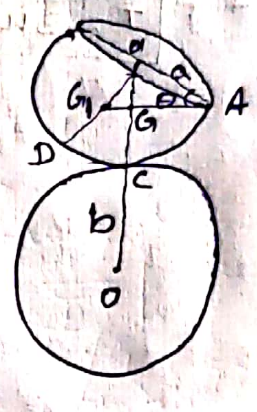
2012 - Statics

Q 1 ⇒ A heavy hemispherical shell of radius a has particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius b at the highest point. Prove that if $\frac{b}{a} > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

Sol^n let O' be the centre of base of hemispherical shell of radii a.



Centre of Gravity of hemisphere at G_1.

$$O'G_1 = \frac{O'D}{2} = \frac{a}{2}.$$

let weight is placed at A.

let G be the centre of Gravity of the combined body.

For equilibrium the line $OCGO'$ must be vertical.

let $CG = h$, $\rho_1 = a$, $\rho_2 = b$.

Equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{i.e.} \quad \frac{1}{h} > \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{h} > \frac{a+b}{ab} \quad ; \quad h < \frac{ab}{a+b}$$

The value of h depends on the weight of the particle attached at A. ~~So the equilibrium will be stable~~

Now h will be maximum if $O'G$ is minimum.

if $O'G$ is perpendicular to AG_1.

let $\angle O'AG_1 = \theta$, $\tan\theta = \frac{O'G_1}{O'A} = \frac{a/2}{a} = 1/2$

$$\sin\theta = 1/\sqrt{5}$$

∴ minimum value of $O'G = O'A \sin\theta = a \times \frac{1}{\sqrt{5}} = \frac{a}{\sqrt{5}}$

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if $\frac{a(\sqrt{5}-1)}{\sqrt{5}} < \frac{ab}{a+b}$

if $\frac{\sqrt{5}-1}{\sqrt{5}} < \frac{b}{a+b}$

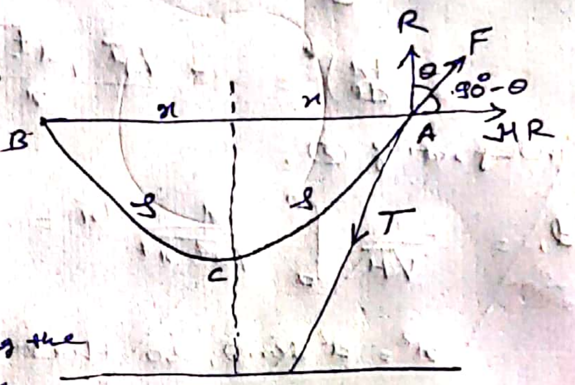
i.e. if $(\sqrt{5}-1)\,b + (\sqrt{5}-1)\,a < \frac{b}{\sqrt{5}}$

$(\sqrt{5}-1)\,a < b$, $\boxed{\frac{b}{a} > \sqrt{5}-1}$

Q2 ⇒ The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is $\mu \log\left[\frac{1+\sqrt{1+\mu^2}}{\mu}\right]$

where μ is the coefficient of friction.

Solⁿ :- Let the end links A & B of a uniform chain slide along a fixed rough horizontal rod.



$R \to$ reaction of the rod at A.

$\mu R \to$ frictional force at A along the rod in the outward direc.

$F \to$ resultant force. ~~R & μR~~.

$\theta \to$ angle b/w F & R (where $\tan\theta = \mu$).

$T \to$ tension at A.

for the point A of the catenary, we have $\psi = \psi_A = \frac{\pi}{2} - \theta$

length of the chain $= 2s = 2c\tan\psi_A = 2c\tan(\pi/2 - \theta)$

$= 2c\cot\lambda = \frac{2c}{\mu}$ $(\because \tan\lambda = \mu)$.

if (x_A, y_A) are the co-ordinates of the point A, then the maximum span $AB = 2x_A$

$= 2c\log(\tan\psi_A + \sec\psi_A)$

$= 2c\log(\tan\psi_A + \sqrt{1+\tan^2\psi_A}$

$= 2c\log\left(\cot\psi_A + \sqrt{1+\cot^2\psi_A}\right)$

$= 2c\log(\cot\theta + \sqrt{1+\cot^2\theta}$

$= 2c\log\left\{\frac{1}{\mu} + \sqrt{1+\frac{1}{\mu^2}}\right\} = 2c\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$

Hence the required ratio $= \frac{2x}{2s} = \dfrac{2c\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}}{2c/\mu}$

$= \mu\log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$

Q1→ A particle moves with an acceleration $\mu\left(x+\frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a from the origin, find its velocity when its distance from the origin is $a/2$.

Solⁿ Given, $\frac{d^2x}{dt^2} = -\mu\left[x+\frac{a^4}{x^3}\right]$ —— ①

multiply both side by $2\left(\frac{dx}{dt}\right)$ and integrating.

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2+\frac{a^4}{x^2}\right]+c$$

when $x=a$, $\frac{dx}{dt}=0$, so that $c=0$

$$\therefore \left(\frac{dx}{dt}\right) = -\frac{\sqrt{\mu}\sqrt{a^4-x^4}}{x} \quad — ②$$

~~integrating ②,~~

$$\cancel{t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{a^4-x^4}}dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{a^4-x^4}}}$$

$\cancel{x^2 = a^2\sin\theta}$, so that $2x\,dx = a^2\cos\theta\,d\theta$, when $x=0$, $\theta=0$; when $x=a$, $\theta=\pi/2$

$$\cancel{\therefore t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}}$$

$$\cancel{= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta = \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}}$$

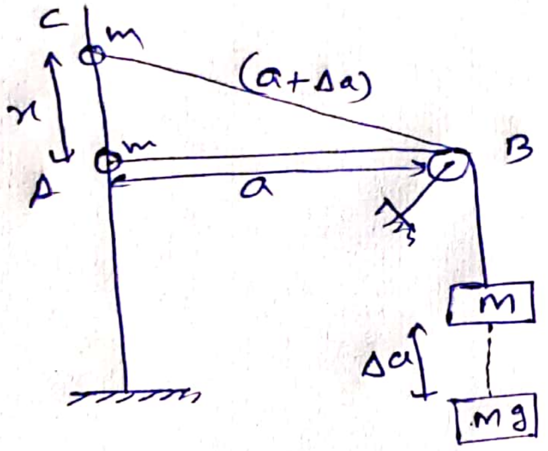
$$\left(\frac{dx}{dt}\right)_{at\ x=\frac{a}{2}} = \frac{-\sqrt{\mu}\sqrt{a^4-\frac{a^4}{16}}}{a/2}$$

$$= -\sqrt{\mu}\,\frac{\frac{\sqrt{15}\,a^2}{4}}{a/2}$$

$$= -\frac{\sqrt{15\mu}}{2}\,a$$

Q2→ A heavy ring of mass m, slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant a from the rod and has a mass M(>m) fastened to its other end. Show that if the ring be dropped from a point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2Mma}{M^2-m^2}$ before coming to rest.

Solⁿ



$$mg\Delta a = mgx$$

$$M\Delta a = mx$$

$$\Delta a = \frac{mx}{M}$$

in ΔABC

$$(a+\Delta a)^2 = a^2 + x^2$$

$$\left(a + \frac{xm}{M}\right)^2 = a^2 + x^2$$

$$\cancel{a^2} + x^2\frac{m^2}{M^2} + \frac{2amx}{M} = \cancel{a^2} + x^2$$

$$x^2\left(\frac{m^2-M^2}{M^2}\right) + \left(\frac{2am}{M}\right)x = 0$$

$$x\left(\frac{m^2-M^2}{M^2}\right) = -\frac{2am}{M}$$

$$\boxed{x = \frac{2amM}{M^2-m^2}}$$