Set – III

MISCELLANEOUS

11. Velocity and Acceleration.

If the scalar variable t be the time and $\mathbf{r}$ be the position vector of a moving particle P with respect to the origin O, then $\delta\mathbf{r}$ is the displacement of the particle in time δt.

The vector $\frac{\delta \mathbf{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If $\mathbf{v}$ represents the velocity vector of the particle at P, then $\mathbf{v} = \lim\limits_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}$.

Since $\frac{d\mathbf{r}}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving, therefore the direction of velocity is along the tangent.

If $\delta\mathbf{v}$ be the change in the velocity $\mathbf{v}$ during the time δt, then $\frac{\delta \mathbf{v}}{\delta t}$ is the average acceleration during that interval. If $\mathbf{a}$ represents the acceleration of the particle at time t, then

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\delta \mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}.$$

Solved Examples

Ex. 1. *If* $\mathbf{r} = (t+1)\,\mathbf{i} + (t^2+t+1)\,\mathbf{j} + (t^3+t^2+t+1)\,\mathbf{k}$ *find* $\frac{d\mathbf{r}}{dt}$ *and* $\frac{d^2\mathbf{r}}{dt^2}$.

Sol. Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant vectors, therefore

$$\frac{d\mathbf{i}}{dt} = 0 = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(t+1)\,\mathbf{i}+\frac{d}{dt}(t^2+t+1)\,\mathbf{j}+\frac{d}{dt}(t^3+t^2+t+1)\,\mathbf{k}$$
$$=\mathbf{i}+(2t+1)\,\mathbf{j}+(3t^2+2t+1)\,\mathbf{k}.$$

Again, $$\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{i}}{dt}+\frac{d}{dt}(2t+1)\,\mathbf{j}+\frac{d}{dt}(3t^2+2t+1)\,\mathbf{k}$$
$$=\mathbf{0}+2\mathbf{j}+(6t+2)\,\mathbf{k}=2\mathbf{j}+(6t+2)\,\mathbf{k}.$$

Ex. 2. *If* $\mathbf{r}=\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+t\,\mathbf{k}$, *find*

(*i*) $\dfrac{d\mathbf{r}}{dt}$, (*ii*) $\dfrac{d^2\mathbf{r}}{dt^2}$, (*iii*) $\left|\dfrac{d\mathbf{r}}{dt}\right|$, (*iv*) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$. **[Agra 1978]**

Sol. Since **i**, **j**, **k** are constant vectors, therefore $\dfrac{d\mathbf{i}}{dt}=\mathbf{0}$ etc.

Therefore

(i) $$\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\sin t)\,\mathbf{i}+\frac{d}{dt}(\cos t)\,\mathbf{j}+\frac{d}{dt}(t)\,\mathbf{k}=\cos t\,\mathbf{i}-\sin t\,\mathbf{j}+\mathbf{k}.$$

(ii) $$\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=\frac{d}{dt}(\cos t)\,\mathbf{i}-\frac{d}{dt}(\sin t)\,\mathbf{j}+\frac{d\mathbf{k}}{dt}$$
$$=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}+\mathbf{0}=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}.$$

(iii) $$\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{[(\cos t)^2+(-\sin t)^2+(1)^2]}=\sqrt{2}.$$

(iv) $$\left|\frac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{[(-\sin t)^2+(-\cos t)^2]}=1.$$

Ex. 3. *If* $\mathbf{r}=(\cos nt)\,\mathbf{i}+(\sin nt)\,\mathbf{j}$, *where* n *is a constant and* t *varies, show that* $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=n\mathbf{k}$

Sol. We have

$$\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos nt)\,\mathbf{i}+\frac{d}{dt}(\sin nt)\,\mathbf{j}=-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j}.$$

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos nt\,\mathbf{i}+\sin nt\,\mathbf{j})\times(-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j})$$
$$=-n\cos nt\sin nt\,\mathbf{i}\times\mathbf{i}+n\cos^2 nt\,\mathbf{i}\times\mathbf{j}$$
$$-n\sin^2 nt\,\mathbf{j}\times\mathbf{i}+n\cos nt\sin nt\,\mathbf{j}\times\mathbf{j}$$
$$=n\cos^2 nt\,\mathbf{k}+n\sin^2 nt\,\mathbf{k}$$
$$[\because \ \mathbf{i}\times\mathbf{i}=\mathbf{0},\ \mathbf{j}\times\mathbf{j}=\mathbf{0},\ \mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]$$
$$=n(\cos^2 nt+\sin^2 nt)\,\mathbf{k}=n\mathbf{k}.$$

Ex. 4. *If* **a**, **b** *are constant vectors,* ω *is a constant, and* **r** *is a vector function of the scalar variable* t *given by*

$$\mathbf{r}=\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b},$$

show that

(i) $\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\mathbf{0}$, *and* (ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\omega\mathbf{a}\times\mathbf{b}$.

[Rohilkhand 1984; Agra 81; Kumayun 82; Madras 83]

Sol. Since **a**, **b** are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=\mathbf{0},\ \frac{d\mathbf{b}}{dt}=\mathbf{0}$$

(i) $\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos\omega t)\,\mathbf{a}+\frac{d}{dt}(\sin\omega t)\,\mathbf{b}$

$=-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b}$.

$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}=-\omega^2\cos\omega t\,\mathbf{a}-\omega^2\sin\omega t\,\mathbf{b}$

$=-\omega^2(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})=-\omega^2\mathbf{r}$.

$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\mathbf{0}$.

(ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})\times(-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b})$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}-\omega\sin^2\omega t\,\mathbf{b}\times\mathbf{a}$ $\quad[\because\ \mathbf{a}\times\mathbf{a}=\mathbf{0},\ \mathbf{b}\times\mathbf{b}=\mathbf{0}]$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}+\omega\sin^2\omega t\,\mathbf{a}\times\mathbf{b}$

$=\omega(\cos^2\omega t+\sin^2\omega t)\,\mathbf{a}\times\mathbf{b}=\omega\mathbf{a}\times\mathbf{b}$.

Ex. 5. *If* $\mathbf{r}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}$, *where* **a** *and* **b** *are constant vectors, then show that* $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}$.

Sol. Since **a**, **b** are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=\mathbf{0},\ \frac{d\mathbf{b}'}{dt}=\mathbf{0}.$$

$\therefore\quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\sinh t)\,\mathbf{a}+\frac{d}{dt}(\cosh t)\,\mathbf{b}$

$=(\cosh t)\,\mathbf{a}+(\sinh t)\,\mathbf{b}$.

$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}=\mathbf{r}$.

Ex. 6. *If* $\mathbf{r}=t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}$, *show that* $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{k}$.

[Utkal 1973]

Sol. We have $\mathbf{r}=t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}$.

$\therefore\quad \frac{d\mathbf{r}}{dt}=3t^2\,\mathbf{i}+\left(6t^2+\frac{2}{5t^3}\right)\mathbf{j}$.

$\therefore\quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=\left[t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}\right]\times\left[3t^2\,\mathbf{i}+\left(6t^2+\frac{2}{5t^3}\right)\mathbf{j}\right]$

$$=t^3\left(6t^2+\frac{2}{5t^3}\right)\mathbf{i}\times\mathbf{j}+3t^2\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}\times\mathbf{i}$$

$[\because \ \mathbf{i}\times\mathbf{i}=0,\ \mathbf{j}\times\mathbf{j}=0]$

$$=\left(6t^5+\frac{2}{5}\right)\mathbf{k}+\left(6t^5-\frac{3}{5}\right)(-\mathbf{k})$$

$[\because \ \mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]$

$$=\left(6t^5+\frac{2}{5}-6t^5+\frac{3}{5}\right)\mathbf{k}=\mathbf{k}.$$

Ex. 7. *If* $\mathbf{r}=e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b}$, *where* $\mathbf{a}, \mathbf{b}$ *are constant vectors, show that* $\dfrac{d^2\mathbf{r}}{dt^2}-n^2\mathbf{r}=0.$

[Agra 1976]

Sol. Given $\mathbf{r}=e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b}$, ...(1)

where $\mathbf{a}, \mathbf{b}$ are constant vectors.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\left[\frac{d}{dt}(e^{nt})\right]\mathbf{a}+e^{nt}\frac{d\mathbf{a}}{dt}+\left[\frac{d}{dt}(e^{-nt})\right]\mathbf{b}+e^{-nt}\frac{d\mathbf{b}}{dt}$$

$$=ne^{nt}\,\mathbf{a}-ne^{-nt}\,\mathbf{b}. \quad \left[\because \ \frac{d\mathbf{a}}{dt}=0=\frac{d\mathbf{b}}{dt},\ \mathbf{a} \text{ and } \mathbf{b} \text{ being constant vectors}\right]$$

Again differentiating with respect to t, we get

$$\frac{d^2\mathbf{r}}{dt^2}=n^2e^{nt}\,\mathbf{a}+n^2e^{-nt}\,\mathbf{b}=n^2(e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b})=n^2\,\mathbf{r}, \text{ from (1).}$$

$$\therefore \quad \frac{d^2\mathbf{r}}{dt^2}-n^2\,\mathbf{r}=0.$$

Ex. 8. *If* $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\dfrac{\mathbf{c}t}{\omega^2}\sin\omega t$, *prove that*

$$\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\frac{2\mathbf{c}}{\omega}\cos\omega t,$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are constant vectors and* ω *is a constant scalar.*

[Meerut 1991, Marathwada 74]

Sol. Given $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\dfrac{\mathbf{c}t}{\omega^2}\sin\omega t$, ...(1)

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are constant vectors and ω is a constant scalar.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\mathbf{a}\,\omega\cos\omega t-\mathbf{b}\,\omega\sin\omega t+\frac{\mathbf{c}}{\omega^2}\sin\omega t+\frac{\mathbf{c}}{\omega^2}t\omega\cos\omega t$$

and $$\frac{d^2\mathbf{r}}{dt^2}=-\mathbf{a}\omega^2\sin\omega t-\mathbf{b}\,\omega^2\cos\omega t+\frac{\mathbf{c}}{\omega^2}\omega\cos\omega t$$

$$+\frac{\mathbf{c}}{\omega^2}\omega\cos\omega t-\frac{\mathbf{c}}{\omega^2}t\,\omega^2\sin\omega t$$

$$=-\omega^2\left(\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\frac{\mathbf{c}t}{\omega^2}\sin\omega t\right)+\frac{2\mathbf{c}}{\omega}\cos\omega t$$

$$=-\omega^2 \mathbf{r}+\frac{2\mathbf{c}}{\omega}\cos \omega t, \text{ from (1).}$$

$$\therefore \quad \frac{d^2\mathbf{r}}{dt^2}+\omega^2 \mathbf{r}=\frac{2\mathbf{c}}{\omega}\cos \omega t.$$

Ex. 9. *Show that* $\mathbf{r}=\mathbf{a}\, e^{mt}+\mathbf{b}\, e^{nt}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are the constant vectors, is the solution of the differential equation*

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}=\mathbf{0}.$$

Hence solve the equation

$$\frac{d^2\mathbf{r}}{dt^2}-\frac{d\mathbf{r}}{dt}-2\mathbf{r}=\mathbf{0}, \text{ where}$$

$\mathbf{r}=\mathbf{i}$ *and* $\dfrac{d\mathbf{r}}{dt}=\mathbf{j}$ *for* $t=0$. **[Kanpur 1977]**

Sol. We have $\mathbf{r}=\mathbf{a}\, e^{mt}+\mathbf{b}\, e^{nt}$, ...(1)

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\mathbf{a}\, me^{mt}+\mathbf{b}\, ne^{nt} \quad \text{...(2)}$$

and $$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{a}\, m^2 e^{mt}+\mathbf{b}\, n^2 e^{nt} \quad \text{...(3)}$$

From (1), (2) and (3), we get

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}$$

$$=\mathbf{a}m^2 e^{mt}+\mathbf{b}\, n^2 e^{nt}-(m+n)\,[\mathbf{a}\, me^{mt}+\mathbf{b}\, ne^{nt}]$$

$$=e^{mt}\,(m^2-m^2-mn+mn)\,\mathbf{a}+e^{nt}\,(n^2-mn-n^2+mn)\,\mathbf{b}$$

$$=0\mathbf{a}+0\mathbf{b}=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Hence $\mathbf{r}=\mathbf{a}\, e^{mt}+\mathbf{b}\, e^{nt}$ is the solution of the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}=\mathbf{0}. \quad \text{...(4)}$$

Putting $m=2$ and $n=-1$ in (1) and (4), we see that the general solution of the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}-\frac{d\mathbf{r}}{dt}-2\mathbf{r}=\mathbf{0} \quad \text{...(5)}$$

is $$\mathbf{r}=\mathbf{a}\, e^{2t}+\mathbf{b}\, e^{-t}, \quad \text{...(6)}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

From (6), $$\frac{d\mathbf{r}}{dt}=\mathbf{a}\, 2e^{2t}-\mathbf{b}\, e^{-t}. \quad \text{...(7)}$$

But it is given that for $t=0$, $\mathbf{r}=\mathbf{i}$ and $\dfrac{d\mathbf{r}}{dt}=\mathbf{j}$.

$\therefore$ from (6) and (7), we have

$$\mathbf{a}+\mathbf{b}=\mathbf{i} \quad \text{...(8)}$$

and $$2\mathbf{a}-\mathbf{b}=\mathbf{j}. \qquad \ldots(9)$$

Adding (8) and (9), we get $3\mathbf{a}=\mathbf{i}+\mathbf{j}$ or $\mathbf{a}=\frac{1}{3}(\mathbf{i}+\mathbf{j})$.

Now from (8), we have $\mathbf{b}=\mathbf{i}-\mathbf{a}=\mathbf{i}-\frac{1}{3}(\mathbf{i}+\mathbf{j})=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}$.

Putting $\mathbf{a}=\frac{1}{3}(\mathbf{i}+\mathbf{j})$ and $\mathbf{b}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}$ in (6), the required solution of the differential equation (5) under the given conditions is

$$\mathbf{r}=\frac{1}{3}(\mathbf{i}+\mathbf{j})e^{2t}+\left(\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}\right)e^{-t}$$

or $$\mathbf{r}=\frac{1}{3}(e^{2t}+2e^{-t})\mathbf{i}+\frac{1}{3}(e^{2t}-e^{-t})\mathbf{j}.$$

Ex. 10. *Prove the following :*

(i) $$\frac{d}{dt}\left[\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right]=\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}.$$

(ii) $$\frac{d}{dt}\left[\mathbf{a}\times\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right]=\mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}.$$

Sol. (i) We have $\frac{d}{dt}\left[\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right]$

$$=\frac{d}{dt}\left(\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}\right)-\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right)$$

$$=\frac{d\mathbf{a}}{dt}\cdot\frac{d\mathbf{b}}{dt}+\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}-\frac{d\mathbf{a}}{dt}\cdot\frac{d\mathbf{b}}{dt}$$

$$=\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}.$$

(ii) $$\frac{d}{dt}\left[\mathbf{a}\times\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right]$$

$$=\frac{d}{dt}\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}\right)-\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right)$$

$$=\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{b}}{dt}+\mathbf{a}\times\frac{d}{dt}\left(\frac{d\mathbf{b}}{dt}\right)-\left[\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\right)\right]\times\mathbf{b}-\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{b}}{dt}$$

$$=\mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}.$$

Ex. 11. *If $\mathbf{r}=t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}$, find at $t=0$, the values of* $\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \left|\frac{d\mathbf{r}}{dt}\right|, \left|\frac{d^2\mathbf{r}}{dt^2}\right|$.

Sol. $\mathbf{r}=t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=2t\,\mathbf{i}-\mathbf{j}+2\mathbf{k} \qquad \ldots(1)$$

and $$\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i} \qquad \ldots(2)$$

From (1) and (2), we have

$$\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{[(2t)^2+(-1)^2+2^2]}=\sqrt{(4t^2+5)} \qquad ...(3)$$

and $$\left|\frac{d^2\mathbf{r}}{dt^2}\right|=|\,2\mathbf{i}\,|=2. \qquad ...(4)$$

Putting $t=0$ in (1), (2), (3) and (4), we have at $t=0$,

$$\frac{d\mathbf{r}}{dt}=-\mathbf{j}+2\mathbf{k},\ \frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i},\ \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{5},\ \left|\frac{d^2\mathbf{r}}{dt^2}\right|=2.$$

Ex. 12. *If* $\mathbf{u}=t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}$ *and* $\mathbf{v}=(2t-3)\,\mathbf{i}+\mathbf{j}-t\mathbf{k}$, *find* $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})$, *when* $t=1$. **[Kanpur 1982]**

Sol. $$\frac{d\mathbf{u}}{dt}=\frac{d}{dt}\left[t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}\right]=2t\mathbf{i}-\mathbf{j}+2\mathbf{k}$$

and $$\frac{d\mathbf{v}}{dt}=\frac{d}{dt}[(2t-3)\,\mathbf{i}+\mathbf{j}-t\,\mathbf{k}]=2\mathbf{i}+0\,\mathbf{j}-\mathbf{k}.$$

$$\therefore\quad \frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}+\mathbf{v}\cdot\frac{d\mathbf{u}}{dt}$$

$$=[t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}]\cdot(2\mathbf{i}-\mathbf{k})+[(2t-3)\,\mathbf{i}+\mathbf{j}-t\,\mathbf{k}]\cdot(2t\,\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

$$=2t^2-(2t+1)+2t\,(2t-3)-1-2t$$

$$=2t^2-2t-1+4t^2-6t-1-2t$$

$$=6t^2-10t-2$$

$$=-6,\text{ when } t=1.$$

Ex. 13. *If* $\mathbf{A}=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$ *and* $\mathbf{B}=\sin t\,\mathbf{i}-\cos t\,\mathbf{j}$, *find*

(*i*) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$; (*ii*) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$; (*iii*) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A})$.

Sol. We have $\frac{d\mathbf{A}}{dt}=10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k}$ and $\frac{d\mathbf{B}}{dt}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}$.

(i) $$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}+\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$$

$$=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\cdot(\cos t\,\mathbf{i}+\sin t\mathbf{j})+(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})\cdot(\sin t\,\mathbf{i}-\cos t\,\mathbf{j})$$

$$=5t^2\cos t+t\sin t+10t\sin t-\cos t$$

$$=(5t^2-1)\cos t+11t\sin t.$$

(ii) We have $\mathbf{A}\times\mathbf{B}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\times(\sin t\,\mathbf{i}-\cos t\,\mathbf{j})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ \sin t & -\cos t & 0\end{vmatrix}$$

$$=-t^3\cos t\,\mathbf{i}-(0+t^3\sin t)\,\mathbf{j}+(-5t^2\cos t-t\sin t)\,\mathbf{k}$$

$= -t^3 \cos t\,\mathbf{i} - t^3 \sin t\,\mathbf{j} - (5t^2 \cos t + t \sin t)\,\mathbf{k}.$

$$\therefore \quad \frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = (t^3 \sin t - 3t^2 \cos t)\,\mathbf{i} - (t^3 \cos t + 3t^2 \sin t)\,\mathbf{j}$$
$$- (10t \cos t - 5t^2 \sin t + \sin t + t \cos t)\,\mathbf{k}$$
$$= t^2 (t \sin t - 3 \cos t)\,\mathbf{i} - t^2 (t \cos t + 3 \sin t)\,\mathbf{j}$$
$$- (11t \cos t - 5t^2 \sin t + \sin t)\,\mathbf{k}.$$

(iii) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A} + \mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}$

$= 2\,(5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k})\cdot(10t\,\mathbf{i} + \mathbf{j} - 3t^2\,\mathbf{k})$

$= 2\,[50\,t^3 + t + 3t^5] = 100t^3 + 2t + 6t^5.$

Ex. 14. *If* $\mathbf{r} = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j} + at \tan \alpha\,\mathbf{k}$, *find*

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right].$$

[Meerut 1991 P, 92; Agra 82, 88; Kanpur 88]

Sol. We have

$$\frac{d\mathbf{r}}{dt} = -a \sin t\,\mathbf{i} + a \cos t\,\mathbf{j} + a \tan \alpha\,\mathbf{k}$$

$$\frac{d^2\mathbf{r}}{dt^2} = -a \cos t\,\mathbf{i} - a \sin t\,\mathbf{j}, \quad \left[\because \frac{d\mathbf{k}}{dt} = 0\right]$$

$$\frac{d^3\mathbf{r}}{dt^3} = a \sin t\,\mathbf{i} - a \cos t\,\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & a \tan \alpha \\ -a \cos t & -a \sin t & 0 \end{vmatrix}$$
$$= a^2 \sin t \tan \alpha\,\mathbf{i} - a^2 \cos t \tan \alpha\,\mathbf{j} + a^2\mathbf{k}.$$

$$\therefore \quad \left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{\left(a^4 \sin^2 t \tan^2 \alpha + a^4 \cos^2 t \tan^2 \alpha + a^4\right)}$$
$$= a^2 \sec \alpha.$$

Also $\left[\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}, \dfrac{d^3\mathbf{r}}{dt^3}\right] = \left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)\cdot\dfrac{d^3\mathbf{r}}{dt^3}$

$= (a^2 \sin t \tan \alpha\,\mathbf{i} - a^2 \cos t \tan \alpha\,\mathbf{j} + a^2\mathbf{k})\cdot(a \sin t\,\mathbf{i} - a \cos t\,\mathbf{j})$

$= a^3 \sin^2 t \tan \alpha\,\mathbf{i}\cdot\mathbf{i} + a^3 \cos^2 t \tan \alpha\,\mathbf{j}\cdot\mathbf{j} \quad [\because \mathbf{i}\cdot\mathbf{j} = 0 \text{ etc.}]$

$= a^3 \tan \alpha\,(\sin^2 t + \cos^2 t) \quad [\because \mathbf{i}\cdot\mathbf{i} = 1 = \mathbf{j}\cdot\mathbf{j}]$

$= a^3 \tan \alpha.$

Ex. 15. *If* $\dfrac{d\mathbf{u}}{dt} = \mathbf{w}\times\mathbf{u}$, $\dfrac{d\mathbf{v}}{dt} = \mathbf{w}\times\mathbf{v}$, *show that*

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

[Meerut 1991S, Kanpur 88]

Sol. We have

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\frac{d\mathbf{u}}{dt}\times\mathbf{v}+\mathbf{u}\times\frac{d\mathbf{v}}{dt}=(\mathbf{w}\times\mathbf{u})\times\mathbf{v}+\mathbf{u}\times(\mathbf{w}\times\mathbf{v})$$
$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{v}\cdot\mathbf{u})\,\mathbf{w}+(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}$$
$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}\quad[\because\quad \mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}]$$
$$=(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}=\mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

Ex. 16. *If* $\mathbf{R}$ *be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt},\text{ where } r=|\,\mathbf{r}\,|.$$

[Kanpur 1987; Agra 83; Garhwal 86]

Sol. We have $\mathbf{r}=r\mathbf{R}$; so that $\mathbf{R}=\frac{1}{r}\,\mathbf{r}$.

$$\therefore\quad \frac{d\mathbf{R}}{dt}=\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}.$$

Hence $\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r}\,\mathbf{r}\times\left(\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right)$

$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}-\frac{1}{r^3}\frac{dr}{dt}\,\mathbf{r}\times\mathbf{r}$$
$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}.\qquad[\because\quad \mathbf{r}\times\mathbf{r}=\mathbf{0}]$$

Ex. 17. *Show that* $\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=(\mathbf{r}\times d\mathbf{r})/r^2$, *where* $\mathbf{r}=r\,\hat{\mathbf{r}}$.

[Rohilkhand 1991, Agra 83]

Sol. We have $\hat{\mathbf{r}}=\frac{1}{r}\,\mathbf{r}$.

$$\therefore\quad d\hat{\mathbf{r}}=d\left(\frac{1}{r}\mathbf{r}\right)=\frac{1}{r}\,d\mathbf{r}+\left(-\frac{1}{r^2}\,dr\right)\mathbf{r}.$$

Hence $\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=\left(\frac{1}{r}\mathbf{r}\right)\times\left[\frac{1}{r}\,d\mathbf{r}-\left(\frac{1}{r^2}\,dr\right)\mathbf{r}\right]$

$$=\frac{1}{r^2}\,\mathbf{r}\times d\mathbf{r}-\left(\frac{1}{r^3}\,dr\right)\mathbf{r}\times\mathbf{r}$$
$$=\frac{\mathbf{r}\times d\mathbf{r}}{r^2},\text{ since }\mathbf{r}\times\mathbf{r}=\mathbf{0}.$$

Ex. 18. *If* $\mathbf{r}$ *is the position vector of a moving point and* r *is the modulus of* $\mathbf{r}$, *show th*

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\,\frac{dr}{dt}.$$

Interpret the relations $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0$ *and* $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}$.

[Rohilkhand 1980]

Sol. We have $\mathbf{r}\cdot\mathbf{r}=|\,\mathbf{r}\,|^2=r^2$.

$$\therefore \quad \frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})=\frac{d}{dt}(r^2)$$

or $$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}=2r\frac{dr}{dt}$$

or $$2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=2r\frac{dr}{dt} \qquad \left[\because \quad \mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right]$$

or $$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\frac{dr}{dt}.$$

Geometrical interpretation of $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}=0$ and $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=\mathbf{0}$.

$\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}=0$ is a necessary and sufficient condition for the vector $\mathbf{r}(t)$ to have constant modulus while $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=\mathbf{0}$ is a necessary and sufficient condition for the vector $\mathbf{r}(t)$ to have constant direction.

Ex. 19. *If the direction of a differentiable vector function $\mathbf{r}(t)$ is constant, show that $\mathbf{r}\times(d\mathbf{r}/dt)=\mathbf{0}$.* **[Kanpur 1982; Rohilkhand 79]**

Or

If $\mathbf{r}(t)$ is a vector of constant direction, show that its derivative is collinear with it. **[Allahabad 1981]**

Sol. Let $\mathbf{r}$ be a vector function of the scalar variable t having a constant direction.

If $\mathbf{R}$ be a unit vector in the direction of $\mathbf{r}$, then $\mathbf{R}$ is a constant vector because it has constant direction as well as constant modulus.

If r be the modulus of $\mathbf{r}$, then $\mathbf{r}=r\mathbf{R}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(r\mathbf{R})=\frac{dr}{dt}\mathbf{R}+r\frac{d\mathbf{R}}{dt}$$

$$=\frac{dr}{dt}\mathbf{R}\left[\because \quad \frac{d\mathbf{R}}{dt}=\mathbf{0},\ \mathbf{R}\text{ being a constant vector}\right]$$

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(r\mathbf{R})\times\left(\frac{dr}{dt}\mathbf{R}\right)$$

$$=r\frac{dr}{dt}(\mathbf{R}\times\mathbf{R})=\mathbf{0} \qquad [\because \quad \mathbf{R}\times\mathbf{R}=\mathbf{0}]$$

Now $\mathbf{r}\times(d\mathbf{r}/dt)=\mathbf{0}$ implies that the vector $d\mathbf{r}/dt$ is collinear with $\mathbf{r}$.

Ex. 20. If $\mathbf{r}\times d\mathbf{r}=\mathbf{0}$, *show that* $\hat{\mathbf{r}}=$ *constant.*

[Kanpur 1987; Rohilkhand 80]

Sol. Let r be the modulus of the vector $\mathbf{r}$.

Then $\mathbf{r}=r\hat{\mathbf{r}}$.

$\therefore \quad d\mathbf{r}=d(r\hat{\mathbf{r}})=dr\,\hat{\mathbf{r}}+r\,d\hat{\mathbf{r}}$.

$\therefore \quad \mathbf{r}\times d\mathbf{r}=(r\hat{\mathbf{r}})\times(dr\,\hat{\mathbf{r}}+r\,d\hat{\mathbf{r}})$

$=(r\,dr)\,\hat{\mathbf{r}}\times\hat{\mathbf{r}}+r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}}$

$=r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}} \qquad [\because \quad \hat{\mathbf{r}}\times\hat{\mathbf{r}}=\mathbf{0}]$

$\therefore \quad \mathbf{r}\times d\mathbf{r}=\mathbf{0} \Rightarrow r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=\mathbf{0}$

$\Rightarrow \hat{\mathbf{r}}\times d\,\hat{\mathbf{r}}=\mathbf{0}$ ...(1)

Since $\hat{\mathbf{r}}$ is of constant modulus, therefore

$\hat{\mathbf{r}}\cdot d\hat{\mathbf{r}}=0.$ ...(2)

From (1) and (2), we get $d\hat{\mathbf{r}}=\mathbf{0}$.

Hence $\hat{\mathbf{r}}$ is a constant vector.

Alternative method.

Let $\mathbf{r}=x\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, then $d\mathbf{r}=dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$.

$$\therefore \quad \mathbf{r}\times d\mathbf{r}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ dx & dy & dz \end{vmatrix}$$

$=(y\,dz-z\,dy)\,\mathbf{i}+(z\,dx-x\,dz)\,\mathbf{j}+(x\,dy-y\,dx)\,\mathbf{k}$.

$\therefore \quad \mathbf{r}\times d\mathbf{r}=\mathbf{0} \Rightarrow (y\,dz-z\,dy)\,\mathbf{i}+(z\,dx-x\,dz)\,\mathbf{j}+(x\,dy-y\,dx)\,\mathbf{k}=\mathbf{0}$

$\Rightarrow \quad y\,dz-z\,dy=0,\ z\,dx-x\,dz=0,\ x\,dy-y\,dx=0$

$$\Rightarrow \quad \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

If $\dfrac{dx}{x}=\dfrac{dy}{y}$, then $\log x=\log y+\log c_1$

or $\qquad x=c_1\,y.$ ...(1)

If $\dfrac{dy}{y}=\dfrac{dz}{z}$, then $\log y+\log c_2=\log z$

or $\qquad z=c_2\,y.$ ...(2)

Now $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(c_1\,y)\,\mathbf{i}+y\,\mathbf{j}+(c_2\,y)\,\mathbf{k}$

$=y\,(c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k}).$

$$\therefore \quad \hat{\mathbf{r}}=\frac{\mathbf{r}}{|\mathbf{r}|}=\frac{y\,(c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k})}{\sqrt{[(c_1\,y)^2+y^2+(c_2\,y)^2]}}$$

$=\dfrac{c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k}}{\sqrt{(c_1^2+1+c_2^2)}}$, which is a constant vector because it is independent of x, y, z.

Ex. 21. *If* **r** *is a unit vector, then prove that*

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\left|\frac{d\mathbf{r}}{dt}\right|.$$

[Rajasthan 1974]

Sol. Since **r** is a unit vector, therefore | **r** | is constant and so **r** is perpendicular to its derivative $d\mathbf{r}/dt$.

Now by the definition of the cross product of two vectors, we have

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\left|\mathbf{r}\right|.\ \left|\frac{d\mathbf{r}}{dt}\right|.\ \sin 90^\circ=1.\ \left|\frac{d\mathbf{r}}{dt}\right|.\ 1=\left|\frac{d\mathbf{r}}{dt}\right|$$

Ex. 22. *If* **e** *is the unit vector making an angle* θ *with x-axis, show that* $d\mathbf{e}/d\theta$ *is a unit vector obtained by rotating* **e** *through a right angle in the direction of* θ *increasing.* [Allahabad 1979]

Sol. Since **e** is the unit vector which makes an angle θ with x-axis, therefore

$$\mathbf{e}=\cos\theta\ \mathbf{i}+\sin\theta\ \mathbf{j}.$$ [Draw figure yourself]

$$\therefore\quad \frac{d\mathbf{e}}{d\theta}=-\sin\theta\ \mathbf{i}+\cos\theta\ \mathbf{j}=\cos\left(\frac{1}{2}\pi+\theta\right)\mathbf{i}+\sin\left(\frac{1}{2}\pi+\theta\right)\mathbf{j},$$

which is a unit vector which makes an angle $\frac{1}{2}\pi+\theta$ with x-axis.

Thus $d\mathbf{e}/d\theta$ is a unit vector perpendicular to the vector **e** in the direction of θ increasing.

Hence $d\mathbf{e}/d\theta$ is a unit vector obtained by rotating **e** through a right angle in the direction of θ increasing.

Ex. 23. *If* **r** *is a vector function of a scalar t and* **a** *is a constant vector, m a constant, differentiate the following with respect to t ;*

(*i*) $R=$ $\mathbf{r}\cdot\mathbf{a}$, (*ii*) $\mathbf{r}\times\mathbf{a}$, (*iii*) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$, (*iv*) $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$,

(*v*) $\mathbf{r}^2+\dfrac{1}{\mathbf{r}^2}$, (*vi*) $m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$, (*vi*) $\dfrac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$, (*viii*) $\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Sol. (i) Let $R=\mathbf{r}\cdot\mathbf{a}$. [Note $\mathbf{r}\cdot\mathbf{a}$ is a scalar]

$$\text{Then } \frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}$$

$$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\mathbf{0}\quad\left[\because\ \frac{d\mathbf{a}}{dt}=\mathbf{0},\text{ as } \mathbf{a} \text{ is constant}\right]$$

$$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+0=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}.$$

(ii) Let $\mathbf{R}=\mathbf{r}\times\mathbf{a}$.

$$\text{Then } \frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}$$

$$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\mathbf{0}\left[\because\quad\frac{d\mathbf{a}}{dt}=\mathbf{0}\right]$$

$$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{0}=\frac{d\mathbf{r}}{dt}\times\mathbf{a}.$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\frac{d\mathbf{r}}{dt}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$

$$=\mathbf{0}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\qquad\left[\because\quad\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}\right]$$

$$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}.$$

(iv) Let $R=\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}$.

Then $\frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}=\left(\frac{d\mathbf{r}}{dt}\right)^2+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}.$

(v) Let $R=\mathbf{r}^2+\frac{1}{\mathbf{r}^2}$.

Then $\frac{dR}{dt}=\frac{d}{dt}(\mathbf{r}^2)+\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right)$

$$=\frac{d}{dt}(r^2)+\frac{d}{dt}\left(\frac{1}{r^2}\right),\text{ where } r=|\mathbf{r}|$$

$$=2r\frac{dr}{dt}-\frac{2}{r^3}\frac{dr}{dt}.$$

(vi) Let $R=m\left(\frac{d\mathbf{r}}{dt}\right)^2$.

Then $\frac{dR}{dt}=m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$

$$=2m\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\qquad\left[\text{Note }\frac{d\mathbf{r}^2}{dt}=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2m\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}.$$

(vii) Let $\mathbf{R}=\frac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d}{dt}(\mathbf{r}+\mathbf{a})+\left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\right)\right\}(\mathbf{r}+\mathbf{a})$

[Note that $\mathbf{r}^2+\mathbf{a}^2$ is a scalar]

$$=\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)^2}\frac{d}{dt}(\mathbf{r}^2+\mathbf{a}^2)\right\}(\mathbf{r}+\mathbf{a})$$

$$=\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d\mathbf{r}}{dt}-\frac{2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}}{(\mathbf{r}^2+\mathbf{a}^2)^2}(\mathbf{r}+\mathbf{a}).$$

$$\left[\because \quad \frac{d\mathbf{a}}{dt}=0,\ \frac{d}{dt}\ \mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt},\ \frac{d}{dt}\ \mathbf{a}^2=0\right]$$

(viii) Let $\mathbf{R}=\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\dfrac{d}{dt}(\mathbf{r}\times\mathbf{a})+\left\{\dfrac{d}{dt}\left(\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\right)\right\}(\mathbf{r}\times\mathbf{a})$

[Note that $\mathbf{r}\cdot\mathbf{a}$ is a scalar quantity]

$$=\frac{1}{\mathbf{r}\cdot\mathbf{a}}\left(\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{a})\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}\right)\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\frac{\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}}{(\mathbf{r}\cdot\mathbf{a})^2}(\mathbf{r}\times\mathbf{a})\qquad\left[\because\quad\frac{d\mathbf{a}}{dt}=0\right]$$

Ex. 24. *If* $\mathbf{r}$ *is a vector function of a scalar* t, r *its module, and* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors, differentiate the following with respect to* t :

(*i*) $r^3\,\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$, (*ii*) $r^2\,\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$, (*iii*) $r^n\,\mathbf{r}$, (*iv*) $(a\mathbf{r}+r\mathbf{b})^2$.

Sol. (i) Let $\mathbf{R}=r^3\,\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{d}{dt}(r^3\,\mathbf{r})+\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}$

$$=3r^2\frac{dr}{dt}\,\mathbf{r}+r^3\,\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}$$

$$=3r^2\frac{dr}{dt}\,\mathbf{r}+r^3\,\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}\qquad\left[\because\quad\frac{d\mathbf{a}}{dt}=0\right]$$

(ii) Let $\mathbf{R}=r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{d}{dt}(r^2\,\mathbf{r})+\left\{\dfrac{d}{dt}(\mathbf{a}\cdot\mathbf{r})\right\}\mathbf{b}+(\mathbf{a}\cdot\mathbf{r})\dfrac{d\mathbf{b}}{dt}$

$$=2r\,\frac{dr}{dt}\,\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{r}+\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}\qquad\left[\because\quad\frac{d\mathbf{b}}{dt}=0\right]$$

$$=2r\,\frac{dr}{dt}\,\mathbf{r}+r^2\,\frac{d\mathbf{r}}{dt}+\left(\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}\qquad\left[\because\quad\frac{d\mathbf{a}}{dt}=0\right]$$

(iii) Let $\mathbf{R}=r^n\,\mathbf{r}$.

Then $\frac{d\mathbf{R}}{dt}=\left(\frac{d}{dt}r^n\right)\mathbf{r}+r^n\frac{d\mathbf{r}}{dt}=\left(nr^{n-1}\frac{dr}{dt}\right)\mathbf{r}+r^n\frac{d\mathbf{r}}{dt}.$

(iv) Let $R=(a\mathbf{r}+r\mathbf{b})^2$. Then

$$\frac{dR}{dt}=2\,(a\mathbf{r}+r\mathbf{b})\cdot\frac{d}{dt}(a\mathbf{r}+r\mathbf{b}) \qquad \left[\text{Note } \frac{d}{dt}\mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2\,(a\mathbf{r}+r\mathbf{b})\cdot\left(\frac{da}{dt}\mathbf{r}+\mathbf{a}\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}+r\frac{d\mathbf{b}}{dt}\right)$$

$$=2\,(a\mathbf{r}+r\mathbf{b})\cdot\left(a\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}\right) \qquad \left[\because \frac{da}{dt}=0,\ \frac{d\mathbf{b}}{dt}=0\right]$$

Ex. 25. *Find*

(*i*) $\frac{d}{dt}\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$; (*ii*) $\frac{d^2}{dt^2}\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$;

(*iii*) $\frac{d}{dt}\left[\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right].$

Sol. **(i)** Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$. Then R is the scalar triple product of three vectors $\mathbf{r}$, $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$. Therefore using the rule for finding the derivative of a scalar triple product, we have

$$\frac{dR}{dt}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$$

$=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$, since scalar triple products having two equal vectors vanish.

(ii) Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$. Then as in part **(i)**

$$\frac{dR}{dt}=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right].$$

Differentiating again, we get

$$\frac{d^2R}{dt^2}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d^3\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]$$

$$=\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right].$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)$. Then $\mathbf{R}$ is the vector triple product of three vectors. Therefore using the rule for finding the derivative of a vector triple product, we have

$$\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$$

$$=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right),$$

since $\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{0}$, being vector product of two equal vectors.

Ex. 26. *If* $\mathbf{a}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}+\theta\mathbf{k}$, $\mathbf{b}=\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j}-3\mathbf{k}$, *and* $\mathbf{c}=2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, *find* $\frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$ *at* $\theta=\frac{\pi}{2}$.

[Kanpur 1987; Rohilkhand 79]

Sol. We have

$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & -\sin\theta & -3\\ 2 & 3 & -3\end{vmatrix}=(3\sin\theta+9)\,\mathbf{i}+(3\cos\theta-6)\mathbf{j}+(3\cos\theta+2\sin\theta)\,\mathbf{k}.$$

$$\therefore\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin\theta & \cos\theta & \theta\\ 3\sin\theta+9 & 3\cos\theta-6 & 3\cos\theta+2\sin\theta\end{vmatrix}$$

$$=(3\cos^2\theta+2\sin\theta\cos\theta-3\theta\cos\theta+6\theta)\,\mathbf{i}+(3\theta\sin\theta+9\theta-3\sin\theta\cos\theta-2\sin^2\theta)\,\mathbf{j}+(-6\sin\theta-9\cos\theta)\,\mathbf{k}.$$

$$\therefore\ \frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$$

$$=(-6\cos\theta\sin\theta+2\cos^2\theta-2\sin^2\theta-3\cos\theta+3\theta\sin\theta+6)\,\mathbf{i}+(3\sin\theta+3\theta\cos\theta+9-3\cos^2\theta+3\sin^2\theta-4\sin\theta\cos\theta)\,\mathbf{j}+(-6\cos\theta+9\sin\theta)\,\mathbf{k}.$$

Putting $\theta=\pi/2$, we get the required derivative

$$=(4+\tfrac{3}{2}\pi)\,\mathbf{i}+15\mathbf{j}+9\mathbf{k}.$$

Ex. 27. *Show that if* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are constant vectors, then* $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ *is the path of a particle moving with constant acceleration.*

Sol. The velocity of the particle $=\frac{d\mathbf{r}}{dt}=2t\mathbf{a}+\mathbf{b}$.

The acceleration of the particle $=\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{a}$.

Thus the point whose path is $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ is moving with constant acceleration.

Ex. 28. *A particle moves along the curve* $x=4\cos t$, $y=4\sin t$, $z=6t$. *Find the velocity and acceleration at time* $t=0$ *and* $t=\frac{1}{2}\pi$. *Find also the magnitudes of the velocity and acceleration at any time* t.

[Kanpur 1980; Agra 81]

Sol. Let **r** be the position vector of the particle at time t.

Then $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=4\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}+6t\,\mathbf{k}$. If **v** is the velocity of the particle at time t and **a** its acceleration at that time then $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=-4\sin t\,\mathbf{i}+4\cos t\,\mathbf{j}+6\mathbf{k}$,

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=-4\cos t\,\mathbf{i}-4\sin t\,\mathbf{j}.$$

Magnitude of the velocity at time $t=|\mathbf{v}|$

$$=\sqrt{(16\sin^2 t+16\cos^2 t+36)}=\sqrt{(52)}=2\sqrt{(13)}.$$

Magnitude of the acceleration

$$=|\mathbf{a}|=\sqrt{(16\cos^2 t+16\sin^2 t)}=4.$$

At $t=0$, $\mathbf{v}=4\mathbf{j}+6\mathbf{k}$, $\mathbf{a}=-4\mathbf{i}$

At $t=\frac{1}{2}\pi$, $\mathbf{v}=-4\,\mathbf{i}+6\mathbf{k}$, $\mathbf{a}=-4\,\mathbf{j}$.

Ex. 29. *A particle moves along the curve $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at $t=0$.* **[Gorakhpur 1985]**

Sol. Let **r** be the position vector of the particle at time t.

Then $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=e^{-t}\,\mathbf{i}+2\cos 3t\,\mathbf{j}+2\sin 3t\,\mathbf{k}$.

If **v** is the velocity of the particle at time t and **a** its acceleration at that ime, then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-e^{-t}\,\mathbf{i}-6\sin 3t\,\mathbf{j}+6\cos 3t\,\mathbf{k},$$

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=e^{-t}\,\mathbf{i}-18\cos 3t\,\mathbf{j}-18\sin 3t\,\mathbf{k}.$$

Putting $t=0$ in the above relations, the velocity at $t=0$ is $-\mathbf{i}+6\mathbf{k}$,

and the acceleration at $t=0$ is $\mathbf{i}-18\mathbf{j}$.

Hence at $t=0$,

the magnitude of velocity $=|-\mathbf{i}+6\mathbf{k}|=\sqrt{[(-1)^2+6^2]}=\sqrt{37}$,

and the magnitude of acceleration

$$=|\mathbf{i}-18\mathbf{j}|=\sqrt{[1^2+(-18)^2]}=\sqrt{(325)}.$$

Ex. 30. *A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$, where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$.*

[Agra 1979; Rohilkhand 81]

Sol. If **r** is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(t^3+1)\,\mathbf{i}+t^2\,\mathbf{j}+(2t+5)\mathbf{k}.$$

Velocity $=\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=3t^2\,\mathbf{i}+2t\mathbf{j}+2\mathbf{k}=3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ at $t=1$.

Acceleration $=\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=6t\mathbf{i}+2\mathbf{j}=6\mathbf{i}+2\mathbf{j}$ at $t=1$.

Now the unit vector in the given direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$

$$=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+\mathbf{j}+3\mathbf{k}|}=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}}=\mathbf{b}, \text{ say.}$$

$\therefore$ the component of velocity in the given direction

$$=\mathbf{v}\cdot\mathbf{b}=\frac{(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{11}{\sqrt{(11)}}=\sqrt{(11)};$$

and the component of acceleration in the given direction

$$=\mathbf{a}\cdot\mathbf{b}=\frac{(6\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{8}{\sqrt{(11)}}.$$

Ex. 31. *A particle moves so that its position vector is given by $\mathbf{r}=\cos\omega t\ \mathbf{i}+\sin\omega t\ \mathbf{j}$ where ω is a constant; show that (i) the velocity of the particle is perpendicular to $\mathbf{r}$, (ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin, (iii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$ is a constant vector.*

Sol. (i) Velocity $\mathbf{v}=\frac{d\mathbf{r}}{dt}=-\omega\sin\omega t\ \mathbf{i}+\omega\cos\omega t\ \mathbf{j}$.

We have $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=(\cos\omega t\ \mathbf{i}+\sin\omega t\ \mathbf{j})\cdot(-\omega\sin\omega t\ \mathbf{i}+\omega\cos\omega t\ \mathbf{j})$

$$=-\omega\cos\omega t\sin\omega t+\omega\sin\omega t\cos\omega t=0.$$

Therefore the velocity is perpendicular to $\mathbf{r}$.

(ii) Acceleration of the particle

$$=\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=-\omega^2\cos\omega t\ \mathbf{i}-\omega^2\sin\omega t\ \mathbf{j}$$

$$=-\omega^2(\cos\omega t\ \mathbf{i}+\sin\omega t\ \mathbf{j})=-\omega^2\mathbf{r}.$$

$\therefore$ Acceleration is a vector opposite to the direction of $\mathbf{r}$ *i.e.* acceleration is directed towards the origin. Also magnitude of acceleration $=|\mathbf{a}|=|-\omega^2\mathbf{r}|=\omega^2 r$ which is proportional to r *i.e.*, the distance of the particle from the origin.

(iii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos\omega t\ \mathbf{i}+\sin\omega t\ \mathbf{j})\times(-\omega\sin\omega t\ \mathbf{i}+\omega\cos\omega t\ \mathbf{j})$

$$=\omega\cos^2\omega t\ \mathbf{i}\times\mathbf{j}-\omega\sin^2\omega t\ \mathbf{j}\times\mathbf{i}\quad[\because \mathbf{i}\times\mathbf{i}=\mathbf{0},\ \mathbf{j}\times\mathbf{j}=\mathbf{0}]$$

$$=\omega\cos^2\omega t\ \mathbf{k}+\omega\sin^2\omega t\ \mathbf{k}\quad[\because \mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}]$$

$$=\omega(\cos^2\omega t+\sin^2\omega t)\ \mathbf{k}=\omega\mathbf{k}, \text{ a constant vector.}$$

Ex. 32. *Find the unit tangent vector to any point on the curve $x=a\cos t,\ y=a\sin t,\ z=bt$.*

Sol. If **r** is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=a\cos t\,\mathbf{i}+a\sin t\,\mathbf{j}+bt\,\mathbf{k}.$$

The vector $\frac{d\mathbf{r}}{dt}$ is also the tangent at the point (x, y, z) to the given curve.

We have $\frac{d\mathbf{r}}{dt}=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\,\mathbf{k}.$

$\therefore \quad \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{(a^2\sin^2 t+a^2\cos^2 t+b^2)}=\sqrt{(a^2+b^2)}.$

Hence the unit tangent vector **t**

$$=\frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}=\frac{-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\mathbf{k}}{\sqrt{(a^2+b^2)}}$$

$$=\frac{1}{\sqrt{(a^2+b^2)}}(-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\mathbf{k}).$$

§ 12. Integration of Vector Functions.

We shall define *integration as the reverse process of differentiation*. Let $\mathbf{f}(t)$ and $\mathbf{F}(t)$ be two vector functions of the scalar t such that $\frac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$.

Then $\mathbf{F}(t)$ is called the *indefinite integral* of $\mathbf{f}(t)$ with respect to t and symbolically we write $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(1)

The function $\mathbf{f}(t)$ to be integrated is called the *integrand*.

If **c** is any *arbitrary constant vector* independent of t, then

$$\frac{d}{dt}\left\{\mathbf{F}(t)+\mathbf{c}\right\}=\mathbf{f}(t).$$

This is equivalent to $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)+\mathbf{c}$. ...(2)

From (2) it is obvious that the integral $\mathbf{F}(t)$ of $\mathbf{f}(t)$ is indefinite to the extent of an additive arbitrary constant **c**. Therefore $\mathbf{F}(t)$ is called the indefinite integral of $\mathbf{f}(t)$. The constant vector **c** is called the *constant of integration*. It can be determined if we are given some initial conditions.

If $\frac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$ for all t in the interval $[a, b]$, then the *definite integral* between the limits $t=a$ and $t=b$ can in such case be written

$$\int_a^b \mathbf{f}(t)\,dt=\int_a^b\left\{\frac{d}{dt}\mathbf{F}(t)\right\}dt$$

$$=\Big[\mathbf{F}(t)+\mathbf{c}\Big]_a^b=\mathbf{F}(b)-\mathbf{F}(a).$$

Theorem. *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then*
$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Proof. Let $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$. ...(1)

Then $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(2)

Let $\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$.

Then from (1), we have
$$\frac{d}{dt}\{F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}\}=\mathbf{f}(t)$$
or
$$\left\{\frac{d}{dt}F_1(t)\right\}\mathbf{i}+\left\{\frac{d}{dt}F_2(t)\right\}\mathbf{j}+\left\{\frac{d}{dt}F_3(t)\right\}\mathbf{k}=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}.$$

Equating the coefficients of **i**, **j**, **k**, we get
$$\frac{d}{dt}F_1(t)=f_1(t),\ \frac{d}{dt}F_2(t)=f_2(t),\ \frac{d}{dt}F_3(t)=f_3(t).$$
$$\therefore\quad F_1(t)=\int f_1(t)\,dt,\ F_2(t)=\int f_2(t)\,dt,\ F_3(t)=\int f_3(t)\,dt.$$
$$\therefore\quad \mathbf{F}(t)=\left\{\int f_1(t)\,dt\right\}\mathbf{i}+\left\{\int f_2(t)\,dt\right\}\mathbf{j}+\left\{\int f_3(t)\,dt\right\}\mathbf{k}.$$

So from (2), we get
$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}.\int f_3(t)\,dt.$$

Note. From this theorem we conclude that the definition of the integral of a vector function implies the definition of integrals of three scalar functions which are the components of that vector function. **Thus in order to integrate a vector function we should integrate its components.**

§ 13. Some Standard Results.

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

1. We have $\dfrac{d}{dt}(\mathbf{r}\cdot\mathbf{s})=\dfrac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\dfrac{d\mathbf{s}}{dt}$

Therefore $\displaystyle\int\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\right)dt=\mathbf{r}\cdot\mathbf{s}+c,$

where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar,

2. We have $\dfrac{d}{dt}(\mathbf{r}^2)=2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$.

Therefore $\displaystyle\int\left(2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}^2+c.$

Here the constant of integration c is a scalar quantity.

3. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2=2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}.$

Therefore we have

$$\int\left(2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\right)\,dt=\left(\frac{d\mathbf{r}}{dt}\right)^2+c.$$

Here the constant of integration c is a scalar quantity.

Also $\qquad \left(\frac{d\mathbf{r}}{dt}\right)^2=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}.$

4. We have $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}.$

$$\therefore\quad \int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)\,dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}.$$

Here the constant of integration $\mathbf{c}$ is a vector quantity since the integrand $\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$ is also a vector quantity.

5. If $\mathbf{a}$ is a constant vector, we have

$$\frac{d}{dt}\,(\mathbf{a}\times\mathbf{r})=\frac{d\mathbf{a}}{dt}\times\mathbf{r}+\mathbf{a}\times\frac{d\mathbf{r}}{dt}=\mathbf{a}\times\frac{d\mathbf{r}}{dt}.$$

Therefore $\int\left(\mathbf{a}\times\frac{d\mathbf{r}}{dt}\right)\,dt=\mathbf{a}\times\mathbf{r}+\mathbf{c}.$

Here the constant of integration $\mathbf{c}$ is a vector quantity.

6. If $r=|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$ then

$$\frac{d}{dt}\,(\hat{\mathbf{r}})=\frac{d}{dt}\left(\frac{1}{r}\,\mathbf{r}\right)=\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\,\frac{dr}{dt}\,\mathbf{r}.$$

Therefore $\int\left(\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\,\frac{dr}{dt}\,\mathbf{r}\right)\,dt=\hat{r}+\mathbf{c}.$

7. If c is a constant scalar and $\mathbf{r}$ a vector function of a scalar t, then obviously $\int c\mathbf{r}\,dt=c\int\mathbf{r}\,dt.$

8. If $\mathbf{r}$ and $\mathbf{s}$ are two vector functions of the scalar t, then obviously $\int(\mathbf{r}+\mathbf{s})\,dt=\int\mathbf{r}\,dt+\int\mathbf{s}\,dt.$

Solved Examples

Ex. 1. *If* $\mathbf{f}\,(t)=(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}$, *find*

(*i*) $\int\mathbf{f}\,(t)\,dt$ *and* (*ii*) $\int_1^2\mathbf{f}\,(t)\,dt.$

Sol. (i) $\int\mathbf{f}\,(t)\,dt=\int\{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$

$$=\mathbf{i}\int (t-t^2)\,dt+\mathbf{j}\int 2t^3\,dt+\mathbf{k}\int -3\,dt$$

$$=\mathbf{i}\left(\frac{t^2}{2}-\frac{t^3}{3}\right)+\mathbf{j}\left(2\,\frac{t^4}{4}\right)+\mathbf{k}\,(-3t)+\mathbf{c},$$

where **c** is an arbitrary constant vector

$$=\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\,\mathbf{j}-3t\mathbf{k}+\mathbf{c}.$$

(ii) $$\int_1^2 \mathbf{f}(t)\,dt=\int_1^2 \{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$$

$$=\mathbf{i}\int_1^2 (t-t^2)\,dt+\mathbf{j}\int_1^2 2t^3\,dt-\mathbf{k}\int_1^2 3dt$$

$$=\mathbf{j}\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_1^2+\mathbf{j}\left[2.\,\frac{t^4}{4}\right]_1^2-3\mathbf{k}\left[\;t\;\right]_1^2=-\frac{5}{6}\,\mathbf{i}+\frac{15}{2}\,\mathbf{j}-3\mathbf{k}.$$

Ex. 2. *Evaluate* $\int_0^1 (e^t\,\mathbf{i}+e^{-2t}\,\mathbf{j}+t\mathbf{k})\,dt.$

Sol. $$\int_0^1 (e^t\,\mathbf{i}+e^{-2t}\,\mathbf{j}+t\mathbf{k})\,dt$$

$$=\mathbf{i}\int_0^1 e^t\,dt+\mathbf{j}\int_0^1 e^{-2t}\,dt+\mathbf{k}\int_0^1 t\,dt$$

$$=\mathbf{i}\left[e^t\right]_0^1+\mathbf{j}\left[-\tfrac{1}{2}.e^{-2t}\right]_0^1+\mathbf{k}\left[\tfrac{1}{2}t^2\right]_0^1$$

$$=(e-1)\,\mathbf{i}-\tfrac{1}{2}\,(e^{-2}-1)\,\mathbf{j}+\tfrac{1}{2}\mathbf{k}.$$

Ex. 3. *If* $\mathbf{f}(t)=t\,\mathbf{i}+(t^2-2t)\,\mathbf{j}+(3t^2+3t^3)\,\mathbf{k}$, *find*

$$\int_0^1 \mathbf{f}(t)\,dt.$$

[Agra 1977]

Sol. $$\int_0^1 \mathbf{f}(t)\,dt=\int_0^1 [t\,\mathbf{i}+(t^2-2t)\,\mathbf{j}+(3t^2+3t^3)\,\mathbf{k}]\,dt$$

$$=\mathbf{i}\int_0^1 t\,dt+\mathbf{j}\int_0^1 (t^2-2t)\,dt+\mathbf{k}\int_0^1 (3t^2+3t^3)\,dt$$

$$=\mathbf{i}\left[\tfrac{1}{2}t^2\right]_0^1+\mathbf{j}\left[\frac{t^3}{3}-t^2\right]_0^1+\mathbf{k}\left[t^3+\frac{3t^4}{4}\right]_0^1$$

$$=\tfrac{1}{2}\,\mathbf{i}+(\tfrac{1}{3}-1)\,\mathbf{j}+(1+\tfrac{3}{4})\,\mathbf{k}=\tfrac{1}{2}\,\mathbf{i}-\tfrac{2}{3}\,\mathbf{j}+\tfrac{7}{4}\mathbf{k}.$$

Ex. 4. *If* $\mathbf{r}=t\,\mathbf{i}-t^2\,\mathbf{j}+(t-1)\,\mathbf{k}$ *and* $\mathbf{s}=2t^2\,\mathbf{i}+6t\,\mathbf{k}$, *evaluate*

(*i*) $\int_0^2 \mathbf{r}\cdot\mathbf{s}\,dt,$ (*ii*) $\int_0^2 \mathbf{r}\times\mathbf{s}\,dt$

[Meerut 1992]

Sol. (i) We have $\mathbf{r}\cdot\mathbf{s}=[t\,\mathbf{i}-t^2\,\mathbf{j}+(t-1)\,\mathbf{k}]\cdot(2t^2\,\mathbf{i}+6t\,\mathbf{k})$

$$=2t^3+6t\,(t-1)=2t^3+6t^2-6t.$$

$$\therefore \quad \int_0^2 \mathbf{r} \cdot \mathbf{s}\, dt = \int_0^2 (2t^3 + 6t^2 - 6t)\, dt$$

$$= \left[\frac{t^4}{2} + 2t^3 - 3t^2\right]_0^2 = 8 + 16 - 12 = 12.$$

(ii) We have $\mathbf{r} \times \mathbf{s} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$

$$= -6t^3\, \mathbf{i} - [6t^2 - 2t^2(t-1)]\, \mathbf{j} + 2t^4\, \mathbf{k}$$

$$= -6t^3\, \mathbf{i} - (8t^2 - 2t^3)\, \mathbf{j} + 2t^4\, \mathbf{k}.$$

$$\therefore \quad \int_0^2 \mathbf{r} \times \mathbf{s}\, dt = \int_0^2 [-6t^3\, \mathbf{i} - (8t^2 - 2t^3)\, \mathbf{j} + 2t^4\, \mathbf{k}]\, dt$$

$$= \mathbf{i} \int_0^2 -6t^3\, dt - \mathbf{j} \int_0^2 (8t^2 - 2t^3)\, dt + \mathbf{k} \int_0^2 2t^4\, dt$$

$$= \mathbf{i} \left[-\frac{3}{2} t^4\right]_0^2 - \mathbf{j} \left[\frac{8t^3}{3} - \frac{t^4}{2}\right]_0^2 + \mathbf{k} \left[\frac{2t^5}{5}\right]_0^2$$

$$= -24\, \mathbf{i} - \left(\frac{64}{3} - 8\right) \mathbf{j} + \frac{64}{5}\, \mathbf{k}$$

$$= -24\, \mathbf{i} - \frac{40}{3}\, \mathbf{j} + \frac{64}{5}\, \mathbf{k}.$$

Ex. 5. *Evaluate* $\int_1^2 (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\, dt$, *where*

$\mathbf{a} = t\, \mathbf{i} - 3\mathbf{j} + 2t\, \mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\, \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\, \mathbf{i} + t\, \mathbf{j} - \mathbf{k}$.

[Garhwal 1977]

Sol. We have $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$= \begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix}$$

$$= \begin{vmatrix} t & -3+2t & 0 \\ 1 & 0 & 0 \\ 3 & t+6 & -7 \end{vmatrix}, \text{ by } C_2 + 2C_1 \text{ and } C_3 - 2C_1$$

$$= -1 . \{-7(2t-3) - 0\},$$

expanding the determinant along R_2

$$= 7(2t-3).$$

$$\therefore \quad \int_1^2 (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\,dt=\int_1^2 7(2t-3)\,dt=7\int_1^2 (2t-3)\,dt$$

$$=7\Big[t^2-3t\Big]_1^2=7\,[(4-6)-(1-3)]=7.(-2+2)=0.$$

Ex. 6. *Evaluate* $\int_1^2 \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\,dt$, *where* $\mathbf{r}=2t^2\,\mathbf{i}+t\,\mathbf{j}-3t^3\,\mathbf{k}.$

[Kanpur 1975]

Sol. Given $\mathbf{r}=2t^2\,\mathbf{i}+t\,\mathbf{j}-3t^3\,\mathbf{k}.$

$\therefore \quad \frac{d\mathbf{r}}{dt}=4t\,\mathbf{i}+\mathbf{j}-9t^2\,\mathbf{k}$ and $\frac{d^2\mathbf{r}}{dt^2}=4\,\mathbf{i}+0\,\mathbf{j}-18\,t\,\mathbf{k}.$

$$\therefore \quad \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=(2t^2\,\mathbf{i}+t\,\mathbf{j}-3t^3\,\mathbf{k})\times(4\,\mathbf{i}+0\,\mathbf{j}-18\,t\mathbf{k})$$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^2 & t & -3t^3 \\ 4 & 0 & -18t \end{vmatrix}$$

$$=-18t^2\,\mathbf{i}-(-36\,t^3+12\,t^3)\,\mathbf{j}-4t\,\mathbf{k}$$

$$=-18t^2\,\mathbf{i}+24\,t^3\,\mathbf{j}-4t\,\mathbf{k}.$$

$$\therefore \quad \int_1^2 \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\,dt=\int_1^2(-18\,t^2\,\mathbf{i}+24\,t^3\,\mathbf{j}-4t\,\mathbf{k})\,dt$$

$$=-18\,\mathbf{i}\int_1^2 t^2\,dt+24\,\mathbf{j}\int_1^2 t^3\,dt-4\,\mathbf{k}\int_1^2 t\,dt$$

$$=-18\,\mathbf{i}\left[\frac{t^3}{3}\right]_1^2+24\,\mathbf{j}\left[\frac{t^4}{4}\right]_1^2-4\,\mathbf{k}\left[\frac{t^2}{2}\right]_1^2$$

$$=-6\,(8-1)\,\mathbf{i}+6(16-1)\,\mathbf{j}-2\,(4-1)\,\mathbf{k}$$

$$=-42\,\mathbf{i}+90\,\mathbf{j}-6\,\mathbf{k}.$$

Ex. 7. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector. Also it is given that when* $t=0$, $\mathbf{r}=\mathbf{0}$ *and* $\frac{d\mathbf{r}}{dt}=\mathbf{u}$.

[Agra 1981]

Sol. Integrating the equation $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$, we get

$\frac{d\mathbf{r}}{dt}=t\mathbf{a}+\mathbf{b}$, where $\mathbf{b}$ is an arbitrary constant vector.

But it is given that when $t=0$, $\frac{d\mathbf{r}}{dt}=\mathbf{u}$.

$$\therefore \quad \mathbf{u} = 0\mathbf{a} + \mathbf{b} \quad \text{or} \quad \mathbf{b} = \mathbf{u}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = t\mathbf{a} + \mathbf{u}.$$

Integrating again with respect to t, we get

$\mathbf{r} = \frac{1}{2}t^2\,\mathbf{a} + t\mathbf{u} + \mathbf{c}$, where $\mathbf{c}$ is constant.

But when $t = 0$, $\mathbf{r} = \mathbf{0}$.

$$\therefore \quad \mathbf{0} = \mathbf{0} + \mathbf{0} + \mathbf{c} \quad \text{or} \quad \mathbf{c} = \mathbf{0}.$$

$$\therefore \quad \mathbf{r} = \tfrac{1}{2}t^2\,\mathbf{a} + t\mathbf{u}.$$

Ex. 8. *Solve the equation* $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$ *where* $\mathbf{a}$ *is a constant vector; given that* $\mathbf{r} = \mathbf{0}$ *and* $\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$ *when* $t = 0$.

Sol. Proceed as in Ex. 7. Ans. $\mathbf{r} = \frac{1}{2}t^2\mathbf{a}$.

Ex. 9. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\dfrac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.* [Agra 1979; Rohilkhand 83]

Sol. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, we get

$\dfrac{d\mathbf{r}}{dt} = \frac{1}{2}t^2\,\mathbf{a} + t\mathbf{b} + \mathbf{c}$, where $\mathbf{c}$ is constant.

Again integrating, we get

$\mathbf{r} = \frac{1}{6}t^3\,\mathbf{a} + \frac{1}{2}t^2\,\mathbf{b} + t\,\mathbf{c} + \mathbf{d}$, where $\mathbf{d}$ is constant.

Ex. 10. *Integrate* $\dfrac{d^2\mathbf{r}}{dt^2} = -n^2\mathbf{r}$.

Sol. We have $\dfrac{d^2\mathbf{r}}{dt^2} = -n^2\mathbf{r}$. ...(1)

Forming the scalar product of each side of (1) with the vector $2\dfrac{d\mathbf{r}}{dt}$, we get $2\,\dfrac{d\mathbf{r}}{dt}\cdot\dfrac{d^2\mathbf{r}}{dt^2} = -2n^2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$.

Now integrating we get

$$\left(\frac{d\mathbf{r}}{dt}\right)^2 = -n^2\mathbf{r}^2 + c, \text{ where } c \text{ is constant.}$$

Ex. 11. *If* $\mathbf{r}\cdot d\mathbf{r} = 0$, *show that* $|\,\mathbf{r}\,| =$ *constant.*

[Agra 1975; Rohilkhand 79]

Sol. We have $\mathbf{r}\cdot d\mathbf{r} = 0 \Rightarrow 2\mathbf{r}\cdot d\mathbf{r} = 0 \Rightarrow d(\mathbf{r}\cdot\mathbf{r}) = 0$

$\Rightarrow d(\mathbf{r}^2)=0 \Rightarrow \mathbf{r}^2=\text{constant} \Rightarrow |\mathbf{r}|^2=\text{constant}$

$\Rightarrow |\mathbf{r}|=\text{constant.}$

Ex. 12. *Find the value of* $\mathbf{r}$ *satisfying the equation*

$$\frac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4\sin t\mathbf{k},$$

given that $\mathbf{r}=2\mathbf{i}+\mathbf{j}$ *and* $d\mathbf{r}/dt=-\mathbf{i}-3\mathbf{k}$ *at* $t=0$.

Sol. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4\sin t\mathbf{k}$, we get

$\dfrac{d\mathbf{r}}{dt}=3t^2\mathbf{i}-8t^3\mathbf{j}-4\cos t\mathbf{k}+\mathbf{b}$, where $\mathbf{b}$ is an arbitrary constant vector.

But it is given that when $t=0$, $d\mathbf{r}/dt=-\mathbf{i}-3\mathbf{k}$.

$\therefore \quad -\mathbf{i}-3\mathbf{k}=-4\mathbf{k}+\mathbf{b}$ or $\mathbf{b}=-\mathbf{i}+\mathbf{k}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=3t^2\mathbf{i}-8t^3\mathbf{j}-4\cos t\mathbf{k}-\mathbf{i}+\mathbf{k}$$

$$=(3t^2-1)\mathbf{i}-8t^3\mathbf{j}+(1-4\cos t)\mathbf{k}.$$

Integrating again w.r.t. t, we get

$$\mathbf{r}=(t^3-t)\mathbf{i}-2t^4\mathbf{j}+(t-4\sin t)\mathbf{k}+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

But is is given that when $t=0$, $\mathbf{r}=2\mathbf{i}+\mathbf{j}$.

$\therefore \quad 2\mathbf{i}+\mathbf{j}=0+\mathbf{c}=\mathbf{c}$.

$\therefore \quad \mathbf{r}=(t^3-t)\mathbf{i}-2t^4\mathbf{j}+(t-4\sin t)\mathbf{k}+2\mathbf{i}+\mathbf{j}$

or $\quad \mathbf{r}=(t^3-t+2)\mathbf{i}+(1-2t^4)\mathbf{j}+(t-4\sin t)\mathbf{k}$

is the required solution of the given differential equation.

Ex. 13. *Show that* $\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$, *where* $\mathbf{c}$ *is an arbitrary constant vector.*

Sol. We have $\dfrac{d}{dt}\left(\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\dfrac{d^2\mathbf{r}}{dt^2}+\dfrac{d\mathbf{r}}{dt}\times\dfrac{d\mathbf{r}}{dt}$

$$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}, \text{ since } \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}.$$

Integrating both sides with respect to t, we get

$\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)+\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.

Ex. 14. *Integrate* $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.*

Sol. We have $\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}=\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}$.

Therefore integrating $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, we get

$$\mathbf{a}\times\frac{d\mathbf{r}}{dt}=t\mathbf{b}+\mathbf{c}, \text{ where } \mathbf{c} \text{ is constant.}$$

Again integrating, we get

$$\mathbf{a}\times\mathbf{r}=\tfrac{1}{2}t^2\mathbf{b}+t\mathbf{c}+\mathbf{d}, \text{ where } \mathbf{d} \text{ is constant.}$$

Ex. 15. *If* $\mathbf{r}(t)=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$, *prove that*

$$\int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

[Meerut 1991, Kanpur 87; Agra 82, 86; Rohilkhand 85]

Sol. We have $\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$.

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right]_1^2.$$

Let us now find $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$. We have $\dfrac{d\mathbf{r}}{dt}=10t\mathbf{i}+\mathbf{j}-3t^2\mathbf{k}$.

$$\therefore\quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\times(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ 10t & 1 & -3t^2\end{vmatrix}=-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}.$$

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}\right]_1^2$$

$$=\left[-2t^3\right]_1^2\mathbf{i}+\left[5t^4\right]_1^2\mathbf{j}-\left[5t^2\right]_1^2\mathbf{k}=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

Ex. 16. *Given that*

$$\mathbf{r}(t)=2\mathbf{i}-\mathbf{j}+2\mathbf{k}, \text{ when } t=2$$
$$=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}, \text{ when } t=3,$$

show that $\displaystyle\int_2^3\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=10.$

[Kanpur 1986; Rohilkhand 84; Agra 83, 87]

Sol. We have $\displaystyle\int\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\tfrac{1}{2}\mathbf{r}^2+c.$

$$\therefore \quad \int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \left[\tfrac{1}{2}\mathbf{r}^2 \right]_2^3.$$

When $t=3$, $\mathbf{r}=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

$\therefore$ when $t=3$, $\mathbf{r}^2=(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})=16+4+9=29$.

When $t=2$, $\mathbf{r}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

$\therefore$ When $t=2$, $\mathbf{r}^2=4+1+4=9$.

$$\therefore \quad \int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \tfrac{1}{2}[29-9]=10.$$

Ex. 17. *The acceleration of a particle at any time $t \geqslant 0$ is given by*

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=12 \cos 2t\, \mathbf{i}-8 \sin 2t\, \mathbf{j}+16t\, \mathbf{k}.$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time. **[Meerut 1991 P; Kerala 74]**

Sol. We have $\dfrac{d\mathbf{v}}{dt}=12 \cos 2t\, \mathbf{i}-8 \sin 2t\, \mathbf{j}+16t\, \mathbf{k}$.

Integrating, we get

$$\mathbf{v}=\mathbf{i}\int 12 \cos 2t\, dt+\mathbf{j}\int -8 \sin 2t\, dt+\mathbf{k}\int 16t\, dt$$

or $\mathbf{v}=6 \sin 2t\, \mathbf{i}+4 \cos 2t\, \mathbf{j}+8t^2\, \mathbf{k}+\mathbf{c}$.

When $t=0$, $\mathbf{v}=\mathbf{0}$.

$$\therefore \quad \mathbf{0}=0\mathbf{i}+4\mathbf{j}+0\mathbf{k}+\mathbf{c}$$

or $$\mathbf{c}=-4\mathbf{j}.$$

$$\therefore \quad \mathbf{v}=\frac{d\mathbf{r}}{dt}=6 \sin 2t\, \mathbf{i}+(4 \cos 2t-4)\, \mathbf{j}+8t^2\mathbf{k}.$$

Integrating, we get

$$\mathbf{r}=\mathbf{i}\int 6 \sin 2t\, dt+\mathbf{j}\int (4 \cos 2t-4)\, dt+\mathbf{k}\int 8t^2\, dt$$

$=-3 \cos 2t\, \mathbf{i}+(2 \sin 2t-4t)\, \mathbf{j}+\frac{8}{3}t^3\, \mathbf{k}+\mathbf{d}$, where $\mathbf{d}$ is constant.

When $t=0$, $\mathbf{r}=\mathbf{0}$.

$\therefore \quad \mathbf{0}=-3\mathbf{i}+0\mathbf{j}+0\mathbf{k}+\mathbf{d}. \qquad \therefore \quad \mathbf{d}=3\mathbf{i}.$

$$\therefore \quad \mathbf{r}=-3 \cos 2t\, \mathbf{i}+(2 \sin 2t-4t)\, \mathbf{j}+\tfrac{8}{3}t^3\, \mathbf{k}+3\mathbf{i}$$

$$=(3-3 \cos 2t)\, \mathbf{i}+(2 \sin 2t-4t)\, \mathbf{j}+\tfrac{8}{3}t^3\mathbf{k}.$$

Ex. 18. *The acceleration of a particle at any time t is $e^t\, \mathbf{i}+e^{2t}\, \mathbf{j}+\mathbf{k}$. Find $\mathbf{v}$, given that $\mathbf{v}=\mathbf{i}+\mathbf{j}$ at $t=0$.*

[Meerut 1991 S; Kanpur 88]

Sol. Given $\dfrac{d\mathbf{v}}{dt}=e^t\ \mathbf{i}+e^{2t}\ \mathbf{j}+\mathbf{k}$, where $\mathbf{v}$ is the velocity vector of the particle at any time t. Integrating with respect to t, we get

$$\mathbf{v}=e^t\ \mathbf{i}+\tfrac{1}{2}\, e^{2t}\ \mathbf{j}+t\mathbf{k}+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

But at $t=0$, it is given that $\mathbf{v}=\mathbf{i}+\mathbf{j}$.

$\therefore\quad \mathbf{i}+\mathbf{j}=\mathbf{i}+\tfrac{1}{2}\mathbf{j}+\mathbf{c}$ or $\mathbf{c}=\tfrac{1}{2}\mathbf{j}$

$\therefore\quad \mathbf{v}=e^t\ \mathbf{i}+\tfrac{1}{2}e^{2t}\ \mathbf{j}+t\mathbf{k}+\tfrac{1}{2}\mathbf{j}$

$\qquad =e^t\,\mathbf{i}+\tfrac{1}{2}\,(e^{2t}+1)\ \mathbf{j}+t\mathbf{k}.$

§ 5. **Level Surfaces.**

Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type

$$f(x, y, z)=c, \text{ (arbitrary constant)}$$

constitute a family of surfaces in three dimensional space. The surfaces of this family are called *level surfaces*. Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called *iso-f-surfaces*.

Theorem 1. *Let $f(x, y, z)$ be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof. Let (x_1, y_1, z_1) be any point of the region R. Then the level surface $f(x, y, z)=f(x_1, y_1, z_1)$ passes through this point.

Now suppose the level surfaces $f(x, y, z)=c_1$ and $f(x, y, z)=c_2$ pass through the point (x_1, y_1, z_1). Then

$$f(x_1, y_1, z_1)=c_1 \text{ and } f(x_1, y_1, z_1)=c_2.$$

Since $f(x, y, z)$ has a unique value at (x_1, y_1, z_1) therefore we have
$$c_1=c_2.$$

Hence only one level surface passes through the point

$$(x_1, y_1, z_1).$$

Theorem 2. *∇f is a vector normal to the surface $f(x, y, z)=c$ where c is a constant.* **[Agra 1968; Kerala 75]**

Proof. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the level surface $f(x, y, z)=c$. Let

$$Q(x+\delta x, y+\delta y, z+\delta z)$$

be a neighbouring point on this surface. Then the position vector of $Q=\mathbf{r}+\delta\mathbf{r}=(x+\delta x)\,\mathbf{i}+(y+\delta y)\,\mathbf{j}+(z+\delta z)\,\mathbf{k}$.

$$\therefore \quad \vec{PQ}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta\mathbf{r}=\delta x\,\mathbf{i}+\delta y\,\mathbf{j}+\delta z\,\mathbf{k}.$$

As $Q \to P$, the line PQ tends to tangent at P to the level surface. Therefore $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz$$

$$=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\nabla f\cdot d\mathbf{r}.$$

Since $f(x, y, z)=$ constant, therefore $df=0$.

$\therefore\quad \nabla f\cdot d\mathbf{r}=0$ so that ∇f is a vector perpendicular to $d\mathbf{r}$ and therefore to the tangent plane at P to the surface

$$f(x, y, z)=c.$$

Hence ∇f is a vector normal to the surface $f(x, y, z)=c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z)=c$ passing through that point.

§ 6. Directional Derivative of a scalar point function.

[Agra 1972; Kolhapur 73; Bombay 70]

Definition. *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\lim\limits_{Q\to P}\dfrac{f(Q)-f(P)}{PQ}$, *if it exists, is called the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of directional derivative. Let P be the point (x, y, z) and let Q be the point $(x+\delta x, y+\delta y, z+\delta z)$. Suppose $PQ=\delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If $\delta f=f(x+\delta x, y+\delta y, z+\delta z)-f(x, y, z)=f(Q)-f(P)$, then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\mathbf{a}$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is $\lim\limits_{Q\to P}\dfrac{f(Q)-f(P)}{PQ}=\lim\limits_{\delta s\to 0}\dfrac{\delta f}{\delta s}=\dfrac{df}{ds}$. It represents the rate of change of f with respect to distance at point P in the direction of unit vector $\hat{\mathbf{a}}$.

Theorem 1. *The directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by*

$$\frac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}.$$

[Allahabad 1982; Poona 70]

Proof. Let $f(x, y, z)$ define a scalar field in the region R. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ denote the position vector of any point $P(x, y, z)$ in this region. If s denotes the distance of P from some fixed point A in the direction of $\hat{\mathbf{a}}$, then δs denotes small element at P in the direction of $\hat{\mathbf{a}}$. Therefore $\frac{d\mathbf{r}}{ds}$ is a unit vector at P in this direction *i.e.* $\frac{d\mathbf{r}}{ds}=\hat{\mathbf{a}}$.

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore$ $\frac{d\mathbf{r}}{ds}=\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}=\hat{\mathbf{a}}$.

$$\text{Now } \nabla f\cdot\hat{\mathbf{a}}=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}\right)$$

$$=\frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds}+\frac{\partial f}{\partial z}\frac{dz}{ds}$$

$=\frac{df}{ds}=$ directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.

Alternative Proof. Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\hat{\mathbf{a}}$. If l, m, n are the direction cosines of the line PQ, then $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}=$ the unit vector in the direction of $PQ=\hat{\mathbf{a}}$. Further if $PQ=\delta s$, then the co-ordinates of Q are $(x+l\delta s, y+m\delta s, z+n\delta s)$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is

$$=\lim_{Q\to P}\frac{f(Q)-f(P)}{PQ}$$

$$=\lim_{\delta s\to 0}\frac{f(x+l\delta s, y+m\delta s, z+n\delta s)-f(x, y, z)}{\delta s}$$

$$=\lim_{\delta s\to 0}\frac{f(x, y, z)+\left(l\delta s\frac{\partial f}{\partial x}+m\delta s\frac{\partial f}{\partial y}+n\delta s\frac{\partial f}{\partial z}\right)+\ldots-f(x, y, z)}{\delta s},$$

on expanding by Taylor's theorem

$$=l\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z}$$

$$=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=\nabla f\cdot\hat{\mathbf{a}}.$$

Theorem 2. *If $\hat{\mathbf{n}}$ be a unit vector normal to the level surface*

$f(x, y, z)=c$ at a point $P(x, y, z)$ and n be the distance of P from some fixed point A in the direction of $\hat{\mathbf{n}}$ so that δn represents element of normal at P in the direction of $\hat{\mathbf{n}}$, then

$$\text{grad } f=\frac{df}{dn}\hat{\mathbf{n}}.$$

[Agra 1971]

Proof. We have grad $f=\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$.

Also grad f is a vector normal to the surface $f(x, y, z)=c$. Since $\hat{\mathbf{n}}$ is a unit vector normal to the surface $f(x, y, z)=c$, therefore let grad $f=A\hat{\mathbf{n}}$, where A is some scalar to be determined.

Now $\frac{df}{dn}=$ directional derivative of f in the direction of $\hat{\mathbf{n}}$

$$=\nabla f\cdot\hat{\mathbf{n}}$$

$$=A\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} \quad [\because \nabla f=\text{grad } f=A\hat{\mathbf{n}}]$$

$$=A.$$

$\therefore$ grad $f=\nabla f=\frac{df}{dn}\hat{\mathbf{n}}$.

Note. If the vector $\hat{\mathbf{n}}$ is in the direction of f increasing, then $\frac{df}{dn}$ is positive. Therefore ∇f is a vector normal to the surface $f(x, y, z)=c$ in the direction of f increasing.

Theorem 3. *Grad f is a vector in the direction of which the maximum value of the directional derivative of f i.e. $\frac{df}{ds}$ occurs.*

[Agra 1971]

Proof. The directional derivative of f in the direction of $\hat{\mathbf{a}}$ is given by $\frac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}$

$$=\left(\frac{df}{dn}\hat{\mathbf{n}}\right)\cdot\hat{\mathbf{a}} \qquad \left[\because \nabla f=\frac{df}{dn}\hat{\mathbf{n}}\right]$$

$$=\frac{df}{dn}(\hat{\mathbf{n}}\cdot\hat{\mathbf{a}})$$

$$=\frac{df}{dn}\cos\theta,$$ where θ is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$.

Now $\frac{df}{dn}$ is fixed. Therefore $\frac{df}{dn}\cos\theta$ is maximum when $\cos\theta$ is maximum *i.e.*, when $\cos\theta=1$. But $\cos\theta$ will be 1 when the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is 0 *i.e.* when $\hat{\mathbf{a}}$ is along the unit normal vector $\hat{\mathbf{n}}$.

Therefore the directional derivative is maximum along the normal to the surface. Its maximum value is

$$=\frac{df}{dn}=|\text{ grad } f\,|.$$

§ 7. Tangent plane and Normal to a level surface.

To find the equations of the tangent plane and normal to the surface $f(x, y, z)=c$.

Let $f(x, y, z)=c$ be the equation of a level surface. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on this surface.

Then $\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$ is a vector along the normal to the surface at P *i.e.* ∇f is perpendicular to the tangent plane at P.

Tangent plane at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q(X, Y, Z)$ on the tangent plane at P to the surface. The vector

$$\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$$

lies in the tangent plane at P. Therefore it is perpendicular to the vector ∇f.

$$\therefore\quad (\mathbf{R}-\mathbf{r})\cdot\nabla f=0$$

or $$[(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}]\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)=0$$

or $$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}+(Z-z)\frac{\partial f}{\partial z}=0, \qquad \ldots(1)$$

is the equation of the tangent plane at P.

Normal at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q(X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \quad (\mathbf{R}-\mathbf{r})\times\nabla f=\mathbf{0} \qquad \ldots(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form.: The vectors

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k} \text{ and } \nabla f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

will be parallel if

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}=p\left(\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of **i**, **j**, **k**, we get

$$X-x=p\,\frac{\partial f}{\partial x},\ Y-y=p\,\frac{\partial f}{\partial y},\ Z-z=p\,\frac{\partial f}{\partial z}$$

or

$$\frac{X-x}{\dfrac{\partial f}{\partial x}}=\frac{Y-y}{\dfrac{\partial f}{\partial y}}=\frac{Z-z}{\dfrac{\partial f}{\partial z}}$$

are the equations of the normal at P.

Solved Examples

Ex. 1. *Find a unit normal vector to the level surface $x^2y+2xz=4$ at the point* $(2, -2, 3)$.

Sol. The equation of the level surface is

$$f(x, y, z)\equiv x^2y+2xz=4.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\nabla(x^2y+2xz)=(2xy+2z)\,\mathbf{i}+x^2\,\mathbf{j}+2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

$\therefore$ $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{|-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}|}=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{\sqrt{(4+16+16)}}=-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}.$$

The vector $-(-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})$ *i.e.*, $\tfrac{1}{3}\mathbf{i}-\tfrac{2}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Ex. 2. *Find the unit normal to the surface $z=x^2+y^2$ at the point* $(-1, -2, 5)$. **[Kanpur 1986]**

Sol. The equation of the given surface is

$$f(x, y, z)\equiv x^2+y^2-z=0.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}=2x\,\mathbf{i}+2y\,\mathbf{j}-\mathbf{k}$.

$\therefore$ at the point $(-1, -2, 5)$, grad $f=-2\mathbf{i}-4\,\mathbf{j}-\mathbf{k}$.

$\therefore$ $-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})$ is a vector along the normal to the given surface at the point $(-1, -2, 5)$.

Hence the required unit normal vector to the surface at this point

$$=\frac{\text{grad } f}{|\text{grad } f|}=\frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{|-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})|}=\frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{\sqrt{(2^2+4^2+1^2)}}$$

$$=\frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{\sqrt{(21)}}.$$

The vector $\dfrac{2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}$ is also a unit normal vector to the given surface at the point $(-1, -2, 5)$.

Ex. 3. *Find the unit normal to the surface*

$$x^4-3xyz+z^2+1=0$$

at the point $(1, 1, 1)$. **[Gorakhpur 1988; Allahabad 79]**

Sol. The given surface is

$$f(x, y, z)\equiv x^4-3xyz+z^2+1=0. \quad \ldots(1)$$

We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$

$$=(4x^3-3yz)\,\mathbf{i}+(-3xz)\,\mathbf{j}+(-3xy+2z)\,\mathbf{k}.$$

Now a vector normal to the surface (1) at the point $(1, 1, 1)$

$=$ grad f at the point $(1, 1, 1)=(4-3)\,\mathbf{i}+(-3)\,\mathbf{j}+(-3+2)\,\mathbf{k}$

$$=\mathbf{i}-3\,\mathbf{j}-\mathbf{k}.$$

$\therefore$ the required unit normal vector $=\dfrac{\text{grad } f}{|\text{grad } f|}$

$$=\frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{|\mathbf{i}-3\mathbf{j}-\mathbf{k}|}=\frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{\sqrt{[1^2+(-3)^2+(-1)^2]}}=\frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{\sqrt{(11)}}.$$

Ex. 4. *Find the unit vector normal to the surface* $x^2-y^2+z=2$ *at the point* $(1, -1, 2)$. **[Ravi Shankar 1981]**

Sol. The given surface is

$$f(x, y, z)\equiv x^2-y^2+z-2=0. \quad \ldots(1)$$

We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$

$$=2x\,\mathbf{i}-2y\,\mathbf{j}+\mathbf{k}.$$

Now a vector normal to the surface (1) at the point $(1, -1, 2)$ $=\text{grad } f$ at the point $(1, -1, 2)=2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}$.

Hence the required unit vector normal to the surface (1) at the point $(1, -1, 2)$

$$=\frac{\text{grad } f}{|\text{grad } f|}=\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{|2\mathbf{i}+2\mathbf{j}+\mathbf{k}|}=\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{(4+4+1)}}=\tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k}).$$

Ex. 5. *Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point $(2, 0, 0)$.*

Sol. The given surface is

$$f(x, y, z)\equiv x^2+y-z-4=0. \qquad \ldots(1)$$

We have $\text{grad } f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}=2x\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

$\therefore$ at the point $(2, 0, 0)$, $\text{grad } f=4\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

Now a vector along the normal to the surface (1) at the point $(2, 0, 0)$

$=\text{grad } f$ at the point $(2, 0, 0)=4\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

Hence the required unit normal to the surface (1) at the point $(2, 0, 0)$

$$=\frac{4\mathbf{i}+\mathbf{j}-\mathbf{k}}{|4\mathbf{i}+\mathbf{j}-\mathbf{k}|}=\frac{4\,\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{(16+1+1)}}=\frac{1}{3\sqrt{2}}(4\,\mathbf{i}+\mathbf{j}-\mathbf{k}).$$

Ex. 6. *Find the directional derivatives of a scalar point function f in the direction of coordinate axes.*

Sol. The grad f at any point (x, y, z) is the vector

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

The directional derivative of f in the direction of $\mathbf{i}$

$$=\text{grad } f\cdot\mathbf{i}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f}{\partial x}.$$

Similarly the directional derivatives of f in the directions of $\mathbf{j}$ are $\mathbf{k}$ are $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Ex. 7. *Find the directional derivative of $f(x, y, z)=x^2yz+4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.*

[Allahabad 1978]

Sol. We have $f(x, y, z)=x^2yz+4xz^2$.

$$\therefore \quad \text{grad } f=(2xyz+4z^2)\,\mathbf{i}+x^2z\,\mathbf{j}+(x^2y+8xz)\,\mathbf{k}$$

$$=8\mathbf{i}-\mathbf{j}-10\mathbf{k} \text{ at the point } (1, -2, -1).$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$, then $\hat{\mathbf{a}}=\dfrac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{(4+1+4)}}=\dfrac{2}{3}\mathbf{i}-\dfrac{1}{3}\mathbf{j}-\dfrac{2}{3}\mathbf{k}$.

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\cdot\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k})=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Ex. 8. *Find the directional derivative of*

$$f(x, y, z)=x^2-2y^2+4z^2$$

at the point $(1, 1, -1)$ *in the direction of* $2\mathbf{i}+\mathbf{j}-\mathbf{k}$. **[Agra 1979]**

Ans. $8/\sqrt{6}$.

Ex. 9. *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* P $(1, 2, 3)$ *in the direction of the line* PQ *where* Q *is the point* $(5, 0, 4)$. **[Agra 1987]**

Sol. Here $\text{grad } f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$

$$=2x\,\mathbf{i}-2y\,\mathbf{j}+4z\,\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k} \text{ at the point } (1, 2, 3).$$

Also $\overrightarrow{PQ}=$ position vector of $Q-$ position vector of P

$$=(5\mathbf{i}+0\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}.$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$, then $\hat{\mathbf{a}}=\dfrac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\dfrac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}$.

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{\mathbf{a}}=(2\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot\left\{\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Ex. 10. *Find the directional derivatives of the function*

$$f=xy+yz+zx$$

in the direction of the vector $2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$ *at the point* $(3, 1, 2)$.

[Rohilkhand 1980, 81; Agra 75]

Sol. We have $f(x, y, z)=xy+yz+zx$.

$$\begin{aligned}\therefore\quad \text{grad } f&=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}\\&=(y+z)\,\mathbf{i}+(z+x)\,\mathbf{j}+(x+y)\,\mathbf{k}\\&=(1+2)\,\mathbf{i}+(2+3)\,\mathbf{j}+(3+1)\,\mathbf{k} \text{ at the point } (3, 1, 2)\\&=3\mathbf{i}+5\mathbf{j}+4\mathbf{k}.\end{aligned}$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}+3\mathbf{j}+6\mathbf{k}$, then

$$\hat{\mathbf{a}}=\frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{|2\mathbf{i}+3\mathbf{j}+6\mathbf{k}|}=\frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{(4+9+36)}}=\frac{1}{7}(2\mathbf{i}+3\mathbf{j}+6\mathbf{k}).$$

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{\mathbf{a}}=(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})\cdot\frac{1}{7}(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})$$

$$=\frac{1}{7}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})$$

$$=\frac{1}{7}(6+15+24)=\frac{45}{7}.$$

Ex. 11. *Find the directional derivative of* $\phi=xy+yz+zx$ *in the direction of the vector* $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ *at* $(1, 2, 0)$. **[Agra 1982]**

Sol. We have $\phi(x, y, z)=xy+yz+zx$.

$$\therefore \quad \text{grad } \phi=(\partial\phi/\partial x)\,\mathbf{i}+(\partial\phi/\partial y)\,\mathbf{j}+(\partial\phi/\partial z)\,\mathbf{k}$$
$$=(y+z)\,\mathbf{i}+(z+x)\,\mathbf{j}+(x+y)\,\mathbf{k}$$
$$=(2+0)\,\mathbf{i}+(0+1)\,\mathbf{j}+(1+2)\,\mathbf{k}$$

at the point $(1, 2, 0)$

$$=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the given vector $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, then

$$\hat{\mathbf{a}}=\frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{|\mathbf{i}+2\mathbf{j}+2\mathbf{k}|}=\frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(1+4+4)}}=\frac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k}).$$

$\therefore$ the required directional derivative

$$=(\text{grad } \phi)\cdot\hat{\mathbf{a}}=(2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot\tfrac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

$$=\tfrac{1}{3}(2+2+6)=\frac{10}{3}.$$

Ex. 12. *Find the directional derivative of*

$$\phi(x, y, z)=x^2yz+4xz^2$$

at the point $(1, -2, 1)$ *in the direction of* $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

[Indore 1983]

Sol. Do yourself. **Ans.** $-13/3$.

Ex. 13. *Obtain the directional derivative of* $\phi=xy^2+yz^2$ *at the point* $(2, -1, 1)$ *in the direction of the vector* $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

[Allahabad 1975]

Sol. Do yourself. **Ans.** -3.

Ex. 14. *Find the directional derivatives of* $\phi=xyz$ *at the point* $(2, 2, 2)$, *in the directions*

(i) $\mathbf{i}$, (ii) $\mathbf{j}$, (iii) $\mathbf{i}+\mathbf{j}+\mathbf{k}$. **[Agra 1981]**

Sol. We have $\phi(x, y, z) = xyz$.

$\therefore$ grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$

$= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ at the point (2, 2, 2).

(i) Directional derivative of ϕ at the point (2, 2, 2) in the direction of the unit vector $\mathbf{i}$

$$= \text{grad } \phi \cdot \mathbf{i} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \mathbf{i} = 4.$$

(ii) Directional derivative of ϕ at the point (2, 2, 2) in the direction of the unit vector $\mathbf{j}$

$$= \text{grad } \phi \cdot \mathbf{j} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \mathbf{j} = 4.$$

(iii) Unit vector $\hat{\mathbf{a}}$ in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{|\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}.$$

$\therefore$ directional derivative of ϕ at the point (2, 2, 2) in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$= \text{grad } \phi \cdot \hat{\mathbf{a}} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} = \frac{4 + 4 + 4}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

Ex. 15. *Find the directional derivative of $\phi = xyz$ at* (1, 2, 3) *in the direction of the vector* $\mathbf{i}$.

Sol. Do yourself. **Ans. 6.**

Ex. 16. *Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$, $z = t^3$ at the point* (1, 1, 1).

Sol. Let $\phi(x, y, z) = xy^2 + yz^2 + zx^2$.

Then grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= (y^2 + 2zx)\,\mathbf{i} + (z^2 + 2xy)\,\mathbf{j} + (x^2 + 2yz)\,\mathbf{k}$

$= 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, at the point (1, 1, 1)

$= 3(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Also for the curve $x = t$, $y = t^2$, $z = t^3$, we have

$$dx/dt = 1,\ dy/dt = 2t,\ dz/dt = 3t^2.$$

At the point (1, 1, 1) on the curve $x = t$, $y = t^2$, $z = t^3$, we have $t = 1$.

Now a vector along the tangent to the above curve at the point (x, y, z)

$= (dx/dt)\,\mathbf{i} + (dy/dt)\,\mathbf{j} + (dz/dt)\,\mathbf{k}$

$= \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$.

Putting $t = 1$, a vector along the tangent to the curve at the point $(1, 1, 1) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of this tangent, then

$$\hat{\mathbf{a}}=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+2\mathbf{j}+3\mathbf{k}|}=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{}}$$

$\therefore$ the required directional derivative

$$=\hat{\mathbf{a}}\cdot\text{grad }\phi \text{ at } (1, 1, 1)$$

$$=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{(14)}}\cdot 3\,(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

$$=\frac{3}{\sqrt{(14)}}(1+2+3)=\frac{18}{\sqrt{(14)}}.$$

Ex. 17. *In what direction the directional derivative of $\phi=x^2y^2z$ from $(1, 1, 2)$ will be maximum and what is its magnitude? Also find a unit normal vector to the surface $x^2y^2z=2$ at the point $(1, 1, 2)$.*

Sol. We know that the directional derivative of ϕ at the point (x, y, z) is maximum in the direction of the normal to the surface $\phi=$constant *i.e.*, in the direction of the vector grad ϕ.

Now grad $\phi=(\partial\phi/\partial x)\,\mathbf{i}+(\partial\phi/\partial y)\,\mathbf{j}+(\partial\phi/\partial z)\,\mathbf{k}$

$$=2xy^2z\,\mathbf{i}+2x^2yz\,\mathbf{j}+x^2y^2\,\mathbf{k}$$

$$=4\mathbf{i}+4\mathbf{j}+\mathbf{k}, \text{ at the point } (1, 1, 2).$$

Hence the directional derivative of ϕ at the point $(1, 1, 2)$ will be maximum in the direction of the vector $4\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

Also the magnitude of this maximum directional derivative

$$=\text{modulus of grad } \phi \text{ at } (1, 1, 2)$$

$$=|4\mathbf{i}+4\mathbf{j}+\mathbf{k}|=\sqrt{(16+16+1)}=\sqrt{(33)}.$$

The unit vector along the normal to the surface $x^2y^2z=2$ at the point $(1, 1, 2)$

$$=\frac{\text{grad }\phi}{|\text{grad }\phi|}, \text{ at } (1, 1, 2)$$

$$=\frac{4\mathbf{i}+4\mathbf{j}+\mathbf{k}}{|4\mathbf{i}+4\mathbf{j}+\mathbf{k}|}=\frac{4\mathbf{i}+4\mathbf{j}+\mathbf{k}}{\sqrt{(33)}}.$$

Ex. 18. *Find the greatest value of the directional derivative of the function $2x^2-y-z^4$ at the point $(2, -1, 1)$.*

Sol. Let $\phi(x, y, z)=2x^2-y-z^4$.

Then grad $\phi=(\partial\phi/\partial x)\,\mathbf{i}+(\partial\phi/\partial y)\,\mathbf{j}+(\partial\phi/\partial z)\,\mathbf{k}$

$$=4x\,\mathbf{i}-\mathbf{j}-4z^3\,\mathbf{k}$$

$$=8\mathbf{i}-\mathbf{j}-4\mathbf{k}, \text{ at the point } (2, -1, 1).$$

Now the greatest value of the directional derivative of ϕ at the point $(2, -1, 1)$

= modulus of grad ϕ at the point $(2, -1, 1)$

$= | 8\mathbf{i}-\mathbf{j}-4\mathbf{k} | = \sqrt{[8^2+(-1)^2+(-4)^2]} = \sqrt{(64+1+16)} = 9.$

Ex. 19. *Find the maximum value of the directional derivative of $\phi = x^2 yz$ at the point* $(1, 4, 1)$. **[Bombay 1970]**

Sol. We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2xyz\,\mathbf{i} + x^2z\,\mathbf{j} + x^2y\,\mathbf{k}$

$= 8\mathbf{i}+\mathbf{j}+4\mathbf{k}$, at the point $(1, 4, 1)$.

Now the maximum value of the directional derivative of $\phi = x^2yz$ at the point $(1, 4, 1)$

$= |$ grad ϕ at the point $(1, 4, 1)\,|$

$= | 8\mathbf{i}+\mathbf{j}+4\mathbf{k} | = \sqrt{(8^2+1^2+4^2)} = \sqrt{(81)} = 9.$

Ex. 20. *Calculate the maximum rate of change and the corresponding direction for the function $\phi = x^2 y^3 z^4$ at the point* $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$. **[Allahabad 1982]**

Sol. The coordinates of the point whose position vector is $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ are $(2, 3, -1)$.

We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$

$= 108\mathbf{i} + 108\mathbf{j} - 432\,\mathbf{k}$, at the point $(2, 3, -1)$

$= 108\,(\mathbf{i}+\mathbf{j}-4\,\mathbf{k})$.

Now the rate of change of ϕ (*i.e.*, the directional derivative of ϕ) at the point $(2, 3, -1)$ is maximum in the direction of the vector grad ϕ at this point *i.e.*, in the direction of the vector $108\,(\mathbf{i}+\mathbf{j}-4\mathbf{k})$.

Also the magnitude of the maximum rate of change = modulus of grad ϕ at the point $(2, 3, -1)$

$= | 108\,(\mathbf{i}+\mathbf{j}-4\mathbf{k}) | = 108\,| \mathbf{i}+\mathbf{j}-4\mathbf{k} |$

$= 108\sqrt{[1^2+1^2+(-4)^2]} = 108\sqrt{(18)} = 324\sqrt{2}.$

Ex. 21. *Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2+by^2+cz^2$ at* $(1, 1, 2)$ *has a maximum magnitude 4 in the direction parallel to y-axis.*

Sol. We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}$

$= 2a\,\mathbf{i} + 2b\mathbf{j} + 4c\,\mathbf{k}$, at the point $(1, 1, 2)$.

Now the directional derivative of ϕ at the point $(1, 1, 2)$ is maximum in the direction of the vector grad ϕ at this point. According to the question this directional derivative is maximum in the direction parallel to y-axis *i.e.*, in the direction parallel to the vector $\mathbf{j}$. So if the direction of the vector $2a\mathbf{i}+2b\mathbf{j}+4c\,\mathbf{k}$

Find the values of the constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum of magnitude 64 in direction parallel to the z-axis. Ans: $a=6, b=24, c=-8$

is parallel to the vector **j**, we must have $2a=0$, $4c=0$ *i.e.*, $a=0$ and $c=0$.

Then grad ϕ at $(1, 1, 2) = 2b\mathbf{j}$.

Also the maximum value of directional derivative

$$= |\text{grad } \phi|.$$

$\therefore$ $4 = |2b\mathbf{j}|$, since according to the question the maximum value of directional derivative is 4.

$\therefore$ $2b=4$ or $b=2$.

Hence $a=0$, $b=2$, $c=0$.

Ex. 22. *In what direction from the point* $(1, 1, -1)$ *is the directional derivative of* $f = x^2 - 2y^2 + 4z^2$ *a maximum? Also find the value of this maximum directional derivative.*

Sol. We have grad $f = 2x\mathbf{i} - 4y\mathbf{j} + 8z\mathbf{k}$

$$= 2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k} \text{ at the point } (1, 1, -1).$$

The directional derivative of f is maximum in the direction of grad $f = 2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}$.

The maximum value of this directional derivative

$= |\text{grad } f| = |2\mathbf{i} - 4\mathbf{j} - 8\mathbf{k}| = \sqrt{(4+16+64)} = \sqrt{(84)} = 2\sqrt{(21)}$.

Ex. 23. *For the function* $f = y/(x^2+y^2)$, *find the value of the directional derivative making an angle* $30°$ *with the positive x-axis at the point* $(0, 1)$. **[Agra 1981]**

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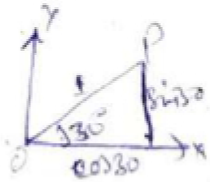

Sol. We have grad $f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j}$

$$= \frac{-2xy}{(x^2+y^2)^2}\mathbf{i} + \frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j} = -\mathbf{j} \text{ at the point } (0, 1).$$

If $\hat{\mathbf{a}}$ is a unit vector along the line which makes an angle $30°$ with the positive x-axis, then

$$\hat{\mathbf{a}} = \cos 30°\,\mathbf{i} + \sin 30°\,\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}.$$

$\therefore$ the required directional derivative is

$$= \text{grad } f \cdot \hat{\mathbf{a}} = (-\mathbf{j}) \cdot \left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) = -\tfrac{1}{2}.$$

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Ex. 24. *What is the greatest rate of increase of* $u = xyz^2$ *at the point* $(1, 0, 3)$? **[Agra 1968]**

Sol. We have $\nabla u = yz^2\,\mathbf{i} + xz^2\,\mathbf{j} + 2xyz\,\mathbf{k}$.

$\therefore$ at the point $(1, 0, 3)$, we have

$$\nabla u = 0\,\mathbf{i} + 9\,\mathbf{j} + 0\,\mathbf{k} = 9\,\mathbf{j}.$$

The greatest rate of increase of u at the point $(1, 0, 3)$

$= $ the maximum value of $\dfrac{du}{ds}$ at the point $(1, 0, 3)$

$= |\nabla u|$, at the point (1, 0, 3)

$= |9\mathbf{j}| = 9.$

Ex. 25. *Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.*

Sol. Let $f(x, y, z)$ be a scalar point function and let $\mathbf{a}$ be a unit vector along a tangent line to the level surface $f(x, y, z) = c$.

We know that ∇f is a normal vector at any point of the surface $f(x, y, z) = c$. Therefore the vectors ∇f and $\mathbf{a}$ are perpendicular.

Now the directional derivative of f in the direction of $\mathbf{a}$

$$= \mathbf{a} \cdot \nabla f = 0.$$

Ex. 26. *Find the equations of the tangent plane and normal to the surface $2xz^2 - 3xy - 4x = 7$ at the point* (1, −1, 2).

Sol. The equation of the surface is

$$f(x, y, z) \equiv 2xz^2 - 3xy - 4x = 7.$$

We have grad $f = (2z^2 - 3y + 4)\,\mathbf{i} - 3x\,\mathbf{j} + 4xz\,\mathbf{k}$

$= 7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$, at the point (1, −1, 2).

$\therefore$ $7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}$ is a vector along the normal to the surface at the point (1, −1, 2).

The position vector of the point (1, −1, 2) is $= \mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

If $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, −1, 2), then the vector $\mathbf{R} - \mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is

$$(\mathbf{R} - \mathbf{r}) \cdot \text{grad } f = 0,$$

i.e. $$\{(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\} \cdot (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0,$$

i.e. $$\{(X-1)\,\mathbf{i} + (Y+1)\,\mathbf{j} + (Z-2)\,\mathbf{k}\} \cdot (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0,$$

i.e. $$7(X-1) - 3(Y+1) + 8(Z-2) = 0.$$

The equations of the normal to the surface at the point (1, −1, 2) are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}} = \frac{Y+1}{\dfrac{\partial f}{\partial y}} = \frac{Z-2}{\dfrac{\partial f}{\partial z}}, \text{ i.e. } \frac{X-1}{7} = \frac{Y+1}{-3} = \frac{Z-2}{8}.$$

Ex. 27. *Find the equations of the tangent plane and normal to the surface $xyz = 4$ at the point* (1, 2, 2). **[Meerut 1991 P, Agra 82]**

Sol. The equation of the surface is

$$f(x, y, z) \equiv xyz - 4 = 0.$$

We have grad $f = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$$=4\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \text{ at the point } (1, 2, 2).$$

$\therefore$ $4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a vector along the normal to the surface at the point $(1, 2, 2)$.

The position vector of the point $(1, 2, 2)$ is $=\mathbf{r}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, 2, 2)$, the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot \text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $4\,(X-1)+2\,(Y-2)+2\,(Z-2)=0,$

i.e. $4X+2Y+2Z=12,$ *i.e.* $2X+Y+Z=6.$

The equations of the normal to the surface at the point $(1, 2, 2)$ are

$$\frac{X-1}{\frac{\partial f}{\partial x}}=\frac{Y-2}{\frac{\partial f}{\partial y}}=\frac{Z-2}{\frac{\partial f}{\partial z}},$$

i.e. $\dfrac{X-1}{4}=\dfrac{Y-2}{2}=\dfrac{Z-2}{2}$, *i.e.*, $\dfrac{X-1}{2}=\dfrac{Y-2}{1}=\dfrac{Z-2}{1}$.

Ex. 28. *Find the equation of the tangent plane to the surface $yz-zx+xy+5=0$, at the point $(1, -1, 2)$.*

Sol. The equation of the given surface is

$$f(x, y, z)\equiv yz-zx+xy+5=0.$$

We have grad $f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$$=(-z+y)\,\mathbf{i}+(z+x)\,\mathbf{j}+(y-x)\,\mathbf{k}$$

$$=-3\mathbf{i}+3\mathbf{j}-2\mathbf{k}, \text{ at the point } (1, -1, 2).$$

$\therefore$ $-3\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ is a vector along the normal to the surface $f(x, y, z)=0$ at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$
$=\mathbf{i}-\mathbf{j}+2\mathbf{k}=\mathbf{r}$, say.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot \text{grad } f=0$$

i.e., $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})\}\cdot(-3\mathbf{i}+3\mathbf{j}-2\mathbf{k})=0$

or $\{(X-1)\,\mathbf{i}+(Y+1)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(-3\mathbf{i}+3\mathbf{j}-2\mathbf{k})=0$

or $-3\,(X-1)+3\,(Y+1)-2\,(Z-2)=0$

or $-3X+3Y-2Z+3+3+4=0$

or $3X-3Y+2Z=10.$

Ex. 29. *Find the equations of the tangent plane and normal to the surface $z=x^2+y^2$ at the point* $(2, -1, 5)$.

Sol. The equation of the given surface is

$$f(x, y, z) \equiv x^2+y^2-z=0. \qquad \ldots(1)$$

We have grad $f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$

$=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$, at the point $(2, -1, 5)$.

$\therefore$ $4\mathbf{i}-2\mathbf{j}-\mathbf{k}$ is a vector along the normal to the surface (1) at the point $(2, -1, 5)$ *i.e.*, perpendicular to the tangent plane to the surface (1) at the point $(2, -1, 5)$.

Hence the equation of the tangent plane to the surface (1) at the point $(2, -1, 5)$ is

$\{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\}\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})=0$

or $\{(x-2)\,\mathbf{i}+(y+1)\,\mathbf{j}+(z-5)\,\mathbf{k}\}\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})=0$

or $4(x-2)-2(y+1)-(z-5)=0$

or $4x-2y-z=5$.

The equations of the normal to the surface (1) at the point $(2, -1, 5)$ are

$$\frac{x-2}{\partial f/\partial x}=\frac{y+1}{\partial f/\partial y}=\frac{z-5}{\partial f/\partial z}$$

i.e., $\dfrac{x-2}{4}=\dfrac{y+1}{2}=\dfrac{z-5}{-1}$.

Ex. 30. *Find the equations of the tangent plane and normal to the surface $x^2+y^2+z^2=25$ at the point* $(4, 0, 3)$.

Sol. The given surface is $f(x, y, z) \equiv x^2+y^2+z^2-25=0$ $\ldots(1)$

We have grad $f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$

$=8\mathbf{i}+0\mathbf{j}+6\mathbf{k}$, at the point $(4, 0, 3)$.

$\therefore$ the direction cosines of the normal to the surface (1) at the point $(4, 0, 3)$ are proportional to $8, 0, 6$.

Hence the equation of the tangent plane to the surface (1) at the point $(4, 0, 3)$ is

$8(x-4)+0(y-0)+6(z-3)=0$ or $4x+3z=25$.

The equations of the normal to the surface (1) at the point $(4, 0, 3)$ are

$$\frac{x-4}{8}=\frac{y-0}{0}=\frac{z-3}{6} \; i.e., \; \frac{x-4}{4}=\frac{y}{0}=\frac{z-3}{3}.$$

Ex. 31. *Given the curve $x^2+y^2+z^2=1$, $x+y+z=1$ (intersection of two surfaces), find the equations of the tangent line at the point* $(1, 0, 0)$. **[Agra 1983]**

Sol. A normal to $x^2+y^2+z^2=1$ at $(1, 0, 0)$ is

$$\text{grad } f_1 = \text{grad } (x^2+y^2+z^2) = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k} = 2\mathbf{i}.$$

A normal to $x+y+z=1$ at $(1, 0, 0)$ is

$$\text{grad } f_2 = \text{grad } (x+y+z) = 1\mathbf{i}+1\mathbf{j}+1\mathbf{k} = \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

The tangent line at the point $(1, 0, 0)$ is perpendicular to both these normals. Therefore it is parallel to the vector

$$(\text{grad } f_1) \times (\text{grad } f_2).$$

Now $(\text{grad } f_1) \times (\text{grad } f_2) = 2\mathbf{i} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})$

$$= 2\mathbf{i} \times \mathbf{j} + 2\mathbf{i} \times \mathbf{k} = 2\mathbf{k} - 2\mathbf{j} = 0\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}.$$

Now to find the equations of the line through the point $(1, 0, 0)$ and parallel to the vector $0\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

The required equations are

$$\frac{X-1}{0} = \frac{Y-0}{-2} = \frac{Z-0}{2},$$

i.e.,
$$X=1, \ \frac{Y}{-1} = \frac{Z}{1}.$$

Ex. 32. *Find the angle between the surfaces* $x^2+y^2+z^2=9$, *and* $z=x^2+y^2-3$ *at the point* $(2, -1, 2)$.

[Meerut 1991S; Kanpur 78, 80]

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at the point.

Let $\quad f_1 = x^2+y^2+z^2$ and $f_2 = x^2+y^2-z$.

Then grad $f_1 = 2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ and grad $f_2 = 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1 = \text{grad } f_1$ at the point $(2, -1, 2)$ and $\mathbf{n}_2 = \text{grad } f_2$ at the point $(2, -1, 2)$. Then

$$\mathbf{n}_1 = 4\mathbf{i}-2\mathbf{j}+4\mathbf{k} \text{ and } \mathbf{n}_2 = 4\mathbf{i}-2\mathbf{j}-\mathbf{k}.$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point $(2, -1, 2)$. If θ is the angle between these vectors then

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| \, |\mathbf{n}_2| \cos\theta$$

or
$$16+4-4 = \sqrt{(16+4+16)}\sqrt{(16+4+1)} \cos\theta.$$

$$\therefore \quad \cos\theta = \frac{16}{6\sqrt{(21)}} \quad \text{or} \quad \theta = \cos^{-1} \frac{8}{3\sqrt{(21)}}.$$

Ex. 33. *Find the angle of intersection at* $(4, -3, 2)$ *of spheres* $x^2+y^2+z^2=29$ *and* $x^2+y^2+z^2+4x-6y-8z-47=0$.

Sol. Let $f_1 = x^2+y^2+z^2-29$ and $f_2 = x^2+y^2+z^2+4x-6y-8z-47$.

Then grad $f_1 = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}$

and grad $f_2 = (2x+4)\,\mathbf{i}+(2y-6)\,\mathbf{j}+(2z-8)\,\mathbf{k}$.

Let $\mathbf{n}_1 = \text{grad } f_1$ at the point $(4, -3, 2)$

and $\mathbf{n}_2 = \text{grad } f_2$ at the point $(4, -3, 2)$. Then

$$\mathbf{n}_1 = 8\mathbf{i}-6\mathbf{j}+4\mathbf{k} = 2\,(4\mathbf{i}-3\mathbf{j}+2\mathbf{k})$$

and
$$\mathbf{n}_2 = 12\mathbf{i}-12\mathbf{j}-4\mathbf{k} = 4\,(3\mathbf{i}-3\mathbf{j}-\mathbf{k}).$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two spheres at the point (4, −3, 2) and the angle θ between these two vectors is the angle of intersection of the two spheres at the point (4, −3, 2).

We have $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{8(12+9-2)}{2\sqrt{(16+9+4)}.4\sqrt{(9+9+1)}}$

$$= \frac{19}{\sqrt{(29)}.\sqrt{(19)}}$$

or $\theta = \cos^{-1}\sqrt{(19/29)}$.

Ex. 34. *Find the constants a and b so that the surface* $ax^2 - byz = (a+2)x$ *will be orthogonal to the surface* $4x^2y + z^3 = 4$ *at the point* (1, −1, 2).

Sol. The given surfaces are

$$f_1 \equiv ax^2 - byz - (a+2)x = 0 \qquad \ldots(1)$$

and $$f_2 \equiv 4x^2y + z^3 - 4 = 0 \qquad \ldots(2)$$

The point (1, −1, 2) obviously lies on the surface (2). It will also lie on the surface (1) if

$$a + 2b - (a+2) = 0 \quad \text{or} \quad 2b - 2 = 0 \quad \text{or} \quad b = 1.$$

Now grad $f_1 = [2ax - (a+2)]\,\mathbf{i} - bz\,\mathbf{j} - by\,\mathbf{k}$

and grad $f_2 = 8xy\,\mathbf{i} + 4x^2\,\mathbf{j} + 3z^2\,\mathbf{k}$.

Then $\mathbf{n}_1$ = grad f_1 at the point $(1, -1, 2) = (a-2)\,\mathbf{i} - 2b\mathbf{j} + b\mathbf{k}$

and $\mathbf{n}_2$ = grad f_2 at the point $(1, -1, 2) = -8\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along the normals to the surfaces (1) and (2) at the point (1, −1, 2). These surfaces will intersect orthogonally at the point (1, −1, 2) if the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular *i.e.*, if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$

or $$-8(a-2) - 8b + 12b = 0 \quad \text{or} \quad b - 2a + 4 = 0 \qquad \ldots(3)$$

But $b = 1$, as already found.

Putting $b = 1$ in (3), we get $a = 5/2$. **Ans.** $a = 5/2, b = 1$.

Ex. 35. *If* **F** *and f are point functions, show that the components of the former tangential and normal to the level surface*

$$f = 0 \text{ are } \frac{\nabla f \times (\mathbf{F} \times \nabla f)}{(\nabla f)^2} \text{ and } \frac{(\mathbf{F} \cdot \nabla f)\,\nabla f}{(\nabla f)^2}.$$

Sol. The unit normal vector to the surface $f = 0$ is

$$= \frac{\nabla f}{|\nabla f|}.$$

∴ The magnitude of the component of **F** along the normal

$$= \mathbf{F} \cdot \frac{\nabla f}{|\nabla f|}$$

$\therefore$ the component of $\mathbf{F}$ along the normal

$$=\left\{\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}\right\}\frac{\nabla f}{|\nabla f|}=\frac{(\mathbf{F}\cdot\nabla f)}{|\nabla f|^2}\nabla f=\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f.$$

Consequently the tangential component of $\mathbf{F}$ is

$$=\mathbf{F}-\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f=\frac{(\nabla f\cdot\nabla f)\,\mathbf{F}-(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$$

$$=\frac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}\ [\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}].$$

§ 12. Invariance.

Theorem 1. *Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator* ∇ *remains invariant.*

Proof. Let O be the fixed origin. Let Ox, Oy, Oz be one system of rectangular axes and Ox', Oy', Oz' be the other system of rectangular axes. Take **i**, **j**, **k** as unit vectors along Ox, Oy, Oz and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ as unit vectors along Ox', Oy', Oz'. Let P be any point in space whose co-ordinates are (x, y, z) or (x', y', z') with respect to the two systems of axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the lines Ox', Oy', Oz' with respect to the co-ordinate axes Ox, Oy, Oz.

The scheme of transformation will be as follows :

$$\left.\begin{aligned} x' &= l_1x+m_1y+n_1z \\ y' &= l_2x+m_2y+n_2z \\ z' &= l_3x+m_3y+n_3z \end{aligned}\right\} \qquad \ldots(1)$$

Also we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where **i**, **j**, **k** are unit vectors along co-ordinate axes. Therefore

$$\left.\begin{aligned} \mathbf{i}' &= l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k} \\ \mathbf{j}' &= l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k} \\ \mathbf{K}' &= l_3\mathbf{i}+m_3\mathbf{j}+n_3\mathbf{k} \end{aligned}\right\} \qquad \ldots(2)$$

If V is any function (vector or scalar) of x, y, z, then

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial V}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial V}{\partial z'}\frac{\partial z'}{\partial x}.$$

$$\therefore \qquad \frac{\partial}{\partial x}\equiv\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}+\frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1$, $\dfrac{\partial y'}{\partial x}=l_2$, $\dfrac{\partial z'}{\partial x}=l_3$.

$$\left.\begin{aligned} &\therefore & \frac{\partial}{\partial x} &\equiv l_1\frac{\partial}{\partial x'}+l_2\frac{\partial}{\partial y'}+l_3\frac{\partial}{\partial z'} \\ &\text{Similarly} & \frac{\partial}{\partial y} &\equiv m_1\frac{\partial}{\partial x'}+m_2\frac{\partial}{\partial y'}+m_3\frac{\partial}{\partial z'} \\ && \frac{\partial}{\partial z} &\equiv n_1\frac{\partial}{\partial x'}+n_2\frac{\partial}{\partial y'}+n_3\frac{\partial}{\partial z'} \end{aligned}\right\} \qquad \ldots(3)$$

Multiplying the equations (3) by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\equiv\mathbf{i}'\frac{\partial}{\partial x'}+\mathbf{j}'\frac{\partial}{\partial y'}+\mathbf{k}'\frac{\partial}{\partial z'}.$$

Theorem 2. *If $\phi(x, y, z)$ is a scalar invariant with respect to a rotation of axes, then grad ϕ is a vector invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorem 1 and obtain the equations (1) and (2).

Now suppose the function $\phi(x, y, z)$ becomes $\phi'(x', y', z')$ after rotation of axes. Then by hypothesis $\phi(x, y, z)=\phi'(x', y', z')$.

By chain rule of differentiation, we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\phi'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\phi'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned}\therefore\quad \frac{\partial\phi}{\partial x}&=l_1\frac{\partial\phi'}{\partial x'}+l_2\frac{\partial\phi'}{\partial y'}+l_3\frac{\partial\phi'}{\partial z'}\\ \text{Similarly}\quad \frac{\partial\phi}{\partial y}&=m_1\frac{\partial\phi'}{\partial x'}+m_2\frac{\partial\phi'}{\partial y'}+m_3\frac{\partial\phi'}{\partial z'}\\ \text{and}\quad \frac{\partial\phi}{\partial z}&=n_1\frac{\partial\phi'}{\partial x'}+n_2\frac{\partial\phi'}{\partial y'}+n_3\frac{\partial\phi'}{\partial z'}\end{aligned}\right\}\quad\ldots(3)$$

Multiplying these equations by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\frac{\partial\phi}{\partial x}+\mathbf{j}\frac{\partial\phi}{\partial y}+\mathbf{k}\frac{\partial\phi}{\partial z}=\mathbf{i}'\frac{\partial\phi'}{\partial x'}+\mathbf{j}'\frac{\partial\phi'}{\partial y'}+\mathbf{k}'\frac{\partial\phi'}{\partial z'}$$

or $$\text{grad }\phi=\text{grad }\phi'.$$

Theorem 3. *If $\mathbf{V}(x, y, z)$ is a vector function invariant with respect to a rotation of axes, then div $\mathbf{V}$ is a scalar invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorems 1 and 2.

Now suppose the function $\mathbf{V}(x, y, z)$ becomes $\mathbf{V}'(x', y', z')$ after rotation of axes. Then by hypothesis

$$\mathbf{V}(x, y, z)=\mathbf{V}'(x', y', z').$$

By chain rule of differentiation, we have

$$\frac{\partial\mathbf{V}}{\partial x}=\frac{\partial\mathbf{V}'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial\mathbf{V}'}{\partial z'}\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned}
\therefore\quad & \frac{\partial \mathbf{V}}{\partial x}=l_1\frac{\partial \mathbf{V}'}{\partial x'}+l_2\frac{\partial \mathbf{V}'}{\partial y'}+l_3\frac{\partial \mathbf{V}'}{\partial z'}\\
\text{Similarly}\quad & \frac{\partial \mathbf{V}}{\partial y}=m_1\frac{\partial \mathbf{V}'}{\partial x'}+m_2\frac{\partial \mathbf{V}'}{\partial y'}+m_3\frac{\partial \mathbf{V}'}{\partial z'}\\
\text{and}\quad & \frac{\partial \mathbf{V}}{\partial z}=n_1\frac{\partial \mathbf{V}'}{\partial x'}+n_2\frac{\partial \mathbf{V}'}{\partial y'}+n_3\frac{\partial \mathbf{V}'}{\partial z'}
\end{aligned}\right\}\quad \ldots(3)$$

Taking dot product of these three equations by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{div }\mathbf{V}=\text{div }\mathbf{V}'.$$

Theorem 4. *If* $\mathbf{V}(x, y, z)$ *is a vector function invariant under a rotation of axes, then curl* $\mathbf{V}$ *is a vector invariant under this rotation.*

Proof. Proceed exactly in the same manner as in theorem 3.

In place of taking dot product of equations (3), take cross product. We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}=\mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'}+\mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'}+\mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{curl }\mathbf{V}=\text{curl }\mathbf{V}'.$$

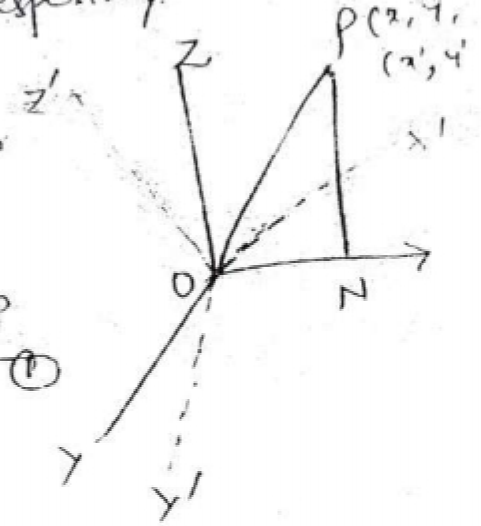
(i-) To change the direction of co-ordinate axes without changing the origin

Let OX, OY, OZ and OX', OY', OZ' be two sets of co-ordinate axes through the common origin O.

Let the direction cosines of OX', OY' and OZ' be l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 respectively referred to OX, OY and OZ.

Then the direction cosines of OX, OY, OZ referred to OX', OY', OZ' are evidently l_1, l_2, l_3, m_1, m_2, m_3 and n_1, n_2, n_3 respectively.

Let the co-ordinates of P be (x, y, z) and (x', y', z') referred to the original axes OX, OY, OZ and the new axes OX', OY', OZ' respectively.



From P draw PN perpendicular to OX.

Then $x = ON$ = Projection of OP on OX ——— ①

Now d.c.s of OX referred to the new axes are l_1, l_2, l_3 and the co-ordinates of P referred to the new axes are (x', y', z').

from ①, we have

$x = l_1(x'-0) + l_2(y'-0) + l_3(z'$

$$\left.\begin{aligned} \Rightarrow \quad x &= l_1 x' + l_2 y' + l_3 z' \\ \text{Similarly,}\quad y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}\right\} \text{--- (2)}$$

Multiplying these relations by l_1, m_1, n_1 respectively and adding we get

$$l_1 x + m_1 y + n_1 z = x' \sum l_1^2 + y' \sum l_1 l_2 + z' \sum l_1 l_3$$

$$= x'(1) + y'(0) + z'(0)$$

$$= x' \quad (\because \sum l_1^2 = 1, \sum l_1 l_2 = 0 = \sum l_1 l_3)$$

$$\left.\begin{aligned} \therefore \quad x' &= l_1 x + m_1 y + n_1 z \\ \text{Similarly}\quad y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \text{--- (3)}$$

The relations (1) express the old co-ordinates x, y, z in terms of the new co-ordinates x', y', z' and the relations (3) express x', y', z' in terms of x, y, z.

The relations are also written conveniently with the help of adjoining table.

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

In this table the horizontal and vertical lines denote the direction cosines of mutually ⊥ar axes.