(15 Marks)

Q124. Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}, n = 1, 2, 3$ .... show that the sequence x_1, x_2, x_3 ...is convergent.

(Year 2017)

(10 Marks)

Q125. Find the Supremum and the infimum of $\frac{x}{\sin x}$ on the interval $\left(0, \frac{\pi}{2}\right)$

(Year 2017)

(10 Marks)

Q126. Let $f(t) = \int_0^t f(x)dx$ where $[x]$ denote the largest integer less than or equal to x

(i) Determine all the real numbers t at which f is differentiable.

(ii) Determine all the real numbers t at which f is continuous but not differentiable.

(Year 2017)

(15 Marks)

Q127. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_n\,(n)$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100.

(Year 2017)

(20 Marks)

124. Let $x_1 = 2$, $x_{n+1} = \sqrt{x_n + 20}$

If $n = 1$, $x_2 = \sqrt{2+20} = \sqrt{22} > 2$

$\Rightarrow x_2 > x_1$

Let us assume that $x_{k+1} > x_k$

Then $x_{k+1} + 20 > x_k + 20$

$\Rightarrow \sqrt{x_{k+1} + 20} > \sqrt{x_k + 20}$

$\Rightarrow x_{k+2} > x_{k+1}$

$\therefore$ By mathemical induction, we can say that

$x_{n+1} > x_n$

$\Rightarrow \langle x_n \rangle$ is an increasing sequenie

Also, $x_1 = 2 < 5$

$x_2 = \sqrt{22} < 5$

Let us assume $x_k < 5$

then $x_k + 20 < 5 + 20$

$\Rightarrow \sqrt{x_k + 20} < \sqrt{25}$

$\Rightarrow x_{k+1} < 5$

$\therefore$ By Mathematical Induction,

$x_n < 5 \quad \forall n \in N$.

Since $\langle x_n \rangle$ is an increasing sequenie and is bounded above,

$\Rightarrow \langle x_n \rangle$ converges to its least upper bound

$\Rightarrow \langle x_n \rangle$ is convergent.

125. $f(x) = \dfrac{x}{\sin x}$ $\qquad x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$$

$$f'(x) = \frac{\tan x - x}{\sin^2 x \cos x}$$

Since $\sin^2 x \cos x > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

Let $g(x) = \tan x - x$; then change in sign of $g(x)$ implies a similar change for $f(x)$.

$g'(x) = \sec^2 x - 1 > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $g(x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow g(x) > g(0)$

$\Rightarrow \tan x - x > 0$

$\Rightarrow \dfrac{\tan x - x}{\sin^2 x . \cos x} > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > 0$ $\qquad \forall x \in \left(0, \frac{\pi}{2}\right)$

$\therefore f$ is increasing on $\left(0, \frac{\pi}{2}\right)$

$\therefore$ Infimum $= f(0) = \lim\limits_{x \to 0} \dfrac{x}{\sin x} = 1$

Supremum $= f\left(\frac{\pi}{2}\right) = \dfrac{\pi/2}{\sin \pi/2} = \dfrac{\pi}{2}$.

Let $\quad f(t) = \int_0^t [x]\, dx,$

where $[x]$ denotes the largest integer less than or equal to x.

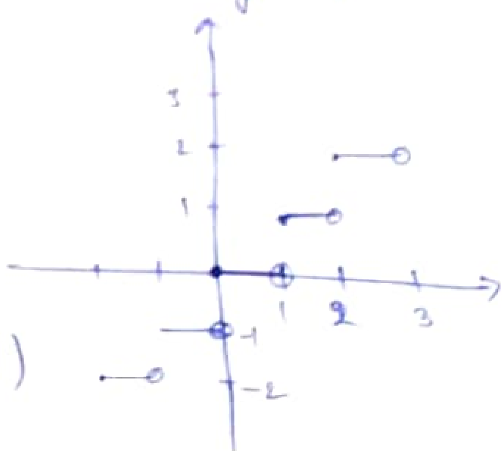
Determine all the real numbers t, at which f is differentiable.

Determine all the real numbers t at which f is continuous but not differentiable.

(15)

Function, $[x]$ is discontinuous at every integer and continuous at non-integer points.

$$[x] = \begin{cases} \vdots & \\ 1, & 1 \le x < 2 \\ 0, & 0 \le x < 1 \\ -1, & -1 \le x < -2 \\ -2, & -2 \le x < -3 \\ \vdots & \end{cases} \quad \text{——(1)}$$



$f(t) = \int_0^t [x]\, dx$ is defined by integrating the greatest integer function from 0 to 1.

~~If $n-1 \le t < n$, for some $n \in \mathbb{Z}$~~

For example

$$f(3.5) = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^{3.5} [x]\,dx$$

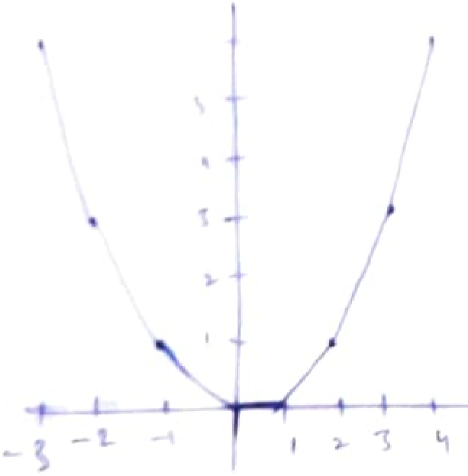
$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^{3.5} 3\,dx$$

$$= 0 + 1 + 2 + 3(0.5) = 4.5$$

$$f(x) = \begin{cases} \vdots \\ x-1 & \text{if } 1 \le x < 2 \\ 0, & \text{if } 0 \le x < 1 \\ -x, & \text{if } -1 \le x < 0 \\ -2x-1, & \text{if } -2 \le x < -1 \\ \vdots \end{cases} \quad \text{--- (2)}$$

Each piece is indeed an anti-derivative of the corresponding piece in the definit of $[x]$ in $-(1)$

Graphing this function,



So the integral of the discontinuous function is, in fact, continuous

We note that there are corners in this graph at integer values of x. Hence the function

$$f(t) = \int_0^t [x]\, dx$$

is continuous for all $x \in \mathbb{R}$ but is not differentiable at the places where the integrand is discontinuous ie at all integer points.

Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100. (20)

First we prove the following Result -

Let $\sum a_n$ be a conditionally convergent series with real-valued terms. Let x and y be given nos in the closed interval $[-\infty, +\infty]$, with $x \leq y$. Then there exist a rearrangement $\sum b_n$ s.t. $\lim_{n\to\infty} \inf t_n = x$ & $\lim_{n\to\infty} \sup t_n = y$

where $t_n = b_1 + \dots + b_n$.

Proof: Discarding those terms of a series which are zero does not affect its convergence or divergence. Hence, we might as well assume that no terms of $\sum a_n$ are zero.

Let p_n denote the n^{th} positive term of $\sum a_n$ and let $-q_n$ denote ~~the~~ its n^{th} negative term.

Then $\sum p_n$ and $\sum q_n$ are both divergent series of positive terms [$\because$ $\sum a_n$ is ~~con~~ not absolutely cgt but converges conditionally only]

we construct two sequences of real nos $\langle x_n \rangle$ and $\langle y_n \rangle$ s.t.

$\lim_{n\to\infty} x_n = x$; $\lim_{n\to\infty} y_n = y$, with $x_n < y_n$, $y_1 > 0$.

The idea of the proof is that we take just enough (say k_1) positive terms so that

$p_1+p_2+\dots+p_{k_1} > y_1$,

followed by just enough (say r_1) negative term s.t.

$p_1+\dots+p_{k_1}-q_1-\dots-q_{r_1} < x_1$.

Next we take just enough positive terms s.t.

$p_1+\dots+p_{k_1}-q_1-\dots-q_{r_1}+p_{k_1+1}+\dots+p_{k_2} > y_2$

followed by just enough further negative terms to satisfy the inequality

$p_1+\dots+p_{k_1}-q_1-\dots-q_{r_1}+p_{k_1+1}+\dots+p_{k_2}-q_{r_1+1}-\dots-q_{r_2} < x_2$

These steps are possible since Σp_n and Σq_n are both divergent series of positive terms. If the process is continued in this way, we obviously obtain a rearrangement of Σa_n.

→ Now we need to show that partial sums of this rearrangement have limit superior y and limit inferior x.

Since Σa_n is convergent $\Rightarrow p_n, q_n \to 0$.

$y_n \to y$ $\therefore$ for any $\varepsilon>0$, $\exists$ n_0 s.t. $y_n < y+\frac{\varepsilon}{2}$ $\forall n \geq n_0$.

We note that the selection of terms from the sequences p_n, q_n to make it just greater than y_n, now removing the last term of type p_n will make the sum less than y_n.

Since $p_n \to 0$, $\therefore$ we will reach a point when this sum from y_n is less than $\varepsilon/2$. It follows that the sum itself will be less than $y_n+\varepsilon/2$.

Thus we can find the infinite sums of the type $\Sigma p - \Sigma q$ s.t. $y_n < \Sigma p_{k_i} - \Sigma q_{k_j} < y_n + \frac{\varepsilon}{2} < y+\varepsilon$

Hence we find a value n_1 such that $t_n < y+\varepsilon$ $\forall n \geq n_1$

We comple the question by taking, $x=y=100$

2017

1

1.(b) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as below :

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1 - x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. 10

2,3,4,5

3.(a) Evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ given that

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y} & \text{if } xy \neq 0 \\ 0 & , \text{ otherwise} \end{cases}$$

10

3.(b) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition $\dfrac{x^2}{4} + \dfrac{y^2}{5} + \dfrac{z^2}{25} = 1.$ 10

3.(c) Prove that $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely convergent. 12

3.(d) Find the volume of the region common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. 8

6

4.(c) Evaluate $\displaystyle\int_{x=0}^{\infty} \int_{y=0}^{x} x e^{-x^2/y} \, dy \, dx$ 8

1.b

A function, $f: \mathbb{R} \to \mathbb{R}$ is defined as below:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$. (10).

Sol: First, let $a \neq \frac{1}{2}$ be any rational number, so that $f(a) = a$

Since in every interval there lies an infinite number of rational and irrational numbers, therfore for each positive integer n, we can choose an irrational number a_n s.t.

$$|a_n - a| < \frac{1}{n}$$

Thus sequence $\langle a_n \rangle$ converges to 'a'.

But, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (1-a_n) = 1-a, \quad a \neq \frac{1}{2}$

$$\therefore \lim_{n\to\infty} f(a_n) \neq f(a), \quad a \neq \frac{1}{2}.$$

Hence, the function is discontinuous at any rational number, other than zero.

In a similary manner, the function $f(x)$ may be shown to be discontinuous at every irrational point.

* Limit of a function (sequential approach) –

A number l is called the limit of a fn 'f' as x tends to c if ~~the~~ sequence $\langle f(x_n) \rangle \to$ ~~l~~ for any sequence, $\langle x_n \rangle \to c$.

~~Let us show~~

It may be seen from above, that the function is continuous at $x = \frac{1}{2}$ ie $a = \frac{1}{2}$.

However, it can be shown to be continuous at $x = \frac{1}{2}$ as follows –

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$, then

Consider, $|f(x) - f(\frac{1}{2})| = |x - \frac{1}{2}| < \varepsilon$

whenever $|x - \frac{1}{2}| < \delta$ & x is rational.

and

$$|f(x) - f(\tfrac{1}{2})| = |(1-x) - \tfrac{1}{2}| = |x - \tfrac{1}{2}| < \varepsilon$$

whenever $|x - \frac{1}{2}| < \delta$ & x is irrational

$\therefore |x - \frac{1}{2}| < \delta \Rightarrow |f(x) - f(\frac{1}{2})| < \varepsilon$

or

$$\lim_{x \to \frac{1}{2}} f(x) = f(\tfrac{1}{2})$$

Hence, the function, f is continuous at $x = \frac{1}{2}$

3(a)

$$f(x,y) = \begin{cases} x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y} & ; \text{ if } xy \neq 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0 \quad -(1)$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$f_y(0,0) = 0 \quad -(2)$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \quad -(3)$$

and $$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad -(4)$$

Now, $$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{h^2\tan^{-1}\left(\frac{k}{h}\right) - k^2\tan^{-1}\left(\frac{h}{k}\right)}{k}$$

$$= \lim_{k\to 0} \frac{h^2\cdot\left(\frac{1}{h}\right)\cdot\left(\frac{1}{1+k^2/h^2}\right)}{1} - 0$$

$$= h. \quad -(5)$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \tan^{-1}\left(\frac{k}{h}\right) - k^2 \tan^{-1}\left(\frac{h}{k}\right)}{h}$$

$$= 0 - \lim_{h\to 0} \frac{k^2 \cdot \left(\frac{1}{k}\right) \cdot \left(\frac{1}{1+h^2/k^2}\right)}{1}$$

$$= -k \quad - (6)$$

Now putting (1), (2), (5) & (6) in (3) & (4)

$$\Rightarrow f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1$$

and $$f_{yx}(0,0) = \lim_{k\to 0} \frac{-k-0}{k} = -1$$

3(b) →

Let $f(x,y,z) = x^2+y^2+z^2 + \lambda\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right)$

$$f_x = 2x + \frac{2\lambda x}{4} = 0 \quad - (1) \quad \Rightarrow x\left(1+\frac{\lambda}{4}\right) = 0 \Rightarrow x=0 \text{ OR } \lambda=-4$$

$$f_y = 2y + \frac{2\lambda y}{5} = 0 \quad - (2) \quad \Rightarrow y\left(1+\frac{\lambda}{5}\right) = 0 \Rightarrow y=0 \text{ OR } \lambda=-5$$

$$f_z = 2z + \frac{2\lambda z}{25} = 0 \quad - (3) \quad \Rightarrow z\left(1+\frac{\lambda}{25}\right) = 0 \Rightarrow z=0 \text{ OR } \lambda=-25$$

Also, multiplying ① by 'x', ② by 'y', ③ by 'z' and adding

we get $x^2+y^2+z^2+\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}\right)=0$

Let $u=x^2+y^2+z^2$

$\Rightarrow u+\lambda(1)=0 \qquad \left[\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \text{ (given)}\right]$

$\Rightarrow u=-\lambda$

$\therefore u=4$ or $u=5$ or $u=25$

Hence minimum value $=4$ & maximum value $=25$.

3.C Prove that $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is convergent but not absolutely convergent. (12)

Sol: Here, point 0 is not a point of infinite discontinuity because $\frac{\sin x}{x} \to 1$ as $x \to 0$.

So, we take $\int_0^{\infty} \frac{\sin x}{x} = \int_0^{1} \frac{\sin x}{x}\,dx + \int_1^{\infty} \frac{\sin x}{x}\,dx$

Now, $\int_0^{1} \frac{\sin x}{x}\,dx$ is a proper integral.

$\Rightarrow$ Convergence of $\int_1^{\infty} \frac{\sin x}{x}\,dx$ at ∞

$\left| \int_1^{\infty} \sin x\,dx \right| = |\cos 1 - \cos X| \leq |\cos 1| + |\cos X| < 2$

So that $\left| \int_1^{X} \sin x\,dx \right|$ is bounded above for all $X \geq 1$

also, $\frac{1}{x}$ is a monotone decreasing function tending to 0 as $x \to \infty$.

Hence, by Dirichlet's test, $\int_1^{\infty} \frac{\sin x}{x}\,dx$ is cgt.

Hence, $\int_0^{\infty} \frac{\sin x}{x}\,dx$ is convergent.

[Dirichlet's test: If ϕ is bounded and monotonic in $[a, \infty)$ and tends to 0 as $x \to \infty$, and $\int_a^{X} f\,dx$ is bounded for $X \geq a$, then $\int_a^{\infty} f\phi\,dx$ is convergent at ∞].

To show $\int_0^\infty \frac{\sin x}{x}\,dx$ is not absolutely cgt.

Consider for $n \geq 1$, the proper integral

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx$$

Now $\forall\, x \in [(r-1)\pi, r\pi]$

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx \geq \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{r\pi}\,dx$$

Putting, $x = (r-1)\pi + y$

$$\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{r\pi}\,dx = \int_0^{\pi} \frac{|\sin((r-1)\pi + y)|\,dy}{r\pi}$$

$$= \frac{1}{r\pi}\int_0^{\pi} \sin y\,dy = \frac{2}{r\pi}$$

Hence, $\int_0^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{r=1}^{n} \frac{2}{r\pi}$

But $\sum_{r=1}^{n} \frac{2}{r\pi}$ is a divergent series.

$\therefore \lim_{n\to 0} \int_0^{n\pi} \frac{|\sin x|}{x}\,dx \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{2}{r\pi} \Rightarrow \lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x}\,dx$ is infinite.

Now, let t be a real numb. $\exists\, n \in \mathbb{N}$ s.t. $n\pi \leq t < (n+1)\pi$

$\therefore \int_0^{t} \frac{|\sin x|}{x}\,dx \geq \int_0^{n\pi} \frac{|\sin x|}{x}\,dx$.

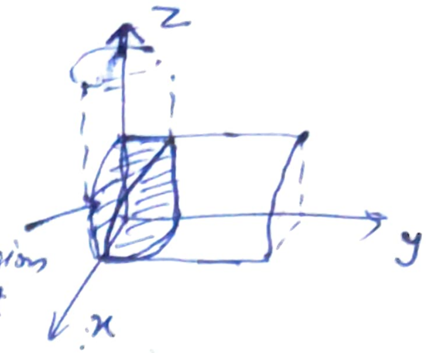
Let $t \to \infty$, so that $n \to \infty$, $\therefore \int_0^{t} \frac{|\sin x|}{x}\,dx \to \infty$

$\Rightarrow \int_0^{\infty} \frac{|\sin x|}{x}\,dx$ does not converge.

~~3(c)~~

3(d)

$$\text{Volume} = 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dx\,dy\,dz$$



Bounded region in I^{st} Octant

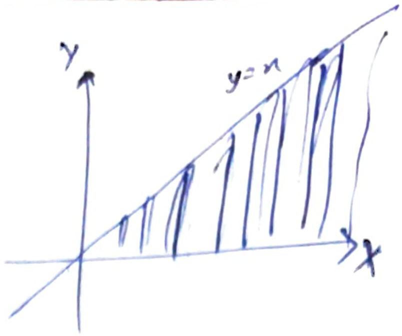
$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy \cdot dx$$

$$= 8\int_{0}^{a} (a^2-x^2)\, dx$$

$$= 8\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{16a^3}{3}.$$

$4(c) \rightarrow \displaystyle\int_{x=0}^{\infty}\int_{y=0}^{x} x e^{-x^2/y}\, dy\, dx$



Changing order of integration

$$\int_{y=0}^{\infty}\int_{x=y}^{\infty} x e^{-x^2/y}\, dx\, dy$$

Let $+\frac{x^2}{y}=t \Rightarrow +\frac{2x\,dx}{y}=dt$

$$= \int_{y=0}^{\infty}\int_{t=+y}^{+\infty} \left(\frac{y}{2}\right)e^{-t}\,dt\,dy$$

$$= \int_{0}^{\infty}\frac{y}{2}\left[-e^{-t}\right]_{t=y}^{\infty} dy = \int_{0}^{\infty}\frac{ye^{-y}}{2}\,dy$$

$$= \frac{1}{2}\left[-ye^{-y}+\int_{0}^{\infty}e^{-y}\,dy\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[-ye^{-y}-e^{-y}\right]_{0}^{\infty}$$

$$= \frac{1}{2}.$$