2 Open Sets, Closed Sets and Countable Sets

1. INTRODUCTION

In this chapter, we shall study the concept of *neighbourhood* of a point, *open and closed sets*, and *limit points* of a set of real numbers and the *Bolzano-Weierstrass theorem*, which is one of the most fundamental theorems of Real Analysis and lays down a sufficient condition for the existence of *limit points* of a set. *We shall be dealing only with real numbers and sets of real numbers unless otherwise stated.*

1.1 Neighbourhood of a Point

A set $N \subseteq \mathbf{R}$ is called the **neighbourhood** of a point a, if there exists an open interval I containing a and contained in N, *i.e.*,

$$a \in I \subseteq N$$

It follows from the definition that an open interval is a neighbourhood of each of its points. Though open intervals containing the point are not the only neighbourhoods of the point but they prove quite adequate for a discussion like ours and are more expressive of the idea of neighbourhoods as understood in ordinary language. We shall, therefore, whenever convenient, take the open interval $]a - \delta, a + \delta[$ where $\delta > 0$ is a neighbourhood of the point a.

Deleted Neighbourhoods

The set $\{x : 0 < |x - a| < \delta\}$, *i.e.*, an open interval $]a - \delta, a + \delta[$ from which the number a itself has been *excluded* or *deleted* is called a *deleted neighbourhood* of a.

Note: For the sake of brevity, we shall write neighbourhood as '**nbd**'.

ILLUSTRATIONS

1. The set **R** of real numbers is the neighbourhood of each of its points.
2. The set **Q** of rationals is not the *nbd* of any of its points.
3. The open interval $]a, b[$ is *nbd* of each of its points.
4. The closed interval $[a, b]$ is the *nbd* of each point of $]a, b[$ but is not a *nbd* of the end points a and b.

5. The null set ϕ is a *nbd* of each of its points in the sense that there is no point in ϕ of which it is not a *nbd*.

Example 1. A non-empty finite set is not a *nbd* of any point.

A set can be a *nbd* of a point if it contains an open interval containing the point. Since an interval necessarily contains an infinite number of points, therefore, in order that a set be a *nbd* of a point it must necessarily contain an infinity of points. Thus a finite set cannot be a *nbd* of any point.

Example 2. Superset of a *nbd* of a point x is also a *nbd* of x. *i.e.*, if N is a *nbd* of a point x and $M \supseteq N$ then M is also a *nbd* of x.

Example 3. Union (finite or arbitrary) of *nbds* of a point x is again a *nbd* of x.

Example 4. If M and N are *nbds* of a point x, then show that $M \cap N$ is also a *nbd* of x.

■ Since M, N are *nbds* of x, $\exists$ open intervals enclosing the points x such that

$$x \in \,]x-\delta_1, x+\delta_1[\, \subseteq M \text{ and } x \in \,]x-\delta_2, x+\delta_2[\, \subseteq N$$

Let $\delta = \min(\delta_1, \delta_2)$. Then

$$]x-\delta, x+\delta[\, \subseteq \,]x-\delta_1, x+\delta_1[\, \subseteq M$$

and

$$]x-\delta, x+\delta[\, \subseteq \,]x-\delta_2, x+\delta_2[\, \subseteq N$$

$$\Rightarrow \quad]x-\delta, x+\delta[\, \subseteq M \cap N$$

$$\Rightarrow \quad M \cap N \text{ is a } nbd \text{ of } x$$

1.2 Interior Points of a Set

A point x is an *interior point* of a set S if S is a *nbd* of x. In other words, x is an interior point of S if $\exists$ an open interval $]a, b[$ containing x and contained in S, *i.e.*, $x \in \,]a, b[\, \subseteq S$.

Thus a set is a neighbourhood of each of its interior points.

Interior of a Set. The set of all interior points of a set is called the **interior** of the set. The interior of a set S is generally denoted by S^i or int S.

Ex. 1. Show that the interior of the set **N** or **I** or **Q** is the null set, but interior of **R** is **R**.

Ex. 2. Show that the interior of a set S is a subset of S, *i.e.*, $S^i \subseteq S$.

1.3 Open Set

A set S is said to be **open** if it is a *nbd* of each of its points, *i.e.*, for each $x \in S$, there exists an open interval I_x such that

$$mx \in I_x \subseteq S.$$

Thus *every point of an open set is an interior point*, so that for an open set S, $S^i = S$.

Evidently, S is open $\Leftrightarrow S = S^i$

Of course the set is **not open** if it is not a *nbd* of at least one of its points or that there is at least one point of the set which is not an interior point.

ILLUSTRATIONS

1. The set **R** of real numbers is an open set.
2. The set **Q** of rationals is not an open set.
3. The closed interval $[a, b]$ is not open, for it is not a neighbourhood of the end points a and b.
4. The null set ϕ is open, for there is no point in ϕ of which it is not a neighbourhood.
5. A non-empty finite set is not open.
6. The set $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is not open.

Ex. Give an example of an open set which is not an interval.

Example 5. Show that every open interval is an open set. Or, every open interval is a *nbd* of each of its points.

- Let x be any point of the given open interval $]a, b[$ so that we have $a < x < b$.

Let c, d be two numbers such that

$$a < c < x, \text{ and } x < d < b$$

so that we have

$$a < c < x < d < b \Rightarrow x \in \,]c, d[\, \subset \,]a, b[.$$

Thus the given interval $]a, b[$ contains an open interval containing the point x, and is, therefore, a *nbd* of x.

Hence, the open interval is a *nbd* of each of its points and is therefore an open set.

Ex. Show that every point of an open interval is its interior point.

Example 6. Show that every open set is a union of open intervals.

- Let S be an open set and x_λ a point of S.

Since S is open, therefore $\exists$ an open interval $I_{x\lambda}$ for each of its points x_λ such that

$$x_\lambda \in I_{x_\lambda} \subseteq S \qquad \forall\, x_\lambda \in S$$

Again the set S can be thought of as the union of singleton sets like $\{x_\lambda\}$, *i.e.*,

$$S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\}, \text{ where } \Lambda \text{ is the index set}$$

$$\therefore \qquad S = \bigcup_{\lambda \in \Lambda} \{x_\lambda\} \subseteq \bigcup_{\lambda \in \Lambda} I_{x_\lambda} \subseteq S$$

$$\Rightarrow \qquad S = \bigcup_{\lambda \in \Lambda} I_{x_\lambda}$$

Theorem 1. *The interior of a set is an open set.*

Let S be a given set, and S^i its interior.

If $S^i = \phi$ then S^i is open.

When $S^i \neq \phi$, and let x be any point of S^i.

As x is an interior point of S, $\exists$ an open interval I_x such that $x \in I_x \subseteq S$.

But I_x, being an open interval, is a *nbd* of each of its points.

$\Rightarrow$ every point of I_x is an interior point of I_x, and $I_x \subseteq S$

$\Rightarrow$ every point of I_x is an interior point of S

$\therefore$ $I_x \subseteq S^i$

$\Rightarrow$ $x \in I_x \subseteq S^i \Rightarrow$ any point x of S^i is interior point of S^i

$\Rightarrow$ S^i is an open set.

Corollary. The interior of a set S is an open subset of S.

Theorem 2. *The interior of a set S is the largest open subset of S.*

Or

The interior of a set S contains every open subset of S.

We know that the interior S^i of a set S is an open subset of S. Let us now proceed to show that any open subset S_1 of S is contained in S^i.

Let x be any point of S_1.

Since an open set is a *nbd* of each of its points, therefore S_1 is a *nbd* of x. But S is a superset of S

$\therefore$ S is also a *nbd* of x

$\Rightarrow$ x is an interior point of S

$\Rightarrow$ $x \in S^i$

Thus, $x \in S_1 \Rightarrow x \in S^i$

$\therefore$ $S_1 \subseteq S^i$

Hence, every open subset of S is contained in its interior S^i.

$\Rightarrow$ S^i, the interior of S, is the largest open subset of S.

Corollary. Interior of a set S is the union of all open subsets of S.

Theorem 3. *The* **union** *of an arbitrary family of open sets is open.*

Let F be the union of an arbitrary family $\mathbf{F} = \{S_\lambda : \lambda \in \Lambda\}$ of open sets, Λ being an index set. To prove that F is open, we shall show that for any point $x \in F$, it contains an open interval containing x.

Let x be any point of F. Since F is the union of the members of $\mathbf{F}$, $\exists$ at least one member, say S_λ of $\mathbf{F}$ which contains x. Again, S_λ being an open set, $\exists$ an open interval I_x such that $x \in I_x \subseteq S_\lambda \subseteq F$.

Thus the set F contains an open interval containing any point x of $F \Rightarrow F$ is an open set.

Theorem 4. *The* **intersection** *of any finite number of open sets is open.*

Let us consider two open sets S and T.

If $S \cap T = \phi$, it is an open set.

If $S \cap T \neq \phi$, let x be any point of $S \cap T$.

Now $x \in S \cap T \Rightarrow x \in S \wedge x \in T$

$\Rightarrow$ S, T are *nbds* of x $[\because S, T \text{ are open}]$

$\Rightarrow$ $S \cap T$ is a *nbd* of x

But since x is any point of $S \cap T$, therefore $S \cap T$ is a *nbd* of each of its points. Hence, $S \cap T$ is open.

The proof may, of course, be extended to a finite number of sets.

Note: The above theorem does not hold for the intersection of *arbitrary* family of open sets.

Consider, for example, the open sets

$$S_n = \left]-\frac{1}{n}, \frac{1}{n}\right[, n \in \mathbf{N}$$

Their intersection is the set $\{0\}$ consisting of the single point 0, and this set is not open.

2. LIMIT POINTS OF A SET

Definition 1. A real number ξ is a **limit point** of a set $S\ (\subset \mathbf{R})$ if every *nbd* of ξ contains an infinite number of members of S.

Thus ξ is a limit point of a set S if for any *nbd* N of ξ, $N \cap S$ is an infinite set.

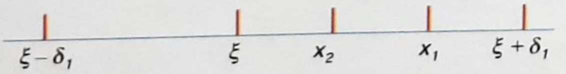
A limit point is also called a *cluster point, a condensation point* or an *accumulation point*.

A limit point of a set may or may not be a member of the set. Further, it is clear from the definition that a finite set cannot have a limit point. Also it is not necessary that an infinite set must possess a limit point. In fact a set may have no limit point, a unique limit point, a finite or an infinite number of limit points. A sufficient condition for the existence of a limit point is provided by *Bolzano-Weierstrass theorem* which is discussed in the next section. The following is another definition of a limit point.

Definition 2. A real number ξ is a **limit point** of a set $S\ (\subseteq \mathbf{R})$ if every *nbd* of ξ contains at least one member of S other than ξ.

The essential idea here is that the points of S different from ξ get 'arbitrarily close' to ξ or 'pile up' at ξ.

Evidently definition 1 implies definition 2. Let us now prove that definition 2 implies definition 1.



Let ξ be a limit point of the set $S\ (\subseteq \mathbf{R})$ such that every *nbd* of ξ contains at least one point of S other than ξ. Let $]\xi - \delta_1, \xi + \delta_1[$ be one such *nbd* of ξ which contains at least one point, say, $x_1 \neq \xi$ of S.

Let $|x_1 - \xi| = \delta_2 < \delta_1$. Now consider the *nbd* $]\xi - \delta_2, \xi + \delta_2[$ of ξ which by def. 2 of a limit point, must have one point, say, x_2 of S other than ξ.

By repeating the argument with the *nbd* $]\xi - \delta_3, \xi + \delta_3[$ of ξ where $\delta_3 = |x_2 - \xi|$ and so on, it follows that the *nbd* $]\xi - \delta_i, \xi + \delta_i[$ of ξ contains an infinity of members of S.

Hence, Def. 2 $\Rightarrow$ Def. 1.

It is instructive to note that a point ξ is not a limit point of a set S if $\exists$ even one *nbd* of ξ containing any point of S other than ξ.

Ex. Give a bounded set having (*i*) no limit point, (*ii*) infinite numbers of limit points.

Derived Sets. The set of all limit points of a set S is called the *derived set* of S and is denoted by S'.

ILLUSTRATIONS

1. The set **I** has no limit point, for a *nbd* $\left]m-\frac{1}{2}, m+\frac{1}{2}\right[$ of $m \in \mathbf{I}$, contains no point of other than m. Thus the derived set of **I** is the null set ϕ.
2. Every point of **R** is a limit point, for every *nbd* of any of its points contains an infinit members of **R**. Therefore $\mathbf{R}' = \mathbf{R}$.
3. Every point of the set **Q** of rationals is a limit point, for, between any two rationals there exis infinite rationals. Further every irrational number is also a limit point of **Q** for between any two irrationals there are infinitely many rationals. Thus every real number is a limit point of **Q**, so that $\mathbf{Q}' = \mathbf{R}$.
4. The set $\left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ has only one limit point, zero, which is not a member of the set.
5. Every point of the closed interval $[a, b]$ is its limit point, and a point not belonging to the interval is not a limit point. Thus the derived set $[a, b]' = [a, b]$.
6. Every point of the open interval $]a, b[$ is its limit point. The end points a, b which are not members of $]a, b[$ are also its limit points. Thus

$$]a, b[' = [a, b]$$

Ex. Obtain the derived sets:

1. $\{x : 0 \le x < 1\}$,
2. $\{x : 0 < x < 1, x \in \mathbf{Q}\}$,
3. $\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{3}, \ldots\right\}$,
4. $\left\{1 + \frac{1}{n} : n \in \mathbf{N}\right\}$,
5. $\left\{\frac{1}{m} + \frac{1}{n} : m \in \mathbf{N}, n \in \mathbf{N}\right\}$.

2.1 A finite set has no limit point. Also we have seen that an infinite set may or may not have limit points. We shall now discuss a theorem which sets out sufficient conditions for a set to have limit points.

Bolzano-Weierstrass Theorem (for sets). *Every infinite bounded set has a limit point.*

Let S be any infinite bounded set and m, M its infimum and supremum respectively. Let P be a set of real numbers is defined as follows:

$x \in P$ iff it exceeds at the most a finite number of members of S.

The set P is non-empty, for $m \in P$. Also M is an upper bound of P, for no number greater than or equal to M can belong to P. Thus the set P is non-empty and is bounded above. Therefore, by the order-completeness property, P has the supremum, say ξ. We shall now show that ξ is a limit point of S.

Consider any *nbd.* $]\xi - \varepsilon, \xi + \varepsilon[$ of ξ, where $\varepsilon > 0$.

Since, ξ is the supremum of P, $\exists$ at least one member say η of P such that $\eta > \xi - \varepsilon$. Now η belongs to P, therefore it exceeds at the most a finite number of members of S, and consequently $\xi - \varepsilon\ (< \eta)$ can exceed at the most a finite number of members of S.

Again as ξ is the supremum of P, $\xi + \varepsilon$ cannot belong to P, and consequently $\xi + \varepsilon$ must exceed an infinite number of members of S.

Now, $\xi - \varepsilon$ exceeds at the most a finite number of members of S and $\xi + \varepsilon$ exceeds infinitely many members of S.

$\Rightarrow$ $]\xi - \varepsilon, \xi + \varepsilon[$ contains an infinite number of members of S

Consequently ξ is a limit point of S.

Note: Boundedness is not necessary in order for an infinite set S to have a limit point. The set $S = \left\{\frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \frac{1}{5}, 5 \ldots\right\}$ is unbounded and infinite and has the limit point 0. The unbounded interval $]a, \infty[$ has infinitely many limit points.

CHAPTER 2

2.2 Example 7. If S and T are subsets of real numbers, then show that

(*i*) $S \subseteq T \Rightarrow S' \subseteq T'$, and

(*ii*) $(S \cup T)' = S' \cup T'$

■ (*i*) If $S' = \phi$, then evidently $S' \subseteq T'$.

When $S' \neq \phi$, let $\xi \in S'$ and N be any *nbd* of ξ.

$\Rightarrow$ N contains an infinite number of members of S.

But $S \subseteq T$

$\therefore$ N contains infinitely many members of T

$\Rightarrow$ ξ is limit point of T, *i.e.*, $\xi \in T'$.

Thus, $\xi \in S' \Rightarrow \xi \in T'$. Hence, $S' \subseteq T'$.

(*ii*) Now, $S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$

and $T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$

Consequently, $S' \cup T' \subseteq (S \cup T)'$ ...(1)

Now we proceed to show that $(S \cup T)' \subseteq S' \cup T'$.

If $(S \cup T)' = \phi$, then evidently $(S \cup T)' \subseteq S' \cup T'$.

When $(S \cup T)' \neq \phi$, let $\xi \in (S \cup T)'$.

Now ξ is a limit point of $(S \cup T)$, therefore, every *nbd* of ξ contains an infinite number of points of $(S \cup T) \Rightarrow$ every *nbd* of ξ contains infinitely many points of S or T or both.

$\Rightarrow \qquad \xi$ is a limit point of S or a limit point of T

$\Rightarrow \qquad \xi \in S' \vee \xi \in T' \;\Rightarrow\; \xi \in S' \cup T'$.

Thus, $\qquad \xi \in (S \cup T)' \;\Rightarrow\; \xi \in S' \cup T'$

Consequently, $(S \cup T)' \subseteq S' \cup T'$

From (1) and (2), it follows that

$$(S \cup T)' = S' \cup T'$$

Thus the derived set of the union = the union of the derived sets.

Aliter. To show that $(S \cup T)' \subseteq S' \cup T'$

We may show that $\xi \notin S' \cup T' \Rightarrow \xi \notin (S \cup T)'$.

Now $\xi \notin S' \cup T'$ implies that ξ does not belong to either.

$\Rightarrow \qquad \xi$ is not a limit point of S or of T

$\therefore \exists$ *nbds* N_1, N_2 of ξ such that N_1 contains no point of S other than ξ and N_2 contain point of T other than possibly ξ.

Again, since $N_1 \cap N_2 \subseteq N_1$, $N_1 \cap N_2 \subseteq N_2$ therefore $\exists$ a *nbd* $N_1 \cap N_2$ of ξ which contain point other than ξ of S or of T and thus of $S \cup T$

$\Rightarrow \qquad \xi$ is not a limit point $S \cup T$

$\Rightarrow \qquad \xi \notin (S \cup T)'$

Thus, $\qquad \xi \notin S' \cup T' \Rightarrow \xi \notin (S \cup T)'$

so that $\qquad (S \cup T)' \subseteq S' \cup T'$

Example 8. (*i*) If S, T are subsets of $\mathbf{R}$, then show that

$$(S \cap T)' \subseteq S' \cap T'$$

(*ii*) Give an example to show that $(S \cap T)'$ and $S' \cap T'$ may not be equal.

■ (*i*) Now $S \cap T \subseteq S \Rightarrow (S \cap T)' \subseteq S'$ and

$$S \cap T \subseteq T \Rightarrow (S \cap T)' \subseteq T'$$

Consequently, $(S \cap T)' \subseteq S' \cap T'$

(*ii*) Let $S =]1, 2[$ and $T =]2, 3[$, so that

$$S \cap T = \phi \Rightarrow (S \cap T)' = \phi' = \phi.$$

Also $\qquad S' = [1, 2], T' = [2, 3]$

$\therefore \qquad S' \cap T' = \{2\}$.

Thus, $\qquad (S \cap T)' \neq S' \cap T'$.

3. CLOSED SETS: CLOSURE OF A SET

3.1 A real number ξ is said to be an **adherent point** of a set S ($\subseteq$ **R**) if every *nbd* of ξ contains at least one point of S.

Evidently an adherent point may or may not belong to the set and it may or may not be a limit point of the set.

It follows from the definition that a number $\xi \in S$ is automatically an adherent point of the set, for every *nbd* of a member of the set contains atleast one member of the set, namely the member itself. Further a number $\xi \notin S$ is an adherent point of S only if ξ is a limit point of S, for every *nbd* of ξ, there contains atleast one point of S which is other than ξ.

Thus the set of adherent points of S consists of S and the derived set S'.

The set of all adherent point of S, called the **closure** of S is denoted by $\tilde{S}$, and is such that

$$\tilde{S} = S \cup S'.$$

ILLUSTRATIONS

1. $\tilde{\mathrm{I}} = \mathrm{I} \cup \mathrm{I}' = \mathrm{I} \cup \phi = \mathrm{I}$.
2. $\tilde{\mathrm{Q}} = \mathrm{Q} \cup \mathrm{Q}' = \mathrm{Q} \cup \mathrm{R} = \mathrm{R}$.
3. $\tilde{\mathrm{R}} = \mathrm{R} \cup \mathrm{R}' = \mathrm{R} \cup \mathrm{R} = \mathrm{R}$
4. $\tilde{\phi} = \phi \cup \phi' = \phi \cup \phi = \phi$.

3.2 Closed Sets

A set is said to be **closed** if each of its limit points is a member of the set.

In other words a set S is *closed* if no limit point of S exists which is not contained in S. In rough terms, a set is closed if its points do not get arbitrarily close to any point outside it.

Thus a set S is *closed* iff

$$S' \subseteq S \text{ or } \tilde{S} = S.$$

Consequently, a *closed set* is also defined as a set S for which

$$\tilde{S} = S.$$

It should be clearly understood that the concept of closed and open sets are neither mutually exclusive nor exhaustive. The word *not closed* should not be considered equivalent to *open*. Sets exist which are both open and closed, or which are neither open nor closed. The set consisting of points of $]a, b]$ is neither open nor closed.

ILLUSTRATIONS

1. $[a, b]$ is a set which is closed but not open.
2. The set $[0, 1] \cup [2, 3]$, which is not an interval, is closed.
3. The null set ϕ is closed for there exists no limit point of ϕ which is not contained in ϕ. As shown earlier, ϕ is also open.
4. The set **R** of real numbers is open as well as closed.
5. The set **Q** is not closed, for $\mathbf{Q}' = \mathbf{R} \not\subseteq \mathbf{Q}$. Also it is not open.

6. $\left\{\frac{1}{n}: n \in \mathbf{N}\right\}$ is not closed, for it has one limit point 0, which is not a member of the set. Also it is not open.
7. Every finite set A is a closed set, for its derived set $A' = \phi \subset A$.
8. A set A which has no limit point coincides with its closure, for $A' = \phi$ and $\tilde{A} = A \cup A' = A$.

3.3 Typical Examples

Example 9. Show that the set $S = \{x: 0 < x < 1, x \in \mathbf{R}\}$ is open but not closed.

■ The set S is the open interval $]0, 1[$.

∴ It contains a *nbd* of each of its points. Hence it is an open set.

Again every point of S is a limit point. The end points 0 and 1 which are not members of the set are also limit points. Thus S is not closed.

Example 10. Show that the set

$$S = \left\{1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \ldots\right\}$$

is neither open nor closed.

■ The members of S heap or cluster near zero on both sides of it and every *nbd* of zero contains an infinite number of points of S. Thus $0 \notin S$ is a limit point $\Rightarrow$ S is not closed.

Again S is not open for it does not contain any *nbd* of any of its points. For example, a *nbd* $\left]\frac{1}{3} - \frac{1}{100}, \frac{1}{3} + \frac{1}{100}\right[$ of $\frac{1}{3}$ is not contained in the set. Hence the set is not open.

Example 11. Show that the set

$$\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$$

is closed but not open.

■ 1 and -1 are the only limit points of the set and are in the set. Therefore, the set is closed.

Again all members of the set (except $1, -1$) are not the interior points of the set. Thus the set is not open.

Hence, the set is closed but not open.

The relationship between closed and open sets is brought out by Theorem 5 that follows and is sometimes taken as the **definition** of a closed set.

3.4 Dense Sets

A subset A of the set of reals $\mathbf{R}$ is said to be *dense* (or *dense* in $\mathbf{R}$ or *everywhere dense*) if every point of $\mathbf{R}$ is a point of A or a limit point of A or equivalently if the closure of A is $\mathbf{R}$.

A set A is said to be *dense in itself* if every point of A is a limit point of A, *i.e.*, if $A \subseteq A'$. A set which is *dense in itself* has no isolated points.

A set A is said to be *nowhere dense* (nondense) relative to $\mathbf{R}$ if no neighbourhood in $\mathbf{R}$ is contained in the closure of A. In other words, the complement of the closure of A is dense in $\mathbf{R}$.

It is clear that if A is an interval or contains an interval then A is not nowhere dense. Because there exists an interval $I \subset \mathbf{R}$ such that $I \cap A \neq \phi$. But there are sets which contain no interval and which fail to be nowhere dense; for example, the set of rationals $\mathbf{Q}$ and the set of irrationals $\mathbf{R} - \mathbf{Q}$.

A set is said to be *perfect* if it is identical with its derived set or equivalently a set which is closed and dense in itself.

ILLUSTRATIONS

1. $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$, $A' = \{0\}$. The set is neither closed nor dense in itself.
2. The set $\mathbf{Q}$ of rationals is dense in itself but not closed.
3. A finite set is closed but not dense in itself.
4. The set $\mathbf{R}$ of real numbers is dense in itself and closed.
5. The set $\{x: a \leq x \leq b\}$ is a perfect set.
6. The set $\mathbf{R}$ of real numbers is a perfect set.
7. The set $A = \left\{0, 1, \frac{1}{2}, ...\right\}$ is nowhere dense in $\mathbf{R}$ since 0 is the only limit point of A and no neighbourhood of 0 in $\mathbf{R}$ is contained in the closure of A.
8. The set $\mathbf{Q}$ of rational numbers and the set of irrationals are dense in $\mathbf{R}$.
9. The empty set is perfect.

3.5 Some Important Theorems

Theorem 5. *A set is closed iff its complement is open.*

Necessary. Let S be a closed set. We shall show that its complement $\mathbf{R} - S = T$ is open. Let x be *any* point of T.

$$x \in T \Rightarrow x \notin S.$$

Also, since S is closed, x cannot be a limit point of S. Thus $\exists$ a *nbd* N of x such that

$$N \cap S = \phi.$$

$\Rightarrow$ $\quad N \subseteq T \Rightarrow$ every point of T is an interior point.

Thus T is an open set.

Sufficient. Let S be a set whose complement $\mathbf{R} - S = T$ is open.

To show that S is closed, we shall show that every limit point of S is in S.

Let, if possible, a limit point ξ of S be not in S so that ξ is in T. As T is open, $\exists$ a *nbd* of ξ contained in T and thus containing no point of S.

$\therefore$ $\quad \exists$ a *nbd* N of ξ which contains no point of S.

$\Rightarrow$ $\quad \xi$ is not a limit point of S, which is a contradiction.

Hence no limit point of S exists which is not in S.

$\therefore$ $\quad S$ is closed.

Theorem 6. *The intersection of an* **arbitrary** *family of closed sets is closed.*

Let F be the intersection set of an arbitrary family $\mathbf{F} = \{S_\lambda : \lambda \in \Lambda\}$ of closed sets, Λ being index set.

If the derived set F' of F is ϕ, *i.e.*, when F is a finite set or an infinite set without limit points, the evidently it is closed.

When $F' \neq \phi$, let $\xi \in F'$, *i.e.* ξ be a limit point of F, so that every *nbd* of ξ contains infinitely many members of F and as such of each member S_λ of the family $\mathbf{F}$ of closed sets.

$\Rightarrow$ ξ is limit point of each closed set S_λ

$\Rightarrow$ ξ belongs to each $S_\lambda \Rightarrow \xi \in \bigcap_{\lambda \in \Lambda} S_\lambda = F$.

Thus the set F is closed.

Note: We have given an independent proof but on taking complements, this theorem follows from theorem 3.

Theorem 7. *The union of* **two** *closed sets is a closed set.*

Let S and T be the two given closed sets and ξ a limit point of F, where $F = S \cup T$.

We have to show that $\xi \in F$, for then, the set F will be closed.

Let if possible $\xi \notin F$, thus $\xi \notin S \wedge \xi \notin T$. Also as S and T are closed sets, the point ξ which does not belong to them, cannot be a limit point of either.

$\therefore$ $\exists$ *nbds* N_1 and N_2 of ξ such that

$$N_1 \cap S = \phi \wedge N_2 \cap T = \phi. \quad \text{...(1)}$$

Let $N_1 \cap N_2 = N$, where $\xi \in N$.

$\therefore$ From (1), it follows that

$$N \cap (S \cup T) = \phi \Rightarrow N \cap F = \phi.$$

Thus, $\exists$ a *nbd* N of ξ which contains no point of F.

$\Rightarrow$ ξ is not a limit point of F, which is a contradiction.

Hence, no point not belonging to F can be its limit point, and consequently $F = S \cup T$ is a closed set.

Remarks:

1. The theorem can be extended to the union of a finite number of sets. So we may restate the theorem as: *The union of a finite number of closed sets is closed.*
2. We have given an independent proof but the theorem follows from theorem 4 on taking complements.
3. The union of an arbitrary family of closed sets may not always be a closed set. For example,

let $S_n = \left[a + \frac{1}{n}, a + 2\right]$, for $n \in \mathbf{N} \wedge a \in \mathbf{R}$.

Then, $\bigcup_{n \in \mathbf{N}} S_n = \left]a, a+2\right]$, which is not a closed set.

Theorem 8. *The derived set of a set is closed.*

Let S' be the derived set of a set S.

We have to show that the derived set S'' of S' is contained in S'.

Now if $S'' = \phi$, *i.e.*, when S' is either a finite set or an infinite set without limit points, then $S'' = \phi \subset S'$ and therefore S' is closed.

When $S'' \neq \phi$, let $\xi \in S''$, *i.e.*, ξ be a limit point of S'.

$\therefore$ Every *nbd N* of ξ contains at least one point $\eta \neq \xi$ of S'.

Again,

$\eta \in S' \Rightarrow \eta$ is a limit point of S

$\Rightarrow$ every *nbd* of η, N being such a *nbd*, contains infinitely many points of S.

Thus every *nbd N* (of ξ) contains an infinitely many points of S.

$\Rightarrow$ ξ is a limit point of S, *i.e.*, $\xi \in S'$.

Consequently $\xi \in S'' \Rightarrow \xi \in S'$

$\therefore$ $S'' \subseteq S'$, *i.e.*, S' is a closed set.

Corollary 1. S'' is closed, and therefore the closure of S' is S', *i.e.*, $\tilde{S} = S'$.

Corollary 2. For every set S the closure $\tilde{S}$ is closed.

We have simply to show that $(\tilde{S})' \subset \tilde{S}$. Now,

$$(\tilde{S})' = (S \cup S')' = S' \cup S'' = S' \subset \tilde{S}. \qquad \text{(Ref. § 2.2 and Theorem 8)}$$

Theorem 9. *A closed set either contains an interval or else is nowhere dense.*

Let A be any closed set and A is not nowhere dense in **R**. Then there is some interval I such that for each interval $I_1 \subseteq I$, we have $I_1 \cap A \neq \phi$. We shall show that $I \subseteq A$.

Let $x \in I$. Then every neighbourhood of x contains within it at least one point of A. This implies that either $x \in A$ or else x is a limit point of A. Since A is closed it contains all its limit points and so $x \in A$.

Theorem 10. *The supremum (infimum) of a bounded non-empty set* **S** $(\subseteq \mathbf{R})$, *when not a member of* **S**, *is a limit point of S.*

Let M be the supremum of the bounded set S $(\subseteq \mathbf{R})$, which must exist by the *order completeness* property of **R**. If $M \notin S$, then for any number $\varepsilon > 0$, however small, $\exists$ at least one member x of S such that

$$M - \varepsilon < x < M.$$

Thus every *nbd* of M contains atleast one member x of the set S other than M. Hence M is a limit point of S.

Corollary. The supremum (infimum) M of a bounded set S is always a member of the closure $\tilde{S}$ of S.

When $M \in S$,

$$M \in S \Rightarrow M \in S \cup S' = \tilde{S} \quad M \in S \Rightarrow M \in S \cup S' = \tilde{S}$$

When $M \notin S$,

$$M \notin S \Rightarrow M \in S' \Rightarrow M \in S \cup S' = \tilde{S}$$

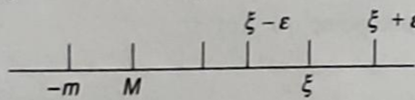
Consequently $M \in \tilde{S}$.

Theorem 11. *The derived set of a bounded set is bounded.*

Let m, M be the bounds of a set S.

It will now be shown that no limit point of S can be outside the interval $[m, M]$.

Let, if possible, $\xi > M$ be a limit point of S, and ε be a positive number such that $\varepsilon < \xi - M$.



Then since M is an upper bound of S, no member of S can lie in the interval $]\xi - \varepsilon, \xi + \varepsilon[$ therefore $\exists$ a *nbd* of ξ which contains no point of S so that ξ cannot be a limit point of S.

Hence, S has no limit point greater than M.

Similarly, it can be shown that no limit point of S is less than m.

Hence, $S' \subseteq [m, M]$.

Corollary. If S is bounded then so is its closure $\bar{S}$.

$$S \subseteq [m, M] \Rightarrow S' \subseteq [m, M] \Rightarrow \bar{S} = S \cup S' \subseteq [m, M].$$

Remark: If supremum M (infimum m) of S is not a member of S, then it is a limit point of S and in view of the above theorem, it is the greatest (least) member of S'.

However, if it is a member of S, then it is not necessarily a limit point of S, so that M (or m) may not be a member of $S' \subseteq [m, M]$. Thus M, m may not always be supremum and infimum of S' but they are always so for $\bar{S} = S \cup S'$.

For example, for the set $S = \left\{-1, 1, -1\frac{1}{2}, 1\frac{1}{2}, -1\frac{1}{3}, 1\frac{1}{3}, \ldots\right\}$

$$m = -1\frac{1}{2},\ M = 1\frac{1}{2},\ S' = \{-1, 1\}$$

$$\inf S' = -1 \neq m,\ \sup S' = 1 \neq M$$

but

$$\inf \bar{S} = m, \sup \bar{S} = M.$$

Theorem 12. *The derived set S' of a bounded infinite set S ($\subseteq \mathbf{R}$) has the smallest and the greatest members.*

Since the set S is bounded, therefore S' is also bounded. Also S' is non-empty, by Bolzano-Weierstrass theorem S has at least one limit point.

Now S' may be finite or infinite.

When $S' (\neq \phi)$ is finite, evidently it has the greatest and the least members.

When S' is infinite, being bounded set of real numbers, by order-completeness property of $\mathbf{R}$, it has the supremum G and the infimum g.

It will now be shown that G, g are limit points of S, *i.e.*,

$$G \in S',\ g \in S'$$

Let us first consider G.

Let $]G - \varepsilon, G + \varepsilon[, \varepsilon > 0$ be any *nbd* of G.

Now G being the supremum of S', $\exists$ at least one member ξ of S' such that $G - \varepsilon < \xi \leq G$.

Thus $]G-\varepsilon, G+\varepsilon[$ is a *nbd* of ξ.

But ξ is a limit point of S, so that $]G-\varepsilon, G+\varepsilon[$ contains an infinite members of S.

$\Rightarrow$ any *nbd* $]G-\varepsilon, G+\varepsilon[$ of G contains an infinite number of members of S.

$\Rightarrow$ G is a limit point of $S \Rightarrow G \in S'$

Similarly, it can be shown that $g \in S'$.

Thus, $G \in S'$ and $g \in S'$, being supremum and infimum of S', are the greatest and the smallest members of S'.

The theorem may be restated as:

Every bounded infinite set has the smallest and the greatest limit points.

The smallest and greatest limit points of a set are called the **lower** and **upper limits of indetermination** or simply the **lower** and **upper limits** respectively of the set.

4. COUNTABLE AND UNCOUNTABLE SETS

An infinite set A is said to be *Countably infinite* (or denumerable or enumerable) if it is equivalent to the set $\mathbf{N}$ of natural numbers.

A set which is either empty, finite or countably infinite is called *countable* otherwise it is *uncountable*.

ILLUSTRATIONS

1. The set of all integers is countable.
2. The set $\{1, 4, 9, 16, ...\}$ is countable.
3. The set P_n of all polynomial functions with integer coefficients is countable.
4. The set of all ordered pairs of integers is countable.
5. The set of all real numbers is uncountable.

Example 12. The set of real numbers in $[0, 1]$ is uncountable.

■ Let the set of all real numbers in $[0, 1]$ be countable, *i.e.*, $\{x : 0 \le x \le 1\} = \{x_1, x_2, \ldots x_n, \ldots\}$. Each real number x_i in $[0, 1]$ has a decimal expansion $0, a_1, a_2, \ldots a_n, \ldots$where a_i, $i \in \mathbf{N}$, are any of the digits 0, 1, 2, ..., 9. We assume that the numbers whose decimal expansion terminate such as .0573 are written as .0573000 ... which is the same as .0572999.... Since all real numbers in $[0, 1]$ are countable, therefore, we can establish a 1–1 correspondence of the members of $[0, 1]$ with the set of positive integers in the following manner:

$1 \leftrightarrow 0.a_{11}a_{12}a_{13}\ldots$

$2 \leftrightarrow 0.a_{21}a_{22}a_{23}\ldots$

$3 \leftrightarrow 0.a_{31}a_{32}a_{33}\ldots$

....

We now construct a number $0.b_1b_2b_3\ldots$ where $** \; b_i = \begin{cases} 4 \text{ if } a_{ii} = 5 \\ \qquad i = 1, 2, 3, \ldots \\ 5 \text{ if } a_{ii} \ne 5 \end{cases}$

(any two digits can be used instead of 4 and 5). Then the number $0.b_1b_2b_3...$ lies between 0 and 1 and is different from the numbers in the above list and therefore cannot be in the list, contradicting the assumption that the set of all real numbers in [0, 1] is countable.

Example 13. The set of rational numbers in [0, 1] is countable.

■ In order to show that the set of rational numbers in [0, 1] is countable, we must show that there exists a 1–1 correspondence between the set of rationals of [0, 1] and the set of positive integers. Arrange the set of rationals according to increasing denominators as

$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, ...$, etc. Then the 1–1 correspondence can be indicated as:

$$\begin{array}{lll} 1 \leftrightarrow 0 & 5 \leftrightarrow \frac{2}{3} & 9 \leftrightarrow \frac{2}{5} \\ 2 \leftrightarrow 1 & 6 \leftrightarrow \frac{1}{4} & 10 \leftrightarrow \frac{3}{5} \\ 3 \leftrightarrow \frac{1}{2} & 7 \leftrightarrow \frac{3}{4} & 11 \leftrightarrow \frac{4}{5} \\ 4 \leftrightarrow \frac{1}{3} & 8 \leftrightarrow \frac{1}{5} & .... \end{array}$$

Theorem 13. If ** *$f: A \to B$ is one-to-one one B is countable then A is countable.*

If A is finite, then there is nothing to prove. Suppose A is infinite. Now A is equivalent to $f(A)$ where $f(A)$ is the range of f, so $f(A)$ is infinite. Also $f(A) \subseteq B$. Therefore B is infinite. By hypothesis B is countable so B is countably infinite. Define a mapping $\phi: \mathbf{N} \to B$ by $\phi(n) = b_n$ for each $n \in \mathbf{N}$. Then $B = \{b_1, b_2, ...\}$. Let n_1 be the first natural number such that $b_{n_1} \in f(A)$. Let n_2 be the first natural number greater than n_1 such that $b_{n_2} \in f(A)$. Again n_3 be the first natural number greater than n_2 such that $b_{n_3} \in f(A)$ and so on.

Thus $f(A) = \{b_{n_1}, b_{n_2}, b_{n_3}, ... b_{n_k}, ...\}$. We now define a mapping $g: f(A) \to \mathbf{N}$ by

$$g(b_{n_k}) = k, \text{ for } k = 1, 2, 3, ...$$

The mapping 'g' is one-to-one and onto for $k \neq j, n_k \neq n_j$. Also $f: A \to B$ and $g: f(A) \to \mathbf{N}$ implies the composition mapping $gof: A \to \mathbf{N}$ is one-one and onto. Thus A is equivalent to $\mathbf{N}$ and hence countable.

Corollary. Every subset of a countable set is countable.

Theorem 14. *The cartesian product of two countable sets is countable.*

Let A and B be any two countable sets. Then $A \times B = \{(a, b): a \in A, b \in B\}$ is their cartesian product. Now if any of the two sets is empty then $A \times B = \phi$ and there is nothing to prove. If one of the sets is finite, say A is finite with m elements then the product of

$$A = \{a_1, a_2, ..., a_m\} \text{ and } B = \{b_1, b_2, ..., b_n, ...\}$$

is

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), ..., (a_1, b_n), ... \\ &(a_2, b_1), (a_2, b_2), ..., (a_2, b_n), ... \\ &\quad \vdots \qquad\quad \vdots \\ &(a_m, b_1), (a_m, b_2), ..., (a_m, b_n), ...\} \end{aligned}$$

which can be seen to be equivalent to **N** by listing the elements as

$(a_1, b_1), (a_2, b_1) \ldots (a_m, b_1); (a_1, b_2), (a_2, b_2) \ldots (a_m, b_2); \ \ldots; (a_1, b_n) (a_2, b_n) \ldots (a_m, b_n); \ldots$

Let A and B be both countably infinite

$$A = \{a_1, a_2, \ldots\}$$
$$B = \{b_1, b_2, \ldots\},$$

then $A \times B$ is equivalent to **N** which can be exhibited as

$(a_1, b_1); (a_2, b_1), (a_1, b_2); (a_3, b_1), (a_2, b_2), (a_1, b_3); (a_4, b_1), (a_3, b_2), (a_2, b_3); (a_1, b_4), \ldots$

The function $f : A \times B \to \mathbf{N}$ is defined as

$$f(a_1, b_1) = 1, f(a_2, b_1) = 2, f(a_1, b_2) = 3 \text{ and so on,}$$

f is one-to-one and onto. Therefore $A \times B$ is countably infinite.

Theorem 15. *A countable union of countable sets is countable.*

Consider the sets $A_i = \{a_{1i}, a_{2i}, a_{3i}, \ldots\}$, $i = 1, 2, 3, \ldots$ Each A_i, $i = 1, 2, 3, \ldots$ is countable. The k th element of A_i is a_{ki}. The elements of the countable union $\bigcup\limits_{i=1}^{\infty} A_i$ of the sets A_i's can be listed as $a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23}, a_{32}, a_{41}, \ldots$ (the order has been taken according to the sum $i + j = k$, $k = 2, 3, \ldots$ i, j being the suffices of the element $a_{ij} \in A_j$). The one-one correspondence between the elements of $\bigcup\limits_{i=1}^{\infty} A_i$ and the set of positive integers is given by

$$\begin{array}{cccccccccc} a_{11} & a_{12} & a_{21} & a_{13} & a_{22} & a_{31} & a_{14} & a_{23} & a_{32} & a_{41} \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \ldots \end{array}$$

Hence, the set $\bigcup\limits_{i=1}^{\infty} A_i$ is countable.

Corollary. The set of all rational numbers is countable.

The set of all rational numbers is the union $\bigcup\limits_{i=1}^{\infty} A_i$, where A_i is the set of rationals which can be written with denominator i. That is $A_i = \left\{\frac{0}{i}, \frac{-1}{i}, \frac{1}{i}, \frac{-2}{i}, \frac{2}{i}, \ldots\right\}$. Each A_i is equivalent to the set of all positive integers and hence countable.

Example 14. The set **R** of real numbers is uncountable.

- Suppose the set **R** is countable. Then $\mathbf{R} = \{x_1, x_2, x_3, \ldots\}$. Enclose each member x_n of **R** in an open interval $I_n = \left]x_n - \frac{1}{2^{n+1}}, x_n + \frac{1}{2^{n+1}}\right[$ of length $\frac{1}{2^n}$, $n = 1, 2, 3, \ldots$. The sum of the lengths of I_n's is $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$. But $x_n \in \mathbf{R}$ and $\mathbf{R} = \bigcup\limits_{n=1}^{\infty} \{x_n\} \subseteq \bigcup\limits_{n=1}^{\infty} I_n$. This implies that the whole real line (whose length is infinite) is contained in the union of intervals whose lengths add up to 1. This is a contradiction. Hence the set **R** is uncountable.