LINEAR ALGEBRA

CSE - 2017

1(a). Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

→ Characteristic equation of A is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0 \Rightarrow 6 - 3\lambda - 2\lambda + \lambda^2 - 2 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 4 = 0 \Rightarrow (\lambda - 1)(\lambda - 4) = 0$

$\therefore \lambda = 1, 4$. Since both eigen values of A are distinct, the matrix A is diagonalizable.

Eigen vector of A corresponding to eigen value $\lambda = 1$

$(A - 1I)X = 0 \Rightarrow \left\{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} X = 0$

$\rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore x + 2y = 0 \Rightarrow x = -2y$

$\therefore X = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\therefore$ Eigen vector corr. to eigen value $\lambda = 1$ is $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Eigen vector of A corresponding to eigen value $\lambda = 4$

$(A - 4I)X = 0 \Rightarrow \left\{\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}\right\} X = 0$

$\rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 + \frac{R_1}{2} \Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore -2x + 2y = 0 \Rightarrow x = y$

$\therefore X = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vector corr. to $\lambda = 4$ is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $P = [X_1 \ X_2] = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$.

Then $P^{-1}AP = D$ where $D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.

1(b). Show that similar matrices have the same characteristic polynomial.

$\longrightarrow$ Let A and B be any two similar matrices. Then $\exists$ a non-singular matrix P such that $B = P^{-1}AP$.

Then $|B-\lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - \lambda P^{-1}P| = |P^{-1}AP - P^{-1}\lambda I P|$

$= |P^{-1}(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P| = \frac{1}{|P|}|A-\lambda I||P|$ $\left[|A^{-1}| = \frac{1}{|A|}\right]$

$\Rightarrow |B-\lambda I| = |A-\lambda I|$

$\therefore$ The characteristic equation of A and B are the same.

$\therefore$ similar matrices have the same characteristic polynomial.

2(d). Suppose U & W are distinct four dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of subspace $U \cap W$.

$\longrightarrow$ Since $\dim U = 4$, $\dim W = 4 \Rightarrow \dim U\cap W \leq 4$ —— ① [Since U & W are distinct]

Since $U, W \subseteq U+W \subseteq V$, then

$\dim U + \dim W - \dim U\cap W = \dim U+W$

$\Rightarrow \quad 4 + 4 - \dim U\cap W = \dim(U+W)$

$\Rightarrow \quad \dim(U\cap W) = 8 - \dim(U+W)$

Now $\dim U \leq \dim(U+W) \leq \dim V$

$\Rightarrow \quad 4 \leq \dim(U+W) \leq 6$

$\Rightarrow \quad 8-4 \geq 8 - \dim(U+W) \geq 8-6$

$\Rightarrow \quad 4 \geq \dim U\cap W \geq 2$

Since $\dim U\cap W \neq 4$ from ①, therefore $2 \leq \dim U\cap W < 4$.

$\therefore$ Possible dimensions of $U\cap W$ = 2 and 3.

3(a) Consider the matrix mapping $A: \mathbb{R}^4 \to \mathbb{R}^3$ where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and dimension of the image of A & those of Kernel of A

→ ~~Standard basis of $\mathbb{R}^4$ = {~~

(i) Image of A: Converting the given matrix into echelon form.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \text{echelon form. } (1)$$

∴ There are only 2 non-zero rows in echelon form,

∴ $\rho(A) = 2 \Rightarrow \dim I(A) = 2$. where $I(A) \equiv$ Image of A.

Also Basis of $I(A)$ is $\{(1,2,3,1), (0,1,2,-3)\}$.

(ii) Kernel of A:

$N(A) = \{(x_1, x_2, x_3, x_4) \,/\, A[(x_1, x_2, x_3, x_4)] = 0\}$

Let $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ which is a homogeneous system of linear equations.

from (1), we can replace the matrix A with its echelon form

$$\therefore \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\to x_2 + 2x_3 - 3x_4 = 0$ & $x_1 + 2x_2 + 3x_3 + x_4 = 0$

$x_2 = -2x_3 + 3x_4$ & $x_1 = -2x_2 - 3x_3 - x_4$

$\Rightarrow x_1 = -2(-2x_3 + 3x_4) - 3x_3 - x_4$

$\Rightarrow x_1 = x_3 - 7x_4$.

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 7x_4 \\ -2x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

∴ Basis of $N(A) = \{(1,-2,1,0), (-7,3,0,1)\}$

$\dim N(A) = 2$

3(b) Prove that distinct non-zero eigenvectors of a matrix are Linearly independent.

→ Let $X_1, X_2, \ldots X_n$ be the ~~eig~~ n-distinct eigen vectors of an $n \times n$ matrix A corresponding to the eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$.

Then, $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$, - - - - - - $AX_n = \lambda_n X_n$. —— ①

Let us assume that the set of n-distinct eigen vectors $X_1, X_2, \ldots X_n$ is Linearly Dependent. Then, we can find 'a' no. 'r' such that $X_1, X_2, \ldots X_r$ are L.I and $X_1, X_2, \ldots X_r, X_{r+1}$ are L.D. eigen vectors.

Since $X_1, X_2, \ldots X_r X_{r+1}$ are L.D. eigen vectors, we have,

$a_1 X_1 + a_2 X_2 + \cdot\cdot + a_r X_r + a_{r+1} X_{r+1} = 0$ where $a_1, a_2, \ldots a_n \in F$ and $a_1, a_2, \ldots a_n$ are not all zeroes —— ②

Premultiplying both sides of ② with A; we have

$a_1 (AX_1) + a_2 (AX_2) + \ldots + a_r (AX_r) + a_{r+1} (AX_{r+1}) = 0$

$\Rightarrow a_1 \lambda_1 X_1 + a_2 \lambda_2 X_2 + \ldots + a_r \lambda_r X_r + a_{r+1} \lambda_{r+1} X_{r+1} = 0$ —— ③

③ − $\lambda_{r+1} \times$ ②

$a_1(\lambda_1 - \lambda_{r+1}) X_1 + a_2(\lambda_2 - \lambda_{r+1}) X_2 + \cdots + a_r(\lambda_r - \lambda_{r+1}) X_r + a_{r+1}(\lambda_{r+1} - \lambda_{r+1}) = 0$

$\rightarrow a_1(\lambda_1 - \lambda_{r+1}) X_1 + a_2(\lambda_2 - \lambda_{r+2}) X_2 + \cdots + a_r (\lambda_r - \lambda_{r+1}) X_r = 0$

Since all the eigen values $X_1, X_2, \ldots X_r$ are distinct, then all the eigen values cannot be equal.

Also, we know that $X_1, X_2, \ldots X_r$ are L.I.

$\therefore a_1 = a_2 = \cdots = a_r = 0$.

Putting in ② $a_{r+1} X_{r+1} = 0 \Rightarrow a_{r+1} = 0$ [since $X_{r+1} \neq 0$]

∴ Our assumption that all the scalars $a_1, a_2, \ldots a_r, a_{r+1}$ are not zeroes is wrong.

∴ The vectors $X_1, X_2, \ldots, X_n$ are L.I. vectors.

4(b). Consider the following system of equations in x, y, z:

$x+2y+2z=1, \quad x+ay+3z=3, \quad x+11y+az=b$

(i) For which values of a does the system have a unique solⁿ?

(ii) For which pair of values (a,b) does the system have more than one solution?

→ The given system can be written as $\begin{bmatrix} 1 & 2 & 2 \\ 1 & a & 3 \\ 1 & 11 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ b \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & a & 3 \\ 1 & 11 & a \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B \begin{bmatrix} 1 \\ 3 \\ b \end{bmatrix}$.

(i) For unique solution: $|A| \neq 0 \Rightarrow \begin{vmatrix} 1 & 2 & 2 \\ 1 & a & 3 \\ 1 & 11 & a \end{vmatrix} \neq 0$

$\rightarrow 1[a^2-33] + 2[3-a] + 2[11-a] \neq 0$

$\rightarrow a^2-4a-5 \neq 0 \Rightarrow (a+1)(a-5) \neq 0$

∴ for values of a other than 5 & -1, the system has a unique solution.

(ii) (For infinitely many solutions) or More than one solution:

The Augmented matrix $(A|B)$ has rank equal to that of A and lesser than the number of unknowns.

$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 1 & a & 3 & 3 \\ 1 & 11 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 3 \\ 0 & 9 & a-2 & b-3 \end{array}\right]$ $\begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & a-2 & 1 & 3 \\ 0 & 0 & -(a-2)^2+9 & 27-(a-2)(b-3) \end{array}\right]$ $R_3 \to 9R_2 - (a-2)R_3$

For infinite solⁿ: $9-(a-2)^2 = 0$ and $27-(a-2)(b-3)=0$

$\rightarrow a-2 = \pm 3$

$\rightarrow a = 2 \pm 3$

$\Rightarrow a = 5, -1$

$a=5$

$27-(5-2)(b-3)=0$

$27-3b+9=0$

$b=12$

$a=-1$

$27+3(b-3)=0$

$9+b-3=0$

$b=-6$

∴ The required pairs of values of a & b for which the system has more thane one solution is $(5,12)$ and $(-1,-6)$