Chapter 12

2009

12.1 Section-A

Question-1(a); Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by $\{t^3+4t^2-t+3, t^3+5t^2+5, 3t^3+10t^2-5t+5\}$ and $\{t^3+4t^2+6, t^3+2t^2-t+5, 2t^3+2t^2-3t+9\}$ respectively. Find
(i) $dim(U+W)$
(ii) $dim(U \cap W)$

[10 Marks]

Solution: Let $S=\{t^3+4t^2-t+3, t^3+5t^2+5, 3t^3+10t^2-5t+5\}=\{\alpha_1,\alpha_2,\alpha_3\}$, and $T=\{t^3+4t^2+6, t^3+2t^2-t+5, 2t^3+2t^2-3t+9\}$=$\{\beta_1,\beta_2,\beta_3\}$
Since U and W are spanned by sets S and T of all polynomials of degree 3.
$\therefore U$ and W are subspaces of vector space $V(R)$ of all real polynomials of degree ≤ 3.

We know that the set $S_1=\{1,t,t^2,t^3\}$ is a standard basis for $V(R)$.
Now, the coordinte vectors of $\alpha_1,\alpha_2,\alpha_3$ wrt above basis S_1 are given by: $(3,-1,4,1)$, $(5,0,5,1)$ and $(5,-5,10,3)$, and the coordinate vectors of β_1,β_2,β_3 wrt above basis S_1 are given by: $(6,0,4,1)$, $(5,-1,2,1)$ and $(9,-3,2,2)$.

(i) Since U and W are 2 subspaces of $V(R)$.
$\therefore U+W$ is also a subspace of $V(R)$.
$\implies$ $U+W$ is the space generated by all the 6 coordinate vectors.
Now, for the matrix A whose rows are given by 6 coordinate vectors abd reduce it to echelon form.

$$A=\begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \\ 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix}$$

On performing row operations, $R_2 \to 3R_2-5R_1$, $R_3 \longrightarrow 3R_3-5R_1$, $R_4 \to R_4-2R_1$, $R_5 \to 3R_5-5R_1$ and $R_6 \to R_6-3R_1$ we get:

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & 2 \\ 0 & -10 & 10 & 4 \\ 0 & 2 & -4 & -1 \\ 0 & 2 & -14 & -2 \\ 0 & 0 & -10 & -1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 2R_2$, $R_4 \longrightarrow 5R_4 - 2R_2$, $R_5 \longrightarrow 5R_5 - 2R_2$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -16 \\ 0 & 0 & -10 & -1 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_6$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & -60 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$, $R_5 \to R_5 - bR_3$

$$A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -10 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form.
$\therefore$ The echelon form of A has 3 non-zero rows, $\implies dim(U+W) = 3...$ **(a)**

(ii) Now, form the matrix A whose rows are coordinate vectors of $'S'$ and reduce it to echelon form.

$$\implies A = \begin{bmatrix} 3 & -1 & 4 & 1 \\ 5 & 0 & 5 & 1 \\ 5 & -5 & 10 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & -10 & 10 & 4 \end{bmatrix} \quad (R_2 \to 3R_2 - 5R_1)$$

$$\implies A \sim \begin{bmatrix} 3 & -1 & 4 & 1 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2$$

$\therefore$ The echelon matrix of A has 2 non-zero rows. $\implies dim(U) = 2$ ... **(b)**
Again, form matrix $'A'$ whose rows are coordinate vectors of T and reduce it to an echelon matrix.

$$A = \begin{bmatrix} 6 & 0 & 4 & 1 \\ 5 & -1 & 2 & 1 \\ 9 & -3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -18 & -24 & 3 \end{bmatrix} \begin{pmatrix} R_2 \to 6R_2 - 5R_1 \\ R_3 \to 6R_3 - 9R_1 \end{pmatrix}$$

$$\implies A \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & -6 & -8 & 1 \end{bmatrix} \quad R_3 \to \frac{1}{3}R_3$$

$$\implies A \sim \begin{bmatrix} 6 & 0 & 4 & 1 \\ 0 & -6 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

It is in echelon form with 2 non-zero rows. $\therefore dim(W) = 2$ ... **(c)**
Now, since $dim(U \cap W) = dim(U) + dim(W) - dim(U + W) = 2 + 2 - 3 = 1$ (Using (a), (b) and (c)) $\therefore dim(U \cap W) = 1$

Question-1(b) Find a linear map $T : R^3 \to R^4$ whose image is generated by $(1, 2, 0, -4)$ and $(2, 0, -1, -3)$.

[10 Marks]

Solution: Given that $R(T)$ is spanned by $\{(1, 2, 0, -4), (2, 0, -1, -3)\}$.
Let us include a vector $(0, 0, 0, 0)$ in this set which will not affect the spanning property so that:
$S = \{(1, 2, 0, -4), (2, 0, -1, -3), (0, 0, 0, 0)\}$ Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the standard basis of R^3.
We know that there exists a transformation $'T'$ such that

$$\begin{aligned} T(\alpha_1) &= (1, 2, 0, -4) \\ T(\alpha_2) &= (2, 0, -1, -3) \\ T(\alpha_3) &= (0, 0, 0, 0) \end{aligned}$$

Now,

$$\begin{aligned} \alpha \in R^3 \Rightarrow \alpha &= (a, b, c) \\ &= a\alpha_1 + b\alpha_2 + c\alpha_3 \end{aligned}$$

Therefore,

$$\begin{aligned} T(\alpha) &= T(a\alpha_1 + b\alpha_2 + c + \alpha_3) \\ &= aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3) \\ &= a(1, 2, 0, -4) + b(2, 0, -1, -3) + c(0, 0, 0, 0) \end{aligned}$$

$\therefore T(a, b, c) = (a + 2b, 2a, -b, -4a - 3b)$ which is the required transformation.

Question-1(c) (i)Find the difference between the maximum and the minimum of the function $\left(a - \frac{1}{a} - x\right)(4 - 3x^2)$ where a is a constant and greater than zero.

[5 Marks]

(ii) If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(\theta h)$, $0 < \theta < 1$. Find θ, when $h = 1$ and $f(x) = (1 - x)^{5/2}$.

[5 Marks]

Solution: (i) Let $f(x) = \left(a - \frac{1}{a} - x\right)(4 - 3x^2)$ ... (a)

where a is a constant and greater than 0.

$\implies f'(x) = \left(a - \frac{1}{a} - x\right)(-6x) + (-1)(4 - 3x^2)$

$\implies f'(x) = -6ax + \frac{6x}{a} + 6x^2 - 4 + 3x^2$

$\implies f'(x) = 9x^2 - 6\left(a - \frac{1}{a}\right)x - 4 \ldots$ (b)

For maxima or minima, $f' = 0$

$$\implies 9x^2 - 6\left(a - \frac{1}{a}\right)x - 4 = 0$$

$$\implies x = \frac{6\left(a-\frac{1}{a}\right) \pm \sqrt{36\left(a-\frac{1}{a}\right)^2 + 36\times 4}}{2\times 9}$$

$$\implies x = \frac{\left(a-\frac{1}{a}\right) \pm \sqrt{\left(a-\frac{1}{a}\right)^2 + 4}}{3}$$

$$\implies x = \frac{\left(a-\frac{1}{a}\right) \pm \left(a+\frac{1}{a}\right)}{3}$$

$$\implies x = \frac{2a}{3}, \frac{-2}{3a}$$

From (b), $f''(x) = 18x - 6\left(a - \frac{1}{a}\right)$ ... (c)

For $x = \frac{2a}{3}$,

$$f''\left(\frac{2a}{3}\right) = 18 \times \frac{2a}{3} - 6\left(a - \frac{1}{a}\right)$$

$$\implies f''\left(\frac{2a}{3}\right) = 6a + \frac{6}{a} > 0 \because a > 0$$

$$\implies f \text{ is minimum at } x = \frac{2a}{3}.$$

$\therefore f_{min} = f\left(\frac{2a}{3}\right) = \left(a - \frac{1}{a} - \frac{2a}{3}\right)\left(4 - 3 \times \frac{4a^2}{9}\right)$

$$\implies f_{min} = \left(\frac{a}{3} - \frac{1}{a}\right)\left(4 - \frac{4a^2}{3}\right)$$

$$\implies f_{min} = \frac{4a}{3} - \frac{4}{a} - \frac{4a^3}{9} + \frac{4a}{3}$$

$$\implies f_{min} = \frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9} \ldots \text{(d)}$$

For $x = \frac{-2}{3a}$,

$$f''\left(-\tfrac{2}{3a}\right) = 18\left(\tfrac{-2}{3a}\right) - 6\left(a - \tfrac{1}{a}\right)$$

$$f''\left(-\tfrac{2}{3a}\right) = \tfrac{-12}{a} - 6a + \tfrac{6}{a} = \tfrac{-6}{a} - 6a < 0$$

$$\implies f \text{ is maximum at } x = \tfrac{-2}{3a}$$

$\therefore f_{max} = f\left(-\tfrac{2}{3a}\right) = 4a - \tfrac{8}{3a} + \tfrac{4}{9a^3} \ldots$ (e)

Subtracting (e) trom (d) we get,

$$\begin{aligned}\left(4a - \frac{8}{3a} + \frac{4}{9a^3}\right) - \left(\frac{8a}{3} - \frac{4}{a} - \frac{4a^3}{9}\right) &= \frac{4a}{3} + \frac{4}{3a} + \frac{4}{a}\left(\frac{1}{a^3} + a^3\right)\\ &= \frac{4}{3}\left(a + \frac{1}{a}\right) + \frac{4}{9}\left(a + \frac{1}{a}\right)\left(a^2 + \frac{1}{a^2} + 1\right)\\ &= \left(a + \frac{1}{a}\right)\left[\frac{4}{3} + \frac{4}{9}\left(a^2 + \frac{1}{a^2} + 1\right)\right]\end{aligned}$$

which is the required answer.

(ii) Given: $f(x) = (1-x)^{5/2} \implies f(h) = (1-h)^{5/2}$
Now,

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \implies f'(0) = -\frac{5}{2}$$

Also,

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta h) = \frac{15}{4}(1-\theta h)^{1/2}$$

$$\therefore f(h) = f(0) + hf'(0) + \tfrac{h^2}{2!}f''(\theta h),\ 0 < \theta < 1$$

$$\implies (1-h)^{5/2} = 1 + h\left(\tfrac{-5}{2}\right) + \tfrac{h^2}{2!}\tfrac{15}{4}(1-\theta h)^{1/2}$$

When $h = 1$,

$$0 = 1 + \tfrac{-5}{2} + \tfrac{1}{2} \times \tfrac{15}{4}(1-\theta)^{1/2}$$

$$\implies 0 = 1 - \tfrac{5}{2} + \tfrac{15}{8} - (1-\theta)^{1/2}$$

$$\implies 0 = \tfrac{3}{8} - (1-\theta)^{1/2}$$

$$\implies (1-\theta)^{1/2} = 3/8$$

$$\implies (1-\theta) = \tfrac{9}{64}$$

$$\implies \theta = \tfrac{55}{64}$$

Question-1(d) Evaluate:

(*i*) $\int_0^{\pi/2} \dfrac{\sin^2 x dx}{\sin x + \cos x}$

[6 Marks]

(*ii*) $\int_1^{\infty} \dfrac{x^2 dx}{(1+x^2)^2}$

[4 Marks]

Solution: (i) $I = \int_0^{\pi/2} \frac{\sin^2 x dx}{\sin x + \cos x}$

$\implies I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right) du}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}$ [Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$] ... (a)

$\implies I = \int_0^{\pi/2} \frac{\cos^2 x dx}{\sin x + \cos x}$ ... (b)

Adding (a) and (b), we get:

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx + \int_0^{\pi} \frac{\cos^2 x}{\sin x + \cos x} dx$$

$$\implies I = \frac{1}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$\implies I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin x/\sqrt{2} + \cos x/\sqrt{2}}$$

$$\implies I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{\pi}{4}\right)}$$

$$\implies I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right) dx.$$

$$\implies \frac{1}{2\sqrt{2}}[\log|\sqrt{x}+1| - \log|\sqrt{2}-1|]$$

$$\implies I = \frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

(ii) $I = \int_1^{\infty} \dfrac{x^2 dx}{(1+x^2)^2} = \lim_{t\to\infty} \int_1^t \dfrac{x^2 dx}{(1+x^2)^2}$

Let $x = \tan\theta \implies dx = \sec^2\theta d\theta$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \frac{\tan^2\theta \sec^2\theta d\theta}{(1+\tan^2\theta)^2}$$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \frac{\tan^2\theta}{\sec^2\theta} d\theta = \int_{\pi/4}^{\tan^{-1} t} \sin^2\theta d\theta$$

$$\implies I = \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1} t} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{1}{2} \times \frac{2\tan\theta}{1+\tan^2\theta}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \lim_{t\to\infty} \frac{1}{2}\left[\theta - \frac{\tan\theta}{1+\tan^2\theta}\right]_{\pi/4}^{\tan^{-1} t}$$

$$\implies I = \frac{1}{2}\lim_{t\to\infty}\left[\left(\tan^{-1} t - \frac{t}{1+1^2}\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$\implies I = \frac{1}{2}\left[\frac{\pi}{2} - 0 - \frac{\pi}{4} + \frac{1}{2}\right]$$

$$\implies I = \frac{1}{2}\left[\frac{1}{2} + \frac{\pi}{4}\right] = \frac{\pi}{8} + \frac{1}{4}$$

$$\implies \int_1^\infty \frac{x^2}{(1+x^2)^2}dx \text{ is convergent and its value is } \frac{\pi}{8} + \frac{1}{4}$$

Question-1(e) Show that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z = 2$ in a circle of radius unity and find the equation of the sphere which has this circle as one of its great circles.

[10 Marks]

Solution: The given sphere is. $x^2 + y^2 + z^2 - x - z - 2 = 0 \ldots$ (i)
and the given plane is $x + 2y - z - 4 = 0 \ldots$ (ii)

Centre of sphere (i) is $C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ and
its radius is $CP = \sqrt{\frac{1}{4} + 0 + \frac{1}{4} + 2} = \sqrt{\frac{5}{2}}$
CA is the perpendicular distance from $C\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$ to plane (ii) and is given by:

$$CP = \frac{\left|\frac{1}{2} + 2(0) - \frac{1}{2} - 4\right|}{\sqrt{1+4+1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

$\therefore$ Radius of circle, $AP = \sqrt{CP^2 - CA^2} = \sqrt{\frac{5}{2} - \frac{3}{2}} = 1$

The plane (ii) meets the sphere (i) in a circle of radius 1. Now, any sphere through the intersection of (i) and (ii) is given by:

$$x^2 + y^2 + z^2 - x + z - 2 + k(x + 2y - z - 4) = 0...(iii)$$

If the circle of intersection of (i) and (ii) is a great circle of sphere (iii), then the centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ lies on plane (ii).

$\therefore \frac{1-k}{k} + 2(-k) - \left(\frac{k-1}{2}\right) - 4 = 0 \implies k = 1$
$\therefore$ required equation of sphere is given by:

$$x^2 + y^2 + z^2 - 2x - 2z + 2x + 2 = 0$$

Question-2(a) Let T be the linear operator on R^3 defined by $T(x, y, z) = (2x, 4x-y, 2x + 3y - z)$
(i) Show that T is invertible.
(ii) Find a formula for T^{-1}.

[10 Marks]

Solution: (i) Let $(x, y, z) \in ker(T)$ be arbitrary.

$$\implies T(x, y, z) = (0, 0, 0)$$

$$\implies (2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$$

$$\implies 2x = 0,\ 4x - y = 0 \text{ and } 2x + 3y - z = 0$$

$$\implies x = 0,\ y = 0,\ z = 0$$

$$\implies ker(T) = \{()0, 0, 0)\}$$

Hence, T is invertible.

(ii) Now we shall find T^{-1}. Since T is invertible, therefore T is onto.
For any $(a, b, c) \in R^3$, there exists some $(x, y, z) \in R^3$ such that

$$T(x, y, z) = (a, b, c).$$

$$\implies (2x, 4x - y, 2x + 3y - z) = (a, b, c)$$

$$\implies 2x = a, 4x - y = b, 2x + 3y - z = c$$

$$\implies a = \tfrac{a}{2}, y = 2a - b$$

From $2x + 3y - z = c$, we get:

$$a\left(\tfrac{a}{2}\right) + 3(2a - b) - z = c$$

$$\implies a + 6a - 3b - z = c$$

$$\implies z = 7a - 3b - c$$

Hence $T(x, y, z) = (a, b, c)$

$$\implies T^{-1}(a, b, c) = (x, y, z)$$

$$\implies T^{-1}(a, b, c) = \left(\tfrac{a}{2}, 2a - b, 7a - 3b - c\right) \forall (a, b, c) \in R^3$$

Question-2(b) Find the rank of the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

[10 Marks]

Solution:

$$A = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$$

Applying the operations $R_2 \to R_2 - R_1$, $R_1 \to R_3 - 2R_1$ and $R_4 \to R_4 - 3R_1$ we get:

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 + 3R_2$ and $R_1 \to R_4 + R_2$ we get;

$$A \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in echelon form. Now, the number of non-zero rows of this echelon form is 2.
$\therefore$ Rank of A is equal to 2.

Question-2(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. **Is A similar to a diagonal matrix? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.**

[10 Marks]

Solution: Characteristic Equation of A is given by:
$\lambda^3 - Trace(A)\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| = 0$ ... (i)
Now, $Trace(A) = 1 - 5 + 4 = 0$,
$C_{11} + C_{22} + C_{33} = (-20 + 18) + (4 - 18) + (-5 + 9) = -2 - 14 + 4 = -12$, and
$|A| = 16$.

$\therefore$, (i) becomes $\lambda^3 + 0\lambda^2 - 12\lambda - 16 = 0$
$\implies \lambda = 4, -2, -2$ (Use calculator for this step)

Now, we find the eigenvectors corresponding to the above eigenvalues:

i) $\lambda = 4$, $(A - 4I)X = 0$

$$\implies \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 + R_3 + 2R_1$ and $R_1 \to \frac{R_1}{-3}$ we get:

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2/12$ and $R_3 \to R_3/12$ we get:

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$,

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, we get:

$$\sim \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$\implies x - \frac{z}{2} = 0$, $y - \frac{z}{2} = 0$
$\implies x = \frac{z}{2}$, $y = \frac{z}{2}$

$$\begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} z/2 \\ z/2 \\ z \end{bmatrix} = z/2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$X_1 = (1, 1, 2)$ is eigenvector for $\lambda = 4$

ii) $\lambda = -2$, $(A + 2I)x = 0$

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\implies x - y + z = 0$
$\implies x = y - z$

$$\therefore \begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\implies x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors for $\lambda = -2$.

Since algebraic multiplicity of each eigenvalue is equal to its geometic multiplicity, hence given matrix A is diagonalizable i.e. similar to some diagonal matrix.

Transformation matrix:

$$p = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Diagonal matrix:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We can verify that $P^{-1}AP = D$, where $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ -2 & 2 & 0 \end{bmatrix}$

Question-2(d) Find an orthogonal transformation of coordinates to reduce the quadratic form $g(x, y) = 2x^2 + 2xy + 2y^2$ to a canonical form.

[10 Marks]

Solution: The matrix form of the given quadratic form is given by:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Now, characteristic equation of A is given by $|A - \lambda I| = 0$

$$\implies \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\implies (\lambda - 2)^2 - 1 = 0$$

$$\implies \lambda^2 - 4\lambda + 3 = 0$$

$$\implies (\lambda - 3)(\lambda - 1) = 0$$

$$\implies \lambda = 1, 3$$

Eigenvector for $\lambda = 1$:

$$(A - 1.I)X = 0$$

$$\implies \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies x + y = 0 \text{ and } x = -y$$

$$\implies v_1(x, y) = (1, -1)$$

Eigenvector for $\lambda = 3$:

$$(A - 3I)X = 0$$

$$\implies \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies x = y$$
$$\implies v_2 = (x, y) = (1, 1)$$

Since the vectors v_1 and v_2 are orthogonal, $\implies v_1 v_2^T = 0$

Modal matrix comprising of eigenvectors is given by $M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\implies$ Normalized modal matrix is given by $N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\implies N^\top = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

To obtain the canonical form, we calculate $N^T(AN)$.

Now,

$$AN = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix}$$
$$\implies N^T(AN) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$\therefore$ Canonical form is given by:

$$y_1^2 + 3y_2^2 = \begin{bmatrix} y_1 & y_2 \end{bmatrix} D \begin{bmatrix} y_1 & y_2 \end{bmatrix}^\top$$

Question-3(a) The adiabatic law for the expansion of air is $PV^{1/4} = K$, where K is a constant. If at a given time the volume is observed to be 50c.c. and the pressure is 30kg per square centimetre, at what rate is the pressure changing if the volume is decreasing at the rate of 2 c.c. per second?

[10 Marks]

Solution: $PV^{1.4} = K \Rightarrow K = 30(50)^{1.4}$ $P = KV^{-1.4}$

$$\begin{aligned} \frac{dP}{dt} &= K(-1.4)\mathrm{V}^{-2\cdot 4} \cdot \frac{dV}{dt} \\ &= -30(50)^{1.4} \frac{1.4}{(50)^{2.4}}(-2) \\ &= \frac{30 \times 1.4 \times 2}{50} \\ &= 1.68 \end{aligned}$$

$\therefore$ The pressure in increasing at the rate of 1.68 $kg/cm^2/sec$.

Question-3(b) Determine the asymptotes of the curve $x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$.

[10 Marks]

Solution: Asymptotes parallel to coordinate axes: As the coefficients of highest degree terms of x and y i. x^3 and y^3 are constants, so the curvve has no asymptotes parallel to $x-axis$ or $y-axis$.
Oblique Asymptotes: Put $x=1, y=m$ in the third, second and first degree terms,

$$\begin{aligned}\phi_3(m) &= 1+m-m^2-m^3\\ \phi_2(m) &= 2m+2m^2\\ \phi_1(m) &= -3+m\end{aligned}$$

Slopes of the asymptotes are real roots of the equation: $\phi_3(n)=0$.

$$\implies 1+m-m^2(1+m)=0$$

$$\implies (1+m)(1-m^2)=0$$

$$\implies m=1,-1,-1$$

For $m=1$:

$$c=-\frac{\phi_2(m)}{\phi_3'(m)}=-\left(\frac{2m+2m^2}{1-2m-3m^2}\right)=\frac{-4}{-4}=1$$

$$\implies y=x+1$$

For, $m=-1$, c is given by:

$$\frac{c^2}{2!}\phi_3''(m)+c\phi_2'(m)+\phi_1(m)=0$$

$$\implies \frac{c^2}{2}(-2-6m)+c(2+4m)+(-3+m)=0$$

$$\implies \frac{c^2}{2}(4)+c(-2)+(-4)=0 \quad (\because m=-1)$$

$$\implies 2c^2-2c-4=0$$

$$\implies c^2-c-2=0$$

$$\implies (c-2)(c+1)=0$$

$$\implies c=2,-1$$

$$\implies y=-x+2,\ y=-x-1$$

Question-3(c) **Evaluate:** $\iint_D x\sin(x+y)dxdy$, **where** D **is the region bounded by** $0 \leqslant x \leqslant \pi$ **and** $0 \leqslant y \leqslant \dfrac{\pi}{2}$.

[10 Marks]

Solution: Let $I = \int_{x=0}^{\pi}\int_{y=0}^{\pi/2} x\sin(x+y)dxdy$

$\implies I = -\int_0^{\pi} x[\cos(x+y)]_{y=0}^{\pi/2}dx$

$\implies I = -\int_0^{\pi} x\left[\cos\left(\frac{\pi}{2}+x\right) - \cos(0+x)\right]dx$

$\implies I = \int_0^{\pi} x(\sin x + \cos x)dx$

$\implies I = \pi - 2$

Question-3(d) **Evaluate** $\iiint(x+y+z+1)^4 dxdydz$ **over the region defined by** $x \geq 0,\ y \geq 0,\ z \geq 0$ and $x+y+z \leq 1$.

[10 Marks]

Solution: Let $I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}(x+y+z+1)^4 dzdydx$

$\implies I = \int_{x=0}^{1}\int_{y=0}^{1-x}\frac{1}{5}\left((x+y+z+1)^5\right]_{z=0}^{1-x-y}dydx$

$\implies I = \frac{1}{5}\int_{x=0}^{1}\int_{y=0}^{1-x}[2^5 - (x+y+1)^5]dydx$

$\implies I = \frac{1}{5}\int_0^1\left[32y - \frac{1}{6}(x+y+1)^6\right]_{y=0}^{y=1-x}dx$

$\implies I = \frac{1}{5}\int_0^1\left[32(1-x) - \frac{1}{6}(2)^6 + \frac{1}{6}(x+1)^6\right]dx$

$\implies I = \frac{1}{5}\int_0^1\left[\frac{64}{3} - 32x + \frac{1}{6}(x+1)^6\right]dx$

$\implies I = \left[\frac{64}{3}x - 16x^2 + \frac{1}{42}(x+1)^7\right]_0^1$

$\implies I = \frac{117}{70}$

Question-4(a) **Obtain the equations of the planes which pass through the point** $(3,0,3)$, **touch the sphere** $x^2+y^2+z^2 = 9$ **and are parallel to the line** $x = 2y = -z$

[10 Marks]

Solution: The given line is $x = 2y = -z \implies \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ ... (i)
Any line parallel to (i) and passing through $(3, 0, 3)$ is given by:

$$\frac{x-3}{2} = \frac{y-0}{1} = \frac{z-3}{-2} \text{ ... (ii)}$$

Now, the general form of the line (ii) is given by:

$$x - 3 = 2y, \; -2y = z - 3$$

$$\implies x - 2y - 3 = 2y + z - 3 \text{ ... (iii)}$$

Now, any plane passing through line (iii) is given by:

$$(x - 2y - 3) + \lambda(2y + z - 3) = 0 \text{ ... (iv)}$$

$$\implies x + (-2 + 2\lambda)y + \lambda z + (-3 - 3\lambda) = 0.$$

Clearly, it will be the tangent plane to the given sphere $x^2 + y^2 + z^2 - 9 = 0$ if the perpendicular distance of the plane form the centre $(0, 0, 0)$ of the sphere is equal to the radius of the sphere, i.e.,

$$\frac{|0 + 0 + 0 - 3 - 3\lambda|}{\sqrt{1 + (-2 + 2\lambda)^2 + \lambda^2}} = 3 \; (\because radius = 3)$$

$$\Rightarrow 1 + \lambda = \sqrt{5 + 5\lambda^2 - 8\lambda}$$

$$\Rightarrow (1 + \lambda)^2 = 5 + 5\lambda^2 - 8\lambda$$

$$\Rightarrow \lambda^2 + 2\lambda + 1 = 5\lambda^2 - 8\lambda + 5$$

$$\Rightarrow 4\lambda^2 - 10\lambda + 4 = 0$$

$$\Rightarrow 2\lambda^2 - 5\lambda + 2 = 0$$

$$\Rightarrow 2\lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$\Rightarrow (2\lambda - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = 2, \tfrac{1}{2}$$

If $\lambda = 2$, then from (iv), we get

$$(x - 2y - 3) + 2(2y + z - 3) = 0$$

$$\implies x + 2y + 2z - 9 = 0 \text{ ... (v)}$$

If $\lambda = \frac{1}{2}$, then from (iv), we get

$$(x - 2y - 3) + \tfrac{1}{2}(2y + z - 3) = 0$$

$$\implies 2x - 2y + z - 9 = 0 \text{ ... (vi)}$$

Therefore, the required planes are given by equation (v) and equation (vi) above.

Question-4(b) The section of a cone whose vertex is P and guiding curve is the eilipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z = 0$ by the plane $x = 0$ is a rectangular hyperbola. Show that the locus of P is $\dfrac{x^2}{a^2} + \dfrac{y^2 + z^2}{b^2} = 1$.

[10 Marks]

Solution: Let the vertex P be (α, β, γ) and given guiding curve the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ ... (i)

Now, the equation of any line through $P(\alpha, \beta, \gamma)$ is given by:

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} ...(ii)$$

It meets the plane $z = 0$

$$\therefore \tfrac{x-\alpha}{l} = \tfrac{y-\beta}{m} = \tfrac{0-\gamma}{n}$$

$$\implies x - \alpha = \tfrac{-l\gamma}{n},\ y - \beta = \tfrac{-m\gamma}{n},\ z = 0$$

$$\implies x = \alpha - \tfrac{l\gamma}{n},\ y = \beta - \tfrac{m\gamma}{n},\ z = 0$$

This point lies on the ellipse given by (i),

$$\therefore \tfrac{1}{a^2}\left(\alpha - \tfrac{l\gamma}{n}\right)^2 + \tfrac{1}{b^2}\left(\beta - \tfrac{m\gamma}{n}\right)^2 = 1 \text{ ... (iii)}$$

Eliminating l, m and n from (ii) and (iii), we get:

$$\tfrac{1}{a^2}\left(\alpha - \tfrac{(x-\alpha)\gamma}{z-\gamma}\right)^2 + \tfrac{1}{b^2}\left(\beta - \tfrac{(y-\beta)\gamma}{z-\gamma}\right)^2 = 1$$

$\frac{1}{a^2}(\alpha z - \gamma x)^2 + \frac{1}{b^2}(\beta z - \gamma y)^2 = (z-\gamma)^2$, which is the required equation of the cone.

This meets the plane $x = 0$.

$$\implies \tfrac{1}{a^2}(\alpha z - 0)^2 + \tfrac{1}{b^2}(\beta z - \gamma y)^2 = (z-\gamma)^2$$

$$\implies \tfrac{\alpha^2 z^2}{a^2} + \tfrac{\beta^2 z^2 + \gamma^2 y^2 - 2\beta\gamma zy}{b^2} = z^2 + \gamma^2 - 2z\gamma$$

This will represent a rectangular hyperbola in $yz - plane$ if coefficient of y^2+ coefficient of $z^2 = 0$.

$$\implies \tfrac{\gamma^2}{b^2} + \tfrac{\alpha^2}{a^2} + \tfrac{\beta^2}{b^2} - 1 = 0$$

$$\implies \tfrac{\alpha^2}{a^2} + \tfrac{\beta^2+\gamma^2}{b^2} - 1 = 0$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is given by:

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1$$

Question-4(c) Prove that the locus of the poles of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is the conicoid $\dfrac{\alpha^2 x^2}{a} + \dfrac{\beta^2 y^2}{b} + \dfrac{\gamma^2 z^2}{c} = 1$.

[10 Marks]

Solution: Let the tangent plane of the conicoid $ax^2 + by^2 + cz = 1$ be given by:

$$lx + my + nz = p \text{ ... (i)}$$

Then, $\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$ ... (ii)

Let (a', b', c') be the pole of the plane (ii) w.r.t

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1.$$

$$\implies a'\alpha x + b'\beta y + c'\gamma z = 1 \text{ ... (iii)}$$

Comparing (i) and (iii), we get:

$$\frac{a'\alpha}{l} = \frac{b'\beta}{m} = \frac{c'\gamma}{n} = \frac{1}{p} \text{ ... (iv)}$$

Eliminating l, m and n from (ii) and (iv), we get:

$$\frac{(a'\alpha p)^2}{a} + \frac{(b'\beta p)^2}{b} + \frac{(c'\gamma p)^2}{c} = p^2$$

$\therefore$ The required locus of (a', b', c') is given by:

$$\frac{(\alpha x)^2}{a} + \frac{(\beta y)^2}{b} + \frac{(\gamma z)}{c}\Big)^2 = 1$$

Question-4(d) Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2 + by^2 + cz^2 = 1$ at its points of intersection with the planes $lx + my + nz = p$ generate the cone $p^2\left(\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{z^2}{c}\right) = \left(\dfrac{lx}{a} + \dfrac{my}{b} + \dfrac{nz}{c}\right)^2$.

[10 Marks]

Solution: Let (α, β, γ) be the point of intersection of the conicoid and the given plane, then we have:

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \ldots \text{ (i),and}$$

$$l\alpha + m\beta + n\gamma = p \ldots \text{ (ii)}$$

Also, the equations of the normals to the given conicoid at (α, β, γ) are:

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$

The equation of the line passing through the origin and parallel to this line is given by:

$$\tfrac{x}{a\alpha} = \tfrac{y}{b\beta} = \tfrac{z}{c\gamma} \ldots \text{ (iii)}$$

From (i) and (ii), we have:

$$a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\tfrac{l\alpha+m\beta+n\gamma^2}{p}\right)^2$$

$$\implies p^2\left(a\alpha^2 + b\beta^2 + c\gamma^2\right) = (l\alpha + m\beta + n\gamma)^2$$

$$\implies p^2\left(\tfrac{a^2\alpha^2}{a} + \tfrac{b^2\beta^2}{b} + \tfrac{c^2\gamma^2}{c}\right) = \left(\tfrac{l(a\alpha)}{a} + \tfrac{m(b\beta)}{b} + \tfrac{n(c\gamma)}{c}\right]^2$$

$$\implies p^2\left[\tfrac{(a\alpha)^2}{a} + \tfrac{(b\beta)^2}{b} + \tfrac{(c\gamma)^2}{c}\right] = \left(\tfrac{l(a\alpha)}{a} + \tfrac{m(b\beta)}{b} + \tfrac{n(c\gamma)}{c}\right]^2$$

Now, eliminating α, β and γ from this equation using (iii), we get:

$$p^2\left(\tfrac{x^2}{a} + \tfrac{y^2}{b} + \tfrac{z^2}{c}\right) = \left(\tfrac{lx}{a} + \tfrac{my}{b} + \tfrac{nz}{c}\right)^2$$

Hence, the line given by (iii) generates the above conicoid.

12.2 Section-B

Question-5(a) Solve: $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$

[10 Marks]

Solution: $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \ldots$ (i)

Let $\tan y = z$

$$\implies \sec^2 y \tfrac{dy}{dx} = \tfrac{dz}{dx}$$

So, equation (i) becomes

$$\tfrac{dz}{dx} + 2xz = x^3$$

$$\implies IF = \int e^{2x} dx = \tfrac{e^x}{2}$$

The general solution of equation (i) is given by:

$$\begin{aligned}
z \cdot \frac{e^{2x}}{2} &= \int x^3 \frac{e^{2x}}{2} \dot{d}x \\
&= \frac{1}{2} \int x^3 e^{2x} dx \\
&= \frac{1}{2} \left[\frac{e^{2x}}{2} \cdot x^3 - \int 3x^2 \frac{e^{2x}}{2} dx \right] \\
&= \frac{e^{2x}}{4} x^3 - \frac{3}{4} \int x^2 e^{2x} dx \\
&= \frac{e^{x}}{4} x^3 - \frac{3}{4} \left[\frac{2x}{2} \cdot x^2 - \int \frac{e^{2x}}{2} \cdot 2x dx \right] \\
&= \frac{e^{2x}}{4} x^3 - \frac{3}{8} x^2 e^{2x} + \int x e^{2x} dx \\
&= \frac{x^3 e^{2x}}{4} - \frac{3}{8} x^2 e^{2x} + \frac{e^{2x}}{2} x - \int \frac{e^{2x}}{2} \alpha \\
&= \frac{x^3 e^{2x}}{4} - \frac{3}{8} x^2 e^{2x} + \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c \\
\tan y \frac{e^{2x}}{2} &= \frac{x^3 e^{2x}}{4} - \frac{3}{8} x^2 e^{2x} + \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + c
\end{aligned}$$

Question-5(b) Find the 2nd order ODE for which e^x and $x^2 e^x$ are solutions.

[10 Marks]

Solution: Let $y_1 = x$ & $y_2 = x^2 e^x$
Wronskian is

$$\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} x & x^2 e^x \\ 1 & x^2 e^x + 2x e^x \end{vmatrix} \\
&= x \left(x^2 e^x + 2x e^x \right) - x^2 e^x \\
&= x^3 e^x + 2x^2 e^x - x^2 e^x \\
&= x^3 e^x + x^2 e^x,
\end{aligned}$$

which is not identically equal to 0 on R in $(-\infty, \infty)$. The general solution of the required differential equation can be written as:

$$y = c_1 y_1 + c_2 y_2$$

,
where C_1, C_2 are arbitrary contants.

$$\Rightarrow y = c_1e^x + c_2x^2e^x \text{ ... (i)}$$

Difrerentiating eq (i) wrt x, we get:

$$\begin{aligned}\frac{dy}{dx} &= c_1e^x + c_2x^2e^x + 2c_2xe^x \\ \Rightarrow y' &= y + 2c_2xe^x \\ \Rightarrow y' - y &= 2c_2xe^x \dots (ii)\end{aligned}$$

Again differentiating eq (ii) w.rt x, we get

$$y'' - y' = 2c_2xe^x + 2c_2e^x = y' - y + 2c_2e^x$$

$$y'' - 2y' + y = 2c_2e^x \ \dots \ \text{(iii)}$$

Now, substituting (iii) in (ii), we get,

$$\begin{aligned}y' - y &= x\left(y'' - 2y' + y\right) \\ &= xy'' - 2xy' + xy\end{aligned}$$

$\Rightarrow x - y'' - (2x + 1)y' + (x + 1)y = 0$,
which is the required differential equation.

Question-5(c) A uniform rectangular board, whose sides are $2a$ and $2b$, rests in limiting equilibrium in contact with two rough pegs in the same horizontal line at a distance d apart. Show that the inclination θ of the side $2a$ to the horizontal is given by the equation $d\cos\lambda[\cos(\lambda + 2\theta)] = a\cos\theta - b\sin\theta$ where λ is the angle of friction.

[10 Marks]

Solution: Let $ABCD$ be the rectangle resting on two pegs P and Q.
Suppose that the resultant of the reactions and the frictional forces at P and Q meet at O. Then, the centre of gravity G of the rectangle must be vertically below O.
Let AN be the perpendicular from A on OG. Suppose that the normals at P and Q meet at O'.
The angles OPO' and OQO' are equal. Hence O, P, Q, O' are concyclic. Again, O', P, A, Q are concyclic. Hence, O, O', A, P are concyclic.
It follows that:

$$\begin{aligned}&\angle OAO' = \lambda \\ &\angle O'AQ = \angle AQP = \angle QAN = \theta \\ &\angle O'OA = \angle O'PA = \pi/2\end{aligned}$$

Also trom the rectangle $O'PAP$, $O'A = PQ$.
We can now find AN in 2 ways:

$$\begin{aligned}AN &= OA\cos(\lambda + 2\theta) \\ &= O'A\cos\lambda\cos(\lambda + 2\theta) \\ &= d\cos\lambda\cos(\lambda + 2\theta)\end{aligned}$$

Again,

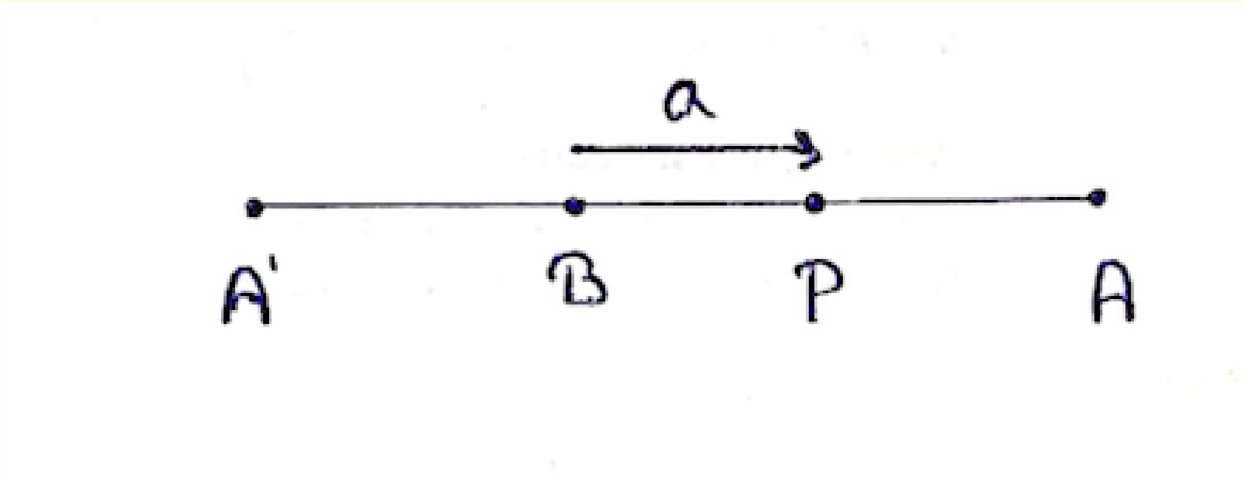
$$\begin{aligned} AN &= AG\cos(\angle GAQ + \angle QAN) \\ &= AG\cos\angle GAQ\cos\theta - AG\sin\angle GAQ\sin\theta \| \\ &= acos\theta - b\sin\theta \end{aligned}$$

Hence, $d\cos\lambda\cos(\lambda+2\theta) = a\cos\theta - b\sin\theta$.

Question-5(d) A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'. The particle is slightly displaced towards one of them, show that the time of small oscillation is $\dfrac{2\pi}{\sqrt{(\mu+\mu')}}$.

[10 Marks]

Solution: Suppose A and A' are two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at B under the attraction of these two centres.



The forces of attraction at B due to the centres. A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions.
As these two forces balance each other, therefore

$$m\mu a = m\mu' a' \text{ ... (i)}$$

Now, suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP = x$.
The attraction at P due to centre A is $m\mu AP = m\mu(a-x)$ in the direction PA, i.e., in the direction of x increasing.
Also, the attraction at P due to centre A' is $m\mu' A'P = m\mu'(a'+x)$ in the direction PA', i.e. in the direction of x decreasing.
Hence, by Newton's 2^{nd} law of motion, the equation of motion of particle at P is given by:

$$m\left(\frac{d^2x}{dt^2}\right) = m\mu(a-x) - m\mu'\left(a'+x\right) \text{ ... (ii)}$$

where the forces in the direction of x increasing has been taken with $+ve$ sign and thee force in the direction of x decreasing has been taken with $-ve$ sign.
Simplifying equation (ii), we get

$$m\left(\frac{d^2x}{dt^2}\right) = m\left(\mu a - \mu x - \mu' a' - \mu' x\right)$$

$$\implies \frac{d^2x}{dt^2} = -\left(\mu + \mu'\right)x \; [\because m\mu a = m\mu' a']$$

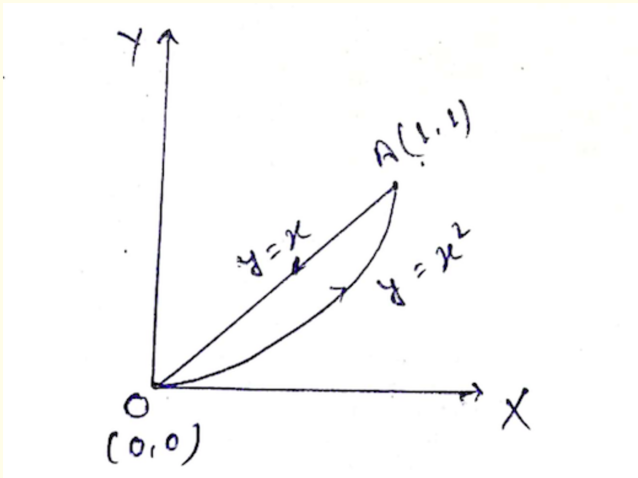
This is the equation of motion with centre at the origin. Hence the motion of particle is SHM with centre at B and its time period is $\frac{2\pi}{\sqrt{\mu+\mu'}}$.

Question-5(e) Verify Green's theorem in the plane for $\oint_c \left[\left(xy + y^2\right)dx + x^2 dy\right]$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

[10 Marks]

Solution: By Green's theorem in plane, we have:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$$



Here, $M = xy + y^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. We have,

$$\begin{aligned}
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy &= \iint_R \left(\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(xy + y^2\right)\right) dxdy \\
&= \iint_R (2x - x - 2y) dxdy \\
&= \iint_R (x - 2y) dxdy \\
&= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y) dydx \\
&= \int_0^1 [xy - y^2]_{y=x^2}^{x} dx \\
&= \int_0^1 \left[x^2 - x^2 - x^3 + x^4\right] dx \\
&= \int_0^1 x^4 - x^3 dx \\
&= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 \\
&= \frac{1}{5} - \frac{1}{4} - 0 \\
&= \frac{4-5}{20} = \frac{-1}{20}
\end{aligned}$$

Now, let us evaluate the line integral along C. The line integral along C = line integral along $y = x^2$ + line integral along $y = X = I_1 + I_2$
Along $y = x^2$, $dy = 2xdx$

$$\begin{aligned}
I_1 &= \int_0^1 \left(x.x^2 + x^4\right) dx + x^2 2xdx \\
I_1 &= \int_0^1 \left(x \cdot x^2 + x^4\right) dx + \int_0^1 x^2 2xdx \\
&= \int_0^1 \left(x^3 + x^4 + 2x^3\right) dx \\
&= \int_0^1 (3x^3 + x^4) dx \\
&= \left[3\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 \\
&= 3\frac{1}{4} + \frac{1}{5} \\
&= \frac{19}{20}
\end{aligned}$$

Along $y = X$, $dy = dx$

$$\begin{aligned}
I_2 &= -\int_0^1 \left(x^2 + x^2\right) dx + x^2 dx \\
&= \int_0^1 3x^2 dx = -\left[x^3\right]_0^1 \\
&= 1
\end{aligned}$$

$\therefore I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$
Hence, the theorem is verified.

Question-6(a) Solve: $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

[10 Marks]

Solution: $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0 \ldots$ (i)

Equation (i) is a homogeneous equation. Comparing equation (i) with $Mdx + Ndy$, we get: $M = y^3 - 2yx^2$, $N = 2xy^2 - x^3$

$$\begin{aligned} Mx + Ny &= xy^3 - 2yx^3 + 2xy^3 - yx^3 \\ &= 3xy^3 - 3yx^3 \\ &= 3xy\left(y^2 - x^2\right) \\ \frac{1}{Mx + Ny} &= \frac{1}{3xy\,(y^2 - x^2)} \end{aligned}$$

Multiplying eq (i) by $\frac{1}{3xy(y^2-x^2)}$, we get

$$\frac{y^3-2yx^2}{3xy(y^2-x^2)}dx + \frac{2xy^2-x^3}{3xy(y^2-x^2)}dy = 0$$

$$\implies \frac{y^2-2x^2}{3x(y^2-x^2)}dx + \frac{2y^2-x^2}{3y(y^2-x^2)}dy = 0 \ldots \text{(ii)}$$

Comparing eq (ii) with $Pdx + Qdy = 0$, we get

$$P = \frac{y^2-2x^2}{3xy^2-3x^3}, \quad Q = \frac{2y^2-x^2}{3y^3-3x^2y}$$

$$\implies \frac{\partial P}{\partial y} = \frac{6xy}{(3y^2-3x^2)^2}, \quad \frac{\partial Q}{\partial x} = \frac{6xy}{(3y^2-3x^2)^2}$$

$$\implies \frac{\partial p}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ Eq (ii) is exact.
Solution is given by:

$$\int_{y=constant} Pdx + \int (\text{ terms in Q containing x})\,dy = c_1$$

$$\implies \int \frac{y^2-2x^2}{3xy^2-3x^3}dx + \int \frac{2}{3y}dy = c_1$$
$$\implies \int \frac{y^2-2x^2}{3x(y^2-x^2)}dx + \int \frac{2dy}{3y} = c_1$$

$$\implies \int \frac{dx}{3x} - \frac{1}{3}\int \frac{xdx}{y^2-x^2} + \frac{2}{3}\int \frac{dy}{y} = c_1$$

Let $y^2 - x^2 = t$

$$\implies -2xdx = dx$$

$$\implies \tfrac{1}{3}\log x + \tfrac{1}{3}\int \tfrac{dt}{2\cdot t} + \tfrac{2}{3}\log y = c_1$$

$$\implies \tfrac{1}{3}\log x + \tfrac{1}{6}\log t + \tfrac{2}{3}\log y = C_1$$

$$\implies 2\log x + \log\left(y^2 - x^2\right) + 4\log y = 6c_1$$

$$\implies \log x^2\left(y^2 - x^2\right)y^4 = \log(c_2).\ \text{where } 6c_1 = \log c_2$$

$$\implies x^2y^4\left(y^2 - x^2\right) = c_2,$$

which is the required solution.

Question-6(b) Solve: $\left(\dfrac{dy}{dx}\right)^2 - 2\dfrac{dy}{dx}\cos hx + 1 = 0$

[8 Marks]

‘

Solution:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx}\cos hx + 1 = 0$$

Let $\dfrac{dy}{dx} = p$

$$p^2 - 2p\cos hx + 1 = 0$$

Solving for p

$$p = \frac{2\cosh x \pm \sqrt{4\cos h^2 x - 4}}{2}$$

$$p = \cosh x \pm \sqrt{\cos h^2 x - 1}$$

$$p = \cosh x \pm \sin hx$$

$$p = \cos m + \sin hx \quad \& \quad p = \cos hx - \sin hx$$

$$\frac{dy}{dx} = \cosh x + \sin hx \quad \& \quad \frac{dy}{dx} = \cos hx - \sin hx$$

integrating above,

$$y = \sin hx + \cos hx + C_1; y = \sin m_x - \cosh x + C_2$$

Hence, general solution is

$$(y - \sin hx - \cos hx - C_1)(y - \sin m_x + \cosh x - C_2) = 0$$

$$\Rightarrow \left(y - \left(\frac{e^x - e^{-x}}{2}\right) - \left(\frac{e^x + e^{-x}}{2}\right) - C_1\right)\left(y - \left(\frac{e^x - e^{-x}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right) - C_2\right) = 0$$

$$\Rightarrow \left(y - \frac{e^x}{2} - \frac{e^x}{2} - C_1\right)\left(y + \frac{e^{-x}}{2} + \frac{e^{-x}}{2} - C_2\right) = 0$$

$$\Rightarrow (y - e^x - C_1) \quad (y + e^{-x} - C_2) = 0$$

Question-6(c) Solve: $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = x^2e^{-x}$

[10 Marks]

Solution: Given: $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = x^2e^{-x}$

$$\implies \left(D^3 + 3D^2 + 3D + 1\right)^2 y = x^2e^{-x}$$

The auxiliary equation is:

$$\begin{aligned} &D^3 + 3D^2 + 3D + 1 = 0 \\ \implies &(D+1)^3 = 0 \\ \implies &D = -1, -1, -1 \end{aligned}$$

CF is given by:

$$y_c = \left(c_1 + c_2x + c_3x^2\right) e^{-x}$$

Now, we calculate PI (particular integral),

$$\begin{aligned} y_p &= \frac{1}{D^3 + 3D^2 + 3D + 1} x^2e^{-x} \\ &= \frac{1}{(D+1)^3} x^2e^{-x} \\ &= e^{-x} \frac{1}{(D-1+1)^3} x^2 \\ &= e^x \frac{1}{D^3} x^2 \\ &= e^x \frac{1}{D^2} \frac{x^3}{3} \\ &= \frac{e^x}{3} \frac{1}{D} \frac{x^4}{4} \\ &= \frac{e^x}{12} \frac{1}{D} x^4 \\ &= \frac{e^x}{12} \frac{x^5}{5} \\ &= \frac{x^5e^x}{60} \end{aligned}$$

The complete solution is given by:

$$\begin{aligned} &y = y_c + y_p \\ \implies &y = \left(c_1 + c_2x + c_3x^2\right) e^{-x} + \tfrac{x^5e^x}{60}. \end{aligned}$$

Question-6(d) Show that e^{x^2} is a solution of $\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + (4x^2-2)\,y = 0$. Find a second independent solution.

[12 Marks]

Solution: Given: $y = e^{x^2} \Rightarrow y' = e^{x^2} \cdot 2x$

$$y'' = 2\left[e^{x^2} + 2x^2 e^{x^2}\right] = 2e^{x^2}\left(1+2x^2\right)$$

Substituting these in the given differential equation, we get:

$$\begin{aligned}\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2-2\right)y &= 2e^{x^2}(1+2x^2) - 4x\cdot e^{x^2}\cdot 2x + \left(4x^2-2\right)e^{x^2} \\ &= \left(4x^2 - 8x^2 + 4x^2\right)e^{x^2} + \left(2e^{x^2} - 2e^{x^2}\right) \\ &= 0\end{aligned}$$

Hence, $y = e^{x^2}$ is a solution of given DE.
$\therefore y = u = e^{x^2}$ is a part of complimentary function of the given DE.

Comparing given DE with

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

$P = -4x$, $Q = 4x^2 - 2$, $R = 0$
Let $y = uv$ be the general solution, then v is obtained by

$$\begin{aligned}&\tfrac{d^2v}{dx^2} + \left(P + \tfrac{2}{u}\cdot\tfrac{du}{dx}\right)\tfrac{dv}{dx} = \tfrac{R}{u} \\ \Longrightarrow\ &P + \tfrac{2}{u}\cdot\tfrac{du}{dx} = -4x + \tfrac{2}{e^{x^2}}\left(2xe^{x^2}\right) = 0 \\ &\therefore \tfrac{d^2v}{dx^2} = 0 \\ \Longrightarrow\ &\tfrac{dv}{dx} = c_1 \\ &v = c_1x + c_2\end{aligned}$$

Hence, the complete solution is:

$$\begin{aligned}&y = uv \\ \Longrightarrow\ &y = e^{x^2}\left(c_1x + c_2\right)\end{aligned}$$

Question-7(a) A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the sphere is in contact. If θ and ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan\phi = \frac{3}{8} + \tan\theta$.

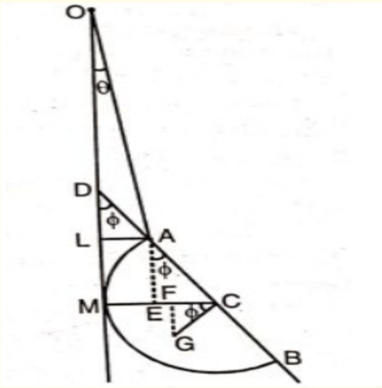
[10 Marks]

Solution: Let O be the point of suspension in the wall, AB the base of the hemisphere, C its centre, G its centre of gravity, M the point of contact of the hemisphere and the wall and OA the string.
Let l be the length of the string OA and let a be the radius of the hemisphere.

$$\therefore \quad CA = a \quad \text{and} \quad CG = \frac{3a}{8}$$

since O is a fixed point, so all the distances will be measured from this point O.

Let d be the depth of G below O.



$$\therefore \quad d = OM + FG = OL + LM + CG\sin\phi$$
$$= l\cos\theta + AC\cos\phi + \frac{3a}{8}\sin\phi$$
$$\Rightarrow \quad d = l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi \quad \ldots (1)$$

The normal reaction at M is perpendicular to the wall.

$$\therefore MC \text{ is horizontal.}$$

Let the system be given a small virtual displacement such that θ becomes $\theta + \delta\theta$ and ϕ becomes $\phi + \delta\phi$
W, the weight of the hemisphere will be the only force doing work. The reaction at M does not appear in the equation of virtual work.

$$\therefore \text{Equation of virtual work is } W.\delta(d) = 0$$

or

$$\delta(d) = 0 \quad [\because W \neq 0]$$

$$or$$

$$\delta\left[l\cos\theta + a\cos\phi + \frac{3a}{8}\sin\phi\right] = 0$$

$$or$$

$$-l\sin\theta\cdot\delta\theta - a\sin\phi\delta\phi + \frac{3a}{8}\cos\phi\delta\phi = 0$$

$$\therefore l\sin\theta.\delta\theta = \left(\frac{3}{8}\cos\phi - \sin\phi\right)\cdot a\delta\phi\ldots(2)$$

Again,

$$\begin{aligned} a = CM &= CE + EM = CE + AL \quad [\because EM = AL] \\ &= CA\sin\phi + OA\sin\theta \\ &= a\sin\phi + l\sin\theta \end{aligned}$$

$$or$$

$$l\sin\theta = a - a\sin\phi$$

Differentiating,

$$l\cos\theta\cdot\delta\theta = -a\cos\phi\delta\phi\ldots(3)$$

Dividing (2) by (3), we get

$$\tan\theta = -\frac{3}{8} + \tan\phi$$

Hence $\tan\phi = \frac{3}{8} + \tan\theta$

Question-7(b) A particle moves with a central acceleration $\mu\left(\gamma + \frac{a^4}{\gamma^3}\right)$ being projected from an apse at a distance a with a velocity $2\sqrt{\mu}a$.
Prove that its path is $\gamma^2(2 + \cos\sqrt{3}\theta) = 3a^2$.

[10 Marks]

Solution: Here, $F = \mu\left(r + \frac{a^4}{r^3}\right) = \mu\left(u^{-1} + a^4u^3\right)\ldots(1)$ where $u = \frac{1}{r}$
Differential equation of central orbit is

$$h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{F}{u^2} = \mu\frac{(u^{-1} + a^4u^3)}{u^2} = \mu\left(u^{-3} + u^4u\right)\ldots(2)$$

Multiplying by $2\frac{du}{d\theta}$, we get

$$h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = 2\mu\left[u^{-3} + a^4u\right]\frac{du}{d\theta}$$

Integrating, we get

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left[-\frac{1}{2}u^{-2}+\frac{a^4u^2}{2}\right]+c$$

or

$$v^2=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left(-u^{-2}+a^4u^2\right)+c\ldots(3)$$

Initially, at an apse,

$$u=\frac{1}{a},\frac{du}{d\theta}=0\text{ and }v=2a\sqrt{\mu}\text{ [Given]}$$

From (3),

$$\therefore\quad 4a^2\mu=\frac{h^2}{a^2}=\mu\left(-a^2+a^2\right)+c$$

$\therefore h^2=4\mu a^4$ and $c=4\mu a^2$

Putting the values of h^2 and c in (3), we get

$$4\mu a^4\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]=\mu\left(-u^{-2}+a^4u^2\right)+4\mu a^2$$

or

$$\begin{aligned}4a^4\left(\frac{du}{d\theta}\right)^2&=-\frac{1}{u^2}+a^4u^2-4a^4u^2+4a^2\\&=\frac{-1+a^4u^4-4a^4u^4+4a^2u^2}{u^2}\\&=\frac{-1-3a^4u^4+4a^2u^2}{u^2}\\&=\frac{-1-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2+\left(\frac{2}{\sqrt{3}}\right)^2}{u^2}\\&=\frac{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}{u^2}\end{aligned}$$

or

$$2a^2\frac{du}{d\theta}=\pm\frac{\left[\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2\right]^{1/2}}{u}$$

or

$$-\frac{2\sqrt{3}a^2udu}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2-\left(\sqrt{3}a^2u^2-\frac{2}{\sqrt{3}}\right)^2}}=\sqrt{3}d\theta$$

[Taking -ve sign] Put $\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}} = t$ so that $2\sqrt{3}a^2udu = dt$

$$\therefore \quad -\frac{dt}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 - t^2}} = \sqrt{3}d\theta$$

Integrating, $\cos^{-1}(t\sqrt{3}) = \sqrt{3}\theta + A \ldots (4)$
Initially, when $u = \frac{1}{a}$, i.e., $t = \frac{1}{\sqrt{3}}, \theta = 0 \therefore \quad A = 0$
From (4),

$$\therefore \quad \cos^{-1}(t\sqrt{3}) = \sqrt{3}\theta$$

$$or$$

$$t\sqrt{3} = \cos\sqrt{30}$$

$$or$$

$$\sqrt{3}\left(\sqrt{3}a^2u^2 - \frac{2}{\sqrt{3}}\right) = \cos\sqrt{3}\theta$$

$$or$$

$$3a^2u^2 - 2 = \cos\sqrt{3}\theta$$

$$or$$

$$3a^2u^2 = 2 + \cos\sqrt{3}\theta$$

Hence, $3a^2 = r^2[2 + \cos\sqrt{3}\theta]$ which is the required path. $\left[\because u = \frac{1}{r}\right]$

Question-7(c) A shell, lying in a straight smooth horizontal tube, suddenly explodes and breaks into portions of masses m and m'. If d is the distance apart of the masses after a time t, show that the work done by the explosion is $\frac{1}{2}\frac{mm'}{m+m'} \cdot \frac{d^2}{t^2}$.

[10 Marks]

Solution: Since, the shell is lying in the tube, its velocity before explosion is zero. Let u_1 and u_2 be the velocities, of the masses m and m' respectively after explosion. Then, the relative velocity of the masses after explosion is $u_1 + u_2$. Since, the tube is smooth and horizontal, $u_1 + u_2$ will remain constant.

$$\therefore (u_1 + u_2)\,t = d - (i)$$

Also, by the principle of conservation of linear momentum, we have,

$$mu_1 - m'u_2 = 0$$
$$\Rightarrow \quad mu_1 = m'u_2 - (ii)$$

Substituting, for u_2 from (ii) in (i) we get,

$$\left(u_1 + \frac{mu_1}{m'}\right) t = d \; u_1 \left(\frac{m' + m}{m'}\right) t = d$$
$$\Longrightarrow u_1 = \frac{dm'}{(m' + m)\, t}$$

from (ii)

$$\therefore u_2 = \frac{m}{m'} u_1$$
$$= \frac{m}{m'} \frac{m'd}{(m + m')\, t}$$
$$u_2 = \frac{md}{(m + m')\, t}$$

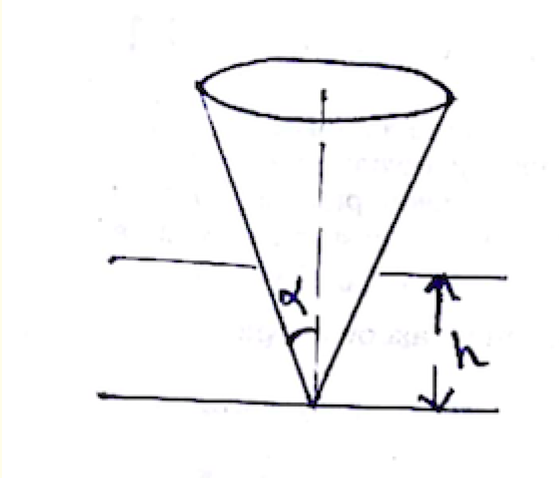
Now, the work done by the explosion = the kinetic energy released due to the explosion

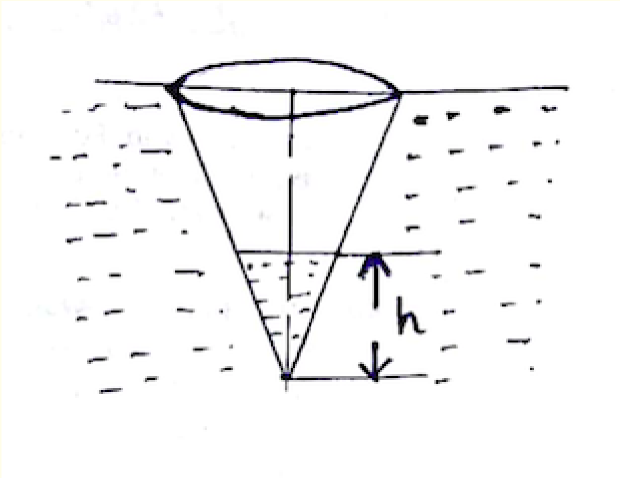
$$= \frac{1}{2} m u_1^2 + \frac{1}{2} m' u_2^2$$
$$= \frac{1}{2} m \left[\frac{m'^2 d^2}{(m + m')^2 t^2}\right] + \frac{1}{2} m' \left[\frac{m^2 d^2}{(m + m')^2 t^2}\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{1}{(m + m')^2} \left[mm'^2 + m'm^2\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{mm'}{(m + m')^2} \left[m' + m\right]$$
$$= \frac{1}{2} \frac{d^2}{t^2} \frac{mm'}{(m + m')}$$

Question-7(d) A hollow conical vessel floats in water with its vertex downwards and a certain depth of its axis immersed. When water is poured into it up to the level originally immersed, it sinks till its mouth is on a level with the surface of the water. What portion of axis was originally immersed?

[10 Marks]

Solution: According to Law of buoyancy,
Upward force on a body = weight of fluid displaced by immersed part of body.





Let W be weight of cone with height H and semi-vertical angle α, then
upward force $= \frac{1}{3}\pi(htan\alpha)^2 \times h\rho g$
[weight of fluid = vol. of body submerged×density ×g]
$\therefore \quad W = \frac{1}{3}\pi h^3 \tan^2 \alpha\rho g - (1)$

Now,

$$\text{Total weight} = \text{Total upward force}$$

$$\Rightarrow W + \text{ Weight of water} = \text{Vol. of body submerged}\times\text{density}\times\text{g}$$

$$\Rightarrow \quad W + \frac{1}{3}\pi(h\tan\alpha)^2 \times h\rho g = \frac{1}{3}\pi(H\tan\alpha)^2 H\rho g$$

where W =Weight of cone,

$$\frac{1}{3}\pi(htan\alpha)^2 \times h\rho g = \text{Weight of water in cone,}$$

$$\frac{1}{3}\pi(H\tan\alpha)^2 H\rho g = \text{Total upward force}$$

$$\Rightarrow 2 \times \frac{1}{3}\pi h^3 \tan^2 \alpha\rho g = \frac{1}{3}\pi H^3 \tan^2 \alpha\rho g \text{ [from (1)]}$$

$$\Rightarrow \quad 2h^3 = H^3$$

$$\Rightarrow \quad \frac{h}{H} = \left(\frac{1}{2}\right)^{1/3}$$

Question-8(a) Show that $\bar{A} = (6xy + z^3)\,\hat{\imath} + (3x^2 - z)\,\hat{\jmath} + (3xz^2 - y)\,\hat{k}$ is irrotational. Find a scalar function ϕ such that $\bar{A} = \operatorname{grad}\phi$.

[10 Marks]

Solution: Given that $\vec{A} = (6xy + z^3)\,\hat{\imath} + (3x^2 - z)\,\hat{\jmath} + (3xz^2 - y)\,k$

$$
\begin{aligned}
curl\vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix} \\
&= \hat{\imath}(-1 + 1) + \hat{\jmath}\left(3z^2 - 3z^2\right) + \hat{k}(6x - 6x) \\
&= \hat{\imath}(0) + \hat{\jmath}(0) + \hat{k}(0) \\
&= 0
\end{aligned}
$$

$\therefore$ The vector $\vec{A}$ is irrotational.

Let $\vec{A} = \operatorname{grad}\phi$ i.e $\vec{A} = \nabla\phi$

$$
\begin{aligned}
\Rightarrow \left(6xy + z^3\right)\hat{\imath} + \left(3x^2 - z\right)\hat{\jmath} + \left(3xz^2 - y\right)\hat{k} &= \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k} \\
\frac{\partial\phi}{\partial x} &= 6xy + z^3 - (i) \\
\frac{\partial\phi}{\partial y} &= 3x^2 - z - (ii) \\
\frac{\partial\phi}{\partial z} &= 3xz^2 - y - (iii)
\end{aligned}
$$

(i) partially w.r.t x treating y, z as constants.

$$\text{(i)} \equiv \phi = 3x^2y + z^3x + f_1(y, z) - (iv)$$

(ii) partially w.r.t y treating x, z as constant.

$$\phi = 3x^2y - zy + f_2(x, z) - (v)$$

iii) partially w.r.t z treating x, y as constant.

$$(iii) \equiv \phi = xz^3 - yz + f_3(x, y) - (vi)$$

(iv), (v), (vi) each represents ϕ.These agree if we choose:

$$f_1(y, z) = -yz, f_2(x, z) = xz^2, f_3(x, y) = 3x^2y$$

$\therefore \phi = 3x^2y + xz^3 - yz + C$ where C is an arbitrary constant.

Question-8(b) Let $\psi(x, y, z)$ be a scalar function. Find grad ψ and $\nabla^2\psi$ in spherical coordinates.

[8 Marks]

Solution: We know that, $\nabla\psi = grad\ \psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}e_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}e_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}e_3 - (i)$

for spherical coordinates (r, θ, ϕ)
$u_1 = r, \quad u_2 = \theta \quad , \quad u_3 = \phi$
$e_1 = e_r, \quad e_2 = e_\theta \quad , \quad e_3 = e_\phi$
$h_1 = h_r, \quad h_2 = h_\theta \quad , \quad h_3 = h_\phi = r\sin\theta$

$\therefore$ From (i)

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi = \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}e_\phi$$

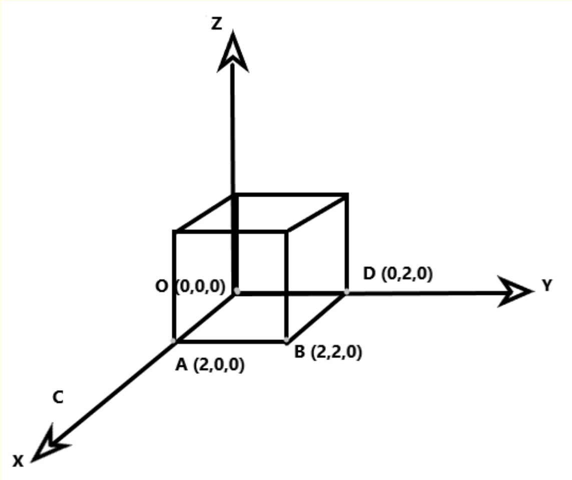
We know that,

$$\begin{aligned}
\nabla^2\Psi &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\cdot\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\cdot\frac{\partial\psi}{\partial u_3}\right)\right] \\
&= \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\left(\frac{(r)(r\sin\theta)}{(1)}\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta(t)}{r}\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{(1)(r)}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\cdot\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned}$$

Question-8(c) Verify Stokes' theorem for $\bar{A} = (y - z + 2)\hat{i} + (yz + 4)\hat{j} - xz\hat{k}$ where $\dot{S}$ is the surface of the cube $x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 2,\ z = 2$ above the xy-plane.

[12 Marks]

Solution: The xy - plane cuts the surface of the cube in a square. Thus, the curve C bounding the surface S is the square.



Say $OABD$, in the xy - plane whose vertices in the xy-plane are the points. $O(0,0), A(2,0), B(2,2), D(0,2)$
Then,

$$\begin{aligned}
\oint \vec{F}\cdot dr &= \int_c [(y-z+2)\hat{\imath} + (yz+4)\hat{\jmath} - xz\hat{k}]\cdot(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}) \\
&= \int_c (y-z+2)dx + (yz+4)dy - xzdz \\
&= \int_c (y+z)dx + 4dy \quad (\because \text{on } c, z=0 \& dz=0) \\
&= \int_{OA} + \int_{AB} + \int_{BD} + \int_{DO} \\
&= I_1 + I_2 + I_3 + I_4 - (i)
\end{aligned}$$

Along OA:

$$\begin{aligned}
y=0, dy=0 \quad &\& \quad x \text{ varies from } 0 \text{ to } 2 \\
\therefore I_1 &= \int_{OA} (y+2)dx + 4dy \\
\Rightarrow &\int_0^2 2\cdot dx = [2x]_0^2 = 4
\end{aligned}$$

Along AB :

$$\begin{aligned}
x=2, dx=0 \quad &\& \quad y \text{ varies from } 0 \text{ to } 2 \\
\therefore I_2 &= \int_{AB} (y+2)dx + 4dy \\
\Rightarrow &\int_0^2 4\cdot dy = [4y]_0^2 = 8
\end{aligned}$$

Along BD:

$$\begin{aligned}
y=2, dy=0 \quad &\& \quad x \text{ varies from } 2 \text{ to } 0 \\
\therefore I_3 &= \int_{BD} (y+2)dx + 4\cdot dy \\
&\int_2^0 4\cdot dx = [4x]_2^0 = -8
\end{aligned}$$

Along DO:

$$x = 0, dx = 0 \quad \& \text{ y varics . From 2 to 0}$$

$$\therefore I_4 = \int_{DO} (y+2)dx + 4 \cdot dy$$

$$\Rightarrow \int_2^0 4 \cdot dy = [4y]_2^0 = -8$$

$$\therefore (i) \equiv \int \vec{F} \cdot dr = I_1 + I_2 + I_3 + I_4$$

$$= 4 + 8 - 8 - 8$$

$$= -4 - (ii)$$

Now,

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$$= \hat{i}(0 - y) + \hat{j}(-1 + z) + \hat{k}(0 - 1)$$

$$= -y\hat{i} + (-1 + z)\hat{j} - \hat{k}$$

$$\hat{n} = \text{ unit normal vector to } S = \hat{k}$$

$$\therefore \quad dS = \frac{dxdy}{|n \cdot \hat{k}|} = dxdy$$

$$(\nabla \times F) \cdot \hat{n} = [(-y\hat{i} + (-1 + z)\hat{j} - \hat{k}] \cdot \hat{k}$$

$$= -1$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} ds = \int_{x=0}^2 \int_{y=0}^2 (-1) dxdy$$

$$= -\int_{x=0}^2 [y]_0^2 dx$$

$$\Rightarrow -2\int_0^2 dx = -2[x]_0^2 = -4 - (iii)$$

$\therefore$ From (2) and (3)

$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS = \int \vec{F} \cdot dr = -4$

Hence the stokes theorem is verified.

Question-8(d) Show that, if $\bar{r} = x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k}$ is a space curve, $\dfrac{d\bar{r}}{ds} \cdot \dfrac{d^2\bar{r}}{ds^2} \times \dfrac{d^3\bar{r}}{ds^3} = \dfrac{\tau}{\rho^2}$, where τ is the torsion and ρ the radius of curvature.

[10 Marks]

Solution: We know that $\tau = \dfrac{d\vec{r}}{ds}$ and $\kappa N = \dfrac{d^2\vec{r}}{ds^2}$,
here, κ is the curvature
Now,

$$\begin{aligned}\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} &= \tau \times \kappa N \\ &= \kappa(\tau \times N) \\ &= \kappa B \quad (\because \tau \times N = B)\end{aligned}$$

$$\therefore \quad \kappa = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$$

$$\begin{aligned}\frac{d^3\vec{r}}{ds^2} &= \frac{d}{ds}\left(\frac{d\vec{r}}{ds^2}\right) \\ &= \frac{d}{ds}(\kappa N) \\ &= \kappa \cdot \frac{dN}{ds} + \frac{d\kappa}{ds}N \\ &= \kappa(\tau B - kT) + \frac{d\kappa}{ds}N \left(\because \frac{dN}{ds} = \tau B - \kappa T\right) \\ &= \kappa\tau B - \kappa^2 T + \frac{d\kappa}{ds}N\end{aligned}$$

$$\begin{aligned}\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) &= T \cdot \left[\kappa N \times \left(\kappa\tau B - \kappa^2 T + \frac{d\kappa}{ds}N\right)\right] \\ &= T \cdot \left[\kappa^2(N \times \tau B) - \kappa^3 \cdot (N \times T) + \kappa \cdot \frac{d\kappa}{ds}(N \times N)\right] \\ &= T \cdot \left[\kappa^2\tau(N \times B) - \kappa^3(-B) + \kappa\frac{d\kappa}{ds}(0)\right] (\because N \times T = -B, N \times N = 0) \\ &= T \cdot \left(\kappa^2\tau(T) + \kappa^3 B\right] \quad (\because N \times B = T) \\ &= \kappa^2\tau(T \cdot T) - \kappa^3(T \cdot B)(\because T \cdot T = 1 and T \cdot B = 0) \\ &= \kappa^2\tau(1) - \kappa^3(0) \\ &= \kappa^2\tau.\end{aligned}$$

We know that, radius of the curvature $'\rho'$ is the reciprocal of curvature κ. i.e

$$\rho = \frac{1}{\kappa} \Rightarrow \kappa = \frac{1}{\rho}$$

$$\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3}\right) = \frac{1}{\rho^2}\tau$$