1}{4} - \frac{4}{3} + 2\right)\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{4}{3} - \frac{9}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left(\frac{5}{12}\right)$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8} \text{ Ans.}$$

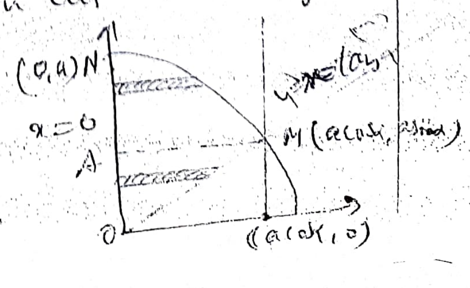
→ Change the order of integration in

$$\int_0^{2a}\int_{x^2/4a}^{3a-x} f(x,y)\,dx\,dy.$$

→ Change the order of integration in the integral

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$ and verify the result when $f(x,y)=1$.

$\boxed{\text{Ans: } \frac{a^2}{2}\left(\frac{\pi}{2}-\alpha\right)}$



E→ Change the order of integration in $\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} V\,dx\,dy$, where V is a function of x and y. 18.

Sol^n: The limits of integration are given by the parabolas $\frac{x^2}{a}=y$ i.e, $x^2=ay$;

$x-x^2/a=y$ i.e, $ax-x^2=ay$

and the lines $x=0$, $x=\frac{1}{2}a$.

Also the equation of parabola $ax-x^2=ay$ may be written as $\left(x-\frac{a}{2}\right)^2=-a\left(y-\frac{a}{4}\right)$

i.e, this parabola has the vertex as the point $\left(\frac{a}{2},\frac{a}{4}\right)$ and its concavity is downwards.

The points of intersection of two parabolas are given as follows:

$ax-x^2=x^2 \Rightarrow x(a-2x)=0$

$\Rightarrow x=0$ or $x=\frac{a}{2}$

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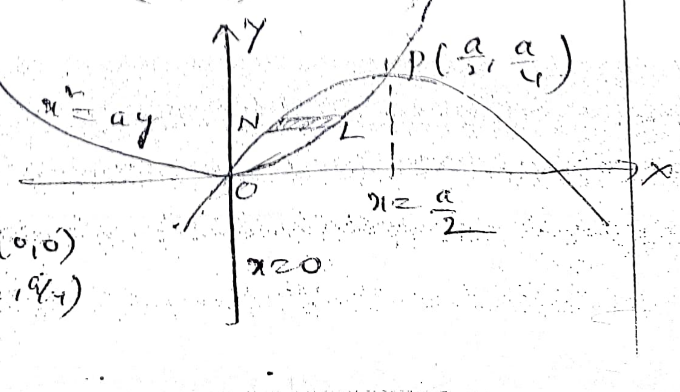
and hence from $x^2=ay$,

we get $y=0$ at $x=0$

$y=\frac{a}{4}$ at $x=\frac{a}{2}$.

Hence the points of intersection of the two parabolas are $(0,0)$, and $\left(\frac{a}{2},\frac{a}{4}\right)$.



Draw the two parabolas $x^2=ay$ and $ax=x^2-ay$ intersecting at the points $O(0,0)$ and $P(a/2, a/4)$

Now draw the lines $x=0$ and $x=\frac{a}{2}$.

Clearly the integral extends to the area ONPLO.

Now take strips of the type NL parallel to the x-axis.

Solving $ay = ax - x^2$

$\therefore x^2 - ax + ay = 0$ for x, we get

$x = \frac{1}{2}\left[a \pm \sqrt{a^2 - 4ay}\right]$

$= \frac{1}{2}\left[a - \sqrt{a^2-4ay}\right]$

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rejecting the +ve sign before square root since x is not greater than $\frac{a}{2}$ for the region of intersection.

In the region ONPLO, the strip NL has the extremities N and L on $ax - x^2 = ay$ and $x^2 = ay$.

Thus the limits of x are from $x = \frac{1}{2}\left(a - \sqrt{a^2-4ay}\right]$ to $x = \sqrt{ay}$.

For limits of y, at O, $y=0$ and at P $y = \frac{a}{4}$.

Hence changing the order of integration we have

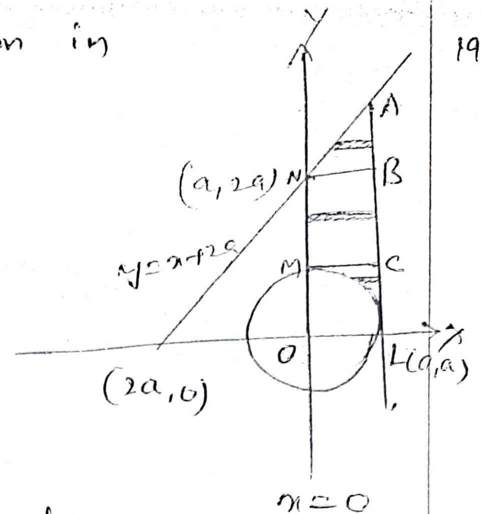
$$\int_0^{a/2}\int_{x^2/a}^{x - x^2/a} v\,dx\,dy = \int_0^{a/4}\int_{\frac{1}{2}(a-\sqrt{a^2-4ay})}^{\sqrt{ay}} v\,dy\,dx.$$

→ Evaluate $\int_0^{\infty}\int_0^{x} x e^{-x^2/y}\,dy\,dx$ by changing the order of integration.

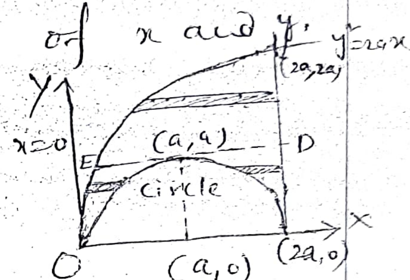
19.

LE → Change the order of integration in

$$\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} \phi(x,y)\,dx\,dy.$$

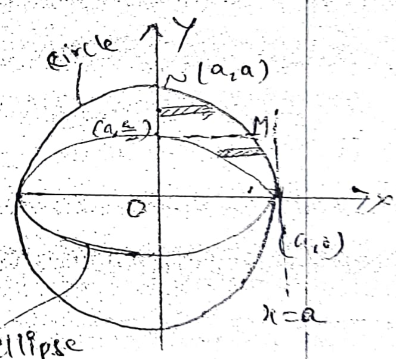


→ Change the order of integration in the double integral $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} V\,dx\,dy$, where V is a function of x and y.

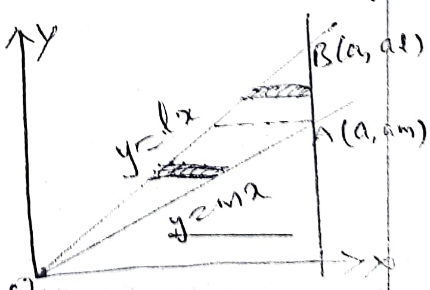


→ Change the order of integration in $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{ax}} \psi(x,y)\,dx\,dy$

→ Change the order of integration in $\int_0^a \int_{b\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} U\,dx\,dy$, where U is a function of x and y.



→ Change the order of integration in $\int_0^a \int_{mx}^{lx} V\,dx\,dy$ where V is a function of x and y.



→ Show that

$$\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} f(x,y)\,dx\,dy = \int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} f(x,y)\,dy\,dx$$

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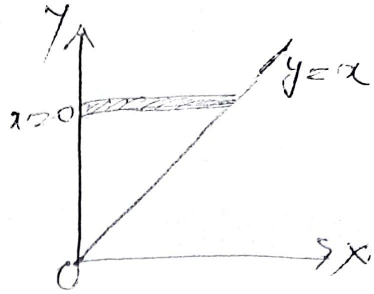
2006) Change the order of integration in the double integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ and hence find the value.

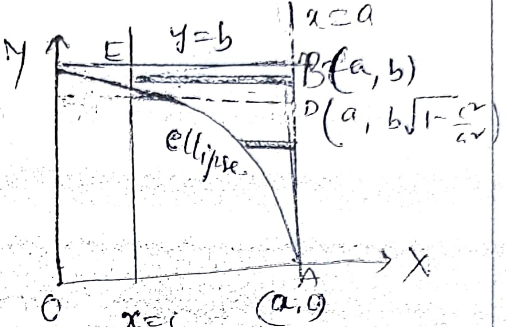
$\boxed{\text{Ans: } 1}$



→ Change the order of integration in

$\int_c^a \int_{\frac{b}{a}\sqrt{a^2-x^2}}^{b} v\,dx\,dy$

where c is less than a.



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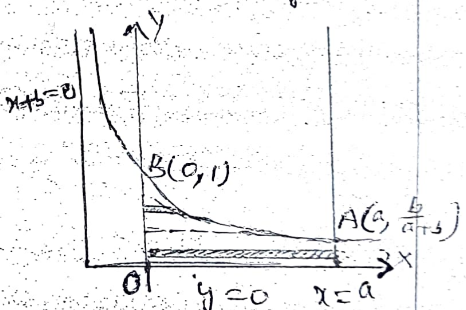
→ Change the order of integration in $\int_0^1 \int_0^{x(2-x)} f(x,y)\,dx\,dy$.

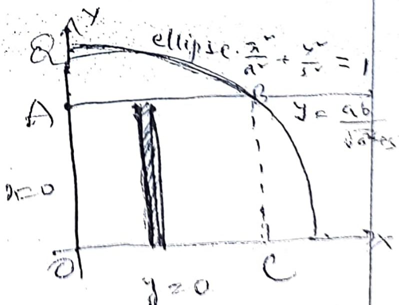
→ Change the order of integration in double integral

$I = \int_0^a \int_0^{b/b+x} v\,dx\,dy$

Ans: $I = \int_0^{b/a+b} \int_0^a v\,dy\,dx + \int_{b/a+b}^1 \int_0^{\frac{b(1-y)}{y}} v\,dy\,dx$.



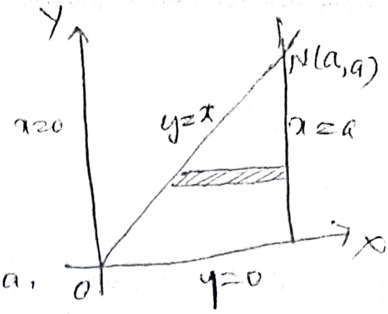
→ Change the order of integration in double integral $\int_0^{ab/\sqrt{a^2+b^2}} \int_a^{\frac{a}{b}\sqrt{b^2-y^2}}$.



→ Change the order of integration in double integral $\int_0^a \int_0^x \frac{\phi'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}}$ and hence find its value.

Sol^n:

The limits of integration are given by the straight lines $y=0$, $y=x$, $x=0$ and $x=a$.



Clearly the region of integration is ONM.

Take strips parallel to the x-axis.

The limits of x are from $x=y$ to $x=a$.

and the limits of y are from $y=0$ to $y=a$.

Hence we have

$$\int_0^a\int_0^x \frac{\phi'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}} = \int_0^a\int_y^a \frac{\phi'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}}$$

To find the value:

Let $x = a\sin^2\theta + y\cos^2\theta$.

$\Rightarrow dx = 2(a-y)\sin\theta\cos\theta\,d\theta$

Also $a-x = a - a\sin^2\theta - y\cos^2\theta$

$= a\cos^2\theta - y\cos^2\theta = (a-y)\cos^2\theta$.

and $x-y = a\sin^2\theta + y\cos^2\theta - y$

$= a\sin^2\theta - y\sin^2\theta = (a-y)\sin^2\theta$.

For limits of θ, when $x=y$, we have

$y = a\sin^2\theta + y\cos^2\theta$.

$\Rightarrow (y-a)\sin^2\theta = 0$

i.e. $\sin\theta = 0$

i.e. $\theta = 0$

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and when $x = a$, we have $a = a\sin^2\theta + y\cos^2\theta$

$\Rightarrow (a-y)\cos^2\theta = 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \pi/2$

Thus the limits of θ are from $\theta = 0$ to $\theta = \pi/2$

we ~~d~~

$\therefore$ The given integral $= \int_0^a \int_y^a \frac{\phi'(y)\,dy\,dx}{\sqrt{(a-x)(x-y)}}$

$= \int_0^a \int_0^{\pi/2} \frac{\phi'(y)\,2(a-y)}{(a-y)\sin\theta\cos\theta} \sin\theta\cos\theta\,dy\,d\theta$

$= 2\int_0^a \int_0^{\pi/2} \phi'(y)\,dy\,d\theta$

$= 2\int_0^a \phi'(y)\,[\theta]_0^{\pi/2}\,dy$

$= 2\frac{\pi}{2} \int_0^a \phi'(y)\,dy$

$= \pi\,[\phi(y)]_0^a$

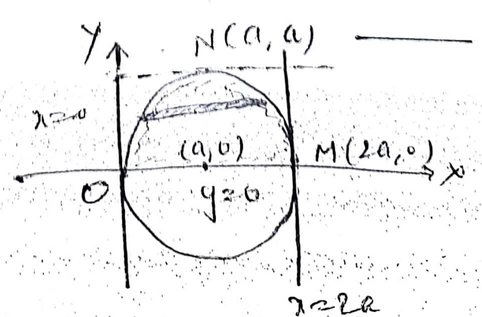
$= \pi\,(\phi(a) - \phi(0))$

$\rightarrow$ Change the order of integration in

$\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \frac{\phi'(y)(x^2+y^2)\,x\,dx\,dy}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$ and hence

evaluate it.

Sol^n:



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4(b) Evaluate $\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\,dx\,dy$.

Sol: Let $\sin x \sin y = \sin\theta$

Then $\sin x \cos y\,dy = \cos\theta\,d\theta$, keeping x constant

When $y=0$, $\sin\theta = 0 \Rightarrow \theta = 0$

& when $y = \pi/2$, $\sin\theta = \sin x \Rightarrow \theta = x$.

Hence θ varies from 0 to x.

$\therefore$ Given integral

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^{x} \sin x \sin^{-1}(\sin\theta)\,dx \cdot \frac{\cos\theta\,d\theta}{\sin x \cos y}$$

$$= \int_0^{\pi/2}\int_0^{x} \sin x\,\theta\,dx \cdot \frac{\cos\theta\,d\theta}{\sin x \cos y}$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta}{\cos y}\,dx\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\,dx\,d\theta}{\sqrt{1-\frac{\sin^2\theta}{\sin^2 x}}} \quad \left(\because \sin x\sin y = \sin\theta \Rightarrow \sin y = \frac{\sin\theta}{\sin x} \Rightarrow \cos y = \sqrt{1-\sin^2 y}\right)$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\sin x\,dx\,d\theta}{\sqrt{\sin^2 x - \sin^2\theta}}$$

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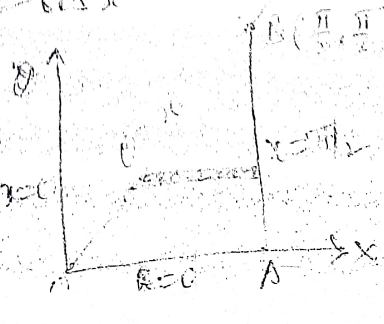
$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\sin x\,dx\,d\theta}{\sqrt{\cos^2\theta - \cos^2 x}}$$

Clearly it is convenient to integrate first w.r.t x. Therefore we shall change



the order of integration.

The limits of integration are given by the straight lines $\theta=0$, $\theta=x$ and $x=0$, $x=\pi/2$.

Clearly the area of integration is OABO.

Consider strips parallel to x-axis.

The limits of x are from θ to $\pi/2$

and limits of θ are from 0 to $\pi/2$.

Hence changing the order of integral, we have

$$\int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\sin x\,dx\,d\theta}{\sqrt{\cos^2\theta-\cos^2 x}} = \int_0^{\pi/2}\int_\theta^{\pi/2} \frac{\theta\cos\theta\sin x}{\sqrt{\cos^2\theta-\cos^2 x}}\,d\theta\,dx$$

$$= \int_0^{\pi/2} \theta\cos\theta\left[-\sin^{-1}\left(\frac{\cos x}{\cos\theta}\right)\right]_\theta^{\pi/2} d\theta$$

$$= \int_0^{\pi/2} \theta\cos\theta\left[-0+\frac{\pi}{2}\right] d\theta$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \theta\cos\theta\, d\theta$$

$$= \frac{\pi}{2}\left[(\theta\sin\theta)_0^{\pi/2} - \int_0^{\pi/2}\sin\theta\, d\theta\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}-0+\left[\cos\theta\right]_0^{\pi/2}\right]$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2}-1\right]$$

$$I = \int_\theta^{\pi/2} \frac{\sin x}{\sqrt{\cos^2\theta-\cos^2 x}}\,dx$$

let $\cos x = t \Rightarrow -\sin x\,dx = dt$

limits $x=\theta \Rightarrow t=\cos\theta$

$x=\pi/2 \Rightarrow t=0$

$$\Rightarrow I = \int_{\cos\theta}^{0} \frac{-dt}{\sqrt{\cos^2\theta-t^2}}$$

$$= \left[-\sin^{-1}\left[\frac{t}{\cos\theta}\right]\right]$$

$$= \left[-\sin^{-1}\left[\frac{\cos x}{\cos\theta}\right]\right]_\theta^{\pi/2}$$

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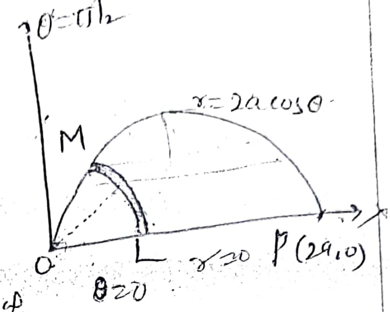
Change of order of integration of polar co-ordinates:

→ Change the order of integration in double integral

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} f(r,\theta)\, d\theta\, dr.$$



Sol: The limits of integration are given by $r=0$ (pole), $r=2a\cos\theta$ (a circle), $\theta=0$ (initial line) and $\theta=\pi/2$ ($\perp$lar to initial line at the pole)

Clearly the region of integration is OPMO.

To change the order of integration, we consider circular arc LM on which θ varies and r remains constant.

Now for limits of θ, the arc LM has its extremities on $\theta=0$ (initial line) and $r=2a\cos\theta$ (circle). $\theta=\cos^{-1}(r/2a)$

Also the limits of r are from $r=0$ to $r=2a$.

Hence

$$\int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)\, d\theta\, dr = \int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)\, dr\, d\theta.$$

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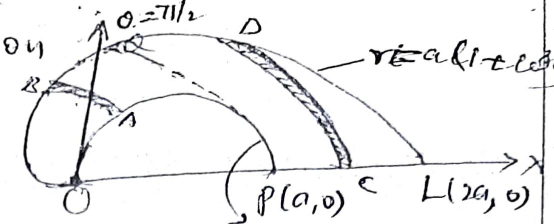
→ Change the order of integration in the system of integrals $\int_0^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\, r\, d\theta\, dr + \int_{\pi/2}^{\pi}\int_{0}^{a(1+\cos\theta)} f(r,\theta)\, r\, d\theta\, dr$

Soln: The limits of integration are given by from $r=a\cos\theta$ to $r=a(1+\cos\theta)$, $\theta=0$ to $\theta=\pi/2$ and $\theta=\pi/2$ to $\theta=\pi$.

i.e., the region of integration is bounded by upper half circle $r = a\cos\theta$, upper half Cardioid $r = a(1+\cos\theta)$ and the initial line.

Clearly OAPLQO is the region of integration.



Now to change the order of integration consider elementary circular arcs (lines AB and CD) about pole 'O' as a centre.

These arcs change their character at P.

Hence the region is divided into two parts namely OAPQBO and QPLQ.

In the region OAPQBO, the extremities of the arc AB lie on $r = a\cos\theta$ and $r = a(1+\cos\theta)$.

Hence θ varies from $\theta = \cos^{-1}(r/a)$ to $\theta = \cos^{-1}\left(\frac{r-a}{a}\right)$

Also r varies from $r=0$ to $r=a$ as $OP = a$.

In the region QPLQ, the extremities of the arc CD lie on $\theta = 0$ and cardioid $r = a(1+\cos\theta)$.

Hence θ varies from $\theta = 0$ to $\theta = \cos^{-1}\frac{r-a}{a}$,

and r varies from $r = a$ to $r = 2a$ as $OL = 2a$

Hence the given integral becomes

$$\int_0^a \int_{\cos^{-1}(r/a)}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\, r\, dr\, d\theta + \int_a^{2a} \int_0^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\, r\, dr\, d\theta$$

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Multiple Integrals and Their Applications

7.1. DOUBLE INTEGRALS

The definite integral $\int_a^b f(x)\,dx$ is defined as the limit of the sum

$$f(x_1)\delta x_1 + f(x_2)\delta x_2 + \ldots\ldots + f(x_n)\delta x_n,$$

where $n \to \infty$ and each of the lengths $\delta x_1, \delta x_2, \ldots\ldots$ tends to zero. A double integral is its counterpart in two dimensions.

Consider a function $f(x, y)$ of the independent variables x, y defined at each point in the finite region R of the xy-plane. Divide R into n elementary areas $\delta A_1, \delta A_2, \ldots\ldots, \delta A_n$. Let (x_r, y_r) be any point within the rth elementary area δA_r. Consider the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots\ldots + f(x_n, y_n)\,\delta A_n, \text{ i.e. } \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r$$

The limit of this sum, if it exists, as the number of sub-divisions increases indefinitely and area of each sub-division decreases to zero, is defined as the *double integral of $f(x, y)$ over the region R* and is written as $\iint_R f(x, y)\,dA$.

Thus
$$\iint_R f(x, y)\,dA = \underset{\substack{n \to \infty \\ \delta A \to 0}}{\text{Lt}} \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \ldots(1)$$

The utility of double integrals would be limited if it were required to take limit of sums to evaluate them. However, there is another method of evaluating double integrals by successive single integrations.

For purposes of evaluation, (1) is expressed as the repeated integral $\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y)\,dx\,dy$.

Its value is found as follows :

(i) *When y_1, y_2 are functions of x and x_1, x_2 are constants,* $f(x, y)$ is first integrated w.r.t. y keeping x fixed between limits y_1, y_2 and then the resulting expression is integrated w.r.t. x within the limits x_1, x_2 *i.e.*

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$$\boxed{I_1 = \int_{x_1}^{x_2} \boxed{\int_{y_1}^{y_2} f(x, y)dy} \; dx}$$

where integration is carried from the inner to the outer rectangle.

Fig. 7·1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

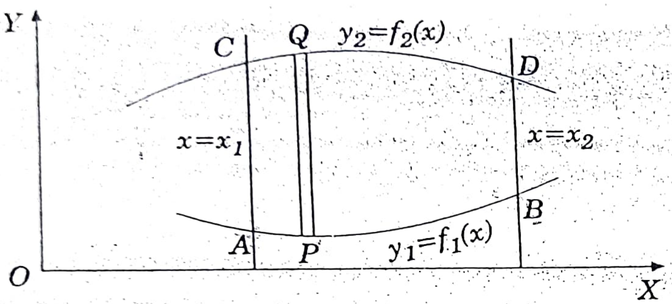

Fig. 7·1.

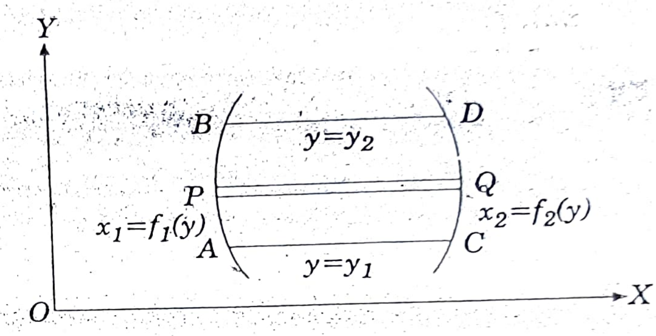

Fig. 7·2.

(ii) *When x_1, x_2 are functions of y and y_1, y_2 are constants,* $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)dx} \; dy}$$ which is geometrically illustrated by Fig. 7·2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(iii) *When both pairs of limits are constants,* the region of integration is the rectangle $ABDC$ (Fig. 7·3).

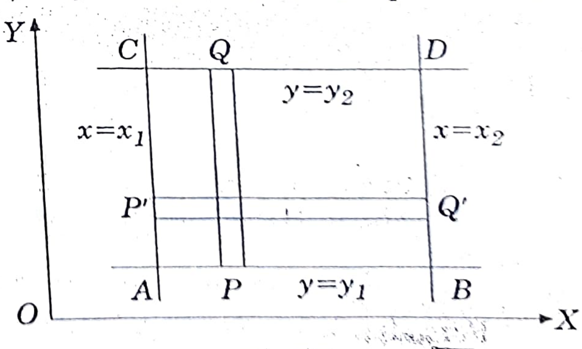

Fig. 7·3.

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7·1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)dxdy.$

Sol.
$$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^2)dy = \int_0^5 \left[x^3y + x\cdot\frac{y^3}{3}\right]_0^{x^2} dx = \int_0^5 \left[x^3\cdot x^2 + x\cdot\frac{x^6}{3}\right]dx$$
$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right)dx = \left|\frac{x^6}{6} + \frac{x^8}{24}\right|_0^5 = 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right] = 18880{\cdot}2 \text{ nearly.}$$

Example 7·2. *Evaluate* $\iint_A xy\,dx\,dy$, *where A is the domain bounded by x-axis, ordinate* $x = 2a$ *and the curve* $x^2 = 4ay$. *(Gulbarga, 1999 S)*

Sol. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Fig. 7·4 shows the domain A which is the area OML.

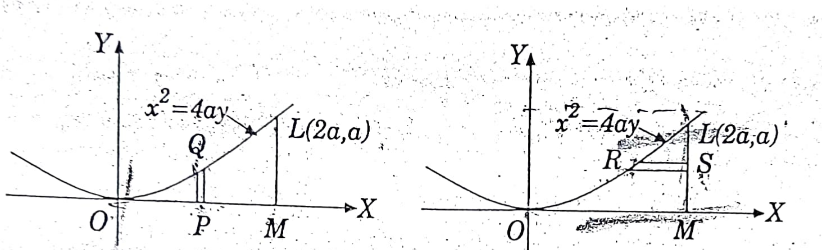

Fig. 7·4.

Integrating first over a vertical strip PQ, *i.e.* w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have
$$\iint_A xy\,dx\,dy = \int_0^{2a} dx \int_0^{x^2/4a} xy\,dy = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$
$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}$$

Otherwise integrating first over a horizontal strip RS, *i.e.* w.r.t. x from $R(x = 2\sqrt{ay})$ on the parabola to $S(x = 2a)$ and then w.r.t. y from $y = 0$ to $y = a$, we get
$$\iint_A xy\,dx\,dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\,dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$
$$= 2a\int_0^a (ay - y^2)dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

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7·2. CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limits of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7·3. *Change the order of integration in the integral*

$$I = \int_{-a}^{a} \int_{0}^{\sqrt{(a^2-y^2)}} f(x, y)\, dx\, dy.$$

Sol. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2 - y^2)}$ (*i.e.* to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7·5).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R $[y = -\sqrt{(a^2 - x^2)}]$ to S $[y = \sqrt{(a^2 - x^2)}]$. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

Thus $\quad I = \int_{0}^{a} dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)\, dy$

or $\quad = \int_{0}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)\, dy\, dx.$

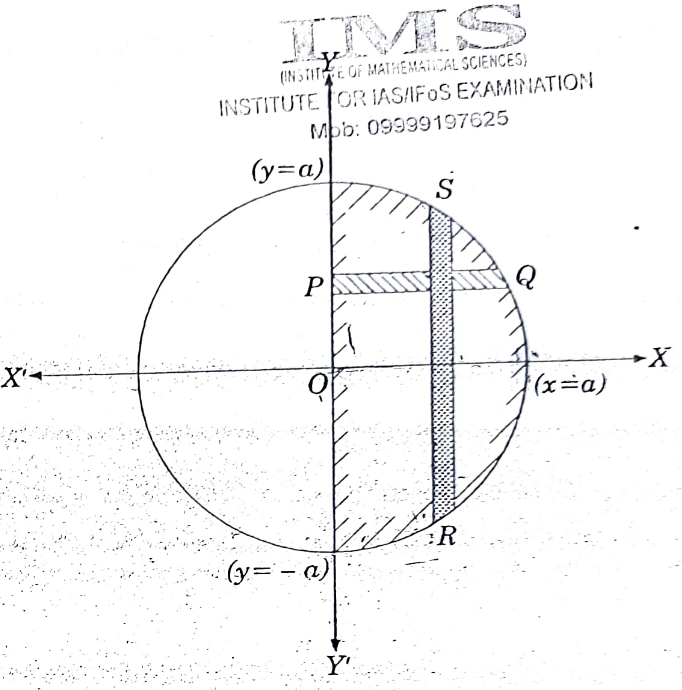

Fig. 7·5.

Example 7·4. *Change the order of integration in* $I = \int_{0}^{1} \int_{x^2}^{2-x} xy\, dx\, dy$ *and hence evaluate the same.* (*Andhra, 1999 ; Gauhati, 1999*)

Sol. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7·6.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line $AC(y = 1)$, *i.e.* the curvilinear triangle OAC and the triangle ABC.

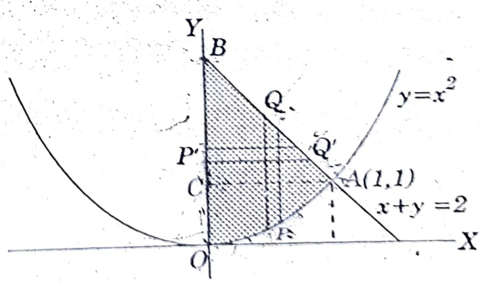

Fig. 7·6.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

$$I_1 = \int_{0}^{1} dy \int_{0}^{\sqrt{y}} xy\, dx.$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_{1}^{2} dy \int_{0}^{2-y} xy\, dx.$$

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Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left|\frac{x^2}{2}\cdot y\right|_0^{\sqrt{y}} + \int_1^2 dy \left|\frac{x^2}{2}\cdot y\right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

7·3. DOUBLE INTEGRALS IN POLAR CO-ORDINATES

To evaluate $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Fig. 7·7 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

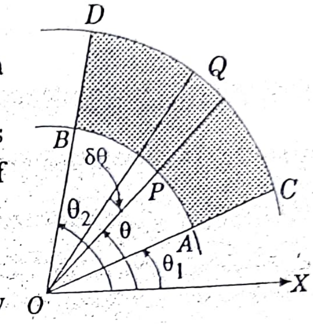

Fig. 7·7.

Then $\int_{r_1}^{r_2} f(r,\theta)\, dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7·5. *Calculate* $\iint r^3\, dr\, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$. *(J.N.T.U., 1999; Marathwada, 1998)*

Sol. Given circles $r = 2\sin\theta$ ...(i)

and $r = 4\sin\theta$ ...(ii)

are shown in Fig. 7·8. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^{\pi} d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 dr = \int_0^{\pi} d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^{\pi} \sin^4\theta\, d\theta = 60 \times 2\int_0^{\pi/2} \sin^4\theta\, d\theta$$

$$= 120 \times \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 22{\cdot}5\,\pi.$$

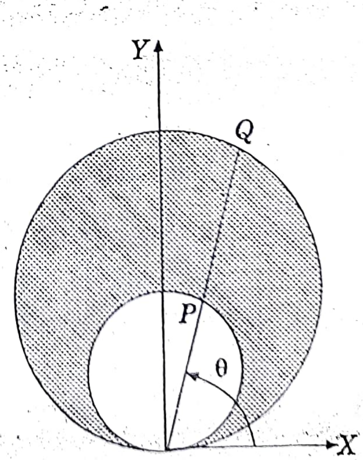

Fig. 7·8.

$+\frac{1}{4} - \frac{1}{6}\Big] = \frac{1}{2}\left(\frac{9}{4} - \frac{8+1}{6}\right) = \frac{1}{2}\left(\frac{9}{4} - \frac{3}{2}\right) = \frac{1}{2}\left(\frac{9-6}{4}\right) = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$

Problems 7.1

Evaluate the following integrals (1—7) :

1. $\int_1^2 \int_1^3 xy^2\, dx\, dy$ (*Madras, 1998 S*)

2. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dx\, dy.$ (*V.T.U., 2000*)

3. $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx.$ (*Osmania, 1999 S*)

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dy\, dx}{1 + x^2 + y^2}.$ (*Madras, 2000*)

5. $\iint xy\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. (*V.T.U., 2001 ; Madras, 2000*)

6. $\iint (x + y)^2\, dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$.

7. $\iint xy(x + y)\, dxdy$ over the area between $y = x^2$ and $y = x$.

Evaluate the following integrals by changing the order of integration (8—16) :

8. $\int_0^1 \int_0^{\sqrt{(1-x^2)}} y^2 dxdy.$ (*Pondicherry, 1998S*)

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9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y)\, dx\, dy.$ (*Anna, 2003 S ; V.T.U., 2003 ; Delhi, 2002*)

10. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$

11. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\, dy\, dx}{\sqrt{(x^2+y^2)}}$ (*Rohtak, 2003 ; I.S.M., 2001*)

Write 12. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log (x^2 + y^2)\, dx\, dy\ (a > 0).$ (*Bhopal, 1998*)

13. $\int_0^a \int_{x/a}^{\sqrt{(x/a)}} (x^2 + y^2)\, dx\, dy.$ (*Marathwada, 1998*)

14. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dy\, dx}{\sqrt{y^4 - a^2x^2}}$

15. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$ (*Madras, 2003 ; V.T.U., 2000*)

16. $\int_0^\infty \int_0^x xe^{-x^2/y}\, dy\, dx.$ (*V.T.U., 2004 ; Delhi, 2002*)

17. Sketch the region of integration of $\int_a^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r, \theta)\, rdr\, d\theta$ and change the order of integration.

18. Evaluate $\iint r \sin\theta\, dr\, d\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

19. Show that $\iint_R r^2 \sin\theta\, dr\, d\theta = 2a^2/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line.

20. Evaluate $\iint \frac{rdrd\theta}{\sqrt{a^2 + r^2}}$ over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

7.4. AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian co-ordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ (Fig. 7.9).

4

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \ \Sigma \, \delta x \, \delta y$.

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \text{ area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

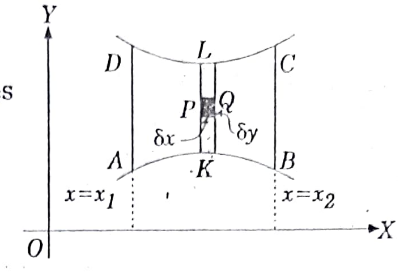

Fig. 7·9.

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \cdot \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

$$= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx \, dy$$

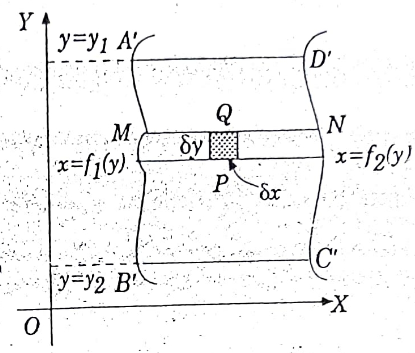

Fig. 7·10.

Similarly dividing the area $A'B'C'D'$ (Fig. 7·10) into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx \, dy$$

(2) *Polar co-ordinates.*

Consider an area A enclosed by a curve whose equation is in polar co-ordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7·11).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \cdot PS = r\delta\theta \cdot \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

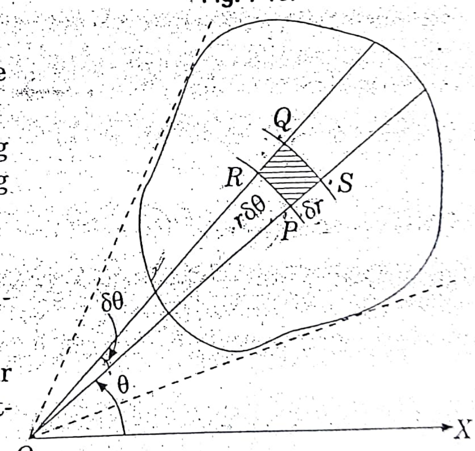

Fig. 7·11.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

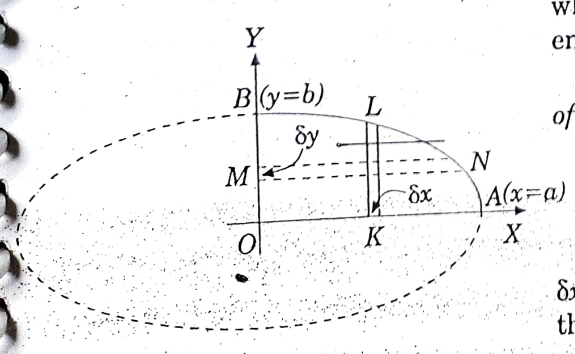

Fig. 7·12.

Example 7·6. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{(V.T.U., 2001 ; Osmania, 2000 S)}$$

Sol. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L \, [y = b\sqrt{(1 - x^2/a^2)}]$ and then x varies from 0 to a (Fig. 7·12).

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$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\, [y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\, dx = \pi ab/4.$$

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x=0)$ to $N[x = a\sqrt{(1-y^2/b^2)}]$ and then y varies from 0 to b.

$$\therefore \text{ required area} = \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\, [x]_0^{a\sqrt{(1-y^2/b^2)}} = \frac{a}{b}\int_0^b \sqrt{(b^2-y^2)}dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7·7. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(Rohtak, 2003)

Sol. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at $O(0, 0)$ and $A(4a, 4a)$. As such for the shaded area between these parabolas (Fig. 7·13) x varies from 0 to $4a$ and y varies from P to Q *i.e.* from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a} \int_{x^2/4a}^{2\sqrt{(ax)}} dy\, dx$$

$$= \int_0^{4a} (2\sqrt{(ax)} - x^2/4a)\, dx$$

$$= \left| 2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

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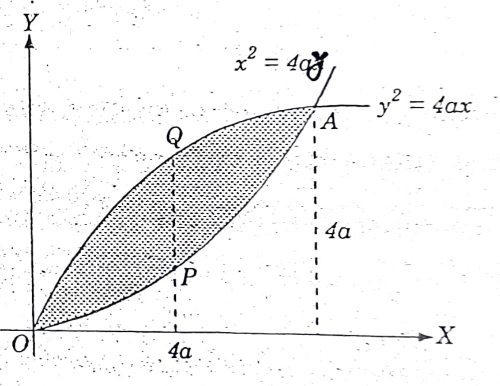

Fig. 7·13.

Example 7·8. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

(Bhopal, 1998)

Sol. The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (Fig. 7·14).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$\therefore$ required area

$$= 2\int_0^{\pi/2} \int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\, dr\, d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta = a^2\int_0^{\pi/2} (2\times\cos^2\theta)\, d\theta = 5\pi a^2/4.$$

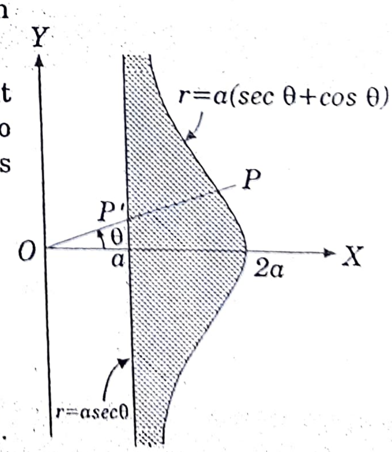

Fig. 7·14.

Problems 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$. *(Andhra, 1999)*
2. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2 + 2)$ and $4y = x^2$. *(Bhopal, 1998)*
3. Find by double integration, the area of the lemniscate $r^2 = a^2 \cos 2\theta$. *(Madras, 2000 S)*
4. Find, by double integration, the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$. *(Assam, 1998)*

7.5. TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots\ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum

$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\, \delta V_r.$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of* $f(x, y, z)$ *over the region* V and is denoted by

$$\iiint f(x, y, z)\, dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz.$$

If x_1, x_2 are constants ; y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or functions of x and y, then this integral is evaluated as follows :

First $f(x, y, z)$ is integrated w.r.t. z between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated w.r.t. y between the limits y_1 and y_2 keeping x constant. The result just obtained is finally integrated w.r.t. x from x_1 to x_2.

$$\text{Thus} \quad I = \boxed{\int_{x_1}^{x_2} \boxed{\int_{y_1(x)}^{y_2(x)} \boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)dz}\, dy}\, dx}$$

where the integration is carried out from the innermost rectangle to the outermost rectangle.

The order of integration may be different for different types of limits.

Example 7·9. *Evaluate* $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz.$

Sol. Integrating first w.r.t. y keeping x and z constant, we have

$$I = \int_{-1}^{1} \int_0^z \left| xy + \frac{y^2}{2} + yz \right|_{x-z}^{x+z} dx\, dz = \int_{-1}^{1} \int_0^z \left[(x+z)(2z) + \frac{1}{2} 4xz \right] dx\, dz$$

$$= 2\int_{-1}^{1} \left| \frac{x^2 z}{2} + z^2 x + \frac{x^2}{2} z \right|_0^z dz$$

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$$= 2\int_{-1}^{1}\left(\frac{z^3}{2}+z^3+\frac{z^3}{2}\right)dz = 4\left|\frac{z^4}{4}\right|_{-1}^{1} = 0.$$

Example 7·10. *Evaluate* $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}} xyz\, dxdydz.$ *(V.T.U., 2003 S)*

Sol. We have

$$I = \int_0^1 x\left[\int_0^{\sqrt{(1-x^2)}} y\left\{\int_0^{\sqrt{(1-x^2-y^2)}} z\, dz\right\} dy\right] dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y \cdot \left|\frac{z^2}{2}\right|_0^{\sqrt{(1-x^2-y^2)}} dy\right\} dx$$

$$= \int_0^1 x\left\{\int_0^{\sqrt{(1-x^2)}} y \cdot \frac{1}{2}(1-x^2-y^2)\, dy\right\} dx = \frac{1}{2}\int_0^1 x\left|(1-x^2)\frac{y^2}{2}-\frac{y^4}{4}\right|_0^{\sqrt{(1-x^2)}} dx$$

$$= \frac{1}{8}\int_0^1 [(1-x^2)^2 \cdot 2x - (1-x^2)^4 \cdot x]\, dx = \frac{1}{8}\int_0^1 (x-2x^3+x^5)\, dx$$

$$= \frac{1}{8}\left|\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right|_0^1 = \frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{48}.$$

Problems 7.3

Evaluate the following integrals :

1. $\int_0^1\int_0^2\int_1^2 x^2yz\, dx\, dy\, dz.$ *(Kottayam, 1996)*
2. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\, dx\, dy\, dz$ *(V.T.U., 2000)*
3. $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{(4z-x^2)}} dy\, dx\, dz$ *(Gauhati, 1999)*
4. $\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$ *(V.T.U., 2004)*
5. $\int_1^e\int_1^{\log y}\int_1^{e^x} \log z\, dz\, dx\, dy.$
6. $\int_0^{\pi/2}\int_0^{a\sin\theta}\int_0^{\frac{a^2-r^2}{a}} r dz\, dr\, d\theta.$ *(Gulbarga, 1999)*

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7.6 VOLUMES OF SOLIDS

(1) Volumes as double integrals. Consider a surface $z = f(x, y)$. Let the orthogonal projection on XY-plane of its portion S' be the area S.

Divide S into elementary rectangles of area $\delta x\, \delta y$ by drawing lines parallel to X and Y-axes. With each of these rectangles as base, erect a prism having its length parallel to OZ.

$\therefore$ Volume of this prism between S and the given surface $z = f(x, y)$ is $z\, \delta x\, \delta y$.

Hence the volume of the solid cylinder on S as base, bounded by the given surface with generators parallel to the Z-axis.

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma\Sigma\, z\,\delta x\,\delta y$$

$$= \iint z\,dx\,dy \quad \text{or} \quad \iint f(x, y)\,dx\,dy$$

where the integration is carried over the area S.

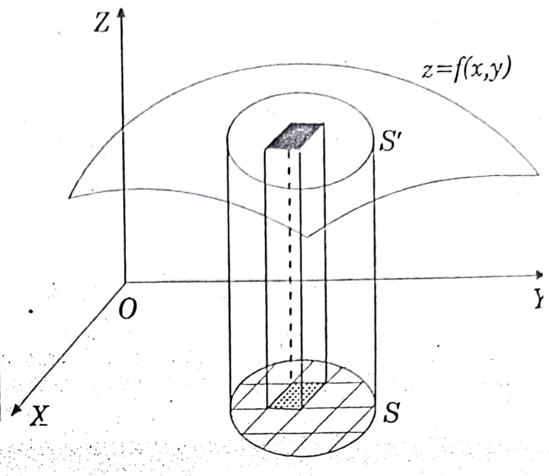

Fig. 7·15.

> **Obs.** While using polar co-ordinates, divide S into elements of area $r\delta\theta\,\delta r$. $\therefore$ replacing $dxdy$ by $rd\theta dr$, we get the required volume $= \iint zr\,d\theta\,dr$.

Example 7·11. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.* (*Madras, 2000 S*)

Sol. From Fig. 7·16, it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the XY-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{(4 - y^2)}$ and y varies from -2 to 2.

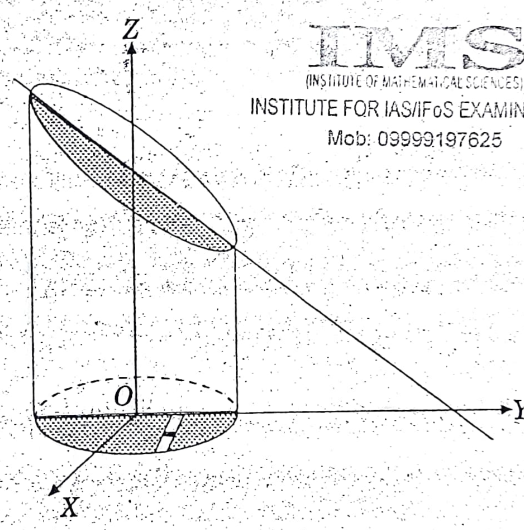

Fig. 7·16.

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$\therefore$ Required volume

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} z\,dx\,dy$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{(4-y^2)}} (4 - y)\,dxdy$$

$$= 2\int_{-2}^{2} (4 - y)\,[x]_0^{\sqrt{(4-y^2)}}\,dy$$

$$= 2\int_{-2}^{2} (4 - y)\sqrt{(4 - y^2)}dy$$

$$= 2\int_{-2}^{2} 4\sqrt{(4 - y^2)}dy - 2\int_{-2}^{2} y\sqrt{(4 - y^2)}dy$$

$$= 8\int_{-2}^{2} \sqrt{(4 - y^2)}dy$$

[The second term vanishes as the integrand is an odd function.]

$$= 8\left|\frac{y\sqrt{(4 - y^2)}}{2} + \frac{4}{2}\sin^{-1}\frac{y}{2}\right|_{-2}^{2} = 16\pi.$$

(2) Volume as triple integral

Divide the given solid by planes parallel to the co-ordinate planes into rectangular parallelopipeds of volume $\delta x\,\delta y\,\delta z$ (Fig. 7·17).

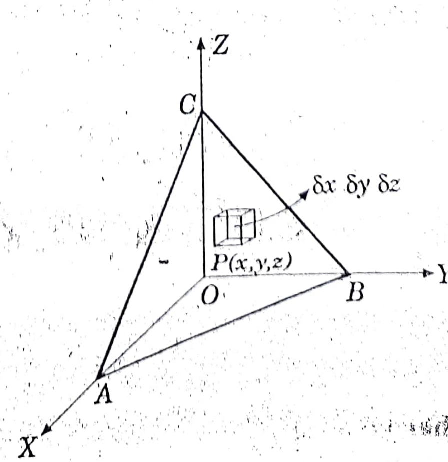

Fig. 7·17.

$\therefore$ the total volume $= \underset{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}}{\text{Lt}} \sum\sum\sum \delta x\, \delta y\, \delta z = \iiint dx\, dy\, dz$

with appropriate limits of integration.

Example 7·12. *Find the volume of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(J.N.T.U., 1998 ; Madras, 1996)

Sol. Let $OABC$ be the positive octant of the given ellipsoid which is bounded by the planes OAB $(z = 0)$, $OBC(x = 0)$, $OCA(y = 0)$ and the surface ABC, *i.e.*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Divide this region R into rectangular parallelopipeds of volume $\delta x \delta y \delta z$. Consider such an element at $P(x, y, z)$. (Fig. 7·18)

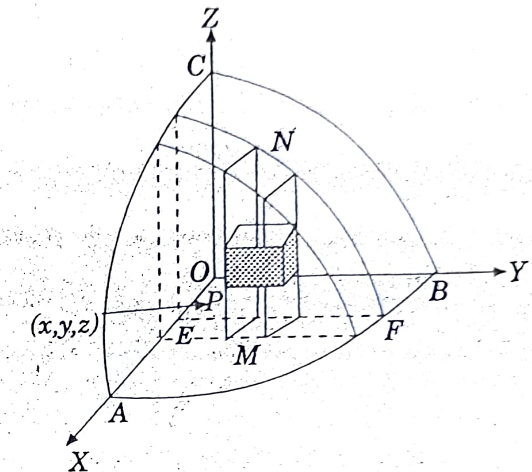

Fig. 7·18.

$\therefore$ the required volume $= 8 \iiint_R dx\, dy\, dz$.

In this region R,

(*i*) z varies from 0 to MN where

$$MN = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}.$$

(*ii*) y varies from 0 to EF, where $EF = b\sqrt{(1 - x^2/a^2)}$ from the equation of the ellipse OAB, *i.e.* $x^2/a^2 + y^2/b^2 = 1$.

(*iii*) x varies from 0 to $OA = a$.

Hence the volume of the whole ellipsoid

$$= 8 \int_0^a \int_0^{b\sqrt{(1-x^2/a^2)}} \int_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}} dx\, dy\, dz$$

$$= 8 \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy\, \left| z \right|_0^{c\sqrt{(1-x^2/a^2-y^2/b^2)}}$$

$$= 8c \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} \sqrt{(1 - x^2/a^2 - y^2/b^2)}\, dy$$

$$= \frac{8c}{b} \int_0^a dx \int_0^{\rho} \sqrt{(\rho^2 - y^2)}\, dy \quad \text{when } \rho = b\sqrt{(1 - x^2/a^2)}.$$

$$= \frac{8c}{b} \int_0^a dx \left[\frac{y\sqrt{(\rho^2 - y^2)}}{2} + \frac{\rho^2}{2} \sin^{-1} \frac{y}{\rho} \right]_0^{\rho} = \frac{8c}{b} \int_0^a \frac{b^2}{2} \left(1 - \frac{x^2}{a^2} \right) \frac{\pi}{2}\, dx$$

$$= 2\pi\, bc \int_0^a \left(1 - \frac{x^2}{a^2} \right) dx = 2\pi\, bc \left| x - \frac{x^3}{3a^2} \right|_0^a = \frac{4\pi abc}{3}.$$

Otherwise. See Problem 22 page 276.

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(3) Volumes of solids of revolution

Consider an elementary area $\delta x \delta y$ at the point $P(x, y)$ of a plane area A. (Fig. 7·19)

As this elementary area revolves about x-axis, we get a ring of volume

$$= \pi[(y + \delta y)^2 - y^2]\delta x = 2\pi y\, \delta x\, \delta y,$$

nearly to the first powers of δy.

Hence the total volume of the solid formed by the revolution of the area A about x-axis

$$= \iint_A 2\pi y\, dx dy.$$

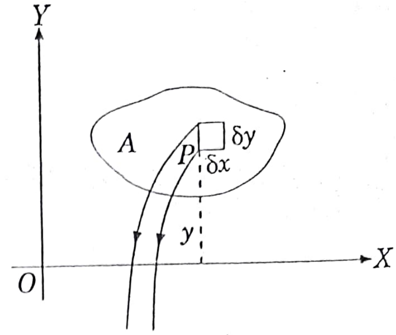

Fig. 7·19.

In polar co-ordinates, the above formula for the volume becomes

$$\iint_A 2\pi r \sin\theta \cdot r d\theta dr, \text{ i.e. } \iint_A 2\pi r^2 \sin\theta\, d\theta\, dr$$

Similarly, the volume of the solid formed by the revolution of the area A about y-axis $= \iint_A 2\pi x\, dx\, dy$.

Example 7·13. *Calculate by double integration, the volume generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about its axis.*

Sol. Reqd. volume

$$= \int_0^{\pi} \int_0^{a(1-\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi \int_0^{\pi} \left|\frac{r^3}{3}\right|_0^{a(1-\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1 - \cos\theta)^3 \cdot \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3} \left|\frac{(1-\cos\theta)^4}{4}\right|_0^{\pi} = \frac{8\pi a^3}{3}$$

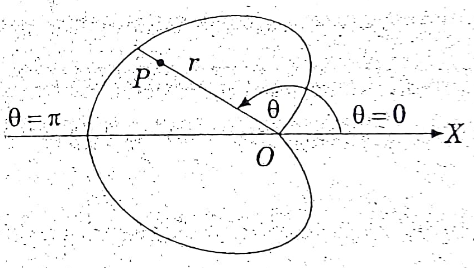

Fig. 7·20.

7·7. CHANGE OF VARIABLES

An appropriate choice of co-ordinates quite often facilitates the evaluation of a double or a triple integral. By changing the variables, a given integral can be transformed into a simpler integral involving the new variables.

(1) In a double integral, let the variables x, y be changed to the new variables u, v by the transformation

$$x = \phi(u, v), y = \psi(u, v)$$

where $\phi(u, v)$ and $\psi(u, v)$ are continuous and have continuous first order derivatives in some region R'_{uv} in the uv-plane which corresponds to the region R_{xy} in the xy-plane. Then

$$\iint_{R_{xy}} f(x, y)\, dxdy = \iint_{R'_{uv}} f[\phi(u, v), \psi(u, v)]\, |J|\, dudv \quad ...(1)$$

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where $$J = \frac{\partial(x, y)}{\partial(u, v)} (\neq 0)$$

is the *Jacobian of transformation** from (x, y) to (u, v) co-ordinates.

* See footnote page 201.

(2) **For triple integrals,** the formula corresponding to (1) is

$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{uvw}} f[x(u, v, w), y(u, v, w), z(u, v, w)]\, |J|\, dudvdw$$

where
$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} \ (\neq 0)$$
is the *Jacobian of transformation* from (x, y, z) to (u, v, w) co-ordinates

Particular cases :

(*i*) ***To change cartesian co-ordinates*** (x, y) ***to polar co-ordinates*** (r, θ), *we have* $x = r\cos\theta, y = r\sin\theta$ *and*

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r$$

[Ex. 5·20, p. 202]

$$\therefore \quad \iint_{R_{xy}} f(x, y)\, dx\, dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\cdot r\, dr\, d\theta.$$

(*ii*) ***To change rectangular co-ordinates*** (x, y, z) ***to cylindrical co-ordinates*** (ρ, ϕ, z) — *Fig. 8·26, we have*

$$x = \rho\cos\phi, \ y = \rho\sin\phi, \ z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho$$

[Ex. 5·20]

Then
$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{\rho\phi z}} f(\rho\cos\phi, \rho\sin\phi, z)\cdot \rho d\rho d\phi dz.$$

(*iii*) ***To change rectangular co-ordinates*** (x, y, z) ***to spherical polar co-ordinates*** (r, θ, ϕ) — *Fig. 8·27, we have*

$$x = r\sin\theta\cos\phi, \ y = r\sin\theta\sin\phi, \ z = r\cos\theta$$

and
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$

[Ex. 5·20]

Then
$$\iiint_{R_{xyz}} f(x, y, z)\, dxdydz = \iiint_{R'_{r\theta\phi}} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\cdot r^2\sin\theta\, drd\theta d\phi.$$

Example 7·14. *Evaluate* $\iint_R (x+y)^2\, dxdy$, *where R is the parallelogram in the xy-plane with vertices (1, 0), (3, 1), (2, 2), (0, 1) using the transformation* $u = x + y$ *and* $v = x - 2y$.

(*Andhra, 1999*)

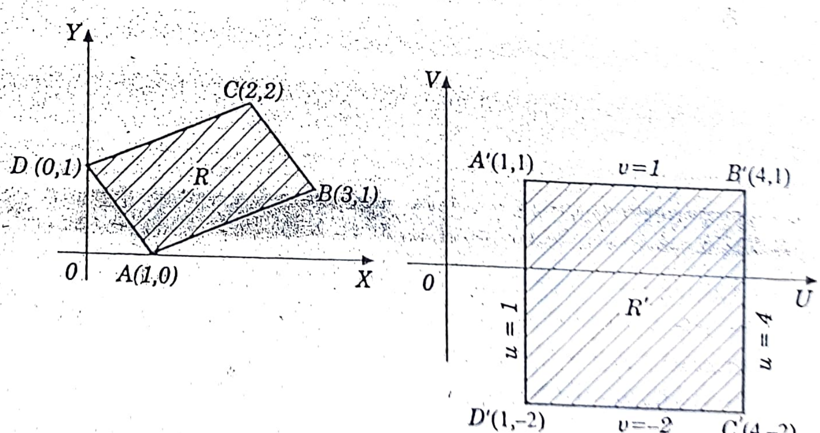

Fig. 7·21.

Sol. The region R, *i.e.* parallelogram $ABCD$ in the xy-plane becomes the region R', *i.e.*, rectangle $A'B'C'D'$ in the uv-plane as shown in Fig. 7·21, by taking

$$u = x + y \quad \text{and} \quad v = x - 2y \qquad \ldots(i)$$

From (*i*), we have $x = \frac{1}{3}(2u + v), y = \frac{1}{3}(u - v)$

$$\therefore \quad \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = -\frac{1}{3}$$

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Hence the given integral $= \iint_{R'} u^2 \, |J| \, du dv$

$$= \int_1^4 \int_{-2}^{1} u^2 \cdot \frac{1}{3} \cdot du \, dv = \frac{1}{3} \left| \frac{u^3}{3} \right|_1^4 \cdot |v|_{-2}^{1} = 21.$$

Example 7·15. *Evaluate* $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dxdy$ *by changing to polar coordinates.*

(*Anna, 2003 ; Assam, 1999*)

Hence show that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$ (*Madras, 2003 ; U.P.T.U., 2003 ; J.N.T.U., 2000*)

Sol. The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$. Hence,

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dxdy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} \, r dr d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left\{ \int_0^\infty e^{-r^2} (-2r) \, dr \right\} d\theta = -\frac{1}{2} \int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta = \pi/4. \quad \text{...}(i)$$

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Also $$I = \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy = \left\{ \int_0^\infty e^{-x^2} dx \right\}^2 \quad \text{...}(ii)$$

Thus, from (*i*) and (*ii*), we have $\int_0^\infty e^{-x^2} = \sqrt{\pi}/2.$ ...(*iii*)

Example 7·16. *Find the volume bounded by the paraboloid* $x^2 + y^2 = az$, *the cylinder* $x^2 + y^2 = 2ay$ *and the plane* $z = 0$.

Sol. The required volume is found by integrating $z = (x^2 + y^2)/a$ over the circle $x^2 + y^2 = 2ay$.

Changing to polar co-ordinates in the xy-plane, we have $x = r \cos \theta, y = r \sin \theta$ so that $z = r^2/a$ and the polar equation of the circle is $r = 2a \sin \theta$.

To cover this circle, r varies from 0 to $2a \sin \theta$ and θ varies from 0 to π. (Fig. 7·22)

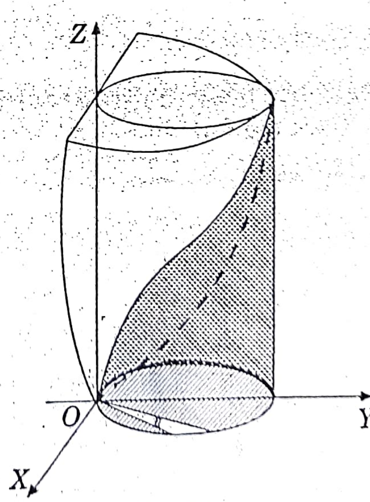

Fig. 7·22.

Hence the required volume

$$= \int_0^\pi \int_0^{2a \sin\theta} z \cdot r \, d\theta \, dr = \frac{1}{a} \int_0^\pi d\theta \int_0^{2a \sin \theta} r^3 \, dr$$

$$= \frac{1}{a} \int_0^\pi d\theta \left| \frac{r^4}{4} \right|_0^{2a \sin\theta} = 4a^3 \int_0^\pi \sin^4 \theta d\theta = \frac{3\pi a^3}{2}.$$

Example 7·17. *Find, by triple integration, the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(*V.T.U., 2000 S*)

Sol. Changing to polar spherical coordinates by putting

$$x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi,\ z = r\cos\theta$$

we have $dx\,dy\,dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also the volume of the sphere is 8 times the volume of its portion in the positive octant for which r varies from 0 to a, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Volume of the sphere

$$= 8\int_0^a \int_0^{\pi/2} \int_0^{\pi/2} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 8\int_0^a r^2\, dr \cdot \int_0^{\pi/2} \sin\theta\, d\theta \cdot \int_0^{\pi/2} d\phi = 8 \cdot \left|\frac{r^3}{3}\right|_0^a \cdot \left| -\cos\theta \right|_0^{\pi/2} \cdot \frac{\pi}{2}$$

$$= 4\pi \cdot \frac{a^3}{3} \cdot (-0+1) = \frac{4}{3}\pi a^3.$$

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Example 7·18. *Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.* (*Rohtak, 2003*)

Sol. The required volume is easily found by changing to cylindrical co-ordinates (ρ, ϕ, z). We therefore, have

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

and
$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a\sin\phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7·23 for which z varies from 0 to $\sqrt{(a^2-\rho^2)}$, ρ varies from 0 to $a\sin\phi$ and ϕ varies from 0 to π.

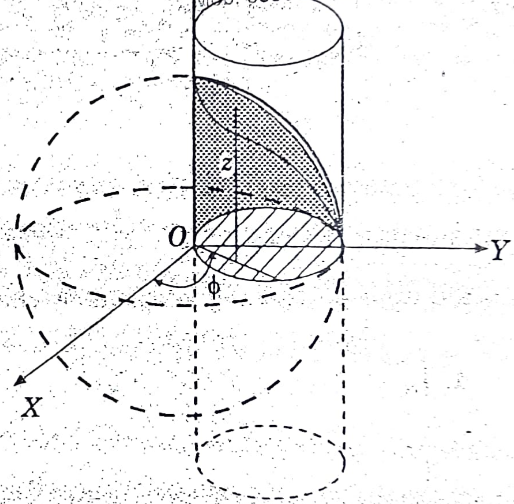

Fig. 7·23.

Hence the required volume

$$= 2\int_0^{\pi} \int_0^{a\sin\phi} \int_0^{\sqrt{(a^2-\rho^2)}} \rho\, dz\, d\rho\, d\phi$$

$$= 2\int_0^{\pi} \int_0^{a\sin\phi} \rho\sqrt{(a^2-\rho^2)}\, d\rho\, d\phi = 2\int_0^{\pi} \left| -\frac{1}{3}(a^2-\rho^2)^{3/2} \right|_0^{a\sin\phi} d\phi$$

$$= \frac{2a^3}{3}\int_0^{\pi} (1-\cos^3\phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7·19. *Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_{\sqrt{(x^2+y^2)}}^1 \frac{dz\, dy\, dx}{\sqrt{(x^2+y^2+z^2)}}$

Sol. We change to spherical polar co-ordinates (r, θ, ϕ), so that

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

and
$$J = r^2\sin\theta,\ x^2+y^2+z^2 = r^2.$$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ unded by the plane $z = 1$ in the positive octant (Fig. 7·24). Hence θ varies from 0 to $\pi/4$, r varies m 0 to $\sec\theta$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Given integral becomes

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta}\frac{1}{r}\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}d\phi\int_0^{\pi/4}\left|\frac{r^2}{2}\right|_0^{\sec\theta}\sin\theta\,d\theta$$

$$=\frac{\pi}{2}\int_0^{\pi/4}\frac{\sec^2\theta}{2}\sin\theta\,d\theta$$

$$=\frac{\pi}{4}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta$$

$$=\frac{\pi}{4}\left|\sec\theta\right|_0^{\pi/4}=\frac{(\sqrt{2}-1)\pi}{4}.$$

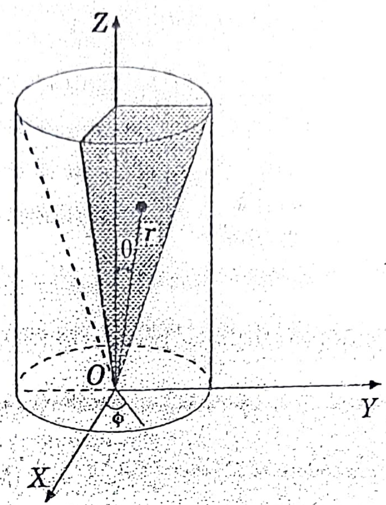

Fig. 7·24.

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Problems 7.4

valuate the following integrals by changing to polar co-ordinates :

1. $\int_0^a\int_0^{\sqrt{(a^2-y^2)}}(x^2+y^2)\,dy\,dx.$

(Andhra, 1998 ; Delhi, 1997)

2. $\int_0^a\int_y^a\frac{x^2\,dx\,dy}{\sqrt{(x^2+y^2)}}.$

(Marathwada, 1998)

3. $\int_0^{4a}\int_{y^2/4a}^{y}\frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

4. $\iint xy\,(x^2+y^2)^{n/2}\,dx\,dy$ over the positive quadrant of $x^2+y^2=4$, supposing $n+3>0$.

5. Transform the following to cartesian form and hence evaluate $\int_0^{\pi}\int_0^{a}r^3\sin\theta\cos\theta\,dr\,d\theta$.

6. By using the transformation $x+y=u, y=uv$, show that $\int_0^1\int_0^{1-x}e^{y/(x+y)}\,dy\,dx=\frac{1}{2}(e-1)$.

Evaluate the following integrals by changing to spherical co-ordinates :

7. $\int_0^1\int_0^{\sqrt{(1-x^2)}}\int_0^{\sqrt{(1-x^2-y^2)}}\frac{dx\,dy\,dz}{\sqrt{(1-x^2-y^2-z^2)}}$

(Madras, 1998 ; Marathwada, 1998 S ; Punjab, 1997)

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8. $\iiint z^2\,dx\,dy\,dz$, taken over the volume bounded by the surfaces $x^2+y^2=a^2$, $x^2+y^2=z$ and $z=0$. (J.N.T.U., 1999)

9. Find the volume bounded by the xy-plane, the cylinder $x^2+y^2=1$ and the plane $x+y+z=3$. (I.S.M., 2001)

10. Find the volume bounded by the xy-plane, the paraboloid $2z=x^2+y^2$ and the cylinder $x^2+y^2=4$.

R-2004 11. Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.

12. Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

13. Find the volume enclosed by the cylinders $x^2+y^2=2ax$ and $z^2=2ax$. (Delhi, 1997)

14. Find the volume of the cylinder $x^2+y^2-2ax=0$, intercepted between the paraboloid $x^2+y^2=2az$ and the xy-plane.

1194-17 15. Find the volume bounded by the cylinder $x^2+y^2=4$ and the hyperboloid $x^2+y^2-z^2=1$.

16. Find the volume of the region bounded by $z=x^2+y^2$, $z=0$, $x=-a$, $x=a$ and $y=-a$, $y=a$. (Kanpur, 1996)

2008 P-I 17. Prove, by using a double integral that the volume generated by the revolution of the cardioid $r=a(1+\cos\theta)$ about its axis is $8\pi a^3/3$. (V.T.U., 2000)

18. Using triple integration, find the volume of the sphere $x^2+y^2+z^2=a^2$. (V.T.U., 2003)

19. Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$. [See Fig. 7·28] (Bangalore, 1998 S)

20. Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. (Burdwan, 2003)

21. *Find the volume of the solid surrounded by the surface $(x/a)^{2/3}+(y/b)^{2/3}+(z/c)^{2/3}=1$.*

[Sol. Changing the variables, x, y, z to X, Y, Z where

$$(x/a)^{1/3}=X,\ (y/b)^{1/3}=Y,\ (z/c)^{1/3}=Z$$

i.e. $$x=aX^3,\ y=bY^3,\ z=cZ^3 \text{ so that}$$

$$J=\partial(x,y,z)/\partial(X,Y,Z)=27\,abc\,X^2Y^2Z^2.$$

$$\therefore \text{Reqd. volume}=\iiint dx\,dy\,dz=27\,abc\iiint X^2Y^2Z^2\,dX\,dY\,dZ$$

taken throughout the sphere $X^2+Y^2+Z^2=1$. ... (i)

Now change X, Y, Z to spherical polar coordinates r, θ, ϕ so that $X=r\sin\theta\cos\phi$, $Y=r\sin\theta\sin\phi$, $Z=r\cos\theta$, and $\partial(X,Y,Z)/\partial(r,\theta,\phi)=r^2\sin\theta$. To describe the positive octant of the sphere (i), r varies from 0 to 1, θ from 0 to $\pi/2$ and ϕ from 0 to $\pi/2$.

$$\therefore \text{Reqd. volume}=27\,abc\times 8\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} r^2\sin^2\theta\cos^2\phi\times r^2\sin^2\theta\sin^2\phi\cdot r^2\cos^2\theta\cdot r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=216\,abc\int_0^1 r^8\,dr\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\ =4\pi\,abc/35]$$

22. Work out example 7·12 by changing the variables.

7·8. AREA OF A CURVED SURFACE

Consider a point P of the surface $S: z=f(x,y)$.

Let its projection on the xy-plane be the region A. Divide it into area elements by drawing lines parallel to the axes of X and Y. (Fig. 7·25).

On the element $\delta x \delta y$ as base, erect a cylinder having generators parallel to OZ and meeting the surface S in an element of area δS.

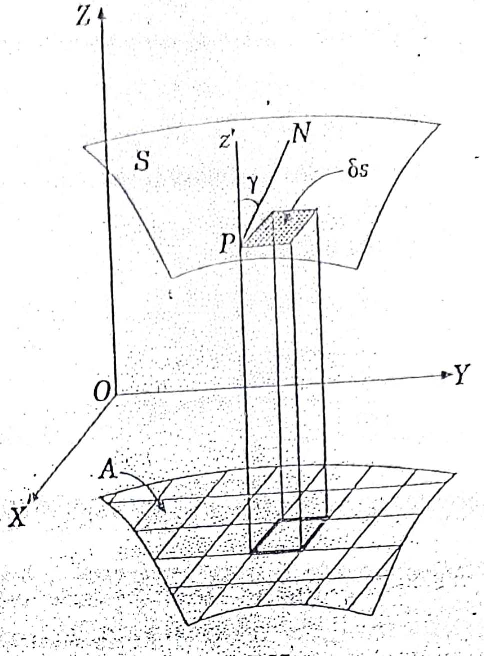

Fig. 7·25.

As $\delta x \delta y$ is the projection of δS on the xy plane,

$\therefore\ \delta x\ \delta y = \delta S \cdot \cos\gamma$, where γ is the angle between the xy-plane and the tangent plane to S at P, *i.e.* it is the angle between the Z-axis and the normal to S at $P (= \angle Z'PN)$.

Now since the direction cosines of the normal to the surface $F(x, y, z) = 0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}.$$

$\therefore$ the direction cosines of the normal to $S[F = f(x, y) - z]$ are proportional to $-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$ and those of the z-axis are 0, 0, 1.

Hence $\cos\gamma = \dfrac{1}{\sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}}$ $\quad\therefore\quad \delta S = \dfrac{\delta x\ \delta y}{\cos\gamma} = \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\ \delta x\ \delta y$

$$\text{Hence } S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right]}\, dx\, dy$$

Similarly, if B and C be the projections of S on the yz-and zx-planes respectively, then

$$S = \iint_B \sqrt{\left[\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1\right]}\, dy\, dz \text{ and } S = \iint_C \sqrt{\left[\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1\right]}\, dz\, dx.$$

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Example 7·20. *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

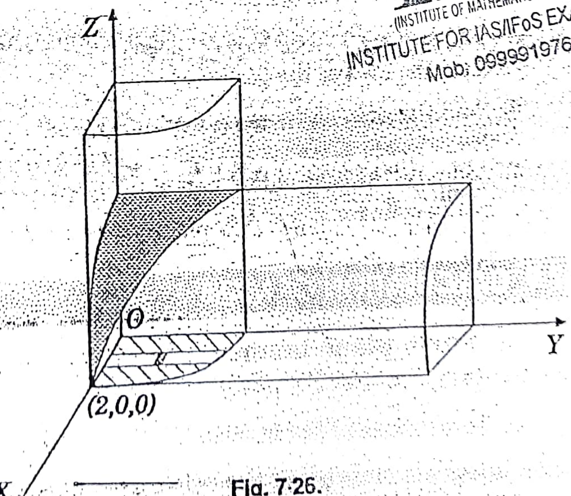

Fig. 7·26.

Sol. Fig. 7·26 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $x^2 + z^2 = 4$, $\quad$ ...(*i*)

we have $\dfrac{\partial z}{\partial x} = -\dfrac{x}{z}, \dfrac{\partial z}{\partial y} = 0$,

so that $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1 = \dfrac{x^2 + z^2}{z^2} = \dfrac{4}{4 - x^2}$.

Hence the required surface area $= 8$ (surface area of the upper portion of (*i*) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8\int_0^2 \int_0^{\sqrt{(4-x^2)}} \frac{2}{\sqrt{(4-x^2)}}\, dx\, dy = 16\int_0^2 dx = 32 \text{ sq. units.}$$

Example 7·21. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Sol. Fig. 7·27 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

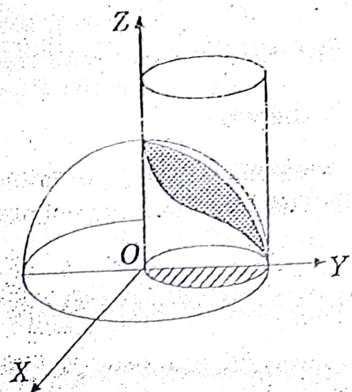

Fig. 7·27.

For the sphere

$$x^2 + y^2 + z^2 = 9, \ \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \quad \text{when } x = r\cos\theta, y = r\sin\theta.$$

Using polar co-ordinates, the required area is found by integrating $3/\sqrt{(9 - r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{(9 - r^2)}}\, r\,d\theta\, dr = -6\int_0^{\pi/2} \left|\frac{\sqrt{(9 - r^2)}}{1/2}\right|_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2} (1 - \cos\theta)\, d\theta = 36\,|\theta - \sin\theta|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

Problems 7.5

1. Show that the surface area of the sphere $x^2 + y^2 + z^2 = a^2$ is $4\pi a^2$. *(Bangalore, 1998 S ; Madras, 1996)*
2. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$. *(Punjab, 1997)*
3. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$.
4. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane. *(Kanpur, 1998)*
5. Compute the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$. *(Burdwan, 2003)*

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-Miscellaneous content

5.10. Some important sequences.

1. The sequence $\{(1+\frac{1}{n})^n\}$ is a monotone increasing sequence, bounded above.

Let $u_n = (1+\frac{1}{n})^n$. Then $u_{n+1} = (1+\frac{1}{n+1})^{n+1}$.

Let us consider $n+1$ positive numbers $1+\frac{1}{n}, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n}$ (n times) and 1.

Applying A.M.> G.M., we have $\frac{n(1+\frac{1}{n})+1}{n+1} > (1+\frac{1}{n})^{\frac{n}{n+1}}$

or, $(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n$, i.e., $u_{n+1} > u_n$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is a monotone increasing sequence.

$$\begin{aligned} \text{Now } u_n &= 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2.1}{n!}\cdot\frac{1}{n^n} \\ &= 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots\frac{2}{n}\cdot\frac{1}{n} \\ &< 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} \text{ for all } n \geq 2. \end{aligned}$$

We have $n! > 2^{n-1}$ for all $n > 2$. Utilising this

$1+1+\frac{1}{2!}+\cdots+\frac{1}{n!} < 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$, for $n > 2$.

Also $1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+2[1-(\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$.

It follows that $u_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{u_n\}$ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence bounded above, is convergent. The limit of the sequence is denoted by e. Since $u_1 = 2$, it follows that $2 \leq u_n < 3$ for all $n \geq 2$.

2. The sequence $\{x_n\}$ where $x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$ is a monotone increasing sequence, bounded above. And $\lim x_n = e$.

$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0$ for all $n \geq 1$. So $x_{n+1} > x_n$ for all $n \geq 1$.

This shows that the sequence $\{x_n\}$ is a monotone increasing sequence.

$$\begin{aligned} x_n &= 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!} \\ &< 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}, \text{ for } n \geq 3, \text{ since } n! > 2^{n-1} \text{ for all } n \geq 3. \end{aligned}$$

Again $1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}} = 1+2[1-(\frac{1}{2})^n] < 3$ for all $n \in \mathbb{N}$

It follows that $x_n < 3$ for all $n \in \mathbb{N}$, proving that the sequence $\{x_n\}$ is bounded above.

Thus the sequence $\{x_n\}$ being a monotone increasing sequence bounded above, is convergent.

Let $u_n = (1+\frac{1}{n})^n$.

$$\begin{aligned} \text{Then } u_n &= 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots 2.1}{n!}\cdot\frac{1}{n^n} \\ &= 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots\frac{2}{n}\cdot\frac{1}{n} \end{aligned}$$

Therefore $\lim u_n \leq \lim x_n$ (since both the limits exist).

or, $e \leq \lim x_n$ ... ... (A).

Let us choose a natural number m. Then for each $n > m$,

$u_n = 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots\frac{2}{n}\cdot\frac{1}{n}$

$> 1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})$.

Keeping m fixed, let $n \to \infty$. Then

$\lim\limits_{n\to\infty} u_n \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!}$

or, $e \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!}$

or, $x_m \leq e$.

The inequality holds for all natural numbers m.

Proceeding to limit as $m \to \infty$, $\lim\limits_{m\to\infty} x_m \leq e$ ... ... (B)

From (A) and (B), $\lim x_n = e$.

3. The sequence $\{(1+\frac{1}{n})^{n+1}\}$ is a monotone decreasing sequence with limit e.

Let $v_n = (1+\frac{1}{n})^{n+1}$.

Let us consider $n+2$ positive numbers $1-\frac{1}{n+1}, 1-\frac{1}{n+1}, \ldots, 1-\frac{1}{n+1}$ [($n+1$) times] and 1.

Applying A.M.> G.M., we have $\frac{(n+1)(1-\frac{1}{n+1})+1}{n+2} > (1-\frac{1}{n+1})^{\frac{n+1}{n+2}}$

or, $(\frac{n+1}{n+2})^{n+2} > (\frac{n}{n+1})^{n+1}$

or, $(\frac{n+1}{n})^{n+1} > (\frac{n+2}{n+1})^{n+2}$

or, $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$

i.e., $v_n > v_{n+1}$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{v_n\}$ is a monotone decreasing sequence.

Again $v_n = 1+\frac{n+1}{n}+\frac{(n+1)n}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{1}{n^{n}} > 1$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{v_n\}$ is bounded below.

Hence the sequence $\{v_n\}$ is convergent.

Let $u_n = (1+\frac{1}{n})^n$. Then $v_n - u_n = (1+\frac{1}{n})^n\cdot\frac{1}{n}$

and $\lim(v_n - u_n) = \lim(1+\frac{1}{n})^n\cdot\frac{1}{n} = 0$.

This implies $\lim v_n = \lim u_n$, since both the limits exist.

As $\lim u_n = e$, it follows that $\lim v_n = e$.

Note. $v_n - u_n = (1+\frac{1}{n})^n\cdot\frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Since ... <

By Cantor's theorem on nested intervals, $\bigcap_{n=1} I_n$ is a singleton set and the set is $\{e\}$.

Worked Examples.

1. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{2}$ and $u_{n+1} = \sqrt{2u_n}$ for all $n \geq 1$ converges to 2.

The sequence is $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\cdots\}$

$u_{n+1}^2 - u_n^2 = 2(u_n - u_{n-1})$

or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = 2(u_n - u_{n-1})$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots\cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $\boxed{2u_n = u_{n+1}^2} > u_n^2$ for all $n \in \mathbb{N}$.

That is, $u_n^2 - 2u_n < 0$ for all $n \in \mathbb{N}$

or, $u_n(u_n - 2) < 0$ for all $n \in \mathbb{N}$.

But $u_n > 0$. Therefore $u_n < 2$ for all $n \in \mathbb{N}$.

This shows that the sequence $\{u_n\}$ is bounded above and therefore it is convergent.

Let $\lim u_n = l$. By definition, $u_{n+1}^2 = 2u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 2l$. Therefore l is either 0 or 2. But l cannot be 0 since the sequence $\{u_n\}$ is monotone increasing and $u_1 = \sqrt{2} > 1$.

Therefore $l = 2$. That is, the sequence converges to 2.

2. Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$ and $u_{n+1} = \sqrt{7 + u_n}$ for all $n \geq 1$ converges to the positive root of the equation $x^2 - x - 7 = 0$.

The sequence is $\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \cdots\cdots\}$

$u_{n+1}^2 - u_n^2 = u_n - u_{n-1}$.

or, $(u_{n+1} + u_n)(u_{n+1} - u_n) = u_n - u_{n-1}$.

Since $u_n > 0$ for all n, $u_{n+1} >$ or $< u_n$ according as $u_n >$ or $< u_{n-1}$.

But $u_2 > u_1$. Consequently, $u_3 > u_2, u_4 > u_3, \cdots\cdots$ and therefore $\{u_n\}$ is a monotone increasing sequence.

Again $u_n^2 < u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$

Since $u_n \geq 0$ for all $n \in \mathbb{N}$, $u_n - \alpha > 0$. Consequently, $u_n < \beta$ for all $n \in \mathbb{N}$.

This proves that the sequence $\{u_n\}$ is bounded above and therefore the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$. By definition, $u_{n+1}^2 = 7 + u_n$ for all $n \in \mathbb{N}$.

Taking limit as $n \to \infty$, we have $l^2 = 7 + l$.

Therefore $(l - \alpha)(l - \beta) = 0$.

But $l \neq \alpha$, since each element of the sequence is positive and $\alpha < 0$. Therefore $l = \beta$. That is, the sequence converges to the positive root of the equation $x^2 - x - 7 = 0$.

3. Let $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$; $v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n, n \geq 2$. Show that $\{u_n\}$ is a monotone decreasing sequence and $\{v_n\}_{n=2}^{\infty}$ is a monotone increasing one and they converge to the same limit.

$u_{n+1} - u_n = \frac{1}{n+1} - \log(n+1) + \log n = \frac{1}{n+1} - \log(1 + \frac{1}{n})$.

$v_{n+1} - v_n = \frac{1}{n} - \log(n+1) + \log n = \frac{1}{n} - \log(1 + \frac{1}{n})$.

As the sequence $\{(1 + \frac{1}{n})^{n+1}\}$ is a strictly monotone decreasing sequence converging to e, $(1 + \frac{1}{n})^{n+1} > e$ for all $n \in \mathbb{N}$.

Therefore $\log(1 + \frac{1}{n}) > \frac{1}{n+1}$ for all $n \in \mathbb{N}$

or, $u_{n+1} < u_n$ for all $n \in \mathbb{N}$.

This shows that $\{u_n\}$ is a strictly monotone decreasing sequence.

As the sequence $\{(1 + \frac{1}{n})^n\}$ is a strictly monotone increasing sequence converging to e, $(1 + \frac{1}{n})^n < e$ for all $n \in \mathbb{N}$.

Therefore $\log(1 + \frac{1}{n}) < \frac{1}{n}$ for all $n \in \mathbb{N}$

or, $v_{n+1} > v_n$ for all $n \geq 2$.

Therefore the sequence $\{v_n\}$ is a strictly monotone increasing sequence.

Again $\frac{1}{n} > \log\frac{n+1}{n} = \log(n+1) - \log n$.

Therefore $1 > \log 2 - \log 1$, $\frac{1}{2} > \log 3 - \log 2, \cdots, \frac{1}{n} > \log(n+1) - \log n$.

So we have $1 + \frac{1}{2} + \cdots + \frac{1}{n} > \log(n+1) > \log n$.

Hence $u_n > 0$ for all $n \in \mathbb{N}$.

Therefore $\{u_n\}$ is a monotone decreasing sequence bounded below.

Hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = \gamma$.

Now $u_n - v_n = \frac{1}{n}$ for $n \geq 2$. Therefore $\lim v_n = \gamma$.

Thus the sequences $\{u_n\}$ and $\{v_n\}$ converge to the same limit γ.

Note 1. This limit γ is called **Euler's constant.**

Since $u_1 = 1$ and $\{u_n\}$ is a strictly monotone decreasing sequence, $\gamma < 1$. Since $v_2 = 1 - \log 2 = 1 - .69315 > .3$ and $\{v_n\}$ is a monotone increasing sequence, $\gamma > .3$. Therefore $.3 < \gamma < 1$.

The approximation of γ upto six places of decimal is given by $\gamma = 0.577215$.

Note 2. $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ is denoted by γ_n. Then the sequence $\{\gamma_n\}$ converges to γ and $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \gamma_n + \log n$.

Evaluation of the limit of some sequences can be done by the help of Euler's constant.

For example, if $s_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, then

$$\begin{aligned}
\lim s_n &= \lim[(1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2n}) - (1 + \tfrac{1}{2} + \cdots + \tfrac{1}{n})] \\
&= \lim[(\gamma_{2n} + \log 2n) - (\gamma_n + \log n)] \\
&= \lim[\gamma_{2n} - \gamma_n + \log 2] \\
&= \log 2, \text{ since } \lim \gamma_{2n} = \lim \gamma_n = \gamma.
\end{aligned}$$

4. Two sequences $\{x_n\}, \{y_n\}$ are defined by

$x_{n+1} = \frac{1}{2}(x_n + y_n), y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0, y_1 > 0$.

Prove that both the sequences converge to a common limit.

Case 1. Let $x_1 \neq y_1$.

$x_2 = \frac{1}{2}(x_1 + y_1) > \sqrt{x_1 y_1} = y_2$.

Let us assume that $x_k > y_k$.

Then $x_{k+1} = \frac{1}{2}(x_k + y_k) > \sqrt{x_k y_k} = y_{k+1}$.

$x_k > y_k$ implies $x_{k+1} > y_{k+1}$ and $x_2 > y_2$.

By the principle of induction, $x_n > y_n$ for all $n \geq 2$.

$x_{n+1} = \frac{1}{2}(x_n + y_n) < \frac{1}{2}(x_n + x_n) = x_n$ for all $n \geq 2$

$y_{n+1} = \sqrt{x_n y_n} > \sqrt{y_n . y_n} = y_n$ for all $n \geq 2$.

So we have $y_2 < y_3 < y_4 < \cdots \cdots < x_4 < x_3 < x_2$.

Therefore the sequence $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and the sequence $\{y_n\}_{n=2}^{\infty}$ is a monotone increasing sequence bounded above. Hence both the sequences are convergent.

Let $\lim x_n = l$, $\lim y_n = m$.

$\cdots = \frac{1}{2}(x_n + y_n)$ for all $n \in \mathbb{N}$.

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ converge to a common limit.

Case 2. Let $x_1 = y_1$.

In this case $x_n = y_n = x_1$ for all $n \in \mathbb{N}$.

Therefore $\{x_n\}$ and $\{y_n\}$ both converge to the same limit x_1.

5. If $u_1 > 0$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$, prove that the sequence $\{u_n\}$ converges to 3.

$u_n^2 - 2u_n u_{n+1} + 9 = 0$. This is a quadratic equation in u_n having real roots. Therefore $4u_{n+1}^2 - 36 \geq 0$.

This implies $u_{n+1} \geq 3$ for all $n \geq 1$, since $u_{n+1} > 0$ for all $n \geq 1$.

$$\begin{aligned}
u_n - u_{n+1} &= u_n - \tfrac{1}{2}(u_n + \tfrac{9}{u_n}) = \tfrac{1}{2}(u_n - \tfrac{9}{u_n}) \\
&= \tfrac{1}{2}(\tfrac{u_n^2 - 9}{u_n}) \geq 0 \text{ for all } n \geq 2.
\end{aligned}$$

Therefore $u_{n+1} \leq u_n$ for all $n \geq 2$.

This shows that the sequence $\{u_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence bounded below and hence the sequence $\{u_n\}$ is convergent.

Let $\lim u_n = l$.

$u_{n+1} = \frac{1}{2}(u_n + \frac{9}{u_n})$ for $n \geq 1$. Proceeding to limit as $n \to \infty$, we have $l = \frac{1}{2}(l + \frac{9}{l})$. This gives $l = 3$, since $l > 0$.

6. $x_1 \geq 2$ and $x_{n+1} = 1 + \sqrt{x_n - 1}$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ is a monotone decreasing sequence and $\lim x_n = 2$.

Case 1. $x_1 = 2$.

For $k \geq 1$, $x_k = 2 \Rightarrow x_{k+1} = 2$. By the principle of induction, $x_n = 2$ for all $n \in \mathbb{N}$ and therefore the sequence $\{x_n\}$ coverges to 2.

Case 2. $x_1 > 2$.

$x_{n+1} - 1 = \sqrt{x_n - 1} = (x_{n-1} - 1)^{\frac{1}{2^2}} = \cdots = (x_1 - 1)^{\frac{1}{2^n}} > 1$.

Therefore $x_{n+1} > 2$ for all $n \in \mathbb{N}$.

$(x_{n+1} - 1)^2 - (x_n - 1)^2 = x_n - x_{n-1}$

or, $x_{n+1} - x_n = \frac{1}{x_{n+1} + x_n - 2}(x_n - x_{n-1})$.

Because $x_{n+1} + x_n - 2 > 0$, $x_{n+1} >$ or $< x_n$ according as $x_n >$ or $< x_{n-1}$.

Since $x_1 - 1 > 1$, $x_2 - 1 = \sqrt{x_1 - 1} < x_1 - 1$ and this gives $x_2 < x_1$.

This implies $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. That is, the the sequence $\{x_n\}$ is a decreasing sequence bounded below. So the sequence $\{x_n\}$ is convergent.

$\lim (x_{n+1} - 1) = \lim (x_1 - 1)^{\frac{1}{2^n}} = 1$. Therefore $\lim x_n = 2$.

(iii) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n + v_n\}$ is convergent.

(iv) Give an example of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent.

2. Find $\sup\{u_n\}$ and $\inf\{u_n\}$ where

(i) $u_n = (-1)^n + \cos\frac{n\pi}{4}$, (ii) $u_n = \frac{(-1)^n}{n} + \sin\frac{n\pi}{2}$.

3. Let $\{u_n\}$ be a bounded sequence and $\lim v_n = 0$. Prove that $\lim u_n v_n = 0$. Utilise this to prove that

(i) $\lim \frac{\sin n}{n} = 0$, (ii) $\lim \frac{(-1)^n n}{n^2+1} = 0$, (iii) $\lim (-1)^n a_n = 0$ if $\lim a_n = 0$.

4. Let $\{u_n\}, \{v_n\}$ be two real sequences with $\lim u_n = l$, $\lim v_n = m$. If $x_n = \max\{u_n, v_n\}$, $y_n = \min\{u_n, v_n\}$ prove that the sequence $\{x_n\}$ converges to $\max\{l, m\}$ and the sequence $\{y_n\}$ converges to $\min\{l, m\}$.

[Hint. $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$, $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ for all $a, b \in \mathbb{R}$.]

5. If $\{u_n\}$ be a bounded sequence and $x_r = \min\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, $y_r = \max\{u_r, u_{r+1}, u_{r+2}, \cdots\}$, for $r \geq 1$, prove that $\{x_n\}$ and $\{y_n\}$ are both monotone convergent sequences.

If $\lim x_n = \lim y_n = l$ prove that the sequence $\{u_n\}$ converges to l.

[Hint. $x_n \leq x_{n+1}$ and $y_n \geq y_{n+1}$ for all n, $x_n \leq u_n \leq y_n$ for all n.]

6. Prove that the sequence $\{u_n\}$ is a null sequence.

(i) $u_n = \frac{n!}{n^n}$, (ii) $u_n = \frac{4^{3n}}{3^{4n}}$, (iii) $u_n = \frac{b^n}{n!}, b > 1$.

7. Use Sandwich theorem to prove that

(i) $\lim(\sqrt{n+1} - \sqrt{n}) = 0$, (ii) $\lim(2^n + 3^n)^{1/n} = 3$,

(iii) $\lim\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+n)^2}\right] = 0$, (iv) $\lim \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} = 0$.

[Hint. (iv) Let $u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$. Then $u_n < \frac{2.4\ldots 2n}{3.5\ldots(2n+1)}$, since $\frac{n}{n+1} < \frac{n+1}{n+2}$ for all $n \geq 1$. Therefore $u_n^2 = u_n . u_n < \frac{1}{2n+1}$ for all $n \geq 1$.]

8. If $0 < u_1 < 1$ and $u_{n+1} = 1 - \sqrt{1 - u_n}$ for $n \geq 1$, prove that

(i) the sequence $\{u_n\}$ converges to 0 and (ii) $\lim \frac{u_{n+1}}{u_n} = \frac{1}{2}$.

[Hint. $1 - u_{n+1} = (1 - u_n)^{1/2} = (1 - u_{n-1})^{1/2^2} = \cdots = (1 - u_1)^{1/2^n}$.]

10. Prove that the sequence $\{u_n\}$ defined by

(i) $u_1 = \sqrt{3}$ and $u_{n+1} = \sqrt{3u_n}$ for $n \geq 1$, converges to 3;

(ii) $u_1 = \sqrt{6}$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \geq 1$, converges to 3.

11. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \sqrt{6 + u_n}$ for $n \geq 1$. Show that

(i) the sequence $\{u_n\}$ is monotone increasing if $0 < u_1 < 3$;

(ii) the sequence $\{u_n\}$ is monotone decreasing if $u_1 > 3$.

Find $\lim u_n$.

12. A sequence $\{u_n\}$ is defined by $u_1 > 0$ and $u_{n+1} = \frac{3(1+u_n)}{3+u_n}$ for $n \geq 1$. Prove that

(i) the sequence $\{u_n\}$ is a decreasing sequence if $u_1 > 1$;

(ii) the sequence $\{u_n\}$ is an increasing sequence if $0 < u_1 < 1$;

(iii) $\lim u_n = 1$ in both cases.

13. Prove that the sequence $\{u_n\}$ defined by

(i) $0 < u_1 < u_2$ and $u_{n+2} = \frac{2u_{n+1} + u_n}{3}$ for $n \geq 1$, converges to $\frac{u_1 + 3u_2}{4}$;

(ii) $0 < u_1 < u_2$ and $u_{n+2} = \frac{u_{n+1} + 2u_n}{3}$ for $n \geq 1$, converges to $\frac{2u_1 + 3u_2}{5}$;

(iii) $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1} u_n}$ for $n \geq 1$, converges to $\sqrt[3]{u_1 u_2^2}$;

(iv) $0 < u_1 < u_2$ and $\frac{2}{u_{n+2}} = \frac{1}{u_{n+1}} + \frac{1}{u_n}$ for $n \geq 1$, converges to $3/(\frac{1}{u_1} + \frac{2}{u_2})$.

14. If $s_1 > 0$ and $s_{n+1} = \frac{1}{2}(s_n + \frac{4}{s_n})$ for $n \geq 1$, prove that the sequence $\{s_n\}$ is a monotone decreasing sequence bounded below and $\lim s_n = 2$.

15. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = \sqrt{x_n y_n}$ and $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$, converge to a common limit.

16. Prove that the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_{n+1} = \frac{1}{2}(x_n + y_n)$, $\frac{2}{y_{n+1}} = \frac{1}{x_n} + \frac{1}{y_n}$ for $n \geq 1, x_1 > 0, y_1 > 0$, converge to a common limit l where $l^2 = x_1 y_1$.

17. Prove that the sequence $\{\gamma_n\}$ where $\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$ is convergent. Hence find

(i) $\lim[1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}]$, (ii) $\lim[\frac{1}{1.3} + \frac{1}{2.5} + \cdots + \frac{1}{n(2n+1)}]$.

18. Let S be a non-empty subset of $\mathbb{R}$ having a limit point l. Show that there exists a sequence $\{u_n\}$ of distinct elements of S such that $\lim u_n = l$.

19. Let S be an infinite subset of $\mathbb{R}$ that is bounded above and let $\sup S \notin S$. Show that there exists a monotone increasing sequence $\{u_n\}$ with $u_n \in S$ such that $\lim u_n = \sup S$.

5.11. Subsequence.

Let $\{u_n\}$ be a real sequence and $\{r_n\}$ be a strictly increasing sequence of natural numbers, i.e., $r_1 < r_2 < r_3 < \cdots < r_n < \cdots$ Then the sequence $\{u_{r_n}\}$ is said to be a *subsequence* of the sequence $\{u_n\}$. The elements of the subsequence $\{u_{r_n}\}$ are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots$

Let $r : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers such that $r_1 < r_2 < \cdots < r_n < \cdots$ and $u : \mathbb{N} \to \mathbb{R}$ be a real sequence. Then the composite mapping $uor : \mathbb{N} \to \mathbb{R}$ is said to be a ***subsequence*** of the real sequence u. The elements of the subsequence uor are $u_{r_1}, u_{r_2}, \ldots, u_{r_n}, \ldots$

Examples.

1. Let $u_n = \frac{1}{n}$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots\ldots\}$
   $= \{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\}$ is a subsequence of $\{\frac{1}{n}\}$.
2. Let $u_n = \frac{1}{n}$ and $r_n = 2n - 1$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_1, u_3, u_5, \ldots\ldots\}$
   $= \{1, \frac{1}{3}, \frac{1}{5}, \ldots\ldots\}$ is a subsequence of $\{\frac{1}{n}\}$.
3. Let $u_n = (-1)^n$ and $r_n = 2n$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{u_2, u_4, u_6, \ldots\ldots\}$
   $= \{1, 1, 1, \ldots\ldots\}$ is a subsequence of $\{(-1)^n\}$.
4. Let $u_n = 1 + 1/n$ and $r_n = n^2$ for all $n \in \mathbb{N}$.
   Then $\{u_{r_n}\} = \{1+1, 1+\frac{1}{2^2}, 1+\frac{1}{3^2}, \ldots\ldots\}$ is a subsequence of $\{1+\frac{1}{n}\}$.

Theorem 5.11.1. *If a sequence $\{u_n\}$ converges to l then every subsequence of $\{u_n\}$ also converges to l.*

Proof. Let $\{r_n\}$ be a strictly increasing sequence of natural numbers. Then $\{u_{r_n}\}$ is subsequence of the sequence $\{u_n\}$.

Let $\epsilon > 0$. Since $\lim u_n = l$, there exists a natural number k such that $l - \epsilon < u_n < l + \epsilon$ for all $n \geq k$.

Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n \geq k$ for all $n \geq k_0$.

Therefore $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k_0$. This shows that $\lim\limits_{n\to\infty} u_{r_n} = l$.

Note. If there exist two different subsequences $\{u_{r_n}\}$ and $\{u_{k_n}\}$ of a sequence $\{u_n\}$ such that $\{u_{r_n}\}$ and $\{u_{k_n}\}$ converge to two different limits, then the sequence $\{u_n\}$ is not convergent.

If a sequence $\{u_n\}$ has a divergent subsequence then $\{u_n\}$ is divergent.

Worked Examples.

1. Prove that $\lim(1 + \frac{1}{2n})^n = \sqrt{e}$.

Let $u_n = (1 + \frac{1}{n})^n$, $v_n = (1 + \frac{1}{2n})^{2n}$ and $w_n = (1 + \frac{1}{2n})^n$ for all $n \in \mathbb{N}$. $\{u_n\}$ is a convergent sequence and $\lim u_n = e$.

Since $v_n = u_{2n}$ for all $n \in \mathbb{N}$, $\{v_n\}$ is a subsequence of $\{u_n\}$ and therefore $\lim v_n = e$.

$w_n = \sqrt{v_n}$ for all $n \in \mathbb{N}$. Therefore $\lim w_n = \lim \sqrt{v_n} = \sqrt{e}$.

2. Prove that the sequence $\{(-1)^n\}$ is divergent.

Let $u_n = (-1)^n$, $v_n = u_{2n}$, $w_n = u_{2n-1}$.
Then $\{v_n\}$ is the subsequence $\{1, 1, 1, \ldots\ldots\}$ and $\lim v_n = 1$,
$\{w_n\}$ is the subsequence $\{-1, -1, -1, \ldots\ldots\}$ and $\lim w_n = -1$.
Since two different subsequences converge to two different limits, the sequence $\{u_n\}$ is divergent.

Theorem 5.11.2. *If the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ of a sequence $\{u_n\}$ converge to the same limit l then the sequence $\{u_n\}$ is convergent and $\lim u_n = l$.*

Proof. Let us choose $\epsilon > 0$. Since $\lim u_{2n} = l$, there exists a natural number k_1 such that $|u_{2n} - l| < \epsilon$ for all $n \geq k_1$.

Since $\lim u_{2n-1} = l$, there exists a natural number k_2 such that $|u_{2n-1} - l| < \epsilon$ for all $n \geq k_2$.

Let $k = \max\{k_1, k_2\}$. Then k is a natural number and for all $n \geq k$, $l - \epsilon < u_{2n} < l + \epsilon$ and $l - \epsilon < u_{2n-1} < l + \epsilon$.

That is, $l - \epsilon < u_n < l + \epsilon$ for all $n \geq 2k - 1$.

As $2k - 1$ is a natural number, it follows that $\lim u_n = l$.

Note 1. If two subsequences of a sequence converge to the same limit l, the sequence $\{u_n\}$ may not be convergent.

For example, let $u_n = \sin\frac{n\pi}{4}$.

Then the subsequence $\{u_{8n-7}\}$ is $\{\sin\frac{\pi}{4}, \sin\frac{9\pi}{4}, \sin\frac{17\pi}{4}, \ldots\}$

Worked Examples (continued.)

4. Prove that the sequence $\{u_n\}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \frac{1}{2}(u_n + u_{n+1})$, is convergent.

$u_3 - u_1 = \frac{u_1+u_2}{2} - u_1 = \frac{(u_2-u_1)}{2} > 0$, i.e., $u_1 < u_3$.
$u_3 - u_2 = \frac{u_1+u_2}{2} - u_2 = \frac{(u_1-u_2)}{2} < 0$, i.e., $u_3 < u_2$.
So $u_1 < u_3 < u_2$.
Similarly, $u_3 < u_4 < u_2$, $u_3 < u_5 < u_4$; $u_5 < u_6 < u_4, \cdots$;
The inequalities give
$u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the sequence $\{u_{2n-1}\}$ is a monotone increasing sequence bounded above, u_2 being an upper bound; and the sequence $\{u_{2n}\}$ is a monotone decreasing sequence bounded below, u_1 being a lower bound.

So both the sequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ are convergent.
Let $\lim u_{2n} = l$, $\lim u_{2n-1} = m$.
Now $2u_{2n+2} = u_{2n} + u_{2n+1}$ for all $n \in \mathbb{N}$.
Proceeding to limit as $n \to \infty$, we have $2l = l + m$, i.e., $l = m$.
Thus the subsequences $\{u_{2n}\}$ and $\{u_{2n-1}\}$ converge to the same limit l and therefore the sequence $\{u_n\}$ is convergent.

5. A sequence $\{u_n\}$ is defined by $u_n > 0$ and $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.
(i) Prove that the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.
(ii) Find $\lim u_n$.

$u_{n+1} - u_n = \frac{6}{1+u_n} - u_n = \frac{6-u_n-u_n^2}{1+u_n} = \frac{(2-u_n)(3+u_n)}{1+u_n}$.
Therefore $u_n < 2 \Rightarrow u_n < u_{n+1}$; $u_n > 2 \Rightarrow u_n > u_{n+1}$.
Again $u_n < 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} > 2$; $u_n > 2 \Rightarrow u_{n+1} = \frac{6}{1+u_n} < 2$.
It follows that
$u_n < 2 \Rightarrow u_n < 2 < u_{n+1}$; $u_n > 2 \Rightarrow u_{n+1} < 2 < u_n$ ... (i)

$u_{n+2} - u_n = \frac{6(1+u_n)}{7+u_n} - u_n = \frac{6-u_n-u_n^2}{7+u_n} = \frac{(2-u_n)(3+u_n)}{7+u_n}$.
$u_n < 2 \Rightarrow u_n < u_{n+2}$; $u_n > 2 \Rightarrow u_n > u_{n+2}$ ... (ii)

Therefore $u_1 < u_3 < u_5 < \cdots < u_6 < u_4 < u_2$.

This shows that the subsequence $\{u_{2n-1}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n}\}$ is a monotone decreasing sequence, bounded below. Hence both the subsequences are convegent.
Let $\lim u_{2n-1} = l$, $\lim u_{2n} = m$.
We have $u_{2n} = \frac{6}{1+u_{2n-1}}$, $u_{2n+1} = \frac{6}{1+u_{2n}}$ for all $n \in \mathbb{N}$.
Taking limit as $n \to \infty$, we have $m = \frac{6}{1+l}$, $l = \frac{6}{1+m}$.
Therefore $l = m$ and the subsequences $\{u_{2n-1}\}$ and $\{u_{2n}\}$ converge to a common limit.

Case 2. $u_1 > 2$.
From (i) and (ii) it follows that $u_2 < u_4 < u_6 < \cdots < u_5 < u_3 < u_1$.
The subsequence $\{u_{2n}\}$ is a monotone increasing sequence, bounded above and the subsequence $\{u_{2n-1}\}$ is a monotone decreasing sequence, bounded below.
Hence both the sequences are convergent.

Proceeding as in case 1, it can be shown that they converge to a common limit.

(ii) Let the limit be l. We have $u_{n+1} = \frac{6}{1+u_n}$ for all $n \in \mathbb{N}$.
Taking limit as $n \to \infty$, we have $l = \frac{6}{1+l}$. This gives $l = 2$, or $l = -3$.
As $u_n > 0$ for all $n \in \mathbb{N}$, $l \neq -3$. Therefore $\lim u_n = 2$.

Theorem 5.11.3. Every subsequence of a monotone increasing (decreasing) sequence of real numbers is monotone increasing (decreasing).

Proof. Let $\{u_n\}$ be a monotone increasing sequence. Then for any two natural numbers p, q with $p > q$, $u_p \geq u_q$.
Let $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers. This implies $r_{n+1} > r_n$ for all $n \in \mathbb{N}$.
$r_{n+1} > r_n \Rightarrow u_{r_{n+1}} \geq u_{r_n}$ for all n.
This proves that $\{u_{r_n}\}$ is a monotone increasing subseqeunce.

Similar proof for a monotone decreasing sequence $\{u_n\}$.

Theorem 5.11.4. A monotone sequence of real numbers having a convergent subsequence with limit l, is convergent with limit l.

Proof. Let $\{u_n\}$ be a monotone increasing sequence and $\{u_{r_n}\}$ be a subsequence of $\{u_n\}$ such that $\lim u_{r_n} = l$.

Since $\{u_n\}$ is a monotone increasing sequence, the subsequence $\{u_{r_n}\}$ is also monotone increasing, by Theorem 5.11.3.

Since $\{u_{r_n}\}$ is a convergent sequence, it is bounded above.

We assert that the sequence $\{u_n\}$ is bounded above. If not, let $\{u_n\}$ be unbounded above. Then being a monotone increasing sequence it must diverge to ∞ and therefore for a pre-assigned positive number G, however large, there must exist a natural number k such that $u_n > G$ for all $n \geq k$. Since $\{r_n\}$ is a strictly increasing sequence of natural numbers, there exists a natural number k_0 such that $r_n > k$ for all $n \geq k_0$. Consequently, $u_{r_n} > G$ holds for all $n \geq k_0$.

Since G is arbitrary, the sequence $\{u_{r_n}\}$ must diverge to ∞, a contradiction. So our assertion is established and the sequence ~~$\{u_n\}$~~ is bounded above.

Thus the sequence $\{u_n\}$ being a monotone increasing sequence, bounded above, is convergent.

Let $\lim u_n = m$. Then $\{u_{r_n}\}$ being subsequence of $\{u_n\}$, converges to m, by Theorem 5.11.1. Therefore $l = m$. This completes the proof.

Theorem 5.11.5. A monotone sequence of real numbers having a divergent subsequence is properly divergent.

Proof. Let $\{u_n\}$ be a montone increasing sequence having a divergent subsequence $\{u_{r_n}\}$. Since the sequence $\{u_n\}$ is monotone increasing, the subsequence $\{u_{r_n}\}$ is also monotone increasing and therefore it is a properly divergent subsequence. Consequently, the subsequence $\{u_{r_n}\}$ is unbounded above. Hence the sequence $\{u_n\}$ must be unbounded above and therefore it is properly divergent.

Similar proof if $\{u_n\}$ be a monotone decreasing sequence.

Worked Example.

1. Prove that the sequence $\{u_n\}$ is divergent where $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

$u_{n+1} - u_n = \frac{1}{n+1} > 0$ for all n. Therefore the sequence $\{u_n\}$ is a monotone increasing sequence.

Let $r_n = 2^n$. Then $\{r_n\}$ is a strictly increasing sequence of natural numbers and so the sequence $\{u_{r_n}\}$ is a subsequence of $\{u_n\}$.

$$u_{r_n} = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n}$$
$$= 1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \cdots + \tfrac{1}{8}) + \cdots + (\tfrac{1}{2^{n-1}+1} + \cdots + \tfrac{1}{2^n})$$
$$> 1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \cdots$$

$$= 1 + \tfrac{1}{2} + 2\tfrac{1}{2^2} + 2^2.\tfrac{1}{2^3} + \cdots + 2^{n-1}.\tfrac{1}{2^n}$$
$$= 1 + \tfrac{n}{2}.$$

Let $v_n = 1 + \frac{n}{2}$. Then $u_{r_n} > v_n$ for all $n > 2$ and $\lim v_n = \infty$. Therefore $\lim u_{r_n} = \infty$.

Thus the sequence $\{u_n\}$ is a monotone increasing sequence having a properly divergent subsequence $\{u_{r_n}\}$ and therefore the sequence $\{u_n\}$ is properly divergent.

Theorem 5.11.6. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{u_n\}$ be a sequence of real numbers. An element u_k is said to be a *peak* of the sequence $\{u_n\}$ if $u_k \geq u_n$ for all $n > k$, i.e., u_k is greater than ~~or equal~~ to all subsequent elements beyond u_k. A sequence may or may not have a peak or else it may have a finite or an infinite number of peaks.

We consider the following cases.

Case 1. Let the sequence $\{u_n\}$ have infinitely many peaks.

Let the peaks be $u_{r_1}, u_{r_2}, u_{r_3}, \ldots$ (u_{r_1} being the first peak, u_{r_2} being the second,...). Then $u_{r_1} \geq u_{r_2} \geq u_{r_3} \geq \cdots$

The subsequence $\{u_{r_1}, u_{r_2}, u_{r_3}, \ldots\}$ is a monotone decreasing subsequence.

Case 2. Let the sequence have either no peak or a finite number of peaks.

Let the peaks be arranged in ascending order of the subscripts as $u_{r_1}, u_{r_2}, \ldots, u_{r_m}$. Let $s_1 = r_m + 1$. Then u_{s_1} is not a peak and there is no peak beyond the element u_{s_1}.

Since u_{s_1} is not a peak, there is an $s_2 \in \mathbb{N}$ with $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$.

Since u_{s_2} is not a peak, there is an $s_3 \in \mathbb{N}$ with $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$.

Proceeding thus we obtain natural numbers s_i such that $s_1 < s_2 < s_3 < \cdots$ and $u_{s_1} < u_{s_2} < u_{s_3} < \cdots$

Clearly, the subsequence $\{u_{s_n}\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

This completes the proof.

Examples

1. Let $u_n = \sin \frac{n\pi}{[illegible]}$ ...

2. Let $u_n = n^{(-1)^n}$. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}, \ldots\}$.

Here the sequence $\{u_n\}$ has no peak.

u_1 is not a peak. Let $s_1 = 1$. Since u_{s_1} is not a peak, there is a natural number $s_2 > s_1$ such that $u_{s_2} > u_{s_1}$. Here $s_2 = 2$.

Since u_{s_2} is not a peak, there is a natural number $s_3 > s_2$ such that $u_{s_3} > u_{s_2}$. Here $s_3 = 4$.

By similar arguments, $s_4 = 6, s_5 = 8, \ldots$

Thus $\{u_1, u_2, u_4, u_6, u_8, \ldots\}$ is a monotone increasing subsequence of the sequence $\{u_n\}$.

5.12. Subsequential limit.

Let $\{u_n\}$ be a real sequence. A real number l is said to be a *subsequential limit* of the sequence $\{u_n\}$ if there exists a subsequence of $\{u_n\}$ that converges to l.

Theorem 5.12.1. A real number l is a subsequential limit of a sequence $\{u_n\}$ if and only if every neighbourhood of l contains infinitely many elements of the sequence $\{u_n\}$.

Proof. Let l be a subsequential limit of the sequence $\{u_n\}$. Then there exists a subsequence $\{u_{r_n}\}$ such that $\lim\limits_{n\to\infty} u_{r_n} = l$.

Let us choose a positive ϵ. Then there exists a natural number k such that $l - \epsilon < u_{r_n} < l + \epsilon$ for all $n \geq k$.

Therefore $l - \epsilon < u_n < l + \epsilon$ for infinitely many values of n.

Since ϵ is arbitrary, every neighbourhood of l contains infinite number of elements of the sequence $\{u_n\}$.

limit point

Conversely, let the sequence $\{u_n\}$ be such that for each pre-assigned positive ϵ the ϵ-neighbourhood of l contains infinitely many elements of the sequence.

Let $\epsilon = 1$. Then $l - 1 < u_n < l + 1$ for infinitely many values of n. Therefore the set $S_1 = \{n : l - 1 < u_n < l + 1\}$ is an infinite subset of the set $\mathbb{N}$. By the well ordering property of the set $\mathbb{N}$, S_1 has a least element, say r_1. Therefore $l - 1 < u_{r_1} < l + 1$.

Continuing thus, we obtain a strictly increasing sequence of natural numbers $\{r_1, r_2, r_3, \ldots\}$ such that $l - \frac{1}{n} < u_{r_n} < l + \frac{1}{n}$ for all $n \in \mathbb{N}$.

By Sandwich theorem, $\lim u_{r_n} = l$.

In other words the subsequence $\{u_{r_n}\}$ converges to l.

That is, l is a subsequential limit of the sequence $\{u_n\}$.

Note. The limit of a sequence, if it exists, is also a subsequential limit of the sequence.

Theorem 5.12.2. Bolzano-Weierstrass theorem.

Every bounded sequence of real numbers has a convergent subsequence. (i.e. at least one limit point).

Proof. Let $\{u_n\}$ be a bounded sequence. Then there is a closed and bounded interval, say $I = [a, b]$, such that $u_n \in I$ for every $n \in \mathbb{N}$.

Let $c = \frac{a+b}{2}$ and $I' = [a, c], I'' = [c, b]$. Then at least one of the intervals I' and I'' contains infinitely many elements of $\{u_n\}$.

Let $I_1 = [a_1, b_1]$ be such an interval. Then $I_1 \subset I$ and $|I_1| =$ the length of the interval $= \frac{1}{2}(b - a)$.

Let $c_1 = \frac{a_1+b_1}{2}$ and $I_1' = [a_1, c_1], I_1'' = [c_1, b_1]$. Then at least one of the intervals I_1' and I_1'' contains infinitely many elements of $\{u_n\}$. Let $I_2 = [a_2, b_2]$ be such an interval.

Then $I_2 \subset I_1$ and $|I_2| = \frac{1}{2}|I_1|$.

Continuing thus, we obtain a sequence of closed and bounded intervals $\{I_n\}$ such that

(i) $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$;

(ii) $|I_n| = \frac{1}{2^n}(b - a)$ and therefore $\lim\limits_{n\to\infty} |I_n| = 0$; and

(iii) each I_n contains infinitely many elements of $\{u_n\}$.

By Cantor's theorem on nested intervals, there exists a unique point α such that $\alpha \in \bigcap\limits_{n=1}^{\infty} I_n$.

We prove that α is a subsequential limit of the sequence $\{u_n\}$.

Let us choose $\epsilon > 0$. There exists a natural number k such that $0 < \frac{b-a}{2^k} < \epsilon$. That is, $|I_k| < \epsilon$.

Since $\alpha \in I_k$ and $|I_k| < \epsilon$, I_k is entirely contained in the neighbourhood $(\alpha - \epsilon, \alpha + \epsilon)$ and consequently, the ϵ-neighbourhood of α contains infinitely many elements of $\{u_n\}$.

Since ϵ is arbitrary, each neighbourhood of α contains infinitely many elements of $\{u_n\}$. Therefore α is a subsequential limit of $\{u_n\}$.

Therefore there exists a subsequence of $\{u_n\}$ that converges to α. In other words, $\{u_n\}$ has a convergent subsequence.
This completes the proof.

Note. Another version of the theorem is –

Every bounded sequence of real numbers has a subsequential limit.

Examples.

Qn 1. The sequence $\{u_n\}$ where $u_n = \sin \frac{n\pi}{2}, n \geq 1$ is a bounded sequence since $|u_n| \leq 1$ for all $n \geq 1$.

(i) The subsequence $\{u_1, u_5, u_9, \ldots \ldots\}$, i.e., $\{u_{4n-3}\}$ is a convergent subsequence that converges to 1.

(ii) The subsequence $\{u_2, u_4, u_6, \ldots \ldots\}$, i.e., $\{u_{2n}\}$ is a convergent subsequence that converges to 0.

(iii) The subsequence $\{u_1, u_3, u_5, \ldots \ldots\}$, i.e., $\{u_{2n-1}\}$ is a divergent subsequence.

Note. The example 1(iii) shows that a bounded sequence may have a divergent subsequence.

2. The sequence $\{u_n\}$ where $u_n = n^{(-1)^n}$ is an unbounded sequence. The sequence is $\{1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \ldots \ldots\}$.

(i) The sequence $\{u_{2n}\}$ is a properly divergent subsequence.

(ii) The sequence $\{u_{2n-1}\}$ is a convergent subsequence.

Note. The example 2(ii) shows that an unbounded sequence may have a convergent subsequence. ⟹ Converse of Bolzano Weierstrass
This is not true.

have to apply the mean value theorem more than once as in example 16. Also, one may have to consider different functions as in example 16.

Example 14. *Applying Lagrange's mean value theorem to the function defined by $f(x) = \log(1 + x)$, for all $x \geq 0$, show that*

$$0 < [\log(1 + x)]^{-1} - x^{-1} < 1, \text{ whenever } x > 0.$$

Solution. Let x be any positive real number. Then the function f, defined by

$$f(x) = \log(1 + x),$$

is continuous on $[0, x]$ and derivable on $]0, x[$.

Therefore, by Lagrange's mean value theorem, there exists a number θ between 0 and 1 such that

$$f(x) - f(0) = xf'(\theta x),$$

i.e.,

$$\log(1 + x) = \frac{x}{1 + \theta x} \qquad \ldots(1)$$

Since $0 < \theta < 1$ and $x > 0$, therefore,

$$\frac{x}{1 + x} < \frac{x}{1 + \theta x} < x. \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{x}{1 + x} < \log(1 + x) < x,$$

or

$$\frac{1 + x}{x} > [\log(1 + x)]^{-1} > x^{-1},$$

or

$$0 < [\log(1 + x)]^{-1} - x^{-1} < 1.$$

Example 15. *Show that* $\sin x$ *lies between*

$$x - \frac{x^3}{6} \text{ and } x - \frac{x^3}{6} + \frac{x^5}{120}.$$

Solution. If $x = 0$, the statement is obviously true; for then each of the expressions

$$\sin x,\ x - \frac{x^3}{6} \text{ and } x - \frac{x^3}{6} + \frac{x^5}{120}$$

has the value 0.

We shall now consider the cases $x > 0$ and $x < 0$ respectively.

First, let $x > 0$.

By applying Taylor's theorem to the function f defined by $f(x) = \sin x$ in $[0, x]$ and writing the remainder after three terms, we have

$$\sin x = x - \frac{x^3}{6}\cos(\theta x), \text{ where } 0 < \theta < 1.$$

Since $-1 \leq -\cos(\theta x)$, whatever θx may be, and since $x > 0$, therefore,

$$x - \frac{x^3}{6} \leq x - \frac{x^3}{6}\cos(\theta x),$$

i.e.,

$$x - \frac{x^3}{6} \leq \sin x. \qquad \ldots(1)$$

Again, by applying Taylor's theorem to the function f in $[0, x]$ and writing remainder after five terms, we have

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120}\sin(\theta_1 x) \text{ where } 0 < \theta_1 < 1.$$

Since $\sin(\theta_1 x) \leq 1$ whatever $\theta_1 x$ may be, and since $x > 0$, therefore

$$x - \frac{x^3}{6} + \frac{x^5}{120}\sin(\theta_1 x) \leq x - \frac{x^3}{6} + \frac{x^5}{120},$$

i.e.

$$\sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}. \qquad \ldots(2)$$

From (1) and (2), we find that

$$x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120} \qquad \ldots(3)$$

whenever $x > 0$.

Let now $x < 0$. If we set $y = -x$, then $y > 0$. From (3), we have

$$y - \frac{y^3}{6} \leq \sin y \leq y - \frac{y^3}{6} + \frac{y^5}{120} \qquad \ldots(4)$$

Putting $y = -x$ in (4), we have

$$x - \frac{x^3}{6} \geq \sin x \geq x - \frac{x^3}{6} + \frac{x^5}{120}$$

whenever $x < 0$.

Thus we find that the given statement is true for all values of x.

Example 16. *If $f''(x) > 0$ for all $x \in \mathbf{R}$, then show that*

$$f\left(\frac{1}{2}(x_1 + x_2)\right) \leq \frac{1}{2}[f(x_1) + f(x_2)],$$

for every pair of real numbers x_1 and x_2.

Solution. If $x_1 = x_2$, the result is obvious. If $x_1 \neq x_2$, let us suppose that $x_1 < x_2$.

Applying Lagrange's mean value theorem to the function f in the intervals $\left[x_1, \frac{1}{2}(x_1 + x_2)\right]$ and $\left[\frac{1}{2}(x_1 + x_2), x_2\right]$, we have

$$f((x_1 + x_2)/2) - f(x_1) = \left[\frac{1}{2}(x_1 + x_2) - x_1\right] f'(c_1), \qquad \ldots(1)$$

Example 13. *If $0 < x < 1$, show that*

$$2x < \log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right).$$

7-(d)

Deduce that

$$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e.e^{\frac{1}{12n(n+1)}}$$

Solution. To show that

$$2x < \log\frac{1+x}{1-x},$$

consider the function f defined on $[0, 1[$ by setting

$$f(x) = 2x - \log\frac{1+x}{1-x},$$

for all x in $[0, 1[$.

If c be any real number in $[0, 1[$, then f is continuous in $[0, c]$ and derivable in $]0, c[$. Also,

$$f'(x) = -\frac{2x^2}{1-x^2},$$

so that $f'(x) < 0$ in $[0, c[$. This shows that f is strictly decreasing in $[0, c]$. In particular, $f(c) < f(0)$. Since $f(0) = 0$, this means that

$$2x - \log\frac{1+x}{1-x} < 0, \text{ when } x = c.$$

$$\log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right),$$

consider the function g defined by setting

$$g(x) = \log\frac{1+x}{1-x} - 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right).$$

Then $$g'(x) = -\frac{4}{3}\cdot\frac{x^4}{(1-x^2)^2}.$$

Since $g'(x) < 0$ for all x in $]0, 1[$ and since $g(0) = 0$, therefore, it follows in the same manner as above that

$$g(x) < 0 \text{ whenever } 0 < x < 1,$$

that is $$\log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right), \quad \ldots(2)$$

whenever $0 < x < 1$.

From (1) and (2), we have

$$2x < \log\frac{1+x}{1-x} < 2x\left(1+\frac{1}{3}\cdot\frac{x^2}{1-x^2}\right), \quad \ldots(3)$$

whenever $0 < x < 1$.

Putting $x = \dfrac{1}{2n+1}$ in (3), we have

$$\frac{2}{2n+1} < \log\frac{n+1}{n} < \frac{2}{2n+1}\left(1+\frac{1}{12n(n+1)}\right),$$

or $$1 < \left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right) < 1+\frac{1}{12n(n+1)}$$

so that $$e < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$$

which is the desired inequality.

Instead of applying theorems 7-1 to 7-4, one may as well directly apply the mean value theorem to some suitably chosen function in order to establish a given inequality. The following examples illustrate this point.

Instead of applying the mean value theorem, one may have to apply Taylor's theorem (or Maclaurin's theorem) as in example 15, or one may

$$f(x_2) - f((x_1 + x_2)/2) = \left[x_2 - \tfrac{1}{2}(x_1 + x_2)\right] f'(c_2), \qquad \ldots(2)$$

where c_1 and c_2 are some real numbers such that

$$x_1 < c_1 < \tfrac{1}{2}(x_1 + x_2) < c_2 < x_2.$$

Subtracting both sides of (2) from the corresponding sides of (1), we have

$$2f\left(\tfrac{1}{2}(x_1 + x_2)\right) - f(x_1) - f(x_2) = \tfrac{1}{2}(x_2 - x_1)\left[f'(c_1) - f'(c_2)\right] \ldots(3)$$

Again, applying Lagrange's mean value theorem to the function f' in the interval $[c_1, c_2]$, we find that

$$f'(c_2) - f'(c_1) = (c_2 - c_1) f''(d), \qquad \ldots(4)$$

where d is a suitable real number in $[c_1, c_2]$.

Since $c_2 - c_1 > 0$ and $f''(d) > 0$, therefore it follows from (4), that

$$f'(c_2) - f'(c_1) > 0. \qquad \ldots(5)$$

Since $x_2 > x_1$, therefore, from (3) and (5), we have

$$2f\left(\tfrac{1}{2}(x_1 + x_2)\right) - f(x_1) - f(x_2) < 0,$$

i.e.,

$$f\left(\tfrac{1}{2}(x_1 + x_2)\right) < \tfrac{1}{2}\left[f(x_1) + f(x_2)\right].$$

Remark. The reader should satisfy himself that in the above example, the hypotheses of the mean value theorem are satisfied each time the theorem is applied.

PROBLEMS

1. Show that the function f, defined on $\mathbb{R}$ by
$$f(x) = x^3 + 3x^2 + 3x - 8, \text{ for all } x \in \mathbb{R},$$
is increasing in every interval.

2. Show that the function f, defined on $\mathbb{R}$ by
$$f(x) = 9 - 12x + 6x^2 - x^3, \text{ for all } x \in \mathbb{R},$$
is decreasing in every interval.

3. Separate the intervals in which the function f, defined on $\mathbb{R}$ by
$$f(x) = 2x^3 - 15x^2 + 36x - 7, \text{ for all } x \in \mathbb{R};$$
is increasing or decreasing.

Establish the following inequalities by examining the sign of the derivative of an appropriate function:

4. $x - x^3/3 < \tan^{-1} x$, if $x > 0$.

5. $x - x^3/6 < \sin x < x$, if $x > 0$.
6. $e^x > 1 + x + x^2/2$, if $x > 0$.
7. $e^{-x} > 1 - x$, whenever $x > 0$.

Use the mean value theorem to establish the following inequalities :

8. $|\sin x - \sin y| \leq |x - y|$, for all x and y in $\mathbb{R}$.

9. $\dfrac{x}{1 + x^2} < \tan^{-1} x < x$, if $x > 0$.

10. $|\tan^{-1} x - \tan^{-1} y| < |x - y|$, for all x and y in $\mathbb{R}$.

11. $\left|\dfrac{\cos ax - \cos bx}{x}\right| \leq |a - b|$, if $x \neq 0$.

12. $(1 + x)^{1/2} < 1 + \frac{1}{2}x$, if $-1 < x < 0$ or $x > 0$.

13. $x < \sin^{-1} x < \dfrac{x}{\sqrt{1 - x^2}}$, if $0 < x < 1$.

14. $\dfrac{x - a}{x} < \log \dfrac{x}{a} < \dfrac{x - a}{a}$, if $0 < a < x$.

15. $(1 + x)^p < 1 + px$, if $x > 0$ or $-1 < x < 0$, and $0 < p < 1$.

16. $e^a(x - a) < e^x - e^a < e^x(x - a)$, if $a < x$.

17. $\dfrac{x}{1 + x} < \log(1 + x) < x$, if $x > -1$ and $x \neq 0$.

Use Taylor's theorem to establish the following inequalities :

18. $\cos x \geq 1 - \dfrac{x^2}{2}$, for all $x \in \mathbb{R}$.

19. $x - \dfrac{x^3}{6} < \sin x < x$, if $x > 0$.

20. If $1 + x + \dfrac{x^2}{2} < e^x < 1 + x + \dfrac{x^2}{2} e^x$, if $x > 0$.

Prove :

21. $x - \dfrac{x^2}{2} < \log(1 + x) < x - \dfrac{x^3}{2(1 + x)}$, if $x > 0$.

22. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3(1 + x)} < \log(1 + x) < x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$, if $x > 0$.

Example 2. If $f(x)$ be defined on $[a, b]$ such that for $\delta > 0$,
$$|f(x) - f(y)| \le |x - y|^{1+\delta} \ \forall\, x, y \in [a, b],$$ then $f(x)$ is constant on $[a, b]$.

Solution. For $\varepsilon > 0$, when $|x - y| < \varepsilon^{1/\delta}$,
$$|f(x) - f(y)| \le |x - y|^{1+\delta}$$
$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} < \varepsilon, \text{ i.e. } f'(x) = 0 \ \forall\, x \in [a, b].$$

Hence, $f(x)$ is constant on $[a, b]$.

Example 3. If $f(x)$ be real value and differentiable on R and
$$f(x + y) = \frac{f(x) + f(y)}{1 - f(x) f(y)},$$
then $f(x) = \tan(x f'(0))$.

Solution. Here $x = y = 0$, gives $f(0) = 0$, and $y = -x$ gives $f(-x) = -f(x)$ and
$$\frac{f(y - x)}{y - x} = \frac{f(y) - f(x)}{y - x} \cdot \frac{1}{1 + f(x) f(y)}, \text{ as } y \to x$$
$$\Rightarrow f'(0) = f'(x)/(1 + f^2(x))$$
$$\Rightarrow \frac{df}{1 + f^2} = f'(0)\, dx$$
$$\Rightarrow \tan^{-1} f(x) = x f'(0) + a, \text{ and for } x = 0 \text{ it gives } a = 0.$$

Hence, $\tan^{-1} f(x) = x f'(0)$, i.e. $f(x) = \tan(x f'(0))$.

Example 4. The sequence x_n defined by $x_{n+1} = f(x_n)$, where $f(x)$ is continuously differentiable and $|f'| \le r < 1$, is convergent and $x = f(x)$ has unique real value solution to which x_n converges.

Solution. Here $|f(x_n) - f(x_{n-1})| = |f'(c)|\, |x_n - x_{n-1}|$, where c lies between x_n and x_{n+1}
$$\Rightarrow |x_{n+1} - x_n| \le r\, |x_n - x_{n-1}| \le \ldots \le r^{n-1} |x_2 - x_1|$$
$$\Rightarrow |x_{n+p} - x_n| \le |x_2 - x_1| \{r^{n+p-2} + r^{n+p-3} + \ldots + r^{n-1}\}$$
$$< |x_2 - x_1| \frac{1}{1 - r} r^{n-1} \to 0, \text{ as } n \to \infty, \forall\, p \ge 1.$$
$\Rightarrow x_n$ converges.

Let $\lim x_n = x$, then letting $n \to \infty$, $x_{n+1} = f(x_n) \Rightarrow x = f(x)$, where x is the unique real value limit of x_n. Hence, $x = f(x)$ has unique real value solution to which x_n converges.

Cauchy's mean value theorem

Theorem 2. If functions f and g are continuous on $[a, b]$, derivable and $g'(x) \ne 0 \ \forall\, x \in (a, b)$ then there

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Hence the result.

Example 4. If each $a, b > 0$ and each $p, q > 1$ and $1/p + 1/q = 1$, then

$$\frac{a^p}{p} + \frac{b^q}{q} \ge ab,$$

where the equality occurs only when $a^p = b^q$.

The above result is equivalent to that of the preceding Example 3, for

$$x = \frac{1}{p}, \quad \alpha = \frac{qa^p}{qa^p + pb^q}.$$

Independent proof of the above example is left as an exercise.

In Ex. 3, on replacing x by $x_1/(x_1 + x_2)$ and α by $a_1/(a_1 + a_2)$, $\forall\ x_1, x_2, a_1$ and $a_2 \in R^+$, we get:

$$\frac{x_1^{x_1/(x_1+x_2)} \cdot x_2^{x_2/(x_1+x_2)}}{(x_1 + x_2)} \ge \frac{a_1^{x_1/(x_1+x_2)} \cdot a_2^{x_2/(x_1+x_2)}}{(a_1 + a_2)}$$

i.e.,

$$\left(\frac{x_1}{a_1}\right)^{x_1} \left(\frac{x_2}{a_2}\right)^{x_2} \ge \left(\frac{x_1 + x_2}{a_1 + a_2}\right)^{x_1 + x_2} \quad \ldots (*)$$

where the equality occurs only when $\frac{x_1}{x_1 + x_2} = \frac{a_1}{a_1 + a_2}$, *i.e.*, only when $\frac{x_1}{a_1} = \frac{x_2}{a_2}$.

An obvious extension of $(*)$ $\forall\ x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in R^+$ is

$$\left(\frac{x_1}{a_1}\right)^{x_1} \cdots \left(\frac{x_n}{a_n}\right)^{x_n} \ge \left(\frac{x_1 + \ldots + x_n}{a_1 + \ldots + a_n}\right)^{x_1 + \ldots + x_n},$$

$\frac{x_n^2}{a_n}$, $\forall\ x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in R^+$ we get:

$$\left(\frac{a_1}{x_1}\right)^{x_1} \cdots \left(\frac{a_n}{x_n}\right)^{x_n} \ge \left(\frac{x_1 + \ldots + x_n}{x_1^2/a_1 + \ldots + x_n^2/a_n}\right)^{x_1 + \ldots + x_n},$$

where the equality occurs only when $\frac{x_1}{x_1^2/a_1} = \ldots = \frac{x_n}{x_n^2/a_n}$, *i.e.* only when $\frac{x_1}{a_1} = \ldots = \frac{x_n}{a_n}$.

On combining the preceding two inequalities, $\forall\ x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in R^+$, we get

$$\left(\frac{x_1^2/a_1 + \ldots + x_n^2/a_n}{x_1 + \ldots + x_n}\right)^{x_1 + \ldots + x_n} \ge \left(\frac{x_1}{a_1}\right)^{x_1} \cdots \left(\frac{x_n}{a_n}\right)^{x_n} \ge \left(\frac{x_1 + \ldots + x_n}{a_1 + \ldots + a_n}\right)^{x_1 + \ldots + x_n} \quad \ldots (**)$$

where the equality occurs only when $\frac{x_1}{a_1} = \ldots = \frac{x_n}{a_n}$.

Replacement of a_1 by $x_1/a_1, \ldots, a_n$ by x_n/a_n in $(**)$, it equivalently gives Roger's form of the generalised Law of Means:

Theorem 2. $\forall\ x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in R^+$;

$$\left(\frac{x_1 a_1 + \ldots + x_n a_n}{x_1 + \ldots + x_n}\right)^{x_1 + \ldots + x_n} \ge a_1^{x_1} \ldots a_n^{x_n} \ge \left(\frac{x_1 + \ldots + x_n}{x_1/a_1 + \ldots + x_n/a_n}\right)^{x_1 + \ldots + x_n} \quad \ldots (*)$$

where the equality occurs only when $a_1 = \ldots = a_n$.

The above is weighted means property viz. A.M. $\ge$ G.M. $\ge$ H.M.

By putting a_1/x_1 for a_1, and so on, the first part of Roger's generalised law of means $(*)$ gives that the maximum value of

$$\left(\frac{a_1}{x_1}\right)^{x_1} \cdots \left(\frac{a_n}{x_n}\right)^{x_n} \text{ occurs only when } \frac{a_1}{x_1} = \ldots = \frac{a_n}{x_n}$$

and then this value is $\left(\frac{a_1 + \ldots + a_n}{x_1 + \ldots + x_n}\right)^{x_1 + \ldots + x_n}$

Similarly, when $a_1 + \dots + a_n$ is fixed then $a_1^{x_1} \dots a_n^{x_n}$ has the maximum value only when $\frac{a_1}{x_1} = \dots = \frac{a_n}{x_n}$ and then this value is $x_1^{x_1} \dots x_n^{x_n} \left(\frac{a_1 + \dots + a_n}{x_1 + \dots + x_n} \right)^{x_1 + \dots + x_n}$

Note that (*) readily extends to non-negative x_is provided at least one x_i is positive.

Holder's Inequality

Theorem 3. If $a_1, \dots, a_n; b_1, \dots, b_n$ are non-negative reals then

$$\left\{ \sum_{r=1}^{n} a_r^p \right\}^{1/p} \left\{ \sum_{r=1}^{n} b_r^q \right\}^{1/q} \geq \sum_{r=1}^{n} a_r b_r, \qquad \dots (*)$$

where positive reals p, q are such that $1/p + 1/q = 1$; and the equality sign occurs only when $a_r^p = \lambda b_r^q$ for some $\lambda > 0$, or when $a_r = 0 \vee b_r = 0$ (for $r = 1, \dots, n$).

Proof. In Example 3, by putting

$$a = \frac{a_r}{(a_1^p + \dots + a_n^p)^{1/p}}, \quad b = \frac{b_r}{(b_1^q + \dots + b_n^q)^{1/q}}, \quad (r = 1, \dots, n)$$

successively and adding the corresponding sides, we get

$$\frac{1}{p} \left(\frac{a_1^p + \dots + a_n^p}{a_1^p + \dots + a_n^p} \right) + \frac{1}{q} \left(\frac{b_1^q + \dots + b_n^q}{b_1^q + \dots + b_n^q} \right) \geq \frac{a_1 b_1 + \dots + a_n b_n}{(a_1^p + \dots + a_n^p)^{1/p} (b_1^q + \dots + b_n^q)^{1/q}}$$

i.e., $(a_1^p + \dots + a_n^p)^{1/p} (b_1^q + \dots + b_n^q)^{1/q} \geq a_1 b_1 + \dots + a_n b_n$,

where as per the condition of Example 4 itself, the equality sign occurs only when $a_r^p = \lambda b_r^q$ ($\lambda > 0$); or when $a_r = 0 \vee b_r = 0$ ($r = 1, \dots, n$).

Note that for $p = q = 2$ the Holder's inequality (*) is known as the Cauchy's inequality.

Minkowski's Inequality

Theorem 4. If $a_1, \dots, a_n : b_1, \dots, b_n$ are non-negative reals then

$$\left(\sum_{r=1}^{n} a_r^t \right)^{1/t} + \left(\sum_{r=1}^{n} b_r^t \right)^{1/t} \geq \left\{ \sum_{r=1}^{n} (a_r + b_r)^t \right\}^{1/t}$$

where $t > 1$, and the equality occurs only when $a_r = \lambda b_r$ ($\lambda > 0$); or where all $a_r = 0 \vee b_r = 0$ ($r = 1, \dots, n$).

Proof. On applying Holder's inequality to the two group of reals a_r, $(a_r + b_r)^{p-1}$ and b_r, $(a_r + b_r)^{p-1}$ ($p > 1$), we get

$$\left(\sum_1^n a_r^p \right)^{1/p} \left(\sum_1^n (a_r + b_r)^p \right)^{1/q} \geq \sum_1^n a_r (a_r + b_r)^{p-1},$$

$$\left(\sum_1^n b_r^p \right)^{1/p} \left(\sum_1^n (a_r + b_r)^p \right)^{1/q} \geq \sum_1^n b_r (a_r + b_r)^{p-1},$$

where $1/p + 1/q = 1$. Now, on adding the corresponding sides, we have

$$\left\{ \left(\sum_1^n a_r^p \right)^{1/p} + \left(\sum_1^n b_r^p \right)^{1/p} \right\} \left(\sum_1^n (a_r + b_r)^p \right)^{1/q} \geq \sum_1^n (a_r + b_r)^p,$$

i.e.

$$\left(\sum_1^n a_r^p \right)^{1/p} + \left(\sum_1^n b_r^p \right)^{1/p} \geq \left(\sum_1^n (a_r + b_r)^p \right)^{1/p},$$

where p $(= t) > 1$, and equality sign occurs only when

$$a_r^p = \lambda_1 (a_r + b_r)^p \wedge b_r^p = \lambda_2 (a_r + b_r)^p,$$

i.e. when $a_r = \lambda b_r$, $\lambda > 0$; or when all $a_r = 0 \vee b_r = 0$ ($r = 1, \dots, n$).

Sometimes the first mean value theorem itself can be directly used more conveniently to establish certain functional inequalities.

Example 5. (*i*) $\frac{x}{1 + x^2} < \tan^{-1} x < x$, $\forall x > 0$;

(*ii*) $|\tan^{-1} x - \tan^{-1} y| < |x - y|$, $\forall$ unequal $x, y \in \mathbb{R}$.

Solution. (*i*) For $x > 0$, applying the first mean value theorem on $[0, x]$ to $\tan^{-1} x$, we get

$$\tan^{-1} x - \tan^{-1} 0 = (x - 0) \frac{1}{1 + c^2}, \text{ where } x > c > 0.$$

And since $\frac{x}{1 + x^2} < \frac{x}{1 + c^2} < x$, therefore, for $x > 0$, we get (*i*).

(*ii*) For $x \neq y$, on applying the first mean value theorem, we get

$$\tan^{-1} y - \tan^{-1} x = (y - x) \frac{1}{1 + c^2}, \quad \text{when } y > c > x;$$

and

$$\tan^{-1} x - \tan^{-1} y = (x - y) \frac{1}{1 + c^2}, \quad \text{when } x > c > y.$$

... inequalities, sometimes repeated of the theorem 1 § 10.6 is required. A slight adjustment in the choice of $f(x)$ save labour many times.

$\tan x$ x