IFS Mathematics Main Exam., 2015

PAPER-I

Instructions: *There are EIGHT questions in all, out of which FIVE are to be attempted. Questions Nos. 1 and 5 are compulsory. Out of the remaining SIX questions, THREE are to be attempted selecting at least ONE question from each of the two Sections A and B. Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off. All questions carry equal marks. The number of marks carried by a question/part is indicated against it. Answers must be written in ENGLISH only. Unless otherwise mentioned, symbols and notations have their usual standard meanings. Assume suitable data, if necessary and indicate the same clearly.*

Section-A

1. (*a*) Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$.

(*b*) Let G be the linear operator $\mathbb{R}^3$ defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$.

Find the matrix representation of G relative to the basis S $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$.

(*c*) Let $f(x)$ be a real-valued function defined on the interval $(-5, 5)$ such that $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4+1}\,dt$ for all $x \in (-5, 5)$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

(*d*) For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}\,dt$. Evaluate $f(e) + f\left(\frac{1}{e}\right)$.

(*e*) The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

2. (*a*) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of $U \cap W$.

(*b*) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

(*c*) Consider the three-dimensional region R bounded by $x + y + z = 1$, $y = 0$, $z = 0$.

Evaluate $\iiint_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$.

(*d*) Find the area enclosed by the curve in which the plane $z = 2$ cuts the ellipsoid

$$\frac{x^2}{25} + y^2 + \frac{z^2}{5} = 1$$

3. (*a*) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}.$$

(*b*) If $\sqrt{x+y}+\sqrt{y-x}=c$, find $\frac{d^2y}{dx^2}$.

(*c*) A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum.

(*d*) Find the equation of the plane containing the straight line $y + z = 1$, $x = 0$ and parallel to the straight line $x - z = 1$, $y = 0$.

4. (*a*) Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

(*b*) Find the locus of the variable straight line that always intersects $x = 1$, $y = 0$; $y = 1$, $z = 0$; $z = 1$, $x = 0$.

(*c*) Find the locus of the poles of chords which are normal to the parabola $y^2 = 4ax$.

(*d*) Evaluate $\lim\limits_{x\to 0}\left(\frac{2+\cos x}{x^3 \sin x}-\frac{3}{x^4}\right)$.

Section-B

5. (*a*) Reduce the differential equation $x^2p^2 + yp(2x + y) + y^2 = 0$, $p = \frac{dy}{dx}$ to Clairaut's form. Hence, find the singular solution of the equation.

(*b*) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point.

(*c*) Find the curvature and torsion of the curve $x = a \cos t$, $y = a \sin t$, $z = bt$.

(*d*) A cylindrical vessel on a horizontal circular base of radius a is filled with a liquid of density w with a height h. If a sphere of radius c and density greater than w is suspended by a thread so that it is completely immersed, determine the increase of the whole pressure on the curved surface.

(*e*) Solve the differential equation

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{(1-x)^2}.$$

6. (*a*) Solve $x\frac{d^2y}{dx^2}-\frac{dy}{dx}-4x^3y=8x^3\sin x^2$ by changing the independent variable.

(*b*) The forces P, Q and R act along three straight lines $y = b$, $z = -c$, $z = c$, $x = -a$ and $x = a$, $y = -b$ respectively. Find the condition for these forces to have a single resultant force. Also, determine the equations to its line of action.

(*c*) Solve $\left(D^4+D^2+1\right)y=e^{-x/2}\cos\left(\frac{x\sqrt{3}}{2}\right)$, where $D = \frac{d}{dx}$.

(*d*) Examine if the vector field defined by

$$\vec{F}=2xyz^3\hat{i}+x^2z^3\hat{j}+3x^2yz^2\hat{k}$$

is irrotational. If so, find the scalar potential ϕ such that $\vec{F}=\text{grad }\phi$.

7. (*a*) Determine the length of an endless chain which will hang over a circular pulley or radius a so as to be in contact with two-thirds of the circumference of the pulley.

(*b*) Using divergence theorem, evaluate

$$\iint_S \left(x^3 dydz + x^2 ydzdx + x^2 zdydx\right)$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

(*c*) A particle of mass m is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, determine the distance fallen through in time t.

8. (*a*) An ellipse is just immersed in water with its major axis vertical. If the centre of pressure coincides with the focus, determine the eccentricity of the ellipse.

(*b*) If $\vec{F} = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$, evaluate

$$\iint_S (\nabla \times \vec{F}).\hat{n}dS,$$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

(*c*) A particle moves with a central acceleration which varies inversely as the cube of the distance.

If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path.

ANSWERS

$$\lim_{x\to 0}\frac{\sin x + x.\cos x}{24.\sin x + 24x.\cos x + 72x\cos x - 36x^2.\sin x - 36x^2\sin x - 12x^3\cos x - 4x^3\cos x + x^4.\sin x}$$

$$\lim_{x\to 0}\frac{\cos x + \cos x - x.\sin x}{24\cos x + 96\cos x - 96x.\sin x - 144x.\sin x - 72x^2.\cos x - 48x^2.\cos x + 16x^3.\sin x + 4x^3\sin x + x^4\cos x}$$

$$= \frac{1+1-0}{24+96-0} = \frac{2}{120} = \frac{1}{60}$$

5. (*a*) We have $x^2p^2 + yp(2x + y) + y^2 = 0$ ...(*i*)

Let, $u = y$ and $v = xy$

$\therefore \quad \frac{du}{dx} = \frac{dy}{dx} = p$

And, $\frac{dv}{dx} = x\frac{dy}{dx} + y.1 = xp + y$

$\therefore \quad \frac{dv}{du} = \frac{\frac{dv}{dx}}{\frac{du}{dx}} = \frac{xp+y}{p}$

Let $\quad q = \frac{dv}{du}$

$\therefore \quad q = \frac{xp+y}{p}$

Or, $\quad p = \frac{y}{q-x}$

Plug in value of p in (*i*), we have,

$\Rightarrow x^2.\frac{y^2}{(q-x)^2} + y.\frac{y}{q-x}(2x + y) + y^2 = 0$

$\Rightarrow x^2y^2 + y^2(2x + y)(q - x) + y^2(q - x)^2 = 0$

$\Rightarrow x^2 + (2xq - 2x^2 + yq - xy) + (q^2 + x^2 - 2qx) = 0$

$\Rightarrow q^2 + yq - xy = 0 \Rightarrow q^2 + uq - v = 0$

$\therefore \quad v = qu + q^2$ ...(*ii*)

This is a Clairaut's equations.

Replacing q by c, the solution of (*ii*) is

$v = cu + c^2$

$\Rightarrow \quad xy = cy + c^2.$

This is the general solution of (*i*).

Let, $f(x, y, p) = x^2p^2 + yp(2x + y) + y^2$

This is quadratic in p

$\therefore \quad$ Disc. $= 0$

$\Rightarrow \quad y^2(2x + y)^2 - 4.x^2.y^2 = 0$

$\Rightarrow y^2(4x^2 + y^2 + 4xy - 4x^2) = 0$

$\Rightarrow \quad y^3(y + 4x) = 0$ ...(*iii*)

Let, $\quad \phi(x, y, c) = c^2 + cy - xy$

This is quadratic in c

$\therefore \quad$ Disc. $= 0$

$\Rightarrow \quad y^2 - 4.1.(-xy) = 0$

$\Rightarrow \quad y(y + 4x) = 0$ ...(*iv*)

Using (*iii*), the p-discriminant relation ($ET^2C = 0$) can be written as

$y(y + 4x).y^2.1 = 0$

Using (*iv*), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$y(y + 4x).1^2.1^3 = 0$

$\therefore \quad E = 0$

$\Rightarrow \quad y(y + 4x) = 0$ ...(*v*)

$\Rightarrow \quad y = 0$ or $y + 4x = 0$

$y + 4x = 0$

$\Rightarrow \quad \frac{dy}{dx} + 4 = 0$

Or, $\quad p = -4$

Plug in $p = -4$ in equation (1)

$\Rightarrow x^2(-4)^2 + (-4x)(-4)(2x - 4x) + (-4x)^2 = 0$

$\Rightarrow 16x^2 - 32x^2 + 16x^2 = 0$, which is true.

$\therefore y + 4x = 0$ is singular solution of the given equation.

5. (*b*) Let, OA $= a$ be the natural length of the string whose one end O is fixed. Let B be the position of equilibrium of a particle of mass m attached to the other end A of the string and let AB $= e$. Then at B,

the weight of the particle = the tension T_0 in the string.

$\therefore \quad mg = \lambda(e/a) = (mg)(e/a)$ as

$$\lambda = mg \text{ (given)} \qquad ...(i)$$

Thus, here $\quad e = a \qquad ...(ii)$

Now, the particle is pulled down to a point C such that OC = $4a$ (given) and then let go. The particle will start to move towards B from rest from C.

Let P be its position after time t, where BP $= x$.

At P, the forces acting on the particle are its weight mg acting vertically downwards and tension $T = \lambda(x + e)/a$ acting vetically upwards.

Then, the equation of motion of the particle at P is

$$\frac{md_x^2}{dt^2} = mg - T = mg - \lambda(e + x)/a$$

$$= mg - \lambda(e/a) - \lambda(x/a)$$

Or, $$\frac{md_x^2}{dt^2} = -\lambda(x/a)$$

i.e., $$\frac{d_x^2}{dt^2} = -(\lambda/am)x, \text{ using } (i)$$

Or, $$\frac{d_x^2}{dt^2} = -(mg/am)x$$

$$= -(g/a)x,$$

as $$\lambda = mg \qquad ...(iii)$$

Or, $$v\left(\frac{dv}{dx}\right) = -(g/a)x$$

Or, $$2v dv = -(2g/a)x\, dx$$

Integrating, $v^2 = -(g/a)x^2 + K$,

Where, K being a constant $\qquad ...(iv)$

At the point C, when

$$x = BC = OC - OB$$

$$= 4a - (a + e)$$

$$= 4a - (a + a)$$

i.e., $$x = 2a,\ v = 0.$$

Hence, (iv)

reduces to $0 = -(g/a)(2a)^2 + K$

So that, $\quad K = 4ag.$

From (iv), $\quad v^2 = 4ag - (g/a)x^2$

Or, $\quad (dx/dt) = (g/a)(4a^2 - x^2) \qquad ...(v)$

When the particle reaches A, let the velocity of the particle be V. Then, putting $x = -a$ and $v = V$ in (v), we get

$$V^2 = (g/a)(4a^2 - a^2)$$

Or, $$V = (3ag)^{1/2} \qquad ...(vi)$$

From (v), $$\frac{dx}{dt} = \left(\frac{g}{a}\right)^{1/2}(4a^2 - x^2)$$

Or, $$dt = \left(\frac{g}{a}\right)^{1/2}\frac{dx}{(4a^2 - x^2)^{1/2}} \qquad ...(vii)$$

Where we have taken negative sign on R.H.S. due to the fact that in moving from C towards B, x decreases as t increases.

Let, t_1 be the time taken from C to A.

Then integrating (vii) between $t = 0$ to $t = t_1$ and corresponding limits $x = -2a$ to $x = -a$, we get,

$$\int_0^{t_1} dt = \left(\frac{a}{g}\right)^{1/2}\int_{2a}^{-a}\frac{dx}{(4a^2 - x^2)^{1/2}}$$

$$= \left(\frac{a}{g}\right)^{1/2}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

Or, $$t_1 = (a/g)^{1/2}\{\cos^{-1}(-1/2) - \cos^{-1}\}$$

$$= (a/g)^{1/2}\{\pi - \cos^{-1}(1/2)\}$$

Thus, $$t_1 = (a/g)^{1/2}(\pi - \pi/3)$$

$$= (a/g)^{1/2}(2\pi/3)$$

Thus, particle has velocity V in upward direction and it moves above A. But the string becomes slack in upward motion from A so the S.H.M. ceases at A and the particle moves vertically upwards freely under gravity till its velocity V is destroyed.

Let t_2 be the time taken by the particle from A till its velocity V becomes zero.

Then, using formula

$$v = u - gt, \text{ we get}$$

$$0 = V - gt_2$$

Or, $t_2 = V/g$

$= (3ag)^{1/2}/g$

$= (3a/g)^{1/2}$

Conditions being the same, the particle will take time t_2 in falling freely back to A.

Again from A to C the time taken by the particle will be t_1 (which was taken by it to move from C to A).

So, the required time

$= 2(t_1 + t_2) = 2\{a/g)^{1/2}\ (2\pi/3) + (3a/g)^{1/2}\}$

$= (a/g)^{1/2}\ \left(4\pi/3+2\sqrt{3}\right)$

5. (*c*) Curvature of the curve: Replacing the parameter t by $s = t\sqrt{a^2+b^2}$, we obtain

$$x = a\cos\frac{s}{\sqrt{a^2+b^2}},$$

$$y = a\sin\frac{s}{\sqrt{a^2+b^2}},$$

$$z = \frac{bs}{\sqrt{a^2+b^2}},$$

And, from the formula (*)

$$K = \frac{a}{a^2+b^2} = \text{const},$$

$$e = \frac{a^2+b^2}{a} = \text{const}.$$

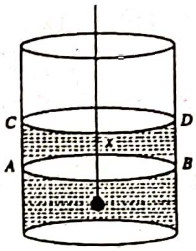
The same result can be obtained without changing the parameter from the formula.

torsion of the curve :

$$\tau = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a^2\sin^2 t + a^2\cos^2 t + b^2\right)\left(a^2\cos^2 t + a^2\sin^2 t\right) - \left(a^2\sin t\cos t - a^2\cos t\sin t\right)^2}$$

$$= \frac{b}{a^2+b^2}$$

5. (*d*) Let the level of the liquid in the vessel be AB before the immersion of the sphere. After the sphere is immersed, let the level of the liquid be CD. If x be the increased height when the level is raised the AC = BD = x.



Since the volume of the liquid displaced by the sphere must be equal to the volume of the sphere, so we have

$$\pi a^2 x = \frac{4}{3}\pi c^3$$

$$\Rightarrow \qquad x = \frac{4}{3}\left(\frac{c^3}{a^2}\right)$$

Now, the whole pressure on the curved surface before immersion

$$= P_1 = 2\pi ah.\frac{1}{2}h.wg$$

$$= \pi ah^2 wg.$$

Whole pressure on the curved surface after immersion = P_2

$$= 2\pi a\ (h + x).\frac{1}{2}(h + x)wg$$

$$= \pi a(h + x)^2 wg.$$

$\therefore$ Increase of whole pressure on the curved surface = $P_2 - P_1$

$$= \pi awg[(h + x)^2 - h^2]$$

$$= \pi awg(x^2 + 2hx)$$

$$= \pi awg\ x(x + 2h)$$

$$= \pi awg\frac{4}{3}\cdot\left(\frac{c^3}{a^2}\right)\left(\frac{4}{3}\cdot\frac{c^3}{a^2}+2h\right)$$

$$= \frac{8\pi}{3a}wgc^3\left(h+\frac{2c^3}{3a^2}\right).$$

5. (*e*) $x^2\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=\dfrac{1}{(1-x)^2}$

Auxiliary equation:

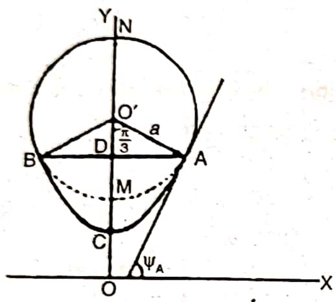
$$(D(D - 1) + 3D + 1)y = 0$$

$$(D^2 + 2D + 1)y = 0$$
$$(D + 1)^2 = 0$$
$$y_1 = (C_1 + C_2 \log x)e^{-\log x}$$
$$y_1 = (C_1 + C_2 \log x)\frac{1}{x}$$

and general solution

$$y(x) = \frac{(C_1 + C_2 \log x)}{x} - \frac{\log(1-x)}{x}$$

7. (*a*) Let, ANBMA be the circular pulley of radius a and ANBCA the endless chain hanging over it.



Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain.

$$= \frac{2}{3}\text{ (circumference of the pulley)}$$

$$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a.$$

Let, the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point *i.e.*, the vertex, CO′N the axis and OX the directrix of this catenary.

Let, OC = c = the parameter of the catenary. The tangent at A will be perpendicular to the radius O′A

∴ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B)$$

$$= \frac{1}{2}\left(\frac{1}{3}.2\pi\right) = \frac{\pi}{3}$$

From the triangle AO′D, we have

$$DA = O'A \sin\frac{1}{3}\pi = a\sqrt{3/2}.$$

∴ From $x = c \log(\tan\psi + \sec\psi)$, for the point A, we have

$$x = DA = c\log(\tan\psi_A + \sec\psi_A)$$

$$\text{Or, } \frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$

$$= c\log(\sqrt{3}+2)$$

$$\therefore \quad c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}.$$

From $s = c\tan\psi$ applied for the point A, we have arc CA $= c\tan\psi_A = c\tan\frac{1}{3}\pi = c\sqrt{3}$

$$= \frac{3a}{2\log(2+\sqrt{3})}$$

Hence, the total length of the chain

= arc ABC + length of the chain in contact with the pulley

$$= 2\cdot(\text{arc CA}) + \frac{4}{3}\pi a$$

$$= 2\frac{3a}{2\log(2+\sqrt{3})} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}.$$

7. (*b*) By divergence theorem, we have,

$$\iint_S \left(x^3 dy.dz + x^2 y\, dz.dx + x^2.z\, dy.dx\right)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx.dy.dz$$

Where, V is the volume enclosed by S.

Here, $F_1 = x^3$, $F_2 = x^2 y$ and $F_3 = x^2.z$

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 + x^2 + x^2$$

$$= 5x^2$$

$\therefore$ The given surface integral

$$= \iiint_v 5x^2.dx.dy.dz$$

$$= 5\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2.\cos^2\theta.r^2.\sin\theta.dr.d\theta.d\phi$$

$$= 5.\left(\frac{1}{5}\right).\left(\frac{2}{3}\right)(2\pi) = \frac{4\pi}{3}.$$

7. (*c*) Let, the particle start from rest from a fixed point O. Let, P be the position of the particle at any time t such that OP = x. Let, v be its velocity at P.

Here, the force of resistance is μv (given) which acts in vertical upward direction.

The weight mg of the particle acts in vertically downward direction.

Then, the equation of motion of the particle at any time t is

$$m\ddot{x} = mg - \mu v$$

Or, $$\frac{dv}{dt} = g - \left(\frac{\mu}{m}\right)v$$

Or, $$dt = \left\{\frac{1}{\frac{(g-\mu v)}{m}}\right\}dv$$

On Integration, we have,

$$t = -\left(\frac{m}{\mu}\right)\log\left(g - \frac{\mu v}{m}\right) + A,$$

where A is a constant

Initially at Point O, when $t = 0$, $v = 0$,

Hence,

$$A = \left(\frac{m}{\mu}\right)\log g$$

$$\therefore \quad t = -\left(\frac{m}{\mu}\right)\log\left(g - \frac{\mu v}{m}\right) + \left(\frac{m}{\mu}\right)\log g$$

$$= -\left(\frac{m}{\mu}\right)\log\left(1 - \frac{\mu v}{gm}\right)$$

Or, $$\log\left(1 - \frac{\mu v}{gm}\right) = -\frac{\mu t}{m}$$

Or, $$1 - \left(\frac{\mu}{gm}\right)v = e^{-\mu t/m}$$

Or, $$v = \frac{dx}{dt} = \left(\frac{gm}{\mu}\right)\left\{1 - e^{-\frac{\mu t}{m}}\right\}$$

Or, $$dx = \left(\frac{gm}{\mu}\right)\left(1 - e^{-\frac{\mu t}{m}}\right)dt$$

On integration, we have

$$x = \left(\frac{gm}{\mu}\right)\left\{t + \left(\frac{m}{\mu}\right)e^{-\mu t/m}\right\} + B,$$

where B is a constant ...(*i*)

Initially at Point O,

when $t = 0$, $x = 0$, so

$$B = -\frac{gm^2}{\mu^2}$$

Then,

$$x = \left(\frac{gm}{\mu}\right)\left\{t + \left(\frac{m}{\mu}\right)e^{-\mu t/m}\right\} - \frac{gm^2}{\mu^2}$$

or $$x = \left(\frac{gm^2}{\mu^2}\right)\left\{e^{-(\mu/m)t} - 1 + \frac{\mu t}{m}\right\}.$$

8. (*a*) Take the major axis and minor axis respectively as the axes of x and y.

Then the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(1)$$

By symmetry it is clear that the C.P. $(\bar{x}, \bar{y})$ will lie on the line AOA′ *i.e.*, x-axis.

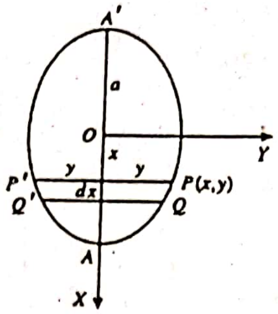
$\therefore \quad \bar{y} = 0$

Take an elementary strip PQQ′P′ at a depth x below O, the centre of the ellipse, and of width dx.

Then dS = area of the elementary strip

$= 2y\ dx$,

p = intensity of pressure at any point of the strip = $\rho g(a + x)$, where ρ is the density of the liquid.



$\therefore$ $\bar{x}$ = depth of the C.P. of the ellipse below

Point, O = $\dfrac{\int x\,p\,dS}{\int p\,dS}$, between suitable limits.

$$\frac{\int_{-a}^{a} x\rho g(a+x)2y\,dx}{\int_{-a}^{a} \rho g(a+x)2y\,dx} = \frac{\int_{-a}^{a} xy(a+x)\,dx}{\int_{-a}^{a} y(a+x)\,dx}$$

The parametric equations of the ellipse (1) are

$$x = a\cos t,\ y = b\sin t.$$

$$\therefore \quad dx = -a\sin t\,dt$$

Also when, $x = a$, $\cos t = 1 \Rightarrow t = 0$ and when $x = -a$, $\cos t = -1 \Rightarrow t = \pi$.

$$\therefore\ \bar{x} = \frac{\int_{\pi}^{0} a\cos t\cdot b\sin t(a+a\cos t)(-a\sin t\,dt)}{\int_{\pi}^{0} b\sin t(a+a\cos t)(-a\sin t\,dt)}$$

$$= \frac{a\int_0^{\pi}(\cos t\sin^2 t+\cos^2 t\sin^2 t)dt}{\int_0^{\pi}(\sin^2 t+\cos t\sin^2 t)dt}$$

$$= \frac{a\left[\int_0^{\pi}\cos t\sin^2 t\,dt+\int_0^{\pi}\cos^2 t\sin^2 t\,dt\right]}{\int_0^{\pi}\sin^2 t\,dt+\int_0^{\pi}\cos t\sin^2 t\,dt}$$

$$= \frac{a\left[0+2\int_0^{\pi/2}\cos^2 t\sin^2 t\,dt\right]}{2\int_0^{\pi/2}\sin^2 t\,dt+0}$$

$$= \frac{a\left(\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right)}{\frac{1}{2}\cdot\frac{\pi}{2}} = \frac{a}{4}$$

Now, the C.P. of the ellipse will coincide with the focus,

if $\bar{x} = ae$ *i.e.*,

if $\dfrac{a}{4} = ae$

$$e = \frac{1}{4}.$$

8. (*b*) $\vec{F} = y\hat{i} + (x-2xz)\hat{j} - xy\,\hat{k}$

The surface $x^2 + y^2 + z^2 = a^2$ meets the plane $z = 0$ in a circle given by $x^2 + y^2 = a^2$, $z = 0$.

Let S_1 be the plane region bounded by the circle C.

If S′ be the surface consisting of the surfaces S and S_1, then S′ is a closed surface. Let V be the region bounded by S′.

If n denotes outward drawn unit vector of S′, then on the plane surface S_1, we have $n = -k$.

Now, by application of Gauss divergence theorem, we have

$$\iint_{S'} \text{Curl F}\cdot n\,ds = 0$$

Or, $\iint_{S} \text{Curl F}\cdot n\,ds + \iint_{S'} \text{Curl F}\cdot n\,ds = 0$

{$\because$ S′ consist of S and S_1}

Or, $\iint_{S} \text{Curl F}\cdot n\,ds - \iint_{S'} \text{Curl F}\cdot k\,ds = 0$

{$\because$ on S, $n = -k$}

Or, $\iint_{S} \text{Curl F}\cdot n\,ds = \iint_{S_1} \text{Curl F}\cdot k\,ds$

Now, Curl F = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$

$$= (-x-2x)\hat{i} - (-y)\hat{j} + (1-2z-1)\cdot\hat{k}$$

$$\therefore \text{Curl F} = -3x\hat{i} + y\hat{j} - 2z\hat{k}$$

$$\therefore \text{Curl F}\cdot\hat{k} = \left(-3x\hat{i} + y\hat{j} - 2z\hat{k}\right)\cdot\hat{k} = -2z$$

$$\therefore \iint_s \text{Curl F}\cdot n\, ds = \iint_{s_1} -2z\cdot ds$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} -2(0)\cdot r\cdot dr\cdot d\theta$$

$$= 0$$

8. (*c*) Here, the central acceleration varies inversely as the cube of the distance *i.e.*,

$$P = \frac{\mu}{r^3} = \mu u^3, \text{ where } \mu \text{ is a constant.}$$

If V is the velocity of a particle along a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

Or, $$V = \sqrt{\left(\frac{\mu}{a^2}\right)}$$

$\therefore$ The velocity of proection $v_1 = \sqrt{2}V$

$$= \sqrt{\left(\frac{2\mu}{a^2}\right)}$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$, and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

$$= \mu u^2 + A, \qquad \text{...(1)}$$

Where, A is a constant

But initially when $r = a$ *i.e.*,

$$u = \frac{1}{a}, \frac{du}{d\theta} = 0 \text{ (at an apse), and}$$

$$v = v_1 = \sqrt{\left(\frac{2\mu}{a^2}\right)}.$$

$\therefore$ From equation, we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$$\therefore h^2 = 2\mu \text{ and } A = \frac{\mu}{a^2}$$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

Or, $$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2$$

$$= \frac{1 - a^2u^2}{a^2}$$

Or, $$\sqrt{2}a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)}$$

Or, $$\frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}$$

On integration, we have,

$$\left(\frac{\theta}{\sqrt{2}}\right) + B = \sin^{-1}(au), \text{ where B is a constant.}$$

But initially, when $u = \frac{1}{a}$, $\theta = 0$.

$$\therefore \quad B = \sin^{-1} 1 = \frac{1}{2}\pi.$$

$$\therefore \quad \left(\frac{\theta}{\sqrt{2}}\right) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$$

For, $$au = \frac{a}{r}$$

$$\frac{a}{r} = \sin\left\{\frac{1}{2}\pi + \left(\frac{\theta}{\sqrt{2}}\right)\right\}$$

or, $a = r\cos\left(\frac{\theta}{\sqrt{2}}\right)$,

which is the required equation of the path.