Fluid mechanics — PAPER II

(1) Continuity equation $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$

(2) Surface $F(x,y,z,t) = 0$ will be a possible boundary surface if it identically satisfy $\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$ and the same value of u, v, w also satisfies continuity eqn $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

(3) Normal velocity $= \dfrac{u\frac{\partial F}{\partial x} + v\,\partial F/\partial y + w\,\partial F/\partial z}{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2 + (\partial F/\partial z)^2}} = \dfrac{\bar{q}\cdot\nabla F}{|\nabla F|}$

(4) Fluid velocity $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k} = (u, v, w)$

(5) streamline / line of flow $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

(6) pathline $\frac{d\bar{r}}{dt} = \bar{q}$ $\quad \frac{dx}{dt} = u(x,y,z,t),\ \frac{dy}{dt} = v(x,y,z,t),\ \frac{dz}{dt} = w(x,y,z,t)$

(7) In case of steady motion, streamlines and pathlines are identical

(8) velocity potential (ϕ), $\bar{q} = -\nabla\phi$, $\nabla\times\bar{q} = 0$

potential kind or conservative or irrotational $u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y},\ w = -\frac{\partial\phi}{\partial z}$

(9) Vorticity vector $\bar{\Omega} = \nabla\times\bar{q}$, $\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

(10) Vortexline $\bar{\Omega}\times d\bar{r} = 0$ $\Rightarrow \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$, Irrotation $\bar{\Omega} = 0$

(11) Spherical polar coordinates (r, θ, ϕ) $\quad x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$0 \le r \le a,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$

(12) Surfaces orthogonal to streamlines $u\,dx + v\,dy + w\,dz = 0$

(13) Equation of motion $\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = F - \frac{1}{\rho}\frac{\partial p}{\partial r'}$

(14) Equation of motion of an incompressible fluid flow

$X \to$ body force

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Steady state $\frac{\partial u}{\partial t} = 0$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

(15) Pressure $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$

(16) Complex potential $w = \phi + i\psi$, $\phi_x = \psi_y$, $\phi_y = -\psi_x$

(17) magnitude of velocity $= \left|\frac{dw}{dz}\right| = \sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2} = \sqrt{u^2+v^2} = q$

(18) Streamlines are given by $\psi = \text{const}$

(19) $z = -c\cos\omega$, $\cos(iu) = \cosh u$, $\sin(iu) = i\sinh u$

(20) Bernoulli's eqn. $\frac{p}{\rho} + \frac{q^2}{2} = C$

(21) $$\frac{x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2} = \frac{a^2+b^2}{b^2-a^2}\left(\frac{1}{x^2+a^2} - \frac{1}{x^2+b^2}\right) - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}$$

(22) total Head $H = \frac{p}{\rho g} + \frac{v^2}{2g} + z$

(23) $$\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{B} - \frac{1}{\rho}\times\nabla p + \nu\nabla^2\bar{q}$$

conservative $B = -\nabla V$

↓

Navier-Stokes equation for incompressible fluid with constant viscosity.

(24) $$\frac{\partial \bar{q}}{\partial t} - \bar{q}\times(\nabla\times\bar{q}) = F - \frac{1}{\rho}\nabla p \cancel{+} - \frac{1}{2}\nabla q^2$$

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Mechanics

① Moment of inertia $I = \int r^2 \, dm$

② Radius of gyration (K), $I = MK^2$

③ Dirichlet's Theorem $x, y, z \geq 0$, $x+y+z \leq 1$

$$\iiint x^{l-1} y^{m-1} z^{n-1} \, dx\,dy\,dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma(l+m+n+1)}$$

④ Routh's table : Moment of inertia about an axis of symmetry

$$= \text{Mass} \times \frac{\text{sum of squares of} \perp \text{semi axis}}{3, 4 \text{ or } 5}$$

3 : rectangular (+ rod)

4 : Elliptical (+ circle)

5 : ellipsoidal (+ sphere)

⑤ Parallel axes theorem $I = I_G + Ma d^2$

⑥ Impulse of a Force $I = \int_{t_1}^{t_2} F \, dt = m(v_2 - v_1)$

⑦ Lagrange's equation $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$

or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$, $L = T - V$

⑧ Generalised momentum $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha}$

⑨ Hamiltonian form of equation

$\dot{p}_i = -\partial H / \partial q_i$, $\dot{q}_i = +\partial H / \partial p_i$

or $\frac{dp_i}{-\partial H / \partial q_i} = \frac{dq_i}{+\partial H / \partial p_i} = dt$

⑩ $H = \sum p_i \dot{q}_i - L = 2T - (T - V) = T + V = E$, $H = H(q, p) = H(x, p)$

⑪ Spherical pendulum (r, θ, ϕ), $T = \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

$V = -mgl\cos\theta$

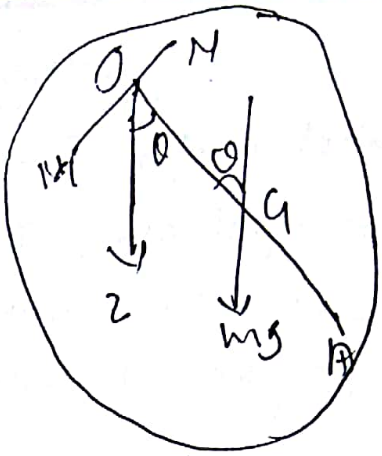
⑫ 1-D harmonic oscillator $T = \frac{1}{2} m\dot{x}^2$, $V = \frac{k}{2} x^2$, $m\ddot{x} + kx = 0$

⑬ Spherical polar coordinate $T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2)$

⑭ SHM in straight line $\ddot{x} = -\mu x$

$T = \frac{1}{2} m\dot{x}^2$, $V = \frac{1}{2} m \mu x^2$

(15) compound pendulum

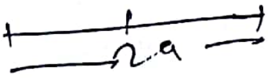


(16) Particle smooth inclined plane

$T = \frac{1}{2} m \dot{x}^2$, $V = -mgx \sin\alpha$

$\ddot{x} = g \sin\alpha$

(17) Moment of inertia



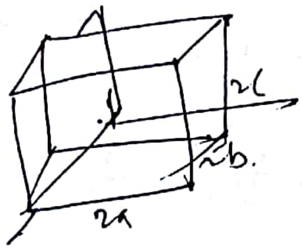
$\frac{1}{3} Ma^2$

$\frac{4}{3} Ma^2$

$I_{xx} = \frac{1}{3} Mb^2$

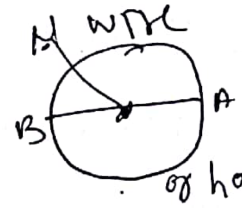
$I_{yy} = \frac{1}{3} Ma^2$

$I_{zz} = \frac{1}{3} M(a^2+b^2)$



$I_{xx} = \frac{1}{3} M(b^2+c^2)$

wire

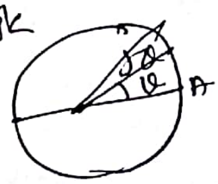


of hoop

$I_{AB} = \frac{1}{2} Ma^2$

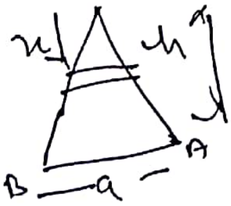
$I_{ON} = Ma^2$

disk



$I_{AB} = \frac{1}{4} Ma^2$

$I_{ON} = \frac{1}{2} Ma^2$



$I_{AB} = \frac{1}{6} Mh^2$

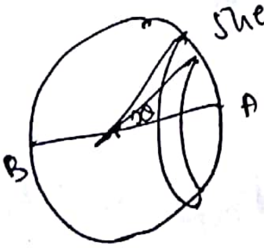
elliptical disk

$I_{xx} = \frac{1}{4} Mab^2$

$I_{yy} = \frac{1}{4} Ma^2$

$I_{oz} = \frac{1}{4} M(a^2+b^2)$

shell



$I_{AB} = \frac{2}{3} Ma^2$

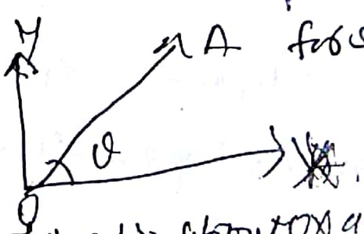
solid sphere $= \frac{2}{5} Ma^2$

ellipsoid $I_{ox} = \frac{1}{5} M(b^2+c^2)$

MI of the lamina about OA

$= A \cos^2\theta + B \sin^2\theta - F \sin 2\theta$

A: I_{ox}, B: I_{oy}, F = Product of inertia about OX and OY



F = 0 for symmetry

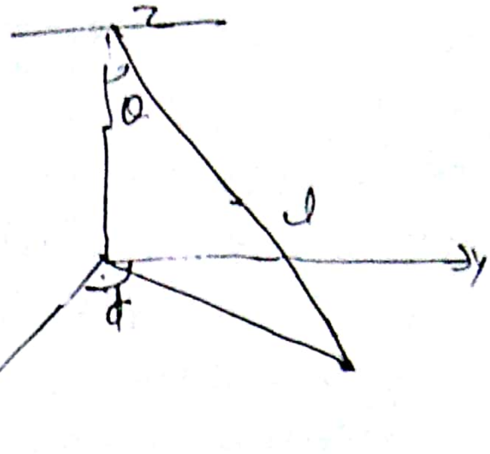
MI of circular hoop $= \frac{1}{2} Ma^2$

(18) Spherical pendulum

$x = l \sin\theta \cos\phi$

$y = l \sin\theta \sin\phi$

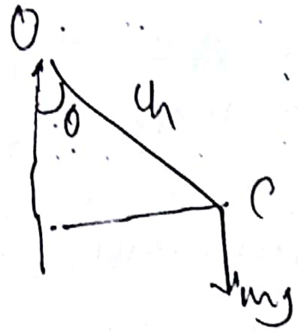
$z = l \cos\theta$



$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

$V = -mgl\cos\theta$

(19) compound pendulum



(20) Hamilton's principle $\int_{t_0}^{t} \delta L \, dt = 0$

(21) $H = \sum p_i \dot{q}_i - L = T + V = E$ (total energy)

(22) $\frac{dH}{dt} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \frac{dq_i}{dt} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{dp_i}{dt}$

(23) Euler's theorem on homogeneous function.

$\sum \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$ $L = T - V$

(24) $\frac{1}{2} m (k^2 \dot{\theta}^2 + \quad)$ K radius of gyration Sphere $K^2 = \frac{2}{5} a^2$

rod $K = \frac{a^2}{3}$ (2a)

$\theta = A \cos(nt + B)$

(25) $(\theta + 3\phi)$ principle coordinates $\theta + 3\phi = X$

$\Rightarrow D^2(\theta + 3\phi) = K(\theta + 3\phi)$ $12\theta - 3\phi = Y$

(26) For circular hoop $\boxed{K^2 = a^2}$ $I = Ma^2$

remember for different body

Modern Algebra

① Ring : $a, b, c \in R$, $(R, +, \cdot)$

$\boxed{A}$ $a+b \in R$, $a+b = b+a$, $a+(b+c) = (a+b)+c$, $a+0 = a$, $a+(-a) = 0$

$\boxed{M}$ $a \cdot b \in R$, $a(b \cdot c) = (a \cdot b) \cdot c$, $a \cdot (b+c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$

commutative ring $a \cdot b = b \cdot a$, $\forall$ all $a, b \in R$

boolean ring $x^2 = x$ $\forall$ all $x \in R$

Nilpotent $a^n = 0$, idempotent $a^2 = a$

ring with unity $e \cdot a = a \cdot e$ $\forall$ all $a \in R$

② Subring: $a - b \in S$, $ab \in S$ $\forall a, b \in S$ OR or $(S, +, \cdot)$ is a ring

③ Field: $a, b, c \in R$

$\boxed{A}$ $a+b \in R$, $a+b = b+a$, $a+(b+c) = (a+b)+c$, $a+0 = a$, $a+(-a) = 0$

$\boxed{M}$ $a \cdot b \in R$, $a \cdot b = b \cdot a$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, $a \cdot 1 = a$, $a \cdot a^{-1} = 1$ $a \cdot (b+c) = a \cdot b + a \cdot c$

④ Ideals in a ring R $a \in S$, $r \in R$ right left

$a \in S$, $b \in S \Rightarrow a - b \in S$, $ab \in S$ + $a \in S$, $r \in R \Rightarrow ar \in S$, $ra \in S$

⑤ characteristic of a ring $na = 0$ (least n) for all $a \in R$

⑥ Ring of integer modulo n, $Z_n = \{0, 1, 2 \dots n-1\}$, prime

⑦ . Division ring (skew field) of $(R, +, \cdot) \rightarrow$ Nonzero elements form a group w.r.t ⊙

⑧ commutative division ring $\rightarrow$ called a field

⑨ Ring of gaussian integer $Z[i] = \{m + ni ; m, n \in Z\}$

⑩ zero divisor $a \neq 0 \in R$ such that $\exists\, b \neq 0 \in R$ s.t. $ab = 0$

⑪ Integral domain : commutative ring with unity $e \neq 0$ called ID if it has no zero divisor $ab = 0 \Rightarrow a = 0$ or $b = 0$

or $a \neq 0$, $b \neq 0 \Rightarrow ab \neq 0$ $\forall a, b \in R$

⑫ Euclidean Domain (ED) : ID is called ED if $a \neq 0 \in R \Rightarrow$ non-negative integer $d(a)$ such that $d(a) \leq d(ab)$ $\forall a \neq 0, b \neq 0$

$\exists\, t, r \in R$ s.t. $a = tb + r \Rightarrow r = 0$ or $d(r) < d(b)$

⑬ ED $\Rightarrow$ PID $\Rightarrow$ UFD

⑭ Permutation of order m in S_n $\quad {}^nC_m \cdot \frac{m!}{m}$

(15) Homomorphism of ring $(R,+,\cdot)$, $(R', \oplus, \otimes)$, $f: R \to R'$

$f(a+b) = f(a) \oplus f(b)$, $f(a\cdot b) = f(a) \otimes f(b)$, $\forall\, a, b \in R$

monomorphism = homomorphism + one-to-one

isomorphism = homomorphism + one-to-one + onto

(16) Group : $a, b, c \in G$, $(G, *)$

$a*b \in G$, $a*(b*c) = (a*b)*c$, $a*e = e*a = a$, $a*a' = a'*a = e$

abelian/commutative group $a*b = b*a$, order of a group

Semigroup $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

Subgroup $H < G \Rightarrow ab^{-1} \in H\ \forall a, b \in H$ OR $ab \in H$, $a^{-1} \in H$

cyclic group $x = a^n = aaa\cdots$, $G = \langle a \rangle$ or (a)

Normal subgroup $gng^{-1} \in H$ for $g \in G$, $n \in H$

(17) Euler ϕ function $\phi(n) =$ no. of +ive integers less than n and relatively prime to n. $\phi(p) = p-1$

$\phi(1) = 1$, $\phi(4) = 2$

(18) Left coset $H < G$, $a \in G$ $\hookrightarrow aH = \{ah : h \in H\}$

Right coset $Ha = \{ha : h \in H\}$

(19) Lagrange theorem $o(H) \mid o(G)$, $H < G$

(20) $o(HK) = \dfrac{o(H)\, o(K)}{o(H \cap K)}$

(21) G/N is abelian (iff) $\Leftrightarrow$ $NxNy = NyNx\ \forall x, y \in G$

(22) Prime ideal : ideal P of R for which $a \in R$, $b \in R$; $ab \in P \Rightarrow a \in P$ or $b \in P$

(23) Irreducible element : commutative ring with unity, nonzero, nonunit $b \in R$ s.t. $b = ab \Rightarrow a$ is unit or b is unit

(24) Prime element $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

(25) Quotient ring $\dfrac{Z}{A} = \{A, 1+A, 2+A, \ldots\}$

(26) An element irreducible + not prime $\longrightarrow$ Not a ED
e.g. 3 $\in Z[\sqrt{-5}]$

(27) $S_3 = \{i, \sigma_1, \sigma_2, \sigma_3, t_1, t_2\}$, $S_3 = \{1, 2, 3\}$

(28) Klein's Group : $G = \{e, a, b, ab\}$, $a^2 = b^2 = e$, $ab = ba$ (not cyclic).
$O(G) = 4$, order of any element of $G \neq 4$

(29) Equivalence class $cl(a) = \{x \in A, x \sim a\}$

(30) Subgroup of $\frac{Z}{(12)}$ are $\frac{2}{(12)}, \frac{3}{(12)}, \frac{4}{(12)}, \frac{6}{(12)}, \frac{12}{(12)}$

$\frac{2}{(12)} = \{(12), 2+(12), 4+(12), 6+(12), 8+(12), 10+(12)\}$

(31) Cayley's theorem : Every group G is isomorphic to a permutation group

(32) In S_n, there are $\frac{1}{r} \frac{n!}{(n-r)!}$ distinct cycles of length r

(33) Z_6 is an integral domain

(34) Sum of two ideal of a ring R i.e.
$A+B = \{a+b : a \in A, b \in B$ is an ideal of R

(35) Product of two ideal of R
$AB = \{a_1 b_1 + a_2 b_2 + \cdots, a_i \in A, b_i \in B\}$ is an ideal of R.

(36) Division ring is a simple ring [use to prove simple ring]

(37) $f : R \to R' \Rightarrow \ker f = \{a \in R : f(a) = 0' \in R'\}$

(38) Anti homomorphism $f : R \to R'$ if
$f(x+y) = f(x) + f(y)$ & $f(xy) = f(y) f(x)$ $\forall x, y \in R$

(39) Quotient ring $\frac{R}{U} = \{a+U : a \in R\}$ defined when U is an ideal of R.

(40) Fundamental theorem of homomorphism
$f : R \to R'$ $\frac{R}{\ker f} \simeq R'$

(41) If R is commutative ring with unity then an ideal M is maximal ideal iff R/M is a field.

(42) For prime p $\frac{Z}{(p)}$ is a field

(43) An integral domain (ID) with unity is called a PID if each ideal of R is a principal ideal, $A = (a) = \{ar : r \in R\}$ $\forall a \in A$

(44) An ID with unity is called UFD if
(i) every nonzero, nonunit of R is expressible as a product of irreducible elements of R
(ii) Each irreducible element is prime.

(45) binary composition, $a * b \in S$, $\forall a \in S, b \in S$

(46) Number of generators of a finite cyclic group of order n is $\phi(n)$ $\quad \phi(6) = 2$

(47) $G = \{I, A, B, C\}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

finite abelian group which is not cyclic

(48) Not cyclic: no element of G has order $n = o(G)$

(49) $i_G(H) = \frac{o(G)}{o(H)}$

(50) If G is a finite group and $a \in G$, the order of a divides $o(G)$

(51) Every subgroup of a cyclic group is normal, since a cyclic group is abelian.

(52) Quaternion Group of order 8
$G = \{\pm 1, \pm i, \pm j, \pm k\}$, $i^2 = j^2 = k^2 = 1$, $i \cdot j = k$

(53) Any subgroup of index two is normal

(54) $S_3 = \{I, (12), (13), (23), (123), (132)\}$
$A_3 = \{I, (123), (132)\}$

(57) $A_4 = \{I, (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)$

(58) $o(G/H) = o(G)/o(H)$

(59) Not simple: $\exists$ exist a normal subgroup (nontrivial)

(60) $\frac{S_3}{A_3} = \{A_3, A_3(12)\}$

(61) Automorphism: $f: G \to G$ if f is homomorphism one-one onto

(62) A field has no proper ideals
(63) Division ring is ~~not~~ a simple ring
(64) If $f: R \to R'$ is a homomorphism, then
$f(0) = 0'$, $f(-a) = -f(a)$
(65) Nonprime element: $p \mid ab$ but $p \nmid a$, $p \nmid b$
(66) Coprime element: Two nonzero elements of a PID R are said to be relatively prime or coprime, if their gcd is a unit of R.
If a is any element of group G, then
(67) $$o(a^n) = \frac{o(a)}{(n, o(a))} = \frac{o(a)}{\gcd(n, o(a))}$$
order of disjoint permutation $(124)(357)$ $= \text{lcm}(3,3) = 3$
(68) even or odd permutation $(135) = (15)(13)$ even
(69) Go with basic approach
(70) Number of generators of a finite cyclic group of order n is $\phi(n)$, euler function.
(71) Dihedral Group
$G = \{x^i y^j : i = 0, 1;\ j = 0, 1, 2, \ldots n-1;$ where $x^2 = e$, $y^n = e$, and $xy = y^{-1}x\}$
(72) To prove field, prove $1 \in R$, $a^{-1} \in R$
(73) If S be an ideal of ring R, and $1 \in S$, then $S = R$
(74) If A and B are two ideals of a ring R, then $A \cup B$ is an ideal of R iff $A \subseteq B$ or $B \subseteq A$.
(75) Simple ring: Ring R is called simple ring, if (i) $\exists$ $a, b \in R$ s.t $ab \neq 0$ (ii) R has no proper ideals. i.e. $\{0\}$ and R are only ideals of R.
(76) Division ring is a simple ring (use it to prove)
(77) Commutative ring with unity is a field iff it has no proper ideals.
(78) order of an integral domain $n = p^m$, p prime if exist $m \in \mathbb{N}$
(79) $R_1 \times R_2$. $(a, b)(x, y) = (a \cdot x, b \cdot y)$

(80) Imbedding of rings: A ring R is said to be imbedded in a ring R', if $\exists$ an isomorphism of R onto R' i.e. $\exists$ $f: R \to R'$ s.t. (i) f is a homomorphism (ii) f is one-to-one

Also, R' is an extension ring or over ring of R.

(81) Every ring can be embedded in a ring with unity.

(82) Every integral domain can be embedded in a field.

(83) Quotient field: QF of an integral domain D may be defined as a pair (F, f), where F is a field and $f: D \to F$ is an isomorphism such that each $x \in F$ is expressible as the equivalence class $[f(a), f(b)]$ or $\frac{f(a)}{f(b)}$ for some $a, b \in D$ and $b \neq 0$.

(84) If R is a commutative ring with unity, then an ideal M of R is maximal ideal iff R/M is a field.

(85) For each prime p, $Z/(p)$ is a field

$(p) = \{px : x \in Z\}$

(86) HCF: Let R be a commutative ring ~~with unity~~ and $a \neq 0, b \neq 0 \in R$. A nonzero $d \in R$ is called HCF or GCD of a and b if $d = (a, b)$

(i) $d|a$, $d|b$

(ii) whenever $c \neq 0 \in R$ is such that $c|a$, $c|b$, then $c|d$

(87) Divisibility: Let R be a commutative ring, $a \neq 0 \in R$, $b \in R$. then we say 'a divides b' or $a|b$ if $\exists$ $c \in R$ such that $\boxed{b = ac}$ e.g. HCF of 4, 6 in Z

(88) Let R be commutative ring and $a \neq 0$, $b \neq 0 \in R$. A nonzero $c \in R$ is called LCM of a and b if $c = [a, b]$

(i) $a|c$, $b|c$

(ii) whenever $x \neq 0 \in R$ is such that $a|x$, $b|x$ then $c|x$

(89) Associates: Let R be commutative ring with unity. Two elements $a, b \in R$ are called associates if $\exists$ a unit $u \in R$ s.t. $b = ua$

written as: $a \sim b$ (associate)

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(90) unit: Let R be a CR1. an element $u \neq 0 \in R$ is called a unit iff u divides 1. $u \neq 0 \in R$ is called a unit iff $\exists$ $c \in R$ s.t

$$1 = uc = cu \quad \text{i.e. } u \neq 0 \in R \text{ is a unit iff } u^{-1} \in R$$

(91) Coprime/relatively prime element: Two nonzero elements of a PID R are said to be coprime if their g.c.d. is a unit of R. $(a,b) = 1$

(92) Two elements a and b of a PID R are coprime iff $\exists$ $x, y \in R$ s.t. $ax + by = 1$ $\quad d = (a,b)$

(93) Let R be a PID, which is not a field. Then an ideal $A = (a)$ is a maximal ideal iff 'a' is an irreducible element of R.

(94) $(a,b)\,[a,b] = ab$ $\quad$ ~~i.e.~~ GCD. LCM = a-b

(95) Polynomial ring: Let R be a ring. The ring of polynomial in the indeterminate x, $R[x]$

$$R[x] = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n;\ a_i \in R,\ n \geq 0 \in \mathbb{Z}\}$$

$f(x) = a_0 + a_1x + \cdots a_nx^n \in R[x]$, $g(x) = b_0 + b_1x + \cdots b_nx^n \in R[x]$

equality, sum, product $f(x)g(x)$

$a_i = b_i$, $\quad c_i = a_i + b_i$ $\quad c_i = a_0b_i + a_1b_{i-1} + \cdots a_ib_0$

(i) If R is commutative, then $R[x]$ also commutative

(ii) If R has unity 1, then $R[x]$ also has unity 1, where $1 = 1 + 0x + 0x^2 + \cdots$

(iii) If F is a field, then $F[x]$ is a commutative ring with unity but $F[x]$ not a field..

(iv) If R is an integral domain, then $\deg(f(x)g(x)) = \deg f(x) + \deg(g(x))$

(v) If R is an ID, then any irreducible element of R is an irreducible element of $R[x]$.

(vi) every ring R is isomorphic to a subring of $R[x]$ OR every ring R can be embedded in the polynomial ring $R[x]$.

(96) Division algorithm: (Theorem)
If $f(x), g(x) \neq 0 \in F[x]$ (field), then $\exists\ t(x), r(x) \in F[x]$ s.t.
$f(x) = t(x)\, g(x) + r(x)$
where $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(97) If F is a field, then $F[x]$ is a ED. → (degree)

(98) ~~Ideal A = (p)~~ Ideal $A = \langle p(x) \rangle \in F[x]$ is a maximal ideal
iff $p(x)$ is an irreducible element of $F[x]$.

(99) $Q[x]$ is different from Q → degree → units
↳ $f(x)\, g(x) = x^2 + 2x + 3$ (e.g.) irreducible / MI

(100) UFD: an ID is called a UFD, if
(i) every nonzero, nonunit element of R is expressible as a finite product of irreducible elements of R.
(ii) every irreducible element of R is prime.

(101) An element in a UFD is prime iff it is irreducible

(102) Irreducible polynomial: Let R be ID. A polynomial $f(x) \in R[x]$ is said to be irreducible over R, if whenever
$f(x) = g(x)\, h(x)$; $g(x), h(x) \in R[x]$, then
either $\deg(g(x)) > 0$ or $\deg(h(x)) > 0$
Further $f(x) \in R[x]$ is said to be reducible over R if
$f(x) = g(x) \cdot h(x)$, when $\deg g(x) > 0$, $\deg f(x) > 0$

(103) Any irreducible element of R is an irreducible element in $R[x]$

(104) Eisenstein's criteria of irreducibility over Q (sufficient cond)
$f(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n$, polynomial with integer coefficients
Let p be a prime s.t.
$p | a_0,\ p | a_1 \dots p | a_{n-1}$, $p \nmid a_n$, $p^2 \nmid a_0$
then $f(x)$ is irreducible over Q

(105) Eisenstein's criterion is not necessary for the irreducibility of any polynomial in $Z[x]$ over Q
e.g. ~~x~~+1

(106) If G is an infinite group, then no. of elements of order n would be multiple of $\phi(n)$
or $\phi(n) | n$ ~~n = …~~ here n = no. of elements

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(107) Prime ideal: An ideal P of a ring R is called a prime ideal, if for any $a \in R, b \in R$;

$ab \in P \Rightarrow$ either $a \in P$ or $b \in P$

(108) $A_n \subset A_m$ for $n < m$ $(A_4 \subset A_5)$

(109) $\phi(100) = \phi(2^2 \cdot 5^2) = (2-1)(2^1) \cdot (5-1)(5^1) = 40$

$\phi(40) = \phi(2^3 \cdot 5^1) = (2-1)(2^2) \cdot (5-1)(5^0) = 16$

(110) no of elements of order 2 in S_n

$= {}^nC_2 + \frac{1}{2!}\, {}^nC_2\, {}^{n-2}C_2 + \frac{1}{3!}\, {}^nC_2\, {}^{n-2}C_2\, {}^{n-4}C_2 + \cdots$

(111) No. of permutations of order r in S_n

$= \frac{1}{r}\, {}^nP_r = \frac{r!}{r}\, {}^nC_r$

(112) Product of disjoint cycles is commutative $\alpha\beta = \beta\alpha$ S_{10}

(113) no. of elements of order 4 in A_6 $\cdot\, {}^6C_4 \cdot \angle 3 = 90$

No. of elements of order 5 in A_6 ${}^6C_5 \cdot \angle 4 = 144$

(114) Derived set, limit point

(115)

(116) Centralizer of an element z of a group G is the set of elements of G which commute with z

$C_G(z) = \{x \in G,\ xz = zx\}$

(117) $\frac{K}{\phi(n)} =$ No. of subgroup of order n, where K is the no. of elements of order n. e.g. $\frac{8}{\phi(3)} = \frac{8}{2} = 4$

(118) Principle of induction

(119) Subgroup → nonempty

Irreducibility over $Q[x]$, $f(x) \cdot f(x+1)$, contradiction, deg

(120) Let $f(x) = a_0 + a_1 x \ldots$

(121) Content of polynomial: Let R be U.F.D. Let f be any nonzero polynomial in $R[x]$. The content of f is

$c(f) = \gcd(a_0, a_1, \ldots a_m)$

(122) Primitive: A polynomial in $R[x]$ (R being U.F.D) is said to be primitive, if its content is unit.

e.g. content of $4x^2 + 6x + 2 = 2$

(123) Z is U.F.D.

(124) Gauss's lemma
A noncontant polynomial $f(x) \in Q[x]$ is primitive if it has integer coefficients, the g.c.d. of its coefficients is 1, and its leading coefficient is +ive.

(125) Products of primitive polynomials is primitive

(126) ϕ is well defined
homomorphism
$x = y \Rightarrow \phi(x) = \phi(y)$

(127) First Sylow's Theorem (simple group)

(128) Group of order ~~n~~ not simple $\Rightarrow \exists$ a nontrivial normal subgroup of G

(129) $\phi(n)$ = total no. of generators of cyclic group of order n.

e.g.

(130) $\frac{G}{Z(G)} \cong Z_2 \oplus Z_2$.

(131) $\langle x^4+4 \rangle = \{(x^4+4) f(x) : f(x) \in Q[x]\}$

(132)

Complex Analysis

Paper II

① $\cos\theta + \cos(\theta+\alpha) + \cdots \cos(\theta+n\alpha) = \dfrac{\sin(n+1)\frac{\alpha}{2}}{\sin\alpha/2}\cos\left(\theta+\frac{n}{2}\alpha\right)$

② $\sin\theta + \sin(\theta+\alpha) + \cdots \sin(\theta+n\alpha) = \dfrac{\sin(n+1)\alpha/2}{\sin\alpha/2}\sin\left(\theta+\frac{n}{2}\alpha\right)$

③ Limits $\lim_{z\to z_0} f(z) = l$, $|f(z)-l| < \varepsilon$ whenever $0<|z-z_0|<\delta$

④ Continuity $\lim_{z\to z_0} f(z) = f(z_0)$ ① ② ③

⑤ uniform continuity: $f(z)$ is uniformly continuous in a region if for any $\varepsilon>0$ we can find $\delta>0$ such that $|f(z_1)-f(z_2)|<\varepsilon$ whenever $|z_1-z_2|<\delta$. Also, if $f(z)$ is continuous in a closed and bounded region. Then it is uniformly continuous there.

⑥ Limit of a sequence $|u_n - l| < \varepsilon$ for all $n>M$ or $\lim_{n\to\infty} u_n = l$

⑦ Infinite series $S_n = u_1+u_2+\cdots$ $u_n = \sum_{n=1}^{n} u_n$.

if $\lim_{n\to\infty} S_n = S$ exist → convergent series.

⑧ $f'(z) = \lim_{\Delta z\to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z} = \dfrac{dw}{dz} = \lim_{\Delta z\to 0}\dfrac{\Delta w}{\Delta z}$

Analytic/Regular functions

⑨ CR Equations $w = u+iv$

$u_x = v_y$, $u_y = -v_x$ $\quad u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$

⑩ Harmonic functions $u_{xx}+u_{yy}=0$, $v_{xx}+v_{yy}=0$

⑪ L' Hospital's Rule: if $f(z)$, $g(z)$ analytic and $f(z_0)=g(z_0)=0$

$\lim_{z\to z_0}\dfrac{f(z)}{g(z)} = \dfrac{f'(z_0)}{g'(z_0)}$

⑫ Singularities: Isolated singularities, poles, removal singularities essential singularities, singularities at infinity.

$f(z) = u+iv$ $\quad f'(z) = u_x(z,0) - iu_y(z,0)$

$f'(z) = v_y(z,0) + iv_x(z,0)$

⑬ Milne's Method

⑭ Complex potential $w = \underset{\text{potential}}{\phi} + i\psi_{\text{stream}}$

⑮ Analytic = CReq^n + four partial derivatives continuous

(16) Green's Theorem in plane: If $P(x,y)$, $Q(x,y)$ be continuous and have continuous partial derivatives in region R and on its boundary.

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy$$

(17) Cauchy's Theorem: If $f(z)$ is analytic in R and its boundary C, then

$$\oint_C f(z)\,dz = 0$$

also $\int_a^b f(z)\,dz = F(b) - F(a)$, $F'(z) = f(z)$

(18) Cauchy's Integral Formula: If $f(z)$ be analytic inside and on a simple closed curve C and 'a' is any point inside C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz, \quad f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

C is traversed CCW.

(19) Rouche's Theorem: If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C. Then $f(z)+g(z)$ and $f(z)$ have the same number of zeroes inside C.

(20) bounded + monotonic $\Rightarrow$ convergent

necessary cond^n $\lim_{n\to\infty} u_n = 0$

(21) Comparison test

i) if $\Sigma|v_n|$ converges and $|u_n| \le |v_n|$, then Σu_n converges absolutely.

ii) if $\Sigma|v_n|$ diverges and $|u_n| \ge |v_n|$, then $\Sigma|u_n|$ diverges but Σu_n may or may not converge.

(22) Ratio Test

if $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = L$, then Σu_n converges (absolutely) if $L<1$ and diverges if $L>1$. If $L=1$, the test fails.

(23) Root test

If $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$, then Σu_n converges (absolutely) if $L<1$ and diverges if $L>1$. If $L=1$, test fails.

(24) Integral test: If $f(x) \ge 0$ for $x \ge a$, then $\Sigma f(n)$ converges or diverges according as $\lim_{M\to\infty}\int_a^M f(x)\,dx$ converges or diverges.

(25) Raabe's test

If $\lim_{n\to\infty} n\left(1 - \left|\frac{u_{n+1}}{u_n}\right|\right) = L$, then Σu_n converges (absolutely) if $L>1$ and diverges or converges ~~[illegible]~~ conditionally if $L<1$. If $L=1$, test fails.

(26) Gauss's test: Suppose $\left|\frac{u_{n+1}}{u_n}\right| = 1 - \frac{L}{n} + \frac{C_n}{n^2}$ where $|C_n| < M$ for all $n > N$, then Σu_n converges (absolutely) if $L > 1$ and diverges or converges conditionally if $L \leq 1$

(27) Alternating series test: If $a_n \geq 0$, $a_{n+1} \leq a_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} a_n = 0$ then $a_1 - a_2 + a_3 - \ldots = \Sigma (-1)^{n+1} a_n$ converges

(28) Weierstrass M test: If $|u_n(z)| \leq M_n$, where M_n is independent of z in a region R and ΣM_n converges, then $\Sigma u_n(z)$ is uniformly convergent in R.

(29) Taylor's theorem: If $f(z)$ be analytic inside and on a simple closed curve C. Let a, $a+h$ be two points inside C. Then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots \frac{h^n}{n!} f^{(n)}(a) + \ldots$$

or $$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots \frac{f^{(n)}(a)(z-a)^n}{n!} + \ldots$$

region of convergence $|z-a| < R$

(30) Laurent series

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \ldots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \ldots$$

(31) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$

(32) Residue at $z = a$ for pole of order k

$$\text{Residue} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\}$$

(33) $\oint_C f(z)\,dz = 2\pi i \, \Sigma \text{Residue}$

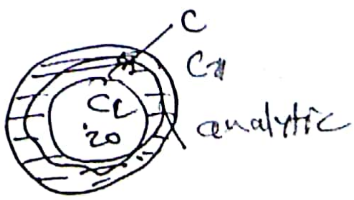
(34) If $\lim_{z\to\infty} z f(z) = 0 \Rightarrow \lim_{R\to\infty} \int f(z) = 0$

(35) $$L^{-1}\left\{ \frac{1}{(s+1)(s-2)^2} \right\} = \frac{1}{2\pi i} \int_C \frac{e^{st}\,ds}{(s+1)(s-2)^2}$$

(36) Cauchy's convergence principle for series

A series $z_1 + z_2 + \ldots$ is convergent iff for every $\varepsilon > 0$ we can find N such that $|z_{n+1} + z_{n+2} + \ldots z_{n+p}| < \varepsilon$ for $n > N$, $p = 1, 2, 3 \ldots$

(37) Radius of convergence $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$



(38) Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

(39) $\frac{1}{1-z} = \begin{cases} \sum_{n=0}^{\infty} z^n & |z|<1 \\ -\sum_{n=1}^{\infty} z^{-n} & |z|>1 \end{cases}$

(40) if $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$, $q(z)$ has simple zero, then

$\text{Residue}_{z \to z_0} = \text{Res}_{z \to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$ e.g. $\lim_{z\to\alpha} \frac{(z-\alpha)}{z^6+1} = \frac{1}{6z^5}$ $(z+1)$

(41) Residue at $z=0$ (or $u=0$) coefficient of $\frac{1}{z}$

at $z=\alpha$, $z-\alpha = u$ coefficient of $\frac{1}{u}$

(42) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$, $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

(43) For $f(z)$ even $\int_{-\infty}^{\infty} f(z)\,dz = 2\int_0^{\infty} f(z)\,dz$

(44) $\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz = 0$

(45) Leibnitz rule $\frac{d}{d\alpha}\int_a^b F(x,\alpha)\,dx = \int_a^b \frac{\partial F}{\partial \alpha}\,dx$

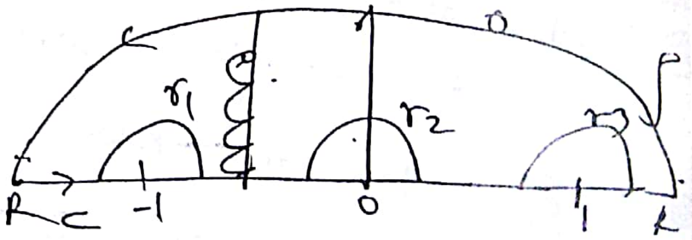
(46) Using the definition of integral $P \to Q$ along x axis + along y-axis

$\prod U_n \to \log U_n$ converges expansion subject.

(47) $\prod \left(\frac{n-2}{n+2}\right) e^{2z/n} \to$ $z\neq 0$, $z=0$ (automatically converge)

(48) Cauchy's residue thm

$\int_C f(z)\,dz = 0$



$\int + \int + \int + \int + \int + \int + \int + \int = 0$

$\int_{\gamma_1} f(z)\,dz = i(0-\pi)$ Resi at $z=-1$ etc.

$= -\pi i$ Res

(49)

Numerical Analysis

① Bisection method $x_l, x_u,\ f(x_l)\cdot f(x_u) < 0,\ x_m = \frac{x_l + x_u}{2}$

② Regula Falsi $x_{r+1} = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

③ Newton Raphson $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$ ε, δ

④ Gaussian elimination $\begin{pmatrix} \diagdown \\ 0 \end{pmatrix}(\ ) = (\ )$

⑤ Gauss Jordan elimination $\begin{pmatrix} \diagdown & 0 \\ 0 & \diagdown \end{pmatrix}(\ ) = (\ )$

⑥ Gauss-Seidel iterative method $x_1 = \text{—},\ x_2 = \text{—},\ x_3 = \text{—}$ → write formula $x_{n+1} =$

⑦ Newton's forward interpolation $P(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots$

$p = \frac{x - x_0}{h},\ f_0 = y_0$

⑧ Newton's backward interpolation $P(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n +$

$p = \frac{x - x_n}{h}.$

⑨ Lagrange's interpolation $f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$

$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right),\quad x_0 < x_1 < x_2 \cdots < x_{n-1} < x_n$

$f(x) \approx f_n(x)$ (polynomial)

⑩ Newton Cotes formula

⑪ Trapezoidal method $n=1$ $\int_a^b f(x)\,dx = \frac{b-a}{2}[f(a) + f(b)]$ $h = b-a$

$h = \frac{b-a}{n}$ $\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(a+ih)\right]$

⑫ Simpson's 1/3rd rule $n=2$ $\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],\ h = \frac{b-a}{2}$

$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + f(x_n) + 4\{f(x_1) + f(x_3) + \cdots\} + 2\{f(x_2) + f(x_4) + \cdots\}\right]$ (

⑬ Simpson's 3/8th rule

$\int_a^b f(x)\,dx = \frac{3h}{8}\left[f(x_0) + f(x_n) + 2\{f(x_3) + f(x_6) + \cdots\} + 3\text{(Rem.)}\right]$

$e = \frac{c}{4!} f^{(4)}(x)$

⑭ Gaussian Quadrature 2 point / 3 point $\int_a^b f(x)\,dx = c_1 f(x_1) + c_2 f(x_2)$ $\equiv$

$b - a = c_1 + c_2$

$\frac{1}{2}(b^2 - a^2) = c_1 x_1 + c_2 x_2$

$\frac{1}{3}(b^3 - a^3) = c_1 x_1^2 + c_2 x_2^2$

$\frac{1}{4}(b^4 - a^4) = c_1 x_1^3 + c_2 x_2^3$

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(15) Euler's method $y_{n+1} = y_n + h f(x_n, y_n)$ $E = f'(x_n, y_n)\frac{h^2}{2!}$

(16) Improved Euler $y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, \bar{y}_{n+1})]$ $\bar{y}_{n+1} = y_n + h f(x_n, y_n)$

(17) Runge Kutta 2nd order

$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$

$k_1 = f(x_n, y_n) \cdot h$

$k_2 = f(x_n + h, y_n + k_1) \cdot h$

$y' = f(x, y)$, $y(x_0) = y_0$

4th order $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(x_n, y_n)$

$k_2 = h f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$

$k_3 = h f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$

$k_4 = h f(x_n + h, y_n + k_3)$

(18) Gauss's formula: $n=1$, one point $\int_{-1}^{1} f(x)dx = 2f(0)$

$n=2$, two point formula, $\int_{-1}^{1} f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

$n=3$, three point formula, $\int_{-1}^{1} f(x)\,dx = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$

(19) Simpson's 3/8th error constant $C = \int_{x_0}^{x_n} f(x)\,dx - \frac{3h}{8}(\quad)$

error term remainder $= \frac{C}{4!} f^{(4)}(x)$

(20) Number system, 1's and 2's complement

(21) Logic gates, universal gate - NOR NAND

(22) Demorgan's law $(x+y)' = x'y'$, $(x \cdot y)' = x' + y'$

(23) sum of product (m), product of sum (M)

(24) Associative law $(p \vee q) \wedge r \Leftrightarrow p \vee (q \wedge r)$

$(p \wedge q) \vee r \Leftrightarrow p \wedge (q \vee r)$

(25) Distributive law $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(26) $p \to q \Leftrightarrow \neg p \vee q \Leftrightarrow \neg q \to \neg p$

(27) Tautology $\cancel{p \wedge \neg p \Leftrightarrow F}$ $p \vee \neg p \Leftrightarrow T$

(28) Contradiction $p \wedge \neg p \Leftrightarrow F$

(29) Equivalence $(p \to q) \wedge (q \to p) \Leftrightarrow (p \leftrightarrow q)$

(30) To get conjunctive normal form of a proposition, construct the disjunctive normal form of its negation and then ~~negate~~ negate again and apply De morgan's law.

(31) $\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) dx$

(32) Newton forward formula [to get acceleration at $t = x_n$]

$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots a_n(x-x_0)\dots(x-$

a_0, a_1, a_2 to be determined fitting the data

(33) Newton backward formula

$P_n(x) = b_0 + b_1(x-x_n) + b_2(x-x_n)(x-x_{n-1}) + \dots b_n(x-x_n)(x-$

$a_0 = y_0$, $a_1 = \frac{y_1 - y_0}{h}$ etc. $a_2 = \frac{\Delta^2 y_0}{2h^2}$, $a_k = \frac{\Delta^k y_0}{k!\,h^k}$

$k = 0, 1, 2, \dots n$

(34) Simpson → flow chart, NR → Algorithm, Euler → Algorithm, Lagrange → flowchar algorithm, NR → flowchart, Simpson → algorithm,

(35) bound on error $|E_2(f;x)| \le \frac{1}{6} M_3 \left(\max_{0 \le x \le 3} |x(x-1)(x-3)| \right) = .354$

$M_3 = \max_{0 \le x \le 3} |f'''(x)|$

(36)

Partial Differential Equations

① $\phi(u,v)=0$, compare $\partial\phi/\partial y \,/\, \partial\phi/\partial x$, $Pp+Qq=R$

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\quad Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z},\quad R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

② Lagrange's method $Pp+Qq=R$ auxilary eqn

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}\qquad u(x,y,z)=c_1,\ v(x,y,z)=c_2,\ \phi(u,v)=0$$

③ Surfaces orthogonal to given surface

$$\frac{dx}{f_x}=\frac{dy}{f_y}=\frac{dz}{f_z}\qquad u=\phi(v)\rightarrow \text{surface}$$

④ complete integral, Particular integral, singular integral, general integral.

⑤ complete integral of $f(x,y,z,p,q)=0$, $\phi(x,y,z,a,b)=0$

⑥ Singular integral $\phi(x,y,z,a,b)=0$, $\partial\phi/\partial a=0=\frac{\partial\phi}{\partial b}$ $\Rightarrow \psi(x,y,z)=0$

OR $f(x,y,z,p,q)=0$, $\partial f/\partial a=0$, $\frac{\partial f}{\partial p}=0$ $\Rightarrow \psi(x,y,z)=0$

⑦ General integral $\phi(x,y,z,a,g(a))=0$; $\partial\phi/\partial a=0 \Rightarrow \psi(x,y,z)=0$

⑧ Charpit's method $dz=p\,dx+q\,dy$, $z=\int(\,)\,dx+\int(\,)\,dy+c$

w.r.t x — not contain x

$f(x,y,z,p,q)=0$ $\quad \frac{dF}{0}=\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}$

Spl case — p,q,z
— p,q,x,y

⑨ $F(D,D')$ homogeneous function of D,D' $\quad \frac{1}{F(D,D')}\phi^{(n)}(ax+by)=\frac{1}{F(a,b)}\phi(ax+by)$

$ax+by=v$, $PI=\frac{1}{F(D,D')}\phi(v)=\frac{1}{F(a,b)}\iiint\phi(v)\,(dv)\,dv\ldots$

$F(a,b)=0$, $PI=\frac{1}{(bD-aD')}\phi(ax+by)=\frac{x^n}{b^n n!}\phi(ax+by)$

⑩ $PI=\frac{1}{F(D,D')}f(x,y)=\frac{1}{(D-m_1D')(D-m_2D')\cdots(D-m_nD')}f(x,y)$

⑪ $\frac{1}{D-mD'}f(x,y)=\int f(x,c-mx)\,dx$, $\quad y+mx=c$

⑫ $PI=\frac{1}{F(D,D')}\sin(ax+by)$ or $\cos(ax+by)$, $\quad D^2=-a^2$, $D'^2=-b^2$, $DD'=-ab$

⑬ $PI=\frac{1}{F(D,D')}V e^{ax+by}=e^{ax+by}\frac{1}{F(D+a,D'+b)}(V)$

(14) $PI = \frac{1}{F(D,D')} e^{ax+by} = \frac{1}{F(D,D')} e^{ax+by} \cdot 1 = e^{ax+by} \frac{1}{F(D+a, D'+b)} (\cdot 1)$

(15) $a_0 x^n D^n + a_1 x^{n-1} D^{n-1} D' + \dots a_n y^n D'^n = f(x,y)$

$x = e^u$, $y = e^v$, $xD = D_1$, $yD' = D_1'$

$x^n y^m D^n D'^m = D_1(D_1 - 1) \dots (D_1 - n + 1) \dots D_1'(D_1' - 1) \dots D_1' - m$

(16) $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

hyperbolic $S^2 - 4RT > 0$ characteristic u, v

parabolic $S^2 - 4RT = 0$ p, q, r, s, t

elliptic $S^2 - 4RT < 0$

(17) canonical form

hyperbolic $\frac{\partial^2 z}{\partial u \partial v} = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v)$

parabolic $\partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v)$, $J = \frac{\partial(u,v)}{\partial(x,y)} \neq 0$

$u, v \to$ independent

elliptic $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta})$

(18) λ - quadratic eqn $R\lambda^2 + S\lambda + T = 0$

(19) characteristic eqn $y' + \lambda(x,y) = 0$

(20) wave eqn $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$ $c^2 = \frac{k}{\rho\sigma}$

(21) Laplace eqn $u_{xx} + u_{yy} + u_{zz} = 0$

(22) poisson eqn $u_{xx} + u_{yy} + u_{zz} = f(x,y)$

(23) separation of variable

(24) clairaut's eqn $z = px + qy + f(p,q)$

(25) substitution $x = r\cos\theta$, $y = r\sin\theta$

(26) $pq = 1 \Rightarrow z = ax + \frac{y}{a} + c$

$p^2 + q^2 = 1 \Rightarrow z = ax + y\sqrt{1 - a^2} + c$ or $z = ax + by + c$

(27)

(28)

(29) canonical form $A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}) = 0$

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right) + T\left(\frac{\partial u}{\partial y}\right)^2$$

$$B = R\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\right) + \frac{S}{2}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right) + T\left(\frac{\partial v}{\partial y}\right)^2$$

(30) $P\,dx + Q\,dy + R\,dz = 0$

$F = (P, Q, R)$

Integrability cond^n $F \cdot (\text{curl } F) = 0$

or $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$

$\Rightarrow$ $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$ $\Rightarrow u = \text{const.}$

(31) check all three possible solutions $(\pm d^2, 0)$ for BVP/PDE

Linear Programming

① Standard form, restricted/unrestricted in sign, slack and surplus variables, decision variables, $x = x' - x''$, $x', x'' \geq 0$ optimal solution.

② $m+n$ variables and n- constraints (equations)

solution, feasible solⁿ (≥ 0), basic solution, basic feasible solⁿ, nonbasic variables, BFS(≥ 0), nondegenerate BFS, degenerate BFS, optimal BFS, unbounded solution ($z \to \infty$).

S.N. of basic solⁿ	Basic variable	Non basic variable	Value of BV	Value of Z	Is solⁿ feasible?	Nondgn?	feasible optimal

③ Simplex method: standard form (slack or artificial variables) → initial simplex table → Initial BFS → min nonnegative ratio → (if all ratios are -ive or infinity → unbounded solⁿ) → key element → elementary row operations → optimal solⁿ (max/min), multiple optimal solⁿ (Z coefficient under at least one non-basic variable to be zero in optimal solⁿ)

unbounded solⁿ: either Z is infinite or values of variables are ∞

④ Big M method: Minimisation problem

⑤ Simplex method: Starting with a basic solution → searching optimal solution in a systematic way

⑥ Transportation problem: Stepping stone (optimal solⁿ) → $m+n-1$ allocation

⑦ Assignment problem → optimality test → n ll lines → optimal.

⑧ Transportation problem: Try to use method such that no. of allocation are $\boxed{m+n-1}$

⑨ Simplex: minimum non-negative ratio, Initial BFS

write $Z - c_1x_1 - c_2x_2 - 0S_1 - 0S_2 = 0$

Minimization $Z = c_1x_1 + c_2x_2 + 0S_1 + 0S_2 + MA_1 + MA_2$ $a_{11}x_1 + a_{12}x_2 - S_1 + A_1 = b$

Big M method $a_{21}x_1 + a_{22}x_2 - S_2 + A_2 = b$

IBFS.. make Z coefficient under A zero to apply simplex

⑩ Transportation: UV method / maximization / minimization

allocate & avoiding   occupied cell / vacant cell ($u_i + v_j$) → cell evaluation $\boxed{c_{ij} - (u_i + v_j)} \leq 0$

change = loop, take shift loop formation → cell evaluation ≥ 0

Real Analysis

① Pointwise convergence: $x \in I$, $\forall \varepsilon > 0$ $\exists m$ s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq m$

$\lim_{n \to \infty} f_n(x) = f(x) \rightarrow$ pointwise limit

② Uniform convergence: If each $\varepsilon > 0$ $\exists m$ +ive (depending on ε only) such that $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq m$ and $\forall x \in I$. f is called the uniform limit of the sequence $\{f_n\}$ on I.

③ Uniform convergence $\Rightarrow$ pointwise convergence,

④ Uniformly bounded sequence $|f_n(x)| < K$ $\forall n \in \mathbb{N}$, $\forall x \in I$
$\hookrightarrow$ uniform bound of functions

⑤ Cauchy's criteria for uniform convergence for sequence of functions
$\forall$ each $\varepsilon > 0$, $\forall$ all $x \in I$ $\exists$ $m > 0$ such that for $p \geq 1$

$|f_{n+p}(x) - f_n(x)| < \varepsilon$ for $n \geq m$

⑥ M_n test for uniform convergence of sequence of functions $\lim_{n \to \infty} f_n(x) = f(x)$, $\forall x \in I$

$M_n = \sup \{|f_n(x) - f(x)|\}$, $f_n(x)$ converges uniformly

iff $\lim_{n \to \infty} M_n = 0$

⑦ Pointwise convergence of sequence $\{S_n\}$ of partial sums
$\sum_{n=1}^{\infty} f_n(x) = f(x)$ $\forall x \in I$

uniform convergence of series of functions

⑧ uniform convergence of series of functions, then $\sum f_n$ is said to converge uniformly on I if the sequence $\{S_n\}$ converges uniformly on I. — uniform limit function
$\varepsilon > 0$ $\exists M$ (depending on ε only) such that

$|S_n(x) - f(x)| < \varepsilon$ for $n \geq m$, $\sum f_n = f$

Cauchy's criterion

$|f_{n+1}(x) + f_{n+2}(x) + \cdots f_{n+p}(x)| < \varepsilon$ $\forall n \geq m$, $p \geq 1$ integer

⑨ Weierstrass's M-test: A series of function $\sum_{n=1}^{\infty} f_n$ converges uniformly (and absolutely) on I if $\exists$ a convergent series $\sum_{n=1}^{\infty} M_n$ of nonnegative terms s.t. $|f_n(x)| \leq M_n$ $\forall n \in \mathbb{N}$, $x \in I$

⑩ If a sequence of continuous functions $\{f_n\}$ is uniformly convergent to a function f on $[a,b]$, then f is continuous on $[a,b]$, [negative test], converse is not true.

(11) If a sequence $\{f_n\}$ converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$ then f is integrable on $[a,b]$ and the sequence $\{\int_a^b f_n(x)\,dx\}$ converges uniformly to $\int_a^b f(x)\,dx$.

(12) $\lim_{n\to\infty} \int_a^b f_n(x)\,dx \neq \int_a^b f(x)\,dx \Rightarrow$ can't converge uniformly

(13) uniform convergence and differentiation:
f_n is differentiable on $[a,b]$ + f_n' is continuous + f_n' converges uniformly to g + f_n converges to f, then f is differentiable on $[a,b]$ and $f'(x) = g(x)$ $\forall x \in [a,b]$.

(14) ________ $f'(x) = \lim_{n\to\infty} f_n'(x)$

(15) Abel's test: $\sum f_n(x)$ is uniformly convergent on $[a,b]$, $\sum g_n(x)$ is monotonically decreasing, $\exists$ K +ve s.t. $|g_n(x)| \le K$, then series $\sum f_n(x)\, g_n(x)$ converges uniformly on $[a,b]$.

(16) Dirichlet's test: $g_n(x)$ is a positive monotonically decreasing sequence converging uniformly to zero on $[a,b]$, then series $\sum f_n(x) g_n(x)$ is uniformly convergent on $[a,b]$.

(17) $\int_0^\infty e^{-z} \cdot z^a \, dz = \sqrt{a+1}$

(18) $a = x_0 \le x_1 \le x_2 \le \dots \le x_n = b$, m_i infimum, M_i supremum

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = M_1 \Delta x_1 + M_2 \Delta x_2 + \dots M_n \Delta x_n$

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = m_1 \Delta x_1 + m_2 \Delta x_2 + \dots m_n \Delta x_n$

$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$, $b \ge a$

$\overline{\int_a^b} f\,dx = \inf U$, $\underline{\int_a^b} f\,dx = \sup L$

if $\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx = \int_a^b f(x)\,dx$ (Riemann integrable)

(19) condition of integrability $U(P,f) - L(P,f) < \epsilon$

(20) $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$, $a \le c \le b$

(21) First MVT: f continuous on $[a,b]$

$\int_a^b f\,dx = f(\xi)(b-a)$.

(22) $\int_a^b fg\,dx = \mu \int_a^b g\,dx$, $\int_a^b fg\,dx = f(\xi) \int_a^b g\,dx$

(23) 2nd MVT: if $\int_a^b f\,dx$, $\int_a^b g\,dx$ both exist and monotonic on $[a,b]$, then $\exists\ \xi \in [a,b]$ s.t.

$$\int_a^b fg\,dx = f(a) \int_a^{\xi} g\,dx + f(b) \int_{\xi}^b g\,dx$$

(24) Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$ ...

(25) $\frac{\partial(u_1,u_2,u_3 \dots u_n)}{\partial(x_1,x_2,x_3,\dots x_n)} = \frac{\partial(u_1,u_2 \dots u_n)}{\partial(y_1,y_2 \dots y_n)} \cdot \frac{\partial(y_1,y_2,y_3 \dots y_n)}{\partial(x_1,x_2,x_3 \dots x_n)}$

(26) $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(-1)^n \partial(F_1,F_2,F_3)}{\partial(x,y,z)} \because \frac{\partial(u,v,w)}{\partial(F_1,F_2,F_3)}$

(27) $d^2F = F_{xx}\,dx^2 + 2F_{xy}\,dx\,dy + F_{yy}\,dy^2$ (Lagrange multiplier)

(28) Differentiability

must satisfy $df = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi$

ϕ and $\psi \to 0$ as $(h,k) \to (0,0)$, $A = f_x(0,0)$, $B = f_y(0,0)$

(29) Sufficient condition for differentiability

(i) f_x is continuous on $[a,b]$ (ii) f_y exist at $[a,b]$

(30) If f_x and f_y are both differentiable at (a,b) then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

(31) $f(\lambda x, \lambda y) = \lambda^n f(x,y)$, $\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

where f is homogeneous function of x, y

(32) $p = \frac{y - xy'}{\sqrt{1+y'^2}}$, length of $\perp$ from centre to the tangent.

(33) term by term differentiation/ integrable possible ← $\{f_n\}$ is uniformly convergent. Converse is not true.

uniform convergence and differentiation: If a sequence of functions

(34) $\langle f_n(x) \rangle$ such that

(i) each f_n is differentiable on $[a,b]$, (ii) each f_n' is continuous on $[a,b]$

(iii) $\{f_n\}$ converges to f on $[a,b]$ (iv) $\{f_n'\}$ converges uniformly to g on $[a,b]$

then f is differentiable and $f'(x) = g(x)$ $\forall x \in [a,b]$

(a) Same for series

(35) for $n>0$ $\lim_{x\to 0^+} x^n \log x = 0$

(36) continuous + bounded $\Rightarrow$ uniformly continuous

(37) Uniformly convergent $\rightarrow$ transfer of property from fns to f : continuity, differentiability, integrability,

(38) Mn test $\rightarrow$ sequence

(39) If the sequence $\{S_n\}$ of partial sums $[S_n = f_1+f_2+\dots f_n]$ converges uniformly on I, then the series Σf_n is uniformly convergent on I

or $|S_n(x) - f(x)| < \varepsilon \quad \forall n \geq m$

$\Sigma f_n = f$ uniform limit f of $\{S_n\}$

(40) If a sequence of continuous functions $\{f_n\}$ is uniformly convergent to f on $[a,b]$, then f is continuous on $[a,b]$ (– ive test for UC.)

(40) discontinuous $\rightarrow$ can't converge uniformly e.g. $f_n(x) = x^n$ $x \in [0,1]$

uniformly convergent $\rightarrow$ continuous

(41) $x=0$ is a point of nonuniform convergence $S_n(x) = \Sigma f_n(x)$ find $n >$ —

(42) Weierstrass M test (series $\rightarrow$ term) $f_n(x) \leq M_n(x)$ (M)

(43) $n! > 2^n$ for $n > 3$

(44) $\cos x + \cos 2x + \dots \cos nx = \dfrac{\cos\left(x + \frac{n-1}{2}x\right)\sin\frac{nx}{2}}{\sin x/2} = \dfrac{\cos\frac{n+1}{2}x \sin nx/2}{\sin x/2}$

(45) $\sin x + \sin 2x + \dots \sin nx = \dfrac{\sin\left(x + \frac{n-1}{2}x\right)\sin\frac{nx}{2}}{\sin x/2} = \dfrac{\sin\frac{n+1}{2}x \sin\frac{nx}{2}}{\sin x/2}$

(46) $\sum_{n=1}^{\infty} \dfrac{x}{n^p + n^q x^2}$ is uniformly convergent if $p+q \geq 2$

(47) Conditionally convergent $\rightarrow$ can be made to converge to any limit or can be made to diverge.

(48) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdot$

$= [1] - \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{5}\right) - \frac{1}{4} + \left(\frac{1}{7} + \frac{1}{9}\right) - \frac{1}{6} + \dots$

oscillate around 0.

(49) f is Riemann integrable = bounded & continuous e.g $(0,1]$ ∴ given fn

(50) $\forall x \in (0,1)$, $\dfrac{1}{\sqrt{4-x^2}} < \dfrac{1}{\sqrt{4-x^2-x^3}} < \dfrac{1}{\sqrt{4-2x^2}}$

(51) $C_n = \sum_{k=0}^{n} \frac{1}{k+1} \frac{1}{n-k+1} \Rightarrow C_n = a_1 b_{n+1} + a_2 b_n + \dots a_n b_1$