Date : 21/07/2019

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

Keep practising

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

$\frac{215}{250}$

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-7: IAS(M)/21-JULY.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: YASH MESHRAM

Roll No.: 115

Test Centre: HOME

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			06	
	(c)			08	38
	(d)			08	
	(e)			08	
2	(a)				
	(b)				
	(c)				
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3	(a)				
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4	(a)			16	
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5	(a)			09	
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7	(a)			17	
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			Total Marks		

$\frac{215}{250}$

SECTION – A

1. (a) (i) The rank of a product of two matrices cannot exceed the rank of either matrix.

(ii) Prove : The zero vector $\mathbf{0} = (0, 0, \ldots\ldots, 0)$ is a solution (the zero solution) of any homogeneous system $AX = 0$. **[10]**

i> $\rho(A) = r_1$ and $\rho(B) = r_2$ and $\rho(AB) = r$
where A's order is $m \times n$ & B' order $n \times l$.
We have a matrix P such that $PA = \begin{bmatrix} C \\ 0 \end{bmatrix}$ and $|P| \neq 0$

where C's order is $r_1 \times n$ and 0 is zero matrix of order $(m-r_1)\times n$

$\therefore PAB = \begin{bmatrix} C \\ 0 \end{bmatrix} B \Rightarrow \rho(PAB) = \rho(AB) = r$

As matrix C has only r_1 rows which are non-zero,
Rank of matrix $\begin{bmatrix} C \\ 0 \end{bmatrix} B \leq r_1 \Rightarrow r \leq r_1 \Rightarrow \rho(AB) \leq \rho(A)$

We have $\rho(AB) = [\rho(AB)^T] = \rho[B^T A^T] \leq \rho(B^T) = \rho(B) = r_2$

$\therefore r_1 \leq r_2$

i.e., $\rho(AB) \leq \rho(B)$

$\therefore \rho(AB) \leq \rho(A)$ & $\rho(AB) \leq \rho(B)$

Hence, rank of product of 2 matrices cannot exceed the rank of either matrix.

ii> The homogeneous system $AX = 0$ can be represented as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

[Here, A is a matrix of order $m\times n$]

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If $X = (x_1, x_2 \ldots\ldots x_n)^T = 0 = (0, 0, \ldots\ldots 0)$,
then we clearly see $AX = 0$ irrespective of the matrix A.

$\therefore$ The zero vector $\vec{0} = (0, 0, \ldots\ldots 0)$ is a solution of any homogeneous system $AX = 0$.

1. (b) Define a basis of a vector space over a field F and the dimension of vector space. What is the dimension of complex vector space over the field of complex numbers ? Give an example of a vector space the dimension of which is not finite. [10]

Consider $S = \{x_1, x_2 \ldots\ldots\ldots x_n\} \subseteq V$ having linearly independent vectors $x_1, x_2 \ldots\ldots x_n$.

If $L(S) = V$, i.e., any vector in V can be written as a linear combination of elements of S, then S is called Basis of V(F) where V(F) is a vectorspace.

Definition of dimension of vector space ??

Dimension of complex vector space over the field of complex numbers is 1 as any field as a vectorspace over itself is one-dimensional.

Example of a vectorspace the dimension of which is not finite:-

Vectorspace of all polynomials in x,

Basis $= \{1, x, x^2 \ldots \ldots x^n \ldots \ldots\}$

$\therefore$ Dimension of vectorspace is infinite

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1. (c) If $z = (x + y) + (x + y)\,\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

[10]

We have $z = x+y + x+y\,\phi\left(\frac{y}{x}\right) = (x+y)\left[1+\phi\left(\frac{y}{x}\right)\right]$ — ①

$\therefore \frac{\partial z}{\partial x} = 1+\phi\left(\frac{y}{x}\right) + (x+y)\left[\phi'\left(\frac{y}{x}\right)\right]\left(-\frac{y}{x^2}\right)$ — ②

and $\frac{\partial z}{\partial y} = 1+\phi\left(\frac{y}{x}\right) + (x+y)\left[\phi'\left(\frac{y}{x}\right)\right]\frac{1}{x}$ — ③

From ②, $\frac{\partial^2 z}{\partial x^2} = -\frac{y}{x^2}\phi'\left(\frac{y}{x}\right) + \left[-\frac{y}{x^2}\phi'\left(\frac{y}{x}\right)\right] + (x+y)\left[\phi''\left(\frac{y}{x}\right)\right]\frac{y^2}{x^4}$
$+ (x+y)\left[\phi'\left(\frac{y}{x}\right)\right]\frac{2y}{x^3}$ — ④

From ③, $\frac{\partial^2 z}{\partial y^2} = \frac{1}{x}\phi'\left(\frac{y}{x}\right) + \frac{1}{x}\phi'\left(\frac{y}{x}\right) + \frac{1}{x^2}(x+y)\phi''\left(\frac{y}{x}\right)$ — ⑤

From ②, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[1+\phi\left(\frac{y}{x}\right) + (x+y)\left(-\frac{y}{x^2}\right)\phi'\left(\frac{y}{x}\right)\right]$

$= \frac{1}{x}\phi'\left(\frac{y}{x}\right) - \frac{y}{x^2}\phi'\left(\frac{y}{x}\right) - \frac{(x+y)}{x^2}\phi'\left(\frac{y}{x}\right) - \frac{y(x+y)}{x^3}\phi''\left(\frac{y}{x}\right)$ — ⑥

From ③, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial x}\left[1+\phi\left(\frac{y}{x}\right)+\frac{1}{x}(x+y)\,\phi'\left(\frac{y}{x}\right)\right]$

$$= -\frac{y}{x^2}\phi'\left(\frac{y}{x}\right) - \frac{(x+y)}{x^2}\phi'\left(\frac{y}{x}\right) + \frac{1}{x}\phi'\left(\frac{y}{x}\right) - \frac{y(x+y)}{x^3}\phi''\left(\frac{y}{x}\right) \quad — ⑦$$

We need to prove :- $x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$

consider $x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) - y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$

$$= \Big[-\frac{y}{x}\phi'\left(\frac{y}{x}\right) - \frac{y}{x}\phi'\left(\frac{y}{x}\right) + \frac{y^2(x+y)}{x^3}\phi''\left(\frac{y}{x}\right) + \frac{2y(x+y)}{x^2}\phi'\left(\frac{y}{x}\right)$$
$$- \phi'\left(\frac{y}{x}\right) + \frac{y}{x}\phi'\left(\frac{y}{x}\right) + \frac{(x+y)}{x}\phi'\left(\frac{y}{x}\right) + \frac{y(x+y)}{x^2}\phi''\left(\frac{y}{x}\right)\Big]$$
$$- \Big[\frac{y}{x}\phi'\left(\frac{y}{x}\right) + \frac{y}{x}\phi'\left(\frac{y}{x}\right) + \frac{y}{x^2}(x+y)\phi''\left(\frac{y}{x}\right) + \frac{y^2}{x^2}\phi'\left(\frac{y}{x}\right)$$
$$+ \frac{y(x+y)}{x^2}\phi'\left(\frac{y}{x}\right) - \frac{y}{x}\phi'\left(\frac{y}{x}\right) + \frac{y^2}{x^3}(x+y)\phi''\left(\frac{y}{x}\right)\Big]$$

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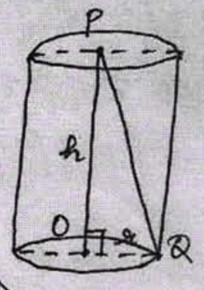
$= 0$

$$\therefore \boxed{x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)}$$

1. (d) Show that the right circular cylinder of the given surface and maximum volume is such that its height is equal to the diameter of its base. **[10]**

Let the height of cylinder $= OP = h$

and radius $= OQ = r$



Surface area of cylinder $= S$ (say)

$\therefore \quad S = 2\pi r(r+h)$ — ①

Hence, Volume of cylinder $= V = \pi r^2 h$ — ②

From ① & ②, $V = \pi r^2\left[\frac{S}{2\pi r} - r\right]$

For maximum volume, $\frac{dV}{dr} = \frac{d}{dr}\left[\frac{Sr}{2} - \pi r^3\right] = 0$

$\therefore \frac{S}{2} - 3\pi r^2 = 0 \Rightarrow r^2 = \frac{S}{6\pi} \Rightarrow r = \left[\frac{S}{6\pi}\right]^{1/2}$ — ③

From ① & ③, $S = 2\pi\left[\frac{S}{6\pi} + \left(\frac{S}{6\pi}\right)^{1/2} h\right]$ {Discard negative value}

$\therefore \frac{2S}{3} = 2\pi\left(\frac{S}{6\pi}\right)^{1/2} h \Rightarrow h = \frac{2}{3}S \times \frac{(6\pi)^{1/2}}{2\pi S^{1/2}}$

$\therefore\ h = \frac{S}{3} \times \frac{6^{1/2}\pi^{1/2}}{\pi\, S^{1/2}} = \frac{S^{1/2} \times 2^{1/2} \times 3^{1/2}}{3^{1/2} \times 3^{1/2} \times \pi^{1/2}} = \left[\frac{2S}{3\pi}\right]^{1/2}$

i.e., $h = \left[\frac{4S}{6\pi}\right]^{1/2} = 2\left[\frac{S}{6\pi}\right]^{1/2} = 2r =$ diameter

Hence, height is equal to the diameter of its base.

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1. (e) Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circles. **[10]**

centre of sphere $x^2+y^2+z^2-x+z-2=0$ is $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ and radius $= \sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{1}{2}\right)^2+2} = \sqrt{\frac{5}{2}}$

Distance of centre from plane $= \frac{\left|\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4\right|}{\sqrt{1^2+2^2+(-1)^2}} = \frac{3}{\sqrt{6}}$

$\therefore$ Radius of circle $= \left[\left(\sqrt{\frac{5}{2}}\right)^2 - \left(\frac{3}{\sqrt{6}}\right)^2\right]^{1/2} = \left[\frac{5}{2} - \frac{3}{2}\right]^{1/2} = 1$

$\therefore$ $\boxed{\text{Radius of circle is unity}}$

Also, equation of sphere passing through circle is

$S + \lambda P = 0$, i.e., $x^2+y^2+z^2-x+z-2+\lambda(x+2y-z-4)=0$

$\therefore\ x^2+y^2+z^2+(\lambda-1)x+2\lambda y-(\lambda-1)z-(2+4\lambda) = 0$

As the circle is a great circle, radius of sphere is the radius of circle.

$\therefore \frac{1}{4}(\lambda-1)^2 + \frac{(2\lambda)^2}{4} + \frac{(\lambda-1)^2}{4} + 2 + 4\lambda = 1^2 = 1$

$\therefore (\lambda-1)^2 + (2\lambda)^2 + (\lambda-1)^2 + 8 + 16\lambda = 4$

$\therefore 6\lambda^2 + 12\lambda + 6 = 0$

$\therefore \lambda^2 + 2\lambda + 1 = 0$

$\therefore \lambda = -1$

$\therefore$ Equation of sphere $\equiv x^2 + y^2 + z^2 - 2x - \cancel{2y} + 2 = 0$ $+2z$

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2. (a) Let W be the solution space of the homogeneous system

$x + 2y - 3z + 2s - 4t = 0$

$2x + 4y - 5z + s - 6t = 0$

$5x + 10y - 13z + 4s - 16t = 0$

Find the dimension and a basis for W. **[15]**

4. (a) Let A be an n × n matrix.

(i) If A has n linearly independent eigenvectors it is diagonalizable. The matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $C^{-1}AC$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.

(ii) If A is diagonalizable then it has n linearly independent eigenvectors. **[18]**

i> A has n linearly independent eigenvectors which are $x_1, x_2 \ldots\ldots x_n$ and corresponding eigen values are $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$.

$\Rightarrow AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2 \ldots\ldots AX_n = \lambda_n X_n$. — ①

Assume $C = [x_1, x_2, x_3 \ldots\ldots x_n]$ and $D = \text{diagonal}[\lambda_1, \lambda_2 \ldots\ldots \lambda_n]$ ②

$$\therefore AC = A[x_1, x_2, \ldots\ldots x_n]$$
$$= [AX_1, AX_2, \ldots\ldots AX_n]$$
$$= [\lambda_1 X_1, \lambda_2 X_2, \ldots\ldots \lambda_n X_n] \quad \rightarrow \text{From ①}$$
$$= [x_1, x_2, \ldots\ldots x_n]\ \text{diagonal}[\lambda_1, \lambda_2 \ldots\ldots \lambda_n]$$

$\therefore AC = CD \quad \rightarrow$ From ②

$\therefore \boxed{D = C^{-1}AC} \Rightarrow$ A is similar to D and hence, A is diagonalizable.

Here, C consists of columns of n linearly independent eigenvectors and D consists of eigenvalues of corresponding eigenvectors in diagonal form.

ii> We have a matrix A which is diagonalizable. Hence, A should be similar to the diagonal matrix where $D = \text{diagonal}[\lambda_1, \lambda_2 \ldots\ldots \lambda_n]$. — ①

From ①, we have $C^{-1}AC = D$ where C is an $n \times n$ invertible matrix and $C = [x_1, x_2 \ldots\ldots x_n]$ — ②

From ②, $AC = CD$, i.e.,

$$A[x_1, x_2, \ldots\ldots x_n] = [x_1, x_2 \ldots\ldots x_n]\ \text{diagonal}[\lambda_1, \lambda_2 \ldots \lambda_n]$$

$$\therefore [AX_1, AX_2 \ldots\ldots AX_n] = [\lambda_1 X_1, \lambda_2 X_2 \ldots\ldots, \lambda_n X_n]$$ ③

From ③, $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2 \ldots\ldots\ldots AX_n = \lambda_n X_n$

$\Rightarrow X_1, X_2 \ldots\ldots X_n$ are eigen vectors of A corresponding to $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ respectively.

$X_1, X_2 \ldots\ldots X_n$ vectors are linearly independent as C is invertible.

$\therefore$ If A is diagonalizable, then it has 'n' linearly independent eigenvectors.

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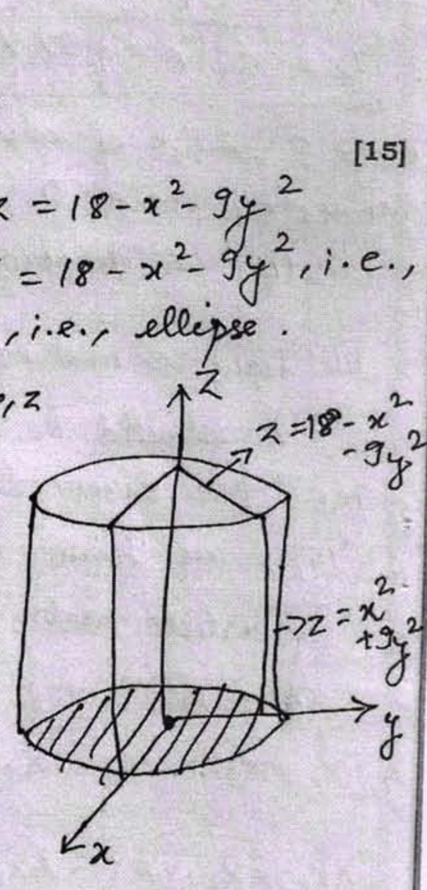
4. (b) The volume bounded by the elliptic paraboloids
$z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$. [15]

Surfaces $S_1 : z = x^2 + 9y^2$ and $S_2 :- z = 18 - x^2 - 9y^2$ intersect each other where $x^2 + 9y^2 = 18 - x^2 - 9y^2$, i.e., $x^2 + 9y^2 = 9$, i.e., $\frac{x^2}{9} + \frac{y^2}{1} = 1$, i.e., ellipse.

To calculate volume, limits of x, y, z are :-



$x^2 + 9y^2 \le z \le 18 - x^2 - 9y^2$,

$-\sqrt{1 - \frac{x^2}{9}} \le y \le \sqrt{1 - \frac{x^2}{9}}$

$-3 \le x \le 3$

$\therefore$ Volume bounded $= \iiint dz\, dy\, dx$

$= \int_{-3}^{3} \int_{-\sqrt{1-\frac{x^2}{9}}}^{\sqrt{1-\frac{x^2}{9}}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz\, dy\, dx$

$$= \int_{-3}^{3} \int_{-\sqrt{1-\frac{x^2}{9}}}^{\sqrt{1-x^2/9}} [18 - 2x^2 - 18y^2]\, dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{1-\frac{x^2}{9}}}^{\sqrt{1-\frac{x^2}{9}}} 2[9 - x^2 - 9y^2]\, dy\, dx$$

$$= \int_{-3}^{3} 2\left[(9-x^2)y - 3y^3\right]_{-\sqrt{1-x^2/9}}^{\sqrt{1-x^2/9}} dx$$

$$= \int_{-3}^{3} 2\left[\frac{2}{3}(9-x^2)\sqrt{9-x^2} - \frac{3\times 2}{27}(9-x^2)^{3/2}\right] dx$$

$$= \int_{-3}^{3} \frac{8}{9}(9-x^2)^{3/2} dx$$

Let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$\therefore$ Volume bounded $= \int_{-\pi/2}^{\pi/2} \frac{8}{9} \times 27\cos^3\theta \times 3\cos\theta\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} 72\cos^4\theta\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{72}{4}(1+\cos 2\theta)^2 d\theta$$

$$= 27\pi$$

$\therefore$ Volume bounded is $\boxed{27\pi}$ cubic units

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4. (c) Prove that in general two generators of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z = 0$ in P and Q, PQ touches the ellipse $(x^2/a^6) + (y^2/b^6) = c^4/(a^4b^4)$. **[17]**

Generator belonging to λ-system :- $\frac{x}{a} - \frac{z}{c} = \lambda\left(1-\frac{y}{b}\right)$ & $\frac{x}{a}+\frac{z}{c} = \lambda^{-1}\left[1+\frac{y}{b}\right]$

$\therefore \frac{x}{a} + \lambda\frac{y}{b} - \frac{z}{c} = \lambda$ and $\lambda\frac{x}{a} - \frac{y}{b} + \lambda\frac{z}{c} = 1$

If the directional ratios of generator are l_1, m_1, n_1, then

$\frac{1}{a}l_1 + \frac{\lambda}{b}m_1 - \frac{1}{c}n_1 = 0$ and $\frac{\lambda}{a}l_1 - \frac{1}{b}m_1 + \frac{\lambda}{c}n_1 = 0$

$\Rightarrow \frac{l_1/a}{\lambda^2-1} = \frac{m_1/b}{-2\lambda} = \frac{n_1/c}{-(1+\lambda^2)} \Rightarrow \frac{l_1}{a(1-\lambda^2)} = \frac{m_1}{2\lambda b} = \frac{n_1}{c(1+\lambda^2)}$ — (1)

Similarly, generator belonging to μ-system are

$\frac{x}{a} - \frac{z}{c} = \mu\left(1+\frac{y}{b}\right)$ and $\frac{x}{a} + \frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right)$

l_2, m_2, n_2 are the directional ratios of generator.

$\therefore \frac{l_2}{a(\mu^2-1)} = \frac{m_2}{2\mu b} = \frac{n_2}{-c(\mu^2+1)}$ — (2)

Generators cut at right angles — (3)

From (1), (2) & (3), $a^2(1-\lambda^2)(\mu^2-1) + 4\mu\lambda b^2 - c^2(1+\lambda^2)(\mu^2+1) = 0$

If λ = constant, we get quadratic in μ, i.e., 2 roots of μ

i.e., 2 generators of μ system will be perpendicular to one generator of λ.

Let the generators of μ-system meet $z=0$ at $P(a\cos\alpha, b\sin\alpha, 0)$ & $Q(a\cos\beta, b\sin\beta, 0)$.

$\therefore$ Generators of μ system :- $\frac{x-a\cos\alpha}{a\sin\alpha} = \frac{y-b\sin\alpha}{-b\cos\alpha} = \frac{z}{c}$

& $\frac{x-a\cos\beta}{a\sin\beta} = \frac{y-b\sin\beta}{-b\cos\beta} = \frac{z}{c}$

Generator at 'θ' :- $\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{-c}$

Above generator is perpendicular to 2 generators of μ-system

$\therefore a^2\sin\alpha\sin\theta + b^2\cos\alpha\cos\theta - c^2 = 0$ and

$a^2\sin\beta\sin\theta + b^2\cos\beta\cos\theta - c^2 = 0$

$\therefore \frac{a^2\sin\theta}{\cos\alpha - \cos\beta} = \frac{b^2\cos\theta}{\sin\beta - \sin\alpha} = \frac{-c^2}{\sin(\alpha-\beta)}$

$$\therefore \frac{a^2\sin\theta}{2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)} = \frac{b^2\cos\theta}{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)} = \frac{-c^2}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{\alpha}\right)}$$

$$\therefore \frac{a^2\sin\theta}{\sin\left(\frac{\alpha+\beta}{2}\right)} = \frac{b^2\cos\theta}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{c^2}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

Equation of line PQ :- $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right), z=0$

$$\therefore \frac{x}{a}\left(\frac{b^2\cos\theta}{c^2}\right) + \frac{y}{b}\left(\frac{a^2\sin\theta}{c^2}\right) = 1, z=0 \quad (4)$$

Differentiate above equation wrt θ,

$$-\frac{xb^2}{ac^2}\sin\theta + \frac{ya^2}{bc^2}\cos\theta = 0 \quad -(5)$$

From (4) & (5), $\frac{x^2.b^4}{a^2c^4}(\sin^2\theta+\cos^2\theta) + \frac{y^2a^4}{b^2c^4}(\cos^2\theta+\sin^2\theta) = 1^2+0^2$

$$\therefore \frac{x^2b^4}{a^2c^4} + \frac{y^2a^4}{b^2c^4} = 1 \quad , \text{i.e.}, \boxed{\frac{x^2}{a^6}+\frac{y^2}{b^6} = \frac{c^4}{a^4b^4}}$$

which is an ellipse on xy plane

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SECTION – B

5. (a) Solve $(D^4 - 4D^2 - 5)\,y = e^x(x + \cos x)$. **[10]**

Auxiliary equation:- $m^4 - 4m^2 - 5 = 0 \Rightarrow m = \pm i, \pm\sqrt{5}$

$\therefore$ C.F. $= c_1 \cosh\sqrt{5}x + c_2 \sinh\sqrt{5}x + c_3\cos x + c_4 \sin x$

P.I. corresponding to $xe^x = \dfrac{1}{D^4 - 4D^2 - 5} xe^x = e^x \dfrac{1}{(D+1)^4 - 4(D+1)^2 - 5} x$

$$= e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 4D^2 - 8D - 4 - 5} x$$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} x = -\frac{e^x}{8}\left[1 + \frac{D}{2} - \frac{D^2}{4} - \frac{D^3}{2} - \frac{D^4}{8}\right]^{-1} x$$

$$= -\frac{e^x}{8}\left\{1 - \left(\frac{D}{2} - \frac{D^2}{4} - \frac{D^3}{2} - \frac{D^4}{8}\right) + \ldots\ldots\right\} x = -\frac{e^x}{8}\frac{(2x-1)}{2}$$

P.I. corresponding to $e^x\cos x = \dfrac{1}{D^4 - 4D^2 - 5} e^x \cos x = e^x \dfrac{1}{(D+1)^4 - 4(D+1)^2 - 5}\cos x$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8}\cos x = e^x \frac{1}{(-1)^2 - 4D - 2 - 4D - 8}\cos x$$

$$= -e^x \frac{1}{8D + 9}\cos x$$

$$= -e^x \frac{(8D - 9)}{64D^2 - 81}\cos x$$

$$= \frac{e^x(9\cos x + 8\sin x)}{-145}$$

$\therefore$ Required solution $= y = $ C.F. $+$ P.I.

$\therefore y = c_1\cosh\sqrt{5}x + c_2\sinh\sqrt{5}x + c_3\cos x + c_4\sin x$
$\quad + \dfrac{e^x}{16}(1 - 2x) - \dfrac{e^x}{145}(9\cos x + 8\sin x)$

where c_1, c_2, c_3, c_4 are arbitrary constants

09

5. (b) Solve the differential equation $(D^2 - 2D + 2)\,y = e^x \tan x$, $D \equiv d/dx$ by method of variation of parameters. **[10]**

Consider $(D^2-2D+2)y = 0$. Auxiliary equation :- $m^2-2m+2=0$

$\therefore m = 1+i, 1-i \Rightarrow C.F. = e^x(c_1 \cos x + c_2 \sin x)$

We have $(D^2-2D+2)y = e^x \tan x$

$\therefore P = -2,\ Q = 2,\ R = e^x \tan x$

Let $y_p = Au + Bv$ where $u = e^x \cos x$ & $v = e^x \sin x$

$$\text{Wronskian}(u,v) = \begin{vmatrix} u & v \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix}$$

$$= e^{2x}[\cancel{\sin x \cos x} + \cos^2 x - \cancel{\cos x \sin x} + \sin^2 x]$$

$$= e^{2x}\cdot 1 = e^{2x} \neq 0$$

By method of variation of parameters,

$$A = -\int \frac{vR}{W}\,dx = -\int \frac{e^x \sin x \cdot e^x \tan x}{e^{2x}}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx = \int (\cos x - \sec x)\,dx = \sin x - \log(\sec x + \tan x)$$

$$B = \int \frac{uR}{W}\,dx = \int \frac{e^x \cos x \cdot e^x \tan x}{e^{2x}}\,dx = \int \sin x\,dx = -\cos x$$

$$\therefore y_p = Au + Bv = [\sin x - \log(\sec x + \tan x)]\,e^x \cos x + (-\cos x)\,e^x \sin x$$

$$= -\log(\sec x + \tan x)\cdot e^x \cos x$$

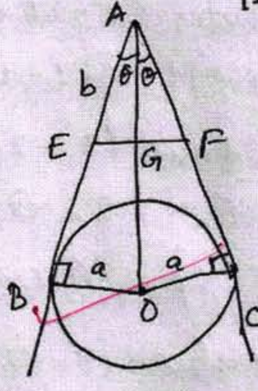
$\therefore y = y_p + C.F.$

$$\therefore \boxed{y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)}$$

where c_1 and c_2 are arbitrary constants

09

5. (c) Two equal rods, AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then b sin³ θ = a cosθ. **[10]**



$AB = 2b = AC$

$AE = b$ (E is midpoint of AB)

Radius of circle $= a$

Giving the rods a small displacement '$\delta\theta$'.

We have $OG = \frac{a}{\sin\theta} - b\cos\theta = a\operatorname{cosec}\theta - b\cos\theta$

Point O remains fixed but G displaces

$\therefore$ By Principle of Virtual Work,

$-2W\,\delta(OG) = 0$ where W = Weight of each rod

$\therefore \delta(OG) = 0$

$\therefore \delta(a\operatorname{cosec}\theta - b\cos\theta) = 0$

$\therefore [a\operatorname{cosec}\theta\cot\theta + b\sin\theta]\,\delta\theta = 0$

$\therefore b\sin\theta = a\operatorname{cosec}\theta\cot\theta \quad [\because \delta\theta \neq 0]$

$\therefore b\sin\theta = \frac{a\cos\theta}{\sin\theta \cdot \sin\theta}$

$\therefore \boxed{b\sin^3\theta = a\cos\theta}$

08

5. (d) Show that the Frenet-Serret formulae can be written in the form $\frac{dT}{ds} = \omega \times T, \frac{dN}{ds} = \omega \times N, \frac{dB}{ds} = \omega \times B$ and determine ω. **[10]**

Serret Frenet formulae :- $\frac{d\vec{T}}{ds} = K\vec{N}$, $\frac{d\vec{B}}{ds} = -\tau\vec{N}$ & $\frac{d\vec{N}}{ds} = \tau\vec{B} - K\vec{T}$

Consider $\frac{d\vec{T}}{ds} = K\vec{N} = 0 + K\vec{N} = \tau\vec{T}\times\vec{T} + K\vec{B}\times\vec{T}$ $[\because \vec{N} = \vec{B}\times\vec{T}]$

$\therefore \frac{d\vec{T}}{ds} = (\tau\vec{T} + K\vec{B})\times\vec{T}$ — ①

Consider $\frac{d\vec{B}}{ds} = -\tau\vec{N} = -0 - \tau\vec{N} = -K\vec{B}\times\vec{B} - \tau\vec{B}\times\vec{T}$

$\therefore \frac{d\vec{B}}{ds} = -\vec{B}\times(K\vec{B} + \tau\vec{T}) = (K\vec{B} + \tau\vec{T})\times\vec{B}$ ② $[\because \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}]$

Consider $\frac{d\vec{N}}{ds} = \tau\vec{B} - K\vec{T} = \cancel{\tau\vec{B} - K\vec{T} + \tau\vec{N} - \tau\vec{N} + K\vec{N} - K\vec{N}}$

$\therefore \frac{d\vec{N}}{ds} = \tau(\vec{T}\times\vec{N}) - K(\vec{N}\times\vec{B}) = \tau(\vec{T}\times\vec{N}) + K(\vec{B}\times\vec{N})$
$= (\tau\vec{T})\times\vec{N} + (K\vec{B})\times\vec{N} = (K\vec{B} + \tau\vec{T})\times\vec{N}$ - ③

From ①, ② & ③ and given form in question, we conclude that $\boxed{\omega = K\vec{B} + \tau\vec{T}}$.

$\therefore$ Serret frenet formulae can be written in the form
$\frac{dT}{ds} = \omega\times T$, $\frac{dN}{ds} = \omega\times N$, $\frac{dB}{ds} = \omega\times B$.

08

5. (e) Show that

$(y^2z^3 \cos x - 4x^3z)\,dx + 2z^3y \sin x\,dy + (3y^2z^2 \sin x - x^4)\,dz$ is an exact differential of some function ϕ and find this function. **[10]**

Above expression is of the form $P\,dx + Q\,dy + R\,dz$ where

$P = (y^2z^3\cos x - 4x^3z)$, $Q = 2z^3y\sin x$ and $R = 3y^2z^2\sin x - x^4$

consider $\frac{\partial^2 P}{\partial z \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial P}{\partial y}\right] = \frac{\partial}{\partial z}\left[2yz^3\cos x\right] = 6yz^2\cos x$ — ①

$\frac{\partial^2 Q}{\partial x \partial z} = \frac{\partial}{\partial x}\left[\frac{\partial Q}{\partial z}\right] = \frac{\partial}{\partial x}\left[6yz^2\sin x\right] = 6yz^2\cos x$ — ②

$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial R}{\partial y}\right] = \frac{\partial}{\partial x}\left[6yz^2\sin x\right] = 6yz^2\cos x$ — ③

From ①, ② & ③, $\frac{\partial^2 P}{\partial z \partial y} = \frac{\partial^2 Q}{\partial x \partial z} = \frac{\partial^2 R}{\partial x \partial y}$

Hence, $\boxed{\text{the given expression is exact differential}}$.

Let $d\phi = P\,dx + Q\,dy + R\,dz$

where $P = \frac{\partial \phi}{\partial x}$, $Q = \frac{\partial \phi}{\partial y}$ and $R = \frac{\partial \phi}{\partial z}$ [By definition of exact function]

$\therefore \frac{\partial \phi}{\partial x} = y^2z^3\cos x - 4x^3z \Rightarrow \phi = y^2z^3\sin x - x^4z + f_1(y,z)$ — ④

$\frac{\partial \phi}{\partial y} = 2z^3y\sin x \Rightarrow \phi = y^2z^3\sin x + f_2(x,z)$ — ⑤

$\frac{\partial \phi}{\partial z} = 3y^2z^2\sin x - x^4 \Rightarrow \phi = y^2z^3\sin x - x^4z + f_3(x,y)$ — ⑥

From ④, ⑤ & ⑥, $f_1(y,z) = c$, $f_2(x,z) = c - x^4z$

and $f_3(x,y) = c$

$\therefore \boxed{\phi = y^2z^3\sin x - x^4z + c}$ where c is an arbitrary constant

09

7. (a) (i) Solve $6 \cos^2 x\,(dy/dx) - y \sin x + 2y^4 \sin^3 x = 0$

(ii) Reduce the equation $x^2p^2 + py\,(2x + y) + y^2 = 0$ where $p = dy/dx$ to Clairaut's form and find its complete primitive and its singular solution. **[6+14=20]**

i) $6\cos^2 x \frac{dy}{dx} - y\sin x + 2y^4 \sin^3 x = 0$

$\therefore \frac{3\cot^2 x}{y^4}\frac{dy}{dx} - \frac{1}{2y^3}\operatorname{cosec} x = -\sin x$

Let $z = \frac{1}{y^3} \Rightarrow \frac{dz}{dx} = -\frac{3}{y^4}\frac{dy}{dx}$

$\therefore -\cot^2 x \frac{dz}{dx} - \frac{\operatorname{cosec} x}{2} z = -\sin x$ or $\cot^2 x \frac{dz}{dx} + \frac{\operatorname{cosec} x}{2} z = \sin x$

$\therefore \frac{dz}{dx} + \frac{\sec x \tan x}{2} z = \sin x \tan^2 x$

Integrating factor $= e^{\int \sec x \tan x/2\, dx} = e^{\sec x/2}$

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$\therefore \frac{d}{dx}\left[e^{\sec x/2} \cdot z\right] = \sin x(\sec^2 x - 1)e^{\sec x/2}$

Integrating wrt x, $z e^{\sec x/2} = \int (\sec x \tan x - \sin x) e^{\sec x/2} dx$

$\therefore z e^{\sec x/2} = \int e^{\sec x/2} \sin x \tan^2 x + C$

$\therefore \boxed{\frac{e^{\sec x/2}}{y^3} = \int e^{\sec x/2} \sin x \tan^2 x + C}$ //

ii) $x^2p^2 + py(2x+y) + y^2 = 0$ and $p = \frac{dy}{dx}$

Let $u = y$ and $v = xy \Rightarrow dy = du$ & $dv = x\,dy + y\,dx$

Let $p' = \frac{dv}{du} = \frac{x\,dy + y\,dx}{dy} = x + y\frac{dx}{dy} = x + \frac{y}{p}$

$\because p' = x + \frac{y}{p} \Rightarrow \frac{y}{p} = p' - x \Rightarrow p = \frac{y}{p'-x}$

Hence, $x^2\frac{y^2}{(p'-x)^2} + \frac{y^2}{(p'-x)}(2x+y) + y^2 = 0$

$\therefore \frac{x^2 + (2x+y)(p'-x) + (p'-x)^2}{(p'-x)} = 0$

$\therefore x^2 + (p'-x)(2x+y) + (p'-x)^2 = 0$

$\therefore \cancel{x^2} + 2\cancel{x}p' + p'y - 2x^2 - xy + p'^2 - 2\cancel{x}p' + \cancel{x^2} = 0$

$\therefore p'y = xy - p'^2$ or $up' = v - p'^2$ or $v = up' + (p')^2$

which is of the Clairut form.

$\therefore$ General solution is $v = uc + c^2$

or $\boxed{xy = cy + c^2}$ where c is an arbitrary constant

$\downarrow$

complete primitive

Consider $xy = cy + c^2$, i.e., $c^2 + yc - xy = 0$

$\therefore$ c-discriminant :- $(y)^2 - 4(1)(-xy) = 0$

$\therefore y^2 + 4xy = 0$

$\therefore y(y+4x) = 0$

$\therefore y = 0$ or $y = -4x$

Consider $y = 0 \Rightarrow p = \frac{dy}{dx} = 0$

$y = 0$ and $p = 0$ satisfy the given differential equation.

Hence, $\boxed{y = 0 \text{ is a singular solution}}$

Consider $y = -4x \Rightarrow p = \frac{dy}{dx} = -4$

$\therefore x^2p^2 + py(2x+y) + y^2$ (at $y=-4x$ & $p=-4$)

$= x^2(16) + 16x(-2x) + 16x^2$

$= 16x^2 - 32x^2 + 16x^2 = 0$

$\therefore y=-4x$ satisfies the given differential equation

Hence, $\boxed{y=-4x \text{ is a singular solution}}$

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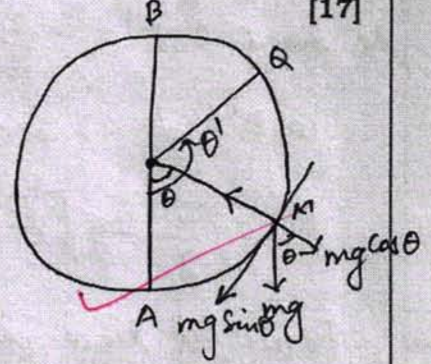
7. (b) A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. show that the reaction between the particle and the wire is zero after a time

$\sqrt{(a/g)}.\log\left(\sqrt{5}+\sqrt{6}\right)$. [17]



Let mass of particle = m and radius = a

Lowest point is A and M is a point at any time 't'

AM = S and $\angle AOM = \theta$

By balancing forces, we get

$m\frac{d^2s}{dt^2} = -mg\sin\theta$ and $\frac{mv^2}{a} = R - mg\cos\theta$

We have $s = a\theta \Rightarrow ma\frac{d^2\theta}{dt^2} = -mg\sin\theta \Rightarrow a\frac{d^2\theta}{dt^2} = -g\sin\theta$

$\therefore 2a\frac{d^2\theta}{dt^2}\frac{d\theta}{dt} = -2ag\sin\theta\frac{d\theta}{dt} \Rightarrow \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$

and $v = \frac{ds}{dt} = a\frac{d\theta}{dt} \Rightarrow v^2 = 2ag\cos\theta + A$

At $\theta = \pi$ (highest point), $v = 0 \Rightarrow 0 = 2ag\cos\pi + A \Rightarrow A = 2ag$

$\therefore v^2 = 2ag(1+\cos\theta)$

Hence, $R = \frac{mv^2}{a} + mg\cos\theta = \frac{m}{a}[2ag(1+\cos\theta)] + mg\cos\theta$

$\therefore R = 2mg + 3mg\cos\theta$

At Q, $R=0$ and $\theta = \theta' \Rightarrow 0 = mg(2+3\cos\theta')$

$\Rightarrow \cos\theta' = -\frac{2}{3}$

We have $v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag(1+\cos\theta) \Rightarrow \frac{d\theta}{dt} = \sqrt{\frac{g}{a}}\sqrt{2(1+\cos\theta)}$

$\therefore \frac{d\theta}{dt} = 2\sqrt{\frac{g}{a}}\sqrt{\cos^2\frac{\theta}{2}} = 2\sqrt{\frac{g}{a}}\cos\frac{\theta}{2}$

$\therefore dt = 2\sqrt{\frac{a}{g}}\sec\frac{\theta}{2}\,d\theta$

Integrating ~~w.r.t~~ above equation, we get

$$\int_0^t dt = 2^{-1}\sqrt{\frac{a}{g}}\int_0^{\theta'}\sec\frac{\theta}{2}\,d\theta$$

$$\therefore t = \frac{1}{2}\sqrt{\frac{a}{g}} \times 2\left[\log\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right]_{\theta=0}^{\theta=\theta'}$$

$$= \sqrt{\frac{a}{g}}\log\left(\sec\frac{\theta'}{2} + \tan\frac{\theta'}{2}\right)$$

We have $\cos\theta' = -\frac{2}{3} \Rightarrow 2\cos^2\frac{\theta'}{2} = 1 - \frac{2}{3} \Rightarrow \cos^2\frac{\theta'}{2} = \frac{1}{6}$

$\therefore \cos\theta' = \frac{1}{\sqrt{6}} \Rightarrow \sec\theta' = \sqrt{6}$ and $\tan\theta' = \sqrt{(\sqrt{6})^2 - 1} = \sqrt{5}$

$\therefore t = \sqrt{\frac{a}{g}}\log[\sqrt{6}+\sqrt{5}]$

$\therefore$ Required time is $\boxed{\sqrt{\frac{a}{g}}\log[\sqrt{6}+\sqrt{5}]}$

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7. (c) Verify Green's theorem in the plane for $\oint_C (2x - y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. [13]

Green's Theorem :- $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 2x - y^3$ and $N = -xy$ and the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

$RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[(-y) - (-3y^2)\right] dx\,dy$

$= \iint [3y^2 - y]\,dx\,dy = \int_0^{2\pi}\int_1^3 [3r^2\sin^2\theta - r\sin\theta]\, r\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{3}{4}\times 80 \times \sin^2\theta - \frac{\sin\theta}{3}\times 26\right] d\theta = \int_0^{2\pi}\left[60\sin^2\theta - \frac{26}{3}\sin\theta\right] d\theta$

$= \int_0^{2\pi}\left[30(1-\cos 2\theta)\right] d\theta - \int_0^{2\pi}\frac{26}{3}\sin\theta\,d\theta$

$= 30\cdot 2\pi - 30\cdot\frac{1}{2}\cdot[0-0] - \frac{26}{3}[1-1]$

$= 60\pi \quad -(1)$

To calculate LHS :- we have $C_1 :- x^2+y^2=1$ & $C_2 :- x^2+y^2=9$

Consider $\oint_{C_1} (2x-y^3)\,dx - xy\,dy = I_1$. Let $x = \cos\theta, y = \sin\theta$
$dx = -\sin\theta\,d\theta$ & $dy = \cos\theta\,d\theta$

$\therefore I_1 = \int_0^{2\pi}[2\cos\theta - \sin^3\theta](-\sin\theta\,d\theta) - \int_0^{2\pi}\cos\theta\sin\theta\cdot\cos\theta\,d\theta$

$= \int_0^{2\pi}[\sin^4\theta - \sin 2\theta]\,d\theta + \left.\frac{\cos^3\theta}{3}\right|_{\theta=0}^{\theta=2\pi}$

$= \frac{3\pi}{4} - 0 + 0 = \frac{3\pi}{4} \quad -(2)$

Consider $I_2 = \oint_{C_2}(2x-y^3)\,dx - xy\,dy$. Let $x = 3\cos\theta, y = 3\sin\theta$
$dx = -3\sin\theta\,d\theta$ & $dy = 3\cos\theta\,d\theta$

$\therefore I_2 = \int_0^{2\pi}[6\cos\theta - 27\sin^3\theta](-3\sin\theta)\,d\theta - \int_0^{2\pi} 9\cos\theta\sin\theta\,(3\cos\theta)\,d\theta$

$= \int_0^{2\pi}\left[81\sin^4\theta - \frac{18}{2}\sin 2\theta\right] d\theta - 27\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta$

$= 81\times\frac{3\pi}{4} - 9(0) - 27(0)$

$= \frac{243\pi}{4} \quad -(3)$

From ② & ③, $\oint\limits_C (2x-y^3)dx - xy\,dy = \oint\limits_{C_2-C_1} (2x-y^3)dx - xy\,dy$

$= I_2 - I_1 = \frac{243\pi}{4} - \frac{3\pi}{4} = \frac{240\pi}{4} = 60\pi$ — ④

From ① & ④, LHS = RHS

$\therefore \oint\limits_C M dx + N dy = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 60\pi$

Hence, Green's Theorem is verified.

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8. (a) By using Laplace transform method solve

$(D^2 + n^2)\,y = a \sin(nt + \alpha)$, $y = Dy = 0$ when $t = 0$. **[17]**

$\therefore y'' + n^2 y = a \sin(nt+\alpha)$ ① and $y(t=0) = 0$ & $Dy(t=0) = 0$

Taking Laplace's transform on both sides of ①, we get

$L\{y''\} + n^2 L\{y\} = a\,L\{\sin nt \cos\alpha + \sin\alpha \cos nt\}$

$\therefore s^2 L\{y\} - s y(0) - y'(0) + n^2 L\{y\} = \frac{an\cos\alpha}{s^2+n^2} + \frac{as\sin\alpha}{s^2+n^2}$

$\therefore (s^2+n^2) L\{y\} - 0 - 0 = \frac{1}{(s^2+n^2)}(an\cos\alpha + as\sin\alpha)$

$\therefore L\{y\} = \frac{an\cos\alpha}{(s^2+n^2)^2} + \frac{as\sin\alpha}{(s^2+n^2)^2}$

Taking Inverse Laplace Transform,

$y = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha \left\{\frac{s}{(s^2+n^2)^2}\right\}$

Consider $L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = -\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+n^2}\right)\right\} = -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{s^2+n^2}\right\}$

Now, we need to find $L^{-1}\{1/s^2+n^2\}$

$\therefore L^{-1}\{s/(s^2+n^2)^2\} = \frac{t}{2n}\sin nt$

Let $f(s) = 1/s^2+n^2$, $g(s) = 1/s^2+n^2$

$\therefore L^{-1}\{f(s)\} = \frac{\sin nt}{n}$ & $L^{-1}\{g(s)\} = \frac{\sin nt}{n}$

$\therefore$ By convolution Theorem, $L^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,du$

$$\therefore L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} = \int_0^t \frac{\sin nu}{n}\cdot\frac{\sin n(t-u)}{n}\,du$$

$$= \frac{1}{2n^2}\int_0^t [\cos n(t-2u) - \cos nt]\,du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_{u=0}^{u=t}$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} + \frac{\sin nt}{2n} - t\cos nt\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right]$$

$$\therefore y = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha\left\{\frac{s}{(s^2+n^2)^2}\right\}$$

$$\therefore \boxed{y = \frac{a}{2n^2}\left[\cos\alpha\sin nt - nt\cos(\alpha+nt)\right]}$$

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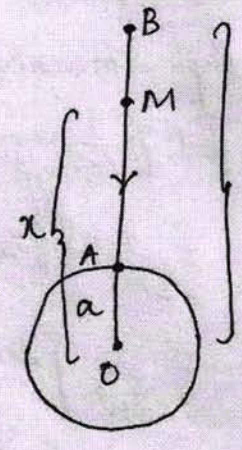
8. (b) Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b(b > a) from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the time to travel from the surface to the centre of the eath is $\sqrt{\left(\frac{a}{b}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface. **[18]**



Equation of motion at M :- $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$

Here, O is centre of earth, $OA = a$ & $OB = b$.

Let $OM = x$.

At A, $\frac{d^2x}{dt^2} = -g \Rightarrow \mu = a^2 g$

$\therefore \frac{d^2x}{dt^2} = -\frac{a^2g}{x^2} \Rightarrow 2\frac{d^2x}{dt^2}\frac{dx}{dt} = -\frac{2a^2g}{x^2}\frac{dx}{dt}$

$\therefore$ Integrating, we get $\left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} + A$

At B, $\frac{dx}{dt} = 0 \Rightarrow 0 = \frac{2a^2g}{b} + A \Rightarrow A = -2a^2g/b$

$\therefore \left(\frac{dx}{dt}\right)^2 = 2a^2g\left[\frac{1}{x} - \frac{1}{b}\right] = V^2$ (say) — ①

Equation of motion inside earth :- $\frac{d^2x}{dt^2} = -\lambda x$

At A, $\frac{d^2x}{dt^2} = -g \Rightarrow \lambda = \frac{g}{a}$ $\quad\therefore \frac{d^2x}{dt^2} = -\frac{g}{a}x$

$\therefore 2\frac{d^2x}{dt^2}\frac{dx}{dt} = -\frac{2g}{a}x\frac{dx}{dt} \Rightarrow \left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + B$ — ②

At A, from ① & ②, $2a^2g\left[\frac{1}{a} - \frac{1}{b}\right] = -\frac{g}{a}\cdot a^2 + B$

$\Rightarrow B = ag\left(\frac{3b-2a}{b}\right) \Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{ag}{b}(3b-2a) - \frac{g}{a}x^2$ — ③

$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}\left[c^2 - x^2\right]$ where $c = \left[\frac{a^2}{b}(3b-2a)\right]^{1/2}$

$\therefore \frac{dx}{dt} = -\sqrt{\frac{g}{a}}\sqrt{c^2-x^2}$ [Negative sign is taken as x is decreasing]

$\therefore dt = -\sqrt{\frac{a}{g}}(c^2-x^2)^{-1/2}dx$ — ④

Integrating ④, we get

$$\int_0^t dt = -\sqrt{\frac{a}{g}}\int_a^0 (c^2-x^2)^{-1/2}\,dx$$

$$= -\sqrt{\frac{a}{g}}\left[\sin^{-1}\frac{x}{c}\right]_{x=a}^{x=0} = \sqrt{\frac{a}{g}}\sin^{-1}\left(\frac{a}{c}\right)$$

$$\therefore \boxed{t = \sqrt{\frac{a}{g}}\sin^{-1}\left\{\sqrt{\frac{b}{3b-2a}}\right\}}$$

We have $\left(\frac{dx}{dt}\right)^2 = \frac{ag}{b}(3b-2a) - \frac{g}{a}x^2$

At $x = 0$ (centre), $\left(\frac{dx}{dt}\right)^2 = \frac{ag}{b}(3b-2a)$

$$\therefore \frac{dx}{dt} = \boxed{V = \sqrt{\frac{ag}{b}(3b-2a)}}$$

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8. (c) Evaluate $\iint_S (\nabla\times F)\cdot \mathbf{n}\,dS$, where

$F = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane. **[15]**

S:- $z = 4-(x^2+y^2)$ above xy plane is not a closed surface.
We take $S_1 = S + S'$ which is a closed surface where
S':- $0 = 4-(x^2+y^2)$, i.e., $x^2+y^2 = 4$ & $z = 0$.
$\therefore S_1$ is a closed surface

Hence, by Divergence Theorem, $\iint_{S_1} (\nabla\times F)\cdot n\,ds = \iiint_V \nabla\cdot(\nabla\times F)\,dV$

But $\nabla\cdot(\nabla\times F) = 0 \Rightarrow \iint_{S_1}(\nabla\times F)\cdot n\,dS = \iiint_V 0\,dV = 0$

i.e., $\iint_S (\nabla\times F)\cdot n\,ds + \iint_{S'}(\nabla\times F)\cdot n\,dS = 0$

$$\therefore \iint_S (\nabla\times F)\cdot n\,ds = -\iint_{S'}(\nabla\times F)\cdot n\,dS$$

$$= -\iint_{S'}(\nabla\times F)\cdot(-\hat{k})\,dS \quad \left[n = -\hat{k} \text{ in } x^2+y^2=4 \text{ and } z=0\right]$$

outward normal

$$= \iint_{S'}(\nabla\times F)\cdot\hat{k}\,dS$$

Consider $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xy+z^2 \end{vmatrix} = 2x\hat{i} - 2y\hat{j} + (3y-1)\hat{k}$

$\therefore (\nabla \times F) \cdot n = (\nabla \times F) \cdot \hat{k} = (3y-1)$

$$\therefore \iint_S (\nabla \times F) \cdot n \, ds = \iint_{S'} (3y-1) \, dS'$$

$$= \iint_{S'} (3y-1) \frac{dx \, dy}{|\hat{k} \cdot \hat{k}|}$$

$$= \iint_{S'} (3y-1) \, dx \, dy$$

$S' :- x^2+y^2 = 4$ ~~let x =~~ converting above integral into polar form, we get

$$\iint_S (\nabla \times F) \cdot n \, ds = \int_0^{2\pi} \int_0^2 (3r \sin\theta - 1) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 [3r^2 \sin\theta - r] \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[3 \times \frac{1}{3} \times \sin\theta \times (8-0) - \frac{1}{2}(4-0)\right] d\theta$$

$$= \int_0^{2\pi} [8 \sin\theta - 2] \, d\theta$$

$$= -8 \cos\theta \Big|_{\theta=0}^{\theta=2\pi} - 2 \times 2\pi$$

$$= 0 - 4\pi$$

$$= -4\pi$$

$$\therefore \boxed{\iint_S (\nabla \times F) \cdot n \, ds = -4\pi}$$

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