Mains Test Series – 2019

TEST NO. 16

Section – A

Ques: 1(a) Prove that a non-empty subset H of an group G, is normal subgroup of G $\Leftrightarrow$ for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$.

Solution:-

Let H be a normal subgroup of G.
then we have to prove that
for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$

Now, we have

$$(gx)(gy)^{-1} = (gx)(y^{-1}g^{-1}) \quad [\because \text{By reversal Law of } G]$$
$$= g(xy^{-1})g^{-1} \quad (\text{by associative prop. of } G)$$
$$\in H \quad [\because xy^{-1} \in H,\ g \in G.]$$

[H is normal subgroup of G.]

Conversely suppose that;
for all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$
we prove that the non-empty subset H of G is a normal subgroup of G.
Now, let $x, y \in H$, then we have

$$xy^{-1} = e\,xy\,e^{-1} \quad [\because e \in G).$$

$$xy^{-1} = (ex)(y^{-1}e^{-1})$$
$$= (ex)(ey)^{-1} \in H$$

[$\because x, y \in H \subseteq G,\ e \in G$
$\Rightarrow (ex)(ey)^{-1} \in H$].

$\therefore\ xy^{-1} \in H$

$\Rightarrow$ H is a subgroup of G.

Again, let $h \in H \subseteq G$, $g \in G$

then we have.

$(gh)(ge)^{-1} \in H$

$\Rightarrow (gh)(e^{-1}g^{-1}) \in H$ [$\because$ By reversal law of G]

$\Rightarrow g(he)g^{-1} \in H$ [$\because e = e^{-1}$]

$\Rightarrow ghg^{-1} \in H$ [$\because he = h$]

$\Rightarrow$ H is normal subgroup of G.

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Ques: 1(b)) Let G be the group of non-zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication?

Solution:- Let us define $f: G \rightarrow R$, where R is a group of all positive real numbers

such that $f(x) = e^{2\pi i a}$ where $a \in G$

Now, f is well defined since

if $a, b \in G$ and

$$\Rightarrow a = b$$
$$\Rightarrow 2\pi i a = 2\pi i b$$
$$\Rightarrow e^{2\pi i a} = e^{2\pi i b}$$
$$\Rightarrow \boxed{f(a) = f(b)}$$

$\therefore$ f is onto since for every real R, $\exists$ $a \in G$

f is homomorphism

$$f(a+b) = e^{2\pi i(a+b)} = e^{2\pi i a} \cdot e^{2\pi i b}$$
$$= f(a) \cdot f(b).$$

Then; f is homomorphism function.

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By 1^{st} fundamental theorem of homomorphism,

$\frac{G}{N}$ is isomorphic to R if $N = $ Kernel

and

Now $\text{Ker} f = \{a \in G \mid e^{2\pi i a} = 1\}$

$= \{a \in G \mid e^{2\pi i a} = e^{2\pi i N}\}$

where N = natural numbers.

$= a \in N$.

Hence, $\boxed{\frac{G}{N} \cong R}$

Hence; G/N is isomorphic to the group of all positive real numbers under multiplication.

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Ques:-1 (c)> Define a compact set. Prove that the range of a continuous function defined on a compact set is compact.

Solution:-

Definition of a Compact Set

Let S be a subset of $\mathbb{R}$. S is said to be a Compact Set if every open cover $\mathcal{G}$ of S has a finite sub-cover. That is, if $\mathcal{G}$ be a family of open sets that covers S then there exists a finite subcollection $\mathcal{G}'$ of $\mathcal{G}$ such that $\mathcal{G}'$ also covers S.

To be explicit, if $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover of $S \subset \mathbb{R}$ then S will be compact if there exists a finite number of indices $\alpha_1, \alpha_2, \ldots\ldots, \alpha_m \in \Lambda$ such that
$S \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots\ldots \cup G_{\alpha_m}$.

(ii) Proof:

Let $\mathcal{G}$ be a family of open intervals $\{I_\alpha : \alpha \in \Lambda\}$, Λ being the index set, such that

$$f(D) \subset \bigcup_{\alpha \in \Lambda} I_\alpha .$$

Then, $\mathcal{G}$ is an open cover of $f(D)$.

Let $c \in D$. Then $f(c) \in f(D)$ and there exists an open interval of the family $\mathcal{G}$, say I_c, such that $f(c) \in I_c$

Since I_c is an open interval, it is open set.

So $f(c)$ is an interior point of I_c and there exists a neighbourhood of $f(c)$, say $N(f(c), \epsilon_c)$ such that

$$N(f(c), \epsilon_c) \subset I_c.$$

Since, f is continuous at c, there exists a $\delta_c > 0$ such that $f(x) \in N(f(c), \epsilon_c)$ for all $x \in N(c, \delta_c) \cap D$.

Clearly, the set of neighbourhoods $\{N(c, \delta_c) : c \in D\}$ covers D.

Since, D is compact, there exists a finite subcollection of the family of the neighbourhoods $\{N(c, \delta_c) : c \in D\}$ which covers D. Therefore there exists a finite number of points $c_1, c_2, \ldots, c_m$ in D such that

$$D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \ldots \cup N(c_m, \delta_{c_m})$$

Let $p \in f(D)$. Then there exists a point $q \in D$, such that $f(q) = p$

Since $D \subset N(c_1, \delta_{c_1}) \cup N(c_2, \delta_{c_2}) \cup \ldots \cup N(c_m, \delta_{c_m})$, $q \in N(c_k, \delta_{c_k})$ for some natural number $k \leq m$.

But $x \in N(c_i, \delta_{c_i}) \Rightarrow f(x) \in N(f(c_i), \epsilon_{c_i}) \subset I_{c_i}$ for each $i = 1, 2, \ldots m$.

As $q \in N(c_k, \delta_{c_k})$, $p \in N(f(c_k), \epsilon_{c_k}) \subset I_{c_k}$

Since, p is arbitrary; $f(D) \subset I_{c_1} \cup I_{c_2} \cup I_{c_3} \cup \ldots \cup I_{c_m}$

Thus, a finite subcollection of $\mathcal{G}$ covers $f(D)$ and therefore $f(D)$ is compact.

This completes the proof.

Alternate proof

Let $\{y_n\}$ be an sequence in $f(D)$. For each $n \in N$, let us choose $x_n \in D$ such that $f(x_n) = y_n$.
Since D is compact and $\{x_n\}$ is a sequence in D, there is a subsequence $\{x_{r_n}\}$ of $\{x_n\}$ such that $\{x_{r_n}\}$ converges to a point, say c, of D.

Since, f is continuous at c, the sequence $\{f(x_{r_n})\}$ converges to $f(c)$. That is, the subsequence $\{y_{r_n}\}$ of $\{y_n\}$ converges to a point $f(c)$ of $f(D)$.

Therefore every sequence in $f(D)$ has a subsequence that converges to a point of $f(D)$.
Consequently, $f(D)$ is compact.

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Ques: 1(d)) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\displaystyle\int_{|z|=1} \frac{z+3}{z^4+az^3}\,dz$; $(|a|>1)$.

Solution:- Given that,

$$\int_{|z|=1} \frac{z+3}{z^4+az^3}\,dz = \int_{|z|=1} \frac{z+3}{z^3(z+a)}\,dz$$

comparing the given integral with

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \int \frac{f(z)}{z^{2+1}}\,dz = \frac{2\pi i}{2!} f^{n}(z_0).$$

Since; $f(z) = \dfrac{z+3}{z+a}$ is analytic in $|z|=1$.

Here; $z_0=0$ and $z_0=0$ is a point inside $|z|=1$.

$\therefore$ We can apply the Cauchy's Integral formula:-

$$\int_{|z|=1} \frac{f(z)}{(z-z_0)^3} = \frac{2\pi i}{2!} f''(z_0) \qquad \text{—— (1)}$$

$$f(z) = \frac{z+3}{z+a} \Rightarrow f'(z) = \frac{(z+a)-(z+3)}{(z+a)^2}$$

$$f'(z) = \frac{a-3}{(z+a)^2} \Rightarrow f''(z) = (a-3)\left[\frac{-2}{(z+a)^3}\right]$$

$$f''(z) = \frac{-2(a-3)}{(z+a)^3}$$

$$\Rightarrow f''(z_0) = f''(0) = \frac{-2(a-3)}{(0+a)^3} = \frac{-2(a-3)}{a^3}.$$

$\therefore$ from ①

$$\int_{|z|=1} \frac{f(z)}{(z+0)^3} dz = \frac{\cancel{2}\pi i}{\cancel{2}} \times \frac{(-2)(a-3)}{a^3}$$

$$\Rightarrow \int_{|z|=1} \frac{f(z)}{(z+0)^3} = \frac{2\pi i(3-a)}{a^3}.$$

$$\therefore \Rightarrow \boxed{\int_{|z|=1} \frac{z+3}{z^3(z+a)} dz = \frac{2\pi i(3-a)}{a^3}}$$

(ii) $\displaystyle\int_{|z|=4} \frac{z^4}{(z-i)^3} dz.$

Solution:- Given that $\displaystyle\int_{|z|=4} \frac{z^4}{(z-i)^3} dz$

Here, $f(z) = z^4$ is analytic in $|z|=4$.

comparing the given integral with $\displaystyle\int_C \frac{f(z)}{(z-z_0)^n} dz$

Since, $f(z) = z^4$, $z_0 = i$ which is inside $|z|=4$.

$\therefore$ We can apply the Cauchy's Integral formula.

$$\int_{|z|=4} \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i}{2!} f''(z_0).$$

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$f(z) = z^4$

$\Rightarrow f'(z) = 4z^3$

$\Rightarrow f''(z) = 12z^2$

$\Rightarrow f''(z_0) = 12(i)^2 = -12.$

$$\therefore \int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz = \frac{\cancel{2}\pi i}{\cancel{2}} \times -12$$

$$\therefore \boxed{\int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz = -12\pi i}$$

required result.

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Q.1 (e) Find all the basic feasible solutions of the following problem.

$$2x_1 + 3x_2 + x_3 + x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$

and choose the one which maximize $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$.

Soln: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^4C_2 = 6$.

The characteristics of the various basic solutions are given below.

No. of basic solutions	Basic variables	Non-basic variables	values of basic variables	Is the solution feasible? (Are all $x_j \geq 0$?)	value of z	Is the solution optimal?
1	x_1, x_2	$x_3 = 0$, $x_4 = 0$	$2x_1 + 3x_2 = 8$, $x_1 - 2x_2 = -3$, $x_1 = 1, x_2 = 2$	Yes	8	
2	x_1, x_3	$x_2 = 0$, $x_4 = 0$	$2x_1 + x_3 = 8$, $x_1 + 6x_3 = -3$, $x_1 = \frac{51}{11}, x_3 = \frac{-14}{11}$	NO	—	—
3.	x_1, x_4	$x_2 = 0$, $x_3 = 0$	$2x_1 + x_4 = 8$, $x_1 - 7x_4 = -3$, $x_1 = \frac{53}{15}$, $x_4 = \frac{14}{15}$	Yes	$\frac{68}{5}$	
4.	x_2, x_3	$x_1 = 0$, $x_4 = 0$	$3x_2 + x_3 = 8$, $-2x_2 + 6x_3 = -3$, $x_2 = \frac{51}{20}, x_3 = \frac{7}{20}$	Yes	$\frac{181}{20}$	
5.	x_2, x_4	$x_1 = 0$, $x_3 = 0$	$x_2 = \frac{53}{19}, x_4 = \frac{-7}{19}$	NO.	—	—
6	x_3, x_4	$x_1 = 0$, $x_2 = 0$	$x_3 + x_4 = 8$, $6x_3 - 7x_4 = -3$, $x_3 = \frac{53}{13}, x_4 = \frac{51}{13}$	Yes.	$\frac{569}{13}$	Yes

Hence the optimal basic feasible solution is $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{53}{13}$, $x_4 = \frac{51}{13}$ and the maximum value of $z = \frac{569}{13} = 43.76$

Ques:- 2(b)(i) If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial y^2} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$

Solution:-

we have;

$$u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$$

which is not a homogenous function.

However, we write.

$$u = x^2 \left\{\tan^{-1}(y/x) - \left(\frac{y}{x}\right)^2 \tan^{-1}\left(\frac{x}{y}\right)\right\} = x^2 f(x/y)$$

$\Rightarrow z$ is a homogeneous function of degree '2'

$\therefore$ By Euler's theorem.

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \quad \text{——— (1)}$$

Differentiate (1) w.r.t x; we get

$$x \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial y \partial x} = 2 \frac{\partial u}{\partial x}.$$

$$x \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}. \quad \text{——— (2)}$$

Multiply 'x' in eqn (2), we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = x \frac{\partial u}{\partial x}. \quad \text{——— (3)}$$

Now differentiate eqn (1), w.r.t y, we have.

$$x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2 \frac{\partial u}{\partial y}.$$

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$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial y}.$$

Multiply above equation with y, we get

$$y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = y\frac{\partial u}{\partial y} \quad \text{——— (4)}$$

Adding (3) and (4), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$\therefore \boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u} \quad \text{from (1)}$$

Hence, the required result.

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Ques: 2(b)(ii) Let f be defined on $[0,1]$ as

$$f(x) = \begin{cases} \sqrt{1-x^2} & ; \text{if } x \text{ is rational} \\ 1-x & ; \text{if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Reimann integrals of 'f' over $[0,1]$?

Solution:-

Given; $$f(x) = \begin{cases} \sqrt{1-x^2} & ; \text{if } x \text{ is rational} \\ 1-x & ; \text{if } x \text{ is irrational} \end{cases}$$

Let $h(x) = \sqrt{1-x^2} - (1-x)$

$h(x) = \sqrt{1-x}\,(\sqrt{1+x} - \sqrt{1-x})$

Clearly, $h(x) > 0$ in $[0,1]$ ——— ①

Let partition, P be $\left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots, \frac{n}{n}=1\right\}$.

$[I_{r-1}, I_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]$; $\delta_r = \frac{1}{n}$

$L(P,f) = \sum m_r \delta_r = \sum (1-x)\delta_r = \sum \frac{1-r/n}{n}$

$\int_{-0}^{1} f(x)dx = \inf(L(P,f)) = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ —②

$U(P,F) = \sum M_r \delta_r = \sum \sqrt{1-x^2}\cdot \delta r = \sum \sqrt{1-\frac{r^2}{n^2}}\cdot\frac{1}{n}$

$\int_0^{-1} f(x)dx = \sup(U(P,f)) = \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$ —③

From ② and ③, we get.

$$\boxed{\operatorname{Inf}(L(P,f)) = \frac{1}{2} \quad \text{and} \quad \operatorname{Sup}(U(P,f)) = \frac{\pi}{4}.}$$

Given; function 'f' is defined on $[0,1]$ by.

$$f(x) = \begin{cases} \sqrt{1-x^2} & ; x \in [0,1]\cap Q \\ 1-x & ; x \in [0,1]-Q \end{cases}$$

f is bounded on $[0,1]$. For all $x \in (0,1)$

$\sqrt{1-x^2} > 1-x$.

Let ; $I = [0,1]$.

$f/(I\cap Q)$ is monotone decreasing on $I\cap Q$,

$f/I-Q$ is monotone decreasing on $I-Q$.

Let us take a partition P_n of $[0,1]$ defined by $P_n = (x_0, x_1, \ldots, x_n)$; where $x_r = \frac{r}{n}$.

Let; $$\left.\begin{array}{l} M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x) \\ m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x) \end{array}\right\} \begin{array}{l} \text{for } f(x); \\ \text{for } r = 1,2,3,\ldots,n. \end{array}$$

Since, $f/(I\cap Q)$ is monotone decreasing on $[x_{r-1}, x_r]\cap Q$

$$\sup_{x\in[x_{r-1},x_r]\cap Q} f(x) = f(x_{r-1}) = \sqrt{1-x_{r-1}^2} = \sqrt{1-\left(\frac{r-1}{n}\right)^2}$$

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Since; $f/(I-Q)$ is monotone decreasing on $[x_{r-1}, x_r]-Q$ and x_{r-1} is rational

$$\sup_{x \in [x_{r-1}, x_r]-Q} f(x) = \lim_{n\to\infty} f(u_n);$$

where $\{u_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $\boxed{x_{r-1} = 1 - x_{r-1} = 1 - \frac{r-1}{n}}$

Since; $1 - \frac{r-1}{n} < \sqrt{1 - \left(\frac{r-1}{n}\right)^2}$

$$\sup_{x \in [x_{r-1}, x_r]} f(x) = \sqrt{1 - \left(\frac{r-1}{n}\right)^2}$$

That is; $\boxed{M_r = \sqrt{1 - \left(\frac{r-1}{n}\right)^2}}$

Since, $f/(I \cap Q)$ is monotone decreasing on $[x_{r-1}, x_r] \cap Q$

$$\inf_{x \in [x_{r-1}, x_r] \cap Q} f(x) = f(x_r) = \sqrt{1 - \left(\frac{r}{n}\right)^2}$$

Since; $f/(I-Q)$ is monotone decreasing on $[x_{r-1}, x_r]-Q$ and x_r is rational

$$\inf_{x \in [x_{r-1}, x_r]} f(x) = \lim_{n\to\infty} f(v_n)$$

where $\{v_n\}$ is a sequence of irrational points in $[x_{r-1}, x_r]$ converging to $\boxed{x_r = 1 - x_r = 1 - \frac{r}{n}}$

Since; $1 - \frac{r}{n} \leq \sqrt{1 - \left(\frac{r}{n}\right)^2}$

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$$\inf_{x \in [x_{r-1}, x_r]} f(x) = 1 - \frac{r}{n}$$

$$\therefore \boxed{m_r = 1 - \frac{r}{n}}$$

$$U(P_n, f) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1})$$

$$= \frac{1}{n}\left\{\sqrt{1-\left(\frac{0}{n}\right)^2} + \sqrt{1-\left(\frac{1}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n-1}{n}\right)^2}\right\}$$

$$L(P_n, f) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1})$$

$$= \frac{1}{n}\left\{\left(1 - \frac{1}{n}\right) + \left(1 - \frac{2}{n}\right) + \cdots + \left(1 - \frac{n}{n}\right)\right\}$$

$$= \frac{1}{n^2}\{1 + 2 + 3 + \cdots + (n-1)\} = \frac{\cancel{n}(n-1)}{2 \cdot n^{\cancel{2}}} = \frac{n-1}{2n}.$$

Let, us considee the sequence of partition $\{P_n\}$ of $[0,1]$.

Here; $\|P_n\| = \frac{1}{n}$ and $\lim_{n \to \infty} \|P_n\| = 0$

$$\int_0^1 f = \lim_{n \to \infty} U(P_n, f) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1 - \left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \sqrt{1 - x^2}\, dx$$

Since $\sqrt{1-x^2}$ is integ. ble on $[0,1]$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1 = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$$

$$\boxed{\therefore \int_0^1 f = \lim_{n \to \infty} U(P_n, f) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1 - \left(\frac{r}{n}\right)^2} = \frac{\pi}{4}}$$

and $\boxed{\int_0^1 f = \lim_{n \to \infty} L(P_n, f) = \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2}}$

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Ques: 2(c)i) Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function $f(z)$ in terms of z ?

Solution:- Given; $u = y^3 - 3x^2y$

$$\frac{\partial u}{\partial x} = -6xy \quad ; \quad \frac{\partial^2 u}{\partial x^2} = -6y.$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = 6y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

Hence, u is harmonic function.

Let, v be the harmonic conjugate of u, then $f(z) = u + iv$ be analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -6xy.$$

Integrate both side w.r.t y, we get

$$v = -\frac{6xy^2}{2} + f(x) = -3xy^2 + f(x).$$

$$\frac{\partial v}{\partial x} = -3y^2 + f'(x). \quad \text{—— (A)}$$

and as $f(z)$ is analytic function.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -[3y^2 - 3x^2]$$

$$= 3x^2 - 3y^2 \quad \text{—— (B)}$$

from (A) and (B).

$$f'(x) = 3x^2$$

Integrate both side.

$$f(x) = x^3 + c.$$

$$\therefore v = -3xy^2 + x^3 + c$$

or

$$\boxed{v = x^3 - 3xy^2 + c}$$ which is required harmonic conjugate.

Now, applying Milne theorem's method

$$f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

$$f'(z) = -6xy - i(3y^2 - 3x^2)$$

$$f'(z) = -6xy + i(3x^2 - 3y^2) \quad \text{—— (C)}$$

Now, replace x by z and y by 0, in eq^n (C), we get.

$$f'(z) = -6z\cdot 0 + i(3z^2 - 0)$$

$$f'(z) = i\,3z^2$$

$$f(z) = i\,z^3 + c_1 \quad \text{[on integrating w.r.t } z\text{]}.$$

$$\boxed{f(z) = z^3 i + c_1}$$

which is required analytic function in terms of z.

Ques:- 2) c) ii) Prove that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta+p^2} = \pi\left[\frac{1-p+p^2}{1-p}\right]$, $0<p<1$.

Solution:- let $\displaystyle I = \int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta+p^2}$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta \; d\theta}{1-2p\cos 2\theta+p^2}$$

$$I = \frac{1}{2} \text{ real part of } \int_0^{2\pi} \frac{1+e^{6i\theta}}{1-p(e^{2i\theta}+e^{-2i\theta})+p^2} d\theta.$$

$$I = \frac{1}{2} \text{ real part of } \int_C \frac{1+z^6}{1-p\left(z^2+\frac{1}{z^2}\right)+p^2} \frac{dz}{iz}.$$

by putting $e^{i\theta} = z \Rightarrow \boxed{dz = iz\,d\theta}$

where 'C' denotes the unit circle $|z|=1$.

$$I = \frac{1}{2} \text{ real part of } \frac{1}{i}\int_C \frac{z(1+z^6)}{(1-pz^2)(z^2-p)} dz.$$

$$I = \frac{1}{2} \text{ real part of } \int_C f(z)\,dz \text{ , where}$$

$$\boxed{f(z) = \frac{z(1+z^6)}{(1-pz^2)(z^2-p)}}$$

Poles of $f(z)$ are given by $(1-pz^2)(z^2-p)=0$

Thus; $z = \pm\sqrt{p}$ and $z = \pm\frac{1}{\sqrt{p}}$ are the simple poles.

The only poles which lies with in C are $z = \pm\sqrt{p}$ as $p < 1$.

$\therefore$ Residue at $z = \sqrt{p}$ is

$$\lim_{z \to \sqrt{p}} \cancel{(z-\sqrt{p})} \cdot \frac{z(1+z^6)}{(1-pz^2)\cancel{(z-\sqrt{p})}(z+\sqrt{p})}$$

$$= \frac{\sqrt{p}\,(1+(\sqrt{p})^6)}{(1-p(\sqrt{p})^2)(\sqrt{p}+\sqrt{p})} = \frac{\cancel{\sqrt{p}}(1+p^3)}{(1-p^2)\cdot 2\cancel{\sqrt{p}}}$$

$$= \frac{1+p^3}{2(1-p^2)}. \quad \text{——— (A)}$$

$\therefore$ Residue at $z = -\sqrt{p}$ is $\Rightarrow \lim_{z \to -\sqrt{p}} \cancel{(z+\sqrt{p})} \cdot \frac{z(1+z^6)}{(1-pz^2)\cancel{(z+\sqrt{p})}(z-\sqrt{p})}$

$$= \frac{\cancel{-\sqrt{p}}(1+p^3)}{(1-p^2)(\cancel{-2\sqrt{p}})} = \frac{1+p^3}{2(1-p^2)} \quad \text{——— (B)}$$

Thus, from (A) & (B), sum of Residues $= \frac{1+p^3}{1-p^2}$

Hence, by Cauchy Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues within the contour.}$$

$$= 2\pi i \cdot \frac{(1+p^3)}{1-p^2}$$

$$I = \frac{1}{2} \text{ real part of } \frac{1}{i}\int_C f(z)\,dz = \frac{1}{\cancel{2}} \times \frac{1}{\cancel{i}} \times \frac{\cancel{2\pi i}(1+p^3)}{1-p^2}$$

$$I = \frac{\pi(1+p^3)}{1-p^2} = \frac{\pi\,\cancel{(1+p)}(1-p+p^2)}{(1-p)\cancel{(1+p)}} = \pi\left[\frac{1-p+p^2}{1-p}\right]$$

Hence the result.

Ques:-3(a) Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A$ for some positive integer $n\}$. Is B an ideal of R? Justify your answer?

Solution:- Let $a, b \in B$

Then $a^m \in A$ and $b^n \in A$, for some positive integers m and n.

Since, R is a commutative ring.

$$(a-b)^{m+n} = a^{m+n} - {}^{m+n}C_1 a^{m+n-1} b + \cdots + (-1)^{m+n} b^{m+n}$$

$$= a^m . a^n - ({}^{m+n}C_1) a^m . a^{n-1} . b + \cdots + (-1)^{m+n} b^m . b^n \in A$$

Since; $a^m \in A$, $b^n \in A$ and A is an ideal of R.

Thus, $(a-b) \in B$.

for any $r \in A$, $a \in B$, we have,

$(ra)^m = r^m a^m$ — since R is commutative.

Again, $r^m a^m \in A$

[$\because$ since $a^m \in A$, $r^m \in R$ & A is an ideal of R).

$\therefore ra \in B$.

Similarly, $ar \in B$.

Hence, B is an ideal of R.

[Ideal: A non-empty subset S of a ring R is called an Ideal of R, if

(i) $(S, +)$ is a subgroup of $(R, +)$

i.e $a, b \in S \Rightarrow a - b \in S$

(ii) $a \in S$ and $r \in R \Rightarrow ar \in S$ and $ra \in S$]

Ques:- 3(b) (A) Show that the series $\sum(-1)^n[\sqrt{n^2+1}-n]$ is conditionally convergent.

Solution:-

The series is $\sum(u_n)(-1)^n$

given $f(n) = \sum(-1)^n[\sqrt{n^2+1}-n]$

By comparing above two equations -

$$u_n = [\sqrt{n^2+1}-n]$$

$$\therefore u_n = \frac{[\sqrt{n^2+1}-n][\sqrt{n^2+1}+n]}{\sqrt{n^2+1}+n} = \frac{((n^2+1)^{1/2})^2 - n^2}{\sqrt{n^2+1}+n}.$$

$$u_n = \frac{1}{\sqrt{n^2+1}+n} = \frac{1}{n\left[\sqrt{1+\frac{1}{n^2}}+1\right]}$$

Let $V_n = \frac{1}{n}$.

$$\lim_{n\to\infty}\frac{u_n}{V_n} = \lim_{n\to\infty}\frac{1}{\left[\sqrt{1+\frac{1}{n^2}}+1\right]} = \frac{1}{1+1} = \frac{1}{2}.$$

$\Rightarrow$ $\sum u_n$ & $\sum V_n$ converges or diverges together

Now, $\sum\frac{1}{n}$ is divergent by P-Test as $P=1$.

Hence $\sum u_n$ is also divergent. $\therefore$ The given series is not absolutely convergent.

Now, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}+n} = 0$; $|u_n| \le \frac{1}{2}$.

$\therefore$ By leibnitz's test, the given series is conditionally convergent

Ques: 3(b) (B) (i) Check whether or not the following functions is Riemann integrable on $[0,1]$:

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(ii) Let $f : [-1,1] \longrightarrow [0,1]$ bedefined by $f(x) = |x|$. Check whether it is Riemann integrable.

Solution:- (i) $f(x) = \begin{cases} \frac{\sin x}{x} & ; \text{if } x \neq 0 \\ 1 & ; \text{if } x = 0 \end{cases}$

$$f(x+h) = \lim_{x \to 0^+} \frac{\sin x}{x} = \underset{x \to 0^+}{\text{lt}} \frac{\cos x}{1} = 1.$$

$$f(x-h) = \lim_{x \to 0^-} \frac{\sin x}{x} = \underset{x \to 0^-}{\text{lt}} \frac{\cos x}{1} = 1.$$

$\therefore f(x-h) = f(x) = f(x+h) = 1.$ at $x = 0$

Hence, the function is continuous on $(0,1]$

Therefore; the function is Reimann integrable on $[0,1]$.

(ii) $f(x) = |x| = \begin{cases} -x & ; x \in [-1,0] \\ x & ; x \in [0,1] \end{cases}$

$$\underset{x \to 0^-}{\text{lt}} f(x) = \underset{x \to 0^-}{\text{lt}} (-x) = 0$$

$$\underset{x \to 0^+}{\text{lt}} f(x) = \underset{x \to 0^+}{\text{lt}} (x) = 0$$

$\therefore f(x-h) = f(x) = f(x+h) = 0.$ at $x = 0$

$\therefore$ The function is continuous on $[-1,1]$.

Hence, the function is Reimann Integrable on $[-1,1]$.

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Q.3 (c) Obtain an optimal basic feasible solution of the following transportation problem.

	D_1	D_2	D_3	D_4	a_i
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

Sol^n → **Step 1:** It is balanced transportation problem. Using VAM. We get the following non-degenerate initial basic feasible solution:

$x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$.

Step : 2 Test of optimality.

Table - 1

	D_1	D_2	D_3	D_4	u_i
O_1	$\boxed{5}$ 19	32 30	60 50	$\boxed{2}$ 10	0
O_2	1 70	-18 30	$\boxed{7}$ 40	$\boxed{2}$ 60	50
O_3	11 40	$\boxed{8}$ 8	70 70	$\boxed{10}$ 20	10
v_j	19	-2	-10	10	

Enter the initial basic feasible solution in the upper left corner of the basic cell. Now, construct u_i and v_j and then find the cell evaluation for unoccupied cell using $D_{ij} = c_{ij} - \bar{c}_{ij}$: $\bar{c}_{ij} = u_i + v_j$, which are shown in circles.

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Table-1 Shows that $D_{22} = -18 < 0$. Hence, the basic feasible solution is not optimal.

Table-2 Cell evaluation -

	D_1	D_2	D_3	D_4
O_1				
O_2			(+) 2	2-2 (-)
O_3			(-) 8-2	(+) 10+2

$\rightarrow$

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7
O_2	70	$\boxed{2}$ 30	$\boxed{7}$ 40	60	9
O_3	40	$\boxed{6}$ 8	70	$\boxed{18}$ 20	18
b_j	5	8	7	14	

The second basic feasible solution :

$x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$, $x_{32} = 6$, $x_{34} = 12$, which is also a non-degenerate solution.

Step-3: Test of optimality of second basic feasible solution.

Table-3 :

	D_1	D_2	D_3	D_4	u_i
O_1	$\boxed{5}$ 19	32 30	42 50	$\boxed{2}$ 10	0
O_2	19 70	$\boxed{2}$ 30	$\boxed{7}$ 40	18 60	32
O_3	11 40	$\boxed{6}$ 8	52 70	$\boxed{12}$ 20	10
v_j	19	-2	8	10	

Table 3 indicates all $D_{ij} > 0$ (number under circle in unoccupied cells). Therefore, the second non-degenerate basic feasible solution is the unique optimal solution and is given by.

$$\langle x_{11}^* = 5,\ x_{14}^* = 2,\ x_{22}^* = 2,\ x_{23}^* = 7,\ x_{32}^* = 6,\ x_{34}^* = 12 \rangle$$

and the optimal cost $= 5\times 19 + 2\times 10 + 2\times 30 + 7\times 40 + 6\times 8 + 12\times 20$
$= 743$

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Q.4 (a)

Show that $d(a) < d(ab)$, where a, b be two non-zero elements of a Euclidean domain R and b is not a unit in R.

Sol^n: By the definition of the Euclidean domain,

we have $d(ab) \geq d(a)$.

Let 'b' be not a unit in R.

Since a and b are non-zero elements of the Euclidean domain R.

$\therefore$ ab is also a non-zero element of R.

Now $a \in R$ and $0 \neq ab \in R$

$\therefore$ by definition of Euclidean domain $\exists$ elements q and r in R such that

$$a = q(ab) + r. \quad \text{---} (1)$$

where either $r = 0$ or $d(r) < d(ab)$.

$$\text{---} (2)$$

If $r = 0$ then

$a = qab$.

$\Rightarrow a - qab = 0$

$\Rightarrow a(1 - qb) = 0$

$\Rightarrow 1 - qb = 0$ ($\because a \neq 0$ and R is an integral domain).

$\Rightarrow qb = 1$.

$\Rightarrow b$ is invertible.

$\Rightarrow b$ is a unit in R.

Thus we get a contradiction.

Hence r cannot be zero.

$\therefore$ we must have

$$d(r) < d(ab).$$

$$\text{i.e., } d(ab) > d(r). \quad \text{——— (2)}.$$

Also from (1),

we have $r = a - qab$

$$= a(1-qb).$$

$$\therefore d(r) = d[a(1-qb)]$$

But $d[a(1-qb)] \geq d(a)$ by the definition.

$$\therefore d(r) \geq d(a). \quad \text{——— (3)}$$

$\therefore$ from (2) & (3)

we have

$$d(a) \leq d(r) < d(ab)$$

$$\Rightarrow d(a) < d(ab).$$

Ques: 4(b)) State the Weierstrass M-test for uniform convergence of an infinite series of functions. Hence, prove that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $(-\infty, \infty)$?

Solution

Statement of the Weierstrass M-test for uniform convergence of an infinite series of functions:

Let $D \subset \mathbb{R}$ and $\sum f_n$ be a series of functions on D to $\mathbb{R}$. Let $\{M_n\}$ be a sequence of positive real numbers such that for all $x \in D$, $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$. If the series $\sum M_n$ be convergent then the series $\sum f_n$ is uniformly and absolutely convergent on D.

Now to prove the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $(-\infty, \infty)$.

Let $f_n(x) = \frac{x}{n(1+nx^2)}$

when $x = 0$; $f_n(x) = 0$

when; $x \neq 0$; $\frac{n}{|x|} + n^2|x| \geq 2n^{3/2}$,

the equality occurs when $|x| = \frac{1}{\sqrt{n}}$

or; $|f_n(x)| \leq \frac{1}{2n^{3/2}}$; the equality occurs when $|x| = \frac{1}{\sqrt{n}}$.

It follows that $|f_n(x)| \leq \frac{1}{2^{n^{3/2}}}$ for all $(-\infty, \infty)$ and for all $n \in \mathbb{N}$.

Let $M_n = \frac{1}{2n^{3/2}}$.

Then for all real x, $|f_n(x)| \leq M_n$ for all $n \in N$.

The series $\sum M_n$ is a convergent series of positive real numbers.

$\therefore$ By Weierstrass's M-test,

$\sum f_n$ is uniformly convergent for all $x \geq 0$.

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Q.4 (c) If $f(z) = u+iv$ is an analytic function of $z = x+iy$ and $u-v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition $f(\pi/2) = \dfrac{3-i}{2}$.

Sol'n: we have $u+iv = f(z)$ $\quad \therefore iu - v = if(z)$

On adding, we have

$u-v+i(u+v) = (1+i)f(z) = F(z)$ say

i.e. $(u-v)+i(u+v) = F(z)$

Let $U = u-v$, and $V = u+v$, then $U+iV = F(z)$ is an analytic function.

$$\text{Here } U = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x} = \frac{\cosh y + \sinh y - \cos x + \sin x}{\cosh y - \cos x}$$

$$= 1 + \frac{\sinh y + \sin x}{\cosh y - \cos x} = \left\{1 - \frac{\sin x + \sinh y}{\cos x - \cosh y}\right\}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{-1 - \sin x \sinh y + \cos x \cosh y}{(\cos x - \cosh y)^2} = \phi_1(x,y)$$

$$\text{and } \frac{\partial U}{\partial y} = \frac{1 - \sin x \sinh y - \cos x \cosh y}{(\cos x - \cosh y)^2} = \phi_2(x,y)$$

By Milne's method we have

$$F'(z) = [\phi_1(z,0) - i\phi_2(z,0)]$$

$$= -\frac{1}{1-\cos z} - i\frac{1}{1-\cos z}$$

$$= -(1+i)\frac{1}{1-\cos z} = -\frac{1}{2}(1+i)\operatorname{cosec}^2\frac{z}{2}$$

Integrating we get

$$F(z) = -\frac{1}{2}(1+i)\int \operatorname{cosec}^2 z/2 + c = (1+i)\cot z/2 + c$$

i.e. $(1+i)f(z) = (1+i)\cot z/2 + c$

$\Rightarrow f(z) = \cot z/2 + c_1$

But when $z = \pi/2$, $f(\pi/2) = \dfrac{3-i}{2}$

$\therefore c_1 = \dfrac{3-i}{2} - 1 = \dfrac{1-i}{2}$

Hence $f(z) = \cot z/2 + \frac{1}{2}(1-i)$

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Q.4 (d) food x contains 6 units of vitamin A per gram and 7 units of Vitamin B per gram and costs 12 paise per gram. food y contains 8 units of Vitamin A per gram and 12 units of Vitamin B and costs 20 paise per gram. The daily minimum requirements of vitamin A and Vitamin B are 100 units and 120 units respectively. find the minimum cost of product mix by simplex method.

Soln: Let x_1 grams of food x and x_2 grams of food Y be purchased. Then the problem can be formulated as: minimize $z=12x_1+20x_2$ subject to the constraints: $6x_1+8x_2 \geq 100$, $7x_1+12x_2 \geq 120$ and

$x_1, x_2 \geq 0$

Introducing the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ and artificial variables $a_1 \geq 0$, $a_2 \geq 0$, the constraints

$$6x_1+8x_2-x_3+a_1=100$$

$$7x_1+12x_2-x_4+a_2=120$$

objective function becomes

$$\text{Max } z' = -12x_1-20x_2+0x_3+0x_4-Ma_1-Ma_2$$

where $z'=-z$.

Now proceeding by usual simplex method, the solution is obtained as given in the table.

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Basic VAR	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	MIN Ratio (X_B/X_K)
a_1	$-M$	100	6	8	-1	0	1	0	100/8
$\leftarrow a_2$	$-M$	120	7	$\boxed{12}$	0	-1	0	1	120/12 $\leftarrow$
	$z' = -220M$		$-(13M+12)$	$(-20M+20)$	M	M	0	0	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	20	$\boxed{4/3}$	0	-1	$2/3$	1	×	60/4 $\leftarrow$
$\rightarrow x_2$	-20	10	$7/12$	1	0	$-1/12$	0	×	120/7
	$z' = -20M - 200$		$-\frac{(4M-1)}{3}$	0	M	$\frac{(2M+5)}{3}$	0	×	$\leftarrow \Delta_j$
$\rightarrow x_1$	-12	15	1	0	$-3/4$	$1/2$	×	×	
x_2	-20	$5/4$	0	1	$7/16$	$-3/4$	×	×	$\leftarrow \Delta_j \geq 0$
	$z' = -205$		0	0	$1/4$	9	×	×	

Since $\Delta_j \geq 0$ an optimal solution is attained.

Hence the optimal solution is:

$x_1 = 15$, $x_2 = 5/4$, $\max z = -(-205) = 205$

Hence 15 grams of food X and 5/4 grams of food Y should be the required product-mix with minimum cost of 205.

— Ans.

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Ques: 5(a)> Solve $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

Solution:- Here given equation is

$$(x^2-yz)p+(y^2-zx)q=z^2-xy.$$

Here the lagrange's auxillary equations are.

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{——} (1)$$

Choosing $(1,-1,0)$ and $(0,1,-1)$ as multipliers in turn each fraction of (1).

So that $\dfrac{dx-dy}{(x-y)(x+y+z)}=\dfrac{dy-dz}{(y-z)(x+y+z)}$

$$\Rightarrow \frac{d(x-y)}{x-y}=\frac{d(y-z)}{y-z}$$

$$\Rightarrow \log(x-y)=\log(y-z)+\log c_1$$

$$\Rightarrow \log\frac{(x-y)}{(y-z)}=\log c_1$$

$$\Rightarrow \boxed{\frac{x-y}{y-z}=c_1} \quad \text{——} (2)$$ c_1 being an arbitrary constant.

Choosing x, y, z as multiplier, each fraction of (1)

$$\Rightarrow \frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-xyz-xyz-xyz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz} = \frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{—— (3)}$$

Again choosing 1, 1, 1 as multiplier, each fraction of (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{—— (4)}$$

from (3) and (4)

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Rightarrow 2(x+y+z)\,d(x+y+z) - (2x\,dx + 2y\,dy + 2z\,dz) = 0$$

Integrating both side.

$$(x+y+z)^2 - (x^2+y^2+z^2) = 2C_2.$$

$$\Rightarrow \cancel{x^2}+\cancel{y^2}+\cancel{z^2}+2xy+2yz+2zx-\cancel{x^2}-\cancel{y^2}-\cancel{z^2} = 2C_2$$

$$\cancel{2}(xy+yz+zx) = \cancel{2}C_2.$$

$$\therefore \boxed{xy+yz+zx = C_2} \text{ —— (5)}$$ C_2 being an arbitrary constant

from (2) and (5), the required general solution is given by —

$$\boxed{\phi\,[xy+yz+zx,\ (x-y)/(y-z)] = 0}$$, ϕ being an arbitrary function.

required solution.

Ques:- 5(b)) Solve $(D^2 - 6DD' + 9D'^2)z = \tan(y+3x)$.

Solution:- Given to solve.

$$(D^2 - 6DD' + 9D'^2)z = \tan(y+3x)$$

By Putting $D=m$ and $D'=1$.

The auxillary equation is

$$(m^2 - 6m + 9) = 0$$

$$\Rightarrow (m-3)^2 = 0$$

$$\Rightarrow m = 3, 3.$$

$$\boxed{\therefore C.F = \phi_1(y+3x) + x\phi_2(y+3x)} \quad \text{---} ①$$

ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{(D-3D')^2}\tan(y+3x)$$

$$= \frac{x^2}{1^2 \times 2!}\tan(y+3x).$$

$$\therefore \boxed{PI = \frac{x^2}{2}\tan(y+3x)} \quad \text{---} ②$$

from ① and ②.

general solution $\Rightarrow y = C.F + PI$

$$\boxed{y = \phi_1(y+3x) + x\phi_2(y+3x) + \frac{x^2}{2}\tan(y+3x)}$$

Required Solution.

Q.5 (c) The current i in an electric circuit is given by $i = 10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i = 2$ amp.

Sol^n: Given function $i = 10e^{-t}\sin 2\pi t$

we have to find t such that $i = 2$ amp

$$\Rightarrow 10e^{-t}\sin 2\pi t = 2$$

let $f(t) = 5\sin 2\pi t - e^{t} = 0$

$$\therefore f'(t) = 10\pi\cos 2\pi t - e^{t}$$

By Newton's method,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$\Rightarrow t_{n+1} = t_n - \frac{5\sin 2\pi t - e^{t}}{10\pi\cos 2\pi t - e^{t}}.$$

taking $t_0 = 0$

$$t_1 = \frac{-(-1)}{10\pi - 1} = 0.03289$$

$$t_2 = 0.03289 - \frac{5\sin 2\pi(0.03289) - e^{0.03289}}{10\pi\cos 2\pi(0.03289) - e^{0.03289}}$$

$$= 0.03289 - \frac{-0.0075}{29.69887}$$

$$= 0.03263 \Rightarrow t_3 = 0.03314$$

$\therefore$ The value of t correct to 3 decimal places is $t = 0.033$ sec.

Ques: 5(d)) Draw a switching circuit that realizes the following switching function. If possible, draw a simpler switching circuit.

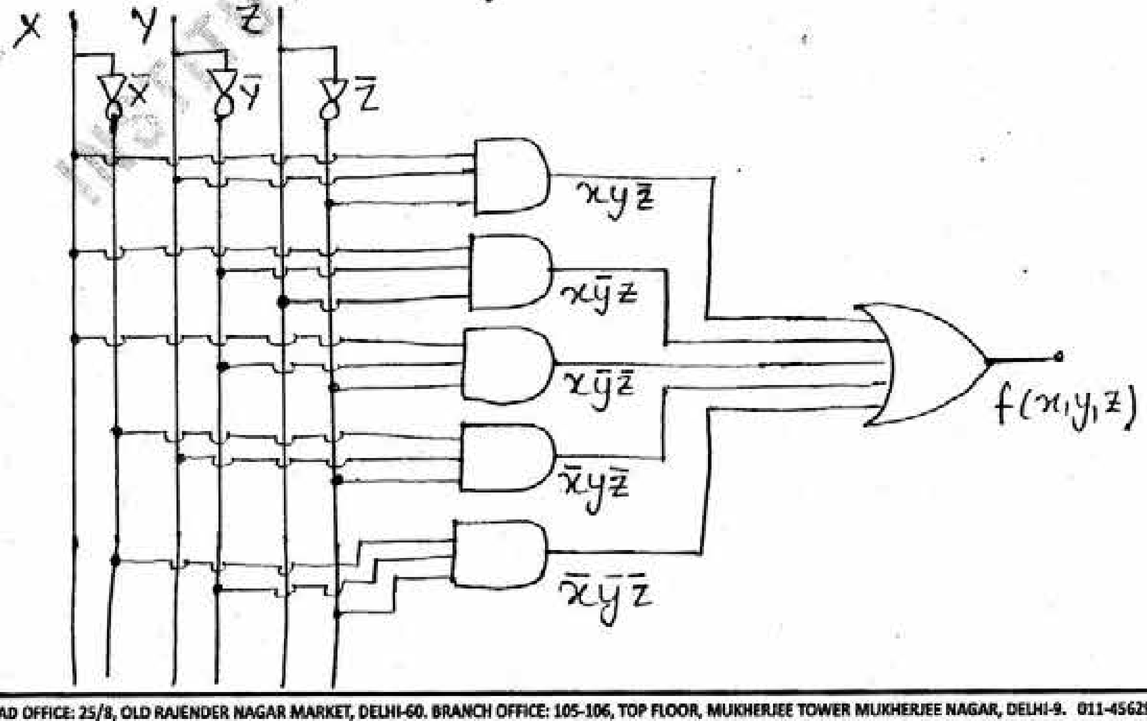
x	y	z	$f(x,y,z)$
1	1	1	0
1	1	0	1.
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

Solution:-

from the above truth table

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

Hence, the switching circuit will be.



f(x, y, z) can be simplied into simpler form so as the switching circuit.

Using Boolean Expression.

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}\bar{z}(y+\bar{y})$$

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}\bar{z} \quad [\because y+\bar{y}=1]$$

$$f(x,y,z) = x\bar{z}[y+\bar{y}] + \bar{x}\bar{z} + x\bar{y}z$$

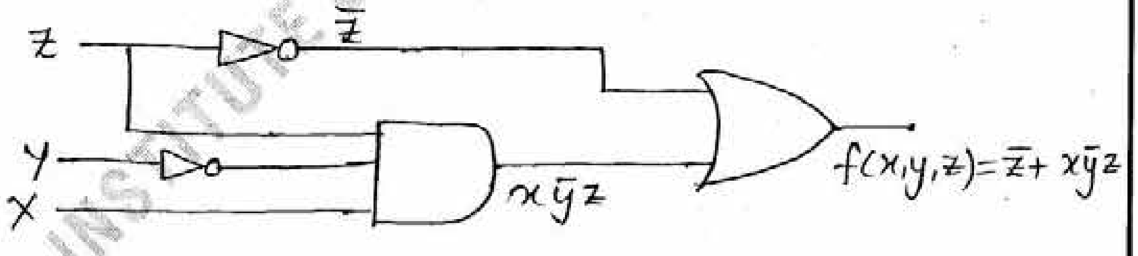
$$f(x,y,z) = x\bar{z} + \bar{x}\bar{z} + x\bar{y}z \quad [\because y+\bar{y}=1]$$

$$f(x,y,z) = \bar{z}(x+\bar{x}) + x\bar{y}z$$

$$\boxed{f(x,y,z) = \bar{z} + x\bar{y}z} \quad [\because x+\bar{x}=1]$$

which is simplified f(x, y, z).

Hence, simpler switching circuit is shown—

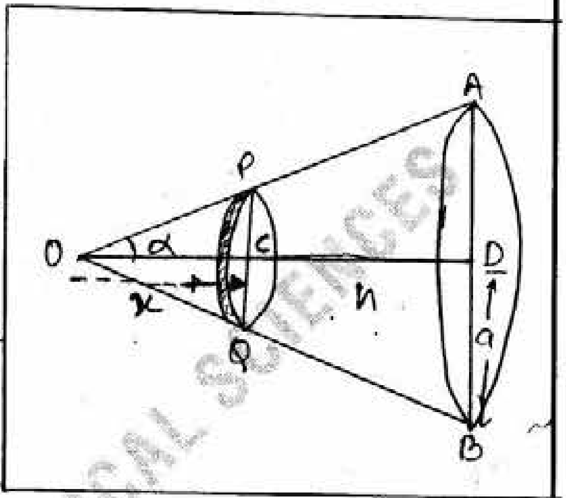


Ques: 5(e)} Find the M.I of a right solid cone of mass M, height h and radius of whose base is 'a', about the axis?

Solution:

Let 'O' be the vertex of the right solid cone of mass 'M', height 'h' and radius of whose base is 'a'.

If 'α' is the semi-vertical angle and ρ be the density of the cone, then



$$\boxed{M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha} \quad \text{——— (1)}$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance of 'x' from the vertex 'O'.

$\therefore$ Mass of the disc ; $\boxed{\delta m = \rho\pi x^2 \tan^2\alpha\,\delta x}$

M.I of this elementary disc about axis OD

$$= \frac{1}{2}\delta m\, CP^2 = \frac{1}{2}(\rho\pi x^2 \tan^2\alpha\,\delta x)\,x^2\tan^2\alpha$$

$$= \frac{1}{2}\rho\pi x^4 \cdot \tan^4\alpha \cdot \delta x .$$

$\therefore$ M.I. of the cone about the axis OD.

$$= \int_0^h \frac{1}{2}\rho\pi x^4 \cdot \tan^4\alpha\, dx = \rho\cdot\frac{\pi}{10}\cdot h^5\tan^4\alpha = \frac{3}{10}Mh^2\tan^2\alpha. \quad \text{(from (1))}$$

$$\boxed{\therefore \text{M.I of the cone about the axis OD} = \frac{3}{10}Ma^2}$$

required solution. [$\because \tan\alpha = a/h$].

Ques: 6(a)) Form a partial differential equation by eliminating the arbitrary function ϕ from

$$\phi(x^2+y^2+z^2,\ z^2-2xy)=0$$

Solution:- Let $u=x^2+y^2+z^2$ and $v=z^2-2xy$.

$\therefore \phi(u,v)=0$

$$\therefore \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

here $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$.

Now; $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=2y$, $\frac{\partial u}{\partial z}=2z$. and.

$\frac{\partial v}{\partial x}=-2y$, $\frac{\partial v}{\partial y}=-2x$, $\frac{\partial v}{\partial z}=2z$

$$\therefore \frac{\partial\phi}{\partial u}(2x+p\,2z)+\frac{\partial\phi}{\partial v}(-2y+p2z)=0$$

$$\therefore \boxed{\frac{\partial\phi}{\partial u}(x+pz)=(y-pz)\frac{\partial\phi}{\partial v}} \quad \text{——— (1)}$$

Now differentiating $\phi(u,v)=0$, w.r.t y.

$$\therefore \frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right]=0$$

$$\therefore \frac{\partial\phi}{\partial u}\left[2y+2zq\right]+\frac{\partial\phi}{\partial v}\left[-2x+2qz\right]=0$$

$$\therefore \boxed{(y+qz)\left(\frac{\partial\phi}{\partial u}\right)=(x-qz)\frac{\partial\phi}{\partial v}} \quad \text{——— (2)}$$

Dividing (1) by (2)

$$\frac{(x+pz)}{(y+qz)}=\frac{(y-pz)}{(x-qz)}$$

$$\therefore\ pz(y+x)-qz(y+x)=y^2-x^2.$$

$$z(p-q)(\cancel{y+x})=(y-x)(\cancel{y+x}) \Rightarrow \boxed{z(p-q)=y-x}$$

Required Solution

Q.6 (b) Find a surface satisfying the equation $D^2 z = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.

Sol^n:- Given that $D^2 z = 6x + 2$

i.e. $\frac{\partial^2 z}{\partial x^2} = 6x + 2$ —— (1)

Integrating (1) w.r.t x, we get

$\frac{\partial z}{\partial x} = 3x^2 + 2x + f(y)$ —— (2)

Integrating (2) w.r.t x, we get

$z = x^3 + x^2 + x f(y) + g(y)$ —— (3)

where $f(y)$ and $g(y)$ are arbitrary functions

The given surface $z = x^3 + y^3$ —— (4)

and the given plane is $x + y + 1 = 0$ —— (5)

Since (3) and (4) touch each other along their section by (5), the values of p and q at any point on (5) must be equal. thus we must have

$3x^2 + 2x + f(y) = 3x^2$ $\left[\because (3) = \frac{\partial z}{\partial x} = 3x^2 \quad (4) = \partial z/\partial x = 3x^2 + 2x + f(y)\right]$

$\Rightarrow f(y) = -2x$ —— (6)

and $x f'(y) + g'(y) = 3y^2$ —— (7) $\left[\because (4) = \frac{\partial z}{\partial y} = 3y^2 \quad (3) = \frac{\partial z}{\partial y} = x f'(y) + g'(y)\right]$

from (5) and (6),

$f(y) = -2x = 2(y+1)$ —— (8) $[\because x + y + 1 = 0 \Rightarrow x = -(y+1)]$

$\Rightarrow f'(y) = 2$

$\therefore (7) = 2x + g'(y) = 3y^2$

$\Rightarrow g'(y) = 3y^2 - 2x = 3y^2 + 2(y+1)$ [$\because$ by eqn (8)]

$\Rightarrow g'(y) = 3y^2 + 2y + 2$

Integrating it,

$g(y) = y^3 + y^2 + 2y + c$ —— (9)

where C is an arbitrary constant.

using ⑧ and ⑨, ③ gives

$$z = x^3 + x^2 + x[2(y+1)] + y^3 + y^2 + 2y + c \quad \text{—— (10)}$$

Now at the point of contact of ④ and ⑩ values of z must be the same and hence we have

$$x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c = x^3 + y^3$$

$$\Rightarrow x^2 + 2x(y+1) + y^2 + 2y + c = 0$$

$$\Rightarrow x^2 + 2x(-x) + (x+1)^2 - 2(x+1) + c = 0$$

$$\Rightarrow -x^2 + x^2 + 2x + 1 - 2x - 2 + c = 0 \quad \left[\because x+y+1=0 \Rightarrow y+1=-x \Rightarrow y = -(x+1)\right]$$

$$\Rightarrow \boxed{c = 1}$$

Putting this in ⑩, the required surface is

$$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + 1$$

$$\Rightarrow z = x^3 + y^3 + \underline{(x+y+1)^2}$$

Q.6 (c) Reduce $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) - y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$ to canonical form and hence solve it.

Soln: Rewriting the given equation $x^2 r - y^2 t = 0$ —— ①

Comparing ① with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0$, $y \neq 0$ and hence ① is hyperbolic. The ~~λ~~ quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x, -y/x$ and hence the corresponding characteristic equations become

$$\frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{y}{x} = 0$$

Integrating these $xy = c_1$ and $x/y = c_2$

In order to reduce ① to its canonical form, we choose $u = xy$ and $v = x/y$ —— ②

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+\frac{1}{y}\frac{\partial z}{\partial v},\ \text{using (2)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}-\frac{x}{y^2}\frac{\partial z}{\partial v},\ \text{using (2)}$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+\frac{1}{y}\frac{\partial z}{\partial v}\right)=y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+\frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=y\left(\frac{\partial^2 z}{\partial u^2}\times y+\frac{\partial^2 z}{\partial v\partial u}\times\frac{1}{y}\right)+\frac{1}{y}\left(\frac{\partial^2 z}{\partial v\partial u}\times y+\frac{\partial^2 z}{\partial v^2}\times\frac{1}{y}\right),$$

$$\therefore\ r=y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}-\frac{x}{y^2}\frac{\partial z}{\partial v}\right)=x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)-\left[-\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+\frac{2x}{y^3}\frac{\partial z}{\partial v}-\frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x\left[\frac{\partial^2 z}{\partial u^2}\times x+\frac{\partial^2 z}{\partial v\partial u}\times\left(-\frac{x}{y^2}\right)\right]+\frac{2x}{y^3}\frac{\partial z}{\partial v}-\frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u\partial v}\times x+\frac{\partial^2 z}{\partial v^2}\times\left(-\frac{x}{y^2}\right)\right]$$

$$\therefore\ t=x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v}+\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r & t in (1), we get

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right)-y^2\left(x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v}+\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)=0$$

$$\Rightarrow 4x^2\frac{\partial^2 z}{\partial u\partial v}-\frac{2x}{y}\frac{\partial z}{\partial v}=0\quad\text{or}\quad 2xy\frac{\partial^2 z}{\partial u\partial v}-\frac{\partial z}{\partial v}=0$$

$$\Rightarrow\quad 2u\left(\frac{\partial^2 z}{\partial u\partial v}\right)-\left(\frac{\partial z}{\partial v}\right)=0,\ \cancel{\text{using (2)}}\ \text{———}\ (3)$$

This is the required canonical form of (1)

Now we find the solution of (1), Multiplying bothsides of (3) by v, we get

$$2uv\frac{\partial^2 z}{\partial u\partial v}-v\frac{\partial z}{\partial v}=0\quad(\text{or})\ (2uv\,DD'-vD')z=0\ \text{———}\ (4)$$

where $D\equiv\partial/\partial u$ and $D'\equiv\partial/\partial v$. we now reduce (4) to a linear equation with Constant Coefficients by usual method.

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Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ & $Y = \log v$ — (5)

Let $D_1 \equiv \frac{\partial}{\partial X}$ and $D_1' = \frac{\partial}{\partial Y}$. Then (4) reduces to

$$(2D_1D_1' - D_1')z = 0 \Rightarrow D_1'(2D_1 - 1)z = 0$$

Its general solution is given by

$$z = e^{X/2}\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u),$$ using (5)

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy),$$ using (2)

where f and ψ_2 are arbitrary functions.

Q.6 (d) Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at $10°C$ and the other end is insulated. Further it is given that $Y(x,0) = 1 - x$, $0 < x < 1$.

Sol^n: Suppose the bar be placed along the x-axis with its one end (which is at $10°C$) at origin and other end at $x = 1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity). Then we are to solve heat equation,

$$\frac{\partial y}{\partial t} = k\left(\frac{\partial^2 y}{\partial x^2}\right) \quad \text{— (1)}$$

with boundary conditions $y_x(1,t) = 0$, $y(0,t) = 10$ — (2)

and initial conditions $y(x,0) = 1 - x$, $0 < x < 1$ — (3)

Let $y(x,t) = u(x,t) + 10$ — (4)

i.e., $u(x,t) = y(x,t) - 10$ — (5)

using (4) or (5), (1), (2) and (3) reduces to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{— (6)}$$

$$u_x(1,t) = 0, \quad u(0,t) = 0 \quad \text{— (7)}$$

and $u(x,0) = y(x,0) - 10 = -(x+9)$ — (8)

Suppose that (6) has solutions of the form $u(x,t) = X(x)T(t)$ — (9)

Substituting this value of u in (6), we get

$$XT' = kX''T \Rightarrow \frac{X''}{X} = \frac{T'}{kT} \quad \text{— (10)}$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ. Hence (10) gives. $X'' - \mu X = 0$ — (11)

and $T' = \mu k T$ ——— (12)

Using (7), (9) gives $X'(1)T(t) = 0$ and $X(0)T(t) = 0$ — (13).

Since $T(t) = 0$ leads to $u \equiv 0$, so we suppose that $T(t) \neq 0$.

Then, from (13), we get $X'(1) = 0$ and $X(0) = 0$ ——— (14)

we now solve (11) under B.C (14), three cases arise.

<u>Case I</u>. Let $\mu = 0$. Then solution of (11) is

$$X(x) = Ax + B \quad \text{———} \quad (15)$$

from (15), $X'(x) = A$ ——— (15)(i)

Using B.C (14), (15) and (15)(i) give $0 = A$ and $0 = B$

so from (15), $X(x) \equiv 0$, which leads to $u \equiv 0$, so reject $\mu = 0$.

<u>Case II</u>. Let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{———} \quad (16)$$

So that $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ——— (16)(i)

Using B.C (14), (16) and (16)(i) give $0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda}$ and $0 = A + B$

These give $A = B = 0$ so that $X(x) \equiv 0$ and hence $u(x) = 0$

So we reject and hence $u(x) \equiv 0$.

<u>Case III</u>: Let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (12) is

$$X(x) = A\cos\lambda x + B\sin\lambda x. \quad \text{———} \quad (17)$$

So that $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ ——— (17)(i)

Using B.C. (14), (17) and (17)(i) give $0 = -A\lambda\sin\lambda + B\lambda\cos\lambda$ and $0 = A$.

These give $A = 0$ and $\cos\lambda = 0$ ——— (18)

where we have taken $B \neq 0$. Since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$..

Now, $\cos\lambda = 0 \Rightarrow \lambda = (2n-1) \times \pi/2 = \frac{1}{2} \times (2n-1)\pi, n = 1, 2, 3 \ldots$

So, $\mu = -\lambda^2 = -\frac{1}{4} \times (2n-1)^2 \pi^2$ ——— (19)

Hence non-zero solutions $X_n(x)$ of (17) are given by

$$X_n(x) = B_n \sin\{(2n-1)\pi x/2\}.$$

Again using (19), (12) reduces to

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$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4} T \Rightarrow \frac{dT}{T} = -C_n^2\, dt \quad \text{—— (20)}$$

where $C_n^2 = \frac{1}{4} \times (2n-1)^2 \pi^2 k$ —— (21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ —— (22)

Thus, $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7), Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{—— (23)}$$

putting $n=0$ in (23) and using (8),

we have $-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}$ —— (24)

Multiplying both sides of (24) by $\sin\frac{(2m-1)\pi x}{2}$ and then integrating w.r.t x from 0 to 1, we get

$$-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\, dx = \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2} \sin\frac{(2m-1)\pi x}{2}\, dx \quad \text{—— (25)}$$

But $\int_0^1 \sin\frac{(2n-1)\pi x}{2} \sin\frac{(2m-1)\pi x}{2}\, dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n. \end{cases}$ —— (26)

using (26), (25) gives $-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\, dx = E_m$

$$\therefore E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}\, dx \quad \text{——}$$

$$\Rightarrow E_n = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{(2n-1)\pi/2}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{(2n-1)^2\pi^2/4}\right\}\right]_0^1$$

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$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \qquad \left[\because \cos(2n-1)\pi/2 = 0 \text{ and } \sin(2n-1)\pi/2 = (-1)^{n-1}\right]$$

——— (27)

Using (23) and (4), the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$$

where C_n and E_n are given by (21) and (27) respectively.

SCIENCES

Ques: 7(a). Convert the following binary numbers to the base indicated:

(i) $(10111011001.101110)_2$ to octal.

Solution:- Binary numbers can be converted into equivalent octal numbers by making groups of 3 digit/bits starting from LSB and moving towards MSB for integer part of the number and then replacing each group of 3 bits by its octal representation.
for fractional part, the grouping of 3 bits are made from the binary point.

Given, $(10111011001.101110)_2$

$$= [\underbrace{010}_{2}\ \underbrace{111}_{7}\ \underbrace{011}_{3}\ \underbrace{001}_{1} . \underbrace{101}_{5}\ \underbrace{110}_{6}]_2$$

$$= (2731.56)_8.$$

(ii) $(10111011001.10111000)_2$ to Hexadecimal.

Solution:- Here, Binary numbers converted into equivalent hexadecimal number by making groups of 4 bits and in the same way from LSB to MSB assign the hexadecimal equivalent of 4 bits.

$$[\underbrace{0101}_{5}\ \underbrace{1101}_{D}\ \underbrace{1001}_{9} . \underbrace{1011}_{B}\ \underbrace{1000}_{8}]_2$$

$$= (5D9.B8)_{16}.$$

(iii) $(0.101)_2$. to decimal

Solution> $(0.101)_2$

$= 1\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3}$

$= 1\times 0.5 + 0 + 1\times 0.125$

$= 0.5 + 0.125$

$= 0.625$

$\Rightarrow \boxed{(0.101)_2 \longleftrightarrow (0.625)_{10}}$

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Q.7 (b) A missile is launched from a ground station. The acceleration during its first 80 second of flight, as recorded, is given in the following table:

t(s)	0	10	20	30	40	50	60	70	80
a(m/s²)	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Compute the velocity of the missile when t = 80s, using Simpson's 1/3rd rule.

Soln since acceleration is defined as the rate of change of velocity

we have $\frac{dv}{dt} = a$ (or) $v = \int_0^{80} a\, dt$

using Simpson's 1/3rd rule, we have

$$v = \frac{h}{3}\left[(y_0+y_8) + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)\right]$$

$$= \frac{10}{3}\left[(30+50.67) + 4(31.63+35.47+40.33+46.69) + 2(33.34+37.75+43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

Therefore the required velocity is given by

$$v = 3.0861 \text{ Km/s}.$$

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Ques: 7(c)) Using fourth order Runge-Kutta method, find the solution of the initial value problem.

$y' = \frac{1}{(x+y)}$, $y(0) = 1$ in the range $0.5 \leq x \leq 2.0$, by taking $h = 0.5$?

Solution:- Formula for Runge-Kutta Method

$$y_{n+1} = y_n + K \quad ; \text{where } K = \frac{1}{6}[K_1 + K_4 + 2(K_2 + K_3)].$$

where;
$$K_1 = h f(x_n, y_n)$$
$$K_2 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}K_1)$$
$$K_3 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}K_2).$$
$$K_4 = h f(x_n + h, y_n + K_3).$$

Given; $y' = \frac{1}{x+y}$ i.e $\frac{dy}{dx} = \frac{1}{x+y}$.

at $x = 0$; $y(0) = 1$; $h = 0.5$

For; $y(0.5)$, $y(1.0)$, $y(1.5)$, $y(2.0)$; we need four iterations.

I[st] iteration:- To find $y(0.5) = y(0) + K.$, $h = 0.5$, $x = 0$, $y(0) = 1.$

$$K_1 = 0.5 f(0, 1) = 0.5 \times 1 = 0.5$$

$$K_2 = 0.5 f(0.25, 1.25) = 0.5 \times \frac{1}{1.5} = \frac{1}{3} = 0.3333.$$

$$K_3 = 0.5 f(0.25, 1.1667) = \frac{0.5}{1.41667} = 0.353$$

$$K_4 = 0.5 f(0.5, 1.353) = \frac{0.5}{1.853} = 0.2698$$

$$y(0.5) = y(0) + K = 1 + \frac{1}{6}\left[0.5 + 0.2698 + 2(0.333 + 0.353)\right]$$

$$y(0.5) = 1 + \frac{1}{6} \times 2.1425 = 1 + 0.3571$$

$$\boxed{y(0.5) = 1.3571}$$

<u>IInd Iteration</u>

$y(0.5) = 1.3571$, $x = 0.5$, $h = 0.5$; $y(1.0) = ?$

$$K_1 = hf(0.5, 1.3571) = 0.5 \times \frac{1}{0.5 + 1.3571}$$

$$K_1 = \frac{0.5}{1.857} = 0.269.$$

$$K_2 = 0.5 f(0.75, 1.4916) = \frac{0.5}{2.2416} = 0.22305$$

$$K_3 = 0.5 f(0.75, 1.4685) = \frac{0.5}{2.21853} = 0.2254$$

$$K_4 = 0.5 f(1, 1.5824) = \frac{0.5}{2.5824} = 0.19362$$

$$y(\ldots) = y(0.5) + \frac{1}{6}\left[K_1 + K_4 + 2(K_2 + K_3)\right]$$

$$y(1.0) = 1.357 + \frac{1}{6}\left[0.269 + 0.194 + 2(0.223 + 0.225)\right]$$

$$y(1.0) = 1.3571 + \frac{1}{6}\left[1.359\right] = 1.357 + 0.2265$$

$$\boxed{\therefore\ y(1.0) = 1.5836}$$

<u>IIIrd Iteration</u>

$y(1.0) = 1.5836$, $h = 0.5$, $x = 1$, $y(1.5) = ?$

$$y(1.5) = y(1.0) + K = y(1.0) + \frac{1}{6}\left[K_1 + K_4 + 2(K_2 + K_3)\right]$$

$K_1 = 0.5 f(1, 1.5836) = \dfrac{0.5}{2.5836} = 0.1935$

$K_2 = 0.5 f(1.25, 1.6804) = \dfrac{0.5}{2.9304} = 0.1706$

$K_3 = 0.5 f(1.25, 1.6689) = \dfrac{0.5}{2.9189} = 0.1713$

$K_4 = 0.5 f(1.5, 1.7549) = \dfrac{0.5}{3.2549} = 0.1536$

$y(1.5) \doteq y(1.0) + \frac{1}{6}(0.1935 + 0.1536 + 2(0.1706 + 0.1713))$

$y(1.5) = 1.5836 + \frac{1}{6}[1.0309] = 1.5836 + 0.1718$

$\boxed{y(1.5) = 1.7554}$

IVth Iteration : $y(1.5) = 1.7554$, $x = 1.5$, $h = 0.5$, $y(2.0) = ?$

$y(2.0) = y(1.5) + \frac{1}{6}(K_1 + K_4 + 2(K_2 + K_3))$.

$K_1 = 0.5 f(1.5, 1.7554) = \dfrac{0.5}{3.2554} = 0.1536$

$K_2 = 0.5 f(1.75, 1.8322) = 0.5/3.5822 = 0.1396$

$K_3 = 0.5 f(1.75, 1.8252) = 0.5/3.5752 = 0.1398$

$K_4 = 0.5 f(2, 1.8952) = 0.5/3.8952 = 0.1284$

$y(2.0) = y(1.5) + \frac{1}{6}(0.1536 + 0.1284 + 2(0.1396 + 0.1398))$

$y(2.0) = 1.7554 + \frac{1}{6} \times 0.8408$

$y(2.0) = 1.7554 + 0.1401$

$\boxed{y(2.0) = 1.8955}$

required Solution.

Hence; $y(0.5) = 1.3571$; $y(1.0) = 1.5836$; $y(1.5) = 1.7554$
$y(2.0) = 1.8955$

Ques: 7(d)) Draw a flow chart to declare the results for the following examination system:

60 candidates takes the examination. Each candidate writes one major and two minor papers.

A candidate is declared to have passed in the examination if she/he gets a minimum of 40 in all the three papers separately and an average of 50 in all the three papers put together.

Remaining candidates fail in the examination with an exemption in major if they obtained 60 and above and exemption in each minor if they obtain 50 and more in that minor.

Solution:-

→ Let, i be the no. of candidates appeared in examination; where $i = 1$ to 60, so repeat it 60 times.

→ let M be the major subject and m_1 and m_2 be the two minor subjects.

→ To be declared passed a candidate has to obtain a minimum of 40 marks in each paper and an average of 50 in all three put together.

i.e $M \geq 40$, $m_1 \geq 40$, $m_2 \geq 40$

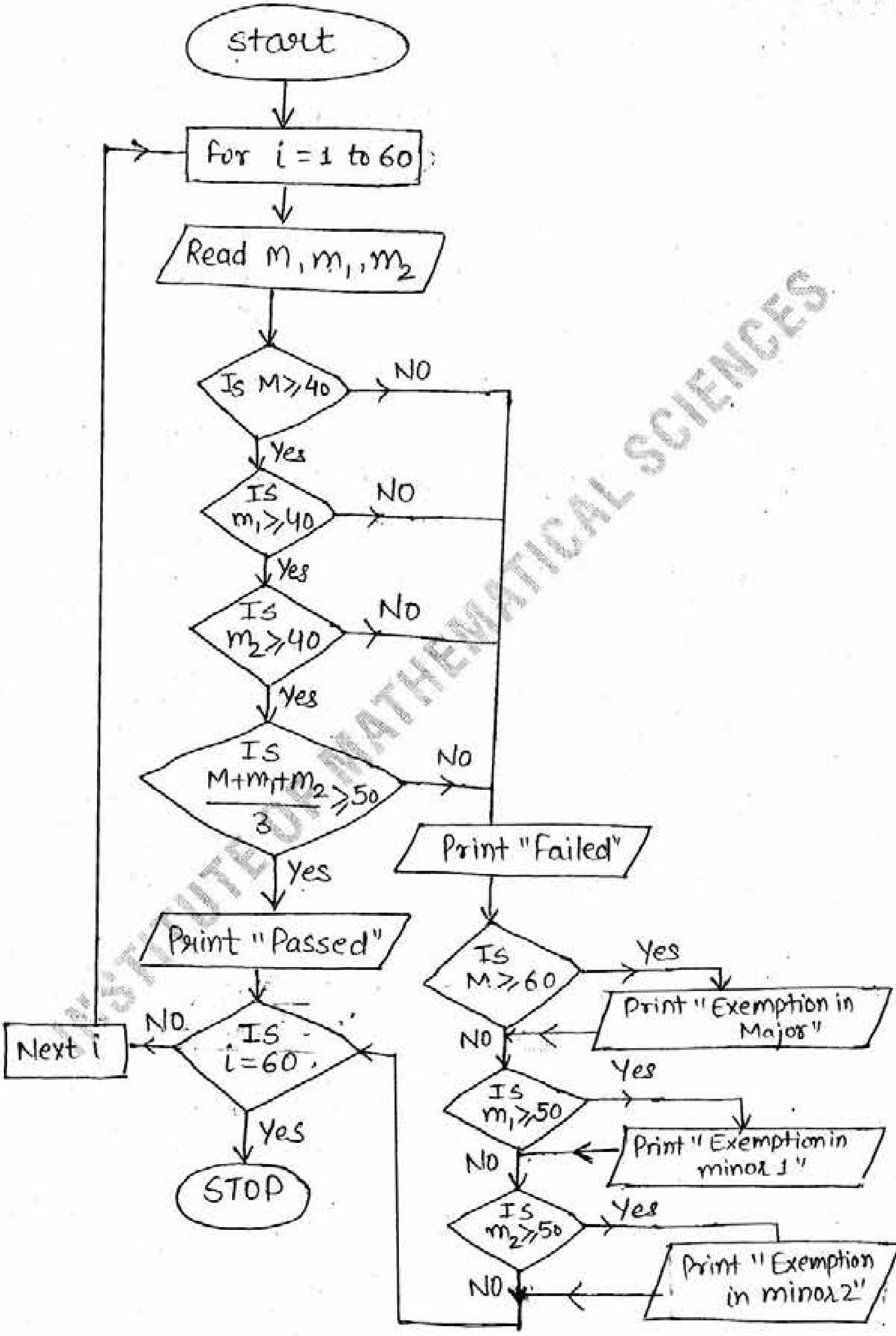
$$\frac{M + m_1 + m_2}{3} \geq 50 \Rightarrow M + m_1 + m_2 \geq 150$$

→ Else fail,

Also; if $M \geq 60$ → Exempted in Major

$m_1 \geq 50$ → Exempted in 1^{st} minor

$m_2 \geq 50$ → Exempted in 2^{nd} minor.

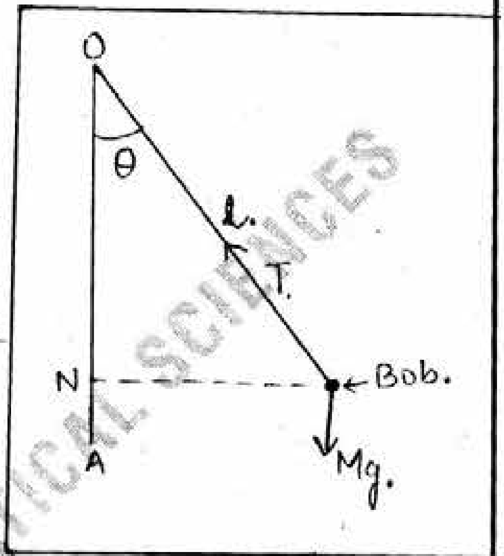


Hence, the required flow chart is drawn.

Ques: 8 (a) Use Hamilton's equations to find the equation of motion of the (i) simple pendulum & (ii) Compound pendulum.

Solution:

(i) For Simple pendulum:



Let 'l' be the length of the pendulum and M the mass of bob. At time t, let 'θ' be the inclination of the string to the downward, vertical. Then, if T and V are the kinetic and potential Energies of the pendulum, then

$$\boxed{T = \frac{1}{2}M(l\dot{\theta})^2 = \frac{1}{2}Ml^2\dot{\theta}^2}$$

and V = work done against $Mg = Mg\,A'B$

$$\boxed{V = Mgl(1-\cos\theta)}$$

$$\therefore \boxed{L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)} \quad \text{——— (1)}$$

Here 'θ' is the only generalised co-ordinate.

$$\boxed{\therefore\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta}} \quad \text{——— (2)}$$

Since, L does not contain t explicitly,

$$\therefore\ H = T + V = \frac{1}{2}Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta$$

$$H = \frac{p_\theta^2}{2Ml^2} + Mgl(1-\cos\theta) \quad \text{—(from (2))}$$

Here, the two Hamilton's equations are –

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \Rightarrow \boxed{\dot{p}_\theta = -Mgl\sin\theta} \quad \text{——— (3)}$$

and $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \boxed{\dot{\theta} = p_\theta/(Ml^2)} \quad \text{——— (4)}$$

Differentiating (4), we get.

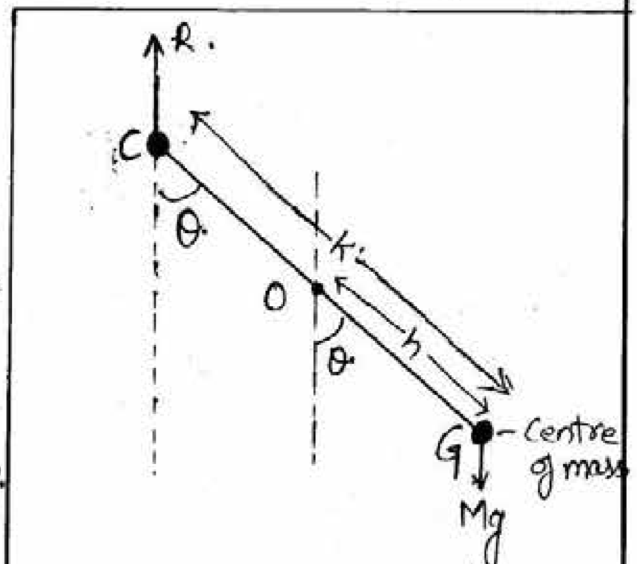
$$\ddot{\theta} = \dot{p}_\theta/(Ml^2) = \frac{-Mgl\sin\theta}{Ml^2} \quad (\text{from } (3))$$

or $$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

which is the equation of motion of a simple pendulum.

ii) For Compound pendulum:-

At time t, let θ be the angle between the vertical plane through the fixed axis (plane fixed in the space) and the plane through the C.G. 'G' and the fixed axis (plane fixed in the body)



Let $OG = h$

If T and V are kinetic and potential energies of the pendulum, then

$$\boxed{T = \frac{1}{2}Mk^2\dot{\theta}^2} \text{ and } \boxed{V = -Mgh\cos\theta.}$$

(Negative sign is taken because G is below the fixed axis.)

$\therefore L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta$

Here 'θ' is the only generalised co-ordinate,

$\therefore P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}Mk^2\dot{\theta}\times 2 = Mk^2\dot{\theta}$ —— (1)

Since, L does not have 't' explicitly,

$\therefore H = T + V = \frac{1}{2}Mk^2\dot{\theta}^2 - Mgh\cos\theta$

$= \frac{1}{2}\cancel{M}\cancel{k^2}\cdot\frac{P_\theta^2}{M^{\cancel{2}}k^{\cancel{4}2}} - Mgh\cos\theta$

$H = T + V = \frac{1}{2Mk^2}P_\theta^2 - Mgh\cos\theta$ —— [from (1)]

Hence, the two Hamilton equations are:-

$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -Mgh\sin\theta$ —— H_1.

$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{1}{Mk^2}P_\theta$ —— H_2

Differentiating (H_2) and substituting from (H_1), we get

$\ddot{\theta} = \frac{1}{Mk^2}\dot{P}_\theta = \frac{1}{Mk^2}(-Mgh\sin\theta)$

or $\boxed{\ddot{\theta} = -\frac{gh}{k^2}\sin\theta}$

which is the equation of motion of a compound pendulum.

Ques: 8(b)) Show that the velocity potential $\phi = \frac{a}{2}x(x^2+y^2-2z^2)$ satisfies the Laplace equation. Also determine the streamlines.

Solution:- We know that the velocity q of the fluid is given by-

$$q = -\nabla\phi = -\left[i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right]\left\{\frac{a}{2}(x^2+y^2-2z^2)\right\}$$

$$q = -(a/2)\times(2x\hat{i}+2y\hat{j}-4z\hat{k}) \quad \text{——} (1)$$

$$q = u\hat{i}+v\hat{j}+w\hat{k} \quad \text{——} (2)$$

Comparing (1) and (2), we get

$$u=-ax \;,\quad v=-ay \;,\quad w=2az$$

The equations of streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \Rightarrow \frac{dx}{-x} = \frac{dy}{-y} = \frac{dz}{2z}$$

or $$\frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z} \quad \text{——} (3)$$

from the first two fractions of (3)

$$\frac{1}{x}dx = \frac{1}{y}dy \Rightarrow \text{Integrating}$$

$$\Rightarrow \log x = \log y + \log c_1 \Rightarrow \boxed{\frac{x}{y} = c_1}$$

or $$\boxed{x = c_1 y} \quad \text{——} (4)$$

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Taking the last two fractions of (3),

$$\frac{2\,dy}{y} = \frac{dz}{-z} \Rightarrow \text{Integrating.}$$

$$2 \cdot \log y = -\log z + \log c_2$$

$$\Rightarrow \log y^2 + \log z = \log c_2$$

$$\boxed{c_2 = y^2 z} \quad \text{——} (5)$$

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that $\phi = \left(\frac{a}{2}\right) \times (x^2 + y^2 - 2z^2)$ —— (6)

from (6); $\frac{\partial \phi}{\partial x} = ax$, $\frac{\partial \phi}{\partial y} = ay$ & $\frac{\partial \phi}{\partial z} = -2az$.

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = a \;;\; \frac{\partial^2 \phi}{\partial y^2} = a \quad \text{and} \quad \frac{\partial^2 \phi}{\partial z^2} = -2a$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = a + a - 2a = 0.$$

or $\boxed{\nabla^2 \phi = 0}$

Showing that ϕ satisfies the Laplace Equation.

Ques: 8(c)) Given that the velocity potential

$$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}.$$ Determine the streamlines.

Solve:- To determine the stream lines, we use the fact that velocity potential ϕ and the stream function ψ satisfy the Cauchy-Riemann equations, namely-

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \quad \text{———} (1)$$

and given;

$$\phi = \frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2} \quad \text{———} (A)$$

$$\frac{\partial\phi}{\partial x} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

$$\therefore \frac{\partial\psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

Integrating it w.r.t. y, we get-

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + f(x) \quad \text{———} (2)$$

$f(x)$ being an arbitrary function of x

$$\frac{\partial\psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} + f'(x) \quad \text{———} (3)$$

Also $\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}$

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and from (A)

$$\frac{\partial\phi}{\partial y}=\frac{y}{(x+a)^2+y^2}-\frac{y}{(x-a)^2+y^2}$$

$$\therefore\ \frac{\partial\psi}{\partial x}=\frac{-\partial\phi}{\partial y}=-\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2}\quad\text{—— (4)}$$

comparing (3) and (4), $f'(x)=0$,

that gives;

$$\psi=\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x-a}$$

$$\psi=\tan^{-1}\frac{[y/(x+a)]-[y/(x-a)]}{1-[y/(x+a)][y/(x-a)]}$$

$$\therefore\ \boxed{\psi=\tan^{-1}\frac{(-2ay)}{x^2+y^2-a^2}}$$

Hence, the streamlines are given by.

$\psi=$ Contant $=\tan^{-1}(-2a/c)$, that is

$$\boxed{x^2+y^2-\ cy=a^2}\Rightarrow \text{as}\begin{bmatrix}cy=0\\ y=0\end{bmatrix}$$

which are circles. when $c=0$, the streamlines is the circle passing through $(a,0)$ and $(-a,0)$..

as $y=0$, it follow x-axis.