SuccessClap

Online Coaching for UPSC MATHEMATICS

QUESTION BANK SERIES

PAPER 1 : 06 Vector Analysis

Content:

SuccessClap : Question Bank for Practice

01 GRADIENT, DIVERGENCE,CURL

(1) Find the directional derivative of $\varphi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2 k. In what direction the directional derivative will be maximum and what is its magnitude? Also find a unit normal to the surface $x^2yz +4xz^2 =6$ at the point (1,-2,-1).

(2) For the function f = $y/(x^2+y^2)$, find the value of the directional derivative making an angle 30° with the positive x – axis at the point (0,1).

(3) If $\varphi(x,y) = \log\sqrt{(x^2+y^2)}$, show that
grad $\varphi = \frac{r-(k.r)k}{(\{r-(k.r)k\}.\{r-(k.r)k\}}$.

(4) If a and b be constant vectors then show that
grad[r a b] = a × b.

(5) Find the equations of the tangent plane and normal to the surface $2xz^2-3xy-4x = 7$ at the point (1,-1,2).

(6) Given the curve $x^2+y^2+z^2=1$, x+y+z=1 (intersection of two surfaces), find the equations of the tangent line at point (1,0,0).

(7) Find the angle between the surfaces $x^2+y^2+z^2 = 9$, and $z = x^2+y^2-3$ at the point (2,-1,2).

(8) If A is a differential vector function and φ is a differentiable scalar function, then
div$(\varphi A) = (grad\varphi).A + \varphi div\ A\ or\ (\nabla.\varphi A) = (\nabla\varphi).A + \varphi(\nabla.A)$.

(9) Find the constants a,b,c so that the vector F =(x+2y+az)i+(bx-3y-z)j +(4x+cy+2z)k is irrotational.

(10) Prove that the curl $(\varphi A) = (grad\varphi) \times A + \varphi curl A \nabla \times (\varphi A) = (\nabla\varphi) \times A + \varphi(\nabla \times A)$.

(11) Prove that $\nabla^2\left(\frac{x}{r^2}\right) = -\frac{2x}{r^4}$.

(12) Prove that div $\hat{r} = \frac{2}{r}$

(13) Prove that div $r^n r = (n+3)r^n$.

(14) Prove that curl $(r^n r) = 0$, i.e., $r^n r$ is irrotational.

(15) Prove that curl $(A\times B) = (B.\nabla)A - BdivA - (A.\nabla)B + A\ div\ B$.

(16) Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$.

(17) If $\nabla^2 f(r) = 0, show\ that\ f(r) = \frac{c_1}{r} + c_{2,} where\ r^2 = x^2+y^2+z^2$ and c_1,c_2 are arbitary constants.

(18) Prove that $\nabla^2\left(\frac{1}{r}\right) = 0$.

(19) Prove that div grad $r^n = n(n+1)\ r^{n-2}$, $\nabla^2 r^n = n(n+1)\ r^{n-2}$.

(20) Prove that curl $(\varphi\text{grad}\ \varphi) = 0$.

(21) If f and g are two scalar point functions, prove that
div $(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$.

(22) Prove that $\nabla.\left(r\nabla\left(\frac{1}{r^3}\right)\right) = \frac{3}{r^4}$.

(23) Prove that $b.\nabla\left(a.\nabla\frac{1}{r}\right) = \frac{3(a.r)(b.r)}{r^5} - \frac{a.b}{r^3}$ where a and b are constant vectors.

(24) If a is a constant vector, prove that curl $\frac{a\times r}{r^3} = -\frac{a}{r^3} + \frac{3r}{r^6}(a\text{-}r)$.

(25) Prove that div $\left\{\frac{f(r)r}{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$.

(26) If a and b are constant vectors, prove that
(i) div $[(r\times a)\times b] = -2b.a$,
(ii) curl $[(r\times a)\times b] = b\times a$.

(27) Show that $\nabla^2\left(\frac{x}{r^3}\right) = 0$

(28) Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator ∇ remains invariant.

(29) If $\varphi(x, y, z)$ is a scalar invariant with respect to a rotation of axes, then grad φ is a vector invariant under this transformation.

(30) If V (x,y,z) is a vector function invariant with respect to a rotation of axes, then div V is a scalar invariant under this transformation.

(31) Prove that
grad (A.B)=(B.∇)$A + (A.\nabla)B + \ B \times curl A + A \times curl\ B$.

(32) Prove that curl of the gradient of $\varphi\ is\ zero\ \nabla \times (\nabla\varphi) = 0, i.e\ curl\ grad\ \varphi = 0$.

(33) Prove that div (A× $B) = B.curlA - A.curlB$
$\nabla.(A \times B) = B.(\nabla \times A) - A.(\nabla \times B)$.

SuccessClap : Question Bank for Practice

02 GREEN, GAUSS, DIVERGENCE

(1). Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if force field F is given by $F = (2x - y + 2z)i + (x + y - z)j + (3x - 2y - 5z)k$.

(2). Evaluate

$$\int_c \{(2xy^3 - y^2\cos x)dx + (1 - 2y\sin x + 3x^2y^2)dy\}$$

Where C is the arc of the parabola $2x = \pi y^2$ from $(0,0)$ to $\left(\frac{1}{2}\pi, 1\right)$.

(3). Evaluate $\iint_s F.n\,ds,$ where $F = yzi + zxj + xyk$ and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

(4). Evaluate $\iint_s F.n\,ds,$ where $F = zi + xj - 3y^2zk$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

(5). Evaluate $\iint_s F.n\,ds,$ where $F = (x + y^2)i - 2xj + 2yzk$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

(6). Evaluate $\iint_s F.n\,ds,$ where $F = yi + 2xj - zk$ and S is the surface of the plane $2x + y = 6$ in the first octant cut-off by the plane $z = 4$.

(7). Evaluate by Green's theorem $\int_c e^{-x}(\sin y dx + \cos y dy)$, C being the rectangle with vertices $(0,0)(\pi, 0)\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

(8). Verify divergence theorem for

$$F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$$

Taken over the rectangular parallelepiped

$$0 < x < a, 0 < y < b, 0 < z < c.$$

(9). Evaluate $\iint_S x^2\, dydz + y^2 dzdx + 2z(xy - x - y)dxdy.$

Where S is the surface of the cube $0 < x < 1, 0 < y < 1, 0 < z < 1.$

(10). By transforming to a triple integral evaluate.

$$I = \iint_S (x^3\, dydz + x^2 ydzdx + x^2 zdxdy)$$

S is the closed surface bounded by the plane $z = 0, z = b$ and the cylinder $x^2 + y^2 = a^2$.

(11). Apply Gauss's divergence theorem to evaluate

$$\iint_S [(x^3 - yz)\, dydz - 2x^2 ydzdx + zdxdy]$$

Over the surface of a cube bounded by the coordinate planes and the plane $x = y = z = a$.

(12). Find $\iint_S A.n\, ds$, where

$$A = (2x + 3z)i - (xz + y)j + (y^2 + 2z)k$$

And S is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3.

(13). Evaluate

$$\iint_S (y^2 z^2\, i + z^2 x^2 j + z^2 y^2 k).n\, ds$$

Where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane.

(14). Evaluate $\iint_S F.n\, ds$, over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$, if F= $4xzi + xyz^2 j + 3zk$

(15). Show that $\iint_S (x^2 i + y^2 j + z^2 k)\, .n\, ds$ vanishes where S denotes the surface of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(16). If $F=(x^2+y-4)i+3xyj+(2xz+z^2)k$, evaluate $\iint_s(\nabla\times F).n\,ds$ where S is the surface of the sphere $x^2+y^2+z^2=16$ above the xy plane.

(17). Evaluate $\iint_s(\nabla\times A).n\,ds$, where $A=(x-z)i+(x^3+yz)j-3xy^2k$ and S is the surface of the cone $z=2-\sqrt{(x^2+y^2)}$ above the xy plane.

(18). Evaluate $\iint_s(ax^2+by^2+cz^2)ds$ Over the sphere $x^2+y^2+z^2=1$ using the divergence theorem.

(19). Verify Stoke's theorem for $F=(2x-y)i-yz^2j-y^2zk$, where S is the upper half surface of the $x^2+y^2+z^2=1$ and C is its boundary.

(20). Verify Stoke's theorem for $F=(x^2+y^2)i-2xyj$ taken round the rectangle bounded by $x=\pm a, y=0, y=b$.

(21). Verify Stoke's theorem for $F=(-y^2)i+x^2j$, where S is the circular disc $x^2+y^2\le 1, z=0$.

(22). Evaluate by Stoke's theorem

$$\oint_c(e^xdx+2ydy-dz)$$

Where C is the curve $x^2+y^2=4, z=2$.

(23). Evaluate

$$\oint_c(xydx+xy^2dy)$$

By Stoke's theorem where C is the square in the xy plane with vertices $(1,0),(-1,0),(0,1),(0,-1)$.

(24). Evaluate by Stoke's theorem

$$\oint_{c}(\sin z dx - \cos x dy + \sin y dz)$$

Where C is the boundary of the rectangle.

$$0 < x < \pi, 0 < y < 1, z = 3.$$

(25). Apply Stoke's theorem to prove that

$$\int_{c}(ydx + zdy + xdz) = -2\sqrt{2}\,\pi a^2$$

Where C is the curve given by

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$$

And begins at the point $(2a, 0,0)$ and goes at first below the plane.

(26). Use Stoke's theorem to evaluate $\iint_{s}(\nabla \times F).n\,ds$ and where $F = yi + (x - 2xz)j - xyk$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above xy plane.

(27). Evaluate the surface integral $\iint_{s}\text{curl}\, F.n\,ds$ by transforming it into a line integral, S being that part of the surface of the paraboloid

$$z = 1 - x^2 - y^2 \text{ for which } z > 0, \text{and } F = yi + zj + xk.$$

(28). If $F = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$, evaluate $\iint_{s}\text{curl}\, F.n\,ds$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ above the plane $z = 0$, Verify Stoke's theorem.

(29). Using Green's theorem evaluate

$$\int_{c}(x^2 - y^2)dx + 2xydy$$

Where C is the curve of the region bounded by $y^2 = x$ and $x^2 = y$.

(30). Evaluate

$$\int_{c}(\sin x - y)dx - \cos x dy,$$

Where C is the triangle with vertices $(0,0)$ $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 1\right)$.

SuccessClap : Question Bank for Practice

03 DIFFERENTIAL GEOMETRY

(1) State and prove servet Frenet formula.

(2) Show

(a) $\dot{v} = \dot{s}\, t$

(b) $\ddot{v} = \ddot{s}\, t + k\, \dot{s}^2\, n$

(c) $\dddot{v} = (\dddot{s} - k^2 s^3) t + \dot{s}(3k\, \ddot{s} + \dot{k}\dot{s}) n + k\, c\, \dot{s}^3 b$

Hence deduce

(d) $n = \dfrac{\dot{s}\ddot{v} - \ddot{s}\dot{v}}{k\dot{s}^3}$

(e) $b = \dfrac{\dot{v} \times \ddot{v}}{k\dot{s}^3}$

(f) $k^2 = \dfrac{|\ddot{r}| - \ddot{s}^2}{\dot{s}^4}$

(g) $c = \dfrac{[\dot{v}\ \ddot{v}\ \dddot{v}]}{k^2 \dot{s}^6}$

(3) For helix r = a cos t + a sin t + bt k a > 0, b ≠ 0. Find curvature at "t".

(4) Let v = v(s) be a curve. Prove

(i) $v' . v'' = 0$

(ii) $v''' = -k^2 t + k' n + kcb$

(iii) $v'.v''' = -k^2$

(iv) $v'' . v''' = kk'$

(v) $v'''' = -3kk' t + (k'' - k^3 - kc^2) n + 2(k'c + c'k) b$

(vi) $v'. v'''' = -3kk'$

(vii) $v'' . v'''' = k(k'' - k^3 - kc^2)$

(viii) $v'''. v'''' = k'. k'' + 2k^3 k' + k^2 cc' + kk'c^2$.

(ix) $[t'\ t''\ t'''] = k^3(kc' - k'c)$

$= k^5 \dfrac{d}{ds}\left(\dfrac{c}{k}\right)$

(x) $[b'\ b''\ b'''] = c^3 (k'c - kc')$

$= c^5 \dfrac{d}{ds}\left(\dfrac{k}{c}\right)$

(5) Find the curvature vector and curvature at t =1 for
v = t i+ ½ t^2 j + 1/3 t^3 k

(6) Prove that a regular curve of class c^m (m ⇒z) is a straight line if and only if its curvature is identically zero.

(7) Show k = $\frac{|\dot{v}\times\ddot{v}|}{|\dot{r}|^3}$

(8) Show radius of curvature at t for
r = a cos t + a sin t j a > 0 is a.

(9) For v = x(t) i + y(t) j k = $\frac{|\dot{x}\ddot{y}-\ddot{x}\dot{y}|}{(\dot{x}^2+\dot{y}^2)^{3/2}}$

(10) Show curvature of v = (t – sin t) + (1- cos t) j + t k is $\frac{(1+4\,sin^4\frac{t}{2})^{1/2}}{(1+4\,sin^2\frac{t}{2})^{3/2}}$

(11) For x = 4a $\cos^3$t , y = 4a $\sin^3$ t, z = 3 c cos2t
Show k = $\frac{a}{6(a^2+c^2)sin2t}$

(12) Show radius of curvature at any point of curve
x^2+ y^2 = a^2, x^2 – y^2 = az is $\frac{(5a^2-4z^2)^{3/2}}{a\sqrt{5a^2+12z^2}}$.

(13) Find equation of asculating plane and curvature at t of $\begin{matrix} x = a\ cos2t \\ y = a\ sin2t \\ z = 2a\sin t \end{matrix}$

is (sin t + sin 2t cos t)x –2$\cos^2$ t y+2z =3a sin t.

k = $\frac{\sqrt{5+3cos^2t}}{2a(1+cos^2t)^{3/2}}$

(14) For curve x = 3t, y = 3t^2, z = 2t^3 show p = $\frac{3}{2}(1+2t^2)^2$.

(15) Find curvature of v = a(t –sin t)i + a(1-cos t)j + bt k

Ans: $\frac{a(b^2+4a^2sin^4\frac{t}{2})^{1/2}}{(b^2+4a^2sin^2\frac{t}{2})^{3/2}}$

(16) For helix x = a cos t

y = a sin t

z = a t cot ∝, show k = $\frac{1}{a}$ $\sin^2 \propto$

(17) For x = t , y = t^2 , z = t^3

$k^2 = \frac{4(9t^4+9t^2+1)}{(9t^4+4t^2+1)^3}$

(18) For x =a(3t-t^3), y = 3at^2, z = a(3t+t^3). Show k = $\frac{1}{3a(1+t^2)^2}$.

(19) Find curvature (i) y = x^2 Ans: $\frac{2}{(1+4x^2)^{3/2}}$

(ii) xy = λ Ans: $\frac{2\lambda x^3}{(x^4+\lambda^2)^{3/2}}$

(20) Find curvature (a) v = a cos t i + b sin t j Ans: $\frac{ab}{(a^2 sin^2 t+b^2 cos^2 t)^{3/2}}$

(b) v = cosh t $\hat{\imath}$ + sinh t j

Ans: $\frac{1}{cosh^2 2t}$

(c) v = t i + $t^{3/2}$ j

Ans: $\frac{6}{\sqrt{t}\ (4+9t)^{3/2}}$

(21) Find torsion for v = a cos t $\hat{\imath}$ + a sint j +bt L

Ans: $\frac{b}{a^2+b^2}$

(22) Show along curve kc = |t'. b'|

(23) Show along curve c = [t n n']

(24) Show c = $\frac{[v'\ v'' v''']}{k^2}$

(25) Show c = $\frac{[\dot{v}\ \ddot{v}\ \dddot{v}]}{|\dot{v}\times\ddot{v}|^2}$

(26) Show v = a cost i + a sint j + bk +a > 0 b$\neq$ 0 *is a plane* curve.

(27) Show r = (t, $\frac{1+t}{t}$, $\frac{1-t^2}{t}$) lies on the plane.

(28) Find torsion v = t i + t^2 j + t^3k Ans: $\frac{3}{1+9t^2+9t^4}$.

(29) Find torsion r = (at – a sin t)i +(a- a cos t)j +bt k

Ans: $\frac{-b}{b^2+4a^2 sin^4\frac{t}{2}}$

(30) For curve x = a tan t, y = a cot t, z = $\sqrt{z}\ a$ log tan t

Prove p = $\sigma = \frac{2\sqrt{2}\ a}{sin^2\ 2t}$

(31) Prove for curve of intersection of $x^2+y^2 = z^2$ and z =a tan''$\frac{y}{x}$

P = $\frac{a(2+\theta^2)^{3/2}}{8+5\theta^2+\theta^4)^{\frac{1}{2}}}$ $\quad \sigma = \frac{a(8+5\theta^2+\theta^4)}{6+\theta^2}$ $\quad where\ y = x \tan\theta$

(32) For a point on curve of intersection x^2-y^2 = c^2 and y = x tanh $\frac{z}{c}. Show\ p = \sigma = \frac{2x^2}{c}$

(33)

(34) Determine the function f(y) so that curve

v = (a cos u, a sin u, f(u)) should be a plane curve.

Ans: f(u) = A sin u+B cos u+C.

(35) If tangent and binomial at a point of curve make angle $\theta\ and\ \varphi\ with$ a fixed drx.

Show $\frac{\sin\theta}{\sin\varphi}\frac{d\theta}{d\varphi} = \frac{-k}{c}$.

(36) If $\frac{dt}{ds} = w \times t, \frac{dn}{ds} = w \times n, \frac{db}{ds} = w \times b. Find\ w.$

Ans: w = c t +kb w: Darboux vector.

(37) Find Direction cosines of

(a) Unit principal normal vector

(b) Unit Binomial vector.

(38) Show that position vector of current point on curve

v = v(s) satisfies eqn $\frac{d}{ds}\left[\sigma\frac{d}{ds}\left(p\frac{d^2v}{ds^2}\right)\right] + \frac{d}{ds}\left[\frac{\sigma}{p}\frac{dr}{ds}\right] + \frac{p}{\sigma}\frac{d^2v}{ds^2}$ =0

(39) Find torsion

(a) v = t i +t^2j+t^3k at t =2

(b) v = (3t-t^3)i+3t^2j+(3t+t^3)k

(c) v =(t-sint)i+(1-cost)j +t k

(d) x = a cost, y= a sin t, z = at tan ∝.

(40) For x =a(3t-t^3), y=3at^2, z =a(3t+t^3).

Show k = c = $\frac{1}{3a(1+t^2)^2}$.

(41) For x =3t, y=3t^2, z =2t^3

k =c= $\frac{z}{3(1+2t^2)^2}$.

(42) Find torsion

(a) x^2+y2=a^2} Intersection point

x^2-y^2 =az

Ans: $\frac{6\sqrt{a^2-z^2}}{5a^2+12z^2}$

(b) x = a cos 2t

y = a sin 2t}

z = 2a sin t

Ans: $\frac{3}{(5sect+3\cos t)}$

(43) Find cuvature and torsion of x =a cost, y = a sin t, z = ct

Ans: $\frac{a}{a^2+c^2}$, $\frac{c}{a^2+c^2}$

(44) (a) Find length of helix for 0 ≤ t ≤ 2π

r = a cos t i+a sin t + bt k

(b)Find length of 0 ≤ t ≤ π

V=4 cosh 2t i+4sinh 2t j +8t k.

(c) Find length of v =t i +$t^{3/2}$j from (0,0,0) to (4,8,0).

(44) (a) Show length of the curve x = 2a(sin t + t$\sqrt{1-t^2}$) ,

$y = 2at^2$, $z = 4at$ between points $t = t_1$ and $t = t_2$ is $4\sqrt{2}\, a\,(t_2 - t_1)$.

(b)Find arc length as a function of θ

$x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\,\theta\right)$, $y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right)$,

$z = 0$

Ans: $\frac{4(a+b)b}{a}\left[1-\cos\frac{a\theta}{2b}\right]$

(c)Find length of the curve given by intersection

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $x = a\cosh\frac{z}{a}$ from (0,0,0) to (x,y,z).

Ans: $\frac{\sqrt{a^2+b^2}}{b}y$

(d)Length of helix v = 3cost i+3 sin j+4 tk

$0 \le t \le 2\pi$

(e)Find length of one complete turn of helix v = (a cost, a sin t, bt) -

$\infty < t < \infty \quad a > 0, b > 0$

(f)Find length of curve in $0 \le t \le \pi$

v = 3 cosh 2t i+3 sin 2t j+6t k.

(g)Find length of curve in t =1 to t =3

v= (sin t+t$\sqrt{1-t^2}$)$i + t^2 j + 2t\ k$

(h)Length of curve by intersection of $x^2-y^2=1$,

x = cosh z from (1,0,0) to (x,y,z).

(45)Find equation of helix v = a cost i+a sint j +ctk

$-\infty < t < \infty$ in terms of arc length "s" as parameter.

(46)Express v = e^t cost i + e^t sint j+e^t k

$-\infty < t < \infty$ in terms of arc length "s".

(47)Find the unit tangent vector t and direction cosines of tangent at a point on circular helix

x= a cos t, y = a sin t, z = bt $\quad -\infty < t < \infty$

(48)Show that the tangent vectors along the curve $v = at\, i+bt^2\, j+t^3\, k$ where $2b^2 = 3a$ make a constant angle with vector $i + k$.