1(e) If the coordinates of a point A & B are respectively $(b\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$ and if the joining line A & B is produced to the point $M(x,y)$ so that $AM : BM = b : a$, then show that $x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = 0$.

→ The line AB is divided externally by the point M in the ratio $b : (-a)$

A $(a\cos\beta, a\sin\beta)$ B $(b\cos\beta, b\sin\beta)$ $M(x,y)$

Then the point M has coordinates:

$$x = \frac{-ab\cos\alpha + ab\cos\beta}{-a+b} = \frac{ab}{b-a}\left[\cos\beta - \cos\alpha\right] = \frac{ab}{b-a}\left[-2\sin\frac{\beta-\alpha}{2}\sin\frac{\beta+\alpha}{2}\right]$$

$$y = \frac{-ab\sin\alpha + ab\sin\beta}{-a+b} = \frac{ab}{b-a}\left[\sin\beta - \sin\alpha\right] = \frac{ab}{b-a}\left[2\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}\right]$$

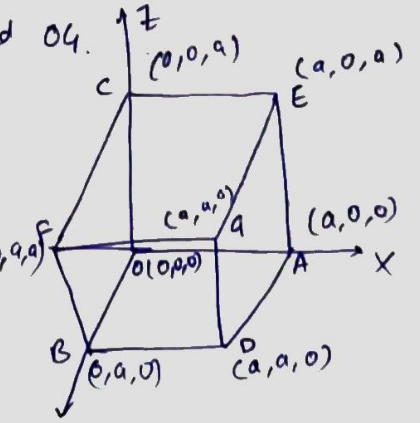
Then $x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{2ab}{b-a}\sin\left(\frac{\beta-\alpha}{2}\right)\left[-\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)\right]$

$$\Rightarrow x\cos\left(\frac{\alpha+\beta}{2}\right) + y\sin\left(\frac{\alpha+\beta}{2}\right) = 0$$

2(c) A line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

→ Let one of the vertices of the cube be the origin & the three axes be its 3 sides through the origin. Let the edge length be a.

Diagonals are CD, AF, BE and OG.



DRs:

(i) CD: $(-a,-a,a) \equiv (-1,-1,1)$

(ii) AF: $(a,-a,-a) \equiv (1,-1,-1)$

(iii) BE: $(a,-a,a) \equiv (1,-1,1)$

(iv) OG: $(a,a,a) \equiv (1,1,1)$

Let the drs of the line which makes angles $\alpha, \beta, \gamma, \delta$ with CD, AF, BF & OG respectively be l, m, n. Then.

$$\cos^2\alpha = \frac{(l-m+n)^2}{3(l^2+m^2+n^2)}, \quad \cos^2\beta = \frac{(l-m-n)^2}{3(l^2+m^2+n^2)}, \quad \cos^2\gamma = \frac{(l-m+n)^2}{3(l^2+m^2+n^2)}$$

$$\cos^2\delta = \frac{(l+m+n)^2}{3(l^2+m^2+n^2)}$$

<u>Adding</u>: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{1}{3(l^2+m^2+n^2)}\left[(l-m+n)^2+(l-m-n)^2+(l-m+n)^2+(l+m+n)^2\right]$

$$= \frac{1}{3(l^2+m^2+n^2)}\left[(l^2+m^2+n^2)4 + 2lm - 2lm - 2lm + 2lm + 2mn - 2mn - 2mn + 2mn + 2nl - 2nl - 2nl + 2nl\right]$$

$$= \frac{4}{3}\,\frac{l^2+m^2+n^2}{l^2+m^2+n^2} = \frac{4}{3}.$$

$$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta = \frac{4}{3}.$$

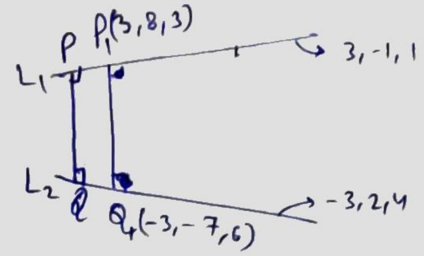
(3) (c) Show that the shortest distance between the straight lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ & $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is $3\sqrt{30}$. Find also the equation of the line of S.D.

~~(x+3)~~

→ Let l, m, n be the dcs of the shortest distance line PQ between the lines:

$L_1: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $L_2: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$

Then $PQ \perp L_1$ & $PQ \perp L_2$



$\therefore 3l - m + n = 0$
$-3l + 2m + 4n = 0$ $\Rightarrow \frac{l}{-6} = \frac{m}{-15} = \frac{n}{3}$

$\Rightarrow \frac{l}{2} = \frac{m}{5} = \frac{n}{-1} = \frac{1}{\sqrt{30}}$

$\therefore l = \frac{2}{\sqrt{30}}, \quad m = \frac{5}{\sqrt{30}}, \quad n = \frac{-1}{\sqrt{30}}$

$S.D. \equiv PQ = l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$

$= \frac{2}{\sqrt{30}}(3+3) + \frac{5}{\sqrt{30}}(8+7) - \frac{1}{\sqrt{30}}(3-6) = 3\sqrt{30}$ units

Now, the eqn of plane through PQ & L_1 is $\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix} = 0$

$\Rightarrow (x-3)(-4) + (y-8)5 + (z-3)17 = 0$

$\rightarrow 4x - 5y - 17z + 79 = 0$ —— ①

Eqn of plane through L_2 & $PQ \equiv \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix} = 0$

$\Rightarrow (x+3)(-22) + (y+7)5 + (z-6)(-19) = 0$

$\Rightarrow 22x - 5y + 19z = 83$ —— ②

Then, the SD line is $4x - 5y - 17z + 79 = 0 = 22x - 5y + 19z - 83$

4(c) A variable plane is parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and intercepts the coordinates axes at A, B & C. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$

→ Any plane parallel to the given plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$ —— ①

$\rightarrow \frac{x}{ak} + \frac{y}{bk} + \frac{z}{ck} = 1$ —— ②

It meets the axes at $(ak,0,0)$, $(0,bk,0)$ & $(0,0,ck)$

Eqn of any sphere through the origin & passing through $A(ak,0,0)$, $B(0,bk,0)$ & $C(0,0,ck)$ be taken as

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0. \quad \text{---} (3)$$

It passes through:

(i) $(0,0,0) \equiv \quad d=0$ (ii) $(ak,0,0) = \quad u=-\frac{ak}{2}$

(iii) $(0,bk,0) \equiv \quad v=-\frac{bk}{2}$ (iv) $(0,0,ck) = \quad w=-\frac{ck}{2}$

$$\therefore (3) \equiv x^2+y^2+z^2-k(ax+by+cz)=0 \quad \text{---} (4)$$

The ~~cone~~ circle is given by intersection of sphere (4) & plane (1)

Eliminating k between (4) & (1):

$$x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$$

$$\Rightarrow \boxed{yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0}$$