# Main's Test Series – 2019

## Test no. 9

## Section-A

**Ques: 1) a)** Determine whether the following matrices have the same column space:-

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 7 \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{pmatrix}.$$

**Solution:-** Given; $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 4 & 3 \\ 1 & 1 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 7 & 12 & 17 \end{bmatrix}$

Observe that A and B have the same column space if and only if the transpose $A^T$ & $B^T$ have the same row space. Thus reduce $A^T$ & $B^T$ to row canonical form.

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 1 \\ 5 & 3 & 7 \end{bmatrix} \sim \underset{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 \to 5R_1}}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -2 & 2 \end{bmatrix}} \sim \underset{R_3 \to R_2 + 2R_2}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad R_2 \to R_2 - R_3 \quad \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \frac{-1}{2} R_3 \quad \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_1 \to R_1 - R_3 \end{array}$$

$$B^T = \begin{bmatrix} 1 & -2 & 7 \\ 2 & -3 & 12 \\ 3 & -4 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 \to 3R_1$ ; $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 2R_2$

$$\boxed{\therefore \ A^T \neq B^T}$$

Since; $A^T$ & $B^T$ have different row spaces, hence $A$ & $B$ have different column spaces.

Ques) 1(b) Determine the conditions for the consistency of the equations
$ax+by+cz=p$ ; $bx+cy+az=q$ & $cx+ay+bz=r$.
where $a, b, c$ are not all zero; solve completely in the case of consistency.

Solve:- given Equations are

$$ax + by + cz = p$$
$$bx + cy + az = q$$
$$cx + ay + bz = r$$

It is in the form of $AX = B$

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

for consistency $|A| = 0$

$$|A| = 3abc - a^3 - b^3 - c^3$$
$$|A| = a^3 + b^3 + c^3 - 3abc = 0$$
$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$$
$$a+b+c = 0 \quad \text{or} \quad a^2+b^2+c^2 = ab+bc+ca.$$

or Both

Now adding all 3 equations

$$(a+b+c)(x+y+z) = p+q+r$$

$$x+y+z = \frac{p+q+r}{a+b+c}$$

Solving we get.

$$z = \frac{(a^2-bc)q+(b^2-ac)r+(c^2-ab)p}{a^3+b^3+c^3-3abc}$$

$$z = \frac{(c^2-ab)q+(a^2-bc)r+(b^2-ac)p}{a^3+b^3+c^3-3abc}$$

$$x = \frac{(b^2-ac)q+(c^2-ab)r+(a^2-bc)p}{a^3+b^3+c^3-3abc}$$

Also $a \neq b$, $a \neq c$

These are the required values of $x, y, z$ for the given equations to be consistent.

Ques: 1(c)> Discuss the continuity and differentiability of the following function at (0,0):

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Solution:-

given $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Let $x = r\cos\theta$ ; $y = r\sin\theta$

$$\therefore \left| f(x,y) - f(0,0) \right| = \left| \frac{r\cos\theta \cdot r^2\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)} - 0 \right|$$

$$= \left| \frac{r^{\cancel{3}}\cos\theta \cdot \sin^2\theta}{\cancel{r^2}\cdot 1} \right| = \left| r\cos\theta \cdot \sin^2\theta \right| \leq |r|$$

Change $\delta = \varepsilon$

$$\therefore \left| f(x,y) - f(0,0) \right| < \varepsilon \quad ; \text{ whenever } 0 < |r| < \delta$$

$$\boxed{\therefore f(x,y) = \frac{xy^2}{x^2+y^2} \text{ is continous at } (0,0)}$$

For Differentiability at (0,0)

$$\Delta = f_x \Delta x + f_y \Delta y + f_1 \Delta x + f_2 \Delta y$$

$$\Delta = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$\because \; f(x,y) = \dfrac{xy^2}{x^2+y^2}$

$$\Delta = \frac{(x+\Delta x)(y+\Delta y)^2}{\left((x+\Delta x)^2+(y+\Delta y)^2\right)} - \frac{xy^2}{x^2+y^2}$$

$$\Delta_{(x,y)\to(0,0)} = \frac{\Delta x\,\Delta y^2}{\Delta x^2+\Delta y^2}$$

$$f_x \text{ at } (0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h-0} = \frac{h\times 0}{h^2+0}\Big/ h = 0$$

$$f_y \text{ at } (0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k-0} = \frac{0\cdot k^2}{k^2+0}\Big/ k = 0$$

$$\frac{\Delta x\,\Delta y^2}{\Delta x^2+\Delta y^2} = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

If Differentiable as

$\Delta x \to 0, \; \Delta y \to 0$

$\varepsilon_1 \to 0, \; \varepsilon_2 \to 0$ $\quad [\because \Delta x = m\, \Delta y]$

$$\varepsilon_1 = \frac{\Delta x\,\Delta y^2}{\Delta x^2+\Delta y^2} = \frac{\cancel{\Delta y^2}\cdot \Delta x}{\cancel{\Delta y^2}\left(1+\Delta x^2/\Delta y^2\right)} = \frac{m}{1+m^2}$$

which depends on the value of $m$.

$\therefore f(x,y)$ is not differentiable in $[0,0]$.

Ques:-1(d)) Find the volume of the portion of the cylinder determined by the equation $x^2-y^2-2ax=0$ which is intercepted between the planes $z = x\tan\alpha$, $z = x\tan\beta$.

Solution:- Clearly the limits of $z$ are from $x\tan\alpha$ to $x\tan\beta$, the limits of $y$ are from $-\sqrt{(2ax-x^2)}$ to $\sqrt{(2ax-x^2)}$ and those of $x$ are from 0 to $2a$.

$$\therefore \text{The volume } (V) = \iiint dx\,dy\,dz$$

$$= \int_0^{2a} 2 \cdot \int_0^{\sqrt{(2ax-x^2)}} [z]_{x\tan\alpha}^{x\tan\beta}\, dx\,dy$$

$$= \int_0^{2a} 2x(\tan\beta - \tan\alpha)\,[y]_0^{\sqrt{(2ax-x^2)}}\, dx$$

$$= 2(\tan\beta - \tan\alpha)\int_0^{2a} x\sqrt{2ax-x^2}\,dx$$

$$= 2(\tan\beta - \tan\alpha)\int_0^{2a} x\sqrt{\{a^2-(a^2-x^2)\}}\,dx$$

Put $a-x = a\sin\theta$, $-dx = a\cos\theta\,d\theta$

$$\therefore V = -2(\tan\beta - \tan\alpha)\,a^2\int_{\pi/2}^{-\pi/2} (1-\sin\theta)\cos\theta\cdot\cos\theta\,d\theta$$

$$= 2(\tan\beta - \tan\alpha)\,a^2\int_{-\pi/2}^{\pi/2} (\cos^2\theta - \cos^2\theta\sin\theta)\,d\theta$$

$$= 2(\tan\beta - \tan\alpha)\cdot 2\cdot a^2\int_0^{\pi/2} \cos^2\theta(1-\sin\theta)\,d\theta$$

$$= 4(\tan\beta - \tan\alpha)\cdot a^2 \cdot \int_0^{\pi/2} \cos^2\theta \cdot a\, d\theta$$

$$= 4a^3(\tan\beta - \tan\alpha)\int_0^{\pi/2} \cos^2\theta\, d\theta.$$

[ $2^{nd}$ Integral vanished from definite integral prop. (V) ]

$$= \cancel{4}^{2}(\tan\beta - \tan\alpha)\cdot a^3 \cdot \frac{\pi}{\cancel{2}}\cdot\frac{1}{2}$$

$$= \frac{\cancel{2}}{\cancel{2}}\pi a^3(\tan\beta - \tan\alpha).$$

$$\boxed{\therefore \text{The volume } V = \pi a^3(\tan\beta - \tan\alpha)}$$

Ques: 1.e) Obtain the equations of the spheres which passes through the circle $y^2+z^2=4$; $x=0$ and are cut by the plane $2x+2y+z=0$ in a circle of radius 3:

Solution:-

The equation of any sphere through the given circle

$$(x^2+y^2+z^2-4)+\lambda x=0$$

$$\Rightarrow \quad x^2+y^2+z^2+\lambda x-4=0 \quad \text{——(i)}$$

Its centre 'C' is $(-\frac{1}{2}\lambda, 0, 0)$ and radius $\sqrt{(\frac{1}{4}\lambda^2+4)}$

let, A be any point on the circumference of the circle in which the sphere (i) is cut by the plane $\boxed{2x+2y+z=0}$

and B be the centre of this circle.

Then; $CA$ = radius of sphere (i) $=\sqrt{\frac{1}{4}\lambda^2+4}$ ——(ii)

and $CB$ = length of prep. from 'C' to the plane $2x+2y+z=0$

ie. $$\boxed{CB=\frac{\cancel{2}(-\frac{1}{\cancel{2}}\lambda)+2.0+1.0}{\sqrt{2^2+2^2+1}}=-\frac{\lambda}{3}} \quad \text{——(iii)}$$

Also given that the radius of the circle is 3

ie $\boxed{BA=3}$

Since ; ABC is a right angled triangle,
with $\angle B = \frac{1}{2}\pi$

so; $CA^2 = CB^2 + BA^2$

or $\left(\frac{1}{4}\lambda^2+4\right) = \frac{1}{9}\lambda^2+9$ from (ii)&(iii)

$\left(\frac{1}{4}-\frac{1}{9}\right)\lambda^2 = 9 - 4$

$\left(\frac{9-4}{36}\right)\lambda^2 = 5$

$\lambda^2 = \frac{\cancel{5}}{\cancel{5}}\times 36$

$\boxed{\lambda = \pm 6}$

$\therefore$ From (i), the required equations of the spheres are

$$\boxed{\therefore\ x^2+y^2+z^2 \pm 6x - 4 = 0}$$

Ques: 2(a) (i) Let $T: R^3 \to R^3$ be the linear mapping defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find a basis and the dimension of the image 'U' of 'T'.

Solution:- given; $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

The image of vectors which span the domain $\mathbb{R}^3$

$$T(1,0,0) = (1,0,1)$$
$$T(0,1,0) = (2,1,1)$$
$$T(0,0,1) = (-1,1,-2)$$

The images span the image U of T; hence form the matrix whose rows are the images vectors and rows reduce to echelon form:

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$ ; $R_3 \to R_3 - R_2$

Thus;

Basis of $U = \{(1,0,1), (0,1,-1)\}$

and Dimension $= 2$.

Ques:- 2(a) ii)) If $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, find out the values of $\alpha, \beta$ such that $(\alpha I+\beta A)^2 = A^2$.

Solution:- given; $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$$A^2 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} = -I \quad \text{—— (i)}$$

$$(\alpha I+\beta A)^2 = \alpha^2 I^2 + \beta^2 A^2 + 2\alpha\beta AI$$

$$= \alpha^2 I + \beta^2(-I) + 2\alpha\beta \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

(from (i)

$$= (\alpha^2-\beta^2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 2\alpha\beta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$(\alpha I+\beta A)^2 = \begin{bmatrix}\alpha^2-\beta^2 & 2\alpha\beta\\ -2\alpha\beta & \alpha^2-\beta^2\end{bmatrix} \quad \text{—— (ii)}$$

given; $(\alpha I+\beta A)^2 = A^2$

$$\begin{bmatrix}\alpha^2-\beta^2 & 2\alpha\beta\\ -2\alpha\beta & \alpha^2-\beta^2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$\alpha^2-\beta^2 = 1$ $\qquad$ $2\alpha\beta = 0$

$\alpha = \pm\sqrt{1+\beta^2}$ $\qquad$ $\alpha\beta = 0$

If $\alpha = +\sqrt{1+\beta^2}$ $\qquad$ $(1+\beta^2)\beta^2 = 0$

If $\alpha = -\sqrt{1+\beta^2}$ $\qquad$ $\Rightarrow (1+\beta^2) = 0 \Rightarrow \beta = \pm i$

$\beta^2 = 0 \Rightarrow \beta = 0$

If $\beta^2 = -1$

$\therefore \alpha = \pm\sqrt{1-1} = 0$

If $\beta^2 = 0$

$\alpha = \pm\sqrt{1} = \pm 1$

Hence;

If $\beta^2 = -1$ $\qquad$ $\alpha = 0, \beta = \pm i$

If $\beta^2 = 0$ $\qquad$ $\alpha = \pm 1, \beta = 0$

Ques: 2(b)(i) If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$ ; prove that

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{x^2-y^2}{x^2+y^2}.$$

Solution: given; $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$

$$\frac{\partial u}{\partial y} = x^{\cancel{2}}\cdot\frac{1}{(1+(y/x)^2)}\cdot\frac{1}{\cancel{x}} - \left[2y\tan^{-1}\left(\frac{x}{y}\right) + y^{\cancel{2}}\cdot\frac{1}{1+(x/y)^2}\left(\frac{-x}{\cancel{y^2}}\right)\right]$$

$$\therefore \frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\left(\frac{x}{y}\right) + \frac{xy^2}{x^2+y^2}.$$

Partially differentiating w.r.t $x$, we obtain

$$\therefore \frac{\partial^2 u}{\partial x\partial y} = \frac{(x^2+y^2).3x^2 - x^3 2x}{(x^2+y^2)^2} - 2\cancel{y}\cdot\frac{1}{(1+(x/y)^2)}\cdot\frac{1}{\cancel{y}}$$
$$+ \frac{y^2(x^2+y^2) - xy^2.2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{x^4+3x^2y^2}{(x^2+y^2)^2} - \frac{2y^2}{x^2+y^2} + \frac{y^2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{x^4+3x^2y^2 - 2y^2(x^2+y^2) + y^2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{x^4 + 3\cancel{x^2}y^2 - 2\cancel{x^2}y^2 - 2y^4 + y^4 - x^2\cancel{y^2}}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{x^4-y^4}{(x^2+y^2)^2} = \frac{\cancel{(x^2+y^2)}(x^2-y^2)}{(x^2+y^2)^{\cancel{2}}}$$

$$\boxed{\frac{\partial^2 u}{\partial x\partial y} = \frac{x^2-y^2}{x^2+y^2}}$$ which is required solution.

Ques: 2 (b) ii) Evaluate $\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan^2 x}$.

Solution:- given function $= f(x) = \lim_{x\to\pi/2} (\sin x)^{\tan^2 x}$

at given limit, it become $1^\infty$ form.

Let $y = (\sin x)^{\tan^2 x}$ ; so that

$$\log y = \tan^2 x \log \sin x$$

$$\lim_{x\to\pi/2} \log y = \lim_{x\to\pi/2} \tan^2 x \cdot \log \sin x$$

$$= \lim_{x\to\pi/2} \frac{\log \sin x}{\cot^2 x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$\Rightarrow$ using L-Hopital rule.

$$\Rightarrow \lim_{x\to\pi/2} \frac{\cancel{\cot x}}{-2\cancel{\cot x} \cdot \operatorname{cosec}^2 x}$$

$$\Rightarrow -\frac{1}{2} \lim_{x\to\pi/2} \cdot \sin^2 x$$

$$= -\frac{1}{2} \cdot (1) = -\frac{1}{2}.$$

$$\therefore \lim_{x\to\pi/2} \log y = \log \lim_{x\to\pi/2} y = -\frac{1}{2}$$

$$\therefore \boxed{\text{Hence, } \lim_{x\to\pi/2} (\sin x)^{\tan^2 x} = e^{-1/2}}$$

Which is required solution.

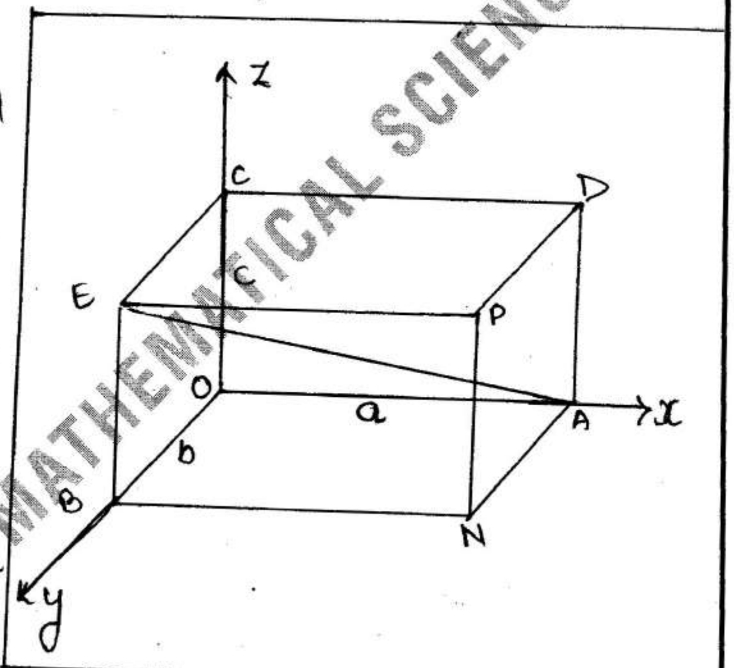
Ques: 2(C)) Prove that the S.D. between the diagonals of rectangular parallelopiped and the edges not meeting it are.

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}.$$

Where, a, b, c are lengths of the edges.

Solution:-

In the parallelopiped shown adjacent.
Let the sides $OA=a$; $OB=b$; $OC=c$
Let, the edges OA, OB and OC be taken as co-ordinates axes.
consider the diagonal AE and edge OB not intersecting this diagonal.




from figure, it is evident that the co-ordinates of A, B and E are $(a,0,0)$, $(0,b,0)$ and $(0,b,c)$ respectively.

The direction ratios of AE are $a-0, 0-b, 0-c$.
i.e, $(a,-b,-c)$ respectively.

$\therefore$ The equations of A.E. and OB are

$$\frac{x-a}{a} = \frac{y-0}{-b} = \frac{z-0}{-c} \quad \text{——— (i)}$$

and $\frac{x}{0} = \frac{y}{b} = \frac{z}{0}$ ——— (ii)

If $l, m, n$ be the d.c.'s of the S.D. between AE and OB, then as the line of S.D. is perpendicular to both AE and OB, therefore, we have

$l \cdot a - m \cdot b - n \cdot c = 0$ and.

$l \cdot 0 + m \cdot b + n \cdot 0 = 0$

Solving, these, we get $\frac{l}{bc} = \frac{m}{0} = \frac{n}{ab} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{b^2c^2+a^2b^2}}$

$\Rightarrow \frac{l}{bc} = \frac{m}{0} = \frac{n}{ab} = \frac{1}{b\sqrt{c^2+a^2}}$

or $l = \frac{c}{\sqrt{c^2+a^2}}$, $m = 0$ ; $n = \frac{a}{\sqrt{c^2+a^2}}$ ——(iii)

Now, the S.D. between AE and OB.

= the projection of the join $A(a,0,0)$ and $O(0,0,0)$ on the line whose d.c.'s are given by (iii)

$= l(a-0) + m(0-0) + n(0-0)$

where $l, m, n$ are given by (iii)

$= ac/\sqrt{a^2+c^2}$.

Similarly, we can find <u>other</u> S.D.'s.

Ques: 3 (a) (i) Show that the vectors $v = (1+i, 2i)$ and $w = (1, 1+i)$ in $\mathbb{C}^2$ are linearly dependent over the complex field 'C' but are linearly independent over the real field $\mathbb{R}$.

Solution

We know that, the two vectors are dependent iff one is a multiple of the other. Since, the first co-ordinate of $w$ is 1,

$v$ can be multiple of $w$ iff $v = (1+i)w$

But $1+i \notin \mathbb{R}$

Hence; $v$ and $w$ are independent over $\mathbb{R}$

Since; $(1+i)w = (1+i)(1, 1+i)$

$$(1+i)w = (1+i, 2i) = v$$

and $(1+i) \in \mathbb{C}$.

They are dependent over 'C'.

3(a) (ii) Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a,b,c) : a+b+c=0\}$. Find a basis and dimension of W.

Solution :-

Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a,b,c) : a+b+c=0\}$.

Note ; $W \neq \mathbb{R}^3$

Since, for example, $(1,2,3) \notin W$

Thus ; $\dim W < 3$

$u_1 = (1,0,-1)$ and $u_2 = (0,1,-1)$

are two independent vectors in W.

Thus; $\boxed{\dim W = 2}$

and $u_1$ & $u_2$ are basis of W.

i.e $\boxed{W = \{(1,0,-1), (0,1,-1)\}}$

3(a) iii) Suppose U and W are distinct 4-D [4-Dimensional] subspaces of a vector space V of dimension 6. Find the possible dimensions of $U \cap W$.

Solution:- Since; U and W are distinct 4 Dimensional subspaces of a vector space

$\Rightarrow$ $U+W$ properly contains U and W.

Hence, $\dim(U+W) > 4$.

But; $\dim(U+W)$ cannot be greater than 6.

Since, $\dim(V) = 6$.

Hence; we have two possibilities.

(i) $\dim(U+W) = 5$

(ii) $\dim(U+W) = 6$

$\therefore \dim(U \cap W) = \dim U + \dim W - \dim(U+W)$

$= 2+2-3 = 1.$

That is, $U \cap W$ is a line through the origin

Ques: 3(b) Prove that $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\,dx = -\frac{\pi}{4}\log 2.$

Solution:-

Let ; $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\,dx$

$$I = \int_{-\pi/4}^{\pi/4} \log\left[\sqrt{2}\sin\left(x+\pi/4\right)\right]dx$$

Now put ; $x + \pi/4 = t$

$dx = dt$

Hence; $I = \int_0^{\pi/2} \log(\sqrt{2}\sin t)\,dt$

$$= \frac{1}{2}\log 2 \int_0^{\pi/2} dt + \int_0^{\pi/2} \log \sin t\,dt$$

$$= \frac{1}{2}\log 2 \cdot (t)_0^{\pi/2} + -\frac{\pi}{2}\log 2.$$

$$= \frac{\pi}{2}\cdot\frac{1}{2}\log 2 - \frac{\pi}{2}\log 2$$

$$\boxed{I = -\frac{\pi}{4}\log 2.}$$ required solution

Ques: 3(c)) Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$, there is at least one real root of the equation $\tan x + 1 = 0$

Solution:-

Given; $e^x \sin x + 1 = 0$

Let, the interval be $[a, b]$

$$f(x) = e^x \sin x + 1$$

where a, b are two real roots of the given equation

(i) Such that $f(a) = f(b) = 0$

(ii) $f(x)$ is continuous in the given interval $[a, b]$

(iii) $f(x)$ is differentiable is the interval $(a, b)$

So; $\exists$ a 'c' such that $f'(c) = 0$

$f(x) = e^x \sin x + 1$

$f'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$

$f'(x) = e^x \cos x [\tan x + 1]$

$f'(c) = e^c \cos c [\tan c + 1]$.

$f'(c) = 0 = e^c \cos c [\tan c + 1]$

$\Rightarrow \tan c + 1 = 0$

So; $\tan x + 1 = 0$ has one real root 'c' between any two real roots of the given equation $e^x \sin x + 1 = 0$.

Ques: 3(d)(i)⟩ Find the limiting points of co-axial systems defined by the spheres

$$x^2+y^2+z^2+2x+2y+4z+2=0 \text{ and}$$
$$x^2+y^2+z^2+x+y+2z+2=0.$$

Solution:-

The equation of the co-axial system of sphere is –

$$(x^2+y^2+z^2+2x+2y+4z+2)+\lambda(x^2+y^2+z^2+x+y+2z+2)=0$$

$$(x^2+y^2+z^2)+\left(\frac{2+\lambda}{1+\lambda}\right)x+\left(\frac{2+\lambda}{1+\lambda}\right)y+\left(\frac{4+2\lambda}{1+\lambda}\right)z+2=0$$

Its centre is $\left[\frac{-(2+\lambda)}{1+\lambda}, -\left(\frac{2+\lambda}{1+\lambda}\right), -\left(\frac{4+2\lambda}{1+\lambda}\right)\right]$ ——(i)

and equating its radius to zero, we get

$$\left(\frac{2+\lambda}{1+\lambda}\right)^2+\left(\frac{2+\lambda}{1+\lambda}\right)^2+\left(\frac{4+2\lambda}{1+\lambda}\right)^2-2=0$$

$$(2+\lambda)^2+(2+\lambda)^2+(4+2\lambda)^2-2(1+\lambda)^2=0$$

$$\Rightarrow {}^{3}\cancel{6}(2+\lambda)^2=\cancel{2}^{1}(1+\lambda)^2$$

$$3(2+\lambda)^2=(1+\lambda)^2$$

$$\Rightarrow 3(\lambda^2+4\lambda+4)=\lambda^2+2\lambda+1$$

$$\Rightarrow 2\lambda^2+10\lambda-11=0$$

$$\lambda=\frac{-10\pm\sqrt{100-88}}{4}=\frac{-5\pm\sqrt{3}}{2}$$

$$\boxed{\lambda=\frac{-5+\sqrt{3}}{2}, \frac{-5-\sqrt{3}}{2}}$$

Substituting the values of $\lambda$ in the coordinates of the above centre, the required limiting points are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \text{ and } \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}\right)$$

Which is required solution

---

Ques: 3(d) ii) If the plane $2x-y+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines. find the value of $c$.

Solution :-

Let, the plane $2x-y+cz=0$

cut the cone $yz+zx+xy=0$ in a line

$$\boxed{\frac{x}{l}=\frac{y}{m}=\frac{z}{n}}$$

Then; $2l-m+cn=0$ and $mn+nl+lm=0$ — (i)

Eliminating 'm' between these relations, we get,

$$(2l+cn)n+nl+l(2l+cn)=0$$

$$2l^2+(c+3)ln+cn^2=0 \quad \text{or}$$

$$\boxed{2\left(\frac{l}{n}\right)^2+(c+m)\frac{l}{n}+c=0} \quad \text{—— (ii)}$$

If roots of this equation are $(l_1/n_1)$ & $(l_2/n_2)$, then

$$\frac{l_1}{n_1}\cdot\frac{l_2}{n_2} = \text{product of the roots} = \frac{c}{2}$$

or. $$\boxed{\frac{l_1 l_2}{c} = \frac{n_1 \cdot n_2}{2}} \quad \text{——— (iii)}$$

Eliminating '$l$' between the relations (i), we get

$$2nm + n(m-cn) + m(m-cn) = 0$$

$$m^2 + (3-c)mn - cn^2 = 0$$

or $$c(n/m)^2 + (c-3)(n/m) - 1 = 0$$

$\therefore$ If the roots of this equation are $\frac{n_1}{m_1}$ & $\frac{n_2}{m_2}$, then

$$\frac{n_1}{m_1}\cdot\frac{n_2}{m_2} = \text{product of the roots} = \frac{-1}{c}$$

Or $$\frac{n_1 \cdot n_2}{1} = \frac{m_1 m_2}{-2c} \quad \text{or} \quad \frac{n_1 n_2}{2} = \frac{m_1 \cdot m_2}{-2c} \quad \text{— (iv)}$$

from (iii) and (iv); we get

$$\frac{l_1 l_2}{1} = \frac{m_1 m_2}{-2c} = \frac{n_1 n_2}{2} = k \text{ (say)}.$$

If the angle between the lines is a right angle, then we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \text{or.}$$

$$c + (-2c) + 2 = 0. \quad \Rightarrow \boxed{c = 2}$$

$\boxed{c = 2}$ which is required value of c.

Ques: 4(a) i) Let T be a linear operator on $\mathbb{R}^3$. which is represented in the standard ordered basis by the matrix.

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$ each vector of which is characteristic vector of T.

Solution:-

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation of T is $\det(A-xI)=0$

i.e. $$\begin{vmatrix} -9-x & 4 & 4 \\ -8 & 3-x & 4 \\ -16 & 8 & 7-x \end{vmatrix} = 0$$

or $$\begin{vmatrix} -1-x & 4 & 4 \\ -1-x & 3-x & 4 \\ -1-x & 8 & 7-x \end{vmatrix} = 0 \qquad C_1 \to C_1+C_2+C_3$$

$$\Rightarrow -(1+x)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-x & 4 \\ 1 & 8 & 7-x \end{vmatrix} = 0$$

Now, $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= -(1+x)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -(1+x) & 0 \\ 0 & 8 & 3-x \end{vmatrix} = 0$$

$$= -(1+x)\cdot 1\cdot\left[-(1+x)(3-x)-0\right]$$

$$\Rightarrow +(1+x)[1+x][3-x] = 0$$

$$(1+x)^2(3-x) = 0$$

Hence, the characteristics values of T are 3, -1, -1.

The characteristics vector corresponding to $x=3$ is given by -

$$(A-3I)X = 0$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} -4 & 4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The above matrix equation yields

$$-x_1 + x_2 = 0 \quad ; \quad -2x_2 + x_3 = 0$$

These equation are satisfied by ; $x_1 = 1, x_2 = 1$
$x_3 = 2$.

An eigen vector corresponding to eigen value $x=3$ is $\Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

The eigen vector corresponding to eigen value $x=-1$, is given by

$$(A+I)X = 0$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ ; $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From, the above matrix equation, we get

$$-2x_1 + x_2 + x_3 = 0$$

Taking; $x_2 = 0$ ; $x_1 = 1, x_3 = 2$

taking; $x_3 = 0$ ; $x_1 = 1$, $x_2 = 2$

Hence, two L.I. characteristic vectors corresponding to the characteristic value $x = -1$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

It is easy to verify that $X_1, X_2, X_3$ are linearly independent over $\mathbb{R}$ and so the set $\{X_1, X_2, X_3\}$ constitutes a basis of $\mathbb{R}^3$.

Hence, T is diagonalizable. Indeed.

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \text{ where } \boxed{P = [X_1\ X_2\ X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}}$$

Required solution.

Ques: 4(a) ii) If 'A' is non-singular, prove that the eigenvalues of $A^{-1}$ are reciprocals of the eigenvalues of 'A'.

Solution:- Let, $\lambda$ be an eigen value of A and X be corresponding eigen vector. Then,

$$AX = \lambda X \Rightarrow X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X \quad \left[\because A \text{ is non-singular} \Rightarrow \lambda \neq 0\right]$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

$\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$

and X is a corresponding eigenvector.

Conversely, suppose that K is an eigenvalue of $A^{-1}$. Since, A is non-singular

$\Rightarrow A^{-1}$ is non-singular and $(A^{-1})^{-1} = A$,

Therefore, it follows from the first part that $1/K$ is an eigenvalue of A.

Thus, each eigenvalue of $A^{-1}$ is reciprocal of some eigenvalue of A.

Ques: 4(b)) The cone $z^2 = x^2 + y^2$ is cut by the plane $z = 1 + x + y$ in a conic section C. Find the points on 'C' that are nearest to, and farthest from, the origin.

Solution:- We model this as a lagrange multiplier problem in which we find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{——} \text{(A)}$$

subject to the two constraints-

$$g(x, y, z) = x^2 + y^2 - z^2 = 0 \quad \text{——} \text{(i)}$$
$$h(x, y, z) = 1 + x + y - z = 0 \quad \text{——} \text{(ii)}$$

gradient equation (A) gives

$$2x\hat{i} + 2y\hat{j} + 2z\hat{k} = \lambda(2x\hat{i} + 2y\hat{j} - 2z\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

Which leads to the scalar equations

$$\left.\begin{array}{l} 2x = 2x\lambda + \mu \\ 2y = 2y\lambda + \mu \end{array}\right\} \Rightarrow x - y = (x - y)\lambda \quad \text{—(iii)}$$

$$\left.\begin{array}{l} 2y = 2y\lambda + \mu \\ 2z = -2z\lambda - \mu \end{array}\right\} \Rightarrow y + z = (y - z).\lambda \quad \text{—(iv)}$$

The equation $\Rightarrow x - y = (x - y)\lambda$

is satisfied if $x = y$ or

if $x \neq y$ and $\lambda = 1$.

The case $\lambda = 1$; does not lead to a point on the cutting plane (i.e. to a point whose co-ordinates satisfy Equation (ii).

for $\lambda = 1$ implies

$$y + z = y - z$$
$$2z = 0 \Rightarrow \boxed{z = 0}$$

and Eq (ii), then gives.

$$\boxed{x^2 + y^2 = 0 \quad ; \quad x = 0, \; y = 0}$$

The point $(0,0,0)$ does not satisfy the constraint (ii).

Therefore; $\lambda \neq 1$; we have $x = y$.

The plane $x = y$ meets the plane $z = 1 + x + y$ in a line that cuts the cone in just two points,

Now, By substituting $y = x$ and $z = 1 + 2x$ into the cone equations:

$$z^2 = x^2 + y^2$$
$$(1 + 2x)^2 = x^2 + x^2$$
$$2x^2 + 4x + 1 = 0$$
$$\boxed{x = -1 \pm \frac{\sqrt{2}}{2}}$$

The points are.

$$\boxed{\begin{aligned} A &= \left(-1 - \sqrt{1/2}, \; -1 - \sqrt{1/2}, \; -1 - \sqrt{2}\right) \\ B &= \left(-1 + \sqrt{1/2}, \; -1 + \sqrt{1/2}, \; -1 + \sqrt{2}\right) \end{aligned}}$$

Now, we know that 'C' is either an ellipse or a hyperbola. If it is an ellipse, we would conclude that 'B' is the point on it nearest the origin, and 'A' the point farthest from the origin. But if it is a hyperbola, then there is no point on it that is farthest away from the origin and the point A & B are the points on the two branches that are nearest the origin.

Ques: 4 (c)) Prove that the normals from $(\alpha, \beta, \gamma)$ to the paraboloid $(x^2/a^2) + (y^2/b^2) = 2z$, lies on the cone $\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$?

Solution:-

The equations of any line through $(\alpha, \beta, \gamma)$ can be taken as

$$\boxed{\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}} \quad \text{—— (1)}$$

If it represents a normal to the given paraboloid –

$$\boxed{(x^2/a^2) + (y^2/b^2) = 2z} \quad \text{—— (2)}$$

at $(x_1, y_1, z_1)$, ~~then~~

$$\boxed{l = x_1/a^2, \quad m = y_1/b^2, \quad n = -1} \quad \text{—— (3)}$$

Also, we know that there are five points such as $(x_1, y_1, z_1)$, the normals at which to be paraboloid (2) passes through the point $(\alpha, \beta, \gamma)$ and as in

$$x_1 = \frac{\alpha}{1 + (r/a^2)} \; ; \; y_1 = \frac{\beta}{1 + r/b^2} \; ; \; z_1 = \gamma + r$$

$\therefore$ from (3) above, we get

$$l = \frac{\alpha/a^2}{1 + r/a^2} \; ; \; m = \frac{\beta/b^2}{1 + r/b^2} \; ; \; n = -1$$

or $l = \dfrac{\alpha}{a^2+r}$ ; $m = \dfrac{\beta}{b^2+r}$ ; $n=-1$

$\dfrac{\alpha}{l} = a^2+r$ ; $\dfrac{\beta}{m} = b^2+r$ ; $\dfrac{1}{n} = -1$

Eliminating '$r$' between these, we get

$$\left(\alpha/l\right)-\left(\beta/m\right) = -(a^2-b^2)(1/n)$$

$$\Rightarrow \boxed{\alpha/l - \beta/m + a^2-b^2/n = 0} \text{ —— } (4)$$

Eliminating $l, m, n$ between (4) & (1), we get the equation of the cone on which the normals to the given paraboloid from $(\alpha, \beta, \gamma)$ lie as

$$\boxed{\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0}$$

Which is required result

## Section - B

Ques: 5 - a) Solve $dy/dx = (x+y+4)/(x-y-6)$.

Solution:-

given - $\frac{dy}{dx} = \frac{x+y+4}{x-y-6}$ —— (1)

Let; $x = X+h$ , $y = Y+k$

so that $\frac{dy}{dx} = \frac{dY}{dX}$ —— (2)

using (2), (1) reduces to

$\frac{dy}{dx} = \frac{(X+Y)+(h+k+4)}{(X-Y)+(h-k-6)}$ —— (3)

we choose h and k, such that

$$\left.\begin{array}{l} h+k+4=0 \text{ i.e } h+k=-4 \\ h-k-6=0 \Rightarrow h-k=6 \end{array}\right] \text{—— (4)}$$

By solving (4); we get

$h = 1$ and $k = -5$

so by (2)

$X = x-1$ ; $Y = y+5$ —— (5)

using (4), (3) reduces to

$\frac{dY}{dX} = \frac{X+Y}{X-Y} = \frac{1+Y/X}{1-Y/X}$ —— (6)

Putting $y = xv$ and $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$

(6) becomes, $v + x\left(\frac{dv}{dx}\right) = \frac{1+v}{1-v}$

$$\Rightarrow \frac{dx}{x} = \frac{1-v}{1+v^2}dv = \frac{dv}{1+v^2} - \frac{v}{1+v^2}dv$$

By Integrating $\Rightarrow$ $\log x = \tan^{-1}v - \frac{1}{2}\log(1+v^2) + \frac{1}{2}\log c$

$$\Rightarrow 2\log x + \log\left(1 + \frac{y^2}{x^2}\right) - \log c = 2\tan^{-1}\frac{y}{x}$$

as $v = y/x$.

$$\Rightarrow \log x^2 + \log\left(\frac{x^2+y^2}{x^2}\right) - \log c = 2\tan^{-1}\frac{y}{x}$$

$$\Rightarrow \log\left(\frac{x^2+y^2}{c}\right) = 2\tan^{-1}(y/x)$$

$$\Rightarrow x^2 + y^2 = ce^{2\tan^{-1}(y/x)}$$

$$\Rightarrow \boxed{(x-1)^2 + (y+5)^2 = ce^{2\tan^{-1}\left(\frac{y+5}{x-1}\right)}}$$

c being an arbitrary constant.

Ques: 5(b)) Find the orthogonal trajectories of
$r = a(1+\cos n\theta)$.

Solution:-

Given family is - $r = a(1+\cos n\theta)$; —— ①
where a is parameter.

Taking logarithm of both sides,

$$\log r = \log a + \log(1+\cos n\theta) \quad \text{—— ②}$$

Differentiating w.r.t $\theta$,

$$\left(\frac{1}{r}\right)\left(\frac{dr}{d\theta}\right) = -(n\sin n\theta)/(1+\cos n\theta) \quad \text{—— ③}$$

which is differential equation of the family of curves (1).

Replacing $\frac{dr}{d\theta}$ by $-r^2(d\theta/dr)$ in ③,

the differential equation of the required trajectories is -

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta}$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}\,d\theta$$

$$\Rightarrow \frac{n\,dr}{r} = \frac{\cancel{2}\cos^{\cancel{2}}(n\theta/2)\,d\theta}{\cancel{2}\sin(n\theta/2)\cancel{\cos(n\theta/2)}}$$

$$\Rightarrow n\frac{dr}{r} = \cot\left(\frac{n\theta}{2}\right)d\theta$$

Integrating the equation

$$n\log r = \frac{2}{n}\times\log\sin(n\theta/2)+\frac{1}{n}\log c$$

$c$ is being arbitrary constant

$$n^2\log r = 2\log\sin\left(\frac{n\theta}{2}\right)+\log c$$

$$r^{n^2} = c\sin^2\left(\frac{n\theta}{2}\right)$$

$$\Rightarrow r^{n^2} = \frac{c}{2}(1-\cos n\theta)$$

$$\boxed{r^{n^2} = b(1-\cos n\theta)}$$

where; $b = c/2$, which is the equation of required orthogonal trajectory with 'b' as parameter.

Ques: 5(c)) If R be the horizontal range and 'h' the greatest height of a projectile, prove that the initial velocity is

$$\left[2g\left(h+\frac{R^2}{16h}\right)\right]^{1/2}.$$

Solution:-

Let, u be the velocity of projection and $\alpha$ the angle of projection. Then

$$R = \frac{2u^2 \sin\alpha \cdot \cos\alpha}{g} \quad \text{——} \ (1)$$

and
$$h = \frac{u^2 \sin^2\alpha}{2g} \quad \text{——} \ (2)$$

To obtain the required values of u, we have to eliminate '$\alpha$' between (1) and (2).

Squaring both sides of (1), we get

$$R^2 = \frac{4u^4 \sin^2\alpha \cdot \cos^2\alpha}{g^2} = \frac{4u^4 \sin^2\alpha(1-\sin^2\alpha)}{g^2}$$

Substituting for $\sin^2\alpha$ from (2), we have

$$R^2 = \frac{4\cancel{u^4}u^2}{g^2} \cdot \frac{2gh}{\cancel{u^2}}\left(1-\frac{2gh}{u^2}\right)$$

$$R^2 = \frac{8\cancel{g}hu^2}{g^{\cancel{2}}}\left(1-\frac{2gh}{u^2}\right) = \frac{8u^2h}{g}\left(1-\frac{2gh}{u^2}\right)$$

$$R^2 = \frac{8u^2h}{g} - 16h^2$$

$$\therefore \boxed{\frac{(8u^2h)}{g} = 16h^2 + R^2}$$

Or $$u^2 = \frac{g}{8h}(16h^2 + R^2)$$

$$\Rightarrow u^2 = \frac{g.\cancel{16}^{2}\cancel{h}}{\cancel{8}\cancel{h}}\left(h + \frac{R^2}{16h}\right)$$

$$\Rightarrow u^2 = \frac{2g}{}\left[h + \frac{R^2}{16h}\right]$$

$$\therefore \boxed{u = \left[2g\left[h + \frac{R^2}{16h}\right]\right]^{1/2}}$$

which is required result.

Ques: 5(d)) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$ and $z = 3t - 5$, where '$t$' is the time. Find the components of its velocity and accelaration at time $t=1$ in the direction $i - 3j + 2k$.

Solution:-

given; $x = 2t^2$, $y = t^2 - 4t$

$z = 3t - 5$

$r = x\hat{i} + y\hat{j} + z\hat{k}$

$r = 2t^2\hat{i} + (t^2 - 4t)\hat{j} + (3t-5)\hat{k}$

velocity $= \frac{dr}{dt} = \frac{d}{dt}\left[2t^2\hat{i} + (t^2-4t)\hat{j} + (3t-5)\hat{k}\right]$

velocity $= \frac{dr}{dt} = 4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}$

$$\boxed{\left.\frac{dr}{dt}\right|_{t=1} = 4\hat{i} + (-2\hat{j}) + 3\hat{k}}$$

Unit vector in direction of $i - 3j + 2k$

$$= \frac{i - 3j + 2k}{\sqrt{1 + 9 + 4}} = \frac{i - 3j + 2k}{\sqrt{14}}$$

The component of the velocity in the given direction is

$$= \frac{(4i + (-2)j + 3k)(i - 3j + 2k)}{\sqrt{14}}$$

$$= \frac{4(1) + (-2)(-3) + (3)(2)}{\sqrt{14}}$$

$$= \frac{16}{\sqrt{14}} = \frac{8\sqrt{14}}{7}$$

Now, Accelaration

$$\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left[4t\hat{i} + (2t-4)\hat{j} + 3\hat{k}\right]$$

$$\boxed{\frac{d^2r}{dt^2} = 4\hat{i} + 2\hat{j}}$$

Then, the component of the accelaration in the given direction is

$$= \frac{(4i+2j)(i-3j+2k)}{\sqrt{14}} = \frac{4\times1 + (2)(-3)}{\sqrt{14}}$$

$$= \frac{4-6}{\sqrt{14}} = -\frac{\sqrt{14}}{7}.$$

$\therefore$ $\boxed{\text{component of accelaration in given direction is} = -\frac{\sqrt{14}}{7}}$

Ques: 5(e) Show that the vector field defined by –

$F = (2xy - z^3)\hat{i} + (x^2+z)\hat{j} + (y - 3xz^2)\hat{k}$ is conservative, and find the scalar potential of F.

Solution:

given; $F = (2xy - z^3)\hat{i} + (x^2+z)\hat{j} + (y-3xz^2)\hat{k}$

To check F (vector field to be conservative)

$$\boxed{\text{Curl } F = \nabla \times F = 0}$$

$$\therefore \text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy - z^3 & x^2+z & y-3xz^2 \end{vmatrix}$$

$$\nabla \times F = \left[\frac{\partial}{\partial y}(y-3xz^2) - \frac{\partial}{\partial z}(x^2+z)\right]\hat{i} - \hat{j}\left[\frac{\partial}{\partial x}(y-3xz^2) - \frac{\partial}{\partial z}(2xy-z^3)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2+z) - \frac{\partial}{\partial y}(2xy-z^3)\right]$$

$$\nabla \times F = [1-1]\hat{i} - [-3z^2 - (-3z^2)]\hat{j} + \hat{k}(2x - 2x)$$

$$\boxed{\nabla \times F = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0}$$

$\therefore$ The vector field F is conservative

Let; $F = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$. Then,

$\frac{\partial\phi}{\partial x} = 2xy - z^3$ ; where

$$\phi = x^2y - z^3x + f_1(y,z) \quad \text{——— (1)}$$

$$\frac{\partial\phi}{\partial y} = x^2 + z$$

Hence; $\phi = x^2 y + zy + f_2(z,x)$ ——— (2)

$$\frac{\partial\phi}{\partial z} = y - 3xz^2$$

$\phi = yz - xz^3 + f_3(x,y)$ ——— (3)

(1), (2), & (3) each represents $\phi$.

These agree, if we choose.

$f_1(y,z) = zy$

$f_2(z,x) = -z^3 x$

$f_3(x,y) = x^2 y$

$\therefore \phi = x^2 y - z^3 x + yz$ to which we can add a constant 'C'

Hence; $\boxed{\phi = x^2 y - z^3 x + yz + C}$

which is required equation.

Ques: 6(a) Solve $(px^2+y^2)(px+y) = (p+1)^2$ by reducing it to Clairaut's form and find its singular solution.

Solution:

given: $(px^2+y^2)(px+y) = (p+1)^2$ —— (A)

Let ; $x+y=u$ and $xy=v$ —— (1)

$$\therefore \frac{dv}{du} = \frac{x\,dy + y\,dx}{dx+dy} = \frac{x(dy/dx)+y}{1+(dy/dx)}$$

or $$P = \frac{xp+y}{1+p} \quad \text{—— (2)}$$

where $P = \frac{dv}{du}$ & $p = \frac{dy}{dx}$

Now, $(px^2+y^2)$ can be written as

$$px^2+y^2 = (px+y)(x+y) - xy(p+1)$$

Hence, the equation can be written as

$$\{(px+y)(x+y) - xy(p+1)\}(px+y) = (p+1)^2$$

Or $$\left(\frac{px+y}{p+1}\right)^2 (x+y) - xy\left(\frac{px+y}{p+1}\right) = 1$$

Or $P^2u - vP = 1$ — by (1) and (2)

Or $\boxed{v = uP - 1/P}$, which is in Clairaut's form.

$\therefore$ Its general solution is

$$v = uc - (1/c)$$

$$\Rightarrow \quad xy = (x+y)c - (1/c)$$

$$\Rightarrow \quad \boxed{c^2(x+y) - cxy - 1 = 0}$$

c being an arbitrary constant.

Its c-discriminant relation is

$B^2 - 4AC = 0$ i.e.

$$(xy)^2 - 4(x+y)\times(-1) = 0$$

$$\boxed{x^2y^2 + 4(x+y) = 0}$$

This relation satisfies (A), and hence it is the singular solution.

Ques: 6) b) Solve

$$(1+x^2)\frac{d^2y}{dx^2} + (1+x)\left(\frac{dy}{dx}\right) + y = 4\cos\log(1+x).$$

Solution:-

Given; $[(1+x)^2\frac{d^2y}{dx^2} + (1+x)\left(\frac{dy}{dx}\right) + y = 4\cos\log(1+x)$

which can be re-written as. — $D = \frac{d}{dx}$

$$[(1+x)^2D^2 + (1+x)D + 1]y = 4\cos\log(1+x) \quad \text{— (1)}$$

Let ; $1+x = e^z \Rightarrow \log(1+x) = z$

$$D_1 = d/dz \quad \text{— (2)}$$

Then; we have

$$(1+x)D = D_1 \; ; \; (1+x)^2D^2 = D_1(D_1-1)$$

and hence (1) gives.

$$y[D_1(D_1-1) + D_1 + 1] = 4\cos z$$

$$\Rightarrow (D_1^2+1)y = 4\cos z \quad \text{— (3)}$$

The auxillary equation is

$$D_1^2 + 1 = 0 \Rightarrow D = \pm i$$

$\therefore$ C.F. $= c_1\cos z + c_2\sin z$

$$\boxed{\therefore \text{C.F} = c_1\cos\log(1+x) + c_2\sin\log(1+x)}$$

— using (2)

where, $c_1$ and $c_2$ are arbitrary constants.

$$P.I = \frac{1}{D_1^2+1} 4\cos z = R.P. \text{ of } \frac{1}{D_1^2+1} 4e^{iz}$$

$\boxed{R.P = \text{Real part}}$

$$= R.P. \text{ of } \frac{1}{D_1^2+1} e^{iz} \cdot 4.$$

$$= R.P. \text{ of } e^{iz} \cdot \frac{1}{(D_1+i)^2+1} \cdot 4.$$

$$= R.P. \text{ of } e^{iz} \frac{1}{D_1^2+2D_1 i} \cdot 4$$

$$= R.P. \text{ of } e^{iz} \cdot \frac{1}{2D_1 i(1+D_1/2i)} \cdot 4.$$

$$= R.P. \text{ of } \frac{e^{iz}}{2i} \cdot \frac{1}{D_1}\left(1+\frac{D_1}{2i}\right)^{-1} \cdot 4$$

$$= R.P. \text{ of } \frac{e^{iz}}{2i} \cdot \frac{1}{D_1}\left(1-\frac{D_1}{2i}+\cdots\right) \cdot 4$$

$$= R.P \text{ of } \frac{e^{iz}}{2i} \times 4z$$

$$= R.P. \text{ of } (-2iz) \times (\cos z + i\sin z) \quad \left[\because \frac{1}{i} = -i\right]$$

$$= 2z \cdot \sin z = 2\log(1+x) \cdot \sin \log(1+x)$$

$$\therefore P.I. = 2\log(1+x) \cdot \sin\log(1+x).$$

$\therefore$ Solution of $y = C.F + P.I.$

$$\boxed{\therefore y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) + 2\log(1+x) \cdot \sin\log(1+x)}$$

which is required solution.

Ques: 6> c> Use the method of variation of parameters to solve $y'' + y = 1/(1+\sin x)$.

Solution:- Given; $y'' + y = \frac{1}{1+\sin x}$ —— ①

Comparing ①, with $y'' + Py' + Qy = R$

hence; Here; $P = 0$, $Q = 1$, $R = \frac{1}{1+\sin x}$ —— ②

Consider: $y'' + y = 0$ or $(D^2+1)y = 0$

$$D^2+1 = 0 \Rightarrow D = \pm i$$

$\boxed{\therefore C.F = C_1 \cos x + C_2 \sin x}$ where; $C_1$ & $C_2$ being arbitrary constants.

Let; $u = \cos x$ and $v = \sin x$

Here; $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$$\boxed{W = \cos^2 x + \sin^2 x = 1 \neq 0}$$

Then;

P.I. of ① $= uf(x) + vg(x)$, where —— ③

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin x}{1+\sin x} dx$$

$$f(x) = -\int \frac{\sin x(1-\sin x)}{1^2 - \sin^2 x} dx = -\int \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$f(x) = -\int (\sec x \tan x - \tan^2 x) dx$$

$$f(x) = -\int \{\sec x \tan x - (\sec^2 x - 1)\}dx$$

$$f(x) = -(\sec x - \tan x + x)$$

$$\boxed{f(x) = \tan x - \sec x - x} \quad \text{——} \; (A)$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos x}{1+\sin x} dx$$

$$\boxed{g(x) = \log(1+\sin x)} \quad \text{——} \; (B)$$

Using (A) & (B), in (3), we get

$$P.I = -\cos x(\sec x - \tan x + x) + \sin x \log(1+\sin x)$$

$$\boxed{\therefore P.I = -1 + \sin x - x\cos x + \sin x \log(1+\sin x)}$$

$\therefore$ Required solution $y = C.F + P.I$

$$\boxed{y = c_1\cos x + c_2 \sin x - 1 + \sin x - x\cos x + \sin x \log(1+\sin x)}$$

which is required solution.

Ques: 6(d)) By using Laplace Transform Method Solve $(D^3-D)y = 2\cos t$ ; $y=3$, $Dy=2$, $D^2y=1$ ; when $t=0$.

Solution:-

given ; $(D^3-D)y = 2\cos t$

Taking laplace transform of both sides of the equation, we have

$$L\{y'''\} - L\{y'\} = 2L\{\cos t\}.$$

$$\Rightarrow p^3 L(y) - p^2 y(0) - p y'(0) - y''(0) - [pL\{y\} - y(0)] = \frac{2p}{p^2+1}$$

$$\Rightarrow (p^3-p)L(y) - 3p^2 - 2p - 1 + 3 = \frac{2p}{p^2+1}$$

$$\Rightarrow p(p^2-1)L(y) = \frac{2p}{p^2+1} + 3p^2 + 2p - 2$$

$$L(y) = \frac{2}{(p^2-1)(p^2+1)} + \frac{3p^2+2p-2}{p(p^2-1)}$$

$$L(y) = \frac{1}{p^2-1} - \frac{1}{p^2+1} + \frac{2}{p} + \frac{3}{2(p-1)} - \frac{1}{2(p+1)}$$

$$\therefore y = L^{-1}\left\{\frac{1}{p^2-1}\right\} - L^{-1}\left\{\frac{1}{p^2+1}\right\} + 2L^{-1}\left(\frac{1}{p}\right) + \frac{3}{2}L^{-1}\left\{\frac{1}{p-1}\right\} - \frac{1}{2}\left\{\frac{1}{p+1}\right\}$$

$$y = \sinh t - \sin t + 2 + \frac{3}{2}e^{t} - \frac{1}{2}e^{-t}$$

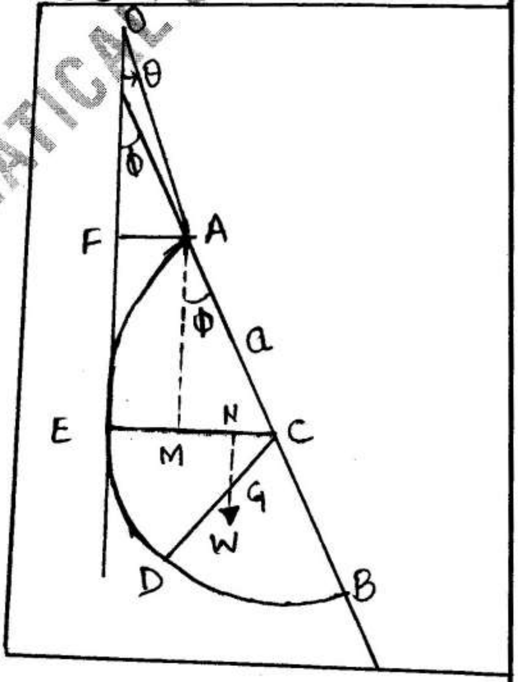
$$y = \sinh t - \sin t + 2 + \frac{e^t + e^{-t}}{2} + 2 \cdot \frac{e^t - e^{-t}}{2}$$

$$y = \sinh t - \sin t + 2 + \cosh t + 2 \sinh t$$

$$\boxed{y = 3\sinh t + \cosh t - \sin t + 2}$$

which is required solution.

Ques: 7(a)) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If $\theta, \phi$ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$?




Solution:

O is a fixed point in the wall to which one end of the string has been attached. Let $l$ be the length of string AO and '$a$' be the radius of the hemisphere the centre of whose base is C.

The weight 'W' of the hemisphere acts at its centre of Gravity G which lies on the symmetrical radius CD and is such that

$$\boxed{CG = \frac{3}{8}a}$$

The hemisphere touches the wall at E. We have $\angle OEC = 90^\circ$, so that EC is horizontal.

The string AO makes an angle '$\theta$' with the wall and the base BA of the hemisphere makes an angle $\phi$ with the wall.

The depth of G below O $= OF + AM + NG$

$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note that $\angle NCG = 90° - \angle ACM = 90° - (90° - \phi) = \phi$]

Given, the system a small displacement in which '$\theta$' changes to $\theta + \delta\theta$, $\phi$ changes to $\phi + \delta\phi$, the point 'O' remains fixed, the length of the string AO does not change so that the work done by its tension is zero and the point G is slightly displaced. The $\angle OEC$ remains $90°$.

The only force that contributes to the equation of virtual work is the weight 'W' of the hemisphere acting at G whose depth below the fixed point 'O' has been found above.

The equation of virtual work is $\rightarrow$

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0$$

or $\Rightarrow$ $-l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$

or $\Rightarrow$ $l\sin\theta\,\delta\theta + a\sin\phi\,\delta\phi - \frac{3}{8}a\cos\phi\,\delta\phi = 0$

$\Rightarrow$ $l\sin\theta\,\delta\theta + a\sin\phi\,\delta\phi = \frac{3}{8}a\cos\phi\,\delta\phi$

$$\boxed{l\sin\theta\,\delta\theta = a\left[\frac{3}{8}\cos\phi - \sin\phi\right]\delta\phi} \quad \text{——} (1)$$

From figure, $EC = a$,

Also $EC = EM + MC = FA + MC$

$$a = l\sin\theta + a\sin\phi$$

Differentiating; $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

$$\Rightarrow \boxed{-l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi} \quad \text{———} \quad (2)$$

Dividing (1) by (2), we get

$$-\tan\theta = \frac{3}{8} - \tan\phi$$

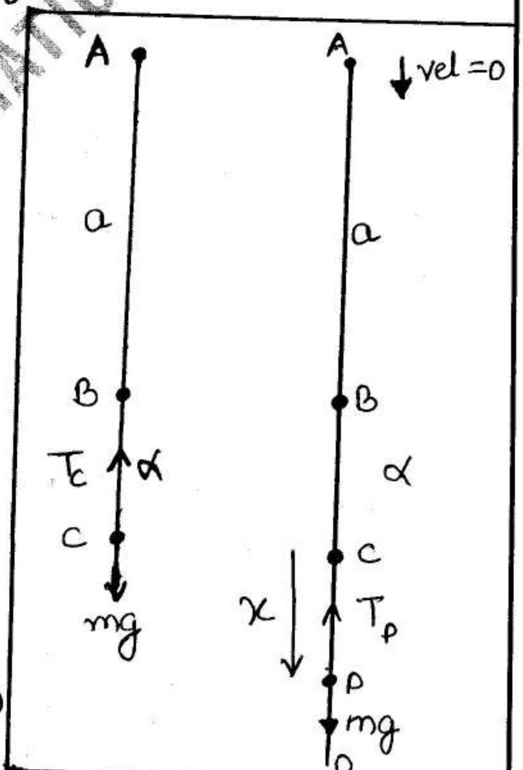
$$\Rightarrow \boxed{\tan\phi = \frac{3}{8} + \tan\theta}$$

which is required solution

Ques: 7(b)) One end of a light elastic string of a natural length 'a' and modulus of elasticity 2mg is attached to a fixed point and the other end to a particle of mass 'm'. The particle initially held at rest at 'A', is let fall. Show that the greatest extension of the string is $\frac{1}{2}a(1+\sqrt{5})$ during the motion and show that the particle will reach back 'A' again after a time

$$(\pi + 2 - \tan^{-1} 2)\sqrt{(2a/g)}.$$

Solution:-




AB = a is the natural length of an elastic string whose one end is fixed at 'A'. Let 'C' be the position of equilibrium of a particle of mass 'm' attached to the other end of the string and BC = d. In the position of equilibrium of the particle at 'C', the tension $T_C = \lambda \frac{d}{a} = 2mg\frac{d}{a}$ in the string AC balances the weight mg of the particle.

$\therefore \boxed{mg = 2mg(d/a)}$ or $\boxed{d = a/2}$ —— (1)

Now, the particle is dropped at rest from A. It falls the distance AB freely under gravity. If $v_1$ be the velocity gained at B, we have $v_1 = \sqrt{(2ga)}$ in the downward direction. When the particle falls below B, the string begins to extend beyond its natural length and the tension begins to operate.

During the fall from B to C the velocity of the particle goes on increasing as the tension remains less than the weight of the particle and when the particle begins to fall below C, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at D.

During the motion of the particle below B, let P be its position after any time $t$, where $CP = x$. If $T_P$ be the tension in the string AP, we have $T_P = \lambda \frac{d+x}{a} = 2mg \frac{\frac{1}{2}a + x}{a}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is–

$$m\frac{d^2x}{dt^2} = mg - T_P = mg - 2mg\frac{\frac{1}{2}a + x}{a} = -\frac{2mg}{a}x$$

$$\boxed{\therefore \frac{d^2x}{dt^2} = -\frac{2g}{a}x} \quad \text{———} \quad (2)$$

which is the equation of a S.H.M. with centre at the point C and amplitude CD.

Multiplying ② by $2(dx/dt)$ and integrating w.r.t 't', we get

$$\boxed{\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}.x^2 + k}\text{ ; where ; } k = \text{constant.}$$

At the point B, the velocity

$$= \frac{dx}{dt} = \sqrt{(2ga)} \text{ and } x = -d = -\frac{a}{2}$$

$$\therefore k = 2ga + \frac{2g}{a}.\frac{a^2}{4} = 2ga + \frac{2ga}{4} = \frac{5ag}{2}$$

$\therefore$ We have $\boxed{\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \frac{5ag}{2}}$ ——— ③

The equation (3) gives the velocity of the particle at any point between B and D. At D,

$$x = CD \text{ and } \frac{dx}{dt} = 0$$

So putting $\frac{dx}{dt} = 0$ in ③, we have

$$0 = -\frac{2g}{a}x^2 + \frac{5ag}{2}$$

$$\Rightarrow x^2 = \frac{5a^2}{4}$$

(or) $\Rightarrow$ $\boxed{x = \frac{a}{2}\sqrt{5} = CD}$

∴ the greatest extension of the string

$$= BC + CD = \frac{1}{2}a + \frac{1}{2}a\sqrt{5} = \frac{1}{2}a(1+\sqrt{5})$$

Now, from (3); we have

$$\boxed{\left(\frac{dx}{dt}\right)^2 = \frac{2g}{a}\left[\frac{5}{4}a^2 - x^2\right]}$$

$$\therefore \frac{dx}{dt} = \sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left[\frac{5}{4}a^2 - x^2\right]}$$

, the +ve sign has been taken because the particle is moving in the direction of $x$ increasing.

Separating the variables, we have

$$\boxed{dt = \sqrt{\frac{a}{2g}}\frac{dx}{\sqrt{\frac{5}{4}a^2 - x^2}}}$$

If $t_1$ is the time from B to D, then

$$\int_0^{t_1} dt = \sqrt{\left(\frac{a}{2g}\right)}\int_{-a/2}^{(a\sqrt{5}/2)} \frac{dx}{\sqrt{\frac{5}{4}a^2 - x^2}}$$

or $$t_1 = \sqrt{\frac{a}{2g}}\left[\sin^{-1}\left\{\frac{x}{a\sqrt{5}/2}\right\}\right]_{-a/2}^{a\sqrt{5}/2}$$

$$t_1 = \sqrt{\frac{a}{2g}}\left[\sin^{-1}1 + \sin^{-1}\frac{1}{\sqrt{5}}\right]$$

$$\boxed{t_1 = \sqrt{\frac{a}{2g}}\left(\frac{\pi}{2} + \tan^{-1}\frac{1}{2}\right)}$$

$$= \sqrt{\left(\frac{a}{2g}\right)\left(\frac{\pi}{2} + \cot^{-1} 2\right)} = \sqrt{\left(\frac{a}{2g}\right)\left(\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1} 2\right)}$$

$$= \sqrt{\left(\frac{a}{2g}\right)\left(\pi - \tan^{-1} 2\right)}$$

And, if $t_2$ is the time from A to B, (while falling freely under gravity), then

$$a = 0 \cdot t_2 + \frac{1}{2} g t_2^2$$

$$\Rightarrow \boxed{t_2 = \sqrt{\frac{2a}{g}}}$$

$\therefore$ the total time to return back to A = 2 (time from A to D).

$$= 2(t_2 + t_1)$$

$$= 2\left[\sqrt{\left(\frac{a}{2g}\right)}\left(\pi - \tan^{-1} 2\right) + \sqrt{\frac{2a}{g}}\right]$$

$\therefore$ The total time to return back to A = 2

$$= \sqrt{\left(\frac{2a}{g}\right)}\left(\pi - \tan^{-1} 2 + 2\right)$$

This proves the required result

Ques: 7(c)) A particle moves with a central accelaration which varies inversely as the cube of the distance. If it be projected from an apse at a distance 'a' from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius 'a', show that the equation to its path is

$$r\cos(\theta/\sqrt{2}) = a.$$

Solution: Here the central accelaration varies inversely as the cube of the distance i.e.

$P = \mu/r^3 = \mu u^3$ ; where $\mu$ is a constant

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

$$\Rightarrow V^2 = \frac{\mu}{a^2} \Rightarrow \boxed{V = \frac{\sqrt{\mu}}{a}} \text{ ——— (1)}$$

∴ the velocity of projection

$$\boxed{V_1 = \sqrt{2}\,V = \sqrt{2\mu/a^2}}$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$\boxed{v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A} \text{ ——— (2)}$$

where A is constant.

But initially when $r=a$; i.e $u=1/a$, $\frac{du}{d\theta}=0$

(at an apse), and $\boxed{v=v_1=\sqrt{2\mu/a^2}}$

$\therefore$ from (2), we have

$$\frac{2\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}+A$$

$\therefore$ $h^2=2\mu$ and $A=\mu/a^2$

Substituting the values of $h^2$ and A in (2), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$$

$$\sqrt{2}\,a\frac{du}{d\theta}=\sqrt{1-a^2u^2}$$

$$\Rightarrow \boxed{\frac{d\theta}{\sqrt{2}}=\frac{a\,du}{\sqrt{1-a^2u^2}}}$$

Integrating; $\boxed{\frac{\theta}{\sqrt{2}}+B=\sin^{-1}(au)}$

where, B is constant.

But initially,

when $u=1/a$, $\theta=0$

$$\boxed{\therefore B=\sin^{-1}1=\pi/2}$$

$\therefore$ $\theta/\sqrt{2}+\pi/2=\frac{1}{2}\sin^{-1}(au)$

or $au=a/r=\sin\left\{\frac{1}{2}\pi+\theta/\sqrt{2}\right\}$

$\Rightarrow \boxed{a=r\cos(\theta/\sqrt{2})}$ which is required equation of the path.

Ques) 8 (a) ) Find (i) the curvature $\kappa$
(ii) the torsion $\tau$ for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$

Solution :

Given that $\vec{r} = \left(t - t^3/3\right)\hat{i} + t^2\hat{j} + \left(t + t^3/3\right)\hat{k}$

$$\frac{d\vec{r}}{dt} = (1-t^2)\,\hat{i} + 2t\,\hat{j} + (1+t^2)\,\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} \Rightarrow -2t\,\hat{i} + 2\hat{j} + 2t\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2}$$

$$= \sqrt{1 + t^4 \cancel{-2t^2} + 4t^2 + 1 + t^4 \cancel{+2t^2}}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{2 + 2t^4 + 4t^2}$$

Now; $\left[\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right] = \begin{vmatrix} i & j & k \\ 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \end{vmatrix}$

$$= i\,(4t^2 - 2 - 2t^2) - j\,(2t \cancel{-2t^3} + 2t \cancel{+2t^3}) + \hat{k}\,(2 - 2t^2 + 4t^2)$$

$$= 2(t^2-1)\,\hat{i} - \hat{j}\,4t + \hat{k}\,2(t^2+1)$$

$$= 2(t^2-1)\,\hat{i} - 4t\,\hat{j} + 2(t^2+1)\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{4(t^2-1)^2 + 16t^2 + 4(t^2+1)^2}$$

$$= 2\sqrt{(t^2-1)^2 + 4t^2 + (t^2+1)^2}$$

$$= 2\sqrt{2 + 2t^4 + 4t^2}$$

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = 2\left|\frac{d\vec{r}}{dt}\right|$$

$$\frac{d^3\vec{r}}{dt^3} = -2\hat{i}+0\hat{j}+2\hat{k}$$

$$\left[\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^3\vec{r}}{dt^3}\right] = \begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -2t & 2 & 2t \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 4\begin{vmatrix} 1-t^2 & 2t & 1+t^2 \\ -t & 1 & t \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 4\left[(1-t^2)(1)-2t(-t+t)+(1+t^2)(1)\right]$$

$$= 4\left[1-\cancel{t^2}+1+\cancel{t^2}\right] = 8$$

$$\text{Curvature} = K = \frac{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{2\cancel{\left|\frac{d\vec{r}}{dt}\right|}}{\left|\frac{d\vec{r}}{dt}\right|^{\cancel{3}\,2}}$$

$$K = \frac{2}{\left(\sqrt{2+2t^4+4t^2}\right)^2} = \frac{2}{2t^4+4t^2+2}$$

$$\boxed{K = \frac{1}{t^4+2t^2+1} = \frac{1}{(t^2+1)^2}}$$

$$\text{Torsion }(\tau) = \frac{\left[\frac{dr}{dt}\cdot\frac{d^2r}{dt^2}\cdot\frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|^2} = \frac{8}{4(2t^4+4t^2+2)}$$

$$\tau = \frac{\cancel{8}}{\cancel{8}(t^4+2t^2+1)} \Rightarrow \boxed{\tau = \frac{1}{(t^2+1)^2}}$$

Hence; $\boxed{\kappa = \tau = \frac{1}{(t^2+1)^2}}$

is the required solution

Ques: 8 (b) (i) In what direction the directional derivative of $\phi = x^2y^2z^2$ from $(1,1,2)$ will be maximum and what is its magnitude? Also find a unit normal vector to the surface $x^2y^2z = 2$ at the point $(1,1,2)$.

Solution:

We know that the directional derivative of $\phi$ at the point $(x,y,z)$ is maximum in the direction of the normal to the surface $\phi$ = constant ; i.e in the direction of the vector grad $\phi$.

Now; $\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$\text{grad}\,\phi = 2xy^2z\,\hat{i} + 2x^2yz\,\hat{j} + x^2y^2\hat{k}$$

$$\text{grad}\,\phi = 4\hat{i} + 4\hat{j} + \hat{k}\text{, at point } (1,1,2)$$

Hence, the directional derivative of $\phi$ at the point $(1,1,2)$ will be maximum in the direction of the vector $4\hat{i}+4\hat{j}+\hat{k}$.

Also, the magnitude of this maximum directional derivative = modulus of grad $\phi$ at $(1,1,2)$

$$= |4\hat{i}+4\hat{j}+\hat{k}| = \sqrt{16+16+1} = \sqrt{33}.$$

The unit vector along the normal to the surface $x^2y^2z = 2$ at point $(1,1,2)$.

$$= \frac{\text{grad } \phi}{|\text{grad } \phi|}, \text{ at } (1,1,2).$$

$$= \frac{4i+4j+k}{|4i+4j+k|} = \frac{4i+4j+k}{\sqrt{16+16+1}}.$$

The unit vector along the normal to the surface $x^2y^2z = 2$ at point $(1,1,2)$

$$= \frac{4i+4j+k}{\sqrt{33}}.$$

Ques) 8 (b) ii) find div F and curl F, where

$$F = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz).$$

Solution:

We have ; $F = \text{grad}\,(x^3+y^3+z^3-3xyz)$

$$= i\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + j\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \hat{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\boxed{F = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}}$$

Now ; div $F = \nabla \cdot F$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy).$$

$$= 6x + 6y + 6z = 6(x+y+z).$$

$$\boxed{\therefore \text{div} F = \nabla \cdot F = 6(x+y+z)}$$

Now, Curl $F = \nabla \times F$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}.$$

$$\text{Cur} F = \hat{i}\left[\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right] + \hat{j}\left[\frac{\partial}{\partial z}(3x^2-3yz) - \frac{\partial}{\partial x}(3z^2-3xy)\right] + \hat{k}\left[\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right]$$

$$\text{Cur} F = \hat{i}[-3x+3x] + \hat{j}[-3y+3y] + \hat{k}[-3z+3z]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

$$\boxed{\text{Curl} F = \nabla \times F = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0}$$ Ans.

Ques: 8 (c) (i) Show that curl $\vec{a}\phi(\vec{r}) = \frac{1}{r}\phi'(r)\vec{r}\times\vec{a}$, where $\vec{a}$ is a constant vector.

Solution:-

we know that;

$$\text{curl}(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi\,\text{curl}\,\vec{A}$$

replacing $\phi$ by $\phi(\vec{r})$ and $\vec{A}$ by $\vec{a}$

Now, we have;

$$\text{curl}[\vec{a}\phi(\vec{r})] = [\nabla\phi(\vec{r})]\times\vec{a} + \phi(\vec{r})\,\text{curl}\,\vec{a}$$

$$= [\phi'(\vec{r})\,\nabla\vec{r}]\times\vec{a} + \phi(\vec{r})\cdot 0.$$

$[\because \vec{a}$ is constant vector $\Rightarrow \text{curl}\,\vec{a} = 0]$

$$= \left[\phi'(r)\cdot\frac{1}{r}\vec{r}\right]\times\vec{a}$$

$$\boxed{\text{Curl}[\vec{a}\phi(\vec{r})] = \frac{1}{r}\phi'(r)\vec{r}\times\vec{a}.}$$

which is required result.

Ques: 8 (c)(ii) Prove that curl ($\phi$ grad $\phi$) = 0.

Solution:- We know that

$$\text{Curl}(\phi A) = \text{grad}\,\phi \times A + \phi\,\text{Curl}\,A$$

Putting grad $\phi$ in place of A, we get

$$\text{curl}(\phi\,\text{grad}\,\phi) = \text{grad}\,\phi \times \text{grad}\,\phi + \phi\,\text{curl}\,\text{grad}\,\phi$$
$$= 0 + \phi \cdot 0$$

$$\boxed{\text{Curl}(\phi\,\text{grad}\,\phi) = 0}$$

Here; grad $\phi$ × grad $\phi$ = 0

since, it is the cross product of two equal vectors.

Also; curl grad $\phi$ = 0

$$\boxed{\therefore\ \text{Curl}[\phi\,\text{grad}\,\phi] = 0}$$

Ques: 8(d) Verify Stoke's theorem for

$F = y\hat{i} + (x - 2xz)j - xy\hat{k}$, where 'S' is the surface of sphere; $x^2+y^2+z^2=a^2$, above the xy-plane.

Solution:-

Given; $F = y\hat{i} + (x-2xz)j - xy\hat{k}$

and the surface of sphere $\Rightarrow x^2+y^2+z^2=a^2$

above x-y plane.

By stoke's Theorem;

$$\boxed{\iint (\text{curl } F)\cdot\hat{n}\, ds = \int F\cdot dr}$$

Now;

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x-2xz)\right] - \hat{j}\left(\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}\cdot y\right] + \hat{k}\left[\frac{\partial}{\partial x}(x-2xz) - \frac{\partial}{\partial y}\cdot y\right]$$

$$= i\left[-x-(-2x)\right] - j\left[-y-0\right] + \hat{k}\left[\cancel{1} - 2z - \cancel{1}\right]$$

$$\boxed{\text{Curl } F = \hat{i} + y\hat{j} - 2z\hat{k}}$$

$$\Rightarrow \hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{\cancel{2}(x\hat{i}+y\hat{j}+z\hat{k})}{\cancel{2}\, a}$$

$[\because x^2+y^2+z^2=a^2]$

$$\boxed{\therefore\ \hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}} \quad \& \quad \frac{dx\,dy}{z/a} = ds$$

as xy plane

$\Rightarrow \iint (\text{curl } F)\cdot\hat{n}\, ds \Rightarrow$

$$= \iint \frac{[x\hat{i}+y\hat{j}-2z\hat{k}]\cdot[xi+yj+z\hat{k}]}{\cancel{a}} \cdot \frac{dx\,dy}{z/\cancel{a}}$$

$$\Rightarrow \iint \frac{[x^2+y^2-2z^2]}{z}\, dx\,dy.$$

$$= \iint \frac{x^2+y^2-2(a^2-x^2-y^2)}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$$

converting it into polar form

$x = a\cos\theta$ , $y = a\sin\theta$ ; $x^2+y^2 = r^2$

$$= \int_0^{2\pi}\int_0^a \frac{3r^2-2a^2}{\sqrt{a^2-r^2}} \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\int r\cdot\frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\right]_0^a d\theta$$

$$= \int_0^{2\pi}\left[\int \frac{3r^3-2a^2r}{\sqrt{a^2-r^2}}\,dr\right]_0^a d\theta$$

$= 0$ - [$\because$ it is an odd function]

$$\boxed{\therefore \iint (\text{curl } F)\cdot\hat{n}\, ds = 0}$$

Now;

$$\int F\cdot dr = \int \left[ y\hat{i} + (x - 2xz)\hat{j} - xy\,\hat{k} \right] \cdot [dx\,\hat{i} + dy\,\hat{j} + dz)\,\hat{k}].$$

$$\int F\cdot dr = \int [y\,dx + (x - 2xz)dy] \quad \left[ \because \text{In } xy \text{ plane } z = 0 \;\therefore dz = 0 \right]$$

$$= \int y\,dx + x\,dy.$$

$$= \int d(xy) = 0$$

$$\boxed{\therefore \int F\cdot dr = 0}$$

$$\boxed{\therefore \int F\cdot dr = \iint (\text{curl} F)\cdot\hat{n}\cdot ds = 0}$$

Hence, Stoke's theorem satisfied.