* DIFFERENTIAL EQUTIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE *, Set-IV

If we denote $\frac{dy}{dx}$ by P then the equation of the form $f(x, y, P) = 0$ where P is not of first degree, is called a differential equation of first order but not of first degree.

The most general form of a differential equation of first order and nth degree is $P^n + A_1 P^{n-1} + A_2 P^{n-2} + \cdots + A_{n-1} P + A_n Y = 0$

i.e $\left(\frac{dy}{dx}\right)^n + A_1 \left(\frac{dy}{dx}\right)^{n-1} + \cdots + A_{n-1}\left(\frac{dy}{dx}\right) + A_n Y = 0$

where $A_1, A_2 \cdots A_n$ are functions of a such equation can be divided into the following categories.

(i) Solvable for P (ii) Solvable for x

(iii) Solvable for y (iv) Clairaut's equation.

(i) Differential equations solvable for P:

Such equations can be resolved into linear factors of first degree. Let the given equation be

$$P^n + A_1 P^{n-1} + A_2 P^{n-2} + \cdots + A_{n-1} P + A_n = 0 \quad \text{——} (1)$$

Let it be resolved into linear factors to give

$$[P - f_1(x,y)][P - f_2(x,y)] \cdots [P - f_n(x,y)] = 0$$

Then $P = f_1(x,y)$, $P = f_2(x,y)$ ---- $P = f_n(x,y)$

These equations on integration give

$$F_1(x, y, C_1) = 0, \quad F_2(x, y, C_2) = 0, \cdots \quad F_n(x, y, C_n) = 0$$

$\therefore$ The solution of (1) is

$$F_1(x, y, C_1) \cdot F_2(x, y, C_2) \cdots F_n(x, y, C_n) = 0$$

But (1) being an equation of first order, its general solution must be contain one arbitrary constant.

Taking $c_1 = c_2 = c_3 = \cdots = c_n = c$

$\therefore$ The general solution (1) is

$$F_1(x, y, c) \cdot F_2(x, y, c) \cdots F_n(x, y, c) = 0$$

Problems:

→ Solve the following differential equations.

(1) $(P - xy)(P - x^2)(P - y^2) = 0$ —— (I)

It is solving for P

$P - xy = 0$ —① ; $P = x^2$ —② ; $P = y^2$ —③

$\frac{dy}{dx} = xy$; $\frac{dy}{dx} = x^2$; $\frac{dy}{dx} = y^2$

$\Rightarrow \log y = \frac{x^2}{2} + c$; $y = \frac{x^3}{3} + c$; $-\frac{1}{y} = x + c$

$\therefore$ General solution of (I) is

$$\left(\log y - \frac{x^2}{2} - c\right)\left(y - \frac{x^3}{3} - c\right)\left(-\frac{1}{y} + x + c\right) = 0$$

→ $(P + y + x)(xP + y + x)(P + 2x) = 0$

→ $P^2 - 5P + 6 = 0$ / $\Rightarrow (P-3)(P-2) = 0$

→ $x + yP^2 = (1 + xy)P$ / $\Rightarrow x + yP^2 - P - xyP = 0 \Rightarrow (x - P) - yP(x - P) = 0$

→ $4y^2P^2 + 2xy(3x+1)P + 3x^3 = 0$

→ $xy^2(P^2 + 2) = 2Py^3 + x^3$ / Add and sub x^2yP

→ $yP^2 + (x - y)P - x = 0$

→ $x^2P^2 - 2xyP + 2y^2 - x^2 = 0$

→ $P^2x^2 = y^2$ / $(Px)^2 - y^2 = 0 \Rightarrow (Px - y)(Px + y) = 0$

(ii) **Differential Equations Solvable for 'x':**

Let the given equation be solvable for 'x' then it can be put in the form

$$x = f(y, P) \quad \text{—— ①}$$

on diff. (1) w.r.t y, we get

$$\frac{dx}{dy} = F\left(y, P, \frac{dP}{dy}\right) \quad \text{(or)} \quad \frac{1}{P} = F\left(y, P, \frac{dP}{dy}\right) \quad \text{—— ②}$$

Let the solution of ② be $\phi(y, p, c) = 0$ —— ③

Eliminating p b/w ① & ③ is given the solution of the given equation.

If it is not possible to eliminate 'p' then the values of 'x' and 'y' interms of p in the form $x = f_1(p, c)$ & $y = f_2(p, c)$ together give the solution.

→ Solve the following differential equations:

(1) $xp^3 = a + bp$

Sol'n: Given that $xp^3 = a + bp$ —— ①

It is solving for 'x'

we have $x = \frac{a}{p^3} + \frac{bp}{p^3}$

$$\Rightarrow x = \frac{a}{p^3} + \frac{b}{p^2} \quad \text{—— ②}$$

Diff. ② w.r.t. 'y' we get

$$\frac{dx}{dy} = \left(\frac{-3a}{p^4} - \frac{2b}{p^3}\right)\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{1}{p}\left(\frac{-3a}{p^3} - \frac{2b}{p^2}\right)\frac{dp}{dy} \qquad \left(\because \frac{dx}{dy} = \frac{1}{p}\right)$$

$$\Rightarrow dy = \left(\frac{-3a}{p^3} - \frac{2b}{p^2}\right)dp$$

$$\Rightarrow y = \frac{3a}{2p^2} + \frac{2b}{p} + c \quad \text{—— ③}$$

It is not possible to eliminate 'p' from ② & ③.

∴ General solution of ① is $x = \frac{a}{p^3} + \frac{b}{p^2}$ &

$$y = \frac{3a}{2p^2} + \frac{2b}{p} + c$$

→ $p^3 - 4xyp + 8y^2 = 0$ —— ①

Sol'n: $4pxyp = p^3 + 8y^2$

$$x = \frac{p^2}{4y} + \frac{2y}{p} \quad \text{—— ②}$$

Diff. w.r.t 'y', we get,

$$\frac{dx}{dy} = \frac{1}{4}\frac{\left[y(2p)\frac{dp}{dy} - p^2\right]}{y^2} + \frac{2\left[p - y\frac{dp}{dy}\right]}{p^2}$$

$$\Rightarrow \frac{1}{p} = \frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \left(\frac{2}{p} - \frac{p^2}{4y^2}\right)$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \frac{1}{p} - \frac{p^2}{4y^2} = 0$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \frac{p}{2y}\left(\frac{2y}{p^2} - \frac{p}{2y}\right) = 0$$

$$\Rightarrow \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\left(\frac{dp}{dy} - \frac{p}{2y}\right) = 0$$

Omitting the first factor which leads to a singular solution, we get.

$$\frac{dp}{dy} - \frac{p}{2y} = 0$$

$$\Rightarrow \frac{dp}{p} = \frac{1}{2}\,\frac{1}{y}\,dy$$

$$\Rightarrow \log p = \frac{1}{2}\log y + \log c$$

$$\Rightarrow p = y^{1/2}c \quad \text{--- (3)}$$

Now eliminating p b/w (2) & (3), we get

$$x = \frac{yc^2}{4y} + \frac{2y}{y^{1/2}c}$$

$$\Rightarrow x = \frac{c^2}{4} + \frac{2y^{1/2}}{c}$$

which is the required general solution of (1).

Note: The factor which does not involve a derivative of p w.r.t 'x' or 'y' will be omitted. Such factor always lead to singular solutions.

→ $x = p^2$

→ $ap^2 + py - x = 0$

→ $y^2\log y = xyp + p^2$

2008 → solve the equation $y - 2xp + yp^2 = 0$ where $p = \frac{dy}{dx}$.

* Differential Equations solvable for 'y':

Let the given equation be $y = f(x, p)$ —— ①

Diff. w.r.t 'x' and writing $\frac{dy}{dx} = p$,

we get an equation of the form

$$p = F\left(x, p, \frac{dp}{dx}\right).$$

This is a differential equation in p & x and we get its solution in the form $\phi(x, p, c) = 0$.

Eliminating p b/w (1) & ② we get the required solution.

If it is not possible to eliminate 'p' then ① & ② can be put in the form $x = f_1(p, c)$,

$y = f_2(p, c)$ and

these two equations together constitute the solution.

Problems:

→ Solve the following Differential Equations:

(1) $y = 3x + \log p$ —— (1)

It is solving for 'y'

Now diff. w.r.t 'x' we get

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx} \qquad \left(\because \frac{dy}{dx} = p\right)$$

$$\Rightarrow \frac{1}{p}\frac{dp}{dx} = p - 3$$

$$\Rightarrow \frac{dp}{p(p-3)} = dx$$

$$\Rightarrow \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right]dp = dx$$

$$\Rightarrow \left[\log(p-3) - \log p\right] = 3x + c$$

$$\Rightarrow \log\left(\frac{p-3}{p}\right) = 3x + c$$

$$\Rightarrow \frac{p-3}{p} = e^{3x+c}$$

$$\Rightarrow 1 - \frac{3}{p} = e^{3x+c}$$

$$\Rightarrow 1 - e^{3x}.e^{c} = \frac{3}{p}$$

$$\Rightarrow p = \frac{3}{1-e^{3x}e^{c}} \quad \text{——} \ (2)$$

Eliminating p from (1) & (2) we get,

$$y = 3x + \log\left[\frac{3}{1-e^{3x}e^{c}}\right]$$

which is required general solution of (1).

→ $x^2 + p^2 x = yp$

→ $x = y - p^2$

→ $p^3 - p(y+3) + x = 0$

→ $y = 2p + 3p^2$

* Clairaut's Equation:

The differential Equation of the form $y = xp + f(p)$ is known as Clairauts equation.

This equation is solved by considering the equation $y = f(x, p)$, solvable for y.

Solution of Clairauts equation:

Let the given equation be

$$y = xp + f(p) \quad \text{——} \ (1)$$

Diff w.r.t 'x' we get

$$\frac{dy}{dx} = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = x\frac{dp}{dx} + p + f'(p)\frac{dp}{dx}$$

$$\Rightarrow (x + f'(p))\frac{dp}{dx} = 0.$$

$\Rightarrow \frac{dp}{dx} = 0 \qquad (x + f'(p)$ is discarded)

$\Rightarrow dp = 0 \Rightarrow \boxed{p = c}$

$\boxed{(1) \equiv y = xc + f(c)}$

which is the required general solution of (1).

Working Rule:

Given equation can be put in the form $y = xp + f(p)$ —— (1)

In order to find its solution

replace 'p' by 'c'.

$\therefore$ The general solution of (1) is $\boxed{y = xc + f(c)}$

Problems:

→ Solve $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$

Sol'n: Given that $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$ —— (1)

Let $\frac{dy}{dx} = p$ then

$y = xp + p^2$ —— (2)

Clearly which is in Clairaut's form

$\therefore$ Replacing p by c in (2), we get

$y = xc + c^2$ which is the required general solution of (1).

→ Solve the following differential Equations:

2005, 1994 (1) $p = \log(px - y)$

(2) $y = xp + \frac{a}{p}$

(3) $\sin(y - xp) = p$ (or) $\sin px \cos y = \cos px \sin y + p$

(4) $(y - xp)(p-1) = p$

(5) $(xp - y)^2 = p^2 - 1 \Rightarrow xp - y = \pm\sqrt{p^2 - 1}$.

* Equations reducible to clairaut's form:

Form I:

$$y^2 = pxy + f\left(\frac{py}{x}\right) \quad \text{---} \quad ①$$

put $x^2 = X$; $y^2 = Y$

$2x\,dx = dX$; $2y\,dy = dY$

$$\Rightarrow \frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx} \Rightarrow \boxed{P = \frac{y}{x}p}$$

$$\Rightarrow \boxed{p = \frac{x}{y}P}$$

$$\therefore ① \equiv Y = \frac{x}{y}P.(xy) + f\left(\frac{x}{y}P\frac{y}{x}\right)$$

$$\Rightarrow Y = x^2P + f(P)$$

$$\Rightarrow \boxed{Y = XP + f(P)} \quad \text{---} \quad ②$$

clearly which is clairaut's form

The general solution of ② is $Y = cX + f(c)$

$\Rightarrow y^2 = cx^2 + f(c)$ is the general solution of ①.

→ solve the following differential equations by using the transformations $x^2 = X$, $y^2 = Y$.

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(1) $x^2(y - px) = yp^2 \Rightarrow x^2y^2 - x^3py = y^2p^2$

$$\Rightarrow y^2 = pxy + \left(\frac{py}{x}\right)^2$$

(2) $(px - y)(py + x) = h^2p$

(3) $(px - y)(x - yp) = 2p$.

*** Form II:**

If $e^{by}(a - bp) = f(pbe^{by - ax})$

then put $e^{ax} = X$, $e^{by} = Y$

Problems:

→ solve $e^{3x}(p - 1) + p^3e^{2y} = 0$ ——— ①

Sol'n: $e^{3x}(1 - p) = p^3e^{2y}$

$$\Rightarrow (1 - p) = p^3e^{2y - 3x}$$

$\Rightarrow e^{y}(1-p) = p^3 e^{3y-3x}$

$\Rightarrow e^{y}(1-p) = (pe^{y-x})^3$

Clearly which is in the form of

$e^{by}(a-bp) = f(pb\, e^{by-ax})$, where $b=1, a=1$

Let $e^x = X$; $e^y = Y$

$e^x dx = dX$; $e^y dy = dY$

$\Rightarrow \frac{dY}{dX} = \frac{e^y}{e^x}\frac{dy}{dx}$

$\Rightarrow P = \frac{e^y}{e^x} p$

$\Rightarrow \boxed{p = \frac{e^x}{e^y} P}$

$\therefore$ (1) $\equiv X^3\left(\frac{e^x}{e^y}P - 1\right) + \left(\frac{e^x}{e^y}P\right)^3 (Y)^2 = 0$

$\Rightarrow X^3\left(\frac{X}{Y}P - 1\right) + \left(\frac{XP}{Y}\right)^3 Y^2 = 0$

$\Rightarrow XP - Y + P^3 = 0$

$\Rightarrow \boxed{Y = XP + P^3}$

which is in Clairaut's form.

$\therefore$ It's general solution is $Y = Xc + c^3$

$\Rightarrow e^y = e^x c + c^3$ is general solution of (1)

* Solve the following differential equations by using the mentioned transformations.

Differential Equation	Transformation.
$y = 2xp + y^2p^3$ (or) $y = 2px + y^{n-1}p^n$	$y^2 = Y \Rightarrow 2y\,dy = dY$ $\Rightarrow 2y\frac{dy}{dx} = \frac{dY}{dx}$ $\Rightarrow 2yp = P$ where $p = \frac{dy}{dx}, P = \frac{dY}{dx}$
$y + px = x^4p^2$	$\frac{1}{x} = X$
$y = 3px + 6y^2p^2$	$y^3 = Y$
$\cos^2 y\, p^2 + \sin x \cos x \cos y\, p - \sin y \cos^2 x = 0$	$\sin x = X$; $\sin y = Y$
$(xp - y)^2 = a(1+p^2)(x^2+y^2)^{3/2}$	$x = r\cos\theta$ $y = r\sin\theta$
$(xp - y)^2 = (x^2 - y^2)\sin^{-1}(y/x)$	$y = vx$
[2005] $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)\cdot(x+py) + (x+py)^2 = 0$	$x + y = u$ $x^2 + y^2 = v$
[2006] $x^2p^2 + 2xyp + y^2(1+p) = 0$	$y = Y$; $xy = Y$ $y = u$; $xy = v$
[2003] $(px^2 + y^2)(px + y) = (p+1)^2$	$x + y = X$; $y = Y$ $xy = Y$. $x + y = u$; $xy = v$
$xp^2 - 2yp + x + 2y = 0$	$x^2 = X$, $y - x = Y$
$y^2(y - xp) = x^4p^2$	$\frac{1}{x} = X$; $\frac{1}{y} = Y$.

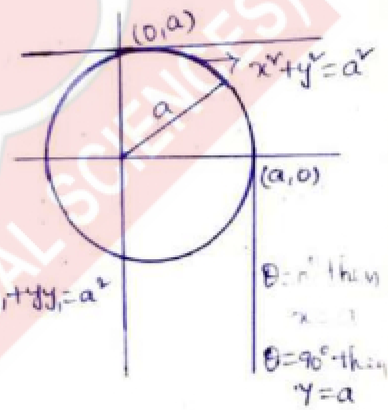
* Singular Solutions:

Def: A solution of a differential equation which is not derived from general solution by giving the particular values to the arbitrary constants, is called a singular solution. The singular solution does not involve any arbitrary constant.

Envelopes:

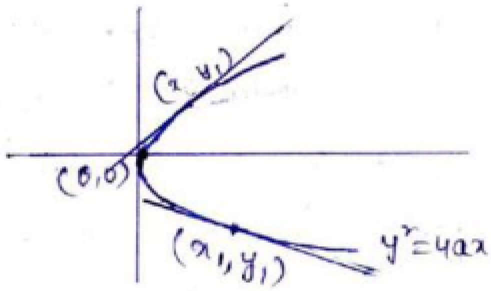
→ Consider the equation $x\cos\theta + y\sin\theta = a$, where 'a' is constant. For different values of θ, the equation represents a family of straight lines touching the circle $x^2+y^2=a^2$.
Here θ is the parametre of the family of straight lines $x\cos\theta + y\sin\theta = a$.

By the above example, the circle which is touched by a family of straight lines, is called the envelope of the family of straight lines.
i.e. the curve E which is touched by a family of curves 'C' is called the envelope of the family of curves 'C'.



Equation of Tangent:

The equation of tangent at (x_1, y_1) to the parabola $y^2=4ax$ is
$$yy_1 = 2a(x+x_1) \quad \text{—— (i)}$$



* Equation of Tangent in terms of slope (m):

The equation of tangent (i) is
$$y = \left(\frac{2a}{y_1}\right)x + \left(\frac{2ax_1}{y_1}\right) \quad \text{—— (ii)}$$
Let the slope of tangent at (x_1, y_1) be 'm'

Now the eq^n of the tangent at $P(x_1, y_1)$ on the parabola $y^2=4ax$ —①
Diff ① w.r.t 'x'
$2yy' = 4a \Rightarrow \frac{dy}{dx} = \frac{4a}{2y_1}$
∴ The eq^n of tangent at (x_1,y_1) on the parabola is $(y-y_1) = \frac{2a}{y_1}(x-x_1)$

$y^2=4x$; $a=1$
$yy_1 = 2(x+x_1)$
$(x_1,y_1) = (1,2)$
$\therefore 2y = 2(x+1)$
$\Rightarrow x-y+1=0$
$x=1,\ y=2$

$(x_1,y_1)=(0,0)$
$\Rightarrow x=0$

$\therefore \frac{2a}{y_1} = m$

$\Rightarrow \boxed{y_1 = \frac{2a}{m}}$

Since (x_1, y_1) lies on parabola $y^2 = 4ax$

$\Rightarrow y_1^2 = 4ax_1 \Rightarrow \frac{4a^2}{m^2} = 4ax_1$

$\Rightarrow \boxed{x_1 = \frac{a}{m^2}}$

$\therefore (x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is the point of contact.

$\therefore$ The equation of tangent at $(x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

is $\boxed{y = mx + \frac{a}{m}}$

Note: By the definition of envelope, $y^2 = 4ax$ is the envelope of the straight lines $y = mx + \frac{a}{m}$; m being parameter.

→ Let us consider the differential equation.

$$y = px + \frac{a}{p} \; ; p = \frac{dy}{dx} \quad \text{——} ①$$

Clearly which is in Clairaut's form

$\therefore$ It's general solution is $y = mx + \frac{a}{m}$; —— ②

'm' is the parameter.

We know that the equation ② is the tangent to the Parabola $y^2 = 4ax$ —— ③

clearly ③ is the envelope of ②

Now we show that $y^2 = 4ax$ is also solution of ①

Diff ③ w.r.t 'x', we get

$2yy' = 4a$

$\therefore \boxed{y' = \frac{2a}{y}}$ &

The whole parabola ③ Covered by ② for different values of 'm'

$\therefore$ ③ is the soln of ①

is $\phi(x,y,c)=0$ then $E(x,y)$ is a factor of both the discriminants (i.e. P & C) and $E(x,y)=0$ must satisfy the differential equation $f(x,y,p)=0$.

Ex: show that $x=0$ is the singular solution of

$$4xp^2=(3x-a)^2$$

Sol'n: the given differential equation is

$$4xp^2-(3x-a)^2=0 \quad \text{——} \quad (1)$$

general solution is $(y+c)^2=x(x-a)^2$ —— (2)

P-discriminant:

$$(0)^2-4\cdot 4x[(3x-a)^2]=0 \qquad (\because b^2-4ac=0)$$

$$\Rightarrow x(3x-a)^2=0 \quad \text{——} \quad (3)$$

C-discriminant:

$$(2)\equiv c^2+2yc+y^2-x(x-a)^2=0$$

$$\Rightarrow (2y)^2-4(1)[y^2-x(x-a)^2]=0$$

$$\Rightarrow 4y^2-4y^2+x(x-a)^2=0$$

$$\Rightarrow x(x-a)^2=0 \quad \text{——} \quad (4)$$

$\therefore x=0$ is the factor of both the discriminants.

and it must be satisfy the given differential Equation.

Since from (1), $4x-\dfrac{(3x-a)^2}{p^2}=0$

which is satisfied by $x=0$

because $\dfrac{dx}{dy}=0$ i.e. $\dfrac{1}{p}=0$

$\therefore x=0$ is the singular solution of (1).

from ③, $\boxed{x = \frac{y^2}{4a}}$

$\therefore$ ① $\equiv y = \left(\frac{2a}{y}\right)\left(\frac{y^2}{4a}\right) + a\left(\frac{y}{2a}\right)$

$\Rightarrow y = y$ (identity)

$\therefore$ ③ is the solution of ①.

→ The equation of the envelope of the family of curves given by the general solution of a differential equation is known as the singular solution. Such a solution does not contain any arbitrary constant. and is not a particular case of the general solution.

It is some times possible to reduce this solution from the general solution by giving the particular values to the arbitrary constants. In such a case the singular solution is also called particular solution.

→ Let $f(x,y,p)=0$ be the differential equation whose solution is $\phi(x,y,c)=0$, then p-discriminant is obtained by eliminating p between $f(x,y,p)=0$ & $\frac{\partial f}{\partial p}=0$.

Also c-discriminant is obtained by eliminating c between $\phi(x,y,c)=0$ & $\frac{\partial \phi}{\partial c}=0$.

→ If $E(x,y)=0$ is a singular solution (envelope) of the differential equation $f(x,y,p)=0$. whose general solution.

→ Each discriminant may have other factor which correspond to other loci associated with the general solution of the given differential Equation. Generally the equations of these loci donot satisfy the differential equation, they are known as extraneous loci.

These are three types of Extraneous loci

(1) Tac-locus (2) Node-locus and (3) Cusp-locus

* Methods for finding the singular solution:

→ To find the singular solution of a diff. equation $f(x,y,P)=0$

(1) Find its general solution $\phi(x,y,c)=0$.

(2) Find P-discriminant

(3) Find C-discriminant

* Now P-discriminant equated to zero may include as a factor.

(1) Envelope i.e. Singular solution once (E)

(2) Cusp-locus once (C)

(3) Tac-locus twice (T^2)

i.e. P-discriminant $= ET^2C$

* C-discriminant equated to zero may include as a factor:

(1) Envelope i.e. singular solution

(2) Cusp-locus thrice (c^3)

(3) Node-locus twice (N^2)

i.e. C-discriminant $= EN^2c^3$

Problems: SetA

(when equations are solvable for P)

Obtain the complete primitive (i.e. g.s) and singular solution of the following equations. Explaing the geometrical significance of the irrelevent factors that present themselves.

(ii) $xp^2 = (x-a)^2$

Sol'n: Given that $xp^2 = (x-a)^2$ —— ①

It is solvable for 'p'.

$\therefore [\sqrt{x}p - (x-a)][\sqrt{x}p + (x-a)] = 0$ —— ②

$\Rightarrow \sqrt{x}p = x-a; \quad \sqrt{x}p = a-x$

$\Rightarrow \frac{dy}{dx} = \sqrt{x} - ax^{-1/2}$ | $\Rightarrow \frac{dy}{dx} = ax^{-1/2} - \sqrt{x}$

$\Rightarrow dy = (\sqrt{x} - ax^{-1/2})dx$ | $\Rightarrow dy = (ax^{-1/2} - \sqrt{x})dx$

$\Rightarrow y = \frac{2}{3}x^{3/2} - 2ax^{1/2} + c$ | $\Rightarrow y = 2ax^{1/2} - \frac{2}{3}x^{3/2} + c$

$\therefore (y - \frac{2}{3}x^{3/2} + 2ax^{1/2} - c)(y - 2ax^{1/2} + \frac{2}{3}x^{3/2} - c) = 0$

$\Rightarrow (y-c)^2 - (\frac{2}{3}x^{3/2} - 2ax^{1/2})^2 = 0$

$\Rightarrow (y-c)^2 = \frac{4}{9}x(x-3a)^2$ —— ③

which is the required general solution of ①

Now P-discriminant:

$0 - 4x[-(x-a)^2] = 0$

$\Rightarrow 4x(x-a)^2 = 0$

$\Rightarrow x(x-a)^2 = 0$ —— ④

C-discriminant:

③ $\equiv y^2 + c^2 - 2yc = \frac{4}{9}x(x-3a)^2$

$\Rightarrow c^2 + (-2y)c + y^2 - \frac{4}{9}x(x-3a)^2 = 0$

$\Rightarrow (-2y)^2 - 4(1)[y^2 - \frac{4}{9}x(x-3a)^2] = 0 \quad (\because b^2 - 4ac = 0)$

$\Rightarrow 4y^2 - 4[y^2 - \frac{4}{9}x(x-3a)^2] = 0$

$\Rightarrow \frac{4}{9}x(x-3a)^2 = 0$

$\Rightarrow x(x-3a)^2 = 0$ —— ⑤

$\therefore x=0$ is the factor of both the discriminants (i.e. P&C)

and $x=0$ satisfies the given diff. equation.

$\therefore x=0$ is a singular solution.

Now $x-a=0$ is a tac-locus because it appears twice in the P-discriminant relation ④,
it does not occur in the C-discriminant relation ⑤ and does not satisfy the differential Equation ①

Now $x-3a=0$ is a node-locus because it appears twice in the C-discriminant relation ⑤, it does not occur in the P-discriminant relation ④ and does not satisfy the diff. equation ①.

(2) $4p^2x(x-a)(x-b) = [3x^2-2x(a+b)+ab]^2$

Given equation is

$$4p^2x(x-a)(x-b)^2 = [3x^2-2x(a+b)+ab]^2 \quad \text{——} \ ①$$

Its general solution is

$$(y+c)^2 = x(x-a)(x-b)$$

$$\Rightarrow c^2+2cy+y^2-x(x-a)(x-b)=0 \quad \text{——} \ ②$$

Now P-discriminant:

$$0-4(4x)(x-a)(x-b)[-(3x^2-2x(a+b)+ab]^2=0$$

$$\Rightarrow x(x-a)(x-b)[3x^2-2x(a+b)+ab]^2=0 \quad \text{——} \ ③$$

C-discriminant:

$$4y^2-4.1[y^2-x(x-a)(x-b)]=0$$

$$\Rightarrow x(x-a)(x-b)=0$$

Here $x=0$ appears once in both the discriminants and it satisfies the equation ①.

i.e. $4x(x-a)(x-b)-\dfrac{[3x^2-2x(a+b)+ab)]^2}{p^2}=0$ $\left[\because x=0 \Rightarrow \dfrac{dx}{dy}=0 = \dfrac{1}{p}=0\right]$

$\therefore x=0$ is the singular solution.

Similarly $(x-a)=0$ and $(x-b)=0$ are also singular solutions.

Now $3x^2-2x(a+b)+ab=0$ is a tac-locus because it appears twice in the p-discriminant.

Again solving it for 'x' we get

$$x = \frac{2(a+b) \pm \sqrt{4(a+b)^2-12ab}}{6}$$

$$\Rightarrow 3x = (a+b) \pm (a^2-ab+b^2)^{1/2} \quad \text{—— (5)}$$

The above mentioned 'tac-locus factors are given by (5)

$\therefore$ There are two tac-loci given by (5)

Set-B

(when equations are solvable for 'x')

→ Solve and examine for singular solutions.

$$p^3 - 4xyp + 8y^2 = 0 \quad \text{—— (1)}$$

Sol'n: it is solvable for 'x',

$$4xyp = p^3 + 8y^2$$

$$\Rightarrow x = \frac{p^2}{4y} + \frac{2y}{p} \quad \text{—— (2)}$$

Diff. w.r.t 'y' we get so on

It's g.s. is $\boxed{y = c(c-x)^2}$ —— (3)

Now p-discriminant:

Diff. (1) partially w.r.t 'p', we get

$$3p^2 - 4xy = 0$$

$$\Rightarrow p^2 = \frac{4xy}{3} \quad \text{—— (4)}$$

Now eliminating p from (1) & (4)

For this (1) $\equiv 8y^2 = p(4xy - p^2)$

$$\Rightarrow 64y^4 = p^2(4xy - p^2)^2$$

$$\Rightarrow 64y^4 = \frac{4xy}{3}\left(4xy - \frac{4xy}{3}\right)^2 \quad (\text{from (4)})$$

$$\Rightarrow 64y^4 = \frac{64xy}{3}\left(\frac{2xy}{3}\right)^2$$

$$\Rightarrow y^4 = \frac{4x^3y^3}{27}$$

$$\Rightarrow 27y^4 = 4x^3y^3$$

$$\Rightarrow y^3\left[27y - 4x^3\right] = 0$$

$$\Rightarrow y^2 \cdot y\left[27y - 4x^3\right] = 0 \quad \text{--- (5)}$$

C-discriminant:

Diff (3) partially w.r.t. 'c' we get

$$0 = (c-x)^2 + 2c(c-x)$$

$$\Rightarrow (c-x)(3c-x) = 0$$

$$\Rightarrow c = x \ \& \ c = x/3$$

If $c = x$ then (3) $\equiv y = 0$ --- (6)

If $c = x/3$ then (3) $\equiv y = \frac{4x^3}{27}$ --- (7)

from (6) & (7) we have

$$y\left(y - \frac{4x^3}{27}\right) = 0$$

$$\Rightarrow y(27y - 4x^3) = 0 \quad \text{--- (8)}$$

$\therefore$ $y = 0$ & $27y - 4x^3 = 0$ are the singular solutions.

Because they both appear once in (5) & (8) and satisfying the given differential equation.

HW → Find the solution of the differential equation $y = 2xp - yp^2$. Also find the singular solution.

Set C:

(when equations are solvable for y)

HW → Solve the differential equation $(8p^3 - 27)x = 12p^2 y$ and investigate whether a singular solution exists.

* Examine the following equations for singular solution and extraneous loci if any

2000 → $y = x - 2ap + ap^2$

2001 → $xp^2 - 2yp + ax = 0$

→ $ap^2 - 2yp + 4x = 0$

Set-D

(when equations are in clairaut's form)

→ Find the complete solution & singular solution of

$$y = px + \sqrt{b^2 + a^2p^2}$$

Sol'n: Given that $y = px + \sqrt{b^2 + a^2p^2}$ ——— (1)

Clearly which is in Clairaut's form

$\therefore$ It's g.s is

$$y = cx + \sqrt{b^2 + a^2c^2} \quad \text{——— (2)}$$

Now from (1) & (2) both P & C – discriminant:

For this (1)

$$(y - px)^2 = b^2 + a^2p^2$$

$$\Rightarrow (x^2 - a^2)p^2 - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow 4x^2y^2 - 4[x^2 - a^2][y^2 - b^2] = 0$$

$$\Rightarrow x^2y^2 - (x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2) = 0$$

$$\Rightarrow x^2b^2 + a^2y^2 - a^2b^2 = 0$$

which is the P- & C- discriminant.

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differential equation.

Note: In case of clairaut's equation P-and C- discriminants are always identical.

Set E:

(Equations Reducible to clairaut's form)

→ Reduce the differential equation $(px-y)(x-py)=2p$ to clairaut's form by the substitution $x^2=u$ and $y^2=v$ and find its complete primitive and its singular solution, if any

Sol'n: Given that $(px-y)(x-py)=2p$ — ①

It's g.s is

$$y = cx - \frac{2c}{1-c} \quad \text{(by previous methods)}$$

(or)

$$x^2c^2 - (x^2+y^2-2)c + y^2 = 0 \quad \text{— ②}$$

$$① \equiv xyp^2 - (x^2+y^2-2)p + 2xy = 0 \quad \text{— ③}$$

Now p-discriminant is:

$$(x^2+y^2-2)^2 - (2xy)^2 = 0 \quad \text{— ④}$$

$$\Rightarrow (x^2+y^2-2-2xy)(x^2+y^2-2+2xy) = 0$$

$$\Rightarrow [(x-y)^2 - (\sqrt{2})^2][(x+y)^2 - (\sqrt{2})^2] = 0$$

$$\Rightarrow (x-y+\sqrt{2})(x-y-\sqrt{2})(x+y+\sqrt{2})(x+y-\sqrt{2}) = 0$$

Now from ②, c-discriminant is

$$(x^2+y^2-2) - (2xy)^2 = 0$$

which is same as ④

∴ It also reduces to ④

∴ p- and c- discriminant relations are coincident here.

$x-y+\sqrt{2}=0$ appears in both the discriminant and satisfies the given diff. equation and hence it is a singular solution.

Similarly,

$x-y-\sqrt{2}=0$, $x+y+\sqrt{2}=0$ and

$x+y-\sqrt{2}=0$ are also singular solutions.

HW → Reduce the equation $xyp^2-(x^2+y^2-1)p+xy=0$ to clairaut's form by the substitution $x^2=u$ and $y^2=v$. Hence show that the equation represents a family of conics touching the four sides of a square.

Sol'n: Given that $xyp^2-(x^2+y^2-1)p+xy=0$ —— ①

It's g.s is $c^2x^2-c(x^2+y^2-1)+y^2=0$ —— ② (by using previous methods)

which represents a family of conics.

from ①: the P-discriminant relation is

$$(x^2+y^2-1)^2-4x^2y^2=0 \quad —— ③$$

from ②: the c-discriminant relation is

$$(x^2+y^2-1)^2-4x^2y^2=0 \quad —— ④$$

∴ from ③ & ④, $(x^2+y^2-1)^2-4x^2y^2=0$ —— ⑤

must be singular solution, because it is present once in both the discriminants.

Again from ⑤, we have

$$(x+y+1)(x+y-1)(x-y+1)(x-y-1)=0$$

$\Rightarrow x+y+1=0$, $x+y-1=0$, $x-y+1=0$ and $x-y-1=0$

are four singular solutions (envelops).

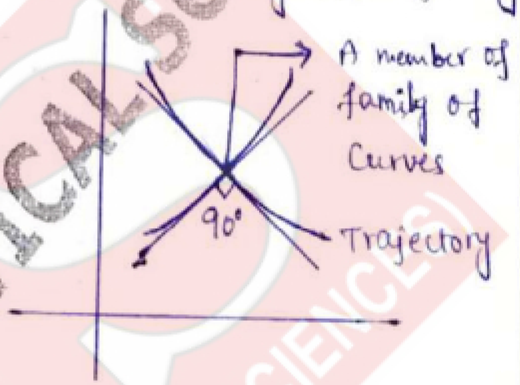
Thus the given diff. equation represents a family of conics given by ② which are touched by the four lines (envelops) mentioned above and the four lines form the four sides of a square.

Orthogonal Trajectories

Definition:

→ A curve which cuts every member of a given family of curves according to a given law, is called a Trajectory of the given family of curves.

→ A trajectory of a family of curves is called an orthogonal trajectory of the family if it cuts every member of the family at right angle.

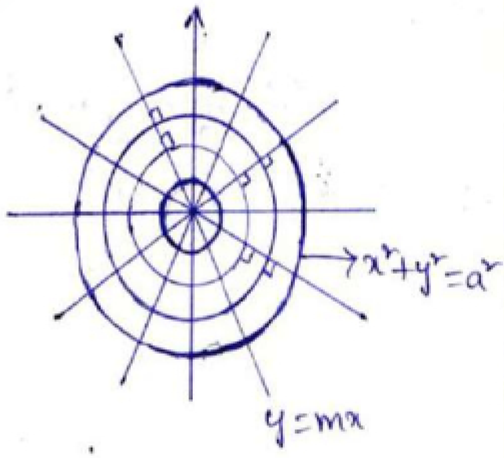


→ A trajectory of a family of curves is called an oblique trajectory of the family if it cuts every member of the given family of curves at an angle $\alpha \neq 90°$.

Ex: Consider the two family of curves $y=mx$ & $x^2+y^2=a^2$ where m & a are parameters.

then $y=mx$ is an orthogonal trajectory of the family of circles $x^2+y^2=a^2$.

Since every line (i.e. $y=mx$) passing through the origin of coordinates is an orthogonal trajectory of the family of the concentric circles (i.e. $x^2+y^2=a^2$)



* **Working Rule for Finding the orthogonal trajectories of the given family of curves in Cartesian coordinates (i.e. $f(x,y,c)=0$):**

Step(1): Form the diff. equation $F\left(x,y,\frac{dy}{dx}\right)=0$ by eliminating 'c' from the given family of curves $f(x,y,c)=0$.

step(2): Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ in $F(x, y, \frac{dy}{dx}) = 0$.

to get the diff equation $F(x, y, -\frac{dx}{dy}) = 0$ of the family of orthogonal trajectories.

step(3): solve the diff. equation

$F(x, y, -\frac{dx}{dy}) = 0$ to get the family of orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of the family of curves $x^2+y^2=a^2$; a is parameter.

sol'n: The given family of curves is

$$x^2+y^2=a^2; \quad a \text{ is parameter.} \qquad \text{——— ①}$$

Diff. w.r.t 'x', we get

$$2x+2yy'=0$$

$$\Rightarrow x+yy'=0 \qquad \text{——— ②}$$

which is the differential equation of the given family of curves ①.

Replacing "y'" by $-1/y'$ in ②, we get the diff. equation of the family of orthogonal trajectories

$$\therefore x - \frac{y}{y'} = 0$$

$$\Rightarrow xy' = y$$

$$\Rightarrow \frac{1}{y}\,dy = \frac{1}{x}\,dx$$

$$\Rightarrow \log y = \log x + \log m$$

$$\Rightarrow \boxed{y = mx}; \quad m \text{ is parameter.}$$

which is the required orthogonal trajectories.

→ Find the orthogonal trajectories of the following family of curves.

(i) $y = ax^2$; a is parameter.

(ii) $3xy = x^3 - a^3$; a is parameter.

(iii) $x^2+y^2=2ax$; a is parameter

(iv) $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$; λ is a parameter.

(v) $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$; λ is a parameter

2005 (vi) $x^2+y^2+2gx+c=0$; g is parameter

1992 → $ay^2=x^3$; a is parameter.

* Self-orthogonal family of curves:

A family of curves is said to be self orthogonal if the diff. equation of the family of curves is same as the diff. eqn of the orthogonal trajectories

(Or)

If each member of a given family of curves intersects all other members orthogonally, then the given family of curves is said to be self-orthogonal.

Problems:

→ Show that the system of confocal and co-axial parabolas $y^2=4a(x+a)$ is self-orthogonal, a being parameter.
—— (1)

Differentiating (1) w.r.t 'x' we get,

$$2yy'=4a \quad \text{—— (2)}$$

Now eliminating 'a' b/w (1) & (2), we get

$$y^2=2yy'\left(x+\frac{yy'}{2}\right)$$

$$\Rightarrow y^2=2xyy'+y^2(y')^2$$

$$\Rightarrow y=2xy'+y(y')^2 \quad \text{—— (3)}$$

which is the differential equation of the given family of parabolas.

Now replacing y' by $-\frac{1}{y'}$ in ③

we get the differential equation of orthogonal trajectories.

$$\therefore y = 2x\left(-\frac{1}{y'}\right) + y\left(-\frac{1}{y'}\right)^2$$

$$\Rightarrow y = \frac{-2x}{y'} + \frac{y}{y'^2}$$

$$\Rightarrow y(y')^2 = -2xy' + y$$

$$\Rightarrow y = 2xy' + y(y')^2 \quad \text{—— ④}$$

which is the same as the differential equation of the given family of parabolas. i.e, ③ & ④ are same.

$\therefore$ the system of Parabolas ① is self orthogonal.

1993 → show that the system of confocal conics

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$ is self-orthogonal; λ is Parameter.

(or)

2003 → show that the orthogonal trajector of a system of confocal ellipses is self-orthogonal.

sol'n: Given equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{—— ①}$$

Diff ① w.r.t 'x' we get

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0 \Rightarrow x[b^2+\lambda] + yy'[a^2+\lambda] = 0$$

$$\Rightarrow b^2x + a^2yy' + \lambda(x+yy') = 0$$

$$\Rightarrow \lambda = \frac{-(b^2x + a^2yy')}{x+yy'}$$

Now $a^2+\lambda = a^2 - \dfrac{(b^2x+a^2yy')}{x+yy'}$

$$= \frac{a^2x + a^2yy' - b^2x - a^2yy'}{x+yy'}$$

$$\Rightarrow \boxed{a^2+\lambda = \frac{(a^2-b^2)x}{x+yy'}}$$

and $$b^2+\lambda = b^2 - \frac{(b^2x+a^2yy')}{x+yy'}$$

$$= \frac{b^2x + b^2yy' - b^2x - a^2yy'}{x+yy'}$$

$$= \frac{-(a^2-b^2)yy'}{x+yy'}$$

$$① \equiv x^2\left[\frac{x+yy'}{(a^2-b^2)x}\right] - y^2\left[\frac{x+yy'}{(a^2-b^2)yy'}\right] = 1$$

$$xy'(x+yy') - y(x+yy') = (a^2-b^2)y'$$

$$\Rightarrow (x+yy')(xy'-y) = (a^2-b^2)y'$$

$$\Rightarrow (x+yy')\left(x-\frac{y}{y'}\right) = (a^2-b^2) \quad \text{--- ②}$$

which is the diff. of the given system of conics ①.

Now replacing y' by $\frac{-1}{y'}$ in ②

we get the differential equation of the family of orthogonally trajectories.

$$\therefore \left(x-\frac{y}{y'}\right)(x+yy') = a^2-b^2 \quad \text{--- ③}$$

② & ③ are same

$\therefore$ The given system of confocal conics is self orthogonal.

* <u>Orthogonal trajectories in Polar co-ordinates (i.e, $f(r,\theta,c)=0$):</u>

<u>working Rule</u>:

<u>Step(1)</u>: Form the diff equation $F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ by eliminating 'c' from the given family of curves $f(r,\theta,c)=0$.

Step (2): Replace $\frac{dr}{d\theta}$ by $\frac{-r^2}{\frac{dr}{d\theta}}$ $\left((\text{or})\ -r^2\frac{d\theta}{dr}\right)$ in $F\left(r,\theta,\frac{dr}{d\theta}\right)=0$ to get the diff. equation $F\left(r,\theta,-r^2\frac{d\theta}{dr}\right)=0$ of the family of orthogonal trajectories.

Step (3): solve the diff. equation $F\left(r,\theta,r^2\frac{d\theta}{dr}\right)=0$ to get the orthogonal trajectories.

Problems:

→ Find the orthogonal trajectories of family of Cardioids $r=a(1-\cos\theta)$; where a is the parameter.

Sol'n: Given family of Cardioids is $r=a(1-\cos\theta)$ —— ①

Taking log on bothsides we get

$$\log r=\log a+\log(1-\cos\theta) \quad \text{—— ②}$$

Diff. ② w.r.t θ, we get,

$$\frac{1}{r}\frac{dr}{d\theta}=0+\frac{1}{1-\cos\theta}(\sin\theta)$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta}=\frac{\sin\theta}{1-\cos\theta} \quad \text{—— ③}$$

which is the diff equation of the given family ①.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in ③, we get the differential equation of the orthogonal trajectories.

$$\therefore \frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right)=\frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow -r\frac{d\theta}{dr}=\frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow -r\frac{d\theta}{dr}=\cot(\theta/2)$$

$$\Rightarrow \frac{dr}{r}=-\tan(\theta/2)d\theta$$

$$\Rightarrow \log r=\frac{-\log(\sec(\theta/2))}{\frac{1}{2}}+c$$

$$\Rightarrow \log(r/c) = \log|\cos^2(\theta/2)|$$

$$\Rightarrow r/c = \cos^2(\theta/2)$$

$$\Rightarrow r/c = \left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow r = \frac{c}{2}(1+\cos\theta)$$

$$\Rightarrow r = b(1+\cos\theta) \quad (\text{Put } c/2 = b)$$

which is the required orthogonal trajectories of the following family of curves.

(1) $r = \frac{2a}{1+\cos\theta}$; a is parameter

(2) $r^n \cos n\theta = a^n$; a is parameter.

* Working rule for finding the oblique trajectories of the given family of curves in Cartisian co-ordinates (i.e. $f(x,y,c)=0$):

Step(1): Form the diff. equation $F(x,y,\frac{dy}{dx})=0$ by eliminating c from the given family of curves. $f(x,y,c)=0$.

Step(2): Replace $\frac{dy}{dx}$ (i.e. p) by $\frac{p+\tan\alpha}{1-p\tan\alpha}$ (or) $\frac{1+p\cot\alpha}{\cot\alpha - p}$ in $F(x,y,\frac{dy}{dx})=0$ to get the diff. equation $F\left(x,y,\frac{p+\tan\alpha}{1-p\tan\alpha}\right)=0$ where $p=\frac{dy}{dx}$. of the family of trajectories.

Step(3): solve the diff. equation $F\left(x,y\ \frac{p+\tan\alpha}{1-p\tan\alpha}\right)=0$ to get the family trajectories.

Problems:

1994 → Find the 45^0 trajectories of the family of curves $xy=c$. where c is parameter.

Sol'n: Given family of curves is $xy=c$ —— ①

Diff. w.r.t 'x', we get

$$xy'+y=0 \quad \text{——} ②$$

$$\text{(or)} \quad xp+y=0$$

Replacing 'p' by $\frac{p+\tan\alpha}{1-p\tan\alpha}$ in ②.

we get,

$$\frac{x(p+\tan\alpha)}{1-p\tan\alpha}+y=0$$

$$\Rightarrow x\left(\frac{p+\tan 45^\circ}{1-p\tan 45^\circ}\right)+y=0$$

$$\Rightarrow x\left(\frac{p+1}{1-p}\right)+y=0$$

$$\Rightarrow x(p+1)+y(1-p)=0$$

$$\Rightarrow (x-y)p+(x+y)=0$$

$$\Rightarrow p=\frac{y+x}{y-x}$$

$$\Rightarrow (y+x)dx+(x-y)dy=0 \quad \text{——} ③$$

Clearly which is the form of $Mdx+Ndy=0$.

where $M=y+x$; $N=x-y$

$$\frac{\partial M}{\partial y}=1 \qquad \frac{\partial N}{\partial x}=1$$

$$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Integrating ③ we get

$$yx+\frac{x^2}{2}-\frac{y^2}{2}=c$$

which is the required trajectory of the given family.

H.W → Determine the 45° trajectories of the family of concentric circles $x^2+y^2=a^2$; a is the parameter solve.

$\rightarrow (xp-y)^2 = (x^2-y^2)\sin^{-1}(y/x)$ ——— (1)

putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\Rightarrow p = v + xP$ where $P = \frac{dv}{dx}$

$\therefore$ (1) $\equiv$

$$[x(v+xP)-y]^2 = (x^2 - x^2v^2)\sin^{-1}v$$

$$\Rightarrow (y + x^2P - y)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow (x^2P)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow x^2P^2 = (1-v^2)\sin^{-1}v$$

$$\Rightarrow P^2 = \frac{1-v^2}{x^2}\sin^{-1}v$$

$$\Rightarrow P = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \frac{dv}{dx} = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \int\frac{1}{\sqrt{1-v^2}\sqrt{\sin^{-1}v}}dv = \pm\int\frac{dx}{x} + C$$

$\sin^{-1}v = t$

$$\Rightarrow \frac{1}{\sqrt{1-v^2}}dv = dt$$

$$\int\frac{dt}{\sqrt{t}} = \pm\log x + C$$

$$\Rightarrow 2t^{1/2} = \pm\log x + C$$

$$\Rightarrow 4t = (\pm\log x + C)^2$$

$$\Rightarrow 4\sin^{-1}v = (\pm\log x + C)^2$$

$$\Rightarrow 4\sin^{-1}(y/x) = (\pm\log x + C)^2$$

which is the required soln.

$\rightarrow xp^2 - (y-x)p - y = 1$ ——— (1)

$\Rightarrow px(p+1) - y(p+1) = 1$

$\Rightarrow (px-y)(p+1) = 1$

$\Rightarrow px - y = \frac{1}{p+1}$

$\Rightarrow y = px - \frac{1}{p+1}$ ——— (2)

Clearly which is in the form of Clairaut's equation.

put $p = c$, we get

$$\boxed{y = cx - \frac{1}{c+1}}$$ ——— (3)

which is the G.S. of (1)

Due to the Clairaut's form of (2), the p-discriminant and c-discriminant relations are same.

$\therefore$ from (1)

p-discriminant is given by

$$(y-x)^2 + 4xy = 0$$

$$\Rightarrow (x+y)^2 = 0$$

which must be the singular solution because it is present in both the discriminants and satisfies the given differential equation.

$\Rightarrow (xp-y)^2 = a(1+p^2)(x^2+y^2)^{3/2}$ ——— (1)

putting $x = r\cos\theta$; $y = r\sin\theta$

$x^2+y^2 = r^2$

$\Rightarrow x\,dx + y\,dy = r\,dr$

$\Rightarrow x + yp = r\frac{dr}{dx}$

and $\frac{y}{x} = \tan\theta$

$\Rightarrow \frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta$

$\Rightarrow xp - y = r^2 d\theta$

Now $\frac{x+yp}{xp-y} = \frac{1}{r}\frac{dr}{d\theta}$

$\Rightarrow \frac{x+yp}{xp-y} = \frac{1}{r}P$ where $\frac{dr}{d\theta} = P$

$\Rightarrow rx + ryp = Ppx - Py$

$\Rightarrow (\cancel{Px - ry})p = rx + Py$

$\Rightarrow p = \frac{rx+Py}{Px-ry}$

$\therefore$ (1)

$\left[x\left(\frac{rx+Py}{Px-ry}\right) - y\right]^2 = a\left[1+\left(\frac{rx+Py}{Px-ry}\right)^2\right](r^2)^{3/2}$

$\Rightarrow (x^2r + \cancel{Pxy} - \cancel{Pxy} + ry^2)^2 = a\left[(Px-ry)^2 + (rx+Py)^2\right](r^3)$

$\Rightarrow (x^2+y^2)^2 r^2 = a\left[(P^2+r^2)(x^2+y^2)\right]r^3$

$\Rightarrow (x^2+y^2) = a(P^2+r^2)r$

$\Rightarrow r^2 = a(P^2+r^2)r$

$\Rightarrow r = a(P^2+r^2)$

$\Rightarrow aP^2 = r - ar^2$

$\Rightarrow P^2 = \frac{r - ar^2}{a}$

$\Rightarrow P = \frac{\pm\sqrt{r(1-ar)}}{\sqrt{a}}$

$\Rightarrow \frac{dr}{d\theta} = \pm\frac{\sqrt{r}\sqrt{1-ar}}{\sqrt{a}}$

$\Rightarrow \int \frac{\sqrt{a}}{\sqrt{r}\sqrt{1-ar}}\,dr = \pm\int d\theta$

put $1 - ar = t$ and $r = \frac{1-t}{a}$

$\Rightarrow -a\,dr = dt$

$\Rightarrow dr = -\frac{1}{a}dt$

$\int \frac{\sqrt{a}\left(-\frac{1}{a}dt\right)}{\sqrt{\frac{1-t}{a}}\sqrt{t}} = \pm\theta + c$

$-\int \frac{dt}{\sqrt{t(1-t)}} = \pm\theta + c$

$-\int \frac{dt}{\sqrt{t-t^2}} = \pm\theta + c$

$-\int \frac{dt}{\sqrt{-(t-\frac{1}{2})^2 + \frac{1}{4}}} = \pm\theta + c$

$\Rightarrow -\sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right) = \pm\theta + c$

$\Rightarrow -\sin^{-1}(2t-1) = \pm\theta + c$

$\Rightarrow -\sin^{-1}\left[2(1-ar)-1\right] = \pm\theta + c$

$\Rightarrow -1 + 2ar = \sin^{-1}(\pm\theta + c)$

$\Rightarrow 2a\sqrt{x^2+y^2} - 1 = \sin^{-1}\left(\pm\tan^{-1}\frac{y}{x} + c\right)$

$\therefore 2a\sqrt{x^2+y^2} - 1 = \sin^{-1}\left(\pm\tan^{-1}\frac{y}{x} + c\right)$

which is the required solution