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Convergence of Improper Integrals

§ 1. Some Definitions.

1. Infinite Interval. The interval whose length (range) is infinite is said to be an *infinite interval.* Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. Bounded Functions. A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \leq f(x) \leq b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x/\{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. Monotonic functions. A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \ \forall \ x, y \in I$$

and **monotonically decreasing if**

$$x > y \Rightarrow f(x) < f(y) \ \forall \ x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.

For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \leq x \leq \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \leq x \leq \pi$.

4. Proper Integral. The definite integral $\int_a^b f(x)\, dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x \, dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x} dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. Improper Integrals. The definite integral $\int_a^b f(x)\, dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. Improper integrals of the first kind or infinite integrals. A definite integral $\int_a^b f(x)\, dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x \, dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded,** we define

$$(i) \quad \int_a^\infty f(x)\, dx = \lim_{x \to \infty} \int_a^x f(x)\, dx,$$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

$$(ii) \quad \int_{-\infty}^b f(x)\, dx = \lim_{x \to \infty} \int_{-x}^b f(x)\, dx,$$

provided that the limit exists finitely.

$$(iii) \quad \int_{-\infty}^\infty f(x)\, dx = \lim_{x_1 \to \infty} \int_{-x_1}^c f(x)\, dx + \lim_{x_2 \to \infty} \int_c^{x_2} f(x)\, dx$$

provided that both these limits exist finitely.

7. Improper integrals of the second kind. A definite integral $\int_a^b f(x)\, dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \leq x \leq b$, is called an improper integral of the second kind.

Thus $\int_0^4 \frac{dx}{(x-2)(x-3)}$ and $\int_0^1 \frac{1}{x^2} dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\, dx,$$

if the range of integration (a, b) is finite and the integrand $f(x)$ is **unbounded at one or more points of the given interval,** we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

§ 2. Convergence of improper integrals.

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\,dx$ as follows :

Definition. *The integral $\int_a^\infty f(x)\,dx$ is said to converge to the value I, if for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding positive number N such that*

$$\left| \int_a^b f(x)\,dx - I \right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Solved Examples

Ex. 1. *Discuss the convergence of the following integrals by evaluating them*

(i) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$, (ii) $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$.

Sol. (i) We have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x \to \infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-1/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{1/2}}{1/2}\right]_1^x = \lim_{x \to \infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x \to \infty} \int_1^x \frac{dx}{x^{3/2}}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-3/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{-1/2}}{-1/2}\right]_1^x = \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}}\right]_1^x$$

$$= \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} + 2\right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Ex. 2. *Evaluate* $\displaystyle\int_1^\infty \frac{dx}{x}$.

Sol. We have

$$\int_1^\infty \frac{dx}{x} = \lim_{x \to \infty} \int_1^x \frac{dx}{x} = \lim_{x \to \infty} [\log x]_1^x$$

$$= \lim_{x \to \infty} [\log x - 0] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

Ex. 3. *Evaluate* $\displaystyle\int_3^\infty \frac{dx}{(x-2)^2}$.

Sol. We have

$$\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{x \to \infty} \int_3^x \frac{dx}{(x-2)^2}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_3^x (x-2)^{-2}\,dx = \lim_{x \to \infty} \left[\frac{(x-2)^{-1}}{-1}\right]_3^x$$

$$= \lim_{x \to \infty} \left[-\frac{1}{x-2}\right]_3^x = \lim_{x \to \infty} \left[-\frac{1}{x-2} + 1\right] = 1,$$

which is a definite real number. Therefore the given integral is convergent and its value is 1.

Ex. 4. *Test the convergence of* $\int_0^\infty e^{-mx}\,dx$, $(m > 0)$.

Sol. We have $\int_0^\infty e^{-mx}\,dx = \lim_{x\to\infty} \int_0^x e^{-mx}\,dx$, (by def.)

$$= \lim_{x\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^x = \lim_{x\to\infty} \left\{-\frac{1}{m}(e^{-mx} - 1)\right\} = -\frac{1}{m}[0-1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Ex. 5. *Test the convergence of* $\int_0^\infty e^{2x}\,dx$.

Sol. We have $\int_0^\infty e^{2x}\,dx = \lim_{x\to\infty} \int_0^x e^{2x}\,dx$, (By def.)

$$= \lim_{x\to\infty} \left[\frac{e^{2x}}{2}\right]_0^x = \frac{1}{2}\lim_{x\to\infty} [e^{2x} - 1] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.* the integral does not exist).

Ex. 6 (a). *Test the convergence of* $\displaystyle\int_0^\infty \frac{4a\,dx}{x^2 + 4a^2}$.

Sol. We have $\displaystyle\int_0^\infty \frac{4a\,dx}{x^2+4a^2} = \lim_{x\to\infty} \int_0^x \frac{4a\,dx}{x^2 + (2a)^2}$, (By def.)

$$= \lim_{x\to\infty} \left[4a \cdot \frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^x = 2\lim_{x\to\infty} \left[\tan^{-1}\frac{x}{2a}\right]_0^x$$

$$= 2 \cdot \lim_{x\to\infty} \left[\tan^{-1}\frac{x}{2a} - 0\right] = 2 \cdot [\tan^{-1}\infty] = 2 \cdot \frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Ex. 6 (b). *Show that* $\displaystyle\int_0^\infty \frac{dx}{(1+x)^{2/3}}$ *is not convergent.*

Sol. We have $\displaystyle\int_0^\infty \frac{dx}{(1+x)^{2/3}} = \lim_{x\to\infty}\int_0^x (1+x)^{-2/3}\,dx$, (By def.)

$$= \lim_{x\to\infty} \left[\frac{(1+x)^{1/3}}{\frac{1}{3}}\right]_0^x = \lim_{x\to\infty} 3\,[(1+x)^{1/3} - 1] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

Ex. 7. *Test the convergence of*

(*i*) $\int_{-\infty}^0 e^x\,dx$; (*ii*) $\int_{-\infty}^0 e^{-x}\,dx$.

Sol. (*i*) We have $\int_{-\infty}^0 e^x\,dx = \lim_{x\to\infty}\int_{-x}^0 e^x\,dx$, (By def.)

$$= \lim_{x\to\infty} \left[e^x\right]_{-x}^0 = \lim_{x\to\infty} [1 - e^{-x}] = [1-0] = 1.$$

Thus the limits exist and is unique and finite; therefore the given integral is convergent.

(*ii*) We have $\int_{-\infty}^0 e^{-x}\,dx = \lim_{x\to\infty}\int_{-x}^0 e^{-x}\,dx$, (By def.)

$$= \lim_{x\to\infty}\left[\frac{e^{-x}}{-1}\right]_{-x}^0 = -\lim_{x\to\infty}[e^0 - e^x] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

Ex. 8. *Test the convergence of*

(*i*) $\int_{-\infty}^0 \sinh x\,dx$; (*ii*) $\int_{-\infty}^0 \cosh x\,dx$.

Sol. (*i*) We have $\int_{-\infty}^0 \sinh x\,dx = \lim_{x\to\infty}\int_{-x}^0 \sinh x\,dx$, (By def.)

$$= \lim_{x\to\infty}\int_{-x}^0 \frac{e^x - e^{-x}}{2}\,dx \qquad \textbf{[Note]}$$

$$= \frac{1}{2}\left[\lim_{x\to\infty}\int_{-x}^0 e^x\,dx - \lim_{x\to\infty}\int_{-x}^0 e^{-x}\,dx\right]$$

$$= \frac{1}{2}\left[\lim_{x\to\infty}\left\{e^x\right\}_{-x}^0 - \lim_{x\to\infty}\left\{\frac{e^{-x}}{-1}\right\}_x^0\right]$$

$$= \frac{1}{2}\left[\lim_{x\to\infty}\{e^0 - e^{-x}\} + \lim_{x\to\infty}\{e^0 - e^x\}\right] = \frac{1}{2}[1 - \infty] = -\infty.$$

Thus the given integral diverges to $-\infty$.

(*ii*) We have

$$\int_{-\infty}^0 \cosh x\,dx = \int_{-\infty}^0 \frac{e^x + e^{-x}}{2}\,dx$$

$$= \tfrac{1}{2}\left[\int_{-\infty}^0 e^x\,dx + \int_{-\infty}^0 e^{-x}\,dx\right] = \tfrac{1}{2}[1+\infty] = \infty.$$

(As proved in Ex. 7; prove it here.)

Thus the given integral diverges to ∞.

Ex. 9. *Test the convergence of* $\int_{-\infty}^\infty e^{-x}\,dx$.

Sol. We have

$$\int_{-\infty}^\infty e^{-x}\,dx = \int_{-\infty}^0 e^{-x}\,dx + \int_0^\infty e^{-x}\,dx$$

$$= +\infty + 1 = \infty. \qquad \text{(Refer Ex. 7. and Ex. 4)}$$

Thus the limit does not exist finitely. Therefore the given integral is divergent (*i.e.*, the integral does not exist).

Ex. 10. *Test the convergence of* $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$. **(Rohilkhand 1984)**

Sol. We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} \int_{-x}^{0} \frac{dx}{1+x^2} + \lim_{x \to \infty} \int_{0}^{x} \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} \left[\tan^{-1} x\right]_{-x}^{0} + \lim_{x \to \infty} \left[\tan^{-1} x\right]_{0}^{x}$$

$$= \lim_{x \to \infty} [0 - \tan^{-1}(-x)] + \lim_{x \to \infty} [\tan^{-1} x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Ex. 11. *Evaluate* $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$.

Sol. We have $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+1}$

$$= \lim_{x_1 \to \infty} \int_{-x_1}^{c} \frac{dx}{(x+1)^2+1} + \lim_{x_2 \to \infty} \int_{c}^{x_2} \frac{dx}{(x+1)^2+1},$$

where c is any real number

$$= \lim_{x_1 \to \infty} \left[\tan^{-1}(x+1)\right]_{-x_1}^{c} + \lim_{x_2 \to \infty} \left[\tan^{-1}(x+1)\right]_{c}^{x_2}$$

$$= \lim_{x_1 \to \infty} [\tan^{-1}(c+1) + \tan^{-1}(1-x_1)]$$

$$+ \lim_{x_2 \to \infty} [\tan^{-1}(x_2+1) - \tan^{-1}(c+1)]$$

$$= \tan^{-1}(c+1) - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1}(c+1)$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Hence the given integral is convergent and its value is π.

Ex. 12. *Evaluate* $\int_{0}^{1} \frac{dx}{\sqrt{x}}$.

Sol. In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_{0}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \left[2\sqrt{x}\right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Ex. 13. *Evaluate* $\int_{0}^{1} \frac{dx}{x^3}$.

Sol. Here the integrand (*i.e.*, $1/x^3$) becomes infinite at the lower limit $x = 0$.

$$\therefore \int_{0}^{1} \frac{dx}{x^3} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{1} \frac{dx}{x^3} = \lim_{\varepsilon \to 0} \left[-\frac{1}{2x^2}\right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{2} + \frac{1}{2\varepsilon^2}\right]$$

$$= -\tfrac{1}{2} + \infty = \infty.$$

Hence the limit does not exist finitely. Therefore the given integral diverges to ∞ and its value does not exist.

Ex. 14. *Evaluate* $\int_{0}^{1} \frac{dx}{\sqrt{(1-x)}}$.

Sol. Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \int_{0}^{1} \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} \left[-2\sqrt{(1-x)}\right]_{0}^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Ex. 15. *Evaluate* $\int_{0}^{1} \frac{dx}{1-x}$.

Sol. Here the integrand [*i.e.*, $1/(1-x)$] becomes infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \int_{0}^{1} \frac{dx}{1-x} = \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} \frac{dx}{1-x} = \lim_{\varepsilon \to 0} \left[-\log(1-x)\right]_{0}^{1-\varepsilon}$$

$$= \lim_{\varepsilon \to 0} [-\log \varepsilon + 0] = -[-\infty] = \infty.$$

Hence the limit does not exist finitely and therefore the given integral does not exist *i.e.*, is divergent.

Ex. 16. *Evaluate* $\int_{-1}^{1} \frac{dx}{x^{2/3}}$.

Sol. Here the integrand bcomes infinite at $x = 0$ and 0 lies between -1 and 1.

$$\therefore \int_{-1}^{1} \frac{dx}{x^{2/3}} = \int_{-1}^{0} \frac{dx}{x^{2/3}} + \int_{0}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{\varepsilon \to 0} \int_{-1}^{0-\varepsilon} \frac{dx}{x^{2/3}} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'+0}^{1} \frac{dx}{x^{2/3}}$$

$$= \lim_{\varepsilon \to 0} \left[3x^{1/3}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[3x^{1/3}\right]_{\varepsilon'}^{1}$$

$$= \lim_{\varepsilon \to 0} [-3\varepsilon^{1/3} + 3] + \lim_{\varepsilon' \to 0} [3 - 3(\varepsilon')^{1/3}] = 3 + 3 = 6.$$

Ex. 17. *Evaluate* $\int_{-1}^{1} \frac{dx}{x^2}$.

Sol. Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \int_{-1}^{1} \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^{1} \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^{1}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'}\right].$$

Since both the limits do not exist finitely, therfore the integral does not exist and is divergent.

Ex. 18. *Evaluate* $\int_{0}^{2a} \frac{dx}{(x-a)^2}$.

Sol. Here the integrand becomes infinite at $x = a$ and $0 < a < 2a$. Proceeding exactly as in Ex. 16, we find that the limits do not exist finitely *i.e.* the given integral is meaningless.

§ 3. Tests for convergence of improper integrals of the first kind *i.e., to test the convergence of improper integrals in which the range of integration is infinite and the integrand is bounded.*

If an integral of the form $\int_a^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ cannot be actually integrated, its convergence is determined with the help of the following tests :

§ 4. Comparison test.

Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable in the interval (a, ∞). Also let $g(x)$ be positive and $|f(x)| \le g(x)$ when $x \ge a$. Then, if $\int_a^{\infty} g(x)\,dx$ is **convergent**, *$\int_a^{\infty} f(x)\,dx$ is also* **convergent.**

Similarly if $|f(x)| \ge g(x)$ for all values of x greater than some number $x_0 > a$ and $\int_a^{\infty} g(x)\,dx$ is **divergent** , *then $\int_a^{\infty} f(x)\,dx$ is also* **divergent.**

Alternative form of the above comparison test.

If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^{\infty} f(x)\,dx$ and $\int_a^{\infty} g(x)\,dx$ either both converge or both diverge.

Note. While applying comparison test, we generally take $g(x) = \frac{1}{x^n}$ *i.e,* $\int_a^{\infty} \frac{dx}{x^n}$ is generally taken as the comparison integral.

Theorem. *The comparison integral $\int_a^{\infty} \frac{dx}{x^n}$, where $a > 0$, is convergent when $n > 1$ and divergent when $n \le 1$.* **(Meerut 1973)**

Proof. By the definition of an improper integral, we have

$$\int_a^{\infty} \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^{x} \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^{x} x^{-n}\,dx$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^{x}, \text{ if } n \ne 1$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]. \quad \text{...(1)}$$

If n > 1, then $1 - n$ is negative and so $n - 1$ is positive.

Therefore in this case $\lim_{x \to \infty} x^{1-n} = \lim_{x \to \infty} \frac{1}{x^{n-1}} = \frac{1}{\infty} = 0.$

Hence from (1), we have

$$\int_a^{\infty} \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}, \text{ if } n > 1.$$

Hence the given integral is convergent when $n > 1$.

If n < 1, then $1 - n$ is positive and so $\lim_{x \to \infty} x^{1-n} = \infty$.

$\therefore$ from (1), we have $\int_a^{\infty} \frac{dx}{x^n} = \infty$.

Hence the given integral is divergent when $n < 1$.

When n = 1, we have $\int_a^{\infty} \frac{dx}{x^n} = \int_a^{\infty} \frac{dx}{x}$

$$= \lim_{x \to \infty} \int_a^{x} \frac{dx}{x} = \lim_{x \to \infty} \left[\log x\right]_a^{x} = \lim_{x \to \infty} [\log x - \log a]$$

$$= \infty - \log a = \infty.$$

Hence the given integral is divergent when $n = 1$.

$\therefore \int_a^{\infty} \frac{dx}{x^n}$ converges when $n > 1$ and diverges when $n \le 1$.

Solved Examples

Ex. 18 (a). *Test the convergence of the integral*

$$\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\,dx.$$

(Meerut 1991; Rohikhand 80)

Sol. Here $f(x) = \dfrac{\cos mx}{x^2 + a^2}$. Let $g(x) = \dfrac{1}{x^2 + a^2}$.

Obviously $g(x)$ is positive in the interval $(0, \infty)$.

We have $$|f(x)| = \left|\frac{\cos mx}{x^2 + a^2}\right| = \frac{|\cos mx|}{x^2 + a^2}$$
$$\leq \frac{1}{x^2 + a^2}, \text{ since } |\cos mx| \leq 1.$$

Thus $|f(x)| \leq g(x)$ when $x \geq 0$.

$\therefore$ by comparison test, $\displaystyle\int_0^\infty \frac{\cos mx}{x^2 + a^2}\,dx$ is convergent if $\displaystyle\int_0^\infty \frac{dx}{x^2 + a^2}$ is convergent.

But $$\int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \int_0^x \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$$
$$= \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right] = \frac{1}{a}\cdot\frac{\pi}{2} = \text{a definite real number.}$$

$\therefore$ $\displaystyle\int_0^\infty \frac{dx}{x^2 + a^2}$ is convergent.

Hence $\displaystyle\int_0^\infty \frac{\cos mx}{x^2 + a^2}\,dx$ is also convergent.

Ex. 18 (b). *Test the convergence of the integral*
$$\int_0^\infty \frac{\cos x}{1 + x^2}\,dx.$$
(Agra 1974)

Sol. Proceed as in Ex. 18 (a) by taking $m = 1$ and $a = 1$.

Ex. 19 (a). *Test the convergence of the integral*
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$
(Meerut 1991P; Rohikhand 80)

Sol. Let $a > 0$. Then we can write
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^a \frac{\sin^2 x}{x^2}\,dx + \int_a^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Since $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{x^2} = 1$, therefore the integrand $\dfrac{\sin^2 x}{x^2}$ is bounded throughout the finite interval $(0, a)$.

So $\displaystyle\int_0^a \frac{\sin^2 x}{x^2}\,dx$ is a proper integral and we need to check the convergence of the integral $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ only.

Here $f(x) = \dfrac{\sin^2 x}{x^2}$. Take $g(x) = \dfrac{1}{x^2}$.

Obviously $g(x)$ is positive in the interval (a, ∞).

We have $$|f(x)| = \left|\frac{\sin^2 x}{x^2}\right| = \frac{\sin^2 x}{x^2}$$
$$\leq \frac{1}{x^2}, \text{ since } \sin^2 x \leq 1.$$

$\therefore$ by comparison test, $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent if $\displaystyle\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\displaystyle\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

$\therefore$ $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Hence $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

Ex. 19 (b). *Show that the integral* $\displaystyle\int_\pi^\infty \frac{\sin^2 x}{x^2}\,dx$ *is convergent.*

(Agra 1975; Meerut 98)

Sol. Proceed as in Ex. 19 (a), by taking $a = \pi$.

Ex. 19 (c). *Test the convergence of the integral*
$$\int_\pi^\infty \frac{\sin x}{x^2}\,dx.$$
(Rohilkhand 1976, 79)

Sol. Let $f(x) = \dfrac{\sin x}{x^2}$. Then $f(x)$ is bounded in the interval (π, ∞). Take $g(x) = \dfrac{1}{x^2}$. Then $g(x)$ is positive in the interval (π, ∞).

We have $$|f(x)| = \left|\frac{\sin x}{x^2}\right| = \frac{|\sin x|}{x^2}$$
$$\leq \frac{1}{x^2}, \text{ since } |\sin x| \leq 1.$$

$\therefore$ by comparison test, $\displaystyle\int_\pi^\infty \frac{\sin x}{x^2}\,dx$ is convergent if $\displaystyle\int_\pi^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\displaystyle\int_\pi^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

$\therefore$ Hence $\displaystyle\int_\pi^\infty \frac{\sin x}{x^2}\,dx$ is also convergent.

Ex. 20. *Show that the integral* $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ *converges, where* $a > 0$.

Sol. Let
$$f(x) = \frac{1}{x\sqrt{(1+x^2)}}.$$
Then $f(x)$ is bounded in the interval (a, ∞). Take $g(x) = 1/x^2$. Then $g(x)$ is positive in the interval (a, ∞). We have
$$|f(x)| = \left|\frac{1}{x\sqrt{(1+x^2)}}\right| = \frac{x}{x^2\sqrt{\{1+(1/x^2)\}}}$$
$$< \frac{1}{x^2}, \text{ since } \frac{1}{\sqrt{\{1+(1/x^2)\}}} < 1.$$

$\therefore$ by comparison test, $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is convergent if $\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is also convergent.

Alternative Method

Here $f(x) = \frac{1}{x\sqrt{(1+x^2)}} = \frac{1}{x^2\sqrt{\{1+(1/x^2)\}}}$. Take $g(x) = \frac{1}{x^2}$.

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{\{1+(1/x^2)\}}} = 1$, which is finite and non-zero. Therefore $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge. But $\int_a^\infty g(x)\,dx = \int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty f(x)\,dx$ i.e., $\int_a^\infty \frac{1}{x\sqrt{(1+x^2)}}\,dx$ is also convergent.

Ex. 21. *Test the convergence of the integral* $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$.

Sol. Let $f(x) = \frac{1}{\sqrt{(x^3+1)}} = \frac{1}{x^{3/2}\sqrt{\{1+(1/x^3)\}}}$.

Take $g(x) = \frac{1}{x^{3/2}}$. We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{\{1+(1/x^3)\}}} = 1$, which is finite and non-zero. Therefore $\int_1^\infty f(x)\,dx$ and $\int_1^\infty g(x)\,dx$ are either both convergent or both divergent. But the comparison integral $\int_1^\infty g(x)\,dx$ i.e., $\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent because here $n = 3/2$ which is > 1. Hence $\int_1^\infty f(x)\,dx$ i.e., $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$ is also convergent.

Ex. 22. *Test the convergence of*

(*i*) $\int_2^\infty \frac{dx}{\sqrt{(x^2-x-1)}}$; (*ii*) $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$.

Sol. (*i*) Let $f(x) = \frac{1}{\sqrt{(x^2-x-1)}} = \frac{1}{\sqrt{\{(x-\frac{1}{2})^2-\frac{5}{4}\}}}$.

Obviously $f(x)$ is bounded in the interval $(2, \infty)$.

We can write $f(x) = \frac{1}{x\sqrt{\{1-(1/x)-(1/x^2)\}}}$. Take $g(x) = \frac{1}{x}$.

We have
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\sqrt{\{1-(1/x)-(1/x^2)\}}} = 1,$$
which is finite and non-zero. Therefore
$$\int_2^\infty f(x)\,dx \text{ and } \int_2^\infty g(x)\,dx$$
either both converge or both diverge.

But the comparison integral
$$\int_2^\infty g(x)\,dx \text{ i.e., } \int_2^\infty \frac{1}{x}\,dx$$
is divergent because here $n = 1$. Hence
$$\int_2^\infty f(x)\,dx \text{ i.e., } \int_2^\infty \frac{dx}{\sqrt{(x^2-x-1)}}$$
is also divergent.

(*ii*) Proceed exactly as in part (*i*). Here also the given integral is divergent.

Ex. 23. *Test the convergence of* $\int_0^\infty \frac{x^3}{(x^2+a^2)^2}\,dx$.

Sol. Let $f(x) = \frac{x^3}{(x^2+a^2)^2}$. Then $f(x)$ is bounded in the interval $(0, \infty)$. Let $b > 0$.

We can write
$$\int_0^\infty \frac{x^3}{(x^2+a^2)^2}\,dx = \int_0^b \frac{x^3}{(x^2+a^2)^2}\,dx + \int_b^\infty \frac{x^3}{(x^2+a^2)^2}\,dx.$$
The first integral on the right hand side is a proper integral because the interval of integration $(0, b)$ is finite and the integrand $f(x)$

is bounded in this interval. So we need to check the convergence of the integral $\int_b^\infty \frac{x^3}{(x^2+a^2)^2}\,dx$ only.

We can write

$$f(x) = \frac{x^3}{x^4\{1+(a^2/x^2)\}^2} = \frac{1}{x\{1+(a^2/x^2)\}^2}.$$

Take $g(x) = 1/x$.

We have $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{\{1+(a^2/x^2)\}^2} = 1$, which is finite and non-zero. Therefore $\int_b^\infty f(x)\,dx$ and $\int_b^\infty g(x)\,dx$ either both converge or both diverge. But the comparison integral $\int_b^\infty g(x)\,dx$ *i.e.*, $\int_b^\infty \frac{1}{x}\,dx$ is divergent. Therefore $\int_b^\infty f(x)\,dx$ is also divergent. Hence $\int_0^\infty \frac{x^3}{(x^2+a^2)^2}\,dx$ is divergent.

Ex. 24. *Test the convergence of*

$$\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx.$$

Sol. We can write

$$\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx = \int_0^1 e^{-x}\frac{\sin x}{x}\,dx + \int_1^\infty e^{-x}\frac{\sin x}{x}\,dx.$$

Since $\lim_{x\to 0} e^{-x}\frac{\sin x}{x} = 1$, therefore the integrand $e^{-x}\frac{\sin x}{x}$ is bounded throughout the finite interval $(0, 1)$. So $\int_0^1 e^{-x}\frac{\sin x}{x}\,dx$ is a proper integral and therefore it is convergent. Thus we need to check the convergence of $\int_1^\infty e^{-x}\frac{\sin x}{x}\,dx$ only.

Let $f(x) = e^{-x}\frac{\sin x}{x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$.

Take $g(x) = e^{-x}$. Then $g(x)$ is positive in the interval $(1, \infty)$.

We have

$$|f(x)| = \left|e^{-x}\frac{\sin x}{x}\right| = e^{-x}\cdot|\sin x|\cdot\frac{1}{x}$$

$$\le e^{-x}, \text{ since } |\sin x| \le 1 \text{ and } \frac{1}{x} \le 1.$$

Thus $|f(x)| \le g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty f(x)\,dx$ is convergent if $\int_1^\infty g(x)\,dx$ is convergent.

Now $\int_1^\infty g(x)\,dx = \int_1^\infty e^{-x}\,dx = \lim_{x\to\infty} \int_1^x e^{-x}\,dx$

$$= \lim_{x\to\infty} \left[-e^{-x}\right]_1^x$$

$$= \lim_{x\to\infty} \left[-e^{-x}+e^{-1}\right] = 0 + e^{-1} = 1/e,$$

which is a definite finite number. Hence $\int_1^\infty g(x)\,dx$ is convergent.

$\therefore$ $\int_1^\infty f(x)\,dx$ is also convergent.

Hence $\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx$ is convergent because the sum of two convergent integrals is also convergent.

Ex. 25. *Test the convergence of the integral*

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx.$$

Sol. We have

$$\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \int_0^\infty \frac{2\sin^2(x/2)}{x^2}\,dx$$

$$= \int_0^\infty \frac{2\sin^2 t}{4t^2}\cdot 2dt, \text{ putting } \frac{x}{2} = t \text{ so that } dx = 2dt$$

$$= \int_0^\infty \frac{\sin^2 t}{t^2}\,dt.$$

Now proceed as in Ex. 19 (a).

Ex. 26. *Show that the integral* $\int_0^\infty e^{-x^2}\,dx$ *is convergent.*

Sol. We have

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx.$$

Obviously $\int_0^1 e^{-x^2}\,dx$ is a proper integral because here the interval of integration $(0, 1)$ is finite and the integrand e^{-x^2} is bounded throughout this interval. Therefore this integral is convergent. So we need to check the convergence of $\int_1^\infty e^{-x^2}\,dx$ only.

Let $f(x) = e^{-x^2}$. Take $g(x) = xe^{-x^2}$ so that $g(x)$ is positive throughout the interval $(1, \infty)$. We have

$$|f(x)| = e^{-x^2} \le xe^{-x^2}, \text{ since } x \ge 1.$$

Thus $|f(x)| \le g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty e^{-x^2}\,dx$ is convergent if $\int_1^\infty xe^{-x^2}\,dx$ is convergent.

Now $\quad \int_1^\infty xe^{-x^2}\,dx = \lim_{x\to\infty}\int_1^x xe^{-x^2}\,dx$

$$= \lim_{x\to\infty}\left(-\tfrac{1}{2}e^{-x^2}\right)_1^x = \lim_{x\to\infty}\left(-\tfrac{1}{2}e^{-x^2}+\tfrac{1}{2}e^{-1}\right)$$

$= \frac{1}{2}e^{-1}$, which is a definite number.

$\therefore$ $\int_1^\infty xe^{-x^2}\,dx$ is convergent and so $\int_1^\infty e^{-x^2}\,dx$ is also convergent.

Hence the given integral $\int_0^\infty e^{-x^2}\,dx$ is also convergent as it is the sum of two convergent integrals.

§ 5. The μ-Test.

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.

If there is a number $\mu > 1$, such that $\lim_{x\to\infty} x^\mu f(x)$ exists, then $\int_a^\infty f(x)\,dx$ is convergent.

If there is a number $\mu \le 1$, such that $\lim_{x\to\infty} x^\mu f(x)$ exists and is non-zero, then $\int_a^\infty f(x)\,dx$ is divergent and the same is true if $\lim_{x\to\infty} x^\mu f(x)$ is $+\infty$ or $-\infty$.

While applying the μ-test, the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand.

Solved Examples

Ex. 27. *Examine the convergence of* $\displaystyle\int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$.

Sol. Let $\quad f(x) = \dfrac{1}{x^{1/3}(1+x^{1/2})} = \dfrac{1}{x^{1/3}x^{1/2}\{1+(1/x^{1/2})\}}$

$$= \frac{1}{x^{5/6}\{1+(1/x^{1/2})\}}.$$

Obviously $f(x)$ is bounded in the interval $(1, \infty)$.
Take $\mu = \frac{5}{6} - 0 = \frac{5}{6}$. We have

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^{5/6}\cdot\frac{1}{x^{5/6}\{1+(1/x^{1/2})\}}$$

$$= \lim_{x\to\infty}\frac{1}{1+(1/x^{1/2})} = 1,$$

which is finite and non-zero. Since $\mu = \frac{5}{6}$ *i.e.*, < 1, it follows from the μ-test that the given integral is divergent.

Ex. 28. *Examine the convergence of* $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)^3}$.

Sol. Let $a > 0$. Then we have

$$\int_0^\infty \frac{x\,dx}{(1+x)^3} = \int_0^a \frac{x\,dx}{(1+x)^3} + \int_a^\infty \frac{x\,dx}{(1+x)^3}.$$

The first integral on the rigth hand side is convergent because it is a proper integral. We observe that in this integral the range of integration $(0, a)$ is finite and the integrand $x/(1+x)^3$ is bounded throughout the interval $(0, a)$. So we need to check the convergence of $\displaystyle\int_a^\infty \frac{x\,dx}{(1+x)^3}$ only.

Let $f(x) = \dfrac{x}{(1+x)^3}$. Then $f(x)$ is bounded in the interval (a, ∞).
Take $\mu = 3 - 1 = 2$. Then

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^2\cdot\frac{x}{(1+x)^3} = \lim_{x\to\infty}\frac{1}{\{1+(1/x)\}^3} = 1,$$

which exists *i.e.*, is equal to a definite real number.

Since $\mu = 2$ *i.e.*, > 1, therefore by μ-test the integral $\displaystyle\int_a^\infty \frac{x\,dx}{(1+x)^3}$ is convergent.

Hence $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)^3}$ is also convergent because it is the sum of two convergent integrals.

Ex. 29. *Show that the following integrals are convergent.*

(*i*) $\displaystyle\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$; $\qquad$ (*ii*) $\displaystyle\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$.

Sol. (*i*) Let $b > 0$. Then we have

$$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx = \int_0^b \frac{x^2}{(a^2+x^2)^2}\,dx + \int_b^\infty \frac{x^2}{(a^2+x^2)^2}\,dx.$$

The first integral on the right hand side is convergent because it is a proper integral. So we need to check the convergence of

$$\int_b^\infty \frac{x^2}{(a^2+x^2)^2}\,dx.$$

Let $f(x) = \dfrac{x^2}{(a^2+x^2)^2}$. Then $f(x)$ is bounded in the interval (b, ∞).
Take $\mu = 4 - 2 = 2$. Then

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} x^2 \cdot \frac{x^2}{(a^2 + x^2)^2} = \lim_{x \to \infty} \frac{1}{\{1 + (a^2/x^2)\}^2} = 1,$$

which is a definite real number.

Since $\mu > 1$, therefore by μ-test $\int_b^\infty \frac{x^2}{(a^2 + x^2)^2} dx$ is convergent.

Hence $\int_0^\infty \frac{x^2 \, dx}{(a^2 + x^2)^2}$ is also convergent because it is the sum of two convergent integrals.

(ii) Apply μ-test by taking $\mu = \frac{3}{2}$.

Ex. 30. *Test the convergence of* $\int_b^\infty \frac{x^{3/2} \, dx}{\sqrt{(x^4 - a^4)}}$, *where* $b > a$.

Sol. Let $f(x) = \frac{x^{3/2}}{\sqrt{(x^4 - a^4)}}$. Then $f(x)$ is bounded in the interval (b, ∞). Take $\mu = 2 - \frac{3}{2} = \frac{1}{2}$. Then

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} x^{1/2} \cdot \frac{x^{3/2}}{x^2 \sqrt{\{1 - (a^4/x^4)\}}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{\{1 - (a^4/x^4)\}}} = 1, \text{ which is finite and non-zero.}$$

Since $\mu < 1$, therefore by μ-test the given integral is divergent.

Ex. 31. *Examine the convergence of* $\int_a^\infty \frac{dx}{x(\log x)^{n+1}}$, *where* $a > 1$.

Sol. Let $\log x = t$ so that $(1/x) \, dx = dt$.

$$\therefore \int_a^\infty \frac{dx}{x(\log x)^{n+1}} = \int_{\log a}^\infty \frac{dt}{t^{n+1}}.$$

Let $f(t) = 1/t^{n+1}$. Then $f(t)$ is bounded in the interval $(\log a, \infty)$. Take $\mu = (n + 1) - 0 = n + 1$. Then

$$\lim_{t \to \infty} t^\mu f(t) = \lim_{t \to \infty} \frac{t^{n+1}}{t^{n+1}} = \lim_{t \to \infty} 1 = 1,$$

which is finite and non-zero.

Therefore by μ-test, the given integral is convergent if

$$\mu > 1 \text{ i.e., } n + 1 > 1 \text{ i.e., } n > 0$$

and divergent if $\mu \leq 1$ *i.e.*, $n + 1 \leq 1$ *i.e.*, $n \leq 0$.

Ex. 32 (a). *Show that the integral* $\int_1^\infty x^{n-1} e^{-x} dx$ *is convergent.*

Sol. Let $f(x) = x^{n-1} e^{-x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} \frac{x^\mu . x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{2!} + \ldots}$$

$= 0$ for all values of μ and n.

Taking $\mu > 1$, we see by μ-test that the integral

$$\int_1^\infty x^{n-1} e^{-x} dx$$

is convergent for all values of n.

Ex. 32 (b). *Show that the integral* $\int_a^\infty x^{n-1} e^{-x} dx$ *is convergent, where* $a > 0$.

Sol. Proceed exactly in the same way as in Ex. 32 (a).

Ex. 33. *Test the convergence of the integral*

$$\int_0^\infty \frac{x^{2m}}{1 + x^{2n}} dx,$$

where m and n are positive integers. **(Meerut 1983)**

Sol. Let $a > 0$. We have

$$\int_0^\infty \frac{x^{2m}}{1 + x^{2n}} dx = \int_0^a \frac{x^{2m}}{1 + x^{2n}} dx + \int_a^\infty \frac{x^{2m}}{1 + x^{2n}} dx.$$

The first integral on the right hand side is a proper integral and so it is convergent. Therefore the given integral is convergent or divergent according as $\int_a^\infty \frac{x^{2m}}{1 + x^{2n}} dx$ is convergent or divergent.

To test the convergence of $\int_a^\infty \frac{x^{2m}}{1 + x^{2n}} dx$, let us take $\mu = 2n - 2m$.

We have $\lim_{x \to \infty} x^\mu \cdot \frac{x^{2m}}{1 + x^{2n}} = \lim_{x \to \infty} x^{2n - 2m} \frac{x^{2m}}{x^{2n} \{1 + (1/x^{2n})\}}$

$$= \lim_{x \to \infty} \frac{1}{1 + (1/x^{2n})} = 1, \text{ which is finite and non-zero.}$$

$\therefore$ by μ-test, the given integral is convergent if $\mu > 1$ *i.e.*, if $2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers. Also by μ-test, the given integral is divergent if $\mu \leq 1$ *i.e.*, if $2n - 2m \leq 1$ *i.e.*, if $n \leq m$ since n and m are positive integers.

§ 6. Abel's test for the convergence of integral of a product.

If $\int_a^\infty f(x) \, dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $x > a$, *then* $\int_a^\infty f(x) \phi(x) \, dx$ *is convergent.*

Solved Examples

Ex. 34. *Test the convergence of*

$$\int_a^\infty (1-e^{-x})\frac{\cos x}{x^2}\,dx, \text{ when } a>0.$$

Sol. Let $f(x)=\dfrac{\cos x}{x^2}$ and $\phi(x)=1-e^{-x}$.

We have $\left|\dfrac{\cos x}{x^2}\right|\le\dfrac{1}{x^2}$ as $|\cos x|\le 1$.

Since $\int_a^\infty \dfrac{1}{x^2}dx$ is convergent, therefore by comparison test $\int_a^\infty \dfrac{\cos x}{x^2}dx$ is also convergent.

Agian $\phi(x)=1-e^{-x}$ is monotonic increasing and bounded function for $x>a$.

Hence by Abel's test $\int_a^\infty (1-e^{-x})\dfrac{\cos x}{x^2}dx$ is convergent.

Ex. 35. *Test the convergence of* $\int_a^\infty e^{-x}\dfrac{\sin x}{x^2}dx$ *where* $a>0$.

Sol. Let $f(x)=\dfrac{\sin x}{x^2}$ and $\phi(x)=e^{-x}$.

Since $\left|\dfrac{\sin x}{x^2}\right|\le\dfrac{1}{x^2}$ and $\int_a^\infty\dfrac{1}{x^2}dx$ is convergent, therefore by comparison test $\int_a^\infty\dfrac{\sin x}{x^2}dx$ is also convergent.

Again e^{-x} is monotonic decreasing and bounded function for $x>a$.

Hence by Abel's test $\int_a^\infty e^{-x}\dfrac{\sin x}{x^2}dx$ is convergent.

§ 7. Dirichlet's test for the convergence of integral of a product.

If $f(x)$ be bounded and monotonic in the interval $a\le x<\infty$ and if $\lim_{x\to\infty} f(x)=0$, then the integral $\int_a^\infty f(x)\,\phi(x)\,dx$ converges provided $\left|\int_a^x \phi(x)\,dx\right|$ is bounded as x takes all finite values.

Solved Examples

Ex. 36. *Test the convergence of the integral*

$$\int_a^\infty \frac{\sin x}{\sqrt{x}}\,dx, \text{ where } a>0.$$

Sol. Let $f(x)=\dfrac{1}{\sqrt{x}}$ and $\phi(x)=\sin x$.

Now $\dfrac{1}{\sqrt{x}}$ is bounded and monotonic decreasing for all $x\ge a$ and $\lim_{x\to\infty}\dfrac{1}{\sqrt{x}}=0$.

Also $\left|\int_a^x\phi(x)\,dx\right|=\left|\int_a^x\sin x\,dx\right|=|\cos a-\cos x|\le 2$, for all finite values of x. [Note that the value of $\cos x$ lies between -1 and 1].

$\therefore$ $\left|\int_a^x\phi(x)\,dx\right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty\dfrac{\sin x}{\sqrt{x}}dx$ is convergent.

Ex. 37. *Show that $\int_0^\infty \sin x^2\,dx$ is convergent.* **(Meerut 1979)**

Sol. We have $\int_0^\infty\sin x^2\,dx=\int_0^1\sin x^2\,dx+\int_1^\infty\sin x^2\,dx$.

But $\int_0^1\sin x^2\,dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of $\int_1^\infty\sin x^2\,dx$. We can write $\int_1^\infty\sin x^2\,dx=\int_1^\infty 2x\,.(\sin x^2)\,.\dfrac{1}{2x}dx.$

Let $f(x)=\dfrac{1}{2x}$ and $\phi(x)=2x\sin x^2$.

The function $f(x)=\dfrac{1}{2x}$ is bounded and monotonic decreasing for all $x\ge1$ and $\lim_{x\to\infty}\dfrac{1}{2x}=0$.

Also $\left|\int_1^x\phi(x)\,dx\right|=\int_1^x 2x\sin x^2\,dx$

$=|\cos 1^2-\cos x^2|\le 2$, for all finite values of x.

$\therefore$ $\left|\int_1^x\phi(x)\,dx\right|$ is bounded for all finite values of x.

Hence by Dirichlet's test

$\int_1^\infty\dfrac{1}{2x}.(\sin x^2)\,2x\,dx$ *i.e.*, $\int_1^\infty\sin x^2\,dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore the integral $\int_0^\infty\sin x^2\,dx$ is convergent.

Ex. 38. *Show that the integral* $\int_0^\infty\dfrac{\sin x}{x}dx$ *is convergent.* **(Meerut 1977)**

Sol. We have $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^a \frac{\sin x}{x}\,dx + \int_a^\infty \frac{\sin x}{x}\,dx$, where $a > 0$.

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the integral $\int_0^a \frac{\sin x}{x}\,dx$ is a proper integral and hence convergent.

Now to test the convergence of $\int_a^\infty \frac{\sin x}{x}\,dx$.

Let $f(x) = 1/x$ and $\phi(x) = \sin x$.

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left|\int_a^x \phi(x)\,dx\right| = \left|\int_a^x \sin x\,dx\right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left|\int_a^x \phi(x)\,dx\right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \frac{\sin x}{x}\,dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty \frac{\sin x}{x}\,dx$ is convergent.

Ex. 39. *Prove that* $\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\,dx$ *is convergent where* $a > 0$.

Sol. We have

$$\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\,dx = \int_a^\infty \frac{\cos \alpha x}{x}\,dx - \int_a^\infty \frac{\cos \beta x}{x}\,dx.$$

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left|\int_a^x \cos \alpha x\,dx\right| = \left|\frac{1}{\alpha}(\sin \alpha x - \sin \alpha a)\right| \leq \frac{2}{|\alpha|}$.

$\therefore$ $\left|\int_a^x \cos \alpha x\,dx\right|$ is bounded for all finite values of x.

Similarly $\left|\int_a^x \cos \beta x\,dx\right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test both the integrals

$$\int_a^\infty \frac{\cos \alpha x}{x}\,dx \text{ and } \int_a^\infty \frac{\cos \beta x}{x}\,dx \text{ are convergent.}$$

Hence the given integral is convergent.

Ex. 40. *Show that the integral*

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx,\ a \geq 0 \text{ is convergent.}$$

(Meerut 1980)

Sol. We have

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx = \int_0^\alpha e^{-ax} \frac{\sin x}{x}\,dx + \int_\alpha^\infty e^{-ax} \frac{\sin x}{x}\,dx,$$

where $\alpha > 0$.

Since $\lim_{x \to 0} e^{-ax} \frac{\sin x}{x} = 1$, the integral $\int_0^\alpha e^{-ax} \frac{\sin x}{x}\,dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of

$$\int_\alpha^\infty e^{-ax} \frac{\sin x}{x}\,dx.$$

Let $f(x) = \frac{e^{-ax}}{x}$ and $\phi(x) = \sin x$.

Obviously the function $f(x) = \frac{1}{x e^{ax}}$ is bounded and monotonic decreasing for all $x \geq \alpha$ and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x e^{ax}} = 0$.

Moreover $\left|\int_\alpha^x \phi(x)\,dx\right| = \left|\int_\alpha^x \sin x\,dx\right| = |\cos \alpha - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left|\int_\alpha^x \phi(x)\,dx\right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test $\int_\alpha^\infty e^{-ax} \frac{\sin x}{x}\,dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx$ is convergent.

§ 8. Absolute Convergence.

The infinite integral $\int_a^\infty f(x)\,dx$ *is said to be absolutely convergent if the integral* $\int_a^\infty |f(x)|\,dx$ *is convergent.*

If the integral $\int_a^\infty f(x)\,dx$ is absolutely convergent, it is necessarily convergent. But if the integral $\int_a^\infty f(x)\,dx$ is convergent, it is not necessarily absolutely convergent. Thus absolute convergence gives a sufficient but not a necessary condition for the convergence of an infinite integral.

Ex. 41. *Show that* $\int_1^\infty \frac{\sin x}{x^4}\,dx$ *is absolutely convergent.*

Sol. The integral $\int_1^\infty \frac{\sin x}{x^4}\,dx$ will be absolutely convergent if $\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin x}{x^4}\right|$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$f(x) = \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \le \frac{1}{x^4}, \text{ since } |\sin x| \le 1.$$

$\therefore$ by comparison test, $\int_1^\infty f(x)\,dx$ is convergent if $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent. But the comparison integral $\int_1^\infty \frac{1}{x^4}\,dx$ is convergent because here $n = 4$ which is > 1.

Hence $\int_1^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

Ex. 42. *Show that* $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\,dx$ *converges absolutely.*

Sol. The integral $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\,dx$ will be absolutely convergent if $\int_0^\infty \left|\frac{\sin mx}{a^2 + x^2}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right| = \frac{|\sin mx|}{a^2 + x^2} \le \frac{1}{a^2 + x^2}, \text{ since } |\sin mx| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\,dx$ is convergent if $\int_0^\infty \frac{1}{a^2 + x^2}\,dx$ is convergent.

But $$\int_0^\infty \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \int_0^x \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$$

$$= \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right] = \frac{1}{a}\cdot\frac{\pi}{2},$$

which is a definite real number.

$\therefore$ $\int_0^\infty \frac{dx}{a^2 + x^2}$ is convergent. Hence $\int_0^\infty f(x)\,dx$ is also convergent and so the given integral is absolutely convergent.

Ex. 43. *Show that the integral* $\int_0^\infty e^{-x}\cos mx\,dx$ *converges absolutley.*

Sol. The integral $\int_0^\infty e^{-x}\cos mx\,dx$ will be absolutely convergent if $\int_0^\infty |e^{-x}\cos mx|\,dx$ is convergent.

Let $f(x) = |e^{-x}\cos mx|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = |e^{-x}\cos mx| = e^{-x}\,|\cos mx| \le e^{-x},$$
since $|\cos mx| \le 1$.

$\therefore$ by comparison test, $\int_0^\infty f(x)\,dx$ is convergent if $\int_0^\infty e^{-x}\,dx$ is convergent.

But $$\int_0^\infty e^{-x}\,dx = \lim_{x \to \infty} \int_0^x e^{-x}\,dx = \lim_{x \to \infty} \left[-e^{-x}\right]_0^x$$

$$= \lim_{x \to \infty} [-e^{-x} + 1] = 1, \text{ which is a definite real number.}$$

$\therefore$ $\int_0^\infty e^{-x}\,dx$ is convergent.

Hence $\int_0^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

§ 9. Tests for convergence of improper integrals of the second kind.

Now we shall make a study of the tests for the convergence of a definite integral of the type $\int_a^b f(x)\,dx$ in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$. It is sufficient to consider the case when $f(x)$ becomes unbounded at $x = a$ and bounded for all other values of x in the interval $[a, b]$. In this case we have

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x)\,dx.$$

In the articles to follow we give a few important tests for the convergence of the above integral.

§ 10. Comparison test.

Consider the improper integral $\int_a^b f(x)\,dx$, *where the range of integration* (a, b) *is finite and* $f(x)$ *is unbounded only at* $x = a$. *Let* $g(x)$ *be* positive in the interval $(a+\varepsilon, b)$ and $|f(x)| \leq g(x)$ in the interval $(a+\varepsilon, b)$. Then $\int_a^b f(x)\,dx$ is convergent if $\int_a^b g(x)\,dx$ is convergent.

Similarly if $|f(x)| \geq g(x)$ for all values of x in the interval $(a+\varepsilon, b)$, then $\int_a^b f(x)\,dx$ is divergent provided $\int_a^b g(x)\,dx$ is divergent.

Alternative form of the above comparison test.

If $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

Note. While applying the above comparison test, we generally take $g(x) = \dfrac{1}{(x-a)^n}$ *i.e.*, $\displaystyle\int_a^b \frac{dx}{(x-a)^n}$ is generally taken as the comparison integral.

Theorem. *The comparison integral* $\displaystyle\int_a^b \frac{dx}{(x-a)^n}$ *is convergent when* $n < 1$ *and divergent when* $n \geq 1$. **(Rohilkhand 1985)**

Proof. We have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b (x-a)^{-n}\,dx$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{1-n}\right]_{a+\varepsilon}^b, \text{ if } n \neq 1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n}\right]. \quad \text{...(1)}$$

If $\mathbf{n < 1}$, then $1-n$ is positive and so $\lim_{\varepsilon \to 0} \varepsilon^{1-n} = 0$. Therefore from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}, \text{ if } n < 1.$$

Hence the given integral converges when $n < 1$.

If $\mathbf{n > 1}$, then $1-n$ is negative and so $n-1$ is positive. Therefore in this case, from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} + \frac{1}{(n-1)\,\varepsilon^{n-1}}\right] = \infty.$$

Hence the given integral diverges when $n > 1$.

When $\mathbf{n = 1}$, we have

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)}$$

$$= \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0} \left[\log(x-a)\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0} [\log(b-a) - \log \varepsilon]$$

$$= \infty, \qquad [\because \log 0 = -\infty].$$

Hence the given integral diverges when $x = 1$.

Solved Examples

Ex. 44. *Show that the integral* $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ *is convergent.*

Sol. In the given integral, the integrand $f(x) = \dfrac{1}{x^{1/3}(1+x^2)}$ is unbounded at the lower limit of integration $x = 0$.

Take $g(x) = 1/x^{1/3}$.

Then $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1$, which is finite and non-zero.

$\therefore$ by comparison test

$$\int_0^1 f(x)\,dx \text{ and } \int_0^1 g(x)\,dx$$

either both converge or both diverge. But the comparison integral $\displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ is convergent because here $n = 1/3$ which is less than 1. Hence the integral $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is also convergent.

Ex. 45. *Test the convergence of the integral* $\displaystyle\int_0^1 \frac{dx}{x^3(1+x^2)}$.

Sol. In the given integral the integrand $f(x) = \dfrac{1}{x^3(1+x^2)}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x^3$.

Then $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1$,

which is finite and non-zero. Therefore, by comparison test,

$$\int_0^1 f(x)\,dx \text{ and } \int_0^1 g(x)\,dx$$

either both converge or both diverge. But the comparison integral $\displaystyle\int_0^1 \frac{dx}{x^3}$ is divergent because here $n = 3$ which is > 1.

Hence the given integral $\displaystyle\int_0^1 \frac{dx}{x^3(1+x^2)}$ is also divergent.

Ex. 46. *Test the convergence of the integral* $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x^4-1)}}$.

Sol. In the given integral the integrand $f(x) = 1/\sqrt{(x^4 - 1)}$ is unbounded at the lower limit of integration $x = 1$.

Take $g(x) = 1/\sqrt{(x^2 - 1)}$.

Then $\lim_{x \to 1} \dfrac{f(x)}{g(x)} = \lim_{x \to 1} \left\{\dfrac{1}{\sqrt{(x^4 - 1)}} \cdot \sqrt{(x^2 - 1)}\right\}$

$= \lim_{x \to 1} \dfrac{1}{\sqrt{(x^2 + 1)}}$

$= 1/\sqrt{2}$, which is finite and non-zero.

Therefore by comparison test,

$$\int_1^2 f(x)\, dx \text{ and } \int_1^2 g(x)\, dx$$

are either both convergent or both divergent.

But $\int_1^2 g(x)\, dx = \int_1^2 \dfrac{dx}{\sqrt{(x^2 - 1)}} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^2 \dfrac{dx}{\sqrt{(x^2 - 1)}}$

$= \lim_{\varepsilon \to 0} \left[\log\{x + \sqrt{(x^2 - 1)}\}\right]_{1+\varepsilon}^2$

$= \lim_{\varepsilon \to 0} [\log(2 + \sqrt{3}) - \log\{1 + \varepsilon + \sqrt{(\varepsilon^2 + \varepsilon)}\}] = \log(2 + \sqrt{3})$,

which is a definite real number.

$\therefore \int_1^2 g(x)\, dx$ is convergent.

Hence $\int_1^2 \dfrac{1}{\sqrt{(x^4 - 1)}}\, dx$ is also convergent.

Ex. 47. *Test the convergence of the integral*

$$\int_0^1 \frac{dx}{(x + 1)\sqrt{(1 - x^2)}}.$$

Sol. In the given integral the integrand $f(x) = \dfrac{1}{(x + 1)\sqrt{(1 - x^2)}}$ is unbounded at the upper limit of integration $x = 1$. Take

$$g(x) = \frac{1}{\sqrt{(1 - x^2)}}.$$

Then $\lim_{x \to 1} \dfrac{f(x)}{g(x)} = \lim_{x \to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$,

which is finite and non-zero.

Therefore, by comparison test.

$$\int_0^1 f(x)\, dx \text{ and } \int_0^1 g(x)\, dx$$

either both converge or both diverge.

But $\int_0^1 g(x)\, dx = \int_0^1 \dfrac{dx}{\sqrt{(1 - x^2)}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \dfrac{dx}{\sqrt{(1 - x^2)}}$

$= \lim_{\varepsilon \to 0} \left[\sin^{-1} x\right]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [\sin^{-1}(1 - \varepsilon)] = \sin^{-1} 1 = \pi/2$,

which is a definite real number.

$\therefore \int_0^1 g(x)\, dx$ is convergent.

Hence $\int_0^1 \dfrac{dx}{(x + 1)\sqrt{(1 - x^2)}}$ is also convergent.

Ex. 48. *Show that the integral* $\int_0^1 \dfrac{\sec x}{x}\, dx$ *is divergent.*

Sol. In the given integral the integrand $f(x) = \dfrac{\sec x}{x}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x$.

Then $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} \left\{\dfrac{\sec x}{x} \cdot x\right\} = \lim_{x \to 0} \sec x = 1$,

which is finite and non-zero.

Therefore, by comparison test,

$$\int_0^1 f(x)\, dx \text{ and } \int_0^1 g(x)\, dx$$

either both converge or both diverge. But the comparison integral $\int_0^1 \dfrac{1}{x}\, dx$ is divergent because here $n = 1$.

Hence the given integral $\int_0^1 \dfrac{\sec x}{x}\, dx$ is also divergent.

Ex. 49. *Test the convergence of the integral* $\int_0^{\pi/2} \dfrac{\cos x}{x^2}\, dx$.

Sol. In the given integral the integrand $f(x) = \dfrac{\cos x}{x^2}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x^2$.

Then $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} \left\{\dfrac{\cos x}{x^2} \cdot x^2\right\} = \lim_{x \to 0} \cos x = 1$,

which is finite and non-zero.

Therefore, by comparison test, $\int_0^{\pi/2} f(x)\, dx$ and $\int_0^{\pi/2} g(x)\, dx$ either both converge or both diverge.

But $\int_0^{\pi/2} g(x)\, dx = \int_0^{\pi/2} \dfrac{1}{x^2}\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/2} \dfrac{1}{x^2}\, dx$

$= \lim_{\varepsilon \to 0} \left[-\dfrac{1}{x}\right]_\varepsilon^{\pi/2} = \lim_{\varepsilon \to 0} \left[-\dfrac{2}{\pi} + \dfrac{1}{\varepsilon}\right] = \infty$.

$\therefore \int_0^{\pi/2} g(x)\, dx$ is divergent.

Hence the given integral $\int_0^{\pi/2} \dfrac{\cos x}{x^2}\, dx$ is also divergent.

Ex. 50. *Show that $\int_0^1 x^{n-1} e^{-x}\, dx$ is convergent if $n > 0$.*

(Rohilkhand 1977)

Sol. If $n \geq 1$, then $\int_0^1 x^{n-1} e^{-x}\, dx$ is a proper integral because the integrand $f(x) = x^{n-1} e^{-x}$ is bounded in the interval $(0, 1)$. So the given integral is convergent when $n \geq 1$.

If $0 < n < 1$, the integrand $f(x) = x^{n-1} e^{-x}$ is unbounded at $x = 0$. Take $g(x) = x^{n-1}$.

Then $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} e^{-x} = 1$, which is finite and non-zero.

$\therefore$ by comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ either both converge or both diverge.

But $\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \left[\dfrac{x^n}{n}\right]_\varepsilon^1$

$= \lim_{\varepsilon \to 0} \left[\dfrac{1}{n} - \dfrac{\varepsilon^n}{n}\right] = \dfrac{1}{n}$, which is a definite real number.

$\therefore \quad \int_0^1 g(x)\, dx$ is convergent.

Hence $\int_0^1 x^{n-1} e^{-x}\, dx$ is also convergent.

Ex. 51. *Show that the integral $\int_0^\infty x^{n-1} e^{-x}\, dx$ is convergent if $n > 0$.*

Sol. We have

$$\int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^\infty x^{n-1} e^{-x}\, dx.$$

Let $I_1 = \int_0^1 x^{n-1} e^{-x}\, dx$ and $I_2 = \int_1^\infty x^{n-1} e^{-x}\, dx$.

The integral I_2 is convergent for all values of n.

[For proof see Ex. 32 (a)]

Also the integral I_1 is convergent if $n > 0$. [For proof see Ex. 50]

Hence the given integral is convergent if $n > 0$ because then it is the sum of two convergent integrals.

§ 11. The μ-Test.

Let $f(x)$ be unbounded at $x = a$ and be bounded and integrable in the arbitrary interval

$$(a + \varepsilon, b), \text{ where } 0 < \varepsilon < b - a.$$

If there is a number μ between 0 and 1 such that

$$\lim_{x \to a+0} (x - a)^\mu f(x) \text{ exists, then } \int_a^b f(x)\, dx$$

is convergent.

If there is a number $\mu \geq 1$ such that $\lim_{x \to a+0} (x - a)^\mu f(x)$ exists and is non-zero, then $\int_a^b f(x)\, dx$ is divergent and the same is true if $\lim_{x \to a+0} (x - a)^\mu f(x) = +\infty$, or $-\infty$.

In case $f(x)$ is unbounded at $x = b$, we should find

$$\lim_{x \to b-0} (b - x)^\mu . f(x),$$

the other conditions of the test remaining the same.

Solved Examples

Ex. 52. *Prove that the integral $\displaystyle\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ converges.*

(Meerut 1982, 98)

Sol. In the given integral the integrand $f(x) = 1/\sqrt{\{x(1-x)\}}$ is unbounded both at $x = 0$ and at $x = 1$. If $0 < a < 1$, we can write

$$\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = \int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} + \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$$
$$= I_1 + I_2, \text{ say.}$$

In the integral I_1 the integrand $f(x)$ is unbounded at the lower limit of integration $x = 0$ and in the integral I_2 the integrand $f(x)$ is unbounded at the upper limit of integration $x = 1$.

To test the convergence of I_1. Take $\mu = \frac{1}{2}$. We have

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} x^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}} = \lim_{x \to 0} \frac{1}{\sqrt{(1-x)}}$$
$$= 1 \text{ i.e., the limit exists.}$$

Since $0 < \mu < \frac{1}{2}$, therefore by μ-test I_1 is convergent.

To test the convergence of I_2. Take $\mu = \frac{1}{2}$. We have

$$\lim_{x \to 1-0} (1 - x)^\mu . f(x) = \lim_{x \to 1-0} (1 - x)^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}}$$
$$= \lim_{x \to 1-0} \frac{1}{\sqrt{x}} = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{(1-\varepsilon)}} = 1.$$

Hence by μ-test I_2 is convergent since $0 < \mu < 1$.

Thus the given integral is the sum of two convergent integrals. Hence the given integral itself is convergent.

Ex. 53. *Test the convergence of $\displaystyle\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\, dx$.*

Sol. Let $f(x) = \dfrac{\log x}{\sqrt{(2-x)}}$ · Then $f(x)$ is unbounded both at $x = 0$ and $x = 2$. If $0 < a < 2$, we can write

$$\int_0^2 \frac{\log x}{\sqrt{(2-x)}}\,dx = \int_0^a \frac{\log x}{\sqrt{(2-x)}}\,dx + \int_a^2 \frac{\log x}{\sqrt{(2-x)}}\,dx$$
$$= I_1 + I_2, \text{ say.}$$

To test the convergence of I_1. We have

$$\lim_{x\to 0} x^\mu f(x) = \lim_{x \to 0} \left\{x^\mu \cdot \frac{\log x}{\sqrt{(x-2)}}\right\} = 0 \text{ if } \mu > 0.$$

Therefore taking μ between 0 and 1, it follows by μ-test that I_1 is convergent.

To test the convergence of I_2. Take $\mu = \frac{1}{2}$. We have

$$\lim_{x\to 2-0} (2-x)^\mu . f(x) = \lim_{x \to 2-0} (2-x)^{1/2} \cdot \frac{\log x}{\sqrt{(2-x)}}$$
$$= \lim_{x\to 2-0} \log x = \lim_{\varepsilon \to 0} \log(2-\varepsilon) = \log 2.$$

$\therefore$ by μ-test I_2 is convergent because $0 < \mu < 1$.

Hence the given integral is also convergent, it being the sum of two convergent integrals.

Ex. 54. *Test the convergence of* $\int_0^1 x^{p-1} e^{-x}\,dx$.

Sol. Let $f(x) = x^{p-1} e^{-x}$ and $I = \int_0^1 x^{p-1} e^{-x}\,dx$.

If $p \geq 1$, $f(x)$ is bounded throughout the interval $(0, 1)$ and so I is a proper integral and hence it is convergent if $p \geq 1$.

If $p < 1$, $f(x)$ is unbounded at $x = 0$. In this case, we have

$$\lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^\mu . x^{p-1} e^{-x} = \lim_{x\to 0} x^{\mu+p-1} e^{-x}$$
$$= 1 \text{ if } \mu + p - 1 = 0 \text{ i.e., } \mu = 1 - p.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < p < 1$, the given integral is convergent and when $\mu \geq 1$ *i.e.*, $p \leq 0$, the given integral is divergent.

Hence I is convergent if $p > 0$ and is divergent if $p \leq 0$.

Ex. 55. *Test the convergence of* $\displaystyle\int_0^{\pi/4} \frac{1}{\sqrt{(\tan x)}}\,dx$.

Sol. Here the integrand $f(x) = 1/\sqrt{(\tan x)}$ is unbounded at $x = 0$. Take $\mu = \frac{1}{2}$.

We have $\displaystyle\lim_{x\to 0} x^\mu . f(x) = \lim_{x\to 0} x^{1/2} \cdot \frac{1}{\sqrt{(\tan x)}}$

$$= \lim_{x\to 0} \sqrt{\left(\frac{x}{\sin x}\right)} \cdot \sqrt{(\cos x)} = 1.1 = 1.$$

Since $0 < \mu < 1$, therefore by μ-test the given integral is convergent.

§ 12. Abel's test.

If $\int_a^b f(x)\,dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $a \leq x \leq b$, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

§ 13. Dirichlet's test.

If $\int_{a+\varepsilon}^b f(x)\,dx$ *be bounded and* $\phi(x)$ *be bounded and monotonic on the interval* $a \leq x \leq b$, *converging to zero as* x *tends to* a, *then* $\int_a^b f(x)\,\phi(x)\,dx$ *converges.*

Miscellaneous Solved Examples

Ex. 56. *Test the convergence of* $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\,dx$.

Sol. We have $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\,dx = \int_0^{\pi/2} \left(\frac{\sin x}{x}\right) \cdot \frac{1}{x^n}\,dx.$

Now $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$. Therefore for $n \leq 0$, the integrand is bounded throughout the interval $(0, \pi/2)$ and so the given integral is a proper integral and hence it is convergent if $n \leq 0$.

If $n > 0$, the integrand is unbounded only at $x = 0$. In this case, we have

$$\lim_{x\to 0} x^\mu \frac{\sin x}{x^{n+1}} = \lim_{x\to 0} \left\{\left(\frac{\sin x}{x}\right) \cdot \frac{x^\mu}{x^n}\right\}$$
$$= \lim_{x\to 0} \left\{x^{\mu-n} \cdot \left(\frac{\sin x}{x}\right)\right\} = 1, \text{ if } \mu - n = 0 \text{ i.e., } \mu = n.$$

$\therefore$ by μ-test if $0 < \mu < 1$ *i.e.*, $0 < n < 1$, the given integral is convergent and if $\mu \geq 1$ *i.e.*, $n \geq 1$, the given integral is divergent.

Hence the given integral is convergent if $n < 1$ and divergent if $n \geq 1$.

Ex. 57. *Test the convergence of* $\displaystyle\int_0^{\pi/2} \frac{\cos x}{x^n}\,dx$.

Sol. When $n \leq 0$, the given integral is a proper integral and hence convergent.

When $n > 0$, the integrand becomes unbounded at $x = 0$.

Let $\qquad f(x) = \dfrac{\cos x}{x^n}$.

Then $\qquad \displaystyle\lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^{\mu-n} \cos x = 1$, if $\mu = n$.

Hence by μ-test it follows that the given integral is convergent when $0 < n < 1$, and divergent when $n \geq 1$.

From the above discussion we conclude that the given integral is convergent when $n < 1$, and divergent when $n \geq 1$.

Ex. 58. *Show that the integral* $\int_0^{\pi/2} \log \sin x \, dx$ *converges.*

Sol. The only point of infinite discontinuity of the integrand is $x = 0$.

Now $\lim_{x \to 0} x^{\mu} \log \sin x$, when $\mu > 0$

$$= \lim_{x \to 0} \frac{\log \sin x}{x^{-\mu}}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\cot x}{-\mu x^{-\mu-1}}$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{x^{\mu+1}}{\tan x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{(\mu+1) x^{\mu}}{\sec^2 x}, \qquad \text{[by L-Hospital's rule]}$$

$$= 0, \text{ if } \mu > 0.$$

Taking μ between 0 and 1, it follows from μ-test that the given integral is convergent.

Ex. 59. *Discuss the convergence of the integral*

$$\int_0^1 x^{n-1} \log x \, dx.$$

(Meerut 1982)

Sol. (i) Since $\lim_{x \to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral, when $n > 1$.

(ii) When $n = 1$, we have

$$\int_0^1 \log x \, dx = \lim_{\varepsilon \to 0} \int_{\varepsilon}^1 \log x \, dx = \lim_{\varepsilon \to 0} \left[x \log x - x\right]_{\varepsilon}^1$$

$$= \lim_{\varepsilon \to 0} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1.$$

$\therefore$ the integral is convergent if $n = 1$.

(iii) Let $n < 1$ and $f(x) = x^{n-1} \log x$.

Then $\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu+n-1} \log x$

$$= 0 \quad \text{if } \mu > 1 - n \qquad \ldots(1)$$

and $$= -\infty \quad \text{if } \mu \leq 1 - n. \qquad \ldots(2)$$

Hence when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying (1). The integral is therefore convergent by μ-test when $0 < n < 1$.

Again when $n \leq 0$, we can take $\mu = 1$ and satisfying (2). Hence by μ-test the integral is divergent when $n \leq 0$.

Therefore from (i), (ii) and (iii), we conclude that the given integral is convergent when $n > 0$ and divergent when $n \leq 0$.

Ex. 60. *Show that the integral*

$$\int_0^{\infty} e^{-a^2 x^2} \cos bx \, dx$$

is absolutely convergent.

Sol. We have

$$\int_0^{\infty} \left| e^{-a^2x^2} \cos bx \right| dx \leq \int_0^{\infty} \left| e^{-a^2x^2} \right| dx$$

$$= \int_0^{\infty} e^{-a^2x^2} dx = \int_0^c e^{-a^2x^2} dx + \int_c^{\infty} e^{-a^2x^2} dx, \text{ where } c > 0.$$

But the integral $\int_0^c e^{-a^2x^2} dx$ is a proper integral and hence convergent.

Also $\int_c^{\infty} e^{-a^2x^2} dx$ is convergent by μ-test, for we have

$$\lim_{x \to \infty} x^{\mu} e^{-a^2x^2} = \lim_{x \to \infty} \frac{x^{\mu}}{1 + a^2x^2 + \frac{a^4x^4}{2!} + \ldots}$$

$$= 0, \text{ for all values of } \mu.$$

Taking $\mu > 1$, we see that $\int_c^{\infty} e^{-a^2x^2} dx$ is convergent.

$\therefore$ $\int_0^{\infty} e^{-a^2x^2} dx$ is convergent.

Hence $\int_0^{\infty} \left| e^{-a^2x^2} \cos bx \right| dx$ is convergent, by comparison test.

From the above discussion it follows that the given integral is absolutely convergent.

Ex. 61. *Discuss the convergence or divergence of the integral*

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx.$$

(Rohilkhand 1978, Meerut 82)

Sol. Let $f(x) = \dfrac{x^{a-1}}{1+x}$. If $b > 0$, we can write

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \int_0^b \frac{x^{a-1}}{1+x} dx + \int_b^{\infty} \frac{x^{a-1}}{1+x} dx$$

$$= I_1 + I_2, \text{ say.}$$

Let $a \geq 1$. Then $f(x)$ is bounded throughout the interval $(0, b)$ and so the integral I_1 is a proper integral and hence it is convergent. To test the convergence of the infinite integral I_2 in this case, we have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{1+x} = \lim_{x \to \infty} \frac{x^{\mu+a-1}}{x+1}$$

$$= 1, \text{ if } \mu + a - 1 = 1$$

i.e., if $\mu = 2 - a$ which is ≤ 1 since $a \geq 1$.

Hence by μ-test I_2 is divergent.

$\therefore$ the given integral is divergent if $a \geq 1$.

Let a < 1. Then in the interval $(0, b), f(x)$ is unbounded only at $x = 0$. Also $f(x)$ is bounded throughout the interval (b, ∞). Therefore in this case I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to 0} \frac{x^{\mu+a-1}}{x+1} = 1 \text{ if } \mu + a - 1 = 0$$

i.e., if $\mu = 1 - a$.

If we take $0 < a < 1$, then we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $a \leq 0$, then $\mu \geq 1$ and so by μ-test I_1 is divergent.

To test the convergence of I_2 when $a < 1$, we have

$$\lim_{x \to \infty} x^{\mu} \cdot \frac{x^{a-1}}{x+1} = \lim_{x \to \infty} \frac{x^{\mu+a+1}}{x+1} = 1, \text{ if } \mu + a - 1 = 1$$

i.e., if $\mu = 2 - a$ which is > 1 since $a < 1$.

Hence by μ-test I_2 is convergent if $a < 1$.

Thus I_2 is convergent if $a < 1$. But I_1 is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

$\therefore$ the given integral is convergent if $0 < a < 1$ and is divergent if $a \leq 0$.

Hence the given integral is convergent if $0 < a < 1$ and is divergent if $a \geq 1$ or if $a \leq 0$.

Ex. 62. *Discuss the convergence of the Beta function*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

(Meerut 1986, Agra 70)

Sol. Let $f(x) = x^{m-1}(1-x)^{n-1}$.

The following different cases arise :

(i) When m and n are both ≥ 1, the integrand $f(x)$ is bounded throughout the interval $(0, 1)$ and so the given integral is a proper integral and is convergent.

(ii) When m and n are both < 1, the integrand $f(x)$ becomes infinite both at $x = 0$ and at $x = 1$. In this case we take $0 < a < 1$ and we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^a x^{m-1}(1-x)^{n-1}\, dx + \int_a^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$= I_1 + I_2, \text{ say.}$$

In the case of the integral I_1, the interval of integration is $(0, a)$ and so the integrand is unbounded at $x = 0$ only. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} . f(x) = \lim_{x \to 0} x^{\mu} . x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 0} x^{\mu+m-1}(1-x)^{n-1}$$

$$= 1, \text{ if } \mu + m - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - m.$$

If we take $0 < m < 1$, we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $m \leq 0$, we have $\mu \geq 1$ and so by μ-test I_1 is divergent.

Again in the case of the integral I_2, the interval of integration is $(a, 1)$ and so the integrand is unbounded at $x = 1$ only. To test the convergence of I_2, we have

$$\lim_{x \to 1-0} (1-x)^{\mu} . f(x) = \lim_{x \to 1-0} (1-x)^{\mu} x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 1-0} (1-x)^{\mu+n-1} x^{m-1}$$

$$= \lim_{\varepsilon \to 0} \{1-(1-\varepsilon)\}^{\mu+n-1}(1-\varepsilon)^{m-1}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{\mu+n-1}(1-\varepsilon)^{m-1} = 1 \text{ if } \mu + n - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - n.$$

If we take $0 < n < 1$, we have $0 < \mu < 1$ and so by μ-test I_2 is convergent. If we take $n \leq 0$, we have $\mu \geq 1$ and so by μ-test I_2 is divergent.

Thus if m and n are both < 1, the given integral is convergent only if $0 < m < 1$ and $0 < n < 1$.

(iii) When $m < 1$ and $n \geq 1$, the integrand $f(x)$ is unbounded only at $x = 0$. In this case by μ-test, the given integral is convergent if $0 < m < 1$ and is divergent if $m \leq 0$.

Again if $m \geq 1$ and $n < 1$, the integrand $f(x)$ is unbounded only at $x = 1$. In this case by μ-test, the given integral is convergent if $0 < n < 1$ and is divergent if $n \leq 0$.

Hence from (i), (ii) and (iii) it follows that the given integral is convergent if both m and n are > 0 and divergent otherwise.

Ex. 63. *Discuss the convergence of the Gamma function*

$$\int_0^{\infty} x^{n-1} e^{-x}\, dx.$$

(Meerut 1983, 92; Indore 79)

Sol. We can write

$$\int_0^{\infty} x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx$$

$$= I_1 + I_2, \text{ say.}$$

Let us first discuss the convergence of I_1.

Let $\qquad f(x) = x^{n-1} e^{-x}$.

If $n \geq 1, f(x)$ is bounded throughout the interval $[0, 1]$ and so I_1 is a proper integral and hence it is convergent if $n \geq 1$.

If $n < 1, f(x)$ is unbounded at $x = 0$. In this case we have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu} . x^{n-1} e^{-x} = \lim_{x \to 0} x^{\mu + n - 1} e^{-x}$$

$$= 1 \text{ if } \mu + n - 1 = 0 \text{ } i.e., \mu = 1 - n.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < n < 1$, the integral I_1 is convergent and when $\mu \geq 1$ *i.e.* $n \leq 0$, the integral I_1 is divergent.

$\therefore$ I_1 is convergent if $n > 0$ and is divergent if $n \leq 0$.

Now let us discuss the convergence of the integral I_2. The function $f(x) = x^{n-1} e^{-x}$ is bounded for all values of x in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} . x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu + n - 1}}{1 + x + \frac{x^2}{2!} + \dots}$$

$$= 0 \text{ for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral

$$I_2 = \int_1^{\infty} x^{n-1} e^{-x} dx \text{ is convergent for all values of } n.$$

Hence the given integral is convergent if $n > 0$ and is divergent if $n \leq 0$.

$\square$