LINEAR ALGEBRA

IFoS - 2010

① Show that the set $P(t) = \{at^2+bt+c \mid a,b,c \in \mathbb{R}\}$ forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space?

→ Let $P(t) = \{at^2+bt+c \mid a,b,c \in \mathbb{R}\}$.

<u>Internal Composition:</u> Let $p_1, p_2 \in P(t)$ such that $p_1(t) = a_1t^2+b_1t+c_1$ & $p_2(t) = a_2t^2+b_2t+c_2$

$$(p_1+p_2)(t) = p_1(t) + p_2(t) = a_1t^2+b_1t+c_1 + a_2t^2+b_2t+c_2$$
$$= (a_1+a_2)t^2 + (b_1+b_2)t + (c_1+c_2) \in P(t).$$

$\therefore p_1+p_2 \in P(t)$. Hence, internal composition is satisfied.

<u>External Composition:</u> Let $p_1 \in P(t)$ such that $p_1(t) = at^2+bt+c$

Let $c_1 \in \mathbb{R}$. Then $(c_1p_1)(t) = c_1p_1(t)$
$$= c_1[at^2+bt+c]$$
$$= (c_1a)t^2 + (c_1b)t + c_1c \in P(t)$$
as $c_1a, c_1b, c_1c \in \mathbb{R}$ [closure in $\mathbb{R}$]

$\therefore$ External Composition is satisfied.

(I) <u>$(P(t),+)$ is a group and is commutative:</u>

(i) <u>Closure:</u> Satisfied (by internal composition).

(ii) <u>Associativity:</u> Addition of polynomials is associative in nature. i.e. $p_1, p_2, p_3 \in P(t)$, then
$$\cancel{p_1(t)+} \; ((p_1+p_2)+p_3)(t) = (p_1+(p_2+p_3))(t).$$

(iii) <u>Existence of Identity:</u> $\exists\, O(t) = 0t^2+0t+0 \in P(t)$ such that $\forall\, p_1 \in P(t)$, $O+p_1 = p_1+O = p_1$.

$\therefore O(t)$ is the identity in $P(t)$.

(iv) <u>Existence of Inverse:</u> For each $p_1(t) = at^2+bt+c \in P(t)$, $\exists\, (-p_1)(t) = (-a)t^2+(-b)t+(-c) \in P(t)$ such that $(p_1+(-p_1))(t) = ((-p_1)+p_1)(t) = O(t)$. Hence, $(-p_1)(t)$ is the inverse of $p_1(t)$ in $P(t)$

(v) Commutativity: Let $p_1, p_2 \in P(t)$ such that $p_1(t) = a_1t^2 + b_1t + c_1$ & $p_2(t) = a_2t^2 + b_2t + c_2$. Then,

$$(p_1+p_2)(t) = (a_1+a_2)t^2 + (b_1+b_2)t + (c_1+c_2) = (a_2+a_1)t^2 + (b_2+b_1)t + (c_2+c_1)$$
$$= (p_2+p_1)(t)$$

$\therefore$ Comm. prop. is satisfied.

$\therefore (P(t), +)$ is an abelian group.

(II) External Properties:

Let $p_1, p_2 \in P(t)$ such that $p_1(t) = a_1t^2 + b_1t + c_1$ & $p_2(t) = a_2t^2 + b_2t + c_2$.
& $a, b \in \mathbb{R}$. Then

(i) $a(p_1+p_2)(t) = a[a_1t^2 + b_1(t) + c_1 + a_2t^2 + b_2t + c_2]$ [Distr. in $\mathbb{R}$]
$= a[a_1t^2 + b_1(t) + c_1] + a[a_2t^2 + b_2t + c_2]$
$= ap_1(t) + ap_2(t)$.

(ii) $(a+b)\,p_1(t) = (a+b)(a_1t^2 + b_1t + c_1)$ [Dist. in $\mathbb{R}$]
$= a(a_1t^2 + b_1t + c_1) + b(a_1t^2 + b_1t + c_1)$
$= ap_1(t) + bp_1(t)$.

(iii) $(ab)\,p_1(t) = a(bp_1(t))$ [Asso. in $\mathbb{R}$]

(iv) $1 \cdot p_1(t) = p_1(t)$ [Identity in $\mathbb{R}$].

$\therefore$ P(t) satisfies all these properties.

Hence P(t) is a vector space.

Basis of $P(t) = \{1, t, t^2\}$

$\dim P(t) = 3$

② Show that the vectors $\alpha_1 = (1,0,-1)$, $\alpha_2 = (1,2,1)$, $\alpha_3 = (0,-3,2)$ form a basis of $\mathbb{R}^3$. Find the eq components of $(1,0,0)$ w.r.t the basis $\{\alpha_1, \alpha_2, \alpha_3\}$

$\rightarrow$ Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

Check: A containing $\alpha_1, \alpha_2, \alpha_3$ has determinant non-zero if $\alpha_1, \alpha_2, \alpha_3$ are L.I.

$|A| = 1.(4+3) - 1(-3) = 10 \neq 0.$

$\therefore \alpha_1, \alpha_2, \alpha_3$ are L.I.

Also, $\dim \mathbb{R}^3 = 3$. WKT, for every n-dimensional vector space, any subset containing L.I. vectors & is of length 'n' forms a basis.

$\therefore S = \{\alpha_1, \alpha_2, \alpha_3\}$ forms a basis of $\mathbb{R}^3$.

Now: $(x,y,z) = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$

$\Rightarrow (x,y,z) = (a+b, \; 2b-3c, -a+b+2c)$

$\Rightarrow$ ~~(x,y,z)~~ $x = a+b$, $y = 2b-3c$, $z = -a+b+2c$

$\Rightarrow$ If $(x,y,z) = (1,0,0)$, then, $x = 1, y = 0, z = 0$.

$1 = a+b$, $\quad 2b-3c = 0$, $\quad -a+b+2c = 0$

$2b = 3c \Longrightarrow a = b+2c = \frac{3}{2}c + 2c = \frac{7}{2}c$.

$a+b = 1$

$\Rightarrow \frac{7}{2}c + \frac{3}{2}c = 1 \Rightarrow 5c = 1 \Rightarrow c = \frac{1}{5}$.

$A = \frac{7}{2}c = \frac{7}{10}$, $\quad B = \frac{3}{2}c = \frac{3}{10}$.

$\therefore (1,0,0) = \frac{7}{10}(1,0,-1) + \frac{3}{10}(1,2,1) + \frac{1}{5}(0,-3,2)$

③ Find the characteristic polynomial of $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$. Verify Cayley-Hamilton theorem for this matrix & hence find inverse

→ Cayley-Hamilton Theorem states that every square matrix satisfies its char. eqn.

Char. eqn of A is given by $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda[-\lambda(3-\lambda)-2] + 1[1] = 0$

$\Rightarrow -\lambda[\lambda^2-3\lambda-2]+1 = 0 \quad \Rightarrow -\lambda^3+3\lambda^2+2\lambda+1=0$

$\Rightarrow \lambda^3-3\lambda^2-2\lambda-1=0$ which is the reqd. char. polynomial — ①

Putting A in LHS of ①

$$A^3-3A^2-2A-I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$-2\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ A satisfies its char. eqn.

Hence, Cayley-Hamilton's Theorem is verified.

Now $A^3-3A^2-2A-I=0. \Rightarrow A^3-3A^2-2A = I$

Pre-multiplying with A^{-1} on both sides

$A^2-3A-2I = A^{-1}$

$$\Rightarrow A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

④ Let $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Char. eqn of A is given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -(4+\lambda) \end{vmatrix} = 0$

$\Rightarrow (5-\lambda)[\lambda^2 - 16 + 12] - 6[6 + 4 + \lambda] - 6[6 - 3(4-\lambda)] = 0$

$\Rightarrow \quad 4 - 8\lambda + 5\lambda^2 - \lambda^3 = 0 \quad \Rightarrow \quad \cancel{\lambda^3 = 0} \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda - 2)^2 = 0 \quad \Rightarrow \lambda = 1, 2, 2.$

Eigen values of A are 1, 2, 2.

Eigen vectors of A corr. to eigen value.

① $\underline{\lambda = 1}$: $(A - I)(X) = 0$

$R_1 \to R_1 + 4R_2,\ R_3 \to R_3 + 3R_2$

$\Rightarrow \begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 6 & 2 \\ -1 & 3 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\qquad\qquad$ $R_3 \to R_3 - \frac{R_2}{2}$

$\sim \begin{bmatrix} -1 & 3 & 2 \\ 0 & 6 & 2 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 & 2 \\ 0 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to$ Echelon form.

$\therefore 6y + 2z = 0 \Rightarrow z = -3y.$

$-x + 3y + 2z = 0 \Rightarrow x = 3y + 2z = -3y$

$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y \\ y \\ -3y \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}. \quad \therefore X_1 = \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$

② $\lambda = 2$: $(A-2I)X = 0 \Rightarrow \begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$R_2 \to R_2 + \frac{R_1}{3}$, $R_3 \to R_3 - R_1$ $\begin{bmatrix} 3 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ → Echelon form.

$\therefore 3x - 6y - 6z = 0 \Rightarrow x = 2y + 2z$.

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y+2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $X_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore$ Eigen vectors of A corr. to eigen value 1 is $\begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}$ and corr. to eigen value 2 are $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Since algebraic multiplicity of each root is equal to its geometric multiplicity, the matrix A is diagonalizable.

Let $P = [X_1 \; X_2 \; X_3]$ & $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -3 & 2 & 2 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$\therefore D = P^{-1}AP$

⑤ Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix}$

→ Reducing to echelon form using elementary row operations.

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - 3R_1$, $R_5 \to R_5 - 4R_1$ $\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \sim$ $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + R_2$, $R_5 \to R_5 - 2R_2$ $\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$

$R_5 \to R_5 - \frac{5}{4}R_3$ $\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is clearly in echelon form.

The matrix has 3 non-zero rows in echelon form.

Hence, Rank of the matrix is 3.