$\Rightarrow$ 2015 PDE

Q.1 $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

$$Pp + Qq = R$$

$P = y^2+z^2-x^2$; $Q = -2xy$ $R = -2xz$

Lagrange's aux. eq^n

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} \quad -(i)$$

taking last two frac^n

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z c_1$$

$$\Rightarrow \boxed{y = z c_1} \Rightarrow \boxed{\frac{y}{z} = c_1}$$

taking multiplier x, y, z

every frac^n of (i) will become

$$\frac{x\,dx + y\,dy + z\,dz}{-x(x^2+y^2+z^2)} = \frac{dy}{-2xy}$$

$$\Rightarrow \frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2} = \frac{dy}{y} \Rightarrow \boxed{\frac{x^2+y^2+z^2}{y} = c_2}$$

$$\boxed{f\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{y}\right) = 0} \text{ — Ans}$$

Q.2 $(D^2 + DD' - 2D'^2)u = e^{x+y}$

Aux. eqn $(m^2 + m - 2)u = 0$ (taking $D = m$, $D' = 1$)

$(m+2)(m-1) = 0$

$m = -2, m = 1$

C.F $u = \phi_1(y-2x) + \phi_2(y+x)$

$$P.I = \frac{1}{(D+2D')(D-D')} e^{x+y} \qquad a=1, b=1$$

$F(D, D') = F(a,b)$
$= F(1,1)$
$=$

$$= \frac{1}{(D-D')}\left[\frac{1}{(1)+(2)} e^{x+y}\right]$$

$$= \frac{1}{3}\left[\frac{1}{D-D'} e^{x+y}\right]$$

$\rightarrow F(a,b) = F(1,1) = 0$

$bD - aD' = D - D'$ is a factor

$$= \frac{1}{3}\frac{x}{(1!)} e^{x+y} \qquad \left[\because \frac{1}{(bD-aD')^n}\phi(ax+by) = \frac{x^n}{n!\, b^n}\phi(ax+by)\right]$$

$\Rightarrow$ gen. soln $=$ C.F + P.I

$= \phi_1(y-2x) + \phi_2(y+x) + \frac{x}{3}e^{x+y}$ — Ans

Q.3 $p\cos(x+y) + q\sin(x+y) = z$

$\Rightarrow Pp + Qq = R$

$P = \cos(x+y)$, $Q = \sin(x+y)$ & $R = z$

Lagrange's aux. eqn

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \quad \text{---(i)}$$

taking first two fraction of (i)

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)}$$

$$\Rightarrow \frac{dy}{dx} = \tan(x+y)$$

$$x+y = t$$

$$\Rightarrow 1+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx} = 1+\tan t \Rightarrow \int \frac{dt}{1+\tan t} = \int dx$$

$$\Rightarrow \int \frac{\cos t \, dt}{\sin t + \cos t} = x + c_1$$

$$\text{Sol:} \quad \frac{\cos t}{\sin t + \cos t} = \frac{1}{2}(\sin t + \cos t) + \frac{1}{2}(\cos t - \sin t)$$

$$= \int \frac{1}{2} dt + \frac{1}{2}\int \frac{\cos t - \sin t}{\sin t + \cos t} dt = x + c_1$$

$$\Rightarrow \frac{t}{2} + \frac{1}{2} \log(\sin t + \cos t) = x + c_1$$

$$\Rightarrow \boxed{x+y + \log(\sin(x+y) + \cos(x+y)) - 2x = c_1} \quad \text{—(ii)}$$

→ taking multiplier 1, 1, 0

$$\frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dz}{z}$$

$$\Rightarrow \int \frac{d(x+y)}{\cos(x+y) + \sin(x+y)} = \int \frac{dz}{z}$$

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$$\int \frac{dt}{\sin t + \cos t} = \int \frac{dt \left(\sec^2 \frac{t}{2}\right)}{2\tan\frac{t}{2} + 1 - \tan^2\frac{t}{2}} = \frac{dz}{z}$$

$$\tan\frac{t}{2} = u$$

$$\sec^2\frac{t}{2}\,dt = 2u\,du$$

$$\int \frac{2u\,du}{1+2u-u^2} = \int \frac{2u-2+2}{1+2u-u^2}\,du = \int \frac{(2u-2)\,du}{1+2u-u^2} + 2\int \frac{du}{1+2u-u^2}$$

$$= -\log(1+2u-u^2) + \frac{2}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+u}{\sqrt{2}-u}\right) = \log z C_2$$

$$\log\left[\left(\frac{\sqrt{2}+u}{\sqrt{2}-u}\right)^{1/2} \Big/ 1+2u-u^2\right] = \log z C_2$$

$$\rightarrow \frac{1}{z}\left\{\frac{\sqrt{2}+\tan\left(\frac{x+y}{2}\right)}{\sqrt{2}-\tan\left(\frac{x+y}{2}\right)\left(1+2\tan\left(\frac{x+y}{2}\right)-\tan\left(\frac{x+y}{2}\right)^2\right)}\right\} = C_2$$

Q.4

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = 0 \quad \text{---(1)}$$

$u(0,t) = u(1,t) = 0, \; t \geq 0$
$u(x,0) = x(1-x) \; 0 \leq x \leq 1$

let $u(x,t) = X(x)\,T(t) \Rightarrow \boxed{X(0) = X(1) = 0}$

$\frac{\partial u}{\partial t} = X(x)\,T'(t)$

$\frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$

$\Rightarrow \quad X(x)T'(t) - X''(x)T(t) + X(x)T(t) = 0$

$\Rightarrow \quad X(x)T'(t) + T(t)[X(x) - X''(x)] = 0$

$\Rightarrow \quad \frac{X''(x) - X(x)}{X(x)} = \frac{T'(t)}{T(t)} = \mu$

$\Rightarrow \quad X''(x) - X(x) - \mu(X(x)) = 0$

$\Rightarrow \quad X''(x) - (\mu+1)X(x) = 0$

i) $\mu = -1$

$\Rightarrow X''(x) = 0$

$X(x) = Ax + B$

$\Rightarrow X(0) = A(0) + B$

$X(1) = A + B$

$\Rightarrow A = B = 0 \Rightarrow X(x) = 0$

$\Rightarrow u(x,t) = 0$

$\downarrow$

does not satisfy $u(x,0)$

rejecting $\mu = -1$

ii) $\mu = -1 + \lambda^2 \; (\lambda \neq 0)$

$X'' - \lambda^2 X(x) = 0$

$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$

$X(0) = c_1 + c_2 = 0$

$X(1) = c_1 e^{\lambda} + c_2 e^{-\lambda} = 0$

$\Rightarrow c_1 = c_2 = 0$

$\Rightarrow X(x) = 0$

$\Rightarrow u(x,t) = 0$

$\downarrow$

rejecting $\mu = -1 + \lambda^2$

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(iii) $\mu = -1-\lambda^2 \quad (\lambda \neq 0)$

$X''(x) + \lambda^2 X(x) = 0$

$X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$

$X(0) = 0 = c_1$

$X(1) = 0 = c_2 \sin \lambda x$

taking $c_2 \neq 0$ $\quad \sin \lambda(1) = 0$

$\Rightarrow \lambda(1) = n\pi$

$\boxed{\lambda = n\pi}$

$T'(t) - \mu T(t) = 0$

$T'(t) + (1+\lambda^2) T(t) = 0$

$T(t) = c_3 e^{-(1+\lambda^2)t}$

$X(x) = c_2 \sin n\pi x$

$u_n(x,t) = E_n \sin n\pi x \, e^{-(1+\lambda^2)t}$

$u(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x \, e^{-(1+\lambda^2)t}$

using initial cond^n

$u(x,0) = x(1-x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x)$

$\Rightarrow E_n = 2\int_0^1 x(1-x) \sin n\pi x$

$= 2\int_0^1 x \sin n\pi x - x^2 \sin n\pi x$

$E_n = \frac{4}{(n\pi)^3}\left[1 - \cos n\pi\right] = \begin{cases} 0 & n = 2m \\ \frac{8}{(n\pi)^3} & n = (2m-1) \end{cases}$

$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{8}{((2m-1)\pi)^3} \sin(2m-1)\pi x \; \cancel{e^{-(1+n^2\pi^2)t}} \; e^{-(1+(2m-1)^2\pi^2)t}}$

Ans

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surface (3) along its section by

Q.1 Reduce to canonical form & find gen. sol^n

$$x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$x^2 r^2 - 2xys + y^2 t^2 + xp + yq = 0 \quad \left\{ r = \frac{\partial^2 u}{\partial x^2} \quad s = \frac{\partial^2 u}{\partial x \partial y} \quad t = \frac{\partial^2 u}{\partial y^2} \quad p = \frac{\partial u}{\partial x} \quad q = \frac{\partial u}{\partial y} \right\}$$

Comparing with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

$$\Rightarrow R = x^2 \quad S = -2xy, \quad T = y^2$$

Quadratic eq^n

$$R\lambda^2 + S\lambda + T = 0$$

$$\Rightarrow x^2\lambda^2 - 2xy\lambda + y^2 = 0$$

$$(x\lambda - y)^2 = 0$$

$$\lambda = \frac{y}{x} \quad \{\text{equal roots}\}$$

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

$$\log y = \log\left(\frac{c_1}{x}\right)$$

$$\Rightarrow \boxed{xy = c_1} \qquad \boxed{v = xy}$$

assuming $\boxed{w = \frac{y}{x}}$ s.t. it is independent of v

$$\Rightarrow p = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} = \frac{\partial u}{\partial v}y + \frac{\partial u}{\partial w}\left(\frac{-y}{x^2}\right)$$

$$p = y\frac{\partial u}{\partial v} - \frac{y}{x^2}\frac{\partial u}{\partial w}$$

$$q = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial y} = x\frac{\partial u}{\partial v} + \frac{1}{x}\frac{\partial u}{\partial w}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[y\frac{\partial u}{\partial v} - \frac{y}{x^2}\frac{\partial u}{\partial w}\right]$$

$$= y\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x} + y\frac{\partial^2 u}{\partial v\partial w}\frac{\partial w}{\partial x} + \frac{2y}{x^3}\frac{\partial u}{\partial w} - \frac{y}{x^2}\frac{\partial^2 u}{\partial w^2}\frac{\partial w}{\partial x} - \frac{y}{x^2}\frac{\partial^2 u}{\partial v\partial w}\frac{\partial v}{\partial x}$$

$$= y^2\frac{\partial^2 u}{\partial v^2} - \frac{y^2}{x^2}\frac{\partial^2 u}{\partial v\partial w} + \frac{2y}{x^3}\frac{\partial u}{\partial w} + \frac{y^2}{x^4}\frac{\partial^2 u}{\partial w^2} - \frac{y^2}{x^2}\frac{\partial^2 u}{\partial v\partial w}$$

$$= y^2\frac{\partial^2 u}{\partial v^2} - \frac{2y^2}{x^2}\frac{\partial^2 u}{\partial v\partial w} + \frac{y^2}{x^4}\frac{\partial^2 u}{\partial w^2} + \frac{2y}{x^3}\frac{\partial u}{\partial w}$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial v} + \frac{1}{x}\frac{\partial u}{\partial w}\right)$$

$$= \frac{\partial u}{\partial v} + x\frac{\partial^2 u}{\partial v^2}\frac{\partial v}{\partial x} + x\frac{\partial^2 u}{\partial w\partial v}\frac{\partial w}{\partial x} - \frac{1}{x^2}\frac{\partial u}{\partial w} + \frac{1}{x}\frac{\partial^2 u}{\partial w^2}\frac{\partial w}{\partial x} + \frac{1}{x}\frac{\partial^2 u}{\partial v\partial w}\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial v} + xy\frac{\partial^2 u}{\partial v^2} - \frac{y}{x}\frac{\partial^2 u}{\partial v\partial w} - \frac{1}{x^2}\frac{\partial u}{\partial w} - \frac{y}{x^3}\frac{\partial^2 u}{\partial w^2} + \frac{y}{x}\frac{\partial^2 u}{\partial v\partial w}$$

$$= xy\frac{\partial^2 u}{\partial v^2} - \frac{y}{x^3}\frac{\partial^2 u}{\partial w^2} - \frac{1}{x^2}\frac{\partial u}{\partial w} + \frac{\partial u}{\partial v}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = x^2\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v\partial w} + \frac{1}{x^2}\frac{\partial^2 u}{\partial w^2}$$

putting value in eq (1)

$$\cancel{x^2y^2\frac{\partial^2 u}{\partial v^2}} - 2y^2\frac{\partial^2 u}{\partial v\partial w} + \frac{y^2}{x^2}\frac{\partial^2 u}{\partial w^2} + \frac{2y}{x}\frac{\partial u}{\partial w} - \cancel{2x^2y^2\frac{\partial^2 u}{\partial v^2}} + \cancel{x^2y^2\frac{\partial^2 u}{\partial v^2}} + \cancel{2y^2\frac{\partial^2 u}{\partial v\partial w}}$$

$$+ \frac{2y^2}{x^2}\frac{\partial^2 u}{\partial w^2} + \frac{2y}{x}\frac{\partial u}{\partial w} - \cancel{2xy\frac{\partial u}{\partial v}}$$

$$+ \frac{y^2}{x^2}\frac{\partial^2 u}{\partial w^2} + \cancel{xy\frac{\partial u}{\partial v}} - \cancel{\frac{y}{x}\frac{\partial u}{\partial w}} + \cancel{xy\frac{\partial u}{\partial v}} + \cancel{\frac{y}{x}\frac{\partial u}{\partial w}} = 0$$

$$\frac{4y^2}{x^2}\frac{\partial^2 u}{\partial w^2} + \frac{4y}{x}\frac{\partial u}{\partial w} = 0$$

$$\Rightarrow \frac{y^2}{x^2}\frac{\partial^2 u}{\partial w^2} + \frac{y}{x}\frac{\partial u}{\partial w} = 0$$

$$\boxed{w^2\frac{\partial^2 u}{\partial w^2} + w\frac{\partial u}{\partial w} = 0}$$ – Ca

$$\boxed{-w\frac{\partial^2 u}{\partial w^2} = +\frac{\partial u}{\partial w}}$$ – Canonical form

Ans

$$\Rightarrow w\frac{\partial q}{\partial w} + q = 0 \qquad \left(q = \frac{\partial u}{\partial w}\right)$$

$$w\frac{\partial q}{\partial w} = -q$$

$$-\frac{dq}{q} = \frac{dw}{w}$$

integrating

$$-\log q = \log w + \phi(v)$$

$$\log qw = -\phi(v)$$

$$qw = e^{-\phi(v)} \qquad \rightarrow \left[e^{-\phi(v)} = \psi_1(v)\right]$$

$$\Rightarrow qw = \psi(v)$$

$$w\frac{\partial u}{\partial w} = \psi(v)$$

$$\int \partial u = \int \psi(v)\frac{dw}{w}$$

$$\rightarrow u = \psi_1(v)\log w + \psi_2(v)$$

$$\boxed{u = \psi_1(xy)\log\left(\frac{y}{x}\right) + \psi_2(xy)}$$

$u = \cancel{\psi_1(v)}$

$u = \cancel{\psi_1\left(\frac{y}{x}\right)}$

Ans