2016 5(c) IFoS

If E be the solid bounded by xy plane and paraboloid $z = 4 - x^2 - y^2$ then evaluate $\iint_S \vec{F} \cdot d\vec{S}$ where S is surface bounding volume E and $\vec{F} = (zx \sin yz + x^3)\hat{i} + \cos yz\,\hat{j} + (3zy^2 - e^{x^2+y^2})\hat{k}$

Using Gauss Divergence Theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V (\operatorname{div} \vec{F})\, dV$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(zx \sin yz + x^3) + \frac{\partial}{\partial y}(\cos yz) + \frac{\partial}{\partial z}(3zy^2 - e^{x^2+y^2})$$

$$= 3(x^2 + y^2) \quad \text{--- (1)}$$

$$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3(x^2+y^2)\, dV$$

converting to cylindrical coordinates

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$\Rightarrow$

Limits
$r = 0$ to 2
$\theta = 0$ to 2π
$z = 0$ to $4 - r^2$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

(1) becomes

$$\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{4-r^2} 3r^2 \cdot r\,dr\,d\theta\,dz$$

$$\Rightarrow 3\int_{r=0}^{2}\int_{\theta=0}^{2\pi} r^3\,dr\,d\theta\,(4-r^2)$$

$$\Rightarrow 3\int_{r=0}^{2} r^3(4-r^2)\,dr \int_{\theta=0}^{2\pi} d\theta$$

$$\Rightarrow 3\left[r^4 - \frac{r^6}{6}\right]_0^2 \cdot 2\pi = \boxed{32\pi}$$

$$\therefore \boxed{\iint_S \vec{F} \cdot d\vec{S} = 32\pi}$$

Ans

2016 6(d) IFoS

Evaluate $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$ for $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k}$

where S is upper half of sphere $x^2+y^2+z^2=1$ bounded by its projection on xy plane.

The sphere meets $z=0$ in circle C given by

$C: \; x^2+y^2=1, \; z=0$

let S_1 be plane region bounded by circle C.

Let S be the surface above xy plane and

S' be the whole surface, i.e., $S' = S + S_1$

Let V be the volume bounded by S'.

on S_1, $\hat{n} = -\hat{k}$

Now,

$$\iint_{S'} \text{curl } \vec{F} \cdot \hat{n}\, ds = \iiint_V (\text{div curl } \vec{F})\, dv = 0$$

[div curl F = 0]

$$\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds + \iint_{S_1} \text{curl } \vec{F} \cdot \hat{n}\, ds = 0$$

$$\Rightarrow \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \text{curl } \vec{F} \cdot \hat{k}\, ds$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix} = \hat{k}$$

$$\therefore \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \iint_{S_1} \hat{k} \cdot \hat{k}\, ds$$

$$= \iint_{S_1} ds$$

[S_1 = area bounded by circle C]

$$= \pi \cdot (1)^2$$

$$\boxed{\iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds = \pi}$$

Ans

2016 7(a) 120S

State Stokes' theorem. Verify the Stokes' theorem for $\vec{F} = x\hat{i} + z\hat{j} + 2y\hat{k}$ where c is curve obtained by intersection of plane $z = x$ and cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected line.

I) Stokes' Theorem

Let S be a piecewise smooth open surface bounded by piecewise smooth closed curve C. Let $\vec{F}(x, y, z)$ be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot d\vec{S}$$

II) $\vec{F} = x\hat{i} + z\hat{j} + 2y\hat{k}$

C is curve of intersection of $z = x$ and $x^2 + y^2 = 1$

Let $x = \cos t \quad y = \sin t \quad z = x = \cos t$

$\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$

$d\vec{r} = (-\sin t\,\hat{i} + \cos t\,\hat{j} - \sin t\,\hat{k})dt$

$\vec{F} = \cos t\,\hat{i} + \cos t\,\hat{j} + 2\sin t\,\hat{k}$

$\vec{F} \cdot d\vec{r} = (-\sin t \cos t + \cos^2 t - 2\sin^2 t)\,dt$

$= (-\sin t \cos t - 3\sin^2 t)\,dt$

$= (1 - \sin t \cos t - 3\sin^2 t)\,dt$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left(1 - \frac{\sin 2t}{2}\right. \quad \text{——— (1)}$$

$= \boxed{-\pi}$

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Now,

$$\text{curl } \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & xy \end{vmatrix} = \hat{i}$$

$\hat{n} \Rightarrow$ unit normal vector to curve $z = x$

$$\hat{n} = \frac{-\hat{i} + \hat{k}}{\sqrt{2}}$$

$$\iint_S (\text{curl } \vec{f} \cdot \hat{n}) ds = \iint_S \frac{-1}{\sqrt{2}} ds$$ [projecting on xy plane]

$$= \iint_{S_1} \frac{-1}{\sqrt{2}} \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_{S_1} \frac{-1}{\sqrt{2}} \frac{dx\,dy}{1/\sqrt{2}}$$

$$= -\iint_{S_1} dx\,dy$$

Here S_1 : $x^2 + y^2 = 1$, $z = 0$

$$\therefore \iint_S \text{curl } \vec{f} \cdot d\vec{s} = -(\text{Area of circle } S_1)$$

$$= \boxed{-\pi} \quad \text{--- (2)}$$

From (1) & (2)

$$\oint_C \vec{f} \cdot d\vec{r} = \iint_S \text{curl } \vec{f} \cdot d\vec{s}$$

Thus, Stokes' Theorem verified.