3(a) $f(x) = x + \frac{1}{x}$ in $[\frac{1}{2}, 3]$

We note that,

i) $f(x)$ is continuous on $[\frac{1}{2}, 3]$.

ii) $f'(x) = 1 - \frac{1}{x^2}$

Hence, $f(x)$ is differentiable on $(\frac{1}{2}, 3)$

So, $f(x)$ satisfy the conditions of the mean value theorem.

Hence there must exist some $c \in (\frac{1}{2}, 3)$ such that

$$f'(c) = \frac{f(3) - f(\frac{1}{2})}{3 - \frac{1}{2}}$$

$$1 - \frac{1}{c^2} = \frac{(3 + \frac{1}{3}) - (2 + \frac{1}{2})}{5/2} = \frac{1}{3}$$

i.e. $$\frac{1}{c^2} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$c^2 = \frac{3}{2} \Rightarrow c = \pm\sqrt{\frac{3}{2}}$$

$$\simeq \pm 1.225$$

Hence, $c = \sqrt{\frac{3}{2}} \in (\frac{1}{2}, 3)$

3(b)

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

To find a similarity transformation that diagonalises the matrix A.

First we find the eigen-values of A. Then we find a transformation matrix P consisting of eigen-vectors of A such that

$$P^{-1} A P = D$$

Characteristic Equation : $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$-(\lambda+1)\left[\lambda(\lambda-2)+1\right] - 2\left[-\lambda+1\right] - 2\left[-1-(\lambda-2)\right] = 0$$

$$(\lambda+1)(\lambda^2 - 2\lambda + 1) - 2\lambda + 2 - 2\lambda + 2 = 0$$

$$(\lambda^3 - 2\lambda^2 + \lambda + \lambda^2 - 2\lambda + 1) - 4\lambda + 4 = 0$$

$$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

[Alternatively, characteristic Eqn is given by:

$$\lambda^3 - D_1 \lambda^2 + D_2 \lambda - D_3 = 0.$$

D_1 = sum of diagonal entries in A

$= -1 + 2 + 0 = 1$

D_2 = Sum of minors of diagonal elements of A
$= M_{11} + M_{22} + M_{33}$
$= (0+1) + (0-2) + (-2-2)$
$= -5$

D_3 = Determinant of matrix A
$= -1(0+1) - 2(0+1) + 2(-1+2)$
$= -1-2-2 = -5$

Hence, characteristic Eqn is
$$\lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

$\Rightarrow \quad (\lambda-1)(\lambda^2-5) = 0$

$\Rightarrow \quad \lambda = 1, \pm\sqrt{5}$

For $\lambda = 1$, $(A-I)X = 0$

i.e.
$$\begin{bmatrix} -2 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -x + y - z = 0$ $\quad \Rightarrow \quad y = 0$ (adding)
$\quad x + y + z = 0$

$\therefore \quad x + z = 0$ or $z = -x$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$ which is eigenvector for $\lambda = 1$

Again for $\lambda = \sqrt{5}$, $(A - \sqrt{5} I)X = 0$

$$\begin{bmatrix} -1-\sqrt{5} & 2 & -2 \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solve it using row-reduced echelon form

$R_1 \to -\frac{(\sqrt{5}-1)}{4} R_1$

$$\begin{bmatrix} 1 & -\frac{(\sqrt{5}-1)}{2} & \frac{\sqrt{5}-1}{2} \\ 1 & 2-\sqrt{5} & 1 \\ -1 & -1 & -\sqrt{5} \end{bmatrix} X = 0$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -(\sqrt{5}-1)/2 & (\sqrt{5}-1)/2 \\ 0 & \frac{3-\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ 0 & -\left(\frac{\sqrt{5}+1}{2}\right) & -\left(\frac{\sqrt{5}+1}{2}\right) \end{bmatrix} X = 0$$

$R_2 \to R_2 \times \frac{2}{(3-\sqrt{5})}$

$R_3 \to R_3 \times \frac{2}{-(\sqrt{5}+1)}$

$$\begin{bmatrix} 1 & -(\sqrt{5}-1)/2 & (\sqrt{5}-1)/2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} X = 0$$

$R_1 \to R_1 + \left(\frac{\sqrt{5}-1}{2}\right) R_2$, $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & \sqrt{5}-1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow \quad x + (\sqrt{5}-1) z = 0 \quad \Rightarrow x = -(\sqrt{5}-1) z$

$y + z = 0 \quad \Rightarrow y = -z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(\sqrt{5}-1) z \\ -z \\ z \end{bmatrix} = -z \begin{bmatrix} \sqrt{5}-1 \\ 1 \\ -1 \end{bmatrix}$$

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Similarly, we find eigen-vector corresponding to eigenvalue, $\lambda = -\sqrt{5}$ which is $\begin{bmatrix} \sqrt{5}+1 \\ -1 \\ 1 \end{bmatrix}$

form the matrix P, whose ith column is eigenvector no. i, that is

$$P = \begin{bmatrix} -1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{—} \ (\star)$$

The diagonal matrix is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Transforming matrix P gives the required similarity transformation which diagonalizes matrix A.

$$P^{-1} A P = D$$

[verify using calci]

****Ex. 5. Show that the straight line whose direction cosines are given by the equations : $ul + vm + wn = 0 \quad al^2 + bm^2 + cn^2 = 0$ are (α) perpendicular if $u^2 (b + c) + v^2 (c + a) + w^2 (a + b) = 0$ and (β) parallel, if $(u^2/a) + (v^2/b) + (w^2/c) = 0$.**

(*Bundelkhand* 96; *Garhwal* 96, 94, 92, 91; *Gorakhpur* 91; *Kanpur* 93; *Kumaum* 94, 92; *Rohilkhand* 90)

Sol. The d.c.'s of the lines are given by

$$ul + vm + wn = 0 \quad \text{and} \quad al^2 + bm^2 + cn^2 = 0$$

Eliminating n between these, we get

$$al^2 + bm^2 + c\,[-(ul + vm)/w]^2 = 0$$

or $$(aw^2 + cu^2)\,l^2 + (bw^2 + cv^2)\,m^2 + 2cuvlm = 0$$

or $$(aw^2 + cu^2)\,(l/m)^2 + 2\,cuv\,(l/m) + (bw^2 + cv^2) = 0, \qquad \text{...(i)}$$

dividing each term by m^2.

(α) Its two roots are l_1/m_1 and l_2/m_2, if the d.c.'s of the two lines be taken as (l_1, m_1, n_1) and (l_2, m_2, n_2).

∴ From (i), we have

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{bw^2 + cv^2}{cu^2 + aw^2}$$

or $$\frac{l_1 l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{cu^2 + aw^2} = \frac{n_1 n_2}{av^2 + bu^2}, \text{ by symmetry.}$$

∴ If the two lines are perpendicular, then we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad i.e. \quad (bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2) = 0.$$

or $$u^2 (b + c) + v^2 (c + a) + w^2 (a + b) = 0 \qquad \text{Hence proved.}$$

(β) If the two lines are parallel, then their d.c.'s are equal are consequently the roots of (i) are equal, the condition for the same being

$$\text{"}b^2 = 4ac\text{"} \quad i.e. \quad (2cuw)^2 = 4\,(aw^2 + cu^2)\,(bw^2 + cv^2)$$

or $$c^2u^2v^2 = abw^4 + acw^2v^2 + bcu^2w^2 + c^2u^2v^2$$

or $$abw^4 + acw^2v^2 + bcu^2w^2 = 0 \text{ or } abw^2 + acv^2 + bcu^2 = 0$$

or $$\frac{w^2}{c} + \frac{v^2}{b} + \frac{u^2}{a} = 0, \text{ dividing each term by } abc.$$

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4 (a) Let the equation of sphere be

$$x^2+y^2+z^2+2fx+2gy+2hx+c=0 \qquad -(1)$$

having centre $C(-f,-g,-h)$ $\qquad -(2)$

As sphere passes through points $(1,1,2)$ and $(1,-1,2)$. Hence these points satisfy eqn (1)

$$\therefore \quad 1^2+1^2+2^2+2f+2g+4h+c=0$$

$$\Rightarrow \quad 2f+2g+4h+c=-6 \qquad -(3)$$

$$1^2+(-1)^2+2^2+2f-2g+4h+c=0$$

$$\Rightarrow \quad 2f-2g+4h+c=-6 \qquad -(4)$$

Subtracting (4) from (3), we get $g=0$ $\quad -(5)$

Also, centre $(-f,-g,-h)$ lies on the line

$$x+y-z-1=0=2x+y-z-2$$

$$\therefore \quad -f-g+h-1=0=-2f-g+h-2$$

or
$$\left.\begin{aligned} -f+h&=1 \\ -2f+h&=2 \end{aligned}\right\} \qquad [\text{from (5)}, g=0]$$

Solving these, we get $f=-1$, $h=0$

from (3),

$$2(-1)+0+0+c=-6$$

$$\Rightarrow \quad c=-4$$

Putting these values in Eqn (1), we get the equation of sphere as

$$x^2+y^2+z^2-2x-4=0, \quad \text{Centre } (1,0,0)$$

[Note: If equations are complex, use Calci to solve it]

4(b) Given,

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

Let us find characteristic ~~polynomial,~~ Equation,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 3 \\ 1 & -\lambda & -2 \\ 4 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[-\lambda+\lambda^2+4] + 1.[1-\lambda+8] + 3[2+4\lambda] = 0$$

$$(-2\lambda+2\lambda^2+8+\lambda^2-\lambda^3-4\lambda) + (-\lambda+9) + (6+12\lambda) = 0$$

$$-\lambda^3+3\lambda^2+5\lambda+23 = 0.$$

Cayley - Hamilton Theorem states that every square matrix satisfies its characteristic equation. Hence

$$-A^3+3A^2+5A+23 = 0. \quad \text{———} (\star)$$

$$|A| = 2(0+4)+1(1+8)+3(2-0) = 8+9+6$$
$$= 23 \neq 0 \implies A^{-1} \text{ exist.}$$

multiplying A^{-1} on both sides of $(\star)$

$$-A^2+3A+5I+23A^{-1} = 0$$

i.e. $$A^{-1} = \frac{1}{23}[A^2-3A-5I]$$

$$A^2 = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1+12 & -2+6 & 6+2+3 \\ 2-8 & -1-4 & 3-2 \\ 8+2+4 & -4+2 & 12-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 4 & 11 \\ -6 & -5 & 1 \\ 14 & -2 & 9 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{23} \left(\begin{bmatrix} 15 & 4 & 11 \\ -6 & -5 & 1 \\ 14 & -2 & 9 \end{bmatrix} - 3 \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{23} \begin{bmatrix} 15-6-5 & 4+3 & 11-9 \\ -6-3 & -5-5 & 1+6 \\ 14-12 & -2-6 & 9-3-5 \end{bmatrix}$$

$$= \frac{1}{23} \begin{bmatrix} 4 & 7 & 2 \\ -9 & -10 & 7 \\ 2 & -8 & 1 \end{bmatrix}$$

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EXAMPLE 5. *Find the area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptote.*

[K.U. 2016; M.D.U 2015,13]

Solution. The given curve is $x^2y^2 = a^2(y^2 - x^2)$ ...(1)

Let us trace the given curve roughly.

1. The curve is symmetrical about both the axes.

2. The curve passes through the origin and the tangents at the origin are given by $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$, which are real and distinct and so the origin is a node.

3. The curve meets both x-axis and y-axis at the origin.

4. The asymptotes parallel to y-axis are given by $a^2 - x^2 = 0$ *i.e.*, $x = \pm a$ and the curve has no other asymptote.

5. From (1), $y^2 = \dfrac{a^2x^2}{(a^2 - x^2)} = \dfrac{a^2x^2}{(a+x)(a-x)}$ ...(2)

L.H.S. of (2) is always positive, thus its R.H.S. must also be positive. If $x < -a$ or $x > a$, then y^2 becomes negative. Hence no portion of the curve lies beyond the lines $x = \pm a$.

Also as x increases from 0 to a, y increases from 0 to ∞. For the portion of the curve in the 1st quadrant, x varies from 0 to a.

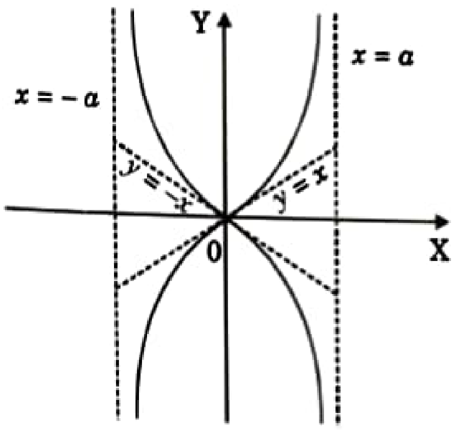

Fig. 10.7

As the curve is symmetrical about both the axes

$\therefore$ Required area between the curve and its asymptotes

$$= 4 \text{ [Area of the curve in the first quadrant and its asymptote]}$$

$$= 4\int_0^a y\,dx$$

$$= 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\,dx \qquad \text{[Using (2)]}$$

Put $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

Now when $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \dfrac{\pi}{2}$

$\therefore$ Required area $= 4\displaystyle\int_0^{\pi/2} \frac{a \cdot a\sin\theta}{a\cos\theta} \cdot a\cos\theta\,d\theta$

$$= 4a^2\int_0^{\pi/2} \sin\theta\,d\theta = 4a^2\left|-\cos\theta\right|_0^{\pi/2}$$

$$= 4a^2(0+1) = 4a^2.$$