6

Hamiltonian Formulation, Transformations And Hamilton-Jacobi Theory

6·1 A-Hamiltonian Formulation

6·1·0 Phase Space.

In Lagrangian formulation, there exist $3n$ equations of motion of the second order for a finite system consisting of n particles in the absence of holonomic constriants. As a matter of fact, for general consideration it is very much essential to write $6n$ partial differential equations of first order instead of $3n$ ~~equations~~ of second order. Hence there must exist $6n$ degrees of freedom or $6n$ dimensional space, usually known as *phase-space* or *Γ-space*. In phase space, a particle is specified by six co-ordinates ; 3 position co-ordinates and 3 momentum co-ordinates where of course momenta are regarded as independent variables like space co-ordinates. The six dimensional space is some times called *μ-space*. The superposition of *μ-space* is *Γ-space*. In short, we may say that *phase-space* is a *6-dimensional space* having co-ordinates $p_1, p_2, p_3, \ldots, p_{3n}$; $q_1, q_2, \ldots$; q_{3n}.

Hamiltonian Formulation. In Lagrangian formulation, it was assumed that the mechanical state of the system is completely defined, once its velocities and geneneralised co-ordinates are specified. It is one of the many possible ways in which the motion of the system may be defined. There does exist another mode of description, much more advantageous then the Lagrangian, in term of velocities and generalised co-ordinates. This mode describes the state of the system in terms of momenta and generalised co-ordinates. Before proceeding further, we write the Lagrangian of the system as follows :

$$L=L\,(q_\alpha, \dot{q}_\alpha, t).$$

Also Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0 \ (\alpha=1, 2, \ldots n) \qquad \ldots(1)$$

$$\ldots(2)$$

writing $p_\alpha=(\partial L/\partial \dot{q}_\alpha)$, we get $\dot{p}_\alpha=(\partial L/\partial q_\alpha)$.

Hence from the former of these sets of equations, we can regard either of the sets of quantities $(q_1, q_2, \ldots, q_n)$ or $(p_1, p_2, \ldots, p_n)$ as functions of the other set.

Now let δ denote the increment in any function of the variables $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ or $(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$, then we get

$$dL = \sum_{\alpha=1}^{n} \left[\frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right] + \frac{\partial L}{\partial t} dt$$

(when L contains t explicity)

$$= \sum_{\alpha=1}^{n} (\dot{p}_\alpha \, dq_\alpha + p_\alpha \, d\dot{q}_\alpha) + \frac{\partial L}{\partial t} dt$$

$$= d\left(\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha \right) + \sum_{\alpha=1}^{n} (\dot{p}_\alpha \, dq_\alpha - \dot{q}_\alpha \, dp_\alpha) + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow d\left\{ \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - L \right\} = \sum_{\alpha=1}^{n} (\dot{q}_\alpha \, dp_\alpha - \dot{p}_\alpha \, dp_\alpha) - \frac{\partial L}{\partial t} dt.$$

Thus, if the quantity $\left(\sum_{\alpha=1}^{n} (p_\alpha \dot{q}_\alpha - L) \right)$, when expressed in terms of $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t)$ be denoted by H, we have

$$dH = \sum_{\alpha=1}^{n} (\dot{q}_\alpha \, dp_\alpha - \dot{p}_\alpha \, dp_\alpha) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \sum_{\alpha=1}^{n} \frac{\partial H}{\partial q_\alpha} dq_\alpha + \sum_{\alpha=1}^{n} \frac{\partial H}{\partial p_\alpha} dp_\alpha + \frac{\partial H}{\partial t} dt$$

$$= \sum_{\alpha=1}^{n} (q_\alpha \, dp_\alpha - p_\alpha \, dp_\alpha) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \ \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}, \ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If H does not contain t explicitly (*i.e.* L does not contain t explicity), we have

$$p_\alpha = -\frac{\partial H}{\partial q_\alpha} \text{ and } \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$ **[Agra 82]**

These equations are called as *Hamilton's equations*, or *Hamilton's canonical equations* and the function H is called *Hamiltonian.*

The total of Hamilton's equations is the same as the total order of Lagrange's equations, namely, $2n$. But whereas Lagrange's equations present us with n equations each of the second order: Hamilton's equations are $2n$ equations each of the first order. Hamilton's equations can also be written as

$$\frac{dp_\alpha}{-\dfrac{\partial H}{\partial q_\alpha}}=\frac{dq_\alpha}{\dfrac{\partial H}{\partial p_\alpha}}=dt$$

6·1-1. *If the Hamiltonian H is independent of t explicitly, prove that it is*

(a) *a constant and is*

(b) *equal to the total energy of the system.* [Agra 88]

Proof. (a) We have

$$\frac{dH}{dt}=\sum_{\alpha=1}^{n}\frac{\partial H}{\partial q_\alpha}\frac{dq_\alpha}{dt}+\sum_{\alpha=1}^{n}\frac{\partial H}{\partial p_\alpha}\frac{dp_\alpha}{dt}=\sum_{\alpha=1}^{n}-(\dot{q}_\alpha)\dot{p}_\alpha+\sum_{\alpha=1}^{n}\dot{q}_\alpha\dot{p}_\alpha$$

$$\left(\because\ \dot{q}_\alpha=-\frac{\partial H}{\partial q_\alpha}\text{ and }\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}\right)$$

$$=0$$

$\Rightarrow$ H=constant, say E

(b) By Euler's theorem on homogeneous functions, we have

$$\Sigma\dot{q}_\alpha\frac{\partial T}{\partial\dot{q}_\alpha}=2T,\text{ where } T \text{ is the K.E. of the system}$$

$$[\because\ T=T\{\dot{q}_1,\ \dot{q}_2,\ \ldots,\ \dot{q}_n\}]$$

But $L=T-V$, $\therefore$ $\dfrac{\partial L}{\partial\dot{q}_\alpha}=\dfrac{\partial(T-V)}{\partial\dot{q}_\alpha}=\dfrac{\partial T}{\partial\dot{q}_\alpha}$

(since V does not depend on $\dot{q}_\alpha$)

or $$\Sigma\,\dot{q}_\alpha\frac{\partial L}{\partial\dot{q}_\alpha}=2T\Rightarrow\Sigma\,\dot{q}_\alpha p_\alpha=2T\left(\because\ p_\alpha=\frac{\partial L}{\partial\dot{q}_\alpha}\right)$$

$$\therefore\ H=\Sigma\,p_\alpha\dot{q}_\alpha-L=2T-(T-V)=T+V=E.$$

6·1-2. Passage from the Hamiltonian to the Lagrangian.

Suppose that we are given a function* $H(q, p, t)$ and are told that the motion of the system satisfies the canonical equations :

$$\dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}\text{ and }\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha}\qquad\ldots(4)$$

Then we want to find a function

$L(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t)$ *i.e.* $L(p, q, t)$

such that this motion also satisfies the equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0.\qquad\ldots(5)$$

Solve the first set of equations in (4) for the p's in terms of the q's, the $\dot{q}$'s and t.

*$H(q_1, \ldots q_2 \ldots, q_n, \ldots, p_1 \ldots, p_n)$ is also written as $H(q, p, t)$.

Then write $L=\sum_{\alpha=1}^{n} \dot{q}_\alpha p_\alpha - H$ and express L as function of the q's, $\dot{q}$'s and t. This is required the Lagrangian.

$$\therefore \quad \frac{\partial L}{\partial \dot{q}_\alpha}=p_\alpha \qquad \left(\text{using } L=\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha - H\right).$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)=\dot{p}_\alpha \text{ and } \frac{\partial L}{\partial q_\alpha}=-\frac{\partial H}{\partial q_\alpha}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=\dot{p}_\alpha+\frac{\partial H}{\partial q_\alpha}=\dot{p}_\alpha-\dot{p}_\alpha=0$$

i.e. L satisfies (5) assuming (4).

Similarly we can prove its converse.

6·1-3. Ignoration of Co-ordinates and Routh's procedure.

If one of the co-ordinates, say q_1, happens to be absent from H, then we have $\left(\frac{\partial H}{\partial q_1}\right)=0$ and hence $\dot{p}_1=-\left(\frac{\partial H}{\partial q_1}\right)=0.$

$\Rightarrow$ p_1=constant$=a_1$, say. Now replacing p_1 by a_1 in the right-hand sides of the equations :

$$\dot{q}_\alpha=\left(\frac{\partial H}{\partial p_\alpha}\right),\quad \dot{p}_\alpha=-\left(\frac{\partial H}{\partial q_\alpha}\right) \quad (\alpha=1, 2, 3, \ldots, n) \qquad \ldots(6)$$

and dropping from the $2n$ equations the two equations for which $\alpha=1$, we have before us a set of canonical equations for the $2(n-1)$ quantities $q_2 \ldots, q_n, p_2, \ldots p_n$. Under such circumstances, the generalised co-ordinate q_1 is said to be ignorable. Further it is obvious that, if there exists one ignorable co-ordinate, the number of degrees of freedom is at once reduced by unity, without loss of the canonical form of the equations of motion. The ignorable co-ordinate q_1 is to be obtained from the equation $\dot{q}_1=\left(\frac{\partial H}{\partial p_1}\right)$.

If the ignorable co-ordinates are m, the number of degrees of freedom is reduced by m.

But $\quad \left(\frac{\partial H}{\partial q_1}\right)=-\left(\frac{\partial L}{\partial q_1}\right)$

Hence $\quad \left(\frac{\partial H}{\partial q_1}\right)=0 \Rightarrow \left(\frac{\partial L}{\partial q_1}\right)=0.$

Thus, if we start from Lagrangian instead of from a Hamiltonian, an Ignorable co-ordinate can be detected through its absence from L.

In order to solve the problems of cyclic and non-cyclic co-ordinates, Routh obtained a method by combining the Lagrangian and Hamiltonian, being called as *Routh's Procedure.*

Let $q_1, q_2, \ldots q_s$ be then cyclic co-ordinates, the Routhian function is defined by the relation

$$R(q_1, q_2, \ldots, q_\alpha, \ldots q_n; p_1, p_2, \ldots p_s, \dot{q}_{s+1}, \ldots \dot{q}_\alpha, \ldots \dot{q}_n, t)$$

$$= \sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - L.$$

$$\therefore \quad dR = \sum_{\alpha=1}^{s} \dot{q}_\alpha dp_\alpha + \sum_{\alpha=1}^{s} d_\alpha d\dot{q}_\alpha - \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha$$

$$- \sum_{\alpha=1}^{n} \frac{\partial L}{\partial q_\alpha} dq_\alpha - \frac{\partial L}{\partial t} d$$

$$\Rightarrow \quad dR = \sum_{\alpha=1}^{s} \dot{q}_\alpha dp_\alpha + \sum_{\alpha=1}^{s} \frac{\partial \Gamma}{\partial \dot{q}_\alpha} d\dot{q}_\alpha - \left\{ \sum_{\alpha=1}^{s} \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha + \right.$$

$$\left. + \sum_{\alpha=s+1}^{n} \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right\} \sum_{\alpha=1}^{n} \frac{\partial L}{\partial q_\alpha} dq_\alpha - \frac{\partial L}{dt} dt \left[\because \; p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \right]$$

$$\Rightarrow \quad dR = \sum_{\alpha=1}^{s} \dot{q}_\alpha dp_\alpha - \sum_{\alpha=s+1}^{n} \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha - \sum_{\alpha=1}^{n} \frac{\partial L}{\partial q_\alpha} dq_\alpha - \frac{\partial L}{\partial t} dt,$$

$$\Rightarrow \quad \frac{\partial R}{\partial p_\alpha} = \dot{q}_\alpha, \; \frac{\partial R}{\partial q_\alpha} = -\dot{p}_\alpha \; [\alpha = 1 \text{ to } s] \left[\because \; \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha \right] \qquad \ldots(7)$$

and
$$\frac{\partial R}{\partial \dot{q}_\alpha} = -\frac{\partial L}{\partial \dot{q}_\alpha}, \; \frac{\partial R}{\partial q_\alpha} = -\frac{\partial L}{\partial q_\alpha} \; [\alpha = s+1 \text{ to } n] \qquad \ldots(8)$$

Equations (7) $\Rightarrow$ Hamilton's equations with R as Hamiltonian whereas equatious (8) $\Rightarrow$ that

$$\left(\frac{d}{dt}\right)\left(\frac{\partial R}{\partial \dot{q}_\beta}\right) - \left(\frac{\partial R}{\partial q_\beta}\right) = 0 \text{ for } \beta = s+1 \text{ to } n,$$

i.e. the co-ordinates $q_{s+1}, q_{s+2}, \ldots q_n$ satisfy Lagrange's equations with R as the Lagrangian function.

Any co-ordinate which is absent in Lagrangian, will not appear in the Routhian R. Here co-ordinates $q_1, q_2, \ldots q_s$ are ignorable, so they will not appear in R and the momenta $p_1, p_2, \ldots p_s$ conjugate to $q_1, q_2, \ldots, q_s$ respectively are constants such that they can be replaced by a set $\{a_1, a_2, \ldots a_s\}$ of constants which are to be determined by the boundary conditions. By this fact, R will contain only $(n-s)$ non cyclic co-ordinates as variables and the generalised velocities corresponding to these $(n-s)$ co-ordinates implying $R = R(q_{s+1}, q_n, \dot{q}_{s+1}, \ldots \dot{q}_n, a_1, a_2 \ldots, a_s, t)$ and hence Lagrange's equations for non-cyclic co-ordinates can be solved as if the cyclic co-ordinates are absent whereas Hamilton's equation can be solved for cyclic co-ordinates.

Thus according to Routhian procedure a problem involving cyclic and non-cyclic co-ordinates can be solved by solving-Lagrangian equations for non-cyclic co-ordinates with R as the Lagrangian function and solving Hamiltonion equations for given cyclic co-ordinates with R as Hamiltonian function.

6·1-4. **Variational Mothods.**

(a) **Tecbniques of Calculus of Variations.**

[Meerut M. Sc. Maths. 92]

The calculus of variations arose out of the quest for the mathematical requirements in the solution of problems like the study of :

(*i*) the path followed by a body falling freely under gravity (brachistochrone) first studied by Newton.

(*ii*) the equilibrium shape of freely hanging homogeneous flexible cord between two horizontal points (catenary), first studied by Bernoulli.

Presently, the technique of calculus of variations, serves as a mathematical preliminary (in the form of Euler-Lagrange's equations) in the study of a wide range of physical problems such as geodesies and minimal surfaces in Riemannian and differential geometries and the various *variational* (*minimal*) *principles* in the different branches of Physics. The technique is co-ordinates invariant and therein lies its great power. First we develop here this technique in a purely mathematical form.

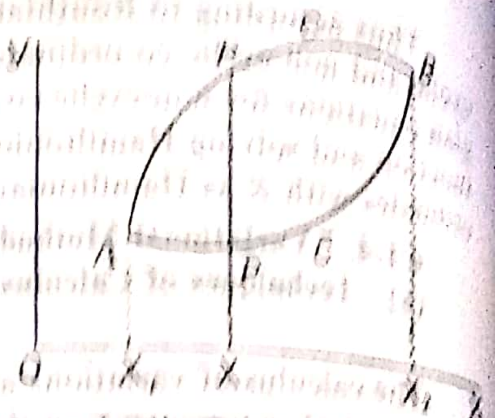
Suppose A and B are fixed points (x_1, y_1), (x_2, y_2) in a cartesian plane. Also suppose that (x, y, y') is a known functional form of the variables $x, y, y' = \left\{\left(\frac{dy}{dx}\right)\right\}$. Then if C is a curve joining A to B and having equation $y=y(x)$, the integral

$$I = \int_{x1}^{x_2} f(x, y, y')\, dx \qquad \ldots(9)$$

has a definite value whenever the function $y(x)$ is prescribed. The value I will change as we vary the form of the curve C through A, B. Consequently we may consider that in general three will be some curve C through these fixed points such that the value of I taken along it is stationary (in general a maximum or minimum compared with the value along neighbouring paths C. Calculus of variations, the branch of mathematics is concerned amongst other things with finding the form of $y(x)$ for which this stationary property holds. Let the path C have the equation $y=y(x)$ and let the equation of a neighbouring curve C' be $y=y(x)+\epsilon\eta(x)$, where ϵ is small and $\eta(x)$ is an arbitary continuous differentiable function of x satisfying

$$\eta(x_1)=0=\eta(x_2),$$

to ensure that the curve passes through A and B.



Now we have $P'X=y(x)+\epsilon\eta(x)$, and $PX=y(x)$

$$\Rightarrow \quad PP'=\epsilon\eta(x).$$

Hence the value of I taken along C' is thus a function of ϵ of the form

$$I(\epsilon)=\int_{x_1}^{x_2} f(x,\, y+\epsilon\eta,\, y'+\epsilon\eta')\, dx. \quad ...(10)$$

But the curve C, for which $\epsilon=0$, makes I stationary so we must have $I'(0)=0$. Now differentiating (10) w.r.t. ϵ given

$$I'(\epsilon)=\int_{x_1}^{x_2} [\eta f_y(x,\, y+\epsilon\eta,\, y'+\epsilon\eta')+\eta' f_{y'}(x,\, y+\epsilon\eta,\, y'+\epsilon\eta')]\, dx,$$

where $\quad f_y(x,\, y+\epsilon\eta,\, y'+\epsilon\eta')$

$$=\left(\frac{\partial}{\partial y}\right) f(x,y,y') \text{ when } y=y+\epsilon\eta,\ y'=y'+\epsilon\eta'$$

with a similar meaning for $f_{y'}(x,\, y+\epsilon\eta,\, y'+\epsilon\eta')$. Hence the stationary condition gives $I'(0)=0$,

$$\Rightarrow \int_{x_1}^{x_2} [\eta f_y(x,y,y')+\eta' f_{y'}(x,y,y')]\, dx=0. \quad ...(11)$$

Now $\displaystyle\int_{x_1}^{x_2} \eta' f_{y'}(x,y,y')\, dx$

$$=\Big[\eta f_{y'}(x,y,y')\Big]_{x_1}^{x_2}-\int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x,y,y')\, dx$$

$$=-\int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x,y,y')\, dx \quad [\because \eta(x_1)=\eta(x_2)=0] \quad ...(12)$$

$$\therefore \int_{x_1}^{x_2} \eta(x)\left[f_y(x,y,y')-\frac{d}{dx} f_{y'}(x,y,y')\right] dx=0. \quad ...(13)$$

[using 11]

But $\eta(x)$ is arbitrary, subject to its being differentiable and vanishing at A, B; (13) implies that

$$f_y(x,y,y')-\left(\frac{d}{dx}\right) f_{y'}(x,y,y')=0. \quad ...(14)$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)-\left(\frac{d}{dx}\right)\left(\frac{\partial f}{\partial y'}\right)=0. \qquad \ldots(15)$$

This is called **Euler's Lagrange's equation.** Equation (15) can be proved from (13) by using the following lemma.

Lemma. *If x_1 and x_2 $(> x_1)$ are fixed constants and $\phi(x)$ is a particular continuous function for $x_1 \leqslant x \leqslant x_2$ and if*

$$\int_{x_1}^{x_2} \eta(x)\,\phi(x)\,dx=0$$

for every choice of continuously differentiable function $\eta(x)$ for which $\eta(x_1)=\eta(x_2)=0$ then $\phi(x)=0$ identically $x_2 \leqslant x \leqslant x_2$.

Proof. Let the lemma be not true for all x in $x_1 \leqslant x \leqslant x_2$. Let x' been point where $\phi(x')\neq 0$ and > 0. But $\psi(x)$ is continuous in $x_1 \leqslant x \leqslant x_2$ and in particular is continuous at $x=x'$. Hence there exists an interval $x'_1 \leqslant x \leqslant x'_2$ around x' where $\phi(x) > 0$. For other x's $\phi(x)$ may not vanish; then the equation

$$\int_{x_1}^{x_2} \eta(x)\,\phi(x)\,dx=0$$

could be written as

$$\int_{x'_1}^{x'_2} \eta(x)\,\phi(x)\,dx=0. \qquad \ldots(16)$$

But $\eta(x)$ is at our disposal; we could choose it as greater than zero for $x'_1 \leqslant x \leqslant x'_2$ and $\eta(x)=0$ for other values of x. Thus $\eta(x)\,\phi(x) > 0$ for $x'_1 \leqslant x \leqslant x'_2$, consequently integrand cannot be zero. Thus we arrive at the contradiction. Hence $\phi(x)=0$ in $x_1 \leqslant x \leqslant x_2$.

Certain remarks about Euler-Lagrange's equation.

(*A*) We have $\left(\frac{\partial f}{\partial y}\right)-\left(\frac{d}{dx}\right)\left(\frac{\partial f}{\partial y'}\right)=0$ ...(17)

where $f=f(x, y, y')$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)-\left[\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial f}{\partial y'}\right)+\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial y'}\right)\left(\frac{dy}{dx}\right)+\left(\frac{\partial}{\partial y'}\right)\left(\frac{\partial f}{\partial y'}\right)\left(\frac{dy'}{dx}\right)\right]=0$$

$$\Rightarrow \left(\frac{\partial f}{\partial y}\right)-\left[\left(\frac{\partial^2 f}{\partial x\,\partial y'}\right)+\left(\frac{\partial^2 f}{\partial y\,\partial y'}\right)\left(\frac{dy}{dx}\right)+\left(\frac{\partial^2 f}{\partial y'^2}\right)\left(\frac{d^2 y}{dx^2}\right)\right]=0$$

$$\Rightarrow \left(\frac{\partial^2 f}{\partial y'^2}\right)\left(\frac{d^2 y}{dx^2}\right)+\left(\frac{\partial^2}{\partial y\,\partial y'}\right)\left(\frac{dy}{dx}\right)\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 f}{\partial x\,\partial y'}\right)=0. \qquad \ldots(18)$$

This is a second order differential equation for determining y as a function of x. The solution will contain two arbitrary constants which could be determined from the conditions :

$$y=y_1 \text{ at } x=x_1 \text{ and } y=y_2 \text{ at } x=x_2. \qquad \ldots(19)$$

(*B*) Consider $\left(\frac{d}{dx}\right)\left[f-y'\left(\frac{\partial f}{\partial y'}\right)\right]=\left(\frac{\partial f}{\partial x}\right)$ ...(20)

$$\Rightarrow \left(\frac{\partial}{\partial x}\right)\left[f-y'\left(\frac{\partial f}{\partial y'}\right)\right]+\left(\frac{\partial}{\partial y}\right)\left[f-y'\left(\frac{\partial f}{\partial y'}\right)\right]\left(\frac{dy}{dx}\right)$$
$$+\left(\frac{\partial}{\partial y'}\right)\left[f-y'\left(\frac{\partial f}{\partial y'}\right)\right]\left(\frac{d^2y}{dx^2}\right)=\left(\frac{\partial f}{\partial x}\right)$$

$$\Rightarrow -y'\left(\frac{\partial^2 f}{\partial x\,\partial y'}\right)+\left(\frac{\partial f}{\partial x}\right)\left(\frac{dy}{dx}\right)-y'\left(\frac{\partial^2 f}{\partial y\,\partial y'}\right)\left(\frac{dy}{dx}\right)$$
$$+\left(\frac{\partial f}{\partial y'}\right)\left(\frac{d^2y}{dx^2}\right)-\left(\frac{\partial f}{\partial y'}\right)\left(\frac{d^2y}{dx^2}\right)-y'\left(\frac{\partial^2 f}{\partial y'^2}\right)\left(\frac{d^2y}{dx^2}\right)=0$$

$$\Rightarrow y'\left[\left(\frac{\partial f}{\partial y}\right)-\left(\frac{\partial^2 f}{\partial x\,\partial y'}\right)-y'\left(\frac{\partial^2 f}{\partial y\,\partial y'}\right)-y''\left(\frac{\partial^2 f}{\partial y'^2}\right)\right]=0$$

so that $\quad f_y-f_{xy'}-y'f_{yy'}-y''f_{y'y'}=0 \quad (\text{if } y'\neq 0)$ ...(21)

which is same as (18) and shows that if Euler's equation is satisfied equation (20) is also satisfied but if (20) is satigfied, Euler's equation may nol be true.

(*C*) If f does not contain x explicitly

i.e. $\qquad I=\int_{x_1}^{x_2} f(y, y')\,dx,$

then from (20), we have

$$\left\{\left(\frac{d}{dx}\right)\right\}\left\{f-y'\left(\frac{\partial f}{\partial y'}\right)\right\}=0 \quad \left\{\because \left(\frac{\partial f}{\partial x}\right)=0\right\}$$

$$\Rightarrow f-y'\left(\frac{\partial f}{\partial y'}\right)=\text{constant}=C \text{ (say)} \qquad ...(22)$$

Obviously (22) is first order differential equation.

(*D*) If f does not contain y explicity, then we have

$$I=\int_{x_1}^{x_2} f(x, y')\,dx.$$

$\therefore$ from the equation

$$\left(\frac{\partial f}{\partial y}\right)-\left(\frac{d}{dx}\right)\left(\frac{\partial f}{\partial y'}\right)=0, \text{ we obtain } \left(\frac{d}{dx}\right)\left(\frac{\partial f}{\partial y'}\right)=0$$

$$\Rightarrow \left(\frac{\partial f}{\partial y'}\right)=C \text{ (say)}. \qquad ...(23)$$

(*E*) If we assume that the end points are not fixed

i.e. $\qquad \eta(x_1)\neq 0 \quad \text{and} \quad \eta(x_2)\neq 0.$

Euler's equation still holds in this case, if $\left(\frac{\partial f}{\partial y'}\right)=0$ where $x=x_1, x_2$.

This becomes the boundary condition of the problem.

(*F*) Euler's equation gives only a necessary condition, even if it is statisfied there may be no extremum.

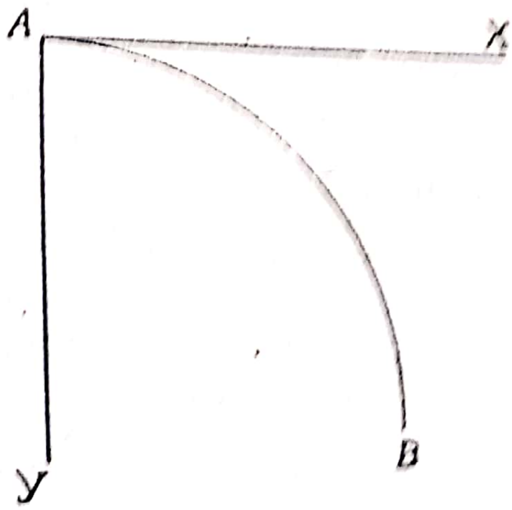
(*G*) The conditions to be imposed on f are such that its partial derivatives are continuous and differentiable.

(*H*) The conditions to be imposed on the curve are that there should be continuous curves.

(b) **Brachistochrone Problem.** **(Meerut M.Sc. Maths. 90)**

Given two points A and B in a vertical plane, to find for the moveable particle M. the path AMB, descending along which by its own gravity and beginig to be urged from the point A, it may in the shortest time reach the point B. If it is tacit in the statement that the particle descends without friction.

Let A be the origin, then velocity of the particle of mass M at any time t is given by



$$v=\left(\frac{ds}{dt}\right)=\sqrt{(2gy)}$$

$$\Rightarrow \quad dt=\left[\frac{ds}{\sqrt{(2gy)}}\right]$$

$$\Rightarrow \quad t=\int_A^B \frac{ds}{\sqrt{(2gy)}}$$

$$=\int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{(2gy)}}\,.\,dx$$

$$=\frac{1}{\sqrt{(2g)}}\int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{(y)}}\,.\,dx$$

$$=\frac{1}{\sqrt{(2g)}}\int_A^B f(y, y')\,dx,$$

where $\quad f(y, y')=\dfrac{\sqrt{(1+y'^2)}}{\sqrt{y}}.$

As x is absent in f, so we have

$$f-y'\left(\frac{\partial f}{\partial y'}\right)=A.$$

$$\Rightarrow \quad \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}-y'\frac{y'}{\sqrt{\{y(1+y'^2)\}}}=A.$$

$$\Rightarrow \quad \frac{1}{\sqrt{\{y\,(1+y'^2)\}}}=\text{constant} \Rightarrow y\,(1+y'^2)=\text{constant}=2C, \text{ say.}$$

then
$$\left(\frac{dy}{dx}\right)=\tan\psi \Rightarrow y'=\tan\psi.$$

$\therefore \quad y\sec^2\psi=2c \quad \Rightarrow \quad y=2c\cos^2\psi=c\,(1+\cos 2\psi).$

Also
$$dx=\cot\psi.dy=-2c\sin 2\psi\cot\psi\,d\psi$$
$$=-4c\cos^2\psi=-2c\,(1+\cos 2\psi)\,d\psi$$

$$\Rightarrow \quad x=a-2c\left(\psi+\frac{\sin 2\psi}{2}\right)=a-2c\,(\psi+\sin\psi\cos\psi).$$

Hence the required path is

$$x=a-c\,(2\psi+\sin 2\psi),\ y=c\,(2+\cos 1\psi)\ \text{(cycloid)} \quad \ldots(24)$$

(c) Extension of variational method.

Suppose the n co-ordinates $q_1, q_2, \ldots, q_n$ are each functions of an independent variable t and that we require the solution of the variational problem

$$I=\int_{t_1}^{t_2} f\,(q_1, q_2, \ldots, q_n\,;\,\dot q_1, \dot q_2, \ldots, \dot q_n\,;\,t)\,dt$$

$$=\text{Stationary value}$$

where f is of known functional form, t_1 and t_2 are fixed, and each q_α is to be determined.

Let $q_\alpha=q_\alpha\,(t)$ give the stationary value to I, and let

$$q_\alpha=q_\alpha\,(t)+\epsilon_\alpha\eta_\alpha\,(t)$$

be the neighbouring path.

$$I\,(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)=\int_{t_1}^{t_2} f\{q_\alpha\,(t)+\epsilon_\alpha\eta_\alpha\,(t),\ \dot q_\alpha\,(t)+\epsilon_\alpha\dot\eta_\alpha\,(t),\ t\}\,dt$$

$$=\int_{t_1}^{t_2} f(q_\alpha, \dot q_\alpha, t)\,dt+\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\left\{\eta_\alpha\,(t)\frac{\partial f}{\partial q_\alpha}+\dot\eta_\alpha\,(t)\frac{\partial f}{\partial \dot q_\alpha}\right\}dt+0\,(\epsilon^2_\alpha)$$

First variation

$$\dot I=\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\left\{\eta_\alpha\,(t)\,\frac{\partial f}{\partial q_\alpha}+\dot\eta_\alpha\,(t)\,\frac{\partial f}{\partial \dot q_\alpha}\right\}dt$$

$$=\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\eta_\alpha\,(t)\,\frac{\partial f}{\partial q_\alpha}\,dt+\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\dot\eta_\alpha\,(t)\,\frac{\partial f}{\partial \dot q_\alpha}\,dt$$

$$=\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\eta_\alpha\,(t)\frac{\partial f}{\partial q_\alpha}\,dt+\Sigma\epsilon_\alpha\left[\left\{\eta_\alpha\,(t)\,\frac{\partial f}{\partial \dot q_\alpha}\right\}_{t_1}^{t_2}-\int_{t_1}^{t_2}\eta_\alpha\,(t)\,\frac{d}{dt}\left(\frac{\partial f}{\partial \dot q_\alpha}\right)dt\right]$$

$$=\Sigma\epsilon_\alpha\int_{t_1}^{t_2}\eta_\alpha\,(t)\,\left[\frac{\partial f}{\partial q_\alpha}-\frac{d}{dt}\left(\frac{\partial f}{\partial \dot q_\alpha}\right)\right]dt$$

$$\{\because\ \ 0=\eta_\alpha\,(t_1)=\eta_\alpha\,(t_2)\}$$

As η's are arbitary and I is stationary, so we have

$$\left(\frac{\partial f}{\partial q_\alpha}\right)-\left(\frac{d}{dt}\right)\left(\frac{\partial f}{\partial \dot q_\alpha}\right)=0 \qquad (\alpha=1, 2, \ldots, n) \qquad \ldots(25)$$

*This problem was set by the Johan Bernoulli in June, 1966 before the scholars of his time. Although Newton had earlier considered least one problem falling within, the province of the calculus of variations, the proposal of Bernoulli brachistochrone Problem marked the real beginning of general interest in the subject. Actually the term *Brachistrochrone* derives from the greek Brachistors, shortest ahd chronos, time.

(d) Hamilton's variational principle.

Variational (minimal) principles has been the greatest fascination of old generation physicsts and in this pursuit several and varied variational principles have been put forth in the various branches of physics, such as, in optics, the Fermat's principle of '*least time*', in mechanics, the Gauss's principle of *least constraint*', the Hertz's principle of '*least curvature*' and most important of all the Hamilton's *variational principle*, which will be our main centre of attraction here in mechanics. The great value of these variational principles, lies in their extreme economy of expression.

Let T, V be the kinetic and potential energies of a conservative holonomic dynamical system defined by n generalised co-ordinates $q_1, q_2, \ldots, q_n$ at time t. Writing $L=T-V$, we know that Lagrange's equations for the motion of the system are :

$$\left(\frac{\partial L}{\partial q_\alpha}\right)-\left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)=0. \qquad (\alpha=1, 2, \ldots, n)$$

Now applylng previous article we can say that these are the n Euler-Lagrange's equations arising from the variational problem.

$$\int_{t_1}^{t_2} L\,(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)\,dt=\text{stationary } (\textit{i.e.} \text{ stat.}) \qquad \ldots(26)$$

where t_1, t_2 are fixed. Thus we have established that :

During the motion of a 'conservative holonomic dynamical system' over a fixed time intervel, the time integral over that interval of a difference between the kinetic and potential energies is stationary. This is Hamilton's principle.

6·1-5 (i) Derivation of Hamilton's equations from the variational principle.

Hamilton's principle implies

$$\delta I=\delta\int_{t_1}^{t_2} L\,dt=0 \Rightarrow \delta\int_{t_2}^{t_1}[\Sigma_\alpha\, p_\alpha\, \dot{q}_\alpha-H\,(q, p, t)]\,dt=0$$

$$[\because L=\Sigma_\alpha\, p_\alpha\, \dot{q}_\alpha-H\,(q, p, t)]$$

This implies $\delta\Sigma_\alpha \int_{q_1}^{q_2} p_\alpha\, dp_\alpha-\delta\int_{t_1}^{t_2} H\,dt=0.$ ...(27)

Equation (27) is called the modified Hamilton's principle.

Now $\delta I=\dfrac{\partial I}{\partial r}dr=dr\dfrac{\partial}{\partial r}\left\{\displaystyle\int_{t_1}^{t_2} \Sigma_\alpha\, p_\alpha\, \dot{q}_\alpha-H\,(q, p, t)\,dt\right\}=0$

where $\delta=dr\left(\dfrac{\partial}{\partial r}\right)$.

$$\Rightarrow dr\int_{t_1}^{t_2}\frac{\partial}{\partial r}[\Sigma_\alpha\, p_\alpha\, \dot{q}_\alpha-H\,(q, p, t)]\,dt=0.$$

Now assume that the system is acted upon by a number of forces represented by $\mathbf{F}_\alpha$.

Let the αth particle of the system be acted upon by a force $\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha$ where $\ddot{\mathbf{r}}_\alpha$ is acceleration vector.

Again by D' Alembert's principle. we have

$\Sigma_\alpha (\mathbf{F}_\alpha - m_\alpha \ddot{\mathbf{r}}_\alpha).\delta\mathbf{r}_\alpha = 0$, *i.e.* $\Sigma_\alpha \mathbf{F}_\alpha.\delta\mathbf{r}_\alpha - \Sigma m_\alpha \ddot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha = 0$ ...(31)

If there is a little variation along the actual and neighbouring paths. then

$$\delta\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \mathbf{r}_\alpha \text{ (say)},$$

$$\Rightarrow \left(\frac{d}{dt}\right)(\delta\mathbf{r}_\alpha) = \left(\frac{d}{dt}\right)(\mathbf{r}'_\alpha - \mathbf{r}_\alpha) = \left(\frac{d\mathbf{r}'_\alpha}{dt}\right) - \left(\frac{d\mathbf{r}_\alpha}{dt}\right)$$

$$= \delta\left(\frac{d\mathbf{r}_\alpha}{dt}\right) = \delta\,(d\,\dot{\mathbf{r}}_\alpha),$$

where dashes have been used for neighbouring paths. ...(32)

But $\ddot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha = \left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha.\left(\frac{d}{dt}\right)(\delta\mathbf{r}_\alpha).$ ...(33)

Using (32), (33) $\Rightarrow \ddot{\mathbf{r}}.\delta\mathbf{r}_\alpha = \left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha \cdot \delta\,(\dot{\mathbf{r}}_\alpha)$ ...(34)

$$\therefore \quad (31) \Rightarrow \sum_\alpha \mathbf{F}_\alpha.\delta\mathbf{r}_\alpha - \sum_\alpha m_\alpha \left[\left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha.\delta\,\dot{\mathbf{r}}_\alpha\right] = 0$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha\, \delta\mathbf{r}_\alpha - \sum_\alpha m_\alpha \left[\left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \tfrac{1}{2}\delta\,(\dot{\mathbf{r}}_\alpha^2)\right] = 0$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha \delta\mathbf{r}_\alpha + \sum_\alpha \tfrac{1}{2} m_\alpha \delta\,(\dot{\mathbf{r}}_\alpha^2) = \Sigma \left(\frac{d}{dt}\right)(m_\alpha\, \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha)$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha\, \delta\mathbf{r}_\alpha + \delta\Big(\sum_\alpha \tfrac{1}{2} m_\alpha\, \dot{\mathbf{r}}_\alpha^2\Big) = \sum_\alpha \left(\frac{d}{dt}\right)(m_\alpha\, \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) \quad ...(35)$$

But $\sum_\alpha \tfrac{1}{2} m_\alpha \dot{\mathbf{r}}^2_\alpha$ = kinetic energy of the system $= T$ and $\sum_\alpha \mathbf{F}_\alpha \cdot \delta\mathbf{r}_\alpha$

= work done by the forces $\mathbf{F}_\alpha$ during displaec-ment $\delta\mathbf{r}_\alpha$
$= \delta W$ (say).

$\therefore$ equation (35) $\Rightarrow \delta W + \delta T = \sum_\alpha \left(\frac{d}{dt}\right)(m_\alpha\, \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha).$ ...(36)

Integrating (36) between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_\alpha \frac{d}{dt}(m_\alpha\, \dot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha)\, dt = \Sigma \int_{t_1}^{t_2} d\,(m_\alpha\, \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha)$$

$$= \sum_\alpha \Big[(m_\alpha\, \dot{\mathbf{r}} \cdot \delta\mathbf{r}_\alpha)\Big]_A^C = 0$$

since $\delta\mathbf{r}_\alpha = 0$ at the end points A and C.

Now assume that the system is acted upon by a number of forces represented by $\mathbf{F}_\alpha$.

Let the αth particle of the system be acted upon by a force $\mathbf{F}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha$ where $\ddot{\mathbf{r}}_\alpha$ is acceleration vector.

Again by D' Alembert's principle. we have

$\Sigma_\alpha (\mathbf{F}_\alpha - m_\alpha \ddot{\mathbf{r}}_\alpha).\delta\mathbf{r}_\alpha = 0$, *i.e.* $\Sigma_\alpha \mathbf{F}_\alpha.\delta\mathbf{r}_\alpha - \Sigma m_\alpha \ddot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha = 0$ ...(31)

If there is a little variation along the actual and neighbouring paths. then $\delta\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \mathbf{r}_\alpha$ (say),

$$\Rightarrow \left(\frac{d}{dt}\right)(\delta\mathbf{r}_\alpha) = \left(\frac{d}{dt}\right)(\mathbf{r}'_\alpha - \mathbf{r}_\alpha) = \left(\frac{d\mathbf{r}'_\alpha}{dt}\right) - \left(\frac{d\mathbf{r}_\alpha}{dt}\right)$$

$$= \delta\left(\frac{d\mathbf{r}_\alpha}{dt}\right) = \delta\,(d\,\dot{\mathbf{r}}_\alpha), \qquad ...(32)$$

where dashes have been used for neighbouring paths.

$$\text{But } \ddot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha = \left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha.\left(\frac{d}{dt}\right)(\delta\mathbf{r}_\alpha). \qquad ...(33)$$

$$\text{Using (32), (33)} \Rightarrow \ddot{\mathbf{r}}.\delta\mathbf{r}_\alpha = \left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha \cdot \delta\,(\dot{\mathbf{r}}_\alpha) \qquad ...(34)$$

$$\therefore \quad (31) \Rightarrow \sum_\alpha \mathbf{F}_\alpha.\delta\mathbf{r}_\alpha - \sum_\alpha m_\alpha \left[\left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \dot{\mathbf{r}}_\alpha.\delta\,\dot{\mathbf{r}}_\alpha\right] = 0$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha\,\delta\mathbf{r}_\alpha - \sum_\alpha m_\alpha \left[\left(\frac{d}{dt}\right)(\dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) - \tfrac{1}{2}\delta\,(\dot{\mathbf{r}}_\alpha^2)\right] = 0$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha \delta\mathbf{r}_\alpha + \sum_\alpha \tfrac{1}{2} m_\alpha \delta\,(\dot{\mathbf{r}}_\alpha^2) = \Sigma \left(\frac{d}{dt}\right)(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha)$$

$$\Rightarrow \sum_\alpha \mathbf{F}_\alpha\,\delta\mathbf{r}_\alpha + \delta\Big(\sum_\alpha \tfrac{1}{2} m_\alpha \dot{\mathbf{r}}_\alpha^2\Big) = \sum_\alpha \left(\frac{d}{dt}\right)(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha) \quad ...(35)$$

But $\sum_\alpha \frac{1}{2} m_\alpha \dot{\mathbf{r}}^2_\alpha$ = kinetic energy of the system $= T$ and $\sum_\alpha \mathbf{F}_\alpha \cdot \delta\mathbf{r}_\alpha$

= work done by the forces $\mathbf{F}_\alpha$ during displaecment $\delta\mathbf{r}_\alpha$ $= \delta W$ (say).

$$\therefore \text{ equation (35)} \Rightarrow \delta W + \delta T = \sum_\alpha \left(\frac{d}{dt}\right)(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha). \qquad ...(36)$$

Integrating (36) between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\,dt = \int_{t_1}^{t_2} \sum_\alpha \frac{d}{dt}(m_\alpha \dot{\mathbf{r}}_\alpha.\delta\mathbf{r}_\alpha)\,dt = \Sigma \int_{t_1}^{t_2} d\,(m_\alpha \dot{\mathbf{r}}_\alpha \cdot \delta\mathbf{r}_\alpha)$$

$$= \sum_\alpha \Big[(m_\alpha \dot{\mathbf{r}} \cdot \delta\mathbf{r}_\alpha)\Big]_A^C = 0$$

since $\delta\mathbf{r}_\alpha = 0$ at the end points A and C.

But we know that for a conservative system,

$\delta W = -\delta V$ where V is potential energy

$$\Rightarrow \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0, \quad i.e. \quad \int_{t_1}^{t_2} \delta (T - V)\, dt = 0,$$

$$\Rightarrow \delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \Rightarrow \delta \int_{t_1}^{t_2} L\, dt = 0.$$

$$\Rightarrow \delta \int_{t_1}^{t_2} L\, dt = \text{Stat} = \text{extremum},$$ which is Hamilton's principle.

Ex. 1 (a). *Use Hamilton's principle to find the equation of motion of one-dimensional harmonic oscillator.*

Solution. The kinetic energy of harmonic oscillator is given by $T = \frac{1}{2} m \dot{x}^2$ and the potential energy of the harmonic oscillator is given by $V = -\int F\, dx = \int kx\, dx = \frac{1}{2} kx^2$.

$\therefore$ the Lagrangian of the system $L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$.

Using Hamilton's principle, we have $\delta \int_{t_1}^{t_2} L\, dt = 0$.

$$\Rightarrow \delta \int_{t_1}^{t_2} (\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2)\, dt = 0 \Rightarrow \int_{t_1}^{t_2} \delta (\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2)\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (m\dot{x}\, \delta\dot{x} - kx\, \delta x)\, dt = 0 \quad \ldots(1)$$

But $\delta \dot{x} = \left(\frac{d}{dt}\right) (\delta x)$.

$$\therefore \quad (1) \Rightarrow \int_{t_1}^{t_2} m\dot{x} \frac{d}{dt} (\delta x)\, dt - \int_{t_1}^{t_2} kx\, \delta x\, dt = 0.$$

$$i.e., \quad \left[m\dot{x}\, \delta x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m \frac{d}{dt} (\dot{x})\, \delta x\, dt - \int_{t_1}^{t_2} kx\, \delta x\, dt = 0 \quad \ldots(2)$$

But $\left[m\dot{x}\, \delta x\right]_{t_1}^{t_2} = 0$

[$\because$ $\delta x = 0$ at fixed points, *i.e.* at instants t_1 and t_2]

$$\therefore \quad (2) \Rightarrow -\int_{t_1}^{t_2} m \frac{d}{dt} (\dot{x})\, \delta x\, dt - \int_{t_1}^{t_2} kx\, \delta x\, dt = 0,$$

$$i.e., \quad \int_{t_1}^{t_2} (m\ddot{x} + kx)\, \delta x\, dt = 0.$$

But δx is arbitrary, hence the above equation is only satisfied if $m\ddot{x} + kx = 0$. This is the equation of motion for one dimensional harmonic oscillator.

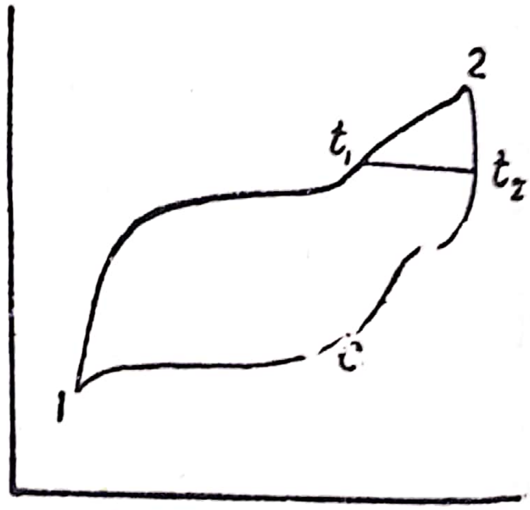
6·1-5 (2). Extension of Hamilton's principle to non-conservative and non holonomic system.

We can generalise Hamiton's principle to include non-conservative forces as well, so that an alternative form of the Lagrange's equation can be obtained. The extension of the Hamilton's principle is

$$\delta I=\delta \int_1^2 (T+W)\, dt=0 \qquad \text{...(1)}$$

where $$W=\Sigma\, \mathbf{F}_l \cdot \mathbf{r}_r. \qquad \text{...(2)}$$

We know that the variations δq_l or $\delta \mathbf{r}_l$ are identical with virtual displacements of the co-ordinates as there is no variation of time. Thus we can consider the varied path in configuration space as built up by a succession of virtual displacements from the actual path of motion along C. Again each virtual displacement takes place at some definite time and at that time the forces acting on the body have definite values. δW represents the amount of work done by the forces on the system during the period of virtual displacement from the actual to the varied path. Thus the Hamilton's principle given by (1) can be said as *the integral of the variation of kinetic energy together with the amount of virtual work involved in the variation must be zero.*



varied path in configuration space by virtual displacement.

We can evaluate the variations $\delta \mathbf{r}_l$ in terms of δq_j by making use of the equations of transformation between $\mathbf{r}$ and q, where each q depends on the path chosen through a parameter say α :

$$\mathbf{r}_l=\mathbf{r}\,\{q_k\,(\alpha, t)\}.$$

We can abbreviate the process by using the equivalence of $\delta \mathbf{r}_l$ with a virtual displacement. We know that

$$\sum_l \mathbf{F}_l \cdot \delta \mathbf{r}_l=\sum_k Q_k\, \delta q_k.$$

Thus (1) may be put as

$$\delta \int_1^2 T\, dt+\int_1^2 \sum_k Q_k\, \delta q_k\, dt=0. \qquad \text{...(3)}$$

Now it can be shown easily that (3) reduces to ordinary form of Hamilton's principle in case Q_k's are derivable from the generalized potential. Therefore the integral of virtual work, under these conditions becomes,

$$\int_1^2 \sum_k Q_k \, \delta q_k \, dt = -\int_1^2 \sum_k \delta q_k \left(\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_k}\right) dt.$$

Now reversing by integration parts procedure ; the above integral can be put as

$$-\int_1^2 \sum_k \left\{\frac{\partial V}{\partial \dot{q}_k} \delta q_k + \frac{\partial V}{\partial q_k} \delta \dot{q}_k\right\} dt = -\delta \int_1^2 V \, dt$$

Thus (3) reduces to

$$\delta \int_1^2 T \, dt - \delta \int_1^2 V \, dt = \delta \int_1^2 (T-V) \, dt = \delta \int_1^2 L \, dt = 0$$

which is Hamilton's principle.

For more general problem, the variation of the first integral in (3) can be written at once, as T like L for conservative systems is a function of q_k and $\dot{q}_k$

$$\therefore \quad \delta \int_1^2 T \, dt = \int_1^2 \sum_k \left\{\frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}\right\} \delta q_k \, dt \qquad \ldots(4)$$

$$\int_1^2 \sum_k \left\{\frac{\partial T}{\partial q_k} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} + Q_k\right\} \delta q_k \, dt = 0 \qquad \ldots(5)$$

Further it is assumed that the constrains are holonomic so the integral (5) can vanish if and only if the separate coefficients vanish, *i.e.*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k \qquad \ldots(6)$$

so that (1) represents the proper extension of Hamilton's principle, which yields that form of Lagrange's equation, in case the forces are not derived from a potential.

We can also extend the Hamilton's principle to cover certain categories of non-holonomic systems as well. While deriving Lagrange's equations from Hamilton's or D'Alemberts principle, application of holonomic constrains is made only in last step when the variatons q_k are considered as independent of each other. While in non-holonomic system the generalized co ordinates are not independent to each other so they cannot be reduced further by means of equation of constraints of the form

$$f[q_1, q_2, \ldots, q_n; t] = 0.$$

Thus q_k's can not be as all independent

We can further treat non-holonomic system, provided the equations of constraint can be put in the form

$$\sum_j a_{ij}\, dq_j + a_{it}\, dt = 0, \qquad \text{...(7)}$$

which is a relation connecting the differentials of q's. Since, in variation process used in Hamilton's principle, the time for each point on the path is taken constant, therefore the virtual displacements occuring in the variation must satisfy the equations of constraint of the form

$$\sum_j a_{ij}\, \delta q_j = 0, \; (j=1, 2, \ldots m) \qquad \text{...(8)}$$

The equations (8) can very well be used to reduce the number of virtual displacement to independent onces. The method used for eliminating these extra virtual displacements is that of Lagrange's *undetermined multipliers*.

Also from (8), we get

$$\lambda_l \sum_j a_{ij}\, \delta q_j = 0, \; (l=1, 2, \ldots m) \qquad (9)$$

where λ_l are some undetermined constants, which in general, are functions of time.

Now first of all summing the equations (9) over l and then integrating the resulting equation from 1 to 2, we get

$$\int_1^2 \sum_{j,l} \lambda_l\, a_{ij}\, \delta q_j\, dt = 0 \qquad \text{...(10)}$$

Again the equations corresponding to (5) in the case of conservative system are given as

$$\int_1^2 dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j = 0 \qquad \text{.. (11).}$$

Combining the equations (10) with (11), we get

$$\int_1^2 dt \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_l \lambda_l a_{lj} \right) \delta q_j = 0 \qquad \text{...(12)}$$

It is to be noted here that δq_j's are still not independent. They are connected by the m relations given by (8). This is, while the first $n-m$ of these equations may be chosen independently, the last m are then fixed by the equations (8). Since the values of λ_l's are at our disposal, we choose them to be such that

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} = 0, \; (j=n-m+1, \ldots n) \qquad \text{.. (13)}$$

which are in the nature of equations of motion for the last m of the q_j variable.

Using the value of λ_l, obtained from (13) in (12), we get

$$\int_1^2 dt \sum_{j=1}^{n-m} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} \right) \delta q_j = 0 \qquad \text{...(14)}$$

Since δq_j's involved here are independent ones, therefore, we have

$$\frac{\partial L}{\partial q_j}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}+\sum_l \lambda_l a_{lj}=0, \ (j=1, 2, \ldots n-m) \quad \ldots(15)$$

Now combining (13) with (15), we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=\sum_l \lambda_l a_{lj}, \ (j=1, 2, \ldots n) \quad \ldots(16)$$

which are known as *Lagrange's equations for non holonomic systems.*

We are now observe that (16) gives us a total of only n-equations, where as the unknown involved are $(n+m)$ is number, namely then n co-ordinates q_j and the m, λ_l's. The additional equations required are exactly the equations of constraint *i.e.*

$$\sum_j a_{lj}\,\dot{q}_j+a_{lt}=0 \quad \ldots(17)$$

The equations (16) together with (17) constitute $(n+m)$ equations for $n+m$ unknowns.

It is to be noted here that equation (7) is not the most general type of non-holonomic constraint. For example, it does not include equations of constraints in the form of inequalities. More over it includes holonomic constraints. An equation of the form

$$f(q_1, q_2, q_3, \ldots q_n;\ t)=0,$$

is known as a holonomic equation of constraint.

This is equivalent to a differential equation

$$\sum_j \frac{\partial f}{\partial q_j}\,dq_j+\frac{\partial f}{\partial t}\,dt=0 \quad \ldots(18)$$

It is of the same form as the equation (7) with the coefficients

$$a_{lj}=\frac{\partial f}{\partial q_j} \quad \text{and} \quad a_{lt}=\frac{\partial f}{\partial t} \quad \ldots(19)$$

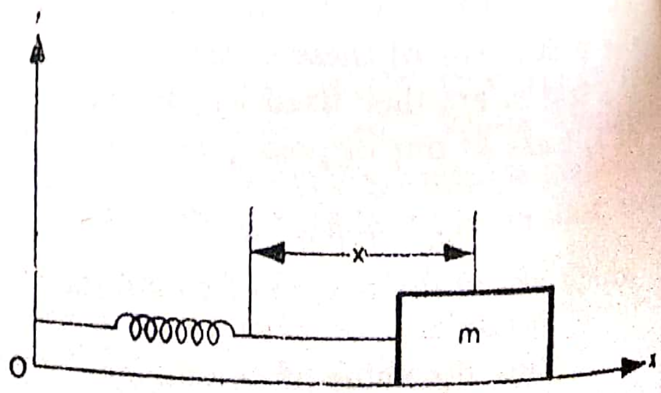
It means that the method of Lagrange's undetermined multipliers can be used also for holonomic constraints, when it is not easy to reduce all the q's to independent co-ordinates.

Ex. 1 (b). *Obtain Hamiltanian and Hamilton's equations for the spring mass arrangement as shown in Fig. Spring is unstretched at* $x=0$.

Sol. Let x be the displacement of the mass at time t, then the kinetic energy of the mass m is given by

$$T=\tfrac{1}{2}m\dot{x}^2$$

Further, let k be the spring constant, then potential energy is



$$\Rightarrow \; L=T-V=\tfrac{1}{2}m\dot{x}^2-\tfrac{1}{2}kx^2. \qquad \ldots(1)$$

$$\Rightarrow \; p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x}; \quad \Rightarrow \; \dot{x}=\frac{p_x}{m}. \qquad \ldots(2)$$

$$\therefore \; H=\Sigma \, p_\alpha \dot{q}_\alpha - L = p_x\dot{x}-L$$

$$\Rightarrow \; H=p_x\left(\frac{p_x}{m}\right)-\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}kx^2, \text{ using (2)}$$

$$=\frac{p_x^2}{m}-\tfrac{1}{2}m\left(\frac{p_x}{m}\right)^2+\tfrac{1}{2}kx^2, \text{ using (1)}$$

$$=\frac{p_x^2}{2m}+\tfrac{1}{2}kx^2, \qquad \ldots(3)$$

$$\therefore \; \frac{\partial H}{\partial p_x}=\frac{p_x}{m}, \; \frac{\partial H}{\partial x}=kx.$$

Thus Hamilton's equations are given by

$$\left.\begin{aligned} \dot{x}&=\frac{\partial H}{\partial p_x} \text{ or } \dot{x}=\frac{p_x}{m}, \\ \dot{p}_x&=-\frac{\partial H}{\partial x} \text{ or } \dot{p}_x=-kx \end{aligned}\right\} \qquad \ldots(4)$$

6·1-6. Principle of least action. (Nagpur M.Sc. Physics 92)

The action A of a dynamical system over the interval $t_1<t<t_2$ is defined to be

$$A=\int_{t_1}^{t_2} 2T\,dt, \text{ where } T \text{ is the K.E. of the system.}$$

When we were discussing Hamilton's principle then we saw that t_1 and t_2 were kept fixed. Now let us assume that t is no longer the independent variable but that the motion is dependent upon some other independent variable say r which is assumed to take fixed values at the end points.

$$\text{Then } \; A=\int_{r_1}^{r_2} 2T\frac{dt}{dr}\,dr \qquad (r_1, r_2 \text{ are fixed}) \qquad \ldots(37)$$

Further assume that the law of conservation of energy $T+V=C$ (const.) holds and that $T-(C-T)=2T-C$.

we have $\quad E=T-V=T-(C-T)=2T-C.$

$\therefore$ L is also independent of time,

where r changes to $r+\delta r$, A changes to $A+\delta A$

$$\text{where } \delta A=\delta\int_{r_1}^{r_2} 2T\frac{dt}{dr}\,dr=\int_{r_1}^{r_2}\left[\delta\,(2T)\frac{dt}{dr}+2T\delta\left(\frac{dt}{dr}\right)\right]dr \qquad \ldots(38)$$

$$\text{But } \delta\left(\frac{dt}{dr}\right)=\frac{d}{dt}\,(\delta t)=\left[\frac{d}{dr}\,(\delta t)\Big/\left(\frac{dt}{dr}\right)\right]$$

and

$$\delta\,(2T)=\delta\,(2T-C)=\delta L$$

$$\therefore \; (38) \Rightarrow \delta A=\int_{r_1}^{r_2}\left[\delta L\frac{dt}{dr}+2T\left\{\frac{d}{dr}(\delta t)\Big/\left(\frac{dt}{dr}\right)\right\}\right]dr \qquad \ldots(39)$$

Now $\delta L=\sum_{\alpha=1}^{n}\frac{\partial L}{\partial q_\alpha}\,\delta q_\alpha+\sum_{\alpha=1}^{n}\frac{\partial L}{\partial \dot{q}_\alpha}\,\delta \dot{q}_\alpha$ ...(40)

$\delta\dot{q}_\alpha=\delta\,(q_\alpha'/t')=(\delta q_\alpha'/t')-(q_\alpha'/t'^2)\,\delta t'$

where to total differentions w.r.t. "r" are denoted by dashes,

$=(1/t')\,(d/dr)\,(\delta q_\alpha)-(q_\alpha'/t'^2)\,(d/dr)\,(\delta t)$

$=(d/dt)\,(\delta q_\alpha)-(q_\alpha'/t')\,(d/dt)\,(\delta t)$

$(d/dt)\,(\delta q_\alpha)-\dot{q}_\alpha\,(d/dt)\,(\delta t).$

$$\therefore\quad \sum_{\alpha=1}^{n}\frac{\partial L}{\partial \dot{q}_\alpha}\,\delta\dot{q}_\alpha=\sum_{\alpha=1}^{n}\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}\,(\delta q_\alpha)-\sum_{\alpha=1}^{n}\left(\dot{q}_\alpha\,\frac{\partial L}{\partial \dot{q}_\alpha}\right)\frac{d}{dt}\,(\delta t). \quad ...(41)$$

$$\text{Now } \sum_{\alpha=1}^{n}\dot{q}_\alpha\,\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}\,(\delta t)=\left(\sum_{\alpha=1}^{n}\dot{q}_\alpha\,\frac{\partial L}{\partial \dot{q}_\alpha}\right)\frac{d}{dt}\,(\delta t)\quad\left(\because\ \frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial L}{\partial \dot{q}_\alpha}\right)$$

$$={}^*2T\,\frac{d}{dt}\,(\delta t).$$

$$\therefore\quad (40)\Rightarrow \partial L=\sum_{\alpha=1}^{n}\frac{\partial L}{\partial \dot{q}_\alpha}\,\delta q_\alpha+\sum_{\alpha=1}^{n}\frac{\partial L}{\partial q_\alpha}\,\delta\dot{q}_\alpha$$

$$=\sum_{\alpha=1}^{n}\left\{\frac{\partial L}{\partial q_\alpha}\,\delta q_\alpha+\frac{\partial L}{\partial \dot{q}_\alpha}\frac{d}{dt}\,(\delta q_\alpha)\right\}-2T\,\frac{d}{dt}\,(\delta t).$$

Now from (39) we get

$$\delta A=\int_{r_1}^{r_2}\sum_{\alpha=1}^{n}\left[\frac{\partial L}{\partial q_\alpha}\,\delta q_\alpha+\frac{\partial L}{\partial q_\alpha}\frac{d}{dt}\,(\delta q_\alpha)\right]\frac{dt}{dr}\,dr$$

$$=\int_{r_1}^{r_2}\sum_{\alpha=1}^{n}\left\{\delta q_\alpha\,\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)+\frac{\partial L}{\partial q_\alpha}\frac{d}{dt}\,(\delta q_\alpha)\right\}\frac{dt}{dr}\,dr$$

$$=\int_{r_1}^{r_2}\sum_{\alpha=1}^{n}\frac{d}{dt}\left\{\delta q_\alpha\,\frac{\partial L}{\partial \dot{q}_\alpha}\right\}\frac{dt}{dr}\,dr=\left[\sum_{\alpha=1}^{n}\delta q_\alpha\,\frac{\partial L}{\partial \dot{q}_\alpha}\right]_{r_1}^{r_2}\quad ...(42)$$

Now if we allow $\delta q_\alpha=0$ ($\alpha=1, 2, ... n$) at each point $r=r_1$, $r=r_2$, then (42) $\Rightarrow$ δA is stationary.

Such a result is known *principle of least action*. Hence we have simply established that A is stationary in character but it can be shown very easily that it is minimum.

6·1-7. Principle of least action in terms of arc length of the particle trajectory. **(Gorakhpur 89)**

Let there be a particle of mass m whose kinetic energy is given by

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*T is a homogeneous quadratic function of the velocities hence by Euler's theorem, we get

$$2T=\sum_{\alpha=1}^{n}\dot{q}_\alpha\,\frac{\partial T}{\partial \dot{q}_\alpha}.$$

$$T=\tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2,$$

where ds is the element of arc traversed by the particle in time dt. ...(43)

Equation (43) $\Rightarrow$ $dt=\sqrt{\left(\frac{m}{2T}\right)}\,ds$,

hence the principle of least action, from equation (37),

$$\Rightarrow \Delta\int_{t_1}^{t_2} 2T\,dt=\Delta\int 2T\sqrt{\left(\frac{m}{2T}\right)}\,ds=0$$

$$\Rightarrow \Delta\int\sqrt{(2mT)}\,ds=0 \Rightarrow \Delta\int\sqrt{[2m\,(E-V)]}\,ds=0,$$

where E is the total energy of the particle

$$\Rightarrow \Delta\int\sqrt{[2m\,(H-V)}\,ds=0.] \qquad \text{[since } E=H]$$

$$\Rightarrow \Delta\int\sqrt{(H-V)}\,ds=0. \qquad ...(44)$$

This gives the principle of least action in terms of arc length of the particle trajectory.

6·1-8. Jacobi's form of the principle of least action.

$$T=\tfrac{1}{2}\sum_{k,l} a_{kl}\,\dot{q}_k\,\dot{q}_l=\tfrac{1}{2}\sum_{kl} a_{kl}\left(\frac{dq_k\,dq_l}{dt^2}\right) \qquad ...(45)$$

Now assume $(d\rho)^2=\sum_{k,l} a_{kl}\,dq_k\,dq_l$ ...(46)

so that (45) $\Rightarrow$ $T=\tfrac{1}{2}\left(\frac{d\rho}{dt}\right)^2$ $\Rightarrow$ $dt=\frac{d\rho}{\sqrt{[(2T)]}}$.

Hence the principle of least action implies

$$\Delta\int_{t_1}^{t_2} 2T\,dt=\Delta\int 2T\frac{d\rho}{\sqrt{(2T)}}=0 \qquad ...(47)$$

$$\Rightarrow \Delta\int\sqrt{(2T)}\,d\rho=0 \Rightarrow \Delta\int\sqrt{[2\,(E-V)]}\,d\rho=0$$

$$\Rightarrow \Delta\int\sqrt{[H-V]}\,d\rho=0. \qquad ...(48)$$

(Jacobi's form of the principle of least action)

Ex. 2. *A particle of mass m falls a given distance z_0 in time $t_0=\sqrt{(2z_0/g)}$ and the distance travelled in time t is given by $z=at+bt^2$, where constants a and b are such that the time t_0 is always the same. Show that the integral $\int_0^{t_0} L\,dt$ is an extremum for real values of coefficients only when $a=0$ and $b=\frac{g}{2}$.*

Solution. Let z be the distance travelled by the particle in time t; then

kinetic energy of the particle $=\frac{1}{2}m\dot{z}^2$.

potential energy of the particle $=-mgz$

$$\therefore \quad L=T-V=\tfrac{1}{2}m\dot{z}^2+mgz.$$

Now by Hamilton's principle, we have

$$\delta \int_0^{t_0} L\,dt=0 \text{ i.e. } \delta \int_0^{t_0} (\tfrac{1}{2}m\dot{z}^2+mgz)\,dt=0$$

$$\Rightarrow \int_0^{t_0} (\tfrac{1}{2}m\dot{z}^2+mgz)\,dt=\text{extremum}$$

$$\Rightarrow \int_0^{t_0} f(z,\dot{z},t)\,dt=\text{extremum}$$

where $\quad f(z,\dot{z},t)=\frac{1}{2}m\dot{z}^2+mgz$

$$\Rightarrow \left(\frac{df}{dz}\right)-mg,\ \left(\frac{df}{d\dot{z}}\right)=m\dot{z}.$$

Now the function f satisfies the Euler-Lagrange's equations implying $\left(\frac{d}{dt}\right)\left(\frac{\partial f}{\partial \dot{z}}\right)-\left(\frac{\partial f}{\partial z}\right)=0 \Rightarrow \left(\frac{d}{dt}\right)(m\dot{z})-mg=0$

$$\Rightarrow m\ddot{z}-mg=0. \qquad ...(1)$$

But we are given.

$$z=at+bt^2 \Rightarrow \dot{z}=a+2bt \quad \text{and} \quad \ddot{z}=2b.$$

$\therefore$ equation (1) $\Rightarrow 2mb-mg=0 \Rightarrow b=\frac{g}{2}$.

But $z=z_0$ when $t=t_0$, hence $z_0=at+bt_0^2$

$$\Rightarrow \left(\frac{gt_0^2}{2}\right)=at_0+bt_0^2=at_0+\left(\frac{gt_0^2}{2}\right)\left[\because \quad z_0=\left(\frac{gt_0^2}{2}\right)=(\text{given})\right]$$

which gives $a=0$ and $b=\frac{g}{2}$.

Ex. 3. *Applying principle of least action to prove that out of all possible paths between two points, the system for which kinetic energy is conserved moves along the path for which the transit time is extremum.*

Solution. By the principle of least action, we have

$$\Delta \int_{t_1}^{t_2} 2T\,dt=0. \qquad ...(1)$$

When kinetic energy of the system is conserved, then

(1) $\Rightarrow \Delta \int_{t_1}^{t_2} dt=0$ i.e. $\Delta\,(t_2-t_1)=0$.

This states that the system moves along the path for which the transit time is extremum.

6·1-9. **Fermat's principle :** *It states that the time taken by a light ray to travel between two points is extremum.*

Proof. Hamilton's principle of least action

$$\Rightarrow \Delta \int_{t_1}^{t_2} 2T\, dt=0 \Rightarrow \Delta(t_2-t_1)=0, \text{ where } T \text{ is constant}$$

$$\Rightarrow t_2-t_1=\text{extremum}=\text{Stat.} \qquad \ldots(49)$$

i.e. the time taken by the light ray to travel between two points is extremum.

6·1-10. **Distinction between Hamilton's principle and the principle of least action.**

In Hamilton's principle, we had seen that the time interval t_2-t_1 was prescribed but in establishing the principle of least action, however, the time interval from one configuration to another was in no way restricted. In the principle of least action, we choose fixed end points $r=r_1$, $r=r_2$ and prescribe the total energy $T+V$, of the system.

Ex. 4. (*a*) *Use the variational method to show that the shortest curve joining two fixed points is a straight line.* **[Meerut 73, 81]**

Solution. (*a*) *First method.* (Using cartesian co-ordinates) :

We have $s=\int_{x_1}^{x_2} \sqrt{(1+y'^2)}\, dx.$

Obviously $f=\sqrt{(1+y'^2)}$ does not contain y explicitly, hence we have $\left(\frac{\partial f}{\partial y'}\right)=\text{constant} \Rightarrow y'=\text{constant} \Rightarrow y=ax+b$

which represents a straight line.

Thus the distance between two points is shortest along the straight line joining the points. We must note that we have proved if for weak variations only through we know that it is true in general. Obviousity it cannot be longest distance.

(*b*) *Second method* (using polar co-ordinates) :

We have $ds=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta$, $\therefore$ $s=\int_{\theta_1}^{\theta_2} \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta.$

Now $f=\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}$ does not contain θ explicity.

$$\therefore f-\frac{dr}{d\theta}\frac{\partial f}{\partial\left(\frac{dr}{d\theta}\right)}=A \Rightarrow \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}-\frac{dr}{d\theta}\left[\frac{\frac{dr}{d\theta}}{\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}}\right]=A$$

$$\Rightarrow \frac{r^2}{\sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}}=A \Rightarrow \frac{r^2 \frac{d\theta}{dr}}{\sqrt{\left\{1+\left(r\frac{d\theta}{dr}\right)^2\right\}}}=\text{constant}$$

$$\Rightarrow r^2 \frac{d\theta}{dr}=\sqrt{(\tan^2 \phi+1)}\ \text{constant}.$$

$$\Rightarrow r \sin \phi=\text{constant} \Rightarrow p=\text{constant},$$

which is true only for straight line in a plane.

Ex. 4. (*b*). *Prove that the geodesics of a spherical surface are great circles, i.e., then circle whose centres lie at the centre of the sphere.* (Rohilkhand 1978, 86)

Or

Prove that the shortest distance between two points on the surface of the spherc is the arc of the great circle connecting them.

Sol. Elementary arc ds on the surface of sphere of radius r, in spherical co-ordinates, is given by

$$ds^2=r^2 (d\theta^2+\sin^2 \theta\, d\phi^2) \quad \text{since } r \text{ is constant}$$

or

$$ds=r (d\theta^2+\sin^2 \theta\, d\phi^2)^{1/2}.$$

Hence, the total distance between two points having co-ordinates (r, θ_1, ϕ_1), (r, θ_2, ϕ_2) is

$$S=\int_{\theta_1, \phi_1}^{\theta_2, \phi_2} ds=\int_{\theta_1}^{\theta_2} r \left\{1+\sin^2 \theta \left(\frac{d\phi}{d\theta}\right)^2\right\}^{1/2} d\theta. \qquad ...(1)$$

Now comparing (1), with Euler Lagrange's equation, we get

$$f=r\left\{1+\sin^2 \theta \left(\frac{d\phi}{d\theta}\right)^2\right\}^{1/2}.$$

If S is to be minimum or maximum, we must have

$$\frac{d}{d\theta}\left[\frac{\partial}{\partial \phi'}\left\{r\left(1+\sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2\right\}^{1/2}\right]-\frac{\partial}{\partial \phi}\left[r\left\{1+\sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2\right\}^{1/2}\right]=0$$

where $\phi'=\frac{\partial \phi}{\partial \theta}$

$$\Rightarrow \frac{d\theta}{d\theta}\left[r\left\{\frac{\partial}{\partial \phi'}(1+\sin^2 \theta\, \phi'^2)^{1/2}\right\}\right]=0$$

$$\Rightarrow \frac{d}{d\theta}\left[\frac{2r \sin^2 \theta\, \phi'}{2 (1+\sin^2 \theta\, \phi'^2)^{1/2}}\right]=0$$

$$\Rightarrow \frac{d}{d\theta}\left[\frac{\sin^2 \theta\, \phi'}{(1+\sin^2 \theta\, \phi'^2)^{1/2}}\right]=0, \ (r\neq 0)$$

whence integrating, we get $\frac{\sin^2 \theta\, \phi'}{(1+\sin^2 \theta)\, \phi'^2}=c$, where c is any constant,

$$\Rightarrow \phi' = \frac{c \operatorname{cosec}^2 \theta}{(1-c^2-c^2 \cot^2 \theta)^{1/2}}.$$

Again, integrating we get $\phi = \alpha - \sin^{-1}(k \cot \theta)$ ...(2)
where α and k are new constants.

Transforming spherical polar co-ordinates into cartesian co-ordinates using transformation equations which are

$x = r \sin \theta \cos \phi,\ y = r \sin \theta \sin \phi,\ z = r \cos \theta,$

we have $\quad ak \cot \theta = r \sin (\alpha - \phi)$

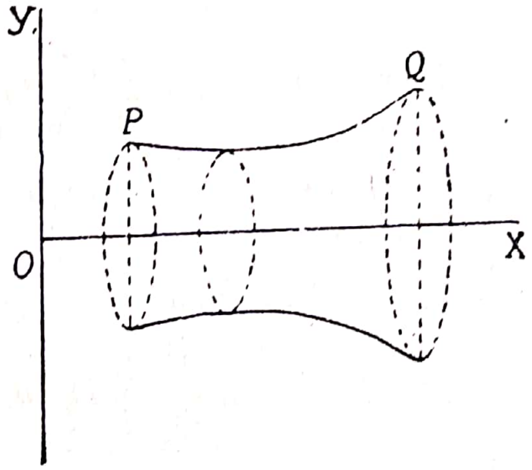
$\quad ka \cos \theta = r \sin (\alpha - \phi) \sin \theta$

or $\quad zk = r [\sin \alpha \cos \phi - \cos \alpha \sin \phi]$

or $\quad = x \sin \alpha - y \cos \alpha.$ ...(3)

It represents a plane passing through the origin and hence cutting the surface of the sphere in great circle there by indicating that the shortest or longest distance between two points on the surface of the sphere is the arc of the circle whose centre lies at the centre of the sphere.

Ex. 5. *Show that the area of the surface of revolution of a curve $y = y(x)$ is $2\pi \int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\ dx$. Hence show that for this to be a minimum, the curve must be a catenary.*



Solution. $dA = 2y\pi\ ds$

$$\Rightarrow A = 2\pi \int y\ ds$$

$$= 2\pi \int_P^Q y\sqrt{(1+y'^2)}\ dx.$$

Here $f = y\sqrt{(1+y'^2)}$ is independent of x; so we get

$$f - y' \frac{\partial f}{\partial y'} = \text{const.}$$

$$\Rightarrow y\sqrt{(1+y'^2)} - y' \frac{yy'}{\sqrt{(1+y'^2)}} = \text{const.}$$

$$\Rightarrow \frac{y}{\sqrt{(1+y'^2)}} = c \text{ say.}$$

$$\Rightarrow y = c \sec \psi \qquad \left[\because \left(\frac{dy}{dx}\right) = \tan \psi = y'\right]$$

$\Rightarrow$ the curve is common catenary.

6·1-11. Derivation of Lagrange's equations from Hamilton's principle.

Method. (1) By Hamilton's principle, we have

$$\int_{t_1}^{t_2} L\,(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n;\ t)\,dt = \text{stationary.}$$

This integral is stationary only when

$$\left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \left(\frac{\partial L}{\partial q_r}\right) = 0,\ r = 1, 2, 3, \ldots, n.$$

These are obviously Lagrange's equations

Method (2). The Lagrangian function is given by

$$L = L\,(q_1, q_2, \ldots q_n;\ \dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots, \dot{q}_n\ ;\ t = L\,(q, \dot{q}, t).$$

If L is not a function of t explicity, then we have

$$L = L\,(q, \dot{q}) \qquad \ldots(50)$$

$$\Rightarrow\ \delta L = \sum_{r=1}^{n} \frac{\partial L}{\partial q_r} + \delta q_r \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r}\,\delta q_r$$

$$\Rightarrow\ \int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r}\,\delta q_r\,dt + \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r}\,\delta \dot{q}_r\,dt$$

$$\Rightarrow\ \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r}\,\delta q_r\,dr + \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r}\,\delta \dot{q}_r\,dt = 0 \qquad \ldots(51)$$

$$\left[\because\ \delta \int_{t_1}^{t_2} L\,dt = 0 \text{ by Hamilton's principle}\right]$$

But $\delta \dot{q}_r = \delta\left(\dfrac{dq_r}{dt}\right) = \dfrac{dq_r}{dt} - \dfrac{dq_r}{dt} = \dfrac{d}{dt}\,(q_r' - q_r) = \dfrac{d}{dt}\,(\delta q_r).$

Hence (51) $\Rightarrow\ \displaystyle\int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r}\,\delta q_r\,dt + \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r}\,\frac{d}{dt}\,(\delta q_r)\,dt = 0$

$$\Rightarrow\ \int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r}\,\delta q_r\,dt + \sum_{r=1}^{n}\left[\left\{\frac{\partial L}{\partial \dot{q}_r}\,\delta q_r\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r\,dt\right] = 0.$$

But by the principle of least action, we have $\delta q_r - 0$ at each end points $\therefore\ \displaystyle\int_{t_1}^{t_2} \sum_{r=1}^{n} \frac{\partial L}{\partial q_r}\,\delta q_r\,dt = \sum_{r=1}^{n} \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r\,dt,$

$$\Rightarrow\ \int_{t_1}^{t_2} \sum_{r=1}^{n} \left\{\frac{\partial L}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\right\}\delta q_r\,dt = 0.$$

But variables being independent, the variation δq_r are independent $\Rightarrow\ \dfrac{\partial L}{\partial q_r} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_r}\right) = 0,\ (r = 1, 2, 3, \ldots, n) \qquad \ldots(52)$

These are Lagrange's equations of motion.

6·1-12. Conservation theorems and symmetry properties.

In many problems a number of first integrals of the equations of motion can be obtained immediately in the form

$$f\,(q, q_2, \ldots, q_n\ ;\ \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n\ ;\ t) = \text{const.} \qquad \ldots(53)$$

which are first order differential equations. Such integrals are of great importance as they give the physical information of the system. Before proceeding further, we shall define the following terms.

Generalised momentum and cyclic co-ordinates. Consider a conservative system *i.e.* a system in which the forces are derivable from a potential function V dependent on position only : then

$$\frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial (T-V)}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha}-\frac{\partial V}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha}$$

($\because$ V is independent of $\dot{q}_\alpha$)

$$\Rightarrow \quad \frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial}{\partial \dot{q}_\alpha}\sum_\alpha \tfrac{1}{2}m_\alpha \dot{q}_\alpha^2=m_\alpha \dot{q}_\alpha=p_\alpha,$$

where p_α is called the *generalised momentum* associated with the *generalised co-ordinate* q_α.

If the Lagrangian of a system is not the function of q_α, then the co-ordinate q_α is known as cyclic co-ordinate or ignorable and as such we have $\quad \left(\frac{\partial L}{\partial q_\alpha}\right)=0.$ ...(54)

Conservations theorem :

(*i*) *For generalised momentum :*

Lagrange's equation $\Rightarrow \left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\left(\frac{\partial L}{\partial q_\alpha}\right)=0,$ ...(55)

when q_α is cyclic, we have

$$\left(\frac{\partial L}{\partial q_\alpha}\right)=0 \Rightarrow \left(\frac{d}{dt}\right)\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)=0 \Rightarrow \left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)=\text{constant}$$

$\Rightarrow p_\alpha=$constant *i.e. the generalised momentum corresponding to a cyclic co-ordinate is conserved.* This equation constitutes a first integral of the form (53) and can be used formally to eliminate the cyclic co-ordinate from the problem which can be solved entirely in terms of remaing generalised co-ordinates.

(*ii*) *For energy.* [see Chapter on Lagrangian Dynamics].

(ii) **Principle of linear and angular momentum.** Consider first a generalised co-ordinate q_α for which a change dq_α represents a translation of the system as a whole in same given direction. As a matter of fact velocities are not affected by a shift in the origin ; so q_α will not apper in T implying $\left(\frac{\partial T}{\partial q_\alpha}\right)=0.$ Hence Lagrange's q_α equation gives $\left(\frac{d}{dt}\right)\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)=Q_\alpha=-\left(\frac{\partial V}{\partial q_\alpha}\right).$ ...(56)

But $\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)=\left(\frac{\partial}{\partial \dot{q}_\alpha}\right)(T-V)=\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right)=\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)$

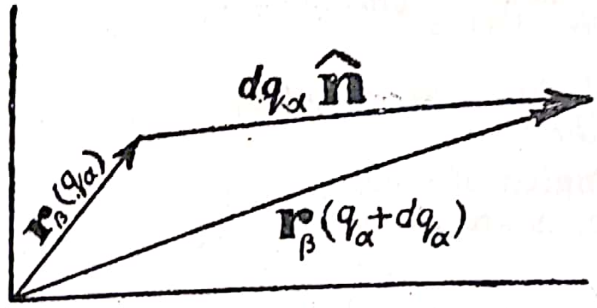
($\because$ V is independent of $\dot{q}_\alpha$)

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)=p_\alpha \ldots (56), \text{ and hence we have}$$

$$\left(\frac{d}{dt}\right)(p_\alpha)=-\left(\frac{\partial V}{\partial q_\alpha}\right)=Q_\alpha \text{ [from (56)]}$$

$$\Rightarrow \dot{p}_\alpha=-\left(\frac{\partial V}{\partial q_\alpha}\right)=Q_\alpha. \qquad \ldots(57)$$

Now we shall show that Q_α represents the component of the total force along the direction of q_α and p_α is the component of the total linear momentum along this d rection, i.e. (57) *is the equation of motion for the total linear momentum.*



Proof. (i) Since dq_α corresponds to a translation of the system along some axis so the vectors $\mathbf{r}_\beta\,(q_\alpha)$ and $\mathbf{r}_\beta\,(q_\alpha+dq_\alpha)$ are related as shown in the adjoining diagram. By the definition of the differential coefficient, we have

$$\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}=\lim_{\delta q_\alpha \to 0} \frac{\mathbf{r}_\beta\,(q_\alpha+dq_\alpha)-\mathbf{r}_\beta\,(q_\alpha)}{dq_\alpha}=\mathbf{r}_\beta'\,(q_\alpha)\,(dq_\alpha/dq_\alpha)$$

$$=\frac{dq_\alpha}{dq_\alpha}\hat{\mathbf{n}}=\hat{\mathbf{n}}, \quad \text{where } \mathbf{r}_\beta'=\mathbf{n} \qquad \ldots(58)$$

where $\hat{\mathbf{n}}$ is the unit vector alon; the direction of translation.

But we have

$$Q_\alpha=\sum_\beta \mathbf{F}_\beta\cdot\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}=\sum_\beta \mathbf{F}_\beta\cdot\hat{\mathbf{n}}=\left\{\sum_\beta \mathbf{F}_\beta\right\}\cdot\hat{\mathbf{n}}=\mathbf{F}\cdot\hat{\mathbf{n}}.$$

This implies *that Q_α represents the component of the total force along the direction of translation of q_α.*

(ii) We have

$$T=\tfrac{1}{2}\sum_\beta \dot{\mathbf{r}}_\beta^2$$

$$\Rightarrow p_\alpha=\frac{\partial T}{\partial \dot{q}_\alpha}=\tfrac{1}{2}\sum_\beta\left(2m_\beta\,\dot{\mathbf{r}}_\beta\cdot\frac{\partial \dot{\mathbf{r}}_\beta}{\partial \dot{q}_\alpha}\right)=\sum_\beta m_\beta \dot{\mathbf{r}}_\beta\cdot\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}$$

$$\Rightarrow p_\alpha=\sum_\beta m_\beta\,\mathbf{v}_\alpha\cdot\delta\mathbf{r}_\beta=\sum_\beta m_\beta\,\mathbf{v}_\beta\cdot\hat{\mathbf{n}}=\hat{\mathbf{n}}\cdot\sum_\beta m_\beta\,\mathbf{v}_\beta$$

$\Rightarrow$ *that p_α is the component of the linear momentum along the direction of $\hat{\mathbf{n}}$.*

When q_α is cyclic co-ordinate, we have $-\frac{\partial V}{\partial q_\alpha}=Q_\alpha=0$ as q_α cannot appear in V. Hence $\dot{p}_\alpha=0 \Rightarrow$ constant, *i.e. If a given component of the total applied force vanishes, the corresponding component of the linear, momentum is conserved.*

[Principle of linear momentum]

To prove that with q_α a rotation co-ordinate, the generalised force Q_α is the component of the total applied torque about the axis of rotation and p_α is the component of the total angular momentum along the same axis.

Proof. (i) Since the rotation of the co-ordinate system can not affect the magnitude of the velocities, so T will not contain q_α implying that $\left(\frac{\partial T}{\partial q_\alpha}\right)=0$. Also V is independent of $\dot{q}_\alpha$, so $\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right)=0$.

Now Lagrange's q_α equation gives

$$\left(\frac{d}{dt}\right)\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)-\left(\frac{\partial T}{\partial q_\alpha}\right)=-\left(\frac{\partial V}{\partial q_\alpha}\right)$$

$$\Rightarrow \left[\frac{d}{dt}\right]\left[\frac{\partial T}{\partial \dot{q}_\alpha}\right]=Q_\alpha=-\left[\frac{\partial V}{\partial q_\alpha}\right].$$

But $$\left\{\frac{\partial L}{\partial \dot{q}_\alpha}\right\}=\left\{\frac{\partial (T-V)}{\partial \dot{q}_\alpha}\right\}=p_\alpha. \qquad \ldots(60)$$

$$\therefore \frac{d}{dt}(p_\alpha)=Q_\alpha=-\left[\frac{\partial V}{\partial q_\alpha}\right]. \qquad \ldots(61)$$

Now by the adjoing diagram, we have

$$|d\mathbf{r}_\beta|=r_\beta \sin\theta\, dq_\alpha$$

and $$\left|\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}\right|=r_\beta \sin\theta$$

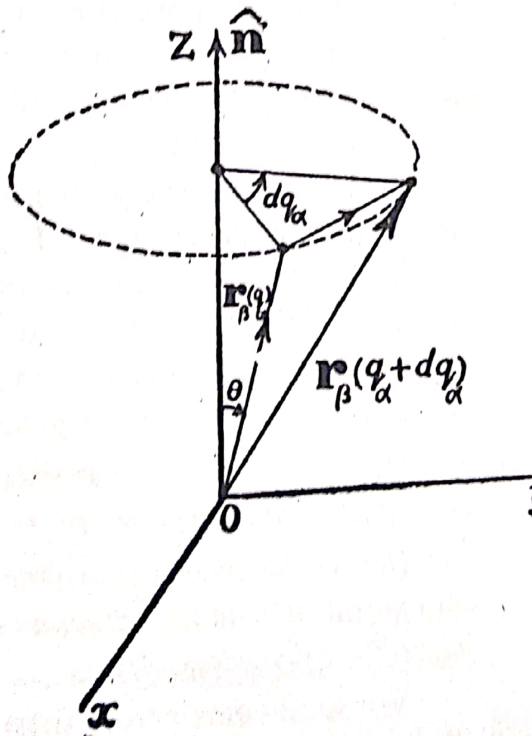
and its direction is normal to $\mathbf{r}_\beta$ and $\hat{\mathbf{n}}$.

$$\therefore \left[\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}\right]=\hat{\mathbf{n}}\times\mathbf{r}_\beta \qquad \ldots(62)$$

Thus $$Q_\alpha=\sum_\beta \mathbf{F}_\beta\cdot\left[\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}\right]$$

$$=\sum_\beta \mathbf{F}_\beta\cdot(\hat{\mathbf{n}}\times\mathbf{r}_\beta)=\sum_\beta \hat{\mathbf{n}}\cdot(\mathbf{r}_\beta\times\mathbf{F}_\beta)$$

$$=\hat{\mathbf{n}}\cdot\sum_\beta(\mathbf{r}_\beta\times\mathbf{F}_\beta)=\hat{\mathbf{n}}\cdot\mathbf{N},$$



Change of p, v. under rotation of the system where N is the moment of force or torque about O.

$\Rightarrow$ Q_α=component of the total applied torque about the axis of rotation.

(ii) We have

$$p_\alpha=\sum_\beta m_\beta \mathbf{v}_\beta \cdot \left[\frac{\partial \mathbf{r}_\beta}{\partial q_\alpha}\right]=\Sigma\, m_\beta\, \mathbf{v}_\beta . \hat{\mathbf{n}} \times \mathbf{r}_\beta$$

$$\Rightarrow\ p_\alpha=\hat{\mathbf{n}}\cdot\sum_\beta (\mathbf{r}_\beta \times \mathbf{v}_\beta\, m)=\mathbf{n}\cdot\sum_\beta (\mathbf{r}\times m\, \mathbf{v}_\beta)=\mathbf{n}\cdot\mathbf{L},$$

where L is the moment of momentum *viz.* angular momentum about O.

$\Rightarrow$ p_α=component of the total angular momentum about the axis of rotation.

When q_α is cyclic co-ordinate, we have $-\left[\frac{\partial V}{\partial q_\alpha}\right]=Q_\alpha=0$ as q_α cannot appear in V. Hence $\dot{p}_\alpha=0=p_\alpha$=constant, *i.e. if the component of the total applied torque ab ut the axis of rotation vanishes then component of the total angular momentum about the axis of rotation is constant.* **[Principle of angular momentum]**

Symmetry properties :

The significance of cyclic translation or rotation co-ordinates in relation to the properties of the system deserves some notice at this point. If any co-ordinate (say q_α) corresponding to a displacement is cyclic then it implies that a translation of the system, as if rigid, has no effect on the problem. In other words it can be said that if the system is invariant under translation along a given direction then the corresponding linear momentum is conserved. Similarly, when a rotation co-ordinate is cyclic then the system is invariant under rotation about the given axis. Hence the conservation theorems of momentum are closely related to the *symmetric properties* of the system as given below :

(*i*) *If the system is symmetric spherically, all the components of angular momentum are conserved.*

(*ii*) *If the system is symmetric about the z-axis then only the component of L in the direction of z i.e. L_z will be conserved. This holds for other axes also.*

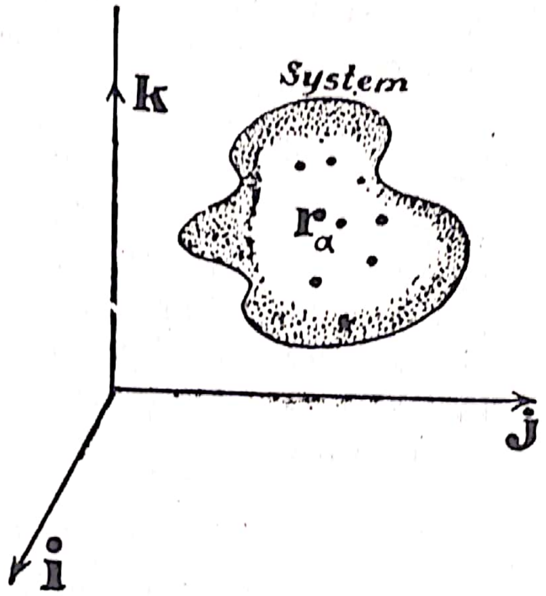
We shall come across these connections between the constants of motion and the symmetry properties several times.

6·1-13. Homogeneity and Isotropy of space and time conservation laws. Here we shall deduce the fact that linear and angular momentum are conserved when the space is homogeneous and isotropic and further due to the homogeneity of time, the total energy is conserved. By the homogeneity and isotropy of space, we mean that the Hamiltonian of an isolated system is invariant under translation and rotation respectively.

Homogeneity of space.

Total linear momentum is conserved.

Consider an infinitesimally small translation of the system by $\vec{\eta}$, which displaces each particle of the system by the amount η. Hence the position vectot $\mathbf{r}_\alpha$ of the α^{th} say, or in other words, we have



$$\delta \mathbf{r}_\alpha = \vec{\eta}. \qquad \ldots(63)$$

Owing to the infinitesimally small translation of the system, the Hamiltonian H of the system is changed by an amount dH.

$$dH = \sum_\alpha (\partial H/\partial \mathbf{r}_\alpha).\delta \mathbf{r}_\alpha = \vec{\eta}.\sum_\alpha (\partial H/\partial \mathbf{r}_\alpha) \qquad \ldots(64)$$

But due to the invariance of H in an homogeneous space, dH must be zero.

i.e. $\quad \sum_\alpha (\partial H/\partial \mathbf{r}_\alpha).\vec{\eta} = 0 \quad \Rightarrow \quad \Sigma (\partial H/\partial \mathbf{r}_\alpha) = 0 \quad (\because \vec{\eta}$ is arbitrary)

Again, we have

$$\dot{\mathbf{p}}_\alpha = -(\partial H/\partial \mathbf{q}_\alpha) \quad i.e. \quad \dot{\mathbf{p}}_\alpha = -(\partial H/\partial \mathbf{r}_\alpha) \qquad (\mathbf{q} = \mathbf{r} \text{ here})$$

$$\Rightarrow \quad \Sigma\, \dot{\mathbf{p}}_\alpha = 0 \quad \Rightarrow \quad \frac{d}{dt}\Sigma\, \mathbf{p}_\alpha = 0 \qquad \ldots(65)$$

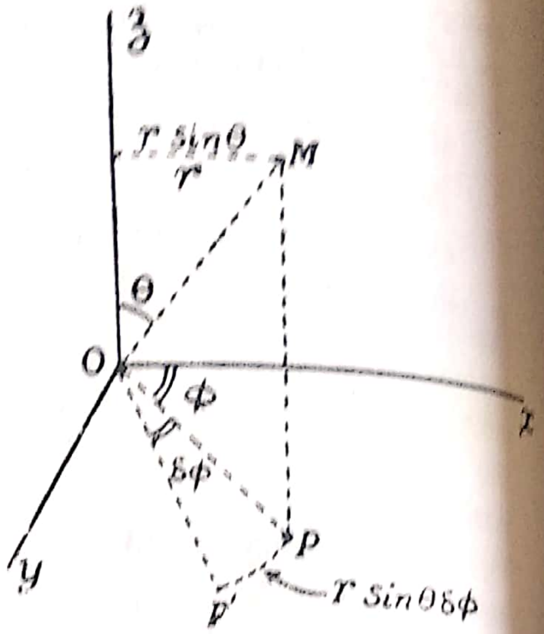
$$\Rightarrow \quad \Sigma\, \mathbf{p}_\alpha = \text{constant vector}$$

$\Rightarrow$ the homogeneity of space ensures conservation of linear momentum.

Isotropy of space.

Total angular momentum is conserved.

Consider an infinitesimally rotation of the system by an amount $\delta\phi$. Let the resiting change in the radius vector $\mathbf{r}$ be $\delta\mathbf{r}$, then we have (as obvious by the adjoining diagram)



$$\delta r = r \sin \delta\phi$$

i.e. $\quad \delta\mathbf{r} = \overrightarrow{\delta\phi} \times \mathbf{r} \qquad \ldots(66)$

where the rotation occurs, the velocity vectors of the particles change their directions and the corresponding change in $\mathbf{v}$ (the velocity vector) is given by

$$\delta\dot{\mathbf{r}} = \overrightarrow{\delta\phi} \times \dot{\mathbf{r}} \quad i.e. \quad \delta\mathbf{v} = \overrightarrow{\delta\phi} \times \mathbf{v} \qquad \ldots(67)$$

Also, the change in the Hamiltoninan is given by

$$dH = \sum_\alpha (\partial H/\partial \mathbf{r}_\alpha) \cdot \delta\mathbf{r}_\alpha + \sum_\alpha (\partial H/\partial \mathbf{v}_\alpha) \cdot \delta\mathbf{v}_\alpha$$

$$= \sum_\alpha (\partial H/\partial \mathbf{r}_\alpha) \cdot \overrightarrow{\delta\phi} \times \mathbf{r}_\alpha + \sum_\alpha (\partial H/\partial \mathbf{p}_\alpha) \cdot \delta\mathbf{p}_\alpha$$

$$= \sum_\alpha (\partial H/\partial \mathbf{r}_\alpha) \cdot \overrightarrow{\delta\phi} \times \mathbf{r}_\alpha + \sum_\alpha (\partial H/\partial \mathbf{p}_\alpha) \cdot \overrightarrow{\delta\phi} \times \mathbf{p}_\alpha \qquad \ldots(68)$$

But $\quad (\partial H/\partial \mathbf{r}_\alpha) = -\dot{\mathbf{p}}_\alpha$ and $(\partial H/\partial \mathbf{p}_\alpha) = \dot{\mathbf{r}}_\alpha$

$$\therefore \quad dH = \sum_\alpha [-\dot{\mathbf{p}}_\alpha \cdot \overrightarrow{\delta\phi} \times \mathbf{r}_\alpha + \dot{\mathbf{r}}_\alpha \cdot \overrightarrow{\delta\phi} \times \mathbf{p}_\alpha]$$

$$= \sum_\alpha [\dot{\mathbf{p}}_\alpha \cdot \mathbf{r}_\alpha \times \overrightarrow{\delta\phi} + \dot{\mathbf{r}}_\alpha \cdot \overrightarrow{\delta\phi} \cdot \mathbf{p}_\alpha]$$

$$= \sum_\alpha [\overrightarrow{\delta\phi} \cdot \dot{\mathbf{p}}_\alpha \times \mathbf{r}_\alpha + \overrightarrow{\delta\phi} \cdot \mathbf{p}_\alpha \times \dot{\mathbf{r}}_\alpha]$$

$$(\because \quad \mathbf{a}.\mathbf{b} \times \mathbf{c} = \mathbf{b}.\mathbf{c} \times \mathbf{a} = \mathbf{c}.\mathbf{a} \times \mathbf{b}) \qquad \ldots(69)$$

$$= \sum_\alpha (\dot{\mathbf{p}}_\alpha \times \mathbf{r}_\alpha + \mathbf{p}_\alpha \times \dot{\mathbf{r}}_\alpha) \cdot \overrightarrow{\delta\phi},$$

Again H is invariant under rotation in an isotropic space and $\overrightarrow{\delta\phi}$ is arbitrary, so we have $dH = 0$.

$$\Rightarrow \quad \dot{\mathbf{p}}_\alpha \times \mathbf{r}_\alpha + \mathbf{p}_\alpha \times \dot{\mathbf{r}}_\alpha = 0$$

$$\Rightarrow \quad \frac{d}{dt}\{\sum_\alpha \mathbf{p}_\alpha \times \mathbf{v}_\alpha\} = 0$$

$$\Rightarrow \quad \Sigma_\alpha\, p_\alpha \times \mathbf{r}_\alpha = \text{constant vector}$$

$$\Rightarrow \quad \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \text{constant vector } \mathbf{L}, \text{ say} \qquad \ldots(70)$$

Homogeneity of space :

Total energy is conserved.

When there exists the concept of homogeneity of space, the Hamiltonian H does not explicity depend on time and hence

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_\alpha \left[\frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha\right]$$

$$= \frac{\partial H}{\partial t} + \sum_\alpha [(-\dot{p}_\alpha \dot{q}_\alpha + \dot{p}_\alpha \dot{q}_\alpha)] = \frac{\partial H}{\partial t} \qquad \ldots(71)$$

But H does not involve explicit time dependence, so we have

$$\frac{\partial H}{\partial t} = 0 \left(\because \frac{dH}{dt} = 0\right) \Rightarrow H \text{ is conserved.}$$

Again $H = T + V$, $\therefore$ $T + V = \text{constant}$ $\Rightarrow$ Total energy is conserved.

6·1-14. Virial Theorem.

The function $Z = \sum_\alpha (\mathbf{F}_\alpha \cdot \mathbf{x}_\alpha)$ introduced by Clausius is known as virial. ...(72)

Let us define

$$P = \Sigma\, p_\alpha q_\alpha, \Rightarrow \frac{dP}{dt} = \sum_\alpha [p_\alpha \dot{q}_\alpha + \dot{p}_\alpha q_\alpha]$$

$$\frac{dP}{dt} = \sum_\alpha p_\alpha \left(\frac{\partial H}{\partial p_\alpha}\right) - \sum_\alpha q_\alpha \cdot \left(\frac{\partial H}{\partial q_\alpha}\right) \qquad \ldots(73)$$

Hence, time average of $\frac{dP}{dt}$ over an interval τ

$$= \frac{\int_0^\tau (dP/dt)\, dt}{\tau} = \frac{P(\tau) - P(0)}{\tau}$$

i.e. $$\frac{P(\tau) - P(0)}{\tau} = \frac{1}{\tau}\int_0^\tau \left[\sum_\alpha \left(p_\alpha \frac{\partial H}{\partial p_\alpha} - q_\alpha \frac{\partial H}{\partial q_\alpha}\right)\right] dt \qquad \ldots(74)$$

But, when the system is periodic, $P(\tau) = P(0)$ if τ is choosen to be the period.

In case of non-periodic systems, for which P has an upper bound, $P(\tau)$ can be made equal to $P(0)$ by choosing τ to be sufficiently large. Thus in both the cases, we have

$$0=\frac{1}{\tau}\int_0^\tau \left\{ \Sigma p_\alpha \frac{\partial H}{\partial p_\alpha} dt - \Sigma q_\alpha \frac{\partial H}{\partial q_\alpha} dt \right\}$$

i.e. Time average of $\Sigma \left\{ p_\alpha \frac{\partial H}{\partial q_\alpha} \right\}$

i.e. $\Sigma p_\alpha \dot{q}_\alpha$ = time average of $\Sigma \left\{ q_\alpha \frac{\partial H}{\partial q_\alpha} \right\}$ *i.e.* $\Sigma - p_\alpha q_\alpha$ (...75)

whence, the virial Z is given by

$$Z=-\sum_\alpha q_\alpha \frac{\partial H}{\partial q_\alpha}$$

$\Rightarrow$ Time average of Z = time average of $[-\Sigma q_\alpha (\partial H/\partial q_\alpha)]$

$=$ time average of $[-\Sigma p_\alpha (\partial H/\partial p_\alpha)]$

$=-[\sum_\alpha p_\alpha (\partial H/\partial p_\alpha)_{T\cdot A}$ say

$=-[\sum_\alpha p_\alpha \dot{q}_\alpha]=[p_\alpha q_\alpha]_{T\cdot A}=-2^*[\Sigma_\alpha T_\alpha]_{T\cdot A}$

$=-2N\left[\frac{\Sigma_\alpha T_\alpha}{N}\right]_{T\cdot A}$

$=-2N\bar{T}$, where $\bar{T}$ = average of T over the time as well as over the N particles ...(76)

In particular, when a particle moves in a potential field $U(r)$, we have

$(Z)_{T\cdot A}=-(p\dot{q})_{T\cdot A}=-(p\dot{r})_{T\cdot A}=(\dot{p}r)_{T\cdot A}=-2[T]_{T\cdot A}$ ...(77)

$\Rightarrow -2[T]_{T\cdot A}=[\dot{p}r]_{T\cdot A}=-[\mathbf{r}.\nabla U]_{T\cdot A}$ ...(78)

$\Rightarrow [T]_{T\cdot A}=\frac{1}{2}[\mathbf{r}.\nabla U]_{T\cdot A}$

Again, let $U_\alpha=r^n \Rightarrow U=Ar^n \Rightarrow \frac{\partial U}{\partial \mathbf{r}}\cdot\mathbf{r}=nU$

$\Rightarrow [T]_{T\cdot A}=\frac{1}{2}n[U]_{T\cdot A}$ ...(79)

When there exists the inverse square law, we have $n=-1$

$\Rightarrow [T]_{T\cdot A}=-\frac{1}{2}[U]_{T\cdot A}$ ...(80)

This is well known form of the *virial theorem*.

6·1-15. Liouville's Theorem.

It states that "*the phase volume occupied by a set of particles is constant*", or in other words, "*the number of particles per unit volume in phase space is constant.*"

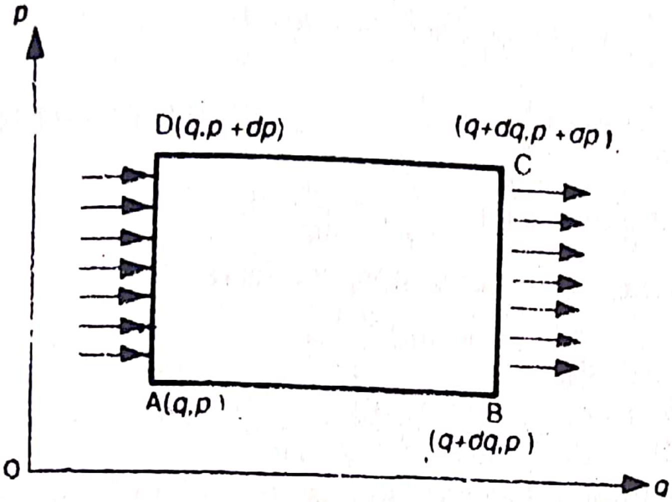
Proof. *Liouville's theorem is very easy to prove for a system having one degree of freedom, but it is complicated for a system having more than one degree of freedom.*

---

*If T is the K. E. of the system, we have

$$2T=\Sigma \dot{q}_\alpha p_\alpha$$

Case (i). **One degree of freedom.** Here, we have a two dimensional phase space and the volume element reduces to the area element $dp\, dq$.



Now let $\rho=[\rho\,(p, q, t)]$ be the number of particles per unit area, in the system. Then the number of points which enter through AD in unit time.

$=\rho\dot{q}\, dp$, where $\dot{q}$ is the velocity of the points entering through AD.

Also, the number of points leaving through AC

$$=\left[\rho\dot{q}+\frac{\partial}{\partial q}(\rho\dot{q})\, dq\right] dp.$$

Whence, increase in the number of particles, remaining in the element $ABCD$

$$=\rho\dot{q}\, dp-\left\{\rho\dot{q}+\frac{\partial}{\partial q}(\rho\dot{q})\, dq\right\} dp=-\frac{\partial}{\partial q}(\rho\dot{q})\, dp\, dq. \qquad ...(1)$$

Similary, the number of particles which enter through AB

$$=\rho\dot{p}\, dq$$

and the number of particles which leave through CD

$$=\left\{\rho\dot{q}+\frac{\partial}{\partial q}(\rho\dot{p})\, dp.\right\}.$$

Thus increase in the number of particles, remaining in the element $ABCD$

$$=\rho\dot{p}\, dq-\left\{\rho\dot{p}+\frac{\partial}{\partial p}(\rho\dot{p})\, dp\right\} dq=-\frac{\partial}{\partial p}(\rho\dot{p})\, dq\, dp. \qquad ...(2)$$

Now increase in number of particles in the element $ABCD$

$$=-\left[\frac{\partial}{\partial q}(\rho\dot{q})+\frac{\partial}{\partial p}(\rho\dot{p})\right] dq\, dp, \text{ [using (1) and (2)]}$$

This must be equal to $\frac{\partial \rho}{\partial t}\, dp,\, dq$, *i.e.* we have

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial q}\, dp\, dq=-\left[\frac{\partial}{\partial q}(\rho \dot{q})+\frac{\partial}{\partial p}(\rho \dot{p})\right] dq\, dp$$

$$\Rightarrow \quad \left[\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial q}(\rho \dot{q})+\frac{\partial}{\partial p}(\rho \dot{p})\right] dp\, dq=0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial q}(\rho \dot{q})+\frac{\partial}{\partial p}(\rho \dot{p})=0 \quad (\because \quad dp,\ dq \text{ are non zeros})$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t}+\rho\frac{\partial \dot{p}}{\partial q}+\dot{q}\frac{\partial \rho}{\partial q}+\rho\frac{\partial \dot{p}}{\partial q}+\dot{q}\frac{\partial \rho}{\partial q}=0. \quad \ldots(3)$$

But, by Hamilton's equations, we have

$$\frac{\partial H}{\partial q}=-\dot{p} \text{ and } \frac{\partial H}{\partial p}=\dot{q}.$$

$$\therefore \quad \frac{\partial \dot{p}}{\partial p}=-\frac{\partial^2 H}{\partial p\, \partial q};\ \frac{\partial \dot{p}}{\partial q}=\frac{\partial^2 H}{\partial q\, \partial p} \quad \ldots(4)$$

But Hamiltonian has continuous second order derivatives, so we have

$$\frac{\partial \dot{p}}{\partial q}=-\frac{\partial \dot{p}}{\partial q}$$

Whence, we get $\frac{\partial \rho}{\partial t}+\dot{q}\frac{\partial \rho}{\partial q}+\dot{p}\frac{\partial \rho}{\partial p}=0 \Rightarrow \frac{d}{dt}\, \rho\,(p, q, t)=0$

$\Rightarrow \quad \rho\,(p, q, t)=\text{constant},$

$\Rightarrow$ density in phase space is constant. This is *Liouville's theorem.*

Case (ii). System having more than one degrees of freedom. Here, the phase volume is

$$dV=dq_1\, dq_2 \ldots dq_\beta; \ldots dp_1\, dp_2 \ldots dp_\beta$$

Proceeding as in the first case, the increase in number of particles in the volume element dV

$$=-\left\{\frac{\partial(\rho \dot{q}_1)}{\partial q_1}+\frac{\partial(\rho \dot{q}_2)}{\partial q_2}+\ldots+\frac{\partial(\rho \dot{q}_2)}{\partial q_\beta}+\ldots+\frac{\partial(\rho \dot{p}_1)}{\partial p_1}+\frac{\partial(\rho \dot{p}_2)}{\partial p_2}\right.$$

$$\left.\ldots+\frac{\partial(\rho \dot{p}_\beta)}{\partial p_\beta}+\ldots\right\} dp_1\, dq_2 \ldots dq_\beta \ldots dp_1\, dp_2 \ldots dp_\beta$$

$$=-\left\{\frac{\partial(\rho \dot{q}_1)}{\partial q_1}+\ldots+\frac{\partial(\rho \dot{q}_\beta)}{\partial q_\beta}+\ldots+\frac{\partial(\rho \dot{p}_1)}{\partial p_1}+\ldots+\frac{\partial(\rho \dot{p}_\beta)}{\partial p_\beta}+\ldots\right\} dV,$$

which should be equal to $\frac{\partial \rho}{\partial t}\, dV$, and hence we obtain

$$\frac{\partial \rho}{\partial t}\, dV=-\left\{\frac{\partial(\rho \dot{q}_1)}{\partial q_1}+\ldots+\frac{\partial(\rho \dot{q}_\beta)}{\partial q_\beta}+\ldots+\frac{\partial(\rho \dot{p}_\beta)}{\partial p_1}\right.$$

$$\left.+\ldots+\frac{\partial(\rho \dot{p}_\beta)}{\partial p_\beta}+\ldots\right\} dV$$

$$=-\sum_\beta \left\{\frac{\partial(\rho \dot{q}_\beta)}{\partial q_\beta}+\frac{\partial(\rho \dot{p}_\beta)}{\partial q_\beta}\right\} dV$$

or
$$\frac{\partial \rho}{\partial t}+\sum_\beta \left\{\frac{\partial(\rho \dot{q}_\beta)}{\partial p_\beta}+\frac{\partial(\rho \dot{p}_\beta)}{\partial p_\beta}\right\}=0$$

$$\frac{\partial \rho}{\partial t}+\sum_\beta \left\{\frac{\partial \rho}{\partial q_\beta}\dot{q}_\beta+\rho\frac{\partial \dot{q}_\beta}{\partial q_\beta}+\frac{\partial \rho}{\partial p_\beta}\dot{p}_\beta+\rho\frac{\partial \dot{p}_\beta}{\partial p_\beta}\right\}=0.$$

Now, using Hamilton's equations $\frac{\partial H}{\partial q_\beta}=-\dot{p}_\beta$ and $\frac{\partial H}{\partial p_\beta}=\dot{q}_\beta$; we oblain

$$\Rightarrow \quad \frac{\partial \dot{p}_\beta}{\partial p_\beta}=\frac{\partial^2 H}{\partial p_\beta\, \partial q_\beta};$$

$$\frac{\partial \dot{p}_\beta}{\partial q_\beta}=-\frac{\partial^2 H}{\partial q_\beta\, \partial p_\beta}$$

But Hamiltonian has second order continuous derivatives, so we get

$$\frac{\partial \dot{p}_\beta}{\partial q_\beta}=-\frac{\partial \dot{q}_\beta}{\partial q_\beta}$$

Whence, we get $\frac{\partial \rho}{\partial t}+\sum_\beta \left\{\frac{\partial \rho}{\partial q_\beta}\dot{q}_\beta+\frac{\partial \rho}{\partial q_\beta}\dot{p}_\beta\right\}=0$

$$\Rightarrow \quad \frac{d\rho}{dt}(q_1, q_2, \ldots, q_\beta, \ldots, p_1, p_2, \ldots, t)=0$$

$\Rightarrow$ density in phase space is constant. *This is Liouville's theorem.*

Ex. 6. *If the particles attract each other according to inverse square law of force, prove that $2T+V=0$, where T is the total kinetic energy of the particles and V the potential energy.*

(Meerut 1981; Agra 63)

Sol. For a conservative system, the force $\mathbf{F}_i$ are derivable from a scalar potential function *i.e.* $\mathbf{F}=-\nabla U$, then we have

$$[T]_{T\cdot A}=\tfrac{1}{2}\left[\sum_i \nabla U \mathbf{r}_i\right]_{T\cdot A}$$

For a single particle moving under a central force, it requces to

$$[T]_{T\cdot A}=\tfrac{1}{2}\left(\frac{\partial U}{\partial r}\right) r$$

Again, if the *force law varies* as r^n, then we have

$$U=kr^{n+1},$$

$$\Rightarrow \quad \frac{\partial U}{\partial r} r=(n+1)\, kr^n.r$$

$$=(n+1)\, kr^{n+1}=(n+1)\, U$$

Thus, we have $[T]_{T\cdot A}=\frac{n+1}{2}[U]_{T\cdot A}$

When the law is that of *inverse square, we have*, $n=-2$

$\Rightarrow \quad [T]_{T\cdot A}=-\dfrac{2+1}{2}[U]_{T\cdot A} \qquad \Rightarrow \quad [T]_{T\cdot A}=-\frac{1}{2}[U]_{T\cdot A}$

$\Rightarrow 2[T]_{T\cdot A}+[U]_{T\cdot A}=0.$

SUPPLEMENTARY PROBLEMS

1. In the problem of the motion of a rigid body, with one point fixed there are no geometrical equations containing time explicity so that $H=T+V=$constant. Making use of the Eulerian angles ϕ, θ, ψ and Euler's geometrical equations, set up the Hamiltonian H and from it deduec Euler's dynamical equations.

2 Consider a system consisting of one particle under the action of conservative forces which are independent of the particle's azimuth about the z-axis of an inertial system. Find the Hamiltonian of the particle in terms of its cartesian co-ordinates with respect to a system of axes rotating uniformly about the z-axis with an angular velocity ω What is the physical significance of the Hamiltonian here? Is it a constant of the motion? Hamiltonian is the sum of two constants. What are they?

3. Obtain the Hamiltonian ol a heavy symmetrical top with one point, fixed Find from it Hamilton's canonical equations of motion. Show how the solution may be reduced to quadratures. [see chap. 7]

4. A particle of mass m moves in a central field of attractive force (α/r^2) $e^{-\beta t}$, where α and β are constants, t is the time, r is the distance from the force centre. Find the Lagrangian, and the Hamiltonian. Is H is total energy? Is it a constant?

5. (a) Construct the Routhian for the two body problem for which

$$L=\frac{\mu}{2}(\dot{r}^2+r^2\dot{\theta}^2)-V(r)$$

(b) There are two cyclic variables in the Lagrangian for a heavy symmetrical top; construct the Routhian

(c) For a simple harmonic oscillator of force constant $-kx$, set up the Hamiltonian.

6. State and explain Hamilton's principle and derive Lagrange's equation of motion from it. Discuss how the result will be modified if the forces are non-conservative. (*Meerut 1975; Agra 1974, 71*)

7. Derive Hamilton's canonical equations of motion from the modified Hamilton's principle. (*Agra 1975*)

8. State Hamilton's principle and derive it by differental method. Using Hamilton's principle, show that a sphere is the solid figure of revolution which has maximum volume for a given area.

9. (a) Derive the Euler-Lagrange equations of motion using the method of calculus of variations. (*Agra 1977*)

(b) A particle moves under the influence of gravity on the frictionless inner surface of a cone given by $x^2+y^2=c^2z^2$. Obtain the equations of motion.

10. A particle is sliding down a frictionless inclined plane of angle θ. Use Lagrange's equation for non-holonomic systems to find the equations of motion.

11. A smooth wire is bent into the form of an inverted cycloid. Find the motion of a particle sliding on this wire under the influence of gravity.

12. Explain the method of Lagrange's undermined multiplier in deriving the equations of motion for a conservative non-holonomic system from Hamilton's principle. Apply that method to solve the problem of a hoop rolling down on inclined plane without slipping. (*Agra 1976*)

13. Find x and y as functions of t so that.

$$J=\int_{t_1}^{t_2}\left[\frac{m}{2}(\dot{x}^2+\dot{y}^2)-mgy\right]dt$$

may have stationary valne. It may be assumed that x and y are given at t_1 and t_2.

6·2 B. Transformations and Brackets.

6·2·1. Point and canonical transformations.

Lagrange's equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0 \quad (\alpha=1, 2, \ldots, n) \qquad \ldots(1)$$

This system of equations is invariant with respect to the choice of the set of any generalised co-ordinates. Hence if the set $\{q_1, q_2, \ldots q_n\}$ of generalised co-ordinates be transformed to any other set of generalised co-ordinates $\{Q_1, Q_2, \ldots, Q_n\}$; then one may write

$$\left.\begin{aligned} Q_1&=Q_1(q_1, q_2, \ldots, q_n)\\ Q_2&=Q_2(q_1, q_2, \ldots, q_n)\\ \ldots&\quad \ldots \quad \ldots \quad \ldots\\ Q_\alpha&=Q_\alpha(q_1, q_2, \ldots, q_n)\\ \ldots&\quad \ldots \quad \ldots \quad \ldots\\ Q_n&=Q_n(q_1, q_2, \ldots q_n)\end{aligned}\right\} \qquad \ldots(2)$$

and the invariance of equations (1) $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_\alpha}\right)-\frac{\partial L}{\partial Q_\alpha}=0 \ (\alpha=1, 2, \ldots n). \qquad \ldots(3)$$

The transformations defined by (2) are known as *Point-Transformations.*

From above, we can also say that the Lagrangian system is covariant with respect to the point transformations. From this covariance, it can also be said that Hamilton's canonical equations will also be covariant

i.e. $$\dot{Q}_\alpha=\frac{\partial}{\partial P_\alpha}H(P_\alpha, Q_\alpha) \text{ and } \dot{P}_\alpha=-\frac{\partial}{\partial Q_\alpha}H(P_\alpha, Q_\alpha) \qquad \ldots(4)$$

provided $$P_\alpha=[\partial/\partial \dot{Q}_\alpha][L(Q_\alpha, \Omega_\alpha)].$$

Another type of transformations is a *canonical transformation* which makes the integration of equations of motion simpler. The principle consists the transformation from one set of variables (q_α, p_α) to another set of variables Q_α, P_α such that the canonical nature is preserved ; *viz.*

$$\dot{q}_\alpha = (\partial/H\partial p_\alpha),\ \dot{p}_\alpha = (-\partial H/\partial q_\alpha) \qquad ...(5)$$

One such transformation in which the formal appearance of Hamilton's canonical equations remains unaltered is given by

$$\left.\begin{array}{l} q_\alpha \to P_\alpha \\ p_\alpha \to -Q_\alpha \end{array}\right\}$$

so equations (5) $\Rightarrow$ $\dot{P}_\alpha = -(\partial H/\partial Q_\alpha)$, $\dot{Q}_\alpha = (\partial H/\partial P_\alpha)$. ...(6)

From equation (6), we can say that Q_α and P_α refer to the position co-ordinates and momentum co-ordinates repectively, yet this is not the case in (*G*). This anamoly can be resolved by realising that q_α and p_α should be treated on an equal footing. When equal status is given to the generalised co-ordinates q_α and generalised momenta p_α in the Hamiltonion formultation, the canonical equations give a wide range of transformations from the q, p basis to Q, P basis

where $\quad Q_\alpha = Q_\alpha\,(q, p, t)$ and $P_\alpha = P_\alpha\,(q, p, t)$. ...(7)

All the transformations of the type (7) do not necessarily preserve the canonical nature of the equations of motion. Now we are going to find that transformation [defined by (7)] which makes the new variable "Q, P" canonical

i.e. $\quad \dot{Q}_\alpha = (\partial H'/\partial P_\alpha)$ and $\dot{P}_\alpha = -(\partial H'/\partial Q_\alpha)$ ...(8)

where $H' = H'\,(Q, P)$, is the new Hamiltonian. The transformations, (7) for which equations (8) are vaild, are said to be *canonical* or sometimes **contact transformation*.

Theorem 1. *If two Hamiltonians differ by a total derivative of a function $F = F\,(p, q, t)$ the form of canonical equations of motion is unaltered.*

Proof. Modified Hamilon's principle is given by

$$\int_{t_1}^{t_2} [\sum_\alpha p_\alpha \dot{q}_\alpha - H\,(p, q)]\, dt = 0. \qquad ...(9)$$

If, we add to the Hamiltonian, a total time derivative of the function $F = F\,(p, q, t)$, equation (9) changes to

*The geometrical significance of this name is that, is two hyper surfaces are tangent at a point in the q—p space, their transforms are also tangent at the corresponding point in the Q—P space.

$$\delta \int_{t_1}^{t_2} [\sum_\alpha p_\alpha \dot{q}_\alpha - H(p, q) + (d/dt) F(p, q, t)]\, dt = 0. \qquad \text{...(10)}$$

Now L. H. S $= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt$

$$+\delta \int_{t_1}^{t_2} d\,[F(p, q)\, t)].$$

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt + \delta \Big[F(p, q, t) \Big]_{t_1}^{t_2}$$

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt + \left[\frac{\partial F}{\partial q_\alpha} \delta q_\alpha + \frac{\partial F}{\partial p_\alpha} \delta p_\alpha \right]_{t_1}^{t_2}$$

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt$$

$$+ \left\{ \frac{\partial F}{\partial q_\alpha} \delta q_\alpha \right\}_{t_1}^{t_2} + \left\{ \frac{\partial F}{\partial p_\alpha} \delta p_\alpha \right\}_{t_1}^{t_2}$$

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt$$

$$+ \left\{ \frac{\partial F}{\partial q_\alpha} \delta q_\alpha \right\}_{t_1}^{t_2} + \left\{ \frac{\partial F}{\partial p_\alpha} \delta p_\alpha \right\}_{t_1}^{t_2}$$

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt + 0$$

[$\because$ variation in q_α and p_α vanishes at the end points]

$$= \delta \int_{t_1}^{t_2} [\Sigma\, p_\alpha \dot{q}_\alpha - H(p, q)]\, dt.$$

Hence Proved.

Theorem 2. *To obtain the conditions that a transformation from (q, p) basis to (Q, P) basis be canonical.*

Proof. Let us effect a transformation from the (q, p) basis to the (Q, P) basis through the relations

$$Q_\alpha = Q_\alpha (q, p, t), \quad P_\alpha = P_\alpha (q, p, t).$$

This changes $H(p, q, t)$ to $H'(P, Q, t)$ say. These two Hamil tonians, *viz.* H and H' will give canonical aquations in (q, p) and (Q, P), provided these differ by a total time derivative of arbitrary function *i.e.* (dF/dt) In this situation, we have

$$\delta \int [\sum_\alpha p_\alpha \dot{q}_\alpha - H(p, q)]\, dt$$

$$= \delta \int [\sum_\alpha P_\alpha \dot{Q}_\alpha - H'(P, Q) + (dF/dt)]\, dt = 0. \qquad \text{...(11)}$$

The function F defined above is known as *generating function of the transformation.* Equation (11) is true

iff $\quad \sum_\alpha p_\alpha \dot{q}_\alpha - H(p, q) = \Sigma\, P_\alpha \dot{Q}_\alpha - H'(P, Q) + (dF/dt) \qquad \text{...(12)}$

As a matter of fact, the transformation is being affected from (q, p) to (Q, P) by the generating function F so F must be the function of p, q, P, Q and t, *i.e.* all $(n+n+n+n+1)=4n+1$ co-ordinates. But, as there exist the following transformation equations; $Q_\alpha=Q_\alpha(q, p, t)$, $P_\alpha=P_\alpha(q, p, t)$, only $2n$ of these are independent and thus F if a function $(2n+1)$ variable only and hence can be expressed in either of the following forms :

$$\underset{\longleftarrow(\mathrm{i})\longrightarrow}{F_1(q, Q, t)}, \quad \underset{\longleftarrow(\mathrm{ii})\longrightarrow}{F_2(q, P, t)}, \quad \underset{\longleftarrow(\mathrm{iii})\longrightarrow}{F_3(p, Q, t)}, \quad \underset{\longleftarrow(\mathrm{iv})\longrightarrow}{F_4(p, P, t)}. \quad \ldots(13)$$

Using (i) alternative form, we get from equation (12),

$$\sum_\alpha p_\alpha \dot{q}_\alpha - H(p, q) = \sum_\alpha P_\alpha \dot{Q}_\alpha - H'(P, Q) + \frac{d}{dt} F_1(q, Q, t) \quad \ldots(13a)$$

$$= \Sigma P_\alpha \dot{Q}_\alpha - H'(P, Q) + \Sigma \left\{\frac{\partial F_1}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial F_1}{\partial Q_\alpha}\dot{Q}_\alpha\right\} + \frac{\partial F}{\partial t} \quad \ldots(14)$$

But, the old and new co-ordinates q_α and Q_α are independent variables so the equation (14) is true, iff the coefficients of $\dot{q}_\alpha$ and $\dot{Q}_\alpha$ vanish separately,

i.e. $p_\alpha=(\partial F_1/\partial q_\alpha)$, $P_\alpha=-(\partial F_1/\partial Q_\alpha)$ and $H'=H+(\partial F_1/\partial t)$. ..(15)

Choosing the second alternative and marking the transition from F_1 to F_2 such that* $(\partial F_2/\partial Q_\alpha)=(\partial F_1/\partial Q_\alpha)+P_\alpha$. ...(16)

Integrating (16) w. r. t. "Q_α", we obtain $F_2=F_1+\Sigma P_\alpha Q_\alpha$, or in more explicit form

$$F_2(q, P, t)=F_1(q, Q, t)+\sum_\alpha P_\alpha Q_\alpha. \quad \ldots(17)$$

Inserting the values of F_1 from (17) in equation (14), we readily get $\sum_\alpha p_\alpha \dot{q}_\alpha - H = \Sigma P_\alpha \dot{Q}_\alpha - H' + (d/dt)\,[F_2(q_2 P, t) = \sum_\alpha P_\alpha Q_\alpha]$

$$= -\Sigma Q_\alpha \dot{P} - H' + \sum_\alpha (\partial F_2/\delta q_\alpha)\dot{q}_\alpha$$

$$+\sum_\alpha (\partial F_2/\partial P_\alpha)\dot{P}_\alpha + (\partial F_2/dt). \quad \ldots(18)$$

With the earlier arguments, we again get ...(19)

$P_\alpha-(\partial F_2/\partial q_\alpha)$, $Q_\alpha=(\partial F_3/\partial P_\alpha)$ and $H'=H+(\partial F_2/\partial t)$.

Now we shall link the third form of F. We know that ...(20)

$$(\partial F_3/\partial q_\alpha)=0 \text{ and also } p_\alpha=(\partial F_1/\partial q_\alpha)-p_\alpha.$$

Integrating (20), w. r. t. "q_α", we get ...(21)

$$F_3(p, Q, t)=F_1(q, Q, t)-\Sigma p_\alpha q_\alpha.$$

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*Since $(\partial F_2/\partial Q_\alpha)=0$, and also $(\partial F_1/\partial F_1/\partial Q_\alpha)+P_\alpha=0$,

we have $(\partial F_2/\partial Q_\alpha)=(\partial F_1/\partial Q_\alpha)+P_\alpha$.

Inserting the value of F_1 form (21) in (24), we readily get

$$\sum_{\alpha} p_\alpha \dot{q}_\alpha - H = \Sigma\, P_\alpha \dot{Q}_\alpha - H' + \frac{d}{dt}\left\{F_3(p, Q, t) + \sum_{\alpha} p_\alpha q_\alpha\right\}$$

$$\Rightarrow -\sum_{\alpha} q_\alpha \dot{p}_\alpha - H = \sum_{\alpha} P_\alpha \dot{Q}_\alpha + \sum_{\alpha} (\partial F_3/\partial p_\alpha)\, \dot{p}_\alpha$$

$$+\Sigma\, (\partial F_3/\partial Q_\alpha)\, \dot{Q}_\alpha + (\partial F_3/\partial t). \qquad \ldots(22)$$

Reasoning in the same manner as before, we get

$$q_\alpha = -(\partial F_3/\partial p_\alpha),\ P_\alpha = -(\partial F_3/\partial Q_\alpha) \text{ and } H' = H + (\partial F_3/\partial t). \qquad \ldots(23)$$

Finally, choosing the generating function of the form $F_4(p, P, t)$, we try to link F_4 with F_1 as follows :

Since $\qquad (\partial F_4/\partial q_\alpha) = 0 = (\partial F_4/\partial Q_\alpha), \qquad \ldots(24)$

and also $\qquad (\partial F_1/\partial q_\alpha) - p_\alpha = 0,\ (\partial F_1/\partial Q_\alpha) + P_\alpha = 0,$

so we have $\qquad F_4(p, P, t) = F_1(q, Q, t) + \Sigma\, P_\alpha Q_\alpha - \Sigma\, p_\alpha q_\alpha. \qquad \ldots(25)$

Now inserting the values of F_1 from equation (25) in equation (14), we readily get

$$\Sigma\, p_\alpha \dot{q}_\alpha - H = \Sigma\, P_\alpha \dot{Q}_\alpha - H' + \frac{d}{dt}\left[F_4(p, P, t) - \Sigma\, P_\alpha Q_\alpha + \Sigma\, p_\alpha q_\alpha\right]$$

$$\Rightarrow -H - \Sigma\, q_\alpha \dot{p}_\alpha = +H' - \Sigma\, Q_\alpha \dot{P}_\alpha + \Sigma\, (\partial F_4/\partial p_\alpha)\, \dot{p}_\alpha$$

$$+\Sigma\, (\partial F_4/\partial P_\alpha)\, \dot{P}_\alpha + (\partial F_4/\partial t). \qquad \ldots(26)$$

$$\Rightarrow q_\alpha = -(\partial F_4/\partial p_\alpha),\ Q_\alpha = (\partial F_4/\partial P_\alpha) \text{ and } H' = H + (\partial F_4/\partial t). \qquad \ldots(27)$$

Condition for a transformation to be canonical.

For a transformation from (q, p) basis to (Q, P) basis to be canonical certain conditions are to be satisfied. One of the conditions is that the expreesion

$$\Sigma\, (P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha) = \text{Exact differential} \qquad \ldots(28)$$

To prove this, we proceed as follows.

If the generating function F does not contain time explicity, we have $H' = H$. But for the transformation to be canonical (13a) must be satisfied which gives

$$\Sigma\, P_\alpha \dot{Q}_\alpha - \Sigma\, p_\alpha \dot{q}_\alpha = (dF/dt),$$

i.e. $$\Sigma\, P_\alpha \frac{dQ_\alpha}{dt} - \Sigma\, p_\alpha \frac{dq_\alpha}{dt} \Rightarrow \frac{dF}{dt} \Rightarrow dF = \Sigma\, P_\alpha\, dQ_\alpha - \Sigma\, p_\alpha\, dq_\alpha.$$

Hence our assertion is complete.

Ex. 1. *Show that the transformation $Q = \sqrt{(2q)}\, e^a \cos p$, $P = \sqrt{(2q)}\, e^{-a} \sin p$ is canonical transformation.*

Solution. We have $dQ=d\ [\sqrt{(2q)}\ e^{a} \cos p]$
$=(2q))^{-1/2} e^{a} \cos p\ dq-(2q)^{1/2} p^{a} \sin p\ dp$

$\Rightarrow P\ dQ-p\ dq=\sin p\ [\cos p\ dq-2q \sin p\ dp]-p\ dq$
$=(\frac{1}{2} \sin 2p-p)\ dq-2q \sin^2 p\ dp$
=Exact differential, provided

$\frac{\partial}{\partial p}\{(\sin 2p/2)-p\}=\frac{\partial}{\partial q}\{-2q \sin^2 p-p\}$, which is true.

Ex. 2. *Show that the transformation $P=\frac{1}{2}(p^2+q^2)$, $Q=\tan^{-1}(q/p)$ is canonical.* [Rohilkhand 79, Agra 82]

Sol. We have $\Sigma p_\alpha\ dq_\alpha-\Sigma p_\alpha\ dQ_\alpha=p\ dq-P\ dQ$

$$\Rightarrow p\ dq-\frac{1}{2}(p^2+q^2)\frac{(p\ dq-q\ dp)}{p^2+q^2}=\frac{1}{2}(p\ dq+q dp)=d\ [\tfrac{1}{2}\ pq]$$

=Exact differential

$\therefore$ Transformation is canonical.

Ex. 3. *Show that the transformation $Q=p$, $P=-q$ is canonical.*

Sol. We have $\Sigma p_\alpha\ dq_\alpha-\Sigma p_\alpha\ dQ_\alpha=p\ dq-P\ dQ$
$=p\ dq+q\ dp\quad (\because\ dQ=dp)=d\ (pq)$
=Exact differential.

$\therefore$ Transformation is canonical.

Ex. 4. *Prove that the transformation $Q=q \tan p$, $P=\log \sin p$ is canonical.*

Solution. We have $p\ dq-P\ dQ$=exact differential etc.

6·2-2. Bilinear Invariant as the condition for canonical transformation.

Theorem. *If a transformation from (q, p) basis to the (Q, P) basis is canonical than the bilinear form $\Sigma\ (\delta p_\alpha\ dq_\alpha-\delta q_\alpha\ dp_\alpha)$ remains invariant.*

Proof. If the transformation from old co-ordinates to the new co-ordinates is canonical; then Hamilton's equations imply

$$dq_\alpha=(\partial H/\partial p_\alpha)\ dt,\ dq_\alpha=(-\partial H/\partial q_\alpha)\ dt,$$
$$dq_\alpha=(\partial H/\partial P_\alpha)\ dt,\ dP_\alpha=(-\partial H/\partial Q_\alpha)\ dt \quad ...(1)$$

$$\Rightarrow \Sigma\delta p_\alpha\left(dq_\alpha-\frac{\partial H}{\partial p_\alpha}\ dt\right)-\Sigma\delta q_\alpha\left(dp_\alpha+\frac{\partial H}{\partial q_\alpha}\right)=0,$$

for arbitrary values of δq_α and δp_α

$$\Rightarrow \sum_\alpha (\delta p_\alpha\ dq_\alpha-\delta q_\alpha)-\delta H\ dt=0 \quad ...(2)$$

Similarly, we also have

$$\Sigma\ (\delta P_\alpha\ dQ_\alpha-\delta Q_\alpha\ dP_\alpha)\ dp_\alpha-\delta H\ dt=0 \quad ...(3)$$

Subtracting (3) from (2), we get

$$\Sigma\ (\delta p_\alpha\ dq_\alpha-\delta q_\alpha\ dq_\alpha)=\Sigma\ (\delta P_\alpha-dQ_\alpha\ dP_\alpha)$$

which shows that $\Sigma\ (\delta p_\alpha\ dq_\alpha-\delta q_\alpha\ dp_\alpha)$=Invariant.

Ex. 5. *Prove that the transformation $Q=p^{-1}$, $P=qp^2$ is canonical using the invariance of Bilinear form.*

Solution. We have $P=qp^2 \Rightarrow \delta P=p^2\,\delta q+2qp\,\delta p$,

$$dP=p^2\,dq+2qp\,dp,\ \delta Q=-p^{-2}\,\delta p,\ dQ=-p^{-2}\,dp$$

$$\therefore\quad \delta P\,dQ-\delta Q\,dP=(p^2\,\delta q+2pq\,\delta p)\,(-p^{-2}\,dp)$$
$$+(p^{-2}\,\delta p)\,(p^2\,dq+2qp+dq)$$
$$=\delta p\,dq-\delta q\,dp$$

$\Rightarrow$ the transformation is canonical.

6·2-3. Generating functions.

By Hamilton's principle, the canonical transformation

$$P_\alpha=P_\alpha\,(p,q,t),\ Q_\alpha=Q_\alpha\,(p,q,t)$$

must satisfy the conditions that

$$\int_{t_1}^{t_2} L\,dt \text{ and } \int_{t_1}^{t_2} M\,dt$$

are both extrema *i.e.* we must simultaneously have

$$\delta\int_{t_1}^{t_2} L\,dt=0 \text{ and } \delta\int_{t_1}^{t_2} M\,dt=0$$

Thus by subtraction, we have $\delta\displaystyle\int_{t_1}^{t_2}(L-M)\,dt=0$ ...(1)

Now (1) can be accomplished, if there exists a function G such that $L-M=\dfrac{dG}{dt}$

i.e. (1) $\Rightarrow \delta\displaystyle\int_{t_1}^{t_2}\frac{dG}{dt}\,dt=0$ or $\delta\displaystyle\int_{t_1}^{t_2}\frac{dG}{dt}\,dt=0=\delta\,\{G\,(t_2)-G\,(t_1)\}=0$

The function G defined in such a way is a said to be the generating function.

Theorem 1. *Suppose that the generating function L is function of the old new position co-ordinates q_α and Q_α respectively as well as the time t, i.e. $\Gamma=\Gamma\,(q_\alpha, Q_\alpha\,t)$.*

Prove that $p_\alpha=\dfrac{\partial\Gamma}{\partial q_\alpha}$, $P_\alpha=-\dfrac{\partial\Gamma}{\partial Q_\alpha}$, $J=\dfrac{\partial\Gamma}{\partial t}+H$

where $\dot{P}_\alpha=-\dfrac{\partial J}{\partial Q_\alpha}$, $Q_\alpha=\dfrac{\partial J}{\partial P_\alpha}$.

Proof. Since Γ is the generating function, we have

$$\frac{\partial\Gamma}{\partial t}=L-M=\Sigma\,\dot{p}_\alpha q_\alpha-\mathrm{H}-\{\Sigma\,\dot{P}_\alpha Q_\alpha\,\{-J\}$$

$$=\Sigma\,p_\alpha\,q_\alpha-\Sigma\,P_\alpha\,\dot{Q}_\alpha+J-H$$

$$\Rightarrow d\Gamma=\sum_{\alpha=1}^{n} p_\alpha\,dp_\alpha-\sum_{\alpha=1}^{n} P_\alpha\,dQ_\alpha+(J-H)\,dt \quad ...(i)$$

But $\Gamma = \Gamma(q_\alpha, Q_\alpha, t)$.

$$\therefore \quad d\Gamma = \sum_{\alpha=1}^{n} \frac{\partial \Gamma}{\partial q_\alpha} dq_\alpha + \sum_{\alpha=1}^{n} \frac{\partial \Gamma}{\partial Q_\alpha} dQ_\alpha + \frac{\partial \Gamma}{\partial t} dt. \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get

$$p_\alpha = \frac{\partial \Gamma}{\partial q_\alpha}, \; P_\alpha = -\frac{\partial \Gamma}{\partial Q_\alpha}, \; J - H = \frac{\partial \Gamma}{\partial t}.$$

Also J is the Hamiltonian in the co-ordinates P_α, Q_α hence Hamilton's equations imply

$$\dot{P}_\alpha = -\frac{\partial J}{\partial Q_\alpha}, \; \dot{Q}_\alpha = \frac{\partial J}{\partial P_\alpha}.$$

Theorem 2. *Let K be a generating function dependent only on q_α, P_α, t. Prove that*

$$P_\alpha = \frac{\partial K}{\partial q_\alpha}, \; Q_\alpha = \frac{\partial K}{\partial P_\alpha}, \; J = \frac{\partial K}{\partial t} + H \quad \textit{where} \quad \dot{P}_\alpha = -\frac{\partial J}{\partial Q_\alpha}, \; \dot{Q}_\alpha = \frac{\partial J}{\partial P_\alpha}.$$

Proof. By theorem (1), we have

$$d\Gamma = \Sigma\, p_\alpha\, dp_\alpha - \Sigma\, P_\alpha\, dQ_\alpha + (J - H)\, dt$$

$$= \Sigma\, p_\alpha\, dp_\alpha - d\,\{\Sigma\, P_\alpha Q_\alpha\} + \Sigma\, Q_\alpha\, dP_\alpha + (J-H)\, dt$$

$$\Rightarrow d\,(\Gamma + \Sigma\, P_\alpha Q_\alpha) = \Sigma\, p_\alpha\, dp_\alpha + \Sigma\, Q_\alpha\, dp_\alpha + (J-H)\, dt$$

$$\Rightarrow dK = \Sigma\, p_\alpha\, dQ_\alpha + \Sigma\, Q_\alpha\, dP_\alpha + (J-H)\, dt,$$

where $\qquad K = \Gamma + \Sigma\, P_\alpha\, Q_\alpha \qquad$ ...(i)

But $\qquad K = K(q_\alpha, p_\alpha, t)$

$$\therefore \quad dK = \Sigma \frac{\partial K}{\partial q_\alpha} dq_\alpha + \Sigma \frac{\partial K}{\partial P_\alpha} dP_\alpha + \frac{\partial K}{\partial t} dt. \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get

$$p_\alpha = \frac{\partial K}{\partial q_\alpha}, \; Q = \frac{\partial K}{\partial P_\alpha}, \; J = \frac{\partial K}{\partial t} + H.$$

Also, as J is the Hamiltonian, we have

$$\dot{P}_\alpha = -\frac{\partial J}{\partial Q_\alpha} \text{ and } \dot{Q}_\alpha = \frac{\partial J}{\partial P_\alpha}.$$

6·2-4. Poincare's Integral Invariants.

Previously we have seen that canonical transformations leave *invariant* the form of Hamilton's equations of motion. Naturally now we are faced with the problem that "*Are there other expressions which are invariant under canonical transformation*". An example of this type was introduced by *Poincar'e* and is called as *integral invariant.*

The integral $$J_1 = \iint_S \sum_\gamma dq_\gamma\, dp_\gamma \qquad \text{...(1a)}$$

where the integral extends over an arbitray two dimensional surface S of the $2n$ dimensional (p, q) phase space. *Now we shall show that J_1 is invariant under canonical transformation :*

Proof. The invariance of J_1 implies that

$$\iint_S \sum_\gamma dq_\gamma\, dp_\gamma = \iint_S \sum_\gamma dQ_\gamma\, dP_\gamma. \qquad \text{...(1b)}$$

But the position of any point on a two dimensional surface is completely determined by two parameters *e.g.* u, v. So we take $d_\gamma = d_\gamma(u, v)$, $p_\gamma = p_\gamma(u, v)$. Again by Jacobian theory we know that

$$dq_\gamma\, dp_\gamma = \frac{\partial(q_\gamma, p_\gamma)}{\partial(u, v)}\, du\, dv \qquad \text{...(2)}$$

where, jacobian of (q_γ, p_γ) w.r.t. $(u, v) = \begin{vmatrix} \partial q_\gamma/\partial u & \partial q_\gamma/\partial v \\ \partial q_\gamma/\partial u & \partial p_\gamma/\partial v \end{vmatrix}$

$$= \frac{\partial(q_\gamma, p_\gamma)}{\partial(u, v)}.$$

Thus equation $(1b)$ gives

$$\iint_S \sum_\gamma \frac{\partial(q_\gamma, p_\gamma)}{\partial(u, v)}\, du\, dv = \iint_S \sum_\gamma \frac{\partial(Q_\alpha, P_\alpha)}{\partial(u, v)}\, du\, dv. \qquad \text{...(3)}$$

The surface S is arbitray, so equation (3) implies

$$\sum_\gamma \frac{\partial(q_\gamma, p_\gamma)}{\partial(u, v)} = \sum_\gamma \frac{\partial(Q_\gamma, P_\gamma)}{\partial(u, v)}. \qquad \text{...(4)}$$

Let us now effect the transformation from the $q-p$ basis to Q, P basis through the generating function $F_2(q, P, t)$. Under this form of the* generating function, we have

$$p_\gamma = (\partial F_2/\partial y_\gamma),\ Q_\gamma = (\partial F_2/\partial P_\gamma)$$

$$\left.\begin{aligned} \Rightarrow \frac{\partial p_\gamma}{\partial u} &= \frac{\partial}{\partial u}\left(\frac{\partial F_2}{\partial q_\gamma}\right) = \sum_\alpha \frac{\partial^2 F_2}{\partial q_\alpha\, \partial q_\gamma} \cdot \frac{\partial q_\gamma}{\partial u} + \sum_\alpha \frac{\partial^2 F_2}{\partial p_\alpha\, \partial q_\gamma}\, \frac{\partial P_\alpha}{\partial u} \\ \text{and} \quad \frac{\partial p_\alpha}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial q_\gamma}\right) = \sum_\alpha \frac{\partial^2 F_2}{\partial q_\alpha\, \partial q_\gamma}\, \frac{\partial q_\alpha}{\partial v} + \sum_\alpha \frac{\partial^2 F_2}{\partial p_\alpha\, \partial q_\gamma}\, \frac{\partial P_\alpha}{\partial v} \end{aligned}\right\} \qquad \text{...(5)}$$

$$\text{Thus } \sum_\gamma \frac{\partial(q_\gamma, p_\gamma)}{\partial(u, v)} = \sum_\gamma \begin{vmatrix} \dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v} \\[2ex] \dfrac{\partial p_\gamma}{\partial u} & \dfrac{\partial p_\gamma}{\partial v} \end{vmatrix}$$

$$= \sum_\gamma \begin{vmatrix} \dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v}. \\[2ex] \displaystyle\sum_\alpha \frac{\partial^2 F_2}{\partial q_\alpha\, \partial q_\gamma}\frac{\partial q_\alpha}{\partial u} + \sum \frac{\partial^2 F_2}{\partial P_\alpha\, \partial q_\gamma} \cdot \frac{\partial P_\alpha}{\partial u} & \displaystyle\sum \frac{\partial^2 F_2}{\partial q_\alpha\, \partial q_\gamma}\frac{\partial q_\alpha}{\partial v} + \sum \frac{\partial^2 F_2}{\partial P_\alpha\, \partial q_\gamma}\frac{\partial P_\alpha}{\partial v} \end{vmatrix}$$

*Theorem 2 P. 306.

$$=\sum_{\alpha}\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\alpha\,\partial q_\gamma}\begin{vmatrix}\dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix}+\sum_{\alpha}\sum_{\gamma}\frac{\partial^2 F_2}{\partial q_\alpha\,\partial q_\gamma}\begin{vmatrix}\dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v}\\[2mm] \dfrac{\partial q_\alpha}{\partial u} & \dfrac{\partial q_\alpha}{\partial v}\end{vmatrix}$$

$$=\sum_{\alpha}\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\alpha\,\partial q_\gamma}\begin{vmatrix}\dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix} \quad \text{(since the second term vanishes)} \qquad \ldots(6)$$

Again, we also have

$$\sum_{\alpha}\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\alpha\,\partial P_\gamma}\begin{vmatrix}\dfrac{\partial P_\gamma}{\partial u} & \dfrac{\partial P_\alpha}{\partial u}\\[2mm] \dfrac{\partial P_\gamma}{\partial v} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix}=0 \qquad \ldots(G)$$

So addition of relation (G) to (6) gives

$$\sum_{\gamma}\frac{\partial(q_\gamma, p_\gamma)}{\partial(u,v)}=\sum_{\alpha}\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\alpha\,\partial q_\gamma}\begin{vmatrix}\dfrac{\partial q_\gamma}{\partial u} & \dfrac{\partial q_\gamma}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix}+\sum_{\gamma}\sum_{\alpha}\frac{\partial^2 F_2}{\partial P_\alpha\,\partial P_\gamma}\begin{vmatrix}\dfrac{\partial P_\gamma}{\partial u} & \dfrac{\partial P_\gamma}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix}$$

$$=\sum_{\alpha}\begin{vmatrix}\sum_{\gamma}\dfrac{\partial^2 F_2}{\partial P_\alpha\,\partial q_\gamma}\dfrac{\partial q_\gamma}{\partial u}+\sum_{\gamma}\dfrac{\partial^2 F_2}{\partial P_\alpha\,\partial P_\gamma}\dfrac{\partial P_\gamma}{\partial u} & \sum_{\gamma}\dfrac{\partial^2 F_2}{\partial P_\alpha\,\partial q_\gamma}\dfrac{\partial q_\gamma}{\partial v}+\sum_{\gamma}\dfrac{\partial^2 F_{2\gamma}}{\partial P_\alpha\,\partial q_\gamma}\dfrac{\partial P}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix} \qquad \ldots(7)$$

But we also have

$$\frac{\partial Q_\alpha}{\partial u}=\frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_\alpha}\right)=\sum_{\gamma}\frac{\partial^2 F_2}{\partial q_\gamma\,\partial P_\alpha}\frac{\partial q_\gamma}{\partial u}+\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\gamma\,\partial P_\alpha}\frac{\partial P_\gamma}{\partial u}$$

and

$$\frac{\partial Q_\alpha}{\partial v}=\frac{\partial}{\partial v}\left(\frac{\partial F_2}{\partial P_\alpha}\right)=\sum_{\gamma}\frac{\partial^2 F_2}{\partial q_\gamma\,\partial P_\alpha}\frac{\partial q_\gamma}{\partial v}+\sum_{\gamma}\frac{\partial^2 F_2}{\partial P_\gamma\,\partial P_\alpha}\frac{\partial P_\gamma}{\partial v}$$

Inserting these values in (7), we get

$$\sum_{\gamma}\frac{\partial(q_\gamma, p_\gamma)}{\partial(u,v)}=\sum_{\alpha}\begin{vmatrix}\dfrac{\partial Q_\alpha}{\partial u} & \dfrac{\partial Q_\alpha}{\partial v}\\[2mm] \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v}\end{vmatrix}=\sum_{\alpha}\frac{\partial(Q_\alpha, \partial P_\alpha)}{\partial(u,v)}=\sum_{\gamma}\frac{\partial(Q_\gamma, P_\gamma)}{\partial(u,v)} \qquad \ldots(8)$$

This shows that the jacobian of Q_γ, P_γ w. r. t. u, v is the same as the Jacobian of q_γ, p_γ, w. r. t. u, v; i.e. the Jacobian is invariant under the above said transformation.

Hence the integral (1*a*) is also invariant.

Note. Continuing this argument, we can say that the integral

$$J_n=\iiint_S \cdots \iint dq_1\, dq_2 \quad dq_n\, dp_1\, dp_2 \ldots dp_n$$

is also invariant where integral extends over an arbitrary n dimensional surface S of the $2n$ dimensional (p, q) phase space

6·2-4. Lagrangian and Poisson Brackets.

(a) Lagrange's Bracket.

If the transformation from (q, p) basis to (Q, P) basis is canonical then by the previous article we know that

$$\sum_\alpha \frac{\partial (q_\alpha, p_\alpha)}{\partial (u, v)}=\sum_\alpha \frac{\partial (Q_\alpha, P_\alpha)}{\partial (u, v)}$$

i.e.
$$\sum_\alpha \begin{vmatrix} \dfrac{\partial q_\alpha}{\partial u} & \dfrac{\partial q_\alpha}{\partial v} \\ \dfrac{\partial p_\alpha}{\partial u} & \dfrac{\partial p_\alpha}{\partial v} \end{vmatrix}=\sum_\alpha \begin{vmatrix} \dfrac{\partial Q_\alpha}{\partial u} & \dfrac{\partial Q_\alpha}{\partial v} \\ \dfrac{\partial P_\alpha}{\partial u} & \dfrac{\partial P_\alpha}{\partial v} \end{vmatrix}$$

$$\Rightarrow \sum_\alpha \left(\frac{\partial q_\alpha}{\partial u}\frac{\partial p_\alpha}{\partial v}-\frac{\partial p_\alpha}{\partial u}\frac{\partial q_\alpha}{\partial v}\right)=\sum_\alpha \left(\frac{\partial Q_\alpha}{\partial u}\frac{\partial P_\alpha}{\partial v}-\frac{\partial P_\alpha}{\partial u}\frac{\partial Q_\alpha}{\partial v}\right). \qquad \ldots(9)$$

An expression of the type [on each side of equation (9)] is called a Lagrange bracket. In other words the Lagrange's bracket of (u, v) w.r.t. the basis (q_α, p_α) is defined by

$$\{u, v\}_{q,p}=\sum_\alpha \left(\frac{\partial q_\alpha}{\partial u}\frac{\partial p_\alpha}{\partial v}-\frac{\partial p_\alpha}{\partial u}\frac{\partial q_\alpha}{\partial v}\right). \qquad \ldots(10)$$

Thus equation (9) $\Rightarrow \{u, v\}_{q,p}=\{u, v\}_{Q,P}$

i.e. Lagrange's bracket remains invariant, if the transformation is canonical. ...(11)

Again, by the definition of Lagrange's bracket, we have

$$\{u, v\}_{q,p}=-\{v, u\}_{q,p} \qquad \ldots(12)$$

(b) Invariance of Lagrange's Brackets as a Condition for Canonical Transformation.

Invariance of bilinear form implies

$$\sum_\alpha (dp_\alpha\, \delta q_\alpha-\delta p_\alpha\, dq_\alpha)=\sum_\alpha (dP_\alpha\, \delta Q_\alpha-\delta P_\alpha\, dQ_\alpha). \qquad \ldots(1)$$

But if the transformation is effected, we can denote $(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ as any functions of the $2n$ variables $(Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n)$ i.e. we can write

$$dp_\alpha=\sum_{\beta=1}^{n} (\partial p_\alpha/\partial Q_\beta)\, dQ_\beta+\sum_{\beta=1}^{n} (\partial p_\alpha/\partial P_\beta)\, dP_\beta$$

$$\delta q_\alpha = \sum_{\gamma=1}^{n} (\partial q_\alpha/\partial Q_\gamma)\,\partial Q_\gamma + \sum_{\gamma=1}^{n} (\partial q_\alpha/\partial P_\gamma)\,\delta P_\gamma$$

$$\delta p_\alpha = \sum_{\beta=1}^{n} (\partial p_\alpha/\partial Q_\beta)\,\delta Q_\beta + \sum_{\beta=1}^{n} (\partial p_\alpha/\partial P_\gamma)\,\delta P_\gamma$$

and
$$dq_\alpha = \sum_{\gamma=1}^{n} (\partial q_\alpha/\partial Q_\gamma)\,dQ_\gamma + \sum_{\gamma=1}^{n} (\partial q_\alpha/\partial P_\gamma)\,dP_\gamma.$$

Inserting these values in (1) and remembering that there are no terms of the type $dP_\alpha\,\delta P_\beta$, dQ_α etc., we at once obtain

$$(Q_\beta, Q_\gamma)_{q,p}=0,\ \{P_\beta, P_\gamma\}_{q,p}=0 \text{ and } \{Q_\beta, P_\gamma\}_{q,p}=\delta_{\beta\gamma} \qquad \ldots(2)$$
$$(\beta, \gamma=1, 2, \ldots n)$$

(c) **Poisson Brackets.**

If the functions u and v depend on position co-ordinates q_α, momenta p_α and time t, the *Poisson Bracket of u and v* is defined as :

$$[u, v]_{q,p}=\sum_\alpha \left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}-\frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right)$$

Now we shall prove the following identities :

(i) $[u, v]=-[v, u]$, (ii) $[u_1+u_2, v]=[u_1, v]=[u_2, v]$

(iii) $[u, q_r]=-(\partial u/\partial p_r)$, (iv) $[u, p_r]=(\partial u/\partial q_r)$ etc.

Proof. (i) We have

$$[u, v]=\sum_\alpha \left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}-\frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right)=-\sum_\alpha \left(\frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}-\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}\right)$$

$$=-\sum_\alpha \left(\frac{\partial v}{\partial q_\alpha}\frac{\partial u}{\partial p_\alpha}-\frac{\partial v}{\partial p_\alpha}\frac{\partial u}{\partial q_\alpha}\right)=-[v, u]_{q,p}$$

$\Rightarrow$ *that the Poisson Bracket does not obey the commutative law of algebra.*

(ii) $$[u_1+u_2, v]=\sum \left\{\frac{\partial (u_1+u_2)}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}-\frac{\partial (u_1+u_2)}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right\}$$

$$=\sum \left\{\frac{\partial u_1}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}-\frac{\partial u_1}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right\}+\sum_\alpha \left\{\frac{\partial u_2}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha}-\frac{\partial u_2}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right\}$$

$\Rightarrow$ *that the Poission Bracket obeys the distributive law of algebra.*

(iii) $$[u, q_r]=\sum_\alpha \left(\frac{\partial u}{\partial q_\alpha}\frac{\partial q_r}{\partial p_\alpha}-\frac{\partial u}{\partial p_\alpha}\frac{\partial q_r}{\partial q_\alpha}\right)=\frac{-\partial u}{\partial p_r}$$

$$\therefore \left[(\partial q_r/\partial q_r)=\begin{matrix}1 & \text{for} & \alpha=r\\ 0 & \text{for} & \alpha\neq r\end{matrix}\right] \text{ and } (\partial q_r/\partial p_\alpha)=0,\ \forall\ \alpha.$$

(iv) $$[u, p_r]=\sum_\alpha \left(\frac{\partial u}{\partial q_\alpha}\frac{\partial p_r}{\partial p_\alpha}-\frac{\partial u}{\partial p_\alpha}\frac{\partial p_r}{\partial p_\alpha}\right)=\frac{\partial u}{\partial q_r}$$

$\therefore\ [(\partial p_r/\partial q_\alpha)=0,\ \forall\ \alpha,\ (\partial p_r/\partial p_\alpha)=1$ for $\alpha=r$
and $=0$ for $\alpha\neq r]$

(v) $[u_1u_2, v]=u_1\,[u_2, v]+u_2\,[u_1, v]$

(vi) $\dfrac{\partial}{\partial t}\,[u, v]=\left[\dfrac{\partial u}{\partial t}, v\right]+\left[u, \dfrac{\partial v}{\partial t}\right]$

(vii) $\dfrac{d}{dt}\,[u, v]=\left[\dfrac{du}{dt}, v\right]+\left[u, \dfrac{dv}{dt}\right]$

where $[u, v]$ is the *Poisson Bracket.*

(viii) *If H is the Hamiltonian, prove that if u is any function depending on position, momenta and time, then*

$$(du/dt)=(\partial u/\partial t)+[u, H]$$

where H is the Hamiltonian.

We have $\quad (du/dt)=(\partial u/\partial t)+\sum\limits_\alpha\left(\dfrac{\partial u}{\partial p_\alpha}\dfrac{\partial p_\alpha}{\partial t}+\dfrac{\partial u}{\partial q_\alpha}\dfrac{dq}{dt}\right)$

$$=(\partial u/\partial t)+\sum_\alpha\left(-\frac{\partial u}{\partial q_\alpha}\frac{\partial H}{\partial q_\alpha}+\frac{\partial u}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha}\right)=(\partial u/\partial t)+[u, H]$$

$$\left[\because\quad \dot q_\alpha=\frac{\partial H}{\partial p_\alpha}\text{ and }\dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}\right]$$

Theorem. *Poisson Bracket of two dynamical variables u and v under canonical transformation is invariant.* **[Rohilkhand 87]**

Proof. Let u and v be any two arbitrary functions of canonical variables (q, p) which are themselves transformed to another set of canonical variables (Q, P), then

$$[u, v]_{q,p}=[u, v]_{Q,P} \qquad \ldots(1)$$

We have $[u, v]_{p,q}=\sum\limits_\beta\left(\dfrac{\partial u}{\partial q_\beta}\dfrac{\partial v}{\partial q_\beta}-\dfrac{\partial u}{\partial p_\beta}\dfrac{\partial v}{\partial q_\beta}\right) \qquad \ldots(2)$

where $\quad \dfrac{\partial v}{\partial p_\alpha}=\sum\limits_\gamma\dfrac{\partial v}{\partial Q_\gamma}\dfrac{\partial Q_\gamma}{\partial p_\beta}+\sum\limits_\alpha\dfrac{\partial P}{\partial p_\beta};$

and $\quad \dfrac{\partial v}{\partial q_\beta}=\sum\limits_\gamma\dfrac{\partial v}{\partial Q_\gamma}\dfrac{\partial Q_\gamma}{\partial q_\beta}+\sum\limits_\gamma\dfrac{\partial v}{\partial P_\gamma}\dfrac{\partial q_\gamma}{\partial q_\beta}.$

Inserting these values in (2), we get

$$[u, v]_{q,p}=\sum_\beta\sum_\gamma\left[\frac{\partial u}{\partial q_\beta}\left(\frac{\partial v}{\partial Q_\gamma}\frac{\partial Q_\gamma}{\partial P_\beta}+\frac{\partial v}{\partial P_\gamma}\frac{\partial P_\gamma}{\partial p_\beta}\right)\right.$$

$$\left.=\sum_\gamma\frac{\partial u}{\partial p_\beta}\left(\frac{\partial v}{\partial Q_\beta}\frac{\partial Q_\gamma}{\partial q_\beta}+\frac{\partial v}{\partial P_\gamma}\frac{\partial P_\gamma}{\partial q_\beta}\right)\right]$$

$$=\sum_\gamma\left\{\frac{\partial v}{\partial Q_\gamma}\,[u, Q_\gamma]_{q,p}+\frac{\partial v}{\partial P_\gamma}\,[u, P_\gamma]_{q,p}\right\} \qquad \ldots(3)$$

Putting Q_γ for u and u for v in (3), we get

$$[Q_\gamma, u]=\sum_\beta \frac{\partial u}{\partial Q_\beta}[Q_\gamma, Q_\beta]+\sum_\beta \frac{\partial u}{\partial P_\beta}[Q_\gamma, P_\beta]=\sum_\beta \frac{\partial u}{\partial P_\beta}\delta_{\beta\gamma} \quad ...(4)$$

In view of relation (iii); *viz.* $[u, q_r]=-(\partial u/\partial p_r)$; we have

$$[u, Q_\gamma]=-(\partial u/\partial P_\gamma). \quad ...(5)$$

Like wise $[P_\gamma, u]=\sum_\beta \frac{\partial u}{\partial Q_\beta}[P_\gamma, Q_\beta]+\sum_\beta \frac{\partial u}{\partial P_\beta}[P_\gamma, P_\beta]$

$$\Rightarrow [P_\gamma, u]=\sum_\beta \frac{\partial u}{\partial Q_\beta}[P_\gamma, Q_\beta] \Rightarrow [u, P_\gamma]=\sum \frac{\partial u}{\partial Q_\beta}[Q_\beta, P_\gamma]$$

$$=\sum_\beta \frac{\partial u}{\partial Q_\beta}\delta_{\beta\gamma}=\frac{\partial u}{\partial Q_\gamma}.$$

Making use of the above results, we at once obtain [from eqyation (3)]

$$[u, v]_{q,p}=\sum_\gamma\left(\frac{\partial u}{\partial Q_\gamma}\frac{\partial v}{\partial P_\gamma}-\frac{\partial u}{\partial P_\gamma}\frac{\partial v}{\partial Q_\gamma}\right)=[u, v]_{Q,P}.$$

$\Rightarrow$ *Poisson bracket is invariant under a C.T.*

6·2-5. Relation between Lagrange's and Poisson Brackets.

Let u_i, $i=1, 2, ..., 2n$ be $2n$ independent functions of $2n$ independent variables q's and p's and conversely q's and p's are functions of u_i $(i=1, 2, ..., 2n)$.

We now proceed with the following assumption.

$$\sum_{i=1}^{2n}\{u_i\, u_j\}[u_i, u_k]=\delta_{jk}. \quad ...(1)$$

Expanding the sum [for the validity of 1], we have

$$\sum_{j=1}^{2n}\{u_i, u_j\}[u_i, u_k]=\sum_{i=1}^{2n}\sum_{l=1}^{n}\sum_{m=1}^{n}\left(\frac{\partial q_l}{\partial u_i}\frac{\partial p_l}{\partial u_j}-\frac{\partial q_l}{\partial u_j}\frac{\partial p_l}{\partial u_i}\right)$$

$$\times\left(\frac{\partial u_i}{\partial q_m}\frac{\partial u_k}{\partial p_m}-\frac{\partial u_k}{\partial q_m}\frac{\partial u_i}{\partial p_m}\right). \quad ...(2)$$

The first term out of the four terms in (2), may be written (*after changing the order of summation*) as

$$\sum_{l,m}^{2n}\frac{\partial p_l}{\partial u_j}\frac{\partial u_k}{\partial p_m}\cdot\sum_{i=1}^{2n}\frac{\partial q_l}{\partial u_i}\frac{\partial u_i}{\partial q_m}. \quad ...(3)$$

But q's are independent, so we have $\sum_{i=1}^{2n}\frac{\partial q_l}{\partial u_i}\frac{\partial u_i}{\partial q_m}=\delta_{lm}$.

Thus the expression in (3) is reduced to

$$\sum_{l,m}\frac{\partial p_l}{\partial u_j}\frac{\partial u_k}{\partial p_m}\delta_{lm}=\sum_m\frac{\partial u_k}{\partial p_m}\frac{\partial p_m}{\partial u_j}. \quad ...(4)$$

For the last term, we again repeat the same procedure and thus obtain.

$$\sum_{l,m} \frac{\partial q_l}{\partial u_j} \cdot \frac{\partial u_k}{\partial q_m} \sum_{i=1}^{2n} \frac{\partial p_l}{\partial u_i} \frac{\partial u_i}{\partial p_m} = \sum_{l,m} \frac{\partial q_l}{\partial q_j} \frac{\partial u_k}{\partial q_m} \delta_m = \sum_m \frac{\partial q_m}{\partial u_j} \frac{\partial u_k}{\partial q_m}. \quad ...(5)$$

Similarly, for the second and the third terms, we have

$$-\sum_{l,m} \frac{\partial q_l \partial u_k}{\partial u_j \partial p_m} \sum_{i=1}^{2n} \frac{\partial p_l}{\partial u_i} \frac{\partial u_i}{\partial q_m} \text{ and } -\sum_{l,m} \frac{\partial p_l}{\partial u_j} \frac{\partial u_k}{\partial q_m} \sum_{i=1}^{2n} \frac{\partial q_l}{\partial u_i} \cdot \frac{\partial u_i}{\partial p_m}. \quad ...(6)$$

But $\displaystyle\sum_{i=1}^{2n} \frac{\partial p_i}{\partial u_i} \frac{\partial u_i}{\partial q_m} = \frac{\partial p_l}{\partial q_m}$ and $\displaystyle\sum_{i=1}^{2n} \frac{\partial q_i}{\partial u_i} \frac{\partial u_i}{\partial p_m} = \frac{\partial q_l}{\partial p_m}$

so both these vanish. This implies that the second and third terms do not contribute anything.

Now, $$\sum_{m=1}^{n} \left(\frac{\partial u_k}{\partial p_m} \frac{\partial p_m}{\partial u_j} + \frac{\partial u_k}{\partial q_m} \frac{\partial q_m}{\partial u_j} \right) = \frac{\partial u_k}{\partial u_j}. \quad ...(7)$$

[adding the first and last]

Thus implying $\displaystyle\sum_{i=1}^{n} \{u_i, u_j\}, [u_i, u_k] = \frac{\partial u_k}{\partial u_j} = \delta_{jk}$.

This relation holds for any arbitrary transformation from (q_i, p_i) to (q_i', p_i') even if the co-ordinates are not canonical.

Ex. 5. *Show that the transformation defined by*

$$q = \sqrt{(2P)} \sin Q, \; p = \sqrt{(2P)} \cos Q$$

is canonical [Using *Poisson Bracket*]

(Rohilkhand 1987)

Solution. The transformation is canonical,

if $$[Q, Q] = [P, P] = 0 \text{ and } [Q, P] = 1,$$

But $$\tan Q = (d/p),\; 2P = q^2 + p^2,$$

$$\sec^2 Q \frac{\partial Q}{\partial q} = \frac{1}{p}, \frac{\partial P}{\partial q} = q \text{ ; and } \sec^2 Q \frac{\partial Q}{\partial p} = -\frac{q}{p^2}, \frac{\partial P}{\partial p} = p$$

$$\Rightarrow [Q, P] = \left[\frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} \right]$$

$$= \left[\frac{1}{p} \cos^2 Q \, p + \frac{q}{p^2} \cos^2 Q . q \right] = \cos^2 Q \left(1 + \frac{q^2}{p^2} \right)$$

$$= \cos^2 Q \,(1 + \tan^2 Q) = 1.$$

As other conditions are already satisfined, so we can say that the transformation defined above is canonical.

Ex. 6. *Show that the transformation*

$$Q=\log\left(\frac{1}{q}\sin p\right),\ P=q\cot p$$

is canonical. What is the generating function F_1?

(Meerut 1982 ; Rohilkhand 84)

Solution. We have

$$e^Q=\frac{1}{q}\sin p,\ P=q\cot p.$$

For P, Q to be canonical we must have

$$[P, P]=[Q, Q]=0 \text{ and } [Q, P]=1.$$

Now $$[Q, P]=\left[\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right]=1$$

$$\because\ \frac{\partial Q}{\partial q}=-\frac{1}{q},\ \frac{\partial P}{\partial p}=\cot p,\text{ and }\frac{\partial Q}{\partial p}=\cot p,\ \frac{\partial P}{\partial p}=-q\operatorname{cosec}^2 p.$$

Hence the transformation is canonical as other conditions are also satisfied.

Further let $F_1(q, Q)$ be the generating function, then we have

$$\frac{\partial F_1}{\partial q}=p,\ \frac{\partial F_1}{\partial Q}=-P,\ \Rightarrow\ \frac{\partial F_1}{\partial q}=\sin^{-1}(qe^Q),\ \frac{\partial F_1}{\partial Q}=-q\cot p$$

$$\Rightarrow\ F_1=\int\sin^{-1}(qe^Q)\,dq+E(Q)$$

where $E(Q)$ depends on Q only.

Again let us suppose,

$$x=\sin^{-1}(qe^Q)\Rightarrow\sin x=qe^Q\Rightarrow\cos x\,dx=e^Q\,dq\Rightarrow dq=\frac{\cos x\,dx}{e^Q}$$

$$\Rightarrow\ F_1=\int\frac{x\cos x\,dx}{e^Q}+E(Q)=e^{-Q}\left[\sin x.x-\int\sin x\,dx\right]+E(Q)$$

$$=e^{-Q}\left[qe^Q\sin^{-1}(qe^Q)+\sqrt{(1-q^2e^{2Q})}\right]+E(Q)$$

$$\Rightarrow\ F_1=e^{-Q}(1-q^2e^{2Q})^{1/2}+q\sin^{-1}(qe^Q)+E(Q).$$

The function E seems to be independent of Q as can be seen by evaluating $\dfrac{\partial F_1}{\partial Q}$. Thus the required generating function is given by

$$F_1(q, Q)=e^{-Q}(1-q^2e^{2Q})^{1/2}+r\sin^{-1}(qe^Q).$$

Ex. 7. *If the transformation equations between two sets of co-ordinates are*

$$p=1\,(1+q^{1/2}\cos p)\,q^{1/2}\sin p,\ Q=\log(1+q^{1/2}\cos p),$$

(*i*) *the transformation is canonical,*

and (*ii*) *the generating function of this transformation is*

$$F_3=-(e^Q-1)^2\tan p$$

(Rohilkhand 1985)

Sol. (i)

$$p\,dq-P\,dQ=p\,dq-2\,(1+q^{1/2}\cos p)\,q^{1/2}\sin p\ \frac{\cos p\,(dq-2q\sin p\,dp)}{2q^{1/2}\,(1+q^{1/2}\cos p)}$$

$$=p\,dq-\sin p\cos p\,dq+2q\sin^2 p\,dp$$
$$=(p-\tfrac{1}{2}\sin 2p)\,dq+q\,(1-\cos 2p)\,dp$$
$$=d\,[q\,(p-\tfrac{1}{2}\sin 2p)].$$

This is an exact differential.

Hence the given transformation is canonical.

(ii) $Q=\log\,(1+q^{1/2}\cos p)$

$e^Q=1+q^{1/2}\cos p \Rightarrow q^{1/2}=\dfrac{e^Q-1}{\cos p} \Rightarrow q=\left\{\dfrac{e^Q-1}{\cos p}\right\}^2,$

$\Rightarrow P=2\,(1+q^{1/2}\cos p)\,q^{1/2}\sin p$ reduces to

$$\Rightarrow P=2\left[1+\frac{e^Q-1}{\cos p}\cdot\cos p\right]\frac{e^Q-1}{\cos p}\sin p=2e^Q\,(e^Q-1)\tan p. \quad \ldots(1)$$

Again, $q=-\dfrac{\partial F_3}{\partial p}$ and $P=-\dfrac{\partial F_3}{\partial Q},$

$$\Rightarrow \frac{\partial F_3}{\partial p}=-\left\{\frac{e^Q-1}{\cos p}\right\}^2=-(e^Q-1)^2\sec^2 p \quad \ldots(2)$$

Also, $\dfrac{\partial F_3}{\partial Q}=-2e^Q\,(e^Q-1)\tan p.$

$\Rightarrow F_3=-\displaystyle\int (e^Q-1)^2\sec^2 p\,dp,$ taking constant of integration to be zero

$$=-(e^Q-1)^2\tan p$$

Also $F_3=-2\displaystyle\int e^Q\,(e^Q-1)\tan p\,dQ,$

constant of integration chosen to be zero.

$$=-(e^Q-1)^2\tan p.$$

$\Rightarrow F_3=-(e^Q-1)^2\tan p.$

Ex. 8. *Find the values of α and β so that the equations*

$$Q=q^\alpha\cos\beta p,\ P=q^2\sin\beta p$$

represent a canonical transformation. Also find the generating function G_3 for this case. **(Rohilkhand 1986, 82)**

Sol. Given transformation is canonical, if

$p\,dq-P\,dQ$ is an exact differential

or if $p\,dq-q^\alpha\sin\beta p\,d\,(q^\alpha\cos\beta p)$ is an exact differential

or if $p\,dq-q^\alpha\sin\beta p\,(\beta q^{\alpha-1}\cos\beta p\,dq-\beta q^\alpha\sin\beta p\,dp)$ is an exact differential

or if $(p-\alpha q^{2\alpha-1}\sin\beta p\cos\beta p)\,dq+\beta q^{2\alpha}\sin^2\beta p\,dp.$ is an exact differential

But, $Mdx+Ndy$ is an exact differential, if we have $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$

Thus the above expression will be exact differential, if we have

$$\frac{\partial}{\partial p}(p-\alpha q^{2\alpha-1}\sin\beta p\cos\beta p)=\frac{\partial}{\partial q}(\beta q^{2\alpha}\sin^2\beta p)$$

or $\quad 1-\alpha p^{2\alpha-1}\beta(\cos^2\beta p-\sin^2\beta p)=2p\beta q^{2\alpha-1}\sin^2\beta p$

$\Rightarrow \quad 1-\alpha q^{2\alpha-1}\beta(1-2\sin^2\beta p)=2\alpha\beta q^{\alpha-1}\sin^2\beta p$

$\Rightarrow \quad 1-\alpha\beta q^{2\alpha-1}+2\alpha\beta q^{2\alpha-1}\sin^2\beta p=2\alpha\beta\, q^{2\alpha-1}\sin^2\beta p$

This is an identity, so equating the coefficients of like terms, we get

$$1-2\alpha\beta\, q^{2\alpha-1}=0 \Rightarrow 2\alpha\beta\, q^{2\alpha-1}=1.$$

This is satisfied for $\beta=2$ and $2\alpha-1=0$, *i.e.*, when $\alpha=\frac{1}{2}$.

Hence $\alpha=\frac{1}{2}$, $\beta=2$ whence to obtain F_3, we have

$$q=-\frac{\partial F_3}{\partial p} \text{ and } P=-\frac{\partial F_3}{\partial Q}.$$

Here $Q=q^\alpha\cos\beta d \Rightarrow q=\left\{\dfrac{Q}{\cos 2p}\right\}^{1/2}$ $(\because \alpha=\frac{1}{2}, \beta=2)$.

Also, $P=q^{1/2}\sin 2q=\dfrac{Q}{\cos 2p}.\sin 2p=Q\tan 2p.$

$$\Rightarrow \quad \frac{\partial F_3}{\partial p}=-\left(\frac{Q}{\cos 2p}\right)^{1/2} \text{ and } \frac{\partial F_3}{\partial Q}=-Q\tan 2p,$$

$$\Rightarrow \quad F_3=-\int\left(\frac{Q}{\cos 2p}\right)^{1/2}dp \text{ and also } F_3=-\int Q\tan 2p\, dQ,$$

$$\Rightarrow \quad F_3=-\tfrac{1}{2}Q^2\tan 2p.$$

Ex. 9. *Show that the generating function for the transformation*

$$p=\frac{1}{Q},\ q=PQ^2 \text{ is } G=\frac{q}{Q}.$$

Sol. We have $\quad p=\dfrac{\partial G}{\partial q},\quad P=-\dfrac{\partial G}{\partial Q}.$ ...(1)

Here $\qquad p=\dfrac{1}{Q}$ and $P=\dfrac{q}{Q^2}$.

$\therefore$ (1) $\Rightarrow \quad \dfrac{1}{Q}=\dfrac{\partial G}{\partial q}$ and $\dfrac{q}{Q^2}=-\dfrac{\partial G}{\partial Q}$.

$\Rightarrow \quad G=\displaystyle\int\frac{dq}{Q}$ and also $G=-\displaystyle\int q\frac{dQ}{Q^2}$ choosing constants of integration to be zero.

Both of these integrals give $G=\dfrac{q}{Q}$.

Ex. 10. *Show that* $\sum_i q_i Q_i$ *generates the exchange transformation in which position co-ordinates and momenta are interchanged.*

Sol. The generating function G is of the first type as

$$G=G_1(q_j, Q_j, t)$$

and then we have $p_j = \frac{\partial G_1}{\partial q_j}$, $P_j = -\frac{\partial G_1}{\partial Q_j}$. ...(1)

Now, when $G_1 = q_j Q_j$, we obtain $\frac{\partial G_1}{\partial Q_j} = Q_j$, and $\frac{\partial G_1}{\partial Q_j} = q_j$. ...(2)

From here, it is clear that

$$p_j = Q_j \text{ and } P_j = -q_j,$$

and hence the position co-ordinates and momenta are interchanged.

Ex. 11. *Show that* $G_3 = -\sum_i Q_j P_j$ *generates the identity transformation.*

Sol. We have

$$q_j = -\frac{\partial G_3}{\partial p_j} \text{ and } P_j = -\frac{\partial G_3}{\partial Q_j}, \text{ and } \bar{H} - H = \frac{\partial G_2}{\partial t} \quad ...(1)$$

and from the given generating function, we have

$$-\frac{\partial G_3}{\partial p_j} = Q_j \text{ and } -\frac{\partial G_3}{\partial Q_j} = p_j \text{ and also } \frac{\partial G_3}{\partial t} = 0. \quad (2)$$

From equations (1) and (2), we have

$$q_j = Q_j \text{ and } P_j = p_j, \text{ also } \bar{H} = H.$$

Thus the new and old position co-ordinates and momenta are similar. Hence, the given function generates idendity transformation.

6·2-6. Equations of motion in Poisson Bracket notation.

Consider a function $F(q, p, t)$ depending on co-ordinates, momenta and time. Then we have

$$(dF/dt) = (\partial F/\partial t) + (\partial F/\partial p)\, \dot{p} + (\partial F/\partial q)\, \dot{q}$$
$$= (\partial F/\partial t) + (\partial F/\partial p)\,(-\partial H/\partial q) + (\partial F/\partial q)\,(\partial H/\partial p)$$
$$[\because \dot{q} = (\partial H/\partial p) \text{ and } \dot{p} = -(\partial H/\partial q)] \quad ...(1)$$
$$= (\partial F/\partial t) + [F, H].$$

But we know that the functions which depend on dynamical variables and remain unaltered during the course of motion of the system are called *integrals of motion*. This shows that the function F is an integral of motion, if ...(2)

$$(dF/dt) = 0 \Rightarrow (\partial F/\partial t) + [F, H] = 0$$

Further, if F is not related with time explicitly, we have

$$(\partial F/\partial t) = 0 \Rightarrow [F, H] = 0,$$

$\Rightarrow$ *that the function F, not related with time explicitly, is an integral of motion, if its P.B. (with H) vanishes.*

Again putting H for F in (1), we get ...(3)

$$(dH/dt) = (\partial H/\partial t) + [H, H] = (\partial H/\partial t).$$

6·2-8. Generator of translatory motion.

If the Hamiltonian H contains an ignorable co-ordinate q_α then
$$(\partial H/\partial q_\alpha)=0.$$

Further, let there exist an infinitesimal displacement
$$\delta q_\alpha=\delta t=\epsilon \text{ say in } q_\alpha$$
H will not be affected and remain invariant. Under this condition we have
$$\delta q_\alpha=Q_\alpha-q_\alpha=(\partial K/\partial p_\alpha)\ \epsilon=\epsilon \Rightarrow (\partial K/\partial p_\alpha)=1$$
and
$$\delta p_\alpha=P_\alpha-p_\alpha=(-\partial K/\partial q_\alpha)\ \epsilon=0.$$

From above it is obvious that if $K=p_\alpha$, only then
$$\delta q_\alpha=\epsilon \text{ and } \delta p_\alpha=0.$$

Thus $K=p_\alpha$ satisfies the above equations of transformation which is identified with the momentum conjugate to the variable q_α. If in particular, we take $q_\alpha=x$ *i.e.* $\delta q_\alpha=\delta x=\epsilon$: also $K=p_\alpha$. Thus the I.C.T. generated by the x-component of the momentum corresponds to a translatory motion of the co-ordinate system parallel to itself in negative direction of x-axis by an amount ϵ.

Also $\delta p_\alpha=0 \Rightarrow$ that the corresponding conjugate momentum is unchanged.

6·2-9. Contact transformation possesses the group property.

Let $(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n)$ be a set of $2n$ variables, and let $(P_1, P_2, \ldots, P_n)$, $(Q_1, Q_2, \ldots, Q_n)$ be $2n$ variables which are defined in terms of the old ones. If the equations connecting the two sets of variable are such that the differential form.
$$\left[\sum_{\alpha=1}^{n} P_\alpha\, dQ_\alpha-\sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha\right]=\text{Perfect differential of a function of}$$
$(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$ then the change from the set of variables $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$ to the other set $(P_1, P_2, \ldots, P_n;$ $Q_1, Q_2, \ldots, Q_n)$ is called a *contact transformation.*

This definition is different from the definition which is most convenient when contact transformations are studied with a view to the applications governing the theory of partial differential equation and geometry. The latter difinition runs as follows ; a contact transformation is transformation from a set of (2n+1) variables
$$(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, z)$$
to another set
$$(P_1, P_2, \ldots, P_n; Q_1, Q_2, \ldots, Q_n, Z)$$
for which the equation

$$dZ-\sum_{\alpha=1}^{n} P_\alpha\, dQ_\alpha=\rho\left[dz-\sum_{\alpha=1}^{n} p_\alpha\, dq_\alpha\right]$$

is satisfied where ρ denotes some function of (p, q, z).

(*i*) From the definition, it is obvious that the result of performing two contact transformations in succession is to obtain a change of variable which is itself a contact transformation.

(*ii*) Out of the whole set, a unique contact transformation exists such that the result of performing this contact transformation with any of the other contact transformations gives the same contact transformation. This is called the **identity**.

(*iii*) It is also evident that if the transformation from (p, q) to (P, Q) is a contact transformation, then the transformation from (P, Q) to (p, q) is also a contact transformation. **Inverse.**

(*iv*) Associate law also holds good.

Hence the set of contact transformation possesses the group property.

6·2-10. Point transformations and extended point transformations.

If the n variables $(Q_1, Q_2, \ldots, Q_n)$ are functions of $(q_1, q_2, \ldots, q_n)$ only, the contact transformation from the variable $(p_1, p_2, \ldots, p_n\,;\, q_1, q_2, \ldots, q_n)$ to the variables $(P_1, P_2, \ldots, P_n\,;\, Q_1, Q_2, \ldots, Q_n)$ is called an extended point transformation and the equations which connect $(q_1, q_2, \ldots, q_n)$ with $(Q_1, Q_2, \ldots, Q_n)$ are said to define a point transformation.

6·2-11. To obtain an analytical expression for a contact transformation.

Let the transformation from variables $(p_1, p_2, \ldots, p_n\,;\, q_1, q_2, \ldots, q_n)$ to the variables $(P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n)$ be a contact transformation, so that

$$\sum_{\alpha=1}^{n} (P_\alpha\, dQ_\alpha - p_\alpha\, dq_\alpha)=dW=\text{complete differential} \qquad \ldots(A)$$

But $Q_1, Q_2, \ldots, Q_n\,;\, P_1, P_2, \ldots, P_n$ are functions of $p_1, p_2, \ldots, p_n\,;\, q_1, q_2, \ldots, q_n$ so it is possible to eliminate $(P_1, P_2, \ldots, P_n\,;\, p_1, p_2, \ldots, p_n)$ completely. Let such relations be *i.e.* $\Omega_\alpha(q_1, q_2, \ldots, q_n\,;\, Q_1, Q_2, \ldots, Q_n)=0 \qquad (\alpha=1, 2, \ldots, k)$ $\ldots$(i)

Differentiating (i), we get

$$d\Omega_\alpha=\frac{\partial \Omega_\alpha}{\partial q_1}\, dq_1+\ldots+\frac{\partial \Omega_\alpha}{\partial Q_n}\, dq_n=\frac{\partial \Omega_\alpha}{\partial Q_1}\, dQ_1+\ldots+\frac{\partial \Omega_\alpha}{\partial Q_n}\, dQ_n=0 \qquad \ldots(ii)$$

$(\alpha=1, 2, \ldots, k)$

But the variations $(dq_1, dq_2, \ldots, dq_n, dQ_1, dQ_2, \ldots, dQ_n)$ in the equation (A) are conditioned by (1), so we must have

$$\left.\begin{aligned} P_\alpha &= \frac{\partial W}{\partial Q_\alpha} + \lambda_1 \frac{\partial \Omega_1}{\partial Q_\alpha} + \ldots + \lambda_k \frac{\partial \Omega_k}{\partial Q_\alpha} \\ p_\alpha &= -\frac{\partial W}{\partial q_\alpha} - \lambda_1 \frac{\partial \Omega_k}{\partial q_\alpha} - \ldots - \lambda_k \frac{\partial \Omega_k}{\partial q_\alpha} \end{aligned}\right\} \quad (\alpha = 1, 2, \ldots, n) \qquad \ldots(B)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are underdetermined multipliers and $W = W(q_1, q_2, \ldots, q_n ; Q_1, Q_2, \ldots, Q_n)$. Equations (1) and (B) are $2n+k$ equations to determine the $(2n+k)$ quantities *viz.*, $(Q_1, Q_2, \ldots, Q_n; P_1, P_2, \ldots, P_n ; \lambda_1, \lambda_2, \ldots, \lambda_k)$ in terms of p's and q's. Hence these equations may be regarded as explicitly formulating the contact transformations, in terms of the functions $(W, \Omega_1, \Omega_2, \ldots, \Omega_k)$ which characterise the transformation.

Converse. Let $(W, \Omega_1, \Omega_2, \ldots, \Omega_k)$ be any $(k+1)$ functions of the variables $(q_1, q_2, \ldots, q_n ; Q_1, Q_2, \ldots, Q_n)$, $k < n$ and furthermore if $(P_1, P_2, \ldots, P_n ; Q_1, Q_2, \ldots, Q_n ; \lambda_1, \lambda_2, \ldots, \lambda_k)$ are defined in terms of $(p_1, p_2, \ldots, p_n ; q_1, q_2, \ldots, q_n)$ by the following equations :

$$P_\alpha = \frac{\partial W}{\partial Q_\alpha} + \lambda_1 \frac{\partial \Omega_1}{\partial Q_\alpha} + \ldots + \lambda_k \frac{\partial \Omega_k}{\partial Q_\alpha} \quad (\alpha = 1, 2, \ldots, n)$$

$$p_\alpha = -\frac{\partial W}{\partial q_\alpha} - \lambda_1 \frac{\partial \Omega_1}{\partial q_\alpha} - \ldots - \lambda_k \frac{\partial \Omega_k}{\partial q_\alpha} \quad (\alpha = 1, 2, \ldots, n)$$

$$\Omega_\alpha (q_1, q_2, \ldots, q_n, Q_2, \ldots, Q_n) = 0 \quad (\alpha = 1, 2, \ldots, k)$$

then the transformation (p, q) to (P, Q) is a contact transformation for the expression $\sum_{\alpha=1}^{n} (P_\alpha \, dQ_\alpha - p_\alpha \, dq_\alpha)$ becomes dW, in virtue of these equations and so is a perfect differential.

6·2·12. Sub-groups of Mathieu transformation and extended point transformations.

If with in a group of transformations, there exist a set of transformations such that the result of performing in succession two transformations of the set is always equivalent to a transformation which also belongs to the set, this set of trarsformations is said to form a sub-group of the group.

A sub-group of the general group of contact-transformations is evidently constituted by those transformation, for which the equation $\sum_{\alpha=1}^{n} P_\alpha \, dQ_\alpha = \sum_{\alpha=1}^{n} p_\alpha \, dq_\alpha$ is satisfied. Such transformations have been studied by Mathieu. *These are essentially the same as* the transformations called "*Homogeneous contact transformations*" in $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$) by Lie. In this case by art 6·2·11, we immediately obtain.

$$\Omega_\alpha (q_1, q_2, \ldots, q_n, Q_1, Q_2, \ldots, Q_n)=0 \qquad (\alpha=1, 2, \ldots, k)$$

$$P_\alpha=\lambda_1 (\partial\Omega_1/\partial Q_\alpha)+\lambda_2 (\partial\Omega_2/\partial Q_\alpha)+\ldots+\lambda_k (\partial\Omega_k/\partial Q_\alpha)$$

$$(\alpha=1, 2, \ldots, n)$$

$$p_\alpha=-\lambda_1 (\partial\Omega_1/\partial q_\alpha)-\lambda_2 (\partial\Omega_2/\partial q_\alpha)\ldots-\lambda_k (\partial\Omega_k/\partial q_\alpha)$$

$$(\alpha=1, 2, \ldots, n)$$

From the form of these equations, it is evident that if ($p_1, p_2, \ldots, p_n$) are each multiplied by any quantity μ, the effect is to multiply ($P_1, P_2, \ldots, P_n$) each by μ ; and therefore ($P_1, P_2, \ldots P_n$) must be homogeneous of the first degree (though not necessarily integral) in ($p_1, p_2, \ldots, p_n$). A sub group with in the group of Mathieu transformations is constituted by those transformations for which ($P_1, P_2, \ldots, P_n$) are not only homogeneous of the first degree in ($p_1, p_2, \ldots, p_n$) but also integral, *i.e.*, linear in them; so that we have equations of the form $P_\alpha=\sum_{k=1}^{n} p_k f_{\alpha k} (q_1, q_2, \ldots, q_n)$

$$(\alpha=1, 2, \ldots, n)$$

Substituting in the equation $\sum_{\alpha=1}^{n} P_\alpha \, dQ_\alpha-\sum_{\alpha=1}^{n} p_\alpha \, dq_\alpha=0$ and then equating the coefficient of p_k, we immediately obtain

$$\sum_{\alpha=1}^{n} f_{\alpha k} (q_1, q_2, \ldots, q_n) \, dQ_\alpha=dq_k \qquad (k=1, 2, \ldots, n)$$

$\Rightarrow$ ($q_1, q_2, \ldots, q_n$) are functions of ($Q_1, Q_2, \ldots, Q_n$) only and $f_{\alpha k}=(\partial q_k/\partial Q_\alpha)$. $\qquad (\alpha, k=1, 2, \ldots n)$

Thus it follows that transformations of this kind are obtained by assigning "n" arbitrary relations connecting the variables ($q_1, q_2, \ldots, q_n$) *with the variables* ($Q_1, Q_2, \ldots, Q_n$) *and then determining* ($P_1, P_2, \ldots, P_n$) *from the equations* $P_\alpha=\sum_{\alpha=1}^{n} p_k (\partial q_k/\partial Q_\alpha)$

$$(\alpha=1, 2, \ldots, n)$$

Obviously these transformations are extended point-transformations.

SUPPLEMENTARY PROBLEMS

1. Prove that the transformation

$$q=\sqrt{\left(\frac{P}{\sqrt{\kappa}}\right)} \sin Q, \quad p=\sqrt{(mP\sqrt{\kappa})} \cos Q$$

is canonical and show that its generating function is $\frac{1}{2}\sqrt{\kappa} q^2 \cot Q$.

2. Prove that the following transformations are canonical ;
(i) when $Q=p$, $P=-q$; (Rohilkhand 1979)
(ii) when $Q=q \tan p$, $P=\log \sin P$. (Rohilkhand 1979)
3. Show that $G_2=\sum_j p_j Q_j$ generates the inversion of phase space i.e. both the new momenta and co-ordinates are the old momenta and co-ordinates with signs reversed.
4. Write note on canonical transformations. (Agra 1979)
5. Define canonical transformations and obtain the transformation equations corresponding to all possible generating functions. (Rohilkhand 1979)
6. Define the canonical transformations. (Rohilkhand 1979)
7. The Hamiltonian of a dynamical system in the (q, p) basis is given by

$$H=\tfrac{1}{2}(p_1^2+p_2^2+\alpha^{-2} q_1^2+\alpha^{-2} q_2^2)$$

A transformation to the new basis (P, Q) is made through

$$Q_1=b_1^2+\alpha^2 p_1^2, \qquad Q_2=\frac{1}{2\alpha^2}(q_1^2+a_2^2+\alpha^2 p_1^2+\alpha^2 p_2^2)$$

$$2\alpha P_1=\tan^{-1}\left(\frac{q_1}{\alpha p_1}\right)-\tan^{-1}\left(\frac{q_2}{\alpha p_1}\right), \quad P_2=\alpha \tan^{-1}\left(\frac{q_2}{\alpha p_2}\right)$$

Show that the transformation is canonical and that the Hamiltonian in the new co-ordinate system is Q_2.

8. Show that the canonical transformation defined by the set of equations

$$q_1=\sqrt{\left(\frac{2Q_1}{k_1}\right)} \cos P_1+\sqrt{\left(\frac{2Q_2}{k_2}\right)} \cos P_2$$

$$q_2=-\sqrt{\left(\frac{2Q_1}{k_1}\right)} \cos P_1+\sqrt{\left(\frac{2Q_2}{k_2}\right)} \cos P_2$$

$$p_1=\tfrac{1}{2}\sqrt{(2k_1Q_1)} \sin P_1+\tfrac{1}{2}\sqrt{(2k_2Q_2)} \sin P_2$$

$$p_2=-\tfrac{1}{2}\sqrt{(2K_1Q_1)} \sin P_1+\tfrac{1}{2}\sqrt{(2K_2Q_2)} \sin P_2$$

changes the Hamiltonian

$$H=p_1^2+p_2^2+\tfrac{1}{8}k_1^2 (q_1-q_2)^2+\tfrac{1}{8}k_2^2 (q_1+q_2)^2 \text{ to } H'=k_1Q_1+k_2Q_2.$$

9. Show that the transformation

$$Q=(2q)^{1/2} k^{-1/2} \cos p, \quad P=(2q)^{1/2} k^{1/2} \sin p$$

is canonical and that the generating function $F_1(q, Q)$ is

$$F_1(q, Q)=\tfrac{1}{2}Q\,(2qk-k^2Q^2)^{1/2}-q \cos^{-1} Q\sqrt{\left(\frac{k}{2q}\right)}.$$

10. Show that the transformation from the cartesian to spherical co-ordinates

$$x=r \sin\theta \cos\phi, \; y=r \sin\theta \sin\phi, \; z=r\cos\theta$$

is generated by the function

$$F_3(q, Q)=-(r \sin\theta \cos\phi\, p_x+r \sin\theta \sin\phi\, p_y+r \cos\theta\, p_z)$$

and obtain the Hamiltonian in the spherical co-ordinates for a particle moving in a spherically symmetric potential $V(r)$ from

$$H(x, y, z, p_x, p_y, p_z)=\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V\{\sqrt{(x^2+y^2+z^2)}\}$$

11. If the Hamiltonian for a particle of charge 'e' and mass m moving in a electromagnetic field is given by

$$H=\frac{1}{2m}(p-eA)^2+e\phi$$

discuss the poission brackets for motion.

12. If the canonical variables are not all independent, but are connected by auxiliary conditions of the form

$$\psi_k(q_j, p_i, t)=0.$$

Show that the canonical equations of motion can be written as

$$\frac{\partial H}{\partial p_j}+\sum_k \lambda_k \frac{\partial \psi_k}{\partial p_k}=\dot{q}_j,\ \frac{\partial H}{\partial q_j}+\sum_k \lambda_k \frac{\partial \psi_k}{\partial q_j}=-\dot{p}_j$$

where λ_k are undermined multipliers.

13. Show that if the Hamiltonian and a quantity G are constants of the motion then $\frac{\partial G}{\partial t}$ must also be a constant,

14. Write note on Poisson brackets. (*Agra 1972, 74*)

6·3. (C) Hamilton-Jacobi Theory.

6·3-1. Hamilton-Jacobi equation.

Consider a dynamical system with Homiltonian $H(q_\alpha, p_\alpha\ t)$. The Hamilton's canonical equations are then

$$\dot{q}_\alpha=(\partial H/\partial p_\alpha),\ \dot{p}_\alpha=(-\partial H/\partial q_\alpha). \quad ...(1)$$

And if we now make the canonical transformation from $q-p$ basis to $Q-P$ basis where the new Hamiltonian is J, then the canonical equations is new co-ordinates are given by

$$\dot{P}_\alpha=(-\partial J/\partial Q_\alpha),\ \dot{Q}_\alpha=(\partial J/\partial P_\alpha) \quad ...(2)$$

and $J=H+(\partial K/\partial t)$, where K is the generating function.

In particular case, when $J\equiv 0$, we get

$\dot{P}_\alpha=0,\ \dot{Q}_\alpha=0 \Rightarrow P_\alpha=\text{constant}=a_\alpha$ say; $Q_\alpha=\text{constant}=-b_\alpha$ say.

Hence P_α, Q_α are ignorable co-ordinates.

Also, we have $H+(\partial K/\partial t)=0 \Rightarrow (\partial K/\partial t)+H\{q_\alpha, p_\alpha\ t\}=0$

$\Rightarrow (\partial K/\partial t)+H\{(\partial K/\partial q_\alpha), q_\alpha, t\}=0$

$(\because\ p_\alpha=(\partial K/\partial q_\alpha)$ theorem 2, P. 275)$\}$

$\Rightarrow (\partial K/\partial t)+H\{(\partial K/\partial q_1), (\partial K/\partial q_2),..., (\partial K/\partial q_n)\ q_1, q_2,..., q_n; t\}=0$ ...(3)

This equation is known as *Hamilton-Jacobi equation.*

Solution of Hamilton-Jacobi equation and Hamilton's principal function.

Given the Hamiltonian $H(p_\alpha, q_\alpha, t)$ of a dynamical system, we can at once write down the Hamilton Jacobi equation

$$(\partial K/\partial t)+H\{(\partial K/\partial q_\alpha), q_\alpha, t\}=0,\ H=-(\partial K/\partial t).$$

Since no derivatives higher than the first order occur, it is an equation of the first order. But it is not a linear equation, because the derivatives occur in degrees higher than the first.

Let $K=K(q_\alpha, t, a_\alpha)+c$ be a complete integral (3); here a_α stands for a set of n constants $(a_1, a_2, \ldots, a_n)$ and c for a single constant. Such a solution of Hamilton-Jacobi equation, denoted by K, is known as Hamilton's characteristic or principal function.

Now write down the equations

$$(\partial K/\partial q_\alpha)=p_\alpha \quad \text{and} \quad (\partial K/\partial a_\alpha)=-b_\alpha=Q_\alpha, \qquad \ldots(4)$$

where b_α stands for a new of n constants.

The n equations $(\partial K/\partial a_\alpha)=-b_\alpha$ constants; $(n+1)$ variables (q_α, t) and the $2n$ constants (a, b) when solved for the q's they give a motion for each choice of the constants.

Then n equations $(\partial K/\partial q_\alpha)=p_\alpha$ give momenta associated with the motion.

We have $K=K(q_1, q_2, \ldots, q_n; a_1, a_2, \ldots, a_n; t)$.

$$\therefore \quad \frac{dK}{dt}=\frac{\partial K}{\partial t}+\sum_{\alpha=1}^{n}\frac{\partial K}{\partial q_\alpha}\dot{q}_\alpha+\sum_{\alpha=1}^{n}\frac{\partial K}{\partial a_\alpha}\dot{a}_\alpha=-H\sum_{\alpha=1}p_\alpha\dot{q}_\alpha=L$$

$\Rightarrow \quad K=\int L\,dt+\text{constant}$ which identifies K with Hamiltonian principal function. This is the reason, why K is known as Hamilton's characteristic or principal function.

Now we are in a position to prove the following theorem :

***Theorem.** *Let K be any complete integral of the Hamilton-Jacobi equation, then each motion, with associated momenta, given by* (3) *is a natural motion, satisfying the canonical equations*

$$\dot{q}_\alpha=(\partial H/\partial p_\alpha), \quad \dot{p}_\alpha=-(\partial H/\partial q_\alpha).$$

Proof. Differentiating (3) partially w.r.t. a_α and q_β respectively

$$\frac{\partial^2 K}{\partial q_\alpha \partial t}+\sum_{\beta=1}^{n}\frac{\partial H}{\partial p_\beta}\frac{\partial^2 K}{\partial a_\alpha\,\partial q_\beta}=0. \qquad \ldots(4)$$

and

$$\frac{\partial^2 K}{\partial q_\alpha\,\partial t}+\frac{\partial H}{\partial q_\alpha}+\sum_{\beta=1}^{n}\frac{\partial H}{\partial q_\beta}\frac{\partial^2 K}{\partial q_\alpha\,\partial q_\beta}=0. \qquad \ldots(5)$$

We also have $(\partial K/\partial a_\alpha)=-b_\alpha$ which defines the q's as functions of t, and t is abitrary.

*This gives the importance to complete integrals of the Hamilton-Jacobi equation.

Differentiating w.r.t. "t", we get

$$\frac{\partial^2 K}{\partial t\,\partial a_\alpha}+\sum_{\beta=1}^{n}\frac{\partial^2 K}{\partial q_\beta\,\partial a_\alpha}\dot{q}_\beta=0. \quad \text{...(7)}$$

which give n linear equations.

These can be solved for $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ and obviously the above n equations are precisely the same in form as the n linear equations (5) which we can solve for

$\{(\partial H/\partial p_1), (\partial H/\partial p_2), \ldots, (\partial H/p_n)\}$. Thus $\dot{q}_\alpha=(\partial H/\partial p_\alpha)$. ...(8)

Again differentiating $(\partial K/\partial q_\alpha)=p_\alpha$ w.r.t. "t", we get

$$\frac{\partial^2 K}{\partial t\,\partial q_\alpha}+\sum_{\beta=1}^{n}\frac{\partial^2 K}{\partial q_\beta\,\partial q_\alpha}\dot{q}_\beta=\dot{p}_\alpha.$$

Now using (6) and (8), we get $\dot{p}_\alpha=-(\partial H/\partial q_\alpha)$.

This completes the proof of the theorem.

6·3-2. Hamilton-Jacobi equation for Hamilton's characteristic function.

When Hamiltonian does not depend explicity on time, the Hamilton Jacobi equation is given by $(\partial K/\partial t)+H\{q_\alpha, (\partial K/\partial q)\}=0$ ...(9)

where the first term is a function of t only, while the second one is a function of q_α and K only. Let its solution be

$$K(q_\alpha, a_\alpha, t)=D\,(q_\alpha, a_\alpha)-tC \quad \text{...(10)}$$

where C is a constant, equal to the total energy of the system and D is the action connected with the principal function K.

Now (9) and (10) $\Rightarrow$ $H\{q_\alpha, (\partial D/\partial q_\alpha)\}=C$ ...(11)

where $D=D\,(q_\alpha, a_\alpha)$, $p_\alpha=(\partial D/\partial q_\alpha)$. ...(12)

It can be shown very easily that D separately generates its own canonical transformation whese properties are quite different from those generated by K. Now consider $F(q, P)$ to be the generating function for that canonical transformation in which C is the constant of motion and the new momenta are all constants of the motion a_α, then the equations of transformations

$$\Rightarrow\ p_\alpha=(\partial D/\partial p_\alpha),\ Q_\alpha=-b_\alpha=(\partial D/\partial P_\alpha)=(\partial D/\partial a_\alpha).$$

Combining these equations we observe that D can be determined if and only if $H(q_\alpha, p_\alpha)=C$. ...(13)

The function D introduced over here is known as *Hamilton's characteristic function.* This is the generator of canonical transformation having all the new co-ordinates cyclic.

Now we have $\dot{P}_\alpha=(\partial J/\partial Q_\alpha)=0$, $P_\alpha=a_\alpha$ i.e. the canonical equations P_α imply that the conjugate momenta to the cyclic co-ordinates are all constants. Further K is the function of q_α, t and n arbitrary constants a_α. Hence using (20) and (21), we can say that constant C introduced in (19) cannot be independent of a_α, but it should be some function of a_α i.e. $C=C\,(a_\alpha)$ or in general form

$$C=\sum_{\alpha=1}^{n} C\,(a_\alpha). \quad \ldots(14)$$

Now putting $C=a_\alpha$ in (10), we get

$$K\,(q_\alpha, a_\alpha, t)=D\,(q_\alpha, a_\alpha)-a_\alpha t$$

$$\Rightarrow (\partial K/\partial a_\alpha)=(\partial D/a_\alpha)-t \text{ i.e. } (\partial D/\partial a_\alpha)=(\partial K/\partial a_\alpha)+t=t-b_\alpha \quad \ldots(15)$$

These n equations obtained by (15) describe parametrically the path of the system as a function of time. When the particular point of the path is considered, we may take $C=a_\alpha$ and hence the equations (15)

$$\left.\begin{aligned} \Rightarrow (\partial D/\partial a_\alpha)&=Q_\alpha=-b_\alpha\ (\alpha=2, 3, 4, \ldots n) \\ (\partial D/\partial a_1)&=Q_1=t-b_1 \qquad [\alpha=1] \end{aligned}\right\}$$

Here the first $(n-1)$ equations describe the path whereas the last one $\Rightarrow$ that what point on the path is occupied at any given instant of time.

6·3-3 *The Hamiltonian being given by* $H=(1/2m)\{p_r^2+(p_\theta^2/r^2)\}-(\lambda/r)$, *use Hamilton Jacobi theory to solve Kepler's problem for a particle in an inverse square central force field.* **[Rohilkhand 89]**

We have $H=(1/2m)\{p_r^2+(p_\theta^2/r^2)\}-(\lambda/r)$. If K is the generating function of the transformation, then we have $p_r=(\partial K/\partial r)$, $p_\theta=(\partial K/\partial\theta)$ while new Hamiltonian J is zero if $H+(\partial K/\partial t)=0$. Hence the Hamiltonian Jacobi equation is

$$(\partial K/\partial t)+(1/2m)\{(\partial K/\partial r)^2-(1/r^2)(\partial K/\partial\theta)^2\}-(\lambda/r)=0. \quad \ldots(16)$$

Let its solution be $K\,(r, \theta, t)=D_1\,(r)+D_2\,(\theta)+D_3\,(t)$.

$$\therefore\ (\partial K/\partial t)=(\partial D_3/\partial t),\ (\partial K/\partial r)=(\partial D_1/\partial r),\ (\partial K/\partial\theta)=(\partial D_2/\partial\theta).$$

$$\therefore\ (1) \Rightarrow (\partial D_3/\partial t)+(1/2m)\{(\partial D_1/\partial r)^2+(1/r^2)(\partial D_2/\partial\theta)^2\}-(\lambda/r)=0$$

i.e. $$\underbrace{(1/2m)\{(\partial D_1/\partial r)^2+(1/r^2)\,\partial D_2/\partial\theta)^2\}-(\lambda/r)}_{\text{function of } r,\ \theta}=\underbrace{-(\partial D_3/\partial t)}_{\text{function of time}}.$$

This relation will hold good for all values of the variables only it the two sides reduce to a constant value, say a.

$$\therefore\ (\partial D_3/\partial t)=-a \Rightarrow D_3=-at.$$

Also $$(1/2m)\{(\partial D_1/\partial r)^2+(1/r^2)(\partial D_2/\partial\theta)^2\}-(\lambda/r)=a$$

$$\Rightarrow \ (\partial D_2/\partial\theta)^2 = r^2 \, \{2ma + 2m\,(\lambda/r) - (\partial D_1/\partial r)^2\}$$

$\leftarrow$function of $\theta\rightarrow$ $\quad\longleftarrow$———function of r———$\rightarrow$

Here L.H.S. depends on θ which the R.H.S. depends on r; so each side must reduce to a constant value say b^2

$$\Rightarrow \ \frac{\partial D_2}{\partial\theta} = b \text{ and } r^2 \left\{ 2ma + 2m\,\frac{\lambda}{r} - \left(\frac{\partial D_1}{\partial}\right)^2 \right\} = b^2$$

$$\Rightarrow \ D_2 = b\theta \text{ and } \frac{\partial D_1}{\partial r} = \int \sqrt{\left\{2ma + \frac{m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr$$

$$\Rightarrow \ D_1 = b\theta \text{ and } D_1 = \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr$$

$$\Rightarrow \ K = D_1 + D_2 + D_3 = \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} + \frac{b^2}{r^2}\right\}}\, dr + b\theta - at$$

Here if the constants b and a are identified with the new momenta P_r and P_θ, then Q_r and Q_θ are given by

$$Q_r = \frac{\partial K}{\partial P_r} = \frac{\partial K}{\partial b} = \frac{\partial}{\partial b} \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr + \theta$$

and $$Q_\theta = \frac{\partial K}{\partial P_\theta} = \frac{\partial S}{\partial a} = \frac{\partial}{\partial a} \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr - t.$$

If Q_r, Q_θ are constants, say k_1 and k_2, then we have

$$\frac{\partial}{\partial b} \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr + \theta = k_1.$$

and $$\frac{\partial}{\partial a} \int \sqrt{\left\{2ma + \frac{2m\lambda}{r} - \frac{b^2}{r^2}\right\}}\, dr - t = k_2.$$

Differentiating these w.r.t. b and a, we get

$$\int \frac{b\, dr}{r^2 \sqrt{\{2ma + (2m\lambda/r) - (b^2/r^2)\}}} = \theta - k_1 \qquad \text{...(17)}$$

and $$\int \frac{m\, dr}{\sqrt{\{2ma + (2m\lambda/r) - (b^2/r^2)\}}} = k^2 + t. \qquad \text{...(18)}$$

To integrate (17), we assume

$$u = \frac{1}{r} \ \Rightarrow \ \theta - k_1 = \int \frac{-(b/u^2)\, du}{\sqrt{\{2ma + 2m\lambda u - b^2u^2\}}}$$

i.e. $$\theta - k_1 = \int \frac{-du}{\sqrt{[(2ma/b^2) + (m\lambda/b^2)^2 - \{u - (m\lambda/b^2)\}^2]}}$$

$$= \cos^{-1}\left[\frac{u - (m\lambda/b^2)}{\{\sqrt{(2mab^2 + m^2\lambda^2)}\}/b^2}\right] = \cos^{-1}\left\{\frac{(b^2u - m\lambda)}{\sqrt{(2mab^2 + m^2\lambda^2)}}\right\}$$

This implies $b^2u^2 - m\lambda = \sqrt{\{2mab^2 + m^2\lambda_2\}} \cos(\theta - k_1)$

$$\Rightarrow \ r = b^2/\{m\lambda + \sqrt{(2mab^2 + m^2\lambda^2)} \cos(\theta - k_1)\}$$

$$\Rightarrow \ \{(b^2/m\lambda) r\} = 1 \sqrt{\{1 + (2ab^2/m\lambda^2)\}} \cos(\theta - k_1)$$

which is an ellipse, parabola or hyperbola according as

$$\sqrt{\{1+(2ab^2/m\lambda^2)\}} < = \text{ or } > 1$$

i.e., if $\quad 1+(2ab^2/m\lambda^2) < = \text{or} > 1$ *i.e.* $a < = \text{or} > 0.$

Hence, if the constant a is identified as the total energy C, then the central orbit will be an ellipse, parabola, or hyperbola according as $\quad C < \text{ or } = > 0.$

6·3-4. Harmonic oscillator problem as an example of the Hamilton-Jacobi method.

If the Hamiltonian for a simpie harmonlc oscillator of mass m is given by $H=(p^2/2m)+(kq^2/2)$; (q, p) being position and momentum co-ordinates of the harmonic oscillator and $k=m\omega^2$, to find the corresponding Hamilton-Jacobi equation and the motion of the oscillator, where k is the spring constant.

We have $H=\frac{1}{2}\{(p^2/m)+kq^2\}$. Now taking k to be the generating function of the transformation, the new Hamiltonian J is zero if $H+(\partial K/\partial t)=0$, where $p=(\partial K/\partial q)$.

This implies $(1/2m)(\partial K/\partial q)^2+\frac{1}{2}kq^2+(\partial K/\partial t)=0$

i.e. $$(\partial K/\partial t)+(1/2m)(\partial K/\partial q)^2+\frac{1}{2}kq^2=0. \quad \ldots(19)$$

Equation (19) is the required Hamilton Jacobi equation for the simple harmonic oscillator. *In order to obtain its solution, we proceed as follows :*

When the Hamiltonian does not contain t expilicitly, the solution of (1) may be taken as

$$K(q;\, a, t)=D(q, a)-ct, \text{ where } D=c(a). \quad \ldots(2)$$

The constance $c(a)$ can be identified with a itself. Hence (20) takes the form $K(q, a, t)=D(q, a)-at. \quad \ldots(1)$

$\Rightarrow (\partial K/\partial t)=-a$ and $(\partial K/\partial q)=(\partial D/\partial q) \Rightarrow (1/2m)(\partial D/\partial q)^2+\frac{1}{2}kq^2-q=0$

or $\quad (\partial D/\partial q)^2=2m(a-\frac{1}{2}kq^2) \Rightarrow p=(\partial D/\partial q)=\sqrt{\{2m(a-\frac{1}{2}kq^2)\}}$,

$$\Rightarrow D=\int \sqrt{\{2m(a-\tfrac{1}{2}kq^2)\}}\, dq \quad \ldots(22)$$

[omitting the irrelevent constant].

Putting this value of D in (21), we get

$$K=\int \sqrt{\{2m(a-\tfrac{1}{2}kq^2)\}}\, dq-at.$$

$$\therefore \quad Q=-b=(\partial K/\partial a)-(\partial/\partial a)\int\left\{\sqrt{[2m(a-\tfrac{1}{2}kq^2)]}\, dq\right\}-at$$

$$=\left\{\sqrt{\left(\frac{m}{2}\right)}\int\frac{dq}{\sqrt{\{a-(kq^2/2)\}}}\right\}-t$$

$$\Rightarrow -a=\left\{\sqrt{\left(\frac{m}{2}\right)}\sqrt{\left(\frac{2}{k}\right)}\sin^{-1}\frac{q\sqrt{k}}{\sqrt{(2a)}}\right\}=t\quad\left(\because\ \frac{\partial K}{\partial a}=-b\right)$$

$$\Rightarrow t-b=\sqrt{\left(\frac{m}{k}\right)}\sin^{-1}\frac{q\sqrt{k}}{\sqrt{(2a)}}\Rightarrow q=\sqrt{\left(\frac{2a}{k}\right)}\sin\left\{\sqrt{\left(\frac{k}{m}\right)}(t-b)\right\}\qquad\ldots(23)$$

This is the required solution. To find the constants, we consider the initial conditions, that at time $t=0$, the particle is stationary implying $p_0=0$. But the particle is displaced from the equilibrium position by an amount q_0; then equation (23).

$$\Rightarrow q_0=-\sqrt{\left(\frac{2a}{k}\right)}\sin\left\{\sqrt{\left(\frac{k}{m}\right)}.b\right\}\text{ and also }p_0=\left(\frac{\partial D}{\partial q}\right)_{t=0}$$

$$=-\{2m\,(a-\tfrac{1}{2}kq_0^2)\}=0$$

$$\Rightarrow a=\tfrac{1}{2}kq_0^2\text{ or }q_0=\sqrt{(2a/k)}\Rightarrow q_0=-q_0\sin\{\sqrt{(ck/m)},b\}$$

$$\Rightarrow -\sin(\pi/2)=(q_0/q_0)\{\sin b\sqrt{(k/m)}\}\Rightarrow -(\pi/2)=b\sqrt{(k/m)}$$

i.e.
$$-b=(\pi/2)\sqrt{(m/k)}$$

$$\Rightarrow q=q_0\sin\sqrt{(k/m)}\{\sqrt{(m/k)},(\pi/2)+t\}$$

$$=q_0\sin[(\pi/2)+\sqrt{(k/m)}\,t]\qquad\ldots(24)$$

$$\Rightarrow q=q_0\cos[\sqrt{(k/m)}\,t]\Rightarrow p=\sqrt{\{2m\,(a-\tfrac{1}{2}kq^2)\}}$$

i.e.
$$p=[2m\,(\tfrac{1}{2}kq_0^2-\tfrac{1}{2}kh_0^2\cos^2\sqrt{(k/m)}\,t\}]$$

$$=\sqrt{(mk)}\,q_0\sin[\sqrt{(k/m)}.t].$$

This gives the motion of the harmonic oscillator.

6·3-5. Separation of variables.

Here, we shall study those systems for which Hamiltonian is one of the integrals of the motion. Writing H-J equation, we have

$$(\partial K/\partial t)+H\,[q_i,(\partial K/\partial q_i)]=0\qquad\ldots(25)$$

The solution of (25) can be written as

$$K=W\,(q,\alpha)-\alpha t,\text{ where }\alpha=H\,[q_i,(\partial K/\partial q_i)]\qquad\ldots(26)$$

Now $(\partial K/\partial q_i)=(\partial W/\partial q_i)\Rightarrow$ * $H\,[q_i,(\partial W/\partial q_i)]=\alpha$ ...(27)

$$H\,[q_i,(\partial W/\partial q)]=\alpha.$$

The $H-J$ equation is said to be separable if there exists a solution of the form

$$W=W_1\,(q_1,\alpha_1,\alpha_2,\ldots,\alpha_n)+\ldots+W_n\,(q_n,\alpha_1,\alpha_2,\ldots,\alpha_n)\qquad\ldots(28)$$

and splits the H-J equation (27) into n-equations of following form

$$[H_i/q_i\,(\partial W_i/\partial q_i),\alpha_i,\alpha_{i+1},\ldots,\alpha_n]=\alpha_i\ (i=1,2,3,\ldots,n)\qquad\ldots(29)$$

Solving (29), one could obtain

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*H-J equation for the Hamiltons characteristic function W is given by

$$H\,[q_j,(\partial W/\partial q_i)]=\alpha.$$

$(\partial W_i/\partial q_i)$, *viz* $(\partial W_i/\partial q_i)=(\partial/\partial q_i\,[W_i\,(b_i, \alpha_1, \alpha_2, \ldots, \alpha_n)]$ ...(30)

$$\Rightarrow \quad W_i=\int (\partial W_i/\partial q_i)\, dq_i \qquad \ldots(31)$$

$$\Rightarrow \quad K=\sum_{i=1}^{n} W_i\,(q_i, \alpha_1, \alpha_2, \ldots \alpha_n)\, \alpha t \qquad \ldots(32)$$

$$\Rightarrow \quad p_k=(\partial K/(\partial q_k),\ \beta_k=(\partial K/\partial x_k),\ \beta+t=(\partial W/\partial \alpha) \qquad \ldots(33)$$

Ex. 7. *To find the motion of a massive particle in a plane under a central force, Hamiltonian H being given by*

$$H=\frac{1}{2\mu}\,[\dot{r}^2+r^2\,\dot{\theta}^2]+V(r)=\frac{1}{2\mu}\,[p_r^{\,2}+(p_\theta/r)^2+V(r)] \qquad \ldots(i)$$

Here H is not involving time t, hence it is a constant of motion and equals to the total energy "E" of the system.

H-J equation $\Rightarrow (\partial K/\partial t)+(\tfrac{1}{2}\mu)\,[(\partial K/\partial r)^2+(1/r)^2+(\partial K/\partial\theta)^2]+V(r)=0$

where $\qquad K=W\,(r, \theta, \alpha_i)-Et \qquad (i=1, 2)$ ...(ii)

Thus H-J equation for Hamilton's characterstic function "W" is given by

$$(1/2\mu)\,[(\partial K/\partial r)^2+(1/r)^2\,(\partial K/\partial\theta)^2]+V\,(r)=E \qquad \ldots(iii)$$

Let the solution of (iii) be

$$W=W_r\,(r)+W_\theta\,(\theta) \qquad \ldots(iv)$$

Using (iv), equation (iii)

$$\Rightarrow (\partial W_r/\partial r)^2+(1/r)^2\,[\partial W_\theta/\partial\theta]^2=2\mu\,(E-V)$$

$$\Rightarrow (\partial W_\theta/\partial\theta)^2=r^2\,[2\mu\,\{E-V\,(r)\}-\{\partial W_r/\partial r\}^2] \qquad \ldots(v)$$

$\leftarrow$function of $\theta\rightarrow$ $\qquad$ $\leftarrow$function of r ———$\rightarrow$

The two sides cannot be equal, unless each requeces to a constant value.

Let us put

$$(\partial W_\theta/\partial\theta)^2=\alpha^2, \text{ say} \Rightarrow (\partial W_\theta/\partial\theta)=\alpha_\theta \Rightarrow W_\theta=\alpha_\theta\,\theta \qquad \ldots(vi)$$

Also $\qquad 2\mu\,\{E-V\,(r)\}-(\partial W_r/\partial r)^2=(\alpha_\theta/r)^2$

$$\Rightarrow (\partial W/\partial r)=[2\mu\,\{K-V\,(r)\}-(\alpha_\theta/r)^3]^{1/2}$$

$$\Rightarrow \quad W_r=\int [2\mu\,\{E-V\,(r)\}-(\alpha_\theta/r)^2]^{1/2}\, dr \qquad \ldots(vii)$$

Thus Hamilton's characteristic function W, is given by

$$W=W_\theta+W_r=\alpha_\theta\theta+\int [2\mu\,(E-V)-(\alpha_\theta/r)^2]^{1/2}\, dr \qquad \ldots(viii)$$

$$\Rightarrow \quad K=\int [2\mu\,(E-V)-(\alpha_\theta/r)^2]^{1/2}\, dr+\alpha_\theta\theta-Et \qquad \ldots(ix)$$

Now using (33), we get

$$\beta+t=(\partial W/\partial E)=(\partial W_r/\partial E)$$

$$\Rightarrow \quad \beta+t=\int \frac{\mu\, dr}{\{2\mu\,(E-V)-(\alpha_0/r)^2\}^{1/2}}. \qquad ...(x)$$

Again, we also have

$$\beta_0=(\partial K/\partial\alpha_0)+(\partial W/\partial\alpha_0)$$

$$=\theta-\int \frac{\alpha_0\, dr}{r^2\sqrt{[2\mu\,(E-V)-(\alpha_0/r)^2]}}$$

$$\Rightarrow \quad \theta=\beta_0+\int \frac{\alpha_0\, dr}{r^2\sqrt{[2\mu\,(E-V)-(\alpha_0/r)^2]}} \qquad ...(xi)$$

Putting $\beta_0=\theta_0$ and changing the variable r to v $(r=u^{-1})$, we obtain

$$\theta=\theta_0-\int \frac{du}{\sqrt{[(2\mu/\alpha_0^2)\,(L-V)-u^2]}} \qquad ...(xiii)$$

This is the equation of the orbit.

Ex. 8. Three dimensional harmonic oscillator.

Hamiltonian is given by

$H=(2m)^{-1}\,(p_x^2+p_y^2+p_z^2]+\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2+\frac{1}{2}k_3z^2.$ ...(i)

Since, H does not contain time explicitly, it is a constant of motion implying $H=E$. Proceeding as in the previous example, H-J equation for Hamilton's characteristic function W is given by

$(1/2m)\,[(\partial W/\partial x)^2+(\partial W/\partial y)^2]+(\partial W/\partial z)^2+\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2+\frac{1}{2}k_3z^2=E$ ...(ii)

$$\Rightarrow \underbrace{\{(\partial W/\partial x)^2+\tfrac{1}{2}k_1x^2\}}_{\leftarrow\text{function of } x\rightarrow}+\underbrace{\{(\partial W/\partial y)^2+\tfrac{1}{2}k_2y^2\}}_{\leftarrow\text{function of } y\rightarrow}=\underbrace{\{(\partial W/\partial z)^2+\tfrac{1}{2}k_3z^2\}}_{\leftarrow\text{function of } z\rightarrow}=2mE$$

Hence, each expression with in the middle brackes must requnce to *a* constant value

i.e. $(\partial W/\partial x)^2+\frac{1}{2}k_1x^2=2m\alpha_x$ say ; $(\partial W/\partial y)^2+\frac{1}{2}k_2y^2=2m\alpha_y$

and $(\partial W/\partial z)^2+\frac{1}{2}k_3z^2=2m\alpha_z$, where $E=\alpha_x+\alpha_y+\alpha_z$.

Now proceed as usual.

Ex. 9. *Motion of a particle to gravitational field, when H equals to*

$$(1/2m)\,[p_x^2+p_y^2]+mgy=E \text{ say.}$$

Solution. Proceeding as usual, Hamilton's characteristic function W satisfies the H-J equation, *viz.*,

$$(1/2m)\,[(\partial W/\partial x)^2+(\partial W/\partial y)^2]+mgy=E \qquad ...(i)$$

$$\Rightarrow \quad \underbrace{[(\partial W/\partial x)^2]}_{\leftarrow \alpha_x^2 \rightarrow}+\underbrace{\{(\partial W/\partial y)^2+2m^2gy\}}_{\leftarrow \alpha_y^2 \rightarrow}=2mE \qquad ...(ii)$$

$$(\partial W/\partial x)=\alpha_x \Rightarrow W=\alpha_x.x \text{ i.e. } W_x=\alpha_x.x.$$

Also, we have

$$(\partial W_y/\partial y)^2+2m^2gy=\alpha_y^2=2mE-\alpha_x^2$$

$$\Rightarrow \quad (\partial W_y/\partial y)^2=(2mE-\alpha_x^2-2m^2gy)$$

$$\Rightarrow \quad (\partial W_y/\partial y)=\sqrt{(2mE-\alpha_x^2-2m^2gy)}$$

Now,
$$W=\int\left(\frac{\partial W_x}{\partial x}\right)dx+\int\left(\frac{\partial W_y}{\partial y}\right)dy$$

$$=\alpha_x.x+\int\sqrt{(2mE-2m^2gy-\alpha_x^2)}\,dy. \quad \ldots\text{(iii)}$$

Thus $K=W-Et$

$$=\alpha_x.x+\int\sqrt{[2mE-2m^2gy-\alpha_x^2]}\,dy-Et. \quad \ldots\text{(iv)}$$

Also $\beta+t=(\partial W/\partial E)$

$$=m\int\frac{dy}{\sqrt{[2m\,(E-mgy)-\alpha_x^2]}}\text{, using (iii)} \quad \ldots\text{(v)}$$

and
$$\beta_x=(\partial K/\partial\alpha_x)=x-\int\frac{\alpha_x\,dy}{\sqrt{[2m(E-mgy)-\alpha_x^2]}}$$

$$=x+\frac{\alpha_x}{m^2g}\,[2mE-\alpha_x^2-2m^2gy]^{1/2} \quad \ldots\text{(vi)}$$

Putting $x_0=\beta_x$ and $2m^2gy_0=2mE-\alpha_x^2$ in (vi), we obtain

$$y=y_0-(m^2g/2\alpha_x^2)\,(x-x_0)^2. \quad \ldots\text{(vii)}$$

Further, from (v) on integration, we get

$$(\beta+t)^2=(2/g)\,(y_0-y). \quad \ldots\text{(viii)}$$

Putting $-t_0=\beta$, equation (viii) $\Rightarrow y-y_0=-\frac{1}{2}g(t-t_0)^2$ ...(ix)

Also, we have $p_x=(\partial W/\partial x)=\alpha_x \Rightarrow m\dot{x}=\alpha_x$

$$\Rightarrow (x-x_0)\,m=\alpha_x\,(t-t_0) \Rightarrow x-x_0=(\alpha_x/m)\,(t-t_0)$$

i.e., we have

$$x-x_0=(\alpha_x/m)\,(t-t_0) \text{ and } (y-y_0)=-(gm^2/2\alpha x^2)\,(x-x_0)^1$$

(Agra 92)

6·3-6. Action angle variables.

When H is not an explicit function of time then E (there exists) a transformation independent of time, generated by $D'\,(q_i\,P_i)$ alone; $P_1, P_2,\ldots$are constants and the new Hamiltonian J equals to its original value *i.e.*, $J(P_i)=H\,(q_i,\ p_i) \Rightarrow Q_1, Q_2,\ldots$are not constants. The transformations generated by the time independent part of K defined as $K'=D'\,(\dot{q}_i, P_i)-ct\ldots(1)$, $D'=D+P_iQ_i$, where $D'\,(\dot{q}, P)$ generates the canonical transformation, leads to an idea of action angle variables or in other words, we generally come across with systems which have their motion periodic. Such types of systems are conveniently dealt with in the $H-J$ procedure. In this method, the constants a_i of integration coming directly in the solution of $H-J$ equation are taken as new momenta which are replaced by suitably defined constants $k_1,\ k_2\ldots$which form a set of independent functions of a_i's. These constants are known as the *action variables*.

Now in order to develop the idea of action and angle variables, we consider a system having one degree of freedom described by a time independent Hamiltonian $H(q, p)$. The canonical transformations generated by $D'(q,P)$ are $p_i=(\partial D'_i/\partial q_i)$, $Q_i=(\partial D'_i/\partial P_i)$ $J=H$ under the condition $\dot{P}_i=-(\partial J/\partial Q_i)$, $\dot{Q}_i=(\partial J/\partial P_i)$.

Now assume that P_i=constant$=k_i$ and let then $Q_i=u_i$, where u_i is the cyclic co-ordinate, then we have

$$J=J(k_i)=H(q_i, p_i)=C \qquad \ldots(34)$$

where

$$\dot{Q}_i=(\partial J/\partial P_i)=(\partial J/\partial v_i)=\dot{u}_i. \qquad \ldots(35)$$

Now, $u_i=(\partial D'/\partial k_i)=(\partial D'/\partial P_i)$, $\dot{u}_i=(\partial J/\partial k_i)=v_i$, say $\dot{k}_i=-(\partial J/\partial u_i)$. Hence $\dot{u}_i=v_i \Rightarrow u_i=v_i\, t+\epsilon$, where ϵ is the constant of integration...(36). This further shows that u increases linearly with time. At this stage. if we consider the case of a number of systems in which p_i and q_i are periodic functions of time, of period T say, then in order to evaluate the action over one period we shall take k_i in the form of a line integral (*termed as phase integral or action variable*) $k_i=\oint p_i\, dq_i$ under the assumptions that the projections of the motion of the selected points in *phase space on p_i, q_i plane are closed curves where the integration is to be carried over a complete period of oscilllation or rotation of q_i, as the case may be. Now let Δu_i be the increment in the value of u_i during one period uf the motion ; then we may write

$$\Delta u_i=\oint du_i=\oint \frac{\partial u_i}{\partial q_i}\, dq_i=\oint\left\{\frac{\partial}{\partial q_i}\left(\frac{\partial D'}{\partial k_i}\right) dq_i\right\}=\frac{\partial}{\partial k_i}\oint \frac{\partial D'}{\partial q_i}\, dq_i$$

$$=\frac{\partial}{\partial k_i}\oint p_i\, dq_i=(\partial k_i/\partial k_i)=1$$

$\Rightarrow$ change in u_i during one full period is unity.

Now taking $\epsilon=0$, $u_i=1$ in $u_i=v_i\, t+\epsilon$, we get $1=v_i\, t \Rightarrow v_i=t^{-1}$ i.e. u_i increases by unity in the time $t=v_i^{-1}$. Thus it follows that v_i gives the frequency of the periodic motion.

Note 1. k_1 is termed as *action variable* and u_i the corrzsponding angle variable.

Note 2. Here the transformation has been generated by

$$D'(q_i, P_i)=D'(q_i, k_i)$$

and the generating function can be expressed by

$$D(q_i, Q_i)=D(q_i, u_i) \Rightarrow D'=D+ku,$$

satisfying

$$p_i=(\partial D/\partial q_i),\ k_i=(-\partial D/\partial n_i).$$

SOME SOLVED PROBLEMS

Ex. 10. *Prove that the transformation* $P=\frac{1}{2}(p^2+q^2)$, $Q=\tan^{-1}(q/p)$ *is canonical.*

Method 1. Let the Hamiltonians in the co-ordinates p, q and P, Q be respectively $H(p, q)$ and $J(P, Q)$ respectively so that $H(p, q)=J(P, Q)$.

But p, q are canonical co-ordinates so we have

$$\dot{p}=-\frac{\partial H}{\partial q},\ \dot{q}=\frac{\partial H}{\partial p} \quad \ldots(1)$$

Also $\dot{p}=\dfrac{\partial p}{\partial P}\dot{P}+\dfrac{\partial p}{\partial Q}\dot{Q},\ \dot{q}=\dfrac{\partial q}{\partial P}\dot{P}+\dfrac{\partial q}{\partial Q}\dot{Q}$ ...(2)

$$\frac{\partial H}{\partial q}=\frac{\partial J}{\partial P}\frac{\partial P}{\partial q}+\frac{\partial J}{\partial Q}\frac{\partial Q}{\partial q},\ \frac{\partial H}{\partial q}=\frac{\partial J}{\partial P}\frac{\partial P}{\partial p}+\frac{\partial J}{\partial Q}\frac{\partial Q}{\partial p} \quad \ldots(3)$$

But $P=\frac{1}{2}(p^2+q^2)$ and $Q=\tan^{-1}(q/p)$

$$\therefore\ \frac{\partial P}{\partial p}=p,\frac{\partial P}{\partial q}=q,\ \frac{\partial Q}{\partial p}=\frac{-p}{p^2+q^2},\ \frac{\partial Q}{\partial q}=\frac{p}{p^2+q^2},$$

$$\begin{cases} \dfrac{dP}{dP}=p\dfrac{\partial p}{\partial P}+q\dfrac{\partial q}{\partial P}\cdot\ \dfrac{\partial Q}{\partial P}=\dfrac{\partial Q}{\partial p}\cdot\dfrac{\partial p}{\partial P}+\dfrac{\partial Q}{\partial q}\cdot\dfrac{\partial p}{\partial P} \\ \qquad =-\dfrac{k}{p^2+q^2}\dfrac{\partial p}{\partial P}+\dfrac{p}{p^2+q^2}\dfrac{\partial q}{\partial P}=\left(p\dfrac{\partial q}{\partial P}-q\dfrac{\partial p}{\partial P}\right)\Big/(p^2+q^2) \\ \dfrac{dQ}{dQ}=\dfrac{\partial Q}{\partial p}\cdot\dfrac{\partial p}{\partial Q}+\dfrac{\partial Q}{\partial q}\cdot\dfrac{\partial q}{\partial Q}=-\dfrac{q}{p^2+q^2}\dfrac{\partial p}{\partial Q}+\dfrac{p}{p^2+q^2}\dfrac{\partial q}{\partial Q} \\ \qquad =\left(p\dfrac{\partial q}{\partial Q}-q\dfrac{\partial p}{\partial Q}\right)\Big/(p^2+q^2)\ ; \\ \dfrac{dQ}{dQ}=\dfrac{\partial P}{\partial p}\dfrac{\partial p}{\partial Q}+\dfrac{\partial P}{\partial p}+\dfrac{\partial P}{\partial p}\dfrac{\partial q}{\partial Q}=p\dfrac{\partial p}{\partial Q}+q\dfrac{\partial q}{\partial Q} \end{cases}$$

$$\Rightarrow \begin{cases} p\dfrac{\partial p}{\partial q}+q\dfrac{\partial q}{\partial P}=1,\left(p\dfrac{\partial q}{\partial P}-q\dfrac{\partial p}{\partial P}\right)\Big/(p^2+q^2)=0 \\ p\dfrac{\partial p}{\partial Q}+q\dfrac{\partial q}{\partial Q}=0,\left(p\dfrac{\partial p}{\partial Q}-q\dfrac{\partial q}{\partial Q}\right)\Big/(p^2+q^2)=0 \end{cases}$$

$$\Rightarrow \left\{\frac{\partial p}{\partial P}=\frac{p}{p^2+q^2},\ \frac{\partial q}{\partial P}=\frac{q}{p^2+q^2},\ \frac{\partial p}{\partial Q}=-q,\frac{\partial q}{\partial Q}=p\right. \quad \ldots(4)$$

$\therefore$ Equations (1) and (2) $\Rightarrow$

$$p=\frac{p}{p^2+q^2}\dot{P}-q\dot{Q},\ \dot{q}=\frac{q}{q^2+p^2}\dot{P}+pQ \quad \ldots(5)$$

$$\frac{\partial H}{\partial q}=q\frac{\partial J}{\partial P}+\frac{p}{p^2+q^2}\frac{\partial J}{\partial Q},\ \frac{\partial H}{\partial p}=p\frac{\partial J}{\partial p}-\frac{q}{p^2+q^2}\frac{\partial J}{\partial Q} \quad \ldots(6)$$

Now from equations (4), (5), (6) we obtain

$$\frac{p}{p^2+q^2}\,\dot{P}-q\,\dot{Q}=-q\frac{\partial J}{\partial P}=\frac{p}{p^2+q^2}\frac{\partial J}{\partial Q}$$

$$\frac{q}{p^2+q^2}\,\dot{P}+p\,\dot{Q}=p\frac{\partial J}{\partial p}-\frac{q}{p^2+q^2}\frac{\partial J}{\partial Q}$$

These imply $\dot{P}=\frac{\partial J}{\partial Q}$, $\dot{Q}=\frac{\partial J}{\partial P}$

showing that P and Q are canonical and hence the transformation is canonical

Ex. 11. *If the Hamiltonian H is independent of t explicitly, then it is*

(*a*) *a constant and is*

(*b*) *equal to the total energy of the system.*

Solution. (*a*) $$\frac{dH}{dt}=\sum_{\alpha=1}^{n}\frac{\partial H}{\partial p_\alpha}\dot{p}_\alpha+\sum_{\alpha=1}^{n}\frac{\partial H}{\partial q_\alpha}\dot{q}_\alpha$$

$$=\sum_{\alpha=1}^{n}\dot{q}_\alpha\,\dot{p}_\alpha+\sum_{\alpha=1}^{n}-\dot{p}_\alpha\,\dot{q}_\alpha=0\left(\because\ \frac{\partial H}{\partial q_\alpha}=\dot{q}_\alpha,\ \frac{\partial H}{\partial q_\alpha}=-\dot{p}_\alpha\right)$$

$\Rightarrow H=\text{constant}=E$ say. ...(1)

(b) By Euler's theorem on homogeneous functions, we have

$$\sum_{\alpha=1}^{n}\dot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}=2T \quad ...(2)$$

But $p_\alpha=\frac{\partial L}{\partial \dot{q}_\alpha}=\frac{\partial (T-V)}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha}-\frac{\partial V}{\partial \dot{q}_\alpha}=\frac{\partial T}{\partial \dot{q}_\alpha}$

$\left(\because\ \frac{\partial V}{\partial \dot{q}_\alpha}=0 \text{ as } V \text{ is independent of } \dot{q}_\alpha\right)$

$\therefore$ (2) $\Rightarrow \sum_{\alpha=1}^{n}\dot{q}_\alpha\,p_\alpha=2T.$

Thus $H=\sum_{\alpha=1}^{n}p_\alpha\,\dot{q}_\alpha-L=2T-L=2T-(T-V)=T+V=E.$

Ex. 12. *A particle of mass m moves in a force field of potential V. Write*

(*a*) *the Hamiltonian, and*

(*b*) *Hamilton's equations in polar co-ordinates.*

Solution. (*a*) K. E. is given by

$T=\frac{1}{2}m\,(\dot{r}^2+r^2\,\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2)$ ...(1)

$\therefore\ L=T-V\ m=(\dot{r}^2+r^2\,\dot{\theta}^2+r^2\sin^2\theta\ \dot{\phi}^2)-V$ ...(2)

We have $p_r=\frac{\partial L}{\partial \dot{r}}=m\dot{r}$, $p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$, $p_\phi=\frac{\partial L}{\partial \dot{\phi}}=mr^2\sin^2\theta\dot{\phi}$

$$\Rightarrow \quad \dot{r}=\frac{p_r}{m}\ ;\ \dot{\theta}=\frac{p_\theta}{mr^2},\ \dot{\phi}=\frac{p_\phi}{mr^2\sin^2\theta} \qquad \ldots(3)$$

Now Hamiltonian is given by

$$H=\Sigma p_\alpha \dot{q}_\alpha-L=p_r\,\dot{r}+p_\theta\,\dot{\theta}+p_\phi\,\dot{\phi}-L$$

$$=\frac{p_r^2}{2m}+\frac{p_\theta^2}{2mr^2}+\frac{p_\phi^2}{2mr^2\sin^2\theta}+V(r,\theta,\phi)$$

= Total energy of the system.

(b) Hamilton's equation are given by

$$\dot{q}_\alpha=\frac{\partial H}{\partial p_\alpha},\ \dot{p}_\alpha=-\frac{\partial H}{\partial q_\alpha}$$

i.e.

$$\left.\begin{aligned} \dot{r}&=\frac{\partial H}{\partial p_r}=\frac{p_r}{m}\\ \dot{\theta}&=\frac{\partial H}{\partial p_\theta}=\frac{p^2}{mr^2}\\ \dot{\phi}&=\frac{\partial H}{\partial p_\phi}=\frac{p_\phi}{mr^2\sin^2\theta}\end{aligned}\right\}\ ;\ \left\{\begin{aligned} \dot{p}_r&=-\frac{\partial H}{\partial r}=\frac{p_\theta^2}{mr^3}+\frac{p_\phi^2}{mr^3\sin^2\theta}-\frac{\partial V}{\partial r}\\ \dot{p}_\phi&=-\frac{\partial H}{\partial \theta}=\frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta}-\frac{\partial V}{\partial \theta}\\ \dot{p}_\phi&=-\frac{\partial H}{\partial \phi}=-\frac{\partial V}{\partial \phi}\end{aligned}\right\}$$

Ex. 13. *A particle of mass m moves in a force field whose potential in spherical co-ordinates is* $V=-\dfrac{\lambda\cos\theta}{r^2}$.

Write the Hamilton-Jacobi equation describing its motion.

Solution. By Ex. 12, we have

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\theta^2}{r^2\sin^2\theta}\right)-\frac{\lambda\cos\theta}{r^2}. \qquad \ldots(1)$$

Writing $p_r=\dfrac{\partial K}{\partial r},\ p_\theta=\dfrac{\partial K}{\partial \theta},\ p_\phi=\dfrac{\partial K}{\partial \phi}$,

$$(1)\Rightarrow H\left(\frac{\partial K}{\partial r},\frac{\partial K}{\partial \theta},\frac{\partial K}{\partial \phi};r,\theta,\phi\right)$$

$$=\frac{1}{2m}\left[\left(\frac{\partial K}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial K}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial K}{\partial \phi}\right)^2\right]-\frac{\lambda\cos\theta}{r^2}.$$

$\therefore$ Hamilton-Jacobi equation is given by

$$\frac{\partial K}{\partial t}+\frac{1}{2m}\left\{\left(\frac{\partial K}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial K}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial K}{\partial \phi}\right)^2\right\}-\frac{\lambda\cos\theta}{r^2}=0. \qquad \ldots(2)$$

Ex. 14. *Find the complete solution of the Hamilton-Jacobi equation of Ex. 13 and indicate how the motion of the particle can be determined.*

Solution.

Part I. Let $K=S_1(r)+S_2(\theta)+S_3(\phi)-Et$.

Putting this value of K in equation (2) of previous example we get

$$-E+\frac{1}{2m}\left\{\left(\frac{\partial S_1}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S_2}{\partial r}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial S_3}{\partial\phi}\right)^2\right\}-\frac{\lambda\cos\theta}{r^2}=0 \quad \ldots(1)$$

But S_1 is a function of r along, so $\frac{\partial S_1}{\partial r}=\frac{dS_1}{dr}$ etc.

$\therefore$ (1) gives $\left\{r^2\left(\frac{dS_1}{dr}\right)-2mEr^2\right\}$

$\leftarrow$function of r alone$\rightarrow$

$$=\left\{-\left(\frac{dS_3}{d\phi}\right)^2-\frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\phi}\right)+2m\lambda\cos\theta\right\}$$

$\leftarrow$function of θ and ϕ

From above it is obvious that L.H.S. is a function of r alone while R.H.S. depends on θ and ϕ. Hence it follows that each side must be a constant, say β_1

Thus $$r^2\left(\frac{dS_1}{dr}\right)^2-2m\,Er^2=\beta_1$$

and $$-\left(\frac{dS_2}{d\theta}\right)^2-\frac{1}{\sin^2\theta}\left(\frac{dS_3}{d\phi}\right)^2+2m\lambda\cos\theta=\beta_1 \quad \ldots(2)$$

$\therefore$ Second equation of (2)

$$\Rightarrow \left(\frac{dS_3}{d\phi}\right)^2=2m\lambda\sin^2\theta\cos\theta-\beta_1\sin^2\theta-\sin^2\theta-\sin^2\theta\left(\frac{dS_2}{d\phi}\right)^2 \quad \ldots(3)$$

Now L.H.S. in (3) depends on ϕ while R.H.S. depends only on θ. Hence each side must be a constant, say β_2.

i.e. $$\left(\frac{dS_3}{d\phi}\right)^2=\beta_3$$

and $$2m\lambda\sin^2\theta\cos\theta-\beta_1\sin^2\theta-\sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2=\beta_2 \quad \ldots(4)$$

But $$p_\phi=\frac{\partial K}{\partial\phi}=\frac{dS_3}{d\phi}=\sqrt{\beta_2}\Rightarrow \beta_2=p_\phi^2.$$

So we have

$$2m\lambda\sin^2\theta\cos\theta-\beta_1\sin^2\theta-\sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2=p_\phi^2. \quad \ldots(5)$$

Now we shall try to solve the following equations :

$$\left.\begin{aligned} r^2\left(\frac{dS_1}{dr}\right)^2-2.n\,Er^2=\beta_1 \quad &\ldots(i)\\ 2m\lambda\sin^2\theta\cos\theta-\beta_1\sin^2\theta\,\sin^2\theta\left(\frac{dS_2}{d\theta}\right)^2=p_\phi^2 \quad &\ldots(ii)\\ p_\phi=\frac{dS_3}{d\phi} \quad &\ldots(iii)\end{aligned}\right\}$$

(i) $\Rightarrow S_1=\int\sqrt{\left(2mE+\frac{\beta_1}{r^2}\right)}\,dr$

where as (ii) $\Rightarrow S_2=\int\sqrt{(2m\lambda\cos\theta-p_\phi^2\operatorname{cosec}^2\theta-\beta_1)}\,d\theta$.

Also (iii) $\Rightarrow S_3=\int p_\phi\,d\phi$,

where we have chosen the positive square roots and omitted arbitrary additive constants.

$\therefore$ Complete solution is

$$K=\int\sqrt{\left(2mE+\frac{\beta_1}{r^2}\right)}\,dr$$
$$+\int\sqrt{(2m\lambda\cos\theta-p_\phi^2\operatorname{cosec}^2\theta-\beta_1)}\,d\theta+\int p_\phi\,d\phi-Et.$$

Part II. Equations of motion are obtained by writting

$$(\partial K/\partial\beta_1)=\gamma_1,\ (\partial K/\partial E)=\gamma_2,\ (\partial K/\partial p_\phi)=\gamma_3.$$

Solving these, we can obtain r, θ, ϕ so as to evuluate the arbitrary constants.

Ex. 15. *Obtain equations of motion of two dimensional harmonic oscillator.*

Case I. Cartesian co-ordinates. K.E. of the two-dimensional

$$T=\tfrac{1}{2}k\,(\dot{x}^2+\dot{y}^2).$$

Also the P.E. is given by

$$V=\tfrac{1}{2}kr^2=\tfrac{1}{2}m\,(x^2+y^2)$$
$$\therefore\quad L=T-V=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2)-\tfrac{1}{2}k\,(x^2+y^2)$$
$$\Rightarrow\quad p_x=\left(\frac{\partial L}{\partial\dot{x}}\right)=m\dot{x},\ p_y=\left(\frac{\partial L}{\partial\dot{y}}\right)=m\dot{y}$$

i.e.,
$$\dot{x}=(p_x/m)\text{ and }\dot{y}=(p_y/m).$$
$$\therefore\quad L=\tfrac{1}{2}m\,[(p_x/m)^2+(p_y/m)^2]-\tfrac{1}{2}k\,(x^2+y^2)$$
$$=(1/2m)\,(p_x^2+p_y^2)-\tfrac{1}{2}k\,(x^2+y^2).$$
$$\therefore\quad H=\Sigma\,p_i\dot{q}_i-L=p_x\dot{x}+p_y\dot{y}-L=p_\alpha\,(p_\alpha/m)+p_\beta\,(p_y/m)-L$$
$$=(p_\alpha^2/m)+(p_y^2/m)-(1/2m)\,(p_x^2+p_y^2)+\tfrac{1}{2}k\,(x^2+y^2)$$
$$=(1/2m)\,(p_x^2+p_y^2)+\tfrac{1}{2}k\,(x^2+y^2).$$
$$(\partial H/\partial p_x)=(p_x/m),\ (\partial H/\partial p_y)=(p_y/m)\Rightarrow(\partial H/\partial x)=kx,$$
$$(\partial H/\partial y)=ky.$$

Now Hamilton's equations for $\dot{x}$ and $\dot{y}$

$\Rightarrow \dot{x}=(\partial H/\partial p_x),\ \dot{y}=(\partial H/\partial p_y)$ i.e. $\dot{x}=(p_x/m),\ \dot{y}=(p_y/m)$.

$$\therefore\quad p_x=m\dot{x},\ q_y=m\dot{y}$$
$$\Rightarrow\ \dot{p}_x=m\ddot{x},\ \dot{p}_y=m\ddot{y}.$$

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But the Hamilton's equations for p_x and p_y

$\Rightarrow \dot{p}_x=-(\partial H/\partial x)$, $\dot{p}_y=-(\partial H/\partial y)$, *i.e.* $\dot{p}_x=-kx$, $\dot{p}_y=-ky$.

These equations $\Rightarrow -kx=m\ddot{x}$ and $-ky=m\ddot{y}$

Thus the equations of motion a of two-dimensional harmonic osillator are given by $m\ddot{x}+kx=0$ and $m\ddot{y}+ky=0$.

Case 2. Polar co-ordinates. We have

$$T=\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2) \text{ and } V=\tfrac{1}{2}kr^2$$

$$\therefore\quad L=T-V=\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2)-\tfrac{1}{2}kr^2$$

$$\Rightarrow\quad p_r=(\partial L/\partial \dot{\theta})=m\dot{r} \text{ and } p_\theta=(\partial L/\partial \dot{\theta})=mr^2\,\dot{\theta})$$

$$\Rightarrow\quad \dot{r}=(p_r/m) \text{ and } \dot{\theta}=(p_\theta/mr^2).$$

$$\therefore\quad H=\Sigma\, p_i\dot{q}_i-L=p_r\dot{r}+p_\theta\dot{\theta}-\{\tfrac{1}{2}m\,(\dot{r}^2+r^2\dot{\theta}^2)-\tfrac{1}{2}kr^2\}$$

$$=p_r\,(p_r/m)+p_\theta\,(p_\theta/mr^2)-\tfrac{1}{2}m\,\{p_r/m)^2+r^2\,(p_\theta/mr^2)^2\}+\tfrac{1}{2}kr^2$$

$$=(1/2m)\,\{p_r^2+(p_\theta^2/r^2)\}+\tfrac{1}{2}kr^2$$

$$\Rightarrow\quad (\partial H/\partial p_r)=(p_r/m),\ (\partial H/\partial p_\theta)=(p_\theta/mr^2)$$

and $\quad(\partial H/\partial r)=-(p_\theta^2/mr^3)+kr,\ (\partial H/\partial \theta)=0.$

Now the two Hamilton's equations for $\dot{r}$ and $\dot{\theta}$ are

$$\dot{r}=(\partial H/\partial p_r) \Rightarrow \dot{r}=(p_r/m)$$

and $\quad \dot{\theta}=(\partial H/\partial p_\theta) \Rightarrow \dot{\theta}=(p_\theta/mr^2).$

Similarly the two equations for $\dot{p}_r$ and $\dot{p}_\theta$ are given by

$$\dot{p}_r=-(\partial H/\partial r) \Rightarrow \dot{p}_r=(p_\theta^2/mr^2)-kr$$

and $\quad \dot{p}_\theta=-(\partial H/\partial \theta) \Rightarrow \dot{p}_\theta=0.$

These are the four equations of motion of the first order

Ex. 16. *Obtain the Hamiltonian and Hamilton's equations for a charged particle, in an electromagnetic field.*

(Agra M.Sc. Physics 1989)

Solution. We have $L=\tfrac{1}{2}mv^2-e\phi+(e/c)\,(\mathbf{v}\cdot\mathbf{A})$ ...(*i*)

$\therefore\quad p_i=(\partial L/\partial \dot{r}_i)=(\partial L/\partial v_i)=mv_i+(e/c)\,A_i$ ...(*ii*)

[using (*i*)]

So $\quad \mathbf{p}=\Sigma p_i=\sum_i\{mv_i+(e/c)\,A_i\}=m\mathbf{v}+(e/c)\,\mathbf{A}.$ ...(*iii*)

$$\therefore\quad H=\sum_i p_i\dot{q}_i-L=\sum_i p_i\dot{r}_i-L=\Sigma\, p_iv_i-L$$

$$=\sum_i\{mv_i+(e/c)\,A_i\}\,v_i-[\tfrac{1}{2}mv^2-e\phi+(e/c)\,(\mathbf{v}\cdot\mathbf{A})]$$

$$=\sum_i\{mv_i^2+(e/c)\,A_iv_i\}-\tfrac{1}{2}mv^2+e\phi-(e/c)\,(\mathbf{v}\cdot\mathbf{A})$$

$$=mv^2+(e/c)\,(\mathbf{A}\cdot\mathbf{v})-\tfrac{1}{2}mv^2+e\phi-(e/c)\,(\mathbf{v}\cdot\mathbf{A})=\tfrac{1}{2}mv^2+e\phi$$

$\therefore \quad H=\frac{1}{2}m\,\{(\mathbf{p}/m)-(e/mc)\,\mathbf{A}\}^2+e\phi=(1/2m)\,\{\mathbf{p}-(e/c)\,\mathbf{A}\}^2+e\phi.$ ...(iv)

Now the canonical equations are

$$\dot{\mathbf{r}}=(\partial H/\partial \mathbf{p}), \quad \text{and} \quad \dot{\mathbf{p}}=-\nabla H.$$

Using equation (iv), we obtain

$(\partial H/\partial \mathbf{p})=(1/m)\,\{\mathbf{p}-(e/c)\,\mathbf{A}\}$ and $\nabla H=e\nabla\phi-(e/c)\,\nabla\,(\mathbf{v}\cdot\mathbf{A})$.

Hence the Hamiltonian equations of motion are given by

$\dot{\mathbf{r}}=\mathbf{v}=(1/m)\,\{\mathbf{p}-(e/c)\,\mathbf{A}\}$ and $\dot{\mathbf{p}}=-e\nabla\phi+(e/c)\,\nabla\,(\mathbf{v}\cdot\mathbf{A})$.

Ex. 17. *A particle of mass m moves in a force field of potential V.*

(*a*) *Write the Hamiltonian, and*

(*b*) *Hamilton's equations in cartesian co-ordinates.*

Solution. (*a*) We have

$$T=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$\Rightarrow L=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)-V\,(x,y,z), \quad ...(1)$$

$$\therefore \quad p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x},\ p_y=\frac{\partial L}{\partial \dot{y}}=m\dot{y},\ p_z=\frac{\partial L}{\partial \dot{z}}=m\dot{z}$$

$$\Rightarrow \dot{x}=\frac{p_x}{m},\ \dot{y}=\frac{p_y}{m},\ \dot{z}=\frac{p_z}{m}.$$

Thus $\quad H=\Sigma\, p_x\dot{q}_x-L=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}$

$$-\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)+V\,(x,y,z)$$

$$=p_x\frac{p_x}{m}+p_y\cdot\frac{p_y}{m}+p_z\cdot\frac{p_z}{m}-\tfrac{1}{2}m\left(\frac{p_x^2}{m^2}+\frac{p_y^2}{m^2}+\frac{p_z^2}{m^2}\right)+V\,(x,y,z)$$

$$=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{p_z^2}{2m}+V\,(x,y,z)=\text{Total energy of the system.}$$

(*b*) Hamilton's equations are

$$\left.\begin{aligned} \dot{p}_x&=-\frac{\partial H}{\partial x}\\ \dot{p}_y&=-\frac{\partial H}{\partial y}\\ \dot{p}_z&=-\frac{\partial H}{\partial z}\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \dot{x}&=\frac{\partial H}{\partial p_x}\\ \dot{y}&=\frac{\partial H}{\partial p_y}\\ \dot{z}&=\frac{\partial H}{\partial p_z}\end{aligned}\right\}$$

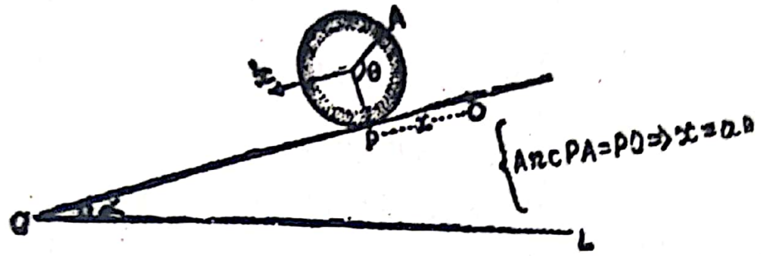
$$\Rightarrow \dot{p}_x=-\frac{\partial V}{\partial x},\ \dot{p}_y=-\frac{\partial V}{\partial y},\ \dot{p}_z=-\frac{\partial V}{\partial z}$$

$$\dot{x}=\frac{p_x}{m},\ \dot{y}=\frac{p_y}{m},\ \dot{z}=\frac{p_z}{m}.$$

Ex. 18. *A sphere rolls down a rough inclined plane, if x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration.*

Solution. We have

$$T=\tfrac{1}{2}m\,(\dot{x}^2+k^2\dot{\theta}^2)=\tfrac{1}{2}m\left(\dot{x}^2+\frac{2}{5}a^2\,\dot{\theta}^2\right) \qquad \left(\because\ k^2=\frac{2a^2}{5}\right)$$



$$=\tfrac{1}{2}m\left(\dot{x}^2+\frac{2}{5}\dot{x}^2\right)=\frac{9}{10}m\dot{x}^2, \quad ...(1);\quad V=-mgx\sin\alpha. \quad ...(2)$$

$$\therefore\quad L=T-V=\frac{7}{10}m\dot{x}^2+mgx\sin\alpha. \quad ...(3)$$

Now $\quad p_x=\dfrac{\partial L}{\partial \dot{x}}=\dfrac{7}{5}m\dot{x}\ \Rightarrow\ \dot{x}=\dfrac{5p_x}{7m}.$

Thus $\quad H=-L+p_x\dot{x}=-\dfrac{7}{10}m\dot{x}^2-mgx\sin\alpha+p_x\cdot\dfrac{5p_x}{7m}$

$$=-\frac{7}{10}m\left(\frac{25p_x^2}{49m^2}\right)-mgx\sin\alpha+\frac{5}{7m}p_x^2$$

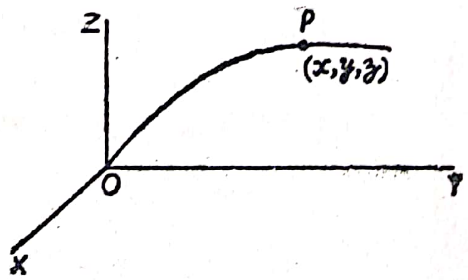
$$=\frac{5}{14}\frac{p_x^2}{m}-mgx\sin\alpha. \quad ...(4)$$

$\therefore$ One of the Hamilton's equations gives

$$\dot{p}_x=-\frac{\partial H}{\partial x}=mg\sin\alpha\ \Rightarrow\ \frac{7}{5}m\ddot{x}=mg\sin\alpha\ \Rightarrow\ \ddot{x}=\frac{5g}{7}\sin\alpha.$$

Ex. 19. *Use Hamilton's equations to find the equations of motion of a projectile in space.*

Solution. Regarding the projectile as a particle and axes attached to the earth as inertial, we obtain



$T=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)$;

$V=mgz$

$$\Rightarrow\quad L=T-V=\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz.$$

Now $\quad p_x=\dfrac{\partial L}{\partial \dot{x}}=m\dot{x},\ p_y=m\dot{y},\ p_z=m\dot{z}$

$$\Rightarrow \quad \dot{x}=\frac{p_x}{m},\ \dot{y}=\frac{p_y}{m},\ \dot{z}=\frac{p_z}{m}.$$

Thus $\quad H=-L+\Sigma p_x\,\dot{x}=-\{\tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz\}$

$$+\left\{\frac{p_x^2}{m}+\frac{p_y^2}{m}+\frac{p_z^2}{m}\right\}$$

$$=-\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+mgz+\frac{2p_x^2}{2m}+\frac{2p_y^2}{2m}+\frac{2p_z^2}{2m}$$

$$=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{p_z^2}{2m}+mgz=\text{Total energy of the system.}$$

$\therefore$ Hamilton's equation are :

$$\left.\begin{aligned}
\dot{p}_x&=-\frac{\partial H}{\partial x}=0\Rightarrow m\ddot{x}=0 &&\ldots(1)\\
\dot{p}_\beta&=-\frac{\partial H}{\partial y}=0\Rightarrow m\ddot{y}=0 &&\ldots(2)\\
\dot{p}_z&=-\frac{\partial H}{\partial z}=-mg\Rightarrow m\ddot{z}=-mg &&\ldots(3)
\end{aligned}\right\}$$

and

$$\left.\begin{aligned}
\dot{x}&=\frac{\partial H}{\partial p_x}=\frac{p_x}{m}\Rightarrow p_x=m\dot{x} &&\ldots(4)\\
\dot{y}&=\frac{\partial H}{\partial p_y}=\frac{p_y}{m}\Rightarrow p_y=m\dot{y} &&\ldots(5)\\
\dot{z}&=\frac{\partial H}{\partial p_z}=\frac{p_z}{m}\Rightarrow p_z=m\dot{z} &&\ldots(6)
\end{aligned}\right\}$$

$\therefore$ Equations of motion are

$$m\ddot{x}=0=m\ddot{y}\quad\text{and}\quad m\ddot{z}=-mg.$$

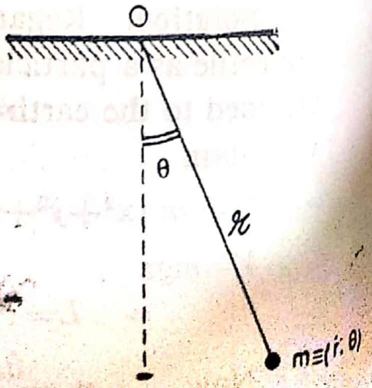
Ex. 20. *A mass m its constrained by a wire of length l to swing in an arc. Deduce the equation of motion.*

OR

Find the equation of motion of a simple pendulum.

Solution. Let the point of support 0, be at the origin. In polar co-ordinates, the Lagrangian of the system is given by

$$L=T-V=\frac{m}{2}(\dot{r}^2+r^2\dot{\theta}^2)+mgr\ \cos\theta,$$



$$\Rightarrow\quad \left.\begin{aligned}
&\frac{\partial L}{\partial \dot{r}}=m\dot{r},\ \frac{\partial L}{\partial r}=mr\dot{\theta}^2+mg\cos\theta,\\
&\frac{\partial T}{\partial \dot{\theta}}=mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta}=-mgr\sin\theta.
\end{aligned}\right\}\ldots(1)$$

Thus Lagrange's equations are

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{r}}\right\}-\frac{\partial L}{\partial r}=A_r\lambda\ ;$$

$$\frac{d}{dt}\left\{\frac{\partial T}{\partial \dot{\theta}}\right\}-\frac{\partial L}{\partial \theta}=A_\theta\lambda,$$

$$\ldots(2)$$

where A_r and A_θ are constraint coefficients and λ is the Lagragian multiplier.

But the constraint condition is $r-l=0 \Rightarrow dr=0$,

$$\Rightarrow A=1 \text{ and } A_\theta=0.$$

Whence equations of motion are given by

$$\frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2-mg\cos\theta=\lambda, \text{ and } \frac{d}{dt}(mr^2\dot{\theta})+mgr\sin\theta=0$$

Now, putting $r=l$, we obtain

$$ml\dot{\theta}^2+mg\cos\theta+\lambda=0 \text{ and } ml\ddot{\theta}+mgl\sin\theta=0, \qquad (\because \dot{r}=0)$$

Ex. 21. *A particle of mass m moves under gravity on a smooth sphere of radius r. Find the equations of motion in cartesian co-ordinates with the origin at the centre of sphere and z-axis vertically upwards.*

Sol. Equation of constraint is given by

$$x^2+y^2+z^2=r^2$$

$$\Rightarrow \quad 2x\,dx+2y\,dy+2z\,dz=0.$$

$$\Rightarrow \quad x\,dx+y\,dy+z\,dz=0.$$

Thus, we have $A_x=x$, $A_y=y$ and $A_z=z$.

Now, T=kinetic energy of the system$=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$, and potential energy$=mgz=V$, say

$$\Rightarrow \quad L=T-V=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz,$$

$$\Rightarrow \quad \left.\begin{aligned} &\frac{\partial L}{\partial \dot{x}}=m\dot{x},\ \frac{\partial L}{\partial x}=0\ ;\ \frac{\partial L}{\partial \dot{y}}=m\dot{y},\ \frac{\partial L}{\partial y}=0,\\ &\frac{\partial L}{\partial \dot{z}}=m\dot{z},\ \frac{\partial L}{\partial z}=-mg \end{aligned}\right\}$$

Hence Lagrange's equations are given by

$$\left.\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=A_x\lambda\ ;\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\frac{\partial L}{\partial y}=A_y\lambda\\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=A_z\lambda \end{aligned}\right\}$$

$$\Rightarrow \quad m\ddot{x}=x\lambda,\quad m\ddot{y}=y\lambda \quad \text{and} \quad m\ddot{z}+mg=z\lambda.$$

These are equations of motion.

Ex. 22. *A particle moves under the iufluence of gravity on the frictionless inner surface of the pasaboloid of revolution $x^2+y^2=az$. Obtain the equations of motion.*

Sol. Let the particle of mass m move on the frictionless inner surface of the paraboloid of revolution and let it be at a point P at any instant t. Let the cylindrical co-ordinates of P be (r, θ, z).

Now, kinetic energy of the particle, is given by

$$T=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2).$$

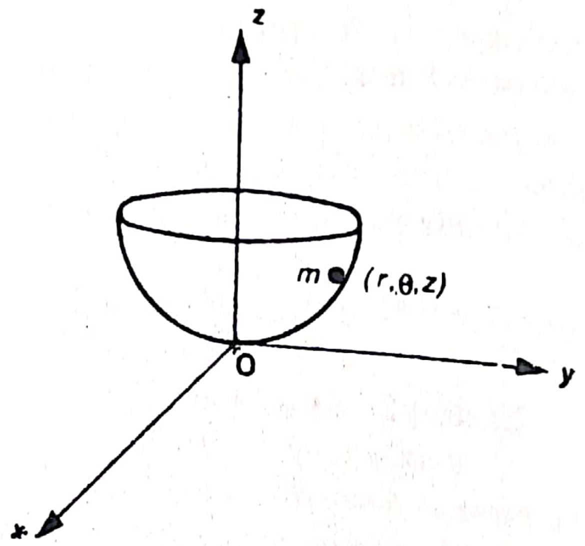
Also, the potential energy, $V=mgz$.

$$\Rightarrow \quad L=T-V=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)-mgz. \quad \ldots(1)$$

Again, eqn. of the paraboloid of revolution is given by

$$x^2+y^2=az.$$

But $x^2+y^2=r^2$; therefore the paraboloid of revolution is expressed as



$$r^2=az \quad \Rightarrow \quad r^2-az=0.$$

This gives the condition of constraint ; viz $2r\,dr-a\,dz=0$.

$$\Rightarrow \quad A_r=2r,\ A_\theta=0 \text{ and } A_z=-a,$$

Whence, $\dfrac{\partial L}{\partial \dot{r}}=m\dot{r}$, $\dfrac{\partial L}{\partial r}=mr\dot{\theta}^2$, $\dfrac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$, $\dfrac{\partial L}{\partial \theta}=0$,

$\dfrac{\partial L}{\partial \dot{z}}=m\dot{z}$, $\dfrac{\partial L}{\partial z}=-mg$.

Now, Lagrange's equations are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \lambda}\right)-\frac{\partial L}{\partial r}=A_r\lambda,\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=A_\theta\lambda,$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=A_z\lambda.$$

$\Rightarrow \quad m\ddot{r}-mr\dot{\theta}^2=2r\lambda$; $\dfrac{d}{dt}(mr^2\dot{\theta})=0$; $\Rightarrow \quad mr^2\ddot{\theta}+2m\dot{r}r\dot{\theta}=0$

and $m\ddot{z}+mg=-a\lambda$.

Ex. 23. *A particle moves under the influence of gravity on the frictionless inner surface of the elliptic paraboloid given by*

$$bx^2+cy^2=az,$$

where a, b, c are positive constants. Obtain the equations of motion.

Sol. Paraboioid is $bx^2+cy^2=az$.

Hence the equation of the constraint is

$$2bx\,dx+cy\,dy-a\,dz=0.$$

$\Rightarrow$ $A_x=2bx$, $A_y=2cy$ and $A_z=-a$.

Also, Lagrangian of the particle is given by

$$L=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz.$$

$$\Rightarrow \left.\begin{aligned}&\frac{\partial L}{\partial \dot{x}}=m\dot{x},\ \frac{\partial L}{\partial x}=0\ ;\ \frac{\partial L}{\partial \dot{y}}=m\dot{y},\ \frac{\partial L}{\partial y}=0,\\ &\frac{\partial L}{\partial \dot{z}}=m\dot{z},\ \frac{\partial L}{\partial z}=-mg.\end{aligned}\right]$$

Thus, Lagrange's equations are

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{x}}\right\}-\frac{\partial L}{\partial x}=A_x\lambda,\quad \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{y}}\right\}-\frac{\partial L}{\partial y}=A_y\lambda,$$

and
$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{z}}\right\}-\frac{\partial L}{\partial z}=A_z\lambda,$$

$$\Rightarrow \quad m\ddot{x}=2bx\lambda,\ m\ddot{y}=2cy\lambda,\ m\ddot{z}+mg=-a\lambda.$$

Ex. 24. *Find the equation of motion of a hoop rolling without slipping on an inclined plane and hence find its acceleration and frictional force of constraint.* **(Rohilkhand 1986; Agra 93)**

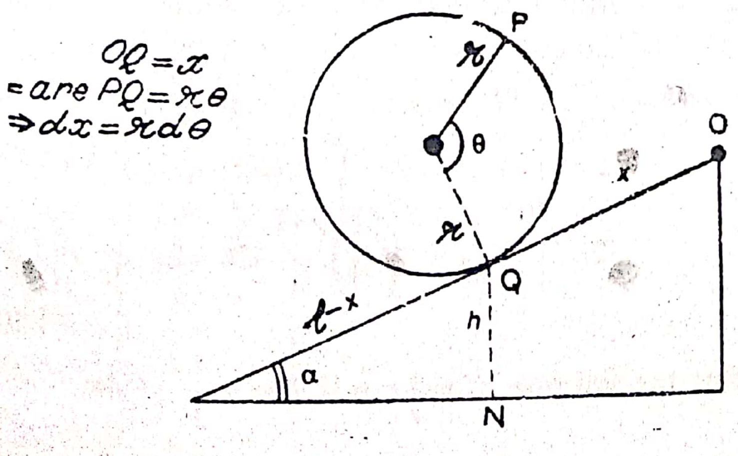
Sol. Let α be the inclination of the inclined plane of length l with the horizontal and let the hoop of radius r roll down from a point O without slipping ; then the equation of the constraint is given by

$$dx=r\,d\theta$$

$$\Rightarrow \quad r\,d\theta-dx=0. \qquad \text{...(1)}$$

Since, there is only one equation of constraint, only one Lagrangian multiplier is required. The coefficients in the constraint equation are

$$A_\theta=r,\ A_x=-1. \qquad \text{...(2)}$$



Whence kinetic energy of the hoop, T=kinetic energy of the motion of centre of mass+kinetic energy of motion relative to centre of mass, $=\frac{1}{2}M\dot{x}^2+\frac{1}{2}Mr^2\dot{\theta}^2$.

Also, potential energy of the hoop, V is given by

$$V=Mgh=Mg\,(l-x)\sin\alpha$$

$$\Rightarrow \quad L=T-V=\tfrac{1}{2}M\,(\dot{x}^2+r^2\dot{\theta}^2)-Mg\,(l-x)\sin\alpha \qquad \text{...(3)}$$

Here are only two generalised co-ordinates x and θ, hence. There will be two Lagrangian equations.

$$\text{Viz}: \quad \frac{\partial L}{\partial \dot{x}}=M\dot{x},\quad \frac{\partial L}{\partial x}=Mg\sin\alpha\quad \frac{\partial L}{\partial \dot{\theta}}=Mr^2\dot{\theta},\ \frac{\partial L}{\partial \theta}=0. \qquad \text{...(4)}$$

Now, Lagrangian equation in x is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=A_x\lambda \quad \Rightarrow \quad M\ddot{x}-Mg\sin\alpha+\lambda=0. \qquad \text{...(5)}$$

Further, Lagrangian equation in θ is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=A_\theta\lambda, \quad \Rightarrow \quad Mr^2\ddot{\theta}-r\lambda=0. \qquad \text{...(6)}$$

Whence equations (5) and (6) denote Lagrangian equations.

Again (1) $\Rightarrow$ $r\ddot{\theta}=\ddot{x}$. $\Rightarrow$ $M\ddot{x}=\lambda$. [using (6)]

$\Rightarrow$ $M\ddot{x}-Mg\sin\alpha+M\ddot{x}=0$ $\Rightarrow$ $\ddot{x}=\dfrac{g\sin\alpha}{2}$

$\Rightarrow$ $\ddot{\theta}=(g\sin\alpha/2r)$ ($\because$ $x=r\theta$). Thus we canclude that the hoop rolls down the inclined plae with one half of the acceleration it would have acquired in slipping down the friction less inclined plane. Further, frictional force of constraint λ is $\lambda=(Mg\sin\alpha/2)$.

Ex. 25. *Show that a sphere is the solid figure of revolution which has maximum volume for a given surface area.* **(Rohilkhand 84)**

Sol. The volume of a sphere may be supposed to be made up of a large number of discs while the surface area may be supposed as the collection of large number of rings.

Now, the surface area of sphere is given by

$$2\pi\int y\,ds=2\pi\int y\,(dx^2+dy^2)^{1/2}$$

$$=2\pi\int y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx=2\pi\int y\left[1+y'^2\right]^{1/2}dx \qquad \text{...(1)}$$

where $y'=(dy/dx)$.

But the volume of sphere is $V=\pi\int y^2\,dx$. ...(2)

Now, Combining (1) and (2), we have

$$\pi \int y^2\, dx + \lambda\, 2\pi \int y\, (1+y'^2)^{1/2}\, dx = \text{constant}$$

$$\Rightarrow \quad \int \left[y^2 + \lambda y\, (1+y'^2)^{1/2} \right] dx = \text{constant}$$

$$\Rightarrow \quad y^2 + \lambda y\, (1+y'^2)^{1/2} = f = \text{extremum (say)} \qquad \ldots(3)$$

As f does not depend on x explicitly, we have

$$\left(y' \frac{\partial f}{\partial y'} - f \right) = \text{constant}. \qquad \ldots(4)$$

Again, $\qquad f = y^2 + \lambda y\, (1+y'^2)^{1/2}.$

$$\Rightarrow \quad \frac{\partial f}{\partial y'} = 0 + \lambda y . \tfrac{1}{2}\, (1+y'^2)^{-1/2} \times 2y' = \frac{\lambda y\, y'}{(1+y'^2)^{1/2}}.$$

Thus equations (4) $\Rightarrow$

$$\left[y' . \frac{\lambda y\, y'}{(1+y'^2)^{1/2}} - y^2 - \lambda y\, (1+y'^2)^{1/2} \right] = \text{const. } C.$$

But $y=0$ at $x=0$, this gives $C=0$.

$$\Rightarrow \quad \frac{\lambda y\, y'^2}{(1+y'^2)^{1/2}} - y^2 - \lambda y\, (1+y'^2)^{1/2} = 0$$

$$\Rightarrow \quad \lambda y\, y'^2 - y^2\, (1+y'^2)^{1/2} - \lambda y - \lambda y\, y'^2 = 0$$

$$\Rightarrow \quad -y^2\, (1+y'^2)^{1/2} - \lambda y = 0.$$

$$\Rightarrow \quad y\, (1+y'^2)^{1/2} - \lambda = 0$$

$$\Rightarrow \quad y\, (1+y'^2)^{1/2} = \lambda.$$

$$\Rightarrow \quad y^2\, (1+y'^2) = \lambda^2.$$

$$\Rightarrow \quad y' = \frac{dy}{dx} = \frac{\sqrt{(\lambda^2 - y^2)}}{y}.$$

On integrating, we get

$$-\sqrt{(\lambda^2 - y^2)} = x - x_0, \quad \Rightarrow \quad (x-x_0)^2 + y^2 = \lambda^2.$$

This represents a sphere with centre at x_0 on x-axis and radius λ. Hence we say that for the values of area (A) and volume (V) of sphere, f is extremum. In other words, we can say that the sphere is the solid figure of revolution, which has maximum volume for a given surface area.

SUPPLEMENTARY PROBLEMS

1. A particle of mass m moves in a force field or potential V.
   (a) Write the Hamiltonian and
   (b) Hamilton's equations in rectangular co-ordinates (x, y, z).

Ans. (a) $H = (p_x^2 + p_y^2 + p_z^2)/2m + V(x, y, z)$

(b) $\dot{x} = p_x/m,\ \dot{y} = p_y/m,\ \dot{z} = p_z/m,$

$$\dot{p}_x = -\frac{\partial V}{\partial x},\ \dot{p}_y = -\frac{\partial V}{\partial y},\ \dot{p}_z = -\frac{\partial V}{\partial z}.$$

2. Use Hamilton's equations to obtain the motion of a particle of mass m down a frictionless inclined plane of angle α.
3. Use Hamilton's equations to obtain the motion of a projectile launched with speed v_0 at angle α with the horizontal.
4. A particle of mass m in a force field having potential $V(\rho, \phi, z)$ where ρ, ϕ, z are cylinderical co-ordinates. Give
(a) the Hamilton's and
(b) Hamilton's equations for the particle.

Ans. (a) $H=\{p_\rho^2+(p_\phi^2/\rho)+p_z^2\}/2m+V(\rho, \phi, z)$,

(b) $\dot{\rho}=(p_\rho/m)$, $\dot{\phi}=(p_\phi/mr^2)$, $\dot{z}=(p_z/m)$,

$$\dot{p}_\rho=(p_\phi^2/m^3)-\frac{\partial V}{\partial \rho},\ \dot{p}_\phi=-\frac{\partial V}{\partial \phi},\ \dot{p}_z=-\frac{\partial V}{\partial z}.$$

5. A particle of mass m moves on the inside of a frictionless vertical cone having equation $x^2+y^2=z^2 \tan^2 \alpha$.
(a) Write the Hamiltonian and
(b) Hamilton's equations using cylindrical co-ordinates.

Ans. (a) $\dfrac{p_\rho^2 \sin^2 \alpha}{2m}+\dfrac{p_\phi^2}{2m\rho^2}+mg\rho \cot \alpha$.

(b) $\dot{\rho}=\dfrac{p_\rho \sin^2 \alpha}{m}$, $\dot{p}_\rho=\dfrac{p_\phi^2}{m\rho^3}-mg \cot \alpha$.

6. What is meant by a holonomic system. Obtain Hamilton's canonical equations for a holonomic system. (*Agra 1971*)
Show that, if the Lagrangian does not depend upon time explicitly, the Hamiltonian is equal to the total energy of the system.
7. State and prove Hamilton's equations of motion and use them to obtain the motion of a projectile launched with speed v_0 at an angle α with the Horizontal. (*Rohilkhand 1978*)
8. Define generalised momenta. Derive the Hamilton's canonical equations of motion and explain the significance of Hamiltonian function. What are cyclic coordinates ? (*Agra 1974*)
9. (a) Obtain the Hamiltonian for a double pendulum.
(b) A particle mass m moves in a force field of potential

$$V=-\frac{k \cos \theta}{r^2};$$

find (a) the Hamiltonian, and (b) the Hamilton's equations is spherical polar co-ordinates.
10. Obtain the Hamilton's canonical equations of motion from the Lagrangian equations using Lagrange transformations. Define a cyclic coordinate and prove that such a coordinate is ignorable in the Hamiltonian formulation. Discuss the physical significance of the Hamiltonian.
11. Prove that the transformation $Q=q \tan p$, $P=\log \sin p$ is canonical (*Agra 1977*)
12. Let u be a generating function dependent only on Q_α, P_α, t. Prove that

$$P_\alpha=-\frac{\partial u}{\partial Q_\alpha},\ q_\alpha=-\frac{\partial u}{\partial p_\alpha},\ H'=\frac{\partial u}{\partial t}+H.$$

13. Let v be a generating function dependent only on the old and new momenta p_α and P_α respectively and the time t. Prove that

$$P_\alpha = -\frac{\partial v}{\partial P_\alpha}, \quad Q_\alpha = \frac{\partial v}{\partial p_\alpha}, \quad H' = \frac{\partial v}{\partial t} + H$$

14. (a) Set up the Hamilton-Jacobi equation for the motion of a particle sliding down a frictionltes inclined plane of angle α.

(b) Solve the Hamilton-Jacobi equation in (a) and thus determine the motion of the particle.

15. Work the problem of a projectile launched with speed v_0 at angle a with the horizontal by using Hamilton-Jacobi method.

16. State Hamilton's Principle
By using Lagrange's equations or otherwise, establish the principle for a conservative holonomic system.
A particle of unit mass moves along OX under a constant force f, starting, from rest at the origin at time $t=0$. If T and V are the kinetic and potential energies of the particle, calculate

$$\int_0^{t_0} (T-V)\, dt.$$

Evaluate this integral for the varied motion (in the Hamiltonian sense) in which the position of the particle is given by

$$x = \tfrac{1}{2} ft^2 + \epsilon\, ft\, (t - t_0),$$

where ϵ is a constant; and show that the result is in agreement with Hamilton's Principle. What are the essential features if the varied motion that ensure this agreement ?

Ans. Integral$=\frac{1}{3} f^2 t^3 (1+\epsilon^2)$. This is minimum for $\epsilon=0$, approximately. The varied motion is such that x has the same values as for the unvaried motion at the end points $t=0$, $t=t_0$.

17. Prove that the transformation $Q=p$, $P=-q$ is canonical.

18. If $[F, G]$ is the poisson bracket, prove that

(a) $[F_1 F_2, G] = F_1 [F_2, G] + F_2 [F_1, G]$

(b) $\dfrac{\partial}{\partial t}[F, G] = \left[\dfrac{\partial F}{\partial t}, G\right] + \left[F, \dfrac{\partial G}{\partial t}\right]$

(c) $\dfrac{d}{dt}[F, G] = \left[\dfrac{dF}{dt}, G\right] + \left[F, \dfrac{dG}{dt}\right]$

19. Prove Jacobi's identity for Poisson bracket.

$$[F_1, [F_2, F_3]] + [F_2, [F_3, F_1]] + [F_3, [F_1, F_2]] = 0.$$

20. An atom consists of an electron of charge $-e$ moving in a central force field F about a nucleus of charge Ze such that

$$\mathbf{F} = -\frac{Ze^2\, \mathbf{r}}{r^3}$$

where r is the position vector of the electron relative to the nucleus and Z is the atomic number.

In Bohr's Quantum theory of the atom the phase integrals are integral multiples of Planck's constant h i.e.,

$$\oint p_r\, dr = n_1 h,\ \oint p_\theta\, d\theta = n_2 h$$

using these equations, prove that there will be only a discrete set of energies given by

$$E_n = -\frac{2\pi^2 m Z^2 e^4}{n^2 h^2}$$

where $n = n_1 + n_2 = 1, 2, 3, 4 \ldots$ is called the orbital quantum number.

21. For a system with Lagrangian

$$L = \tfrac{1}{2} \sum_{\rho=1}^{n} \sum_{\sigma=1}^{n} a_{\rho\sigma}(q)\, \dot{q}_\rho\, \dot{q}_\sigma - V(q)$$

where $a_{\rho\sigma}(q) = a_{\sigma\rho}(q)$, show that the Hamiltonian is

$$H = \tfrac{1}{2} \sum_{\rho=1}^{n} \sum_{\sigma=1}^{n} a_{\rho\sigma}(q)\, p_\rho\, p_\sigma + V(q)$$

where $a^{\rho\sigma}$ are determined by the equations

$$\sum_{r=1}^{n} a^{r} a_{\sigma r} = \begin{cases} 1 \text{ if } \rho = \sigma \\ 0 \text{ if } \rho \neq \sigma \end{cases}$$

(This means that $a^{\rho\sigma}$ is the cofactor, with proper sign, of $a_{\rho\sigma}$ in the determinant formed by the elements $a_{\rho\sigma}$ divided by the determinant).

22. Apply the result of exercise 16 to the particular case where

$$L = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 - \dot{q}_2^2) - V(q)$$

Ans. $H = \tfrac{2}{3}(p_1^2 - p_1 p_2 + p_2^2) + V(q)$.

23. A particle moves in space with the Lagrangian

$$L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V + \dot{x}A + \dot{y}B + \dot{z}C$$

where V, A, B, C are given function of x, y, z show that the equations of motion are

$$m\ddot{x} = -\frac{\partial V}{\partial x} + \dot{y}\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) - \dot{z}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)$$

and two similar equations [These are essentially the equations of motion of a charged particle in an electromagnetic field].

Find the Hamiltonian H, and obtain the canonical equations of motion.

Ans. $H = (1/2m)\,[(p_x - A)^2 + (p_y - B)^2 + (p_z - C)^2] + V$.

24. A dynamical system has kinetic energy.

$$T = \tfrac{1}{2}(a\dot{q}_1^2 + 2a\, \dot{q}_1\dot{q}_2 + b\dot{q}_2^2)$$

and potential energy

$$V = \tfrac{1}{2}(Aq_1^2 + 2Hq_1q_2 + Bq_2^2).$$

Additional generalized forces

$$Q_1 = -k_{11}\dot{q}_1 - k_{12}\dot{q}_2,\ Q_2 = -k_{12}\dot{q}_1 = k_{12}\dot{q}_2$$

are applied. All the coefficients a, h, b, A, H, B and the k, s are constants Show that the energy sum $T+V$ decreases steadily during any motion provided.

$$k_{11} > 0 \; 4k_{11}\, k_{22} > (k_{12}+k_{21})^2.$$

25. Show that if the amplitude of oscillation is small, the energy of a simple pendulum is given by $E=J\nu$.

26. Using the method of action-angle variables, show that the frequency of a simple pendulum of lenght l assuming oscillation is $2\pi \sqrt{\left(\frac{g}{l}\right)}$.

27. Outline Hamilton-Jacobi theory and apply it to solve the problem of one dimensional harmonic oscillator. (*Rohilkhand 1976; Agra 73*)

28. State and prove Hamilton-Jacobi equation for Hamilton's principal function and explain how it can be used to solve Kepler's problem for a particle in an inverse square central force field.

(*Rohilkhand 1978 ; Agra 71*)