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Krishna's Series

Integral Calculus

(Fully Solved)

(For Degree Honours Students of all Indian Universities and for Various Competitive Examinations like P.C.S., I.A.S. and Engineering Students)

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Elementary Integration

§ 1. Definition.

The process inverse to differentiation is defined as integration.

Thus if $\frac{d}{dx}F(x) = f(x)$, we say that $F(x)$ is an *integral* or a *primitive* of $f(x)$ and, in symbols, we write

$$\int \mathbf{f(x)\, dx = F(x)}.$$

The letter x in dx denotes that the integration is to be performed with respect to the variable x.

The process of determining an integral of a function is called **integration** and the function to be integrated is called **integrand.**

§ 2. Constant of integration.

As the differential coefficient of a constant is zero, we have

$$\frac{d}{dx}[F(x) + c] = f(x), \text{ if } \frac{d}{dx}F(x) = f(x);$$

therefore $\int \mathbf{f(x)\, dx = F(x) + c}.$

This constant c is called a *constant of integration* and can take any constant value. Also $\int f(x)\, dx$ is called the **Indefinite integral** of $f(x)$ w.r.t. 'x' ; for by giving different values to the constant of integration the indefinite nature is preserved.

For example, we know that

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{(1-x^2)}} \quad \text{and} \quad \frac{d}{dx}(-\cos^{-1}x) = \frac{1}{\sqrt{(1-x^2)}};$$

it follows, when we omit the constant of integration, that $\int \frac{1}{\sqrt{(1-x^2)}}\, dx$ is equal to $\sin^{-1}x$ and also equal to $-\cos^{-1}x$.

But it is wrong to conclude from above that $\sin^{-1}x$ is equal to $-\cos^{-1}x$. The correct inference is that the two integrals, given above differ in their constant of integration.

The correct result, as we see from trigonometry, is that

$$\sin^{-1}x = \tfrac{1}{2}\pi - \cos^{-1}x.$$

The arbitrary constant of integration may be imaginary also. Generally such a constant is added to make the result real.

§ 3. Two simple Theorems.

Theorem I. *The integral of the product of a constant and a function is equal to the product of the constant and the integral of the function..*

Thus if λ is a constant, then $\int \lambda \mathbf{f}(\mathbf{x})\, \mathbf{dx} = \lambda \int \mathbf{f}(\mathbf{x})\, \mathbf{dx}$.

Proof. Let $\int f(x)\, dx = F(x)$. Then $\frac{d}{dx} F(x) = f(x)$.

By differential calculus, $\frac{d}{dx}\{\lambda F(x)\} = \lambda \frac{d}{dx} F(x) = \lambda f(x)$.

$\therefore$ by definition of integral

$$\int \lambda f(x)\, dx = \lambda F(x) = \lambda \int f(x)\, dx.$$

Theorem II. *The integral of a sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the functions. Symbolically*

$$\int [\mathbf{f}_1(\mathbf{x}) \pm \mathbf{f}_2(\mathbf{x}) \pm \ldots \pm \mathbf{f}_n(\mathbf{x})]\, \mathbf{dx}$$
$$= \int \mathbf{f}_1(\mathbf{x})\, \mathbf{dx} \pm \int \mathbf{f}_2(\mathbf{x})\, \mathbf{dx} \pm \ldots\ldots \pm \int \mathbf{f}_n(\mathbf{x})\, \mathbf{dx}.$$

Proof. Let $\int f_1(x)\, dx = F_1(x), \int f_2(x)\, dx = F_2(x), \ldots\ldots,$

$$\int f_n(x)\, dx = F_n(x).$$

Clearly $\frac{d}{dx}\{F_1(x) \pm F_2(x)\} = \frac{d}{dx} F_1(x) \pm \frac{d}{dx} F_2(x)$

$$= f_1(x) \pm f_2(x).$$

Hence from the definition of the integral, we have

$$\int \{f_1(x) \pm f_2(x)\}\, dx = F_1(x) \pm F_2(x)$$
$$= \int f_1(x)\, dx \pm \int f_2(x)\, dx.$$

Thus repeating the above process to other functions, we have

$$\int \{f_1(x) \pm f_2(x) \pm \ldots\ldots \pm f_n(x)\}\, dx$$
$$= \int f_1(x)\, dx \pm \int f_2(x)\, dx \pm \ldots\ldots \pm \int f_n(x)\, dx.$$

§ 4. Hyperbolic Functions.

Following fundamental properties of hyperbolic functions should be **committed to memory** by the students as we shall make frequent use of them during the study of integral calculus.

$\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$

$\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$,

$\coth x = (e^x + e^{-x})/(e^x - e^{-x})$

$\operatorname{sech} x = 2/(e^x + e^{-x})$, $\operatorname{cosech} x = 2/(e^x - e^{-x})$

$\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$\operatorname{cosech}^2 x = \coth^2 x - 1$, $\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$.

Logarithmic values of inverse hyperbolic functions.

(i) $\sinh^{-1} x = \log [x + \sqrt{(x^2 + 1)}]$;

$\sinh^{-1}(x/a) = \log [\{x + \sqrt{(x^2 + a^2)}\}/a]$

(ii) $\cosh^{-1} x = \log [x + \sqrt{(x^2 - 1)}]$;

$\cosh^{-1}(x/a) = \log [\{x + \sqrt{(x^2 - a^2)}\}/a]$

(iii) $\tanh^{-1} x = \frac{1}{2} \log \{(1 + x)/(1 - x)\}$;

$\tanh^{-1} (x/a) = \frac{1}{2} \log \{(a + x)/(a - x)\}, (x < a)$

(iv) $\coth^{-1} x = \frac{1}{2} \log \{(x + 1)/(x - 1)\}$;

$\coth^{-1} (x/a) = \frac{1}{2} \log \{(x + a)/(x - a)\}, (x > a)$.

§ 5. Fundamental Formulae.

We have read in differential calculus that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)x^n}{(n+1)} = x^n.$$

Thus $$\int x^n \, dx = \frac{x^{n+1}}{n+1}, \quad (n \neq -1).$$

The above formula is very important and shall be frequently used in this book. This may by remembered like this :

"*To find the integral of x^n w.r.t. 'x', increase the index (power) of x by one (unity) and divide by the increased index.*"

Thus $$\int x^3 \, dx = \frac{x^4}{4}, \quad \int x^{5/2} \, dx = \frac{x^{(5/2)+1}}{(5/2)+1} = \frac{2}{7} x^{7/2};$$

$$\int \frac{1}{x^5} dx = \int x^{-5} \, dx = \frac{x^{-5+1}}{-5+1} = -\frac{1}{4} x^{-4} = -\frac{1}{4x^4};$$

$$\int \frac{1}{x^{1/2}} dx = \int x^{-1/2} \, dx = \frac{x^{(-1/2)+1}}{(-1/2)+1} = 2x^{1/2} = 2\sqrt{x};$$

and $$\int dx = \int 1 \, . \, dx = \int x^0 \, dx = \frac{x^{0+1}}{0+1} = x.$$

Thus $\int a \, dx = ax$ *i.e., the integral of a constant is equal to the constant multiplied by the variable.*

However if $n = -1$, we have

$$\int x^{-1} \, dx = \int \frac{1}{x} dx = \log x, \qquad \left[\because \frac{d}{dx} \log x = \frac{1}{x}\right].$$

Standard results. These are derived directly from the standard results of differential calculus. **These are fundamental formulae and should be committed to memory.**

(i) $\int x^n \, dx = \dfrac{x^{n+1}}{n+1}$, (ii) $\int \dfrac{1}{x} dx = \log x$,

(iii) $\int \sin x \, dx = -\cos x$, (iv) $\int \cos x \, dx = \sin x$,

(v) $\int \sec^2 x \, dx = \tan x$, (vi) $\int \operatorname{cosec}^2 x \, dx = -\cot x$,

(vii) $\int \sec x \tan x \, dx = \sec x$, (viii) $\int e^x \, dx = e^x$,

(ix) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$,

(x) $\int a^x \, dx = a^x/(\log_e a)$, (xi) $\int \dfrac{1}{\sqrt{(1 - x^2)}} dx = \sin^{-1} x$,

(xii) $\int \frac{1}{1+x^2} dx = \tan^{-1} x$,

(xiii) $\int \frac{1}{x\sqrt{(x^2-1)}} dx = \sec^{-1} x$,

(xiv) $\int \cosh x \, dx = \sinh x$,

(xv) $\int \sinh x \, dx = \cosh x$,

(xvi) $\int \text{sech}^2 x \, dx = \tanh x$,

(xvii) $\int \text{cosech}^2 x \, dx = -\coth x$,

(xviii) $\int \text{sech}\, x \tanh x \, dx = -\text{sech}\, x$,

(xix) $\int \text{cosech}\, x \coth x \, dx = -\text{cosech}\, x$.

In the above formulae the *constant of integration* has been omitted. In future also we shall not add the constant of integration. It will remain understood that we have added it.

Examples on fundamental formulae.

Ex. 1. *Integrate* (*i*) $(3x+4x^2)$, (*ii*) $(5x+7)/x$.

Sol. (i) Let $I = \int (3x+4x^2)\, dx$.

Then $I = \int 3x\, dx + \int 4x^2\, dx = \frac{3}{2} \cdot x^2 + \frac{4}{3}x^3$.

(ii) Here $I = \int \{(5x+7)/x\}\, dx$

$= \int 5\, dx + \int (7/x)\, dx = 5x + 7 \log x$.

Ex. 2. *Integrate* $e^x + 2 \sin x - 3 \cos x$.

Sol. Here $I = \int (e^x + 2\sin x - 3\cos x)\, dx$

$= \int e^x dx + 2\int \sin x\, dx - 3\int \cos x\, dx$

$= e^x - 2\cos x - 3\sin x$.

Ex. 3. *Integrate* $10^x + 3e^x + x^3$.

Sol. Here $I = \int (10^x + 3e^x + x^3)\, dx$

$= \int 10^x dx + 3\int e^x dx + \int x^3 dx = \{10^x/(\log_e 10)\} + 3e^x + \frac{1}{4}x^4$.

Ex. 4. *Integrate* $5\cos x + 2\sec^2 x - 10$.

Sol. Here $I = \int (5\cos x + 2\sec^2 x - 10)\, dx$

$= \int 5\cos x\, dx + 2\int \sec^2 x\, dx - \int 10\, dx$

$= 5\sin x + 2\tan x - 10x$.

Ex. 5. *Integrate* $\{6/\sqrt{(1-x^2)}\} + 3\sec^2 x$.

Sol. Here $I = \int [\{6/\sqrt{(1-x^2)}\} + 3\sec^2 x]\, dx$

$= \int \{6/\sqrt{(1-x^2)}\}\, dx + 3\int \sec^2 x\, dx = 6\sin^{-1} x + 3\tan x$.

Ex. 6. *Integrate* $\sec x \tan x - 5\,\text{cosec}^2 x$.

Sol. Here $I = \int (\sec x \tan x - 5\,\text{cosec}^2 x)\, dx$

$= \int \sec x \tan x\, dx - 5\int \text{cosec}^2 x\, dx = \sec x + 5\cot x$.

Ex. 7. *Integrate* $\{(2\cos x)/(3\sin^2 x)\} + 1$.

Sol. Here $I = \int \left(\frac{2\cos x}{3\sin^2 x} + 1\right) dx = \int \frac{2\cos x}{3\sin^2 x} dx + \int 1 \,.\, dx$

Sol. Here $I = \int \left(\frac{2\cos x}{3\sin^2 x} + 1\right) dx = \int \frac{2\cos x}{3\sin^2 x} dx + \int 1 \, . dx$

$= \frac{2}{3} \int \operatorname{cosec} x \cot x \, dx + \int 1 \, . dx = -\frac{2}{3} \operatorname{cosec} x + x.$

Ex. 8. *Integrate* $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Sol. Here $I = \int \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) dx$

$= \int 1 \, . dx + \int x \, . dx + \int \frac{1}{2} x^2 \, dx + \int \frac{1}{6} x^3 \, dx + \ldots$

$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$

Ex. 9. *Integrate* $(2x^3 + 3x - 7)/x^{2/3}$.

Sol. Here $I = \int (2x^3 + 3x - 7) \, x^{-2/3} \, dx$

$= \int (2x^{7/3} + 3x^{1/3} - 7x^{-2/3}) \, dx$

$= \frac{2x^{(7/3)+1}}{(7/3)+1} + \frac{3x^{(1/3)+1}}{(1/3)+1} - \frac{7x^{(-2/3)+1}}{(-2/3)+1}$

$= \frac{3}{5} x^{10/3} + \frac{9}{4} x^{4/3} - 21 x^{1/3}.$

Ex. 10. *Integrate* $\frac{a}{x^2} + \frac{b}{x} + c$.

Sol. Here $I = \int \left(\frac{a}{x^2} + \frac{b}{x} + c\right) dx = \int \frac{a}{x^2} dx + \int \frac{b}{x} dx + \int c \, dx$

$= \int a x^{-2} \, dx + \int \frac{b}{x} dx + \int c \, dx$

$= \frac{a x^{-1}}{-1} + b \log x + cx = -\frac{a}{x} + b \log x + cx.$

Ex. 11. *Integrate* $(x^2 + 8)^2 / x^4$.

Sol. Here $I = \int \frac{(x^2+8)^2}{x^4} dx = \int \frac{(x^4 + 16x^2 + 64)}{x^4} dx$

$= \int (1 + 16x^{-2} + 64x^{-4}) \, dx = x + 16 \frac{x^{-2+1}}{-2+1} + 64 \frac{x^{-4+1}}{-4+1}$

$= x - (16/x) - 64/(3x^3).$

Ex. 12. *Integrate* $(x + a)^3 / \sqrt{x}$.

Sol. Here $I = \int \frac{(x+a)^3}{\sqrt{x}} dx = \int \left\{\frac{(x^3 + 3ax^2 + 3a^2x + a^3)}{\sqrt{x}}\right\} dx$

$= \int \{x^{5/2} + 3ax^{3/2} + 3a^2 x^{1/2} + a^3 x^{-1/2}\} \, dx$

$= \frac{2}{7} x^{7/2} + 3 \cdot \frac{2}{5} a x^{5/2} + 3 \cdot \frac{2}{3} a^2 x^{3/2} + 2a^3 . x^{1/2}.$

Ex. 13. *Integrate* $\frac{1}{x^{3/4}} + \frac{1 - x^4}{1 - x} + \sec x \tan x$.

Sol. Here

$I = \int x^{-3/4} \, dx + \int (1 + x + x^2 + x^3) \, dx + \int \sec x \tan x \, dx,$

$I = \int x^{-3/4}\,dx + \int (1 + x + x^2 + x^3)\,dx + \int \sec x \tan x\,dx,$

$[\because (1 - x^n)/(1 - x) = 1 + x + x^2 + x^3 + \ldots + x^{n-1}]$

$\therefore \quad I = 4x^{1/4} + x + (x^2/2) + (x^3/3) + (x^4/4) + \sec x.$

Ex. 14. *Integrate*

$$\frac{5\cos^3 x + 2\sin^3 x}{2\sin^2 x\cos^2 x} + \sqrt{(1 + \sin 2x)} + \frac{1 + 2\sin x}{\cos^2 x} + \frac{1 - \cos 2x}{1 + \cos 2x}.$$

Sol. The given expression may be written as

$$\frac{5\cos x}{2\sin^2 x} + \frac{\sin x}{\cos^2 x} + \sqrt{(\cos^2 x + \sin^2 x + 2\sin x\cos x)}$$

$$+ \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{2\sin^2 x}{2\cos^2 x}$$

$$= \tfrac{5}{2}\operatorname{cosec} x\cot x + \sec x\tan x + \cos x + \sin x + \sec^2 x$$

$$+ 2\sec x\tan x + 2(\sec^2 x - 1)$$

$$= \tfrac{5}{2}\operatorname{cosec} x\cot x + 3\sec x\tan x + \cos x + \sin x + 3\sec^2 x - 2.$$

Now integrating, we get

$$I = \tfrac{5}{2}\int \operatorname{cosec} x\cot x\,dx + 3\int \sec x\tan x\,dx$$

$$+ \int \cos x\,dx + \int \sin x\,dx + 3\int \sec^2 x\,dx - 2\int dx$$

$$= -\tfrac{5}{2}\operatorname{cosec} x + 3\sec x + \sin x - \cos x + 3\tan x - 2x.$$

§ 6. Extended forms of fundamental formulae.

Suppose we know that $\int f(x)\,dx = F(x)$ and we want to find $\int f(ax + b)\,dx$.

Let $I = \int f(ax + b)\,dx$.

Put $ax + b = t$ so that $a\,dx = dt$.

Then $I = \int f(t)\dfrac{dt}{a} = \dfrac{1}{a}\int f(t)\,dt = \dfrac{1}{a}F(t) = \dfrac{1}{a}F(ax + b).$

Thus if $\int f(x)\,dx = F(x)$, then $\int f(ax + b)\,dx = \dfrac{1}{a}F(ax + b).$

From this we conclude the following results :

(i) $\int (ax + b)^n\,dx = \dfrac{1}{a}\dfrac{(ax + b)^{n+1}}{(n + 1)},$

(ii) $\int \dfrac{1}{(ax + b)^n}\,dx = -\dfrac{1}{a(n - 1)(ax + b)^{n-1}},$

(iii) $\int \dfrac{1}{ax + b}\,dx = \dfrac{1}{a}\log(ax + b),$

(iv) $\int e^{ax+b}\,dx = \dfrac{1}{a}e^{ax+b}$ and $\int a^{px+q}\,dx = \dfrac{1}{p}\dfrac{a^{px+q}}{\log_e a},$

(v) $\int \sin(ax + b)\,dx = -\dfrac{1}{a}\cos(ax + b),$

(vii) $\int \sec^2(ax + b)\,dx = \dfrac{1}{a}\tan(ax + b),$

(vii) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)$,

(viii) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)$,

(ix) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)$,

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)$,

(xi) $\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}$ and $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

To evaluate $\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx$.

Put $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$.

Also $a^2 - x^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta$.

Thus
$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \int \frac{1}{a\cos\theta}\,a\cos\theta\,d\theta$$
$$= \int 1\,.\,d\theta = \theta = \sin^{-1}(x/a).$$

And to evaluate $\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx$.

Put $x = a\sinh\theta$; so that $dx = a\cosh\theta\,d\theta$. Also we have
$$a^2 + x^2 = a^2(1+\sinh^2\theta) = a^2\cosh^2\theta.$$

Thus
$$\int \frac{1}{\sqrt{(a^2+x^2)}}\,dx = \int \frac{1}{a\cosh\theta}\,a\cosh\theta\,d\theta$$
$$= \int 1\,.\,d\theta = \theta = \sinh^{-1}(x/a) = \log\left\{\frac{x+\sqrt{(x^2+a^2)}}{a}\right\}$$
$$= \log\{x+\sqrt{(x^2+a^2)}\} - \log a = \log\{x+\sqrt{(x^2+a^2)}\},$$

omitting the constant term $-\log a$ because it may be added to the constant of integration c which we usually do not write.

Similarly,
$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \cosh^{-1}(x/a) = \log\{x+\sqrt{(x^2-a^2)}\}$$
$$\int \sqrt{(a^2-x^2)}\,dx = \frac{x\sqrt{(a^2-x^2)}}{2} + \frac{a^2}{2}\sin^{-1}(x/a)$$
$$\int \sqrt{(a^2+x^2)}\,dx = \frac{x\sqrt{(a^2+x^2)}}{2} + \frac{a^2}{2}\sinh^{-1}(x/a)$$
$$\int \sqrt{(x^2-a^2)}\,dx = \frac{x\sqrt{(x^2-a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}(x/a).$$

Examples on § 6.

Ex. 15. *Integrate* $\sqrt{(2x+3)}$.

Sol. Here $I = \int \sqrt{(2x+3)}\,dx = \frac{1}{2}\frac{(2x+3)^{3/2}}{\frac{3}{2}} = \frac{1}{3}(2x+3)^{3/2}$.

Ex. 16. *Integrate* $\sin(2x + \frac{1}{4}\pi)$.

Sol. Here $I = \int \sin(2x + \frac{1}{4}\pi)\,dx = \frac{1}{2}[-\cos(2x + \frac{1}{4}\pi)]$.

***Ex. 17.** *Integrate* $\int \frac{1}{4 + (2 - 3x)^2}\,dx$.

Sol. We know that

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Here in place of x we have $(2 - 3x)$; therefore after applying the standard result we shall divide in the end by the coefficient of x in $(2 - 3x)$ *i.e.*, by -3. Also here $a^2 = 4$; $\therefore\ a = 2$. Thus the required integral

$$I = -\frac{1}{3}\cdot\frac{1}{2}\tan^{-1}\left(\frac{2-3x}{2}\right) = -\frac{1}{6}\tan^{-1}\left(\frac{2-3x}{2}\right).$$

Ex. 18. *Integrate* $\frac{1}{\sqrt{[7 - (\frac{1}{2}x - 3)^2]}}$.

Sol. We know that

$$\int \frac{1}{\sqrt{(a^2 - x^2)}}\,dx = \sin^{-1}\frac{x}{a}.$$

Here in place of x we have $(\frac{1}{2}x - 3)$; therefore after applying the standard result we shall divide in the end by the coefficient of x in $\frac{1}{2}x - 3$ *i.e.*, by $\frac{1}{2}$. Also here $a^2 = 7$; $\therefore\ a = \sqrt{7}$.

Thus the required integral $I = \int \frac{1}{\sqrt{[7 - (\frac{1}{2}x - 3)^2]}}\,dx$

$$= \frac{1}{(\frac{1}{2})}\sin^{-1}\left\{\frac{(\frac{1}{2}x - 3)}{\sqrt{7}}\right\} = 2\sin^{-1}\{(x-6)/2\sqrt{7}\}.$$

***Ex. 19.** *Integrate* $1/(1 + \sin x)$.

Sol. Here $I = \int \frac{1}{1 + \sin x}\,dx = \int \frac{dx}{1 - \cos(\frac{1}{2}\pi + x)}$

$$= \int \frac{dx}{2\sin^2(\frac{1}{4}\pi + \frac{1}{2}x)} = \frac{1}{2}\int \operatorname{cosec}^2(\frac{1}{2}x + \frac{1}{4}\pi)\,dx$$

$$= -\cot(\frac{1}{2}x + \frac{1}{4}\pi).$$

Ex. 20. *Integrate* $(1 + \sin x)/(1 - \cos x)$.

Sol. Here $I = \int \frac{1 + \sin x}{1 - \cos x}\,dx = \int \frac{1 + 2\sin\frac{1}{2}x\cos\frac{1}{2}x}{2\sin^2\frac{1}{2}x}\,dx$

$= \frac{1}{2}\int \operatorname{cosec}^2 \frac{1}{2}x\,dx + \int \cos\frac{1}{2}x\,dx$

$= -\cot\frac{1}{2}x + 2\log(\sin\frac{1}{2}x)$. $\quad [\because \int \cot x\,dx = \log(\sin x)]$

Ex. 21. *Integrate* $\dfrac{1}{\sqrt{(2x^2+3x+4)}}$.

Sol. We have $2x^2+3x+4 = 2[x^2+\frac{3}{2}x+2]$

[Note that we have made the coefficient of x^2 as 1]

$$= 2\left[\left(x+\frac{3}{4}\right)^2 + 2 - \frac{9}{16}\right] = 2\left[\left(x+\frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2\right].$$

Hence the given integral

$$I = \int \frac{1}{\sqrt{(2x^2+3x+4)}}\,dx = \int \frac{dx}{\sqrt{2}.\sqrt{[(x+\frac{3}{4})^2+(\sqrt{23}/4)^2]}}$$

$$= \frac{1}{\sqrt{2}}\sinh^{-1}\left\{\frac{x+(\frac{3}{4})}{\sqrt{(23)}/4}\right\} = \frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{4x+3}{\sqrt{23}}\right).$$

Ex. 23. *Integrate* $\dfrac{1}{\sqrt{(-2x^2+3x+4)}}$.

Sol. Here $-2x^2+3x+4 = -2[x^2-\frac{3}{2}x-2]$

$$= -2\left[\left(x-\frac{3}{4}\right)^2 - \left(\frac{\sqrt{41}}{4}\right)^2\right] = 2\left[\left(\frac{\sqrt{41}}{4}\right)^2 - \left(x-\frac{3}{4}\right)^2\right].$$

$$\therefore \quad I = \int \frac{1.dx}{\sqrt{(-2x^2+3x+4)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\{(\sqrt{41}/4)^2-(x-\frac{3}{4})^2\}}}$$

$$= \frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{x-(\frac{3}{4})}{\sqrt{(41)}/4}\right\} = \frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{4x-3}{\sqrt{(41)}}\right\}.$$

§ 7. Methods of Integration.

There are various methods of integration by which we can reduce the given integral to one of the fundamental or known integrals. Following are the four principal methods of integration :

(*i*) *Integration by substitution,*
(*ii*) *Integration by parts,*
(*iii*) *Integration by decomposition into sum,*
(*iv*) *Integration by successive reduction.*

(i) Integration by substitution. A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the **Method of substitution.**

Let $I = \int f(x)\,dx$; then by differentiation w.r.t. x, we have

$$\frac{dI}{dx} = f(x). \text{ Now put } x = \phi(t), \text{ so that } \frac{dx}{dt} = \phi'(t).$$

Then $\frac{dI}{dt}=\frac{dI}{dx}\cdot\frac{dx}{dt}=f(x).\phi'(t)=f\{\phi(t)\}\,\phi'(t)$, for $x=\phi(t)$.

This gives, $I=\int f\{\phi(t)\}.\phi'(t)\,dt$.

Rule to remember.

To evaluate $\int f\{\phi(x)\}.\phi'(x)\,dx$,

put $\phi(x)=t$ and $\phi'(x)\,dx=dt$,

where $\phi'(x)$ is the differential coefficient of $\phi(x)$ w.r.t. x.

Important. The success of the method of substitution depends on choosing the substitution $x=\phi(t)$ so that the new integrand $f\{\phi(t)\}.\phi'(t)$ is of a form whose integral is known. This is done by guess rather than in according with some rule. However, try to put that expression of x equal to t whose differential coefficient is multiplied with dx.

Solved Examples

Ex. 1. *Evaluate* $\int x\sec^2 x^2\,dx$.

Sol. Put $x^2=t$, so that $2x\,dx=dt$ or $x\,dx=\frac{1}{2}dt$.

$\therefore\ \int x\sec^2 x^2\,dx=\frac{1}{2}.\int \sec^2 t\,dt=\frac{1}{2}\tan t=\frac{1}{2}\tan x^2$.

Ex. 2. *Evaluate* $-\int\frac{\cos^{-1}x}{\sqrt{(1-x^2)}}dx$.

Sol. Put $\cos^{-1}x=t$, so that $-\frac{1}{\sqrt{(1-x^2)}}dx=dt$.

$\therefore$ the given integral $=\int t\,dt=\frac{1}{2}t^2=\frac{1}{2}(\cos^{-1}x)^2$.

Ex. 3. *Evaluate* $\int[1/(cx+d)^4]\,dx$.

Sol. Put $cx+d=t$; then $c\,dx=dt$, or $dx=dt/c$.

Thus the given integral

$$=\int(cx+d)^{-4}dx=\int t^{-4}\frac{dt}{c}=\frac{1}{c}\int t^{-4}dt=\frac{1}{c}\cdot\frac{t^{-3}}{-3}$$

$$=-\frac{1}{3c\,(cx+d)^3}.$$

Ex. 4. *Evaluate* $\int 20^{5x}dx$.

Sol. Put $5x=t$; then $5dx=dt$ or $dx=\frac{1}{5}dt$.

$$\therefore\quad I=\int 20^{5x}dx=\int\frac{1}{5}\cdot 20^t\,dt$$

$$=\frac{1}{5}\frac{20^t}{\log 20}=\frac{1}{5}\frac{20^{5x}}{\log 20}.$$

Ex. 5. *Evaluate* $\int\frac{dx}{\sqrt{\{1-(cx+d)^2\}}}$.

Sol. Put $cx+d=t$, so that $c\,dx=dt$. .

Hence the given integral

$$= \int \frac{1}{c} \frac{dt}{\sqrt{(1-t^2)}} = \frac{1}{c} \sin^{-1} t = \frac{1}{c} \sin^{-1}(cx+d).$$

Ex. 6. *Evaluate* $\int \{1/(c^2+b^2y^2)\}\, dy$.

Sol. Put $by = t$; $\therefore$ $b\,dy = dt$ or $dy = (1/b)\,dt$.

$$\therefore \quad I = \frac{1}{b}\int \frac{dt}{c^2+t^2} = \frac{1}{bc}\tan^{-1}\frac{t}{c} = \frac{1}{bc}\tan^{-1}\left(\frac{by}{c}\right).$$

Ex. 7. *Evaluate* $\int \{(\cos ax)/(\sin^2 ax)\}\, dx$.

Sol. We have $I = \int \frac{\cos ax\, dx}{\sin^2 ax} = \int \frac{\cos ax\, dx}{\sin ax \sin ax}$

$= \int \cot ax . \operatorname{cosec} ax\, dx$.

Now put $ax = t$; $\therefore$ $a dx = dt$ or $dx = (1/a)\,dt$.

Then $I = \frac{1}{a}\int \cot t \operatorname{cosec} t\, dt = -\frac{1}{a}\operatorname{cosec} t = -\frac{1}{a}\operatorname{cosec} ax$.

Ex. 8. *Evaluate* $\int -\frac{\operatorname{cosec}^2 x}{\sqrt{(\cot^2 x - 16)}}\, dx$.

Sol. Put $\cot x = t$, so that $-\operatorname{cosec}^2 x\, dx = dt$.

Hence $I = \int \frac{dt}{\sqrt{(t^2-16)}} = \cosh^{-1}\frac{t}{4} = \cosh^{-1}\left(\frac{\cot x}{4}\right)$.

Ex. 9. *Evaluate* $\int e^x \cos e^x\, dx$.

Sol. Put $e^x = t$, so that $e^x\, dx = dt$.

$\therefore$ $I = \int e^x \cos e^x\, dx = \int \cos t\, dt = \sin t = \sin e^x$.

Ex. 10. *Evaluate* $\int \sin^2 x \cos x\, dx$.

Sol. Put $\sin x = t$, so that $\cos x\, dx = dt$.

$\therefore$ $I = \int \sin^2 x \cos x\, dx = \int t^2\, dt = \frac{1}{3}t^3 = \frac{1}{3}\sin^3 x$.

Ex. 11. *Evaluate* $\int [4x^3/(1+x^8)]\, dx$.

Sol. Put $x^4 = t$ so that $4x^3\, dx = dt$.

$\therefore$ $I = \int [4x^3/(1+x^8)]\, dx = \int [1/(1+t^2)]\, dt = \tan^{-1} t$

$= \tan^{-1} x^4$.

§ 8. Three important forms of integrals.

1. $\int \frac{f'(x)}{f(x)}\, dx = \log f(x)$.

Put $f(x) = t$; differentiating we have $f'(x)\, dx = dt$.

$$\therefore \quad \int \frac{f'(x)}{f(x)}\, dx = \int \frac{dt}{t} = \log t = \log f(x).$$

Thus (**Remember**)

the integral of a fraction whose numerator is the exact derivative of its denominator is equal to the logarithm of its denominator.

For example $\int \frac{4x^3}{1+x^4}\,dx = \log(1+x^4)$,

as in this case numerator is the exact derivative of the denominator.

Similarly $\int \frac{e^x}{1+e^x}\,dx = \log(1+e^x)$.

Integrals of tan x, cot x, sec x and cosec x.

(i) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx$,

adjusting the numerator as the exact diff. coeffi. of the denominator

$$= -\log\cos x = \log(\cos x)^{-1} = \log(\sec x).$$

(ii) Similarly, $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \log(\sin x)$.

(iii) $\int \operatorname{cosec} x\,dx = \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x} = \int \frac{\sec^2\frac{1}{2}x\,dx}{2\tan\frac{1}{2}x}$,

[dividing Nr. and Dr. by $\cos^2\frac{1}{2}x$]

$= \log(\tan\frac{1}{2}x)$, $\quad(\because \frac{1}{2}\sec^2\frac{1}{2}x$ is the diff. coeffi. of $\tan\frac{1}{2}x]$.

(iv) $\int \sec x\,dx = \int \operatorname{cosec}(\frac{1}{2}\pi + x)\,dx$.

Now proceeding as in the case of $\int \operatorname{cosec} x\,dx$, we have

$$\int \sec x\,dx = \log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right).$$

***Alternative Method.**

$$\int \sec x\,dx = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\,dx,$$

[We have multiplied the Nr. and Dr. both by $(\sec x + \tan x)$]

$= \int \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$, [Here Nr. is the diff. coeffi. of Dr.]

$= \log(\sec x + \tan x)$.

****2.** $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}$, **when** $n \neq -1$.

(Power formula)

Putting $f(x) = t$, so that $f'(x)\,dx = dt$, we get

$$\int [f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1},\ (\text{for } n \neq -1) = \frac{[f(x)]^{n+1}}{n+1}.$$

Thus **Remember :** *If the integrand consists of the product of a constant power of a function $f(x)$ and the derivative $f'(x)$ of $f(x)$, to obtain the integral we increase the index by unity and then divide by the increased index.* This is known as **Power formula.** The students are advised to have a lot of practice of applying this formula.

3. $\int \mathbf{f}'(\mathbf{ax}+\mathbf{b})\,\mathbf{dx} = \dfrac{\mathbf{f}(\mathbf{ax}+\mathbf{b})}{\mathbf{a}}$.

Solved Examples

Ex. 1. *Integrate* (i) $\dfrac{ax+b}{ax^2+2bx+c}$. (ii) $\dfrac{ax^{n-1}}{x^n+b}$.

Sol. (i) Let $I = \displaystyle\int \frac{ax+b}{ax^2+2bx+c}\,dx$.

[Put $ax^2+2bx+c=t$, so that $(2ax+2b)\,dx = dt$].

$\therefore \quad I = \dfrac{1}{2}\displaystyle\int \frac{dt}{t} = \frac{1}{2}\log t = \frac{1}{2}\log(ax^2+2bx+c)$.

(ii) Let $I = \displaystyle\int \frac{ax^{n-1}\,dx}{x^n+b} = \frac{1}{n}\int \frac{dt}{t}$,

[Putting $x^n+b=t$, so that $nx^{n-1}\,dx = dt$]

$= (a/n)\,.\log(t) = (a/n)\,.\log(x^n+b)$.

Ex. 2. *Integrate* (i) $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, (ii) $\dfrac{10x^9+10^x.\log_e 10}{10^x+x^{10}}$.

Sol. (i) Let $I = \displaystyle\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$.

Now putting $e^x+e^{-x}=t$, so that $(e^x-e^{-x})\,dx = dt$,
we have $I = \int (1/t)\,dt = \log t = \log(e^x+e^{-x})$.

(ii) Here $I = \displaystyle\int \frac{10x^9+10^x.\log_e 10}{10^x+x^{10}}\,dx$.

Now putting $10^x+x^{10}=t$, and $(10^x\log_e 10+10x^9)\,dx = dt$,
we have $I = \int (1/t)\,dt = \log t = \log(10^x+x^{10})$.

Ex. 3. *Integrate* (i) $\dfrac{\operatorname{cosec}^2 x}{1+\cot x}$, (ii) $\dfrac{1}{(1+x^2)\tan^{-1}x}$.

Sol. (i) Here $I = \int \{\operatorname{cosec}^2 x/(1+\cot x)\}\,dx$.
Putting $1+\cot x = t$, so that $-\operatorname{cosec}^2 x\,dx = dt$,
we have $I = -\int (1/t)\,dt = -\log t = -\log(1+\cot x)$.

(ii) Here $I = \displaystyle\int \frac{dx}{(1+x^2)\tan^{-1}x}$.

Putting $\tan^{-1}x = t$, so that $[1/(1+x^2)]\,dx = dt$, we have
$I = \int (1/t)\,dt = \log t = \log(\tan^{-1}x)$.

Ex. 4. *Integrate*

(i) $\dfrac{\sin x}{a+b\cos x}$, (ii) $\dfrac{\sin x\cos x}{a\cos^2 x+b\sin^2 x}$.

Sol. (i) Here $I = \int \frac{\sin x \, dx}{a + b \cos x} = -\frac{1}{b} \int \frac{-b \sin x}{a + b \cos x} dx$.

Putting $a + b \cos x = t$, so that $-b \sin x \, dx = dt$, we have

$I = (-1/b) \int (1/t) \, dt$

$= -(1/b) . \log (t) = -(1/b) . \log (a + b \cos x)$.

(ii) Let $I = \int \left(\frac{\sin x \cos x}{a \cos^2 x + b \sin^2 x} \right) dx$.

Put $a \cos^2 x + b \sin^2 x = t$,

so that $(-2a \cos x \sin x + 2b \sin x \cos x) \, dx = dt$,

or $\{2 (b - a) \sin x \cos x\} \, dx = dt$.

We have $I = \frac{1}{2 (b - a)} \int \frac{dt}{t} = \frac{1}{2 (b - a)} \log t$

$= \frac{1}{2 (b - a)} \log (a \cos^2 x + b \sin^2 x)$.

Ex. 5. *Integrate*

(i) $\frac{\sin x}{1 + \cos^2 x}$, (ii) $\frac{\sin (a + b \log x)}{x}$.

Sol. (i) Here $I = \int \frac{\sin x}{1 + \cos^2 x} dx$.

Putting $\cos x = t$, so that $-\sin x \, dx = dt$, we have

$I = -\int dt/(1 + t^2) = -\tan^{-1} t = -\tan^{-1} (\cos x)$.

(ii) Here $I = \int [\{\sin (a + b \log x)\}/x] \, dx$.

Putting $a + b \log x = t$, so that $(b/x) \, dx = dt$, we have

$I = \frac{1}{b} \int \sin t \, dt = -\frac{1}{b} \cos t = -\frac{1}{b} \cos (a + b \log x)$.

Ex. 6. *Integrate*

(i) $\frac{1}{x \cos^2 (1 + \log x)}$ (ii) $\frac{1}{x (1 + \log x)^m}$.

Sol. (i) Here $I = \int dx/\{x \cos^2 (1 + \log x)\}$.

Putting $1 + \log x = t$, so that $(1/x) \, dx = dt$, we have

$I = \int dt/\cos^2 t = \int \sec^2 t \, dt = \tan t = \tan (1 + \log x)$.

(ii) Here $I = \int dx/\{x (1 + \log x)^m\}$.

Putting $1 + \log x = t$, so that $(1/x) \, dx = dt$, we have

$I = \int \frac{dt}{t^m} = \frac{t^{-m+1}}{-m+1} = \frac{(1 + \log x)^{-m+1}}{(1 - m)}$

$= \frac{1}{(1 - m)} (1 + \log x)^{1 - m}$.

Ex. 7. *Integrate* (i) $\frac{e^{\tan^{-1} x}}{1 + x^2}$ **(Meerut 1986 S, 88)**

(ii) $\dfrac{\sin(\tan^{-1}x)}{1+x^2}$.

Sol. (i) Putting $\tan^{-1}x = t$, so that $[1/(1+x^2)]\,dx = dt$, we have

$$I = \int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^t\,.\,dt = e^t = e^{\tan^{-1}x}.$$

(ii) Putting $\tan^{-1}x = t$ so that $[1/(1+x^2)]\,dx = dt$, we have

$$I = \int \frac{\sin(\tan^{-1}x)}{1+x^2}\,dx = \int \sin t\,dt = -\cos t = -\cos(\tan^{-1}x).$$

Ex. 8. *Integrate* (i) $x\cos^3 x^2 . \sin x^2$ (ii) $x^3\tan^4 x^4 . \sec^2 x^4$.

Sol. (i) Here $I = \int x\cos^3 x^2 \sin x^2\,dx$.

First, putting $x^2 = t$, so that $2x\,dx = dt$, we have

$$I = \tfrac{1}{2}\int \cos^3 t \sin t\,dt.$$

Now putting $\cos t = u$, so that $-\sin t\,dt = du$, we have

$$I = -\tfrac{1}{2}\int u^3\,du = -\tfrac{1}{2}\frac{u^4}{4} = -\frac{1}{8}u^4$$

$$= -\tfrac{1}{8}\cos^4 t = -\tfrac{1}{8}\cos^4 x^2. \qquad [\because t = x^2]$$

Note. The students should also solve this problem by making the single substitution $\cos x^2 = t$.

(ii) Here $I = \int x^3 \tan^4 x^4 \sec^2 x^4\,dx$.

Putting $\tan x^4 = t$, so that $(\sec^2 x^4)\,.\,4x^3\,dx = dt$, we have

$$I = \int \frac{t^4}{4}\cdot dt = \tfrac{1}{4}\left(\frac{t^5}{5}\right) = \frac{(\tan x^4)^5}{20}.$$

Ex. 9. *Integrate* $\dfrac{x^3\tan^{-1}x^4}{1+x^8}$.

Sol. Putting $\tan^{-1}x^4 = t$ so that $\dfrac{1}{1+x^8}\cdot 4x^3\,dx = dt$, we have

$$I = \int \frac{x^3\tan^{-1}x^4}{1+x^8}\,dx = \tfrac{1}{4}\int t\,dt = \tfrac{1}{8}t^2 = \tfrac{1}{8}(\tan^{-1}x^4)^2.$$

Ex. 10. *Evaluate* $\displaystyle\int \frac{e^x(1+x)}{\sin^2(xe^x)}\,dx$. **(Meerut 1982)**

Sol. Putting $xe^x = t$ so that $(e^x + xe^x)\,dx = dt$

or $e^x(1+x)\,dx = dt$, we have

$$I = \int \frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t\,dt = -\cot t = -\cot(xe^x).$$

Ex. 11. *Integrate* (i) $\dfrac{1}{(e^x+1)}$, (ii) $\dfrac{1}{(e^x-1)}$.

(Meerut 1972)

Sol. (i) Here $I = \int \frac{dx}{e^x + 1} = \int \frac{e^{-x}}{1 + e^{-x}} dx,$

[Multiplying Nr. and Dr. both by e^{-x}]

$= -\int \frac{-e^{-x}}{1 + e^{-x}} dx = -\log(1 + e^{-x}),$

[$\because$ Nr. is diff. coeffi. of Dr.]

$= -\log\left(\frac{1 + e^x}{e^x}\right) = -[\log(1 + e^x) - \log e^x]$

$= x \log e - \log(1 + e^x) = x - \log(1 + e^x).$

(ii) Similarly $\int \frac{dx}{e^x - 1} = \log(1 - e^{-x}).$

Ex. 12. *Integrate* (i) $\frac{\cot x}{\log(\sin x)}$, (ii) $\frac{\tan x}{\log(\sec x)}$.

Sol. (i) Here $\frac{d}{dx}(\log \sin x) = \frac{1}{\sin x} \cos x = \cot x.$

$\therefore \quad I = \int \frac{\cot x \, dx}{\log \sin x} = \log(\log \sin x),$

[$\because$ Nr. is diff. coeffi. of Dr.]

(ii) Similarly, we have

$$\int \frac{\tan x \, dx}{\log \sec x} = \log(\log \sec x).$$

****Ex. 13.** *Integrate* (i) $\sqrt{(1 + \sin x)}$, (ii) $\frac{1}{\sqrt{(1 + \sin x)}}$

(Meerut 1984 P)

Sol. (i) We have

$I = \int \sqrt{(1 + \sin x)}\, dx = \int \sqrt{\{1 - \cos(\frac{1}{2}\pi + x)\}}\, dx$

$= \int \sqrt{[2 \sin^2(\frac{1}{4}\pi + \frac{1}{2}x)]}\, dx = \sqrt{2} \int \sin(\frac{1}{2}x + \frac{1}{4}\pi)\, dx.$

Now putting $\frac{1}{2}x + \frac{1}{4}\pi = t$, so that $\frac{1}{2} dx = dt$ or $dx = 2\, dt$, we have

$I = \sqrt{2} \int 2 \sin t \, dt = -2\sqrt{2} \cos t = -2\sqrt{2} \cos(\frac{1}{2}x + \frac{1}{4}\pi).$

(ii) Here $I = \int \frac{dx}{\sqrt{(1 + \sin x)}} = \frac{1}{\sqrt{2}} \int \operatorname{cosec}(\frac{1}{2}x + \frac{1}{4}\pi)\, dx,$

[Proceeding as in part (i)]

Now putting $\frac{1}{2}x + \frac{1}{4}\pi = t$ so that $dx = 2\, dt$, we have

$I = \frac{2}{\sqrt{2}} \int \operatorname{cosec} t \, dt = \sqrt{2} \log \tan(\frac{1}{2}t)$

$= \sqrt{2} \log \tan(\frac{1}{4}x + \frac{1}{8}\pi).$

Ex. 14. *Integrate* (i) $\frac{\sin x}{\sqrt{(1 + \sin x)}}$ (Meerut 1985)

(ii) $\sqrt{(1-\cos x)}$.

Sol. (i) Here $I=\int \frac{\sin x\, dx}{\sqrt{(1+\sin x)}}=\int \frac{(1+\sin x)-1}{\sqrt{(1+\sin x)}}dx$

$$=\int \sqrt{(1+\sin x)}\,dx-\int \frac{1}{\sqrt{(1+\sin x)}}dx$$

$$=-2\sqrt{2}\cos\left(\tfrac{1}{2}x+\tfrac{1}{4}\pi\right)-\sqrt{2}\log\tan\left(\tfrac{1}{4}x+\tfrac{1}{8}\pi\right),$$

[Proceeding as in Ex. 13 parts (i) and (ii)].

(ii) Here $I=\int \sqrt{(1-\cos x)}\,dx=\sqrt{2}\int \sin\tfrac{1}{2}x\,dx$

$$=-\sqrt{2}\frac{\cos(x/2)}{1/2}=-2\sqrt{2}\cos(x/2).$$

Ex. 15. *Integrate*

(i) $\dfrac{\sec x}{a+b\tan x}$ (ii) $\dfrac{\sec x}{\sqrt{3}+\tan x}$ **(Meerut 1989)**

Sol. (i) Let $I=\int \frac{\sec x}{a+b\tan x}dx=\int \frac{dx}{a\cos x+b\sin x}$.

Now let $a=r\sin\phi$ and $b=r\cos\phi$. This gives

$$r=\sqrt{(a^2+b^2)},\text{ and }\phi=\tan^{-1}(a/b).$$

$$\therefore\quad I=\int \frac{dx}{r\sin(x+\phi)}=\frac{1}{\sqrt{(a^2+b^2)}}\int \operatorname{cosec}(x+\phi)\,dx$$

$$=\frac{1}{\sqrt{(a^2+b^2)}}\log\tan\left(\tfrac{1}{2}x+\tfrac{1}{2}\phi\right)$$

$$=\frac{1}{\sqrt{(a^2+b^2)}}\log\tan\left(\frac{1}{2}x+\frac{1}{2}\tan^{-1}\frac{a}{b}\right).$$

(ii) We have $\int \frac{\sec x\, dx}{\sqrt{3}+\tan x}=\int \frac{\sec x\, dx}{\sqrt{3}+(\sin x/\cos x)}$

$$=\int \frac{dx}{\sqrt{3}\cos x+\sin x}=\int \frac{dx}{2\left[(\sqrt{3}/2)\cos x+\frac{1}{2}\sin x\right]}$$

$$=\tfrac{1}{2}\int \frac{dx}{\sin\left(x+\frac{1}{3}\pi\right)}=\tfrac{1}{2}\int \operatorname{cosec}\left(x+\tfrac{1}{3}\pi\right)dx$$

$$=\tfrac{1}{2}\log\tan\left[\tfrac{1}{2}x+(\pi/6)\right].$$

Ex. 16. *Integrate*

(i) $\dfrac{\cos 2x}{\sin x}$, (ii) $\dfrac{\cos 2x}{\cos x}$.

Sol. (i) $I=\int \frac{\cos 2x}{\sin x}dx=\int \frac{1-2\sin^2 x}{\sin x}dx$

$[\because \cos 2x=1-2\sin^2 x]$

$$=\int (\operatorname{cosec} x-2\sin x)\,dx=\log\tan\left(\tfrac{1}{2}x\right)+2\cos x.$$

(ii) Here $I=\int \frac{\cos 2x}{\cos x}dx=\int \frac{2\cos^2 x-1}{\ldots}dx$

(ii) Here $I = \int \frac{\cos 2x}{\cos x} dx = \int \frac{2\cos^2 x - 1}{\cos x} dx,$

$[\because \cos 2x = 2\cos^2 x - 1]$

$= \int (2\cos x - \sec x)\, dx = 2\sin x - \log \tan\left(\tfrac{1}{2}x + \tfrac{1}{4}\pi\right).$

Ex. 17. *Integrate* $1/(1 + 3\sin^2 x)$.

Sol. Dividing Nr. and Dr. by $\cos^2 x$, we have

$$I = \int \frac{dx}{1 + 3\sin^2 x} = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{(1 + \tan^2 x) + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{1 + 4\tan^2 x}.$$

Now putting $2\tan x = t$ so that $2\sec^2 x\, dx = dt$, we have

$$I = \tfrac{1}{2} \int \frac{dt}{1 + t^2} = \tfrac{1}{2}\tan^{-1} t = \tfrac{1}{2}\tan^{-1}(2\tan x).$$

Ex. 18. *Evaluate the following integrals* :

(i) $\int \frac{d\theta}{a^2 \sin^2\theta + b^2\cos^2\theta}$ **(Meerut 1985 P)**

(ii) $\int \frac{dx}{4\sin^2 x + 5\cos^2 x}$ **(Meerut 1989)**

Sol. **(i)** Dividing Nr. and Dr. by $\cos^2\theta$, we have

$$I = \int \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta} = \int \frac{\sec^2\theta\, d\theta}{a^2\tan^2\theta + b^2}$$

$$= \frac{1}{a^2}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta + (b^2/a^2)}$$

$$= \frac{1}{a^2}\int \frac{dt}{t^2 + (b^2/a^2)}$$ [Putting $t = \tan\theta$, so that $dt = \sec^2\theta\, d\theta$]

$$= \frac{1}{a^2}\cdot\frac{a}{b}\tan^{-1}\left(\frac{t}{b/a}\right) = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan\theta\right).$$

(ii) Let $I = \int \frac{dx}{4\sin^2 x + 5\cos^2 x}$

$= \int \frac{\sec^2 x\, dx}{4\tan^2 x + 5}$, dividing the Nr. and Dr. by $\cos^2 x$

$= \int \frac{dt}{4t^2 + 5}$, putting $\tan x = t$ so that $\sec^2 x\, dx = dt$

$$= \tfrac{1}{4}\int \frac{dt}{t^2 + (\sqrt{5}/2)^2} = \tfrac{1}{4}\frac{1}{(\sqrt{5}/2)}\tan^{-1}\frac{t}{\sqrt{5}/2}$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2t}{\sqrt{5}} = \frac{1}{2\sqrt{5}} \tan^{-1} \frac{2 \tan x}{\sqrt{5}}.$$

Ex. 19. *Evaluate* $\int [(\cos x)/(a^2 + b^2 \sin^2 x)]\, dx$.

Sol. Putting $b \sin x = t$ so that $b \cos x\, dx = dt$, we have

$$I = \int \frac{\cos x\, dx}{a^2 + b^2 \sin^2 x} = \frac{1}{b} \int \frac{dt}{a^2 + t^2} = \frac{1}{b} \cdot \frac{1}{a} \tan^{-1} \left(\frac{t}{a}\right)$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{b \sin x}{a}\right).$$

Ex. 20. *Evaluate* $\int \frac{\cot(\log x)}{x}\, dx$.

Sol. Putting $\log x = t$ so that $(1/x)\, dx = dt$, we have the given integral

$$I = \int \cot t\, dt = \log \sin t = \log \{\sin(\log x)\}.$$

§ 9. Some more standard Integrals.

(i) To evaluate $\int [1/\sqrt{(a^2 + x^2)}]\, dx$.

For complete solution of this problem see § 6, page 7. The result is

$$\int \frac{dx}{\sqrt{(a^2 + x^2)}} = \sinh^{-1} \left(\frac{x}{a}\right) = \log \{x + \sqrt{(x^2 + a^2)}\}.$$

(ii) To evaluate $\int \frac{dx}{\sqrt{(a^2 - x^2)}}$.

For complete solution of this problem see § 6, page 7. The result is

$$\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1} \left(\frac{x}{a}\right).$$

(iii) To evaluate $\int \frac{dx}{\sqrt{(x^2 - a^2)}}$.

Put $x = a \cosh \theta$ so that $dx = a \sinh \theta\, d\theta$.

Then the given integral $= \int \frac{dx}{\sqrt{(x^2 - a^2)}} = \int \frac{a \sinh \theta\, d\theta}{\sqrt{(a^2 \cosh^2 \theta - a^2)}}$

$$= \int \frac{a \sinh \theta\, d\theta}{a\sqrt{(\cosh^2 \theta - 1)}} = \int \frac{\sinh \theta\, d\theta}{\sinh \theta} = \int d\theta = \theta = \cosh^{-1}(x/a)$$

$$= \log \left[\frac{x}{a} + \sqrt{\left\{\left(\frac{x}{a}\right)^2 - 1\right\}}\right] = \log \left\{\frac{x + \sqrt{(x^2 - a^2)}}{a}\right\}$$

$$= \log \{x + \sqrt{(x^2 - a^2)}\} - \log a = \log \{x + \sqrt{(x^2 - a^2)}\},$$

because the constant term $-\log a$ may be added to the constant of integration c which we usually do not write.

Thus $\int \frac{dx}{\sqrt{(x^2 - a^2)}} = \cosh^{-1}(x/a) = \log \{x + \sqrt{(x^2 - a^2)}\}.$

(iv) To evaluate $\int \sqrt{(x^2+a^2)}\,dx$.

Put $x = a\sinh\theta$ so that $dx = a\cosh\theta\,d\theta$.

Then the given integral $= \int \sqrt{(a^2\sinh^2\theta + a^2)}\,.\,a\cosh\theta\,d\theta$

$= \int a^2\cosh^2\theta\,d\theta = \int \frac{1}{2}a^2(1+\cosh 2\theta)\,d\theta,$

$[\because \cosh 2\theta = 2\cosh^2\theta - 1]$

$= \frac{1}{2}a^2 \int (1+\cosh 2\theta)\,d\theta = \frac{1}{2}a^2[\theta + \frac{1}{2}\sinh 2\theta]$

$= \frac{1}{2}a^2[\theta + \sinh\theta\cosh\theta],$ $[\because \sinh 2\theta = 2\sinh\theta\cosh\theta]$

$= \frac{1}{2}a^2[\theta + \sinh\theta\sqrt{(1+\sinh^2\theta)}]$

$$= \frac{a^2}{2}\left[\sinh^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{\left(1+\frac{x^2}{a^2}\right)}\right]$$

$$= \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\cdot\frac{x}{a^2}\sqrt{(a^2+x^2)}.$$

Thus $\int \sqrt{(x^2+a^2)}\,dx = \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

$= \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{a^2}{2}\log\{x + \sqrt{(x^2+a^2)}\}.$

(v) To evaluate $\int \sqrt{(a^2-x^2)}\,dx$. **(Kashmir 1983)**

Put $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$.

Then the given integral $= \int a\cos\theta\,.\,a\cos\theta\,d\theta$

$= \int a^2\cos^2\theta\,d\theta = \frac{1}{2}a^2\int 2\cos^2\theta\,d\theta = \frac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta$

$= \frac{1}{2}a^2(\theta + \frac{1}{2}\sin 2\theta) = \frac{1}{2}a^2(\theta + \sin\theta\cos\theta)$

$= \frac{1}{2}a^2[\theta + \sin\theta\sqrt{(1-\sin^2\theta)}]$

$$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^2\cdot\frac{x}{a}\sqrt{\left(1-\frac{x^2}{a^2}\right)}.$$

Thus $\int \sqrt{(a^2-x^2)}\,dx = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a).$

(vi) To evaluate $\int \sqrt{(x^2-a^2)}\,dx$.

Put $x = a\cosh\theta$ so that $dx = a\sinh\theta\,d\theta$.

Then the given integral $= \int \sqrt{(a^2\cosh^2\theta - a^2)}\,a\sinh\theta\,d\theta$

$= \int a^2\sinh^2\theta\,d\theta = \int \frac{1}{2}a^2(\cosh 2\theta - 1)\,d\theta,$

$[\because 2\sinh^2\theta = \cosh 2\theta - 1]$

$= \frac{1}{2}a^2[\frac{1}{2}\sinh 2\theta - \theta] = \frac{1}{2}a^2[\sinh\theta\cosh\theta - \theta]$

$= \frac{1}{2}a^2[\sqrt{(\cosh^2\theta - 1)}\cosh\theta - \theta]$

$$= \frac{1}{2}a^2\left[\sqrt{\left(\frac{x^2}{a^2}-1\right)}\cdot\frac{x}{a} - \cosh^{-1}\frac{x}{a}\right]\cdot\quad\left[\because \cosh\theta = \frac{x}{a}\right]$$

Thus $\int \sqrt{(x^2-a^2)}\,dx = \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\cosh^{-1}(x/a)$

$= \frac{1}{2}x\sqrt{(x^2-a^2)} - \frac{1}{2}a^2\log\{x+\sqrt{(x^2-a^2)}\}$.

Solved Examples

Ex. 1. *Evaluate* $\displaystyle\int \frac{e^y\,dy}{\sqrt{(1+e^{2y})}}$.

Sol. Put $e^y = t$ so that $e^y\,dy = dt$.

$\therefore$ the given integral

$= \int \{1/\sqrt{(1+t^2)}\}\,dt = \sinh^{-1} t = \sinh^{-1}(e^y)$.

Ex. 2. *Evaluate* $\int \cos x\sqrt{(4-\sin^2 x)}\,dx$.

Sol. Put $\sin x = t$ so that $\cos x\,dx = dt$.

Then the given integral $= \int \sqrt{(4-t^2)}\,dt = \int \sqrt{(2^2-t^2)}\,dt$

$= \dfrac{1}{2}t\sqrt{(4-t^2)} + \dfrac{2^2}{2}\sin^{-1}(t/2)$

$= \frac{1}{2}\sin x.\sqrt{(4-\sin^2 x)} + 2\sin^{-1}(\frac{1}{2}\sin x)$.

Ex. 3. *Evaluate* $\int \sec x\tan x\sqrt{(\sec^2 x+1)}\,dx$.

Sol. Put $\sec x = t$ so that $\sec x\tan x\,dx = dt$.

Then the given integral $= \int \sqrt{(t^2+1)}\,dt$

$= \frac{1}{2}t\sqrt{(t^2+1)} + \frac{1}{2}\sinh^{-1} t$

$= \frac{1}{2}\sec x\sqrt{(\sec^2 x+1)} + \frac{1}{2}\sinh^{-1}(\sec x)$.

Ex. 4. *Evaluate* $\displaystyle\int \frac{x}{\sqrt{(x^4+4)}}\,dx$.

Sol. Put $x^2 = t$; $\therefore$ $2x\,dx = dt$.

$\therefore$ the given integral $= \displaystyle\frac{1}{2}\int \frac{1}{\sqrt{(t^2+4)}}\,dt$

$= \frac{1}{2}\sinh^{-1}(t/2) = \frac{1}{2}\sinh^{-1}(x^2/2)$.

Ex. 5. *Evaluate* $\int \{x^2/\sqrt{(x^6-9)}\}\,dx$.

Sol. Put $x^3 = t$, $\therefore$ $3x^2\,dx = dt$.

$\therefore$ the given integral $= \displaystyle\frac{1}{3}\int \frac{dt}{\sqrt{(t^2-9)}}$

$= \frac{1}{3}\cosh^{-1}(t/3) = \frac{1}{3}\cosh^{-1}(x^3/3)$.

Ex. 6. *Evaluate* $\int x\sqrt{(x^4+9)}\,dx$.

Sol. Put $x^2 = t$; $\therefore$ $2x\,dx = dt$.

$\therefore$ the given integral $= \frac{1}{2}\int \sqrt{(t^2+3^2)}\,dt$

$= \frac{1}{2}[\frac{1}{2}t\sqrt{(t^2+9)} + \frac{9}{2}\sinh^{-1}(t/3)]$

$= \frac{1}{4}x^2\sqrt{(x^4+9)} + \frac{9}{4}\sinh^{-1}(x^2/3)$.

Ex. 7. *Evaluate* $\int x^2 \sqrt{(x^6-1)}\,dx$.

Sol. Put $x^3 = t$; $\therefore$ $3x^2\,dx = dt$.

$\therefore$ the given integral $= \frac{1}{3}\int \sqrt{(t^2-1)}\,dt$

$$= \frac{1}{3}\left[\frac{t}{2}\sqrt{(t^2-1)} - \frac{1}{2}\cosh^{-1}\left(\frac{t}{1}\right)\right]$$

$$= \frac{1}{3}\left[\frac{x^3}{2}\sqrt{(x^6-1)} - \frac{1}{2}\cosh^{-1}x^3\right]$$

$$= \tfrac{1}{6}x^3\sqrt{(x^6-1)} - \tfrac{1}{6}\cosh^{-1}x^3.$$

§ 10. Integral of the product of two functions.

Integration by parts. Let u and v be two functions of x. Then we have from differential calculus

$$\frac{d}{dx}(uv) = u\cdot\frac{dv}{dx} + v\cdot\frac{du}{dx} \quad ...(1)$$

Integrating both sides of (1) with respect to x, we have

$$uv = \int u\cdot\frac{dv}{dx}\,dx + \int v\cdot\frac{du}{dx}\,dx.$$

By transposition, we have

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx. \quad ...(2)$$

Now put $u = f_1(x)$ and $v = \int f_2(x)\,dx$, so that $\frac{dv}{dx} = f_2(x)$.

Then from (2), we have

$$\int \mathbf{f_1(x)\,f_2(x)\,dx = f_1(x)\,.\int f_2(x)\,dx}$$
$$-\int\left[\left\{\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{f_1(x)}\right\}\cdot\int \mathbf{f_2(x)\,dx}\right]\mathbf{dx}$$

i.e., the integral of the product of two functions

= first function × integral of second function

− integral of {diff. coeffi. of first function × Integral of second function}.

Note 1. Care must be taken in choosing the first function and the second function. Obviously we must take that function as the second function whose integral is well known to us. Thus to evaluate $\int x\log x\,dx$ we shall take x as the second function because we so far do not know the integral of $\log x$. But to evaluate $\int x\sin x\,dx$ we must take $\sin x$ as the second function and x as the first function. Here if we take x as the second function, then the new integral will become more complicated. Thus to evaluate integrals of the type $\int x^2 e^x\,dx$ $\int x^3\cos x\,dx$ etc., the function of the type x^n must be taken as the first function. In certain cases we can take unity (*i.e.*, 1) as the second function. Thus to evaluate $\int \log x \, dx$ we shall take 1 as the second function. To evaluate $\int e^x \sin x \, dx$ we can take either e^x or $\sin x$ as the second function.

Note 2. The formula of integration by parts can be applied more than once if necessary.

Note 3. Integration by parts as applied to the functions of the type $e^x [f(x) + f'(x)]$.

Let $I = \int e^x [f(x) + f'(x)] \, dx = \int e^x f(x) \, dx + \int e^x f'(x) \, dx$.

Integrating the first integral by parts regarding e^x as the 2nd function, we have

$$I = [f(x) e^x - \int f'(x) e^x dx] + \int f'(x) e^x dx = e^x f(x).$$

[Note that we have left the other integral unchanged because the last two integrals cancel each other].

§ 11. Successive integration by parts.

If u is a function of the type $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, where n is a positive integer, the following formula for successive integration by parts can be applied. While writing this formula the successive differential coefficients of u have been denoted by u', u'', u''' etc., while the successive integrals of v have been denoted by v_1, v_2, v_3 etc. Thus

$$\int uv \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots.$$

The process of successive integration by parts will be continued till on being differentiated successively the differential coefficient of u becomes zero. The following examples will make the process clear.

Example 1. *Evaluate* $\int x^5 e^x \, dx$.

Sol. Here e^x will be successively integrated and x^5 will be successively differentiated. Thus applying successive integration by parts, the given integral

$$I = x^5 e^x - (5x^4) e^x + (20x^3) e^x - (60x^2) e^x + (120x) e^x - 120 e^x,$$

the process of successive integration by parts terminates because the differential coefficient of 120 is zero.

$$\therefore \quad I = e^x (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120).$$

Example 2. *Evaluate* $\int x^4 \sin x \, dx$.

Sol. Applying successive integration by parts, the given integral

$$I = x^4 . (-\cos x) - (4x^3) . (-\sin x) + (12x^2) . (\cos x) - (24x) . (\sin x) + (24) . (-\cos x)$$

$$= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x.$$

Remark. While applying successive integration by parts the successive differential coefficients and the successive integrals must at the first stage be put within brackets.

Example 3. *Evaluate* $\int x^3 e^{-x} dx$

Example 3. *Evaluate* $\int x^3 e^{-x}\,dx$.

Sol. Applying successive integration by parts, the given integral

$$I = (x^3).(-e^{-x}) - (3x^2).(e^{-x}) + (6x).(-e^{-x}) - (6).(e^{-x})$$
$$= -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}.$$
$$= -(x^3 + 3x^2 + 6x + 6)\,e^{-x}.$$

§ 12. Integrals of e^{ax} cos bx and e^{ax} sin bx.

Let $I = \int e^{ax} \sin bx\,dx$.

Integrating by parts taking $\sin bx$ as the second function, we get

$$I = -\frac{e^{ax}\cos bx}{b} - \int ae^{ax}\left(-\frac{\cos bx}{b}\right)dx$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx.$$

Again integrating by parts taking $\cos bx$ as the second function, we get

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[\frac{e^{ax}\sin bx}{b} - \int ae^{ax}\frac{\sin bx}{b}dx\right]$$

or $$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$

or $$I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx) - \frac{a^2}{b^2}I. \qquad \left[\because \int e^{ax}\sin bx\,dx = I\right]$$

Transposing the term $-\frac{a^2}{b^2}I$ to the left hand side, we get

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$$

or $$\frac{1}{b^2}(a^2 + b^2)\,I = \frac{1}{b^2}e^{ax}(a\sin bx - b\cos bx).$$

$$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx).$$

Thus $\int e^{ax}$ **sin bx dx** $= \frac{e^{ax}}{a^2 + b^2}$ **(a sin bx − b cos bx).**

Similarly $\int e^{ax}$ **cos bx dx** $= \frac{e^{ax}}{a^2 + b^2}$ **(a cos bx + b sin bx).**

Alternative forms of $\int e^{ax}$ **sin bx dx and** $\int e^{ax}$ **cos bx dx.**

We have $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx).$

Put $a = r\cos\theta$ and $b = r\sin\theta$. Then

$$r = \sqrt{(a^2 + b^2)} \quad \text{and} \quad \theta = \tan^{-1}(b/a).$$

Now we have

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{r^2}(r\cos\theta \sin bx - r\sin\theta\cos bx)$$

$$= \frac{e^{ax}}{r}\sin(bx-\theta).$$

Thus $\int e^{ax}\sin bx\, dx = \dfrac{e^{ax}}{\sqrt{(a^2+b^2)}}\sin\left(bx - \tan^{-1}\dfrac{b}{a}\right).$

Similarly $\int e^{ax}\cos bx\, dx = \dfrac{e^{ax}}{\sqrt{(a^2+b^2)}}\cos\left(bx - \tan^{-1}\dfrac{b}{a}\right).$

Solved Examples

Ex.1. *Integrate* (i) $x \log x$, (ii) $(\log x)/x^2$, (iii) $x^n \log x$.

Sol. (i) Here x should be taken as the second function because the integral of $\log x$ cannot be easily written down.

We have $\int x \log x\, dx = \int (\log x)\, x\, dx$

$= (\log x)\cdot \frac{1}{2}x^2 - \int (1/x)\cdot\frac{1}{2}x^2\, dx,$ (Integrating by parts)

$= \frac{1}{2}x^2\log x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2\log x - \frac{1}{2}\cdot(x^2/2)$

$= \frac{1}{4}x^2\log x^2 - \frac{1}{4}x^2\log e,$ $[\because \log e = 1]$ **Note**

$= \frac{1}{4}x^2\log(x^2/e).$

(ii) We have $\int [(\log x)/x^2]\, dx = \int (\log x)(1/x^2)\, dx$

$= (\log x)(-1/x) - \int (1/x).(-1/x)\, dx,$

[Integrating by parts taking $1/x^2$ as the second function]

$= -(1/x)\log x - (1/x) = -(1/x)(\log x + \log e)$ **Note**

$= -(1/x)\log(x\,e).$

(iii) We have $\int x^n \log x\, dx = \int (\log x).x^n\, dx$

$$= (\log x)\cdot\frac{x^{n+1}}{n+1} - \int \frac{1}{x}\cdot\frac{x^{n+1}}{n+1}\, dx,$$

[Integrating by parts taking x^n as the second function]

$$= (\log x)\cdot\frac{x^{n+1}}{n+1} - \int\frac{x^n}{n+1}\, dx$$

$$= (\log x)\cdot\frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}.$$

Ex. 2. *Integrate* (i) $\tan^{-1}x$, (ii) $\cot^{-1}x$, (iii) $\sin^{-1}x$.

Sol. (i) As there is only one function here, unity should be taken as the 2nd function. We have $\int \tan^{-1}x\, dx = \int (\tan^{-1}x).1\, dx$.

Integrating by parts regarding 1 as the second function, we have

$\int (\tan^{-1}x).1\, dx = (\tan^{-1}x).x - \int \{1/(1+x^2)\}\, x\, dx$

$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}\, dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2),$$

($\because$ Nr. $2x$ is the diff. coeffi. of the Dr. $1+x^2$)

(ii) We have $\int \cot^{-1} x\,dx = \int (\cot^{-1} x).1\,dx$

$= (\cot^{-1} x).x - \int \{-1/(1+x^2)\}.x\,dx,$

[Integrating by parts taking unity as the second function]

$= x\cot^{-1} x + \frac{1}{2}\int \{2x/(1+x^2)\}\,dx = x\cot^{-1} x + \frac{1}{2}\log(1+x^2).$

(iii) We have $\int \sin^{-1} x\,dx = \int (\sin^{-1} x).1\,dx$

$= (\sin^{-1} x).x - \int \{1/\sqrt{(1-x^2)}\}\,x\,dx$

$= x\sin^{-1} x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$ Note

$= x\sin^{-1} x + \frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2},$

[By power formula; see § 8 on page 12]

$= x\sin^{-1} x + (1-x^2)^{1/2}.$

Ex. 3. *Integrate* (i) $x^2 e^{mx}$, (ii) $x^2 \sin x$, (iii) $x^2 \cos 2x$.

Sol. (i) Integrating by parts taking e^{mx} as the second function, we have $\int x^2 e^{mx}\,dx = x^2\cdot\frac{e^{mx}}{m} - \int 2x\cdot\frac{e^{mx}}{m}\,dx$

$= \frac{x^2}{m}e^{mx} - \frac{2}{m}\int x.e^{mx}\,dx = \frac{x^2}{m}e^{mx} - \frac{2}{m}\left\{x\cdot\frac{e^{mx}}{m} - \int 1.\frac{e^{mx}}{m}\,dx\right\},$

[Again integrating by parts]

$= \frac{x^2}{m}e^{mx} - \frac{2}{m^2}x\,e^{mx} + \frac{2}{m^3}e^{mx}$

$= e^{mx}m^{-3}(m^2x^2 - 2mx + 2).$

(ii) Integrating by parts taking $\sin x$ as second function, we have

$\int x^2\sin x\,dx = -x^2\cos x - \int 2x(-\cos x)\,dx$

$= -x^2\cos x + 2\int x\cos x\,dx$

$= -x^2\cos x + 2x\sin x - 2\int \sin x\,dx$

$= -x^2\cos x + 2x\sin x + 2\cos x.$

Similarly $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x.$

(iii) Integrating by parts taking $\cos 2x$ as second function, we have

$\int x^2\cos 2x\,dx = x^2.(\frac{1}{2}\sin 2x) - \int 2x.\frac{1}{2}\sin 2x\,dx$

$= \frac{1}{2}x^2\sin 2x - \int x\sin 2x\,dx$

$= \frac{1}{2}x^2\sin 2x - \{x.\frac{1}{2}(-\cos 2x) + \frac{1}{2}\int \cos 2x\,dx\},$

(Again integrating by parts taking $\sin 2x$ as the 2nd function)

$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{2}\int \cos 2x\,dx$

$= \frac{1}{2}x^2\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x$

$= \frac{1}{2}(x^2 - \frac{1}{2})\sin 2x + \frac{1}{2}x\cos 2x.$

Ex. 4. *Integrate* (i) $\log x$ (M.U. 81)

$= \frac{1}{2}(x^2 - \frac{1}{2}) \sin 2x + \frac{1}{2} x \cos 2x.$

Ex. 4. *Integrate* **(i)** $\log x$, **(M.U. 81)**
(ii) $(\log x)^2$, **(iii)** $x^n (\log x)^2$.

Sol. (i) As there is only one function here, unity should be taken as the second function. We have $\int \log x \, dx = \int (\log x) . 1 \, dx$
$= (\log x) . x - \int (1/x) . x \, dx = x \log x - \int 1 . dx = x \log x - x$
$= x(\log x - 1) = x \log (x/e).$

(ii) We have $\int (\log x)^2 dx = \int (\log x)^2 . 1 \, dx$
$= (\log x)^2 . x - \int \{(2 \log x) . (1/x)\} . x \, dx,$
[Integrating by parts taking 1 as the second function]
$= x (\log x)^2 - 2 \int (\log x) . 1 \, dx$

$$= x (\log x)^2 - 2 \left[(\log x) . x - \int \frac{1}{x} \cdot x \, dx \right],$$

[Again integrating by parts taking 1 as the 2nd function]
$= x (\log x)^2 - 2x \log x + 2 \int 1 . dx$
$= x (\log x)^2 - 2x \log x + 2x.$

(iii) We have $\int x^n (\log x)^2 dx = \int (\log x)^2 . x^n dx$

$$= (\log x)^2 \frac{x^{n+1}}{n+1} - \int \left\{ (2 \log x) . \frac{1}{x} \right\} \frac{x^{n+1}}{n+1} dx,$$

[Integrating by parts taking x^n as the second function]

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{n+1} \int (\log x) . x^n dx$$

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{(n+1)} \left[(\log x) . \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} \cdot dx \right],$$

[Again integrating by parts taking x^n as the second function]

$$= \frac{1}{(n+1)} x^{n+1} . (\log x)^2 - \frac{2}{(n+1)^2} (\log x) . x^{n+1} + \frac{2}{(n+1)^2} \int x^n dx$$

$$= \frac{1}{(n+1)} x^{n+1} (\log x)^2 - \frac{2}{(n+1)^2} (\log x) . x^{n+1} + \frac{2}{(n+1)^3} x^{n+1}$$

$$= x^{n+1} \left[\frac{(\log x)^2}{(n+1)} - \frac{2 \log x}{(n+1)^2} + \frac{2}{(n+1)^3} \right].$$

Ex. 5. *Integrate* **(i)** $e^x \sin x$ **(ii)** $e^x \cos x$,
(iii) $e^{2x} \sin x$, **(iv)** $e^{3x} \cos 4x$.

Sol. **(i)** Integrating by parts taking $\sin x$ as the second function, we have

$$\int e^x \sin x\, dx = -e^x \cos x + \int e^x \cos x\, dx.$$

Now taking $\cos x$ as the second function we again apply integration by parts to the integral on the right hand side. Thus, we get

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx.$$

Transposing the last term on the right hand side to the left and dividing by 2, we get

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x).$$

(ii) Here $\int e^x \cos x\, dx = e^x . \sin x - \int e^x \sin x\, dx,$

[Integrating by parts taking $\cos x$ as the second function]

$$= e^x \sin x - [e^x . (-\cos x) - \int e^x (-\cos x)\, dx],$$

[Again integrating by parts taking $\sin x$ as the second function]

$$= e^x (\sin x + \cos x) - \int e^x \cos x\, dx.$$

Transposing the last term on the right hand side to the left and dividing by 2, we get

$$\int e^x \cos x\, dx = \tfrac{1}{2} e^x (\sin x + \cos x).$$

(iii) We have

$$\int e^{2x} \sin x\, dx = e^{2x} . (-\cos x) - \int 2e^{2x} . (-\cos x)\, dx,$$

[Integrating by parts taking $\sin x$ as the second function]

$$= -e^{2x} \cos x + 2 \int e^{2x} \cos x\, dx$$

$$= -e^{2x} \cos x + 2 [e^{2x} . \sin x - \int 2e^{2x} \sin x\, dx]$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x\, dx.$$

Transposing and dividing by 5, we get

$$\int e^{2x} \sin x\, dx = \tfrac{1}{5} e^{2x} [2 \sin x - \cos x].$$

(iv) Integrating by parts taking $\cos 4x$ as the second function, we have

$$\int e^{2x} \cos 4x\, dx + e^{3x} . \left(\frac{\sin 4x}{4}\right) - \int (3e^{3x}) . \left(\frac{\sin 4x}{4}\right) dx$$

$$= \tfrac{1}{4} e^{3x} \sin 4x - \tfrac{3}{4} \int e^{3x} \sin 4x\, dx$$

$$= \frac{1}{4} e^{3x} \sin 4x - \frac{3}{4} \left[e^{3x} . \left(-\frac{\cos 4x}{4}\right) - \int 3e^{3x} . \left(-\frac{\cos 4x}{4}\right) dx \right]$$

$$= \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x - \tfrac{9}{16} \int e^{3x} \cos 4x\, dx.$$

Transposing the last term on the right hand side to the left, we have

$$(1 + \tfrac{9}{16}) \int e^{3x} \cos 4x\, dx = \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x$$

or $$\tfrac{25}{16} \int e^{3x} \cos 4x\, dx = \tfrac{1}{4} e^{3x} \sin 4x + \tfrac{3}{16} e^{3x} \cos 4x$$

or $$\int e^{3x} \cos 4x\, dx = \tfrac{4}{25} e^{3x} \sin 4x + \tfrac{3}{25} e^{3x} \cos 4x$$

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$= \frac{1}{25} e^{3x} (4 \sin 4x + 3 \cos 4x)$.

Alternative solution. $\int e^{3x} \cos 4x \, dx$ is of the form

$\int e^{ax} \cos bx \, dx$, where $a = 3$ and $b = 4$.

Now $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{\sqrt{(a^2 + b^2)}} \cos \left(bx - \tan^{-1} \dfrac{b}{a} \right)$.

[See § 12, page 24]

$\therefore \quad \int e^{3x} \cos 4x \, dx = \dfrac{e^{3x}}{\sqrt{(9 + 16)}} \cos \left(4x - \tan^{-1} \dfrac{4}{3} \right)$

$= \dfrac{e^{3x}}{5} \cos (4x - \tan^{-1} \frac{4}{3})$.

Ex. 6. *Evaluate* $\int e^x (n \cos x + \sin x) \, dx$. **(Meerut 1977, 86)**

Sol. $\int e^x (n \cos x + \sin x) \, dx = n \int e^x \cos x \, dx + \int e^x \sin x \, dx$.

Now using results of Ex. 5 (i), and (ii), we get the required value of the given integral

$= \frac{1}{2} n e^x (\cos x + \sin x) + \frac{1}{2} e^x (\sin x - \cos x)$.

Ex. 7 (a). *Evaluate* $\displaystyle\int \frac{x e^x}{(x + 1)^2} \, dx$.

(Kashmir 1983; Delhi 80; Meerut 82, 84 P, 90)

Sol. We have $\displaystyle\int \frac{x e^x}{(x + 1)^2} \, dx = \int (x e^x) \frac{1}{(x + 1)^2} \, dx$.

Integrating by parts taking $\dfrac{1}{(x + 1)^2}$ as the second function and $x e^x$ as the first function, we have

$$\int \frac{x e^x}{(x + 1)^2} \, dx = (x e^x) \left(-\frac{1}{x + 1} \right) - \int (e^x + x e^x) \left(-\frac{1}{x + 1} \right) dx,$$

$\left[\text{Note that the integral of } \dfrac{1}{(x + 1)^2} \text{ is } -\dfrac{1}{x + 1} \right]$

$$= -\frac{x e^x}{x + 1} + \int e^x (x + 1) \cdot \frac{1}{x + 1} \, dx$$

$$= -\frac{x e^x}{x + 1} + \int e^x \, dx = -\frac{x e^x}{x + 1} + e^x$$

$$= e^x \left[1 - \frac{x}{x + 1} \right] = e^x \frac{x + 1 - x}{x + 1} = \frac{e^x}{x + 1}.$$

Alternative solution.

We have $\displaystyle\int \frac{x e^x}{(x + 1)^2} \, dx = \int e^x \frac{(x + 1) - 1}{(x + 1)^2} \, dx$

$$= \int e^x \left[\frac{1}{x + 1} - \frac{1}{(x + 1)^2} \right] dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where } f(x) = \frac{1}{x+1}$$
$$= \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$
$$= e^x f(x) - \int e^x f'(x)\, dx + \int e^x f'(x)\, dx,$$
applying integration by parts to the first integral taking e^x as the second function

$$= e^x f(x) = e^x \frac{1}{x+1}.$$

Ex. 7 (b). *Evaluate* $\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx.$

Sol. We have $\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx$

$$= \int e^x \frac{(x^2-1)+2}{(x+1)^2}\, dx = \int e^x \frac{(x+1)(x-1)+2}{(x+1)^2}\, dx$$
$$= \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right] dx$$
$$= \int e^x \left[\frac{(x+1)-2}{x+1} + \frac{2}{(x+1)^2}\right] dx$$
$$= \int e^x \left[\left\{1 - \frac{2}{x+1}\right\} + \frac{2}{(x+1)^2}\right] dx$$
$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where } f(x) = 1 - \frac{2}{x+1} \text{ so that } f'(x) = \frac{2}{(x+1)^2}$$
$$= e^x f(x) = e^x \left[1 - \frac{2}{x+1}\right] = e^x \frac{x-1}{x+1}.$$

Ex. 7 (c). *Evaluate* $\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx.$

Sol. We have $\int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx = \int e^x \frac{(1+x^2)-2x}{(1+x^2)^2}\, dx$

$$= \int e^x \left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right] dx$$
$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where } f(x) = \frac{1}{1+x^2} \text{ and } f'(x) = \frac{-2x}{(1+x^2)^2}$$
$$= e^x f(x) = e^x \cdot \frac{1}{1+x^2}.$$

Ex. 7 (d). *Evaluate* $\int e^x \frac{x^2+3x+3}{(x+2)^2} dx$.

Sol. We have $\int e^x \frac{x^2+3x+3}{(x+2)^2} dx$

$$= \int e^x \frac{(x+2)(x+1)+1}{(x+2)^2} dx$$

$$= \int e^x \left[\frac{x+1}{x+2} + \frac{1}{(x+2)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)] dx$, where

$f(x) = \frac{x+1}{x+2} = \frac{(x+2)-1}{x+2} = 1 - \frac{1}{x+2}$ so that $f'(x) = \frac{1}{(x+2)^2}$

$$= e^x \frac{x+1}{x+2}.$$

Ex. 7 (e). *Evaluate* $\int e^x \frac{x^3-x+2}{(x^2+1)^2} dx$.

Sol. We have $\int e^x \frac{x^3-x+2}{(x^2+1)^2} dx$

$$= \int e^x \frac{(x^2+1)(x+1)-x^2-2x+1}{(x^2+1)^2} dx$$

$$= \int e^x \left[\frac{x+1}{x^2+1} + \frac{1-x^2-2x}{(x^2+1)^2}\right] dx$$

$= \int e^x [f(x) + f'(x)] dx$, where $f(x) = \frac{x+1}{x^2+1}$ so that

$$f'(x) = \frac{1.(x^2+1) - 2x(x+1)}{(x^2+1)^2} = \frac{1-x^2-2x}{(x^2+1)^2}$$

$$= e^x f(x) = e^x \frac{x+1}{x^2+1}.$$

Ex. 7 (f). *Evaluate* $\int \frac{\log x}{(1+\log x)^2} dx$.

Sol. Put $\log x = t$. $\therefore x = e^t$ and $dx = e^t dt$.

Then the given integral

$$I = \int \frac{t e^t}{(1+t)^2} = \int e^t \frac{(1+t)-1}{(1+t)^2} dt$$

$$= \int e^t \left[\frac{1}{1+t} - \frac{1}{(1+t)^2}\right] dt$$

$= \int e^t [f(t) + f'(t)] dt$, where $f(t) = \frac{1}{1+t}$

$$= e^t f(t) = e^t \frac{1}{1+t} = \frac{x}{1+\log x}.$$

Ex. 8 (a). *Evaluate* $\int \frac{e^x (1+\sin x)}{1+\cos x} dx.$

Sol. The given integral $I = \int \frac{e^x\, dx}{1+\cos x} + \int \frac{e^x \sin x\, dx}{(1+\cos x)}$

$$= \int \frac{e^x\, dx}{2\cos^2 (x/2)} + \int \frac{e^x\, 2\sin (x/2) \cos (x/2)}{2\cos^2 (x/2)} dx$$

$$= \tfrac{1}{2} \int e^x \sec^2 (x/2)\, dx + \int e^x \tan (x/2)\, dx$$

$$= \int e^x \left[\tan \frac{x}{2} + \frac{1}{2} \sec^2 \frac{x}{2}\right] dx.$$

Since $\frac{d}{dx} (\tan \frac{1}{2} x) = \frac{1}{2} \sec^2 \frac{x}{2}$, therefore this *integral is of the type* $\int e^x [f(x) + f'(x)]\, dx$.

To evaluate this integral, integrating $e^x \tan (x/2)$ by parts regarding e^x as the second function, we get

$$I = e^x \tan \frac{x}{2} - \int \frac{1}{2} e^x. \sec^2 \frac{x}{2} dx + \int \frac{1}{2} e^x \sec^2 \frac{x}{2} dx = e^x \tan \frac{x}{2},$$

because the last two integrals cancel each other.

Ex. 8 (b). *Evaluate* $\int e^x \frac{1-\sin x}{1-\cos x} dx.$ **(Meerut 1980, 83, 87)**

Sol. We have

$$\int e^x \frac{1-\sin x}{1-\cos x} dx = \int e^x \left[\frac{1}{1-\cos x} - \frac{\sin x}{1-\cos x}\right] dx$$

$$= \int e^x \left[\frac{1}{2\sin^2 \frac{1}{2} x} - \frac{2\sin \frac{1}{2} x \cos \frac{1}{2} x}{2\sin^2 \frac{1}{2} x}\right] dx$$

$$= \int e^x [\tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2} x - \cot \tfrac{1}{2} x]\, dx$$

$$= \int e^x [(-\cot \tfrac{1}{2} x) + \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2} x]\, dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where}$$

$$f(x) = -\cot \tfrac{1}{2} x \text{ so that } f'(x) = \tfrac{1}{2} \operatorname{cosec}^2 \tfrac{1}{2} x$$

$$= e^x f(x) = e^x (-\cot \tfrac{1}{2} x) = -e^x \cot \tfrac{1}{2} x.$$

Ex. 8 (c). *Evaluate* $\int e^x \frac{2+\sin 2x}{1+\cos 2x} dx.$

Sol. We have $\int e^x \frac{2+\sin 2x}{1+\cos 2x} dx$

$$= \int e^x \left[\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right] dx$$

$$= \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right] dx$$

$$= \int e^x [\sec^2 x + \tan x]\, dx$$

$$= \int e^x [\tan x + \sec^2 x]\, dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where } f(x) = \tan x$$

$$= e^x f(x) = e^x \tan x.$$

Ex. 8 (d). *Evaluate* $\int e^x (\cot x + \log \sin x)\, dx$.

Sol. We have $\int e^x (\cot x + \log \sin x)\, dx$

$$= \int e^x (\log \sin x + \cot x)\, dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where}$$

$$f(x) = \log \sin x \text{ so that } f'(x) = (1/\sin x)\cos x = \cot x$$

$$= e^x f(x) = e^x \log \sin x.$$

Ex. 8 (e). *Evaluate* $\int e^x [\log(\sec x + \tan x) + \sec x]\, dx$.

Sol. The given integral

$$I = \int e^x \sec x\, dx + \int e^x \log(\sec x + \tan x)\, dx$$

$$= e^x \log(\sec x + \tan x) - \int e^x \log(\sec x + \tan x)\, dx$$

$$+ \int e^x \log(\sec x + \tan x)\, dx,$$

(applying integration by parts to the first integral taking e^x as the second function and keeping the second integral as it is)

$$= e^x \log(\sec x + \tan x).$$

Ex. 8 (f). *Evaluate* $\int e^x \left[\frac{1 + \sqrt{(1-x^2)}\sin^{-1} x}{\sqrt{(1-x^2)}}\right] dx$.

Sol. The given integral

$$I = \int e^x \left[\frac{1}{\sqrt{(1-x^2)}} + \sin^{-1} x\right] dx$$

$$= \int e^x \left[\sin^{-1} x + \frac{1}{\sqrt{(1-x^2)}}\right] dx$$

$$= \int e^x [f(x) + f'(x)]\, dx, \text{ where}$$

$$f(x) = \sin^{-1} x \text{ so that } f'(x) = \frac{1}{\sqrt{(1-x^2)}}$$

$$= e^x f(x) = e^x \cdot \frac{1}{\sqrt{(1-x^2)}}.$$

Ex. 9. *Evaluate* $\int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$.

Sol. Put $\sin^{-1} x = t$ (or $x = \sin t$); $\therefore \frac{1}{\sqrt{(1-x^2)}} dx = dt$.

Sol. Put $\sin^{-1} x = t$ (or $x = \sin t$); $\therefore \frac{1}{\sqrt{(1-x^2)}} dx = dt$.

$$\therefore \int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} dx = \int (\sin t) \cdot t\, dt = \int t \sin t\, dt$$
$$= t \cdot (-\cos t) - \int 1 \cdot (-\cos t) \cdot dt = -t \cos t + \sin t$$
$$= -t\sqrt{(1-\sin^2 t)} + \sin t$$
$$= -\sin^{-1} x \cdot \sqrt{(1-x^2)} + x.$$

Ex. 10. *Evaluate* $\int \frac{x^2 \tan^{-1} x}{1+x^2} dx$. **(Meerut 1990 S)**

Sol. Put $\tan^{-1} x = t$ (or $x = \tan t$); $\therefore \frac{1}{1+x^2} dx = dt$.

$$\therefore \int \frac{x^2 \tan^{-1} x}{1+x^2} dx = \int t \tan^2 t\, dt = \int t (\sec^2 t - 1)\, dt \quad \text{[Note]}$$
$$= \int t \sec^2 t\, dt - \int t\, dt$$
$$= t \tan t - \int \tan t\, dt - \tfrac{1}{2} t^2$$
$$= t \tan t - \log \sec t - \tfrac{1}{2} t^2 = t \tan t - \log \sqrt{(1+\tan^2 t)} - \tfrac{1}{2} t^2$$
$$= x \tan^{-1} x - \log \sqrt{(1+x^2)} - \tfrac{1}{2} (\tan^{-1} x)^2, \quad [\because x = \tan t].$$

Ex. 11. *Evaluate* $\int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}}$.

Sol. Put $\sin^{-1} x = t$, *i.e.*, $x = \sin t$ so that $dx = \cos t\, dt$.

$$\therefore \int \frac{\sin^{-1} x\, dx}{(1-x^2)^{3/2}} = \int \frac{t}{\cos^3 t} \cdot \cos t\, dt = \int t \cdot \sec^2 t\, dt$$
$$= t \tan t - \int 1 \cdot \tan t\, dt$$
$$= t \tan t + \log \cos t, \quad \left[\because \int \tan t\, dt = -\log \cos t\right]$$
$$= t \frac{\sin t}{\cos t} + \log \cos t = t \frac{\sin t}{\sqrt{(1-\sin^2 t)}} + \log \{\sqrt{(1-\sin^2 t)}\}$$
$$= (\sin^{-1} x) \cdot \frac{x}{\sqrt{(1-x^2)}} + \log \sqrt{(1-x^2)}, \quad [\because x = \sin t]$$
$$= \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} + \tfrac{1}{2} \log (1-x^2).$$

Ex. 12. *Find* $\int \frac{2x \sin^{-1} (x^2)}{\sqrt{(1-x^4)}} dx$.

Sol. Put $\sin^{-1} x^2 = t$, so that $\frac{2x\, dx}{\sqrt{(1-x^4)}} = dt$.

$\therefore$ the given integral $= \int t\, dt = \frac{1}{2} t^2 = \frac{1}{2} (\sin^{-1} x^2)^2$.

Ex. 13. *Evaluate* $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx$.

Sol. Put $\tan^{-1} x = t$, so that $x = \tan t$. Also $dx = \sec^2 t \, dt$.

$\therefore$ the given integral $= \int \frac{(\tan t)\, t \sec^2 t \, dt}{(1 + \tan^2 t)^{3/2}} = \int \frac{t \tan t \sec^2 t \, dt}{\sec^3 t}$

$$= \int \frac{t \tan t}{\sec t} dt = \int t \sin t \, dt.$$

Now integrating by parts regarding $\sin t$ as the second function, the given integral $= - t \cos t + \int \cos t \, dt = - t \cos t + \sin t$

$$= - \frac{\tan^{-1} x}{\sqrt{(1 + x^2)}} + \frac{x}{\sqrt{(1 + x^2)}} = \frac{x - \tan^{-1} x}{\sqrt{(1 + x^2)}}.$$

Ex. 14. *Evaluate* $\int x^3 e^{x^2} dx$.

Sol. The given integral $I = \int x^2 . e^{x^2} . x \, dx$.

Put $x^2 = t$ so that $2x \, dx = dt$ or $x \, dx = \frac{1}{2} dt$.

$\therefore$ the given integral $I = \frac{1}{2} \int t . e^t \, dt$.

Now integrating by parts regarding e^t as the 2nd function, we have

$$I = \tfrac{1}{2} t . e^t - \tfrac{1}{2} \int e^t \, dt = \tfrac{1}{2} t . e^t - \tfrac{1}{2} e^t$$

$$= \tfrac{1}{2} e^t (t - 1) = \tfrac{1}{2} e^{x^2} (x^2 - 1).$$

Ex. 15. *Evaluate* $\int \frac{x^2 \, dx}{(x \sin x + \cos x)^2}$. **(Delhi 1979; Meerut 84 S)**

Sol. Let $I = \int \frac{x^2}{(x \sin x + \cos x)^2} dx$

$$= \int x^2 (x \sin x + \cos x)^{-2} dx.$$

Here $\frac{d}{dx} (x \sin x + \cos x) = \sin x + x \cos x - \sin x = x \cos x$.

So we adjust the given integral in the form

$$I = \int \frac{x^2}{x \cos x} \{(x \sin x + \cos x)^{-2} (x \cos x)\} \, dx$$

$$= \int \left(\frac{x}{\cos x} \right) \{(x \sin x + \cos x)^{-2} (x \cos x)\} \, dx.$$

Now by power formula, the integral of

$(x \sin x + \cos x)^{-2} (x \cos x)$ is $\{(x \sin x + \cos x)^{-1}\}/(-1)$

i.e., $- 1/(x \sin x + \cos x)$. So applying to I integration by parts taking $(x \sin x + \cos x)^{-2} (x \cos x)$ as the second function, we get

$$I = \left(\frac{x}{\cos x} \right) \left(- \frac{1}{x \sin x + \cos x} \right)$$

$$- \int \left[\left\{ \frac{d}{dx} \left(\frac{x}{\cos x} \right) \right\} \cdot \left\{ - \frac{1}{x \sin x + \cos x} \right\} \right] dx$$

$$= -\frac{x}{\cos x\,(x \sin x + \cos x)} + \int \left(\frac{\cos x + x \sin x}{\cos^2 x}\right) \cdot \left(\frac{1}{x \sin x + \cos x}\right) dx$$

$$= -\frac{x}{\cos x\,(x \sin x + \cos x)} + \int \sec^2 x \, dx$$

$$= -\frac{x}{\cos x\,(x \sin x + \cos x)} + \tan x = \frac{\sin x}{\cos x} - \frac{x}{\cos x\,(x \sin x + \cos x)}$$

$$= \frac{\sin x\,(x \sin x + \cos x) - x}{(x \sin x + \cos x) \cos x} = \frac{\sin x \cos x - x\,(1 - \sin^2 x)}{(x \sin x + \cos x) \cos x}$$

$$= \frac{\sin x \cos x - x \cos^2 x}{(x \sin x + \cos x) \cos x} = \frac{\cos x\,(\sin x - x \cos x)}{\cos x\,(x \sin x + \cos x)}$$

$$= \frac{\sin x - x \cos x}{x \sin x + \cos x}.$$

Ex. 16. *Evaluate* $\int \sin^{-1} [2x/(1 + x^2)]\, dx$.

Sol. Put $x = \tan \theta$, so that $dx = \sec^2 \theta \, d\theta$.

Then the given integral

$$= \int \sin^{-1} \left(\frac{2x}{1 + x^2}\right) dx = \int \left[\sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right)\right] \sec^2 \theta \, d\theta$$

$$= \int \sin^{-1} (\sin 2\theta) \sec^2 \theta \, d\theta = \int 2\theta \sec^2 \theta \, d\theta = 2 \int \theta \sec^2 \theta \, d\theta$$

$$= 2\,[\theta \tan \theta - \int 1 . \tan \theta \, d\theta] = 2\,[\theta \tan \theta - \log \sec \theta]$$

$$= 2\,[(\tan^{-1} x) . x - \log \sqrt{(1 + x^2)}].$$

Ex. 17. *Evaluate* (i) $\int \cos^{-1} \frac{1 - x^2}{1 + x^2} dx$,

(ii) $\int \tan^{-1} \frac{2x}{1 - x^2} dx$.

Sol. Put $x = \tan \theta$, so that $dx = \sec^2 \theta \, d\theta$.

Then each of the two given integrals becomes $= \int \theta \sec^2 \theta \, d\theta$

$$= 2\,[x \tan^{-1} x - \log \sqrt{(1 + x^2)}]. \qquad \text{[Proceed as in Ex. 16]}$$

Ex. 18. *Evaluate* $\int \sin^{-1} \sqrt{\left(\frac{x}{a + x}\right)}\, dx$.

(Meerut 1985, 90 S; Lucknow 81)

Sol. Put $x = a \tan^2 \theta$, so that $dx = 2a \tan \theta \sec^2 \theta \, d\theta$.

Then the given integral

$$= \int \left\{\sin^{-1} \sqrt{\left(\frac{a \tan^2 \theta}{a \sec^2 \theta}\right)}\right\} 2a \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a \int \{\sin^{-1} (\sin \theta)\} . \tan \theta \sec^2 \theta \, d\theta = 2a \int \theta \tan \theta \sec^2 \theta \, d\theta$$

$$= 2a\left[\theta \cdot \frac{\tan^2\theta}{2} - \int 1 \cdot \frac{\tan^2\theta}{2}\right],$$ [Integrating by parts taking $\tan\theta\sec^2\theta$ as the second function. Note that by power formula the integral of $\tan\theta\sec^2\theta$ is $\frac{1}{2}\tan^2\theta$]

$$= a\left[\theta\tan^2\theta - \int(\sec^2\theta - 1)\,d\theta\right] = a\left[\theta\tan^2\theta - \tan\theta + \theta\right]$$

$$= a\left[\left\{\tan^{-1}\sqrt{\left(\frac{x}{a}\right)}\right\}\cdot\frac{x}{a} - \sqrt{\left(\frac{x}{a}\right)} + \tan^{-1}\sqrt{\left(\frac{x}{a}\right)}\right],$$

[$\because \tan^2\theta = x/a$]

$$= \{\tan^{-1}\sqrt{(x/a)}\}\{a(x/a) + a\} - a\sqrt{(x/a)}$$

$$= (a + x)\tan^{-1}\sqrt{(x/a)} - \sqrt{(ax)}.$$

Ex. 19. *Evaluate* $\displaystyle\int \frac{\log(\sec^{-1}x)\,dx}{x\sqrt{(x^2-1)}}$.

Sol. Put $\sec^{-1}x = t$ so that $\dfrac{1}{x\sqrt{(x^2-1)}}dx = dt$.

Then the given integral

$$= \int \log t\,dt = \int(\log t).1\,dt = (\log t).t - \int \frac{1}{t}t\,dt$$

$$= t\log t - \int dt = t\log t - t = t(\log t - 1) = t.(\log t - \log e)$$

$$= t\log\left(\frac{t}{e}\right) = (\sec^{-1}x)\log\left(\frac{\sec^{-1}x}{e}\right).$$

Ex. 20. *Evaluate* $\displaystyle\int \frac{e^{\sec^{-1}x}\,dx}{x\sqrt{(x^2-1)}}$.

Sol. Put $\sec^{-1}x = t$, so that $\dfrac{1}{x\sqrt{(x^2-1)}}dx = dt$.

Then the given integral $= \int e^t\,dt = e^t = e^{\sec^{-1}x}$.

Ex. 21. *Evaluate* $\displaystyle\int \frac{\tan x}{\log\sec x}dx$.

Sol. Put $\log\sec x = t$, so that $\tan x\,dx = dt$.

$\left[\text{Note that } \dfrac{d}{dx}(\log\sec x) = \tan x\right]$

Then the given integral $= \displaystyle\int \frac{1}{t}dt = \log t = \log(\log\sec x)$.

Ex. 22. *Evaluate* $\displaystyle\int \frac{\log(\log x)}{x}dx$.

Sol. Put $\log x = t$, so that $(1/x)\,dx = dt$.

Then the given integral $= \int \log t\,dt = \int(\log t).1\,dt$

$$= (\log t).t - \int\frac{1}{t}\cdot t\,dt = t\log\left(\frac{t}{e}\right) = (\log x).\log\left(\frac{\log x}{e}\right).$$

$$= (\log t) . t - \int \frac{1}{t} \cdot t \, dt = t \log\left(\frac{t}{e}\right) = (\log x) . \log\left(\frac{\log x}{e}\right).$$

Ex. 23. *Evaluate* $\int \frac{\tan x}{\sqrt{(a + b \tan^2 x)}} dx, (b > 0)$.

Sol. The given integral $I = \int \frac{\sin x \, dx}{\sqrt{(a \cos^2 x + b \sin^2 x)}}$

$$= \int \frac{\sin x \, dx}{\sqrt{\{b - (b - a) \cos^2 x\}}}.$$

Now put $\sqrt{(b - a)} \cos x = t$ so that $-\sqrt{(b - a)} \sin x \, dx = dt$.

Then $I = -\frac{1}{\sqrt{(b - a)}} \int \frac{dt}{\sqrt{(b - t^2)}} = \frac{1}{\sqrt{(b - a)}} \cos^{-1}\left(\frac{t}{\sqrt{b}}\right)$

$$= \frac{1}{\sqrt{(b - a)}} \cos^{-1}\left\{\frac{\sqrt{(b - a)} \cos x}{\sqrt{b}}\right\}$$

$$= \frac{1}{\sqrt{(b - a)}} \cos^{-1}\left\{\sqrt{\left(\frac{b - a}{b}\right)} \cos x\right\}.$$

Ex. 24. *Evaluate* $\int \cosh 2x \sin 2x \, dx$.

Sol. Let $I = \int \cosh 2x \sin 2x \, dx$. Integrating by parts taking $\sin 2x$ as the second function, we get

$$I = (\cosh 2x)(-\tfrac{1}{2} \cos 2x) - \int (2 \sinh 2x)(-\tfrac{1}{2} \cos 2x) \, dx$$

$$= -\tfrac{1}{2} \cosh 2x \cos 2x + \int \sinh 2x \cos 2x \, dx.$$

Again integrating by parts taking $\cos 2x$ as the second function, we get

$$I = -\tfrac{1}{2} \cosh 2x \cos 2x + [(\sinh 2x)(\tfrac{1}{2} \sin 2x) - \int (2 \cosh 2x)(\tfrac{1}{2} \sin 2x) \, dx]$$

$$= -\tfrac{1}{2} \cosh 2x \cos 2x + \tfrac{1}{2} \sinh 2x \sin 2x - \int \cosh 2x \sin 2x \, dx$$

$$= -\tfrac{1}{2} \cosh 2x \cos 2x + \tfrac{1}{2} \sinh 2x \sin 2x - I.$$

$\therefore \quad 2I = -\tfrac{1}{2} \cosh 2x \cos 2x + \tfrac{1}{2} \sinh 2x \sin 2x$

or $\quad I = -\tfrac{1}{4} \cosh 2x \cos 2x + \tfrac{1}{4} \sinh 2x \sin 2x.$

Ex. 25. *Evaluate* $\int \sinh 2x \sin 2x \, dx$.

Sol. Let $I = \int \sinh 2x \sin 2x \, dx$. Integrating by parts taking $\sin 2x$ as the second function, we get

$$I = (\sinh 2x) . (-\tfrac{1}{2} \cos 2x) - \int (2 \cosh 2x)(-\tfrac{1}{2} \cos 2x) \, dx$$

$$= -\tfrac{1}{2} \sinh 2x \cos 2x + \int \cosh 2x \cos 2x \, dx.$$

Again integrating by parts taking $\cos 2x$ as the second function, we get

$$I = -\tfrac{1}{2} \sinh 2x \cos 2x + [(\cosh 2x)(\tfrac{1}{2} \sin 2x) - \int (2 \sinh 2x)(\tfrac{1}{2} \sin 2x) \, dx]$$

$$I = -\tfrac{1}{2}\sinh 2x \cos 2x + [(\cosh 2x)(\tfrac{1}{2}\sin 2x) - \int (2\sinh 2x)(\tfrac{1}{2}\sin 2x)\,dx]$$

$$= -\tfrac{1}{2}\sinh 2x\cos 2x + \tfrac{1}{2}\cosh 2x \sin 2x - \int \sinh 2x \sin 2x\,dx$$

$$= -\tfrac{1}{2}\sinh 2x\cos 2x + \tfrac{1}{2}\cosh 2x\sin 2x - I. \qquad \textbf{(Note)}$$

$\therefore$ $$2I = -\tfrac{1}{2}\sinh 2x\cos 2x + \tfrac{1}{2}\cosh 2x \sin 2x$$

or $$I = \tfrac{1}{4}\cosh 2x \sin 2x - \tfrac{1}{4}\sinh 2x \cos 2x.$$

Ex. 26. *Evaluate* $\int e^x \sin x \cos x \cos 2x\,dx$.

Sol. The given integral $= \frac{1}{2}\int e^x \sin 2x \cos 2x\,dx$

$$= \tfrac{1}{4}\int e^x \sin 4x\,dx = \frac{1}{4}\cdot\frac{e^x}{1^2+4^2}[1.\sin 4x - 4\cos 4x],$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{4\times 17}e^x[\sin 4x - 4\cos 4x].$$

Ex. 27. *If u and v are functions of x, prove that*

$$\int u\frac{d^2v}{dx^2}dx = u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2u}{dx^2}dx.$$

Sol. Integrating by parts regarding u as first function and (d^2v/dx^2) as second function, we have

$$\int u\frac{d^2v}{dx^2}dx = u\frac{dv}{dx} - \int\frac{du}{dx}\cdot\frac{dv}{dx}dx$$

$$= u\frac{dv}{dx} - \left[\frac{du}{dx}v - \int\frac{d^2u}{dx^2}\cdot v\,dx\right],$$

[Again integrating by parts regarding (dv/dx) as the second function]

$$= u\frac{dv}{dx} - v\frac{du}{dx} + \int v\frac{d^2u}{dx^2}dx.$$

Ex. 28. *Evaluate* $\int x\sin^{-1}\left\{\frac{1}{2}\sqrt{\left(\frac{2a-x}{a}\right)}\right\}dx$.

Sol. Put $\frac{1}{2}\sqrt{\left(\frac{2a-x}{a}\right)} = \sin\theta$ *i.e.*, $2a - x = 4a\sin^2\theta$ *i.e.*,

$x = 2a(1 - 2\sin^2\theta)$ *i.e.*, $x = 2a\cos 2\theta$, so that

$dx = -4a\sin 2\theta\,d\theta$.

$\therefore$ the given integral

$$= \int (2a\cos 2\theta)\sin^{-1}(\sin\theta)\times(-4a\sin 2\theta)\,d\theta$$

$$= -4a^2\int \theta\, 2\sin 2\theta\cos 2\theta\,d\theta = -4a^2\int\theta\sin 4\theta\,d\theta$$

$$= -4a^2[\theta.(-\tfrac{1}{4}\cos 4\theta) - \int 1.(-\tfrac{1}{4}\cos 4\theta)\,d\theta]$$

$$= a^2 [\theta \cos 4\theta - \int \cos 4\theta \, d\theta] = a^2 [\theta \cos 4\theta - \tfrac{1}{4} \sin 4\theta]$$
$$= a^2 [\theta (2 \cos^2 2\theta - 1) - \tfrac{1}{4} \cdot 2 \cos 2\theta \sin 2\theta]$$
$$= a^2 [\theta (2 \cos^2 2\theta - 1) - \tfrac{1}{2} \cos 2\theta \sqrt{(1 - \cos^2 2\theta)}]$$
$$= a^2 \left[\left(\tfrac{1}{2} \cos^{-1} \frac{x}{2a} \right) \left(2 \cdot \frac{x^2}{4a^2} - 1 \right) - \frac{1}{2} \cdot \frac{x}{2a} \sqrt{\left(1 - \frac{x^2}{4a^2}\right)} \right]$$
$$= a^2 \left[\tfrac{1}{2} \left(\cos^{-1} \frac{x}{2a} \right) \left(\frac{x^2 - 2a^2}{2a^2} \right) - \frac{x}{8a^2} \sqrt{(4a^2 - x^2)} \right]$$
$$= \tfrac{1}{8} \left[2 (x^2 - 2a^2) \cos^{-1} \left(\frac{x}{2a} \right) - x \sqrt{(4a^2 - x^2)} \right].$$

Ex. 29. *Evaluate* $\int \tan^{-1} \sqrt{\left(\frac{1 - x}{1 + x} \right)} \, dx$.

Sol. Put $x = \cos \theta$ so that $dx = - \sin \theta \, d\theta$.

$\therefore$ the given integral $= \int \left\{ \tan^{-1} \sqrt{\left(\frac{1 - \cos \theta}{1 + \cos \theta} \right)} \right\} (- \sin \theta) \, d\theta$

$$= - \int \{ \tan^{-1} (\tan \tfrac{1}{2} \theta) \} \sin \theta \, d\theta$$
$$= - \int \tfrac{1}{2} \theta \sin \theta \, d\theta = - \tfrac{1}{2} \int \theta \sin \theta \, d\theta$$
$$= - \tfrac{1}{2} [\theta . (- \cos \theta) - \int (- \cos \theta) \, d\theta] = \tfrac{1}{2} [\theta \cos \theta - \int \cos \theta \, d\theta]$$
$$= \tfrac{1}{2} [\theta \cos \theta - \sin \theta] = \tfrac{1}{2} [x \cos^{-1} x - \sqrt{(1 - x^2)}].$$

Ex. 30. *Evaluate* $\int \frac{x + \sin x}{1 + \cos x} \, dx$. **(Meerut 1982 P, 83, 87)**

Sol. We have

$$\int \frac{x + \sin x}{1 + \cos x} \, dx = \int \frac{x + 2 \sin (x/2) \cos (x/2)}{1 + 2 \cos^2 (x/2) - 1} \, dx$$
$$= \tfrac{1}{2} \int x \sec^2 (x/2) \, dx + \int \tan(x/2) \, dx$$
$$= \tfrac{1}{2} \left[x \frac{\tan (x/2)}{(1/2)} - \int 1 \cdot \frac{\tan (x/2)}{(1/2)} \, dx \right] + \int \tan (x/2) \, dx$$
$$= x \tan (x/2) - \int \tan (x/2) \, dx + \int \tan (x/2) \, dx = x \tan (x/2).$$

Ex. 31. *Evaluate :*

(i) $\int x^2 \tan^{-1} x \, dx$ **(ii)** $\int x^3 \tan^{-1} x \, dx$.

Sol. (i). We have $\int x^2 \tan^{-1} x \, dx$

$$= \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{3} \cdot \frac{1}{1 + x^2} \, dx,$$

integrating by parts taking x^2 as the second function

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x (x^2 + 1) - x}{1 + x^2} \, dx, \quad [\because x^3 = x (x^2 + 1) - x]$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int x \, dx + \frac{1}{3} \cdot \frac{1}{2} \int \frac{2x}{1 + x^2} \, dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}\,dx$$

$$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(1+x^2).$$

(ii) We have $\int x^3\tan^{-1}x\,dx$

$$=\frac{x^4}{4}\tan^{-1}x-\int\frac{x^4}{4}\cdot\frac{1}{1+x^2}\,dx,$$

integrating by parts taking x^3 as the second function

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int\left[x^2-1+\frac{1}{1+x^2}\right]dx,$$

dividing x^4 by x^2+1, we get x^2-1 as the quotient and 1 as the remainder

$$=\frac{1}{4}\left[x^4\tan^{-1}x-\frac{x^3}{3}+x-\tan^{-1}x\right]$$

$$=\frac{1}{4}\left[(x^4-1)\tan^{-1}x-\frac{x^3}{3}+x\right].$$

Ex. 32. *Evaluate :*

(i) $\int\frac{\log(1+x)}{\sqrt{(1+x)}}\,dx$ (ii) $\int\log(x+2)^{x+2}\,dx.$

Sol. (i) We have

$$\int\frac{\log(1+x)}{\sqrt{(1+x)}}\,dx=\int[\log(1+x)](1+x)^{-1/2}\,dx$$

$$=[\log(1+x)]\cdot\frac{(1+x)^{1/2}}{\frac{1}{2}}-\int\frac{1}{1+x}\cdot\frac{(1+x)^{1/2}}{\frac{1}{2}}\,dx,$$

integrating by parts taking $(1+x)^{-1/2}$ as the second function

$$=2\sqrt{(1+x)}\log(1+x)-2\int(1+x)^{-1/2}\,dx$$

$$=2\sqrt{(1+x)}\log(1+x)-2\cdot\frac{(1+x)^{1/2}}{\frac{1}{2}}$$

$$=2\sqrt{(1+x)}\,[\log(1+x)-2].$$

(ii) We have $\int\log(x+2)^{x+2}\,dx=\int(x+2)\log(x+2)\,dx$

$$=\frac{(x+2)^2}{2}\log(x+2)-\int\frac{(x+2)^2}{2}\cdot\frac{1}{x+2}\,dx,$$

integrating by parts taking $(x+2)^1$ as the second function

$$=\frac{(x+2)^2}{2}\log(x+2)-\frac{1}{2}\int(x+2)\,dx$$

$$=\frac{(x+2)^2}{2}\log(x+2)-\frac{1}{2}\frac{(x+2)^2}{\cdot 2}$$

$(x+2)^2$

$$= \frac{(x+2)^2}{4}[2\log(x+2) - 1].$$

Ex. 33. *Evaluate :*

(i) $\int x \sin^{-1} x \, dx$ (ii) $\int x^2 \sin^{-1} x \, dx$.

Sol. (i) We have $\int x \sin^{-1} x \, dx$

$$= \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{(1-x^2)}} dx,$$

integrating by parts taking x as the second function

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{(1-x^2)-1}{\sqrt{(1-x^2)}} dx$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \sqrt{(1-x^2)}\, dx - \frac{1}{2}\int \frac{dx}{\sqrt{(1-x^2)}}$$

$$= \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{x}{2}\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1} x\Big] - \frac{1}{2}\sin^{-1} x$$

$$= \tfrac{1}{2}x^2 \sin^{-1} x + \tfrac{1}{4}x\sqrt{(1-x^2)} - \tfrac{1}{4}\sin^{-1} x.$$

(ii) We have $\int x^2 \sin^{-1} x \, dx$

$$= \frac{x^3}{3}\sin^{-1} x - \int \frac{x^3}{3} \cdot \frac{1}{\sqrt{(1-x^2)}} dx,$$

integrating by parts taking x^2 as the second function

$$= \frac{x^3}{3}\sin^{-1} x + \frac{1}{3}\int \frac{x(1-x^2) - x}{\sqrt{(1-x^2)}} dx$$

$$= \frac{x^3}{3}\sin^{-1} x + \frac{1}{3}\int x\sqrt{(1-x^2)}\, dx - \frac{1}{3}\int \frac{x}{\sqrt{(1-x^2)}} dx$$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{6}\int (1-x^2)^{1/2}(-2x)\, dx$$

$$+ \frac{1}{6}\int (1-x^2)^{-1/2}(-2x)\, dx$$

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{6} \cdot \frac{(1-x^2)^{3/2}}{3/2} + \frac{1}{6} \cdot \frac{(1-x^2)^{1/2}}{1/2},$$

intregrating by using power formula

$$= \frac{x^3}{3}\sin^{-1} x - \frac{1}{9}(1-x^2)^{3/2} + \frac{1}{3}(1-x^2)^{1/2}$$

$$= \frac{x^3}{3}\sin^{-1} x + \frac{1}{9}(1-x^2)^{1/2}[3 - (1-x^2)]$$

$$= \frac{x^3}{3}\sin^{-1} x + \frac{2+x^2}{9}\sqrt{(1-x^2)}.$$

Ex. 34. *Evaluate :*

(i) $\int \frac{x^3 \sin^{-1} x}{(1-x^2)^{3/2}} dx$ (ii) $\int \cos 2x \log \frac{\cos x + \sin x}{\cos x - \sin x} dx$.

Sol. (i) Let $I = \int \frac{x^3 \sin^{-1} x}{(1-x^2)^{3/2}} dx$.

Sol. (i) Let $I = \int \frac{x^3 \sin^{-1} x}{(1-x^2)^{3/2}} dx$.

Put $x = \sin\theta$ or $\sin^{-1} x = \theta$. Then $x = \cos\theta \, d\theta$.

$$\therefore \quad I = \int \frac{\sin^3\theta}{\cos^3\theta}\theta\cos\theta\, d\theta = \int \frac{\theta\sin\theta\,(1-\cos^2\theta)}{\cos^2\theta} d\theta$$
$$= \int \theta\,[\sec\theta\tan\theta - \sin\theta]\, d\theta$$
$$= \theta(\sec\theta + \cos\theta) - \int 1.(\sec\theta + \cos\theta)\, d\theta,$$

integrating by parts taking $\sec\theta\tan\theta - \sin\theta$ as the second function

$$= \theta(\sec\theta + \cos\theta) - \log(\sec\theta + \tan\theta) - \sin\theta$$
$$= \theta\left(\frac{1}{\cos\theta} + \cos\theta\right) - \log\left[\frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right] - \sin\theta$$
$$= \theta\,\frac{(1+\cos^2\theta)}{\cos\theta} - \log\frac{1+\sin\theta}{\sqrt{(1-\sin^2\theta)}} - \sin\theta$$
$$= \theta\,\frac{(1+1-\sin^2\theta)}{\sqrt{(1-\sin^2\theta)}} - \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)^{1/2} - \sin\theta$$
$$= \theta\,\frac{2-\sin^2\theta}{\sqrt{(1-\sin^2\theta)}} - \tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} - \sin\theta$$
$$= \frac{2-x^2}{\sqrt{(1-x^2)}}\sin^{-1}x - x - \tfrac{1}{2}\log\frac{1+x}{1-x}.$$

(ii) We have $\int \cos 2x \log \frac{\cos x + \sin x}{\cos x - \sin x} dx$

$$= \int \cos 2x \log\frac{1+\tan x}{1-\tan x} dx = \int \cos 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) dx$$
$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right)$$
$$- \int \tfrac{1}{2}\sin 2x \cdot \frac{1}{\tan\left(\frac{1}{4}\pi + x\right)}\sec^2\left(\tfrac{1}{4}\pi + x\right) dx,$$

integrating by parts taking $\cos 2x$ as the second function

$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) - \int \frac{\sin 2x}{2\sin\left(\frac{1}{4}\pi + x\right)\cos\left(\frac{1}{4}\pi + x\right)} dx$$
$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) - \int \frac{\sin 2x}{\sin\left(\frac{1}{2}\pi + 2x\right)} dx$$
$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) - \int \frac{\sin 2x}{\cos 2x} dx$$
$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) - \int \tan 2x\, dx$$
$$= \tfrac{1}{2}\sin 2x \log\tan\left(\tfrac{1}{4}\pi + x\right) - \tfrac{1}{2}\log\sec 2x.$$

Ex. 35. *Evaluate :*

(i) $\int \sin x \log(\sec x + \tan x)\, dx$.

(ii) $\int x \log [x + \sqrt{(x^2 + a^2)}]\, dx$.

Sol. (i) We have $\int \sin x \log (\sec x + \tan x)\, dx$

$$= -\cos x \log (\sec x + \tan x) - \int (-\cos x) . \sec x\, dx,$$

integrating by parts taking $\sin x$ as the second function and observing that $\frac{d}{dx} \log (\sec x + \tan x) = \sec x$

$$= -\cos x \log (\sec x + \tan x) + \int dx$$

$$= x - \cos x \log (\sec x + \tan x).$$

(ii) We have $\int x \log [x + \sqrt{(x^2 + a^2)}]\, dx$

$$= \frac{x^2}{2} \log [x + \sqrt{(x^2 + a^2)}] - \int \frac{x^2}{2} \cdot \frac{1}{\sqrt{(x^2 + a^2)}}\, dx,$$

integrating by parts taking x as the second function and observing that $\frac{d}{dx} \log [x + \sqrt{(x^2 + a^2)}] = \frac{1}{\sqrt{(x^2 + a^2)}}$

$$= \frac{x^2}{2} \log [x + \sqrt{(x^2 + a^2)}] - \frac{1}{2} \int \frac{(x^2 + a^2) - a^2}{\sqrt{(x^2 + a^2)}}\, dx$$

$$= \frac{x^2}{2} \log [x + \sqrt{(x^2 + a^2)}] - \frac{1}{2} \int \sqrt{(x^2 + a^2)}\, dx + \frac{a^2}{2} \int \frac{dx}{\sqrt{(x^2 + a^2)}}$$

$$= \frac{x^2}{2} \log [x + \sqrt{(x^2 + a^2)}] - \frac{1}{2} \left[\frac{x}{2} \sqrt{(x^2 + a^2)} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}\right]$$

$$= \frac{x^2}{2} \log [x + \sqrt{(x^2 + a^2)}] - \frac{x}{4} \sqrt{(x^2 + a^2)} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{a^2}{4} \sinh^{-1} \frac{x}{a}.$$

Ex. 36. *Evaluate :*

(i) $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$.

(ii) $\int \frac{\cosh x + \sinh x . \sin x}{1 + \cos x}\, dx$.

(iii) $\int \frac{x + \sqrt{(1 - x^2)} \sin^{-1} x}{\sqrt{(1 - x^2)}}\, dx$.

Sol. (i) We have $\int \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx$

$$= \int \left(\frac{1}{\log x}\right) \cdot 1\, dx - \int \frac{1}{(\log x)^2}\, dx$$

$$= x \cdot \frac{1}{\log x} - \int x \cdot \frac{-1}{(\log x)^2} \frac{1}{x}\, dx - \int \frac{1}{(\log x)^2}\, dx,$$

(applying integration by parts to the first integral taking 1 as the second function and keeping the second integral as it is)

$$= \frac{x}{\log x} + \int \frac{1}{(\log x)^2} dx - \int \frac{1}{(\log x)^2} dx = \frac{x}{\log x}.$$

(ii) We have $\int \frac{\cosh x + \sinh x \,.\, \sin x}{1 + \cos x} dx$

$$= \int \frac{\cosh x}{1 + \cos x} dx + \int \sinh x \cdot \frac{\sin x}{1 + \cos x} dx$$

$$= \int \frac{\cosh x}{2 \cos^2 \frac{1}{2} x} dx + \int \sinh x \cdot \frac{2 \sin \frac{1}{2} x \cos \frac{1}{2} x}{2 \cos^2 \frac{1}{2} x} dx$$

$$= \int (\cosh x) \,.\, \tfrac{1}{2} \sec^2 \tfrac{1}{2} x \, dx + \int \sinh x \,.\, \tan \tfrac{1}{2} x \, dx$$

$$= (\cosh x) \,.\, \tan \tfrac{1}{2} x - \int \sinh x \,.\, \tan \tfrac{1}{2} x \, dx + \int \sinh x \,.\, \tan \tfrac{1}{2} x \, dx$$

(applying integration by parts to the first integral taking $\frac{1}{2} \sec^2 \frac{1}{2} x$ as the second function and keeping the second integral as it is)

$$= \cosh x \,.\, \tan \tfrac{1}{2} x.$$

(iii) We have $\int \frac{x + \sqrt{(1 - x^2)} \sin^{-1} x}{\sqrt{(1 - x^2)}} dx$

$$= \int \frac{x}{\sqrt{(1 - x^2)}} dx + \int \sin^{-1} x \, dx$$

$$= x \sin^{-1} x - \int 1 . \sin^{-1} x \, dx + \int \sin^{-1} x \, dx,$$

applying integration by parts to the first integral taking $1/\sqrt{(1 - x^2)}$ as the second function and keeping the second integral as it is

$$= x \sin^{-1} x.$$

§ 13. Integration by partial fractions.

Sometimes it is convenient to break up a given function into the sum of a number of suitable functions (*e.g.*, partial fractions) and then to integrate each function separately. This method is particularly useful in the case of rational algebraic fractions and certain trigonometrical products.

Note. Here we intend to study only elementary problems based upon partial fractions. However in the following chapter a detailed treatment of partial fractions will be given.

Integration of $\frac{1}{x^2 - a^2}$, $(x > a)$.

We know that $\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{1}{2a} \left(\frac{1}{x - a} - \frac{1}{x + a} \right)$.

Hence $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \left\{ \int \frac{dx}{x - a} - \int \frac{dx}{x + a} \right\}$

$$= (1/2a) \{ \log (x - a) - \log (x + a) \}.$$

Thus $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{x-a}{x+a}, x > a.$

Integration of $\frac{1}{a^2 - x^2}$, $(x < a)$.

We know that $\frac{1}{a^2 - x^2} = \frac{1}{(a-x)(a+x)}$.

Let $\frac{1}{(a-x)(a+x)} \equiv \frac{A}{a-x} + \frac{B}{a+x}$. Then $A = \frac{1}{2a}, B = \frac{1}{2a}$.

$\therefore \quad \frac{1}{(a-x)(a+x)} = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$.

$$\therefore \quad \int \frac{1}{(a^2 - x^2)} dx = \int \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right] dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx\right]$$

$$= \frac{1}{2a}\left[-\log(a-x) + \log(a+x)\right] = \frac{1}{2a}\log\frac{a+x}{a-x}.$$

Thus $\int \frac{1}{(a^2 - x^2)} dx = \frac{1}{2a} \log \frac{a+x}{a-x}, x < a.$

Note. We can also evaluate $\int \frac{dx}{a^2 - x^2}$ by substituting $x = a \tanh \theta$. Then we shall get the formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}.$$

§ 14. Reduction formulae.

(Application of successive integration by parts).

A formula which connects an integral with another integral of the same type in which the integral is of lower degree but is more readily integrable, is called a reduction formula. Generally we apply integration by parts again and again and finally the function is integrated by some standard method.

Solved Examples

Ex. 1 (a). *Evaluate* $\int \frac{(x-1)\,dx}{(x-3)(x-2)}$. **(Meerut 1986 P)**

Sol. Let

$$\frac{x-1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{A(x-2) + B(x-3)}{(x-3)(x-2)}.$$

Clearly $x - 1 \equiv A(x-2) + B(x-3)$ ...(1)

To find A, put $x = 3$ on both sides of (1) and we have

$$3 - 1 = A(3-2); \quad \therefore \quad A = 2.$$

To find B, put $x = 2$ on both sides of (1) and we have

$$2 - 1 = B(2-3); \quad \therefore \quad B = -1.$$

$$\therefore \quad \frac{x-1}{(x-3)(x-2)} = \frac{2}{x-3} - \frac{1}{x-2}.$$

Thus $\displaystyle\int \frac{(x-1)\,dx}{(x-3)(x-2)} = \int \frac{2\,dx}{x-3} - \int \frac{1 \cdot dx}{x-2}$

$$= 2\log(x-3) - \log(x-2) = \log\left[(x-3)^2/(x-2)\right].$$

Ex. 1 (b). *Evaluate* $\displaystyle\int \frac{x+1}{(x-1)(x-4)}\,dx.$ **(Delhi 1982)**

Sol. Let $\dfrac{x+1}{(x-1)(x-4)} = \dfrac{A}{x-1} + \dfrac{B}{x-4}$

$\Rightarrow \quad x + 1 \equiv A(x-4) + B(x-1).$

$\therefore \quad A = -\frac{2}{3}$ and $B = \frac{5}{3}$.

Now $\displaystyle\int \frac{(x+1)\,dx}{(x-1)(x-4)} = -\frac{2}{3}\int \frac{dx}{x-1} + \frac{5}{3}\int \frac{dx}{x-4}$

$$= -\tfrac{2}{3}\log(x-1) + \tfrac{5}{3}\log(x-4).$$

Ex 2. *Evaluate* $\displaystyle\int \frac{2x\,dx}{(x-1)(x+3)}.$

Sol. Let $\dfrac{2x}{(x-1)(x+3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x+3)}.$

Clearly $2x \equiv A(x+3) + B(x-1)$. ...(1)

Comparing the coefficients of x and the constant terms on both sides of (1), we get

$$2 = A + B \quad ...(2), \quad \text{and} \quad 3A - B = 0 \quad ...(3)$$

Solving (2) and (3), we get $A = \frac{1}{2}$ and $B = \frac{3}{2}$.

Thus $\displaystyle\int \frac{2x\,dx}{(x-1)(x+3)} = \int \frac{1}{2(x-1)}\,dx + \int \frac{3}{2(x+3)}\,dx$

$$= \tfrac{1}{2}\log(x-1) + (3/2)\log(x+3) = \tfrac{1}{2}\log(x-1) + \tfrac{1}{2}\log(x+3)^3$$

$$= \tfrac{1}{2}\log\{(x-1)(x+3)^3\}.$$

Note. An easy way to find the constants A and B etc., corresponding to linear non-repeated factors is like this :

The factor below A is $x - 1$. The equation $x - 1 = 0$ gives $x = 1$. Now suppress $x - 1$ in the given fraction $\dfrac{2x}{(x-1)(x+3)}$ and put $x = 1$ in the remaining fraction $\dfrac{2}{(x+3)}$ to get A. Thus $A = \dfrac{2 \times 1}{(1+3)} = \frac{1}{2}$. Similarly $B = \dfrac{2 \times (-3)}{(-3-1)} = \dfrac{-6}{-4} = \dfrac{3}{2}$.

Ex. 3. *Evaluate* $\int \frac{x}{x^2+x-6} dx$.

Sol. Let $\frac{x}{x^2+x-6} = \frac{x}{(x+3)(x-2)} = \frac{A}{(x+3)} + \frac{B}{(x-2)}$.

$\therefore \quad x \equiv A(x-2) + B(x+3)$

or $\quad x \equiv x(A+B) + 3B - 2A$. ...(1)

Comparing the coefficients of x and the constant terms on both sides of (1), we get $A + B = 1$ and $3B - 2A = 0$. Solving these we get $A = \frac{3}{5}$ and $B = \frac{2}{5}$.

Thus $\int \frac{x\,dx}{x^2+x-6} = \frac{3}{5}\int \frac{dx}{x+3} + \frac{2}{5}\int \frac{dx}{x-2}$

$= \frac{3}{5}\log(x+3) + \frac{2}{5}\log(x-2) = \frac{1}{5}\log\{(x+3)^3(x-2)^2\}$.

****Ex. 4.** *Evaluate* $\int \frac{1}{(e^x-1)^2} dx$. **(Meerut 1976)**

Sol. We have $\int \frac{1}{(e^x-1)^2} dx = \int \frac{e^x}{e^x(e^x-1)^2} dx$,

[multiplying the Nr. and Dr. by e^x]

$= \int \frac{dt}{t(t-1)^2}$, putting $e^x = t$ so that $e^x\,dx = dt$.

Now $\frac{1}{t(t-1)^2} \equiv \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$,

[on resolving into partial fractions]

$\therefore \quad 1 \equiv A(t-1)^2 + Bt(t-1) + Ct$. ...(1)

To find A, putting $t = 0$ on both sides of (1), we get $A = 1$.

To find C, put $t = 1$ and we get $C = 1$.

Thus $1 \equiv (t-1)^2 + Bt(t-1) + t$.

Comparing the coefficients of t^2 on both sides, we get

$$0 = 1 + B \quad \text{or} \quad B = -1.$$

$\therefore \quad \frac{1}{t(t-1)^2} = \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}$.

Hence $\int \frac{dt}{t(t-1)^2} = \int \frac{1}{t}dt - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2}$

$= \log t - \log(t-1) - \{1/(t-1)\}$

$= \log e^x - \log(e^x-1) - \{1/(e^x-1)\}$

$= x - \log(e^x-1) - \{1/(e^x-1)\}$.

Ex. 5. *Evaluate* $\int \frac{x}{(x^2-a^2)(x^2-b^2)} dx$.

Sol. Put $x^2 = t$ so that $2x\,dx = dt$.

Then $\int \frac{x\,dx}{(x^2-a^2)(x^2-b^2)} = \frac{1}{2}\int \frac{dt}{(t-a^2)(t-b^2)}$.

Now let $\frac{1}{(t-a^2)(t-b^2)} = \frac{A}{(t-a^2)} + \frac{B}{(t-b^2)}$.

$\therefore \quad 1 \equiv A(t-b^2) + B(t-a^2)$

or $\quad 1 \equiv t(A+B) - (a^2B + b^2A)$,

giving $A + B = 0$ and $Ab^2 + Ba^2 = -1$.

Solving these we get $A = 1/(a^2-b^2)$ and $B = -1/(a^2-b^2)$.

Thus the given integral

$$= \frac{1}{2}\left[\int \frac{dt}{(a^2-b^2)(t-a^2)} - \int \frac{dt}{(a^2-b^2)(t-b^2)}\right]$$

$$= \frac{1}{2(a^2-b^2)}[\log(t-a^2) - \log(t-b^2)]$$

$$= \frac{1}{2(a^2-b^2)}\log\left[\frac{t-a^2}{t-b^2}\right] = \frac{1}{2(a^2-b^2)}\log\left[\frac{x^2-a^2}{x^2-b^2}\right].$$

§ 15. Definite Integrals (Simple).

If $\int f(x)\,dx = F(x)$, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

is called the *definite integral* of $f(x)$ between the limits a and b. We call b the upper limit of x and a the lower limit.

If in place of $F(x)$ we take $F(x) + c$ as the value of the integral, we have

$$\int_a^b f(x)\,dx = \Big[F(x) + c\Big]_a^b$$

$$= \{F(b) + c\} - \{F(a) + c\} = F(b) - F(a).$$

Thus the value does not depend on the constant c and so in the evaluation of a definite integral the constant of integration does not play any role.

Substitutions in the case of definite integrals.

Substitutions are made in definite integrals also to reduce them to easily integrable forms. While doing so we should not forget to change the limits of the new variable accordingly in the light of the substitution.

Important Note 1. *While evaluating definite integrals principal values of inverse circular functions are considered.*

Note 2. *While evaluating a definite integral if we come across a product of two functions we can safely use 'Integration by parts.'*

Note 3. *Definite integrals with infinite limits can be evaluated in the ordinary way.*

Solved Examples

Ex. 1. *Evaluate* $\int_2^3 2x^4\,dx$.

Sol. We have $\int_2^3 2x^4\,dx = 2\left[\frac{x^5}{5}\right]_2^3 = \frac{2}{5}\left[x^5\right]_2^3$

$= \frac{2}{5}[3^5 - 2^5] = \frac{2}{5}\cdot(243 - 32) = \frac{2}{5}\cdot 211 = \frac{422}{5}$.

Ex. 2. *Evaluate* $\int_0^{\pi/2} \cos^2 x\,dx$.

Sol. We have $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2} = \frac{1}{2}[(\frac{1}{2}\pi + \frac{1}{2}\sin\pi) - (0 + \frac{1}{2}\sin 0)]$

$= \frac{1}{2}[(\frac{1}{2}\pi + 0) - 0] = \frac{1}{4}\pi$.

Ex. 3. *Evaluate* $\int_1^\infty \frac{dx}{(1 + x^2)}$.

Sol. We have

$$\int_1^\infty \frac{dx}{(1 + x^2)} = \left[\tan^{-1} x\right]_1^\infty = [\tan^{-1}\infty - \tan^{-1} 1]$$
$$= \tfrac{1}{2}\pi - \tfrac{1}{4}\pi = \tfrac{1}{4}\pi.$$

Ex. 4. *Evaluate* $\int_0^1 \frac{(\tan^{-1} x)^3}{1 + x^2}\,dx$.

Sol. Put $\tan^{-1} x = t$ so that $\{1/(1 + x^2)\}\,dx = dt$.

$\therefore$ the given integral

$$= \int_0^{\pi/4} t^3\,dt = \left[\frac{t^4}{4}\right]_0^{\pi/4} = \frac{1}{4}\left[(\pi/4)^4 - 0\right] = \frac{1}{4}\left(\frac{\pi}{4}\right)^4.$$

Note. The lower limit of t is zero for when $x = 0$, $t = \tan^{-1} 0 = 0$. Similarly the upper limit of t is $\pi/4$ for when $x = 1, t = \tan^{-1} 1 = \pi/4$.

Ex. 5. *Evaluate* $\int_0^a x^2 \sin x^3\,dx$.

Sol. Put $x^3 = t$ so that $3x^2\,dx = dt$.

Also when $x = 0, t = 0$ and when $x = a, t = a^3$.

Hence $\int_0^a x^2 \sin x^3\,dx = \frac{1}{3}\int_0^{a^3} \sin t\,dt = \frac{1}{3}\left[-\cos t\right]_0^{a^3}$

$= \frac{1}{3}[-\cos a^3 - (-\cos 0)] = \frac{1}{3}[-\cos a^3 + 1] = \frac{1}{3}[1 - \cos a^3]$.

Ex. 6. *Evaluate* $\int_0^1 \frac{5x^3\,dx}{\sqrt{(1 - x^8)}}$.

Sol. Put $x^4 = t$ so that $4x^3\,dx = dt$.

Also when $x = 0, t = 0$ and when $x = 1, t = 1$.

Hence $\displaystyle\int_0^1 \frac{5x^3\,dx}{\sqrt{(1-x^8)}} = \frac{5}{4}\int_0^1 \frac{dt}{\sqrt{(1-t^2)}} = \frac{5}{4}\left[\sin^{-1} t\right]_0^1$

$= \frac{5}{4}[\sin^{-1} 1 - \sin^{-1} 0] = \frac{5}{4}[\frac{1}{2}\pi - 0] = \frac{5}{8}\pi.$

Ex. 7. *Evaluate* $\displaystyle\int_1^2 \frac{(1+\log x)^4}{x}\,dx.$

Sol. Put $1 + \log x = t$ so that $(1/x)\,dx = dt$.

Also when $x = 1, t = 1 + \log 1 = 1$

and when $x = 2, t = (1 + \log 2)$.

Hence $\displaystyle\int_1^2 \frac{(1+\log x)^4}{x}\,dx = \int_1^{(1+\log 2)} t^4\,dt = \left[\frac{t^5}{5}\right]_1^{(1+\log 2)}$

$= \frac{1}{5}[(1+\log 2)^5 - 1^5] = (1+\log 2)^5 - \frac{1}{5}.$

Ex. 8. *Evaluate* $\displaystyle\int_0^{\pi/3} \frac{\cos x\,dx}{3+4\sin x}.$

Sol. Put $\sin x = t$ so that $\cos x\,dx = dt$.

Also when $x = 0, t = \sin 0 = 0$ and when $x = \pi/3$, $t = \sin\frac{1}{3}\pi = \sqrt{3}/2$.

Hence $\displaystyle\int_0^{\pi/3} \frac{\cos x\,dx}{3+4\sin x} = \int_0^{\sqrt{3}/2} \frac{dt}{3+4t} = \frac{1}{4}\left[\log(3+4t)\right]_0^{\sqrt{3}/2}$

$= \frac{1}{4}[\log\{3+4(\sqrt{3}/2)\} - \log(3+4\times 0)]$

$= \frac{1}{4}[\log(3+2\sqrt{3}) - \log 3]$

$= \frac{1}{4}\left[\log\left(\frac{3+2\sqrt{3}}{3}\right)\right] = \frac{1}{4}\left[\log\left(1+\frac{2}{\sqrt{3}}\right)\right].$

Ex. 9. *Evaluate* $\displaystyle\int_1^3 \frac{\cos(\log x)}{x}\,dx.$

Sol. Put $\log x = t$ so that $(1/x)\,dx = dt$.

Also when $x = 1, t = \log 1 = 0$ and when $x = 3, t = \log 3$.

Hence $\displaystyle\int_1^3 \frac{\cos(\log x)\,dx}{x} = \int_0^{\log 3} \cos t\,dt = \left[\sin t\right]_0^{\log 3}$

$= [\sin(\log 3) - \sin 0] = \sin(\log 3).$

Ex. 10. *Evaluate* $\int_0^2 x^2 e^{2x}\,dx$.

Sol. Integrating by parts regarding e^{2x} as the second function, we have

$$\int_0^2 x^2 e^{2x}\,dx = \left[\frac{x^2 e^{2x}}{2}\right]_0^2 - \int_0^2 2x \cdot \frac{e^{2x}}{2}\,dx$$

$$= 2e^4 - \int_0^2 xe^{2x}\,dx = 2e^4 - \left\{\left(x\frac{e^{2x}}{2}\right)_0^2 - \int_0^2 \frac{1}{2}e^{2x}\,dx\right\},$$

[again integrating by parts]

$$= 2e^4 - \left[\frac{xe^{2x}}{2}\right]_0^2 + \int_0^2 \frac{e^{2x}}{2}\,dx$$

$$= 2e^4 - e^4 + \left[\frac{e^{2x}}{4}\right]_0^2 = e^4 + \frac{e^4}{4} - \frac{1}{4}, \qquad [\because e^0 = 1]$$

$$= \tfrac{1}{4}[4e^4 + e^4 - 1] = \tfrac{1}{4}[5e^4 - 1].$$

Ex. 11. *Evaluate* $\int_0^{\pi/2} \sin^3 x\,dx$.

Sol. The given integral

$$I = \int_0^{\pi/2} \sin^2 x \sin x\,dx = \int_0^{\pi/2} (1 - \cos^2 x)\sin x\,dx. \quad \textbf{(Note)}$$

Now put $\cos x = t$ so that $-\sin x\,dx = dt$.
Also when $x = 0, t = 1$ and when $x = \pi/2, t = 0$.

Hence $I = -\int_1^0 (1 - t^2)\,dt = \int_0^1 (1 - t^2)\,dt,$

$$\left[\text{Note that } \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= \left[t - \tfrac{1}{3}t^3\right]_0^1 = 1 - \tfrac{1}{3} = \tfrac{2}{3}.$$

Ex. 12. *Evaluate* $\int_0^{\pi} \cos^3 x\,dx$.

Sol. We know that $\cos 3x = 4\cos^3 x - 3\cos x$.
Therefore $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$.

Now $\int_0^{\pi} \cos^3 x\,dx = \int_0^{\pi} \frac{1}{4}(\cos 3x + 3\cos x)\,dx$

$$= \tfrac{1}{4}\left[\tfrac{1}{3}\sin 3x + 3\sin x\right]_0^{\pi}$$

$$= \tfrac{1}{4}[(\tfrac{1}{3}\sin 3\pi + 3\sin\pi) - (\tfrac{1}{3}\sin 0 + 3\sin 0)] = \tfrac{1}{4}[0 - 0] = 0.$$

Ex. 13. *Evaluate* $\int_0^{\pi/2} \cos^4 x\,dx$.

Sol. The given integral $I = \frac{1}{4}\int_0^{\pi/2} (2\cos^2 x)^2\,dx$

$$= \tfrac{1}{4}\int_0^{\pi/2} (1 + 2\cos 2x)^2\,dx$$

$$= \tfrac{1}{4}\int_0^{\pi/2} (1 + 2\cos 2x + \cos^2 2x)\,dx \;.$$

$$= \tfrac{1}{4}\int_0^{\pi/2} [1 + 2\cos 2x + \tfrac{1}{2}(2\cos^2 2x)]\,dx$$

$$=\tfrac{1}{4}\int_0^{\pi/2}[1+2\cos 2x+\tfrac{1}{2}(2\cos^2 2x)]\,dx$$
$$=\tfrac{1}{4}\int_0^{\pi/2}\{1+2\cos 2x+\tfrac{1}{2}(1+\cos 4x)\}\,dx$$
$$=\tfrac{1}{4}\int_0^{\pi/2}[\tfrac{3}{2}+2\cos 2x+\tfrac{1}{2}\cos 4x]\,dx$$
$$=\tfrac{1}{4}\left[\tfrac{3}{2}x+\frac{2\sin 2x}{2}+\tfrac{1}{2}\frac{\sin 4x}{4}\right]_0^{\pi/2}$$
$$=\tfrac{1}{4}\left[\left\{\frac{3}{2}\times\frac{\pi}{2}+\sin\left(2\times\frac{\pi}{2}\right)+\tfrac{1}{8}\sin\left(4\times\frac{\pi}{2}\right)\right\}-0\right]$$
$$=\tfrac{1}{4}[\tfrac{3}{4}\pi+\sin\pi+(1/8)\sin 2\pi]=\tfrac{3}{16}\pi.$$

Ex. 14. *Evaluate* $\displaystyle\int_0^1\frac{1-x}{1+x}\,dx.$

Sol. We have $\displaystyle\int_0^1\frac{1-x}{1+x}\,dx=\int_0^1\frac{-(1+x)+2}{1+x}\,dx$

$$=\int_0^1\left[-1+\frac{2}{1+x}\right]dx=\left[-x+2\log(1+x)\right]_0^1$$
$$=\{-1+2\log 2\}-\{-0+2\log 1\}$$
$$=-1+2\log 2.$$

Ex. 15. *Evaluate* $\displaystyle\int_1^2\frac{dx}{x(1+x^4)}.$

Sol. We have $\displaystyle\int_1^2\frac{dx}{x(1+x^4)}=\int_1^2\frac{x\,dx}{x^2(1+x^4)}.$

Now put $x^2=t$ so that $2x\,dx=dt$.

Also when $x=1, t=1$ and when $x=2, t=4$.

Thus $\displaystyle\int_1^2\frac{dx}{x(1+x^4)}=\tfrac{1}{2}\int_1^4\frac{dt}{t(1+t^2)}.$

Now let $\displaystyle\frac{1}{t(1+t^2)}=\frac{A}{t}+\frac{Bt+C}{1+t^2}.$

Clearly $1\equiv A+At^2+Bt^2+Ct$. Comparing the coefficients of like powers of t on both sides, we have

$A=1, C=0$ and $A+B=0$. Therefore $B=-1$.

$$\therefore\quad\frac{1}{t(1+t^2)}=\frac{1}{t}-\frac{t}{1+t^2}.$$

Hence the given integral $\displaystyle=\tfrac{1}{2}\int_1^4\left[\frac{1}{t}-\frac{t}{1+t^2}\right]dt$

$$=\tfrac{1}{2}\left[\left(\log t\right)_1^4-\tfrac{1}{2}\left\{\log(t^2+1)\right\}_1^4\right]$$

$$=\tfrac{1}{2}[\{\log 4-\log 1\}-\tfrac{1}{2}(\log 17-\log 2)]$$

$= \frac{1}{2}[\{\log 4 - \log 1\} - \frac{1}{2}(\log 17 - \log 2)]$

$= \frac{1}{2}[2\log 2 - \frac{1}{2}\log 17 + \frac{1}{2}\log 2]$, $\quad [\because \log 1 = 0]$

$= \log 2 - \frac{1}{4}\log 17 + \frac{1}{4}\log 2 = \frac{5}{4}\log 2 - \frac{1}{4}\log 17$

$= \frac{1}{4}\log 2^5 - \frac{1}{4}\log 17 = \frac{1}{4}\log 32 - \frac{1}{4}\log 17 = \frac{1}{4}\log(32/17)$.

Ex. 16. *Evaluate* $\int_0^{\pi/2} e^x(\sin x + \cos x)\,dx$.

Sol. Given integral $I = \int_0^{\pi/2} e^x \sin x\,dx + \int_0^{\pi/2} e^x \cos x\,dx$.

Integrating the second integral by parts taking $\cos x$ as the second function, we get

$$I = \int_0^{\pi/2} e^x \sin x\,dx + \left[(e^x \sin x)_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx + (e^x \sin x)_0^{\pi/2} - \int_0^{\pi/2} e^x \sin x\,dx$$

$$= (e^x \sin x)_0^{\pi/2} = e^{\pi/2}\sin(\pi/2) - e^0 \sin 0$$

$$= (e^{\pi/2}.1) - (1 \times 0) = e^{\pi/2}.$$

Ex. 17. ***Integrate and evaluate***

(i) $\int_0^{\pi/2} \sin^2 x\,dx$, (ii) $\int_0^{\pi/4} \tan^2 x\,dx$, (iii) $\int_0^1 xe^x\,dx$.

Sol. (i) We have $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 2\sin^2 x\,dx$

$$= \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2}[(\tfrac{1}{2}\pi - \tfrac{1}{2}\sin\pi) - 0] = \frac{1}{2}(\tfrac{1}{2}\pi - 0) = \tfrac{1}{4}\pi.$$

(ii) We have $\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)\,dx$

$$= \left[\tan x\right]_0^{\pi/4} - \left[x\right]_0^{\pi/4} = \tan\frac{\pi}{4} - \frac{\pi}{4} = 1 - \frac{\pi}{4}.$$

(iii) We have $\int_0^1 xe^x\,dx = \left[x.e^x\right]_0^1 - \int_0^1 1.e^x\,dx$

$$= (1.e^1) - (0.e^0) - \int_0^1 e^x\,dx$$

$$= e - \left[e^x\right]_0^1 = e - [e - e^0] = 0 + 1 = 1, \quad [\because e^0 = 1].$$

Ex. 18. *Show that*

$$\int_a^b \frac{\log x}{x}\,dx = \frac{1}{2}\log\left(\frac{b}{a}\right)\log(ab).$$

Sol. Put $\log x = t$ so that $(1/x)\,dx = dt$.
Also when $x = a, t = \log a$ and when $x = b, t = \log b$.

$\therefore \int_a^b \frac{\log x}{x}\,dx = \int_{\log a}^{\log b} t\,dt = \left[\frac{t^2}{2}\right]^{\log b}$

$$\therefore \int_a^b \frac{\log x}{x}\,dx = \int_{\log a}^{\log b} t\,dt = \left[\frac{t^2}{2}\right]_{\log a}^{\log b}$$

$= \frac{1}{2}[(\log b)^2 - (\log a)^2] = \frac{1}{2}[(\log b - \log a).(\log b + \log a)]$

$= \frac{1}{2}[\log (b/a).\log (ab)]$.

Ex. 19. *Evaluate* $\int_a^{\infty} \frac{dx}{x^4}$.

Sol. Given integral $= \lim_{b \to \infty} \int_a^b \frac{dx}{x^4} = \lim_{b \to \infty} \left[\frac{-1}{3x^3}\right]_a^b$

$= \lim_{b \to \infty} \left[\frac{-1}{3b^3} + \frac{1}{3a^3}\right] = \frac{1}{3a^3}$. $\qquad \left[\because \lim_{b \to \infty} \frac{-1}{3b^3} = 0\right]$

Ex. 20. *Evaluate* $\int_a^{\infty} \frac{dx}{\sqrt{x}}$.

Sol. Given integral $= \lim_{b \to \infty} \int_a^b \frac{dx}{\sqrt{x}} = \lim_{b \to \infty} \left[2\sqrt{x}\right]_a^b$

$= \lim_{b \to \infty} 2(\sqrt{b} - \sqrt{a})$.

Since $\lim_{b \to \infty} 2\sqrt{b}$ is not finite *i.e.*, is not a definite number, therefore the integral under consideration is meaningless.

Ex. 21. *Prove that*

$$\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x\,dx}{(1-x^2)^{3/2}} = \frac{\pi}{4} - \frac{1}{2}\log 2. \qquad \textbf{(Lucknow 1983)}$$

Sol. Put $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$.

Also when $x = 0, \sin\theta = 0$ or $\theta = 0$;

and when $x = 1/\sqrt{2}, \sin\theta = 1/\sqrt{2}$ or $\theta = \frac{1}{4}\pi$.

Thus the given integral $= \int_0^{\pi/4} \theta \sec^2\theta\,d\theta$

$= \left[\theta \tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4} 1.\tan\theta\,d\theta$, (Integrating by parts)

$= \left(\frac{\pi}{4}\tan\frac{\pi}{4}\right) - (0 \tan 0) - \left[\log\sec\theta\right]_0^{\pi/4}$

$= \frac{\pi}{4}\cdot 1 - [\log\sec\frac{1}{4}\pi - \log\sec 0] = \frac{1}{4}\pi - (\log\sqrt{2} - \log 1)$

$= \frac{1}{4}\pi - (\log\sqrt{2} - 0) = \frac{1}{4}\pi - \frac{1}{2}\log 2$.

Ex. 22. *Prove that* $\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx = \frac{7}{9}$.

Sol. Let $I = \int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx$.

Sol. Let $I=\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$.

Put $x=\sin\theta$ or $\sin^{-1}x=\theta$. Then $dx=\cos\theta\, d\theta$.

When $x=0$, we have $\sin\theta=0$ giving $\theta=0$ and when $x=1$, we have $\sin\theta=1$ giving $\theta=\pi/2$.

$$\therefore \quad I=\int_0^{\pi/2} \frac{\theta \sin^3\theta}{\cos\theta}\cdot\cos\theta\, d\theta=\int_0^{\pi/2}\theta\sin^3\theta\, d\theta$$

$$=\int_0^{\pi/2}\theta\cdot\frac{3\sin\theta-\sin 3\theta}{4}\, d\theta, \qquad [\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$$

$$=\frac{1}{4}\int_0^{\pi/2}\theta.(3\sin\theta-\sin 3\theta)\, d\theta$$

$$=\frac{1}{4}\left[\theta.\left(-3\cos\theta+\frac{1}{3}\cos 3\theta\right)\right]_0^{\pi/2} - \frac{1}{4}\int_0^{\pi/2} 1.\left(-3\cos\theta+\tfrac{1}{3}\cos 3\theta\right)d\theta,$$

integrating by parts

$$=\frac{1}{4}\cdot 0-\frac{1}{4}\left[-3\sin\theta+\frac{1}{9}\sin 3\theta\right]_0^{\pi/2}$$

$$=-\tfrac{1}{4}\left[-3\sin\tfrac{1}{2}\pi+\tfrac{1}{9}\sin(3\pi/2)\right]=-\tfrac{1}{4}\left[-3-\tfrac{1}{9}\right]=\tfrac{7}{9}.$$

Ex. 23. *Prove that* $\int_0^1 x(\tan^{-1}x)^2\, dx=\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\log 2.$

(Meerut 1988 S)

Sol. We have $\int_0^1 x(\tan^{-1}x)^2\, dx$

$$=\left[\frac{x^2}{2}(\tan^{-1}x)^2\right]_0^1-\int_0^1\frac{x^2}{2}\,\frac{2\tan^{-1}x}{1+x^2}\,dx,$$

integrating by parts taking x as the second function

$$=\tfrac{1}{2}(\tan^{-1}1)^2-\int_0^1\frac{(x^2+1)-1}{1+x^2}\tan^{-1}x\, dx$$

$$=\frac{\pi^2}{32}-\int_0^1\tan^{-1}x\, dx+\int_0^1(\tan^{-1}x)\cdot\frac{1}{1+x^2}\,dx$$

$$=\frac{\pi^2}{32}-\left[x\tan^{-1}x\right]_0^1+\int_0^1 x\cdot\frac{1}{1+x^2}\,dx+\left[\frac{(\tan^{-1}x)^2}{2}\right]_0^1,$$

applying integration by parts to the first integral taking 1 as the second function and power formula to the second integral

$$=\frac{\pi^2}{32}-[1.\tan^{-1}1]+\frac{1}{2}\left[\log(1+x^2)\right]_0^1+\frac{\pi^2}{32}$$

$$=\frac{\pi^2}{16}-\frac{\pi}{4}+\frac{1}{2}[\log 2-\log 1]=\frac{\pi}{4}\left(\frac{\pi}{4}-1\right)+\frac{1}{2}\log 2.$$

Ex. 24. *Evaluate*

(i) $\int_2^4 \frac{x^2+x}{2x+1}dx$ **(Meerut 1988 S)**

(ii) $\int_0^1 \frac{ab}{[(a-b)x+b]^2}dx.$ **(Meerut 1989 P)**

Sol. (i) We have $I=\int_2^4 \frac{x^2+x}{2x+1}dx$

$$=\int_2^4\left[\frac{1}{2}x+\frac{1}{4}-\frac{1/4}{2x+1}\right]dx,$$

dividing the Nr. by the Dr. we get $\frac{1}{2}x+\frac{1}{4}$ as the quotient and $-\frac{1}{4}$ as the remainder

$$=\left[\frac{x^2}{4}+\frac{x}{4}-\frac{1}{8}\log(2x+1)\right]_2^4$$

$$=4+1-\tfrac{1}{8}\log 9-1-\tfrac{1}{2}+\tfrac{1}{8}\log 5=\tfrac{7}{2}-\tfrac{1}{8}\log\left(\tfrac{9}{5}\right).$$

(ii) Let $I=\int_0^1 \frac{ab}{[(a-b)x+b]^2}dx.$

Put $(a-b)x+b=t$. Then $(a-b)\,dx=dt$.

When $x=0, t=b$ and when $x=1, t=a$.

$$\therefore \quad I=\int_b^a\left[\frac{ab}{t^2}\cdot\frac{1}{a-b}\right]dt=\frac{ab}{a-b}\int_b^a\frac{1}{t^2}dt$$

$$=\frac{ab}{a-b}\left[-\frac{1}{t}\right]_b^a=\frac{ab}{a-b}\left[-\frac{1}{a}+\frac{1}{b}\right]=\frac{ab}{a-b}\left[\frac{1}{b}-\frac{1}{a}\right]$$

$$=\frac{ab}{a-b}\cdot\frac{a-b}{ab}=1.$$

□

2

Integration of Rational Fractions

§ 1. Rational Fractions.

A fraction whose numerator and denominator are both rational and algebraic functions is defined as a rational fraction.

Thus $\dfrac{f(x)}{\phi(x)} = \dfrac{a_0 x^m + a_1 x^{m-1} + \ldots + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x + b_n}$,

in which $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n$ are constants and m and n are positive integers, is a rational algebraic fraction. Such fractions can always be integrated by splitting the given fraction into *partial fractions.*

Case I : Integration of fractions with non-repeated factors only in the denominator.

Working Rule : (i) The degree of the Nr. $f(x)$ must be less than the degree of the Dr. $\phi(x)$ and if not so, then divide $f(x)$ by $\phi(x)$ till the remainder is of a lower degree than $\phi(x)$.

(ii) Now break the denominator $\phi(x)$ into linear and quadratic factors.

(iii) Express the fraction as the sum of partial fractions. For example the partial fractions of the fraction

$\dfrac{x^3 - 4x^2 + 5x + 7}{(x-a)(x-b)(lx^2 + mx + n)}$ will be written as

$$\frac{A}{x-a} + \frac{B}{x-b} + \frac{Cx + D}{lx^2 + mx + n}.$$

(iv) To obtain the partial fraction corresponding to the factor $(x-a)$ in the denominator we put $(x-a) = 0$ and we get the value of x i.e., $x = a$. Now we put $x = a$ everywhere in the given fraction except in the factor $(x-a)$ itself. Thus the value of A is obtained.

Solved Examples

Ex. 1. *Integrate* $(x^2+1)/(x^2-1)$.

Sol. We have $\displaystyle\int \frac{(x^2+1)}{(x^2-1)} dx = \int \frac{(x^2-1+2)}{(x^2-1)} dx$ **(Note)**

$$= \int \frac{(x^2-1)}{(x^2-1)} dx + \int \frac{2}{(x^2-1)} dx = \int dx + 2\int \frac{dx}{x^2-1}$$

$$= x + 2 \cdot \frac{1}{2 \times 1} \log \frac{x-1}{x+1} = x + \log \frac{x-1}{x+1}.$$

Ex. 2. *Integrate* $(x+1)/(x^3+x^2-6x)$.

Ex. 2. *Integrate* $(x+1)/(x^3+x^2-6x)$.

Sol. Here $\dfrac{x+1}{x^3+x^2-6x}=\dfrac{x+1}{x(x-2)(x+3)}\equiv\dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{x+3}$, (say).

To find A suppress x in the given fraction and put $x=0$ in the remaining fraction. Thus $A=\dfrac{0+1}{(0-2)(0+3)}=-\dfrac{1}{6}$.

To find B suppress $(x-2)$ in the given fraction and put $x=2$ in the remaining fraction. Thus $B=\dfrac{2+1}{2(2+3)}=\dfrac{3}{10}$.

Similarly $C=\dfrac{-3+1}{-3(-3-2)}=-\dfrac{2}{15}$.

Thus $\dfrac{x+1}{x(x-2)(x+3)}=-\dfrac{1}{6x}+\dfrac{3}{10(x-2)}-\dfrac{2}{15(x+3)}$.

Obviously

$$\int\frac{(x+1)}{x(x-2)(x+3)}=-\int\frac{1.\,dx}{6x}+\int\frac{3\,dx}{10(x-2)}-\int\frac{2\,dx}{15(x+3)}$$

$$=-\tfrac{1}{6}\log x+\tfrac{3}{10}\log(x-2)-\tfrac{2}{15}\log(x+3).$$

Ex. 3. *Integrate* $x^2/\{(x+1)(x-2)(x+3)\}$.

Sol. Let $\dfrac{x^2}{(x+1)(x-2)(x+3)}\equiv\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{x+3}$.

We have $A=\dfrac{(-1)^2}{(-1-2)(-1+3)}=\dfrac{1}{(-3)\times(2)}=-\dfrac{1}{6}$,

$B=\dfrac{2^2}{(2+1)(2+3)}=\dfrac{4}{15}$, and

$C=\dfrac{(-3)^2}{(-3+1)(-3-2)}=\dfrac{9}{-2\times(-5)}=\dfrac{9}{10}$.

$\therefore$ required integral

$$=\int\left[-\frac{1}{6(x+1)}+\frac{4}{15(x-2)}+\frac{9}{10(x+3)}\right]dx$$

$$=-\tfrac{1}{6}\log(x+1)+\tfrac{4}{15}\log(x-2)+\tfrac{9}{10}\log(x+3)$$

$$=\tfrac{9}{10}\log(x+3)+\tfrac{4}{15}\log(x-2)-\tfrac{1}{6}\log(x+1).$$

Ex. 4. *Integrate* $x^2/\{(x-1)(3x-1)(3x-2)\}$.

Sol. Let $\dfrac{x^2}{(x-1)(3x-1)(3x-2)}\equiv\dfrac{A}{(x-1)}+\dfrac{B}{(3x-1)}+\dfrac{C}{(3x-2)}$.

Here $A=\dfrac{1^2}{(3\times1-1)(3\times1-2)}=\tfrac{1}{2}$,

$B=\dfrac{(1/3)^2}{(1/3-1)\{3.(1/3)-2\}}=\dfrac{1/9}{2/3}=\tfrac{1}{6}$

and $$C = \frac{(2/3)^2}{(2/3-1)\{3.(2/3)-1\}} = -\tfrac{4}{3}.$$

$$\therefore \text{ required integral } = \int \left[\frac{1}{2(x-1)} + \frac{1}{6(3x-1)} - \frac{4}{3(3x-2)}\right] dx$$

$$= \tfrac{1}{2}\log(x-1) + \tfrac{1}{18}\log(3x-1) - \tfrac{4}{9}\log(3x-2).$$

Ex. 5. *Integrate* $x/\{(x-a)(x-b)(x-c)\}$.

Sol. Let $$\frac{x}{(x-a)(x-b)(x-c)} \equiv \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}.$$

Now to find the partial fraction corresponding to the factor $(x-a)$ in the denominator, put $x=a$ everywhere in the given fraction except in the factor $(x-a)$ itself.

Thus we get $A = a/\{(a-b)(a-c)\}$.

Similarly we get $B = b/\{(b-c)(b-a)\}$

and $C = c/\{(c-a)(c-b)\}$.

Now $$\int \frac{x}{(x-a)(x-b)(x-c)} dx = \int \left[\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}\right] dx$$

$$= A\log(x-a) + B\log(x-b) + C\log(x-c)$$

$$= \frac{a}{(a-b)(a-c)}\log(x-a) + \frac{b}{(b-c)(b-a)}\log(x-b)$$

$$+ \frac{c}{(c-a)(c-b)}\log(x-c) = \Sigma\left[\frac{a\log(x-a)}{(a-b)(a-c)}\right].$$

Ex. 6. *Integrate*

$$\{(x-a)(x-b)(x-c)\}/\{(x-\alpha)(x-\beta)(x-\gamma)\}.$$

(Meerut 1984 S)

Sol. Since the numerator is not of a lower degree than the denominator, we first divide Nr. by Deno. Dividing the numerator by the denominator orally we see that the quotient is 1. So let

$$\frac{(x-a)(x-b)(x-c)}{(x-\alpha)(x-\beta)(x-\gamma)} \equiv 1 + \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)} + \frac{C}{(x-\gamma)}.$$

Then $$A = \frac{(\alpha-a)(\alpha-b)(\alpha-c)}{(\alpha-\beta)(\alpha-\gamma)},\ B = \frac{(\beta-a)(\beta-b)(\beta-c)}{(\beta-\alpha)(\beta-\gamma)},$$

and $$C = \frac{(\gamma-a)(\gamma-b)(\gamma-c)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Now the given integral $$I = \int \left[1 + \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}\right] dx$$

$$= x + A\log(x-\alpha) + B\log(x-\beta) + C\log(x-\gamma)$$

$$= x + \Sigma\left[\frac{(\alpha-a)(\alpha-b)(\alpha-c)}{(\alpha-\beta)(\alpha-\gamma)}\log(x-\alpha)\right],$$

[On putting the values of A, B and C].

Ex. 7. *Integrate* $(x^2+x+2)/\{(x-2)(x-1)\}$.

Sol. Here since numerator is not of a lower degree than the denominator, we first divide the numerator by the denominator. As we see orally the quotient is 1. So let

$$\frac{x^2+x+2}{(x-2)(x-1)} \equiv 1 + \frac{A}{x-2} + \frac{B}{x-1}.$$

We have $A = \dfrac{2^2+2+2}{(2-1)} = 8$, $B = \dfrac{1^2+1+2}{(1-2)} = -4$.

Hence $\displaystyle\int \frac{x^2+x+2}{(x-2)(x-1)}\,dx = \int \left\{1 + 4\left(\frac{2}{x-2} - \frac{1}{x-1}\right)\right\} dx$

$= x + 4\{2\log(x-2) - \log(x-1)\}$

$= x + 4\log\{(x-2)^2/(x-1)\}$.

Ex. 8 (a). *Integrate* $x^3/\{(x-1)(x-2)(x-3)\}$.

Sol. Here since the numerator is not of a lower degree than the denominator, we divide the numerator by the denominator till the remainder is of lesser degree than the denominator. Wo orally see that the quotient is 1.

We need not find out the actual value of the remainder because ultimately we have to break the fraction into partial fractions. Note that the denominators of the partial fractions depend only upon the denominator of the given fraction. So let

$$\frac{x^3}{(x-1)(x-2)(x-3)} \equiv 1 + \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

We have $A = \dfrac{1^3}{(1-2)(1-3)} = \frac{1}{2}$, $B = \dfrac{8}{(2-1)(2-3)} = -8$,

and $\qquad C = \dfrac{3^3}{(3-1)(3-2)} = \dfrac{27}{2}$.

$$\therefore \quad \frac{x^3}{(x-1)(x-2)(x-3)} = 1 + \frac{1}{2(x-1)} - \frac{8}{(x-2)} + \frac{27}{2(x-3)}.$$

Hence $\displaystyle\int \frac{x^3\,dx}{(x-1)(x-2)(x-3)}$

$\displaystyle= \int 1\,.\,dx + \int \frac{dx}{2(x-1)} - \int \frac{8\,dx}{(x-2)} + \int \frac{27\,dx}{2(x-3)}$

$= x + \frac{1}{2}\log(x-1) - 8\log(x-2) + (27/2)\log(x-3)$.

Ex. 8 (b). *Evaluate* $\displaystyle\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$. **(Meerut 1988)**

Sol. We have $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = \dfrac{(y+1)(y+2)}{(y+3)(y+4)}$, where $y = x^2$.

Now let $\dfrac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \dfrac{A}{y+3} + \dfrac{B}{y+4}$,

resolving into partial fractions.

$(-3+1)(-3+2)$

We have $A = \dfrac{(-3+1)(-3+2)}{(-3+4)} = 2,$

$$B = \frac{(-4+1)(-4+2)}{(-4+3)} = -6.$$

$\therefore$ $\dfrac{(y+1)(y+2)}{(y+3)(y+4)} = 1 + \dfrac{2}{y+3} - \dfrac{6}{y+4}.$

$\therefore$ the given integral $I = \displaystyle\int \left[1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}\right] dx$

$$= \int dx + 2\int \frac{dx}{x^2+3} - 6\int \frac{dx}{x^2+4}$$

$$= x + 2 \cdot \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 6 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2}$$

$$= x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3 \tan^{-1}\left(\frac{x}{2}\right).$$

Ex. 8 (c). *Evaluate* $\displaystyle\int \frac{(x^2+a^2)(x^2+b^2)}{(x^2+c^2)(x^2+d^2)} dx.$ **(Meerut 1988 P)**

Sol. Proceed as in Ex. 8 (b).

Ans. The given integral

$$I = x + \frac{(a^2-c^2)(b^2-c^2)}{(d^2-c^2)} \cdot \frac{1}{c} \tan^{-1}\left(\frac{x}{c}\right) + \frac{(a^2-d^2)(b^2-d^2)}{(c^2-d^2)} \cdot \frac{1}{d} \tan^{-1}\left(\frac{x}{d}\right).$$

Case II. Repeated linear factors.

If the denominator contains a repeated linear factor, the application of long division method is simpler to evaluate the partial fractions.

Working Rule — Put the repeated linear factor equal to y, (say). Find x in terms of y and then transform x everywhere in terms of y. The repeated factor $(1/y^r)$ should be taken out. Arrange the numerator and the denominator in **ascending powers** of y and then divide the numerator by the denominator till y^r comes as a factor of the remainder.

Thus the fraction should be written as

$$\frac{1}{y^r}\left[\textbf{Quotient} + \frac{\textbf{Remainder}}{\textbf{divisor}}\right].$$

After removing the brackets express y in terms of x.

Note. When repeated factor is of degree two only, we can directly find out the partial fractions and need not apply the long division method.

Solved Examples

Ex. 1. (a). *Evaluate* $\int \frac{dx}{(x-1)^2(x^2+4)}$.

Sol. Let $\frac{1}{(x-1)^2(x^2+4)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4}$.

$\therefore \quad 1 \equiv A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2$ ...(1)

Putting $x = 1$ on both sides of (1), we get $B = 1/5$.

Now to get the values of C and D we put $x^2 = -4$ on both sides of (1) and we get

$$1 \equiv (Cx+D)(x^2-2x+1)$$

or $\quad 1 \equiv (Cx+D)(-4-2x+1)$. $\quad [\because x^2 = -4]$

or $\quad 1 \equiv (Cx+D)(-2x-3)$

or $\quad 1 \equiv -2Cx^2 - 3Cx - 2Dx - 3D$

or $\quad 1 \equiv 8C - 3Cx - 2Dx - 3D$, $\quad [\because x^2 = -4]$.

Now equating the coefficients of x and constant terms on both sides, we get

$$-3C - 2D = 0, 8C - 3D = 1.$$

Solving these we get $C = 2/25, D = -3/25$.

Putting the values of B, C and D in (1), we get

$$1 \equiv A(x-1)(x^2+4) + \tfrac{1}{5}(x^2+4) + (\tfrac{2}{25} - \tfrac{3}{25}x)(x-1)^2. \quad ...(2)$$

Equating the coefficients of x^3 on both sides of (2), we get

$$0 = A + \tfrac{2}{25} \quad \text{or} \quad A = -\tfrac{2}{25}.$$

$$\therefore \int \frac{dx}{(x-1)^2(x^2+4)} = \frac{-2}{25}\int \frac{dx}{x-1} + \frac{1}{5}\int \frac{dx}{(x+1)^2} + \frac{1}{25}\int \frac{2x-3}{x^2+4}\,dx$$

$$= -\frac{2}{25}\log(x-1) - \frac{1}{5(x-1)} + \frac{1}{25}\int \frac{2x\,dx}{x^2+4} - \frac{3}{25}\int \frac{dx}{x^2+4}$$

$$= -\frac{2}{25}\log(x-1) - \frac{1}{5(x-1)} + \frac{1}{25}\log(x^2+4) - \frac{3}{50}\tan^{-1}\frac{x}{2}.$$

Ex. 1 (b). *Evaluate* $\int \frac{(x^2+x+1)\,dx}{(x+1)^2(x+2)}$. **(Delhi 1983)**

Sol. Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$.

$\therefore \quad x^2+x+1 \equiv A(x+1)(x+2) + B(x+2) + C(x+1)^2$ ...(1)

Putting $x = -1$ on both sides of (1), we get $B = 1$.

Again putting $x = -2$ on both sides of (1), we get $C = 3$.

Putting the values of B and C in (1), we get

$$x^2+x+1 \equiv A(x+1)(x+2) + (x+2) + 3(x+1)^2 \quad ...(2)$$

Equating the coefficients of x^2 on both sides of (2), we get

$$1 = A + 3, \quad \text{or} \quad A = -2.$$

$$\therefore \quad \int \frac{(x^2+x+1)\,dx}{(x+1)^2(x+2)}$$
$$= \int \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right] dx$$
$$= -2\log(x+1) - \frac{1}{x+1} + 3\log(x+2)$$
$$= \log\frac{(x+2)^3}{(x+1)^2} - \frac{1}{x+1}.$$

Ex. 2. *Evaluate* $\int \frac{dx}{x^3(x-1)^2(x+1)}$.

Sol. Let $\frac{1}{x^3(x-1)^2(x+1)}$

$$\equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x-1)} + \frac{E}{(x-1)^2} + \frac{F}{(x+1)}.$$

$$\therefore \quad 1 \equiv Ax^2(x-1)^2(x+1) + Bx(x-1)^2(x+1)$$
$$+ C(x-2)^2(x+1)$$
$$+ Dx^3(x-1)(x+1) + Ex^3(x+1) + Fx^3(x-1)^2 \quad \text{...(1)}$$

Putting $x = 0, 1$ and -1 successively on both sides of (1), we get $C = 1, E = \frac{1}{2}$ and $F = -\frac{1}{4}$.

With these values of C, E and F, (1) becomes

$$1 \equiv Ax^2(x-1)^2(x+1) + Bx(x-1)^2(x+1) + (x-1)^2(x+1)$$
$$+ Dx^3(x-1)(x+1) + \tfrac{1}{2}x^3(x+1) - \tfrac{1}{4}x^3(x-1)^2. \quad \text{...(2)}$$

To obtain A, B, D, equating the coefficients of x^3, x^4, x^5 on both sides of (2), we get

$$-A - B + 1 - D + \tfrac{1}{2} - \tfrac{1}{4} = 0, -A + B + \tfrac{1}{2} + \tfrac{1}{2} = 0 \text{ and}$$
$$A + D - \tfrac{1}{4} = 0.$$

Solving these, we get $A = 2, B = 1$ and $D = -\frac{7}{4}$.

$$\therefore \quad \int \frac{1}{x^3(x-1)^2(x+1)}\,dx$$
$$= \int \left[\frac{2}{x} + \frac{1}{x^2} + \frac{1}{x^3} - \frac{7}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)}\right] dx$$
$$= 2\log x - \frac{1}{x} - \frac{1}{2x^2} - \frac{7}{4}\log(x-1) - \frac{1}{2(x-1)} - \frac{1}{4}\log(x+1).$$

Ex. 3. *Integrate* $x^3/\{(x+1)^4(x+2)(x-1)\}$.

Sol. First we shall break the given fraction into partial fractions. Putting $x + 1 = y$, we get

$$\frac{x^3}{(x+1)^4(x+2)(x-1)} = \frac{1}{y^4}\,\frac{(-1+y)^3}{(1+y)(-2+y)}$$

$$\frac{x^3}{(x+1)^4(x+2)(x-1)}=\frac{1}{y^4}\frac{(-1+y)^3}{(1+y)(-2+y)}$$

$$=\frac{1}{y^4}\cdot\left[\frac{-1+3y-3y^2+y^3}{-2-y+y^2}\right],$$ on arranging the Nr. and Dr. in ascending powers of y.

Now we divide the numerator by the denominator till y^4 is a factor of the remainder. The actual division has been shown below :

$$\begin{array}{r|l|l}
-2-y+y^2 & -1+3y-3y^2+y^3 & \frac{1}{2}-\frac{7}{4}y+\frac{21}{8}y^2-\frac{43}{16}y^3 \\
 & -1-\frac{1}{2}y+\frac{1}{2}y^2 & \\ \hline
 & \frac{7}{2}y-\frac{7}{2}y^2+y^3 & \\
 & \frac{7}{2}y+\frac{7}{4}y^2-\frac{7}{4}y^3 & \\ \hline
 & -\frac{21}{4}y^2+\frac{11}{4}y^3 & \\
 & -\frac{21}{4}y^2-\frac{21}{8}y^3+\frac{21}{8}y^4 & \\ \hline
 & \frac{43}{8}y^3-\frac{21}{8}y^4 & \\
 & \frac{43}{8}y^3+\frac{43}{16}y^4-\frac{43}{16}y^5 & \\ \hline
 & -\frac{85}{16}y^4+\frac{43}{16}y^5 &
\end{array}$$

$\therefore$ the given fraction

$$=\frac{1}{2y^4}-\frac{7}{4y^3}+\frac{21}{8y^2}-\frac{43}{16y}+\frac{-\frac{85}{16}+\frac{43}{16}y}{-2-y+y^2}$$

$$=\frac{1}{2(x+1)^4}-\frac{7}{4(x+1)^3}+\frac{21}{8(x+1)^2}-\frac{43}{16(x+1)}+\frac{-85+43(x+1)}{16(x+2)(x-1)}.$$

Also by resolving into partial fractions the last term

$$\frac{43x-42}{16(x+2)(x-1)}=\frac{8}{3(x+2)}+\frac{1}{48(x-1)}.$$

Hence the required integral of the given fraction

$$=-\frac{1}{6(x+1)^3}+\frac{7}{8(x+1)^2}-\frac{21}{3(x+1)}-\frac{43}{16}\log(x+1)+\tfrac{8}{3}\log(x+2)+\tfrac{1}{48}\log(x-1)$$

Ex. 4. *Integrate* $x/\{(x-1)^3(x-2)\}$. **(Kanpur 1980; Vikram 76)**

Sol. Putting $(x-1)=y$ or $x=y+1$, we get

$$\frac{x}{(x-1)^3(x-2)}=\frac{y+1}{y^3(y+1-2)}=\frac{y+1}{y^3(y-1)}=\frac{1+y}{y^3(-1+y)},$$

[Note that we have arranged the Nr. and the Dr. in

[Note that we have arranged the Nr. and the Dr. in ascending powers of y]

$$=\frac{1}{y^3}\left[-1-2y-2y^2+\frac{2y^3}{-1+y}\right], \text{ by actual division}$$

$$=-\frac{1}{y^3}-\frac{2}{y^2}-\frac{2}{y}+\frac{2}{(y-1)}$$

$$=-\frac{1}{(x-1)^3}-\frac{2}{(x-1)^2}-\frac{2}{(x-1)}+\frac{2}{(x-2)}, \quad [\because y=x-1].$$

Hence the required integral of the given fraction

$$=-\int\frac{dx}{(x-1)^3}-\int\frac{2\,dx}{(x-1)^2}-\int\frac{2\,dx}{(x-1)}+\int\frac{2\,dx}{(x-2)}$$

$$=\frac{1}{x(x-1)^2}+\frac{2}{(x-1)}-2\log(x-1)+2\log(x-2).$$

Ex. 5. *Integrate* $(x^2+2)/\{(x-1)(x-2)^3\}$. **(Meerut 1984 P)**

Sol. Putting $x-2=y$ or $x=y+2$, we get

$$\frac{x^2+2}{(x-1)(x-2)^3}=\frac{(y+2)^2+2}{(y+2-1)y^3}=\frac{y^2+4y+6}{(y+1)y^3}=\frac{1}{y^3}\left[\frac{6+4y+y^2}{1+y}\right]$$

$$=\frac{1}{y^3}\left[6-2y+3y^2-\frac{3y^3}{1+y}\right], \text{ by actual division i.e., by dividing}$$

$6+4y+y^2$ by $1+y$ till y^3 is a factor of the remainder

$$=\frac{6}{y^3}-\frac{2}{y^2}+\frac{3}{y}-\frac{3}{1+y}$$

$$=\frac{6}{(x-2)^3}-\frac{2}{(x-2)^2}+\frac{3}{(x-2)}-\frac{3}{(x-1)}.$$

Hence the required integral of the given fraction

$$=-\frac{3}{(x-2)^2}+\frac{2}{(x-2)}+3\log(x-2)-3\log(x-1).$$

Ex. 6. *Integrate* $(3x+1)/\{(x-1)^3(x+1)\}$.

Sol. Putting $x-1=y$ so that $x=1+y$, we get

$$\frac{3x+1}{(x-1)^3(x+1)}=\frac{3(1+y)+1}{y^3(2+y)}=\frac{4+3y}{y^3(2+y)},$$

arranging the Nr. and the Dr. in asending powers of y

$$=\frac{1}{y^3}\left[2+\tfrac{1}{2}y-\tfrac{1}{4}y^2+\tfrac{1}{4}\frac{y^3}{2+y}\right], \text{ by actual division}$$

$$=\frac{2}{y^3}+\frac{1}{2y^2}-\frac{1}{4y}+\tfrac{1}{4}\cdot\frac{1}{(2+y)}$$

$$=\frac{2}{(x-1)^3}+\frac{1}{2(x-1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x+1)}.$$

Hence the required integral of the given fraction

$$= \int \left[\frac{2}{(x-1)^3} + \frac{1}{2(x-1)^2} - \frac{1}{4(x-1)} + \frac{1}{4(x+1)}\right] dx$$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} - \tfrac{1}{4}\log(x-1) + \tfrac{1}{4}\log(x+1)$$

$$= \frac{-1}{(x-1)^2} - \frac{1}{2(x-1)} + \tfrac{1}{4}\log\frac{x+1}{x-1}.$$

§ 2. Integration of $1/(ax^2+bx+c)$.

To evaluate such integrals put the denominator in the form $a\{(x+\alpha)^2 \pm \beta^2\}$ and then integrate.

Ex. 7. *Integrate* $1/(9x^2-12x+8)$.

Sol. We have $\displaystyle\int \frac{dx}{9x^2-12x+8} = \frac{1}{9}\int \frac{dx}{x^2-\frac{4}{3}x+\frac{8}{9}}$,

making the coeff. of x^2 in the denominator as 1

$$= \frac{1}{9}\int \frac{dx}{(x^2-\frac{4}{3}x+\frac{4}{9}+\frac{8}{9}-\frac{4}{9})} = \frac{1}{9}\int \frac{dx}{(x-\frac{2}{3})^2+\frac{4}{9}}$$

$$= \frac{1}{9}\int \frac{dx}{(x-\frac{2}{3})^2+(\frac{2}{3})^2}$$

$$= \frac{1}{9}\cdot\frac{3}{2}\cdot\tan^{-1}\frac{(x-\frac{2}{3})}{2/3} = \frac{1}{6}\tan^{-1}\frac{3x-2}{2}.$$

***Ex. 8.** *Evaluate* $\int_0^1 \{1/(1-x+x^2)\}dx$.

Sol. Dr. $= 1-x+x^2 = (x-\frac{1}{2})^2 + (\sqrt{3}/2)^2$.

$$\therefore \quad \int_0^1 \frac{dx}{1-x+x^2} = \int_0^1 \frac{dx}{(x-\frac{1}{2})^2+(\sqrt{3}/2)^2}$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{3}/2}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^1 = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right],$$

$[\because \tan^{-1}(-x) = -\tan^{-1}x]$

$$= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{2\pi}{3\sqrt{3}}.$$

Ex. 9 (a). *Integrate* $1/(2x^2+x+1)$.

Sol. We have

$$\int \frac{dx}{2x^2+x+1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2+\frac{1}{2}-\frac{1}{16}}$$

$$= \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2+\frac{7}{16}} = \frac{1}{2}\int \frac{dx}{(x+\frac{1}{4})^2+(\sqrt{7}/4)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{7}}\tan^{-1}\frac{x+\frac{1}{4}}{(\sqrt{7})/4} = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{4x+1}{\sqrt{7}}\right).$$

Ex. 9. (b) *Evaluate* $\int \frac{dx}{2x^2+3x+5}$. **(Meerut 1988 P)**

Ans. $\frac{2}{\sqrt{31}}\cdot\tan^{-1}\left[\frac{4x+3}{\sqrt{31}}\right]$.

Ex. 10. *Evaluate* $\int_{-\infty}^{\infty} \{1/(x^2+2x+2)\}\,dx$.

Sol. We have $\int_{-\infty}^{\infty}\frac{dx}{x^2+2x+2} = \int_{-\infty}^{\infty}\frac{dx}{(x+1)^2+1}$

$$= \tan^{-1}\left[\frac{(x+1)}{1}\right]_{-\infty}^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(-\infty) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$$

$$= \pi/2 + \pi/2 = \pi.$$

Ex. 11. *Integrate* $1/(2x^2+x-1)$.

Sol. We have $\int \frac{dx}{(2x^2+x-1)} = \frac{1}{2}\int\frac{dx}{(x^2+\frac{1}{2}x-\frac{1}{2})}$

$$= \frac{1}{2}\int\frac{dx}{(x+\frac{1}{4})^2-\frac{1}{2}-\frac{1}{16}} = \frac{1}{2}\int\frac{dx}{(x+\frac{1}{4})^2-\frac{9}{16}} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{3}\log\frac{x+\frac{1}{4}-\frac{3}{4}}{x+\frac{1}{4}+\frac{3}{4}}$$

$$= \frac{1}{3}\log\left\{\frac{2x-1}{2(x+1)}\right\} = \frac{1}{3}\log\frac{2x-1}{x+1} - \frac{1}{3}\log 2$$

$= \frac{1}{3}\log\{(2x-1)/(x-1)\}$, omitting the constant term $-\frac{1}{3}\log 2$ which may be added to constant of integration.

Ex. 12 (a). *Integrate* $1/(x^2-3x+2)$.

Sol. We have

$$\int\frac{dx}{(x^2-3x+2)} = \int\frac{dx}{(x-\frac{3}{2})^2+2-\frac{9}{4}} = \int\frac{dx}{(x-\frac{3}{2})^2-\frac{1}{4}}$$

$$= \int\frac{dx}{(x-\frac{3}{2})^2-(\frac{1}{2})^2} = \frac{1}{2.(\frac{1}{2})}\log\left[\frac{(x-\frac{3}{2})-(\frac{1}{2})}{(x-\frac{3}{2})+(\frac{1}{2})}\right] = \log\left[\frac{(x-2)}{(x-1)}\right].$$

Ex. 12. (b) *Evaluate* $\int\frac{x}{x^4+x^2+1}dx$. **(Meerut 1984, 85 S)**

Sol. Put $x^2 = t$, so that $2x\,dx = dt$.

Then the given integral $I = \frac{1}{2}\int\frac{dt}{t^2+t+1}$

$$=\tfrac{1}{2}\int \frac{dt}{(t+\frac{1}{2})^2+(3/4)}=\tfrac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{t+\frac{1}{2}}{\sqrt{3}/2}\right]$$

$$=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right).$$

§ 3. Integration of $(px+q)/(ax^2+bx+c)$.

To integrate such integrals break the given fraction into two fractions such that in one the numerator is the differential coefficient of the denominator, and in the other the numerator is merely a constant. Thus

$$\int \frac{(px+q)\,dx}{ax^2+bx+c}=\int \frac{(p/2a)(2ax+b)+q-\{(pb)/2a\}}{ax^2+bx+c}\,dx$$

$$=\frac{p}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx+\int \frac{q-\{(pb)/(2a)\}}{ax^2+bx+c}\,dx$$

$$=\frac{p}{2a}\log(ax^2+bx+c)+\int \frac{q-\{(pb)/(2a)\}}{ax^2+bx+c}\,dx.$$

The 2nd integral can now be easily evaluated.

Ex. 13. *Integrate* $x/(x^2+x-6)$. **(Meerut 1972)**

Sol. Let $I=\int \frac{x}{x^2+x-6}\,dx.$

Here $\frac{d}{dx}$ (denominator) $=\frac{d}{dx}(x^2+x-6)=2x+1.$

$$\therefore\quad I=\int \frac{\frac{1}{2}(2x+1)-\frac{1}{2}}{x^2+x-6}\,dx=\tfrac{1}{2}\int \frac{2x+1}{x^2+x-6}\,dx-\tfrac{1}{2}\int \frac{dx}{x^2+x-6}$$

$$=\tfrac{1}{2}\log(x^2+x-6)-\tfrac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2-6-\frac{1}{4}}$$

$$=\tfrac{1}{2}\log(x^2+x-6)-\tfrac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2-\frac{25}{4}}$$

$$=\tfrac{1}{2}\log(x^2+x-6)-\tfrac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2-(\frac{5}{2})^2}$$

$$=\tfrac{1}{2}\log(x^2+x-6)-\tfrac{1}{2}\cdot\frac{1}{2\cdot(\frac{5}{2})}\log\frac{x+\frac{1}{2}-\frac{5}{2}}{x+\frac{1}{2}+\frac{5}{2}}$$

$$=\tfrac{1}{2}\log(x^2+x-6)-\frac{1}{10}\log\frac{x-2}{x+3}.$$

Ex. 14. *Integrate* $3x/(x^2-x-2)$.

Sol. Here $(d/dx)(x^2-x-2)=2x-1.$

$$\therefore\quad \int \frac{3x\,dx}{\ldots}=\int \frac{\frac{3}{2}(2x-1)+\frac{3}{2}}{\ldots}\,dx$$ **(Note)**

$$\therefore \int \frac{3x\,dx}{(x^2-x-2)} = \int \frac{\frac{3}{2}(2x-1)+\frac{3}{2}}{(x^2-x-2)}\,dx \qquad \textbf{(Note)}$$

$$= \frac{3}{2}\int \frac{(2x-1)\,dx}{(x^2-x-2)} + \frac{3}{2}\int \frac{dx}{(x^2-x-2)}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2-(\frac{3}{2})^2}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\cdot\frac{1}{2\left(\frac{3}{2}\right)}\log\left\{\frac{(x-\frac{1}{2})-(\frac{3}{2})}{(x-\frac{1}{2})+(\frac{3}{2})}\right\}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{1}{2}\log\{(x-2)/(x+1)\}.$$

Ex. 15. *Integrate* $(3x+1)/(2x^2-2x+3)$.

Sol. Here $(d/dx)(2x^2-2x+3) = 4x-2$.

$$\therefore \quad I = \int \frac{3x+1}{2x^2-2x+3}\,dx = \int \frac{\frac{3}{4}(4x-2)+1+\frac{3}{2}}{(2x^2-2x+3)}\,dx \qquad \textbf{(Note)}$$

$$= \frac{3}{4}\int \frac{4x-2}{2x^2-2x+3}\,dx + \frac{5}{2}\int \frac{1}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{2.2}\int \frac{dx}{x^2-x+(3/2)}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{(x-\frac{1}{2})^2+(3/2)-(1/4)}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\int \frac{dx}{(x-\frac{1}{2})^2+(\sqrt{5}/2)^2}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{4}\frac{1}{(\sqrt{5}/2)}\tan^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{5}/2)}\right\}$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right).$$

Ex. 16. *Integrate* $(5x-2)/(1+2x+3x^2)$.

Sol. Here $(d/dx)(1+2x+3x^2) = 6x+2$.

$$\therefore \quad I = \int \frac{5x-2}{1+2x+3x^2}\,dx = \int \frac{\frac{5}{6}(6x+2)-2-\frac{5}{3}}{3x^2+2x+1}\,dx$$

$$= \int \frac{\frac{5}{6}(6x+2)-\frac{11}{3}}{3x^2+2x+1}\,dx$$

$$= \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1}\,dx - \frac{11}{3}\int \frac{1}{3x^2+2x+1}\,dx$$

$$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}.$$

Now $\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}} = \int \frac{dx}{(x+\frac{1}{3})^2+\frac{1}{3}-\frac{1}{9}} = \int \frac{dx}{(x+\frac{1}{3})^2+\frac{2}{9}}$

$= \int \frac{9\,dx}{(3x+1)^2+(\sqrt{2})^2} = \frac{9}{3\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right).$

Hence the required integral

$$I = \tfrac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\cdot\frac{9}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)$$

$$= \tfrac{5}{6}\log(3x^2+2x+1) - \tfrac{11}{6}\sqrt{2}\tan^{-1}\{(3x+1)/\sqrt{2}\}.$$

Ex. 17. *Integrate* $x^2/(x^4+x^2+1)$. **(Meerut 1984 S)**

Sol. Here we shall give a very interesting method for evaluating the integral of a fraction in which the denominator is of degree 4 and numerator is either of degree 2 or is constant. Moreover the odd powers of x occur neither in the numerator nor in the denominator.

Let $I = \int \frac{x^2}{x^4+x^2+1}\,dx,$ [Note the form of the integrand]

$= \int \frac{1}{x^2+1+(1/x^2)}\,dx,$ dividing the numerator and the denominator both by x^2.

Now the denominator $x^2+1+\frac{1}{x^2}$ can be written either as $\left(x-\frac{1}{x}\right)^2+3$ or as $\left(x+\frac{1}{x}\right)^2-1$. The diff. coeff of $x-\frac{1}{x}$ is $1+\frac{1}{x^2}$ and that of $x+\frac{1}{x}$ is $1-\frac{1}{x^2}$. So we write

$$I = \tfrac{1}{2}\int \frac{(1+1/x^2)+(1-1/x^2)}{x^2+1+(1/x^2)}\,dx \qquad \textbf{(Note)}$$

$$= \tfrac{1}{2}\int \frac{(1+1/x^2)\,dx}{(x-1/x)^2+3} + \tfrac{1}{2}\int \frac{(1-1/x^2)\,dx}{(x+1/x)^2-1}.$$

In the first integral put $x-\frac{1}{x}=t$ so that $\left(1+\frac{1}{x^2}\right)dx = dt$, and in the second integral put $x+\frac{1}{x}=z$ so that $\left(1-\frac{1}{x^2}\right)dx = dz$.

$$\therefore \quad I = \tfrac{1}{2}\int \frac{dt}{t^2+(\sqrt{3})^2} + \tfrac{1}{2}\int \frac{dz}{z^2-1}$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + \tfrac{1}{2}\frac{1}{2\times 1}\log\frac{z-1}{z+1}$$

$$= \frac{1}{2\sqrt{3}}\tan^{-1}\left\{\frac{(x-1/x)}{\sqrt{3}}\right\} + \tfrac{1}{4}\log\frac{(x+1/x)-1}{(x+1/x)+1}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1}\left\{\frac{x^2-1}{(\sqrt{3})\,x}\right\} + \frac{1}{4}\log\frac{x^2-x+1}{x^2+x+1}.$$

Ex. 18. *Evaluate* $\displaystyle\int_0^1 \frac{(x-3)\,dx}{x^2+2x-4}$. **(Lucknow 1980)**

Sol. We have $\displaystyle\int_0^1 \frac{(x-3)\,dx}{x^2+2x-4} = \int_0^1 \frac{3-x}{4-2x-x^2}\,dx$

$$= \int_0^1 \frac{\{4-\frac{1}{2}(2x+2)\}}{(4-2x-x^2)}\,dx, \qquad \left[\because \frac{d}{dx}(\text{denominator}) = -(2x+2)\right]$$

$$= \int_0^1 \frac{4\,dx}{(4-2x-x^2)} - \frac{1}{2}\int_0^1 \frac{(2x+2)\,dx}{(4-2x-x^2)}$$

$$= \int_0^1 \frac{4\,dx}{4-(x^2+2x)} + \frac{1}{2}\left[\log(4-2x-x^2)\right]_0^1$$

$$= \int_0^1 \frac{4\,dx}{(\sqrt{5})^2-(x+1)^2} + \frac{1}{2}[\log 1 - \log 4]$$

$$= 4 \cdot \frac{1}{2\sqrt{5}}\left[\log\frac{\sqrt{5}+(x+1)}{\sqrt{5}-(x+1)}\right]_0^1 - \frac{1}{2}\log(2^2)$$

$$= \frac{2}{\sqrt{5}}\left[\log\left\{\frac{\sqrt{5}+2}{\sqrt{5}-2}\right\} - \log\left\{\frac{\sqrt{5}+1}{\sqrt{5}-1}\right\}\right] - \log 2$$

$$= \frac{2}{\sqrt{5}}\left[\log\left\{\frac{(\sqrt{5}+2)(\sqrt{5}-1)}{(\sqrt{5}-2)(\sqrt{5}+1)}\right\}\right] - \log 2$$

$$= \frac{2}{\sqrt{5}}\left[\log\left\{\frac{3+\sqrt{5}}{3-\sqrt{5}}\right\}\right] - \log 2$$

$$= \frac{2}{\sqrt{5}}\left[\log\left\{\frac{(3+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}\right\}\right] - \log 2$$

$$= \frac{2}{\sqrt{5}}\left[\log\left(\frac{3+\sqrt{5}}{2}\right)^2\right] - \log 2 = \frac{4\sqrt{5}}{5}\left[\log\left(\frac{3+\sqrt{5}}{2}\right)\right] - \log 2.$$

Ex. 19. *Evaluate* $\displaystyle\int_0^1 \frac{x^3\,dx}{(x^2+1)(x^2+7x+12)}$.

Sol. We have

$$\int_0^1 \frac{x^3\,dx}{(x^2+1)(x^2+7x+12)} = \int_0^1 \frac{x^3\,dx}{(x^2+1)(x+4)(x+3)}.$$

Let $\displaystyle\frac{x^3}{(x^2+1)(x+4)(x+3)} \equiv \frac{A}{(x+4)} + \frac{B}{(x+3)} + \frac{Cx+D}{(x^2+1)}$. ...(1)

We have $\displaystyle A = \frac{(-4)^3}{\{(-4)^2+1\}(-4+3)} = \frac{-64}{-17} = \frac{64}{17}$,

and $B = \dfrac{(-3)^3}{\ldots} = \dfrac{-27}{\ldots}$.

and $$B=\frac{(-3)^3}{\{(-3)^2+1\}\{-3+4\}}=\frac{-27}{10}.$$

Now putting $x=0$ on both sides of (1), we get $0=\frac{A}{4}+\frac{B}{3}+D.$

Therefore $D=-\frac{A}{4}-\frac{B}{3}=-\frac{16}{17}+\frac{9}{10}=-\frac{7}{170}.$

Again multiplying both sides of (1) by x and taking limit when $x\to\infty$, we get

$1=A+B+C$ so that $C=1-A-B=1-\frac{64}{17}+\frac{27}{10}=\frac{-11}{170}.$

The students must note carefully the method we have adopted to find the values of C and D. This method is very helpful to find the values of two constants when the values of all other constants have been found by some other methods.

$\therefore$ the required integral

$$I=\int_0^1\left\{\frac{64}{17(x+4)}-\frac{27}{10(x+3)}-\frac{1}{170}\left(\frac{11x+7}{x^2+1}\right)\right\}dx$$

$$=\frac{64}{17}\{\log(x+4)\}_0^1-\frac{27}{10}\{\log(x+3)\}_0^1-\frac{1}{170}\int_0^1\frac{11x+7}{(x^2+1)}dx$$

$$=\frac{64}{17}\log\frac{5}{4}-\frac{27}{10}\log\frac{4}{3}-\frac{1}{170}\int_0^1\frac{11x+7}{(x^2+1)}dx.$$

Now $$\int_0^1\frac{11x+7}{x^2+1}dx=11\int_0^1\frac{x}{x^2+1}dx+7\int_0^1\frac{dx}{x^2+1}$$

$$=\frac{11}{2}\int_0^1\frac{2x}{x^2+1}dx+7\int_0^1\frac{dx}{x^2+1}$$

$$=\frac{11}{2}\left[\log(x^2+1)\right]_0^1+7\left[\tan^{-1}x\right]_0^1=\frac{11}{2}\log 2+\frac{7}{4}\pi.$$

$\therefore$ required integral $=\frac{64}{17}\log\frac{5}{4}-\frac{27}{10}\log\frac{4}{3}-\frac{11}{340}\log 2-\frac{7\pi}{680}.$

Ex. 20. *Integrate* $1/(x^3-1)$.

Sol. We have $\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}.$

Let $$\frac{1}{(x-1)(x^2+x+1)}\equiv\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} \qquad \text{...(1)}$$

Then $A=\frac{1}{1^2+1+1}=\frac{1}{3}.$

Now putting $x=0$ on both sides of (1), we get $1=-A+C$ so that $C=-1+A=-1+\frac{1}{3}=-(2/3).$

Also multiplying both sides of (1) by x and taking limit when

Also multiplying both sides of (1) by x and taking limit when $x \to \infty$, we get $0 = A + B$ so that $B = -A = -(1/3)$.

[Note that $\lim_{x \to \infty} \dfrac{x}{(x-1)(x^2+x+1)}$

$$= \lim_{x \to \infty} \frac{1}{x^2\{1-(1/x)\}\{1+(1/x)+(1/x^2)\}} = 0,$$

$$\lim_{x \to \infty} \frac{Ax}{x-1} = \lim_{x \to \infty} \frac{A}{\{1-(1/x)\}} = A, \lim_{x \to \infty} \frac{Bx^2+Cx}{x^2+x+1}$$

$$= \lim_{x \to \infty} \frac{x^2[B+(C/x)]}{x^2[1+(1/x)+(1/x^2)]} = \lim_{x \to \infty} \frac{B+(C/x)}{1+(1/x)+(1/x^2)} = B\Bigg].$$

$$\therefore \int \frac{1}{x^3-1}\,dx = \int \left[\frac{1}{3(x-1)} - \frac{1}{3}\frac{x+2}{x^2+x+1}\right] dx$$

$$= \tfrac{1}{3}\int \frac{1}{x-1}\,dx - \tfrac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$= \tfrac{1}{3}\int \frac{1}{x-1}\,dx - \tfrac{1}{3}\int \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{x^2+x+1}\,dx$$

$$= \tfrac{1}{3}\log(x-1) - \tfrac{1}{6}\int \frac{2x+1}{x^2+x+1}\,dx - \tfrac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+(\sqrt{3}/2)^2}$$

$$= \tfrac{1}{3}\log(x-1) - \tfrac{1}{6}\log(x^2+x+1) - \frac{1}{2.(\sqrt{3}/2)}\tan^{-1}\left[\frac{(x+\frac{1}{2})}{\sqrt{3}/2}\right]$$

$$= \tfrac{1}{3}\log(x-1) - \tfrac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

Ex. 21. *Integrate* $(1-3x)/\{(1+x^2)(1+x)\}$.

Sol. Let $\dfrac{(1-3x)}{(1+x^2)(1+x)} = \dfrac{A}{(1+x)} + \dfrac{Bx+C}{(x^2+1)}$.

$\therefore \quad 1-3x \equiv A(x^2+1)+(Bx+C)(1+x)$.

Now putting $x=-1$, we get $A=2; x=0, C=-1$ and $x=1$, $B=-2$.

$$\therefore \int \frac{(1-3x)\,dx}{(1+x^2)(1+x)} = \int \left[\frac{2}{(1+x)} - \frac{(2x+1)}{(x^2+1)}\right] dx$$

$$= 2\log(1+x) - \int \frac{2x}{(x^2+1)}\,dx - \int \frac{1}{(x^2+1)}\,dx$$

$$= 2\log(1+x) - \log(x^2+1) - \tan^{-1}x$$

$$= \log(1+x)^2 - \log(x^2+1) - \tan^{-1}x$$

$$= \log\{(1+x)^2/(1+x^2)\} - \tan^{-1}x.$$

§ 3. Integration of $1/(x^2+k)^n$.

§ 3. Integration of $1/(x^2+k)^n$.

This function is integrated by the method of successive reduction. To obtain a reduction formula, we integrate $1/(x^2+k)^{n-1}$ by parts, taking unity as the second function.

Thus $$\int \frac{1}{(x^2+k)^{n-1}} \cdot 1\,dx = \frac{x}{(x^2+k)^{n-1}} - \int x \cdot \frac{-(n-1)}{(x^2+k)^n} \cdot 2x\,dx$$

or $$I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\int \frac{(x^2+k)-k}{(x^2+k)^n}\,dx,$$

$[\because x^2 = (x^2+k)-k]$

or $$I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)\left[\int \frac{dx}{(x^2+k)^{n-1}} - k\int \frac{dx}{(x^2+k)^n}\right]$$

or $$I_{n-1} = \frac{x}{(x^2+k)^{n-1}} + 2(n-1)I_{n-1} - 2k(n-1)I_n.$$

$$\therefore \quad 2k(n-1)I_n = \frac{x}{(x^2+k)^{n-1}} + \{2(n-1)-1\}I_{n-1}$$

or $$2k(n-1)I_n = \frac{x}{(x^2+k)^{n-1}} + (2n-3)I_{n-1}.$$ Hence

$$\int \frac{dx}{(x^2+k)^n} = \frac{x}{2k(n-1)(x^2+k)^{n-1}} + \frac{(2n-3)}{2k(n-1)}\int \frac{dx}{(x^2+k)^{n-1}}.$$

Above is the reduction formula for $\int [1/(x^2+k)^n]\,dx$. By repeated application of this formula the integral shall reduce to that of $\dfrac{1}{(x^2+k)}$ which is $\dfrac{1}{\sqrt{k}}\tan^{-1}\left(\dfrac{x}{\sqrt{k}}\right)$.

***Ex. 22.** *Integrate* $1/(x^2+3)^3$. **(Lucknow 1984)**

Sol. By the reduction formula of § 3, we get

$$\int \frac{dx}{(x^2+3)^3} = \frac{x}{12(x^2+3)^2} + \frac{3}{12}\int \frac{dx}{(x^2+3)^2},$$

[Putting $n=3$ and $k=3$ in the formula]

$$= \frac{x}{12(x^2+3)^2} + \frac{1}{4}\left\{\frac{x}{6(x^2+3)} + \frac{1}{6}\int \frac{dx}{(x^2+3)}\right\},$$

on applying the same reduction formula by putting $n=2$ and $k=3$

$$= \frac{x}{12(x^2+3)^2} + \frac{x}{24(x^2+3)} + \frac{1}{24\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}.$$

Note. Before applying the reduction formula of § 3, the students must first derive it.

Ex. 23. *Evaluate* $\int [1/(x^2+4)^3]\, dx$.

Sol. By the reduction formula of § 3, (putting $n = 3, k = 4$) we get

$$I = \int \frac{dx}{(x^2+4)^3} = \frac{x}{(6-2) \cdot 4\,(x^2+4)^2} + \frac{(2.3-3)}{2.(3-1)\,4} \int \frac{dx}{(x^2+4)^2}$$

$$= \frac{x}{16\,(x^2+4)^2} + \frac{3}{16} \int \frac{dx}{(x^2+4)^2}.$$

Again applying the same reduction formula (putting $n = 2, k = 4$), we get

$$I = \frac{x}{16\,(x^2+4)^2} + \frac{3}{16} \cdot \frac{x}{2.(2-1) \cdot 4\,(x^2+4)}$$

$$+ \frac{3}{16} \cdot \frac{(2.2-3)}{2.(2-1)} \int \frac{dx}{(x^2+4)}$$

$$= \frac{x}{16\,(x^2+4)^2} + \frac{3x}{128\,(x^2+4)} + \frac{3}{32} \int \frac{dx}{x^2+4}$$

$$= \frac{x}{16\,(x^2+4)^2} + \frac{3x}{128\,(x^2+4)} + \frac{3}{32.2} \tan^{-1}\left(\frac{x}{2}\right)$$

$$= \frac{x}{16\,(x^2+4)^2} + \frac{3x}{128\,(x^2+4)} + \frac{3}{64} \tan^{-1}\left(\frac{x}{2}\right).$$

§ 4. To integrate $(px+q)/(ax^2+bx+c)^n$.

The above integral can be evaluated by breaking it into a sum of two integrals such that in the first integral the numerator is the differential coefficient of (ax^2+bx+c) and in the second integral there is no term of x in the numerator. For this we have to find numbers L and M such that $(px+q) = L\,(2ax+b) + M$.

Thus we write

$$px + q = \frac{p}{2a}(2ax+b) + q - \frac{pb}{2a}.$$

Ex. 24. *Integrate* $(x+2)/(2x^2+4x+3)^2$. **(Meerut 1986)**

Sol. Here $(d/dx)\,(2x^2+4x+3) = 4x+4$.

$$\therefore \int \frac{(x+2)\,dx}{(2x^2+4x+3)^2} = \int \frac{\frac{1}{4}(4x+4)+2-1}{(2x^2+4x+3)^2}\,dx$$

$$= \tfrac{1}{4} \int \frac{(4x+4)\,dx}{(2x^2+4x+3)^2} + \tfrac{1}{4} \int \frac{(2-1)\,dx}{(x^2+2x+\frac{3}{2})^2}$$

$$= \tfrac{1}{4} \int (2x^2+4x+3)^{-2}\,(4x+4)\,dx + \tfrac{1}{4} \int \frac{dx}{(x^2+2x+\frac{3}{2})^2}$$

$$= -\frac{1}{4(2x^2+4x+3)} + \tfrac{1}{4}\int \frac{dx}{\{(x+1)^2+\frac{1}{2}\}^2}.$$

Now put $x+1=t$ and then applying the reduction formula of § 3, we get

$$I = -\frac{1}{4(2x^2+4x+3)} + \tfrac{1}{4}\left[\frac{(x+1)}{(x+1)^2+\frac{1}{2}} + \sqrt{2}\tan^{-1}\{\sqrt{2}(x+1)\}\right].$$

Ex. 25. *Integrate* $(2x+3)/(x^2+2x+3)^2$.

Sol. Here $(d/dx)(x^2+2x+3) = 2x+2$.

$$\therefore \quad I = \int \frac{(2x+3)\,dx}{(x^2+2x+3)^2} = \int \frac{(2x+2+1)\,dx}{(x^2+2x+3)^2}$$

$$= \int \frac{(2x+2)\,dx}{(x^2+2x+3)^2} + \int \frac{dx}{(x^2+2x+3)^2}$$

$$= -\frac{1}{(x^2+2x+3)} + \int \frac{dx}{\{(x+1)^2+2\}^2}. \qquad \text{...(1)}$$

Now let $I_1 = \int \frac{dx}{[(x+1)^2+2]^2}$.

Put $x+1=\sqrt{2}\tan t$, so that $dx = \sqrt{2}\sec^2 t\,dt$.

$$\therefore \quad I_1 = \int \frac{\sqrt{2}\sec^2 t\,dt}{(2\tan^2 t+2)^2} = \frac{\sqrt{2}}{4}\int \cos^2 t\,dt = \frac{\sqrt{2}}{4}\int \tfrac{1}{2}(1+\cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{8}[t+\tfrac{1}{2}\sin 2t] = \frac{\sqrt{2}}{8}[t+\sin t\cos t].$$

Now $\tan t = \dfrac{x+1}{\sqrt{2}}$.

Therefore $\sin t = \dfrac{x+1}{\sqrt{\{(x+1)^2+2\}}} = \dfrac{x+1}{\sqrt{(x^2+2x+3)}}$,

and $\qquad \cos t = \dfrac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$;

also $\qquad t = \tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)$.

Hence $\quad I_1 = \dfrac{\sqrt{2}}{8}\tan^{-1}\left(\dfrac{x+1}{\sqrt{2}}\right)$

$$+ \frac{\sqrt{2}}{8}\cdot\frac{x+1}{\sqrt{(x^2+2x+3)}}\cdot\frac{\sqrt{2}}{\sqrt{(x^2+2x+3)}}$$

$$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{4}\frac{x+1}{(x^2+2x+3)}.$$

$$\therefore \quad I = -\frac{1}{x^2+2x+3} + \frac{1}{4}\frac{x+1}{x^2+2x+3} + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right),$$

from (1)

$$= \frac{x+1-4}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$= \frac{x-3}{4(x^2+2x+3)} + \frac{\sqrt{2}}{8} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right).$$

Note. Ex. 24 can also be solved by the method we used in solving Ex. 25.

Ex. 26. *Evaluate* $\displaystyle\int_0^\infty \frac{(3+3x+x^2)}{(2+2x+x^2)^2}\,dx.$

Sol. Here $(d/dx)(x+2x+x^2) = 2+2x$.

Now $3+3x+x^2 = (2+2x+x^2)+(x+1)$

$$= (2+2x+x^2) + \tfrac{1}{2}(2x+2). \qquad \textbf{(Note)}$$

Thus $\displaystyle\int_0^\infty \frac{(3+3x+x^2)\,dx}{(2+2x+x^2)^2}$

$$= \int_0^\infty \frac{(2+2x+x^2)\,dx}{(2+2x+x^2)^2} + \frac{1}{2}\int_0^\infty \frac{(2+2x)\,dx}{(2+2x+x^2)^2}$$

$$= \int_0^\infty \frac{dx}{(x^2+2x+2)} + \tfrac{1}{2}\int_0^\infty \frac{(2+2x)\,dx}{(x^2+2x+2)^2}$$

$$= \int_0^\infty \frac{dx}{\{(x+1)^2+1\}} + \tfrac{1}{2}\left[-\frac{1}{(x^2+2x+2)}\right]_0^\infty$$

$$= \left[\tan^{-1}(x+1)\right]_0^\infty + \tfrac{1}{2}[0-(-\tfrac{1}{2})]$$

$$= (\tan^{-1}\infty - \tan^{-1}1) + \tfrac{1}{4} = \tfrac{1}{2}\pi - \tfrac{1}{4}\pi + \tfrac{1}{4} = \tfrac{1}{4}(\pi+1).$$

Integration of rational functions by substitution.

Ex. 27. *Integrate* $1/(1+3e^x+2e^{2x})$. **(Magadh 1977)**

Sol. Multiplying numerator and denominator by e^{-x}, we have

$$I = \int \frac{dx}{1+3e^x+2e^{2x}} = \int \frac{e^{-x}\,dx}{e^{-x}+3+2e^x}.$$

Now put $y = e^{-x}$ so that $dy = -e^{-x}\,dx$.

Thus

$$I = \int \frac{-dy}{y+3+2(1/y)} = \int \frac{-y\,dy}{y^2+3y+2} = \int \frac{-\frac{1}{2}(2y+3)+\frac{3}{2}}{y^2+3y+2}\,dy$$

$$= -\tfrac{1}{2}\int \frac{(2y+3)\,dy}{y^2+3y+2} + \frac{3}{2}\int \frac{dy}{(y+\frac{3}{2})^2-(\frac{1}{2})^2}$$

$$= -\tfrac{1}{2}\log(y^2+3y+2) + \tfrac{3}{2}\frac{1}{2\cdot(\frac{1}{2})}\log\left|\frac{(y+\frac{3}{2})-\frac{1}{2}}{(y+\frac{3}{2})+\frac{1}{2}}\right|$$

$$= -\tfrac{1}{2}\log(e^{-2x} + 3e^{-x} + 2) + \tfrac{3}{2}\log\left\{\frac{y+1}{y+2}\right\}$$

$$= -\tfrac{1}{2}\log\left\{\frac{1+3e^x+2e^{2x}}{e^{2x}}\right\} + \tfrac{3}{2}\log\left\{\frac{e^{-x}+1}{e^{-x}+2}\right\}$$

$$= -\tfrac{1}{2}\log(1+3e^x+2e^{2x}) + \tfrac{1}{2}\log(e^{2x}) + \tfrac{3}{2}\log\left(\frac{1+e^x}{1+2e^x}\right)$$

$$= -\tfrac{1}{2}\log\{(1+e^x)(1+2e^x)\} + \tfrac{1}{2}(2x)$$
$$+ \tfrac{3}{2}\log(1+e^x) - \tfrac{3}{2}\log(1+2e^x)$$

$$= \log(1+e^x) - 2\log(1+2e^x) + x.$$

Ex. 28 (a). *Integrate* $(x^2+1)/(x^4+1)$.

(Lucknow 1980; Magadh 75; Rajasthan 75; Meerut 88 P)

Sol. Let $I = \int \frac{x^2+1}{x^4+1}\,dx$.

Here both the numerator and the denominator do not contain odd powers of x. Also the numerator is of degree 2 and the denominator is of degree 4. So dividing the numerator and the denominator by x^2, we get

$$I = \int \frac{1+(1/x^2)}{x^2+(1/x^2)}\,dx$$

$$= \int \frac{1+(1/x^2)}{[x-(1/x)]^2+2}\,dx, \quad \left[\text{Note that } \frac{d}{dx}\left\{x-\frac{1}{x}\right\} = 1+\frac{1}{x^2}\right].$$

Now put $x-(1/x) = t$ so that $\{1+(1/x^2)\}\,dx = dt$.

$$\therefore \quad I = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left\{\frac{x-(1/x)}{\sqrt{2}}\right\}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right).$$

Ex. 28 (b). *Integrate* $\frac{x^2-1}{x^4+1}$. **(Meerut 1983S, 88, 91)**

Sol. Let $I = \int \frac{x^2-1}{x^4+1}\,dx$

$$= \int \frac{1-(1/x^2)}{x^2+(1/x^2)}\,dx,$$ dividing the numerator and the denominator by x^2

$$= \int \frac{1-(1/x^2)}{[x+(1/x)]^2-2}\,dx, \quad \left[\text{Note that } \frac{d}{dx}\left\{x+\frac{1}{x}\right\} = 1-\frac{1}{x^2}\right]$$

Now put $x+(1/x) = t$ so that $\{1-(1/x^2)\}\,dx = dt$.

$$\therefore \quad I=\int \frac{dt}{t^2-2}=\int \frac{dt}{t^2-(\sqrt{2})^2}$$
$$=\frac{1}{2\sqrt{2}}\log\frac{t-\sqrt{2}}{t+\sqrt{2}}=\frac{1}{2\sqrt{2}}\log\frac{x+(1/x)-\sqrt{2}}{x+(1/x)+\sqrt{2}}$$
$$=\frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}.$$

***Ex. 29 (a).** *Integrate* $(x^2-1)/(x^4+x^2+1)$. **(Meerut 1982S, 89 S)**

Sol. We have $I=\int \frac{x^2-1}{x^4+x^2+1}\,dx,$

[Note the form of the integrand]

$=\int \frac{1-(1/x^2)}{x^2+1+(1/x^2)}\,dx$, dividing the numerator and the denominator by x^2

$$=\int \frac{1-(1/x^2)}{\{x+(1/x)\}^2-1}\,dx.$$

$\left[\text{Note that } \frac{d}{dx}\{x+(1/x)\}=1-(1/x^2)\right]$

Now put $x+(1/x)=t$, so that $\{1-(1/x^2)\}\,dx=dt$.

$$\therefore \quad I=\int \frac{dt}{t^2-1}=\tfrac{1}{2}\log\frac{t-1}{t+1}=\tfrac{1}{2}\log\frac{x+(1/x)-1}{x+(1/x)+1}$$
$$=\tfrac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}.$$

Ex. 29 (b). *Integrate* $\int \frac{x^2+1}{x^4-x^2+1}$. **(Meerut 1983)**

Sol. Let $I=\int \frac{x^2+1}{x^4-x^2+1}\,dx$

$=\int \frac{1+(1/x^2)}{x^2-1+(1/x^2)}\,dx$, dividing the numerator and the denominator by x^2

$$=\int \frac{1+(1/x^2)}{[x-(1/x)]^2+1}\,dx.$$

Now put $x-(1/x)=t$, so that $\{1+(1/x^2)\}dx=dt$.

$$\therefore \quad I=\int \frac{dt}{t^2+1}=\tan^{-1}t=\tan^{-1}\{x-(1/x)\}.$$

Ex. 30. *Integrate* $x^2/(x^4+a^4)$.

Sol. We have $I=\int \frac{x^2}{x^4+a^4}\,dx=\int \frac{1}{\{x^2+(a^4/x^2)\}}\,dx,$

dividing the numerator and the denominator by x^2

$$=\tfrac{1}{2}\int \frac{\{1-(a^2/x^2)\}+\{1+(a^2/x^2)\}}{x^2+(a^4/x^2)}\,dx$$

$$=\tfrac{1}{2}\int \left[\frac{1-(a^2/x^2)}{\{x+(a^2/x)\}^2-2a^2}+\frac{1+(a^2/x^2)}{\{x-(a^2/x)\}^2+2a^2}\right]dx.$$

In the first integral, put

$$\{x+(a^2/x)\}=t \text{ so that } \{1-(a^2/x^2)\}\,dx=dt,$$

and in the second integral put

$$x-(a^2/x)=z \text{ so that } \{1+(a^2/x^2)\}\,dx=dz.$$

$$\therefore\quad I=\tfrac{1}{2}\left[\int \frac{dt}{t^2-2a^2}+\int \frac{dz}{z^2+2a^2}\right]$$

$$=\tfrac{1}{2}\left[\frac{1}{2a\sqrt{2}}\log\frac{t-a\sqrt{2}}{t+a\sqrt{2}}+\frac{1}{a\sqrt{2}}\tan^{-1}\frac{z}{a\sqrt{2}}\right]$$

$$=\frac{1}{4a\sqrt{2}}\log\left[\frac{\{x+(a^2/x)-a\sqrt{2}\}}{\{x+(a^2/x)+a\sqrt{2}\}}\right]+\frac{1}{2a\sqrt{2}}\tan^{-1}\frac{\{x-(a^2/x)\}}{a\sqrt{2}}$$

$$=\frac{\sqrt{2}}{8a}\log\left\{\frac{x^2-\sqrt{2}ax+a^2}{x^2+\sqrt{2}ax+a^2}\right\}+\frac{\sqrt{2}}{4a}\tan^{-1}\left\{\frac{x^2-a^2}{\sqrt{2}ax}\right\}.$$

***Ex. 31 (a).** *Integrate* $1/(x^4+8x^2+9)$.

Sol. We have $I=\displaystyle\int \frac{1}{x^4+8x^2+9}\,dx$

$$=\int \frac{1/x^2}{x^2+8+(9/x^2)}\,dx,$$ dividing the numerator and the denominator by x^2

$$=\frac{1}{6}\int \frac{\{1+(3/x^2)\}-\{1-(3/x^2)\}}{x^2+8+(9/x^2)}\,dx$$

$$=\frac{1}{6}\int \frac{1+(3/x^2)}{\{x-(3/x)\}^2+14}\,dx-\frac{1}{6}\int \frac{1-(3/x^2)}{\{x+(3/x)\}^2+2}\,dx \quad \text{(Note)}$$

In the first integral put $x-(3/x)=t$ so that $\{1+(3/x^2)\}\,dx=dt$, and in the second integral put $x+(3/x)=z$ so that $\{1-(3/x^2)\}\,dx=dz$.

Then $I=\dfrac{1}{6}\displaystyle\int \frac{dt}{t^2+14}-\frac{1}{6}\int \frac{dz}{z^2+2}$

$$=\frac{1}{6\sqrt{(14)}}\tan^{-1}\frac{t}{\sqrt{(14)}}-\frac{1}{6\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}$$

$$=\frac{1}{6\sqrt{(14)}}\tan^{-1}\frac{x-(3/x)}{\sqrt{(14)}}-\frac{1}{6\sqrt{2}}\tan^{-1}\frac{x+(3/x)}{\sqrt{2}}$$

$$= \frac{1}{6\sqrt{(14)}} \tan^{-1} \frac{x^2 - 3}{x\sqrt{(14)}} - \frac{1}{6\sqrt{2}} \tan^{-1} \frac{x^2 + 3}{x\sqrt{2}}.$$

Ex. 31 (b). *Evaluate* $\int \frac{dx}{x^4 + x^2 + 1}$. **(Meerut 1982, 84; Delhi 81)**

Sol. Let $I = \int \frac{dx}{x^4 + x^2 + 1}$

$= \int \frac{1/x^2}{x^2 + 1 + 1/x^2} dx$, dividing the numerator and the denominator by x^2

$$= \tfrac{1}{2} \int \frac{(1 + 1/x^2) - (1 - 1/x^2)}{x^2 + 1 + 1/x^2} dx$$

$$= \tfrac{1}{2} \int \frac{(1 + 1/x^2)}{(x - 1/x)^2 + 3} dx - \tfrac{1}{2} \int \frac{(1 - 1/x^2)}{(x + 1/x)^2 - 1} dx.$$

In the first integral put $x - 1/x = t$ so that $(1 + 1/x^2)\, dx = dt$, and in the second integral put $x + 1/x = z$ so that $(1 - 1/x^2)\, dx = dz$.

Then $I = \tfrac{1}{2} \int \frac{dt}{t^2 + 3} - \tfrac{1}{2} \int \frac{dz}{z^2 - 1}$

$$= \tfrac{1}{2} \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} - \tfrac{1}{2} \cdot \tfrac{1}{2} \log \frac{z - 1}{z + 1}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x - 1/x}{\sqrt{3}} \right) - \tfrac{1}{4} \log \frac{x + (1/x) - 1}{x + (1/x) + 1}$$

$$= \frac{1}{2\sqrt{3}} \tan^{-1} \frac{x^2 - 1}{\sqrt{3}x} - \tfrac{1}{4} \log \frac{x^2 - x + 1}{x^2 + x + 1}.$$

Ex. 32. *Integrate* $1/\{x (x^2 + 1)^3\}$. **(Meerut 1988S)**

Sol. We have $I = \int \frac{1}{x (x^2 + 1)^3} dx = \int \frac{x\, dx}{x^2 (x^2 + 1)^3}$.

Now put $x^2 + 1 = t$ so that $2x\, dx = dt$.

$\therefore \quad I = \tfrac{1}{2} \int \frac{dt}{(t - 1)\, t^3} = \tfrac{1}{2} \int \frac{1}{t^3} \cdot \left[\frac{1}{-1 + t} \right] dt.$

Now to resolve $\frac{1}{(t - 1)\, t^3}$ into partial fractions divide 1 by $(-1 + t)$ till t^3 is a factor of the remainder. Then

$$I = \tfrac{1}{2} \int \frac{1}{t^3} \cdot \left[-1 - t - t^2 + \frac{t^3}{t - 1} \right] dt$$

$$= -\tfrac{1}{2} \int \left[\frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{t} - \frac{1}{(t - 1)} \right] dt$$

$$= -\tfrac{1}{2} \left\{ -\frac{1}{2t^2} - \frac{1}{t} + \log t - \log (t - 1) \right\}$$

$$= 1/(4t^2) + 1/(2t) + \tfrac{1}{2} \log \{(t-1)/t\}$$

$$= \frac{1}{4(x^2+1)^2} + \frac{1}{2(x^2+1)} + \tfrac{1}{2} \log \frac{x^2}{x^2+1}.$$

Ex. 33. *Integrate* $1/\{x(x^5+1)\}$.

Sol. We have $I = \int \frac{1}{x(x^5+1)} dx = \int \frac{x^{5-1}}{x^5(x^5+1)} dx.$ **(Note)**

Now put $x^5 = t$ so that $5x^{5-1} dx = dt$.

$\therefore$ required integral $I = \frac{1}{5} \int \frac{dt}{t(t+1)} = \frac{1}{5} \int \left[\frac{1}{t} - \frac{1}{(t+1)}\right] dt$

$$= (1/5) [\log t - \log (t+1)] = (1/5) . \log \{t/(t+1)\}$$

$$= (1/5) . \log \{x^5/(x^5+1)\}, \qquad [\because t = x^5].$$

***Ex. 34.** *Integrate* $1/(x^4 + a^4)$. **(Meerut 1980)**

Sol. We have $I = \int \frac{dx}{x^4+a^4} = \int \frac{1/x^2}{x^2 + (a^4/x^2)} dx,$

dividing the numerator and the denominator by x^2

$$= \frac{1}{2a^2} \int \frac{\{1 + (a^2/x^2)\} - \{1 - (a^2/x^2)\}}{x^2 + (a^4/x^2)} dx \qquad \textbf{(Note)}$$

$$= \frac{1}{2a^2} \int \frac{1 + (a^2/x^2)}{\{x - (a^2/x)\}^2 + 2a^2} dx - \frac{1}{2a^2} \int \frac{\{1 - (a^2/x^2)\} dx}{\{x + (a^2/x)\}^2 - 2a^2}.$$

In the first integral put

$x - (a^2/x) = t$ so that $\{1 + (a^2/x^2)\} dx = dt$,

and in the second integral put

$x + (a^2/x) = z$ so that $\{1 - (a^2/x^2)\} dx = dz$. Then

$$I = \frac{1}{2a^2} \int \frac{dt}{t^2 + 2a^2} - \frac{1}{2a^2} \int \frac{dz}{z^2 - 2a^2}$$

$$= \frac{1}{2a^2 . a\sqrt{2}} \tan^{-1} \left(\frac{t}{a\sqrt{2}}\right) - \frac{1}{2a^2 . 2a\sqrt{2}} \log \frac{z - a\sqrt{2}}{z + a\sqrt{2}}$$

$$= \frac{1}{2a^3\sqrt{2}} \tan^{-1} \frac{x - (a^2/x)}{a\sqrt{2}} - \frac{1}{4a^3\sqrt{2}} \log \frac{x + (a^2/x) - a\sqrt{2}}{x + (a^2/x) + a\sqrt{2}}$$

$$= \frac{1}{2a^3\sqrt{2}} \tan^{-1} \left(\frac{x^2 - a^2}{xa\sqrt{2}}\right) - \frac{1}{4a^3\sqrt{2}} \log \frac{x^2 - \sqrt{2}ax + a^2}{x^2 + \sqrt{2}ax + a^2}.$$

****Ex. 35.** *Integrate* $1/(x^4 + 1)$.

(Meerut 1983, 85, 88 P; Agra 71; Jiwaji 75; Delhi 75)

Sol. Proceed as in Ex. 34. Here $a = 1$. The answer is

$$\int \frac{dx}{x^4+1} = \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x^2-1}{x\sqrt{2}} - \frac{1}{4\sqrt{2}} \log \frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}.$$

Ex. 36. *Integrate* $1/\{x(x^n + 1)\}$.. **(Meerut 1980, 91S)**

Sol. We have $I = \int \frac{dx}{\dots} = \int \frac{x^{n-1} dx}{\dots},$

Sol. We have $I = \int \frac{dx}{x(x^n+1)} = \int \frac{x^{n-1}\,dx}{x^n(x^n+1)}$,

multiplying the numerator and the denominator by x^{n-1}.

Now put $x^n = t$ so that $nx^{n-1}\,dx = dt$. Then

$$I = \frac{1}{n}\int \frac{dt}{t(t+1)} = \frac{1}{n}\int \left[\frac{1}{t} - \frac{1}{t+1}\right] dt = \frac{1}{n}[\log t - \log(t+1)]$$

$$= \frac{1}{n}\log\frac{t}{t+1} = \frac{1}{n}\log\left(\frac{x^n}{x^n+1}\right).$$

Ex. 37. *Evaluate* $\int_1^2 \frac{dx}{x(1+2x)^2}$.

Sol. Let $I = \int_1^2 \frac{dx}{x(1+2x)^2} = \int_1^2 \frac{2\,dx}{2x(1+2x)^2}$.

Put $1 + 2x = y$ so that $2\,dx = dy$.

Also when $x = 1, y = 3$ and when $x = 2, y = 5$.

$$\therefore \quad I = \int_3^5 \frac{dy}{y^2(y-1)} = \int_3^5 \frac{1}{y^2}\cdot\left\{\frac{1}{-1+y}\right\} dy.$$

Now to resolve $\frac{1}{y^2(y-1)}$ into partial fractions we divide the numerator 1 by the denominator $-1+y$ till y^2 is a factor of the remainder.

Thus $\frac{1}{-1+y} = -1 - y + \frac{y^2}{-1+y}$. Hence

$$I = \int_3^5 \frac{1}{y^2}\left[-1 - y + \frac{y^2}{y-1}\right] dy = \int_3^5 \left[-\frac{1}{y^2} - \frac{1}{y} + \frac{1}{y-1}\right] dy$$

$$= \left[\frac{1}{y}\right]_3^5 - \left[\log y\right]_3^5 + \left[\log(y-1)\right]_3^5$$

$$= \left(\frac{1}{5} - \frac{1}{3}\right) - (\log 5 - \log 3) + (\log 4 - \log 2)$$

$$= -\frac{2}{15} - \log 5 + \log 3 + \log 4 - \log 2$$

$$= -\frac{2}{15} + \log\left(\frac{3\times 4}{5\times 2}\right) = -\frac{2}{15} + \log\left(\frac{6}{5}\right).$$

****Ex. 38.** *Evaluate* $\int \frac{\sin x}{\sin 4x}\,dx$. **(Meerut 1980, 85, 88, 90P, 91S)**

Sol. We have $I = \int \frac{\sin x}{\sin 4x}\,dx = \int \frac{\sin x\,dx}{2\sin 2x\cos 2x}$

$$= \int \frac{\sin x\,dx}{4\sin x\cos x\cos 2x} = \frac{1}{4}\int \frac{dx}{\cos x\cos 2x} = \frac{1}{4}\int \frac{\cos x\,dx}{\cos 2x\cos^2 x}$$

$$= \frac{1}{4}\int \frac{\cos x\,dx}{(1-\sin^2 x)(1-2\sin^2 x)}.$$ **(Note)**

Now put $\sin x = t$ so that $\cos x\,dx = dt$.

$$=\frac{1}{4}\int \frac{\cos x\, dx}{(1-\sin^2 x)(1-2\sin^2 x)}. \quad \textbf{(Note)}$$

Now put $\sin x = t$ so that $\cos x\, dx = dt$.

$$\therefore \quad I=\frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}=\frac{1}{4}\int \frac{dt}{(t^2-1)(2t^2-1)}$$

$$=\frac{1}{4}\int \left[\frac{1}{(t^2-1)}-\frac{2}{(2t^2-1)}\right] dt, \text{ resolving into partial fractions}$$

$$=\frac{1}{4}\int \frac{dt}{(t^2-1)}-\frac{1}{4}\int \frac{dt}{t^2-(\frac{1}{2})}$$

$$=\frac{1}{4}\cdot\frac{1}{2}\log\left(\frac{t-1}{t+1}\right)-\frac{1}{4}\cdot\frac{1}{2.(1/\sqrt{2})}\log\left[\frac{t-(1/\sqrt{2})}{t+(1/\sqrt{2})}\right] \quad \textbf{(Note)}$$

$$=\frac{1}{8}\log\left(\frac{t-1}{t+1}\right)-\frac{1}{4\sqrt{2}}\log\left(\frac{t\sqrt{2}-1}{t\sqrt{2}+1}\right)$$

$$=\frac{1}{8}\log\left(\frac{\sin x-1}{\sin x+1}\right)-\frac{1}{4\sqrt{2}}\log\left(\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right).$$

Ex. 39. *Evaluate* $\int_0^{\pi/4} \sqrt{(\cot\theta)}\, d\theta.$ **(Meerut 1982 S; Delhi 74)**

Sol. Let $I=\int_0^{\pi/4}\sqrt{(\cot\theta)}\, d\theta.$

Put $\cot\theta = z^2$ so that $-\operatorname{cosec}^2\theta\, d\theta = 2z\, dz$

or $$d\theta=\frac{-2z\,dz}{\operatorname{cosec}^2\theta}=\frac{-2z\,dz}{1+\cot^2\theta}=\frac{-2z\,dz}{1+z^4}.$$

Also when $\theta=0, z=\infty$ and when $\theta=\pi/4, z=1$.

$$\therefore \quad I=\int_\infty^1 \frac{z(-2z)\,dz}{z^4+1}=\int_1^\infty \frac{2z^2}{z^4+1}dz,$$

$$\left[\because \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=\int_1^\infty \frac{2}{z^2+(1/z^2)}dz,$$

dividing the numerator and the denominator by z^2

$$=\int_1^\infty \frac{\{1+(1/z^2)\}+\{1-(1/z^2)\}}{z^2+(1/z^2)}dz$$

$$=\int_1^\infty \frac{\{1+(1/z^2)\}\,dz}{\{z-(1/z)\}^2+2}+\int_1^\infty \frac{\{1-(1/z^2)\}\,dz}{\{z+(1/z)\}^2-2}.$$

In the first integral put $z-(1/z)=t$ so that $[1+(1/z^2)]\,dz=dt$. The corresponding limits for t are 0 to ∞. In the second integral put $z+(1/z)=u$ so that $[1-(1/z^2)]\,dz=du$. The limits for u are from 2 to ∞. Hence

$$I = \int_0^\infty \frac{dt}{t^2+2} + \int_2^\infty \frac{du}{u^2-2}$$

$$= \frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{t}{\sqrt{2}}\right]_0^\infty + \frac{1}{2\sqrt{2}}\left[\log\frac{u-\sqrt{2}}{u+\sqrt{2}}\right]_2^\infty$$

$$= \frac{1}{\sqrt{2}}[\tan^{-1}\infty - \tan^{-1}0]$$

$$+ \frac{1}{2\sqrt{2}}\left[\lim_{u\to\infty}\log\frac{u-\sqrt{2}}{u+\sqrt{2}} - \log\frac{2-\sqrt{2}}{2+\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{2}-0\right) + \frac{1}{2\sqrt{2}}\left[\lim_{u\to\infty}\log\left\{\frac{1-(\sqrt{2}/u)}{1+(\sqrt{2}/u)}\right\}\right.$$

$$\left. - \log\left\{\frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}+1)}\right\}\right]$$

$$= \frac{\pi}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\left[\log 1 - \log\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\log\left\{\frac{(\sqrt{2}-1)(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\right\} = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\log(\sqrt{2}-1)^2$$

$$= \frac{\pi}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\log(\sqrt{2}-1) = \frac{\pi\sqrt{2}}{4} - \frac{\sqrt{2}}{2}\log(\sqrt{2}-1).$$

****Ex. 40.** *Integrate* $\int_0^{\pi/4} \sqrt{(\tan\theta)}\, d\theta$.

(Meerut 1981 S, 82 P, 83 S, 84 S, 85 P; Jiwaji 77)

Sol. Let $I = \int_0^{\pi/4} \sqrt{(\tan\theta)}\, d\theta$.

Put $\tan\theta = z^2$ so that $\sec^2\theta\, d\theta = 2z\, dz$

or $$d\theta = \frac{2z}{\sec^2\theta}dz = \frac{2z}{1+\tan^2\theta}dz = \frac{2z}{1+z^4}dz.$$

Also when $\theta = 0$, $z = 0$ and when $\theta = \pi/4$, $z = 1$.

$$\therefore \quad I = \int_0^1 \frac{z\,.\,2z}{1+z^4}dz = \int_0^1 \frac{2z^2\,dz}{z^4+1} = \int_0^1 \frac{2dz}{z^2+(1/z^2)},$$

dividing the numerator and the denominator by z^2

$$= \int_0^1 \frac{\{1+(1/z^2)\}+\{1-(1/z^2)\}}{z^2+(1/z^2)}dz$$

$$= \int_0^1 \frac{\{1+(1/z^2)\}\,dz}{\{z-(1/z)\}^2+2} + \int_0^1 \frac{\{1-(1/z^2)\}\,dz}{\{z+(1/z)\}^2-2}.$$

In the first integral put $z-(1/z)=t$ so that $\{1+(1/z^2)\}\,dz = dt$. The corresponding limits for t are $-\infty$ to 0. In the second integral put $z+(1/z)=u$ so that $\{1-(1/z^2)\}\,dz = du$. The corresponding limits for u are ∞ to 2. Hence

$$I = \int^0 \frac{dt}{\quad} + \int^2 \frac{du}{\quad} = \int^0 \frac{dt}{\quad} - \int^\infty \frac{du}{\quad}$$

$$I=\int_{-\infty}^{0}\frac{dt}{t^2+2}+\int_{\infty}^{2}\frac{du}{u^2-2}=\int_{-\infty}^{0}\frac{dt}{t^2+2}-\int_{2}^{\infty}\frac{du}{u^2-2}$$

$$=\frac{1}{\sqrt2}\left[\tan^{-1}\frac{t}{\sqrt2}\right]_{-\infty}^{0}-\frac{1}{2\sqrt2}\left[\log\frac{u-\sqrt2}{u+\sqrt2}\right]_{2}^{\infty}$$

$$=\frac{1}{\sqrt2}[\tan^{-1}0-\tan^{-1}(-\infty)]-\frac{1}{2\sqrt2}\left[\lim_{u\to\infty}\log\frac{u-\sqrt2}{u+\sqrt2}-\log\frac{2-\sqrt2}{2+\sqrt2}\right]$$

$$=\frac{1}{\sqrt2}\left[0-\left(-\frac{\pi}{2}\right)\right]-\frac{1}{2\sqrt2}\left[\lim_{u\to\infty}\log\frac{u\{1-(\sqrt2/u)\}}{u\{1+(\sqrt2/u)\}}-\log\frac{\sqrt2(\sqrt2-1)}{\sqrt2(\sqrt2+1)}\right]$$

$$=\frac{\pi}{2\sqrt2}-\frac{1}{2\sqrt2}\left[\log 1-\log\left\{\frac{(\sqrt2-1)(\sqrt2-1)}{(\sqrt2+1)(\sqrt2-1)}\right\}\right]$$

$$=\frac{\pi}{2\sqrt2}-\frac{1}{2\sqrt2}[0-\log(\sqrt2-1)^2]$$

$$=\frac{\pi}{2\sqrt2}+\frac{2}{2\sqrt2}\log(\sqrt2-1)=\frac{\pi\sqrt2}{4}+\frac{\sqrt2}{2}\log(\sqrt2-1).$$

***Ex. 41.** *Evaluate* $\displaystyle\int_0^{\pi/2}\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$.

(Lucknow 1982, 79; Delhi 81; Allahabad 73; Gorakh. 70)

Sol. Put $\sin x=y$ so that $\cos x\,dx=dy$.
Also when $x=0$, $y=\sin 0=0$; when $x=\frac12\pi$, $y=\sin\frac12\pi=1$.
Thus the given integral I

$$=\int_0^{\pi/2}\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)}$$

$$=\int_0^1\frac{dy}{(1+y)(2+y)}$$

$$=\int_0^1\left[\frac{1}{1+y}-\frac{1}{2+y}\right]dy,$$

[Resolving the integrand into partial fractions]

$$=\Big[\log(1+y)-\log(2+y)\Big]_0^1$$

$$=[(\log 2-\log 3)-(\log 1-\log 2)]$$

$$=(2\log 2-\log 3)=\log(4/3),\qquad [\because \log 1=0].$$

Ex. 42. *Evaluate* $\displaystyle\int_0^{\pi/2}\frac{\cos x\,dx}{(1+\sin x)(2+\sin x)(3+\sin x)}$.

Sol. Put $\sin x=y$ so that $\cos x\,dx=dy$.

Also when $x = 0$, $y = \sin 0 = 0$; when $x = \frac{1}{2}\pi$, $y = \sin \frac{1}{2}\pi = 1$.

Thus the given integral $I = \int_0^1 \frac{dy}{(1+y)(2+y)(3+y)}$

$$= \int_0^1 \left[\frac{1}{2(1+y)} - \frac{1}{(2+y)} + \frac{1}{2(3+y)}\right] dy, \text{ (by partial fractions)}$$

$$= \left[\tfrac{1}{2}\log(1+y) - \log(2+y) + \tfrac{1}{2}\log(3+y)\right]_0^1$$

$$= (\tfrac{1}{2}\log 2 - \log 3 + \tfrac{1}{2}\log 4) - (\tfrac{1}{2}\log 1 - \log 2 + \tfrac{1}{2}\log 3)$$

$$= \tfrac{1}{2}\log 2 - \tfrac{3}{2}\log 3 + \tfrac{1}{2}\log 2^2 + \log 2, \qquad [\because \log 1 = 0]$$

$$= \tfrac{1}{2}\log 2 - \tfrac{3}{2}\log 3 + (2 \cdot \tfrac{1}{2})\log 2 + \log 2 = \tfrac{5}{2}\log 2 - \tfrac{3}{2}\log 3.$$

***Ex. 43.** *Integrate* $(1 + \sin x)/\{\sin x\,(1 + \cos x)\}$. **(Kanpur 1978)**

Sol. Here $\dfrac{(1+\sin x)}{\sin x\,(1+\cos x)}$

$$= \frac{1}{\sin x\,(1+\cos x)} + \frac{\sin x}{\sin x\,(1+\cos x)}$$

$$= \frac{\sin x}{\sin^2 x\,(1+\cos x)} + \frac{1}{(1+\cos x)} \qquad \textbf{(Note)}$$

$$= \frac{\sin x}{(1-\cos^2 x)(1+\cos x)} + \frac{1}{2\cos^2 \frac{1}{2}x}.$$

$$\therefore \quad I = \int \frac{(1+\sin x)\,dx}{\sin x\,(1+\cos x)}$$

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(1+\cos x)}\,dx + \tfrac{1}{2}\int \sec^2 \frac{x}{2}\,dx.$$

In the first integral put $\cos x = t$ so that $-\sin x\,dx = dt$. Then

$$I = -\int \frac{dt}{(1-t^2)(1+t)} + \tan \frac{x}{2}$$

$$= -\int \frac{dt}{(1-t)(1+t)^2} + \tan \frac{x}{2} \qquad \text{...(1)}$$

Now let $\dfrac{1}{(1-t)(1+t)^2} = \dfrac{A}{1-t} + \dfrac{B}{1+t} + \dfrac{C}{(1+t)^2}$

or $\quad 1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$. ...(2)

Putting $t = 1$ on the both sides of (2), we get $A = \frac{1}{4}$

and putting $t = -1$, we get $C = \frac{1}{2}$.

Also by comparing the coefficients of t^2 on both sides of (2), we get

$$0 = A - B. \quad \therefore \quad B = A = \tfrac{1}{4}.$$

$$\therefore \quad \frac{1}{(1-t)(1+t)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1+t)} + \frac{1}{2(1+t)^2}.$$

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$$\therefore \quad \frac{1}{(1-t)(1+t)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1+t)} + \frac{1}{2(1+t)^2}.$$

Hence the required integral

$$I = -\int \left[\frac{1}{4(1-t)} + \frac{1}{4(t+1)} + \frac{1}{2(1+t)^2}\right] dt + \tan\frac{1}{2}x, \quad \text{[from (1)]}$$

$$= \frac{1}{4}\log(1-t) - \frac{1}{4}\log(1+t) + \frac{1}{2}\cdot\frac{1}{(1+t)} + \tan\frac{1}{2}x$$

$$= -\tfrac{1}{4}\log\{(1+t)/(1-t)\} + \tfrac{1}{2}\{1/(1+t)\} + \tan\tfrac{1}{2}x$$

$$= -\tfrac{1}{4}\log\left[\frac{1+\cos x}{1-\cos x}\right] + \frac{1}{2(1+\cos x)} + \tan\frac{1}{2}x, \quad [\because t = \cos x]$$

$$= -\tfrac{1}{4}\log(\cot^2\tfrac{1}{2}x) + \frac{1}{2.\,2\cos^2\frac{1}{2}x} + \tan\tfrac{1}{2}x$$

$$= -\tfrac{1}{2}\log(\cot\tfrac{1}{2}x) + \tfrac{1}{4}\sec^2\tfrac{1}{2}x + \tan\tfrac{1}{2}x.$$

Ex. 44. *Integrate* $1/\{\sin x\,(3+\cos^2 x)\}$.

Sol. Here $I = \displaystyle\int \frac{dx}{\sin x\,(3+\cos^2 x)} = \int \frac{\sin x\,dx}{\sin^2 x\,(3+\cos^2 x)}$ **(Note)**

$$= \int \frac{\sin x\,dx}{(1-\cos^2 x)(3+\cos^2 x)}.$$

Now put $\cos x = t$ so that $-\sin x\,dx = dt$. Then

$$I = -\int \frac{dt}{(1-t^2)(3+t^2)}.$$

Now to resolve $\dfrac{1}{(1-t^2)(3+t^2)}$ into partial fractions we shall ragard it as a fraction in t^2. So let

$$\frac{1}{(1-t^2)(3+t^2)} \equiv \frac{A}{1-t^2} + \frac{B}{3+t^2}.$$

[Note that $1-t^2$ and $3+t^2$ are factors of first degree in t^2]

Then $A = \dfrac{1}{(3+1)} = \dfrac{1}{4}$, $B = \dfrac{1}{1-(-3)} = \dfrac{1}{4}$.

Hence $I = -\displaystyle\int \left[\frac{1}{4(1-t^2)} + \frac{1}{4(t^2+3)}\right] dt$

$$= -\frac{1}{8}\log\frac{1+t}{1-t} - \frac{1}{4\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}$$

$$= \frac{1}{8}\log\left(\frac{1+t}{1-t}\right)^{-1} - \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$$

$$= \frac{1}{8}\log\frac{1-t}{1+t} - \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$$

$$= \frac{1}{8} \log \frac{1-t}{1+t} - \frac{1}{4\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right)$$

$$= \frac{1}{8} \log \left(\frac{1-\cos x}{1+\cos x}\right) - \frac{1}{4\sqrt{3}} \tan^{-1}\left\{\frac{\cos x}{\sqrt{3}}\right\}$$

$$= \tfrac{1}{8} \log \{\tan^2 (\tfrac{1}{2}x)\} - \{1/(4/\sqrt{3})\} \tan^{-1} \{(\cos x)/\sqrt{3}\}$$

$$= \tfrac{1}{4} \log \tan (\tfrac{1}{2}x) - \{1/(4/\sqrt{3})\} \tan^{-1} \{(\cos x)/\sqrt{3}\}.$$

Ex. 45. *Integrate* $1/(\sin x + \sin 2x)$. **(Meerut 1991 S)**

Sol. We have $I = \int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x + 2 \sin x \cos x}$

$$= \int \frac{dx}{\sin x\,(1 + 2\cos x)}$$

$$= \int \frac{\sin x\, dx}{\sin^2 x\,(1 + 2\cos x)} = \int \frac{\sin x\, dx}{(1 - \cos^2 x)(1 + 2\cos x)}.$$

(Note)

Now putting $\cos x = t$, so that $-\sin x\, dx = dt$, we get

$$I = -\int \frac{dt}{(1-t^2)(1+2t)} = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

$$= -\int \left[\frac{1}{6(1-t)} - \frac{1}{2(1+t)} + \frac{4}{3(1+2t)}\right] dt,$$

[by partial fractions]

$$= \tfrac{1}{6} \log (1-t) + \tfrac{1}{2} \log (1+t) - \tfrac{2}{3} \log (1+2t)$$

$$= \tfrac{1}{6} \log (1 - \cos x) + \tfrac{1}{2} \log (1 + \cos x) - \tfrac{2}{3} \log (1 + 2\cos x).$$

Ex. 46. *Integrate* $\sec x/(1 + \operatorname{cosec} x)$.

Sol. We have $I = \int \frac{\sec x\, dx}{(1 + \operatorname{cosec} x)} = \int \frac{(1/\cos x)\, dx}{[1 + (1/\sin x)]}$

$$= \int \frac{\sin x\, dx}{\cos x\,(1 + \sin x)} = \int \frac{\sin x \cos x\, dx}{(1 - \sin^2 x)(\sin x + 1)}.$$ **(Note)**

Now putting $\sin x = t$ so that $\cos x\, dx = dt$, we get

$$I = \int \frac{t\,.\,dt}{(1-t^2)(t+1)} = \int \frac{t}{(1-t)(1+t)^2}\, dt$$ **(Note)**

$$= \int \left[\frac{1}{4(1-t)} + \frac{1}{4(1+t)} - \frac{1}{2(1+t)^2}\right] dt,$$

(by partial fractions)

$$= -\tfrac{1}{4} \log (1-t) + \tfrac{1}{4} \log (1+t) + \tfrac{1}{2} \{1/(1+t)\}$$

$$= \tfrac{1}{4} \log \{(1+t)/(1-t)\} + \tfrac{1}{2} \{1/(1+t)\}$$

$$= \tfrac{1}{4} \log \{(1 + \sin x)/(1 - \sin x)\} + 1/\{2(1 + \sin x)\}.$$

**Ex. 47. *Evaluate* $\displaystyle\int_0^{\pi/4} \frac{dx}{\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x}$.

Sol. Let $I = \displaystyle\int_0^{\pi/4} \frac{dx}{\cos^4 x + \cos^2 x \sin^2 x + \sin^4 x}$

$= \displaystyle\int_0^{\pi/4} \frac{\sec^4 x\, dx}{1 + \tan^2 x + \tan^4 x}$, dividing the numerator and the denominator by $\cos^4 x$

$= \displaystyle\int_0^{\pi/4} \frac{(1 + \tan^2 x) \sec^2 x\, dx}{\tan^4 x + \tan^2 x + 1}$, $[\because 1 + \tan^2 x = \sec^2 x]$.

Now put $\tan x = t$ so that $\sec^2 x\, dx = dt$.

Also when $x = 0$, $t = \tan 0 = 0$ and when $x = \pi/4$, $t = \tan \frac{1}{4}\pi = 1$.

$\therefore \quad I = \displaystyle\int_0^1 \frac{1 + t^2}{t^4 + t^2 + 1}\, dt$, [Note the form of the integrand]

$= \displaystyle\int_0^1 \frac{[1 + (1/t^2)]\, dt}{t^2 + 1 + (1/t^2)}$, dividing the numerator and the denominator by t^2

$= \displaystyle\int_0^1 \frac{[1 + (1/t^2)]\, dt}{\{t - (1/t)\}^2 + 3}$. **(Note)**

Now put $t - (1/t) = y$ so that $\{1 + (1/t^2)\}\, dt = dy$.

Also when $t = 0$, $y = -\infty$ and when $t = 1$, $y = 0$.

$\therefore I = \displaystyle\int_{-\infty}^0 \frac{dy}{y^2 + 3} = \frac{1}{\sqrt{3}} \left[\tan^{-1} \frac{y}{\sqrt{3}}\right]_{-\infty}^0$

$= (1/\sqrt{3})\, [\tan^{-1} 0 - \tan^{-1} (-\infty)]$

$= \dfrac{1}{\sqrt{3}} [0 - (-\frac{1}{2}\pi)] = \dfrac{\pi}{2\sqrt{3}} = \dfrac{\sqrt{3}}{6}\pi$.

Ex. 48. *Show that*

$$\int_0^\infty \frac{x^2\, dx}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)} = \frac{\pi}{2(a + b)(b + c)(c + a)}.$$

Sol. We have $\dfrac{x^2}{(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)}$

$= \dfrac{y}{(y + a^2)(y + b^2)(y + c^2)}$, where $y = x^2$.

Let $\dfrac{y}{(y + a^2)(y + b^2)(y + c^2)} \equiv \dfrac{A}{y + a^2} + \dfrac{B}{y + b^2} + \dfrac{C}{y + c^2}$

or $y \equiv A(y + b^2)(y + c^2) + B(y + a^2)(y + c^2) + C(y + a^2)(y + b^2)$ ...(1)

Now putting $y = -a^2$ on both sides of (1), we get

$A = -a^2/\{(b^2 - a^2)(c^2 - a^2)\}$.

$$A = -a^2/\{(b^2-a^2)(c^2-a^2)\}.$$

Also putting $y = -b^2$, we get

$$B = -b^2/\{(a^2-b^2)(c^2-b^2)\},$$

and putting $y = -c^2$, we get

$$C = -c^2/\{(a^2-c^2)(b^2-c^2)\}.$$

Now $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}$

$$= \int_0^\infty \frac{A\,dx}{(x^2+a^2)} + \int_0^\infty \frac{B\,dx}{(x^2+b^2)} + \int_0^\infty \frac{C\,dx}{(x^2+c^2)}$$

$$= A\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty + B\left[\frac{1}{b}\tan^{-1}\frac{x}{b}\right]_0^\infty + C\left[\frac{1}{c}\tan^{-1}\frac{x}{c}\right]_0^\infty$$

$$= \frac{A}{a}\cdot\frac{\pi}{2} + \frac{B}{b}\cdot\frac{\pi}{2} + \frac{C}{c}\cdot\frac{\pi}{2} = \frac{\pi}{2}\left(\frac{A}{a}+\frac{B}{b}+\frac{C}{c}\right)$$

$$= \frac{\pi}{2}\left[\frac{-a}{(b^2-a^2)(c^2-a^2)} + \frac{-b}{(a^2-b^2)(c^2-b^2)} + \frac{-c}{(a^2-c^2)(b^2-c^2)}\right]$$

$$= \frac{\pi}{2}\left[\frac{a}{(a^2-b^2)(c^2-a^2)} + \frac{b}{(a^2-b^2)(b^2-c^2)} + \frac{c}{(c^2-a^2)(b^2-c^2)}\right]$$

$$= \frac{\pi}{2}\left[\frac{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}\right]$$

$$= \frac{\pi}{2}\left[\frac{(a-b)(b-c)(c-a)}{(a^2-b^2)(b^2-c^2)(c^2-a^2)}\right] = \frac{\pi}{2(a+b)(b+c)(c+a)}.$$

Ex. 49. *Prove that*

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+ax+a^2)(x^2+bx+b^2)} = \frac{2\pi(a+b)}{(\sqrt{3})\,ab\,(a^2+ab+b^2)}.$$

Sol. Let $\displaystyle\frac{1}{(x^2+ax+a^2)(x^2+bx+b^2)}$

$$\equiv \frac{Ax+B}{(x^2+ax+a^2)} + \frac{Cx+D}{(x^2+bx+b^2)} \qquad \text{...(1)}$$

$\therefore \quad 1 \equiv (Ax+B)(x^2+bx+b^2) + (Cx+D)(x^2+ax+a^2)$ ...(2)

To get the values of A and B, we put $x^2 = -ax - a^2$ on both sides of (1). Thus we get

$1 \equiv (Ax+B)(-ax-a^2+bx+b^2)$

or $\quad 1 \equiv (Ax+B)\{x(b-a)+b^2-a^2\}$

or $\quad 1 \equiv Ax^2(b-a) + Ax(b^2-a^2) + Bx(b-a) + B(b^2-a^2)$

or $\quad 1 \equiv A(b-a)(-ax-a^2) + Ax(b^2-a^2) + Bx(b-a) + B(b^2-a^2)$ ...(3)

Now comparing the coefficients of x and constant terms on both sides of (2), we get

$$0 = -Aa\,(b-a) + A\,(b^2 - a^2) + B\,(b-a) \text{ and}$$
$$1 = -Aa^2\,(b-a) + B\,(b^2 - a^2).$$

The first of these two equations gives

$$-Aa + A\,(b+a) + B = 0 \text{ so that } B = -bA.$$

Putting this value of B in the second of these equations, we get

$$1 = -Aa^2\,(b-a) - bA\,(b^2 - a^2)$$

or $\quad 1 = A\,(b-a)\,(-b^2 - ab - a^2)$

or $\quad 1 = A\,(a-b)\,(a^2 + ab + b^2)$

or $\quad A = 1/(a^3 - b^3)$. Therefore $B = -bA = -b\,(a^3 - b^3)$.

Now comparing the coefficients of x^3 and constant terms on both sides of (2), we get $0 = A + C$, and $1 = Bb^2 + Da^2$. Therefore $C = -A = -1/(a^3 - b^3)$, $D = (1/a^2)\{1 - Bb^2\} = a/(a^3 - b^3)$.

Hence putting the values of A, B, C and D in (1), we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + ax + a^2)\,(x^2 + bx + b^2)}$$

$$= \frac{1}{(a^3 - b^3)} \int_{-\infty}^{\infty} \left[\frac{(x-b)}{(x^2 + ax + a^2)} - \frac{(x-a)}{(x^2 + bx + b^2)} \right] dx$$

$$= \frac{1}{(a^3 - b^3)} \left[\int_{-\infty}^{\infty} \frac{(x-b)\,dx}{(x^2 + ax + a^2)} - \int_{-\infty}^{\infty} \frac{(x-a)\,dx}{(x^2 + bx + b^2)} \right] \qquad \text{...(4)}$$

Now let $I_1 = \displaystyle\int \frac{(x-b)\,dx}{(x^2 + ax + a^2)} = \int \frac{\frac{1}{2}(2x + a) - b - \frac{1}{2}a}{(x^2 + ax + a^2)}\,dx$

(Note)

$$= \tfrac{1}{2} \int \frac{(2x+a)\,dx}{(x^2 + ax + a^2)} - \frac{(a+2b)}{2} \int \frac{dx}{[x + (a/2)]^2 + (3a^2/4)}$$

$$= \tfrac{1}{2} \log\,(x^2 + ax + a^2) - \frac{(a+2b)}{2} \cdot \frac{2}{a\sqrt{3}} \tan^{-1} \left[\frac{x + (a/2)}{a\sqrt{3}/2} \right]$$

$$= \tfrac{1}{2} \log\,(x^2 + ax + a^2) - \frac{(a+2b)}{a\sqrt{3}} \tan^{-1} \left(\frac{2x + a}{a\sqrt{3}} \right).$$

Similarly $I_2 = \displaystyle\int \frac{(x-a)\,dx}{(x^2 + bx + b^2)}$

$$= \tfrac{1}{2} \log\,(x^2 + bx + b^2) - \frac{(b+2a)}{b\sqrt{3}} \tan^{-1} \left(\frac{2x + b}{b\sqrt{3}} \right).$$

$\therefore$ from (4), the required integral

$$= \frac{1}{(a^3 - b^3)} \Big[I_1 - I_2 \Big]_{-\infty}^{\infty}$$

$$= \frac{1}{(a^3 - b^3)} \left[\tfrac{1}{2} \log (x^2 + ax + a^2) - \left(\frac{a + 2b}{a\sqrt{3}} \right) \tan^{-1} \left(\frac{2x + a}{a\sqrt{3}} \right) \right.$$
$$\left. - \tfrac{1}{2} \log (x^2 + bx + b^2) + \left(\frac{b + 2a}{b\sqrt{3}} \right) \tan^{-1} \left(\frac{2x + b}{b\sqrt{3}} \right) \right]_{-\infty}^{\infty}$$

$$= \frac{1}{(a^3 - b^3)} \left[\left\{ \tfrac{1}{2} \log \left(\frac{x^2 + ax + a^2}{x^2 + bx + b^2} \right) \right\}_{-\infty}^{\infty} \right.$$
$$\left. - \left(\frac{a + 2b}{a\sqrt{3}} \right) \left\{ \tan^{-1} \left(\frac{2x + a}{a\sqrt{3}} \right) \right\}_{-\infty}^{\infty} + \left(\frac{b + 2a}{b\sqrt{3}} \right) \left\{ \tan^{-1} \left(\frac{2x + b}{b\sqrt{3}} \right) \right\}_{-\infty}^{\infty} \right]$$

$$= \frac{1}{(a^3 - b^3)} \left[\left\{ \tfrac{1}{2} \log \frac{1 + (a/x) + (a^2/x^2)}{1 + (b/x) + (b^2/x^2)} \right\}_{-\infty}^{\infty} \right.$$
$$- \left(\frac{a + 2b}{a\sqrt{3}} \right) \{ \tan^{-1} \infty - \tan^{-1} (-\infty) \}$$
$$\left. + \left(\frac{b + 2a}{b\sqrt{3}} \right) \{ \tan^{-1} \infty - \tan^{-1} (-\infty) \} \right]$$

$$= \frac{1}{a^3 - b^3} \left[\{ \tfrac{1}{2} \log 1 - \tfrac{1}{2} \log 1 \} - \left(\frac{a + 2b}{a\sqrt{3}} \right) \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \right.$$
$$\left. + \left(\frac{b + 2a}{b\sqrt{3}} \right) \left(\frac{\pi}{2} + \frac{\pi}{2} \right) \right]$$

$$= \frac{\pi}{(a^3 - b^3)} \left[\frac{b + 2a}{b\sqrt{3}} - \frac{a + 2b}{a\sqrt{3}} \right] = \frac{\pi}{(a^3 - b^3)} \left[\frac{ab + 2a^2 - ab - 2b^2}{ab\sqrt{3}} \right]$$

$$= \frac{2\pi (a^2 - b^2)}{(a^3 - b^3)\, ab\sqrt{3}} = \frac{2\pi (a + b)}{\sqrt{3}\, ab\, (a^2 + ab + b^2)}.$$

Ex. 50. *Evaluate* $\int x^2 \log (1 - x^2)\, dx$ *and deduce that*

$$\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \ldots = \frac{8}{2} - \frac{2}{3} \log 2.$$

Sol. Integrating by parts regarding x^2 as the second function, we get

$$\int \{\log (1 - x^2)\} . x^2\, dx$$
$$= \{\log (1 - x^2)\} \cdot \frac{x^3}{3} - \int \frac{-2x . x^3}{(1 - x^2) . 3}\, dx$$
$$= \tfrac{1}{3} x^3 \log (1 - x^2) + \tfrac{2}{3} \int \frac{1 - (1 - x^4)}{(1 - x^2)}\, dx \qquad \textbf{(Note)}$$
$$= \tfrac{1}{3} x^3 \log (1 - x^2) + \tfrac{2}{3} \int \frac{dx}{1 - x^2} - \tfrac{2}{3} \int (1 + x^2)\, dx$$
$$= \tfrac{1}{3} x^3 \log (1 + x) + \tfrac{1}{3} x^3 \log (1 - x) + \tfrac{1}{3} \log \{(1 + x)/(1 - x)\}$$
$$- \tfrac{2}{3} \{x + (x^3/3)\}$$

$$= \tfrac{1}{3}(x^3+1)\log(1+x) + \tfrac{1}{3}(x^3-1)\log(1-x) - \tfrac{2}{3}\{x + (x^3/3)\}.$$

$$\therefore \int_0^1 x^2 \log(1-x^2)\,dx = \Big[\tfrac{1}{3}(x^3+1)\log(1+x) + \tfrac{1}{3}(x^3-1)\log(1-x) - \tfrac{2}{3}\{x+(x^3/3)\}\Big]_0^1$$

$$= \tfrac{2}{3}\log 2 - \tfrac{8}{9}. \qquad \text{...(1)}$$

Note that $\lim_{x\to 1}(x^3-1)\log(1-x)$

$$= \lim_{x\to1}(x^2+x+1)\cdot\lim_{x\to1}(x-1)\log(1-x)$$

$$= 3\cdot\lim_{x\to1}\frac{\log(1-x)}{1/(x-1)}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= 3\cdot\lim_{x\to1}\frac{-1/(1-x)}{-1/(x-1)^2} = 3\cdot\lim_{x\to1}(1-x) = 0.$$

Again $\displaystyle\int_0^1 x^2\log(1-x^2)\,dx = \int_0^1 x^2\left(-x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \dots\right)dx$

$$= -\int_0^1\left(x^4 + \frac{x^6}{2} + \frac{x^8}{3} + \dots\right)dx$$

$$= -\left[\frac{x^5}{5} + \frac{x^7}{2.7} + \frac{x^9}{3.9} + \dots\right]_0^1 = -\left[\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \dots\right] \qquad \text{...(2)}$$

Equating the two values of the given integral from (1) and (2), we get

$$\frac{1}{1.5} + \frac{1}{2.7} + \frac{1}{3.9} + \dots = \frac{8}{9} - \frac{2}{3}\log 2.$$

Ex. 51. *Show that the sum of the infinite series*

$$\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots, (a>0, b>0)$$

can be expressed in the form $\displaystyle\int_0^1 \frac{t^{a-1}\,dt}{1+t^b}$

and hence prove that

$$1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} + \dots = \frac{1}{3}\left[\frac{\pi}{\sqrt{3}} + \log 2\right].$$

Sol. We have $\displaystyle I = \int_0^1 \frac{t^{a-1}\,dt}{1+t^b} = \int_0^1 t^{a-1}(1+t^b)^{-1}\,dt.$

On expanding $(1+t^b)^{-1}$ by binomial theorem, we have

$$I = \int_0^1 t^{a-1}(1 - t^b + t^{2b} - t^{3b} + \dots\infty)\,dt$$

$$= \int_0^1 (t^{a-1} - t^{a+b-1} + t^{a+2b-1} - t^{a+3b-1} + \dots)\, dt$$

$$= \left[\frac{t^a}{a} - \frac{t^{a+b}}{a+b} + \frac{t^{a+2b}}{a+2b} - \frac{t^{a+3b}}{a+3b} + \dots\right]_0^1$$

$$= \left[\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \dots\right] \cdot \text{ This proves the first part.}$$

In this result if we put $a = 1$ and $b = 3$, we have

$$\int_0^1 \frac{dt}{1+t^3} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \dots$$

Now $\displaystyle\int_0^1 \frac{dt}{1+t^3} = \int_0^1 \frac{dt}{(1+t)(1-t+t^2)}$

$$= \int_0^1 \left[\frac{1}{3(1+t)} - \frac{1}{3}\frac{(t-2)}{1-t+t^2}\right] dt,$$

[On resolving into partial fractions]

$$= \left[\tfrac{1}{3}\log(1+t) - \tfrac{1}{6}\log(t^2 - t + 1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2t-1}{\sqrt{3}}\right]_0^1 \quad \textbf{(Note)}$$

$$= \left[\tfrac{1}{3}\log 2 - 0 + \frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}\right] - \left[0 - 0 + \frac{1}{\sqrt{3}}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\log 2 + \frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}},$$

$[\because \tan^{-1}(-x) = -\tan^{-1}x]$

$$= \frac{1}{3}\log 2 + \frac{1}{\sqrt{3}}\left[\frac{\pi}{6} + \frac{\pi}{6}\right] = \frac{1}{3}\left[\log 2 + \frac{\pi}{\sqrt{3}}\right] \cdot \qquad \textbf{Proved.}$$

Ex. 52. *Evaluate* $\displaystyle\int_0^1 \frac{x^2+1}{x^4+x^2+1}\,dx$, *and deduce that*

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \dots = \frac{\pi}{2\sqrt{3}}.$$

Sol. Let $I = \displaystyle\int \frac{x^2+1}{x^4+x^2+1}\,dx = \int \frac{1+(1/x^2)}{x^2+1+(1/x^2)}\,dx.$

Now putting $x - (1/x) = z$ so that $\{1 + (1/x^2)\}\,dx = dz$

and $x^2 + 1 + (1/x^2) = \{x - (1/x)\}^2 + 3 = z^2 + 3$, we get

$$I = \int \frac{dz}{z^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\{x - (1/x)\}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2-1}{x\sqrt{3}}.$$

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$$\therefore \int_0^1 \frac{x^2+1}{x^4+x^2+1} = \frac{1}{\sqrt{3}}\left[\tan^{-1}\frac{x^2-1}{x\sqrt{3}}\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1} 0 - \tan^{-1}(-\infty)\right] = \frac{1}{\sqrt{3}}\left[0 - \left(-\tfrac{1}{2}\pi\right)\right] = \frac{\pi}{2\sqrt{3}} \quad \text{...(1)}$$

Again $\displaystyle\int_0^1 \frac{1+x^2}{1+x^2+x^4}\,dx = \int_0^1 \frac{(1-x^4)\,dx}{(1+x^2+x^4)(1-x^2)}$ **(Note)**

$$= \int_0^1 \frac{1-x^4}{1-x^6}\,dx = \int_0^1 (1-x^4)(1-x^6)^{-1}\,dx$$

$$= \int_0^1 (1-x^4)(1+x^6+x^{12}+\dots)\,dx$$

$$= \int_0^1 (1-x^4+x^6-x^{10}+x^{12}-\dots)\,dx$$

$$= \left[x - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \dots\right]_0^1$$

$$= 1 - \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{11} + \tfrac{1}{13} - \dots \quad \text{...(2)}$$

Comparing (1) and (2), we get

$$1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \dots = \frac{\pi}{2\sqrt{3}}.$$

□