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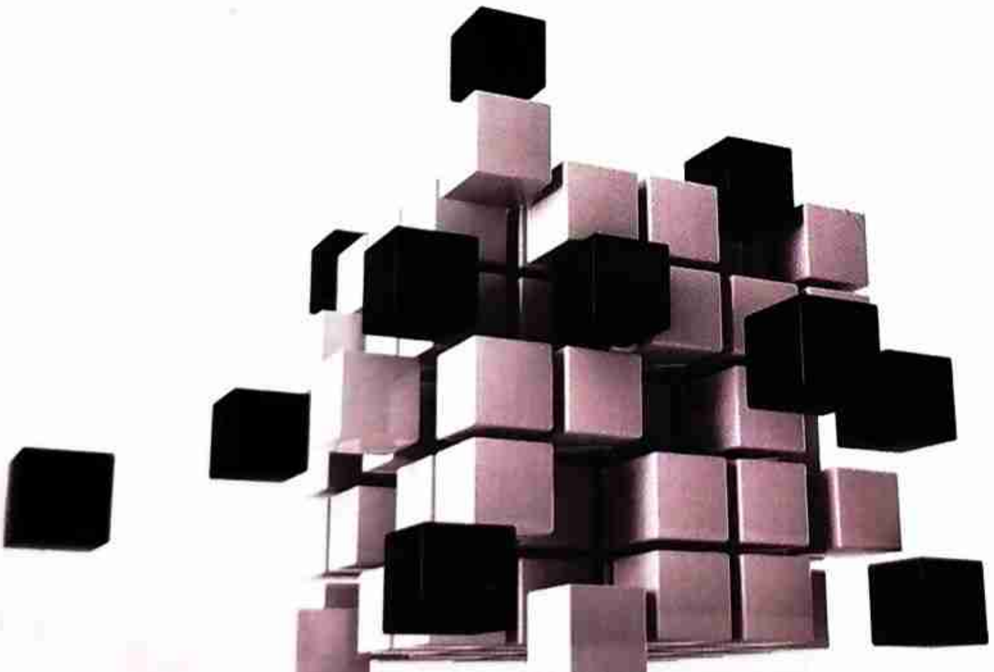

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Differential Calculus

(For Degree and Honours Students of All Indian Universities, various Competitive Examinations like P.C.S., I.A.S. and Engineering Students)

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Jai Shri Radhey Shyam

Dedicated

to

Lord

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Authors & Publishers

Preface to the First Edition

This book on Differential Calculus has been written for the use of the students of Degree and Honours classes of Indian Universities. The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The articles have been explained in details in a nice manner and all the examples have been completely solved. We have tried to solve each problem in an elegant and more interesting way. Students should follow the solutions very carefully and they should try to reproduce them when they do the problems independently. The book contains almost all the questions set at the various examinations held by Indian Universities and it covers the syllabi of all the Universities.

Authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book will be gratefully accepted.

Authors

Preface to the Latest Edition

It gives us a great pleasure in bringing out this latest edition of the book. The book has been thoroughly revised and a number of new examples selected from recent examination papers of various Universities have been added.

We hope that the students will find the persent edition great improvement over the previous one. Suggestions for the further improvement of the book will be gratefully accepted.

Authors

Detailed Contents

UNIT – II

Note : The chapter 1 of Unit-I, chapter 1 of Unit-II and two chapters of Unit-III of this book are not in the syllabus of C.C.S. University, Meerut. The chapters 2 to 9 cover the syllabus of Unit I and the chapters 2 to 7 of Unit-II cover the syllabus of Unit II.

1

Differentiation

§ 1. Derivative (or differential coefficient) of a function.

Let $y = f(x)$ be a function of x defined in the interval (a, b). For a small increment δx in x, let the corresponding increment in the value of y be δy. The quotient $\delta y/\delta x$ is called the *average rate of change* of y with respect to x in the interval $(x, x + \delta x)$. If the increment ratio $\delta y/\delta x$ tends to a finite limit as $\delta x \to 0$, then this limit is called the **differential coefficient** of y with respect to x and is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx} f(x) \text{ or } f'(x) \text{ or } Df(x) \text{ or } Dy \text{ etc.}$$

Thus $\dfrac{dy}{dx}$ **= the diff. coeff. of y w.r.t. x =** $\displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

The diff. coeff. dy/dx is also called the *rate of change* of y with respect to x.

Now if $y = f(x)$ and $y + \delta y = f(x + \delta x)$, we have

$$\delta y = f(x + \delta x) - f(x).$$

$\therefore \quad \dfrac{dy}{dx} = \displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$, provided the limit exists. The process of finding the diff. coeff. of a given function $f(x)$ is called **differentiation.**

The differential coefficient of $f(x)$ *for* $x = a$ *is defined as* $\displaystyle\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, *provided the limit exists.* It is denoted by $\left(\dfrac{dy}{dx}\right)_{x = a}$ or $(y')_a$ or $f'(a)$.

§ 2. Some standard results.

To find the differential coefficients of some standard functions from the first principles.

1. Differential coefficient of a constant. Let $f(x) = c$, where c is a constant. Then $f(x + \delta x) = c$, because $f(x)$ takes the same value c for every value of x.

Now by the definition of a diff. coeff., we have

$$\frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \frac{c - c}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{0}{\delta x} = \lim_{\delta x \to 0} 0 = 0.$$

Thus, **the diff. coeff. of a constant is zero.**

2. Diff. coeff. of the product of a constant and a function.

We have, $\frac{d}{dx}\{cf(x)\} = \lim_{\delta x \to 0} \frac{cf(x+\delta x) - cf(x)}{\delta x}$

$$= c \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = c \frac{d}{dx} f(x).$$

3. Derivative of a sum or a difference of two functions.

Let $f(x) = f_1(x) \pm f_2(x)$.

Then $f(x+\delta x) = f_1(x+\delta x) \pm f_2(x+\delta x)$.

$$\therefore \quad \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{\{f_1(x+\delta x) \pm f_2(x+\delta x)\} - \{f_1(x) \pm f_2(x)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{f_1(x+\delta x) - f_1(x)}{\delta x} \pm \lim_{\delta x \to 0} \frac{f_2(x+\delta x) - f_2(x)}{\delta x}$$

$$= \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x).$$

4. Differential coefficient of the product of two functions.

(Delhi 1982)

Let $y = f_1(x) f_2(x)$. Then $y + \delta y = f_1(x+\delta x) \cdot f_2(x+\delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f_1(x+\delta x) f_2(x+\delta x) - f_1(x) \cdot f_2(x)}{\delta x},$$

[by the def. of a diff. coeff.]

$$= \lim_{\delta x \to 0} \left[f_1(x+\delta x) \cdot \frac{f_2(x+\delta x) - f_2(x)}{\delta x} + f_2(x) \frac{f_1(x+\delta x) - f_1(x)}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} f_1(x+\delta x) \frac{f_2(x+\delta x) - f_2(x)}{\delta x}$$

$$+ \lim_{\delta x \to 0} f_2(x) \cdot \frac{f_1(x+\delta x) - f_1(x)}{\delta x}$$

$$= f_1(x) \cdot \frac{d}{dx} f_2(x) + f_2(x) \cdot \frac{d}{dx} f_1(x).$$

$\therefore$ **the diff. coeff. of the product of two functions**

= **(first function) × (diff. coeff. of the second function)**

+ **(second function) × (diff. coeff. of the first function).**

5. Differential coefficient of the quotient of two functions.

Let $y = f_1(x)/f_2(x)$. Then $y + \delta y = f_1(x+\delta x)/f_2(x+\delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\left[\frac{f_1(x+\delta x)}{f_2(x+\delta x)} - \frac{f_1(x)}{f_2(x)} \right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f_1(x+\delta x) \cdot f_2(x) - f_1(x) \cdot f_2(x+\delta x)}{\delta x \cdot f_2(x+\delta x) \cdot f_2(x)} \right]$$

$$= \lim_{\delta x \to 0} \left[\frac{f_1(x+\delta x) f_2(x) - f_1(x) f_2(x) - f_1(x) f_2(x+\delta x) + f_1(x) \cdot f_2(x)}{\delta x \cdot f_2(x+\delta x) \cdot f_2(x)} \right]$$

$$= \lim_{\delta x \to 0} \left[\frac{f_2(x) \cdot \left\{ \dfrac{f_1(x+\delta x) - f_1(x)}{\delta x} \right\} - f_1(x) \cdot \left\{ \dfrac{f_2(x+\delta x) - f_2(x)}{\delta x} \right\}}{f_2(x+\delta x) f_2(x)} \right]$$

$$= \frac{f_2(x) \cdot \dfrac{d}{dx} f_1(x) - f_1(x) \cdot \dfrac{d}{dx} f_2(x)}{[f_2(x)]^2}.$$

$\therefore$ **the diff. coeff. of the quotient of two functions**

$$= \frac{\textbf{(diff. coeff. of Nr.) (Dr.) − (Nr.) (diff. coeff. of Dr.)}}{\textbf{square of the denominator}}.$$

6. $(\mathbf{d/dx})\, \mathbf{x}^n = \mathbf{n}\mathbf{x}^{n-1}$.

Let $y = x^n$. Then $y + \delta y = (x + \delta x)^n$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x} = \lim_{\delta x \to 0} x^n \left[\frac{\left(1 + \dfrac{\delta x}{x}\right)^n - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} x^n \left[\frac{1 + n\dfrac{\delta x}{x} + \dfrac{n(n-1)}{2!} \dfrac{(\delta x)^2}{x^2} + \ldots - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!} \frac{\delta x}{x^2} + \text{terms containing higher powers of } \delta x \right]$$

$= x^n (n/x) = nx^{n-1}$.

7. $(\mathbf{d/dx})\, \mathbf{e}^x = \mathbf{e}^x$.

Let $y = e^x$. Then $y + \delta y = e^{x+\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{x+\delta x} - e^x}{\delta x}$$

$$= \lim_{\delta x \to 0} e^x \left[\frac{e^{\delta x} - 1}{\delta x} \right] = \lim_{\delta x \to 0} e^x \left[\frac{1 + \delta x + (\delta x)^2/2! + \ldots - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} e^x \left[1 + \frac{\delta x}{2!} + \text{terms containing higher powers of } \delta x \right] = e^x.$$

8. $(\mathbf{d}\ \mathbf{dx})\, \mathbf{a}^x = \mathbf{a}^x \log_e \mathbf{a}$.

Let $y = a^x$. Then $y + \delta y = a^{x+\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x}$$

$$= \lim_{\delta x \to 0} a^x \left[\frac{a^{\delta x} - 1}{\delta x} \right]$$

$$= \lim_{\delta x \to 0} a^x \left[\frac{1 + \delta x \log a + \dfrac{(\delta x)^2 (\log a)^2}{2!} + \ldots - 1}{\delta x} \right]$$

$= \lim_{\delta x \to 0} a^x [\log a + (\delta x/2!) (\log a)^2 + \text{terms containing higher powers of } \delta x]$

$= a^x \log a.$

9. $\mathbf{(d/dx) \log_e x = 1/x.}$

Let $y = \log x$. Then $y + \delta y = \log (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\log(x + \delta x) - \log x}{\delta x} = \lim_{\delta x \to 0} \frac{\log \{(x + \delta x)/x\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\log \left(1 + \dfrac{\delta x}{x}\right)}{\delta x} = \lim_{\delta x \to 0} \frac{\dfrac{\delta x}{x} - \dfrac{(\delta x)^2}{2x^2} + \dots}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{1}{x} - \frac{\delta x}{2x^2} + \dots\right] = \frac{1}{x}.$$

10. $\mathbf{\dfrac{d}{dx} \log_a x = \dfrac{1}{x \log_e a}.}$

We have $\log_a x = \log_e x \,.\, \log_a e = \log_a e \,.\, \log_e x$, where $\log_a e$ is simply a constant.

$$\therefore \quad \frac{d}{dx}(\log_a x) = (\log_a e) \frac{d}{dx}(\log_e x) = (\log_a e) \frac{1}{x} = \frac{1}{x} \cdot \log_a e$$

$$= \frac{1}{x \log_e a}, \text{ since } \log_a e \,.\, \log_e a = 1.$$

11. $\mathbf{(d/dx) \sin x = \cos x.}$

Let $y = \sin x$. Then $y + \delta y = \sin (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin (x + \delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2 \cos \{x + (\delta x/2)\} \sin (\delta x/2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos \{x + (\delta x/2)\} \sin (\delta x/2)}{(\delta x/2)}$$

$$= \lim_{\delta x \to 0} \cos \left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin (\delta x/2)}{(\delta x/2)}$$

$$= (\cos x) \cdot 1 = \cos x.$$

12. $\mathbf{(d/dx) \cos x = -\sin x.}$

Let $y = \cos x$. Then $y + \delta y = \cos (x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\cos (x + \delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2 \sin \{x + (\delta x/2)\} \sin (\delta x/2)}{\delta x}$$

$$= \lim_{\delta x \to 0} -\sin \left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin (\delta x/2)}{(\delta x/2)} = (-\sin x) \cdot 1 = -\sin x.$$

13. $\mathbf{(d/dx)\tan x = \sec^2 x.}$ **(Delhi 1982)**

Let $y = \tan x$. Then $y + \delta y = \tan(x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\tan(x + \delta x) - \tan x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left[\dfrac{\sin(x + \delta x)}{\cos(x + \delta x)} - \dfrac{\sin x}{\cos x}\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x + \delta x)\cos x - \sin x \cos(x + \delta x)}{\delta x \cos(x + \delta x) \cdot \cos x}$$

$$= \lim_{\delta x \to 0} \frac{\sin\{(x + \delta x) - x\}}{\delta x \cdot \cos(x + \delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\sin \delta x}{\delta x} \cdot \frac{1}{\cos(x + \delta x)\cos x}\right] = \frac{1}{\cos^2 x} = \sec^2 x.$$

14. $\mathbf{(d/dx)\cot x = -\operatorname{cosec}^2 x.}$

For proof proceed as in the case of $\tan x$.

15. $\mathbf{(d/dx)\sec x = \sec x \tan x.}$

Let $y = \sec x$. Then $y + \delta y = \sec(x + \delta x)$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sec(x + \delta x) - \sec x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\dfrac{1}{\cos(x + \delta x)} - \dfrac{1}{\cos x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos x - \cos(x + \delta x)}{\delta x \cdot \cos(x + \delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \frac{2\sin(x + \frac{1}{2}\delta x)\sin(\delta x/2)}{\delta x \cdot \cos(x + \delta x)\cos x}$$

$$= \lim_{\delta x \to 0} \left[\frac{\sin(x + \frac{1}{2}\delta x)}{\cos(x + \delta x)\cos x} \cdot \frac{\sin(\delta x/2)}{(\delta x/2)}\right]$$

$$= \frac{\sin x}{\cos x \cos x} = \sec x \tan x.$$

16. $\mathbf{(d/dx)\operatorname{cosec} x = -\operatorname{cosec} x \cot x.}$

For proof proceed as in the case of $\sec x$.

17. $$\mathbf{\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1 - x^2)}}}.$$

Let $y = \sin^{-1} x$. Then $x = \sin y$ and so $x + \delta x = \sin(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sin(y + \delta y) - \sin y$.

$$\therefore \quad 1 = \frac{\sin(y + \delta y) - \sin y}{\delta x}, \qquad \text{[on dividing both sides by } \delta x]$$

or $$1 = \frac{\sin(y + \delta y) - \sin y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sin(y+\delta y) - \sin y}{\delta y} \cdot \lim_{\delta y \to 0} \frac{\delta y}{\delta x},$$

$[\because \delta y \to 0 \text{ when } \delta x \to 0]$

$$= \left[\lim_{\delta y \to 0} \frac{2\cos(y + \frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y}\right] \cdot \frac{dy}{dx}$$

$$= \left[\lim_{\delta y \to 0} \cos(y + \tfrac{1}{2}\delta y) \cdot \frac{\sin(\frac{1}{2}\delta y)}{\delta y/2}\right] \frac{dy}{dx}$$

$$= (\cos y) \, . \, (dy/dx).$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1 - \sin^2 y)}} = \frac{1}{\sqrt{(1 - x^2)}}.$$

18. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cos^{-1} x}) = -\dfrac{\mathbf{1}}{\sqrt{(\mathbf{1 - x^2})}}.$

For proof proceed exactly as in the case of $\sin^{-1} x$.

19. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\tan^{-1} x}) = \dfrac{\mathbf{1}}{\mathbf{1 + x^2}}.$

Let $y = \tan^{-1} x$. Then $x = \tan y$ and so $x + \delta x = \tan(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\qquad \delta x = \tan(y + \delta y) - \tan y.$

$$\therefore \quad 1 = \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\tan(y+\delta y) - \tan y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \left[\lim_{\delta y \to 0} \left\{\frac{\sin(y+\delta y)}{\cos(y+\delta y)} - \frac{\sin y}{\cos y}\right\} \Big/ \delta y\right] \cdot \frac{dy}{dx}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y+\delta y)\cos y - \cos(y+\delta y)\sin y}{\delta y \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\sin(y + \delta y - y)}{\delta y \, . \cos(y+\delta y)\cos y}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin \delta y}{\delta y} \cdot \frac{1}{\cos(y+\delta y)\cos y}\right]$$

$$= \frac{dy}{dx} \cdot \frac{1}{\cos^2 y} = \frac{dy}{dx} \cdot \sec^2 y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

20. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cot^{-1} x}) = -\dfrac{\mathbf{1}}{\mathbf{1 + x^2}}.$

For proof proceed exactly as in the case of $\tan^{-1} x$.

21. $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{(x^2-1)}}$.

Let $y = \sec^{-1} x$. Then $x = \sec y$ and so $x + \delta x = \sec(y + \delta y)$. As $\delta x \to 0$, δy also $\to 0$.

Now $\delta x = \sec(y+\delta y) - \sec y$.

$$\therefore \quad 1 = \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \frac{\delta y}{\delta x}.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$1 = \lim_{\delta y \to 0} \frac{\sec(y+\delta y) - \sec y}{\delta y} \cdot \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{1}{\cos(y+\delta y)} - \frac{1}{\cos y}\right] \Big/ \delta y$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{\cos y - \cos(y+\delta y)}{\delta y \cdot \cos y \cos(y+\delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \frac{2\sin(y + \frac{1}{2}\delta y)\sin(\frac{1}{2}\delta y)}{\delta y \cdot \cos y \cos(y+\delta y)}$$

$$= \frac{dy}{dx} \cdot \lim_{\delta y \to 0} \left[\frac{\sin(y+\frac{1}{2}\delta y)}{\cos y \cos(y+\delta y)} \cdot \frac{\sin(\frac{1}{2}\delta y)}{\frac{1}{2}\delta y}\right]$$

$$= \frac{dy}{dx} \cdot \frac{\sin y}{\cos y \cos y} = \frac{dy}{dx} \cdot \sec y \tan y.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2-1)}}.$$

22. $\dfrac{d}{dx}(\operatorname{cosec}^{-1} x) = -\dfrac{1}{x\sqrt{(x^2-1)}}$.

For proof proceed as in the case of $\sec^{-1} x$.

§ 3. Differential coefficient of a function of a function.

Consider the function log sin x. Here $\log(\sin x)$ is a function of $\sin x$ whereas $\sin x$ is itself a function of x. Thus we have the case of a function of a function.

Let $y = f\{\phi(x)\}$.

Put $t = \phi(x)$. Then $t + \delta t = \phi(x + \delta x)$. As $\delta x \to 0$, δt also $\to 0$.

We have

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta t} \cdot \frac{\delta t}{\delta x} = \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta t}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta t}{\delta x}\right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\delta y}{\delta t}\right) \cdot \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x}\right) \cdot \text{since } \delta t \to 0 \text{ when } \delta x \to 0$$

$$= \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Thus, *if y is a function of t and t is a function of x, then y is a function of x and we have*

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}.$$

Similarly, *if y is a function of u, u is a function of v and v is a function of x, then y is also a function of x, and we have*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

This is known as the **chain rule of differentiation**. We can extend it still further.

Cor. To prove that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$.

Let $y = f(x)$. Then $y + \delta y = f(x + \delta x)$. As $\delta x \to 0$, δy also $\to 0$. We have

$$\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} = 1.$$

Taking limits of both sides when $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta y} \right) = 1 \text{ or } \left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \right) \left(\lim_{\delta x \to 0} \frac{\delta x}{\delta y} \right) = 1$$

or $\left(\lim_{\delta x \to 0} \frac{\delta y}{\delta x} \right) \cdot \left(\lim_{\delta y \to 0} \frac{\delta x}{\delta y} \right) = 1$, since $\delta y \to 0$ as $\delta x \to 0$

or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$. Thus $\frac{dy}{dx} = \frac{1}{dx/dy}$.

§ 4. Hyperbolic functions.

These are defined as follows :

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}); \cosh x = \tfrac{1}{2}(e^x + e^{-x});$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}};$$

$$\operatorname{sech} x = \frac{1}{\cosh x}; \text{ and } \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Remember the following relations between hyperbolic functions.

(i) $\cosh^2 x - \sinh^2 x = 1$; (ii) $\sinh 2x = 2 \sinh x \cosh x$;

(iii) $1 - \tanh^2 x = \operatorname{sech}^2 x$; (iv) $\coth^2 x - 1 = \operatorname{cosech}^2 x$;

(v) $\cosh 2x = \cosh^2 x + \sinh^2 x$;

(vi) $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$.

To get any of these relations first write the corresponding relation between circular functions. Then convert each circular function into the corresponding hyperbolic function and also change the sign of the term which contains the product of two sines.

Derivatives of hyperbolic functions.

We have, $\frac{d}{dx}\sinh x = \frac{d}{dx}\{\frac{1}{2}(e^x - e^{-x})\} = \frac{1}{2}(e^x + e^{-x}) = \cosh x.$

Similarly we can find the derivatives of other hyperbolic functions. Remember the following results.

$$\frac{d}{dx}\sinh x = \cosh x, \frac{d}{dx}\cosh x = \sinh x, \frac{d}{dx}\tanh x = \text{sech}^2 x,$$

$$\frac{d}{dx}\coth x = -\text{cosech}^2 x, \frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x,$$

$$\frac{d}{dx}\text{cosech}\, x = -\text{cosech}\, x \coth x.$$

We note that these results are similar to the corresponding results on circular funcions. The only change is that here the derivatives of $\sinh x$, $\cosh x$ and $\tanh x$ have + ive sign placed before them and the derivatives of the remaining three functions have − ive sign placed before them.

§ 5. Inverse hyperbolic functions.

Let $y = \sinh^{-1} x$. Then $\sinh y = x$. Therefore $\cosh y = \sqrt{(\sinh^2 y + 1)} = \sqrt{(x^2 + 1)}$. Adding these, we get $\sinh y + \cosh y = x + \sqrt{(x^2 + 1)}$

or $\frac{1}{2}(e^y - e^{-y}) + \frac{1}{2}(e^y + e^{-y}) = x + \sqrt{(x^2 + 1)}$ or $e^y = x + \sqrt{(x^2 + 1)}$.

$\therefore \quad y = \log[x + \sqrt{(x^2 + 1)}]$.

Thus $\quad \sinh^{-1} x = \log[x + \sqrt{(x^2 + 1)}]$.

Similarly, $\quad \cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]$;

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}; \coth^{-1} x = \frac{1}{2}\log\frac{x+1}{x-1};$$

$$\text{sech}^{-1} x = \log\frac{1+\sqrt{(1-x^2)}}{x}; \text{cosech}^{-1} x = \log\frac{1+\sqrt{(1+x^2)}}{x}.$$

Derivatives of Inverse Hyperbolic functions.

Let $\sinh^{-1} x = y$. Then $x = \sinh y$.

$$\therefore \quad \frac{d}{dx}(x) = \frac{d}{dx}(\sinh y) \text{ or } 1 = \left[\frac{d}{dy}(\sinh y)\right]\frac{dy}{dx} = \cosh y\,\frac{dy}{dx}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh^2 y + 1)}} = \frac{1}{\sqrt{(x^2+1)}}.$$

$$\therefore \quad \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{(x^2+1)}}.$$

Similarly, $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{(x^2 - 1)}}$;

$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}; \frac{d}{dx}\coth^{-1} x = -\frac{1}{x^2 - 1},$$

$$\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{1}{x\sqrt{(1-x^2)}};\ \frac{d}{dx}\operatorname{cosech}^{-1} x = -\frac{1}{x\sqrt{(1+x^2)}}.$$

§ 6. Some more methods of differentiation.

(a) Logarithmic differentiation. When a function consists of the product or the quotient of a number of functions, we take the logarithm and differentiate. This process, called *logarithmic differentiation*, is also useful when a funcion of x is raised to a power which is itself a function of x.

(b) Implict functions. If the relation between y and x is represented by an equation from which y cannot be easily expressed in terms of x, then y is called an *implicit function* of x. But if x is given directly in terms of x, then y is called an *explicit function* of x.

To find dy/dx in the case of implicit functions, differentiate each term of the given equation w.r.t. 'x' and solve the resulting equation for dy/dx.

Note. Here the value of dy/dx shall usually contain both x and y.

(c) Parametric equations. Sometimes x and y are both expressed in terms of a third variable, say, t. This variable is called a parameter and the equations thus given are known as parametric equations. In the case of parametric equations $x = f_1(t)$ and $y = f_2(t)$, we find dy/dx in the following manner :

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy/dt}{dx/dt}.$$

(d) Trigonometrical transformations. Sometimes it is easy to differentiate after making some trigonometrical transformations.

Following formulae of trigonometry are of frequent use :

(i) $\cos x = 2\cos^2(x/2) - 1 = 1 - 2\sin^2(x/2)$;

(ii) $\sin x = 2\tan\frac{1}{2}x/(1+\tan^2\frac{1}{2}x)$;

(iii) $\cos x = (1-\tan^2\frac{1}{2}x)/(1+\tan^2\frac{1}{2}x)$;

(iv) $\tan x = 2\tan\frac{1}{2}x/(1-\tan^2\frac{1}{2}x)$;

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$;

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$;

(vii) $2\tan^{-1}x = \tan^{-1}2x/(1-x^2)$;

(viii) $3\tan^{-1}x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$;

(ix) $\sin 3x = 3\sin x - 4\sin^3 x$;

(x) $\cos 3x = 4\cos^3 x - 3\cos x$.

(e) Differentiation of a function w.r.t. a function.

Suppose we have to differentiate $f(x)$ w.r.t. $\phi(x)$ · Put $\phi(x) = t$.

Then

$$\frac{d f(x)}{d \phi(x)}=\frac{d f(x)}{dt}=\frac{d f(x)}{dx}\cdot\frac{dx}{dt}=\frac{d f(x)}{dx}\Big/\frac{dt}{dx}.$$

Hence $\dfrac{d f(x)}{d \phi(x)}=\dfrac{d f(x)}{dx}\Big/\dfrac{d \phi(x)}{dx}$.

§ 7. List of Standard results to be Committed to Memory.

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}\tan x=\sec^2 x$$

$$\frac{d}{dx}e^x=e^x \qquad \frac{d}{dx}\cot x=-\operatorname{cosec}^2 x$$

$$\frac{d}{dx}\log_e x=\frac{1}{x} \qquad \frac{d}{dx}\sec x=\sec x\tan x$$

$$\frac{d}{dx}\log_a x=\frac{1}{x}\log_a e \qquad \frac{d}{dx}\operatorname{cosec} x=-\operatorname{cosec} x\cot x$$

$$\frac{d}{dx}a^x=a^x\log_e a \qquad \frac{d}{dx}\sinh x=\cosh x$$

$$\frac{d}{dx}\sin x=\cos x \qquad \frac{d}{dx}\cosh x=\sinh x$$

$$\frac{d}{dx}\cos x=-\sin x \qquad \frac{d}{dx}\tanh x=\operatorname{sech}^2 x$$

$$\frac{d}{dx}\coth x=-\operatorname{cosech}^2 x \qquad \frac{d}{dx}\cot^{-1}x=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}\operatorname{sech} x=-\operatorname{sech} x\tanh x \qquad \frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\operatorname{cosech} x=-\operatorname{cosech} x\coth x \qquad \frac{d}{dx}\operatorname{cosec}^{-1}x=-\frac{1}{x\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{(1-x^2)}} \qquad \frac{d}{dx}\sinh^{-1}x=\frac{1}{\sqrt{(x^2+1)}}$$

$$\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{(1-x^2)}} \qquad \frac{d}{dx}\cosh^{-1}x=\frac{1}{\sqrt{(x^2-1)}}$$

$$\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2} \qquad \frac{d}{dx}\tanh^{-1}x=\frac{1}{1-x^2}$$

Solved Examples

Ex. 1. *Find the differential coefficients of the following w.r.t. x.*

(i) $\sqrt{(2x-x^{-2})}$ **(ii)** $\dfrac{\sqrt{(5-2x)}}{2x+1}$ **(iii)** $\log\dfrac{2}{x}$

(iv) $\sqrt{\{(1+\sin x)/(1-\sin x)\}}$ **(v)** $\dfrac{ax^2+b}{ax^2-b}+\dfrac{ax^2-b}{ax^2+b}$

(vi) $\log(\log x)$.

Sol. **(i)** $\frac{d}{dx}(2x - x^{-2})^{1/2} = \frac{1}{2}(2x - x^{-2})^{-1/2}(2 + 2x^{-3})$

$= (2x - x^{-2})^{-1/2}(1 + x^{-3}) = \frac{1 + x^{-3}}{\sqrt{(2x - x^{-2})}}.$

(ii) $\frac{d}{dx}\left\{\frac{\sqrt{(5 - 2x)}}{2x + 1}\right\} = \frac{d}{dx}\{(5 - 2x)^{1/2} \cdot (2x + 1)^{-1}\}$

$= \frac{1}{2}(5 - 2x)^{-1/2}(-2)(2x + 1)^{-1} - (2x + 1)^{-2} \cdot 2 \cdot (5 - 2x)^{1/2}$

$= \frac{-1}{(2x + 1)\sqrt{(5 - 2x)}} - \frac{2\sqrt{(5 - 2x)}}{(2x + 1)^2}$

$= \frac{-(2x + 1) - 2(5 - 2x)}{(2x + 1)^2\sqrt{(5 - 2x)}} = \frac{2x - 11}{(2x + 1)^2\sqrt{(5 - 2x)}}.$

(iii) $\frac{d}{dx}\log\frac{2}{x} = \frac{d}{dx}(\log 2 - \log x) = -\frac{1}{x}.$

(iv) We have $\sqrt{\left(\frac{1 + \sin x}{1 - \sin x}\right)} = \sqrt{\left\{\frac{(1 + \sin x)(1 + \sin x)}{(1 - \sin x)(1 + \sin x)}\right\}}$

$= \sqrt{\left\{\frac{(1 + \sin x)^2}{1 - \sin^2 x}\right\}} = \frac{1 + \sin x}{\cos x} = \sec x + \tan x.$

$\therefore \quad \frac{d}{dx}\sqrt{\left(\frac{1 + \sin x}{1 - \sin x}\right)} = \frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x.$

(v) $\frac{d}{dx}\left(\frac{ax^2 + b}{ax^2 - b} + \frac{ax^2 - b}{ax^2 + b}\right) = \frac{d}{dx}\left(\frac{2a^2x^4 + 2b^2}{a^2x^4 - b^2}\right)$

$= \frac{8a^2x^3(a^2x^4 - b^2) - 4a^2x^3(2a^2x^4 + 2b^2)}{(a^2x^4 - b^2)^2}$

$= -\frac{16a^2b^2x^3}{(a^2x^4 - b^2)^2}.$

(vi) $\frac{d}{dx}\{\log(\log x)\} = \frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log x}.$

Ex. 2. *Find the differential coefficients of the following w.r.t. x.*

(i) $7\sin x + 2\log x - e^x + (x^2 - 7x + 4)$

(ii) $(\cos x) \cdot (\log x)$ **(iii)** $e^{\sin^{-1} x}$

(iv) $e^{ax}\cos(bx + c)$ **(v)** $\log \sec x$

(vi) $a^{2x}\sinh 2x$ **(vii)** $\log\frac{1 + \sqrt{x}}{1 - \sqrt{x}}$

(viii) $\frac{e^x}{1 + x}$ **(ix)** $\frac{\tan x}{x + e^x}$

(x) $\log[\sqrt{(1 + \log x)} - \sin x]$ **(xi)** $\tan(\log \tan^{-1}\sqrt{x})$.

Sol. **(i)** Let $y = 7\sin x + 2\log x + e^x + (x^2 - 7x + 4)$.

Then $\frac{dy}{dx} = 7.\frac{d}{dx}(\sin x) + 2\frac{d}{dx}(\log x) - \frac{d}{dx}(e^x) + \frac{d}{dx}(x^2 - 7x + 4)$

$= 7\cos x + 2/x - e^x + 2x - 7.$

(ii) Let $y = (\cos x) \cdot (\log x)$.

Then $\dfrac{dy}{dx} = (\cos x)\dfrac{d}{dx}(\log x) + (\log x)\dfrac{d}{dx}(\cos x)$

$= (\cos x)(1/x) + (\log x)(-\sin x)$

$= (1/x)\cos x - (\sin x)\log x.$

(iii) Let $y = e^{\sin^{-1} x}$. Then $\dfrac{dy}{dx} = e^{\sin^{-1} x}\dfrac{d}{dx}\sin^{-1} x$

$= e^{\sin^{-1} x}. \dfrac{1}{\sqrt{(1-x^2)}} \cdot$

(iv) Let $y = e^{ax}\cos(bx + c)$.

Then $\dfrac{dy}{dx} = e^{ax}. \dfrac{d}{dx}\cos(bx + c) + \cos(bx + c) \cdot \dfrac{d}{dx}e^{ax}$

$= e^{ax}[\{-\sin(bx + c)\}b] + \cos(bx + c). e^{ax}. a$

$= e^{ax}[a\cos(bx + c) - b\sin(bx + c)].$

(v) Let $y = \log\sec x$. Then $\dfrac{dy}{dx} = \dfrac{1}{\sec x} \cdot \dfrac{d}{dx}(\sec x)$

$= \dfrac{1}{\sec x}\sec x\tan x = \tan x.$

(vi) Let $y = a^{2x}\sinh 2x$. Then $\dfrac{dy}{dx} = a^{2x}. \dfrac{d}{dx}\sinh 2x + \sinh 2x \cdot \dfrac{d}{dx}a^{2x}$

$= 2a^{2x}\cosh 2x + (\sinh 2x) \cdot (a^{2x}\log a) \cdot 2$

$= 2a^{2x}[\cosh 2x + (\log a)\sinh 2x].$

(vii) Let $y = \log\dfrac{1 + \sqrt{x}}{1 - \sqrt{x}} = \log(1 + \sqrt{x}) - \log(1 - \sqrt{x})$.

Then $\dfrac{dy}{dx} = \dfrac{1}{1 + \sqrt{x}} \cdot \dfrac{d}{dx}(1 + \sqrt{x}) - \dfrac{1}{1 - \sqrt{x}} \cdot \dfrac{d}{dx}(1 - \sqrt{x})$

$= \dfrac{1}{2\sqrt{x}}\left[\dfrac{1}{1 + \sqrt{x}} + \dfrac{1}{1 - \sqrt{x}}\right]$

$= \dfrac{1}{2\sqrt{x}} \cdot \dfrac{2}{1 - x} = \dfrac{1}{(1 - x)\sqrt{x}}.$

(viii) Let $y = e^x/(1 + x)$.

Then $\dfrac{dy}{dx} = \dfrac{(1 + x) \cdot \dfrac{d}{dx}e^x - e^x \cdot \dfrac{d}{dx}(1 + x)}{(1 + x)^2}$

$= \dfrac{(1 + x)e^x - e^x \cdot 1}{(1 + x)^2} = \dfrac{x e^x}{(1 + x)^2}.$

(ix) Let $y = \tan x/(x + e^x)$.

Then $\dfrac{dy}{dx} = \dfrac{(x + e^x)\sec^2 x - (\tan x)(1 + e^x)}{(x + e^x)^2}$

$= \dfrac{x\sec^2 x + e^x(\sec^2 x - \tan x) - \tan x}{(x + e^x)^2}$

(x) Let $y = \log [\sqrt{(1 + \log x)} - \sin x]$.

Then $$\frac{dy}{dx} = \frac{1}{\sqrt{(1 + \log x)} - \sin x} \cdot \left[\frac{1}{2x\sqrt{(1 + \log x)}} - \cos x\right].$$

(xi) Let $y = \tan (\log \tan^{-1} \sqrt{x})$.

Then $$\frac{dy}{dx} = \sec^2 (\log \tan^{-1} \sqrt{x}) \left[\frac{1}{\tan^{-1} (\sqrt{x})} \cdot \frac{1}{1 + x} \cdot \frac{1}{2\sqrt{x}}\right]$$
$$= \frac{\sec^2 (\log \tan^{-1} \sqrt{x})}{(2\sqrt{x}) (1 + x) \tan^{-1} (\sqrt{x})}.$$

Problems involving logarithmic differentiation, implicit functions and parametric equations.

Ex. 3. *Find* (dy/dx), *when*

(i) $y = x^{(x^x)}$ **(Delhi 1983,79)**

(ii) $y = (\tan x)^{\log x} + (\cot x)^{\sin x}$,

(iii) $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$, **(iv)** $y = x^x + x^{\sin x}$.

Sol. (i) Taking logarithm of both sides, we get
$$\log y = (x^x) \log x.$$

Now differentiating both sides w.r.t. x, we get
$$\frac{1}{y}\frac{dy}{dx} = x^x . \frac{1}{x} + \log x . \frac{d}{dx}(x^x). \qquad ...(1)$$

Let $z = x^x$. Then $\log z = x \log x$.

Differentiating w.r.t. x, we have
$$\frac{1}{z}\frac{dz}{dx} = x\frac{1}{x} + \log x. \quad \therefore \quad \frac{dz}{dx} = x^x (1 + \log x).$$

Now (1) gives, $\dfrac{dy}{dx} = x^{(x^x)} \cdot [x^{x-1} + x^x (\log x) \cdot (1 + \log x)]$.

(ii) We have $y = (\tan x)^{\log x} + (\cot x)^{\sin x}$.

Let $(\tan x)^{\log x} = u$ and $(\cot x)^{\sin x} = v$.

Then $y = u + v$. $\therefore$ $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Now $\log u = (\log x) (\log \tan x)$.
$$\therefore \quad \frac{1}{u} \cdot \frac{du}{dx} = (\log x) \cdot \frac{\sec^2 x}{\tan x} + \frac{1}{x} \log \tan x$$

or $$\frac{du}{dx} = (\tan x)^{\log x}. [\cot x \sec^2 x \log x + (1/x) \log \tan x].$$

Again $\log v = (\sin x) (\log \cot x)$.
$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\cot x} (- \operatorname{cosec}^2 x) + \cos x \log \cot x$$

or $$\frac{dv}{dx} = (\cot x)^{\sin x} [\cos x \log \cot x - \sec x].$$

Hence $$\frac{dy}{dx} = (\tan x)^{\log x} [\cot x \sec^2 x \log x + (1/x) \log \tan x] + (\cot x)^{\sin x} [\cos x \log \cot x - \sec x].$$

(iii) Here $y = (\cot x)^{\cot x} + (\cosh x)^{\cosh x}$.

Let $(\cot x)^{\cot x} = u$ and $(\cosh x)^{\cosh x} = v$.

Then $y = u + v$. Therefore $dy/dx = du/dx + dv/dx$.

Now $\log u = \cot x . \log \cot x$.

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \cot x . \frac{1}{\cot x}(-\operatorname{cosec}^2 x) - \operatorname{cosec}^2 x . \log \cot x$$

or $\dfrac{du}{dx} = -\operatorname{cosec}^2 x\,(1 + \log \cot x)$.

Again $\log v = \cosh x \log \cosh x$.

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = \cosh x \cdot \frac{1}{\cosh x} \cdot \sinh x + \sinh x \log \cosh x$$

or $\dfrac{dv}{dx} = v \sinh x\,(1 + \log \cosh x)$.

Hence

$$\frac{dy}{dx} = -u \operatorname{cosec}^2 x\,(1 + \log \cot x) + v \sinh x\,(1 + \log \cosh x)$$

$$= -(\cot x)^{\cot x} \operatorname{cosec}^2 x\,(1 + \log \cot x) + (\cosh x)^{\cosh x} \sinh x\,(1 + \log \cosh x).$$

(iv) We have $y = x^x + (x)^{\sin x}$.

Let $x^x = u$ and $(x)^{\sin x} = v$.

Then $y = u + v$ and $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Now $\log u = x \log x$.

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \log x \text{ or } \frac{du}{dx} = x^x(1 + \log x).$$

Again $\log v = (\sin x) \log x$.

$$\therefore \quad \frac{1}{v} \cdot \frac{dv}{dx} = (\sin x) . \frac{1}{x} + (\cos x) \log x$$

or $\dfrac{dv}{dx} = x^{\sin x} \cdot \left[\dfrac{1}{x}\sin x + (\cos x) \log x\right]$.

Hence $\dfrac{dy}{dx} = x^x . (1 + \log x) + (x)^{\sin x}\left[\dfrac{1}{x}\sin x + (\cos x) \log x\right]$.

Ex. 4. *Find* $\dfrac{dy}{dx}$ *if*

(*i*) $y = (x)^{\tan x} + (\sin x)^{\cos x}$. **(Delhi 1980)**

(*ii*) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$.

Sol. **(i)** Let $u = (x)^{\tan x}$ and $v = (\sin x)^{\cos x}$.

$\therefore \quad y = u + v \Rightarrow (dy/dx) = (du/dx) + (dv/dx)$. ...(1)

Now $u = (x)^{\tan x} \Rightarrow \log u = \tan x \log x$.

$\therefore$ differentiating w.r.t. x, we have

$(1/u)(du/dx) = (\sec^2 x) \cdot \log x + (\tan x)/x$

or $\quad \dfrac{du}{dx} = (x)^{\tan x}\left[\sec^2 x \log x + \dfrac{\tan x}{x}\right]\cdot$

Similarly $\dfrac{dv}{dx} = (\sin x)^{\cos x}\left[(-\sin x)\cdot \log \sin x + (\cos x)\cdot \cot x\right]\cdot$

$\therefore$ from (1), we get

$$(dy/dx) = (du/dx) + (dv/dx)$$
$$= (x)^{\tan x}[\sec^2 x \log x + (\tan x)/x]$$
$$+ (\sin x)^{\cos x}[\cos x \cot x - \sin x \cdot \log \sin x]\cdot$$

(ii) Proceed exactly as in part (i).

Ex. 5. *Find (dy/dx) if $\sin y = \log_{\sin x} \cos x$.* **(Delhi 1980)**

Sol. We have

$$\sin y = \log_{\sin x} \cos x = \frac{\log_e \cos x}{\log_e \sin x}\cdot \qquad \textbf{[Note]}$$

Differentiating both sides w.r.t. x, we have

$$(\cos y)\cdot \frac{dy}{dx} = \frac{(-\tan x)\cdot \log \sin x - (\cot x)\cdot \log \cos x}{(\log_e \sin x)^2}\cdot$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(\tan x)\log \sin x + (\cot x)\log \cos x}{(\cos y)(\log \sin x)^2}\cdot$$

Ex. 6. *Find (dy/dx), when*

(i) $x = a(t - \sin t), y = a(1 - \cos t)$.

(ii) $x = a(\cos t + \log \tan t/2), y = a \sin t$.

(iii) $y = \tan^{-1}\dfrac{2t}{1-t^2}, x = \sin^{-1}\dfrac{2t}{1+t^2}$.

(iv) $x = a\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}, y = at\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}\cdot$

Sol. **(i)** We have $x = a(t - \sin t), y = a(1 - \cos t)$.

$$\therefore \quad \frac{dx}{dt} = a(1 - \cos t) \text{ and } \frac{dy}{dt} = a \sin t.$$

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a(1-\cos t)} = \dfrac{2 \sin t/2 \cos t/2}{2 \sin^2 t/2} = \cot t/2.$

(ii) Here $\quad x = a(\cos t + \log \tan t/2), y = a \sin t$.

$$\therefore \quad \frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan t/2}(\sec^2 t/2)\cdot \tfrac{1}{2}\right]$$

$$= a\left(-\sin t + \frac{1}{2 \sin t/2 \cos t/2}\right)$$

$$= a\left[\frac{1}{\sin t} - \sin t\right] = a\left[\frac{1 - \sin^2 t}{\sin t}\right] = \frac{a \cos^2 t}{\sin t}.$$

Again $(dy/dt) = a \cos t$.

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = a \cos t \cdot \dfrac{\sin t}{a \cos^2 t} = \tan t.$

(iii) Let $t = \tan\theta$. Then

$$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}\tan 2\theta = 2\theta = 2\tan^{-1} t.$$

$\therefore \quad (dy/dt) = 2/(1+t^2).$

Again $\quad x = \sin^{-1}\dfrac{2t}{1+t^2} = \sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta}$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} t.$$

$\therefore \quad (dx/dt) = 2/(1+t^2).$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2/(1+t^2)}{2/(1+t^2)} = 1.$

(iv) Here $x = a\sqrt{\left\{\dfrac{t^2-1}{t^2+1}\right\}}$ and $y = at\sqrt{\left\{\dfrac{t^2-1}{t^2+1}\right\}}$.

We have, $\log y = \log a + \log t + \frac{1}{2}\log(t^2-1) - \frac{1}{2}\log(t^2+1).$

$$\therefore \quad \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{t}{t^2-1} - \frac{t}{t^2+1} = \frac{t^4+2t^2-1}{t(t^4-1)}$$

or $\quad \dfrac{dy}{dt} = a\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}\cdot\dfrac{t^4+2t^2-1}{(t^4-1)}.$

Again $\log x = \log a + \frac{1}{2}\log(t^2-1) - \frac{1}{2}\log(t^2+1).$

$$\therefore \quad \frac{1}{x}\frac{dx}{dt} = \frac{t}{t^2-1} - \frac{t}{t^2+1} = \frac{2t}{t^4-1}$$

or $\quad \dfrac{dx}{dt} = a\sqrt{\left(\dfrac{t^2-1}{t^2+1}\right)}\cdot\dfrac{2t}{t^4-1}.$

Now $\quad \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t^4+2t^2-1}{2t}.$

****Ex. 7 (a).** *If $\sin y = x\sin(a+y)$, prove that*

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

(Delhi 1982,75,72)

(b) *If $x^y = e^{x-y}$, prove that*

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

(Delhi 1981, 76: Meerut 81S)

(c) *If $x = e^{\tan^{-1}\{(y-x^2)/x^2\}}$, find (dy/dx), expressing it as a function of x only.*

(d) *If $x/(x-y) = \log\{a/(x-y)\}$, prove that* $(dy/dx) = 2-(x/y)$.

(Delhi 1983, 74)

Sol. (a) We have $x = \sin y/\sin(a+y)$.

Differentiating both sides w.r.t. x, we get

$$1 = \frac{\cos y\sin(a+y) - \sin y\cos(a+y)}{\sin^2(a+y)}\cdot\frac{dy}{dx}$$

or $\quad 1 = \dfrac{\sin\{(a+y)-y\}}{\sin^2(a+y)}\dfrac{dy}{dx}\cdot$ Hence $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

(b) We have, $x^y = e^{x-y}$.

$\therefore \quad y\log x = (x-y)\log e = x - y$ or $y + y\log x = x$

or $\quad y = x/(1+\log x)$. Hence $\dfrac{dy}{dx} = \dfrac{1\,(1+\log x) - x\cdot(1/x)}{(1+\log x)^2}$

$= \log x/(1+\log x)^2$.

(c) We have $x = e^{\tan^{-1}\{(y-x^2)/x^2\}}$.

Therefore $\log x = \tan^{-1}\dfrac{y-x^2}{x^2}\log e$

or $\quad \dfrac{y-x^2}{x^2} = \tan\log x$ or $y = x^2 + x^2\tan\log x$.

$\therefore \quad \dfrac{dy}{dx} = 2x + x^2(\sec^2\log x)\cdot\dfrac{1}{x} + 2x\cdot\tan\log x$

$= x\,[2 + \sec^2\log x + 2\tan\log x]$.

(d) We have $x/(x-y) = \log\{a/(x-y)\}$.

Differentiating w.r.t. x, we have

$$\frac{d}{dx}\left(\frac{x}{x-y}\right) = \frac{d}{dx}\left(\log\frac{a}{x-y}\right) = \frac{d}{dx}[\log a - \log(x-y)]$$

or $\quad \dfrac{1\cdot(x-y) - \{1-(dy/dx)\}\cdot x}{(x-y)^2} = \left[0 - \dfrac{1}{(x-y)}\left(1 - \dfrac{dy}{dx}\right)\right]$

or $\quad (x-y) - x + x\,(dy/dx) = -(x-y) + (x-y)(dy/dx)$

or $\quad y\,(dy/dx) = 2y - x$ *i.e.*, $dy/dx = 2 - (x/y)$. **Proved.**

Ex. 8. *Find (dy/dx) if $y = \tan^{-1}\{(ax-b)/(bx+a)\}$.* **(Delhi 1980)**

Sol. We have $y = \tan^{-1}\left(\dfrac{ax-b}{bx+a}\right)$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{1+(ax-b)^2/(bx+a)^2}\cdot\dfrac{d}{dx}\left(\dfrac{ax-b}{bx+a}\right)$

or $\quad \dfrac{dy}{dx} = \dfrac{(bx+a)^2}{(bx+a)^2+(ax-b)^2}\cdot\dfrac{a\,(bx+a) - b\,(ax-b)}{(bx+a)^2}$

$$= \frac{a^2+b^2}{(a^2+b^2)x^2 + (a^2+b^2)} = \frac{1}{1+x^2}.$$

2

Successive Differentiation

§ 1. Definition.

If $y = f(x)$ be a differentiable function of x, then dy/dx is called the *first differential coefficient of y w.r.t. x.* If this derivative is again a differentiable function, then its derivative *i.e.* $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the *second differential coefficient of y w.r.t. x* and is denoted by d^2y/dx^2. Similarly, the differential coefficient of $\frac{d^2y}{dx^2}$ *i.e.*, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called the *third diff. coeff. of y w.r.t. x* and is denoted by d^3y/dx^3. In general, the *nth differential coefficient* of y w.r.t. x is denoted by d^ny/dx^n and is given by $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$. We usually represent the diff. operator d/dx by the symbol D. Then we can write $\frac{d^ny}{dx^n} = \left(\frac{d}{dx}\right)^n y = D^n y$. If $y = f(x)$, the nth diff. coeff. of y w.r.t. x is represented by any of the following ways :

$$\frac{d^ny}{dx^n}, D^n y, y_n, f^{(n)}(x), f^n(x), D^n f(x), f_n(x) \text{ etc.}$$

If $y = f(x)$, then the nth diff. coeff. of y_r is the $(n + r)$ th diff. coeff. of y *i.e.* $D^n y_r = D^{n+r} y = y_{n+r}$. In particular,

$$D^n y_2 = D^{n+2} y = y_{n+2}.$$

The value of the n^{th} diff. coeff. of $y = f(x)$ at $x = a$ is denoted by $(y_n)_{x=a}$ or by $(y_n)_a$ or by $f^{(n)}(a)$.

§ 2. Standard results.

(i) **If $\mathbf{y = (ax + b)^m}$**, then $y_1 = ma(ax + b)^{m-1}$;

$y_2 = m(m-1)a^2(ax + b)^{m-2}$, $y_3 = m(m-1)(m-2)a^3(ax + b)^{m-3}$, and so on.

In general,

$$y_n = m(m-1)(m-2)\ldots\{m-(n-1)\}a^n(ax + b)^{m-n}.$$

Thus

$$\mathbf{D^n(ax + b)^m = m(m-1)(m-2)\ldots(m-n+1)a^n(ax + b)^{m-n}}.$$

If m is a positive interger, the above result can be written in a compact form by using the factorial notation. Thus in this case $D^n(ax + b)^m$

$$= \frac{m(m-1)\dots(m-n+1)(m-n)(m-n-1)\dots 1}{(m-n)(m-n-1)\dots 2.1} a^n (ax+b)^{m-n}$$

$$= \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}.$$

Note 1. If m is a + ive integer, then $D^m (ax+b)^m$
$= (m!/0!) \cdot a^m (ax+b)^0 = m!\, a^m$. In particular, $D^m x^m = m!$.

Note 2. If m is a + ive integer, the m^{th} diff. coeff. of $(ax+b)^m$ is constant. Therefore the $(m+1)^{th}$ and all the higher differential coefficients of $(ax+b)^m$ are zero.

Note 3. If m is a negative integer, say $m = -p$, where p is a positive integer, then

$$D^n (ax+b)^{-p} = (-p)(-p-1)(-p-2)\dots\dots \{-p-(n-1)\}\, a^n (ax+b)^{-p-n}$$
$$= (-1)^n p(p+1)\dots(p+n-1)\, a^n (ax+b)^{-p-n}$$
$$= (-1)^n \frac{(p+n-1)!}{(p-1)!} a^n (ax+b)^{-p-n}.$$

(ii) If $\mathbf{y = (ax+b)^{-1}}$, then $y_1 = (-1)\, a\, (ax+b)^{-2}$, $y_2 = (-1)(-2)\, a^2 (ax+b)^{-3}$, $y_3 = (-1)(-2)(-3)\, a^3 (ax+b)^{-4}$, and so on.

In general, $y_n = (-1)(-2)(-3)\dots(-n)\, a^n (ax+b)^{-(n+1)}$.

Thus $\mathbf{D^n (ax+b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}}$.

(iii) If $\mathbf{y = e^{ax+b}}$, then $y_1 = a\, e^{ax+b}, y_2 = a^2 e^{ax+b}, y_3 = a^3 e^{ax+b}$, and so on.

Thus $\mathbf{D^n e^{ax+b} = a^n e^{ax+b}}$. In particular, $\mathbf{D^n e^{ax} = a^n e^{ax}}$.

(Meerut 1992)

(iv) If $\mathbf{y = a^x}$, then $y_1 = a^x \log a, y_2 = a^x (\log a)^2, y_3 = a^x (\log a)^3$, and so on.

Thus $\mathbf{D^n a^x = a^x (\log a)^n}$.

(v) If $\mathbf{y = \log (ax+b)}$, then $y_1 = a/(ax+b) = a\,(ax+b)^{-1}$, $y_2 = (-1)\, a^2 (ax+b)^{-2}, y_3 = (-1)(-2)\, a^3 (ax+b)^{-3}$, and so on.

In general, $y_n = (-1)(-2)\dots\{-(n-1)\}\, a^n (ax+b)^{-n}$.

Thus $\mathbf{D^n \log (ax+b) = \dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}}$. **(Meerut 1991)**

In particular, $\mathbf{D^n \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}}$.

(vi) If $\mathbf{y = \sin (ax+b)}$, then

$y_1 = a \cos (ax+b) = a \sin (ax+b+\frac{1}{2}\pi)$. Comparing y with y_1, we find that

$$y_2 = a^2 \sin (ax+b+\tfrac{1}{2}\pi+\tfrac{1}{2}\pi) = a^2 \sin (ax+b+\tfrac{2}{2}\pi),$$

$y_3 = a^3 \sin(ax + b + \frac{3}{2}\pi)$, and so on.

Thus $\mathbf{D^n \sin(ax + b) = a^n \sin(ax + b + \frac{1}{2} n\pi)}$. **(Meerut 1996)**

(vii) Similarly, $\mathbf{D^n \cos(ax + b) = a^n \cos(ax + b + \frac{1}{2} n\pi)}$.

(Meerut 1996 BP)

(viii) **If** $\mathbf{y = e^{ax} \sin(bx + c)}$, then

$y_1 = ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c)$

$= e^{ax}[a \sin(bx + c) + b \cos(bx + c)]$.

Putting $a = r\cos\phi$ and $b = r\sin\phi$, we get

$y_1 = re^{ax} \sin(bx + c + \phi)$, where $r^2 = a^2 + b^2$ and $\phi = \tan^{-1}(b/a)$.

Similarly, $y_2 = r^2 e^{ax} \sin(bx + c + 2\phi)$, and so on.

Thus $\mathbf{D^n \{e^{ax} \sin(bx + c)\} = r^n e^{ax} \sin(bx + c + n\phi)}$, **where**

$\mathbf{r = (a^2 + b^2)^{1/2}}$ **and** $\mathbf{\phi = \tan^{-1}(b/a)}$.

Similarly, $\mathbf{D^n \{e^{ax} \cos(bx + c)\} = r^n e^{ax} \cos(bx + c + n\phi)}$, **where**

$\mathbf{r = (a^2 + b^2)^{1/2}}$ **and** $\mathbf{\phi = \tan^{-1}(b/a)}$.

Solved Examples

Ex. 1. *Find the n^{th} differential coefficients of*

(i) *$\log[(ax + b)(cx + d)]$*, **(ii)** *$\sin ax \cos bx$*,

(iii) *$\sin^3 x$*, **(iv)** *$\cos^4 x$*,

(v) *$\cos x \cos 2x \cos 3x$*, **(vi)** *$e^{ax} \sin bx \cos cx$*

(Meerut 1988 S)

(vii) *$e^x \sin^2 x$*, **(viii)** *$e^{ax} \cos^2 x \sin x$*,

(ix) *$\cos^2 x \sin^3 x$*. **(Kashmir 1983)**

(x) *$\sin^2 x \sin 2x$*, **(xi)** *$\sin^5 x \cos^3 x$*,

(xii) *$\sin x \cos 3x$*, **(xiii)** *$e^{2x} \sin^3 x$*. **(Meerut 1988 S)**

Sol. (i) Let $y = \log[(ax + b)(cx + d)]$

$= \log(ax + b) + \log(cx + d)$.

We know that $D^n \log(ax + b) = (-1)^{n-1}(n-1)!\, a^n (ax + b)^{-n}$

[See result (v) of § 2]

$$\therefore \quad y_n = (-1)^{n-1}(n-1)!\, a^n (ax + b)^{-n} + (-1)^{n-1}(n-1)!\, c^n (cx + d)^{-n}$$

$$= (-1)^{n-1}(n-1)! \left[\frac{a^n}{(ax + b)^n} + \frac{c^n}{(cx + d)^n}\right].$$

(ii) Let $y = \sin ax \cos bx = \frac{1}{2}[2 \sin ax \cos bx]$

$= \frac{1}{2}[\sin(a + b)x + \sin(a - b)x]$.

We know that $D^n \sin(ax + b) = a^n \sin(ax + b + \frac{1}{2}n\pi)$.

$$\therefore y_n = \frac{1}{2}[(a + b)^n \sin\{(a + b)x + \tfrac{1}{2}n\pi\} + (a - b)^n \sin\{(a - b)x + \tfrac{1}{2}n\pi\}]$$

(iii) Let $y = \sin^3 x = \frac{1}{4}(3 \sin x - \sin 3x)$,

$[\because \sin 3x = 3 \sin x - 4 \sin^3 x]$

$= \frac{3}{4} \sin x - \frac{1}{4} \sin 3x.$

Now $D^n \sin (ax + b) = a^n \sin (ax + b + \frac{1}{2} n\pi)$.

$\therefore \quad y_n = \frac{3}{4} \sin \left(x + \frac{n\pi}{2}\right) - \frac{1}{4} \cdot 3^n \sin \left(3x + \frac{n\pi}{2}\right)$.

(iv) Let $y = \cos^4 x = (\cos^2 x)^2 = [\frac{1}{2}(1 + \cos 2x)]^2$

$= \frac{1}{4}(1 + 2 \cos 2x + \cos^2 2x) = \frac{1}{4}[1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x)]$

$= \frac{1}{4}(\frac{3}{2} + 2 \cos 2x + \frac{1}{2} \cos 4x) = \frac{3}{8} + \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x.$

Now $D^n \cos (ax + b) = a^n \cos (ax + b + \frac{1}{2} n\pi)$.

$\therefore \quad y_n = 0 + \frac{1}{2} \cdot 2^n \cos (2x + \frac{1}{2} n\pi) + \frac{1}{8} \cdot 4^n \cos (4x + \frac{1}{2} n\pi)$

$= 2^{n-1} \cos (2x + \frac{1}{2} n\pi) + 2^{2n-3} \cos (4x + \frac{1}{2} n\pi).$

(v) Let $y = \cos x \cos 2x \cos 3x = \frac{1}{2} \cos x (2 \cos 2x \cos 3x)$

$= \frac{1}{2} \cos x (\cos 5x + \cos x) = \frac{1}{4}(2 \cos x \cos 5x + 2 \cos^2 x)$

$= \frac{1}{4}(\cos 6x + \cos 4x + \cos 2x + 1).$

$\therefore \quad y_n = \frac{1}{4}\{6^n \cos (6x + \frac{1}{2} n\pi) + 4^n \cos (4x + \frac{1}{2} n\pi)$

$+ 2^n \cos (2x + \frac{1}{2} n\pi)\}.$

(vi) Let $y = e^{ax} \sin bx \cos cx = \frac{1}{2} e^{ax} (2 \sin bx \cos cx)$

$= \frac{1}{2} e^{ax} [\sin (bx + cx) + \sin (bx - cx)]$

$= \frac{1}{2} [e^{ax} \sin (b + c) x + e^{ax} \sin (b - c) x].$

Now $D^n \{e^{ax} \sin (bx + c)\}$

$= (a^2 + b^2)^{n/2} e^{ax} \sin \{bx + c + n \tan^{-1} (b/a)\}.$

$\therefore y_n = \frac{1}{2} [\{a^2 + (b + c)^2\}^{n/2} e^{ax} \sin \{(b + c) x + n \tan^{-1} (b + c)/a\}$

$+ \{a^2 + (b - c)^2\}^{n/2} e^{ax} \sin \{(b - c) x + n \tan^{-1} (b - c)/a\}].$

(vii) Let $y = e^x \sin^2 x = \frac{1}{2} e^x (2 \sin^2 x) = \frac{1}{2} e^x (1 - \cos 2x)$

$= \frac{1}{2} [e^x - e^x \cos 2x].$

Now $\quad D^n e^{ax} = a^n e^{ax}.$

Also $D^n e^{ax} \cos (bx + c) = r^n e^{ax} \cos (bx + c + n\phi)$, where

$r = (a^2 + b^2)^{1/2}$ and $\phi = \tan^{-1} (b/a)$.

$\therefore \quad y_n = \frac{1}{2} [e^x - r^n e^x \cos (2x + n\phi)],$

where $r = \sqrt{(1^2 + 2^2)} = \sqrt{5}$ and $\phi = \tan^{-1} \frac{2}{1} = \tan^{-1} 2.$

(viii) Let $y = e^{ax} \cos^2 x \sin x = \frac{1}{2} e^{ax} (1 + \cos 2x) \sin x$

$= \frac{1}{2} e^{ax} \sin x + \frac{1}{2} e^{ax} \cos 2x \sin x$

$= \frac{1}{2} e^{ax} \sin x + \frac{1}{4} e^{ax} (2 \cos 2x \sin x)$

$= \frac{1}{2} e^{ax} \sin x + \frac{1}{4} e^{ax} (\sin 3x - \sin x) = \frac{1}{4} [e^{ax} \sin x + e^{ax} \sin 3x].$

Now using the formula for $D^n\{e^{ax}\sin(bx+c)\}$, we get

$$y_n = \tfrac{1}{4}[(a^2+1)^{n/2} e^{ax} \sin\{x + n\tan^{-1}(1/a)\}$$
$$+ (a^2+9)^{n/2} e^{ax} \sin\{3x + n\tan^{-1}(3/a)\}].$$

(ix) Let $y = \cos^2 x \sin^3 x = \tfrac{1}{2}(1+\cos 2x)\tfrac{1}{4}(3\sin x - \sin 3x)$

$$= \tfrac{1}{8}(3\sin x - \sin 3x + 3\sin x\cos 2x - \sin 3x\cos 2x)$$
$$= \tfrac{1}{8}[3\sin x - \sin 3x + \tfrac{3}{2}(2\sin x\cos 2x) - \tfrac{1}{2}(2\sin 3x\cos 2x)]$$
$$= \tfrac{1}{8}[3\sin x - \sin 3x + \tfrac{3}{2}(\sin 3x - \sin x) - \tfrac{1}{2}(\sin 5x + \sin x)]$$
$$= \tfrac{1}{16}(6\sin x - 2\sin 3x + 3\sin 3x - 3\sin x - \sin 5x - \sin x)$$
$$= \tfrac{1}{16}(2\sin x + \sin 3x - \sin 5x).$$

Now using the standard formula $D^n \sin(ax+b)$, we get

$$y_n = \tfrac{1}{16}[2\sin(x + \tfrac{1}{2}n\pi) + 3^n\sin(3x + \tfrac{1}{2}n\pi) - 5^n\sin(5x + \tfrac{1}{2}n\pi)].$$

(x) Let $y = \sin^2 x \sin 2x = \tfrac{1}{2}(1-\cos 2x)\sin 2x$

$$= \tfrac{1}{2}(\sin 2x - \sin 2x\cos 2x)$$
$$= \tfrac{1}{2}(\sin 2x - \tfrac{1}{2}\cdot 2\sin 2x\cos 2x) = \tfrac{1}{2}(\sin 2x - \tfrac{1}{2}\sin 4x)$$
$$= \tfrac{1}{4}(2\sin 2x - \sin 4x).$$

Then $y_n = \tfrac{1}{4}[2\cdot 2^n\sin(2x + \tfrac{1}{2}n\pi) - 4^n\sin(4x + \tfrac{1}{2}n\pi)]$

$$= 2^{n-1}\sin(2x + \tfrac{1}{2}n\pi) - 4^{n-1}\sin(4x + \tfrac{1}{2}n\pi).$$

(xi) Let $z = \cos x + i\sin x$.

Then $z^{-1} = (\cos x + i\sin x)^{-1} = \cos x - i\sin x$.

Therefore $z + z^{-1} = 2\cos x$ and $z - z^{-1} = 2i\sin x$.

Also by De-Moivre's theorem, $z^m = (\cos x + i\sin x)^m$

$= \cos mx + i\sin mx$ and $z^{-m} = \cos mx - i\sin mx$.

$\therefore\quad z^m + z^{-m} = 2\cos mx$ and $z^m - z^{-m} = 2i\sin mx$.

Now $(2i\sin x)^5(2\cos x)^3 = (z - z^{-1})^5.(z + z^{-1})^3$

$= (z^8 - z^{-8}) - 2(z^6 - z^{-6}) - 2(z^4 - z^{-4}) + 6(z^2 - z^{-2})$

$= 2i\sin 8x - 2(2i\sin 6x) - 2(2i\sin 4x) + 6(2i\sin 2x)$.

$\therefore\quad \sin^5 x\cos^3 x = 2^{-7}[\sin 8x - 2\sin 6x - 2\sin 4x + 6\sin 2x]$.

Now let $y = \sin^5 x\cos^3 x$. Then using the standard result for $D^n\sin(ax+b)$, we get

$$y_n = 2^{-7}[8^n\sin(8x + \tfrac{1}{2}n\pi) - 2.6^n\sin(6x + \tfrac{1}{2}n\pi)$$
$$- 2.4^n\sin(4x + \tfrac{1}{2}n\pi) + 6.2^n\sin(2x + \tfrac{1}{2}n\pi)].$$

(xii) Let $y = \sin x\cos 3x = \tfrac{1}{2}(\sin 4x - \sin 2x)$.

Then $y_n = \tfrac{1}{2}\{4^n\sin(4x + \tfrac{1}{2}n\pi) - 2^n\sin(2x + \tfrac{1}{2}n\pi)\}$.

(xiii) Let $y = e^{2x}\sin^3 x$.

We know that $\sin 3x = 3\sin x - 4\sin^3 x$.

$\therefore\quad 4\sin^3 x = 3\sin x - \sin 3x$

or $\sin^3 x = (1/4)(3 \sin x - \sin 3x)$.

$\therefore \quad y = \frac{1}{4} e^{2x} [3 \sin x - \sin 3x]$

$= \frac{3}{4} e^{2x} \sin x - \frac{1}{4} e^{2x} \sin 3x.$

$\therefore \quad y_n = \frac{3}{4} [(2^2 + 1^2)^{1/2}]^n e^{2x} \sin [x + n \tan^{-1} (1/2)]$

$- \frac{1}{4} [(2^2 + 3^2)^{1/2}]^n e^{2x} \sin [2x + n \tan^{-1} (3/2)].$

[Use of partial fractions]

Ex. 2. *Find the nth derivatives of*

(i) $\dfrac{x}{1 + 3x + 2x^2}$ **(ii)** $\dfrac{1}{1 - 5x + 6x^2}$

(Kanpur 1980; Meerut 86)

(iii) $\dfrac{17x^2 + 26x - 42}{6x^3 - 25x^2 - 29x + 20}$ **(iv)** $\dfrac{x^2}{(x - a)(x - b)}$

(v) $\dfrac{1}{(x - 1)^3 (x - 2)}$ **(vi)** $\dfrac{x^4}{(x - 1)(x - 2)}$

(Gorkhpur 1989; Meerut 97)

(vii) $\dfrac{1}{x^2 - a^2}$ **(viii)** $\dfrac{1}{a^2 - x^2}$ **(Agra 1983)**

(ix) $\dfrac{x^2}{(x + 2)(2x + 3)}$ **(Meerut 1988)**

Sol. (i) Let $y = \dfrac{x}{1 + 3x + 2x^2} = \dfrac{x}{(1 + x)(1 + 2x)}$.

Resolving into partial fractions, we get

$$y = \frac{-1}{(1 - 2)(1 + x)} + \frac{-\frac{1}{2}}{(1 - \frac{1}{2})(1 + 2x)} = \frac{1}{x + 1} - \frac{1}{2x + 1}.$$

Now using the standard result for $D^n (ax + b)^{-1}$, we get

$$y_n = \frac{(-1)^n n!}{(x + 1)^{n+1}} - \frac{(-1)^n n! \, 2^n}{(2x + 1)^{n+1}}$$

$$= (-1)^n n! \left[\frac{1}{(x + 1)^{n+1}} - \frac{2^n}{(2x + 1)^{n+1}} \right].$$

(ii) Let $y = \dfrac{1}{6x^2 - 5x + 1} = \dfrac{1}{(3x - 1)(2x - 1)}$

$= \dfrac{2}{2x - 1} - \dfrac{3}{3x - 1}$, (on resolving into partial fractions)

$= 2(2x - 1)^{-1} - 3(3x - 1)^{-1}$.

Now $D^n (ax + b)^{-1} = (-1)^{-n} n! \, a^n (ax + b)^{-n-1}$.

$\therefore \quad y_n = 2(-1)^n n! \, 2^n (2x - 1)^{-n-1}$

$- 3(-1)^n n! \, 3^n (3x - 1)^{-n-1}$

$= (-1)^n n! \, [2^{n+1} (2x - 1)^{-n-1} - 3^{n+1} (3x - 1)^{-n-1}].$

(iii) Let $y =$ the given fraction. Resolving into partial fractions, we get

$$y = \frac{1}{2x-1} - \frac{2}{3x+4} + \frac{3}{x-5} \cdot$$

$$\therefore \quad y_n = \frac{(-1)^n n!\, 2^n}{(2x-1)^{n+1}} - \frac{2(-1)^n n!\, 3^n}{(3x+4)^{n+1}} + \frac{3(-1)^n . n!}{(x-5)^{n+1}}$$

$$= (-1)^n . n! \left\{ \frac{2^n}{(2x-1)^{n+1}} - \frac{2 \cdot 3^n}{(3x+4)^{n+1}} + \frac{3}{(x-5)^{n+1}} \right\} \cdot$$

(iv) Let $y = \dfrac{x^2}{(x-a)(x-b)} \cdot$ Since the given fraction is not a proper one, therefore we should first divide the numerator by the denominator before resolving it into partial fractions. Here we observe orally that the quotient will be 1. So let

$$\frac{x^2}{(x-a)(x-b)} \equiv 1 + \frac{A}{x-a} + \frac{B}{x-b} \cdot$$

Then $\quad A = \dfrac{a^2}{a-b}$ and $B = \dfrac{b^2}{b-a} \cdot$

Hence $y = 1 + \dfrac{a^2}{(a-b)(x-a)} + \dfrac{b^2}{(b-a)(x-b)}$

$$= 1 + \frac{a^2}{(a-b)}(x-a)^{-1} - \frac{b^2}{(a-b)}(x-b)^{-1}.$$

Now differentiating both sides n times, we get

$$y_n = \frac{a^2}{(a-b)}(-1)^n n!\,(x-a)^{-n-1} - \frac{b^2}{(a-b)}(-1)^n n!\,(x-b)^{-n-1}$$

$$= \frac{(-1)^n n!}{(a-b)} \left[\frac{a^2}{(x-a)^{n+1}} - \frac{b^2}{(x-b)^{n+1}} \right] \cdot$$

(v) Let $y = \dfrac{1}{(x-1)^3 (x-2)} \cdot$ To resolve y into partial fractions we put $x - 1 = z$. Then $y = \dfrac{1}{z^3} \cdot \dfrac{1}{z-1} = \dfrac{1}{z^3} \cdot \dfrac{1}{-1+z}$, arranging the numerator and the denominator both in ascending powers of z.

Now dividing the numerator 1 by the denominator $-1+z$ till z^3 is a common factor in the remainder, we get

$$y = \frac{1}{z^3}\left[-1 - z - z^2 + \frac{z^3}{-1+z}\right] = -\frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} + \frac{1}{-1+z}$$

$$= -\frac{1}{(x-1)^3} - \frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2}$$

$$= -(x-1)^{-3} - (x-1)^{-2} - (x-1)^{-1} + (x-2)^{-1}.$$

Now $D^n (ax+b)^{-p} = (-p)(-p-1)(-p-2) \ldots$

$$\ldots \{-p-(n-1)\}\, a^n (ax+b)^{-p-n}$$

$$= \frac{(-1)^n (p+n-1)!}{(p-1)!} a^n (ax+b)^{-p-n}.$$ [See Note 3, page 20]

$$\therefore \quad y_n = -\frac{(-1)^n (3+n-1)!}{(3-1)!}(x-1)^{-3-n}$$

$$-\frac{(-1)^n (2+n-1)!}{(2-1)!} \cdot (x-1)^{-2-n}$$

$$-(-1)^n n!\,(x-1)^{-n-1} + (-1)^n n!\,(x-2)^{-n-1}$$

$$= (-1)^n \left[-\frac{(n+2)!}{2!}(x-1)^{-3-n} - \frac{(n+1)!}{1!}(x-1)^{-2-n} - n!\,(x-1)^{-n-1} + n!\,(x-2)^{-n-1}\right]$$

$$= (-1)^{n+1} n! \left[\frac{(n+2)(n+1)}{2(x-1)^{n+3}} + \frac{n+1}{(x-1)^{n+2}} + \frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right].$$

(vi) Let $y = \dfrac{x^4}{(x-1)(x-2)} = \left[x^2 + 3x + 7 + \dfrac{15x-14}{(x-1)(x-2)}\right]$, dividing N^r by the D^r

$= \left[x^2 + 3x + 7 + \dfrac{16}{x-2} - \dfrac{1}{x-1}\right]$, on resolving into partial fractions.

Then $y_n = 16(-1)^n \cdot n!\,(x-2)^{-1-n} - (-1)^n n!\,(x-1)^{-1-n}$, if $n > 2$

$$= (-1)^n \cdot n!\,[16(x-2)^{-n-1} - (x-1)^{-n-1}].$$

(vii) Let $y = \dfrac{1}{x^2-a^2} = \dfrac{1}{(x-a)(x+a)} = \dfrac{1}{2a}\left[\dfrac{1}{x-a} - \dfrac{1}{x+a}\right]$

$$= \frac{1}{2a}\left[(x-a)^{-1} - (x+a)^{-1}\right].$$

Then $y_n = \dfrac{1}{2a}(-1)^n n!\,\{(x-a)^{-n-1} - (x+a)^{-n-1}\}$.

(viii) Let $\quad y = \dfrac{1}{a^2-x^2} = \dfrac{1}{(a-x)(a+x)} = \dfrac{1}{2a}\left[\dfrac{1}{a-x} + \dfrac{1}{a+x}\right]$

$$= \frac{1}{2a}\left[\frac{1}{x+a} - \frac{1}{x-a}\right] = \frac{1}{2a}\left[(x+a)^{-1} - (x-a)^{-1}\right].$$

Then $\quad y_n = \dfrac{1}{2a}(-1)^n n!\,\{(x+a)^{-n-1} - (x-a)^{-n-1}\}$.

(ix) Let $y = x^2 / [(x+2)(2x+3)]$.

The given fraction is not a proper one. If we divide the numerator by the denominator, we observe orally that the quotient will be $1/2$. So let

$$\frac{x^2}{(x+2)(2x+3)} \equiv \tfrac{1}{2} + \frac{A}{x+2} + \frac{B}{2x+3}.$$

Then $A = -4$ and $B = 9/2$.

Hence $y = \tfrac{1}{2} - \dfrac{4}{x+2} + \dfrac{9}{2(2x+3)}$

$$= \tfrac{1}{2} - 4(x+2)^{-1} + \tfrac{9}{2}(2x+3)^{-1}.$$

$$\therefore \quad y_n = -4(-1)^n n!\,(x+2)^{-n-1} + \tfrac{9}{2}\cdot(-1)^n n!\,.2^n (2x+3)^{-n-1}$$

$$= (-1)^n n! \left[\frac{9\cdot 2^{n-1}}{(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}}\right].$$

Ex. 3. *Prove that the value of the nth differential coefficient of $x^3/(x^2-1)$ for $x = 0$, is zero if n is even, and is $-n!$ if n is odd and greater than* 1.

Sol. Let $y = \dfrac{x^3}{x^2-1} = x + \dfrac{x}{x^2-1} = x + \dfrac{x}{(x-1)(x+1)}$

$$= x + \frac{1}{(1+1)(x-1)} + \frac{-1}{(-1-1)(x+1)}$$

$$= x + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}.$$

$\therefore$ When $n > 1$, we have

$$y_n = \frac{(-1)^n n!}{2}\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right].$$

Putting $x = 0$ in the expression for y_n, we get

$$(y_n)_0 = \frac{(-1)^n n!}{2}\left[\frac{1}{(-1)^{n+1}} + \frac{1}{1}\right] = \frac{(-1)^n n!}{2}\left[\frac{1}{(-1)^{n+1}} + 1\right].$$

When n is even, $(y_n)_0 = \dfrac{n!}{2}\left[\dfrac{1}{(-1)} + 1\right] = \dfrac{n!}{2}\cdot 0 = 0.$

When n is odd, $(y_n)_0 = -\dfrac{n!}{2}[1+1] = -n!.$

Ex. 4. *Find the nth differential coefficient of* $1/(x^2+a^2)$.

(Lucknow 1983; Gorakhpur 82; Magadh 84)

Sol. We have $y = \dfrac{1}{x^2+a^2} = \dfrac{1}{(x+ia)(x-ia)}$

$$= \frac{1}{2ia}\left\{\frac{1}{x-ia} - \frac{1}{x+ia}\right\}.$$

$$\therefore \quad y_n = \frac{1}{2ia}\cdot(-1)^n n!\left\{\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right\}.$$

Let $x = r\cos\phi$ and $a = r\sin\phi$, so that $\phi = \tan^{-1}(a/x)$.
Then

$$y_n = \frac{(-1)^n n!}{2ia} \left\{ \frac{1}{(r\cos\phi - ir\sin\phi)^{n+1}} - \frac{1}{(r\cos\phi + ir\sin\phi)^{n+1}} \right\}$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}} \{(\cos\phi - i\sin\phi)^{-(n+1)} - (\cos\phi + i\sin\phi)^{-(n+1)}\}$$

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}} [\{\cos(n+1)\phi + i\sin(n+1)\phi\}$$

$$- \{\cos(n+1)\phi - i\sin(n+1)\phi\}],$$

(by De-Moivre's theorem)

$$= \frac{(-1)^n n!}{2ia\, r^{n+1}} 2i\sin(n+1)\phi = \frac{(-1)^n n!}{a \cdot a^{n+1}} \sin(n+1)\phi \sin^{n+1}\phi,$$

since $r = a/\sin\phi$

$$= (-1)^n n!\, a^{-(n+2)} \sin(n+1)\phi \sin^{n+1}\phi,$$

where $\phi = \tan^{-1}(a/x)$.

Ex. 5. *Find the nth differential coefficient of $\tan^{-1}(x/a)$.*

(Meerut 1998, 96P, 83, 82, 81; Jhansi 1988; Avadh 1988)

Sol. Let $y = \tan^{-1}(x/a)$. Then $y_1 = \dfrac{a}{x^2 + a^2} = \dfrac{a}{(x + ia)(x - ia)}$

$$= \frac{a}{(-ia - ia)(x + ia)} + \frac{a}{(ia + ia)(x - ia)}$$

$$= -\frac{1}{2i(x + ia)} + \frac{1}{2i(x - ia)} = \frac{1}{2i}\left[(x - ia)^{-1} - (x + ia)^{-1}\right].$$

Now differentiating both sides $(n - 1)$ times w.r.t. 'x', we get

$$y_n = \frac{1}{2i}\left[(-1)^{n-1}(n-1)!\,(x - ia)^{-n}\right.$$

$$\left. - (-1)^{n-1}(n-1)!\,(x + ia)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[(x - ia)^{-n} - (x + ia)^{-n}\right].$$

Put $x = r\cos\phi$ and $a = r\sin\phi$. Then

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i} [(r\cos\phi - ir\sin\phi)^{-n}$$

$$- (r\cos\phi + ir\sin\phi)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}\left[(\cos\phi - i\sin\phi)^{-n}\right.$$

$$\left. - (\cos\phi + i\sin\phi)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n} [(\cos n\phi + i\sin n\phi)$$

$$- (\cos n\phi - i\sin n\phi)]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} r^{-n}\, 2i\sin n\phi = (-1)^{n-1}(n-1)!\, r^{-n}\sin n\phi$$

$= (-1)^{n-1}(n-1)!\,(a/\sin\phi)^{-n}\sin n\phi$, since $r = a/\sin\phi$

$= (-1)^{n-1}(n-1)!\, a^{-n}\sin^n\phi \sin n\phi$, where $\phi = \tan^{-1}(a/x)$.

Ex. 6. *Find the n^{th} derivative of $\tan^{-1}\{2x/(1-x^2)\}$.*

(Lucknow 1982)

Sol. Let $y = \tan^{-1}\{2x/(1-x^2)\} = 2\tan^{-1}x$.

Then $$y_1 = \frac{2}{1+x^2} = \frac{2}{(x-i)(x+i)} = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right],$$ on resolving into partial fractions.

Now differentiating both sides $(n-1)$ times w.r.t. 'x', we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{i}\left[(x-i)^{-n} - (x+i)^{-n}\right].$$

Putting $x = r\cos\phi$ and $1 = r\sin\phi$, we have

$$y_n = \frac{(-1)^{n-1}(n-1)!}{i}[r^{-n}(\cos\phi - i\sin\phi)^{-n} - r^{-n}(\cos\phi + i\sin\phi)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{i}r^{-n}[(\cos n\phi + i\sin n\phi) - (\cos n\phi - i\sin n\phi)]$$

$= 2(-1)^{n-1}(n-1)!\,r^{-n}\sin n\phi$

$= 2(-1)^{n-1}(n-1)!\,(1/\sin\phi)^{-n}\sin n\phi$, since $r = 1/\sin\phi$

$= 2(-1)^{n-1}(n-1)!\,\sin^n\phi\sin n\phi$, where $\phi = \tan^{-1}(1/x)$.

Ex. 6 (a). *If $y = \sin^{-1}\{2x/(1+x^2)\}$, prove that*

$y_n = 2(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$, *where* $\theta = \cot^{-1}x$.

(Rohilkhand 1990)

Sol. Put $x = \tan\phi$. Then

$$y = \sin^{-1}\left(\frac{2\tan\phi}{1+\tan^2\phi}\right) = \sin^{-1}\sin 2\phi = 2\phi$$

$= 2\tan^{-1}x$. Now proceed as in Ex. 6.

Ex. 6 (b). *If* $y = \tan^{-1}\left\{\frac{\sqrt{(1+x^2)}-1}{x}\right\}$, *show that*

$$y_n = \tfrac{1}{2}(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta,$$

where $\theta = \cot^{-1}x$.

(Meerut 1982, 85)

Sol. Let $y = \tan^{-1}\left\{\frac{\sqrt{(1+x^2)}-1}{x}\right\}$.

Put $x = \tan\phi$. Then

$$y = \tan^{-1}\left[\frac{\sqrt{(1+\tan^2\phi)}-1}{\tan\phi}\right] = \tan^{-1}\frac{\sec\phi - 1}{\tan\phi}$$

$$= \tan^{-1}\frac{1-\cos\phi}{\sin\phi} = \tan^{-1}\frac{2\sin^2(\phi/2)}{2\sin(\phi/2)\cos(\phi/2)}$$

$= \tan^{-1}\tan(\phi/2) = \phi/2 = \tfrac{1}{2}\tan^{-1}x$.

$\therefore\quad y_1 = 1/\{2(1+x^2)\}$.

Now proceeding as in Ex. 6, we get

$y_n = \tfrac{1}{2}(-1)^{n-1}(n-1)!\,\sin^n\theta\sin n\theta$, where $\theta = \cot^{-1}x$.

Ex. 7. *Find the n^{th} derivative of* $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$

Sol. Let

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}\left\{\frac{1+x}{1-1\,.\,x}\right\} = \tan^{-1}1 + \tan^{-1}x.$$

Then $y_1 = 1/(1+x^2)$. Now proceeding as in Ex. 6, we have

$$y_n = (-1)^{n-1}(n-1)!\sin^n\phi\sin n\phi, \text{ where } \phi = \tan^{-1}(1/x).$$

Ex. 7 (a). *If* $y = x(a^2+x^2)^{-1}$, *prove that*

$y_n = (-1)^n n!\, a^{-n-1}\sin^{n+1}\phi\cos(n+1)\phi$, *where* $\phi = \tan^{-1}(a/x)$

(Meerut 1982S, 83S; Rohilkhand 88; Gorakhpur 82)

Sol. We have $y = \dfrac{x}{a^2+x^2} = \dfrac{x}{(x-ia)(x+ia)}$

$= \frac{1}{2}\left[\dfrac{1}{(x-ia)} + \dfrac{1}{(x+ia)}\right]$, on resolving into partial fractions.

Differentiating both sides n times w.r.t. 'x', we get

$$y_n = \tfrac{1}{2}[(-1)^n n!(x-ia)^{-n-1} + (-1)^n n!(x+ia)^{-n-1}]$$
$$= \tfrac{1}{2}(-1)^n n![(x-ia)^{-n-1} + (x+ia)^{-n-1}].$$

Putting $x = r\cos\phi$ and $a = r\sin\phi$, we get

$$y_n = \tfrac{1}{2}(-1)^n n![r^{-n-1}(\cos\phi - i\sin\phi)^{-n-1} + r^{-n-1}(\cos\phi + i\sin\phi)^{-n-1}]$$
$$= \tfrac{1}{2}(-1)^n n!\,r^{-n-1}[\{\cos(n+1)\phi + i\sin(n+1)\phi\} + \{\cos(n+1)\phi - i\sin(n+1)\phi\}]$$
$$= \tfrac{1}{2}(-1)^n n!\,r^{-n-1}\,2\cos(n+1)\phi$$
$$= (-1)^n n!\,(a/\sin\phi)^{-n-1}\cos(n+1)\phi, \text{ since } r = a/\sin\phi$$
$$= (-1)^n.n!\,a^{-n-1}\sin^{n+1}\phi\cos(n+1)\phi,$$

where $\phi = \tan^{-1}(a/x)$.

Ex. 8. (a) *If* $y = \sin^{-1}x$, *prove that*

$(1-x^2)(d^2y/dx^2) = x\cdot(dy/dx)$.

(b) *Find* d^2y/dx^2 *for the cycloid whose equation is*

$x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$.

(c) *If* $x = a(t - \sin t)$ *and* $y = a(1+\cos t)$, *prove that*

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\operatorname{cosec}^4\left(\frac{t}{2}\right).$$

(Meerut 1985)

(d) *If* $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$, *find* d^2y/dx^2.

(Meerut 1991P)

(e) *If* $x = 2\cos t - \cos 2t, y = 2\sin t - \sin 2t$, *find the value of* (d^2y/dx^2) *at* $t = \pi/2$.

(f) *If* $y = \sin(\sin x)$, *prove that*

$(d^2y/dx^2) + (dy/dx)\tan x + y\cos^2 x = 0$. **(Kashmir 1983)**

Sol. (a) We have $y = \sin^{-1}x$; $\therefore\ dy/dx = 1/\sqrt{(1-x^2)}$

or $(1-x^2)(dy/dx)^2 = 1.$

Differentiating again with respect to x, we get

$$(1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 0$$

or $$(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx}\cdot \qquad \left[\because \frac{dy}{dx} \neq 0\right]\cdot$$

(b) We have $x = a(\theta - \sin\theta)$; $\therefore (dy/d\theta) = a(1-\cos\theta)$.

Also $y = a(1-\cos\theta)$; $\therefore (dy/d\theta) = a\sin\theta$.

Now $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = \cot\frac{\theta}{2}\cdot$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot\theta/2) = \left[\frac{d}{d\theta}(\cot\theta/2)\right]\frac{d\theta}{dx}$$

$$= -\tfrac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{a(1-\cos\theta)}$$

$$= -\tfrac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\cdot\frac{1}{2a\sin^2\theta/2} = -\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}\cdot$$

(c) Proceed as in (b).

(d) We have $x = a(\cos\theta + \theta\sin\theta)$.

$\therefore \quad dx/d\theta = a(-\sin\theta + \sin\theta + \theta\cos\theta) = a\theta\cos\theta.$

Also $\quad y = a(\sin\theta - \theta\cos\theta)$.

$\therefore \quad dy/d\theta = a(\cos\theta - \cos\theta + \theta\sin\theta) = a\theta\sin\theta.$

Now $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta) = \left(\frac{d}{d\theta}(\tan\theta)\right)\frac{d\theta}{dx}$$

$$= \sec^2\theta\cdot\frac{1}{a\theta\cos\theta} = \frac{1}{a}\frac{\sec^3\theta}{\theta}\cdot$$

(e) We have $x = 2\cos t - \cos 2t$; $\therefore \dfrac{dx}{dt} = -2\sin t + 2\sin 2t.$

Also $y = 2\sin t - \sin 2t$; $\quad\therefore \quad \dfrac{dy}{dt} = 2\cos t - 2\cos 2t.$

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t}$$

$$= \frac{2\sin 3t/2\cdot\sin t/2}{2\cos 3t/2\cdot\sin t/2} = \tan\frac{3t}{2}\cdot$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan 3t/2) = \left[\frac{d}{dt}(\tan 3t/2)\right]\frac{dt}{dx}$$

$$= \frac{3}{2}\sec^2\frac{3t}{2}\cdot\frac{1}{2\sin 2t - 2\sin t}\cdot$$

When $t = \dfrac{\pi}{2}$, $\dfrac{d^2y}{dx^2} = \dfrac{3}{2}\sec^2\dfrac{3\pi}{4}\cdot\dfrac{1}{2\sin\pi - 2\sin\pi/2}\cdot$

$\therefore \quad \left(\dfrac{d^2y}{dx^2}\right)_{\text{at } t = \pi/2} = \dfrac{3}{2} \cdot 2 \cdot \dfrac{1}{(0-2)} = -\dfrac{3}{2}.$

(f) We have $\quad y = \sin(\sin x)$. ...(1)

$\therefore \quad \dfrac{dy}{dx} = [\cos(\sin x)] \cos x.$

Differentiating again, we have

$$(d^2y/dx^2) = [-\sin(\sin x)] \cos^2 x - [\cos(\sin x)] \sin x$$
$$= -y \cos^2 x - [\cos(\sin x)] \cos x \cdot (\sin x / \cos x),$$

[$\because$ from (1), $y = \sin(\sin x)$]

$$= -y \cos^2 x - (dy/dx) \cdot \tan x.$$

[$\because$ from (2), $dy/dx = \{\cos(\sin x)\} \cos x$]

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} \tan x + y \cos^2 x = 0.$$

Ex. 9. (i) *If $y = A \sin mx + B \cos mx$, prove that*
$y_2 + m^2 y = 0.$ **(Indore 1984)**

(ii) *If $y = e^{ax} \sin bx$, prove that*
$y_2 - 2ay_1 + (a^2 + b^2) y = 0.$ **(Gorakhpur 1981)**

(iii) *If $y = \sin nx + \cos nx$, show that*
$y_r = n^r \{1 + (-1)^r \sin 2nx\}^{1/2}.$

(Meerut 1983, 83P, 95, 98; Lucknow 83; Gorakhpur 84)

Sol. (i) We have, $\quad y = A \sin mx + B \cos mx.$ ...(1)

Differentiating both sides w.r.t. 'x' we get

$$y_1 = Am \cos mx - Bm \sin mx.$$

Again $y_2 = -Am^2 \sin mx - Bm^2 \cos mx$

$$= -m^2 (A \sin mx + B \cos mx) = -m^2 y, \text{ from (1).}$$

$\therefore \quad y_2 + m^2 y = 0.$

(ii) Here $y = e^{ax} \sin bx.$ ...(1)

$\therefore \quad y_1 = e^{ax} b \cos bx + a \cdot e^{ax} \sin bx.$

Replacing $e^{ax} \sin bx$ by y, we get

$y_1 = be^{ax} \cos bx + ay.$ ...(2)

Differentiating both sides of (2) w.r.t. 'x', we get

$$y_2 = abe^{ax} \cos bx - b^2 e^{ax} \sin bx + ay_1$$

$= a(y_1 - ay) - b^2 y + ay_1,\quad$ [$\because$ from (2), $be^{ax} \cos bx = y_1 - ay$ and from (1), $e^{ax} \sin bx = y$]

$$= 2ay_1 - (a^2 + b^2) y.$$

$\therefore \quad y_2 - 2ay_1 + (a^2 + b^2) y = 0.$

(iii) Here $y = \sin nx + \cos nx.$

$\therefore \quad y_r = n^r \sin(nx + \tfrac{1}{2} r\pi) + n^r \cos(nx + \tfrac{1}{2} r\pi)$

$= n^r [\{\sin (nx + \frac{1}{2}r\pi) + \cos (nx + \frac{1}{2}r\pi)\}^2]^{1/2}$ **(Note)**

$= n^r [\sin^2 (nx + \frac{1}{2}r\pi) + \cos^2 (nx + \frac{1}{2}r\pi)$

$+ 2 \sin (nx + \frac{1}{2}r\pi) \cos (nx + \frac{1}{2}r\pi)]^{1/2}$

$= n^r [1 + \sin 2 (nx + \frac{1}{2}r\pi)]^{1/2} = n^r [1 + \sin (2nx + r\pi)]^{1/2}$

$= n^r [1 + (-1)^r \sin 2nx]$, since $\sin (r\pi + \theta) = (-1)^r \sin \theta$.

Ex. 10. *Prove that the value when $x = 0$ of $D^n (\tan^{-1} x)$ is $0, (n-1)!$ or $-(n-1)!$ according as n is of the form $2p$, $4p+1$ or $4p+3$ respectively.* **(Kanpur 1983)**

Sol. Let $y = \tan^{-1} x$. Then

$$y_1 = \frac{1}{1+x^2} = \frac{1}{(x-i)(x+i)} = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

or $$y_1 = (1/2i)[(x-i)^{-1} - (x+i)^{-1}].$$

Differentiating both sides $(n-1)$ times w.r.t. 'x', we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}[(x-i)^{-n} - (x+i)^{-n}].$$

Putting $x = 0$ in the expression for y_n, we get

$$(y_n)_{x=0} = \frac{(-1)^{n-1}(n-1)!}{2i}[(-i)^{-n} - i^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}[(-1)^{-n} i^{-n} - i^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i} i^{-n}[(-1)^{-n} - 1].$$

Now we shall consider the **three cases.**

(i) If n is of the form $2p$, we have

$$(y_n)_0 = \frac{(-1)^{2p-1}(n-1)!}{2i} i^{-2p}[(-1)^{-2p} - 1]$$

$$= -\frac{(n-1)!}{2i} i^{-2p}[1-1] = -\frac{(n-1)!}{2i} i^{-2p} . 0 = 0.$$

(ii) If n is of the form $4p+1$, we have

$$(y_n)_0 = \frac{(-1)^{4p}(n-1)!}{2i} i^{-(4p+1)}[(-1)^{-(4p+1)} - 1]$$

$$= \frac{(n-1)!}{2 . i^{4p+2}}[-1-1],$$

$$[\because (-1)^{-(4p+1)} = 1/(-1)^{4p+1} = 1/(-1) = -1]$$

$$= \frac{(n-1)!}{2 . i^{4p+2}}(-2) = -\frac{(n-1)!}{i^{4p} . i^2} = -\frac{(n-1)!}{(i^4)^p i^2} = -\frac{(n-1)!}{1^p (-1)},$$

$$[\because i^4 = 1 \text{ and } i^2 = -1]$$

$= (n-1)!$.

(iii) If n is of the form $4p + 3$, we have

$$(y_n)_0 = \frac{(-1)^{4p+2}(n-1)!\, i^{-(4p+3)}}{2i}[(-1)^{-(4p+3)} - 1]$$

$$= \frac{(n-1)!}{2 \cdot i^{4p+4}}[-1-1] = \frac{(n-1)!}{2 \cdot i^{4p+4}}(-2)$$

$$= -\frac{(n-1)!}{(i^4)^{p+1}} = -\frac{(n-1)!}{1^{p+1}} = -(n-1)!.$$

**Ex. 11. *If $p^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$, prove that*

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}.$$

(Meerut 1983, 83S, 82P; Delhi 81)

Sol. We have $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$. ...(1)

Differentiating both sides of (1) w.r.t. θ, we have

$$2p\,(dp/d\theta) = -2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta$$

or $$p\,(dp/d\theta) = (b^2 - a^2)\sin\theta\cos\theta. \quad ...(2)$$

Now differentiating both sides of (2) w.r.t. θ, we have

$$p\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = (b^2 - a^2)(\cos^2\theta - \sin^2\theta)$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - (a^2\cos^2\theta + b^2\sin^2\theta)$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - p^2. \quad \text{[from (1)]}$$

$$\therefore \quad p\frac{d^2p}{d\theta^2} + p^2 = (b^2\cos^2\theta + a^2\sin^2\theta) - \left(\frac{dp}{d\theta}\right)^2$$

$$= (b^2\cos^2\theta + a^2\sin^2\theta) - \frac{(b^2 - a^2)^2\sin^2\theta\cos^2\theta}{p^2},$$

substituting for $dp/d\theta$ from (2)

$$= \frac{1}{p^2}[p^2(b^2\cos^2\theta + a^2\sin^2\theta) - (b^2 - a^2)^2\sin^2\theta\cos^2\theta]$$

$$= \frac{1}{p^2}[(a^2\cos^2\theta + b^2\sin^2\theta)(b^2\cos^2\theta + a^2\sin^2\theta) - (b^2 - a^2)^2\sin^2\theta\cos^2\theta]$$

$$= \frac{1}{p^2}[a^2b^2\cos^4\theta + a^2b^2\sin^4\theta + 2a^2b^2\sin^2\theta\cos^2\theta]$$

$$= \frac{a^2b^2}{p^2}(\cos^2\theta + \sin^2\theta)^2 = \frac{a^2b^2}{p^2}.$$

Thus $$p\frac{d^2p}{d\theta^2} + p^2 = \frac{a^2b^2}{p^2}.$$

Dividing both sides by p, we have

$$\frac{d^2p}{d\theta^2} + p = \frac{a^2b^2}{p^3}.$$

Ex. 12. *Find the n^{th} derivative of $1/(1 + x + x^2 + x^3)$ and show that for $x = 0$ it is zero when n is of the form $4p + 2$ or $4p + 3$ and is n ! or $- n$! according as n is of the form $4p + 1$.*

Sol. Let $y = \dfrac{1}{1 + x + x^2 + x^3} = \dfrac{1}{(1 + x)(x^2 + 1)}$

$= \dfrac{1}{2(x+1)} - \dfrac{x-1}{2(x^2+1)}$,

(on resolving into partial fractions)

$= \dfrac{1}{2(x+1)} - \dfrac{x-1}{2(x+i)(x-i)}$

$= \dfrac{1}{2(x+1)} - \frac{1}{2}\left[\dfrac{-i-1}{(-i-i)(x+i)} + \dfrac{i-1}{(i+i)(x-i)}\right]$

$= \dfrac{1}{2(x+1)} - \dfrac{1+i}{4i(x+i)} + \dfrac{1-i}{4i(x-i)}$.

Differentiating n times w.r.t. 'x', we get

$$y_n = (-1)^n n!\left[\frac{1}{2(x+1)^{n+1}} - \frac{1+i}{4i(x+i)^{n+1}} + \frac{1-i}{4i(x-i)^{n+1}}\right]$$

Putting $x = 0$ in the expression for y_n, we get

$$(y_n)_0 = (-1)^n n!\left[\frac{1}{2} - \frac{1+i}{4 \,.\, i^{n+2}} + \frac{1-i}{4(-1)^{n+1} i^{n+2}}\right].$$

Now we shall discuss the **different cases.**

(i) When $n = 4p + 2$, we have $(-1)^n = (-1)^{4p+2} = 1$,
$(-1)^{n+1} = (-1)^{4p+3} = -1$,
$i^{n+2} = i^{4p+4} = i^{4(p+1)} = 1^{p+1} = 1$.

$\therefore \quad (y_n)_0 = n!\left[\frac{1}{2} - \dfrac{1+i}{4} - \dfrac{1-i}{4}\right] = (n!) \,.\, 0 = 0.$

(ii) When $n = 4p + 3$, we have $(-1)^n = -1$, $(-1)^{n+1} = 1$,
$i^{n+2} = i^{4p+5} = i^{4p+4} i = i$.

$\therefore \quad (y_n)_0 = -n!\left[\frac{1}{2} - \dfrac{1+i}{4i} + \dfrac{1-i}{4i}\right] = -(n!) \,.\, 0 = 0.$

(iii) When $n = 4p$, we have $(-1)^n = 1$, $(-1)^{n+1} = -1$,
$i^{n+2} = i^{4p+2} = i^{4p} i^2 = -1$.

$\therefore \quad (y_n)_0 = n!\left[\frac{1}{2} + \dfrac{1+i}{4} + \dfrac{1-i}{4}\right] = n!.$

(iv) When $n = 4p + 1$, we have $(-1)^n = -1$, $(-1)^{n+1} = 1$,
$i^{n+2} = i^{4p+3} = i^{4p} i^3 = i^3 = -i$.

$\therefore \quad (y_n)_0 = -n!\left[\frac{1}{2} + \dfrac{1+i}{4i} - \dfrac{1-i}{4i}\right] = -n!.$

**§ 3. Leibnitz's Theorem.

This theorem helps us to find the nth differential coefficient of the product of two functions. The statement of the theorem is as follows :

If u and v are any two functions of x such that all their desired differential coefficients exist, then the nth differential coefficient of their product is given by

$$D^n(uv) = (D^n u) . v + {}^nC_1 D^{n-1} u\, Dv + {}^nC_2 D^{n-2} u . D^2 v + \dots$$
$$\dots + {}^nC_r D^{n-r} u . D^r v + \dots + u\, D^n v.$$

(Kanpur 1988; Allahabad 87; Agra 84; Delhi 81, 80; Bihar 85; Gorakhpur 89, 86; Luck. 83, 81; Meerut 83, 93; Ranchi 85; Indore 91)

Proof. We shall prove the theorem by mathematical induction. By actual differentiation, we have

$$D(uv) = (Du) . v + u . Dv. \quad \text{...(1)}$$

From (1) we see that the theorem is true for $n = 1$.

Now assume that the theorem is true for a particular value of n. Then we have

$$D^n(uv) = (D^n u) . v + {}^nC_1 D^{n-1} u\, Dv + {}^nC_2 D^{n-2} u . D^2 v + \dots$$
$$+ {}^nC_r D^{n-r} u\, D^r v + {}^nC_{r+1} D^{n-r-1} u\, D^{r+1} v + \dots + u\, D^n v. \quad \text{...(2)}$$

Differentiating both sides of (2) with respect to x, we get

$$D^{n+1}(uv) = \{(D^{n+1}u) . v + D^n u\, Dv\}$$
$$+ \{{}^nC_1 D^n u . Dv + {}^nC_1 D^{n-1} u . D^2 v\}$$
$$+ \{{}^nC_2 D^{n-1} u . D^2 v + {}^nC_2 D^{n-2} u . D^3 v\} + \dots$$
$$\dots + \{{}^nC_r D^{n-r+1} u . D^r v + {}^nC_r D^{n-r} u . D^{r+1} v\}$$
$$+ \{{}^nC_{r+1} D^{n-r} u . D^{r+1} v + {}^nC_{r+1} D^{n-r-1} u . D^{r+2} v\} + \dots$$
$$\dots + \{Du\, D^n v + u\, D^{n+1} v\}.$$

Rearranging the terms, we get

$$D^{n+1}(uv) = (D^{n+1} u) . v + (1 + {}^nC_1)(D^n u\, Dv)$$
$$+ ({}^nC_1 + {}^nC_2) D^{n-1} u . D^2 v$$
$$+ \dots + ({}^nC_r + {}^nC_{r+1})(D^{n-r} u\, D^{r+1} v) + \dots + u\, D^{n+1} v. \quad \text{...(3)}$$

But from algebra, ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$. Therefore

$1 + {}^nC_1 = {}^nC_0 + {}^nC_1 = {}^{n+1}C_1$, ${}^nC_1 + {}^nC_2 = {}^{n+1}C_2$, and so on.

Hence (3) gives

$$D^{n+1}(uv) = (D^{n+1} u) . v + {}^{n+1}C_1 (D^n u) . Dv$$
$$+ {}^{n+1}C_2 (D^{n-1} u) . (D^2 v) + \dots$$
$$+ {}^{n+1}C_{r+1} D^{n-r} u . D^{r+1} v + \dots + u . D^{n+1} v. \quad \text{...(4)}$$

From (4) we see that if the theorem is true for any value of n it is also true for the next value of n. But we have already seen that the theorem is true for $n = 1$. Hence it must be true for $n = 2$ and so for $n = 3$, and so on. Thus the theorem is true for all positive integral values of n.

Important Note. While applying Leibnitz's theorem if we observe that one of the two functions is such that all its differential coefficients after a certain stage become zero, then we should take that function as the second function.

Ex. 13. *Find the nth differential coefficient of $x^3 \cos x$.*

(Meerut 1991S)

Sol. Since the fourth and higher derivatives of x^3 are all zero, therefore for the sake of convenience we shall take x^3 as the second function. Applying Leibnitz's theorem, we have

$$D^n [(\cos x)\, x^3] = (D^n \cos x) \,.\, x^3 + {}^nC_1 \,(D^{n-1} \cos x) \,.\, (Dx^3)$$
$$+ {}^nC_2 \,(D^{n-2} \cos x) \,.\, (D^2x^3) + {}^nC_3 \,(D^{n-3} \cos x) \,.\, (D^3x^3),$$

since all other terms become zero

$$= [\cos (x + \tfrac{1}{2} n\pi) \,.\, x^3 + n\, [\cos \{x + \tfrac{1}{2} (n-1)\, \pi\}] \,.\, 3x^2$$
$$+ \frac{n\,(n-1)}{1\,.\,2} [\cos \{x + \tfrac{1}{2} (n-2)\, \pi\}] \,.\, 6x$$
$$+ \frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3} [\cos \{x + \tfrac{1}{2} (n-3)\, \pi\}] \,.\, 6$$
$$= x^3 \cos (x + \tfrac{1}{2} n\pi) + 3nx^2 \sin (x + \tfrac{1}{2} n\pi)$$
$$- 3n\,(n-1)\, x \cos (x + \tfrac{1}{2} n\pi) - n\,(n-1)\,(n-2) \sin (x + \tfrac{1}{2} n\pi)$$
$$= [x^3 - 3n(n-1)\, x] \cos (x + \tfrac{1}{2} n\pi)$$
$$+ [3x^2 n - n\,(n-1)\,(n-2)] \sin (x + \tfrac{1}{2} n\pi).$$

Ex. 14. *Find the n^{th} differential coefficient of $x^{n-1} \log x$.*

(Agra 1984; Kanpur 85)

Sol. Let $y = x^{n-1} \log x$ ...(1)

Then $y_1 = x^{n-1} (1/x) + (n-1) \,.\, x^{n-2} \,.\, \log x$.

Multiplying both sides by x, we have

$$xy_1 = x^{n-1} + (n-1)\, x^{n-1} \log x$$

or $$xy_1 = x^{n-1} + (n-1)\, y. \qquad ...(2)$$

$[\because$ from (1), $y = x^{n-1} \log x]$

Differentiating both sides of (2) $(n-1)$ times, we have

$$D^{n-1} (y_1 x) = D^{n-1} x^{n-1} + (n-1)\, D^{n-1} y$$

or $(D^{n-1} y_1) \,.\, x + {}^{n-1}C_1 \,(D^{n-2} y_1) \,.\, 1 = (n-1)\,! + (n-1)\, y_{n-1}$

or $xy_n + (n-1)\, y_{n-1} = (n-1)\,! + (n-1)\, y_{n-1}$

or $xy_n = (n-1)\,!$ or $y_n = (n-1)\,!/x$.

Ex. 15. *Find the n^{th} derivative of $x^3 \log x$.*

Sol. By Leibnitz's theorem, we have

$$D^n (uv) = (D^n u) \,.\, v + {}^nC_1 \,(D^{n-1} u)\, (Dv) + {}^nC_2 \,(D^{n-2} u)\, (D^2 v) + \ldots$$

Taking $u = \log x$ and $v = x^3$, we find that

$D^n u = (-1)^{n-1}(n-1)!/x^n,$
$D^{n-1} u = (-1)^{n-2}(n-2)!/x^{n-1}$ and $Dv = 3x^2,$
$D^{n-2} u = (-1)^{n-3}(n-3)!/x^{n-2}$ and $D^2 v = 6x,$
$D^{n-3} u = (-1)^{n-4}(n-4)!/x^{n-3}$ and $D^3 v = 6.$

Therefore by Leibnitz's theorem, we have

$$D^n(x^3 \log x) = \frac{(-1)^{n-1}(n-1)!}{x^n} x^3 + {}^nC_1 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 3x^2 + {}^nC_2 \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} 6x + {}^nC_3 \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} 6,$$

since all other terms become zero

$$= \frac{(-1)^{n-1}(n-4)!}{x^{n-3}} [(n-1)(n-2)(n-3) - 3n(n-2)(n-3) + 3n(n-1)(n-3) - n(n-1)(n-2)]$$

$$= (-1)^{n-1}(n-4)!\, x^{3-n} [(n-1)(n-2)(n-3-n) + 3n(n-3)(n-1-n+2)]$$

$$= 6(-1)^n (n-4)!\, x^{3-n}.$$

Ex. 16. *Find the n^{th} differential coefficient of $e^x \log x$.*

(Garhwal 1983)

Sol. By Leibnitz's theorem, we have

$$D^n(e^x \log x) = (D^n e^x) . \log x + {}^nC_1 (D^{n-1} e^x) . (D \log x) + {}^nC_2 (D^{n-2} e^x)(D^2 \log x) + \dots + e^x D^n \log x$$

$$= e^x . \log x + {}^nC_1 e^x . \left(\frac{1}{x}\right) + {}^nC_2 e^x \left(-\frac{1}{x^2}\right) + {}^nC_3 e^x . \frac{2!}{x^3} + \dots + \dots + e^x(-1)^{n-1} . (n-1)! . x^{-n}$$

$$= e^x [\log x + {}^nC_1 x^{-1} - {}^nC_2 x^{-2} + {}^nC_3 2!\, x^{-3} - \dots + (-1)^{n-1} . (n-1)!\, x^{-n}].$$

***Ex. 16. (a)** *If $y = x^2 e^x$, prove that*

$$\frac{d^n y}{dx^n} = \tfrac{1}{2} n(n-1) \frac{d^2 y}{dx^2} - n(n-2) \frac{dy}{dx} + \tfrac{1}{2}(n-1)(n-2) y.$$

(Agra 1983; Gorakhpur 81)

Sol. We have $y = x^2 e^x$. ...(1)

Differentiating (1) n times by Leibnitz's theorem, we get

$$y_n = e^x . x^2 + {}^nC_1 e^x . 2x + {}^nC_2 e^x . 2$$

$$= x^2 e^x + 2nx\, e^x + n(n-1) e^x. \quad ...(2)$$

Also differentiating (1) only once w.r.t. x, we have

$$y_1 = e^x . x^2 + 2xe^x \text{ or } y_1 = y + 2xe^x. \quad ...(3)$$

Now differentiating (3) w.r.t. x, we have

$$y = y_1 + 2xe^x + 2e^x$$

$$= y_1 + (y_1 - y) + 2e^x, \qquad [\because \text{ from (3), } y_1 - y = 2xe^x]$$

or $\quad y_2 = 2y_1 - y + 2e^x.$ ...(4)

Hence substituting for x^2e^x, $2xe^x$ and e^x respectively from (1), (3) and (4) in (2), we get

$$y_n = y + n(y_1 - y) + \tfrac{1}{2}n(n-1)(y_2 - 2y_1 + y)$$
$$= \tfrac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \tfrac{1}{2}(n-1)(n-2)y$$

i.e.,
$$\frac{d^ny}{dx^n} = \tfrac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \tfrac{1}{2}(n-1)(n-2)y.$$

Ex. 17. *If $y = x^n \log x$, show that $y_{n+1} = n!/x$.*

Sol. We have $y = x^n \log x$.

$\therefore \quad y_1 = x^n(1/x) + nx^{n-1}\log x \quad$ or $\quad xy_1 = x^n + nx^n \log x$

or $\quad xy_1 = x^n + ny.$

Now differentiating both sides n times and using Leibnitz's theorem, we get

$$D^n(y_1x) = D^nx^n + nD^ny$$

or $\quad (D^ny_1).x + {}^nC_1(D^{n-1}y_1).(Dx) = n! + ny_n$

or $\quad y_{n+1}.x + ny_n.1 = n! + ny_n$ or $y_{n+1}.x = n!.$

Therefore $\quad y_{n+1} = n!/x.$

Ex. 18. *If $y = x^2e^x \cos x$, find y_n.*

Sol. Let $u = e^x \cos x$ and $v = x^2$. Then $y = uv$. Differentiating n times by Leibnitz's theorem, we get

$$y_n = D^n(uv) = (D^nu)v + {}^nC_1(D^{n-1}u).(Dv) + {}^nC_2(D^{n-2}u).(D^2v) + \ldots$$

Now $D^nu = D^n(e^x \cos x) = r^ne^x \cos(x + n\phi)$, where

$$r = \sqrt{(1+1)} = \sqrt{2} \text{ and } \phi = \tan^{-1}(1/1) = \tan^{-1}1 = \pi/4.$$

Thus $\quad D^nu = 2^{n/2}e^x\cos(x + \tfrac{1}{4}n\pi)$. Also $Dv = Dx^2 = 2x$,

$D^2v = 2$ = constant and so the third and the higher derivatives of v all vanish. Hence

$$y_n = [2^{n/2}e^x\cos(x + \tfrac{1}{4}n\pi)].x^2 + n[2^{(n-1)/2}e^x\cos\{x + \tfrac{1}{4}(n-1)\pi\}].2x + \tfrac{1}{2}n(n-1)[2^{(n-2)/2}e^x\cos\{x + \tfrac{1}{4}(n-2)\pi\}].2$$
$$= e^x[2^{n/2}x^2\cos(x + \tfrac{1}{4}n\pi) + 2nx\,2^{(n-1)/2}\cos\{x + \tfrac{1}{4}(n-1)\pi\} + n(n-1)2^{(n-2)/2}\cos\{x + \tfrac{1}{4}(n-2)\pi\}].$$

Ex. 18. (a) *Find the nth derivative of $x^2e^x\cos^3 x$.*

Sol. Let $y = x^2e^x\cos^3 x$. Since $\cos 3x = 4\cos^3 x - 3\cos x$, therefore $\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$.

Hence $\quad y = \tfrac{1}{4}x^2e^x(3\cos x + \cos 3x)$

$$= \tfrac{3}{4}(e^x\cos x)x^2 + \tfrac{1}{4}(e^x\cos 3x).x^2.$$

Now differentiating n times by Leibnitz's theorem, we get

$$y_n = \tfrac{3}{4}[\{r^n e^x \cos(x + n\phi)\} . x^2 + {}^nC_1 \{r^{n-1} e^x \cos(x + \overline{n-1}\phi)\} . 2x + {}^nC_2 \{r^{n-2} e^x \cos(x + \overline{n-2}\phi)\} . 2]$$
$$+ \tfrac{1}{4}[r_1^n e^x \cos(3x + n\phi_1)\} . x^2 + {}^nC_1 \{r_1^{n-1} e^x \cos(3x + \overline{n-1}\phi_1)\} . 2x + {}^nC_2 \{r_1^{n-2} e^x \cos(3x + \overline{n-2}\phi_1)\} . 2],$$

where $r = \sqrt{2}$, $\phi = \tan^{-1} 1 = \pi/4$; $r_1 = \sqrt{10}$, $\phi_1 = \tan^{-1} 3$.

Ex. 19. *If $y = x^2 \tan^{-1} x$, find y_n.*

Sol. Let $u = \tan^{-1} x$ and $v = x^2$. Then $y = uv$. Differentiating n times by Leibnitz's theorem, we get

$$y_n = D^n(uv) = (D^n u) . v + {}^nC_1 (D^{n-1} u) . Dv + {}^nC_2 (D^{n-2} u) . (D^2 v) + \dots$$

Now $D^n u = D^n (\tan^{-1} x) = (-1)^{n-1} (n-1)! \sin^n \theta \sin n\theta$,

where $\theta = \tan^{-1}(1/x)$.

Also $Dv = Dx^2 = 2x$, $D^2 v = 2$ = constant and so the third and the higher derivaties of v all vanish.

Hence

$$y_n = [(-1)^{n-1} (n-1)! \sin^n \theta \sin n\theta] . x^2 + {}^nC_1 [(-1)^{n-2} (n-2)! \sin^{n-1} \theta \sin (n-1)\theta] . 2x + {}^nC_2 [(-1)^{n-3} (n-3)! \sin^{n-2} \theta \sin (n-2)\theta] . 2$$
$$= (-1)^{n-1} (n-3)! [(n-1)(n-2) x^2 \sin^n \theta \sin n\theta - 2nx (n-2) \sin^{n-1} \theta \sin (n-1)\theta + n(n-1) \sin^{n-2} \theta \sin (n-2)\theta], \text{ where } \theta = \tan^{-1}(1/x).$$

****Ex. 20.** *If $y = a \cos(\log x) + b \sin(\log x)$, show that*

$$x^2 y_2 + xy_1 + y = 0,$$

and $\quad x^2 y_{n+2} + (2n+1) xy_{n+1} + (n^2+1) y_n = 0.$

(Rohilkhand 1987; Delhi 82; Meerut 94P, 91P, 90S, 88, 85; Kanpur 78; Gorakhpur 80; Agra 82, 81)

Sol. We have $\quad y = a \cos(\log x) + b \sin(\log x)$. ...(1)

$\therefore \quad y_1 = -(a/x) \sin(\log x) + (b/x) \cos(\log x)$,

or $\quad xy_1 = -a \sin(\log x) + b \cos(\log x)$. ...(2)

Differentiating (2) with respect to x, we have

$xy_2 + y_1 = -(a/x) \cos(\log x) - (b/x) \sin(\log x)$

or $\quad x^2 y_2 + xy_1 = -y$ [from (1)]

or $\quad x^2 y_2 + xy_1 + y = 0$. ...(3)

Differentiating (3) n times by Leibnitz's theorem, we have

$D^n(x^2 y_2) + D^n(xy_1) + D^n(y) = 0$

or $\quad (D^n y_2) \cdot x^2 + {}^nC_1 (D^{n-1} y_2) \cdot (Dx^2) + {}^nC_2 (D^{n-2} y_2) \cdot (D^2 x^2)$
$$+ (D^n y_1) \cdot x + {}^nC_1 (D^{n-1} y_1) \cdot (Dx) + D^n y = 0$$

or $$\left[y_{n+2} x^2 + n \cdot y_{n+1} \cdot 2x + \frac{n(n-1)}{2!} y_n \cdot 2\right] + [y_{n+1} \cdot x + n y_n \cdot 1] + y_n = 0$$

or $$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0.$$

Ex. 20. (a) *If $y = \cos(\log x)$, prove that*
$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0. \qquad \textbf{(Meerut 1986)}$$

Sol. Proceed as in Ex. 20.

***Ex. 21.** *If $y = e^{a \sin^{-1} x}$, show that*
$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0.$$

(Rohilkhand 1982; Meerut 90, 94, 98; Garhwal 87; Bundelkhand 84; Indore 83; Gorakhpur 83; Allahabad 80; Bihar 88)

Sol. We have $y = e^{a \sin^{-1} x}$.

Therefore $y_1 = e^{a \sin^{-1} x} \cdot a/\sqrt{(1-x^2)}$

or $\quad y_1 \cdot \sqrt{(1-x^2)} = a e^{a \sin^{-1} x} = ay, \qquad$ [replacing $e^{a \sin^{-1} x}$ by y]

or $\quad y_1^2 (1-x^2) = a^2 y^2. \qquad$ ...(1)

Differentiating (1) w.r.t. 'x', we have
$$2y_1 y_2 (1-x^2) + y_1^2 (-2x) = 2a^2 y y_1$$

or $$2y_1 [y_2 (1-x^2) - y_1 x - a^2 y] = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get
$$y_2 (1-x^2) - y_1 x - a^2 y = 0. \qquad \text{...(2)}$$

Differentiating (2) n times by Leibnitz's theorem, we have
$$D^n [y_2 (1-x^2)] - D^n (y_1 x) - a^2 D^n y = 0$$

or $$\left[y_{n+2} \cdot (1-x^2) + n y_{n+1} (-2x) + \frac{n(n-1)}{2!} y_n (-2)\right] - [y_{n+1} x + n y_n \cdot 1] - a^2 y_n = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2+a^2) y_n = 0.$$

Ex. 22. *If $y = (x^2-1)^n$, prove that*
$$(x^2-1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0.$$

(Allahabad 1981; Gorakhpur 86; Rohilkhand 82; Agra 86; Jhansi 89)

Hence if $P_n = \dfrac{d^n}{dx^n} (x^2-1)^n$, show that
$$\frac{d}{dx}\left\{(1-x^2) \frac{dP_n}{dx}\right\} + n(n+1) P_n = 0.$$

(Kanpur 1988)

Sol. We have $y = (x^2-1)^n$. Therefore $y_1 = n(x^2-1)^{n-1} \cdot 2x$

or $\quad (x^2-1) y_1 = n(x^2-1)^n \cdot 2x = 2nxy, \quad$ [replacing $(x^2-1)^n$ by y]

or $\qquad (x^2-1)y_1 - 2nxy = 0.$ ...(1)

Differentiating (1) $(n+1)$ times by Leibnitz's theorem, we have

$$D^{n+1}[y_1(x^2-1)] - 2n\,D^{n+1}(yx) = 0$$

or $\qquad y_{n+2}(x^2-1) + (n+1)y_{n+1}\cdot 2x + \dfrac{(n+1)n}{2!}\cdot y_n \cdot 2$

$$- 2nxy_{n+1}\cdot x - 2n(n+1)y_n \cdot 1 = 0$$

or $\qquad (x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0,$ ...(2)

giving the first result. From (2), we get

$$(x^2-1)D^2y_n + 2x\,Dy_n - n(n+1)y_n = 0. \qquad ...(3)$$

Putting $\quad y_n = \dfrac{d^n}{dx^n}(x^2-1)^n = P_n$, (3) becomes

$$(x^2-1)D^2P_n + 2x\,D(P_n) - n(n+1)P_n = 0$$

or $\qquad -(1-x^2)D^2(P_n) + 2x\,D(P_n) - n(n+1)P_n = 0$

or $\qquad -\dfrac{d}{dx}\{(1-x^2)D(P_n)\} - n(n+1)P_n = 0$

or $\qquad \dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}P_n\right\} + n(n+1)P_n = 0.$

Ex. 23. *If* $y^{1/m} + y^{-1/m} = 2x$, *prove that*

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

(Rohilkhand 1991; Meerut 83S; Luck. 81; Kanpur 85)

Sol. We have $\quad y^{1/m} + y^{-1/m} = 2x.$

Multiplying both sides by $y^{1/m}$, we get

$$y^{2/m} + 1 = 2xy^{1/m} \quad \text{or} \quad y^{2/m} - 2xy^{1/m} + 1 = 0.$$

$$\therefore \quad y^{1/m} = \frac{2x \pm \sqrt{(4x^2-4)}}{2} = x \pm \sqrt{(x^2-1)}$$

or $\qquad y = [x \pm \sqrt{(x^2-1)}]^m.$ ...(1)

$$\therefore \quad y_1 = m[x \pm \sqrt{(x^2-1)}]^{m-1}\left\{1 \pm \frac{x}{\sqrt{(x^2-1)}}\right\}$$

$$= \pm \frac{my}{\sqrt{(x^2-1)}}[x \pm \sqrt{(x^2-1)}]^m$$

$$= \pm \frac{my}{\sqrt{(x^2-1)}}\cdot \text{from (1).}$$

Squaring both sides, we get

$y_1^2(x^2-1) = m^2y^2$. Differentiating again, we get

$$2y_1y_2(x^2-1) + 2xy_1^2 = 2m^2yy_1$$

or $\qquad 2y_1[y_2(x^2-1) + xy_1 - m^2y] = 0$

or $\qquad y_2(x^2-1) + xy_1 - m^2y = 0$, since $2y_1 \neq 0.$ ...(2)

Differentiating (2) n times by Leibnitz's theorem, we get

$$D^n\{y_2(x^2-1)\} + D^n(y_1x) - m^2D^ny = 0$$

or $y_{n+2}\cdot(x^2-1)+{}^nC_1 y_{n+1}\,2x+{}^nC_2 y_n\,.\,2+y_{n+1}\,.\,x$
$$+\,{}^nC_1 y_n\,.\,1-m^2y_n=0$$

or $(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0.$

Ex. 24. *If* $y=(\sin^{-1}x)/\sqrt{(1-x^2)}$, *prove that*
$$(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2y_{n-1}=0.$$
(Delhi 1983; Gorakhpur 81)

Sol. We have $y\sqrt{(1-x^2)}=\sin^{-1}x.$

Differentiating both sides, we get
$$y_1\sqrt{(1-x^2)}+y\cdot\tfrac{1}{2}\frac{-2x}{\sqrt{(1-x^2)}}=\frac{1}{\sqrt{(1-x^2)}}$$
or $\quad y_1\,(1-x^2)-yx-1=0.$ ...(1)

Differentiating (1) n times by Leibnitz's theorem, we get
$$y_{n+1}\,(1-x^2)+{}^nC_1 y_n\,(-2x)+{}^nC_2 y_{n-1}\,(-2)$$
$$-\,y_n\,.\,x-{}^nC_1 y_{n-1}\,.\,1=0$$
or $(1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2y_{n-1}=0.$

****Ex. 25.** *If* $y=(\sin^{-1}x)^2$, *prove that*
$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0.$$
(Kumayun 1983)

Differentiate the above equation n times w.r.t. 'x'.

(Delhi 1980; Ranchi 86; Garhwal 83; Lucknow 82; Meerut 95)

Sol. We have $y=(\sin^{-1}x)^2.$

Therefore $y_1=2\,(\sin^{-1}x)/\sqrt{(1-x^2)},$

or $\quad y_1^2=\dfrac{4\,(\sin^{-1}x)^2}{1-x^2}=\dfrac{4y}{1-x^2},\qquad [\because (\sin^{-1}x)^2=y]$

or $\quad y_1^2\,(1-x^2)-4y=0.$ Differentiating again, we get
$$2y_1y_2\,(1-x^2)-2xy_1^2-4y_1=0$$
or $\quad 2y_1\,[y_2\,(1-x^2)-xy_1-2]=0.$

Cancelling $2y_1$, since $2y_1\neq 0$, we get
$$y_2\,(1-x^2)-y_1x-2=0, \qquad ...(1)$$
proving the first result.

Differentiating (1) n times by Leibnitz's theorem, we get
$$y_{n+2}\,(1-x^2)+{}^nC_1 y_{n+1}\cdot(-2x)+{}^nC_2 y_n\,.\,(-2)-y_{n+1}\,.\,x$$
$$-\,{}^nC_1 y_n\,.\,1=0$$
or $(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-n^2y_n=0.$

Ex. 26. *If* $\cos^{-1}(y/b)=\log\,(x/n)^n$, *prove that*
$$x^2y_{n+2}+(2n+1)\,xy_{n+1}+2n^2y_n=0.$$
(Kanpur 1986; Gorakhpur 84; Jhansi 88; Indore 90; Meerut 94, 96)

Sol. Here $\cos^{-1}(y/b) = \log(x/n)^n = n \log(x/n) = n(\log x - \log n)$.
Differentiating both sides with respect to x, we have

$$-\frac{1}{\sqrt{\{1-(y^2/b^2)\}}}\frac{y_1}{b} = \frac{n}{x} \quad \text{or} \quad -\frac{y_1}{\sqrt{(b^2-y^2)}} = \frac{n}{x}$$

or $\quad y_1^2 x^2 = n^2(b^2 - y^2)$.

Differentiating again, we get

$$2y_1 y_2 x^2 + 2xy_1^2 = -2n^2 y y_1$$

or $\quad y_2 x^2 + y_1 x + n^2 y = 0, \quad$ since $2y_1 \neq 0$. ...(1)

Differentiating (1) n times by Leibnitz's theorem, we get

$$y_{n+2} . x^2 + {}^nC_1 y_{n+1} . (2x) + {}^nC_2 y_n . (2) + y_{n+1} . x + {}^nC_1 y_n + n^2 y_n = 0$$

or $\quad x^2 y_{n+2} + (2n+1) xy_{n+1} + 2n^2 y_n = 0$.

Ex. 27. *If* $y = e^{\tan^{-1} x}$, *prove that*

$$(1+x^2) y_{n+2} + [2(n+1)x - 1] y_{n+1} + n(n+1) y_n = 0.$$

(Agra 1985; Meerut 76, 96BP; Ranchi 82)

Sol. We have $\quad y = e^{\tan^{-1} x}$.

Therefore $\quad y_1 = e^{\tan^{-1} x} . \dfrac{1}{(1+x^2)} = \dfrac{y}{(1+x^2)}$

or $\quad y_1(1+x^2) - y = 0$. ...(1)

Differentiating (1) $(n+1)$ times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^{n+1}C_1 y_{n+1} . 2x + {}^{n+1}C_2 y_n . 2 - y_{n+1} = 0$$

or $\quad (1+x^2) y_{n+2} + [2(n+1)x - 1] y_{n+1} + (n+1) n y_n = 0$.

Ex. 28. *Prove that*

$$D^n\left(\frac{\sin x}{x}\right) = \{P \sin(x + \tfrac{1}{2}n\pi) + Q \cos(x + \tfrac{1}{2}n\pi)\}/x^{n+1},$$

where $\quad P = x^n - n(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4} - \ldots,$
and $\quad Q = nx^{n-1} - n(n-1)(n-2)x^{n-3} + \ldots$.

Sol. Differentiating n times by Leibnitz's theorem taking $\sin x$ as first function and x^{-1} as second function, we get

$$D^n[(\sin x) . x^{-1}] = [\sin(x + \tfrac{1}{2}n\pi)] . (x^{-1}) + {}^nC_1[\sin\{x + \tfrac{1}{2}(n-1)\pi\}][(-1)x^{-2}]$$
$$+ {}^nC_2[\sin\{x + \tfrac{1}{2}(n-2)\pi\}].[(-1)(-2)x^{-3}]$$
$$+ {}^nC_3[\sin\{x + \tfrac{1}{2}(n-3)\pi\}].[(-1)(-2)(-3)x^{-4}]$$
$$+ {}^nC_4[\sin\{x + \tfrac{1}{2}(n-4)\pi\}]. \times [(-1)(-2)(-3)(-4)x^{-5}] + \ldots$$
$$= \frac{1}{x^{n+1}}[x^n \sin(x + \tfrac{1}{2}n\pi) + nx^{n-1}\cos(x + \tfrac{1}{2}n\pi) - n(n-1)x^{n-2} \times \sin(x + \tfrac{1}{2}n\pi) - n(n-1)(n-2)x^{n-3}\cos(x + \tfrac{1}{2}n\pi)$$

$$+ n(n-1)(n-2)(n-3)x^{n-4}\sin(x+\tfrac{1}{2}n\pi) + \ldots]$$

$$= \frac{1}{x^{n+1}}[\{x^n - n(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4} - \ldots\}$$
$$\times \sin(x+\tfrac{1}{2}n\pi) + \{nx^{n-1} - n(n-1)(n-2)x^{n-3} + \ldots\}$$
$$\times \cos(x+\tfrac{1}{2}n\pi)]$$

$$= \frac{1}{x^{n+1}}[P\sin(x+\tfrac{1}{2}n\pi) + Q\cos(x+\tfrac{1}{2}n\pi)].$$

Ex. 29. *Prove that*

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = \frac{(-1)^n.(n)!}{x^{n+1}}[\log x - 1 - \tfrac{1}{2} - \tfrac{1}{3} - \ldots - 1/n].$$

(Lucknow 1980; Delhi 81; Kanpur 89)

Sol. We have $y = (1/x).\log x = (x^{-1}).\log x.$ ...(1)

Differentiating (1) n times by Leibnitz's theorem taking x^{-1} as first function, we have

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\log x + {}^nC_1\frac{(-1)^{n-1}(n-1)!}{x^n}\cdot\frac{1}{x}$$
$$+ {}^nC_2\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\cdot(-1)\cdot\frac{1}{x^2}$$
$$+ {}^nC_3\frac{(-1)^{n-3}(n-3)!}{x^{n-2}}\cdot(-1)(-2)\cdot\frac{1}{x^3}$$
$$+ \ldots + \frac{1}{x}\cdot\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\log x + n\cdot\frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$
$$+ \frac{n(n-1)}{1.2}\cdot\frac{(-1)^{n-1}(n-2)!}{x^{n+1}} + \frac{n(n-1)(n-2)}{1.2.3}\cdot$$
$$\frac{(-1)^{n-1}(n-3)!}{x^{n+1}}.1.2 + \ldots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right].$$

Note. Solve this question also by differentiating (1) n times by Leibnitz theorem taking $\log x$ as first function.

Ex. 30. *If* $I_n = \frac{d^n}{dx^n}(x^n \log x)$, *prove that*

$$I_n = nI_{n-1} + (n-1)!;$$ **(Meerut 1983, 87, 95, 97; Agra 87)**

hence show that

$$I_n = n!(\log x + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + 1/n).$$

Sol. We have, $I_n = \frac{d^n}{dx^n}[x^n\log x] = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n\log x)\right]$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[nx^{n-1}\log x + x^n \cdot \frac{1}{x}\right]$$

$$= n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$= n\,I_{n-1} + (n-1)\,!. \quad \textbf{Proved.} \qquad ...(1)$$

We have just proved that $I_n = n\,I_{n-1} + (n-1)\,!$.

Dividing both sides by $n\,!$, we have

$$\frac{I_n}{n\,!} = \frac{I_{n-1}}{(n-1)\,!} + \frac{1}{n}. \qquad ...(2)$$

Changing n to $n-1$ in the above relation (2), we have

$$\frac{I_{n-1}}{(n-1)\,!} = \frac{I_{n-2}}{(n-2)\,!} + \frac{1}{n-1}.$$

Putting this value of $\dfrac{I_{n-1}}{(n-1)\,!}$ in (2), we have

$$\frac{I_n}{n\,!} = \frac{I_{n-2}}{(n-2)\,!} + \frac{1}{n-1} + \frac{1}{n}.$$

Thus making repeated use of the reduction formula (2), we ultimately have

$$\frac{I_n}{n\,!} = \frac{I_1}{1\,!} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

But $I_1 = \dfrac{d}{dx}(x\log x) = x \cdot \dfrac{1}{x} + \log x = \log x + 1.$

$$\therefore \quad \frac{I_n}{n\,!} = \log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

or $$I_n = n\,!\left(\log x + 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right).$$

Ex. 31. *By forming in two different ways the n^{th} derivative of x^{2n}, prove that*

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \dots = \frac{(2n)\,!}{(n\,!)^2}.$$

(I.C.S. 1995)

Sol. $\because y = x^{2n}, \quad \therefore \quad y_n = \dfrac{(2n)\,!}{(2n-n)\,!}\cdot x^{2n-n} = \dfrac{(2n)\,!}{n\,!}\cdot x^n.$ ...(1)

Again, $\quad y = x^n . x^n.$ ...(2)

Differentiating (2) n times by Leibnitz's theorem, we get

$$y_n = n\,!\,x^n + {}^nC_1\frac{n\,!}{1\,!}x\,.\,nx^{n-1} + {}^nC_2\frac{n\,!}{2\,!}x^2.\,n(n-1)x^{n-2}$$

$$+ {}^nC_3\frac{n\,!}{3\,!}x^3.\,n(n-1)(n-2)x^{n-3} + \dots + {}^nC_n x^n.\,n\,!$$

$$= n\,!\,x^n\left[1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+\ldots\right] \quad ...(3)$$

Equating the two values of y_n obtained in (1) and (3), we get

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}+\ldots=\frac{(2n)\,!}{(n\,!)^2}.$$

To find the nth derivative for x = 0.

Ex. 32. *If $y=(\sin^{-1}x)^2$, find $(y_n)_0$.* **(Delhi 1982; Lucknow 82)**

Sol. We have $y=(\sin^{-1}x)^2$. ...(1)

Differentiating both sides of (1) w.r.t. x, we get

$$y_1=(2\sin^{-1}x)\cdot\frac{1}{\sqrt{(1-x^2)}} \quad ...(2)$$

Squaring both sides of (2) and multiplying by $(1-x^2)$,

we have $(1-x^2)\,y_1^2=4\,(\sin^{-1}x)^2=4y$. $[\because y=(\sin^{-1}x)^2]$

or $(1-x^2)\,y_1^2-4y=0$. ...(3)

Differentiating both sides of (3) w.r.t. x, we get

$(1-x^2)\,2y_1y_2-2xy_1^2-4y_1=0$ or $2y_1\,[(1-x^2)\,y_2-xy_1-2]=0$.

Cancelling $2y_1$, since $2y_1\neq 0$, we get

$$(1-x^2)\,y_2-xy_1-2=0. \quad ...(4)$$

Differentiating (4) n times by Leibnitz's theorem, we get

$$[(1-x^2)\,y_{n+2}+{}^nC_1\,y_{n+1}(-2x)+{}^nC_2\,y_n(-2)]$$
$$-[xy_{n+1}+{}^nC_1\,y_1]-0=0$$

or $(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-n^2y_n=0$. ...(5)

Putting $x=0$ in (1), we get $(y)_0=(\sin^{-1}0)^2=0^2=0$.

Putting $x=0$ in (2), we get $(y_1)_0=(2\sin^{-1}0)\,.1=0$.

Putting $x=0$ in (4), we get $(y_2)_0-2=0$ *i.e.*, $(y_2)_0=2$.

[Note that y_2 is a function of x. So on putting $x=0$ in (4), y_2 becomes $(y_2)_0$].

Also putting $x=0$ in (5), we get

$$(y_{n+2})_0=n^2\,(y_n)_0.$$

Now there arise two cases.

Case I. When n is odd. Putting $n=1, 3, 5, \ldots$ in (6), we get $(y_3)_0=1^2\,(y_1)_0=1^2\,.\,0=0$, $(y_5)_0=3^2\,(y_3)_0=3^2\,.\,0=0$,
$(y_7)_0=5^2\,(y_5)_0=5^2\,.\,0=0$, and so on. Thus when n is odd, $(y_n)_0=0$.

Case II. When n is even. Putting $n=2, 4, 6, \ldots$ in (6), we get $(y_4)_0=2^2(y_2)_0=2^2\,.\,2$, $(y_6)_0=4^2(y_4)_0=4^2\,.\,2^2\,.\,2$, $(y_8)_0=6^2\,(y_6)_0$
$=6^2\,.\,4^2\,.\,2^2\,.\,2$, and so on. Thus when n is even, we have
$(y_n)_0=(n-2)^2\,(n-4)^2\ldots 6^2\,.\,4^2\,.\,2^2\,.\,2$.

****Ex. 33.** *If $y = \sin^{-1} x$, find $(y_n)_0$.*

(Allahabad 1982; Meerut 81; Agra 80; Vikram 88)

Sol. We have $$y = \sin^{-1} x. \quad ...(1)$$

$\therefore$ $$y_1 = \frac{1}{\sqrt{(1-x^2)}} \quad ...(2)$$

or $$(1-x^2)\, y_1^2 - 1 = 0. \quad ...(3)$$

Differentiating (3) w.r.t. x, we get

$$(1-x^2)\, 2y_1 y_2 - 2xy_1^2 = 0 \quad \text{or} \quad 2y_1\,[(1-x^2)\, y_2 - xy_1] = 0.$$

Cancelling $2y_1$, since $2y_1$ is not identically equal to zero,

we get $$(1-x^2)\, y_2 - xy_1 = 0. \quad ...(4)$$

Differentiating (4) n times by Leibnitz's theorem, we get

$$y_{n+2}\,(1-x^2) + {}^nC_1\, y_{n+1} \cdot (-2x) + {}^nC_2\, y_n \cdot (-2)$$
$$- y_{n+1} \cdot x - {}^nC_1\, y_n \cdot 1 = 0$$

or $$y_{n+2}(1-x^2) - (2n+1)\, xy_{n+1} - n^2 y_n = 0. \quad ...(5)$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 1 \quad \text{and} \quad (y_2)_0 - 0\,(y_1)_0 = 0 \text{ i.e., } (y_2)_0 = 0.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 = n^2\, (y_n)_0. \quad ...(6)$$

(6) is a reduction formula which expresses $(y_{n+2})_0$ in terms of $(y_n)_0$. Putting $n - 2$ in place of n in (6), we get

$$(y_n)_0 = (n-2)^2\, (y_{n-2})_0$$
$$= (n-2)^2\, (n-4)^2\, (y_{n-4})_0,$$

$$[\because \text{ from (6), } (y_{n-2})_0 = (n-4)^2\, (y_{n-4})_0]$$

Now there arise two cases.

Case I. When n is odd. Putting $n = 1, 3, 5, ...$ in (6), we have

$$(y_3)_0 = 1^2 (y_1)_0 = 1^2 . 1, \quad [\because (y_1)_0 = 1]$$

$(y_5)_0 = 3^2 (y_3)_0 = 3^2 . 1^2 . 1$, $(y_7)_0 = 5^2 (y_5)_0 = 5^2 . 3^2 . 1^2 . 1$, and so on.

Thus if n is odd, we have

$$(y_n)_0 = (n-2)^2\, (n-4)^2 \ldots 5^2 . 3^2 . 1^2 . 1.$$

Case II. When n is even. Putting $n = 2, 4, 6, ...$ in (6), we have

$$(y_4)_0 = 2^2 (y_2)_0 = 0, \quad [\because (y_2)_0 = 0]$$

$(y_6)_0 = 4^2\, (y_4)_0 = 4^2 . 0 = 0$, $(y_8)_0 = 6^2\, (y_6)_0 = 0$, and so on.

Thus if n is even, we have $(y_n)_0 = 0$.

****Ex. 34.** *If $y = \sin\, (m \sin^{-1} x)$, find $(y_n)_0$.*

(Kashmir 1983; Avadh 87; Rohilkhand 83, 81;
Meerut 88P, 89P, 91, 93; Gorakhpur 82;
Lucknow 80; Delhi 81)

Sol. We have $\quad y = \sin(m \sin^{-1} x)$. ...(1)

Differentiating, we get $y_1 = [\cos(m \sin^{-1} x)] \cdot \dfrac{m}{\sqrt{(1-x^2)}}$ ...(2)

Squaring both sides of (2) and multiplying by $(1-x^2)$, we get

$$(1-x^2) y_1^2 = m^2 \cos^2 (m \sin^{-1} x)$$

or $$(1-x^2) y_1^2 = m^2 [1 - \sin^2 (m \sin^{-1} x)]$$

or $$(1-x^2) y_1^2 = m^2 (1-y^2), \qquad [\because y = \sin(m \sin^{-1} x)]$$

or $$(1-x^2) y_1^2 + m^2 y^2 - m^2 = 0. \qquad ...(3)$$

Differentiating (3), we get

$$(1-x^2)\, 2y_1 y_2 - 2xy_1^2 + 2m^2 y y_1 = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get

$$(1-x^2) y_2 - xy_1 + m^2 y = 0. \qquad ...(4)$$

Differentiating (4) n times by Leibnitz's theorem, we get

$$(1-x^2) y_{n+2} + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n (-2) - xy_{n+1} - {}^nC_1 y_n + m^2 y_n = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 - m^2) y_n = 0. \qquad ...(5)$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = \sin(m \sin^{-1} 0) = \sin 0 = 0, \quad (y_1)_0 = (\cos 0) \,.\, m = m,$$

$$(y_2)_0 + m^2 (y)_0 = 0 \text{ i.e., } (y_2)_0 = -m^2 (y)_0 = -m^2 \,.\, 0 = 0.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 = (n^2 - m^2)(y_n)_0. \qquad ...(6)$$

Putting $n - 2$ in place of n in the reduction formula (6), we get

$$(y_n)_0 = \{(n-2)^2 - m^2\}(y_{n-2})_0$$

$$= \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}(y_{n-4})_0, \text{ since from (6), we have } (y_{n-2})_0 = \{(n-4)^2 - m^2\}(y_{n-4})_0.$$

Now there arise two cases :

Case I. When n is odd. Putting $n = 1, 3, 5, \ldots$ in (6), we have

$$(y_3)_0 = (1^2 - m^2)(y_1)_0 = (1^2 - m^2) m,$$

$$(y_5)_0 = (3^2 - m^2)(y_3)_0 = (3^2 - m^2)(1^2 - m^2) m,$$

$$(y_7)_0 = (5^2 - m^2)(y_5)_0 = (5^2 - m^2)(3^2 - m^2)(1^2 - m^2) m,$$

and so on.

Thus if n is odd, we have

$$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \ldots (3^2 - m^2)(1^2 - m^2) m.$$

Case II. When n is even. Putting $n = 2, 4, 6, \ldots$ in (6), we have $(y_4)_0 = (2^2 - m^2)(y_2)_0 = 0$, $(y_6)_0 = (4^2 - m^2)(y_4)_0 = 0$, and so on.

Thus if n is even, we have $(y_n)_0 = 0$.

Ex. 35. *If* $y = \cos(m \sin^{-1} x)$, *find* $(y_n)_0$. **(Meerut 1982, 87P, 96P)**

Sol. We have $\quad y = \cos(m \sin^{-1} x)$. ...(1)

Differentiating, we get

$$y_1 = [-\sin(m \sin^{-1} x)] \cdot [m/\sqrt{(1-x^2)}]. \quad ...(2)$$

Squaring both sides of (2) and multiplying by $(1-x^2)$, we get

$$(1-x^2)y_1^2 = m^2 \sin^2(m \sin^{-1} x) = m^2[1-\cos^2(m \sin^{-1} x)]$$
$$= m^2(1-y^2).$$

$$\therefore \quad (1-x^2)y_1^2 + m^2y^2 - m^2 = 0. \quad ...(3)$$

Differentiating (3), we get

$$(1-x^2)\,2y_1y_2 - 2xy_1^2 + 2m^2yy_1 = 0$$

or $\quad 2y_1[(1-x^2)y_2 - xy_1 + m^2y] = 0.$

Cancelling $2y_1$, since $2y_1$ is not identically equal to zero, we get

$$(1-x^2)y_2 - xy_1 + m^2y = 0. \quad ...(4)$$

Differentiating (4) n times by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1}$$
$$- {}^nC_1 y_n + m^2y_n = 0$$

or $\quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0. \quad ...(5)$

Putting $x = 0$ in (1), (2) and (4), we get

$(y)_0 = \cos 0 = 1$, $(y_1)_0 = 0$, $(y_2)_0 + m^2(y)_0 = 0$ *i.e.*, $(y_2)_0 = -m^2(y)_0$
$= -m^2$.

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 = (n^2-m^2)(y_n)_0. \quad ...(6)$$

Putting $n-2$ in place of n in the reduction formula (6), we get

$(y_n)_0 = \{(n-2)^2 - m^2\}(y_{n-2})_0$
$= \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}(y_{n-4})_0$, since from (6),
we have $(y_{n-2})_0 = \{(n-4)^2 - m^2\}(y_{n-4})_0$.

Now there arise two cases.

Case I. When n is odd, we have

$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \ldots (3^2 - m^2)(1^2 - m^2)(y_1)_0$
$= 0$, since $(y_1)_0 = 0$.

Case II. When n is even, we have

$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \ldots (4^2 - m^2)(2^2 - m^2)(y_2)_0$
$= -\{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\} \ldots (4^2 - m^2)(2^2 - m^2)m^2$,
since $(y_2)_0 = -m^2$.

Ex. 36. *If $y = \tan^{-1} x$, find $(y_n)_0$.* **(Delhi 1981; Meerut 90P, 94P)**

Sol. We have $\quad y = \tan^{-1} x$. ...(1)

$$\therefore \quad y_1 = 1/(1+x^2), \quad ...(2)$$

or $\quad (1+x^2)y_1 - 1 = 0. \quad ...(3)$

Differentiating (3), we get

$$(1+x^2)\,y_2 + 2xy_1 = 0. \qquad ...(4)$$

Differentiating (4) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + ny_{n+1}(2x) + \frac{n(n-1)}{2!}y_n \,.\, 2 + 2xy_{n+1} + 2ny_n = 0$$

or $$(1+x^2)\,y_{n+2} + 2(n+1)\,xy_{n+1} + n(n+1)\,y_n = 0. \qquad ...(5)$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 = -\{(n+1)\,n\}\,(y_n)_0. \qquad ...(6)$$

Putting $n-2$ in place of n in the reduction formula (6), we get

$$(y_n)_0 = -\{(n-1)(n-2)\}\,(y_{n-2})_0$$
$$= [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)\}]\,(y_{n-4})_0,$$

since from (6), we have $(y_{n-2})_0 = -\{(n-3)(n-4)\}\,(y_{n-4})_0$.

Now there arise two cases.

Case I. When n is even, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)\}] \ldots [-(3)(2)]\,(y_2)_0$$
$$= 0, \text{ since } (y_2)_0 = 0.$$

Case II. When n is odd, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)\}] \ldots [-(4)(3)]\,[-(2)(1)]\,(y_1)_0$$
$$= (-1)^{(n-1)/2}\,(n-1)\,!, \text{ since } (y_1)_0 = 1.$$

Ex. 36 (a). *If $y = \tan^{-1} x$, find the value of $(y_7)_0$ and $(y_8)_0$.*

(Rohilkhand 1985)

Sol. Proceed as in Ex. 36 to get the relation (6).

Putting $n = 1, 2, 3, 4, 5, 6$ in (6), we get

$(y_3)_0 = -(2\,.\,1)\,(y_1)_0 = -2\,!,\ (y_4)_0 = -(3\,.\,2)\,(y_2)_0 = 0,$

$(y_5)_0 = -(4\,.\,3)\,(y_3)_0 = -(4\,.\,3)\,.\,(-2\,!) = 4\,!,$

$(y_6)_0 = -(5\,.\,4)\,(y_4)_0 = 0,$

$(y_7)_0 = -(6\,.\,5)\,(y_5)_0 = -(6\,.\,5)\,(4\,!) = -6\,!,$

$(y_8)_0 = -(7\,.\,6)\,(y_6)_0 = 0.$

$\therefore\quad (y_7)_0 = -6\,! = -720$ and $(y_8)_0 = 0.$

Ex. 37. *If $y = e^{a \sin^{-1} x}$, find y_n at $x = 0$.*

(Agra 1982; Jiwaji 88; Meerut 84, 88, 90; Rohilkhand 88)

Sol. Proceeding as in Ex. 21 on page 41, we have

$$y = e^{a \sin^{-1} x} \qquad ...(1)$$

$$y_1 = e^{a \sin^{-1} x} \cdot a/\sqrt{(1-x^2)}, \qquad ...(2)$$

$$(1-x^2)\, y_2 - xy_1 - a^2 y = 0, \qquad ...(3)$$

and $$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+a^2)\, y_n = 0. \qquad ...(4)$$

Putting $x = 0$ in (1), (2), (3) and (4), we have

$$(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2,$$

and $$(y_{n+2})_0 = (n^2 + a^2)\,(y_n)_0. \qquad ...(5)$$

Now there arise two cases.

Case I. When n is odd.

Putting $n = 1, 3, 5, ...$ in (5), we have

$(y_3)_0 = (1^2 + a^2)\,(y_1)_0 = (1^2 + a^2)\,.\,a,$

$(y_5)_0 = (3^2 + a^2)\,(y_3)_0 = (3^2 + a^2)\,(1^2 + a^2)\,.\,a,$

$(y_7)_0 = (5^2 + a^2)\,(y_5)_0 = (5^2 + a^2)\,(3^2 + a^2)(1^2 + a^2)\,.\,a$, and so on.

Thus if n is odd, we have

$$(y_n)_0 = [(n-2)^2 + a^2] \ldots (3^2 + a^2)\,(1^2 + a^2)\,a.$$

Case II. When n is even.

Putting $n = 2, 4, 6, ...$ in (5), we have

$(y_4)_0 = (2^2 + a^2)\,(y_2)_0 = (2^2 + a^2)\,.\,a^2,$

$(y_6)_0 = (4^2 + a^2)\,(y_4)_0 = (4^2 + a^2)\,(2^2 + a^2)\,.\,a^2,$

$(y_8)_0 = (6^2 + a^2)\,(y_6)_0 = (6^2 + a^2)\,(4^2 + a^2)\,(2^2 + a^2)\,a^2,$

ans so on.

Thus if n is even, we have

$$(y_n)_0 = [(n-2)^2 + a^2] \ldots (4^2 + a^2)\,(2^2 + a^2)\,a^2.$$

Ex. 38. $y = e^{m \cos^{-1} x}$, *find* $(y_n)_0$.

(Meerut 1970, 84, 88S, 94; Delhi 84; Gorakhpur 83)

Sol. Proceed as in Ex. 37. The answer is :

When n is odd,

$$(y_n)_0 = [(n-2)^2 + m^2] \ldots [3^2 + m^2]\,[1^2 + m^2]\,.\,(-m e^{m\pi/2});$$

and when n is even,

$$(y_n)_0 = [(n-2)^2 + m^2] \ldots (2^2 + m^2)\,(m^2 e^{m\pi/2}).$$

****Ex. 39 (a).** *If* $y = [x + \sqrt{(1+x^2)}]^m$, *find the value of the* n^{th} *differential coefficient of y for* $x = 0$.

(Rohilkhand 1980; Agra 85; Gorakhpur 86; Jiwaji 88; Indore 83; Kanpur 80; Lucknow 86; Meerut 85, 92)

Sol. Here $$y = [x + \sqrt{(1+x^2)}]^m. \qquad ...(1)$$

$$\therefore \quad y_1 = m\,[x + \sqrt{(1+x^2)}]^{m-1} \cdot \left[1 + \tfrac{1}{2} \frac{2x}{\sqrt{(1+x^2)}}\right]$$

$$= \frac{m}{\sqrt{(1+x^2)}}\,[x + \sqrt{(1+x^2)}]^m = \frac{my}{\sqrt{(1+x^2)}} \qquad ...(2)$$

or $\quad y_1^2(1+x^2)-m^2y^2=0.$

Differentiating again, we get

$$2y_1y_2(1+x^2)+2xy_1^2-2m^2yy_1=0$$

or $\quad y_2(1+x^2)+xy_1-m^2y=0, \qquad \text{...(3)}$

cancelling $2y_1$, since $2y_1 \neq 0$.

Again differentiating (3) n times, we get

$$y_{n+2}(1+x^2)+{}^nC_1 \cdot 2xy_{n+1}+{}^nC_2 \cdot 2y_n+xy_{n+1}$$
$$+{}^nC_1 \cdot y_n-m^2y_n=0$$

or $\quad y_{n+2}(1+x^2)+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0. \qquad \text{...(4)}$

Putting $x=0$ in (1), (2), (3) and (4), we have

$$(y)_0=1, (y_1)_0=m, (y_2)_0=m^2,$$

and $\quad (y_{n+2})_0+(n^2-m^2)\cdot(y_n)_0=0$

i.e., $\quad (y_{n+2})_0=(m^2-n^2)\cdot(y_n)_0. \qquad \text{...(5)}$

Case I. When n is odd.

Putting $n=$ 1, 3, 5, 7, ... in (5), we get

$$(y_3)_0=(m^2-1^2)(y_1)_0=(m^2-1^2)\cdot m,$$

$(y_5)_0=(m^2-3^2)(y_3)_0=(m^2-3^2)(m^2-1^2)\cdot m$, and so on.

Hence when n is odd, we have

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-3^2)(m^2-1^2)m.$$

Case II. When n is even.

Putting $n=$ 2, 4, 6, ... in (5), we get

$$(y_4)_0=(m^2-2^2)(y_2)_0=(m^2-2^2)\cdot m^2,$$

$(y_6)_0=(m^2-4^2)(y_4)_0=(m^2-4^2)(m^2-2^2)\cdot m^2$, and so on.

Hence when n is even, we have

$$(y_n)_0=\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-2^2)\cdot m^2.$$

Ex. 39 (b). *Find $y_n(0)$, when $y=\log[x+\sqrt{(1+x^2)}]$.*

(Meerut 1989S)

Sol. Here $\quad y=\log[x+\sqrt{(1+x^2)}]. \qquad \text{...(1)}$

$$\therefore \quad y_1=\frac{1}{x+\sqrt{(1+x^2)}}\cdot\left[1+\tfrac{1}{2}\cdot\frac{1}{\sqrt{(1+x^2)}}\cdot 2x\right]=\frac{1}{\sqrt{(1+x^2)}} \qquad \text{...(2)}$$

Squaring both sides of (2), we get

$$(1+x^2)y_1^2-1=0.$$

Differentiating again, we get

$$2y_1y_2(1+x^2)+2xy_1^2=0$$

or $\quad 2y_1[y_2(1+x^2)+xy_1]=0$

or $\quad y_2(1+x^2)+xy_1=0, \qquad \text{...(3)}$

since $y_1 \neq 0$.

Differentiating (3) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^nC_1 \cdot y_{n+1} 2x + {}^nC_2 \cdot y_n \cdot 2 + y_{n+1} \cdot x + {}^nC_1 \cdot y_n \cdot 1 = 0$$

or $$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0. \quad \text{...(4)}$$

Putting $x = 0$ in (1), (2), (3) and (4), we have

$$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0,$$

and $$(y_{n+2})_0 + n^2 (y_n)_0 = 0$$

i.e., $$(y_{n+2})_0 = -n^2 (y_n)_0. \quad \text{...(5)}$$

Putting $n - 2$ in place of n on both sides of (5), we get

$$(y_n)_0 = -(n-2)^2 (y_{n-2})_0$$
$$= [-(n-2)^2][-(n-4)^2](y_{n-4})_0, \text{ since from (5),}$$
$$\text{we have } (y_{n-2})_0 = -(n-4)^2 (y_{n-4})_0.$$

Now there arise two cases :

Case I. When n is odd, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2][-(n-6)^2][-3^2][-1^2](y_1)_0$$
$$= (-1)^{(n-1)/2}(n-2)^2(n-4)^2 \ldots 3^2 . 1^2, \text{ since } (y_1)_0 = 1.$$

Case II. When n is even, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2] \ldots (y_2)_0$$
$$= 0, \text{ since } (y_2)_0 = 0.$$

Ex. 39 (c). *If* $y = [\log \{x + \sqrt{(1+x^2)}\}]^2$, *prove that* $(y_{n+2})_0 = -n^2 (y_n)_0$, *hence find* $(y_n)_0$.

(Kanpur 1983; Rohilkhand 82; Meerut 93, 95, 97)

Sol. Here $y = [\log \{x + \sqrt{(1+x^2)}\}]^2$.

$$\therefore y_1 = [2 \log \{x + \sqrt{(1+x^2)}\}] \cdot \frac{1}{x + \sqrt{(1+x^2)}} \cdot \left[1 + \frac{1}{2} \cdot \frac{1}{\sqrt{(1+x^2)}} \cdot 2x\right]$$
$$= \frac{2}{\sqrt{(1+x^2)}} \log \{x + \sqrt{(1+x^2)}\}. \quad \text{...(2)}$$

Squaring both sides of (2), we get

$$(1+x^2)y_1^2 = 4[\log \{x + \sqrt{(1+x^2)}\}]^2 = 4y, \text{ from (1)}$$

or $$(1+x^2)y_1^2 - 4y = 0.$$

Differentiating again, we get

$$2y_1y_2(1+x^2) + 2xy_1^2 - 4y_1 = 0$$

or $$2y_1[y_2(1+x^2) + xy_1 - 2] = 0$$

or $$y_2(1+x^2) + xy_1 - 2 = 0, \quad \text{...(3)}$$

since $y_1 \neq 0$.

Differentiating (3) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^nC_1 \cdot y_{n+1} \cdot 2x + {}^nC_2 y_n \cdot 2 + xy_{n+1} + {}^nC_1 y_n \cdot 1 = 0$$

or $$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0. \quad \text{...(4)}$$

Putting $x = 0$ in (1), (2), (3) and (4), we get

$$(y)_0 = 0, (y_1)_0 = 0, (y_2)_0 = 2,$$

and $\qquad (y_{n+2})_0 + n^2(y_n)_0 = 0$

i.e., $\qquad (y_{n+2})_0 = -n^2(y_n)_0. \qquad ...(5)$

Putting $n-2$ in place of n on both sides of (5), we get

$$(y_n)_0 = -(n-2)^2(y_{n-2})_0$$
$$= [-(n-2)^2][-(n-4)^2](y_{n-4})_0,$$

since from (5), we have $(y_{n-2})_0 = -(n-4)^2(y_{n-4})_0$.

Now there arise two cases :

Case I. When n is odd, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2] \ldots (y_1)_0 = 0, \text{ since } (y_1)_0 = 0.$$

Case II. When n is even, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2][-(n-6)^2] \ldots [-4^2][-2^2](y_2)_0$$
$$= (-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2 \ldots 4^2 . 2^2 . 2.$$

Ex. 40. *If $y = \frac{1}{2}(\tan^{-1}x)^2$, show that*

$$(y_{n+2})_0 + 2n^2(y_n)_0 + n(n-1)^2(n-2)(y_{n-2})_0 = 0.$$

Sol. Here $\quad y = \frac{1}{2}(\tan^{-1}x)^2; \quad \therefore \quad y_1 = \dfrac{\tan^{-1}x}{1+x^2}.$

$$\therefore \qquad (1+x^2)y_1^2 = 2y.$$

Differentiating again, we have

$$(1+x^2)^2 2y_1y_2 = 2(1+x^2) . 2xy_1^2 = 2y_1$$

or $\qquad (1+2x^2+x^4)y_2 + 2(x+x^3)y_1 - 1 = 0$, since $2y_1 \neq 0$.

Differentiating n times by Leibnitz's theorem, we have

$$y_{n+2}(1+2x^2+x^4) + ny_{n+1}(4x+4x^3) + \frac{n(n-1)}{2}y_n(4+12x^2)$$
$$+ \frac{n(n-1)(n-2)}{6}y_{n-1}(24x) + \frac{n(n-1)(n-2)(n-3)}{24}y_{n-2} . (24)$$
$$+ 2y_{n+1} . (x+x^3) + 2ny_n . (1+3x^2) + \frac{2n(n-1)}{2}y_{n-1} . (6x)$$
$$+ \frac{2n(n-1)(n-2)}{6}y_{n-2} . (6) = 0$$

or $\quad (1+x^2)^2y_{n+2} + (4n+2)(x+x^3)y_{n+1} + 2n^2(3x^2+1)y_n$

$$+ 2n(n-1)(2n-1)xy_{n-1} + n(n-1)^2(n-2)y_{n-2} = 0. \qquad ...(1)$$

Putting $x = 0$ in (1), we get

$$(y_{n+2})_0 + 2n^2(y_n)_0 + n(n-1)^2(n-2)(y_{n-2})_0 = 0.$$

Ex. 41. *If $x + y = 1$, prove that*

$$\frac{d^n}{dx^n}(x^n y^n) = n! . \{y^n - ({}^nC_1)^2 y^{n-1} . x + ({}^nC_2)^2 . y^{n-2} . x^2 + \ldots$$
$$+ (-1)^n x^n\}.$$

(Kanpur 1987; Agra 88; Gorakhpur 84)

Sol. Given $x + y = 1$; $\therefore$ $y^n = (1 - x)^n$.

Thus $\dfrac{d^n}{dx^n}(x^n y^n) = \dfrac{d^n}{dx^n}\{x^n (1 - x)^n\} = D^n[x^n \,.\, (1 - x)^n]$

$$= (D^n x^n)(1 - x)^n + {}^nC_1 (D^{n-1} x^n) \,.\, D(1 - x)^n$$
$$+ {}^nC_2 (D^{n-2} x^n) \,.\, D^2 (1 - x)^n + \ldots + x^n D^n (1 - x)^n$$

$$= n\,!\,(1 - x)^n + {}^nC_1 \frac{n\,!}{1\,!} x \{-n (1 - x)^{n-1}\}$$
$$+ {}^nC_2 \frac{n\,!}{2\,!} \,.\, x^2 \{(-1)^2 \,.\, n(n-1)(1 - x)^{n-2}\} + \ldots + x^n (-1)^n \,.\, n\,!$$

$$\left[\text{Note that } D^r x^n = \frac{n\,!}{(n - r)\,!} x^{n-r}\right]$$

$$= n\,!\left[(1 - x)^n - n \,.\, {}^nC_1 x (1 - x)^{n-1} + \frac{n(n-1)}{2\,!} \,.\, {}^nC_2 x^2 (1 - x)^{n-2} + \ldots + (-1)^n x^n\right]$$

$$= n\,!\,[(1 - x)^n - ({}^nC_1)^2 x (1 - x)^{n-1} + ({}^nC_2)^2 x^2 (1 - x)^{n-2} + \ldots + (-1)^n x^n],$$

$$\text{since } n = {}^nC_1, \ \frac{n(n-1)}{2\,!} = {}^nC_2, \text{ etc.}$$

$$= n\,!\,[y^n - ({}^nC_1)^2 x y^{n-1} + ({}^nC_2)^2 x^2 y^{n-2} - \ldots + (-1)^n x^n].$$

Ex. 42. *Prove that the nth differential coefficient of $x^n (1 - x)^n$ is equal to*

$$n\,!\,(1 - x)^n \left\{1 - \frac{n^2}{1^2} \frac{x}{1 - x} + \frac{n^2 (n-1)^2}{1^2 \,.\, 2^2} \frac{x^2}{(1 - x)^2} - \ldots\right\}.$$

(Rohilkhand 1989)

Sol. For complete solution of the question refer Ex. 41 above.

Ex. 43. *If $Y = sX$ and $Z = tX$, all the variables being functions of x, prove that*

$$D \equiv \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix},$$

where the suffices denote the order of differentiation with respect to x.

Sol. We have $Y_1 = sX_1 + s_1 X$

and $Y_2 = sX_2 + 2s_1 X_1 + s_2 X$.

Similarly $Z_1 = tX_1 + t_1 X$

and $Z_2 = tX_2 + 2t_1 X_1 + t_2 X$.

Hence $\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$

$$= \begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + s_1X & tX_1 + t_1X \\ X_2 & sX_2 + 2s_1X_1 + s_2X & tX_2 + 2t_1X_1 + t_2X \end{vmatrix}$$

$$= \begin{vmatrix} X & 0 & 0 \\ X_1 & s_1X & t_1X \\ X_2 & 2s_1X_1 + s_2X & 2t_1X_1 + t_2X \end{vmatrix},$$

operating $C_2 - sC_1$ and $C_3 - tC_1$

$$= X \begin{vmatrix} s_1X & t_1X \\ 2s_1X_1 + s_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ 2s_1X_1 + s_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ s_2X & t_2X \end{vmatrix}, \text{ operating } R_2 - 2X_1R_1$$

$$= X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}.$$

□

3
Expansions of Functions

§ 1. Taylor's series.

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[a, a+h]$. Then for every positive integral value of n, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad \ldots(1)$$

where $R_n = \dfrac{h^n}{n!}f^{(n)}(a+\theta h), (0 < \theta < 1)$.

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^n}{n!}f^{(n)}(a) + \ldots} \quad \ldots(2)$$

The series given in (2) is known as **Taylor's infinite series** for the expansion of $f(a+h)$ as a **power series** in h.

§ 2. Maclaurin's series.

Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n, \quad \ldots(1)$$

where $R_n = \dfrac{x^n}{n!}f^{(n)}(\theta x), (0 < \theta < 1)$.

Suppose $R_n \to 0$, as $n \to \infty$. Then taking limits of both sides of (1) when $n \to \infty$, we get

$$\mathbf{f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots} \quad \ldots(2)$$

The series given in (2) is known as **Maclaurin's infinite series** for the expansion of $f(x)$ as a **power series** in x. Maclaurin's series is a particular case of Taylor's series. If in Taylor's series we put $a = 0$ and $h = x$ we get Maclaurin's series.

Maclaurin's expansion of $f(x)$ fails if any of the functions $f(x)$, $f'(x), f''(x), \ldots$ becomes infinite or discontinuous at any point of the interval $[0, x]$ or if R_n does not tend to zero as $n \to \infty$.

§ 3. Formal expansions of functions.

We have seen that for the validity of the expansion of a function $f(x)$ as an infinite Maclaurin's series, it is necessary that $R_n \to 0$ as $n \to \infty$. But to examine the behaviour of R_n as $n \to \infty$ is a difficult job because in many cases it is not possible to find a general expression for the nth derivatives of the functions to be expanded. So in this chapter we shall simply obtain *formal expansion* of a function $f(x)$ without showing that $R_n \to 0$ as $n \to \infty$.

Such an expansion will not give us any idea of the range of values of x for which the expansion is valid. To obtain such an expansion of $f(x)$ we have only to calculate the values of its derivatives for $x = 0$ and substitute them in the infinite Maclaurin's series

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \ldots\ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots.$$

For the convenience of the students we shall now give *formal proofs* of Maclaurin's and Taylor's theorems without bothering about the nature of R_n as $n \to \infty$.

Maclaurin's theorem.

(Gorakhpur 87; Meerut 84; Magadh 84; Bihar 82; Kashmir 83)

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$. Assuming that $f(x)$ can be expanded as an infinite power series in x, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

Proof. Suppose $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \ldots$ ...(1)

Let the expansion (1) be differentiable term by term any number of times. Then by successive differentiation, we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \ldots,$$
$$f''(x) = 2.1 A_2 + 3.2 A_3 x + 4.3 A_4 x^2 + \ldots,$$
$$f'''(x) = 3.2.1 A_3 + 4.3.2 A_4 x + \ldots, \text{ and so on.}$$

Putting $x = 0$ in each of these relations, we get

$$f(0) = A_0, f'(0) = A_1, f''(0) = 2!A_2, f'''(0) = 3!A_3, \ldots$$

Substituting theses values of $A_0, A_1, A_2, \ldots$ in (1), we get

$$\mathbf{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots}$$

This is **Maclaurin's theorem.** If we denote $f(x)$ by y, then Maclaurin's theorem can be written in either of the following ways :

$$\mathbf{y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots}$$

or $$\mathbf{y = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots}$$

Taylor's theorem. **(Gorakhpur 1988; Bihar 82; Vikram 88; Meerut 85; Magadh 87; Jiwaji 90)**

Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[a, a+h]$. Assuming that $f(a+h)$ can be expanded as an infinite power series in h, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots$$

Proof. Suppose $f(a+h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots$ ...(1)

Let the expansion (1) be differentiable term by term any number of times w.r.t. h. Then by successive differentiation w.r.t. h, we have

$$f'(a+h) = A_1 + 2A_2 h + 3A_3 h^2 + \ldots,$$
$$f''(a+h) = 2.1A_2 + 3.2A_3 h + \ldots,$$
$$f'''(a+h) = 3.2.1A_3 + \ldots, \text{ and so on.}$$

Putting $h = 0$ in each of the above relations, we get

$f(a) = A_0, f'(a) = A_1, f''(a) = 2!\,A_2, f'''(a) = 3!\,A_3$, and so on.

$$\therefore \quad A_0 = f(a), A_1 = f'(a), A_2 = \frac{1}{2!} f''(a), A_3 = \frac{1}{3!} f'''(a),$$

and so on.

Substituting these values of $A_0, A_1, A_2, A_3, \ldots$ in (1), we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots}$$

This is **Taylor's theorem**. Another useful form is obtained on replacing h by $(x - a)$. Thus

$$\mathbf{f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots,}$$

which is an expansion of $f(x)$ as a power series in $(x - a)$.

Note. If we expand $f(x+h)$, by Taylor's theorem, as a power series in h, then the result is as follows :

$$\mathbf{f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots + \frac{h^n}{n!} f^{(n)}(x) + \ldots}$$

§ 4. Some Important Expansions.

1. Expansion of e^x. *(Exponential series).*

Let $f(x) = e^x$. Then $f(0) = e^0 = 1$;

$f^{(n)}(x) = e^x$ so that $f^{(n)}(0) = e^0 = 1$, where $n = 1, 2, 3, \ldots$

Substituting these values in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

we get $\quad \mathbf{e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!} + \ldots}$

2. Expansion of sin x. (*Sine series*)

Let $f(x) = \sin x$. Then $f(0) = 0$,

$f'(x) = \cos x$ so that $f'(0) = 1$,

$f''(x) = -\sin x$ so that $f''(0) = 0$,

$f'''(x) = -\cos x$ so that $f'''(0) = -1$, and so on.

In general, $f^n(x) = \sin(x + \frac{1}{2}n\pi)$ so that $f^n(x) = \sin(\frac{1}{2}n\pi)$.

When $n = 2m$, $f^n(0) = \sin m\pi = 0$ and when $n = 2m + 1$,

$$f^n(0) = \sin\{\tfrac{1}{2}(2m+1)\pi\} = \sin(m\pi + \tfrac{1}{2}\pi)$$
$$= (-1)^m \sin(\tfrac{1}{2}\pi) = (-1)^m.$$

Substituting these values in Maclaurin's series, we get

$$\sin x = 0 + x.1 + 0 + \frac{x^3}{3!}(-1) + 0 + \ldots$$
$$+ 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \ldots$$

or $\quad \mathbf{\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + (-1)^m \dfrac{x^{2m+1}}{(2m+1)!} + \ldots}$ **(Meerut 1985)**

Similarly, we may obtain the *Cosine series* :

$$\mathbf{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + \ldots}$$

3. Expansion of log (1 + x). **(Meerut 1976, 89; Luck. 83)**

Let $f(x) = \log(1 + x)$. Then $f(0) = \log 1 = 0$;

$$f^n(x) = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$$

so that $\quad f^n(0) = (-1)^{n-1}(n-1)!$, where $n = 1, 2, 3, \ldots$

$\therefore \quad f'(0) = (-1)^{1-1}(1-1)! = 1$,

$f''(0) = (-1)^{2-1}(2-1)! = -1!$,

$f'''(0) = (-1)^{3-1}(3-1)! = 2!$,

$f^{iv}(0) = (-1)^{4-1}(4-1)! = 3!$, and so on,

Substituting the values of $f(0)$, $f'(0)$, $f''(0)$, etc. in Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots,$$

we get $\quad \log(1 + x) = 0 + x.1 - \dfrac{x^2}{2!}\cdot 1! + \dfrac{x^3}{3!}\cdot 2! - \dfrac{x^4}{4!}\cdot 3! + \ldots$

$$+ \frac{x^n}{n!}(-1)^{n-1}(n-1)! + \ldots$$

or $$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

Note. The function $\log x$ does not possess Maclaurin's series expansion because it is not defined at $x = 0$.

4. Expansion of $(1 + x)^n$. (*Binomial series*).

Let $f(x) = (1+x)^n$. Then $f(0) = 1$;

$$f^m(x) = n(n-1)(n-2)\ldots(n-m+1)(1+x)^{n-m}$$

so that $f^m(0) = n(n-1)\ldots(n-m+1)$, where $m = 1, 2, 3, \ldots$

$\therefore$ $f'(0) = n, f''(0) = n(n-1), f'''(0) = n(n-1)(n-2)$ and so on.

Substituting the values of $f(0), f'(0), f''(0)$ etc. in Maclaurin's series for $f(x)$, we get

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-m+1)}{m!}x^m + \ldots$$

Solved Examples

Ex. 1. *Expand the following by Maclaurin's theorem :*

(i) a^x

(ii) $\tan x$ **(Meerut 1981; Vikram 87; Agra 81; Kashmir 84)**

(iii) $\log \sec x$ **(Rohilkhand 1991; Meerut 82, 86, 88, 95, 98)**

(iv) $\log \cos x$ **(v)** $\sec x$

(vi) $\log(\sec x + \tan x)$.

Sol. (i) Let $f(x) = a^x$. Then $f(0) = a^0 = 1$,

$f'(x) = a^x \log a$ so that $f'(0) = \log a$,

$f''(x) = a^x (\log a)^2$ so that $f''(0) = (\log a)^2$,

$f'''(x) = a^x (\log a)^3$ so that $f'''(0) = (\log a)^3$, and so on.

In general, $f^n(x) = a^x (\log a)^n$ so that $f^n(0) = (\log a)^n$.

Now by Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots + \frac{x^n}{n!}f^n(0) + \ldots$$

$$\therefore \quad a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots + \frac{x^n}{n!}(\log a)^n + \ldots$$

(ii) Let $y = \tan x$. Then $(y)_0 = \tan 0 = 0$,

$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ so that

$$(y_1)_0 = 1 + (y)_0^2 = 1 + 0 = 1,$$

$y_2 = 2yy_1$ so that $(y_2)_0 = 2\,(y)_0\,(y_1)_0 = 2 \times 0 \times 1 = 0,$

$y_3 = 2y_1y_1 + 2yy_2 = 2y_1^2 + 2yy_2$ so that $(y_3)_0 = 2 \times 1^2 + 0 = 2,$

$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$ so that

$$(y_4)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$

$y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6y_2^2 + 8y_1y_3 + 2yy_4$ so that

$$(y_5)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \frac{x^4}{4\,!}(y_4)_0 + \frac{x^5}{5\,!}(y_5)_0 + \ldots$$

$$\therefore \quad \tan x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot 2 + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot 16 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

(iii) Let $y = \log \sec x$. Then $(y)_0 = \log \sec 0 = \log 1 = 0,$

$y_1 = \dfrac{1}{\sec x}\cdot \sec x \tan x = \tan x$ so that $(y_1)_0 = 0,$

$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2$ so that $(y_2)_0 = 1 + (y_1)_0^2 = 1,$

$y_3 = 2y_1y_2$ so that $(y_3)_0 = 2\,(y_1)_0\,(y_2)_0 = 0,$

$y_4 = 2y_2^2 + 2y_1y_3$ so that $(y_4)_0 = 2 \times 1^2 + 0 = 2,$

$y_5 = 4y_2y_3 + 2y_2y_3 + 2y_1y_4 = 6y_2y_3 + 2y_1y_4$ so that

$$(y_5)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$

$y_6 = 6y_3^2 + 6y_2y_4 + 2y_2y_4 + 2y_1y_5 = 6y_3^2 + 8y_2y_4 + 2y_1y_5$ so that

$$(y_6)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \frac{x^4}{4\,!}(y_4)_0 + \ldots$$

$$\therefore \quad \log \sec x = 0 + x\,.\,0 + \frac{x^2}{2\,!}\cdot 1 + \frac{x^3}{3\,!}\cdot 0 + \frac{x^4}{4\,!}\cdot 2 + \frac{x^5}{5!}\cdot 0$$

$$+ \frac{x^5}{6\,!}\cdot 16 + \ldots$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$$

(iv) Let $y = \log \cos x$. Then $(y)_0 = 0,$

$y_1 = \dfrac{-\sin x}{\cos x} = -\tan x \quad$ so that $(y_1)_0 = 0,$

$y_2 = -\sec^2 x = -(1 + \tan^2 x) = -(1 + y_1^2) = -1 - y_1^2$

so that $(y_2)_0 = -1 - 0 = -1,$

$y_3 = -2y_1y_2$ so that $(y_3)_0 = 0,$

$y_4 = -2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = -2(-1)^2 - 0 = -2$,
$(y_5)_0 = 0$, $(y_6)_0 = -16$, and so on.

[For calculation of $(y_5)_0$ and $(y_6)_0$ proceed as in part (iii)]

Now substituting these values in Maclaurin's theorem, we get

$$\log \cos x = 0 + x \,.\, 0 + \frac{x^2}{2!}\cdot(-1) + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot(-2) + \frac{x^5}{5!}\cdot 0 + (x^6/6!)\,.\,(-16) + \ldots$$

$$= -(x^2/2) - (x^4/12) - (x^6/45) + \ldots$$

(v) Let $y = \sec x$. Then $(y)_0 = \sec 0 = 1$,

$y_1 = \sec x \tan x$ so that $(y_1)_0 = 1 \times 0 = 0$,

$y_2 = \sec x \sec^2 x + \sec x \tan x \tan x = \sec^3 x + \sec x \tan^2 x$

$= \sec^3 x + \sec x(\sec^2 x - 1) = 2\sec^3 x - \sec x = 2y^3 - y$

so that $(y_2)_0 = 2 \times 1^3 - 1 = 1$,

$y_3 = 6y^2y_1 - y_1$ so that $(y_3)_0 = 0 - 0 = 0$,

$y_4 = 6y^2y_2 + 12yy_1^2 - y_2$ so that $(y_4)_0 = 6 \times 1^2 \times 1 + 0 - 1 = 5$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots$$

$$\therefore \quad \sec x = 1 + x\,.\,0 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}\cdot 5 + \ldots$$

$$= 1 + (x^2/2!) + (5x^4/4!) + \ldots$$

(vi) Let $y = \log(\sec x + \tan x)$. Then $(y)_0 = \log(1 + 0) = 0$,

$$y_1 = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \text{ so that } (y_1)_0 = \sec 0 = 1,$$

$y_2 = \sec x \tan x$ so that $(y_2)_0 = 1 \times 0 = 0$,

$y_3 = \sec x \sec^2 x + \sec x \tan^2 x = \sec^3 x + \sec x(\sec^2 x - 1)$

$= 2\sec^3 x - \sec x = 2y_1^3 - y_1$ so that $(y_3)_0 = 2 \times 1^3 - 1 = 1$,

$y_4 = 6y_1^2y_2 - y_2$ so that $(y_4)_0 = 0$,

$y_5 = 6y_1^2y_3 + 12y_1y_2^2 - y_3$ so that $(y_5)_0 = 6 \times 1^2 \times 1 + 0 - 1 = 5$,

and so on.

Substituting these values in Maclaurin's theorem, we get

$$\log(\sec x + \tan x) = 0 + x\,.\,1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}\cdot 1 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 5 + \ldots$$

$$= x + (x^3/6) + (x^5/24) + \ldots$$

**Ex. 2. *Use Maclaurin's formula to show that*

$$e^x \sec x = 1 + x + \frac{2x^2}{2\,!} + \frac{4x^3}{3\,!} + \dots$$

(Garhwal 1983; Kanpur 80; Gorakhpur 83)

Sol. Let $y = e^x \sec x$. Then $(y)_0 = 1$,

$y_1 = e^x \sec x + e^x \sec x \tan x = y + y \tan x$ so that $(y_1)_0 = 1$,

$y_2 = y_1 + y_1 \tan x + y \sec^2 x$ so that $(y_2)_0 = 1 + 0 + 1 = 2$,

$y_3 = y_2 + y_2 \tan x + 2y_1 \sec^2 x + 2y \sec^2 x \tan x$ so that

$(y_3)_0 = 2 + 2 = 4$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^x \sec x = 1 + x + \frac{2x^2}{2\,!} + \frac{4x^3}{3\,!} + \dots$$

Ex. 3. *Expand the following functions by Maclaurin's theorem :*

***(i)** $e^{\sin x}$ **(Meerut 1990S; Gorakhpur 86; G.N.U. 81; K.U. 89)**

(ii) $e^{x \cos x}$

(iii) $e^x \log (1 + x)$

(iv) $\log (1 + \sin x)$ **(Gorakhpur 1989; Kanpur 86, Meerut 92)**

Sol. (i) Let $y = e^{\sin x}$. Then $(y)_0 = e^{\sin 0} = e^0 = 1$,

$y_1 = e^{\sin x} \cos x = y \cos x$ so that $(y_1)_0 = (y)_0 \cos 0 = 1 \times 1 = 1$,

$y_2 = y_1 \cos x - y \sin x$ so that $(y_2)_0 = 1 \times 1 - 1 \times 0 = 1$,

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$\quad = y_2 \cos x - 2y_1 \sin x - y \cos x$ so that $(y_3)_0 = 1 - 0 - 1 = 0$,

$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x$ so that $(y_4)_0 = -3$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \frac{x^4}{4\,!}(y_4)_0 + \dots$$

$$\therefore \quad e^{\sin x} = 1 + x\,.\,1 + \frac{x^2}{2\,!} \cdot 1 + \frac{x^3}{3\,!} \cdot 0 + \frac{x^4}{4\,!} \cdot (-3) + \dots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots$$

(ii) Let $y = e^{x \cos x}$. Then $(y)_0 = e^0 = 1$,

$$y_1 = e^{x \cos x} (\cos x - x \sin x) = y\,(\cos x - x \sin x)$$

so that $(y_1)_0 = 1\,.\,(1 - 0) = 1$,

$y_2 = y_1 (\cos x - x \sin x) + y\,(-2 \sin x - x \cos x)$ giving

$(y_2)_0 = 1\,.\,(1 - 0) + 1\,.\,0 = 1$,

$y_3 = y_2 (\cos x - x \sin x) + y_1 (-2 \sin x - x \cos x)$

$\quad + y_1 (-2 \sin x - x \cos x) + y\,(-3 \cos x + x \sin x)$

$$= y_2(\cos x - x\sin x) + 2y_1(-2\sin x - x\cos x) + y(-3\cos x + x\sin x)$$

so that $(y_3)_0 = 1.(1-0) + 0 + 1.(-3) = -2$,

$$y_4 = y_3(\cos x - x\sin x) + 3y_2(-2\sin x - x\cos x) + 3y_1(-3\cos x + x\sin x) + y(4\sin x + x\cos x)$$

giving $(y_4)_0 = -2.(1-0) + 0 + 3.1.(-3+0) + 0 = -11$,

$$y_5 = y_4(\cos x - x\sin x) + 4y_3(-2\sin x - x\cos x) + 6y_2(-3\cos x + x\sin x) + 4y_1(4\sin x + x\cos x) + y(5\cos x - x\sin x)$$

so that $(y_5)_0 = -11 + 6.1.(-3) + 5 = -24$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^{x\cos x} = 1 + x.1 + \frac{x^2}{2!}\cdot 1 + \frac{x^3}{3!}\cdot(-2) + \frac{x^4}{4!}\cdot(-11) + \frac{x^5}{5!}\cdot(-24) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} + \ldots$$

(iii) Let $y = e^x \log(1+x)$. Then $(y)_0 = e^0 \log 1 = 1 \times 0 = 0$,

$$y_1 = e^x \log(1+x) + e^x(1+x)^{-1} = y + e^x(1+x)^{-1}$$

so that $(y_1)_0 = 0 + 1 = 1$,

$y_2 = y_1 + e^x(1+x)^{-1} - e^x(1+x)^{-2}$ so that $(y_2)_0 = 1 + 1 - 1 = 1$,

$$y_3 = y_2 + e^x(1+x)^{-1} - e^x(1+x)^{-2} - e^x(1+x)^{-2} + 2e^x(1+x)^{-3}$$

$$= y_2 + e^x(1+x)^{-1} - 2e^x(1+x)^{-2} + 2e^x(1+x)^{-3}$$

so that $(y_3)_0 = 1 + 1 - 2 + 2 = 2$,

$y_4 = y_3 + e^x(1+x)^{-1} - 3e^x(1+x)^{-2} + 6e^x(1+x)^{-3} - 6e^x(1+x)^{-4}$ so that $(y_4)_0 = 2 + 1 - 3 + 6 - 6 = 0$,

$y_5 = y_4 + e^x(1+x)^{-1} - 4e^x(1+x)^{-2} + 12e^x(1+x)^{-3} - 24e^x(1+x)^{-4} + 24e^x(1+x)^{-5}$ so that

$(y_5)_0 = 0 + 1 - 4 + 12 - 24 + 24 = 9$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$e^x \log(1+x) = 0 + x.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!}\cdot 2 + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\cdot 9 + \ldots$$

$$= x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{9x^5}{5!} + \ldots$$

(iv) Let $y = \log(1 + \sin x)$. Then $(y)_0 = 0$,

$y_1 = \dfrac{\cos x}{1 + \sin x}$ so that $(y_1)_0 = 1$,

$$y_2 = \frac{-\sin x\,(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$$= -\frac{1}{1+\sin x} \text{ so that } (y_2)_0 = -1,$$

$$y_3 = \frac{\cos x}{(1+\sin x)^2} = \frac{\cos x}{1+\sin x} \cdot \frac{1}{1+\sin x} = -y_1 y_2$$

so that $(y_3)_0 = -1 \,.\, (-1) = 1$,

$y_4 = -y_1 y_3 - y_2^2$ so that $(y_4)_0 = -1 \,.\, 1 - (-1)^2 = -1 - 1 = -2$,

$y_5 = -y_1 y_4 - y_2 y_3 - 2y_2 y_3 = -y_1 y_4 - 3y_2 y_3$ so that

$(y_5)_0 = -1 \,.\, (-2) - 3 \,.\, (-1) \,.\, 1 = 2 + 3 = 5$, and so on.

Substituting these values in Maclaurin's theorem, we get

$$\log(1+\sin x) = 0 + x \,.\, 1 + (x^2/2!) \,.\, (-1) + (x^3/3!) \,.\, 1 + (x^4/4!) \,.\, (-2) + (x^5/5!) \,.\, 5 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + \ldots$$

Ex. 4. *Expand by Maclaurin's theorem $e^x/(1+e^x)$ as far as the term x^3.*

Sol. Let $y = \dfrac{e^x}{1+e^x} = \dfrac{1+e^x-1}{1+e^x} = 1 - \dfrac{1}{1+e^x}$.

Then $(y)_0 = \dfrac{e^0}{1+e^0} = \frac{1}{2}$,

$y_1 = 0 + \dfrac{e^x}{(1+e^x)^2} = \dfrac{e^x}{1+e^x}\,\dfrac{1}{1+e^x} = y\,(1-y) = y - y^2$ so that

$(y_1)_0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$,

$y_2 = y_1 - 2yy_1$ so that $(y_2)_0 = \frac{1}{4} - 2 \,.\, \frac{1}{2} \,.\, \frac{1}{4} = 0$,

$y_3 = y_2 - 2y_1^2 - 2yy_2$ so that $(y_3)_0 = 0 - 2 \,.\, (1/4)^2 - 0 = -1/8$,

and so on.

Substituting these values in Maclaurin's theorem, we get

$$\frac{e^x}{1+e^x} = \frac{1}{2} + x \,.\, \frac{1}{4} + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-1/8) + \ldots$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{48}x^3 + \ldots$$

Ex. 5. *Apply Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e(1+e^x)$ to the term containing x^4.*

(Gorakhpur 1981; Kanpur 88; Ranchi 84)

Sol. Let $y = \log_e(1+e^x)$. Then $(y)_0 = \log_e(1+e^0) = \log_e 2$,

$$y_1 = \frac{e^x}{1+e^x} = \frac{(1+e^x)-1}{1+e^x} = 1 - \frac{1}{1+e^x} \text{ so that } (y_1)_0 = 1 - \tfrac{1}{2} = \tfrac{1}{2},$$

$$y_2 = 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)} \cdot \frac{1}{1+e^x} = y_1(1-y_1) = y_1 - y_1^2$$

so that $(y_2)_0 = \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$,

$y_3 = y_2 - 2y_1y_2$ so that $(y_3)_0 = \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = 0$,

$y_4 = y_3 - 2y_2^2 - 2y_1y_3$ so that $(y_4)_0 = 0 - 2 \cdot (\frac{1}{4})^2 - 0 = -1/8$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \ldots$$

$\therefore \quad \log(1+e^x)$

$$= \log 2 + x \cdot \tfrac{1}{2} + \frac{x^2}{2!} \cdot \tfrac{1}{4} + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-1/8) + \ldots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \ldots$$

Ex. 6. *Find the first four terms in the expansion of* $\log(1 + \tan x)$ *in powers of* x. **(Magadh 1986)**

Sol. Let $y = \log(1 + \tan x)$. Then $(y)_0 = \log(1 + \tan 0) = 0$.

Now $e^y = 1 + \tan x$. Differentiating both sides w.r.t. x, we get

$$e^y y_1 = \sec^2 x. \quad \ldots(1)$$

Putting $x = 0$ on both sides of (1), we get

$$e^0 (y_1)_0 = 1 \text{ or } (y_1)_0 = 1.$$

Differentiating (1), we get

$$e^y y_1^2 + e^y y_2 = 2\sec^2 x \tan x$$

or $$e^y(y_1^2 + y_2) = 2\sec^2 x \tan x. \quad \ldots(2)$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 0 \quad \text{or} \quad (y_2)_0 = -1.$$

Differentiating (2), we get

$$e^y y_1(y_1^2 + y_2) + e^y(2y_1y_2 + y_3) = 2\sec^4 x + 4\sec^2 x \tan^2 x$$

or $$e^y(y_1^3 + 3y_1y_2 + y_3) = 2\sec^4 x + 4\sec^2 x \tan^2 x. \quad \ldots(3)$$

Putting $x = 0$ in (3), we get

$$1 + 3 \cdot 1 \cdot (-1) + (y_3)_0 = 2 \text{ or } (y_3)_0 = 4.$$

Differentiating (3), we get

$$e^y y_1(y_1^3 + 3y_1y_2 + y_3) + e^y(3y_1^2y_2 + 3y_2^2 + 3y_1y_3 + y_4)$$
$$= 8\sec^4 x \tan x + 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x$$

or $$e^y(y_1^4 + 6y_1^2y_2 + 4y_1y_3 + 3y_2^2 + y_4)$$
$$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x. \quad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$1 + 6 \cdot 1 \cdot (-1) + 4 \cdot 1 \cdot 4 + 3 \cdot (-1)^2 + (y_4)_0 = 0$$

or $\quad (y_4)_0 + 14 = 0$ or $(y_4)_0 = -14.$

Now substituting these values in Maclaurin's theorem, we get

$$\log(1+\tan x) = 0 + x \cdot 1 + \frac{x^2}{2!}\cdot(-1) + \frac{x^3}{3!}\cdot 4 + \frac{x^4}{4!}\cdot(-14) + \ldots$$

$$= x - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{7}{12}x^4 + \ldots$$

Ex. 7. *Apply Maclaurin's theorem to obtain terms upto x^4 in the expansion of $\log(1+\sin^2 x)$.* **(Rohilkhand 1979)**

Sol. Let $y = \log(1+\sin^2 x)$. Then $(y)_0 = 0$.

Now $\quad e^y = 1 + \sin^2 x.$

Differentiating, we get

$$e^y y_1 = 2\sin x\cos x = \sin 2x. \qquad \ldots(1)$$

Putting $x = 0$ in (1), we get

$$e^0 (y_1)_0 = 0 \text{ or } (y_1)_0 = 0.$$

Differentiating (1), we get

$$e^y (y_1^2 + y_2) = 2\cos 2x. \qquad \ldots(2)$$

Putting $x = 0$ in (2), we get

$$(y_2)_0 = 2.$$

Differentiating (2), we get

$$e^y [(y_1^2 + y_2) y_1 + 2y_1 y_2 + y_3] = -4\sin 2x$$

or $$e^y (y_1^3 + 3y_1 y_2 + y_3) = -4\sin 2x. \qquad \ldots(3)$$

Putting $x = 0$ in (3), we get

$$(y_3)_0 = 0.$$

Differentiating (3), we get

$$e^y y_1 (y_1^3 + 3y_1 y_2 + y_3) + e^y (3y_1^2 y_2 + 3y_2^2 + 3y_1 y_3 + y_4) = -8\cos 2x$$

or $$e^y (y_1^4 + 6y_1^2 y_2 + 4y_1 y_3 + 3y_2^2 + y_4) = -8\cos 2x. \qquad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$3 \cdot 2^2 + (y_4)_0 = -8 \text{ or } (y_4)_0 = -20.$$

Now substituting these values in Maclaurin's theorem, we get

$$\log(1+\sin^2 x) = 0 + x \cdot 0 + \frac{x^2}{2!}\cdot 2 + \frac{x^3}{3!}\cdot 0 + \frac{x^4}{4!}(-20) + \ldots$$

$$= x^2 - \tfrac{5}{6}x^4 + \ldots$$

****Ex. 8.** *Show that*

(i) $$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \ldots + 2^{n/2}\cos\frac{n\pi}{4}\cdot\frac{x^n}{n!} + \ldots$$

(Meerut 1974, 96; Kanpur 85; Rohilkhand 88, 89; Jhansi 88)

(ii) $e^x \sin x = x + x^2 - \frac{2}{3!}x^3 - \frac{2^2}{5!}x^5 - \dots + \sin(\frac{1}{4}n\pi)\frac{2^{n/2}}{n!}x^n + \dots$

(Meerut 1983, 95, 96 BP; Avadh 87; Lucknow 82; Gorakhpur 81; Agra 80, 84, 89)

Sol. **(i)** Let $y = e^x \cos x$. Then $(y)_0 = e^0 \cos 0 = 1$,

$y_1 = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$

so that $(y_1)_0 = 1(1 - 0) = 1$,

$y_2 = e^x (\cos x - \sin x) + e^x (-\sin x - \cos x) = -2e^x \sin x$

so that $(y_2)_0 = 0$,

$y_3 = -2e^x \sin x - 2e^x \cos x = -2e^x (\sin x + \cos x)$

so that $(y_3)_0 = -2$,

$y_4 = -2e^x (\sin x + \cos x) - 2e^x (\cos x - \sin x)$

$= -4e^x \cos x = -2^2 y$ so that $(y_4)_0 = -2^2$,

$y_5 = -2^2 y_1$ so that $(y_5)_0 = -2^2$,

$y_6 = -2^2 y_2$ so that $(y_6)_0 = 0$,

$y_7 = -2^2 y_3$ so that $(y_7)_0 = 2^3$, and so on.

In general

$y_n = (1 + 1)^{n/2} \cos(x + n \tan^{-1} 1) = 2^{n/2} \cos(x + n\pi/4)$

so that $(y_n)_0 = 2^{n/2} \cos(\frac{1}{4}n\pi)$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \dots + \frac{x^n}{n!}(y_n)_0 + \dots$$

$$= 1 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!} \cdot (-2^2) + \frac{x^5}{5!} \cdot (-2^2)$$

$$+ \frac{x^6}{6!} \cdot 0 + \frac{x^7}{7!} \cdot 2^3 + \dots + \frac{x^n}{n!} 2^{n/2} \cos(\tfrac{1}{4}n\pi) + \dots$$

$$= 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} - \frac{2^2x^5}{5!} + \frac{2^3x^7}{7!} + \dots + 2^{n/2} \cos(\tfrac{1}{4}n\pi)\frac{x^n}{n!} + \dots$$

(ii) Proceed as in part (i).

****Ex. 9.** *Apply Maclaurin's theorem to prove that*

(i) $e^{ax} \sin bx = bx + abx^2 + \frac{3a^2b - b^3}{3!}x^3 + \dots$

$$+ \frac{(a^2 + b^2)^{n/2}}{n!} x^n \sin\{n \tan^{-1}(b/a)\} + \dots$$

(Meerut 1984)

(ii) $e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2}x^2 + \frac{a(a^2 - 3b^2)}{3!}x^3 + \dots$

$$+ \frac{(a^2 + b^2)^{n/2}}{n!} x^n \cos\{n \tan^{-1}(b/a)\} + \dots$$

(Meerut 1998, 84; 83S, 77; Gorakhpur 89; Allahabad 84; Kurukshetra 82; Magadh 83; Delhi 88; G.N.U. 85)

Hence deduce that

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2!} \cos 2\alpha + \frac{x^3}{3!} \cos 3\alpha + \ldots$$

(Kanpur 1982)

Sol. (i) Let $y = e^{ax} \sin bx$. Then $(y)_0 = e^0 \sin 0 = 0$,

$y_1 = ae^{ax} \sin bx + be^{ax} \cos bx = ay + be^{ax} \cos bx$

so that $(y_1)_0 = b$,

$y_2 = ay_1 + abe^{ax} \cos bx - b^2 e^{ax} \sin bx$

$= ay_1 - b^2 y + abe^{ax} \cos bx$ so that $(y_2)_0 = ab - 0 + ab = 2ab$,

$y_3 = ay_2 - b^2 y_1 + a^2 be^{ax} \cos bx - ab^2 e^{ax} \sin bx$

$= ay_2 - b^2 y_1 - ab^2 y + a^2 be^{ax} \cos bx$

so that $(y_3)_0 = 2a^2 b - b^3 + a^2 b = 3a^2 b - b^3$, and so on.

In general,

$y_n = (a^2 + b^2)^{n/2} \sin \{bx + n \tan^{-1} (b/a)\}$

so that $(y_n)_0 = (a^2 + b^2)^{n/2} \sin \{n \tan^{-1} (b/a)\}$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!} \cdot (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \ldots + \frac{x^n}{n!} (y_n)_0 + \ldots$$

$$= 0 + \frac{x}{1!} \cdot b + \frac{x^2}{2!} \cdot (2ab) + \frac{x^3}{3!} (3a^2 b - b^3) + \ldots$$

$$+ \frac{x^n}{n!} (a^2 + b^2)^{n/2} \sin \{n \tan^{-1} (b/a)\} + \ldots$$

$$= bx + abx^2 + \frac{3a^2 b - b^3}{3!} x^3 + \ldots$$

$$+ \frac{(a^2 + b^2)^{n/2}}{n!} x^n \sin \{n \tan^{-1} (b/a)\} + \ldots$$

(ii) Let $y = e^{ax} \cos bx$. Then $(y)_0 = e^0 \cos 0 = 1$,

$y_1 = ae^{ax} \cos bx - be^{ax} \sin bx = ay - be^{ax} \sin bx$ so that $(y_1)_0 = a$,

$y_2 = ay_1 - abe^{ax} \sin bx - b^2 e^{ax} \cos bx = ay_1 - b^2 y - abe^{ax} \sin bx$

so that $(y_2)_0 = a^2 - b^2$,

$y_3 = ay_2 - b^2 y_1 - a^2 be^{ax} \sin bx - ab^2 e^{ax} \cos bx$

$= ay_2 - b^2 y_1 - ab^2 y - a^2 be^{ax} \sin bx$

so that $(y_3)_0 = a (a^2 - b^2) - b^2 a - ab^2 = a (a^2 - 3b^2)$, and so on.

In general, $y_n = (a^2 + b^2)^{n/2} \cos \{bx + n \tan^{-1} (b/a)\}$ so that

$(y_n)_0 = (a^2 + b^2)^{n/2} \cos \{n \tan^{-1} (b/a)\}$.

Substituting these values in Maclaurin's theorem, we get

$$e^{ax}\cos bx = 1 + ax + \frac{a^2 - b^2}{2\,!}x^2 + \frac{a\,(a^2 - 3b^2)}{3\,!}x^3 + \dots$$
$$+ \frac{(a^2 + b^2)^{n/2}}{n\,!}x^n \cos\{n \tan^{-1}(b/a)\} + \dots$$

Deduction. Putting $a = \cos\alpha$ and $b = \sin\alpha$, we get
$(y_n)_0 = (\cos^2\alpha + \sin^2\alpha)^{n/2}\cos(n\tan^{-1}\tan\alpha) = \cos n\alpha$ so that
$(y_1)_0 = \cos\alpha$, $(y_2)_0 = \cos 2\alpha$, $(y_3)_0 = \cos 3\alpha$, etc.

$$\therefore\quad e^{x\cos\alpha}\cos(x\sin\alpha)$$
$$= 1 + x\cos\alpha + (x^2/2\,!)\cos 2\alpha + (x^3/3\,!)\cos 3\alpha + \dots$$

Ex. 10. *Prove that*

$$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \dots$$

(Rohilkhand 1980, 87S)

Sol. Let $y = \sin^{-1}x$. Then $y_1 = 1/\sqrt{(1 - x^2)}$

or $\quad (1 - x^2)\,y_1^2 - 1 = 0.$

Differentiating again, we get

$$(1 - x^2)\,2y_1y_2 - 2xy_1^2 = 0$$

or $\quad (1 - x^2)\,y_2 - xy_1 = 0$, since $2y_1 \neq 0$.

Now differentiating n times by Leibnitz's theorem, we get

$$(1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - n^2y_n = 0.$$

Putting $x = 0$ in the above relations, we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

and $\quad (y_{n+2})_0 = n^2\,(y_n)_0. \qquad \dots(1)$

Putting $n = 1, 2, 3, \dots$ in (1), we get
$(y_3)_0 = 1^2\,(y_1)_0 = 1^2$, $(y_4)_0 = 2^2\,(y_2)_0 = 0$, $(y_5)_0 = 3^2\,(y_3)_0 = 3^2 . 1^2$,
$(y_6)_0 = 4^2\,(y_4)_0 = 0$, $(y_7)_0 = 5^2\,(y_5)_0 = 5^2 . 3^2 . 1^2$, and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \dots$$

$$\therefore\quad \sin^{-1}x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot 1^2 + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot 3^2 . 1^2$$
$$+ \frac{x^6}{6\,!}\cdot 0 + \frac{x^7}{7\,!}\cdot 5^2 . 2^2 . 1^2 + \dots$$

$$= x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \dots$$

Ex. 11. *Expand $\tan^{-1}x$ by Maclaurin's theorem. Write also the general term.* **(Delhi 1981; Bundelkhand 82; Kashmir 87)**

Sol. Let $y = \tan^{-1}x$. Proceeding as in Ex. 36 on page 50, we get

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0,$$

and $$(y_{n+2})_0 = -\{(n+1)\,n\}\,(y_n)_0. \qquad ...(1)$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$(y_3)_0 = -(2.1)(y_1)_0 = -2\,!,\ (y_4)_0 = -(3.2)(y_2)_0 = 0,$$
$$(y_5)_0 = -(4.3)(y_3)_0 = -(4.3).(-2\,!) = 4\,!, \text{ etc.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots$$

$$\therefore \quad \tan^{-1} x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot(-2\,!) + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot(4\,!) + \ldots$$

$$= x - (x^3/3) + (x^5/5) - \ldots$$

The general term in this expansion is $(x^n/n\,!)\,(y_n)_0$.

So we need the value of $(y_n)_0$. Putting $(n-2)$ in place of n in (1), we get

$$(y_n)_0 = -\{(n-1)(n-2)\}(y_{n-2})_0$$
$$= [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)]\,(y_{n-4})_0.$$

Now there arise two cases :

Case I. When n is even, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)]\ldots[-(3)(2)]\,(y_2)_0$$
$$= 0, \text{ since } (y_2)_0 = 0.$$

Case II. When n is odd, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}]\,[-\{(n-3)(n-4)\}]\ldots [-(4)(3)]\,[-(2)(1)]\,(y_1)_0$$
$$= (-1)^{(n-1)/2}(n-1)\,!, \text{ since } (y_1)_0 = 1.$$

Thus in the expansion of $\tan^{-1} x$, the coefficient of x^n is 0 if n is even and is $\dfrac{(-1)^{(n-1)/2}(n-1)\,!}{n\,!}$ *i.e.*, $\dfrac{(-1)^{(n-1)/2}}{n}$ if n is odd.

$$\therefore \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{[(2n-1)-1]/2}}{2n-1}x^{2n-1} + \ldots$$
$$= x - \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \ldots$$

Ex. 12. *Expand* $\log\{1 - \log(1-x)\}$ *in powers of* x *by Maclaurin's theorem as far as the term* x^3.

By substituting $x/(1+x)$ *for* x *deduce the expansion of* $\log\{1 + \log(1+x)\}$ *as far as the term in* x^3.

Sol. Let $y = \log\{1 - \log(1-x)\}$. Then $(y)_0 = 0$.

Now $e^y = 1 - \log(1-x)$. Differentiating, we get

$$e^y y_1 = (1-x)^{-1}. \qquad ...(1)$$

Putting $x = 0$ in (1), we get $(y_1)_0 = 1$.

Differentiating (1), we get

$$e^y y_1^2 + e^y y_2 = (1-x)^{-2}$$

or $$e^y (y_1^2 + y_2) = (1-x)^{-2} \qquad \text{...(2)}$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 1 \quad \text{or} \quad (y_2)_0 = 0.$$

Differentiating (2), we get

$$e^y (y_1^3 + 3y_1 y_2 + y_3) = 2(1-x)^{-3} \qquad \text{...(3)}$$

Putting $x = 0$ in (3), we get

$$1 + (y_3)_0 = 2 \quad \text{or} \quad (y_3)_0 = 1.$$

Substituting these values in Maclaurin's theorem, we get

$$\log\{1 - \log(1-x)\} = 0 + x \, . \, 1 + \frac{x^2}{2\,!} \cdot 0 + \frac{x^3}{3\,!} \cdot 1 + \ldots$$

$$= x + (x^3/6) + \ldots \qquad \text{...(A)}$$

Now substituting $x/(1+x)$ for x on both sides of (A), we get

$$\log\left\{1 - \log\left(1 - \frac{x}{1+x}\right)\right\} = \frac{x}{1+x} + \frac{1}{6}\left(\frac{x}{1+x}\right)^3 + \ldots$$

or $$\log\{1 + \log(1+x)\} = x(1+x)^{-1} + (1/6)x^3(1+x)^{-3} + \ldots$$

$$= x\left\{1 + (-1)x + \frac{(-1)(-2)}{1\,.\,2}x^2 + \ldots\right\} + \frac{1}{6}x^3\{1 + (-3)x + \ldots\},$$

on expanding by binomial theorem

$$= (x - x^2 + x^3 + \ldots) + (\tfrac{1}{6}x^3 + \ldots)$$

$$= x - x^2 + \tfrac{7}{6}x^3 + \ldots$$

Ex. 13. *Expand $\{x + \sqrt{(1+x^2)}\}^m$ in ascending powers of x and find the general term also.* **(Agra 1978; Kanpur 78)**

Sol. Let $y = \{x + \sqrt{(1+x^2)}\}^m$. Proceeding as in Ex. 39 (a) on page 52, we get

$$(y)_0 = 1, (y_1)_0 = m, (y_2)_0 = m^2,$$

and $$(y_{n+2})_0 = (m^2 - n^2)(y_n)_0. \qquad \text{...(1)}$$

Now putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$(y_3)_0 = (m^2 - 1^2)(y_1)_0 = (m^2 - 1^2)\,m,$$

$$(y_4)_0 = (m^2 - 2^2)(y_2)_0 = (m^2 - 2^2)\,m^2,$$

$$(y_5)_0 = (m^2 - 3^2)(y_3)_0 = (m^2 - 3^2)(m^2 - 1^2)\,m,$$

$$(y_6)_0 = (m^2 - 4^2)(y_4)_0 = (m^2 - 4^2)(m^2 - 2^2)\,m^2, \text{ etc.}$$

In general,

if n is odd, $(y_n)_0 = \{m^2 - (n-2)^2\} \ldots (m^2 - 3^2)(m^2 - 1^2)\,m$

and if n is even, $(y_n)_0 = \{m^2 - (n-2)^2\} \ldots (m^2 - 4^2)(m^2 - 2^2)\,m^2$ ...(2)

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \dots + \frac{x^n}{n!}(y_n)_0 + \dots$$

$$\therefore \quad \{x + \sqrt{(1+x^2)}\}^m = 1 + mx + \frac{m^2}{2!}x^2 + \frac{m(m^2-1^2)}{3!}x^3 + \frac{m^2(m^2-2^2)}{4!}x^4 + \frac{m(m^2-1^2)(m^2-3^2)}{5!}x^5 + \dots$$

The general term $= \dfrac{x^n}{n!}(y_n)_0$, where $(y_n)_0$ is given by (2).

Ex. 14. *Expand log* $\{x + \sqrt{(1+x^2)}\}$ *in ascending powers of x and find the general term.* **(Meerut 1991; K.U. 85; Bihar 84)**

Sol. Let $y = \log\{x + \sqrt{(1+x^2)}\}$. Then

$$y_1 = \frac{1}{x + \sqrt{(1+x^2)}} \cdot \left\{1 + \frac{2x}{2\sqrt{(x^2+1)}}\right\} = \frac{1}{\sqrt{(x^2+1)}} \cdot$$

Therefore $y_1^2(x^2+1) - 1 = 0.$

Differentiating again, we get

$$(x^2+1)\,2y_1y_2 + 2xy_1^2 = 0$$

or $\quad (x^2+1)\,y_2 + xy_1 = 0$, since $2y_1 \neq 0$.

Now differentiating n times by Leibnitz's theorem, we get

$$(x^2+1)\,y_{n+2} + ny_{n+1}\,.\,2x + \frac{n(n-1)}{2!}y_n\,.\,2 + xy_{n+1} + ny_n = 0$$

or $\quad (x^2+1)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0.$

Putting $x = 0$ in the above relations, we get

$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0$, and

$$(y_{n+2})_0 = -n^2(y_n)_0. \qquad \dots(1)$$

Now putting $n = 1, 3, 5, \dots$ in (1), we get

$(y_3)_0 = -1^2(y_1)_0 = -1^2, (y_5)_0 = (-3^2)(y_3)_0 = (-3^2)(-1^2) = 3^2\,.\,1^2,$
$(y_7)_0 = (-5^2)(y_5)_0 = (-5^2)(-3^2)(-1^2) = -5^2\,.\,3^2\,.\,1^2$, etc.

In general, if n is odd, we have

$$(y_n)_0 = \{-(n-2)^2\}\{-(n-4)^2\}\dots(-5^2)(-3^2)(-1^2)$$
$$= (-1)^{(n-1)/2}(n-2)^2(n-4)^2\dots5^2\,.\,3^2\,.\,1^2. \qquad \dots(2)$$

Again, putting $n = 2, 4, 6 \dots$ in (1), we get

$(y_4)_0 = -2^2(y_2)_0 = 0, (y_6)_0 = -4^2(y_4)_0 = 0$, etc.

Thus, if n is even, we have $(y_n)_0 = 0$.

Now by Maclaurin's theorem, we have

$$\log[x + \sqrt{(1+x^2)}] = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$$
$$= 0 + x\,.\,1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}(-1^2) + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}(3^2\,.\,1^2) + \dots$$

$$= x - \frac{x^3}{3!} 1^2 + \frac{x^5}{5!}(3^2 . 1^2) - \frac{x^7}{7!}(5^2 . 3^2 . 1^2) + \ldots$$

The general term $= (x^n/n!) (y_n)_0$, where $(y_n)_0$ is given by (2) when n is odd and $(y_n)_0 = 0$, when n is even.

Putting $2n - 1$ in place of n in (2), we find that

$$(y_{2n-1})_0 = (-1)^{n-1} (2n-3)^2 \ldots 5^2 . 3^2 . 1^2.$$

Thus $\quad \log [x + \sqrt{(1 + x^2)}]$

$$= x - 1^2 . \frac{x^3}{3!} + 1^2 . 3^2 . \frac{x^5}{5!} - 1^2 . 3^2 . 5^2 . \frac{x^7}{7!} + \ldots$$

$$+ (-1)^{(n-1)} 1^2 . 3^2 . 5^2 \ldots (2n-3)^2 . \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

****Ex. 15.** *Expand $e^{a \sin^{-1} x}$ by Maclaurin's theorem and find the general term. Hence show that*

$$e^\theta = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots$$

(Meerut 1982, 84, 94; Allahabad 87; Rohilkhand 78; Agra 77; Magadh 77; Ranchi 75; Indore 73; Lucknow 83, 79; Kanpur 87, 89)

Sol. Let $y = e^{a \sin^{-1} x}$. Proceeding as in Ex. 37 on page 51, we get

$$y(0) = 1, y_1(0) = a, y_2(0) = a^2,$$

and $\quad y_{n+2}(0) = (n^2 + a^2) y_n(0)$. ...(1)

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$y_3(0) = (1^2 + a^2) y_1(0) = (1^2 + a^2) a, y_4(0) = (2^2 + a^2) y_2(0)$$
$$= (2^2 + a^2) a^2, y_5(0) = (3^2 + a^2) y_3(0) = (3^2 + a^2)(1^2 + a^2) a,$$
$$y_6(0) = (4^2 + a^2) y_4(0) = (4^2 + a^2)(2^2 + a^2) a^2, \text{ etc.}$$

In general,

$$y_n(0) = \begin{cases} a(1^2 + a^2)(3^2 + a^2) \ldots [(n-2)^2 + a^2] \text{ if } n \text{ is odd} \\ a^2 (2^2 + a^2)(4^2 + a^2) \ldots [(n-2)^2 + a^2] \text{ if } n \text{ is even.} \end{cases}$$

Substituting these values in Maclaurin's expansion

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots,$$

we get

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{2!} x^2 + \frac{a(1^2 + a^2)}{3!} x^3 + \frac{a^2 (2^2 + a^2)}{4!} x^4 + \ldots \quad \ldots(2)$$

The general term is $(x^n/n!) y_n(0)$, where $y_n(0)$ is as given above.

Now putting $x = \sin\theta$ and $a = 1$ in (2), we get

$$e^\theta = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots$$

Ex. 16. *Use Maclaurin's theorem to show that*

$$e^{m \cos^{-1} x} = e^{m\pi/2}\left[1 - mx + \frac{m^2}{2!}x^2 - \frac{m(1^2+m^2)}{3!}x^3 + \frac{m^2(2^2+m^2)}{4!}x^4 - \ldots\right]$$

(Agra 1983; Delhi 82)

Sol. Proceed as in Ex. 15.

Ex. 17. *Expand sin $(m \sin^{-1} x)$ by Maclaurin's theorem as far as x^5. Hence expand sin $m\theta$ in powers of sin θ.*

(Meerut 1978, 93; G.N.U. 72; Kanpur 88; Rohilkhand 83, 85, 90; Lucknow 75)

Sol. Let $y = \sin(m \sin^{-1} x)$. Proceeding as in Ex. 34 on page 49, we get

$$y(0) = 0, y_1(0) = m, y_2(0) = 0,$$

and $$y_{n+2}(0) = (n^2 - m^2)\, y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$y_3(0) = (1^2 - m^2)\, y_1(0) = (1^2 - m^2)\, m,$$

$$y_4(0) = (2^2 - m^2)\, y_2(0) = 0,$$

$$y_5(0) = (3^2 - m^2)\, y_3(0) = (3^2 - m^2)(1^2 - m^2)\, m, \text{ etc.}$$

Substituting these values in Maclaurin's theorem, we get

$$\sin(m \sin^{-1} x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots$$

$$= mx + \frac{m(1^2 - m^2)}{3!}x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}x^5 + \ldots$$

Putting $x = \sin\theta$ on both sides, we get

$$\sin m\theta = m \sin\theta + \frac{m(1^2 - m^2)}{3!}\sin^3\theta + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!}\sin^5\theta + \ldots$$

Ex. 18. *Show that*

$$(\sin^{-1} x)^2 = \frac{2}{2!}x^2 + \frac{2.2^2}{4!}x^4 + \frac{2.2^2.4^2}{6!}x^6 + \ldots$$

(Meerut 1983, 84, 86)

Deduce that

$$\theta^2 = 2 \cdot \frac{\sin^2\theta}{2!} + 2^2 \cdot \frac{2\sin^4\theta}{4!} + 2^2 . 4^2 \frac{2\sin^6\theta}{6!} + \ldots$$

(Rajasthan 1987; Meerut 86 P)

Sol. Let $y = (\sin^{-1} x)^2$. Proceeding as in Ex. 32 on page 47, we get

$$y(0) = 0, y_1(0) = 0, y_2(0) = 2,$$

and $$y_{n+2}(0) = n^2 y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$y_3(0) = 1^2 y_1(0) = 0, y_4(0) = 2^2 y_2(0) = 2^2 . 2,$$
$$y_5(0) = 3^2 y_3(0) = 0, y_6(0) = 4^2 y_4(0) = 4^2 . 2^2 . 2, \text{ etc.}$$

Hence by Maclaurin's theorem, we get

$$(\sin^{-1} x)^2 = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \dots$$
$$= \frac{2}{2!} x^2 + \frac{2.2^2}{4!} x^4 + \frac{2.2^2 . 4^2}{6!} x^6 + \dots$$

Now putting $\sin^{-1} x = \theta$ or $x = \sin\theta$ on both sides, we get the required expansion for θ^2.

Ex. 19. (a) *By Maclaurin's theorem or otherwise find the expansion of $y = \sin(e^x - 1)$ upto and including the term in x^4.*

(Gorakhpur 1982, 88)

(b) *Also show that $x = y - \frac{1}{2} y^2 + \dots$* **(Allahabad 1973)**

Sol. (a) Let $y = \sin(e^x - 1)$. Then $(y)_0 = \sin 0 = 0$,

$$y_1 = [\cos(e^x - 1)]. e^x \text{ so that } (y_1)_0 = (\cos 0). e^0 = 1,$$
$$y_2 = [\cos(e^x - 1)]. e^x - [\sin(e^x - 1)] . e^{2x} = y_1 - ye^{2x}$$
$$\text{so that } (y_2)_0 = (y_1)_0 - (y)_0 e^0 = 1 - 0 = 1,$$
$$y_3 = y_2 - y_1 e^{2x} - 2y e^{2x} \text{ so that } (y_3)_0 = 1 - 1 - 0 = 0,$$
$$y_4 = y_3 - y_2 e^{2x} - 4y_1 e^{2x} - 4y e^{2x} \text{ so that } (y_4)_0 = -5, \text{ etc.}$$

Hence by Maclaurin's theorem, we get

$$\sin(e^x - 1) = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$$
$$= 0 + x . 1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-5) + \dots$$
$$= x + \tfrac{1}{2} x^2 - \tfrac{5}{24} x^4 + \dots$$

(b) We have

$$y = \sin(e^x - 1) \Rightarrow e^x - 1 = \sin^{-1} y \Rightarrow e^x = 1 + \sin^{-1} y. \quad \dots(1)$$

Differentiating (1) w.r.t. 'y', we get

$$e^x . x_1 = 1/\sqrt{(1 - y^2)}, \text{ where } x_1 = dx/dy. \quad \dots(2)$$

From (2), we get $(1 - y^2) x_1^2 = e^{-2x}$.

Differentiating it w.r.t. 'y', we get

$$(1 - y^2) 2x_1 x_2 - 2yx_1^2 = e^{-2x}(-2x_1)$$

or $\quad (1 - y^2) x_2 - yx_1 = -e^{-2x}$, since $2x_1 \neq 0$. $\quad \dots(3)$

From (1), we have $x = \log(1 + \sin^{-1} y)$. $\quad \dots(4)$

Now putting $y = 0$ in (4), (2) and (3), we get

$(x)_0 = \log(1 + 0) = 0, e^0 . (x_1)_0 = 1/\sqrt{(1 - 0)}$ giving $(x_1)_0 = 1$,

$(x_2)_0 = -e^0 = -1$. [Note that $(x)_{y=0} = 0$]

Hence by Maclaurin's theorem, we get

$$x = (x)_0 + y\,(x_1)_0 + (y^2/2\,!)\,(x_2)_0 + \dots$$
$$= 0 + y\,.\,1 + (y^2/2\,!)\,(-1) + \dots = y - \tfrac{1}{2}y^2 + \dots$$

***Ex. 20. (a)** *If* $y = (\sin^{-1} x)/\sqrt{(1-x^2)}$, *where* $-1 < x < 1$ *and* $-\pi/2 < \sin^{-1} x < \pi/2$,

prove that $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$

Also if $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$,

prove that $(n+1)\,a_{n+1} = n\,a_{n-1}$ *and hence obtain the general term of the expansion.* **(Lucknow 1976)**

(b) *Expand* $(\sin^{-1} x)/\sqrt{(1-x^2)}$ *in powers of x upto three terms.*

Sol. (a) Here $y = (\sin^{-1} x)/\sqrt{(1-x^2)}$. ...(1)

$\therefore$ $y^2 (1-x^2) = (\sin^{-1} x)^2$.

Differentiating w.r.t. x, we get

$$2yy_1 (1-x^2) - 2xy^2 = 2\,(\sin^{-1} x)/\sqrt{(1-x^2)} = 2y.$$

Since $2y \neq 0$, therefore $y_1 (1-x^2) - xy = 1$

i.e. $y_1(1-x^2) - xy - 1 = 0.$ ...(2)

Differentiating (2) n times by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2) + ny_n(-2x) + \frac{n(n-1)}{2!}\,y_{n-1}\cdot(-2) - x\,y_n - n\,y_{n-1} = 0$$

or $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$ ...(3)

Now putting $x = 0$ in (1), (2) and (3), we get

$(y)_0 = 0,\ (y_1)_0 = 1$ and $(y_{n+1})_0 = n^2\,(y_{n-1})_0$. ...(4)

By Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + (x^2/2\,!)\,(y_2)_0 + \dots + (x^n/n\,!)\,(y_n)_0 + \dots$$

Also we are given that

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

Comparing the coefficients of x^n in the two expansions for y, we get

$$a_n = (y_n)_0 / n\,!.$$

$$\therefore \quad \frac{a_{n+1}}{a_{n-1}} = \frac{(y_{n+1})_0}{(n+1)\,!} + \frac{(y_{n-1})_0}{(n-1)\,!} = \frac{(y_{n+1})_0}{(y_{n-1})_0}\cdot\frac{(n-1)\,!}{(n+1)\,!}$$

$$= n^2 \cdot \frac{1}{n\,(n+1)}, \qquad \text{from (4)}$$

$$= \frac{n}{n+1}.$$

$\therefore$ $(n+1)\,a_{n+1} = n\,a_{n-1}$ **Proved.**

or $$a_{n+1} = \frac{n}{n+1}\,a_{n-1}. \qquad ...(5)$$

Now $a_0 = (y)_0 = 0$, $a_1 = (y_1)_0 = 1$. Putting $n = 1, 3, 5, \ldots$ in (5), we get $a_2 = \frac{1}{2} a_0 = 0$, $a_4 = \frac{3}{4} a_2 = 0$, $a_6 = \frac{5}{6} a_4 = 0$, etc.

Thus $a_n = 0$ if n is even *i.e.*, $a_{2m} = 0$.

Again putting $n = 2, 4, 6, \ldots$ in (5), we get

$$a_3 = \tfrac{2}{3} a_1 = \tfrac{2}{3}, a_5 = \tfrac{4}{5} \cdot a_3 = \tfrac{4}{5} \cdot \tfrac{2}{3}, a_7 = \tfrac{6}{7} a_5 = \tfrac{6}{7} \cdot \tfrac{4}{5} \cdot \tfrac{2}{3}, \text{ etc.}$$

In general, if n is odd, we have

$$a_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

Thus $$a_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

(b) As found in part (a), we have $a_0 = 0$, $a_1 = 1$, $a_2 = 0$, $a_3 = 2/3$, $a_4 = 0$, $a_5 = \frac{2}{3} \cdot \frac{4}{5}$, etc.

$$\therefore \quad \frac{\sin^{-1} x}{\sqrt{(1-x^2)}} = x + \tfrac{2}{3} x^3 + \frac{2.4}{3.5} x^5 + \ldots$$

Ex. 20. (c) *If* $y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots$, *prove that* $(n+1)(n+2)\, a_{n+2} = n^2 a_n$. **(Meerut 1990 P)**

Sol. Let $y = \sin^{-1} x$. ...(1)

Then $$y_1 = \frac{1}{\sqrt{(1-x^2)}}. \qquad \text{...(2)}$$

$$\therefore \quad (1-x^2) y_1^2 - 1 = 0.$$

Differentiating again, we get

$$(1-x^2)\, 2y_1 y_2 - 2xy_1^2 = 0 \text{ or } 2y_1 [(1-x^2) y_2 - xy_1] = 0$$

or $$(1-x^2) y_2 - xy_1 = 0, \qquad \text{...(3)}$$

since $2y_1 \neq 0$.

Now differentiating (3) n times by Leibnitz's theorem, we get

$$(1-x^2) y_{n+2} + n \,.\, y_{n+1} \cdot (-2x) + \frac{n(n-1)}{1.2} y_n \cdot (-2) - y_{n+1} \,.\, x - ny_n \,.\, 1 = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0. \qquad \text{...(4)}$$

Putting $x = 0$ in (4), we get

$$(y_{n+2})_0 = n^2 (y_n)_0. \qquad \text{...(5)}$$

By Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!} (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \ldots + \frac{x^n}{n!} (y_n)_0 + \ldots$$

Also we are given that

$$y = \sin^{-1} x = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

Equating the coefficients of x^n in the two expansions for y, we get

$$a_n = \frac{(y_n)_0}{n\,!}\,.$$

$$\therefore \quad \frac{a_{n+2}}{a_n} = \frac{(y_{n+2})_0}{(n+2)\,!} \cdot \frac{n\,!}{(y_n)_0} = \frac{(y_{n+2})_0}{(y_n)_0} \cdot \frac{1}{(n+2)\,(n+1)}$$

$$= \frac{n^2}{(n+2)\,(n+1)}, \text{ substituting for } \frac{(y_{n+2})_0}{(y_n)_0} \text{ from (5).}$$

Hence $(n+1)\,(n+2)\,a_{n+2} = n^2\,a_n$.

Ex. 20 (d). *If $y = \sin\log(x^2 + 2x + 1)$, prove that*

$$(x+1)^2 y_{n+2} + (2n+1)\,(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as for as x^6.

(Allahabad 1989; Agra 85)

Sol. Here $y = \sin\log(x^2 + 2x + 1) = \sin\log(x+1)^2$ ...(1)

$$\therefore \quad y_1 = [\cos\log(x+1)^2] \cdot \frac{1}{(x+1)^2} \cdot 2\,(x+1)$$

$$= [\cos\log(x+1)^2] \cdot \frac{2}{x+1}\,. \quad \text{...(2)}$$

Squaring both sides of (2), we get

$$(x+1)^2 y_1^2 = 4\cos^2\log(x+1)^2 = 4\,[1 - \sin^2\log(x+1)^2]$$

$$= 4\,(1 - y^2)$$

or $(x+1)^2 y_1^2 + 4y^2 - 4 = 0.$ ...(3)

Differentiating (3), we get

$$(x+1)^2\,2y_1 y_2 + 2\,(x+1)\,y_1^2 + 8yy_1 = 0$$

or $2y_1\,[(x+1)^2 y_2 + (x+1)\,y_1 + 4y] = 0$

or $(x+1)^2 y_2 + (x+1)\,y_1 + 4y = 0,$ ...(4)

since $2y_1 \neq 0$.

Differentiating (4) n times by Leibnitz's theorem, we get

$$(x+1)^2 y_{n+2} + {}^nC_1 \cdot y_{n+1} \cdot 2\,(x+1) + {}^nC_2 \cdot y_n \cdot 2$$

$$+ (x+1)\,y_{n+1} + {}^nC_1 \cdot y_n \cdot 1 + 4y_n = 0$$

or $(x+1)^2 y_{n+2} + (2n+1)\,(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0.$ ...(5)

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 2,\ (y_2)_0 + (y_1)_0 + 4(y)_0 = 0 \text{ or } (y_2)_0 = -2.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 + (2n+1)\,(y_{n+1})_0 + (n^2+4)\,(y_n)_0 = 0$$

or $(y_{n+2})_0 = -\,[(2n+1)\,(y_{n+1})_0 + (n^2+4)\,(y_n)_0]$. ...(6)

Now putting $n = 1, 2, 3, 4$ in (6), we get

$$(y_3)_0 = -\,[3\,(y_2)_0 + 5\,(y_1)_0] = -\,[3\,.\,(-2) + 5\,.\,2] = -4,$$

$$(y_4)_0 = -\,[5\,(y_3)_0 + 8\,(y_2)_0] = -\,[5\,.\,(-4) + 8\,.\,(-2)] = 36,$$

$(y_5)_0 = -[7(y_4)_0 + 13(y_3)_0] = -[7.(36) + 13.(-4)] = -200,$

and $(y_6)_0 = -[9(y_5)_0 + 20(y_4)_0] = -[9.(-200) + 20.(36)] = 1080.$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \dots$$

$$= 0 + \frac{x}{1}\cdot 2 + \frac{x^2}{2}\cdot(-2) + \frac{x^3}{1.2.3}\cdot(-4) + \frac{x^4}{1.2.3.4}\cdot 36$$

$$+ \frac{x^5}{1.2.3.4.5}\cdot(-200) + \frac{x^6}{1.2.3.4.5.6}\cdot(1080) + \dots$$

$$= 2x - x^2 - \frac{2}{3}x^3 + \frac{3}{2}x^4 - \frac{5}{3}x^5 + \frac{3}{2}x^6 + \dots$$

Examples on Taylor's Series

Ex. 21. *Expand* $\sin^{-1}(x+h)$ *in powers of* x *as far as the term in* x^3.

(Ranchi 1971; Kashmir 71)

Sol. First we observe that we are to expand $\sin^{-1}(x+h)$ in ascending powers of x. So let $f(h) = \sin^{-1} h$. Then

$$f(h+x) = \sin^{-1}(h+x).$$

Thus we are to expand $f(h+x)$ in powers of x. So by Taylor's theorem, we have

$$f(h+x) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \dots \quad \dots(1)$$

Now $f(h) = \sin^{-1} h$. Therefore

$$f'(h) = \frac{1}{\sqrt{(1-h^2)}} = (1-h^2)^{-1/2},$$

$$f''(h) = h(1-h^2)^{-3/2},$$

$$f'''(h) = (1-h^2)^{-3/2} + h(-3/2)(1-h^2)^{-5/2}(-2h)$$

$$= (1-h^2)^{-3/2} + 3h^2(1-h^2)^{-5/2} = (1-h^2)^{-5/2}[(1-h^2) + 3h^2]$$

$$= (1-h^2)^{-5/2}(1+2h^2), \text{ etc.}$$

Substituting these values in (1), we have

$$\sin^{-1}(h+x) = \sin^{-1} h + (1-h^2)^{-1/2}x + (x^2/2!)h(1-h^2)^{-3/2}$$
$$+ (x^3/3!)(1-h^2)^{-5/2}(1+2h^2) + \dots$$

Ex. 22. *Show that* $\log(x+h) = \log h + \dfrac{x}{h} - \dfrac{x^2}{2h^2} + \dfrac{x^3}{3h^3} - \dots$

(Rohilkhand 1981; Gorakhpur 80; Vikram 75; Indore 72)

Sol. First we observe that we are to expand $\log(x+h)$ in ascending pwoers of x. So let $f(h) = \log h$. Then

$$f(h+x) = \log(h+x).$$

Now proceed in Ex. 21.

Here $\quad f(h) = \log h, f'(h) = 1/h, f''(h) = -1/h^2,$
$\quad f'''(h) = 2/h^3, \dots$ etc.

Substituting these values in Taylor's expansion, we get

$$\log(h+x)=\log h+\frac{x}{h}-\frac{x^2}{2h^2}+\frac{x^3}{3h^3}-\ldots$$

***Ex. 23.** *Expand log sin (x + h) in powers of h by Taylor's theorem.*

(Agra 1982, 88; Kanpur 79; Meerut 98)

Sol. First we observe that we are to expand $\log\sin(x+h)$ in powers of h. So let $f(x)=\log\sin x$. Then

$$f(x+h)=\log\sin(x+h).$$

Expanding $f(x+h)$ by Taylor's theorem in powers of h, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots \qquad \ldots(1)$$

Now $f(x)=\log\sin x$. Therefore $f'(x)=\dfrac{1}{\sin x}\cdot\cos x=\cot x$,

$$f''(x)=-\operatorname{cosec}^2 x,\ f'''(x)=2\operatorname{cosec}^2 x\cot x,\text{ etc.}$$

Substituting these values in (1), we get

$$\log\sin(x+h)=\log\sin x+h\cot x-(h^2/2!)\operatorname{cosec}^2 x + (2h^3/3!)\operatorname{cosec}^2 x\cot x+\ldots$$

Ex. 23. (a) *Prove that*

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{2!}\sin x-\ldots$$

(Gorakhpur 1987)

Sol. First we observe that we are to expand $\sin(x+h)$ in powers of h. So let $f(x)=\sin x$. Then

$$f(x+h)=\sin(x+h).$$

Expanding $f(x+h)$ by Taylor's theorem in powers of h, we have

$$\sin(x+h)=f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$$

$$=\sin x+h\cos x+\frac{h^2}{2!}(-\sin x)+\frac{h^3}{3!}(-\cos x)+\ldots$$

$$=\sin x+h\cos x-\frac{h^2}{2!}\sin x-\frac{h^3}{3!}\cos x+\ldots$$

Ex. 24. *Use Taylor's theorem to prove that*

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\,\frac{\sin\theta}{1}-(h\sin\theta)^2\frac{\sin 2\theta}{2}+(h\sin\theta)^3\frac{\sin 3\theta}{3}-\ldots+(-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n}+\ldots,$$

where $\theta=\cot^{-1}x$. **(Allahabad 1980; Rohilkhand 86, 88; Jhansi 88; Lucknow 74; Kanpur 71; Meerut 94P)**

Sol. First we observe that we are to expand $\tan^{-1}(x+h)$ in ascending powers of h. So let $f(x)=\tan^{-1}x$. Then

$f(x+h)=\tan^{-1}(x+h)$. Expanding $f(x+h)$ in powers of h by Taylor's theorem we have

$$f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\ldots+\frac{h^n}{n!}f^n(x)+\ldots \qquad \ldots(1)$$

Now $f(x)=\tan^{-1}x$. Therefore $f^n(x)=D^n\tan^{-1}x$
$=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$, where $\theta=\cot^{-1}x$.

[See Ex. 5 on page 28].

Putting $n=1,2,3,\ldots$ in it, we get

$f'(x)=\sin\theta\sin\theta, f''(x)=-1!\sin^2\theta\sin 2\theta,$
$f'''(x)=2!\sin^3\theta\sin 3\theta$, etc.

Substituting these values in (1), we have

$$\tan^{-1}(x+h)=\tan^{-1}x+h\sin\theta\sin\theta-\frac{h^2}{2!}\sin^2\theta\sin 2\theta$$
$$+\frac{h^3}{3!}2!\sin^3\theta\sin 3\theta-\ldots+\frac{h^n}{n!}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta+\ldots$$
$$=\tan^{-1}x+h\sin\theta\frac{\sin\theta}{1}-(h\sin\theta)^2\frac{\sin 2\theta}{2}$$
$$+(h\sin\theta)^3\frac{\sin 3\theta}{3}-\ldots+(-1)^{n-1}(h\sin\theta)^n\frac{\sin n\theta}{n}+\ldots$$

Ex. 25. *Prove that $f(mx)$ is equal to*

$$f(x)+(m-1)xf'(x)+(1/2!)(m-1)^2x^2f''(x)+\ldots.$$

(Kanpur 1988; Agra 86)

Sol. First we observe that we are to expand $f(mx)$ in powers of $(m-1)x$. We can write $f(mx)=f\{x+(m-1)x\}$.

Expanding $f\{x+(m-1)x\}$ in powers of $(m-1)x$ by Taylor's theorem, we get

$$f\{x+(m-1)x\}=f(x)+(m-1)xf'(x)$$
$$+(1/2!)(m-1)^2x^2f''(x)+\ldots$$

$$\therefore \quad f(mx)=f(x)+(m-1)xf'(x)+(1/2!)(m-1)^2x^2f''(x)+\ldots$$

Ex. 26. *Prove that*

(i) $f\left(\frac{x^2}{1+x}\right)=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x)^2}\frac{1}{2!}f''(x)-\ldots$

(Rohilkhand 1987)

(ii) $f(x)=f(0)+xf'(x)-\frac{x^2}{2!}f''(x)+\frac{x^3}{3!}f'''(x)-\ldots$

Sol. (i) First we observe that we are to expand $f\left(\frac{x^2}{1+x}\right)$ in powers of $\left(-\frac{x}{1+x}\right)$. We can write $f\left(\frac{x^2}{1+x}\right)=f\left[x+\left(-\frac{x}{1+x}\right)\right]$.

Now expanding $f\left[x+\left(-\frac{x}{1+x}\right)\right]$ by Taylor's theorem in powers of $-\frac{x}{1+x}$, we get

$$f\left(\frac{x^2}{1+x}\right)=f(x)+\left(-\frac{x}{1+x}\right)f'(x)+\frac{1}{2!}\left(-\frac{1}{1+x}\right)^2 f''(x)+\ldots$$

$$=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x)^2}\frac{1}{2!}f''(x)-\ldots$$

(ii) We can write $f(0)=f[x+(-x)]$. Now expanding $f[x+(-x)]$ by Taylor's theorem in powers of $-x$, we get

$$f(0)=f[x+(-x)]=f(x)+(-x)f'(x)+\frac{(-x)^2}{2!}f''(x)+\frac{(-x)^3}{3!}f'''(x)+\ldots$$

or $$f(0)=f(x)-xf'(x)+\frac{x^2}{2!}f''(x)-\frac{x^3}{3!}f'''(x)+\ldots.$$

$$\therefore\quad f(x)=f(0)+xf'(x)-\frac{x^2}{2!}f''(x)+\frac{x^3}{3!}f'''(x)-\ldots,$$

by transposition.

Ex. 27. *Expand* $2x^3+7x^2+x-1$ *in powers of* $(x-2)$.

(Meerut 1981 S; Jhansi 89)

Sol. Let $f(x)=2x^3+7x^2+x-1$. We can write

$$f(x)=f[2+(x-2)].$$

Now expanding $f[2+(x-2)]$ by Taylor's theorem in powers of $x-2$, we get

$$f(x)=f[2+(x-2)]=f(2)+(x-2)f'(2)+(1/2!)(x-2)^2f''(2)+\ldots \quad \ldots(1)$$

Now $f(x)=2x^3+7x^2+x-1$

so that $f(2)=2.2^3+7.2^2+2-1=45, f'(x)=6x^2+14x+1$

so that $f'(2)=53, f''(x)=12x+14$ so that $f''(2)=38$,

$f'''(x)=12$ so that $f'''(2)=12$,

$f^{iv}(x)=0$ so that $f^{iv}(2)=0$. Obviously $f^n(2)=0$ when $n\geq 4$. Now substituting these values in (1), we get

$$f(x)=45+(x-2).53+\frac{(x-2)^2}{2!}\cdot 38+\frac{(x-2)^3}{3!}\cdot 12$$

$$=45+53(x-2)+19(x-2)^2+2(x-2)^3.$$

****Ex. 28.** *Expand* $\sin x$ *in powers of* $(x-\frac{1}{2}\pi)$.

(Rohilkhand 1982; Agra 84; Kanpur 80, 79; Vikram 77; Meerut 88 P; Gorakhpur 76)

Sol. Let $f(x)=\sin x$. We want to expand $f(x)$ in powers of $x-\frac{1}{2}\pi$. We can write $f(x)=f[\frac{1}{2}\pi+(x-\frac{1}{2}\pi)]$. Now expanding $f[\frac{1}{2}\pi+(x-\frac{1}{2}\pi)]$ by Taylor's theorem in powers of $(x-\frac{1}{2}\pi)$, we get

$$f(x) = f[\tfrac{1}{2}\pi + (x - \tfrac{1}{2}\pi)] = f(\pi/2) + (x - \tfrac{1}{2}\pi) f'(\pi/2)$$
$$+ \frac{1}{2!}(x - \tfrac{1}{2}\pi)^2 f''(\pi/2) + \frac{1}{3!}(x - \tfrac{1}{2}\pi)^3 f'''(\pi/2) + \dots \quad \dots(1)$$

Now $f(x) = \sin x$. Therefore $f(\pi/2) = \sin \pi/2 = 1$,
$f'(x) = \cos x$ giving $f'(\pi/2) = \cos \pi/2 = 0$,
$f''(x) = -\sin x$ so that $f''(\pi/2) = -\sin \pi/2 = -1$,
$f'''(x) = -\cos x$ so that $f'''(\pi/2) = -\cos \pi/2 = 0$, etc.
Substituting these values in (1), we get

$$\sin x = 1 + (x - \tfrac{1}{2}\pi) . 0 + \frac{1}{2!}(x - \tfrac{1}{2}\pi)^2 . (-1)$$
$$+ \frac{1}{3!}(x - \tfrac{1}{2}\pi)^3 . 0 + \dots$$
$$= 1 - (1/2!)(x - \tfrac{1}{2}\pi)^2 + \dots$$

Ex. 29. *Expand e^x in powers of* $(x - 1)$.

Sol. Let $f(x) = e^x$. Then, we have

$$e^x = f(x) = f[1 + (x - 1)]$$

[$\because$ we are to expand $f(x)$ in powers of $(x - 1)$]

$$= f(1) + (x - 1) f'(1) + (1/2!)(x - 1)^2 f''(1)$$
$$+ (1/3!)(x - 1)^3 f'''(1) + \dots,$$

on expanding $f[1 + (x - 1)]$ by Taylor's theorem in powers of $(x - 1)$.

Now $f(x) = e^x$. Therefore $f(1) = e^1 = e$, $f'(x) = e^x$ so that $f'(1) = e, f''(x) = e^x$ so that $f''(1) = e$, and so on.

Substituting these values in the above expansion, we get

$$e^x = e + (x - 1) e + (1/2!)(x - 1)^2 e + (1/3!)(x - 1)^3 e + \dots$$
$$= e [1 + (x - 1) + (1/2!)(x - 1)^2 + (1/3!)(x - 1)^3 + \dots].$$

Ex. 30. *Expand $\tan^{-1} x$ in powers of* $(x - \tfrac{1}{4}\pi)$.

(Meerut 1980, 85, 96P)

Sol. Let $f(x) = \tan^{-1} x$. Then, we have

$$\tan^{-1} x = f(x) = f[\tfrac{1}{4}\pi + (x - \tfrac{1}{4}\pi)],$$

[$\because$ we have to expand $f(x)$ in powers of $(x - \tfrac{1}{4}\pi)$]

$$= f(\pi/4) + (x - \tfrac{1}{4}\pi) f'(\pi/4) + \frac{1}{2!}(x - \tfrac{1}{4}\pi)^2 f''(\pi/4) + \dots,$$

on expanding $f[\tfrac{1}{4}\pi + (x - \tfrac{1}{4}\pi)]$ by Taylor's theorem in powers of $(x - \tfrac{1}{4}\pi)$.

Now $f(x) = \tan^{-1} x$. Therefore

$$f(\pi/4) = \tan^{-1}(\pi/4), f'(x) = \frac{1}{1 + x^2}$$

so that $f'(\pi/4) = 1/(1 + \pi^2/16)$, $f''(x) = -2x/(1 + x^2)^2$ so that

$$f''(\pi/4) = -\pi/\{2(1+\pi^2/16)^2\} \text{ and so on.}$$

Substituting these values in the above expansion, we get

$$\tan^{-1} x = \tan^{-1}(\pi/4) + (x - \tfrac{1}{4}\pi)/(1+\pi^2/16)$$
$$- \pi(x - \tfrac{1}{4}\pi)^2/\{4(1+\pi^2/16)^2\} + \dots$$

Ex. 31. *Expand log sin x in powers of* $(x-a)$.

(Vikram 1988; Gurunanak 84)

Sol. Let $f(x) = \log \sin x$. We can write $f(x) = f[a + (x-a)]$. Expanding $f[a+(x-a)]$ by Taylor's theorem in powers of $(x-a)$, we get

$$f(x) = f(a) + (x-a) f'(a) + (1/2!)(x-a)^2 f''(a)$$
$$+ (1/3!)(x-a)^3 f'''(a) + \dots \quad \dots(1)$$

Now $f(x) = \log \sin x$. Therefore $f(a) = \log \sin a$,

$f'(x) = (1/\sin x) \cdot \cos x = \cot x$, giving $f'(a) = \cot a$

$f''(x) = -\operatorname{cosec}^2 x$ so that $f''(a) = -\operatorname{cosec}^2 a$,

$f'''(x) = 2\operatorname{cosec}^2 x \cot x$ so that $f'''(a) = 2\operatorname{cosec}^2 a \cot a$,

and so on.

Substituting these values in (1), we get

$$\log \sin x = \log \sin a + (x-a)\cot a - \frac{(x-a)^2}{2!}\operatorname{cosec}^2 a$$
$$+ \frac{(x-a)^3}{3!} 2\operatorname{cosec}^2 a \cot a + \dots$$

Ex. 32. *Expand log sin x in powers of* $(x-2)$.

(Meerut 1990, 91 P, 97, 98)

Sol. Proceed as in Ex. 31. Put $a = 2$.

□

4

Partial Differentiation

§ 1. Functions of two or more independent variables.

Functions in which several independent variables occur are very common. For example, the volume of a rectangular box depends upon three variables, *viz.* its length, breadth and depth. Similarly the area of a triangle depends upon two variables, *viz.*, the base and the altitude. In both these examples we observe that any of the variables may vary independently of the others.

A function of x, y and z may be written as $f(x, y, z)$ or $\phi(x, y, z)$, etc. Similarly, a function of x and y is generally denoted by the symbols $f(x, y)$ or $\phi(x, y)$ etc.

If a derivative of a function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a **partial derivative.** The operation of finding the partial derivatives of a function of more than one independent variables is called **partial differentiation.** The symbols $\partial/\partial x$, $\partial/\partial y$, etc., are used to denote such differentiations and the expressions $\partial u/\partial x$, $\partial u/\partial y$, etc., are respectively called the **partial differential coefficients** of u w.r.t. x, y, etc. Thus if $u = f(x, y, z)$, then the partial differential coefficient of u w.r.t. x *i.e.*, $\partial u/\partial x$ is obtained by differentiating u w.r.t. x keeping y and z as constants. Sometimes $\partial f/\partial x$ is also denoted by f_x.

Second order partial differential coefficients. If $u = f(x, y)$, then $\partial u/\partial x$ or f_x and $\partial u/\partial y$ or f_y are themselves functions of x and y and can be again differentiated partially.

We call $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right), \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right), \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$ and $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)$

as the second order partial derivatives of u and these are respectively denoted by $\dfrac{\partial^2 u}{\partial x\,\partial y}, \dfrac{\partial^2 u}{\partial y\,\partial x}, \dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial^2 u}{\partial y^2}$.

Note. If $u = f(x, y)$ and its partial derivatives are continuous (as is true in all ordinary cases), the order of differentiation is immaterial *i.e.* $\dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{x}\,\partial \mathbf{y}} = \dfrac{\partial^2 \mathbf{u}}{\partial \mathbf{y}\,\partial \mathbf{x}}$.

Solved Examples

Ex. 1. *Verify that* $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{\partial^2 z}{\partial y\,\partial x}$ *for the following functions :*

(i) $z = x \log y$ **(Kanpur 1988)** **(ii)** $z = \sin^{-1}(x/y)$
(iii) $z = a \tan^{-1}(x/y)$ **(iv)** $z = x^y + y^x$
(v) $z = e^{ax} \sin by$ **(vi)** $z = \log(y \sin x + x \sin y)$.

Sol. **(i)** We have $z = x \log y$.

Differentiating z partially w.r.t. x taking y as constant, we have $\partial z/\partial x = \log y$.

Now $$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(\log y) = \frac{1}{y} \cdot$$

Again differentiating z partially w.r.t. y taking x as constant, we have $\partial z/\partial y = x/y$.

Now $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{x}{y}\right) = \dfrac{1}{y}$, treating y as constant.

Hence $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

(ii) We have $z = \sin^{-1}(x/y)$.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{\sqrt{\{1-(x/y)^2\}}} \cdot \frac{1}{y} = \frac{1}{\sqrt{(y^2-x^2)}} = (y^2-x^2)^{-1/2}.$$

Now $$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\{(y^2-x^2)^{-1/2}\}$$

$$= -\tfrac{1}{2}(y^2-x^2)^{-3/2} \,.\, 2y = \frac{-y}{(y^2-x^2)^{3/2}} \cdot$$

Again $$\frac{\partial z}{\partial y} = \frac{1}{\sqrt{\{1-(x/y)^2\}}} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y}(y^2-x^2)^{-1/2},$$

and $$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= -\frac{1}{y}(y^2-x^2)^{-1/2} - \left(\frac{x}{y}\right) \cdot \{-\tfrac{1}{2}(y^2-x^2)^{-3/2}\}\,.\,(-2x)$$

$$= -\frac{1}{y(y^2-x^2)^{1/2}} - \frac{x^2}{y(y^2-x^2)^{3/2}} = -\frac{(y^2-x^2)+x^2}{y(y^2-x^2)^{3/2}}$$

$$= \frac{-y}{(y^2-x^2)^{3/2}} \cdot$$

Hence $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

(iii) We have $z = a \tan^{-1}(x/y)$.

$$\therefore \quad \frac{\partial z}{\partial x} = a\,\frac{1}{1+(x^2/y^2)} \cdot \frac{1}{y} = \frac{ay}{x^2+y^2},$$

and $$\frac{\partial^2 z}{\partial y\, \partial x} = a \cdot \frac{1 \cdot (x^2+y^2) - y\,.\,2y}{(x^2+y^2)^2} = \frac{a(x^2-y^2)}{(x^2+y^2)^2} \cdot$$

Again $$\frac{\partial z}{\partial y} = a \cdot \frac{1}{1+(x^2/y^2)} \cdot \left(-\frac{x}{y^2}\right) = \frac{-ax}{x^2+y^2},$$

and $$\frac{\partial^2 z}{\partial x\, \partial y} = -a \cdot \frac{1\,.\,(x^2+y^2) - x\,.\,2x}{(x^2+y^2)^2} = -a \cdot \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{a(x^2-y^2)}{(x^2+y^2)^2}$$

Thus $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

(iv) We have $z = x^y + y^x$.

$\therefore \quad \partial z/\partial x = yx^{y-1} + y^x . \log y$

and $\quad \dfrac{\partial^2 z}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}[yx^{y-1} + y^x \log y]$

$= 1 . x^{y-1} + yx^{y-1} . \log x + xy^{x-1} \log y + y^x . (1/y)$

$= x^{y-1}(1 + y \log x) + y^{x-1}(1 + x \log y)$.

Again $\dfrac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$ and $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}[x^y \log x + xy^{x-1}]$

$= yx^{y-1} . \log x + x^y . (1/x) + 1 . y^{x-1} + xy^{x-1} \log y$

$= x^{y-1}(1 + y \log x) + y^{x-1}(1 + x \log y)$.

Thus $\partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

(v) We have $z = e^{ax} \sin by$.

$\therefore \quad \partial z/\partial x = ae^{ax} \sin by$,

$\dfrac{\partial^2 z}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}(ae^{ax} \sin by) = ae^{ax} . b \cos by = ab\, e^{ax} \cos by.$

Again $\partial z/\partial y = e^{ax} . b \cos by$;

$\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}(be^{ax} \cos by) = ab\, e^{ax} \cos by.$

Thus $\partial^2 z/\partial y\, \partial x = \partial^2 z/\partial x\, \partial y$.

(vi) We have $z = \log(y \sin x + x \sin y)$.

$\therefore \quad \dfrac{\partial z}{\partial x} = \dfrac{y \cos x + \sin y}{y \sin x + x \sin y}$,

and $\partial^2 z/\partial y\, \partial x$

$$= \frac{(y \sin x + x \sin y)(\cos x + \cos y) - (y \cos x + \sin y)(\sin x + x \cos y)}{(y \sin x + x \sin y)^2}$$

$$= \frac{x \sin y \cos x + y \sin x \cos y - \sin x \sin y - xy \cos x \cos y}{(y \sin x + x \sin y)^2}.$$

Again $\quad \dfrac{\partial z}{\partial y} = \dfrac{\sin x + x \cos y}{y \sin x + x \sin y}$

and $\partial^2 z/\partial x\, \partial y$

$$= \frac{(y \sin x + x \sin y)(\cos x + \cos y) - (\sin x + x \cos y)(y \cos x + \sin y)}{(y \sin x + x \sin y)^2}$$

$$= \frac{x \sin y \cos x + y \sin x \cos y - \sin x \sin y - xy \cos x \cos y}{(y \sin x + x \sin y)^2}.$$

$\therefore \quad \partial^2 z/\partial x\, \partial y = \partial^2 z/\partial y\, \partial x$.

Ex. 2. *If* $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$, *find* $\dfrac{\partial^2 u}{\partial x\, \partial y}$.

(Andhra 1986; Meerut 89P; Kashmir 88; Utkal 89)

Sol. We have

$$\frac{\partial u}{\partial y} = x^2 . \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x} - 2y \tan^{-1}\frac{x}{y} - y^2 \frac{1}{1 + (x/y)^2} \cdot \left(-\frac{x}{y^2}\right)$$

$$= \frac{x^3}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y} = x - 2y \tan^{-1}\frac{x}{y}.$$

Now $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = 1 - 2y \dfrac{1}{1+(x/y)^2} \cdot \dfrac{1}{y} = 1 - \dfrac{2y^2}{x^2+y^2}$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}.$$

Ex. 3. *If $z = f(y/x)$, show that $x(\partial z/\partial x) + y(\partial z/\partial y) = 0.$*

(Garhwal 1983; Bundelkhand 82; U.P. P.C.S. 94)

Sol. We have

$\partial z/\partial x = [f'(y/x)](-y/x^2)$, (diff. partially w.r.t. x).

$\therefore \quad x(\partial z/\partial x) = -(y/x) f'(y/x).$ ...(1)

Again $\partial z/\partial y = [f'(y/x)] \cdot (1/x)$, (diff. partially w.r.t. y)

$\therefore \quad y(\partial z/\partial y) = (y/x) f'(y/x).$ ...(2)

Adding (1) and (2), we get $x(\partial z/\partial x) + y(\partial z/\partial y) = 0.$

****Ex. 4 (a)** *If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$*

(Meerut 1982S, 94P, 95, 96P; Agra 82; Allahabad 84; Gorakhpur 85; Lucknow 82; Kashmir 84; Rohilkhand 91)

Sol. Here $\dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{\{1-(x/y)^2\}}} \cdot \dfrac{1}{y} + \dfrac{1}{1+(y/x)^2} \cdot \left(-\dfrac{y}{x^2}\right)$,

(treating y as constant)

$$= \frac{1}{\sqrt{(y^2-x^2)}} - \frac{y}{(x^2+y^2)}.$$

$\therefore \quad x\dfrac{\partial u}{\partial x} = \dfrac{x}{\sqrt{(y^2-x^2)}} - \dfrac{xy}{x^2+y^2}.$ ...(1)

Again $\quad \dfrac{\partial u}{\partial y} = \dfrac{1}{\sqrt{\{1-(x/y)^2\}}}\left(-\dfrac{x}{y^2}\right) + \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x}$,

(treating x as constant)

$$= -\frac{x}{y\sqrt{(y^2-x^2)}} + \frac{x}{x^2+y^2}.$$

$\therefore \quad y\dfrac{\partial u}{\partial y} = -\dfrac{x}{\sqrt{(y^2-x^2)}} + \dfrac{xy}{x^2+y^2}.$ ...(2)

Adding (1) and (2), we have $x(\partial u/\partial x) + y(\partial u/\partial y) = 0.$

Ex. 4 (b) *Find the value of*

$\dfrac{1}{a^2}\dfrac{\partial^2 z}{\partial x^2} + \dfrac{1}{b^2}\dfrac{\partial^2 z}{\partial y^2}$ *when* $a^2x^2 + b^2y^2 - c^2z^2 = 0.$ **(Meerut 1991)**

Sol. We have $\quad z^2 = \dfrac{a^2}{c^2}x^2 + \dfrac{b^2}{c^2}y^2.$ ...(1)

Differentiating (1) partially with respect to x taking y as constant, we have

$$2z\frac{\partial z}{\partial x}=2\frac{a^2}{c^2}x \text{ or } \frac{\partial z}{\partial x}=\frac{a^2}{c^2}\cdot\frac{x}{z}.$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{a^2}{c^2}x\cdot\frac{1}{z}\right)=\frac{a^2}{c^2}\cdot\frac{1}{z}+\frac{a^2}{c^2}x\cdot\left(-\frac{1}{z^2}\right)\frac{\partial z}{\partial x}$$

$$=\frac{a^2}{c^2z}-\frac{a^2x}{c^2z^2}\cdot\left(\frac{a^2x}{c^2z}\right)=\frac{a^2}{c^2z}-\frac{a^4x^2}{c^4z^3}.$$

$$\therefore \quad \frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2z}-\frac{a^2x^2}{c^4z^3} \qquad \text{...(2)}$$

Again differentiating (1) partially with respect to y taking x as constant, we have

$$2z\frac{\partial z}{\partial y}=2\frac{b^2}{c^2}y \quad \text{or} \quad \frac{\partial z}{\partial y}=\frac{b^2}{c^2}\cdot\frac{y}{z}.$$

$$\therefore \quad \frac{\partial^2 z}{\partial y^2}=\frac{b^2}{c^2}\cdot\frac{1}{z}+\frac{b^2}{c^2}y\cdot\left(-\frac{1}{z^2}\right)\frac{\partial z}{\partial y}$$

$$=\frac{b^2}{c^2z}-\frac{b^2y}{c^2z^2}\cdot\left(\frac{b^2y}{c^2z}\right)=\frac{b^2}{c^2z}-\frac{b^4y^2}{c^4z^3}.$$

$$\therefore \quad \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2z}-\frac{b^2y^2}{c^4z^3}. \qquad \text{...(3)}$$

Adding (2) and (3), we get

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2}+\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}=\frac{2}{c^2z}-\frac{a^2x^2+b^2y^2}{c^4z^3}$$

$$=\frac{2}{c^2z}-\frac{c^2z^2}{c^4z^3}=\frac{2}{c^2z}-\frac{1}{c^2z}=\frac{1}{c^2z}.$$

Ex. 4 (c) *If* $u=\tan^{-1}\frac{xy}{\sqrt{(1+x^2+y^2)}}$, *show that*

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{3/2}}.$$

(Agra 1987)

Sol. We have

$$\frac{\partial u}{\partial x}=\frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\cdot$$

$$y\cdot\frac{1\cdot\sqrt{(1+x^2+y^2)}-x\cdot\frac{1}{2}(1+x^2+y^2)^{-1/2}(-2x)}{1+x^2+y^2}$$

$$=\frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\cdot y\cdot\frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$=\frac{y(1+y^2)}{(1+x^2)(1+y^2)(1+x^2+y^2)^{1/2}}=\frac{1}{1+x^2}\cdot\frac{y}{(1+x^2+y^2)^{1/2}}.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{1}{1+x^2} \cdot \frac{1 \,.\, (1+x^2+y^2)^{1/2} - y \,.\, \frac{1}{2}(1+x^2+y^2)^{-1/2} \,.\, 2y}{1+x^2+y^2}$$

$$= \frac{1}{1+x^2} \cdot \frac{1+x^2+y^2-y^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{1}{1+x^2} \cdot \frac{1+x^2}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$

***Ex. 5.** *If* $z = f(x+ay) + \phi(x-ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2 (\partial^2 z/\partial x^2)$.

(Rohilkhand 1981; Bihar 87; Gorakhpur 80; Meerut 90, 85)

Sol. We have $z = f(x+ay) + \phi(x-ay)$.

$\therefore \quad \partial z/\partial x = f'(x+ay) + \phi'(x-ay)$, (diff. partially w.r.t. 'x')

and $\quad \partial^2 z/\partial x^2 = f''(x-ay) + \phi''(x-ay)$. ...(1)

Again, $\partial z/\partial y = af'(x+ay) - a\phi'(x-ay)$.

$\therefore \quad \partial^2 z/\partial y^2 = a^2 f''(x+ay) + a^2 \phi''(x-ay)$. ...(2)

From (1) and (2), we get $\partial^2 z/\partial y^2 = a^2 (\partial^2 z/\partial x^2)$.

Ex. 6. *If* $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, *show that*

$x(\partial u/\partial x) + y(\partial u/\partial y) = \tan u$. **(Kanpur 1980; Allahabad 84)**

Sol. We have $\sin u = (x^2+y^2)/(x+y)$.

$\therefore \quad \log \sin u = \log(x^2+y^2) - \log(x+y)$. ...(1)

Differentiating (1) partially w.r.t. x, we get

$$\frac{1}{\sin u} \cos u \,.\, \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x+y}.$$

$$\therefore \quad (\cot u)\, x \frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y}. \qquad ...(2)$$

Again differentiating (1) partially w.r.t. y, we get

$$\frac{\cos u}{\sin u} \cdot \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{x+y}.$$

$$\therefore \quad (\cot u) \,.\, y \frac{\partial u}{\partial y} = \frac{2y^2}{x^2+y^2} - \frac{y}{x+y}. \qquad ...(3)$$

Adding (2) and (3), we get

$$(\cot u)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = \frac{2x^2+2y^2}{x^2+y^2} - \frac{x+y}{x+y} = 2 - 1 = 1.$$

$\therefore \quad x(\partial u/\partial x) + y(\partial u/\partial y) = 1/\cot u = \tan u$.

Ex. 7. *If* $z = xf(x+y) + y\phi(x+y)$, *prove that* $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) = 2(\partial^2 z/\partial x\, \partial y)$.

Sol. We have $z = xf(x+y) + y\phi(x+y)$.

$\therefore \quad \partial z/\partial x = f(x+y) + xf'(x+y) + y\phi'(x+y)$,

and $\partial^2 z/\partial x^2 = f'(x+y) + f'(x+y) + xf''(x+y) + y\phi''(x+y)$

$= 2f'(x+y) + xf''(x+y) + y\phi''(x+y)$.

Also $\partial z/\partial y = xf'(x+y) + \phi(x+y) + y\phi'(x+y)$

and $\partial^2 z/\partial y^2 = xf''(x+y) + \phi'(x+y) + \phi'(x+y) + y\phi''(x+y)$
$= 2\phi'(x+y) + xf''(x+y) + y\phi''(x+y)$.

Again $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$
$= \partial/\partial x \{xf'(x+y) + \phi(x+y) + y\phi'(x+y)\}$
$= f'(x+y) + xf''(x+y) + \phi'(x+y) + y\phi''(x+y)$.

Now $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2)$
$= 2[f'(x+y) + \phi'(x+y) + xf''(x+y) + y\phi''(x+y)]$
$= 2(\partial^2 z/\partial x \partial y)$.

***Ex. 8.** *If $z = (x^2+y^2)/(x+y)$, show that*

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

(Kanpur 1978; Meerut 85; Agra 80; Allahabad 7?)

Sol. We have $z = (x^2+y^2)/(x+y)$.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{(x+y).2x - (x^2+y^2).1}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2},$$

and

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y - (x^2+y^2).1}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}.$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{(x^2-y^2+2xy)-(y^2-x^2+2xy)}{(x+y)^2}\right]^2$$

$$= \left[\frac{2(x^2-y^2)}{(x+y)^2}\right]^2 = 4\left[\frac{x-y}{x+y}\right]^2.$$

Also

$$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{x^2-y^2+2xy}{(x+y)^2} - \frac{y^2-x^2+2xy}{(x+y)^2}$$

$$= \frac{(x^2+y^2+2xy) - x^2 + y^2 - 2xy - y^2 + x^2 - 2xy}{(x+y)^2} = \frac{x^2-2xy+y^2}{(x+y)^2}$$

$$= \left(\frac{x-y}{x+y}\right)^2.$$

Hence

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

****Ex. 9.** *If $1/u = \sqrt{(x^2+y^2+z^2)}$, show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u,$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

(Allahabad 1981; Meerut 85, 87, 98; Agra 79; Garhwal 77; Kumayun 83; Gorakhpur 83; Vikram 85; Jhansi 89)

Sol. We have $u = (x^2+y^2+z^2)^{-1/2}$.

$\therefore \quad \partial u/\partial x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.(2x) = -x(x^2+y^2+z^2)^{-3/2}$,

and $\partial^2 u/\partial x^2 = x.\frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x) - 1(x^2+y^2+z^2)^{-3/2}$

$$= (x^2 + y^2 + z^2)^{-5/2} [3x^2 - (x^2 + y^2 + z^2)]$$
$$= u^5 (2x^2 - y^2 - z^2).$$

Similarly, by symmetry $\partial u/\partial y = -y (x^2 + y^2 + z^2)^{-3/2}$,

$$\partial^2 u/\partial y^2 = u^5 (2y^2 - x^2 - z^2), (\partial u/\partial z) = -z (x^2 + y^2 + z^2)^{-3/2},$$

and $$\partial^2 u/\partial z^2 = u^5 (2z^2 - x^2 - y^2).$$

Now $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= -x^2 (x^2 + y^2 + z^2)^{-3/2} - y^2 (x^2 + y^2 + z^2)^{-3/2} - z^2 (x^2 + y^2 + z^2)^{-3/2}$$

$$= -(x^2 + y^2 + z^2)^{-3/2} (x^2 + y^2 + z^2) = -(x^2 + y^2 + z^2)^{-1/2} = -u.$$

Also $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= u^5 (2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2) = u^5 . 0 = 0.$$

Ex. 10. *If $u = \log (x^2 + y^2 + z^2)$, show that*

$$x \frac{\partial^2 u}{\partial y \, \partial z} = y \frac{\partial^2 u}{\partial z \, \partial x} = z \frac{\partial^2 u}{\partial x \, \partial y}.$$

Sol. We have $u = \log (x^2 + y^2 + z^2)$.

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{2z}{x^2 + y^2 + z^2}, \qquad \text{[treating } x \text{ and } y \text{ as constants]}$$

and $$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial z} \right) = 2z . [-(x^2 + y^2 + z^2)^{-2} . 2y]$$
$$= -4yz/(x^2 + y^2 + z^2)^2.$$

Now $$x \frac{\partial^2 u}{\partial y \, \partial z} = -4xyz/(x^2 + y^2 + z^2)^2.$$

By symmetry, $$y \frac{\partial^2 u}{\partial z \, \partial x} = z \frac{\partial^2 u}{\partial x \, \partial y} = -\frac{4xyz}{(x^2 + y^2 + z^2)^2}.$$

Hence $$x \frac{\partial^2 u}{\partial y \, \partial z} = y \frac{\partial^2 u}{\partial z \, \partial x} = z \frac{\partial^2 u}{\partial x \, \partial y}.$$

****Ex. 11.** *If $u = \log (x^3 + y^3 + z^3 - 3xyz)$, show that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$$

and $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x + y + z)^2}.$$

(Meerut 1996, 94P, 83S, 81; Bundelkhand 78; Gorakhpur 75; Rohilkhand 83, 88, 89; Lucknow 80; Alld. 80; Kanpur 86)

Sol. We have $u = \log (x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}, \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

and $$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}.$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3\,(x^2+y^2+z^2-yz-zx-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3\,(x^2+y^2+z^2-yz-zx-xy)}{(x+y+z)\,(x^2+y^2+z^2-yz-zx-xy)}=\frac{3}{x+y+z}\cdot \quad ...(1)$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$

$=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$, from (1)

$$=3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$=3\left[\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}+\frac{-1}{(x+y+z)^2}\right]=\frac{-9}{(x+y+z)^2}\cdot$$

Ex. 12. *If $z=\tan(y+ax)+(y-ax)^{3/2}$, find the value of* $(\partial^2 z/\partial x^2)-a^2\,(\partial^2 z/\partial y^2)$.

Sol. Here $z=\tan(y+ax)+(y-ax)^{3/2}$.

$\therefore \quad (\partial z/\partial x)=\{\sec^2(y+ax)\}\,.\,a+\frac{3}{2}(y-ax)^{1/2}\,.\,(-a)$,

and $(\partial^2 z/\partial x^2)=2a^2\tan(y+ax)\sec^2(y+ax)+\frac{3}{4}a^2(y-ax)^{-1/2}$.

Again $(\partial z/\partial y)=\sec^2(y+ax)+\frac{3}{2}(y-ax)^{1/2}$,

and $(\partial^2 z/\partial y^2)=2\sec^2(y+ax)\tan(y+ax)+\frac{3}{4}(y-ax)^{-1/2}$.

Thus $(\partial^2 z/\partial x^2)-a^2\,(\partial^2 z/\partial y^2)=0$.

Ex. 13. *If $x=r\cos\theta$, $y=r\sin\theta$, prove that*

***(a)** $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$,

(Kanpur 81; Vikram 83; Gorakhpur 82)

(b) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x\partial y}\right)^2$,

(c) $(\partial r/\partial x)^2+(\partial r/\partial y)^2=1$. **(Meerut 1983)**

Sol. (a). We have $x=r\cos\theta, y=r\sin\theta$.

Therefore $r^2=x^2+y^2$ ...(1)

Now $2r\,(\partial r/\partial x)=2x$; [diff. (1) partially w.r.t. 'x']

$\therefore \quad \partial r/\partial x=x/r$. ...(2)

Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^2 r}{\partial x^2}=\frac{r\,.\,1-x\,.\,\partial r/\partial x}{r^2}=\frac{r-x\,.\,x/r}{r^2}, \quad \text{[using (2)]}$$

$$=\frac{r^2-x^2}{r^3}=\frac{(x^2+y^2)-x^2}{r^3}=\frac{y^2}{r^3}\cdot \quad ...(3)$$

Again differentiating (1) partially w.r.t. 'y', we get

$$2r\frac{\partial r}{\partial y}=2y;\quad \therefore \quad \frac{\partial r}{\partial y}=\frac{y}{r}\cdot \quad ...(4)$$

Differentiating (4) partially w.r.t. y, we get

$$\frac{\partial^2 r}{\partial y^2}=\frac{r\,.\,1-y\,.\,\partial r/\partial y}{r^2}=\frac{r-y\,.\,y/r}{r^2},\qquad \left[\because \text{ from (4)}, \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{r^2-y^2}{r^3}=\frac{(x^2+y^2)-y^2}{r^3}=\frac{x^2}{r^3}. \qquad ...(5)$$

Adding (3) and (5), we get

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}+\frac{x^2}{r^3}=\frac{x^2+y^2}{r^3}=\frac{1}{r}.$$

Also $\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]=\frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]=\frac{x^2+y^2}{r^3}=\frac{r^2}{r^3}=\frac{1}{r}.$

$$\therefore \quad \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$$

(b) Differntiating (4) partially w.r.t. 'x', we get

$$\frac{\partial^2 r}{\partial x\,\partial y}=-\frac{y}{r^2}\cdot\frac{\partial r}{\partial x}=-\frac{xy}{r^3}, \qquad \left[\because \text{ from (2)}, \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

Now $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\frac{y^2}{r^3}\cdot\frac{x^2}{r^3},$ [from (3) and (5)]

$$=\frac{x^2y^2}{r^6}=\left(\frac{-xy}{r^3}\right)^2=\left(\frac{\partial^2 r}{\partial x\,\partial y}\right)^2.$$

(c) From (2) and (4), on squaring and adding, we get

$$\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2=\frac{x^2}{r^2}+\frac{y^2}{r^2}=\frac{x^2+y^2}{r^2}=\frac{r^2}{r^2}=1.$$

Ex. 14. *If $x=r\cos\theta, y=r\sin\theta$, prove that*

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \textit{ except when } x=0, y=0.$$

(Meerut 1983, 87; Gorakhpur 81; Vikram 87; G.N.U. 88)

Sol. Given $x=r\cos\theta, y=r\sin\theta$.

$\therefore \quad r^2=x^2+y^2; \theta=\tan^{-1}(y/x)$.

Now $\frac{\partial\theta}{\partial x}=\frac{\partial}{\partial x}\{\tan^{-1}(y/x)\}=\frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right]=-\frac{y}{x^2+y^2}.$

$$\therefore \quad \frac{\partial^2\theta}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right)=-y\cdot\frac{-2x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2}. \qquad ...(1)$$

Also $\frac{\partial\theta}{\partial y}=\frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{(x^2+y^2)}.$

$$\therefore \quad \frac{\partial^2\theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial y}\right)=x\cdot\frac{-2y}{(x^2+y^2)^2}=-\frac{2xy}{(x^2+y^2)^2}. \qquad ...(2)$$

Adding (1) and (2), we get $(\partial^2\theta/\partial x^2)+(\partial^2\theta/\partial y^2)=0$.

But at $x=0, y=0$ both $\partial^2\theta/\partial x^2$ and $\partial^2\theta/\partial y^2$ are of the indeterminate form 0/0.

$\therefore \quad (\partial^2\theta/\partial x^2)+(\partial^2\theta/\partial y^2)=0$, excpet when $x=0, y=0$.

Ex. 14 (a). *If $u=\tan^{-1}(y/x)$, then verify that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(Meerut 1991 P, Gorakhpur 87)

Sol. For complete solution of this question refer Ex. 14.

****Ex. 15.** *If $u = f(r)$ where $r^2 = x^2 + y^2$, show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

(Delhi 1982; Lucknow 82; Rohilkhand 82; Kanpur 85, 89; Meerut 97, 94, 82 S, 77; Allahabad 82, 77)

Sol. Differentiating $r^2 = x^2 + y^2$ partially w.r.t. x and y, we get

$$2r \frac{\partial r}{\partial x} = 2x \text{ or } \frac{\partial r}{\partial x} = \frac{x}{r};\ 2r \frac{\partial r}{\partial y} = 2y \text{ or } \frac{\partial r}{\partial y} = \frac{y}{r} \cdot \qquad \text{...(1)}$$

Now $u = f(r)$. Therefore $\dfrac{\partial u}{\partial x} = \{f'(r)\} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} f'(r)$, [from (1)]

and
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[x \cdot \frac{1}{r} \cdot f'(r)\right]$$

$$= 1 \cdot \frac{1}{r} \cdot f'(r) + \{x f'(r)\}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \frac{x}{r}\{f''(r)\}\frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2} \cdot \frac{x}{r} f'(r) + \frac{x^2}{r^2} f''(r),$$

[$\because$ from (1), $\partial x / \partial r = x/r$]

$$= (1/r) f'(r) - (x^2/r^3) f'(r) + (x^2/r^2) f''(r). \qquad \text{...(2)}$$

Similarly, by symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r). \qquad \text{...(3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r} f'(r) - \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r)$$

$$= (2/r) f'(r) - (r^2/r^3) f'(r) + (r^2/r^2) f''(r), \qquad [\because r^2 = x^2 + y^2]$$

$$= (2/r) f'(r) - (1/r) f'(r) + f''(r) = f''(r) + (1/r) f'(r).$$

Ex. 16. *If $u = x\phi(y/x) + \psi(y/x)$, prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

(Agra 1978; Alld. 84)

Sol. We have $u = x\phi(y/x) + \psi(y/x)$. ...(1)

Differentiating (1) partially w.r.t. x and y, we get

$(\partial u / \partial x) = x\{\phi'(y/x)\} \cdot (-y/x^2) + \phi(y/x) + \{\psi'(y/x)\} \cdot (-y/x^2)$,

and $(\partial u / \partial y) = x\{\phi'(y/x)\} \cdot (1/x) + \{\psi'(y/x)\} \cdot (1/x)$.

$\therefore \quad x(\partial u / \partial x) + y(\partial u / \partial y) = x\phi(y/x)$. ...(2)

Now differentiating (2) partially w.r.t. x and y respectively, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = x\left\{\phi'\left(\frac{y}{x}\right)\right\}\left(-\frac{y}{x^2}\right) + \phi\left(\frac{y}{x}\right),$$

and $$x\frac{\partial^2 u}{\partial x \partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=x\left\{\phi'\left(\frac{y}{x}\right)\right\}\cdot\frac{1}{x}.$$

Multiplying these equations by x and y respectively and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\phi\,(y/x)$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0, \qquad \text{[from (2)]}.$$

Ex. 17. *If* $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

(Meerut 1983; Kanpur 82; Gorakhpur 87)

Sol. From the given equation we observe that u is a function of three independent variables x, y and z. Differentiating the given equation partially w.r.t. 'x', we get

$$\frac{2x}{a^2+u}-\left\{\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x}=0.$$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{2x/(a^2+u)}{\Sigma\,\{x^2/(a^2+u)^2\}}.$$

Similarly, by symmetry, we can write the values of $\partial u/\partial y$ and $\partial u/\partial z$.

Now $$\left(\frac{\partial u}{\partial x}\right)^2=\frac{4x^2/(a^2+u)^2}{[\Sigma\,\{x^2/(a^2+u)^2\}]^2}.$$

$$\therefore \quad \Sigma(\partial u/\partial x)^2=\frac{4\Sigma\,\{x^2/(a^2+u)^2\}}{[\Sigma\,\{x^2/(a^2+u)^2\}]^2}$$

$$=\frac{4}{\Sigma\,\{x^2/(a^2+u)^2\}} \qquad \text{...(1)}$$

Again $$2x\frac{\partial u}{\partial x}=\frac{4x^2/(a^2+u)}{\Sigma\,\{x^2/(a^2+u)^2\}}.$$

$$\therefore \quad 2\Sigma x\,(\partial u/\partial x)=\frac{4\Sigma\,\{x^2/(a^2+u)\}}{\Sigma\,\{x^2/(a^2+u)^2\}}=\frac{4}{\Sigma\,\{x^2/(a^2+u)^2\}}. \qquad \text{...(2)}$$

[$\because \Sigma\,\{x^2/(a^2+u)\}=1$, from the given relation]

Now from (1) and (2), we have

$$\Sigma\,(\partial u/\partial x)^2=2\Sigma\,\{x\,(\partial u/\partial x)\}.$$

Ex. 18. *If* $u=\log r$, *where* $r^2=(x-a)^2+(y-b^2)+(z-c)^2$, *show that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}.$$

Sol. We have $r^2=(x-a)^2+(y-b)^2+(z-c)^2$. ...(1)

Differentiating (1) partially w.r.t. 'x', we have

$$2r\frac{\partial r}{\partial x} = 2(x-a) \text{ or } \frac{\partial r}{\partial x} = \frac{1}{r}\left(\frac{x-a}{r}\right) \qquad ...(2)$$

Now $u = \log r$. $\therefore \quad \dfrac{\partial u}{\partial x} = \dfrac{1}{r}\dfrac{\partial r}{\partial x} = \dfrac{1}{r}\left(\dfrac{x-a}{r}\right)$. [from (2)]

Thus $(\partial u/\partial x) = (x-a)/r^2$.

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x-a}{r^2}\right) = \frac{r^2(1) - (x-a) \cdot 2r(\partial r/\partial x)}{r^4}$$

$$= \frac{r^2 - 2(x-a)^2}{r^4}, \qquad \left[\because \text{ from (2), } \frac{\partial r}{\partial x} = (x-a)/r\right]$$

Similarly, by symmetry

$$\frac{\partial^2 u}{\partial y^2} = \frac{r^2 - 2(y-b)^2}{r^4} \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = \frac{r^2 - 2(z-c)^2}{r^4}.$$

Hence $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{3r^2 - 2\{(x-a)^2 + (y-b)^2 + (z-c)^2\}}{r^4}$

$$= \frac{3r^2 - 2r^2}{r^4}, \{\text{ using (1)}\} = \frac{r^2}{r^4} = \frac{1}{r^2}.$$

Ex. 19. *If $u = e^{xyz}$, show that*

$$\partial^3 u/\partial x\, \partial y\, \partial z = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

(Lucknow 1983; Meerut 84; Gorakhpur 81)

Sol. Here $u = e^{xyz}$. $\therefore \quad \partial u/\partial z = xy\, e^{xyz}$.

Now $\dfrac{\partial^2 u}{\partial y\, \partial z} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial z}\right) = \dfrac{\partial}{\partial y}(xy\, e^{xyz}) = x\dfrac{\partial}{\partial y}(y\, e^{xyz})$

$$= x\,[y \cdot xz\, e^{xyz} + e^{xyz}] = e^{xyz}(x^2yz + x).$$

Again $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial y\, \partial z}\right) = \dfrac{\partial}{\partial x}[e^{xyz}(x^2yz + x)]$

$$= e^{xyz}(2xzy + 1) + yz\, e^{xyz}(x^2yz + x)$$
$$= e^{xyz}[2xyz + 1 + x^2y^2z^2 + xyz]$$
$$= e^{xyz}[1 + 3xyz + x^2y^2z^2].$$

Ex. 20. *If $x^x y^y z^z = c$, show that at $x = y = z$,*

$$\partial^2 z/\partial x \partial y = -\{x \log(ex)\}^{-1}.$$

(Agra 1983; Meerut 1982, 84, 88 P, Alld. 80, 76)

Sol. We have $x^x \cdot y^y \cdot z^z = c$. From this equation we observe that we can regard z as a function of two independent variables x and y. Taking logarithms of both sides of the given equation, we get

$$x\log x + y\log y + z\log z = \log c. \qquad ...(1)$$

Now differentiating (1) partially w.r.t. x taking y as constant, we have

$$x \cdot \frac{1}{x} + 1 \cdot \log x + \left[z \cdot \frac{1}{z} + 1 \cdot \log z\right]\frac{\partial z}{\partial x} = 0.$$

[Note that z is not a constant but is a function of x and y]

$$\therefore \qquad \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}. \qquad ...(2)$$

Similarly differentiating (1) partially w.r.t. y, we have

$$\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}. \qquad ...(3)$$

Now $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[-\left(\dfrac{1+\log y}{1+\log z}\right)\right]$, [from (3)]

$$= -(1+\log y)\cdot\frac{\partial}{\partial x}[(1+\log z)^{-1}]$$

$$= -(1+\log y)\cdot\left[-(1+\log z)^{-2}\cdot\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

$$= \frac{(1+\log y)}{z(1+\log z)^2}\cdot\left[-\left(\frac{1+\log x}{1+\log z}\right)\right], \qquad \text{[from (2)].}$$

Hence, when $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3},$$

[putting $y = z = x$ in the value of $(\partial^2 z/\partial x \partial y)$]

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)}, \qquad [\because \log e = 1]$$

$$= -\frac{1}{x\log(ex)} = -\{x\log(ex)\}^{-1}.$$

Ex. 21. (a) *If* $u = (1 - 2xy + y^2)^{-1/2}$, *prove that*

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$$

(Meerut 1982)

Sol. Here $u = (1 - 2xy + y^2)^{-1/2}$.

$$\therefore \quad \frac{\partial u}{\partial x} = -\tfrac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3,$$

and
$$\frac{\partial u}{\partial y} = -\tfrac{1}{2}(1 - 2xy + y^2)^{-3/2}.(-2x + 2y) = (x - y)u^3.$$

Now $\dfrac{\partial}{\partial x}\left\{(1-x^2)\dfrac{\partial u}{\partial x}\right\} = \dfrac{\partial}{\partial x}\{(1-x^2).yu^3\}$

$$= y(-2x)u^3 + y(1-x^2).3u^2\frac{\partial u}{\partial x} = -2xy\,u^3 + 3y(1-x^2)u^2.yu^3$$

$$= -2xy\,u^3 + 3y^2u^5(1-x^2). \qquad ...(1)$$

Also $\dfrac{\partial}{\partial y}\left\{y^2\dfrac{\partial u}{\partial y}\right\} = \dfrac{\partial}{\partial y}\{y^2(x-y)u^3\} = \dfrac{\partial}{\partial y}\{(y^2x - y^3)u^3\}$

$$= (2xy - 3y^2)u^3 + (y^2x - y^3).3u^2\frac{\partial u}{\partial y}$$

$$= (2xy - 3y^2)u^3 + y^2(x-y).3u^2.(x-y)u^3$$

$$= (2xy - 3y^2)u^3 + y^2(x-y)^2.3u^5$$

$$= 2xy\,u^3 + 3y^2u^5[(x-y)^2 - u^{-2}]$$

$$= 2xy\,u^3 + 3y^2 u^5 [(x-y)^2 - (1 - 2xy + y^2)],$$

$$[\because u^{-2} = 1 - 2xy + y^2]$$

$$= 2xy\,u^3 + 3y^2 u^5 [x^2 - 1] = 2xy\,u^3 - 3y^2 u^5 (1 - x^2). \quad ...(2)$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial u}{\partial y}\right\} = 0.$$

Ex. 21. (b) *If $\theta = t^n e^{-r^2/4t}$, what value of n will make*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}?$$

(Meerut 1989)

Sol. We have $\dfrac{\partial \theta}{\partial r} = t^n \,.\, e^{-r^2/4t} \,.\left(-\dfrac{2r}{4t}\right) = -\dfrac{r}{2} t^{n-1} e^{-r^2/4t}.$

$\therefore \quad r^2 \dfrac{\partial \theta}{\partial r} = -\frac{1}{2} r^3 t^{n-1} e^{-r^2/4t}.$

$\therefore \quad \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = -\dfrac{3r^2}{2} t^{n-1} e^{-r^2/4t} - \frac{1}{2} r^3 t^{n-1} e^{-r^2/4t} \,.\left(-\dfrac{2r}{4t}\right)$

$= -\frac{3}{2} r^2 t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^4 t^{n-2} e^{-r^2/4t}.$

$\therefore \quad \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = -\frac{3}{2} t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$

Also $\dfrac{\partial \theta}{\partial t} = nt^{n-1} e^{-r^2/4t} + t^n e^{-r^2/4t} \,.\, \dfrac{r^2}{4t^2}$

$= nt^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t}.$

Now $\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \theta}{\partial r}\right) = \dfrac{\partial \theta}{\partial t}$

$\Rightarrow \quad -\frac{3}{2} t^{n-1} e^{-r^2/4t} + \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t} = nt^{n-1} e^{-r^2/4t}$

$+ \frac{1}{4} r^2 t^{n-2} e^{-r^2/4t}$

$\Rightarrow \quad -\frac{3}{2} t^{n-1} e^{-r^2/4t} = nt^{n-1} e^{-r^2/4t},$

for all possible values of r and t

$\Rightarrow \quad n = -\frac{3}{2}.$

§ 2. Homogeneous Functions.

An expression in which every term is of the same degree is called a homogeneous function. Thus

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots + a_{n-1} xy^{n-1} + a_n y^n$$

is a homogeneous function of x and y of degree n. This can also be written as

$$x^n \left\{a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \ldots + a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n\right\}$$

or $\quad x^n f(y/x)$, where $f(y/x)$ is some function of (y/x).

Note 1. To test whether a given function f (x, y) is homogeneous or not we put tx for x and ty for y in it.

If we get $f(tx, ty) = t^n f(x, y)$,
the function $f(x, y)$ is homogeneous of degree n; otherwise $f(x, y)$ is not a homogeneous function.

Note 2. If u is a homogeneous function of x and y of degree n then $\partial u/\partial x$ and $\partial u/\partial y$ are also homogeneous functions of x and y each being of degree n − 1.

Let $u = x^n f(y/x)$.

[$\because$ u is a homogeneous function of x and y of degree n]

Then $\dfrac{\partial u}{\partial x} = nx^{n-1} f(y/x) + x^n \{f'(y/x)\} . (-y/x^2)$

$= x^{n-1} [n f(y/x) - (y/x) f'(y/x)]$

$= x^{n-1}$. [some function of y/x]

= a homogeneous function of x and y of degree $(n - 1)$.

Similarly, $\dfrac{\partial u}{\partial y} = x^n \{f'(y/x)\} \cdot \dfrac{1}{x} = x^{n-1} f'(y/x)$

$= x^{n-1}$. (some function of y/x)

= a homogeneous function of x and y of degree $(n - 1)$.

**§ 3. Euler's Theorem on homogeneous functions.

If u is a homogeneous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

(Meerut 1982, 84, 86, 87S, 88, 88S, 89, 92, 96 BP; Agra 81; Gorakhpur 86, 88; Kanpur 78; Lucknow 79; Alld. 82, 78)

Proof. Since u is a homogeneous function of x and y of degree n, therefore u may be put in the form

$$u = x^n f(y/x). \qquad ...(1)$$

Differentiating (1) partially w.r.t. 'x', we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x^n f(y/x)]$$

$$= [f(y/x)]\, nx^{n-1} + x^n [f'(y/x)] (-y/x^2).$$

$$\therefore \quad x\frac{\partial u}{\partial x} = nx^n f(y/x) - x^{n-1} y . f'(y/x). \qquad ...(2)$$

Again differentiating (1) partially w.r.t. 'y', we have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^n f(y/x)] = x^n [f'(y/x)] \cdot \frac{1}{x} = x^{n-1} f'(y/x).$$

$$\therefore \quad y\frac{\partial u}{\partial y} = y . x^{n-1} f'(y/x). \qquad ...(3)$$

Adding (2) and (3), we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f(y/x)=n\,u. \qquad \text{[from (1)]}$$

Hence $\mathbf{x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=n\,u.}$

Note. Euler's theorem can be extended to a homogeneous function of any number of variables. *Thus if $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function of $x_1, x_2, \ldots, x_n$ of degree n, then*

$$\mathbf{x_1\frac{\partial f}{\partial x_1}+x_2\frac{\partial f}{\partial x_2}+\ldots+x_n\frac{\partial f}{\partial x_n}=nf.}$$

Solved Examples

Ex. 22. *Verify Euler's theorem in the following cases :*

(i) $u=x^4-3x^3y+5x^2y^2+4xy^3-2y^4$.

(ii) $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$, **(Meerut 1989P)**

(iii) $u=\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$,

(iv) $u=axy+byz+czx$, **(Meerut 1987)**

(v) $u=x^n\log(y/x)$, **(Meerut 1989P)**

(vi) $u=1/\sqrt{(x^2+y^2)}$. **(Meerut 1989)**

Sol. (i) We have $u=x^4-3x^3y+5x^2y^2+4xy^3-2y^4$. Obviously u is a homogeneous function of x and y of degree 4. So by Euler's theorem, we must have $x(\partial u/\partial x)+y(\partial u/\partial y)=4u$. Let us verify it.

We have

$$(\partial u/\partial x)=4x^3-9x^2y+10xy^2+4y^3,$$

and
$$(\partial u/\partial y)=-3x^3+10x^2y+12xy^2-8y^3.$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x(4x^3-9x^2y+10xy^2+4y^3)$$
$$+y(-3x^3+10x^2y+12xy^2-8y^3)$$
$$=4(x^4-3x^3y+5x^2y^2+4xy^3-2y^4)$$

$=4u$. This verifies Euler's theorem.

(ii) We have $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$ which is obviously a homogeneous function of x and y of degree $4-3$ *i.e.*, 1. Note that each term in the numerator is of degree 4 while each term in the denominator is of degree 3

In order to verify Euler's theorem we are to show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\,.\,u=u\,.$$

Now $\log u=\log x+\log(x^3-y^3)-\log(x^3+y^3)$. ...(1)

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{3x^2}{x^3-y^3}-\frac{3x^2}{x^3+y^3} \qquad ...(2)$$

and $$\frac{1}{u}\frac{\partial u}{\partial y}=0-\frac{3y^2}{x^3-y^3}-\frac{3y^2}{x^3+y^3}. \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$$
$$=1+3-3=1.$$

$\therefore \quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=u.$ This verifies Euler's theorem.

(iii) Here u is a homogeneous function of x and y of degree $\frac{1}{4}-\frac{1}{5}$ *i.e.*, $\frac{1}{20}$. So by Euler's theorem we must have

$$x(\partial u/\partial x)+y(\partial u/\partial y)=\tfrac{1}{20}u.$$

Let us verify it. We have

$$\log u=\log(x^{1/4}+y^{1/4})-\log(x^{1/5}+y^{1/5}).$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{\frac{1}{4}x^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{\frac{1}{5}x^{-4/5}}{x^{1/5}+y^{1/5}}$$

and $$\frac{1}{u}\frac{\partial u}{\partial y}=\frac{\frac{1}{4}y^{-3/4}}{x^{1/4}+y^{1/4}}-\frac{\frac{1}{5}y^{-4/5}}{x^{1/5}+y^{1/5}}.$$

$$\therefore \quad \frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{1}{4}\frac{x^{1/4}+y^{1/4}}{x^{1/4}+y^{1/4}}-\frac{1}{5}\frac{x^{1/5}+y^{1/5}}{x^{1/5}+y^{1/5}}=\frac{1}{4}-\frac{1}{5}=\frac{1}{20}.$$

$\therefore \quad x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{20}u.$ This verifies Euler's theorem.

(iv) We have $u=axy+byz+czx$, which is a homogeneous function of x, y and z of degree 2. So in order to verify Euler's theorem, we must show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=2u.$

Now $\dfrac{\partial u}{\partial x}=ay+cz,\ \dfrac{\partial u}{\partial y}=ax+bz,$ and $\dfrac{\partial u}{\partial z}=by+cx.$

$$\therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x(ay+cz)+y(ax+bz)+z(by+cx)$$

$=2(axy+byz+czx)=2u.$ This verifies Euler's theorem.

(v) Here u is a homogeneous function of x and y of degree n. So by Euler's theorem we must have

$$x(\partial u/\partial x)+y(\partial u/\partial y)=nu.$$

Now do the verification yourself.

(vi) Here $u=\dfrac{1}{\sqrt{(x^2+y^2)}}=\dfrac{1}{x\sqrt{[1+(y/x)^2]}}=x^{-1}\cdot\dfrac{1}{\sqrt{[1+(y/x)^2]}}$ is a homogeneous function of x and y of degree -1. Now proceed yourself.

Ex. 23. *If* $u = x^2y^2/(x+y)$, *show that*

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = 3u.$$

Sol. We have $u = \dfrac{x^2y^2}{x+y} = \dfrac{x^3\,(y/x)^2}{[1+(y/x)]} = x^3 f(y/x)$, say.

Thus u is a homogeneous function of x and y of degree 3.

Therefore by Euler's theorem, we have $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u.$

****Ex. 24.** *If* $u = \sin^{-1}\{(x^2+y^2)/(x+y)\}$, *show that*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u.$$

(Meerut 1981, 88, 98; Lucknow 83; Delhi 80, 74; Gorakhpur 89; Agra 82, 81; Kanpur 80; Magadh 74; Jodhpur 76; Alld. 78; U.P. P.C.S. 90)

Sol. We have given one method for solving this question in Ex. 6 page 93. Here we shall give another method using Euler's theorem.

We have $\sin u = (x^2+y^2)/(x+y) = v$, say.

Obviously $v = (x^2+y^2)/(x+y)$ is a homogeneous function of x and y of degree $2-1$ *i.e.*, 1. Therefore by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1\,.\,v = v. \qquad \text{...(1)}$$

Now $v = \sin u$.

Therefore $\dfrac{\partial v}{\partial x} = \cos u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y} = \cos u\dfrac{\partial u}{\partial y}\cdot$

Putting these values in (1), we get

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = v \quad \text{or} \quad \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{v}{\cos u} = \frac{\sin u}{\cos u} = \tan u, \qquad [\because v = \sin u.]$$

Ex. 24. (a). *If* $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *show that*

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \tfrac{1}{2}\tan u.$$

(Rohilkhand 1990; Agra 82; Kanpur 88)

Sol. We have

$$\sin u = (x+y)/(\sqrt{x}+\sqrt{y}) = v, \text{ say.}$$

Then v is a homogeneous function of x and y of degree $(1-\frac{1}{2})$ *i.e.* $\frac{1}{2}$. Applying Euler's theorem for v, we have

$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = \tfrac{1}{2}v$$

or $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \tfrac{1}{2}\sin u, \qquad [\because v = \sin u]$$

or $$x\cos u\,(\partial u/\partial x) + y\cos u\,(\partial u/\partial y) = \tfrac{1}{2}\sin u$$

or $$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \tfrac{1}{2}\tan u.$$

Ex. 24 (b). *If* $u = \cos^{-1}\{(x+y)/(\sqrt{x}+\sqrt{y})\}$, *show that* $x(\partial u/\partial x) + y(\partial u/\partial y) + \frac{1}{2}\cot u = 0$.

Sol. Proceed exactly as in Ex. 24 (a).

Ex. 24 (c). *If* $u = \log \dfrac{x^3+y^3}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2$.

(Meerut 1990 P)

Sol. We have $e^u = \dfrac{x^3+y^3}{x+y} = v$, say.

Obviously $v = (x^3+y^3)/(x+y)$ is a homogeneous function of x and y of degree $3-1$ *i.e.*, 2. Therefore by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2 \, . \, v = 2v \qquad ...(1)$$

Now $v = e^u$. $\therefore$ $\dfrac{\partial v}{\partial x} = e^u \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y} = e^u \dfrac{\partial u}{\partial y}$.

Putting these values in (1), we get

$$xe^u \frac{\partial u}{\partial x} + y\,e^u \frac{\partial u}{\partial y} = 2e^u$$

or $$e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2e^u \text{ or } x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2.$$

****Ex. 25.** *If* $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, *show that*

$$x(\partial u/\partial x) + y(\partial u/\partial y) = \sin 2u.$$

(Meerut 1982, 83, 88 P, 90 S; Delhi 83, 81; Luck. 80; Gorakhpur 89; Rohilkhand 81)

Sol. We have $\tan u = (x^3+y^3)/(x+y) = v$, say. Then v is a homogeneous function of x and y of degree $3-1$ *i.e.*, 2. Therefore by Euler's theorem, we have

$$x(\partial v/\partial x) + y(\partial v/\partial y) = 2v. \qquad ...(1)$$

Now $\qquad v = \tan u$.

$$\therefore \quad \frac{\partial v}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}.$$

Putting these values in (1), we get

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2v$$

or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2v}{\sec^2 u} = \frac{2\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$$

Ex. 25 (a). *If* $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, *show that*

$x(\partial u/\partial x) + y(\partial u/\partial y) = \sin 2u$. **(Kanpur 1985; Meerut 91S, 93)**

Sol. Proceed exacly as in Ex. 25.

Ex. 25 (b). *If* $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, *then prove that*
$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \frac{1}{2}\sin 2u.$ **(Gorakhpur 1982; Allahabad 82)**

Sol. Proceed exactly as in Ex. 25.

Ex. 26 (a). *If* $u = \sin^{-1}\{(\sqrt{x}-\sqrt{y})/(\sqrt{x}+\sqrt{y})\}$, *show that*
$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$
(Rohilkhand 1987; Alld. 77; Kurukshetra 83)

Sol. We have $\sin u = (\sqrt{x}-\sqrt{y})/(\sqrt{x}+\sqrt{y}) = v$, say. Then v is a homogeneous function of x and y of degree $\frac{1}{2}-\frac{1}{2}$ *i.e.*, 0.

Therefore by Euler's theorem, we have $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = 0\,.\,v = 0.$ ...(1)

Now $v = \sin u$.
$$\therefore \quad \frac{\partial v}{\partial x} = \cos u\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y} = \cos u\frac{\partial u}{\partial y}.$$
Putting these values in (1), we get
$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0 \text{ or } \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0$$
or $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, \text{ since } \cos u \neq 0. \text{ Hence } \frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}.$$

Ex. 26 (b). *Use Euler's theorem to show that if*
$u = \tan^{-1}(y/x)$, *then*
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$
(Meerut 1989 S)

***Ex. 27.** *If u be a homogeneous function of x and y of degree n, show that*
$$x\,(\partial^2 u/\partial x^2) + y\,(\partial^2 u/\partial x\,\partial y) = (n-1)\,(\partial u/\partial x),$$
and
$$x\,(\partial^2 u/\partial x\,\partial y) + y\,(\partial^2 u/\partial y^2) = (n-1)\,(\partial u/\partial y).$$
Hence deduce that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n\,(n-1)\,u.$$
(Allahabad 1981; Vikram 83; Agra 84; Gorakhpur 83; I.C.S. 95; Jiwaji 85; Kurukshetra 88)

Sol. By Euler's theorem, we have
$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = nu \qquad \text{...(1)}$$
Differentiating (1) partially w.r.t. x, we get
$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x}$$
$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = (n-1)\frac{\partial u}{\partial x} \qquad \text{...(2)}$$

Proved.

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$x \frac{\partial^2 u}{\partial x \, \partial y} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y}. \quad \text{...(3)}$$

Proved.

Now multiplying (2) by x and (3) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= (n-1) \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = (n-1) nu = n(n-1) u.$$

Ex. 28. *If* $u = x\phi(y/x) + \psi(y/x)$, *prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

(Rohilkhand 1986; Agra 85)

Sol. Let $u = z_1 + z_2$, where $z_1 = x\,\phi(y/x)$ and $z_2 = \psi(y/x)$.

Obviously z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree zero. Now

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \frac{\partial}{\partial x}(z_1 + z_2) + y \frac{\partial}{\partial y}(z_1 + z_2)$$

$$= \left(x \frac{\partial z_1}{\partial x} + y \frac{\partial z_1}{\partial y} \right) + \left(x \frac{\partial z_2}{\partial x} + y \frac{\partial z_2}{\partial y} \right) = 1 \, . \, z_1 + 0 \, . \, z_2.$$

(by Euler's theorem).

Thus $\quad x(\partial u/\partial x) + y(\partial u/\partial y) = z_1 \quad \text{...(1)}$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \, \partial y} = \frac{\partial z_1}{\partial x}, \quad \text{...(2)}$$

and
$$x \frac{\partial^2 u}{\partial x \, \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{\partial z_1}{\partial y}. \quad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x \frac{\partial z_1}{\partial x} + y \frac{\partial z_1}{\partial y}$$

or
$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + z_1 = 1 \, . \, z_1,$$

[$\because x(\partial u/\partial x) + y(\partial u/\partial y) = z_1$ by (1), and $x(\partial z_1/\partial x) + y(\partial z_1/\partial y) = 1 \, . \, z_1$ by Euler's theorem]

or
$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Ex. 29. *If* $z = xy\,f(y/x)$, *show that* $x(\partial z/\partial x) + y(\partial z/\partial y) = 2z$. *Show also that if* z *is a constant,*

$$\frac{f'(y/x)}{f(y/x)} = \frac{x\{y + x(dy/dx)\}}{y\{y - x(dy/dx)\}}.$$

Sol. We have, $z = x^2 . (y/x) f(y/x)$, so that z is a homogeneous function of x and y of degree 2.

Hence by Euler's theorem, we have

$$x(\partial z/\partial x) + y(\partial z/\partial y) = 2z.$$

If z be a constant, then differentiating

$z = xy f(y/x)$ logarithmically, w.r.t. x, we get

$$0 = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + \frac{f'(y/x)}{f(y/x)} \cdot \frac{x(dy/dx) - y}{x^2}$$

$$= \frac{y + x(dy/dx)}{xy} + \frac{f'(y/x)}{f(y/x)} \cdot \frac{x(dy/dx) - y}{x^2}.$$

Hence $\dfrac{f'(y/x)}{f(y/x)} = \dfrac{x\{y + x(dy/dx)\}}{y\{y - x(dy/dx)\}}$.

Ex. 30. *If* $u = \tan^{-1}\dfrac{x^3 + y^3}{x - y}$, *find the value of*

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

(U.P. P.C.S. 1993)

Sol. First proceed as in Ex. 25 on page 107. Thus we get

$$x(\partial u/\partial x) + y(\partial u/\partial y) = \sin 2u \qquad \text{...(1)}$$

Now differentiating (1) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u \cdot \frac{\partial u}{\partial x}, \qquad \text{...(2)}$$

and

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\,\frac{\partial u}{\partial y}. \qquad \text{...(3)}$$

Multiplying (2) by x, (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \sin 2u = 2\cos 2u \sin 2u, \text{ [by (1)]}$$

or

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$$

Ex. 31. *If* $f(x, y, z)$ *is a homogeneous function of the* n^{th} *degree in* x, y, z, *prove that*

$$x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + z^2\frac{\partial^2 f}{\partial z^2} + 2yz\frac{\partial^2 f}{\partial y\,\partial z} + 2zx\frac{\partial^2 f}{\partial z\,\partial x} + 2xy\frac{\partial^2 f}{\partial x\,\partial y}$$

$$= n(n-1)f(x, y, z).$$

Sol. Here $f(x, y, z)$ is a homogeneous function of the n^{th} degree in x, y, z. Therefore $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ are homogeneous functions of the $(n-1)$th degree in x, y, z. So using Euler's theorem for $\partial f/\partial x$, we have

$$x \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) + z \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = (n-1)\frac{\partial f}{\partial x}$$

or
$$x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y \partial x} + z \frac{\partial^2 f}{\partial z \partial x} = (n-1)\frac{\partial f}{\partial x}. \qquad \text{...(1)}$$

Similarly, $$x \frac{\partial^2 f}{\partial x \partial y} + y \frac{\partial^2 f}{\partial y^2} + z \frac{\partial^2 f}{\partial y \partial z} = (n-1)\frac{\partial f}{\partial y} \qquad \text{...(2)}$$

and
$$x \frac{\partial^2 f}{\partial z \partial x} + y \frac{\partial^2 f}{\partial y \partial z} + z \frac{\partial^2 f}{\partial z^2} = (n-1)\frac{\partial f}{\partial z} \qquad \text{...(3)}$$

Multiplying (1) by x, (2) by y and (3) by z and adding, we get

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + 2yz \frac{\partial^2 f}{\partial y\, \partial z} + 2zx \frac{\partial^2 f}{\partial z\, \partial x} + 2xy \frac{\partial^2 f}{\partial x\, \partial y}$$

$$= (n-1)\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z}\right)$$

$$= (n-1)\, n f(x,y,z) = n(n-1) f(x,y,z).$$

§ 4. Total Derivatives.

If $u = f(x,y)$, where $x = \phi_1(t)$ and $y = \phi_2(t)$, then

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dt}}.$$

Here du/dt is called the **total differential coefficient** of u with respect to t while $\partial u/\partial x$ and $\partial u/\partial y$ are partial derivatives of u.

Proof. Let t be given a small increment δt, and let the corresponding changes in u, x and y be δu, δx and δy respectively. We have then

$$u = f(x,y), \qquad \text{...(1)}$$

and
$$u + \delta u = f(x + \delta x, y + \delta y). \qquad \text{...(2)}$$

$$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - f(x,y)$$

$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)]$$

$$+ [f(x, y + \delta y) - f(x,y)]. \qquad \textbf{[Note]}$$

$$\therefore \quad \frac{\delta u}{\delta t} = \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t}\right\} + \left\{\frac{f(x, y + \delta y) - f(x,y)}{\delta t}\right\}$$

$$= \left\{\frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x}\right\} \cdot \frac{\delta x}{\delta t}$$

$$+ \left\{\frac{f(x, y + \delta y) - f(x,y)}{\delta y}\right\} \cdot \frac{\delta y}{\delta t}. \qquad \text{...(3)}$$

Let $\delta t \to 0$ so that $\delta x \to 0$ and $\delta y \to 0$.

Now $$\lim_{\delta t \to 0} \frac{\delta u}{\delta t} = \frac{du}{dt}, \lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} \text{ and } \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Also $$\lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x,y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y},$$

and $$\lim_{\delta x \to 0} \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}.$$

Since δx and δy tend to zero with δt and the functions involved are all supposed to be continuous, therefore the limit (3) becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

In the same way if $u = f(x, y, z)$, where x, y, z are all functions of some variable t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$

This result can be extended to any number of variables.

Cor. 1. *If u be a function of x and y, where y is a function of x, then*

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

This result follows immediately by taking $t = x$ in the formula of § 4.

Cor. 2. *If $u = f(x, y)$ and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then*

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1} \quad \text{and} \quad \frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}.$$

Cor. 3. *If x and y are connected by an equation of the form $f(x, y) = 0$, then*

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

Since $f(x, y) = 0$, therefore by cor. 1, we get

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx},$$

from which the required result follows.

Ex. 32. (a). *If $(\tan x)^y + y^{\cot x} = a$, find dy/dx.*

Sol. Let $f(x, y) = (\tan x)^y + y^{\cot x} - a$. Then

$$(\partial f/\partial x) = y(\tan x)^{y-1} . \sec^2 x + y^{\cot x} . \log y . (-\operatorname{cosec}^2 x),$$

and $$(\partial f/\partial y) = (\tan x)^y \log \tan x + (\cot x) . y^{\cot x - 1}$$

Now we are given that $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y(\tan x)^{y-1} \sec^2 x - y^{\cot x} . \log y . \operatorname{cosec}^2 x}{(\tan x)^y \log \tan x + \cot x . y^{\cot x - 1}}.$$

Ex. 32. (b). *If $x^3y^3 + 3x \sin y = e^y$, find dy/dx.*

Sol. Let $f(x, y) = x^3y^3 + 3x \sin y - e^y$. Then we have $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(3x^2y^3 + 3\sin y)}{3x^3y^2 + 3x\cos y - e^y}.$$

Ex. 32. (c). *If $x^y + y^x = a^b$, find dy/dx.* **(Meerut 1981, 82)**

Sol. Let $f(x, y) = x^y + y^x - a^b$. Then we have $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

Ex. 32 (d). *If* $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a\,(x-y)$, *prove that*

$$\frac{dy}{dx} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

(Meerut 1987)

Sol. It is given that $\dfrac{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}{x-y} = a.$

Let $f(x, y) = \dfrac{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}}{x-y} - a$. Then $f(x, y) = 0$.

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{\dfrac{\{\frac{1}{2}(1-x^2)^{-1/2}.(-2x)\}(x-y) - 1.\{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\}}{(x-y)^2}}{\dfrac{\{\frac{1}{2}(1-y^2)^{-1/2}.(-2y)\}(x-y) - (-1).\{\sqrt{(1-x^2)} + \sqrt{(1-y^2)}\}}{(x-y)^2}}$$

$$= -\frac{\dfrac{-x(x-y)}{\sqrt{(1-x^2)}} - \sqrt{(1-x^2)} - \sqrt{(1-y^2)}}{\dfrac{-y(x-y)}{\sqrt{(1-y^2)}} + \sqrt{(1-x^2)} + \sqrt{(1-y^2)}}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{-x^2 + xy - (1-x^2) - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-yx + y^2 + \sqrt{(1-x^2)}\sqrt{(1-y^2)} + 1 - y^2}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}} \cdot \frac{xy - 1 - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-xy + 1 - \sqrt{(1-x^2)}\sqrt{(1-y^2)}}$$

$$= \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

Ex. 33. *If* $u = \log\{(x^2+y^2)/xy\}$, *find* du.

Sol. We have $\quad u = \log(x^2+y^2) - \log x - \log y.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x} = \frac{2x^2 - x^2 - y^2}{x(x^2+y^2)} = \frac{x^2-y^2}{x(x^2+y^2)},$$

and
$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y} = \frac{2y^2 - x^2 - y^2}{y(x^2+y^2)} = \frac{(y^2-x^2)}{y(x^2+y^2)}.$$

Now
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{(x^2-y^2)}{x(x^2+y^2)}dx + \frac{(y^2-x^2)}{y(x^2+y^2)}dy$$

$$= \frac{x^2-y^2}{xy(x^2+y^2)}(y\,dx - x\,dy).$$

Ex. 34. *If* $u = \sin(x^2+y^2)$, *where* $a^2x^2 + b^2y^2 = c^2$, *find* du/dx.

Sol. We have $\quad \dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}.$ ...(1)

Now $\quad u = \sin(x^2+y^2).$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x \cos (x^2 + y^2) \quad \text{and} \quad \frac{\partial u}{\partial y} = 2y \cos (x^2 + y^2).$$

Since $a^2x^2 + b^2y^2 = c^2$,

therefore $\qquad 2a^2x + 2b^2y\,(dy/dx) = 0$ or $dy/dx = -(a^2x)/(b^2y)$.

$$\therefore \quad \text{from (1)}, \frac{du}{dx} = 2x \cos (x^2 + y^2) - [2y \cos (x^2 + y^2)]\,[(a^2x)/(b^2y)]$$

$$= [2x \cos (x^2 + y^2)]\,[1 - (a^2/b^2)].$$

Ex. 35. (a) *If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that*

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

(Meerut 1983, 87; Alld. 79, 78)

(b) *If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at point of contact*

$$\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial x}.$$

Sol. (a) From $f(x, y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$. ...(1)

From $\phi(y, z) = 0$, we have $\dfrac{dz}{dy} = -\dfrac{\partial \phi/\partial y}{\partial \phi/\partial z}$. ...(2)

Multiplying the respective sides of (1) and (2), we have

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}\right) \Big/ \left(\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}\right)$$

or
$$\frac{dz}{dx} \cdot \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

(b) The curves will touch if at their common point they have the same value of dy/dx. Now for the curve $f(x, y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$ and for the curve $\phi(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{\partial \phi/\partial x}{\partial \phi/\partial y}.$$

$\therefore$ the two curves touch if at their common point, we have

$$\frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-\partial \phi/\partial x}{\partial \phi/\partial y} \quad i.e., \quad \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial x}.$$

Ex. 36. Formula for the second differential coefficient of an implicit function. *If $f(x, y) = 0$ be an implicit function of x and y, find a formula for d^2y/dx^2.*

Sol. We have $\qquad \dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{p}{q}$, ...(1)

where $\partial f/\partial x$ and $\partial f/\partial y$ have been denoted by p and q respectively. Also using the notations

$$r = \frac{\partial^2 f}{\partial x^2}, \; s = \frac{\partial^2 f}{\partial x \partial y} \text{ and } t = \frac{\partial^2 f}{\partial y^2}, \text{ we have from (1),}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{p}{q}\right)=-\frac{q\,(dp/dx)-p\,(dq/dx)}{q^2} \qquad ...(2)$$

But $\dfrac{dp}{dx}=\dfrac{\partial p}{\partial x}+\dfrac{\partial p}{\partial y}\cdot\dfrac{dy}{dx}=r+s\left(-\dfrac{p}{q}\right)=\dfrac{qr-sp}{q}$,

and $\dfrac{dq}{dx}=\dfrac{\partial q}{\partial x}+\dfrac{\partial q}{\partial y}\cdot\dfrac{dy}{dx}=s+t\left(-\dfrac{p}{q}\right)=\dfrac{qs-pt}{q}$.

Substituting in (2) the values of dp/dx and dq/dx, we get

$$\frac{d^2y}{dx^2}=-\left[q\left(\frac{qr-ps}{q}\right)-p\left(\frac{qs-pt}{q}\right)\right]\cdot\left[\frac{1}{q^2}\right]$$

$$=-(q^2r-2pqs+p^2t)/q^3. \qquad \textbf{(Remember)}$$

Ex. 37. *Prove that* $\dfrac{d^2y}{dx^2}+\dfrac{2a^2x^2}{y^5}=0$, *where* $y^3-3ax^2+x^3=0$.

(Delhi 1983)

Sol. Let $f(x,y)\equiv y^3-3ax^2+x^3=0$.

Then $p=\partial f/\partial x=-6ax+3x^2,\ q=\partial f/\partial y=3y^2$,

$r=\partial^2 f/\partial x^2=-6a+6x,\ s=\partial^2 f/\partial x\partial y=0,\ t=\partial^2 f/\partial y^2=6y$.

Now $\dfrac{d^2y}{dx^2}=-\dfrac{q^2r-2pqs+p^2t}{q^3}$, refer Ex. 36

$$=-\frac{(-6a+6x)(3y^2)^2+6y(-6ax+3x^2)^2}{(3y^2)^3}$$

$$=-\frac{2(-a+x)y^3+2(-2ax+x^2)^2}{y^5}$$

$$=-\frac{2x(y^3+x^3-3ax^2)-2ay^3+8a^2x^2-2ax^3}{y^5} \qquad \textbf{(Note)}$$

$$=-\frac{-2a(y^3+x^3)+8a^2x^2}{y^5}, \qquad [\because\ y^3+x^3-3ax^2=0]$$

$$=-\frac{-2a(3ax^2)+8a^2x^2}{y^5}=-\frac{2a^2x^2}{y^5}.$$

Hence $\dfrac{d^2y}{dx^2}+\dfrac{2a^2x^2}{y^5}=0$.

Ex. 38 (a). *If* $ax^2+2hxy+by^2+2gx+2fy+c=0$, *find* d^2y/dx^2.

(b) *If* $ax^2+2hxy+by^2=1$, *find* d^2y/dx^2. **(Delhi 1983)**

Sol. **(a)** Let $F(x,y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$.

Then with usual notation *i.e.*, $\partial F/\partial x=p$, $\partial F/\partial y=q$, etc., we have

$p=2(ax+hy+g),\ q=2(hx+by+f)$,

$r=2a,\ s=2h$ and $t=2b$. Then from Ex. 36, we get

$$\frac{d^2y}{dx^2}=-8\,[(hx+by+f)^2a-2(ax+hy+g)(hx+by+f)h$$
$$+b(ax+hy+g)^2]/\{8(hx+by+f)^3\}$$

$$= \frac{-[(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh]}{(hx+by+f)^3}$$

$$= \frac{-[(ab-h^2)(-c)+af^2+bg^2-2fgh]}{(hx+by+f)^3}$$

$$= [abc+2fgh-af^2-bg^2-ch^2]/(hx+by+f)^3.$$

(b) Proceed exactly as in part (a).

Ex. 39. *If* $u = f(y-z, z-x, x-y)$, *prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

(Meerut 1983 S, 77; Lucknow 79)

Ex. 40. *If* $z = f(u, v)$, *where* $u = x^2-2xy-y^2$ *and* $v = y$, *show that*

$$(x+y)\frac{\partial z}{\partial x}+(x-y)\frac{\partial z}{\partial y}=(x-y)\frac{\partial z}{\partial v}.$$

(Agra 1983)

Ex. 41. *If z be a function of x and y and*
$x = e^u + e^{-v}$, $y = e^{-u} - e^v$, *prove that*

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}.$$

Ex. 42. *Transform the equation* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ *into polar coordinates.*

(Allahabad 1980; Gorakhpur 82; Kanpur 86)

Ex. 43. *Transform* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ *into polars and show that* $u = (Ar^n + Br^{-n})\sin n\theta$ *satisfies the above equation.*

For complete solutions of exercises 39 to 43 refer chapter 9 on change of independent variables.

□

5

Indeterminate Forms

§ 1. Definition.

If $f(x)$ and $\phi(x)$ be any two functions of x, then we know that

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)}, \text{ provided } \lim_{x \to a} \phi(x) \neq 0.$$

In case $\phi(x) \to 0$ but $\lim f(x) \neq 0$, the fraction tends to $+\infty$ or $-\infty$ or it may not have any limit.

When $f(x)$ and $\phi(x)$ both tend to zero as $x \to a$, then $\lim_{x \to a} \frac{f(x)}{\phi(x)}$, when written in the form $\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} \phi(x)}$ reduces to the form 0/0, which is meaningless.

A fraction whose numerator and denominator both tend to zero as $x \to a$ is called the **indeterminate form 0/0**. It has no definite value. The other indeterminate forms are ∞/∞, $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0 and ∞^0.

If a fraction $f(x)/\phi(x)$ takes the indeterminate form 0/0 when $x \to a$ it does not mean that $\lim_{x \to a} \frac{f(x)}{\phi(x)}$ will not exist. For example the fraction $\frac{x^2 - a^2}{x - a}$ takes the indeterminate form $\frac{0}{0}$ when $x \to a$. But $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x - a} = \lim_{x \to a} (x + a) = 2a$, showing that the limit exists in this case. Thus $\lim_{x \to a} \frac{f(x)}{\phi(x)}$ may exist even if the fraction $\frac{f(x)}{\phi(x)}$ takes the indeterminate form 0/0. In this chapter we shall give methods to find the limits of the functions which take indeterminate forms.

§ 2. The form 0/0. L'Hospital's Rule.

If $f(x)$ and $\phi(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = \phi(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)},$$

provided, the latter limit exists, finite or infinite.

(Gorakhpur 1982; Delhi 76; Agra 74; Punjab 76)

Proof. We have by Taylor's theorem,

$$\lim_{x \to a} \frac{f(x)}{\phi(x)}$$

$$= \lim_{x \to a} \frac{f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + R_1}{\phi(a) + (x-a)\phi'(a) + \frac{(x-a)^2}{2!}\phi''(a) + \dots + R_2},$$

where $R_1 = \frac{(x-a)^n}{n!} f^n \{a + \theta_1 (x-a)\},\ 0 < \theta_1 < 1$

and $R_2 = \frac{(x-a)^n}{n!} \phi^n \{a + \theta_2 (x-a)\},\ 0 < \theta_2 < 1.$

Now, since $f(a) = 0$ and $\phi(a) = 0$, we get, after dividing both numerator and denominator by $(x-a)$,

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(a) + (x-a)\{(1/2!) f''(a) + \dots\}}{\phi'(a) + (x-a)\{(1/2!)\phi''(a) + \dots\}}$$

$$= \frac{f'(a)}{\phi'(a)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

This proves the theorem which is generally known as *L' Hospital's Rule.*

Note. If $f'(a) = f''(a) = \dots = f^{n-1}(a) = 0$

and $\phi'(a) = \phi''(a) = \dots = \phi^{n-1}(a) = 0$

but $f^n(a)$ and $\phi^n(a)$ are not both zero, then by repeated application of Hospital's rule, we have

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f^n(x)}{\phi^n(x)}.$$

Rule : If the limit of f $(x)/\phi$ (x) as $x \to a$ takes the form 0/0, differentiate the numerator and denominator separately w.r.t x and obtain a new function f $'(x)/\phi'(x)$. Now as $x \to a$, if it again takes the form 0/0, differentiate the numerator and denominator again w.r.t. x and repeat the above process, till indeterminate form persists.

Caution. Before applying Hospital's rule at any stage be sure that the form is 0/0. Do not go on applying this rule even if the form is not 0/0.

Remember. *While applying L'Hospital's rule we are not to differentiate $f(x)/\phi(x)$ as a fraction. The numerator and denominator must be differentiated separately.*

§ 3. Method of expansion (Algebraic Methods).

In many cases the limit of an indeterminate form can be easily obtained by using some well known algebraic and trigonometrical expansions. We can also make use of some well-known limits in order to solve the problems or to shorten the work. The following expansions should be remembered :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots, \ |x| < 1$$

$$a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \ |x| < 1$$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right), \ |x| < 1$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots; \ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots; \ \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

Also **remember** that

$$\log 1 = 0, \ \log e = 1, \ \log \infty = \infty; \ \log 0 = -\infty.$$

Sometimes the use of the following limits shortens the work :

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1,$ **(ii)** $\lim_{x \to 0} \cos x = 1,$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1,$ **(iv)** $\lim_{x \to 0} (1+x)^{1/x} = e,$

(v) $\lim_{x \to 0} (1+nx)^{1/x} = e^n.$ **(vi)** $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e,$

(vii) $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a.$

Solved Examples

Ex. 1. *Evaluate* (*a*) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x},$

(*b*) $\lim_{x \to 0} \frac{1 - \sqrt{(1-x^2)}}{x^2},$

(c) $\lim_{x \to 0} \dfrac{xe^x - \log(1+x)}{x^2}$, **(Agra 1986)**

(d) $\lim_{x \to 0} \dfrac{\log(1-x^2)}{\log \cos x}$, **(Meerut 1977)**

(e) $\lim_{x \to 1} \dfrac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$ **(Meerut 1983)**

Sol. (a) We have

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{n(1+x)^{n-1}}{1}, \qquad \text{[by L'Hospital's rule]}$$

$$= n.$$

Aliter. We have

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = \lim_{x \to 0} \frac{1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots - 1}{x}$$

$$= \lim_{x \to 0} \left\{ n + \frac{n(n-1)}{2!}x + \ldots \right\} = n.$$

(b) $\lim_{x \to 0} \dfrac{1 - \sqrt{(1-x^2)}}{x^2}$, [form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{0 - \{\frac{1}{2}(1-x^2)^{-1/2}(-2x)\}}{2x}$$

$$= \lim_{x \to 0} \frac{x/\sqrt{(1-x^2)}}{2x} = \lim_{x \to 0} \frac{1}{2\sqrt{(1-x^2)}} = \frac{1}{2}.$$

(c) $\lim_{x \to 0} \dfrac{xe^x - \log(1+x)}{x^2}$, [form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{xe^x + e^x - \{1/(1+x)\}}{2x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{xe^x + e^x + e^x + \{1/(1+x)^2\}}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}.$$

(d) $\lim_{x \to 0} \dfrac{\log(1-x^2)}{\log \cos x}$. [form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{-2x/(1-x^2)}{-\sin x/\cos x} = \lim_{x \to 0} \left(\frac{2}{1-x^2} \cdot \frac{x}{\tan x} \right)$$

$$= \left(\lim_{x \to 0} \frac{2}{1-x^2} \right) \left(\lim_{x \to 0} \frac{x}{\tan x} \right) = 2 \times 1 = 2.$$

(e) We have

$$= \lim_{x \to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}, \quad \text{[form 0/0 so we shall apply L'Hospital's rule]}$$

$$= \lim_{x \to 1} \frac{5x^4 - 6x^2 - 8x + 9}{4x^3 - 6x^2 + 2}, \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 1} \frac{20x^3 - 12x - 8}{12x^2 - 12x}, \quad \text{[form 0/0, } \therefore \text{ again apply L'Hospital's rule]}$$

$$= \lim_{x \to 1} \frac{60x^2 - 12}{24x - 12} = \frac{60 - 12}{24 - 12} = \frac{48}{12} = 4.$$

Ex. 2. *Evaluate*

(a) $\lim_{x \to 0} \frac{\sin x}{x}$, **(b)** $\lim_{x \to 0} \frac{\tan x}{x}$, **(c)** $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

Sol. **(a)** $\lim_{x \to 0} \frac{\sin x}{x}$ [form 0/0]

$$= \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

(b) Proceeding as in part (a), we get

$$\lim_{x \to 0} \frac{\tan x}{x} = 1.$$

(c) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$, [form 0/0]

$$= \lim_{x \to 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}.$$

Ex. 3. *Evaluate* $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$.

Sol. We have $\lim_{x \to 0} \frac{a^x - 1}{b^x - 1}$ [form 0/0]

$$= \lim_{x \to 0} \frac{a^x \log a}{b^x \log b} = \frac{\log a}{\log b} = \log_b a.$$

Ex. 4. *Evaluate* $\lim_{x \to 0} \frac{x - \sin x}{x^3}$. **(Rohilkhand 1983; Meerut 94)**

Sol. Here $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ [form 0/0 so we shall apply L'Hospital's rule]

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} \quad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Aliter. $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \lim_{x \to 0} \frac{x - [x - (x^3/3\,!) + (x^5/5\,!) - ...]}{x^3}$

$$= \lim_{x \to 0} \frac{(x^3/6) - (x^5/120) + \dots}{x^3}$$

$$= \lim_{x \to 0} \left(\frac{1}{6} - \frac{x^2}{120} + \dots\right) = \frac{1}{6}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x}$.

Sol. $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x} = = \lim_{x \to 0} \left\{\frac{x - \sin x}{x^3} \cdot \left(\frac{x}{\tan x}\right)^3\right\}$ **(Note)**

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3} \cdot \lim_{x \to 0} \left(\frac{x}{\tan x}\right)^3$$

$$= \lim_{x \to 0} \frac{x - \sin x}{x^3}, \qquad \left[\because \lim_{x \to 0} \frac{x}{\tan x} = 1\right]$$

$= \frac{1}{6}$ [Proceeding as in Ex. 4.].

Ex. 6. *Evaluate* $\lim_{x \to 0} \frac{x - \tan x}{x^3}$.

(Delhi 1975; Kanpur 72; Kashmir 72; Meerut 73)

Sol. $\lim_{x \to 0} \frac{x - \tan x}{x^3}$, [form 0/0]

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-2 \sec x \,.\, \sec x \tan x}{6x}$$

$$= \lim_{x \to 0} -\frac{\sec^2 x}{3} \cdot \frac{\tan x}{x} = -\frac{1}{3} \times 1 = -\frac{1}{3}.$$

Aliter. $\lim_{x \to 0} \frac{x - \tan x}{x^3}$

$$= \lim_{x \to 0} \frac{x - [x + (x^3/3) + (2x^5/15) + ...]}{x^3}$$

$$= \lim_{x \to 0} \left(-\frac{1}{3} - \frac{2}{15}x^2 - \dots\right) = -\frac{1}{3}.$$

Ex. 7. (a). *Evaluate* $\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$.

(Meerut 1988S)

Sol. We have $\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$, [form 0/0]

$$= \lim_{x \to 0} \frac{\sin x}{6x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Ex. 7 (b). *Evaluate* $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$. **(Meerut 1991)**

Sol. $\lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

$$= \lim_{x \to 0} \left[\frac{x^2 + 2\cos x - 2}{x^4} \cdot \frac{x^3}{\sin^3 x} \right]$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x} \right)^3 \cdot \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 + 2\cos x - 2}{x^4}, \qquad \left[\because \lim_{x \to 0} \frac{x}{\sin x} = 1 \right]$$

$$= \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3},$$

by L'Hospital's rule for the form 0/0

$$= \lim_{x \to 0} \frac{2 - 2\cos x}{12x^2} \qquad \text{[form again 0/0]}$$

$$= \lim_{x \to 0} \frac{2\sin x}{24x} = \frac{1}{12} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{12} \cdot 1 = \frac{1}{12}.$$

Ex. 8. *Evaluate* $\lim_{x \to 0} \frac{1 - \cos x}{x \log(1 + x)}$.

Sol. $\lim_{x \to 0} \frac{1 - \cos x}{x \log(1 + x)}$, [form 0/0]

$$= \lim_{x \to 0} \frac{\sin x}{\log(1 + x) + \{x/(1 + x)\}}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{1 + x} + \frac{1 + x - x}{(1 + x)^2}} = \lim_{x \to 0} \frac{\cos x}{\frac{1}{1 + x} + \frac{1}{(1 + x)^2}} = \frac{1}{2}.$$

Ex. 9. *Evaluate* $\lim_{x \to 1} \frac{\log x}{x - 1}$. **(Agra 1983; Meerut 95, 98)**

Sol. $\lim_{x \to 1} \frac{\log x}{x - 1}$ [form 0/0]

$$= \lim_{x \to 1} \frac{1/x}{1} = 1.$$

Ex. 10. *Evaluate* $\lim_{x \to 0} \frac{a \sin x - \sin ax}{x (\cos x - \cos ax)}$.

Sol. $\lim_{x \to 0} \frac{a \sin x - \sin ax}{x (\cos x - \cos ax)}$, [form 0/0]

$$= \lim_{x \to 0} \frac{a \cos x - a \cos ax}{(\cos x - \cos ax) + x(-\sin x + a \sin ax)}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-a \sin x + a^2 \sin ax}{(-\sin x + a \sin ax) + (-\sin x + a \sin ax) + x(-\cos x + a \cos ax)}$$

$$= \lim_{x \to 0} \frac{-a \sin x + a^2 \sin ax}{2(a \sin ax - \sin x) + x(a^2 \cos ax - \cos x)}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-a \cos x + a^3 \cos ax}{2(a^2 \cos ax - \cos x) + (a^2 \cos ax - \cos x) + x(-a^3 \sin ax + \sin x)}$$

$$= \lim_{x \to 0} \frac{-a \cos x + a^3 \cos ax}{3(a^2 \cos ax - \cos x) + x(\sin x - a^3 \sin ax)}$$

$$= \frac{-a + a^3}{3(a^2 - 1)} = \frac{a}{3} \frac{(a^2 - 1)}{(a^2 - 1)} = \frac{a}{3}.$$

Ex. 11. *Evaluate* $\lim_{x \to 0} \frac{e^{ax} - e^{-ax}}{\log(1 + bx)}$.

Sol. $\lim_{x \to 0} \frac{e^{ax} - e^{-ax}}{\log(1 + bx)}$, [form 0/0]

$$= \lim_{x \to 0} \frac{ae^{ax} + ae^{-ax}}{b/(1 + bx)} = \frac{a + a}{b} = \frac{2a}{b}.$$

Ex. 12. *Evaluate* $\lim_{x \to 0} \frac{\log(1 + x^3)}{\sin^3 x}$.

Sol. $\lim_{x \to 0} \frac{\log(1 + x^3)}{\sin^3 x} = \lim_{x \to 0} \left\{ \frac{\log(1 + x^3)}{x^3} \cdot \left(\frac{x}{\sin x} \right)^3 \right\}$

$$= \lim_{x \to 0} \frac{\log(1 + x^3)}{x^3}, \text{ since } \lim_{x \to 0} \frac{x}{\sin x} = 1,$$

[form again 0/0]

$$= \lim_{x \to 0} \frac{3x^2/(1 + x^3)}{3x^2} = \lim_{x \to 0} \frac{1}{1 + x^3} = 1.$$

Ex. 13. *Evaluate* $\lim_{x \to 0} \frac{\sin x \sin^{-1} x}{x^2}$. **(Rohilkhand 1979; Raj. 79)**

Sol. $\lim_{x \to 0} \frac{\sin x \sin^{-1} x}{x^2} = \lim_{x \to 0} \left(\frac{\sin^{-1} x}{x} \cdot \frac{\sin x}{x} \right)$

$$= \lim_{x \to 0} \frac{\sin^{-1} x}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1/\sqrt{(1 - x^2)}}{1} = 1.$$

Ex. 14. *Evaluate* $\lim_{x \to 0} \frac{\log(1 + kx^2)}{1 - \cos x}$.

Sol. $\lim_{x \to 0} \frac{\log(1 + kx^2)}{1 - \cos x}$, [form 0/0]

$$= \lim_{x \to 0} \frac{2kx/(1+kx^2)}{\sin x} = \lim_{x \to 0} \left(\frac{2k}{1+kx^2} \cdot \frac{x}{\sin x} \right)$$
$$= 2k \times 1 = 2k.$$

Ex. 15. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x - \sin x}$.

Sol. We have $\lim_{x \to 0} \dfrac{e^x - e^{x \cos x}}{x - \sin x}$, [Form 0/0]

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)}{1 - \cos x}, \text{ by Hospital's Rule}$$

[Form again 0/0]

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)^2 - e^{x \cos x}(-2 \sin x - x \cos x)}{\sin x},$$

by Hospital's rule

$$= \lim_{x \to 0} \frac{e^x - e^{x \cos x}(\cos x - x \sin x)^2 + e^{x \cos x}(2 \sin x + x \cos x)}{\sin x},$$

[form 0/0]

$$= \lim_{x \to 0} \frac{\begin{matrix} e^x - e^{x \cos x}(\cos x - x \sin x)^3 - 2e^{x \cos x}(\cos x - x \sin x) \times \\ (-2 \sin x - x \cos x) + e^{x \cos x}(\cos x - x \sin x) . (2 \sin x \\ + x \cos x) + e^{x \cos x}(3 \cos x - x \sin x) \end{matrix}}{\cos x}$$

$$= \frac{1 - 1(1-0)^3 - 2 \times 1 \times 1 \times 0 + 1 \times 1 \times 0 + 1 \times (3 - 0)}{1}$$

$$= \frac{1 - 1 + 3}{1} = 3.$$

Ex. 16. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$. **(Kanpur 1979; Meerut 94P, 96P)**

Sol. Here $\text{Nr.} = e^x - e^{\sin x} = e^x - e^{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$

$$= e^x - e^x . e^{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}$$

$$= e^x (1 - e^z), \text{ where } z = -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$= e^x \left[1 - \left(1 + z + \frac{z^2}{2!} + \dots \right) \right] = -e^x \left[z + \frac{z^2}{2!} + \dots \right]$$

$$= -e^x \left[\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) + \frac{1}{2!} \left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)^2 + \dots \right]$$

$$= -e^x \left[-\frac{x^3}{6} + \frac{x^5}{120} - \dots \right] = x^3 e^x \left(\frac{1}{6} - \frac{x^2}{120} + \dots \right).$$

Also Denom. $= x - \sin x = x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots\right)$

$$= \frac{x^3}{6} - \frac{x^5}{120} + \dots = x^3\left(\frac{1}{6} - \frac{x^2}{120} + \dots\right).$$

$$\therefore \lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} \frac{x^3 \cdot e^x [(1/6) - (x^2/120) + \dots]}{x^3 [(1/6) - (x^2/120) + \dots]}$$

$$= \lim_{x \to 0} \frac{e^x [(1/6) - (x^2/120) + \dots]}{(1/6) - (x^2/120) + \dots} = \frac{1/6}{1/6} = 1.$$

Note. The students can also solve Ex. 15 by the method we adopted in Ex. 16.

Ex. 17. *Evaluate* $\displaystyle\lim_{x \to b} \frac{x^b - b^x}{x^x - b^b}$. **(Meerut 1983, Rohilkhand 87)**

Sol. Here the form is 0/0. To differentiate the Dr. we shall require the diff. coeff. of x^x. Let $y = x^x$; then $\log y = x \log x$.

Differentiating, $\dfrac{1}{y} \cdot \dfrac{dy}{dx} = x \cdot \dfrac{1}{x} + \log x$

or $\dfrac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$. $[\because y = x^x]$.

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x).$$

$$\therefore \lim_{x \to b} \frac{x^b - b^x}{x^x - b^b} = \lim_{x \to b} \frac{bx^{b-1} - b^x \log b}{x^x(1 + \log x) - 0}, \text{ by Hospital's rule}$$

$$= \frac{b \cdot b^{b-1} - b^b \log b}{b^b(1 + \log b)} = \frac{b^b(1 - \log b)}{b^b(1 + \log b)} = \frac{1 - \log b}{1 + \log b}.$$

****Ex. 18.** *Evaluate* $\displaystyle\lim_{a \to b} \frac{a^b - b^a}{a^a - b^b}$.

Sol. Here $a \to b$. Therefore, while differentiating we shall regard a as variable and b as constant. **(Note)**

We have $\displaystyle\lim_{a \to b} \frac{a^b - b^a}{a^a - b^b}$, [form 0/0]

$$= \lim_{a \to b} \frac{ba^{b-1} - b^a \log b}{a^a(1 + \log a) - 0}, \quad \left[\because \frac{d}{da}(a^a) = a^a(1 + \log a)\right].$$

$$= \frac{b \cdot b^{b-1} - b^b \log b}{b^b(1 + \log b)} = \frac{b^b(1 - \log b)}{b^b(1 + \log b)} = \frac{1 - \log b}{1 + \log b}.$$

Ex. 19. *Evaluate* $\displaystyle\lim_{x \to 0} \frac{[e^x + \log\{(1 - x)/e\}]}{\tan x - x}$.

(Rohilkhand 1986, Gorakhpur 82; Kanpur 79; Lucknow 75)

Sol. $$\lim_{x \to 0} \frac{e^x + \log\left(\dfrac{1 - x}{e}\right)}{\tan x - x} = \lim_{x \to 0} \frac{e^x + \log(1 - x) - \log e}{\tan x - x}$$

(Note)

$$= \lim_{x \to 0} \frac{e^x + \log(1-x) - 1}{\tan x - x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \left(-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots\right) - 1}{\left(x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots\right) - x}$$

$$= \lim_{x \to 0} \frac{-\frac{1}{6}x^3\,(1 + \text{terms containing } x \text{ and its higher powers})}{\frac{1}{3}x^3\,(1 + \text{terms containing } x \text{ and its higher powers})}$$

$$= -\tfrac{1}{2}.$$

Ex. 20. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}.$

(Agra 1982, Meerut 82, 96 BP, 98; Vikram 77)

Sol. $\displaystyle \lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2} \qquad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - 1/(1+x)}{2x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-\sin x - \sin x - x \cos x + 1/(1+x)^2}{2} = \tfrac{1}{2}.$$

Ex. 21. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x}.$

Sol. $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^2 \sin x}, \qquad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \left\{\frac{e^x - e^{-x} - 2x}{x^3} \cdot \left(\frac{x}{\sin x}\right)\right\}, \qquad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x^3}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{3x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{6x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}}{6} = \frac{1+1}{6} = \frac{1}{3}.$$

Ex. 22. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}.$

(Meerut 1990, 97; Agra 85; Delhi 86; Kashmir 74; Punjab 72)

Sol. $\displaystyle \lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}, \qquad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \left\{\frac{\cosh x - \cos x}{x^2} \cdot \left(\frac{x}{\sin x}\right)\right\}, \qquad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{2x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{2} = \frac{1 + 1}{2} = 1.$$

Ex. 23. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - \log (e + ex)}{x^2}$.

Sol. $\lim_{x \to 0} \dfrac{e^x - \log (e + ex)}{x^2} = \lim_{x \to 0} \dfrac{e^x - \log \{e (1 + x)\}}{x^2}$

$$= \lim_{x \to 0} \frac{e^x - \log e - \log (1 + x)}{x^2}$$

$$= \lim_{x \to 0} \frac{e^x - 1 - \log (1 + x)}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - \{1/(1 + x)\}}{2x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + \{1/(1 + x)^2\}}{2} = \frac{1 + 1}{2} = 1.$$

Ex. 24. *Evaluate* $\lim_{x \to 0} \dfrac{e^x - 2 \cos x + e^{-x}}{x \sin x}$.

Sol. $\lim_{x \to 0} \dfrac{e^x - 2 \cos x + e^{-x}}{x \sin x}$

$$= \lim_{x \to 0} \left\{\frac{e^x - 2 \cos x + e^{-x}}{x^2} \cdot \frac{x}{\sin x}\right\}, \qquad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{e^x - 2 \cos x + e^{-x}}{x^2}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + 2 \sin x - e^{-x}}{2x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x + 2 \cos x + e^{-x}}{2} = \frac{1 + 2 + 1}{2} = 2.$$

Ex. 25. *Evaluate* $\lim_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^3}$, $\qquad \left[\text{form } \dfrac{0}{0}\right]$

Sol. Required limit,

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - x - x^2}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} x^3 + \text{terms containing powers of } x \text{ higher than } 3}{x^3}$$

$$= \lim_{x \to 0} \left(\tfrac{1}{3} + \text{terms containing } x \text{ and its higher powers}\right) = \tfrac{1}{3}.$$

Ex. 26. *Evaluate* $\lim_{x \to 1} \frac{x^x - x}{1 - x + \log x}$. **(Kanpur 1979)**

Sol. $\lim_{x \to 1} \frac{x^x - x}{1 - x + \log x}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x \to 1} \frac{x^x (1 + \log x) - 1}{-1 + (1/x)}$, $\left\{\because \frac{d}{dx}(x^x) = x^x (1 + \log x)\right\}$

[form again 0/0]

$= \lim_{x \to 1} \frac{x^x (1/x) + x^x (1 + \log x) . (1 + \log x)}{(-1/x^2)}$

$= \frac{1 + 1 \ (1 + 0) \ (1 + 0)}{-1} = -2.$

Ex. 27. *Evaluate* $\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$. **(Indore 1974)**

Sol. $\lim_{x \to \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x \to \frac{1}{2}} \frac{2 \cos \pi x . (-\pi \sin \pi x)}{2e^{2x} - 2e}$

$= \lim_{x \to \frac{1}{2}} \frac{-\pi \sin 2\pi x}{2e^{2x} - 2e}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x \to \frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{4e^{2x}} = \frac{-2\pi^2 (-1)}{4e} = \frac{\pi^2}{2e}.$

Ex. 28. *Evaluate* $\lim_{\theta \to 0} \frac{\sin \theta - \theta \cos \theta}{\sin \theta - \theta}$.

Sol. $\lim_{\theta \to 0} \frac{\sin \theta - \theta \cos \theta}{\sin \theta - \theta}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{\theta \to 0} \frac{\cos \theta - \cos \theta + \theta \sin \theta}{\cos \theta - 1} = \lim_{\theta \to 0} \frac{\theta \sin \theta}{\cos \theta - 1}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{\theta \to 0} \frac{\theta \cos \theta + \sin \theta}{-\sin \theta}$, $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{\theta \to 0} \frac{\cos \theta - \theta \sin \theta + \cos \theta}{-\cos \theta} = \frac{1 - 0 + 1}{-1} = -2.$

Ex. 29. *Evaluate* $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log (1 - x)}$. **(Gorakhpur 1989)**

Sol. $\lim_{x \to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log (1 - x)}$ $\left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - x - x^2}{x^2 + x\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)}$$

$$= \lim_{x \to 0} \frac{\{x + x^2 - \frac{1}{6}x^3 + \frac{1}{2}x^3 + \text{terms containing powers of } x \text{ higher than } 3\} - x - x^2}{-\frac{1}{2}x^3 + \text{terms containing powers of } x \text{ higher than } 3}$$

$$= \lim_{x \to 0} \frac{x^3 [\frac{1}{3} + \text{terms containing } x \text{ and its higher powers}]}{x^3 [-\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}]}$$

$$= \frac{\frac{1}{3}}{-\frac{1}{2}} = -\frac{2}{3}.$$

Ex. 30. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$.

(Meerut 1977, 96 BP; Rohilkhand 88)

Sol. $\displaystyle \lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$, $\qquad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x \to 0} \frac{\frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x^5} = \lim_{x \to 0} \left(\frac{1}{5!} - \frac{x^2}{7!} + \ldots\right) = \frac{1}{5!} = \frac{1}{120}.$$

Ex. 31. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{a^x - 1 - x \log a}{x^2}$.

Sol. $\displaystyle \lim_{x \to 0} \frac{a^x - 1 - x \log a}{x^2}$, $\qquad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{a^x \log a - \log a}{2x}, \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{a^x (\log a)^2}{2} = \tfrac{1}{2} (\log a)^2.$$

****Ex. 32.** *Evaluate* $\displaystyle \lim_{x \to 0} \frac{(1 + x)^{1/x} - e}{x}$.

(Rohilkhand 1990; Kanpur 86; Gorakhpur 87; Allahabad 81; Delhi 76; G.N.U. 74; Agra 88; Bihar 73; U.P. P.C.S. 97; Magadh 71; Vikram 76)

Sol. Here $\displaystyle \lim_{x \to 0} \frac{(1 + x)^{1/x} - e}{x}$ is of the form $\dfrac{0}{0}$ because $\displaystyle \lim_{x \to 0} (1 + x)^{1/x} = e$. First we shall obtain an expansion for $(1 + x)^{1/x}$ in ascending powers of x. Let $y = (1 + x)^{1/x}$. Then

$$\log y = \frac{1}{x} \log (1 + x) = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$= 1 + z, \quad \text{where } z = -(x/2) + (x^2/3) - \ldots$$

$$\therefore \quad y = e^{1+z} = e \,.\, e^z = e \cdot \left(1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \ldots\right)$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \ldots\right)^2 + \ldots\right]$$

$$= e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8}x^2 + \text{terms containing powers of } x \text{ higher than } 3\right]$$

$$= e\left[1 - \tfrac{1}{2}x + \tfrac{11}{24}x^2 + \ldots\ldots\right].$$

Now $$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right] - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left[-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right]}{x}$$

$$= \lim_{x \to 0} e\left[-\tfrac{1}{2} + \tfrac{11}{24}x + \ldots\ldots\right] = -\tfrac{1}{2}e.$$

Ex. 33. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$.

Sol. $$\lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x \sin x}$$

$$= \lim_{x \to 0} \left\{\frac{e^x - e^{-x} - 2\log(1+x)}{x^2} \cdot \frac{x}{\sin x}\right\} \qquad \textbf{(Note)}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} - 2\log(1+x)}{x^2}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2/(1+x)}{2x}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + 2/(1+x)^2}{2} = \frac{1 - 1 + 2}{2} = 1.$$

Ex. 34. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

Sol. $$\lim_{x \to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{2\cos 2x + 4\sin x \cos x - 2\cos x}{-\sin x - 2\cos x\,(-\sin x)}$$

$$= \lim_{x \to 0} \frac{2\cos 2x + 2\sin 2x - 2\cos x}{-\sin x + \sin 2x}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 4\cos 2x + 2\sin x}{-\cos x + 2\cos 2x} = \frac{-0 + 4 + 0}{-1 + 2} = 4.$$

**Ex. 35. *Evaluate* $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$.

(Meerut 1982, 83, 85, 88, 94P)

Sol. Here $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2}$, [form 0/0]

$$= \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \dots\dots\right] - e + \frac{1}{2}ex}{x^2} \quad \text{(See Ex. 32)}$$

$$= \lim_{x \to 0} \frac{e\,[(11/24)\,x^2 + \text{terms containing higher powers of } x]}{x^2}$$

$$= \lim_{x \to 0} e\,[(11/24) + \text{terms containing } x \text{ and its higher powers}]$$

$= (11/24)\,e$.

Ex. 36. *Evaluate* $\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$.

(Kumayun 1983; Agra 81; Bundelkhand 78; U.P. P.C.S. 94)

Sol. $\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$, [form 0/0]

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[\{1 + x + (x^2/2\,!) + \dots\dots\} - 1]^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[x + (x^2/2) + \dots\dots]^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{x^{3/2}\,[1 + (x/2) + \dots\dots]^{3/2}}$$

$$= \lim_{x \to 0} \frac{\tan x}{x\,[1 + (x/2) + \dots\dots]^{3/2}}$$

$$= \lim_{x \to 0} \left[\frac{1}{[1 + (x/2) + \dots\dots]^{3/2}} \cdot \frac{\tan x}{x}\right]$$

$= 1 \cdot 1 = 1.$

Ex. 37. *Evaluate* $\lim_{\to 0} \frac{\tan x - x}{x^2 \tan x}$.

(Meerut 1990P, 91S)

Sol. $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x} = \lim_{x \to 0} \left[\frac{\tan x - x}{x^3} \quad \frac{x}{\tan x}\right]$, **(Note)**

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{2 \sec x \,.\, \sec x \tan x}{6x}$$

$$= \lim_{x \to 0} \left(\frac{1}{3} \cdot \sec^2 x \cdot \frac{\tan x}{x}\right) = \frac{1}{3} \cdot 1.\,1 = \frac{1}{3}.$$

Ex. 38. *Evaluate* $\lim_{x \to +\infty} \dfrac{a^{1/x} - b^{1/x}}{\log \{x/(x-1)\}}$.

Sol. We have $\lim_{x \to +\infty} \dfrac{a^{1/x} - b^{1/x}}{\log \{x/(x-1)\}}$

$$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log [x/x \{1 - (1/x)\}]} = \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log [1/\{1 - (1/x)\}]}$$

$$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{\log 1 - \log \{1 - (1/x)\}}$$

$$= \lim_{x \to +\infty} \frac{a^{1/x} - b^{1/x}}{-\log \{1 - (1/x)\}}, \qquad [\text{form } 0/0]$$

$$= \lim_{x \to +\infty} \frac{(a^{1/x} \log a)(-1/x^2) - (b^{1/x} \log b)(-1/x^2)}{-1/\{1 - (1/x)\} \,.\, (1/x^2)},$$

by Hospital's rule

$$= \lim_{x \to +\infty} \frac{b^{1/x} \log b - a^{1/x} \log a}{-1/\{1 - (1/x)\}},$$

cancelling $1/x^2$ from the Nr. and the Dr.

$$= \frac{b^0 \log b - a^0 \log a}{-1/(1-0)} = \frac{\log b - \log a}{-1} = \log a - \log b = \log (a/b).$$

Ex. 39. *Evaluate* $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$.

Sol. We have $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$, [form 0/0]

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\dots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\dots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{3} + \frac{1}{3!}\right) x^3 + \left(\frac{2}{15} - \frac{1}{5!}\right) x^5 + \dots\dots}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} x^3 + (1/8) x^5 + \dots\dots}{x^3} = \lim_{x \to 0} \left(\tfrac{1}{2} + \tfrac{1}{8} x^2 + \dots\dots\right) = \tfrac{1}{2}.$$

Ex. 40. *Evaluate* $\lim_{x \to 0} \left\{\dfrac{a^x - b^x}{x}\right\}$.

(Rohilkhand 1991; Kashmir 84; Ranchi 74; Vikram 78; Meerut 94)

Sol. We have $\lim_{x \to 0} \left\{\dfrac{(a^x - b^x)}{x}\right\}$, [form 0/0]

$$= \lim_{x \to 0} \frac{a^x \log a - b^x \log b}{1}, \qquad \text{by Hospital's rule}$$

$$= \log a - \log b = \log (a/b).$$

Ex. 41. *Evaluate* $\lim_{x \to 0} \dfrac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$.

(Meerut 1983S; Lucknow 74)

Sol. $\lim_{x \to 0} \dfrac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$, [form 0/0]

$$= \lim_{x \to 0} \frac{5\left(x - \frac{x^3}{3!} + \ldots\right) - 7\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} + 3\left\{(3x) - \frac{(3x)^3}{3!} + \ldots\right\}}{(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots) - x}$$

$$= \lim_{x \to 0} \frac{x^3\left(-\frac{5}{6} + \frac{28}{3} - \frac{27}{2} + \text{higher powers of } x\right)}{\left(\frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\right)}$$

$$= \lim_{x \to 0} \frac{-5 + \text{terms containing } x \text{ and its higher powers}}{\left(\frac{1}{3} + \frac{2}{15}x^2 + \ldots\right)}$$

$$= -\frac{5}{1/3} = -15.$$

****Ex. 42.** *Find the values of a and b in order that* $\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$ *may be equal to* 1.

(Agra 1980; Meerut 1981, 83, 85, 90S, 92; Garhwal 77; G.N.U. 72; Kashmir 71; Kanpur 87; Raj. 73)

Sol. We have $\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3}$,

[form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x \to 0} \frac{1 + a \cos x - ax \sin x - b \cos x}{3x^2}. \quad \ldots(1)$$

Now the denominator of (1) $\to 0$ as $x \to 0$. Therefore if the numerator of (1) does not tend to 0 as $x \to 0$, then the given limit cannot be equal to 1. Hence for the given limit to be equal to 1 the numerator of (1) must also $\to 0$ as $x \to 0$.

$\therefore \quad 1 + a - b = 0 \quad$ or $\quad a - b = -1. \quad \ldots(2)$

Now if $1 + a - b = 0$, then (1) takes the form 0/0. Hence by applying L' Hospital's rule to (1), the given limit is equal to

$$\lim_{x \to 0} \frac{-a \sin x - a \sin x - ax \cos x + b \sin x}{6x}$$

$$= \lim_{x \to 0} \frac{-ax \cos x + (b - 2a) \sin x}{6x}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-a \cos x + ax \sin x + (b - 2a) \cos x}{6},$$

[by L'Hospital's rule]

$$= \frac{-a + b - 2a}{6} = \frac{b - 3a}{6} = 1, \text{ (as given).}$$

$\therefore \quad b - 3a = 6. \quad \ldots(3)$

Adding (2) and (3), we have $-2a = 5$ or $a = -5/2$.

$\therefore \quad b = a + 1 = (-5/2) + 1 = -3/2.$

Hence $a = -5/2, \quad b = -3/2.$

Ex. 43. *Find the values of a and b in order that*

$\lim_{x \to 0} \dfrac{x(1 - a\cos x) + b \sin x}{x^3}$ *may be equal to* $\frac{1}{3}$. **(Delhi 1977)**

Sol. Proceed as in Ex. 42. The answer is $a = \frac{1}{2}$ and $b = -\frac{1}{2}$.

Ex. 44. *Find the values of a, b, c so that*

$$\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2.$$

(Meerut 1982, 96)

Sol. Here the given limit

$$= \lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x}, \qquad \left[\text{form } \frac{a - b + c}{0}\right].$$

$\therefore$ for the given limit to be equal to 2, we must have

$$a - b + c = 0. \qquad \text{...(1)}$$

Now applying L'Hospital's rule for the form 0/0, we have the given limit

$$= \lim_{x \to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}, \qquad \left[\text{form } \frac{a - c}{0}\right].$$

$\therefore$ for the given limit to be equal to 2, we must have

$$a - c = 0. \qquad \text{...(2)}$$

Now again applying L' Hospital's rule for the form 0/0, we have the given limit

$$= \lim_{x \to 0} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} = \frac{a + b + c}{2}.$$

$\therefore$ for the given limit to be equal to 2, we must have

$$(a + b + c)/2 = 2 \text{ i.e., } a + b + c = 4. \qquad \text{...(3)}$$

Solving (1), (2) and (3), we get $a = 1$, $b = 2$, $c = 1$.

Ex. 45. *If* $\lim_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ *be finite, find the value of 'a' and the limit.*

Sol. We have $\lim_{x \to 0} \dfrac{\sin 2x + a\sin x}{x^3}$, [form 0/0]

$$= \lim_{x \to 0} \frac{2\cos 2x + a\cos x}{3x^2} \qquad \text{...(1)}$$

Now the denominator of (1) $\to 0$ as $x \to 0$.

But if the numerator of (1) does not tend to zero as $x \to 0$, then the given limit becomes infinite. Therefore for the given limit to be finite the numerator of (1) must $\to 0$ as $x \to 0$.

Hence $\quad 2 + a = 0$ or $a = -2$.

With this value of a, we get from (1), the given limit

$$= \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6} = \frac{-8+2}{6} = -\frac{6}{6} = -1.$$

Form II : ∞/∞.

If $f(x)$ and $\phi(x)$ be two functions such that

$$\lim_{x \to a} f(x) = \infty \quad \textbf{and} \quad \lim_{x \to a} \phi(x) = \infty, \textbf{ then}$$

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}, \textbf{ provided the limit exists.}$$

Proof. We have, $\lim_{x \to a} \dfrac{f(x)}{\phi(x)}$

$$= \lim_{x \to a} \left[\frac{1}{\phi(x)}\right] \Big/ \left[\frac{1}{f(x)}\right], \qquad \text{[form 0/0]}$$

$$= \lim_{x \to a} \left[-\frac{1}{\{\phi(x)\}^2}\phi'(x)\right] \Big/ \left[-\frac{1}{\{f(x)\}^2} f'(x)\right],$$

[By L' Hospital's rule]

$$= \left[\lim_{x \to a} \frac{\phi'(x)}{f'(x)}\right] \left[\lim_{x \to a} \frac{f(x)}{\phi(x)}\right]^2 \qquad ...(1)$$

Now three cases arise :

Case I. *When* $\lim_{x \to a} f(x)/\phi(x)$ *is neither zero nor infinite..*

In this case dividing both sides of (1) by

$\left[\lim_{x \to a} \{f(x)/\phi(x)\}\right]^2$, we get

$$\left[\lim_{x \to a} \frac{f(x)}{\phi(x)}\right]^{-1} = \lim_{x \to a} \frac{\phi'(x)}{f'(x)}$$

or
$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Case II. *When* $\lim_{x \to a} \left\{f(x)/\phi(x)\right\} = 0.$

In this case, we have

$$1 + \lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \left[1 + \frac{f(x)}{\phi(x)}\right]$$

$$= \lim_{x \to a} \frac{\phi(x) + f(x)}{\phi(x)}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to a} \frac{\phi'(x) + f'(x)}{\phi'(x)},$$

[by case I, since the form is ∞/∞ and the limit is equal to $1+0$ *i.e.*, 1 which is neither zero nor infinite]

$$= \lim_{x \to a} \left[1 + \frac{f'(x)}{\phi'(x)}\right] = 1 + \lim_{x \to a} \frac{f'(x)}{\phi'(x)}. \qquad ...(2)$$

Subtracting 1 from both sides of (2), we get

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Case III. When $\lim_{x \to a} \{f(x)/\phi(x)\}$ is infinite. In this case,

$$\lim_{x \to a} \frac{\phi(x)}{f(x)} = 0 = \lim_{x \to a} \frac{\phi'(x)}{f'(x)}, \qquad \text{[by case II].}$$

$$\therefore \quad \lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}.$$

Note. The above proposition is also true when $x \to \infty$ or $-\infty$ in place of a.

Important. It is interesting to note that in both cases when the form is ∞/∞ or $0/0$ the rule of evaluating the limit by differentiating the numerator and denominator separately holds good. But while evaluating $\lim_{x \to a} \{f(x)/\phi(x)\}$, when it is of the form ∞/∞, it is sometimes necessary to change it into the form $0/0$, otherwise the process of differentiating the numerator and the denominator will never end.

Remember. $\log 0 = -\infty$, and $\log \infty = \infty$.

Solved Examples

Ex. 1. *Evaluate* $\lim_{x \to \infty} \dfrac{x^2+2x}{5-3x^2}$.

Sol. We have, $\lim_{x \to \infty} \dfrac{x^2+2x}{5-3x^2}$, [form ∞/∞]

$$= \lim_{x \to \infty} \frac{2x+2}{-6x}, \qquad \text{[form } \infty/\infty\text{]}$$

$$= \lim_{x \to \infty} \frac{2}{-6} = -\frac{1}{3}.$$

Aliter. $\lim_{x \to \infty} \dfrac{x^2+2x}{5-3x^2} = \lim_{x \to \infty} \dfrac{x^2\{1+(2/x)\}}{x^2\{(5/x^2)-3\}}$

$$= \lim_{x \to \infty} \frac{1+(2/x)}{(5/x^2)-3} = -\tfrac{1}{3}.$$

Ex. 2. *Evaluate* $\lim_{x \to 0} \dfrac{\log x}{\cot x}$. **(Kanpur 1988; Meerut 98)**

Sol. We have, $\lim_{x \to 0} \dfrac{\log x}{\cot x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x}, \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-2\sin x \cos x}{1} = \frac{-2 \times 0 \times 1}{1} = 0.$$

Ex. 3. *Evaluate* $\lim_{x \to 0} \dfrac{\log \sin 2x}{\log \sin x}$. **(Meerut 1994)**

Sol. We have $\lim_{x \to 0} \dfrac{\log \sin 2x}{\log \sin x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/\sin 2x)(2\cos 2x)}{(1/\sin x)\cos x} = \lim_{x \to 0} \frac{2\cot 2x}{\cot x}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{2\tan x}{\tan 2x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{2\sec^2 x}{2\sec^2 2x}, \qquad \text{(By L' Hospital's rule)}$$

$$= 1.$$

Ex. 4. *Evaluate* $\lim_{x \to \infty} \left(\dfrac{\log x}{x}\right)$

Sol. We have $\lim_{x \to \infty} \dfrac{\log x}{x}$, [form ∞/∞]

$$= \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \left(\frac{1}{x}\right) = 0.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \dfrac{\log \sin x}{\cot x}$.

Sol. $\lim_{x \to 0} \dfrac{\log \sin x}{\cot x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{\cos x/\sin x}{-\operatorname{cosec}^2 x} = \lim_{x \to 0} \left(-\frac{\cos x}{\sin x}\sin^2 x\right)$$

$$= \lim_{x \to 0} (-\sin x \cos x) = 0.$$

Ex. 6. *Evaluate* $\lim_{x \to \pi/2} \dfrac{\log (x - \frac{1}{2}\pi)}{\tan x}$. **(Meerut 1989P, 97)**

Sol. We have $\lim_{x \to \frac{1}{2}\pi} \dfrac{\log (x - \frac{1}{2}\pi)}{\tan x}$, [form ∞/∞]

$$= \lim_{x \to \pi/2} \frac{1/(x - \frac{1}{2}\pi)}{\sec^2 x} = \lim_{x \to \pi/2} \frac{\cos^2 x}{x - \frac{1}{2}\pi}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-2\cos x \sin x}{1} = \lim_{x \to \pi/2} (-\sin 2x) = 0.$$

Ex. 7. *Evaluate* $\lim_{x \to 0} \dfrac{\log \sin x}{\log x}$.

Sol. We have $\lim_{x \to 0} \dfrac{\log \sin x}{\log x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{\cos x/\sin x}{1/x} = \lim_{x \to 0} x \cot x = \lim_{x \to 0} \frac{x}{\tan x} = 1.$$

Ex. 8. *Evaluate* $\lim_{x \to 0} \dfrac{\log x^2}{\cot x^2}$.

Sol. We have $\lim_{x \to 0} \dfrac{\log x^2}{\cot x^2}$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/x^2) \cdot 2x}{(-\operatorname{cosec}^2 x^2)(2x)} = \lim_{x \to 0} \left(-\frac{\sin^2 x^2}{x^2}\right)$$

$$= \lim_{x \to 0} \left\{\left(\frac{\sin x^2}{x^2}\right)(-\sin x^2)\right\}$$ **(Note)**

$$= \lim_{x \to 0} (-\sin x^2) = 0. \qquad \left[\because \lim_{x \to 0} \frac{\sin x^2}{x^2} = 1\right]$$

Ex. 9. *Evaluate* $\lim_{x \to 0} \dfrac{\log \tan x}{\log x}$.

Sol. We have $\lim_{x \to 0} \dfrac{\log \tan x}{\log x}$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{\sec^2 x/\tan x}{1/x} = \lim_{x \to 0} \left\{\left(\frac{x}{\tan x}\right)\left(\frac{1}{\cos^2 x}\right)\right\} = 1 \times 1 = 1.$$

Ex. 10. *Evaluate* $\lim_{x \to \infty} x^m e^{-x}$.

(K.U. 1977)

Sol. We have $\lim_{x \to \infty} x^m e^{-x} = \lim_{x \to \infty} \dfrac{x^m}{e^x}$, [form ∞/∞]

$$= \lim_{x \to \infty} \frac{m x^{m-1}}{e^x}, \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to \infty} \frac{m(m-1)x^{m-2}}{e^x}, \qquad \text{[form } \infty/\infty]$$

....

$$= \lim_{x \to \infty} \frac{m(m-1)(m-2)\ldots 3 \cdot 2 \cdot 1}{e^x} = \lim_{x \to \infty} \frac{m!}{e^x}$$

$$= 0, \qquad [\because e^x \to \infty \text{ when } x \to \infty].$$

Ex. 11. *Evaluate* $\lim_{x \to 0+} \dfrac{\operatorname{cosec} x}{\log x}$.

(Agra 1978)

Sol. $\lim_{x \to 0+} \dfrac{\operatorname{cosec} x}{\log x}$, [form ∞/∞]

$$= \lim_{x \to 0+} \frac{-\operatorname{cosec} x \cot x}{1/x} = \lim_{x \to 0+} -\left[\left(\frac{x}{\tan x}\right) \operatorname{cosec} x\right]$$

$$= \lim_{x \to 0+} (-\operatorname{cosec} x) = -\infty.$$

Ex. 12. *Evaluate* $\lim\limits_{x \to \infty} \dfrac{3x+4}{\sqrt{(2x^2+5)}}$. **(K.U. 1975)**

Sol. We have $\lim\limits_{x \to \infty} \dfrac{3x+4}{\sqrt{(2x^2+5)}}$, [form ∞/∞]

$$= \lim_{x \to \infty} \frac{x\,[3+(4/x)]}{x\,[\sqrt{\{2+(5/x^2)\}}]} = \lim_{x \to \infty} \frac{3+(4/x)}{\sqrt{\{2+(5/x^2)\}}} = \frac{3}{\sqrt{2}}.$$

Ex. 13. *Evaluate* $\lim\limits_{x \to \infty} \dfrac{\log(x-a)}{\log(e^x - e^a)}$.

Sol. $\lim\limits_{x \to a} \dfrac{\log(x-a)}{\log(e^x - e^a)}$, [form ∞/∞]

$$= \lim_{x \to a} \frac{1/(x-a)}{\{1/(e^x-e^a)\}\,e^x} = \lim_{x \to a} \frac{e^x - e^a}{e^x\,(x-a)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to a} \frac{e^x}{e^x\,(x-a) + e^x} = \lim_{x \to a} \frac{e^x}{e^x\,[(x-a)+1]}$$

$$= \lim_{x \to a} \frac{1}{x-a+1} = 1.$$

Ex. 14. *Evaluate* $\lim\limits_{x \to 0} \dfrac{\log(\tan^2 2x)}{\log(\tan^2 x)}$. **(Magadh 1972)**

Sol. We have $\lim\limits_{x \to 0} \dfrac{\log(\tan^2 2x)}{\log(\tan^2 x)}$

$$= \lim_{x \to 0} \frac{2\log(\tan 2x)}{2\log(\tan x)}, \quad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{(1/\tan 2x)\,.\,2\sec^2 2x}{(1/\tan x)\sec^2 x} = \lim_{x \to 0} \frac{2\tan x \cos^2 x}{\tan 2x \cos^2 2x}$$

$$= \lim_{x \to 0} \frac{2\sin x \cos x}{\sin 2x \cos 2x} = \lim_{x \to 0} \frac{\sin 2x}{\sin 2x \cos 2x}$$

$$= \lim_{x \to 0} \frac{1}{\cos 2x} = \frac{1}{1} = 1.$$

Ex. 15. *Evaluate* $\lim\limits_{x \to 0} \dfrac{\log\log(1-x^2)}{\log\log\cos x}$. **(Meerut 1984)**

Sol. We have, $\lim\limits_{x \to 0} \dfrac{\log\log(1-x^2)}{\log\log\cos x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{\dfrac{1}{\log(1-x^2)} \cdot \dfrac{1}{1-x^2} \cdot (-2x)}{\dfrac{1}{\log\cos x} \cdot \dfrac{1}{\cos x} \cdot (-\sin x)}$$

$$= 2\lim_{x \to 0} \frac{x\cos x \log\cos x}{\sin x\,.\,(1-x^2)\log(1-x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1 - x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}, \qquad \text{[form 0/0]}$$

$$= 2 \lim_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\frac{1}{1 - x^2} \cdot (-2x)} = 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1 - x^2}{\cos x} \right)$$

$$= 1.$$

***Ex. 16.** *Evaluate* $\lim_{x \to \pi/2} \frac{\tan 5x}{\tan x}$. **(Agra 1988)**

Sol. $\lim_{x \to \pi/2} \frac{\tan 5x}{\tan x}$ [form ∞/∞]

$$= \lim_{x \to \pi/2} \left(\frac{\sin 5x}{\sin x} \cdot \frac{\cos x}{\cos 5x} \right)$$

$$= \lim_{x \to \pi/2} \frac{\sin 5x}{\sin x} \cdot \lim_{x \to \pi/2} \frac{\cos x}{\cos 5x}$$

$$= 1 \cdot \lim_{x \to \pi/2} \frac{\cos x}{\cos 5x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-\sin x}{-5 \sin 5x}, \text{ by L'Hospital's rule}$$

$$= \frac{-1}{-5} = \frac{1}{5}.$$

§ 5. Form III. $0 \times \infty$.

This form can be easily reduced to the form $0/0$ or to the form ∞/∞.

Let $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} \phi(x) = \infty$.

Then we can write $\lim_{x \to a} f(x) \cdot \phi(x)$

$$= \lim_{x \to a} \frac{f(x)}{1/\phi(x)}, \qquad \text{[form 0/0]}$$

or

$$= \lim_{x \to a} \frac{\phi(x)}{1/f(x)}. \qquad \text{[form } \infty/\infty\text{]}$$

Thus $\lim_{x \to a} f(x) \cdot \phi(x)$ is reduced to the form $0/0$ or ∞/∞ which can now be evaluated by L' Hospital's Rule or otherwise.

Solved Examples

Ex. 1. *Evaluate* $\lim_{x \to \infty} \left\{ x \tan \left(\frac{1}{x} \right) \right\}$. **(Meerut 1982S)**

Sol. We have $\lim_{x \to \infty} [\tan(1/x)] \,.\, x,$ [form $0 \times \infty$]

$$= \lim_{x \to \infty} [\tan(1/x)]/[1/x], \quad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{[(-1/x^2)\sec^2(1/x)]}{[-1/x^2]} = \lim_{x \to \infty} \sec^2(1/x) = 1.$$

Ex. 2. *Evaluate* $\lim_{x \to 1} (1-x) \,.\, \tan \frac{\pi x}{2}.$ **(Meerut 1983, 97; Agra 85)**

Sol. We have $\lim_{x \to 1} (1-x) \,.\, \tan \frac{\pi x}{2},$ [form $0 \times \infty$]

$$= \lim_{x \to 1} \frac{1-x}{\cot(\pi x/2)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{-1}{-\frac{\pi}{2} \cdot \operatorname{cosec}^2 \frac{\pi x}{2}} = \lim_{x \to 1} \frac{2}{\pi} \sin^2 \frac{\pi x}{2} = \frac{2}{\pi}.$$

Ex. 3. *Evaluate* $\lim_{x \to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x.$

Sol. We have $\lim_{x \to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x$ [form $\infty \times 0$]

$$= \lim_{x \to 1} \frac{\log x}{\cos(\pi/2x)}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{1/x}{-[\sin(\pi/2x)] \,.\, (-\pi/2x^2)} = \lim_{x \to 1} \frac{2x}{\pi} \operatorname{cosec} \frac{\pi}{2x} = \frac{2}{\pi}.$$

Ex. 4. *Evaluate* $\lim_{x \to \infty} (a^{1/x} - 1)\, x.$ **(Meerut 1980, 88P)**

Sol. We have $\lim_{x \to \infty} (a^{1/x} - 1)\, x$ [form $0 \times \infty$]

$$= \lim_{x \to \infty} \frac{a^{1/x} - 1}{1/x}, \quad \text{[form 0/0]}$$

$$= \lim_{x \to \infty} \frac{a^{1/x} \,.\, \log a \,.\, (-1/x^2)}{-1/x^2}$$

$$= \lim_{x \to \infty} a^{1/x} \log a = a^0 \log a = \log a.$$

Ex. 5. (a) *Evaluate* $\lim_{x \to 0} x \log x.$

(Kanpur 1985, Agra 86; Meerut 79, 95)

Sol. $\lim_{x \to 0} x \log x,$ [form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log x}{1/x}, \quad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{1/x}{-(1/x^2)} = \lim_{x \to 0} (-x) = 0.$$

Ex. 5. (b) *Evaluate* $\lim_{x \to 0} x^m (\log x)^n$, *where* m, n *are positive integers.* **(Allahabad 1971)**

Sol. We have $\lim_{x \to 0} x^m (\log x)^n$, [form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{(\log x)^n}{x^{-m}}, \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{n (\log x)^{n-1} (1/x)}{-m x^{-m-1}} = \lim_{x \to 0} \left[-\frac{n}{m} \cdot \frac{(\log x)^{n-1}}{x^{-m}} \right],$$

[form ∞/∞ if $n > 1$]

$$= \lim_{x \to 0} \left(-\frac{n}{m} \right) \cdot \frac{(n-1)(\log x)^{n-2} . (1/x)}{-m x^{-m-1}}$$

$$= \lim_{x \to 0} (-1)^2 \frac{n(n-1)}{m^2} \cdot \frac{(\log x)^{n-2}}{x^{-m}}, \qquad \text{[form } \infty/\infty \text{ if } n > 2]$$

$$= \lim_{x \to 0} (-1)^n \cdot \frac{n(n-1)(n-2) \ldots \text{upto } n \text{ factors}}{m^n} \cdot \frac{(\log x)^{n-n}}{x^{-m}},$$

[by repeated application of the above process]

$$= \lim_{x \to 0} (-1)^n \frac{n!}{m^n} \cdot x^m = (-1)^n \frac{n!}{m^n} \cdot \lim_{x \to 0} x^m = 0.$$

Ex. 6. *Evaluate* $\lim_{x \to \infty} 2^x \sin \left(\frac{a}{2^x} \right)$.

Sol. We have $\lim_{x \to \infty} 2^x \sin \left(\frac{a}{2^x} \right)$, [form $\infty \times 0$]

$$= \lim_{x \to \infty} \frac{\sin (a . 2^{-x})}{2^{-x}}, \qquad \text{[form } 0/0]$$

$$= \lim_{x \to \infty} \frac{\{\cos (a . 2^{-x})\} . a . 2^{-x} (\log 2) . (-1)}{2^{-x} (\log 2) . (-1)}$$

$$= \lim_{x \to \infty} a \cos \left(\frac{a}{2^x} \right) = a \cos 0 = a \quad 1 = a.$$

Ex. 6. (a) *Evaluate* $\lim_{x \to \infty} a^x \sin \left(\frac{b}{a^x} \right)$, $a > 1$. **(Gorakhpur 1988)**

Sol. Proceed as in Ex. 6. **Ans.** b.

§ 6. Form IV : [$\infty - \infty$].

When $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} \phi(x) = \infty$, we have

$$= \lim_{x \to a} [f(x) - \phi(x)], \qquad \text{[form } \infty - \infty]$$

$$= \lim_{x \to a} \left[\frac{1}{1/f(x)} - \frac{1}{1/\phi(x)} \right]$$

$$=\lim_{x \to a} \frac{\{1/\phi(x)\} - \{1/f(x)\}}{\{1/f(x)\} \cdot \{1/\phi(x)\}}. \qquad \text{[form 0/0]}$$

Now this can be evaluated by applying L' Hospital's Rule.

Solved Examples

**Ex. 1. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$.

(Agra 1987; Raj. 77; Gurunanak 73; Ranchi 76; Meerut 86, 88P, 96; Rohilkhand 87; Vikram 75; Delhi 75)

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$ [form $\infty - \infty$]

$$=\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}, \qquad \text{[form 0/0]}$$

$$=\left(\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}\right) \cdot \left(\lim_{x \to 0} \frac{x^2}{\sin^2 x}\right), \qquad \textbf{(Note)}$$

$$=\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}, \qquad \left[\because \lim_{x \to 0} \frac{x^2}{\sin^2 x} = 1\right]$$

$$=\lim_{x \to 0} \frac{1}{x^4} \left[\left(x - \frac{x^3}{3!} + \dots\right)^2 - x^2\right]$$

$$=\lim_{x \to 0} \frac{1}{x^4} \left[\left\{x^2 - \frac{1}{3}x^4 + \dots\right\} - x^2\right]$$

$$=\lim_{x \to 0} \left[-\tfrac{1}{3} + \text{terms containing } x \text{ and its higher powers}\right]$$

$$=-\tfrac{1}{3}.$$

Ex. 2. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$. **(Agra 1979; Utkal 72)**

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$, [form $\infty - \infty$]

$$=\lim_{x \to 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x}\right) = \lim_{x \to 0} \frac{\sin x - x\cos x}{x \sin x}$$

$$=\lim_{x \to 0} \left(\frac{\sin x - x \cos x}{x^2} \cdot \frac{x}{\sin x}\right)$$

$$=\lim_{x \to 0} \frac{\sin x - x\cos x}{x^2}, \qquad \text{[form 0/0]}$$

$$=\lim_{x \to 0} \frac{\cos x - \cos x + x \sin x}{2x} = \lim_{x \to 0} \left(\frac{1}{2}\sin x\right) = 0.$$

Ex. 3. (a) *Evaluate* $\lim_{x \to \pi/2} (\sec x - \tan x)$.

(Vikram 1970; Indore 70)

Sol. We have $\lim_{x \to \pi/2} (\sec x - \tan x)$, [form $\infty - \infty$]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \lim_{x \to \pi/2} \cot x = 0.$$

(b) $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\sin x}{\cos x} = \lim_{x \to 0} \tan x = 0.$$

Ex. 4. *Evaluate* $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$.

Sol. We have $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \frac{x - e^x + 1}{x(e^x - 1)}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - e^x}{e^x - 1 + xe^x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}.$$

Ex. 5. *Evaluate* $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$

Sol. We have $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \left(\frac{\sin^2 x - x^2 \cos^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x}\right)$$

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2 \cos^2 x}{x^4}$$

$$= \lim_{x \to 0} \frac{[x - (x^3/3!) + \ldots]^2 - x^2[1 - (x^2/2!) + (x^4/4!) - \ldots]^2}{x^4}$$

$$= \lim_{x \to 0} \frac{[x^2 - (2x^4/3!) + \ldots] - x^2(1 - x^2 + \ldots)}{x^4}$$

$$= \lim_{x \to 0} \frac{x^2 - (x^4/3) + \ldots - x^2 + x^4 + \ldots}{x^4}$$

$$= \lim_{x \to 0} \frac{(\frac{2}{3})x^4 + \text{terms containing higher powers of } x}{x^4} = \frac{2}{3}.$$

Ex. 6. *Evaluate* $\displaystyle \lim_{x \to 0} \frac{\cot x - (1/x)}{x}$. **(Kanpur 1986; Gorakhpur 89)**

Sol. We have $\displaystyle \lim_{x \to 0} \frac{\cot x - (1/x)}{x}$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \left(\frac{x \cos x - \sin x}{x^3} \cdot \frac{x}{\sin x} \right)$$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x^3}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \cos x}{3x^2}$$

$$= \lim_{x \to 0} \left(-\frac{1}{3} \cdot \frac{\sin x}{x} \right) = -\frac{1}{3} \cdot 1 = -\frac{1}{3}.$$

Ex. 7. *Evaluate* $\displaystyle \lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x} \right]$.

Sol. We have $\displaystyle \lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x} \right]$, [form $\infty - \infty$]

$$= \lim_{x \to 1} \left[\frac{x \log x - x + 1}{(x-1) \log x} \right], \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \left[\frac{x \,.\, (1/x) + \log x - 1}{(x-1)(1/x) + \log x} \right]$$

$$= \lim_{x \to 1} \frac{\log x}{1 - (1/x) + \log x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 1} \frac{1/x}{(1/x^2) + (1/x)} = \frac{1}{1+1} = \frac{1}{2}.$$

Ex. 8. *Evaluate* $\displaystyle \lim_{x \to 0} \left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2} \right]$.

Sol. $\displaystyle \lim_{x \to 0} \left[\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2} \right]$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left[\frac{x - (1+x) \log(1+x)}{x^2(1+x)} \right], \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - (1+x) \,.\, \{1/(1+x)\} - \log(1+x)}{2x + 3x^2}$$

$$= \lim_{x \to 0} \frac{-\log(1+x)}{2x + 3x^2}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-\{1/(1+x)\}}{2 + 6x} = -\frac{1}{2}.$$

Ex. 9. *Evaluate* $\displaystyle \lim_{x \to 0} \left(\operatorname{cosec}^3 x - \frac{1}{x^3} \right)$.

Sol. We have $\lim_{x \to 0} \left(\operatorname{cosec}^3 x - \frac{1}{x^3}\right)$, [form $\infty - \infty$]

$$= \lim_{x \to 0} \left(\frac{1}{\sin^3 x} - \frac{1}{x^3}\right) = \lim_{x \to 0} \frac{x^3 - \sin^3 x}{x^3 \sin^3 x}$$

$$= \lim_{x \to 0} \left\{\left(\frac{x^3 - \sin^3 x}{x^6}\right) \cdot \left(\frac{x}{\sin x}\right)^3\right\}$$

$$= \lim_{x \to 0} \frac{x^3 - \sin^3 x}{x^6} = \lim_{x \to 0} \frac{x^3 - \{x - (x^3/3\,!) + (x^5/5\,!) - \dots\}^3}{x^6}$$

$$= \lim_{x \to 0} \frac{x^3 - [x + \{-(x^3/3\,!) + (x^5/5\,!) - \dots\}]^3}{x^6}$$

$$= \lim_{x \to 0} \frac{x^3 - [x^3 + 3x^2\{-(x^3/3\,!) + (x^5/5\,!)\} + \dots]}{x^6}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x^5 + \text{terms containing powers of } x \text{ higher than } 5}{x^6}$$

$$= \lim_{x \to 0} \frac{x^5\,[\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}]}{x^6}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} + \text{terms containing } x \text{ and its higher powers}}{x} = \infty.$$

Ex. 10. *Evaluate* $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log (1 + x)\right]$ **(Gorakhpur 1981)**

Sol. We have $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log (1 + x)\right]$

$$= \lim_{x \to 0} \frac{x - \log (1 + x)}{x^2}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{1 - \{1/(1 + x)\}}{2x} = \lim_{x \to 0} \frac{1 + x - 1}{2x\,(1 + x)}$$

$$= \lim_{x \to 0} \frac{1}{2\,(1 + x)} = \frac{1}{2}.$$

Ex. 11. *Evaluate* $\lim_{x \to \pi/2} \left(\sec x - \frac{1}{1 - \sin x}\right)$.

(Vikram 1978; Meerut 89S)

Sol. We have $\lim_{x \to \pi/2} \left(\sec x - \frac{1}{1 - \sin x}\right)$, [form $\infty - \infty$]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{1}{1 - \sin x}\right) = \lim_{x \to \pi/2} \left[\frac{1 - \sin x - \cos x}{\cos x\,(1 - \sin x)}\right], \qquad \text{[form 0/0]}$$

$$= \lim_{x \to \pi/2} \left[\frac{-\cos x + \sin x}{-\sin x\,(1 - \sin x) + \cos x\,(-\cos x)}\right]$$

$$= \lim_{x \to \pi/2} \left[\frac{\sin x - \cos x}{-\sin x + \sin^2 x - \cos^2 x} \right] = \frac{1}{-1+1} = \infty.$$

§ 7. The forms $0^0, 1^\infty, \infty^0$.

Suppose $\lim_{x \to a} [f(x)]^{\phi(x)}$ takes any one of these three forms.

Then let $y = \lim_{x \to a} [f(x)]^{\phi(x)}$.

Taking logarithm of both sides, we get

$$\log y = \lim_{x \to a} \phi(x) \, . \log f(x).$$

Now in any of the above three cases $\log y$ takes the form $0 \times \infty$ which is changed to the form [0/0] or [∞/∞] whichever is convenient and then its limit is evaluated by L' Hospital's Rule or by using standard Expansions.

Solved Examples

Ex. 1. (a) *Evaluate* $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$.

(Kanpur 1988; Rohilkhand 82; Allahabad 72; Gorakhpur 74; Meerut 77, 79 S)

Sol. Let $y = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$. [form 1^∞]

$\therefore \quad \log y = \lim_{x \to \infty} \left\{ x \log \left(1 + \frac{a}{x}\right) \right\}$, [form $\infty \times 0$]

$= \lim_{x \to \infty} \frac{\log \{1 + (a/x)\}}{1/x}$, [form 0/0]

$= \lim_{x \to \infty} \frac{[1/\{1 + (a/x)\}] \, . \, (-a/x^2)}{-(1/x^2)} = \lim_{x \to \infty} \frac{a}{1 + (a/x)} = a.$

$\therefore \quad y = e^a$ or $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x = e^a$.

Ex. 1. (b) *Evaluate* $\lim_{x \to 0} (1 + x)^{1/x}$.

(Kanpur 1987; Gorakhpur 87)

Sol. Let $y = \lim_{x \to 0} (1 + x)^{1/x}$, [form 1^∞].

$\therefore \quad \log y = \lim_{x \to 0} \frac{\log (1 + x)}{x}$, [form 0/0]

$= \lim_{x \to 0} \frac{1/(1 + x)}{1} = \lim_{x \to 0} \frac{1}{1 + x} = 1.$

$\therefore \quad y = e^1 = e$ or $\lim_{x \to 0} (1 + x)^{1/x} = e$.

Ex. 2. *Evaluate* $\lim_{x \to 0} (\cos x)^{\cot^2 x}$.

(Allahabad 1987; Rohilkhand 88, 89; Gorakhpur 86)

Sol. Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$. [form 1^∞]

$\therefore \quad \log y = \lim_{x \to 0} (\cot^2 x) \cdot (\log \cos x)$, [form $\infty \times 0$]

$= \lim_{x \to 0} \dfrac{\log \cos x}{\tan^2 x}$, [form 0/0]

$= \lim_{x \to 0} \dfrac{\{(1/\cos x) \cdot (-\sin x)\}}{2 \tan x \cdot \sec^2 x}$, [by L' Hospital's rule]

$= \lim_{x \to 0} \dfrac{-\tan x}{2 \tan x \cdot \sec^2 x} = \lim_{x \to 0} \dfrac{-1}{2 \sec^2 x} = -\dfrac{1}{2}$.

$\therefore \quad y = e^{-1/2}$ *i.e.*, $\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{-1/2}$.

Ex. 3. *Evaluate* $\lim_{x \to \pi/2} (\sin x)^{\tan x}$.

(Kanpur 1989; Agra 84, 87; Gorakhpur 79; Meerut 88 S)

Sol. Let $y = \lim_{x \to \pi/2} (\sin x)^{\tan x}$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to \pi/2} \tan x \cdot \log \sin x$, [form $\infty \times 0$]

$= \lim_{x \to \pi/2} \dfrac{\log \sin x}{\cot x}$, [form 0/0]

$= \lim_{x \to \pi/2} \dfrac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x}$, [by L' Hospital's rule]

$= \lim_{x \to \pi/2} (-\sin x \cos x) = 0$.

$\therefore \quad y = e^0 = 1$.

Ex. 4. *Evaluate* $\lim_{x \to \infty} (1 + x^{-2})^x$.

Sol. Let $y = \lim_{x \to \infty} \left(1 + \dfrac{1}{x^2}\right)^x$, [form 1^∞].

$\therefore \quad \log y = \lim_{x \to \infty} x \log \left(1 + \dfrac{1}{x^2}\right)$, [form $\infty \times 0$]

$= \lim_{x \to \infty} x \left[\dfrac{1}{x^2} - \dfrac{1}{2x^4} + \dots\right]$, [Expanding $\log \{1 + (1/x^2)\}$]

$= \lim_{x \to \infty} \left[\dfrac{1}{x} - \dfrac{1}{2x^3} + \dots\right] = 0$.

$\therefore \quad y = e^0 = 1$.

Ex. 5. *Evaluate* $\lim_{x \to 0} (\cos x)^{1/x^2}$

(Meerut 1991 P; Kanpur 76)

Sol. Let $y = \lim_{x \to 0} (\cos x)^{1/x^2}$ [form 1^∞].

$\therefore \quad \log y = \lim_{x \to 0} \left(\frac{1}{x^2}\right) (\log \cos x),$ [form $\infty \times 0$]

$= \lim_{x \to 0} \frac{\log \cos x}{x^2},$ [form 0/0]

$= \lim_{x \to 0} \frac{(1/\cos x)(-\sin x)}{2x},$ [by L' Hospital's rule]

$= \lim_{x \to 0} \left(\frac{-\tan x}{2x}\right),$ [form 0/0]

$= \lim_{x \to 0} \left(\frac{-\sec^2 x}{2}\right),$ [by L' Hospital's Rule]

$= -\frac{1}{2}. \qquad \therefore y = e^{-1/2}.$

Ex. 6. *Evaluate* $\lim_{x \to 0} (\cos x)^{1/x}.$

(Magadh 1970)

Sol. Let $y = \lim_{x \to 0} (\cos x)^{1/x}.$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to 0} \left(\frac{1}{x}\right) (\log \cos x),$ [form $\infty \times 0$]

$= \lim_{x \to 0} \frac{\log \cos x}{x},$ [form 0/0]

$= \lim_{x \to 0} \frac{(1/\cos x)(-\sin x)}{1} = \lim_{x \to 0} (-\tan x) = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 7. *Evaluate* $\lim_{x \to 1} (x)^{1/(x-1)}.$

(Vikram 1977)

Sol. Let $y = \lim_{x \to 1} x^{1/(x-1)}.$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to 1} \frac{1}{(x-1)} \log x = \lim_{x \to 1} \frac{\log x}{(x-1)},$ $\left[\text{form } \frac{0}{0}\right]$

$= \lim_{x \to 1} \frac{1/x}{1} = 1.$ Hence $y = e^1 = e.$

****Ex. 8.** *Evaluate* $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

(Allahabad 1987; Gurunanak 73; Magadh 76; Bihar 72; Kanpur 85; Meerut 94P, 96P)

Sol. Let $y = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x}.$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x} \log \frac{\tan x}{x}$

$$= \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{x + (x^3/3) + (2x^5/15) + \dots}{x}\right\}\right],$$

(writing the expansion for $\tan x$)

$$= \lim_{x \to 0} \frac{1}{x} \log \left[1 + \frac{x^2}{3} + \frac{2x^4}{15} + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \log (1 + z), \text{ where } z = \frac{x^2}{3} + \frac{2x^4}{15} + \dots$$

$$= \lim_{x \to 0} \frac{1}{x} \left[z - \frac{z^2}{2} + \dots\right], \qquad [\text{Expanding } \log (1 + z)]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\left(\frac{x^2}{3} + \frac{2x^4}{15} + \dots\right) - \frac{1}{2}\left(\frac{x^2}{13} + \frac{2x^4}{15} + \dots\right)^2 + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{3} + \left(\frac{2}{15} - \frac{1}{18}\right) x^4 + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$$

$$= \lim_{x \to 0} \left[\frac{x}{3} + \frac{7}{90} x^3 + \dots\right] = 0.$$

$\therefore \quad y = e^0 = 1.$

Ex. 9. *Evaluate* $\displaystyle \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

(Meerut 1981, 85, 87; Avadh 87; Kanpur 77; Rohilkhand 81; Gorakhpur 80)

Sol. Proceeding as in Ex. 8, we have

$$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{3} + \frac{7}{90} x^2 + \dots\right] = \frac{1}{3}$$

$\therefore \quad y = e^{1/3}.$

Ex. 10. *Evalu[illegible]* $\displaystyle \lim_{x \to 0+} \left(\frac{t[illegible]}{x}\right)$ **. P.C.S. 1995[illegible] [illegible]eerut 79)**

Sol. Proceeding as in Ex. 8, we [illegible]

$$\log y = \lim_{x \to 0+} \frac{1}{x^3} \left[\frac{x^2}{3} + \frac{7}{90} x^4 + \dots\right]$$

$$= \lim_{x \to 0+} \left[\frac{1}{3x} + \frac{7}{90} x + \dots\right] = +\infty.$$

$\therefore \quad y = e^{\infty} = \infty.$

Ex. 11. *Evaluate* $\displaystyle \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$

(K.U. 1977; Pbi 75)

Sol. Let $y = \displaystyle \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$, $\left[\text{form } 1^{\infty}, \text{ since } \displaystyle \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$.

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \frac{\sin x}{x}\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \log \left\{\frac{x - (x^3/3\,!) + (x^5/5\,!) - \dots}{x}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \log \left(1 - \frac{x^2}{3\,!} + \frac{x^4}{5\,!} - \dots\right)$$

$$= \lim_{x \to 0} \frac{1}{x} \log \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \log (1 - z), \text{ where } z = \frac{x^2}{6} - \frac{x^4}{120} + \dots$$

$$= \lim_{x \to 0} \frac{1}{x} \left(- z - \frac{z^2}{2} - \dots\right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left[- \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right) - \tfrac{1}{2} \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right)^2 - \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[- \frac{x^2}{6} + \left(\frac{x^4}{120} - \frac{x^4}{72}\right) + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[- \frac{x^2}{6} - \frac{x^4}{180} + \dots\right]$$

$$= \lim_{x \to 0} \left[- \frac{x}{6} - \frac{x^3}{180} + \dots\right] = 0.$$

$$\therefore \quad y = e^0 = 1.$$

****Ex. 12.** *Evaluate* $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

(Kashmir 1975; Ranchi 75; Meerut 89; Andhra 71)

Sol. Proceeding as in Ex. 11, we have

$$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[- \frac{x^2}{6} - \frac{x^4}{180} + \dots\right]$$

$$= \lim_{x \to 0} \left[- \frac{1}{6} - \frac{x^2}{180} + \dots\right] = - \frac{1}{6}.$$

$$\therefore \quad y = e^{-1/6}.$$

Ex. 13. *Evaluate* $\lim_{x \to 0+} \left(\frac{\sin x}{x}\right)^{1/x^3}$

Sol. Proceeding as in Ex. 11, we have

$$\log y = \lim_{x \to 0+} \frac{1}{x^3} \cdot \left[- \frac{x^2}{6} - \frac{x^4}{180} + \dots\right] = - \infty.$$

$$\therefore \quad y = e^{-\infty} = 0.$$

Ex. 14. *Evaluate* $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x}$

Sol. Let $y = \lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x}$, [form 1^∞].

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{\sinh x}{x}\right\}\right].$$

Now $\qquad \sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x} \log \left\{\frac{x + (x^3/3!) + (x^5/5!) + \dots}{x}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x} \log \left[1 + \frac{x^2}{6} + \frac{x^4}{120} + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \log (1 + z), \text{ where } z = \frac{x^2}{6} + \frac{x^4}{120} + \dots$$

$$= \lim_{x \to 0} \frac{1}{x} \left[z - \frac{z^2}{2} + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\left(\frac{x^2}{6} + \frac{x^4}{120} + \dots\right) - \frac{1}{2}\left(\frac{x^2}{6} + \frac{x^4}{120} + \dots\right)^2 + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{6} + \left(\frac{1}{120} - \frac{1}{72}\right) x^4 + \dots\right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{x^2}{6} - \frac{1}{180} x^4 + \dots\right]$$

$$= \lim_{x \to 0} \left[\frac{x}{6} - \frac{1}{180} x^3 + \dots\right] = 0.$$

$$\therefore \quad y = e^0 = 1.$$

Ex. 15. *Evaluate* $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$

(Garhwal 1983; Gorakhpur 73)

Sol. Proceeding as in Ex. 14, we have

$$\log y = \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{6} - \frac{1}{180} x^4 + \dots\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{6} - \frac{1}{180} x^2 + \dots\right] = \frac{1}{6}.$$

$$\therefore \quad y = e^{1/6}.$$

Ex. 16. *Evaluate* $\lim_{x \to 0} \left[\frac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$

(Meerut 1984)

Sol. Let $y = \lim_{x \to 0} \left[\frac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$, $\qquad$ [form 1^∞].

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2(\cosh x - 1)}{x^2}\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2}{x^2}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots - 1\right)\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left\{\frac{2}{x^2}\left(\frac{x^2}{2} + \frac{x^4}{24} + \dots\right)\right\}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left[1 + \left(\frac{x^2}{12} + \ldots\right)\right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \left[\left(\frac{x^2}{12} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{12} + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \left[\frac{x^2}{12} + \text{higher powers of } x\right]$$

$$= \lim_{x \to 0} \left[\frac{1}{12} + \text{terms containing } x \text{ and its higher powers}\right] = \frac{1}{12}.$$

$\therefore \quad y = e^{1/12}.$

Ex. 17. *Evaluate* $\lim_{x \to 0} (a^x + x)^{1/x}$. **(Kanpur 1980)**

Sol. Let $y = \lim_{x \to 0} (a^x + x)^{1/x}$, [form 1^∞]

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log (a^x + x)\right]$$

$$= \lim_{x \to 0} \frac{\log (a^x + x)}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\{1/(a^x + x)\}\,(a^x \log a + 1)}{1}$$

$$= \lim_{x \to 0} \frac{a^x \log a + 1}{a^x + x} = \frac{\log a + 1}{1 + 0} = \log a + 1$$

$= \log a + \log e = \log (ae),$ [$\because$ $1 = \log e$]

$\therefore \quad y = ae.$

Ex. 18. *Evaluate* $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ **(Rohilkhand 1982, 76)**

Sol. Let $y = \lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$, [form 1^∞].

$$\therefore \quad \log y = \lim_{x \to 0} \left[\frac{1}{x} \log \left\{\frac{a^x + b^x}{2}\right\}\right]$$

$$= \lim_{x \to 0} \frac{\log \{(a^x + b^x)/2\}}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\log (a^x + b^x) - \log 2}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{\dfrac{a^x \log a + b^x \log b}{a^x + b^x}}{1} = \lim_{x \to 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

$$= \frac{\log a + \log b}{1 + 1} = \tfrac{1}{2} \log (ab) = \log \sqrt{(ab)}.$$

$\therefore \quad y = \sqrt{(ab)}.$

Ex. 19 (a). *Evaluate* $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$ **(G.N.U. 1977; Delhi 70)**

Sol. Let $y = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan(\pi x/2a)}$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \left[\log\left(2 - \frac{x}{a}\right)\right],$ [form $\infty \times 0$]

$= \lim_{x \to a} \left[\left\{\log\left(2 - \frac{x}{a}\right)\right\} \Big/ \cot\left(\frac{\pi x}{2a}\right)\right],$ [form 0/0]

$= \lim_{x \to a} \left[\left(-\frac{1}{a}\right)\left(2 - \frac{x}{a}\right)^{-1}\right] \Big/ \left[\left\{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\right\}\frac{\pi}{2a}\right]$

$= \lim_{x \to a} \frac{1}{a} \cdot \frac{2a}{\pi} \cdot \frac{1}{\{2 - (x/a)\}} \sin^2\left(\frac{\pi x}{2a}\right) = \frac{2}{\pi}.$

$\therefore \quad y = e^{2/\pi}.$

Ex. 19 (b). *Evaluate* $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}.$

(Rohilkhand 1978; Raj. 78)

Sol. Let $y = \lim_{x \to \pi/4} [(\tan x)^{\tan 2x}]$ [form 1^∞]

$\therefore \quad \log y = \lim_{x \to \pi/4} [(\tan 2x) . (\log \tan x)]$ [form $\infty \times 0$]

$= \lim_{x \to \pi/4} \left[\frac{\log \tan x}{\cot 2x}\right],$ [form 0/0]

$= \lim_{x \to \pi/4} \left[\frac{(1/\tan x) . \sec^2 x}{-2 \operatorname{cosec}^2 (2x)}\right] = \lim_{x \to \pi/4} \left[\frac{\sec^2 x . \sin^2 (2x)}{-2 \tan x}\right]$

$= \frac{(\sqrt{2})^2 . (1)^2}{-2 . 1} = -1.$

$\therefore \quad y = e^{-1} = 1/e.$

Ex. 20. *Evaluate* $\lim_{x \to \infty} \frac{Ax^n + Bx^{n-1} + Cx^{n-2} + \dots}{ax^m + bx^{m-1} + cx^{m-2} + \dots}.$

Sol. Given limit $= \lim_{x \to \infty} \frac{x^n [A + (B/x) + (C/x^2) + \dots]}{x^m [a + (b/x) + (c/x^2) + \dots]}$

$= \lim_{x \to \infty} \left(x^{n-m} . \frac{A}{a}\right).$

Now, if $n > m$, $\lim_{x \to \infty} \left(\frac{A}{a} x^{n-m}\right) = \infty;$

if $n = m$, $\lim_{x \to \infty} \frac{A}{a} x^{n-m} = \lim_{x \to \infty} \frac{A}{a} = \frac{A}{a};$

if $n < m$, $\lim_{x \to \infty} \frac{A}{a} x^{n-m} = \lim_{x \to \infty} \left(\frac{A}{a} \cdot \frac{1}{x^{m-n}}\right) = 0.$

Ex. 21. *Evaluate* $\lim_{x \to 0} x^x.$

(Meerut 1979, 93)

Sol. Let $y = \lim_{x \to 0} x^x$. [form 0^0]

$\therefore \quad \log y = \lim_{x \to 0} x \log x$, [form $0 \times \infty$]

$= \lim_{x \to 0} \frac{\log x}{(1/x)}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{1/x}{(-1/x^2)} = \lim_{x \to 0} (-x) = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 22. *Evaluate* $\lim_{x \to 0} \frac{x^x - 1}{x}$.

Sol. $\lim_{x \to 0} \frac{x^x - 1}{x}$, $\left[\text{form } 0/0, \quad \because \lim_{x \to 0} x^x = 1\right]$

$= \lim_{x \to 0} \frac{d/dx\,(x^x)}{1}$, [by L' Hospital's Rule]

$= \lim_{x \to 0} \left\{\frac{d}{dx}(x^x)\right\}$

Now let $y = x^x$; then $\log y = x \log x$. Now differentiating,

$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + 1 \,.\, \log x = (1 + \log x).$$

$\therefore \quad \frac{dy}{dx} = y\,(1 + \log x)$. Thus $\frac{d}{dx}(x^x) = x^x\,(1 + \log x)$.

Hence $\lim_{x \to 0} \frac{x^x - 1}{x} = \lim_{x \to 0} [x^x\,(1 + \log x)]$

$= 1\,(1 - \infty) = -\infty.$ $\left[\because \lim_{x \to 0} x^x = 1\right]$

Ex. 23. *Evaluate* $\lim_{x \to 0} (\sin x)^{\tan x}$

Sol. Let $y = \lim_{x \to 0} (\sin x)^{\tan x}$ [form 0^0]

$\therefore \quad \log y = \lim_{x \to 0} (\tan x) \,.\, (\log \sin x)$, [form $0 \times \infty$]

$= \lim_{x \to 0} \frac{\log \sin x}{\cot x}$, [form ∞/∞]

$= \lim_{x \to 0} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x}$, [form ∞/∞]

$= \lim_{x \to 0} \left[-\left(\frac{\sin^2 x}{\tan x}\right)\right]$, [form 0/0]

$= \lim_{x \to 0} \left[\frac{-2 \sin x \cos x}{\sec^2 x}\right] = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 24. *Evaluate* $\lim_{x \to \infty} \left(\frac{1}{x}\right)^{1/x}$ **(Agra 1974)**

Sol. Let $y = \lim_{x \to \infty} \left(\frac{1}{x}\right)^{1/x}$ [form 0^0]

$\therefore \quad \log y = \lim_{x \to \infty} \frac{1}{x} \log \left(\frac{1}{x}\right)$, [form $0 \times \infty$]

$= \lim_{x \to \infty} \frac{-\log x}{x}$,

[$\because \log(1/x) = \log 1 - \log x = -\log x$]

$= -\lim_{x \to \infty} \frac{\log x}{x}$, [form ∞/∞]

$= -\lim_{x \to \infty} \frac{(1/x)}{1} = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 25. *Evaluate* $\lim_{x \to \infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$.

Sol. Let $y = \lim_{x \to \infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$. [form 0^0]

$\therefore \quad \log y = \lim_{x \to \infty} \left(\frac{1}{x}\right) \log (\frac{1}{2}\pi - \tan^{-1} x)$

$= \lim_{x \to \infty} \frac{\log (\frac{1}{2}\pi - \tan^{-1} x)}{x}$, [form ∞/∞]

$= \lim_{x \to \infty} \frac{\{1/(\frac{1}{2}\pi - \tan^{-1} x)\} . \{-1/(1+x^2)\}}{1}$

$= \lim_{x \to \infty} \left\{\frac{-1/(1+x^2)}{(\frac{1}{2}\pi - \tan^{-1} x)}\right\}$, [form 0/0]

$= \lim_{x \to \infty} \frac{\{1/(1+x^2)^2\} . 2x}{-1/(1+x^2)}$, [by L' Hospital's rule]

$= \lim_{x \to \infty} \left\{\frac{-2x}{1+x^2}\right\}$, [form ∞/∞]

$= \lim_{x \to \infty} \left\{\frac{-2}{2x}\right\} = 0.$

$\therefore \quad y = e^0 = 1.$

Ex. 26. *Evaluate* $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$.

(Agra 1978, 75; Rajasthan 77; Indore 72)

Sol. Let $y = \lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$. [form ∞^0]

$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{\log x} (\log \operatorname{cosec} x)$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/\operatorname{cosec} x)(-\operatorname{cosec} x \cot x)}{1/x}$$

$$= \lim_{x \to 0} \left(\frac{-x}{\tan x}\right), \qquad \text{[form } 0/0]$$

$$= \lim_{x \to 0} \left(\frac{-1}{\sec^2 x}\right) = -1.$$

$\therefore \quad y = e^{-1} = 1/e.$

Ex. 27. *Evaluate* $\lim_{x \to 0} (\cot x)^{\sin x}$. **(Allahabad 1974; Andhra 71)**

Sol. Let $y = \lim_{x \to 0} (\cot x)^{\sin x}$. [form ∞^0]

$$\therefore \quad \log y = \lim_{x \to 0} (\sin x) \cdot (\log \cot x), \qquad [\text{form } 0 \times \infty]$$

$$= \lim_{x \to 0} \left(\frac{\log \cot x}{\operatorname{cosec} x}\right), \qquad [\text{form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{(1/\cot x) \cdot (-\operatorname{cosec}^2 x)}{-\operatorname{cosec} x \cot x} = \lim_{x \to 0} \left(\frac{\operatorname{cosec} x}{\cot^2 x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{\cos^2 x}\right) = 0.$$

$\therefore \quad y = e^0 = 1.$

Ex. 28. *Evaluate* $\lim_{x \to \frac{1}{2}\pi} (\sec x)^{\cot x}$. **(Gorakhpur 1971)**

Sol. Let $y = \lim_{x \to \frac{1}{2}\pi} (\sec x)^{\cot x}$. [form ∞^0]

$$\therefore \quad \log y = \lim_{x \to \pi/2} (\cot x) \cdot (\log \sec x), \qquad [\text{form } 0 \times \infty]$$

$$= \lim_{x \to \pi/2} \frac{\log \sec x}{\tan x}, \qquad [\text{form } \infty/\infty]$$

$$= \lim_{x \to \pi/2} \frac{(1/\sec x) \cdot \sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\tan x}{\sec^2 x} = \lim_{x \to \pi/2} (\sin x \cos x) = 0.$$

$\therefore \quad y = e^0 = 1.$

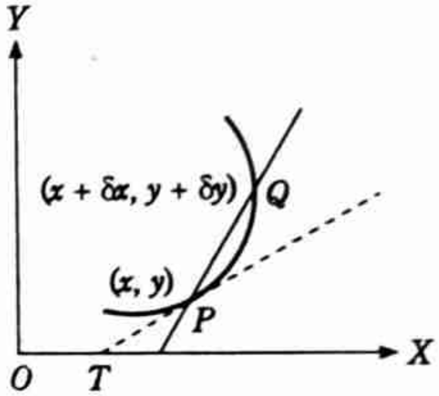
□

6

Tangents and Normals

§ 1. Tangent.

Definition. *The tangent to a plane curve at the point $P(x, y)$ on it is defined as the limiting position of the chord PQ as the point $Q(x+\delta x, y+\delta y)$ approaches the point P, provided such a limiting position exists.*



Equation of the tangent. Let P be any point (x, y) on the curve $y=f(x)$, and Q a neighbouring point $(x+\delta x, y+\delta y)$ on it. The point Q may be taken on either side of P. The equation of the chord PQ is

$$Y-y=\frac{(y+\delta y)-y}{(x+\delta x)-x}(X-x),$$ [Here (X, Y) are the current coordinates of any point on the chord]

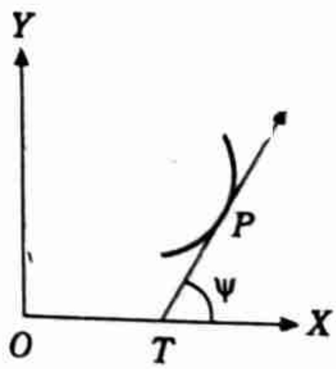
or $$Y-y=(\delta y/\delta x)(X-x). \qquad ...(1)$$

Now as $Q \to P$, $\delta x \to 0$, $\delta y/\delta x \to dy/dx$, and the chord PQ tends to the tangent at P. Therefore taking limit of (1) as $Q \to P$, we see that the equation of the tangent to the curve $y=f(x)$ at the point (x, y) is

$$\mathbf{Y-y=(dy/dx)(X-x)} \qquad ...(2)$$

Geometrical meaning of dy/dx. The equation (2) of the tangent at any point $P(x, y)$ on the curve $y=f(x)$ may be written as

$$Y=\left(\frac{dy}{dx}\right)X+\left(y-x\frac{dy}{dx}\right),$$



which is of the form $Y=mX+c$. On comparing the two equations, we have $m=dy/dx$. Therefore if ψ is the angle which the tangent to the curve $y=f(x)$ at the point (x, y) on it makes with the positive direction of x-axis then

$$\mathbf{dy/dx=\tan\psi}.$$

Remember. *The differential coefficient dy/dx at any point (x, y) on the curve $y=f(x)$ is equal to $\tan\psi$, where ψ is the angle which the positive direction of the tangent at P to the curve makes with the positive direction of the axis of x.*

Note 1. The equation of the tangent at any point (x_1, y_1) on the curve $y = f(x)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$. Here (x, y) are the current coordinates of any point on the tangent.

Note 2. If the equations of the curve be given in the parametric form, say, $x = f(t)$ and $y = \phi(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}.$$

Therefore the equation of the tangent at any point 't' on the curve is given by

$$Y - \phi(t) = \frac{\phi'(t)}{f'(t)} [X - f(t)].$$

Note 3. If the equation of a curve is given in the form $f(x, y) = 0$, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}.$$

§ 2. Tangents parallel and perpendicular to the x-axis.

The tangent at any point is *parallel* to the x-axis if and only if $\tan \psi = 0$ *i.e.*, if and only if $dy/dx = 0$. Again the tangent at any point is *perpendicular* to the x-axis if and only if dy/dx is infinite or if and only if $dx/dy = 0$.

§ 3. Normal.

Definition. *The normal to a curve at any point P on it is the straight line which passes through P and which is perpendicular to the tangent to the curve at P.*

Equation of the normal. Let P be any given point (x, y) on the curve $y = f(x)$. The gradient of the tangent at (x, y) to the curve $= dy/dx$. Therefore the gradient of the normal at (x, y) to the curve $= \frac{-1}{dy/dx} = -\frac{dx}{dy}$.

Hence the equation of the normal at (x, y) to the curve is

$$\mathbf{Y - y = -(dx/dy)(X - x)}$$

or

$$\mathbf{(dy/dx)(Y - y) + (X - x) = 0.}$$

§ 4. Angle of intersection of two curves.

The angle of intersection of two curves is defined as the angle betwen their tangents at their point of intersection.

Let the equations of the two curves be

$$f(x, y) = 0 \quad \text{...(1),} \qquad \text{and} \quad \phi(x, y) = 0. \qquad \text{...(2)}$$

Let $(dy/dx)_1$ stand for the dy/dx of the curve (1) and $(dy/dx)_2$ for the dy/dx of the curve (2).

Suppose (x_1, y_1) is a point of intersection of (1) and (2). Let m_1 and m_2 be the gradients (or slopes) of the tangents at the point (x_1, y_1) to the curves (1) and (2) respectively. Then

$$m_1 = (dy/dx)_1 \text{ at the point } (x_1, y_1)$$

and $$m_2 = (dy/dx)_2 \text{ at the point } (x_1, y_1).$$

Suppose θ is the angle of intersection of (1) and (2) at the point (x_1, y_1).

If $m_1 = m_2$, the angle of intersection θ is $0°$. In this case the two curves have the same tangent at the point (x_1, y_1) and thus the two curves touch each other at the point (x_1, y_1).

If $m_1 = \infty, m_2 = 0$ or if $m_1 = 0, m_2 = \infty$, the angle of intersection θ is $90°$.

If $m_1 m_2 = -1$, again the angle of intersection is $90°$.

In all other cases, the acute angle θ between the tangents is given by

$$\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| .$$

If the angle of intersection of two curves is $90°$, we say that the curves *intersect orthogonally.*

Important. *The curves* (1) *and* (2) *intersect orthogonally at the point* (x_1, y_1) *if*

$$(\mathbf{dy/dx})_1 \cdot (\mathbf{dy/dx})_2 = -1, \text{ at the point } (x_1, y_1).$$

Note. If (1) and (2) intersect at more than one points, we should find their angle of intersection at each point.

Solved Examples

Ex. 1. *Find the equation of the tangent at the point* (x_1, y_1) *to the ellipse*

$$x^2/a^2 + y^2/b^2 = 1.$$ **(Meerut 1975, 84, 94P)**

Sol. Differentiating the equation of the given curve, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}.$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2 x_1}{a^2 y_1}.$$

Hence the required equation of the tangent at the point (x_1, y_1) is

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \quad \text{or} \quad a^2 y_1 (y - y_1) = -b^2 x_1 (x - x_1)$$

or $\quad b^2xx_1 + a^2yy_1 = b^2x_1^2 + a^2y_1^2$

or $\quad \dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}$ [dividing each side by a^2b^2]

or $\quad (xx_1/a^2) + (yy_1/b^2) = 1,$

[$\because$ the point (x_1, y_1) lies on $(x^2/a^2) + (y^2/b^2) = 1$].

Ex. 2. *Find the equation of the normal at the point* (x_1, y_1) *to the ellipse*

$$(x^2/a^2) + (y^2/b^2) = 1.$$

Sol. Proceeding as in Ex. 1,we have $dy/dx = -(b^2x/a^2y)$.

Therefore $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = -\dfrac{b^2x_1}{a^2y_1}$.

Hence the equation of the normal at the point (x_1, y_1) is

$$-(b^2x_1/a^2y_1)(y - y_1) + (x - x_1) = 0$$

or
$$(x - x_1) = (b^2x_1/a^2y_1)(y - y_1)$$

or
$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2}.$$

Ex. 3. *Find the equation of the tangent at the point 't' to the curve*

$$x = a \sin^3 t, y = b \cos^3 t.$$ **(Meerut 1972, 77, 84 P)**

Sol. We have $dx/dt = 3a \sin^2 t \cos t$,

and $\quad dy/dt = -3b \cos^2 t \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3b \cos^2 t \sin t}{3a \sin^2 t \cos t} = -\frac{b \cos t}{a \sin t}.$$

Hence the required equation of the tangent at the point 't' is

$$y - b \cos^3 t = -\frac{b \cos t}{a \sin t}(x - a \sin^3 t)$$

or
$$bx \cos t + ay \sin t = ab \sin t \cos t (\sin^2 t + \cos^2 t)$$
$$= ab \sin t \cos t.$$

Ddividing each side by $ab \sin t \cos t$, we get

$$\frac{x}{a \sin t} + \frac{y}{b \cos t} = 1 \quad \text{or} \quad (x/a).\operatorname{cosec} t + (y/b).\sec t = 1.$$

Ex. 4. *Find the equation of the normal at the point 't' on the curve* $x = a \sin^3 t, y = b \cos^3 t$. **(Meerut 1977, 84 P)**

Sol. Proceeding as in Ex. 3, we get

$$dy/dx = -(b \cos t)/(a \sin t).$$

Therefore the gradient of the normal at the point 't'

$$= -(dx/dy) = (a \sin t)/(b \cos t).$$

Hence the required equation of the normal at the point 't' is

$$y - b\cos^3 t = \frac{a\sin t}{b\cos t}(x - a\sin^3 t)$$

or $$by\cos t - b^2\cos^4 t = ax\sin t - a^2\sin^4 t$$

or $$ax\sin t - by\cos t = a^2\sin^4 t - b^2\cos^4 t.$$

Ex. 5. *Find the equation of the tangent and the normal at the point 't' on the curve $x = a(t + \sin t), y = a(1 - \cos t)$.* **(Meerut 1985)**

Sol. We have

$$dx/dt = a(1 + \cos t), dy/dt = a\sin t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t} = \tan\tfrac{1}{2}t.$$

$\therefore$ the equation of the tangent at the point 't' is

$$y - a(1 - \cos t) = (\tan\tfrac{1}{2}t)\{x - a(t + \sin t)\}$$

or $$y - 2a\sin^2\tfrac{1}{2}t = (\tan\tfrac{1}{2}t)(x - at) - a\sin t\tan\tfrac{1}{2}t$$

or $$y - 2a\sin^2\tfrac{1}{2}t = (x - at)\tan\tfrac{1}{2}t - 2a\sin^2\tfrac{1}{2}t$$

or $$y = (x - at)\tan\tfrac{1}{2}t.$$

Again the equation of the normal at the point 't' is

$$y - a(1 - \cos t) = -\frac{1}{\tan\frac{1}{2}t}\{x - a(t + \sin t)\}$$

or $$(y - 2a\sin^2\tfrac{1}{2}t)\tan\tfrac{1}{2}t = -x + a(t + \sin t)$$

or $$x + y\tan\tfrac{1}{2}t = a(t + \sin t + 2\sin^2\tfrac{1}{2}t\tan\tfrac{1}{2}t).$$

***Ex. 6.** *Prove that $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y.*

(Kashmir 1983; Delhi 76)

Sol. Differentiating the equation of the curve $y = be^{-x/a}$, we get

$$\frac{dy}{dx} = be^{-x/a}\left(-\frac{1}{a}\right) = -\left(\frac{b}{a}\right)e^{-x/a}.$$

The given curve crosses the axis of y where $x = 0$. So putting $x = 0$ in the equation of the curve, we get $y = b.e^0 = b.1 = b$.

Thus the given curve crosses the axis of y at the point $(0, b)$.

Now $$\left(\frac{dy}{dx}\right)_{(0,b)} = -\left(\frac{b}{a}\right)e^0 = -\frac{b}{a}.$$

Hence the equation of the tangent to the curve at the point $(0, b)$ is

$$y - b = (-b/a)(x - 0) \quad \text{or} \quad ay - ab = -bx \text{ or } ay + bx = ab$$

or $\dfrac{x}{a} + \dfrac{y}{b} = 1$, dividing each side by ab.

Ex. 7. *Prove that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $(x/a) + (y/b) = 2$ at the point (a, b) whatever be the value of n.*

(Meerut 1983, 84, 91, 92)

Sol. Differentiating the equation of the curve, we get

$$n\left(\frac{x}{a}\right)^{n-1}\cdot\frac{1}{a}+n\left(\frac{y}{b}\right)^{n-1}\cdot\frac{1}{b}\frac{dy}{dx}=0 \quad\text{or}\quad \frac{dy}{dx}=-\frac{b}{a}\left(\frac{bx}{ay}\right)^{n-1}.$$

$$\therefore\quad \left(\frac{dy}{dx}\right)_{\text{at }(a,b)}=-\frac{b}{a}\left(\frac{ba}{ab}\right)^{n-1}=-\frac{b}{a}.$$

Hence equation of the tangent to the curve at the point (a, b) is $y-b=(-b/a)(x-a)$ or $bx+ay=2ab$ or $(x/a)+(y/b)=2$, which is free from n. Hence the required result follows.

Ex. 8. *Find the points on the curve $y=x/(1-x^2)$, where the tangent is inclined at an angle $\pi/4$ to the x-axis.*

Sol. The given curve is $\quad y=x/(1-x^2)$. ...(1)

Differentiating (1), we get

$$\frac{dy}{dx}=\frac{(1-x^2)-x(-2x)}{(1-x^2)^2}=\frac{1+x^2}{(1-x^2)^2}.$$

If the tangent to (1) at the point (x, y) makes an angle $\pi/4$ with the axis of x, we have $dy/dx=\tan\pi/4=1$.

$\therefore\quad (1+x^2)/(1-x^2)^2=1,\quad$ or $\quad x^4-3x^2=0\;$ or $\;x^2(x^2-3)=0.$

$\therefore\quad x=0, \pm\sqrt{3}$. Putting $x=0$, $\sqrt{3}$ and $-\sqrt{3}$ in (1), we get respectively $y=0, -\sqrt{3}/2$ and $\sqrt{3}/2$. Hence the required points are $(0, 0)$, $(\sqrt{3}, -\sqrt{3}/2)$ and $(-\sqrt{3}, \sqrt{3}/2)$.

Ex. 9. (a) *Find the tangents to the curve $y=x^3-2x^2+x-2$ which are (i) parallel to the axis of x, (ii) parallel to the straight line $y=x$ bisecting the angle between the coordinate axes.*

(b) *Show that the abscissae of the points on the curve $y=x(x-2)(x-4)$ where the tangents are parallel to the axis of x are given by $x=2\pm(2/\sqrt{3})$.*

(c) *Find the equations of the tangent and normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it cuts the axis of x.*

(Meerut 1979, 85, 86; Avadh 87)

Sol. (a) The given curve is

$$y=x^3-2x^2+x-2. \qquad ...(1)$$

Differentiating (1), we get $\quad dy/dx=3x^2-4x+1$.

(i) If the tangent to (1) at the point (x, y) is parallel to x-axis we have $dy/dx=0$ *i.e.*, $3x^2-4x+1=0$ *i.e.*, $x=1$, $1/3$. Putting $x=1$, $1/3$ in (1), we get respectively $y=-2$, $-50/27$.

Hence the required tangents are $y=-2$ and $y=-50/27$.

(ii) If the tangent to (1) at the point (x, y) is parallel to the line $y=x$ whose gradient is 1, we have $dy/dx=1$ *i.e.*, $3x^2-4x+1=1$ *i.e.*, $x(3x-4)=0$ *i.e.*, $x=0, 4/3$. From (1), $x=0$ gives $y=-2$ and $x=4/3$ gives $y=-50/27$. Thus the tangents to the curve (1) at the points $(0, -2)$ and $(4/3, -50/27)$ are parallel to the line $y=x$. The equations of these tangents are $y - (-2) = 1\,(x - 0)$ and $y - (-50/27) = 1\,(x - 4/3)$ *i.e.*, $x - y = 2$ and $x - y = 86/27$.

(b) The given curve is $y = x\,(x-2)\,(x-4) = x^3 - 6x^2 + 8x$. We have $dy/dx = 3x^2 - 12x + 8$. The tangent to the given curve at the point (x, y) is parallel to the axis of x if $dy/dx = 0$ *i.e.*, $3x^2 - 12x + 8 = 0$ *i.e.*, $x = \{12 \pm \sqrt{(144 - 96)}\}/6 = 2 \pm (2/\sqrt{3})$. Hence the abscissae of the required points are $2 \pm (2/\sqrt{3})$.

(c) The given curve cuts the axis of x at the point $(7, 0)$.

Differentiating the equation of the curve, we get

$$\frac{dy}{dx}(x^2 - 5x + 6) + y\,(2x - 5) - 1 = 0.$$

Putting $x = 7$ and $y = 0$ in it we have

$$\left(\frac{dy}{dx}\right)_{(7,\,0)} = \frac{1}{20}\,.$$

$\therefore$ equation of tangent to the given curve at the point $(7, 0)$ is

$$y - 0 = (1/20)\,(x - 7) \quad i.e., \quad 20y - x + 7 = 0.$$

Also the equation of normal is

$$y - 0 = -20\,(x - 7) \quad i.e., \quad y + 20x - 140 = 0.$$

Ex. 10. *Show that the normal to the curve* $5x^5 - 10x^3 + x + 2y + 6 = 0$ *at* $(0, -3)$ *is also a tangent at two other points of the curve.*

Sol. The given curve is

$$5x^5 - 10x^3 + x + 2y + 6 = 0. \qquad \text{...(1)}$$

Differentiating (1), we get

$$25x^4 - 30x^2 + 1 + 2\,(dy/dx) = 0. \qquad \text{...(2)}$$

Putting $x = 0$ and $y = -3$ in (2), we get $1 + 2\,(dy/dx)_{(0,\,-3)} = 0$ *i.e.*, $(dy/dx)_{(0,\,-3)} = -1/2$. Therefore the gradient of the normal to (1) at the point $(0, -3)$ is 2 and the equation of this normal is

$$y + 3 = 2\,(x - 0) \quad \text{or} \quad y = 2x - 3. \qquad \text{...(3)}$$

Now if the normal (3) is to be a tangent to (1) at two other points of (1), it should meet (1) at two pairs of coincident points. Solving (1) and (3), we get $5x^5 - 10x^3 + x + 2\,(2x - 3) + 6 = 0$ or $x\,(x - 1)^2\,(x + 1)^2 = 0$. Thus $x = 1$ and $x = -1$ give two pairs of coincident points at which (3) meets (1). Hence (3) touches (1) at the points where $x = 1$ and -1.

Ex. 11. *Find the coordinates of the points on the curve* $y = x^2 + 3x + 3$ *the tangent at which passes through the origin.*

Sol. From the equation of the curve, we have $dy/dx = 2x + 3$.

$\therefore$ the equation of the tangent to the given curve at the point (x_1, y_1) is $y - y_1 = (2x_1 + 3)\,(x - x_1)$. If this tangent passes through the origin $(0, 0)$, we have $0 - y_1 = (2x_1 + 3)\,(0 - x_1)$

i.e., $$y_1 = 2x_1^2 + 3x_1. \quad ...(1)$$

Since the point (x_1, y_1) lies on the given curve, therefore

$$y_1 = x_1^2 + 3x_1 + 4. \quad ...(2)$$

From (1) and (2), we have $2x_1^2 + 3x_1 = x_1^2 + 3x_1 + 4$ or $x_1^2 = 4$ or $x_1 = \pm 2$. Putting $x_1 = 2$ and -2 in (1), we get $y_1 = 14$ and 2 respectively. Hence the required points are (2,14) and (− 2, 2).

Ex. 12. *Prove that all points of the curve $y^2 = 4a\{x + a \sin(x/a)\}$ at which the tangent is parallel to the axis of x lie on a parabola.*

Sol. The given curve is

$$y^2 = 4a\{x + a \sin(x/a)\}. \quad ...(1)$$

Differentiating (1), we get

$$2y\,(dy/dx) = 4a\{1 + \cos(x/a)\}$$

or $$dy/dx = (2a/y)\{1 + \cos(x/a)\}.$$

Suppose that the tangent to (1) at the point (x_1, y_1) is parallel to x-axis. Then dy/dx at $(x_1, y_1) = 0$

i.e., $$(2a/y_1)\{1 + \cos(x_1/a)\} = 0$$

i.e., $$\cos(x_1/a) = -1. \quad ...(2)$$

Also (x_1, y_1) lies on (1). Therefore we have

$$y_1^2 = 4a\{x_1 + a \sin(x_1/a)\} = 4a(x_1 + 0),$$

$$[\because \text{ from (2)}, \cos(x_1/a) = -1 \Rightarrow \sin(x_1/a) = 0]$$

or $$y_1^2 = 4ax_1.$$

Hence the required locus of (x_1, y_1) is $y^2 = 4ax$ which is a parabola.

Ex. 13. *Find the points at which the tangent to the curve*

$$ax^2 + 2hxy + by^2 = 1$$

is (i) parallel to and (ii) perpendicular to the axis of x.

Sol. Differentiating the equation of the curve, we get

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\left(\frac{ax + hy}{hx + by}\right).$$

(i) If the tangent is parallel to x-axis, we have $dy/dx = 0$

i.e., $$ax + hy = 0. \quad ...(1)$$

(ii) If the tangent is $\perp$ to x-axis, we have $dx/dy = 0$

i.e., $$hx + by = 0. \quad ...(2)$$

Hence the required points are obtained by solving

$ax^2 + 2hxy + by^2 = 1$ with (1) and (2) respectively.

Ex. 14. *Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on $x^2y^2 = x^2 - y^2$.*

(Meerut 1990 P, 91S; Lucknow 83; Kanpur 81)

Sol. Differentiating the equation of the given curve $y = \sin x$, we get $dy/dx = \cos x$.

Let (x_1, y_1) be the point of contact of a tangent drawn from the origin to the curve $y = \sin x$. We have dy/dx at $(x_1, y_1) = \cos x_1$. Therefore the equation of the tangent to the curve $y = \sin x$ at the point (x_1, y_1) on it is

$$y - y_1 = \cos x_1 \,.\, (x - x_1).$$

Since this tangent passes through the origin *i.e.*, the point $(0, 0)$. therefore

$$0 - y_1 = \cos x_1 \,.\, (0 - x_1) \quad \text{or} \quad y_1 = x_1 \cos x_1$$

or $$(y_1/x_1) = \cos x_1 \qquad \text{...(1)}$$

Again the point (x_1, y_1) lies on the curve $y = \sin x$. Therefore

$$y_1 = \sin x_1. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$(y_1/x_1)^2 + y_1^2 = 1 \quad \text{or} \quad x_1^2 y_1^2 = x_1^2 - y_1^2.$$

Hence generalising we get the locus of (x_1, y_1) as

$$x^2 y^2 = x^2 - y^2.$$

Ex. 15. *In the curve $x^m y^n = a^{m+n}$, prove that the portion of the tangent intercepted between the axes is divided at its point of contact into segments which are in constant ratio.*

Sol. The given curve is $\quad x^m y^n = a^{m+n}$. ...(1)

Differentiating (1) after taking logarithm of both sides, we get

$$(m/x) + (n/y)(dy/dx) = 0$$

or $$dy/dx = (-my)/(nx).$$

$\therefore$ the equation of the tangent to (1) at the point (x_1, y_1) is

$$y - y_1 = -(my_1/nx_1)(x - x_1)$$

or $$nx_1 y - nx_1 y_1 = -my_1 x + mx_1 y_1$$

or $$my_1 x + nx_1 y = (m + n) x_1 y_1$$

or $$\frac{x}{\{(m+n)x_1\}/m} + \frac{y}{\{(m+n)y_1\}/n} = 1. \qquad \text{...(2)}$$

The tangent (2) meets the x-axis at the point $\left(\frac{m+n}{m} x_1, 0\right)$ and the y-axis at the point $\left(0, \frac{m+n}{n} y_1\right)$ · Let the point of contact (x_1, y_1) divide the line joining these points in the ratio $k : 1$. Then

$$x_1 = \frac{1 \cdot \left(\frac{m+n}{m} x_1\right) + k \,.\, 0}{k+1} = \frac{(m+n)x_1}{m(k+1)}.$$

Since $x_1 \neq 0$, therefore $1 = \dfrac{m+n}{m(k+1)}$ or $mk + m = m + n$ or $k = n/m =$ constant. Hence the result.

Ex. 16. (a) *If $p = x \cos \alpha + y \sin \alpha$ touches $(x/a)^m + (y/b)^m = 1$, prove that $p^{m/(m-1)} = (a \cos \alpha)^{m/(m-1)} + (b \sin \alpha)^{m/(m-1)}$.*

Sol. The equation of the curve is $(x/a)^m + (y/b)^m = 1$. ...(1)

Differentiating (1), we get

$$m(x/a)^{m-1}(1/a) + m(y/b)^{m-1}(1/b)(dy/dx) = 0$$

or
$$\frac{dy}{dx} = -\frac{(1/a)(x/a)^{m-1}}{(1/b)(y/b)^{m-1}}.$$

Hence the equation of the tangent at (x, y) to (1) is

$$Y - y = -\frac{(1/a)(x/a)^{m-1}}{(1/b)(y/b)^{m-1}}(X - x)$$

or
$$\frac{X}{a}\left(\frac{x}{a}\right)^{m-1} + \frac{Y}{b}\left(\frac{y}{b}\right)^{m-1} = \left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1, \quad \text{from (1).} \quad ...(2)$$

Now suppose the straight line

$$X \cos \alpha + Y \sin \alpha = p, \quad ...(3)$$

touches the curve (1) at the point (x, y). Then (2) and (3) are the equations of the same straight line and so comparing the coefficients, we get

$$\frac{\cos \alpha}{(1/a)(x/a)^{m-1}} = \frac{\sin \alpha}{(1/b)(y/b)^{m-1}} = \frac{p}{1}.$$

$\therefore \quad (x/a)^{m-1} = (a \cos \alpha)/p$ and $(y/b)^{m-1} = (b \sin \alpha)/p$.

Hence $\left(\dfrac{a \cos \alpha}{p}\right)^{m/(m-1)} + \left(\dfrac{b \sin \alpha}{p}\right)^{m/(m-1)} = \left(\dfrac{x}{a}\right)^m + \left(\dfrac{y}{b}\right)^m = 1,$

[$\because$ (x, y) lies on (1)].

Thus we get the required condition.

****Ex. 16 (b)** *If $p = x \cos \alpha + y \sin \alpha$ touches the curve*

$$(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1,$$

prove that $p^n = (a \cos \alpha)^n + (b \sin \alpha)^n$.

(Delhi 1983, 81, 79; Allahabad 80; Meerut 86 P, 91 P)

Sol. The equation of the curve is

$$(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)} = 1. \quad ...(1)$$

Differentiating (1), we get

$$\frac{1}{a^{n/(n-1)}} \cdot \left(\frac{n}{n-1}\right) x^{1/(n-1)} + \frac{1}{b^{n/(n-1)}}\left(\frac{n}{n-1}\right) y^{1/(n-1)} \cdot \frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{b^{n/(n-1)}}{a^{n/(n-1)}} \cdot \frac{x^{1/(n-1)}}{y^{1/(n-1)}}.$$

Hence the equation of the tangent at (x, y) to the curve (1) is

$$Y - y = -\frac{b^{n/(n-1)}}{a^{n/(n-1)}} \cdot \frac{x^{1/(n-1)}}{y^{1/(n-1)}}(X - x)$$

or $$\frac{X x^{1/(n-1)}}{a^{n/(n-1)}} + \frac{Y y^{1/(n-1)}}{b^{n/(n-1)}} = \left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)}$$

$$= 1, \text{ from } (1). \qquad ...(2)$$

Now suppose the straight line

$$X \cos\alpha + Y \sin\alpha = p, \qquad ...(3)$$

touches the curve (1) at the point (x, y). Then (2) and (3) are the equations of the same straight line. Therefore comparing the coefficients, we get

$$\frac{\cos\alpha}{(x^{1/(n-1)}/a^{n/(n-1)})} = \frac{\sin\alpha}{(y^{1/(n-1)}/b^{n/(n-1)})} = \frac{p}{1}$$

or $\quad a\cos\alpha = p\,(x/a)^{1/(n-1)}\quad$ and $\quad b\sin\alpha = p\,(y/b)^{1/(n-1)}$.

Raising both sides to the power n and adding, we get

$$(a\cos\alpha)^n + (b\sin\alpha)^n = p^n\{(x/a)^{n/(n-1)} + (y/b)^{n/(n-1)}\}$$

$$= p^n \cdot 1, \qquad [\because (x, y) \text{ lies on } (1)]$$

$$= p^n, \text{ which is the required condition.}$$

Ex. 17. *Prove that the condition that $x\cos\alpha + y\sin\alpha = p$ touches the curve $x^m y^n = a^{m+n}$ is*

$$p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \cos^m\alpha \sin^n\alpha.$$

(Delhi 1980, Meerut 77, 84; Lucknow 77)

Sol. The given curve is $\quad x^m y^n = a^{m+n}$. $\qquad ...(1)$

Taking logarithm of both sides of (1), we have

$$m\log x + n\log y = (m+n)\log a.$$

Differentiating it, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\left(\frac{m}{n}\right)\left(\frac{y}{x}\right).$$

Suppose the straight line

$$x\cos\alpha + y\sin\alpha = p \qquad ...(2)$$

touches the curve (1) at the point (x_1, y_1).

The equation of the tangent to (1) at the point (x_1, y_1) is

$$y - y_1 = -\frac{m y_1}{n x_1}(x - x_1)$$

or $$(mx/x_1) + (ny/y_1) = (m+n). \qquad ...(3)$$

Since the equations (2) and (3) represent the same straight line, therefore comparing the coefficients, we get

$$\frac{\cos\alpha}{(m/x_1)} = \frac{\sin\alpha}{(n/y_1)} = \frac{p}{(m+n)}$$

i.e., $$x_1 = \frac{mp}{(m+n)\cos\alpha} \text{ and } y_1 = \frac{np}{(m+n)\sin\alpha}. \qquad ...(4)$$

Now the point (x_1, y_1) lies on (1). Therefore we have or $\left(\dfrac{mp}{(m+n)\cos\alpha}\right)^m \cdot \left(\dfrac{np}{(m+n)\sin\alpha}\right)^n = a^{m+n},$ from (4)

or $m^m n^n p^{m+n} = a^{m+n}(m+n)^{m+n}\cos^m\alpha\sin^n\alpha.$

Ex. 18. *Prove that the curves $y = e^{-ax}\sin bx$, $y = e^{-ax}$ touch at the points for which $bx = 2m\pi + \frac{1}{2}\pi$, where m is an integer.*

Sol. The given curves are

$$y = e^{-ax}\sin bx \quad ...(1)$$

and

$$y = e^{-ax} \quad ...(2)$$

Solving the equations (1) and (2) for x, we get

$$e^{-ax}\sin bx = e^{-ax}$$

or $\sin bx = 1.$ $[\because e^{-ax} \neq 0$ for any real number $x]$

Now the general solution of the equation $\sin bx = 1$ is given by $bx = 2m\pi + \frac{1}{2}\pi$, where m is any integer.

$\therefore$ at the points where the curves (1) and (2) meet, we have

$$bx = 2m\pi + \tfrac{1}{2}\pi.$$

Now $\left(\dfrac{dy}{dx}\right)_1 = -ae^{-ax}\sin bx + be^{-ax}\cos bx$

and $\left(\dfrac{dy}{dx}\right)_2 = -ae^{-ax}.$

At the points where the curves (1) and (2) meet, we have $\sin bx = 1$ and so $\cos bx = 0$.

$\therefore$ at the points where the two curves meet, we have

$$\left(\frac{dy}{dx}\right)_1 = -ae^{-ax} = \left(\frac{dy}{dx}\right)_2$$

i.e., the two curves have the same tangent at these points.

Hence the two curves touch at the points for which $bx = 2m\pi + \frac{1}{2}\pi$.

Ex. 19. *Prove that normal of the curve $y^2 = 4ax$ touches the curve $27ay^2 = 4(x-2a)^3$.* **(Kanpur 1979)**

Sol. The equation of the normal at any point $(am^2, -2am)$ of the parabola

$$y^2 = 4ax \text{ is } y = mx - 2am - am^3. \quad ...(1)$$

The parametric equations of the curve $27ay^2 = 4(x-2a)^3$ can be written as

$$y = 2at^3, x = 2a + 3at^2.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6at^2}{6at} = t.$$

$\therefore$ the equation of tangent at the point 't' is

$$y - 2at^3 = t(x - 2a - 3at^2)$$

or

$$y = tx - 2at - at^3,$$

which is identical with (1) if $t = m$.

Hence the normal at $(am^2, -2am)$ to the parabola $y^2 = 4ax$ is a tangent at $(2a + 3am^2, 2am^3)$ to the curve

$$27ay^2 = 4(x - 2a)^3.$$

Ex. 20. *If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is*

$$y \cos \phi - x \sin \phi = a \cos 2\phi.$$

(Meerut 1983, 87, 90S, 96 BP)

Sol. The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

Differentiating (1), we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Hence the gradient of the normal to (1) at the point (x, y)

$$= -\frac{dx}{dy} = \left(\frac{x}{y}\right)^{1/3} = \tan \phi \text{ (given).}$$

$\therefore \quad x^{1/3}/y^{1/3} = \sin \phi / \cos \phi$.

or $$\frac{x^{1/3}}{\sin \phi} = \frac{y^{1/3}}{\cos \phi} = \frac{\sqrt{(x^{2/3} + y^{2/3})}}{\sqrt{(\sin^2 \phi + \cos^2 \phi)}} = \frac{\sqrt{(a^{2/3})}}{1}, \text{ from (1).}$$

$\therefore \quad x^{1/3} = a^{1/3} \sin \phi$ or $x = a \sin^3 \phi$.

Similarly we have $y = a \cos^3 \phi$.

Thus $(a \sin^3 \phi, a \cos^3 \phi)$ is the point on (1) the normal at which makes an angle ϕ with the axis of x. Hence the required equation of the normal is

$$y - a \cos^3 \phi = \tan \phi \, (x - a \sin^3 \phi)$$

or $$y \cos \phi - a \cos^4 \phi = x \sin \phi - a \sin^4 \phi$$

or $$y \cos \phi - x \sin \phi = a (\cos^4 \phi - \sin^4 \phi)$$

$$= a (\cos^2 \phi + \sin^2 \phi)(\cos^2 \phi - \sin^2 \phi)$$

$$= a \cos 2\phi.$$

Ex. 21 (a) *Find the angle of intersection of the curves*

$$x^2 - y^2 = a^2 \text{ and } x^2 + y^2 = a^2\sqrt{2}.$$ **(Delhi 1979)**

Sol. The given curves are

$$x^2 - y^2 = a^2, \qquad ...(1)$$

and $$x^2 + y^2 = a^2\sqrt{2}. \qquad ...(2)$$

Solving (1) and (2) to get their points of intersection, we have

$$x^2 = \frac{a^2(\sqrt{2} + 1)}{2} \text{ and } y^2 = \frac{a^2(\sqrt{2} - 1)}{2}.$$

Hence $$x^2y^2 = \frac{a^4(2 - 1)}{4} = \frac{a^4}{4}.$$

Differentiating the equation (1), we get $dy/dx = (2x)/(2y) = x/y$ and differentiating the equation (2), we get $dy/dx = -x/y$. If θ is the angle of intersection of (1) and (2), at their point of intersection (x, y), then

$$\theta = \tan^{-1}\left|\frac{x/y-(-x/y)}{1+(x/y)(-x/y)}\right| = \tan^{-1}\left|\frac{2xy}{y^2-x^2}\right|$$

$$= \tan^{-1}\left|\frac{\sqrt{(4x^2y^2)}}{y^2-x^2}\right| = \tan^{-1}\left|\frac{\sqrt{a^4}}{-a^2}\right|,$$

[$\because$ at the point of intersection (x, y), we have $x^2-y^2=a^2$ and $4x^2y^2=a^4$]

$$= \tan^{-1}|-1| = \tan^{-1}1 = \pi/4.$$

Ex. 21 (b). *Find the angles of intersection of the parabolas* $y^2 = 4ax$ *and* $x^2 = 4by$. **(Meerut 1980, 91)**

Sol. The given curves are $y^2 = 4ax$, ...(i)

and $x^2 = 4by$. ...(ii)

Solving (i) and (ii), we get on eliminating y

$$x^4 = 64ab^2x \quad \text{or} \quad x(x^3 - 64ab^2) = 0.$$

$\therefore \quad x = 0$ and $4a^{1/3}b^{2/3}$.

Substituting these values of x in (ii), we get

$y = 0$ for $x = 0$

and $y = 4a^{2/3}b^{1/3}$ for $x = 4a^{1/3}b^{2/3}$.

Therefore $(0, 0)$ and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ are the two points of intersection of (i) and (ii).

Differentiating (i), we get

$$2y\frac{dy}{dx} = 4a \quad i.e. \quad \frac{dy}{dx} = \frac{2a}{y}.$$

Differentiating (ii), we have

$$2x = 4b\frac{dy}{dx} \quad i.e., \quad \frac{dy}{dx} = \frac{x}{2b}.$$

Angle of intersection at (0, 0).

$$\frac{dy}{dx} \text{ of (i) at } (0, 0) = \infty$$

and $$\left(\frac{dy}{dx}\right) \text{ of (ii) at } (0, 0) = 0.$$

$\therefore$ the angle of intersection at $(0, 0)$ is $90°$.

Angle of intersection at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$.

$$\frac{dy}{dx} \text{ of (i) at } (4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3}) = \frac{a^{1/3}}{2b^{1/3}},$$

and $$\frac{dy}{dx} \text{ of (ii) at } (4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3}) = \frac{2a^{1/3}}{b^{1/3}}.$$

Therefore if θ is the acute angle between the tangents to the two curves at the point $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$, then

$$\theta = \tan^{-1}\left|\frac{\dfrac{2a^{1/3}}{b^{1/3}} - \dfrac{a^{1/3}}{2b^{1/3}}}{1 + \dfrac{2a^{1/3}}{b^{1/3}} \cdot \dfrac{a^{1/3}}{2b^{1/3}}}\right| = \tan^{-1}\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}.$$

****Ex. 22.** *Show that the condition that the curves* $ax^2 + by^2 = 1$ *and* $a'x^2 + b'y^2 = 1$ *should intersect orthogonally is that*

$$1/a - 1/b = 1/a' - 1/b'.$$

(Kumayun 1983; Meerut 88, 89S, 91P, 92; Lucknow 82; Avadh 87)

Sol. The given curves are

$$ax^2 + by^2 = 1, \qquad ...(1)$$

and $$a'x^2 + b'y^2 = 1. \qquad ...(2)$$

Let (x_1, y_1) be a point of intersection of (1) and (2). Then (x_1, y_1) satisfies both (1) and (2). Therefore

$$ax_1^2 + by_1^2 - 1 = 0 \qquad ...(3)$$

and $$a'x_1^2 + b'y_1^2 - 1 = 0 \qquad ...(4)$$

Solving (3) and (4), we get

$$\frac{x_1^2}{-b + b'} = \frac{y_1^2}{-a' + a} = \frac{1}{ab' - a'b}. \qquad ...(5)$$

Differentiating (1), we get

$$2ax + 2by\,(dy/dx) = 0 \quad \text{or} \quad dy/dx = -ax/(by).$$

Again differentiating (2), we get $dy/dx = -a'x/(b'y)$.

Now (1) and (2) intersect orthogonally if at their point of intersection (x_1, y_1), we have

$$(dy/dx)_1 \cdot (dy/dx)_2 = -1$$

or $$\left(-\frac{ax_1}{by_1}\right)\left(-\frac{a'x_1}{b'y_1}\right) = -1$$

or $$aa'x_1^2 + bb'y_1^2 = 0 \qquad ...(6)$$

Putting the values of x_1^2 and y_1^2 from (5) in (6), we get

$$aa'\frac{(-b + b')}{ab' - a'b} + bb'\frac{(a - a')}{ab' - a'b} = 0$$

or $$aa'(b' - b) = bb'(a' - a)$$

or $$\frac{b' - b}{bb'} = \frac{a' - a}{aa'} \quad \text{or} \quad \frac{1}{b} - \frac{1}{b'} = \frac{1}{a} - \frac{1}{a'}$$

or $$1/a - 1/b = 1/a' - 1/b'$$

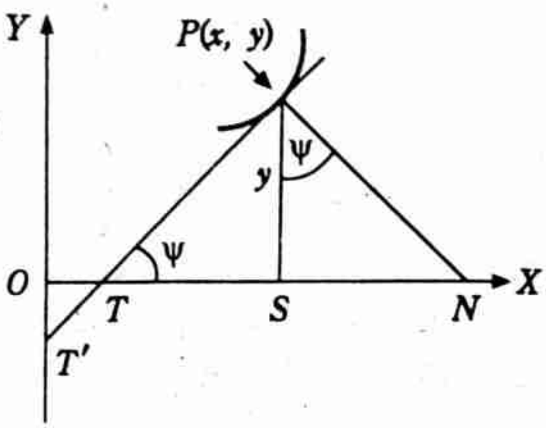
as the required condition.

§ 5. Length of Cartesian Tangent, Normal, Subtangent and Subnormal.

Let $P(x, y)$ be any point on the curve $y = f(x)$. Let the tangent and normal at P on the curve meet the x-axis in T and N respectively and let PS be the ordinate of the point P. Then ST is called the **sub-tangent** and NS the **sub-normal** at the point P of the curve. If ψ be the angle which the tangent at P makes with x-axis, then

$$\angle PTS = \angle SPN = \psi$$

and $$\tan\psi = \frac{dy}{dx}.$$



Now at the point P of the curve, we get

length of tangent

$$= PT = y \operatorname{cosec}\psi$$
$$= y\sqrt{(1+\cot^2\psi)}$$
$$= \mathbf{y}\sqrt{\left\{1+\left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^2\right\}},$$

length of sub-tangent

$$= TS = y\cot\psi = \mathbf{y}\,\frac{\mathbf{dx}}{\mathbf{dy}} = \frac{\mathbf{y}}{(\mathbf{dy/dx})},$$

length of normal $= PN = y\sec\psi = y\sqrt{(1+\tan^2\psi)}$

$$= \mathbf{y}\sqrt{\left\{1+\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2\right\}},$$

and **length of sub-normal**

$$= SN = y\tan\psi = \mathbf{y}\,(\mathbf{dy/dx}).$$

Intercepts made by the tangent on the coordinates axes. The equation of the tangent at the point $P(x, y)$ is $Y - y = (dy/dx)(X - x)$.

It meets the x-axis where $Y = 0$ *i.e.*, where $0 - y = (dy/dx)(X - x)$ or X *i.e.*, $OT = x - \dfrac{y}{dy/dx}\cdot$ Again the tangent meets the y-axis, where $X = 0$ *i.e.*, where $Y - y = (dy/dx)(0 - x)$ or Y *i.e.*, $OT' = y - x\,(dy/dx)$. Hence **the intercept cut off by the tangent on x-axis** $= \mathbf{x} - \dfrac{\mathbf{y}}{\mathbf{dy/dx}} = \mathbf{x} - \mathbf{y}\dfrac{\mathbf{dx}}{\mathbf{dy}}$, **and the intercept cut off by the tangent on y-axis** $= \mathbf{y} - \mathbf{x}\,(\mathbf{dy/dx})$.

Solved Examples

Ex. 23. *Find the subtangent, subnormal, normal, tangent and the intercepts on the axes at the point 't' on the cycloid* $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$.

Sol. We have $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a\,(1+\cos t)}$$

$$= \frac{2a\sin\frac{1}{2}t\cos\frac{1}{2}t}{2a\cos^2\frac{1}{2}t} = \tan\tfrac{1}{2}t.$$

$\therefore \quad \tan\psi = dy/dx = \tan\frac{1}{2}t$ *i.e.*, $\psi = \frac{1}{2}t$.

Now **subtangent** $= \dfrac{y}{dy/dx} = \dfrac{a(1-\cos t)}{\tan\frac{1}{2}t} = \dfrac{2a\sin^2\frac{1}{2}t}{\tan\frac{1}{2}t}$

$= 2a\sin\frac{1}{2}t\cos\frac{1}{2}t = a\sin t$;

subnormal $= y\,.\,(dy/dx) = a(1-\cos t)\tan\frac{1}{2}t = 2a\sin^2\frac{1}{2}t\tan\frac{1}{2}t$;

length of tangent $= y\operatorname{cosec}\psi = a(1-\cos t)\operatorname{cosec}\frac{1}{2}t, \quad [\because \psi = \frac{1}{2}t]$

$= 2a\sin^2\frac{1}{2}t\operatorname{cosec}\frac{1}{2}t = 2a\sin\frac{1}{2}t$;

length of normal $= y\sec\psi = a(1-\cos t)\sec\frac{1}{2}t$

$= 2a\sin^2\frac{1}{2}t\sec\frac{1}{2}t = 2a\sin\frac{1}{2}t\tan\frac{1}{2}t$;

intercept on x-axis $= x - \dfrac{y}{dy/dx} = a(t+\sin t) - \dfrac{a(1-\cos t)}{\tan\frac{1}{2}t}$

$= a(t+\sin t) - 2a\sin^2\frac{1}{2}t\cot\frac{1}{2}t$

$= a(t+\sin t) - a\sin t = at$;

and **intercept on y-axis** $= y - x\,(dy/dx)$

$= a(1-\cos t) - a(t+\sin t)\tan\frac{1}{2}t$

$= 2a\sin^2\frac{1}{2}t - at\tan\frac{1}{2}t - a\sin t\tan\frac{1}{2}t$

$= 2a\sin^2\frac{1}{2}t - at\tan\frac{1}{2}t - 2a\sin\frac{1}{2}t\cos\frac{1}{2}t\tan\frac{1}{2}t$

$= 2a\sin^2\frac{1}{2}t - at\tan\frac{1}{2}t - 2a\sin^2\frac{1}{2}t$

$= -at\tan\frac{1}{2}t$.

Ex. 24. *Show that the subnormal at any point of a parabola is of constant length and the subtangent varies as the abscissa of the point of contact.* **(Garhwal 1983; Meerut 93)**

Sol. Equation of the parabola is $y^2 = 4ax$. Differentiating it, we have

$$2y\,(dy/dx) = 4a \quad \text{or} \quad dy/dx = 2a/y.$$

Now subtangent $= \dfrac{y}{dy/dx} = \dfrac{y}{2a/y} = \dfrac{y^2}{2a} = \dfrac{4ax}{2a} = 2x.$

Therefore subtangent $\propto x$.

Again subnormal $= y\,(dy/dx) = y\,(2a/y) = 2a$ which is a constant.

Ex. 25. *Prove that for the curve $y = be^{x/a}$, the subtangent is of constant length and the sub-normal varies as the square of the ordinate.* **(Meerut 1983, 96P)**

Sol. The given curve is $y = be^{x/a}$.

Differentiating it, we have

$$dy/dx = (b/a)\,e^{x/a} = y/a. \qquad \textbf{(Note)}$$

$\therefore \quad$ subtangent $= \dfrac{y}{dy/dx} = \dfrac{y}{y/a} = a =$ cosntant.

Also subnormal $= y \cdot \dfrac{dy}{dx} = y \cdot \dfrac{y}{a} = \dfrac{1}{a} y^2$.

$\therefore$ subnormal $\propto y^2$ *i.e.*, square of the ordinate.

Ex. 26. *In the catenary $y = a \cosh (x/a)$, prove that the length of the portion of the normal intercepted between the curve and the x-axis is y^2/a.* **(Jiwaji 1970; Magadh 76; Utkal 70)**

Sol. Differentiating the equation of the given catenary, we get

$$dy/dx = a \{\sinh (x/a)\} . (1/a) = \sinh (x/a).$$

Now the length of the portion of the normal intercepted between the curve and the x-axis = the length of the normal

$$= y \sec \psi = y \sqrt{(1 + \tan^2 \psi)} = y \sqrt{\{1 + (dy/dx)^2\}}$$

$$= y\sqrt{\{1 + \sinh^2 (x/a)\}} = y \cosh (x/a)$$

$$= y . (y/a) = y^2/a. \qquad [\because y = a \cosh (x/a)]$$

Ex. 27. *In the curve $x^{m+n} = a^{m-n} y^{2n}$, prove that the mth power of the subtangent varies as the nth power of the subnormal.*

Sol. The given curve is $\quad x^{m+n} = a^{m-n} y^{2n}$. ...(1)

Taking logarithm of both sides of (1), we get

$$(m + r) \log x = (m - n) \log a + 2n \log y.$$

Now differentiating w.r.t. x, we get

$$\frac{m+n}{x} = 0 + \frac{2n}{y} \cdot \frac{dy}{dx} \quad i.e. \quad \frac{dy}{dx} = \left(\frac{m+n}{2n}\right) \cdot \frac{y}{x}. \qquad ...(2)$$

Now we want to prove that (subtangent)$^m \propto$ (subnormal)^n

or (subtangent)^m/(subnormal)^n = constant.

We have (subtangent)^m/(subnormal)^n

$$= \frac{[y/(dy/dx)]^m}{[y(dy/dx)]^n} = \frac{y^{m-n}}{(dy/dx)^{m+n}} = y^{m-n} \cdot \frac{(2n)^{m+n} x^{m+n}}{(m+n)^{m+n} y^{m+n}}, \qquad \text{[from (2)]}$$

$$= \frac{(2n)^{m+n}}{(m+n)^{m+n}} \cdot \frac{x^{m+n}}{y^{2n}} = \frac{(2n)^{m+n}}{(m+n)^{m+n}} a^{m-n}, \text{ from (1)}$$

$$= \text{constant}.$$

Hence the required result follows.

Ex. 28. *In the tractrix $x = a (\cos t + \log \tan \frac{1}{2} t)$, $y = a \sin t$ prove that the portion of the tangent intercepted between the curve and x-axis is of constant length.*

(Meerut 1983, 88; Gorakhpur 76; Delhi 74; Alld. 81; Agra 80)

Sol. Differentiating the given parametric equation of the curve w.r.t. t, we get

$$\frac{dx}{dt} = a \left\{- \sin t + \frac{1}{\tan \frac{1}{2} t} \cdot (\sec^2 \tfrac{1}{2} t) \cdot \tfrac{1}{2}\right\}$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t},$$

and $\qquad dy/dt = a\cos t.$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = (a\cos t)\cdot\frac{\sin t}{a\cos^2 t} = \tan t.$$

Hence $\qquad \tan\psi = dy/dx = \tan t, \quad$ or $\quad \psi = t.$

Now the length of the portion of the tangent intercepted between the curve and x-axis = the length of the tangent

$= y \operatorname{cosec}\psi = a\sin t \operatorname{cosec} t, \qquad [\because \psi = t, \text{ and } y = a\sin t]$
$= a$ *i.e.*, constant.

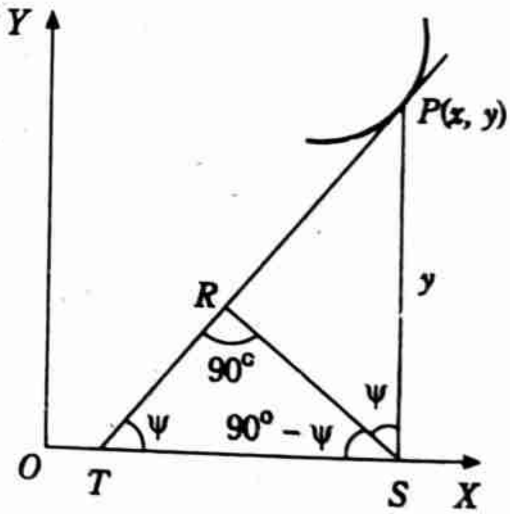
Ex. 29. *Prove that for the catenary $y = c\cosh(x/c)$, the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length.* **(Delhi 1970; Gorakhpur 70)**

Sol. Differentiating the given equation of the curve, we get

$$dy/dx = \sinh(x/c).$$

$$\therefore \quad \tan\psi = dy/dx = \sinh(x/c).$$

Let $P(x, y)$ be any point on the catenary $y = c\cosh(x/c)$.



Let the tangent at P meet the x-axis in T and let PS be the ordinate of the point P *i.e.*, $PS = y$. Also let R be the foot of the perpendicular drawn from S on the tangent at P. If ψ be the angle which the tangent at P makes with the x-axis, then

$$\angle PTS = \angle PSR = \psi.$$

We are to find the length of SR.

We have,

$$SR = PS\cos\psi = y\cos\psi = y/\sec\psi = \frac{y}{\sqrt{(1+\tan^2\psi)}}$$

$$= \frac{c\cosh(x/c)}{\sqrt{\{1+\sinh^2(x/c)\}}} \qquad [\text{putting for } y \text{ and } \tan\psi]$$

$$= \frac{c\cosh(x/c)}{\cosh(x/c)} = c = \text{constant}.$$

Ex. 30. *If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, show that*

$$(x_1^2/a^2) + (y_1^2/b^2) = 1.$$

(Meerut 1977, 84)

Sol. The given curve is $(x/a)^{2/3} + (y/b)^{2/3} = 1$. ...(1)

The coordinates of any point (x, y) on (1) may be taken as

$x = a \cos^3 t, y = b \sin^3 t$, where t is the parameter. **(Note)**

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\frac{b \sin t}{a \cos t}.$$

The equation of the tangent at the point 't' to the curve (1) is

$$y - b \sin^3 t = -\frac{b \sin t}{a \cos t}(x - a \cos^3 t)$$

or $$bx \sin t + ay \cos t = ab \sin t \cos t (\cos^2 t + \sin^2 t)$$
$$= ab \sin t \cos t$$

or $$\frac{bx \sin t}{ab \sin t \cos t} + \frac{ay \cos t}{ab \sin t \cos t} = 1,$$

[dividing each side by $ab \sin t \cos t$]

or $$\frac{x}{a \cos t} + \frac{y}{b \sin t} = 1. \qquad ...(2)$$

If x_1 and y_1 are the intercepts made by the straight line (2) on the axes of x and y respectively, then

$$x_1 = a \cos t \text{ and } y_1 = b \sin t.$$

[Note that we have compared (2) with the equation of a straight line in intercepts form $x/a + y/b = 1$].

Now $$x_1^2/a^2 + y_1^2/b^2 = (a^2 \cos^2 t)(1/a^2) + (b^2 \sin^2 t)(1/b^2)$$
$$= \cos^2 t + \sin^2 t = 1.$$

****Ex. 31.** *If the tangent to the curve $x^{1/2} + y^{1/2} = a^{1/2}$ at any point on it cuts the axes OX, OY at P, Q respectively prove that*

$$OP + OQ = a.$$ **(Delhi 1979, 76; Kanpur 76; Agra 75)**

Sol. The given curve is $(x/a)^{1/2} + (y/a)^{1/2} = 1$. ...(1)

We see that the point $x = a \cos^4 t, y = a \sin^4 t$ satisfies the equation (1) for all values of t. Therefore the coordinates of any point (x, y) on (1) may be taken as

$x = a \cos^4 t, y = a \sin^4 t$, where t is the parameter.

Now $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4a \sin^3 t \cos t}{-4a \cos^3 t \sin t} = -\frac{\sin^2 t}{\cos^2 t}.$$

The equation of the tangent at the point 't' to (1) is

$$y - a \sin^4 t = -\frac{\sin^2 t}{\cos^2 t}(x - a \cos^4 t)$$

or $$x \sin^2 t + y \cos^2 t = a \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)$$
$$= a \sin^2 t \cos^2 t$$

or $$\frac{x \sin^2 t}{a \sin^2 t \cos^2 t} + \frac{y \cos^2 t}{a \sin^2 t \cos^2 t} = 1,$$

dividing each side by $a \sin^2 t \cos^2 t$

or $$\frac{x}{a\cos^2 t} + \frac{y}{a\sin^2 t} = 1. \qquad ...(2)$$

If OP and OQ are the intercepts made by the straight line (2) on the axes of x and y respectively, then

$$OP = a\cos^2 t \quad \text{and} \quad OQ = a\sin^2 t.$$

Hence $$OP + OQ = a\cos^2 t + a\sin^2 t = a.$$

Ex. 32. *Show that the subtangent at any point of the curve $x^m y^n = a^{m+n}$ varies as the abscissa.* **(Delhi 1981; Meerut 89)**

Sol. The given curve is $x^m y^n = a^{m+n}$ ...(1)

Taking logarithm of both sides of (1), we get

$$m\log x + n\log y = (m+n)\log a.$$

Differentiating, we have

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{my}{nx}.$$

Now subtangent $= \dfrac{y}{dy/dx} = y\left(\dfrac{-nx}{my}\right) = -\dfrac{nx}{m}$

$= (\text{some constant}) \, . \, x.$

$\therefore$ Subtangent varies as the abscissa x.

Ex. 33. *Show that in the case of the curve $\beta y^2 = (x+\alpha)^3$, the square of the sub-tangent varies as the subnormal.*

(Delhi 1980; Meerut 87 S)

Sol. Here $\dfrac{dy}{dx} = \dfrac{3(x+\alpha)^2}{2\beta y}$.

Also subtangent $= \dfrac{y}{dy/dx} = \dfrac{2\beta y^2}{3(x+\alpha)^2} = \dfrac{2(x+\alpha)^3}{3(x+\alpha)^2} = \dfrac{2}{3}(x+\alpha)$

and subnormal $= y \cdot \dfrac{dy}{dx} = \dfrac{3(x+\alpha)^2}{2\beta}$.

$$\therefore \quad \frac{(\text{subtangent})^2}{\text{subnormal}} = \frac{4(x+\alpha)^2/9}{3(x+\alpha)^2/2\beta} = \frac{8\beta}{27} = \text{constant}.$$

Thus $(\text{subtangent})^2 \propto \text{subnormal}$.

Ex. 34. *Find the abscissa of the point on the curve $ay^2 = x^3$, the normal at which cuts off equal intercepts from the axes.*

Sol. Differentiating the equation of the given curve, we get $dy/dx = 3x^2/(2ay)$. If the normal to the curve at the point (x, y) cuts off equal intercepts from the coordinates axes, then it makes an angle of 45° or 135° with the axis of x. Therefore the gradient of the normal is equal to tan 45° or tan 135°. Hence we have

$$-dx/dy = 1 \quad \text{or} \quad -1,$$

[$\because$ gradient of the normal $= -dx/dy$]

i.e., $$dy/dx = -1 \quad \text{or} \quad 1.$$

Squaring we get $(dy/dx)^2 = 1$

or $\{3x^2/(2ay)\}^2 = 1$ or $4a^2y^2 = 9x^4$.

Solving this with the equation of the curve $ay^2 = x^3$, we get

$$4ax^3 = 9x^4 \quad \text{or} \quad x = 0, 4a/9.$$

But at $x = 0$, we have $y = 0$ and the normal to the curve at $(0, 0)$ does not cut off any intercept from the coordinate axes. Hence the abscissa of the required point $= 4a/9$.

Ex. 35. *Prove that in the ellipse $x^2/a^2 + y^2/b^2 = 1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent.*

(Delhi 1982)

Sol. Differentiating the equation of the given ellipse, we get

$$dy/dx = (-b^2x)/(a^2y).$$

Therefore the tangent to the ellipse at the point (x, y) is

$$Y - y = (-b^2x/a^2y)(X - x)$$

or $$b^2xX + a^2yY = b^2x^2 + a^2y^2 = a^2b^2. \quad \text{...(1)}$$

[$\because$ from the equation of the ellipse, we have $b^2x^2 + a^2y^2 = a^2b^2$].

Now p = the length of the perpendicular drawn from (0, 0) to the tangent (1)

$$= \frac{a^2b^2}{\sqrt{(b^4x^2 + a^4y^2)}}.$$

Also the length of the normal at the point $(x, y) = y \sec \psi$

$$= y\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = y\sqrt{\left(1 + \frac{b^4x^2}{a^4y^2}\right)} = \frac{\sqrt{(a^4y^2 + b^4x^2)}}{a^2}.$$

We have to prove that the length of the normal is proportional to $1/p$ *i.e.*, (the length of the normal) = (some constant). $(1/p)$

or (the length of the normal). p = some constant.

Now, (the length of the normal). p

$$= \frac{\sqrt{(a^4y^2 + b^4x^2)}}{a^2} \cdot \frac{a^2b^2}{\sqrt{(b^4x^2 + a^4y^2)}} = b^2 = \text{constant}.$$

Hence the required result follows.

****Ex. 36.** *Prove that for the curve $x^{2/3} + y^{2/3} = a^{2/3}$, the portion of the tangent intercepted between the axes is of constant length.*

(Meerut 1983; Gorakhpur 72; Delhi 82, 77)

Sol. The equation of the given curve is

$$x^{2/3} + y^{2/3} = a^{2/3}. \quad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get

$$(2/3)x^{-1/3} + (2/3)y^{-1/3}(dy/dx) = 0$$

or $$dy/dx = (-x^{-1/3})/(y^{-1/3}).$$

Hence the equation of the tangent to (1) at the point (x, y) is

$$Y - y = -\frac{x^{-1/3}}{y^{-1/3}}(X - x)$$

or $\qquad Yy^{-1/3} + Xx^{-1/3} = x^{2/3} + y^{2/3} = a^{2/3}$, from (1)

or $\qquad \dfrac{X}{x^{1/3}a^{2/3}} + \dfrac{Y}{y^{1/3}a^{2/3}} = 1.$ ...(2)

If the tangent (2) cuts off the intercepts OP and OQ from the axes of x and y respectively, then

$OP = x^{1/3}a^{2/3}$, and $OQ = y^{1/3}a^{2/3}$. **(Meerut 1988 P)**

Now the length of the tangent intercepted between the axes

$= PQ = \sqrt{(OP^2 + OQ^2)} = \sqrt{(x^{2/3}a^{4/3} + y^{2/3}a^{4/3})}$

$= a^{2/3}\sqrt{(x^{2/3} + y^{2/3})} = a^{2/3}\sqrt{a^{2/3}}$,

[$\because$ the point (x, y) lies on (1)]

$= a^{2/3}a^{1/3} = a =$ constant.

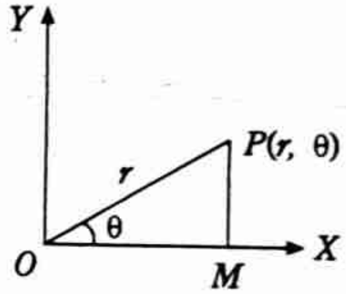
§ 6. Polar Coordinates.

If P be any point on the xy-plane then its position can also be indicated by stating :

(i) its distance r from a fixed point O, and

(ii) the inclination θ of the line OP to a fixed straight line OX called the **initial line.**

The fixed point O is called the **pole** and (r, θ) are called the **polar coordinates** of the point P.



The distance $OP = r$ is called the **radius vector** of P, and $\angle XOP = \theta$ is called the **vectorial angle** of P.

If r and θ are given, there is one and only one point which will have the coordinates (r, θ). If (x, y) be the cartesian coordinates of P, then the formulae of conversion are

$$x = r\cos\theta, y = r\sin\theta.$$

Also, we have $x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$.

§ 7. Angle between Radius vector and Tangent.

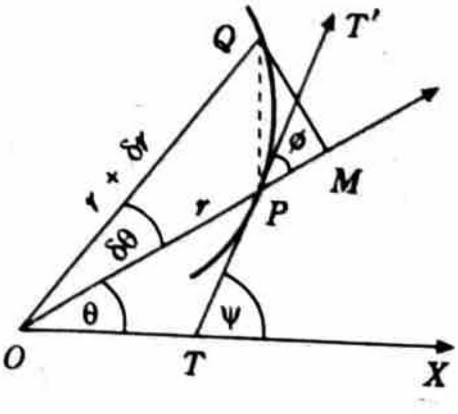
(Meerut 1990, 96 BP; Lucknow 81, 77; Kanpur 74, 72; Delhi 76, 74; Alld. 79)

Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points on the curve $r = f(\theta)$. The line TPT' is tangent to this curve at P. Also let ϕ be the angle between the tangent at P and the radius vector OP. We have to find ϕ.

Draw QM perpendicular to OP. As $Q \to P$, we have $\delta r \to 0, \delta\theta \to 0$, the chord PQ tends to the tangent at P and the

$$\angle QPM \to \phi.$$

Now $\tan\phi = \lim_{\delta\theta \to 0} \tan\angle QPM$



$$= \lim_{\delta\theta \to 0} \frac{QM}{PM} = \lim_{\delta\theta \to 0} \frac{QM}{OM - OP}$$

$$= \lim_{\delta\theta \to 0} \frac{(r + \delta r) \sin \delta\theta}{(r + \delta r) \cos \delta\theta - r}$$

$$= \lim_{\delta\theta \to 0} \frac{(r + \delta r) \{(\delta\theta) - (\delta\theta)^3/3\,! + \ldots\}}{(r + \delta r) \{1 - (\delta\theta)^2/2\,! + \ldots\} - r}$$

$$= \lim_{\delta\theta \to 0} \frac{r\delta\theta}{\delta r},$$ neglecting small quantities of the second and higher order

$$= r\,(d\theta/dr).$$

Hence $$\mathbf{\tan \phi = r \frac{d\theta}{dr} \text{ or } \cot \phi = \frac{1}{r} \frac{dr}{d\theta}}.$$

Note 1. From the figure, we have an important relation

$$\psi = \theta + \phi.$$

Note 2. Important. If the equation of a curve is of the form $r = f(\theta)$, then differentiating w.r.t. θ after taking logarithm of both sides, we at once get $\cot \phi$.

§ 8. Angle of intersection of two polar curves.

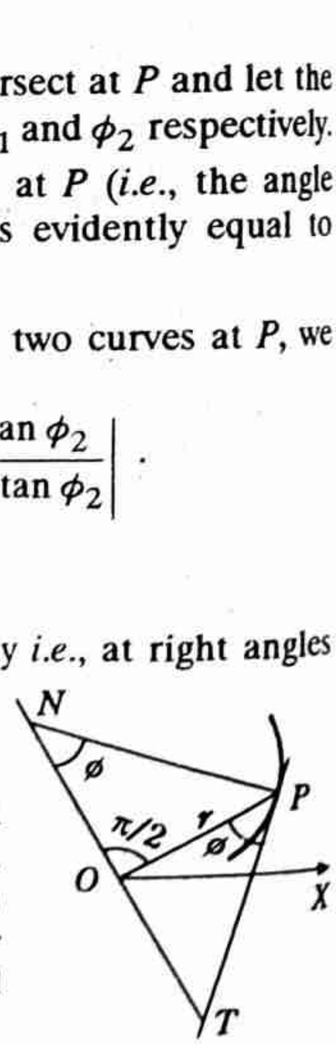
Let the two curves $r = f(\theta)$ and $r = F(\theta)$ intersect at P and let the values of ϕ at the point P for the two curves be ϕ_1 and ϕ_2 respectively. Then the angle of intersection of the two curves at P (*i.e.*, the angle between the tangents to the two curves at P) is evidently equal to $|\phi_1 - \phi_2|$ or $\phi_1 \sim \phi_2$.

If α is the acute angle of intersection of the two curves at P, we have

$$\tan \alpha = |\tan(\phi_1 - \phi_2)| = \left| \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right|.$$

$$\therefore \quad \alpha = \tan^{-1} \left| \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right|.$$

In particular, the two curves cut orthogonally *i.e.*, at right angles if $\tan \phi_1 \tan \phi_2 = -1$ or if $\cot \phi_1 \cot \phi_2 = -1$.



§ 9. Polar Subtangent and Polar Subnormal.

Let the tangent and normal at any point $P(r, \theta)$ on the curve meet the straight line through the pole perpendicular to the radius vector OP in T and N respectively. Then OT is called the **polar subtangent** and ON is called the **polar subnormal at** P.

Thus polar subtangent

$$= OT = OP \tan \phi$$

$$= r \tan \phi = r \left(r \frac{d\theta}{dr} \right) = \mathbf{r^2 \frac{d\theta}{dr}},$$

(Agra 1978)

and **polar subnormal**

$$= ON = OP \cot \phi = r \cot \phi = r \frac{1}{r} \frac{dr}{d\theta} = \mathbf{\frac{dr}{d\theta}}.$$

Also **length of polar tangent**

$$= PT = OP \sec \phi = OP \sqrt{(1 + \tan^2 \phi)}$$

$$= \mathbf{r \sqrt{\left[1 + r^2 \left(\frac{d\theta}{dr} \right)^2 \right]}},$$

and **length of polar normal**

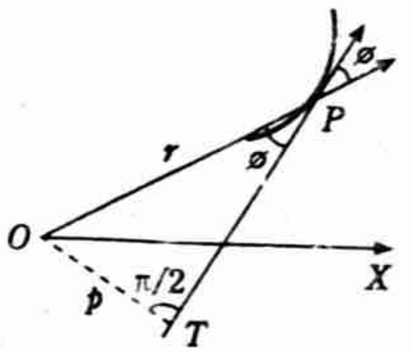
$$= PN = OP \operatorname{cosec} \phi = OP \sqrt{(1 + \cot^2 \phi)}$$

$$= r \sqrt{\left[1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right]} = \mathbf{\sqrt{\left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right]}}.$$

§ 10. Length of the perpendicular from pole to tangent.

Let p be the length of perpendicular OT drawn from the pole O to tangent at any point $P(r, \theta)$ on the curve

$$r = f(\theta).$$



We have $\angle OPT = \phi$. From the right angled triangle OPT, we have

$$\frac{p}{r} = \sin \phi$$

or $$\mathbf{p = r \sin \phi} \qquad \ldots(1)$$

From (1), we have

$$1/p^2 = (1/r^2) \operatorname{cosec}^2 \phi$$

$$= (1/r^2)(1 + \cot^2 \phi)$$

$$= \frac{1}{r^2} \left[1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right] = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2.$$

Thus $$\mathbf{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2}.$$

(Kanpur 1985)

Let $u = 1/r$. Then $\dfrac{du}{d\theta} = -\dfrac{1}{r^2} \dfrac{dr}{d\theta}$.

$$\therefore \qquad \mathbf{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta} \right)^2}.$$

§ 11. Pedal equation.

The relation between p and r for a given curve is called its pedal equation where r is the radius vector of any point on the curve and p is the length of perpendicular from the pole to the tangent at that point.

The pedal equation of a curve is usually written as $p = f(r)$ or $r = f(p)$ or $f(p, r) = 0$.

Case I. To form the pedal equation of a curve whose cartesian equation is given. Let

$$f(x, y) = 0, \qquad ...(1)$$

be the cartesian equation of the given curve.

The equation of the tangent at any point $P(x, y)$ on the curve (1) is

$$Y - y = (dy/dx)(X - x)$$

or $$Y - (dy/dx) \cdot X + \{x \cdot (dy/dx) - y\} = 0.$$

$$\therefore \quad p = \left[x \frac{dy}{dx} - y\right] \Big/ \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \qquad ...(2)$$

Also $$r^2 = OP^2 = x^2 + y^2. \qquad ...(3)$$

Eliminating x and y from the equations (1), (2) and (3), we obtain the required pedal equation of the curve.

Case II. To form the pedal equation of a curve whose polar equation is given.

Let $$f(r, \theta) = 0, \qquad ...(1)$$

be the given polar equation of a curve.

We have $$p = r \sin \phi, \qquad ...(2)$$

and $$\cot \phi = \frac{1}{r} \frac{dr}{d\theta}. \qquad ...(3)$$

Eliminating θ and ϕ between (1), (2) and (3), we get the required pedal equation of the curve.

Important. Sometimes we do not get the value of ϕ from equation (3) in a convenient form. Then instead of using the relations (2) and (3), we can use the single relation

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2. \qquad ...(4)$$

Now eliminating θ between (1) and (4), we get the required pedal equation.

Solved Examples

Ex. 37. *Find the angle at which the radius vector cuts the curve*

$$r = a(1 - \cos \theta). \qquad \textbf{(Meerut 1983)}$$

Sol. The given curve is

$$r = a(1 - \cos \theta). \qquad ...(1)$$

Taking logarithm of both sides of (1), we get

$$\log r = \log a + \log(1 - \cos \theta).$$

Now differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-(-\sin\theta)}{1-\cos\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

$\therefore \quad \cot\phi = \cot\frac{1}{2}\theta$ or $\phi = \theta/2$.

****Ex. 38.** *Show that in the equiangluar spiral $r = ae^{\theta\cot\alpha}$, the tangent is inclined at a constant angle α to the radius vector.*

(Rohilkhand 1979; Meerut 79, 84; Gorakhpur 75)

Sol. The curve is

$$r = ae^{\theta\cot\alpha}. \quad \text{...(1)}$$

$\therefore \quad \dfrac{dr}{d\theta} = ae^{\theta\cot\alpha}\,.\cot\alpha = r\cot\alpha.$ $\quad [\because$ from (1), $r = ae^{\theta\cot\alpha}]$

Now $\quad \tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{dr/d\theta} = \dfrac{r}{r\cot\alpha} = \tan\alpha.$

$\therefore \quad \phi = \alpha =$ constant.

Ex. 39 (a). *Find the angle ϕ for the curve*

$$a\theta = (r^2-a^2)^{1/2} - a\cos^{-1}(a/r).$$

(Agra 1983; Kanpur 80, 89; Meerut 71, 77, 85)

Sol. The given curve is

$$a\theta = (r^2-a^2)^{1/2} - a\cos^{-1}(a/r). \quad \text{...(1)}$$

Differentiating w.r.t. r, we get

$$a\frac{d\theta}{dr} = \tfrac{1}{2}(r^2-a^2)^{1/2}.\,2r + \frac{a}{\sqrt{\{1-(a/r)^2\}}}\cdot\left(-\frac{a}{r^2}\right)$$

$$= \frac{r}{\sqrt{(r^2-a^2)}} - \frac{a^2}{r\sqrt{(r^2-a^2)}} = \frac{r^2-a^2}{r\sqrt{(r^2-a^2)}} = \frac{\sqrt{(r^2-a^2)}}{r}.$$

$\therefore \quad r\dfrac{d\theta}{dr} = \dfrac{\sqrt{(r^2-a^2)}}{a}$ or $\tan\phi = \dfrac{\sqrt{(r^2-a^2)}}{a}$.

Now $\quad \cos\phi = (\sec\phi)^{-1} = \{1+\tan^2\phi\}^{-1/2}$

$$= \left\{1 + \frac{r^2-a^2}{a^2}\right\}^{-1/2} = \left(\frac{r^2}{a^2}\right)^{-1/2} = a/r.$$

$\therefore \quad \phi = \cos^{-1}(a/r)$.

Note. If not required otherwise, we can also write

$$\phi = \tan^{-1}\{\sqrt{(r^2-a^2)}/a\}.$$

Ex. 39 (b). *Find the angle at which the radius vector cuts the curve $l/r = 1 + e\cos\theta$.*

(Meerut 1977, 84, 85)

Sol. Taking logarithm of both sides of the equation of the curve, we get

$$\log l - \log r = \log(1 + e\cos\theta).$$

Differentiating w.r.t. θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{(1+e\cos\theta)}(-e\sin\theta).$$

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+\cos\theta}, \quad \text{or} \quad \tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

or $\qquad \phi = \tan^{-1}[(1+e\cos\theta)/e\sin\theta]$.

Ex. 40. *If ϕ be the angle between tangent to a curve and the radius vector drawn from the origin of coordinates to the point of contact, prove that*

$$\tan\phi = \frac{x\,(dy/dx) - y}{x + y\,(dy/dx)}.$$

(Meerut 1982, 88)

Sol. We have $\psi = \theta + \phi$. $\therefore \ \phi = \psi - \theta$.

$$\therefore \quad \tan\phi = \tan(\psi - \theta) = \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$$

$$= \frac{(dy/dx) - (y/x)}{1 + (dy/dx)(y/x)} = \frac{x\,(dy/dx) - y}{1 + y\,(dy/dx)}.$$

Ex. 41. *Prove that the normal at any point (r, θ) to the curve $r^n = a^n \cos n\theta$ makes an angle $(n+1)\,\theta$ with the initial line.*

(Kanpur 1987)

Sol. The given curve is

$$r^n = a^n \cos n\theta. \qquad \text{...(1)}$$

Taking logarithm of both sides of (1), we get

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating with respect to θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}\cdot(-n\sin n\theta) = -n\tan n\theta$$

or $\qquad \cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta = \cot\left(\tfrac{1}{2}\pi + n\theta\right).$

$\therefore \quad \phi = \tfrac{1}{2}\pi + n\theta.$

If ψ is the angle which the tangent at any point (r, θ) to the curve (1) makes with the initial line, then

$$\psi = \theta + \phi = \theta + \tfrac{1}{2}\pi + n\theta = \tfrac{1}{2}\pi + (n+1)\theta.$$

The slope of the tangent at (r, θ)

$$= \tan\psi = \tan\left[\tfrac{1}{2}\pi + (n+1)\,\theta\right] = -\cot(n+1)\,\theta.$$

$\therefore$ the solpe of the normal to (1) at the point (r, θ)

$$= -\frac{1}{-\cot(n+1)\,\theta} = \tan(n+1)\,\theta.$$

Hence the normal to (1) at the point (r, θ) makes an angle $(n+1)\theta$ with the axis of x *i.e.*, with the initial line.

Ex. 42 (a). *Find the polar subtangent and polar subnormal for the curve $r = a\theta$.* **(Lucknow 1980; Kanpur 78)**

Sol. The equation of the curve is $r = a\theta$. Therefore

$$dr/d\theta = a. \quad \text{Also} \quad d\theta/dr = 1/a.$$

Now polar subtangent $= r^2 (d\theta/dr) = a^2\theta^2/a = a\theta^2 = r^2/a$,
and polar subnormal $= dr/d\theta = a =$ constant.

Ex. 42 (b). *Find the length of the polar tangent and polar normal for the curve $r = a(1 + \cos\theta)$.*

Sol. The equation of the curve is

$$r = a(1 + \cos\theta). \qquad ...(1)$$

Taking logarithm of both sides of (1), we get

$$\log r = \log a + \log(1 + \cos\theta).$$

Now differentiating w.r.t. θ, we get

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1 + \cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= -\tan\tfrac{1}{2}\theta = \cot(\pi/2 + \theta/2).$$

$$\therefore \quad \phi = \pi/2 + \theta/2.$$

Now the length of polar tangent

$$= r\sec\phi = r\sec(\pi/2 + \theta/2)$$

$$= r\operatorname{cosec}\tfrac{1}{2}\theta = a(1 + \cos\theta)\operatorname{cosec}\tfrac{1}{2}\theta$$

$$= 2a\cos^2\tfrac{1}{2}\theta\operatorname{cosec}\tfrac{1}{2}\theta.$$

Again the length of polar normal

$$= r\operatorname{cosec}\phi = r\operatorname{cosec}(\pi/2 + \theta/2)$$

$$= r\sec\theta/2 = a(1 + \cos\theta)\sec\tfrac{1}{2}\theta$$

$$= 2a\cos^2\tfrac{1}{2}\theta\sec\tfrac{1}{2}\theta = 2a\cos\tfrac{1}{2}\theta.$$

Ex. 43. *Find the polar subtangent for the following curves :*

(*a*) $l/r = 1 + e\cos\theta$ **(Meerut 1998)**

(*b*) $r = ae^{\theta\cot\alpha}$

(*c*) $r = a(1 + \cos\theta)$ **(Agra 1976)**

(*d*) $r = a(1 - \cos\theta)$ **(Meeurt 1991; Agra 84; Rohilkhand 91)**

(*e*) $2a/r = 1 - \cos\theta$. **(Agra 1978)**

Sol. **(a)** Differentiating w.r.t. θ, we have

$$(-l/r^2)(dr/d\theta) = -e\sin\theta.$$

$$\therefore \quad \frac{1}{r^2}\frac{dr}{d\theta} = \frac{e\sin\theta}{l}.$$

Now polar subtangent $= r^2(d\theta/dr) = l/(e\sin\theta)$.

(b) We have $dr/d\theta = ae^{\theta\cot\alpha}.\cot\alpha = r\cot\alpha$.

$\therefore$ polar subtangent

$$= r^2\frac{d\theta}{dr} = r^2 \cdot \frac{1}{dr/d\theta} = r^2 \cdot \frac{1}{r\cot\alpha} = r\tan\alpha.$$

(c) We have $dr/d\theta = -a\sin\theta$.

$\therefore$ polar subtangent

$$= r^2 \frac{d\theta}{dr} = a^2 (1 + \cos\theta)^2 \cdot \left(-\frac{1}{a \sin\theta}\right) = -a \frac{(2\cos^2 \frac{1}{2}\theta)^2}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}$$

$$= -2a \cos^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta, \quad \text{or} \quad = 2a \cos^2 \tfrac{1}{2}\theta \cot \tfrac{1}{2}\theta$$

on neglecting the negative sign because the polar subtangent is a length.

(d) We have $dr/d\theta = a \sin\theta$.

$\therefore$ polar subtangent

$$= r^2 \frac{d\theta}{dr} = a^2 (1 - \cos\theta)^2 \cdot \frac{1}{a \sin\theta} = \frac{a^2 (2 \sin^2 \frac{1}{2}\theta)^2}{2a \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}$$

$$= 2a \sin^2 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta.$$

(e) Differentiating w.r.t. θ, we get

$$(-2a/r^2)(dr/d\theta) = \sin\theta.$$

$\therefore$ $\dfrac{1}{r^2}\dfrac{dr}{d\theta} = -\dfrac{1}{2a}\sin\theta$. Now polar subtangent

$$= r^2 (d\theta/dr) = -2a/\sin\theta = -2a \operatorname{cosec}\theta.$$

Since the polar subtangent is a length, therefore neglecting the – ive sign, we get polar subtangent $= 2a \operatorname{cosec}\theta$.

Ex. 44. *Find the angle of intersection of the curves*

(*a*) $r = a(1 + \cos\theta), r = b(1 - \cos\theta)$.

(Rohilkhand 1989; Delhi 78; Meerut 88S; Kanpur 78)

(*b*) $r = a \cos\theta$, $2r = a$.

Sol. **(a)** The given curves are $r = a(1 + \cos\theta)$ ...(1)

and $r = b(1 - \cos\theta)$. ...(2)

Differentiating (1) logarithmically, we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1 + \cos\theta} = -\frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \cos^2 \frac{1}{2}\theta}.$$

$\therefore$ $\cot\phi_1 = (1/r)(dr/d\theta) = -\tan \frac{1}{2}\theta = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$;

so that $\phi_1 = \frac{1}{2}\pi + \frac{1}{2}\theta$.

Again differentiating (2) logarithmically, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}.$$

$\therefore$ $\cot\phi_2 = \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \sin^2 \frac{1}{2}\theta} = \cot \frac{1}{2}\theta$; so that $\phi_2 = \frac{1}{2}\theta$.

Now the angle of intersection of (1) and (2) $= |\phi_1 - \phi_2|$

$$= \tfrac{1}{2}\pi + \tfrac{1}{2}\theta - \tfrac{1}{2}\theta = \tfrac{1}{2}\pi.$$

(b) For the first curve, $dr/d\theta = -a \sin\theta$.

$\therefore$ $\tan\phi_1 = r\dfrac{d\theta}{dr} = a\cos\theta \cdot \dfrac{1}{(-a\sin\theta)} = -\cot\theta = \tan(\frac{1}{2}\pi + \theta)$;

so that $\phi_1 = \frac{1}{2}\pi + \theta$.

For the second curve, $dr/d\theta = 0$.

$\therefore \quad \tan\phi_2 = r\dfrac{d\theta}{dr} = \dfrac{r}{dr/d\theta} = \dfrac{a/2}{0} = \infty$; so that $\phi_2 = \frac{1}{2}\pi$.

Now the angle of intersection of the two curves

$$= \phi_1 \sim \phi_2 = (\tfrac{1}{2}\pi + \theta) - \tfrac{1}{2}\pi = \theta,$$

where θ is to be found at the point of intersection of the two curves.

Solving the equations of the two curves for θ, we get

$$\cos\theta = \tfrac{1}{2} \quad \text{or} \quad \theta = \tfrac{1}{3}\pi.$$

Thus at the point of intersection, we have $\theta = \frac{1}{3}\pi$. Hence at the point of intersection $\theta = \frac{1}{3}\pi$, the required angle of intersection $= \pi/3$.

Ex. 45 (a). *Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.* **(Garhwal 1983; Rohilkhand 86; Kanpur 85)**

Sol. The given curves are

$$r^n = a^n \cos n\theta \quad ...(1)$$

and

$$r^n = b^n \sin n\theta. \quad ...(2)$$

From the equation of the curve (1), on taking logarithm, we get

$$n \log r = n \log a + \log \cos n\theta.$$

Now differentiating both sides w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta} = -n \tan n\theta.$$

$\therefore \quad (1/r)(dr/d\theta) = -\tan n\theta \quad$ or $\quad \cot\phi_1 = -\tan n\theta$.

Again, from equation (2), we get $n \log r = n \log b + \log \sin n\theta$.

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} = n \cot n\theta.$$

$\therefore \quad (1/r)(dr/d\theta) = \cot n\theta \quad$ or $\quad \cot\phi_2 = \cot n\theta$.

Now at the point of intersection (r, θ) of (1) and (2), we have $\cot\phi_1 . \cot\phi_2 = (-\tan n\theta) . (\cot n\theta) = -1$. Hence the curves (1) and (2) intersect orthogonally.

Ex. 45 (b). *Show that the curves $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$ cut orthogonally.* **(Delhi 1976; Meerut 97)**

Sol. The given curves are

$$r = a(1 + \sin\theta) \quad ...(1)$$

and

$$r = a(1 - \sin\theta). \quad ...(2)$$

From (1), on taking logarithm, we get

$$\log r = \log a + \log(1 - \sin\theta).$$

Now differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=0+\frac{\cos\theta}{1+\sin\theta}=\frac{\cos\theta}{1+\sin\theta}.$$

$$\therefore \quad \cot\phi_1=\frac{\cos\theta}{1+\sin\theta}.$$

Again, from equation (2), we get

$$\log r=\log a+\log(1-\sin\theta).$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{-\cos\theta}{1-\sin\theta} \text{ or } \cot\phi_2=\frac{-\cos\theta}{1-\sin\theta}.$$

Now $$\cot\phi_1 \cdot \cot\phi_2=\left(\frac{\cos\theta}{1+\sin\theta}\right)\cdot\left(\frac{-\cos\theta}{1-\sin\theta}\right)$$

$$=\left(\frac{-\cos^2\theta}{1-\sin^2\theta}\right)=\left(\frac{-\cos^2\theta}{\cos^2\theta}\right)=-1.$$

Hence the curves (1) and (2) intersect orthogonally.

Ex. 46. *Find the angle of intersection of the parabolas*

$$r=a/(1+\cos\theta) \quad \textit{and} \quad r=b/(1-\cos\theta).$$

(Kanpur 1978; Rohilkhand 76; Meerut 87, 96P)

Sol. The given curves are

$$r=a/(1+\cos\theta) \qquad ...(1)$$

and $$r=b/(1-\cos\theta). \qquad ...(2)$$

From (1), on taking logarithm, we get

$$\log r=\log a-\log(1+\cos\theta).$$

Differentiating both sides w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=-\frac{(-\sin\theta)}{1+\cos\theta}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}=\tan\tfrac{1}{2}\theta.$$

$$\therefore \quad \cot\phi_1=\tan\tfrac{1}{2}\theta=\cot\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\theta\right);$$

or $$\phi_1=\pi/2-\theta/2.$$

Again taking logarithm of both sides of (2), we get

$$\log r=\log b-\log(1-\cos\theta).$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta}=-\frac{(-\sin\theta)}{1-\cos\theta}=\frac{-\sin\theta}{1-\cos\theta}$$

$$=\frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=-\cot\tfrac{1}{2}\theta.$$

$$\therefore \quad \cot\phi_2=-\cot\tfrac{1}{2}\theta=\cot\left(\pi-\tfrac{1}{2}\theta\right);$$

or $$\phi_2=\pi-\tfrac{1}{2}\theta.$$

Now the angle of intersection of (1) and (2)

$$=\phi_1\sim\phi_2=\left(\pi-\tfrac{1}{2}\theta\right)-\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\theta\right)=\pi/2.$$

Hence (1) and (2) intersect orthogonally.

Note. We can also prove the result by showing that

$$\cot \phi_1 \cdot \cot \phi_2 = -1.$$

Ex. 47. *Find the angle between the tangent and the radius vector in the case of the curve $r^n = a^n \sec(n\theta + \alpha)$, and prove that this curve is intersected by the curve $r^n = b^n \sec(n\theta + \beta)$ at an angle which is independent of a and b.* **(Rohilkhand 1987; Kanpur 79; Meerut 86)**

Sol. The given curves are

$$r^n = a^n \sec(n\theta + \alpha) \quad ...(1)$$

and $$r^n = b^n \sec(n\theta + \beta) \quad ...(2)$$

Taking logarithm of both sides of (1), we get

$$n \log r = n \log a + \log \sec(n\theta + \alpha).$$

Differentiating with respect to θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sec(n\theta + \alpha)} \cdot n \sec(n\theta + \alpha) \tan(n\theta + \alpha)$$

or $$\frac{1}{r}\frac{dr}{d\theta} = \tan(n\theta + \alpha)$$

or $$\cot \phi = \cot\left[\tfrac{1}{2}\pi - (n\theta + \alpha)\right]$$

or $$\phi = \tfrac{1}{2}\pi - (n\theta + \alpha),$$

which gives the angle between the tangent and the radius vector in the case of the curve (1).

Let $$\phi_1 = \tfrac{1}{2}\pi - (n\theta + \alpha).$$

If ϕ_2 denotes the angle between the tangent and the radius vector in the case of the curve (2), then proceeding as above, we have

$$\phi_2 = \tfrac{1}{2}\pi - (n\theta + \beta).$$

$\therefore$ the angle of intersection of (1) and (2) $= \phi_1 - \phi_2 = \beta - \alpha$ which is independent of a and b.

***Ex. 48.** *Find the angle of intersection between the pair of curves*

$$r = 6 \cos\theta \quad \text{and} \quad r = 2(1 + \cos\theta).$$ **(Meerut 1972)**

Sol. The given curves are

$$r = 6 \cos\theta \quad ...(1)$$

and $$r = 2(1 + \cos\theta) \quad ...(2)$$

From (1), on taking logarithm, we get

$$\log r = \log 6 + \log \cos\theta.$$

$$\therefore \quad \cot \phi_1 = \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$= \cot\left(\tfrac{1}{2}\pi + \theta\right). \text{ Thus } \phi_1 = \tfrac{1}{2}\pi + \theta.$$

Again from (2), on taking logarithm, we get

$$\log r = \log 2 + \log(1 + \cos\theta).$$

$$\therefore \quad \cot \phi_2 = \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$= -\tan\tfrac{1}{2}\theta = \cot(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta).$$

Thus $\quad \phi_2 = \frac{1}{2}\pi + \frac{1}{2}\theta.$

Now the angle of intersection of (1) and (2)

$$= \phi_1 \sim \phi_2 = (\tfrac{1}{2}\pi + \theta) - (\tfrac{1}{2}\pi + \tfrac{1}{2}\theta) = \tfrac{1}{2}\theta,$$

where θ is the vectorial angle of the point of intersection of (1) and (2).

Now to get θ for the point of intersection of (1) and (2), we have on eliminating r between (1) and (2),

$$6\cos\theta = 2(1+\cos\theta) \quad \text{or} \quad 1+\cos\theta = 3\cos\theta$$

or $\quad 2\cos\theta = 1 \quad$ or $\quad \cos\theta = \frac{1}{2}$ or $\theta = \frac{1}{3}\pi.$

$\therefore$ the required angle of intersection $= \frac{1}{2}(\frac{1}{3}\pi) = \pi/6.$

Ex. 49. *Find the angle of intersection of the curves*

$$r^2 = 16\sin 2\theta \quad \text{and} \quad r^2 \sin 2\theta = 4.$$ **(Meerut 1990)**

Sol. The given curves are

$$r^2 = 16\sin 2\theta, \qquad \text{...(1)}$$

and $$r^2 \sin 2\theta = 4. \qquad \text{...(2)}$$

From (1), $\; 2\log r = \log 16 + \log \sin 2\theta.$

Therefore $\; (2/r)(dr/d\theta) = 2\cos 2\theta/\sin 2\theta$

or $\quad \cot\phi_1 = (1/r)(dr/d\theta) = \cot 2\theta.$ Thus $\phi_1 = 2\theta.$

From (2), $\; 2\log r + \log\sin 2\theta = \log 4.$

Therefore $\; \dfrac{2}{r}\dfrac{dr}{d\theta} + \dfrac{2\cos 2\theta}{\sin 2\theta} = 0$

or $$\cot\phi_2 = \frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta = \cot(\pi - 2\theta).$$

Thus $\quad \phi_2 = \pi - 2\theta.$

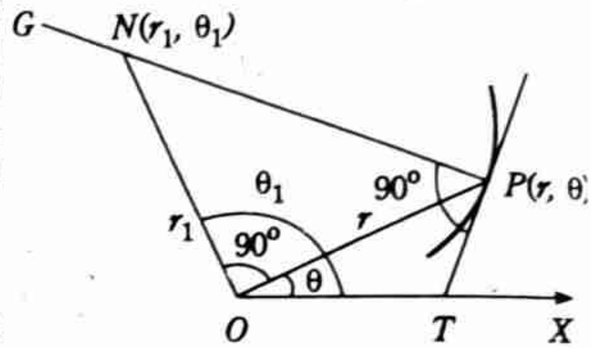
Now the angle of intersection of (1) and (2) $= \phi_1 \sim \phi_2 = (\pi - 2\theta) - 2\theta = \pi - 4\theta$, where θ is to be found at the point where (1) and (2) intersect.

Eliminating r between (1) and (2), we get $\sin^2 2\theta = \frac{1}{4}\cdot$ Therefore $\sin 2\theta = \pm\frac{1}{2}\cdot$ But $\sin 2\theta = -\frac{1}{2}$ is inadmissible because it gives imaginary values of r from (1) and (2). Now $\sin 2\theta = \frac{1}{2}$ gives $2\theta = \frac{1}{6}\pi$ or $\theta = \pi/12.$

Hence the angle of intersection of (1) and (2) at the point $\theta = \pi/12$ is $\pi - 4(\pi/12)$ *i.e.*, $2\pi/3.$

Ex. 50. *Prove that the locus of the extremity of the polar subnormal of the curve* $r = f(\theta)$ *is* $r = f'(\theta - \pi/2)$. *Hence show that the locus of* *the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral.* **(Kanpur 1988; Meerut 88)**

Sol. First part. Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ and PT and PG be the tangent and normal respectively at P. Let the straight line perpendicular to OP through O meet PG in $N(r_1, \theta_1)$.



Now, $r_1 = ON =$ polar subnormal of the curve $r = f(\theta)$ at the point P

$$= dr/d\theta = f'(\theta). \qquad ...(1)$$

Also $$\theta_1 = \theta + \pi/2; \quad \therefore \quad \theta = \theta_1 - \pi/2. \qquad ...(2)$$

Eliminating θ between (1) and (2), we get

$$r_1 = f'(\theta_1 - \pi/2).$$

$\therefore$ the locus of the point (r_1, θ_1) is

$$r = f'(\theta - \pi/2). \qquad ...(3)$$

Second part. Comparing the equation of the curve $r = ae^{m\theta}$ with the equation $r = f(\theta)$, we have $f(\theta) = ae^{m\theta}$.

$\therefore \quad f'(\theta) = ame^{m\theta}$.

$\therefore \quad f'(\theta - \pi/2) = ame^{m(\theta - \pi/2)} = ame^{m\theta} \cdot e^{-m\pi/2}$
$= \lambda e^{m\theta}$, where $\lambda = ame^{-m\pi/2}$.

From (3), the required locus is $r = \lambda e^{m\theta}$, which is also an equilangular spiral.

Ex. 51. *Prove that for the parabola $2a/r = (1 - \cos\theta)$,*

(a) $\phi = \pi - \frac{1}{2}\theta$, **(b)** $p = a \operatorname{cosec} \frac{1}{2}\theta$, **(Meerut1991S)**

(c) $p^2 = ar$. **(Kanpur 1979; Agra 86, Jhansi 88)**

Sol. The given curve is $2a/r = 1 - \cos\theta$. ...(1)

Taking logarithm of both sides of (1), we get

$$\log 2a - \log r = \log(1 - \cos\theta).$$

Differentiating w.r.t. θ, we get

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-(-\sin\theta)}{1 - \cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = -\cot\tfrac{1}{2}\theta = \cot(\pi - \tfrac{1}{2}\theta).$$

Hence $\phi = \pi - \frac{1}{2}\theta$. **[The result (a) proved]**

Now $p = r\sin\phi = r\sin(\pi - \frac{1}{2}\theta) = r\sin\frac{1}{2}\theta$.

From (1), we have

$$r = 2a/(1 - \cos\theta) = 2a/(2\sin^2 \tfrac{1}{2}\theta) = a/(\sin^2 \tfrac{1}{2}\theta).$$

$$\therefore \quad p = (a/\sin^2 \tfrac{1}{2}\theta) \,.\, \sin \tfrac{1}{2}\theta = a/(\sin \tfrac{1}{2}\theta) = a \operatorname{cosec} \tfrac{1}{2}\theta.$$

This proves the result (b).

Now $p = r \sin \frac{1}{2}\theta$ and $r = a/(\sin^2 \frac{1}{2}\theta)$. Eliminating θ between these we get

$$r = \frac{a}{p^2/r^2} = \frac{ar^2}{p^2}; \quad \therefore \quad p^2 = \frac{ar^2}{r} = ar.$$

This proves the result (c).

**Ex. 52. *For the cardioid $r = a(1 - \cos\theta)$, prove that*

(a) $\phi = \frac{1}{2}\theta$. **(Meerut 1981, 83, 84, 86, 96; Rohilkhand 91)**

(b) $2ap^2 = r^3$. **(Garhwal 1983; Meerut 1990 P, 91, 96; Agra 85)**

Sol. The given curve is

$$r = a(1 - \cos\theta). \qquad \text{...(1)}$$

$$\therefore \quad dr/d\theta = a \sin\theta.$$

(a) We have

$$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1 - \cos\theta)}{a \sin\theta} = \frac{2a \sin^2 \frac{1}{2}\theta}{2a \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} = \tan \theta/2.$$

$\therefore \quad \phi = \theta/2.$

(b) We have

$$p = r \sin\phi = r \sin\theta/2, \qquad [\because \phi = \theta/2]$$

Now from (1), $r = 2a \sin^2 \frac{1}{2}\theta = 2a\,(p^2/r^2)$, $\qquad (\because\ p = r \sin \frac{1}{2}\theta)$

$\therefore \quad 2ap^2 = r^3.$

Ex. 53. *Find the pedal equation of the parabola*

$y^2 = 4a(x + a)$. **(Kanpur 1989; Lucknow 1981; Agra 79)**

Sol. The given curve is

$$y^2 = 4a(x + a) \qquad \text{...(1)}$$

Differentiating (1), we get

$$2y\, dy/dx = 4a \quad \text{or} \quad dy/dx = 2a/y.$$

Therefore the equation of the tangent at (x, y) to (1) is

$$Y - y = (2a/y)(X - x)$$

or

$$(2a/y) X - Y + y - (2a/y) x = 0. \qquad \text{...(2)}$$

$\therefore p =$ the length of the perpendicular from $(0, 0)$ on (2)

$$= \left[y - \frac{2ax}{y}\right] \Big/ \left(1 + \frac{4a^2}{y^2}\right)^{1/2} = (y^2 - 2ax)/\sqrt{(y^2 + 4a^2)}$$

$$= \frac{4a(x + a) - 2ax}{\sqrt{[4a(x + a) + 4a^2]}}, \qquad [\because \text{from (1)}, y^2 = 4a(x + a)]$$

$$= \frac{2ax + 4a^2}{\sqrt{[4a(x + 2a)]}} = \frac{2a(x + 2a)}{\sqrt{[4a(x + 2a)]}} = \sqrt{[a(x + 2a)]}.$$

Also $r^2 = x^2 + y^2 = x^2 + 4a(x + a) = (x + 2a)^2$; $\therefore$ $r = (x + 2a)$.

Now $p^2 = a(x + 2a) = ar$. Hence $p^2 = ar$ is the required pedal equation of (1).

Ex. 54. *Show that the pedal equation of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *is* $1/p^2 = (1/a^2) + (1/b^2) - (r^2/a^2b^2)$.

(Jhansi 1989; Kanpur 1976; Gorakhpur 78, 73; Lucknow 79; Meerut 87)

Sol. The equation of the curve is $(x^2/a^2) + (y^2/b^2) = 1$. ...(1)

The co-ordinates (x, y) of any point P on (1) may be taken as $x = a\cos t$, $y = b\sin t$, where t is the parameter.

These give $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{b\cos t}{-a\sin t}$.

Hence the equation of the tangent to (1) at the point 't' is

$$y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$$

or $$bx\cos t + ay\sin t - ab(\sin^2 t + \cos^2 t) = 0$$

or $$ab - bx\cos t - ay\sin t = 0. \qquad ...(2)$$

$\therefore$ p = the length of perpendicular from (0, 0) to (2)

$$= ab/\sqrt{(a^2\sin^2 t + b^2\cos^2 t)}.$$

$\therefore$ $$1/p^2 = (a^2\sin^2 t + b^2\cos^2 t)/a^2b^2 \qquad ...(3)$$

Also $$r^2 = x^2 + y^2 = a^2\cos^2 t + b^2\sin^2 t$$
$$= a^2(1 - \sin^2 t) + b^2(1 - \cos^2 t)$$
$$= a^2 + b^2 - a^2\sin^2 t - b^2\cos^2 t. \qquad ...(4)$$

Eliminating t between (3) and (4), we obtain the pedal equation of (1). From (4), we get

$$a^2\sin^2 t + b^2\cos^2 t = (a^2 + b^2) - r^2.$$

Substituting this value in (3), we get

$$1/p^2 = \{(a^2 + b^2) - r^2\}/a^2b^2$$

or $$1/p^2 = (1/a^2) + (1/b^2) - (r^2/a^2b^2),$$

which is the required pedal equation of the ellipse (1).

Ex. 55. *Find the pedal equation of the ellipse* $l/r = 1 + e\cos\theta$.

(Rohilkhand 1982; Agra 82, 87; Kanpur 73; Meerut 98)

Sol. The given curve is

$$l/r = 1 + e\cos\theta. \qquad ...(1)$$

Differentiating (1) w.r.t. θ, we get

$$\frac{-l}{r^2}\frac{dr}{d\theta} = -e\sin\theta \quad \text{or} \quad \frac{1}{r^2}\frac{dr}{d\theta} = \frac{e\sin\theta}{l} \qquad ...(2)$$

Now $1/p^2 = (1/r^2) + (1/r^4)(dr/d\theta)^2$.

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{e^2\sin^2\theta}{l^2}, \qquad \text{[from (2)]}$$

$$= \frac{1}{r^2} + \frac{e^2}{l^2}(1 - \cos^2\theta) = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{e^2}{l^2}\cos^2\theta.$$

But from (1), $e \cos \theta = (l - r)/r$.

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{l^2}\left(\frac{l-r}{r}\right)^2$$

$$= \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{l^2 r^2}(l^2 - 2lr + r^2) = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{r^2} + \frac{2l}{l^2 r} - \frac{1}{l^2}.$$

Hence $$\frac{1}{p^2} = \frac{1}{l^2}\left(e^2 + \frac{2l}{r} - 1\right),$$

which is the required pedal equation.

Ex. 56. *Find the pedal equations of the curves*

(*i*) $r^n = a^n \sin n\theta$ **(Rohilkhand 1983, 90)**

(*ii*) $r^n = a^n \cos n\theta$. **(Meerut 1990 S, 97; Delhi 81; Agra 88)**

Sol. (i) The given curve is $r^n = a^n \sin n\theta$. ...(1)

Taking logarithm of both sides of (1), we get

$$n \log r = n \log a + \log \sin n\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} = n \cot n\theta.$$

$\therefore \quad \cot \phi = (1/r)\, dr/d\theta = \cot n\theta, \quad$ or $\quad \phi = n\theta.$

Now $\quad p = r \sin \phi = r \sin n\theta.$ ...(2)

Eliminating θ between (1) and (2), we get the pedal equation of (1). From (2), $\sin n\theta = p/r$. Substituting this value in (1), we get $r^n = a^n (p/r)$ or $r^{n+1} = pa^n$, which is the required pedal equation.

(ii) The given curve is $r^n = a^n \cos n\theta$. ...(1)

Taking logarithm, we get $n \log r = n \log a + \log \cos n\theta$.

Differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} = -n \tan n\theta.$$

$\therefore \quad \cot \phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta = \cot(\frac{1}{2}\pi + n\theta); \quad$ or $\phi = \frac{1}{2}\pi + n\theta.$

Now $\quad p = r \sin \phi = r \sin(\frac{1}{2}\pi + n\theta) = r \cos n\theta.$ ...(2)

From (2), we have $\cos n\theta = p/r$. Substituting this value in (1), we get $r^n = a^n (p/r)$ or $r^{n+1} = pa^n$, which is the required pedal equation.

Important Note. To find the pedal equation of a polar curve it is often convenient to find the angle ϕ and then to use the relation $p = r \sin \phi$ (Refer Ex. 56). But if the angle ϕ cannot be obtained in a convenient form, we use the relation $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. (Refer Ex. 55).

Ex. 57. *Find the pedal equation of the curve*

$$r = ae^{\theta \cot \alpha}.$$ **(Meerut 1973; Utkal 70)**

Sol. The given curve is $r = ae^{\theta \cot \alpha}$ ...(1)

Taking logarithm of both sides of (1), we get

$$\log r = \log a + \theta \cot \alpha \log e = \log a + \theta \cot \alpha. \quad [\because \log e = 1]$$

Differentiating w.r.t. θ, we get

$$(1/r)\, dr/d\theta = 0 + \cot \alpha = \cot \alpha.$$

$\therefore \quad \cot \phi = (1/r)\, dr/d\theta = \cot \alpha$; or $\phi = \alpha$.

Now $p = r \sin \phi = r \sin \alpha$. Hence $p = r \sin \alpha$ is the required pedal equation.

***Ex. 58.** *Find the pedal equation of the following curves :*

(*a*) $x^2 + y^2 = 2ax$.

(*b*) $x^{2/3} + y^{2/3} = a^{2/3}$. **(Meerut 1984)**

(*c*) $x^2 - y^2 = a^2$.

(*d*) $r^2 = a^2 \cos 2\theta$.

(*e*) $r^2 \cos 2\theta = a^2$. **(Delhi 1982; Rohilkhand 85)**

(*f*) $r = a \operatorname{sech} n\theta$.

(*g*) $r = a\,(1 + \cos \theta)$. **(Rohilkhand 1982)**

(*h*) $r^m \cos m\theta = a^m$. **(Lucknow 1977; Indore 70)**

Sol. **(a)** The given curve is $x^2 + y^2 = 2ax$. Changing the equation to polar co-ordinates by putting $x = r \cos \theta$ and $y = r \sin \theta$, we get

$$r^2 = 2ar \cos \theta \quad \text{or} \quad r = 2a \cos \theta. \qquad ...(1)$$

$$\therefore \quad dr/d\theta = -\, 2a \sin \theta.$$

Now $$\tan \phi = r\frac{d\theta}{dr} = \frac{2a \cos \theta}{-\, 2a \sin \theta} = -\cot \theta = \tan\left(\tfrac{1}{2}\pi + \theta\right);$$

so that $$\phi = \tfrac{1}{2}\pi + \theta.$$

We have $p = r \sin \phi = r \sin\left(\tfrac{1}{2}\pi + \theta\right) = r \cos \theta = r\,(r/2a)$, from (1).

Hence $2ap = r^2$ is the required pedal equation.

(b) The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The co-ordinates (x, y) of any point P on (1) may be taken as

$$x = a \cos^3 t, \quad y = a \sin^3 t,$$

where t is the parameter. These give

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a \sin^2 t \cos t}{-\, 3a \cos^2 t \sin t} = -\frac{\sin t}{\cos t}.$$

Hence the equation of the tangent to (1) at the point 't' is

$$y - a \sin^3 t = -\frac{\sin t}{\cos t}\,(x - a \cos^3 t)$$

or $$x \sin t + y \cos t = a \sin t \cos t\,(\cos^2 t + \sin^2 t)$$
$$= a \sin t \cos t. \qquad ...(2)$$

$\therefore \quad p =$ the length of the perpendicular from (0, 0) to (2)

$$= \frac{a \sin t \cos t}{\sqrt{(\sin^2 t + \cos^2 t)}} = a \sin t \cos t. \qquad ...(3)$$

Now $r^2 = x^2 + y^2 = a^2 \cos^6 t + a^2 \sin^6 t = a^2\,[(\cos^2 t)^3 + (\sin^2 t)^3]$

$$= a^2 [(\cos^2 t + \sin^2 t)^3 - 3 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)]$$

$$= a^2 [1 - 3 (p^2/a^2) . 1], \qquad [\because \text{ from (3)}, \cos^2 t \sin^2 t = p^2/a^2]$$

$$= a^2 - 3p^2.$$

Hence the required pedal equation is $r^2 = a^2 - 3p^2$.

(c) The given curve is $x^2 - y^2 = a^2$. Changing to polar co-ordinates, the equation becomes

$$r^2 \cos^2 \theta - r^2 \sin^2 \theta = a^2 \quad \text{or} \quad r^2 \cos 2\theta = a^2. \qquad ...(1)$$

Taking logarithm of (1), we get

$$2 \log r + \log \cos 2\theta = 2 \log a.$$

Differentiating w.r.t. θ, we have

$$\frac{2}{r} \frac{dr}{d\theta} - 2 \frac{\sin 2\theta}{\cos 2\theta} = 0 \quad \text{or} \quad \frac{1}{r} \frac{dr}{d\theta} = \tan 2\theta.$$

$$\therefore \quad \cot \phi = (1/r)(dr/d\theta) = \tan 2\theta = \cot (\tfrac{1}{2}\pi - 2\theta);$$

so that $$\phi = \tfrac{1}{2}\pi - 2\theta.$$

Now $p = r \sin \phi = r \sin (\frac{1}{2}\pi - 2\theta) = r \cos 2\theta$

$$= r (a^2/r^2) = a^2/r, \qquad [\because \text{ from (1)}, \cos 2\theta = a^2/r^2]$$

Hence the required pedal equation is $pr = a^2$.

(d) The given curve is $\quad r^2 = a^2 \cos 2\theta. \qquad ...(1)$

Taking logarithm of (1), we get

$$2 \log r = 2 \log a + \log \cos 2\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{2}{r} \frac{dr}{d\theta} = 0 - 2 \frac{\sin 2\theta}{\cos 2\theta} \quad \text{or} \quad \frac{1}{r} \frac{dr}{d\theta} = - \tan 2\theta.$$

$$\therefore \quad \cot \phi = \frac{1}{r} \frac{dr}{d\theta} = - \tan 2\theta = \cot (\tfrac{1}{2}\pi + 2\theta);$$

so that $$\phi = \tfrac{1}{2}\pi + 2\theta.$$

Now $\quad p = r \sin \phi = r \sin (\frac{1}{2}\pi + 2\theta) = r \cos 2\theta = r (r^2/a^2)$, from (1).

Hence the required pedal equation is $pa^2 = r^3$.

(e) See part (c) of this Example.

(f) The given curve is $\quad r = a \operatorname{sech} n\theta \qquad ...(1)$

Differentiating (1), $dr/d\theta = - an \operatorname{sech} n\theta \tanh n\theta$.

Now $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 = \frac{1}{r^2} + \frac{1}{r^4} a^2 n^2 \operatorname{sech}^2 n\theta \tanh^2 n\theta$$

$$= \frac{1}{r^2} + \frac{n^2}{r^4} (a^2 \operatorname{sech}^2 n\theta)(1 - \operatorname{sech}^2 n\theta)$$

$$[\because \operatorname{sech}^2 n\theta = 1 - \tanh^2 n\theta]$$

$$= \frac{1}{r^2} + \frac{n^2}{r^4} \cdot r^2 \left(1 - \frac{r^2}{a^2} \right), \qquad [\text{by (1)}]$$

$$= \frac{1}{r^2} + \frac{n^2}{r^2} - \frac{n^2}{a^2} = (n^2+1)\frac{1}{r^2} - \frac{n^2}{a^2}.$$

$\therefore$ the pedal equation is $\dfrac{1}{p^2} = \dfrac{(n^2+1)}{r^2} - \dfrac{n^2}{a^2}$ which is of the form

$$1/p^2 = (A/r^2) + B.$$

(g) The given curve is $r = a\,(1 + \cos\theta)$. ...(1)

Taking logarithm, we get $\log r = \log a + \log(1 + \cos\theta)$.

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta} = \frac{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta} = -\tan\tfrac{1}{2}\theta.$$

$\therefore \quad \cot\phi = (1/r)(dr/d\theta) = -\tan\frac{1}{2}\theta = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$;

so that $\quad \phi = \frac{1}{2}\pi + \frac{1}{2}\theta.$

Now $p = r\sin\phi = r\sin(\frac{1}{2}\pi + \frac{1}{2}\theta) = r\cos\frac{1}{2}\theta$. ...(2)

Eliminating θ between (1) and (2), we get the required pedal equation.

From (1), we have $r = 2a\cos^2\frac{1}{2}\theta = 2a\,(p/r)^2$,

[$\because$ from (2), $\cos\frac{1}{2}\theta = p/r$].

Hence the required pedal equation is $r^3 = 2ap^2$.

(h) The given curve is $\quad r^m \cos m\theta = a^m$ ...(1)

Taking log, we get $m \log r + \log\cos m\theta = m\log a$.

Differentiating w.r.t. θ, we have $\dfrac{m}{r}\dfrac{dr}{d\theta} - \dfrac{m\sin m\theta}{\cos m\theta} = 0.$

$\therefore \quad \cot\phi = (1/r)(dr/d\theta) = \tan m\theta = \cot(\frac{1}{2}\pi - m\theta)$;

so that $\quad \phi = \frac{1}{2}\pi - m\theta.$

Now $\quad p = r\sin\phi = r\sin(\frac{1}{2}\pi - m\theta) = r\cos m\theta = r\,(a^m/r^m)$,

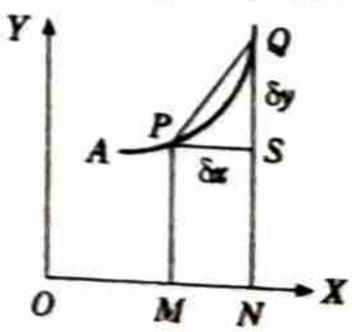
[by (1)]

Hence the required pedal equation is

$$p = a^m/r^{m-1} \quad \text{or} \quad pr^{m-1} = a^m.$$

§ 12. Differential Coefficient of arc length. Cartesian Co-ordinates.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighbouring points on the curve $y = f(x)$. Let the arc length $AP = s$, measured from a fixed point A on the curve. Then



(i) $\dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$,

(ii) $\dfrac{ds}{dy} = \sqrt{\left\{1 + \left(\dfrac{dx}{dy}\right)^2\right\}}$,

(iii) $ds/dt = \sqrt{\{(dx/dt)^2 + (dy/dt)^2\}}$,

(iv) $\sin\psi = dy/ds$, (v) $\cos\psi = dx/ds$,

(vi) $(dx/ds)^2 + (dy/ds)^2 = 1$.

Proof. Let $s + \delta s$ denote the length of the arc AQ, so that arc $PQ = \delta s$. Also let $PM = y$ and $QN = y + \delta y$ be the ordinates of P and Q. Then if PS be the perpendicular from P on QN, we have $PS = \delta x$ and $QS = \delta y$. Join PQ. We have

$$(\text{Chord } PQ)^2 = PS^2 + SQ^2 = (\delta x)^2 + (\delta y)^2 \qquad ...(1)$$

Dividing (1) throughout by $(\delta x)^2$, we get

$$\left(\frac{\text{chord } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

or
$$\left(\frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{\text{arc } PQ}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

or
$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2. \qquad [\because \text{arc } PQ = \delta s]$$

Taking limit of both sides as $Q \to P$, *i.e.*, $\delta x \to 0$, we get

$$\lim_{Q \to P} \left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \lim_{\delta x \to 0} \left(\frac{\delta s}{\delta x}\right)^2 = \lim_{\delta x \to 0} \left[1 + \left(\frac{\delta y}{\delta x}\right)^2\right]$$

or
$$(ds/dx)^2 = 1 + (dy/dx)^2,$$

$$\left[\because \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1, \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}, \text{etc.}\right]$$

Thus $\quad ds/dx = \pm\sqrt{\{1 + (dy/dx)^2\}}$,

where positive or negative sign is to be taken before the radical sign according as s increases as x increases or decreases. Hence if s increases as x increases, we have

$$\mathbf{ds/dx = \sqrt{\{1 + (dy/dx)^2\}}}. \qquad \textbf{(Lucknow 1977)}$$

Cor. 1. Dividing (1) throughout by $(\delta y)^2$ and proceeding to limits, we get

$$\mathbf{ds/dy = \pm\sqrt{\{1 + (dx/dy)^2\}}},$$

where + ive or − ive sign is to be taken before the radical sign according as s increases as y increases or decreases.

Cor. 2. If the equations of the curve be given in the parametric form $x = f(t)$, $y = \phi(t)$, then s is obviously a function of t. In this case dividing (1) throughout by $(\delta t)^2$ and proceeding to limits, we get

$$\mathbf{ds/dt = \pm\sqrt{\{(dx/dt)^2 + (dy/dt)^2\}}},$$

where + ive or − ive sign is to be taken before the radical sign according as s increases as t increases or decreases.

Cor. 3. We have

$$\cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{(1 + \tan^2\psi)}} = \frac{1}{\sqrt{\{1 + (dy/dx)^2\}}} = \frac{1}{ds/dx} = \frac{dx}{ds}.$$

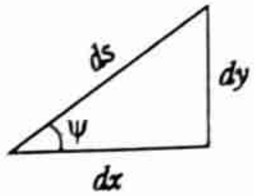
Again $\sin\psi$

$$= \frac{1}{\operatorname{cosec} \psi} = \frac{1}{\sqrt{(1 + \cot^2 \psi)}} = \frac{1}{\sqrt{\{1 + (dx/dy)^2\}}} = \frac{1}{ds/dy} = \frac{dy}{ds}.$$

Now $\cos^2 \psi + \sin^2 \psi = 1$.

Therefore $(dx/ds)^2 + (dy/ds)^2 = 1$.

Note. It is easy to remember all the above results with the help of the adjoining hypothetical figure.



§ 13. Differential coefficient of Arc length. Polar Co-ordinates.

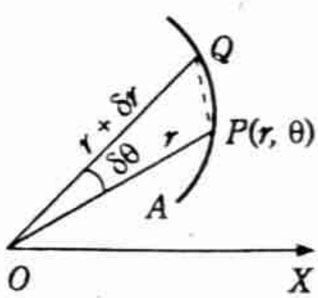
For the curve $r = f(\theta)$, we have

(i) $\dfrac{\mathbf{ds}}{\mathbf{d\theta}} = \sqrt{\left\{\mathbf{r}^2 + \left(\dfrac{\mathbf{dr}}{\mathbf{d\theta}}\right)^2\right\}}$, **(Meerut 1983; Allahabad 82; Kanpur 86)**

(ii) $\dfrac{\mathbf{ds}}{\mathbf{dr}} = \sqrt{\left\{1 + \mathbf{r}^2 \left(\dfrac{\mathbf{d\theta}}{\mathbf{dr}}\right)^2\right\}}$, (iii) $\cos \phi = \dfrac{\mathbf{dr}}{\mathbf{ds}}$,

(iv) $\sin \phi = \mathbf{r} \dfrac{\mathbf{d\theta}}{\mathbf{ds}}$, (v) $\left(\dfrac{\mathbf{dr}}{\mathbf{ds}}\right)^2 + \left(\mathbf{r} \dfrac{\mathbf{d\theta}}{\mathbf{ds}}\right)^2 = 1.$

Proof. Let s be the length of the arc AP of the curve $r = f(\theta)$. Here A is a fixed point on the curve and P is any point (r, θ). Take a point $Q (r + \delta r, \theta + \delta\theta)$ on the curve in the neighbourhood of P such that arc $AQ = s + \delta s$. Then arc $PQ = \delta s$. Also $\delta\theta \to 0$ and $\delta r \to 0$, as $Q \to P$. Join PQ.



From the ΔOPQ, we have

(chord $PQ)^2 = OP^2 + OQ^2 - 2OP \,.\, OQ \cos \angle QOP$

$$= r^2 + (r + \delta r)^2 - 2r (r + \delta r) \cos \delta\theta$$
$$= (\delta r)^2 + 2r\delta r (1 - \cos \delta\theta) + 2r^2 (1 - \cos \delta\theta)$$
$$= (\delta r)^2 + 4r \,.\, \delta r \sin^2 \tfrac{1}{2} \delta\theta + 4r^2 \sin^2 \tfrac{1}{2} \delta\theta.$$

Dividing by $(\delta\theta)^2$, we get

$$\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \cdot \left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r \cdot \left(\frac{\sin \frac{1}{2} \delta\theta}{\frac{1}{2} \delta\theta}\right)^2 \delta r + r^2 \left(\frac{\sin \frac{1}{2} \delta\theta}{\frac{1}{2} \delta\theta}\right)^2.$$

Taking limit of both sides when $Q \to P$, we get

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2, \qquad \left[\because \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1\right]$$

Hence $\dfrac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}}$.

Cor. 1. $\dfrac{ds}{dr} = \dfrac{ds}{d\theta} \cdot \dfrac{d\theta}{dr} = \sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}} \cdot \dfrac{d\theta}{dr}$

$$= \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}.$$

Cor. 2. $\sin \phi = \dfrac{1}{\operatorname{cosec} \phi} = \dfrac{1}{\sqrt{(1 + \cot^2 \phi)}} = \dfrac{\tan \phi}{\sqrt{(1 + \tan^2 \phi)}}$

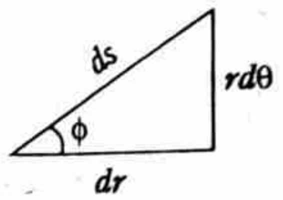
$$= \frac{\tan \phi}{\sqrt{\{1 + r^2 (d\theta/dr)^2\}}} = \frac{r\,(d\theta/dr)}{ds/dr} = \mathbf{r}\,\frac{\mathbf{d\theta}}{\mathbf{ds}}.$$

Cor. 3. $\cos \phi = \dfrac{1}{\sec \phi} = \dfrac{1}{\sqrt{(1 + \tan^2 \phi)}} = \dfrac{1}{\sqrt{\{1 + r^2 (d\theta/dr)^2\}}}$

$$= \frac{1}{ds/dr} = \frac{\mathbf{dr}}{\mathbf{ds}}.$$

Cor. 4. We have $\cos^2 \phi + \sin^2 \phi = 1$.

$$\therefore \quad \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2 = 1.$$



Note. All the above results can be easily remembered from the adjoining hypothetical figure.

Solved Examples

Ex. 59. *For the ellipse $x = a \cos t, y = b \sin t$, prove that*
$$ds/dt = a\sqrt{(1 - e^2 \cos^2 t)}. \qquad \textbf{(Meerut 1993)}$$

Sol. Here $dx/dt = -a \sin t$ and $dy/dt = b \cos t$.

We have

$$ds/dt = \sqrt{\{(dx/dt)^2 + (dy/dt)^2\}} = \sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)}$$
$$= \sqrt{\{a^2 \sin^2 t + a^2 (1 - e^2) \cos^2 t\}},$$

[$\because$ for the given ellipse, $b^2 = a^2 (1 - e^2)$]

$$= a\sqrt{\{(\sin^2 t + \cos^2 t) - e^2 \cos^2 t\}} = a\sqrt{(1 - e^2 \cos^2 t)}.$$

Ex. 60. *If $r^m = a^m \cos m\theta$, prove that* $\dfrac{ds}{d\theta} = \dfrac{a^m}{r^{m-1}}$.

Sol. We have $\quad r^m = a^m \cos m\theta.$ ...(1)

Taking logarithm of both sides of (1), we get

$$m \log r = m \log a + \log \cos m\theta.$$

Differentiating w.r.t. θ, we get

$$\frac{m}{r}\frac{dr}{d\theta} = 0 - \frac{m \sin m\theta}{\cos m\theta}, \quad i.e. \quad \frac{dr}{d\theta} = -r \tan m\theta.$$

Now $\dfrac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}} = \sqrt{(r^2 + r^2 \tan^2 m\theta)} = r \sec m\theta$

$$= \frac{r}{\cos m\theta} = \frac{r}{r^m/a^m}, \qquad \left[\because \text{ from (1), } \cos m\theta = \frac{r^m}{a^m}\right]$$

$$= r \cdot \frac{a^m}{r^m} = \frac{a^m}{r^{m-1}}.$$

Ex. 61. *For the cycloid $x = a(1 - \cos t)$, $y = a(t + \sin t)$, find ds/dt, ds/dx and ds/dy.* **(Meerut 1994)**

Sol. Differentiating w.r.t. t, we have

$$dx/dt = a \sin t, dy/dt = a(1 + \cos t).$$

Now $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \sqrt{\{(a \sin t)^2 + a^2(1 + \cos t)^2\}}$$

$$= a\sqrt{\{(2 \sin \tfrac{1}{2}t \cos \tfrac{1}{2}t)^2 + (2 \cos^2 \tfrac{1}{2}t)^2\}}$$

$$= 2a \cos \tfrac{1}{2}t \sqrt{(\sin^2 \tfrac{1}{2}t + \cos^2 \tfrac{1}{2}t)} = 2a \cos \tfrac{1}{2}t.$$

Also $$\frac{ds}{dx} = \frac{ds}{dt} \cdot \frac{dt}{dx} = \frac{2a \cos \frac{1}{2}t}{dx/dt} = \frac{2a \cos \frac{1}{2}t}{a \sin t} = \frac{2a \cos \frac{1}{2}t}{2a \sin \frac{1}{2}t \cos \frac{1}{2}t}$$

$$= \operatorname{cosec} \tfrac{1}{2}t,$$

and $$\frac{ds}{dy} = \frac{ds}{dt} \cdot \frac{dt}{dy} = \frac{2a \cos \frac{1}{2}t}{dy/dt} = \frac{2a \cos \frac{1}{2}t}{a(1 + \cos t)} = \frac{2a \cos \frac{1}{2}t}{2a \cos^2 \frac{1}{2}t}$$

$$= \sec \tfrac{1}{2}t.$$

Ex. 62. *For any curve, prove that*

(*a*) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$, **(Kanpur 1988)**

(*b*) $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{(r^2 - p^2)}}$. **(Kanpur 1975)**

Sol. **(a)** We know that $r\dfrac{d\theta}{ds} = \sin \phi$ or $\dfrac{d\theta}{ds} = \dfrac{\sin \phi}{r}$.

$$\therefore \quad \frac{ds}{d\theta} = \frac{r}{\sin \phi} = \frac{r}{p/r} = \frac{r^2}{p}, \qquad [\because p = r \sin \phi]$$

(b) We know that $dr/ds = \cos \phi$.

$$\therefore \quad \frac{ds}{dr} = \frac{1}{\cos \phi} = \frac{1}{\sqrt{(1 - \sin^2 \phi)}} = \frac{1}{\sqrt{\{1 - (p/r)^2\}}} = \frac{r}{\sqrt{(r^2 - p^2)}}.$$

Ex. 63. *Prove that for any curve,*

$$\sin^2 \phi\,(d\phi/d\theta) + r\,(d^2r/ds^2) = 0. \qquad \textbf{(Meerut 1998)}$$

Sol. We know that $dr/ds = \cos \phi$. Differentiating both sides w.r.t. s, we get

$$d^2r/ds^2 = -\sin \phi\,(d\phi/ds) = -\sin \phi\,(d\phi/d\theta)\,.\,(d\theta/ds).$$

Multiplying both sides by r, we get

$$r\,(d^2r/ds^2) = -\sin \phi\,(d\phi/d\theta)\,.\,r\,(d\theta/as)$$

or $$r\,(d^2r/ds^2) = -\sin \phi\,(d\phi/d\theta) \sin \phi, \qquad [\because r\,(d\theta/ds) = \sin \phi]$$

or $$r\,(d^2r/ds^2) + \sin^2 \phi\,(d\phi/d\theta) = 0.$$

Ex. 64. *For the curve $r^m = a^m \cos m\theta$, prove that*

(a) $ds/d\theta = a\,(\sec m\theta)^{(m-1)/m}$, **(Vikram 1970)**

(b) $ds/d\theta$ *varies inversely as $(m - 1)^{th}$ power of r.*

(c) $a^{2m}\dfrac{d^2r}{ds^2}+mr^{2m-1}=0.$ **(Lucknow 1980; Kanpur 86)**

Sol. (**a**) Proceeding as in Ex. 60, we get

$$\frac{ds}{d\theta}=r\sec m\theta=(a^m\cos m\theta)^{1/m}.\sec m\theta$$
$$=a(\cos m\theta)^{1/m}.\sec m\theta=a(\sec m\theta)^{-1/m}\sec m\theta$$
$$=a(\sec m\theta)^{1+(-1/m)}=a(\sec m\theta)^{(m-1)/m}.$$

(**b**) Proceeding as in Ex. 60, we get

$$\frac{ds}{d\theta}=\frac{a^m}{r^{m-1}}=(\text{some constant})\cdot\frac{1}{r^{m-1}}\cdot$$

Therefore $ds/d\theta$ varies inversely as $(m-1)$th power of r.

(**c**) Proceeding as in Ex. 60, we get $\dfrac{1}{r}\cdot\dfrac{dr}{d\theta}=-\tan m\theta.$

Therefore $\quad \cot\phi=\dfrac{1}{r}\dfrac{dr}{d\theta}=-\tan m\theta=\cot\left(\dfrac{\pi}{2}+m\theta\right)\cdot$

Hence $\phi=\frac{1}{2}\pi+m\theta$. Now we know that $\dfrac{dr}{ds}=\cos\phi.$

Differentiating w.r.t. s, we get

$$\frac{d^2r}{ds^2}=-\sin\phi\frac{d\phi}{ds}=-\sin\phi\frac{d\phi}{d\theta}\frac{d\theta}{ds}$$

$$=-\sin\phi\frac{d\phi}{d\theta}\frac{\sin\phi}{r}, \qquad [\because r\,(d\theta/ds)=\sin\phi]$$

$$=-\frac{\sin^2\phi}{r}\frac{d\phi}{d\theta}=-\frac{\sin^2(\frac{1}{2}\pi+m\theta)}{r}\cdot m$$

$[\because \phi=\frac{1}{2}\pi+m\theta, \text{ and so } d\phi/d\theta=m]$

$$=-\frac{m\cos^2 m\theta}{r}=-\frac{m}{r}\cdot\left(\frac{r^m}{a^m}\right)^2, \qquad [\because r^m=a^m\cos m\theta]$$

$$=-\frac{mr^{2m-1}}{a^{2m}}\cdot$$

$$\therefore \quad a^{2m}\frac{d^2r}{ds^2}+mr^{2m-1}=0.$$

Ex. 65. *Show that for the curve*

$$\theta=\cos^{-1}\frac{r}{k}-\frac{\sqrt{(k^2-r^2)}}{r}, \quad r\frac{ds}{dr} \textit{ is constant.}$$

Sol. Let $\quad r=k\cos t.$

Then $\quad \theta=\cos^{-1}(\cos t)-\dfrac{\sqrt{(k^2-k^2\cos^2 t)}}{k\cos t}=t-\tan t.$

These give $dr/dt=-k\sin t$ and $d\theta/dt=1-\sec^2 t.$

Now $$r\frac{d\theta}{dr} = r\frac{d\theta/dt}{dr/dt} = k\cos t \cdot \frac{1-\sec^2 t}{-k\sin t}$$

$$= -\cot t \,.\, (-\tan^2 t) = \tan t.$$

$$\therefore \quad \frac{ds}{dr} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}} = \sqrt{(1+\tan^2 t)} = \sec t$$

$$= \frac{1}{\cos t} = \frac{1}{r/k} = \frac{k}{r}\cdot$$

Hence $r\,(ds/dr) = k =$ constant.

Ex. 66. *For the curve $r = ae^{\theta\cot\alpha}$, prove that s/r =constant, s being measured from the origin.* **(Lucknow 1982; Meerut 95)**

Sol. We have $r = ae^{\theta\cot\alpha}$.

$\therefore \quad dr/d\theta = ae^{\theta\cot\alpha}\,.\,\cot\alpha = r\cot\alpha.$

Now $$\frac{ds}{dr} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}} = \sqrt{\left\{1 + r^2\cdot\frac{1}{(r\cot\alpha)^2}\right\}}$$

$$= \sqrt{(1+\tan^2\alpha)} = \sec\alpha.$$

Integrating w.r.t. r, we get $s = r\sec\alpha + C$, where C is constant of integration. Since s is measured from the origin, therefore $s = 0$ when $r = 0$. This gives $0 = 0 + C$ i.e., $C = 0$.

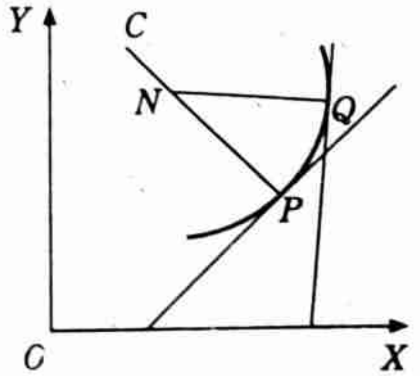
Hence $s = r\sec\alpha$ or $s/r = \sec\alpha =$ constant.

□

7
Curvature

§ 1. Definition. [Lucknow 1982, 77; Raj. 77]

Let P and Q be two neighbouring points on a given curve, such that the arc PQ is concave towards its chord. Let the normals at P and Q intersect in N.



If as $Q \to P, N$ tends to the definite position C, then C is called the **centre of curvature** of the curve at the point P.

The distance CP is called the radius of carvature of the curve at P and is usually denoted by ρ.

The circle with its centre at C and radius CP is called the **circle of curvature** of the curve at P.

Any chord of this circle drawn through the point P is called a **chord of curvature.**

The reciprocal of the distance $CP\ (= \rho)$ is called the **curvature** of the curve at P. **(Gorakhpur 1989)**

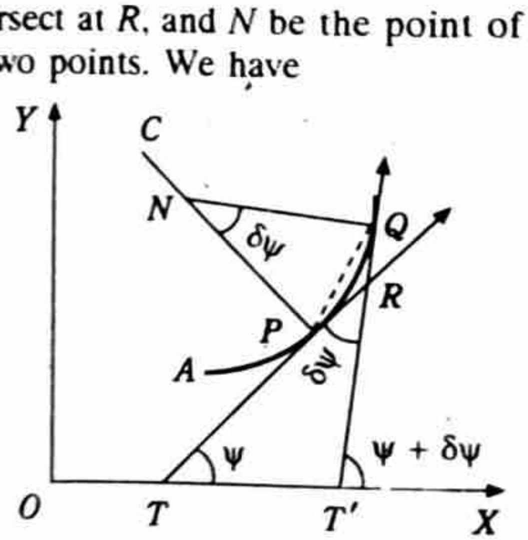
§ 2. Radius of curvature of intrinsic curves.

(Gorakhpur 1989; Kanpur 86; Indore 87; Vikram 82)

Let A be a fixed point on the curve and let P and Q be two neighbouring points on the curve and the tangents at these two points make angles ψ and $\psi + \delta\psi$ respectively with a fixed line, say, x-axis. Also let arc $AP = s$ and arc $AQ = s + \delta s$, so that arc $PQ = \delta s$.

Let the tangents at P and Q intersect at R, and N be the point of intersection of the normals at these two points. We have

$$\angle PNQ = \angle TRT' = \delta\psi.$$



Suppose $N \to C$ as $Q \to P$. Then the radius of curvature at

$$P = \rho = \lim_{Q \to P} PN.$$

From the $\Delta\, PNQ$, we have

$$\frac{PN}{\sin NQP} = \frac{\text{chord } PQ}{\sin PNQ} = \frac{\text{chord } PQ}{\sin \delta\psi}.$$

$$\therefore \quad PN = \frac{\text{chord } PQ}{\sin \delta\psi} \sin NQP$$

$$= \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin NQP.$$

As $Q \to P$, $\delta\psi \to 0, \delta s \to 0$, chord $PQ \to$ tangent at P, $QN \to$ normal at P and consequently $\angle NQP \to \frac{1}{2}\pi$.

$$\therefore \quad \rho = \lim_{Q \to P} PN$$

$$= \left(\lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ}\right) \cdot \left(\lim_{\delta\psi \to 0} \frac{\delta s}{\delta\psi}\right) \cdot \left(\lim_{\delta\psi \to 0} \frac{\delta\psi}{\sin \delta\psi}\right) \cdot \left(\lim_{Q \to P} \sin NQP\right)$$

$$= 1 \cdot \frac{ds}{d\psi} \cdot 1 \,.\, \sin \frac{\pi}{2} = \frac{ds}{d\psi} \cdot \text{ Hence } \rho = \frac{ds}{d\psi}.$$

Note. The relation between s and ψ is called the intrinsic equation of a curve. Therefore $\rho = ds/d\psi$ is known as intrinsic formula for radius of curvature.

Cor. The curvature of the curve at P is defined as the reciprocal of the radius of curvature at P. Hence the curvature at $P = 1/\rho = d\psi/ds$.

**§ 3. Radius of curvature for cartesian curves.

(Garhwal 1983; Meerut 91S, 81, 77; Indore 73; Gorakhpur 82, 86; Magadh 76; Rohilkhand 82; Bundelkhand 77; Lucknow 82, 77, 75; Agra 85)

Let $y = f(x)$ be the equation of the curve. The inclination ψ of the tangent at any point to the axis of x is given by $\tan \psi = dy/dx$.

Differentiating w.r.t. s, we get

$$\sec^2 \psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2} \cdot \frac{dx}{ds} = \frac{d^2y}{dx^2} \cos \psi. \quad \left[\because \frac{dx}{ds} = \cos \psi\right]$$

$$\therefore \quad \frac{ds}{d\psi} = \frac{\sec^2 \psi}{(d^2y/dx^2) \,.\, \cos \psi} = \frac{\sec^3 \psi}{d^2y/dx^2}$$

$$= \frac{\{1 + \tan^2 \psi\}^{3/2}}{d^2y/dx^2} = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}.$$

Hence $\quad \rho = \dfrac{\mathbf{ds}}{\mathbf{d\psi}} = \dfrac{\mathbf{\{1 + (dy/dx)^2\}^{3/2}}}{\mathbf{d^2y/dx^2}}.$

Note 1. The value of ρ is + ive or − ive according as d^2y/dx^2 is positive or negative. However in numerical problems we shall be required to find only the length of the radius of curvature and we shall not be concerned with its sign. So we should ingnore the negative sign whenever we get a negative value for ρ.

Note 2. From the definition, it is obvious that the value of ρ depends on the curve and not on the co-ordinate axes chosen. Therefore interchanging x and y in the above formula for ρ, we obtain

$$\rho = \frac{\{1 + (dx/dy)^2\}^{3/2}}{d^2x/dy^2}.$$

This formula is specially useful when dy/dx is infinite, *i.e.*, when the tangent is parallel to the x-axis.

Definition. *If at any point of a curve $d^2y/dx^2 = 0$, the point is called* **a point of inflexion.**

Solved Examples

Ex. 1. *Find the radius of curvature at any point (s, ψ) of the curves*

(i) $s = a\,(e^{m\psi} - 1)$, **(ii)** $s = c \log \sec \psi$,

(iii) $s = a \log \cot\left(\dfrac{\pi}{4} - \dfrac{\psi}{2}\right) + a\dfrac{\sin\psi}{\cos^2\psi}$. **(Lucknow 1978)**

Sol. (i) We have, $s = a\,(e^{m\psi} - 1)$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = ame^{m\psi}.$$

(ii) We have $s = c \log \sec \psi$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = c \cdot \frac{1}{\sec\psi} \cdot \sec\psi \tan\psi = c\tan\psi.$$

(iii) We have $s = a \log \cot\left(\dfrac{\pi}{4} - \dfrac{\psi}{2}\right) + a \sin\psi \sec^2\psi$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = a\frac{1}{\cot\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)} \cdot \left\{-\operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{\psi}{2}\right)\right\} \cdot \left(-\frac{1}{2}\right)$$

$$+ a\sin\psi \,.\, 2\sec\psi\sec\psi\tan\psi + a\cos\psi\sec^2\psi$$

$$= \frac{a}{2\sin\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)\cos\left(\frac{1}{4}\pi - \frac{1}{2}\psi\right)} + \frac{2a\sin^2\psi}{\cos^3\psi} + \frac{a}{\cos\psi}$$

$$= \frac{a}{\sin\left(\frac{1}{2}\pi - \psi\right)} + \frac{2a\sin^2\psi}{\cos^3\psi} + \frac{a}{\cos\psi}$$

$$= \frac{2a}{\cos\psi} + \frac{2a\sin^2\psi}{\cos^3\psi} = \frac{2a}{\cos^3\psi}(\cos^2\psi + \sin^2\psi) = 2a\sec^3\psi.$$

Ex. 2. *Find the radius of curvature at the point (s, ψ) on the following curves:*

(i) $s = 4a \sin\psi$, **(ii)** $s = c \tan\psi$,

(iii) $s = 8a \sin^2 \frac{1}{6}\psi$.

Sol. (i) We have $s = 4a\sin\psi$.

$\therefore \quad \rho = ds/d\psi = 4a\cos\psi$.

(ii) We have $s = c\tan\psi$.

$\therefore \quad \rho = ds/d\psi = c\sec^2\psi$.

(iii) We have $s = 8a \sin^2 \frac{1}{6}\psi$.

$\therefore \quad \rho = ds/d\psi = 8a \cdot (2 \sin \frac{1}{6}\psi \cos \frac{1}{6}\psi) \cdot \frac{1}{6} = (4a/3) \sin \frac{1}{3}\psi$.

Ex. 3 (a). *Find the radius of curvature at the point (x, y) on the parabola $y^2 = 4ax$.*

Sol. The given curve is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2y \frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{2a}{y}\right) = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2} \cdot \frac{2a}{y} = -\frac{4a^2}{y^3}.$$

$$\text{Now } \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1 + \frac{4a^2}{y^2}\right)^{3/2}}{-4a^2/y^3} = -\frac{(y^2 + 4a^2)^{3/2}}{y^3} \cdot \frac{y^3}{4a^2}$$

$$= -\frac{1}{4a^2}(4ax + 4a^2)^{3/2}, \quad \text{from (1)}$$

$$= -\frac{1}{4a^2} \cdot a^{3/2} \cdot 8 \cdot (x + a)^{3/2} = \frac{2}{\sqrt{a}}(x + a)^{3/2},$$

neglecting the – ive sign because ρ is a length.

Ex. 3 (b). *Find the radius of curvature at (x, y) on the curve*

$$a^2y = x^3 - a^3. \quad \textbf{(Meerut 1977, 84S)}$$

Sol. We have $a^2y = x^3 - a^3$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a^2}(3x^2)\,;\, \frac{d^2y}{dx^2} = \frac{6x}{a^2}.$$

$$\therefore \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (3x^2/a^2)^2]^{3/2}}{6x/a^2}$$

$$= \frac{(a^4 + 9x^4)^{3/2}}{a^6} \cdot \frac{a^2}{6x} = \frac{(a^4 + 9x^4)^{3/2}}{6a^4 x}.$$

Ex. 4 (a). *Find the radius of curvature at (x, y) of the curve*

$$y = \tfrac{1}{2} \cdot c \cdot (e^{x/c} + e^{-x/c}) = c \cosh (x/c). \quad \textbf{(Meerut 1987 S)}$$

Sol. We have $y = c \cosh (x/c)$.

$$\therefore \quad \frac{dy}{dx} = c\left(\sinh \frac{x}{c}\right) \cdot \frac{1}{c} = \sinh\left(\frac{x}{c}\right),$$

and $\quad d^2y/dx^2 = (1/c) \cosh (x/c)$.

$$\therefore \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + \sinh^2 (x/c)]^{3/2}}{(1/c) \cosh (x/c)} = \frac{c\,[\cosh^2 (x/c)]^{3/2}}{\cosh (x/c)}$$

$$= c \cosh^2 (x/c) = c\,(y/c)^2 = y^2/c.$$

Ex. 4 (b). *If C be the centre of curvature corresponding to any point $P(x, y)$ on the curve $y = a \cosh (x/a)$ and G is the intersection of the normal at P and x-axis, then show that $PC = PG$.*

Sol. The given curve is $y = a \cosh (x/a)$. ...(1)

If ρ be the radius of curvature of the curve (1) at the point $P(x, y)$, then $\rho = PC$.

Proceeding as in Ex. 4 (a), we get $\rho = a \cosh^2 (x/a) = PC$.

Also length of normal at $P = PG = y \sec \psi = y \sqrt{(1 + \tan^2 \psi)}$
$= y \sqrt{\{1 + (dy/dx)^2\}} = y \sqrt{\{1 + \sinh^2 (x/a)\}} = a \cosh (x/a) \cosh (x/a)$
$= a \cosh^2 (x/a) = PC$.

Ex. 5. *Find the radius of curvature at any point (x, y) of the curves*

(i) $ay^2 = x^3$, (ii) $xy = c^2$,

(iii) $y = c \log \sec (x/c)$, **(Lucknow 1974)**

(iv) $x^{2/3} + y^{2/3} = a^{2/3}$. **(Meerut 1991 P; Kanpur 79)**

Sol. (i) The equation of the given curve is $ay^2 = x^3$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2ay \frac{dy}{dx} = 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2}{2ay} = \frac{3x^2}{2a (x^3/a)^{1/2}}, \qquad \text{from (1)}$$
$$= (3/2\sqrt{a})\, x^{1/2}.$$

Differentiating again, $\dfrac{d^2y}{dx^2} = \dfrac{3}{2\sqrt{a}} \cdot \dfrac{1}{2} x^{-1/2} = \dfrac{3}{4\sqrt{a} \,.\, x^{1/2}}$.

$$\text{Now } \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{\{1 + (9x/4a)\}^{3/2}}{3/(4\sqrt{a}x^{1/2})}$$
$$= \frac{(4a + 9x)^{3/2}}{8a^{3/2}} \cdot \frac{4\sqrt{a}x^{1/2}}{3} = \frac{x^{1/2} (4a + 9x)^{3/2}}{6a}.$$

(ii) We have $xy = c^2$ or $y = c^2/x$.

$$\therefore \quad \frac{dy}{dx} = -\frac{c^2}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2c^2}{x^3}.$$

$$\therefore \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + (c^4/x^4)]^{3/2}}{2c^2/x^3}$$
$$= \frac{(x^4 + c^4)^{3/2}}{2c^2x^3} = \frac{(x^4 + x^2y^2)^{3/2}}{2c^2x^3}, \qquad [\because xy = c^2]$$
$$= \frac{(x^2 + y^2)^{3/2}}{2c^2} = \frac{r^3}{2c^2}, \text{ where } r^2 = x^2 + y^2.$$

(iii) We have $y = c \log \sec (x/c)$.

$$\therefore \quad \frac{dy}{dx} = \frac{c}{\sec (x/c)} \cdot \sec \left(\frac{x}{c}\right) \cdot \tan \left(\frac{x}{c}\right) \cdot \frac{1}{c} = \tan \left(\frac{x}{c}\right)$$

Also $\qquad d^2y/dx^2 = (1/c) \sec^2 (x/c)$.

$$\therefore \quad \rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[1 + \tan^2 (x/c)]^{3/2}}{(1/c) \sec^2 x/c} = \frac{c \sec^3 (x/c)}{\sec^2 (x/c)}$$
$$= c \sec (x/c).$$

(iv) The given curve is $\quad x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The co-ordinates (x, y) of any point on (1) may be taken as $x = a \cos^3 t$, $y = a \sin^3 t$, where t is the parameter.

$\therefore \quad dx/dt = -3a \cos^2 t \sin t$ and $dy/dt = 3a \sin^2 t \cos t$.

Now $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}=-\tan t.$

Also $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}(-\tan t)=\left\{\frac{d}{dt}(-\tan t)\right\}\cdot\frac{dt}{dx}$

$=-\sec^2 t\,.\,(dt/dx)$

$=-\sec^2 t\cdot\frac{1}{-3a\cos^2 t\sin t},\qquad\left[\because\ \frac{dx}{dt}=-3a\cos^2 t\sin t\right]$

$=-(1/3a)\sec^4 t\operatorname{cosec} t.$

$\therefore\quad \rho$ at the point 't' $=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}=\frac{(1+\tan^2 t)^{3/2}}{(1/3a)\sec^4 t\operatorname{cosec} t}$

$=\frac{3a\sec^3 t}{\sec^4 t\operatorname{cosec} t}=3a\cos t\sin t.$

But $\cos^3 t=x/a$ and $\sin^3 t=y/a$. Therefore $\cos t=(x/a)^{1/3}$ and $\sin t=(y/a)^{1/3}$.

Hence ρ at the point (x,y)

$$=3a\,(x/a)^{1/3}(y/a)^{1/3}=3a^{1/3}x^{1/3}y^{1/3}.$$

Alternative solution. Differentiating both sides of (1) with respect to x, we get

$$\tfrac{2}{3}x^{-1/3}+\tfrac{2}{3}y^{-1/3}\,.\,(dy/dx)=0$$

or $$\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-y^{1/3}x^{-1/3}.$$

$\therefore\quad \frac{d^2y}{dx^2}=-\left(\frac{1}{3}y^{-2/3}\frac{dy}{dx}\right)\cdot x^{-1/3}+y^{1/3}\cdot\tfrac{1}{3}x^{-4/3}$

$=-\tfrac{1}{3}y^{-2/3}x^{-1/3}.\,(-y^{1/3}x^{-1/3})+\tfrac{1}{3}y^{1/3}x^{-4/3}$

$=\tfrac{1}{3}y^{-1/3}x^{-2/3}+\tfrac{1}{3}y^{1/3}x^{-4/3}$

$=\tfrac{1}{3}x^{-4/3}y^{-1/3}(x^{2/3}+y^{2/3})=\tfrac{1}{3}x^{-4/3}y^{-1/3}a^{2/3}.$

$\therefore\quad \rho$ at the point $(x,y)=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$

$=\frac{[1+(y^{2/3}/x^{2/3})]^{3/2}}{\frac{1}{3}x^{-4/3}y^{-1/3}a^{2/3}}=\frac{3\,(x^{2/3}+y^{2/3})^{3/2}}{(x^{2/3})^{3/2}x^{-4/3}y^{-1/3}a^{2/3}}$

$=\frac{3\,(a^{2/3})^{3/2}}{x\,x^{-4/3}y^{-1/3}a^{2/3}}=\frac{3a}{x^{-1/3}y^{-1/3}a^{2/3}}=3a^{1/3}x^{1/3}y^{1/3}.$

Ex. 6 (a). *Find the curvature at the point* $(3a/2,3a/2)$ *of the curve* $x^3+y^3=3axy$. **(Meerut 1979, 83 S, 89 P)**

Sol. The curve is $\quad x^3+y^3=3axy.$ ...(1)

Differentiating both sides of (1) w.r.t. x, we get

$$3x^2+3y^2\,(dy/dx)=3ay+3ax\,(dy/dx)$$

or $$x^2+y^2\,(dy/dx)=ay+ax\,(dy/dx).\qquad ...(2)$$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{x^2 - ay}{ax - y^2}$, so that $\left(\dfrac{dy}{dx}\right)_{(\frac{3}{2}a,\, \frac{3}{2}a)} = -1.$

Again, differentiating both sides of (2) w.r.t. x, we get

$$2x + 2y\left(\frac{dy}{dx}\right)^2 + y^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\frac{dy}{dx} + ax\frac{d^2y}{dx^2}$$

or $$(ax - y^2)\frac{d^2y}{dx^2} = 2x + 2y\left(\frac{dy}{dx}\right)^2 - 2a\frac{dy}{dx}. \qquad \text{...(3)}$$

Putting $x = 3a/2$, $y = 3a/2$ and $(dy/dx)_{(3a/2,\, 3a/2)} = -1$ in (3), we get

$$\left(\frac{d^2y}{dx^2}\right)_{(3a/2,\, 3a/2)} = -\frac{32}{3}\cdot\frac{1}{a} = -32/(3a).$$

Hence the radius of curvature ρ at $(3a/2, 3a/2)$

$$= \left[\frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}\right]_{(3a/2,\, 3a/2)}$$

$$= \frac{(1+1)^{3/2}}{(-32/3a)} = -\frac{3a}{8\sqrt{2}}.$$

$\therefore$ Curvature at $(3a/2, 3a/2) = 1/\rho = -8\sqrt{2}/3a.$

Neglecting the negative sign, the value of curvature at

$$(3a/2, 3a/2) = 8\sqrt{2}/3a.$$

Ex. 6 (b). *Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.* **(Kanpur 1988; Magadh 87)**

Sol. We have $y = e^x$.

Therefore $dy/dx = e^x$, and $d^2y/dx^2 = e^x$.

$$\therefore \quad \rho \text{ at } (x, y) = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{(1 + e^{2x})^{3/2}}{e^x}.$$

The curve $y = e^x$ crosses the y-axis (*i.e.*, the straight line $x = 0$) at the point $(0, 1)$.

$$\therefore \quad \rho \text{ at } (0, 1) = \frac{(1 + e^0)^{3/2}}{e^0} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}.$$

•**Ex. 7.** *Find the radius of curvature at any point (x_1, y_1) of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.* **(Kanpur 1980; Meerut 72, 84P)**

Sol. The given curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$. ...(1)

Differentiating (1) w.r.t. x, we have

$$\tfrac{1}{2}x^{-1/2} + \tfrac{1}{2}y^{-1/2}(dy/dx) = 0$$

or $$dy/dx = -y^{1/2}/x^{1/2} = -y^{1/2}x^{-1/2}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\left(\frac{1}{2}y^{-1/2}\frac{dy}{dx}\right)x^{-1/2} - y^{1/2}\left(-\frac{1}{2}x^{-3/2}\right)$$

$$= -\tfrac{1}{2}y^{-1/2}x^{-1/2}(-y^{1/2}x^{1/2}) + \tfrac{1}{2}y^{1/2}x^{-3/2}$$

$$= \frac{1}{2x} + \frac{y^{1/2}}{2x^{3/2}} = \frac{1}{2x}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right) = \frac{1}{2x}\frac{x^{1/2}+y^{1/2}}{x^{1/2}} = \frac{\sqrt{a}}{2x^{3/2}}.$$

$$\therefore \quad \rho \text{ at } (x_1, y_1) = \frac{[1 + (y_1/x_1)]^{3/2}}{\sqrt{a}/(2x_1^{3/2})} = \frac{2(x_1 + y_1)^{3/2}}{\sqrt{a}}.$$

Ex. 8. *Find the radius of curvature at the point* $(\frac{1}{4}, \frac{1}{4})$ *of the curve* $\sqrt{x} + \sqrt{y} = 1$. **(Gorakhpur 1982; Kanpur 85)**

Sol. Proceed exactly as in Ex. 7. Here $a = 1$ and instead of (x_1, y_1) the point is $(\frac{1}{4}, \frac{1}{4})$. The answer is $1/\sqrt{2}$.

Ex. 9. *In the curve* $y = ae^{x/a}$, *prove that* $\rho = a \sec^2\theta \operatorname{cosec}\theta$, *where* $\theta = \tan^{-1}(y/a)$.

Sol. We have $dy/dx = a\,(e^{x/a})\,(1/a) = e^{x/a} = y/a$;

and $\quad d^2y/dx^2 = (1/a)\,(dy/dx) = (1/a)\,(y/a) = y/a^2$.

$$\therefore \quad \rho = \frac{[1 + (y^2/a^2)]^{3/2}}{y/a^2} = \frac{(a^2 + y^2)^{3/2}}{ay} = \frac{(a^2 + a^2\tan^2\theta)^{3/2}}{a \cdot a\tan\theta},$$

[$\because$ given $y = a\tan\theta$]

$$= a\sec^3\theta\cot\theta = a\sec^2\theta\operatorname{cosec}\theta.$$

Ex. 10. *In the ellipse* $(x^2/a^2) + (y^2/b^2) = 1$, *show that the radius of curvature at an end of the major axis is equal to the semi-latus rectum of the ellipse.* **(Garhwal 1977)**

Sol. The ellipse is $(x^2/a^2) + (y^2/b^2) = 1$. ...(1)

Differentiating, $\quad \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\left(\dfrac{x}{y}\right)$.

Differentiating again, $\dfrac{d^2y}{dx^2} = -\dfrac{b^2}{a^2}\left[\dfrac{y \cdot 1 - x\,(dy/dx)}{y^2}\right]$

$$= -\frac{b^2}{a^2y^2}\left[y - x\left(-\frac{b^2x}{a^2y}\right)\right]$$

$$= -\frac{b^2}{a^2y^2}\left(\frac{a^2y^2 + b^2x^2}{a^2y}\right) = -\frac{b^2}{a^2y^2}\left(\frac{a^2b^2}{a^2y}\right),$$

[$\because$ from (1), $a^2y^2 + b^2x^2 = a^2b^2$]

$$= -b^4/a^2y^3.$$

$$\therefore \quad \rho \text{ at the point } (x, y) = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

$$= \frac{[1 + (-b^2x/a^2y)^2]^{3/2}}{b^4/(a^2y^3)}, \text{ neglecting the } - \text{ive sign}$$

$$= \frac{(a^4y^2 + b^4x^2)^{3/2}}{a^4b^4}.$$

Now the co-ordinates of one end of major axis are $(a, 0)$.

$$\therefore \quad [\rho]_{\text{at }(a,0)} = \frac{[a^4 \cdot 0 + b^4a^2]^{3/2}}{a^4b^4} = \frac{b^6a^3}{a^4b^4} = \frac{b^2}{a}$$

= semi latus-rectum of the ellipse.

**Ex. 11. *Prove that for the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1, \rho=\dfrac{a^2b^2}{p^3}$, *p being the perpendicular from the centre upon the tangent at* (x, y).

(Delhi 1983, 80; Rohilkhand 88; Agra 76; Meerut 71; Kanpur 72; Allahabad 72; Lucknow 83, 80, 70; Indore 73; Bhopal 71)

Sol. Proceeding as in Ex. 10, we get

$$\rho \text{ at the point } (x, y)=\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}. \quad \text{...(1)}$$

Now the equation of the tangent to the given ellipse at (x, y) is

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1.$$

$\therefore$ p = the length of the perpendicular from the centre $(0, 0)$ to the tangent $=\dfrac{1}{\sqrt{(x^2/a^4+y^2/b^4)}}=\dfrac{a^2b^2}{\sqrt{(b^4x^2+a^4y^2)}}$.

$$\therefore \quad p^3=\frac{a^6b^6}{(b^4x^2+a^4y^2)^{3/2}} \quad \text{or} \quad (b^4x^2+a^4y^2)^{3/2}=\frac{a^6b^6}{p^3}.$$

$\therefore$ substituting it in (1), we get $\rho=\dfrac{a^6b^6/p^3}{a^4b^4}=\dfrac{a^2b^2}{p^3}$.

*Ex. 12. *If* ρ, ρ' *be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that*

$$\{(\rho)^{2/3}+(\rho')^{2/3}\}(ab)^{2/3}=(a^2+b^2).$$

(Meerut 1983 S; Gorakhpur 76; Indore 74; Kanpur 77, 72)

Sol. Let the equation of the ellipse be

$$x^2/a^2+y^2/b^2=1. \quad \text{...(1)}$$

Let CP and CQ be a pair of conjugate semi-diameters of (1), where C is the centre of the ellipse. Let 't' be the eccentric angle of the point P. Then by co-ordinate geometry, the eccentric angle of the point Q is '$t+\frac{1}{2}\pi$'.

Now in terms of the eccentric angle 't' the co-ordinates (x, y) of the point P are given by

$$x=a\cos t \quad \text{and} \quad y=b\sin t.$$

$$\therefore \quad dx/dt=-a\sin t,\ dy/dt=b\cos t.$$

Hence $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{b}{a}\cot t\right)=\left\{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\right\}\cdot\frac{dt}{dx}$$

$$=\frac{b}{a}\operatorname{cosec}^2 t\cdot\frac{1}{-a\sin t}=-\frac{b}{a^2}\operatorname{cosec}^3 t.$$

$\therefore$ if ρ be the radius of curvature at the point P, *i.e.*, at the point 't', we have

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1+\frac{b^2\cos^2 t}{a^2\sin^2 t}\right)^{3/2}}{(-b/a^2)\operatorname{cosec}^3 t}$$
$$= -\frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}.$$

Neglecting the negative sign, we have

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \quad \text{or} \quad ab\,.\,\rho = (a^2\sin^2 t + b^2\cos^2 t)^{3/2}$$

or $\quad (ab)^{2/3}\rho^{2/3} = a^2\sin^2 t + b^2\cos^2 t.$ ...(2)

If ρ' be the radius of curvature at the point Q, *i.e.*, at the point '$t + \frac{1}{2}\pi$', then replacing ρ by ρ' and t by $t + \frac{1}{2}\pi$ in (2), we get

$$(ab)^{2/3}\rho'^{2/3} = a^2\sin^2(\tfrac{1}{2}\pi + t) + b^2\cos^2(\tfrac{1}{2}\pi + t)$$
$$= a^2\cos^2 t + b^2\sin^2 t. \quad ...(3)$$

Adding (2) and (3), we get

$$(ab)^{2/3}\rho^{2/3} + (ab)^{2/3}\rho'^{2/3} = a^2 + b^2$$

or $\quad (ab)^{2/3}(\rho^{2/3} + \rho'^{2/3}) = a^2 + b^2.$

Ex. 13. *If CP, CD be a pair of conjugate semi-diameters of an ellipse, prove that the radius of curvature at P is CD^3/ab, a and b being the lengths of the semi-axes of the ellipse.* **(Meerut 1982, 84, 85)**

Sol. Proceed exactly as in Ex. 12. Name the point 'Q' as the point 'D'.

If ρ be the radius of curvature at the point P, we get

$$\rho = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab}.$$

Now the eccentric angle of the point D is '$t + \frac{1}{2}\pi$'. Therefore the co-ordinates of D are $\{a\cos(\frac{1}{2}\pi + t), b\sin(\frac{1}{2}\pi + t)\}$, *i.e.*, $(-a\sin t, b\cos t)$. Also C is the point $(0, 0)$. Therefore

$$CD = (a^2\sin^2 t + b^2\cos^2 t)^{1/2} \quad \text{or} \quad CD^3 = (a^2\sin^2 t + b^2\cos^2 t)^{3/2}.$$

Hence $\quad \rho = CD^3/ab.$

Note. Examples 10 and 11 can also be done by taking the co-ordinates of any point $P(x, y)$ on the given ellipse as $x = a\cos t$, $y = b\sin t$. Then as proved in Ex. 12, we have ρ at the point 't'

$$= (a^2\sin^2 t + b^2\cos^2 t)^{3/2}/(ab).$$

Now at an extremity of the major axis, we have $t = 0$. Therefore at an extremity of the major axis, we have

$$\rho = (a^2\sin^2 0 + b^2\cos^2 0)^{3/2}/ab = b^3/ab = b^2/a$$
$$= \text{the length of semi-latus rectum.}$$

Ex. 14. *Find the radius of curvature of $y = 4\sin x - \sin 2x$ at $x = \frac{1}{2}\pi$.* **(Kanpur 1974)**

Sol. We have $\quad y = 4\sin x - \sin 2x.$

$\therefore$ $dy/dx = 4\cos x - 2\cos 2x$ and $d^2y/dx^2 = -4\sin x + 4\sin 2x$.

$\therefore$ at $x = \frac{1}{2}\pi$,

$$dy/dx = 4\cos\tfrac{1}{2}\pi - 2\cos\pi = 2,$$

and $$d^2y/dx^2 = -4\sin\tfrac{1}{2}\pi + 4\sin\pi = -4.$$

$$\therefore \quad \rho \text{ at } (x = \pi/2) = \frac{(1+4)^{3/2}}{-4}$$

$= (5\sqrt{5}/4)$, neglecting the $-$ ive sign.

Ex. 15 (a). *Show that in the parabola $y^2 = 4ax$, the radius of curvature ρ at any point P is twice the part of the normal intercepted between the curve and the directrix.*

(b) *Also prove that ρ^2 varies as $(SP)^3$, where S is the focus.*

(Meerut 1981 S, 84)

Sol. **(a)** We have $y^2 = 4ax$.

$$\therefore \quad 2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{(4ax)^{1/2}} = \frac{\sqrt{a}}{\sqrt{x}}.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \sqrt{a}\,.\,(-\tfrac{1}{2})\,x^{-3/2} = \frac{-\sqrt{a}}{2x^{3/2}}.$$

$\therefore$ at any point $P(x, y)$, we have

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = -\frac{[1 + (a/x)]^{3/2}}{-\sqrt{a}/(2x^{3/2})}$$

$= (2/\sqrt{a})\,(x + a)^{3/2}$, neglecting the $-$ ive sign.

Now the equation of the normal to the given parabola at the point $P(x, y)$ is

$$\frac{dy}{dx}(Y - y) + (X - x) = 0 \quad \text{or} \quad \frac{2a}{y}(Y - y) + (X - x) = 0,$$

$$\left[\because \frac{dy}{dx} = \frac{2a}{y}\right]$$

or $$yX + 2aY = xy + 2ay. \qquad ...(1)$$

Also directrix of the parabola is $X = -a$. ...(2)

Now the co-ordinates of the point of intersection of (1) and (2) are obtained by solving (1) and (2) for X and Y.

From equations (1) and (2), we get

$$-ay + 2aY = xy + 2ay, \qquad [\text{putting } X = -a \text{ in (1)}]$$

or $$Y = (1/2a)(xy + 3ay) = (y/2a)(x + 3a).$$

$\therefore$ the point of intersection of the normal and the directrix is

$$[-a, (y/2a)(x + 3a)].$$

Therefore the length of the normal intercepted between the curve and the directrix

= the distance between (x, y) and $\{-a, (y/2a)(x + 3a)\}$

$$= \sqrt{\left[(x+a)^2 + \left\{y - \frac{y}{2a}(x+3a)\right\}^2\right]}$$

$$= \frac{1}{2a}\sqrt{[4a^2(x+a)^2 + y^2(x+a)^2]}$$

$$= \frac{1}{2a}\cdot(x+a)\,.\,\sqrt{(4a^2+y^2)} = \frac{1}{2a}\cdot(x+a)\sqrt{(4a^2+4ax)},$$

$$[\because y^2 = 4ax]$$

$$= (x+a)^{3/2}/a^{1/2}.$$

As already proved, ρ at $(x, y) = 2(x+a)^{3/2}/\sqrt{a}$

$= 2 \times$ the part of normal intercepted between the curve and directrix.

(b) The focus S is $(a, 0)$.

$\therefore \quad SP = \sqrt{[(x-a)^2 + (y-0)^2]} = \sqrt{\{(x-a)^2 + 4ax\}},$

$$[\because y^2 = 4ax]$$

$$= \sqrt{\{(x+a)^2\}} = x + a.$$

But $\rho = \dfrac{2}{\sqrt{a}}\cdot(x+a)^{3/2} = \dfrac{2}{\sqrt{a}}\cdot(SP)^{3/2}. \quad \therefore \quad \rho^2 \propto (SP)^3.$

Ex. 16. *Prove that* $1/\rho = (d^2y/dx^2)\cos^3\psi$. **(Agra 1980)**

Sol. Proceeding as in § 3 on page 207, we get

$$\sec^2\psi\,\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\cos\psi.$$

$$\therefore \quad \sec^2\psi\cdot\frac{1}{\rho} = \frac{d^2y}{dx^2}\cos\psi, \qquad \left[\because \rho = \frac{ds}{d\psi}\right]$$

or

$$1/\rho = \cos^3\psi(d^2y/dx^2).$$

Ex. 17. *For the curve* $y = \dfrac{ax}{a+x}$, *if* ρ *is the radius of curvature at any point* (x, y), *show that*

$$(2\rho/a)^{2/3} = (y/x)^2 + (x/y)^2.$$

(Meerut 1990 P; Kumayun 83; Agra 88)

Sol. We have $\quad y = \dfrac{ax}{a+x}$ ...(1)

$$\therefore \quad \frac{dy}{dx} = a\,\frac{(a+x)-x}{(a+x)^2} = \frac{a^2}{(a+x)^2} = a^2(a+x)^{-2},$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -2a^2(a+x)^{-3} = \frac{-2a^2}{(a+x)^3} = \frac{-2a}{(ax/y)^3} = -\frac{2y^3}{ax^3}.$$

Now
$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^4}{(a+x)^4} = 1 + \frac{a^4}{(ax/y)^4} = 1 + \frac{y^4}{x^4} = \frac{x^4+y^4}{x^4}.$$

$$\therefore \quad \rho = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{[(x^4+y^4)/x^4]^{3/2}}{(2y^3/ax^3)},$$

(negative sign being neglected)

$$= \frac{a}{2}\,\frac{(x^4+y^4)^{3/2}}{x^6\,(y^3/x^3)} = \frac{a}{2}\,\frac{(x^4+y^4)^{3/2}}{x^3y^3}.$$

Hence $\left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^4 + y^4}{x^2y^2} = \frac{x^4}{x^2y^2} + \frac{y^4}{x^2y^2} = \frac{x^2}{y^2} + \frac{y^2}{x^2} = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$

Ex. 18. *Prove that for the curve*

$$y = a \log \cot \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right) + a \sin\psi \sec^2\psi, \ \rho = 2a \sec^3\psi$$

and hence prove that $d^2y/dx^2 = 1/2a$, *and that this differential equation is satisfied by the parabola* $x^2 = 4ay$. **(Gorakhpur 1977)**

Sol. We have $s = a \log \cot \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right) + a \sin\psi \sec^2\psi$

$$= a \log \cot \left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right) + a \sec\psi \tan\psi.$$

$$\therefore \quad \rho = \frac{ds}{d\psi}$$

$$= \frac{a\left\{-\operatorname{cosec}^2\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right)\right\}}{\cot\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right)} \cdot \left(-\tfrac{1}{2}\right) + a\left(\tan^2\psi \sec\psi + \sec^3\psi\right)$$

$$= \frac{a}{2\sin\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right)\cos\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\psi\right)} + a\sec\psi\left(\sec^2\psi + \tan^2\psi\right)$$

$$= \frac{a}{\sin\left(\tfrac{1}{2}\pi - \psi\right)} + a\sec\psi\left(\sec^2\psi + \tan^2\psi\right)$$

$$= a\sec\psi + a\sec\psi\left(\sec^2\psi + \tan^2\psi\right)$$

$$= a\sec\psi\left(1 + \tan^2\psi + \sec^2\psi\right)$$

$$= a\sec\psi\left(\sec^2\psi + \sec^2\psi\right) = 2a\sec^3\psi.$$

But $\rho = [1 + (dy/dx)^2]^{3/2}/(d^2y/dx^2) = (1 + \tan^2\psi)^{3/2}/(d^2y/dx^2)$

$$= \sec^3\psi/(d^2y/dx^2).$$

$$\therefore \quad 2a\sec^3\psi = \frac{\sec^3\psi}{d^2y/dx^2}; \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{1}{2a}. \qquad \text{...(1)}$$

Now for the parabola $x^2 = 4ay$,

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}; \quad \therefore \quad \frac{d^2y}{dx^2} = \frac{1}{2a} \qquad \text{...(2)}$$

We see that the differential equations (1) and (2) are the same. Hence the differential equation (1) is satisfied by the parabola

$$x^2 = 4ay.$$

Ex. 19. *Prove that for the curve*

$$s = a(\sec^3\psi - 1), \quad \rho = 3a\tan\psi\sec^3\psi$$

and hence prove that $3a\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = 1$. *Also prove that this differential equation is satisfied by the curve* $27ay^2 = 8x^3$.

Sol. We have $s = a(\sec^3\psi - 1)$.

$$\therefore \quad \rho = ds/d\psi = a(3\sec^2\psi\sec\psi\tan\psi) = 3a\tan\psi\sec^3\psi. \qquad \text{...(1)}$$

Also $\rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{(1 + \tan^2\psi)^{3/2}}{d^2y/dx^2} = \dfrac{\sec^3\psi}{d^2y/dx^2}.$

$[\because dy/dx = \tan\psi]$

$$\therefore \quad \frac{d^2y}{dx^2}=\frac{\sec^3\psi}{\rho}=\frac{\sec^3\psi}{3a\tan\psi\sec^3\psi}, \qquad \text{from (1)}$$

$$=\frac{1}{3a\,(dy/dx)}. \quad \therefore \quad 3a\frac{dy}{dx}\frac{d^2y}{dx^2}=1. \qquad ...(2)$$

Now for the curve $27ay^2=8x^3$, we have on differentiating w.r.t. x, $54ay\,(dy/dx)=24x^2$

$$\text{or} \quad \frac{dy}{dx}=\frac{4x^2}{9ay}, \quad \text{or} \quad \left(\frac{dy}{dx}\right)^2=\frac{16x^4}{81a^2y^2}=\frac{16x^4}{81a\,(8x^3/27)}.$$

$$[\because ay^2=8x^3/27]$$

$$\text{Thus} \quad \left(\frac{dy}{dx}\right)^2=\frac{(16x^4)\,.\,27}{(81a)\,.\,8x^3}=\frac{2x}{3a} \qquad ...(3)$$

Differentiating w.r.t. x, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}=\frac{2}{3a} \quad \text{or} \quad 3a\frac{dy}{dx}\frac{d^2y}{dx^2}=1. \qquad ...(4)$$

The differential equations (2) and (4) are the same. Hence the differential equation (2) is satisfied by the curve $27ay^2=8x^3$.

§ 4. Radius of curvature for parametric curves.

(Kanpur 1978; Meerut 77)

Let the curve be defined by means of parametric equations

$$x=f(t) \quad \text{and} \quad y=\phi(t).$$

$$\text{Then} \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{y'}{x'}.$$

(Here dashes or accents denote differentiation w.r.t. 't'.)

$$\text{Also,} \ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{y'}{x'}\right)=\left\{\frac{d}{dt}\left(\frac{y'}{x'}\right)\right\}\frac{dt}{dx}$$

$$=\frac{x'y''-y'x''}{(x')^2}\cdot\frac{1}{x'}, \qquad \left[\because \frac{dx}{dt}=x' \text{ and } \frac{dt}{dx}=\frac{1}{x'}\right]$$

$$=\frac{x'y''-y'x''}{(x')^3}.$$

$$\therefore \quad \rho=\frac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}=\frac{[1+(y'/x')^2]^{3/2}}{(x'y''-y'x'')/(x')^3}$$

$$=\frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}, \quad \text{where } x'y''-y'x''\neq 0.$$

Note. Curvature $=\dfrac{1}{\rho}=\dfrac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}.$

Solved Examples

***Ex. 20 (a).** *Find the radius of curvature at any point of the cycloid $x=a\,(t+\sin t)$, $y=a\,(1-\cos t)$.* **(Meerut 1981, 88S, 93, 98; Gorakhpur 80, 78; Delhi 87; Rohilkhand 89)**

Sol. We have $x=a\,(t+\sin t)$, $y=a\,(1-\cos t)$.

$\therefore \quad dx/dt = a\,(1 + \cos t),\ dy/dt = a \sin t.$

Now $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \sin t}{a\,(1 + \cos t)} = \dfrac{2 \sin \frac{1}{2} t \cos \frac{1}{2} t}{2 \cos^2 \frac{1}{2} t} = \tan \frac{1}{2} t.$

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\tan \frac{1}{2} t) = \left\{\dfrac{d}{dt}(\tan \frac{1}{2} t)\right\} \cdot \dfrac{dt}{dx}$

$= \frac{1}{2} \sec^2 \frac{1}{2} t \cdot \dfrac{1}{a\,(1 + \cos t)} = \frac{1}{2} \sec^2 \frac{1}{2} t \cdot \dfrac{1}{2a \cos^2 \frac{1}{2} t} = \dfrac{1}{4a} \sec^4 \frac{1}{2} t.$

Hence $\rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{(1 + \tan^2 \frac{1}{2} t)^{3/2}}{(1/4a) \sec^4 \frac{1}{2} t} = \dfrac{4a \sec^3 \frac{1}{2} t}{\sec^4 \frac{1}{2} t}$

$= 4a \cos \frac{1}{2} t.$

Note. This question can also by done by using the formula of § 4. Thus here

$x' = dx/dt = a\,(1 + \cos t),\ x'' = d^2x/dt^2 = -a \sin t,$
$y' = dy/dt = a \sin t,\ y'' = d^2y/dt^2 = a \cos t.$

Now $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \dfrac{[a^2 (1 + \cos t)^2 + a^2 \sin^2 t]^{3/2}}{a\,(1 + \cos t)\, a \cos t - a \sin t\,(-a \sin t)}$

$= a\,\dfrac{[2\,(1 + \cos t)]^{3/2}}{1 + \cos t} = (2\sqrt{2})\, a\,(1 + \cos t)^{1/2}$

$= (2\sqrt{2})\, a \,.\, (2 \cos^2 \frac{1}{2} t)^{1/2} = 4a \cos \frac{1}{2} t.$

Ex. 20 (b). *Find the radius of curvature at the vertex of the cycloid* $x = a\,(\theta + \sin \theta),\ y = a\,(1 - \cos \theta).$

Sol. Proceeding as in Ex. 20 (a), we get $\rho = 4a \cos \frac{1}{2} \theta.$

At the vertex $\theta = 0$ and $\rho = 4a \cos 0 = 4a.$

Ex. 20 (c). *Prove that in the curve*

$4x = \theta + \sin \theta,\ 4y = 1 - \cos \theta,\ \rho = \cos \frac{1}{2} \theta.$ **(Rohilkhand 1976)**

Sol. Proceed exactly as in Ex. 20 (a).

Ex. 21. *Find the radius of curvature at any point 'θ' of the curve*

(i) $x = a \cos^3 \theta,\ y = a \sin^3 \theta.$ **(Meerut 1975, 77, 85, 86; Rohilkhand 86)**

(ii) $x = a \cos \theta,\ y = b \sin \theta.$ **(Gorakhpur 1972)**

(iii) $x = a\,(\theta - \sin \theta),\ y = a\,(1 - \cos \theta).$

(iv) $x = a\,(\cos \theta + \log \tan \frac{1}{2} \theta),\ y = a \sin \theta.$

Sol. (i) For complete solution refer Example 5, part (iv) on page 210.

(ii) For complete solution refer Ex. 12 on page 214.

(iii) We have $x = a\,(\theta - \sin \theta),\ y = a\,(1 - \cos \theta).$

$\therefore \quad dx/d\theta = a\,(1 - \cos \theta) = 2a \sin^2 \frac{1}{2} \theta,$

and $\quad dy/d\theta = a \sin \theta = 2a \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta.$

Now $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{2a \sin \frac{1}{2}\theta \,.\, \cos \frac{1}{2}\theta}{2a \sin^2 \frac{1}{2}\theta} = \cot \frac{1}{2}\theta.$

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(\cot \frac{1}{2}\theta) = \dfrac{d}{d\theta}(\cot \frac{1}{2}\theta) \cdot \dfrac{d\theta}{dx}$

$$= -\tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2}\theta \cdot \frac{1}{2a \sin^2 \frac{1}{2}\theta} = -\frac{1}{4a} \operatorname{cosec}^4 \tfrac{1}{2}\theta.$$

$$\therefore \quad \rho = \frac{[1 + \cot^2 \frac{1}{2}\theta]^{3/2}}{-(1/4a) \operatorname{cosec}^4 \frac{1}{2}\theta} = -\frac{4a}{\operatorname{cosec} \frac{1}{2}\theta} = 4a \sin \tfrac{1}{2}\theta.$$

(Negative sign being neglected).

(iv) We have $x = a(\cos\theta + \log \tan \frac{1}{2}\theta),\ y = a \sin\theta.$

$$\therefore \quad \frac{dx}{d\theta} = -a \sin\theta + a \cdot \frac{1}{\tan \frac{1}{2}\theta} \cdot \tfrac{1}{2} \sec^2 \tfrac{1}{2}\theta$$

$$= -a \sin\theta + \frac{a}{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta} = -a \sin\theta + \frac{a}{\sin\theta} = \frac{a \cos^2\theta}{\sin\theta}$$

and $dy/d\theta = a \cos\theta.$

Now $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{a \cos\theta}{a \cos^2\theta/\sin\theta} = \tan\theta.$

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}(\tan\theta) \cdot \dfrac{d\theta}{dx} = \sec^2\theta \cdot \dfrac{\sin\theta}{a\cos^2\theta} = \dfrac{\sin\theta}{a \cos^4\theta}.$

$$\therefore \quad \rho = \frac{(1 + \tan^2\theta)^{3/2}}{\sin\theta/(a \cos^4\theta)} = \frac{a \cos^4\theta}{\sin\theta} \cdot \sec^3\theta = a\cos\theta.$$

Ex. 22. *If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of the parabola $y^2 = 4ax$, prove that*

$$(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}.$$

Sol. Let the extremities of a focal chord be the points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$.

Then by co-ordinate geometry, $t_1 t_2 = -1.$ ...(1)

Also $x = at^2, y = 2at$ are the parametric equations of the parabola $y^2 = 4ax$.

We have $x' = dx/dt = 2at, \quad y' = dy/dt = 2a,$
$x'' = d^2x/dt^2 = 2a, \quad y'' = d^2y/dt^2 = 0.$

$$\therefore \quad \rho \text{ at the point '}t\text{'} = \frac{\{(x')^2 + (y')^2\}^{3/2}}{(x'y'' - y'x'')},$$

[See formula of § 4, page 219]

$$= \frac{[4a^2t^2 + 4a^2]^{3/2}}{2at \,.\, 0 - 2a \,.\, 2a} = \frac{8a^3(1 + t^2)^{3/2}}{-4a^2}$$

$= 2a(1 + t^2)^{3/2},$ [Negative sign being neglected].

$\therefore \quad \rho_1$ = radius of curvature at the point $P(at_1^2, 2at_1)$

$= 2a\,(1 + t_1^2)^{3/2},$

and ρ_2 = radius of curvature at the point $Q\,(at_2^2, 2at_2) = 2a\,(1 + t_2^2)^{3/2}$.

Hence $\quad (\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}\left[\dfrac{1}{1+t_1^2} + \dfrac{1}{1+t_2^2}\right]$

$$= (2a)^{-2/3}\left[\frac{t_1^2 + t_2^2 + 2}{t_1^2 + t_2^2 + 1 + 1}\right] \qquad [\because t_1^2 t_2^2 = 1]$$

$= (2a)^{-2/3}$.

§ 5. Radius of curvature for pedal curves.

(Meerut 1983, 90, 91, 96, 96P, 96 BP, 97; Gorakhpur 80, 77; Rohilkhand 82, 91; Agra 81, 79; Kanpur 77)

We have $\quad p = r \sin \phi$.

Differentiating w.r.t. r, we get

$$dp/dr = \sin \phi + r \cos \phi\,(d\phi/dr)$$

$$= r\frac{d\theta}{ds} + r\frac{d\phi}{dr}\frac{dr}{ds}, \qquad \left[\because \sin \phi = r\frac{d\theta}{ds} \text{ and } \cos \phi = \frac{dr}{ds}\right]$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right) = r\frac{d}{ds}(\theta + \phi) = r\frac{d\psi}{ds}, \qquad [\because \psi = \theta + \phi]$$

$$= r\,.\,(1/\rho), \qquad [\because \rho = ds/d\psi].$$

Thus $\quad \dfrac{dp}{dr} = \dfrac{r}{\rho}\cdot$ Therefore $\boldsymbol{\rho = r\dfrac{dr}{dp}}$.

Solved Examples

Ex. 23. *Find the radius of curvature at any point of the ellipse*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}.$$

(Delhi 1982)

Sol. Differentiating the given equation w.r.t. 'p', we have

$$-\frac{2}{p^3} = -\frac{2r}{a^2b^2}\frac{dr}{dp}; \quad \therefore \quad \rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3}.$$

Ex. 24. *Find the radius of curvature at the point* (p, r) *on the following curves:*

(i) $p^2 = ar$, **(ii)** $r^3 = 2ap^2$,

(iii) $r^3 = a^2p$. **(Meerut 1977)**

Sol. (i) We have $\quad p^2 = ar.$ ...(1)

Differentiating (1) w.r.t. p, we get

$$2p = a\frac{dr}{dp}; \quad \therefore \quad \frac{dr}{dp} = \frac{2p}{a}.$$

Now $\quad \rho = r\dfrac{dr}{dp} = \dfrac{2pr}{a} = \dfrac{2p}{a}\left(\dfrac{p^2}{a}\right)$, from (1)

$= 2p^3/a^2$.

(ii) We have $\quad r^3 = 2ap^2.$ ...(1)

Differentiating (1) w.r.t. p, we have

$$3r^2\frac{dr}{dp} = 4ap \quad \text{or} \quad r\frac{dr}{dp} = \frac{4ap}{3r}.$$

$$\therefore \qquad \rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\sqrt{\left(\frac{r^3}{2a}\right)}, \qquad \text{from (1)}$$

$$= \tfrac{2}{3}\sqrt{(2ar)}.$$

(iii) We have $\quad r^3 = a^2 p.$ ...(1)

Differentiating (1) w.r.t. p, we get

$$3r^2(dr/dp) = a^2; \quad \therefore \quad \rho = r(dr/dp) = a^2/3r.$$

Ex. 25. *Find the radius of curvature at the point* (p, r) *on the spiral* $p^2 = r^4/(r^2 + a^2)$.

Sol. The equation of the curve is $p^2 = r^4/(r^2 + a^2)$. ...(1)

$$\therefore \quad \frac{1}{p^2} = \frac{r^2 + a^2}{r^4} = \frac{r^2}{r^4} + \frac{a^2}{r^4} = \frac{1}{r^2} + \frac{a^2}{r^4}.$$

Now differentiating both sides w.r.t. r, we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4a^2}{r^5} = -2\cdot\frac{r^2 + 2a^2}{r^5}.$$

$$\therefore \quad \frac{dp}{dr} = p^3\frac{(r^2 + 2a^2)}{r^5} = (p^2)^{3/2}\frac{(r^2 + 2a^2)}{r^5} \qquad \textbf{(Note)}$$

$$= \left(\frac{r^4}{r^2 + a^2}\right)^{3/2}\frac{(r^2 + 2a^2)}{r^5}, \qquad \text{from (1)}$$

$$= \frac{r(r^2 + 2a^2)}{(r^2 + a^2)^{3/2}}.$$

Hence $\rho = r\dfrac{dr}{dp} = r\cdot\dfrac{(r^2 + a^2)^{3/2}}{r(r^2 + 2a^2)} = \dfrac{(r^2 + a^2)^{3/2}}{r^2 + 2a^2}.$

Ex. 26. *Find the radius of curvature of the curve* $p = r^{m+1}/a^m$; *show that the radius of curvature varies inversely as the* $(m-1)^{th}$ *power of the radius vector.*

Sol. We have $\quad a^m p = r^{m+1}.$ ...(1)

Differentiating (1) w.r.t. p, we get

$$a^m = (m+1)r^m\frac{dr}{dp}. \quad \therefore \quad \frac{dr}{dp} = \frac{a^m}{(m+1)r^m}.$$

Now $\rho = r\dfrac{dr}{dp} = \dfrac{ra^m}{(m+1)r^m} = \dfrac{a^m}{(m+1)r^{m-1}}.$

$\therefore \rho \propto \dfrac{1}{r^{m-1}}$ *i.e.*, the radius of curvature varies inversely as the $(m-1)^{th}$ power of the radius vector.

§ 6. Radius of curvature for polar curves.

(Gorakhpur 1981, 79; Gurunanak 71; Meerut 70)

We know that $$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. 'r', we get

$$-\frac{2}{p^3}\cdot\frac{dp}{dr}=-\frac{2}{r^3}-\frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2+\frac{1}{r^4}2\left(\frac{dr}{d\theta}\right)\cdot\frac{d}{dr}\left(\frac{dr}{d\theta}\right)$$

$$=-\frac{2}{r^3}-\frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2+\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)\left\{\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)\right\}\cdot\frac{d\theta}{dr}$$

$$=-\frac{2}{r^3}-\frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2+\frac{2}{r^4}\frac{d^2r}{d\theta^2}. \qquad \left[\because \frac{dr}{d\theta}\cdot\frac{d\theta}{dr}=1\right]$$

$$\therefore \quad \frac{1}{p^3}\frac{dp}{dr}=\frac{1}{r^3}+\frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^4}\frac{d^2r}{d\theta^2}.$$

Now $$\rho=r\frac{dr}{dp}=\frac{r\,(1/p^3)}{(1/r^3)+(2/r^5)\,(dr/d\theta)^2-(1/r^4)\,(d^2r/d\theta^2)}$$

$$=\frac{r^6\,[(1/r^2)+(1/r^4)\,(dr/d\theta)^2]^{3/2}}{r^2+2\,(dr/d\theta)^2-r\,(d^2r/d\theta^2)}.$$

$$[\because \text{from (1)},\ 1/p^3=\{(1/r^2)+(1/r^4)\,(dr/d\theta)^2\}^{3/2}].$$

$$\therefore \quad \rho=\frac{\mathbf{[r^2+(dr/d\theta)^2]^{3/2}}}{\mathbf{r^2+2\,(dr/d\theta)^2-r\,(d^2r/d\theta^2)}}.$$

Cor. If we put $r=1/u$, we have

$$\frac{dr}{d\theta}=-\frac{1}{u^2}\frac{du}{d\theta} \quad \text{and} \quad \frac{d^2r}{d\theta^2}=-\frac{1}{u^2}\frac{d^2u}{d\theta^2}+\frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2.$$

$$\therefore \quad \rho=\frac{[r^2+(dr/d\theta)^2]^{3/2}}{r^2+2\,(dr/d\theta)^2-r\,(d^2r/d\theta^2)}$$

$$=\frac{\left[\dfrac{1}{u^2}+\dfrac{1}{u^4}\left(\dfrac{du}{d\theta}\right)^2\right]^{3/2}}{\dfrac{1}{u^2}+2\cdot\dfrac{1}{u^4}\left(\dfrac{du}{d\theta}\right)^2-\dfrac{1}{u}\left[-\dfrac{1}{u^2}\cdot\dfrac{d^2u}{d\theta^2}+\dfrac{2}{u^3}\left(\dfrac{du}{d\theta}\right)^2\right]}$$

$$=\frac{1}{u^6}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]^{3/2}\Big/\left[\frac{1}{u^2}+\frac{1}{u^3}\frac{d^2u}{d\theta^2}\right].$$

Hence $$\rho=\frac{\mathbf{[u^2+(du/d\theta)^2]^{3/2}}}{\mathbf{u^3\,[u+d^2u/d\theta^2]}}.$$

Important Note. We see that the pedal formula for ρ is simpler than the polar formula. Therefore if we are given the equation of a curve in polar form, it is often convenient to change it first to pedal equation and then to find ρ with the help of pedal formula.

Asymptotes parallel to y-axis are $x^2 - 1 = 0$ or $x = \pm 1$.

(v) When $0 < x < 1, y$ is negative and less than -1.

At $x = 0, y = -1$. When $x \to 1$ from the left, $y \to -\infty$ and when $x \to 1$ from the right $y \to \infty$.

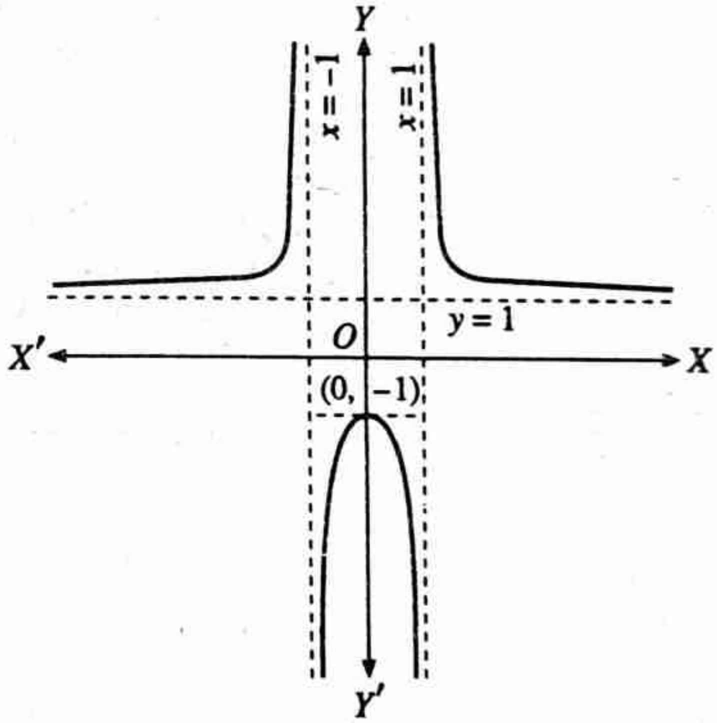
When $x > 1, y$ is positive and greater than 1.

Hence when $0 < x < 1$, the curve is in the IVth quadrant below the line $y = -1$

and when $x > 1$, the curve is in the Ist quadrant above the line $y = 1$.

(iv) Special points :

$x = 0$	$1/2$	2	3	$\to \infty$
$y = -1$	$-5/3$	$5/3$	$5/4$	$\to 1$



$\therefore$ the curve is as shown in the figure.

Ex. 8. *Trace the curve* $ay^2 = x^2 (a - x)$.

(Agra 1982; Gorakhpur 78; Meerut 92)

Sol. We have $x^3 + a (y^2 - x^2) = 0$. ...(1)

(i) The curve is symmetrical about x-axis.

(ii) The curve passes through $(0, 0)$.

Tangents at the origin are $y^2 - x^2 = 0$ or $y = \pm x$.

(iii) When $x = 0, y = 0$; when $y = 0, x = 0, a$.

Hence the curve cuts the coordinates axes at the points $(0, 0)$, $(a, 0)$.

Differentiating (1), $2ay\,(dy/dx) = 2ax - 3x^2$.

At $(a, 0)$, $dy/dx = -\infty$; $\therefore$ tangent at $(a, 0)$ is parallel to y-axis.

(iv) No asymptotes.

(v) From (1), $y = \pm x\sqrt{\{1 - (x/a)\}}$.

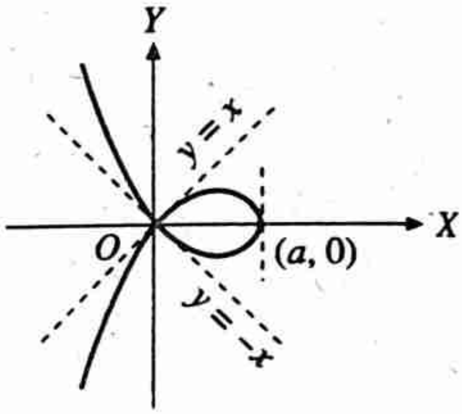
At $x = 0, y = 0$.

When $0 < x < a$, y is real and is numerically less than x. Hence the curve exists in the region $0 < x < a$ and in the first quadrant it lies below the line $y = x$.

When $x > a$, y is imaginary and therefore the curve does not lie on the right hand side of the line $x = a$.

When $x < 0$, y is real and is numerically greater than x. Hence the curve exists when $x < 0$ and in the third quadrant it lies below the line $y = x$.

When $x \to -\infty, y \to \pm\infty$.



First trace the curve in the Ist and IIIrd quadrants and then by symmetry about x-axis in IInd and IVth quadrants. The shape is as shown in the figure.

Ex. 9. *Trace the curve* $9ay^2 = x(x - 3a)^2$. **(Gorakhpur 1980)**

Sol. We have $9ay^2 = x(x - 3a)^2 = x(x^2 - 6ax + 9a^2)$. ...(1)

(i) The curve is symmetrical about x-axis.

(ii) The curve passes through $(0, 0)$. Tangent at origin is $x = 0$ *i.e.*, the y-axis.

(iii) When $y = 0, x = 0$ and $3a$; when $x = 0, y = 0$.

Hence the curve cuts the coordinate axes at the points $(0, 0)$ and $(3a, 0)$.

Transferring the origin to $(3a, 0)$ the equation of the curve becomes

$$9ay^2 = (x + 3a)(x + 3a - 3a)^2,$$

(putting $x + 3a$ for x and $y + 0$ for y)

or $$9ay^2 = (x + 3a)x^2.$$

Tangents at the new origin are $9ay^2 = 3ax^2$ or $y = \pm(1/\sqrt{3})x$.

Hence there is a node at the new origin *i.e.*, at the point $(3a, 0)$ on (1) and the two branches of the curve cross at this point.

Tangents at $(3a, 0)$ are parallel to the lines $y = \pm x/\sqrt{3}$ each of which is inclined at an angle of 30° to the x-axis.

Again, the curve is $y = (x - 3a)x^{1/2}/(3a^{1/2})$.

$\therefore\quad dy/dx = \{1/(3a^{1/2})\}\{x^{1/2} + (x - 3a)(\frac{1}{2}x^{-1/2})\}$

$$= (3x - 3a)/(6x^{1/2}a^{1/2}).$$

Therefore at $x = a$, $(dy/dx) = 0$ *i.e.*, the tangent to the curve is parallel to the x-axis.

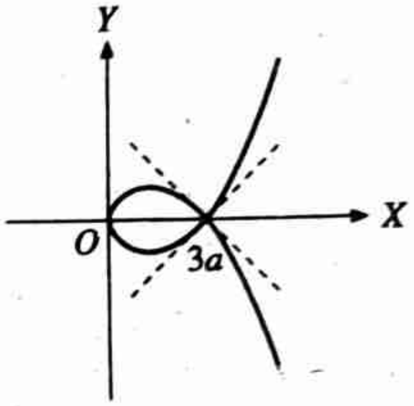
(iv) No asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = \{x(x-3a)^2\}/9a.$$

When $x = 0, y^2 = 0$ and when $x = 3a, y^2 = 0$. When $0 < x < 3a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region. When $x > 3a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty$, $y^2 \to \infty$.

When $x < 0, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in the region $x < 0$. Thus the curve does not lie on the left hand side of the y-axis.



(vi) Special points :

x	0	a	$3a$	$4a$	$9a$	$\to \infty$
y	0	$\pm \frac{2}{3}a$	0	$\pm \frac{2}{3}a$	$\pm 6a$	$\to \pm \infty$

The curve is as shown in the figure.

Ex. 10. *Trace the curve* $x^2y^2 = (1+y)^2(4-y^2)$. ...(1)

Sol. (i). The curve is symmetrical about y-axis.

(ii) The curve does not pass through $(0, 0)$.

(iii) When $x = 0, y = -1, \pm 2$; when $y = 0$ we get no value of x.

Hence the curve cuts the coordinate axes at the points $(0, -1), (0, \pm 2)$.

Shifting the origin to the point $(0, -1)$ the equation of the curve becomes

$$x^2(y-1)^2 = \{1 + (y-1)\}^2 \{4 - (y-1)^2\}$$

or $$x^2(y^2 - 2y + 1) = y^2(3 + 2y - y^2).$$

Therefore the tangents at this new origin are $x^2 = 3y^2$ *i.e.*, $y = \pm(1/\sqrt{3})x$. Thus the given curve has a node at the point $(0, -1)$ and the tangents to the two branches of the curve at this point make angles $\pm \frac{1}{6}\pi$ with x-axis.

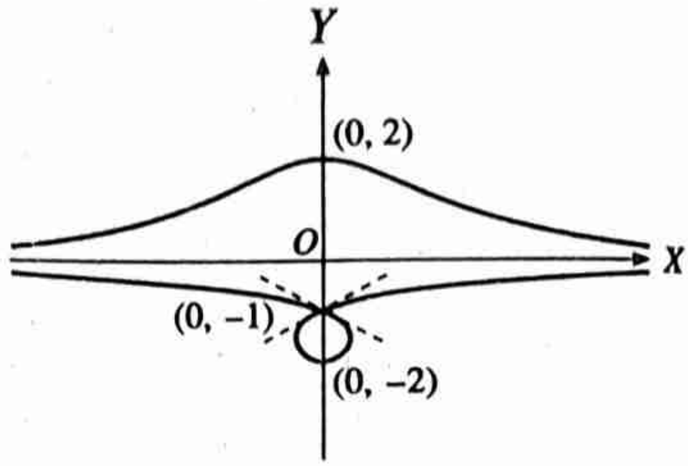
Now shifting the origin to the point $(0, 2)$ the given equation of the curve becomes

$$x^2(y+2)^2 = \{1 + (y+2)\}^2 \{4 - (y+2)^2\}$$

or $$x^2(y^2 + 4y + 4) = (3+y)^2(-y^2 - 4y).$$

Therefore the tangent at this new origin is $y = 0$. Thus at the point $(0, 2)$ on the given curve the tangent is parallel to x-axis.

Again shifting the origin to the point $(0, -2)$ the given equation of the curve becomes $x^2(y-2)^2 = \{1+(y-2)\}^2\{4-(y-2)^2\}$ or $x^2(y-2)^2 = (y-1)^2(-y^2+4y)$. Therefore the tangent at this new origin is $y = 0$. Thus at the point $(0, -2)$ on the given curve the tangent is parallel to x-axis.



(iv) Asymptotes prallel to x-axis are given by $y^2 = 0$ *i.e.*, x-axis is an asymptote and we note that the curve has no other asymptotes.

(v) Solving the equation of the curve for x, we get

$$x^2 = \{(1+y)^2(4-y^2)\}/y^2.$$

When $y \to 0, x^2 \to \infty$. When $y = 2, x = 0$. When $0 < y < 2, x$ is real and so the curve exists in this region. When $y > 2, x$ is imaginary and so the curve does not exist in this region.

When $y = -1, x = 0$ and when $y = -2$ again $x = 0$. When $-2 < y < 0, x$ is real and so the curve exists in this region. But when $y < -2, x$ is imaginary and so the curve does not exist in this region.

Tracing the curve from the above data the curve is as shown in the figure.

Ex. 11. *Trace the curve* $a^3y^2 = (x-a)^4(x-b), a > b$. ...(1)

Sol. (i). The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x = a, b$; when $x = 0, y$ is imaginary.

Hence the curve cuts the coordinate axes at the points $(a, 0)$ and $(b, 0)$.

Shifting the origin to the point $(a, 0)$ the equation of the curve becomes $a^3y^2 = x^4(x+a-b)$. Therefore the tangents at this new origin are $a^3y^2 = 0$ *i.e.*, $y = 0, y = 0$. Thus the new x-axis is tangent at this new origin and this new origin may be a cusp.

Again shifting the origin to the point $(b, 0)$ the equation of the curve becomes $a^3y^2 = (x+b-a)^4 x$.

Therefore the tangent at this new origin is $x = 0$. Thus the new y-axis is tangent at this new origin.

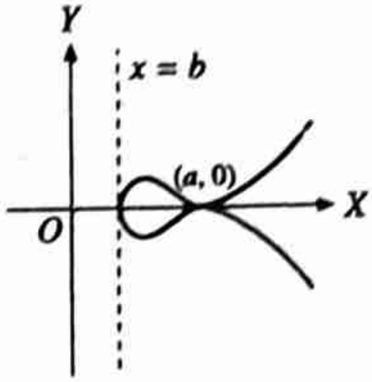
(iv) The curve has no asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = \frac{(x-a)^4 (x-b)}{a^3}.$$

When $x = 0, y$ is imaginary and when $x = b, y = 0$. When $0 < x < b, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x = a, y = 0$ and when $b < x < a, y$ is real and so the curve exists in this region. When $x > a, y$ is real and so the curve exists in this region and when $x \to \infty, y^2 \to \infty$.



When $x < 0, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

Hence the curve is as shown in the figure.

***Ex. 12.** *Trace the curve* $xy^2 = 4a^2(2a - x)$. **(Witch of Agnesi)**

Sol. (i). The curve is symmetrical about the axis of x.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, 2a - x = 0$ or $x = 2a$.

$\therefore$ the curve crosses the x-axis at $(2a, 0)$.

When $x = 0$, we do not get any value of y and so the curve does not meet the y-axis.

Shifting the origin to $(2a, 0)$ the equation to the curve becomes

$$(x + 2a)y^2 = 4a^2(2a - x - 2a)$$

or $$y^2x + 2ay^2 + 4a^2x = 0. \qquad \text{...(1)}$$

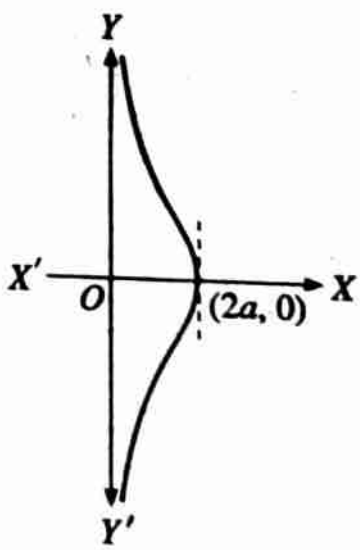
Equating to zero, the lowest degree terms in (1), the equation of tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

Thus the tangent to the given curve at the point $(2a, 0)$ is parallel to y-axis.

(iv) The asymptote of the curve parallel to y-axis is $x = 0$ *i.e.*, the y-axis and we note that the curve has no other real asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = \{4a^2(2a - x)\}/x.$$



When $x \to 0$ from the right $y^2 \to \infty$ showing that the line $x = 0$ is an asymptote. When $x = 2a, y = 0$. When $0 < x < 2a, y$ is real and so the curve exists in this region. When $x > 2a, y$ is imaginary and the curve does not exist for $x > 2a$. When $x < 0, y$ is again imaginary and so the curve does not exist in the region where x is –ive. When x decreases from $2a$ to $0, y^2$ increases from 0 to ∞. Thus the shape of the curve is as shown in the figure.

Ex. 13. *Trace the curve* $x^2y^2 = a^2(x^2 + y^2)$. **(Meerut 1989 S, 90)**

Sol. (i) The curve is symmetrical about both the axes.

(ii) The curve passes through the origin.

Equating to zero, the lowest degree terms of the given curve, the tangents at the origin are $x^2 + y^2 = 0$ or $y^2 = -x^2$ *i.e.*, two imaginary tangents. Hence (0, 0) is a conjugate point *i.e.*, an isolated point.

(iii) The curve does not meet the coordinate axes.

(iv) Equating to zero the coefficients of highest powers of x and y, the asymptotes parallel to the coordinate axes are $x = \pm a$ and $y = \pm a$.

(v) Solving the given equation for y, we get

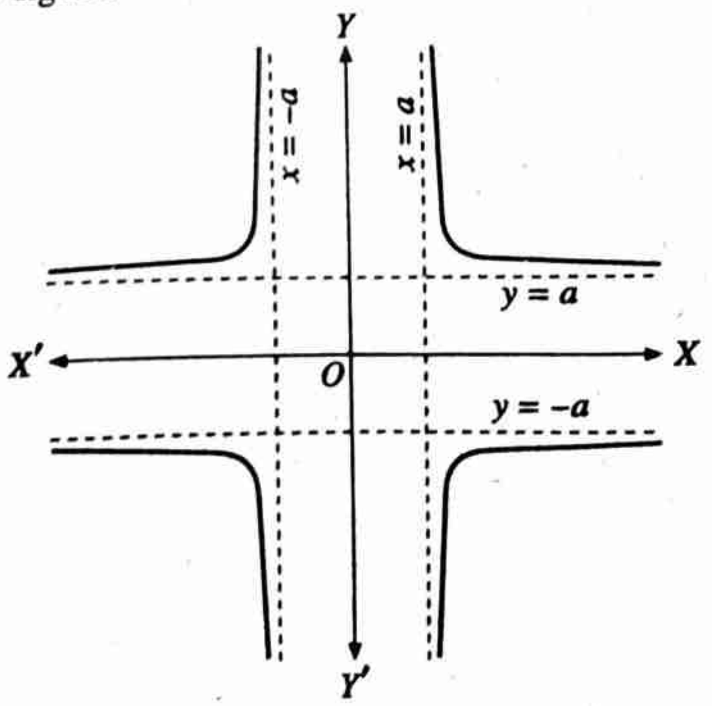
$$y^2 = a^2x^2/(x^2 - a^2).$$

When $0 < x < a, y^2$ is negative *i.e.*, y is imaginary and so the curve does not exist in this region. When $x \to a$ from the right, $y^2 \to \infty$.

When $x > a, y$ is real and so the curve exists in this region. When $x \to \infty, y^2 \to a^2$ showing the fact that the lines $y = \pm a$ are asymptotes of the curve.

Similarly, we find that the curve does not exist in the region $0 < y < a$. It exists in the region where $y > a$ and as $y \to \infty$, $x^2 \to a^2$ showing the fact that the lines $x = \pm a$ are asymptotes of the curve.

Combining all these facts we see that the shape of the curve is as shown in the figure.



**Ex. 14. *Trace the curve $a^2y^2 = x^2(a^2 - x^2)$.* (Delhi 1982)

Sol. **(i)** The curve is symmetrical about both the axes.

(ii) The curve passes through the origin.

Equating to zero the lowest degree terms in the given equation, tangents at the origin are given by $a^2y^2 - a^2x^2 = 0$ or $y = \pm x$. These being real and distinct, origin is a node.

(iii) When $y = 0, x = 0, \pm a$ *i.e.*, the curve crosses the x-axis at $(0,0), (a,0)$ and $(-a,0)$. When $x = 0, y = 0$ and so the curve meets the y-axis only at the origin. Shifting the origin to $(a, 0)$, the equation of the curve becomes

$$a^2y^2 = (x+a)^2[a^2 - (x+a)^2]$$

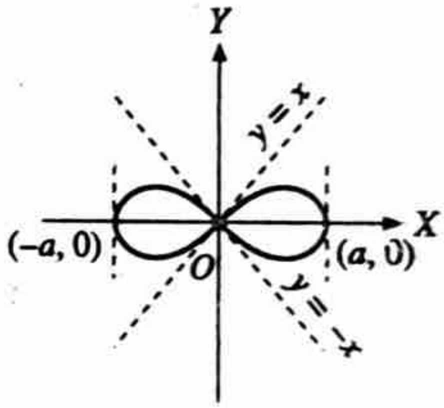
or
$$a^2y^2 = (x+a)^2[-2ax - x^2].$$

$\therefore$ The tangent at the new origin $(a, 0)$ is $x = 0$ *i.e.*, the new y-axis.

(iv) Solving the equation of the curve for y, we have

$$y^2 = \{x^2(a^2 - x^2)\}/a^2.$$

When $x = 0, y = 0$ and when $x = a, y = 0$. When $0 < x < a, y$ is real and so the curve exists in this region. When $x > a, y^2$ is negative or y is imaginary and so the curve does not exist in the region where $x > a$.



(v) No asymptotes.

Thus the shape of the curve is as shown in the figure.

Ex. 15. *Trace the curve $a^2y^2 = x^3(2a - x)$.*

Sol. **(i)** The curve is symmetrical about x-axis.

(ii) The curve passes through the origin and tangents at the origin are $y^2 = 0$ *i.e.*, $y = 0, y = 0$. So a cusp is expected at the origin.

(iii) When $y = 0, x = 0, 2a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$ and $(2a, 0)$. When $x = 0, y = 0$ and so the curve meets the y-axis only at the origin. Shifting the origin to $(2a, 0)$, the equation of the curve transforms to

$$a^2y^2 = (x + 2a)^3(2a - x - 2a) = -x(x + 2a)^3.$$

$\therefore$ tangent at the new origin $(2a, 0)$ is $x = 0$ *i.e.*, the new y-axis.

(iv) No asymptotes.

(v) Solving for y, the equation of curve is

$$y^2 = x^3(2a - x)/a^2.$$

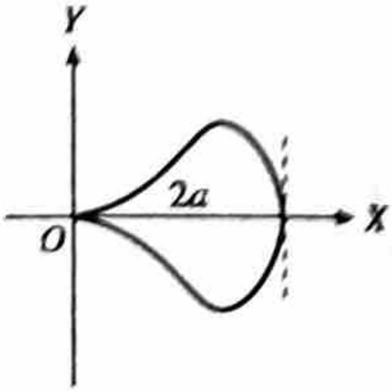
When $x = 0, y = 0$ and when $x = 2a, y = 0$. When $0 < x < 2a, y^2$ is $+$ive *i.e.*, y is real and so the curve exists in this region.

When $x > 2a, y^2$ is negative *i.e.*, y is imaginary and so the curve does not exist for $x > 2a$.

When $x < 0, y^2$ is negative *i.e.*, y is imaginary and so the curve does not exist in the region where x is –ive.

Thus the curve exists only for values of x from $x = 0$ to $x = 2a$.

Thus the shape of the curve is as shown in the figure.



***Ex. 16.** *Trace the curve* $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

(Delhi 1980; Meerut 86 S, 89, 96)

or $$x^2(x^2 + y^2) = a^2(x^2 - y^2). \quad ...(1)$$

(Agra 1983)

Sol. (i) The curve is symmetrical about both the axes.

(ii) The curve passes through the origin. The tangents at the origin are $y^2 = x^2$ or $y = \pm x$. These being real and distinct the origin is a node.

(iii) When $y = 0, x = 0, \pm a$ *i.e.*, the curve crosses the x-axis at $(0, 0), (a, 0)$ and $(-a, 0)$. When $x = 0, y = 0$ and so the curve cuts the y-axis only at the origin.

Shifting the origin to $(a, 0)$, the equation of the curve transforms to

$$y^2\{a^2 + (x + a)^2\} = (x + a)^2\{a^2 - (x + a)^2\}$$

or $$y^2(x^2 + 2ax + 2a^2) = (x + a)^2(-x^2 - 2ax). \quad ...(2)$$

Equating to zero, the lowest degree terms in (2), the tangent at the new origin is given by $x = 0$, *i.e.*, the new y-axis.

(iv) Solving for y, we get $y^2 = x^2(a^2 - x^2)/(a^2 + x^2)$.

When $x = 0, y = 0$ and when $x = a, y = 0$. When $0 < x < a$, y is real and so the curve exists in this region.

When $x > a, y^2$ is negative *i.e.*, y is imaginary and so the curve does not exist for $x > a$.

(v) No asymptotes.

Combining all these facts, we see that the shape of the curve is as shown in the figure of Ex. 14.

***Ex. 17.** *Trace the curve* $y^2(2a - x) = x^3$. **(Cissoid)**

(Gorakhpur 1976; Rohilkhand 76; Delhi 81; Meerut 84, 88, 97)

Sol. We note the following particulars about the curve :

(i) It is symmetrical about the axis of x, since the powers of y that occur are all even.

(ii) The curve passes through the origin and the tangents at the origin are $2ay^2 = 0$ *i.e.*, $y = 0, y = 0$ are two coincident tangents at the origin. Therefore the origin may be a cusp.

(iii) The curve meets the coordinate axes only at the origin.

(iv) Solving the equation of the curve for y, we get

$$y^2 = x^3/(2a - x).$$

When $x = 0, y^2 = 0$. When $x \to 2a, y^2 \to \infty$. Therefore $x = 2a$ is an asymptote of the curve.

When $0 < x < 2a, y^2$ is positive *i.e.*, y is real. Therefore the curve exists in this region.

When $x > 2a, y^2$ is negative *i.e.*, y is imaginary. Therefore the curve does not exist in the region $x > 2a$. When $x < 0, y^2$ is negative. Therefore the curve does not exist in the region $x < 0$. Since the curve exists in the neighbourhood of origin where $x > 0$, therefore there is a single cusp at the origin.

(v) The curve has an asymptote parallel to y-axis and it is the line $x - 2a = 0$ *i.e.*, $x = 2a$. Putting $y = m$ and $x = 1$ in the third degree terms in the equation of the curve, we get $\phi_3(m) = m^2 + 1$. The roots of the equation $m^2 + 1 = 0$ are imaginary, therefore $x = 2a$ is the only real asymptote of the curve.

(iv) For the branch of the curve lying above the x-axis, we have

$$y = x^{3/2}/\sqrt{(2a - x)}.$$

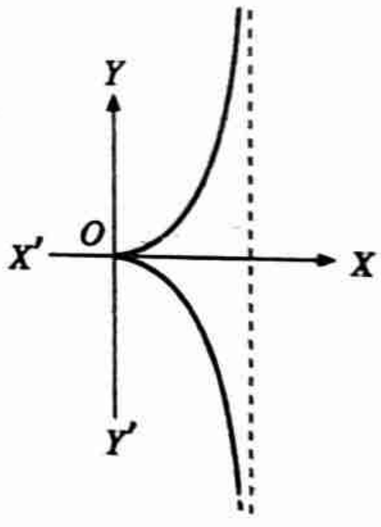
$$\therefore \qquad \frac{dy}{dx} = \frac{(3a - x)\sqrt{x}}{(2a - x)^{3/2}},$$

which vanishes when $x = 0$, or $3a$.

But $x = 3a$ is outside the range of admissible values of x. Therefore dy/dx vanishes at no admissible value of x except $x = 0$.

When $0 < x < 2a, dy/dx$ is positive. Therefore in this region y increases continuously as x increases.

Combining all these facts, we see that the shape of the curve is as shown in the adjoining figure.



***Ex. 18 (a).** *Trace the curve* $y^2(a + x) = x^2(a - x)$.

(Lucknow 1983; Kanpur 79; Meerut 91P; Agra 78)

Sol. (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. The tangents at origin are $a(y^2 - x^2) = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve.

(iii) The curve intersects the x-axis where $y = 0$ *i.e.*,

$$x^2 (a - x) = 0.$$

Therefore the curve intersects the x-axis at $(0, 0)$, $(a, 0)$.

The curve intersects the y-axis only at origin.

(iv) **Tangent at (a, 0)**. Shifting the origin to $(a, 0)$ the equation of the curve becomes

$$y^2 (2a + x) = (x + a)^2 \{a - (x + a)\}$$

or $$y^2 (2a + x) = -x (x^2 + 2ax + a^2).$$

Equating to zero the lowest degree terms, we get $x = 0$ (*i.e.*, the new y-axis) as the tangent at the new origin. Thus the tangent at $(a, 0)$ is perpendicular to x-axis.

(v) Solving the equation of the curve for y, we get

$$y^2 = x^2 (a - x)/(x + a).$$

When $x = 0, y^2 = 0$ and when $x = a$, $y^2 = 0$.

When $0 < x < a, y^2$ is positive. Therefore the curve exists in this region.

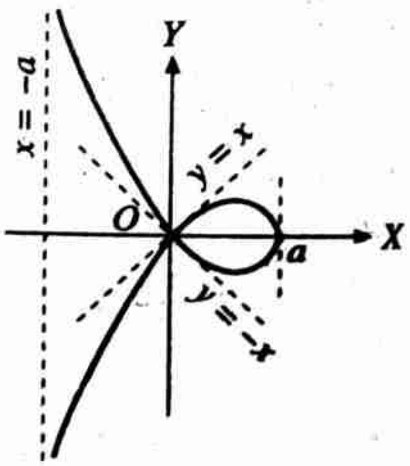
When $x > a, y^2$ is negative. Therefore the curve does not exist in the region $x > a$.

When $x \to -a, y^2 \to \infty$. Therefore $x = -a$ is an asymptote of the curve.

When $-a < x < 0, y^2$ is positive. Therefore the curve exists in this region.

When $x < -a, y^2$ is negative. Therefore the curve does not exist in the region $x < -a$.

(vi) The curve has an asymptote parallel to x-axis and it is $x + a = 0$. Putting $y = m$ and $x = 1$ in the highest *i.e.*, third degree terms in the equation of the curve, we get $\phi_3 (m) = m^2 + 1$. The roots of the equation $\phi_3 (m) = 0$ are imaginary. Therefore $x = -a$ is the only real asymptote of the curve.



(vii) For the portion of the curve lying in the first quadrant, we have

$$y = x \sqrt{\{(a - x)/(a + x)\}}$$

$$= x \left(1 - \frac{x}{a}\right)^{1/2} \Big/ \left(1 + \frac{x}{a}\right)^{1/2}.$$

When $0 < x < a, y$ is less than x. Therefore the curve lies below the line $y = x$ which is tangent at the origin.

For the portion of the curve lying in the second quadrant, we have

$$y = -x \frac{\left(1 - \frac{x}{a}\right)^{1/2}}{\left(1 + \frac{x}{a}\right)^{1/2}}, x < 0.$$

When $-a < x < 0, y$ is greater than the numerical value of x. Therefore the curve lies above the tangent $y = -x$.

Hence the shape of the curve is as shown in the adjoining figure.

Ex. 18 (b). *Trace the curve* $y^2 (a + x) = x^2 (3a - x)$.

(Lucknow 1981; Meerut 94P)

Sol. Proceed exactly as in Ex. 18 (a).

Ex. 19. *Trace the curve*

$$y^2 (x^2 + y^2) + a^2 (x^2 - y^2) = 0.$$

(Rohilkhand 1981; Meerut 91S)

Sol. (i) The curve is symmetrical about both the axes.

(ii) It passes through the origin and $a^2 (x^2 - y^2) = 0$ *i.e.*, $y = \pm x$ are the two tangents at the origin. Therefore the origin is a node.

(iii) The curve intersects the x-axis only at origin. It intersects the y-axis at $(0, 0)$, $(0, a)$ and $(0, -a)$.

(iv) Shifting the origin to $(0, a)$, the equation of the curve becomes

$$(y + a)^2 \{x^2 + (y + a)^2\} + a^2 \{x^2 - (y + a)^2\} = 0$$

or

$$(y^2 + 2ay + a^2) \{x^2 + y^2 + 2ay + a^2\} + a^2 (x^2 - y^2 - 2ay - a^2) = 0.$$

Equating to zero the lowest degree terms, we get

$$2a^3y + 2a^3y - 2a^3y = 0 \text{ i.e., } y = 0$$

as the tangent at the new origin. Thus the new x-axis is tangent at the new origin.

We need not find the tangent at $(0, -a)$ as the curve is symmetrical about x-axis.

(v) Solving the equation of the curve for x, we get

$$x^2 = y^2 (a^2 - y^2)/(a^2 + y^2).$$

When $y = 0, x^2 = 0$ and when $y = a, x^2 = 0$.

When $0 < y < a, x^2$ is positive. Therefore the curve exists in the region $0 < y < a$. When $y > a, x^2$ is negative. Therefore the curve does not exist in the region $y > a$.

We need not consider the negative values of y as the curve is symmetrical about x-axis.

(vi) The asymptotes parallel to x-axis are given by $a^2 + y^2 = 0$ *i.e.*, $y = \pm ia$. Also $\phi_4 (m) = m^2 (1 + m^2)$. Its roots are

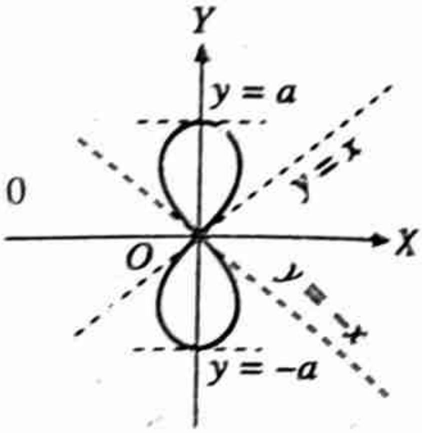
$m = 0, 0, i, -i$. The asymptotes corresponding to $m = 0$ are imaginary. Hence all the four asymptotes are imaginary.

(vii) In the positive quadrant, we have

$$x = y\,(a^2 - y^2)^{1/2}/(a^2 + y^2)^{1/2},\ y > 0$$

or $$x = y\left(1 - \frac{y^2}{a^2}\right)^{1/2} \Big/ \left(1 + \frac{y^2}{a^2}\right)^{1/2}.$$

When $0 < y < a$, we see that x is less than y. Therefore the curve lies above the line $y = x$ which is tangent at the origin.



Combining all these facts, we see that the shape of the curve is as shown in the adjoining figure.

Ex. 20 (a). *Trace the curve* $x^2(x^2 - 4a^2) = y^2(x^2 - a^2)$.

(Gorakhpur 1981)

Sol. (i) Symmetry about both the axes.

(ii) The curve passes through the origin and

$$a^2y^2 - 4a^2x^2 = 0 \text{ i.e., } y = \pm 2x$$

are the tangents at the origin. Therefore origin is a node on the curve.

(iii) The curve cuts the x-axis at $(0, 0), (2a, 0), (-2a, 0)$. It cuts the y-axis only at the origin.

(iv) Shifting the origin to $(2a, 0)$, the equation of the curve becomes

$$(x + 2a)^2(x^2 + 4ax) = y^2(x^2 + 4ax + 3a^2).$$

The equation to the tangent at the new origin is $16a^3x = 0$ *i.e.*, $x = 0$. Thus the new y-axis is tangent at the new origin.

(v) Solving the equation of the curve for y, we get

$$y^2 = x^2(x^2 - 4a^2)/(x^2 - a^2).$$

When $x = 0, y^2 = 0$.

When $x \to a, y^2 \to \infty$ *i.e.*, $x = a$ is an asymptote of the curve.

When $0 < x < a, y^2$ is positive *i.e.*, the curve exists in this region.

When $x = 2a, y^2 = 0$.

When $a < x < 2a, y^2$ is negative *i.e.*, the curve does not exist in this region.

When $x > 2a, y^2$ is positive *i.e.*, the curve exists in this region. When $x \to \infty, y^2 \to \infty$. We need not consider the negative values of x as the curve is symmetrical about the y-axis.

(vi) The asymptotes of the curve parallel to y-axis are given by $x^2 - a^2 = 0$. Thus $x = \pm a$ are two asymptotes of the curve.

Also the equation of the curve can be written as

$$x^2(y^2 - x^2) - a^2y^2 + 4a^2x^2 = 0.$$

$\therefore$ $$\phi_4(m) \equiv m^2 - 1 = 0 \text{ gives } m = \pm 1.$$

Also $\phi_3(m) = 0$.

For $m = \pm 1, c$ is given by $c\,\phi'_4(m) + \phi_3(m) = 0$

i.e., by $2cm = 0$.

When $m = 1, c = 0$. Also when $m = -1, c = 0$.

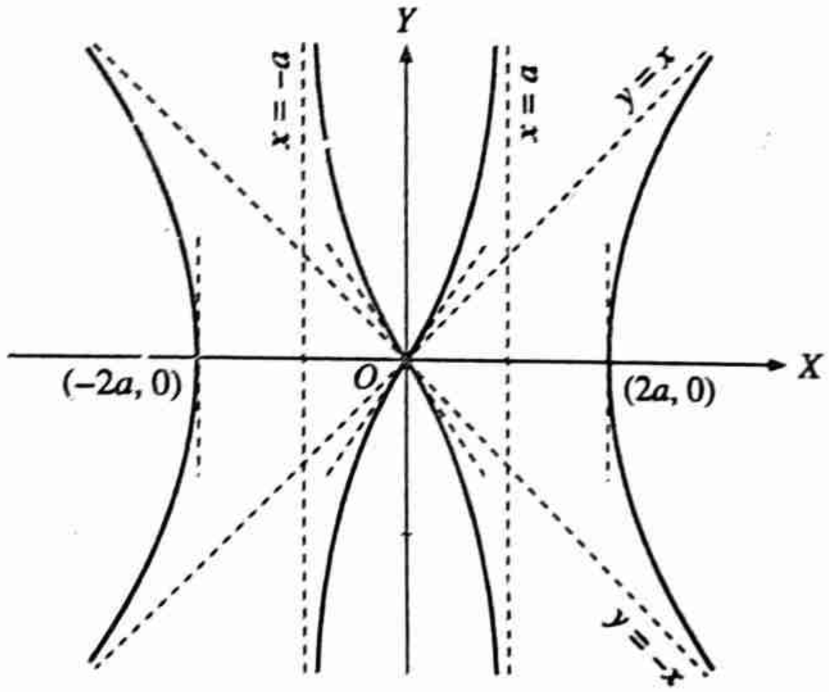
Therefore $y = \pm x$ are two oblique asymptotes of the curve.

(vii) In the positive quadrant, we have

$$y^2 = \frac{x^2(4a^2 - x^2)}{(a^2 - x^2)}, \; 0 < x < a$$

or $$y = 2x\left(1 - \frac{x^2}{4a^2}\right)^{1/2} \Big/ \left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

When $0 < x < a, y$ is greater than $2x$. Therefore the curve lies above the line $y = 2x$ which is tangent at the origin.



Combining all these facts we see that the shape of the curve is as shown in the above figure.

Ex. 20 (b). *Trace the curve* $y^2(1 - x^2) = x^2(1 + x^2)$.

(Meerut 1983 S, 90 S)

Sol. **(i)** The given curve is symmetrical about both the axes.

(ii) The curve passes through the origin. The tangents at origin are $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, the origin is a node on the curve.

(iii) The curve cuts the x-axis where $y = 0$. Putting $y = 0$ in the equation of the curve we get $x^2(1 + x^2) = 0$. The only real value of x satisfying this equation is $x = 0$. So the curve cuts the x-axis only at the origin. Similarly we observe that the curve cuts the y-axis only at the origin.

(iv) Solving the equation of the curve for y, we get

$$y^2 = \frac{x^2(1 + x^2)}{1 - x^2}.$$

When $x = 0, y^2 = 0$ *i.e.*, $y = 0$.

When $x \to 1, y^2 \to \infty$. Therefore $x = 1$ is an asymptote of the curve.

When $0 < x < 1, y^2$ is positive and so y is real.

Therefore the curve exists in this region.

When $x > 1, y^2$ is negative and so y is imaginary. Therefore the curve does not exist in the region where $x > 1$.

We need not consider the negative values of x as the curve is symmetrical about the y-axis and so it can be drawn by symmetry in the region where $x < 0$.

(v) The asymptotes of the curve parallel to y-axis are given by $1 - x^2 = 0$. Thus $x = \pm 1$ are two asymptotes of the curve.

To find the other asymptotes, if there are any, the equation of the curve can be written as

$$y^2x^2 + x^4 + x^2 - y^2 = 0.$$

Putting $y = m$ and $x = 1$ in the highest degree *i.e.*, 4 degree terms of the equation of the curve, we get $\phi_4(m) = m^2 + 1$. The equation $\phi_4(m) = 0$ *i.e.*, $m^2 + 1 = 0$ gives no real values of m. So the curve has no other asymptotes.

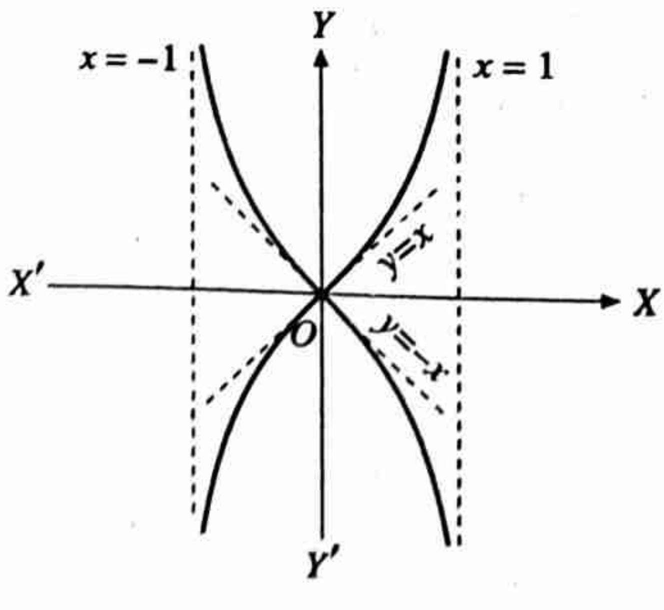
(vi) In the positive quadrant, we have

$$y^2 = x^2 \frac{1 + x^2}{1 - x^2}, \; 0 < x < 1$$

or $$y = x\sqrt{\left(\frac{1 + x^2}{1 - x^2}\right)}.$$

When $0 < x < 1, y > x$. Therefore the curve lies above the line $y = x$ which is tangent at the origin.

Combining all these facts the shape of the curve is as shown in the figure.



Ex. 20 (c). *Trace the curve* $y^2(a-x) = x^2(x-2a)$. **(Garhwal 1983)**

Sol. Proceed as in Ex. 20 (a) and Ex. 20 (b).

This question has also been solved in the latter portion of this chapter in Ex.2 after § 2.

****Ex. 21.** *Trace the curve* $x^3 + y^3 = 3axy$. **(Folium of Descartes).**

(Meerut 1982, 84 R, 89 P, 91, 94; Bihar 74)

Sol. (i) The curve is symmetrical about the line $y = x$, since its equation remains unchanged on interchanging x and y.

(ii) The curve passes through the origin and the tangents at origin are $3axy = 0$ *i.e.*, $x = 0, y = 0$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve.

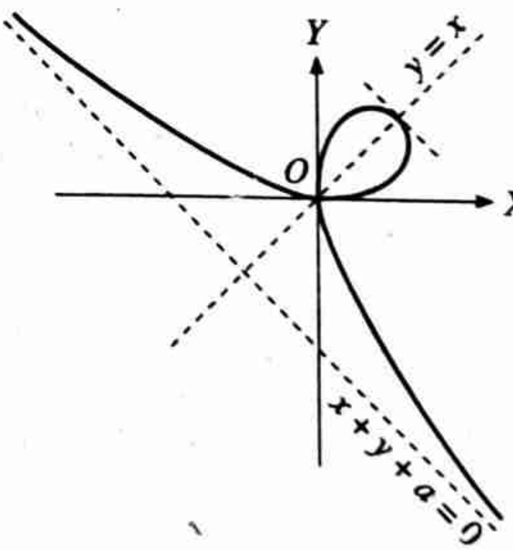
(iii) The curve intersects the coordinate axes only at $(0, 0)$.

(iv) From the equation of the curve we see that x and y cannot be both negative because then the left hand side of the equation of the curve becomes negative while the right hand side becomes positive. Therefore the curve does not exist in the third quadrant.

(v) The curve meets the line $y = x$ at the point $(3a/2, 3a/2)$. From the equation of the curve, we have

$$\frac{dy}{dx} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}.$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $\frac{dy}{dx} = -1$. Therefore the tangent at this point makes an angle of $135°$ with the positive direction of x-axis.



(vi) Asymptotes.

$$\phi_3(m) = m^3 + 1.$$

The only real root of the equation $\phi_3(m) = 0$

i.e., $\quad m^3 + 1 = 0$, is $m = -1$.

Also $\phi_2(m) = -3am$.

For $m = -1$, c is given by $c(3m^2) - 3am = 0$.

$\therefore$ when $m = -1, c = -a$

Hence $y = -x - a$ is the only real asymptote of the curve.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Ex. 22. *Trace the curve* $y^3 + x^3 = a^2x$.

(Agra 1979; Kanpur 77; Gorakhpur 77; Meerut 93P, 98)

Sol. (i) If we change the signs of x and y both, the equation of the curve does not change. Therefore the curve is symmetrical in opposite quadrants.

(ii) The curve passes through the origin and the tangent at origin is $x = 0$ *i.e.*, y-axis.

(iii) The curve cuts the x-axis where $y = 0$ *i.e.*, $x(x^2 - a^2) = 0$. Thus the curve cuts the x-axis at $(0, 0), (a, 0), (-a, 0)$.

The curve intersects the y-axis only at the origin.

(iv) From the equation of the curve, we have

$$\frac{dy}{dx} = \frac{a^2 - 3x^2}{3y^2}.$$

At $(a, 0)$, $\dfrac{dy}{dx} = -\infty$ *i.e.*, the tangent is perpendicular to x-axis.

Also at $(-a, 0)$, $\dfrac{dy}{dx} = -\infty$ *i.e.*, the tangent is perpendicular to x-axis.

(v) $\dfrac{dy}{dx} = 0$ at $x = \pm \dfrac{a}{\sqrt{3}}$. Therefore the tangents at these points are parallel to the x-axis.

(vi) Solving the equation of the curve for y, we get

$$y^3 = x(a^2 - x^2).$$

When $x = 0, y^3 = 0$ and when $x = a, y^3 = 0$.

When $0 < x < a$, y^3 is positive *i.e.*, y is positive in this region.

When $x > a$, y^3 is negative *i.e.*, y is negative in this region.

When $x \to \infty, y^3 \to -\infty$ *i.e.*, $y \to -\infty$.

We need not consider the negative values of x as there is symmetry in opposite quadrants.

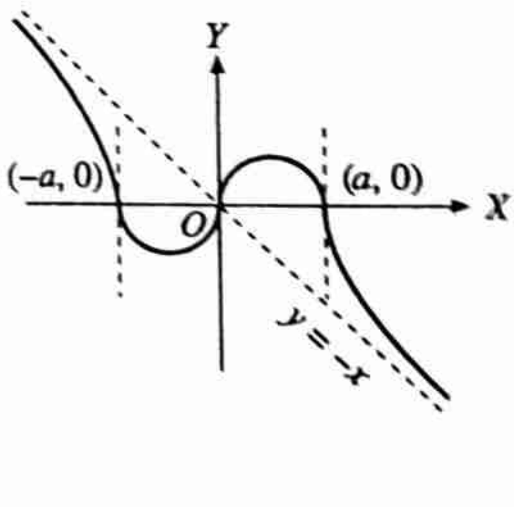
Asymptotes. $\phi_3(m) = m^3 + 1, \phi_2(m) = 0$.

The only real root of $m^3 + 1 = 0$ is $m = -1$. Also c is given by

$$c(3m^2) + 0 = 0.$$

When $m = -1, c = 0$.

Hence $y = -x$ is the only real asymptote of the curve.



Combining all these facts the shape of the curve is as shown in the figure.

Ex. 23. *Trace the curve* $y^2(x+3a)=x(x-a)(x-2a)$.

(Meerut 1983, 93)

Sol. The given curve is $y^2(x+3a)=x(x-a)(x-2a)$. ...(1)

(i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the origin and the tangent at the origin is $x=0$ *i.e.*, the y-axis.

(iii) When $y=0, x=0, a, 2a$; when $x=0, y=0$. Therefore the curve meets the x-axis at the points $(0,0), (a,0), (2a,0)$ and it meets the y-axis only at the origin.

Shifting the origin to the point $(a,0)$, the equation of the curve becomes $y^2(x+4a)=(x+a)x(x-a)$ *i.e.*, $y^2(x+4a)=x(x^2-a^2)$. Therefore the tangent at the new origin is $x=0$ *i.e.*, the new y-axis.

Again shifting the origin to the point $(2a,0)$ the equation of the curve becomes $y^2(x+5a)=(x+2a)(x+a)x$. Therefore the tangent at this new origin is $x=0$ *i.e.*, the new y-axis.

(iv) Solving the equation of the curve for y, we have

$$y^2=\frac{x(x-a)(x-2a)}{x+3a}.$$

When $x=0, y^2=0$ and when $x=a, y^2=0$. When $0<x<a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and it has a loop between the lines $x=0$ and $x=a$. When $x=2a, y^2=0$ and when $a<x<2a, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x>2a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and when $x\to\infty, y^2\to\infty$.

When $x\to-3a$ (from the left), $y^2\to\infty$ and so on its left side the line $x=-3a$ is an asymptote of the curve.

When $-3a<x<0, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

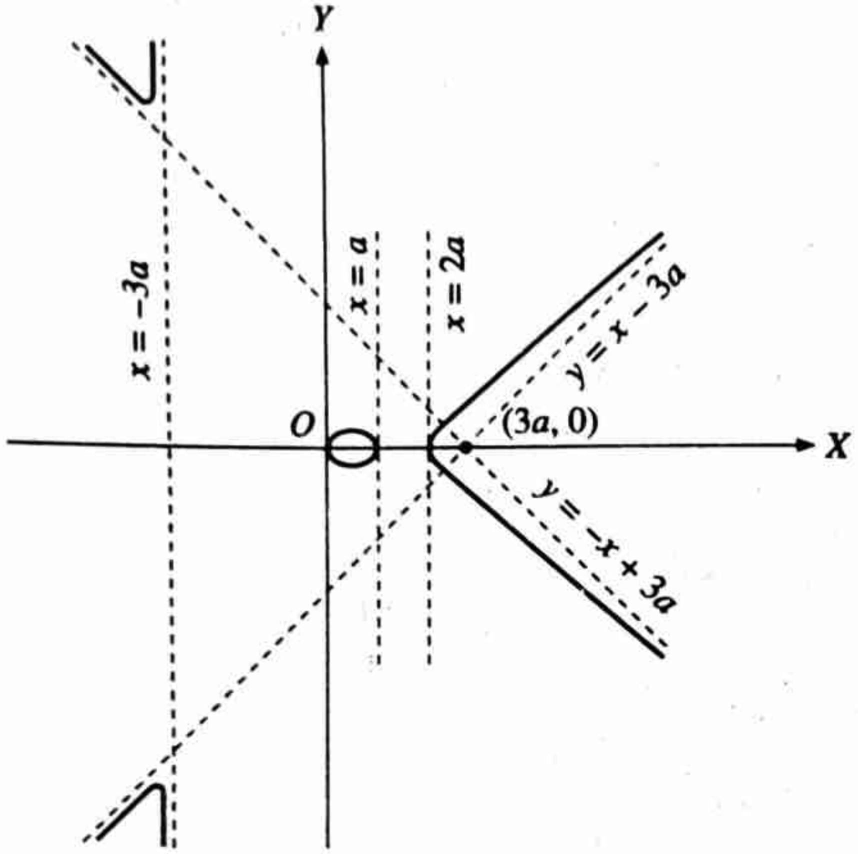
When $x<-3a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and when $x\to-\infty, y^2\to\infty$.

(v) Asymptotes. The curve has an asymptote parallel to x-axis *i.e.*, the line $x+3a=0$. Putting $y=m$ and $x=1$ in the third degree terms in the equation of the curve, we get $\phi_3(m)=m^2-1$. The equation $\phi_3(m)=0$ gives $m^2-1=0$ *i.e.*, $m=\pm 1$. Also $\phi_2(m)=3am^2+3a$.

Now c is given by

$$c\,\phi'_3(m)+\phi_2(m)=0 \text{ i.e., } c\,(2m)+3am^2+3a=0.$$

When $m = 1, c = -3a$ and when $m = -1, c = 3a$. Hence the oblique asymptotes of the curve are $y = x - 3a$ and $y = -x + 3a$. Both these lines pass through the point $(3a, 0)$ and they make angles $\pm 45^\circ$ with the x-axis.



Combining all these facts the shape of the curve is as shown in the above figure.

Ex. 24. *Trace the curve* $x(x-2a)y^2 = a^2(x-a)(x-3a)$.

(Lucknow 1980)

Sol. **(i)** The curve is symmetrical about the axis of x.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x = a, 3a$; when $x = 0$, we do not get any value of y.

Hence the curve cuts the coordinate axes at the points $(a, 0)$, $(3a, 0)$.

Transferring the origin to the points $(a, 0)$, $(3a, 0)$ respectively, we find that the tangents at these points are parallel to the y-axis.

(iv) Asymptotes parallel to x-axis : $y^2 - a^2 = 0$ or $y = \pm a$.

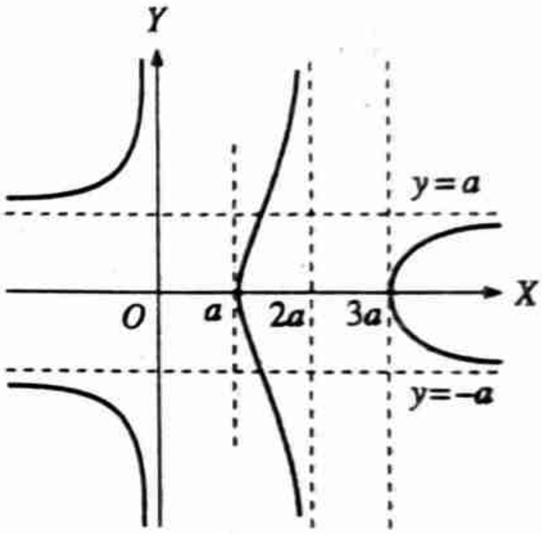
Asymptotes parallel to y-axis : $x(x-2a) = 0$ or $x = 0, x = 2a$.

(v) Solving the equation of the curve for y, we have

$$y^2 = \frac{a^2(x-a)(x-3a)}{x(x-2a)}.$$

When $0 < x < a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x = a$, $y^2 = 0$.

When $a < x < 2a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x \to 2a$ (from the left), $y^2 \to \infty$ showing that on its left side the line $x = 2a$ is an asymptote of the curve. When $x = 3a$, $y^2 = 0$ and when $2a < x < 3a$, y^2 is –ive and so the curve does not exist in this region. When $x > 3a$, y^2 is +ive and so the curve exists in this region. When $x \to \infty$, $y^2 \to a^2$ showing that the lines $y = \pm a$ are asympotes of the curve. Also when $x > 3a$, we observe that $y^2 < a^2$. When $x < 0$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x \to 0$ (from the left), $y^2 \to \infty$ and so on its left side the line $x = 0$ is an asymptote of the curve. When $x \to -\infty$, $y^2 \to a^2$ showing that the lines $y = \pm a$ are asymptotes of the curve. Also when $x < 0$, we observe that $y^2 > a^2$. Combining all these facts the shape of the curve is as drawn in the figure .



Ex. 25. *Trace the curve* $y^2x = a^2(x - a)$. ...(1)

Sol. **(i)** The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x = a$ *i.e.*, the curve crosses the x-axis at $(a, 0)$.

When $x = 0$, we do not get any value of y and so the curve does not meet the y-axis.

Shifting the origin to $(a, 0)$ the equation of the curve transforms to

$$y^2(x + a) = a^2x. \qquad ...(2)$$

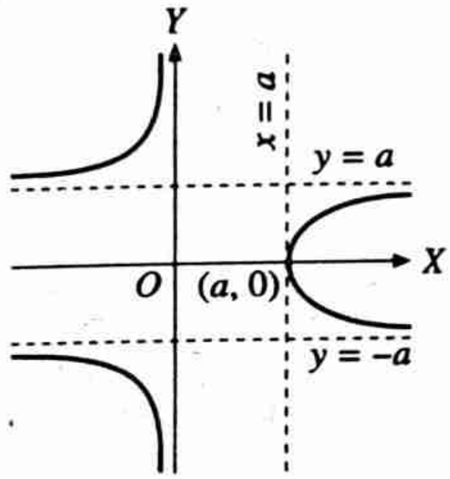
Equating to zero the lowest degree terms in (2), the tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

(iv) Equating to zero, the coefficients of highest powers of x and y in (1), we get $y = \pm a$ and $x = 0$ as the asymptotes parallel to the coordinate axes. Also these are the only asymptotes of the curve.

(v) From (1), solving for y, we get

$$y^2 = a^2(x - a)/x.$$

When $x = a, y^2 = 0$ and when $0 < x < a, y^2$ is $-$ ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x > a, y^2$ is $+$ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty, y^2 \to a^2$ showing that the lines $y = \pm a$ are asymptotes of the curve. Also when $x > a$, we observe that $y^2 < a^2$. When $x \to 0$ (from the left), $y^2 \to \infty$ showing that the line $x = 0$ (on its left side) is asymptote of the curve. When $x < 0, y^2$ is $+$ive *i.e.*, y is real and so the curve exists in this region and when $x \to -\infty, y^2 \to a^2$ showing that the lines $y = \pm a$ are asymptotes of the curve. Also when $x < 0$, we observe that $y^2 > a^2$.



Combining all these facts we see that the shape of the curve is as shown in the figure.

Ex. 26 (a). *Trace the curve* $y^2(x^2 - 1) = x$. ...(1)

(Lucknow 1979)

Sol. (i) The curve is symmetrical about the x-axis.

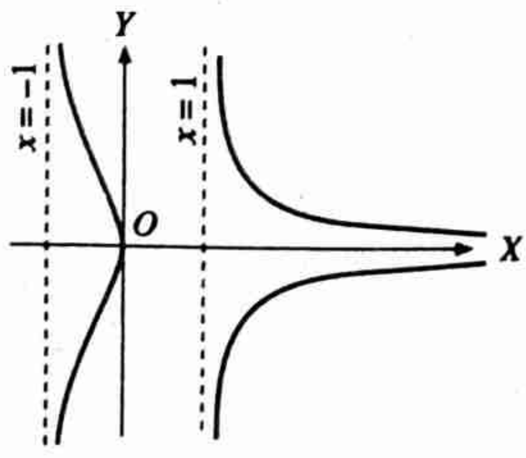
(ii) The curve passes through the origin. Equating to zero the lowest degree terms in (1), the equation of tangent at the origin is $x = 0$ *i.e.*, y-axis.

(iii) The curve meets the coordinate axes only at $(0, 0)$.

(iv) Asymptotes parallel to coordinates axes are obtained by equating to zero the coefficients of the highest powers of x and y in (1). Thus $y = 0, y = 0$ and $x = \pm 1$ are the asymptotes of the given curve and these are the only asymptotes of the curve because the curve is of degree 4 and so it cannot have more than four asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = x/(x^2 - 1)$$
$$= x/\{(x-1)(x+1)\}.$$



When $0 < x < 1, y^2$ is $-$ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x > 1, y^2$ is $+$ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty, y^2 \to 0$ showing that the line $y = 0$ is an asymptote of the curve. When $x \to 1$ (from the right) $y^2 \to \infty$ showing that the line $x = 1$

(on its right side) is an asymptote of the curve. When $x \to -1$ (but from the right), $y^2 \to \infty$ showing that the line $x = -1$ is an asymptote of the curve. When $-1 < x < 0, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region. When $x < -1, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Ex. 26 (b). *Trace the curve* $x^2 = y^2 (x + 1)^2$. **(Lucknow 1982)**

Sol. **(i)** The given curve is symmetrical about the x-axis.

(ii) The curve passes through the origin. Equating to zero the lowest degree terms in the equation of the curve, the tangents at the origin are given by $x^2 - y^2 = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, the origin is a node on the curve.

(iii) The curve cuts the x-axis where $y = 0$. Putting $y = 0$ in the equation of the curve, we get $x^2 = 0$, whose only solution is $x = 0$. So the curve cuts the x-axis only at the origin. In a similar way we observe that the curve meets the y-axis only at the origin.

(iv) Solving the equation of the curve for y, we get

$$y^2 = \frac{x^2}{(x + 1)^2}.$$

When $x = 0, y^2 = 0$ *i.e.*, $y = 0$.

When $x > 0, y^2$ is positive *i.e.*, y is real.

So the curve exists in the region where $x > 0$. When $x \to \infty, y^2 \to 1$ showing that the lines $y = \pm 1$ are asymptotes of the curve.

When $x \to -1, y^2 \to \infty$ showing that the line $x = -1$ is an asymptote of the curve.

When $-1 < x < 0, y^2$ is positive and so y is real. Therefore the curve exists in this region.

When $x < -1, y^2$ is positive and so y is real. Therefore the curve exists in the region where $x < -1$.

When $x \to -\infty, y^2 \to 1$ showing that the lines $y = \pm 1$ are asymptotes of the curve.

When $x > 0, y^2 < x^2$ *i.e.*, the numerical value of y is less than the numerical value of x showing that in the region $x > 0$ the curve lies below the line $y = x$ and above the line $y = -x$. Also in this region $y^2 < 1$ *i.e.*, $-1 < y < 1$ showing that in this region the curve lies between the lines $y = -1$ and $y = 1$.

When $-1 < x < 0, y^2 > x^2$ showing that in this region the curve lies below the line $y = x$ and above the line $y = -x$.

Also when $x < -1$, we observe that $y^2 > 1$ *i.e.*, the numerical value of y is > 1. Therefore in the region $x < -1$, the curve lies above the line $y = 1$ and below the line $y = -1$.

The line $x = -1$ is an asymptote of the curve on its right side and also on its left side because $y^2 \to \infty$ when $x \to -1$ both from the right as well as from the left.

Taking help from all these facts draw the shape of the curve.

Ex. 27. *Trace the curve* $y^2 (x - a) = x^2 (x + a)$. ...(1)

(Meerut 1985)

Sol. **(i)** The curve is symmetrical about x-axis.

(ii) The curve passes through the origin.

Equating to zero, the lowest degree terms of the given equation of the curve, the tangents at the origin are $x^2 + y^2 = 0$ *i.e.*, two imaginary tangents. Therefore $(0, 0)$ is a conjugate point.

From (1), when $y = 0$ we get $x = 0, -a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$ and $(-a, 0)$. Also $x = 0$, gives $y = 0$ *i.e.*, the curve meets the y-axis only at the origin and that too is an isolated point.

Shifting the origin to the point $(-a, 0)$, the equation of the curve transforms to

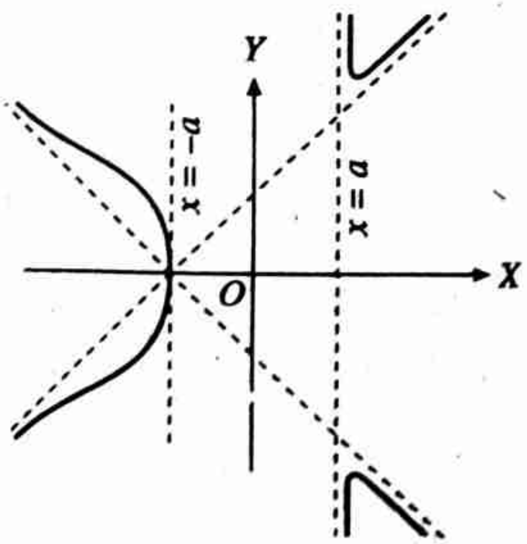
$$y^2 (x - 2a) = (x - a)^2 x. \qquad \text{...(2)}$$

Equating to zero, the lowest degree terms in (2), we get the tangent at the new origin as $x = 0$ *i.e.*, the new y-axis.

Solving the equation of the curve for y, we have

$$y^2 = \{x^2 (x + a)\}/(x - a).$$

When $x = 0, y^2 = 0$ and when $0 < x < a, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x \to a$ (but from the right), $y^2 \to \infty$ showing that on its right side the line $x = a$ is an asymptote of the curve.



When $x > a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty, y^2 \to \infty$.

When $x = -a, y^2 = 0$ and when $-a < x < 0, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x < -a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to -\infty, y^2 \to \infty$.

Asymptotes. The curve has an asymptote parallel to y-axis and it is

$x - a = 0$. We have $\phi_3(m) = m^2 - 1$ and $\phi_2(m) = -am^2 - a$. The equation $\phi_3(m) = 0$ gives $m = \pm 1$. Now c is given by $c\phi'_3(m) + \phi_2(m) = 0$ *i.e.*, $c(2m) + (-am^2 - a) = 0$. When $m = 1, c = a$ and when $m = -1, c = -a$. Thus $y = x + a$ and $y = -x - a$ are two oblique asymptotes of the curve. Both these lines pass through the point $(-a, 0)$ and they make angles $\pm 45°$ with the x-axis.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Ex. 28. *Trace the curve* $y^2 = (x - a)(x - b)(x - c)$, *where* $a > b > c$. **(Meerut 1992)**

Sol. **(i)** The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin.

(iii) From the equation of the curve when $y = 0$, we get $x = a, b, c$ *i.e.*, the curve crosses the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$. Again putting $x = 0$ in the equation of the curve we get

$y^2 = -abc$ which gives imaginary values of y and so the curve does not cut the y-axis.

Shifting the origin to $(a, 0)$ the equation of the curve transforms to

$$y^2 = x(x + a - b)(x + a - c). \qquad \text{...(2)}$$

Equating to zero, the lowest degree terms in (2) we find that the tangent at the new origin is $x = 0$ *i.e.*, the new y-axis. Thus on the given curve the tangent at $(a, 0)$ is parallel to the y-axis.

Similarly the tangents at $(b, 0)$ and $(c, 0)$ are the lines through them parallel to the y-axis.

(iv) The curve has no asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = (x - a)(x - b)(x - c).$$

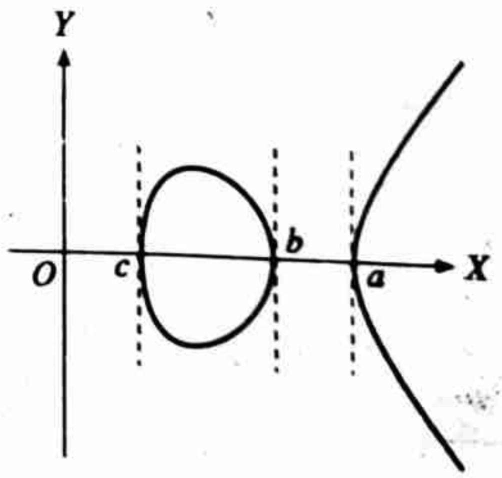
When $x < c$, y^2 is –ive, *i.e.*, y is imaginary and so the curve does not exist in the region $x < c$ *i.e.*, to the left of the line $x = c$.

When $x = c, y = 0$.

When $c < x < b$ (where $b > c$), y^2 is positive *i.e.*, y is real and so the curve exists in this region.

When $x = b, y = 0$.

This implies that there is a loop between $x = c$ and $x = b$.



When $b < x < a$ (where $a > b$), y^2 is negative *i.e.*, y is imaginary and so the curve does not exist between the lines $x = b$ and $x = a$.

At $x = a, y = 0$. When $x > a$, y^2 is +ive and so the curve exists in the region $x > a$. Also as $x \to \infty, y^2 \to \infty$.

(vi) As $x \to \infty$, $(dy/dx) \to \infty$ *i.e.*, the curve ultimately becomes parallel to y-axis.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Ex. 29. *Show that the curve $(a^2 + x^2) y = a^2 x$ has three points of inflexion and trace it.*

Sol. We have $(a^2 + x^2) y = a^2 x$ or $y = a^2 x/(a^2 + x^2)$. ...(1)

$$\therefore \quad \frac{dy}{dx} = a^2 \left\{ \frac{1 \cdot (a^2 + x^2) - x \cdot 2x}{(a^2 + x^2)^2} \right\} = \frac{a^2 (a^2 - x^2)}{(a^2 + x^2)^2},$$

and
$$\frac{d^2y}{dx^2} = a^2 \left\{ \frac{-2x (a^2 + x^2)^2 - (a^2 - x^2) \cdot 2 (a^2 + x^2) \cdot 2x}{(a^2 + x^2)^4} \right\}$$

$$= \frac{-2a^2 \{a^2 x + x^3 + 2a^2 x - 2x^3\}}{(a^2 + x^2)^3} = \frac{-2a^2 x (3a^2 - x^2)}{(a^2 + x^2)^3}.$$

For a point of inflexion, we must have

$$\frac{d^2y}{dx^2} = 0 \text{ which gives } x = 0, 3a^2 = x^2 \text{ i.e., } x = \pm \sqrt{3} a.$$

Now $(a^2 + x^2)^3 \dfrac{d^2y}{dx^2} = -6a^4 x + 2a^2 x^3$.

Differentiating, we have

$$(a^2 + x^2)^3 \frac{d^3y}{dx^3} + 3 (a^2 + x^2)^2 \cdot 2x \cdot \frac{d^2y}{dx^2} = -6a^4 + 6a^2 x^2. \quad ...(2)$$

Now at $x = 0, \pm \sqrt{3} a$, we have $d^2y/dx^2 = 0$. So from (2), we see that at $x = 0, \pm \sqrt{3} a$, $d^3y/dx^3 \neq 0$.

Hence the curve has three points of inflexion at $x = 0, \pm \sqrt{3} a$.

Tracing. **(i)** There is symmetry in opposite quadrants because by putting $-x$ for x and $-y$ for y, the equation of the curve remains unchanged.

(ii) The curve passes through the origin and the tangent at the origin is $a^2 y - a^2 x = 0$ *i.e.*, $y = x$.

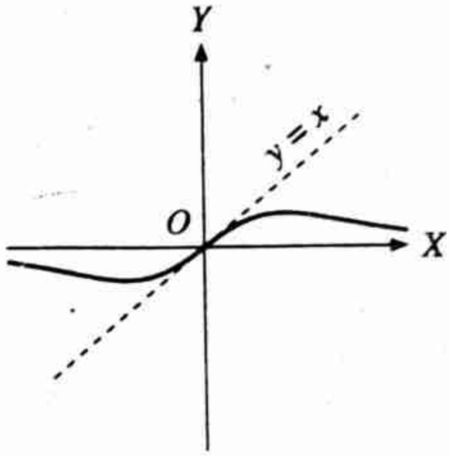
(iii) When $x = 0, y = 0$; when $y = 0, x = 0$. Hence the curve cuts the coordinate axes only at the origin.

(iv) Asymptotes. The curve is of degree 3 and so it can have at the most three asymptotes real or imaginary. An asymptote parallel to x-axis is $y = 0$ *i.e.*, x-axis. Also the asymptotes parallel to y-axis are given by $x^2 + a^2 = 0$ which gives two imaginary asymptotes. Thus x-axis is the only real asymptote of the curve.

(v) From (1), we see that when x is +ive, y is +ive and when x is –ive, y is –ive.

Hence the curve lies only in the Ist and IIIrd quadrants.

When $x \to \infty$, $y \to 0$ showing the fact that the line $y = 0$ is an asymptote of the curve.



(vi) We have $dy/dx = 0$ when $x = \pm a$. Thus at the points $x = \pm a$ the tangent to the curve is parallel to the axis of x.

(vii) Special points :

x	0	a	$2a$	$\to \infty$
y	0	$a/2$	$2a/5$	$\to 0$.

First trace the curve in the Ist quadrant and then by symmetry in the IIIrd quadrant. The shape of the curve is as shown in the figure.

Ex. 30. *Trace the curve* $y^2 x^2 = x^2 - a^2$. **(Meerut 1990 P)**

Sol. (i) The curve is symmetrical about both the axes.

(ii) The curve does not pass through the origin.

(iii) The curve cuts the x-axis where $y = 0$. Putting $y = 0$ in the equation of the curve, we get $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$. Thus the curve cuts the x-axis at the points $(a, 0)$ and $(-a, 0)$.

The curve cuts the y-axis where $x = 0$. Putting $x = 0$ in the equation of the curve we get no values of y and so the curve does not cut y-axis.

(iv) Tangent at (a, 0). Shifting the origin to $(a, 0)$, the equation of the curve becomes

$$y^2 (x + a)^2 = (x + a)^2 - a^2$$

or $$y^2 (x^2 + 2ax + a^2) = x^2 + 2ax.$$

$\therefore$ the tangent at the new origin is the line $2ax = 0$ *i.e.*, the line $x = 0$ *i.e.*, the new y-axis.

(v) Solving the equation of the curve for y, we have

$$y^2 = \frac{x^2 - a^2}{x^2} = 1 - (a^2/x^2).$$

When $x \to 0$, $y^2 \to -\infty$.

When $x = a$, $y^2 = 0$ *i.e.*, $y = 0$.

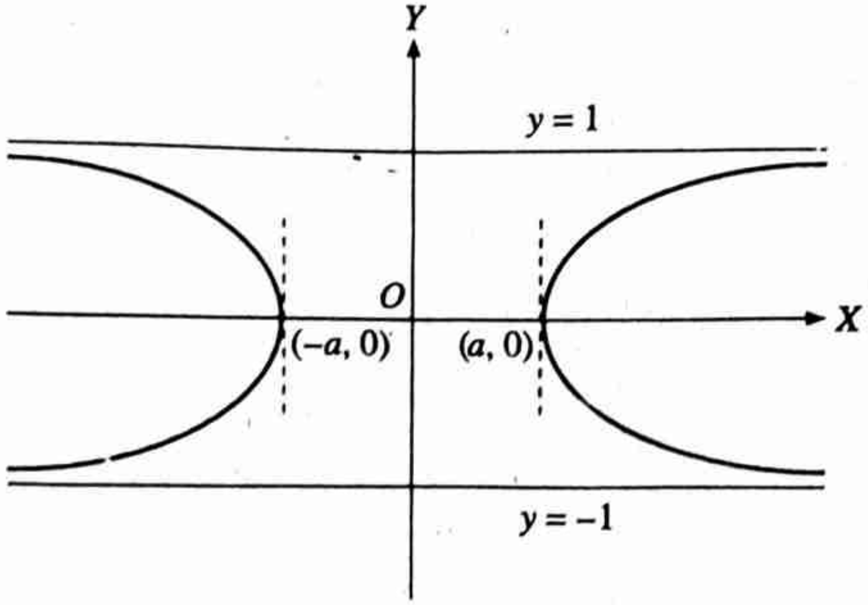
When $0 < x < a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x > a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region.

When $x \to \infty$, $y^2 \to 1$ and so the lines $y = \pm 1$ are asymptotes of the curve.

(vi) Asymptotes. The given curve is of degree four and so it cannot have more than four asymptotes, real or imaginary.

The asymptotes parallel to y-axis are given by $x^2 = 0$ *i.e.*, $x = 0, x = 0$. But they do not exist because the curve does not exist in the neighbourhood of origin.



The asymptotes parallel to x-axis are given by $y^2 - 1 = 0$ *i.e.*, $y = \pm 1$. Thus the only real asymptotes of the curve are $y = \pm 1$. The shape of the curve is as shown in the figure.

§ 2. Polar Equations. Procedure for Tracing.

1. Symmetry.

(i) If the equation of the curve does not change by changing the sign of θ, then the curve is symmetrical about the initial line.

(ii) If the equation of the curve remains unchanged by changing r into $-r$, then the curve is symmetrical about the pole and the pole is the centre of the curve.

2. Some Special points on the Curve. The curve will pass through the pole if for some value of θ the value of r comes out to be zero. Also if $r = 0$ when $\theta = \alpha$, then usually the line $\theta = \alpha$ will be a tangent to the curve at the pole.

We should find the values of θ for which $r = 0$, or r is maximum, or r is minimum, or $r \to \infty$.

3. Solve the equation of the curve for r and consider how r varies as θ increases from 0 to $+\infty$, and also as θ decreases from 0 to

$-\infty$. We should pay special attention to the values of θ found in the paragraph 2.

We should form a table of corresponding values of r and θ which would give us a number of points on the curve. Plotting these points we shall find the shape of the curve.

In the polar equations in which only periodic functions ($\sin\theta, \cos\theta, \tan\theta$ etc.) occur, the values of θ from 0 to 2π (or sometimes some multiple or sub-multiple of 2π) need be considered, as the remaining values of θ do not give any new branch of the curve.

4. Regions where the curve dose not exist. If r is imaginary when $\alpha < \theta < \beta$, then the curve does not exist in the region bounded by the lines $\theta = \alpha$ and $\theta = \beta$.

5. Asymptotes. Find the asymptotes if the curve possesses an infinite branch. If $r \to \infty$ as $\theta \to \alpha$, we should not assume that $\theta = \alpha$ is an asymptote. The asymptote might be parallel to the line $\theta = \alpha$ or even might not exist at all. The asymptotes should be found by the method given in the chapter on Asymptotes.

6. Find $\tan\phi$ *i.e.*, $r\, d\theta/dr$ which will indicate the direction of the tangent at any point. If for $\theta = \alpha$, ϕ comes out to be zero, then the line $\theta = \alpha$ will be a tangent to the curve at the point $\theta = \alpha$. If for $\theta = \alpha, \phi$ comes out to be $\pi/2$, then at the point $\theta = \alpha$, the tangent will be perpendicualr to the radius vector $\theta = \alpha$.

7. Important. It is sometimes convenient to change the equation from the polar form to the cartesian form. Remember that the relations between the cartesian and polar coordinates are

$$x = r\cos\theta, y = r\sin\theta.$$

Solved Examples

Ex. 1. *Trace the curve* $r = 2a\cos\theta$. **(Circle)**

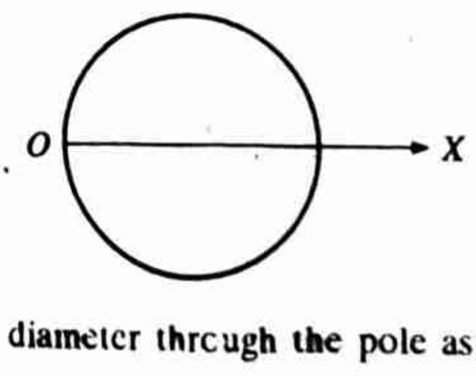
Sol. We have $\quad r = 2a\cos\theta$. ...(1)

Multiplying both sides by r, we have $r^2 = 2ar\cos\theta$.

Changing to cartesians, we get

$$x^2 + y^2 = 2ax.$$

$(\because r^2 = x^2 + y^2 \text{ and } x = r\cos\theta)$.



This is the equation of a circle with centre $(a, 0)$ and the radius a. Thus $r = 2a\cos\theta$ is the equation of a circle passing through the pole and the diameter thrcugh the pole as initial line.

Ex. 2. *Trace the curve* $r = a(\cos\theta + \sec\theta)$

(Meerut 1986 S)

Sol. We have $r = a(\cos\theta + \sec\theta)$. Multiplying both sides by r, we get

$$r^2 = a\left(r\cos\theta + \frac{r^2}{r\cos\theta}\right).$$

Changing to cartesian form, the equation becomes

$$x^2 + y^2 = a\left\{x + \frac{(x^2+y^2)}{x}\right\}$$

or $$y^2(x-a) = x^2(2a-x). \quad \text{...(1)}$$

Now we shall trace the curve (1) by the method we used for tracing the curves whose cartesian equations were given.

Tracing. **(i)** The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. Tangents at the origin are given by $-ay^2 - 2ax^2 = 0$ or $y^2 + 2x^2 = 0$

i.e., the tangents at (0, 0) are imaginary and so origin is a conjugate point.

(iii) When $y = 0, x = 0, 2a$ *i.e.*, the curve meets the x-axis at the points (0, 0) and $(2a, 0)$ out of which (0, 0) is an isolated point. When $x = 0, y = 0$ *i.e.*, the curve meets the y-axis only at the origin and that too is an isolated point.

Shifting the origin to $(2a, 0)$ the equation of the curve becomes

$$y^2(x+a) = (x+2a)^2(-x).$$

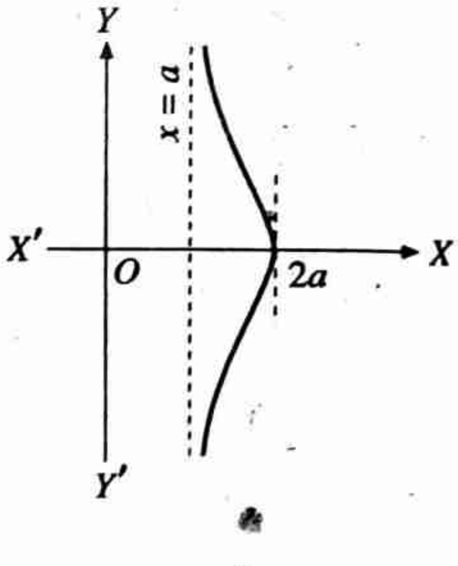
Therefore the tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

(iv) $x = a$ is an asymptote parallel to y-axis and the curve has no other real asymptotes.

(v) Solving the equation of the curve for y, we have

$y^2 = \{x^2(2a-x)\}/(x-a)$.

When $x = 0, y^2 = 0$. When $0 < x < a, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $a < x < 2a, y^2$ is +ive *i.e.*, y is real and so the curve exists in this region. When $x \to a$ (from the right) $y^2 \to \infty$ showing that the line $x = a$ (on its right side) is an asymptote of the curve. When $x = 2a, y^2 = 0$ and when $x > 2a, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in the region where $x > 2a$.



When $x < 0, y^2$ is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

(vi) As x decreases from $2a$ to a, y^2 increases from 0 to ∞.

Taking all these facts into consideration, the shape of the curve is as shown in the above figure.

***Ex. 3.** *Trace the curve* $r = a \cos 3\theta$. **(Meerut 1990)**

Sol. (i) The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $\cos 3\theta = 0$

i.e., $3\theta = \pm \frac{1}{2}\pi, \pm \frac{3}{2}\pi, \pm \frac{5}{2}\pi$, etc. **(Note)**

or $\theta = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi, \pm \frac{5}{6}\pi$, etc.

Thus the lines $\theta = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi$ etc. are the tangents to the curve at the pole.

(iii) Differentiating the equation of the curve, we get

$$(dr/d\theta) = -3a \sin 3\theta.$$

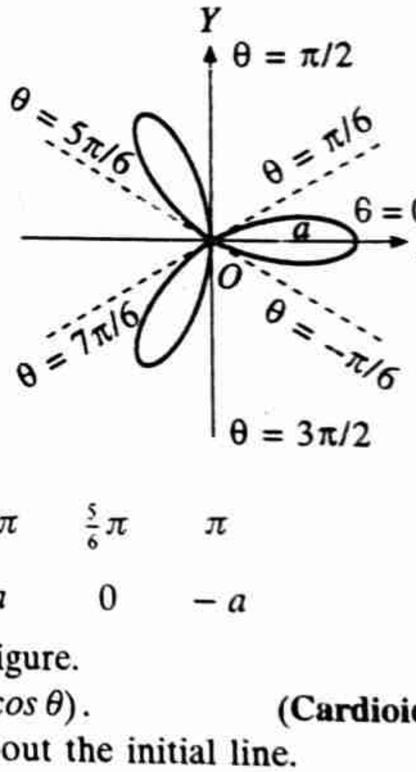
Therefore $\cot \phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{a \cos 3\theta} \cdot (-3a \sin 3\theta) = -3 \tan 3\theta$.

Now $\phi = 90^\circ$,

when $\tan 3\theta = 0$

i.e., $3\theta = 0 \quad \pi \quad 2\pi \quad 3\pi$

or $\theta = 0 \quad \frac{1}{3}\pi \quad \frac{2}{3}\pi \quad \pi$.



At all these points the tangent to the curve is perpendicular to the radius vector. Also at each of these points the numerical value of r is a which is the greatest value of the radius vector for this curve.

(iv) Table showing corresponding values of θ and r :

$\theta = 0$	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π
$r = a$	0	$-a$	0	a	0	$-a$

Hence the curve is as shown in the figure.

****Ex. 4.** *Trace the curve* $r = a(1 - \cos\theta)$. **(Cardioid)**

Sol. (i) The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $1 - \cos\theta = 0$ or $\cos\theta = 1$ or $\theta = 0$. Therefore the line $\theta = 0$ is tangent to the curve at the pole.

(iii) We have $dr/d\theta = a \sin\theta$.

Therefore $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a \sin\theta}{a(1 - \cos\theta)} = \cot \frac{1}{2}\theta$.

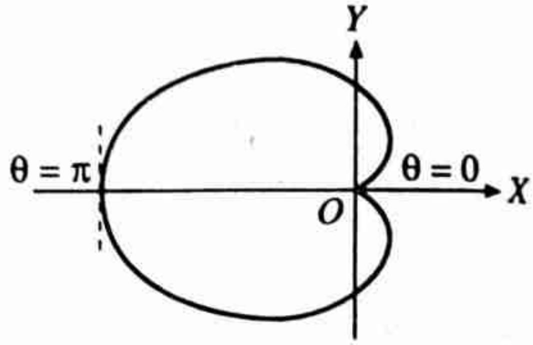
Now $\phi = 90^\circ$ when $\frac{1}{2}\theta = \frac{1}{2}\pi$ *i.e.,* $\theta = \pi$. Thus at the point $\theta = \pi$ the tangent to the curve is perpendicular to the radius vector.

(iv) Table showing corresponding values of θ and r :

$\theta = 0$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	π
$r = 0$	$\frac{1}{2}a$	a	$\frac{3}{2}a$	$2a$

Thus as θ increases from 0 to π, r also increases from 0 to $2a$.

Hence the curve is as shown in the figure.



Ex. 5. *Trace the curve* $r^2 = a^2 \cos 2\theta$.

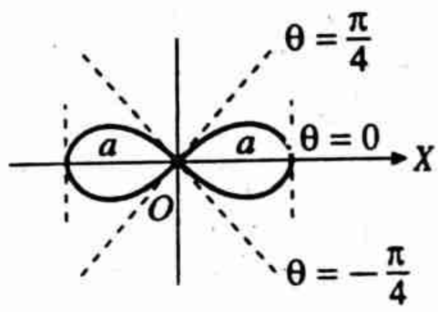
(Delhi 1981; Meerut 84S, 89P, 94)

Sol. (i) The curve is symmetrical about the initial line and it is also symmetrical about the pole.

(ii) When $r = 0, \cos 2\theta = 0$ or $2\theta = \pm \frac{1}{2}\pi$ *i.e.*, $\theta = \pm \frac{1}{4}\pi$.

Therefore the lines $\theta = \pm \frac{1}{4}\pi$ are the tangents to the curve at the pole.

When $\theta = 0, r = a$. Also the greatest value of the radius vector of this curve is a.



(iii) Table showing variation of r as θ varies from 0 to π :

$\theta = 0$	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$r^2 = a^2$	$\frac{1}{2}a^2$	0	–ive	0	$\frac{1}{2}a^2$	a^2
$r = \pm a$	$\pm a/\sqrt{2}$	0	imaginary	0	$\pm a/\sqrt{2}$	$\pm a$.

From above it is clear that the curve does not exist for values of θ lying between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

Hence the shape of the curve is as shown in the above figure.

***Ex. 6.** *Trace the curve* $r = a(1 + \cos\theta)$. **(Cardioid)**

(Meerut 1983S, 95, 98; Lucknow 77)

Sol. (i) The curve is symmetrical about the initial line since its equation remains unchanged by writing $-\theta$ in place of θ.

(ii) $r = 0$, when $\cos\theta = 1$ *i.e.*, $\theta = \pi$,

r is maximum when $\cos\theta = 1$, *i.e.*, $\theta = 0$. Then $r = 2a$.

Also r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. Then $r = 0$.

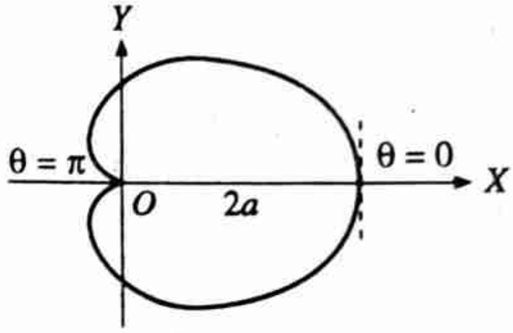
(iii) $(dr/d\theta) = -a\sin\theta$. When $0 < \theta < \pi$, $(dr/d\theta)$ is throughout negative. Therefore r decreases continuously as θ increases from 0 to π.

(iv) Also $\tan\phi = r\dfrac{d\theta}{dr} = -\dfrac{a(1+\cos\theta)}{a\sin\theta} = -\cot\dfrac{\theta}{2}$.

$\phi = 0$ when $\theta = \pi$. Then $r = 0$.

Therefore the line $\theta = \pi$ is tangent to the curve at the pole.

$\phi = 90^\circ$ when $\theta = 0$. Then $r = 2a$. Therefore the tangent at $\theta = 0$ is perpendicular to the radius vector $\theta = 0$.



(v) Since r is never greater than $2a$, therefore the curve will have no asymptotes.

(vi) The following table gives the corresponding values of θ and r.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$2a$	$3a/2$	a	$a/2$	0

The portion of the curve lying in the region $\pi < \theta < 2\pi$ can be drawn by symmetry. Hence the shape of the curve is as shown in the figure.

Ex. 7. *Trace the curve* $r = a\cos 2\theta$. **(Meerut 1984 P, 90P, 95BP)**

Sol. (i) The curve is symmetrical about the initial line.

(ii) $r = 0$, when $\cos 2\theta = 0$, *i.e.*, $2\theta = \pm\pi/2$ *i.e.*, $\theta = \pm\pi/4$.

Therefore the lines $\theta = \pm\pi/4$ are tangents to the curve at the pole.

r is maximum when $\cos 2\theta = 1$. Then $\theta = 0$ and $r = a$.

(iii) $\tan\phi = r\dfrac{d\theta}{dr} = a\cos 2\theta \cdot \dfrac{1}{-2a\sin 2\theta} = -\dfrac{1}{2}\cot 2\theta$.

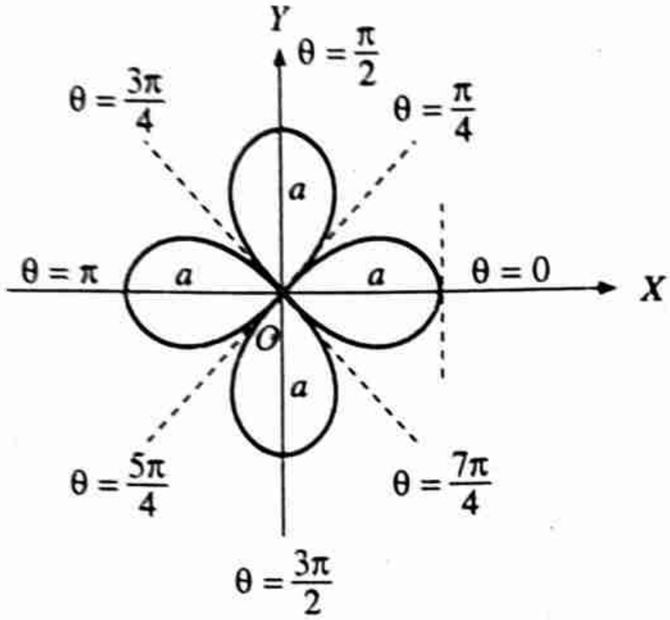
$\phi = 90^\circ$ when $2\theta = 0$ *i.e.*, $\theta = 0$. Therefore at the point $\theta = 0$, the tangent is perpendicular to the radius vector $\theta = 0$.

(iv) The following table gives the corresponding values of θ and r:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5}{6}\pi$	π
r	a	$\frac{1}{2}a$	0	$-\frac{1}{2}a$	$-a$	$-\frac{1}{2}a$	0	$\frac{1}{2}a$	a

The variation of θ from π to 2π need not be considered because of symmetry about the initial line.

Hence the curve is as shown in the figure. The curve consists of four similar loops, all lying within a circle of radius a and centre at the pole.



Important. The above curve is a particular case of the curves of the type $r = a \cos n\theta$ which have n loops when n is odd and $2n$ loops when n in even.

Ex. 8. *Trace the curve $r = a \sin 3\theta$.*

(Delhi 1983; Kanpur 76; Meerut 85 S, 91)

Sol. (i) The curve is not symmetrical about the initial line.

(ii) $r = 0$ when $\sin 3\theta = 0$ *i.e.*, $3\theta = 0, \pi$, *i.e.*, $\theta = 0, \pi/3$.

Therefore the lines $\theta = 0$ and $\theta = \pi/3$ are tangents to the curve at the pole.

Also r is maximum when $\sin 3\theta = 1$ *i.e.*, $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$.

The maximum value of r is a.

(iii) $\tan \phi = r \dfrac{d\theta}{dr} = -\dfrac{1}{3} \tan 3\theta$.

$\phi = 90^\circ$ when $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$.

Therefore at the point $\theta = \pi/6$, tangent is perpendicular to the radius vector $\theta = \pi/6$.

(iv) The following table gives the corresponding values of θ and r :

3θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π	$\frac{9\pi}{2}$	5π	$\frac{11\pi}{2}$	6π
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0	a	0	$-a$	0

(ii) By equating the coefficients of the two highest powers of x to zero find m and c.

(iii) Substitute these values of m and c in $y = mx + c$.

(iv) If a value of m, makes the coefficient of x^{n-1} identically zero, find c from the equation obtained by equating to zero the coefficient of x^{n-2}; and so on.

A short cut method.

(i) Put $x = 1$ and $y = m$ in the highest *i.e.* nth degree terms and get $\phi_n(m)$. Put $\phi_n(m) = 0$ and solve the resulting equation for m. Let $m = m_1, m_2, \ldots, m_n$ be its roots.

(ii) Form $\phi_{n-1}(m)$ by putting $x = 1$ and $y = m$ in the terms of degree $n - 1$. Then the values of c, say $c_1, c_2, \ldots, c_n$ are found by substituting $m = m_1, m_2, \ldots, m_n$, in turn, in the formula

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}.$$

The asymptotes then are

$$y = m_1 x + c_1;\ y = m_2 x + c_2; \ldots$$

Solved Examples

Ex. 1. *Find the asymptotes of the curve*

$$y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0.$$

(Lucknow 1982; Meerut 77)

Sol. Putting $y = mx + c$, we have

$$(mx + c)^3 - x^2(mx + c) - 2x(mx + c)^2 + 2x^3 - 7x(mx + c) + 3(mx + c)^2 + 2x^2 + 2x + 2(mx + c) + 1 = 0$$

or $$x^3(m^3 - m - 2m^2 + 2) + x^2(3m^2c - c - 4mc - 7m + 3m^2 + 2) + \ldots = 0.$$

Equating the coefficient of the highest degree term (*i.e.*, x^3) to zero, we get

$$m^3 - m - 2m^2 + 2 = 0$$

or $$(m - 1)(m + 1)(m - 2) = 0.$$

Therefore, $m = 1, -1, 2$.

Now equating the coefficient of x^2 to zero, we get

$$c(3m^2 - 4m - 1) + 3m^2 - 7m + 2 = 0$$

or $$c = -(3m^2 - 7m + 2)/(3m^2 - 4m - 1).$$

Therefore, when $m = 1$, $c = -1$; when $m = -1$, $c = -2$; and when $m = 2$, $c = 0$.

Substituting these pairs of values of m and c in $y = mx + c$, the required asymptotes are

$$y = x - 1,\ y = -x - 2 \text{ and } y = 2x.$$

Ex. 2. *Find the asymptotes of the curve*

$$x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 + 1 = 0.$$

(Agra 1987; Meerut 77, 84 S, 92)

Sol. Putting $y = mx + c$, we have

$$x^3 - 2(mx + c)^3 + 2x^2(mx + c) - x(mx + c)^2 + x(mx + c) - (mx + c)^2 + 1 = 0$$

or $$x^3(-2m^3 - m^2 + 2m + 1) + x^2(-6m^2c - 2mc + 2c + m - m^2) + \ldots = 0.$$

Equating to zero the coefficient of x^3, we have

$$2m^3 + m^2 - 2m - 1 = 0, \quad i.e, \quad (m^2 - 1)(2m + 1) = 0.$$

$\therefore \quad m = 1, -1, -\frac{1}{2}.$

Equating to zero the coefficient of x^2, we have

$$-2(3m^2 + m - 1)c - (m^2 - m) = 0,$$

i.e. $$c = (m - m^2)/2(3m^2 + m - 1).$$

Therefore, when $m = 1$, $c = 0$; when $m = -1$, $c = -1$; and when $m = -\frac{1}{2}$, $c = \frac{1}{2}$.

$\therefore$ the asymptotes are $y = x$, $y = -x - 1$ and $y = -\frac{1}{2}x + \frac{1}{2}$;

or $x - y = 0$, $x + y + 1 = 0$ and $x + 2y - 1 = 0$.

Ex. 3. *Find the asymptotes of the curve* $x^3 + y^3 - 3axy = 0$.

(Rohilkhand 1982; Delhi 82; Magadh 76; Meerut 88, 97; Gorakhpur 87)

Sol. Putting $x = 1$ and $y = m$ in the third and second degree terms separately, we get

$$\phi_3(m) = 1 + m^3, \quad \text{and} \quad \phi_2(m) = -3am.$$

Now we solve the equation $\phi_3(m) = (1 + m)(1 - m + m^2) = 0$.

Obviously $m = -1$ is the only real root of this equation.

To determine c, we have the formula

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-3am}{3m^2} = \frac{a}{m}.$$

Putting $m = 1$, we get $c = -a$.

Hence the only asymptote of the curve is

$$y = -x - a \quad \text{or} \quad x + y + a = 0.$$

Ex. 4. *Find the asymptotes of*

$$4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 - 7 = 0.$$

Sol. Putting $x = 1$ and $y = m$ in the third degree and second degree terms separately, we have

$$\phi_3(m) = 4 - m - 4m^2 + m^3; \; \phi_2(m) = 3 + 2m - m^2.$$

Now solving the equation $\phi_3(m) = 4 - m - 4m^2 + m^3 = 0$

i.e., $(4 - m)(1 - m^2) = 0$, we get $m = 1, -1, 4$.

Also $$\phi_3'(m) = -1 - 8m + 3m^2.$$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{3 + 2m - m^2}{1 + 8m - 3m^2}.$$

Putting $m = 1$, we get $c = \frac{2}{3}$;

when $m = -1$, $c = 0$ and when $m = 4$, $c = \frac{1}{3}$.

Hence the asymptotes are

$$y = x + \tfrac{2}{3};\ y = -x \ \text{ and } \ y = 4x + \tfrac{1}{3}.$$

Ex. 5. *Find the asymptotes of*

$$x^3 - y^3 - x^2 - 5y^2 - 11x - 5y + 7 = 0.$$

Sol. Putting $x = 1$, $y = m$ in the third degree and second degree terms separately, we have

$$\phi_3(m) = 1 - m^3 \quad \text{and} \quad \phi_2(m) = -1 - 5m^2.$$

$$\therefore \quad \phi_3'(m) = -3m^2.$$

Now $\phi_3(m) = 1 - m^3 = (1 - m)(1 + m + m^2) = 0$ gives $m = 1$ as the only real root.

Also $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-1 - 5m^2}{-3m^2}.$$

Putting $m = 1$, we have $c = -2$.

Hence the only asymptote of the curve is $y = x - 2$.

Ex 6. *Find all asymptotes of the curve*

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

(Meerut 1984, 94P, 97; Gorakhpur 82; Agra 88)

Sol. Putting $x = 1$ and $y = m$ in the third and second degree terms separately, we have

$$\phi_3(m) = 3 + 2m - 7m^2 + 2m^3 \quad \text{and} \quad \phi_2(m) = 7m^2 - 14m.$$

$$\therefore \quad \phi_3'(m) = 2 - 14m + 6m^2.$$

Putting $\phi_3(m) = 0$, we get $3 + 2m - 7m^2 + 2m^3 = 0$

or $(m - 1)(2m^2 - 5m - 3) = 0$ or $(m - 1)(2m + 1)(m - 3) = 0$.

$$\therefore \quad m = 1, 3, -\tfrac{1}{2}.$$

Also $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{14m - 7m^2}{2 - 14m + 6m^2}.$$

Putting $m = 1$, we get $c = -\frac{7}{6}$;

when $m = 3$, $c = -1$, and when $m = -\frac{1}{2}$, $c = -\frac{5}{6}$.

Hence asymptotes are

$$y = x - \tfrac{7}{6};\ y = 3x - 1 \text{ and } y = -\tfrac{1}{2}x - \tfrac{5}{6},$$

or $$6x - 6y - 7 = 0,\ y = 3x - 1 \text{ and } 6y + 3x + 5 = 0.$$

Ex. 7. *Find the asymptotes of the curve*

$$2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y - 3 = 0.$$

Sol. Here $\phi_3(m) = 2 + 3m - 3m^2 - 2m^3$,

$\phi_3'(m) = 3 - 6m - 6m^2$ and $\phi_2(m) = 3 - 3m^2$; $\phi_3(m) = 0$, gives $m = 1, -1/2, -2$.

Also $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{3m^2 - 3}{3 - 6m - 6m^2}.$$

When $m = 1, c = 0$; when $m = -\frac{1}{2}, c = -\frac{1}{2}$;

and when $m = -2, c = -1$.

$\therefore \quad y = x, y = -\frac{1}{2}x - \frac{1}{2}$ and $y = -2x - 1$

are the required asymptotes.

Ex. 8. *Find the asymptotes of the curve $y^3 = x^3 + ax^2$.*

(Kumayun 1983; Meerut 89)

Sol. The given curve is $y^3 - x^3 - ax^2 = 0$.

Putting $y = m$ and $x = 1$ in the highest *i.e.*, third degree terms in the equation of the curve, we get $\phi_3(m) = m^3 - 1$.

Solving the equation $\phi_3(m) = 0$,

i.e., $m^3 - 1 = 0$, *i.e.*, $(m - 1)(m^2 + m + 1) = 0$, we get $m = 1$ as the only real root.

The other two roots are imaginary.

Again putting $y = m$ and $x = 1$ in the second degree terms in the equation of the curve, we get $\phi_2(m) = -a$.

Now c is given by $c\,\phi_3'(m) + \phi_2(m) = 0$, *i.e.*, $c(3m^2) - a = 0$.

Putting $m = 1$, we get $3c - a = 0$ or $c = a/3$.

Hence the only real asymptote of the curve is

$$y = x + (a/3).$$

Ex. 9. *Find the asymptotes of*

$$x^3 - 2y^3 + xy(2x - y) + y(x - 1) + 1 = 0.$$

Sol. Putting $x = 1, y = m$ in the third and second degree terms separately, we have

$$\phi_3(m) = 1 - 2m^3 + 2m - m^2 \text{ and } \phi_2(m) = m.$$

$$\therefore \quad \phi_3'(m) = -6m^2 + 2 - 2m.$$

Now the slopes of the asymptotes are given by the equation $\phi_3(m) = 0$

or $$(1 - 2m^3 + 2m - m^2) = 0 \text{ or } (1 - m^2)(1 + 2m) = 0.$$

$$\therefore \quad m = 1, -1, -\tfrac{1}{2}.$$

Also we know that $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{m}{6m^2 + 2m - 2}.$$

$\therefore \quad$ when $m = 1, c = \frac{1}{6}$; when $m = -1, c = -\frac{1}{2}$

and when $m = -\frac{1}{2}$, $c = \frac{1}{3}$.

Hence the asymptotes are

$$y = x + \tfrac{1}{6};\ y = -x - \tfrac{1}{2} \text{ and } y = -\tfrac{1}{2}x + \tfrac{1}{3}$$

i.e. $6x - 6y + 1 = 0$; $2x + 2y + 1 = 0$ and $6y + 3x - 2 = 0$.

Ex. 10. *Find all the asymptotes of the curve*

$$y^3 - 3x^2y + xy^2 - 3x^3 + 2y^2 + 2xy + 4x + 5y + 6 = 0.$$

(Magadh 1970)

Sol. Putting $x = 1$ and $y = m$ in the third and second degree terms separately, we have

$$\phi_3(m) = m^3 - 3m + m^2 - 3 \text{ and } \phi_2(m) = 2(m^2 + m).$$

$$\therefore\quad \phi_3'(m) = 3m^2 + 2m - 3.$$

Equating $\phi_3(m)$ to zero, we get $m^3 + m^2 - 3m - 3 = 0$

or $m^2(m + 1) - 3(m + 1) = 0$ or $(m + 1)(m^2 - 3) = 0$

i.e., $m = -1, +\sqrt{3}, -\sqrt{3}$.

Also we know that $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = -\dfrac{2(m^2 + m)}{3m^2 + 2m - 3}$.

$\therefore$ when $m = 1$, $c = -2$; when $m = \sqrt{3}$, $c = -1$;

and when $m = -\sqrt{3}$, $c = -1$.

Hence the asymptotes are

$$y = x - 2;\ y = \pm x\sqrt{3} - 1.$$

§ 4. Non-existence of Asymptotes.

If $\phi_n(m) = 0$, gives one or more values of m, such that they make $\phi_n'(m) = 0$, whereas $\phi_{n-1}(m) \neq 0$, then from the equation $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$, we get $c = +\infty$ or $-\infty$ and this corresponds to the case when the tangent goes farther and farther away from the origin as $x \to \infty$. Therefore for such values of m, we shall get no asymptotes.

Ex. 11. *Find the asymptotes of the curve* $y^3 = x^2 + 3x$.

Sol. Putting $x = 1$, $y = m$ in the third degree and second degree terms separately, we have

$$\phi_3(m) = m^3 \text{ and } \phi_2(m) = -1.$$

$$\therefore\quad \phi_3'(m) = 3m^2.$$

Now $\phi_3(m) = 0$ *i.e.*, $m^3 = 0$ gives $m = 0, 0, 0$.

Also $c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = -\dfrac{-1}{3m^2}$ which is ∞ for $m = 0$.

Hence $y^3 = x^2 + 3x$ has no asymptotes.

§ 5. Case of parallel asymptotes.

(i) **Two parallel asymptotes.** Suppose the equation $\phi_n(m) = 0$ gives us two equal values of m. This repeated value of m makes $\phi_n'(m) = 0$. In case it does not make $\phi_{n-1}(m)$ equal to zero, the value of c determined by the equation $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$ comes out to be infinite, and hence the asymptotes corresponding to this value of m do not exist. Thus for the existence of the asymptotes corresponding to this value of m, it is necessary that it must make $\phi_{n-1}(m)$ equal to zero. But then the equation $c\phi_n'(m) + \phi_{n-1}(m) = 0$ from which c is usually determined reduces to the identity

$$0 \cdot c + 0 = 0,$$

and we cannot find the value of c in this way. To determine c in this case, we equate to zero the coefficient of x^{n-2} in the equation (3) of § 3, and we get the equation

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0.$$

This equation is a quadratic in c. It gives us two values of c, say c_1 and c_2, corresponding to that repeated value of m. The corresponding asymptotes are $y = mx + c_1$ and $y = mx + c_2$, which are obviously parallel.

(ii) **Three parallel asymptotes.** If the equation $\phi_n(m) = 0$ gives us three equal values of m, then this repeated value of m makes $\phi_n'(m)$ and $\phi_n''(m)$ equal to zero. For the existence of the corresponding asymptotes it must make $\phi_{n-1}(m)$ equal to zero. If it also makes $\phi_{n-1}'(m)$ and $\phi_{n-2}(m)$ equal to zero, then the equation to determine c reduces to the identity

$$0 \cdot c^2 + 0 \cdot c + 0 = 0,$$

and we cannot find the values of c in this way. To determine c in this case, we equate to zero the coefficient of x^{n-3} in the equation (3) of § 3, and we get the equation

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0.$$

This equation gives us three values of c corresponding to that repeated value of m and accordingly we get three parallel asymptotes. In a similar way we can discuss the case of more than three parallel asymptotes.

Solved Examples

Ex. 12. *Find the asymptotes of* $y^3 + x^2y + 2xy^2 - y + 1 = 0$.

(Mysore 1971; U.P. P.C.S. 92)

Sol. Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = m^3 + m + 2m^2, \quad \text{and} \quad \phi_2(m) = 0.$$

$$\therefore \quad \phi_3'(m) = 3m^2 + 4m + 1.$$

Now the slopes of the asymptotes are given by the equation $\phi_3(m) = 0$

i.e., $m^3 + 2m^2 + m = 0$ *i.e.*, $m(m^2 + 2m + 1) = 0$

i.e., $m(m + 1)^2 = 0$.

$\therefore \quad m = 0, -1, -1.$

Again, c, is given by the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3m^2 + 4m + 1} \qquad \text{...(i)}$$

When $m = 0$, we have $c = -\frac{0}{1} = 0$, and the corresponding asymptote is $y = 0 \cdot x + 0$ *i.e.*, $y = 0$.

When $m = -1$, the equation (1) gives $c = -0/0$ which is indeterminate form. So the equation (1) fails to give c when $m = -1$. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation of the curve, we get $\phi_1(m) = -m$. Also $\phi_3''(m) = 6m + 4$, $\phi_2'(m) = 0$.

Therefore for $m = -1$, c is given by the equation

$$\tfrac{1}{2}c^2(6m + 4) + c \cdot 0 - m = 0 \text{ i.e., } (3m + 2)c^2 - m = 0.$$

Putting $m = -1$ in this equation, we get

$$-c^2 + 1 = 0 \quad \text{or} \quad c^2 = 1 \quad \text{or} \quad c = \pm 1.$$

Hence $y = -x + 1$ and $y = -x - 1$ are two parallel asymptotes corresponding to the slope $m = -1$.

$\therefore$ all the three asymptotes of the curve are

$$y = 0, y + x - 1 = 0 \text{ and } y + x + 1 = 0.$$

***Ex. 13.** *Find the asymptotes of* $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$.

(Rohilkhand 1991; Gorakhpur 88; Agra 86; Meerut 76, 86, 91S, 93)

Sol. Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = 1 + 3m - 4m^3 \text{ and } \phi_2(m) = 0.$$

(Note that there are no second degree terms.)

$\therefore \qquad \phi_3'(m) = 3 - 12m^2.$

Now the slopes of the asymptotes are given by the equation $\phi_3(m) = 0$ *i.e.*, $1 + 3m - 4m^3 = 0$ *i.e.*, $(1 - m)(1 +$ $= 0$.

$\therefore \quad m = 1, -\frac{1}{2}, -\frac{1}{2}.$

Again, c, is given by the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3 - 12m^2}. \qquad \ldots(1)$$

When $m = 1$, we have $c = 0$, and the corresponding asymptote is $y = x + 0$ *i.e.*, $y = x$.

When $m = -\frac{1}{2}$, the equation (1) fails to give c. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation of the curve, we get $\phi_1(m) = -1 + m$. Also $\phi_3''(m) = -24m$ and $\phi_2'(m) = 0$. Hence for $m = -\frac{1}{2}$, c is to be given by

$$\tfrac{1}{2}c^2(-24m) + 0 + m - 1 = 0.$$

Putting $m = -\frac{1}{2}$ in this equation, we get

$$6c^2 - \tfrac{3}{2} = 0 \quad \text{or} \quad c^2 = \tfrac{1}{4} \quad \text{or} \quad c = \pm\tfrac{1}{2}.$$

Hence $y = -\frac{1}{2}x + \frac{1}{2}$ and $y = -\frac{1}{2}x - \frac{1}{2}$ are two parallel asymptotes corresponding to the slope $m = -\frac{1}{2}$.

Therefore the required asymptotes are

$$y = x \quad \text{and} \quad x + 2y = \pm 1.$$

Ex. 14. *Find all the asymptotes of the curve*

$$(x + y)^2(x + 2y + 2) = x + 9y + 2.$$

(Kanpur 1979; Rohilkhand 90; Meerut 83 S)

Sol. The equation of the given curve can be written as

$$(x + y)^2(x + 2y) + 2(x + y)^2 - (x + 9y) - 2 = 0.$$

Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we have

$$\phi_3(m) = (1 + m)^2(1 + 2m) \text{ and } \phi_2(m) = 2(1 + m)^2.$$

$$\therefore \quad \phi_3'(m) = 2(1 + m)(1 + 2m) + 2(1 + m)^2.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \quad \textit{i.e.}, \quad (1 + m)^2(1 + 2m) = 0.$$

$\therefore \quad m = -\frac{1}{2}, -1, -1.$

To determine c, we have the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2(1+m)^2}{2(1+m)(1+2m)+2(1+m)^2} \qquad ...(1)$$

When $m = -\frac{1}{2}$, we have

$$c = -\frac{2(1-\frac{1}{2})^2}{2(1-\frac{1}{2})(1-1)+2(1-\frac{1}{2})^2} = -1,$$

and the corresponding asymptote is $y = -\frac{1}{2}x - 1$ *i.e.*, $2y + x + 2 = 0$.

When $m = -1$, the equation (1) fails to give c, because then the value of c takes the indeterminate form 0/0. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Now $\phi_3''(m) = 2(1+2m) + 4(1+m) + 4(1+m)$, $\phi_2'(m) = 4(1+m)$. Also putting $y = m$ and $x = 1$ in the first degree terms of the equation of the curve, we get $\phi_1(m) = -(1+9m)$. Hence for $m = -1$, c is given by

$$\tfrac{1}{2}c^2\{2(1+2m) + 8(1+m)\} + c\{4(1+m)\} - (1+9m) = 0.$$

Putting $m = -1$ in this equation, we get

$$-c^2 + 8 = 0 \quad \text{or} \quad c = \pm 2\sqrt{2}.$$

Hence $y = -x + 2\sqrt{2}$ and $y = -x - 2\sqrt{2}$ are two parallel asymptotes corresponding to the slope $m = -1$.

$\therefore$ the required asymptotes are

$$2y + x + 2 = 0 \quad \text{and} \quad y + x = \pm 2\sqrt{2}.$$

Ex. 15. *Find all the asymptotes of the curve*

$$(x+y)^2(x+2y+2) = x + 9y - 2.$$

Sol. Proceed exactly as in Ex. 14. The required asymptotes are $2y + x + 2 = 0$ and $y + x = \pm 2\sqrt{2}$.

§ 6. Asymptotes parallel to the co-ordinate axes.

(i) Asymptotes parallel to y-axis of a rational algebraic curve.

Asymptotes parallel to y-axis cannot be determined by the method given in § 3 because the equation to a straight line parallel to y-axis cannot be put in the form $y = mx + c$. Here we shall discuss the method of finding asymptotes parallel to y-axis.

Let the equation of the curve, when arranged in descending powers of y, be

$$y^m\phi(x) + y^{m-1}\phi_1(x) + y^{m-2}\phi_2(x) + \ldots = 0, \qquad ...(1)$$

where $\phi(x)$, $\phi_1(x)$, $\phi_2(x)$ etc., are polynomials in x.

Dividing the equation (1) by y^m, we get

$$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots = 0. \qquad \ldots(2)$$

If $x = k$ is an asymptote parallel to y-axis of (1), then obviously $k = \lim_{y \to \infty} x$, where (x, y) lies on the curve (1). Therefore taking limit of (2) as $x \to \infty$ and remembering that $x \to k$ as $y \to \infty$, we get $\phi(k) = 0$.

Therefore k is a root of the equation $\phi(x) = 0$.

If k_1, k_2 etc., be the roots of $\phi(x) = 0$, then the asymptotes of (1) parallel to y-axis are $x = k_1, x = k_2$, etc.

From algebra, we know that if k_1 is a root of $\phi(x) = 0$, then $x - k_1$ must be a factor of $\phi(x)$. Also $\phi(x)$ is the coefficient of the highest power of y *i.e.*, y^m in the equation of the curve. Hence we have the following simple rule :

The asymptotes parallel to the axis of y are obtained by equating to zero the coefficient of the highest power of y in the equation of the curve. In case the coefficient of highest power of y, is a constant or if its linear factors are all imaginary, there will be no asymptotes parallel to y-axis.

(ii) Asymptotes parallel to x-axis. Proceeding as above, we have the following rule for finding the asymptotes parallel to x-axis of a rational algebraic curve :

The asymptotes parallel to the axis of x are obtained be equating to zero the coefficient of the highest power of x, in the equation of the curve. In case the coefficient of the highest power of x, is a constant or its linear factors are all imaginary, there will be no asymptotes parallel to x-axis.

§ 7. Total number of asymptotes of a curve.

The number of asymptotes, real or imaginary, of an algebraic curve of the nth degree cannot exceed n.

The slopes of the asymptotes which are not parallel to y-axis are given as the roots of the equation $\phi_n(m) = 0$ which is of degree n at the most. If the equation of the curve possesses some asymptotes parallel to y-axis, then we can easily see that the degree of $\phi_n(m) = 0$ will be smaller than n by atleast the same number. In general, one value of m gives only one value of c. In case the equation for determining c is a quadratic, the equation $\phi_n(m) = 0$ has two equal roots. Similarly if the equation for determining c is a cubic, the equation $\phi_n(m) = 0$ has three equal roots.

Hence *a curve of degree n cannot have more than n asymptotes*. But the number of real asymptotes can be less than n. Some roots of the equation $\phi_n(m) = 0$ may come out to be imaginary or even corresponding to a real value of m the value of c may come out to be infinite.

***§ 8. Complete working rule for finding the asymptotes of rational algebraic curves.**

(i) A curve of degree n cannot have more than n asymptotes real or imaginary.

(ii) Equating to zero the coefficient of the highest power of y in the equation of the curve, we get asymptotes parallel to y-axis. Similarly equating to zero the coefficient of the highest power of x in the equation of the curve, we get asymptotes parallel to x-axis.

If $y = mx + c$ is an asymptote not parallel to y-axis, then the values of m and c are found as follows :

(iii) Putting $y = m$ and $x = 1$ in the highest *i.e.*, nth degree terms in the equation of the curve, we get $\phi_n(m)$. Solving the equation $\phi_n(m) = 0$, we get the slopes of the asymptotes. If some values of m are imaginary, we reject them.

(iv) Corresponding to a value of m, the value of c is given by the equation

$$c\,\phi_n'(m) + \phi_{n-1}(m) = 0,$$

where $\phi_{m-1}(m)$ is obtained by putting $y = m$ and $x = 1$ in the $(n-1)$th degree terms in the equation of the curve. The asymptotes corresponding to $m = 0$ are already found in (ii). So we need not find the value of c corresponding to $m = 0$.

(v) If corresponding to two equal values of m, the equation for determining c, given in (iv), reduces to the identity $0 \,.\, c + 0 = 0$, then the values of c are given by

$$\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m) = 0.$$

(vi) Similarly, if three values of m are equal and the equation for determining c, given in (v), reduces to the identity

$$0 \,.\, c^2 + 0 \,.\, c + 0 = 0,$$

then the corresponding values of c are given by

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0.$$

Solved Examples

Ex. 16 (a). *Find the asymptotes parallel to the coordinate axes of the curve* $x^2(x-y)^2 + a^2(x^2 - y^2) - a^2xy = 0$.

Sol. Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \quad i.e., \quad x = \pm a.$$

The coefficient of the highest power of x (*i.e.*, of x^4) is merely a constant. Hence there is no asymptote parallel to x-axis.

Ex. 16 (b). *Find the asymptotes of the curve*

$$(a^2/x^2) - (b^2/y^2) = 1. \qquad \textbf{(Meerut 1980)}$$

Sol. The equation of the given curve can be written as

$$a^2y^2 - b^2x^2 = x^2y^2 \quad \text{or} \quad x^2y^2 - a^2y^2 + b^2x^2 = 0.$$

Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \quad i.e., \quad x = \pm a.$$

Also equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by

$$y^2 + b^2 = 0,$$

which gives two imaginary asymptotes.

Thus all the four possible asymptotes of the curve have been found and the only real asymptotes are $x = \pm a$.

Ex. 17 (a). *Find the asymptotes of the curve*

$$y^2(x^2 - a^2) = x. \qquad \textbf{(Meerut 1985 S, 86 P, 91 P)}$$

Sol. Since the given curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \quad i.e., \quad x = \pm a.$$

Again equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by

$$y^2 = 0 \; i.e, \; y = 0, y = 0 \qquad \text{(two coincident asymptotes).}$$

Thus all the four asymptotes have been found.

$\therefore$ The asymptotes are $x = \pm a, y = 0$.

Ex. 17 (b). *Find the asymptotes of the curve*

$$x^2y^2 = a^2(x^2 + y^2). \qquad \textbf{(Meerut 1980, 94, 96 BP)}$$

Sol. The given curve is of degree 4, so it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \quad i.e., \quad x = \pm a.$$

Again equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by

$$y^2 - a^2 = 0 \; i.e., \; y = \pm a.$$

Thus all the four asymptotes have been found and they are

$$x = \pm a, y = \pm a.$$

**Ex. 17 (c). *Find the asymptotes of the curve*

$$y^2 (x^2 - a^2) = x^2 (x^2 - 4a^2).$$

(Meerut 1972, 85, 88S, 90S, 96)

Sol. The given curve is

$$y^2 (x^2 - a^2) = x^2 (x^2 - 4a^2)$$

or $$y^2x^2 - x^4 - a^2y^2 + 4a^2x^2 = 0. \qquad ...(1)$$

Since the curve (1) is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \text{ i.e., } x = \pm a. \qquad \text{(Two asymptotes)}$$

The coefficient of the highest power of x *i.e.*, of x is simply a constant and so there is no asymptote parallel to x-axis.

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.*, fourth degree terms in the equation (1) and we get

$$\phi_4 (m) = m^2 - 1.$$

The slopes of the asymptotes are given by the equation

$$\phi_4 (m) = 0 \text{ i.e., } m^2 - 1 = 0.$$

$$\therefore \quad m = \pm 1.$$

Again putting $y = m$ and $x = 1$ in the next highest *i.e.*, third degree terms, we get

$\phi_3 (m) = 0$. [Note that there are no terms of degree 3 in the equation of the curve and so $\phi_3 (m) = 0$.]

Now c is given by the equation $c \phi_4' (m) + \phi_3 (m) = 0$

i.e., $$c (2m) + 0 = 0 \text{ i.e., } 2cm = 0.$$

When $m = 1, c = 0$; and when $m = -1, c = 0$.

$\therefore$ the asymptotes are $y = 1x + 0$ and $y = (-1) x + 0$

i.e., $$y = x \quad \text{and} \quad y = -x.$$

Hence all the four asymptotes of the curve are $x = \pm a, y = \pm x$.

Ex. 17 (d). *Find the asymptotes of the curve*

$$y^2 (x - 2) = x^2 (y - 1).$$ **(Meerut 1991, 96P)**

Sol. The given curve is

$$y^2 (x - 2) - x^2 (y - 1) = 0$$

or $$y^2x - x^2y - 2y^2 + x^2 = 0. \qquad ...(1)$$

Since the curve (1) is of degree 3, it cannot have more than three asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptote parallel to y-axis is given by

$$x - 2 = 0 \text{ i.e., } x = 2.$$

Again equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptote parallel to x-axis is given by

$$y - 1 = 0 \text{ i.e., } y = 1.$$

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.*, third degree terms in the equation (1) and we get

$$\phi_3(m) = m^2 - m.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \text{ i.e., } m^2 - m = 0 \text{ i.e., } m(m-1) = 0.$$

$\therefore \quad m = 0, 1.$

Again putting $y = m$ and $x = 1$ in the second degree terms, we get $\phi_2(m) = -2m^2 + 1.$

Now c is given by the equation

$$c\,\phi_3'(m) + \phi_2(m) = 0$$

i.e., $$c(2m-1) - 2m^2 + 1 = 0.$$

The asymptote corresponding to $m = 0$ is parallel to x-axis and has already been found. So no need of finding c for $m = 0$.

When $m = 1$, we have $c - 1 = 0$ *i.e.*, $c = 1$ and so the corresponding asymptote is $y = 1x + 1$ *i.e.*, $y = x + 1$. Hence all the three asymptotes of the given curve are $x = 2$, $y = 1$ and $y = x + 1$.

Ex. 17 (e). *Find the asymptotes of the curve*

$$x^2y^3 + x^3y^2 = x^3 + y^3.$$ **(Rohilkhand 1989)**

Sol. The given curve is

$$x^2y^3 + x^3y^2 - x^3 - y^3 = 0 \qquad ...(1)$$

The curve (1) is of degree 5, so it cannot have more than five asymptotes.

The asymptotes parallel to y-axis are given by

$$x^2 - 1 = 0 \text{ i.e., } x = \pm 1.$$

The asymptotes parallel to x-axis are given by

$$y^2 - 1 = 0 \text{ i.e., } y = \pm 1.$$

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest i.e., fifth degree terms in the equation (1) and we get

$$\phi_5(m) = m^3 + m^2.$$

The slopes of the asymptotes are given by the equation

$$\phi_5(m) = 0 \text{ i.e., } m^3 + m^2 = 0 \text{ i.e., } m^2(m+1) = 0.$$

$$\therefore \quad m = 0, 0, -1.$$

Again putting $y = m$ and $x = 1$ in the next highest *i.e.*, fourth degree terms in the equation (1), we get

$\phi_4(m) = 0.$ [Note that there are no fourth degree terms in the equation (1) and so $\phi_4(m) = 0$.]

Now c is given by the equation

$$c\,\phi_5'(m) + \phi_4(m) = 0$$

i.e., $$c\,(3m^2 + 2m) + 0 = 0. \qquad ...(2)$$

The asymptotes corresponding to $m = 0$ are parallel to x-axis and have already been found and so no need of finding c for $m = 0$.

When $m = -1$, we have from (2), $c = 0$ and so the corresponding asymptote is $y = -1x + 0$ *i.e.,* $y = -x$.

Hence all the five asymptotes of the given curve are

$$x = \pm 1, y = \pm 1, y = -x.$$

Ex. 18. *Show that the asymptotes of the curve*

$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$

form a square through two of whose angular points the curve passes.

(Gorakhpur 1976; Rohilkhand 80; Meerut 83, 80; U.P. P.C.S. 94)

Sol. Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.,* of y^2), we get the asymptotes parallel to y-axis as $x^2 - a^2 = 0$, *i.e.,* $x = \pm a$.

Also equating to zero the coefficient of the highest power of x (i.e., of x^2), we get the asymptotes parallel to x-axis as $y^2 - a^2 = 0$ *i.e.,* $y = \pm a$.

Thus all the four asymptotes of the curve have been found and they are $x = \pm a$, $y = \pm a$. These lines form a square whose sides are parallel to the axes each of length $2a$.

The angular points of the square are (a, a), $(a, -a)$, $(-a, a)$ and $(-a, -a)$. It is evident that the curve passes through two angular points $(a, -a)$ and $(-a, a)$.

Ex. 19. *Find the asymptotes of the curve*

$$x^2(x^2 + y^2) = a^2(y^2 - x^2).$$

Sol. The equation of the curve is $x^2(x^2 + y^2) - a^2(y^2 - x^2) = 0$. Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.,* of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \text{ i.e., } x = \pm a.$$

The coefficient of the highest power of x, *i.e.,* (of x^4) is merely a constant. Hence there is no asymptote parallel to x-axis.

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.,* fourth degree terms and we get

$$\phi_4(m) = 1 + m^2.$$

The roots of the equation $\phi_4(m) = 0$, *i.e.*, $1 + m^2 = 0$ are imaginary and consequently the corresponding asymptotes are imaginary.

Hence the only real asymptotes of the curve are $x = \pm a$.

Ex. 20. *Find the asymptotes of the curve*

$$x^2(x-y)^2 + a(x^2 - y^2) - a^2xy = 0.$$

Sol. Since the given curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get the asymptotes parallel to y-axis as $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$. But there is no asymptote parallel to x-axis because the coefficient of the highest power of x (*i.e.*, of x^4) is merely a constant.

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.*, fourth degree terms of the equation of the curve, and we get $\varphi_4(m) = (1-m)^2$.

The slopes of the asymptotes are given by the equation

$$\phi_4(m) = 0 \text{ i.e., } (1-m)^2 = 0. \quad \therefore \quad m = 1, 1.$$

Now $\phi_3(m) = 0$ because there are no third degree terms in the equation of the curve. To determine c we have the equation

$$c\,\phi_4'(m) + \phi_3(m) = 0$$

i.e., $$c\{-2(1-m)\} + 0 = 0. \quad ...(1)$$

For $m = 1$, the equation (1) reduces to the identity $c \cdot 0 + 0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2!}\phi_4''(m) + \frac{c}{1!}\phi_3'(m) + \phi_2(m) = 0. \quad ...(2)$$

Now $\phi_4''(m) = 2$, $\phi_3'(m) = 0$ and $\phi_2(m) = -a^2m$.

Putting these values in (2), we get $c^2 - a^2m = 0$.

Putting $m = 1$ in this equation, we get $c^2 - a^2 = 0$ *i.e.*, $c = \pm a$.

Hence $y = x + a$ and and $y = x - a$ are two parallel asymptotes corresponding to the slope $m = 1$.

$\therefore$ the requird asymptotes are

$$x = \pm a,\ y = x + a \text{ and } y = x - a.$$

Ex. 21. (a). *Find the asymptotes of the curve*

$$x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0.$$

(Lucknow 1980, 70; Gorakhpur 79; Meerut 93, 98)

Sol. The given curve is of degree 3. So it cannot have more than three asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get the asymptote parallel to y-axis as $x = 0$. Also there is no asymptote parallel to x-axis because the coefficient of x^2 is merely a constant.

Now we proceed to find the remaining oblique asymptotes.

Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = 1 - 2m + m^2 \quad \text{and} \quad \phi_2(m) = 1 - m.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \text{ i.e., } 1 - 2m + m^2 = 0 \text{ i.e., } (1 - m)^2 = 0. \quad \therefore m = 1, 1.$$

To determine c, we have the equation

$$c\,\phi_3'(m) + \phi_2(m) = 0 \text{ i.e., } c(-2 + 2m) + (1 - m) = 0. \quad \text{...(1)}$$

For $m = 1$, the equation (1) reduces to the identity $c \cdot 0 + 0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2!}\phi_3''\, m + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Now $\phi_3''(m) = 2, \phi_2'(m) = -1$ and $\phi_1(m) = 0$ because there are no first degree terms in the equation of the curve. So for $m = 1$, c is to be given by

$$\tfrac{1}{2}c^2 \cdot (2) + c \cdot (-1) + 0 = 0 \text{ i.e., } c^2 - c = 0 \text{ i.e., } c(c - 1) = 0.$$

$\therefore \quad c = 0, 1.$

Hence $y = x + 0$ and $y = x + 1$ are two parallel asymptotes corresponding to the slope $m = 1$.

$\therefore$ the required asymptotes are $x = 0, y = x$ and $y = x + 1$.

Ex. 21 (b). *Find the asymptotes of the curve*

$$x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0. \quad \textbf{(Meerut 1988P, 90P)}$$

Sol. Proceed as in Ex. 21 (a).

Ans. $x = 0, x + y = 0, x + y - 1 = 0.$

Ex. 22. *Find the asymptotes of the following curves :*

(a) $x^2y - xy^2 + 3x^2 - 2y^2 = 0;$

(b) $x^3 - xy^2 + 6y^2 = 0.$

Sol. (a) The given curve is of degree 3. So it cannot have more than 3 asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get the asymptote parallel to y-axis as $-x - 2 = 0$ *i.e.*, $x = -2$. Also equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptote parallel to x-axis is given by $y + 3 = 0$, *i.e.*, $y = -3$.

Now we proceed to find the remaining oblique asymptotes.

Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = m - m^2 \text{ and } \phi_2(m) = 3 - 2m^2.$$

The slopes of the asymptotes are given by the equation

$\phi_3(m) = 0$ *i.e.*, $m - m^2 = 0$ *i.e.*, $m(1 - m) = 0$. $\therefore m = 0, 1$.

To determine c, we have the equation

$$c\,\phi_3'(m) + \phi_2(m) = 0 \text{ i.e., } c(1 - 2m) + 3 - 2m^2 = 0. \quad ...(1)$$

The asymptote corresponding to $m = 0$ (*i.e.*, parallel to x-axis) has already been found. So putting $m = 1$ in (1), we get

$c(-1) + 1 = 0$ *i.e.*, $c = 1$ and the corresponding asymptote is $y = x + 1$.

$\therefore$ the required asymptotes are $x = -2$, $y = -3$, and $y = x + 1$.

(b) The given curve is of degree 3 and so it cannot have more than three asymptotes. The coefficients of the highest powers of y and x are $(-x + 6)$ and 1 respectively. The asymptote parallel to y-axis is given by $-x + 6 = 0$, *i.e.*, $x = 6$. Also there is no asymptote parallel to x-axis.

For oblique asymptotes, we have

$$\phi_3(m) = 1 - m^2 \quad \text{and} \quad \phi_2(m) = 6m^2.$$

The equation $\phi_3(m) = 0$ gives $m = 1, -1$.

To determine c, we have the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{6m^2}{-2m} = \frac{3m^2}{m} = 3m.$$

When $m = 1$, $c = 3$ and when $m = -1$, $c = -3$.

Hence the oblique asymptotes are $y = x + 3$ and $y = -x - 3$.

$\therefore$ the required asymptotes are $x = 6$, $y = x + 3$ and $y = -x - 3$.

Ex. 23 (a). *Find all the asymptotes of the curve*

$$y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 - 1 = 0.$$

(Gorakhpur 1989; Meerut 87, 91P)

Sol. Putting $y = m$ and $x = 1$ in the highest i.e., third degree terms of the equation of the curve, we get

$$\phi_3(m) = m^3 - m^2 - m + 1.$$

The slopes of the asymptotes are given by

$\phi_3(m) = 0$ *i.e.*, $m^3 - m^2 - m + 1 = 0$ *i.e.*, $(m - 1)^2(m + 1) = 0$.

$\therefore \quad m = 1, 1, -1.$

Now putting $y = m$ and $x = 1$ in the next highest i.e., second degree terms, we get $\phi_2(m) = 1 - m^2$.

To determine c, we have $c\phi_3'(m) + \phi_2(m) = 0$

i.e., $\quad c(3m^2 - 2m - 1) + (1 - m^2) = 0. \quad ...(1)$

When $m = -1$, we have $c = 0$ and the corresponding asymptote is

$$y = -x + 0 \text{ i.e., } y + x = 0.$$

When $m = 1$, the equation (1) reduces to the identity

$$c \cdot 0 + 0 = 0$$

and we cannot determine c from it. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms in the equation of the curve, we get $\phi_1(m) = 0$, since there are no first degree terms.

Hence for $m = 1$, c is to be given by

$$(c^2/2)(6m - 2) + c(-2m) = 0$$

i.e., $$(3m - 1)c^2 - 2mc = 0.$$

For $m = 1$, this becomes $2c^2 - 2c = 0$ i.e., $c(c - 1) = 0$.

$\therefore$ $c = 0$ and 1.

Hence $y = x + 1$ and $y = x + 0$ are two parallel asymptotes corresponding to the slope $m = 1$.

$\therefore$ the required asymptotes are

$$y + x = 0,\ y - x = 0,\ y - x - 1 = 0.$$

Ex. 23 (b). *Find the asymptotes of the curve*

$$x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0.$$

(Meerut 1991, 95, 96P, 98)

Sol. The given curve is of degree 3, so it cannot have more than three asymptotes.

There are no asymptotes parallel to y-axis because the coefficient of the highest power of y i.e., of y^3 is merely a constant. Similarly there are no asymptotes parallel to x-axis.

Now putting $y = m$ and $x = 1$ in the highest i.e., third degree terms in the equation of the curve, we get

$$\phi_3(m) = 1 + m - m^2 - m^3.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \text{ i.e., } 1 + m - m^2 - m^3 = 0$$

or $$(1 + m) - m^2(1 + m) = 0$$

or $$(1 + m)(1 - m^2) = 0$$

or $$(1 + m)^2(1 - m) = 0.$$

$\therefore$ $m = 1, -1, -1$.

Again putting $y = m$ and $x = 1$ in the second degree terms in the equation of the curve, we get

$\phi_2(m) = 0$, since there are no second degree terms.

To determine c, we have the equation

$$c\phi_3'(m) + \phi_2(m) = 0 \text{ i.e., } c(1 - 2m - 3m^2) + 0 = 0. \quad ...(1)$$

For $m = 1$, the equation (1) gives $-4c = 0$ or $c = 0$ and the corresponding asymptote is $y = 1 . x + 0$ i.e., $y = x$.

For $m = -1$, the equation (1) reduces to the identity $c . 0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Now putting $y = m$ and $x = 1$ in the first degree terms in the equation of the curve, we get

$$\phi_1(m) = -3 - m.$$

So for $m = -1$, c is to be given by the equation

$$\tfrac{1}{2}c^2 . (-2 - 6m) + 0 . c - 3 - m = 0$$

or $$c^2(1 + 3m) + 3 + m = 0.$$

Putting $m = -1$ in this equation, we get

$$-2c^2 + 2 = 0 \quad \text{or} \quad c^2 = 1 \quad \text{or} \quad c = \pm 1.$$

$\therefore$ the asymptotes corresponding to $m = -1$ are

$$y = -1x + 1 \text{ and } y = -1x - 1.$$

Hence all the three asymptotes of the given curve are

$$y = x, y = -x + 1 \text{ and } y = -x - 1.$$

Ex. 24. *Find all the asymptotes of the curve*

$$x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0.$$

(Gorakhpur 1973)

Sol. Putting $y = m$ and $x = 1$ in the third and second degree terms separately, we get

$$\phi_3(m) = 1 - m - m^2 + m^3 \text{ and } \phi_2(m) = 2 - 4m^2 + 2m.$$

$$\therefore \quad \phi_3'(m) = -1 - 2m + 3m^2.$$

Now the slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \text{ i.e., } 1 - m - m^2 + m^3 = 0 \text{ i.e., } (1 - m)^2(1 + m) = 0.$$

$$\therefore \quad m = 1, 1, -1.$$

Again c is given by the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{2 - 4m^2 + 2m}{-1 - 2m + 3m^2} = \frac{2 - 4m^2 + 2m}{1 + 2m - 3m^2}. \quad ...(1)$$

When $m = -1$, we have from (1), $c = 1$. The corresponding asymptote is $y = -x + 1$ i.e., $x + y = 1$.

When $m = 1$, the equation (1) gives $c = 0/0$ and thus the equation (1) fails to give c for $m = 1$.

In this case c is to be determined from the equation

$$(c^2/2!)\phi_3''(m) + (c/1!)\phi_2'(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation of the curve, we get $\phi_1(m) = 1 + m$. Also $\phi_3''(m) = -2 + 6m$, and $\phi_2'(m) = -8m + 2$. Hence for $m = 1$, c is to be given by

$$\tfrac{1}{2}c^2(-2 + 6m) + c(-8m + 2) + 1 + m = 0.$$

Putting $m = 1$ in this equation, we get

$$2c^2 - 6c + 2 = 0 \quad i.e., \quad c^2 - 3c + 1 = 0.$$

$$\therefore \qquad c = \frac{3 \pm \sqrt{(9-4)}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Hence the two parallel asymptotes corresponding to the slope $m = 1$ are $y = x + \frac{1}{2}(3 \pm \sqrt{5})$.

$\therefore$ the required asymptotes are

$$x + y = 1, 2x - 2y + 3 \pm \sqrt{5} = 0.$$

§ 9. Asymptotes by expansion.

To show that $y = mx + c$, is an asymptote of the curve

$$y = mx + c + (A/x) + (B/x^2) + (C/x^3) + \dots, \qquad \text{...(1)}$$

where the series $(A/x) + (B/x^2) + (C/x^3) + \dots$ is convergent for sufficiently large values of x.

Differentiating (1), we get

$$\frac{dy}{dx} = m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots$$

$\therefore$ the equation of the tangent to (1) at (x, y) is

$$Y - y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)(X - x)$$

or $$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)X + y - \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)x$$

or $$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)X + c + \frac{2A}{x} + \frac{3B}{x^2} + \dots, \qquad \text{...(2)}$$

substituting the value of y from (1).

Now when $x \to \infty$, equation (2) tends to the equation $Y = mX + c$.

Hence $\mathbf{y = mx + c}$ is the asymptote of the curve

$$\mathbf{y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots}$$

Ex. 25. *Find the asymptotes of the hyperbola*

$$(x^2/a^2) - (y^2/b^2) = 1.$$

Sol. The given curve may be written as $y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$.

$$\therefore \quad y = \pm \frac{b}{a}x\left(1 - \frac{a^2}{x^2}\right)^{1/2}$$

$$= \pm \frac{b}{a} x \left[1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \ldots\right], \qquad \text{(by binomial theorem)}$$

$$= \pm \frac{b}{a} \left[x - \frac{a^2}{2x} - \frac{a^4}{8x^3} - \ldots\right].$$

Hence by § 9, the asymptotes of the curve are $y = \pm (b/a) x$.

Ex. 26. *Find the asymptotes of the curve, $y^3 = x^2 (x - a)$.*

Sol. The given curve may be written as

$$y = x \left(1 - \frac{a}{x}\right)^{1/3} = x \left[1 - \frac{a}{3x} - \frac{a^2}{9x^2} - \ldots\right],$$

(by binomial theorem)

or $$y = x - \frac{a}{3} - \frac{a^2}{9x} - \ldots$$

Hence by § 9, the asymptote of the curve is $y = x - \frac{1}{3} a$.

§ 10. Alternative methods of finding asymptotes of algebraic curves.

The methods of finding asymptotes of algebraic curves already discussed are sufficient to determine the asymptotes. But the following methods of finding asymptotes are sometimes found to be more convenient than the previous methods.

Let the equation of the curve be of degree n in x and y. Obviously, the asymptotes of the curve are parallel to the lines obtained by equating to zero the linear factors of the highest degree terms in its equation.

Case I. Let $(y - m_1 x)$ be a non-repeated linear factor of the n^{th} degree terms in the given equation of the curve.

Then the equation of the curve can be put into the form

$$(y - m_1 x) F_{n-1} + P_{n-1} = 0, \qquad \ldots(1)$$

where F_{n-1} contains only terms of degree $(n - 1)$, and P_{n-1} contains terms of various degrees, not higher than $(n - 1)$. Obviously, one of the values of m (slope of an asymptote) in this case is m_1 and so there is an asymptote $y - m_1 x = c_1$, provided we can find a finite value of c_1.

Also $c_1 = \lim_{x \to \infty, y/x \to m_1} (y - m_1 x)$, where (x, y) lies on (1).

But when (x, y) lies on (1),

$$y - m_1 x = -\frac{P_{n-1}}{F_{n-1}}. \qquad \therefore \quad c_1 = \lim_{x \to \infty, y/x \to m_1} \left(-\frac{P_{n-1}}{F_{n-1}}\right).$$

Hence $y - m_1x + \lim_{x \to \infty, y/x \to m_1} \left(\frac{P_{n-1}}{F_{n-1}}\right) = 0$, is an asymptote of the curve (1).

Thus dividing (1) by F_{n-1} and taking limit as $x \to \infty$, $y/x \to m_1$, we get an asymptote of (1). Similarly we can find asymptotes corresponding to the other non-repeated linear factors.

Case II. Let $(y - m_1x)^2$ be a factor of the nth degree terms of the equation of the curve and $(y - m_1x)$ be not a factor of the $(n-1)^{th}$ degree terms.

Then, proceeding as in Case I, we find that

$$\lim_{x \to \infty, y/x \to m_1} (y - m_1x)^2 \text{ is either } + \infty \text{ or } - \infty.$$

Hence there is no asymptote corresponding to the factor $(y - m_1x)^2$ in this case.

Case III. Let the equation to the curve be of the form $(y - m_1x)^2 F_{n-2} + (y - m_1x) G_{n-2} + P_{n-2} = 0$, where F_{n-2} and G_{n-2} contain only terms of $(n-2)^{th}$ degree and P_{n-2} contains terms none of which is of a degree higher than $(n-2)$.

Dividing the equation of the curve by F_{n-2}, we have

$$(y - m_1x)^2 + (y - m_1x) \cdot \frac{G_{n-2}}{F_{n-2}} + \frac{P_{n-2}}{F_{n-2}} = 0.$$

Taking limits as $x \to \infty$ and $y/x \to m_1$, we get an equation of the form $(y - m_1x)^2 + \lambda (y - m_1x) + \mu = 0$, which on being solved for $(y - m_1x)$ gives us two parallel asymptotes of the form $y = m_1x + c_1$ and $y = m_1x + c_2$.

Case IV. Let the nth degree terms contain $(y - m_1x)^3$ or a higher power of $(y - m_1x)$, as a factor.

In this case we can proceed exactly in the same way as in case III, to find the asymptotes.

Remark. If the equation to the curve is of the form,

$$(ax + by + c) F_{n-1} + P_{n-1} = 0,$$

where F_{n-1} and P_{n-1} contain terms none of which is of a degree higher than $n - 1$, and F_{n-1} contains at least one term of degree $n - 1$, a little consideration will show that the asymptote corresponding to the factor $ax + by + c$ is

$$(ax + by + c) + \lim_{x \to \infty, y/x \to -a/b} \left(\frac{P_{n-1}}{F_{n-1}}\right) = 0.$$

A similar modification can be made in cases III and IV.

Working Rule. In order to find the asymptotes (not parallel to y-axis) of any given curve,

1. Factorise the highest degree terms or (the terms including all the highest degree terms) into real linear factors.

2. Proceed as in Case I or Cases II, III and IV according as a factor $(y - m_1 x)$ or $(ax + by + c)$ is a non-repeated one or a repeated one, and get asymptotes.

Solved Examples

Ex. 27. *Find all the asymptotes of the curve*

$$(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0. \qquad \textbf{(Meerut 1979)}$$

Sol. Factorising the highest degree terms into real linear factors, the given equation of the curve may be written as

$$(x + y)(x - y)(x + 2y) + (x^2 - y^2) + (x + y + 1) = 0.$$

The slope m of the asymptote corresponding to the factor $x + y$ is -1.

Hence the asymptote corresponding to the factor $(x + y)$ is

$$x + y = \lim_{x \to \infty, y/x \to -1} \frac{-(x^2 - y^2) - (x + y + 1)}{(x + 2y)(x - y)}, \quad [\because m = -1]$$

$$= \lim_{x \to \infty, y/x \to -1} \frac{(y^2 - x^2)}{(x - y)(x + 2y)} - \lim_{x \to \infty, y/x \to -1} \frac{x + y + 1}{(x - y)(x + 2y)}$$

$$= \lim_{x \to \infty, y/x \to -1} \frac{\{(y^2/x^2) - 1\}}{\{1 - (y/x)\}\{1 + 2(y/x)\}} - \lim_{x \to \infty, y/x \to -1} \frac{\frac{1}{x} + \frac{y}{x} \cdot \frac{1}{x} + \frac{1}{x^2}}{\{1 - (y/x)\}\{1 + 2(y/x)\}},$$

on dividing the numerator and denominator by x^2

$$= \frac{1 - 1}{(1 + 1)(1 - 2)} - 0 = 0.$$

Thus $x + y = 0$ is one asymptote of the curve. Again the asymptote corresponding to the factor $(x - y)$ is

$$x - y = \lim_{x \to \infty, y/x \to 1} \frac{(y^2 - x^2)}{(x + y)(x + 2y)} - \lim_{x \to \infty, y/x \to 1} \frac{x + y + 1}{(x + y)(x + 2y)}, \quad [\because m = 1]$$

$$= \lim_{x \to \infty, y/x \to 1} \frac{(y^2/x^2) - 1}{\{1 + (y/x)\}\{1 + 2(y/x)\}}$$

$$= \lim_{x \to \infty, y/x \to 1} \frac{\frac{1}{x} + \frac{y}{x} \cdot \frac{1}{x} + \frac{1}{x^2}}{\{1 + (y/x)\}\{1 + 2(y/x)\}}$$

$$= \frac{(1-1)}{(1+1)(1+2)} = 0 = 0.$$

Thus $x - y = 0$ is another asymptote of the curve. The third asymptote of the curve is

$$x + 2y = \lim_{x \to \infty, y/x \to -\frac{1}{2}} \frac{(y^2 - x^2) + \text{terms of degree lower than } 2}{(x-y)(x+y)}$$

$$= \lim_{x \to \infty, y/x \to -\frac{1}{2}} \frac{\{(y^2/x^2) - 1\} + \text{terms which} \to 0}{\{1 - (y/x)\}\{1 + (y/x)\}}$$

$$= \frac{(\frac{1}{4} - 1) + 0}{(1 + \frac{1}{2})(1 - \frac{1}{2})} = \frac{-\frac{3}{4}}{\frac{3}{4}} = -1.$$

Therefore $x + 2y + 1 = 0$ is the third asymptote of the curve.

Important Remark. It should be noted that while taking limit we should reject all the terms in the numerator whose degree is lower than the degree of denominator. All such terms $\to 0$ as $x \to \infty$.

****Ex. 28.** *Find the asymptotes of the curve*

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy = 1.$$

(Meerut 1983; Agra 80; Lucknow 78, 71; Gorakhpur 71)

Sol. Factorising the highest degree terms, the given equation can be re-written as

$$(x+y)(x-y)(y+2x)(y-2x) = 6x^3 - 5x^2y - 3xy^2 + 2y^3 + x^2 - 3xy + 1.$$

$\therefore$ the asymptotes are parallel to the lines $x + y = 0$, $x - y = 0$, $y + 2x = 0$ and $y - 2x = 0$.

Asymptote parallel to $x + y = 0$ is

$$x + y = \lim_{x \to \infty, y/x \to -1} \left[\frac{6x^3 - 5x^2y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x-y)(y+2x)(y-2x)}\right], \quad [\because m = -1]$$

$$= \lim \left[\frac{6 - 5(y/x) - 3(y^2/x^2) + 2(y^3/x^3) + (1/x) - 3(y/x^2) + (1/x^3)}{(1 - y/x)(2 + y/x)(-2 + y/x)}\right],$$

on dividing the numerator and denominator by x^3

$$= \frac{6 + 5 - 3 - 2}{2 \cdot 1 \cdot (-3)} = \frac{6}{-6} = -1.$$

Hence $x + y + 1 = 0$ is one asymptote of the curve.

The second asymptote parallel to $x - y = 0$ is

$$x - y = \lim_{x \to \infty, y/x \to 1} \left[\frac{6x^3 - 5x^2y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x+y)(y+2x)(y-2x)}\right].$$

Taking limits as above, we get

$x - y = 0$ as the second asymptote of the curve.

The third asymptote parallel to $y + 2x = 0$ is

$$y + 2x = \lim_{x \to \infty, y/x \to -2} \left[\frac{6x^3 - 5x^2y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x - y)(x + y)(y - 2x)} \right].$$

Taking limits as above, we get

$y + 2x = -1$, as the third asymptote of the curve.

The fourth asymptote parallel to $(y - 2x)$ is

$$y - 2x = \lim_{x \to \infty, y/x \to 2} \left[\frac{6x^3 - 5x^2y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x - y)(x + y)(y + 2x)} \right].$$

Taking limits as above, we get

$y - 2x = 0$, as the fourth asymptote of the curve.

Hence the required asymptotes are

$x + y + 1 = 0$, $x - y = 0$, $2x + y + 1 = 0$ and $2x - y = 0$.

Ex. 29. *Find all the asymptotes of the curve*

$$(x^2 - y^2)(x + 2y) = y^2 - y + 1. \qquad \textbf{(Utkal 1970)}$$

Sol. Factorising the highest degree terms, the given equation can be re-written as $(x - y)(x + y)(x + 2y) = y^2 - y + 1$.

$\therefore$ the asymptotes are parallel to the lines $x - y = 0$, $x + y = 0$ and $x + 2y = 0$.

The asymptote parallel to $x - y = 0$ is

$$x - y = \lim_{x \to \infty, y/x \to 1} \left[\frac{y^2 - y + 1}{(x + y)(x + 2y)} \right]$$

$$= \lim_{x \to \infty, y/x \to 1} \left[\frac{(y/x)^2 - (y/x)(1/x) + (1/x^2)}{\{1 + (y/x)\}\{1 + 2(y/x)\}} \right],$$

on dividing the numerator and the denominator by x^2

$$= \frac{1 - 0 + 0}{2 \cdot (1 + 2)} = \frac{1}{6}.$$

Hence $x - y = \frac{1}{6}$ is an asymptote of the curve.

The second asymptote parallel to $x + y = 0$ is

$$x + y = \lim_{x \to \infty, y/x \to -1} \left[\frac{y^2 - y + 1}{(x - y)(x + 2y)} \right]$$

$$= \lim_{x \to \infty, y/x \to -1} \left[\frac{(y^2/x^2) - (y/x)(1/x) + (1/x^2)}{\{1 - (y/x)\}\{1 + 2(y/x)\}} \right] = -\frac{1}{2}.$$

Thus $x + y = -\frac{1}{2}$ is the second asymptote of the curve.

The third asymptote parallel to $x + 2y = 0$ is

$$x + 2y = \lim_{x \to \infty, y/x \to -\frac{1}{2}} \left[\frac{y^2 - y + 1}{(x + y)(x - y)} \right]$$

$$= \lim_{x \to \infty, y/x \to -\frac{1}{2}} \left[\frac{(y/x)^2 - (y/x)(1/x) + (1/x)^2}{\{1 + (y/x)\}\{1 - (y/x)\}} \right]$$

$$= \frac{(-\frac{1}{2})^2 - (-\frac{1}{2}) \cdot 0 + 0}{\{1 + (-\frac{1}{2})\}\{1 - (-\frac{1}{2})\}} = \frac{\frac{1}{4}}{\frac{1}{2} \cdot \frac{3}{2}} = \frac{1}{3}.$$

Thus $x + 2y = \frac{1}{3}$ is the third asymptote of the curve.

Hence the required asymptotes are

$$x - y = \frac{1}{6}, x + y = -\frac{1}{2}, x + 2y = \frac{1}{3}.$$

Ex. 30. *Find all the asymptotes of $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$.*

(Agra 1979; Meerut 72, 85, 88 S, 90 S; Jiwaji 70)

Sol. Factorising the highest degree terms, the given equation can be re-written as

$$x^2(y - x)(y + x) + 4a^2x^2 - y^2a^2 = 0.$$

Equating the coefficient of the highest power of y (*i.e.*, of y^2) to zero, we get $x^2 - a^2 = 0$ or $x = \pm a$ as the asymptotes parallel to y-axis. The asymptote corresponding to the factor $(y - x)$ is

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{a^2y^2 - 4a^2x^2}{x^2(y + x)}\right]$$

$$= \lim_{x \to \infty, y/x \to 1} \left[\frac{a^2(y^2/x^2)(1/x) - 4a^2(1/x)}{(y/x + 1)}\right] = 0,$$

(on taking limit).

$\therefore$ $y - x = 0$ is an asymptote of the curve.

Again the asymptote corresponding to the factor $(y + x)$ is

$$y + x = \lim_{x \to \infty, y/x \to -1} \left[\frac{a^2y^2 - 4a^2x^2}{x^2(y - x)}\right] = 0, \qquad \text{(as above).}$$

Hence the required asymptotes are

$$x = \pm a, y = \pm x.$$

Ex. 31. *Find the asymptotes of the curve*

$$x^2y - xy^2 + xy + y^2 + x - y = 0. \qquad \textbf{(Meerut 1990)}$$

Sol. Factorising the highest degree terms, the given equation can be re-written as $xy(y - x) = xy + y^2 + x - y$.

Equating to zero the coefficient of the highest powers of x and y we get $y = 0$ and $x = 0$ as asymptotes parallel to the co-ordinate axes.

Now the asymptote corresponding to the factor $(y - x)$ is

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{xy + y^2 + x - y}{xy}\right]$$

$$= \lim_{x \to \infty, y/x \to 1} \left[\frac{(y/x) + (y^2/x^2) + (1/x) - (y/x)(1/x)}{(y/x)}\right].$$

Taking limits, we have $y - x = 2$ or $y = x + 2$.

Hence the required asymptotes are $x = 0$, $y = 0$ and $y = x + 2$.

Ex. 32. *Find the asymptotes of the curve*

$$(y - x)(y - 2x)^2 + (y + 3x)(y - 2x) + 2x + 2y - 1 = 0.$$

(Meerut 1984)

Sol. The asymptotes corresponding to the factor $(y-2x)^2$ are given by

$$(y-2x)^2+(y-2x)\lim_{x\to\infty,\, y/x\to 2}\left[\frac{y+3x}{y-x}\right]+\lim_{x\to\infty,\, y/x\to 2}\left[\frac{2x+2y-1}{y-x}\right]=0$$

or $$(y-2x)^2+(y-2x)\lim_{x\to\infty,\, y/x\to 2}\left[\frac{(y/x)+3}{(y/x)-1}\right]+\lim_{x\to\infty,\, y/x\to 2}\left[\frac{2+2\,(y/x)-(1/x)}{(y/x)-1}\right]=0$$

or $(y-2x)^2+5\,(y-2x)+6=0$

i.e., $y-2x=\frac{1}{2}[-5\pm\sqrt{(25-24)}]=\frac{1}{2}(-5\pm 1)$

i.e., $y-2x=\frac{1}{2}(-5+1)$ and $y-2x=\frac{1}{2}(-5-1)$.

$\therefore$ $y=2x-2$ and $y=2x-3$ are the asymptotes corresponding to the factor $(y-2x)^2$.

Also the asymptote corresponding to the factor $(y-x)$ is

$$y-x=\lim_{x\to\infty,\, y/x\to 1}\left[\frac{-(y+3x)(y-2x)-2x-2y+1}{(y-2x)^2}\right]$$

$$=\lim_{x\to\infty,\, y/x\to 1}\left[\frac{-\{(y/x)+3\}\{(y/x)-2\}-2\,(1/x)-2\,(y/x^2)+(1/x^2)}{\{(y/x)-2\}^2}\right]=4,$$

on taking limits.

Hence $y=x+4$ is another asymptote of the given curve

Ex. 33. *Find the asymptotes of the curve*

$$(y-x)^2x-3y\,(y-x)+2x=0.$$

Sol. The given equation is of third degree. So there are at most three asymptotes.

Dividing each term by x and taking limits, the asymptotes corresponding to the factor $(y-x)^2$ are given by

$$(y-x)^2-3\,(y-x)\lim_{x\to\infty,\, y/x\to 1}\left(\frac{y}{x}\right)+\lim_{x\to\infty,\, y/x\to 1}\left(\frac{2x}{x}\right)=0$$

or $(y-x)^2-3\,(y-x)+2=0.$

Hence $y-x=\frac{1}{2}[3\pm\sqrt{(9-8)}]$ *i.e.*, $=2$, or 1.

Therefore, $y-x=2$ and $y-x=1$ are the two asymptotes.

Also equating to zero the coefficient of the highest power of y, we get the asymptote parallel to y-axis as $x=3$.

Hence the required asymptotes are

$$x=3,\ y-x=2 \text{ and } y-x=1.$$

Ex. 34. *Find the asymptotes of the curve*

$$(x-y)^2(x^2+y^2)-10\,(x-y)\,x^2+12y^2+2x+y=0.$$

(Agra 1983)

Sol. The given curve is

$$(x-y)^2(x^2+y^2)-10(x-y)x^2+12y^2+2x+y=0.$$

The equation of the curve is of fourth degree and so there are at most four asymptotes. Dividing each term by the coefficient x^2+y^2 of $(x-y)^2$ and taking limits, the asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2-10(x-y)\lim_{x\to\infty,\,y/x\to1}\frac{x^2}{x^2+y^2}+\lim_{x\to\infty,\,y/x\to1}\frac{12y^2+2x+y}{x^2+y^2}=0$$

or $$(x-y)^2-10(x-y)\lim_{x\to\infty,\,y/x\to1}\frac{1}{1+(y/x)^2}+\lim_{x\to\infty,\,y/x\to1}\frac{12(y/x)^2+(2/x)+(y/x)(1/x)}{1+(y/x)^2}=0,$$

on dividing the Nr. and Dr. by x^2

or $(x-y)^2-10(x-y)\,.\,(1/2)+(12/2)=0$

or $(x-y)^2-5(x-y)+6=0$

i.e., $x-y=\dfrac{5\pm\sqrt{(25-24)}}{2}=\dfrac{5\pm1}{2}=2$ or 3

i.e., $x-y=2$ and $x-y=3$.

Again $x^2+y^2=(x+iy)(x-iy)$ implies that the remaining linear factors of the fourth degree terms in the equation of the curve are imaginary and so the other two asymptotes are imaginary.

Ex. 35. *Find all the asymptotes of the curve*

$$x^2(x+y)(x-y)^2+ax^2(x-y)-a^2y^3=0.$$

(Lucknow 1988; Meerut 71)

Sol. The given equation is of fifth degree. So there are at most five asymptotes. Dividing each term by the coefficient of $(x-y)^2$ and taking the limits, the asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2+(x-y)\lim_{x\to\infty,\,y/x\to1}\left[\frac{a}{x+y}\right]-\lim_{x\to\infty,\,y/x\to1}\left[\frac{a^2y^3}{x^2(x+y)}\right]=0$$

or $$(x-y)^2+(x-y)\lim\left[\frac{a/x}{1+y/x}\right]-\lim\left[\frac{a^2(y^3/x^2)}{1+(y/x)}\right]=0.$$

Taking limits, we get $(x-y)^2+(x-y)\,.\,0-\frac{1}{2}a^2=0$

or $$x-y=\pm\frac{a}{\sqrt2}$$ as the two asymptotes.

Also the asymptote corresponding to the factor $(x+y)$ is

$$x + y = \lim_{x \to \infty, y/x \to -1} \left[\frac{ax^2 (y - x) + a^2 y^3}{x^2 (x - y)^2}\right]$$

$$= \lim_{x \to \infty, y/x \to -1} \left[\frac{a \{(y/x) - 1\} (1/x) + a^2 (y^3/x^3) (1/x)}{\{1 - (y/x)\}^2}\right].$$

Taking limits we have $x + y = 0$, as an asymptote.

Also equating to zero the coefficient of the highest power of y, we get the aysmptotes parallel to y-axis as $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$.

Hence the required asymptotes are

$$x = \pm a, x - y = \pm \frac{a}{\sqrt{2}} \text{ and } x + y = 0.$$

Ex. 36. *Find the asymptotes of the curve*

$$(y - x)(y - 2x)^2 + (y + 3x)(y - 2x) + 2x + 2y - 1 = 0.$$

(Meerut 1974, 73)

Sol. The given equation is of third degree and so there are atmost three asymptotes. Dividing each term by the coefficient of $(y - 2x)^2$ and taking limits, the asymptotes corresponding to the factor $(y - 2x)^2$ are given by

$$(y - 2x)^2 + (y - 2x) \lim_{x \to \infty, y/x \to 2} \left[\frac{(y + 3x)}{(y - x)}\right] + \lim_{x \to \infty, y/x \to 2} \left[\frac{2x + 2y - 1}{y - x}\right] = 0$$

or $$(y - 2x)^2 + (y - 2x) \lim \left[\frac{(y/x) + 3}{(y/x) - 1}\right] + \lim \left[\frac{2 + 2(y/x) - 1/x}{(y/x) - 1}\right] = 0.$$

Taking limits, we have

$$(y - 2x)^2 + 5(y - 2x) + 6 = 0$$

giving $$y - 2x = \tfrac{1}{2}[-5 \pm 1] = -2, \text{ or } -3.$$

Thus $y - 2x + 2 = 0$ and $y - 2x + 3 = 0$ are the two asymptotes.

Also the asymptote corresponding to the factor $(y - x)$ is

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{-(y + 3x)(y - 2x) - 2x - 2y + 1}{(y - 2x)^2}\right]$$

$$= \lim_{x \to \infty, y/x \to 1} \left[\frac{-(y/x + 3)(y/x - 2) - 2(1/x) - 2(y/x^2) + (1/x^2)}{\{(y/x) - 2\}^2}\right].$$

Taking limits, we have $y - x = 4$ as another asymptote.

Hence the required asymptotes of the curve are

$$x - y + 4 = 0, y - 2x + 2 = 0 \text{ and } y - 2x + 3 = 0.$$

Ex. 37. *Find the asymptotes of the curve*

$$x^2 (x - y)^2 + 9(x^2 - y^2) = 9xy.$$

(Agra 1970)

Sol. The given equation can be re-written as

$$x^2 (x - y)^2 + 9(x - y)(x + y) = 9xy.$$

Dividing each term by the coefficient of $(x-y)^2$ (*i.e.*, x^2) and taking limits, the asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2 + 9(x-y) \lim_{x \to \infty, y/x \to 1} \left[\frac{(x+y)}{x^2}\right] - \lim_{x \to \infty, y/x \to 1} \left[\frac{9xy}{x^2}\right] = 0$$

or $(x-y)^2 = 9$. Then $x - y = \pm 3$ are two asymptotes.

Also equating to zero the coefficient of highest power of y, we get the asymptotes parallel to y-axis as $x^2 - 9 = 0$ *i.e.*, $x = \pm 3$.

Hence the required asymptotes are

$$x = \pm 3 \quad \text{and} \quad x - y = \pm 3.$$

***Ex. 38.** *Find the asymptotes of the curve*

$$(x+y)^2 (x + 2y + 2) = x + 9y - 2. \qquad \textbf{(Meerut 1983 S)}$$

Sol. The given curve is of degree 3. So it cannot have more than three asymptotes. Obviously the asymptotes of the curve are parallel to the lines $x + y = 0$ and $x + 2y + 2 = 0$.

Asymptotes parallel to the line $x + y = 0$ are given by

$$(x+y)^2 = \lim_{x \to \infty, y/x \to -1} \frac{x + 9y - 2}{x + 2y + 2}$$

$$= \lim_{x \to \infty, y/x \to -1} \frac{1 + 9(y/x) - 2/x}{1 + 2(y/x) + 2/x} = \frac{1-9}{1-2} = 8.$$

Thus $x + y = \pm \sqrt{8} = \pm 2\sqrt{2}$ are two asymptotes of the curve.

The asymptote parallel to the line $x + 2y + 2 = 0$ is given by

$$x + 2y + 2 = \lim_{x \to \infty, y/x \to -\frac{1}{2}} \frac{x + 9y - 2}{(x+y)^2}$$

$$= \lim_{x \to \infty, y/x \to -\frac{1}{2}} \frac{1/x + 9(y/x)(1/x) - 2/x^2}{(1 + y/x)^2} = 0.$$

Hence the required asymptotes are

$$x + y = \pm \sqrt{2} \quad \text{and} \quad x + 2y + 2 = 0.$$

Ex. 39. *Find all the asymptotes of the curve* $y^2(x - b) = x^3 + a^3$.

Sol. The given curve is $x(y - x)(y + x) = a^3 + by^2$.

$\therefore$ the required asymptotes are

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{a^3 + by^2}{x(y+x)}\right]$$

$$= \lim_{x \to \infty, y/x \to 1} \left[\frac{(a^3/x^2) + b(y^2/x^2)}{(y/x + 1)}\right] = \tfrac{1}{2} b,$$

and $$y + x = \lim_{x \to \infty, y/x \to -1} \left[\frac{a^3 + by^2}{x(y-x)}\right] = -\tfrac{1}{2} b.$$

Also equating the coefficient of y^2 to zero, we get $x - b = 0$ as an asymptote parallel to y-axis.

$\therefore \quad \pm y = x + \frac{1}{2}b$ and $x = b$ are the required asymptotes.

Ex. 40. *Find all the asymptotes of the curve*

$$(y-a)^2(x^2-a^2) = x^4 + a^4.$$ **(Rohilkhand 1987; Kanpur 89)**

Sol. The given curve is

$$y^2(x^2-a^2) - 2ay(x^2-a^2) + a^2(x^2-a^2) = x^4 + a^4$$

or $\quad x^2(y-x)(y+x) - 2ax^2y + a^2(x^2-y^2) + 2a^3y - 2a^4 = 0.$

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get $x^2 - a^2 = 0$ or $x = \pm a$ as the asymptotes parallel to y-axis.

The other asymptotes are

$$y - x = \lim_{x\to\infty, y/x\to 1}\left[\frac{2ax^2y - a^2(x^2-y^2) - 2a^3y + 2a^4}{x^2(y+x)}\right]$$

$$= \lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{2a(y/x) - a^2\{(1/x) - (y^2/x^3)\} - 2a^3(y/x^3) + 2a^4(1/x^3)}{[1+(y/x)]}\right]$$

$= a,$

and $$y + x = \lim_{x\to\infty, y/x\to -1}\left[\frac{2ax^2y - a^2(x^2-y^2) - 2a^3y + 2a^4}{x^2(y-x)}\right]$$

$$= \lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{2a(y/x) - a^2\{(1/x) - (y^2/x^3)\} - 2a^3(y/x^3) + 2a^4(1/x^3)}{(y/x)-1}\right]$$

$= a.$

$\therefore$ the required asymptotes are $x = \pm a$ and $y = \pm x + a.$

Ex. 41. *Find the asymptotes of the curve*

$$x(y-3)^3 = 4y(x-1)^2.$$ **(Agra 1982)**

Sol. The equation of the given curve can be written as

$$xy(y-2x)(y+2x) = 9xy^2 - 12yx^2 - 15xy + 27x - 4y.$$

Here the coefficient of the highest power of $y = x$.

Therefore $x = 0$ is an asymptote parallel to y-axis.

Again the coefficient of the highest power of $x = 4y$.

Therefore $y = 0$ is an asymptote parallel to x-axis.

The other asymptotes are given by

$$y - 2x = \lim_{x\to\infty, y/x\to 2}\left[\frac{9xy^2 - 12yx^2 - 15xy + 27x - 4y}{xy(y+2x)}\right]$$

$= \frac{3}{2}$; *i.e.*, $2y - 4x = 3$;

and $$y + 2x = \lim_{x\to\infty, y/x\to -2}\left[\frac{9xy^2 - 12yx^2 - 15xy + 27x - 4y}{xy(y-2x)}\right]$$

$= \frac{15}{2}$ *i.e.*, $2y + 4x = 15.$

Ex. 42. *Find the asymptotes of the curve*

$$x(y^2 - x^2) = 2ay^2 - a^3.$$

Sol. The given curve is $(x-2a)y^2 - x^3 + a^3 = 0.$ ...(1)

The asymptote parallel to y-axis is $x - 2a = 0$ or $x = 2a$.

The equation (1) can be written as

$$x(y-x)(y+x) = 2ay^2 - a^3.$$

Hence the other asymptotes are

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{2ay^2 - a^3}{x(y+x)}\right] = \frac{2a}{2} = a,$$

and
$$y + x = \lim_{x \to \infty, y/x \to -1} \left[\frac{2ay^2 - a^3}{x(y-x)}\right] = \frac{2a}{-2} = -a.$$

$\therefore$ the required asymptotes are $x = 2a, y = x + a, y = -x - a$.

Ex. 43. *Find all the asymptotes of the curve*

$$(a+x)^2(b^2+x^2) = x^2y^2.$$ **(Raj. 1977)**

Sol. The given curve may be written as

$$x^2(y-x)(y+x) = 2ax^3 + (a^2+b^2)x^2 + 2ab^2x + a^2b^2.$$

$\therefore$ the asymptotes are

$x^2 = 0$ *i.e.*, $x = 0, x = 0$ (the asymptotes parallel to y-axis)

and
$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{2ax^3 + (a^2+b^2)x^2 + 2ab^2x + a^2b^2}{x^2(y+x)}\right]$$

$= a$, (dividing the numerator and the denominator by x^3 and taking the limits),

and
$$y + x = \lim_{x \to \infty, y/x \to -1} \left[\frac{2ax^3 + (a^2+b^2)x^2 + 2ab^2x + a^2b^2}{x^2(y-x)}\right]$$

$= -a$.

$\therefore$ the required asymptotes are $x = 0, y = x + a, y = -x - a$.

Ex. 44. *Find the asymptotes of the curve*

$$y^2(x-2) = x^2(y-1).$$ **(Meerut 91; Gorakhpur 83)**

Sol. The given curve may be written as

$$xy(y-x) - 2y^2 + x^2 = 0.$$

Here $y - 1 = 0$ is the asymptote parallel to x-axis,
and $x - 2 = 0$ is the asymptote parallel to y-axis.
The other asymptote is given by

$$y - x = \lim_{x \to \infty, y/x \to 1} \left[\frac{2y^2 - x^2}{xy}\right]$$

$= 1$, [dividing the numerator and the denominator by x^2 and taking the limits].

$\therefore$ the required asymptotes are $x = 2, y = 1$ and $y = x + 1$.

Ex. 45. *Find the asymptotes of the curve*

$$y(y+x)^2 - y + 1 = 0.$$

Sol. Here $y = 0$ is an asymptote parallel to x-axis.

The asymptotes corresponding to the factor $(y+x)^2$ are given by

$$(y+x)^2 = \lim_{x \to \infty, y/x \to -1} \left[\frac{y-1}{y}\right]$$

$$= \lim_{x \to \infty, y/x \to -1} \left[\frac{(y/x) - (1/x)}{(y/x)}\right] = 1.$$

$\therefore$ the required asymptotes are $y + x = \pm 1$, and $y = 0$.

Ex. 46. *Find all the asymptotes of the curve*

$$(x - y + 1)(x - y + 2)(x + y) = 8x - 1. \qquad \textbf{(Meerut 1981)}$$

Sol. The given curve may be written as

$$(x + y)(x - y)^2 - (x^2 - y^2) - 10x - 2y + 1 = 0.$$

The asymptote corresponding to the factor $(x + y)$ is

$$x + y = \lim_{x \to \infty, y/x \to -1} \left[\frac{8x - 1}{(x - y + 1)(x - y - 2)}\right]$$

$= 0$, [dividing the numerator and the denominator by x^2 and taking the limits].

Also the asymptotes corresponding to the factor $(x - y)^2$ are given by

$$(x - y)^2 - (x - y) = \lim_{x \to \infty, y/x \to 1} \left[\frac{10x + 2y - 1}{x + y}\right] = 6$$

or $x - y = \frac{1}{2}[1 \pm \sqrt{(1 + 24)}] = 3$ or -2

i.e., $y = x - 3$ and $y = x + 2$.

$\therefore$ the required asymptotes are

$$y + x = 0, y = x - 3 \text{ and } y = x + 2.$$

Ex. 47. *Find all the asymptotes of the curve*

$$(x^2 - y^2)^2 = 4y^2 - y.$$

The asymptotes corresponding to the factors $(x - y)^2$ and $(x + y)^2$ are

$$(x - y)^2 = \lim_{x \to \infty, y/x \to 1} \left[\frac{4y^2 - y}{(x + y)^2}\right] = 1, \text{ i.e., } x - y = \pm 1,$$

and $$(x + y)^2 = \lim_{x \to \infty, y/x \to -1} \left[\frac{4y^2 - y}{(x - y)^2}\right] = 1, \text{ i.e., } x + y = \pm 1.$$

$\therefore$ the required asymptotes are $x - y = \pm 1$ and $x + y = \pm 1$.

Ex. 48. *Find the asymptotes of the curve*

$$x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0. \qquad \textbf{(Meerut 1988 P, 90 P)}$$

Sol. The given curve may be written as

$$x(x + y)^2 - x(x + y) + 2 = 0.$$

Here an asymptote parallel to y-axis is $x = 0$.

The asymptotes corresponding to the factor $(x + y)^2$ are given by

$$(x + y)^2 - (x + y) + \lim_{x \to \infty, y/x \to -1} \left[\frac{2}{x}\right] = 0$$

or $$(x + y)^2 - (x + y) + 0 = 0$$

or $$(x + y)(x + y - 1) = 0.$$

$\therefore$ the required asymptotes are

$$x = 0, x + y = 0 \text{ and } x + y - 1 = 0.$$

Ex. 49. *Find the asymptotes of the curve*

$$x^2(x-y)^2 + a^2(x^2-y^2) = a^2xy.$$

Sol. The equation of the curve is

$$x^2(x-y)^2 + a^2(x-y)(x+y) = a^2xy.$$

The asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2 + a^2(x-y) \lim_{x\to\infty,\, y/x\to 1}\left[\frac{x+y}{x^2}\right] = \lim_{x\to\infty,\, y/x\to 1}\frac{a^2xy}{x^2}$$

or $\qquad (x-y)^2 = a^2 \quad$ or $\quad x-y = \pm a.$

Also $x = \pm a$ are asymptotes parallel to y-axis.

$\therefore$ the required asymptotes are $x = \pm a$ and $x - y = \pm a$.

Ex. 50. *Find all the asymptotes of the curve*

$$y^2(x-2) = x^2(y-1) \quad \text{or} \quad xy(y-x) = 2y^2 - x^2.$$

(Gorakhpur 1970)

Sol. Equating the coefficients of the highest powers of x and y to zero, we get

$y - 1 = 0$ as an asymptote parallel to x-axis

and $\qquad x - 2 = 0$ as an asymptote parallel to y-axis.

The other asymptote corresponding to the factor $(y-x)$ is

$$y - x = \lim_{x\to\infty,\, y/x\to 1}\left[\frac{2y^2-x^2}{xy}\right] = 1.$$

$\therefore$ the required asymptotes are $y = 1$, $x = 2$ and $y - x = 1$.

Ex. 51. *Find the asymptotes of the curve*

$$x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$$

Sol. The equation of the curve may be written as

$$(x-y)(x+2y)^2 = x - y - 3.$$

The asymptote corresponding to the factor $(x-y)$ is

$$x - y = \lim_{x\to\infty,\, y/x\to 1}\left[\frac{x-y-3}{(x+2y)^2}\right] = 0.$$

Also the asymptotes corresponding to the factor $(x+2y)^2$ are given by

$$(x+2y)^2 = \lim_{x\to\infty,\, y/x\to -\frac{1}{2}}\left[\frac{x-y-3}{x-y}\right] = 1$$

or $\qquad x + 2y = \pm 1.$

$\therefore$ the required asymptotes are $x - y = 0$ and $x + 2y = \pm 1$.

§ 11. Asymptotes by Inspection.

If the equation of a curve is of the form $F_n + F_{n-2} = 0$, where F_n is of degree n (i.e., contains terms of degree n and may also contain terms of lower degree) and F_{n-2} is of degree $(n-2)$ at the most, and if $F_n = 0$ can be broken up into n linear factors so as to represent n straight *lines no two of which are parallel or coincident, then $F_n = 0$ gives all the asymptotes of the curve.*

Ex. 52. *Find the asymptotes of the curve*

$$xy(x^2-y^2)(x^2-4y^2)+3xy(x^2-y^2)+x^2+y^2-7=0.$$

(Meerut 1983)

Sol. The given equation can be put in the form

$$[xy(x^2-y^2)(x^2-4y^2)]+[3xy(x^2-y^2)+x^2+y^2-7]=0.$$

This equation is of the form $F_n + F_{n-2} = 0$, where F_n can be broken up into n linear factors so as to represent n straight lines no two of which are parallel or coincident.

$\therefore$ all the asymptotes are given by $F_n = 0$

or $$xy(x-y)(x+y)(x-2y)(x+2y)=0.$$

Then by inspection, the asymptotes are

$$x=0,\ y=0,\ x-y=0,\ x+y=0,\ x-2y=0 \text{ and } x+2y=0.$$

Ex. 53. *Find the asymptotes of the curve*

$$x^3-4xy^2+x^2y-4y^3-2x^3+8y^2+3x+4y-5=0.$$

Sol. The given equation can be put in the form

$$(x-2y)(x+2y)(x+y-2)+(3x+4y-5)=0.$$

By inspection, the asymptotes are

$$x-2y=0,\ x+2y=0,\ x+y-2=0.$$

§ 12. Intersection of a curve and its asymptotes.

To prove that any asymptote of an algebraic curve of the n^{th} degree cuts the curve in $(n-2)$ points.

A straight line $$y = mx + c \quad ...(1)$$
cuts the curve of the n^{th} degree

$$x^n \phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + x^{n-2}\phi_{n-2}(y/x) + \dots = 0 \quad ...(2)$$

in n points real or imaginary.

Eliminating y from (1) and (2), we get

$$x^n \phi_n(m+c/x) + x^{n-1}\phi_{n-1}(m+c/x) + x^{n-2}\phi_{n-2}(m+c/x) + \dots = 0.$$

Expanding each term by Taylor's therorem and arranging in descending powers of x, we get

$$x^n \phi_n(m) + [c\phi_n'(m) + \phi_{n-1}(m)]x^{n-1} + \left[\frac{c^2}{2!}\phi_n''(m) + \frac{c}{1!}\phi_{n-1}'(m) + \phi_{n-2}(m)\right]x^{n-2} + \dots = 0. \quad ...(3)$$

The equation (3) gives the abscissae of the points of intersection of (1) and (2).

If $y = mx + c$ is an asymptote of (2), then $\phi_n(m) = 0$ and $c\phi_n'(m) + \phi_{n-1}(m) = 0$.

Consequently (3) reduces to an equation of $(n-2)^{th}$ degree in x and therefore the asymptote (1) cuts the curve (2) in $(n-2)$ points.

Cor. *The, n, asymptotes of a curve of the n^{th} degree cut it in $n(n-2)$ points. In general, a curve of the degree $n-2$, or less, can be made to pass through these $n(n-2)$ points.*

Ex. 54. *Form the equation of a curve which has $x=0$, $y=0$, $y=x$ and $y=-x$ four asymptotes and which passes through the point (a,b) and cuts its asymptotes again in eight points lying upon the circle $x^2+y^2=a^2$.*

Sol. The combined equation of the asymptotes is

$$xy(y^2-x^2)=0. \qquad \text{...(1)}$$

The required curve passes throught the points of intersection of the asymptotes and the given circle $x^2+y^2-a^2=0$.

Let the equation of the curve be

$$xy(y^2-x^2)+\lambda(x^2+y^2-a^2)=0. \qquad \text{...(2)}$$

Now this curve also passes through the point (a,b).

$\therefore \quad ab(b^2-a^2)+\lambda(a^2+b^2-a^2)=0,$

i.e., $\quad \lambda=(a/b)(a^2-b^2)$.

Substituting this value of λ in (2), we have

$$bxy(y^2-x^2)+a(a^2-b^2)(x^2+y^2-a^2)=0$$

as the required equation of the curve.

Ex. 55. *Find the equation of the cubic which has the same asymptotes as the curve $x^3-6x^2y+11xy^2-6y^3+x+y+1=0$ and which touches the axis of y at the origin and passes through the point $(3,2)$.* **(Rohilkhand 1986; Lucknow 80; Kanpur 78; Meerut 69)**

Sol. The given curve is

$$(x^3-6x^2y+11xy^2-6y^3)+(x+y)+1=0.$$

Here $\phi_3(m)=1-6m+11m^2-6m^3$

$$=(1-m)(1-2m)(1-3m).$$

Therefore $\quad \phi_3(m)=0$ gives $m=1,\frac{1}{2},\frac{1}{3}\cdot$ Also $\phi_2(m)=0$.

$$\therefore \quad c=-\frac{\phi_2(m)}{\phi_3'(m)}=0, \text{ for all the three values of } m.$$

$\therefore \quad$ the asymptotes of the given curve are

$$y=x;\ y=\tfrac{1}{2}x \quad \text{and} \quad y=\tfrac{1}{3}x.$$

Hence the combined equation of the asymptotes is

$$(x-y)(x-2y)(x-3y)=0. \qquad \text{...(1)}$$

Now the most general equation of any curve, having these asymptotes, is of the form

$$(x-y)(x-2y)(x-3y)+F_1=0, \qquad \text{[See § 12, Cor.]}$$

where F_1 is of the first degree in x and y (say $F_1=ax+by+c$).

If the curve $(x-y)(x-2y)(x-3y)+ax+by+c=0$, passes through the origin $(0, 0)$, then $c=0$ and the equation of the curve becomes

$$(x-y)(x-2y)(x-3y)+ax+by=0. \qquad ...(2)$$

Equating to zero, the lowest degree terms in (2), we get $ax+by=0$, as the equation of the tangent at origin.

But $x=0$ *i.e.*, y-axis is given to be the tangent at origin.

Therefore $b=0$ and hence the equation of the curve is

$$(x-y)(x-2y)(x-3y)+ax=0 \qquad ...(3)$$

It passes through the point $(3, 2)$, (given).

$\therefore \quad (3-2)(3-4)(3-6)+3a=0$; or $a=-1$.

Therefore the required equation of the curve is

$$(x-y)(x-2y)(x-3y)-x=0, \qquad \text{[putting } a=-1 \text{ in (3)]}$$

or $$x^3-6x^2y+11xy^2-6y^3-x=0.$$

Ex. 56. *Find the equation of the cubic which has the same asymptotes as the curve* $x^3-6x^2y+11xy^2-6y^3+4x+5y+7=0$ *and which passes through the points* $(0, 0)$, $(2, 0)$ *and* $(0, 2)$.

Sol. The given curve is

$$(x-y)(x-2y)(x-3y)+(4x+5y+7)=0. \qquad ...(1)$$

By inspection,

$(x-y)(x-2y)(x-3y)=0$, is the combined equation of the asymptotes of (1).

Now the most general equation of the curve, having these asymptotes, is

$$(x-y)(x-2y)(x-3y)+ax+by+c=0. \qquad ...(2)$$

If (2) passes through the points $(0, 0)$, $(2, 0)$ and $(0, 2)$, then

$c=0$; $8+2a=0$ or $a=-4$; $-48+2b=0$ or $b=24$.

$\therefore$ from (2), the required equation of the curve is

$$x^3-6x^2y+11xy^2-6y^3-4x+24y=0.$$

Ex. 57. *Show that asymptotes of the cubic*

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0$$

cut the curve again in three points which lie on the straight line

$$x-y+1=0.$$ **(Meerut 1982 P, 88P)**

Sol. The given curve is

$$x^3-2y^3+xy(2x-y)+y(x-y)+1=0. \qquad ...(1)$$

$\therefore \quad \phi_3(m)=1-2m^3+2m-m^2=0$ gives $m=1, -1, -\frac{1}{2}$.

Also $\phi_3'(m)=-6m^2+2-2m$ and $\phi_2(m)=m-m^2$.

$$\therefore \quad c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\frac{(m-m^2)}{(2-2m-6m^2)}.$$

When $m=1$, $c=0$; when $m=-1$, $c=-1$;

and when $m=-\frac{1}{2}$, $c=\frac{1}{2}$.

$\therefore$ the asymptotes of (1) are $y = x, y = -x - 1$

and $y = -\frac{1}{2}x + \frac{1}{2}$.

Hence the combined equation of the asymptotes of (1) is

$$(x - y)(x + y + 1)(x + 2y - 1) = 0$$

or $$x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 - x + y = 0. \quad \text{...(2)}$$

Subtracting (2) from (1), we get

$$x - y + 1 = 0,$$

which shows that the points of intersection of the curve and its asymptotes lie on the straight line $x - y + 1 = 0$.

Also the three asymptotes cut the cubic in $n(n-2)$, *i.e.*, $3(3-2) = 3$ points. As shown above, these, three points lie on the straight line $x - y + 1 = 0$.

Ex. 58. *Find the equation of the cubic curve whose asymptotes are $x + a = 0$, $y - a = 0$ and $x + y + a = 0$ and which touches the axis of x at the origin and passes through the point $(-2a, -2a)$.*

Sol. The asymptotes are

$$x + a = 0, y - a = 0 \text{ and } x + y + a = 0.$$

$\therefore$ the combined equation of the asymptotes is

$$(x + a)(y - a)(x + y + a) = 0.$$

Now the equation of any curve, having these asymptotes, is of the form

$$(x + a)(y - a)(x + y + a) + F_1 = 0, \quad \text{[See § 12 cor.]}$$

where F_1 is of 1st degree in x and y (say $F_1 = bx + cy + d$).

So let the equation of the required curve be

$$(x + a)(y - a)(x + y + a) + (bx + cy + d) = 0.$$

But it is given that the curve passes through the origin (0, 0), therefore $d = a^3$.

Thus the equation of the curve reduces to

$$(x + a)(y - a)(x + y + a) + bx + cy + a^3 = 0$$

or $$xy(x + y) + a(y^2 - x^2 + xy) - 2a^2x + bx + cy = 0.$$

Now equating the lowest degree terms to zero the equation of the tangent at origin is $(-2a^2 + b)x + cy = 0$.

Also given that the x-axis (*i.e.*, $y = 0$) is tangent at the origin.

$\therefore$ $-2a^2 + b = 0$ or $b = 2a^2$.

$\therefore$ the equation of the curve is

$$xy(x + y) + a(y^2 - x^2 + xy) + cy = 0.$$

This equation passes through $(-2a, -2a)$. (given)

$\therefore$ $4a^2(-4a) + a(4a^2 - 4a^2 + 4a^2) + c(-2a) = 0$ or $c = -6a^2$.

Therefore the required equation is

$$xy(x + y) + a(y^2 - x^2 + xy) - 6a^2y = 0.$$

Ex. 59 (a). *Show that the eight points of the curve*

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

and its asymptotes lie on a rectangular hyperbola. **(Meerut 1982)**

Sol. The equation of the given curve is

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0 \qquad ...(1)$$

or $$(x^2 - y^2)(x^2 - 4y^2) + x^2 - y^2 + x + y + 1 = 0$$

or $$(x - y)(x + y)(x - 2y)(x + 2y) + x^2 - y^2 + x + y + 1 = 0.$$

$\therefore$ by inspection, the combined equation of the asymptotes of (1) is $(x - y)(x + y)(x - 2y)(x + 2y) = 0$

or $$x^4 - 6x^2y^2 + 4y^4 = 0. \qquad ...(2)$$

Now each asymptote of (1) will cut it in $4 - 2$ *i.e.*, 2 points. Therefore the four asymptotes will cut it in 4×2 *i.e.*, 8 points.

Subtracting (2) from (1), we get

$$x^2 - y^2 + x + y + 1 = 0. \qquad ...(3)$$

The curve (3) passes through the eight points of intersection of (1) and (2). Also the conic (3) is a rectangular hyperbola because in its equation the sum of the coefficients of x^2 and y^2 is zero.

Hence the eight points of intersection of (1) and (2) lie on a rectangular hyperbola.

Ex. 59 (b). *Show that the four asymptotes of the curve*

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in eight points which lie on the circle $x^2 + y^2 = 1$.

(Gorakhpur 1989; Meerut 83 S; Agra 80; I.C.S. 97)

Sol. The given curve is

$$P \equiv (x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0 \qquad ...(1)$$

Here $\phi_4(m) = (1 - m^2)(m^2 - 4) = -4 + 5m^2 - m^4.$

The slopes of the asymptotes are given by the equation

$$\phi_4(m) = (1 - m^2)(m^2 - 4) = 0.$$

$\therefore$ $$m = \pm 1, \pm 2.$$

Also $$\phi_3(m) = 6 - 5m - 3m^2 + 2m^3,$$

and $$\phi_4'(m) = 10m - 4m^3.$$

Now c is given by the equation

$$c\,\phi_4'(m) + \phi_3(m) = 0$$

i.e., $$c(10m - 4m^3) + 6 - 5m - 3m^2 + 2m^3 = 0$$

i.e., $$c = \frac{6 - 5m - 3m^2 + 2m^3}{4m^3 - 10m}.$$

When $m = 1, c = 0$; when $m = -1, c = 1$;
when $m = 2, c = 0$; and when $m = -2, c = 1$.

Thus the asymptotes of the curve (1) are

$$y = x,\ y = -x + 1,\ y = 2x \text{ and } y = -2x + 1$$

The combined equation of the asymptotes is

$$(y-x)(y+x-1)(y-2x)(y+2x-1)=0$$

or $[(y^2-x^2)-y+x][(y^2-4x^2)-y+2x]=0$

or $(y^2-x^2)(y^2-4x^2)-y^3+2xy^2+x^2y-2x^3-y^3+4x^2y$
$+xy^2-4x^3+y^2-2xy-xy+2x^2=0$

or $Q\equiv(y^2-x^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3$
$+y^2-3xy+2x^2=0.$ ...(2)

Now each asymptote of (1) will cut it in $4-2$ *i.e.*, 2 points. Therefore the four asymptotes will cut it in 4×2 *i.e.*, 8 points.

Now taking $\lambda=1$, $P+\lambda Q=0$ gives $x^2+y^2-1=0$ *i.e.*, $x^2+y^2=1$, which is the equation of a circle.

Hence the eight points of intersection of (1) and (2) lie on the circle $x^2+y^2=1$

Ex. 60. *Find the asymptotes of the curve*

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

and show that they pass through the points of intersection of the curve with the ellipse $x^2+4y^2=4$. **(I.C.S. 1996)**

Sol. The given curve may be written as

$$(4x^4+4y^4-17x^2y^2)-4(4xy^2-x^3)+2x^2-4=0. \quad ...(1)$$

Here $\phi_4(m)=4m^4-17m^2+4=0$ gives $m=\pm\frac{1}{2},\pm 2$.

Also $\quad c=-\dfrac{\phi_3(m)}{\phi_4'(m)}=\dfrac{4(4m^2-1)}{16m^3-34m}$.

When $m=1/2, c=0$, when $m=-1/2, c=0$,
when $m=2, c=1$ and when $m=-2, c=-1$.

Thus the asymptotes of (1) are

$$y=\tfrac{1}{2}x,\ y=-\tfrac{1}{2}x,\ y=2x+1 \text{ and } y=-2x-1.$$

$\therefore$ the combined equation of the asymptotes is

$$(x-2y)(x+2y)(2x-y+1)(2x+y+1)=0$$

or $$(4x^4+4y^4-17x^2y^2)-4(4xy^2-x^3)+(x^2-4y^2)=0. \quad ...(2)$$

Subtracting (2) from (1), we get

$$x^2+4y^2-4=0. \quad ...(3)$$

Also the four asymptotes cut the curve into $n(n-2)$ *i.e.*, $4(4-2)=8$ points. These 8 points of intersection lie on the ellipse (3).

§ 13. Asymptotes of the polar curves.

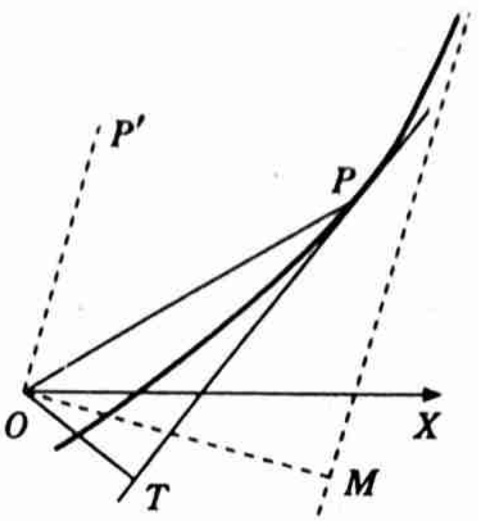
If α be a root of the equation $f(\theta)=0$, then

$$r\sin(\theta-\alpha)=1/f'(\alpha)$$

is an asymptote of the curve

$$1/r=f(\theta) \quad \text{or} \quad v=f(\theta), \text{ where } v=1/r.$$

Let P be any point (r,θ) on the curve $1/r=f(\theta)$. ...(1)



As $P \to \infty$, $r \to \infty$ and consequently $f(\theta) \to 0$.

Let OT be the perpendicular to the radius vector OP. Then OT (*i.e.*, the polar subtangent of the curve at P) is given by

$$OT = r^2 \frac{d\theta}{dr} = -\frac{1}{f'(\theta)}.$$

$$\left[\because \text{ from (1), } -\frac{1}{r^2}\frac{dr}{d\theta} = f'(\theta)\right]$$

Now let $\theta \to \alpha$. Then $f(\theta) \to 0$.

$$[\because f(\alpha) = 0]$$

Therefore $r \to \infty$, $PT \to$ to an asymptote, and

$$OT \to \left(r^2 \frac{d\theta}{dr}\right)_{\theta = \alpha} \quad \text{or} \quad OT \to -\frac{1}{f'(\alpha)} \text{ if } f'(\alpha) \neq 0.$$

Also OP and PT will tend to become parallel, and the angle OTP will tend to a right angle and OT will tend to OM where OM is perpendicular to the asymptote. Hence, the asymptote is the straight line parallel to the radius vector $\theta = \alpha$ and situated at a distance $\left(r^2 \frac{d\theta}{dr}\right)_{\theta = \alpha}$ from O.

Now the polar equation of the straight line, the perpendicular p on which from the origin makes an angle β with the initial line, is $r \cos(\theta - \beta) = p$. **[Remember]**

Here, for the asymptote, $p = OM = \left(r^2 \frac{d\theta}{dr}\right)_{\theta = \alpha} = -\frac{1}{f'(\alpha)}$,

and $\quad \beta = -\left(\frac{1}{2}\pi - \alpha\right) = \alpha - \frac{1}{2}\pi$.

Hence the equation of the asymptote is

$$r \cos\{\theta - (\alpha - \tfrac{1}{2}\pi)\} = -1/f'(\alpha)$$

i.e., $$\mathbf{r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}}.$$

Ex. 61. *Find the asymptotes of the curve $r \sin n\theta = a$.*

(Gorakhpur 1981; Meerut 95)

Sol. The equation of the curve can be written as

$$1/r = (1/a) \sin n\theta = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ if $\sin n\theta = 0$ *i.e.*, $n\theta = m\pi$ (where m is any integer).

$\therefore \quad \theta = (m\pi/n) = \alpha$, say.

Also $\quad f'(\theta) = (1/a) . n \cos n\theta$.

$\therefore \quad f'(\alpha) = (1/a) . n \cos n\alpha = (1/a) . n \cos\left(n \cdot \frac{m\pi}{n}\right)$

$$= (1/a) \cdot n \cdot \cos m\pi.$$

$\therefore$ the asymptotes are $r \sin\left(\theta - \dfrac{m\pi}{n}\right) = \dfrac{a}{n \cos m\pi}$,

where m is any integer.

Ex. 62. *Find the asymptotes of the curve*

$r\theta = a.$ **(Garhwal 1983; Meerut 80, 96, 97)**

Sol. The equation of the given curve can be written as

$$1/r = \theta/a = f(\theta), \text{ say.}$$

Now $\quad f(\theta) = 0$ if $\theta/a = 0$ *i.e.*, $\theta = 0 = \alpha$, say.

Also $\quad f'(\theta) = 1/a$, so that

$$f'(\alpha) = f'(0) = 1/a, \quad \text{or} \quad 1/f'(\alpha) = a.$$

Now the asymptote corresponding to $\theta = \alpha$ is given by

$$r \sin(\theta - \alpha) = 1/f'(\alpha)$$

i.e., $\quad r \sin(\theta - 0) = a, \qquad [\because \text{ here } \alpha = 0]$

i.e., $\quad r \sin\theta = a.$

Ex. 63. *Find the asymptotes of the curve $r = a/(1 - \cos\theta)$.*

Sol. The equation to the curve can be written as

$$\frac{1}{r} = \frac{1}{a}(1 - \cos\theta) = f(\theta), \text{ say.}$$

Now $f(\theta) = 0 \Rightarrow 1 - \cos\theta = 0$ or $\cos\theta = 1$

i.e., $\quad \theta = 2k\pi$, where k is any integer.

Also $\quad f'(\theta) = \dfrac{1}{a} \sin\theta.$

$$\therefore \quad f'(2k\pi) = \frac{1}{a} \sin(2k\pi) = 0.$$

Hence $\dfrac{1}{f'(2k\pi)} = \infty$

and consequently there is no asymptote of the given curve.

Ex. 64. *Find the asymptotes of the curve*

$r\theta \cos\theta = a \cos 2\theta.$ **(Agra 1985)**

Sol. The equation of the given curve can be written as

$$\frac{1}{r} = \frac{\theta \cos\theta}{a \cos 2\theta} = f(\theta), \text{ say.}$$

$$\therefore \quad f'(\theta) = \frac{1}{a}\left[\frac{\cos 2\theta \cdot \{\cos\theta - \theta \cdot \sin\theta\} + 2\theta \cos\theta \cdot \sin 2\theta}{\cos^2 2\theta}\right].$$

Now $f(\theta) = 0 \Rightarrow \theta = 0$ or $\cos\theta = 0$ *i.e.*, $\theta = (2k + 1) \cdot \frac{1}{2}\pi.$

If $\quad \theta = 0 = \alpha$ (say), then $f'(\alpha) = 1/a.$

$\therefore \quad r \sin(\theta - 0) = a$

or $\quad r \sin\theta = a$ is the corresponding asymptote.

When $\quad \theta = (2k + 1) \cdot \frac{1}{2}\pi = \alpha$ (say), we have

$$\cos\alpha = 0, \sin 2\alpha = 0, \sin\alpha = (-1)^k$$

and $\cos 2\alpha = \cos(2k\pi + \pi) = \cos\pi = -1.$

$$\therefore \quad f'(\alpha) = \frac{1}{a}\left[\frac{-\{(-1)^k (2k+1) \cdot \frac{1}{2}\pi}{1}\right]$$

$$= (-1)^{k+1}(2k+1)\cdot\frac{\pi}{2a}\cdot$$

The corresponding asymptote is

$$r\sin\left\{\theta - (2k+1)\cdot\frac{\pi}{2}\right\} = \frac{2a}{(-1)^{k+1}.(2k+1)\pi}$$

or $(-1)^k r\{\cos(k\pi - \theta)\}(2k+1)\pi = 2a,$

or $(-1)^{2k} r\cos\theta = 2a/\{(2k+1)\pi\}.$

$\therefore$ the required asymptotes are $r\sin\theta = a$

and $r\cos\theta = 2a/\{(2k+1)\pi\}.$

Ex. 65. *Find the asymptotes of the curve*

$$r \sin\theta = 2\cos 2\theta.$$ **(Meerut 1989 S, 92, 94, 96 BP)**

Sol. The equation to the curve can be written as

$$\frac{1}{r} = \frac{\sin\theta}{2\cos 2\theta} = f(\theta), \text{ say.}$$

Now $f(\theta) = 0 \Rightarrow \sin\theta = 0$ *i.e.*, $\theta = k\pi = \alpha$ (say).

Here k is any integer.

Also $$f'(\theta) = \tfrac{1}{2}\frac{\cos 2\theta\cos\theta - \sin\theta.(-2\sin 2\theta)}{\cos^2 2\theta}.$$

$$\therefore \quad \frac{1}{f'(\alpha)} = \frac{2\cos^2(2k\pi)}{\cos 2k\pi.\cos k\pi + 2\sin k\pi.\sin 2k\pi}$$

$$= \frac{2}{\cos k\pi}, \qquad [\because \cos 2k\pi = 1]$$

$$= 2/(-1)^k. \qquad [\because \cos k\pi = (-1)^k]$$

$\therefore$ the required asymptotes are given by

$r\sin(\theta - k\pi) = 2/(-1)^k$

or $-r\sin(k\pi - \theta) = 2/(-1)^k$

or $-r\{(-1)^{k-1}\sin\theta\} = 2/(-1)^k$

or $r\sin\theta = 2.$

Ex. 66. *Find the asymptotes of the curve* $r = a\ \text{cosec}\ \theta + b.$

Sol. The equation of the curve can be written as

$$\frac{1}{r} = \frac{1}{a\ \text{cosec}\ \theta + b} = \frac{\sin\theta}{a + b\sin\theta} = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$, if $\sin\theta = 0$ *i.e.*, $\theta = n\pi$, (where n is any integer).

Also $$f'(\theta) = \frac{\cos\theta(a + b\sin\theta) - b\sin\theta\cos\theta}{(a + b\sin\theta)^2}.$$

$$\therefore \quad f'(n\pi) = \frac{\cos n\pi(a + b\sin n\pi) - b\sin n\pi\cos n\pi}{(a + b\sin n\pi)^2} = \frac{\cos n\pi}{a}.$$

$\therefore$ the asymptotes are given by $r \sin(\theta - n\pi) = \dfrac{a}{\cos n\pi}$

or $\quad r(\sin\theta \cos n\pi - \cos\theta \sin n\pi) = \dfrac{a}{\cos n\pi}$

or $\quad r \sin\theta \cos^2 n\pi - 0 = a \quad$ or $\quad r \sin\theta = a.$

Ex. 67. *Find the asymptotes of the curve*

$$r = 2a/(1 - 2\cos\theta.)$$

Sol. The equation of the curve can be written as

$$\frac{1}{r} = \frac{1}{2a}(1 - 2\cos\theta) = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ if $1 - 2\cos\theta = 0$ *i.e.*, $2\cos\theta = 1$ or $\cos\theta = \frac{1}{2}$ *i.e.*, $\theta = 2n\pi \pm \frac{1}{3}\pi$, where n is any integer.

Also $f'(\theta) = \dfrac{1}{2a}(2\sin\theta) = \dfrac{1}{a}\sin\theta.$

$$\therefore \quad f'\left(2n\pi \pm \frac{\pi}{3}\right) = \frac{1}{a}\sin\left(2n\pi \pm \frac{\pi}{3}\right) = \pm\frac{1}{a}\sin\frac{\pi}{3} = \pm\frac{\sqrt{3}}{2a}\cdot$$

$\therefore$ the asymptotes are given by

$$r \sin\left\{\theta - \left(2n\pi \pm \frac{\pi}{3}\right)\right\} = \pm\frac{2a}{\sqrt{3}}$$

or $\quad r \sin\theta \cos\left(2n\pi \pm \dfrac{\pi}{3}\right) - r\cos\theta \,.\, \sin\left(2n\pi \pm \dfrac{\pi}{3}\right) = \pm\dfrac{2a}{\sqrt{3}}$

or $\quad (r\sin\theta)\cos\dfrac{\pi}{3} \mp r\cos\theta \sin\dfrac{\pi}{3} = \pm\dfrac{2a}{\sqrt{3}}$

i.e., $\quad r\sin\theta\cos\dfrac{\pi}{3} - r\cos\theta\sin\dfrac{\pi}{3} = \dfrac{2a}{\sqrt{3}}$

and $\quad r\sin\theta\cos\dfrac{\pi}{3} + r\cos\theta\sin\dfrac{\pi}{3} = -\dfrac{2a}{\sqrt{3}}$

i.e., $\quad r\sin\left(\theta - \dfrac{\pi}{3}\right) = \dfrac{2a}{\sqrt{3}}$ and $r\sin\left(\theta + \dfrac{\pi}{3}\right) = -\dfrac{2a}{\sqrt{3}}\cdot$

Ex.68. *Show that the curve $r = a\sec m\theta + b\tan m\theta$ has two sets of asymptotes; members of the first set touching one fixed circle, and those of the other, another fixed circle.*

Sol. The equation of the curve can be written as

$$\frac{1}{r} = \frac{1}{a\sec m\theta + b\tan m\theta} = \frac{\cos m\theta}{a + b\sin m\theta} = f(\theta), \text{ say.}$$

$$\therefore \quad f'(\theta) = \frac{(a + b\sin m\theta)(-m\sin m\theta) - \cos m\theta\,(bm\cos m\theta)}{(a + b\sin m\theta)^2}\cdot$$

Now $f'(\theta) = 0 \Rightarrow \cos m\theta = 0, \quad$ or $\quad m\theta = (2k+1)\cdot\frac{1}{2}\pi,$

(k is any integer)

or $\theta = (2k+1)\cdot\dfrac{\pi}{2m} = \alpha$, say.

Obviously $\cos m\alpha = 0$.

Also $\sin m\alpha = \sin\{(2k+1)\frac{1}{2}\pi\} = \sin(k\pi + \frac{1}{2}\pi) = \cos k\pi = (-1)^k$.

$$\therefore\quad f'(\alpha) = \frac{\{a + b(-1)^k\}\{-m(-1)^k\}}{\{a + b(-1)^k\}^2} = \frac{(-1)^{k+1}.m}{\{a + b(-1)^k\}}.$$

$\therefore$ the asymptotes are given by

$$r\sin\left[\theta - \left\{(2k+1)\cdot\frac{\pi}{2m}\right\}\right] = \frac{a + b(-1)^k}{(-1)^{k+1}.m}. \qquad \text{...(1)}$$

Case I. When k is odd, the first set of asymptotes are

$$r\sin\left\{\theta - \frac{(2k+1)\pi}{2m}\right\} = \frac{a-b}{m}, \qquad [\because (-1)^k = -1]$$

or $$r\left[\sin\theta\cos\frac{(2k+1)\pi}{2m} - \cos\theta\sin\frac{(2k+1)\pi}{2m}\right] = \frac{a-b}{m},$$

or $$x\sin\frac{(2k+1)\pi}{2m} - y\cos\frac{(2k+1)\pi}{2m} + \frac{a-b}{m} = 0,$$

[since $r\cos\theta = x$ and $r\sin\theta = y$]

and these asymptotes touch the circle $x^2 + y^2 = \left(\dfrac{a-b}{m}\right)^2$, which is fixed.

Case II. When k is even, the second set of asymptotes are

$$r\sin\left(\theta - \frac{2k+1}{2m}\cdot\pi\right) = -\frac{a+b}{m}, \qquad [\because (-1)^k = 1]$$

or $$y\cos\left(\frac{2k+1}{2m}\cdot\pi\right) - x\sin\left(\frac{2k+1}{2m}\cdot\pi\right) = -\frac{a+b}{m}$$

or $$x\sin\left(\frac{2k+1}{2m}\cdot\pi\right) - y\cos\left(\frac{2k+1}{2m}\cdot\pi\right) - \left(\frac{a+b}{m}\right) = 0$$

and these touch another fixed circle

$$x^2 + y^2 = \left(\frac{a+b}{m}\right)^2.$$

§ 14. Circular asymptotes.

Definition. *Let the equation of a curve be $r = f(\theta)$.*

If $\lim\limits_{\theta\to\infty} f(\theta) = l$, then the circle $r = l$ is called the circular asymptote of the curve $r = f(\theta)$.

Ex. 69. *Find the circular asymptote of the curve*

$$r = a\cdot\frac{\theta}{\theta - 1}. \qquad \textbf{(Meerut 1981)}$$

Sol. The circular asymptote is given by

$r = a\lim\limits_{\theta\to\infty}\dfrac{\theta}{\theta-1} = a$. Thus $r = a$ is the circular asymptote.

Ex. 70. *Find the circular asymptote of the curve*

$$r\,(e^{\theta}-1)=a\,(e^{\theta}+1).$$

(Agra 1981)

Sol. The given equation is

$$r=\frac{a\,(e^{\theta}+1)}{(e^{\theta}-1)}=f(\theta),\text{ say.}$$

$\therefore$ the circular asymptote is given by

$$r=a\lim_{\theta\to\infty}\frac{e^{\theta}+1}{e^{\theta}-1},\qquad\left[\text{form }\frac{\infty}{\infty}\right]$$

$$=a\lim_{\theta\to\infty}\frac{1+e^{-\theta}}{1-e^{-\theta}}=a.$$

Hence $r=a$ is the circular asymptote.

Ex. 71. *Find the circular asymptote of the curve*

$$r=\frac{3\theta^2+2\theta+1}{2\theta^2+\theta+1}\cdot$$

Sol. The given equation is

$r=\dfrac{3+(2/\theta)+(1/\theta^2)}{2+(1/\theta)+(1/\theta^2)}$, dividing numerator and denominator by θ^2.

Taking limits when $\theta\to\infty$, we see that the circular asymptote is $r=\frac{3}{2}\cdot$

□

9

Change of Independent Variables

§ 1. Introduction.

If there be an expression involving two variables x and y and containing the differential coefficients dy/dx, d^2y/dx^2, etc, it is sometimes desirable to change the independent variable into the dependent variable or to change the independent variable from x to some third variable z of which x is a known function. A similar transformation may be made in the case of several independent variables which are connected to a set of other variables by means of given relations.

§ 2. To change the independent variable into the dependent variable.

If we wish to make y the independent variable instead of x, we have

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{1}{dx/dy} = \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^{-1}, \qquad \text{...(1)}$$

and $$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)^{-1} = -\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\cdot\left(\frac{dy}{dx}\right)$$

$$= -\left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^{-3}\frac{\mathbf{d^2x}}{\mathbf{dy^2}}. \qquad \text{...(2)}$$

Similarly $$\frac{\mathbf{d^3y}}{\mathbf{dx^3}} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[-\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}\right], \text{ from (2)}$$

$$= -\frac{d}{dy}\left[\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}\right]\frac{dy}{dx}$$

$$= -\left[-3\left(\frac{dx}{dy}\right)^{-4}\frac{d^2x}{dy^2}\cdot\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^{-3}\frac{d^3x}{dy^3}\right]\frac{dy}{dx}$$

$$= -\left[-3\left(\frac{dx}{dy}\right)^{-4}\left(\frac{d^2x}{dy^2}\right)^2 + \left(\frac{dx}{dy}\right)^{-3}\frac{d^3x}{dy^3}\right]\left(\frac{dx}{dy}\right)^{-1}$$

$$= 3\left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^{-5}\left(\frac{\mathbf{d^2x}}{\mathbf{dy^2}}\right)^2 - \left(\frac{\mathbf{dx}}{\mathbf{dy}}\right)^{-4}\frac{\mathbf{d^3x}}{\mathbf{dy^3}},$$

and so on.

§ 3. To change the independent variable x into another variable z, where x = f (z).

We have $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \left(\dfrac{dx}{dz}\right)^{-1} \dfrac{d}{dz}(y).$

The operator d/dx is thus equivalent to the operator $\left(\dfrac{dx}{dz}\right)^{-1} \dfrac{d}{dz}$

and we write $\dfrac{d}{dx} \equiv \left(\dfrac{dx}{dz}\right)^{-1} \dfrac{d}{dz}$.

Now $$\frac{\mathbf{d^2y}}{\mathbf{dx^2}} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{dx}{dz}\right)^{-1} \frac{d}{dz}\left[\left(\frac{dx}{dz}\right)^{-1} \frac{dy}{dz}\right]$$
$$= \left(\frac{dx}{dz}\right)^{-1}\left[-\left(\frac{dx}{dz}\right)^{-2} \frac{d^2x}{dz^2}\frac{dy}{dz} + \left(\frac{dx}{dz}\right)^{-1}\frac{d^2y}{dz^2}\right]$$
$$= \left(\frac{\mathbf{dx}}{\mathbf{dz}}\right)^{-3}\left[\frac{\mathbf{dx}}{\mathbf{dz}}\frac{\mathbf{d^2y}}{\mathbf{dz^2}} - \frac{\mathbf{d^2x}}{\mathbf{dz^2}}\frac{\mathbf{dy}}{\mathbf{dz}}\right].$$

Similarly $$\frac{\mathbf{d^3y}}{\mathbf{dx^3}} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$
$$= \left(\frac{dx}{dz}\right)^{-1}\frac{d}{dz}\left[\left(\frac{dx}{dz}\right)^{-3}\left\{\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right\}\right]$$
$$= \left(\frac{dx}{dz}\right)^{-1}\left[\left(\frac{dx}{dz}\right)^{-3}\left\{\frac{d^2x}{dz^2}\frac{d^2y}{dz^2} + \frac{dx}{dz}\frac{d^3y}{dz^3} - \frac{d^3x}{dz^3}\frac{dy}{dz} - \frac{d^2x}{dz^2}\frac{d^2y}{dz^2}\right\}\right.$$
$$\left. - 3\left(\frac{dx}{dz}\right)^{-4}\frac{d^2x}{dz^2}\left\{\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right\}\right]$$
$$= \left(\frac{\mathbf{dx}}{\mathbf{dz}}\right)^{-5}\left[\left(\frac{\mathbf{dx}}{\mathbf{dz}}\right)\left\{\frac{\mathbf{dx}}{\mathbf{dz}}\frac{\mathbf{d^3y}}{\mathbf{dz^3}} - \frac{\mathbf{d^3x}}{\mathbf{dz^3}}\frac{\mathbf{dy}}{\mathbf{dz}}\right\} - 3\frac{\mathbf{d^2x}}{\mathbf{dz^2}}\left\{\frac{\mathbf{dx}}{\mathbf{dz}}\frac{\mathbf{d^2y}}{\mathbf{dz^2}} - \frac{\mathbf{d^2x}}{\mathbf{dz^2}}\frac{\mathbf{dy}}{\mathbf{dz}}\right\}\right],$$
and so on.

Solved Examples

Ex. 1. *Show that the differential equation* $d^2x/dy^2 = a$ *may be written in the form* $(d^2y/dx^2) + a\,(dy/dx)^3 = 0$.

Sol. We have $$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}. \qquad \text{...(1)}$$

$$\therefore \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}, \qquad \text{from (1)}$$
$$= \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\} \cdot \frac{dx}{dy} = -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx}\right)^{-1} = -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}.$$

Hence the differential equation $d^2x/dy^2 = a$ becomes
$$-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2} = a \quad \text{or} \quad \frac{d^2y}{dx^2} = -a\left(\frac{dy}{dx}\right)^3$$
or $$(d^2y/dx^2) + a\,(dy/dx)^3 = 0.$$

Ex. 2. *Show that the equation* $\dfrac{dy}{dx}\cdot\dfrac{d^3y}{dx^3}-3\left(\dfrac{d^2y}{dx^2}\right)^2=0$ *can be written in the form* $(d^3x/dy^3)=0$. **(Meerut 1990P)**

Sol. Proceeding as in § 2, we have

$$\frac{dy}{dx}=\left(\frac{dx}{dy}\right)^{-1};\quad \frac{d^2y}{dx^2}=-\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2};$$

and $$\frac{d^3y}{dx^3}=3\left(\frac{dx}{dy}\right)^{-5}\left(\frac{d^2x}{dy^2}\right)^2-\left(\frac{d^3x}{dy^3}\right)\left(\frac{dx}{dy}\right)^{-4}.$$

The given equation therefore becomes

$$\left(\frac{dx}{dy}\right)^{-1}\left[3\left(\frac{dx}{dy}\right)^{-5}\left(\frac{d^2x}{dy^2}\right)^2-\left(\frac{d^3x}{dy^3}\right)\left(\frac{dx}{dy}\right)^{-4}\right]-3\left[-\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}\right]^2=0$$

or $$3\left(\frac{dx}{dy}\right)^{-6}\left(\frac{d^2x}{dy^2}\right)^2-\left(\frac{d^3x}{dy^3}\right)\left(\frac{dx}{dy}\right)^{-5}-3\left(\frac{dx}{dy}\right)^{-6}\left(\frac{d^2x}{dy^2}\right)^2=0$$

or $$-\left(\frac{d^3x}{dy^3}\right)\left(\frac{dx}{dy}\right)^{-5}=0\quad\text{or}\quad\frac{d^3x/dy^3}{(dx/dy)^5}=0\quad\text{or}\quad\frac{d^3x}{dy^3}=0.$$

Ex. 3 (*a*). *Transform the equation* $x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}+n^2y=0$ *by the substitution* $x=1/z$. **(Meerut 1992)**

(*b*) *Change the independent variable from x to z in the equation* $x^4(d^2y/dx^2)+a^2y=0$ *where* $x=1/z$.

Sol. (a) We have $\dfrac{dy}{dx}=\dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$.

But $x=\dfrac{1}{z}\Rightarrow\dfrac{dx}{dz}=-\dfrac{1}{z^2}=-x^2$, so that $\dfrac{dz}{dx}=-\dfrac{1}{x^2}$.

$$\therefore\quad \frac{dy}{dx}=-\frac{1}{x^2}\frac{dy}{dz}\quad\text{or}\quad x^2\frac{dy}{dx}=-\frac{dy}{dz}.$$

$$\therefore\quad x^2\frac{d}{dx}\equiv-\frac{d}{dz}.$$

$$\therefore\quad x^2\frac{d}{dx}\left(x^2\frac{dy}{dx}\right)=-\frac{d}{dz}\left(-\frac{dy}{dz}\right)\qquad\textbf{(Note)}$$

or $$x^4\frac{d^2y}{dx^2}+2x^3\frac{dy}{dx}=\frac{d^2y}{dz^2}.$$

Hence the given equation becomes $(d^2y/dz^2)+n^2y=0$.

Note. Here we solved the problem by first establishing the fact that the two operators $x^2\dfrac{d}{dx}$ and $-\dfrac{d}{dz}$ have the same effect on y. Then we proceeded in such a way that we atonce got the expression $x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}$ transformed to $\dfrac{d^2y}{dz^2}$. However, the problem can also be solved by direct calculations as shown below.

Alternative solution.

We have $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = -z^2 \dfrac{dy}{dz}$ $\left[\because \dfrac{dx}{dz} = -\dfrac{1}{z^2} \text{ or } \dfrac{dz}{dx} = -z^2\right]$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{dz}\left(\frac{dy}{dx}\right)\right\} \cdot \frac{dz}{dx} = \frac{d}{dz}\left(-z^2 \frac{dy}{dz}\right) \cdot (-z^2)$$

$$= \left[-2z \frac{dy}{dz} - z^2 \frac{d^2y}{dz^2}\right](-z^2) = 2z^3 \frac{dy}{dz} + z^4 \frac{d^2y}{dz^2}.$$

Hence the given equation reduces to

$$\frac{1}{z^4}\left(z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}\right) + \frac{2}{z^3}\left(-z^2 \frac{dy}{dz}\right) + n^2y = 0$$

or $\quad (d^2y/dz^2) + n^2y = 0.$

(b) Proceeding as in the alternative solution of part (a), the given equation transforms to

$$\frac{1}{z^4}\left(z^4 \frac{d^2y}{dz^2} + 2z^3 \frac{dy}{dz}\right) + a^2y = 0 \quad \text{or} \quad \frac{d^2y}{dz^2} + \frac{2}{z}\frac{dy}{dz} + a^2y = 0.$$

***Ex. 4** (*a*). *Transform the equation* $(1+x^2)^2 \dfrac{d^2y}{dx^2} + 2x(1+x^2)\dfrac{dy}{dx}$ $+ y = 0$ *by the substitution* $x = \tan z$. **(Meerut 1990 S, 91S, 93)**

Sol. We have $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$.

But $\quad x = \tan z \Rightarrow \dfrac{dx}{dz} = \sec^2 z = 1 + \tan^2 z = 1 + x^2,$

so that $\dfrac{dz}{dx} = \dfrac{1}{1+x^2}$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1+x^2}\frac{dy}{dz} \quad \text{or} \quad (1+x^2)\frac{dy}{dx} = \frac{dy}{dz}.$$

$$\therefore \quad (1+x^2)\frac{d}{dx} \equiv \frac{d}{dz}.$$

$$\therefore \quad (1+x^2)\frac{d}{dx}\left[(1+x^2)\frac{dy}{dx}\right] = \frac{d}{dz}\left(\frac{dy}{dz}\right) \qquad \textbf{(Note)}$$

or $\quad (1+x^2)^2 \dfrac{d^2y}{dx^2} + 2x(1+x^2)\dfrac{dy}{dx} = \dfrac{d^2y}{dz^2}.$

Hence the given equation becomes $(d^2y/dz^2) + y = 0$.

Alternative solution.

We have $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx} = \cos^2 z \dfrac{dy}{dz}$,

$[\because dx/dz = \sec^2 z \text{ or } dz/dx = \cos^2 z]$

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\cos^2 z \dfrac{dy}{dz}\right) = \left\{\dfrac{d}{dz}\left(\cos^2 z \dfrac{dy}{dz}\right)\right\}\dfrac{dz}{dx}$

$= [\cos^2 z\,(d^2y/dz^2) + 2\cos z\,(-\sin z)\,(dy/dz)]\cos^2 z$

$= \cos^4 z \,.\, (d^2y/dz^2) - 2 \sin z \cos^3 z \,.\, (dy/dz)$.

Substituting the values of $x, dy/dx$ and d^2y/dx^2 in the given equation, it becomes,

$$(1 + \tan^2 z)^2 [\cos^4 z \, (d^2y/dz^2) - 2 \sin z \cos^3 z \, (dy/dz)] + 2 \tan z \, (1 + \tan^2 z) \cos^2 z \, (dy/dz) + y = 0$$

or $(d^2y/dz^2) - 2 \tan z \, (dy/dz) + 2 \tan z \, (dy/dz) + y = 0$

or $(d^2y/dz^2) + y = 0$.

Ex. 4 (b). *Transform the equation*

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + a^2y = 0, \textit{ where } x = \sin \theta.$$ **(Meerut 1988)**

Sol. We have $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \dfrac{d\theta}{dx}$.

But $x = \sin \theta$. $\therefore$ $(dx/d\theta) = \cos \theta$ or $(d\theta/dx) = \sec \theta$.

$\therefore$ $(dy/dx) = (dy/d\theta) \sec \theta$.

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \left[\frac{d}{d\theta} \left(\frac{dy}{dx} \right) \right] \cdot \frac{d\theta}{dx}$$

$$= \left[\frac{d^2y}{d\theta^2} \sec \theta + \frac{dy}{d\theta} \cdot \sec \theta \tan \theta \right] \sec \theta$$

$$= \sec^2 \theta \, (d^2y/d\theta^2) + \sec^2 \theta \tan \theta \, (dy/d\theta).$$

$\therefore$ the given differential equation becomes

$$\cos^2 \theta \left[\sec^2 \theta \frac{d^2y}{d\theta^2} + \sec^2 \theta \tan \theta \frac{dy}{d\theta} \right] - \sin \theta \cdot \frac{dy}{d\theta} \frac{1}{\cos \theta} + a^2y = 0$$

or
$$\frac{d^2y}{d\theta^2} + \tan \theta \frac{dy}{d\theta} - \tan \theta \frac{dy}{d\theta} + a^2y = 0$$

or $(d^2y/d\theta^2) + a^2y = 0$.

Ex. 4 (c). *Change the independent variable from x to θ in the equation :*

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + a^2y = 0, \textit{ where } x = \cos \theta.$$ **(Meerut 1988 S)**

Sol. Do yourself. **Ans.** $(d^2y/d\theta^2) + a^2y = 0$.

Ex. 4 (d). *Change the independent variable from x to z in the equation* $\dfrac{d^2y}{dx^2} + xy \dfrac{dy}{dx} + \sec^2 x = 0$, *where* $x = \tan^{-1} z$. **(Meerut 1988 P)**

Sol. Do yourself.

Ans. $(1 + z^2) \dfrac{d^2y}{dz^2} + 2z \dfrac{dy}{dz} + (\tan^{-1} z) \, y \dfrac{dy}{dz} + 1 = 0$.

Ex. 5. *Transform the equation*

$$\sin^2 2z \, (d^2y/dz^2) + \sin 4z \, (dy/dz) + 4y = 0$$

by putting $\tan z = e^x$. **(Meerut 1975, 91 P)**

Sol. We have $\dfrac{dy}{dz} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dz}$.

But $\tan z = e^x$. Therefore $\sec^2 z\,(dz/dx) = e^x = \tan z$

i.e., $$\frac{dz}{dx} = \frac{\tan z}{\sec^2 z} = \sin z \cos z = \tfrac{1}{2}\sin 2z, \quad \text{or} \quad \frac{dx}{dz} = \frac{2}{\sin 2z}.$$

$$\therefore \quad \frac{dy}{dz} = \frac{2}{\sin 2z}\frac{dy}{dx} \quad \text{or} \quad \sin 2z \frac{dy}{dz} = 2\frac{dy}{dx}.$$

$$\therefore \quad \sin 2z \frac{d}{dz} \equiv 2\frac{d}{dx}.$$

$$\therefore \quad \sin 2z \frac{d}{dz}\left(\sin 2z \frac{dy}{dz}\right) = 2\frac{d}{dx}\left(2\frac{dy}{dx}\right) \qquad \textbf{(Note)}$$

or $$\sin^2 2z \frac{d^2y}{dz^2} + 2\sin 2z \cos 2z \frac{dy}{dz} = 4\frac{d^2y}{dx^2}.$$

Hence the given equation becomes

$$4\frac{d^2y}{dx^2} + 4y = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0.$$

Ex. 6. *Show that the equation* $x^2(d^2y/dx^2) + x(dy/dx) + y = 0$ *becomes* $(d^2y/dz^2) + y = 0$ *by substituting* e^z *for* x. **(Meerut 1990)**

Sol. We have $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$$

But $e^z = x$. Therefore $e^z \dfrac{dz}{dx} = 1$ or $\dfrac{dz}{dx} = \dfrac{1}{e^z} = \dfrac{1}{x}$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \text{or} \quad x\frac{dy}{dx} = \frac{dy}{dz}.$$

$$\therefore \quad x\frac{d}{dx} \equiv \frac{d}{dz}.$$

$$\therefore \quad x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right) \qquad \textbf{(Note)}$$

or $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}.$$

Hence the given equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \text{ becomes } \frac{d^2y}{dz^2} + y = 0$$

by substituting e^z for x.

Ex. 7. *Change the independent variable in the equation*

$$(a^2 - x^2)\frac{d^2y}{dx^2} - \frac{a^2}{x}\frac{dy}{dx} + \frac{x^2y}{a^2} = 0$$

given $$a^2z^2 = a^2 - x^2.$$

Sol. Here $a^2z^2 = a^2 - x^2$.

$$\therefore \quad 2a^2z\frac{dz}{dx} = -2x \quad \textit{i.e.,} \quad \frac{dz}{dx} = -\frac{x}{a^2z}.$$

Now $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = -\frac{x}{a^2z}\cdot\frac{dy}{dz},$$

and $$\frac{d^2y}{dx^2} = -\frac{x}{a^2z}\cdot\frac{d^2y}{dz^2}\cdot\left(\frac{dz}{dx}\right) - \frac{1}{a^2}\left(\frac{1}{z} - \frac{x}{z^2}\frac{dz}{dx}\right)\frac{dy}{dz}$$
$$= \frac{x^2}{a^4z^2}\cdot\frac{d^2y}{dz^2} - \frac{1}{a^2}\left(\frac{1}{z} + \frac{x^2}{a^2z^3}\right)\frac{dy}{dz}$$
$$= \frac{x^2}{a^4z^2}\cdot\frac{d^2y}{dz^2} - \frac{1}{a^2z^3}\cdot\frac{dy}{dz}. \qquad [\because a^2z^2 = a^2 - x^2]$$

Putting the values of $a^2 - x^2$, dy/dx and d^2y/dx^2 in the given equation, we get
$$a^2z^2\left(\frac{x^2}{a^4z^2}\cdot\frac{d^2y}{dz^2} - \frac{1}{a^2z^3}\cdot\frac{dy}{dz}\right) - \frac{a^2}{x}\left(-\frac{x}{a^2z}\right)\frac{dy}{dz} + \frac{x^2}{a}y = 0$$
or $$\frac{x^2}{a^2}\frac{d^2y}{dz^2} + \frac{x^2}{a}y = 0 \quad \text{or} \quad \frac{d^2y}{dz^2} + ay = 0.$$

Ex. 8. *If the equation*
$$x^n\frac{d^ny}{dx^n} + A_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + A_{n-1}x\frac{dy}{dx} + A_ny = 0$$
be transformed by the substitution $x = e^\theta$, to show that all the coefficients in the transformed equation shall be constants.

Sol. Here $x = e^\theta$, $\therefore$ $\dfrac{dx}{d\theta} = e^\theta = x$, or $\dfrac{d\theta}{dx} = \dfrac{1}{x}$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx} = \dfrac{1}{x}\cdot\dfrac{dy}{d\theta}$. Therefore $x\dfrac{dy}{dx} = \dfrac{dy}{d\theta}$.

The operator $x\,(d/dx)$ is thus equivalent to the operator $d/d\theta$.
Let $d/d\theta$ be denoted by D.
Then we have
$$x\frac{d}{dx}\left[x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\right] = x^n\frac{d^ny}{dx^n} + (n-1)x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$
or $$x^n\frac{d^ny}{dx^n} = \left(x\frac{d}{dx} - n + 1\right)x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}$$
$$= (D - n + 1)\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}.$$

Now putting $n = 2, 3, 4, \ldots$ etc., we get
$$x^2\frac{d^2y}{dx^2} = (D-1)\,x\frac{dy}{dx} = (D-1)\,Dy$$
$$x^3\frac{d^3y}{dx^3} = (D-2)\,x^2\frac{d^2y}{dx^2} = (D-2)(D-1)\,Dy$$
$$\ldots\ldots \qquad \ldots\ldots \qquad \ldots\ldots$$
$$x^n\frac{d^ny}{dx^n} = (D-n+1)(D-n+2)\ldots(D-1)\,Dy$$
$$= D(D-1)(D-2)\ldots(D-n+1)\,y.$$

The given equation shall therefore be transformed into one with constant coefficients by substituting these values of
$x\,(dy/dx)$, $x^2\,(d^2y/dx^2)$, etc.

§ 4. Change of both the dependent and independent variables.

Sometimes both the dependent and independent variables are to be changed by means of given relations. Such transformations will be clear from the following examples.

Ex. 9. *Transform the polar formula* $\tan \phi = r\,(d\theta/dr)$ *to cartesians taking x as independent variable.* **(Meerut 1987, 91)**

Sol. We know that $x = r\cos\theta,\ y = r\sin\theta$

so that $r^2 = x^2 + y^2,\ \theta = \tan^{-1}(y/x)$.

Therefore $$2r\frac{dr}{dx} = 2x + 2y\frac{dy}{dx} \quad \text{or} \quad r\frac{dr}{dx} = x + y\frac{dy}{dx}. \qquad \ldots(1)$$

Also $$\frac{d\theta}{dx} = \frac{1}{1 + (y/x)^2}\cdot\frac{x\,(dy/dx) - y\,.1}{x^2} = \frac{1}{x^2 + y^2}\left\{x\cdot\frac{dy}{dx} - y\right\} \qquad \ldots(2)$$

Now $$\tan\phi = r\frac{d\theta}{dr} = \left(r\frac{d\theta}{dx}\right)\bigg/\left(\frac{dr}{dx}\right)$$

$$= \frac{r\,\{1/(x^2 + y^2)\}\,\{x\,(dy/dx) - y\}}{(1/r)\,\{x + y\,(dy/dx)\}}, \text{ from (1) and (2).}$$

Hence $$\tan\phi = \frac{x\,(dy/dx) - y}{x + y\,(dy/dx)}. \qquad [\because r^2 = x^2 + y^2]$$

Ex. 10. *Transform the formula* $\rho = \dfrac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}$ *into polar coordinates.*

Sol. We have $x = r\cos\theta,\ y = r\sin\theta$; therefore $r^2 = x^2 + y^2$, $\theta = \tan^{-1}(y/x)$.

Now $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\right\}\frac{d\theta}{dx} = \frac{1}{dx/d\theta}\cdot\frac{d}{d\theta}\left(\frac{dy}{dx}\right)$$

$$= \left(\frac{1}{dx/d\theta}\right)\frac{d}{d\theta}\left\{\frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}\right\}$$

$$= \frac{r^2 + 2\,(dr/d\theta)^2 - r\,(d^2r/d\theta^2)}{\{(dr/d\theta)\cos\theta - r\sin\theta\}^3},$$

[On simplifying after performing the differentiation]

Substituting the values of dy/dx and d^2y/dx^2 in the formula for ρ, we get

$$\rho = \frac{[\{(dr/d\theta)\cos\theta - r\sin\theta\}^2 + \{(dr/d\theta)\sin\theta + r\cos\theta\}^2]^{3/2}}{r^2 + 2\,(dr/d\theta)^2 - r\,(d^2r/d\theta^2)}$$

or $$\rho = \frac{\{r^2 + (dr/d\theta)^2\}^{3/2}}{r^2 + 2\,(dr/d\theta)^2 - r\,(d^2r/d\theta^2)}.$$

§ 5. Transformation in the case of two independent variables.

We recall the following results already established in the chapter on partial differentiation.

(1) If $u = f(x, y)$, then $du = \dfrac{\partial u}{\partial x} dx + \dfrac{\partial u}{\partial y} dy.$

(2) If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

(3) If $u = f(x, y)$ where $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$, then

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} &= \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{t}_1} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{t}_1} \\ \text{and} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2} &= \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{t}_2} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{t}_2} \end{aligned}\right\} \quad \text{...(A)}$$

The values of $\dfrac{\partial x}{\partial t_1}, \dfrac{\partial y}{\partial t_1}, \dfrac{\partial x}{\partial t_2}$ and $\dfrac{\partial y}{\partial t_2}$ are to be found from the given relations $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$. Substituting these values in the equations (A), we get the values of $\partial u / \partial t_1$ and $\partial u / \partial t_2$ in terms of $\partial u / \partial x$ and $\partial u / \partial y$.

If the equations $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$ are easily solvable for t_1 and t_2 in terms of x and y, say $t_1 = F_1(x, y)$ and $t_2 = F_2(x, y)$, then we can find $\dfrac{\partial t_1}{\partial x}, \dfrac{\partial t_1}{\partial y}, \dfrac{\partial t_2}{\partial x}, \dfrac{\partial t_2}{\partial y}$ and we can make use of the following formulae :

$$\left.\begin{aligned} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} &= \frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} \cdot \frac{\partial \mathbf{t}_1}{\partial \mathbf{x}} + \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2} \cdot \frac{\partial \mathbf{t}_2}{\partial \mathbf{x}} \\ \text{and} \quad \frac{\partial \mathbf{u}}{\partial \mathbf{y}} &= \frac{\partial \mathbf{u}}{\partial \mathbf{t}_1} \cdot \frac{\partial \mathbf{t}_1}{\partial \mathbf{y}} + \frac{\partial \mathbf{u}}{\partial \mathbf{t}_2} \cdot \frac{\partial \mathbf{t}_2}{\partial \mathbf{y}} \end{aligned}\right\} \quad \text{...(B)}$$

The above formulae (A) and (B) can easily be extended to the case of more than two independent variables.

Important Remark. In the transformations (A) and (B), we should note carefully which variables are to be regarded as constants while performing partial differentiations. There are two groups of variables : one of x and y and the other of t_1 and t_2. There is independence between x and y and also between t_1 and t_2. Thus while finding $\partial x / \partial t_1$, t_2 is to be regarded as a constant. Since x is a function of t_1 and t_2 both, therefore the students are warned against any such assumption [illegible]

$$\frac{\partial x}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} = 1.$$

Ex. 11 (a). *If* $z = e^u f(v)$, $u = ax + by$, $v = ax - by$ *show that*

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

(Meerut 1981; 87 S)

Sol. It is given that

$$z = e^u f(v), \quad \text{...(1)}$$

and $$u = ax + by, \quad v = ax - by. \quad \text{...(2)}$$

We have $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \text{...(3)}$$

and $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}. \quad \text{...(4)}$$

From the given relations (2), we have

$$\frac{\partial u}{\partial x} = a, \ \frac{\partial u}{\partial y} = b, \ \frac{\partial v}{\partial x} = a, \ \frac{\partial v}{\partial y} = -b.$$

Substituting these values in (3) and (4), we get

$$\frac{\partial z}{\partial x} = a\frac{\partial z}{\partial u} + a\frac{\partial z}{\partial v}, \quad \text{...(5)}$$

and $$\frac{\partial z}{\partial y} = b\frac{\partial z}{\partial u} - b\frac{\partial z}{\partial v}. \quad \text{...(6)}$$

Multiplying (5) by b and (6) by a and adding, we get

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\frac{\partial z}{\partial u}.$$

But from (1), $$\frac{\partial z}{\partial u} = e^u f(v) = z.$$

$\therefore$ $$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

***Ex. 11 (b).** *If* z *is a function of* x *and* y *and* $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, *prove that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

(Meerut 1980; Rohilkhand 87)

Sol. We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}, \quad \text{...(1)}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}. \quad \text{...(2)}$$

Given, $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$.

Differentiating partially $(\partial x/\partial u) = e^u$, $(\partial x/\partial v) = -e^{-v}$,

and $(\partial y/\partial u) = -e^{-u}$, $(\partial y/\partial v) = -e^v$.

$\therefore$ $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} e^u + \frac{\partial z}{\partial y}(-e^{-u}), \quad \text{[from (1)]}$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v}), \quad \text{[from (2)]}$$

Subtracting, we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} + (e^v - e^{-u})\frac{\partial z}{\partial y} = x\,(\partial z/\partial x) - y\,(\partial z/\partial y).$$

Ex. 11 (c). *Given that z is a function of u and v, where*

$u = x^2 - y^2 - 2xy$ *and* $v = y$

prove that the equation $(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y} = 0$ *is transformed into* $\dfrac{\partial z}{\partial v} = 0.$

(Meerut 1970, 88)

Sol. We have $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}\cdot$

From $u = x^2 - 2xy - y^2$ and $v = y$, we have

$$\partial u/\partial x = 2x - 2y \quad \text{and} \quad \partial v/\partial x = 0.$$

$$\therefore \quad \partial z/\partial x = 2\,(x-y)\,(\partial z/\partial u). \quad \text{...(1)}$$

Again $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}(-2x - 2y) + \frac{\partial z}{\partial v}\cdot 1$$

$$= -2\,(x+y)\,(\partial z/\partial u) + (\partial z/\partial v). \quad \text{...(2)}$$

Now $(x+y)\,(\partial z/\partial x) + (x-y)\,(\partial z/\partial y)$

$$= (x+y)\,2\,(x-y)\frac{\partial z}{\partial u} + (x-y)\left[-2\,(x+y)\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right]\cdot$$

[from (1) and (2)]

$$= [2\,(x^2 - y^2) - 2\,(x^2 - y^2)]\frac{\partial z}{\partial u} + (x-y)\frac{\partial z}{\partial v}$$

$$= (x-y)\,(\partial z/\partial v).$$

$\therefore$ the differential equation

$(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y} = 0$ is equivalent to

$(x-y)\dfrac{\partial z}{\partial v} = 0$ or equivalent to $\dfrac{\partial z}{\partial v} = 0,$ because $x \neq y.$

***Ex. 12.** *Prove that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial \xi^2} + \dfrac{\partial^2 u}{\partial \eta^2},$

where $x = \xi\cos\alpha - \eta\sin\alpha,\ y = \xi\sin\alpha + \eta\cos\alpha.$

[Allahabad 1982, 78; Gorakhpur 75]

Sol. We have $$x = \xi\cos\alpha - \eta\sin\alpha, \quad \text{...(1)}$$

and $$y = \xi\sin\alpha + \eta\cos\alpha. \quad \text{...(2)}$$

From (1), $\partial x/\partial\xi = \cos\alpha,\ \partial x/\partial\eta = -\sin\alpha;$

and from (2), $\partial y/\partial\xi = \sin\alpha,\ \partial y/\partial\eta = \cos\alpha.$

Now $\dfrac{\partial u}{\partial \xi}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial \xi}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial \xi}=\cos\alpha\dfrac{\partial u}{\partial x}+\sin\alpha\dfrac{\partial u}{\partial y}$

or $\dfrac{\partial}{\partial \xi}(u)=\left(\cos\alpha\dfrac{\partial}{\partial x}+\sin\alpha\dfrac{\partial}{\partial y}\right)u.$

$\therefore$ the operator $\partial/\partial\xi$ is equivalent to the operator

$$\cos\alpha\,(\partial/\partial x)+\sin\alpha\,(\partial/\partial y).$$

$$\therefore\quad \frac{\partial^2 u}{\partial \xi^2}=\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi}\right)=\left(\cos\alpha\frac{\partial}{\partial x}+\sin\alpha\frac{\partial}{\partial y}\right)\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right)$$

$$=\cos\alpha\frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right)+\sin\alpha\frac{\partial}{\partial y}\left(\cos\alpha\frac{\partial u}{\partial x}+\sin\alpha\frac{\partial u}{\partial y}\right)$$

$$=\cos^2\alpha\frac{\partial^2 u}{\partial x^2}+2\sin\alpha\cos\alpha\frac{\partial^2 u}{\partial x\,\partial y}+\sin^2\alpha\frac{\partial^2 u}{\partial y^2}. \qquad ...(3)$$

Similarly show that $\dfrac{\partial u}{\partial \eta}=\dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \eta}+\dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \eta}$

$$=\left(-\sin\alpha\frac{\partial}{\partial x}+\cos\alpha\frac{\partial}{\partial y}\right)u.$$

Consequently $\dfrac{\partial^2 u}{\partial \eta^2}=\dfrac{\partial}{\partial \eta}\left(\dfrac{\partial u}{\partial \eta}\right)=\sin^2\alpha\dfrac{\partial^2 u}{\partial x^2}-2\sin\alpha\cos\alpha\dfrac{\partial^2 u}{\partial x\,\partial y}$

$$+\cos^2\alpha\frac{\partial^2 u}{\partial y^2} \qquad ...(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \eta^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

***Ex. 13 (a).** *If $u=f(y-z,z-x,x-y)$, prove that*

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0.$$

(Meerut 1977, 83; Lucknow 79; Agra 88; Rohilkhand 86; Kanpur 89)

Sol. We have $u=f(y-z,z-x,x-y)$.

Let $y-z=A$, $z-x=B$, and $x-y=C$.

Then $u=f(A,B,C)$ where A,B and C are functions of x,y and z.

Now $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial A}\cdot\dfrac{\partial A}{\partial x}+\dfrac{\partial u}{\partial B}\cdot\dfrac{\partial B}{\partial x}+\dfrac{\partial u}{\partial C}\cdot\dfrac{\partial C}{\partial x}$

$$=\frac{\partial u}{\partial A}\cdot(0)+\frac{\partial u}{\partial B}\cdot(-1)+\frac{\partial u}{\partial C}\cdot(1)=-\frac{\partial u}{\partial B}+\frac{\partial u}{\partial C} \qquad ...(1)$$

Similarly $\dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial A}\cdot\dfrac{\partial A}{\partial y}+\dfrac{\partial u}{\partial B}\cdot\dfrac{\partial B}{\partial y}+\dfrac{\partial u}{\partial C}\cdot\dfrac{\partial C}{\partial y}$

$$=\frac{\partial u}{\partial A}\cdot(1)+\frac{\partial u}{\partial B}\cdot(0)+\frac{\partial u}{\partial C}\cdot(-1)=\frac{\partial u}{\partial A}-\frac{\partial u}{\partial C} \qquad ...(2)$$

and $\dfrac{\partial u}{\partial z}=\dfrac{\partial u}{\partial A}\cdot\dfrac{\partial A}{\partial z}+\dfrac{\partial u}{\partial B}\cdot\dfrac{\partial B}{\partial z}+\dfrac{\partial u}{\partial C}\cdot\dfrac{\partial C}{\partial z}$

$$= \frac{\partial u}{\partial A} \cdot (-1) + \frac{\partial u}{\partial B} \cdot (1) + \frac{\partial u}{\partial C} \cdot (0) = -\frac{\partial u}{\partial A} + \frac{\partial u}{\partial B} \qquad \text{...(3)}$$

Adding (1), (2) and (3), we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Ex. 13 (b). *If $u = f(r, s, t)$ and $r = x/y$, $s = y/z$, $t = z/x$, prove that* $x\,(\partial u/\partial x) + y\,(\partial u/\partial y) + z\,(\partial u/\partial z) = 0$. **(Rohilkhand 1985; I.C.S. 96)**

Sol. Here u is a function of r, s and t where r, s and t are functions of x, y and z.

We have $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s} \cdot \dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t} \cdot \dfrac{\partial t}{\partial x}$

$$= \frac{\partial u}{\partial r} \cdot \frac{1}{y} + \frac{\partial u}{\partial s} \cdot 0 + \frac{\partial u}{\partial t} \cdot \left(-\frac{z}{x^2}\right).$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}. \qquad \text{...(1)}$$

Also $\quad \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s} \cdot \dfrac{\partial s}{\partial y} + \dfrac{\partial u}{\partial t} \cdot \dfrac{\partial t}{\partial y}$

$$= \frac{\partial u}{\partial r} \cdot \left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s} \cdot \frac{1}{z} + \frac{\partial u}{\partial t} \cdot 0.$$

$$\therefore \quad y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}. \qquad \text{...(2)}$$

Again $\dfrac{\partial u}{\partial z} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial z} + \dfrac{\partial u}{\partial s} \cdot \dfrac{\partial s}{\partial z} + \dfrac{\partial u}{\partial t} \cdot \dfrac{\partial t}{\partial z}$

$$= \frac{\partial u}{\partial r} \cdot 0 + \frac{\partial u}{\partial s} \cdot \left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t} \cdot \left(\frac{1}{x}\right).$$

$$\therefore \quad z\frac{\partial u}{\partial z} = -\frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}. \qquad \text{...(3)}$$

Adding (1), (2) and (3), we get

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) + z\,(\partial u/\partial z) = 0.$$

Ex. 14. *Given that $f(x, y)$ has continuous partial derivatives of the first two orders and $x + y = (u + v)^3$, $x - y = (u - v)^3$, then show that*

$$9(x^2 - y^2)\left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) = (u^2 - v^2)\left(\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2}\right).$$

Sol. Here $\quad x + y = (u + v)^3$, $x - y = (u - v)^3$.

Solving, we get $x = u^3 + 3uv^2$, $y = 3u^2v + v^3$. ...(1)

Now $\quad \dfrac{\partial f}{\partial u} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

$$= \frac{\partial f}{\partial x}(3u^2 + 3v^2) + \frac{\partial f}{\partial y}(6uv). \qquad \text{from (1)}$$

Thus $\quad \partial f/\partial u = 3(u^2 + v^2)(\partial f/\partial x) + 6uv\,(\partial f/\partial y)$. ...(2)

Similarly, show that

$$\partial f/\partial v = 6uv\,(\partial f/\partial x) + 3(u^2 + v^2)(\partial f/\partial y). \qquad \text{...(3)}$$

Adding (2) and (3), we get

$$\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}=3\,(u^2+v^2+2uv)\,\frac{\partial f}{\partial x}+3\,(u^2+v^2+2uv)\,\frac{\partial f}{\partial y}$$

$$=3\,(u+v)^2\,(\partial f/\partial x)+3\,(u+v)^2\,(\partial f/\partial y).$$

$$\therefore\quad (u+v)\left(\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}\right)=3\,(u+v)^3\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)$$

$$=3\,(x+y)\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right).$$

$$\therefore\quad (u+v)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\equiv 3\,(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right). \qquad ...(4)$$

Similarly subtracting (3) from (2) and then multiplying both sides by $u-v$, we get

$$(u-v)\left(\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\right)=3\,(x-y)\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right). \qquad ...(5)$$

Now from (4) and (5), we get

$$(u+v)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left[(u-v)\left(\frac{\partial f}{\partial u}-\frac{\partial f}{\partial v}\right)\right]$$

$$=3\,(x+y)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left[3\,(x-y)\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)\right]$$

$$=9\,(x+y)\left[\left\{(x-y)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial x\,\partial y}\right)+\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)\right\}\right.$$

$$\left.+\left\{(x-y)\left(\frac{\partial^2 f}{\partial x\,\partial y}-\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)\right\}\right]$$

$$=9\,(x+y)\,(x-y)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right)=9\,(x^2-y^2)\left(\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}\right).$$

§ 6. Transformation from cartesian to polar co-ordinates and vice-versa.

Case I. To transform the Laplace equation $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$ into polar co-ordinates. **(Very Important)**

(Allahabad 1980; Meerut 79, 80, 82, 86; Gorakhpur 77; Kanpur 86)

Here V is a function of x and y, where

$$x=r\cos\theta \quad\text{and}\quad y=r\sin\theta.$$

Obviously V is a function of r and θ also, and we have

$$r=\sqrt{(x^2+y^2)} \quad\text{and}\quad \theta=\tan^{-1}(y/x).$$

We have $\dfrac{\partial r}{\partial x}=\dfrac{1}{2\sqrt{(x^2+y^2)}}\cdot 2x=\dfrac{r\cos\theta}{r}=\cos\theta,$

and $$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y^2/x^2)}\left[-\frac{y}{x^2}\right]=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}.$$

Also $\dfrac{\partial r}{\partial y} = \dfrac{1}{2\sqrt{(x^2+y^2)}} \cdot 2y = \dfrac{r\sin\theta}{r} = \sin\theta,$

and $\dfrac{\partial \theta}{\partial y} = \dfrac{1}{1+(y^2/x^2)} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta}{r}.$

Now $\dfrac{\partial V}{\partial x} = \dfrac{\partial V}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial V}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x} = \cos\theta \cdot \dfrac{\partial V}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial V}{\partial \theta}$

$$= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right] V.$$

$\therefore \quad \dfrac{\partial}{\partial x} \equiv \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta}.$

Again $\dfrac{\partial V}{\partial y} = \dfrac{\partial V}{\partial r} \cdot \dfrac{\partial r}{\partial y} + \dfrac{\partial V}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial y} = \sin\theta \cdot \dfrac{\partial V}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial V}{\partial \theta}.$

$\therefore \quad \dfrac{\partial}{\partial y} \equiv \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta}.$

Now $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial V}{\partial x}\right]$

$$= \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right]\left[\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right]$$

$$= \cos\theta \frac{\partial}{\partial r}\left[\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial V}{\partial \theta}\right]$$

$$= \cos\theta\left[\cos\theta \frac{\partial^2 V}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial r\,\partial \theta}\right]$$

$$- \frac{\sin\theta}{r}\left[\cos\theta \frac{\partial^2 V}{\partial \theta\,\partial r} - \sin\theta \frac{\partial V}{\partial r} - \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 V}{\partial \theta^2}\right]$$

$$= \cos^2\theta \frac{\partial^2 V}{\partial r^2} - 2\frac{\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 V}{\partial r\,\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\sin^2\theta}{r}\cdot\frac{\partial V}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial V}{\partial \theta}.$$

Again

$$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \sin\theta \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}\right)$$

$$= \sin\theta \left(\sin\theta \frac{\partial^2 V}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta}\right)$$

$$+ \frac{\cos\theta}{r}\left(\sin\theta \frac{\partial^2 V}{\partial\theta\,\partial r} + \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 V}{\partial\theta^2}\right)$$

$$= \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial V}{\partial\theta}.$$

Therefore adding, we get

$$\frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{V}}{\partial \mathbf{y}^2} = \frac{\partial^2 \mathbf{V}}{\partial \mathbf{r}^2} + \frac{1}{\mathbf{r}}\frac{\partial \mathbf{V}}{\partial \mathbf{r}} + \frac{1}{\mathbf{r}^2}\frac{\partial^2 \mathbf{V}}{\partial\theta^2}.$$

Hence the transformed equation is $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\theta^2} = 0.$

Case II. To transform the equation $\dfrac{\partial^2 \mathbf{V}}{\partial \mathbf{r}^2} + \dfrac{1}{\mathbf{r}}\dfrac{\partial \mathbf{V}}{\partial \mathbf{r}} + \dfrac{1}{\mathbf{r}^2}\dfrac{\partial^2 \mathbf{V}}{\partial\theta^2} = 0$ into Cartesian Coordinates.

Here V is a function of r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. Obviously V is a function of x and y also.

We have $\dfrac{\partial x}{\partial r} = \cos\theta = \dfrac{x}{r}$, and $\dfrac{\partial y}{\partial r} = \sin\theta = \dfrac{y}{r}\cdot$

Also $\dfrac{\partial x}{\partial\theta} = -r\sin\theta = -y$ and $\dfrac{\partial y}{\partial\theta} = r\cos\theta = x.$

Now $\dfrac{\partial V}{\partial r} = \dfrac{\partial V}{\partial x}\cdot\dfrac{\partial x}{\partial r} + \dfrac{\partial V}{\partial y}\cdot\dfrac{\partial y}{\partial r} = \dfrac{x}{r}\dfrac{\partial V}{\partial x} + \dfrac{y}{r}\dfrac{\partial V}{\partial y}\cdot$

$\therefore\quad r\dfrac{\partial V}{\partial r} = \left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\right)V.$

Thus $\quad r\dfrac{\partial}{\partial r} \equiv x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\cdot$

Again $\dfrac{\partial V}{\partial\theta} = \dfrac{\partial V}{\partial x}\cdot\dfrac{\partial x}{\partial\theta} + \dfrac{\partial V}{\partial y}\cdot\dfrac{\partial y}{\partial\theta} = -y\dfrac{\partial V}{\partial x} + x\dfrac{\partial V}{\partial y}$

$$= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)V.$$

Thus $\quad \dfrac{\partial}{\partial\theta} \equiv x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}\cdot$

Therefore we have

$$r\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right)$$

i.e., $r^2\dfrac{\partial^2 V}{\partial r^2} + r\dfrac{\partial V}{\partial r} = x\dfrac{\partial}{\partial x}\left(x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y}\right) + y\dfrac{\partial}{\partial y}\left(x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y}\right)$

$$= x\left(x\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x} + y\frac{\partial^2 V}{\partial x\,\partial y}\right) + y\left(x\frac{\partial^2 V}{\partial x\,\partial y} + y\frac{\partial^2 V}{\partial y^2} + \frac{\partial V}{\partial y}\right)$$

$$= x^2 \frac{\partial^2 V}{\partial x^2} + 2xy \frac{\partial^2 V}{\partial x \, \partial y} + y^2 \frac{\partial^2 V}{\partial y^2} + x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y}. \qquad \text{...(1)}$$

Again $\quad \dfrac{\partial^2 V}{\partial \theta^2} = \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial V}{\partial \theta}\right)$

$$= \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\left(x \frac{\partial V}{\partial y} - y \frac{\partial V}{\partial x}\right)$$

$$= x \frac{\partial}{\partial y}\left(x \frac{\partial V}{\partial y} - y \frac{\partial V}{\partial x}\right) - y \frac{\partial}{\partial x}\left(x \frac{\partial V}{\partial y} - y \frac{\partial V}{\partial x}\right)$$

$$= x\left(x \frac{\partial^2 V}{\partial y^2} - y \frac{\partial^2 V}{\partial y \, \partial x} - \frac{\partial V}{\partial x}\right) - y\left(x \frac{\partial^2 V}{\partial x \, \partial y} + \frac{\partial V}{\partial y} - y \frac{\partial^2 V}{\partial x^2}\right)$$

$$= x^2 \frac{\partial^2 V}{\partial y^2} - 2xy \frac{\partial^2 V}{\partial x \, \partial y} + y^2 \frac{\partial^2 V}{\partial x^2} - x \frac{\partial V}{\partial x} - y \frac{\partial V}{\partial y}. \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$r^2 \frac{\partial^2 V}{\partial r^2} + r \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

$$= r^2 \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right).$$

$$\therefore \quad \frac{\partial^2 \mathbf{V}}{\partial \mathbf{r}^2} + \frac{\mathbf{1}}{\mathbf{r}} \frac{\partial \mathbf{V}}{\partial \mathbf{r}} + \frac{\mathbf{1}}{\mathbf{r}^2} \frac{\partial^2 \mathbf{V}}{\partial \theta^2} = \frac{\partial^2 \mathbf{V}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{V}}{\partial \mathbf{y}^2}.$$

Hence the transformed equation is

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = 0.$$

Remember. $\dfrac{\partial x}{\partial r} \times \dfrac{\partial r}{\partial x} \neq 1.$

Ex. 15. *If V be function of r alone, $r^2 = x^2 + y^2 + z^2$, show that* $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = \dfrac{d^2 V}{dr^2} + \dfrac{2}{r}\dfrac{dV}{dr}.$

Sol. Since V is a function of r alone, we have

$$\partial V/\partial x = (dV/dr) \, . \, (\partial r/\partial x).$$

Also differentiating the equation $r^2 = x^2 + y^2 + z^2$ partially w.r.t. x, we get

$$2r \, (\partial r/\partial x) = 2x \quad \text{or} \quad \partial r/\partial x = x/r.$$

$$\therefore \quad \partial V/\partial x = (dV/dr) \, . \, (x/r).$$

Now $\quad \dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial x}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{dV}{dr} \cdot \dfrac{x}{r}\right]$

$$= \frac{dV}{dr} \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{x}{r} \frac{\partial}{\partial x}\left(\frac{dV}{dr}\right)$$

$$= \frac{dV}{dr}\left[\frac{1}{r} + x\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)\right] + \frac{x}{r}\left(\frac{d^2 V}{dr^2} \cdot \frac{\partial r}{\partial x}\right)$$

$$= \frac{1}{r}\frac{dV}{dr} - \frac{x}{r^2} \cdot \frac{dV}{dr} \cdot \left(\frac{x}{r}\right) + \frac{x}{r} \cdot \frac{d^2 V}{dr^2} \cdot \left(\frac{x}{r}\right), \qquad \left[\because \ \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$=\frac{1}{r}\frac{dV}{dr}-\frac{x^2}{r^3}\frac{dV}{dr}+\frac{x^2}{r^2}\frac{d^2V}{dr^2}.$$

Similarly, by symmetry

$$\frac{\partial^2 V}{\partial y^2}=\frac{1}{r}\frac{dV}{dr}-\frac{y^2}{r^3}\frac{dV}{dr}+\frac{y^2}{r^2}\frac{d^2V}{dr^2},$$

and

$$\frac{\partial^2 V}{\partial z^2}=\frac{1}{r}\frac{dV}{dr}-\frac{z^2}{r^3}\frac{dV}{dr}+\frac{z^2}{r^2}\frac{d^2V}{dr^2}.$$

Adding these, we get

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{3}{r}\frac{dV}{dr}-\frac{1}{r}\frac{dV}{dr}+\frac{d^2V}{dr^2}, \qquad [\because x^2+y^2+z^2=r^2]$$

$$=\frac{2}{r}\frac{dV}{dr}+\frac{d^2V}{dr^2}.$$

Ex. 16. *If $u=f(r)$, where $r^2=x^2+y^2$, then prove that*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

(Meerut 1983, 82 S, 77; Rohilkhand 82; Allahabad 82)

Sol. For complete solution of this problem refer Ex. 15. page 98.

Ex. 17. *If $x=r\cos\theta, y=r\sin\theta, z=f(x,y)$, prove that*

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta$$

(Garhwal 1983; Meerut 79)

and

$$\frac{\partial^2 (r^n\cos n\theta)}{\partial x\,\partial y}=-n(n-1)r^{n-2}\sin(n-2)\theta.$$

Sol. Since $x=r\cos\theta$ and $y=r\sin\theta$, therefore $r^2=x^2+y^2$ and $\theta=\tan^{-1}(y/x)$.

From these, we have

$$2r\frac{\partial r}{\partial x}=2x \quad\text{or}\quad \frac{\partial r}{\partial x}=\frac{x}{r}=\frac{r\cos\theta}{r}=\cos\theta;$$

$$2r\frac{\partial r}{\partial y}=2y \quad\text{or}\quad \frac{\partial r}{\partial y}=\frac{y}{r}=\frac{r\sin\theta}{r}=\sin\theta.$$

Also $$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r},$$

and

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+(y^2/x^2)}\cdot\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}.$$

Now $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\cos\theta\frac{\partial z}{\partial r}-\frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}$$ **(Proved)**

or

$$\frac{\partial z}{\partial x}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)z.$$

$$\therefore\quad \frac{\partial}{\partial x}\equiv\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \qquad ...(1)$$

Similarly $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\sin\theta\frac{\partial z}{\partial r}+\frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}$$

$$= \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) z.$$

$$\therefore \quad \frac{\partial}{\partial y} \equiv \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \qquad ...(2)$$

Now taking $z = r^n \cos n\theta$, we get

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(r^n \cos n\theta) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(r^n \cos n\theta), \quad \text{[from (2)]}$$

$$= \sin\theta \frac{\partial}{\partial r}(r^n \cos n\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(r^n \cos n\theta)$$

$$= \sin\theta \,.\, nr^{n-1}\cos n\theta - \frac{\cos\theta}{r}\cdot r^n \,.\, n \sin n\theta.$$

Thus $$\frac{\partial}{\partial y}(r^n \cos n\theta) = nr^{n-1}(\sin\theta\cos n\theta - \cos\theta \sin n\theta)$$

$$= nr^{n-1}\sin(1-n)\theta.$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}(r^n \cos n\theta)\right\} = \frac{\partial}{\partial x}[nr^{n-1}\sin(1-n)\theta]$$

or $$\frac{\partial^2(r^n \cos n\theta)}{\partial x\,\partial y} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)[nr^{n-1}\sin(1-n)\theta],$$

[from (1)]

$$= \cos\theta\frac{\partial}{\partial r}[nr^{n-1}\sin(1-n)\theta] - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}[nr^{n-1}\sin(1-n)\theta]$$

$$= \cos\theta\,[n(n-1)r^{n-2}\sin(1-n)\theta]$$

$$- \frac{\sin\theta}{r} nr^{n-1}(1-n)\cos(1-n)\theta$$

$$= n(n-1)r^{n-2}[\sin(1-n)\theta\cos\theta + \cos(1-n)\theta\sin\theta]$$

$$= n(n-1)r^{n-2}\sin\{(1-n)\theta+\theta\}$$

$$= -n(n-1)r^{n-2}\sin(n-2)\theta.$$

Ex. 18. *If $x = r\cos\theta$, $y = r\sin\theta$, $z = f(x, y)$, then prove that*

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial\theta}\right)^2.$$

(Gorakhpur 1977; Allahabad 79)

Sol. Proceeding as in Ex. 17, we get

$$\frac{\partial z}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}, \qquad ...(1)$$

and $$\frac{\partial z}{\partial y} = \sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial\theta} \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = (\cos^2\theta + \sin^2\theta)\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}(\sin^2\theta + \cos^2\theta)\left(\frac{\partial z}{\partial\theta}\right)^2$$

$$= (\partial z/\partial r)^2 + (1/r^2)(\partial z/\partial\theta)^2.$$

Miscellaneous Examples on change of variables

•**Ex. 19.** *Transform the equation*

$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$, *by the substitution* $x = \frac{1}{2}\{z + (1/z)\}$.

Sol. We have $dy/dx = (dy/dz) \cdot (dz/dx)$.

But $x = \frac{1}{2}\left\{z + \frac{1}{z}\right\}$ gives $\frac{dx}{dz} = \frac{1}{2}\left\{1 - \frac{1}{z^2}\right\} = \frac{z^2-1}{2z^2}$

i.e., $$\frac{dz}{dx} = \frac{2z^2}{z^2-1}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2z^2}{z^2-1}\frac{dy}{dz}.$$

$$\therefore \quad (1-x^2)\frac{dy}{dx} = \left[1 - \frac{1}{4}\left\{z+\frac{1}{z}\right\}^2\right]\frac{2z^2}{z^2-1}\frac{dy}{dz}$$

$$= \left[\frac{4z^2-(z^2+1)^2}{4z^2}\right]\frac{2z^2}{z^2-1}\frac{dy}{dz}$$

$$= \frac{4z^2 - z^4 - 2z^2 - 1}{2(z^2-1)}\frac{dy}{dz} = \frac{-(z^4-2z^2+1)}{2(z^2-1)}\frac{dy}{dz}$$

$$= -\frac{(z^2-1)^2}{2(z^2-1)}\frac{dy}{dz} = -\frac{1}{2}(z^2-1)\frac{dy}{dz}.$$

$$\therefore \quad \frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] = \frac{d}{dx}\left\{-\frac{1}{2}(z^2-1)\frac{dy}{dz}\right\}$$

$$= -\frac{1}{2}(z^2-1)\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} - \frac{1}{2}\cdot 2z\frac{dz}{dx}\frac{dy}{dz}$$

$$= -\frac{1}{2}(z^2-1)\frac{2z^2}{z^2-1}\frac{d^2y}{dz^2} - z\cdot\frac{2z^2}{z^2-1}\frac{dy}{dz} = -z^2\frac{d^2y}{dz^2} - \frac{2z^3}{z^2-1}\frac{dy}{dz}.$$

Hence the given differential equation becomes

$$-z^2\frac{d^2y}{dz^2} - \frac{2z^3}{(z^2-1)}\frac{dy}{dz} + n(n+1)y = 0$$

or $$z^2(z^2-1)\frac{d^2y}{dz^2} + 2z^3\frac{dy}{dz} - n(n+1)(z^2-1)y = 0.$$

Ex. 20. *If* $x^2 + z^2 = 1$, *show that the equation*

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

becomes $z(z^2-1)\frac{d^2y}{dz^2} + (2z^2-1)\frac{dy}{dz} - n(n+1)zy = 0.$

Sol. We have $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$

But $x^2 + z^2 = 1$ gives $2x + 2z\frac{dz}{dx} = 0$ *i.e.*, $\frac{dz}{dx} = -\frac{x}{z}.$

$$\therefore \quad dy/dx = -(x/z)(dy/dz).$$

$\therefore \quad (1-x^2)\dfrac{dy}{dx} = z^2 \cdot \left(-\dfrac{x}{z}\dfrac{dy}{dz}\right) = -xz\dfrac{dy}{dz}, \qquad [\because 1-x^2 = z^2].$

$\therefore \quad \dfrac{d}{dx}\left\{(1-x^2)\dfrac{dy}{dx}\right\} = \dfrac{d}{dx}\left\{-xz\dfrac{dy}{dz}\right\}$

$= -z\dfrac{dy}{dz} - x\dfrac{dz}{dx}\cdot\dfrac{dy}{dz} - xz\dfrac{d^2y}{dz^2}\cdot\dfrac{dz}{dx}$

$= -z\dfrac{dy}{dz} - x\left(-\dfrac{x}{z}\right)\dfrac{dy}{dz} - xz\left(-\dfrac{x}{z}\right)\dfrac{d^2y}{dz^2}, \qquad \left[\because \dfrac{dz}{dx} = -\dfrac{x}{z}\right]$

$= -z\dfrac{dy}{dz} + \dfrac{x^2}{z}\dfrac{dy}{dz} + x^2\dfrac{d^2y}{dz^2} = \dfrac{x^2-z^2}{z}\dfrac{dy}{dz} + (1-z^2)\dfrac{d^2y}{dz^2}$

$= \dfrac{1-2z^2}{z}\dfrac{dy}{dz} + (1-z^2)\dfrac{d^2y}{dz^2}, \qquad [\because x^2 = 1-z^2]$

Hence the given differential equation becomes

$$\frac{1-2z^2}{z}\frac{dy}{dz} + (1-z^2)\frac{d^2y}{dz^2} + n(n+1)y = 0$$

or $$z(z^2-1)\frac{d^2y}{dz^2} + (2z^2-1)\cdot\frac{dy}{dz} - n(n+1)zy = 0.$$

Ex. 21. *Show that the form of the equation*

$$x^2(d^2y/dx^2) + x(dy/dx) + y = 0$$

remains unchanged by the substitution $x = 1/z$. **(Meerut 1989 P)**

Sol. We have $\dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx}$.

But $\qquad x = \dfrac{1}{z} \Rightarrow \dfrac{dx}{dz} = -\dfrac{1}{z^2} \Rightarrow \dfrac{dz}{dx} = -z^2.$

$\therefore \quad dy/dx = -z^2(dy/dz).$

$\therefore \quad x\dfrac{dy}{dx} = -xz^2\dfrac{dy}{dz} = -\dfrac{1}{z}\cdot z^2\dfrac{dy}{dz} = -z\dfrac{dy}{dz}.$

$\therefore \quad x(d/dx) \equiv -z(d/dz).$

$\therefore \quad x\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = -z\dfrac{d}{dz}\left(-z\dfrac{dy}{dz}\right)$

or $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz}.$$

Hence the differential equation $x^2(d^2y/dx^2) + x(dy/dx) + y = 0$ becomes $z^2(d^2y/dz^2) + z(dy/dz) + y = 0$. We observe that the form of the equation remains unchanged.

Ex. 22. *Transform d^2y/dx^2 to the new variables u and v, taking u as independent variable, given $x = 1/v$, $y = uv$.*

Sol. We have $x = \dfrac{1}{v} \Rightarrow \dfrac{dx}{du} = -\dfrac{1}{v^2}\dfrac{dv}{du}$;

also $\quad y = uv \Rightarrow \dfrac{dy}{du} = v + u\dfrac{dv}{du}.$

Now $\dfrac{dy}{dx}=\dfrac{dy/du}{dx/du}=\dfrac{v+u\,(dv/du)}{(-1/v^2)\,(dv/du)}=-\dfrac{v^3+uv^2\,(dv/du)}{dv/du}$.

Again $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=\dfrac{d}{du}\left(\dfrac{dy}{dx}\right)\cdot\dfrac{du}{dx}$

$$=\frac{d}{du}\left[-\frac{v^3+uv^2\,(dv/du)}{dv/du}\right]\cdot\left(-v^2\frac{du}{dv}\right)$$

$$=\left(v^2\frac{du}{dv}\right)\frac{(dv/du)\,\{4v^2\,(dv/du)+2uv\,(dv/du)^2+uv^2\,(d^2v/du^2)\}-(d^2v/du^2)\,\{v^2+uv^2\,(dv/du)\}}{(dv/du)^2}$$

$$=v^2\left(\frac{dv}{du}\right)^{-3}\left[4v^2\left(\frac{dv}{du}\right)^2+2uv\left(\frac{dv}{du}\right)^3-v^3\frac{d^2v}{du^2}\right]$$

$$=4v^4\left(\frac{dv}{du}\right)^{-1}+2uv^3-v^5\left(\frac{dv}{du}\right)^{-3}\frac{d^2v}{du^2}.$$

Ex. 23. *If $z=f(x,y)$, $x^2=uv$ and $y^2=u/v$, change the independent variables to u, v in the equation*

$$x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+2y\frac{\partial z}{\partial y}=0.$$

Sol. We have $\left(x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}\right)\left(x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}\right)$

$$=x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}-xy\frac{\partial^2 z}{\partial x\,\partial y}-xy\frac{\partial^2 z}{\partial x\,\partial y}+y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$$

$$=x^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\cdot$$

$\therefore$ the given differential equation may be written as

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)\left(x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\right)-\left(x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\right)=0. \qquad ...(1)$$

Now $x^2=uv$, $y^2=u/v \Rightarrow u=xy$ and $v=x/y$.

$\therefore$ $\dfrac{\partial u}{\partial x}=y,\ \dfrac{\partial u}{\partial y}=x,\ \dfrac{\partial v}{\partial x}=1/y,\ \dfrac{\partial v}{\partial y}=-x/y^2.$

We have $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=y\dfrac{\partial z}{\partial u}+\dfrac{1}{y}\dfrac{\partial z}{\partial v}\cdot$

Also $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=x\dfrac{\partial z}{\partial u}-\dfrac{x}{y^2}\dfrac{\partial z}{\partial v}\cdot$

$\therefore$ $x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}=xy\dfrac{\partial z}{\partial u}+\dfrac{x}{y}\dfrac{\partial z}{\partial v}-xy\dfrac{\partial z}{\partial u}+\dfrac{x}{y}\dfrac{\partial z}{\partial v}$

$$=2\frac{x}{y}\frac{\partial z}{\partial v}=2v\frac{\partial z}{\partial v}\cdot$$

$\therefore$ $x\dfrac{\partial}{\partial x}-y\dfrac{\partial}{\partial y}\equiv 2v\dfrac{\partial}{\partial v}\cdot$

Hence the differential equation (1) becomes

$$2v\frac{\partial}{\partial v}\left(2v\frac{\partial z}{\partial v}\right)-2v\frac{\partial z}{\partial v}=0$$

or $\quad 4v^2\dfrac{\partial^2 z}{\partial v^2}+4v\dfrac{\partial z}{\partial v}-2v\dfrac{\partial z}{\partial v}=0 \quad$ or $\quad 4v^2\dfrac{\partial^2 z}{\partial v^2}+2v\dfrac{\partial z}{\partial v}=0$

or $\quad 2v\dfrac{\partial^2 z}{\partial v^2}+\dfrac{\partial z}{\partial v}=0.$

Ex. 24. *If $x=e^\theta$, $y=e^\phi$, prove that*

$$e^{2\theta}\frac{\partial^2 V}{\partial x^2}+e^{2\phi}\frac{\partial^2 V}{\partial y^2}+e^\theta\frac{\partial V}{\partial x}+e^\phi\frac{\partial V}{\partial y}=\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial \phi^2}.$$

(Meerut 1984)

Sol. We have

$$\frac{\partial V}{\partial \theta}=\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \theta}=e^\theta\frac{\partial V}{\partial x}+0\cdot\frac{\partial V}{\partial y}=x\frac{\partial V}{\partial x}.$$

$$\therefore \quad \frac{\partial}{\partial \theta}\equiv x\frac{\partial}{\partial x}.$$

$$\therefore \quad \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \theta}\right)=x\frac{\partial}{\partial x}\left(x\frac{\partial V}{\partial x}\right)$$

or
$$\frac{\partial^2 V}{\partial \theta^2}=x^2\frac{\partial^2 V}{\partial x^2}+x\frac{\partial V}{\partial x}. \qquad \text{...(1)}$$

Again $\dfrac{\partial V}{\partial \phi}=\dfrac{\partial V}{\partial x}\cdot\dfrac{\partial x}{\partial \phi}+\dfrac{\partial V}{\partial y}\cdot\dfrac{\partial y}{\partial \phi}=0\cdot\dfrac{\partial V}{\partial x}+e^\phi\dfrac{\partial V}{\partial y}=y\dfrac{\partial V}{\partial y}.$

$$\therefore \quad \frac{\partial}{\partial \phi}\equiv y\frac{\partial}{\partial y}.$$

$$\therefore \quad \frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\right)=y\frac{\partial}{\partial y}\left(y\frac{\partial V}{\partial y}\right)$$

or
$$\frac{\partial^2 V}{\partial \phi^2}=y^2\frac{\partial^2 V}{\partial y^2}+y\frac{\partial V}{\partial y}. \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial \phi^2}=x^2\frac{\partial^2 V}{\partial x^2}+x\frac{\partial V}{\partial x}+y^2\frac{\partial^2 V}{\partial y^2}+y\frac{\partial V}{\partial y}$$

$$=e^{2\theta}\frac{\partial^2 V}{\partial x^2}+e^{2\phi}\frac{\partial^2 V}{\partial y^2}+e^\theta\frac{\partial V}{\partial x}+e^\phi\frac{\partial V}{\partial y}.$$

Ex. 25. *If $x=r\cos\theta$, $y=r\sin\theta$, prove that*

(*a*) $\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2=1,$ **(Meerut 1983)**

(*b*) $\dfrac{\partial r}{\partial x}=\dfrac{\partial x}{\partial r},\ \dfrac{1}{r}\dfrac{\partial x}{\partial \theta}=r\dfrac{\partial \theta}{\partial x},$ **(Meerut 1987)**

(*c*) $(\partial^2\theta/\partial x^2)+(\partial^2\theta/\partial y^2)=0,$ **(Meerut 1983, 87)**

(*d*) $(\partial^2 r/\partial x^2)\cdot(\partial^2 r/\partial y^2)=(\partial^2 r/\partial x\partial y)^2,$

(*e*) $\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}=\dfrac{1}{r}\left\{\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial r}{\partial y}\right)^2\right\}.$ **(Meerut 1981S; Rohilkhand 79)**

Sol. We have $x = r\cos\theta$, and $y = r\sin\theta$. ...(1)

$\therefore$ $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$. ...(2)

(a) From (2), we have

$$2r\frac{\partial r}{\partial x} = 2x; \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

Also $$2r\frac{\partial r}{\partial y} = 2y; \quad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta.$$

$\therefore$ $(\partial r/\partial x)^2 + (\partial r/\partial y)^2 = \cos^2\theta + \sin^2\theta = 1.$

(b) From (1), we have $\partial x/\partial r = \cos\theta$. But $\partial r/\partial x = \cos\theta$.

Therefore $\partial x/\partial r = \partial r/\partial x$.

From (1), we have $\partial x/\partial\theta = -r\sin\theta$;

$\therefore$ $(1/r)(\partial x/\partial\theta) = -\sin\theta$.

From (2), we have $$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-r\sin\theta}{r^2}$$
$$= -(1/r)\sin\theta;$$

$\therefore$ $r(\partial\theta/\partial x) = -\sin\theta$.

Hence $(1/r)(\partial x/\partial\theta) = r(\partial\theta/\partial x)$.

Note. For further solution of the problem refer Ex. 13 and Ex. 14 on pages 96 and 97.

Ex. 26. *Show that the equation*

$$xy(\partial^2u/\partial x^2 - \partial^2u/\partial y^2) - (x^2-y^2)\,\partial^2u/\partial x\partial y = 0$$

becomes $r\,\partial^2u/\partial r\,\partial\theta - \partial u/\partial\theta = 0$ *when transformed to polars.*

Sol. Proceeding as in § 6, case II page 319, we have

$$r\frac{\partial u}{\partial r} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial\theta} = x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}.$$

$$\therefore \quad r\frac{\partial^2u}{\partial r\,\partial\theta} = r\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial\theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}\right)$$
$$= x^2\frac{\partial^2u}{\partial x\,\partial y} + x\frac{\partial u}{\partial y} - xy\frac{\partial^2u}{\partial x^2} + xy\frac{\partial^2u}{\partial y^2} - y\frac{\partial u}{\partial x} - y^2\frac{\partial^2u}{\partial x\,\partial y}$$
$$= (x^2-y^2)\frac{\partial^2u}{\partial x\,\partial y} - xy\left(\frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2}\right) + x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x}$$
$$= (x^2-y^2)\frac{\partial^2u}{\partial x\,\partial y} - xy\left(\frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2}\right) + \frac{\partial u}{\partial\theta}.$$

$$\therefore \quad r\frac{\partial^2u}{\partial r\,\partial\theta} - \frac{\partial u}{\partial\theta} = -\left[xy\left(\frac{\partial^2u}{\partial x^2} - \frac{\partial^2u}{\partial y^2}\right) - (x^2-y^2)\frac{\partial^2u}{\partial x\,\partial y}\right].$$

Hence the differential equation $r\dfrac{\partial^2u}{\partial r\,\partial\theta} - \dfrac{\partial u}{\partial\theta} = 0$ is equivalent to

$$xy(\partial^2u/\partial x^2 - \partial^2u/\partial y^2) - (x^2-y^2)(\partial^2u/\partial x\,\partial y) = 0.$$

Ex. 27. *If* $x = r\cos\theta$, $y = r\sin\theta$ *and* $r = e^z$, *prove that*

$$x^2\frac{\partial^2u}{\partial x^2} + 2xy\frac{\partial^2u}{\partial x\,\partial y} + y^2\frac{\partial^2u}{\partial y^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} - 1\right)u = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - 1\right)u,$$

and $x^2 \dfrac{\partial^2 u}{\partial y^2} - 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial \theta^2} + r \dfrac{\partial u}{\partial r} = \dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial u}{\partial z}$.

Sol. Since $r = e^z$, therefore $z = \log r$.

Now $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial z} \cdot \dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial z} \cdot \dfrac{1}{r}$.

$\therefore \quad r \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial z}$ *i.e.,* $r \dfrac{\partial}{\partial r} \equiv \dfrac{\partial}{\partial z}$. ...(1)

Now proceeding as in § 6, Case II, page 319 we have

$$r \frac{\partial u}{\partial r} = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x}.$$

$$\therefore \quad r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} - 1\right) u = \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - u\right)$$

$$= x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = r \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} - 1\right) u = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - 1\right) u,$$

from (1).

Also
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)\left(x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x}\right)$$

$$= x^2 \frac{\partial^2 u}{\partial y^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial x^2} - x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y}$$

$$= x^2 \frac{\partial^2 u}{\partial y^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial x^2} - r \frac{\partial u}{\partial r}.$$

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial y^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \theta^2} + r \frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial u}{\partial z}.$$

Ex. 28. *Transform* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ *into polars and show that* $u = (Ar^n + Br^{-n}) \sin n\theta$ *satisfies the above equation.*

Sol. For the first part see § 6, Case I, page 317.

Second Part. The equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ is transformed into

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

Now if we take $u = (Ar^n + Br^{-n}) \sin n\theta$, then

$\partial u/\partial r = n\,(Ar^{n-1} - Br^{-n-1}) \sin n\theta$,

$\partial u/\partial \theta = (Ar^n + Br^{-n}) \,.\, n \cos n\theta$,

$\partial^2 u/\partial r^2 = n\,[A(n-1)\, r^{n-2} + B\,(n+1)\, r^{-n-2}] \sin n\theta$

and
$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = -\,(Ar^n + Br^{-n})\, n^2 \sin n\theta.$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r}$$

$$= n\,[A\,(n-1)\,r^{n-2} + B\,(n+1)\,r^{-n-2}]\sin n\theta$$
$$- \frac{1}{r^2}(Ar^n + Br^{-n})\,n^2 \sin n\theta + \frac{1}{r}\,n\,(Ar^{n-1} - Br^{-n-1})\sin n\theta$$
$$= [A\,\{n^2 - n - n^2 + n\}\,r^{n-2} + B\,\{n^2 + n - n^2 - n\}\,r^{-n-2}]\sin n\theta = 0.$$

Hence $u = (Ar^n + Br^{-n})\sin n\theta$ satisfies the given equation.

Ex. 29. *If* $x + y = 2e^{\theta}\cos\phi$ *and* $x - y = 2\sqrt{(-1)}\,e^{\theta}\sin\phi$, *show that*

$$\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial\phi^2} = 4xy\,\frac{\partial^2 V}{\partial x\,\partial y}.$$

(Kanpur 1987; U.P. P.C.S. 97)

Sol. We have $x = e^{\theta}(\cos\phi + i\sin\phi) = e^{\theta}\,.\,e^{i\phi} = e^{\theta + i\phi}$

and $y = e^{\theta}(\cos\phi - i\sin\phi) = e^{\theta}\,.\,e^{-i\phi} = e^{\theta - i\phi}.$

Now $$\frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial\theta} = \frac{\partial V}{\partial x}e^{\theta + i\phi} + \frac{\partial V}{\partial y}e^{\theta - i\phi}$$
$$= x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}$$

and $$\frac{\partial V}{\partial\phi} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial\phi} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial\phi} = \frac{\partial V}{\partial x}\cdot i\,.\,e^{\theta + i\phi} - \frac{\partial V}{\partial y}\,i\,.\,e^{\theta - i\phi}$$
$$= i\left(x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y}\right).$$

$$\therefore\quad \frac{\partial V}{\partial\theta} + i\frac{\partial V}{\partial\phi} = 2y\frac{\partial V}{\partial y}\quad\text{and}\quad \frac{\partial V}{\partial\theta} - i\frac{\partial V}{\partial\phi} = 2x\frac{\partial V}{\partial x}.$$

Hence $$\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial\phi^2} = \left(\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi}\right)\left(\frac{\partial V}{\partial\theta} - i\frac{\partial V}{\partial\phi}\right)$$
$$= \left(2y\frac{\partial}{\partial y}\right)\left(2x\frac{\partial V}{\partial x}\right) = 4xy\frac{\partial^2 V}{\partial x\,\partial y}.$$

Ex. 30. *If* $x = e^{v}\sec u$, $y = e^{v}\tan u$ *and* ϕ *is a function of* x *and* y *show that*

$$\cos u\left(\frac{\partial^2\phi}{\partial u\,\partial v} - \frac{\partial\phi}{\partial u}\right) = xy\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2\phi}{\partial x\,\partial y}.$$

Sol. We have

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial\phi}{\partial x}\cdot e^{v}\sec u\tan u + \frac{\partial\phi}{\partial y}\,.\,e^{v}\sec^2 u.$$

$$\therefore\quad \cos u\frac{\partial\phi}{\partial u} = e^{v}\tan u\frac{\partial\phi}{\partial x} + e^{v}\sec u\frac{\partial\phi}{\partial y} = y\frac{\partial\phi}{\partial x} + x\frac{\partial\phi}{\partial y}. \qquad ...(1)$$

Hence $$\cos u\frac{\partial}{\partial u} \equiv y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Similarly show that

$$\frac{\partial\phi}{\partial v} = x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}.$$

Now $$\cos u\frac{\partial^2\phi}{\partial u\,\partial v} = \cos u\frac{\partial}{\partial u}\left(\frac{\partial\phi}{\partial v}\right) = \left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right)$$

$$= xy\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right\}+(x^2+y^2)\frac{\partial^2\phi}{\partial x\,\partial y}+y\frac{\partial\phi}{\partial x}+x\frac{\partial\phi}{\partial y}\cdot$$

$$\therefore \quad xy\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right\}+(x^2+y^2)\frac{\partial^2\phi}{\partial x\,\partial y}=\cos u\,\frac{\partial^2\phi}{\partial u\,\partial v}-\cos u\,\frac{\partial\phi}{\partial u},$$

[from (1)]

$$=\cos u\left\{\frac{\partial^2\phi}{\partial u\,\partial v}-\frac{\partial\phi}{\partial u}\right\}.$$

Ex. 31. *Transform the equation*

$$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z$$

by putting $x = u\cos v,\ y = u\sin v$.

Sol. The given equation may be written as

$$\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\left(y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}\right)+z=0.$$

Now $$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$=\frac{\partial z}{\partial x}\cdot(-u\sin v)+\frac{\partial z}{\partial y}\cdot u\cos v$$

$$= x\,(\partial z/\partial y)-y\,(\partial z/\partial x).$$

The operator $\dfrac{\partial}{\partial v}$ is therefore equivalent to $x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}\cdot$

Hence the given equation is transformed into

$$\partial/\partial v\,(\partial z/\partial v)+z=0,\quad \text{or}\quad (\partial^2 z/\partial v^2)+z=0.$$

Ex. 32. *Transform the equation*

$$\frac{\partial^2 z}{\partial x^2}+2xy^2\frac{\partial z}{\partial x}+2(y-y^3)\frac{\partial z}{\partial y}+x^2y^2z=0$$

by the substitution $x = uv,\ y = 1/v$.

Sol. We have $u = xy$ and $v = 1/y$.

Now $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial z}{\partial u}\cdot y+\frac{\partial z}{\partial v}\cdot 0=y\frac{\partial z}{\partial u}\cdot$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=y\frac{\partial}{\partial u}\left(y\frac{\partial z}{\partial u}\right)=y\left[\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial u}+y\frac{\partial^2 z}{\partial u^2}\right]$$

$$=y^2\,(\partial^2 z/\partial u^2). \qquad [\because \partial y/\partial u=0]$$

Also $$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}-\frac{1}{y^2}\frac{\partial z}{\partial v}\cdot$$

Therefore the given equation is transformed into

$$y^2\frac{\partial^2 z}{\partial u^2}+2xy^2\cdot y\frac{\partial z}{\partial u}+2(y-y^3)\left(x\frac{\partial z}{\partial u}-\frac{1}{y^2}\frac{\partial z}{\partial v}\right)+x^2y^2z=0$$

or $\quad \dfrac{1}{v^2}\dfrac{\partial^2 z}{\partial u^2} + 2uv \cdot \dfrac{1}{v^3} \cdot \dfrac{\partial z}{\partial u} + 2\left(\dfrac{1}{v} - \dfrac{1}{v^3}\right)\left(uv\dfrac{\partial z}{\partial u} - v^2\dfrac{\partial z}{\partial v}\right) + u^2 z = 0$

or $\quad \dfrac{\partial^2 z}{\partial u^2} + 2uv^2\dfrac{\partial z}{\partial u} + 2(v - v^3)\dfrac{\partial z}{\partial v} + u^2v^2 z = 0.$

Ex. 33. *Change the variables x and y in the equation*

$$y^2\frac{\partial^2 z}{\partial x^2} - x^2\frac{\partial^2 z}{\partial y^2} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

to u and v where $u = x^2 + y^2$ *and* $v = 2xy$ *and show that the new equation is*

$$\left(u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\right)\frac{\partial z}{\partial v} = 0.$$

Sol. The given equation can be written as

$$\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\left(y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right) = 0.$$

Now proceed as in Ex. 23 on page 325.

Ex. 34. *If* $z = f(u, v)$, $u = e^x \cos y$, $v = e^x \sin y$, *show that*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$ **(Meerut 1981S)**

Sol. We have $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}$

$$= \frac{\partial z}{\partial u}e^x \cos y + \frac{\partial z}{\partial v}e^x \sin y = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}.$$

$\therefore \quad \partial/\partial x \equiv u(\partial/\partial u) + v(\partial/\partial v).$

Now proceed as in § 6, case I on page 317.

Ex. 35. *Transform the equation*

$$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0$$

into polar coordinates.

Sol. Here V is a function of x, y and z, where

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

Let $r\sin\theta = u$. Then $x = u\cos\phi$ and $y = u\sin\phi$.

As proved in § 6, case I, page 317, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2}. \quad ...(1)$$

Adding $\partial^2 V/\partial z^2$ to both sides of (1), we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial u^2} + \frac{1}{u}\frac{\partial V}{\partial u} + \frac{1}{u^2}\frac{\partial^2 V}{\partial \phi^2} \quad ...(2)$$

Now $\quad z = r\cos\theta \quad$ and $\quad u = r\sin\theta. \quad ...(3)$

$\therefore \quad$ by § 6, case I, we have

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial u^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}. \quad ...(4)$$

Also $$\frac{\partial V}{\partial u}=\frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial u}+\frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial u}\cdot$$

From (3), we have $r^2=z^2+u^2$ and $\theta=\tan^{-1}(u/z)$.

$$\therefore\quad 2r\frac{\partial r}{\partial u}=2u,\quad\text{or}\quad \frac{\partial r}{\partial u}=\frac{u}{r}=\frac{r\sin\theta}{r}=\sin\theta.$$

Also $$\frac{\partial \theta}{\partial u}=\frac{1}{1+(u^2/z^2)}\cdot\frac{1}{z}=\frac{z}{z^2+u^2}=\frac{r\cos\theta}{r^2}=\frac{1}{r}\cos\theta.$$

$$\therefore\quad \frac{\partial V}{\partial u}=\sin\theta\frac{\partial V}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial V}{\partial \theta}\cdot$$

$$\therefore\quad \frac{1}{u}\frac{\partial V}{\partial u}=\frac{1}{u}\sin\theta\frac{\partial V}{\partial r}+\frac{1}{u}\cdot\frac{1}{r}\cos\theta\frac{\partial V}{\partial \theta}$$

$$=\frac{1}{r\sin\theta}\sin\theta\frac{\partial V}{\partial r}+\frac{1}{r\sin\theta}\frac{1}{r}\cos\theta\frac{\partial V}{\partial \theta}\qquad [\because u=r\sin\theta]$$

$$=\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}\cdot\qquad ...(5)$$

With the help of (4) and (5), (2) gives

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$
$$+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\cot\theta\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$
$$[\because u=r\sin\theta]$$
$$=\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}\cdot$$

□

Some Miscellaneous Solved Problems

Ex. 1. *By use of Taylor's theorem expand the following polynomial in powers of* $(x+1)$:

$$f(x) = x^5 + 2x^4 - x^2 + x + 1. \qquad \text{(U.P. P.C.S. 1992)}$$

Sol. We have $\quad f(x) = x^5 + 2x^4 - x^2 + x + 1.$

We can write $\quad f(x) = f[-1 + (x+1)].$

Now expanding $f[-1 + (x+1)]$ by Taylor's theorem in powers of $(x+1)$, we get

$$f(x) = f[-1 + (x+1)] = f(-1) + \frac{(x+1)}{1!} f'(-1) + \frac{(x+1)^2}{2!} f''(-1) + \frac{(x+1)^3}{3!} f'''(-1) + \dots \qquad \dots(1)$$

Now $f(x) = x^5 + 2x^4 - x^2 + x + 1$ so that

$$f(-1) = (-1)^5 + 2(-1)^4 - (-1)^2 + (-1) + 1$$
$$= -1 + 2 - 1 - 1 + 1 = 0,$$

$f'(x) = 5x^4 + 8x^3 - 2x + 1$ so that $f'(-1) = 5 - 8 + 2 + 1 = 0,$

$f''(x) = 20x^3 + 24x^2 - 2$ so that $f''(-1) = -20 + 24 - 2 = 2,$

$f'''(x) = 60x^2 + 48x$ so that $f'''(-1) = 60 - 48 = 12,$

$f^{iv}(x) = 120x + 48$ so that $f^{iv}(-1) = -120 + 48 = -72,$

$f^{v}(x) = 120$ so that $f^{v}(-1) = 120.$

Obviously $f^{(n)}(-1) = 0$ when $n \geq 6$.

Now substituting these values in (1), we get

$$f(x) = 0 + \frac{(x+1)}{1!} \cdot 0 + \frac{(x+1)^2}{2!} \cdot 2 + \frac{(x+1)^3}{3!} \cdot 12 + \frac{(x+1)^4}{4!} \cdot (-72) + \frac{(x+1)^5}{5!} \cdot 120$$

$$= (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5.$$

Ex. 2. *Transform the differential equation*

$$\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2 \cos^5 x$$

into one having z as independent variable where $z = \sin x$. **(I.C.S. 1994)**

Sol. The given differential equation is

$$\frac{d^2y}{dx^2} \cos x + \frac{dy}{dx} \sin x - 2y \cos^3 x = 2 \cos^5 x$$

or $$\frac{d^2y}{dx^2} \sec^2 x + \frac{dy}{dx} \sec^2 x \tan x - 2y = 2(1 - \sin^2 x), \qquad \dots(1)$$

dividing both sides by $\cos^3 x$.

Now $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{dy}{dz} \cdot \cos x. \qquad \left[\because z = \sin x \Rightarrow \frac{dz}{dx} = \cos x \right]$$

$$\therefore \quad \sec x \frac{dy}{dx} = \frac{dy}{dz} \cdot$$

$$\therefore \quad \sec x \frac{d}{dx} \equiv \frac{d}{dz} \cdot$$

$$\therefore \quad \sec x \frac{d}{dx} \left(\sec x \frac{dy}{dx} \right) = \frac{d}{dz} \left(\frac{dy}{dz} \right)$$

or $$\sec^2 x \frac{d^2y}{dx^2} + \sec^2 x \tan x \frac{dy}{dx} = \frac{d^2y}{dz^2} \cdot$$

With the help of the above relations the differential equation (1) transforms to

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2).$$

Ex. 3. *If g is the inverse of f and $f'(x) = \dfrac{1}{1 + x^3}$, prove that*

$$g'(x) = 1 + [g(x)]^3. \qquad \textbf{(I.C.S. 1995)}$$

Sol. It is given that $g = f^{-1}$.

Let $\quad f^{-1}(x) = y.$

Then by def. of inverse function, $x = f(y)$. ...(1)

Also $\quad g = f^{-1} \Rightarrow g(x) = f^{-1}(x)$

$\qquad\qquad\qquad \Rightarrow g(x) = y.$ ...(2)

From (2), $g'(x) = dy/dx$. ...(3)

Differentiating both sides of (1) with respect to x, we get

$$1 = f'(y) \frac{dy}{dx} \cdot$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{1/(1 + y^3)}$$

$$\left[\because f'(x) = \frac{1}{1 + x^3} \Rightarrow f'(y) = \frac{1}{1 + y^3} \right]$$

$$= 1 + y^3 = 1 + [g(x)]^3 \qquad [\because y = g(x), \text{ from } (2)]$$

or $$g'(x) = 1 + [g(x)]^3. \qquad [\because \text{ from } (3), dy/dx = g'(x)]$$

Ex. 4. *Suppose* $f(x) = 17x^{12} - 124x^9 + 16x^3 - 129x^2 + x - 1$. *Determine* $\dfrac{d}{dx}[f^{-1}]$ *at* $x = -1$ *if it exists.* **(I.C.S. 1997)**

Sol. Let $f^{-1}(x) = y$. Then to find dy/dx at $x = -1$.

If $f^{-1}(x) = y$, then by definition of inverse function, we have

$$x = f(y)$$
$$= 17y^{12} - 124y^9 + 16y^3 - 129y^2 + y - 1. \qquad ...(1)$$

$$[\because f(x) = 17x^{12} - 124x^9 + 16x^3 - 129x^2 + x - 1$$

$$\Rightarrow f(y) = 17y^{12} - 124y^9 + 16y^3 - 129y^2 + y - 1]$$

When $x = -1$, we have from (1), $y = 0$.

Now differentiating both sides of (1) with respect to x, we get

$$1 = [17 \,.\, 12y^{11} - 124 \,.\, 9y^8 + 16 \,.\, 3y^2 - 129 \,.\, 2y + 1] \frac{dy}{dx} \cdot \qquad ...(2)$$

Putting $x = -1$ or $y = 0$ on both sides of (2), we get

$$1 = (0 + 1) \left(\frac{dy}{dx}\right)_{\text{at } x = -1} \quad \text{or} \quad \left(\frac{dy}{dx}\right)_{\text{at } x = -1} = 1.$$

Ans. $\frac{d}{dx} [f^{-1}]$ at $x = -1$ is 1.

Ex. 5. *If* $u = \sin^{-1} [(x^2 + y^2)^{1/5}]$, *prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{25} \tan u \, (2 \tan^2 u - 3).$$

(I.C.S. 1997)

Sol. It is given that $u = \sin^{-1} [(x^2 + y^2)^{1/5}]$.

$\therefore \quad \sin u = (x^2 + y^2)^{1/5} = v$, say.

Obviously v is a homogeneous function of x and y of degree $2/5$. Therefore by Euler's theorem, we have

$$x \, (\partial v/\partial x) + y \, (\partial v/\partial y) = \tfrac{2}{5} v. \qquad ...(1)$$

Now $v = \sin u$.

$$\therefore \quad \frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y} \cdot$$

Putting these values in (1), we get

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{2}{5} v = \frac{2}{5} \sin u$$

or

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{5} \tan u. \qquad ...(2)$$

Now differentiating (2) partially w.r.t. x and y respectively, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{5} \sec^2 u \frac{\partial u}{\partial x} \qquad ...(3)$$

and

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{2}{5} \sec^2 u \frac{\partial u}{\partial y} \qquad ...(4)$$

Multiplying (3) by x, (4) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{5} \sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \frac{2}{5} \tan u = \frac{2}{5} \sec^2 u \cdot \frac{2}{5} \tan u$$

[from (2)]

or

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{5} \tan u \left(\frac{2}{5} \sec^2 u - 1\right)$$

$$= \tfrac{2}{25} \tan u \, (2 \sec^2 u - 5)$$

$$= \tfrac{4}{25} \tan u \, [2\,(1 + \tan^2 u) - 5] = \tfrac{4}{25} \tan u \, (2 \tan^2 u - 3).$$

Ex. 6. *If* $V = \log_e \sin \left\{ \dfrac{\pi\,(2x^2 + y^2 + xz)^{1/2}}{2\,(x^2 + xy + 2yz + z^2)^{1/3}} \right\}$,

find $x\,(\partial V/\partial x) + y\,(\partial V/\partial y) + z\,(\partial V/\partial z)$ *when* $x = 0, y = 1, z = 2$.

(U.P. P.C.S. 1991)

Sol. When $x = 0, y = 1, z = 2$, we have

$$V = \log_e \sin(\pi/4) = \log_e (1/\sqrt{2})$$

or $\quad e^V = 1/\sqrt{2}$.

From the given relation, we have

$$\sin^{-1}(e^V) = \frac{\pi\,(2x^2 + y^2 + xz)^{1/2}}{2\,(x^2 + xy + 2yz + z^2)^{1/3}} = u, \text{ say.}$$

Obviously u is a homogeneous function of x, y and z of degree $1 - (2/3)$ *i.e.*, of degree $1/3$.

$\therefore$ by Euler's theorem, we have

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) + z\,(\partial u/\partial z) = \tfrac{1}{3} u. \qquad \text{...(1)}$$

But $\dfrac{\partial u}{\partial x} = \dfrac{1}{\sqrt{(1 - e^{2V})}} \cdot e^V \dfrac{\partial V}{\partial x}, \quad \dfrac{\partial u}{\partial y} = \dfrac{1}{\sqrt{(1 - e^{2V})}} \cdot e^V \dfrac{\partial V}{\partial y},$

$$\frac{\partial u}{\partial z} = \frac{1}{\sqrt{(1 - e^{2V})}} \cdot e^V \frac{\partial V}{\partial z}.$$

Putting these values in (1), we get

$$\frac{e^V}{\sqrt{(1 - e^{2V})}} \left(x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \right) = \frac{1}{3} u = \frac{1}{3} \sin^{-1}(e^V)$$

or

$$x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = \frac{1}{3} \cdot \frac{\sqrt{(1 - e^{2V})} \,.\, \sin^{-1}(e^V)}{e^V}.$$

Putting $x = 0, y = 1, z = 2$ on both sides, we get

$$\left(x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} \right)_{\text{at } x = 0,\, y = 1,\, z = 2}$$

$$= \frac{\tfrac{1}{3} \cdot \sqrt{\{1 - (1/\sqrt{2})^2\}} \,.\, \sin^{-1}(1/\sqrt{2})}{(1/\sqrt{2})}$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{2} \cdot \frac{\pi}{4} = \frac{\pi}{12}.$$

□

1

Maxima and Minima

(Of Functions of a Single Independent Variable)

§ 1. Definitions.

A function $f(x)$ is said to be **maximum** at $x = a$, if there exists a positive number δ such that

$$f(a+h) < f(a)$$

for all values of h, other than zero, in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be **minimum** at $x = a$, if there exists a positive number δ such that

$$f(a+h) > f(a)$$

for all values of h, other than zero, in the interval $(-\delta, \delta)$.

Maximum and minimum values of a function are also called its **extreme values** or **turning values** and the points at which they are attained are called **points of maxima and minima.**

The points at which a function has extreme values are called **turning points.**

§ 2. Properties of maxima and minima.

(i) At least one maximum or one minimum must lie between two equal values of a function.

(ii) Maximum and Minimum values must occur **alternately.**

(iii) There may be several maximum or minimum values of the same function.

(iv) A function $y = f(x)$ is maximum at $x = a$, if dy/dx changes sign from +ive to –ive as x passes through a.

(v) A function $y = f(x)$ is minimum at $x = a$, if dy/dx changes sign from –ive to +ive as x passes through a.

(vi) If the sign of dy/dx does not change while x passes through a, then y is neither maximum nor minimum at $x = a$.

****§ 3. Conditions for maximum or minimum values.**

(Kanpur 1980, 79; Agra 79; Lucknow 76)

Necessary Conditions : *A necessary condition for $f(x)$ to be a maximum or a minimum at $x = a$ is that $f'(a) = 0$.*

Let $f(x)$ be a given function of x and suppose $f(x)$ can be expanded in the neighbourhood of $x = a$ by Taylor's theorem.