Set-III

CAUCHY-EULER EQUTIONS

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

Where $a_1, a_2, \ldots a_n$ are constants and X is a function of x, is called the cauchy-Euler homogenous linear equation of the n^{th} order.

Method of solution:-

Reduce the linear equation

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X$$

into linear equation with constant coeffcients.

The given equation is

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = X \quad \text{--- (1)}$$

put $x = e^z \Rightarrow z = \log x \quad (x>0)$

$$\Rightarrow \frac{dz}{dx} = 1/x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \frac{1}{x}$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow \boxed{x\frac{dy}{dx} = D_1 y}$$

where $D_1 = \frac{d}{dz}$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$= \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{dz}\right]$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\left(\frac{d}{dx}\left(1/x\right)\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dx}\right) + \frac{dy}{dz}\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right) + \frac{dy}{dz}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}(D_1^2 - D_1)y \quad \text{where } D_1 = d/dz$$

$$= \frac{1}{x^2} D_1(D_1-1)y$$

$$\therefore \boxed{x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y}$$

sly $x^3\dfrac{d^3y}{dx^3} = D_1(D_1-1)(D_1-2)y$

$\vdots$

$$x^n\frac{d^ny}{dx^n} = D_1(D_1-1)(D_1-2)\cdots\cdots(D_1-(n-1))y.$$

$\therefore$ From ①, we have

$$\left[D_1(D_1-1)(D_1-2)\cdots\cdots(D_1-(n-1)) + a_1D_1(D_1-1)(D_1-2)\cdots\cdots(D_1-(n-2)) + \cdots\cdots + a_{n-1}D_1 + a_n\right]y = e^z$$

clearly which is a linear equation with constant coefficients and is therefore solvable for y interms of z.

→ If $y = F(z)$ is its solution then putting $z = \log x$, then the required solution is

$$y = F(\log x)$$

Problems

→ Solve the following:

(1) $(x^2D^2 - xD + 2)y = x\log x$

Sol:- Given that $(x^2D^2 - xD + 2)y = x\log x \longrightarrow ①$

Putting $x = e^z \Rightarrow z = \log x$

Let $D_1 = \dfrac{d}{dz}$ then

From ①, we have

$$(D_1(D_1-1) - D_1 + 2)y = e^z z$$

$$\Rightarrow (D_1^2 - 2D_1 + 2)y = ze^z \longrightarrow ②$$

A.E of ② is

$$D_1^2-2D_1+2=0$$

$$D_1 = \frac{2\pm\sqrt{4-4(1)2}}{2(1)}$$

$$= \frac{2\pm\sqrt{-4}}{2}$$

$$= \frac{2\pm 2i}{2} = 1\pm i$$

$$\therefore \boxed{y_c = e^z(c_1\cos z + c_2 \sin z)}$$

P.I of ② is $\frac{1}{D_1^2-2D_1+2}(e^z z)$

$$= e^z\left[\frac{1}{(D_1+1)^2-2(D_1+1)+2}(z)\right]$$

$$= e^z\left[\frac{1}{D_1^2+1+2D_1-2D_1-2+2}\, z\right]$$

$$= e^z\left[\frac{1}{D_1^2+1}\, z\right]$$

$$= e^z(1+D_1^2)^{-1} z$$

$$= e^z(1-D_1^2+\cdots)z$$

$$= e^z(z-0)$$

$$= ze^z$$

$$\therefore \boxed{y_p = ze^z}$$

$\therefore$ G.S of ② is $y = y_c + y_p$

$$y = e^z(c_1\cos z + c_2\sin z) + ze^z$$

$$\Rightarrow y = x\left(c_1\cos(\log x) + c_2\sin(\log x)\right) + x\log x.$$

which is the required general solution of ①

H.W

(2) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Note:- $x^n \frac{d^ny}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X$

Let $X = x^m$ then convert only LHS interms of Z by using $D_1 = \frac{d}{dz}$ but doesnot change RHS $X = x^m$ interms of Z

$\therefore$ The given equation reduces to $f(D_1)y$.

For this special case, we use the following formula directly. $\frac{1}{f(D_1)}x^m = \frac{1}{f(m)}x^m$, provided $f(m) \neq 0$

Problem:

$\rightarrow$ solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x$ —— (1)

Sol: Let $x = e^z \Rightarrow z = \log x$ and

let $D_1 = \frac{d}{dz}$

then $[D_1(D_1-1) - 4D_1 + 6]y = x$

$\Rightarrow [D_1^2 - D_1 - 4D_1 + 6]y = x$

$\Rightarrow [D_1^2 - 5D_1 + 6]y = x$ —— (2)

A.E of (2) is $D_1^2 - 5D_1 + 6 = 0$

$\Rightarrow D_1 = 2, 3$

$\therefore \boxed{y_c = c_1 e^{2z} + c_2 e^{3z}}$

$y_p = \frac{1}{D_1^2 - 5D_1 + 6}(x^1) = \frac{1}{1-5+6}(x)$

$= \frac{1}{2}(x)$

$\therefore$ The g.s (1) is $y = y_c + y_p$

$y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}(x)$

$\Rightarrow \boxed{y = c_1 x^2 + c_2 x^3 + x/2}$

which is the required g.s of (1)

Note:- An alternative method for getting P.I of Cauchy-Euler equation without changing the R.H.S interms of Z.

Let $F(D_1)y = f(x)$ where $D_1 = \frac{d}{dz}$.

then $\frac{1}{D_1-\alpha} f(x) = x^{\alpha}\int x^{-\alpha-1} f(x)\,dx$.

and $\frac{1}{D_1+\alpha} f(x) = x^{-\alpha}\int x^{\alpha-1} f(x)\,dx$.

To evaluate P.I, we first factorize $F(D_1)$ into linear factors and then one of the following methods can be used.

Method (I): If the operator $\frac{1}{F(D_1)}$ into partial fractions then

$$P.I = \frac{1}{F(D_1)} f(x) = \left[\frac{A_1}{(D_1-\alpha_1)} + \frac{A_2}{(D_1-\alpha_2)} + \cdots + \frac{A_n}{D_1-\alpha_n}\right] f(x).$$

Method (II): $$P.I = \frac{1}{(D_1-\alpha_1)(D_1-\alpha_2)\cdots(D_1-\alpha_n)} f(x)$$

where the operations indicated by factors are to be taken in succession, begining with the first on the right.

Problems:-

→ Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x$.

Sol:- Given that $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x + \sin x$ ——— (1)

Let $x = e^z \Rightarrow z = \log x$

then (1) $= [D_1(D_1-1) + 4D_1 + 2]y = x + \sin x$

$\Rightarrow [D_1^2 + 3D_1 + 2]y = x + \sin x$ ——— (2)

Now A.E of (2) is $m^2 + 3m_1 + 2 = 0$

$\Rightarrow m = -2, -1$

$\therefore \boxed{y_c = c_1 e^{-2z} + c_2 e^{-z}}$

$$P.I = \frac{1}{D_1^2+3D_1+2}(x+\sin x) = \frac{1}{(D_1+2)(D_1+1)}(x+\sin x)$$

$$= \frac{1}{(D_1+2)(D_1+1)}x + \frac{1}{(D_1+2)(D_1+1)}\sin x \quad \text{——— (3)}$$

Now $\frac{1}{(D_1+2)(D_1+1)} \sin x = \frac{1}{D_1+2}\left[\frac{1}{D_1+1}\sin x\right]$

$= \frac{1}{D_1+2}\left[x^{-1}\int x^{1-1}\sin x\,dx\right]$

$= \frac{1}{D_1+2}\left[\frac{-1}{x}\cos x\right]$

$= x^{-2}\int x^{2-1}\left(\frac{-1}{x}\cos x\right)dx$

$= -x^{-2}\int \cos x\,dx$

$= -\frac{1}{x^2}\sin x$

$\therefore$ G.S of ② is $y = y_C + y_C$

$y = (c_1 e^{-2z} + c_2 e^{-z}) - \frac{1}{x^2}\sin x + \frac{x}{6}$

$\Rightarrow \boxed{y = c_1 x^{-2} + c_2 x^{-1} + \frac{x}{6} - \frac{1}{x^2}\sin x.}$

which is reqd g.s of ①

* solve the following diff eqns :-

→ $(x^2D^2 + xD - 1)y = x^2 e^{2x}$ Ans: $y = c_1 x + c_2 x^{-1} + \frac{1}{8}(e^{2x})(2 - x^{-1})$

→ $(x^2D^2 + xD - 1)y = x^2 e^{x}$ Ans: $y = c_1 x + c_2 x^{-1} + e^{x}(1 - x)$

→ $x^3\frac{d^3y}{dx^3} + 2x^2\left(\frac{d^2y}{dx^2}\right) + 2y = 10\left(x + \frac{1}{x}\right)$

96 → $(x^3D^3 + 3x^2D^2 + xD + 1)y = x\log x$

1990 → $x^2\left(\frac{d^2y}{dx^2}\right) + 2x\left(\frac{dy}{dx}\right) + y = 0$

→ $x^2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = e^x$

* LEGENDRE'S LINEAR EQUATIONS:

An equation of the form

$(a+bx)^n \frac{d^ny}{dx^n} + A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + A_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots$

$+ A_{n-1}(ax+b)\frac{dy}{dx} + A_n y = Q(x).$

where $A_1, A_2, \ldots\ldots A_n$ are constants and $Q(x)$ is a function of 'x' is called Legendre's linear equation of the n^{th} order.

* Method of solution:-

Reduce the linear equation

$$(a+bx)^n \frac{d^n y}{dx^n} + A_1 (a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + A_2 (a+bx)^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots$$

$$+ A_{n-1}(ax+b)\frac{dy}{dx} + A_n y = Q(x) \quad \text{——— (1)}$$

where $A_1, A_2, \ldots\ldots A_{n-1}, A_n$ are constants. into linear equation with constant coeffficients.

putting $a+bx = e^z \Rightarrow z = \log(a+bx)$

$$\Rightarrow \frac{dz}{dx} = b\left(\frac{1}{a+bx}\right).$$

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

$$= \frac{dy}{dz}\left[b\frac{1}{a+bx}\right]$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$\Rightarrow \boxed{(a+bx)\frac{dy}{dx} = bD_1 y} \quad \text{Where } D_1 = \frac{d}{dz}$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$$= \frac{d}{dx}\left[\frac{dy}{dz}\,\frac{b}{a+bx}\right]$$

$$= \frac{b}{a+bx}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\frac{d}{dx}\left(\frac{b}{a+bx}\right).$$

$$= \frac{b}{a+bx}\frac{d}{dz}\left(\frac{dy}{dx}\right) + \frac{dy}{dz}\left(\frac{-b^2}{(a+bx)^2}\right)$$

$$= \frac{b}{a+bx}\frac{d}{dz}\left[\frac{dy}{dz}\cdot\frac{b}{a+bx}\right] - \frac{b^2}{(a+bx)^2}\frac{dy}{dz}.$$

$$= \frac{b^2}{(a+bx)^2}\frac{d^2y}{dz^2} - \frac{b^2}{(a+bx)^2}\frac{dy}{dz}$$

$$= \frac{b^2}{(a+bx)^2}(D_1^2 - D_1)y$$

$$= \frac{b^2}{(a+bx)^2} D_1(D_1-1)y.$$

$$\therefore \boxed{(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y}$$

$$\text{s}^{ly}\ (a+bx)^3 \frac{d^3y}{dx^3} = b^3 D_1(D_1-1)(D_1-2)y$$

$$\vdots$$

$$(a+bx)^n \frac{d^ny}{dx^n} = b^n\left[D_1(D_1-1)(D_1-2)\cdots(D_1-(n-1))\right]y$$

$\therefore$ from (1), we have,

$$b^n(D_1(D_1-1)(D_1-2)\cdots\cdots(D_1-(n-1)))+$$
$$b^{n-1}A_1(D_1(D_1-1)(D_1-2)\cdots(D_1-(n-2)))+\cdots\cdots+A_{n-1}bD_1$$
$$+A_n]y = Q\left(\frac{e^z-a}{b}\right)$$

clearly which is a linear equation with constant coefficients and hence is solvable for y interms of z.

Working rule:-

putting $a+bx = e^z \Rightarrow z = \log(a+bx)$

Let $D_1 = \frac{d}{dz}$ then

$$(a+bx)\frac{dy}{dx} = bD_1y$$

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2D_1(D_1-1)y \text{ etc.}$$

problems.

$\rightarrow$ Solve the following:

(1) $[(3x+2)^2D^2+3(3x+2)D-36]y = 3x^2+4x+1$

Sol: Given that

$$\left[(3x+2)^2D^2+3(3x+2)D-36\right]y = 3x^2+4x+1 \quad \text{——— (1)}$$

Putting $3x+2 = e^z \Rightarrow z = \log(3x+2)$

Let $D_1 = \frac{d}{dz}$ then

$$(3x+2)\frac{dy}{dx} = 3D_1 y$$

$$(3x+2)\frac{d^2y}{dx^2} = 3^2 D_1(D_1-1)y.$$

From (1), we have

$$\left[3^2 D_1(D_1-1)+3(3)D_1-36\right]y = \frac{e^{2z}-1}{3}$$

$$\Rightarrow 9\left[D_1^2-\cancel{D_1}+\cancel{D_1}-4\right]y = \frac{e^{2z}-1}{3}$$

$$\Rightarrow \left[D_1^2-4\right]y = \frac{e^{2z}-1}{27} \quad \text{——— (2)}$$

A.E of (2) is $m^2-4=0$

$m^2=4 \Rightarrow m=\pm 2$

$$\therefore \boxed{y_c = c_1e^{-2z}+c_2e^{+2z}}$$

P.I of (2) is $\frac{1}{D_1^2-4}\left(\frac{e^{2z}-1}{27}\right) = \frac{1}{27}\left[\frac{1}{D_1^2-4}(e^{2z}-1)\right]$

$$= \frac{1}{27}\left[\frac{1}{D_1^2-4}e^{2z}-\frac{1}{D_1^2-4}(1)\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{1}{(D_1+2)^2-4}(1)-\frac{1}{-4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{1}{D_1^2+2D_1}(1)+\frac{1}{4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{z}{2D_1+2}(1)+\frac{1}{4}\right]$$

$$= \frac{1}{27}\left[e^{2z}\frac{z}{2}(1)+\frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{ze^{2z}}{2}+\frac{1}{4}\right]$$

$$= \frac{1}{54}\left[ze^{2z}+\frac{1}{2}\right]$$

$\therefore$ G.S of (2) is $y = y_c + y_p$

$$y = (c_1 e^{-2z} + c_2 e^{2z}) + \frac{1}{54}(z e^{2z} + 1/2)$$

$$\Rightarrow y = \left[c_1 \frac{1}{(3x+2)^2} + c_2 (3x+2)^2\right] + \frac{1}{54}\left[\log(3x+2).(3x+2)^2 + \frac{1}{2}\right]$$

which is the reqd g.s of ①

$\xrightarrow{HW}$ $(x+1)^2 y_2 - 3(x+1) y_1 + 4y = x^2$

$\rightarrow$ $[(5+2x)^2 D^2 - 6(5+2x) D + 8] y = 0$

$\xrightarrow[HW]{1997}$ $(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$

$y(0) = 0,\ y'(0) = 2$

$\xrightarrow{HW}$ $[(x+1)^4 D^3 + 2(x+1)^3 D^2 - (D+1)^2 D + (x+1)] y = \frac{1}{x+1}$

* Linear Differential Equations of the second order with variable coefficients:

An equation of the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) y = R(x)$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of 'x' is called linear diff. equation of the second order with variable co-effcients.

$\rightarrow$ The linear diff. equation of the second order with variable coefficients can be solved by the following methods.

(1) Change of the dependent variable, when a part of the complementary function (C.F) is known.

(2) Change of the dependent variable and removal of the first derivative (or) reduction to normal form.

(3) change of the independent variable.

(4) variation of parameters.

1. To solve $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ by changing of the dependent variable when a part of the C.F is known:-

Given equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \quad \text{——— (1)}$$

and its linear homogenous equation is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{——— (2)}$$

Let $y = u(x)$ be a known solution of the C.F of (1)

Then. $y = u(x)$ is a solution of (2),

$$\frac{d^2y}{dx^2} + P(x)\frac{du}{dx} + Q(x)u(x) = 0 \quad \text{——— (3)}$$

Let $y = uv$ be the g.s of (1) where $u = u(x)$, $v = v(x)$

then
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{——— (4)}$$

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} \quad \text{——— (5)}$$

$\therefore$ (1) $\equiv$
$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P(x)\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) \ldots$$
$$+ Q(x)uv = R(x).$$

$$\Rightarrow v\left[\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)u\right] + u\left[\frac{d^2v}{dx^2} + P(x)\frac{dv}{dx}\right]$$
$$+ 2\frac{du}{dx}\frac{dv}{dx} = R(x)$$

$$\Rightarrow v(0) + u\left(\frac{d^2v}{dx^2} + P(x)\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R(x) \text{ (by (3))}$$

$$\Rightarrow u\frac{d^2v}{dx^2} + uP(x)\frac{dv}{dx} + 2\frac{du}{dx}\frac{dv}{dx} = R(x)$$

$$\Rightarrow u\frac{d^2v}{dx^2} + \left(uP(x) + 2\frac{du}{dx}\right)\frac{dv}{dx} = R(x).$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(P(x) + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R(x)}{u} \longrightarrow (6)$$

Let $\frac{dv}{dx} = v$ then $\frac{d^2v}{dx^2} = \frac{dv}{dx}$

$$\therefore ⑥ \equiv \frac{dv}{dx} + \left(P(x) + \frac{2}{u}\frac{du}{dx}\right)v = \frac{R(x)}{u} \quad \text{———} ⑦$$

Clearly this is linear equation in v of first order

$$\therefore I.F = e^{\int\left(P(x)+\frac{2}{u}\frac{du}{dx}\right)dx}$$

$$= e^{\int P(x)dx + \int \frac{2}{u}du}$$

$$= e^{\int P(x)dx + 2\log u}$$

$$\boxed{I.F = u^2 e^{\int P(x)dx}}$$

$\therefore$ The g.s of ⑦ is

$$v.u^2 e^{\int P(x)dx} = \int\left[\frac{R}{u}u^2 e^{\int P(x)dx}\right]dx + c$$

$$\Rightarrow \quad vu^2 e^{\int P(x)dx} = \int\left[Ru\, e^{\int P(x)dx}\right]dx + c$$

$$\Rightarrow \quad \frac{dv}{dx}u^2 e^{\int P(x)dx} = \int\left[Ru\, e^{\int P(x)dx}\right]dx + c \qquad \left(\because v = \frac{dv}{dx}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = \frac{e^{-\int P(x)dx}}{u^2}\left[\int\left(Ru\, e^{\int P(x)dx}\right)dx\right] + \frac{c\,e^{-\int P(x)dx}}{u^2}$$

$$\Rightarrow \quad v = \int\left[\left[\frac{1}{u^2}e^{-\int P(x)dx}\int\left(Ru\,e^{\int P dx}\right)dx + \frac{c\,e^{-\int P(x)dx}}{u^2}\right]dx + c_2\right.$$

$\therefore$ From ①, we have

$$y = c_2 u + u\int\left[\frac{1}{u^2}e^{-\int P(x)dx}\int\left(Ru\,e^{\int P dx}\right)dx + \left[\frac{c\,e^{-\int P(x)dx}}{u^2}\right]dx\right.$$

Since this solution includes the known solution $y = u(x)$ and it contains two arbitrary constants, It is the g.s of ①

* Methods for finding one integral (Solution) in C.F by inspection i.e. Solution of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

Method (1):

If $y=e^{ax}$ is a solution of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \quad \text{——— (i)}$$

then $\frac{dy}{dx}=ae^{ax}$, $\frac{d^2y}{dx^2}=a^2e^{ax}$

$\therefore$ (i) $\equiv a^2e^{ax}+P(x)=e^{ax}+Q(x)e^{ax}=0$

$\Rightarrow a^2+aP(x)+Q(x)=0 \quad (\because e^{ax}\neq 0)$ ——— (ii)

$\therefore$ If $y=e^{ax}$ is a solution of (i) then $a^2+aP(x)+Q(x)=0$

putting $a=1$:

$\therefore y=e^{x}$ is a solution of (i) then $1+P(x)+Q(x)=0$

putting $a=-1$:

$\therefore y=e^{-x}$ is a solution of (i)

then $1-P+Q=0$

$\therefore$ if $1+P+Q=0$ then $y=e^{x}$ is a solution of (i)

if $1-P+Q=0$ then $y=e^{-x}$ is a solution of (i).

i.e a part of the C.F of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)=R(x)$

Method (2):

If $y=x^m$ is a solution of $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

then $\frac{dy}{dx}=mx^{m-1}$, $\frac{d^2y}{dx^2}=m(m-1)x^{m-2}$

$\therefore$ (1) $\equiv m(m-1)x^{m-2}+P(x)mx^{m-1}+Q(x)x^m=0$

$\Rightarrow m(m-1)+P(x)mx+Q(x)+x^2=0$

$\therefore$ If $y=x^m$ is a solution of (i) then

$m(m-1)+P(x)mx+Q(x)x^2=0$

putting $m=1$ in the above

$\therefore y=x$ is a solution of (i) then

$0+xP(x)+x^2Q(x)=0$

$$\boxed{P + xQ = 0} \quad (\because x \neq 0)$$

putting $m = 2$

$\therefore y = x^2$ is a solution of (i) then

$$2(2-1) + 2P.x + Q.x^2 = 0$$

$$\Rightarrow \boxed{2 + 2P.x + Q.x^2 = 0}$$

Conversely suppose that $P + Q.x = 0$ then $y = x$ is solution of (i)

and $2 + 2Px + Q.x^2 = 0$ then $y = x^2$ is solution of (i)

Working rule:

step(1): write the given equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\_ = R(x)$$

step(2): Find one solution $y = u(x)$ of C.F by using the following formulas.

Condition satisfied	one solⁿ of C.F is
(1) $a^2 + aP + Q = 0$	$u = e^{ax}$
(i) $1 + P + Q = 0$	$u = e^{x}$
(ii) $1 - P + Q = 0$	$u = e^{-x}$
(2) $m(m-1) + Pmx + Qx^2 = 0$	$u = x^m$
(i) $P + Qx = 0$	$u = x$
(ii) $2 + 2Px + Qx^2$	$u = x^2$

Step(3): Assume the g.s of given equation is $y = uv$ where u is obtained by step(2) and v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}.$$

Problems: Solve the following:

$\rightarrow xy'' - (2x-1)y' + (x-1)y = 0$ ——— ①

$\Rightarrow \dfrac{d^2y}{dx^2} - (2 - 1/x)y' + \left(1 - \dfrac{1}{x}\right)y = 0$ ——— ②

Comparing ① with

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$$

$$P=-\left(2-\frac{1}{x}\right);\ Q=\left(1-\frac{1}{x}\right) \text{ and } R=0$$

Now here

$$1+P+Q=1-2+\frac{1}{x}+1-\frac{1}{x}$$
$$=0$$

$\therefore\ y=u=e^x$ is a part of C.F of ①

Let $y=uv$ — ② be the g.s of ①

then v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u} \quad \text{— ③}$$

Since $u=e^x \Rightarrow \frac{du}{dx}=e^x$;

$$P+\frac{2}{u}\frac{du}{dx}=-2+\frac{1}{x}+2e^{-x}e^{x}$$
$$=-2+\frac{1}{x}+2$$
$$=\frac{1}{x}$$

$\therefore$ From ③, we have

$$\frac{d^2v}{dx^2}+\left(\frac{1}{x}\right)\frac{dv}{dx}=0 \quad \text{— ④}$$

Take $\frac{dv}{dx}=V \Rightarrow \frac{d^2v}{dx^2}=\frac{dV}{dx}$

$$\therefore\ ④ \equiv \frac{dV}{dx}+\frac{1}{x}V=0$$

$$\Rightarrow \frac{dV}{V}=-\frac{1}{x}dx$$

$$\Rightarrow \log(Vx)=\log c_1$$

$$\Rightarrow Vx=c_1$$

$$\Rightarrow \frac{dv}{dx}x=c_1 \quad \left(\because V=\frac{dv}{dx}\right)$$

$$\Rightarrow dv=\frac{c_1}{x}dx$$

$$\Rightarrow \boxed{v=c_1\log x+c_2}$$

$\therefore$ ② $\equiv$ $y=e^x(c_1\log x+c_2)$ is the reqd g.s of ①.

→ $y'' + y = \sec x$ given that $\cos x$ is a part of C.F

Sol:- Given that $\frac{d^2y}{dx^2} + y = \sec x$ ——— ①

and $y = u = \cos x$ is a part of C.F of ①

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = 0;\ Q = 1;\ R = \sec x.$

Let $y = uv$ be the g.s of ①

then v is obtained by

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \text{ — ②}$$

since $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$

$$\therefore P + \frac{2}{u}\frac{du}{dx} = 0 + \frac{2}{\cos x}(-\sin x)$$

$$= -2\tan x$$

Now from ②, we have

$$\frac{d^2v}{dx^2} - 2\tan x\frac{dv}{dx} = \frac{\sec x}{\cos x}$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\tan x\frac{dv}{dx} = \sec^2 x \text{ — ③}$$

Let $\frac{dv}{dx} = V \Rightarrow \frac{d^2v}{dx^2} = \frac{dV}{dx}$

$$\therefore ③ \equiv \frac{dV}{dx} - 2\tan x\, V = \sec^2 x \text{ —— ④}$$

$$\text{I.F} = e^{-\int 2\tan x\, dx}$$

$$= e^{+2\log(\cos x)}$$

$$= \cos^2 x$$

$\therefore$ G.s of ④ is $V\cos^2 x = \int \sec^2 x \cos^2 x\, dx + c_1$

$$= x + c_1$$

$$\therefore V\cos^2 x = x + c_1$$

$$\Rightarrow \frac{dv}{dx}\cos^2 x = x + c_1 \quad \left(\because V = \frac{dv}{dx}\right)$$

$$\Rightarrow dv = (x + c_1)\sec^2 x\, dx.$$

$$\Rightarrow v = \int (x+c_1)\sec^2 x\, dx + c_2$$
$$= (x+c_1)\tan x + \log(\cos x) + c_2$$
$$= \frac{(x+c_1)\sin x + \cos x \log(\cos x) + c_2 \cos x}{\cos x}$$

$\therefore$ G.S of ① is $y = uv$

$\Rightarrow y = (x+c_1)\sin x + \cos x \log(\cos x) + c_2 \cos x$

$\xrightarrow{H.W}$ $y'' - (2x+1)y' + (x+1)y = x^3 e^x$

$\xrightarrow{H.W}$ $(1-x^2)y_2 + xy_1 - y = x(1-x^2)^{3/2}$

$\xrightarrow{H.W}$ $xy_2 - y_1 - 4x^3 y = 4x^5$, given that $y = e^{x^2}$ is a solution if that left hand side is equated to zero.

Ans: $\boxed{y = c_1 e^{-x^2} + c_2 e^{x^2} + x^2}$

② To solve : $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ by changing the dependent variable and removal of the first derivative.

2000 (or) Reduce the diff. equation $y'' + Py' + Qy = R$ where P, Q, R are functions of x, to the form $\frac{d^2v}{dx^2} + Iv = S$ which is known as the normal form of the given equation.

Sol:- Given equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Let $y = uv$ —② be the g.s of ① where u, v, are fns of x.

Now $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ ——③

and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$ ——④

① $\Rightarrow u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{du}{dx} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$

$\Rightarrow u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$ ——⑤

To remove the first derivative $\frac{dv}{dx}$ in ⑤

choose u s.t $Pu + 2\frac{du}{dx} = 0$ ——⑥

$\Rightarrow \frac{du}{dx} = -\frac{P}{2}u$

$\Rightarrow \frac{du}{u} = -\frac{1}{2} P(x)\,dx$

$\Rightarrow \log u = -\frac{1}{2}\int P(x)\,dx$

$\Rightarrow \boxed{u = e^{-\frac{1}{2}\int P\,dx}}$

$⑥ \equiv \boxed{\frac{du}{dx} = -\frac{1}{2} Pu}$ —— ⑦

$$\frac{d^2u}{dx^2} = -\frac{1}{2} P\frac{du}{dx} - \frac{1}{2} u\frac{dP}{dx}$$

$$\boxed{\frac{d^2u}{dx^2} = -\frac{1}{2} P\left(-\frac{1}{2} Pu\right) - \frac{1}{2} u\frac{dP}{dx}} \quad \text{—— ⑧}$$

$⑤ \equiv u\frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)uv = R$

$\Rightarrow \frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)v = \frac{R}{u}$

$\Rightarrow \boxed{\frac{d^2v}{dx^2} + Iv = S}$ —— ⑨

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$, $S = \frac{R}{u}$ which is the reqd normal form of ①

The g.s of ① is $y = uv$

where $u = e^{-\frac{1}{2}\int P\,dx}$ & v is given by ⑨

Working rule:

Step(1): Write the given equation in the standard form

$y'' + Py' + Qy = R$

Step(2): To remove the first derivative we choose $u = e^{-\frac{1}{2}\int P\,dx}$.

Step(3): Assume the g.s of the given eqn is $y = uv$ then u is given by step(2) and v is given by the normal form. $\frac{d^2v}{dx^2} + Iv = S$

Where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$ and $S = \frac{R}{u}$

Problems

→ Solve the following:

$\boxed{2000}$ ① $y'' - 4xy' + (4x^2-1)y = -3e^{x^2}\sin 2x$.

Sol: Comparing ① with $y'' + P(x)y' + Q(x)y = R(x)$

$P = -4x$; $Q = 4x^2-1$; $R = -3e^{x^2}\sin 2x$

To remove the first derivative

We choose $u = e^{-\frac{1}{2}\int P\,dx}$

$= e^{-\frac{1}{2}\int(-4x)\,dx}$

$= e^{x^2}$ —— ②

Let $y = uv$ —③ be the g.s of ①

then v is given by the normal form $\frac{d^2v}{dx^2} + Iv = S$ —— ④

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$; $S = \frac{R}{u}$

Now $I = 4x^2 - 1 - \frac{1}{4}(16x^2) - \frac{1}{2}(-4)$

$= 1$

and $S = -3\sin 2x$

∴ ④ ≡ $\frac{d^2v}{dx^2} + v = -3\sin 2x$

$\Rightarrow (D^2+1)v = -3\sin 2x$ —— ⑤

A.E is $D^2+1=0$

$\Rightarrow D = \pm i$

∴ C.F $= c_1\cos 2x + c_2\sin 2x$

P.I $= \frac{1}{D^2+1}(-3\sin 2x)$

$= \frac{-3}{-3}\sin 2x$

$= \sin 2x$

∴ G.S of ⑤ is $v = c_1\cos 2x + c_2\sin 2x + \sin 2x$.

∴ ③ ≡ $y = e^{x^2}(c_1\cos 2x + c_2\sin 2x) + \sin 2x$

which is the reqd g.s of ①

H.W → $y'' - 2\tan x \cdot y' + 5y = 0$.

1997 → Make use of the transformation $y(x) = v(x)\sec x$ to obtain the solution of $y'' - 2y'\tan x + 5y = 0$, $y(0) = 0$, $y'(0) = \sqrt{6}$.

(3) To solve $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Qy = R(x)$ by changing the independent variable.

Given eqn is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ —— (1)

Let the independent variable 'x' be changed into another independent variable z.

where z is a function of 'x'

i.e let $z = f(x)$

Now $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dx}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right).$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2}$$

(1) ≡ $\frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + P\frac{dy}{dz}\cdot\frac{dz}{dx} + Qy = R$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

$$\Rightarrow \frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \left(\frac{Q}{\left(\frac{dz}{dx}\right)^2}\right)y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \text{ —— (2)}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (3)

$Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (4)

and $R_1 = \dfrac{R_1}{\left(\dfrac{dz}{dx}\right)^2}$ ——— (5)

Here P_1, Q_1, R_1 are functions of x. But these functions of z by using $z = f(x)$

We now chose z s.t $P_1 = 0$ and $Q_1 = \pm a^2$ (constant)

Case(i) If $P_1 = 0$ then (3) $\equiv \dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx} = 0$

$$\Rightarrow \frac{d^2z}{dx^2} \Big/ \frac{dz}{dx} = -P$$

$$\Rightarrow \frac{\frac{d}{dx}\left(\frac{dz}{dx}\right)}{\frac{dz}{dx}} = -P$$

$$\Rightarrow \log\left(\frac{dz}{dx}\right) = -\int P\,dx$$

$$\Rightarrow \frac{dz}{dx} = e^{-\int P dx}$$

$$\Rightarrow \boxed{z = \int e^{-\int P dx}\, dx}$$

Case(ii): If $Q_1 = \pm a^2$ (real constant) then from (4), we get

$$\frac{Q}{(dz/dx)^2} = \pm a^2 \Rightarrow \pm Q = a^2\left(\frac{dz}{dx}\right)^2$$

$$\Rightarrow a\frac{dz}{dx} = \sqrt{\pm Q}$$

$$\Rightarrow a\int dz = \int \sqrt{\pm Q}\, dx$$

(+ve or −ve sign is taken to make the expression under the radical sign +ve).

Problems:

→ Solve the following:

(1) $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3\sin x^2$ ——— (1)

Sol: $\dfrac{d^2y}{dx^2} - \dfrac{1}{x}\dfrac{dy}{dx} - 4x^2y = 8x^2\sin x^2$ ——— (2)

comparing ① with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = R(x)$.

$P = \frac{1}{x}$; $Q = -4x^2$; $R = 8x^2 \sin x^2$

Changing the independent variable from x to z

Where z is a function of 'x' i.e $z = f(x)$

∴ The given equation ① transformed into

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \text{— ②}$$

Where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$

$Q_1 = \dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$; $R_1 = \dfrac{R}{\left(\frac{dz}{dx}\right)^2}$

Choosing z s.t $\dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = $ constant $= -1$ (say)

$$\Rightarrow \left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \frac{dz}{dx} = 2x.$$

$$\Rightarrow \boxed{z = x^2}$$

Now $P_1 = \dfrac{2 + \left(\frac{1}{x}\right)(2x)}{4x^2} = 0$

$R_1 = \dfrac{8x^2 \sin x^2}{4x^2} = 2\sin x^2$
$= 2\sin z$.

$$\therefore ② \equiv \frac{d^2y}{dz^2} + 0 - y = 2\sin z$$

$$\Rightarrow \frac{d^2y}{dz^2} - y = 2\sin z \quad \text{— ③}$$

$\Rightarrow (D^2 - 1)\,y = 2\sin z$ where $D = \frac{d}{dz}$

A.E is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore \boxed{C.F = c_1 e^{z} + c_2 e^{-z}}$

$$P.I = 2 \cdot \frac{1}{D^2-1} \sin z$$

$$= 2 \cdot \frac{1}{-1-1} \sin z = -\sin z$$

$\therefore$ G.S of ③ is $y = y_c + y_p$

$$y = c_1 e^{z} + c_2 e^{-z} - \sin z.$$

$$\therefore \boxed{y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2}$$

which is the reqd g.s of ①.

$\rightarrow$ $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y\cos^3 x = \cos^5 x$

Sol: $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} - 2(\cos^2 x)\, y = 2\cos^4 x$ —— ①

$P = \tan x$; $Q = -2\cos^2 x$; $R = 2\cos^4 x$.

changing the independent variable z from 'x' to new independent variable z. where z is a function of 'x' i.e $z = f(x)$.

$\therefore$ The given equation is transformed into

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \quad \text{—— ②}$$

Where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$; $Q_1 = \dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$; $R_1 = \dfrac{R}{\left(\dfrac{dz}{dx}\right)^2}$

Let us choose z s.t $-\dfrac{2\cos^2 x}{\left(\dfrac{dz}{dx}\right)^2}$

$= \text{constant}$

$= -2$ (say)

$P_1 = 0$, $R_1 = 2\cos^2 x$

$= 2(1-\sin^2 x)$

$= 2(1-z)$

$\therefore$ ② $\equiv \dfrac{d^2y}{dz^2} + 0 - 2y = 2(1-z)$

$\Rightarrow \dfrac{d^2y}{dz^2} - 2y = 2(1-z)$

$\boxed{\text{continue next soln}}$.

$\xrightarrow{\text{H.W}}$ $y'' + \left(\frac{2}{x}\right)y' + \left(\frac{a^2}{x^4}\right)y = 0$ / $Q_1 = a^2$

$\xrightarrow{\text{H.W}}$ $x^4y'' + 2x^3y' + n^2y = 0$ / $Q_1 = n^2$

$\xrightarrow{\text{H.W}}$ $x^6y'' + 3x^5y' + a^2y = \frac{1}{x^2}$ / $Q_1 = a^2$

→ Transform the diff. equation $xy'' - y' + 4x^3y = x^5$ into z as independent variable where $z = x^2$ and solve it.

Sol: Given that

$$xy'' - y' + 4x^3y = x^5 \quad \text{——— ①}$$

and $z = x^2 \Rightarrow \dfrac{dz}{dx} = 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$

$= \dfrac{dy}{dz}(2x)$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{dx}\left(\dfrac{dy}{dz} \cdot 2x\right)$

$= \dfrac{dy}{dz}(2) + 2x\dfrac{d}{dx}\left(\dfrac{dy}{dz}\right)$

$= 2\dfrac{dy}{dz} + 2x\dfrac{d}{dz}\left(\dfrac{dy}{dz}(2x)\right)$

$= 2\dfrac{dy}{dz} + (2x)^2\dfrac{d^2y}{dz^2}$

$\therefore$ ① $\equiv x\left[2\dfrac{dy}{dz} + 4x^2\dfrac{d^2y}{dz^2}\right] - 2x\dfrac{dy}{dz} + 4x^3y = x^5$

$\Rightarrow 4z\dfrac{d^2y}{dz^2} + 4z = z^2$

$\Rightarrow \dfrac{d^2y}{dz^2} + y = \dfrac{1}{4}z \quad$ ——— ②

$\boxed{\text{Continue next soln}}$

*→ An equation of the form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + P_n(x)\,y = Q(x)$$

where $P_1(x), P_2(x) \ldots\ldots P_n(x)$ & $Q(x)$ functions of x, is called a linear diff. equation of order 'n'.

This can be divided into two types:

(i) Homogenous linear diff. eqn.

(ii) Non-Homogenous linear diff. eqn.

→ If $Q(x)=0$ then ① is called homo.

→ If $Q(x)\neq 0$ then ① is called non-homo.

<u>The wronskeian Condition</u>:-

If $y=y_1(x), y=y_2(x), \ldots\ldots y=y_n(x)$ are solutions of $f(D)y=0$ then

$$W(y_1, y_2, \ldots y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

→ If $W(y_1, y_2, \ldots y_n)(x)\neq 0$ then the 'n' solutions are L.I

→ If $W(y_1, y_2, \ldots y_n)(x) = 0$ then the 'n' solutions are L.D.

<u>Note</u>:- The n^{th} order linear equation $f(D)y=0$ possesses 'n' distinct solutions which are L.I

<u>Ex</u>:- e^x, e^{2x}, e^{3x}

Let $y_1=e^x$, $y_2=e^{2x}$, $y_3=e^{3x}$ then

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{2x} \end{vmatrix}$$

$$= e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= 2e^{6x} \neq 0$$

$\therefore W \neq 0$

$\therefore$ The functions e^x, e^{2x}, e^{3x} are L.I.

$\therefore y_1 = e^x, y_2 = e^{2x}, y_3 = e^{3x}$ are L.I solutions of $f(D)y=0$

Ex:-(2) e^{2x}, e^{2x}, e^{-x}

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^{2x} & e^{2x} & e^{-x} \\ 2e^{2x} & 2e^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} & e^{-x} \end{vmatrix}$$

$$= e^{3x}\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & 4 & 1 \end{vmatrix}$$

$$= e^{3x}[6 - 6 + 0]$$

$$= 0$$

$\therefore y = y_1, y_2, y_3$ are not L.I solns of $f(D)y=0$.

Note:- The n^{th} order linear equation $f(D)y=0$ does not posses all solutions are distinct. which are L.D.

④ Method of variation of parameters:-

To solve $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ by the method of variation of parameters.

Given eqn is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Its homogenous eqn is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{——— ②}$$

Let $y_c = c_1 u(x) + c_2 v(x)$ be the g.s of ② and hence it is c.f of ①

Since $y = u(x)$, $y = v(x)$ are L.I solutions of ②

$$\therefore \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \quad \text{——— ③}$$

Let $y_p = A\, u(x) + B\, v(x)$ be particular integral of ①

which is obtained from c.f of ① by replacing c_1 & c_2 by A and B respectively

which are fns of 'x' Diff ④ w.r.t 'x', we get

$$\frac{dy_p}{dx} = A\frac{du}{dx} + u\frac{dA}{dx} + v\frac{dB}{dx} + B\frac{dv}{dx}$$

Now choosing A & B s.t

$$u\frac{dA}{dx} + v\frac{dB}{dx} = 0 \quad \text{——— ⑤}$$

Then $\frac{dy_p}{dx} = A\frac{du}{dx} + B\frac{dv}{dx} \longrightarrow ⑥$

Diff ⑥ w.r.t 'x', we get,

$$\frac{d^2y_p}{dx^2} = A\frac{d^2u}{dx^2} + \frac{dA}{dx}\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\cdot\frac{dv}{dx} \quad \text{——— ⑦}$$

$\therefore$ ① $\equiv$

$$A\frac{d^2u}{dx^2} + \frac{dA}{dx}\frac{du}{dx} + B\frac{d^2v}{dx^2} + \frac{dB}{dx}\frac{dv}{dx} + P\left(A\frac{du}{dx} + B\frac{dv}{dx}\right) + Q(Au + Bv) = R$$

$$\Rightarrow A\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + B\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right] + \left(\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx}\right) = R$$

$$= A(0) + B(0) + \left(\frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx}\right) = R \ (\text{from } ③)$$

$$\Rightarrow \frac{du}{dx}\frac{dA}{dx} + \frac{dv}{dx}\frac{dB}{dx} = R \quad \text{——} \quad (8)$$

Solving (5) & (8)

$$\frac{\frac{dA}{dx}}{vR} = \frac{\frac{dB}{dx}}{-uR} = \frac{-1}{uv'-vu'} \qquad \text{where } u' = \frac{du}{dx}, \; v' = \frac{dv}{dx}$$

$$\Rightarrow \frac{dA}{dx} = \frac{-vR}{uv'-vu'};$$

$$\frac{dB}{dx} = \frac{uR}{uv'-vu'}$$

$$\Rightarrow A = -\int \frac{vR}{uv'-vu'}\,dx,$$

$$B = \int \frac{uR}{uv'-vu'}\,dx$$

After integration the constant is not added.

($\because$ A & B are involved in y_p)

$\therefore$ Substituting the values of A & B in (4)

$\therefore$ The g.s of (1) is $y = y_c + y_p$

$$\Rightarrow y = c_1 u(x) + c_2 v(x) + A\,u(x) + B\,v(x)$$

Note:- (1) Since the form of y_c & y_p is the same. But the constants which occur in y_c are changed into functions of the independent variable x in y_p.

For this reason the method of finding the P.I is called the method of variation of parameters.

(2) The above method can be extended to linear equations of order higher than the two.

(3) The above method is applicable to linear equations with constant coefficients and also variable co-efficients.

(4) W.K.T the given linear eqn of second order can be solved when part of C.F. is known.

Therefore the above method is surely superior to the variation of parameters.

Since this method requires a complete knowledge of the C.F. instead of one solution of it.

Hence the method of variation of parameters should be used only when specifically asked to solve by this method.

Working rule:

(1) write the given equation in the standard form
$y'' + Py' + Qy = R$.

(2) Find the solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.
Let it be $y_c = c_1 u(x) + c_2 v(x)$

(3) Let the P.I of the given eqn be $y_P = A(x)u + B(x)v$
where A & B are function of x.

(4) Find $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$
$= u\frac{dv}{dx} - v\frac{du}{dx}$.

(5) Find A and B by using, $A = \int \frac{-vR}{uv' - vu'}dx$ & $B = \int \frac{uR}{uv' - vu'}dx$

(6) The g.s of the given eqn is $y = y_c + y_P$
$\Rightarrow \boxed{y = (c_1 u + c_2 v) + (Au + Bv)}$

→ solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Sol: Given that $(D^2 + a^2)y = \tan ax$ —— ①

A.E of ① is $D^2 + a^2 = 0$
$\Rightarrow D^2 = -a^2$
$\Rightarrow D = \pm ai$
$\therefore \boxed{y_c = c_1 \cos ax + c_2 \sin ax}$

Let $y_P = A\cos ax + B\sin ax$ be a P.I of ① where A & B fns of 'x'.

then $u(x) = \cos x$; $v(x) = \sin ax$; $R(x) = \tan ax$

Now $uv' - vu' = \cos ax\,(a\cos ax)$
$= + \sin ax\,(a\sin ax)$
$= a(\sin^2 ax + \cos^2 ax)$
$= a$

Now $A = \int \frac{-vR}{uv' - vu'}\,dx$

$= \frac{-1}{a}\int \left[\frac{1-\cos^2 ax}{\cos ax}\right]dx$

$= \frac{-1}{a}\int [\sec ax - \cos ax]\,dx.$

$= \frac{-1}{a}\left[\frac{\log|\sec ax + \tan ax|}{a} - \frac{\sin ax}{a}\right]$

$= -\frac{1}{a^2}\left[\log|\sec ax + \tan ax| - \sin ax\right]$

and $B = \int \frac{uR}{uv' - vu'}\,dx$

$= \frac{-1}{a^2}\cos ax.$

$\therefore y_p = \frac{1}{a^2}\left[\sin ax - \log|\sec x + \tan ax - \cos ax\right]$

$\therefore$ The g.s of ① is $y = y_c + y_p$.

$\Rightarrow y = (c_1 \cos ax + c_2 \sin ax) + \frac{1}{a^2}\left[\sin ax - \log|\cancel{\sec ax + \tan ax}| - \cos ax\right]$

→ Solve $[(x-1)D^2 - xD + 1]y = (x-1)^2$ by the method of Variation of parameters.

Sol: Given eqn is

$$(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = (x-1)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{x}{(x-1)}\frac{dy}{dx} + \frac{1}{(x-1)}y = (x-1)^2 \quad \text{—— ①}$$

Comparing ① with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$.

$P = \frac{-x}{x-1}$; $Q = \frac{1}{x-1}$; $R = (x-1)$.

Now the homo. eqn of ① is

$$\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{(x-1)}y = 0$$

Now $1+P+Q=0$

$\therefore y = e^x$ is a part of C.F of ②.

Let $y_c = uv$ be the g.s of ②, where $u = e^x$

then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0$ —— ③

Now since $u = e^x \Rightarrow \frac{du}{dx} = e^x$

$$\therefore P + \frac{2}{u}\frac{du}{dx} = \frac{-x}{x-1} + 2e^{-x}e^{x}$$

$$= \frac{-x}{x-1} + 2$$

$$= \frac{x-2}{x-1}$$

$$\therefore \frac{d^2v}{dx^2} + \left(\frac{x-2}{x-1}\right)\frac{dv}{dx} = 0 \quad \text{—— ④}$$

Let $\frac{dv}{dx} = V$ then

$$\frac{dV}{dx} + \left(1 - \frac{1}{x-1}\right)V = 0.$$

$\Rightarrow \log V = \log(x-1) - x + \log c_1$

$\Rightarrow \log\left(\frac{V}{(x-1)c_1}\right) = -x$

$\Rightarrow V = c_1(x-1)e^{-x}$

$\Rightarrow \frac{dv}{dx} = c_1(x-1)e^{-x}$

$\Rightarrow v = -c_1 e^{-x}(x+1) + c_1 e^{-x} + c_2$

$y_c = e^x\left[-c_1 e^{-x}(x+1) + c_1 e^{-x} + c_2\right]$

$= -c_1(x-1) - c_1 + c_2 e^x$

$\therefore \boxed{y_c = c_1 x + c_2 e^x}$

Let $y_p = Au + Bv$ be a P.I of ① where A & B are fns of x

& $u = x$; $v = e^x$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - vu'$

$= xe^x - e^x$

$= e^x(x-1)$

$\therefore A = \int \frac{-vR}{uv' - vu'}\, dx = \int -\frac{e^x(x-1)}{e^x(x-1)}\, dx$

$= -x$

and $B = \int \frac{uR}{uv' - vu'}\, dx = \int \frac{x(x-1)}{e^x(x-1)}\, dx$

$= \int x e^{-x}\, dx$

$= -e^{-x}(x+1)$

$\therefore y_p = -x(x) - e^{-x}(x+1)e^x$

$= -x^2 - (x+1)$

$= -(1 + x + x^2)$

$\therefore$ G.S of ① is

$y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 x + c_2 e^x - (1 + x + x^2)}$

* Apply the method of variation of parameters to solve the following diff. eqns.

$\xrightarrow{1999}$ $y'' + n^2 y = \sec nx$

$\xrightarrow{2000}$ $x\frac{dy}{dx} - y = (x-1)\left(\frac{d^2y}{dx^2} - x + 1\right)$

$\xrightarrow{2001}$ $y'' + 4y = 4\tan 2x$

$\xrightarrow{2002}$ $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$ with $y(0) = 0$ and $\left(\frac{dy}{dx}\right)_{x=0} = 0$

$\xrightarrow{2003}$ $x^2 y'' - 4xy' + 6y = x^4 \sec^2 x$

$\xrightarrow{2005}$ $x^2 y'' - 2xy' + 2y = x\log x$; $x > 0$.

$\xrightarrow{2008}$ Use the method of variation of parameters to find the general solution of $x^2 y'' - 4xy' + 6y = -x^4 \sin x$

$\xrightarrow{\text{IFoS-2008}}$ $x^2 y'' + xy' - y = x^2 e^x$.