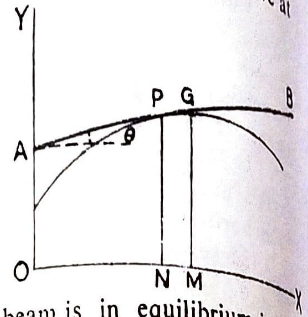
*Ex. 69. A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any position, find the equation of the curve.* **[Kanpur 80]**

Sol. Let $AB$ be the beam of length $2a$ touching the curve at $P$ and resting with its end $A$ in contact with the vertical wall $OY$. Take the wall $OY$ as the $y$-axis and a fixed horizontal line $OX$ as the $x$-axis. The weight $W$ of the beam acts at its middle point $G$. Let $z$ be the height of $G$ above the fixed horizontal line $OX$ i.e., $MG=z$. Suppose the beam makes an angle $\theta$ with the horizontal. The beam is in equilibrium in all positions. If we give the beam a small displacement in which $\theta$ changes to $\theta+\delta\theta$, then the equation of virtual work is




$$-W\delta(MG)=0, \text{ i.e., } \delta(z)=0,$$

for the reactions of the wall and the curve do not work.

$\therefore$ $z=\text{constant}=h$ (say).

Hence the coordinates of $G$ are $(a\cos\theta, h)$.

Now the straight line $AB$ passes through the point $G$ $(a\cos\theta, h)$ and makes an angle $\theta$ with the $x$-axis. Therefore the equation of $AB$ is $y-h=\tan\theta\,(x-a\cos\theta)$

i.e., $x\tan\theta-y=a\sin\theta-h$, ...(1)

where $\theta$ is the parameter.

Since $AB$ touches the curve, therefore the curve is the envelope of $AB$ for varying values of $\theta$.

Differentiating (1) partially with respect to $\theta$, we have

$$x\sec^2\theta=a\cos\theta, \text{ i.e., } x=a\cos^3\theta. \quad ...(2)$$

If we now eliminate $\theta$ between (1) and (2), we get the envelope of (1) i.e., the curve upon which the beam rests.

Putting $x=a\cos^3\theta$ in (1), we get

$$y-h=a\cos^3\theta\tan\theta-a\sin\theta=-a\sin\theta\,(1-\cos^2\theta)=-a\sin^3\theta \quad ...(3)$$

From (2) and (3), $x^{2/3}+(y-h)^{2/3}=(a\cos^3\theta)^{2/3}+(-a\sin^3\theta)^{2/3}$

$$=a^{2/3}(\cos^2\theta+\sin^2\theta)=a^{2/3}.$$

Hence the equation of the curve is $x^{2/3}+(y-a)^{2/3}=a^{2/3}$, which is an astroid.

# Strings In Two Dimensions

**(Catenary)**

## § 1. Introduction.

In the present chapter we shall consider the equilibrium of perfectly flexible strings. All those strings which offer no resistance on bending at any point are called **flexible strings**. In such cases, the resultant action across any section of the string consists of a single force whose line of action is along the tangent to the curve formed by the string. The normal section of the string is taken to be so small that it may be regarded as a curved line. A chain with short and perfectly smooth links approximates a flexible string.

## § 2. The Catenary. **[Kanpur 76]**

*When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called a* **catenary**.

**A Uniform or Common Catenary. [Raj. T.D.C. 80].** If the weight per unit length of the suspended flexible string or chain is constant, then the catenary is called the **uniform or common catenary**.

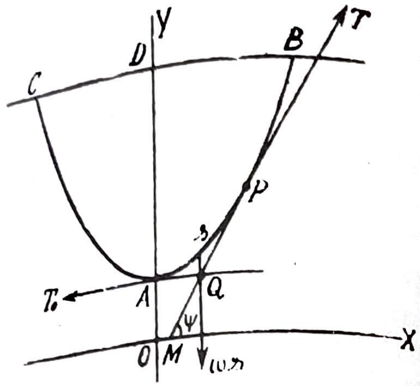
Note. Here we shall frequently use the word catenary for the common catenary *i.e.* the word catenary will always mean the common catenary in this chapter.

## § 3. Intrinsic equation of the common catenary.

**[Raj. T.D.C. 78, 80; Kanpur 80; Agra 80; Rohilkhand 86, 88; Luck. 77, 79]**

Let the uniform flexible string $BAC$ hang in the form of a uniform catenary with $A$ as its lowest point. Let $P$ be any point on the portion $AB$ of the string and $s$ the distance of $P$ from $A$ measured along the arc length of the string. If $w$ is the weight per unit length of the string, then the weight of the portion $AP$ will be $ws$ and will act vertically downwards through the centre of gravity of $AP$.

The portion $AP$ of the string is in equilibrium under the action of the following three forces :




(Fig. 1)

(i) The weight $ws$ of the string $AP$ acting vertically downwards through its centre of gravity,

(ii) The tension $T_0$ at the lowest point $A$ acting along the tangent to the curve at $A$ which is horizontal,

and (iii) The tension $T$ at $P$ acting along the tangent to the curve at $P$ inclined at an angle $\psi$ to the horizontal.

Since the string $AP$ is in equilibrium under the action of the three forces acting in the same vertical plane therefore the line of action of the weight $ws$ must pass through the point $Q$ which is the point of intersection of the lines of action of the tensions $T_0$ and $T$.

Resolving the forces acting on $AP$ horizontally and vertically, we have

$$T \cos \psi = T_0, \quad \ldots(1)$$

and

$$T \sin \psi = ws. \quad \ldots(2)$$

Dividing (2) by (1), we have

$$\tan \psi = \frac{ws}{T_0} \quad \ldots(3)$$

Now let

$$T_0 = wc, \quad \ldots(4)$$

*i.e.*, let the tension at the lowest point be equal to the weight of the length $c$, of the string, then from (3) we have

$$\tan \psi = \frac{s}{c}$$

or

$$s = c \tan \psi, \quad \ldots(5)$$

which is the *intrinsic equation of the common catenary*.

**Remark 1.** From the equation (1) it is clear that the horizontal component of the tension at every point of the catenary is the same and is equal to $T_0$, the tension at the lowest point.



**Remark 2.** From the equation (2) we conclude that the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and that point.

**Remark 3.** From the relation (4) it follows that the tension at the lowest point is equal to the weight of the string of length $c$.

## § 4. Cartesian equation of the common catenary.

**[Allahabad 78, 79; Rohilkhand 79, 81. 83, 85, 86; Lucknow 81; Agra 84, 86; Kanpur 83, 86, 87; Meerut 90, 90P]**

The intrinsic equation of the common catenary is [see § 3]

$$s = c \tan \psi, \quad \ldots(1)$$

where $\psi$ is the angle which the tangent at any point $P$ of the catenary makes with some horizontal line to be taken as the axis of $x$ and $s$ is the arc length of the catenary measured from the vertex $A$ to the point $P$.

We know that $dy/dx = \tan \psi$.

$\therefore$ from (1), we have

$$s = c \frac{dy}{dx}.$$

Differentiating both sides with respect to $x$, we have

$$\frac{ds}{dx} = c \frac{d^2y}{dx^2}$$

or

$$\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = c \frac{d^2y}{dx^2}$$

Putting $\dfrac{dy}{dx} = p$, so that $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$, we get

$$\sqrt{(1+p^2)} = c \frac{dp}{dx}$$

or

$$\frac{dx}{c} = \frac{dp}{\sqrt{(1+p^2)}}.$$

Integrating, we have

$$\frac{x}{c} + A = \sinh^{-1}(p) = \sinh^{-1}\left(\frac{dy}{dx}\right). \quad \ldots(2)$$

where $A$ is the constant of integration.

Now if we choose the vertical line throught the lowest point $A$ of the catenary as the axis of $y$, then at the point $A$, we have

$$x = 0 \text{ and } dy/dx = 0$$

because the tangent at the point $A$ is horizontal *i.e.*, parallel to the axis of $x$.

from (2), we have $A=0$.

$$\therefore \quad \frac{x}{c}=\sinh^{-1}\left(\frac{dy}{dx}\right)$$

or

$$\frac{dy}{dx}=\sinh\left(\frac{x}{c}\right).$$

Integrating both sides with respect to $x$, we have

$$y=c\cosh\left(\frac{x}{c}\right)+B,$$

where $B$ is a constant of integration.

If we take the origin $O$ at a depth $c$ below the lowest point $A$ of the catenary, then at $A$, we have $x=0$ and $y=c$.

$\therefore$ from (3), we have $B=0$.

$$\therefore \quad y=c\cosh\left(\frac{x}{c}\right), \qquad \ldots(4)$$

which is the *cartesian equation of the common catenary.*

§ 5. **Definitions.**

1. **Axis of the catenary.** Since $\cosh(x/c)$ is an even function of $x$, therefore the curve is symmetrical about the axis of $y$ which is along the vertical through the lowest point of catenary. *This vertical line of symmetry is called the* **axis of catenary.**

2. **Vertex of the catenary.** *The lowest point $A$ of the common catenary at which the tangent is horizontal is called the* **vertex of the catenary.**

3. **Parameter of rhe catenary.** *The quantity $c$ occuring in the cartesian equation $y=c\cosh(x/c)$ of the catenary is called the* **parameter of the catenary.**

4. **Directrix of the catenary.** *The horizontal line at a depth $c$ below the lowest point i.e., the axis of $x$, is called the* **directrix of the catenary.**

5. **Span and Sag.** Let the string be suspended from the two points $B$ and $C$ in the same horizontal line. *Then the distance $BC$ is called the* **span of a catenary** *and the depth $DA$ (see fig. of § 3) of the lowest point below $BC$ is called the* **sag of the catenary.**

§ 6. **Some important relations for the common catenary.**

1. **Relation between x and s.**

[Rohilkhand 85; Raj. T.D.C. 81; Kanpur 85, 87; Agra 88]

For a catenary, we have

$$s=c\tan\psi=c\frac{dy}{dx}. \qquad \left[\because \frac{dy}{dx}=\tan\psi\right]$$



$$\therefore \quad \frac{dy}{dx}=\frac{s}{c}.$$

Also $\quad y=c\cosh(x/c). \qquad \ldots(1)$

Differentiating, we have

$$\frac{dy}{dx}=\sinh\left(\frac{x}{c}\right) \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{s}{c}=\sinh(x/c),$$

or

$$s=c\sinh(x/c),$$

which is the relation between $x$ and $s$.

2. **Relation between y and s.**

[Raj. T.D.C. 81; Kanpur 84; Rohilkhand 85, 88]

For a catenary, we have

$$y=c\cosh(x/c).$$

Also $\quad s=c\sinh(x/c). \qquad$ [see relation (3)]

$\therefore$ squaring and subtracting, we have

$$y^2-s^2=c^2[\cosh^2(x/c)-\sinh^2(x/c)]=c^2$$

or

$$y^2=c^2+s^2, \qquad \ldots(4)$$

which is the relation between $y$ and $s$.

3. **Relation between y and $\psi$.**

[Gorakhpur 77; Garhwal 76; Agra 80, 87; Kanpur 81, 87; Raj. T.D.C. 79; Rohilkhand 82]

For any curve, we have

$$dy/ds=\sin\psi.$$

$$\therefore \quad \frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\sin\psi.\frac{d}{d\psi}(c\tan\psi) \qquad [\because s=c\tan\psi]$$

$$=c\sin\psi.\sec^2\psi=c\sec\psi\tan\psi.$$

Thus

$$\frac{dy}{d\psi}=c\sec\psi\tan\psi.$$

Integrating, we get $y=c\sec\psi+A$, where $A$ is a constant of integration.

But when $\quad y=c,\ \psi=0; \quad \therefore \ A=0.$

Hence $\quad y=c\sec\psi, \qquad \ldots(5)$

which is the relation between $y$ and $\psi$.

**Aliter.** From relation (4), we have $y^2=c^2+s^2$.

Also $\quad s=c\tan\psi.$

$$\therefore \quad y^2=c^2+s^2=c^2+c^2\tan^2\psi=c^2(1+\tan^2\psi)=c^2\sec^2\psi$$

or

$$y=c\sec\psi.$$

4. **Relation between x and $\psi$.** [Garhwal 76; Alld. 79; Agra 84, 87; Gorakhpur 72, 75, 80; Kanpur 84]

For any curve, we have $dx/ds = \cos\psi$.

$$\therefore \quad \frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi} = \cos\psi \frac{d}{d\psi}(c \tan\psi) \qquad [\because s = c \tan\psi]$$

$$= \cos\psi \,(c \sec^2\psi)$$

or $$\frac{dx}{d\psi} = c \sec\psi.$$

Integrating, we get

$$x = c \log(\sec\psi + \tan\psi) + B,$$

where $B$ is a constant of integration.

But when $x=0, \psi=0$; $\therefore B=0$.

Hence $$x = c \log(\sec\psi + \tan\psi), \qquad ...(6)$$

which is the relation between $x$ and $\psi$.

**Note.** The equations (5) and (6) together form the parametric equations of the catenary, $\psi$ being the parameter.

**5. Relation between tension and ordinate.**

**[Rohilkhand 82; Allad. 78; Agra 85; Raj. T.D.C. 79(S); Gorakhpur 81; Kanpur 77]**

From § 3, we have

$$T \cos\psi = T_0 \quad \text{and} \quad T_0 = wc$$

$$\therefore \quad T = T_0 \sec\psi = wc \sec\psi.$$

But $$y = c \sec\psi.$$

Hence $$T = wy, \qquad ...(7)$$

which is the relation between $T$ and $y$.

*The relation* (7) *shows that the tension at any point of a catenary varies as the height of the point above the directrix.*

**6. Radius of curvature at any point of a catenary.**

**[Rohilkhand 82]**

For a catenary, we have $s = c \tan\psi$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = c \sec^2\psi. \qquad ...(8)$$

**Illustrative Examples**

**Ex. 1.** *If the normal at any point $P$ of a catenary meets the directrix at $Q$, show that $PQ=\rho$.*

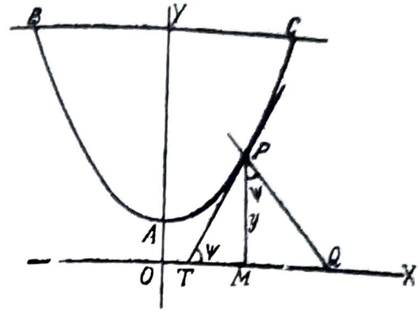
**Sol.** If the tangent at the point $P$ makes an angle $\psi$ with the axis of $x$, then the normal $PQ$ at $P$ will make an angle $\psi$ with the vertical through $P$.

Let $PM$ be the ordinate of the point $P$. Then from the right angled triangle $PMQ$, we have

$$PQ = PM \sec\psi = y \sec\psi$$

$$= c \sec^2\psi \qquad [\because y = c \sec\psi]$$






(Fig. 2)

$= \rho$, because for the catenary $s = c \tan\psi$,

$$\rho = ds/d\psi = c \sec^2\psi.$$

**Ex. 2.** *If $T$ be the tension at any point $P$ of a catenary, and $T_0$ that at the lowest point $A$, prove that*

$$T^2 - T_0^2 = W^2,$$

*$W$ being the weight of the arc $AP$ of the catenary.*

**[Rohilkhand 81,83; Agra 85]**

**Sol.** Let arc $AP=s$ and $\psi$ be the inclination of the tangent at $P$ to the horizontal. If $w$ is the weight per unit length of the string, then $W$= weight of the arc $AP = ws$.

If $T$ is the tension at the point $P$ of the catenary and $T_0$ that at the lowest point $A$, then we have

$$T\cos\psi = T_0 \quad \text{and} \quad T \sin\psi = ws = W.$$

Squaring and adding, we have

$$T^2 = T_0^2 + W^2 \quad \text{or} \quad T^2 - T_0^2 = W^2.$$

**Ex. 3.** *Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their weights.*

**Sol.** Let $T_1$ and $T_2$ be the tensions at the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively of the chain.

Then from $T=wy$, we have

$$T_1 = wy_1 \quad \text{and} \quad T_2 = wy_2.$$

$$\therefore \quad T_2 - T_1 = w(y_2 - y_1)$$

or $$T_2 - T_1 \propto (y_2 - y_1). \qquad [\because w \text{ is constant}]$$

Hence the difference of the tensions at $P$ and $Q$ varies as the difference of their heights $y_1$ and $y_2$.

**Ex. 4** *Show that for a common catenary* $x=c\log\left(\frac{y+s}{c}\right)$.

**[Agra 85; Gorakhpur 76; Rohilkhand 77]**

**Sol.** The parametric equations of a catenary are

$x=c\log(\sec\psi+\tan\psi)$, ...(1)

and $y=c\sec\psi$. ...(2)

Also for a catenary,

$s=c\tan\psi$. ...(3)

From (2) and (3), we have

$\sec\psi=y/c$ and $\tan\psi=s/c$.

Substituting in (1), we get

$x=c\log\left(\frac{y}{c}+\frac{s}{c}\right)$, or $x=c\log\left(\frac{y+s}{c}\right)$.

**Ex. 5.** *If* $(\bar{x}, \bar{y})$ *be the coordinates of the centre of gravity of the arc measured from the vertex upto the point* $P(x, y)$, *prove that*

$\bar{x}=x-c\tan(\psi/2)$, $\bar{y}=\frac{1}{2}(c/\cos\psi+x\cot\psi)$.

**Sol.** The parametric equations of a catenary are

$x=c\log(\sec\psi+\tan\psi)$ and $y=c\sec\psi$.

Also for a catenary $s=c\tan\psi$. $\therefore$ $ds/d\psi=c\sec^2\psi$.

We have

$$\bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_0^\psi x\frac{ds}{d\psi}\,d\psi}{\int_0^\psi \frac{ds}{d\psi}\,d\psi}\qquad [\because \text{ At the vertex, } \psi=0]$$

$$=\frac{\int_0^\psi c\log(\sec\psi+\tan\psi)\,c\sec^2\psi\,d\psi}{\int_0^\psi c\sec^2\psi\,d\psi}$$

$$=\frac{c^2\left[\left\{\log(\sec\psi+\tan\psi).\tan\psi\right\}_0^\psi-\int_0^\psi\frac{1}{(\sec\psi+\tan\psi)}(\sec\psi\tan\psi+\sec^2\psi)\tan\psi\,d\psi\right]}{c\left[\tan\psi\right]_0^\psi}$$

[Integrating the numerator by parts taking $\sec^2\psi$ as 2nd function]

$$=\frac{c\left[\tan\psi\log(\sec\psi+\tan\psi)-\int_0^\psi\sec\psi\tan\psi\,d\psi\right]}{\tan\psi}$$



$$=\frac{c\left[\tan\psi\log(\sec\psi+\tan\psi)-\left\{\sec\psi\right\}_0^\psi\right]}{\tan\psi}=\frac{x\tan\psi-c(\sec\psi-1)}{\tan\psi}.$$

$$=x-\frac{c(\sec\psi-1)}{\tan\psi}=x-\frac{c(1-\cos\psi)}{\sin\psi}=x-c\frac{2\sin^2\frac{1}{2}\psi}{2\sin\frac{1}{2}\psi\cos\frac{1}{2}\psi}$$

$=x-c\tan\frac{1}{2}\psi$,

$$\text{and } \bar{y}=\frac{\int y\,ds}{\int ds}=\frac{\int_0^\psi y\frac{ds}{d\psi}\,d\psi}{\int_0^\psi\frac{ds}{d\psi}\,d\psi}=\frac{\int_0^\psi c\sec\psi.c\sec^2\psi\,d\psi}{\int_0^\psi c\sec^2\psi\,d\psi}$$

$$=\frac{c^2\left[\left\{\frac{1}{2}\sec\psi\tan\psi\right\}_0^\psi+\frac{1}{2}\int_0^\psi\sec\psi\,d\psi\right]}{c\left[\tan\psi\right]_0^\psi}$$

$$\left[\because \int\sec^n\theta\,d\theta=\frac{\sec^{n-2}\theta\tan\theta}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}\theta\,d\theta\right]$$

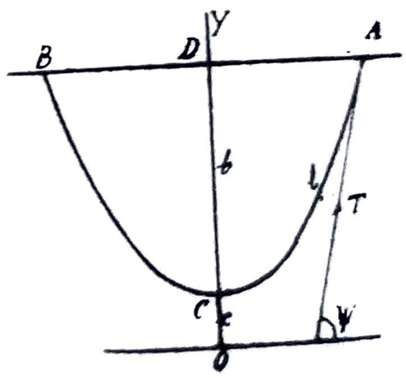
$$=\frac{c\left[\frac{1}{2}\sec\psi\tan\psi+\frac{1}{2}\left\{\log(\sec\psi+\tan\psi)\right\}_0^\psi\right]}{\tan\psi}$$

$$=\frac{1}{2\tan\psi}\left[c\sec\psi\tan\psi+c.\log(\sec\psi+\tan\psi)\right]$$

$$=\frac{1}{2\tan\psi}\left[c\sec\psi\tan\psi+x\right]=\frac{1}{2}\left[\frac{c}{\cos\psi}+x\cot\psi\right].$$

**Ex. 6.** *A rope of length 2l feet is suspended between two points at the same level, and the lowest point of the rope is b feet below the points of suspension. Show that the horizontal component of the tension is* $w(l^2-b^2)/2b$, *w being the weight of the rope per foot of its length.* **[G.N.U. 77]**

**Sol.** Let the rope $ACB$ of length $2l$ be suspended between two




(Fig. 3)

points $A$ and $B$ and the same level and let $C$ be its lowest point. It is given that the sag $CD=b$.

Let $OC=c$ be the parameter of the catenary.

For the point $A$ of the catenary, we have $s=l$ and $y=c+b$.

Substituting these values in the formula $y^2=c^2+s^2$, we have

$$(c+b)^2=c^2+l^2$$

or $$c^2+2cb+b^2=c^2+l^2$$

or $$2cb=l^2-b^2, \text{ or } c=(l^2-b^2)/2b.$$

Now the horizontal component of the tension at any of the catenary is constant and is equal to $wc$, where $w$ is the weight per unit length of the string.

$\therefore$ here the horizontal component of the tension $=\dfrac{w(l^2-b^2)}{2b}$.

**Ex. 7.** *A uniform chain of length $l$, is to be suspended from two points $A$ and $B$, in the same horizontal line so that either terminal tension is $n$ times that at the lowest point. Show that the span $AB$ must be*

$$\frac{l}{\sqrt{(n^2-1)}} \log\{n+\sqrt{(n^2-1)}\}.$$

**[Agra 85; Gorkhpur 79, 82]**

**Sol.** Draw figure as in Ex. 6 on page 9.

Let the uniform chain $ACB$ of length $l$ be suspended from two points $A$ and $B$ in the same horizontal line.

Let $(x_A, y_A)$ be the coordinates of the point $A$ and $\phi_A$ be the angle which the tangent at $A$ makes with the $x$-axis.

If $T$ is the tension at the terminal point $A$ and $T_0$ that at the lowest point, then given that $T=nT_0$

But $$T=wy_A \text{ and } T_0=wc.$$

$$\therefore\ wy_A=nwc$$

or $$wc\sec\psi_A=nwc. \qquad [\because y=c\sec\psi]$$

$$\therefore\ \sec\psi_A=n. \qquad \text{...(1)}$$

Now at the point $A$, $s=\text{arc } CA=\frac{1}{2}l$.

$\therefore$ from $s=c\tan\psi$, we have

$$\tfrac{1}{2}l=c\tan\psi_A$$

or $$c=\frac{l}{2\tan\psi_A}=\frac{l}{2\sqrt{(\sec^2\psi_A-1)}}=\frac{l}{2\sqrt{(n^2-1)}}. \qquad \text{...(2)}$$

Hence the span $AB=2DA=2x_A$

$$=2c\log(\sec\psi_A+\tan\psi_A)$$



$$=2c\log\{\sec\psi_A+\sqrt{(\sec^2\psi_A-1)}\}$$

$$=\frac{l}{\sqrt{(n^2-1)}}\log\{n+\sqrt{(n^2-1)}\}.$$

[Substituting for $\sec\psi_A$ from (1) and for $c$ from (2)]

**Ex. 8.** *A uniform chain of length $l$, is suspended from two points $A$, $B$ in the same horizontal line. If the tension at $A$ is twice that at the lowest point show that the span $AB$ is*

$$\frac{l}{\sqrt{3}}\log(2+\sqrt{3}).$$

**[Kanpur 80]**

**Sol.** Proceed exactly as in the preceding example 7.

Here $n=2$.

**Ex. 9.** *A uniform chain of length $l$, which can just bear a tension of $n$ times its weight, is stretched between two points on the same horizontal line. Show that the least possible sag in the middle is $l\{n-\sqrt{n^2-\frac{1}{4}}\}$.* **[Raj T.D.C. 79, 81; Kanpur 84; Rohilkhand 86]**

**Sol.** (Refer figure of Ex. 6 on Page 9).

Let the uniform chain of given length $l$ be stretched between two points $A$ and $B$ in the same horizontal line. Obviously the sag in the middle is least when the span $AB$ is maximum. Since the chain can just bear a tension of $n$ times its weight, therefore in the case of maximum span $AB$ the terminal tension at the end $A$ should be equal to $nlw$, where $w$ is the weight per unit length of the chain. So, for the least possible sag, if $T$ is the tension at the end $A$, we have

$$T=nwl. \qquad \text{...(1)}$$

If $(x_A, y_A)$ are the coordinates of the point $A$, then

$$T=wy_A. \qquad \text{...(2)}$$

$\therefore$ from (1) and (2), we have

$nwl=wy_A$, or $y_A=nl$. Thus for the point $A$, $y=y_A=nl$. ...(3)

Now arc $CA=\frac{1}{2}l$, *i.e.*, for the point $A$, $s=\frac{1}{2}l$.

Using the formula $y^2=c^2+s^2$ for the point $A$, we have

$$n^2l^2=c^2+\tfrac{1}{4}l^2,$$

or $$c^2=l^2\left(n^2-\tfrac{1}{4}\right), \text{ or } c=l\sqrt{\left(n^2-\tfrac{1}{4}\right)} \qquad \text{...(4)}$$

$\therefore$ least possible sag in the middle

$=CD=OD-OC=y_A-c$

$=nl-l\sqrt{(n^2-\frac{1}{4})}$ [Substituting for $y_A$ from (3) and for $c$ from (4)]

$=l\{n-\sqrt{(n^2-\frac{1}{4})}\}$.

**Ex. 10.** *A given length, 2s, of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is* $\sqrt{(b^2-s^2)}\log\left(\dfrac{b+s}{b-s}\right)$.

**[Raj. T. D. C. 81; Agra 79, 86]**

**Sol.** [Refer figure of Ex. 6 on page 9].

Let a uniform chain of given length $2s$ be hung between two points $A$ and $B$ at the same level. Let $w$ be the weight per unit length of the chain. Then $bw$ is the weight of a length $b$ of the chain. From the formula $T=wy$, we observe that the tension in the chain is maximum at the terminal point $A$. Since the tension in the chain is not to exceed $bw$, therefore in the case of maximum span $AB$ if $T$ is the tension at $A$, we must have $T=bw$. ...(1)

If $(x_A, y_A)$ are the coordinates of the point $A$, then $T=wy_A$. ...(2)

From (1) and (2), we have

$$wy_A=bw, \quad \text{or} \quad y_A=b. \qquad \text{...(3)}$$

Now using the formula $y^2=c^2+s^2$ for the point $A$, we have

$$b^2=c^2+s^2, \quad \text{or} \quad c^2=b^2-s^2,$$

$$\therefore \quad c=\sqrt{(b^2-s^2)}. \qquad \text{...(4)}$$

Now the greatest span $=AB=2x_A$

$=2c\log(\sec\psi_A+\tan\psi_A)$, $\psi_A$ being the inclination of the tangent at $A$ to the x-axis

$$=2c\log\left(\frac{c\sec\psi_A+c\tan\psi_A}{c}\right)$$

$$=2c\log\left(\frac{y_A+s}{c}\right) \qquad [\because y_A=c\sec\psi_A \text{ and } s=c\tan\psi_A]$$

$$=2\sqrt{(b^2-s^2)}\log\left[\frac{b+s}{\sqrt{(b^2-s^2)}}\right], \text{ putting for } y_A \text{ from (3) and for } c \text{ from (4)}$$

$$=2\sqrt{(b^2-s^2)}\log\left[\frac{(b+s)}{\sqrt{\{(b+s)(b-s)\}}}\right]$$

$$=\sqrt{(b^2-s^2)}\log\left(\frac{b+s}{b-s}\right)^{1/2}$$

$$=\sqrt{(b^2-s^2)}\log\left(\frac{b+s}{b-s}\right).$$

**Ex. 11.** *If α, β be the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary, and l the length of the portion, show that the height of one extremity above the other is* $\dfrac{l\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, *the two extremities being on one side of the vertex of the catenary.*

**[Rohilkhand 88, 89; Gorakhpur 81; Agra 87; Luck. 80]**



**Sol.** Let $\alpha$, $\beta$ be the inclinations to the horizontal of the tangents at the extremities $P$ and $Q$ ($P$ lying above $Q$) of a portion $PQ$ of a common catenary, the points $P$ and $Q$ being on the same side of the vertex of the catenary. Draw the figure by taking an arc $PQ$ of a catenary lying only on one side of the vertex.

If $y_P$ and $y_Q$ are the ordinates of the points $P$ and $Q$ respectively, then from $y=c\sec\psi$, we have

$$y_P=c\sec\alpha \quad \text{and} \quad y_Q=c\sec\beta$$

$$[\because \psi=\alpha \text{ at } P \text{ and } \psi=\beta \text{ at } Q]$$

$\therefore$ Height of the extremity $P$ above the other extremity $Q$

$$=y_P-y_Q=c(\sec\alpha-\sec\beta). \qquad \text{...(1)}$$

If $C$ is the lowest point (*i.e.*, the vertex) of the catenary of which $PQ$ is an arc, then from the formula $s=c\tan\psi$, we have

$$\text{arc } CP=c\tan\alpha \quad \text{and} \quad \text{arc } CQ=c\tan\beta.$$

$\therefore$ $l=$ length of the arc $PQ=$ arc $CP-$ arc $CQ$

$$=c(\tan\alpha-\tan\beta)$$

or

$$c=\frac{l}{(\tan\alpha-\tan\beta)}.$$

$\therefore$ from (1), the required height

$$=\frac{l(\sec\alpha-\sec\beta)}{(\tan\alpha-\tan\beta)}=\frac{l(\cos\beta-\cos\alpha)}{(\sin\alpha\cos\beta-\cos\beta\sin\alpha)}$$

$$=\frac{2l\sin\frac{1}{2}(\beta+\alpha)\sin\frac{1}{2}(\alpha-\beta)}{\sin(\alpha-\beta)}$$

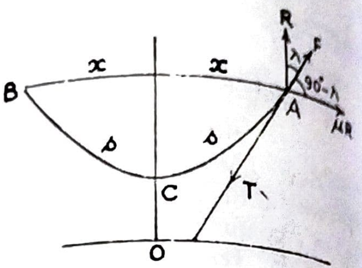
$$=\frac{2l\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)}=\frac{l\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

**Ex. 12.** *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is*

$$\mu\log\left\{\frac{1+\sqrt{(1-\mu^2)}}{\mu}\right\}$$

*where μ is the coefficient of friction.*

**[Kanpur 85; Raj. T.D.C. 78, Meerut 90; Rohilkhand 88]**




(Fig. 4)

**Sol.** Let the end links $A$ and $B$ of a uniform chain slide along a fixed rough horizontal rod. If $AB$ is the maximum span, then $A$ and $B$ are in the state of limiting equilibrium. Let $R$ be the reaction of the rod at $A$ acting perpendicular to the rod. Then the frictional force $\mu R$ will act at $A$ along the rod in the outward direction $BA$ as shown in the figure. The resultant $F$ of the forces $R$ and $\mu R$ at $A$ will make an angle $\lambda$ (where $\tan \lambda = \mu$) with the direction of $R$. For the equilibrium of $A$ the resultant $F$ of $R$ and $\mu R$ at $A$ will be equal and opposite to the tension $T$ at $A$.

Since the tension at $A$ acts along the tangent to the chain at $A$, therefore the tangent to the catenary at $A$ makes an angle $\psi_A = \frac{1}{2}\pi - \lambda$ to the horizontal.

Thus for the point $A$ of the catenary, we have $\psi = \psi_A = \frac{1}{2}\pi - \lambda$.

$\therefore$ the length of the chain

$$= 2s = 2c \tan \psi_A = 2c \tan (\tfrac{1}{2}\pi - \lambda)$$

$$= 2c \cot \lambda = \frac{2c}{\mu}. \quad [\because \tan \lambda = \mu]$$

If $(x_A, y_A)$ are the coordinates of the point $A$, then the maximum span $AB = 2x_A$

$$= 2c \log (\tan \psi_A + \sec \psi_A)$$

$$= 2c \log \{\tan \psi_A + \sqrt{(1 + \tan^2 \psi_A)}\}$$

$$= 2c \log \{\cot \lambda + \sqrt{(1 + \cot^2 \lambda)}\} \qquad [\because \psi_A = \tfrac{1}{2}\pi - \lambda]$$

$$= 2c \log \left\{\frac{1}{\mu} + \sqrt{\left(1 + \frac{1}{\mu^2}\right)}\right\} = 2c \log \left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Hence the required ratio

$$= \frac{2x}{2s} = \frac{2c \log \left\{\dfrac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}}{(2c/\mu)}$$

$$= \mu \log \left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}.$$

**Ex. 13.** *If the ends of a uniform inextensible string of length $l$ hanging freely under gravity slide on a fixed rough horizontal rod whose coefficient of friction is $\mu$, show that at most they can rest at a distance $\mu l \log \left\{\dfrac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}$.*

[Lack. 75]



**Sol.** [Refer figure of Ex. 12 on page 13]

Proceed as in the last example 12.

At $A$, $\psi = \frac{1}{2}\pi - \lambda$.

Also at the point $A$, $s = \text{arc } CA = \frac{1}{2}l$.

Using the formula $s = c \tan \psi$ for the point $A$, we have

$$\tfrac{1}{2}l = c \tan (\tfrac{1}{2}\pi - \lambda) = c \cot \lambda = c/\mu.$$

$$\therefore \quad c = \mu l/2.$$

$\therefore$ the required maximum span $AB$

$$= 2x = 2c \log (\tan \psi + \sec \psi)$$

$$= 2c \log [\tan \psi + \sqrt{(2 + \tan^2 \psi)}]$$

$$= 2c \log [\cot \lambda + \sqrt{(1 + \cot^2 \lambda)}]$$

$$[\because \text{ for } A, \psi = \tfrac{1}{2}\pi - \lambda]$$

$$= 2c \log \left[\frac{1}{\mu} + \sqrt{\left(1 + \frac{1}{\mu^2}\right)}\right]$$

$$[\because \cot \lambda = 1/\tan \lambda = 1/\mu]$$

$$= \mu l \log \left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\} \qquad [\because c = \mu l/2]$$

**Ex. 14.** *The extremities of a heavy string of length $2l$ and weight $2lw$, are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to $lw$. Show that the distance apart of the rings is*

$$2l \log (1 + \sqrt{2}).$$

**Sol.** [Refer figure of Ex. 12 on page 13]

Here the length of the string $ACB = 2l$ and its weight $= 2lw$.

$\therefore$ the weight per unit length of the string $= w$.

Here it is given that the small ring at $A$ is acted on by a horizontal force $lw$. For the equilibrium of the small ring at $A$, the horizontal force $lw$ must balance the horizontal component of the tension at $A$. But the horizontal component of the tension at any point of the string is equal to $wc$. So we must have

$$lw = wc, \quad \text{or} \quad c = l.$$

Since arc $CA = l$, therefore for the point $A$, $s = l$. So using the formula $s = c \tan \psi$ for the point $A$, we have

$$l = c \tan \psi, \quad \text{or} \quad \tan \psi = l/c = l/l = 1.$$

$$\therefore \quad \text{for } A, \psi = \psi_A = 45°.$$

If $(x_A, y_A)$ are the corrdinates of the point $A$, then the distance between the rings $=AB=2x_A$

$$=2c \log (\tan \psi_A+\sec \psi_A)$$

$$=2l \log (\tan 45^\circ+\sec 45^\circ)=2l \log (1+\sqrt{2}).$$

**Ex. 15.** *A uniform chain of length $2l$ is suspended by its ends which are on the same horizontal level. The distance apart $2a$ of the ends is such that the lowest point of the chain is at a distance $a$ vertically below the ends. Prove that if $c$ be the distance of the lowest point from the directrix of the catenary, then*

$$\frac{2a^2}{l^2-a^2}=\log \left(\frac{l+a}{l-a}\right) \text{ and } \tanh \frac{a}{c}=\frac{2al}{l^2+a^2}.$$ **[Gorakhpur 76]**

**Sol.** [Refer figure of Ex. 6 on page 9]

Here arc $CA=l$, $AB=2a$, $CD=a$ and $OC=c$.

If $(x_A, y_A)$ are the coordinates of the point $A$, then

$x_A=DA=\frac{1}{2}AB=a$ and $y_A=OC+CD=c+a$.

At $A$, $s=$ arc $CA=l$

$\therefore$ Using the formula $y^2=c^2+s^2$ for the point $A$, we have

$(c+a)^2=c^2+l^2$, or $c^2+2ac+a^2=c^2+l^2$.

$$\therefore \quad c=\frac{l^2-a^2}{2a}. \qquad \ldots(1)$$

Also using the formula $s=c \tan \psi$ for the point $A$, we have

$$l=c \tan \psi.$$

$$\therefore \text{ at } A, \tan \psi=\frac{l}{c}=\frac{2al}{l^2-a^2}.$$

Now at $A$, we have $x=a$. So using the formula

$$x=c \log (\tan \psi+\sec \psi),$$

for the point $A$, we have

$$a=c \log \{\tan \psi+\sqrt{(1+\tan^2 \psi)}\}$$

or

$$a=\frac{l^2-a^2}{2a} \log \left[\frac{2al}{l^2-a^2}+\sqrt{\left\{1+\frac{4a^2l^2}{(l^2-a^2)^2}\right\}}\right]$$

or

$$\frac{2a^2}{l^2-a^2}=\log \left\{\frac{2al}{l^2-a^2}+\frac{l^2+a^2}{l^2-a^2}\right\}=\log \left(\frac{2al+l^2+a^2}{l^2-a^2}\right)$$

$$=\log \frac{(l+a)^2}{(l+a)(l-a)}.$$

$$\therefore \quad \frac{2a^2}{l^2-a^2}=\log \left(\frac{l+a}{l-a}\right).$$



Again for a catenary $s=c \sinh (x/c)$. Since at $A$, $s=l$ and $x=x_A=a$, therefore we have

$$l=c \sinh \left(\frac{a}{c}\right), \text{ or } \sinh \frac{a}{c}=\frac{l}{c}=\frac{2al}{l^2-a^2}.$$

$$\therefore \quad \tanh \frac{a}{c}=\frac{\sinh \frac{a}{c}}{\cosh \frac{a}{c}}=\frac{\sinh \frac{a}{c}}{\sqrt{\left(\sinh^2 \frac{a}{c}+1\right)}}$$

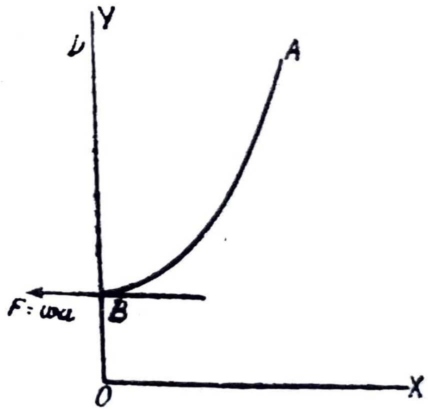
$$=\frac{\frac{2al}{l^2-a^2}}{\sqrt{\left\{\frac{4a^2l^2}{(l^2-a^2)^2}+1\right\}}}=\frac{2al}{l^2+a^2}.$$

**Ex. 16.** *A heavy uniform string, of length $l$, is suspended from a fixed point $A$, and its other end $B$ is pulled horizontally by a force equal to the weight of a length $a$ of the string. Show that the horizontal and vertical distances between $A$ and $B$ are*

$$a \sinh^{-1} (l/a) \text{ and } \sqrt{(l^2+a^2)}-a,$$

*respectively.*

**Sol.** In the equilibrium position the arc $AB$ will represent half of the arc of the complete catenary with $B$ as its lowest point.




(Fig. 5)

The horizontal force $F=aw$ by which the end $B$ is pulled is equal to the tension $T_0$ at the lowest point $B$.

$\therefore$ $aw=T_0=wc$, so that $c=a$.

Let $(x_A, y_A)$ be the coordinates of the point $A$.

We have arc $BA=l$ i.e., at $A$, $s=$ arc $BA=l$.

∴ from $s=c\sinh(x/c)$, we have

$$l=a\sinh(x_A/a) \qquad [\because c=a]$$

∴ $x_A=a\sinh^{-1}(l/a)$ = the horizontal distance between $B$ and $A$.

Again from $y^2=s^2+c^2$, we have

$$y_A^2=l^2+a^2$$

But $y_A=OD=OB+BD=c+BD=a+BD$.

∴ $(a+BD)^2=l^2+a^2$, or $a+BD=\sqrt{(l^2+a^2)}$.

or $BD=\sqrt{(l^2+a^2)}-a$ = the vertical distance between $B$ and $A$.

Hence the horizontal and vertical distances between $A$ and $B$ are $a\sinh^{-1}(l/a)$ and $\sqrt{(l^2+a^2)}-a$ respectively.

**Ex. 17.** *A box kite is flying at a height $h$ with a length $l$ of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite the inclination of the wire to the ground is*

$$2\tan^{-1}(h/l),$$

*and that its tensions there and at the ground are*

$$w\frac{l^2+h^2}{2h} \text{ and } w\frac{l^2-h^2}{2h},$$

*where $w$ is the weight of the wire per unit of length.* **[Luck. 80; Kanpur 87]**

**Sol.** Let $AB$ be the wire. $B$ the vertex of the catenary on the ground and $A$ the position of the kite.

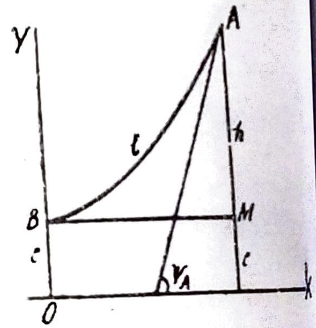
The height of $A$, above $B$ is $h$ i.e., $AM=h$.

We have the ordinate of the point $A=y_A=c+h$ and at $A$, $s=\text{arc } BA=l$.

∴ from $y^2=c^2+s^2$,

we have $y_A^2=c^2+l^2$,

or $(c+h)^2=c^2+l^2$, or $c^2+ch+h^2=c^2+l^2$.




(Fig. 6)

$$\therefore \quad c=\frac{l^2-h^2}{2h} \text{ and } y_A=c+h=\frac{l^2-h^2}{2h}+h=\frac{l^2+h^2}{2h}.$$

If the tangent at $A$ is inclined at an angle $\psi_A$ to the ground, then from $s=c\tan\psi$, we have

$$l=c\tan\psi_A$$

or $$\tan\psi_A=\frac{l}{c}=\frac{2hl}{l^2-h^2}=\frac{2(h/l)}{1-(h/l)^2}.$$



$$\therefore \quad \psi_A=\tan^{-1}\left\{\frac{2(h/l)}{1-(h/l)^2}\right\}=2\tan^{-1}(h/l)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

∵ for a catenary $T=wy$,

∴ the tension at the point $A=T_A=wy_A=w\dfrac{l^2+h^2}{2h}$

and the tension at the point $B=T_B=wy_B=wc=w\dfrac{l^2-h^2}{2h}$

**Ex. 18.** *$A$ is the lowest point of a uniform thread hanging from two fixed points, $B$ and $C$. Let $a$, $b$ be the heights of $A$ and $B$ above the directrix of the catenary formed by the thread. Show that the length of the thread between $A$ and $B$ equals $\sqrt{(b^2-a^2)}$.*

**Sol.** The lowest point $A$ of the thread is the vertex of the catenary in which the thread hangs.

Let the length of the thread between the lowest point $A$ and the point $B=l$.

Then for the point $B$, we have $s=s_B=l$.

The height of the lowest point $A$ above the directrix $OX$ i.e., the $x$-axis $=c=a$ (given)

and the height of the point $B$ above the directrix $OX$
= the ordinate of the point $B=y_B=b$ (given)

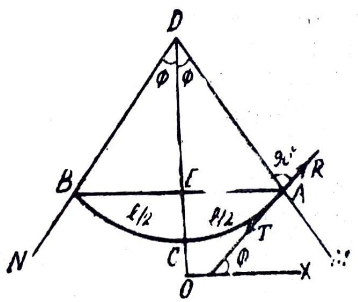
∴ applying the formula $y^2=c^2+s^2$ for the point $B$, we have

$$y_B^2=a^2+s_B^2 \quad \text{i.e.,} \quad b^2=a^2+l^2.$$

$$\therefore \quad l=\sqrt{(b^2-a^2)}.$$

**Ex. 19.** *The end links of a uniform chain of length $l$ can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles $\phi$ to the vertical. Prove that the sag in the middle is $\frac{1}{2}l\tan\frac{1}{2}\phi$.* **[Gorakhpur 73]**

**Sol.** The end links $A$ and $B$ of a uniform chain $ACB$ slide on two smooth rods $DM$ and $DN$ which are inclined in opposite directions at equal angles $\phi$ to the vertical as shown in the figure. The point $C$ is the vertex of the catenary in which the chain hangs.




(Fig. 7)

Let us consider the equilibrium of the link at $A$. Two forces are acting upon it : (i) the reaction $R$ of the rod $DM$ acting perpendicular to the rod, and (ii) the tension $T$ of the chain acting tangentially to the chain at $A$. For the equilibrium of the link at $A$ these two forces must be equal and opposite and their lines of action must coincide. Therefore the tangent at $A$ to the catenary makes an angle $\phi$ with the horizontal or say with the directrix $OX$ of the catenary.

Thus at $A$, we have $\psi=\psi_A=\phi$.

Also at $A$, we have $s=s_A=\text{arc } CA=\frac{1}{2}l$.

Using the formula $s=c\tan\psi$ for the point $A$, we have

$s_A=c\tan\psi_A$, or $\frac{1}{2}l=c\tan\phi$, or $c=\frac{1}{2}l\cot\phi$.

Now the sag in the middle $=EC=OE-OC=y_A-c$ $\quad [\because\ y=c\sec\psi]$

$=c\sec\psi_A-c$

$=c(\sec\phi-1)=\frac{1}{2}l\cot\phi(\sec\phi-1)=\frac{1}{2}l\dfrac{1-\cos\phi}{\sin\phi}$ $\quad [\because\ \psi_A=\phi]$

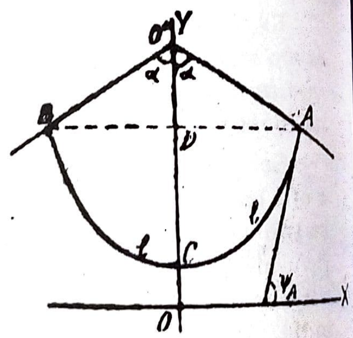
$=\frac{1}{2}l\dfrac{2\sin^2\frac{1}{2}\phi}{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}=\frac{1}{2}l\tan\frac{1}{2}\phi.$

**Ex. 20.** *A uniform heavy chain is fastened at its extremities to two rings of equal weight, which slide on smooth rods intersecting in a vertical plane, and inclined at the same angle $\alpha$ to the vertical, find the condition that the tension at the lowest point may be equal to half the weight of the chain, and in that case, show that the vertical distance of the rings from the point of intersection of the rods is*

$$l\cot\alpha\log(\sqrt{2}+1),$$

*where $2l$ is the length of the chain.*

**Sol.** Let the rods inclined at the same angle $\alpha$ to the vertical intersect at the point $O'$. Since the rods are inclined at the same angles, rings are of the same weight and the chain is uniform, hence in equilibrium the positions of the rings will be symmetrical with respect to the vertical line through the point $O'$. Let $A$ and $B$ be the positions of the rings in equilibrium. Clearly $AB$ is horizontal. Let $O'D$ be perpendicular from $O'$ on $AB$, $OO'$ the axis of the catenary and $OX$ the directrix. Let $OC=c$.




(Fig. 8)



If $w$ is the weight per unit length of the chain, then its weight $=2wl$.

Then according to the question, if the tension at the lowest point is equal to half the weight of the chain, *i.e.*, if $T_0=wc=wl$, we have $c=l$.

For the point $A$ of the catenary, we have $s=s_A=\text{arc } CA=l$.

If the tangent at $A$ is inclined at an angle $\psi_A$ to the horizontal, then from $s=c\tan\psi$, we have

$s_A=c\tan\psi_A$, or $l=c\tan\psi_A$

or $l=l\tan\psi_A$ $\quad [\because\ c=l]$

or $\tan\psi_A=1$, or $\psi_A=\frac{1}{4}\pi$.

Hence the condition that the tension at the lowest point may be equal to half the weight of the chain is that the tangents at the ends $A$ and $B$ of the chain will make an angle $\frac{1}{4}\pi$ to the horizontal.

Now using the formula $x=c\log(\tan\psi+\sec\psi)$ for the point $A$, we have

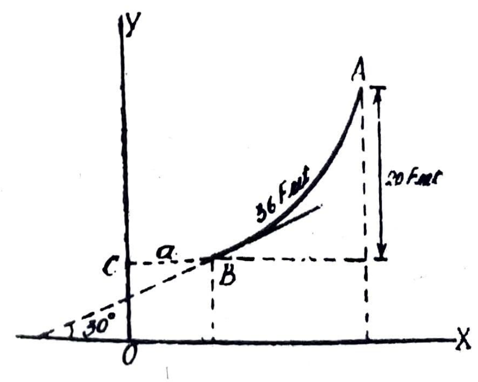
$$x_A=DA=c\log(\tan\tfrac{1}{4}\pi+\sec\tfrac{1}{4}\pi)=l\log(1+\sqrt{2}).$$

Hence the vertical distance of the rings from the point of intersection $O'$ of the rods

$$=O'D=DA\cot\alpha=l\cot\alpha\log(1+\sqrt{2}).$$

**Ex. 21.** *A boat is towed by means of a rope attached to a ship and the lower end of the rope makes an angle of 30° to the horizontal. If the length of the rope is 36 feet, and the upper end is 20 feet higher than the lower end, find the resistance of the water to the motion of the boat, the weight of each foot of the rope being ten ounces.*

**Sol.** Let $AB$ be the rope of length 36 feet with the lower end




(Fig. 9)

$B$ making an angle of 30° to the horizontal. Let $C$ be the lowest point i.e., the vertex, $OX$ the directrix and $c$ the parameter of the catenary for which $AB$ is an arc.

If arc $CB=a$ feet, then $a=c \tan 30°=c/\sqrt{3}$.

Let $y_A$ and $y_B$ be the ordinates of the ends $A$ and $B$ respectively.

Then $\qquad y_A-y_B=20$ ft. ...(1)

But from $y=c \sec \psi$, we have

$$y_B=c \sec 30°=2c/\sqrt{3} \text{ feet.}$$

$$\therefore \quad y_A=y_B+20=\left(\frac{2c}{\sqrt{3}}+20\right) \text{ feet.}$$

Also $s_A=\text{arc } CA=\text{arc } CB+\text{arc } BA=a+36=\left(\frac{c}{\sqrt{3}}+36\right)$ feet.

Now using the formula $y^2=c^2+s^2$ for the point $A$, we have

$$y_A^2=c^2+\left(\frac{c}{\sqrt{3}}+36\right)^2$$

or $$\left(\frac{2c}{\sqrt{3}}+20\right)^2=c^2+\left(\frac{c}{\sqrt{3}}+36\right)^2$$

or $$\frac{4}{3}c^2+\frac{80}{\sqrt{3}}c+400=c^2+\frac{1}{3}c^2+\frac{72}{\sqrt{3}}c+1296$$

or $$\frac{8}{\sqrt{3}}c=1296-400. \quad \therefore \quad c=112\sqrt{3} \text{ feet.}$$

Now $w$ = weight of one foot of the rope $=\frac{10}{16}=\frac{5}{8}$ lbs.

The resistance due to the water will act horizontally and therefore will be equal to

$$T_0=wc=\tfrac{5}{8}\times 112\sqrt{3}=70\times 1{\cdot}732=121{\cdot}2 \text{ lbs. wt.}$$

**Ex. 22.** *A weight $W$ is suspended from a fixed point by a uniform string of length $l$ and weight $w$ per unit length. It is drawn aside by a horizontal force $P$. Show that in the position of equilibrium, the distance of $W$ from the vertical through the fixed point is*

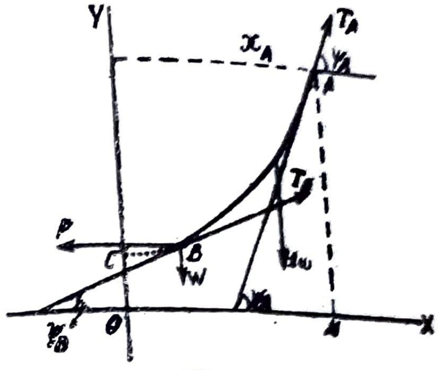
$$\frac{P}{w}\left\{\sinh^{-1}\left(\frac{W+lw}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right\}. \qquad \text{(Luck. 78)}$$

**Sol.** The end $A$ of the string $AB$ of length $l$ is attached to the fixed point $A$ and a weight $W$ hanging at the other end $B$ of the string is drawn aside by a horizontal force $P$.

Let us first consider the equilibrium of the end $B$ of the string. There are three forces acting on it : (i) the horizontal force $P$ applied at $B$, (ii) the weight $W$ suspended at $B$ and acting



vertically downwards, and (iii) the tension $T_B$ of the string $BA$ acting along the tangent to the string at $B$ which makes an angle $\psi_B$ with $OX$.




(Fig. 10)

Resolving these forces horizontally and vertically, we have

$$T_B \cos \psi_B=P \qquad ...(1)$$

and $$T_B \sin \psi_B=W. \qquad ...(2)$$

Dividing (2) by (1), we have

$$\tan \psi_B=W/P. \qquad ...(3)$$

Since the horizontal component of the tension at any point of the string is constant and is equal to $wc$, therefore

$$T_B \cos \psi_B=wc=P, \text{ so that}$$

$$c=P/w. \qquad ...(4)$$

Now let us consider the equilibrium of the whole string $AB$. The forces acting on it are : (i) the horizontal force $P$ applied at $B$, (ii) the weight $W$ suspended at $B$, (iii) the weight $lw$ of the string $AB$ acting vertically downwards through the centre of gravity of the string $AB$, and (iv) the tension $T_A$ at the end $A$ acting along the tangent to the string at $A$ which makes an angle $\psi_A$ with $OX$. Resolving these forces horizontally and vertically, we have

$$T_A \cos \psi_A=P \qquad ...(5)$$

and $$T_A \sin \psi_A=W+lw. \qquad ...(6)$$

Dividing (6) by (5), we have

$$\tan \psi_A=(W+lw)/P. \qquad ...(7)$$

Now the distance of the weight $W$ from the vertical $AM$ through the fixed point $A$

= (the $x$-coordinate of $A$) − (the $x$-coordinate of $B$)

$= x_A - x_B$.

But for a catenary, we have

$$s = c \sinh (x/c).$$

$$\therefore \quad x = c \sinh^{-1}\left(\frac{s}{c}\right) = c \sinh^{-1}\left(\frac{c \tan \psi}{c}\right) \qquad [\because \ s = c \tan \psi]$$

$$= c \sinh^{-1} (\tan \psi).$$

$$\therefore \quad x_A - x_B = c \sinh^{-1} (\tan \psi_A) - c \sinh^{-1} (\tan \psi_B)$$

$$= \frac{P}{w}\left\{ \sinh^{-1}\left(\frac{W + lw}{P}\right) - \sinh^{-1}\left(\frac{W}{P}\right)\right\},$$

substituting for $c$ from (4), for $\tan \psi_A$ from (7) and for $\tan \psi_B$ from (3).

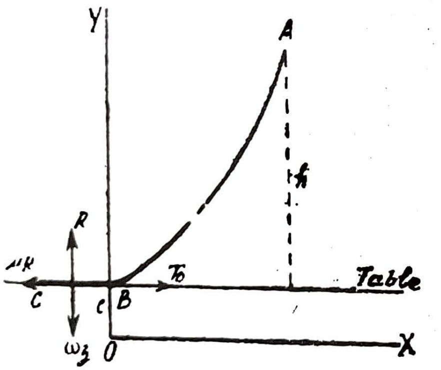
This gives the required distance of $W$ from the vertical through the fixed point $A$.

**Ex. 23.** *A length $l$ of a uniform chain has one end fixed at a height $h$ above a rough table, and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length of the table is*

$$l + \mu h - \sqrt{\{(\mu^2 + 1) h^2 + 2\mu l h\}},$$

*where $\mu$ is the coefficient of friction.* **[Rohilkhand 87]**

**Sol.** Let one end of a uniform chain $ABC$ of length $l$ be fixed at $A$ at a height $h$ above the rough table and let the portion $BC$ of the chain rest on the table.




(Fig. 11)

Let $BC = z$.

The chain rests in limiting equilibrium with the portion $AB$ in the form of an arc of a catenary with $B$ as its vertex *i.e.*, the lowest point. Let us consider the limiting equilibrium of the portion $BC$ of the chain lying on the table.



The forces acting on the portion $BC$ of the chain are : (i) its weight $wz$ acting vertically downwards, (ii) the normal reaction $R$ of the table acting perpendicular to the table, (iii) the force of limiting friction $\mu R$ along the table in the direction $BC$ which is opposite to the direction $CB$ in which the chain $BC$ will begin to slip, and (iv) the tension $T_0$ in the chain at $B$.

For the equilibrium of the chain $BC$, resolving these forces horizontally and vertically, we have

$$R = wz \quad \text{and} \quad \mu R = T_0 = wc.$$

$$\therefore \quad \mu w z = wc, \quad \text{or} \quad c = \mu z.$$

Length of the arc $AB = s_A = l - z$,

and the ordinate of the point $A = y_A = c + h$.

$\therefore$ using the formula $y^2 = c^2 + s^2$ for the point $A$, we have

$$y_A^2 = c^2 + s_A^2$$

or $$(c + h)^2 = c^2 + (l - z)^2$$

or $$z^2 - 2lz + (l^2 - h^2) - 2ch = 0$$

or $$z^2 - 2(l + \mu h) z + (l^2 - h^2) = 0. \qquad [\because \ c = \mu z]$$

$$\therefore \quad z = \frac{2(l + \mu h) \pm \sqrt{[4(l + \mu h)^2 - 4(l^2 - h^2)]}}{2.1}$$

or $$z = (l + \mu h) \pm \sqrt{[(l + \mu h)^2 - (l^2 - h^2)]}.$$

The plus sign will give $z > l$, which is impossible. Therefore neglecting the plus sign, the required length $z$ of the chain on the table is given by

$$z = (l + \mu h) - \sqrt{[(l + \mu h)^2 - (l^2 - h^2)]}$$

$$= (l + \mu h) - \sqrt{[(\mu^2 + 1) h^2 + 2\mu l h]}.$$

**Ex. 24.** *One extremity of a uniform string is attached to a fixed point and the string rests partly on a smooth inclined plane ; prove that the directrix of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.*

*If $\alpha$ is the inclination of the plane, $\beta$ the inclination of the tangent at the fixed extremity, and $l$ the whole length of the string, prove that the length of the portion in contact with the plane is*

$$\frac{l \cos \beta}{\cos \alpha \cos (\beta - \alpha)}.$$

**Sol.** Let $ABC$ be a string of length $l$ of which the portion $BC$ is on the plane inclined at an angle $\alpha$ to the horizontal. Let the length of the portion $BC$ of the string be $a$. Then the length of the portion $AB$ of the string hanging in the form of an arc of a catenary is $l-a$.

Let us consider the equilibrium of the portion $BC$ of the string lying on the inclined plane. There are three forces acting on it: (i) its weight $aw$ acting vertically downward through its centre of gravity, (ii) the normal reaction $R$ of the inclined plane acting perpendicular to the plane, and (iii) the tension $T_B$ of the string at $B$ acting along the tangent at $B$ to the string which is along the ling $CB$ lying in the inclined plane. Resolving these forces along the inclined plane, we have

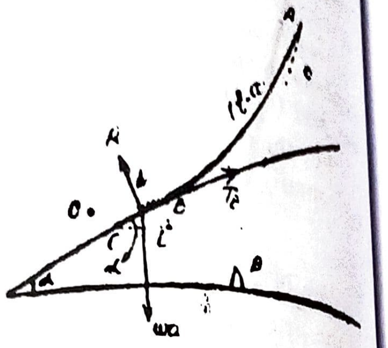



Fig. 12

$$T_B = wa \sin \alpha. \quad ...(1)$$

Let $BL$ be the perpendicular from $B$ on the horizontal line through $C$. Then $BL = BC \sin \alpha = a \sin \alpha$.

$\therefore$ from (1), we have $T_B = w.BL$.

But for a catenary, from $T = wy$, we have

$$T_B = wy_B,$$

where $y_B$ is the vertical distance of the point $B$ from the directrix of the catenary.

Thus we have $\quad wBL = wy_B$,

so that $\quad y_B = BL$.

Hence the directrix of the catenary $AB$ is the horizontal line $CL$ through the exremity $C$ of the portion of the string which rests on the inclined plane.

**Second Part.** Let $O$ be the lowest point *i.e.*, the vertex of the catenary of which $AB$ is a part. The inclinations to the horizontal of the tangents at $B$ and $A$ to the string are $\alpha$ and $\beta$ respectively.

Then from $s = c \tan \psi$, we have

$$\text{arc } OB = c \tan \alpha \text{ and arc } OA = c \tan \beta;$$

$$\therefore \text{ arc } AB = \text{arc } OA - \text{arc } OB = c(\tan \beta - \tan \alpha)$$

or $$l - a = c(\tan \beta - \tan \alpha). \quad ...(2)$$

From (1), we have

$$wa \sin \alpha = T_B$$



$$= wy_B = w.c \sec \alpha \qquad [\because \text{ from } y = c \sec \psi, \text{ we have } y_B = c \sec \alpha]$$

$$\therefore \quad a = c \sec \alpha \operatorname{cosec} \alpha. \quad ...(3)$$

Diving (2) by (3), we have

$$\frac{l-a}{a} = \frac{c(\tan \beta - \tan \alpha)}{a \sec \alpha \operatorname{cosec} \alpha}$$

or $$\frac{l-a}{a} = \left(\frac{\sin \beta}{\cos \beta} - \frac{\sin \alpha}{\cos \alpha}\right) \sin \alpha \cos \alpha = \frac{\sin(\beta-\alpha) \sin \alpha}{\cos \beta}$$

or $(l-a) \cos \beta = a \sin(\beta - \alpha) \sin \alpha$

or $l \cos \beta = a[\cos \beta + \sin(\beta-\alpha) \sin \alpha]$

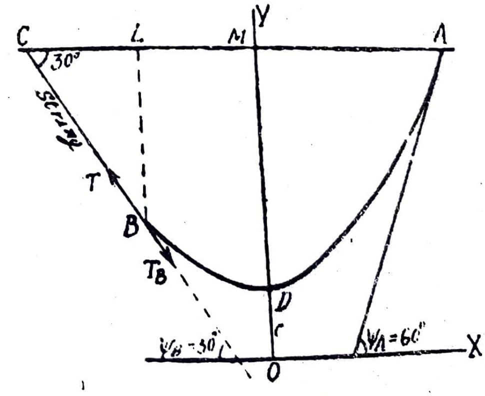
$$= c[\cos \beta + (\sin \beta \cos \alpha - \cos \beta \sin \alpha) \sin \alpha]$$
$$= a[\sin \beta \cos \alpha \sin \alpha + (1 - \sin^2 \alpha) \cos \beta]$$
$$= a[\sin \beta \cos \alpha \sin \alpha + \cos^2 \alpha \cos \beta]$$
$$= a \cos \alpha [\sin \alpha \sin \beta + \cos \alpha \cos \beta]$$
$$= a \cos \alpha \cos(\beta - \alpha).$$

$$\therefore \quad a = \frac{l \cos \beta}{\cos \alpha \cos(\beta - \alpha)}.$$

**Ex. 25.** *A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles 60° and 30° respectively with the horizontal. Prove that the ratio of these lengths is* $(\sqrt{3}-1) : 1$.

**[Gorakhpur 82; Kanpur 81; Agra 80, 86]**

**Sol.** Let the lengths of the heavy uniform chain $AB$ and the light string $BC$ be $l$ and $a$ respectively.




(Fig. 13)

The chain $AB$ being heavy will hang in the form of an arc of a catenary while the string $BC$ being light will hang in the form of a straight line. Since the tension $T_B$ of the chain at $B$ will be balanced by the tension $T$ in the string, therefore the string $BC$ will be along the tangent at the point $B$ of the chain. Let $D$ be the lowest point *i.e.*, the vertex of the catenary and $OX$ the directrix such that $OD=c$. If the tangents at $A$ and $B$ are inclined at angles $\psi_A$ and $\psi_B$ to the horizontal, then given that

$$\psi_A=60^\circ \quad \text{and} \quad \psi_B=30^\circ.$$

Let $y_A$ and $y_B$ be the ordinates of the points $A$ and $B$ respectively. Then from $y=c\sec\psi$, we have

$$y_B=c\sec\psi_B=c\sec 30^\circ=2c/\sqrt{3}$$

and

$$y_A=c\sec\psi_A=c\sec 60^\circ=2c.$$

Let $BL$ be the perpendicular from $B$ on $AC$.

Then $$BL=BC\sin 30^\circ=\tfrac{1}{2}a. \qquad [\because BC=a]$$

$$\therefore a=2BL=2(y_A-y_B)=2\left(2c-\frac{2c}{\sqrt{3}}\right)=\frac{4c}{\sqrt{3}}(\sqrt{3}-1).$$

If the length of the arc $DA$ be $s_1$ and that of the arc $DB$ be $s_2$, then from $s=c\tan\psi$, we have

$$s_1=c\tan 60^\circ=c\sqrt{3}, \text{ and } s_2=c\tan 30^\circ=c/\sqrt{3}.$$

$\therefore$ $l=$ the length of the chain $ADB$

$$=s_1+s_2=c\sqrt{3}+c/\sqrt{3}=4c/\sqrt{3}.$$

Hence the ratio of the lenghts of the string and the chain

$$=\frac{a}{l}=\frac{(4c/\sqrt{3})(\sqrt{3}-1)}{(4c/\sqrt{3})}=\frac{\sqrt{3}-1}{1}=(\sqrt{3}-1):1.$$

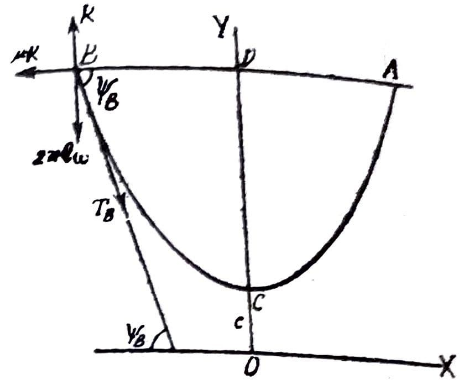
**Ex. 26.** *A heavy chain, of length $2l$, has one end tied at $A$ and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through $A$. If the weight of the ring be $n$ times the weight of the chain, show that its greatest possible distance from $A$ is $\frac{2l}{\lambda}\log\{\lambda+\sqrt{(1+\lambda^2)}\}$, where $1/\lambda=\mu(2n+1)$ and $\mu$ is the coefficient of friction.*

**[Agra 85; Kanpur 78, 81]**

**Sol.** Let one end of a heavy chain of length $2l$ be fixed at $A$ and the other end be attached to a small heavy ring which can slide on a rough horizontal rod $ADB$ through $A$. Let $B$ be the position of limiting equilibrium of the ring when it is at greatest possible distance from $A$.

In this position of limiting equilibrium the forces acting on the ring are : (i) the weight $2nlw$ of the ring acting vertically downwards, (ii) the normal reaction $R$ of the rod, (iii) the force of limiting friction $\mu R$ of the rod acting in the direction $AB$, and (iv) the tension $T_B$ in the string at $B$ acting along the tangent to the string at $B$.






(Fig. 14)

For the equilibrium of the ring at $B$, resolving the forces acting on it horizontally and vertically, we have

$$\mu R=T_B\cos\psi_B \qquad \ldots(1)$$

and

$$R=2nlw+T_B\sin\psi_B, \qquad \ldots(2)$$

where $\psi_B$ is the angle of inclination of the tangent at $B$ to the horizontal.

Let $C$ be the lowest point of the catenary formed by the chain, $OX$ be the directrix and $OC=c$ be the parameter. We have arc $CB=s_B=l$. By the formula $T\cos\psi=wc$, we have $T_B\cos\psi_B=wc$. Also by the formula $T\sin\psi=ws$, we have

$$T_B\sin\psi_B=ws_B=wl.$$

Putting these values in (1) and (2), we have

$$\mu R=wc \text{ and } R=2nlw+wl=(2n+1)wl.$$

$$\therefore \mu(2n+1)wl=wc \quad \text{or} \quad \mu(2n+1)l=c.$$

But it is given that $\mu(2n+1)=1/\lambda$.

$$\therefore l/\lambda=c \qquad \ldots(3)$$

Using the formula $s=c\tan\psi$ for the point $B$, we have

$$l=c\tan\psi_B;$$

$$\therefore \tan\psi_B=l/c=\lambda. \qquad \ldots(4)$$

Now the required greatest possible distance of the ring from $A$

$$=AB=2DB=2x_B$$

$$=2c\log(\sec\psi_B+\tan\psi_B) \qquad [\because x=c\log(\sec\psi+\tan\psi)]$$

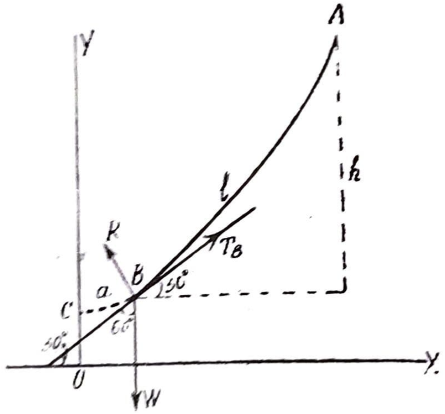
$$=2c\log[\tan\psi_B+\sqrt{(1+\tan^2\psi_B)}]$$

$$=\frac{2l}{\lambda}\log\left[\lambda+\sqrt{(1+\lambda^2)}\right]$$

[$\because$ from (3), $c=l/\lambda$ and from (4), $\tan\psi_B=\lambda$]

**Ex. 27.** *A uniform inextensible string, of length $l$ and weight $wl$, carries at one end B, a particle of weight W which is placed on a smooth plane inclined at 30° to the horizontal. The other end of the string is attached to a point A, situated at a height h above the horizontal through B and in the vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the, tangent at B to the catenary lying in the inclined plane if*
$$\frac{W}{w}=\frac{(l-h)\,(l-h)}{(h-\frac{1}{2}l)}.$$

**Sol.** Let $AB$ be the string of weight $wl$ and length $l$, carrying a particle of weight $W$ at the end $B$ which is placed on the plane inclined at an angle 30° to the horizontal. The other end of the string is attached to the fixed point $A$, at a height $h$ above the




(Fig. 15)

horizontal through $B$. Let the particle rest in equilibrium with the tangent at $B$ to the catenary lying in the inclined plane. And let $C$ be the lowest point and $OX$ the directrix of the catenary of which $AB$ is a part and let the arc $CB=a$.

Let $T_B$ be the tension at $B$ and $y_B$ be the ordinate of the point $B$.

The tension $T_B$ at $B$ is inclined at an angle $\psi_B=30°$ to the horizontal.

The particle of weight $W$ at $P$ is in equilibrium under the action of three forces :



(i) the weight $W$ acting vertically downwards,

(ii) the normal reaction $R$ of the inclined plane and

(iii) the tension $T_B$ of the string at $B$ acting along the tangent at $B$.

Resolving these forces along the inclined plane, we have

$T_B=$the component of weight $W$ along the inclined plane$=W\cos 60°$.

$$\therefore\quad T_B=\tfrac{1}{2}W.$$

Now
$$y_B=c\sec\psi_B=c\sec 30°=2c/\sqrt{3}$$
and
$$T_B=wy_B=2wc/\sqrt{3}.$$
$$\therefore\quad \frac{W}{2}=\frac{2wc}{\sqrt{3}},\quad\text{or}\quad c=\frac{\sqrt{3}W}{4w}.$$

Also from $s=c\tan\psi$, at $B$, we have $a=c\tan\psi_B$

or
$$a=c\tan 30°=\frac{\sqrt{3}W}{4w}\cdot\frac{1}{\sqrt{3}}=\frac{W}{4w}.$$

Now at $B$, $s=a$, $y=y_B$

and at $A$, $s=a+l$, $y=y_B+h$.

$\therefore$ from $y^2=c^2+s^2$, we have

$$y_B^2=c^2+a^2\ \text{and}\ (y_B+h)^2=c^2+(a+l)^2.$$

Subtracting, we have
$$h^2+2hy_B=l^2+2al$$

or
$$h^2+2h.\frac{2c}{\sqrt{3}}=l^2+2.\frac{W}{4w}l\quad[\because\ y_B=2c/\sqrt{3},\ \text{and}\ a=W/4w]$$

or
$$h^2+\frac{4h}{\sqrt{3}}.\frac{\sqrt{3}W}{4w}=l^2+\frac{Wl}{2w}\qquad\left[\because\ c=\frac{\sqrt{3}W}{4w}\right]$$

or
$$\frac{W}{w}.\left(h-\frac{l}{2}\right)=l^2-h^2$$

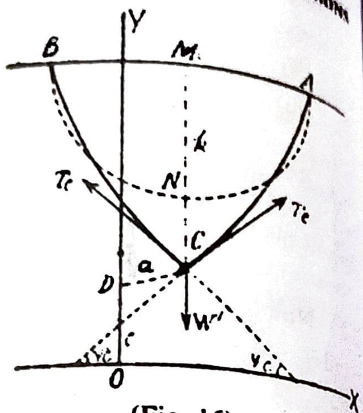
or
$$\frac{W}{w}=\frac{(l^2-h^2)}{(h-\frac{1}{2}l)}=\frac{(l-h)\,(l+h)}{(h-\frac{1}{2}l)}.$$

**Ex. 28.** *A uniform chain, of length l and weight W, hangs between two fixed points at the same level, and a weight W′ is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of support is*

$$\frac{k}{2l}W+\frac{l}{4k}W'+\frac{l}{8k}W.$$

**[Raj. T.D.C. 79 (S); Gorakhpur 81; Rohilkhand 80; Allad 73]**

**Sol.** Let a string of length $l$ and weight $W$ suspended from two points $A$ and $B$ at the same level hang freely under gravity in the form of the catenary $ANB$.




(Fig. 16)

When a weight $W'$ is attached at the middle point $N$ of the string then it will descend downwards to $C$, and the two portions $AC$ and $BC$ of the string each of length $\frac{1}{2}l$ will be the parts of two equal catenaries. Let $D$ be the lowest point, *i.e.*, the vertex and $OX$ the directrix of the catenary of which $AC$ is an arc.

The weight per unit length of the chain $= w = W/l$.

If $T_C$, $T_C$ are the tensions at the point $C$ in the string $CA$ and $CB$ acting along the tangents at $C$, then resolving vertically the forces acting at $C$, we have

$$2T_C \sin \psi_C = W', \qquad ...(1)$$

where the tangents at $C$ to the arcs $AC$ and $BC$ are inclined at an angle $\psi_C$ to the horizontal.

But from $T \sin \psi = ws$, we have

$$T_C \sin \psi_C = wa, \text{ where arc } DC = a.$$

$\therefore$ from (1), we have

$$2wa = W', \text{ or } a = \frac{W'}{2w} = \frac{W'l}{2W}. \qquad ...(2)$$

Let $y_A$ and $y_C$ be the ordinates of the points $A$ and $C$ respectively.

Now at $A$, $s = s_A = \text{arc } DA = \text{arc } DC + \text{arc } CA = a + \frac{1}{2}l$
and $y = y_A$;
at $C$, $s = s_C = \text{arc } DC = a$ and $y = y_C$.

Since sag in the middle $= CM = k$,

$\therefore$ $y_C + k = y_A$ or $y_C = y_A - k$.

$\therefore$ from $y^2 = c^2 + s^2$, we have

$$y_A^2 = c^2 + (a + \tfrac{1}{2}l)^2 \text{ and } y_C^2 = c^2 + a^2.$$

Subtracting, we have

$$y_A^2 - y_C^2 = al + \tfrac{1}{4}l^2$$



$$y_A^2 - (y_A - k)^2 = al + \tfrac{1}{4}l^2 \qquad [\because y_C = y_A - k]$$

or

$$2ky_A - k^2 = al + \tfrac{1}{4}l^2 = \frac{W'l^2}{2W} + \frac{l^2}{4}$$

or $\qquad [\because a = W'l/2W \text{ from (2)}]$

$$y_A = \frac{k}{2} + \frac{l^2W'}{4kW} + \frac{l^2}{8k}$$

Hence the pull (*i.e.*, the tension) at either point of support $A$ or $B$

$$= T_A = wy_A = \frac{W}{l}\left(\frac{k}{2} + \frac{l^2W'}{4kW} + \frac{l^2}{8k}\right) \qquad \left[\because w = \frac{W}{l}\right]$$

$$= \frac{k}{2l}W + \frac{l}{4k}W' + \frac{l}{8k}W.$$

**Ex. 29 (a).** *A uniform chain of length $l$ and weight $W$ hangs between two fixed points at the same level and a weight is suspended from its middle point so that the total sag in the middle is $h$. Show that if $P$ is the pull on either point of support, the weight suspended is*

$$\frac{4h}{l}P - \left(\frac{1}{2} + \frac{2h^2}{l^2}\right)W. \qquad \textbf{[Rohilkhand 85]}$$

**Sol.** Proceed exactly as in the last Ex. 28.

If $W'$ is the weight suspended at the middle point of the chain, then the pull $P$ at either point of support is given by

$$P = \frac{h}{2l}W + \frac{l}{4h}W' + \frac{l}{8h}W.$$

$$\therefore \frac{l}{4h}W' = P - \left(\frac{h}{2l} + \frac{l}{8h}\right)W.$$

or

$$W' = \frac{4h}{l}P - \left(\frac{2h^2}{l^2} + \frac{1}{2}\right)W.$$

**Ex. 29. (b).** *A uniform chain of length $2l$ and weight $W$, is suspended from two points $A$ and $B$, in the same horizontal line. A load $P$ is now suspended from the middle point $D$ of the chain and the depth of this point below $AB$ is found to be $h$. Show that each terminal tension is* $\frac{1}{2}\left[P\frac{l}{h} + W\frac{h^2+l^2}{2hl}\right]$.

**[I.A.S. 79; Allad. 76; Kanpur 83]**

**Sol.** Peoceed as in Ex. 28.

**Ex. 30.** *A uniform string of weight $W$ is suspended from two points at the same level and a weight $W'$ is attached to its lowest point. If $\alpha$ and $\beta$ are now the inclinations to the horizontal of the tangents at the highest and lowest points, prove that*

$$\frac{\tan \alpha}{\tan \beta} = 1 + \frac{W}{W'}.$$

**[Raj. T.D.C. 80; Luck. 81; Kanpur 86; Agra 8**

**Sol.** [Refer figure of Ex. 28, on page 32. In this question the inclinations of the tangents at $A$ and $C$ to the horizontal are $\alpha$ and $\beta$ respectively].

If $l$ is the length of the string and $w$ the weight per unit length, then $w=W'/l$.

Let $T_C$, $T_C$ be the tensions at the point $C$ in the strings $CA$ and $CB$ acting along the tangents at $C$ to the arcs $CA$ and $CB$.

Resolving vertically the forces acting at $C$, we have

$$2T_C \sin \beta = W'. \quad ...(1)$$

[$\because$ the tangent at $C$ is inclined at an angle $\beta$ to the horizontal]

Since the horizontal component of the tension at each point of a catenary is equal to $wc$, where $c$ is the parameter of the catenary, therefore

$$T_C \cos \beta = wc. \quad ...(2)$$

Dividing (1) by (2), we have

$$2 \tan \beta = \frac{W'}{wc}. \quad ...(3)$$

Let the tension at each point of support $A$ and $B$ be $T$ acting along the tangents at these points which are inclined at an angle $\alpha$ to the horizontal.

Considering the equilibrium of the whole string and resolving the forces vertically, we have

$2T \sin \alpha =$ the total weight of the system acting vertically downwards,

or $$2T \sin \alpha = W+W'. \quad ...(4)$$

Also the horizontal component of tension $T$ at $A$,

$$T \cos \alpha = wc. \quad ...(5)$$

Dividing (4) by (5) we have

$$2 \tan \alpha = \frac{W+W'}{wc}. \quad ...(6)$$

Now dividing (6) by (3), we have

$$\frac{\tan \alpha}{\tan \beta} = \frac{W+W'}{W'}$$

or $$\frac{\tan \alpha}{\tan \beta} = 1 + \frac{W}{W'}.$$

**Ex. 31.** *A uniform chain, of length $2l$ and weight $2w$, is suspended from two points in the same horizontal line. A load $w$ is now suspended from the middle point of the chain and the depth of this point below the horizontal line is $h$. Show that the terminal tension is*

$$\frac{1}{2} w \cdot \frac{h^2+2l^2}{hl}.$$

**[Rohilkhand 83, 86, 89; Luck. 79, 86, 89]**



**Sol.** [Refer figure of Ex. 28, page 32].

Let a weight $w$ be suspended at the middle point $C$ of the spring suspended from two points $A$ and $B$ in the same horizontal line.

Let $MC=h$ (Depth of the middle point $C$ of the chain below the horizontal line $AB$).

If $w'$ is the weight per unit length of the chain, then

$$w' = \frac{2w}{2l} = \frac{w}{l}.$$

Let $T_C$, $T_C$ be the tensions at the point $C$ in the strings $CA$ and $CB$ acting along the tangents at $C$.

Let the tangents at the point $C$ be inclined at an angle $\psi_C$ to the horizontal.

Resolving vertically the forces acting at $C$, we have

$$2T_C \sin \psi_C = w. \quad ...(1)$$

Let $D$ be the vertex of the catenary of which $AC$ is an arc.

If arc $DC=a$, then from $T \sin \psi = w's$, we have

$$T_C \sin \psi_C = w'a. \quad ...(2)$$

$\therefore$ from (1) and (2), we have

$$2w'a = w, \quad \text{or} \quad a = \frac{w}{2w'} = \frac{l}{2}. \quad \left[\because w' = \frac{w}{l}\right]$$

Let $y_A$ and $y_C$ be the ordinates of the points $A$ and $C$ respectively.

Then at $C$, $y=y_C$, $s=$ arc $DC=a=l/2$

and at $A$, $y=y_A$, $s=$ arc $DA=$ arc $DC+$ arc $CA=\frac{1}{2}l+l=\frac{3}{2}l$.

$\therefore$ from $y^2=s^2+c^2$, we have

$$y_C^2 = \tfrac{1}{4}l^2+c^2 \quad \text{and} \quad y_A^2 = \tfrac{9}{4}l^2+c^2.$$

Subtracting, we have

$$y_A^2 - y_C^2 = 2l^2. \quad ...(3)$$

But $$y_C = y_A - CM = y_A - h.$$

$\therefore$ from (3), we have

$$y_A^2 - (y_A-h)^2 = 2l^2$$

or $$2hy_A = 2l^2 + h^2$$

or $$y_A = \frac{h^2+2l^2}{2h}.$$

The terminal tension $T_A$ at $A$

$$=w\frac{h^2+2l^2}{2h}=\frac{1}{2}w\frac{(h^2+2l^2)}{hl}$$

*Ex. 32. A heavy string of uniform density and thickness is suspended from two given points in the same horizontal plane. A weight, n times that of the string, is attached to its lowest point; show that if θ, φ be the inclinations to the vertical of the tangents at the highest and lowest points of the string*

$$\tan\phi=(1+n)\tan\theta.$$

**[Gorakhpur 77]**

**Sol.** [Refer figure of Ex. 28, Page 32].

Let $W$ be the weight and $2l$ the length of the string suspended from two points $A$ and $B$ in the same horizontal line. Then the weight attached at middle point $C$ of the string is $Wn$. The weight per unit length $w$ of the string is given by

$$w=W/2l.$$

If $\psi_C$ and $\psi_A$ are the angles of inclination of the tangents at $C$ and $A$ to the horiznntal respectively, then

$$\psi_C=\tfrac{1}{2}\pi-\phi,\ \psi_A=\tfrac{1}{2}\pi-\theta.$$

For the equilibrium of the point $C$, resolving the forces acting on it vertically, we have

$$2T_C\sin\psi_C=Wn, \quad \ldots(1)$$

where $T_C$ is the tension at $C$ in each of the strings $AC$ and $BC$.

Also we have $T_A\cos\psi_C=wc$ where $c$ is the parameter of the catenary of which $AC$ is an arc.

$$\therefore\ T_C\cos\psi_C=\frac{W}{2l}c. \quad \ldots(2)$$

$[\because\ w=W/2l]$

Dividing (1) by (2), we have

$$2\tan\psi_C=\frac{2l}{nc},\ \text{or}\ \tan(\tfrac{1}{2}\pi-\phi)=\frac{l}{nc},\ \text{or}\ \cot\phi=\frac{l}{nc}$$

$$\therefore\ c=(l/n)\tan\phi.$$

Now from $s=c\tan\psi$, we have

$$\text{arc } DC=c\tan\psi_C \text{ and arc } DA=c\tan\psi_A,$$

where $D$ is the vertex of the catenary of which $AC$ is an arc.

Subtracting, we have

$$\text{arc } DA-\text{arc } DC=c(\tan\psi_A-\tan\psi_C$$

or $$\text{arc } CA=c[\tan(\tfrac{1}{2}\pi-\theta)-\tan(\tfrac{1}{2}\pi-\phi)]$$

or $$l=(l/n)\tan\phi.[\cot\theta-\cot\phi]$$



or $$n=\tan\phi\cot\theta-1,$$

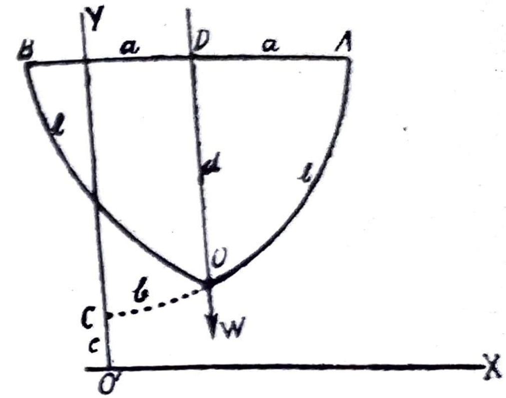
or $$1+n=\tan\phi\cot\theta$$

or $$\tan\phi=(1+n)\tan\theta.$$

*Ex. 33 (a). A and B are two points in the same horizontal line distant 2a apart. AO, OB are two equal heavy strings tied together at O and carrying a weight at O. If l is the length of each string, d the depth of O below AB, show that the parameter c of the catenary in which either string hangs is given by*

$$l^2-d^2=2c^2[\cosh(a/c)-1].$$

**Sol.** Let $C$ be the vertex of the catenary of which $AO$ is an arc and $c$ be its parameter. Let $O'X$ be the directrix and $O'Y$ the axis of this catenary. Referred to $O'X$ and $O'Y$ as the coordinate axes, let $(x_1, y_1)$ and $(x_2, y_2)$ be the coordinates of the points $O$ and $A$ respectively and let arc $CO=b$. Then



(Fig. 17)

$$\text{arc } CA=\text{arc } CO+\text{arc } OA$$
$$=b+l.$$

Given that

$OD=d$ and $AB=2a$.

so that $AD=a$.

We hav $y_2=y_1+d$ and $x_2=x_1+DA=x_1+a$.

$$b=c\sinh\frac{x_1}{c}, \quad \text{[for the point } O]$$

and $$b+l=c\sinh\frac{x_2}{c} \quad \text{[for the point } A]$$

Subtracting, we have

$$l=c\left(\sinh\frac{x_1+a}{c}-\sinh\frac{x_1}{c}\right). \quad [\because\ x_2=x_1+a]$$

Also from $y=c\cosh\dfrac{x}{c}$, we have

$$y_1=c\cosh\frac{x_1}{c} \text{ and } y_2=c\cosh\frac{x_2}{c}.$$

Subtracting, we have

$$y_2-y_1=c\left(\cosh\frac{x_2}{c}-\cosh\frac{x_1}{c}\right),$$

or
$$d=c\left(\cosh\frac{x_1+a}{c}-\cosh\frac{x_1}{c}\right).$$

$$\therefore l^2-d^2=c^2\left[\left(\sinh\frac{x_1+a}{c}-\sinh\frac{x_1}{c}\right)^2-\left(\cosh\frac{x_1+a}{c}-\cosh\frac{x_1}{c}\right)^2\right]$$

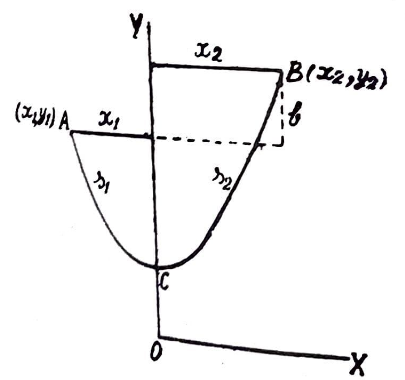
$$=c^2\left[-\left(\cosh^2\frac{x_1+a}{c}-\sinh^2\frac{x_1+a}{c}\right)-\left(\cosh^2\frac{x_1}{c}-\sinh^2\frac{x_1}{c}\right)\right.$$
$$\left.+2\left\{\cosh\frac{x_1+a}{c}.\cosh\frac{x_1}{c}-\sinh\frac{x_1+a}{c}.\sinh\frac{x_1}{c}\right\}\right]$$

$$=c^2\left[-1-1+2\cosh\left(\frac{x_1+a}{c}-\frac{x_1}{c}\right)\right]$$
$$=2c^2\,[\cosh(a/c)-1].$$

**Ex. 33 (b).** *A uniform chain of length $l$ hangs between two points $A$ and $B$ which are at a horizontal distance $a$ from one another, with $B$ at a vertical distance $b$ above $A$. Prove that the parameter of the catenary is given by*
$$2c\sinh(a/2c)=\sqrt{(l^2-b^2)}.$$

*Prove also that, if the tensions at $A$ and $B$ are $T_1$ and $T_2$ respectively,*
$$T_1+T_2=W\sqrt{\left(1+\frac{4c^2}{l^2-b^2}\right)}$$
*and*
$$T_2-T_1=Wb/l,$$
*where $W$ is the weight of the chain.* **[Rohilkhand 82]**

**Sol.** A uniform chain of length $l$ and weight $W$ hangs between two points $A$ and $B$. Let $C$ be the vertex, $OX$ the directrix, $OY$ the axis and $c$ the parameter of the catenary in which the chain




(Fig. 18)



hangs. Let $(x_1, y_1)$ and $(x_2, y_2)$ be the coordinates of the points $A$ and $B$ respectively and let arc $CA=s_1$ and arc $CB=s_2$.

We have $\quad s_1+s_2=l.$

Since the horizontal distance between $A$ and $B$ is $a$, therefore
$$x_1+x_2=a.$$

Again since the vertical distance of $B$ above $A$ is $b$, therefore
$$y_2-y_1=b.$$

Let $w$ be the weight per unit length of the chain. Then
$$W=lw, \quad\text{or}\quad w=W/l.$$

By the formula $s=c\sinh(x/c)$, we have
$$s_1=c\sinh(x_1/c) \text{ and } s_2=c\sinh(x_2/c)]$$
$$\therefore \quad l=s_1+s_2=c\,[\sinh x_1/c)+\sinh(x_2/c)]. \quad \ldots(1)$$

Again by the formula $y=c\cosh(x/c)$, we have
$$y_1=c\cosh(x_1/c) \text{ and } y_2=c\cosh(x_2/c).$$
$$\therefore \quad b=y_2-y_1=c\,[\cosh(x_2/c)-\cosh(x_1/c)] \quad \ldots(2)$$

Squaring and subtracting (1) and (2), we have
$$l^2-b^2=c^2\,[-\{\cosh^2(x_1/c)-\sinh^2(x_1/c)\}-\{\cosh^2(x_2/c)-\sinh^2(x_2/c)\}$$
$$+2\{\cosh(x_1/c)\cosh(x_2/c)+\sinh(x_1/c)\sinh(x_2/c)\}]$$
$$=c^2\,[-1-1+2\cosh(x_1/c+x_2/c)]$$
$$=c^2\,[-2+2\cosh\{(x_1+x_2)/c\}]$$
$$=2c^2\left\{\cosh\frac{a}{c}-1\right\}=c^2\left\{1+2\sinh^2\frac{a}{2c}-1\right\}$$
$$=4c^2\sinh^2\frac{a}{2c} \quad \ldots(3)$$

$\therefore$ $c$ is given by
$$2c\sinh(a/2c)=\sqrt{(l^2-b^2)}.$$

[Remember that
$$\cosh(\alpha+\beta)=\cosh\alpha\cosh\beta+\sinh\alpha\sinh\beta$$
and
$$\cosh 2\alpha=1+2\sinh^2\alpha]$$

Now let $T_1$ and $T_2$ be the tensions at the points $A$ and $B$ respectively. Then by the formula $T=wy$, we have
$$T_1=wy_1,\; T_2=wy_2.$$
$$\therefore \quad T_2-T_1=w\,(y_2-y_1)=wb=(W/l)\,b=Wb/l.$$

Also $T_1+T_2=w\,(y_1+y_2)=\dfrac{W}{l}(y_1+y_2)=W\dfrac{y_1+y_2}{s_1+s_2}$
$$=W\,\frac{c\cosh(x_1/c)+c\cosh(x_2/c)}{c\sinh(x_1/c)+c\sinh(x_2/c)}$$

$$= W \frac{\cosh(x_1/c)+\cosh(x_2/c)}{\sinh(x_1/c)+\sinh(x_2/c)}$$

$$= W \frac{2\cosh\frac{1}{2}(x_1/c+x_2/c)\cosh\frac{1}{2}(x_1/c-x_2/c)}{2\sinh\frac{1}{2}(x_1/c+x_2/c)\cosh\frac{1}{2}(x_1/c-x_2/c)}$$

$$= W\coth\left(\frac{x_1+x_2}{2c}\right)=W\coth\frac{a}{2c}$$

$$= W\sqrt{\left[1+\text{cosech}^2\frac{a}{2c}\right]}$$

$[\because \coth^2\alpha=1+\text{cosech}^2\alpha]$

$$= W\sqrt{\left[1+\frac{4c^2}{l^2-b^2}\right]}$$

substituting for $\text{cosech}^2(a/2c)$ from (3).

**Ex. 34.** *Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is*

$$a\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}$$

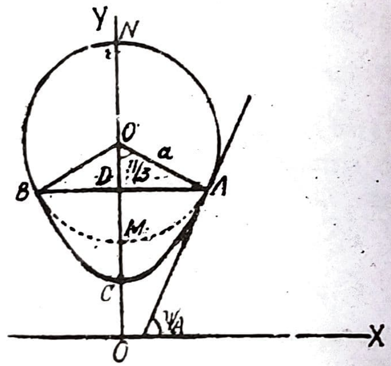
**[Gorakhpur 80; Agra 84; Luck. 79; Kanpur 87, 88, Meerut 90 P]**

**Sol.** Let $ANBMA$ be the circular pulley of radius $a$ and $ANBCA$ the endless chain hanging over it.

Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion $ANB$ of the chain

$=\frac{2}{3}$ (circumference of the pulley)

$=\frac{2}{3}(2\pi a)=\frac{4}{3}\pi a.$



(Fig. 20)

Let the remaining portion of the chain hang in the form of the catenary $ACB$, with $AB$ horizontal. $C$ is the lowest point *i.e.*, the vertex, $CO'N$ the axis and $OX$ the directrix of this catenary.

Let $OC=c=$the parameter of the catenary.

The tangent at $A$ will be perpendicular to the radius $O'A$

∴ If the tangent at $A$ is inclined at an angle $\psi_A$ to the horizontal, then

$$\psi_A=\angle AO'D=\tfrac{1}{2}(\angle AO'B)=\tfrac{1}{2}(\tfrac{1}{3}.2\pi)=\tfrac{1}{3}\pi.$$



From the triangle $AO'D$, we have

$$DA=O'A\sin\tfrac{1}{3}\pi=a\sqrt{3}/2.$$

∴ from $x=c\log(\tan\psi+\sec\psi)$, for the point $A$, we have

$$x=DA=c\log(\tan\psi_A+\sec\psi_A)$$

or $$\frac{a\sqrt{3}}{2}=c\log\left(\tan\frac{\pi}{3}+\sec\frac{\pi}{3}\right)=c\log(\sqrt{3}+2).$$

$$\therefore\ c=\frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $s=c\tan\psi$ applied for the point $A$, we have

$$\text{arc } CA=c\tan\psi_A=c\tan\tfrac{1}{3}\pi=c\sqrt{3}=\frac{3a}{2\log(2+\sqrt{3})}$$

Hence the total length of the chain

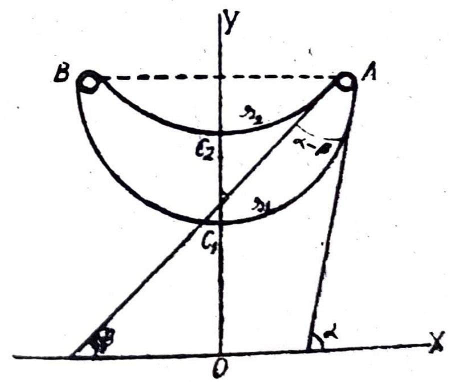
$=$arc $ABC$+length of the chain in contact with the pulley

$$=2.(\text{arc } CA)+\frac{4}{3}\pi a$$

$$=2\cdot\frac{3a}{2\log(2+\sqrt{3})}+\frac{4}{3}\pi a=a\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}$$

**Ex. 35.** *An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs.*

**Sol.** Let an endless uniform chain hang over two smooth pegs $A$ and $B$ in the same horizontal line. The two portions of



(Fig. 19)

the chain will hang in the form of two catenaries $AC_1B$ and $AC_2B$ with $C_1, C_2$ as their lowest point (vertices). Let $c_1$ and $c_2$ be the parameters of the two catenaries.

Also let $2s_1$ and $2s_2$ be the lengths of the portions $AC_1B$ and $AC_2B$ of the chain.

Let $\alpha$ and $\beta$ be the inclinations to the horizontal of the tangents of the two catenaries $AC_1B$ and $AC_2B$ at $A$. Since the tension of a chain does not change while passing over a smooth peg, therefore the tension at the point $A$ in each of the strings $AC_1B$ and $AC_2B$ is the same. Let $y_1$ and $y_2$ be the heights of the point $A$ of the two catenaries $AC_1B$ and $AC_2B$ above their corresponding directrices. Then using the formula $T=wy$ for each of these catenaries for the point $A$, we have

$$wy_1=wy_2, \text{ so that } y_1=y_2.$$

Therefore the two catenaries $AC_1B$ and $AC_2B$ have the same directrix. Let it be $OX$. The common axis of the two catenaries is the line $OY$ passing through their vertices $C_1$ and $C_2$.

Now using the formula $y=c\sec\psi$ for the two catenaries for the point $A$, we have $c_1\sec\alpha=c_2\sec\beta$, so that

$$c_2=\frac{c_1\sec\alpha}{\sec\beta}=\frac{c_1\cos\beta}{\cos\alpha}.$$

The distance between the vertices $C_1$ and $C_2$ of the two catenaries $=OC_2-OC_1$

$$=c_2-c_1=\frac{c_1\cos\beta}{\cos\alpha}-c_1=c_1\left(\frac{\cos\beta-\cos\alpha}{\cos\alpha}\right).$$

Again using the formula $s=c\tan\psi$ for the two catenaries for the point $A$, we have

$$s_1=c_1\tan\alpha \text{ and } s_2=c_2\tan\beta.$$

$\therefore$ half the length of the chain $=s_1+s_2$

$$=c_1\tan\alpha+c_2\tan\beta=c_1\tan\alpha+\frac{c_1\cos\beta}{\cos\alpha}\tan\beta=c_1\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha}\right).$$

Hence the required ratio

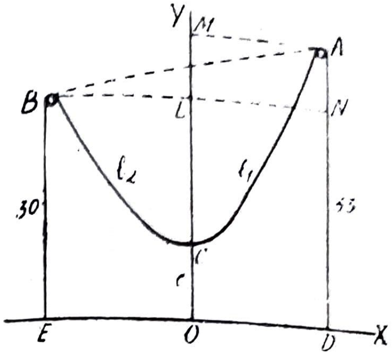
$$=\frac{c_2-c_1}{s_1+s_2}=\frac{\cos\beta-\cos\alpha}{\sin\alpha+\sin\beta}=\frac{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}=\tan\frac{1}{2}(\alpha-\beta).$$

**Ex. 36. (a)** *A heavy uniform string, 90 inches long, hangs over two smooth pegs at different heights. The parts which han are of lengths 30 and 33 inches. Prove that the vertex*



*nary divides the whole string in the ratio 4 : 5, and find the distance between the pegs.* **[Raj. T.D.C. 78; Gorakhpur 74]**

**Sol.** Let the heavy string $DACBE$ of length 90 inches hang




(Fig. 21)

over two smooth pegs $A$ and $B$ at different heights. Let the portion $ACB$ hang in the form of a catenary with $C$ as its lowest point i.e. vertex. Let $OX$ be the directrix and $c$ the parameter of this catenary. The portions $AD$ and $BE$ of the string hang vertically and are of lengths 33 and 30 inches respectively. Now the tension of a string remains unaltered while passing over a smooth peg. Therefore the tension at the point $A$ in the strings $AD$ and $AC$ is the same. But the tension at the point $A$ due to the strings $AD$ is $w.AD$, where $w$ is the weight per unit length of the string. Also by the formula $T=wy$, the tension at the point $A$ in the string $AC$ is $wy_A$, where $y_A$ is the height of the point $A$ above the directrix of the catenary $ACB$. So we have $w.AD=w.y_A$, so that $y_A=AD$, Therefore the free end $D$ of the string lies on the directrix $OX$ of the catenary $ACB$. Similarly the free end $E$ of the string also lies on the directrix of the catenary $ACB$. Thus for the catenary $ACB$, we have

$$y_A=AD=33 \text{ inches, and } y_B=BE=30 \text{ inches.}$$

Now let the lengths of the strings $CA$ and $CB$ be $l_1$ and $l_2$ inches respectively.

Then $$AD+BE+l_1+l_2=90$$

or $$33+30+l_1+l_2=90$$

or $$l_1+l_2=27. \qquad \ldots(1)$$

From $y^2=c^2+s^2$, we have

$$y_A^2=c^2+l_1^2 \text{ and } y_B^2=c^2+l_2^2$$

or $$33^2=c^2+l_1^2 \text{ and } 30^2=c^2+l_2^2.$$

Subtracting, we have

$$l_1^2-l_2^2=33^2-30^2=(33-30)(33+30)$$

or $$(l_1-l_2)(l_1+l_2)=3\cdot 63$$

$$\therefore\quad l_1-l_2=\frac{3\times 63}{l_1+l_2}=\frac{3\times 63}{27}. \qquad \ldots(2)$$

or $$l_1-l_2=7.$$

Solving (1) and (2), we have

$$l_1=17 \text{ inches and } l_2=10 \text{ inches.}$$

$\therefore$ the lengths of the string on the two sides of the vertex C are

$$CB+BE=l_2+BE=10+30=40 \text{ inches,}$$

and $$CA+AD=l_1+AD=17+33=50 \text{ inches.}$$

Hence the required ratio=40 : 50=4 : 5.

Now from $y_A^2=c^2+l_1^2$, we have $33^2=c^2+17^2$

or $$c^2=33^2-17^2=(33-17)(33+17)=16\times 50.$$

$$\therefore\quad c=20\sqrt{2}.$$

If $(x_A, y_A)$ and $(x_B, y_B)$ are the coordinates of the points $A$ and $B$ respectively, then from $y=c\cosh(x/c)$, we have

$$y_A=c\cosh\frac{x_A}{c} \text{ and } y_B=c\cosh\frac{x_B}{c}.$$

$$\therefore\quad x_A=c\cosh^{-1}\frac{y_A}{c} \text{ and } x_B=c\cosh^{-1}\frac{y_B}{c}.$$

Hence the horizontal distance between the pegs

$$=x_A+x_B=c\left[\cosh^{-1}\frac{y_A}{c}+\cosh^{-1}\frac{y_B}{c}\right]$$

$$=20\sqrt{2}\left[\cosh^{-1}\left(\frac{33}{20\sqrt{2}}\right)+\cosh^{-1}\left(\frac{30}{20\sqrt{2}}\right)\right]$$

$$=20\sqrt{2}\left[\cosh^{-1}\left(\frac{33}{40}\sqrt{2}\right)+\cosh^{-1}\left(\frac{3}{4}\sqrt{2}\right)\right]$$

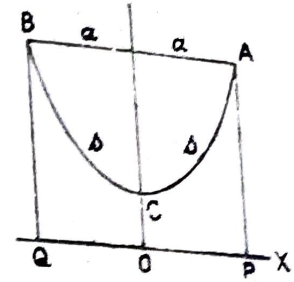
**Ex. 36 (b).** *A string hangs over two smooth pegs which are at the same level, its free ends hanging vertically. Prove that when the string is of shortest possible length, the parameter of the catenary is equal to half the distance between the pegs, and find the whole length of the string.*

**Sol.** Suppose $A$ and $B$ are two smooth pegs at the same level and at a distance $2a$ apart i.e., $AB=2a$. A string hangs over the





pegs $A$ and $B$. The portions $AP$ and $BQ$ of the string hang vertically and the portion $ACB$ hangs in the form of a catenary whose vertex is $C$ and directrix is $OX$. Let $w$ be the weight per unit length of the string.




(Fig. 22)

Now the tension of the string remains unaltered while passing over the smooth peg at $A$. Therefore the tension at the point $A$ due to the string $AP$ is equal to the tension at the point $A$ due to the string $ACB$ hanging in the form of a catenary. But the tension at the point $A$ due to the string $AP$ is equal to the weight $w\ AP$ of the string $AP$ and by the formula $T=wy$, the tension at $A$ in the catenary $ACB$ is equal to $w\,y_A$, where $y_A$ is the height of the point $A$ above the directrix $OX$ of the catenary $ACB$. So we have $w\,y_A=w\ AP$ i.e., $y_A=AP$. Therefore the free end $P$ of the string lies on the directrix $OX$ of the catenary $ACB$. Similarly the other free end $Q$ of the string also lies on the directrix $OX$.

Let $c$ be the parameter of the catenary $ACB$ i.e., let $OC=c$. For the point $A$ of the catenary $ACB$, we have

$$y=y_A=AP \text{ and } x=x_A=a$$

By the formula $s=c\sinh(x/c)$, for the point $A$, we have

$$s=\text{arc } CA=c\sinh(a/c).$$

Also by the formula $y=c\cosh(x/c)$, for the point $A$, we have

$$y=y_A=AP=c\cosh(a/c).$$

Hence the total length of the string, say $l$, is given by

$$l=2(\text{arc } CA+y_A)=2\{c\sinh(a/c)+c\cosh(a/c)\}$$

$$=2c\{\tfrac{1}{2}(e^{a/c}-e^{-a/c})+\tfrac{1}{2}(e^{a/c}+e^{-a/c})\}=2c\,e^{a/c}. \qquad \ldots(1)$$

Now $l$ is a function of $c$. For a maximum or a minimum of $l$, we must have $dl/dc=0$.

From (1), we have

$$\frac{dl}{dc}=2e^{a/c}+2c\,e^{a/c}\cdot\left(-\frac{a}{c^2}\right)=2e^{a/c}\left(1-\frac{a}{c}\right)$$

Putting $dl/dc=0$, we get $2e^{a/c}\{1-(a/c)\}=0$, $\qquad [\because e^{a/c}\neq 0]$

or $$1-(a/c)=0$$

or $$c=a,$$

Now $$\frac{d^2l}{dc^2}=2e^{a/c}\cdot\frac{a}{c^2}+\left(1-\frac{a}{c}\right)\cdot 2e^{a/c}\left(-\frac{a}{c^2}\right)$$

$$=2e/a, \text{ when } c/a=+\text{ive}.$$

$\therefore$ $l$ is minimum when $c=a$.

Thus when the string is of shortest possible length, we have $c=a=\frac{1}{2}(2a)$ = half the distance between the pegs.

Now putting $c=a$ in (1), the length $l$ of the string in this case is given by
$$l=2a\,e^{a/a}=2ae.$$

**Ex. 36 (c)** *A heavy string hangs over two fixed small smooth pegs. The two ends of the string are free and the central portion hangs in a catenary. Show that the free ends of the string are on the directrix of the catenary. If the two pegs are on the same level and distant 2a apart, show that equilibrium is impossible unless the string is equal to or greater than 2ae.* **[Rohilkhand 87]**

**Sol.** For the first part of the question draw figure as in Ex. 36 (a) and proceed in the same way.

For the second part of the question draw figure as in Ex. 36 (b) and proceed in the same way.

The least possible length for the string to be in equilibrium comes out to be $2ae$. Therefore the equilibrium is impossible unless the string is equal to or greater than $2ae$.

## § 7. Approximations to the common catenary. [Kanpur 75]

1. The cartesian equation of the common catenary is
$$y=c\cosh(x/c)=\tfrac{1}{2}c\,(e^{x/c}+e^{-x/c})$$
$$=\frac{c}{2}\left[\left\{1+\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\dots\right\}\right.$$
$$\left.+\left\{1-\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2-\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4\dots\dots\right\}\right]$$
$$=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\dots\dots\right]. \quad (1)$$

Now if $x/c$ is small, then neglecting the powers of $x/c$ higher than two, the equation (1) reduces to
$$y=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right],$$
or
$$x^2=2c\,(y-c).$$
which is the equation of a parabola of latus rectum $2c$ or $2T_0/w$.

*Thus if $x$ is small compared to $c$, the common catenary coincides very nearly with a parabola of latus rectum $2c$ or $2T_0/w$ and vertex at the point $(0, c)$.*

Examples of such a case are the electric transmission wires and the telegraphic wires tightly streched between the poles. Besides such cases of tightly stretched strings, even in the case of a common catenary not tightly stretched if we consider the portion of the curve near the vertex, $x$ is small compared to $c$.

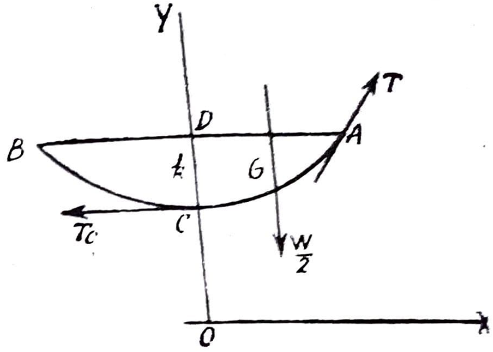


2. When $x$ is large *i.e.* at points far removed from the lowest point, $x/c$ is large and so $e^{-x/c}$ becomes very small, hence
$$y=c\cosh(x/c)=\tfrac{1}{2}c\,(e^{x/c}+e^{-x/c})$$
behaves as $y=\frac{1}{2}c\,e^{x/c}$, which is an exponential curve.

*Hence at points far removed from the lowest point, a common catenary behaves as an exponential curve.*

## § 8. Sag of tightly stretched wires.

Consider a tightly stretched wire which appears nearly a straight line, as for example a telegraphic wire stretched tightly between the poles.



(Fig. 23)

Let $A$ and $B$ be two points in a horizontal line between which a wire is stretched tightly. Let $C$ be the lowest point of the catenary formed by the wire. Let $W$ be the weight and $l$ the length of the wire $ACB$. Also let $T_0$ be the horizontal tension at the lowest point $C$. The portion $CA$ of the wire is in equilibrium under the action of the following forces :

(i) the tension $T_0$ acting horizontally at the point $C$.

(ii) the tension $T$ at $A$ acting along the tangent at $A$,

and (iii) the weight $\frac{1}{2}W$ of the wire $CA$ acting vertically downwards through its centre of gravity $G$.

Since the wire is tightly stretched, the distance of the centre of gravity $G$ of the wire $AC$ from the vertical line through $A$ will be approximately equal to $\frac{1}{2}AC$ *i.e.* $\frac{1}{4}l$. Let $k$ be the sag $CD$ and $a$ the span $AB$ of the catenary.

Taking moments of the forces acting on the portion $CA$, about $A$, we have $T_0.k=\frac{1}{2}W.\frac{1}{4}l$
$$T_0=\frac{lW}{8k}. \quad \dots(1)$$

Now we calculate *the increase in the length of the wire on account of the sag.*

For a catenary, we have
$$s=c\sinh(x/c)=\tfrac{1}{2}c\,(e^{x/c}-e^{-x/c})$$

$$=\frac{c}{2}\left[\left\{1+\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\ldots\right\}-\left\{1-\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2-\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4\ldots\ldots\right\}\right]$$

$$=c\left[\frac{x}{c}+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{5!}\left(\frac{x}{c}\right)^5+\ldots\ldots\right]. \qquad \ldots(2)$$

The radius of curvature $\rho$ of the catenary is given by

$$\rho=c\sec^2\psi.$$

At the vertex, $\psi=0$ and so $\rho=c$ at the vertex. This shows that if the curve is flat near its vertex $C$ so that $\rho$ is large at the vertex, then $c$ will be very large and $x$ will be small as compared to $c$. Thus for a tightly stretched wire $x/c$ is very small.

Thus retaining only the first two terms in (2), we have

$$s=c\left[\frac{x}{c}+\frac{1}{3!}\left(\frac{x}{c}\right)^3\right]=x+\frac{x^3}{6c^2}$$

or $$s-x=\frac{x^3}{6c^2}.$$

But $c=T_0/w$. $[\because T_0=wc]$

$$\therefore \quad s-x=\frac{w^2x^3}{6T_0^2}.$$

Now putting $x=\frac{1}{2}a$, where $a$ is the span $AB$, we have

$$s-\frac{a}{2}=\text{arc } CA-DA=\frac{a^3w^2}{48T_0^2}.$$

$\therefore$ *total increase in the length of the wire due to sagging*

$$=\text{arc } ACB-\text{span } AB=2s-a=\frac{a^3w^2}{24T_0^2}.$$

**Illustrative Examples**

**Ex. 37.** *Show that the maximum tension in a wire which weighs ·15 lb. per yard and hangs with a sag of 1 foot in a horizontal span of 100 feet is about $62\frac{1}{2}$ lbs. wt.*

**Sol.** Refer figure of § 8 on page 47.

From the formula $T=wy$ it is obvious that the maximum tension in the wire will be at the extremities $A$ or $B$.

Here $w=\cdot15/3=\cdot05$ lb. per foot, and span $AB=100$ feet.

The sag $CD=k=1$ foot.

If $T$ is the maximum tension in the wire at $A$, then

$$T=wy_A, \qquad \ldots(1)$$

where $y_A$ is the ordinate of the point $A$





For a catenary, we have

$$y=c\cosh(x/c)=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\ldots\right] \qquad \ldots(2)$$

Since sag 1 foot is very small compared to the span 100 feet, hence the wire is tightly stretched and $x/c$ is very small.

$\therefore$ neglecting higher powers of $x/c$ in (2), we have

$$y=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right]=c+\frac{x^2}{2c} \qquad \ldots(3)$$

For the point $A$, $x=DA=\frac{1}{2}AB=50$ feet. Therefore from (3), we have

$$y_A=c+\frac{(50)^2}{2c}.$$

But $$y_A=c+k=c+1.$$

$$\therefore \quad c+\frac{(50)^2}{2c}=c+1 \quad \text{or} \quad c=1250.$$

$$\therefore \quad y_A=c+1=1250+1=1251 \text{ feet}.$$

Hence from (3), the required maximum tension

$$T=wy_A=(\cdot05)\times(1251)=62\cdot55 \text{ lbs. wt.}$$
$$=62\tfrac{1}{2} \text{ lbs. wt. nearly.}$$

**Ex. 38.** *A telegraph is constructed of No. 8 iron wire which weighs 7·3 lbs. per 100 feet; the distance between the posts is 150 feet and the wire sags 1 foot in the middle. Show that it is screwed up to a tension of about 205 lbs. wt.*

**Sol.** Here the sag $k=1$ foot is small as compared to the span 150 feet, hence the wire is tightly stretched between the posts.

Here $w=\dfrac{7\cdot3}{100}=\cdot073$ lbs. per foot.

$\because$ wire is tightly stretched, $\quad\therefore$ $x/c$ is very small.

$$\therefore \quad y=c\cosh(x/c)$$

$$=c\left[1+\frac{1}{2!}\left\{\frac{x}{c}\right\}^2+\frac{1}{4!}\left\{\frac{x}{c}\right\}^4+\ldots\ldots\right]$$

$$=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right], \text{ neglecting higher powers of } x/c$$

$$=c+\frac{x^2}{2c}. \qquad \ldots(1)$$

If $(x_A, y_A)$ are the coordinates of the extremity $A$ of the wire, then

$$x_A=\tfrac{150}{2}=75 \text{ feet and } y_A=c+k=(c+1) \text{ feet}. \qquad \ldots(2)$$

Also from (1), $y_A=c+\dfrac{x_A^2}{2c}$ or $c+1=c+\dfrac{x_A^2}{2c}$,

or $$1=\frac{x_A^2}{2c}.$$

$\therefore \quad c=\frac{1}{2}x_A^2=\frac{1}{2}(75)^2=2812\cdot5.$

Hence the required tension at the extremity

$T=wy_A=(\cdot073)\times(c+1)$

$=(\cdot073)\times(2812\cdot5+1)$

$=\cdot073\times2813\cdot5=205$ lbs. wt. nearly.

**Ex. 39.** *The ends of a tightly stretched cable weighing ½ lb. per yard are fixed to two points on a level 80 yards apart and the cable has a sag, at the middle, of 1 foot 4 inches. Find the tension in the wire at the lowest point.*

**Sol.** Here the cable is tightly stretched.

$\therefore \quad y=c\cosh(x/c)=c+x^2/2c, \qquad \ldots(1)$

neglecting higher powers of $x/c$ which is small.

Also here $w=\frac{1}{6}$ lb. per foot, span $=80\times3=240$ feet

and sag $k=\frac{4}{3}$ feet.

At the extremity of the wire,

$x=120$ feet and $y=c+k=(c+\frac{4}{3})$ feet.

$\therefore$ from (1), we have $c+\frac{4}{3}=c+\frac{(120)^2}{2c}$

or $\frac{4}{3}=\frac{(120)^2}{2c} \quad \therefore \quad c=\frac{(120)^2\times3}{8}=45\times120.$

Hence the tension in the wire at the lowest point

$T_0=wc=\frac{1}{6}\times45\times120=900$ lbs. wt.

**Ex. 40.** *A heavy uniform string 155 ft. long, is suspended from two points A and B, 150 ft. apart on the same horizontal plane. Show that the tension at the lowest point is approximately equal to 1·08 times the weight of the string.*

**Sol.** If $W$ is the weight of the string and $w$ is the weight per foot of the string, then $w=(W/155)$.

$\therefore$ tension at the lowest point,

$T_0=wc=(W/155)\,c. \qquad \ldots(1)$

Now from $s=c\sinh(x/c)$, we have

$$s=c\left[\frac{x}{c}+\frac{1}{3\,!}\left(\frac{x}{c}\right)^3\right],$$

neglecting higher powers of $x/c$ which is small because the string is tightly stretched. $\therefore \quad s-x=(x^3/6c^2)$

or $\qquad 2s-2x=\dfrac{x^3}{3c^2}. \qquad \ldots(2)$

But at the extremity of the string, we have

$2s=155$ ft., $2x=150$ ft., *i.e.*, $x=75$ ft.



$\therefore$ from (2), we have

$$155-150=\frac{(75)^3}{3c^2}$$

or $\qquad c^2=\dfrac{(75)^3}{3\times5}=5\times(75)^2; \quad \therefore \quad c=75\sqrt{5}.$

Hence from (1)

$T_0=(W/155)\times75\sqrt{5}=1\cdot08W$ approximately.

**Ex. 41.** *A uniform measuring chain of length $l$ is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation $k$ above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $(8k^2/3l)$.*

**Sol.** Let $l$ be the length of the chain, $2a$ the breadth of the river, and $k$ the sag. It is required to show that

$l-2a=8k^2/3l$ nearly.

Since the chain is tightly stretched, therefore $x/c$ is small. So neglecting higher powers of $x/c$, the equation $y=c\cosh(x/c)$ of the catenary approximates to $y=c+x^2/2c$.

At the extremity of the chain, $x=a$, and $y=c+k$.

$\therefore \quad c+k=c+(a^2/2c)$, giving $c=(a^2/2k)$.

Now we have $s=c\sinh\dfrac{x}{c}=c\left[\dfrac{x}{c}+\dfrac{1}{3\,!}\dfrac{x^3}{c^3}\right]$ nearly.

At an extremity of the chain, $s=\frac{1}{2}l$ and $x=a$.

$\therefore \quad \dfrac{l}{2}=c\left(\dfrac{a}{c}+\dfrac{a^3}{6c^3}\right)$, or $\dfrac{l}{2}-a=\dfrac{a^3}{6c^2}$

or $\qquad l-2a=\dfrac{a^3}{3c^2}=\dfrac{a^3}{3}\cdot\dfrac{4k^2}{a^4} \qquad \left[\because \; c=\dfrac{a^2}{2k}\right]$

$=4k^2/3a.$

Since the string is tightly stretched,

$\therefore \quad a=l/2$ approximately.

Hence $\qquad l-2a=\dfrac{4k^2}{3}\cdot\dfrac{2}{l}=\dfrac{8k^2}{3l}.$

**Ex. 42.** *A chain is suspended in a vertical plane from two fixed supports A and B, which lie in a horizontal line 462 feet apart. If the tangent to the chain at A be inclined at an angle $\tan^{-1}(3/4)$ to the horizon, find the length of the chain (take $\log_e 2=\cdot693$).*

**Sol.** Here the angle of inclination of the tangent at $A$,

$\psi_A=\tan^{-1}\frac{3}{4}$

or $\tan\psi_A=\frac{3}{4} \quad \therefore \quad \sec\psi_A=(1+\tan^2\psi_A)=\sqrt{(1+9/16)}=5/4.$

If the $x$-coordinate of the extremity $A$ is $x_A$, then

$x_A=\frac{1}{2}\times462=231.$

$$\therefore \quad c=\tfrac{1}{2}x_A^2=\tfrac{1}{2}(75)^2=2812{\cdot}5.$$

Hence the required tension at the extremity

$$T=wy_A=({\cdot}073)\times(c+1)$$
$$=({\cdot}073)\times(2812{\cdot}5+1)$$
$$={\cdot}073\times2813{\cdot}5=205 \text{ lbs. wt. nearly.}$$

**Ex. 39.** *The ends of a tightly stretched cable weighing ½ lb. per yard are fixed to two points on a level 80 yards apart and the cable has a sag, at the middle, of 1 foot 4 inches. Find the tension in the wire at the lowest point.*

Sol. Here the cable is tightly stretched.

$$\therefore \quad y=c\cosh(x/c)=c+x^2/2c, \qquad \ldots(1)$$

neglecting higher powers of $x/c$ which is small.

Also here $w=\frac{1}{6}$ lb. per foot, span $=80\times3=240$ feet

and sag $k=\frac{4}{3}$ feet.

At the extremity of the wire,

$x=120$ feet and $y=c+k=(c+\frac{4}{3})$ feet.

$\therefore$ from (1), we have $c+\dfrac{4}{3}=c+\dfrac{(120)^2}{2c}$

or $\dfrac{4}{3}=\dfrac{(120)^2}{2c}$ $\quad\therefore\quad c=\dfrac{(120)^2\times3}{8}=45\times120.$

Hence the tension in the wire at the lowest point

$$T_0=wc=\tfrac{1}{6}\times45\times120=900 \text{ lbs. wt.}$$

**Ex. 40.** *A heavy uniform string 155 ft. long, is suspended from two points A and B, 150 ft. apart on the same horizontal plane. Show that the tension at the lowest point is approximately equal to 1·08 times the weight of the string.*

Sol. If $W$ is the weight of the string and $w$ is the weight per foot of the string. then $\quad w=(W/155).$

$\therefore$ tension at the lowest point,

$$T_0=wc=(W/155)\,c. \qquad \ldots(1)$$

Now from $s=c\sinh(x/c)$, we have

$$s=c\left[\frac{x}{c}+\frac{1}{3\,!}\left(\frac{x}{c}\right)^3\right],$$

neglecting higher powers of $x/c$ which is small because the string is tightly stretched. $\quad\therefore\quad s-x=(x^3/6c^2)$

or $$2s-2x=\frac{x^3}{3c^2}. \qquad \ldots(2)$$

But at the extremity of the string, we have

$2s=155$ ft., $2x=150$ ft., *i.e.*, $x=75$ ft.



$\therefore$ from (2), we have

$$155-150=\frac{(75)^3}{3c^2}$$

or $$c^2=\frac{(75)^3}{3\times5}=5\times(75)^2; \quad\therefore\quad c=75\sqrt{5}.$$

Hence from (1)

$$T_0=(W/155)\times75\sqrt{5}=1{\cdot}08W \text{ approximately.}$$

**Ex. 41.** *A uniform measuring chain of length $l$ is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation $k$ above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $(8k^2/3l)$.*

**Sol.** Let $l$ be the length of the chain, $2a$ the breadth of the river, and $k$ the sag. It is required to show that

$$l-2a=8k^2/3l \text{ nearly.}$$

Since the chain is tightly stretched, therefore $x/c$ is small. So neglecting higher powers of $x/c$, the equation $y=c\cosh(x/c)$ of the catenary approximates to $y=c+x^2/2c$.

At the extremity of the chain, $x=a$, and $y=c+k$.

$\therefore\quad c+k=c+(a^2/2c)$, giving $c=(a^2/2k)$.

Now we have $s=c\sinh\dfrac{x}{c}=c\left[\dfrac{x}{c}+\dfrac{1}{3\,!}\dfrac{x^3}{c^3}\right]$ nearly.

At an extremity of the chain, $s=\frac{1}{2}l$ and $x=a$.

$$\therefore\quad \frac{l}{2}=c\left(\frac{a}{c}+\frac{a^3}{6c^3}\right), \text{ or } \frac{l}{2}-a=\frac{a^3}{6c^2}$$

or $$l-2a=\frac{a^3}{3c^2}=\frac{a^3}{3}\cdot\frac{4k^2}{a^4} \qquad \left[\because\ c=\frac{a^2}{2k}\right]$$
$$=4k^2/3a.$$

Since the string is tightly stretched,

$$\therefore\quad a=l/2 \text{ approximately.}$$

Hence $$l-2a=\frac{4k^2}{3}\cdot\frac{2}{l}=\frac{8k^2}{3l}.$$

Ex. 42. *A chain is suspended in a vertical plane from two fixed supports A and B, which lie in a horizontal line 462 feet apart. If the tangent to the chain at A be inclined at an angle $\tan^{-1}(3/4)$ to the horizon, find the length of the chain (take $\log_e 2={\cdot}693$).*

Sol. Here the angle of inclination of the tangent at $A$,

$$\psi_A=\tan^{-1}\tfrac{3}{4}$$

or $\tan\psi_A=\frac{3}{4}\quad\therefore\quad \sec\psi_A=(1+\tan^2\psi_A)=\sqrt{(1+9/16)}=5/4.$

If the $x$-coordinate of the extremity $A$ is $x_A$, then

$$x_A=\tfrac{1}{2}\times462=231.$$

$\therefore$ from $x=c\log(\tan\psi+\sec\psi)$, we have

$$x_A=c\log(\tan\psi_A+\sec\psi_A),$$

$$231=c\log\left(\tfrac{3}{4}+\tfrac{5}{4}\right)=c\log 2=\cdot693c.$$

or $\therefore$ $c=\dfrac{331}{\cdot693}=\dfrac{1000}{3}$.

Hence the length of the chain

$$=2s_A=2c\tan\psi_A=2\times\frac{1000}{3}\times\frac{3}{4}=500\text{ ft.}$$

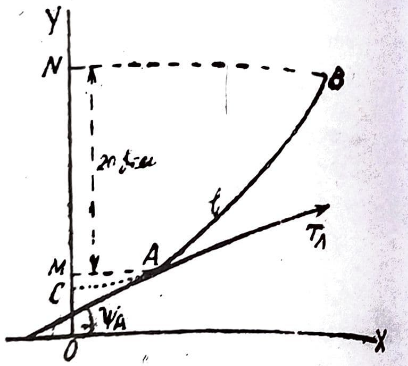
**Ex. 43.** *A uniform chain has its ends fixed at A and B where B is 20 ft. above the level of A, and no part of the chain is below A. At A the chain is inclined at* $\sec^{-1}(5/3)$ *to the horizontal, and the tension there is equal to the wt. of 100 ft. of chain. Prove that the length of the chain is 23 ft. 11 inch. to the nearest inch.*

**Sol.** Let $AB$ be the chain of length $l$ feet with the end $B$ at a height 20 ft. above the end $A$. The tension $T_A$ at the point $A$ acting along the tangent at $A$ is inclined at an angle $\psi_A=\sec^{-1}(5/3)$ to the horizontal.

We have. $\sec\psi_A=5/3$

and $\tan\psi_A=\sqrt{(\sec^2\psi_A-1)}$

$=4/3$.




(Fig. 24)

If $w$ is the weight of one foot length of the chain, then according to the question, $T_A=100w$.

Let $C$ be the vertex and $OX$ the directrix of the catenary of which $AB$ is an arc.

Let $y_A$ and $y_B$ be the ordinates of the point $A$ and $B$ respectively and let arc $CA=a$ feet.

From $T=wy$, we have $T_A=wy_A$.

But $T_A=100w$.

$\therefore$ $wy_A=100w$ or $y_A=100$ feet.

Also we have $y_A=c\sec\psi_A$

$\therefore$ $c=y_A\cos\psi_A=100.\frac{3}{5}=60$ feet.

$\therefore$ from $s=c\tan\psi$, for the point $A$, we have

arc $CA=a=c\tan\psi_A=60.\frac{4}{3}=80$.

Now $y_B=y_A+MN=100+20=120$

and arc $CB=a+l=80+l$.



$\therefore$ from $y^2=s^2+c^2$, we have

$$y_B^2=(l+80)^2+60^2$$

$$120^2=(l+80)^2+60^2$$

or $(l+80)^2=120^2-60^2=(120-60)(120+60)=60\times180$

or $\therefore$ $l+80=\pm60\sqrt{3}$

or $l=-80+60\sqrt{3}$, neglecting the negative sign otherwise $l$ will be negative

or $l=(60\times1\cdot732-80)$ ft.$=23\cdot92$ ft.$=23$ ft. 11 in. nearly.

**Ex. 44.** *A kite is flown with 600 ft. of string from the hand to the kite, and a spring balance held in hand shows a pull equal to the weight of 100 ft. of the string, inclined at 30° to the horizontal. Find the vertical height of the kite above the hand.* **[Agra 78]**

**Sol.** [Refer figure 24 of Ex. 43 on page 52].

Let $AB$ be the string of length 600 ft. with the kite at $B$. If $w$ is the weight of one ft. length of the string, then the hand at $A$ experiences a pull of 100 $w$ *i.e.*, the tension at $A$, $T_A=100w$.

Let $C$ be the vertex and $OX$ the directrix of the catenary of which $AB$ is an arc. If the tangent at $A$ is inclined at an angle $\psi_A$ to the horizontal, then according to the question $\psi_A=30°$.

Let $y_A$ and $y_B$ be the ordinates of the points $A$ and $B$ respectively.

From $T=wy$, we have $T_A=wy_A$.

But $T_A=100w$.

$\therefore$ $100w=wy_A$, or $y_A=100$ ft.

Also we have $y_A=c\sec\psi_A$.

$\therefore$ $100=c\sec30°$ or $c=100\cos30°=50\sqrt{3}$ ft.

From $s=c\tan\psi$, for the point $A$, we have

arc $CA=a$ (say)$=c\tan\psi_A=50\sqrt{3}.\tan30°=50$ ft.

Now $y_B=y_A+MN=100+MN$

and arc $CB=$ arc $CA+$ arc $AB=50+600=650$ ft.

$\therefore$ from $y^2=s^2+c^2$, we have

$$y_B^2=650^2+(50\sqrt{3})^2$$

or $(100+MN)^2=50^2\times(13^2+3)=50^2\times172=50^2\times2^2\times43$.

$\therefore$ $100+MN=\pm100\sqrt{(43)}$

or $MN=100[\sqrt{(43)}-1]$, neglecting the negative sign, otherwise $MN$ is negative

$=555\cdot7$ ft. nearly.

**Ex. 45.** *A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of the same height, as will produce the least possible tension at the posts.*

$$l=(d/\lambda)\sinh\lambda,$$

Show that where $\lambda$ is given by the equation $\lambda \tanh \lambda=1$.

**[Rohilkhand 78; Kanpur 83; Luck. 77, 79; Agra 87]**

**Sol.** Let $T$ be the tension at either of the posts. Then

$$T=wy=wc\cosh(x/c)=wc\cosh(d/2c).$$

[Since $d$ is the distance between the posts, $\therefore$ at either of the posts $x=d/2$.]

The tension $T$ is a function of the parameter $c$ of the catenary.

We have

$$\frac{dT}{dc}=w\left(\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}\right)$$

and

$$\frac{d^2T}{dc^2}=\frac{wd^2}{4c^3}\cosh\frac{d}{2c}, \text{ after simplification.}$$

For maximum or minimum of $T$,

$$(dT/dc)=0,$$

i.e.,

$$w\left(\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}\right)=0$$

or

$$\frac{d}{2c}\tanh\frac{d}{2c}=1$$

or

$$\lambda\tanh\lambda=1, \text{ where } \lambda=d/2c.$$

Since $d^2T/dc^2$ is positive, therefore $T$ is minimum for $c=d/2\lambda$, $\lambda$ being given by the equation $\lambda\tanh\lambda=1$.

Now for either of the posts, $x=\frac{1}{2}d$ and $s=\frac{1}{2}l$.

$\therefore$ from $s=c\sinh(x/c)$, we have

$$\tfrac{1}{2}l=c\sinh(d/2c)$$

or

$$l=(2.d/2\lambda)\sinh\lambda=(d/\lambda)\sinh\lambda,$$

for producing the least possible tension at the posts.

**Ex. 46.** *If the length of a uniform chain suspended between posts at the same level is adjusted so that the tension at the posts of support is a minimum for that particular span* $2d$, *show that the equation to determine* $c$ *is* $\coth(d/c)=(d/c)$. **[Kanpur 82]**

**Sol. Hint.** Proceed as in Ex. 45.

**Ex. 47.** *A uniform chain is hung up from two points at the same level and distant* $2a$ *apart. If* $z$ *is the sag at the middle, show that* $z=c\{\cosh(a/c)-1\}$.

*If* $z$ *is small compared with* $a$, *show that*

$$2cz=a^2 \text{ nearly.}$$ **[Kanpur 78]**

**Sol.** [Refer figure of § 8 on page 47].

Let $ACB$ be the uniform chain, $C$ the lowest point *i.e.*, the



vertex of the catenary formed by the chain and $OX$ its directrix.

We have,

$$y=c\cosh(x/c).$$

At $A$,

$$x=\tfrac{1}{2}AB=a.$$

Therefore if $y=y_A$ for the point $A$, then

$$y_A=c\cosh(a/c).$$

$\therefore$ sag at the middle,

$$z=OD-OC=y_A-c=c\cosh(a/c)-c=c\,[\cosh(a/c)-1].$$

Now expanding $\cosh(a/c)$ in powers of $a/c$, we have

$$z=c\left[1+\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots-1\right]$$

$$=c\left[\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots\right]$$

$$=\frac{1}{2!}\frac{a^2}{c}+\frac{1}{4!}\frac{a^4}{c^3}+\ldots\ldots$$

If $z$ is small compared to $a$, then $c$ must be large. Therefore neglecting the higher powers of $1/c$ in the above expansion, we have

$$z=\frac{1}{2!}\cdot\frac{a^2}{c}$$

or

$$2cz=a^2.$$

**Ex. 48.** *A telegraph wire is supported by two poles distant 40 yards apart. If the sag be one foot and the weight of the wire half an ounce per foot, show that the horizontal pull on each pole is* $\frac{1}{2}$ *cwt. nearly.*

**Sol.** Proceed as in the last Ex. 47. The sag $z$ at the middle is given by

$$2cz=a^2 \text{ (nearly).}$$

Here $z=1$ ft. and $a=\frac{1}{2}(40\times3)=60$ ft.

$$\therefore\quad c=\frac{a^2}{2z}=\frac{60^2}{2.1}=1800 \text{ ft.}$$

Now the required horizontal pull at $A$

$$=T_0=wc=\frac{1}{32}\cdot1800 \text{ lbs.}\quad\left[\because\ w=\frac{1}{32}\text{ lbs}\right]$$

$$=\frac{1800}{32\times112}\text{ cwt. nearly}=\tfrac{1}{2}\text{ cwt. nearly.}$$

**Ex. 49.** *A uniform chain, of length* $2l$, *has its ends attached to two points in the same horizontal line at a distance* $2a$ *apart. If* $l$ *is only a little greater than* $a$, *show that the tension of the chain is approximately equal to the weight of a length*

$$\sqrt{\left[\frac{a^3}{6(l-a)}\right]}$$

*of the chain; and that the sag or depression of the lowesr point of*

Show that

$$l=(d/\lambda)\sinh\lambda,$$

*where $\lambda$ is given by the equation $\lambda\tanh\lambda=1$.*

[**Rohilkhand 78; Kanpur 83; Luck. 77, 79; Agra 87**]

**Sol.** Let $T$ be the tension at either of the posts. Then

$$T=wy=wc\cosh(x/c)=wc\cosh(d/2c).$$

[Since $d$ is the distance between the posts, $\therefore$ at either of the posts $x=d/2$.]

The tension $T$ is a function of the parameter $c$ of the catenary. We have

$$\frac{dT}{dc}=w\left(\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}\right)$$

and

$$\frac{d^2T}{dc^2}=\frac{wd^2}{4c^3}\cosh\frac{d}{2c},\text{ after simplification.}$$

For maximum or minimum of $T$,

$$(dT/dc)=0,$$

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$\therefore$ from $s=c\sinh(x/c)$, we have

$$\tfrac{1}{2}l=c\sinh(d/2c)$$

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*If $z$ is small compared with $a$, show that*

$$2cz=a^2\text{ nearly.}$$

[**Kanpur 78**]

**Sol.** [Refer figure of § 8 on page 47].

Let $ACB$ be the uniform chain, $C$ the lowest point *i.e.*, the



vertex of the catenary formed by the chain and $OX$ its directrix.

We have, $y=c\cosh(x/c)$.

At $A$, $x=\frac{1}{2}AB=a$.

Therefore if $y=y_A$ for the point $A$, then

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$$z=OD-OC=y_A-c=c\cosh(a/c)-c=c[\cosh(a/c)-1].$$

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$$z=c\left[1+\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots-1\right]$$

$$=c\left[\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots\right]$$

$$=\frac{1}{2!}\frac{a^2}{c}+\frac{1}{4!}\frac{a^4}{c^3}+\ldots\ldots$$

If $z$ is small compared to $a$, then $c$ must be large. Therefore neglecting the higher powers of $1/c$ in the above expansion, we have

$$z=\frac{1}{2!}\frac{a^2}{c}$$

or

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**Sol.** Proceed as in the last Ex. 47. The sag $z$ at the middle is given by

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Here $z=1$ ft. and $a=\frac{1}{2}(40\times 3)=60$ ft.

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Now the required horizontal pull at $A$

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**Ex. 49.** *A uniform chain, of length $2l$, has its ends attached to two points in the same horizontal line at a distance $2a$ apart. If $l$ is only a little greater than $a$, show that the tension of the chain is approximately equal to the weight of a length*

$$\sqrt{\left[\frac{a^3}{6(l-a)}\right]}$$

*of the chain; and that the sag or depression of the lowesr point of* the chain below its ends is nearly $\frac{1}{2}\sqrt{\{6a\,(l-a)\}}$.

[Kanpur 74; Luck. 78, 80; Agra 86]

**Sol.** [Refer figure of § 8 on page 47]

Since the chain is tightly stretched, the tension at any point shall be the same nearly.

$\therefore$ the tension in the chain $= T = T_0$ (tension at the lowest point)

or $\quad T = wc.$ $\qquad$ ...(1)

Here span $= 2a$ and the length of the chain $= 2l$.

$\therefore$ at either of the supports, $x = a$ and $s = l$.

$\therefore$ from $s = c \sinh (x/c)$, we have

$$l = c \sinh \frac{a}{c} = c\left[\frac{a}{c} + \frac{1}{3\,!}\left(\frac{a}{c}\right)^3 + \ldots\ldots\right] \qquad \ldots(2)$$

Since the chain is tightly stretched, hence $c$ is large *i.e.*, $a/c$ is very small.

$\therefore$ neglecting higher powers of $a/c$ in (2), we have

$$l = c\left[\frac{a}{c} + \frac{1}{3\,!}\left(\frac{a}{c}\right)^3\right] = a + \frac{a^3}{6c^2}$$

or $\quad \dfrac{a^3}{6c^2} = l - a$, or $c = \sqrt{\left\{\dfrac{a^3}{6\,(l-a)}\right\}}$.

$\therefore$ from (1), we have $T = w\sqrt{\left\{\dfrac{a^3}{6\,(l-a)}\right\}}$.

Hence the tension of the chain is approximately equal to the weight of a length $\sqrt{\left\{\dfrac{a^3}{6\,(l-a)}\right\}}$ of the chain.

At the support, $x = a$. $\therefore$ from $y = c \cosh (x/c)$,

at the support $\quad y = y_A = c \cosh (a/c)$.

$\therefore$ sag in the middle $= y_A - c = c \cosh (a/c) - c$

$$= c\left[1 + \frac{1}{2\,!}\left(\frac{a}{c}\right)^2 + \frac{1}{4\,!}\left(\frac{a}{c}\right)^4 + \ldots - 1\right]$$

$= c\,[a^2/2c^2]$ nearly, neglecting higher powers of $a/c$

$= (a^2/2c)$

$$= \frac{a^2}{2}\sqrt{\left\{\frac{6\,(l-a)}{a^3}\right\}} = \frac{1}{2}\sqrt{\{6a\,(l-a)\}}.$$

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# Strings in Two Dimensions

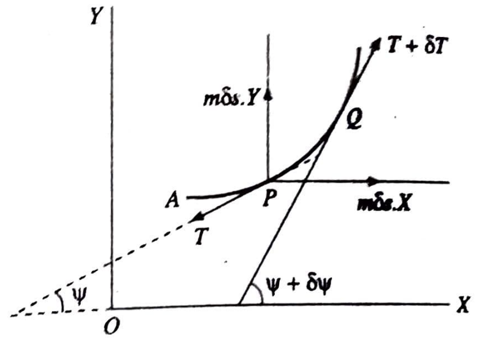
**(Catenary of uniform strength and strings resting on a smooth and rough plane curve)**

## §1. Introduction

We have studied the problems on unifrom strings hanging freely in the shape of a common catenary. Here we shall study the cases of strings which do not hang freely under gravity but are acted by some forces other than gravity as well.

## §2. *To discuss equilibrium of a string in a plane under the action of given forces.*

***Proof :*** Let $O$ be the origin with $OX$ as $x$-axis and $OY$ as $y$-axis perpendicular to each other.




Let $A$ be a fixed point on the curve formed by the string and let $P\,(x, y)$ and $Q\,(x + \delta x, y + \delta y)$ be two neighbouring points on the curve such that arc $AP = s$, arc $AQ = s + \delta s$ so that arc $PQ = \delta s$.

Also let the tensions in the string at $P$ and $Q$ be $T$ and $T + \delta T$ respectively. Let $m$ denote mass per unit length of the string. If $X$ and $Y$ be sums of resolved parts of forces per unit mass of the string parallel to the axes, then the components of forces on $PQ$ parallel to the axes are $m\delta s.X$ and $m\delta s.Y$.

The resolved parts of $T$ parallel to the axes are $T \cos \psi$, $T \sin \psi$. These values will be functions of $s$. Hence suppose that

$$T \cos \psi = T\,\frac{dx}{ds} = f\,(s) \quad \text{as } \cos \psi = \frac{dx}{ds}.$$

This implies that resolved part of $T + \delta T$ along $x$-axis

$= f\,(s + \delta s)$