PDE – III

Linear partial Differential Eqns with constant coefficients:

The general linear partial differential eqn of an order higher than the first:

A PDE in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients all being constants (or) fns of x & y, is called a linear PDE.

The general form of such an eqn can be written in the form

$$\left(\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_n\frac{\partial^n z}{\partial y^n}\right)$$

$$+\left(B_0\frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1\frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + B_2\frac{\partial^{n-1} z}{\partial x^{n-3}\partial y^2} + \cdots + B_{n-1}\frac{\partial^{n-1} z}{\partial y^{n-1}}\right)$$

$$+ \cdots + \left(M_0\frac{\partial z}{\partial x} + M_1\frac{\partial z}{\partial y}\right) + N_0 z = f(x,y) \quad \text{--- (1)}$$

where the co-efficients $A_1, A_2, \ldots A_n$; $B_0, B_1, \ldots B_{n-1}$; $\ldots M_0, M_1, N_0$ are constants (or) functions of x & y.

If the coefficients of various terms are constants then (1) is called a linear PDE with constants co-efficients.

→ If all the derivatives appearing in (1) are of the same order then the resulting eqn is called a linear homogeneous PDE with constant coefficients and it is of the form

$$\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2\frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + A_n\frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{--- (2)}$$

where $A_0, A_1, \ldots A_n$ are constants.

Denoting the operators $\frac{\partial}{\partial x}$ by D ; $\partial/\partial y$ by D'

$\therefore$ ② $= [D^n + A_1 D^{n-1}D' + A_2 D^{n-2}D'^2 + A_3 D^{n-3}D'^3 + \cdots\cdots + A_n D'^n]z = f(x,y)$

$\Rightarrow F(D,D')z = f(x,y)$ ——— ③

where $F(D,D') = D^n + A_1 D^{n-1}D' + A_2 D^{n-2}D'^2 + \cdots\cdots + A_n D'^n$.

Note:- $F(D,D')$ is a homogeneous function in D, D' of degree n.

* **Solution of a linear homogeneous partial Differential eqn with constant coefficients:**

→ If u is the complementary function (C.F) and z' a particular integral (P.I) of a linear PDE $F(D,D')z = f(x,y)$ then $u + z'$ is a g.s of the linear PDE.

→ If $u_1, u_2, \ldots u_n$ are solns of the homogeneous linear PDE $F(D,D')z = 0$ then $\sum_{r=1}^{n} c_r u_r$ is also a soln, where $c_1, c_2, \ldots c_n$ are arbitrary constants.

* **Determination of the C.F of the linear homogeneous PDE with constant coefficients $F(D,D')z = f(x,y)$:**

Let $F(D,D')z = f(x,y)$ be the given ~~linear~~ linear homo. PDE with constant coeff.

then $[D^n + A_1 D^{n-1}D' + A_2 D^{n-2}D'^2 + \cdots\cdots + A_n D'^n]z = f(x,y)$ ——— ①

where $A_1, A_2, \ldots\ldots A_n$ are constants.

The complimentary function (C.F) of (1) is the g.s of $F(D,D')z=0$

i.e $\left[D^n + A_1 D^{n-1}D' + A_2 D^{n-2}D'^2 + \cdots + A_n D'^n\right]z = 0.$ ⟶ (2)

$\Rightarrow \left[(D-m_1D')(D-m_2D')(D-m_3D')\cdots\cdots(D-m_nD')\right]z = 0$ ⟶ (3)

where $m_1, m_2, \ldots m_n$ are some constants.

The soln of any one of the eqns

$(D-m_1D')z=0, \; (D-m_2D')z=0, \ldots (D-m_nD')z=0$ ⟶ (4)

is also a soln of (3).

<u>We now show that the general soln of $(D-mD')z=0$ is $z=\phi(y+mx)$,</u> where ϕ is an arbitrary fn.

Now $(D-mD')z=0 \Rightarrow \dfrac{\partial z}{\partial x} - m\dfrac{\partial z}{\partial y} = 0.$

$\Rightarrow p + (-m)q = 0.$ ⟶ (5)

clearly which is in Lagrange's form

$Pp + Qq = R$

∴ The Lagrange's auxiliary eqns of (5) are

$\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{0}$ ⟶ (6).

Now taking first two fractions of (6)

we get $\dfrac{dx}{1} = \dfrac{dy}{-m} \Rightarrow dy = -m\,dx$

Integrating, we get

$y = -mx + c$

$\Rightarrow \boxed{y+mx = c}$ ⟶ (7)

Now taking third fraction of (6) we get

$dz = 0$

$\Rightarrow \boxed{z = c_1}$ ⟶ (8)

$\therefore$ from (7) & (8), the g.s of (5) is $z = \phi(y+mx)$

where ϕ is arbitrary function

$\therefore$ we assume that a soln of (2) is of the form $z = \phi(y+mx)$ ——— (9).

where ϕ is arbitrary fn & m is const

Now from (9),

$$Dz = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\phi(y+mx)\right]$$
$$= m\,\phi'(y+mx)$$

$$D^2 z = \frac{\partial^2 z}{\partial x^2} = m^2\,\phi''(y+mx).$$

$$\vdots$$

$$D^n z = \frac{\partial^n z}{\partial x^n} = m^n\,\phi^{(n)}(y+mx).$$

and $D'z = \frac{\partial z}{\partial y} = \phi'(y+mx)$

$$D'^2 z = \frac{\partial^2 z}{\partial y^2} = \phi''(y+mx)$$

$$\vdots$$

$$D'^n z = \frac{\partial^n z}{\partial y^n} = \phi^{(n)}(y+mx).$$

Also, in general, $D^r D'^s z = \frac{\partial^{r+s} z}{\partial x^r \partial y^s}$

$$= m^r\,\phi^{(r+s)}(y+mx).$$

$\therefore$ (2) $\equiv \left(m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n\right)\phi^{(n)}(y+mx) = 0$

This is true if m is a root of the eqn $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0$ ——— (10)

The eqn (10) is called the auxiliary eqn (A.E) and is obtained by putting $D=m$, $D'=1$ in (2) $F(D,D')=0$

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In general, the eqn (10) can give 'n' roots, say $m = m_1, m_2, \ldots, m_n$.

— Each value of 'm' will give a soln of (2).

→ If all the roots of the auxiliary eqn (10) are distinct, the g.s of (2) i.e the C.F of (1) is

$$z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \cdots + \phi_n(y+m_nx) \quad \text{——— (11).}$$

i.e $z = \sum_{r=1}^{n} \phi_r(y+m_rx)$; $r = 1, 2, \ldots n$.

Case of equal roots:

If the auxiliary eqn (10) has two equal roots i.e $m_1 = m_2$

i.e $m = m_1, m_1$

Now consider the eqn

$$(D-mD')(D-mD')z = 0 \quad \text{——— (12)} \quad (\text{from (5)})$$

putting $(D-mD')z = u$ ——— (13) in (12), we get

$$(D-mD')u = 0$$

∴ the soln of which is $u = \phi(y+mx)$.

$$\therefore (D-mD')z = \phi(y+mx) \quad (\text{from (13)})$$

$$p - mq = \phi(y+mx) \quad \text{——— (14)}$$

Lagrange's auxiliary eqns of (14) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y+mx)} \quad \text{——— (15)}$$

Taking the first two fractions of (15), we get

$dy = -m\,dx \Rightarrow \boxed{y + mx = a}$ ——— (16) where a is arbitrary const

Taking first & last fractions (6) of (15), we get

$$\frac{dx}{1} = \frac{dz}{\phi(c)} \quad (\text{from } (16))$$

$$\Rightarrow dz = \phi(c)\, dx$$

$$\Rightarrow z = x\,\phi(c) + b$$ where b is arbitrary.

$$\Rightarrow \boxed{z = x\,\phi(y+mx) + b}$$

Since b is arbitrary,

taking $b = \phi_1(c)$

$\therefore$ the soln of (12) is

$$z = x\,\phi(y+mx) + \phi_1(c)$$

$$\boxed{z = x\,\phi(y+mx) + \phi_1(y+mx)}$$

where ϕ & ϕ_1 are arbitrary fns.

Note:

Proceeding in the same way,

If the auxiliary eqn (10) has r equal roots

then the C.F of (1) is

$$z = \phi_1(y+mx) + x\,\phi_2(y+mx) + x^2\,\phi_3(y+mx) + \cdots + x^{r-1}\phi_r(y+mx).$$

Working rule for finding C.F:

Step 1: Write down the given eqn in standard form

$$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n)z = f(x,y) \quad (1)$$

Step (2): Replacing D by m and D' by 1 in the coefficient of z, we obtain the A.E for (1) as $m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n = 0.$ (2)

Step(3): solve (2) for 'm'.

Some cases will arise:

Case(i): Let $m = m_1, m_2, \ldots m_n$ (distinct roots)

then C.F of (1) $= \phi_1(y+m_1x) + \phi_2(y+m_2x) + \ldots \ldots + \phi_n(y+m_nx)$.

where $\phi_1, \phi_2, \ldots \phi_n$ are arbitrary fns.

Case(ii): Let $m = m'$ (repeated n times)

then C.F of (1) $=$
$\phi_1(y+m'x) + x\phi_2(y+m'x) + x^2\phi_3(y+m'x) + \ldots \ldots + x^{n-1}\phi_n(y+m'x)$.

Case (iii) corresponding to a non-repeated factor D on LHS of (1), the part of C.F is taken as $\phi(y)$.

Case(iv): corresponding to a repeated factor D^m on LHS of (1), the part of C.F is taken as $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \ldots \ldots + x^{m-1}\phi_m(y)$.

Case(v): corresponding to a non-repeated factor D' on LHS of (1), the part of C.F is taken as $\phi(x)$.

Case(vi): corresponding to a repeated factor D'^m on LHS of (1), the part of C.F is taken as

$\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \ldots \ldots y^{m-1}\phi_m(x)$.

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* Alternative working rule for finding C.F

Let the given diff. eqn be $F(D,D')z = f(x,y)$.

Factorise $F(D,D')$ into linear factors of the form $(bD - aD')$.

Then we use the following results:

(i) corresponding to each non-repeated factor $(bD - aD')$, the part of C.F is taken as $\phi(by + ax)$.

(ii) corresponding to a repeated factor $(bD - aD')^m$, the part of C.F is taken as $\phi_1(by+ax) + x\phi_2(by+ax) + x^2\phi_3(by+ax) + \cdots + x^{m-1}\phi_m(by+ax)$.

(iii) corresponding to a non-repeated factor D, part of C.F is taken as $\phi(y)$

(iv) corresponding to a repeated factor D^m the part C.F is taken as $\phi_1(y) + x\phi_2(y) + x^2\phi_3(y) + \cdots + x^{m-1}\phi_m(y)$

(v) correspond to a non-repeated factor D', the part of C.F is taken as $\phi(x)$.

(vi) corresponding to a repeated factor D'^m, the part of C.F is taken as $\phi_1(x) + y\,\phi_2(x) + \cdots + y^{m-1}\phi_m(x)$.

Note:- $p = \partial z/\partial x$; $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$; $s = \dfrac{\partial^2 z}{\partial x \partial y}$; $t = \dfrac{\partial^2 z}{\partial y^2}$.

Problems

→ solve $2r+5s+2t=0$.

Sol Given that $2r+5s+2t=0$. —— (1)

W.K.T $r=\frac{\partial^2 z}{\partial x^2}$; $s=\frac{\partial^2 z}{\partial x\partial y}$ & $t=\frac{\partial^2 z}{\partial y^2}$
$= D^2 z$ $\quad = DD'z$ $\quad = D'^2 z$

∴ (1) ≡

$$[2D^2+5DD'+2D'^2]z=0. \quad \text{—— (2)}$$

A.E of (2) is
$2m^2+5m+2=0$ $\quad (\because D=m, D'=1)$.

$\Rightarrow (2m+1)(m+2)=0$

$\Rightarrow m=-1/2, -2$

∴ The g.s of (1) is

$$z=\phi_1(y-\tfrac{1}{2}x)+\phi_2(y-2x).$$

where ϕ_1 & ϕ_2 arbitrary fns.

→ solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$. —— (1)

$(D^2+D'^2)z=0$ —— (2)

A.E of (2) is $m^2+1=0$.

$\Rightarrow m^2=-1$

$\Rightarrow \boxed{m=\pm i}$

∴ The g.s of (1) is

$$z=\phi_1(y+ix)+\phi_2(y-ix).$$

→ solve $r+t+2s=0$

Given that $r+t+2s=0$ —— (1)

$\Rightarrow \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+2\frac{\partial^2 z}{\partial x\partial y}=0$.

$\Rightarrow (D^2+2DD'+D'^2)z=0$ —— (2)

$\therefore$ A.E of (1) is

$m^2 + 2m + 1 = 0.$

$\Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1.$

$\therefore$ G.S of (1) is $z = \phi_1(y-x) + x\phi_2(y-x).$

1987 → solve $r = a^2 t$

→ solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0.$

→ solve $(D^2 - DD')z = 0.$

Sol: Given that $(D^2 - DD')z = 0$ —(1)

A.E of (1) is

$m^2 - m = 0.$

$\Rightarrow m(m-1) = 0.$

$\Rightarrow m = 0;\ m = 1.$

$\therefore$ G.S of (1) is

$z = \phi_1(y) + \phi_2(y+x).$

where ϕ_1 & ϕ_2 arbitrary fns.

→ solve $DD'z = 0.$ —(1)

$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = 0$

$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 0.$

$\Rightarrow \frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0.$

$\Rightarrow z = \phi_1(y)$ and $z = \phi_2(x)$

$\therefore$ The G.S of (1) is $z = \phi_1(y) + \phi_2(x)$

$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{0}$

$dy = 0$, $y = c_1$

$dz = 0$, $z = c_2$

$z = \phi(y)$

→ solve $D'^2 z = 0.$ —(1)

$\frac{\partial^2 z}{\partial y^2} = 0$

A.E is $m'^2 = 0$ (or $m' = 0$)

$m' = 0, 0$

G.S is $z = \phi_1(x) + y\phi_2(x)$

→ solve $D^2 z = 0$ —(1)

A.E is $m^2 = 0$ $\Rightarrow m = 0, 0$ $\therefore$ G.S of (1) is $z = \phi_1(y) + x\phi_2(y)$

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Particular Integral:

Let us consider an equation

$$F(D,D')z = f(x,y) \quad \text{—— (1)}$$

then P.I of (1) is denoted by

$$\frac{1}{F(D,D')} f(x,y).$$

Note: (1) $\frac{1}{D}$ means Integration partially w.r.t 'x'

$\frac{1}{D'}$ means integration partially w.r.t 'y'.

(2) $D[\phi(ax+by)] = \frac{\partial}{\partial x}[\phi(ax+by)]$

$= a\,\phi'(ax+by).$

$D'[\phi(ax+by)] = \frac{\partial}{\partial y}[\phi(ax+by)]$

$= b[\phi'(ax+by)].$

In general,

$D^r[\phi(ax+by)] = a^r\phi^{(r)}(ax+by),$

$D'^s[\phi(ax+by)] = b^s\phi^{(s)}(ax+by)$

and $D^r D'^s[\phi(ax+by)] = a^r b^s \phi^{(r+s)}(ax+by).$

$$\boxed{\frac{\partial^{r+s}\phi}{\partial x^r \partial y^s} = D^r D'^s\phi = \frac{\partial^r\phi}{\partial x^r}\cdot\frac{\partial^s\phi}{\partial y^s}}$$

Working rule:

→ To find P.I of an eqn $F(D,D')z = \phi(ax+by)$ where $F(D,D')$ is a homogeneous function of D, D' of degree n, proceed as follows:

(i) When $F(a,b) \neq 0$,

we have $P.I = \frac{1}{F(D,D')}\phi(ax+by)$

$$= \frac{1}{F(a,b)}\int\int\int\int\cdots\int\phi(v)\,dv\,dv\cdots\cdots dv.$$

i.e $P.I = \frac{1}{F(D,D')}\phi(ax+by)$

$= \frac{1}{F(a,b)} \times$ n^{th} integral of $\phi(v)$ w.r.t v

where $v = ax+by$.

(ii) <u>when $F(a,b) = 0$,</u>

Now $F(a,b)=0$ iff $(bD-aD')$ is a factor of $F(D,D')$.

we have $P.I = \frac{1}{F(D,D')}\phi(ax+by)$

$$= \frac{1}{(bD-aD')^n}\phi(ax+by)$$

$$= \frac{x^n}{b^n n!}\phi(ax+by).$$

→ To find P.I of an eqn $F(D,D')z = x^m y^n$ (or a rational integral algebraic function of x & y).

We have $P.I = \frac{1}{F(D,D')}V$ where $V = x^m y^n$

It evaluated by expanding the symbolic function $\frac{1}{F(D,D')}$ in an infinite series of ascending powers of D' or D

Note:-

If $n < m$, $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D'}{D}$, whereas if $m < n$, $\frac{1}{F(D,D')}$ should be expanded in powers of $\frac{D}{D'}$.

problems:

→ solve $4r-4s+t = 16\log(x+2y)$.

Sol. Given that $4r-4s+t = 16\log(x+2y)$.

$$\Rightarrow 4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$$

$$\Rightarrow [4D^2 - 4DD' + D'^2]z = 16\log(x+2y) \quad \text{—— (1)}$$

A.E of (1) is

$$4m^2 - 4m + 1 = 0.$$

$$\Rightarrow (2m-1)^2 = 0$$

$$\Rightarrow m = 1/2, 1/2.$$

$$\therefore C.F = \phi_1(y + x/2) + x\phi_2(y + x/2).$$

$$= \phi_1[\tfrac{1}{2}(2y+x)] + x\phi_2[\tfrac{1}{2}(2y+x)]$$

$$\boxed{C.F = \psi_1(2y+x) + x\psi_2(2y+x)}.$$

where ψ_1 & ψ_2 are arbitrary fns

$\therefore$ G.S of (1) is

$z = C.F + P.I$

Now $P.I = \frac{1}{4D^2 - 4DD' + D'^2} 16\log(x+2y)$

$$= 16\left[\frac{1}{(2D-D')^2}\log(x+2y)\right]$$

$$= 16\left[\frac{x^2}{2^2 \cdot 2!}\log(x+2y)\right] \quad (\because F(a,b) = 0)$$

$$= 2x^2\log(x+2y)$$

$\frac{1}{2^2(D-\frac{1}{2}D')^2}$, $\frac{x^2}{2!}$

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→ solve $(D^2+3DD'+2D'^2)z = x+y$ —— (1)

Sol: $\boxed{C.F = \phi_1(y-x)+\phi_2(y-2x)}$

where ϕ_1 & ϕ_2 are arbitrary fns.

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{1^2+3(1)(1)+2(1)^2}\int\int v\,dv\,dv$$ where $v = x+y$.

$$= \frac{1}{6}\int \frac{v^2}{2}\,dv$$

$$= \frac{1}{6}\,\frac{v^3}{6}$$

$$= \frac{1}{36}(x+y)^3$$

$\therefore$ G.S of (1) is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-x)+\phi_2(y-2x)+\frac{1}{36}(x+y)^3.$$

→ solve $r-2s+t = \sin(2x+3y)$

96 → solve $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$.

Sol Given ~~that~~

$(D^2+D'^2)z = \cos mx.\cos ny$ —— (1)

~~A.E~~ is $m^2+1^2=0$

$\Rightarrow m = \pm i$

$\therefore \boxed{C.F = \phi_1(y+ix)+\phi_2(y-ix)}$

$$P.I = \frac{1}{D^2+D'^2}\cos mx.\cos ny$$

$$= \frac{1}{D^2+D'^2}\,\frac{1}{2}\left[\cos(mx+ny)+\cos(mx-ny)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+D'^2}\cos(mx+ny)+\frac{1}{D^2+D'^2}\cos(mx-ny)\right]$$ —— (2)

Now $\frac{1}{D^2+D'^2}\cos(mx+ny) = \frac{1}{m^2+n^2}\iint \cos u \, du \, du$ where $u = mx+ny$

$= \frac{1}{m^2+n^2}\int \sin u \, du$

$= \frac{-1}{m^2+n^2}\cos u$

$= \frac{-1}{m^2+n^2}\cos(mx+ny).$

and $\frac{1}{D^2+D'^2}\cos(mx-ny) = \frac{-1}{m^2+n^2}\cos(mx-ny).$

$\therefore$ (2) $\equiv$ P.I $= \frac{1}{D^2+D'^2}\cos mx \cos ny$

$= \frac{1}{2}\left[\frac{-1}{m^2+n^2}\left(\cos(mx+ny) + \cos(mx-ny)\right)\right]$

$= \frac{-1}{m^2+n^2}\cos mx \cdot \cos ny.$

$\therefore$ G.S of (1) is $z = C.F + P.I$

$= \phi_1(y+ix) + \phi_2(y-ix)$
$- \frac{1}{m^2+n^2}\cos mx \cdot \cos ny.$

Q91 $\longrightarrow$ Solve $r + 5s + 6t = (y-2x)^{-1}$

As usual find C.F

Sol: $P.I = \frac{1}{D^2+5DD'+6D'^2}(y-2x)^{-1}.$

$= \frac{1}{(D+2D')(D+3D')}(y-2x)^{-1}$

$= \frac{1}{D+2D'}\left[\frac{1}{D+3D'}(y-2x)^{-1}\right]$

$$= \frac{1}{D+2D'}\left[\frac{1}{-2+3(1)}\int v^{-1}dv\right] \quad (16)$$

where $v = y-2x$

$$= \frac{1}{D+2D'}\left[\log v\right]$$

$$= \frac{1}{D+2D'}\log(y-2x)$$

$$= \frac{x}{(1)\,1!}\log(y-2x) \qquad (\because F(a,b)=0).$$

$$= \underline{\underline{x\log(y-2x)}}$$

$\therefore$ G.S of (1) is $\underline{Z = C.F + P.I}$

→ solve $(D^2-2DD'+D'^2)Z = e^{x+2y}$.

→ solve $\log s = x+y$.

i.e $s = e^{x+y}$.

1994 → solve $(D^2+3DD'+2D'^2)Z = x+y$ by expanding the particular integral (P.I) in ascending powers of D as well as in ascending powers of D'.

Sol Given that $(D^2+3DD'+2D'^2)Z = x+y$ —— (1)

A.E of (1) is $m^2+3m+2=0$

$\Rightarrow m = -2, -1$

$$\therefore \boxed{C.F = \phi_1(y-2x)+\phi_2(y-x).}$$

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{2D'^2\left[1+\left(\frac{D^2}{2D'^2}+\frac{3}{2}\frac{D}{D'}\right)\right]}(x+y)$$

$$= \frac{1}{2D'^2}\left[1+\left(\frac{D^2}{2D'^2}+\frac{3}{2}\frac{D}{D'}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{2D'^2}\left[1-\left(\frac{D^2}{2D'^2}+\frac{3}{2}\frac{D}{D'}\right)+\cdots\right](x+y)$$

$$= \frac{1}{2D'^2}\left[(x+y-\tfrac{3}{2}y)\right] = \frac{1}{2D'^2}\left(x-\frac{y}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{D'}\left[xy-\frac{y^2}{4}\right]$$

$$= \frac{1}{2}\left[\frac{xy^2}{2}-\frac{y^3}{12}\right] = \frac{xy^2}{4}-\frac{y^3}{24}$$

$\therefore$ G.S of (1) is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-2x)+\phi_2(y-x)+\frac{xy^2}{4}-\frac{y^3}{24},$$

Again, by expanding in ascending powers of D', we have

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y) = \frac{1}{D^2\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]}(x+y)$$

$$= \frac{yx^2}{2}-\frac{x^3}{3}.$$

$\therefore$ G.S of (1) is $z = C.F + P.I$

$$\Rightarrow z = \phi_1(y-2x)+\phi_2(y-x)+\frac{yx^2}{2}-\frac{x^3}{3},$$

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→ solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$. —— ①

Sol $\boxed{C.F = \phi_1(y+3x) + x\phi_2(y+3x)}$

$$P.I = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2 + 36xy)$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right)\right]^{-1}(12x^2 + 36xy)$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right) + \left(\frac{6D'}{D} - 9\left(\frac{D'}{D}\right)^2\right)^2 + \cdots\right](12x^2 + 36xy)$$

$$= \frac{1}{D^2}\left[(12x^2 + 36xy) + \frac{6}{D}(36x)\right]$$

$$= \frac{1}{D^2}\left[12x^2 + 36xy + 6(\overset{18}{\cancel{36}})\frac{x^2}{\cancel{2}}\right]$$

$$= \frac{1}{D}\left[4x^3 + 18x^2y + 36x^3\right]$$

$$= x^4 + 6x^3y + 9x^4$$

$$= 10x^4 + 6x^3y$$

∴ G.S of ① is $z = C.F + P.I$

* A general method finding the P.I :

consider the eqn $(D - mD')z = \phi(x,y)$. —— ①

$\Rightarrow p - mq = \phi(x,y)$

∴ Lagrange's A.E's are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x,y)} \quad \text{—— ②}$$

Taking first two fractions of ②, we get

$dy + m\,dx = 0 \Rightarrow \boxed{y + mx = a}$ (a constant).

Again taking first & last fractions of ②, we get

$$dz = \phi(x, y)\,dx$$

$$\Rightarrow dz = \phi(x, a - mx)\,dx \quad (\because y + mx = a)$$

Integrating we get

$$\boxed{z = \int \phi(x, a - mx)\,dx.}$$

$$① \equiv \frac{1}{D - mD'}\phi(x, y) = z$$

$$= \int \phi(x, a - mx)\,dx \longrightarrow ③$$

where after integration the constant a is to be replaced by y+mx (since the P.I doesnot contain any arbitrary constant)

Now If the given eqn is $F(D, D')z = \phi(x, y)$

where $F(D, D') = (D - m_1D')(D - m_2D') \ldots \ldots (D - m_nD')$

then $P.I = \dfrac{1}{F(D, D')}\phi(x, y)$

$$\Rightarrow P.I = \frac{1}{(D - m_1D')(D - m_2D') \ldots (D - m_nD')}\phi(x, y)$$

which can be evaluated by the repeated application of the above method (i.e ③).

Problems

$\rightarrow$ solve $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = \sin x$

The given eqn is $(D + D')z = \sin x$ —①

A.E is $m + 1 = 0$

$\Rightarrow m = -1.$

$\therefore \boxed{C.F = \phi_1(y - x)}$

Now $P.I = \frac{1}{D+D'} \sin x$

$= \frac{1}{D+D'} \sin(x+0y)$ $y-x=a$

$= \int \sin[x+0(a+x)]\,dx$

$= \int \sin x\,dx$

$= -\cos x.$

$\therefore$ G.S of (1) is $z = C.F + P.I$

1992 → $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\sin x$

Given eqn is $(D^2+DD'-6D'^2)z = y\sin x$ ——(1)

A.E is $m^2+m-6=0 \Rightarrow m = 2, -3.$

$\therefore \boxed{C.F = \phi_1(y+2x) + \phi_2(y-3x).}$

$P.I = \frac{1}{D^2+DD'-6D'^2}(y\sin x) = \frac{1}{(D-2D')(D+3D')}(y\sin x)$

$= \frac{1}{D-2D'}\left[\frac{1}{D-(-3)D'}(y\sin x)\right]$

$= \frac{1}{D-2D'}\left[\int (a+3x)\sin x\,dx\right]$ where $y-3x=a$

$= \frac{1}{D-2D'}\left[-y\cos x + 3\sin x\right]$

$= \int\left[-(b-2x)\cos x + 3\sin x\right]dx$ where $y+2x=b$

$= -y\sin x - \cos x.$

$\therefore$ G.S of (1) is $z = C.F + P.I$

2000 → Solve $(D^2-DD'-2D'^2)z = e^{2x+3y} + e^{2x+4y}$.

2008, 12M → Solve $(D^2+DD'-6D'^2)z = y\cos x$.

→ Solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y$.

2004 → Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$

* Non-homogeneous linear partial Differential eqns with constant coefficients:

A linear partial differential eqn which is not homogeneous is called a non-homogeneous linear eqn.

consider the diff. eqn $F(D,D')z = f(x,y)$ —— (1).

when $F(D,D')$ is a homogeneous function in D, D' it can always be resolved into linear factors. But the result is not always true, when $F(D,D')$ is non-homogeneous.

Now we classify linear differential operators $F(D,D')$ into two types

These are: (i) $F(D,D')$ is reducible if it can be written as the product of linear factors of the form $D+aD'+b$, with a, b as constants.

(ii) $F(D,D')$ is irreducible if it cannot be so written.

→ (i) C.F of non-homogeneous linear eqn when $F(D,D')$ can be resolved into linear factors:

The C.F of non-homo. linear eqn (1) is the g.s of the eqn $F(D,D')z=0$ —— (2).

Let us consider a simple non-homo. eqn $(D-mD'-k)z=0$ —— (3)

$\Rightarrow p-mq=kz$ —— (4)

Lagrange's A.E's are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{kz} \quad \text{—— (5)}$$

(22)

Taking the first two fractions of (5), we get

$dy + m\,dx = 0 \Rightarrow \boxed{y + mx = a}$ (const).

Again taking the first & the last fraction of (5), we get

$$\frac{dz}{z} = k\,dx$$

$$\Rightarrow \log z = kx + \log b$$

$$\Rightarrow \boxed{z = b e^{kx}}$$

$$\Rightarrow z = e^{kx}\,\phi(a) \quad (\because b = \phi(a))$$

$$\Rightarrow \boxed{z = e^{kx}\,\phi(y+mx)}.$$

which is the soln of (3).

→ If $F(D,D')$ can be factorised into non-repeated linear factors $(D-m_1D'-k_1), (D-m_2D'-k_2), \ldots, (D-m_nD'-k_n)$ then the eqn (2) becomes $[(D-m_1D'-k_1)(D-m_2D'-k_2)\ldots(D-m_nD'-k_n)]z = 0$ ——(6)

$\therefore$ G.S of (2) is

$$z = e^{k_1x}\,\phi_1(y+m_1x) + e^{k_2x}\,\phi_2(y+m_2x) + \ldots + e^{k_nx}\,\phi_n(y+m_nx).$$

Note:- If the eqn is $(\alpha D + \beta D' + \gamma)z = 0$ then

$$\cancel{z = e^{(-\gamma x/\alpha)}\,\phi(\beta x - \alpha y)}. \quad z = e^{-\frac{\gamma x}{\alpha}}\,\phi(\alpha y - \beta x)$$

→ $F(D,D')$ has repeated factors

(i) If $(D-mD'-k)$ occurs twice then the g.s of (2) is

$$z = e^{kx}\left[\phi_1(y+mx) + x\,\phi_2(y+mx)\right].$$

(ii) If $(D-mD'-k)$ occurs r times then the g.s of (2) is

$$z = e^{kx}\left[\phi_1(y+mx) + x\,\phi_2(y+mx) + \ldots + x^{r-1}\phi_r(y+mx)\right].$$

(ii) **When linear factors of $F(D,D')$ are not possible:**

In case $F(D,D')$ is irreducible, i.e. it cannot be resolved into linear factor in D & D', the above methods of finding the complementary function fail.

In such cases a trial method is used to find solns.

Ex:- solve $(2D^4 - 3D^2D' + D'^2)z = 0$

$$\Rightarrow (2D^2 - D')(D^2 - D')z = 0 \quad \text{---(1)}$$

Let $z = Ae^{hx+ky}$ ---(2) be the soln corresponding to $(D^2 - D')z = 0$ ---(3)

$$\Rightarrow D^2\left[Ae^{hx+ky}\right] - D'\left[Ae^{hx+ky}\right] = 0$$

$$\Rightarrow Ah^2e^{hx+ky} - AKe^{hx+ky} = 0$$

$$\Rightarrow A(h^2 - K)e^{hx+ky} = 0$$

$$\Rightarrow h^2 - K = 0 \quad (\because Ae^{hx+ky} \neq 0)$$

$$\Rightarrow \boxed{K = h^2}$$

putting $K = h^2$ in (2), we get

$$z = Ae^{hx+h^2y}$$

since all values of h satisfy the eqn (3),

$\therefore$ the more general soln of (3) is

$$\boxed{z = \sum Ae^{hx+h^2y}} \quad \text{---(4)}$$

Sly the g.s of $(2D^2 - D')z = 0$ is

$$\boxed{z = \sum B e^{h'x + 2h'^2 y}} \quad 5$$

$\therefore$ The most g.s of the given eqn (1)

is $z = \sum A e^{hx + h^2 y} + \sum B e^{h'x + 2h'^2 y}$.

→ solve $(D - D'^2)z = 0$

→ solve $(D^2 - D'^2 + D - D')z = 0$

Sol Given that $(D^2 - D'^2 + D - D')z = 0$ —(1)

$\Rightarrow (D - D')(D + D' + 1)z = 0.$

$\therefore$ There are distinct factors.

$\therefore$ G.s of (1) is

$z = e^{0x}\phi_1(y+x) + e^{-x}\phi_2(y-x).$

→ solve $DD'(D - 2D' - 3)z = 0.$ —(1)

$D - mD' - k$

$+D' - 0$

Sol There are ~~three~~ distinct factors.

$\therefore$ The g.s of (1) is

$z = \phi_1(y) + \phi_2(x) + e^{3x}\phi_3(y+2x).$

Particular integral:

The complete soln of $F(D,D')z = f(x,y)$

is $z = C.F + P.I$

where $P.I = \frac{1}{F(D,D')} f(x,y)$, ——— ①

The methods of obtaining particular integrals of non-homo. PDE's are very similar to those of ordinary linear eqns with constant coefficients.

→ We now give some cases of finding the particular integrals.

case (I) If $f(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$.

then $P.I = \frac{1}{F(D,D')} e^{ax+by}$

$$= \frac{1}{F(a,b)} e^{ax+by}.$$

case (II) If $f(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

then $P.I = \frac{1}{F(D,D')} \sin(ax+by)$ is evaluated by putting $D^2 = -a^2$, $DD' = -ab$ & $D'^2 = -b^2$,

provided the denominator is not zero.

case (III) If $f(x,y) = x^m y^n$ then

$P.I = \frac{1}{F(D,D')}(x^m y^n) = [F(D,D')]^{-1}(x^m y^n)$

which can be evaluated after expanding $[F(D,D')]^{-1}$ in ascending powers of D or D'.

Case (iv) If $f(x,y) = e^{ax+by} \cdot V$ where V is fn. of x & y.

then $P.I = \frac{1}{F(D,D')} e^{ax+by} \cdot V$

$= e^{ax+by} \cdot \frac{1}{F(D+a, D'+b)} V.$

1991 → solve $s + p - q = z + xy$

Given that $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - z = xy$ —— (1)

$\Rightarrow (DD' + D - D' - 1)z = xy$

$\Rightarrow (D-1)(D'+1)z = xy$ —— (2)

∴ There are two distinct linear factors

in $(D-1)(D'+1)z = 0$.

∴ C.F of (1) is

$e^{x}\phi_1(y) + e^{-y}\phi_2(x)$. where ϕ_1 & ϕ_2 are arbitrary fns.

Now $P.I = \frac{1}{(D-1)(D'+1)}(xy)$

$= -(1-D)^{-1}(1+D')^{-1}(xy)$

$= -\left[(1+D+D^2+D^3+\cdots)(1-D'+D'^2-D'^3+\cdots)\right]xy$

$= -\left[1 - D' + D - DD' + \cdots\right](xy)$

$= -\left[xy - x(1) + y(1) - (1)\right]$

$= -\left[xy - x + y - 1\right]$

∴ the reqd g.s is $z = C.F + P.I$

i.e $z = e^{x}\phi_1(y) + e^{-y}\phi_2(x) - (xy + y - x - 1)$

1998 → solve $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

1998 → solve $r - s + 2q - z = x^2y^2$

2005 → solve $(D - 3D' - 2)^2 z = 2e^{2x}\sin(y+3x)$.

2007 → solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$

→ solve $r - s + p = 1$.

→ solve $(D - 3D' - 2)^2 z = 2e^{2x}\tan(y+3x)$. —— (1).

$$C.F = e^{2x}\left[\phi_1(y+3x) + x\phi_2(y+3x)\right].$$

$$P.I = \frac{1}{(D-3D'-2)^2} 2e^{2x+0y}\tan(y+3x)$$

$$= 2e^{2x}\cdot\frac{1}{\left[D+2-3(D'+0)-2\right]^2}\tan(y+3x)$$

$$= 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$$= 2e^{2x}\cdot\frac{x^2}{1^2\cdot 2!}\tan(y+3x).\quad \left(\because \frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{b^n\, n!}\phi(ax+by)\right).$$

$$= x^2e^{2x}\tan(y+3x).$$

$\therefore$ The reqd g.s of (1) is $z = C.F + P.I$

* **Eqns reducible to linear form with constant coefficients:**

A PDE having variable co-efficients can sometimes be reduced to an eqn with constant coefficients by suitable substitutions.

Reduce an eqn of the form

$$A_0 x^n \frac{\partial^n z}{\partial x^n} + A_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + A_2 x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots$$

$$\cdots\cdots = f(x,y). \text{ Into a} \quad \text{——— (1)}$$

coefficients. linear eqn with constant

Note: In the eqn (1), the term $\frac{\partial^n z}{\partial x^r \partial y^{n-r}}$ is multiplied by the variable expression $x^r y^{n-r}$.

To transform the eqn (1),

putting $x = e^X$, $y = e^Y$

$$\Rightarrow \boxed{X = \log x}; \boxed{Y = \log y}$$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{1}{x}\cdot\frac{\partial z}{\partial X}$

$$\therefore \boxed{x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}}$$

$$\therefore x\,\partial/\partial x = \partial/\partial X = D \text{ (say)} \quad \text{——— (2)}$$

Now $x\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right) = x^2\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial x}$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial x^2} = \left(x\frac{\partial}{\partial x} - 1\right)x\frac{\partial z}{\partial x}$$

$$= (D-1)Dz$$

$$= D(D-1)z \quad \text{——— (3)}$$

In general $x^n \frac{\partial^n z}{\partial x^n} = D(D-1)(D-2)\cdots\cdots(D-n+1)z$ ——— (4).

Now $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial Y}$

$$\Rightarrow \boxed{y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}}$$

$\therefore\ y\,\partial/\partial y = \partial/\partial Y = D'$ (say)

$\text{sly}\ \ y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$

In general, $y^n\frac{\partial^n z}{\partial y^n} = D'(D'-1)\cdots\cdots(D'-n+1)z$

Also $xy\frac{\partial^2 z}{\partial x\partial y} = DD'z$

and $x^m y^n \frac{\partial^{m+n} z}{\partial x^m \partial y^n} = D(D-1)\cdots(D-m+1)\,D'(D'-1)\cdots\cdots(D'-n+1)z$

These substitutions reduce the eq (1) to an eqn having constant coefficients and now it can be easily solved by the known methods of homo & non-homo. linear eqns with constant coefficients.

Problems

1987 → Solve $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$. ——— (1)

Sol putting $x = e^X$, $y = e^Y$ and denoting the operators $\frac{\partial}{\partial X}$ & $\frac{\partial}{\partial Y}$ by D & D'

$\therefore (1) \equiv [D(D-1) + 2DD' + D'(D'-1)]z = 0$ ———

$\Rightarrow [D^2 - D + 2DD' + D'^2 - D']z = 0$

$\Rightarrow (D+D')(D+D'-1)z = 0$ ——— (2).

$\therefore$ G.S of (2) is $z = \phi_1(Y-X) + e^X\phi_2(Y-X)$

$$\therefore z = \phi_1(\log y - \log x) + x\,\phi_2(\log y - \log x)$$

$$= \phi_1\left(\log\left(\tfrac{y}{x}\right)\right) + x\,\phi_2\left(\log\left(\tfrac{y}{x}\right)\right) \quad (\because x = e^X \ \& \ y = e^Y)$$

$$= f_1\left(\tfrac{y}{x}\right) + x\,f_2\left(\tfrac{y}{x}\right).$$

which is the reqd g.s of (1).

1987 → solve $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} = xy$ —(1).

putting $x = e^X$; $y = e^Y$

and denoting the operators $\frac{\partial}{\partial X}$ & $\frac{\partial}{\partial Y}$ by D & D'.

$$\therefore (1) \equiv [D(D-1) - D'(D'-1)]z = e^{X+Y}$$

$$\Rightarrow [D^2 - D'^2 - D + D']z = e^{X+Y}$$

$$\Rightarrow (D - D')(D + D' - 1)z = e^{X+Y} \quad \text{—(2).}$$

$$\therefore C.F = \phi_1(Y+X) + e^X \phi_2(Y-X)$$

$$= \phi_1(\log y + \log x) + x\,\phi_2(\log y - \log x)$$

$$= \phi_1(\log(xy)) + x\,\phi_2\left(\log\left(\tfrac{y}{x}\right)\right) \quad (\because x = e^X \ \& \ y = e^Y)$$

$$= f_1(xy) + x\,f_2\left(\tfrac{y}{x}\right).$$

Now $P.I = \dfrac{1}{(D-D')(D+D'-1)} e^{X+Y}$

$$= \frac{1}{(D-D')(1+1-1)} e^{X+Y}$$

$$= \frac{1}{D-D'} e^{X+Y}$$

$$= \frac{X}{1!} e^{X+Y}$$

$$= (\log x)\,xy$$

$\therefore$ G.S of (1) is $z = C.F + P.I$

2001 → solve the eqn $x^2 \dfrac{\partial^2 z}{\partial x^2} - y^2 \dfrac{\partial^2 z}{\partial y^2} + x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = x^2 y$ by reducing it to the eqn with const coefficients.

1993 / 1997 → solve $x^2 r - y^2 t + px - qy = \log x$.

Cauchy's Problem for Second Order Partial Differential equation. Characteristic equation and characteristic Curves (or simply characteristics) of the second order Partial Differential Equations.

Cauchy's Problem. Consider the second order Partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{——— (1)}$$

in which R, S, and T are functions of x and y only. The Cauchy's Problem consists of the Problem of determining the solution of (1) such that on a given space Curve C it takes on prescribed values of z and $\partial z/\partial n$, where n is the distance measured along the normal to the Curve.

As an example of Cauchy's Problem for the second order Partial differential equation, consider the following Problem:

To determine solution of $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ with the following data prescribed on the x-axis: $z(x,0) = f(x)$, $z_y(x,0) = g(x)$.

Observe that y-axis is the normal to the given Curve (x-axis here).

Characteristic equations and Characteristic Curves.

Corresponding to (1), Consider the λ-quadratic.

$$R\lambda^2 + S\lambda + T = 0 \quad \text{——— (2)}$$

when $S^2 - 4RT \geq 0$, (2) has real roots. then, the ordinary differential equation $\left(\frac{dy}{dx}\right) + \lambda(x,y) = 0$ —— (3)

are called the characteristic equations.

the solutions of (3) are known as characteristic curves or simply the characteristics of the second order partial differential equation (1).

Now, consider the following three cases.

Case(i): If $S^2 - 4RT > 0$ (i.e if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations

$$\left(\frac{dy}{dx}\right) + \lambda_1(x,y) = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right) + \lambda_2(x,y) = 0$$

solving these we get two distinct families of characteristics.

Case(ii). If $S^2 - 4RT = 0$ (i.e (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). solving it, we get only one family of characteristics.

Case(iii): If $S^2 - 4RT < 0$ (i.e. (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. thus we get two families of complex characteristics when (1) is elliptic.

2009 12M → Find the characteristics of $y^2 r - x^2 t = 0$

sol'n: Given $y^2 r - x^2 t = 0$ —— (1)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

here $R = y^2$, $S = 0$ and $T = -x^2$.

Then $S^2 - 4RT = 0 - 4 \cdot y^2(-x^2)$

$$= 4x^2y^2 > 0$$

and hence (1) is hyperbolic everywhere except on the coordinate axes $x=0$ and $y=0$.

the λ - quadratic is $R\lambda^2 + S\lambda + T = 0$ (or)

$$y^2\lambda^2 - x^2 = 0 \quad \text{—— (2)}$$

solving (2), $\lambda = x/y, -x/y$ (two distinct real roots)

corresponding characteristic equations are

$$(dy/dx) + (x/y) = 0 \quad \text{and} \quad (dy/dx) - (x/y) = 0$$

$$x\,dx + y\,dx = 0 \quad \text{and} \quad x\,dx - y\,dy = 0$$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$.

which are the required families of characteristics.

Here these are families of circles and hyperbolas respectively.

→ Find the characteristics of $x^2 r + 2xys + y^2 t = 0$.

sol'n: Given $x^2 r + 2xys + y^2 t = 0$ —— ①

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$,

here $R = x^2$, $S = 2xy$ and $T = y^2$.

Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

and hence ① is parabolic everywhere.

the λ-quadratic is $R\lambda^2 + S\lambda + T = 0$ or $x^2\lambda^2 + 2xy\lambda + y^2 = 0$

—— ②

solving (2), $(x\lambda+y)^2=0$ so that $\lambda=-y/x, -y/x$
(equal roots)

the characteristic equation is $(dy/dx)-(y/x)=0$

(or) $(1/y)\,dy-(1/x)dx=0$ giving $y/x=c_1$ (or) $y=c_1x$.

which is the required family of characteristics.
Here it represents a family of straight lines passing through the origin.

H.W

→ Find the characteristics of $4r+5s+t+p+q-2=0$

[Ans. $y-x=c_1$, and $y-(x/4)=c_2$]

→ Find the characteristics of $(\sin^2 x)\,r+(2\cos x)s-t=0$.

[Ans: $y+\operatorname{cosec}x-\cot x=c_1$, $y+\operatorname{cosec}x+\cot x=c_2$]