IAS MATHEMATICS (OPT.)

PAPER - II : REAL ANALYSIS (2007 to 2000)

IAS-2007

P-II 12M 2007

8.

1(c). Show that the function given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives ~~fx~~ and f_y exist at $(0,0)$.

Solⁿ: Let $(x,y) \to (0,0)$ along the straight line $y = mx$.

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{mx^2}{x^2(1+2m^2)}$$

$$= \frac{m}{1+2m^2}$$ which depends on m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exist

$\therefore f(x,y)$ is not continuous at $(0,0)$.

Now $f_x(0,0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$

$= \lim_{h\to 0} \frac{0-0}{h}$

$= 0$

$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$

$= \lim_{k \to 0} \frac{0-0}{k}$

$= 0$

$\therefore f$ is possesses both ~~the~~ partial derivatives at $(0,0)$.

$\therefore f$ is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$

12M II 2007

1(d). Using Lagrange's mean value theorem, show that $|\cos b - \cos a| \le |b-a|$.

Sol^n: If $a=b$, there is nothing to prove.

If $a<b$, then

Consider the function

$f(x) = \cos x$ on $[a,b]$

Clearly, f is continuous on $[a,b]$ and derivable on (a,b).

$\therefore$ By Lagrange's mean value theorem, there exists $\alpha \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\alpha)$$

$$\Rightarrow \frac{\cos b - \cos a}{b-a} = -\sin\alpha$$

Since $|\sin\alpha| \le 1$

$$\therefore \left|\frac{\cos b - \cos a}{b-a}\right| \le 1$$

$$\Rightarrow |\cos b - \cos a| \le |b-a|$$

If $a>b$, then in a similar manner, we have

$$|\cos a - \cos b| \le |a-b|$$

$$\Rightarrow |\cos b - \cos a| \le |b-a| \quad (\because |-a| = |a|)$$

Hence for all $a, b \in \mathbb{R}$

$$|\cos b - \cos a| \le |b-a|.$$

2011 → 2007 P-II

3(a). Given a positive real number 'a' and any natural number n, there exists one and only one positive real number ξ such that $\xi^n = a$.

Sol^n: Let 'a' be a given positive number.

we shall show that there exists a unique positive real number ξ such that $\xi^n = a$.

Consider the function f defined by

$$f(x) = x^n, \ n \in \mathbb{N}.$$

Suppose that $a < 1$.

The function f is continuous in $[0,1]$ and $f(0) = 0$, $f(1) = 1$.

Of course $0 < a < 1$.

By the Intermediate value theorem, there exists $\xi \in [0,1]$ such that

$$\xi^n = a$$

If η is also a positive number that $\eta^n = a$, we have

$$\xi < \eta \Rightarrow \xi^n < \eta^n \Rightarrow a < a$$

$$\xi > \eta \Rightarrow \xi^n > \eta^n \Rightarrow a > a$$

It follows that $\xi = \eta$.

Now suppose that $a > 1$ so that $\frac{1}{a} < 1$ and as shown above there exists ξ such that

$$\xi^n = \frac{1}{a} \Rightarrow \left(\frac{1}{\xi}\right)^n = a$$

[Notice for Students: The unique positive real number satisfying the equation $x^n = a$; $a > 0$ and $n \in \mathbb{N}$ is called the nth root of a and is denoted by $a^{1/n}$.]

IAS-2006

12M 2006 P-II Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ 11.

1(c). Sol^n: Here $f(x) = \frac{1}{x^{1/2}(1-x)^{1/2}}$

and 0 & 1 are the only points of infinite discontinuity of f on $[0,1]$.

we may write

$$\int_0^1 f(x)dx = \int_0^c f(x)\,dx + \int_c^1 f(x)\,dx \quad \text{—— (1)}$$

where $0<c<1$.

<u>To test the convergence of $\int_0^c f(x)\,dx$ at $x=0$:</u>

Take $g(x) = \frac{1}{x^{1/2}}$

$$\lim_{x\to \frac{1}{2}^+} \frac{f(x)}{g(x)} = \lim_{x\to \frac{1}{2}^+} \frac{1}{(1-x)^{1/2}}$$

$$= \frac{1}{\sqrt{1/2}} = \sqrt{2}$$

which is non-zero and finite.

$\therefore$ By comparison test, the integrals $\int_0^c f(x)dx$ and $\int_0^c g(x)dx$ converge (or) diverge together.

But $\int_0^c g(x)\,dx = \int_0^c \frac{dx}{x^{1/2}}$ is convergent $\left(\because n = \frac{1}{2}<1\right)$

$\therefore \int_0^c f(x)dx$ is convergent.

(it is of the form $\int_a^b \frac{1}{(x-a)^n}dx$ is cgt if $n<1$)

To test the convergence of $\int_c^1 f(x)\,dx$ at $x=1$.

Take $g(x) = \frac{1}{(1-x)^{1/2}}$

$$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{x^{1/2}} = 1$$ which is non-zero and finite.

$\therefore$ By comparison test, the integrals $\int_c^1 f(x)\,dx$ and $\int_c^1 g(x)\,dx$ converge or diverge together.

But $\int_c^1 g(x)\,dx = \int_c^1 \frac{1}{(1-x)^{1/2}}\,dx$ is convergent $(\because n = 1/2 < 1)$.

$\therefore \int_c^1 f(x)\,dx$ is convergent.

Hence, from (1),

$\int_0^1 f(x)\,dx$ is convergent.

12 → 2006 P-II

1(d). prove that the function f defined by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational.} \end{cases}$$

is no where continuous.

Sol^n: Let $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational.} \end{cases}$

Let $x = c \in \mathbb{R}$ then c is either rational or irrational number.

If 'c' is a rational number:

Let (x_n) be a sequence of irrational numbers that converges to 'c'.

Since $f(x_n) = -1 \; \forall n$ and $f(c) = 1$ ($\because x_n$ is irrational)

$\therefore \lim\limits_{n\to\infty} f(x_n) = -1 \neq f(c)$.

$\therefore (f(x_n))$ doesnot converge to $f(c)$.

$\therefore f(x)$ is not continuous at the rational number 'c'.

If 'c' is an irrational number:

Let (x_n) be a sequence of rational numbers that converges to 'c'.

Since $f(x_n) = 1 \; \forall n$ and $f(c) = -1$

P-II 2006 20M

3(a). A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. prove that there is at least one value ξ, $a < \xi < b$ for which $f''(\xi) < 0$

Sol'n: f is twice differentiable on $[a, b]$

$\Rightarrow$ f', f'' exist on $[a, b]$

$\Rightarrow$ f, f' are differentiable on $[a, b]$

$\therefore$ f, f' are continuous functions on $[a, b]$

since $a < c < b$, applying Lagrange's Mean value theorem to on the intervals $[a, c]$ and $[c, b]$ We get

$$\frac{f(c) - f(a)}{c - a} = f'(\xi_1)$$

Where $a < \xi_1 < c$ and

$$\frac{f(b) - f(c)}{b - c} = f'(\xi_2) \text{ where } c < \xi_2 < b$$

But $f(a) = f(b) = 0$

$$\therefore f'(\xi_1) = \frac{f(c)}{c-a} \text{ and}$$

$$f'(\xi_2) = \frac{-f(c)}{b-c} \text{ Where}$$

$$a < \xi_1 < c < \xi_2 < b$$

Again f' is Continuous and derivable on $[\xi_1, \xi_2]$

$\therefore$ By Lagrange's Mean value theorem we have

$$\frac{f'(\xi_2) - f'(\xi_1)}{\xi_2 - \xi_1} = f''(\xi)$$

where $\xi_1 < \xi < \xi_2$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$,

We get

$$f''(\xi) = \frac{\frac{-f(c)}{b-c} - \frac{f(c)}{c-a}}{\xi_2 - \xi_1}$$

$$= \frac{-f(c)}{\xi_2 - \xi_1}\left[\frac{1}{b-c} - \frac{1}{c-a}\right]$$

$$= \frac{-f(c)}{\xi_2 - \xi_1}\left[\frac{b-a}{b-c}\right]$$

Since $a < \xi_1 < c < \xi_2 < b$ and $f(c) > 0$

$\therefore f''(\xi) < 0$ where $a < \xi < b$

12.

2011, 2006 P-II

3(b). Show that the function given by

$$f(x,y)=\begin{cases}\dfrac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(i) is continuous at (0,0)

(ii) possesses partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

sol^n: Given that

$$f(x,y)=\begin{cases}\dfrac{x^3+2y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

sol^n: putting $x=r\cos\theta$, $y=r\sin\theta$;

we obtain

$$|f(x,y)-f(0,0)|=\left|\frac{r^3\cos^3\theta+2r^3\sin^3\theta}{r^2\cos^2\theta+r^2\sin^2\theta}-0\right|$$

$$=\left|\frac{r^2(r\cos^3\theta+2r\sin^3\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right|$$

$$=|(r\cos\theta+2r\sin\theta)|$$

$$=|r(\cos\theta+2\sin\theta)|$$

$$\leq r|\cos\theta+2\sin\theta|$$

$$\leq r[|\cos\theta|+2|\sin\theta|]$$

$$\leq r[1+2(1)]$$

$$=3r$$

$$=3\sqrt{x^2+y^2}$$

Let $\epsilon > 0$ be given and $\delta = \epsilon/3$

Then
$|f(x, y) - f(0,0)| < \epsilon$, whenever $\sqrt{x^2+y^2} < \epsilon/3 = \delta$

$\therefore |f(x,y) - f(0,0)| < \epsilon$, whenever $\sqrt{x^2+y^2} < \delta$.

$\therefore$ The given function $f(x,y)$ is continuous at $(0,0)$.

Now $$f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3+0}{h^2} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^3}{h^3} = 1$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{0+2k^3}{0+k^2} - 0}{k}$$

$$= \lim_{k \to 0} \frac{2k^3}{k^3}$$

$$= 2$$

$\therefore f(x, y)$ possesses partial derivatives at $(0,0)$.

IAS-2005

3(a). 30M

If f' and g' exist for every $x \in [a,b]$ and if $g'(x)$ doesnot vanish anywhere in (a,b), such that there exists c in (a,b) show that $\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$

Sol'n Let us consider

$$\phi(x) = f(x)g(x) - f(a)g(x) - g(b)f(x) \quad \forall x \in [a,b]$$

Since f' and g' exists in $[a,b]$

$\therefore$ f and g are derivable functions on $[a,b]$

$\therefore$ f and g are continuous functions on $[a,b]$

$\therefore$ $\phi(x)$ is continuous and derivable on $[a,b]$

and $\phi(a) = -f(a)g(b)$

$\phi(b) = -f(a)g(b)$

$\therefore \phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies the conditions of Rolle's theorem on $[a,b]$

$\therefore \exists$ at least one point $c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x)g(x) + f(x)g'(x) - f(a)g'(x) - g(b)f'(x)$

$\Rightarrow \phi'(c) = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$

$\Rightarrow 0 = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$

$(\because \phi'(c) = 0)$

$$g'(c)[f(c)-f(a)] + f'(c)[g(c)-g(b)] = 0$$

$$\Rightarrow g'(c)[f(c)-f(a)] = f'(c)[g(b)-g(c)]$$

$$\Rightarrow \frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

$$[\because g'(x) \neq 0 \ \forall \ x \in (a,b)]$$

IAS-2004

12M

1(c). → Show that the function $f(x)$ defined as $f(x)=\frac{1}{2^n}$, $\frac{1}{2^{n+1}}\le x\le\frac{1}{2^n}$, $n=0,1,2,----$, $f(0)=0$ is integrable in $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = 2/3$.

Sol^n: $f(x)=\frac{1}{2^n}$ when $\frac{1}{2^{n+1}}<x\le\frac{1}{2^n}$, $n=0,1,2----$

$$=\frac{1}{2^0}=1 \quad \text{when } \frac{1}{2^1}<x\le\frac{1}{2^0}=1$$

$$=\frac{1}{2} \quad \text{when } \frac{1}{2^2}<x\le\frac{1}{2^1}$$

$$=\frac{1}{2^2} \quad \text{when } \frac{1}{2^3}<x\le\frac{1}{2^2}$$

$$\vdots$$

$$=\frac{1}{2^{n-1}} \quad \text{when } \frac{1}{2^n}<x\le\frac{1}{2^{n-1}}$$

$$\vdots$$

$$0 \quad \text{when } x=0.$$

⇒ f is bounded and continuous on $[0,1]$ except at the points

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ---$

The set of points discontinuity of f on $[0,1]$ is $\{\frac{1}{2}, \frac{1}{2^2}, ---\}$ which has only one limit point '0'.

Since the set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore f$ is integrable on $[0,1]$.

Now $$\int_{1/2^n}^1 f(x)\,dx = \int_{1/2}^1 f(x)dx + \int_{1/2^2}^{1/2} f(x)dx + \int_{1/2^3}^{1/2^2} f(x)dx + --- + \int_{1/2^n}^{1/2^{n-1}} f(x)dx$$

$$=\left(1-\tfrac{1}{2}\right)+\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{2^2}\right)+\tfrac{1}{2^3}\left(\tfrac{1}{2^2}-\tfrac{1}{2^3}\right)+----- + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)$$

$$=\frac{1}{2}\left[1+\frac{1}{2^2}+\left(\frac{1}{2^2}\right)^2+---+\left(\frac{1}{2^2}\right)^{n-1}\right]$$

$$=\tfrac{1}{2}\left[\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}\right]$$

$$= \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Taking limit when $n \to \infty$, we get

$$\lim_{n\to\infty} \int_{\frac{1}{2^n}}^{1} f(x)\,dx = \lim_{n\to\infty} \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

$$\Rightarrow \int_0^1 f(x)\,dx = 2/3$$

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12M 2004 P-II **1(d).** Show that the function defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational.} \end{cases}$$

is continuous only at $x = 0$

Sol^n: Let x be any real number.
For each $n \in \mathbb{N}$, $\exists$ a rational number 'a_n' and an irrational number 'b_n' such that

$$x - \frac{1}{n} < a_n < x + \frac{1}{n} \text{ and}$$

$$x - \frac{1}{n} < b_n < x + \frac{1}{n} \quad \forall n$$

$$\Rightarrow |a_n - x| < \frac{1}{n} \text{ and } |b_n - x| < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow \lim_{n\to\infty} a_n = x \text{ and } \lim_{n\to\infty} b_n = x$$

$$\Rightarrow \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n \quad \text{—— (1)}$$

If 'f' is continuous at 'x' then, we must have

$$\lim_{n\to\infty} f(a_n) = f(x) = \lim_{n\to\infty} f(b_n)$$

But $f(a_n) = -a_n$ and $f(b_n) = b_n$.

$$\therefore \lim_{n\to\infty} -a_n = f(x) = \lim_{n\to\infty} b_n$$

$$\Rightarrow -x = f(x) = x$$

$$\Rightarrow -x = x$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow \boxed{x = 0}$$

$\therefore$ 0 is the only possible point of continuous and discontinuous at every other point.

Now we show that f is continuous at $x=0$

At $x=0$, $f(0)=0$

Let $\epsilon>0$ be given,

for a rational number 'x',

we have

$|f(x)-f(0)| = |-x-0| = |x|$

for an irrational number x, we have

$|f(x)-f(0)| = |x-0| = |x|$

In either case,

$|f(x)-f(0)| = |x| < \epsilon$

whenever $|x|<\epsilon$

Choosing $\delta=\epsilon$, then

$|f(x)-f(0)| < \epsilon$ whenever $|x|<\delta$.

$\Rightarrow$ f is continuous at $x=0$.

IAS-2003

12M 2003 1(c). Let 'a' be a +ve real number (i.e $a>0$) and $\{x_n\}$ a sequence of rational numbers such that $\underset{n\to\infty}{Lt}\, x_n = 0$

Show that $\underset{n\to\infty}{Lt}\, a^{x_n} = 1$

Sol^n: Given that $\{x_n\}$ a sequence of rational numbers such that $\underset{n\to\infty}{Lt}\, x_n = 0$

Let the sequence $\{x_n\} = \{\frac{1}{n}\}$

then we show that

$$\underset{n\to\infty}{Lt}\, a^{1/n} = 1$$

Case(i): Let $c = 1$ then

$$(c^{1/n}) = (1, 1, 1 \ldots)$$

$$\therefore \underset{n\to\infty}{Lt}\,(c^{1/n}) = 1$$

Case(ii): Let $c > 1$

then $c^{1/n} = 1 + d_n$ for some $d_n > 0$

$$\boxed{c^{1/n} - 1 = d_n}$$

and $c = (1+d_n)^n \geq (1+nd_n)\ \forall n$

(by Bernoulli's inequality)

$\Rightarrow c \geq 1 + nd_n\ \forall n$

$\Rightarrow \frac{c-1}{n} \geq d_n\ \forall n$ —①

Now we have,

Rough Idea
If $c=2$ then $c^{1/n} = 2^{1/n}$
$= 2^1, 2^{1/2}, 2^{1/3}, \ldots$
$= 2, \sqrt{2}, \sqrt[3]{2}, \ldots$
$= 2, 1.414, \ldots = 1+1, 1+0.414 \to 1$
$= 1+d_n;\ d_n>0$

$|c^{1/n} - 1| = d_n$

$\leq \frac{c-1}{n} \; \forall n$ (by ①)

$= (c-1)\frac{1}{n}$ —②

If $c=3$ then $c^{1/n} = 3^{1/n}$

$= 3^1, 3^{1/2}, 3^{1/3}, \ldots$

$= 3, \sqrt{3}, \sqrt[3]{3} \ldots$

$= 3, 1.732, \ldots \longrightarrow 1$

$= 1+2, 1+0.732 \ldots$

$= 1+d_n; \; d_n > 0$

Since $\text{Lt}\left(\frac{1}{n}\right) = 0$

and $(c-1) > 0$ $(\because c>1)$

$\therefore \underset{n\to\infty}{\text{Lt}} \left(c^{1/n}\right) = 1$

Case(iii): Let $0 < c < 1$

Rough idea

$c = 0.5$

$= \frac{1}{2}$

$c^{1/n} = \left(\frac{1}{2}\right)^{1/n}$

$= \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^{1/2}, \left(\frac{1}{2}\right)^{1/3} \ldots$

$= \frac{1}{1+1}, \frac{1}{1+0.414}, \ldots$

$= \frac{1}{1+h_n}; \; h_n > 0$

then $c^{1/n} = \frac{1}{1+h_n}$ for some $h_n > 0$

$\Rightarrow c = \frac{1}{(1+h_n)^n}$ —③

By Bernoulli's inequality,

$(1+h_n)^n \geq 1 + nh_n \; \forall n \in N$

$\frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} \; \forall n$ —④

from ③ & ④

$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+nh_n} \; \forall n$

$\Rightarrow c \leq \frac{1}{1+nh_n} < \frac{1}{nh_n} \; \forall n$ —⑤

Now we have,

$0 < c < \frac{1}{nh_n} \; \forall n$

$\Rightarrow \quad 0 < ch_n < \frac{1}{n} \; \forall n$

$\Rightarrow \quad 0 < h_n < \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \; \forall n$ —⑥

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Now we have,

$$\left| c^{1/n} - 1 \right| = \left| \frac{1}{1+h_n} - 1 \right|$$

$$= \left| \frac{-h_n}{1+h_n} \right|$$

$$= \frac{h_n}{1+h_n} < h_n \ \forall n$$

$$< \left(\frac{1}{c}\right)\left(\frac{1}{n}\right) \forall n$$

(by ⑥)

Since $\underset{n\to\infty}{Lt} \frac{1}{n} = 0$

& $c > 0$

$\Rightarrow \frac{1}{c} > 0$

$\therefore \underset{n\to\infty}{Lt} c^{1/n} = 1$

IAS-2002

2002 P-II
12m

1(c). Prove that the integral $\int_0^\infty x^{m-1}e^{-x}\,dx$ is convergent iff $m>0$.

Proof: If $m\geq 1$, the integrand $x^{m-1}e^{-x}$ is continuous at $x=0$.

If $m<1$, the integrand $\frac{e^{-x}}{x^{1-m}}$ has infinite discontinuity at $x=0$.

Thus we have to examine the convergence at 0 and ∞ both. Consider any positive number, say 1, and examine the convergence of

$$\int_0^1 x^{m-1}e^{-x}\,dx \quad \text{and} \quad \int_1^\infty x^{m-1}e^{-x}\,dx \text{ at 0 and } \infty$$

respectively.

Convergence at 0, when $m<1$

Let $f(x) = \frac{e^{-x}}{x^{1-m}}$

Take $g(x) = \frac{1}{x^{1-m}}$

Then $\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1$

which is non-zero finite.

Also $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{1-m}}$ is convergent iff $1-m<1$ i.e. $m>0$.

$\therefore$ By comparison test

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{e^{-x}}{x^{1-m}}\,dx = \int_0^1 x^{m-1}e^{-x}\,dx$$

is convergent at $x=0$ if $m>0$.

Convergence at ∞

we know that $e^x > x^{m+1}$ whatever value m may have

$\therefore\ e^{-x} < x^{-m-1}$

and $x^{m-1}e^{-x} < x^{m-1} \cdot x^{-m-1} = \frac{1}{x^2}$

since $\int_1^{\infty} \frac{1}{x^2}\,dx$ is Convergent at ∞.

$\therefore \int_1^{\infty} x^{m-1}e^{-x}\,dx$ is Convergent at ∞ for every value of m.

Now $\int_0^{\infty} x^{m-1}e^{-x}\,dx = \int_0^{1} x^{m-1}e^{-x}\,dx + \int_1^{\infty} x^{m-1}e^{-x}\,dx$

$\therefore \int_0^{\infty} x^{m-1}e^{-x}\,dx$ Converges iff $m>0$.

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1(d). If $\lim_{n\to\infty} a_n = l$, then Prove that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$

Proof: Let $b_n = a_n - l$ then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n - l$$
$$= l - l$$
$$= 0$$

$\therefore \lim_{n\to\infty} b_n = 0$

i.e. $b_n \to 0$ as $n \to \infty$ —— (1)

Since $a_n = b_n + l \ \forall n$

$$\therefore x_n = \frac{(b_1+l)+(b_2+l)+\cdots+(b_n+l)}{n}$$
$$= \frac{(b_1+b_2+\cdots+b_n)+nl}{n}$$
$$= \frac{b_1+b_2+\cdots+b_n}{n} + l$$

$\therefore$ In order to Prove that $x_n \to l$,

For this we are enough to show that

$$\frac{b_1+b_2+\cdots+b_n}{n} \to 0 \quad \text{—— (2)}$$

from (1), $\lim b_n = 0$ i.e. $b_n \to 0$ as $n \to \infty$

$\therefore$ Given $\epsilon > 0$, $\exists\, m \in I^+$ such that

$$|b_n - 0| < \frac{\epsilon}{2} \ \forall\, n \geq m.$$

$$\Rightarrow |b_n| < \frac{\epsilon}{2} \ \forall\, n \geq m \quad \text{—— (3)}$$

Also the sequence $\{b_n\}$ is Convergent.

$\therefore \{b_n\}$ is bounded.

$\therefore \exists\, M > 0$ such that $|b_n| \leq M \ \forall n$ —— (4)

Now let us prove (2),

we have

$$\left|\frac{b_1+b_2+\cdots+b_n}{n}-0\right| = \left|\frac{b_1+b_2+\cdots+b_n}{n}\right|$$

$$=\frac{|b_1+b_2+\cdots+b_n|}{|n|}$$

$$=\frac{1}{n}\left[|b_1+b_2+\cdots+b_m+b_{m+1}+b_{m+2}+\cdots+b_n|\right]$$

$(\because n\in N \Rightarrow |n|=n)$

$$\leq \frac{1}{n}\left[(|b_1|+|b_2|+\cdots+|b_m|)+(|b_{m+1}|+|b_{m+2}|+\cdots+|b_n|)\right]$$

$$< \frac{1}{n}\left[(M+M+\cdots+M\ (m \text{ times}))+\left(\frac{\epsilon}{2}+\frac{\epsilon}{2}+\cdots+\frac{\epsilon}{2}(n-m \text{ times})\right)\right] \quad \forall\, n\geq m$$

(using (3) & (4))

$$\Rightarrow \left|\frac{b_1+b_2+\cdots+b_n}{n}-0\right| < \frac{1}{n}\left[mM+(n-M)\frac{\epsilon}{2}\right] \quad \forall\, n\geq m$$

$$\Rightarrow \left|\frac{b_1+b_2+\cdots+b_n}{n}-0\right| < \frac{mM}{n}+\frac{(n-m)\epsilon}{2n} \quad \forall\, n\geq m$$

$$< \frac{mM}{n}+\frac{\epsilon}{2} \quad \left(\because \frac{n-m}{n}=1-\frac{m}{n}<1\right)\forall\, n\geq m \quad \text{--- (5)}$$

Now $\frac{mM}{n}<\frac{\epsilon}{2}$ if $\frac{n}{mM}>\frac{2}{\epsilon}$

$\Rightarrow$ if $n>\frac{2mM}{\epsilon}$

If P is a natural number $>\frac{2mM}{\epsilon}$ then $n\geq P$.

Let $q=\text{Max}\{P,m\}$

$\therefore$ from (5), $\left|\frac{b_1+b_2+\cdots+b_n}{n}-0\right| < \frac{\epsilon}{2}+\frac{\epsilon}{2} \quad \forall\, n\geq q$

$=\epsilon$

$$\therefore \left|\frac{b_1+b_2+\cdots+b_n}{n}-0\right| <\epsilon \quad \forall\, n\geq q.$$

$\frac{b_1+b_2+\cdots+b_n}{n}\to 0$ as $n\to\infty$

$x_n\to l$ as $n\to\infty$

i.e. $\lim_{n\to\infty} x_n = l$ Hence the theorem.