Problem 32. *λ denoting a variable parameter, and f a given function, find the condition that $f(x, y, \lambda) = 0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.*

Solution. Suppose $f(x, y, \lambda) = 0$ represents stream lines for different values of λ. Solving this equation, we get

$$\lambda = F(x, y). \qquad \ldots (1)$$

We also know that $\psi =$ const. represents stream lines. So we can suppose that (1) and $\phi = c$ both represent the same stream lines. It means that

$$\psi = \psi(\lambda). \quad \text{Now } \frac{\partial \psi}{\partial x} = \frac{d\psi}{d\lambda}\frac{\partial \lambda}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\cdot\frac{\partial^2 \lambda}{\partial x^2} \qquad \dots (3)$$

But the motion is irrotational and so $\nabla^2 \psi = 0$.

i.e.,
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

In view of (3), this becomes

$$\left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial x^2}\right] + \left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial y^2}\right] = 0.$$

or
$$\frac{d^2 \psi}{d\lambda^2}\left[\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left[\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}\right] = 0$$

This is the required condition.

27 (8c)

(c) Prove that

$$\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$$

is a possible form for the bounding surface of a liquid and find the velocity components. 1

Problem 17. Show that $\dfrac{x^2}{a^2}\tan^2 t + \dfrac{y^2}{b^2}\cot^2 t - 1 = 0$ ... (1)

is a possible form of boundary surface and find an expression for normal velocity.

Solution : To show that $F = 0$ is a possible form of boundary surface, we have to show that

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0 \qquad \text{... (2)}$$

Putting the values of various terms, we get

$$u\,\frac{2x}{a^2}\tan^2 t + v\,.\,\frac{2y}{b^2}\cot^2 t + w.0 + \left(\frac{2x^2}{a^2}\tan t\sec^2 t - \frac{2y^2}{b^2}\cot t\ \operatorname{cosec}^2 t\right) = 0$$

or $$\frac{2x}{a^2}\tan^2 t\left(u + \frac{x\sec^2 t}{\tan t}\right) + \frac{2y}{b^2}\cot^2 t\left(v - \frac{y\operatorname{cosec}^2 t}{\cot t}\right) = 0.$$

Thus (2) will be satisfied if we take

$$u + \frac{x\sec^2 t}{\tan t} = 0, \quad v - y\frac{\operatorname{cosec}^2 t}{\cot t} = 0,$$

i.e., $$u = \frac{-x}{\sin t\cos t}, \quad v = \frac{y}{\sin t\cos t}.$$

This will be a justifiable step if the equation of continuity, namely

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ is satisfied.}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{\sin t\cos t}+\frac{1}{\sin t\cos t}+0=0.$$

Hence (1) is a possible form of boundary surface.

Second Part. Normal velocity $=\dfrac{-\partial F/\partial t}{|\nabla F|}$

$$=\frac{-\left(\dfrac{2x}{a^2}\tan t\sec^2 t-\dfrac{2y}{b^2}\cot t\operatorname{cosec}^2 t\right)}{\left[\left(\dfrac{2x}{a^2}\tan^2 t\right)^2+\left(\dfrac{2y}{b^2}\cot^2 t\right)^2\right]^{1/2}}$$

$$=-\frac{(b^2x\tan t\sec^2 t-a^2t\cot t\operatorname{cosec}^2 t)}{(b^4x^2\tan^4 t+a^4y^2\cot^4 t)^{1/2}}$$

24 (6b)

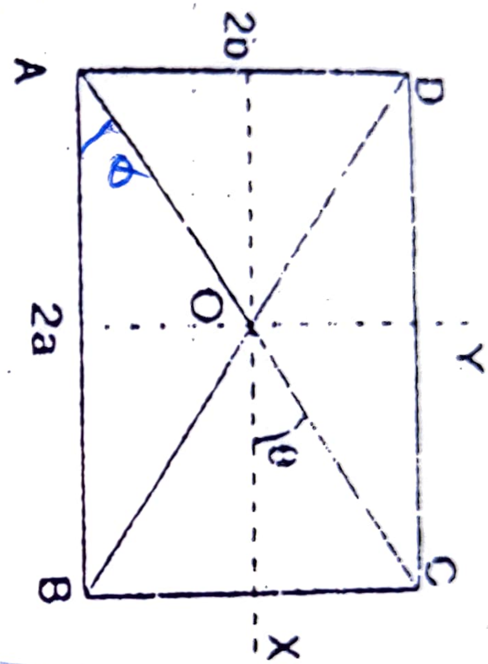
(b) Show that the moment of inertia of a uniform rectangular mass M and sides $2a$ and $2b$ about a diagonal is $\frac{2Ma^2b^2}{3(a^2+b^2)}$. 10

Ex. 17. *Show that M.I. of a rectangle of mass M and sides 2a, 2b about a diagonal is* $\frac{2M}{3}\frac{a^2b^2}{a^2+b^2}$.

Deduce that in case of a square.

Sol. Let $ABCD$ be a rectangle of mass M and $AB = 2a$, $BC = 2b$

Then M.I. of rectangle about $OX = A = \frac{1}{3}Mb^2$,



and M.I. of rectangle about $OY = B = \frac{1}{3}Ma^2$.

P.I. of the rectangle about OX and $OY = F = 0$. *(By symmetry)*

If diagonal AC make an angle θ with AB, then

$$\cos\theta = \frac{AB}{AC} = \frac{2a}{\sqrt{(4a^2+4b^2)}} = \frac{a}{\sqrt{(a^2+b^2)}}$$

and $$\sin\theta = \frac{BC}{AB} = \frac{b}{\sqrt{(a^2+b^2)}}$$

$\therefore$ M.I. of the rectangle about AC

$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$ (see equation (1), § 1.16)

$$= \frac{1}{3}Mb^2 \cdot \frac{a^2}{a^2+b^2} + \frac{1}{3}Ma^2 \cdot \frac{b^2}{a^2+b^2} - 0 = \frac{2M}{3} \cdot \frac{a^2b^2}{a^2+b^2}.$$

Deduction. For a square, $2b = 2a$.

$\therefore$ M.I. of square about AC

$$= \frac{2M}{3} \cdot \frac{a^4}{a^2+a^2} = \frac{1}{3}Ma^2.$$

25 (7c)

(c) A uniform rod OA of length $2a$ is free to turn about its end O, revolves with uniform angular velocity ω about a vertical axis OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or

$$\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$$

1

which will determine the motion of the rod.

5(ii) Ex. ~~5~~ A *uniform rod* OA, *of length* $2a$, *free to turn about its end* O, *revolves with uniform angular velocity* ω *about the vertical* OZ *through* O, *and is inclined at a constant angle* α *to* OZ, *show that the value of* α *is either zero or* $\cos^{-1}(3g/4a\omega^2)$.

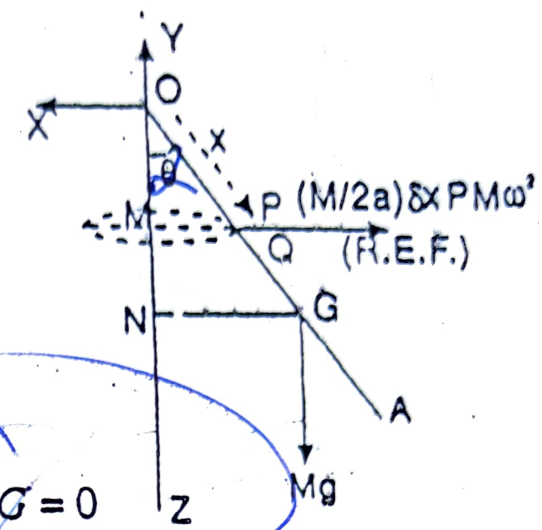
[MeerutTDC 92, 94(P), 95(BP) ; Rohilkhand 83]

Sol. Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let $PQ = \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\dfrac{M}{2a}\delta x$.

This element PQ will make a circle in the horizontal plane with radius PM $(= x \sin \alpha)$ and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\dfrac{M}{2a}\delta x \cdot PM \cdot \omega^2$ along PM.

Thus the reversed effective force on the element PQ is

$\frac{M}{2a}\,\delta x\,.\,x\sin\alpha\,.\,\omega^2$ along MP.



Now by D' Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and rection at O are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\sum\left(\frac{M}{2a}\,\delta x\,.\,\omega^2 x\sin\alpha\right).\,OM - Mg\,.\,NG = 0$$

or $$\int_0^{2a}\frac{M}{2a}\,\omega^2x^2\sin\alpha\cos\alpha\,dx$$

$$-Mg\,.\,a\sin\alpha = 0,\qquad (\because\ OM = x\cos\alpha)$$

or $$\frac{M}{2a}\,\omega^2\,.\left\{\frac{1}{3}(2a)^3\right\}.\sin\alpha\cos\alpha - Mg\,a\sin\alpha = 0$$

or $$Mg\,a\sin\alpha\left(\frac{4a}{3g}\,\omega^2\cos\alpha - 1\right) = 0$$

$\therefore$ either $\sin\alpha = 0$ i.e. $\alpha = 0$

or $$\frac{4a}{3g}\,\omega^2\cos\alpha - 1 = 0, \text{ i.e. } \cos\alpha = \frac{3g}{4a\omega^2}$$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$

Note. If $\omega^2 < \frac{3g}{4a}$, then $\cos\alpha > 1$, $\therefore$ in this case $\cos\alpha = \frac{3g}{4a\omega^2}$ gives an impossible value of α i.e. when $\omega^2 < \frac{3g}{4a}$, then $\alpha = 0$ is the only possible value of α.

26 (8b)

(b) A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

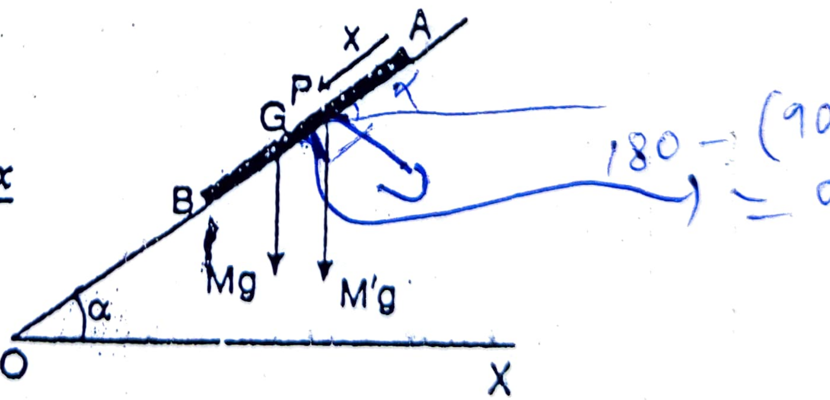
$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

where a is the length of the plank.

Ex. 9. *A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M′, starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time* $\sqrt{\left\{\dfrac{2M'a}{(M+M')\,g\sin\alpha}\right\}}$, *where a is the length of the plane.*

[Meerut, 84, 85, 87, 89, TDC 94(R), 97; Kanupr 82;]

Sol. Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an angle α to the horizon. A man of mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance $AP=x$ in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C. G. of the plank and the man from A in this position, then



$$\bar{x}=\frac{M\,.\,AG+M'\,.\,AP}{M+M'}=\frac{M\,.\,(a/2)+M'x}{M+M'}$$

Differentiating twice w. r. t., 't', we get

$$\ddot{\bar{x}}=\frac{M'}{M+M'}\ddot{x}, \qquad \text{...(1)}$$

Now the total weight $(M+M')g$ will act vertically downwards at the C. G. of the system.

∴ The equation of motion of the C. G. of the system is given by

$$(M + M')\ \ddot{\bar{x}} = (M + M')\ g \sin \alpha. \qquad \ldots(2)$$

$\therefore$ From (1) and (2), we get

$$M'\ \ddot{x} = (M + M')\ g \sin \alpha.$$

Integrating, we get $M'\ \dot{x} = (M + M')\ g \sin \alpha\ .t + c_1$.

But initially when $t = 0, \dot{x} = 0 \quad \therefore \quad c_1 = 0$.

$\therefore\ M'\ \dot{x} = (M + M')\ g \sin \alpha\ .\ t.$

Integrating again, we get $M'x = M + M' \quad g \sin \alpha\ .\frac{1}{2}t^2 + c_2$.

Intitially when $t = 0, x = 0. \quad \therefore \quad C_2 = 0.$

$\therefore\ M'\ x = (M + M')\ g \sin \alpha\ .\ \frac{1}{2}t^2.$

or
$$t = \sqrt{\left\{\frac{2M'x}{(M + M')\ g \sin \alpha}\right\}}$$

Putting $x = AB = a$, the time to reach the other and B of the plank is given by

$$t = \sqrt{\left\{\frac{2M'x}{(M + M')\ g \sin \alpha}\right\}}$$