Mains Test Series – 2018

Test – 07 (Paper-I)

Answer key

1(a) Find a homogeneous system whose solution set W is generated by $\{(1,-2,0,3), (1,-1,-1,4), (1,0,-2,5)\}$.

Sol^n: Let $v=(x,y,z,t)$. Form the matrix M whose first rows are the given vectors and whose last row is v; and then row reduce to echelon form:

$$M = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 1 & -1 & -1 & 4 \\ 1 & 0 & -2 & 5 \\ x & y & z & t \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & -2 & 2 \\ 0 & 2x+y & z & -3x+t \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2x+y+z & -5x-y+t \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The original first three rows show that W has dimension 2. Thus $v \in W$ if and only if the additional row does not increase the dimension of the row space. Hence we set the last two entries in the third row on the right equal to 0 to obtain the required homogeneous system.

$$2x+y+z=0$$
$$5x+y-t=0$$

1(b) Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{bmatrix}$ a symmetric matrix. Find the non-singular matrix P such that P^TAP is diagonal. find P^TAP.

Soln. Let us form the block matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + (-2)R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right] \quad \begin{array}{l} C_2 \to C_2 + (-2)C_1 \\ C_3 \to C_3 + 3C_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right] \quad R_3 \to R_3 + (-2)R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right] \quad C_3 \to C_3 + (-2)C_2$$

$\therefore$ A has been diagonalized.

$\therefore$ Set $P = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ and then

$$P^TAP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

1(c) Show that $\int_0^{\infty} \frac{\log(1+x^2)}{(1+x^2)} dx = \pi \log 2$

Ans. Let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$$\therefore \int_0^{\infty} \log(1+x^2) \frac{dx}{1+x^2} = \int_0^{\pi/2} \log(1+\tan^2\theta) \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2} \log \sec^2\theta\, d\theta$$

$$= -2\int_0^{\pi/2} \log \cos\theta\, d\theta$$

$$= -2\left[-\frac{\pi}{2}\log 2\right] = \pi \log 2$$

1(d) Show that the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by setting
$$f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & \text{when } xy \neq 0 \\ x\sin 1/x, & \text{when } x\neq 0, y=0 \\ y\sin 1/y, & \text{when } x=0, y\neq 0 \\ 0, & \text{when } x=0=y \end{cases}$$
is cts but not differentiable at $(0,0)$.

Sol^n. We have

$$|f(x,y)-f(0,0)| = \left|x\sin\frac{1}{x} + y\sin\frac{1}{y} - 0\right|$$
$$\leq |x|\left|\sin\frac{1}{x}\right| + |y|\left|\sin\frac{1}{y}\right|$$
$$\leq |x|+|y| \qquad (\because |\sin\theta| \leq 1)$$

Let $\epsilon>0$ be given. Choose $\delta = \epsilon/2$

Then $|f(x,y)-f(0,0)| < \epsilon$ if $|x|<\delta$, $|y|<\delta$

Hence the given function is cts at $(0,0)$.

Now, $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{h\sin\frac{1}{h} - 0}{h}$$

$= \lim_{h\to 0} \sin\frac{1}{h}$, which does not exist.

III^{ly}, $f_y(0,0)$ does not exist.

$\Rightarrow f$ is not differentiable.

1(e) Prove that the four planes $my+nz=0$, $nz+lx=0$, $lx+my=0$, $lx+my+nz=p$ form a tetrahedron whose volume is $\frac{2p^3}{3lmn}$.

Soln. Solving the given equations taking three planes at a time, we get the vertices of the tetrahedron as $(0,0,0)$, $\left(\frac{-p}{l},\frac{p}{m},\frac{p}{n}\right)$, $\left(\frac{p}{l},\frac{-p}{m},\frac{p}{n}\right)$ and $\left(\frac{p}{l},\frac{p}{m},\frac{-p}{n}\right)$.

With these points as vertices, the volume V of the tetrahedron is given by

$$V=\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -p/l & p/m & p/n & 1 \\ p/l & -p/m & p/n & 1 \\ p/l & p/m & -p/n & 1 \end{vmatrix}$$

$$=\frac{-p^3}{6lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=\frac{p^3}{6lmn}\times 4$$

$$=\frac{2}{3}\frac{p^3}{lmn}$$

Thus, the four planes $my+nz=0$, $nz+lx=0$, $lx+my=0$, $lx+my+nz=p$ form a tetrahedron whose volume is $\frac{2p^3}{3lmn}$.

2(a) → Let $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$. Is A similar over the field R to a diagonal matrix? Is A similar over the field C to a diagonal matrix?

Sol'n: Given $A = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$

The characteristic polynomial of A is

$$|xI - A| = \begin{vmatrix} x-6 & 3 & 2 \\ -4 & x+1 & 2 \\ -10 & 5 & x+3 \end{vmatrix} = (x-2)(x^2+1)$$

The characteristic values of A are $2, \pm i$.

Let A be similar to a diagonal matrix over R. Then there exists an invertible matrix P such that;

$P^{-1}AP = \text{diag}(a,b,c)$, where a, b, c are eigen values of A and $a,b,c \in R$. But the characteristic values of A are $2, \pm i \in C$. So we arrive at a contradiction. Hence A is not similar over the field R to a diagonal matrix. Since the characteristic values of A are $2, \pm i$, which are all distinct; so A is similar over C to a diagonal matrix.

Q(b) (i) If A and B are two square matrices, then the matrices AB and BA have the same characteristic roots.

Sol: In case either of the two matrices, A or B is non-singular, then the proof is quite simple for we may write

$$AB = B^{-1}(BA)B$$

$$\text{or } AB = A(BA)A^{-1}$$

so that by the preceding result AB, BA have the same characteristic roots.

If r be the rank of A, then there exist two non-singular matrices P and Q such that,

$$PAQ = \text{Diag.}[I_r, O]$$

We have, $PABP^{-1} = (PAQ)(Q^{-1}BP^{-1})$

Let $Q^{-1}BP^{-1} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

where C_{11} is $r \times r$.

$$\therefore PABP^{-1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ 0 & 0 \end{bmatrix}$$

Again $Q^{-1}BAQ = (Q^{-1}BP^{-1})(PAQ)$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & 0 \\ C_{21} & 0 \end{bmatrix}$$

Thus the characteristic roots of AB and BA are the same as those of C_{11} along with $(n-r)$ roots each equal to 0.

2(b)(ii) Show that the characteristic roots of A^θ are the conjugates of the characteristic roots of A.

Sol^n: We have $|A^\theta - \bar{\lambda} I| = |(A-\lambda I)^\theta|$

$$= |\overline{A-\lambda I}|$$

$$\therefore |A^\theta - \bar{\lambda} I| = 0 \text{ iff } \overline{|A-\lambda I|} = 0$$

$$\Rightarrow |A^\theta - \bar{\lambda} I| = 0 \text{ iff } |A-\lambda I| = 0$$

or $\bar{\lambda}$ is an eigenvalue of A^θ, if and only if, λ is an eigenvalue of A.

2(C) Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8abc}{3\sqrt{3}}$.

Ans. We have to find the greatest value of $8xyz$ subject to the conditions

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad x>0, y>0, z>0 \qquad \text{—(1)}$$

Let us consider a function F of three independent variables x, y, z where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\therefore dF = \left(8yz+\frac{2x\lambda}{a^2}\right)dx+\left(8zx+\frac{2y\lambda}{b^2}\right)dy+\left(8xy+\frac{2z\lambda}{c^2}\right)dz$$

At stationary points,

$$8yz+\frac{2x\lambda}{a^2}=0,\quad 8zx+\frac{2y\lambda}{b^2}=0,\quad 8xy+\frac{2z\lambda}{c^2}=0 \qquad \text{—(2)}$$

Multiplying by x, y, z resp. and adding,

$$24xyz + 2\lambda = 0 \quad \text{or } \lambda = -12xyz \qquad [\text{Using (1)}]$$

Hence from (2), $x = a/\sqrt{3}$, $y = b/\sqrt{3}$, $z = c/\sqrt{3}$ and

$$\therefore \lambda = -4abc/\sqrt{3}$$

Again, $d^2F = 2\lambda\left(\frac{dx^2}{a^2}+\frac{dy^2}{b^2}+\frac{dz^2}{c^2}\right)+16z\,dx\,dy+16x\,dy\,dz+16y\,dz\,dx$

$$= -\frac{8abc}{\sqrt{3}}\sum\frac{1}{a^2}dx^2+\frac{16}{\sqrt{3}}\sum c\,dx\,dy \qquad \text{—(3)}$$

Now from eq^ns (1), we have,

$$\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}+\frac{z\,dz}{c^2}=0 \quad \text{or } \frac{dx}{a}+\frac{dy}{b}+\frac{dz}{c}=0$$

Hence squaring,

$$\Sigma \frac{dx^2}{a^2} + 2\Sigma \frac{dx\,dy}{ab} = 0$$

or $$abc\,\Sigma \frac{dx^2}{a^2} = -2\Sigma c\,dx\,dy$$

$$\therefore d^2F = -\frac{16}{\sqrt{3}}\,abc\,\Sigma \frac{dx^2}{a^2}$$

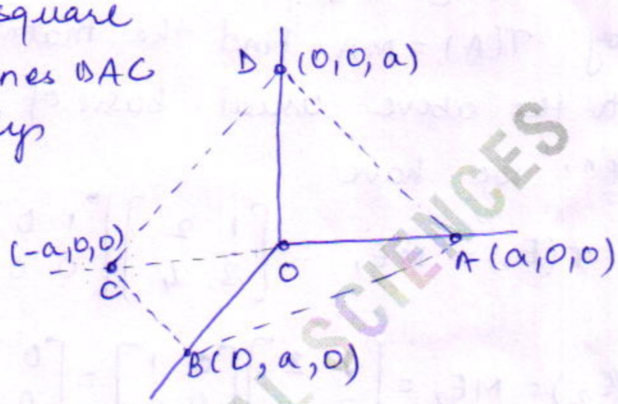
which is always negative.

Hence, $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is a point of maxima and the max. value of $8xyz$ is $\frac{8abc}{3\sqrt{3}}$.

2(d) A square ABCD of diagonal $2a$ is folded along the diagonal AC, so that planes DAC, BAC are at right angles. Show that the shortest distance between DC and AB is then $\frac{2a}{\sqrt{3}}$.

Sol^n. Let O be the centre of square and OA axis of x. Planes DAC and BAC are mutually at right angles.

Take OB and OD as axes of y and z.



Then coordinates of A, B, C, D are $(a,0,0)$, $(0,a,0)$, $(-a,0,0)$ and $(0,0,a)$ respectively.

Equations of AB are $\frac{x-a}{a}=\frac{y}{-a}=\frac{z}{0}$ and of DC are $\frac{x}{a}=\frac{y}{0}=\frac{z-a}{a}$. Thus, a plane containing DC and parallel to AB is

$$\begin{vmatrix} x & y & z-a \\ a & 0 & a \\ a & -a & 0 \end{vmatrix} = 0$$

$$\Rightarrow x+y+z+a=0$$

S.D. = perpendicular distance of this plane from a point $(a,0,0)$ on AB

$$= \frac{a+a}{\sqrt{(1+1+1)}} = \frac{2a}{\sqrt{3}}.$$

3(a) The vectorspace V of 2×2 matrices over R and the following usual basis E of V:

$$E = \left\{ E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

let $M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and T be the linear operator on V defined by $T(A) = MA$. Find the matrix representation of T relative to the above usual basis of V.

Sol'n: we have

$$T(E_1) = ME_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4$$

$$T(E_2) = ME_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4$$

$$T(E_3) = ME_3 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4$$

$$T(E_4) = ME_4 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4$$

Hence

$$[T]_E = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

[Since $\dim V = 4$, any matrix representation of a linear operator on V must be a 4-square matrix].

3(b)(i) If $z = xyf(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$, and if z is a constant then

$$\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}$$

Sol: Given that $z = xy\,f(y/x)$ — ①

Differentiating equation ① partially w.r.t. x and y respectively, we get

$$\frac{\partial z}{\partial x} = yf(y/x) + xy\,f'(y/x)\left(-y/x^2\right)$$

$$= y\,f(y/x) - \frac{y^2}{x}f'(y/x) \quad \text{— ②}$$

& $$\frac{\partial z}{\partial y} = x\,f(y/x) + xy\,f'(y/x)\left(\frac{1}{x}\right)$$

$$= x\,f(y/x) + y\,f'(y/x) \quad \text{— ③}$$

multiply ② by x & ③ by y, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\,f(y/x) - y^2\cancel{f'(y/x)} + xyf(y/x) + y^2\cancel{f'(y/x)}$$

$$= 2xy\,f(x/y)$$

$$= 2z.$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

If z is constant:

$$z = xy\,f(y/x)$$

Taking logarithms on both sides, we have

$$\log xy f(y/x) = \log z$$

$$\Rightarrow \log x + \log y + \log f(y/x) = \log z$$

Differentiating w.r.t x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + \frac{1}{f(y/x)} f'(y/x)\left(-\frac{y}{x^2} + \frac{1}{x}\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{f'(y/x)}{f(y/x)}\left[\frac{-y + x\frac{dy}{dx}}{x^2}\right] + \left[\frac{y + x\frac{dy}{dx}}{xy}\right] = 0$$

$$\Rightarrow \frac{f'(y/x)}{f(y/x)} = -\frac{\left[y + x\frac{dy}{dx}\right]/xy}{\left[-y + x\frac{dy}{dx}\right]/x^2}$$

$$= \frac{x\left[y + x\frac{dy}{dx}\right]}{\left[y - \frac{dy}{dx}\right]y}$$

$$= \frac{x\left[y + x\frac{dy}{dx}\right]}{y\left[y - x\frac{dy}{dx}\right]}$$

3(b)(ii) Change the order of integration in the double integral $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$ and hence find the value.

Ans: The limits of integration are given by the straight line $y=x$

$y=\infty, x=0$ and $x=\infty$

i.e. the region of integration is bounded by $y=x, x=0$ and infinite boundary.

Hence taking the strips parallel to y-axis.

the limits for x are from $x=0$ to $x=y$ and the limits for y are from $y=0$ to $y=\infty$.

Hence changing the order of integration, we have,

$$\int_0^\infty \int_{y=x}^{\infty} \frac{1}{y} e^{-y/2}\,dy\,dx = \int_{y=0}^{\infty}\int_{x=0}^{y} \frac{1}{y} e^{-y/2}\,dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{1}{y} e^{-y/2}\cdot [x]_{x=0}^{y}\,dy$$

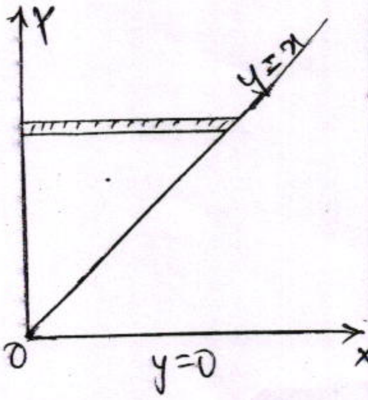
$$= \int_{y=0}^{\infty} \frac{1}{y} e^{-y/2}(y)\,dy$$

$$= \int_{y=0}^{\infty} e^{-y/2}\,dy$$

$$= \left.\frac{e^{-y/2}}{-1/2}\right]_0^\infty$$

$$= -2\left[e^{-y/2}\right]_0^\infty$$

$$= -2[0-1] = 2$$



3(C). Show that the spheres $x^2+y^2+z^2=64$ and $x^2+y^2+z^2+12x+4y-6z+48=0$ touch internally and find their point of contact.

Soln. Two spheres will touch internally if the difference of their radii is equal to the distance between their centres. The distance between two centres viz., $(0,0,0)$ and $(6,-2,3)$ is equal to 7.

Radius of first sphere $=\sqrt{64}=8$

and radius of second sphere $=\sqrt{36+4+9-48}$
$=1$

$\therefore$ difference of radii $=8-1$
$=7=$ distance b/w centres

Hence, two spheres touch internally.

Let (α,β,γ) be their point of contact. Then tangent planes to two spheres at this point are

$$\alpha x+\beta y+\gamma z=64$$

and, $\alpha x+\beta y+\gamma z-6(x+\alpha)+2(y+\beta)-3(z+\gamma)+48=0$

comparing the two, we have

$$\frac{\alpha-6}{\alpha}=\frac{\beta+2}{\beta}=\frac{\gamma-3}{\gamma}=\frac{-6\alpha+2\beta-3\gamma+48}{-64}=k\ (\text{say})$$

$\Rightarrow \alpha-6=\alpha k$; $\beta+2=\beta k$, $\gamma-3=k\gamma$;

$$-6\alpha+2\beta-3\gamma+48=-64k$$

$$\text{or } \alpha=\frac{6}{1-k},\ \beta=\frac{-2}{1-k},\ \gamma=\frac{3}{1-k}$$

Substituting these values in the fourth relation, we get $k=1/8$.

Hence, the point of contact is

$$\left(\frac{48}{7},\frac{-16}{7},\frac{24}{7}\right).$$

3(d). Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Sol^n. Let one of the lines of intersection be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

This line lies on given cone and plane, hence

$$mn+nl+lm=0 \quad \text{——} \quad (1)$$

and, $$al+bm+cn=0 \quad \text{——} \quad (2)$$

Putting the value of n from (2) in (1), we get,

$$(m+l)\left(-\frac{al+bm}{c}\right)+lm=0$$

$$\Rightarrow al^2+(a+b-c)lm+bm^2=0$$

$$\Rightarrow a\left(\frac{l}{m}\right)^2+(a+b-c)\frac{l}{m}+b=0$$

Let $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ be ~~the~~ two roots, then

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{b}{a}$$

$$\Rightarrow \frac{l_1 l_2}{1/a}=\frac{m_1 m_2}{1/b}=\frac{n_1 n_2}{1/c} \quad \text{(by Symmetry)}$$

The angle between the lines will be a right angle if $l_1l_2+m_1m_2+n_1n_2=0$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$$

4(a). Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a linear transformation. Let U be a finite dimensional then $\rho(T) + \nu(T) = \dim U$

i.e. $\text{rank}(T) + \text{nullity}(T) = \dim U$

Proof: The null space $N(T)$ is a subspace of finite dimensional space $U(F)$.

$\Rightarrow N(T)$ is finite dimensional.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis of $N(T)$.

$\therefore \dim N(T) = \nu(T) = k$

$\therefore T(\alpha_1) = \hat{0},\ T(\alpha_2) = \hat{0} \ldots T(\alpha_k) = \hat{0}$ —①

As, S is LI it can be extended to form a basis of $U(F)$.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \theta_1, \theta_2, \ldots, \theta_m\}$ be the extended basis of $U(F)$

$\therefore \dim U = k+m$

Now we show that the set of images of additional vectors $S_2 = \{T(\theta_1), T(\theta_2), \ldots, T(\theta_m)\}$ is a basis of $R(T)$.

Clearly, $S_2 \subseteq R(T)$

(i) To prove S_2 is LI.

Let $a_1, a_2, \ldots, a_m \in F$ such that

$a_1 T(\theta_1) + a_2 T(\theta_2) + \ldots + a_m T(\theta_m) = \hat{0}$

$\Rightarrow T(a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m) = \hat{0}$ ($\because T$ is LT)

$\Rightarrow a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m \in N(T)$

But each vector in $N(T)$ is a l.c. of elements of basis 'S',

$\therefore$ for some $b_1, b_2, \ldots, b_k \in F$,

Let $a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_k\alpha_k$

$\Rightarrow a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m - b_1\alpha_1 - b_2\alpha_2 - \ldots - b_k\alpha_k = 0$

$\Rightarrow a_1 = 0, a_2 = 0, \ldots, a_m = 0, b_1 = 0, b_2 = 0, \ldots, b_k = 0$

($\because S_1$ is LI)

$\Rightarrow S_2$ is LI set

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space $R(T)$, then $\exists\ \alpha \in U$ st $T(\alpha) = \beta$

Now, $\alpha \in U \Rightarrow$ there exist c's, d's $\in F$ such that

$\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \ldots + d_m\theta_m$

$T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \ldots + d_m\theta_m)$

$= c_1T(\alpha_1) + c_2T(\alpha_2) + \ldots + c_kT(\alpha_k) + d_1T(\theta_1) + d_2T(\theta_2) + \ldots + d_mT(\theta_m)$

$\Rightarrow \beta = d_1T(\theta_1) + d_2T(\theta_2) + \ldots + d_mT(\theta_m)$ ($\because$ by ①)

$\Rightarrow \beta \in L(S_2)$

$\therefore S_2$ is a basis of $R(T)$

and $\dim R(T) = m$

$\therefore \dim R(T) + \dim N(T) = m + k = \dim U$

$\Rightarrow \rho(T) + \nu(T) = \dim U$

4(b) (i) Show that $\frac{2}{\pi} < \frac{\sin x}{x} < 1$, $0 < x < \pi/2$

Sol^n Let $f(x) = \begin{cases} \frac{\sin x}{x} & ; x \neq 0 \\ 1 & ; x = 0 \end{cases}$

f is cts in $[0, \pi/2]$ and derivable in $]0, \pi/2[$

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

Let $F(x) = x\cos x - \sin x$, $x \in [0, \pi/2]$

$F'(x) = \cos x - x\sin x - \cos x$

$= -x\sin x < 0$, $x \in]0, \pi/2[$

$\Rightarrow F$ is strictly decreasing in $[0, \pi/2]$

$\Rightarrow F(x) < F(0) = 0$, $x \in [0, \pi/2]$

$\Rightarrow f'(x) < 0$, $x \in]0, \pi/2]$

$\Rightarrow f$ is strictly decreasing in $[0, \pi/2]$

$\Rightarrow f(0) > f(x) > f(\pi/2)$ for $0 < x < \pi/2$

$\Rightarrow 1 > \frac{\sin x}{x} > \frac{1}{\pi/2}$

$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $0 < x < \pi/2$

4(b) (ii) Test for convergence the integrals

$$\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx$$

Sol^n, Let $f(x) = \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}$

$= \frac{\tan^{-1}x}{x^{1/3}(1-x^{-4})^{1/3}}$ $(\sim x^{-1/3}$ at $\infty)$

and $g(x) = \frac{1}{x^{1/3}}$,

so that $\frac{f(x)}{g(x)} = \frac{\tan^{-1}x}{(1+x^{-4})^{1/3}} \to \frac{\pi}{2}$ as $x \to \infty$

Hence, $\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ behave alike.

Since $\int_1^\infty \frac{dx}{x^{1/3}}$ diverges, therefore $\int_1^\infty \frac{x \tan^{-1}x\,dx}{(1+x^4)^{1/3}}$ also diverges.

4(c) Show that the function f defined by
$f(x) = x^p(1-x)^q \quad \forall\ x \in R$
where p, q are positive integers has a maximum value for $x = \frac{p}{p+q}$ for all p, q.

Sol^n We have,

$$f(x) = x^p(1-x)^q$$

$$\Rightarrow f'(x) = px^{p-1}(1-x)^q - qx^p(1-x)^{q-1}$$

$$= x^{p-1}(1-x)^{q-1}(p - x(p+q))$$

$$f'(x) = 0 \Rightarrow x = 0, 1, \frac{p}{p+q}$$

Again $f''(x) = (p-1)x^{p-2}(1-x)^{q-1}[p - x(p+q)]$

$$-(q-1)x^{p-1}(1-x)^{q-2}[p - x(p+q)] - (p+q)x^{p-1}(1-x)^{q-1}$$

$$\Rightarrow f''\left(\frac{p}{p+q}\right) = -(p+q)\left(\frac{p}{p+q}\right)^{p-1}\left(\frac{q}{p+q}\right)^{q-1}$$

< 0 where p and q are integers

Thus the function has a max. value at $x = \frac{p}{p+q}$ for all integers p and q and the max. value is $p^p q^q / (p+q)^{p+q}$.

4(d) Show that the points of intersection R, S of the generators of opposite systems drawn through the points

$(a\cos\theta, b\sin\theta, 0)$, $(a\cos\phi, b\sin\phi, 0)$ of the principal elliptic section of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

are $\left(a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \pm c\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right)$

Soln. Let $R(x_1, y_1, z_1)$ be either of the two points of intersection of the generators.

The tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1$$

at R meets the plane $z=0$ of the principal elliptic section in the line $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$, $z=0$

which is the line joining the points P, Q whose eqn is known to be

$$\frac{x\cos\frac{1}{2}(\theta+\phi)}{a}+\frac{y\sin\frac{1}{2}(\theta+\phi)}{b}=\cos\frac{1}{2}(\theta-\phi), z=0$$

Comparing these eqns, we obtain

$$x_1=a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}, \quad y_1=b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)}$$

Also, we have $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1$

Substituting these values of x_1 and y_1 in this relation, we obtain

$$z_1=\pm c\tan\frac{1}{2}(\theta-\phi)=\pm c\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta-\phi)}$$

Hence, the result.

5(a) → Solve $(2x^2+3y^2-7)x\,dx-(3x^2+2y^2-8)y\,dy=0$

Sol^n Given $(2x^2+3y^2-7)x\,dx-(3x^2+2y^2-8)y\,dy=0$ —(1)

Let $x^2=u$ and $y^2=v$ so that $2x\,dx=du$ and $2y\,dy=dv$ —(2)

From (1) and (2), $(2u+3v-7)du-(3u+2v-8)dv=0$

$$\frac{dv}{du}=\frac{(2u+3v-7)}{(3u+2v-8)} \quad \text{—(3)}$$

Taking $u=U+h$, $v=V+k$ so that $\frac{dv}{du}=\frac{dV}{dU}$ —(4)

the given eq^n becomes $\frac{dV}{dU}=\frac{2U+3V+(2h+3k-7)}{3U+2V+(3h+2k-8)}$ —(5)

Choose h, k so that $2h+3k-7=0$ and $3h+2k-8=0$ —(6)

solving (6), we get $h=2$, $k=1$ so that from (4),

we have

$U=u-2$ and $V=v-1$

or $U=x^2-2$ and $V=y^2-1$, by (2) —(7)

and (5) becomes, $\frac{dV}{dU}=\frac{2U+3V}{3U+2V}=\frac{2+3(V/U)}{3+2(V/U)}$ —(8)

Take $\frac{V}{U}=w$, ie $V=wU$,

$$\therefore \frac{dV}{dU}=w+U\frac{dw}{dU} \quad \text{—(9)}$$

From (8) and (9), $w+U\frac{dw}{dU}=\frac{2+3w}{3+2w}$

$$\text{or} \quad U\frac{dw}{dU}=\frac{2(1-w^2)}{3+2w}$$

or $2\frac{dU}{U}=\frac{3+2w}{1-w^2}dw=\left[\frac{3}{1-w^2}-\frac{-2w}{1-w^2}\right]dw$

Integrating, $2\log U=\frac{3}{2}\log\frac{1+w}{1-w}-\log(1-w^2)+\frac{1}{2}\log c$

or $4\log U=3\log\left(\frac{1+w}{1-w}\right)-2\log(1-w^2)+\log c$

$$\log \frac{U^4}{c} = \log\left(\frac{1+\omega}{1-\omega}\right)^3 - \log(1-\omega^2)^2$$

$$\log \frac{U^4}{c} = \log\left[\left(\frac{1+\omega}{1-\omega}\right)^3 \cdot \frac{1}{(1-\omega^2)^2}\right]$$

$$\Rightarrow \frac{U^4}{c} = \frac{(1+\omega)^3}{(1-\omega)^5 (1+\omega)^2}$$

$$\text{or } (1-\omega)^5 U^4 = c(1+\omega)$$

$$\text{or } \left(1-\frac{V}{U}\right)^5 U^4 = c\left(1+\frac{V}{U}\right)$$

$$\text{or } (U-V)^5 = c(U+V)$$

$$\text{or } (x^2-y^2-1)^5 = c(x^2+y^2-3) \quad (\text{by } (7))$$

5(b) Find the family of curves whose tangents from the angle of $\frac{\pi}{4}$ with the hyperbola $xy = c$.

Sol^n The given family of curves is $xy = c$. —(1)

Differentiating (1), $y + xp = 0$ —(2)

Replacing p by $\frac{p + \tan(\pi/4)}{1 - p\tan(\pi/4)}$ i.e. $\frac{p+1}{1-p}$

in (2) the differential equation of the desired family of curves is

$$y + \frac{p+1}{1-p}x = 0 \text{ or } p = \frac{y+x}{y-x} \text{ or } \frac{dy}{dx} = \frac{y/x + 1}{y/x - 1} \quad —(3)$$

Let $y/x = v$, i.e. $y = xv$ so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad —(4)$$

From (3), $v + x\frac{dv}{dx} = \frac{v+1}{v-1}$

$$x\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{v+1-v^2+v}{v-1}$$

$$= -\left(\frac{v^2-2v-1}{v-1}\right)$$

$$\Rightarrow 2\frac{dx}{x} = -\frac{2(v-1)}{v^2-2v-1}dv$$

Integrating, $2\log x = -\log(v^2-2v-1) + \log c$

or $\log x^2 + \log(v^2-2v-1) = \log c$

$\Rightarrow x^2(v^2-2v-1) = c$

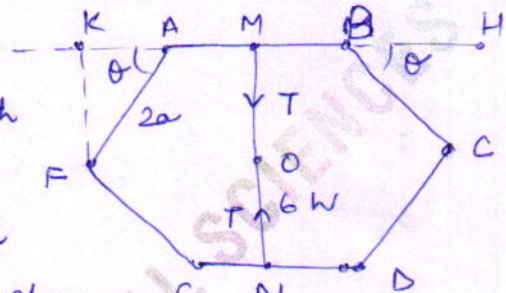
$\Rightarrow x^2\left(\frac{y^2}{x^2} - \frac{2y}{x} - 1\right) = c$

$\Rightarrow y^2 - 2xy - x^2 = c$

which is the required family of curves.

5(C) Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is 3W.

Sol. ABCDEF is a hexagon formed of six equal rods each of weight W and say of length 2a. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string.



Let T be the tension in the string MN. The total weight 6W of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN.

Let $\angle FAK = \theta = \angle CBH$

Give the system a small spherical displacement about the vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rod AB, BC etc. remains fixed, the length MN changes and the point O also changes.

We have, $MN = 2MO = 2KF = 2AF\sin\theta = 4a\sin\theta$

Also, the depth of O below the fixed line AB

$= MO = 2a\sin\theta$

By the principle of virtual work, we have,

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

$$\Rightarrow -4aT\cos\theta\,\delta\theta + 12aW\cos\theta\,\delta\theta = 0$$

$\Rightarrow 4a\,[-T+3W]\cos\theta\,\delta\theta = 0$

$\Rightarrow -T+3W=0$ $\quad [\because \delta\theta \neq 0$ and $\cos\theta \neq 0]$

$\Rightarrow T = 3W$

5(d) A particle whose mass is m is acted upon by a force $m\mu\left[x+\frac{a^4}{x^3}\right]$ towards origin. If it starts from rest at a distance a. Show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$.

Sol^n. Given $\frac{d^2x}{dt^2} = -\mu\left[x+\frac{a^4}{x^3}\right]$, — (1)

the -ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2+\frac{a^4}{x^2}\right]+C$$

When $x=a$, $\frac{dx}{dt}=0$, so that $C=0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4-x^4}{x^2}\right]$$

When or $\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{a^4-x^4}}{x}$ — (2)

the -ive sign is taken because the particle is moving in the direction of x decreasing.

If t_1 be the time taken to reach the origin, then integrating (2), we get,

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4-x^4)}}dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{a^4-x^4}}$$

Put $x^2 = a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$.

When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$

$$\therefore t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}$$

$$= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2} = \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}$$

5(e) a) In what direction from the point $(2,1,-1)$ is the directional derivative of $\phi = x^2 y z^3$ a maximum?

(b) What is the magnitude of this maximum?

Sol^n.

$$\nabla \phi = \nabla (x^2 y z^3)$$
$$= 2xyz^3 \, i + x^2 z^3 \, j + 3x^2 y z^2 \, k$$
$$= -4i - 4j + 12k \quad \text{at } (2,1,-1)$$

(a) The directional derivative is a maximum in the direction

$$\nabla \phi = -4i - 4j + 12k$$

(b) The magnitude of this maximum is $|\nabla \phi|$

$$= \sqrt{(-4)^2 + (-4)^2 + (12)^2}$$
$$= \sqrt{176}$$
$$= 4\sqrt{11}$$

6(a) Solve $e^{3x}(p-1)+p^3e^{2y}=0$ where $p=\frac{dy}{dx}$

Sol^n

$(p-1)e^{3x}=-p^3e^{2y}$

or $(1-p)=p^3e^{2y-3x}$

$e^y(1-p)=p^3e^{3(y-x)}$

or $e^y(1-p)=(pe^{y-x})^3$, which is of form II

Let $e^x=u$, $e^y=v$ [$\because a=1, b=1$ in our case]

$\therefore \frac{e^y\,dy}{e^x\,dx}=\frac{dv}{du}$ or $\frac{v}{u}p=P$

or $p=\frac{uP}{v}$, where $p=\frac{dy}{dx}$ and $P=\frac{dv}{du}$

Putting these in the given eq^n,

$e^{3x}(p-1)+p^3e^{2y}=0$, we get

$u^3\left(\frac{uP}{v}-1\right)+\frac{u^3P^3}{v^3}v^2=0$

or $uP-v+P^3=0$ or $v=uP+P^3$

which is of Clairaut's form and,

hence its general solution is

$v=uc+c^2$

or $y^2=cx^2+c^2$

6(b) Find the values of λ for which all solutions of $x^2\left(\frac{d^2y}{dx^2}\right) - 3x\left(\frac{dy}{dx}\right) - \lambda y = 0$ tend to zero as $x \to \infty$.

Soln. Given: $(x^2D^2 - 3xD + \lambda)y = 0$, $D \equiv \frac{d}{dx}$ —(1)

Let $x = e^z$ so that $z = \log x$.

Also, let $D_1 \equiv \frac{d}{dz}$ —(2)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$, and eq (1) reduces to, $\{D_1(D_1-1) + 3D_1 - 1\}y = 0$

or $(D_1^2 + 2D_1 - \lambda)y = 0$ —(3)

Its auxiliary equations is $D_1^2 + 2D_1 - \lambda = 0$, giving

$$D_1 = \{-2 \pm (4+4\lambda)^{1/2}\}/2 = (-1) \pm (1+\lambda)^{1/2}$$

where, $\lambda \geq -1$ —(4)

Hence the required gen. solⁿ is given by

$$y = c_1 e^{-[1-(1+\lambda)^{1/2}]z} + c_2 e^{-\{1+(1+\lambda)^{1/2}\}z} \quad \text{—(5)}$$

$$\Rightarrow y = c_1 x^{-[1-(1+\lambda)^{1/2}]} + c_2 x^{-\{1+(1+\lambda)^{1/2}\}}, \text{ using (2)}$$

Since all solutions (4) must tend to zero as $x \to \infty$, λ must be chosen to satisfy the following conditions.

$1-(1+\lambda)^{1/2} > 0$ or $(1+\lambda)^{1/2} < 1$ so that $\lambda < 0$ —(6)

(4) and (6) $\Rightarrow$

$$-1 \leq \lambda < 0,$$

which are required values of λ.

6(C) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting ~~fr~~ at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Soln. Let a particle start from rest from the cusp A of the cycloid. Proceeding, ~~as~~ or the velocity v of the particle at any point P, at time t, is given by,

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}(16a^2 - s^2)$$

or $\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\sqrt{16a^2-s^2}$, the negative sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore dt = -2\sqrt{(a/g)}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

The vertical height of the cycloid is 2a. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y = a$. Putting $y = a$ in the equation $s^2 = 8ay$, we get $s^2 = 8a^2$ or $s = 2\sqrt{2}a$

$\therefore$ integrating ① from $s = 4a$ to $s = 2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by,

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{(16a^2-s^2)}} = 2\sqrt{(a/g)}\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{4a}^{2\sqrt{2}a}$$

$= 2\sqrt{(a/g)}\left[\cos^{-1}\left(\frac{2\sqrt{2}a}{4a}\right) - \cos^{-1} 1\right]$

$= 2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}} - \cos^{-1} 1\right]$

$= 2\sqrt{(a/g)}\left[\frac{1}{4}\pi - 0\right] = \frac{1}{2}\pi\sqrt{a/g}$

Again integrating (1) from $s = 2\sqrt{2}a$ to $s = 0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by,

$$t_2 = -2\sqrt{(a/g)}\int_{s=2\sqrt{2}a}^{0} \frac{ds}{\sqrt{16a^2 - s^2}}$$

$$= 2\sqrt{(a/g)} \cdot \left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}$$

$$= 2\sqrt{(a/g)}\left[\cos^{-1} 0 - \cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{(a/g)}\left[\frac{1}{2}\pi - \frac{1}{4}\pi\right] = \frac{1}{2}\pi\sqrt{(a/g)}$$

Hence $t_1 = t_2$ i.e., the time occupied in falling down the first half of the vertical height is equal ~~to~~ to the time of falling down the second half.

6(d) (i) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Sol^n. The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,i + 2y\,j + 2z\,k$$
$$= 4i - 2j + 4k$$

A normal to $z=x^2+y^2-3$ or $x^2+y^2-z=3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,i + 2y\,j - k = 4i - 2j - k$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$, where θ is the required angle.

Then,

$$(4i-2j+4k)\cdot(4i-2j-k)$$
$$= |4i-2j+4k|\,|4i-2j-k|\cos\theta$$
$$\Rightarrow 16+4-4 = \sqrt{4^2+(-2)^2+4^2}\cdot\sqrt{4^2+(-2)^2+(-1)^2}\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \arccos 0.5819 = \cos^{-1}(0.5819)$
$= 54^\circ 25'$.

(ii) Find $\operatorname{curl}(\vec{r}\, f(r))$ where $f(r)$ is differentiable.

Sol^n: $\operatorname{curl}(\vec{r}\, f(r)) = \nabla\times(\vec{r}\, f(r))$

$$= \nabla\times(x f(r)\, i + y f(r)\, j + z f(r)\, k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$= \left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)i + \left(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\right)j + \left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)k$

But $\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right) = \frac{\partial f}{\partial r}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)$

$= \frac{f'(r)x}{\sqrt{x^2+y^2+z^2}} = \frac{f'x}{r}$

Similarly, $\frac{\partial f}{\partial y} = \frac{f'y}{r}$ and $\frac{\partial f}{\partial z} = \frac{f'z}{r}$

Then the result $= \left(z\frac{f'y}{r} - y\frac{f'z}{r}\right)i + \left(x\frac{f'z}{r} - z\frac{f'x}{r}\right)j$

$+ \left(y\frac{f'x}{r} - x\frac{f'y}{r}\right)k$

$= 0$

7(a) Reduce the equation

$$2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\frac{dy}{dx}$$

to homogeneous form by making the substitution $y=z^2$ and hence solve it.

Soln. Given $2x^2y\frac{d^2y}{dx^2}+4y^2=x^2\left(\frac{dy}{dx}\right)^2+2xy\left(\frac{dy}{dx}\right)$ — (1)

and $y=z^2$ — (2)

From (2), $\frac{dy}{dx}=2z\frac{dz}{dx}$ and $\frac{d^2y}{dx^2}=2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}$ — (3)

Using (2) and (3), (1) reduces to

$$2x^2z^2\left\{2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}\right\}+4z^4$$

$$=x^2\cdot 4z^2\left(\frac{dz}{dx}\right)^2+2xz^2\cdot 2z\frac{dz}{dx}$$

$$\Rightarrow x^2\frac{d^2z}{dx^2}-x\frac{dz}{dx}+z=0$$

or $(x^2D^2-xD+1)z=0$

Let $x=e^t$ so that $t=\log x$ and let $D_1=\frac{d}{dt}$ — (5)

then $xD=D_1$ and $x^2D^2=D_1(D_1-1)$ — (6)

Using (5) and (6), (4) reduces to

$[D_1(D_1-1)-D_1+1]z=0$

or $(D_1^2-2D_1+1)z=0$ — (7)

The aux. eqn of (7) is $(D_1-1)^2=0$,

giving $D_1=1,1$

$\therefore$ The general soln of (4) is.

$$z=(c_1+c_2t)e^t=(c_1+c_2\log x)x \quad \text{by (5)}$$

From (2), $y=z^2=(c_1+c_2\log x)^2x^2$,

which is the required general soln.

7(b) Solve $\left(\frac{dx}{dt}\right)-\left(\frac{dy}{dt}\right)+3x=\sin t$, $\frac{dx}{dt}+y=\cos t$, given that $x=1$, $y=0$ for $t=0$.

Ans. Let $D=\frac{d}{dt}$, then

$\left(\frac{dx}{dt}\right)-\left(\frac{dy}{dt}\right)+3x=\sin t$ or $(D+3)x-Dy=\sin t$ — ①

and, $\frac{dx}{dt}+y=\cos t$ or $Dx+y=\cos t$ — ②

Operating ② by D and adding it to ①, we get

$\{(D+3)+D^2\}x=\sin t+D\cos t$

or $(D^2+D+3)x=0$, giving — ③

$D=\{-1\pm(1-12)^{1/2}\}/2=(-1/2)\pm i(\sqrt{11}/2)$

Solution of ③ is

$x=e^{-t/2}\{c_1\cos(t\sqrt{11}/2)+c_2\sin(t\sqrt{11}/2)\}$ — ④

$\frac{dx}{dt}=-(1/2)e^{-t/2}\{c_1\cos(t\sqrt{11}/2)+c_2\sin(t\sqrt{11}/2)\}$
$+e^{-t/2}\{-(c_1\sqrt{11}/2)\sin(t\sqrt{11}/2)+(c_2\sqrt{11}/2)$
$\cos(t\sqrt{11}/2)\}$ — ⑤

From ②, $y=\cos t-\frac{dx}{dt}$

or $y=\cos t+(1/2)e^{-t/2}\{c_1\cos(t\sqrt{11}/2)+$
$c_2\sin(t\sqrt{11}/2)\}-e^{-t/2}\{-(c_1\sqrt{11}/2)$
$\sin(t\sqrt{11}/2)+(c_2\sqrt{11}/2)\cos(t\sqrt{11}/2)\}$ — ⑥

Given that $y=0$ for $t=0$. So the above eqn gives

$0=1+(1/2)c_1-(c_2\sqrt{11}/2)$ — ⑦

Again, given that $x=1$ for $t=0$. So ④ gives $c_1=1$.

With this value of c_1, ⑦ gives $c_2=3/\sqrt{11}$.

So, ④ and ⑥ give

$x=e^{-t/2}[\cos(t\sqrt{11}/2)+(3/\sqrt{11})\sin(t\sqrt{11}/2)]$ — ⑦

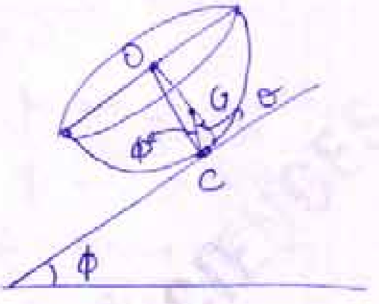
and $y=\cos t+(1/2)e^{-t/2}[\cos(t\sqrt{11}/2)+(3/\sqrt{11})$
$\sin(t\sqrt{11}/2)]-e^{-t/2}\{-(\sqrt{11}/2)\sin(t\sqrt{11}/2)+(3/2)\cos(t\sqrt{11}/2)\}$

or $y = \cos t - e^{-t/2} \cos(t\sqrt{11}/2) + e^{-t/2}(3/2\sqrt{11} + \sqrt{11}/2)$
$\sin(t\sqrt{11}/2)$ —— (8)

The reqd. soln is given by (7) and (8).

7(c) A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ from equilibrium. If ϕ be less than this value, is the equilibrium stable?

Sol^n Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the incline plane, then $OC = r$. Let G be the centre of gravity of the hemisphere



Then $OG = 3r/8$. In the position of eq^m the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is $\perp$ to the horizontal, therefore $\angle OCG = \phi$.
suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical.
From ΔOGC, we have,

$$\frac{OG}{\sin\phi} = \frac{OC}{\sin\theta} \Rightarrow \frac{3r/8}{\sin\phi} = \frac{r}{\sin\theta}$$

$$\therefore \sin\theta = \frac{8}{3}\sin\phi \text{ or } \theta = \sin^{-1}\left(\frac{8}{3}\sin\phi\right)$$

giving the position of equilibrium of the hemisphere.
Since $\sin\theta < 1$, therefore $\frac{8}{3}\sin\phi < 1$

$$\text{i.e. } \sin\phi < \frac{8}{3}$$

$$\text{i.e. } \phi < \sin^{-1}\left(\frac{3}{8}\right).$$

Now, let $CG = h$. Then

$$\frac{h}{\sin(\theta-\phi)} = \frac{3r/8}{\sin\phi}, \text{ so that } h = \frac{3r\sin(\theta-\phi)}{8\sin\phi}$$

Here, $\rho_1 = r$ and $\rho_2 = \infty$

The equilibrium will be stable if

$$h < \frac{\rho_1 \rho_2 \cos\phi}{\rho_1 + \rho_2}.$$

i.e. $\frac{1}{h} > \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sec\phi \Rightarrow \frac{1}{h} > \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \sec\phi$

$\Rightarrow \frac{1}{h} > \frac{1}{r} \sec\phi$ [$\because \rho_1 = r, \rho_2 = \infty$]

$\Rightarrow h < r \cos\phi$

$\Rightarrow \frac{3r \sin(\theta - \phi)}{8 \sin\phi} < r \cos\phi$ [Substituting for h]

$\Rightarrow 3 \sin(\theta - \phi) < 8 \cos\phi \sin\phi$

$\Rightarrow 3 \sin\theta \cos\phi - 3 \cos\theta \sin\phi < 8 \sin\phi \cos\phi$

$\Rightarrow 8 \sin\phi \cos\phi - 3 \sin\phi \sqrt{\left(1 - \frac{64}{9} \sin^2\phi\right)} < 8 \sin\phi \cos\phi$ [$\because \sin\theta = \frac{8}{3} \sin\phi$]

$\Rightarrow -\sin\phi \sqrt{(9 - 64 \sin^2\phi)} < 0$

$\Rightarrow \sin\alpha \sqrt{(9 - 64 \sin^2\phi)} > 0$ —— (2)

But from (1),

$\sin\alpha < \frac{3}{8} \Rightarrow 64 \sin^2\alpha < 9$

i.e. $\sqrt{9 - 64\sin^2\alpha}$ is a positive real number. Therefore the relation (2) is true. Hence, the eq^m is stable.

7(d) Prove $\iiint_V (\phi \nabla^2\psi - \psi\nabla^2\phi)\, dv = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot dS$

Soln. Let $A = \phi\nabla\psi$ in the divergence theorem. Then

$$\iiint_V \nabla\cdot(\phi\nabla\psi)\,dV = \iint_S (\phi\nabla\psi)\cdot\hat{n}\,dS$$
$$= \iint_S (\phi\nabla\psi)\cdot dS$$

But $\nabla\cdot(\phi\nabla\psi) = \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi)$
$$= \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)$$

Thus, $\iiint_V \nabla\cdot(\phi\nabla\psi)\,dV = \iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV$

or $\iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV = \iint_S (\phi\nabla\psi)\cdot dS$ —— ①

which proves Green's first identity. Interchanging ϕ and ψ in ①, we have,

$$\iiint_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]\,dV = \iint_S (\psi\nabla\phi)\cdot dS \quad \text{—— ②}$$

Subtracting ② from ①, we have,

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot ds \quad \text{—— ③}$$

which is Green's second identity or symmetrical theorem. In the proof we have assumed that ϕ and ψ are scalar functions of position with continuous derivatives of the second order at least.

8(a) Solve $(D^2+D)y = t^2+2t$ where $y(0)=4$, $y'(0)=-2$ by using Laplace transformation.

Sol'n: Taking the Laplace tranform of bothsides of the given equation, we have

$$L\{y''\} + L\{y'\} = L\{t^2\} + 2L\{t\}$$

$$\Rightarrow p^2 L\{y\} - py\{0\} - y'(0) + pL\{y\} - y(0) = \frac{2!}{p^3} + \frac{2}{p^2}$$

$$\Rightarrow (p^2+p)L\{y\} - 4p + 2 - 4 = \frac{2}{p^3} + \frac{2}{p^2}$$

$$\Rightarrow p(p+1)L\{y\} = 4p + 2 + \frac{2}{p^3} + \frac{2}{p^2}$$

$$\Rightarrow L\{y\} = \frac{4p^4 + 2p^3 + 2p + 2}{p^4(p+1)}$$

$$= \frac{2}{p^4} + \frac{2}{p} + \frac{2}{p+1}$$

$$\therefore y = 2L^{-1}\left\{\frac{1}{p^4}\right\} + 2L^{-1}\left\{\frac{1}{p}\right\} + 2L^{-1}\left\{\frac{1}{p+1}\right\}$$

$$\Rightarrow y = \frac{1}{3}t^3 + 2 + 2e^{-t}$$

which is the required solution.

8(b) If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are $\alpha, \alpha-\beta, \alpha-2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that: $v_3 t_1 = v_1 t_2$

and, $\frac{1}{v_1}+\frac{1}{v_3} = \frac{2\cos\beta}{v_2}$.

Sol^n Since the horizontal velocity of a projectile remains constant throughout the motion, therefore,

$v_1 \cos\alpha = v_2\cos(\alpha-\beta) = v_3\cos(\alpha-2\beta)$

considering the vertical motion from P to Q and then from Q to R and using the formula $v = u + ft$, we get

$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - g t_1$ — (2)

and, $v_3 \sin(\alpha-2\beta) = v_2\sin(\alpha-\beta) - g t_2$ — (3)

From (2) and (3), we have,

$$\frac{t_1}{t_2} = \frac{v_1\sin(\alpha-v_2)\sin(\alpha-\beta)}{v_2\sin(\alpha-\beta) - v_3\sin(\alpha-2\beta)}$$

$$= \frac{v_1\sin\alpha - \frac{v_1\cos\alpha}{\cos(\alpha-\beta)}\sin(\alpha-\beta)}{v_3\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}\sin(\alpha-\beta) - v_3\sin(\alpha-2\beta)}$$

[substituting suitable for v_2 from (1)]

$$= \frac{v_1[\sin\alpha\cos(\alpha-\beta) - \cos\alpha\sin(\alpha-\beta)]}{v_3[\sin(\alpha-\beta)\cos(\alpha-2\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)]}$$

$$= \frac{v_1\sin\{\alpha-(\alpha-\beta)\}}{v_3\sin\{(\alpha-\beta)-(\alpha-2\beta)\}} = \frac{v_1\sin\beta}{v_3\sin\beta} = \frac{v_1}{v_3}$$

i.e. $v_3 t_1 = v_1 t_2$.

This proves the first result.

Again from (1), we have,

$$\frac{1}{v_1} = \frac{1}{v_2}\frac{\cos\alpha}{\cos(\alpha-\beta)} \quad \text{and} \quad \frac{1}{v_3} = \frac{1}{v_2}\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

$$\therefore \frac{1}{v_1} + \frac{1}{v_3} = \frac{1}{v_2}\,\frac{\cos\alpha + \cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

$$= \frac{1}{v_2}\,\frac{2\cos(\alpha-\beta)\cos\beta}{\cos(\alpha-\beta)}$$

$$= \frac{2\cos\beta}{v_2}$$

This proves the second result

8(c) (i) Derive an expression for $\nabla\phi$ in orthogonal curvilinear coordinates.

Soln. Let $\nabla\phi = f_1\vec{e_1} + f_2\vec{e_2} + f_3\vec{e_3}$, where f_1, f_2, f_3 are to be determined.

Since $d\vec{r} = \frac{\partial\vec{r}}{\partial u_1}du_1 + \frac{\partial\vec{r}}{\partial u_2}du_2 + \frac{\partial\vec{r}}{\partial u_3}du_3$

$= h_1\vec{e_1}\,du_1 + h_2\vec{e_2}\,du_2 + h_3\vec{e_3}\,du_3$

We have:

(1) $d\phi = \nabla\phi\cdot d\vec{r} = h_1 f_1\,du_1 + h_2 f_2\,du_2 + h_3 f_3\,du_3$

But (2) $d\phi = \frac{\partial\phi}{\partial u_1}du_1 + \frac{\partial\phi}{\partial u_2}du_2 + \frac{\partial\phi}{\partial u_3}du_3$

Equating (1) and (2),

$$f_1 = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1},\quad f_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2},\quad f_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$$

Then, $$\nabla\phi = \frac{\vec{e_1}}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\vec{e_2}}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\vec{e_3}}{h_3}\frac{\partial\phi}{\partial u_3}$$

This indicates the operator equivalence

$$\nabla \equiv \frac{\vec{e_1}}{h_1}\frac{\partial}{\partial u_1} + \frac{\vec{e_2}}{h_2}\frac{\partial}{\partial u_2} + \frac{\vec{e_3}}{h_3}\frac{\partial}{\partial u_3}$$

which reduces to the usual expression for the operator ∇ in rectangular coordinates.

(ii) Express $\nabla\times\vec{A}$ in spherical coordinates

(iii) Express $\nabla^2\psi$ in spherical coordinates

Soln (ii) Here, $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$, $e_1 = e_r$, $e_2 = e_\theta$, $e_3 = e_\phi$; $h_1 = h_r = 1$, $h_2 = h_\theta = r$, $h_3 = h_\phi = r\sin\theta$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$= \frac{1}{1\times r\times r\sin\theta}\begin{vmatrix} e_r & re_\theta & r\sin\theta\, e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\left\{\frac{\partial}{\partial\theta}(r\sin\theta\, A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right\}e_r + \left\{\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta\, A_\phi)\right\}re_\theta + \left\{\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right\}r\sin\theta\, e_\phi\right]$$

(iii) $$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{1\times r\times r\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{(r\sin\theta)(1)}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$$

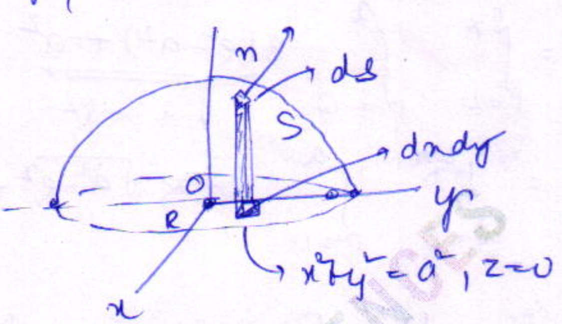
$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

8(d) If $\vec{F} = y\,i + (x-2xz)\,j - xy\,k$, evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ above the xy plane.

Ans.

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$



$$= x\,i + y\,j - 2z\,k$$

A normal to $x^2+y^2+z^2=a^2$ is

$$\nabla(x^2+y^2+z^2) = 2x\,i + 2y\,j + 2z\,k$$

Thus the unit normal $\hat{n}$ of the figure above is given by,

$$\hat{n} = \frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{xi+yj+zk}{a}$$

Since $x^2+y^2+z^2=a^2$.

The projection of S on the xy plane in the region R bounded by the circle $x^2+y^2=a^2$, $z=0$.

Then

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_R (\nabla\times\vec{F})\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot k|}$$

$$= \iint_R (xi+yj-2zk)\cdot\left(\frac{xi+yj+zk}{a}\right)\frac{dx\,dy}{z/a}$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

using the fact that $z=\sqrt{a^2-x^2-y^2}$. To evaluate the double integral, transform to polar coordinates (ρ,ϕ), where $x=\rho\cos\phi$, $y=\rho\sin\phi$ and $dy\,dx$ is replaced by $\rho\,d\rho\,d\phi$.

The double integral becomes,

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi$$

$$=\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi$$

$$=\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left(-3\rho\sqrt{a^2-\rho^2}+\frac{a^2\rho}{\sqrt{a^2-\rho^2}}\right)d\rho\,d\phi$$

$$=\int_{\phi=0}^{2\pi}\left[(a^2-\rho^2)^{3/2}-a^2\sqrt{a^2-\rho^2}\right]_{\rho=0}^{a}d\phi$$

$$=\int_{\phi=0}^{2\pi}(a^3-a^3)\,d\phi$$

$$=0$$