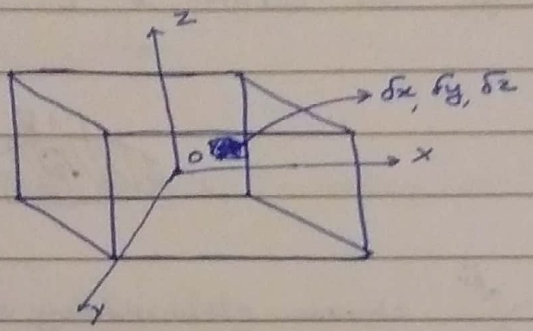
has edge of length $2a, 2b, 2c$

Moment of inertia of a rectangular parallelopiped about an axis through its centre & parallel to one of its edges 6 $\underline{2a}$

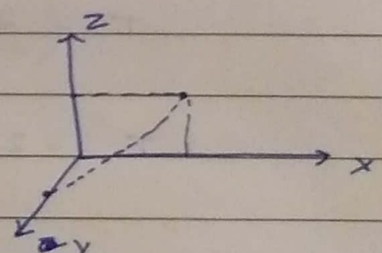


$$\rho = \frac{M}{\text{volume}} = \frac{M}{2a \cdot 2b \cdot 2c} = \frac{M}{8abc}$$

taking an elementary cube of sides $\delta x, \delta y, \delta z$

$\therefore$ ~~Am~~ $\rho = \dfrac{\delta m}{\delta x \, \delta y \, \delta z}$

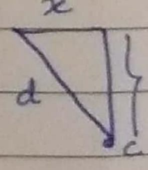
$$\delta m = \rho \, \delta x \, \delta y \, \delta z$$



centre of point of elementary parallelopedia from origin will be $d^2 = x^2 + y^2 + z^2$

$\therefore$ but is distance from x axis will be $d^2 - x^2$

$\therefore d^2 - x^2 = y^2 + z^2$



$\therefore \delta I = (d^2 - x^2)\, \delta m$

$\delta I = (y^2 + z^2)\, \rho \, \delta x \, \delta y \, \delta z$

then required distance from axis $= d^2 - x^2$

$$I = \rho \iiint (y^2 + z^2)\, dx\, dy\, dz$$

$$= \rho \int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (y^2 + z^2)\, dx\, dy\, dz$$

$$= \rho \; \cancel{(2b)(2c)} \; 2a \int_{-c}^{c}\int_{-b}^{b} (y^2+z^2)\, dy\, dz$$

$$= 2a\rho \int_{-c}^{c} \left(\frac{y^3}{3} + z^2 y \Big|_{-b}^{b} \right)$$

$$= 2a\rho \left(\frac{2b^3}{3} + 2z^2 b \right)$$

$$= 2a\rho \cdot 2b \int_{-c}^{c} \left(\frac{b^2}{3} + z^2 \right) dz$$

$$= 4ab\rho \left(\frac{b^2}{3} z + \frac{z^3}{3} \right) \Big|_{-c}^{c}$$

$$= 4ab\rho \left(\frac{b^2 2c}{3} + \frac{2c^3}{3} \right)$$

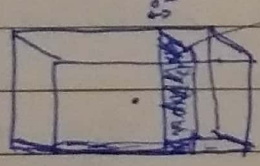
$$= \frac{8abc}{3} \rho \,(b^2+c^2)$$

$$= \frac{M}{8\cancel{abc}} \cdot \frac{8\cancel{abc}}{3} (b^2+c^2)$$

$$= \frac{M}{3} (b^2+c^2)$$

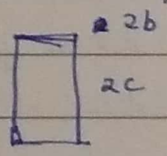
Alternative method

δx



→ slice (this will become a plate)

plate.



$\therefore$ MI of plate through its centre is $\frac{\delta M}{3}(b^2+c^2)$

$\therefore \quad \frac{\delta m}{2b \cdot 2c\, \delta x} = \rho$

$\delta m = 4bc\rho\, \delta x$

$\therefore \quad \delta I = \frac{4}{3} bc\rho (b^2+c^2)\, \delta x$

$$I = \frac{4}{3} bc \rho (b^2+c^2) \int_{-a}^{a} dx$$

$$I = \frac{4}{3} bc \rho (b^2+c^2) 2a$$

$$= \frac{8bca}{3} \rho (b^2+c^2) \qquad \rho = \frac{M}{V} = \frac{M}{2a\, 2b \cdot 2c} = \frac{M}{8abc}$$

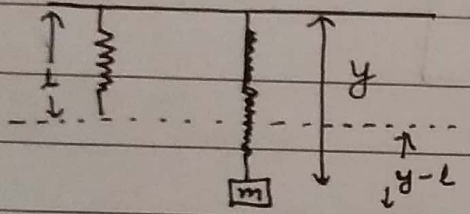
$$= \frac{8abc}{3} \cdot \frac{M}{8abc} (b^2+c^2)$$

$$= \frac{M}{3} (b^2+c^2)$$

Consider a mass m on the end of a spring of natural length l and spring constant k. Let y be the vertical co-ordinate of the mass as measured from the top of the spring. Assume that the mass ~~of the~~ can only move up and down in the vertical direction, show that

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y-l)^2 + mgy$$

Also determine and solve the corresponding Euler langrange Equations of motion



∴ it has spring constant k which will act in upward direction

and gravity force will be downward

& displacement of mass is $(y-l)$ ∴ ~~[illegible]~~

∴ ~~[illegible]~~

$$v^2 = \dot{x}^2 + \dot{y}^2$$
$$= 0^2 + \dot{y}^2$$
$$v^2 = \dot{y}^2$$

∴ ~~[illegible]~~

∴ $P.E = \frac{1}{2}k(y-l)^2 - mgy$

$K.E = \frac{1}{2}m\dot{y}^2$

$\therefore \; L = K.E - P.E$

$$L = \frac{1}{2} m \dot{y}^2 - \frac{1}{2} k (y - l)^2 + mgy$$

$\therefore$ Langrange Equation

$$\therefore \; \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{\partial}{\partial t}\left(m \dot{y}\right) - \left(-\frac{1}{2} k \, 2(y - l) + mg\right) = 0$$

$$m \ddot{y} + k(y - l) - mg = 0$$

$$m \ddot{y} = mg - k(y - l)$$

$$\ddot{y} = g - \frac{k}{m}(y - l)$$

Fluid Dynamics

IFoS - 2017

8(b) Find the streamlines and path lines of the two-dimensional velocity field $u = \frac{x}{1+t}$, $v = y$, $w = 0$.

Soln: We have $u = \frac{x}{1+t}$, $v = y$, $w = 0$

Step I To determine streamlines

Streamlines are the solution of d.eqn given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Putting value of u, v, w, we get

$$\underset{(i)}{\frac{(1+t)\,dx}{x}} = \underset{(ii)}{\frac{dy}{y}} = \underset{(iii)}{\frac{dz}{0}} \quad \text{——}(1)$$

$\Rightarrow$ from (i) and (ii), we have

$$\left(\frac{1+t}{x}\right) dx = \frac{dy}{y}$$

Integrating, we get

$\Rightarrow (1+t)\log x = \log y + \log C_1$

$$\Rightarrow \boxed{x^{1+t} = C_1 y} \quad \text{—}(2)$$

from (ii) and (iii)

$$\frac{dy}{y} = \frac{dz}{0} \Rightarrow dz = 0$$

Integrating, $\boxed{z = C_2}$ —(3)

These two equations (2) and (3) represent streamlines

Step II To determine path lines

path lines are the solutions of differential equations given by

$$\frac{dx}{dt} = \frac{x}{1+t}, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = 0$$

This $\Rightarrow$ $\frac{dx}{x} = \frac{dt}{1+t}$, $\frac{dy}{y} = dt$ & $dz = 0$

Integrating, $\log x = \log(1+t) + \log K_1$; $\log y = t - \log K_2$; $z = K_3$

or, $x = K_1(1+t)$, $y = K_2 e^t$, $z = K_3$

or, $\boxed{y = K_2 e^{\left(\frac{x}{K_1} - 1\right)}}$, $\boxed{z = K_3}$

These two equations represent path lines.

when K_1, K_2, K_3 are A. constants.

Fluid Dynamics

IFoS - 2017

Q. (c) The velocity in the flow field is given by

$$\vec{q} = \hat{i}(az - by) + \hat{j}(bx - cz) + \hat{k}(cy - ax)$$

where a, b, c are non-zero constants. Determine the equations of the vortex lines.

Soln: Let $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$. Then we have

$$u = az - by, \quad v = bx - cz, \quad w = cy - ax \quad \text{--- (1)}$$

Let $\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$ be the vorticity vector.

Then $\Omega = \text{curl}\, \vec{q}$

$$\Rightarrow W = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ az - by & bx - cz & cy - ax \end{vmatrix} = \hat{i}(c+c) - \hat{j}(-a - a) + \hat{k}(b+b)$$

$$= 2c\,\hat{i} + 2a\,\hat{j} + 2b\,\hat{k}$$

$\therefore \Omega_x = 2c, \quad \Omega_y = 2a, \quad \Omega_z = 2b$

The eqn of the vortex lines are

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$

i.e, $$\frac{dx}{2c} = \frac{dy}{2a} = \frac{dz}{2b} \quad \text{--- (2)}$$

Taking the first two members of (2) and integrating we get

$$\boxed{ax - cy = c_1}, \quad c_1 \text{ being an A-constant} \quad \text{--- (3)}$$

Now, Taking last two members and integrating, we get

$$\boxed{by - az = c_2} \quad c_2 \text{ being an A-constant.} \quad \text{--- (4)}$$

The required vortex lines are the straight lines of the intersection of (3) and (4).

Scanned by CamScanner