MAINS TEST SERIES-2021

TEST-8 (BATCH-II) &

TEST-18 (BATCH-I)

FULL SYLLABUS (PAPER-II)

Answer Key

1.(a) Assume that the equation $xyz=1$ holds in a group G. Does it follow that $yzx=1$? That $yxz=1$? justify your answer.

Solⁿ: $xyz=1$ implies that $x(yz)=1$.

Let $yz=a$. Then we have $xa=1$ and $ax=1$

Since 'a' is invertible and $a^{-1}=x$.

It follows that $(yz)x=1$.

Hence $yzx=1$.

on the other hand, if $xyz=1$, it is not always true that $yxz=1$.

To see this, let G be the group of 2×2 real matrices and

let $x=\begin{pmatrix}1&2\\0&2\end{pmatrix}$, $y=\begin{pmatrix}0&1\\2&1\end{pmatrix}$ and $z=\begin{pmatrix}-1/2&3/4\\1&-1\end{pmatrix}$.

Then $xyz=\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$ in G.

But $yxz=\begin{pmatrix}2&-2\\5&-9/2\end{pmatrix}\neq 1$.

1.(b) Let $R=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in Z_2 \right\}$

with ordinary matrix addition and multiplication modulo 2. Show that

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} r \,\middle|\, r \in R \right\}$$

is not an ideal of R.

Soln: $S=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} r \,\middle|\, r \in R \right\}$

$$=\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a,b,c,d \in Z_2 \right\}$$

$$=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b \in Z_2 \right\}$$

For any $s_1, s_2 \in S$ we have $s_1 - s_2 \in S$ but for any $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ in R

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ax & bx \\ az & wb \end{bmatrix}$$

Choose $a=1, b=1, x=1, y=1, z=1, w=1$ then

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin S.$$

Hence S is not an ideal.

1.(c) Show that the sequence $\{f_n\}$, where

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq 1/n \\ -n^2 x + 2n, & 1/n \leq x \leq 2/n \\ 0, & 2/n \leq x \leq 1 \end{cases}$$

is not uniformly convergent on $[0, 1]$.

Sol^n: The sequence converges to f, where $f(x)=0$, for all $x \in [0,1]$. Each function f_n and f are continuous on $[0,1]$.

Also $$\int_0^1 f_n\,dx = \int_0^{1/n} n^2 x\,dx + \int_{1/n}^{2/n} (-n^2 x + 2n)\,dx + \int_{2/n}^1 0\,dx = 1.$$

But $$\int_0^1 f\,dx = 0$$

$$\therefore \lim_{n\to\infty} \int_0^1 f_n\,dx \neq \int_0^1 f\,dx$$

So (by theorem) the sequence $\{f_n\}$ cannot converge uniformly on $[0,1]$.

1.(d) → Verify Cauchy's Theorem and integrating e^{iz} along the boundary of the triangle with vertices at the points $1+i$, $-1+i$ and $-1-i$.

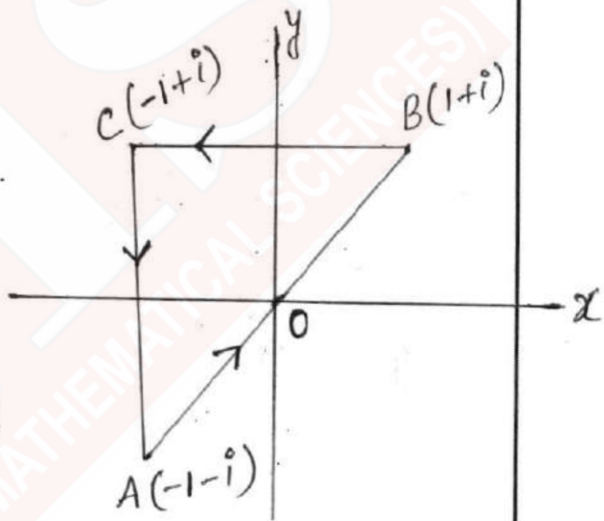
<u>Solⁿ</u>: $f(z)=e^{iz}$ is analytic within and upon closed contour ΔABC and so by Cauchy's theorem

$$\int_C f(z)\,dz = 0 \quad \text{——— (1)}$$

Let $I=\int_C f(z)\,dz = I_1(\vec{AB})+I_2(\vec{BC})+I_3(\vec{CA})$ ——— (2)

Then $I=\int_C e^{iz}\,dz$

$$=\left(\frac{e^{iz}}{i}\right)_C \quad \text{——— (3)}$$



For I_1, equation of $\vec{AB}$ is

$y=x$, $z=x+iy=(1+i)x$

$$I_1(\vec{AB})=\left[\frac{e^{iz}}{i}\right]_C$$

$$=\left[\frac{e^{i(1+i)x}}{i}\right]_{x=-1}^{x=1} = \frac{1}{i}\left[e^{(i-1)}-e^{-(i-1)}\right]$$

For I_2, equation of $\vec{BC}$ is $y=1$, $z=x+iy=x+i$

$$I_2=\left[\frac{e^{iz}}{i}\right]_C = \left[\frac{e^{i(x+i)}}{i}\right] = \left[\frac{e^{ix-1}}{i}\right]_{x=1}^{x=-1}$$

$$=\frac{1}{i}\left(e^{-i-1}-e^{i-1}\right)$$

For I_3, equation of $\vec{CA}$ is $x=-1$, $z=x+iy=-1+iy$

$$I_3 = \left[\frac{e^{iz}}{i}\right]_C = \left[\frac{e^{i(-1+iy)}}{i}\right] = \left[\frac{e^{ix-1}}{i}\right]_{y=1}^{y=-1}$$

or $I_3 = \frac{1}{i}\left(e^{-i+1} - e^{-i-1}\right)$

$$\therefore I = I_1 + I_2 + I_3$$

$$= \frac{1}{i}\left[\left(e^{i-1} - e^{1-i}\right) + \left(e^{-i-1} - e^{i-1}\right) + \left(e^{-i+1} - e^{-i-1}\right)\right]$$

$$= 0$$

or $\int_C f(z)\,dz = 0$ ——— (4)

(1) & (4) $\Rightarrow$ cauchy's theorem is verified.

1.(e) A company has 5 jobs to be done. The following matrix shows the return in rupees on assigning ith $(i = 1,2,3,4,5)$ machine to the jth job $(j = A,B,C,D,E)$. Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solⁿ: Step 1. Converting from Maximization to Minimization: since the highest element is 14, the following reduced cost (opportunity loss of maximum profit) matrix is obtained.

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	2

Step 2. Now following the usual procedure of solving an assignment problem, an optimal assignment is obtained in the following table:

1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
$\cancel{0}$	13	$\cancel{0}$	5	$\boxed{0}$
5	1	7	$\boxed{0}$	5
3	$\boxed{0}$	9	4	5
$\boxed{0}$	3	3	1	5

This table gives the optimum assignment as:

$1 \to C,\ 2 \to E,\ 3 \to D,\ 4 \to B,\ 5 \to A$

with maximum profit of 50.

2.a(i) Let G be a group such that the intersection of all its subgroups which are different from $\{e\}$ is a subgroup different from identity. Prove that every element in G has finite order.

Sol^n: Let $\bigcap_{\{e\}\neq H\leq G} H = K \neq \{e\}$.

Let $e\neq a\in G$. Then consider the subgroups $H_i = \langle a^i\rangle$ for $i=1,2,3\cdots$ are subgroups of G.

for each $i\in N$ such that $H_i\neq\{e\}$

we have $K\leq H_i$. Hence K is cyclic as every subgroup of a cyclic group is cyclic.

Hence $K=\langle a^n\rangle$ for some fixed positive integer n.

Since $\langle a^n\rangle\leq\langle a^i\rangle$ if and only if $a^n=a^{ik}$ if and only if $n=ik$ if and only if $i|n$,

We obtain $i|n$ for each $i=1,2,3,\cdots\cdots$ with $H_i\neq\{e\}$.

As n is a fixed given positive integer it has only finitely many divisors.

$\because K\neq e$ we have only finitely many $H_i\neq e$

i.e. So there exists j, such that

$$H_j=\langle a^j\rangle = e$$

i.e. $$a^j=e.$$

2. a(ii) Let V be the set of real numbers, and for $a, b \in$ real numbers, $a \neq 0$, let $\tau_{a,b}: V \to V$ defined by $\tau_{a,b}(x) = ax + b$. Let $G = \{\tau_{a,b} \mid a, b \text{ real } a \neq 0\}$ and $N = \{\tau_{1b} \in G\}$

(a) Prove that G is a group with respect to composition of maps.

(b) Prove that N is a normal subgroup of G and that G/N is isomorphic to a group of non-zero real numbers under multiplication.

Solⁿ: (a) G is closed with respect to composition of maps.

Let $g, h \in G$. Then there exist $a, b, c, d \in \mathbb{R}$, $a \neq 0 \neq c$ such that $g = \tau_{a,b}$, $h = \tau_{c,d}$.

Then for any $x \in \mathbb{R}$, we have

$$(\tau_{a,b} \circ \tau_{c,d})(x) = \tau_{a,b}(cx + d) = a(cx + d) + b$$
$$= acx + (ad + b)$$

$\because$ with $a \neq 0 \neq c$ we have $ac \neq 0$, $ac \in \mathbb{R}$, $ad + b \in \mathbb{R}$, we see that

$$(\tau_{a,b} \circ \tau_{c,d})(x) = \tau_{ac, ad+b}(x)$$
for any $x \in \mathbb{R}$.

i.e. $\tau_{a,b} \circ \tau_{c,d} = \tau_{ac, ad+b} \in G$.

This means that $g \cdot h \in G$ for any $g, h \in G$.
This binary operation on G is associative, since composition of maps is associative.

$\tau_{1,0}$ is the identity function on $\mathbb{R}$ and hence is the identity element of G. For any $0 \neq a \in \mathbb{R}$, $b \in \mathbb{R}$

$$\tau_{a,b} \circ \tau_{a^{-1}, -a^{-1}b} = \tau_{aa^{-1}, a(-a^{-1}b)+b} = \tau_{1,0}$$

$$\tau_{a^{-1}, -a^{-1}b} \circ \tau_{a,b} = \tau_{a^{-1}a, a^{-1}(b)-a^{-1}b} = \tau_{1,0}$$

Thus, $\tau_{a^{-1}\,b} = (\tau_{a,b})^{-1}$.

i.e. every element of G has an inverse in G.
Hence G is a group with respect to this operation.

(b) observe that the map $\psi : G \to \mathbb{R}-\{0\}$ defined by $\psi(\tau_{a,b}) = a$ is a homomorphism from the group G onto the multiplicative group of real numbers:

$$\psi(\tau_{a,b} \circ \tau_{c,d}) = \psi(\tau_{ac, ad+b}) = ac$$
$$= \psi(\tau_{a,b})\,\psi(\tau_{c,d})$$

for any $\tau_{a,b}, \tau_{c,d} \in G$.

$$K_\psi = \{\tau_{a,b} \mid \psi(\tau_{a,b}) = 1\} = \{\tau_{1,b} \mid b \in \mathbb{R}\} = N$$

Hence N is a normal subgroup of G.
$\therefore$ by isomorphism theorem G/N is isomorphic to the multiplicative group of real numbers.

2.(b) Let $u_n(x) = x^2(x^{1/(2n-1)} - x^{1/(2n-3)}) \sin(1/x)$ for $x \geq 0$

$u_n(0) = 0$, for any positive integer greater than unity and

$u_1(x) = x^3 \sin(1/x)$ for $x \lessgtr 0$, $u_1(0) = 0$.

Show that $\sum_{n=1}^{\infty} u_n(x)$ converges for all values of x to $S(x)$, where $S(x) = x^2 \sin(1/x)$ for $x \lessgtr 0$ and $S(0) = 0$. Also show that f is discontinuous at $x = 0$, that $\sum_{n=1}^{\infty} u_n'(x)$ is not uniformly convergent in any interval including the origin, and that $S'(x) = \sum_{n=1}^{\infty} u_n'(x)$ for all values of x.

Solⁿ: We have, when $x \neq 0$

$$u_1(x) = x^3 \sin(1/x) = x^2(x-0)\sin(1/x)$$

$$u_2(x) = x^2(x^{1/3} - x)\sin(1/x),$$

$$u_3(x) = x^2(x^{1/5} - x^{1/3})\sin(1/x),$$

$$\cdots\cdots\cdots$$

$$u_n(x) = x^2(x^{1/(2n-1)} - x^{1/(2n-3)})\sin(1/x)$$

then $S_n(x) = \sum_{n=1}^{n} u_1(x) = x^2 \sin(1/x) \cdot x^{1/(2n-1)}$.

$\therefore S(x) = \lim_{n\to\infty} S_n(x) = x^2 \sin(1/x)$ when $x \neq 0$;

and $S(x) = 0$ when $x = 0$.

Now, when $x \neq 0$, we have

$$S'(x) = 2x\sin(1/x) - \cos(1/x).$$

$\because \lim_{n\to\infty} \cos(1/x)$ does not exist, $S(x)$ is discontinuous at $x=0$,

$$\sum_1^\infty u_n'(x) = \lim_{n\to\infty} \sum_1^n u_n'(x) = \lim_{n\to\infty} \left[\frac{d}{dx}\left\{x^2 \sin\frac{1}{x} - x^{1/(2n-1)}\right\}\right]$$

$$= \lim_{n\to\infty}\left[\frac{4n-1}{2n-1} x^{2n/(2n-1)} \sin\frac{1}{x} - x^{1/(2n-1)} \cos\frac{1}{x}\right]$$

$$= 2x \sin(1/x) - \cos(1/x)$$

and $S'(x) = 2x\sin(1/x) - \cos(1/x)$

Hence $S'(x) = \sum_1^\infty u_n'(x)$ when $x \neq 0$.

obviously, $S'(0) = \sum_1^\infty u_n'(0)$

Since the sum function $\{2x\sin(1/x) - \cos(1/x)\}$ of the series $\sum u_n'(x)$ is discontinuous at $x=0$, the Series $\sum u_n'(x)$ is non-uniformly convergent near $x=0$.

2·C(i) → Show that the function e^z has an isolated essential singularity at $z=\infty$.

Solⁿ: Let $f(z)=e^z$. —— ①

The behaviour of $f(z)$ at $z=\infty$ is the same as the behaviour of $f(1/z)$ at $z=0$.

$$① \Rightarrow f(1/z) = e^{1/z} = 1+\frac{1}{z}+\frac{1}{z^2}\cdot\frac{1}{2!}+\cdots\cdots$$

$$= 1+\sum_{n=1}^{\infty}\frac{1}{z^n\, n!}$$

or $$f(1/z) = 1+\sum_{n=1}^{\infty}\frac{1}{(z-0)^n\, n!}$$

This is Laurent's expansion of $f(1/z)$ about the point $z=0$. This expansion contains an infinite number of terms in the negative power of z.

Hence, by def., $z=0$ is an essential singularity of $f(1/z)$.

consequently $f(z)$ has essential singularity at $z=\infty$.

2. c(ii) By using contour integration evaluate

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, \text{ where } a>b>0.$$

Soln: Let $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = 4\int_0^{2\pi} \frac{d\theta}{(2a+2b\cos\theta)^2}$

Take c as unit circle $|z|=1$ and so $z=e^{i\theta}$,

$$dz = ie^{i\theta}d\theta, \quad \frac{dz}{iz} = d\theta,$$

$$2\cos\theta = e^{i\theta}+e^{-i\theta} = z+z^{-1}$$

or, $$2\cos\theta = \frac{z^2+1}{z}$$

$$I = \frac{4}{i}\int_C \frac{dz}{z\left[2a+b\frac{(z^2+1)}{z}\right]^2}$$

$$= \frac{4}{ib^2}\int_C \frac{z\,dz}{(2cz+z^2+1)^2}$$

or, $$I = \frac{4}{ib^2}\int_C f(z)\,dz \quad \text{——} (1)$$

where $c = \frac{a}{b}$

and $f(z) = \frac{z}{(z^2+2cz+1)^2}$. For poles $(z^2+2cz+1)^2=0$.

$$\Rightarrow z^2+2cz+1=0 \Rightarrow z = \frac{-2c\pm\sqrt{4c^2-1}}{2}$$

$$= -c\pm\sqrt{(c^2-1)}$$

Take $\alpha = -c+\sqrt{(c^2-1)}$, $\beta = -\left(c+\sqrt{c^2-1}\right)$

Then $\alpha\beta = 1$, $|\alpha|<1$, $|\beta|>1$.

$\therefore\ z=\alpha$ is only pole of order 2 inside C.

$$f(z)=\frac{z}{[(z-\alpha)(z-\beta)]^2}=\frac{\phi(z)}{(z-\alpha)^2}$$

where $\phi(z)=\dfrac{z}{(z-\beta)^2}$, $\text{Res}(z=\alpha)=\dfrac{\phi'(\alpha)}{\lfloor 1}$ ——— ②

$$\phi'(z)=\frac{(z-\beta)^2-2(z-\beta)z}{(z-\beta)^4}=\frac{(z-\beta)-2z}{(z-\beta)^3}=-\frac{(\beta+z)}{(z-\beta)^3}$$

$$\Rightarrow \phi'(\alpha)=\frac{-(\alpha+\beta)}{(\alpha-\beta)^3}=-\frac{(-2c)}{(2\sqrt{c^2-1})^3}=\frac{2c}{8(c^2-1)^{3/2}}$$

$$=\frac{2(a/b)}{8\left[\frac{a^2}{b^2}-1\right]^{3/2}}=\frac{ab^2}{4(a^2-b^2)^{3/2}}$$

$$\Rightarrow \text{Res}(\alpha)=\frac{ab^2}{4(a^2-b^2)^{3/2}},\ \text{according to ②}$$

$$\Rightarrow \int_C f(z)\,dz=2\pi i\ \text{Res}(z=\alpha)=\frac{2\pi i ab^2}{4(a^2-b^2)^{3/2}}$$

Putting this in ①,

$$I=\frac{4}{ib^2}\cdot\frac{2\pi i ab^2}{4(a^2-b^2)^{3/2}}=\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

3.a(i) → Prove that if a group G of order 28 has a normal subgroup of order 4, then G is abelian.

Sol^n: Let G be a group of order 28.

By Cauchy's Theorem G has a subgroup H of order 7.

Let N the given normal subgroup of order 4.

$\because \frac{|G|}{|H|} = 4$ we have a homomorphism from a group G into symmetric group on the cosets of H in G. So by isomorphism theorem G/K_ψ is isomorphic to a subgroup of S_4.

If $|K_\psi| = 1$, then by Lagrange theorem $|G| \,|\, |S_4|$ which is impossible,

$\because |G| = 28$ and $|S_4| = 24$.

Hence $K_\psi \neq \{e\}$ but $K_\psi = \bigcap_{g \in G} gHg^{-1} \leq H$.

The group H is a cyclic group of prime order 7 and $K_\psi \neq \{e\}$ implies $K_\psi = H$.

But kernel of a homomorphism is always a normal subgroup. Hence H is normal in G.

Now consider $H \cap N$ since this is a subgroup of H and N, so $|H \cap N|$ divides 7 and 4.

Hence $H \cap N = \{e\}$.

Now for any $h \in H$, $n \in N$, $h^{-1}n^{-1}hn \in H \cap N = \{e\}$.

Hence $hn = nh$.

Now for any $g \in G$, there exist $h \in H$ and $n \in N$ such that $g = hn$.

Let $g_1, g_2 \in G$, so there exists $h_1, h_2 \in H$, $n_1, n_2 \in N$, $g_1 g_2 = (h_1 n_1)(h_2 n_2) = h_1 (h_2 n_1) n_2$

$= (h_2 h_1)(n_2 n_1)$ as H is abelian and N is abelian (we use the fact that every group of order 4 is abelian).

So $g_1 g_2 = (h_2 h_1)(n_2 n_1) = h_2 (h_1 n_2) n_1$

$= h_2 (n_2 h_1) n_1 = (h_2 n_2) h_1 n_1$

$= g_2 g_1$

Hence G is abelian.

3.a(ii) Let R and S be commutative rings with unity. If ψ is homomorphism from R onto S and the characteristic of R is nonzero, prove that the characteristic of S divides the characteristic of R.

Soln: Let n be a characteristic of R.

Let s be any element of S. Then there exist $r \in R$ such that $\psi(r) = s$.

But then $0 = \psi(n \cdot r) = \psi((n \cdot 1) \cdot r) = \psi(n \cdot 1)\psi(r) = n \cdot s$

We used here that $\psi(1_R) = 1_S$ as ψ is onto.

Hence for any $s \in S$, $ns = 0$

i.e. characteristic of S divides characteristic of R as characteristic of S is the minimal positive integer satisfying

$$na = 0 \ \forall a \in S.$$

3.(b)(i) Is the union of an arbitrary collection of closed sets closed? Justify your answer.

Sol^n: The union of an arbitrary collection of closed sets may not be closed.

Example: Let $S_n = \left[\frac{1}{n}, 1\right] \ \forall \ n \in N$

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cdots\cdots \cup S_n$$

$$= \{1\} \cup \left\{\tfrac{1}{2}, 1\right\} \cup \left\{\tfrac{1}{3}, 1\right\} \cup \cdots\cdots$$

$$= (0, 1]$$

which is not closed set.

Because, '0' is a limit point.

and $D(S) = \{0\} \not\subseteq S_n$.

So, union of arbitrary collection of closed set need not be closed.

3(b) (ii) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

Sol^n:

Let $\varepsilon = \frac{1}{2}$ and δ be any +ve number such that for $n > \pi/\delta^2$

$$\left| \sqrt{\frac{n\pi}{2}} - \sqrt{\frac{(n+1)\pi}{2}} \right| < \delta$$

$\therefore$ taking $x_1 = \sqrt{\frac{n\pi}{2}}$ and $x_2 = \sqrt{\frac{(n+1)\pi}{2}}$, as any two points of the interval $[0, \infty[$

$$|f(x_1) - f(x_2)| = \left| \sin\frac{n\pi}{2} - \sin\frac{(n+1)\pi}{2} \right| = 1 > \varepsilon$$

$$|x_1 - x_2| < \delta$$

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

Q.3) C) Determine the optimum basic feasible solution to the following transportation problem.

	A	B	C	Available
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Required	4	2	2	

Solution:-

The initial basic feasible solution can be easily obtained by two different methods as follows:

(i) By Lowest Cost Method

	1 (30)		1
2 (90)	1 (45)		3
2 (250)		2 (50)	4
4	2	2	

Cost $z = 1\times30 + 2\times90 + 1\times45 + 2\times250 + 2\times50$

Cost $z = ₹855.$

(ii) By Vogel's Method :

1 (50)	(30)	(220)	1
3 (90)	(45)	(170)	3
(250)	2 (200)	2 (50)	4
4	2	2	

$\sum a_i = \sum b_j = 8$

$\therefore$ The problem is balanced.

No. of allocation $= m+n-1 = 5$

Cost $z = 1\times50 + 3\times90 + 2\times200 + 2\times50$
$= 50 + 270 + 400 + 100$

Cost $z = ₹820$

Since, number of allocation = 5 and the cells allocated in Vogel's method is 4. Hence, the Initial basic solution of LC method is used; i.e

	A	B	C	
I		1(30)		1
II	2(90)	1(45)		3
III	2(250)		2(50)	4
	4	2	2	

cost z = ₹ 855.

Now, to check the optimality of the solution using u-v method.

As, the max. no. of basic cells exists in 2^{nd} & 3^{rd} rows, we can start by putting either $u_2=0$ or $u_3=0$

Let, us put $u_2 = 0$

For basic cells we know

$\Delta_{ij} = u_i + v_j - c_{ij} = 0 \Rightarrow u_i + v_j = c_{ij}$

$u_2 + v_1 = 90 \Rightarrow v_1 = 90$

$u_2 + v_2 = 45 \Rightarrow v_2 = 45$

$u_3 + v_1 = 250 \Rightarrow u_3 = 160 \Rightarrow$

$u_3 + v_3 = 50 \Rightarrow v_3 = -110$

$u_1 + v_2 = 30 \Rightarrow u_1 = -15$

	A	B	C		u_{ij}
I		1(30)		1	-15
II	2(90)	1(45)		3	0
III	2(250)		2(50)	4	160
	4	2	2		
v_{ij}	90	45	-110		

For non-basic cells:

$\Delta_{ij} = c_{ij} - (u_i + v_j)$.

$\Delta_{11} = 50 - (-15+90) = -25$

$\Delta_{13} = 220 - (-15+(-110)) = 345$

$\Delta_{23} = 170 - (0-110) = 280$

$\Delta_{32} = 200 - (205) = -5$

Table of c_{ij}

50		220
		170
	200	

$\Downarrow$

Table of Δ_{ij} $\Rightarrow$

-25		345
		280
	-5	

Here, most negative all are non-negative hence solution is not optimal.

Hence, most negative Δ_{ij} term is (-25) at cell (1,1). So, we need to alot maximum to it.

∴ As this necessiciate the shifting of 1 unit to the cell (1,1) from cell (1,2), as directed by the closed loop in below table:

Hence, cell (1,1) enters the solution and cell (1,2) leaves it, and it become empty or non-basic cell.

$+\theta$	$1-\theta$		1
$2-\theta$	$1+\theta$		3
			4
4	2	2	

Here : $\min[1-\theta, 2-\theta] = 1-\theta$

$1-\theta = 0$

$\boxed{\theta = 1 \text{ unit}}$

Now; the Improved basic table is

	A	B	C	
I	1(50)			1
II	1(90)	2(45)		3
III	2(250)		2(50)	4
	4	2	2	

Cost of $Z = 1\times50 + 1\times90 + 2\times45 + 2\times250 + 2\times50$

$= 50+90+90+500+100$

$\boxed{\text{Cost } Z = ₹830}$

Now, check the optimality of this solution.

Here; colum 1 has maximum basic cells.

Hence; either V_1 or V_2 or $V_3 = 0$

Let us take $V_1 = 0$

$V_1 + U_1 = 50 \Rightarrow U_1 = 50$

$V_1 + U_2 = 90 \Rightarrow U_2 = 90$

$V_1 + U_3 = 250 \Rightarrow U_3 = 250$

$V_2 + U_2 = 45 \Rightarrow V_2 = -45$

$V_3 + U_3 = 50 \Rightarrow V_3 = -200$

Matrix $[C_{ij}]$ for empty / non-basic cell

	30	220
		170
	200	

$\Rightarrow$ Matrix $(u_i + v_j)$ for empty cells

				u_i
		5	-150	50
			-110	90
		205		250
v_j	0	-45	-200	

Hence, by solving

Matrix for $[\Delta_{ij} = C_{ij} - (u_i + v_j)] \Rightarrow$

Δ_{ij}

	25	370
		280
	-5	

Here, all are not non-negative, hence solution is not optimal.

Hence, most negative term of Δ_{ij} is -5 at cell $(3,2)$. So, we need to alot maximum to it.

$\therefore$ Second iteration.

Since, the cell $(3,2)$ enters the solution while cell $(3,1)$ leaves the solution. $(2,2)$ may also selected for leaving. Shown in table.

Here; $2 - \theta = 0$

$\boxed{\theta = 2}$

$1+\theta$	$2-\theta$	
$2-\theta$	$+\theta$	

Hence, improved basic cell table is –

	A	B	C	
I	1(50)			1
II	3(90)	0(45)		3
III		2(200)	2(50)	4
	4	2	2	

Cost $Z = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50$

Cost $Z = 50 + 270 + 400 + 100$

Cost $Z = 820$ = Cost Z of Vogel's IBFS.

Again proceed as earlier to test the next improved solution for optimality.

It has been observed that all cell evaluation are now non-negative. Hence, the solution under test is optimal.

$$\boxed{Z = ₹820}$$

This solution was initially obtained by Vogel's method.

4.(a) → Let R be a commutative ring with unit element; Prove that every maximal ideal of R is a prime ideal.

Soln: Let R be a commutative ring with unit element and let M be a maximal ideal.
Then R/M is a field in particular it is an integral domain.

Hence by theorem (Let R be a commutative ring; an ideal P of R is said to be a prime ideal of R if $ab \in P$, for $a, b \in R$ implies that $a \in P$ or $b \in P$. then P is a prime ideal of R if and only if R/P is an integral domain.)

M is a prime ideal.

4.(c) → Prove that function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \; z \neq 0, \; f(0) = 0$$

is continuous and that Cauchy-Riemann equations are satisfied at the origin yet $f'(z)$ does not exist at $z=0$.

Solⁿ: $u + iv = f(z) = \dfrac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}, \; z \neq 0$

This $\Rightarrow$ $u = \dfrac{x^3-y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$, where $x \neq 0, y \neq 0$.

Step I. To prove that $f(z)$ is continuous everywhere. When $z \neq 0$, u and v both are rational functions of x and y with non-zero denominators. It follows that u, v and therefore also $f(z)$ are continuous functions everywhere except at $z=0$.

To test the continuity of u, v at $z=0$, we change u, v to polar co-ordinates:

$$u = r(\cos^3\theta - \sin^3\theta), \quad v = r(\cos^3\theta + \sin^3\theta).$$

As $z \to 0$, $r \to 0$

Evidently $\lim\limits_{r \to 0} u = 0 = \lim\limits_{r \to 0} v$

This $\Rightarrow$ $\lim\limits_{z \to 0} f(z) = 0$

This $\Rightarrow$ $\lim\limits_{z \to 0} f(z) = 0 = f(0)$.

$\Rightarrow f(z)$ is continuous at $z=0$.

Finally, $f(z)$ is continuous everywhere.

Step II. To show that Cauchy-Riemann equations are satisfied at $z=0$.

$$f(0)=0 \Rightarrow u(0,0)+iv(0,0)=0$$

$$\Rightarrow u(0,0)=0=v(0,0)$$

Recall that $\left(\frac{\partial u}{\partial x}\right)_{z=0} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h}$

$$= \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x-0}{x} = 1.$$

$$\left(\frac{\partial u}{\partial y}\right)_{z=0} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y-0}{y} = -1.$$

$$\left(\frac{\partial v}{\partial x}\right)_{z=0} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{z=0} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y-0}{y} = 1.$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z=0$.

Step III. To prove that $f'(0)$ does not exist.

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{f(z)-0}{z}$$

$$= \lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Let $z\to 0$ along the path $y=x$,

then $$f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{(x^2+x^2)(x+ix)} = \frac{i}{1+i}$$

Let $z\to 0$ along x-axis, then

$$f'(0) = \lim_{x\to 0} \frac{x^3-0+i(x^3+0)}{(x^2+0)(x+i0)} \quad \because y=0$$

$$= 1+i$$

$$f'(0) = \begin{cases} i/(1+i) & \text{along the path } y=x \\ 1+i & \text{along the path } y=0. \end{cases}$$

Since the values of $f'(0)$ are not unique along different paths.

Hence $f'(0)$ does not exist. As a result of which $f(z)$ is not analytic at $z=0$.

4.(d) Using the simplex method solve the problem:

Minimize $z = x_1 + x_2$,

Subject to $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

and $x_1, x_2 \geq 0$.

Sol^n^: $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$

and $\text{Min } z = \frac{31}{13}$.

5(a) (i) Form a partial differential equation by eliminating the function ϕ from $lx+my+nz = \phi(x^2+y^2+z^2)$.

Sol^n Given

$$lx+my+nz = \phi(x^2+y^2+z^2). \quad \text{--- (1)}$$

Differentiating (1) partially with respect to x and y, we get

$$l+n\left(\frac{\partial z}{\partial x}\right) = \phi'(x^2+y^2+z^2)\cdot\left\{2x+2z\left(\frac{\partial z}{\partial x}\right)\right\} \quad \text{--- (2)}$$

and $$m+n\left(\frac{\partial z}{\partial y}\right) = \phi'(x^2+y^2+z^2)\cdot\left\{2y+2z\left(\frac{\partial z}{\partial y}\right)\right\} \quad \text{--- (3)}$$

Dividing (2) by (3), $\phi'(x^2+y^2+z^2)$ is eliminated and we get

$$\frac{l+n\left(\frac{\partial z}{\partial x}\right)}{m+n\left(\frac{\partial z}{\partial y}\right)} = \frac{2\left\{x+z\left(\frac{\partial z}{\partial x}\right)\right\}}{2\left\{y+z\left(\frac{\partial z}{\partial y}\right)\right\}}$$

or $$(ny-mz)\left(\frac{\partial z}{\partial x}\right)+(lz-nx)\left(\frac{\partial z}{\partial y}\right) = mx-ly,$$

which is the required partial differential equation.

5.(a)(ii) Find the integral surface of $x^2p+y^2q+z^2=0$, $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$ which passes through the hyperbola $xy=x+y$, $z=1$.

Solution:-

Given, $x^2p+y^2q+z^2=0$

$$x^2p+y^2q=-z^2 \quad \text{——} (1)$$

Given curve; $xy=x+y$; $z=1$ —— (2)

Here Lagrange's auxillary equations are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2} \quad \text{——} (3)$$

from first and third fractions:

$$\frac{dx}{x^2}+\frac{dz}{z^2}=0$$

$$\Rightarrow -\frac{1}{x}-\frac{1}{z}=-c_1 \Rightarrow \frac{1}{x}+\frac{1}{z}=c_1 \quad \text{——} (4)$$

from 2nd and 3rd fractions:-

$$\frac{dy}{y^2}+\frac{dz}{z^2}=0$$

$$\Rightarrow \frac{1}{y}+\frac{1}{z}=c_2 \quad \text{——} (5)$$

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z}=c_1+c_2$

$$\Rightarrow \frac{x+y}{xy}+\frac{2}{z}=c_1+c_2 \Rightarrow \frac{xy}{xy}+\frac{2}{z}=c_1+c_2$$

(from (2))

$$1+\frac{2}{z}=c_1+c_2 \Rightarrow 1+2=c_1+c_2 \text{ [from (2)]}$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{2}{z}=3 \Rightarrow \boxed{yz+2xy+xz=3xyz}$$ required solution.

5.(b) Find a complete integral of $px+qy=z(1+pq)^{1/2}$.

Sol^n: Given $f(x,y,z,p,q)=px+qy-z(1+pq)^{1/2}=0$ —(1)

Here usual Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

$$\Rightarrow \frac{dp}{p-p(1+pq)^{1/2}}=\frac{dq}{q-q(1+pq)^{1/2}}=\cdots \text{ so that } \frac{dp}{p}=\frac{dq}{q} \quad —(2)$$

$$\Rightarrow \log p=\log a+\log q \Rightarrow p=aq \quad —(3)$$

Using (3),(1) $\Rightarrow q(ax+y)=z(1+aq^2)^{1/2} \Rightarrow q^2[(ax+y)^2-az^2]$

$$\therefore q=\frac{z}{[(ax+y)^2-az^2]^{1/2}} \text{ and } p=aq=\frac{az}{[(ax+y)^2-az^2]^{1/2}}$$

Substituting these values in $dz=pdx+qdy$, we have

$$dz=\frac{z(adx+dy)}{\sqrt{\{(ax+y)^2-az^2\}}} \Rightarrow \frac{dz}{z}=\frac{adx+dy}{\sqrt{[(ax+y)^2-az^2]}}$$

Let $ax+y=\sqrt{a}\,u$ so that $adx+dy=\sqrt{a}\,du$.

$$\therefore \frac{dz}{z}=\frac{\sqrt{a}\,du}{\sqrt{(au^2-az^2)}} \Rightarrow \frac{du}{dz}=\frac{\sqrt{(u^2-z^2)}}{z}=\sqrt{\left\{\left(\frac{u}{z}\right)^2-1\right\}},$$

which is linear homogeneous equation. To solve it, we put

$$\frac{u}{z}=v \Rightarrow u=vz \text{ so that } \frac{du}{dz}=v+z\frac{dv}{dz}$$

$$\therefore v+z\frac{dv}{dz}=\sqrt{(v^2-1)} \Rightarrow \frac{dz}{z}=\frac{dv}{\sqrt{(v^2-1)}-v}$$

$$\Rightarrow \left(\frac{1}{z}\right)dz=-\left[\sqrt{(v^2-1)}+v\right]dv, \text{ on rationalization.}$$

Integrating, $\log z=-\left[\frac{v}{2}\sqrt{(v^2-1)}-\frac{1}{2}\log\{v+\sqrt{(v^2-1)}\}\right]-\frac{v^2}{2}+b,$

where, $v=u/z=(ax+y)/z\sqrt{a}$.

5(c) The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, Calculate the time required for the bacteria concentration to be 0.5.

Sol's. Given that $C = 4e^{-2t} + e^{-0.1t}$.

Now the bacteria concentration to be 0.5 $C = 0.5$

ie, $0.5 = 4e^{-2t} + e^{-0.1t}$

i.e. $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$

Now we have to find the time required for the bacteria concentration to be 0.5.

$\therefore$ we want to find the roots of the equation $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$.

By Newton Raphson method,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$f(t) = 4e^{-2t} + e^{-0.1t} - 0.5$$

$$f'(t) = -8e^{-2t} - 0.1e^{-0.1t}$$

Let the initial time be $t_0 = 0$.

Then $t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$

$= 0 - \frac{4.5}{-8.1}$

$= 0.5555$

$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = 0.5555 - \frac{1.7628}{-2.72843}$

$= 1.20168$

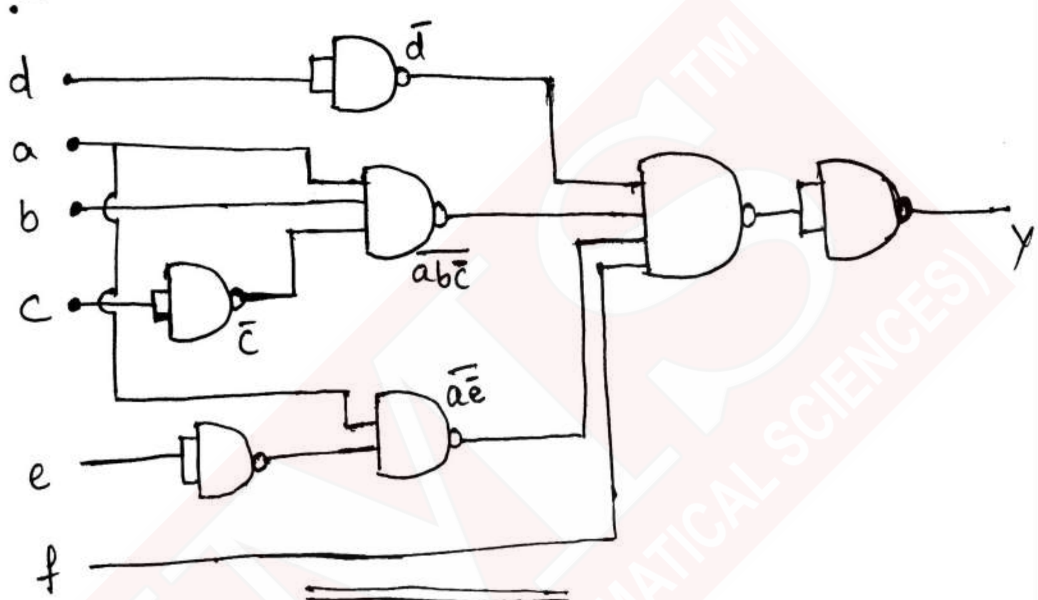
Continue in this way we get $t = 6.932$.

5(d) (i) Realize the following expression by using NAND gates only.

$$g = (\bar{a} + \bar{b} + c)\bar{d}\,(\bar{a} + e)f$$

where $\bar{x}$ denotes the complement of x

Sol^n:-



$$Y = \overline{\overline{(\overline{ab\bar{c}})\,\bar{d}\,(\overline{a\bar{e}})f}}$$

$$= \overline{\overline{(\bar{a} + \bar{b} + \bar{\bar{c}})\,\bar{d}\,(\bar{a} + \bar{\bar{e}})f}}$$

$$= \overline{(\overline{\bar{a} + \bar{b} + c}) + \bar{\bar{d}} + (\overline{\bar{a} + e}) + \bar{f}}$$

$$= (\bar{a} + \bar{b} + c)\,\bar{d}\,(\bar{a} + e)f$$

5.d(ii) Find the decimal equivalent of $(357.32)_8$

Sol^n: $(357.32)_8 = 3\times8^2 + 5\times8^1 + 7\times8^0 + 3\times8^{-1} + 2\times8^{-2}$

$= 192 + 40 + 7 + 0.375 + 0.03125$

$= 239.40625$

$\therefore\ (357.32)_8 = (239.40625)_{10}$

5.(e) The velocity potential function ϕ is given by $\phi = -(xy^3/3) - x^2 + (x^3y/3) + y^2$. Determine the velocity components in x and y directions and show that ϕ represents a possible case of flow.

Sol^n: Here $u = -\frac{\partial \phi}{\partial x} = (y^3/3) + 2x - x^2y$,

$$v = \frac{-\partial \phi}{\partial y} = xy^2 - (x^3/3) - 2y.$$

$$\therefore \frac{\partial u}{\partial x} = 2 - 2xy$$

and $\frac{\partial v}{\partial y} = 2xy - 2$.

Hence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$,

Showing that the continuity equation is satisfied. So ϕ represents a possible case of flow.

6(a) Prove that for the equation $z+px+qy-1-pqx^2y^2=0$ the characteristic strips are given by

$x=(B+Ce^{-t})^{-1}$, $y=(A+De^{-t})^{-1}$, $z=E-(AC+BD)e^{-t}$

$p=A(B+Ce^{-t})^2$, $q=B(A+De^{-t})^2$.

where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z=0$, $x=y$.

Sol: Here $f=z+px+qy-1-pqx^2y^2=0$ —— (1)

The characteristic equations of the given partial differential equation (1) are given by

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-qx^2y^2 \quad \text{—— (2)}$$

$$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-px^2y^2 \quad \text{—— (3)}$$

$$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-qx^2y^2)+q(y-px^2y^2)$$
$$=px+qy-2pqx^2y^2 \quad \text{—— (4)}$$

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-(p-2pqxy^2)-p\cdot 1=-2p(1-qxy^2) \quad \text{—— (5)}$$

& $$\frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-(q-2pqx^2y)-q\cdot 1=-2q(1-px^2y) \quad \text{—— (6)}$$

from (2) & (5), we have

$$\frac{1}{x}\frac{dx}{dt}=-\frac{1}{2p}\frac{dp}{dt} \Rightarrow \frac{2}{x}\frac{dx}{dt}+\frac{1}{p}\frac{dp}{dt}=0$$
$$\Rightarrow 2\log x+\log p=\log A$$
$$\Rightarrow x^2p=A \quad \text{—— (7)}$$

Again from (3) & (6), we have

$$\frac{1}{y}\frac{dy}{dt}=-\frac{1}{2q}\frac{dq}{dt} \Rightarrow \frac{2}{y}\frac{dy}{dt}+\frac{1}{q}\frac{dq}{dt}=0$$
$$\Rightarrow 2\log y+\log q=B$$
$$\Rightarrow y^2q=B \quad \text{—— (8)}$$

from ② & ⑧, we have $\frac{dx}{dt} = x - Bx^2$

$$\Rightarrow \frac{1}{x^2}\frac{dx}{dt} = \frac{1}{x} - B$$

putting $\frac{1}{x} = v \Rightarrow -\frac{1}{x^2}\frac{dx}{dt} = \frac{dv}{dt}$

$$\Rightarrow \frac{dv}{dt} + v = B.$$

- which LDE with

$$If = e^{\int 1 dt} = e^t.$$

$$\therefore ve^t = c + \int Be^t dt = c + Be^t.$$

$$\Rightarrow \frac{1}{x}e^t = c + Be^t.$$

$$\Rightarrow x = (B + Ce^{-t})^{-1} \quad \text{———} ⑨$$

Again from ③ & ⑦,

we have $\frac{dy}{dt} = y - Ay^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dt} = \frac{1}{y} - A$

putting $\frac{1}{y} = u \Rightarrow -\frac{1}{y^2}\frac{dy}{dt} = \frac{du}{dt}$

$$\Rightarrow \frac{du}{dt} + u = A.$$

which is LDE

with $If = e^{\int 1 dt} = e^t$,

$$\therefore ue^t = D + \int Ae^t dt = D + Ae^t$$

$$\Rightarrow \frac{1}{y}e^t = D + Ae^t$$

$$\Rightarrow y = (A + De^{-t})^{-1}$$

Using ⑦, ⑧, ⑨ & ⑩ from ④, we have

$$\frac{dz}{dt} = \frac{A}{x} + \frac{B}{y} - 2AB$$

$$= A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$$

$$= (AC + BD)e^{-t}$$

Integrating, we get $z = E - (AC + BD)e^{-t}$.

Thus, the characteristic strips for equation ① are

(39)

given by. $x=(B+Ce^{-t})^{-1}$, $y=(A+De^{-t})^{-1}$, $z=E-(AC+BD)e^{-t}$

$p=A(B+Ce^{-t})^2$, $q=B(A+De^{-t})^2$. ——— (11)

The parametric equation of the given line can be taken as

$x=f_1(\lambda)=\lambda$, $y=f_2(\lambda)=\lambda$, $z=0$, λ being a parameter.

Initial values of x,y,z are $x=x_0=\lambda$, $y=y_0=\lambda$, $z=z_0=0$ when $t=0$.

The corresponding initial values of p_0 and q_0 are given by

$f_3'(\lambda)=p_0 f_1'(\lambda)+q_0 f_2'(\lambda)$

$\Rightarrow 0=p_0(1)+q_0(1) \Rightarrow p_0+q_0=0 \Rightarrow q_0=-p_0$

and $z_0+p_0x_0+q_0y_0-1-p_0q_0x_0^2y_0^2=0$ (from (1))

$\Rightarrow 0+\lambda(p_0+q_0)-1-p_0q_0\lambda^4=0$

$\Rightarrow p_0q_0\lambda^4=-1. \Rightarrow p_0^2=\frac{1}{\lambda^4} \Rightarrow p_0=\frac{1}{\lambda^2}$, $q_0=-\frac{1}{\lambda^2}$

Using initial values in characteristic strip given by (11), we have

$x_0=\lambda=(B+C)^{-1}$, $y_0=(A+D)^{-1} \Rightarrow B+C=\frac{1}{\lambda}$, $A+D=\frac{1}{\lambda}$.

$\therefore p_0=\frac{1}{\lambda^2}=A(B+C)^2=A\left(\frac{1}{\lambda}\right)^2 \Rightarrow A=1$.

$q_0=-\frac{1}{\lambda^2}=B(A+D)^2=B\left(\frac{1}{\lambda}\right)^2 \Rightarrow B=-1$.

$\therefore C=\frac{1}{\lambda}+1$, $D=\frac{1}{\lambda}-1$.

$z_0=0=E-(AC+BD) \Rightarrow E=1\left(\frac{1}{\lambda}+1\right)-\left(\frac{1}{\lambda}-1\right)=2$

$\therefore x=\left\{-1+\left(\frac{1}{\lambda}+1\right)e^{-t}\right\}^{-1}$ $y=\left\{1+\left(\frac{1}{\lambda}-1\right)e^{-t}\right\}^{-1}$, $z=2(1-e^{-t})$

The required integral surface is obtained by eliminating parameters λ and t from x,y and z

Here $\frac{1}{x}+1=\left(\frac{1}{\lambda}+1\right)e^{-t}$, $\frac{1}{y}-1=\left(\frac{1}{\lambda}-1\right)e^{-t}$. $\Rightarrow \frac{1}{x}-\frac{1}{y}+2=2e^{-t}$.

$\therefore z=2-\left(\frac{1}{x}-\frac{1}{y}+2\right)=-\frac{1}{x}+\frac{1}{y} \Rightarrow \boxed{xyz=x-y}$

which is the required integral surface.

6.(b) → Solve the equations:

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss - Seidal iteration method.

Solution:

Rewriting the given equations as,

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \quad \text{—(i)}$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \quad \text{—(ii)}$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \quad \text{—(iii)}$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3. \quad \text{—(iv)}$$

First iteration

Putting $x_2=0$, $x_3=0$, $x_4=0$ in (i) we get $x_1 = 0.3$

Putting $x_1=0.3, x_3=0, x_4=0$ in (ii) we get $x_2=1.56$

Putting $x_1=0.3$, $x_2=1.56$, $x_4=0$ in (iii) we get $x_3=2.886$

Putting $x_1=0.3$, $x_2=1.56$, $x_3=2.886$ in (iv) we get $x_4=-0.1368$

Second iteration

Putting $x_2=1.56$, $x_3=2.886$, $x_4=-0.1368$, in (i) $\Rightarrow x_1 = 0.8869$

Putting $x_1=0.8869$, $x_3=2.886$, $x_4=-0.1368$, in (ii) $\Rightarrow x_2 = 1.9523$

Putting $x_1=0.8869$, $x_2=1.9523$, $x_4=-0.1368$; in (iii) $\Rightarrow x_3 = 2.9566$

Putting $x_1=0.8869$, $x_2=1.9523$, $x_3=2.9566$; in (iv) $\Rightarrow x_4=-0.0248$

third iteration

Putting ~~$x_2=0.8869$~~

$x_2=1.9523$, $x_3=2.9566$, $x_4=-0.0248$ in (i) $\Rightarrow x_1 = 0.9836$

$x_1=0.9836$, $x_3=2.9566$, $x_4=-0.0248$ in (ii) $\Rightarrow x_2 = 1.9899$

$x_1=0.9836$, $x_2=1.9899$, $x_4=-0.0248$ in (iii) $\Rightarrow x_3 = 2.9924$

$x_1=0.9836$, $x_2=1.9899$, $x_3=2.9924$ in (iv) $\Rightarrow x_4=-0.0042$

Fourth iteration:

Putting

$x_2 = 1.9899,\ x_3 = 2.9924,\ x_4 = -0.0042$ in (i) $\Rightarrow x_1 = 0.9968$

$x_1 = 0.9968,\ x_3 = 2.9924,\ x_4 = -0.0042$ in (ii) $\Rightarrow x_2 = 1.9982$

$x_1 = 0.9968,\ x_2 = 1.9982,\ x_4 = -0.0042$ in (iii) $\Rightarrow x_3 = 2.9987$

$x_1 = 0.9968,\ x_2 = 1.9982,\ x_3 = 2.9987$ in (iv) $\Rightarrow x_4 = -0.0008$

fifth iteration

Putting

$x_2 = 1.9982,\ x_3 = 2.9987,\ x_4 = -0.0008$ in (i) $\Rightarrow x_1 = 0.9994$

$x_1 = 0.9994,\ x_3 = 2.9987,\ x_4 = -0.008$ in (ii) $\Rightarrow x_2 = 1.9997$

$x_1 = 0.9994,\ x_2 = 1.9997,\ x_4 = -0.008$ in (iii) $\Rightarrow x_3 = 2.9997$

$x_1 = 0.9994,\ x_2 = 1.9997,\ x_3 = 2.9997$ in (iv) $\Rightarrow x_4 = -0.0001$

Sixth iteration

Putting

$x_2 = 1.9997,\ x_3 = 2.9997,\ x_4 = -0.0001$ in (i) $\Rightarrow x_1 = 0.9999$

$x_1 = 0.9999,\ x_3 = 2.9997,\ x_4 = -0.0001$ in (ii) $\Rightarrow x_2 = 1.9999$

$x_1 = 0.9999,\ x_2 = 1.9999,\ x_4 = -0.0001$ in (iii) $\Rightarrow x_3 = 2.9999$

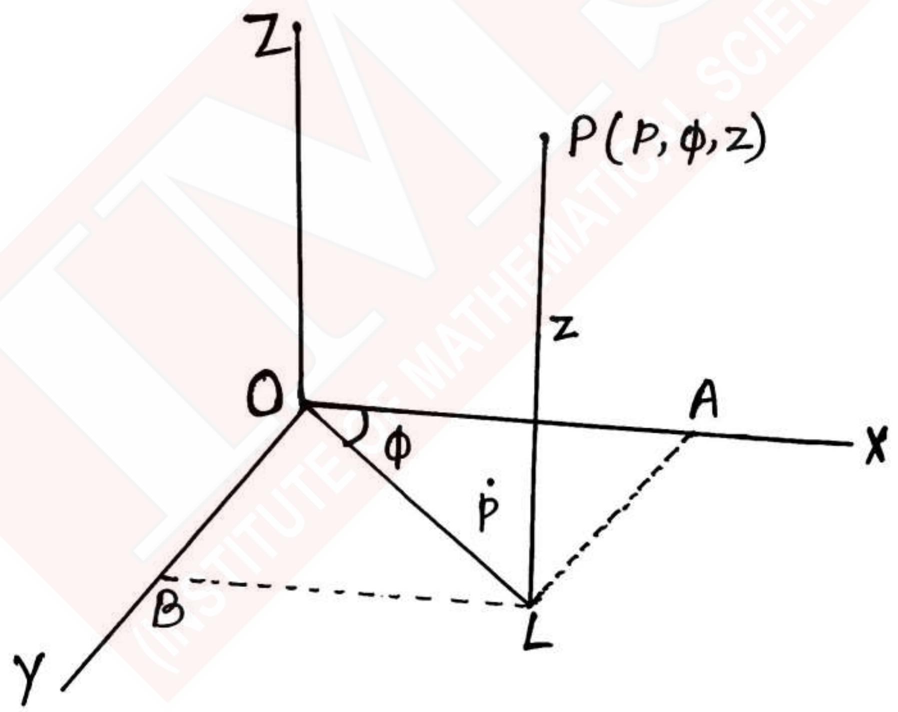
$x_1 = 0.9999,\ x_2 = 1.9999,\ x_3 = 2.9999$ in (IV) $\Rightarrow x_4 = -0.0001$

Hence the solution for the given equations is

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 0 \end{aligned}}$$

6.(C) A particle of mass m moves in a conservative forces fields. Find (i) the Lagrangian function and (ii) the equation of motion in cylindrical co-ordinates (ρ, ϕ, z).

Sol: Let P be the position of the particle of mass m whose cylindrical coordinates reffered to axes OX, OY, OZ are (ρ, ϕ, z).



$\therefore$ If (x, y, z) are its cartesian coordinates, then

$$x = OA = \rho\cos\phi,$$

$$y = OB = \rho\sin\phi, \quad z = z.$$

If $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along OX, OY, OZ respectively, then

$$\vec{OP} = \vec{r} = p\cos\phi\,\vec{i} + p\sin\phi\,\vec{j} + z\vec{k}$$

If $\hat{p}_1$ and $\hat{\phi}_1$ are the unit vectors in the directions of p and ϕ increasing respectively, then

$$\hat{p}_1 = \frac{\partial\vec{r}}{\partial p} \Big/ \left|\frac{\partial\vec{r}}{\partial p}\right|$$

$$= \frac{\cos\phi\,\vec{i} + \sin\phi\,\vec{j}}{\sqrt{(\cos^2\phi + \sin^2\phi)}}$$

$$= \cos\phi\,\vec{i} + \sin\phi\,\vec{j}$$

$$\hat{\phi}_1 = \frac{\partial\vec{r}}{\partial\phi} \Big/ \left|\frac{\partial\vec{r}}{\partial\phi}\right|$$

$$= \frac{-p\sin\phi\,\vec{i} + p\cos\phi\,\vec{j}}{\sqrt{(p^2\sin^2\phi + p^2\cos^2\phi)}}$$

$$= -\sin\phi\,\vec{i} + \cos\phi\,\vec{j}.$$

Now $v = \dot{\vec{r}} = (\dot{p}\cos\phi - p\sin\phi\,\dot{\phi})\vec{i} + (\dot{p}\sin\phi + p\cos\phi\,\dot{\phi})\vec{j} + \dot{z}\vec{k}$

$$= \dot{p}(\cos\phi\vec{i} + \sin\phi\vec{j}) + p\dot{\phi}(-\sin\phi\vec{i} + \cos\phi\vec{j}) + \dot{z}\vec{K}$$

$$= (\dot{p})\hat{P}_1 + (p\dot{\phi})\hat{\phi}_1 + \dot{z}\vec{K}$$

$$\therefore v^2 = \dot{p}^2 + (p\dot{\phi})^2 + \dot{z}^2$$

Total K.E.,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{p}^2 + p^2\dot{\phi}^2 + \dot{z}^2)$$

Let $V = V(p, \phi, z)$ be the potential function.

$\therefore$ (i) Lagrangian function,

$$L = T - V$$

i.e. $$L = \frac{1}{2}m(\dot{p}^2 + p^2\dot{\phi}^2 + \dot{z}^2) - V(p,\phi,z).$$

(ii) Lagrange's p equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0$$

or $$\frac{d}{dt}(m\dot{p}) - \left(mp\dot{\phi}^2 - \frac{\partial V}{\partial p}\right) = 0$$

i.e. $$m\ddot{p} - mp\dot{\phi}^2 = -\frac{\partial V}{\partial p} \quad \text{——— (1)}$$

Lagrange's ϕ equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

or $$\frac{d}{dt}(m\rho^2\dot{\phi}) - \left(-\frac{\partial V}{\partial \phi}\right) = 0$$

or $$\frac{d}{dt}(m\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi} \quad \text{——— (2)}$$

and Lagrange's z equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

or $$\frac{d}{dt}(m\dot{z}) - \left(-\frac{\partial V}{\partial z}\right) = 0$$

or $$m\ddot{z} = -\frac{\partial V}{\partial z} \quad \text{——— (3)}$$

7.(a) Reduce to canonical form and solve
$$r-2s+t+p-q=e^{x}(2y-3)-e^{y}.$$

Soln: Given $r-2s+t+p-q-e^{x}(2y-3)-e^{y}=0$ —— (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$,
here $R=1$,
$$S=-2\ T=1 \quad \text{So that } S^2-4RT=0,$$
Showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$

reduces to $\lambda^2-2\lambda+1=0$ or $(\lambda-1)^2=0$

So that $\lambda=1,1$ (equal roots)

So the corresponding characteristics equation is
$dy/dx+1=0$ Integrating it, $x+y=c_1$

Choose $u=x+y$ and $v=y$ —— (2)

where we have chosen $v=y$ in such a manner that u and v are independent functions as verified below
$$J=\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=1\neq 0.$$

Now, $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}$, using (2) —— (3)

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v} \quad \text{—— (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)=\frac{\partial^2 z}{\partial u^2} \text{ by (3)} \quad \text{—— (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right) = \left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right), \text{ using } ④$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \quad ⑥$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \quad ⑦$$

using ②,③,④,⑤,⑥ and ⑦ in ①, we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}$$
$$-\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^{v}$$

or $\partial^2 z/\partial v^2 - \partial z/\partial v = e^{u-v}(2v-3) - e^{v}$ ⑧

which is the required canonical form of ①

Let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$.

Then ⑧ can be re-written as

$$D'(D'-1)z = e^{u-v}(2v-3) - e^{v} \quad ⑨$$

$$C.F. = \phi(u) + e^{v}\psi(u) = \phi(x+y) + e^{y}\psi(x+y), \text{ by } ②$$

<u>P.I. corresponding to $e^{u-v}(2v-3)$</u>

$$= \frac{1}{D'(D'-1)} e^{u+(-1)v}(2v-3)$$

$$= e^{u+(-1)v}\frac{1}{(D'-1)(D'-1-1)}(2v-3)$$

$$= (1/2)e^{u-v}(1-D')^{-1}(1-D'/2)^{-1}(2v-3)$$

$$= (1/2)\, e^{u-v}\left(1+D'+\cdots\right)\left(1+D'/2+\cdots\right)(2v-3)$$

$$= (1/2)\, e^{u-v}\left(1+3D'/2+\cdots\right)(2v-3)$$

$$= (1/2)\, e^{u-v}(2v-3+3)$$

$$= ve^{u-v} = ye^{x+y-y} = ye^{x}, \text{ using } ②.$$

P.I. corresponding to $-e^{v}$

$$= \frac{1}{D'(D'-1)}(-e^{v}) = \frac{1}{D'-1}\,\frac{1}{D'}e^{v} = -\frac{1}{D'-1}e^{v}.1$$

$$= -e^{v}\frac{1}{D'+1-1}\cdot 1 = -e^{v}(1/D')1$$

$$= -e^{v}v = -e^{y}y, \text{ using } ②$$

Hence the required general solution is given by

$$y = \phi(x+y) + e^{y}\,\psi(x+y) + ye^{x} - ye^{y}$$

$$y = \phi(x+y) + e^{y}\,\psi(x+y) + ye^{x} - (x+y)e^{y} + xe^{y}$$

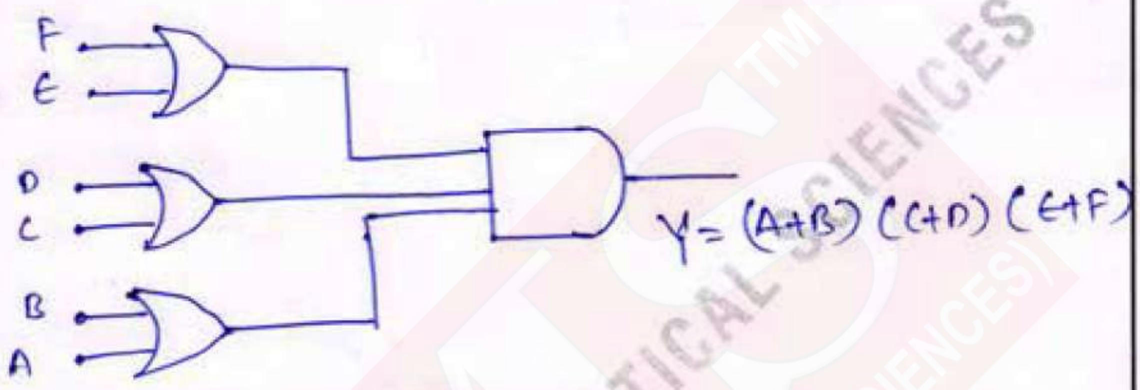
or $y = \phi(x+y) + e^{y}\{\phi(x+y) + (x+y)\} + ye^{x} + xe^{y}$

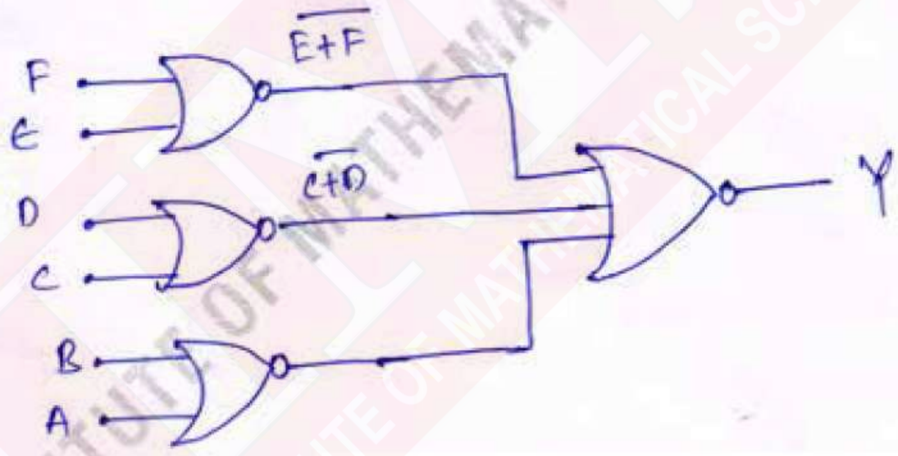
or $y = \phi(x+y) + e^{y}F(x+y) + ye^{x} + xe^{y}$,

where ϕ and F are arbitrary functions and $F(x+y) = \phi(x+y) + x + y$.

7(b) (i) Draw AND-OR logic circuit for the expression $(A+B)(C+D)(E+F)$.

Sol^n: (a) Shows AND-OR logic circuit





—— Ans.

7.(b) (i) Use Runge-Kutta method of fourth order to numerically solve the initial value problem.

$$10\frac{dy}{dx} = x^2+y^2, \; y(0)=1$$

and find y in the interval $0 \le x \le 0.4$ taking $h=0.1$

Sol^n:- Given that $\frac{dy}{dx} = \frac{x^2+y^2}{10}$

To find $y(0.1)$:-

$h=0.1$; $x_0=0$, $y_0=1$

$K_1 = hf(x_0,y_0) = (0.1)\frac{1}{10} = 0.01$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right) = 0.01012.$

$K_3 = h\left[f\left(x_0+h/2, y_0+\frac{K_2}{2}\right)\right] = 0.01012$

$K_4 = hf(x_0+h, y_0+K_3) = 0.0103$

$y(0.1) = y_0 + \frac{K_1+2K_2+2K_3+K_4}{6}$

$= 1.0101$

To find $y(0.2)$:

$x_1 = x_0+h = 0+0.1 = 0.1$, $y_1 = 1.0101$, $h=0.1$

$K_1 = hf(x_1,y_1) = 0.1f(0.1, 1.0101) = 0.0103$

$K_2 = 0.01053$

$K_3 = 0.01053$

$K_4 = 0.0108$

$y(0.2) = 1.0101 + 1/6\,(K_1+2K_2+2K_3+K_4)$

$= 1.0206.$

To find $y(0.3)$.

$x_2 = 0.2$, $y_2 = 1.0206$, $h = 0.1$

$k_1 = hf(x_2, y_2) = 0.0108$

$k_2 = hf(x_2 + h/2, y_2 + \frac{k_1}{2}) = 0.0111$.

$k_3 = hf(x_2 + \frac{h_2}{2}, y_2 + \frac{k_2}{2}) = 0.0111$

$k_4 = hf(x_2 + h, y_2 + k_3) = 0.0115$.

$y(0.3) = 1.0317$.

To find $y(0.4)$

$x_3 = 0.3$, $y_3 = 1.0317$, $h = 0.1$

$k_1 = 0.0115$

$k_2 = 0.01198$

$k_3 = 0.01199$.

$k_4 = 0.01249$.

$y(0.4) = 1.04369$

7.(c) A solid homogeneous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane, show that the smaller sphere will make complete revolution if, when it is in its lowest position, the pressure on it is greater than 34/7 times its own weight.

Sol^n: Let O be the centre and a the radius of fixed hollow sphere. Let C be the centre, M the mass and b the radius of the sphere rolling inside this fixed sphere. A time t let the line CB fixed in moving sphere make an angle ϕ to the vertical then let the line OC joining centres make an angle θ to the vertical where initially B coincided with A.

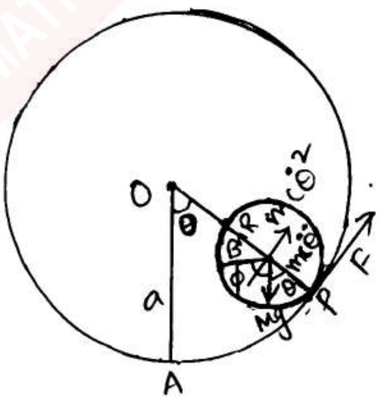
Since there is no slipping.

$\therefore$ Arc AP = Arc PB

$\Rightarrow a\theta = b(\phi+\theta)$

$\therefore b\dot{\phi} = (a-b)\dot{\theta} = c\dot{\theta}$ —— ①

where $c = a-b$.



Since C describes circle of radius OC = $a-b=c$ (say), about C,

$\therefore$ the equations of motion are

$$Mc\dot{\theta}^2 = R - Mg\cos\theta \quad \text{—— ②}$$

and $$Mc\ddot{\theta} = F - Mg\sin\theta \quad \text{—— ③}$$

The coordinates (x_c, y_c) of C referred to the horizontal and vertical lines through O as axes are given by

$x_c = c\sin\theta$ and $y_c = c\cos\theta$

$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$,

At time t, K.E of the moving sphere

$$= \frac{1}{2}Mk^2\dot{\phi}^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}Mc^2\dot{\theta}^2$$

$$= \frac{1}{5}Mc^2\dot{\theta}^2 + \frac{1}{2}Mc^2\dot{\theta}^2 = \frac{7}{10}Mc^2\dot{\theta}^2 \quad [\text{from } (1)]$$

The coordinates (x_c, y_c) of C referred to the horizontal and vertical lines through O as axes are given by

$x_c = c\sin\theta$ and $y_c = c\cos\theta$

$$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$$

At time t, K.E of the moving sphere

$$= \frac{1}{2}Mk^2\dot{\phi}^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}Mc^2\dot{\theta}^2$$

$$= \frac{1}{5}Mc^2\dot{\theta}^2 + \frac{1}{2}Mc^2\dot{\theta}^2 = \frac{7}{10}Mc^2\dot{\theta}^2 \quad [\text{from } (1)]$$

If ω is the initial angular velocity i.e. $\dot{\theta} = \omega$, then the initial K.E at $t=0$ is $\frac{7}{10}Mc^2\omega^2$.

$\therefore$ The energy equation gives

Change in K.E = workdone by the gravity

$$\frac{7}{10}Mc^2\dot{\theta}^2 - \frac{7}{10}Mc^2\omega^2 = -Mg(c - c\cos\theta) \Rightarrow c\dot{\theta}^2 = c\omega^2 - \frac{10}{7}g(1-\cos\theta) \quad \text{—— (4)}$$

$$\therefore \text{from } (2),\ R = Mg\cos\theta + M\left[c\omega^2 - \frac{10}{7}g(1-\cos\theta)\right] \quad \text{—— (5)}$$

The sphere will make complete revolution if $R = 0$ when $\theta = \pi$.

$$\therefore \text{from } (5) \quad 0 = Mg\cos\pi + M\left[c\omega^2 - \frac{10}{7}g(1-\cos\pi)\right]$$

$$\Rightarrow c\omega^2 = g\,\frac{20}{7}g \Rightarrow \omega^2 = \frac{27g}{7c}.$$

$\therefore \omega = \sqrt{\left(\frac{27g}{7c}\right)}$ is the least velocity of ω to make the complete revolution.

Now at the lowest position when $\theta = 0$, $\dot{\theta} = \omega = \sqrt{\left(\frac{27g}{7c}\right)}$, then from (5),

$$R = Mg + M\left[c.\frac{27g}{7c} - \frac{10}{7}g(1-1)\right] = \frac{34}{7}Mg.$$

$\therefore$ when the sphere makes complete revolution, then reaction at the lowest position is greater than $\frac{34}{7}$ times its own weight.

8(a) (i) An insulated rod of length 'l' has its ends A and B maintained at $0°c$ and $100°c$ respectively until steady state conditions prevail. If B is suddenly reduced to $0°c$ and maintained at $0°c$, find the temperature at a distance x from A at time t.

(ii) find also the temperature if the change consists of raising the temperature of A to $20°c$ and reducing that of B to $80°c$.

Sol^n part(a) the Equation for conduction of heat is

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial x^2}\right) \longrightarrow (1)$$

prior to temperature, change at the end B when $t=0$ the heat flow was independent of time (steady state condition, for which $\frac{\partial u}{\partial t}=0$). when u depends only on x (1) reduces to

$$\frac{d^2u}{dx^2}=0 \text{ ; so that } u = c_1x+c_2 \longrightarrow (2)$$

Given $u=0$ for $x=0$ and $u=100$ for $x=l$,

$\therefore$ (2) given $0=c_2$ and $100 = lc_1+c_2$

$\therefore$ (2) becomes $u = (100/l)x$

Hence the boundary conditions for the subsequent flow are

$$u(0,t)=0 \quad \text{for all } t \longrightarrow (3)$$

$$u(l,t)=0 \quad \text{for all } t \longrightarrow (4)$$

and the initial condition is

$$u(x,0) = (100/l)\, x \longrightarrow (5)$$

Now, proceeding as Ex:2 the most general solution of (1) satisfying (3) and (4) is

$$u(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{l}\right) e^{-kn^2\pi^2 t/l^2} \longrightarrow (6)$$

Substituting $t=0$ in (6) and using (5) we get

$$\frac{100x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

which is fourier series. So E_n is given by

$$E_n = \frac{2}{l}\int_0^l \frac{100x}{l} \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - \left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l$$

$$\Rightarrow \frac{200}{l^2}\left(\frac{-l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

$\therefore$ from (6) the required solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l}\, e^{-kn^2\pi^2 t/l^2}.$$

(ii) Here the initial condition is (5) as before. The new boundary conditions are

$$u(0,t) = 20 \text{ for all } t \rightarrow (7)$$

and $$u(l,t) = 80 \text{ for all } t \rightarrow (8)$$

Note that in part (a), the two boundary values were both zero and the required solution was easily obtained. In the present part (b), the two boundary values are non-zero, so we modify the procedure as follows:

we split the temperature function $u(x,t)$ into two parts as

$$u(x,t) = u_1(x) + u_2(x,t) \rightarrow (9)$$

where $u_1(x)$ is a solution of (1) involving x only and satisfying (7) and (8); $u_2(x,t)$ is then a function defined by (9). Hence $u_1(x)$ is a steady state solution of the form (2) and $u_2(x,t)$ may be treated as a transient part of the solution, which decreases with increase of t.

Since $u_1(x) = 20$ for $x=0$ and $u_1(x) = 80$ for $x = l$, (2) given

$$u_1(x) = 20 + (60/l)x \rightarrow (10)$$

putting $x=0$ in (9) and using (7) we get.

$$u_2(0,t) = u(0,t) - u_1(0) = 20 - 20 = 0 \rightarrow (11)$$

∴ Next, putting $x = l$ in (9) and using (8) we get

$$u_2(l,t) = u(l,t) - u_1(l) = 80 - 80 = 0. \longrightarrow (12)$$

Again $u_2(x,0) = u(x,0) - u_1(x)$

$$= (100/l)x - \left[(60/l)x + 20\right]$$

So $u_2(x,0) = (40/l)x - 20 \longrightarrow (13)$

Hence the boundary conditions and initial condition to the transient solution $u_2(x,t)$ are given by (11), (12) and (13).

So we now solve

$$\partial u_2/\partial t = K\left(\partial^2 u_2/\partial x^2\right)$$

Subject to boundary conditions (11) and (12) and initial conditions (13) as in Example 2 and obtain

$$u_2(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} e^{-Kn^2\pi^2 t/l^2} \longrightarrow (14)$$

when E_n is given by

$$E_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{40}{n\pi}(1+\cos n\pi) = -\frac{40}{n\pi}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0, & \text{when 'n' is odd} \\ -80/n\pi, & \text{when n is even.} \end{cases}$$

Taking $n = 2m$, $E_n = -40/m\pi$

from (14) we have

$$u_2(x,t) = \frac{-40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l} e^{-4Km^2\pi^2 t/l^2} \hookrightarrow (15)$$

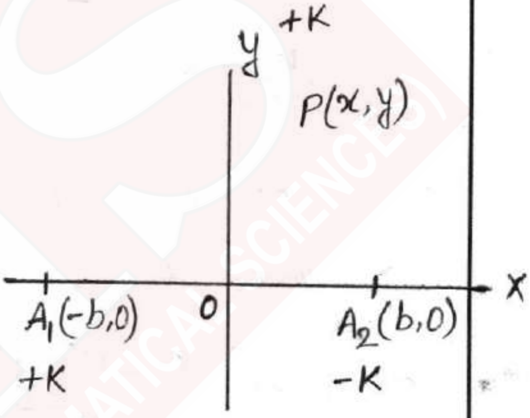
$$u(x,t) = 20 + \frac{60x}{l} - \frac{40}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi x}{l} e^{-4Km^2\pi^2 t/l^2}$$ Ans

8.(C) An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance $2b$. Show that the paths of the fluid particles relative to the vortices can be represented by the equation

$$\log\left\{\frac{(x-b)^2+y^2}{(x+b)^2+y^2}\right\}+\frac{x}{b}=c,$$

O is the middle point of the join which is taken as x-axis.

Sol^n: Let the vortices of strengths $+K$, $-K$ be at $A_1(-b,0)$, $A_2(b,0)$ such that A_1A_2 is along x-axis. The complex potential due to this vortex pair at $P(x,y)$ is



$$W=\frac{iK}{2\pi}\log(z+b)-\frac{iK}{2\pi}\log(z-b)$$

or $\phi+i\psi=\frac{iK}{2\pi}\left[\log(x+b+iy)-\log(x-b+iy)\right]$.

Equating imaginary parts from both sides,

$$\psi=\frac{K}{4\pi}\left[\log\{(x+b)^2+y^2\}-\log\{(x-b)^2+y^2\}\right] \quad \text{—— (1)}$$

The vortex pair will move along a line parallel to y-axis with velocity

$$\frac{K}{2\pi(A_1A_2)}=\frac{K}{2\pi(2b)}=\frac{K}{4\pi b}.$$

To reduce the system to rest, we have to

Superimpose a velocity $(-K/4\pi b)$ parallel to y-axis. If ψ' be the stream function due to this disturbance, then

$$\frac{-K}{4\pi b} = v = -\frac{\partial \phi'}{\partial y} = \frac{\partial \psi'}{\partial x}.$$

$$\therefore \; \psi' = -\frac{Kx}{4\pi b}.$$

The streamlines relative to the vortex system are given by ψ = constant. i.e.,

$$\frac{K}{4\pi}\left[\log\{(x+b)^2+y^2\} - \log\{(x-b)^2+y^2\}\right] - \frac{Kx}{4\pi b} = \text{const.}$$

or $-\log\{(x+b)^2+y^2\} + \log\{(x-b)^2+y^2\} + \frac{x}{b} = \text{const.}$ ——— (2)

Changing into polar co-ordinates,

or $\log\left\{\frac{(x-b)^2+y^2}{(x+b)^2+y^2}\right\} + \frac{x}{b} = \text{const.}$

Proved.

Deduction. For the second statement take A_1A_2 as y-axis and OY as x-axis, make the corresponding changes everywhere, i.e., in place of x write y, in place of y write x. The result at once follows from (2).