Previous Years' Papers (Solved)

IFS Mathematics Main Exam., 2015

PAPER-I

Instructions: *There are EIGHT questions in all, out of which FIVE are to be attempted. Questions Nos. 1 and 5 are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B. Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off. All questions carry equal marks. The number of marks carried by a question/part is indicated against it. Answers must be written in ENGLISH only. Unless otherwise mentioned, symbols and notations have their usual standard meanings. Assume suitable data, if necessary and indicate the same clearly.*

Section-A

1. (*a*) Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$.

(*b*) Let G be the linear operator $\mathbb{R}^3$ defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$.

Find the matrix representation of G relative to the basis $S\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.

(*c*) Let $f(x)$ be a real-valued function defined on the interval $(-5, 5)$ such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1}\, dt$ for all $x \in (-5, 5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

(*d*) For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t} dt$. Evaluate $f(e) + f\left(\frac{1}{e}\right)$.

(*e*) The tangent at $(a \cos \theta, b \sin \theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

2. (*a*) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of $U \cap W$.

(*b*) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

(*c*) Consider the three-dimensional region R bounded by $x + y + z = 1$, $y = 0$, $z = 0$.

Evaluate $\iiint\limits_R (x^2 + y^2 + z^2)\, dx\, dy\, dz$.

(*d*) Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid

$$\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$$

3. (a) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}.$$

(b) If $\sqrt{x+y}+\sqrt{y-x}=c$, find $\dfrac{d^2y}{dx^2}$.

(c) A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum.

(d) Find the equation of the plane containing the straight line $y + z = 1$, $x = 0$ and parallel to the straight line $x - z = 1$, $y = 0$.

4. (a) Find a 3 × 3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

(b) Find the locus of the variable straight line that always intersects $x = 1, y = 0$; $y = 1, z = 0$; $z = 1, x = 0$.

(c) Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$.

(d) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{2+\cos x}{x^3 \sin x} - \dfrac{3}{x^4}\right)$.

Section-B

5. (a) Reduce the differential equation $x^2p^2 + yp(2x + y) + y^2 = 0$, $p = \dfrac{dy}{dx}$ to Clairaut's form. Hence, find the singular solution of the equation.

(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point.

(c) Find the curvature and torsion of the curve $x = a \cos t$, $y = a \sin t$, $z = bt$.

(d) A cylindrical vessel on a horizontal circular base of radius a is filled with a liquid of density w with a height h. If a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface.

(e) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$$

6. (a) Solve $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$ by changing the independent variable.

(b) The forces P, Q and R act along three straight lines $y = b$, $z = -c$, $z = c$, $x = -a$ and $x = a$, $y = -b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action.

(c) Solve $(D^4 + D^2 + 1)y = e^{-x/2}\cos\left(\dfrac{x\sqrt{3}}{2}\right)$, where $D = \dfrac{d}{dx}$.

(d) Examine if the vector field defined by $\vec{F} = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ is irrotational. If so, find the scalar potential ϕ such that $\vec{F} = \text{grad}\ \phi$.

7. (a) Determine the length of an endless chain which will hang over a circular pulley or radius a so as to be in contact with two-thirds of the circumference of the pulley.

(b) Using divergence theorem, evaluate

$$\iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdydx\right)$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

(c) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t.

8. (a) An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse.

(b) If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate

$$\iint_S (\nabla \times \vec{F}).\hat{n}dS,$$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

(c) A particle moves with a central acceleration which varies inversely as the cube of the distance.

If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path.

ANSWERS

PAPER-I

Section-A

1. (a) Let, upper triangular matrix $A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$.

$$\text{Now, } A^2 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} x^2 & xy + yz \\ 0 & z^2 \end{bmatrix}.$$

$$A^3 = A^2.A = \begin{bmatrix} x^2 & xy + yz \\ 0 & z^2 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$= \begin{bmatrix} x^3 & x^2 y + xyz + yz^2 \\ 0 & z^3 \end{bmatrix}$$

$$\text{From question, } A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$

$$i.e., \begin{bmatrix} x^3 & x^2 y + xyz + yz^2 \\ 0 & z^3 \end{bmatrix} = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$

$$x^3 = 8 \quad \Rightarrow x = 2$$

$$z^3 = 27 \quad \Rightarrow z = 3$$

$$\therefore x^2 y + xyz + yz^2 = -57$$

$$4y + 6y + 9y = -57$$

$$19y = -57$$

$$y = -3$$

$$\therefore \quad A = \begin{bmatrix} 2 & -3 \\ 0 & 3 \end{bmatrix}$$

1. (b)

$$G(1, 1, 1) = (2 \times 1 + 1, 1 - 4, 3)$$
$$= (3, -3, 3)$$
$$G(1, 1, 0) = (2 \times 1 + 0, 1 - 4, 3)$$
$$= (2, -3, 3)$$
$$G(1, 0, 0) = (0, 1, 3) = (0, 1, 3)$$

$$\therefore \quad G = \begin{vmatrix} 3 & 2 & 0 \\ -3 & -3 & 1 \\ 3 & 3 & 3 \end{vmatrix}$$

1. (e) Equation of the tangent at $(a \cos \theta, b \sin \theta)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\Rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \qquad ...(i)$$

The joint equation of the lines joining the points of intersection of (i) and the auxillary circle $x^2 + y^2 = a^2$ to the origin, which is the center of the circle, is

$$x^2 + y^2 = a^2\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2$$

Since, these lines are at right angles

Co-efficient of x^2 + Co-efficient of $y^2 = 0$

$$\Rightarrow 1-a^2\left(\frac{\cos^2\theta}{a^2}\right)+1-a^2\left(\frac{\sin^2\theta}{b^2}\right) = 0$$

$$\Rightarrow \sin^2\theta\left(1-\frac{a^2}{b^2}\right)+1 = 0$$

$\Rightarrow \sin^2\theta\ (b^2 - a^2) + b^2 = 0$

$\Rightarrow \sin^2\theta\ [a^2(1 - e^2) - a^2] + a^2(1 - e^2) = 0$

$\Rightarrow (1 + \sin^2\theta)\ a^2e^2 = a^2$

$\Rightarrow e = (1 + \sin^2\theta)^{-1/2}$

4. (*a*) Let, a non-zero vector is r, then r is orthogonal to given two row vector

Cross product two row = 3rd row

$$\text{Here, row } 1 = \left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right] = \frac{1}{3}[1, 2, 2]$$

$$\text{row } 2 = \left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}[0, 1, -1]$$

third row vector

$= (2 \times (-1) - 2 \times 1), (0 - (-1)), (1 - 0)$

$= (-4, 1, 1)$

$$\text{third row } r = \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}, \frac{1}{\sqrt{18}}$$

$$\therefore \text{Required matrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

4. (*b*) The plane passes through given lines are, respectively, given by

$x - 1 + py = 0$

$y - 1 + qz = 0$

$z - 1 + rx = 0$

These planes will intersect in a lines if

$$\begin{vmatrix} 1 & p & 0 & -1 \\ 0 & 1 & q & -1 \\ r & 0 & 1 & -1 \end{vmatrix} = 0$$

i.e., if

$$\begin{vmatrix} 1 & p & 0 \\ 0 & 1 & q \\ r & 0 & 1 \end{vmatrix} = 0$$

$$1 + pqr = 0$$

$$1+\left(\frac{1-x}{y}\right)\left(\frac{1-y}{z}\right)\left(\frac{1-z}{x}\right) = 0$$

$$xyz + (1 - x)(1 - y)(1 - z) = 0$$

4. (*c*) Any normal to the parabola $y^2 = 4ax$ is ... (*i*)

$$y = mx - 2am - am^3 \quad ...(ii)$$

Let, (x_1, y_1) be the pole of (2) with respect to (*i*), then (*ii*) is the polar of (x_1, y_1) w.r.t. (*i*) *i.e.*

$$yy_1 = 2a(x + x_1) \quad ...(iii)$$

comparing (*ii*) & (*iii*), we get $2a/m = y_1/1 = 2ax_1)/(-2am - am^3)$

Hence, we get $\quad x_1 = -2a - am^2 \quad ...(iv)$

And, $\quad y_1 = 2a/m \quad ...(v)$

Eliminating m between (*iv*) & (*v*) we get

$$y_1^2(x_1 + 2a) + 4a^3 = 0$$

$\therefore$ The required locus of (x_1, y_1) is

$$(x + 2a)y^2 - 4a^3 = 0$$

4. (*d*) $\lim\limits_{x\to 0}\left(\dfrac{x(2+\cos x)-3\sin x}{x^4.\sin x}\right)$

This is in $\dfrac{0}{0}$ form, by L' Hospital rule

$$\lim_{x\to 0}\frac{(2+\cos x)-x.\sin x-3\cos x}{4x^3.\sin x+x^4.\cos x}$$

$$\lim_{x\to 0}\frac{2\sin x-\sin x-x.\cos x}{12x^2\sin x+4x^3\cos x+4x^3.\cos-x^4\sin x}$$

$$\lim_{x\to 0}\frac{\cos x-\cos x+x\sin x}{24x.\sin x+12x^2.\cos x+24x^2\cos x-8x^3\sin x-4x^3\sin x-x^4\cos x} = \frac{1}{60}$$

4.d) $L = \lim_{x \to 0} \left(\frac{2+\cos x}{x^3 \sin x} - \frac{3}{x^4} \right)$

$= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^4 \sin x}$

$= \lim_{x \to 0} \frac{x(2+\cos x) - 3\sin x}{x^5} \cdot \frac{x}{\sin x}$

$= \lim_{x \to 0} \frac{2x + x\cos x - 3\sin x}{x^5} \cdot 1$ $\left(\frac{0}{0} \text{ form} \quad \lim_{x \to 0} \frac{x}{\sin x} = 1 \right)$

$= \lim_{x \to 0} \frac{2 + \cos x - x\sin x - 3\cos x}{5x^4}$ $\left(\frac{0}{0} \text{ form} \right)$ L-Hospital

$= \lim_{x \to 0} \frac{2\sin x - \sin x - x\cos x}{20x^3}$ $\left(\frac{0}{0} \text{ form} \right)$

$= \lim_{x \to 0} \frac{\cancel{\cos x} - \cancel{\cos x} + x\sin x}{60x^2}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{60} = \frac{1}{60}$

DATE

1.c) Let $f(x)$ be a real valued function defined on $(-5, 5)$ such that

$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt, \quad \forall x \in (-5,5).$$

Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$. (8)

Sol:
$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(t)}$$

Where $f(t) = x$.

Here,
$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\, dt \quad \text{—(1)}$$

Differentiating both sides w.r.t x.

$$-e^{-x} f(x) + e^{-x} f'(x) = 0 + \sqrt{x^4+1}$$

Put, $x = 0$

$$-f(0) + f'(0) = 1$$

Also, from (1), $f(0) = 2 + 0 \Rightarrow \boxed{f(0) = 2}$

$$\therefore f'(0) = 3.$$

$$\therefore \left.\frac{d}{dx} f^{-1}(x)\right|_{x=2} = \frac{1}{f'(0)} = \frac{1}{3}.$$

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1.(d) For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t}\,dt$.

Evaluate $f(e) + f\left(\frac{1}{e}\right)$. (8)

Sol: $I_1 = f(e) = \int_1^e \frac{\log t}{1+t}\,dt$

$$I_2 = f\left(\frac{1}{e}\right) = \int_1^{1/e} \frac{\log t}{1+t}\,dt$$

$$= \int_1^e \frac{\log(1/y)}{1+\frac{1}{y}} \cdot \left(-\frac{dy}{y^2}\right) \quad \left[\text{Putting } t = \frac{1}{y}\right.$$

$$= \int_1^e \frac{\log y}{1+y} \cdot \frac{dy}{y} = \int_1^e \frac{\log t}{(1+t)} \cdot \frac{dt}{t}$$

$$\therefore I_1 + I_2 = f(e) + f\left(\frac{1}{e}\right)$$

$$= \int_1^e \frac{\log t}{1+t} + \frac{\log t}{1+t} \cdot \frac{dt}{t}$$

$$= \int_1^e \frac{\log t}{1+t}\left(1+\frac{1}{t}\right)dt$$

$$= \int_1^e \frac{\log t}{t}\,dt = \left.\frac{(\log t)^2}{2}\right|_1^e = \frac{1}{2}.$$

DATE

2(a) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$. (10)

$$\text{Dim}(U+W) = \text{Dim}(U) + \text{Dim}(W) - \text{Dim}(U \cap W)$$
$$= 4 + 4 - \text{Dim}(U \cap W)$$

$$\therefore \text{Dim}(U \cap W) = 8 - \text{Dim}(U+W)$$

$U+W$ is a subspace of V

$$\therefore \dim(U+W) \leq \dim(V)$$

$$\therefore \dim(U+W) \leq 6$$

$\Rightarrow$ $\text{Dim}(U \cap W) \geq 8-6$ ie $\dim(U \cap W) \geq 2$

Also, $U \cap W$ is a subspace of U

$$\therefore \dim(U \cap W) \leq \dim(U)$$

ie $\dim(U \cap W) \leq 4$

Hence, Possible values of $\dim(U \cap W)$ are 2, 3 or 4.

[Result : Intersection of two subspaces is a subspace.]

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2(b) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution:

$$x+2y-3z=a$$
$$2x+6y-11z=b$$
$$x-2y+7z=c$$

(10)

$AX = B$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & -5a+2b+c \end{bmatrix}$$

Now this system has solution if

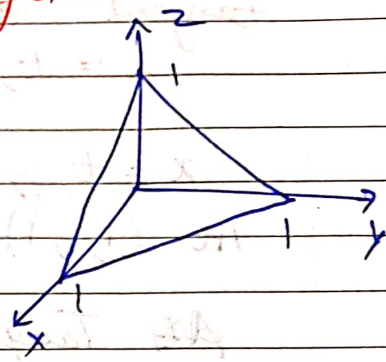
$$\text{Rank}(A:B) = \text{Rank}(A) = 2$$

which is possible only if

$$\boxed{-5a+2b+c=0}$$

2(c) Consider the three-dimensional region R bounded by $x+y+z=1$, $y=0$, $z=0$, $x=0$. Evaluate

$$\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz.$$



Let R be the region bounded by the given tetrahedron.

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left. z(x)^2+(y^2)z + \frac{z^3}{3} \right|_0^{1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (\cancel{x^2}+y^2)(1-x-y) + \frac{1}{3}(1-x-y)^3\,dy\,dx$$
$$x^2(1-x-y)+$$

$$= \int_0^1 \left. x^2(1-x)y - \frac{x^2y^2}{2} + (1-x)\frac{y^3}{3} - \frac{y^4}{4} - \frac{(1-x-y)^4}{12} \right|_0^{1-x} dx$$

$$= \int_0^1 x^2(1-x)^2 - \frac{1}{2}x^2(1-x)^2 + \frac{1}{3}(1-x)^4 - \frac{1}{4}(1-x)^4 + \frac{1}{12}(1-x)^4\,dx$$

$$= \int_0^1 \frac{1}{2}(x^2+x^4-2x^3) + \frac{2}{12}(1-x)^4\,dx$$

$$= \left. \frac{1}{2}\left(\frac{x^3}{3}+\frac{x^5}{5}-\frac{2x^4}{4}\right) - \frac{1}{30}(1-x)^5 \right|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right) - \frac{1}{30}(0-1) = \frac{1}{20}.$$

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2(d) Find the area enclosed by the curve in which the plane $z=2$ cuts the ellipsoid,

$$\frac{x^2}{25}+y^2+\frac{z^2}{5}=1. \qquad (10).$$

The intersection of plane $z=2$ with the ellipsoid is given by

$$\frac{x^2}{25}+y^2+\frac{(2)^2}{5}=1 \Rightarrow \frac{x^2}{25}+y^2=\frac{1}{5}$$

ie $$\frac{x^2}{5}+\frac{y^2}{1/5}=1 \quad \text{(say } S\text{) in space } z=2.$$

The area enclosed by this curve is an ellipse.

We take projection on xy-plane

$$A=\iint_D \sqrt{z_x^2+z_y^2+1}\ dA \qquad \left(\begin{array}{l} z=2 \\ z_x=0 \\ z_y=0 \end{array}\right)$$

$$=\int_{-\sqrt{5}}^{\sqrt{5}}\int_{-\frac{1}{5}\sqrt{5-x^2}}^{\frac{1}{5}\sqrt{5-x^2}} 1\cdot dy\,dx \qquad \left(\begin{array}{l}D \text{ is region of} \\ \text{projection on } xy\text{-plane}\end{array}\right)$$

$$=\int_{-\sqrt{5}}^{\sqrt{5}} \frac{2}{5}\sqrt{5-x^2}\ dx \qquad \begin{array}{l}\text{Put } x=\sqrt{5}\sin\theta \\ dx=\sqrt{5}\cos\theta d\theta\end{array}$$

$$=\frac{2}{5}\times 2\int_0^{\pi/2}\sqrt{5-5\sin^2\theta}\ \sqrt{5}\cos\theta d\theta = 4\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$=4\times\frac{1}{2}\times\frac{\pi}{2}=\pi.$$

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3(a) Find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$ (10)

Minimal Polynomial of a Matrix :- is a monic polynomial of least degree such that $P(A) = 0$.

First let us find the characteristic Polynomial

$$\begin{vmatrix} 4-\lambda & -2 & 2 \\ 6 & -3-\lambda & 4 \\ 3 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(3+\lambda)(\lambda-3)+8\right) + 2\left(18-6\lambda-12\right)$$
$$+2\left(-12+9+3\lambda\right) = 0.$$

$$\lambda^3 - 4\lambda^2 + 5\lambda + 10 = 0.$$

$$\Rightarrow (\lambda+1)(\lambda^2-5\lambda+10) = 0.$$

Hence, we have 3 possibilities for minimal polynomial, $(\lambda+1)$, $(\lambda^2-5\lambda+10)$ or $\lambda^3-4\lambda^2+5\lambda+10$.

Let us check one by one.

Clearly $A+I \neq 0$

$$A^2-5A+10I = \begin{bmatrix} 0 & 4 & -4 \\ -12 & 14 & -8 \\ -6 & 4 & 2 \end{bmatrix} \neq 0$$

By Caley-Hamilton theorem, $A^3-4A^2+5A+10I=0$

Hence, minimal polynomial is $x^3-4x^2+5x+10$.

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3(b) If $\sqrt{x+y} + \sqrt{y-x} = c$, find $\frac{d^2y}{dx^2}$. (10)

$$\left(\sqrt{y+x} + \sqrt{y-x}\right)^2 = c^2$$

$$(y+\cancel{x}) + (y-\cancel{x}) + 2\sqrt{y^2-x^2} = c^2$$

$$2y - c^2 = 2\sqrt{y^2-x^2}$$

$$\cancel{4y^2} - 4c^2y + c^4 = 4(\cancel{y^2} - x^2)$$

$$-4c^2y = -4x^2 - c^4$$

$$y = \frac{1}{c^2}x^2 + \frac{c^2}{4}$$

Differentiating w.r.t x

$$\frac{dy}{dx} = \frac{2}{c^2}x$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{2}{c^2}}$$

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3(c) A rectangular box, open at the top, is said to have a volume of 32 cubic meters. Find the dimensions of the box so that total surface is a minimum. (10).

$$V = xyz = 32 \quad \text{(given)}$$

$S = xy + 2yz + 2zx$, where x, y, z are dimensions.

$$S = xy + 2y \cdot \frac{32}{xy} + 2x \cdot \frac{32}{xy}$$

$$= xy + 64\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} \quad ; \quad \frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$r = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, \quad s = \frac{\partial^2 S}{\partial x \partial y} = 1, \quad t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

for stationary points, $\frac{\partial S}{\partial x} = 0$, $\frac{\partial S}{\partial y} = 0$

$$y - \frac{64}{x^2} = 0 \quad \& \quad x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2} \Rightarrow x \cdot \left(\frac{64}{x^2}\right)^2 = 64 \Rightarrow x = 4$$

$$\Rightarrow y = 4$$

$\therefore$ $(4,4)$ is stationary point

Also, $rt - s^2 = 4 - 1 = 3 > 0$ & $r > 0$

$\therefore$ $(4,4)$ is point of minima.

$\therefore$ $x = 4$, $y = 4$, $z = \frac{32}{4 \times 4} = 2$.

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DATE

3(d) Find the equation of the plane containing the st line $y+z=1$, $x=0$ and parallel to the st line $x-z=1$, $y=0$. (10).

Eqn of plane through the line $y+z-1=0$, $x=0$ is

$$\lambda x+y+z-1=0.$$

Other line $x-z=1$, $y=0$

ie. $$\frac{x}{1}=\frac{z+1}{1}=\frac{y}{0}$$

Plane is parallel to this line

$$\therefore \quad \lambda\cdot 1+1\cdot 1+1\cdot 0=0$$

$$\boxed{\lambda=-1}$$

Hence eqn of plane : $-x+y+z-1=0$.